About influence of some parameters of mechanism of planetary type for change in its potential energy

N I Bondarenko\(^1\), K B Obnosov\(^2\) and A V Panshina\(^3\)

\(^1\)lector, Department of Theoretical Mechanics, Bauman Moscow State Technical University, Moscow, 105005, Russia

E-mail: \(^1\)colia.bond@yandex.ru \(^2\)KOB1-Naz@km.ru \(^3\)panalv@mail.ru

Abstract. The article considers a mechanism located in a uniform field of gravity and consisting of a carrier, load, counterweight and satellite, counterweight and satellite rolling without slipping on a fixed cylindrical surface. The positions of the stable and unstable equilibrium of the mechanism are determined using the nonlinear function of potential energy. The features of changing the function of potential energy are revealed depending on the characteristics of the mechanism, including whether the radius of the satellite is rational, irrational, or transcendental.

1. Introduction

In the study of free vibrations of mechanical systems, it is necessary to calculate their potential energy, since, in accordance with the Lagrange – Dirichlet theorem [1,2], such oscillations occur near the stable equilibrium of the system, which are also determined using the potential energy function. But in this case, it seems appropriate to study the dependence of the potential energy of a real mechanical system on its geometric characteristics, since they can change over time during the functioning of the system. This is especially important if the potential energy function of a mechanical system is nonlinear, and its geometric dimensions can be not only rational numbers, but also irrational and transcendental ones. This is discussed below on the example of a mechanism that can be part of planetary gearboxes, automobile differentials, final drive planetary gears of heavy vehicles, receiving devices of cotton harvesters [3].

2. Formulation of the Problem

A mechanism located in the vertical plane (Figure 1) containing a satellite 1 of radius \(r\) and of mass \(M_1\) is considered. The satellite can roll without slipping along a fixed cylindrical surface of radius \(R\). The axis \(A\) of the satellite is fixed by a cylindrical hinge with a rod \(L\) and of mass \(M_2\). The coupling at point \(O\) can rotate around a fixed horizontal axis passing through this point. When assembling the mechanism or when changing the radius \(r\) of the satellite 1, rod 2 slides along the coupling, and then is attached to it with a screw. The satellite 1 is rigidly attached to a weightless bar \(5\) with a load 3 of mass \(M_3\), which is fixed on this rod at a distance \(AB=13\).
The counterweight 4 of mass $M_4$ is on the rod 2 at a distance $OD=l_4$. In the middle $C$ of the rod 2 is located its center of mass. Assuming the thickness and density of the material of the satellite 1 to be constant values and taking into account the fact that its mass with a radius of $r_0=0.2$ m is equal to $m_0=0.2$ kg, we obtain the dependence of the satellite mass on its radius $m_1(r)=m_0 r^2/r_0^2$.

Other initial mass and geometric characteristics of the mechanism are:

$$M_2=M_4=0.5 \text{ kg}; \quad M_3=0.4 \text{ kg}; \quad R=0.65 \text{ m}; \quad L=0.8 \text{ m}; \quad l_3=0.2 \text{ m}; \quad l_4=0.15 \text{ m}.$$  \hspace{1cm} (1)

To study the potential energy of the P mechanism, we compose its expression. The mechanism under consideration \((4–7)\) has one degree of freedom. As a generalized coordinate, we take the angle $\phi$ of the rod deviation from the vertical. The Figure 2 shows the mechanism at the current time. The superscript index * indicates the position of the points $N, A, B$ when $\phi=0$.

