Emergent infinite-randomness fixed points from the extensive random bipartitions of the spin-1 Affleck-Kennedy-Lieb-Tasaki topological state

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Quantum entanglement under an extensive bipartition can reveal the critical boundary theory of a topological phase in the parameter space. In this study we demonstrate that the infinite-randomness fixed point for spin-1/2 degrees of freedom can emerge from an extensive random bipartition of the spin-1 Affleck-Kennedy-Lieb-Tasaki chain. The nested entanglement entropy of the ground state of the reduced density matrix exhibits a logarithmic scaling with an effective central charge $\tilde{c} = 0.72 \pm 0.02 \approx \ln 2$. We further discuss, in the language of bulk quantum entanglement, how to understand all phase boundaries and the surrounding Griffiths phases for the antiferromagnetic Heisenberg spin-1 chain with quenched disorder and dimerization.

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Entanglement spectrum under an extensive bipartition of a topological ground state has recently emerged as a novel approach to study the quantum phase transition between the topological phase and its trivial counterpart \[1–7\]. The so-called bulk entanglement spectrum (BES) reveals the boundary theory in the corresponding parameter space of a model system, rather than the edge states along the fictitious boundary in real space \[8, 9\]. These studies suggest that a triangular correspondence among the bulk theory, the edge theory, and the critical theory may exist generically for a topological phase \[6, 10\].

An instructive example is the spin-1 Haldane gapped phase \[11\], whose fixed-point properties are captured by the valence-bond solid picture of the Affleck-Kennedy-Lieb-Tasaki (AKLT) model Hamiltonian \[12\]. The corresponding exact ground state wave function can be expressed as a matrix product state (MPS), indicating that the fundamental entities of the spin-1 chain is actually the fractionalized spinons carrying spin 1/2 that form singlets across adjacent sites. The spin-1/2 object can be observed as the edge mode at the end of the chain, or at the end of a partition in the entanglement study. The degeneracy due to the fractionalized spin in the entanglement spectrum of the antiferromagnetic Heisenberg spin-1 chain is key to understand the Haldane phase as a symmetry protected topological phase \[13\]. Under a symmetric extensive bipartition with disjoint segments, the segment end spins coalesce into an emergent critical spin-1/2 chain with a central charge $c = 1 \pm 2$ \[2\], which describes the collapse of the Haldane phase under imposed dimerization (marked by the stars Fig. 1).

Meanwhile, spin chains can be interesting with a random probability distribution of the bond couplings between neighboring spins, which may be broaden without limit as the system is coarse-grained \[13\]. Such a system is governed by the infinite-randomness fixed point (IRFP), whose ground state is characterized by a random pattern of spin singlets formed over large spatial separations \[15, 16\]. For the antiferromagnetic Heisenberg spin-1 chain the Haldane phase is stable against weak disorder \[17, 18\] and weak dimerization \[19\], but the emergent critical spin-1/2 chain is not \[15, 16\]. As illustrated in Fig. 1 the Haldane phase and two dimer phases in the presence of quenched disorder are separated by the spin-1/2 random singlet (RS\textsubscript{1/2}) boundaries, which merges into a single spin-1 random singlet (RS\textsubscript{1}) boundary between the dimer phases. In addition, unusual Griffiths effects, characterized by two continuously varying dynamical exponents, appear near the boundaries \[20\].

In this paper we will show that disorder physics can also be revealed in the bulk entanglement study under an extensive random bipartition of the spin-1 chain.
ment length, the nearest-neighbor couplings in the bulk entanglement Hamiltonian exhibits a power-law distribution, which is precisely the fixed-point solution for random antiferromagnetic spin-$1/2$ chains under strong-disorder renormalization-group (SDRG) transformation \[^{[16]}\]. Therefore, the ground state of the entanglement Hamiltonian realizes the RS$_{1/2}$ state, which we further prove by fitting the ensemble-averaged nested entanglement entropy to a logarithmic scaling with an effective central charge $c = 0.72 \pm 0.02 \approx \ln 2$. We discuss how to vary the random bipartition to explore the phase diagram of the antiferromagnetic Heisenberg spin-1 chain with quenched disorder and dimerization.

