Disjunctive cuts for
Mixed-Integer Conic Optimization

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Abstract. Motivated by the recent success of conic formulations for Mixed-Integer Convex Optimization (MI-CONV), we investigate the impact of disjunctive cutting planes for Mixed-Integer Conic Optimization (MI-CONIC). We show that conic strong duality, guaranteed by a careful selection of a novel normalization in the conic separation problem, as well as the numerical maturity of interior-point methods for conic optimization allow to solve the theoretical and numerical issues encountered by many authors since the late 90s. As a result, the proposed approach allows algorithmic flexibility in the way the conic separation problem is solved and the resulting cuts are shown to be computationally effective to close a significant amount of gap for a large collection of instances.

Keywords: Mixed-Integer Convex Optimization · Disjunctive Cutting Planes · Conic Optimization.

1 Introduction

MI-CONV is a fundamental class of Mixed-Integer Non-Linear Optimization problems with applications such as risk management, non-linear physics (e.g., power systems and chemical engineering) and logistics, just to mention a few. Because of such a relevance, classical algorithms for Mixed-Integer Linear Optimization (MILP) have been successfully extended to MI-CONV, like Branch and Bound \cite{10} or Benders decomposition \cite{20}; others like the Outer Approximation scheme \cite{17} have been designed specifically for MI-CONV. In addition, several software tools are available for solving general MI-CONV problems, see, e.g., the recent comparison in \cite{25}. Finally, some specific classes of MI-CONV problems, like Mixed-Integer (Convex) Quadratically Constrained Quadratic Optimization (MIQCQP) problems are now supported by the major commercial solvers.

This paper builds on two specific aspects that we consider fundamental for solving MI-CONV problems. First, given that cutting planes are instrumental to solving MILP, a number of authors have looked at various approaches to compute cuts for MI-CONV problems and, nowadays, linear cutting planes are part of the arsenal of some MI-CONV solvers. Despite this (partial) success, some fundamental questions in this area are left unanswered. Second, recent experience has shown that conic formulations of MI-CONV problems display enviable properties (tractability, numerical stability, etc.) that make them preferable, from the solving viewpoint, to generic MI-CONV formulations.
Therefore, building on (i) cutting planes and (ii) conic formulations, we answer the (somehow) natural question of what one can gain in terms of cutting planes by using a problem’s conic structure. In the process of doing so, we answer several questions left open by previous works on the topic. In particular, the paper focuses on disjunctive cuts for MI-CONIC. In the remainder of this section, we review the literature on the subject and outline our main contributions.

**Episode I: MI-CONV awakens.** The work on disjunctive cutting planes for MI-CONV (re)started already in the late 90s with two fundamental contributions [12,31]. More precisely, Ceria and Soares [12] show that disjunctive convex problems can be formulated as a single convex problem in a higher dimensional space, and hint that this could serve to generate cutting planes using sub-gradient information at the optimum. Around the same time, Stubbs and Mehrotra [31] make the separation of disjunctive cuts for MI-CONV explicit by (i) solving one Non-Linear Programming (NLP) problem, and (ii) identifying a sub-gradient that yields a violated cut. The latter is done by taking a gradient (under regularity assumptions), or by solving a linear system (under the assumption that the objective function of the former problem is polyhedral). Those assumptions and the use of perspective functions lead to differentiability issues that made the results of the computational investigation in [31] numerically disappointing.

The numerical difficulties encountered in [31] have slowed down the development of the area for a number of years until the renewed interest and the practical approaches of the last decade [8,22]. More precisely, Kilinc et al. [22] note that “A simple strategy for generating lift-and-project cuts for a MINLP problem is to solve a CGLP 3 based on a given polyhedral outer approximation of the relaxed feasible region [...]. The key question to be answered [...] is which points to use to define the polyhedral relaxation.” (from [22], Sec. 3).

The distinction in how to answer the above question is the difference between [8] and [22]. Namely, Bonami [8] solves one NLP and uses the solution to get an outer approximation that provably yields a violated cut if any exists. Instead, Kilinc et al. [22] iteratively refine an outer approximation by solving a sequence of LPs until a violated cut, if any, is separated by solving the associated CGLP 4.

The outer approximation approaches in [8,22] are, to the best of our knowledge, the state of the art for the implementation of disjunctive cuts for MI-CONV and, especially, for MIQCQPs, see e.g., their implementation in CPLEX starting from version 12.6.2. However, despite the impressive practical improvement with respect to the early attempts [31], questions were left on the table, which we answer in the present paper.

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3 The so-called Cut Generating Linear Program, or CGLP for short, is the LP proposed by Balas [4] to separate disjunctive cuts for MILP. Such an LP is defined in a higher dimensional space with roughly the double of the variables of the original problem.

