PIECEWISE SMOOTHNESS FOR LINEAR ELLIPTIC SYSTEMS
WITH PIECEWISE SMOOTH COEFFICIENTS

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Abstract. Li and Vogelius, and Li and Nirenberg obtained piecewise $C^{1,\gamma}$ regularity for linear elliptic problems with piecewise $C^{\gamma}$-coefficients which come from composite materials. In this paper, we obtain piecewise $C^{m+1,\gamma}$-regularity for linear elliptic systems with piecewise $C^{m,\gamma}$-coefficients which also come from composite materials. This answers the open problem suggested by Li and Vogelius, and Li and Nirenberg.

1. Introduction and main result

1.1. Introduction. We consider linear elliptic systems arising from composite materials such as fiber reinforced materials. Assume that $\Omega \subset \mathbb{R}^2$ is an open bounded $C^{m,\gamma}$-domain, and $\{\Omega^k\}_{k=1}^{k_0}$ are the mutually disjoint $C^{m,\gamma}$-subdomains in $\Omega$ which are allowed to touch each other. We denote $\Omega^0 = \Omega \setminus \bigcup_{k=1}^{k_0} \Omega^k$. Let $u$ be a weak solution of a linear elliptic equation

$$D_\alpha \left[ A_{ij}^{\alpha\beta} D_{\beta} u^j \right] = D_\alpha F^i \quad \text{in } \Omega,$$

where $A_{ij}^{\alpha\beta} \in C^{m,\gamma}(\Omega^k)$ and $F^i_\alpha \in C^{m,\gamma}(\Omega^k)$ for any $1 \leq \alpha, \beta \leq n$, $1 \leq i, j \leq N$ and $k \in [0, k_0]$. With our assumption, $A_{ij}^{\alpha\beta}$ and $F^i_\alpha$ are allowed to be discontinuous across the boundary of the subdomains $\{\partial \Omega^k\}_{k=1}^{k_0}$. In this paper, we prove that the weak solution $u$ is locally piecewise $C^{m+1,\gamma}$, meaning that $u$ is $C^{m+1,\gamma}(\tilde{\Omega} \cap \Omega^k)$ for any compact subset $\tilde{\Omega} \subset \subset \Omega$ and $k \in [0, k_0]$.

An application of our result is the following. Suppose that the fibers and resin of a fiber reinforced plastic (FRP) is assumed to be linearly elastic, and described by a linear elliptic equation. Then due to the discontinuity of the coefficients or elasticity, the gradient of the weak solution might be discontinuous on the boundary of each materials. Nevertheless, our result shows that the weak solution on each materials (fiber, resin) can be smooth, even if the fibers are allowed to touch each other. To prove our result, we generalize the difference quotient method to curves instead of traditional fixed directions.

One of long standing problem for the regularity theory of composite material is establishing a new regularity theory when the subdomains $\{\Omega^k\}_{k=1}^{k_0}$ are allowed to touch each other. Under this assumption, the flattening argument could not be applied at the points where two subdomains touch each other, and the difficulty arises. With this geometric assumption, local Lipschitz regularity can be derived as in $[3, 8, 9]$. But better regularity than Lipschtiz regularity is not feasible, because discontinuity of the gradient might occur on the boundary of subdomains. However, from the observations made in the previous literatures, one can consider so called piecewise regularity results. For example, if the distance between the boundary of
subdomains are bigger than some positive constant, then one may use the flattening argument to prove local piecewise $C^{m,\gamma}$-regularity. But this estimate depends on the distance between the boundary of subdomains and could not be applied for composite materials. When two subdomains are allowed to touch each other, which might happen for composite materials, the best result known in the previous literatures is piecewise $C^{1,\gamma}$-regularity by Vogelius and Li [9], and Li and Nirenberg [8]. In that papers, they used observation that in small scales, the boundaries of two touching subdomains are almost flat near the touching point, and such geometric structures could be viewed as linear laminates. Indeed, [9] and [8] proved that the solution is $C^{1,\gamma}(\tilde{\Omega} \cap \Omega^k)$ for any compact subset $\tilde{\Omega} \subset \Omega$ and $k \in [0,k_0]$ by using the regularity theory of linear laminates such as Chipot, Kinderlehrer and Vergara-Caffarelli [4], and in fact their estimates does not depend on the distance between the boundary of subdomains. However, the method in [9] and [8] could not be extended to higher derivatives $C^{m,\gamma}$ with $m \geq 2$, and in [8] they suggest an open question that piecewise $C^{m,\gamma}$-regularity even holds for $m \geq 2$. On the other-hand, regularity results for higher derivatives have been obtained for special geometric structures with dimension 2, when the domain $\Omega$ is a disk, and two subdomains are also disks. In that case, [9] proved $C^{m,\gamma}$ for $m \geq 2$ by using conformal mapping theory, but unfortunately their method could not be extended to higher dimensions. We also refer to [10] for a similar result by using Green functions.

With the observation in this paragraph, it has been expected that the conjecture about piecewise $C^{m,\gamma}$-regularity for $m \geq 2$ seems true, and in this paper we prove that this conjecture indeed holds.

To prove our main result, we generalize the difference quotient method. As far as we are concerned, the difference quotient method have been applied for a fixed direction, but we generalize this method by applying to curves. In fact, we use so called a ‘flow’ to obtain higher regularity, because the computation of difference quotient method is complicated for the higher regularity and we will use an approximation argument for obtaining higher regularity.

If a curve does not across the boundary of subdomains, then the coefficients remain regular along that curve, even if the coefficients are discontinuous between different subdomains. So we choose a suitable flow so that if we restrict the domain of the flow to some subdomain $\Omega^k$, then the range of that flow still remains in the same subdomain $\Omega^k$. To explain our new method, let’s consider our model case. Let $\Omega = \{ -\frac{\pi}{2}, \frac{\pi}{2} \} \cap \{ x \in R^2 : \varphi_+(x) = (x^1)^2 \text{ and } \varphi_-(x) = -(x^2)^2 \text{ where } x = (x^1, x^2) \}$. Then $\varphi_+$ and $\varphi_-$ touches each other at the origin $(0,0)$. We choose a function $\sigma : [-1,1] \times [-1,1] \times [-1,1] \rightarrow [-4,4]$ such that

$$\sigma(x, t) = \frac{[\varphi_+(x^2 + t) - \varphi_+(x^2)](x_1 - \varphi_-(x^2))}{\varphi_+(x^2) - \varphi_-(x^2)} + \frac{[\varphi_-(x^2 + t) - \varphi_-(x^2)](x_1 - \varphi_+(x^2))}{\varphi_+(x^2) - \varphi_-(x^2)}.$$

Then $t \mapsto x + \sigma(x, t, t)$ can be viewed as a flow in $Q^1_t := \{ (x_1, x_2) \in [-1,1] \times [-1,1] : \varphi_-(x^2) \leq x_1 \leq \varphi_+(x^2) \}$. And we find that $\partial_t[f(x_1, \sigma, t)]|_{t=0}$ is the first difference quotient or the first derivative along the curve $t \mapsto \sigma(x, t)$.

However, this approach only works for piecewise $C^{2,\gamma}$-regularity. The reason is that if two subdomains touches each other, then the first derivative of two boundaries coincide at that touching point, but the higher derivatives might not coincide at that touching point. To explain, we calculate 2-nd derivative of $h \in C^\infty(R^2)$ along the curves $\{(\varphi_+(x^2), x^2) : x^2 \in [-1,1]\}$ and $\{(\varphi_-(x^2), x^2) : x^2 \in [-1,1]\}$ at the origin by using the following calculation:

$$\partial^2_t h(\varphi(t), t) = \varphi'(x^2 + t)^2 D_{11} h(\varphi(t), t) + 2 \varphi'(x^2 + t)^2 D_{12} h(\varphi(t), t) + D_{22} h(\varphi(t), t) + \varphi''(x^2 + t) D_{11} h(\varphi(t), t).$$
Since $\varphi''_+(0) = 2 \neq \varphi''_-(0) = -2$ and $\varphi'_-(0) = \varphi'_+(0) = 0$, $D_t h(0,0) \neq 0$ implies $\partial_t^2 |h(\varphi_+(t),t)||t=0 \neq \partial_t^2 |h(\varphi_-(t),t)||t=0$. On the other-hand, it is possible to obtain $C^{2,\gamma}$ by just using the approach in the previous paragraph because

$\partial_t |h(\varphi_+(t),t)||t=0 = \partial_t |h(\varphi_-(t),t)||t=0$.

By our model case, we see that there are jumps of the higher order derivatives on the boundary of subdomains. In fact, this jumps appears even if $h$ is smooth which means that the jumps of the higher order derivatives are mainly due to the geometry of the composite domains. Because of this reason, we obtained geometric property of composite domains in [7, 2]. In [7], the author and Jang prove that the boundary of the subdomains are almost parallel if they are sufficiently close, even if the subdomains are fractal, say Reifenberg flat domains. Also [2] answers an question for the typical examples of the composite domains. If $C^{1,\gamma}$-domains satisfy certain inclusion conditions, they form a composite domains.

To overcome the difficulty for handling the jumps of the higher order derivatives on the boundary of subdomains, we extend functions defined in a subdomain. We briefly explain this idea by an example. Let $u$ be a function which might have jumps on the boundary of subdomains. Let $u^k$ be an extension of $u$ from $\Omega^k$ to $\Omega$ such that $u^k$ is regular in $\Omega \setminus \Omega^k$. Then we see that $\sum_k u^k$ has no jumps on the boundary of the subdomains. So by considering $\sum_k u^k$ instead of $u$, jumps of the function is removed and the function became easier for obtaining piecewise regularity. The value of $u$ and $\sum_k u^k$ might not be the same in $\Omega$, but $u - \sum_k u^k$ is piecewise regular because $u^k$ is an extension of $u$ in $\Omega^k$ and defined to be regular in $\Omega \setminus \Omega^k$. If a suitable perturbations could be done to $\sum_k u^k$ in each subdomains, then one can obtain piecewise regularity of $u$. For partial differential equations, the functions can be differentiated. But the ideas in the previous paragraphs still work for our problem.

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1.2. Main result. We introduce the following notations in this paper.

1. A $(n-1)$-dimensional point is denoted as $x' = (x^2, \ldots, x^n)$.

2. $Q_r'(y') = \left\{ x' = (x^2, \ldots, x^n) \in \mathbb{R}^{n-1} : \max_{2 \leq i \leq n} \left| x^i - y^i \right| < r \right\}$ is the open cube in $\mathbb{R}^{n-1}$ with center $y'$ and size $r$. Also we denote $Q_r = Q_r'(0')$.

3. $Q_r(y) = \left\{ x \in \mathbb{R}^n : \max_{1 \leq i \leq n} \left| x^i - y^i \right| < r \right\} = (y^1 - r, y^1 + r) \times Q_r(y')$ is the open cube in $\mathbb{R}^n$ with center $y$ and size $r$. Also we denote $Q_r = Q_r(0)$.

4. For a function $g(x)$ in $\mathbb{R}^n$,

$$ (g)_U = \int_U g(x) \, dx = \frac{1}{|U|} \int_U g(x) \, dx, $$

where $U$ is an open subset in $\mathbb{R}^n$ and $|U|$ is the $n$-dimensional Lebesgue measure of $U$.

5. $h|_U$ is the restriction of $h$ in $U$.

6. For an open set $U \subset \mathbb{R}^n$, $h \in C^0(\overline{U})$ means that $Du, \ldots, D^k u$ exist in $U$ and are uniformly continuous in $U$.

To handle the index in higher order derivatives, we use the following notations.
Here, Remark 1.2. For the graphs (1.1)
where
For the composite cube (Definition 1.3.
and
for any \( Q \), the graph functions (Definition 1.1.
follows.
We remark that there can be only one element in \( K \) in \( Q \). In \( Q \), the component or the subregions will be denoted as \( k \). For any \( k \geq 1 \), the component or the subregions will be denoted as \( q \). p
For the sets \( Q \), we define composite domain by using composite cubes inside the cube, we use the following natural definition.
We introduce the notations for representing the subregions and the boundary of the subregions. In composite materials, there exists a coordinate system such that the boundaries of the subregions become almost flat graphs in sufficiently small scale, see for instance [1]. So from now on, we assume that the boundaries of the regions are represented by the graphs in \( Q \). We assume that the cube \( Q \) is divided into the components or the subregions by using \( C^{1,\gamma} \)-graph functions.
In \( Q \), the component or the subregions will be denoted as \( Q^k(z) \) with the set \( K = \{ k, \ldots, k_+ \} \). Also let \( K_+ = \{ k_-, 1, \ldots, k_+ \} \) be the \( C^{1,\gamma} \)-graph functions.
We remark that there can be only one element in \( K \). To coincide with the previous definition in [1], we define composite domain by using \( C^{1,\gamma} \)-graph functions as follows.

**Definition 1.1.** For the sets \( K = \{ k, k_-, 1, \ldots, k_+ \}, K_+ = K \cup \{ k_+ + 1 \} \) and the graph functions \( \varphi_k \in C^{1,\gamma} (Q^k(z)) \) \( k \in K_+ \), \( (Q_r(z), \{ \varphi_k : k \in K_+ \}) \) is called a composite cube if
\[
\varphi_k(x') \leq \varphi_{k+1}(x') \quad (x \in Q_r(z), k \in K),
\]
and
\[
Q_r(z) = \bigcup_{k \in K} Q_r^k(z),
\]
where
\[
(1.1) \quad Q_r^k(y) := \{ (x^1, x') \in Q_r(y) : \varphi_k(x') < x^1 \leq \varphi_{k+1}(x') \} \quad (k \in K).
\]
Here, \( \bigcup_k U_k \) denotes disjoint union meaning that \( \bigcup U_k \) is the union of the sets \( \{ U_k : k \in K \} \) and that \( \{ U_k : k \in K \} \) are mutually disjoint.

**Remark 1.2.** For the graphs \( \{ (\varphi_k(x'), x') : x' \in Q^k_r(z') \} \) \( k \in K_+ \) in Definition 1.1, the top \( \{ (\varphi_{k+1}(x'), x') : x' \in Q^k_r(z') \} \) and the bottom \( \{ (\varphi_k(x'), x') : x' \in Q^k_r(z') \} \) ones should be outside of \( Q_r(z) \), because otherwise we have that \( Q_r(z) \neq \bigcup_{k \in K} Q_r^k(z) \).

For the composite cubes inside the cube, we use the following natural definition.

**Definition 1.3.** For the composite cube \( (Q_r(z), \{ \varphi_k : k \in K_+ \}) \), we denote
\[
Q_r^k(y) := \{ (x^1, x') \in Q_r(y) : \varphi_k(x') < x^1 \leq \varphi_{k+1}(x') \} \quad (k \in K),
\]
for any \( Q_r(y) \subset Q_r(z) \).
In view of [1], there exists a coordinate system such that the graphs are almost flat. So we also assume that
\[
\|Dx^k\varphi_k\|_{L^\infty(Q^k_\gamma)} \leq \frac{1}{20n} \quad (k \in K_+).
\]

For a technical reason, we will focus on the set \(Q_5\). Let \(K_- (\subset K)\) be the set of graph functions which intersect \(Q_5\). Then we see that
\[
K_- \subseteq K \subseteq K_+.
\]

From the definition of \(K_-\), we discover that
\[
k \in K_- \iff Q_5 \cap \{(\varphi_k(x'), x') : x' \in Q^k_7\} \neq \emptyset,
\]
which implies that
\[
k \in K_- \iff Q^k_7 \neq \emptyset \quad \text{and} \quad Q^{k-1}_5 \neq \emptyset.
\]

It follows from (1.2) and (1.3) that
\[
\{(\varphi_k(x'), x') : x' \in Q^k_7\} \subset \partial Q^k_7 \cap \partial Q^{k-1}_5 (\subset Q_6) \quad (k \in K_-).
\]

**Remark 1.4.** With the assumption that \(a_{ij}, u, F \in C^{m,\gamma} (Q^k_7) \) \((k \in K)\), we will prove that \(u \in C^{m+1,\gamma} (Q^k_7) \) \((k \in K)\). The most of the calculations in this paper are performed in \(Q_5\) but \(Q^k_5 \) \((k \in K)\) might not be disconnected. So with (1.5), we consider an extended domain \(Q_7\) to overcome this difficulty.

For \(A^{\alpha,\beta, k}_{ij} : \mathbb{R}^n \to \mathbb{R} \) \((1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq N, k \in K)\), assume that
\[
\lambda |\zeta|^2 \leq \sum_{1 \leq i, j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A^{\alpha,\beta, k}_{ij}(x)|\zeta|^2 \quad (x \in \mathbb{R}^n, \zeta \in \mathbb{R}^n, k \in K),
\]

\[
|A^{\alpha,\beta, k}_{ij}(x)| \leq \Lambda \quad (x \in \mathbb{R}^n, 1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq N, k \in K)
\]

and
\[
A^{\alpha,\beta, k}_{ij} \in C^{m,\gamma} (\mathbb{R}^n) \quad (1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq N, k \in K),
\]

for some \(\Lambda \geq \lambda > 0\) and \(\gamma \in (0,1/4)\). Let \(A^{\alpha,\beta}_{ij} : Q_7 \to \mathbb{R}\) as
\[
A^{\alpha,\beta}_{ij} = \sum_{k \in K} A^{\alpha,\beta, k}_{ij}(x)\zeta
\]

for any \(1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq N\) and \(k \in K\). Then by (1.6), (1.7) and (1.8),
\[
|A^{\alpha,\beta, k}_{ij}(x)| \leq \Lambda \quad (x \in \mathbb{R}^n, 1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq N)
\]

and
\[
A^{\alpha,\beta}_{ij} \in C^{m,\gamma} (Q^k_7) \quad (1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq N, k \in K).
\]

Let \((Q_7(z), \{\varphi_k : k \in K_+}\})\) be a compostie cube with the the assumption that
\[
\varphi_k \in C^{m+1,\gamma} (Q^k_7) \quad (k \in K_+).
\]

Then let \(u \in W^{1,2}(Q_7)\) be a weak solution of
\[
\sum_{1 \leq \alpha \leq n} D_\alpha \left[ \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A^{\alpha,\beta}_{ij} D_\beta u^j \right] = \sum_{1 \leq \alpha \leq n} D_\alpha F^\alpha \quad \text{in} \ Q_7,
\]

with the assumption that
\[
u, F \in C^{m,\gamma} (Q^k_7) \quad (k \in K).
In this paper, we might omit the summation by using Einstein notation.

To prove piece-wise Hölder continuity of \(D^{m+1}u\), we will first show in Theorem 1.6 that the following \(U^p : Q_5 \to \mathbb{R}^{Nn}\) is Hölder continuous in \(Q_5\). For any \(p' \geq 0\) with \(|p'| = m\), set \(U^p : Q_5 \to \mathbb{R}^{Nn}\) as

\[
U^p_{\alpha} = G^p_{\alpha} + \sum_{1 \leq \alpha \leq n} \pi_{\alpha} \left[ \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right)^{p'} D^{(q'\cdot p'\cdot q')} \left[ \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A^\beta_{ij} \partial Q D_{\beta} u^j - F^\alpha \right] \right]
\]

with

\[
U^p_{\beta} = G^p_{\beta} + \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right)^{p'} D^{(q'\cdot p'\cdot q')} \left( D_{\beta} u^1 + \pi_{\beta} D^{(q'\cdot p'\cdot q')} D_{\beta} u^1 \right)
\]

for any \(2 \leq \beta \leq n\) and \(1 \leq i \leq N\), where \(G^p : Q_5 \to \mathbb{R}^{Nn}\) is defined as

\[
G^p_{\alpha} = \sum_{k \in K} \sum_{1 \leq \alpha \leq n} G^p_{\alpha} \left[ U_{\alpha}|_{k-1} - U_{\alpha}|_{k} \right] H_k
\]

and

\[
G^p_{\beta} = \sum_{k \in K} \varphi_{\beta}^{k} \left[ u|_{k-1} - u|_{k} \right] H_k
\]

for any \(2 \leq \beta \leq n\) and \(1 \leq i \leq N\). Here, \(H_k : Q_7 \to \mathbb{R} (k \in K)\) is Heaviside function that

\[
H_k(x) = \begin{cases} 
1 & \text{if } x \in Q^k_7 \text{ with } l \geq k, \\
0 & \text{if } x \in Q^l_7 \text{ with } l < k.
\end{cases}
\]

Also with (1.5), the linear operators \(G^p_{\alpha}^{k} : C^{m-1,\gamma}(\partial Q_5^{k} \cap \partial Q_5^{k-1}) \to C^{\gamma}(Q_5^k)\) \((k \in K_-)\) and \(\varphi_{\beta}^{k} : C^{m,\gamma}(\partial Q_5^{k} \cap \partial Q_5^{k-1}) \to C^{\gamma}(Q_5^k)\) \((k \in K_-)\) are defined as

\[
G^p_{\alpha}^{k} h(x') = \sum_{|\xi| \leq |p'| - 1} \pi_{\alpha} p^\rho_{\xi} \partial Q \left( \varphi_{k}(x'), x' \right)
\]

and

\[
H^p_{\alpha}^{k} h(x') = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right)^{p'} D^{q'} \partial Q \left( \varphi_{k}(x'), x' \right)
\]

for any \(1 \leq \alpha \leq n\) and \(k \in K\), where the notation

\[
u|_{k} \in C^{m,\gamma}(Q^k_5) \quad \text{and} \quad U^p_{\alpha}|_{k} \in C^{m-1,\gamma}(Q^k_5)
\]

denote the extension of

\[
u \in C^{m,\gamma}(Q_{5}) \quad \text{and} \quad U^p_{\alpha} \in C^{m-1,\gamma}(Q_{5})
\]

for any \(1 \leq \alpha \leq n\) and \(1 \leq i \leq N\), which exist by the assumption (1.15).

**Remark 1.5.** By the definition, \(G^p_{\alpha}^{k} h(x')\) and \(\varphi_{\beta}^{k} h(x')\) depends only on \(x'\)-variables. So \(G^p\) depends only on \(x'\)-variables in \(Q_5^k\) and are Hölder continuous in each \(Q_5^k\) \((k \in K)\) and might have big jumps on \(\partial Q_5^k \cap \partial Q_5^{k+1}\) \((k \in K_-)\).

We will prove that \(U^p\), \(\pi_{\alpha}\) and \(\pi_{\beta}\) are Hölder continuous in \(Q_5\). So one can only show that \(D^{m+1}u\) is piece-wise Hölder continuous due to vector-valued the function \(G^p\) which might have big jumps on \(\partial Q_5^k \cap \partial Q_5^{k+1}\) \((k \in K_-)\).
Theorem 1.6. Let $u$ be a weak solution of (1.14). Then for any $p' \geq 0'$ with $|p| = m$, we have that $U^{p'} \in C^\gamma (Q_3)$ with the estimate

$$
\| U^{p'} \|_{L^\infty (Q_r(z))} \leq c r^{-2} E^2,
$$

$$
\| U^{p'} \|_{L^\infty (Q_r(z))}^2 \leq c \left[ \int_{Q_{2r}(z)} |U^{p'}|^2 \, dx + E^2 \right]
$$

and

$$
\begin{aligned}
\left[ U^{p'} \right]_{C^\gamma (Q_r(z))}^2 &\leq c \left[ \frac{1}{r^{2\gamma}} \int_{Q_{2r}(z)} \left( U^{p'} - (U^{p'})_{Q_2r(z)} \right)^2 \, dx + \sum_{|p'| = m} \left( U^{p'} \right)_{Q_2r(z)} \right] + E^2 \\
&\leq c \left[ E + \sum_{|p'| = m} \left( U^{p'} \right)_{Q_2r(z)} \right]
\end{aligned}
$$

for any $Q_{2r}(z) \subset Q_5$ where $E = \sum_{k \in K} \left[ \| u \|_{C^{m,\gamma} (Q_2^k)} + \| F \|_{C^{m,\gamma} (Q_2^k)} \right]$.

$D^{m+1}u$ and $U^{p'}$ will be compared by using the following proposition.