The spin-1 AKLT parent Hamiltonian on a periodic chain of length $L$ is defined by \[^{[12]}\]

$$H_{\text{AKLT}} = \sum_{i=1}^{L} J \left[ s_i \cdot s_{i+1} + \frac{1}{3} (s_i \cdot s_{i+1})^2 \right],$$

whose exact ground state can be expressed as an MPS

$$|\Psi_{\text{AKLT}}\rangle = \sum_{\{s_i\}} \text{Tr}\left( A^{[s_1]} A^{[s_2]} \cdots A^{[s_L]} \right) |s_1 s_2 \cdots s_L\rangle,$$

for $J > 0$, where the local physical spin $s_i = -1, 0, +1$ and $A^{[s]}$ are local $2 \times 2$ matrices given, e.g., in Ref. \[^{[2]}\]. In the thermodynamic limit, the spin-spin correlation function decays exponentially with a correlation length $\xi = 1/\ln 3 \approx 0.91$, and any two spins that are separated by an even number of the lattice sites are antiferromagnetically correlated.

To study the quantum entanglement of the AKLT state, we can divide the chain into A and B partitions, and define the entanglement Hamiltonian $H_E$ through the reduced density matrix

$$\rho_A = \text{Tr}_B (|\Psi_{\text{AKLT}}\rangle \langle \Psi_{\text{AKLT}}|) \equiv e^{-H_E}.$$  

In a common practice, as illustrated in Fig. 2a, A is an open spin segment of length $l \geq 2$. The MPS representation dictates that $\rho_A$ contains four eigenvalues: a singlet and a triplet, as required by the SU(2) symmetry. Explicitly, the singlet and triplet eigenvalues are \[^{[21, 22]}\]

$$\Lambda_0 = \frac{1}{4} \left( 1 + 3 \left(-\frac{1}{3}\right)^l \right),$$

$$\Lambda_\alpha = \frac{1}{4} \left( 1 - \left(-\frac{1}{3}\right)^l \right), \alpha = 1, 2, 3.$$  

The corresponding entanglement spectrum can thus be generated by $H_E = J(l) \tau_L \cdot \tau_R$, where

$$J(l) = \ln \left[ \frac{1 + 3 \left(-\frac{1}{4}\right)^l}{1 - \left(-\frac{1}{4}\right)^l} \right]$$

is the effective (entanglement) coupling between two fractionized spin-$1/2$s $\tau_L$ and $\tau_R$ at the segment ends. Hence, they are coupled antiferromagnetically for even $l$, and ferromagnetically for odd $l$. We note that the bulk degrees of freedom of the spin-1 segment are not appearing at finite energies in the entanglement spectrum. If we swap A and B, we obtain the identitical entanglement Hamiltonian, which means that the coupling between the two end spins is independent of their being connected by a segment in A or by that in B.

On the other hand, the bulk entanglement Hamiltonian $H_E$, whose eigenvalues form the BES, is associated with the extensive bipartitions that break the spin chain into A and B sets of alternating segments (see Fig. 2b for an example). The word “bulk” emphasizes that the boundaries between A and B spread out the whole system, and the couplings between the boundary spin-$1/2$s are relevant. Under the uniform bipartition in the previous BES studies \[^{[2, 4, 5]}\], the resulting bulk entanglement Hamiltonian describes an effective uniform spin-$1/2$ chain, which is governed by a CFT with central charge $c = 1$. The theory is, however, unstable against arbitrarily small disorder.
A naive thinking to study the effect of quenched disorder is to randomize the bond couplings in the pure Hamiltonian, such that we modify the ground state wave function, hence the reduced density matrix. The AKLT case is, however, an exception. The entanglement spectrum is immune to weak bond randomness [i.e., if we replace uniform $J$ with random $J_i$ in Eq. (1)] as the exact ground state of the random-bond AKLT model is identical to that of the pure model [12]. This is a vivid example that the perturbation can influence thermal excitations but not the excitations in the entanglement spectrum.

This motivates us to enforce disorder by introducing an extensive random bipartition for the AKLT state, as illustrated in Fig. 2. For the initial simplicity, we assume that the number of lattice sites in each segment is even. After tracing out the degrees of freedom in every other segments, we expect a bulk entanglement Hamiltonian that describes the physics of segment end spin-1/2s with random antiferromagnetic couplings. On physical ground we may then expect that the low-energy physics is governed by the IRFP. Normally, the fixed point is revealed by a real-space decimation process, commonly referred to as the SDRG, developed by Dasgupta and Ma [13] and by Fisher [14]. In general, the SDRG process in the initial stage depends strongly on the probability distribution of the random couplings, thus often drives the energy scale of interest to be exponentially small than the strongest bond in the bare Hamiltonian [23]. Surprisingly, this is not the case here because of the uncorrelated locations of the segment ends (which we impose) and the finite correlation length in the topological phase.