4 An exception to the lack of papers in the first decade of 2000 is that of Zhu and Kuno [33]. In fact, [33] also proposes an outer approximation scheme where (i) an NLP relaxation is solved, (ii) an outer approximation is built through the fractional solution, and (iii) a cut is obtained by solving the associated CGLP. However, this approach is not guaranteed to get a violated cut if one exists, see Example 1 in [22].
**Episode II: Attack of the cones.** Conic optimization is viewed as more stable and tractable than general convex optimization [7], and can tackle a wide range of problems thanks to modeling tools such as disciplined convex optimization [21]. In particular, [27] recently showed that all convex instances in MINLPLib can be formulated using only a handful of cones; writing those problems as MI-CONIC problems led to significant computational improvements. Major commercial solvers have supported Mixed-Integer Second Order Cone Programming for some time and more general MI-CONIC problems are now supported by a number of solvers, e.g., Mosek and Pajarito [14,26,27]. However, while cutting planes for MI-CONIC have been extensively studied [1,2,3,6,13,15,23,24,28,29,30,32], they are rarely used by these solvers (for example, neither Mosek nor Gurobi generate cuts from non-linear information).

**Episode III: Return of the disjunction.** In this paper, we study disjunctive cutting planes for MI-CONIC problems. Our end goal is to obtain practical and efficient tools for the separation of those cuts and we show the conic context allows us do so. We do it by extending Balas’ CGLP [4] into a Cut Generating Conic Program (CGCP) (see also [13]). Our contributions are.

1. We study the role of the normalization constraint in CGCP, and propose a conic normalization that guarantees strong duality. In doing so, we answer some questions that were raised in previous works on MI-CONV. Namely,
   - With respect to [8], we can select the right normalization to overcome issues associated with potential lack of constraint qualification.
   - With respect to [22], because everything is conic, we do not need (i) to pay attention at avoiding generating linearization cuts at points outside the domain where the non-linear functions are known to be convex, (ii) to deal with non-differentiable functions, and (iii) boundedness assumptions on the value of the constraints and their gradients.

   This also opens the door to a number of algorithmic techniques for separating disjunctive cuts, i.e., for solving the CGCP.

2. We show that, under a well-posed condition, all valid inequalities for a disjunctive conic set can be represented using (conic) Farkas multipliers. This is a major difference with respect to [8,22] that have no such tractable representation. Thus, we give a constructive proof of the existence result in [11], namely, that for a well-posed MI-CONIC problem, any valid linear inequality can be derived from some valid outer approximation of the problem.

3. Finally, we provide computational results on the effectiveness of the proposed approach and we show the advantages of the conic representation.

**Episode III** above summarizes the main contributions of the paper. The remainder of it is as follows. In Section 2 we introduce some required notation while Section 3 formalizes the CGCP. In Section 4 we discuss the theoretical and numerical properties of the normalization in CGCP while Section 5 characterizes valid inequalities for disjunctive conic sets. Section 6 presents an extract of our computational experiments and some concluding remarks are in Section 7.

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5 Personal communication with Gurobi and Mosek developers.
2 Notations

Let $\mathcal{X} \subseteq \mathbb{R}^n$, we denote by $\text{int}(\mathcal{X})$, $\partial(\mathcal{X})$, $\text{cl}(\mathcal{X})$, $\text{conv}(\mathcal{X})$ and $\text{cone}(\mathcal{X})$ the interior, boundary, closure, convex hull and conical hull of $\mathcal{X}$, respectively.

The dual cone of $\mathcal{X}$ is $\mathcal{X}^* = \{ u \in \mathbb{R}^n | u^T x \geq 0, \forall x \in \mathcal{X} \}$ and its polar is $\mathcal{X}^\# = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} | \alpha^T x \geq \beta, \forall x \in \mathcal{X} \}$. Each element of $\mathcal{X}^\#$ corresponds to a valid inequality for $\mathcal{X}$, and we have $\mathcal{Y}^\# \subseteq \mathcal{X}^\#$ if $\mathcal{X} \subseteq \mathcal{Y}$, and $(\mathcal{X} \cup \mathcal{Y})^\# = \mathcal{X}^\# \cap \mathcal{Y}^\#$.

A cone $\mathcal{K} \subseteq \mathbb{R}^n$ is pointed if it does not contain a line or, equivalently, if $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. Proper cones are closed, convex, pointed cones with non-empty interior and, if $\mathcal{K}$ is proper, then so is $\mathcal{K}^*$. A proper cone $\mathcal{K}$ induces a partial (resp. strict partial) ordering on $\mathbb{R}^n$, denoted $\succeq_\mathcal{K}$ (resp. $\succ_\mathcal{K}$) and defined by

$$ x \succeq_\mathcal{K} y \iff x - y \in \text{int}(\mathcal{K}), \quad x \succ_\mathcal{K} y \iff x - y \in \text{int}(\mathcal{K}). $$

Then, we refer to $Ax \succeq_\mathcal{K} b$ (resp. $Ax \succ_\mathcal{K} b$) as a conic (resp. strict conic) inequality. The system $Ax \succeq_\mathcal{K} b$ is strongly feasible if it has a solution that strictly satisfies all non-polyhedral inequalities, strongly infeasible if $A^T u = 0$, $b^T u > 0$ for some $u \in \mathcal{K}^*$, and weakly feasible (resp. weakly infeasible) if it is feasible (resp. infeasible) but not strongly feasible (resp. strongly infeasible).

