GRavitational RADIATION BY MAGNETIC FIELD: APPLICATION TO MILLISECOND MAGNETARS

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ABSTRACT

We investigate the direct contribution of the magnetic field to the gravitational wave generation. To do so, we study the post-Newtonian energy-momentum tensor of the magnetized fluid and the post-Newtonian expansion of the gravitational potential in the wave zone. We show that the magnetic field appears even in the first post-Newtonian order of the multipole moment tensor. Then, we derive an explicit relativistic correction containing the magnetic field contribution to the well-known quadrupole formula. As an application of this derivation, we explicitly prove that the magnetic field of millisecond magnetars can be a promising source of the gravitational waves. We show that this type of gravitational wave is strong enough to be detected by the next generation of detectors.

Keywords: Gravitational waves, magnetohydrodynamics, post-Newtonian theory

1. INTRODUCTION

Recently, a new episode in astronomy and astrophysics have been started after detecting the gravitational waves (GWs) generated from the binary black hole and binary neutron star mergers by LIGO and Virgo (Abbott et al. 2016a,b, 2017c,a,b). Among the astrophysical candidates for sources of GWs, the coalescence of the binary systems produces the strongest GW. The current gravitational-wave detectors can only observe these types of waves. Nevertheless, it is expected by constructing and progressively developing space-based and ground-based interferometers, weaker signals can be discovered in the near future (Sathyaprakash et al. 2012; Amaro-Seoane et al. 2017; Sato et al. 2017).

Deformed neutron stars are another source for GWs. This deformation can be induced by several phenomena. For example, the existence of the strong magnetic field inside the neutron star is believed to be flux frozen and independent of the spin of the star (Bocquet et al. 1995; Bonazzola & Gourgoulhon 1996; Ioka 2001). This class of highly magnetized neutron stars is named magnetar. The Soft Gamma Repeaters (SGRs) and the Anomalous X-ray Pulsars (AXPs) are categorized as magnetar candidates (Duncan & Thompson 1992; Thompson & Duncan 1993; Woods & Thompson 2004). The order of magnitude of the magnetic field is estimated about $10^{15}$G on the surface of these systems (Melatos 1999; Makishima et al. 2014). In Usov (1992), Nakamura (1998), Kluzniak & Ruderman (1998), and Wheeler et al. (2000), it is shown that the maximum strength of the magnetic field inside a neutron star can be of the order $\sim 10^{18}$G.

In the dipole model for magnetars, the inclination between the magnetic dipole moment and the rotation axis of the magnetar can induce the magnetic ellipticity and consequently produce GW.

In fact, the asymmetric magnetic pressure and boundary conditions of the magnetic field at the stellar surface distort the magnetar. Therefore, the rotation of the deformed star around the non-principal axes generates GW, for more detail see Bonazzola & Gourgoulhon (1996) and Cutler (2002). We call it the “indirect” effect of the magnetic field on GWs.

On the other hand, the super-strong magnetic field around magnetars appears as an asymmetric “mass-energy” distribution. Therefore, in principle, it can “directly” contribute to GWs. More specifically, a strong magnetic field can appear not only in the equation of motion of the system’s components and consequently in GWs, as an indirect role, but also in the multipole moment of the source.

In this letter, we utilize the well-known post-Newtonian (PN) approximation to study the direct contribution of the magnetic field to GWs. It is shown that GW can be appropriately explained by applying the PN theory to the higher-order corrections (Poisson & Will 2014; Blanchet 2014). In this context, the famous Einstein quadrupole formula is the leading term of the gravitational-wave signals. The important step to derive GW is to calculate the PN multipole moment of the source. To do so, one should have enough information about the energy-momentum tensor, $T^{\alpha\beta}$, of the system to the required PN corrections. Therefore, by considering the role of the magnetic field in the PN expansion of the energy-momentum tensor, one can find the direct influence of the magnetic field on GW.

Here, by utilizing the results of Nazari & Roshan (2018), hereafter NR, where the PN energy-momentum tensor is obtained in the presence of the electromagnetic field, for the first time in the literature, we derive the magnetic field contribution to the PN expansion of the multipole moment and consequently to the PN waveform of the gravitational-wave signals. In the present work, we restrict ourselves to the first PN order, hereafter 1PN, and leave the higher-order contributions of the magnetic field, due to the multipole moments and the wave-zone part, for future works.