Proposition 1.7. For any $z \in Q_5$, we have that

$$
| D^{m+1}u(z) | \leq c \left[ E + \sum_{|p'| = m} | U^{p'} (z) | \right] \leq c \left[ E + | D^{m+1}u(z) | \right].
$$

In additional, for any $w, z \in Q_3^k (k \in K)$, we have that

$$
| D^{m+1}u(w) - D^{m+1}u(z) | \leq c \sum_{|p'| = m} \left[ | U^{p'} (w) - U^{p'} (z) | + | w - z |^\gamma | U^{p'} (z) | \right] + c | w - z |^\gamma \sum_{k \in K} \left[ \| u \|_{C^{m,\gamma} (Q_2^k)} + \| F \|_{C^{m,\gamma} (Q_2^k)} \right].
$$

Theorem 1.8. For a weak solution $u$ of (1.14), we have that $u \in C^{m+1,\gamma} (Q_3^k)$ ($k \in K$) and $D^{m+1}u \in L^\infty (Q_3^k)$ ($k \in K$) with the estimate

$$
\frac{1}{|Q_r|} \int_{Q_r^k(z)} | D^{m+1}u |^2 \, dx \leq c r^{-2} E^2,
$$

$$
\| D^{m+1}u \|_{L^\infty (Q_3^k(z))}^2 \leq c \left[ \sum_{k \in K} \frac{1}{|Q_r|} \int_{Q_{2r}(z)} | D^{m+1}u |^2 \, dx + E^2 \right]
$$

and

$$
\begin{aligned}
\left[ D^{m+1}u \right]_{C^\gamma (Q_3^k(z))}^2 &\leq \frac{c}{r^{2\gamma}} \left[ \sum_{k \in K} \frac{1}{|Q_r|} \int_{Q_{2r}(z)} | D^{m+1}u |^2 \, dx + E^2 \right] \\
&\leq c \left[ \sum_{k \in K} \left[ \| u \|_{C^{m,\gamma} (Q_2^k)} + \| F \|_{C^{m,\gamma} (Q_2^k)} \right] \right].
\end{aligned}
$$

Remark 1.9. As in [1], one can in fact find a vector-valued function related to $(m+1)$-th derivatives which $C^\gamma (Q_2)$. This function has big jumps on the boundary of the subregions.
2. Geometric settings and the related properties

With the assumption (1.13) for some \( m \geq 0 \) and \( \gamma \in (0, 1/4] \), let \((Q_T, \{\varphi_k : k \in K_+\})\) be a composite cube. For the approximation argument, we assume that

\[
\varphi_k(x') + \delta \leq \varphi_{k+1}(x') \quad (x \in Q_T, \ k \in K),
\]

for some \( \delta > 0 \). The desired estimate does not depend on \( \delta > 0 \) and one can use an approximation argument to handle the case that \( \delta = 0 \). Since \((Q_T, \{\varphi_k : k \in K_+\})\) is a composite cube, recall from Definition 1.1 that

\[
Q_T = \bigcup_{k \in K} Q_k^k.
\]

2.1. Time derivatives of the flow. To estimate the higher derivatives for linear elliptic systems in composite cubes, we generalize the difference quotient from lines (or the fixed directions) to general curves. To do it, we set the flow \( \psi(x, t') \) as follows. For the graph functions, \( \{\varphi_k : k \in K_+\} \), set \( T_k : Q_T \to [0, 1] \ (k \in K) \) as

\[
T_k(x^1, x') = \frac{x^1 - \varphi_k(x')}{\varphi_{k+1}(x') - \varphi_k(x')} \quad \text{in} \quad Q_k^k.
\]

Remark 2.1. For any \( l > k \), \( T_k = 1 \) in \( Q_l^k \) and for any \( l < k \), \( T_k = 0 \) in \( Q_l^k \). So with (2.1), \( T_k \) linearly increases from \((\varphi_k(x'), x')\) to \((\varphi_{k+1}(x'), x')\) in \( Q_k^k \).

For any \( k \in K \), define \( \psi : Q_6 \times Q_1' \to \mathbb{R} \) so that

\[
\psi(x, t') = [\varphi_{k+1}(x' + t') - \varphi_{k+1}(x')] \cdot T_k(x) + [\varphi_k(x' + t') - \varphi_k(x')] \cdot [1 - T_k(x)]
\]
in \( Q_6^k \times Q_1' \). If the flow starts at \( x \in Q_6^k \) then the flow \( x + (\psi(x, t'), t') \) at time \( t' \in Q_1^k \) still remains at the same region \( Q_6^k \) as in the following lemma.

Lemma 2.2. For any \( (x, t') \in Q_6^k \times Q_1' \), we have that

\[
x + (\psi(x, t'), t') = (x^1 + \psi(x, t'), x' + t') \in Q_6^k.
\]

Proof. For any \( (x, t') \in Q_6 \times Q_1' \), we have from (1.2) that

\[
\left| \{(x^1 + \psi(x, t'))\} \right| \\
\leq |x^1| + |[\varphi_{k+1}(x' + t') - \varphi_{k+1}(x')] \cdot T_k(x) + [\varphi_k(x' + t') - \varphi_k(x')] \cdot [1 - T_k(x)]| \\
\leq |x^1| + |t'| < 7,
\]

which implies that

\[
\{x + (\psi(x, t'), t') : (x, t') \in Q_6^k \times Q_1'\} \subset Q_7.
\]

Also from (2.3) and (2.4), one can check that for any \( (x, t') \in Q_6^k \times Q_1' \),

\[
x^1 + \psi(x, t')
\]

\[
= [\varphi_{k+1}(x' + t') - \varphi_k(x' + t')]x^1 + \varphi_k(x' + t')\varphi_{k+1}(x') - \varphi_{k+1}(x' + t')\varphi_k(x').
\]

If \( x \in Q_6^k \) then \( \varphi_k(x') < x^1 \leq \varphi_{k+1}(x') \). So for any \( (x, t') \in Q_6^k \times Q_1' \), we find that \( \varphi_k(x' + t') < x^1 + \psi(x, t') \leq \varphi_{k+1}(x' + t') \). With (2.5), this proves the lemma. □

Fix \( p' = (p_1, \ldots, p_n) \geq 0 \) with \( |p'| = m \). Then we consider the following \( m \)-th order time derivative of \( \psi(x, t') \):

\[
\partial_{t'}^p h|_{t' = 0} := \partial_{t'}^{p_1} \cdots \partial_{t'}^{p_n} [h(x + (\psi(x, t'), t'))]|_{t' = 0} \quad \text{in} \quad Q_6.
\]

We calculate (2.6) as follows.

For any \( i \in \{1, \ldots, n\} \), to denote the first time derivative of the flow, set

\[
\pi_\alpha(x) := D_1 \varphi_{k+1}(x') \cdot T_k(x) + D_1 \varphi_k(x') \cdot [1 - T_k(x)]
\]
for any $x \in Q_k^k$ ($k \in K$). Then we find from (2.4) that

$$\pi_\alpha(x) = \partial_t \psi(x,t)|_{t=0} \quad (x \in Q_6).$$

In addition, we set

$$\pi' = (\pi_2, \ldots, \pi_n) \quad \text{and} \quad \pi = (-1, \pi') = (-1, \pi_2, \ldots, \pi_n) \quad \text{in} \ Q_7.$$

Also we denote the power of the first time derivative of the flow as

$$\pi'^\gamma := \pi'^{q_2} \cdots \pi'^{q_n} \quad \text{in} \ Q_7.$$

**Lemma 2.3.** For any $p' \geq 0'$ with $|p'| = m$, there exists a polynomial $\hat{P}^{p'}_\xi$ : $\mathbb{R} \left( \binom{n-1}{1} \times \binom{n}{2} \times \cdots \times \binom{n+m-2}{m} \right) \to \mathbb{R}$ such that

$$\partial_{t'}^{p'} h|_{t'=0'} = \sum_{0' \leq q' \leq p'} \left( \binom{p'}{q'} \right) \pi'^{q'} D^{(|q'|-q')}_1 h + \sum_{|q'| \leq |p'|-1} \hat{P}^{p'}_\xi D_{\xi} h \quad \text{in} \ Q_6,$$

where $\hat{P}^{p'}_\xi : Q_7 \to \mathbb{R}$ is defined as

$$\hat{P}^{p'}_\xi = \hat{P}^{p'}_\xi \left( T_k D_{x'} \phi_{k+1} + [1 - T_k] D_{x'} \phi_k, \ldots, T_k D_{x'} \phi_{k+1} + [1 - T_k] D_{x'} \phi_k \right) \quad \text{in} \ Q_7^k \quad \text{for any} \ k \in K.$$

**Proof.** By the definition of $\partial_{t'}^{p'} h|_{t'=0'}$ in (2.6), we obtain that

$$\partial_{t'}^{p'} h|_{t'=0'} = \partial_{t'}^{p_2} \cdots \partial_{t'}^{p_n} \left( h(x + (\psi(x,t'),t')) \right)|_{t'=0'}$$

$$= \sum_{0' \leq q' \leq p'} \left( \binom{p_2}{q_2} \cdots \binom{p_n}{q_n} \partial_{t_2} \psi|_{t'=0'} \right)^{q_2} \cdots \left( \partial_{t_n} \psi|_{t'=0'} \right)^{q_n} D_{x'}^{q_2} D_{t'}^{q_2} \cdots D_{x'}^{q_n} D_{t'}^{q_n} h$$

$$+ \sum_{|q'| \leq |p'|-1} \hat{P}^{p_2, \ldots, p_n} (\partial_{t_2} \psi|_{t'=0'}, \ldots, \partial_{t_n} \psi|_{t'=0'}, \ldots, \partial_{t_2} \psi|_{t'=0'}) D_{\xi} h$$

in $Q_6$, for some polynomial $\hat{P}^{p_2, \ldots, p_n} : \mathbb{R} \left( \binom{n-1}{1} \times \binom{n}{2} \times \cdots \times \binom{n+m-2}{m} \right) \to \mathbb{R}$. The number of the choice for the first derivative is $\binom{n-1}{1} + 1 - 1$, the second derivative is $\binom{n-1}{2} + 1 - 1$, etc., the $m$-th derivative is $\binom{n-1}{m} + 1 - 1$. We simplify the above equality. Recall that

$$D_{x'}^{q_2} = D_{x'}^{q_2} \cdots D_{x'}^{q_n}, \quad |p'| = p_2 + \cdots + p_n$$

and

$$\left( \binom{p'}{q'} \right) = \binom{p_2}{q_2} \cdots \binom{p_n}{q_n}.$$

From (2.7) and (2.10), we obtain that for any $p' \geq q' \geq 0'$,

$$\sum_{0' \leq q' \leq p'} \left( \binom{p_2}{q_2} \cdots \binom{p_n}{q_n} \partial_{t_2} \psi|_{t'=0'} \right)^{q_2} \cdots \left( \partial_{t_n} \psi|_{t'=0'} \right)^{q_n} D_{x'}^{q_2} D_{t'}^{q_2} \cdots D_{x'}^{q_n} D_{t'}^{q_n} h$$

$$= \sum_{0' \leq q' \leq p'} \left( \binom{p'}{q'} \right) \pi'^{q'} D^{(|q'|)-q'} h.$$
2.2. Decay estimate of the graph functions and \( \pi' \) in (2.9). To handle the two non-crossing graph functions \( \varphi_k (k \in K_+) \), we use following result in [1], which naturally holds from our geometric settings (also see [9, Section 5] or [8, Section 4]).

**Lemma 2.4 ([1]).** Suppose that \( \varphi_k, \varphi_1 : C^{\gamma} (Q'_{\rho, 0}) \to \mathbb{R} \) satisfy that

\[
[D^x \varphi_k]_{C^{\gamma} (Q'_{\rho, 0})}, [D^x \varphi_1]_{C^{\gamma} (Q'_{\rho, 0})} \leq c_1,
\]

and

\[
\| \varphi_k \|_{L^\infty (Q'_{\rho, 0})}, \| \varphi_1 \|_{L^\infty (Q'_{\rho, 0})} \leq c_2,
\]

then we have the following lemma from \([\text{Lemma 2.5}]\). For any \( x' \in Q'_{\rho} \),

\[
|D^x \varphi_k (x') - D^x \varphi_k (x')| \leq 3 \rho^{-1} (\rho^{1+\gamma} c_1 + 2 c_2)^{\frac{1}{1+\gamma}} |\varphi_1 (x') - \varphi_k (x')|^{\frac{1}{1+\gamma}}
\]

for any \( x' \in Q'_{\rho} \).

Let \( (Q_7, \{ \varphi_k : k \in K_+ \}) \) be a composite cube. By applying Lemma 2.4 to \( Q_7 \), we obtain from (1.13) and (2.1) that

\[
|D^x \varphi_k (x') - D^x \varphi_1 (x')| \leq c |\varphi_k (x') - \varphi_1 (x')|^{\frac{1}{2}} (x' \in Q_6', k, l \in K_+).
\]

Also we have from (1.13) that

\[
|D^x \varphi_k (x') - D^x \varphi_k (y')| \leq c |x' - y'|^{\frac{1}{2}} (x', y' \in Q_6', k \in K_+).
\]

Then we have the following lemma from [1].

**Lemma 2.5 ([1]).** For a composite cube \( (Q_7, \{ \varphi_k : k \in K_+ \}) \) and the corresponding \( \pi' \) in (2.9), we have that

\[
|\pi' (y) - \pi' (z)| \leq c |y - z|^{\frac{1}{2}} (y, z \in Q_6).
\]

**Proof.** The lemma follows by applying (2.13) and (2.14) to [1, Lemma 2.3]. \( \square \)

2.3. Estimate of the integrals related to \( \pi \) in (2.9). To use the weak formation, we will apply integration by parts formula. To do it, we obtain the estimates related to \( D\pi \) in Lemma 2.7 and (2.17) where \( \pi \) is defined in (2.9).

**Lemma 2.6.** With the assumptions (2.1) and (1.2), we have that

\[
|D\pi (x) | \leq c \left[ 1 + |\varphi_{k+1} (x') - \varphi_k (x')|^{-\frac{1}{2}} \right] (x \in Q_6', k \in K_+).
\]

**Proof.** Since \( \pi = (\pi_1, \cdots, \pi_n) \) and \( \pi_1 = -1 \), we estimate \( D\pi_{\alpha} \) \( (i = 2, \cdots, n) \). Fix \( i \in \{2, \cdots, n\} \) and let \( k \in K_+ \). By Definition 1.1,

\[
\varphi_k (x') \leq x^1 \leq \varphi_{k+1} (x') (x \in Q_6^k).
\]

With (2.3) and (2.7), we find from (2.13) that

\[
|D_i \pi_{\alpha} | = \left| \frac{D_i \varphi_{k+1} - D_i \varphi_k}{\varphi_{k+1} - \varphi_k} \right| \leq \frac{c}{|\varphi_{k+1} - \varphi_k|^{\frac{1}{2}}} \text{ in } Q_6^k.
\]

It only remains to estimate \( D_j \pi_{\alpha} \) \( (j = 2, \cdots, n) \) in \( Q_6^k \). By (2.3) and (2.7)

\[
\pi_{\alpha} (x) = \frac{D_i \varphi_{k+1} (x') |x^1 - \varphi_k (x')|}{\varphi_{k+1} (x') - \varphi_k (x')} + \frac{D_i \varphi_k (x') |\varphi_{k+1} (x') - x^1|}{\varphi_{k+1} (x') - \varphi_k (x')}.
\]
for any \( x \in Q_0^\epsilon \), which implies that
\[
D_j \pi_\alpha(x) = \frac{D_j \varphi_{k+1}(x') (x_1 - \varphi_k(x'))}{\varphi_{k+1}(x') - \varphi_k(x')} + \frac{D_j \varphi_k(x') (x_1 - x')}{\varphi_{k+1}(x') - \varphi_k(x')}
\]
\[
= \frac{D_j \varphi_{k+1}(x') D_j \varphi_k(x') - D_j \varphi_k(x') D_j \varphi_{k+1}(x')}{\varphi_{k+1}(x') - \varphi_k(x')}
\]
for any \( x \in Q_0^\epsilon \) and \( j = 2, \cdots, n \). Then by the following calculation,
\[
\frac{D_j \varphi_{k+1}(x') D_j \varphi_k(x') - D_j \varphi_k(x') D_j \varphi_{k+1}(x')}{\varphi_{k+1}(x') - \varphi_k(x')}
\]
the lemma follows from (2.13) and (2.15), because \( i \in \{2, \cdots, n\} \) and \( k \in K \) were arbitrary chosen. \( \square \)

**Lemma 2.7.** With the assumptions (2.1) and (1.2), we have that
\[
\int_{Q_r(z)} |D\pi|^2 \eta^2 \, dx \leq c \epsilon^2 \int_{Q_r(z)} |D\eta|^2 \, dx + c \epsilon^{-2} r^2 \int_{Q_r(z)} \eta^2 \, dx \quad (\eta \in C_0^\infty (Q_r(z)))
\]
for any \( Q_r(z) \subset Q_0 \) and \( \epsilon \in (0, 1] \).

**Proof.** Fix \( \eta \in C_0^\infty (Q_r(z)) \) and \( k \in K \). Then by Lemma 2.6 and Young’s inequality,
\[
\int_{Q_r(z)} |D\pi|^2 \eta^2 \, dx \leq c \int_{Q_r(z)} \eta^2 + \eta^2 \chi_{Q_0^\epsilon (z)} \, dx 
\]
\[
= c \int_{Q_r(z)} \eta^2 \, dx + \int_{Q_r(z)} \int_{Q_r(z)} \eta^2 (x_1, x') \, dx \, dx' \]
\[
\leq c \int_{Q_r(z)} \eta^2 \, dx + \int_{Q_r(z)} \int_{Q_r(z)} \frac{\eta^2 (x_1, x')}{\varphi_{k+1} - \varphi_k} \, dx \, dx'.
\]
Since \( \eta \in C_0^\infty (Q_r(z)) \), by the following calculation
\[
\int_{Q_r(z)} \frac{\eta^2 (x_1, x')}{\varphi_{k+1} - \varphi_k} \, dx \, dx' = \int_{Q_r(z)} \frac{\eta^2 (y_1, x')}{\varphi_{k+1} - \varphi_k} \, dy \, dx 
\]
\[
\leq 2 \int_{Q_r(z)} \frac{|D_y \eta (y_1, x')| |\eta (y_1, x')|}{\varphi_{k+1} - \varphi_k} \, dy \, dx 
\]
\[
\leq 2 \int_{Q_r(z)} |D_y \eta (y_1, x')| \, dy \, dx 
\]
Since \( Q_r(z) = (z_1 - r, z_1 + r) \times Q_r'(z') \), we obtain from Young’s inequality that
\[
\int_{Q_r(z)} \int_{Q_r(z)} \frac{\eta^2 (x_1, x')}{\varphi_{k+1} - \varphi_k} \, dx \, dx' \]
\[
\leq 2 \int_{Q_r(z)} \int_{Q_r(z)} |D_y \eta (y_1, x')| \, dy \, dx 
\]
\[
\leq c \epsilon^2 \int_{Q_r(z)} |D\eta|^2 \, dx + c \epsilon^{-2} r^2 \int_{Q_r(z)} \eta^2 \, dx.
\]
Since \( k \in K \) was arbitrary chosen, by combining the above estimates,
\[
\sum_{k \in K} \int_{Q_k^l(z)} |D^n u|^2 \eta^2 \, dx \leq c c \epsilon^2 \int_{Q_{n+1}(z)} |D^{n+1} u|^2 \eta^2 \, dx.
\]
By Poincaré’s inequality,
\[
\int_{Q_{n+1}(z)} \eta^2 \, dx \leq \left( \int_{Q_{n+1}(z)} \eta^2 \, dx \right)^\frac{1}{2} \left( \int_{Q_{n+1}(z)} \eta^2 \, dx \right)^\frac{1}{2} \leq r \left( \int_{Q_{n+1}(z)} |D^{n+1} u|^2 \eta^2 \, dx \right)^\frac{1}{2} \left( \int_{Q_{n+1}(z)} \eta^2 \, dx \right)^\frac{1}{2} \leq c^2 r \int_{Q_{n+1}(z)} |D^{n+1} u|^2 \eta^2 \, dx.
\]
Since \( \epsilon \in (0, 1] \) and \( r \in (0, 6] \), by combining the above two estimates, we find that
\[
\int_{Q_{n+1}(z)} |D^n u|^2 \eta^2 \, dx \leq c c \epsilon^2 \int_{Q_{n+1}(z)} |D^{n+1} u|^2 \eta^2 \, dx + c^2 \epsilon^2 \int_{Q_{n+1}(z)} \eta^2 \, dx.
\]
Since \( \eta \in C_0^\infty (Q_{n+1}(z)) \) and \( \epsilon \in (0, 1] \) were arbitrary chosen, the lemma follows. \( \square \)

2.4. Heavyside type functions. We define \( H_k : Q_5 \to \mathbb{R} \) \( (k \in K) \) so that
\[
H_k(x) = \begin{cases} 1 & \text{if } x \in Q_5^k \text{ with } l \geq k, \\ 0 & \text{if } x \in Q_5^k \text{ with } l < k. \end{cases}
\]
Then we have that
\[
\chi_{Q_5^k} = H_k - H_{k+1} \text{ in } Q_5.
\]

3. Perturbation

From now on, assume (1.13) for some \( m \geq 1 \). With the assumption (1.8), (1.10), (1.13) and (1.2), let \( u \) be a weak solution of (1.14) which is
\[
D_\alpha \left[ A^{ij}_{\alpha \beta} D_j u^i \right] = D_i F^i_\alpha \text{ in } Q_7.
\]
To estimate \( D^{m+1} u \), assume that
\[
u \in C^{m, \gamma} (Q_5^k, \mathbb{R}^N) \quad (k \in K).
\]
The higher order derivatives of \( u \) in \( Q_5^k \) will be perturbed with respect to the points on the graphs \( \{(x', x') : x' \in Q_5^k \} \) \( (k \in K) \). In view of (1.9),
\[
A^{ij}_{\alpha \beta}, F^i_\alpha \in C^{m, \gamma} (Q_6^k, \mathbb{R}^N) \text{ is a restriction of } A^{ij}_{\alpha \beta}, F^i_\alpha \in C^{m, \gamma} (\mathbb{R}^n)
\]
for any \( k \in K, \ 1 \leq \alpha, \beta \leq n \text{ and } 1 \leq i, j \leq N \). With (2.1), one can extend \( u \in C^{m, \gamma} (Q_6^k, \mathbb{R}^N) \) to \( u_k \in C^{m, \gamma} (\partial Q_6^k \cap \partial Q_5^{k-1}, \mathbb{R}^N) \) for any \( k \in K \) so that
\[
\|u\|_{C^{m, \gamma} (\partial Q_6^k \cap Q_5^{k-1}, \mathbb{R}^N)} \leq c \|u\|_{C^{m, \gamma} (Q_5^k \cap Q_5^{k+1}, \mathbb{R}^N)}.
\]
Here, we remark that the (higher order) derivatives of \( u_k \) and \( u_{k+1} \) might not coincide on \( \partial Q_5^k \cap \partial Q_5^{k+1} \) \( (k \in K) \). To obtain \( m \)-th time derivative, fix the index
\[
p' = (p_2, \ldots, p_m) \geq 0 \quad \text{with } |p'| = p_2 + \cdots + p_m = m.
\]
With an approximation argument by using (2.1), we may assume that
\begin{equation}
\tag{3.6}
u \in C^{m+1,\gamma}_{\partial Q_k^m} \left( Q_k^m, \mathbb{R}^N \right) \quad (k \in K).
\end{equation}

We remark that the term \( \|D^{m+1}u\|_{L^{\infty}(Q_k^m)} \) (for fixed \( Q_k^m \subset Q_k \)) might appear in the estimates. But because the coefficient on \( \|D^{m+1}u\|_{L^{\infty}(Q_k^m)} \) can be sufficiently small, this term will be removed in Lemma 5.8.

### 3.1. Perturbation on the equation

In this subsection, we obtain a perturbation on the equation related to higher order derivatives (which will be done in Lemma 3.8). Recall \( T_k(x) \) from (2.3). For any \( i \in \{1, \ldots, n\} \), set \( \psi_i : Q_0 \times Q'_1 \rightarrow \mathbb{R} \) as

\begin{equation}
\tag{3.7}
\psi_i(x, t') = T_k(x) D_i \varphi_{k+1}(x' + t') + [1 - T_k(x)] D_i \varphi_k(x' + t')
\end{equation}

for any \((x, t') \in Q_0^k \times Q'_1 \) \( (k \in K) \). Then we find from (3.7) that
\begin{equation}
\tag{3.8}
\psi_i(x, t') = 0
\end{equation}

for any \((x, t') \in Q_0^k \times Q'_1 \). Also it follows from (2.3) and (3.7) that
\begin{equation}
\tag{3.9}
\psi_i((\varphi_k(x'), x'), t') = D_i \varphi_k(x' + t')
\end{equation}

for any \((x', t') \in Q_0^k \times Q'_1 \) and \( k \in K \).

To obtain piecewise regularity, we use the weak formulation in the following way. We differentiate the equation on the boundary \( \partial Q_0^k \cap \partial Q_0^{k+1} \) by using the flow (3.7). Then we apply integration by parts formula with respect to each \( \partial Q^k_0 \) \((k \in K)\).