Finally, we say that $\mathcal{C} = \{ x | Ax \succeq_\mathcal{K} b \}$ is well-posed if $Ax \succeq_\mathcal{K} b$ is strongly feasible or strongly infeasible, and ill-posed otherwise. Then, one can obtain valid inequalities for $\mathcal{C}$ using Farkas multipliers, namely, $(A^T u, b^T u) \in \mathcal{C}^\#, \forall u \in \mathcal{K}^*$. If $\mathcal{C}$ is well-posed, then all valid inequalities for $\mathcal{C}$ can be written this way [7]; we relax this assumption to state a general result in Theorem 3 of Appendix A.

3 Split cuts for MI-CONIC

We consider an MI-CONIC problem in the form

$$(P) \quad \min_x \{ c^T x \mid Ax \succeq_\mathcal{K} b, x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \},$$

where $A \in \mathbb{R}^{m \times n}$ has full column rank, $p \leq n$, and $\mathcal{K}$ is a proper cone. Note that, since the zero cone $\{0\}$ is not proper, this latter point requires that equality constraints be written as two inequalities. We denote by $\mathcal{X}$ the feasible set of $P$, and by $\mathcal{C} := \{ x \in \mathbb{R}^n \mid Ax \succeq_\mathcal{K} b \}$ the domain of its continuous relaxation $\bar{P}$, which we assume is strongly feasible.

Let $\bar{x} \in \mathbb{R}^n$ be a point to separate, we want to find $(\alpha, \beta) \in \mathcal{X}^\#$ with $\alpha^T \bar{x} < \beta$. To do so, we begin by considering a split relaxation of $\mathcal{X}$

$$ \mathcal{S} = \{ Ax \succeq_\mathcal{K} b, -\pi^T x \geq -\pi_0 \} \lor \{ Ax \succeq_\mathcal{K} b, \pi^T x \geq \pi_0 + 1 \}, $$

where $\pi \in \mathbb{R}^{n-p}$. Then, we refer to $\mathcal{S}$ as two inequalities. We denote by $A$ where...
where \((\pi, \pi_0)\) is integer, \(\pi_j = 0, \forall j \notin \{1, \ldots, p\}\) and \(0 < \pi^T \bar{x} - \pi_0 < 1\). Thus, \(\mathcal{X} \subseteq \mathcal{S}\), \(\bar{x} \notin \mathcal{S}\), and \(\mathcal{S}^\# \subset \mathcal{X}^\#\). Then, it follows from conic duality that any solution to the system

\[
\begin{align*}
\alpha &= A^T u - u_0 \pi, \\
\alpha &= A^T v + u_0 \pi, \\
\beta &\leq b^T u - u_0 \pi_0, \\
\beta &\leq b^T v + v_0 (\pi_0 + 1), \\
u, v \in K^*,
\end{align*}
\]

yields a valid inequality \((\alpha, \beta) \in \mathcal{S}^\#\). A most violated such cut is obtained by solving the cut generating conic problem \(\text{(CGCP)}\)

\[
\min_{\alpha, \beta, u, v, u_0, v_0} \left\{ \alpha^T \bar{x} - \beta \mid (4) - (8) \right\}
\]

whose dual is the membership conic problem \(\text{(MCP)}\)

\[
\max_{y, z, y_0, z_0} 0 \\
\text{s.t.} \quad Ay \succeq_K y_0 b, \\
Az \succeq_K z_0 b, \\
\pi^T y \leq y_0 \pi_0, \\
\pi^T z \geq z_0 (\pi_0 + 1), \\
y + z = \bar{x}, \\
y_0 + z_0 = 1, \\
y_0, z_0 \geq 0.
\]

First, any feasible solution of \(\text{CGCP}\) with negative objective value corresponds to a violated cut. The feasible set of \(\text{CGCP}\) is in fact an unbounded cone: any violated cut yields an unbounded ray. Second, since the two sides of the disjunction have identical recession cones, \(\text{(10)-(16)}\) describes \(\text{conv}(\mathcal{S})\), which is closed \([7]\). Thus, \(\text{MCP}\) asks whether \(\bar{x}\) belongs to \(\text{conv}(\mathcal{S})\), while \(\text{CGCP}\) looks for a Farkas separation proof. Finally, \(\text{CGCP-MCP}\) are both conic problems that can be solved with, e.g., an interior-point method.

