Global well-posedness and scattering for the fourth order nonlinear Schrödinger equations with small data

Hua Zhang

School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China
E-mail: zhanghuamaths@163.com (H. Zhang)

Abstract: For $n \geq 3$, we study the Cauchy problem for the fourth order nonlinear Schrödinger equations, for which the existence of the scattering operators and the global well-posedness of solutions with small data in Besov spaces $B^s_{2,1}(\mathbb{R}^n)$ are obtained. In one spatial dimension, we get the global well-posedness result with small data in the critical homogeneous Besov spaces $\dot{B}^{s}_{2,1}$. As a by-product, the existence of the scattering operators with small data is also obtained. In order to show these results, the global version of the estimates for the maximal functions and the local smoothing effects on the fourth order Schrödinger semi-groups are established.

Keywords: Fourth order nonlinear Schrödinger equations, the Cauchy problem, estimates for the maximal functions, global well-posedness, small data.

MSC 2000: 35Q55, 35G25, 35A07.

1 Introduction

In the present paper, we consider the Cauchy problem for the fourth order nonlinear Schrödinger equations with derivatives (4NLS)

$$iu_t + \Delta^2 u - \varepsilon \Delta u = F((\partial_x^{|\alpha|} u)_{|\alpha| \leq 3}, (\partial_x^{|\alpha|} \bar{u})_{|\alpha| \leq 3}), \quad u(0, x) = u_0(x),$$

where $\varepsilon \in \{0, 1\}$, $u$ is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\Delta u = -\mathcal{F}^{-1} |\xi|^2 \mathcal{F} u, \quad \Delta^2 u = \mathcal{F}^{-1} |\xi|^4 \mathcal{F} u,$$

$F : \mathbb{C}^{\frac{n+2n^2+n^2}{3}+\frac{1}{2}} \rightarrow \mathbb{C}$ is a polynomial of the form

$$F(z) = P(z_1, ..., z^{\frac{1}{3}n^3+2n^2+n^2}) = \sum_{m+1 \leq |\beta| \leq M+1} c_\beta z^\beta, \quad c_\beta \in \mathbb{C} \quad (1.2)$$

$m, M \in \mathbb{N}$ will be given below.

The fourth order nonlinear Schrödinger equation, including its special forms, arise in deep water wave dynamics, plasma physics, optical communications (see [6]). A large amount of work has been devoted to the Cauchy problem of dispersive equations, such as [1, 3–5, 7, 8, 10, 12–16, 18–27, 29] and references therein. In [25], by using the method of Fourier restriction norm, Segata studied a special fourth order nonlinear Schrödinger equation in one dimensional space. And the results have been improved in [13, 26].

In order to study the influence of higher order dispersion on solitary waves, instability and the collapse phenomena, Karpman introduced a class of nonlinear
Schrödinger equations (see [14])

\[ i\Psi_t + \frac{1}{2} \Delta \Psi + \frac{\gamma}{2} \Delta^2 \Psi + f(|\Psi|^2)\Psi = 0. \]

In [1], Ben-Artzi, Koch and Saut discussed the sharp space-time decay properties of fundamental solutions to the linear equation

\[ i\Psi_t - \varepsilon \Delta \Psi + \Delta^2 \Psi = 0, \quad \varepsilon \in \{-1, 0, 1\}. \]

In [8], Guo and Wang considered the existence and scattering theory for the Cauchy problem of nonlinear Schrödinger equations with the form

\[ iu_t + (-\Delta)^m u + f(u) = 0, \quad u(0, x) = \varphi(x). \quad (1.3) \]

where \( m \geq 1 \) is an integer. Pecher and Wahl in [24] proved the existence of classical global solutions of (1.3) for the space dimensions \( n \leq 7m \) for the case \( m \geq 1 \).

In [10], Hao, Hsiao and Wang discussed the local well-posedness of the Cauchy problem (1.3) for the large initial data for \( m = 2 \) and \( f(u) = P((\partial^2_x u)_{|\alpha| \leq 2}, (\partial^2_x \bar{u})_{|\alpha| \leq 2}) \).

In [11], Hao, Hsiao and Wang considered the equation

\[ i\partial_t u = \Delta^2 - \varepsilon \Delta u + P((\partial^2_x u)_{|\alpha| \leq 2}, (\partial^2_x \bar{u})_{|\alpha| \leq 2}), \quad u(0, x) = \varphi(x). \quad (1.4) \]

in multi-dimensional cases with \( \varepsilon = -1, 0, 1 \), where they obtained the local well-posedness for the Cauchy problem (1.4) for the large initial data.

In [20], Kenig, Ponce and Vega studied the nonlinear Schrödinger equation of the form

\[ \partial_t u = i\Delta u + P(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \]

and proved that the corresponding Cauchy problem is locally well-posed for small data in the Sobolev spaces \( H^s(\mathbb{R}^n) \) and in its weighted version by pushing forward the linear estimates associated with the Schrödinger group \( \{e^{it\Delta}\}_{t \in \mathbb{R}} \) and by introducing suitable function spaces where these estimates act naturally. They also studied generalized nonlinear Schrödinger equations in [21] and quasi-linear Schrödinger equations in [22]. In one dimensional case, the smallness assumption on the size of the data was removed by Hayashi and Ozawa [12] by using a change of variables to obtain an equivalent system with a nonlinear term independent of \( \partial_x u \), where the new system could be treated by the standard energy method. In [4], Chihara was able to remove the size restriction on the data in any dimensions by using an invertible classical pseudo-differential operator of order zero.

In [33], Wang and Wang discussed

\[ iu_t + \Delta_{\pm} u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0, x) = u_0(x) \quad (1.5) \]

where \( \Delta_{\pm} u = \sum_{i=1}^n \varepsilon_i \partial^2_{x_i} u \) and \( \varepsilon_i \in \{-1, 1\} \). They established an estimate for the global maximal function and proved (1.5) is global well-posed. Moreover, the existence of the scattering operators to (1.5) with small data in Besov space \( B^s_{2,1}(\mathbb{R}^n) \) are obtained.

In this paper, we mainly use the dispersive smoothing effects of the linear Schrödinger equation(cf. Kenig, Ponce and Vega [20, 21]). The crucial point is that the fourth order Schrödinger semi-groups has the following local smoothing effects for \( n \geq 2 \);

\[ \sup_{\alpha} \|e^{it(\Delta^2 - \varepsilon \Delta)}u_0\|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \lesssim \|u_0\|_{H^{\frac{n}{2}}} \quad (1.6) \]
Moreover, for $\|u\|_{B^s_{2,1}} \leq \tilde{\delta}$ for $n \geq 3$, or $\|u\|_{B^s_{2,1} \cap H^{1/2} \leq \delta}$ for $n = 3, 4$, where $\delta > 0$ is a suitably small number, then (1.1) has a unique global solution $u \in C(\mathbb{R}, B^s_{2,1}) \cap X_0$, where

$$X_0 = \left\{ u : \|D^\beta u\|_{L^2_t L^s_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \delta, |\beta| \leq 3 \right\}.$$ 

Moreover, for $n \geq 5$, the scattering operator of (1.1) carries the ball $\{u : \|u\|_{B^s_{2,1}} \leq \delta\}$ into $B^s_{2,1}$.

(2) If $s + 9/2 \in \mathbb{N}$ and $\|u_0\|_{H^s} \leq \delta$ for $n \geq 3$, or $\|u_0\|_{H^s \cap H^{1/2} \leq \delta}$ for $n = 3, 4$, where $\delta > 0$ is a suitably small number, then (1.1) has a unique global solution $u \in C(\mathbb{R}, H^s) \cap X$, where

$$X = \left\{ u : \|D^\beta u\|_{L^\infty_t L^s_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \delta, |\beta| \leq s + 3/2 \right\}.$$ 

Moreover, for $n \geq 5$, the scattering operator of (1.1) carries the ball $\{u : \|u\|_{H^s} \leq \delta\}$ into $H^s$.

Next, we consider one spatial dimension case. Denote $s_k = \frac{1}{2} - \frac{4}{k}, \quad \tilde{s}_k = \frac{1}{2} - \frac{1}{k}$.

**Theorem 1.2.** Let $n = 1$, $\varepsilon = 0$, $M \geq m \geq 8$, and $u_0 \in B^{3+\tilde{s}_m}_{2,1} \cap B^{s_m}_{2,1}$. Assume that there exists a small $\delta > 0$ such that $\|u_0\|_{B^{3+\tilde{s}_m}_{2,1} \cap B^{s_m}_{2,1}} \leq \delta$. Then (1.1) has a global solution $u \in X = \{u \in \mathcal{S}'(\mathbb{R}^{1+1}) : \|u\|_X \lesssim \delta\}$, where

$$\|u\|_X = \sup_{s_m \leq s \leq \tilde{s}_m} \sum_{i=0}^3 \sum_{j \in \mathbb{Z}} \|\partial_x^i \Delta_j u\|_s \text{ for } m > 8,$$

$$\|u\|_X = \sum_{i=0}^3 \|\partial_x^i u\|_{L^\infty_t L^2_x \cap L^{10}_{t,x}} + \sup_{s_0 \leq s \leq \tilde{s}_M} \sum_{j \in \mathbb{Z}} \|\partial_x^j \Delta_j u\|_s \text{ for } m = 8,$$

$$\|\Delta_j v\|_s := 2^s (\|\Delta_j v\|_{L^\infty_t L^2_x \cap L^{10}_{t,x}} + 2^{s_0} \|\Delta_j v\|_{L^4_x L^2_t}) + 2^{(s-s_m)} \|\Delta_j v\|_{L^\infty_t L^4_x} + 2^{(s-s_m)} \|\Delta_j v\|_{L^\infty_t L^6_x}.$$
Recall that the norm on homogeneous Besov spaces $\dot{B}_{2,1}^s$ can be defined in the following way:

$$\|f\|_{\dot{B}_{2,1}^s} = \sum_{j=-\infty}^{\infty} 2^{sj} \left( \int_{2^j |\xi| < 2^{j+1}} |\mathcal{F} f(\xi)|^2 d\xi \right)^{1/2}.$$ 

### 1.2 Notations

Throughout this paper, we will always use the following notations. $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ stand for the Schwartz space and its dual space, respectively. We denote by $L^p(\mathbb{R}^n)$ the Lebesgue spaces with norms $\| \cdot \|_p := \| \cdot \|_{L^p(\mathbb{R}^n)}$. The Bessel potential space is defined by $H^s_p(\mathbb{R}^n) := (I - \Delta)^{-s/2} L^p(\mathbb{R}^n)$, $\dot{H}^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$, $\dot{H}^s(\mathbb{R}^n) = (-\Delta)^{-s/2} L^2(\mathbb{R}^n)$.

For any quasi-Banach space $X$, we denote by $X^*$ its dual space, by $L^p(I, X)$ the Lebesgue-Bochner space, $\|f\|_{L^p(I, X)} := (\int_I \|f(t)\|^p_X dt)^{1/p}$. If $X = L^p(\Omega)$, then we write $L^p(I, L^p(\Omega)) = L^p_t L^p_x(I \times \Omega)$ and $L^p_{t,x}(I \times \Omega) = L^p_t L^p_x(I \times \Omega)$. Let $Q_\alpha$ be the unit cube with center at $\alpha \in \mathbb{Z}^n$, i.e., $Q_\alpha = \alpha + Q_0$, $Q_0 = \{x = (x_1, \ldots, x_n) : -1/2 \leq x_i < 1/2\}$. We also need the function space $\ell^q_\alpha(L^p_t L^q_x(I \times Q_\alpha))$ with the norm

$$\|f\|_{\ell^q_\alpha(L^p_t L^q_x(I \times Q_\alpha))} := \left( \sum_{\alpha \in \mathbb{Z}^n} \|f\|_{L^p_t L^q_x(I \times Q_\alpha)}^q \right)^{1/q}.$$

We denote by $\mathcal{F}$ ($\mathcal{F}^{-1}$) the (inverse) Fourier transform for the spatial variables; by $\mathcal{F}_t$ ($\mathcal{F}_t^{-1}$) the (inverse) Fourier transform for the time variable and by $\mathcal{F}_{t,x}$ ($\mathcal{F}_{t,x}^{-1}$) the (inverse) Fourier transform for both time and spatial variables, respectively. If there is no additional explanation, we always denote by $\varphi_k(\cdot)$ the dyadic decomposition functions as in (1.9); and by $\sigma_k(\cdot)$ the uniform decomposition functions as in (1.11). $u \ast v$ and $u \ast v$ will stand for the convolution on time and on spatial variables, respectively, i.e.,

$$(u \ast v)(t, x) = \int_{\mathbb{R}} u(t - \tau, x) v(\tau, x) d\tau, \quad (u \ast v)(t, x) = \int_{\mathbb{R}^n} u(t, x - y) v(t, y) dy.$$

Symbols $\mathbb{R}, \mathbb{N}$ and $\mathbb{Z}$ will stand for the sets of real numbers, natural numbers and integers, respectively. $c < 1$, $C > 1$ will denote positive universal constants, which may be different at different places. $a \lesssim b$ stands for $a \leq Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We denote by $p'$ the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. For any $a > 0$, we denote by $[a]$ the minimal integer that is larger than or equals to $a$. $B(x, R)$ will denote the ball in $\mathbb{R}^n$ with center at $x$ and radial $R$. We denote $S_c(t) = e^{it(\Delta^2 + \Delta)}$ and $S_c f = \int_0^1 S_c(t - \tau) f(\tau) d\tau$.