In view of (3.6), for any \( 1 \leq \alpha \leq n, 1 \leq i \leq N \) and \( k \in K \), set
\begin{equation}
\tag{3.10}
U_{\alpha}^{i,k} = \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{\alpha \beta}^{ij,k} D_{\beta} u_{\alpha}^{i,k} - F_{\alpha}^{i,k} \in C^{m,\gamma} (Q_0^k),
\end{equation}

and
\begin{equation}
\tag{3.11}
\partial_{tt'}^\alpha U_{\alpha}^{i,k} \big|_{t' = 0^+} = \partial_{t'}^\alpha \left[ U_{\alpha}^{i,k}(x + (\psi(x, t'), t')) \right] \big|_{t' = 0^+} \quad \text{in} \quad Q_0^k.
\end{equation}

Since \( D_\alpha \left[ A_{ij}^{\alpha \beta} D_{\beta} u_{\alpha}^{i,k} \right] = D_\alpha F_{\alpha}^{i,k} \) in \( Q_7 \), we find that for any \( 1 \leq i \leq N \),
\begin{equation}
\tag{3.12}
\sum_{1 \leq \alpha \leq n} D_\alpha U_{\alpha}^{i,k} = \sum_{1 \leq \alpha \leq n} D_\alpha \left[ \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{\alpha \beta}^{ij,k} D_{\beta} u_{\alpha}^{i,k} - F_{\alpha}^{i,k} \right] = 0 \quad \text{in} \quad Q_0^k.
\end{equation}

**Lemma 3.1.** Let \( \vec{n}_k = (n_1^k, n_2^k, \ldots, n_n^k) \) be the outward normal vector on \( \partial Q_0^k \). Then for any \( \alpha = 2, \ldots, n \) and \( k \in K \), we have that
\begin{equation}
\tag{3.13}
n_k^\alpha = -D_\alpha \varphi_k n_k^1, \quad n_k^\alpha = -D_\alpha \varphi_k n_{k+1}^1 \quad \text{and} \quad n_k^1 = -n_{k+1}^1 \quad \text{on} \quad \partial Q_0^k \cap \partial Q_0^{k+1}.
\end{equation}

Also for any \( i = 2, \ldots, n \) and \( k \in K \), we have that
\begin{equation}
\tag{3.14}
n_k^i = -\psi(x, 0') n_k^1 \quad \text{and} \quad n_{k+1}^i = -\psi(x, 0') n_{k+1}^1 \quad \text{on} \quad \partial Q_0^k \cap \partial Q_0^{k+1}.
\end{equation}

**Proof.** (3.13) holds from Definition 1.1. (3.14) follows from (3.9) and (3.13). \( \square \)

**Lemma 3.2.** With the assumption (3.10), we have that
\begin{equation}
\sum_{1 \leq \alpha \leq n} \left[ U_{\alpha}^{i,k} n_{k+1}^\alpha + U_{\alpha}^{i,k+1} n_{k}^\alpha \right] = 0 \quad \text{in} \quad \partial Q_0^k \cap \partial Q_0^{k+1} \quad (k \in K).
\end{equation}
Proof. Fix $\eta \in C_c^\infty(Q_0, \mathbb{R}^N)$. By (3.10) and the weak formulation,

$$
\sum_{k \in K} \int_{Q_6} \sum_{1 \leq \alpha \leq n} U_{\alpha}^{i,k} D_{\alpha} \eta^i \, dx
$$

(3.15)

$$
= \int_{Q_6} \sum_{1 \leq \alpha \leq n} \left[ \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{\alpha\beta}^{i,j} D_\beta U^{j,k} - F_i \right] D_\alpha \eta^i \, dx = 0.
$$

for any $1 \leq i \leq N$. Apply the integration by parts. Then by (3.12) and (3.15),

$$
\sum_{k \in K} \int_{\partial Q_6} \sum_{1 \leq \alpha \leq n} \eta^i U_{\alpha}^{i,k} n^\alpha_k \, dS = \sum_{k \in K} \int_{Q_6} \sum_{1 \leq \alpha \leq n} [D_\alpha U_{\alpha}^{i,k} \eta^i + U_{\alpha}^{i,k} D_\alpha \eta^i] \, dx = 0,
$$

for any $1 \leq i \leq N$. Since $\eta \in C_c^\infty(Q_0, \mathbb{R}^N)$, we find that

$$
0 = \sum_{k \in K} \int_{Q_6} \sum_{1 \leq \alpha \leq n} \left[ \eta^i U_{\alpha}^{i,k} n^\alpha_k + \eta^i U_{\alpha}^{i,k+1} n_{k+1}^\alpha \right] (\varphi_k(x'), x') \, dx',
$$

(3.16)

for any $1 \leq i \leq N$. In view of (2.1), fix $z \in \partial Q_6^k \cap \partial Q_6^{k+1}$ satisfying that $Q_r(z) \subset Q_6^k \cup Q_6^{k+1}$. Then we obtain from (1.13) and (3.10) that

$$
|U_{\alpha}^{i,k} n^\alpha_k + U_{\alpha}^{i,k+1} n_{k+1}^\alpha| (\varphi_k(x'), x') \quad \text{is bounded and continuous in } Q_r'(z')
$$

(3.17)

for any $1 \leq \alpha \leq n$, $1 \leq i \leq N$ and $k \in K$.

With (3.17), for $m = 1, 2, \cdots$ and $1 \leq i \leq N$, choose $\eta^{i,1}_m : [z^1 - r, z^1 + r] \to [0, 1]$ with $\eta^{i,1}_m \in C_\infty^c(z^1 - r, z^1 + r)$ and $\eta^{i}_m : Q_r'(z') \to \mathbb{R}$ with $\eta^i_m \in C_\infty^c(Q_r'(z'))$ so that

$$
\eta^{i,1}_m = 1 \quad \text{in} \quad [z^1 - (1 - 2^{-m})r, z^1 + (1 - 2^m)r] \quad (m = 1, 2, \cdots),
$$

and

$$
\eta^{i}_m(z') \xrightarrow{m \to \infty} \sum_{1 \leq \alpha \leq n} \left[ U_{\alpha}^{i,k} n^\alpha_k + U_{\alpha}^{i,k+1} n_{k+1}^\alpha \right] (\varphi_k(x'), x') \quad \text{in} \quad L^2(Q_r'(z')).
$$

Take $\eta^i_m = \eta^{i,1}_m \cdot \eta^i_m \in C_\infty^c(Q_r(z))$ instead of $\eta^i$ in (3.16). Then by letting $m \to \infty$,

$$
0 = \int_{Q_r'(z')} \sum_{1 \leq \alpha \leq n} \left[ \int_{Q_6} \sum_{1 \leq \alpha \leq n} U_{\alpha}^{i,k} n^\alpha_k + U_{\alpha}^{i,k+1} n_{k+1}^\alpha \right] (\varphi_k(x'), x') \, dx'.
$$

So we find from (3.17) that that

$$
\sum_{1 \leq \alpha \leq n} \left[ U_{\alpha}^{i,k}(z) n^\alpha_k(z) + U_{\alpha}^{i,k+1}(z) n_{k+1}^\alpha(z) \right] = 0.
$$

Since $z \in \partial Q_6^k \cap \partial Q_6^{k+1}$ was arbitrary chosen, the lemma follows. \hfill $\Box$

**Lemma 3.3.** For any $1 \leq i \leq N$ and $\eta \in C_\infty^c(Q_5, \mathbb{R}^N)$, we have that

$$
0 = \sum_{k \in K} \int_{\partial Q_5^k} \partial_{\nu'} \left( U_{\alpha}^{i,k}(x + (\psi, t')) \right) \bigg|_{\nu = 0} \eta^i n^1_k \, dS
$$

$$
- \sum_{k \in K} \int_{\partial Q_5^k} \partial_{\nu'} \left( \psi_{\alpha}(x, t') U_{\alpha}^{i,k}(x + (\psi, t')) \right) \bigg|_{\nu = 0} \eta^i n^1_k \, dS,
$$

where $\nu_k = (n^1_k, n^2_k, \cdots, n^n_k)$ is the outward normal vector on $\partial Q_5^k$.

**Proof.** It follows from Lemma 3.2 that for any $1 \leq i \leq N$ and $k \in K$,

$$
U_{\alpha}^{i,k} = \sum_{2 \leq \alpha \leq n} D_\alpha \varphi_k U_{\alpha}^{i,k} = U_{\alpha}^{i,k+1} - \sum_{2 \leq \alpha \leq n} D_\alpha \varphi_k U_{\alpha}^{i,k+1} \quad \text{on} \quad \partial Q_6^k \cap \partial Q_6^{k+1}.
$$
Fix $x = (\varphi_k(x'), x') \in \partial Q_k \cap \partial Q_{k+1}^\pm (k \in K)$. Then for any $t' \in Q_1$, we have that
\[
U_1^{i,k}(\varphi_k(x' + t'), x' + t') - \sum_{2 \leq \alpha \leq n} \sum_{2 \leq \alpha \leq n} D_\alpha \varphi_k(x' + t') U_\alpha^{i,k}(\varphi_k(x' + t'), x' + t')
= U_1^{i,k+1}(\varphi_k(x' + t'), x' + t') - \sum_{2 \leq \alpha \leq n} D_\alpha \varphi_k(x' + t') U_\alpha^{i,k+1}(\varphi_k(x' + t'), x' + t'),
\]
for any $1 \leq i \leq N$. Since $x = (\varphi_k(x'), x') \in \partial Q_k \cap \partial Q_{k+1}^\pm (\subset Q_6)$, we find from (3.9) that
\[
\psi_\alpha(x, t) = \psi_\alpha(\varphi_k(x'), x', t') = D_\alpha \varphi_k(x' + t') \quad (t' \in Q_1, 1 \leq \alpha \leq n).
\]
Also from (2.4) and that $x = (\varphi_k(x'), x') \in \partial Q_k \cap \partial Q_{k+1}^\pm (\subset Q_6)$, we find that
\[
x + (\psi(x, t'), t') = (\varphi_k(x' + t'), x' + t') \quad (t' \in Q_1).
\]
So by combining the above three equality,
\[
U_1^{i,k}(x + (\psi, t')) - \sum_{2 \leq \alpha \leq n} \psi_\alpha(x, t') U_\alpha^{i,k}(x + (\psi, t'))
= U_1^{i,k+1}(x + (\psi, t')) - \sum_{2 \leq \alpha \leq n} \psi_\alpha(x, t') U_\alpha^{i,k+1}(x + (\psi, t'))
\]
for any $t' \in Q_1$ and $1 \leq i \leq N$. Then by using that $\partial^\mu \varphi_k = \partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_\alpha} \varphi_k$, we find that
\[
\partial^\mu \left[ U_1^{i,k}(x + (\psi, t')) - \sum_{2 \leq \alpha \leq n} \psi_\alpha(x, t') U_\alpha^{i,k}(x + (\psi, t')) \right]_{t' = 0^+}
= \partial^\mu \left[ U_1^{i,k+1}(x + (\psi, t')) - \sum_{2 \leq \alpha \leq n} \psi_\alpha(x, t') U_\alpha^{i,k+1}(x + (\psi, t')) \right]_{t' = 0^+}.
\]
Since $x = (\varphi_k(x'), x') \in \partial Q_k \cap \partial Q_{k+1}^\pm (k \in K)$ was arbitrary chosen and $\eta \in C^\infty_c(Q_5, \mathbb{R}^N)$, the lemma holds from that $n_k = -n_{k+1}$ for any $k \in K$. \hfill \Box

Let $\tilde{n}_k = (n_{1,k}, \cdots, n_{N,k})$ be the outward normal vector on $\partial Q_k$. For any $\eta \in C^\infty_c(Q_5, \mathbb{R}^N)$, we have from Lemma 3.3 that
\[
I^i + II^i + III^i = 0 \quad (1 \leq i \leq N),
\]
where
\[
I^i = \sum_{k \in K} \int_{\partial Q_k} \partial^\mu \varphi_k \left( U_1^{i,k}(x + (\psi, t')) \right)_{t' = 0^+} \eta^i n_k \, dS,
\]
\[
II^i = -\sum_{k \in K} \sum_{2 \leq \alpha \leq n} \int_{\partial Q_k} \partial^\mu \varphi_k \left( \psi_\alpha(x, t') - \psi_\alpha(x, 0') U_\alpha^{i,k}(x + (\psi, t')) \right)_{t' = 0^+} \eta^i n_k \, dS,
\]
\[
III^i = -\sum_{k \in K} \sum_{2 \leq \alpha \leq n} \int_{\partial Q_k} \partial^\mu \left( \psi_\alpha(x, 0') U_\alpha^{i,k}(x + (\psi, t')) \right)_{t' = 0^+} \eta^i n_k \, dS.
\]
Then by Lemma 3.1, for any $\eta^i \in C^\infty_c(Q_5)$,
\[
III^i = -\sum_{k \in K} \sum_{2 \leq \alpha \leq n} \int_{\partial Q_k} \partial^\mu \left( \psi_\alpha(x, 0') U_\alpha^{i,k}(x + (\psi, t')) \right)_{t' = 0^+} \eta^i n_k \, dS
= \sum_{k \in K} \sum_{2 \leq \alpha \leq n} \int_{\partial Q_k} \partial^\mu \left( U_\alpha^{i,k}(x + (\psi, t')) \right)_{t' = 0^+} \eta^i n_k \, dS.
\]
Thus

\[(3.19) \quad I^i + III^i = \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \int_{\partial Q_{k}^{\epsilon}} \partial_{\nu}^i \left[ U_{\alpha}^{i,k}(x + (\psi, t')) \right] \bigg|_{t' = 0} dS. \]

We first estimate \( I^i \) and \( III^i \) \((1 \leq i \leq N)\). Recall from (2.11) in Lemma 2.3 that

\[(3.20) \quad \partial_{\nu}^i h \bigg|_{t' = 0} = \sum_{0' \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \pi^{q'} D^{(q', p-q')} h + \sum_{|\xi| \leq |p'|-1} P^\xi_{\xi} (x) D_{\xi} h \quad \text{in} \quad Q_5. \]

With (3.10), we find from (3.19) and (3.20), for any \( \eta \in C_0^\infty (Q_5, \mathbb{R}^N) \),

\[(3.21) \quad I^i + III^i = \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \int_{\partial Q_{k}^{\epsilon}} \sum_{0' \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \pi^{q'} D^{(q', p-q')} U_{\alpha}^{i,k} \eta \ n_k \ dS
+ \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \int_{Q_5} \sum_{|\xi| \leq |p'|-1} P^\xi_{\xi} (x) D_{\xi} U_{\alpha}^{i,k} \eta \ n_k \ dS
:= IV_1^i + IV_2^i. \]

for any \( 1 \leq i \leq N \).

We estimate \( IV_1^i \) by applying the integration by parts with respect to \( x^1 \)-variable. In view of (1.8), (1.15) and (3.6), for the simplicity of the calculation, we set

\[(3.22) \quad E_{z, \ell} = \sum_{k \in K} \left[ \|D^{m+1} u\|_{L^\infty (Q_{\ell}^{\epsilon})} + \|u\|_{C^{m, \gamma} (Q_{\ell}^{\epsilon})} + \|F\|_{C^{m, \gamma} (Q_{\ell}^{\epsilon})} \right], \]

for any \( Q_{z} (z) \subset Q_5 \). To handle \( IV_2^i \), we obtain the following Lemma 3.4. In Lemma 3.4, \( f_k \) only need to be defined on the set

\[ \partial Q_5^{\epsilon} \cap \left( \bigcup_{\ell \in \{(k,k+1) \cap K_\epsilon \}} \{ (\varphi_\ell (x'), x') : x \in Q_5' \} \right) \subset \overline{Q_5}^{\epsilon}, \]

(\text{where the inclusion holds from (1.3))} but we define on the set \( \overline{Q_6}^{\epsilon} \) for the simplicity.

**Lemma 3.4.** Under the assumption (1.3), suppose that \( g_k : Q_6^{\epsilon} \rightarrow \mathbb{R} \) and \( h : Q_6^{\epsilon} \rightarrow \mathbb{R} \) satisfies that \( g_k, h \in C^\gamma \left( \overline{Q_6}^{\epsilon} \right) \) for any \( k \in K \). Then for any \( \eta \in C_0^\infty (Q_5) \),

\[ \sum_{k \in K} \int_{\partial Q_5^{\epsilon}} g_k h n_k \ dS = \sum_{k \in K} \int_{Q_5} G_k [g_{k-1} - g_k, h] H_k D_1 \eta \ dx, \]

where the linear operator \( G_k : C^\gamma (\partial Q_6^{\epsilon} \cap \partial Q_6^{\epsilon-1}) \times C^\gamma (\partial Q_6^{\epsilon} \cap \partial Q_6^{\epsilon-1}) \rightarrow C^\gamma (Q_5) \) is defined as

\[ G_k [g, h] (x) = [gh] (\varphi_k (x'), x'). \]

**Proof.** By the definition of \( K^- \) in (1.3),

\[ \sum_{k \in K} \int_{\partial Q_5^{\epsilon}} g_k h n_k \ dS = - \sum_{k+1 \in K} \int_{Q_5 \cap \{(\varphi_{k+1} (x'), x') : x' \in Q_5' \}} g_k h n_k \ dS
- \sum_{k \in K} \int_{Q_5 \cap \{(\varphi_k (x'), x') : x' \in Q_5' \}} g_k h n_k \ dS. \]
By using (2.18) and integration by parts with respect to $x^1$-variable,

$$
\sum_{k+1 \in K_-} \int_{Q_k \cap \{ (\varphi_{k+1}(x'), x') : x' \in Q_k^r \}} g_k h \eta n^1_k \, dS
= \sum_{k+1 \in K_-} \int_{Q_k} [g_k h] (\varphi_{k+1}(x'), x') D_1 \eta H_{k+1} \, dx
= \sum_{k \in K_-} \int_{Q_k} [g_{k-1} h] (\varphi_k(x'), x') D_1 \eta H_k \, dx,
$$

and

$$
\sum_{k \in K_-} \int_{Q_k \cap \{ (\varphi_k(x'), x') : x' \in Q_k^r \}} g_k h \eta n^1_k \, dS
= \sum_{k \in K_-} \int_{Q_k} [g_k h] (\varphi_k(x'), x') D_1 \eta (1 - H_k) \, dx,
$$

where we used that $\eta \in C^\infty_0(Q_5)$. So the lemma holds by combining the above three equality and that $\sum_{k \in K_-} \int_{Q_k} [g_k h] (\varphi_k(x'), x') D_1 \eta \, dx = 0$. □

We now estimate $I^i + III^i$ in Lemma 3.5 and $II^i$ in Lemma 3.6. We remark that $G_1$ in Lemma 3.5 and Lemma 3.6 are Hölder continuous in each subregion $Q^k_5$ ($k \in K$). Moreover, $G_{1,k}[h]$ only depends on $x^2$-variables and $G_1$ might have big jumps on the graphs $\{ (\varphi_k(x'), x') : x' \in Q^r_5 \} (k \in K_-).

**Lemma 3.5.** For any $\eta \in C^\infty_c(Q, ; R^N)$ with $Q_r(z) \subset Q_5$, we have that

$$
\left| I^i + III^i - \sum_{k \in K} \int_{Q^i_r(z)} \left( \frac{\partial^i}{\partial \eta^{i_1}} \right)^j D_1 \left[ D((|\eta'|)^{(i_1, i_2)} U_{i_1, i_2}^n) \right] + G_1^i D_1 \eta^i \, dx \right|
\leq cr^4 \int_{Q_r(z)} |\eta|^2 + |\eta'|^2 + E^2_{z,r} \, dx,
$$

for any $1 \leq i \leq N$ and $\epsilon \in (0, 1]$, where

$$
G_1^i = \sum_{k \in K_0} \sum_{1 \leq \alpha \leq n} G_{1,k}^i \left[ U_{\alpha}^{i,k-1} - U_{\alpha}^{i,k} \right] H_k
$$

with the linear operator $G_{1,k}^i : C^{m-1,\gamma}(\partial Q^i_0 \cap \partial Q^{i-1}_0) \rightarrow C^{\gamma}(Q^i_5) (k \in K_-)$ defined as

$$
G_{1,k}^i[h](x') = \sum_{|\xi| \leq |\eta'| - 1} \pi_{\alpha} P_{\xi}^i D_1 h \left( \varphi_k(x'), x' \right).
$$

**Proof.** With respect to $x^1$-variable, use integration by parts to parts to $IV^i$ in (3.21). Then

$$
IV^i_1 = \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \int_{Q_5^i} \sum_{0 \leq \rho \leq \rho'} \left( \frac{\partial^i}{\partial \eta^{i_1 \rho}} \right)^j D_1 \left[ \pi^{i_2 \rho} \right] D((|\eta'|)^{(i_1, i_2)} U_{i_1, i_2}^n) \, dx
+ \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \int_{Q_5^i} \sum_{0 \leq \rho \leq \rho'} \left( \frac{\partial^i}{\partial \eta^{i_1 \rho}} \right)^j D_1 \left[ \pi^{i_2 \rho} \right] D((|\eta'|)^{(i_1, i_2)} U_{i_1, i_2}^n) \, dx.
$$
By that $Q_\tau(z) \subset Q_5$,

$$
IV_1^\tau = \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \int_{Q_\tau(z)} \sum_{|\alpha| \leq n} \sum_{0 \leq \eta, \xi \leq \rho_\tau} \left( \frac{\partial P_\xi}{\partial \tau} \right) D_\alpha \left[ \pi^\tau \right] D_\alpha \left[ \pi^\tau \right] D_\alpha \left[ \pi^\tau \right] U_{\alpha,\xi}^{i,k} \eta^j \, dx
$$

(3.25)

$$
+ \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \int_{Q_\tau(z)} \sum_{0 \leq \eta, \xi \leq \rho_\tau} \left( \frac{\partial P_\xi}{\partial \tau} \right) \eta^j \, dx.
$$

From (1.2) and (2.7), we have that $|\pi_2|, \cdots, |\pi_n| \leq c$. So by Young's inequality,

$$
\left| \sum_{k \in K} \int_{Q_\tau(z)} \sum_{1 \leq \alpha \leq n} \sum_{0 \leq \eta, \xi \leq \rho_\tau} \left( \frac{\partial P_\xi}{\partial \tau} \right) \eta^j \, dx \right|
$$

\leq c \int_{Q_\tau(z)} |D\pi|E_{x,r} |\eta| \, dx

\leq c \int_{Q_\tau(z)} r^*E_{x,r}^2 + r^{-\frac{q}{q'}} |D\pi|^2 |\eta|^2 \, dx,

where the definition of $E_{x,r}$ in (3.22) is used. So by (3.25) and Lemma 2.7,

$$
IV_1^\tau - \sum_{k \in K} \int_{Q_\tau(z)} \sum_{1 \leq \alpha \leq n} \sum_{0 \leq \eta, \xi \leq \rho_\tau} \left( \frac{\partial P_\xi}{\partial \tau} \right) \eta^j \, dx
$$

\leq cr^* \int_{Q_\tau(z)} \epsilon |D\eta|^2 + c^{-2} |\eta|^2 + E_{x,r}^2 \, dx.

We next estimate $IV_2$. By (2.9) and Lemma 3.1, we find that

$$n_\alpha^i = -\pi_\alpha n_\alpha^i \quad \text{on} \quad \partial Q_5^k \cap \partial Q_5^{k+1},$$

for any $i = 1, 2, \cdots, n$ and $k \in K$. So by (3.21),

$$
IV_2^\tau = \sum_{k \in K} \int_{\partial Q_5^k} \sum_{1 \leq \alpha \leq n} \sum_{|\alpha| = 1} \sum_{|\eta| = 1} P_{\eta}^\xi(x) D_\alpha U_{\alpha,\xi}^{i,k} \eta^j \, dS
$$

$$
= - \sum_{k \in K} \int_{\partial Q_5^k} \sum_{1 \leq \alpha \leq n} \sum_{|\alpha| = 1} \sum_{|\eta| = 0} P_{\eta}^\xi(x) D_\alpha U_{\alpha,\xi}^{i,k} \eta^j \, dS.
$$

With (2.1), we have from (3.10), (2.7) and the definition of $P_{\eta}^\xi$ in Lemma 2.3 that

$$D_\xi U_{\alpha,\xi}^{i,k}, \pi_\alpha P_{\eta}^\xi \in C^\tau \left( \partial Q_5^k \cap \partial Q_5^{k-1} \right) \quad (k \in K, i \in \{1, \cdots, n\}),$$

for any $|\xi| \leq |\eta| - 1$. So take $g_k = D_\xi U_{\alpha,\xi}^{i,k}$ and $h = \pi_\alpha P_{\eta}^\xi$ in Lemma 3.4 to find that

$$IV_2^\tau = - \sum_{k \in K} \int_{Q_5^k} G_1 \, dS \, dx$$

with

$$G_1 = \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \sum_{|\alpha| = 1} G_{\xi} \left[ D_\xi U_{\alpha,\xi}^{i,k} - \pi_\alpha P_{\eta}^\xi \right] H_k$$

where $G_k$ is defined in Lemma 3.4. So by comparing the definition of $G_k$ in Lemma 3.4 and $G_{\eta}^k$ in (3.24), the lemma follows form that $IV_1^\tau + IV_2^\tau = I' + III'$ in (3.21) (1 \leq i \leq N) and that $Q_\tau(z) \subset Q_5$.