### 4 The role of normalization

In order to truncate the feasible set of \(\text{CGCP}\), one can consider a normalization constraint. While its impact on the characteristics and numerical properties of the resulting cut has been well-studied in MILP \([5, 18, 9]\), in the conic setting, we show that the normalization also influences whether strong duality holds for the \(\text{CGCP-MCP}\) pair. Let us emphasize that this latter point only arises in non-linear settings, and has received little attention in the literature despite its fundamental importance for numerical stability and efficiency.

A first normalization, referred to as trivial in what follows, is

\[
u_0 + v_0 \leq 1,
\]

\[\text{(17)}\]
for which the separation-membership pair becomes

\[
\begin{align*}
(CGCP_{tr}) \min & & \alpha^T \bar{x} - \beta \\
& & \text{s.t. } (1)-(3), \\
& & (17),
\end{align*}
\]

\[
\begin{align*}
(MCP_{tr}) \max & & \eta \\
& & \text{s.t. } (10)-(11), (13)-(16) \\
& & \pi^Ty - y_0\pi_0 \leq -\eta, \\
& & \pi^Tz - z_0(\pi_0 + 1) \geq \eta, \\
& & \eta \leq 0.
\end{align*}
\]

In MILP, Gomory Mixed-Integer cuts associated with the simplex tableau of which \(\bar{x}\) is a basic solution are optimal solutions of \(CGLP_{tr}\). Nevertheless, Lemma 1 shows the limitations of the trivial normalization for MI-CONIC.

**Lemma 1.** Let \(\bar{x} \in \mathbb{R}^n\).

1. If \(\bar{x}\) is strongly feasible for \(\bar{P}\), then \(CGCP_{tr}\) and \(MCP_{tr}\) are strongly feasible.
2. If \(\bar{x}\) is weakly feasible for \(\bar{P}\), then \(MCP_{tr}\) is weakly feasible.
3. If \(\bar{x}\) is infeasible for \(\bar{P}\), then \(CGCP_{tr}\) is unbounded and \(MCP_{tr}\) is infeasible.

The proof is in Appendix A. Let us emphasize that Case 2 and Case 3 in Lemma 1 cannot be overlooked. Indeed, on one hand, solving \(\bar{P}\) with an interior-point method typically yields a point \(\bar{x}\) that lies at the boundary of the feasible set. Thus, unless no non-linear constraint is active at \(\bar{x}\), this corresponds to Case 2; we illustrate this situation in Example 1. On the other hand, Case 3 arises in outer approximation-based algorithms, since they generally yield points that are infeasible with respect to some or all non-linear constraints.

Previous works on MI-CONV \cite{31,13,22} have considered the \(\alpha\)-normalization, which consists in bounding the norm of \(\alpha\), i.e., imposing

\[
\|\alpha\| \leq 1,
\]

with \(\|\cdot\|\) a suitable norm whose dual norm is denoted by \(\|\cdot\|_*\). This yields

\[
\begin{align*}
(CGCP_\alpha) \min & & \alpha^T \bar{x} - \beta \\
& & \text{s.t. } (1)-(3), \\
& & \|\alpha\| \leq 1,
\end{align*}
\]

\[
\begin{align*}
(MCP_\alpha) \max & & -\eta \\
& & \text{s.t. } (10)-(13), (15)-(16) \\
& & y + z = x, \\
& & \eta \geq \|\bar{x} - x\|_*.
\end{align*}
\]

Solving \(CGCP_\alpha\) yields a deepest cut, i.e., one that maximizes the distance from \(\bar{x}\) to the cut, while \(MCP_\alpha\) searches \(z \in \text{conv}(S)\) that minimizes \(\|\bar{x} - x\|_*\). Consequently, unless \(S = \emptyset\), \(MCP_\alpha\) is feasible, thus \(CGCP_\alpha\) has bounded objective value. Nevertheless, an optimal solution to \(CGCP_\alpha\) may not exist, as illustrated in Example 1. Furthermore, note that, for non-polyhedral \(\|\cdot\|\), \(13\) adds to the non-linearity of \(CGCP_\alpha\) and \(MCP_\alpha\), though both remain conic problems\footnote{\(\|\cdot\|\) is represented by \(\tilde{K} = \{(t,x)|t \geq \|x\|\}\), and \(\|\cdot\|_*\) by \(\tilde{K}^* = \{(s,u)|s \geq \|u\|_*\}\).}. In addition, using \(13\) may yield \(\alpha, \beta\) that are not extreme rays of \(D^\pi\), i.e., that correspond to dominated inequalities. Finally, in general, a deepest cut may be fully dense \cite{10}, which is not desirable for computational efficiency.
We now propose a \textit{linear}, conic-based normalization that overcomes all the drawbacks above. For $\rho \in \text{int}(K)$, note that $\|u\|_{\rho} = \rho^T u$ is a norm on $K^*$. Thus, we introduce the \textit{conic normalization}

