Area coverage of radial Lévy flights with periodic boundary conditions

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We consider the time evolution of two-dimensional Lévy flights in a finite area with periodic boundary conditions. From simulations we show that the fractal path dimension \( d_f \) and thus the degree of area coverage grows in time until it reaches the saturation value \( d_f = 2 \) at sufficiently long times. We also investigate the time evolution of the probability density function and associated moments in these boundary conditions. Finally we consider the mean first passage time as function of the stable index. Our findings are of interest to assess the ergodic behavior of Lévy flights, to estimate their efficiency as stochastic search mechanisms and to discriminate them from other types of search processes.

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I. INTRODUCTION

Lévy flights are random walk processes, in which the lengths of individual jumps are distributed according to a probability density \( \lambda(x) \) of the asymptotic power-law form \([1,4]\)

\[
\lambda(x) \approx \frac{\sigma^\alpha}{|x|^{1+\alpha}}, \quad 0 < \alpha < 2,
\]

where \( \sigma \) is a scaling factor of physical dimension length, \([\sigma] = \text{cm}\). The resulting motion is spatially scale-free due to the divergence of the jump length variance, \( \int x^2 \lambda(x) dx \). Lévy flights were popularized by Benoît Mandelbrot, who named them after his teacher, French mathematician Paul Lévy \([5]\). They have been mainly applied in the modeling of search processes, following the original idea by Klafter and Shlesinger \([6]\): while regular random walks in one or two dimensions have a high probability to return to already visited sites, Lévy flights combine thorough local search with occasional long excursions, leading them to areas which likely have not been visited before. This strategy reduces unnecessary oversampling and thus represents an advantage to the searcher. Indeed, trajectories consistent with the long-tailed jump length pattern \([1]\) have been reported for various animal species \([7,10]\). Even for molecular search, Lévy flights may emerge from the topology of the search space \([11]\). Due to a mix of modern means of transportation also human motion behavior is characterized by travel lengths with power-law distribution \([12]\). Long-tailed distributions of relocation lengths also change significantly the distribution pattern of diseases, as regular diffusion fronts are broken by, for instance, long-distance air travel, thus carrying the disease to completely decoupled places \([13]\).

For land-based animals typical search processes are essentially two-dimensional. In many cases we may also neglect the vertical dimension for many birds or fishes, when the lateral extension is considerably larger than the maximal height/depth difference of the trajectory. In such a two-dimensional search space, given its fractal dimension \( d_f = \alpha \) a single Lévy flight trajectory cannot fully cover the search space in absence of boundaries. Thus, some small area element in the search space may be hit by one single trajectory while it may be missed by another. In many cases, however, the search space is bounded. For instance, animals only search for food in their own territory, or the business travel patterns of an individual are confined to a certain country or continent. Moreover it is often relevant to have information about the area coverage of a Lévy flight process at finite times. For instance, a predator often does not cover its entire territory on a single day. It is therefore of interest to explore the time evolution of the area coverage of Lévy flights: how long does it take for the animal to efficiently explore its entire patch, or a disease to reach every little town in a country? How large is the area coverage at a given time? These questions of area coverage are directly connected with the ergodic properties of Lévy flights.

Incomplete area coverage of Lévy flights is also of interest in view of their ergodic properties. Ergodicity and its violation have recently received considerable interest, following the possibility to record single trajectories of molecules or small tracers in biological matter \([14]\). Thus, for anomalous diffusion processes with underlying scale-free motion a weak ergodicity breaking is observed \([15,16]\), such that the mean squared displacement obtained from the time average over a single trajectory is a random variable and significantly different from the corresponding ensemble average, as shown for the subdiffusive continuous time random walk process \([17,21]\). This behavior contrasts the ergodic behavior of other anomalous diffusion processes such as fractional Brownian motion \([22,23]\). While weak ergodicity violation occurs for processes with diverging characteristic time scales, ergodicity is also violated for Lévy flights in infinite media, due to the incomplete area coverage even for infinitely long trajectories \([24]\). Confined Lévy flights, in contrast, are
In this paper, we consider radial Lévy flights in two-dimensional square areas with periodic boundary conditions. We investigate the time evolution of the effective fractal dimension of the flights for varying stable index $\alpha$. As a complementary measure for the area coverage we analyze the time evolution of the mean squared displacement up to its saturation. Moreover we analyze the time evolution of the probability density function and its moments. We also pursue the question on the typical time for the Lévy flight to first reach or pass the boundary of the square interval.

