Information divergence-based low probability of intercept waveform detection for multi-antenna intercept receivers

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Abstract: Based on the asymptotic spectral distribution of sample covariance matrix, a new low probability of interception (LPI) signal detection method for multi-antenna intercept receiver is proposed via reforming the noise sequence into a Wishart matrix. The authors deduce the asymptotic cumulative distribution function (CDF) of eigenvalues of sample covariance matrix. Also a numerical Kullback–Leibler divergence of the empirical spectral CDF based on test samples from the deduced asymptotic CDF is established, which is treated as the test statistic. By comparing with other signal detection methods, simulation results show the new LPI signal detection method is superior and robust.

1 Introduction

With the increasing complexity and variability of hostile environment, the property of low probability of interception (LPI) has been rapidly developed as an important performance indicator of radar [1–4]. The basic methods to enhance the LPI performance of radar waveforms are mainly based on the utilisations of low peak powers, wide frequency bandwidth and long time interval [5–7]. This paper will investigate the detection method of the following common LPI waveforms [4, 8]: linear frequency modulation (LFM) waveform and phase modulation (PM) waveforms including Frank code, P1, P2, P3 and P4 codes. Unlike active radars, interceptors cannot implement matched filtering to improve signal-to-noise ratio (SNR) without prior knowledge of the received signals. Thus, various time-frequency analyses are utilised for passive detection [9–11], such as short time Fourier transformation, Wigner–Ville distribution, Choi–Williams distribution and so on. Besides, pattern recognition methods including Hough transform [12] and Radon transform [13] are also applied to the detection of LFM waveform. These approaches can be classified as the coherent method, which depend on the specific modulation methods. Correspondently, the non-coherent method does not need any information about the structure of the received signals, which makes this kind of detection methods is generic. Therein, the most classical way is the energy detection method, which determines the presence of the transmitted signals based on the energy level of the received signals and background noise. In this paper, we will study a non-coherent method based on the information divergence.

For the passive radar with one receiving antenna, the weak LPI signal detection problem can be formulated as a binary hypothesis test between null ($\mathcal{H}_0$, the LPI signal $x(t)$ is absent) and alternative ($\mathcal{H}_1$, the LPI signal $x(t)$ is present) hypotheses

$$\mathcal{H}_0: x(t) = a(t),$$
$$\mathcal{H}_1: x(t) = s(t) + a(t).$$

(1)

where $x(t)$ is the received signal, $a(t)$ is the background noise, $n = 1, 2, ..., N$, and $N$ is the sample size. The common method for solving the hypothesis is normality test [14], such as Wilk–Shapiro test, D’Agostino test, Lilliefors test and Jarque–Bera (JB) test. However, generally, the performance of intercept receiver with multiple antennas is superior to that with one receiving antenna. The binary hypothesis test in (1) is instinctively turned into a multiple hypothesis test

$$\mathcal{H}_i: x(t) = a(t),$$
$$\mathcal{H}_i: x(t) = s(t) + a(t).$$

(2)

where $i = 1, 2, ..., M$, and denotes the $i$th receiving antenna. For dealing with the multiple hypothesis test, the classical method is the likelihood ratio test. However, the exact expressions of likelihood function $p(x|\mathcal{H}_i)$ are difficult to obtain, where $x$ is the aggregation of $x(t) = [x_1(t), x_2(t), ..., x_M(t)]^T$. Therefore, most studies are based on the assumptions of independence and Gaussianity of the received signal samples $x_i(t)$. Therein, maximum–minimum eigenvalue (MME) detection is one of the most effective detection methods [15], which uses eigenvalues of covariance matrix as the test statistic. Nevertheless, the MME method discards most of eigenvalue information for utilising only the maximum and minimum eigenvalues, which can somewhat degrade the detection performance.

Since the effective means of comparing random data sets is to utilise their associated probability density functions (PDFs), and Kullback–Leibler divergence (KLD) has been confirmed to be a powerful tool to measure the information of multivariate data, in this paper, we utilise a numerical KLD of the empirical spectral cumulative distribution function (CDF) based on received signals from the asymptotic spectral CDF of covariance matrix to deal with weak LPI signal detection problem. Simulation results verify our proposed detection method outperforms all other compared methods in multi-antenna passive radar.