**Lemma 3.6.** For any $\eta \in C^\infty_c \left( Q_\tau(z), \mathbb{R}^N \right)$ with $Q_\tau(z) \subset Q_5$, we have that

$$II' = - \int_{Q_\tau(z)} G_1 \, dS \, dx,$$
for any $1 \leq i \leq N$, where

$$G^i_k = \sum_{k \in K_{-1}} \sum_{1 \leq \alpha \leq n} G^k_{\alpha} \left[ U^{i,k}_{\alpha} - U^{i,k}_{\alpha} \right] H_k,$$

with the linear operator $G^k_{\alpha} : C^{m-1,1} \left( \partial Q^k_{\alpha} \cap \partial Q^k_{\alpha} \right) \to C^{\gamma} (Q^k_{\alpha}) \ (k \in K_{-})$ defined as

$$G^k_{\alpha} h(x) = \sum_{t \leq n} \sum_{0' < q' \leq p'} \left( \frac{p'}{q'} \right) D^q \alpha \varphi_k(x') D^{q'} \left[ h(\varphi_k(x'), x') \right].$$

Proof. Recall from (3.18) that

$$II^i = - \sum_{k \in K} \sum_{2 \leq \alpha \leq n} \int_{\partial Q^k_{\alpha}} \partial^q \left( [\psi_{\alpha}(x, t') - \psi_{\alpha}(x, 0')] U^{i,k}_{\alpha} (x + (\psi, t')) \right) \bigg|_{t' = 0'} \eta^i_n k dS.$$

By a direct calculation,

$$II^i = - \sum_{k \in K} \int_{\partial Q^k_{\alpha}} \sum_{2 \leq \alpha \leq n} \sum_{0' < q' \leq p'} \left( \frac{p'}{q'} \right) \partial^q \psi_{\alpha}(x, t') \bigg|_{t' = 0'} \partial^{q'} \left[ U^{i,k}_{\alpha} (x + (\psi, t')) \right] \bigg|_{t' = 0'} \eta^i_n k dS.$$

We now compute the terms in the above integral. From (3.9), we have that

$$(3.26) \quad \partial^q \psi_{\alpha} \left( [\varphi_k(x'), x'), t') \right) \bigg|_{t' = 0'} = \partial^q \alpha \varphi_k(x' + t') \bigg|_{t' = 0'} = D^{q'} \alpha \varphi_k(x'),$$

or any $i = 2, \cdots, n$ and $k \in K_{-}$. By (2.4), for any $k \in K_{-}$, we have that

$$(\varphi_k(x'), x') + (\psi((\varphi_k(x'), x'), t'), t') = (\varphi_k(x'), x') + (\varphi_k(x' + t') - \varphi_k(x', t'))$$

$$= (\varphi_k(x' + t'), x' + t'),$$

which implies that

$$(3.27) \quad \partial^{q'} \left[ h(\varphi_k(x'), x') + (\psi((\varphi_k(x'), x'), t'), t') \right] \bigg|_{t' = 0'} = \partial^{q'} \left[ h(\varphi_k(x'), x') \right].$$

With (3.26) and (3.27), for any $0' < q' \leq p'$, take $g_k = \partial^{q'} \left[ U^{i,k}_{\alpha} (x + (\psi, t')) \right]$ and $h = \partial^q \psi_{\alpha}(x, t') \bigg|_{t' = 0'}$ in Lemma 3.4 to find that

$$(3.28) \quad II^i = - \sum_{k \in K_{-}} \int_{\partial Q^k_{\alpha}} \sum_{2 \leq \alpha \leq n} G^k_{\alpha} \left[ U^{i,k}_{\alpha} - U^{i,k}_{\alpha} \right] H_k D_1 \eta^i dS,$$

for the operator $G^k_{\alpha} : C^{m-1,1} \left( \partial Q^k_{\alpha} \right) \to C^{\gamma} (Q^k_{\alpha})$ defined as

$$G^k_{\alpha} h(x) = \sum_{0' < q' \leq p'} \left( \frac{p'}{q'} \right) D^{q'} \alpha \varphi_k(x') D^{q'} \left[ h(\varphi_k(x'), x') \right].$$

The lemma holds by that $\eta \in C^\infty_c (Q_r(z), \mathbb{R}^N)$ and $D_1 \varphi_k(x') = 0 \ (k \in K_{-}). \quad \Box$

By combining Lemma 3.5 and Lemma 3.6, we obtain the following lemma.

**Lemma 3.7.** For any $\eta \in C^\infty_c (Q_r(z), \mathbb{R}^N)$ with $Q_r(z) \subset Q_5$, we have that

$$\left| \sum_{k \in K} \int_{Q_r(z)} \sum_{0' < q' \leq p'} \left( \frac{p'}{q'} \right) \eta^{q'} D^{q'}(q' \cdot q') U^{i,k}_{\alpha} D_\alpha \eta^i dS - \int_{Q_r(z)} G^i d Q_1 d Q_1 dS \right|$$

$$\leq c_2 \int_{Q_r(z)} \epsilon |D\eta|^2 + c_2^2 |\epsilon|^2 + E^2_{z, r} dS,$$
for any $1 \leq i \leq N$ and $\epsilon \in (0,1]$, where

\begin{equation}
G_i^t = \sum_{k \in K_{+}} \sum_{1 \leq \alpha \leq n} G_{\alpha}^{k} \left[ U^{i,k-1}_{\alpha} - U^{i,k}_{\alpha} \right] H_k
\end{equation}

with the linear operator $G_{\alpha}^{k} : C^{m-1,\gamma} \left( \partial Q_k^6 \cap \partial Q_k^{6-1} \right) \to C^{\gamma} \left( Q_k^5 \right)$ $(k \in K_{+})$ defined as

\[
G_{\alpha}^{k}[h](x') = \sum_{|\ell| \leq |\nu'| - 1} \left[ \pi_{\alpha} P^{\ell}_{x^\nu} D_{\xi} h \right] (\varphi_k(x'), x')
\]

\[+ \sum_{0' < q' \leq \nu'} \left( \frac{p'}{q'} \right) D^{q'} D_{\alpha} \varphi_k(x') D^{q'-q'} \left[ h(\varphi_k(x'), x') \right].
\]

**Proof.** Since $I + II + III = 0$, by ($3.21$), Lemma 3.5 and Lemma 3.6,

\[
\left| \sum_{k \in K_{+}} \int_{Q_k^5(z)} \sum_{0' \leq q' \leq \nu'} \left( \frac{p'}{q'} \right) \pi^{q'} D_{\alpha} \left[ D^{(q',p',q')} U^{i,k}_{\alpha} \eta \right] \right| dx - \int_{Q_k^5(z)} G_1^t D_1 \eta^t dx \leq c r^t \int_{Q_k(z)} \epsilon |D\eta|^2 + \epsilon^{-2} |\eta|^2 + E_{z,r}^2 \, dx.
\]

By a direct calculation,

\[
D_{\alpha} \left[ D^{(q',p',q')} U^{i,k}_{\alpha} \eta \right] = D^{(q',p',q')} D_{\alpha} U^{i,k}_{\alpha} \eta + D^{(q',p',q')} U^{i,k}_{\alpha} D_{\alpha} \eta^t.
\]

From ($3.10$) and ($3.12$), we have that $D_{\alpha} U^{i,k}_{\alpha} = 0$ in $Q_k^5(z)$. So the lemma holds. \(\Box\)

So we obtain the desired estimate of this subsection.

**Lemma 3.8.** For any $\eta \in C^{\infty}_c (Q_{r}(z), \mathbb{R}^N)$ with $Q_{r}(z) \subset Q_5$, we have that

\[
\left| \int_{Q_r(z)} \sum_{1 \leq \alpha \leq n} \left[ \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq \alpha} A^{ij}_{\alpha} \sum_{0' \leq q' \leq \nu'} \left( \frac{p'}{q'} \right) \pi^{q'} D^{(q',p',q')} D_{\beta} u^{i,k} + \nu^{i}_{\alpha} \right] D_{\alpha} \eta^t - G_1^t D_1 \eta^t dx \right|
\]

\[
\leq c r^{n+1} \left[ \|D^{m+1} u\|_{L^\infty(Q_r(z))} + \sum_{k \in K_{+}} \left[ \|u\|_{C^{m,\gamma}(Q_k^5)} + \|F\|_{C^{m,\gamma}(Q_k^5)} \right] \right]
\]

\[+ c r^t \int_{Q_r(z)} \epsilon |D\eta|^2 + \epsilon^{-2} |\eta|^2 \, dx,
\]

for any $1 \leq i \leq N$, with $G_1^t : Q_5 \to \mathbb{R}$ in Lemma 3.7 and

\[
\nu_{\alpha}^t = \sum_{k \in K_{+}} \sum_{0' \leq q' \leq \nu'} \left( \frac{p'}{q'} \right) \pi^{q'} \left[ D^{(q',p',q')} U^{i,k}_{\alpha} - \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq \alpha} A^{ij}_{\alpha} D^{(q',p',q')} D_{\beta} u^{i,k} \right] \chi_{Q_k^5}
\]

for any $1 \leq \alpha \leq n$ and $1 \leq i \leq N$. Moreover, we have that

\begin{equation}
\|G_1^t\|_{C^{\gamma}(Q_5^6)} + \sum_{k \in K_{+}} \|\nu^t_{\alpha}\|_{C^{\gamma}(Q_5^6)} \leq c \sum_{k \in K_{+}} \left[ \|u\|_{C^{m,\gamma}(Q_k^5)} + \|F\|_{C^{m,\gamma}(Q_k^5)} \right],
\end{equation}

for any $1 \leq \alpha \leq n$ and $1 \leq i \leq N$.

**Proof.** With ($3.22$) and ($2.19$), the estimate holds from Lemma 3.7. With ($3.10$), we find from ($3.2$) and ($3.29$) in Lemma 3.7 that ($3.30$) holds. \(\Box\)
3.2. Choosing test function. We will choose a suitable test function in $Q_r(z)$ by using $\eta$ in (3.37) (which will be used in Lemma 5.1 and Lemma 5.2) With (2.1), set $Z_k : Q_T \rightarrow \mathbb{R} \ (k \in K)$ as

$$Z_k(x) := \begin{cases} \varphi_{k+1}(x') & \text{if } x^1 > \varphi_{k+1}(x'), \\ x^1 & \text{if } \varphi_k(x') < x^1 \leq \varphi_{k+1}(x'), \\ \varphi_k(x') & \text{if } x^1 \leq \varphi_k(x'), \end{cases}$$

(3.31)

which implies that

$$\begin{cases} \varphi_{k+1}(x') & \text{if } l > k, \\ x^1 & \text{if } l = k, \\ \varphi_k(x') & \text{if } k > l. \end{cases}$$

(3.32)

We use the perturbation on $Q_r^k$ with respect to the point $z_k (\in \mathbb{R}^n)$ defined as follows. For any $k \in K$ and fixed $z \in Q_5$, set

$$z_k = (Z_k(z), z').$$

(3.33)

To use the perturbation, with (1.2), we will check that $z_k \in Q_{6r}^k$ in Corollary 3.10. However, $z_k$ depends on $z$. So later in Lemma 3.14, we will handle the points $z_k$ ($k \in K$) by using the graph functions $\varphi_k$ ($k \in K$) which does not depend on $z$. In fact, we use perturbation on $Q_{6r}^k (z)$ with respect to the point $z_k$ in (3.33). So with (1.2), we will check that $z_k \in Q_{6r}^k (z)$ in Lemma 3.9.

**Lemma 3.9.** For any $Q_{6r}^k (z) \subset Q_6$ and the corresponding points $z_k$ ($k \in K$) in (3.32), we have that

$$Q_{6r}^k (z) \neq \emptyset \quad \Rightarrow \quad z_k \in Q_{6r}^k (z) \quad (k \in K).$$

Moreover, for any $x \in Q_{6r}^k (z)$, there exists a path belongs to $Q_{6r}^k (z) \cup \{z_k\}$ with length less than $cr$ which connects between $x$ and $z_k$.

**Proof.** Fix $k \in K$. Assume that $Q_{6r}^k (z) \subset Q_6$ and $Q_{6r}^k (z) \neq \emptyset$. Then we have that $Q_{6r}^k \supset Q_{6r}^k (z) \neq \emptyset$, and it follows from Corollary 3.10 that $z_k \in Q_{6r}^k$. So we only need to prove that $z_k \in Q_{6r}^k (z)$.

Since $Q_{6r}^k (z) \neq \emptyset$, we have that $\inf_{Q_{6r}^k (z')} \varphi_k \leq z^1 + 5r$ and $\sup_{Q_{6r}^k (z')} \varphi_{k+1} \geq z^1 - 5r$.

Otherwise we have that $Q_{6r}^k (z) = \emptyset$. By (1.2),

$$\varphi_k(x') < z^1 + \frac{11r}{2} \quad \text{and} \quad \varphi_{k+1}(x') > z^1 - \frac{11r}{2} \quad (x' \in Q_{6r}^k (z')).$$

(3.35)

To prove that $z_k \in Q_{6r}^k (z)$, we consider three cases. (1) Assume that $z^1 > \varphi_{k+1}(z')$. Then from (1.2) and (3.35), we have that $z^1 - 6r < \varphi_{k+1}(z') < z^1$, which implies that $z_k = (Z_k(z), z') = (\varphi_{k+1}(z'), z') \in Q_{6r}^k (z)$. (2) Assume that $\varphi_k(z') \leq z^1 \leq \varphi_{k+1}(z')$. Then we have from (3.31) and (3.33) that $z_k = z \in Q_{6r}^k (z)$. (3) Assume that $z_1 < \varphi_k(z')$. Then from (1.2) and (3.33), we have that $z^1 < \varphi_k(z') < z^1 + 6r$, which implies that $z_k = (Z_k(z), z') = (\varphi_k(z'), z') \in Q_{6r}^k (z)$. Since $k \in K$ was arbitrary chosen, (3.34) holds.

We choose a path connecting $x \in Q_{6r}^k (z)$ and $z_k \in Q_{6r}^k (z)$ as follows. Set

$$l(s) = \left( \min \left\{ z^1 + \frac{11r}{2}, \varphi_{k+1}(sz' + (1 - s)x'), sz' + (1 - s)x' \right\} \right) (s \in [0, 1])$$

The path is a connected path of (1) the line connecting $x$ and $l(0)$, (2) the path $l(s)$ ($s \in [0, 1]$) connecting $l(0)$ and $l(1)$, (3) the line connecting $l(1)$ and $z_k$. Since $x \in Q_{6r}^k (z)$ and $z_k \in Q_{6r}^k (z)$, one can show that this new path belongs to $Q_{6r}^k (z) \cup \{z_k\}$ and has length less than $50nr$. □
The following corollary is a special case of Lemma 3.9.

**Corollary 3.10.** For any \( z \in Q_5 \) and the corresponding points \( z_k \) \((k \in K)\) in (3.32), we have that
\[
Q^5_k \neq \emptyset \quad \Rightarrow \quad z_k \in \overline{Q^5_k} \quad (k \in K).
\]

**Proof.** By taking \( z = 0 \) and \( r = 1 \) in Corollary 3.10, the lemma follows. \( \square \)

Recall from (2.11) that
\[
(3.36) \quad \left. \frac{\partial^p h}{\partial z'} \right|_{z=0} = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} D(|q'|,|p'-q'|)h + \sum_{|\xi| \leq |p'|-1} P^p_{\xi} (x) D_\xi h \quad \text{in} \quad Q_5.
\]
For any \( z \in Q_5 \) and \( p' \geq 0' \) with \(|p'| = m\), the test function will be chosen by using the following \( \eta : Q_5 \to \mathbb{R}^N \) defined as
\[
\eta = \left. \frac{\partial^p u}{\partial z'} \right|_{z=0} - \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} D(|q'|,|p'-q'|) u_k(z_k)
- \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{|\xi| \leq |p'|-1} P^p_{\xi} (z_k, x') D_\xi u_k(z_k)
+ \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} D(|q'|,|p'-q'|) u_k(z_k, x')
+ \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{|\xi| \leq |p'|-1} P^p_{\xi} (z_k, x') D_\xi u_k(z_k, x').
\]
It follows from (3.36) that
\[
\tilde{\eta} = I + II + III \quad \text{in} \quad Q_5,
\]
where
\[
I = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \left[ \pi^{q'} D(|q'|,|p'-q'|) u - \sum_{k \in K, Q^5_k \neq \emptyset} \pi^{q'} (z_k, x') D(|q'|,|p'-q'|) u_k(z_k) \chi_{Q^5_k} \right],
\]
\[
II = - \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} (z_k, x') D(|q'|,|p'-q'|) u_k(z_k) \chi_{Q^5 \setminus Q^5_k}
+ \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} (\varphi_k(x'), x') D(|q'|,|p'-q'|) u_k(\varphi_k(x'), z')
\]
and
\[
III = - \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{|\xi| \leq |p'|-1} P^p_{\xi} (z_k, x') D_\xi u_k(z_k, x') \chi_{Q^5 \setminus Q^5_k}
+ \sum_{k \in K, Q^5_k \neq \emptyset} \sum_{|\xi| \leq |p'|-1} P^p_{\xi} (\varphi_k(x'), x') D_\xi u_k(\varphi_k(x'), x').
\]
Due to the approximation argument, we have that \( \delta > 0 \) in (2.1) and (3.6) hold. So we find that \( \eta \) in (3.37) is weakly differentiable in \( Q_5 \) and that \( D\eta \in L^\infty(Q_5) \).

**Lemma 3.11.** \( \eta \) in (3.37) is weakly differentiable in \( Q_5 \) and that \( D\eta \in L^\infty(Q_5) \).

**Proof.** We only need to check that \( \frac{\partial^p u}{\partial z'} \big|_{z=0} \) is weakly differentiable in \( Q_5 \) and the derivative is bounded in \( Q_5 \). Fix \( \alpha \in \{1, \ldots, n\} \).
In view of (3.6), we have that $\partial^\alpha u_k|_{t=0} \in C^1(\overline{Q_k^5}, \mathbb{R}^N)$. Thus for any $\varphi \in C_0^\infty(Q_k, \mathbb{R}^N)$,

$$\int_{Q_k} \partial^\alpha u_k|_{t=0} D_i \varphi \, dx = \sum_{k \in K} \int_{Q_k^5} \partial^\alpha u_k|_{t=0} D_i \varphi \, dx$$

$$= \sum_{k \in K} \int_{Q_k^5} \partial^\alpha u_k|_{t=0} \, dx$$

$$= -\sum_{k \in K} \int_{Q_k^5} D_i \left( \partial^\alpha u_k \right) |_{t=0} \varphi \, dx + \sum_{k \in K} \int_{\partial Q_k^5} \partial^\alpha u_k|_{t=0} \varphi \, n^\alpha_k \, dS,$$

where $\bar{n}_k = (n_k^1, n_k^2, \cdots, n_k^n)$ is the outward normal vector on $\partial Q_k^5$. From that $u_k = u_{k-1}$ on $\partial Q_k^5 \cap \partial Q_{k-1}^5$, we obtain that $\partial^\alpha u_k|_{t=0} = \partial^\alpha u_{k-1}|_{t=0}$ on $\partial Q_k^5 \cap \partial Q_{k-1}^5$. So from that $n_k^i = -n_{k-1}^i$ ($k \in K$), we obtain that

$$\sum_{k \in K} \int_{\partial Q_k^5} \partial^\alpha u_k|_{t=0} \varphi \, n^\alpha_k \, dS = 0.$$

Thus

$$\int_{Q_k} \partial^\alpha u_k|_{t=0} D_i \varphi \, dx = -\sum_{k \in K} \int_{Q_k^5} D_\alpha \left( \partial^\alpha u_k \right) |_{t=0} \varphi \, dx$$

$$= -\int_{Q_k} \left[ \sum_{k \in K} D_\alpha \left( \partial^\alpha u_k \right) |_{t=0} \chi_{Q_k^5} \right] \varphi \, dx.$$

With (3.6), we discover that $\partial^\alpha u_k|_{t=0}$ is weakly differentiable in $Q_5$ and that $D_\alpha \left( \partial^\alpha u_k \right) |_{t=0} \chi_{Q_k^5} \in L^\infty(Q_5)$. Since $i \in \{1, \cdots, n\}$ was arbitrary chosen, the lemma holds.

Recall from (2.18) that

$$x \in Q_5^k \implies H_k(x) = \begin{cases} 1 & \text{if } l \geq k, \\ 0 & \text{if } l < k. \end{cases}$$

To handle $I$, $II$ and $III$, we obtain three following lemmas related to (3.39).

**Lemma 3.12.** For any $h : Q_5 \to \mathbb{R}$ and $k \in K$, we have that

$$h_Z(x, x') \chi_{Q_k^5} = (1 - H_k) h(\varphi_k(x'), x') + H_{k+1} h(\varphi_{k+1}(x'), x').$$

**Proof.** Fix $k \in K$. From (3.39), we find that

$$\sum_{l < k, l \in K} h(Z_l(x', x')) \chi_{Q_5^l} = \sum_{l < k, l \in K} h(\varphi_{l}(x'), x') \chi_{Q_5^l} = (1 - H_k) h(\varphi_k(x'), x') \text{ in } Q_5.$$

Also from (3.39), we find that

$$\sum_{l > k, l \in K} h(Z_l(x', x')) \chi_{Q_5^l} = \sum_{l > k, l \in K} h(\varphi_{l-1}(x'), x') \chi_{Q_5^l} = H_{k+1} h(\varphi_{k+1}(x'), x') \text{ in } Q_5.$$

Since $k \in K$ was arbitrary chosen, the lemma follows from the above equalities.

With Lemma 3.13, we only need to handle the indices in $K_-$ in (1.3) not all the indices in $K$. Here, $K_-$ represents the index of graph functions intersecting $Q_5$. Let $Q_k^5 \neq \emptyset$.

**Lemma 3.13.** For any $k \in K$ with $Q_k^5 \neq \emptyset$, we have that

$$k + 1 \notin K_- \implies H_{k+1} = 0 \text{ in } Q_5,$$

and

$$k \notin K_- \implies H_k = 1 \text{ in } Q_5.$$
Proof. Let \( K_- = \{ l_-, l_- + 1, \cdots, l_+ \} \). From (1.3) and that \( Q_k^c \neq \emptyset \), we find that
\[
(3.42) \quad k \in \{ l_- - 1, l_-, l_- + 1, \cdots, l_+ \}.
\]
To prove (3.40), assume that \( k + 1 \notin K_- \). Then from (3.42) and that \( k + 1 \notin K_- \), we obtain that \( k + 1 = l_+ + 1 \). Since \( K_- = \{ l_-, l_- + 1, \cdots, l_+ \} \), we have from Definition 1.1 that \( H_{k+1} = 0 \) in \( Q_5^c \).

To prove (3.41), assume that \( k \notin K_- \). Then by (3.42) and that \( k \notin K_- \), we find that \( k = l_- - 1 \). Since \( K_- = \{ l_-, l_- + 1, \cdots, l_+ \} \), we have that \( H_k = 1 \) in \( Q_5 \). \( \Box \)

We use perturbations with respect to the points \( z_k (k \in K) \). In view of Lemma 3.12 and Lemma 3.14, \( z_k \) can be handled by the graph functions \( \varphi_k \) for any \( k \in K \).

**Lemma 3.14.** For any \( k \in K \), we have that
\[
(3.43) \quad H_k(x) \neq 1 \text{ for some } x \in Q_r(z) \quad \implies \quad |z_k - (\varphi_k(z'), z')| < 2r,
\]
and
\[
(3.44) \quad H_{k+1}(x) \neq 0 \text{ for some } x \in Q_r(z) \quad \implies \quad |z_k - (\varphi_{k+1}(z'), z')| < 2r.
\]

Proof. We first prove (3.43). Suppose that \( H_k(x) \neq 1 \) for some \( x \in Q_r(z) \). Then we have from (1.1) and (3.39) that \( x' \leq \varphi_k(x') \).

(1) Assume that \( \varphi_k(z') \geq z' \). Then from (3.32) and (3.33), we have that \( z_k = (\varphi_k(z'), z') \), and so (3.43) holds.

(2) Assume that \( \varphi_k(z') < z' \). Then we have from (3.32) and (3.33) that \( z_k = z \). Since \( x' \leq \varphi_k(x') \), choose any point \( y = (y^1, y^2) \in Q_r(z) \) on the line connecting \( z \) and \( x \) such that \( y^1 = \varphi_k(y^2) \). Then by (1.2) and that \( z_k = z \),
\[
|\varphi_k(z') - z_k^1| = \frac{|z' - y^1|}{10} + |y^1 - z^1|.
\]

Since \( y = (y^1, y^2) \in Q_r(z) \) is on the line connecting \( z \) and \( x \), we have that \( |z' - y^1| \leq |z' - x| < 2nr \) and \( |z^1 - y^1| \leq |z^1 - x^1| < r \), and so (3.43) holds.

(3) Assume that \( z' > \varphi_{k+1}(z') \). Then we have that \( z_k = (\varphi_{k+1}(z'), z') \). Since \( x' > \varphi_{k+1}(x') \), choose any point \( y = (y^1, y^2) \in Q_r(z) \) on the line connecting \( z \) and \( x \) such that \( y^1 = \varphi_{k+1}(y^2) \). Then by (1.2) and that \( z_k = (\varphi_{k+1}(z'), z') \),
\[
|\varphi_k(z') - z_k^1| = \frac{|z' - y^1|}{10} + |y^1 - t^1| + \frac{|t' - z'|}{10}.
\]

Since \( y, t \in Q_r(z) \) are on the line connecting \( z \) and \( x \), we have that \( |z' - y^1|, |t' - z'| \leq |z' - x| < 2nr \) and \( |z^1 - y^1| \leq |z^1 - x^1| < r \), and so (3.44) holds.

We now prove (3.44). Suppose that \( H_{k+1}(x) \neq 0 \) for some \( x \in Q_r(z) \). Then we have from (1.1) and (3.39) that \( x' > \varphi_{k+1}(x') \).

(1) Assume that \( \varphi_{k+1}(z') < z' \). Then from (3.32) and (3.33), we have that \( z_k = (\varphi_{k+1}(z'), z') \), and so (3.44) holds.