We then apply these PN multipole moments to obtain GWs generated by a single rotating magnetar. We show that the magnetic field of the fast spinning magnetar can be a promising source of GWs in the early stages of its life, provided that the magnetic and rotation axes are not parallel. Next, we use two simple models for the magnetic structure of magnetars and analytically display how the magnetic field can emit GW. Finally, estimating...
the characteristic strain of GW for millisecond magnetar, we show that the generated GW is strong enough to be observed by near future detectors.

Throughout this letter, the Greek indices vary from 0 to 3. Also, the Latin indices stand for the coordinates x, y, and z. And \( \eta^{\alpha\beta} = \text{diag}(-1, 1, 1, 1) \) is the Minkowskian metric.

2. MAGNETIC FIELD’S CONTRIBUTION TO THE MULTIPOLE MOMENTS

In this section, we assemble the essential relations needed to study GWs in the presence of the magnetic field. Because of the presence of electromagnetic fields, \( T^{\alpha\beta} \) includes both the matter and field contributions, \( T^{\alpha\beta} = T^{\alpha\beta}_{\text{fluid}} + T^{\alpha\beta}_{\text{field}} \). For the matter portion, we assume that fluid is perfect. So we have 
\[
T^{\alpha\beta}_{\text{fluid}} = (\rho + \epsilon/c^2 + p/c^2) u^\alpha u^\beta + pg^{\alpha\beta},
\]
where \( \rho \) is the mass density, \( \epsilon \) is the proper internal energy density, \( p \) is the thermal pressure, and \( u^\alpha = (c, v) \) is the velocity field in which \( \gamma = u^0/c \). Furthermore, the contribution of electromagnetic fields to the energy-momentum tensor in terms of the electromagnetic field tensor \( F^{\alpha\beta} \), is expressed as 
\[
T^{\alpha\beta}_{\text{field}} = 1/\mu_0 \left( F^{\alpha\mu} F^{\beta}_\mu - 1/4 \mathbf{g}^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right),
\]
where \( \mu_0 \) is the vacuum permeability. By utilizing the components of the PN metric, one can obtain the covariant components of the electromagnetic field tensor and consequently find the energy-momentum tensor, see equations (30)-(32) in NR. Let us only write the magnetic terms in \( T^{\alpha\beta}_{\text{field}} \): 
\[
T^{\alpha\beta}_{\text{field}} \simeq B^2/2\mu_0 + \cdots \quad \text{and} \quad T^{jk} = -1/\mu_0 \left[ B_j B^k + (1/2B^2) \delta^{jk} \right] + \cdots.
\]
Of course we keep all the terms in our calculations. Since \( T^{\alpha\beta}_{\text{field}} \) is traceless, then there are magnetic contributions in \( T^{00} \) and \( T^{jk} \) with the same PN orders.

In order to find the gravitational-wave field, the main task is to obtain the gravitational potentials. In the PN context, the gravitational potentials \( h^{jk} \) in the far-away wave zone where the radiative aspects are important is given by
\[
h^{jk} = \frac{2G}{c^4 R_d} \frac{\partial^2}{\partial \tau^2} \left\{ Q^{jk} + Q^{jka} N_a + Q^{jkab} N_a N_b \right\} + \frac{1}{3} Q^{jkab} N_a N_b N_c + \cdots + \frac{2G}{c^4 R_d} \left\{ P^{jk} + P^{jka} N_a \right\},
\]
in which \( Q^{jk} \), \( Q^{jka} \), \( Q^{jkab} \), and \( Q^{jkabc} \) are the multipole moments, and \( P^{jk} \) and \( P^{jka} \) are surface integrals given by Eqs (11.119a)-(11.120b) in Poisson & Will (2014), respectively. In above relation, \( N = x/R_d \) is a unit vector in the \( x \) direction, \( R_d = |x| \) is the distance to the field point \( x \), and \( \tau = t - R_d/c \) is the retarded time. As mentioned previously, the goal of this letter is to survey the direct contribution of the electromagnetic fields to GW. So, in the following, we estimate the order of magnitude of the multipole moments and surface integrals, and specify the terms containing electromagnetic effects that appear in the gravitational potential.