Since the lengths of the arcs $\measuredangle PN$ and $\measuredangle PN*$ are equal, then the angle $\psi$ of rotation of satellite 1 has the form

$$\psi = \theta - \phi = \frac{R-r}{r} \phi.$$ 

The active forces acting on the mechanism are only gravity, therefore, the potential energies of a homogeneous gravity field of the satellite 1 is

$$\Pi_1 = M_1(r) \cdot g \cdot (R-r) \cdot (1-\cos \phi)$$

of the rod 2 is

$$\Pi_2 = M_2 \cdot g \cdot (R-r-L)/2 \cdot (1-\cos \phi)$$

of the load 3 is

$$\Pi_3 = M_3 \cdot g \cdot \left[ OA \cdot (1-\cos \phi) + AB \cdot (1-\cos \psi) \right] =$$

$$= M_3 \cdot g \cdot \left[ (R-r) \cdot (1-\cos \phi) + l_3 \left( 1-\cos \left( \frac{R-r}{r} \phi \right) \right) \right]$$

of the counterweight 4 is

$$\Pi_4 = -M_4 \cdot g \cdot l_4 \cdot (1-\cos \phi)$$

The potential energy of the whole mechanism has the form
The potential energy function $\Pi(\phi)$ will be periodic at $k = 2\pi n r / (R-r)$, or at $\phi = 2\pi k r / (R-r)$. From here

$$\frac{k}{n+k} = \frac{r}{R},$$

Since $k$ and $n$ are positive integers, the fraction $\frac{k}{n+k}$ is a positive rational number, and therefore the ratio $\frac{r}{R}$ should also be a positive rational number.

Thus, under $M_3 \cdot l_3 \neq 0$ and under the above established condition for the relation $\frac{r}{R}$, the period of the function $\Pi(\phi)$ will be

$$T = 2\pi k,$$

where

$$k = \frac{r}{R-r} \cdot n,$$

provided that $k$ and $n$ are positive integers.

Further we will consider the mechanism in which the centers of mass of the rod and satellite are located on one side of the coupling $O$ (Figure 1), that is, $r \leq 0.25$ m.

Let’s carry out the analysis of the function $\Pi(\phi)$ depending on the position of the load 3 (distance $l_3 \neq 0$), but with $r = 0.2$ m and unchanged other characteristics (1) of the mechanism.

Then from (4) we obtain

$$k = \frac{r}{R-r} \cdot n = \frac{0.2}{0.45} \cdot n = \frac{4}{9} \cdot n.$$}

Hence, $k = 4$ and, therefore, for any $l_3 \neq 0$, the period of variation of the potential energy function is $T = 2\pi k = 8\pi$ rad (four turns of the rod).

In Figure 3 shows the graph of $\Pi(\phi)$ for $l_3 = 0.1$ m. It can be seen that for such value of $l_3$, for one period of change in potential energy, the mechanical system is four times in the positions of stable (local minima of $\Pi(\phi)$) and unstable (local maxima $\Pi(\phi)$) equilibrium.

If $l_3 = 0.15$ m, then on the graph of $\Pi(\phi)$ (Figure 4) at $\phi = 4\pi$ rad there is a local maximum, and to the left ($\phi = (4\pi - 4\pi 9) = 11,1701$ rad) and to the right ($\phi = (4\pi + 4\pi 9) = 13,9626$ rad) from it there are two
local minima. With this value \( l_3 \) for one period of change in \( \Pi(\phi) \), the mechanical system will visit the positions of stable and unstable equilibrium five times.

**Figure 3.** The plot of \( \Pi(\phi) \) at \( l_3 = 0.1 \) m.

**Figure 4.** The graph of \( \Pi(\phi) \) at \( l_3 = 0.15 \) m.

From the analysis of function (2) for the extremum, it follows that the transformation of a stable equilibrium position into an unstable (bifurcation) occurs at \( l_3 = 0.1086 \) m.

In the case of placing a load of mass \( M_3 \) on the rim of the satellite \( (l_3 = r = 0.2) \), the graph \( \Pi(\phi) \) is presented in Figure 5.

**Figure 5.** The graph of \( \Pi(\phi) \) at \( l_3 = r = 0.2 \) m.

It is seen that the number of equilibrium positions of the mechanical system has not changed compared to the previous case.

