Peaks within peaks and the possible two-peak structure of the $P_c(4457)$: the effective field theory perspective

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The LHCb pentaquarks — the $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$ — have been theorized to be $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ S-wave molecules. Here we explore the possibility that two of these pentaquarks — the $P_c(4440)$ and $P_c(4457)$ — contain in addition a $\Lambda(2595)\bar{D}$ component in P-wave. We will analyze the effects of this extra channel within two effective field theories: the first one will be a standard contact-range effective field theory and the second one will include the non-diagonal pion dynamics connecting the $\Sigma_c\bar{D}^*$ and $\Lambda(2595)\bar{D}$ channels, which happens to be unusually long-ranged. The impact of the coupled-channel dynamics between the $\Sigma_c\bar{D}^*$ and $\Lambda(2595)\bar{D}$ components is modest at best for the $P_c(4440)$ and $P_c(4457)$, which will remain to be predominantly $\Sigma_c\bar{D}^*$ molecules. However, if the quantum numbers of the $P_c(4457)$ are $J^P = \frac{1}{2}^-$, the coupled-channel dynamics is likely to induce the binding of a $\Lambda(2595)\bar{D}$ S-wave molecule (coupled to $\Sigma_c\bar{D}^*$ in P-wave) with $J^P = \frac{1}{2}^+$ and a mass similar to the $P_c(4457)$. If this is the case, the $P_c(4457)$ could actually be a double peak containing two different pentaquark states.

I. INTRODUCTION

The discovery of three pentaquark peaks — the $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$ — by the LHCb collaboration [1] raises the question of what is their nature. A commonly invoked explanation is that they are $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ bound states [2–9], which comes naturally from the closeness of the pentaquark peaks to the corresponding baryon-meson thresholds and also from the existence of theoretical predictions predating their observation [10–16]. Yet the evidence that they are molecular is mostly circumstantial at the moment and other explanations might very well be possible [17–21].

In this manuscript we will explore a modified molecular interpretation of the $P_c(4440)$ and $P_c(4457)$ pentaquarks and the consequences it entails. Of course the fundamental idea will still be that these two pentaquarks are hadronic bound states, but besides the standard $\Sigma_c\bar{D}^*$ interpretation we will also consider the existence of a $\Lambda_c(2595)\bar{D}$ ($\Lambda_c\bar{D}$ from now on) component for the $P_c(4440)$ and $P_c(4457)$. In the isospin-symmetric limit the $\Sigma_c\bar{D}^*$ and $\Lambda_c\bar{D}$ threshold are located at 4462.2 and 4459.5 MeV, respectively, very close to the masses of the $P_c(4440)$ and $P_c(4457)$. Thus it is natural to wonder whether the $\Lambda_c\bar{D}$ channel plays a role in the description of the pentaquarks.

This idea was originally proposed by Burns [8], who conjectured that the $\Lambda_c\bar{D}$ component might be important for the binding of molecular pentaquarks. Later it was realized that the pion-exchange dynamics mediating the $D^*\Sigma_c \rightarrow \Lambda_c\bar{D}$ transition is unusually long-ranged and in practice takes the form of a $1/r^2$ potential [22]. This is indeed a really interesting potential in the sense that it can display discrete scale invariance when attractive enough [23–25], which in turn opens the possibility of the existence of hadronic molecules for which there is a geometric spectrum reminiscent of the Efimov effect in the three-boson system [26]. For the hidden charm pentaquarks the strength of the $1/r^2$ potential is probably not enough to trigger a geometric molecular spectrum [22], yet this might very well happen in other two-hadron molecular systems. Recently, Burns and Swanson have considered the $D^*\Sigma_c \rightarrow \Lambda_c\bar{D}$ pion-exchange dynamics beyond its long-distance behavior, leading to the conclusion that the $P_c(4457)$ might not be a $\frac{1}{2}^− D^*\Sigma_c$ S-wave molecular state but a $\frac{1}{2}^+ D\Lambda_c$ one instead [8].

The present manuscript delves further into the consequences that a $\Lambda_c\bar{D}$ component will have for the pentaquark spectrum. For this we formulate two effective field theories (EFTs): a pionless EFT and a half-pionful EFT. By half-pionful we denote an EFT which includes the unusually long-ranged pion dynamics of the $D^*\Sigma_c \rightarrow \Lambda_c\bar{D}$ transition, for which the characteristic length scale is between 10 and 20 fm, but does not include the pion dynamics of the $\bar{D}\Sigma_c$ system, which has a range in between 1 and 2 fm. We find that the addition of the $\Lambda_c\bar{D}$ channel is inconsequential if the quantum numbers of the $P_c(4440)$ and $P_c(4457)$ molecular pentaquarks are $\frac{1}{2}^−$ and $\frac{3}{2}^−$, respectively. However, if the quantum numbers of the $P_c(4457)$ pentaquark are $\frac{3}{2}^−$ instead, then the existence of a partner state with a similar

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mass and quantum numbers $\frac{1}{2}^+$ is very likely. That is, the $P_c(4457)$ might be a double peak, as happened with the original $P_c(4450)$ pentaquark discovered in 2015.