The paper is organized as follows. In section II, we briefly review the mathematical foundation of Lévy flights. In section III, we consider radial two-dimensional Lévy flights in square boxes and numerically study their area coverage in terms of their fractal dimension, their probability density function, evolution of moments, and mean first passage times. Section IV summarizes the results.

II. LÉVY FLIGHTS

For jump length distributions of the type (1), the variance $\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \lambda(x) dx$ diverges, while fractional moments $\langle |x|^{\delta} \rangle$ of order $0 < \delta < \alpha$ exist [3]. This property directly carries over to the random motion itself and is responsible for several peculiar phenomena. Due to this lack of a finite variance Lévy flights have a fractal graph dimension $d_f = \alpha$ [4]. This property causes the sample path of a Lévy flight to be characterized by occasional long jumps, connecting clusters of more localized jumps. Unvisited holes exist on all scales within the sample path. This scale-free behavior gives rise to the fact that clusters host smaller clusters, as shown in Fig. 1. Moreover, the discontinuous jumps on arbitrarily large scales induce a principal discrepancy between first passage and first arrival events. For instance, while their first passage behavior on a semi-infinite domain obeys the Sparre Andersen universality [20, 25], their first arrival behavior is significantly reduced with decreasing $\alpha$ [20]. As Lévy flights strongly overshoot a point target [27]. On finite domains the first passage behavior is also modified [25, 29]. Another interesting effect is the occurrence of multimodal distributions for Lévy flights in the presence of steeper than harmonic potentials [30, 31].

Analytically, the statistics of the Lévy flight are dictated by the detailed distribution of individual jump lengths. However, on large scales—when large search spaces are explored for sufficiently long periods of time—we obtain a universal description. Displacements on large time scales are characterized by Lévy $\alpha$-stable laws, which in the symmetric case are defined in terms of their characteristic function $\langle e^{\imath k x} \rangle$, [2–5, 32],

$$\lambda(k) \equiv \int_{-\infty}^{\infty} \lambda(x) e^{\imath k x} dx = \exp (-\sigma^\alpha |k|^\alpha). \quad (2)$$

Again, $\sigma$ is a typical length scale. From the details of the microscopic jump length distribution, only the tail property [1] is directly passed on to the asymptotic stable law [2] of macroscopic displacements, including in particular the exact tail exponent $\alpha$. Lévy stable laws emerge as the limit of the sum of independent, identically distributed random variables with diverging variance, by virtue of the generalized central limit theorem [2, 3, 5, 32]. For $0 < \alpha < 2$ this leads to the asymptotic power-law [1], while in the limit $\alpha = 2$, we recover a Gaussian distribution for $\lambda(x)$ with finite moments of all orders.

Here, we consider two-dimensional Lévy flights in square boxes of edge length $2a$, with periodic boundary conditions. This means that each time the particle crosses one edge of the box, it will enter from the opposite edge. All Lévy flights start from the origin in the center of the square box. For each step in the two-dimensional random walk we draw a (signed, symmetric) flight distance $r$, with distribution $\lambda(r)$ as in (2), and independently a flight direction $\theta$, uniformly distributed within $[0; \pi]$. The projection of displacements onto the x-axis is thus $\Delta x = r \cos(\theta)$. We call such processes radial two-dimensional Lévy flights. Such radial Lévy flights have been used to analyze search processes (see, e.g., Ref. [1]). We note that they are slightly different from two-dimensional Lévy flights defined in terms of the two-dimensional characteristic function $\exp(-|\sigma^*|^\alpha |k|^\alpha)$ [32].

In what follows we calculate parameters such as the fractal dimension, the mean first passage time, the mean squared displacement, the probability density function...
and its other moments and follow their temporal evolution.

### III. RESULTS AND DISCUSSION

#### A. Fractal dimension

The fractal dimension $d_f$ indicates how completely a fractal graph fills the available embedding space. For a Lévy flight evolving in our finite area the value of the fractal dimension will initially be smaller than two. Starting from the value zero for the first (few) point(s) of the trajectory, if the area is sufficiently large the value of $d_f$ is expected to be the value of the stable index, $d_f = \alpha$, as long as the boundary effects do not come into play. In our simulations we do not see a saturation close to $d_f = \alpha$, as we are interested in the finite size effects and therefore choose quite small domains. Due to this finiteness of the domain the Lévy flight will eventually cover the entire area and thus the fractal dimension is expected to saturate at the value $d_f = 2$. This corresponds to the observation of ergodicity for bounded Lévy flights.