With increasing distance \( l_3 \) to the value \( l_3 = 0.2025 \) m (load outside the satellite) on the graph \( \Pi(\phi) \) at \( \phi_1 \approx 2.2543 \) rad and \( \phi_2 = 8\pi - \phi_1 \approx 22.8784 \) rad simultaneously begin to form (bifurcation) two new local minimums – new positions of stable equilibrium of a mechanical system appear.
The graph $\Pi(\varphi)$ at $l_3=0.23$ m is shown in Figure 6. With this value of $l_3$, for four turns of the rod ($0<\varphi\leq 8\pi$ rad), the mechanical system will visit the positions of stable and unstable equilibrium seven times.

![Figure 6](image)

*Figure 6. The graph of $\Pi(\varphi)$ at $l_3=0.23$ m.*

This situation persists with a further increase in $l_3$ up to the value $l_3=0.265$ m, when at $\varphi_1=7,8557$ rad and $\varphi_2=8\pi - \varphi_1=17,277$ rad, two more local minima appear (bifurcation) on the graph $\Pi(\varphi)$ with increasing $l_3$. Now the mechanical system in four turns of the rod nine times will visit the positions of stable and unstable balance.

With a further increase in $l_3$, the number of equilibrium positions does not change; only the extreme values of $\Pi(\varphi)$ change (Figure 7, $l_3=0.28$ m).

![Figure 7](image)

*Figure 7. The graph of $\Pi(\varphi)$ at $l_3=0.28$ m.*

With a significant increase of $l_3$ (for example, up to $l_3=25r=5$ m), the second member in expression (2) becomes predominant. And although in this case (Fig. 8) the period of the change in potential energy still remains equal to $8\pi$ rad, the extreme values of $\Pi(\varphi)$ approach each other in magnitude and therefore the graph of $\Pi(\varphi)$ is close to the harmonic plot with a period of $8\pi/9$ rad.

Let’s study the dependence of the potential energy of the mechanism on the radius $r$ of the satellite, taking $l_3=0.2$ m and leaving other characteristics of the mechanism in the form (1). We restrict the study to the cases $l_3>r$. 
Calculations of the function $I\!(\varphi)$ confirm that in this case too, its period $T$ is determined by formulas (3) and (4), while the number $z$ of the positions of the stable and unstable equilibrium of the mechanism for the period $T$ is

$$z = k \left( \frac{R}{r} - 1 \right),$$

and the period of the second harmonic is

$$T_1 = \frac{T}{z}.$$

Here are some examples.

In Figure 9 shows the dependence $I\!(\varphi)$ at $r=0.05$ m. As can be seen, the characteristics of $I\!(\varphi)$ are completely determined by formulas (3)–(6):

$$k = \frac{r}{R-r} \cdot n = \frac{0.05}{0.65-0.05} \cdot n = \frac{1}{12} \cdot n; \quad \Rightarrow \quad k=1.$$  

The period of the function $I\!(\varphi)$ is $T=2\pi k = 2\pi$ rad.

$$z = \left( \frac{0.65}{0.05} - 1 \right) = 12$$  
and  
$$T_1 = \frac{2\pi}{12} = \frac{\pi}{6} \text{ rad.}$$

Figure 10 shows a graph of $I\!(\varphi)$ at $r=0.15$ m. It fully corresponds to the values $k=3$; $T=6\pi$ rad; $z=10$; $T_1 = \frac{6\pi}{10}$ rad.
For $r=0.175$ m, the dependence $\Pi(\varphi)$ is shown in Figure 11. Here $k=7$; $T=14\pi$ rad; $z=19$; $T_1=\frac{14\pi}{19}$ rad.

So far, cases have been considered when the geometric characteristics of the structure were strictly constant and were given by rational numbers. But during the operation of any mechanical system, its parts wear out, their sizes change and, in principle, can take on values of both irrational (for example) and transcendental (for example, $e$ or $\pi$) numbers. Therefore, it is of interest to trace the change in the period of the function of the potential energy of the mechanism when the value of any parameter passes through these numbers.

As such a parameter, we choose the radius of the satellite, considering it as before $r<0.25$ m.

Table 1 shows the values of the periods $T$ of the function of the potential energy of the mechanism, calculated in accordance with (3), depending on the radius $r$ of the satellite with the other characteristics of the system unchanged given in (1).