\begin{equation}
\|u\|_{\rho} + \|v\|_{\rho} + u_0 + v_0 \leq 1,
\end{equation}

which is indeed a linear inequality. Then, the CGCP-MCP pair writes

\begin{equation}
(\text{CGCP}_{cn}) \quad \min \alpha^T \bar{x} - \beta
\end{equation}

s.t. \begin{align*}
& (4) - (8), \quad \text{(19)}
\end{align*}

\begin{equation}
(\text{MCP}_{cn}) \quad \max \eta
\end{equation}

s.t. \begin{align*}
& Ay - y_0 b \succeq \kappa \eta \rho, \\
& Az - z_0 b \succeq \kappa \eta \rho, \\
& \pi^T y - y_0 \pi_0 \leq -\eta, \\
& \pi^T z - z_0 (\pi_0 + 1) \geq \eta, \\
& (14) - (16) \\
& \eta \leq 0.
\end{align*}

First, (19) bounds the norm of all Farkas multipliers $u, v, u_0, v_0$. Thus, $\alpha$ is bounded, $\beta$ is upper bounded and, in turn, the optimal value of $\text{CGCP}_{cn}$ is finite. Second, (19) corresponds to adding artificial slacks to the conic constraints in $\text{MCP}_{cn}$. This is akin to Bonami’s approach for MI-CONV in [8]. Third, letting $K = \mathbb{R}^n_+$ and $\rho = (1, \ldots, 1)^T$, (19) reduces to the well-known \textit{standard normalization} in MILP [4, 5, 18]; a similar connection was made in [8] as well. Finally, while in [8], the author mentions that “\textit{We cannot claim that a constraint qualification always holds at the optimum},” in the conic setting, we are able to prove strong duality for $\text{CGCP}_{cn}$-MCP$_{cn}$, as shown by Theorem 1 below.

\textbf{Theorem 1.} \textit{CGCP}_{cn} and \textit{MCP}_{cn} are strongly feasible, for arbitrary $\bar{x}$.

The proof is in Appendix A. Importantly, Theorem 1 holds regardless of (i) the choice of $\bar{x}$, and (ii) whether each disjunction is well-posed. Example 1 below illustrates that numerical issues in CGCP can in fact be duality issues, and demonstrates that the conic normalization (19) does indeed alleviate them.

\textbf{Example 1.} Let $L^3$ denote the (self-dual) second-order cone in $\mathbb{R}^3$, and

\begin{align*}
\mathcal{X} &= \{(x_1, x_2) \in \mathbb{Z}^2 \mid (0, x_1, x_2)^T \succeq_{L^3} (-1/2, 1/2, 1)^T\}.
\end{align*}

Let $\bar{x} = (1/2, 1/2), \pi = (1, 0)$ and $\pi_0 = 0$. Then, system (4)-(8) writes

\begin{align*}
\alpha_1 &= u_2 - u_0, & \alpha_1 &= v_2 + v_0, \\
\alpha_2 &= u_3, & \alpha_2 &= v_3, \\
\beta &\leq \frac{1}{2} u_1 + \frac{1}{2} u_2 + u_3, & \beta &\leq -\frac{1}{2} v_1 + \frac{1}{2} v_2 + v_3 + v_0, \\
u, u_0 &\in L^3 \times \mathbb{R}_+, & v, v_0 &\in L^3 \times \mathbb{R}_+.
\end{align*}

We solve CGCP with normalizations (17), (18), (19) using Mosek 9.0. For the former two, Mosek encounters numerical issues; a feasible solution of CGCP is nonetheless available,
which we retrieve to obtain a valid cut. We display the cuts in Figure 1 and report additional statistics in Table 2 of Appendix B.

First, for CGCP$_{tr}$, Mosek terminates after 39 iterations due to slow progress. Since $\bar{x}$ is weakly feasible, MCP$_{tr}$ has empty interior; this causes duality failure and, in turn, numerical issues. Indeed, Table 2 shows that $\alpha, \beta, u, v$ diverge while $u_0, v_0$ are bounded. Thus, in the limit, $u_0, v_0$ become negligible, i.e., we have $\alpha \approx A^T u$ and $\beta \approx b^T u$: the obtained “cut” does not cut anything, since it is in fact a valid inequality for $C$.