By considering the definition of the multipole moments, we can symbolically indicate the \( c^n \) orders as 
\[
Q^{jk} = c^0 A_1 + c^{-2} A_2 + \cdots \quad \text{in which the magnetic field has a portion in order } c^{-2}, 
Q^{jka} = c^{-1} B_1 + c^{-3} B_2 + \cdots \quad \text{in which the electric and magnetic fields both contribute at order } c^{-3}, 
Q^{jkab} = c^{-2} C_1 + c^{-4} C_2 + \cdots \quad \text{in which the magnetic field has portions in orders } c^{-2} \text{ and } c^{-4} \text{ and the electric field has only a portion of order } c^{-4}, \quad \text{and} \quad Q^{jkabc} = c^{-3} D_1 + c^{-5} D_2 + \cdots \quad \text{in which the magnetic field has portions in orders } c^{-3} \text{ and } c^{-5} \text{ and the electric field contribution is of order } c^{-5}.
\]
For the surface integrals, we see that \( P^{jk} = c^0 E_1 + c^{-2} E_2 + \cdots \) where the magnetic field has portions in orders \( c^3 \) and \( c^{-2} \) and \( P^{jka} = c^{-1} F_1 + c^{-3} F_2 + \cdots \) where the magnetic field contributes at orders \( c^{-1} \) and \( c^{-3} \).

Here, as a good approximation, we assume that, in the definition of \( P^{jk} \) and \( P^{jka} \), the domain of the integrals, \( M^1 \), is sufficiently large so that it contains the matter distribution and its associated electromagnetic fields. Therefore, the matter distribution, as well as the electromagnetic fields, can be ignored on its boundary, \( \partial M \). Consequently, the magnetic field contributions due to the surface integrals \( P^{jk} \) and \( P^{jka} \) evaluated on \( \partial M \) may be automatically omitted from the gravitational potential. Keeping this fact in mind and considering the estimations listed above, we deduce that the magnetic field plays a role at least in 1PN order in the multipole moments \( Q^{jk} \) and \( Q^{jkab} \). In the present work, as a preliminary evaluation of the direct influence of the magnetic field on the gravitational-wave fields, we only endeavor to compute the dominant terms in the definition of \( Q^{jk} \) and \( Q^{jkab} \).

Therefore, by considering the above-mentioned points and keeping the dominant terms, Eq. (1) reduces to
\[
h^{jk} = \frac{2G}{c^4 R_d} \left( \tilde{T}^{jk} + \tilde{Q}^{jkab} N_a N_b \right),
\]
in which \( \tilde{T}^{jk} = T^{jk}_M + T^{jk}_F + O(c^{-2}) \) and \( \tilde{Q}^{jkab} = Q^{jkab} + \delta^{jk} T^{jk}_F + O(c^{-2}) \). Here, overdots represent the second derivative with respect to the retarded time \( \tau \); and \( T^{jk}_M \), \( T^{jk}_F \), and \( \tilde{Q}^{jkab} \) are given by
\[
T^{jk}_M (\tau) = \int_M \rho (\tau, x) x^j x^k d^3 x,
\]
\[
T^{jk}_F (\tau) = \frac{1}{2\mu_0 c^2} \int_M B^j (\tau, x) x^j x^k d^3 x,
\]
\[
\tilde{Q}^{jkab} (\tau) = -\frac{1}{\mu_0 c^2} \int_M B^j (\tau, x) B^k (\tau, x) x^a x^b d^3 x,
\]
respectively. Therefore, we carry out our calculation beyond the famous quadrupole formula \( h^{jk}_M = (2G/c^4 R_d) \tilde{T}^{jk}_M \) by incorporating \( T^{jk}_F \) and \( \tilde{Q}^{jkab} \) into this relation. It should be stressed that some part of \( Q^{jkab} \) which is proportional to \( \delta^{jk} \), has no contribution to the transverse-tracefree piece of \( h^{jk} \), i.e., to the gravitational-wave field. So, hereafter, we drop this term and reduce \( Q^{jkab} \) to \( \tilde{Q}^{jkab} \). In summary, to find the direct contribution of the magnetic field to GWs, the transverse-tracefree piece of \( h^{jk}_F = (2G/c^4 R_d) \left( \tilde{T}^{jk}_F + \tilde{Q}^{jkab} N_a N_b \right) \) must be calculated. In Sec. 3, we attempt to obtain this part of the gravitational-wave field for a single rotating magnetar.