### 1.3 Besov spaces

Let us recall that Besov spaces $B_{p,q}^s := B_{p,q}^s(\mathbb{R}^n)$ are defined as follows (cf. [2,28]). Let $\psi : \mathbb{R}^n \to [0, 1]$ be a smooth radial bump function adapted to the ball $B(0, 2)$:

$$\psi(\xi) = \begin{cases} 
1, & |\xi| \leq 1, \\
\text{smooth}, & |\xi| \in (1, 2), \\
0, & |\xi| \geq 2. 
\end{cases} \quad (1.8)$$

$^1\mathbb{R}^n$ will be omitted in the definitions of various function spaces if there is no confusion.
We write $\delta(\cdot) := \psi(\cdot) - \psi(2\cdot)$ and

$$\varphi_j := \delta(2^{-j}\cdot) \quad \text{for} \quad j \geq 1; \quad \varphi_0 := 1 - \sum_{j \geq 1} \varphi_j.$$  \hfill (1.9)

We say that $\Delta_j := \mathcal{F}^{-1}(\varphi_j \mathcal{F})$, $j \in \mathbb{N} \cup \{0\}$ are the dyadic decomposition operators. Besov spaces $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$ are defined in the following way:

$$B^s_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} = \left( \sum_{j=0}^{\infty} 2^{sjq} \|\Delta_j f\|_p^q \right)^{1/q} < \infty \right\}. \hfill (1.10)$$

Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ and $\rho : \mathbb{R}^n \to [0,1]$ be a smooth radial bump function adapted to the ball $B(0,\sqrt{n})$, say $\rho(\xi) = 1$ as $|\xi| \leq \sqrt{n}/2$, and $\rho(\xi) = 0$ as $|\xi| \geq \sqrt{n}$. Let $\rho_k$ be a translation of $\rho$: $\rho_k(\xi) = \rho(\xi - k)$, $k \in \mathbb{Z}^n$. We write (see [31–34])

$$\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n. \hfill (1.11)$$

We can define the space $\ell^1_{\Delta} \ell^q_{\alpha}(L_t^p L_x^q(I \times Q_\alpha))$ with the following norm:

$$\|f\|_{\ell^1_{\Delta} \ell^q_{\alpha}(L_t^p L_x^q(I \times Q_\alpha))} := \sum_{j=0}^{\infty} 2^{sj} \left( \sum_{\alpha \in \mathbb{Z}^n} \|\Delta_j f\|_{L_t^p L_x^q(I \times Q_\alpha)}^q \right)^{1/q}. \hfill (1.12)$$

A special case is $s = 0$, $\ell^1_{\Delta} \ell^q_{\alpha}(L_t^p L_x^q(I \times Q_\alpha)) = \ell^1 \ell^q(\mathbb{R}^n)$ and $\ell^1 \ell^q(\mathbb{R}^n) = \ell^1 \ell^0(\mathbb{R}^n) = \ell^1 \ell^0(I \times Q_\alpha)$. The rest of this paper is organized as follows. In Section 2, we give the details of the estimates for the maximal function in certain function spaces. Section 3 is devoted to consider the spatial local versions for the Strichartz estimates and giving some remarks on the estimates of the local smoothing effects. In Sections 4-5, we prove our main Theorems 1.1-1.2 respectively.

## 2 Estimates for the maximal function

We give some estimates for the maximal function related to the fourth order Schrödinger semi-groups, an earlier time-local version is due to Kenig, Ponce and Vega [19]. We give some time-global versions by using a different approach. These estimates are crucial in the proof of Theorems 1.1-1.2 respectively.

### 2.1 Time-local version

Recall that $S_\varepsilon(t) = e^{it(\Delta^2 - \varepsilon \Delta)} = \mathcal{F}^{-1}e^{-it(|\xi|^4 + \varepsilon |\xi|^2)}\mathcal{F}$, where

$$|\xi|^4 = \left( \sum_{j=1}^{n} \xi_j^2 \right)^2, \quad |\xi|^2 = \sum_{j=1}^{n} \xi_j^2.$$ 

Hao, Hsiao and Wang [11] established the following maximal function estimate

$$\left( \sum_{\alpha \in \mathbb{Z}^n} \|S_\varepsilon(t)u_0\|_{L_t^\infty L_x^2([0,T] \times Q_\alpha)}^2 \right)^{1/2} \lesssim C(T)\|u_0\|_{H^s},$$

where $s > n + 1/2$, $T \in (0,1]$. 


2.2 Time-global version

Recall that we have the following equivalent norm on Besov spaces [2, 28]:

**Lemma 2.1.** Let \( 1 \leq p, q \leq \infty, \sigma > 0, \sigma \notin \mathbb{N} \). Then we have

\[
\|f\|_{B^\sigma_{p,q}} \sim \sum_{|\beta| \leq |\sigma|} \|D^\beta f\|_{L^p(\mathbb{R}^n)} + \sum_{|\beta| \leq |\sigma|} \left( \int_{\mathbb{R}^n} |h|^{-n-q(\sigma)} \|\Delta_h D^\beta f\|_{L^q(\mathbb{R}^n)}^q dh \right)^{1/q},
\]

where \( \Delta_h f = f(\cdot + h) - f(\cdot) \), \([\sigma]\) denotes the minimal integer that is larger than or equals to \( \sigma \), \( \{\sigma\} = \sigma - [\sigma] \).

Taking \( p = q \) in Lemma 2.1 we get

\[
\|f\|_{B^\sigma_{p,p}} \sim \sum_{|\beta| \leq |\sigma|} \|D^\beta f\|_{L^p(\mathbb{R}^n)} + \sum_{|\beta| \leq |\sigma|} \left( \int_{\mathbb{R}^n} |h|^{-n-p(\sigma)} \|\Delta_h D^\beta f\|_{L^p(\mathbb{R}^n)}^p dh \right)^{1/p}.
\]

**Lemma 2.2.** Let \( 1 < p < \infty, s > 1/p \). Then we have

\[
\left( \sum_{\alpha \in \mathbb{Z}^n} \|u\|_{L^\infty_x(L^p_{L^q}(\mathbb{R} \times Q_\alpha))} \right)^{1/p} \lesssim \|(I - \partial^2_t)^{s/2} u\|_{L^p(\mathbb{R}, B^s_{p,p}(\mathbb{R}^n))}.
\]

The proof of Lemma 2.2 can be found in [33].

For the case \( \varepsilon = 1 \), we recall some results of Guo and Wang [9]. For \( S_1(t) = e^{it(\Delta^2 - \Delta)} \), we have

**Proposition 2.3.** Assume \( 2 \leq p \leq \infty, \frac{1}{p} + \frac{1}{p'} = 1, 1 \leq q \leq \infty, \delta = \frac{1}{2} - \frac{1}{p}, -2n\delta \leq s' - s \), then

\[
\|S_1(t)g\|_{B^s_{p,q}} \lesssim k(t)\|g\|_{B^{s'}_{p',q}},
\]

where

\[
k(t) = \begin{cases} |t|^{\frac{1}{2} \min(s'-s-2n\delta,0)}, & 0 < t \leq 1, \\ |t|^{-n\delta}, & t > 1. \end{cases}
\]

In particular, if we choose \( s' = s = 0 \) in the above proposition, then we have

\[
\|S_1(t)g\|_{B^0_{p,q}} \lesssim k(t)\|g\|_{B^0_{p',q}},
\]

where

\[
k(t) = \begin{cases} |t|^{-\frac{1}{2} n\delta}, & 0 < t \leq 1, \\ |t|^{-n\delta}, & t > 1. \end{cases}
\]

Combining (2.1) with the Strichartz estimate in [9], we have an especial version proposition of Guo, Peng and Wang [9].

**Proposition 2.4.** Let \( A_1 f = \int_0^t S_1(t - \tau) f(\tau) d\tau \), for \( 2 \leq p, q \leq \infty, \frac{n}{2} (\frac{1}{2} - \frac{1}{p}) < \frac{2}{q} < n(\frac{1}{2} - \frac{1}{p}) \),

\[
\|S_1(t) f\|_{L^q_t L^p_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|f\|_{L^2},
\]

\[
\|A_1 f\|_{L^q_t L^p_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|f\|_{L^{q'}_{t'} L^{p'}_{x'}(\mathbb{R} \times \mathbb{R}^n)},
\]

\[
\|A_1 f\|_{L^q_t L^p_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|f\|_{L^{q'}_{t'} L^{p'}_{x'}(\mathbb{R} \times \mathbb{R}^n)},
\]

\[
\|A_1 f\|_{L^q_t L^p_x(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|f\|_{L^{q'}_{t'} L^{p'}_{x'}(\mathbb{R} \times \mathbb{R}^n)}.
\]
When $\varepsilon = 1$, the endpoint Strichartz estimate also hold. In fact, we have

**Proposition 2.5.** Let $n \geq 5$, $2 \leq p$, $\rho \leq 2n/(n-4)(2 \leq p$, $\rho < \infty$, if $n = 4$), $4/\gamma(\cdot) = n(1/2 - 1/\cdot)$. We have

\[
\|S_1(t)u_0\|_{L^{\gamma(p)}(\mathbb{R},L^{\rho}(\mathbb{R}^n))} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}, \quad (2.2)
\]

\[
\|\mathcal{A}_1F\|_{L^{\gamma(p)}(\mathbb{R},L^{\rho}(\mathbb{R}^n))} \lesssim \|F\|_{L^{\gamma(p)}(\mathbb{R},L^{\rho}(\mathbb{R}^n))}. \quad (2.3)
\]

**Proof.** Because the proof is similar to Keel and Tao [17], we only give the sketch. Let $2^{**} = 2n/(n-4)$, we need to prove

\[
\|S_1(t)u_0\|_{L^2(\mathbb{R},L^{2^{**}})} \lesssim \|u_0\|_{L^2} \quad (2.4)
\]

let $u_0 = \varphi$, for $\psi \in C^\infty$. According to dual estimate, we need to prove

\[
| \int_{-T}^T (S_1(t)\varphi,\psi(t))dt | \lesssim \|\varphi\|_{L^2}\|\psi\|_{L^2(-T,T;L^{2^{**}})} \quad (2.5)
\]

By the $TT^*$ method and symmetry, (2.5) is in turn equivalent to the bilinear form estimate

\[
|L(F,G)| \lesssim \|F\|_{L^2(-T,T;L^{2^{**}})}\|G\|_{L^2(-T,T;L^{2^{**}})} \quad (2.6)
\]

Here $L(F,G) = \int \int_D (S_1(t-s)F(s),G(t))dsdt, D = \{(s,t) \in [-T,T]^2, s \leq t\}$.

Now we take a procedure which is suit our proof, one can see Wang [30] for details. We decompose $L(F,G)$ dyadically as $\sum_j L_j(F,G)$. Here

\[
L_j(F,G) = \int \int_{D_j} (S_1(t-s)F(s),G(t))dsdt \quad (2.7)
\]

For simplicity, we assume $F,G \in C_0^\infty(I), I = [-T,T]$. From Proposition 2.3 for $\frac{1}{p} + \frac{1}{p'} = 1, 2 \leq p \leq \infty$, we have

\[
\|S_1(t)u_0\|_p \lesssim (|t|^\frac{2}{p} + |t|^\frac{2}{p'} - 1)\|u_0\|_{p'} \quad (2.8)
\]

Using (2.8), similar to lemma 4.1 in [17], we can prove

\[
|L_j(F,G)| \lesssim ((2^jT)^{\beta_1(a,b)} + (2^jT)^{\beta_2(a,b)})^{-1}\|F\|_{L^2_j(L^{\rho})}\|G\|_{L^2_j(L^{\rho'})} \quad (2.9)
\]

holds for all $j \in \mathbb{Z}$ and all $(\frac{1}{2}, \frac{1}{2})$ in a neighbourhood of $(\frac{1}{2}, \frac{n}{2})$.

Here $\beta_1(a,b) = \frac{2}{\gamma(b)} + \frac{1}{2\gamma(\frac{2}{a})} - 1$ and $\beta_1(a,b) = \frac{4}{\gamma(b)} + \frac{1}{2\gamma(\frac{2}{a})} - 1$. For later bilinear interpolation, we need

**Lemma 2.6. ( [2], Section 3.13.5(b)) If $A_0, A_1, B_0, B_1, C_0, C_1$ are Banach spaces, and the bilinear operator $T$ is bounded from

\[
T: A_0 \times B_0 \to C_0
\]

\[
T: A_0 \times B_1 \to C_1
\]

\[
T: A_1 \times B_0 \to C_1
\]

then whenever $0 < \theta_0, \theta_1 < 1, 1 \leq p, q, r \leq \infty$ are such that $1 \leq \frac{1}{p} + \frac{1}{q}$ and $\theta = \theta_0 + \theta_1$, one has

\[
T: (A_0, A_1)_{\theta_0, pr} \times (B_0, B_1)_{\theta_1, qr} \to (C_0, C_1)_{\theta, r}
\]
To get the result we need, let \( A_0 = B_0 = L_1^2(L_x^{a_0}) \) and \( A_1 = B_1 = L_1^2(L_x^{a_1}) \). We should make the following equations hold

\[
\begin{align*}
\frac{1}{(2^r)^s} &= \frac{1-\theta_1}{a_0} + \frac{\theta_1}{a_1}, \\
\theta_1 + \theta_2 &= \theta.
\end{align*}
\]

There are enough space to choose \( \theta_1, \theta_2 \). For example, we can select \( \theta_1 = \theta_2 = \frac{1}{3} \) to make the above equations hold.

