II. INTRODUCTION

Bilayer quantum Hall systems at total filling factor $\nu_T = 1$ exhibit one of the most interesting many-body correlation effects: spontaneous interlayer phase coherence, which is solely caused by the Coulomb interaction in the limit of zero interlayer tunneling. Recently this phenomenon has received drastic experimental support by Spielman et al., who discovered a strong enhancement in the zero-bias interlayer tunneling conductance for interlayer distance $d < d_c \approx 1.83 l_0$ in the regime of very little tunneling, where $l_0 = \sqrt{hc/eB}$ is the magnetic length. These experiments strongly indicated a presence of spontaneous interlayer coherence at small interlayer distances.

The ground state in the limit of zero interlayer distance can be shown to be spontaneously interlayer phase coherent due to layer symmetry. One of the most convenient ways to see this is to use the pseudospin representation in which the top (bottom) layer is denoted as pseudospin $\uparrow$ ($\downarrow$). In the limit of zero interlayer separation, there is exact pseudospin SU(2) symmetry so that our bilayer system at $\nu_T = 1$ with pseudospin can be exactly mapped onto a single layer system $\nu = 1$ with real spin. Then, it is straightforward to see that the ground state should be a pseudospin ferromagnet in order to reduce the Coulomb energy cost, i.e., Hund’s rule. On the other hand, it has been known that, in the limit of large interlayer separation, the ground state is composed of two split composite fermion Fermi seas.

Therefore, it is clear that there should be a pseudospin phase transition at a critical interlayer distance $d_c$, since the ground states at small and large interlayer separations are not adiabatically connected. However, it is not at all clear what happens near $d_c$. In spite of intense theoretical efforts in the past, the true nature of phase transition near $d_c$ has been elusive. Analytical approaches based on Hartree-Fock theories were not able to reliably treat strong quantum fluctuations near the phase transition, which led to the erroneous prediction of broken translational symmetry. On the other hand, analytic approaches utilizing effective field theories were not directly based on the microscopic Hamiltonian of the many-body Coulomb interaction. So its validity should be justified by other means such as exact diagonalization. In particular, mapping of the original many-body Coulomb Hamiltonian to a spin model Hamiltonian is not expected to be valid near the critical interlayer separation. Previous numerical studies based on exact diagonalization, however, did not take into account fundamental fluctuations of interlayer number difference due to intrinsic, quantum-mechanical uncertainty in layer indices. Consequently those studies could not provide any direct information related to spontaneous interlayer phase coherence.

It is crucial at this point to distinguish between pseudospin ferromagnetic order and interlayer phase coherence. There is no fundamental reason to believe that they are identical because the interlayer phase coherence is a necessary condition, but not a sufficient condition for pseudospin ferromagnetism. We will show in this article that, at intermediate interlayer distances, there is another sort of long-range interlayer phase coherence due to pseudospin spiral ordering. It should be emphasized that this pseudospin spiral order is spontaneous in that it exists solely because of the Coulomb interaction without any external parallel magnetic field. The formation of pseudospin spiral order has been considered previously in the presence of parallel magnetic fields.

This article is organized as follows. We will begin by defining the Hamiltonian of bilayer quantum Hall systems in Sec. II. The pseudospin magnetization is computed as a function of interlayer distance $d$ in the limit of zero interlayer tunneling in Sec. III where the intrinsic quantum-mechanical fluctuation of interlayer number difference is fully taken into account, based on the analogy with superconductivity. Defined as the point where the pseudospin ferromagnetic order is destroyed, the critical interlayer distance $d_c$ is reliably determined. In Sec. IV we focus on the nature of the instability which destroys pseudospin ferromagnetic order at $d_c$. We will show that the instability is due to the formation of pseudospin spiral order which is manifested in the lowest energy excitations. By computing a new order parameter for pseudospin spiral, we will show in Sec. V that the ground state itself has the pseudospin spiral order. Finally, it is argued in Section VI that a signature of the phase transition from pseudospin ferromagnetic order to

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Spontaneous Pseudospin Spiral Order in the Bilayer Quantum Hall Systems

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(Dated: November 21, 2018)

Using exact diagonalization of bilayer quantum Hall systems at total filling factor $\nu_T = 1$ in the torus geometry, we show that there is a new long-range interlayer phase coherence due to spontaneous pseudospin spiral order at interlayer distances larger than the critical value at which the pseudospin ferromagnetic order is destroyed. We emphasize the distinction between the interlayer phase coherence and the pseudospin ferromagnetic order.