(2) Assume that \( \varphi_k(z') < z' \). Then we have from (3.32) and (3.33) that \( z_k = z \). Since \( x' > \varphi_{k+1}(x') \), choose any point \( y = (y^1, y^2) \in Q_r(z) \) on the line connecting \( z \) and \( x \) such that \( y^1 = \varphi_{k+1}(y^2) \). Then by (1.2) and that \( z_k = z \),
\[
|\varphi_{k+1}(z') - z_k^1| = \frac{|z' - y^1|}{10} + |y^1 - z^1|.
\]

Since \( y = (y^1, y^2) \in Q_r(z) \) is on the line connecting \( z \) and \( x \), we have that \( |z' - y^1| \leq |z' - x| < 2nr \) and \( |z^1 - y^1| \leq |z^1 - x^1| < r \), and so (3.44) holds.

(3) Assume that \( z' \leq \varphi_k(z') (\varphi_k(z')) \). Then we have that \( z_k = (\varphi_k(z'), z') \). Since \( x' > \varphi_{k+1}(x') \), choose any point \( y, t \in Q_r(z) \) on the line connecting \( z \) and \( x \)
such that \( y^1 = \varphi_{k+1}(y') \) and \( t^1 = \varphi_k(t') \). Then by (1.2) and that \( z_k = (\varphi_k(z'), z') \),
\[
|\varphi_{k+1}(z') - z_k^1| \leq |\varphi_{k+1}(z') - \varphi_{k+1}(y')| + |\varphi_{k+1}(y') - \varphi_k(t')| + |\varphi_k(t') - \varphi_k(z')|
\leq \frac{|z' - y'|}{10n} + |y^1 - t^1| + \frac{|t' - z'|}{10n}.
\]
Since \( y, t \in Q_r(z) \) are on the line connecting \( z \) and \( x \), we have that \( |z' - y'|, |t' - z'| \leq |z' - x'| < 2nr \) and \( |y^1 - t^1| \leq |z^1 - x^1| < r \), and so (3.44) holds. □

Set \( S = \cup_{k \in K^+} \{(\varphi_k(x'), x') \in x' \in Q_y^k\} \). Then \( |S| = 0 \). By using Corollary 3.10, Lemma 3.12 and Lemma 3.13 and Lemma 3.14, we estimate \( DI, D(II) \) and \( D(III) \) in the set \( Q_r(z) \setminus S \) as in the following three lemmas.

**Lemma 3.15.** We have the following estimate for \( I \) in (3.38):
\[
\int_{Q_y(z) \setminus S} \left| DI - \sum_{0' \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \pi^q D(|q'|, p' - q') Du \right|^2 dx \leq cr^{n+1} ||D^{m+1}u||^2_{L^\infty(Q_{2r}(z))}.
\]

**Proof.** From (1.1) and (3.31), we have that \( (Z_k(x), x') = x \) for any \( x \in Q_y^k \) \((k \in K)\).
It follows that \( \pi^q (Z_k(x), x') = \pi^q (x) \) for any \( x \in Q_y^k \) \((k \in K)\).
So by (3.38),
\[
I = \sum_{0' \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \left[ \pi^q D(|q'|, p' - q')u - \pi^q D(|q'|, p' - q') u_k(z_k) \right] \quad \text{in} \quad Q_y^k(z) \quad \text{(k \in K)}.
\]
It follows that
\[
\sum_{k \in K} \int_{Q_y^k(z) \setminus S} \left| DI - \sum_{0' \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \pi^q D(|q'|, p' - q') Du \right|^2 dx
\]
\[
= \sum_{k \in K} \int_{Q_y^k(z) \setminus S} \left| \sum_{0' \leq q' \leq p'} D \left[ \pi^q \right] \left[ D(|q'|, p' - q') - D(|q'|, p' - q') u_k(z_k) \right] \right|^2 dx.
\]
Since \( Q_{2r}(z) \subset Q_5 \) and \( D^{m}u_k \in C^\gamma \left( Q_{2r}^5(z) \right) \), we have from Lemma 3.9 that
\[
|D^{m}u - D^{m}u_k(z_k)| \leq cr ||D^{m+1}u||_{L^\infty(Q_y^k(z))} \quad \text{in} \quad Q_y^k(z) \quad \text{(k \in K)},
\]
by separating the cases that \( Q_y^k(z) = \emptyset \) and \( Q_y^k(z) \neq \emptyset \). It follows that
\[
\int_{Q_y(z) \setminus S} \left| DI - \sum_{0' \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \pi^q D(|q'|, p' - q') Du \right|^2 dx
\]
\[
\leq cr^2 ||D^{m+1}u||^2_{L^\infty(Q_{2r}(z))} \int_{Q_y(z)} \left| D \left[ \pi^q \right] \right|^2 dx,
\]
and so the lemma follows from (2.17). □

**Lemma 3.16.** For any \( \alpha \in \{1, \cdots, n\} \), the next estimate holds for \( II \) in (3.38):
\[
\left| D_\alpha(II) + \sum_{k \in K^-} \mathbb{H}^k_{\alpha} [u_{k-1} - u_k] H_k \right| \leq cr^\gamma \sum_{k \in K^-} \|u_k\|_{C^{m, \gamma}(Q_y^k)},
\]
in \( Q_y(z) \setminus S \) where the linear operator \( \mathbb{H}^k_{\alpha} : C^{m, \gamma} (\partial Q_y^k \cap \partial Q_y^{k-1}) \to C^\gamma (Q_y^k) \) \((k \in K^-)\) defined as
\[
\mathbb{H}^k_{\alpha} [h](x') = \sum_{0' \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) D_j \left[ \pi^q (\varphi_k(x'), x') \right] D(|q'|, p' - q') h(\varphi_k(x'), x').
\]
Proof. In Lemma 3.12, we take \( h = \sum_{0' \leq q' \leq p'} (p'_q) \pi' D((q'|p'-q') u_k(z_k). Then
\[
II = - \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) (1 - H_k) \pi'(\varphi_k(x'), x') D((q'|p'-q') u_k(z_k)
- \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) H_{k+1} \pi'(\varphi_{k+1}(x'), x') D((q'|p'-q') u_k(z_k)
+ \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) \pi' (\varphi_k(x'), x') D((q'|p'-q') u_k(\varphi_k(z'), z'),
\]
in \( Q_r(z). \) Fix \( \alpha \in \{1, \cdots, n\}. \) We find from Lemma 3.13 that
\[
D_\alpha (II) = II_1 + II_2 + II_3 \quad \text{in} \quad Q_r(z) \setminus S,
\]
where
\[
II_1 = - \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) (1 - H_k) D_\alpha \left[ \pi'(\varphi_k(x'), x') \right] D((q'|p'-q') u_k(z_k)
+ \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) (1 - H_k) D_\alpha \left[ \pi'(\varphi_k(x'), x') \right] D((q'|p'-q') u_k(\varphi_k(z'), z'),
\]
\[
II_2 = \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) H_k D_\alpha \left[ \pi'(\varphi_k(x'), x') \right] D((q'|p'-q') u_k(\varphi_k(z'), z'),
\]
and
\[
II_3 = - \sum_{k+1 \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) H_{k+1} D_\alpha \left[ \pi'(\varphi_{k+1}(x'), x') \right] D((q'|p'-q') u_k(z_k)
= - \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) H_k D_\alpha \left[ \pi'(\varphi_k(x'), x') \right] D((q'|p'-q') u_{k-1}(z_{k-1}),
\]
in \( Q_r(z) \setminus S. \) In view of (2.7), we discover that
\[
\left| D \left[ \pi'(\varphi_k(x'), x') \right] \right| \leq c \quad \text{in} \quad Q'_5,
\]
By Corollary 3.10, if \( Q_k^\circ \neq \emptyset \) for some \( k \in K \) then \( z_k \in \overline{Q}_6^k. \) Also we have from (1.5) that \( (\varphi_k(z'), z') \in \overline{Q}_6^k \) for any \( k \in K_- \). So by Lemma 3.14 and (3.45),
\[
|II_1| \leq c \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (1 - H_k) \left| D((q'|p'-q') u_k(z_k) - D((q'|p'-q') u_k(\varphi_k(z'), z')) \right|
\leq c r^\gamma \sum_{k \in K, Q_k^\circ \neq \emptyset \neq \emptyset} \| u_k \|_{C^{\alpha, \gamma}} (Q_k^\circ)
\]
in \( Q_r(z) \setminus S. \)
From (1.5), we have that \( (\varphi_k(x'), x') \in \overline{Q}_6^k \) for any \( x' \in Q'_r(z') \) and \( k \in K_- \). So it follows from (3.45) that
\[
|II_2| = \sum_{k \in K, Q_k^\circ \neq \emptyset, 0' \leq q' \leq p'} (p'_q) H_k D_\alpha \left[ \pi'(\varphi_k(x'), x') \right] D((q'|p'-q') u_k(\varphi_k(x'), x'))
\leq c r^\gamma \sum_{k \in K, Q_k^\circ \neq \emptyset \neq \emptyset} \| u_k \|_{C^{\alpha, \gamma}} (Q_k^\circ),
\]
in \( Q_r(z) \setminus S. \)
By Corollary 3.10, if $Q_{k}^{k-1} \neq \emptyset$ for some $k-1 \in K$ then $z_{k-1} \in Q_{k}^{k-1}$. Also we have from (1.5) that $(\varphi_{k}(x'), x') \in Q_{k}^{k-1}$ for any $x' \in Q_{k}(z')$ and $k \in K_{-}$. So by Lemma 3.14 and (3.45) that

$$\left| H_{\alpha} + \sum_{k \in K_{-}, Q_{k}^{k-1} \neq \emptyset} \sum_{0 \leq q' \leq q} \frac{p_{k}^{j}}{q} H_{k} D_{\alpha} \left[ \varphi_{k}(x'), x' \right] \right| D(q'|p' - q') u_{k-1}(\varphi_{k}(x'), x') \leq cr^{\gamma} \sum_{k \in K_{-}, Q_{k}^{k-1} \neq \emptyset} \| u_{k} \|_{C^{m, \gamma}(\overline{Q_{k}^{k-1}})},$$

in $Q_{r}(z) \setminus S$.

Since $\alpha \in \{1, \cdots, n\}$ was arbitrary, the lemma holds from (1.4). \hfill \Box

**Lemma 3.17.** For any $\alpha \in \{1, \cdots, n\}$, the following holds for III in (3.38):

$$D_{\alpha}(III) = - \sum_{k \in K_{-}} H_{k}^{b}[u_{k-1} - u_{k}] H_{k}$$

in $Q_{r}(z) \setminus S$ with the linear operator $H_{k}^{b} : C^{m, \gamma}(\partial Q_{k}^{k-1} \cap \partial Q_{k}^{k-1}) \to C^{\gamma}(Q_{k}^{k-1})$ $(k \in K_{-})$ defined as

$$H_{k}^{b}[h](x) = \sum_{|\xi| \leq |p'| - 1} D_{\alpha} \left[ P_{\xi}^{\beta} \left[ \varphi_{k}(x'), x' \right] \right] D_{\xi} h(\varphi_{k}(x'), x').$$

**Proof.** In Lemma 3.12, we take $h = \sum_{|\xi| \leq |p'| - 1} P_{\xi}^{\beta}(Z_{k}, x') D_{\xi} u_{k}(Z_{k}, x')$. Then

$$III = - \sum_{k \in K, Q_{k}^{k-1} \neq \emptyset} \sum_{|\xi| \leq |p'| - 1} (1 - H_{k}) P_{\xi}^{\beta} \left[ \varphi_{k}(x'), x' \right] D_{\xi} u_{k}(\varphi_{k}(x'), x')$$

$$- \sum_{k \in K, Q_{k}^{k-1} \neq \emptyset} \sum_{|\xi| \leq |p'| - 1} H_{k+1} P_{\xi}^{\beta} \left[ \varphi_{k+1}(x'), x' \right] D_{\xi} u_{k}(\varphi_{k+1}(x'), x')$$

$$+ \sum_{k \in K_{-}, Q_{k}^{k-1} \neq \emptyset} \sum_{|\xi| \leq |p'| - 1} P_{\xi}^{\beta} \left( \varphi_{k}(x'), x' \right) D_{\xi} u_{k}(\varphi_{k}(x'), x')$$

in $Q_{r}(z) \setminus S$. Fix $\alpha \in \{1, \cdots, n\}$. Then from Lemma 3.13, we find that

$$D_{\alpha}(III) = III_{1} + III_{2} \text{ in } Q_{5} \setminus S,$$

where

$$III_{1} = \sum_{k \in K_{-}, Q_{k}^{k-1} \neq \emptyset} \sum_{|\xi| \leq |p'| - 1} H_{k} D_{\alpha} \left[ P_{\xi}^{\beta} \left[ \varphi_{k}(x'), x' \right] \right] D_{\xi} u_{k}(\varphi_{k}(x'), x'),$$

and

$$III_{2} = - \sum_{k+1 \in K_{-}, Q_{k}^{k-1} \neq \emptyset} \sum_{|\xi| \leq |p'| - 1} H_{k+1} D_{\alpha} \left[ P_{\xi}^{\beta} \left[ \varphi_{k+1}(x'), x' \right] \right] D_{\xi} u_{k}(\varphi_{k+1}(x'), x'),$$

in $x \in Q_{r}(z) \setminus S$. Since $\alpha \in \{1, \cdots, n\}$ was arbitrary chosen, the lemma follows from (1.4). \hfill \Box

In the following lemma, $G_{\alpha}$ $(\alpha \in \{1, \cdots, N\})$ does not depend on $Q_{r}(z)$. 

Lemma 3.18. For any $Q_{2r}(z) \subset Q_5$ and $\alpha \in \{1, \cdots, n\}$, the following
\[
\int_{Q_r(z)} \left| D_\alpha \eta - \sum_{0^r \leq q' \leq p'} \left( \frac{p'}{q'} \right)^\gamma \pi^{q'} D^{(|q'|, p'-q')} D_\alpha u + G_\gamma \right|^2 dx 
\leq cr^{\eta+2\gamma} \left[ \left\| D^{m+1}u \right\|^2_{L^\infty(Q_{2r}(z))} + \sum_{k \in K_{-}} \left\| u \right\|_{C^{m,\gamma}(Q_k)} \right] ,
\]
holds for $\eta$ in (3.37) where $G_\gamma = \sum_{k \in K_{-}} \mathbb{H}_\alpha^k [u_{k-1} - u_k] H_k$
with the linear operator $\mathbb{H}_\alpha^k : C^{m,\gamma}(\partial Q_6^k \cap \partial Q_6^{k-1}) \to C^\gamma(Q_5^k)$ ($k \in K_{-}$) defined as
\[
\mathbb{H}_\alpha^k [h](x) = \sum_{0^r \leq q' \leq p'} \left( \frac{p'}{q'} \right)^\gamma \pi^{q'} D^{(|q'|, p'-q')} \left[ \pi^{q'} D^{(|q'|, p'-q')} \right] h(x)
+ \sum_{|\xi| \leq p'-1} \left( \frac{p'}{q'} \right)^\gamma \pi^{q'} D^{(|q'|, p'-q')} \left[ \pi^{q'} D^{(|q'|, p'-q')} \right] h(x).
\]
Proof. We first estimate $\eta$. From (3.11), $\eta$ is weakly differentiable in $Q_r(z)$ with the estimate $D_\eta \in L^\infty(Q_r(z))$, which implies that
\[
\int_{S} \left| D_\alpha \eta - \sum_{0^r \leq q' \leq p'} \left( \frac{p'}{q'} \right)^\gamma \pi^{q'} D^{(|q'|, p'-q')} D_\alpha u + \sum_{k \in K_{-}} \mathbb{H}_\alpha^k [u_{k-1} - u_k] H_k \right|^2 dx = 0.
\]
With (3.38), we obtain from Lemma 3.15, Lemma 3.16 and Lemma 3.17 that
\[
\int_{Q_r(z) \setminus S} \left| D_\alpha \eta - \sum_{0^r \leq q' \leq p'} \left( \frac{p'}{q'} \right)^\gamma \pi^{q'} D^{(|q'|, p'-q')} D_\alpha u + \sum_{k \in K_{-}} \mathbb{H}_\alpha^k [u_{k-1} - u_k] H_k \right|^2 dx 
\leq cr^{\eta+2\gamma} \left[ \left\| D^{m+1}u \right\|^2_{L^\infty(Q_{2r}(z))} + \sum_{k \in K_{-}} \left\| u \right\|_{C^{m,\gamma}(Q_k)} \right] .
\]
So the lemma follows from (3.4).

To obtain the piece-wise regularity, we will use the following lemma.

Lemma 3.19. For $\mathbb{H}_\alpha^k$ and $G_\gamma$ in Lemma 3.18, we have that
\[
\sum_{k \in K_{-}} \left\| \mathbb{H}_\alpha^k [u_{k-1} - u_k] \right\|_{C^{m,\gamma}(Q_k)} + \sum_{k \in K} \left\| G_\gamma \right\|_{C^{m,\gamma}(Q_k)} \leq c \sum_{k \in K} \left\| u \right\|_{C^{m,\gamma}(Q_k)},
\]
for any $\alpha \in \{1, \cdots, n\}$.

Proof. By that $\pi^{q'} (\varphi_k(x'), x') = [D_2 \varphi_k(x')]^{q_2} \cdots [D_n \varphi_k(x')]^{q_n}$ ($x' \in Q_5^k$, $k \in K$),
the lemma holds from (1.3), (3.3), (3.4) and the definition of $P_{\xi}^m$ in Lemma 2.3. \qed
4. Higher order derivatives

In the main equation, we assume that the coefficients are piece-wise continuous, and so the gradient of the weak solution might have big jumps on each \( \partial Q^k \cap \partial Q^{k-1} \) \( (k \in K) \). To show that \( D^{m+1} u \) is piece-wise Hölder continuous, we first show that \( U^p' (|p'| = m) \) defined in (4.1) are Hölder continuous. Then we will prove that \( D^{m+1} u \) is piece-wise Hölder continuous by comparing \( D^{m+1} u \) and \( U^p' \). Here, \( U^p' \) is defined by using the perturbation results in Lemma 3.8 and Lemma 3.18.

For the simplicity of the calculation, we abuse the notation \( D_{\xi} u = D_{\xi} u_k \) in \( Q^k \). Recall from (2.9) that

\[
\pi_1 = -1, \quad \pi' = (\pi_2, \cdots, \pi_n), \quad \pi = (\pi_1, \pi') = (-1, \pi_2, \cdots, \pi_n) \quad \text{in} \quad Q^k.
\]

With the assumption (3.6), for any \( p' \geq 0' \) with \( |p'| = m \), set \( G^p_1 \) as in Lemma 3.8 and \( G^p_2, \cdots, G^p_n \) as in Lemma 3.18. Then define \( U^p' : Q_5 \to \mathbb{R}^{N_n} \) and \( G^p' : Q_5 \to \mathbb{R}^{N_n} \) as follows:

\[
(4.1) \quad U^p' = \begin{pmatrix} U_1^{1,p'} & \cdots & U_n^{1,p'} \\ \vdots & \ddots & \vdots \\ U_1^{N,p'} & \cdots & U_n^{N,p'} \end{pmatrix} \quad \text{and} \quad G^p' = \begin{pmatrix} G_1^{1,p'} & \cdots & G_n^{1,p'} \\ \vdots & \ddots & \vdots \\ G_1^{N,p'} & \cdots & G_n^{N,p'} \end{pmatrix},
\]

where

\[
U_1^{i,p'} = G_1^{i,p'} + \sum_{1 \leq \alpha \leq n} \pi_\alpha \left[ \sum_{p'' \leq p' \leq p'} \frac{p'}{q'} \pi^{q'} \partial^{(|q'|,|p'|-q')} \left[ \sum_{1 \leq \beta \leq N} \sum_{1 \leq \delta \leq n} A_{i,j}^{\beta \delta} D_\beta u^j - F^{i}_{\alpha} \right] \right]
\]
in \( Q_5 \), and

\[
(4.2) \quad U_\beta^{i,p'} = G_\beta^{i,p'} + \sum_{p'' \leq p' \leq p'} \frac{p'}{q'} \pi^{q'} \left[ D^{(|q'|,|p'|-q')} D_\beta u^i + \pi_\beta D^{(|q'|,|p'|-q')} D_1 u^i \right]
\]
in \( Q_5 \) for any \( \beta \in \{2, \cdots, n\} \) and \( i \in \{1, \cdots, N\} \). Recall from Lemma 3.7 and (3.10) that

\[
(4.3) \quad G_1^{i,p'} = \sum_{k \in K} \sum_{1 \leq \alpha \leq n} \mathcal{G}_\alpha^{p',k} [U_\alpha^{i-1,k} - U_\alpha^{i,k}] H_k \quad \text{in} \quad Q_5
\]

and

\[
U_\alpha^{i,k} = \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{i,j}^{\beta \delta} D_\beta u^j - F^{i}_{\alpha} \quad \text{in} \quad Q_5
\]

for any \( 1 \leq \alpha \leq n, 1 \leq i \leq N \) and \( k \in K \) with the linear operator \( \mathcal{G}^{p',k}_\alpha : C^{m-1,\gamma} (\partial Q^k \cap \partial Q^{k-1}) \to C^{\gamma} (Q^k) \) \( (k \in K) \) defined as

\[
\mathcal{G}^{p',k}_\alpha [h](x') = \sum_{|\xi| \leq |p'|+1} \partial_\xi h \left( \varphi_k(x'), x' \right) + \sum_{p'' < p' \leq p'} \left[ \frac{p'}{q'} \partial^{p'} D_\nu \varphi_k(x') \partial^{p''-q'} \left[ h(\varphi_k(x'), x') \right] \right]
\]

for any \( x' \in Q^k_1 \), \( 1 \leq \alpha \leq n \) and \( k \in K \). Also recall from Lemma 3.18 that

\[
(4.4) \quad G^p_\beta = \sum_{k \in K} \mathcal{H}^{p',k}_\beta [u_{k-1} - u_k] H_k \quad \text{in} \quad Q_5 \quad (\beta = 2, 3, \cdots, n)
\]
with the linear operator \( \mathbb{H}_{\beta}^{p',k} : C^{m,\gamma}(\partial Q_0^k \cap \partial Q_0^{k-1}) \to C^\gamma(Q^k_\delta) \) \((k \in K_\omega)\) defined as
\[
\mathbb{H}_{\beta}^{p',k}[h](x') = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) D_\beta \left[ \pi^{q'}(\varphi_k(x'), x') \right] D((q'|p'-q'))h(\varphi_k(x'), x') \\
+ \sum_{|q| \leq |p'| - 1} D_\beta \left[ \frac{p'}{q} \pi^{q}(\varphi_k(x'), x') \right] D_q h(\varphi_k(x'), x'),
\]
for any \( x' \in Q^k_\delta, 2 \leq \beta \leq n \) and \( k \in K \). Then by Lemma 3.8 and Lemma 3.19,
\[
\left\| G^{p'} \right\|_{C^\gamma(Q^k_\delta)} \leq c \sum_{k \in K} \left[ \left\| u \right\|_{C^{m,\gamma}(Q^k_\delta)} + \left\| F \right\|_{C^{m,\gamma}(Q^k_\delta)} \right] \quad \text{in } Q_5.
\]
(4.5)

To handle \( U^{i,p}_1 \), we show Lemma 4.1. In view of Lemma 3.8, set \( \nu^{p'} : Q_5 \to \mathbb{R}^{Nn} \) as
\[
\nu^{p'} = \begin{pmatrix} \nu_1^{1, p'} & \cdots & \nu_n^{1, p'} \\
\vdots & \ddots & \vdots \\
\nu_1^{N, p'} & \cdots & \nu_n^{N, p'} \end{pmatrix} \quad \text{in } Q_5,
\]
(4.6)
where
\[
\nu_\alpha^{i, p'} = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} D((q'|p'-q')) \left( \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{ij}^{a\beta} D_\beta u^j - F_\alpha \right)
- \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{ij}^{a\beta} D((q'|p'-q')) D_\beta u^j,
\]
for any \( 1 \leq \alpha \leq n \) and \( 1 \leq i \leq N \). For the simplicity, set \( \tilde{u}^{p'} : Q_5 \to \mathbb{R}^{Nn} \) as
\[
\tilde{u}^{p'} = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^{q'} D((q'|p'-q')) D_\beta u^j \quad \text{in } Q_5,
\]
(4.7)
for any \( 1 \leq \alpha \leq n \) and \( 1 \leq j \leq N \). By recalling (4.6) and that \( U^{i,p}_1 \) in (4.1),
\[
U^{i,p}_1 = G^{i,p}_1 + \sum_{1 \leq \alpha \leq n} \pi_\alpha \left[ \nu_\alpha^{i, p'} + \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{ij}^{a\beta} \tilde{u}^{i, p'} \right] \quad \text{in } Q_5,
\]
(4.8)
and
\[
U^{i,p}_1 = G^{i,p}_1 + \sum_{1 \leq \beta \leq n} \sum_{1 \leq \alpha \leq n} A_{ij}^{a\beta} \pi_\alpha \zeta_j \geq \lambda \left| (\pi_1, \cdots, \pi_n) \right|^2 |\zeta|^2 \geq \lambda |\zeta|^2 \quad \text{in } Q_5.
\]
(4.9)
From (1.10) and that \( \pi_1 = -1 \), we have that
\[
\sum_{1 \leq i, j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A_{ij}^{a\beta} \pi_\alpha \pi_\beta \tilde{u}^{i, p'} \geq \lambda \left| (\pi_1, \cdots, \pi_n) \right|^2 |\zeta|^2 \geq \lambda |\zeta|^2 \quad \text{in } Q_5.
\]

Lemma 4.1. For \( U^{i,p}_1 \) and \( G^{i,p}_1 \) in (4.1) \((1 \leq \alpha \leq n, 1 \leq i \leq N)\), we have that
\[
\sum_{1 \leq j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A_{ij}^{a\beta} \pi_\alpha \pi_\beta \tilde{u}^{i, p'} = G^{i,p}_1 - U^{i,p}_1 + \sum_{1 \leq \alpha \leq n} \pi_\alpha \left[ \nu_\alpha^{i, p'} + \sum_{1 \leq j \leq N} \sum_{2 \leq \beta \leq n} A_{ij}^{a\beta} \left( U^{i,p}_1 - G^{i,p}_1 \right) \right] \quad \text{in } Q_5,
\]
for any $1 \leq i \leq N$.