Moreover, we also need

\[
\begin{align*}
\beta_2(a_0, a_0)(1 - \eta) + \beta_2(a_0, a_1)\eta &= 0, \\
\eta_1 + \eta_2 &= \eta.
\end{align*}
\]

the situation is similar, we omit it. \( \square \)

For the semi-group \( S_0(t) \), we have the following Strichartz estimate (cf. [17]):

**Proposition 2.7.** Let \( n \geq 4, 2 \leq p, \rho \leq 2n/(n - 4)(2 \leq p, \rho < \infty, if \ n = 4), 4/\gamma(\cdot) = n(1/2 - 1/\gamma) \). We have

\[
\begin{align*}
\|S_0(t)u_0\|_{L_t^\gamma(\mathbb{R}, L_x^p(\mathbb{R}^n))} &\lesssim \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{2.10} \\
\|\partial_0 F\|_{L_t^\gamma(\mathbb{R}, L_x^p(\mathbb{R}^n))} &\lesssim \|F\|_{L_t^\gamma(\mathbb{R}, L_x^p(\mathbb{R}^n))}. \tag{2.11}
\end{align*}
\]

If both \( p \) and \( \rho \) are equal to \( 2n/(n - 4) \), then (2.10) and (2.11) are said to be the endpoint Strichartz estimates. Using Propositions 2.4-2.7, we have

**Proposition 2.8.** Let \( 2^* = 2 + 8/n \). For any \( p \geq 2^*, s > n/2 \), we have

\[
\left( \sum_{\alpha \in \mathbb{Z}^n} \|S_\varepsilon(t)u_0\|_{L_{t,x}^p(\mathbb{R}^n)}^p \right)^{1/p} \lesssim \|u_0\|_{H^s}.
\]

**Proof.** For short, we write \( (\partial_t) = (I - \partial_0^2)^{1/2} \). By Lemma 2.2, for any \( s_0 > 1/2^* \),

\[
\left( \sum_{\alpha \in \mathbb{Z}^n} \|S_\varepsilon(t)u_0\|_{L_{t,x}^p(\mathbb{R}^n)}^p \right)^{1/p} \lesssim \left( \sum_{\alpha \in \mathbb{Z}^n} \|S_\varepsilon(t)u_0\|_{L_{t,x}^{2^*}(\mathbb{R}^n)}^{2^*} \right)^{1/2^*} \lesssim \|\langle \partial_t \rangle^{s_0} S_\varepsilon(t)u_0\|_{L_t^{2^*}(\mathbb{R}, B_{2^*,2^*}^{s_0}(\mathbb{R}^n))}. \tag{2.12}
\]

We have

\[
\|\langle \partial_t \rangle^{s_0} S_\varepsilon(t)u_0\|_{L_t^{2^*}(\mathbb{R}, B_{2^*,2^*}^{s_0}(\mathbb{R}^n))} = \sum_{k=0}^{\infty} 2^{ns_0k2^*} \|\langle \partial_t \rangle^{s_0} \Delta_k S_\varepsilon(t)u_0\|_{L_{t,x}^{2^*}(\mathbb{R}^{1+n})}.
\]

Using the dyadic decomposition to the time-frequency, we obtain that

\[
\|\langle \partial_t \rangle^{s_0} \Delta_k S_\varepsilon(t)u_0\|_{L_t^{2^*}(\mathbb{R}^{1+n})} \lesssim \sum_{j=0}^{\infty} \|\mathcal{F}_{t,x}^{-1} \langle \tau \rangle^{s_0} \varphi_j(\tau) \mathcal{F}_{t,x} e^{it(\xi^4 + \varepsilon|\xi|^2)} \varphi_k(\xi) \mathcal{F}_{t,x}u_0\|_{L_{t,x}^{2^*}(\mathbb{R}^{1+n})}.
\]

For convenience, we let \( h_\varepsilon(\xi) = |\xi|^4 + \varepsilon|\xi|^2 \). Observing that

\[
(\mathcal{F}_{t}^{-1} \langle \tau \rangle^{s_0} \varphi_j(\tau)) * e^{ith_\varepsilon(\xi)} = ce^{ith_\varepsilon(\xi)}(\mathcal{F}_{t}^{h_\varepsilon(\xi)} \varphi_j(h_\varepsilon(\xi)) \langle \mathcal{F}_{t}^{h_\varepsilon(\xi)} \rangle^{s_0}, \tag{2.13}
\]
and using the Strichartz’s inequality and Plancherel’s identity, we get
\[
\| (\partial_t)^{s_0} \triangle_k S_\varepsilon(t) u_0 \|_{L_t^2 x^2 (R^{1+n})} \lesssim \sum_{j=0}^{\infty} \| S_\varepsilon(t) F_{-1} (h_\varepsilon(\xi)) \psi_j (h_\varepsilon(\xi)) \psi_k (\xi) F_{x} u_0 \|_{L_t^2 x^2 (R^{1+n})} \\
\lesssim \sum_{j=0}^{\infty} \| F_{-1} (h_\varepsilon(\xi)) \psi_j (h_\varepsilon(\xi)) \psi_k (\xi) F_{x} u_0 \|_{L_t^2 x^2 (R^n)} \\
\lesssim 2^{4 j s_0} \sum_{j=0}^{\infty} \| F_{-1} (h_\varepsilon(\xi)) \psi_j (h_\varepsilon(\xi)) \psi_k (\xi) F_{x} u_0 \|_{L_t^2 x^2 (R^n)}. \tag{2.14}
\]

Combining (2.12) with (2.13), together with Minkowski’s inequality, we have for any \( \rho > 0 \),
\[
\sum_{j=0}^{\infty} \| F_{-1} \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{B_{2,2}^{s_0+4 \rho}} \lesssim \sum_{j=0}^{\infty} \| F_{-1} \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{H^{(n+4) s_0 + 4 \rho}} \\
\lesssim \left( \sum_{j=0}^{\infty} 2^{4 j \rho} \| F_{-1} \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{H^{(n+4) s_0 + 4 \rho}}^2 \right)^{1/2}. \tag{2.15}
\]

In view of \( H^{(n+4) s_0} \subset B_{2,2}^{s_0} \) and Hölder’s inequality, we have for any \( \rho > 0 \),
\[
\sum_{j=0}^{\infty} \| F_{-1} \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{B_{2,2}^{s_0+4 \rho}} \lesssim \sum_{j=0}^{\infty} \| F_{-1} \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{H^{(n+4) s_0 + 4 \rho}} \\
\lesssim \left( \sum_{j=0}^{\infty} 2^{4 j \rho} \| F_{-1} \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{H^{(n+4) s_0 + 4 \rho}}^2 \right)^{1/2}. \tag{2.16}
\]

By Plancherel’s identity and the fact that \( \text{supp} \ \psi_j (h_\varepsilon(\xi)) \subset \{ \xi : h_\varepsilon(\xi) \in [2^{j-1}, 2^{j+1}] \} \), we easily see that
\[
\left( \sum_{j=0}^{\infty} 2^{4 j \rho} \| F_{-1} \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{H^{(n+4) s_0 + 4 \rho}}^2 \right)^{1/2} \lesssim \left( \sum_{j=0}^{\infty} \| (h_\varepsilon(\xi))^\rho \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{H^{(n+4) s_0 + 4 \rho}}^2 \right)^{1/2} \\
\lesssim \left( \sum_{j=0}^{\infty} \| \psi_j (h_\varepsilon(\xi)) F_{x} u_0 \|_{H^{(n+4) s_0 + 4 \rho}}^2 \right)^{1/2} \\
\lesssim \| u_0 \|_{H^{(n+4) s_0 + 4 \rho}}. \tag{2.17}
\]

Taking \( s_0 \) such that \( (n + 4) s_0 + 4 \rho < s \), from (2.14)+ (2.17) we have the result, as desired.

By the sharp inclusion \( H^s \subset L^\infty \) for \( s > n/2 \), it is obvious that Proposition 2.8 is optimal in the sense that it does not hold for \( s = n/2 \).

Using the ideas as in Lemma 2.2 and Proposition 2.8, we can show

**Proposition 2.9.** Let \( 2^* \leq p, q, r \leq \infty \), \( s_0 > 1/2^* - 1/q \) and \( s_1 > n(1/2^* - 1/r) \). Then we have
\[
\left( \sum_{\alpha \in \mathbb{Z}^n} \| S_\varepsilon(t) u_0 \|_{L_t^p (R, L^r (Q_\alpha))}^p \right)^{1/p} \lesssim \| u_0 \|_{H^{s_1 + 4 s_0}}. \tag{2.18}
\]
In particular, for any \( q, p \geq 2^{*}, s > n/2 - 4/q \), it holds
\[
\left( \sum_{\alpha \in \mathbb{Z}^n} \| S_{\varepsilon}(t)u_0 \|^p_{L^1_t(L^\infty(Q_{\alpha}))} \right)^{1/p} \lesssim \| u_0 \|_{H^s}.
\]

**Sketch of the Proof.** In view of \( \ell^2 \subset \ell^p \), it suffices to consider the case \( p = 2^{*} \). Using the inclusions \( H^s_p(\mathbb{R}) \subset L^q(\mathbb{R}) \) and \( B^s_{p,p}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n) \), we have
\[
\| u \|_{L^q(\mathbb{R}, L^r(Q_{\alpha}))} \lesssim \| (I - \partial_t^2)^{s_0/2} \sigma_{\alpha} u \|_{L^p(\mathbb{R}, B^s_{p,p}(\mathbb{R}^n))}.
\]  
(2.19)

Using the same way as in Lemma 2.2 we can show that
\[
\left( \sum_{\alpha \in \mathbb{Z}^n} \| u \|^p_{L^q(\mathbb{R}, L^r(Q_{\alpha}))} \right)^{1/p} \lesssim \| (I - \partial_t^2)^{s_0/2} u \|_{L^p(\mathbb{R}, B^s_{p,p}(\mathbb{R}^n))}.
\]  
(2.20)

One can repeat the procedures as in the proof of Lemma 2.2 to conclude that
\[
\sum_{\alpha \in \mathbb{Z}^n} \| (I - \partial_t^2)^{s_0/2} \sigma_{\alpha} S_{\varepsilon}(t)u \|^p_{L^p(\mathbb{R}, B^s_{p,p}(\mathbb{R}^n))} \lesssim \sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j(h_{\varepsilon}(\xi)) \mathcal{F}u_0 \|_{H^{s_1+4\sigma_0}(\mathbb{R}^n)}.
\]  
(2.21)

Applying an analogous way as in the proof of Proposition 2.8 we can get
\[
\sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j(h_{\varepsilon}(\xi)) \mathcal{F}u_0 \|_{H^{s_1+4\sigma_0}(\mathbb{R}^n)} \lesssim \| u_0 \|_{H^{s_1+4\sigma_0+4\sigma}}.
\]  
(2.22)

Combining (2.20) with (2.22), we immediately get (2.18).

\[ \square \]

### 3 Global-local estimates on time-space

#### 3.1 Time-global and space-local Strichartz estimates

We need to make some modifications to the Strichartz estimates, which are global in the time variable and local in spatial variables. We always denote by \( S_{\varepsilon}(t) \) and \( \mathcal{A}_{\varepsilon} \) the fourth order Schrödinger semi-group and the integral operator as before.

**Proposition 3.1.** For \( n \geq 5 \), we have
\[
\sup_{\alpha \in \mathbb{Z}^n} \| S_{\varepsilon}(t)u_0 \|_{L^2_t(L^\infty(\mathbb{R} \times Q_{\alpha}))} \lesssim \| u_0 \|_2,
\]
(3.1)
\[
\sup_{\alpha \in \mathbb{Z}^n} \| \mathcal{A}_{\varepsilon} F \|_{L^2_t(\mathbb{R} \times Q_{\alpha})} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \| F \|_{L^1_t L^2_x(\mathbb{R} \times Q_{\alpha})},
\]
(3.2)
\[
\sup_{\alpha \in \mathbb{Z}^n} \| \mathcal{A}_{\varepsilon} F \|_{L^2_t L^2_x(\mathbb{R} \times Q_{\alpha})} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \| F \|_{L^1_t L^2_x(\mathbb{R} \times Q_{\alpha})}.
\]
(3.3)

**Proof.** In view of Hölder’s inequality and the endpoint Strichartz estimate, we have
\[
\| S_{\varepsilon}(t)u_0 \|_{L^2_t(\mathbb{R} \times Q_{\alpha})} \lesssim \| S_{\varepsilon}(t)u_0 \|_{L^2_t L^{2n/(n-4)}(\mathbb{R} \times Q_{\alpha})}
\lesssim \| S_{\varepsilon}(t)u_0 \|_{L^2_t L^{2n/(n-4)}(\mathbb{R} \times \mathbb{R}^n)}
\lesssim \| u_0 \|_{L^2(\mathbb{R}^n)}.
\]

Using the above ideas and the following Strichartz estimate
\[
\| \mathcal{A}_{\varepsilon} F \|_{L^2_t L^{2n/(n-4)}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| F \|_{L^1_t L^2(\mathbb{R} \times \mathbb{R}^n)},
\]
\[
\| \mathcal{A}_{\varepsilon} F \|_{L^2_t L^{2n/(n-4)}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| F \|_{L^2_t L^{2n/(n-4)}(\mathbb{R} \times \mathbb{R}^n)},
\]
one can easily get (3.2) and (3.3).

\[ \square \]
Remark 3.2. Using Proposition 2.4, we can verify that Proposition 3.1 also holds for \( n = 3, 4 \) when only consider the case \( \epsilon = 1 \). For example, taking \( q = 2, p = 8 \) in Proposition 2.4 when \( n = 3 \) we can get the result desired.