The natural recipe to place the segment ends is to specify a fixed average length of the segments and to assume that the segment ends are located independently. If we suppose that the average length of the segment is large, one can show that the probability distribution of the segment length $l$ (assumed to be continuous for simplicity) satisfies the following differential equation

$$\frac{dP(l)}{dl} = -\frac{P(l)}{l},$$

where the constant $l$ is the average length of the segments. We can further assume that the segments have a minimum length of $l_0$, such that

$$P(l) \sim \left(\frac{1}{l}\right)e^{-(l-l_0)/\xi},$$

The discreteness of the segment length is not important when the average segment length is sufficiently long. As often emerged in the minimum mathematical model of the life expectancy problem, the exponential form of the probability distribution simply means that the occurrence of the next segment end is independent of the location of the previous one.

As we discussed above, the energy scale that couples two adjacent segment end spin-1/2s depends on the length of the segment. When the segment length $l$ (measured by the number of sites) is sufficiently long the effective coupling [Eq. (4)] asymptotically approaches

$$J(l) \simeq (-1)^l J_0 e^{-l/\xi},$$

where $\xi = 1/\ln 3$ is the correlation length and $J_0 = 4$ the energy unit. For even $l$ the effective couplings are all antiferromagnetic. We point out, though, including ferromagnetic bonds can lead to a different SDRG fixed point with large spins formed in a random-walk fashion [22]. Couplings between further neighboring end spin-1/2s can be neglected as they are exponentially small compared to those between nearest neighbors.

The length scales in the two previous exponential laws should in general be different. The former is the average segment length enforced externally, while the latter is the internal correlation length associated with the topological phase. Together, the probability distribution of the effective nearest-neighbor couplings reads

$$P(J) = \frac{1}{\Omega} \left(\frac{\Omega}{J}\right)^{1+\Gamma},$$

where $\Gamma = l/\xi$ and $\Omega = J_0 e^{-l_0/\xi}$. One can readily recognize Eq. (8) as the celebrated fixed-point solution of the power-law bond distribution in the SDRG formulated by Fisher [16].

The result is striking. We have not done any SDRG calculation, yet we obtained the SDRG fixed-point solution. In Fisher’s formulation Eq. (8) is the solution of the master equation for the SDRG flow of the distribution function,

$$\frac{\partial P(\beta)}{\partial \Gamma} = \frac{\partial P(\beta)}{\partial \beta}$$

$$+ P(0) \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 P(\beta_1) P(\beta_2) \delta_{\beta_1+\beta_2-\beta},$$

where we follow Fisher to introduce the dimensionless scaling variable $\beta = \ln(\Omega/J)$. The differential equation states that during the bond decimation the flow of the bond distribution has two contributions: (i) a shift in $\beta$ due to the reduction of the UV cutoff $\Omega$, and (ii) the replacement of decimated bonds by the effective couplings generated through the second-order perturbation theory as the logarithmic RG flow parameter $\Gamma$ changes. In the present context the UV cutoff $\Omega$ is set by the ratio of the shortest segment length $l_0$ and the average segment length, while the RG flow parameter $\Gamma$ is the ratio of the external average segment length to the internal correlation length $\xi$. Consequently, the random bipartition effectively sets up the bond distribution that satisfies the SDRG flow equation.

When approaching the fixed point in SDRG, the width of the bond distribution function Eq. (8) ($\sim 1/J$) grows without limit, which means that the system flow to infinite randomness. In our setup this points our primary interest to the regime where $l$ is large, hence our assumption that the discreteness of the segment length is not important and our simplification of the length dependence of the effective couplings $J(l)$ between the segment end spins are justified. In the same limit the choice of $l_0$, which sets the UV cutoff, is also not important. To demonstrate that the IRFP is indeed accessible even in small systems, we study the nested entanglement entropy, a straightforward generalization of the uniform case [2], and compare it to the entanglement entropy of random $S = 1/2$
Heisenberg chains, which is controlled by an effective central charge \( \tilde{c} = \ln 2 \) \[^{25}\]. For this purpose, we first compute \( \rho_A \), whose matrix product representation is illustrated in Fig. \[^{23}\], and its ground state. We further divided the subsystem \( A \) with \( n \) segments of a total \( L_A \) spin-1s into left partition (\( p \) segments) and right partition (\( n - p \) segments)

\[
\begin{align*}
&l_1, l_2, \ldots, l_p, \bar{l}_p+1, \ldots, l_n \\
&\Downarrow \sum_{i=1}^{l} = L_A - l
\end{align*}
\]  