Second, when solving CGCP$_{\alpha}$, Mosek terminates after 46 iterations due to slow progress, but a valid cut with $\alpha, \beta \approx (0, 1), 1$ is retrieved. Here, while the deepest cut is $x_2 \geq 1$, this valid inequality cannot be obtained with finite Farkas multipliers, a consequence of both disjunctive terms being ill-posed. Hence, $(\alpha_1, \alpha_2, \beta) = (0, 1, 1)$ is not feasible for CGCP but, by Theorem 3, there exists a (diverging) sequence of multipliers yielding cuts that become arbitrarily close to $x_2 \geq 1$. This indeed happens when solving CGCP$_{\alpha}$: Table 2 shows that $u, v, u_0, v_0$ diverge while the final value of $\alpha, \beta$ is almost equal to $(0, 1), 1$.

Finally, Mosek solves CGCP$_{cn}$ to optimality in 7 iterations with no numerical issues, and Table 2 shows that no multiplier diverges. This stark contrast with CGCP$_{tr}$ and CGCP$_{\alpha}$, in both stability and speed, is the result of having enforced strong duality for CGCP$_{cn}$, as per Theorem 1.

5 Linear relaxations are enough, revisited

We now study valid inequalities for disjunctive conic sets, starting with an extension of Proposition 1.4.3 in [7] to disjunctive sets.

Lemma 2. Let $D = \bigvee_{h=1}^H \{A_h x \succeq_{K_h} b_h\}$ with each $D_h = \{A_h x \succeq_{K_h} b_h\}$ well-posed. Then $D^\# = \bigwedge_{h=1}^H \{\alpha = A_h^T u_h, \beta \leq b_h^T u_h, u_h \in K_h^\star\}$.

Proof. Follows from $D^\# = \bigcap_{h=1}^H D_h^\#$ and each $D_h$ being well-posed. \qed

Lemma 2 allows to give a constructive proof to Theorem 2 which was initially stated in [11], though only with an existence proof.
**Theorem 2 (Adapted from Theorem 3 of [11]).** Let \( P \) be given by (2) and \( L \leq U \in \mathbb{R} \) such that \( C \subseteq [L, U]^n \). Assume that, for every \( h \in \mathbb{Z}^p \cap [L, U]^p \), \( \{ Ax \geq b, B_h x \geq d_h \} \) is well-posed, where \( B_h x \geq d_h \) encodes \( \{ x_j = h_j, j = 1, ..., p \} \).

Then, if \((\alpha, \beta) \in \mathcal{X}^\# \), there exists \( \tilde{A} \) and \( \tilde{b} \) such that \( C \subseteq \{ \tilde{A} x \geq \tilde{b} \} \) and \((\alpha, \beta) \in \tilde{\mathcal{X}}^\# \), where \( \tilde{\mathcal{X}} = \{ x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} | \tilde{A} x \geq \tilde{b} \} \).

**Proof.** Let \( \mathcal{H} = \mathbb{Z}^p \cap [L, U]^p \) and, for \( h \in \mathcal{H} \), define \( \mathcal{D}_h = \{ Ax \geq b, B_h x \geq d_h \} \). Then, let \( \mathcal{D} = \bigvee_h \mathcal{D}_h \) and note that, by construction, we have \( \mathcal{X} = \mathcal{D} \).

Let \((\alpha, \beta) \in \mathcal{X}^\# = \mathcal{D}^\# \). Thus, by Lemma 2 there exists \( u_h \in K^*, v_h \geq 0 \) such that \( \alpha = A^T u_h + B_h^T v_h \) and \( \beta \leq b^T u_h + d_h^T v_h, \forall h \in \mathcal{H} \). Then, consider

\[
(\tilde{P}) \quad \min_{x} \quad c^T x
\]

s.t.

\[
\begin{align*}
(u_h^T A) x & \geq u_h^T b, \quad \forall h \in \mathcal{H}, \\
L & \leq x_j \leq U, \quad \forall j \in \{1, ..., n\}, \\
x & \in \mathbb{Z}^p \times \mathbb{R}^{n-p},
\end{align*}
\]

with feasible set \( \tilde{\mathcal{X}} \). By construction, \( \tilde{P} \) is an outer approximation of \( P \). Then, define \( \tilde{\mathcal{D}} = \bigvee_h \{ (u_h^T A) x \geq u_h^T b, B_h x \geq d_h \} \). It is straightforward to verify that \((\alpha, \beta) \in \tilde{\mathcal{D}}^\# \subseteq \tilde{\mathcal{X}}^\# \), which concludes the proof. \( \square \)

Finally, note that the obtained outer-approximation in this proof is exponentially large. This is not a conservative estimate. Indeed, this can be the case if \( \alpha = c \), since there exist instances for which an outer-approximation algorithm requires exponentially many iterations to converge, see, e.g., Section 3 of [27].

### 6 Computational results

We perform computational tests on a set of 211 MI-CONIC instances in the CBLib library [19], grouped in 9 classes (see Table 1). Problems in \( \text{SRyn} \) and \( \text{Syn} \) have exponential cones; others are Mixed-Integer Second-Order Cone problems.