**Proof.** For any $2 \leq \beta \leq n$ and $1 \leq j \leq N$, (4.9) yields that

$$
\tilde{u}^{i,p'} = U_1^{i,p'} - G_1^{i,p'} - \pi_\beta \tilde{u}_1^{i,p'} \quad \text{in } Q_5.
$$

So by (4.8) and that $\pi_1 = -1$,

$$
U_1^{i,p'} = G_1^{i,p'} + \sum_{1 \leq \alpha \leq n} \pi_\alpha \sum_{1 \leq j \leq N} \sum_{2 \leq \beta \leq n} A_{ij}^{\alpha,\beta} \left( U_{\beta}^{i,p'} - G_{\beta}^{i,p'} \right)
$$

$$
+ \sum_{1 \leq \alpha \leq n} \pi_\alpha \left[ \nu^{i,p'} - \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} \pi_\beta A_{ij}^{\alpha,\beta} \tilde{u}_1^{i,p'} \right]
$$

for any $1 \leq i \leq N$, which implies that

$$
\sum_{1 \leq j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A_{ij}^{\alpha,\beta} \pi_\alpha \pi_\beta \tilde{u}_1^{i,p'}
$$

$$
= G_1^{i,p'} - U_1^{i,p'} + \sum_{1 \leq \alpha \leq n} \pi_\alpha \left[ \nu^{i,p'} + \sum_{1 \leq j \leq N} \sum_{2 \leq \beta \leq n} A_{ij}^{\alpha,\beta} \left( U_{\beta}^{i,p'} - G_{\beta}^{i,p'} \right) \right]
$$

for any $1 \leq i \leq N$. So the lemma follows.

To use (4.2) and Lemma 4.1, for fixed $p' \geq 0'$ with $|p'| = m$, set

$$
\mathbb{B}^{p'}[z, w] = \left( \mathbb{B}_1^{p'}[z, w], \cdots, \mathbb{B}_n^{p'}[z, w] \right) \quad \text{(z, w \, \in \, Q_5)}
$$

where

$$
\mathbb{B}_1^{p'}[z, w] = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^q(z) D^{|q'|,|p'-q'|} \tilde{D}D_1 w^j(w)
$$

and

$$
\mathbb{B}_\beta^{p'}[z, w] = \sum_{0' \leq q' \leq p'} \left( \frac{p'}{q'} \right) \pi^q(z) \left[ D^{|q'|,|p'-q'|} D_\beta w^j(w) + \pi_\beta(z) D^{|q'|,|p'-q'|} D_1 w^j(w) \right],
$$

for any $2 \leq \beta \leq n$ and $1 \leq j \leq N$. Then we have that

$$
\mathbb{B}_\beta^{p'}[w, w] = \tilde{u}_\beta^{i,p'}(w) + [1 - \delta_{\beta 1}] \pi_\beta(w) \tilde{u}_1^{i,p'}(w) \quad (1 \leq \beta \leq n, \, 1 \leq j \leq N),
$$

for any $w \in Q_5$. So it follows from Lemma 2.5 that

$$
\left| \mathbb{B}^{p'}[w, w] \right| \leq c \left| \tilde{u}^{p'}(w) \right|
$$

and

$$
\left| \mathbb{B}^{p'}[w, w] - \mathbb{B}^{p'}[z, z] \right| \leq c \left[ \left| \tilde{u}^{p'}(w) - \tilde{u}^{p'}(z) \right| + \left| z - w \right| \right] \left| \tilde{u}^{p'}(z) \right|
$$

for any $w, z \in Q_5$.

**Lemma 4.2.** For any $z, w \in Q_5^k (k \in K)$ and $p' \geq 0'$ with $|p'| = m$, we have that

$$
\left| \mathbb{B}^{p'}[z, w] - \mathbb{B}^{p'}[w, w] \right| \leq c |z - w| \gamma \left| \tilde{u}^{p'}(w) \right|
$$

(4.16)

$$
\left| \mathbb{B}^{p'}[z, z] \right| \leq c \left[ |U^{p'}(z)| + \sum_{k \in K} \left( \|u\|_{C^{m,\gamma}(Q_5^k)} + \|F\|_{C^{m,\gamma}(Q_5^k)} \right) \right],
$$

(4.17)

$$
\left| \mathbb{B}^{p'}[w, w] - \mathbb{B}^{p'}[z, z] \right| \leq c \left[ |U^{p'}(w) - U^{p'}(z)| + |w - z| \gamma |U^{p'}(z)| \right]
$$

(4.18)

$$
+ c |w - z| \gamma \sum_{k \in K} \left( \|u\|_{C^{m,\gamma}(Q_5^k)} + \|F\|_{C^{m,\gamma}(Q_5^k)} \right).$$
In addition, we have that

\[ |\tilde{u}^p| \leq c \left[ |U^p| + |G^p| + |\nu^p| \right] \text{ in } Q_5. \]  

**Proof.** Since \( \pi \in C^7(Q_5) \), we obtain (4.16) from (4.12). By (4.2) and Lemma 4.1,

\[
\sum_{1 \leq i,j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A_{ij}^{\alpha \beta} \pi_{\alpha} \pi_{\beta} \tilde{u}_1^{i,p} \tilde{u}_1^{j,p} \\
= \sum_{1 \leq i \leq N} \tilde{u}_1^{i,p} \left[ G_1^{i,p} - U_1^{i,p} + \sum_{1 \leq \alpha \leq n} \pi_{\alpha} \left( \nu_1^{i,p} + \sum_{1 \leq j, k \leq N} A_{ij}^{\alpha \beta} \left( U_{1,\beta}^{i,p} - G_{1,\beta}^{i,p} \right) \right) \right]
\]

in \( Q_5 \). It follows from (4.11) that

\[
\lambda \left| \tilde{u}_1^{i,p} \right|^2 \leq \sum_{1 \leq i,j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A_{ij}^{\alpha \beta} \pi_{\alpha} \pi_{\beta} \tilde{u}_1^{i,p} \tilde{u}_1^{j,p} \leq c \left| \tilde{u}_1^{i,p} \right| \left[ |U^p| + |G^p| + |\nu^p| \right].
\]

So (4.19) holds from (4.9). Also with (4.14) and (4.19), (4.17) holds from (4.5) and (4.10). To show (4.18), investigate that

\[
\sum_{1 \leq i,j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A_{ij}^{\alpha \beta} (w) \pi_{\alpha} (w) \pi_{\beta} (w) \left[ \tilde{u}_1^{i,p} (w) - \tilde{u}_1^{j,p} (z) \right]
\]

\[
= \sum_{1 \leq i,j \leq N} \sum_{1 \leq \alpha, \beta \leq n} \left[ A_{ij}^{\alpha \beta} (w) \pi_{\alpha} (w) \pi_{\beta} (w) \tilde{u}_1^{i,p} (w) - A_{ij}^{\alpha \beta} (z) \pi_{\alpha} (z) \pi_{\beta} (z) \tilde{u}_1^{i,p} (z) \right]
\]

\[
+ \sum_{1 \leq i,j \leq N} \sum_{1 \leq \alpha, \beta \leq n} \left[ A_{ij}^{\alpha \beta} (z) \pi_{\alpha} (z) \pi_{\beta} (z) \tilde{u}_1^{i,p} (z) - A_{ij}^{\alpha \beta} (w) \pi_{\alpha} (w) \pi_{\beta} (w) \tilde{u}_1^{i,p} (z) \right].
\]

Recall that \( A_{ij}^{\alpha \beta}, \pi_{\alpha} \in C^7 \left( Q^k_5 \right) \) \((k \in K)\). Then we have from Lemma 4.1 that

\[
\sum_{1 \leq i,j \leq N} \sum_{1 \leq \alpha, \beta \leq n} A_{ij}^{\alpha \beta} (w) \pi_{\alpha} (w) \pi_{\beta} (w) \left[ \tilde{u}_1^{i,p} (w) - \tilde{u}_1^{j,p} (z) \right]
\]

\[
\leq c \left| \tilde{u}_1^{i,p} (w) - \tilde{u}_1^{j,p} (z) \right| \left[ |U^p (w) - U^p (z)| + |G^p (w) - G^p (z)| + |\nu^p (w) - \nu^p (z)| \right]
\]

\[
+ c |w - z|^\gamma \left| \tilde{u}_1^{i,p} (w) - \tilde{u}_1^{j,p} (z) \right| \left| \tilde{u}_1^{j,p} (z) \right|,
\]

for any \( w, z \in Q_5^k \) \((k \in K)\). It follows from (4.11) that

\[
\left| \tilde{u}_1^{i,p} (w) - \tilde{u}_1^{j,p} (z) \right| \leq c \left[ |U^p (w) - U^p (z)| + |G^p (w) - G^p (z)| + |\nu^p (w) - \nu^p (z)| \right]
\]

\[
+ c |w - z|^\gamma \left| \tilde{u}_1^{j,p} (z) \right|,
\]

for any \( w, z \in Q_5^k \) \((k \in K)\). So by (4.9) and that \( \pi_{\alpha} \in C^7 \left( Q^k_5 \right) \) \((k \in K)\),

\[
\left| \tilde{u}^p (w) - \tilde{u}^p (z) \right| \leq c \left[ |U^p (w) - U^p (z)| + |G^p (w) - G^p (z)| + |\nu^p (w) - \nu^p (z)| \right]
\]

\[
+ c |w - z|^\gamma \left| \tilde{u}^p (z) \right|,
\]

for any \( w, z \in Q_5^k \) \((k \in K)\). It follows from (4.15) and (4.19) that

\[
\left| B^p [w, w] - B^p [z, z] \right|
\]

\[
\leq c \left[ |U^p (w) - U^p (z)| + |G^p (w) - G^p (z)| + |\nu^p (w) - \nu^p (z)| \right]
\]

\[
+ c |z - w|^\gamma \left| U^p (z) \right| + |G^p (z)| + |\nu^p (z)|,
\]

for any \( w, z \in Q_5^k \) \((k \in K)\). So (4.18) follows from (4.5) and (4.10). \( \square \)
To derive Lemma 4.8 and Lemma 4.12, we obtain the estimates with respect to
$y$-coordinate system as in Lemma 4.3.

**Lemma 4.3.** For any $z \in Q_5$, let $\Psi_z : Q_5 \to \mathbb{R}^n$ be a coordinate transformation
with $y = \Psi_z(x) = (\Psi_1^1(x), \Psi_1(x))$ and $\Phi^{-1}_z = \Psi_z$ such that
$$\Psi_1^1(x) = x^1 - z^1 - \pi'(z) \cdot (x' - z').$$
Then for any $\alpha, \beta \in \{2, \cdots, n\}$, we have that
$$\frac{\partial x^1}{\partial y^1} = 1, \quad \frac{\partial x^1}{\partial y^3} = \pi_\beta(z), \quad \frac{\partial x^\alpha}{\partial y^1} = 0, \quad \frac{\partial x^\alpha}{\partial y^3} = \delta_{ij},$$
and
$$\frac{\partial y^1}{\partial x^1} = 1, \quad \frac{\partial y^1}{\partial x^3} = -\pi_\beta(z), \quad \frac{\partial y^\alpha}{\partial x^1} = 0, \quad \frac{\partial y^\alpha}{\partial x^3} = \delta_{ij}. \quad \text{Proof. Since} \quad x^1 = y^1 + z^1 + \pi'(z) \cdot y' \quad \text{and} \quad x^1 = y' + z', \quad \text{we find that}
$$
$$\frac{\partial x^1}{\partial y^1} = 1, \quad \frac{\partial x^1}{\partial y^3} = \pi_\beta(z), \quad \frac{\partial x^\alpha}{\partial y^1} = 0 \quad \text{and} \quad \frac{\partial x^\alpha}{\partial y^3} = \delta_{\alpha\beta} \quad (\alpha, \beta \in \{2, \cdots, n\}).$$
Since $y^1 = x^1 - \pi'(z) \cdot (x' - z')$ and $y' = x' - z'$, we find that
$$\frac{\partial y^1}{\partial x^1} = 1, \quad \frac{\partial y^1}{\partial x^3} = -\pi_\beta(z), \quad \frac{\partial y^\alpha}{\partial x^1} = 0 \quad \text{and} \quad \frac{\partial y^\alpha}{\partial x^3} = \delta_{\alpha\beta} \quad (\alpha, \beta \in \{2, \cdots, n\}).$$
This completes the proof. \hfill \Box

**Lemma 4.4.** For $z \in Q_5$, let $y = \Psi_z(x)$ and $\Phi^{-1}_z = \Psi_z$ as in Lemma 4.3. Also let
$$\tilde{A}^{\alpha\beta}_{ij}(y) = \sum_{1 \leq s, t \leq n} A^s_{ij}(\Phi_z(y)) \frac{\partial y^s}{\partial x^\alpha} \frac{\partial y^\beta}{\partial x^t}(\Phi_z(y)) \quad \text{in} \quad \Psi_z(Q_5),$$
and
$$\tilde{F}_\alpha(y) = \frac{\partial y^\alpha}{\partial x^\alpha}(\Phi_z(y)) F_\alpha(\Phi_z(y)) \quad \text{in} \quad \Psi_z(Q_5).$$
Then for $\check{u}(y) = u(\Phi_z(y))$, we have that
$$D_{y^\alpha} \left[ \tilde{A}^{\alpha\beta}_{ij} D_{y^\beta} \check{u} \right] = D_{y^\alpha} \check{F}_\alpha \quad \text{in} \quad \Psi_z(Q_5),$$
and \(\tilde{A}^{\alpha\beta}_{ij}, \check{u}, \check{F} \in C^{m,\gamma}(\Psi_z(Q^5_5)) \) for any \( k \in K \) with the estimate that
$$\|\check{u}\|_{C^{m,\gamma}(\Psi_z(Q^5_5))} + \|\check{F}\|_{C^{m,\gamma}(\Psi_z(Q^5_5))} \leq c \left[ \|\check{u}\|_{C^{m,\gamma}(Q^5_5)} + \|\check{F}\|_{C^{m,\gamma}(Q^5_5)} \right].$$
Proof. Since $D_{x^\alpha} [a_{ij} D_{x^ij} u - F^i] = 0$ in $Q_5$, by changing variables,
$$D_{y^\alpha} \left[ \tilde{A}^{\alpha\beta}_{ij}(\Phi_z(y)) \frac{\partial y^s}{\partial x^\alpha} \frac{\partial y^\beta}{\partial x^t} D_{y^\beta} \check{u} \right] = D_{y^\alpha} \left[ \frac{\partial y^\alpha}{\partial x^\alpha} F_\alpha \right] \quad \text{in} \quad \Psi_z(Q_5).$$
With (3.2) and Lemma 4.3, the lemma holds by (4.21) and that $\check{u}(y) = u(\Phi(y))$. \hfill \Box

**Lemma 4.5.** For a fixed $z \in Q_5$, let $y = \Psi_z(x)$, $\Phi^{-1}_z = \Psi_z$ and $\check{u}(y) = u(\Phi_z(y))$ as in Lemma 4.3. Then for any $w \in Q_5$ and $p' \geq 0$ with $|p'| = m$, we have that
$$D_{y^\beta} D_{y^\alpha} \check{u}(\Psi_z(w)) = B_{ij}^\beta [z, w] \quad (\beta \in \{1, \cdots, n\}).$$
Proof. We prove the lemma for a fixed $p' \geq 0$ with $|p'| = m$. By Lemma 4.3,
$$\frac{\partial x^1}{\partial y^1} = 1, \quad \frac{\partial x^1}{\partial y^3} = \pi_\beta(z), \quad \frac{\partial x^\alpha}{\partial y^1} = 0 \quad \text{and} \quad \frac{\partial x^\alpha}{\partial y^3} = \delta_{\alpha\beta} \quad (\alpha, \beta \in \{2, \cdots, n\}).$$
Fix $w \in Q_5$. By the chain rule,
\[
D_y^p \hat{u}(\Psi_z(w)) = \sum_{0 \leq q' \leq p'} \left( \frac{p'_i}{q'_i} \right) \left( \frac{\partial x}{\partial y} \right)^{q'_i} \left( \frac{\partial x^2}{\partial y^2} \right)^{q'_i-1} \cdots \left( \frac{\partial x^n}{\partial y^n} \right)^{q'_i-n} D_x^{|q'|,|p'-q'|} u(w).
\]

Then we apply the chain rule again to find that
\[
D_y D_y^p \hat{u}(\Psi_z(w)) = \sum_{0 \leq q' \leq p'} \left( \frac{p'_i}{q'_i} \right) \pi^{q'}(z) D_x^{|q'|,|p'-q'|} u(w) = \mathbb{B}^p[z, w],
\]
and
\[
D_y D_y D_y^p \hat{u}(\Psi_z(w)) = \sum_{0 \leq q' \leq p'} \left( \frac{p'_i}{q'_i} \right) \pi^{q'}(z) \left[ D_x^{|q'|,|p'-q'|} u(w) + \pi_{\beta}(z) D_x^{|q'|,|p'-q'|} u(w) \right] = \mathbb{B}^p[z, w],
\]
for any $\beta \in \{2, \cdots, n\}$. Since $w$ was arbitrary chosen, the lemma follows.

We will estimate $|D^{m+1}u|$ in Lemma 4.7.

Lemma 4.6. For a fixed $z \in Q_5$, let $y = \Psi_z(x)$, $\Phi_z^{-1} = \Psi_z$ and $\hat{u}(y) = u(\Phi_z(y))$ as in Lemma 4.3. Then we have that
\[
|D^{m+1}_y \hat{u}(\Psi_z(z))| \leq c \left[ \sum_{|p'|=m} |\mathbb{B}^p[z, z]| + \sum_{k \in K} \left[ \|u\|_{C^{m, \gamma}(Q_{5}^k)} + \|F\|_{C^{m, \gamma}(Q_{5}^k)} \right] \right].
\]

Proof. Fix nonnegative integers $l_1$ and $l_2$ with $l_1 + l_2 = m - 1$. We differentiate (4.22) in Lemma 4.4 by $D_{y}^{l_1} D_{y}^{l_2}$ to find that for any $i \in \{1, \cdots, N\},$
\[
\sum_{1 \leq i \leq N} \tilde{A}_{ij} \left( D_{y}^{l_1} D_{y}^{l_2} D_{y^i y^i} \hat{u}^{i} \right) \leq c \left[ D_{y}^{l_1+1} D_{y}^{l_2+1} \hat{u} + \sum_{k \in K} \left[ \|\hat{u}\|_{C^{m, \gamma}(\Psi(Q_{5}^k))} + \|\hat{F}\|_{C^{m, \gamma}(\Psi(Q_{5}^k))} \right] \right]
\]
in $\Psi(Q_5)$, which implies that
\[
\sum_{1 \leq i \leq N} \tilde{A}_{ij} \left( D_{y}^{l_1} D_{y}^{l_2} D_{y^i y^i} \hat{u}^{i} \right) \left( D_{y}^{l_1} D_{y}^{l_2} D_{y^i y^i} \hat{u}^{i} \right) \leq c \left[ D_{y}^{l_1+1} D_{y}^{l_2+1} \hat{u} + \sum_{k \in K} \left[ \|\hat{u}\|_{C^{m, \gamma}(\Psi(Q_{5}^k))} + \|\hat{F}\|_{C^{m, \gamma}(\Psi(Q_{5}^k))} \right] \right]
\]
in $\Psi(Q_5)$. So it follows from (1.10) that
\[
|D_{y}^{l_1} D_{y}^{l_2} D_{y^i y^i} \hat{u}^{i}| \leq c \left[ D_{y}^{l_1+1} D_{y}^{l_2+1} \hat{u} + \sum_{k \in K} \left[ \|\hat{u}\|_{C^{m, \gamma}(\Psi(Q_{5}^k))} + \|\hat{F}\|_{C^{m, \gamma}(\Psi(Q_{5}^k))} \right] \right].
\]
It follows that for any \( l_1 \) and \( l_2 \) with \( l_1 + l_2 = m - 1 \) were arbitrary chosen, we find from (4.23) in Lemma 4.4 that

\[
|D_y^{m+1} \hat{u}(\Psi)(z)| \leq c \left| D_y^{m+1} \hat{u}(\Psi)(z) \right| + \sum_{k \in K} \left[ \|u\|_{C^{m,\gamma}(Q^k)} + \|F\|_{C^{m,\gamma}(Q^k)} \right].
\]

So the lemma holds from Lemma 4.5. \( \square \)

**Lemma 4.7.** For a fixed \( z \in Q_5 \), let \( y = \Psi(z) \), \( \Phi^{-1} = \Psi \) and \( \hat{u}(y) = u(\Phi)(y) \) as in Lemma 4.3. Then we have that

\[
|D^{m+1}u(z)| \leq c \left[ \sum_{|p'| = m} |B^{p'}[z, z]| + \sum_{k \in K} \left[ \|u\|_{C^{m,\gamma}(Q^k)} + \|F\|_{C^{m,\gamma}(Q^k)} \right] \right].
\]

**Proof.** We claim that

(4.24) \[ |D^{m+1}u(z)| \leq c |D^{m+1} \hat{u}(\Psi)(z)|. \]

By the chain rule, for any \( p' \geq 0 \) with \( |p'| \leq m+1 \), we have from Lemma 4.3 that

\[
D^{m+1-|p'|}D_x^{p'}u(z) = \sum_{0 \leq q' \leq p'} \frac{p'}{q'} \left( \frac{\partial y}{\partial x^1} \right)^{p_1-q_1} \cdots \left( \frac{\partial y}{\partial x^n} \right)^{p_n-q_n} \times D^{m+1-|p'|}D_y^{|p'|} \hat{u}(\Psi)(z) = \sum_{0 \leq q' \leq p'} \left( \frac{p'}{q'} \right) (-1)^{|p'|-|q'|} n^{p'-q'}(z)D^{m+1-|p'|} \hat{u}(\Psi)(z).
\]

It follows that for any \( p' \geq 0 \) with \( |p'| \leq m+1 \),

\[
|D^{m+1-|p'|}D_x^{p'}u(z)| \leq c |D^{m+1} \hat{u}(\Psi)(z)|.
\]

So the claim (4.24) holds. The lemma follows from (4.24) and Lemma 4.6. \( \square \)

**Lemma 4.8.** For any \( z \in Q_5 \), we have that

\[
|D^{m+1}u(z)| \leq c \left[ \sum_{|p'| = m} |U^{p'}(z)| + \sum_{k \in K} \left[ \|u\|_{C^{m,\gamma}(Q^k)} + \|F\|_{C^{m,\gamma}(Q^k)} \right] \right].
\]

**Proof.** The lemma follows from (4.17) in Lemma 4.2 and Lemma 4.7. \( \square \)

We next estimate \( |D^{m+1}u(w) - D^{m+1}u(z)| \) for any \( w, z \in Q^k \) (\( k \in K \)).

**Lemma 4.9.** For a fixed \( z \in Q_5 \), let \( y = \Psi(z) \), \( \Phi^{-1} = \Psi \) and \( \hat{u}(y) = u(\Phi)(y) \) as in Lemma 4.3. Then for any \( w \in Q_5 \), we have that

\[
|D_y D^{m}_y \hat{u}(\Psi)(w) - D_y D^{m}_y \hat{u}(\Psi)(z)| \leq c \sum_{|p'| = m} \left[ |\mathbb{B}^{p'}[w, w] - \mathbb{B}^{p'}[z, z]| + |w - z|^{p'} |\mathbb{B}^{p'}[z, z]| \right] + c |w - z|^{p'} \sum_{k \in K} \left[ \|u\|_{C^{m,\gamma}(Q^k)} + \|F\|_{C^{m,\gamma}(Q^k)} \right].
\]
Proof. We have from Lemma 4.5 that
\[ D_y D_y^p \hat{u}(\Psi_z(w)) - D_y D_y^p \hat{u}(\Psi_z(z)) = B^{p'}[z, w] - B^{p'}[z, z], \]
for any \( w \in Q_3 \) and \( p' \geq 0 \) with \( |p'| = m \). It follows that
\[ |D_y D_y^p \hat{u}(\Psi_z(w)) - D_y D_y^p \hat{u}(\Psi_z(z))| \leq |B^{p'}[z, w] - B^{p'}[w, w]| + |B^{p'}[w, w] - B^{p'}[z, z]|, \]
for any \( w \in Q_3 \) and \( p' \geq 0 \) with \( |p'| = m \). We find from (4.16) in Lemma 4.2 that
\[ |B^{p'}[z, w] - B^{p'}[w, w]| \leq c|z - w|^\gamma|D^{m+1}u(w)| \]
for any \( w \in Q_3 \) and \( p' \geq 0 \) with \( |p'| = m \). By combining the above two estimates, the lemma follows from Lemma 4.7. \( \square \)

The proof of the following lemma is almost parallel to that of Lemma 4.6.