PACS numbers: 73.43.-f, 73.21.-b
our predicted pseudospin spiral order may have been already observed experimentally.

II. HAMILTONIAN

Let us begin by writing the Hamiltonian for the bilayer quantum Hall systems:

\[ H = H_t + \hat{P}_{LLL} V_{Coul} \hat{P}_{LLL} \]  

where \( \hat{P}_{LLL} \) is the lowest Landau level (LLL) projection operator. \( V_{Coul} \) represents the usual Coulomb interaction between electrons:

\[ \frac{V_{Coul}}{e^2/d_0} = \sum_{i,j \in \uparrow} \frac{1}{r_{ij}} + \sum_{k,l \in \downarrow} \frac{1}{r_{kl}} + \sum_{i \in \uparrow, k \in \downarrow} \frac{1}{\sqrt{r_{ik}^2 + (d/l_0)^2}} \]  

(2)

where \( l_0 = \sqrt{\hbar/eB} \) is the magnetic length, \( d \) is the interlayer distance, and \( r_{ij} \) is the lateral separation between the \( i \)-th and \( j \)-th electrons. In the above we have used a pseudospin representation to distinguish the top (\( \uparrow \)) and the bottom (\( \downarrow \)) layers. In general, we define the pseudospin operator as follows:

\[ \hat{S} = \frac{1}{2} \sum_m \epsilon_m^\dagger \sigma_{ab} c_{mb}^\dagger c_{mb} \]  

(3)

where \( \sigma \) is the usual Pauli matrix, and \( m \) denotes the LLL orbital index. Note that \( S_z \) is half the electron number difference between the two layers, and \( S_x \) is associated with interlayer tunneling. We take the real spin to be fully polarized.

The tunneling Hamiltonian \( H_t \) in Eq. (1) can be written as:

\[ H_t = \frac{t}{2} \sum_m \left( c_m^\dagger c_m + c_m^\dagger c_m \right) = -t S_x, \]  

(4)

where \( t \) is the single particle interlayer tunneling gap. Although Eq. (4) is valid for general \( t \), we are interested only in the limit of zero interlayer tunneling, i.e. \( t/(e^2/d_0) \to 0 \), which is appropriate when considering spontaneous interlayer coherence (note that the \( t \to 0 \) limit is not the same as the \( t = 0 \) situation). We analyze the Hamiltonian in Eq. (1) by using exact diagonalization (via a modified Lanczos method) in the torus geometry\(^{13}\).

III. SPONTANEOUS PSEUDOSPIN MAGNETIZATION

As mentioned in the beginning, it will be shown that there is a new long-range interlayer phase coherence due to pseudospin spiral order at sufficiently large \( d \). But, first, we should determine the critical interlayer distance, \( d_c \), at which pseudospin ferromagnetic order terminates. Of course, the most natural order parameter for the pseudospin ferromagnetic order is the pseudospin magnetization in the \( x \)-direction: \( \langle S_x \rangle \).

It is tempting to apply exact diagonalization techniques to finite systems with fixed number of electrons in each layer. But, \( S_x \) changes the interlayer number difference by unity (\( \Delta S_z = \pm 1 \)) so that the ground state expectation value is precisely zero when computed naively without any explicit interlayer tunneling. This problem has been addressed in an \textit{ad hoc} manner by introducing explicit interlayer tunneling\(^{14}\), which, however, severely obscures the effect of spontaneous phase coherence and may produce misleading results. The real solution is to realize that there is intrinsic, quantum-mechanical uncertainty in the layer index of electrons at small interlayer distance (even in the limit of zero tunneling) so that the true ground state \( |\psi\rangle \) is a linear combination of various states with different \( S_z \):

\[ |\psi\rangle = \sum_M \lambda_M |\phi_M\rangle, \]