3. ROTATING MAGNETAR

\footnote{1 M is a sphere with an arbitrary radius R which is \( R < \lambda_c \). Here, \( \lambda_c \) is the characteristic wavelength of the gravitational waves generated by the source.}
Here, we investigate the magnetar which is highly magnetized and its magnetic field magnitude is about $\sim 10^{15}$ G. In the following, by introducing a simple model, the vacuum rotating dipole model, we attempt to describe the magnetic field configuration inside and outside the neutron star. Utilizing this model, we display how GW can change in the presence of such magnetic field shape. Then we compare the results obtained from this simple model with a more realistic model where the magnetar has a corotating dense magnetosphere. This comparison reveals that the results of the simple model are reliable and can be used to interpret GWs emitted from the magnetic field of the real magnetar up to 1PN order.

3.1. Simple model

We launch our calculation by choosing a model simply describing the configuration of the magnetic field inside and outside of magnetars. This simple model is introduced in Bonazzola & Gourgoulhon (1996). We consider a uniformly magnetized sphere with a constant magnetization $\mu_m$ given by

$$\mu_m = \mu_m e_r$$

in which $\mu_m$ is a constant parameter and $e_r$ is a constant unit vector indicating the direction of $\mu_m$. For the magnetic dipole moment, we have $m = (4\pi/3) a^3 \mu_m$, where $a$ is the radius of the star. This magnetized star has the uniform magnetic field inside and the dipole field outside the star (Jackson 2007). So we have

$$B_m = \frac{2}{3} \mu_m \mu_m$$

for $r < a$, (5a)

$$B_{\text{out}}(x) = \frac{\mu_0}{4\pi} \frac{3n (m \cdot n) - m}{r^3}$$

for $r > a$, (5b)

where $n = x/r$ is the unit vector in the direction $x$, and $r = |x|$. Also, we assume that this sphere rigidly rotates around one of its principal axes with the angular velocity $\Omega$. Here, without loss of generality, the $z$-axis is chosen to be parallel to the rotation axis.

In order to find GWs emitted by the magnetic field of this system, as a first task, we should derive the field contribution of the quadrupole and 4-pole moments, i.e., $T_F^{jk}$ and $Q_F^{jkab}$, respectively. We first find these tensors in the coordinate frame $(x, y, z)$ that rotates with the star. To do so, we rewrite the magnetization $\mu_m$ in the Cartesian coordinate system and then insert Eqs. (5a) and (5b) within Eqs. (3b) and (3c). It should be noted that to carry out the integration over sphere $M$, we separate the domain of the integral into the interior and exterior regions of the star. The only remarkable point is that, after deriving the radial integral related to the star exterior, we can freely discard $R$-dependent terms. In fact, in the modern approach of PN approximation, it is shown that the $R$-dependent terms can be discarded. Next, by using the appropriate coordinate transformation, we obtain the components of $T_F^{jk}$ and $Q_F^{jkab}$ in the non-rotating frame which is shown by $(x', y', z')$.

According to Eq (2), the next task is to evaluate the second time derivatives of $T_F^{jk}$ and $Q_F^{jkab}$. After some manipulations, we then obtain the components of the gravitational potential $h^{jk}$. As a final task in this subsection, we should find the transverse-tracefree piece of this tensor. To do so, we recall two independent polarizations of the gravitational-wave tensor $h^{jk}$, i.e., $h_+ = h_{xx} + h_{yy}$ and $h_\times = h_{xy} - h_{yx}$. It is common to introduce a “detector-adapted” frame $(X, Y, Z)$. $h_+$ and $h_\times$ can be produced in terms of bases $e_X$ and $e_y$ that build the transverse subspace as follows: $h_+ = \frac{1}{2}(e_X^j e_X^k - e_Y^j e_Y^k) h_{jk}$ and $h_\times = \frac{1}{2}(e_X^j e_Y^k + e_Y^j e_X^k) h_{jk}$ (Poisson & Will 2014). It is assumed that the origin of this frame coincides with the origin of non-rotating frame. Also, the $Z$-axis represents the direction of the GWs propagation toward the detector. The angle between $Z$ and $x'$-axes is shown by the inclination angle $i$. Moreover, the $X$-axis is the intersection of the plane of the sky with the equatorial plane of the star. It is considered that the $x'$-axis is parallel to the long axis of the deformed star. At $t = 0$ when $x' = x$, the angle between $X$ and $x'$ axes is exhibited by $\omega$.