(9)

with \( l \) and \( L_A - l \) spins, respectively. Under this nested bipartition of \( A \), we calculate the nested entanglement entropy \( s(l, L_A) \) for the ground state of \( \rho_A \). We average over random realizations of the segment lengths in \( A \) and \( B \), along with the same number of segments and the same pattern of the nested bipartition, and plot the averaged \( s(l, L_A) \) as a function of sample averaged \( \ln g(l, L_A) = \ln[(L_A/\pi) \sin(\pi l/L_A)] \) in Fig. \[^{3}\] for samples with 4-12 segments of random length prescribed by the discrete version of Eq. \[^{6}\] with \( l_0 = 2 \) and \( \bar{l} = 5 \gg \xi \). Depending on segment number, we choose 2,000-10,000 random realizations. For comparison we also show data for the uniform bipartition with two spins in each segment \[^{2}\]. The data in the random case can be fitted by

\[
\begin{align*}
\tilde{c} = \frac{1}{3} \ln \left( \frac{L_A}{\pi} \sin \left( \frac{\pi l}{L_A} \right) \right),
\end{align*}
\]  

(10)

where the effective central charge \( \tilde{c} = 0.72 \pm 0.02 \) is in excellent agreement with the expected value of \( \ln 2 \approx 0.693 \), as opposed to \( c = 1 \) in the uniform case \[^{2}\]. The result is remarkable as the largest system contains a mere 12 segments with average segment length \( \bar{l} = 5 \) and no SDRG is involved. The numerical result also demonstrates that the longer-range couplings among the segment end spin-1/2s and the correlations between the couplings are indeed irrelevant \[^{16}\].

The distribution of the segment lengths can introduce additional relevant perturbations for the emergent degrees of freedom. The distribution of the segment lengths can introduce additional relevant perturbations for the emergent degrees of freedom. One example is that the difference between the average segment lengths in \( A \) and \( B \) can introduce an effective dimerization \( \delta \) for the RS\(_{1/2} \) state. Unlike the pure case, \( \delta \) can be continuously tuned in the random case. The resulting bulk entanglement Hamiltonian effectively describes random spin-1/2 chains with weak but continuously varying bond dimerization. This can lead to critical behavior but with finite spin correlation length controlled by \( \delta \), the characteristics of a Griffiths phase \[^{26}\]. On the other hand, the random antiferromagnetic \( S = 1 \) chains with enforced dimerization also contain spin-1 degrees of freedom, which, together with the spin-1/2 degrees of freedom, result in Griffiths phases with two independent dynamical exponents. The entanglement analogy, thus, requires that partition \( A \) (or \( B \)) should contain segments with odd length. We leave the details aside but point out that one can compare the extensive random bipartition case to the domain-wall description of Damle and Huse \[^{27}\], hence the relevant physics will follow.

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It is interesting to point out that a spin-1 random singlet phase without dimerization \[^{17}\] can also be accessed, e.g., by a bipartition (see Fig. \[^{23}\]) that contains single sites in \( A \) (creating effective spin-1s) and segments with random but even length in \( B \) (providing random antiferromagnetic couplings between effective spin-1s). This, however, does not fit the scenario that we discussed above, because the RS\(_{1} \) state, as illustrated in Fig. \[^{1}\] is a critical line that is separated from the Haldane phase (including the Griffiths region) by a multicritical point. The multicritical point has an emergent permutation symmetry corresponding to the interchange of the Haldane phase and the dimer phases that meet at the point \[^{27}\]. We propose the intriguing possibility that a special family of the extensive random bipartitions with both even and odd seg-

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\[^{1}\] Such entanglement entropy calculations are also known for the critical XXZ chain \[^{4}\].

\[^{2}\] SDRG (scale-dependent renormalization group).

\[^{3}\] C = \ln 2.

\[^{4}\] Cfr. Damle and Huse (2007) Phys. Rev. Lett. 98, 156402.
ment lengths in one partition (e.g., in A) can also access the multicutical point.

In summary, we have discussed how to understand the critical phase boundaries of the antiferromagnetic Heisenberg spin-1 chain in the presence of quenched disorder and dimerization in terms of quantum entanglement by applying extensive bipartitions to the AKLT state that represents the topological Haldane phase. In particular, the calculation of the effective central charge provides a striking example how efficient quantum entanglement can access the critical information of a topological quantum phase transition.

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