As follows from the table, the period value substantially depends on the radius of the satellite. Moreover, if the value of $r$ is a rational number, then with its increase, the period $T$ can both increase and decrease. For example an increase of $r=0.156$ m by only 0.1 mm leads to an increase of the period $T$ from $12\pi$ rad to $3122\pi$ rad, that is, more than 260 times, and an increase $r=0.162$ m by 0.5 mm leads to a decrease in $T$ from $162\pi$ rad to $2\pi$ rad, that is, 81 times.

If the value of $r$ approaches an irrational or transcendental number, then the value of the period $T$ only increases. So with an increase in radius of $r$ from 0.157 m to 0.15707 m ($0.05\pi \approx 0.157079...$), the period $T$ increases from $314\pi$ rad to $31414\pi$ rad, that is, almost 100 times. With an increase in radius of $r$ from 0.163 m to 0.16309 m ($0.06 e \approx 0.163096...$), the period $T$ increases from $326\pi$ rad to $32618\pi$ rad, that is, practically also 100 times. With an increase in radius of $r$ from 0.1732 m to
0.173205 m (0.17320508...) the period $T$ increases from $866\pi$ rad to $69282\pi$ rad, that is, more than 80 times.

| $r$, m | $\frac{T}{\pi}$, rad | $r$, m | $\frac{T}{\pi}$, rad |
|--------|-----------------|--------|-----------------|
| 0.15   | 6               | 0.163  | 326             |
| 0.155  | 62              | 0.16309| 32618           |
| 0.156  | 12              | 0.1631 | 3262            |
| 0.1561 | 3122            | 0.164  | 164             |
| 0.1562 | 1562            | 0.172  | 172             |
| 0.1565 | 626             | 0.173  | 346             |
| 0.1569 | 3138            | 0.1731 | 3462            |
| 0.157  | 314             | 0.1732 | 866             |
| 0.15707| 31414           | 0.173205| 69282         |
| 0.1571 | 3142            | 0.1733 | 3466            |
| 0.162  | 162             | 0.174  | 174             |
| 0.1625 | 2               | 0.175  | 14              |
| 0.1629 | 3252            | 0.2    | 8               |

Further numerical calculations showed that the more precisely the value of $r$ approaches the irrational or transcendental number, the greater the value of the period $T$ of the potential energy function of the mechanism.

3. Conclusion

Thus, the study of the movement of the mechanism in a nonlinear setting showed a high sensitivity of the potential energy $\Pi(\varphi)$ of the mechanism to a change in its geometric characteristics. An interesting feature of $\Pi(\varphi)$ was revealed: if the geometric characteristics of the mechanism are set by rational numbers, then $\Pi(\varphi)$ is a periodic function with a precisely defined period; if the geometric characteristics of the mechanism are given by numbers irrational or transcendental, then the period of the function $\Pi(\varphi)$ tends to infinity.

References

[1] Babakov I 2004 Teoriya kolebaniy [Theory of oscillations] (Moscow: Drofa)
[2] Kurs teoreticheskoy mekaniki 2017 [The course of theoretical mechanics] Under the general. ed. Kolesnikov K.S. (Moscow: MGTU publishing House. N.E. Bauman)
[3] Kraynev A 1987 Slovar' – spravochnik po mekhanizmam [Dictionary - a directory of mechanisms] (Moscow: Mechanical engineering)
[4] Aldoshin G 2013 Teoriya lineynykh i nelineynykh kolebaniy [Theory of linear and nonlinear oscillations] (St. Petersburg: Lan)
[5] Bondarenko N, Obnosov K and Panshina A 2015 Yestestvennyye i tekhnicheskiye nauki [Natural and Technical Sciences] 3 18–22
[6] Obnosov K, Panshina A 2016 Spravochnik. Inzhenernyy zhurnal. S prilozheniyem [Handbook. Engineering Journal. With attachment] 3 8-17
[7] Bondarenko N, Obnosov K and Panshina A 2017 Uspekhi sovremennoy nauki [The successes of modern science Vol 7] 3 78-82