After some simplifications, we find the plus polarization as

$$h_+ = h_{0M} H_{xM} + h_{0F} H_{xF},$$

where

$$h_{0M} = \frac{4G c M \Omega^2}{r^4 R^4},$$

$$h_{0F} = \frac{32 G A_0 \Omega^2}{21 c^4 R^4},$$

are the matter and field parts of the GW amplitude, respectively. Here, $A_0 = \frac{1}{2} \frac{\Omega}{\geff} \mu_0 a^2 q^2$ and $\geff = (I_1 - I_2) / I_3$ is known as the ellipticity parameter measuring the deformation of the body in which $I_1 = I_{yy} + I_{zz}$, $I_2 = I_{xx} + I_{zz}$, and $I_3 = I_{xx} + I_{yy}$ are the principal moments of inertia. Eq. (6b) is one of our main results in this letter. Also, $H_{+M}$ and $H_{+F}$ are the matter and field portions of the scale-free polarizations of the GW propagation given by

$$H_{+M} = \frac{1}{2} \left(1 + \cos^2 i\right) \cos 2(\Omega t + \omega),$$

$$H_{+F} = -\sin \theta_0 \left[ \sin \theta_0 \left(1 + \cos^2 i\right) \cos 2(\Omega t + \omega + \phi_0) + \cos \theta_0 \cos i \sin \theta \sin(\Omega t + \omega + \phi_0) \right],$$

in which $\theta_0$ and $\phi_0$ are the angles between the magnetization and $z$ and $x'$-axes in the rotating frame, respectively. For the cross polarization, we also arrive at $h_\times = h_{0M} H_{xM} + h_{0F} H_{xF}$, in which two scale-free polarizations $H_{xM}$ and $H_{xF}$ are introduced as

$$H_{xM} = \cos \phi \sin \omega \sin 2(\Omega t + \omega),$$

$$H_{xF} = \sin \theta_0 \left[ \cos \theta_0 \sin \omega \sin(\Omega t + \omega + \phi_0) - 2 \sin \theta_0 \cos \omega \sin 2(\Omega t + \omega + \phi_0) \right].$$

3.2. Realistic model

In this subsection, we attempt to describe GWs emitted from a more realistic rotating magnetar by applying the well-known model already introduced in

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2 For the matter part of the quadrupole moment, i.e., $T_M^{jk}$ see chapter 11 of Poisson & Will (2014).

3 For proof see Poisson & Will (2014).
Goldreich & Julian (1969) (hereafter GJ). It is shown that these systems are not floating in the vacuum, and they have a dense magnetosphere rotating with the same frequency as the spinning star. GJ divide the space around the star into three zones and study the magnetic field configuration in each region. These regions include the light cylinder with the radius \( r \sin \theta = c/\Omega \) and the height \( z = \pm c/\Omega \), the wind zone, the region between the light cylinder and a sphere with a radius \( r \sim D/10 \) where \( D \) is the radius of the supernova shell, and the boundary zone, from the radius \( D/10 \) to \( D \) beyond which there is the interstellar material. It is shown that, in the light cylinder, the major component is the poloidal one approximately determined by the dipole magnetic field and, in the wave zone, the toroidal component dominates given by equation (14) in GJ.

In GJ, the magnetic field of the rotating neutron star is considered to be symmetric around the rotational axis, i.e., \( \theta_0 = 0 \) and \( \mu_m = \mu_m e_z \) in our previous calculations. Although this symmetric configuration cannot radiate GW, we obtain the field piece of the mass quadrupole-moment tensor, \( T_F^{jk} \), and estimate the magnitude of its components. By applying this model, one can obtain

\[
B_m = \frac{2}{3} \mu_0 \mu_m \quad \text{for } r < a, \quad (9a)
\]
\[
B_p = \frac{\mu_0}{3} \frac{\mu_m a^3}{r^3} \left( 3 \cos^2 \theta + 1 \right)^{1/2} \quad \text{for } a < r < b, \quad (9b)
\]
\[
B_t = -B_p \left( \frac{r}{b} \right)^2 \sin^2 \theta - 1 \right)^{1/2} \quad \text{for } r > b, \quad (9c)
\]

for the dominant components of the magnetic field in each region\(^4\). It is straightforward to obtain the non-vanishing components of the above tensor as

\[
X_F^{xx} = 8A_0 \left( 1 - \frac{38 a}{21 b} \right), \quad (10a)
\]
\[
X_F^{yy} = 8A_0 \left( 1 - \frac{38 a}{21 b} \right), \quad (10b)
\]
\[
X_F^{zz} = 4A_0 \left( 3 - \frac{38 a}{7 b} \right). \quad (10c)
\]

Here, as mentioned before, we freely omit the \( R \)-dependent terms.