(5)

where \( |\phi_M\rangle \) is the lowest energy state with \( S_z = M \), \( \lambda_M \) is a sharply peaked function of \( M \) with width \( \mathcal{O}(\sqrt{N}) \), and \( N \) is the number of electrons. It is important to remember that even the Halperin (1,1,1) state, \( |\psi_{(1,1,1)}\rangle \), has \( \lambda_M \propto \exp(-2M^2/N) \), which can be proved by analyzing the following, explicit wavefunction including both the orbital and pseudospin parts\(^{15}\):

\[ |\psi_{(1,1,1)}\rangle = \prod_m \left[ \frac{1}{\sqrt{2}} (c_m^\dagger + c_m^\dagger) \right] |0\rangle. \]

\( |\psi_{(1,1,1)}\rangle \) is the exact ground state at \( d = 0 \). Because it can be exactly mapped onto the BCS wavefunction, it is fruitful to consider an analogy with the BCS theory for general \( d \). Following the BCS theory, one can compute the pseudospin magnetization \( S_x \) as follows:

\[ \langle \psi | S_x | \psi \rangle = \sum_{M,M'} \lambda_M \lambda_{M'} \langle \phi_M | S_x | \phi_{M'} \rangle \approx \langle \phi_{M'+1} | S_x | \phi_{M'} \rangle \]

\[ \approx 2 \langle \phi_{M'+1} | S_x | \phi_{M'} \rangle \]

(7)

where the last two step are justified in the thermodynamic limit since \( \lambda_M \) is sharply peaked at \( M = M^* \). Although \( \langle S_x \rangle \) should be exactly \( N/2 \) at \( d = 0 \) in the thermodynamic limit due to the exact pseudospin \( SU(2) \) symmetry, there are some finite-system size corrections. After some algebra on the (1,1,1) state, Eq. (7) shows that \( \langle \psi_{(1,1,1)} | S_x | \psi_{(1,1,1)} \rangle \) is \( (N+1)/2 \) for \( N \) odd \( (M^* = -1/2) \), and is \( \sqrt{N(N+2)}/2 \) for \( N \) even \( (M^* = 0) \). Therefore, the pseudospin magnetization should be scaled as \( N + 1 \) \((\sqrt{N(N+2)})\) for \( N \) odd (even).

Fig. 1 shows the pseudospin magnetization per particle (defined in Eq. (5)) as a function of \( d/l_0 \) for various numbers of electrons. As conventional in finite system analysis, the critical interlayer distance can be obtained from the inflection point of \( \langle S_x \rangle \): \( d_c \approx 1.5d_0 \). Although \( d_c \) was estimated to be around 1.5\( d_0 \) previously in various approaches, it should be emphasized that the pseudospin
magnetization ($S_z$) is computed for the first time in this article without any interlayer tunneling. And therefore $d_c$ is reliably determined without any ambiguity.

IV. PSEUDOSPIN SPIRAL INSTABILITY

We now address the nature of instability causing the destruction of pseudospin ferromagnetic order. This question is, in turn, directly related to the nature of the new ground state at $d > d_c$. To be specific, we consider the lowest energy excitations which are responsible for the ground state instability. Fig. 2 shows the dispersion curves of the lowest energy excitations at $d/l_0 = 0.5$ and 1.5 for a finite system with the total number of electrons $N = 13$ and the number of pseudospin-up electrons $N_\uparrow = 7$ (or $N_\downarrow = 6$). There are several points to be emphasized.

First, there is clearly a linearly dispersing Goldstone mode at small interlayer distances, such as $d/l_0 = 0.5$. However, within numerical accuracy regarding the discreteness of finite-system wavevectors, the Goldstone mode seems to vanish for $d \gtrsim d_c$, which coincides with the destruction of pseudospin ferromagnetic order. Consistent with the conclusion from $\langle S_z \rangle$, this shows that the ground state undergoes a phase transition around $d = d_c$.

Second, contrary to the prediction of the random phase approximation (RPA) theory, there is no softening of the dispersion curve (roton) at any finite momentum for any interlayer distance. Remember that the RPA theory predicts a roton around $k_{l_0} \approx 1.4$, which was believed to cause an instability toward charge density wave order. This already shows that previous theories have some serious flaws in describing the phase transition at $d = d_c$. We will show in the next paragraph that there are completely different low-energy excitations at large $d/l_0$.

Third, several previous numerical studies were unfortunately based on the exact diagonalization of $N = 8$ system, which, under careful investigation, can be shown to predict a Wigner crystal at large $d$ rather than a quantum Hall liquid, which is a finite-system artifact of boundary condition and special number of electrons.