2 Analysis of LPI radar waveforms detection

LPI waveforms are designed to reduce the detection, identification and location performance of enemy intercept receivers, and finally to achieve that the enemy interception range is less than the radar detection range. Common wideband modulation techniques used by LPI waveforms include frequency modulation and phase modulation. Therein, LFM waveform is one of the most typical frequency modulation waveforms. The general envelope formulas of LFM pulse and PM pulse can be written as
$$x(t) = \begin{cases} \frac{1}{\sqrt{T}} \text{rect}(\frac{t}{T}) \cos(\omega_k t), & \text{LFM} \\ \frac{1}{\sqrt{T}} \sum_{m=1}^{M} \cos(\phi_m) \text{rect}(\frac{t-(m-1)t_b}{t_b}), & \text{PM} \end{cases}$$ (3)

where $k = \pm (B/T)$ is the frequency slope, $B$ is the frequency band, $T$ is the pulse duration, $\{\phi_m, m = 1, 2, ..., M\}$ is the phase code sequence, and $t_b$ is the duration of each code element.

For these LPI waveforms shown in (3), their energy are spread in their frequency bands ($B = kT$ and $B = 1/t_b$). Therefore, the energy in each channel of the most advanced channelised intercept receiver (or in each frequency point) is at a very low level. In order to accurately identify the intercept signal, each frequency point occupied by radar waveforms must be detected. So, traditional energy-based detection methods will not work for LPI waveforms. Similarly, the maximum eigenvalue of intercept signal which represents the energy characteristic also do not work for LPI waveforms. At this point, to unlock more features of LPI waveforms, the rest of the eigenvalues besides the maximum one should be exploited for the detection of LPI waveforms. The next section will present how to use the PDF of eigenvalues to perform LPI waveform detection.

### 3 Information divergence detector

From the perspective of probability distributions, the multiple hypothesis test in (2) can be equivalently written as

$$H_0^i$: $x_i \sim N_x(\mu_{x_i}, \Sigma)$
$$H_i$: $x_i \sim N_x(\mu_{x_i}, \Sigma)$ (4)

where $x_i = [x_i(1), x_i(2), ..., x_i(N)]^T$, $i = 1, 2, ..., M$, the symbol $(\cdot)^T$ denotes the transpose, $\Sigma$ denotes the $N \times N$ identity matrix, and $\sigma$ is the noise power.

Suppose the $M$ receiving antennas are independent from each other. The received signals $x_i$, $i = 1, 2, ..., M$ can be written in the matrix form $X = (\alpha_i)[x_i, x_i, ..., x_M]$, and the covariance matrix can be defined as

$$S = \frac{1}{M} XX^T.$$ (5)

According to the definition of Wishart matrix and M–P Law [16], the covariance matrix $S$ has a Wishart distribution, under the null hypothesis $H_0$, that is $S \sim \mathcal{W}(M_0, M)$, $\mathcal{W}$ means a Wishart distribution. As $N, M \rightarrow \infty$ and $N/M \rightarrow \gamma \in (0, 1)$, the empirical spectral distribution $p^S(x)$ of covariance matrix $S$ converges weakly to the Marchenko–Pastur distribution with density

$$p^S(x) = \frac{2[(b-x)(x-a)]}{\pi \sigma^2} \text{I}_{[a,b]}(x),$$ (6)

where $a = (1 - \sqrt{\gamma})^2$ and $b = (1 + \sqrt{\gamma})^2$.

Unlike the energy detector in non-coherent methods, here, we use the information divergence level between the empirical spectral distribution $p^S$ and the asymptotic spectral distribution $p^\gamma$ instead of the energy level between the received signal and the background noise to process the LPI signal detection problem. The test statistic of information divergence detector can be defined as

$$D^{KL}(p^\gamma \mid \mid p^S) = \int_{a}^{b} p^\gamma(x) \ln \left( \frac{p^\gamma(x)}{p^S(x)} \right) dx$$ (7)

where $D^{KL}(p \mid \mid q)$ is the KLD of the empirical spectral distribution $p^S$ from the asymptotic spectral distribution $p^\gamma$.