Since the endpoint Strichartz estimates are used in the above proof, Proposition 3.1 only holds for \( n \geq 5 \). It is not clear for us whether (3.1) holds or not for \( n = 3, 4 \) when consider \( S_0(t) \). It is why we need an additional condition that \( u_0 \in \dot{H}^{-3/2} \) is small in the case \( n = 3, 4 \). However, we have the following (cf. [19])

Proposition 3.3. Let \( n = 3, 4 \). Then we have for any \( 1 \leq r < 6/5 \) for \( n = 3 \), or \( 1 \leq r < 4/3 \) for \( n = 4 \)

\[
\sup_{\alpha \in \mathbb{Z}^n} \| S_\epsilon(t)u_0 \|_{L^2_t L^{2r}_x(\mathbb{R} \times Q_\alpha)} \lesssim \min(\| (\Delta)^{-3/4} u_0 \|_{L^2(\mathbb{R}^n)}, \| u_0 \|_{L^2 \cap L^r(\mathbb{R}^n)}). \tag{3.4}
\]

Observing (3.4) is strictly weaker than (3.1) in the low frequencies.

Proof. By Remark 3.2 and Lemma 3.6 it suffices to show that

\[
\sup_{\alpha \in \mathbb{Z}^n} \| S_0(t)u_0 \|_{L^2_t L^r_x(\mathbb{R} \times Q_\alpha)} \lesssim \| u_0 \|_{L^2 \cap L^r(\mathbb{R}^n)}. \tag{3.5}
\]

Using the unitary property in \( L^2 \) and the \( L^p-L^{p'} \) decay estimate of \( S_0(t) \), we have

\[
\sup_{\alpha \in \mathbb{Z}^n} \| S_0(t)u_0 \|_{L^2_x(\mathbb{R}^n)} \lesssim (1 + |t|)^{-3/4} \| u_0 \|_{L^2 \cap L^r(\mathbb{R}^n)}. \tag{3.6}
\]

Taking the \( L^2_t \) norm in both sides of (3.6), we immediately get (3.5).

Proposition 3.4. Let \( n = 3, 4 \). Then we have

\[
\sup_{\alpha \in \mathbb{Z}^n} \| A_\epsilon F \|_{L^2_t L^r_x(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \| F \|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)}. \tag{3.7}
\]

Proof. Firstly, the case \( \epsilon = 1 \) holds by Remark 3.2. Secondly, we notice that

\[
\sup_{\alpha \in \mathbb{Z}^n} \| S_0(t)u_0 \|_{L^2_x(\mathbb{R}^n)} \lesssim (1 + |t|)^{-3/4} \| u_0 \|_{L^2 \cap L^r(\mathbb{R}^n)}. \tag{3.8}
\]

It follows that

\[
\| A_0 F \|_{L^2_x(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}} (1 + |\tau|)^{-3/4} \| F(\tau) \|_{L^1 \cap L^2(\mathbb{R}^n)} d\tau.
\]

Using Young’s inequality, one has that

\[
\| A_0 F \|_{L^2_x(\mathbb{R} \times Q_\alpha)} \lesssim \| F \|_{L^1(\mathbb{R}, L^2 \cap L^2(\mathbb{R}^n))}. \tag{3.9}
\]

In view of Hölder’s inequality, (3.9) yields the result, as desired.

Remark 3.5. For \( n = 2 \), we can’t establish the similar result as in Proposition 3.4. So our results can’t cover this case.
3.2 Note on the time-global and space-local smoothing effects

Kenig, Ponce and Vega [18, 19] obtained the local smoothing effect estimates for the Schrödinger semi-group \( e^{it\Delta} \), and their results can also be developed to the fourth order Schrödinger semi-group \( e^{it(\Delta^2-c\Delta)} \). On the basis of their results and Proposition 3.1, we can obtain a time-global version of the local smoothing effect estimates with the inhomogeneous differential operator \((I - \Delta)^{1/2}\) instead of the homogeneous one \(\nabla\), which is useful to control the low frequencies parts of the nonlinearity.

**Lemma 3.6** (cf. [18]). Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( \phi \) be a \( C^1(\Omega) \) function such that \( \nabla \phi(\xi) \neq 0 \) for any \( \xi \in \Omega \). Assume that there is a \( N \in \mathbb{N} \) such that for any \( \xi := (\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1} \) and \( r \in \mathbb{R} \), the equation \( \phi(\xi_1, ..., \xi_k, x, \xi_{k+1}, ..., \xi_{n-1}) = r \) has at most \( N \) solutions. For \( a(x, s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}) \) and \( f \in \mathcal{S}'(\mathbb{R}^n) \), we denote

\[
W(t)f(x) = \int_\Omega e^{i(t\phi(\xi)+x\xi)}a(x, \phi(\xi))\hat{f}(\xi)d\xi,
\]

then for \( n \geq 2 \), we have

\[
\|W(t)f(x)\|_{L^1_t,L^2_x(\mathbb{R} \times B(0,R))} \leq CNR^{1/2}\|\nabla \phi\|^{-1/2}_{L^2(\Omega)} \|f\|_{L^2(\mathbb{R} \times B(0,R))}, \quad (3.8)
\]

**Corollary 3.7.** Let \( n \geq 5 \), \( S_\varepsilon(t) = e^{it(\Delta^2-c\Delta)} \). We have

\[
\sup_{\alpha \in \mathbb{Z}^n} \|S_\varepsilon(t)u_0\|_{L^2_tL^2_x(\mathbb{R} \times \Omega)} \lesssim \|u_0\|_{H^{3/2}},
\]

\[
\|\mathcal{A}_\varepsilon f\|_{L^\infty(\mathbb{R},H^{3/2})} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \|f\|_{L^2_tL^2_x(\mathbb{R} \times \Omega)},
\]

(3.10)

For \( n = 2, 3, 4 \), \( (3.10) \) also holds if one substitutes \( H^{3/2} \) by \( \dot{H}^{3/2} \).

**Proof.** Let \( \Omega = \mathbb{R}^n \setminus B(0,1) \), \( \phi(\xi) = |\xi|^4 + \varepsilon |\xi|^2 \) and \( \psi \) be as in (1.8), \( a(x, s) = 1 - \psi(s) \) in Lemma 3.6. Taking \( W(t) := S_\varepsilon(t)\mathcal{A}^{-1}(1-\psi)\mathcal{F} \), from (3.8) we have

\[
\sup_{\alpha \in \mathbb{Z}^n} \|S_\varepsilon(t)\mathcal{F}^{-1}(1-\psi)\mathcal{F}u_0\|_{L^2_tL^2_x(\mathbb{R} \times \Omega)} \lesssim \|\psi|^{-3/2}u_0\|_{L^2_\varepsilon(\mathbb{R}^n \setminus B(0,1))}, \quad (3.11)
\]

Using Proposition 3.1 we have

\[
\|S_\varepsilon(t)\mathcal{F}^{-1}\psi\mathcal{F}u_0\|_{L^2_tL^2_x(\mathbb{R} \times \Omega)} \lesssim \|\mathcal{F}^{-1}\psi\mathcal{F}u_0\|_{L^2_\varepsilon(\mathbb{R}^n)} \lesssim \|u_0\|_{L^2_\varepsilon(\mathbb{R}^n \setminus B(0,2))}, \quad (3.12)
\]

Combining (3.11) with (3.12) we have (3.9), as desired. (3.10) is the dual version of (3.9).

Using the method of Kenig, Ponce and Vega [19] and a modification of Hao, Hsiao and Wang [11], we can prove the following local smoothing effect estimates for the inhomogeneous part of the solutions of the fourth order Schrödinger equation.

Now we turn to consider the inhomogeneous Cauchy problem:

\[
i\partial_tu = \Delta^2u - \varepsilon\Delta u + F(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n,
\]

\[
u(0, x) = 0,
\]

(3.13)

(3.14)

with \( F \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^n) \) and \( \varepsilon = 0, 1 \). We have the following estimate on the local smoothing effects in this inhomogeneous case.
Proposition 3.8 (Local smoothing effect: inhomogeneous case). For any multi-index \( \nu, |\nu| = 3 \), the solution \( u(t, x) \) of the Cauchy problem \((3.13)-(3.14)\) satisfies
\[
\sup_{\alpha \in \mathbb{Z}^n} \|D^\nu u(t, x)\|_{L^2_t L^2_x(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \|F\|_{L^2_t L^2_x(\mathbb{R} \times Q_\alpha)}.
\]
(3.15)

Proof. Separating
\[
F = \sum_{\alpha \in \mathbb{Z}^n} F \chi_{Q_\alpha} = \sum_{\alpha \in \mathbb{Z}^n} F_\alpha,
\]
and
\[ u = \sum_{\alpha \in \mathbb{Z}^n} u_\alpha, \]
where \( u_\alpha(t, x) \) is the corresponding solution of the Cauchy problem
\[
i \partial_t u_\alpha = \Delta^2 u_\alpha - \varepsilon \Delta u_\alpha + F_\alpha(t, x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n,
\]
(3.16)
\[
u_\alpha(0, x) = 0,
\]
(3.17)
we formally take Fourier transform in both variables \( t \) and \( x \) in the equation \((3.16)\) and obtain
\[
\hat{u}_\alpha(\tau, \xi) = \frac{\hat{F}_\alpha(\tau, \xi)}{\tau - \varepsilon |\xi|^2 - |\xi|^4}, \quad \text{for each } \alpha \in \mathbb{Z}^n.
\]
By the Plancherel theorem in the time variable, we can get
\[
\sup_{\alpha \in \mathbb{Z}^n} \|D^\nu \mathcal{F} u_\beta(t, x)\|_{L^2_t L^2_x(\mathbb{R} \times Q_\alpha)} = \sup_{\alpha \in \mathbb{Z}^n} \|D^\nu \mathcal{F} u_\beta(t, x)\|_{L^2_t L^2_x(\mathbb{R} \times Q_\alpha)}
\]
\[
= \sup_{\alpha \in \mathbb{Z}^n} \|D^\nu \mathcal{F}^{-1} \left( \frac{\hat{F}_\beta(\tau, \xi)}{\tau - \varepsilon |\xi|^2 - |\xi|^4} \right) \|_{L^2_t L^2_x(\mathbb{R} \times Q_\alpha)}
\]
\[
= \sup_{\alpha \in \mathbb{Z}^n} \|\mathcal{F}^{-1} \left( \frac{\xi^\nu}{\tau - \varepsilon |\xi|^2 - |\xi|^4} \hat{F}_\beta(\tau, \xi) \right) \|_{L^2_t L^2_x(\mathbb{R} \times Q_\alpha)}
\]
\[
= \sup_{\alpha \in \mathbb{Z}^n} \left( \int_{Q_\alpha} \int_{-\infty}^{\infty} |\mathcal{F}^{-1} \left( \frac{\xi^\nu}{\tau - \varepsilon |\xi|^2 - |\xi|^4} \hat{F}_\beta(\tau, \xi) \right) |^2 d\tau dx \right)^{1/2}.
\]
(3.18)

In order to continue the above estimate, we introduce the following estimate.

Lemma 3.9. Let \( \mathcal{M}f = \mathcal{F}^{-1} m(\xi) \mathcal{F} f \) and \( m(\xi) = \frac{\xi^\nu}{1-\varepsilon\tau - \frac{1}{2}|\xi|^2 - |\xi|^4} \) for \( \tau > 0 \) and \( \varepsilon \in \{0, 1\}, |\nu| = 3 \), where \( \mathcal{F}(\mathcal{F}^{-1}) \) denotes the Fourier (inverse, respectively) transform in \( x \) only. Then, we have
\[
\sup_{\alpha \in \mathbb{Z}^n} \left( \int_{Q_\alpha} |\mathcal{M}(g\chi_{Q_\beta})|^2 dx \right)^{1/2} \lesssim CR \left( \int_{Q_\beta} |g|^2 dx \right)^{1/2}.
\]

where \( R \) is the size of \( Q_\alpha \).