To determine the time evolution of the fractal dimension of the Lévy flight under consideration we calculate the box counting dimension of the trajectory. In this method, the number of boxes $N$ containing a point of the fractal set (here, the points of visitation of our trajectory) is counted for a given box size $c^2$. For an underlying fractal geometry and sufficiently small $c$ a power law relation of the form $N(c) \sim c^{-d_f}$ is expected. To validate our method we first determined the fractal dimension of a number of unbounded Lévy flights, for which the value $d_f = \alpha$ is expected. The results are summarized in Table I. We find that the fractal dimension is reproduced quite reliably. However, the number of necessary random walk steps to obtain sufficiently good results increases dramatically with growing $\alpha$.

In Fig. 2 we plotted the time-dependent fractal dimension calculated via the two-dimensional box-counting method for confined Lévy flights with different values of $\alpha$ ($\alpha = 0.7, 1.2, 1.7, \text{ and } 2$) for fixed box size $1,000 \times 1,000$. As expected for all $\alpha$ values the fractal dimension reaches the Euclidean embedding dimension, $d_f \to 2$. We see that for growing $\alpha$ indeed the slope of the curve $d_f(t)$ is smaller, such that the stationary limit is reached slower. Interestingly, for the Gaussian limit $\alpha \to 2$ an intermediate plateau emerges, as indicated by the arrow in the inset of Fig. 2. By analysis of a large number of Lévy flights for this $\alpha$ value we made sure that this effect is not an artifact. The significance of this observation is also underlined by the relatively small statistical sample error bars shown in Fig. 2. Thus, close to $\alpha = 2$ care should be generally taken when determining the fractal graph dimension of a Lévy flight process.

The relaxation time to the ergodic behavior of full area coverage of a Lévy flight may reach considerable values. For a searching animal it is therefore imperative to have a reasonable field of vision, i.e., to be able to scan an area around each point of the trajectory. Even more efficient is constant scanning as discussed in Ref. [7], such that the scanned on-the-fly area represents a sausage. Without such provisions the scanning of the area by the Lévy flight remains sparse a fortiori, unless extremely long search times are invested.

#### B. Probability density function

We now turn to the probability density function of the process. Fig. 3 shows the time evolution of the unidirectional probability density function $P(x,t)$ for Lévy flights with stable indices $\alpha = 0.7$ and $1.7$. For comparison we also present graphs for Brownian motion, $\alpha = 2$. Starting with the sharp initial condition $P_0(x) = \lim_{t \to 0} P(x,t) = \delta(x)$, the probability density function successively broadens, until it reaches the state of equidistribution, $P_{st}(x) = 1/(2a)$. We observe that the specific characteristics of the different cases $0 < \alpha < 1$, $1 < \alpha < 2$, and $\alpha = 2$ remain discernible up to relatively long times. Thus up to some 10% of the overall

| Input $\alpha$ | Number of steps | Output $\alpha$ |
|----------------|-----------------|-----------------|
| 0.7            | 10,000          | 0.68 ± 0.06     |
| 1.2            | 50,000          | 1.16 ± 0.04     |
| 1.7            | 10,000,000      | 1.65 ± 0.03     |

**TABLE I:** Value of the stability index $\alpha$ of free Lévy flights. We list the input $\alpha$ used to generate the trajectory and the output value for $\alpha$ as determined from the box counting method. The number of steps is an indication for the number of steps of the simulated Lévy flight necessary to determine $\alpha$ to reasonable accuracy.
simulations time, it is still possible to distinguish the basic shapes of the probability density functions for these cases. The overall simulations time needed to reach saturation is, however, dramatically different, as already observed in the labels shown in Fig. 3. This Figure also demonstrates how fitting to a stable law, Eq. (2), yields an approximation of $\alpha$. The fits in this figure show excellent agreement with simulation data.

Thus, from fitting probability density functions alone we could conjecture that the initial dynamics of the searcher starting far off the edges of the box are essentially unaffected by boundary conditions: at small times, the spatial distribution can be approximated by the stable law from an unbounded Lévy flight. However, the effect of boundaries is known to be highly non-local if the random walk is governed by long-tailed jump statistics. Due to the non-negligible probability for extremely long excursions, the searcher probes the nature of the boundaries from the very start of its motion. For example, this leads to modifications on the associated diffusion equations [35, 36]. Fig. 4 highlights the specific tail properties of the early-time distribution. Despite the almost perfect fit quality of stable distributions along virtually the whole search space, there are considerable deviations in the far tails near the boundaries. In particular, the distinct heavy-tail property, Eq. (1), is already lost at this early stage of the random walk.