Since it is difficult to obtain the empirical spectral distribution function $p^S(x)$ based on the eigenvalues $\{\lambda_j, j = 1, ..., N\}$ of the covariance matrix $S$ and the analytical solution of the definite integral operation in (7), we denote $c_p, p = 0, 1, 2, ..., P$ as the partition points in the interval $[a, b]$, and $a = c_0 < c_1 < c_2 < \cdots < c_P = b$, and the test statistic $D^{KL}(p^\gamma \mid \mid p^S)$ can be numerically recast as

$$D^{KL}(p^\gamma \mid \mid p^S) = \int_{a}^{b} p^\gamma(x) \ln \left( \frac{p^\gamma(x)}{p^S(x)} \right) dx$$ (8)

$$\approx \sum_{p=0}^{P} (P(c_p) - P(c_{p-1})) \ln \left( \frac{P(c_p) - P(c_{p-1})}{P(c_{p-1})} \right)$$ (8)

$$\text{as } P \rightarrow \infty \text{ and } \max_{1 \leq p \leq P} \left| \frac{P(c_p) - P(c_{p-1})}{P(c_{p-1})} \right| \rightarrow 0$$

where $P(x)$ and $P^S(x)$ are the CDFs of $p^\gamma(x)$ and $p^S(x)$, respectively, and

$$p^S(x) = \frac{1}{\#D} \# \{j \in N; \lambda_j \leq x\},$$ (9)

where, $\#D$ denotes the cardinality of the set $D$. The expression of asymptotic spectral CDF $P^\gamma(x)$ is calculated as follows.

According to (6), the asymptotic spectral CDF of sample covariance matrix $S$ can be calculated as

$$P^\gamma(x) = \begin{cases} 0, & x \leq a \\ \frac{1}{2\pi \sigma^2} \int_{a}^{x} u^{-1} \sqrt{(b-u)(a-u)} du, & a < x \leq b \\ 1, & x > b \end{cases}$$ (10)

When $a < x \leq b$, by setting $u = 1 + y + z$, we have

$$\frac{1}{2\pi \sigma^2} \int_{a}^{b} u^{-1} \sqrt{(b-u)(a-u)} du$$

$$= \frac{1}{2\pi \sigma^2} \int_{a}^{z} (1 + y + z)^{-1} \sqrt{4y + z^2} dz$$

$$= \frac{1}{2\pi \sigma^2} \sum_{j=0}^{\infty} (\frac{1}{2\pi \sigma^2})^j \int_{a}^{z} \sqrt{4y + z^2} dz$$ (11)

$$(\text{since } |z| < 1 + y)$$

$$= \frac{2}{\pi \sigma^2} \sum_{j=0}^{\infty} (\frac{1}{2\pi \sigma^2})^j \int_{a}^{z} \sqrt{4y + z^2} dz$$

$$(\text{by setting } z = 2\sqrt{\gamma})$$

When $a < x \leq 1 + y$

$$\int_{a}^{1} \sqrt{1-r} dr$$

$$= \frac{1}{2} \int_{a}^{1} (1-\gamma)^{1/2} (1-a) dr$$ (12)

$$= \frac{1}{2} \int_{a}^{1} \sqrt{1-r} \frac{1}{2\pi \sigma^2} \int_{a}^{z} (1 + y + z)^{-1} \sqrt{4y + z^2} dz$$

$$= \frac{1}{2} \int_{a}^{1} \sqrt{1-r} \frac{1}{2\pi \sigma^2} \int_{a}^{z} \sqrt{4y + z^2} dz$$

$$= \frac{1}{2} \int_{a}^{1} \sqrt{1-r} \frac{1}{2\pi \sigma^2} \int_{a}^{z} \sqrt{4y + z^2} dz$$