Fourth, these lowest-excitation energies are exactly zero in the $d \to \infty$ limit where there is no interaction between electrons with different pseudospins. However, the most crucial aspect of the dispersion curve is that there are completely new low-energy excitations due to the pseudospin spiral order, which are denoted by arrows in Fig. 2. The lowest energy state in the momentum channel $k_{l_0} = \sqrt{2\pi/N(N_\uparrow,0)}$ (denoted by arrows in Fig. 2) begins to be ripped out of the well-defined dispersion curve at $d/l_0 \approx 0.5$, and it becomes completely separated from all the other excitations at $d/l_0 \gtrsim 1.5$ so that it becomes really the lowest energy excitation among all momentum channels. The same behavior is found in all studied systems with different values of $N$ and $N_\uparrow$, with exception of the $N = 8$ system due to a finite-size artifact as mentioned in the above. Note that there are three other degenerate excitations in the momentum channels $k_{l_0} = \sqrt{2\pi/N(N_\uparrow,0)}$, $\sqrt{2\pi/N(0,N_\downarrow)}$, and $\sqrt{2\pi/N(0,N_\uparrow)}$ due to reflection symmetries and periodic boundary conditions of torus geometry. These peculiar excitations are completely different from usual rotons because (i) they form isolated points rather than a part of conventional dispersion curve, and (ii) their momentum increases as $|k|l_0 \propto \sqrt{N}$, which diverges in the thermodynamic limit. We will show later that these peculiar properties are precisely the properties of pseudospin spiral state.

In order to understand why these excitations are the pseudospin spiral states, it would be best if we first ex-
amine the following pseudospin spiral wavefunction:

$$|\psi_{\text{spiral}}(n)\rangle = \prod_m \left[ \frac{1}{\sqrt{2}} \left( c_{m+n,\uparrow}^\dagger + c_{m,\downarrow}^\dagger \right) \right]|0\rangle$$

(8)

where $m$ denotes the linear momentum in the Landau gauge, i.e., $p_x = 2\pi m/L$ and $L$ is the linear system size (note that $L = l_0 \sqrt{2\pi N}$ at $\nu_T = 1$). We will show later in Fig. 3 by explicitly computing the overlap between $|\psi_{\text{spiral}}(n = 1)\rangle$ and the lowest-energy excitations (denoted by arrows in Fig. 2) that the lowest-energy excitations are the pseudospin spiral states. But, first, we would like to emphasize that the pseudospin spiral state $|\psi_{\text{spiral}}(n)\rangle$ is the exact ground state in the presence of parallel magnetic field. A surprising result of this article is that, even without parallel magnetic field, spiral states are the lowest energy excitations which cause the instability of pseudospin ferromagnetic order.

To elucidate the physical meaning of pseudospin spiral state, examine Eq. (5) which reveals that $|\psi_{\text{spiral}}\rangle$ contains diagonal interlayer correlations between the states with $p_x = 2\pi (m + n)/L$ and $2m/L$. Remember that the interlayer correlation between $p_x = 2\pi (m + n)/L$ and $2m/L$ is identical to the diagonal interlayer correlation between electrons with different pseudospins which are separated laterally in the $y$-direction by $n l_0 \sqrt{2\pi N}$. In turn, this diagonal interlayer correlation is physically equivalent to the pseudospin spiral order with spiraling period of $L/n$ because the pseudospin part of wavefunction is essentially given by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i \frac{2\pi m}{L}} & 1 \end{pmatrix}.$$ 

(9)

Pseudospin spiral excitations (and therefore the diagonal interlayer correlation) are schematically depicted in Fig. 4.

Now we compute the momentum of pseudospin spiral states by directly applying the magnetic translation operator onto $|\psi_{\text{spiral}}(n)\rangle$. The explicit algebra shows that the momentum of $\psi_{\text{spiral}}(n)$ is $|k|_0 = n N_\uparrow \sqrt{2\pi N}/N$ which is exactly the momentum of the lowest-energy excitation seen in Fig. 2 if $n = 1$. Note that the momentum of this pseudospin spiral state is actually proportional to its spiraling period. That is the reason why the momentum of pseudospin spiral state with long spiraling period diverges in the thermodynamic limit. Similarly, we can explain why the pseudospin spiral state occurs in an isolated momentum channel. Consider two pseudospin spiral states with long period, say $n = 1$ and $n = 2$, whose momenta differ by a factor proportional to $\sqrt{N}/l_0$. Therefore, the pseudospin spiral states with two different long spiraling periods do not occur in adjacent momenta.