The manuscript is structured as follows. In Sect. II, we explain how to describe the $D\Sigma_c^+$, $D\Sigma_c^*$ and $D\Lambda_{c1}$ interactions within a pionless contact-range EFT. In Sect. III, we introduce the half-pionful theory, in which we include the pion exchange transition potential in the $D\Sigma_c^* - D\Lambda_{c1}$ channel. In Sect. IV, we revisit the description of the pion exchange transition potential in the $D\Sigma_c^* - D\Lambda_{c1}$ channel. In Sect. V, we present our conclusions.

II. PIONLESS THEORY

In this section we will derive the lowest-order contact-range interaction for the $D\Sigma_c^*$-$D\Lambda_{c1}$ system. For this we will find convenient to use the light-quark notation explained in detail in Ref. [27], but which has been previously used in the literature, e.g. in Refs. [16, 28]. In contrast with the standard superfield notation (see for instance Ref. [23] for a clear exposition) in which we combine heavy hadrons with the same light-quark spin into a unique superfield, in the light-quark notation we simply write the interactions in terms of the light-quark spin degrees of freedom within the heavy hadrons. Of course both notations are equivalent, but for non-relativistic problems the light-quark notation is easier to use.

A. The $D\Sigma_c$ and $D^*\Sigma_c$ channels

The $\bar{D}$ and $D^*$ charmed antimesons are $\bar{Q}q$ states where the light-quark $q$ and heavy antiquark $\bar{Q}$ are in S-wave. From heavy-quark spin symmetry (HQSS) we expect the heavy-antiquark to effectively behave as a static color source, which in practical terms means that the wave function of the light quark is independent of the total spin of the S-wave heavy meson. That is, the light-quark wave function (the "brown muck") of the $\bar{D}$ and $D^*$ charmed antimesons is the same (modulo corrections coming from the heavy-antiquark mass $m_Q$, which scale as $\Lambda_{QCD}/m_Q$, with $\Lambda_{QCD} \sim (200 - 300)$ MeV the QCD scale). Two possible formalisms to express this symmetry are the standard heavy-superfield notation and the light-quark notation. In the former, we combine the $\bar{D}$ and $D^*$ field into a single superfield $[23]$

$$ H_{\bar{Q}} = \frac{1}{\sqrt{2}} \left[ \bar{D} + \sigma \cdot \vec{D}^* \right], $$

where the superfield is well-behaved with respect to heavy-antiquark rotations

$$ H_{\bar{Q}} \rightarrow e^{iS_{\mu} \theta} H_{\bar{Q}}, $$

with $S_{\mu}$ representing the heavy-antiquark spin operator and $\theta$ the rotation axis and angle. Thus the combination of $H_{\bar{Q}}^L$ and $H_{\bar{Q}}$ superfields in the Lagrangian effectively results in invariance with respect to heavy-antiquark rotations, i.e. to heavy-antiquark spin.

Conversely, in the light-subfield (or light-quark) notation, we prescind of writing down the heavy antiquark explicitly and instead express everything in terms of the effective light-quark degrees of freedom within the charmed antimeson and the light quark spin operator:

$$ \bar{D}, \bar{D}^* \rightarrow q_L, \vec{S}_L, $$

where $q_L$ represents an effective light-quark subfield, i.e. a field with the quantum numbers of the light quark within the charmed antimeson [1]. Then we write down explicit rules for transforming the light-quark spin operator into charmed antimeson spin operators

$$ \langle \bar{D} | \vec{S}_L | \bar{D} \rangle = 0, $$

$$ \langle \bar{D} | \vec{S}_L | \bar{D}^* \rangle = \vec{c}_1, $$

$$ \langle \bar{D}^* | \vec{S}_L | \bar{D} \rangle = \vec{S}_1, $$

where $\vec{c}_1$ is the polarization vector of the $\bar{D}^*$ meson and $\vec{S}_1$ the spin-1 matrices.