Analyzing the time evolution of moments reveals further distinct peculiarities of the bounded motion, as we show in the following.

C. Moment analysis

First, consider the mean squared displacement $\langle x^2(t) \rangle$, which is another common approach to assess the area coverage of the process. While this quantity diverges for a free Lévy flight or a Lévy flight in an harmonic external potential [4, 31], it is already finite in steeper than harmonic external potentials [2]. In our finite area the mean squared displacement is thus a valid measure for the motion. Figure 5 shows the time dependence of the one-dimensional variance, for different values of the stable index $\alpha$. The saturation of the mean squared displacement is comparatively sharp, such that we can read off a typical time scale $\tau$ from Fig. 4 but a larger number of $3 \times 10^6$ of random walks were needed to produce precise tail data (continuous red curve).

FIG. 3: Time evolution of the unidirectional probability distribution function $P(x, t)$ of confined Lévy flights with $\alpha = 0.7$ and 1.7, as well as Brownian motion with $\alpha = 2$ in a box of edge length, $2a = 1,000$, for $\sigma = \sqrt{2}$. In each case 10,000 sample trajectories were used to create an individual curve for $P(x, t)$. Fits (dashed black lines) correspond to stable distributions as in Eq. (2).

FIG. 4: Tail analysis for the probability distribution function $P(x, t)$ of confined Lévy flights at an early stage of the random walk. The stable fits (dashed black curve) were performed at the same time instants as in Fig. 3 but a larger number of $3 \times 10^6$ of random walks were needed to produce precise tail data (continuous red curve).
Note that the early diffusion process is "normal" in the sense of ensemble average $\langle \alpha \rangle$. Here, 10,000 sample trajectories have been used for the expected stationary value $a$. The horizontal black line represents the initial dynamics which we see in Fig. 6: the associated diffusion coefficient $D = \langle x^2(t) \rangle/(2t)$ depends on the size of the system. Enlarging the box increases the (finite) variance of individual jump lengths, thereby increasing $D_x$.

To get the full picture we compute arbitrary moments of absolute displacements from our simulations. We generally find an initial power law time dependence, $\langle |x|^{\delta} (t) \rangle \simeq t^{\delta}$, before the occurrence of saturation. More precisely, from the data for the scaling exponent $e(\delta)$ displayed in Fig. 7 we expect that in the asymptotic limit $1 \ll t \ll \tau$, $\langle |x|^{\delta} (t) \rangle \simeq$ $t^{\delta}$, $\delta > 0$. The theoretical prediction (black dotted lines, Eq. (3)) is valid only in the asymptotic limit $1 \ll t \ll \tau \simeq a^\alpha$. For this simulation, $2a = 10^3$, $t \in \{10, \ldots, 30\}$. The calculation of higher order moments exploits the far tails of the distribution, requiring a relatively large number of $10^8$ trajectories.

We note that this type of moment scaling is actually a familiar one in the theory of Lévy flights. It also occurs when truncating the distribution $\delta$ for the jump lengths at some fixed distance $0 < \delta < \alpha$, or when considering the effects of a finite ensemble. Roughly speaking, the truncation distance is analogous to the largest value drawn from a finite ensemble; in our system this role is played by the finite edge length of the box. In all such cases, the initial dynamics resemble an ordinary, unbounded Lévy flight, albeit with finite moments of the type $\delta$.

In particular, we conclude that the second moment of bounded motion mimics "normal" diffusion, but the associated diffusion coefficient must depend on the system size parameter $a$: at early times, moments with $0 < \delta < \alpha$ are adopted from the unbounded Lévy flight and are thus independent of $a$. At late times, all moments saturate at $\langle |x|^{\delta}(\infty) \rangle \simeq a^\delta$. The typical time scale for the interaction contrasts the properties of an unbounded Lévy flight, where $x^2$ scales like $t^{2/\alpha}$. In fact, from measuring the mean squared displacement alone one could assume the motion to be confined Brownian. Thus, the effect of the boundaries goes beyond ensuring the finiteness of the second moment. Interestingly, the boundary characteristics also have a distinct quantitative effect on the initial dynamics, which we see in Fig. 7: the associated diffusion coefficient $D_x = \langle x^2(t) \rangle/(2t)$ depends on the size of the system. Enlarging the box increases the (finite) variance of individual jump lengths, thereby increasing $D_x$.
with the boundaries therefore scales as
\[ \tau \approx a^\alpha, \quad \text{but also} \quad \tau \approx a^2/(2D_x). \] (4)

This necessarily implies that
\[ D_x = D_\perp(a) \approx a^{2-\alpha}. \] (5)

The above relation is in nice agreement with our power law fits in Fig. 6. The exponent also consistently coincides with the one found for the dependence on the truncation parameter in truncated, but unbounded, Lévy flights as anticipated in Ref. [39].