Lemma 4.10. For a fixed \( z \in Q_3 \), let \( y = \Psi_z(x) \), \( \Phi_z^{-1} = \Psi_z \) and \( \hat{u}(y) = u(\Phi_z(y)) \) as in Lemma 4.3. Then for any \( w \in Q_3 \) (\( k \in K \)), we have that
\[ D_y^{m+1} \hat{u}(\Psi_z(w)) - D_y^{m+1} \hat{u}(\Psi_z(z)) \]
\[ \leq c \sum_{|p'| = m} \left( |B^{p'}[w, w] - B^{p'}[z, z]| + |w - z|^\gamma |B^{p'}[z, z]| \right) \]
\[ + c|w - z|^\gamma \sum_{k \in K} \left( \|u\|_{C^{m, \gamma}(\Phi_k)} + \|\hat{F}\|_{C^{m, \gamma}(\Phi_k)} \right). \]

Proof. Fix \( w, z \in Q_3 \) (\( k \in K \)), nonnegative integers \( l_1 \) and \( l_2 \) with \( l_1 + l_2 = m - 1 \). We differentiate (4.22) in Lemma 4.4 by \( D_y^{l_1} D_y^{l_2} \) to find that
\[ \left| \sum_{1 \leq j \leq N} \left[ \hat{A}_{ij}^{l_1} (\Psi_z(w)) D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(w)) - \hat{A}_{ij}^{l_1} (\Psi_z(z)) D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(z)) \right] \right| \]
\[ \leq c|w - z|^\gamma \left| \|D_y^{m+1} \hat{u}(\Psi_z(z))\| + \|\hat{u}\|_{C^{m, \gamma}(\Phi_k)} + \|\hat{F}\|_{C^{m, \gamma}(\Phi_k)} \right| \]
\[ + c|D_y^{l_1+1} D_y^{l_2+1} \hat{u}(\Psi_z(w)) - D_y^{l_1+1} D_y^{l_2+1} \hat{u}(\Psi_z(z))| \].

By the triangle inequality,
\[ \left| \sum_{1 \leq j \leq N} \hat{A}_{ij}^{l_1} (\Psi_z(w)) \left[ D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(w)) - D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(z)) \right] \right| \]
\[ \leq \left| \sum_{1 \leq j \leq N} \left[ \hat{A}_{ij}^{l_1} (\Psi_z(w)) D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(w)) - \hat{A}_{ij}^{l_1} (\Psi_z(z)) D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(z)) \right] \right| \]
\[ + \left| \hat{A}_{ij}^{l_1} (\Psi_z(z)) - \hat{A}_{ij}^{l_1} (\Psi_z(w)) \right| \left| D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(z)) \right|. \]

Since \( \hat{A}_{ij}^{l_1} (\Psi_z(w)) \geq cA_{ij} \) and \( \hat{A}_{ij}, \hat{F} \in C^{m, \gamma} (\Psi_z(Q_k)) \) for any \( k \in K \), it follows that
\[ \left| \sum_{1 \leq j \leq N} \hat{A}_{ij}^{l_1} (\Psi_z(w)) \left[ D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(w)) - D_y^{l_1} D_y^{l_2} D_y^{1+y}_y \hat{u}_y (\Psi_z(z)) \right] \right| \]
\[ \leq c|w - z|^\gamma \left| D_y^{m+1} \hat{u}(\Psi_z(z))\| + \|\hat{u}\|_{C^{m, \gamma}(\Phi_k)} + \|\hat{F}\|_{C^{m, \gamma}(\Phi_k)} \right| \]
\[ + c|D_y^{l_1+1} D_y^{l_2+1} \hat{u}(\Psi_z(w)) - D_y^{l_1+1} D_y^{l_2+1} \hat{u}(\Psi_z(z))| \].
We multiply $D_{y}^{l_{1}} D_{y}^{l_{2}} D_{y}^{l_{3}} \hat{u}(\Psi_{z}(w)) - D_{y}^{l_{1}} D_{y}^{l_{2}} D_{y}^{l_{3}} \hat{u}(\Psi_{z}(z))$ on each side and then sum it over $i \in \{1, \cdots, N\}$. By applying (1.10), we obtain that
\[
\left| D_{y}^{l_{1}+2} \hat{u}(\Psi_{z}(w)) - D_{y}^{l_{1}+2} \hat{u}(\Psi_{z}(z)) \right| \\
\leq c \left| w - z \right|^{\gamma} \sum_{k} \left| \left| D_{y}^{l_{1}+1} \hat{u}(\Psi_{z}(z)) \right| + \left| \hat{u} \right|_{C^{\infty}(\Psi_{z}(Q_{k}))} + \left| \hat{F} \right|_{C^{\infty}(\Psi_{z}(Q_{k}))} \right| \\
+ c \left| D_{y}^{l_{1}+1} D_{y}^{l_{1}+1} \hat{u}(\Psi_{z}(w)) - D_{y}^{l_{1}+1} D_{y}^{l_{1}+1} \hat{u}(\Psi_{z}(z)) \right|. 
\]
Since nonnegative integers $l_{1}$ and $l_{2}$ with $l_{1} + l_{2} = m - 1$ were arbitrary chosen,
\[
\left| D_{y}^{m+1} \hat{u}(\Psi_{z}(w)) - D_{y}^{m+1} \hat{u}(\Psi_{z}(z)) \right| \\
\leq c \left| D_{y} D_{y}^{m} \hat{u}(\Psi_{z}(w)) - D_{y} D_{y}^{m} \hat{u}(\Psi_{z}(z)) \right| \\
+ c \left| w - z \right|^{\gamma} \left( \left| D_{y} D_{y}^{m} \hat{u}(\Psi_{z}(z)) \right| + \left| \hat{u} \right|_{C^{\infty}(\Psi_{z}(Q_{k}))} + \left| \hat{F} \right|_{C^{\infty}(\Psi_{z}(Q_{k}))} \right). 
\]
Since $w, z \in Q_{k}^{\delta}$ ($k \in K$) were arbitrary chosen, the lemma follows from Lemma 4.6 and Lemma 4.9.

\textbf{Lemma 4.11.} For any $z, w \in Q_{k}^{\delta}$ ($k \in K$), we have that
\[
\left| D_{x}^{m+1} u(w) - D_{x}^{m+1} u(z) \right| \leq c \sum_{|p'|=m} \left| \left| B_{p'}^{\mu} [w, w] - B_{p'}^{\mu} [z, z] \right| + \left| w - z \right|^{\gamma} \left| B_{p'}^{\mu} [z, z] \right| \right| \\
+ c \left| w - z \right|^{\gamma} \sum_{k \in K} \left( \left| u \right|_{C^{\infty}(\hat{\Psi}_{k})} + \left| F \right|_{C^{\infty}(\hat{\Psi}_{k})} \right). 
\]

\textbf{Proof.} For a fixed $z, w \in Q_{k}^{\delta}$ ($k \in K$), let $y = \Psi_{z}(x)$, $\Phi_{z}^{-1} = \Psi_{z}$ and $\hat{u}(y) = u(\Phi_{z}(y))$ as in Lemma 4.3. We claim that
\[
\left| D_{x}^{m+1} u(w) - D_{x}^{m+1} u(z) \right| \leq c \left| D_{y}^{m+1} \hat{u}(\Psi_{z}(w)) - D_{y}^{m+1} \hat{u}(\Psi_{z}(z)) \right|. 
\]
By the chain rule, for any $p' \geq 0$ with $|p'| \leq m + 1$, we have from Lemma 4.3 that
\[
\left| D_{x}^{m+1-|p'|} D_{x}^{p'} u(w) \right| = \sum_{0 \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \left( \begin{array}{c} \partial y^{1} \\ \partial x^{2} \end{array} \right)^{q'} \left( \begin{array}{c} \partial y^{2} \\ \partial x^{2} \end{array} \right)^{q'} \cdots \left( \begin{array}{c} \partial y^{m} \\ \partial x^{m} \end{array} \right)^{q'} \left( \begin{array}{c} \partial y^{n} \\ \partial x^{n} \end{array} \right)^{q'} \\
\times \left| D_{y}^{m+1-|p'|} D_{y}^{p'} \hat{u}(\Psi(w)) \right| \\
= \sum_{0 \leq q' \leq p'} \left( \begin{array}{c} p' \\ q' \end{array} \right) \left( -1 \right)^{|p'|} \left( \begin{array}{c} \partial y^{1} \\ \partial x^{2} \end{array} \right)^{q'} \left( \begin{array}{c} \partial y^{2} \\ \partial x^{2} \end{array} \right)^{q'} \cdots \left( \begin{array}{c} \partial y^{n} \\ \partial x^{n} \end{array} \right)^{q'} \\
\times \left| D_{y}^{m+1-|p'|} D_{y}^{p'} \hat{u}(\Psi(w)) \right| \\
\times \left| D_{y}^{m+1-|p'|} D_{y}^{p'} \hat{u}(\Psi(w)) \right| \\
\times \left| D_{y}^{m+1-|p'|} D_{y}^{p'} \hat{u}(\Psi(w)) \right| \\
\times \left| D_{y}^{m+1-|p'|} D_{y}^{p'} \hat{u}(\Psi(w)) \right|. 
\]
It follows that for any $p' \geq 0$ with $|p'| \leq m + 1,$
\[
\left| D_{x}^{m+1-|p'|} D_{x}^{p'} u(w) - D_{x}^{m+1-|p'|} D_{x}^{p'} u(z) \right| \leq c \left| D_{y}^{m+1} \hat{u}(\Psi_{z}(w)) - D_{y}^{m+1} \hat{u}(\Psi_{z}(z)) \right|. 
\]
So the claim (4.25) follows. Since $z, w \in Q_{k}^{\delta}$ ($k \in K$) were arbitrary chosen, the lemma holds from (4.25) and Lemma 4.10.

\textbf{Lemma 4.12.} For any $z, w \in Q_{k}^{\delta}$ ($k \in K$), we have that
\[
\left| D_{x}^{m+1} u(w) - D_{x}^{m+1} u(z) \right| \leq c \sum_{|p'|=m} \left( \left| U_{p'}^{\mu} [w, w] - U_{p'}^{\mu} [z, z] \right| + \left| w - z \right|^{\gamma} \left| U_{p'}^{\mu} [z, z] \right| \right) \\
+ c \left| w - z \right|^{\gamma} \sum_{k \in K} \left( \left| u \right|_{C^{\infty}(\hat{\Psi}_{k})} + \left| F \right|_{C^{\infty}(\hat{\Psi}_{k})} \right). 
\]

\textbf{Proof.} With Lemma 4.11, the lemma holds from (4.17) and (4.18) in Lemma 4.2.

We now obtain Proposition 1.7.

\textbf{Proof of Proposition 1.7.} Proposition 1.7 holds by Lemma 4.8 and Lemma 4.12. □
5. Proof of the main theorem

5.1. L²-estimate of $D^{m+1}u$. We obtain $L^2$-type estimate of $D^{m+1}u$. We remark that we have the term $r^\gamma \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}$ in the right-hand side of Lemma 3.18 and Lemma 5.1, but this term $r^\gamma \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}$ will be removed by using the term $r^\gamma$ later in Lemma 5.8 and Lemma 5.9.

To simplify the computation, we set

$$
E = \sum_{k \in K} \left( \|u\|_{C^{m,\gamma}(Q_k^2)} + \|F\|_{C^{m,\gamma}(Q_k^2)} \right).
$$

Then we find from (4.5) and (4.10) that

$$
\sum_{k \in K} \sum_{|p'|=m} \left[ \|G^{p'}\|_{C^{\gamma}(Q_k^2)} + \|v^{p'}\|_{C^{\gamma}(Q_k^2)} \right] \leq E.
$$

To compare $D^{m+1}u$ and $U^{p'}$, we will use (5.3) and (5.4). To perturb the equation, we use (5.6). The test function will be handled by (5.8).

By Lemma 4.8 and (5.1),

$$
|D^{m+1}u(z)| \leq c \left[ E + \sum_{|p'|=m} \left| U^{p'}(z) \right| \right] \leq c \left[ E + |D^{m+1}u(z)| \right] \quad (z \in Q_5).
$$

Also by Lemma 4.12 and (5.1), for any $z, w \in Q_5^k$ with $k \in K$,

$$
|D^{m+1}u(w) - D^{m+1}u(z)| \leq c \left| w - z \right| E + \sum_{|p'|=m} \left| U^{p'}(w) - U^{p'}(z) \right|.
$$

For the simplicity, for any $p' \geq 0'$ with $|p'| = m$, set $\bar{u}^{p'} : Q_5 \to \mathbb{R}^N$ as

$$
\bar{u}^{p'} = \begin{pmatrix}
\bar{u}^{1,p'}_1 & \cdots & \bar{u}^{1,p'}_n \\
\vdots & \ddots & \vdots \\
\bar{u}^{n,p'}_1 & \cdots & \bar{u}^{n,p'}_n
\end{pmatrix}
$$

as in (4.7) so that

$$
\bar{u}^{p'}_{\beta} = \sum_{0' \leq q' \leq p'} \binom{p'}{q'} \pi^{q'} D^{(|q'|,|p' - q'|)} D_{\beta} u^j \quad \text{in} \quad Q_5.
$$

Then by Lemma 3.8, for any $\eta \in C_0^\infty(Q_r(z))$ with $Q_r(z) \subset Q_5$,

$$
\left| \int_{Q_r(z)} \sum_{1 \leq \alpha \leq n} \left[ \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{\alpha j, k} \bar{u}^{j,k}_\beta + \eta^j \right] D_{\alpha} \eta^j - G_1 D_{\eta} \eta^j \left| dx \right| 
\right.
$$

$\leq c r^{n+\frac{1}{2}} \left[ \|D^{m+1}u\|_{L^\infty(Q_r(z))} + \sum_{k \in K} \left( \|u\|_{C^{m,\gamma}(Q_k^2)} + \|F\|_{C^{m,\gamma}(Q_k^2)} \right) \right]
\left. + c r^{\frac{1}{2}} \int_{Q_r(z)} \epsilon |D\eta|^2 + \epsilon^{-2} |\eta|^2 \right] dx,
$$

In view of (4.2), (4.8) tand (4.19), we have that for any $p' \geq 0'$ with $|p'| = m$,

$$
\left| \bar{u}^{p'} \right| \leq c \left[ \left| U^{p'} \right| + \left| v^{p'} \right| + \left| G^{p'} \right| \right] \quad \text{and} \quad \left| U^{p'} \right| \leq c \left[ \left| \bar{u}^{p'} \right| + \left| v^{p'} \right| + \left| G^{p'} \right| \right] \quad \text{in} \quad Q_5.
So it follows from Lemma 4.8 and (5.2) that

$$
|D^{m+1}u(z)| \leq c \left[ E + \sum_{|\alpha'|=m} |\bar{u}_{\alpha'}(z)| \right] \leq c \left[ E + |D^{m+1}u(z)| \right] \quad (z \in Q_5).
$$

For $\eta' : Q_r(z) \to \mathbb{R}^N$ in (3.37) with $Q_{2r}(z) \subset Q_5$, Lemma 3.18 implies that

$$
\int_{Q_r(z)} \left| D_{\beta} \eta' - \bar{u}_{\beta} + [1 - \delta_{\beta}]G_{\beta} \right|^2 \, dx \leq c r^{n+2\gamma} \left[ \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + E^2 \right],
$$

for any $1 \leq \beta \leq n$. Here, with (4.3) and (4.4), we obtain from (1.3), Lemma 3.8 and Lemma 3.19 that

$$
|\eta| \leq c \sum_{k \in K} \left[ \|u\|_{C^{m,\gamma}(Q_{\frac{r}{2}})} + \|F\|_{C^{m,\gamma}(Q_{\frac{r}{2}})} \right] \leq cE \quad \text{in} \ Q_r(z).
$$

**Lemma 5.1.** For any $\phi \in C^\infty_c(Q_r(z))$ with $0 \leq |\phi| \leq 1$, we have that

$$
\int_{Q_r(z)} |D^{m+1}u|^2 \phi^2 \, dx \leq c \left[ r^{n+2\gamma} \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + \int_{Q_r(z)} (1 + |D\phi|)^2 E^2 \, dx \right]
$$

for any $Q_{2r}(z) \subset Q_5$.

*Proof.* Fix $p' \geq 0$ with $|p'| = m$. For the simplicity, let $\bar{u} = \bar{u}_{p'}$ (in (5.5)), $\eta = \eta'$ (in (3.37)) and omit $\sum_{1 \leq i,j \leq N} \sum_{1 \leq \alpha, \beta \leq n}$ in the integral. Then by (5.2) and (5.8),

$$
\int_{Q_r(z)} \left| D_{\beta} \eta - \bar{u}_{\beta} \right|^2 \, dx \leq c r^{2\gamma} \left[ \|D^{m+1}u\|_{L^\infty(Q_r(z))}^2 + E^2 \right].
$$

For (5.6), we choose $\phi^2 \eta$ instead of $\phi$. Since $r \in (0, 1]$ and $0 \leq |\phi| \leq 1$, we obtain from Young’s inequality and (5.9) that for any $\varepsilon \in (0, 1]$,

$$
\left| \int_{Q_r(z)} A_{ij}^\alpha \bar{u}_{\beta} D_{\alpha} \left( [\phi^2 \eta] \right) \, dx \right|
\leq c \int_{Q_r(z)} \left| [G_\beta] + |p'\| \right| D \left( \phi^2 \eta \right) \, dx
+ cr^{2\gamma} \int_{Q_r(z)} \left[ \|D^{m+1}u\|_{L^\infty(Q_r(z))}^2 + E^2 + \varepsilon \left| D \left( \phi^2 \eta \right) \right|^2 + \varepsilon^{-2} \left| \phi^2 \eta \right|^2 \right] \, dx.
$$

By (5.2) and (5.9), $|\phi| + |G_\beta| + |\eta| \leq cE$ in $Q_r(z)$. So by Young’s inequality,

$$
\left| \int_{Q_r(z)} A_{ij}^\alpha \bar{u}_{\beta} D_{\alpha} \left( \phi^2 \eta \right) \, dx \right|
\leq c \int_{Q_r(z)} \left| [G_\beta] + |p'\| \right| D \left( \phi^2 \eta \right) \, dx
\leq c \int_{Q_r(z)} r^{2\gamma} \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + \varepsilon r^{2\gamma} |D\eta|^2 \phi^2 + c(\varepsilon) \left( 1 + |D\phi|^2 \right) E^2 \, dx,
$$

for any $\varepsilon \in (0, 1]$. We estimate the left-hand and right-hand side.

We first estimate the right-hand side. We obtain from Young’s inequality that

$$
\int_{Q_r(z)} |D\eta|^2 \phi^2 \, dx \leq 2 \int_{Q_r(z)} \sum_{1 \leq \beta \leq n} \left| D_{\beta} \eta - \bar{u}_{\beta} \right|^2 \phi^2 \, dx + 2 \int_{Q_r(z)} |\bar{u}|^2 \phi^2 \, dx.
$$

From (5.2), we have that $|G| \leq cE$ in $Q_r(z)$. So by (5.8),

$$
\int_{Q_r(z)} |D\eta|^2 \phi^2 \, dx \leq c \int_{Q_r(z)} r^{2\gamma} \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + |\bar{u}|^2 + E^2 \, dx.
$$
By (5.7), $|\bar{u}| \leq c \left[ |D^{m+1}u| + E \right]$ in $Q_r(z)$. So the right-hand side is estimated as
\[
\int_{Q_r(z)} r^{2\gamma} \|D^{m+1}u\|^2_{L^\infty(Q_{2r}(z))} + \epsilon r^{2\gamma} |D\eta|^2 \phi^2 + c(\epsilon) \left(1 + |D\phi|^2\right) E^2 \, dx
\leq c \int_{Q_r(z)} r^{2\gamma} \|D^{m+1}u\|^2_{L^\infty(Q_{2r}(z))} + \epsilon |\bar{u}|^2 \phi^2 + c(\epsilon) \left(1 + |D\phi|^2\right) E^2 \, dx
\]
We now estimate the left-hand side. By that $D(\phi^2 \eta) = 2\phi \eta D\phi + \phi^2 D\eta$,
\[
\left| \int_{Q_r(z)} A^{\alpha\beta}_{ij} \bar{u}_j^\alpha D_\alpha \left(\phi^2 \eta^i\right) \, dx \right|
\geq \left| \int_{Q_r(z)} A^{\alpha\beta}_{ij} \bar{u}_j^\alpha \phi \left(D_\alpha \eta^i\right) \, dx \right| - \left| \int_{Q_r(z)} A^{\alpha\beta}_{ij} \bar{u}_j^\alpha \left(2\phi D_\alpha \phi \eta^i\right) \, dx \right|
\]
By Young’s inequality, we obtain that
\[
\left| \int_{Q_r(z)} A^{\alpha\beta}_{ij} \bar{u}_j^\alpha \left(2\phi D_\alpha \phi \eta^i\right) \, dx \right| \leq \frac{\lambda}{4} \int_{Q_r(z)} |\bar{u}|^2 \phi^2 \, dx + c \int_{Q_r(z)} |\eta|^2 |D\phi|^2 \, dx.
\]
By combining the above three estimate, it follows from (5.9) and (5.10) that
\[
\left| \int_{Q_r(z)} A^{\alpha\beta}_{ij} \bar{u}_j^\alpha D_\alpha \left(\phi^2 \eta^i\right) \, dx \right|
\geq \frac{\lambda}{4} \int_{Q_r(z)} |\bar{u}|^2 \phi^2 \, dx - c \int_{Q_r(z)} r^{2\gamma} \|D^{m+1}u\|^2_{L^\infty(Q_{2r}(z))} + \left(1 + |D\phi|^2\right) E^2 \, dx,
\]
and the estimate for the left-hand side is obtained. So by combining the estimates for left-hand side and right-hand side, we obtain that
\[
\frac{\lambda}{4} \int_{Q_r(z)} |\bar{u}|^2 \phi^2 \, dx \leq c_1 \int_{Q_r(z)} r^{2\gamma} \|D^{m+1}u\|^2_{L^\infty(Q_{2r}(z))} + \epsilon |\bar{u}|^2 \phi^2 + c(\epsilon) \left(1 + |D\phi|^2\right) E^2 \, dx,
\]
for some universal constant $c_1 \geq 1$. Let $\epsilon \in (0, 1]$ be a small universal constant satisfying that $c_1 \epsilon \leq \lambda/8$. Then we have that
\[
\int_{Q_r(z)} |\bar{u}|^2 \phi^2 \, dx \leq c \int_{Q_r(z)} r^{2\gamma} \|D^{m+1}u\|^2_{L^\infty(Q_{2r}(z))} + \left(1 + |D\phi|^2\right) E^2 \, dx.
\]
From (5.7), we have that $|D^{m+1}u| \leq c \left[ |\bar{u}| + E \right]$ in $Q_r(z)$. So the lemma follows. \(\square\)

5.2. Comparison estimate. Fix $Q_{2r}(z) \subset Q_5$ and $p' \geq 0'$ with $|p'| = m$. Suppose that $v \in W^{1,2}(Q_r(z))$ is the weak solution of
\[
D_\alpha \left[ A^{\alpha\beta}_{ij} D_\beta v^i \right] = D_\alpha \left[ A^{\alpha\beta}_{ij} [1 - \delta_{\beta 1}] G^j_{\beta'} + \delta_{\alpha 1} G^i_{1 \beta'} - \nu^i_{\alpha \beta'} \right] \text{ in } Q_r(z),
\]
on $\partial Q_r(z)$,
Lemma 5.2. There exists a small universal constant \( R_1 \in (0, 1] \) such that if \( r \in (0, R_1] \) then for any \( p' \geq 0' \) with \( |p'| = m \) and \( 1 \leq \beta \leq n \), we have that

\[
\int_{Q_r(z)} \left| \sum_{\nu' \leq q' \leq p'} \left( \frac{p'}{q'} \right)^{\beta} D^{|q' - q'|} D_{\beta} u - D_{\beta} v - \frac{1}{2} \right|_{L^\infty(\Omega)}^2 \leq c \gamma \left\| D^{m+1} u \right\|_{L^\infty(\Omega)}^2 + E^2.
\]

Proof. Fix \( p' \geq 0' \) with \( |p'| = m \). Recall \( \tilde{u} = \tilde{u}^{p'} \) in (5.5). We find from (5.6) that

\[
\int_{Q_r(z)} A^{\alpha \beta}_{ij} \tilde{u}^{i,j}_{\alpha} D_{\alpha} \phi^i + G^{i,p'}_{\alpha} D_1 \phi^i + \nu_{\alpha}^{i,p'} D_{\alpha} \phi^i \leq c \gamma \int_{Q_r(z)} \left\| D^{m+1} u \right\|_{L^\infty(\Omega)}^2 + E^2 + |D\phi|^2 dx,
\]

for any \( \phi \in C_C^\infty(\Omega) \). From (5.12), we find that

\[
\int_{Q_r(z)} A^{\alpha \beta}_{ij} \left( D_\beta v^j - \frac{1}{2} \right) G^{i,p'}_{\alpha} + \frac{1}{2} + \nu_{\alpha}^{i,p'} D_{\alpha} \phi^i dx = 0,
\]

for any \( \phi \in C_C^\infty(\Omega) \). So it follows from (5.1) and (5.3) that

\[
\int_{Q_r(z)} A^{\alpha \beta}_{ij} \left( \tilde{u}^i_{\alpha} - D_\beta v^j + \frac{1}{2} \right) G^{i,p'}_{\alpha} \leq c \gamma \left\| D^{m+1} u \right\|_{L^\infty(\Omega)}^2 + E^2 + |D\phi|^2 dx,
\]

for any \( \phi \in C_C^\infty(\Omega) \). From (5.8), we obtain that

\[
\int_{Q_r(z)} |D_{\alpha} \eta^i - \tilde{u}^i_{\alpha} + \frac{1}{2} \right|_{L^\infty(\Omega)}^2 \leq c \gamma \left( \sum_{|p'| = m} \left\| U_{p'} \right\|_{L^\infty(\Omega)}^2 + E^2 \right).
\]

Take \( \phi = \eta - v \). Then by using Young's inequality and the elliptic condition,

\[
\int_{Q_r(z)} \left| \tilde{u}^i_{\alpha} - D_\beta v^j - \frac{1}{2} \right|_{L^\infty(\Omega)}^2 dx \leq c \gamma \left[ \int_{Q_r(z)} \left\| D^{m+1} u \right\|_{L^\infty(\Omega)}^2 + E^2 \right] + c \gamma \int_{Q_r(z)} \left| \tilde{u}^i_{\alpha} + \frac{1}{2} \right|_{L^\infty(\Omega)}^2 dx.
\]

So with (5.5), for the lemma holds for sufficiently small \( R_1 > 0 \). \( \square \)

5.3. Excess decay estimate. We obtain the excess decay estimate of \( U_{p'} \) in Lemma 5.5. Then by Campanato type embedding, we show Hölder continuity of \( U_{p'} \). In the proof of Lemma 5.5, we use the following variation of the technical lemma in [5, Lemma 3.4].