It is important to note that $|\psi_{\text{spiral}}(n = 1)\rangle$ is the pseudospin spiral excitation in the limit of long spiraling period ($= L$), which initiates the destruction of pseudospin ferromagnetic order. Therefore, it may be indicative of a second order phase transition.

As mentioned earlier, Fig. 3 shows the overlap between the pseudospin spiral state $|\psi_{\text{spiral}}(n = 1)\rangle$ in Eq. (5) and the lowest-energy excited state with $|k|_0 = \sqrt{2\pi N N_\uparrow}/20$. As one can see, the overlap is high overall at small $d/l_0$, and is very close to unity ($\simeq 0.97$) especially around $d/l_0 = 0.5$ which coincides with the point where $\langle S_\uparrow \rangle$ begins to deviate from the fully saturated value of $1/2$. This shows that the instability of pseudospin ferromagnetic order is indeed caused by pseudospin spiral order.
It should be noted that, though the overlap is lowered at larger \( d/l_0 \), the low energy excitation is in general adiabatically connected to the pseudospin spiral state. In fact, the reason for lower overlaps at larger \( d/l_0 \) is that the ground state has the pseudospin spiral order. Therefore, it would be satisfactory if, in addition to the diagonal interlayer correlation, \( T_n \) be scaled as \( N^2 \).

The following properties of \( T_n \) are worth mentioning. (i) \( T_n \) is a generalization of pseudospin magnetization \( S_z \) which measures the vertical interlayer correlation: \( T_{n=0} = S_z \). So the relationship between \( T_n \) and \( \psi_{\text{spiral}}(n) \) is analogous to that between \( S_z \) and \( \psi_{(1,1,1)} \).

(ii) When acting upon momentum eigenstates, \( T_n \) alters momentum of the eigenstates, though changes in momentum are negligible in the thermodynamic limit. So, in order to avoid any confusion, we compute \( \langle T_n^2 \rangle \) instead.

(iii) In order to be meaningful in the thermodynamic limit, \( \langle T_n^2 \rangle \) should be zero in the thermodynamic limit. However, there are some finite-system size corrections: \( \langle T_n^2 \rangle \neq 0 \) since there is no interlayer correlation in \( d/l_0 \). Of course, in the thermodynamic limit, \( \langle T_n^2 \rangle \) will be zero at \( d \to \infty \) since there is no interlayer correlation in the two split Fermi seas of composite fermions.

Finally, we would like to discuss the physical origin of pseudospin spiral order. First, let us remind ourselves that the pseudospin ferromagnetic state can be viewed as a condensate of neutral pairs of electrons and correlation holes which are bound directly across the interlayer separation. Now for sufficiently large \( d \) the ground state has the pseudospin spiral order, or equivalently diagonal interlayer correlation, which can be also viewed as a condensate of neutral electron-hole pairs with extended size and higher symmetry such as \( p \)-wave, as compared to the \( s \)-wave-like pair in pseudospin ferromagnet. Now, the origin of spiral order can be identified with the origin of extended pairs which is due to weak pairing. Of course, the pairing between electrons and correlation holes is weakened as \( d \) increases. Eventually the binding is completely lost when the split composite fermion Fermi seas are formed.

VI. CONCLUSIONS

We conclude by mentioning some experimental implications. Recently, Kellogg et al. observed an enhanced longitudinal drag resistance at \( d/l_0 \) as large as 2.6 where the enhanced tunneling conductance is absent\(^{24}\). Theoretically, tunneling is renormalized from the bare single particle tunneling gap \( t \) to the greatly enhanced renormalized tunneling \( t(S_z) \). Therefore, it is natural to assume that the enhanced tunneling conductance becomes absent as soon as the pseudospin ferromagnetic order is
destroyed at \( d = d_c \). On the other hand, we have shown that there is a new interlayer phase coherence due to the pseudospin spiral order for \( d > d_c \) which, we speculate, causes the longitudinal drag anomaly because of remaining interlayer correlation. Regarding the Hall resistance, our pseudospin spiral state is expected to be incompressible, as shown in the dispersion curve. However, the lowest energy gap is estimated to be very small so that the Hall plateau will be difficult to be observed in current temperature ranges and impurity concentrations.