When $1 + y < x \leq b$

$$\int_{a}^{1} \sqrt{1-r} dr + \int_{1}^{x} \sqrt{1-r} dr$$ (13)
wherein
\[
\int_{-\infty}^{\infty} r \sqrt{1 - r^2} \, dr = \left( -\frac{1}{2} \right) \int_{0}^{\infty} \omega^{1/2} (1 - \omega)^{1/2} \, d\omega
\]
(by setting \( t = \sqrt{\omega} \))
\[
= \left( -\frac{1}{2} \right) B\left(\frac{1}{2}, \frac{3}{2}\right)
\] (14)
and
\[
\int_{0}^{\infty} (x - 1 - y)^{2/4} r \sqrt{1 - r^2} \, dr = \frac{1}{2} \int_{0}^{\infty} (x - 1 - y)^{2/4} \omega^{1/2} (1 - \omega)^{1/2} \, d\omega
\]
(by setting \( t = \sqrt{\omega} \))
\[
= \frac{1}{2} B\left(\frac{x - 1 - y}{4y}, \frac{i + 1}{2}, \frac{3}{2}\right).
\] (15)
According to the above equations, the asymptotic spectral CDF of sample covariance matrix \( \mathbf{S} \) can be obtained as
\[
P^*(x) = \begin{cases} 
0, & x \leq a; \\
\frac{1}{x} \sum_{i=0}^{\infty} \left( (1 + y)^{i+1}, (2\sqrt{y})^i \cdot B\left(\frac{i+1}{2}, \frac{3}{2}\right) \right), & a < x \leq 1 + y; \\
\frac{1}{x} \sum_{i=0}^{\infty} \left( (1 + y)^{i+1}, (2\sqrt{y})^i \cdot B\left(\frac{i+1}{2}, \frac{3}{2}\right) \right), & 1 + y < x \leq b; \\
1, & x > b;
\end{cases}
\] (16)
where \( B \) is a beta function, and \( I \) is a regularised incomplete beta function. The diagram of asymptotic spectral CDF of sample covariance matrix \( \mathbf{S} \) is given in Fig. 1, according to (16) with \( y = 3/4 \). As shown in Fig. 1, the CDF is non-decreasing, right-continuous and \( \lim_{x \to -\infty} P^*(x) = 0, \lim_{x \to +\infty} P^*(x) = 1 \). So the CDF \( P^*(x) \) we derived is correct.

The KLD in (8) is always non-negative. Thus, the information divergence detector is a one-tailed test. If the value of test statistic \( D_{\text{KL}}(p' \parallel p^0) \) is no less than the threshold \( \text{Th}(P_{fa}) \), one may reject the null hypothesis \( H_0 \) at the pre-specified false alarm rate \( P_{fa} \) therein, the false alarm rate \( P_{fa} \) is the \((1 - P_{fa})\)-quantile of the distribution of the test statistic under the null hypothesis. However, the null distribution of \( D_{\text{KL}}(p' \parallel p^0) \) is not available for a finite sample size \( N \) and a specific number \( M \) of receiving antenna. Thus, the threshold \( \text{Th}(P_{fa}) \) can be only determined as the corresponding percentage points of the empirical distribution of the test statistic \( D_{\text{KL}}(p' \parallel p^0) \) obtained by Monte Carlo simulations.

4 Numerical results

In this section, we validate the superiority of our proposed information divergence-based detection method presented in Section 3 through comparing with common normality tests for a single antenna, which are Pearson's \( \chi^2 \) test, marked as Chi2; Jarque–Bera test, marked as JB; Lilliefors test, marked as Lillie; and D'Agostino test, marked as Dag, and the superior signal detection method for multi-antenna, which is MME method. Here the common LPI radar waveforms are considered, which include: LFM, Frank, P1, P2, P3 and P4 codes. These test waveforms have 100 MHz bandwidth and 1.438 \( \mu \)s pulse width. The code number of all PM waveforms is 144. We also suppose the sampling frequency of the intercept receiver is 500 MHz.