\(^4\) For more detail, see section III of GJ.
As it is obvious from the above relations, in addition to $A_0$, these components also involve the ratio of the star radius $a$ to the cutoff parameter $b$, i.e., $a/b$. It should be noted that by a similar calculation, one can investigate that the ratio $a/b$ also appears in the components of $Q_{jkab}$. To estimate the role of this term in the above equations, let us evaluate the magnitude of $a/b$.

To do so, we choose a typical neutron star with a radius $a = 12$ km and a rotational period $T = 10$ ms. For this realistic values, the ratio $a/b$ is of order $10^{-2}$. Even for this rapidly rotating neutron star, we see that this term is small. Therefore, one can freely ignore the term including $a/b$ in comparison with the first term in Eqs. (10a)-(10c).

Considering the above arguments and recalling our discussion in Subsec. 3.1, one can conclude that if this realistic system can radiate GW, the field part amplitude of the generated signal will be of the same order as the vacuum model in Eq. (6b). This directly means that our simple rotating neutron star model presented in Subsec. 3.1 is effectively helpful to construct a viable model for a magnetar.

### 3.3. Discussion and application to magnetars

At first glance, it seems that the phase shift $\varphi_0$ is of no physical importance. However, one may note that the wave contributions of the matter and magnetic parts interfere with each other, and $\varphi_0$ controls the final intensity of the outcome. It should be noted that, in this case, $\varphi_0$ is actually the angle between the magnetization and the long axis of the deformed star. The neutron star is not necessarily deformed in the direction of the magnetic field lines and consequently $\varphi_0 \neq 0$. So, under this physical circumstance, the partial interferences may happen.

Another important parameter is the angle between the magnetic dipole $\mu_m$ and the rotation axis of the star, i.e., $\theta_0$. It is clear that if $\theta_0 = 0$, then the magnetic field will not contribute to GW. To see the impact of this parameter, in Figs. 1 and 2, we respectively exhibit Eqs. (7b) and (8b) in terms of $\theta_0$ and dimensionless retarded time $\tau/T$, which is divided to the rotational period of the star $T = 2\pi/\Omega$.

As the last and the main point in this subsection, it should be noted that, at the inclination angle $\iota = \pi/2$, when the angle between the rotation and magnetic axes is about $\theta_0 = \pi/4$ or $\theta_0 = 3\pi/4$, there is a possibility to receive a gravitational radiation from the magnetic star with a maximum magnitude $|H_{\times F}| \approx 0.5$ for the field part of cross polarization. This fact can be easily grasped from the bottom right panel of Fig. 2. On the other hand, under this condition, the gravitational-wave detectors cannot observe any wave generated by the corresponding matter contribution. See Eq. (8a). Therefore, in this special situation, i.e., when $\iota = \pi/2$, every cross-polarization wave that is received from the rotating magnetar is only coming from its magnetic field part and matter part has no portion. This interesting fact helps us to discriminate between two different contributions.
of the matter and magnetic fields to the gravitational-wave signals. Also, this fact moderately occurs for the plus polarization at \( \theta_0 = \pi/2 \) whose magnitude is about \( |H_{+F}| \approx 1 \).

Now, we estimate \( h_{\text{OF}} \) of the rapidly rotating magnetic neutron star known as the millisecond magnetar, e.g., see Usov (1992). By rewriting Eq. (6b) in terms of \( T \) and also using definition \( A_0 \), one can obtain that

\[
h_{\text{OF}} = \frac{32 \pi^3 G B^2 a^5}{105 \mu_0 c^5 T^2 R_d^2}.
\]

Here, we assume that the magnetar is in its early stages of evolution. Therefore, for this system with \( B = 10^{15} \, \text{G} \), \( a = 12 \, \text{km} \) (Dall’Osso et al. 2009), and \( T = 10 \, \text{ms} \), we arrive at \( h_{\text{OF}} \approx 5.5 \times 10^{-28} \). We also assume this system can exist at \( R_d = 1 \, \text{kpc} \). Furthermore, for this system with a moment of inertia of order \( I_3 = (2/5) Ma^2 \approx 10^{38} \, \text{kg} \, \text{m}^2 \) and a typical ellipticity parameter \( \epsilon_\text{M} \approx 10^{-6} \), one can obtain that the matter part of the GW amplitude, \( h_{\text{OM}} \), is of order \( \sim 4.1 \times 10^{-26} \). So, as deduced from these estimations, the field part of GWs released in the early stages of a magnetar’s life can be as much as one-hundredth of the signals due to the deformed rotating neutron stars. Obviously, this amplitude will increase for the models that predict stronger magnetic field inside the rapidly rotating magnetars, see Thompson & Duncan (1993).