We first compute the gap closed by an initial MILP relaxation (MILP), including by generating cuts only at its root node (root). This gives an estimate of how much gap can be closed by “forgetting” some of the non-linear information in the problem. Then, we compute the gap closed by lift-and-project cuts when solving \( \text{CGCP}_{cn} \), with (i) an interior-point method (CGCP) and, (ii) an LP-based inner-approximation algorithm (CGLP), similar to [22].

We solve the continuous relaxations and conic problems with Mosek 9.0, and all (MI)LPs with Gurobi 8.1. Cuts are generated by rounds, with at most three cuts per round, and we only generate rank-1 cuts. All experiments use the conic normalization [19]. We stop if (i) an integer point is found, (ii) no violated cut is found, (iii) we encounter numerical issues in the continuous relaxation, or (iv) a two-hour time limit is reached. Finally, the gap closed is defined as \( \frac{z^* - \tilde{z}}{z^*} \), where \( z^*, \tilde{z}, \tilde{z} \) are the obtained lower bound, the value of the best known integer solution, and the value of the continuous relaxation, respectively.
Table 1. Number of instances and average percentage gap closed for each approach.

| Class  | # | %Gap - OA | %Gap - L&P |
|--------|---|-----------|------------|
|        |   | root      | MILP       | CGCP       | CGLP       |
| CLay   | 12| 3.08      | 34.65      | 13.51      | 0.54       |
| FLay   | 10| 0.00      | 12.11      | 43.72      | 46.95      |
| fmo    | 40| 24.01     | 82.55      | 11.50      | 10.50      |
| RSyn   | 48| 45.51     | 62.25      | 61.87      | 30.65      |
| SLay   | 14| 0.52      | 6.26       | 66.62      | 0.25       |
| Syn    | 51| 11.48     | 15.13      | 89.60      | 55.44      |
| sssd   | 16| 4.29      | 4.30       | 99.77      | 98.53      |
| tls    | 5 | 1.79      | 26.69      | 1.45       | 2.22       |
| UFLquad| 15| 0.00      | 0.19       | 21.96      | 3.00       |
| All    | 211| 9.51     | 21.85      | 40.09      | 17.57      |

The results in Table 1 motivate the following remarks. First, CGCP almost always outperforms CGLP, especially for CLay, RSyn, SLay, Syn and UFLquad. Indeed, we observed that CGLP generates more invalid cuts, causing numerical issues in the continuous relaxation. In addition, slow convergence properties hamper CGLP on problems with highly non-linear cones, e.g., RSyn and Syn.

Second, let us mention that the present results are consistent with those obtained by [8,22] in the context of MI-CONV. We provide a more detailed comparison, when applicable, in Appendix B.

Finally, overall, lift-and-project cuts close 30% more gap than cuts obtained from an initial outer approximation, and 20% more than the initial MILP relaxation: exploiting non-linear information to compute cuts does pay off. More precisely, while MILP clearly outperforms CGCP for CLay, fmo and tls, the reverse is true for FLay, SLay, Syn, sssd and UFLquad. We have no systematic explanation for this contrasted behavior; being able to predict it a priori on a per-instance basis would be of major value to MI-CONIC solvers.

7 Conclusion

Motivated by the recent success of conic formulations for MI-CONV, we have investigated the impact of disjunctive cutting planes for MI-CONIC. We have shown the fundamental role of CGCP normalization and proposed a conic normalization that guarantees strong duality. This allowed us to answer several relevant (especially from the numerical standpoint) questions left from 20 years of development in the area. Finally, working with conic representation allows to experiment with a number of algorithmic techniques for separating disjunctive cuts, i.e., for solving CGCP, and we provided computational evidence of the effectiveness of the proposed approach.
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A Technical results and proofs

**Theorem 3.** Let $C = \{ x \in \mathbb{R}^n \mid Ax \succeq_K b \}$ with $A$ of full column rank. Then

$$C^# = \text{cl} \left( \{(A^T u, b^T u) \mid u \in K^* \} \right),$$

and, if $C$ is well-posed, the above holds without the cl.

**Proof.** Let $F = \{(A^T u, b^T u) \mid u \in K^* \}$. The inclusion $F \subseteq C^#$ is a direct consequence of conic duality, and it follows that $\text{cl}(F) \subseteq \text{cl}(C^#) = C^#$. The rest of the proof follows from Proposition 1.4.2. in [7], by showing that

$$A^T u = \alpha, \quad (20)$$

$$b^T u \geq \beta, \quad (21)$$

$$u \in K^*, \quad (22)$$

is either feasible or weakly infeasible. Finally, the result for well-posed $C$ follows from strong conic duality. $\square$

**Proof of Lemma 3.**

**Case 1:** The point $(A^T u, b^T u, u, 0, 0)$ is strongly feasible for CGCP$_{tr}$ for any $u \in \text{int}(K^*)$. Since $\bar{x}$ is strongly feasible for $\bar{P}$, $(\frac{1}{2} \bar{x}, \frac{1}{2} \bar{x}, \frac{1}{2}, \frac{1}{2}, -1)$ is trivially strongly feasible for MCP$_{tr}$.