### D. Mean first passage time

We finally analyze the mean first passage time \( \langle \tau_1 \rangle \) of our confined Lévy flights to the boundary of the area. On a semi-infinite domain Lévy flights show the universal Sparre Anderson scaling \( \tau \sim \tau_1 \sim a^{-3/2} \) of the distribution of first passage times, so the mean first passage time \( \langle \tau_1 \rangle \) diverges \( \sim a^{3/2} \). On our finite domain it converges and depends crucially on \( \alpha \). We determined the mean first passage time for Lévy flights with different values of \( \alpha \) and for different box sizes (\( a = 250, 500, 1000 \) and \( 2000 \)). From scaling arguments we would expect that the mean first passage time grows like \( \langle \tau_1 \rangle \approx a^\alpha \) as function of the initial distance from the absorbing boundary, see Eq. (4).

Fig. 8 depicts the rescaled mean first passage time \( a^{-\alpha} \langle \tau_1 \rangle \) as function of \( \alpha \). We observe that the scaling relation holds provided we can approximate the Lévy flight by a continuous diffusion process. Conversely, by choosing small values for \( a^\alpha \) we can approximate a random walker that escapes the box after only a few number of steps and the universal scaling (4) breaks down. From the monotonic nature of the graphs we conjecture that – for certain fixed parameters \( \sigma \) and \( a \) – one might find an optimal value for the tail parameter \( \alpha \) which minimizes the mean first passage time.

We note that the analogous first passage time problem in one dimension has been treated elsewhere analytically and numerically \([41, 42]\). A generalization to our two-dimensional problem however is not straightforward: Although we fix the scale \( \sigma = \sqrt{2} \) for individual jump distances \( r \), displacements are still affected by random jump directions \( \theta \). One can argue that in effect, the scale of displacements along the \( x \)-, \( y \)- or radial coordinate is not fixed, but has a non-trivial relation to the tail parameter \( \alpha \). Consequently, such considerations raise the general question of how to appropriately fix a jump length scale for variable stable exponents in multi-dimensional Lévy flights.

### IV. CONCLUSIONS

Area coverage of stochastic processes is an important criteria for the efficiency of search or spreading processes.

Here we investigated the time evolution of the time average of the area coverage in finite territories of Lévy flights. As measures we used the fractal dimension, which relaxes to two at sufficiently long times, and moments of displacements of the Lévy flight. Consistently, we found that both quantities show a relatively immediate saturation behavior, such that a clear time scale can be determined for reaching full area coverage. However, this time scale crucially depends on the value of the stable index \( \alpha \): when \( \alpha \) approaches the limiting value two, this time scale increases significantly. As expected, smaller values for \( \alpha \), allowing for longer jump lengths, effect more efficient area coverage and thus search efficiency. In general, the number of steps necessary to reach full area coverage is relatively large. Thus searchers using this type of strategy will likely not be able to achieve absolute certainty to find a given target randomly. A field of vision, as introduced in the search literature, will mellow this problem and render Lévy flights a very efficient search strategy.

The analysis of the area coverage is completed with the study of the equilibration behavior of the probability density function of the process, allowing one to distinguish different domains of the stable index \( \alpha \) (smaller/larger than one) up to relatively long times. Moreover the mean first passage time \( \tau_1 \) scales with the index \( \alpha \) and the system size approximately in the scaling fashion \( \langle \tau_1 \rangle \approx a^\alpha \).

From the data analysis point of view, we find that studying one single type of above measures is usually not enough to identify a given motion as a bounded Lévy flight. The fractal dimension does have a characteristic behavior for long-tailed jump lengths, but quite extensive data is necessary to achieve reliable results from a box counting algorithm. Although fitting the spatial distributions by stable laws might agree nicely at early stages of the process, their characteristic heavy tails are quickly reshaped by interactions with the boundaries. While the
mean squared displacement indicates normal diffusion, the associated diffusion constant turns out to depend on the system size and lower order moments indicate super-diffusion. Finally, an analysis based on first passage times only yields decisive results if data on a variety of system sizes is available. We conclude that usually a combination of methods is necessary to unambiguously determine the Lévy flight nature of a given process.

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