Proof. Since the proof is similar to Hao, Hsiao and Wang [11], we omit it. \( \square \)
Now, we go back to the proof of Proposition 3.8. We first consider the part when \( \tau > 0 \) in (3.18), i.e.,

\[
\sup_{\alpha \in \mathbb{Z}^n} \left( \int_{Q_0} \int_0^\infty |\mathcal{F}_\xi^{-1} \left( \frac{\xi^\nu}{(\tau - \varepsilon |\xi|)^2 - |\xi|^4} \mathcal{F}_\beta(\tau, \xi) \right) |^2 d\tau dx \right)^{1/2}
\]

\[
= \left( \int_0^\infty \tau \frac{\mathcal{N}}{4} \sup_{\alpha \in \mathbb{Z}^n} \int_{\frac{\tau}{4} Q_\alpha} |\int_{\mathbb{R}^n} e^{i\eta \xi} \frac{\eta^\nu}{1 - \varepsilon \tau^{-1/2} |\eta|^2 - |\eta|^4} \mathcal{F}_\beta(\tau, \tau^{-1/2} \eta) d\eta|^2 d\eta d\tau \right)^{1/2}
\]

\[
= \left( \int_0^\infty \tau \frac{\mathcal{N}}{4} \sup_{\alpha \in \mathbb{Z}^n} \int_{\frac{\tau}{4} Q_\alpha} |\mathcal{M}_\tau F_\beta(\tau, \tau^{-1/2} y)|^2 d\eta d\tau \right)^{1/2}
\]

\[
\lesssim \left( \int_0^\infty \int_{Q_\beta} |\mathcal{F}_\tau F(\tau, x)|^2 dx d\tau \right)^{1/2}
\]

where we have used the changes of variables, Lemma 3.9 and the identity

\[
(\mathcal{F}_\eta^{-1} \mathcal{F}_\beta(\tau, \tau^{-1/2} \eta))(\tau, y) = \tau^{-\frac{1}{2}} \mathcal{F}_\tau F(\tau, \tau^{-1/2} y).
\]

For the part when \( \tau \in (-\infty, 0) \) in (3.18), it is easier to handle since this corresponds to the symbol \( \frac{\eta^\nu}{1 + \varepsilon |\tau|^{-2} |\eta|^2 + |\eta|^4} \), which has no singularity.

Therefore, we obtain, by the Plancherel theorem, the Sobolev embedding theorem and the Hölder’s inequality

\[
\sup_{\alpha \in \mathbb{Z}^n} \|D^\nu u_\beta(t, x)\|_{L^2_{t,x}(\Omega)}
\]

\[
\lesssim \left( \int_0^\infty \int_{Q_\beta} |\mathcal{F}_\tau F(\tau, x)|^2 dx d\tau \right)^{1/2}
\]

\[
\lesssim \left( \int_{Q_\beta} \|F(\cdot, x)\|_{L^2_{t,x}}^2 dx \right)^{1/2} \lesssim \|F\|_{L^2_{t,x}(\Omega)}
\]

which implies the desired result (3.15). In general, the solution \( u(t, x) \) of (3.13) may not vanish at \( t = 0 \), we can dealt with this case by the method of Hao, Hsiao and Wang [10].

\[\square\]

## 4 Proof of Theorem 1.1

**Lemma 4.1. (Sobolev Inequality).** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( \partial \Omega \in C^m \), \( m, \ell \in \mathbb{N} \cup \{0\} \), \( 1 \leq r, p, q \leq \infty \). Assume that

\[
\frac{\ell}{m} \leq \theta \leq 1, \quad \frac{1}{p} - \frac{\ell}{n} = \theta \left( \frac{1}{r} - \frac{m}{n} \right) + \frac{1 - \theta}{q}.
\]

Then we have

\[
\sum_{|\beta| = \ell} \|D^\beta u\|_{L^p(\Omega)} \lesssim \|u\|_{L^p(\Omega)}^{1-\theta} \|u\|_{W^{m,q}(\Omega)}^\theta,
\]

where \( \|u\|_{W^{m,q}(\Omega)} = \sum_{|\beta| \leq m} \|D^\beta u\|_{L^r(\Omega)} \).
Proof of Theorem 1.1. For simplicity, we first consider a simple case \( s = \frac{n}{2} + \frac{1}{2} \) and there is no difficult to generalize the proof to the case \( s > \frac{n}{2} + \frac{1}{2} \) with \( s + 1/2 \in \mathbb{N} \). We assume without loss of generality that

\[
F((\partial_x^\alpha u)_{|\alpha| \leq 3}, (\partial_x^\alpha \bar{u})_{|\alpha| \leq 3}) = F((\partial_x^\alpha u)_{|\alpha| \leq 3}) = \sum_{\Lambda_{R_0,R_1,R_2,R_3}} C_{R_0,R_1,R_2,R_3} u^{R_0} (D^{\alpha_1} u)^{R_1} (D^{\alpha_2} u)^{R_2} (D^{\alpha_3} u)^{R_3},
\]

where \( R_i, \alpha_i \) \((i = 1, 2, 3)\) are multi-indices, \(|\alpha_i| = i \) \((i = 1, 2, 3)\) and

\[
\Lambda_{R_0,R_1,R_2,R_3} = \{(R_0, R_1, R_2, R_3) : m + 1 \leq R_0 + |R_1| + |R_2| + |R_3| \leq M + 1\}.
\]

Since we only use the Sobolev norm, \( u \) and \( \bar{u} \) have the same norm. Hence, the general cases can be handled in the same way. Denote

\[
\lambda_1(v) := \|v\|_{\mathbb{L}^\infty_t(L^2_{x,\mathbb{R}}(\mathbb{R} \times \mathbb{Q}_0))},
\]

\[
\lambda_2(v) := \|v\|_{\mathbb{L}^3_t(L^6_{x,\mathbb{R}}(\mathbb{R} \times \mathbb{Q}_0))},
\]

\[
\lambda_3(v) := \|v\|_{\mathbb{L}^\infty_t(L^{2m}_{x,\mathbb{R}}(\mathbb{R} \times \mathbb{Q}_0))}.
\]

Put

\[
\mathcal{G}_n = \left\{ u : \sum_{|\beta| \leq [n/2] + 7} \lambda_1(D^\beta u) + \sum_{|\beta| \leq 3, i = 2, 3} \lambda_i(D^\beta u) \leq \varrho \right\}.
\]

Considering the mapping

\[
\mathcal{T} : u(t) \mapsto S_{\xi}(t)u_0 - i\varphi_z F((\partial_x^\alpha u)_{|\alpha| \leq 3}),
\]

we show that \( \mathcal{T} : \mathcal{G}_n \rightarrow \mathcal{G}_n \) is a contraction mapping for any \( n \geq 3 \).

**Step 1.** For any \( u \in \mathcal{G}_n \), we first estimate \( \lambda_1(D^\beta \mathcal{T} u) \) for \(|\beta| \leq 7 + \frac{n}{2}\). Using Lemma 4.1, the cases \(|\beta| = 1 \) and \(|\beta| = 2\) can be dominated by terms of the cases \(|\beta| = 0 \) and \(|\beta| = 3\). Thus, we only need to consider the cases \(|\beta| = 0 \) and \(|\beta| = 3\). We split it into three cases as the following.

**Case 1.** \( n \geq 3, 3 \leq |\beta| \leq 7 + \frac{n}{2}\). For simplicity, we denote \( \Lambda_{\tilde{R}} = \Lambda_{R_0,R_1,R_2,R_3} \). In view of Corollary 3.7 and Proposition 3.8, we have for any \( \beta \) with \( 3 \leq |\beta| \leq 7 + \frac{n}{2}\) that

\[
\lambda_1(D^\beta \mathcal{T} u) \lesssim \|u_0\|_{H^s} + \sum_{|\beta| \leq 4 + \frac{n}{2}} \sum_{\Lambda_{\tilde{R}}} \|D^{\beta} [u^{R_0} (D^{\alpha_1} u)^{R_1} (D^{\alpha_2} u)^{R_2} (D^{\alpha_3} u)^{R_3}]\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{Q}_0)}.
\]

(4.2)

For simplicity, we only consider the case \( u^{R_0} (D^{\alpha_1} u)^{R_1} (D^{\alpha_2} u)^{R_2} (D^{\alpha_3} u)^{R_3} = u^{R_0} (\partial_{x_1}^3 u)^{R_3} \) in (4.2) and the general case can be treated in an analogous way. So, one can rewrite (4.2) as

\[
\sum_{3 \leq |\beta| \leq 7 + \frac{n}{2}} \lambda_1(D^\beta \mathcal{T} u) \lesssim \|u_0\|_{H^s} + \sum_{|\beta| \leq 4 + \frac{n}{2}} \sum_{\Lambda_{\tilde{R}}} \|D^{\beta} [u^{R_0} (\partial_{x_1}^3 u)^{R_3}]\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{Q}_0)}.
\]

*One can see below for a general treating.*
Observing that
\[ |D^\beta[u^{R_0}(\partial_3^3 u)^{R_3}]| \lesssim \sum_{\beta_1 + \cdots + \beta_R = \beta} |D^{\beta_1} u \cdots D^{\beta_R} u D^{\beta_R+1}(\partial_3^3 u) D^{\beta_R}(\partial_3^3 u)|. \tag{4.3} \]

By Hölder’s inequality, we get
\[
\|D^\beta[u^{R_0}(\partial_3^3 u)^{R_3}]\|_{L^2(Q_n)} \lesssim \sum_{\beta_1 + \cdots + \beta_R = \beta} \prod_{i=1}^{R_0} \|D^{\beta_i} u\|_{L^p_i(Q_n)} \prod_{i=R_0+1}^{R} \|D^{\beta_i}(\partial_3^3 u)\|_{L^p_i(Q_n)}, \tag{4.4}
\]
where
\[ p_i = \begin{cases} 2|\beta|/|\beta_i|, & |\beta_i| \geq 1, \\ \infty, & |\beta_i| = 0. \end{cases} \]

It is clear that for \( \theta_i = |\beta_i|/|\beta| \),
\[ \frac{1}{p_i} = \theta_i(\frac{1}{2} - \frac{|\beta_i|}{n}) + \frac{1 - \theta_i}{n}. \]

Using Sobolev’s inequality, one has that for \( B_\alpha = \{ x : |x - a| \leq \sqrt{n} \} \)
\[
\|D^{\beta_i} u\|_{L^p_i(B_\alpha)} \leq \|D^{\beta_i} u\|_{L^p_i(B_n)} \leq \|u\|_{W^{1-\theta_i}_\infty(B_n)} \|u\|_{W^{\theta_i}_{2i}(B_n)}^{\theta_i}, \quad i = 1, \ldots, R_0, \tag{4.5}
\]
\[
\|D^{\beta_i}(\partial_3^3 u)\|_{L^p_i(B_\alpha)} \leq \|\partial_3^3 u\|_{L^p_i(B_n)} \|\partial_3^3 u\|_{W^{\theta_i}_{2i}(B_n)}^{\theta_i}, \quad i = R_0 + 1, \ldots, R. \tag{4.6}
\]
Since
\[ \sum_{i=1}^{R} \theta_i = 1, \quad \sum_{i=1}^{R} (1 - \theta_i) = \tilde{R} - 1, \]
by \((4.3)\) and \((4.5)\) we have
\[
\|D^\beta[u^{R_0}(\partial_3^3 u)^{R_3}]\|_{L^2(Q_n)} \lesssim \sum_{|\beta| \leq 4 + |\tilde{\beta}|/2} \left( \|u\|_{W^{\theta_i}_{2i}(B_n)} + \|\partial_3^3 u\|_{W^{\theta_i}_{2i}(B_n)} \right) \times \left( \|u\|_{L^p_i(B_n)} + \|\partial_3^3 u\|_{L^p_i(B_n)} \right) \lesssim \sum_{|\gamma| \leq 7 + |\tilde{\beta}|/2} \|D^{\gamma} u\|_{L^2_i(B_n)} \sum_{|\beta| \leq 3} \|D^\beta(\partial_3^3 u)\|_{L^p_i(B_n)}^{\tilde{R}-1}. \tag{4.7}
\]

It follows, from \((4.6)\) and \( \ell^2 \subset \ell^{\tilde{R}-1} \), that
\[
\sum_{|\beta| \leq 4 + |\tilde{\beta}|/2} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta[u^{R_0}(\partial_3^3 u)^{R_3}]\|_{L^2_i(\mathbb{R} \times Q_n)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \|D^{\gamma} u\|_{L^2_i(\mathbb{R} \times B_n)} \sum_{|\beta| \leq 3} \|D^\beta u\|_{L^\infty_i(\mathbb{R} \times B_n)} \lesssim \sum_{|\gamma| \leq 7 + |\tilde{\beta}|/2} \lambda_1(D^{\gamma} u) \sum_{|\beta| \leq 3} \lambda_2(D^\beta u)^{\tilde{R}-1}.
\]