**Case 2:** The latter point above remains feasible for MCP$_{tr}$ if $\bar{x}$ is weakly feasible. Then, if $(y, z, y_0, z_0, \eta)$ is strictly feasible for MCP$_{tr}$, we have

$$A\bar{x} = Ay + Az \succeq_K y_0 b + z_0 b = b, \quad (23)$$

with all non-polyhedral inequalities strictly satisfied. Therefore, $\bar{x}$ is strongly feasible for $\bar{P}$, and the result follows by contraposition.

**Case 3:** Using (23), if MCP$_{tr}$ admits a feasible solution, then $\bar{x} \in C$. Then, since $\bar{x} \not\in C$, there exists $u \in K^*$ with $u^T (A\bar{x} - b) < 0$, thus $(A^T u, b^T u, u, 0, 0)$ is an unbounded ray for CGCP$_{tr}$. $\square$

**Lemma 3.** Let $K$ be a cone with non-empty interior, and $\rho \in \text{int}(K)$. Then

$$\forall x \in \mathbb{R}^n, \exists \lambda \geq 0 : x + \lambda \rho \succ_K 0.$$ 

**Proof.** The case $x = 0$ is trivial, so we assume $x \neq 0$. Since $\rho \in \text{int}(K)$, there exists $\varepsilon > 0$ such that $\rho + t \in \text{int}(K), \forall t \in B(0, \varepsilon)$ where $B(0, \varepsilon)$ is a ball of center 0 and radius $\varepsilon$. Then, let $t = \frac{-\varepsilon x}{\|x\|}$ and $\lambda \geq 0$, we have $\rho + t \in \text{int}(K)$ and

$$x + \lambda \rho \succeq_K (x + \lambda \rho) - \lambda (\rho + t) = x + \lambda t, \quad (24)$$

$$= 0, \quad (25)$$

which concludes the proof. $\square$
Proof of Theorem 1.
Let $\tilde{u} \in \text{int}(K^*)$ and $u = (2\|\tilde{u}\|\rho)^{-1}\tilde{u}$. Then, $(A^T u, b^T u, u, u, 0, 0)$ is strongly feasible for $\text{CGCP}_{cn}$.

For $\text{MCP}_{cn}$, first set $y_0 = z_0 = \frac{1}{2}$, and $y = z = \frac{1}{2}\bar{x}$. Since $\rho \in \text{int}(K)$, by Lemma 3, we can choose $\eta \leq 0$ such that all inequalities in $\text{MCP}_{cn}$ are strictly satisfied. Thus, $(y, z, y_0, z_0, \eta)$ is strongly feasible. \hfill \box

B Computational results

Additional statistics for Example 1 are reported in Table 2. For each normalization (Norm.), we report the number of interior-point iterations (Iter.), the result status of Mosek (Res.) –STALL for MSK RES TRM STALL and OK for MSK RES OK, – the values $\alpha_1, \alpha_2, \beta$, the Euclidean norm of the conic multipliers $u, v$, and the values of $u_0, v_0$ in the retrieved solution.

| Norm. | Iter. | Res. | $\alpha_1$ | $\alpha_2$ | $\beta$ | $\|u\|_2$ | $u_0$ | $\|v\|_2$ | $v_0$ |
|-------|-------|------|------------|------------|--------|----------|------|----------|------|
| (17)  | 39    | STALL| 0.04       | 10557.89   | 5279.21 | 14931.11 | 0.50 | 14931.11 | 0.50 |
| (18)  | 46    | STALL| 0.00       | 1.00       | 1.00    | 17162.73 | 12135.88 | 17156.60 | 12131.55 |
| (19)  | 6     | OK   | 0.00       | 0.25       | 0.19    | 0.44     | 0.19 | 0.44     | 0.19 |

Table 3 displays, when applicable, the gap closed as reported in Table 3 of [22] (KLL), along with column CGCP of Table 1. We do not include fmo and sssd which are not identical in both studies. Since we do not use extended formulations in our experiments, for S Lay and UFLquad, we report the results of [22] that did not exploit separability.

Overall, KLL closes more gap than our vanilla implementation of CGCP, although differences in time limits and numerical tolerances limit the validity of the comparison.

Table 3. Comparison to results from [22]

| Class | CGCP | KLL   |
|-------|------|-------|
| CLay  | 13.51| 40.80 |
| FLay  | 43.72| 50.70 |
| RSyn  | 61.87| 88.70 |
| SLay  | 66.62| 45.90 |
| Syn   | 89.60| 99.80 |
| tls   | 1.45 | 6.30  |
| UFLquad | 21.96| 2.20  |