The author is very grateful to S. Das Sarma for his careful reading of manuscript and his constant support throughout this work. The author is also indebted to E. Demler, S. M. Girvin, A. Kaminiski and V. W. Scarola for their insightful comments. This work was supported by ARDA.

1. I. B. Spielman, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 84, 5808 (2000).
2. J. K. Jain, Phys. Rev. Lett. 63, 199 (1989).
3. V. Kalmeyer and S.-C. Zhang, Phys. Rev. B 46, 9889 (1992).
4. B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B 47, 7312 (1993).
5. V. W. Scarola, and J. K. Jain, Phys. Rev. B 64, 085313 (2001).
6. H. A. Fertig, Phys. Rev. B 40, 1087 (1989).
7. A. H. MacDonald, P. M. Platzman, and G. S. Boebinger, Phys. Rev. Lett. 65, 775 (1990).
8. K. Moon, H. Mori, Kun Yang, S. M. Girvin, A. H. Mac-Donald, L. Zheng, D. Yoshioka, Shou-cheng Zhang, Phys. Rev. B 51, 5138 (1995).
9. Kun Yang, K. Moon, Lofti Belkhir, H. Mori, S. M. Girvin, A. H. MacDonald, L. Zheng, D. Yoshioka, Phys. Rev. B 54, 11644 (1996).
10. S. He, S. Das Sarma, and X. C. Xie, Phys. Rev. B, 47, 4394 (1993).
11. J. Schliemann, S. M. Girvin, and A. H. MacDonald, Phys. Rev. Lett. 86, 1849 (2001).
12. D.-W. Wang, E. Demler, and S. Das Sarma, cond-mat/0303324 (2003).
13. D. Yoshioka, B. I. Halperin, and P. A. Lee, Phys. Rev. Lett. 50, 1219 (1983); F. D. M. Haldane, Phys. Rev. Lett. 55, 2095 (1985).
14. As shown later, the ground state at large \( d \) acquires a pseudospin spiral order which is associated with diagonal interlayer correlation. Since this diagonal interlayer correlation creates a nonlocal correlation between laterally separated electrons (in top layer) and correlation holes (in bottom layer) for sufficiently large \( d \), it is no longer valid to map the original long-range Coulomb Hamiltonian to a usual spin model which is based on the existence of local spin operator. Therefore, there is no fundamental reason for the Goldstone mode at sufficiently large \( d \).
15. Remember that the usual representation of Halperin’s (1,1,1) state, \( \psi_{1,1,1} = \prod_{i<j}(z_i - z_j)\prod_{k<l}(z_k - z_l)\prod_{m\in \uparrow,n\in \downarrow}(z_m - z_n)\prod_p \exp(-|z_p|^2/4) \), depicts only the orbital part of full wavefunction which was given first in Ref.
16. B. I. Halperin, Helv. Phys. Acta 56, 75 (1983).
17. T. Chakraborty and P. Pietiläinen, Phys. Rev. Lett. 59, 2784 (1987).
18. We have proved that for \( N = 8 \) the energy dispersion in the first Brillouin zone has a pattern associated with rectangular-lattice Wigner crystal which is commensurate with rectangular unit cell. Also, this pattern persists even in the \( d \to \infty \) limit.
19. In terms of the viewpoint that the interlayer coherent state is a macroscopic condensate of neutral electron-hole pairs, the pseudospin spiral instability with long spiral period can be interpreted as a gradual change in the nature of neutral pairs from tightly bound s-wave-like pairing to a pairing with extended size and higher symmetry. And this continuous change in the nature of pairing may indicate a second order phase transition which is consistent with the viewpoint from the pseudospin spiral instability with long spiral period.
20. Note that in order to compute the overlap one should project \( |\psi_{\text{spiral}}\rangle \) into the Hilbert space of fixed \( S_z \) since \( |\psi_{\text{spiral}}\rangle \) is a linear combination of various \( S_z \) eigenstates similar to the (1,1,1) state.
21. M. Kellogg, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, cond-mat/0211502 (2002).