For the choice of partition point number \( P \), we perform a number of simulations with values ranging from 3 to 12 recommended by Dahiya and Gurland [17] and find the value of \( P \) almost has no effect on the detection performance. And it should be noted that for a small sample size, it is not feasible to use a large number of subintervals, due to the possibility of zero observations in at least some of them. As a result, we suggest the choice of \( P = 3 \).

4.1 Weak Frank code signal detection for intercept receiver with a single antenna

For a single antenna, the received signal can be split into \( M \) non-overlapping sub-vectors \( x_i, i = 1, 2, ..., M \). Then, the framework of signal detection for multiple antennas can be exploited here to detect signal for a single antenna.

In order to compare the effects of sub-vector number \( M \) and sub-vector length \( N \) on the detection performance, the sample frequency \( f_s \) are set to 2 GHz, the values of \( N \) are set to 10, 25 and 50, respectively, and the corresponding sub-vector number \( M \) are \( M = 288, M = 115 \) and \( M = 57 \).

At a false alarm rate of \( P_{fa} = 0.001 \), Fig. 2 depicts the detection probabilities of the weak Frank code signal with different SNRs. It clearly validates that our proposed normality test outperforms other common normality tests. For \( M = 57 \) and \( N = 50 \), the detection probability of the new normality test is close to 100% until the SNR declines to 0 dB. For \( M = 115 \) and \( N = 25 \), the new normality test still maintain its superiority, which can detect the weak Frank code signals effectively above \(-2 \) dB. In the same way, for \( M = 288 \) and \( N = 10 \), the detection probability is close to 100% until the SNR declines to \(-3 \) dB. In contrast, all compared normality tests fail to detect signal when the SNR is below 4 dB.

![Fig. 1 Diagram of the asymptotic spectral CDF of sample covariance matrix S with y = 3/4](http://creativecommons.org/licenses/by/3.0/)
Although a larger value of $M$ means a better detection performance, although a larger value of $M$ corresponds to a smaller value of sub-vector length $N$. Thus, we should give priority to setting a larger value of $M$ rather than a larger value of $N$.

### 4.2 LPI waveform detection for intercept receiver with multiple antennas

Here the common LPI radar waveforms are considered, which include: LFM, Frank, P1, P2, P3 and P4 codes. We consider the false alarm rate is $P_{fa} = 0.001$. Fig. 3 gives the detection performance of these LPI waveforms for the multi-antenna intercept receiver by using the information divergence detector. As shown in Fig. 3, all of the simulated waveforms can achieve reliability detection, whose detection probability is above 90%, when the SNR is equal or greater than $-12$ dB, with antenna number $M = 500$ and sample size $N = 720$. They almost have the same detection performance. Also the detection performance of P1 and P2 codes is slightly worse than other simulated waveforms.

In Fig. 4, the detection probability maintains constant value 100% over $-7$ dB in all the three experiments. The detection performance with parameters $M = 1000$, $N = 288$ is the best. The SNR of effective detection needs only more than $-9$ dB. The detection performance of parameters $M = 500$, $N = 432$ and parameters $M = 500$, $N = 288$ are almost the same. Both of them can obtain 100% detection probability, when the SNR is more than $-7$ dB. Compared with the MME method, our detection method is superior obviously, which is depicted in Fig. 4.

In regard to the choice of sub-vector number $M$, we run simulations with values 57, 115 and 288. The results in Fig. 2 show that a larger value of $M$ means a better detection performance, although a larger value of $M$ corresponds to a smaller value of sub-vector length $N$. Thus, we should give priority to setting a larger value of $M$ rather than a larger value of $N$.

In Fig. 4, the detection probability maintains constant value 100% over $-7$ dB in all the three experiments. The detection performance with parameters $M = 1000$, $N = 288$ is the best. The SNR of effective detection needs only more than $-9$ dB. The detection performance of parameters $M = 500$, $N = 432$ and parameters $M = 500$, $N = 288$ are almost the same. Both of them can obtain 100% detection probability, when the SNR is more than $-7$ dB. Compared with the MME method, our detection method is superior obviously, which is depicted in Fig. 4.

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