Hence, in view of \((4.2)\) and \((4.7)\) we have
\[
\sum_{3 \leq |\beta| \leq 7 + |\tilde{\beta}|/2} \lambda_1(D^\beta u) \lesssim \|u_0\|_{H^s} + \sum_{\tilde{R} = m+1}^{M+1} \|q\|_{\tilde{R}}.
\]
It follows from Holder's inequality and Proposition 3.3 that
\[
\|u_0\|_{L^2}^2 + \sum_{R = m+1}^{M+1} \sum_{|\beta| \leq 3} \|D^\beta u\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{Q}_0)}^2 \lesssim \sum_{|\gamma| \leq 7 + \lfloor n/2 \rfloor} \|D^\gamma u\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{Q}_0)}^2.
\]

Case 3. \( n = 3, 4, |\beta| = 0 \). By Propositions 3.3 we have
\[
\lambda_1(\mathcal{F} u) \lesssim \|u_0\|_{H^{-3/2}} + \sum_{\alpha \in \mathbb{Z}^n} \|F((\partial_x^\alpha u)_{|\alpha| \leq 3})\|_{L^1_{t}L^2_x(\mathbb{R} \times \mathbb{Q}_0)}.
\]
Using the same way as in Case 2, we have
\[
\lambda_1(\mathcal{F} u) \lesssim \|u_0\|_{H^{-3/2}} + \sum_{R = m+1}^{M+1} \tilde{q}^R. \tag{4.8}
\]

Step 2. We consider the estimates of \( \lambda_2(D^\beta \mathcal{F} u) \) for \( |\beta| \leq 3 \). Using the estimates of the maximal function as in Proposition 2.3, we have for \( |\beta| \leq 3 \) and \( 0 < \rho \ll 1 \)
\[
\lambda_2(D^\beta \mathcal{F} u) \lesssim \|S(t) D^\beta u_0\|_{L^\infty_{t,x}(\mathbb{R} \times \mathbb{Q}_0)} + \|\mathcal{F}_{\rho} D^\beta F((\partial_x^\alpha u)_{|\alpha| \leq 3})\|_{L^\infty_{t,x}(\mathbb{R} \times \mathbb{Q}_0)}
\]
\[
\lesssim \|D^\beta u_0\|_{H^{n/2+\rho}} + \sum_{|\beta| \leq 3} \|D^\beta F((\partial_x^\alpha u)_{|\alpha| \leq 3})\|_{L^1(\mathbb{R}, H^{n/2+\rho})}
\]
\[
\lesssim \|u_0\|_{H^{n/2+3+\rho}} + \sum_{|\beta| \leq 4 + \lfloor n/2 \rfloor} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta F((\partial_x^\alpha u)_{|\alpha| \leq 3})\|_{L^1_{t}L^2_x(\mathbb{R} \times \mathbb{Q}_0)} \times \sum_{|\gamma| \leq 7 + \lfloor n/2 \rfloor} \|D^\gamma u\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{Q}_0)}. \tag{4.9}
\]
Applying the same way as in Step 1, for any \( |\beta| \leq 4 + \lfloor n/2 \rfloor \), we can get
\[
\sum_{\alpha \in \mathbb{Z}^n} \|D^\beta F((\partial_x^\alpha u)_{|\alpha| \leq 3})\|_{L^2_{t,x}(B_\alpha)} \lesssim \sum_{R = m+1}^{M+1} \sum_{|\beta| \leq 3} \|D^\beta u\|_{L^\infty_{t,x}(B_\alpha)}^2 \times \sum_{|\gamma| \leq 7 + \lfloor n/2 \rfloor} \|D^\gamma u\|_{L^2_{t,x}(B_\alpha)}. \tag{4.10}
\]
By Holder's inequality, we have from (4.9) that
\[
\|D^\beta F((\partial_x^\alpha u)_{|\alpha| \leq 3})\|_{L^1_{t}L^2_x(\mathbb{R} \times \mathbb{Q}_0)} \lesssim \sum_{R = m+1}^{M+1} \sum_{|\gamma| \leq 7 + \lfloor n/2 \rfloor} \|D^\gamma u\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{Q}_0)} \times \sum_{|\beta| \leq 3} \|D^\beta u\|_{L^\infty_{t,x}(\mathbb{R} \times \mathbb{Q}_0)}^2 \tag{4.11}
\]

Summating (4.10) over all $\alpha \in \mathbb{Z}^n$, we have for any $|\beta| \leq 4 + [n/2]$

$$
\sum_{\alpha \in \mathbb{Z}^n} \|D^\beta F((\partial_x^\alpha u)|_{|\alpha|\leq 3})\|_{L^1_t L^2_x(\mathbb{R} \times Q_n)} \lesssim \sum_{R=m+1}^{M+1} \sum_{R=m+1}^{M+1} \lambda_1(D^\gamma u) \sum_{|\beta|\leq 3} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta u\|_{L^{R-1}_t L^\infty_x(\mathbb{R} \times B_\alpha)} \lesssim \sum_{R=m+1}^{M+1} \lambda_1(D^\gamma u) \sum_{|\beta|\leq 3} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta u\|_{L^{R-1}_t L^\infty_x(\mathbb{R} \times B_\alpha)} \lesssim \sum_{R=m+1}^{M+1} \lambda_1(D^\gamma u) \sum_{|\beta|\leq 3} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta u\|_{L^{R-1}_t L^\infty_x(\mathbb{R} \times B_\alpha)} \lesssim \sum_{R=m+1}^{M+1} \lambda_1(D^\gamma u) \sum_{|\beta|\leq 3} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta u\|_{L^{R-1}_t L^\infty_x(\mathbb{R} \times B_\alpha)} \lesssim \sum_{R=m+1}^{M+1} \sum q^{R/2}.
$$

Combining (4.8) with (4.11), we get

$$
\sum_{|\beta|\leq 3} \lambda_2(D^\beta \mathcal{T} u) \lesssim \|u_0\|_{H^{n/2+3}_x} + \sum_{R=m+1}^{M+1} q^{R/2}.
$$

**Step 3.** Now, we estimate $\lambda_3(D^\beta \mathcal{T} u)$ for $|\beta| \leq 3$. In view of Proposition [2.9], one has that

$$
\lambda_3(D^\beta \mathcal{T} u) \lesssim \|S_\epsilon(t) D^\beta u_0\|_{L^\infty_t L^2_x(\mathbb{R} \times Q_n)} + \|S_\epsilon D^\beta F((\partial_x^\alpha u)|_{|\alpha|\leq 3})\|_{L^\infty_t L^2_x(\mathbb{R} \times Q_n)} \lesssim \|D^\beta u_0\|_{H^{n/2+2-m}_x} + \sum_{|\beta|\leq 3} \|D^\beta F((\partial_x^\alpha u)|_{|\alpha|\leq 3})\|_{L^1_t H^{n/2+2-m}_x(\mathbb{R} \times Q_n)} \lesssim \|u_0\|_{H^{n/2+3}_x} + \sum_{|\beta|\leq 4 + [n/2]} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta F((\partial_x^\alpha u)|_{|\alpha|\leq 3})\|_{L^1_t L^2_x(\mathbb{R} \times Q_n)};
$$

which reduces to the case as in (4.8).

Therefore, collecting the estimates as in Step 1-3, we have for $n \geq 5$

$$
\sum_{|\beta| \leq [n/2]+7} \lambda_1(D^\beta \mathcal{T} u) + \sum_{|\beta| \leq 3} \sum_{i=2}^{3} \lambda_i(D^\beta \mathcal{T} u) \lesssim \|u_0\|_{H^s} + \sum_{R=m+1}^{M+1} q^{R/2},
$$

and for $n = 3, 4$

$$
\sum_{|\beta| \leq [n/2]+7} \lambda_1(D^\beta \mathcal{T} u) + \sum_{|\beta| \leq 3} \sum_{i=2}^{3} \lambda_i(D^\beta \mathcal{T} u) \lesssim \|u_0\|_{H^s \cap H^{-3/2}} + \sum_{R=m+1}^{M+1} q^{R/2}.
$$

It follows that for $n \geq 5$, $T : \mathcal{D}_n \to \mathcal{D}_n$ is a contraction mapping if both $q$ and $\|u_0\|_{H^s}$ are small enough (similarly for $n = 3, 4$).

Before considering the case $s > n/2 + 9/2$, we first recall a nonlinear mapping estimate (cf. [33]).
Lemma 4.2. Let \( n \geq 2, s > 0, K \in \mathbb{N} \). Let \( 1 \leq p, p_i, q, q_i \leq \infty \) satisfy \( 1/p = 1/p_1 + (K-1)/p_2 \) and \( 1/q = 1/q_1 + (K-1)/q_2 \). We have

\[
\| v_1 \cdots v_K \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))} \lesssim \sum_{k=1}^K \| v_k \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))} \times \prod_{i \neq k, i=1, \ldots, K} \| v_i \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))}. \tag{4.12}
\]

Lemma 4.3. Let \( n \geq 5 \), for any \( s > 0 \) and any multi-index \( \alpha \), we have

\[
\sum_{|\alpha|=0,3} \| S_{\zeta} (t) D^\alpha u_0 \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))} \lesssim \| u_0 \|_{B^{3+3/2}_{2,1}},
\]

\[
\sum_{|\alpha|=0,3} \| \mathcal{A}_\zeta D^\alpha F \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))} \lesssim \| F \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))}.
\]

Proof. In view of Corollary 3.7 and Propositions 3.1 and 3.8 we have the desired results. \(\square\)

Lemma 4.4. Let \( n = 3, 4 \), for any \( s > 0 \) and any multi-index \( \alpha \), we have

\[
\sum_{|\alpha|=0,3} \| S_{\zeta} (t) D^\alpha u_0 \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))} \lesssim \| u_0 \|_{B^{3+3/2}_{2,1} \cap \dot{H}^{-3/2}},
\]

\[
\| \mathcal{A}_\zeta D^\alpha F \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))} \lesssim \| F \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))}, \quad |\alpha| = 3,
\]

\[
\| \mathcal{A}_\zeta D^\alpha F \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))} \lesssim \| F \|_{\ell^1_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))},
\]

Proof. By Propositions 3.3, 3.4 and 3.8 we have the results, as desired. \(\square\)

We now continue the proof of Theorem 1.1 and consider the general case \( s > n/2 + 9/2 \). We write

\[
\lambda_1 (v) := \sum_{i=0,3} \| D^i v \|_{\ell^1_\alpha \ell^{s-3/2}_s \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))},
\]

\[
\lambda_2 (v) := \sum_{i=0,3} \| D^i v \|_{\ell^1_\alpha \ell^s_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))},
\]

\[
\lambda_3 (v) := \sum_{i=0,3} \| D^i v \|_{\ell^1_\alpha \ell^s_\alpha \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))},
\]

\[
\mathcal{D} = \{ v : \sum_{i=1,2,3} \lambda_i (v) \leq \varrho \}. \tag{4.13}
\]

Note \( \lambda_i \) and \( \mathcal{D} \) defined here are different from those in the above. We only give the details of the proof for the case \( n \geq 5 \). The cases \( n = 3, 4 \) can be showed by a slight modification. Let \( \mathcal{F} \) be defined as in (4.11). Using Lemma 4.3 we have

\[
\lambda_1 (\mathcal{F} u) \lesssim \| u_0 \|_{B^{3+3/2}_{2,1}} + \| F \|_{\ell^1_\alpha \ell^{s-3/2}_s \ell^{p_1}_s \ell^{p_2}_s (L^1_t L^q_x (\mathbb{R} \times \mathbb{Q}))}. \tag{4.14}
\]

For simplicity, we write

\[
F = u^{R_0} (D^{\alpha_1} u)^{R_1} (D^{\alpha_2} u)^{R_2} (D^{\alpha_3} u)^{R_3}, \tag{4.15}
\]
Using Lemma 4.1, the terms of $|\alpha| = 1, 2, 3$ are multi-index. By Lemma 4.2, we have
\[
\|F\|_{L^2_{\alpha} L^2_y(R \times R^3)} \\
\lesssim \left( \sum_{|\alpha| = 0}^{3} \|D^\alpha u\|_{L^2_{\alpha} L^2_y(R \times R^3)} \right) \times \left( \sum_{|\beta| = 0}^{3} \|D^\beta u\|_{\bar{L}^2_{\alpha} L^2_y(R \times R^3)} \right)
\]
(4.16)

Using Lemma 4.1, the terms of $|\alpha| = 1, 2$ and $|\beta| = 1, 2$ in the above can be dominated by that of $|\alpha| = 0, 3$ and $|\beta| = 0, 3$, respectively. Therefore, we get
\[
\|F\|_{L^2_{\alpha} L^2_y(R \times R^3)} \\
\lesssim \left( \sum_{|\alpha| = 0, 3} \|D^\alpha u\|_{L^2_{\alpha} L^2_y(R \times R^3)} \right) \times \left( \sum_{|\beta| = 0, 3} \|D^\beta u\|_{\bar{L}^2_{\alpha} L^2_y(R \times R^3)} \right)
\]
Hence, if $u \in \mathcal{D}$, in view of (4.14) and (4.16), we have
\[
\lambda_1(\mathcal{D} u) \lesssim \|u_0\|_{B^2_{2,1}} + \sum_{m+1 \leq |R| \leq M+1} q^R.
\]
(4.17)