Some theories state that the strength of the magnetic field can reach \( \sim 10^{18} \, \text{G} \) (Usov 1992; Nakamura 1998; Kluzniak & Ruderman 1998; Wheeler et al. 2000).

It is also instructive to compute the amplitude produced by the 21 well-known ordinary magnetars (not millisecond) classified in Olausen & Kaspi (2014). According to our results, the strongest amplitude belongs to SGR 1806−20 which is about \( \sim 1.8 \times 10^{-34} \). Therefore, the magnetic part of GWs generated by these magnetars is insignificant.

4. AMPLITUDE AND DETECTABILITY

Here, we investigate the characteristic strain of the gravitational-wave emissions generated by the pure magnetic field part of the millisecond magnetars. Then, we compare it with the sensitivity range of the current and forthcoming generation of gravitational-wave detectors. To do so, we use the relation between the characteristic strain \( h_c \) and the GW amplitude, see equation (35) of Moore et al. (2014). By considering \( h_c \) only for the magnetic contribution in the frequency range \( f \), we have

\[
h_c(f) = \sqrt{\frac{2f^2}{f}} h_{\text{OF}}
\]

in which the dot sign stands for the time derivative.

Taking the typical properties of the millisecond magnetar, we assume that \( B = 10^{15} \, \text{G} \), \( R_d = 1 \, \text{kpc} \), and the time of integration is of the order of a few days. In fact, we consider that this time is equal to the spin-down timescale of the millisecond magnetars \( t_{\text{sd}} \) given by \( t_{\text{sd}} \approx 4.7 \, \text{day} B_1^{-2.7} T_{\text{ms}}^{-2} \) in which \( T_{\text{ms}} = T_0 / 1 \, \text{ms} \) is the scaled birth spin period of the neutron star and \( B_1 = B_{13} / 10^{14} \, \text{G} \) is the surface dipole magnetic field \( B_d \) that is normalized to \( 10^{13} \, \text{G} \). (Metzger et al. 2017). In this case, one may write \( f \approx f / t_{\text{sd}} \), and consequently we have \( h_c(f) \approx 5.0 \times 10^{-27} f^{3/2} \).

Finally, we find that the characteristic strain of the millisecond magnetar is of the order \( 5.0 \times 10^{-24} \lesssim h_c \lesssim 1.6 \times 10^{-22} \) in the frequency band \( 10^2 \, \text{Hz} < f < 10^3 \, \text{Hz} \). This fact is illustrated in Fig. 3. Solid curves indicate the sensitivity of the current and forthcoming detectors like Advanced LIGO (aLIGO), Kamioka Gravitational Wave Detector (KAGRA), and LIGO A+(A+). The characteristic strain of the millisecond magnetars in terms of the frequency is displayed by a blue area in this figure. As shown, this area enters the sensitivity range of the several detectors, i.e., some part of this area lies above the sensitivity curve of the gravitational-wave detectors. This means that the signal-to-noise ratio \( (S/N) \) is large enough to be detected (Moore et al. 2014). So, the millisecond magnetars which are in their early stages of evolution can emit loud signals that are audible to these detectors.

It should be recalled that, as argued in Subsec. 3.3, under the specific circumstance, i.e., when the inclination angle \( \iota = \pi/2 \), the detectors are relatively blind to the matter part of GWs and only receive a part of the signal which directly comes from the magnetic field contribution of the millisecond magnetars. Our results show that after a time of integration, e.g., a few days, the field part of GW emitted from the millisecond magnetar can be detected by the next generation of the detectors.

As the direct contribution of the magnetic field seems to be impressive, in the sequel, it would be interesting to study the black hole-magnetar and magnetar-magnetar binary systems and evaluate the direct contribution of the magnetic field associated with these systems to the gravitational-wave signal (Nazari & Roshan 2020).

ACKNOWLEDGMENTS

We would like to thank Bahram Mashhoon for reading the manuscript and providing us with constructive comments. This work is supported by Ferdowsi University of
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