Regarding the $\Sigma_c$ and $\Sigma_c^*$ charmed baryons, their quark content is $Qqq$ where the qq diquark has light spin $\vec{S}_L = 1$ and the system is in S-wave. The structure of the S-wave charmed baryons is independent of whether the baryon spin is $S = \frac{1}{2} (\Sigma_c)$ or $\frac{3}{2} (\Sigma_c^*)$. In the standard heavy-superfield notation this is taken into account by defining the superfield $[31]$

$$ \vec{S}_Q = \frac{1}{\sqrt{3}} \vec{S}_{\Sigma_c} + \vec{S}_{\Sigma_c^*}, $$

which has good heavy-quark rotation properties, while in the light-quark notation we simply write everything in terms of the light-diquark subfield $a_L$ (i.e. the $qq$ pair) and its light-spin operator

$$ \Sigma_c, \Sigma_c^* \rightarrow a_L, \vec{S}_L, $$

where $a_L$ is the field representing the effective light-diquark degrees of freedom (with quantum numbers $J^P = 1^+$, i.e. an axial vector), with the translation rules

$$ \langle \Sigma_c | \vec{S}_L | \Sigma_c \rangle = \frac{2}{3} \vec{S}_2, $$

$$ \langle \Sigma_c | \vec{S}_L | \Sigma_c^* \rangle = \frac{1}{\sqrt{3}} \vec{S}_2, $$

$$ \langle \Sigma_c^* | \vec{S}_L | \Sigma_c \rangle = \frac{2}{3} \vec{S}_2, $$

where $\vec{S}_2$ are the Pauli matrices as applied to the charmed baryon $\Sigma_c$, $\vec{S}$ a set the matrices representing the spin-1 to spin-$\frac{3}{2}$ transition (which can be consulted in Ref. [31]) and $\vec{S}_2$ the spin-$\frac{3}{2}$ matrices.

1 The $q_L$ field does not represent an actual light-quark field $q$, but the effective field that results from ignoring the heavy-quark spin degree of freedoms within the charmed antimeson.
With these ingredients the interaction between a $\bar{D}$ charmed antimeson and a $\Sigma_c$ charmed baryon can be easily written as

$$L_1 = C_a (q^a_L q_L) (a^a_L a_L) + C_b (q^a_L \bar{q}_L q_L) \cdot (a^a_L \bar{S}_L a_L),$$

which leads to the non-relativistic contact-range potential

$$V_{C1} = C_a + C_b \bar{S}_{L1} \cdot \bar{S}_{L2}.$$  

This potential can be particularized for the two cases of interest for us in the present work, the $D\Sigma_c$ and $\bar{D}^*\Sigma_c$ systems

$$V_{C1}(D\Sigma_c) = C_a,$$

$$V_{C1}(\bar{D}^*\Sigma_c) = C_a + C_b \frac{2}{3} \bar{S}_1 \cdot \bar{S}_2,$$

which we will use for the $P_c(4312)$ and the $\bar{D}^*\Sigma_c$ component of the $P_c(4440)$ and $P_c(4457)$, respectively.

B. The $\bar{D}^*\Sigma_c$-$\bar{D}\Lambda_{c1}$ transition

Now we will consider the $\bar{D}^{(*)}\Sigma^{(*)}_c$ to $\bar{D}^{(*)}\Lambda_{c1}^{(*)}$ transitions, which are necessary for the description of the $\bar{D}\Lambda_{c1}$ component in the $P_c(4440)$ and $P_c(4457)$ pentaquarks. First we will consider the structure of the $\Lambda_{c1}$ and $\Lambda_{c1}^*$ P-wave charmed baryons, which are $Q\bar{q}q$ states in which the spin of the light-quark pair is $S_L = 0$ and their orbital angular momentum is $L_L = 1$, yielding a total angular momentum of $J_L = 1$. In practice this means that there is no substantial difference (except for parity) between the description of the $\Sigma_c$, $\Sigma_c^*$ and $\Lambda_{c1}$, $\Lambda_{c1}^*$ charmed baryons either in terms of heavy-superfield or light-superfield notations. In the superfield notation, we will write

$$\bar{R}_Q = \frac{1}{\sqrt{3}} \bar{\sigma} \Lambda_{c1} + \bar{\Lambda}_{c1}^*,$$

while in the light-quark notation we use

$$\Lambda_{c1}, \Lambda_{c1}^* \rightarrow v_L, \bar{L}_L,$$

with $v_L$ representing the light-diquark pair (with quantum numbers $J^P = 1^-$, i.e. a vector field) and $\bar{L}_L$ the spin-1 matrices, where we use a different notation than in the S-wave charmed-baryon case to indicate that the angular momentum comes from the orbital angular momentum of the light-quark pair. This does not entail any operational difference, with the translation rules being

$$\langle \Lambda_{c1} | \bar{L}_L | \Lambda_{c1} \rangle = \frac{2}{3} \bar{S}_2,$$

$$\langle \Lambda_{c1} | \bar{L}_L | \Lambda_{c1}^* \rangle = \frac{1}{\sqrt{3}} \bar{S}_2,$$

$$\langle \Lambda_{c1}^* | \bar{L}_L | \Lambda_{c1}^* \rangle = \frac{2}{3} \bar{S}_2,$$

which are analogous to these of the $\Sigma_c$, $\Sigma_c^*$ baryons, see Eqs. 9, 11.