In view of the estimate for the maximal function as in Proposition 2.8, one has that
\[
\lambda_2(S_\epsilon(t)u_0) \lesssim \|u_0\|_{B^2_{2,1}},
\]
(4.18)

and for multi-index $|\alpha| = 0, 3$
\[
\|\mathcal{D}^\alpha D^\beta u\|_{L^2_{\alpha} L^2_y(R \times R^3)} \lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}} \|S(t - \tau)(\Delta_j D^\alpha F)(\tau)\|_{L^2_y(R \times R^3)} dt
\]
\[
\lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}} \|((\Delta_j D^\alpha F)(\tau)\|_{H^{s+9/2}(\mathbb{R}^3)} d\tau
\]
\[
\lesssim \sum_{j=0}^{\infty} 2^{(s-3/2)j} \int_{\mathbb{R}} \|((\Delta_j F)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau.
\]
(4.19)
Hence, by (4.17) and (4.18), it follows
\[
\lambda_2(\mathcal{D} u) \lesssim \|u_0\|_{B^2_{2,1}} + \|F\|_{L^2_{\alpha} L^2_y(R \times R^3)}
\]
(4.20)

Similar to (4.19), in view of Proposition 2.9 we have
\[
\lambda_3(\mathcal{D} u) \lesssim \|u_0\|_{B^2_{2,1}} + \|F\|_{L^2_{\alpha} L^2_y(R \times R^3)}.
\]
(4.21)

In view of Lemma 4.1 and 4.2 we have
\[
\|F\|_{L^2_{\alpha} L^2_y(R \times R^3)} \\
\lesssim \left( \sum_{|\beta| = 0, 3} \|D^\beta u\|_{L^2_{\alpha} L^2_y(R \times R^3)} \right) \times \left( \sum_{|\alpha| = 0, 3} \|D^\alpha u\|_{L^2_{\alpha} L^2_y(R \times R^3)} \right)
\]
Hence, if $u \in \mathcal{D}$, we have
\[
\lambda_2(\mathcal{D} u) + \lambda_3(\mathcal{D} u) \lesssim \|u_0\|_{B^2_{2,1}} + \sum_{m+1 \leq |R| \leq M+1} q^R.
\]
Repeating the procedures as in the above, we obtain that there exists a unique $u \in \mathcal{D}$ satisfying the integral equation $\mathcal{D} u = u$, which finishes the proof of Theorem 4.1.
5 Proof of Theorem 1.2

We prove Theorem 1.2 by following some idea as in Wang and Wang [33]. The following is the estimate for the solutions of the fourth order Schrödinger equation, see Kenig, Ponce and Vega [18] and Hao, Hsiao and Wang [10]. Recall that $\triangle_j := \mathcal{F}^{-1}\delta(2^{-j}\cdot)\mathcal{F}$, $j \in \mathbb{Z}$ and $\delta(\cdot)$ are as in Section 1.3.

Lemma 5.1. Let $g \in \mathcal{S}(\mathbb{R})$, $f \in \mathcal{S}(\mathbb{R}^2)$, we have

$$\|\triangle_j S_0(t)g\|_{L^p_tL^q_x} \lesssim \|\triangle_j g\|_2, \quad (5.1)$$
$$\|\triangle_j S_0(t)g\|_{L^p_tL^1_x} \lesssim 2^{j\frac{1}{2}}\|\triangle_j g\|_2, \quad (5.2)$$
$$\|\triangle_j S_0(t)g\|_{L^\infty_tL^2_x} \lesssim 2^{-\frac{3j}{2}}\|\triangle_j g\|_2, \quad (5.3)$$
$$\|\triangle_j \mathcal{A}_0 f\|_{L^p_tL^q_x} \lesssim \|\triangle_j f\|_{L^p_{x,t}}, \quad (5.4)$$
$$\|\triangle_j \mathcal{A}_0 f\|_{L^p_tL^\infty_x} \lesssim 2^{j\frac{1}{2}}\|\triangle_j f\|_{L^p_{x,t}}, \quad (5.5)$$
$$\|\triangle_j \mathcal{A}_0 f\|_{L^p_tL^2_x} \lesssim 2^{-\frac{3j}{2}}\|\triangle_j f\|_{L^p_{x,t}}, \quad (5.6)$$

and

$$\|\triangle_j \mathcal{A}_0 (\partial_x^3 f)\|_{L^p_tL^q_x} \lesssim 2^{\frac{3j}{2}}\|\triangle_j f\|_{L^p_{x,t}}, \quad (5.7)$$
$$\|\triangle_j \mathcal{A}_0 (\partial_x^3 f)\|_{L^p_tL^\infty_x} \lesssim 2^{\frac{3j}{2}}\|\triangle_j f\|_{L^p_{x,t}}, \quad (5.8)$$
$$\|\triangle_j \mathcal{A}_0 (\partial_x^3 f)\|_{L^p_tL^2_x} \lesssim \|\triangle_j f\|_{L^p_{x,t}}. \quad (5.9)$$

For convenience, we write for any Banach function space $X$,

$$\|f\|_{\ell^q_{\triangle}(X)} = \sum_{j \in \mathbb{Z}} 2^{j\gamma} \|\triangle_j f\|_X, \quad \|f\|_{\ell^q_{\triangle}(X)} := \|f\|_{\ell^1_{\triangle}(X)}.$$ 

Now we recall a result of Wang and Wang, see [33].

Lemma 5.2. Let $s > 0$, $1 \leq p, p_i, \gamma, \gamma_i \leq \infty$ satisfy

$$\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + ... + \frac{1}{\gamma_N},$$

then

$$\|u_1 \cdots u_N\|_{\ell^1_{\triangle}(L^p_tL^q_x)} \lesssim \|u_1\|_{\ell^1_{\triangle}(L^{p_1}_tL^{q_1}_x)} \prod_{i=2}^{N} \|u_i\|_{\ell^1_{\triangle}(L^{p_i}_tL^{q_i}_x)} + \|u_2\|_{\ell^1_{\triangle}(L^{p_1}_tL^{q_1}_x)} \prod_{i \neq 2,i=1,...,N} \|u_i\|_{\ell^1_{\triangle}(L^{p_i}_tL^{q_i}_x)} + \cdots + \|u_N\|_{\ell^1_{\triangle}(L^{p_N}_tL^{q_N}_x)} \prod_{i=1}^{N-1} \|u_i\|_{\ell^1_{\triangle}(L^{p_i}_tL^{q_i}_x)}. \quad (5.10)$$

In particular, if $u_1 = \cdots = u_N = u$, then

$$\|u^N\|_{\ell^1_{\triangle}(L^p_tL^q_x)} \lesssim \|u\|_{\ell^1_{\triangle}(L^{p_1}_tL^{q_1}_x)} \prod_{i=2}^{N} \|u\|_{\ell^1_{\triangle}(L^{p_i}_tL^{q_i}_x)}. \quad (5.11)$$

Replace the spaces $L^p_tL^q_x$ and $L^p_tL^q_x$ by $L^{p_i}_tL^{q_i}_x$ and $L^{p_i}_tL^{q_i}_x$, respectively, (5.10) and (5.11) also hold.
Remark 5.3 ([33]). One easily sees that (5.11) can be slightly improved by
\[
\|u^N\|_{L^{1,s}(L^p_t L^q_x)} \lesssim \|u\|_{L^{1,s}(L^p_t L^q_x)} \prod_{i=2}^{N} \|u\|_{L^p_t L^q_x}. \tag{5.12}
\]

In fact, from Minkowski’s inequality it follows that
\[
\|S_t u\|_{L^p_t L^q_x} \lesssim \|u\|_{L^p_t L^q_x}. \tag{5.13}
\]

From Lemma 5.2 and (5.13), we get (5.12).

Proof of Theorem 1.2. We can assume, without loss of generality, that
\[
F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3}) = \sum_{m+1 \leq R \leq M+1} \lambda_{R_0 R_1 R_2 R_3} u_{R_0} u_{R_1} u_{R_2} u_{R_3}. \tag{5.14}
\]

where $\tilde{R}$ is as before.

Step 1. We consider the case $m > 8$. Recall that
\[
\|u\|_X = \sup_{s_m \leq s \leq s_M} \sum_{j=0}^{\infty} \|\partial_x^j u\|_s, \tag{5.15}
\]
\[
\|\Delta_j u\|_s := 2^{sj} \left( \|\Delta_j u\|_{L^\infty_t L^2_x} + 2^{\frac{3j}{2}} \|\Delta_j u\|_{L^2_t L^2_x} \right) + 2^{(s-s_m)j} \|\Delta_j u\|_{L^2_t L^2_x}.
\]

Considering the mapping
\[
\mathcal{T}: u(t) \mapsto S_0(t)u_0 - i\mathcal{A}_0 F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3}),
\]
we will show that $\mathcal{T}: X \rightarrow X$ is a contraction mapping. We have
\[
\|\mathcal{T} u\|_X \leq \|S_0(t)u_0\|_X + \|\mathcal{A}_0 F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3})\|_X. \tag{5.16}
\]
In view of (5.11), (5.2) and (5.3), we have
\[
\|\partial_x^j \Delta_j S_0(t)u_0\|_s \leq 2^{sj} \|\partial_x^j S_0(t)u_0\|_2.
\]

It follows that
\[
\|S_0(t)u_0\|_X \leq \sup_{s_m \leq s \leq s_M} \sum_{i=0}^{\infty} \sum_{j \in \mathbb{Z}} 2^{sj} \|\partial_x^j \Delta_j u_0\|_2 \lesssim \|u_0\|_{\dot{B}^{1+s}_{2,1}} \cap \dot{B}^{s}_{2,1}. \tag{5.17}
\]

We now estimate $\|\mathcal{A}_0 F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3})\|_X$. Using (5.14), (5.3) and (5.6), we have
\[
\|\Delta_j (\mathcal{A}_0 F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3}))\|_s \lesssim 2^{sj} \|\Delta_j F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3})\|_{L^{10/9}_{L^2_t}}. \tag{5.18}
\]

From (5.7), (5.8) and (5.9), it follows that
\[
\|\Delta_j (\mathcal{A}_0 \partial_x^3 F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3}))\|_s \lesssim 2^{sj} 2^{3j/2} \|\Delta_j F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3})\|_{L^{10/9}_{L^2_t}}. \tag{5.19}
\]

Hence, from (5.14), (5.17) and (5.18) we have
\[
\|\mathcal{A}_0 F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3})\|_X
\lesssim 2^{sj} \|\Delta_j F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3})\|_{L^{10/9}_{L^2_t}}
\]
\[
+ 2^{sj} 2^{3j/2} \|\Delta_j F((\partial_x^a u)_{|a|\leq 3}, (\partial_x^a \bar{u})_{|a|\leq 3})\|_{L^{10/9}_{L^2_t}} = I + II. \tag{5.19}
\]
Now we perform the nonlinear estimates. By Lemma 5.2, we have

\[ I \lesssim \sum_{m+1 \leq R \leq M+1} \left( \| u \|_{\mathcal{G}^{1,s}(L_t^{10}; L_{t,x}^{(5(R-1)/4)})} \right)^3 \prod_{i=1}^{3} \| \partial_x^i u \|_{\mathcal{L}_x^{(5(R-1)/4)}}^{R_i} \\
+ \| \partial_x u \|_{\mathcal{G}^{1,s}(L_t^{10}; L_{t,x}^{(5(R-1)/4)})}^{R_1} \prod_{i=0,2,3} \| \partial_x^i u \|_{\mathcal{L}_x^{(5(R-1)/4)}}^{R_i} \\
+ \| \partial_x^2 u \|_{\mathcal{G}^{1,s}(L_t^{10}; L_{t,x}^{(5(R-1)/4)})}^{R_2} \prod_{i=0,1,3} \| \partial_x^i u \|_{\mathcal{L}_x^{(5(R-1)/4)}}^{R_i} \\
+ \| \partial_x^3 u \|_{\mathcal{G}^{1,s}(L_t^{10}; L_{t,x}^{(5(R-1)/4)})}^{R_3} \prod_{i=0}^{2} \| \partial_x^i u \|_{\mathcal{L}_x^{(5(R-1)/4)}}^{R_i} \right). \\
\]

(5.20)

By Sobolev imbedding theorem, we have

\[ I \lesssim \sum_{m+1 \leq R \leq M+1} \left( \sum_{i=0,3} \| \partial_x^i u \|_{\mathcal{G}^{1,s}(L_t^{10}; L_{t,x}^{5\lambda/4})} \right)^2 \left( \sum_{i=0,3} \| \partial_x^i u \|_{\mathcal{L}_x^{(5(R-1)/4)}}^{R_i} \right). \\
\]

(5.21)

For any \( m \leq \lambda \leq M \), let \( \rho = \frac{1}{2} - \frac{16}{53\lambda} \). Observing that the following inclusions hold:

\[ L_t^\infty(\mathbb{R}, \dot{H}^{s_\lambda}) \cap L_t^{10}(\dot{H}^{s_\lambda}) \subset L_t^{5\lambda/4}(\mathbb{R}, \dot{H}^{s_\lambda}) \subset L_t^{5\lambda/4}(\mathbb{R}, \dot{H}^{s_\lambda}). \]

(5.22)