Lemma 5.3. Let \( \phi(t) \) be a nonnegative function on \([0, R]\). If

\[
\phi(p) \leq A \left( \frac{p}{r} \right)^\alpha \phi(r) + Br^\beta \left( \frac{p}{r} \right)^\gamma
\]

holds for any \( 0 < r \leq R \) with \( A, B, \alpha, \beta, \gamma \) nonnegative constants and \( \beta < \alpha \). Then for any \( \gamma \in (\beta, \alpha) \), there exist a positive constant \( c \) depending on \( n, A, \alpha, \beta, \gamma \) such that

\[
\phi(p) \leq c \left( \frac{p}{r} \right)^\gamma \phi(r) + B r^\beta,
\]

for all \( 0 < r \leq p \leq R \).
We have the following excess decay estimate for the reference equations from [1].

**Lemma 5.4.** For $Q_{2r}(z) \subset Q_5$, suppose that $F \in C^0(Q^E_k(z), \mathbb{R}^n)$ for any $k \in K$. Then for the weak solution $v$ of

$$D_\alpha \left[ A_{ij}^{\alpha \beta} D_{\beta} v^j \right] = D_\alpha \tilde{F}_\alpha^i \text{ in } Q_r(z),$$

and $V : Q_r(z) \rightarrow \mathbb{R}^{Nn}$ defined as

$$V = \left( \sum_{1 \leq j \leq N} \sum_{1 \leq \alpha \leq n} \pi_\alpha \left[ \sum_{1 \leq \beta \leq N} \sum_{1 \leq \gamma \leq n} A^{\alpha \beta}_{ij} D_{\beta} v^i - \tilde{F}_\alpha^i \right], D_x v^i + \pi' D_1 v^i \right),$$

we have that for any $0 < \rho \leq r$,

$$\frac{1}{\rho^{2n}} \int_{Q_\rho(z)} |V - (V)_{Q_r(z)}|^2 \, dx \leq c \left( \frac{1}{r^{2n}} \int_{Q_r(z)} |V - (V)_{Q_r(z)}|^2 \, dx + \int_{Q_r(z)} |V|^2 \, dx + \sup_{k \in K} \|F\|_{C^0(Q^E_k(z))}^2 \right).$$

**Proof.** Take $R = 3$ in Theorem 1.7 in [1], then the lemma follows. \qed

**Lemma 5.5.** For any $p' \geq 0'$ with $|p'| = m$, if $Q_{2r}(z) \subset Q_5$ and $0 < \rho \leq 2r$ then

$$\frac{1}{\rho^{2n}} \int_{Q_\rho(z)} \left| U^{p'} - (U^{p'})_{Q_r(z)} \right|^2 \, dx \leq c \left[ \left( \frac{1}{r^{2n}} \int_{Q_r(z)} \left| U^{p'} - (U^{p'})_{Q_r(z)} \right|^2 \, dx + \|D^{m+1} u\|_{L^\infty(Q_{2r}(z))}^2 \right) \right].$$

**Proof.** We only need to prove the case that $0 < r \leq R_1$ where $R_1 \in (0, 1]$ is a universal constant chosen in Lemma 5.2. Otherwise, we have that $0 < \rho \leq R_1 < r(\leq 7)$ or $R_1 < \rho \leq r(\leq 7)$ in which case the lemma can be obtained by using the case that $0 < r \leq R_1$ and that $R_1 \in (0, 1]$ is a universal constant.

We assume that $0 < \rho \leq r$, otherwise we have that $r < \rho \leq 2r$ and one can easily prove the lemma. Fix $p' \geq 0'$ with $|p'| = m$ and $0 < \rho \leq r$. Let $v' \in W^{1,2}(Q_{2r}(z))$ be the weak solution of

$$\begin{cases}
D_\alpha \left[ A_{ij}^{\alpha \beta} D_{\beta} v' \right] = D_\alpha \left[ -A_{ij}^{\alpha \beta} [1 - \delta_{ij}] G^{p'}_{ij} \right] + \delta_{i1} G^{p'}_{1} - \nu_\alpha^{i,p'} \in C^0(Q^E_k(z), \mathbb{R}^n) \text{ in } Q_r(z), \\
v = \eta' \quad \text{on } \partial Q_r(z),
\end{cases}$$

where $\eta$ is defined in (3.37). Here, we have from (5.2) that

(5.13) \hspace{1cm} \tilde{F}_\alpha^i := -A_{ij}^{\alpha \beta} [1 - \delta_{ij}] G^{p'}_{ij} + \delta_{i1} G^{p'}_{i} - \nu_\alpha^{i,p'} \in C^0(Q^E_k(z), \mathbb{R}^n)

with the estimate

(5.14) \hspace{1cm} \|\tilde{F}_\alpha^i\|_{C^0(Q^E_k)} \leq \left[ -A_{ij}^{\alpha \beta} [1 - \delta_{ij}] G^{p'}_{ij} + \delta_{i1} G^{p'}_{i} - \nu_\alpha^{i,p'} \right]_{C^0(Q^E_k(z))} \leq E.

for any $k \in K$ and $i = 1, 2, \ldots, n$. So by Lemma 5.4 and (5.1),

$$\frac{1}{\rho^{2n}} \int_{Q_\rho(z)} \left| U^{p'} - (U^{p'})_{Q_r(z)} \right|^2 \, dx \leq c \left[ \left( \frac{1}{r^{2n}} \int_{Q_r(z)} \left| U^{p'} - (U^{p'})_{Q_r(z)} \right|^2 \, dx + \int_{Q_r(z)} |V|^2 \, dx + \sup_{k \in K} \|F\|_{C^0(Q^E_k(z))}^2 \right) \right].$$

where

$$V^{i,p'} = \left( \sum_{1 \leq \alpha \leq n} \pi_\alpha \left[ \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{ij}^{\alpha \beta} D_{\beta} v' - \tilde{F}_\alpha^i \right], D_x v' + \pi' D_1 v' \right),$$

\(\)
in $Q_{r}(z)$. We compare $U^{p'}$ and $V^{p'}$ as follows. By (4.8) and (4.9),
\[ U_{1}^{i,p'} = G_{1}^{i,p'} + \sum_{1 \leq \alpha \leq n} \pi_{\alpha} \left[ v_{\alpha}^{1,p'} + \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{ij}^{\alpha \beta} \tilde{u}_{\beta}^{i,p'} \right] \quad \text{in } Q_{5}, \]
and
\[ U_{\beta}^{j,p'} = G_{\beta}^{j,p'} + \tilde{u}_{\beta}^{j,p'} + \pi_{\beta} \tilde{u}_{1}^{j,p'} \quad \text{in } Q_{5}, \]
for any $2 \leq \beta \leq n$ and $1 \leq j \leq N$. Since $\pi_{1} = -1$, we obtain from (5.13) that
\[ U_{1}^{i,p'} - V_{1}^{i,p'} = G_{1}^{i,p'} + \sum_{1 \leq \alpha \leq n} \pi_{\alpha} \left[ v_{\alpha}^{1,p'} + \sum_{1 \leq j \leq N} \sum_{1 \leq \beta \leq n} A_{ij}^{\alpha \beta} \tilde{u}_{\beta}^{i,p'} - D_{\beta} v^{j} \right] + F_{1} \]
and
\[ U_{\beta}^{j,p'} - V_{\beta}^{j,p'} = G_{\beta}^{j,p'} + \tilde{u}_{\beta}^{j,p'} + \pi_{\beta} \tilde{u}_{1}^{j,p'} - D_{\beta} v^{j} = \tilde{u}_{\beta}^{j,p'} - D_{\beta} v^{j} - [1 - \delta_{\beta 1}] G_{\beta}^{j,p'} \]
for any $2 \leq \beta \leq n$ and $1 \leq i \leq N$. Since $0 < r \leq R_{1}$, we have from Lemma 5.2 that
\[ \int_{Q_{r}(z)} \left| D_{\alpha} v^{\prime} - \tilde{u}_{\alpha}^{\prime} + [1 - \delta_{\alpha 1}] G_{\alpha}^{\prime} \right|^{2} dx \leq cr^{n+2\gamma} \left[ \|D^{m+1} u\|_{L^{\infty}(Q_{r}(z))} + E^{2} \right], \]
for any $1 \leq \alpha \leq n$. So we obtain that
\[ \int_{Q_{r}(z)} \left| U^{p'} - V^{p'} \right|^{2} dx \leq cr^{n+2\gamma} \left[ \|D^{m+1} u\|_{L^{\infty}(Q_{r}(z))} + E^{2} \right]. \]
Thus
\[ \int_{Q_{r}(z)} \left| U^{p'} - (U^{p'})_{Q_{r}(z)} \right|^{2} dx \]
\[ \leq c \left( \frac{\rho}{r} \right)^{2\gamma} \int_{Q_{r}(z)} \left| U^{p'} - (U^{p'})_{Q_{r}(z)} \right|^{2} dx \]
\[ + c r^{2\gamma} \left( \frac{\rho}{r} \right)^{n} \left( \int_{Q_{r}(z)} \left| U^{p'} \right|^{2} dx + \|D^{m+1} u\|_{L^{\infty}(Q_{r}(z))} + E^{2} \right). \]
By (5.3) and that
\[ \int_{Q_{r}(z)} \left| U^{p'} - (U^{p'})_{Q_{r}(z)} \right|^{2} dx \leq c \int_{Q_{r}(z)} \left| U^{p'} - (U^{p'})_{Q_{r}(z)} \right|^{2} dx, \]
\[ \int_{Q_{r}(z)} \left| U^{p'} - (U^{p'})_{Q_{r}(z)} \right|^{2} dx \leq c \left( \frac{\rho}{r} \right)^{2\gamma} \int_{Q_{r}(z)} \left| U^{p'} - (U^{p'})_{Q_{r}(z)} \right|^{2} dx \]
\[ + c r^{2\gamma} \left( \frac{\rho}{r} \right)^{n} \left[ \|D^{m+1} u\|_{L^{\infty}(Q_{r}(z))} + E^{2} \right]. \]
Since $0 < \rho \leq r$ was arbitrary chosen, the lemma follows from Lemma 5.3. \hfill \Box

So with Campanato type embedding, see for instance [5, Theorem 3.1] or [6, Theorem 2.9], we obtain Hölder continuity of $U$.

**Proposition 5.6.** Suppose that $h \in L^{2}(Q_{r}(z))$ satisfies
\[ \int_{Q_{r}(y)} |h - (h)_{Q_{r}(y)}|^{2} dx \leq M^{2} \rho^{n+2\gamma} \quad (y \in Q_{r}(z), \ \rho \in (0, r]) \]
for some $\gamma \in (0, 1)$. Then we have that $|h|_{C^{\gamma}(Q_{r}(z))} \leq cM$. 

The proof of Lemma 5.7 is an obvious application of Proposition 5.6. In Lemma 5.7, the term \( cr^{2\gamma} \sum_{|q'|=m} \|U^{q'}\|_{L^\infty(Q_{2r}(z))}^2 \) exists but we can handle it by making \( cr^{2\gamma} \) sufficiently small. (See Lemma 5.8)

**Lemma 5.7.** For any \( p' \geq 0' \) with \( |p'| = m \) and \( Q_{2r}(z) \subset Q_5 \), we have that

\[
\|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 \leq c \left( \int_{Q_{2r}(z)} |D^{m+1}u|^2 \, dx + r^{2\gamma} \left( \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + E^2 \right) \right)
\]

and

\[
\left[ U^{p'} \right]_{C^\gamma(Q_{r}(z))}^2 \leq c \left[ \int_{Q_{r}(y)} \left| U^{p'} - (U^{p'})_{Q_{r}(y)} \right|^2 \, dx + \|D^{m+1}u\|_{L^\infty(Q_{r}(y))}^2 + E^2 \right].
\]

**Proof.** Fix \( p' \geq 0' \) with \( |p'| = m \). Choose \( y \in Q_r(z) \) and \( \rho \in (0, r] \). By Lemma 5.5,

\[
\frac{1}{\rho^{2\gamma}} \int_{Q_{\rho}(y)} \left| U^{p'} - (U^{p'})_{Q_{\rho}(y)} \right|^2 \, dx \leq c \left[ \int_{Q_{\rho}(y)} \left| U^{p'} - (U^{p'})_{Q_{\rho}(y)} \right|^2 \, dx + \|D^{m+1}u\|_{L^\infty(Q_{\rho}(y))}^2 + E^2 \right].
\]

Since \( y \in Q_{r}(z) \), we have that

\[
\int_{Q_{r}(y)} \left| U^{p'} - (U^{p'})_{Q_{r}(y)} \right|^2 \, dx \leq c \int_{Q_{2r}(z)} \left| U^{p'} - (U^{p'})_{Q_{2r}(z)} \right|^2 \, dx.
\]

Thus

\[
\frac{1}{\rho^{2\gamma}} \int_{Q_{\rho}(y)} \left| U^{p'} - (U^{p'})_{Q_{\rho}(y)} \right|^2 \, dx \leq c \left[ \frac{1}{\rho^{2\gamma}} \int_{Q_{2r}(z)} \left| U^{p'} - (U^{p'})_{Q_{2r}(z)} \right|^2 \, dx + \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + E^2 \right].
\]

Since \( y \in Q_{r}(z) \) and \( \rho \in (0, r] \) was arbitrary chosen, by taking

\[
M = c \left[ \frac{1}{\rho^{2\gamma}} \int_{Q_{2r}(z)} \left| U^{p'} - (U^{p'})_{Q_{2r}(z)} \right|^2 \, dx + \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + E^2 \right]
\]

in Proposition 5.6, we obtain that

\[
\left[ U^{p'} \right]_{C^\gamma(Q_{r}(z))}^2 \leq c \left[ \frac{1}{\rho^{2\gamma}} \int_{Q_{2r}(z)} \left| U^{p'} - (U^{p'})_{Q_{2r}(z)} \right|^2 \, dx + \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + E^2 \right].
\]

Then from the following inequality

\[
\|U^{p'}\|_{L^\infty(Q_{r}(z))} \leq \left| (U^{p'})_{Q_{r}(z)} \right| + cr^{\gamma} \left[ U^{p'} \right]_{C^\gamma(Q_{r}(z))},
\]

we get that

\[
\|U^{p'}\|_{L^\infty(Q_{r}(z))}^2 \leq c \left[ \int_{Q_{2r}(z)} \left| U^{p'} \right|^2 \, dx + r^{2\gamma} \left( \|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 + E^2 \right) \right].
\]

Since \( p' \geq 0' \) with \( |p'| = m \) was arbitrary chosen, with (5.3), the lemma holds from (5.15) and (5.16).

**Lemma 5.8.** For any \( Q_{2r}(z) \subset Q_5 \), we have that

\[
\|D^{m+1}u\|_{L^\infty(Q_{2r}(z))}^2 \leq c \left[ \int_{Q_{2r}(z)} |D^{m+1}u|^2 \, dx + E^2 \right].
\]
Proof. Fix $r \leq \rho < \tau \leq 2r$ and $p' \geq 0'$ with $|p'| = m$. Then by Lemma 5.7,
\[
\| D^{m+1}u \|^2_{L^\infty(Q_{\rho}(y))} 
\leq c \left[ \int_{Q_{\rho}(y)} |D^{m+1}u|^2 \, dx + (\tau - \rho)^\gamma \left[ \| D^{m+1}u \|^2_{L^\infty(Q_{\tau-\rho}(y))} + E^2 \right] \right],
\]
for any $y \in Q_\rho(z)$. For any $y \in Q_\rho(z)$, we have that $Q_{\tau-\rho}(y) \subset Q_\tau(z) \subset Q_{2r}(z)$. Thus
\[
\| D^{m+1}u \|^2_{L^\infty(Q_{\rho}(y))} 
\leq c \left[ \tau^\gamma \left( \| D^{m+1}u \|^2_{L^\infty(Q_\tau(z))} + E^2 \right) + \frac{1}{(\tau - \rho)^n} \int_{Q_{2r}(z)} |D^{m+1}u|^2 \, dx \right].
\]
Then there exists a small universal constant $\bar{r} \in (0, 3]$ such that if $r \in (0, \bar{r}]$ then
\[
\| D^{m+1}u \|^2_{L^\infty(Q_{\rho}(z))} \leq \frac{1}{2} \| D^{m+1}u \|^2_{L^\infty(Q_\tau(z))} + c \left[ \frac{1}{(\tau - \rho)^n} \int_{Q_{2r}(z)} |D^{m+1}u|^2 \, dx + E^2 \right].
\]
Since $r \leq \rho \leq \tau \leq 2r$ was arbitrary chosen the lemma follows by using a technical argument [5, Lemma 4.3] or [6, Lemma 4.1], we obtain that
\[
\| D^{m+1}u \|^2_{L^\infty(Q_{\rho}(z))} \leq c \left[ \int_{Q_{2r}(z)} |D^{m+1}u|^2 \, dx + E^2 \right],
\]
for any $Q_{2r}(z) \subset Q_\rho$ with $r \in (0, \bar{r}]$.

Also there exists a universal constant $\lambda \in \mathbb{N}$ such that if $\bar{r} \leq \tau \leq 7$ then $Q_\tau(z)$ can be covered with $\lambda$ cubes $\{Q_{\tau}(z_1), \cdots, Q_{\tau}(z_\lambda)\}$ satisfying that $Q_{2r}(z_1), \cdots, Q_{2r}(z_\lambda) \subset Q_{2r}(z)$. So the lemma even holds when $r \geq \bar{r}$.

By using covering argument in [6, Corollary 6.1], we obtain the following result.

Lemma 5.9. For any $Q_{2r}(z) \subset Q_\rho$, we have that
\[
\| D^{m+1}u \|^2_{L^\infty(Q_{2r}(z))} \leq c r^{-2} E^2.
\]

Proof. We claim that
\[
\| D^{m+1}u \|^2_{L^\infty(Q_\tau(z))} \leq c r^{-2} E^2 \quad (Q_{2r}(z) \subset Q_\rho).
\]
If we show (5.17) then the lemma holds by a covering argument.

Fix $r \leq \rho \leq \tau \leq 2r$ and $y \in Q_\rho$. Let $\phi \in C^\infty_c(Q_{\rho}(y))$ be cut-off function with
\[
0 \leq \phi \leq 1, \quad \phi = 1 \text{ in } Q_{\rho}(y) \quad \text{and} \quad |D\phi| \leq c(\tau - \rho)^{-1}.
\]
Since $Q_{\tau-\rho}(y) \subset Q_\rho$, from Lemma 5.1 and Lemma 5.8, we have that
\[
\| D^{m+1}u \|^2_{L^\infty(Q_{\tau-\rho}(y))} \leq c \left[ \int_{Q_{\rho}(y)} |D^{m+1}u|^2 \, dx + E^2 \right] 
\leq c \left[ (\tau - \rho)^\gamma \| D^{m+1}u \|^2_{L^\infty(Q_{\tau-\rho}(y))} + (\tau - \rho)^{-2} E^2 \right].
\]
Since $y \in Q_\rho(z)$ was arbitrary chosen and $Q_{\tau-\rho}(y) \subset Q_\tau(z)$, we find that
\[
\| D^{m+1}u \|^2_{L^\infty(Q_{\tau}(z))} \leq c \left[ (\tau - \rho)^\gamma \| D^{m+1}u \|^2_{L^\infty(Q_{\tau}(z))} + (\tau - \rho)^{-2} E^2 \right].
\]
For a sufficiently small universal constant $\bar{r} \in (0, 1]$, if $r \in (0, \bar{r}]$ then
\[
(5.18) \quad \| D^{m+1}u \|^2_{L^\infty(Q_{\tau}(z))} \leq \frac{1}{2} \| D^{m+1}u \|^2_{L^\infty(Q_{\tau}(z))} + c(\tau - \rho)^{-2} E^2,
\]

because of that \( r \leq \rho < \tau \leq 2r \). Since \( r \leq \rho < \tau \leq 2r \) was arbitrary chosen, by using a technical argument \([5, \text{Lemma 4.3}]\) or \([6, \text{Lemma 4.1}]\),

\[
(5.19) \quad \left\| D^{m+1}u \right\|_{L^\infty(Q_r(z))}^2 \leq cr^{-2}E^2 \quad (Q_{2r}(z) \subset Q_5 \text{ with } r \in (0, \bar{r})).
\]

So the claim \((5.17)\) holds when \( r \in (0, \bar{r}) \).

Since \( \bar{r} \in (0, 1] \) is a universal constant the claim \((5.17)\) when \( r > \bar{r} \) can be proved by a covering argument with \((5.19)\) because \( \bar{r} \in (0, 1] \) is a universal constant. \(\square\)

We are now ready to prove our main theorems.

**Proof of Theorem 1.6.** By using Proposition 1.7, we obtain from Lemma 5.7, Lemma 5.8 and Lemma 5.9 that

\[
\left\| U^{p'} \right\|_{L^\infty(Q_r(z))}^2 \leq cr^{-2}E^2,
\]

\[
\left\| U^{p'} \right\|_{L^\infty(Q_r(z))}^2 \leq c \left[ \frac{1}{r^2} \int_{Q_{2r}(z)} \left| U^{p'} \right|^2 \, dx + E^2 \right]
\]

and

\[
\left[ U^{p'} \right]_{C^1(Q_r(z))}^2 \leq c \left[ \frac{1}{r^2} \int_{Q_{2r}(z)} \left| U^{p'} - \left( U^{p'} \right)_{Q_{2r}(z)} \right|^2 \, dx + \sum_{|q'| = m} \left\| U^{q'} \right\|_{L^\infty(Q_{2r}(z))}^2 + E^2 \right].
\]

for any \( p' \geq 0' \) with \( |p'| = m \) and \( Q_{2r}(z) \subset Q_5 \). So Theorem 1.6 holds. \(\square\)

**Proof of Theorem 1.8.** We find from Theorem 1.6 and Proposition 1.7 that

\[
\frac{1}{|Q_r|} \int_{Q_r(z)} \left| D^{m+1}u \right|^2 \, dx \leq cr^{-2}E^2,
\]

\[
\left\| D^{m+1}u \right\|_{L^\infty(Q_r^\infty(z))}^2 \leq c \left[ \sum_{l \in K} \frac{1}{|Q_{2r}|} \int_{Q_{2r}(z)} \left| D^{m+1}u \right|^2 \, dx + E^2 \right]
\]

and

\[
\left[ D^{m+1}u \right]_{C^1(Q_r^\infty(z))}^2 \leq c \left[ \frac{1}{r^2} \sum_{l \in K} \int_{Q_{2r}(z)} \left| D^{m+1}u \right|^2 \, dx + E^2 \right],
\]

for any \( Q_{2r}(z) \subset Q_5 \) and \( k \in K \). So Theorem 1.8 holds. \(\square\)

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