More precisely, we have

\[ \sum_{j \in \mathbb{Z}} \| \Delta_j u \|_{L_t^2}^{5\lambda/4} \lesssim \sum_{j \in \mathbb{Z}} \| \Delta_j u \|_{L_t^2}^{5\lambda/4}(\mathbb{R}, \dot{H}^{s_\lambda}) \\
\lesssim \sum_{j \in \mathbb{Z}} \| \Delta_j u \|_{L_t^2(\mathbb{R}, \dot{H}^{s_\lambda})}^{8/\lambda} \| \Delta_j u \|_{L_t^\infty(\mathbb{R}, \dot{H}^{s_\lambda})}^{1-8/\lambda} \\
\lesssim \| u \|_{L_t^2}^{8/\lambda} \| \Delta_j u \|_{L_t^2}^{1-8/\lambda} \|
\]

(5.23)

Using (5.22) and noticing that \( s_m \leq s_{R-1} \leq s_M < \tilde{s}_M \), for \( i = 0, 3 \), we have

\[ \| \partial_x^i u \|_{\mathcal{L}_x^{(5\lambda/4)}}^{R_i-1} \lesssim \| \partial_x^i u \|_{\mathcal{G}^{1,s}(L_t^{10}; L_{t,x}^{5\lambda/4})}^{8/\lambda} \| \partial_x^i u \|_{\mathcal{L}_x^{(5\lambda/4)}}^{1-8/\lambda} \lesssim \| u \|_{X}^{R_i-1}. \]

(5.24)

Combining (5.20) with (5.23), we have

\[ I \lesssim \sum_{m+1 \leq R \leq M+1} \| u \|_{X}^{R_i}. \]

(5.24)
Now we estimate the term $II$. By Lemma 5.2, we have

$$II \lesssim \sum_{m+1 \leq R \leq M+1} \left( \|u\|_{L^\infty L^2} \|u\|_{L^2} \prod_{i=1}^{3} \|\partial_x^R \|_{L^2} \right) \sum_{i=0, 3} \|\partial_x^i u\|_{L^2} \left( \sum_{i=0, 3} \|\partial_x^i u\|_{L^2} \right) \lesssim \sum_{m+1 \leq R \leq M+1} \|u\|_{L^X} \left( \sum_{i=0, 3} \|\partial_x^i u\|_{L^2} \right) \lesssim \sum_{m+1 \leq R \leq M+1} \|u\|_{L^X}. \quad (5.26)$$

Collecting (5.19), (5.20), (5.24) and (5.26), we have

$$\|\mathcal{A}_0 F((\partial_x^p u)_{|\alpha| \leq 3}, (\partial_x^p \bar{u})_{|\alpha| \leq 3})\|_X \lesssim \sum_{m+1 \leq R \leq M+1} \|u\|_{L^X}. \quad (5.27)$$

By (5.15), (5.16) and (5.27), it follows

$$\|\mathcal{T} u\|_X \lesssim \|u_0\|_{B^s_{2, 1} \cap B^s_{2, 1}} + \sum_{m+1 \leq R \leq M+1} \|u\|_{L^X}.$$

**Step 2.** We consider the case $m = 8$. Recall that

$$\|u\|_X = \sum_{i=0, 3} \|\partial_x^i u\|_{L^2} + \sup_{s_0 \leq s \leq s_M} \sum_j \|\partial_x^j u\|_s.$$

By (5.1), (5.2) and (5.3), we get

$$\|S_0(t)u_0\|_X \lesssim \|u_0\|_{L^2} + \sup_{s_0 \leq s \leq s_M} \sum_{i=0, 3} \sum_j 2^{3j} \|\partial_x^j u_0\|_2 \lesssim \|u_0\|_{B^s_{2, 1}}.$$

We now estimate $\|\mathcal{A}_0 F((\partial_x^p u)_{|\alpha| \leq 3}, (\partial_x^p \bar{u})_{|\alpha| \leq 3})\|_X$. By Strichartz’s and Hölder’s in-
equality, we have
\[\|\mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} \leq \sum_{9 \leq R \leq M+1} \|F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^{10/9}}\]
\[\leq \sum_{9 \leq R \leq M+1} \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^0} \right) \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^{10/9}} \right) \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^4} \right) \right)^{3/4} \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^4} \right)^{1/4} \leq \sum_{9 \leq R \leq M+1} \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^0} \right) \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^{10/9}} \right) \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^4} \right) \right)^{3/4} \left( \sum_{i=0}^3 \|\partial_x^i u\|_{L_x^4} \right)^{1/4} \right). \tag{5.28}\]

Applying (5.22) and (5.28), it implies that
\[\|\mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} \leq \sum_{9 \leq R \leq M+1} \|u\|_{X}. \tag{5.29}\]

Using Bernstein’s inequality and (5.7), it follows that
\[\|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} \leq \|P_{\leq 1}(\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3}))\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} + \|P_{>1}(\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3}))\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} \leq \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} + \sum_{j \geq 1} 2^{3j/2} \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^2} \leq \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} + \sum_{j \in \mathbb{Z}} 2^{3j/2} \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^2} = III + IV.\]

The estimates of III and IV have been given in (5.29) and (5.25), respectively. We have
\[\|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^\infty L_x^2 \cap L_t^{10/9}} \leq \sum_{9 \leq R \leq M+1} \|u\|_{X}.\]

We have from (5.21)–(5.3) and (5.7)–(5.9) that
\[\sum_{j \in \mathbb{Z}} \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_x^2} \leq \sum_{j \in \mathbb{Z}} 2^{3j/2} \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^2}, \tag{5.30}\]
\[\sum_{j \in \mathbb{Z}} \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_x^2} \leq \sum_{j \in \mathbb{Z}} 2^{3j/2} \|\partial_x^3 \mathcal{A}_0 F((\partial_x^3 u)_{|\alpha| \leq 3}, (\partial_x^2 \bar{u})_{|\alpha| \leq 3})\|_{L_t^2}. \tag{5.31}\]
holds for all $s > 0$. The right hand side in (5.31) has been estimated by (5.25). Thus, it suffices to consider the estimate of the right hand side in (5.30). Let us observe the equality

$$
F((\partial_x^6 u)_{|\alpha| \leq 3}, (\partial_x^6 \bar{u})_{|\alpha| \leq 3})
= \sum_{R=9}^{\tilde{R}} \lambda_{R_0 R_1 R_2 R_3} u_{R_0}^{R_0} u_{R_1}^{R_1} u_{R_2}^{R_3} + \sum_{9 < R \leq M+1} \lambda_{R_0 R_1 R_2 R_3} u_{R_0}^{R_0} u_{R_1}^{R_1} u_{R_2}^{R_2} u_{R_3}^{R_3}
= V + VI.
$$

For any $s_9 \leq s \leq \tilde{s}_M$, $VI$ has been handled in (5.21)-(5.24):

$$
\sum_{9 < R \leq M+1} \sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j F((\partial_x^6 u)_{|\alpha| \leq 3}, (\partial_x^6 \bar{u})_{|\alpha| \leq 3})\|_{L_x^{10/9}} \lesssim \sum_{9 < R \leq M+1} \|u\|_{R}^R.
$$

For the estimate of $V$, we use Remark 5.3 and get for any $s_9 \leq s \leq \tilde{s}_M$,

$$
\sum_{R=9}^{\tilde{R}} \sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j (u_{R_0}^{R_0} u_{R_1}^{R_1} u_{R_2}^{R_2} u_{R_3}^{R_3})\| \lesssim \left( \sum_{i=0}^{3} \|\partial_x^i u\|_{L_x^{10}}^8 \right) \left( \sum_{i=0}^{3} \|\partial_x^i u\|_{L_x^{10}}^{1,4} (L_x^{10}) \right) \lesssim \|u\|_{X}^R.
$$

Summating the estimates above, we obtain

$$
\|\mathcal{T} u(t)\|_{X} \lesssim \|u_0\|_{B^{2+\tilde{s}_M} x} + \sum_{9 \leq R \leq M+1} \|u\|_{R}^R.
$$

Hence, we have the results, as desired. 

**Acknowledgments**

The author would like to express his great thanks to Professor Baoxiang Wang and Doctor Chenchun Hao for their valuable suggestions and frequent encouragement during every stage of this work. The author was partially supported by the National Natural Science Foundation of China (grants numbers 10571004 and 10621061), the 973 Project Foundation of China, grant number 2006CB805902.

**References**

[1] M. Ben-Artzi, H. Koch and J. C. Saut, Dispersion estimates for fourth order Schrödinger equations, C. R. Acad. Sci. Paris Sér. I Math., **330**(2) (2000) 87–92.

[2] J. Bergh and J. Lofstrom, Interpolation Spaces, An Introduction, Springer-Verlag, 1976.

[3] T. Cazenave, An Introduction to Nonlinear Schödinger Equations, Textos de Métodos Matemáticos, 26, Universidade Federal do Rio de Janeiro, 1996.

[4] H. Chihara, Local existence for semilinear Schrödinger equations, Math. Japonica, **42** (1995) 35–52.

[5] P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, J. Amer. Math. Soc., **1**(1988) 413–446.

[6] K. B. Dysthe, Note on a modification to the nonlinear Schrödinger equation for application to deep water waves, Proc. R. Soc. Lond., **A369** (1979) 105–114.
[7] J. Ginibre and Y. Tsutsumi, Uniqueness for the generalized Korteweg-de Vries equations, SIAM J. Math. Anal., 20(1989) 1388–1425.

[8] B. L. Guo and B. X. Wang, The global Cauchy problem and scattering of solutions for nonlinear Schrödinger equations in $H^s$. Diff. Integral Eqns., 15 (9) (2002) 1073–1083.

[9] Z. H. Guo, L. Z. Peng and B. X. Wang, Decay estimate for a class of wave equations, J. Funct. Anal., 254 (2008), 1642–1660.

[10] C. C. Hao, L. Hsiao and B. X. Wang, Wellposedness for the fourth order nonlinear Schrödinger equations, J. Math. Anal. Appl., 320(1) (2006) 246–265.

[11] C. C. Hao, L. Hsiao and B. X. Wang, Wellposedness of the fourth order nonlinear Schrödinger equations in multi-dimension spaces, J. Math. Anal. Appl., 328(1) (2007) 58–83.

[12] N. Hayashi and T. Ozawa, Remarks on nonlinear Schrödinger equations in one space dimension, Diff. Integral Eqns., 2 (1994) 453–461.

[13] Z. H. Huo and Y. L. Jia, The Cauchy problem for the fourth-order nonlinear Schrödinger equation related to the vortex filament, J. Diff. Eqns., 214 (2005) 1–35.

[14] V. I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrödinger-type equations, Phys. Rev. E, 53 (2) (1996) R1336–R1339.

[15] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Studies in applied mathematics, 93–128, Adv. Math. Suppl. Stud., 8, Academic Press, New York, 1983.

[16] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41 (1988) 891–907.

[17] M. Keel and T. Tao, Endpoint Strichartz estimates, Amer. J. Math., 120 (1998), 955–980.

[18] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J., 40 (1991) 33–69.

[19] C. E. Kenig, G. Ponce and L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation, J. Amer. Math. Soc., 4(1991) 323–347.

[20] C. E. Kenig, G. Ponce and L. Vega, Small solutions to nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré, Nonlinear Anal., 10 (1993) 255–288.

[21] C. E. Kenig, G. Ponce and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, Invent. Math., 134 (1998) 489–545.

[22] C. E. Kenig, G. Ponce and L. Vega, The Cauchy problem for quasi-linear Schrödinger equations. Invent. Math., 158 (2004) 343–388.

[23] S. N. Kruzhkov, A. V. Faminskii, Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation, Math. USSR Sbornik, 48 (1984), 93–138.
[24] H. Pecher and W. von Wahl, Time dependent nonlinear Schrödinger equations. Manuscripta Math., 27(2) (1979) 125–157.

[25] J. Segata, Well-posedness for the fourth order nonlinear Schrödinger type equation related to the vortex filament, Diff. Integral Eqs., 16 (7) (2003) 841–864.

[26] J. Segata, Remark on well-posedness for the fourth order nonlinear Schrödinger type equation, Proc. Amer. Math. Soc., 132 (2004) 3559–3568.

[27] P. Sjölin, Regularity of solutions to the Schrödinger equation, Duke Math. J., 55 (1987) 699–715.

[28] H. Triebel, Theory of Function Spaces, Birkhäuser-Verlag, 1983.

[29] L. Vega, Schrödinger equations: pointwise convergence to the initial data, Proc. Amer. Math. Soc., 102 (1988) 874–878.

[30] B. X. Wang, Introduction to nonlinear evolution equations, preprint.

[31] B. X. Wang and C. Y. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, J. Diff. Eqns., 239(2007), 213–250.

[32] B. X. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, J. Diff. Eqns., 232 (2007) 36-73.

[33] B. X. Wang and Y. Z. Wang, Global well posedness and scattering for the elliptic and non-elliptic derivative nonlinear Schrödinger equation with small data, arXiv:0803.2634v1.

[34] B. X. Wang and L. F. Zhao and B. L. Guo, Isometric decomposition operators, function spaces $E^\lambda_{p,q}$ and applications to nonlinear evolution operators, J. Funct. Anal., 233 (2006) 1–39.