With these ingredients we are ready to write the $\bar{D}^{(*)}\Sigma_{c1}^{(*)} \rightarrow \bar{D}^{(*)}\Lambda_{c1}^{(*)}$ transition Lagrangian. We find that at lowest order there are two independent operators mediating the transition, which for convenience we write as

$$L_2 = D_a (q^a_L \bar{q}_L q_L) \cdot (v^a_L \bar{J}_L \times \bar{\nabla} a_L) + C.C.,$$

with $\bar{\nabla} = (\bar{\nabla} - \bar{\nabla})$ and where $\bar{J}_L$ refers to the spin-1 matrices as applied between the light-diquark axial and vector fields within the S- and P-wave charmed baryons. The translation rules for the $\bar{J}_L$ operator happen to be

$$\langle \Sigma_c | \bar{S}_L | \Lambda_{c1} \rangle = \frac{2}{3} \bar{S}_2,$$

$$\langle \Sigma_c^* | \bar{S}_L | \Lambda_{c1}^* \rangle = \frac{1}{\sqrt{3}} \bar{S}_2,$$

$$\langle \Sigma_{c1}^* | \bar{S}_L | \Lambda_{c1} \rangle = \frac{2}{3} \bar{S}_2,$$

which are analogous to Eqs. 9, 11 and 18, 20, except that now the initial and final baryon states are different (either the S- to P-wave baryon transition or vice versa). Other operators choices are possible in the Lagrangian of Eq. 21, but the present one is particularly useful because the $D_a$ term is pion-like, while the $D_b$ term is $\rho$-like: they are similar to what we could get from the exchange of a pion and a $\rho$ respectively, as we explain in Appendix A. The potential we obtain is

$$V_{C2}(1 \rightarrow 2) = +D_a \bar{\sigma}_{L1} \cdot \bar{q} + i D_b \bar{q} \cdot \left( \bar{\sigma}_{L1} \times \bar{J}_{L2} \right),$$

while in the other direction it is

$$V_{C2}(2 \rightarrow 1) = -D_a \bar{\sigma}_{L1} \cdot \bar{q} + i D_b \bar{q} \cdot \left( \bar{\sigma}_{L1} \times \bar{J}_{L2} \right),$$

where $D_a$ and $D_b$ are real in the convention we have used to write the potentials. It is important to notice that $V_{C2}$ is a non-diagonal potential and can be redefined by a phase

$$V_{C2}(1 \rightarrow 2) \rightarrow e^{i\phi} V_{C2}(1 \rightarrow 2),$$

$$V_{C2}(2 \rightarrow 1) \rightarrow e^{-i\phi} V_{C2}(2 \rightarrow 1),$$

in which case the potential is still self-adjoint. In the convention above, the p-space partial wave projection is purely real while the r-space partial wave projection is purely imaginary. To avoid the inconveniences originating from this fact, when working in coordinate space we will automatically add the phase $\phi = \pm \pi$ for the non-diagonal potential to be real.

Phenomenologically we expect the $D_a$ and $D_b$ couplings to represent the exchange of a pseudoscalar and
vector mesons, respectively. However there is no short-
range contribution directly attributable to a pseudoscalar
meson: pion exchange is excessively long-ranged as to be
included in the contact-range potential. For taking this
into account, we will devise a power counting in which
the $D_a$ coupling is a subleading order contribution, while
$D_b$ remains leading. Thus the effective potential we will
use from now on will be

$$V_{C2} = i D_b \bar{q} \cdot \left( \vec{S}_{L1} \times \vec{S}_{L2} \right).$$

(29)

C. The $\bar{D}\Lambda_{c1}$ channel

Finally we consider the $\bar{D}^{(*)}\Lambda_{c1}^{(*)}$ system, which enters
the description of the $P_c^*(4440)$ and $P_c(4457)$ as an addi-
tional (P-wave) component of the wave function. Yet this
meson-baryon system is particularly relevant for a theo-
retical pentaquark with quantum numbers $J^P = \frac{3}{2}^+$, for
which the most important meson-baryon component of the
wave function will be $\bar{D}\Lambda_{c1}$ in S-wave.

The lowest order interaction in the $\bar{D}^{(*)}\Lambda_{c1}^{(*)}$ system hap-
pens to be formally identical to the one for the $D^{(*)}\Sigma_c$ system, that is

$$\mathcal{L}_3 = E_a (q_L^\dagger q_L)(v_L^\dagger v_L) + E_b (q_L^\dagger \vec{S}_L q_L) \cdot (v_L^\dagger \vec{L}_L v_L),$$

(30)

which leads to the potential

$$V_{C3} = E_a + E_b \vec{S}_L \cdot \vec{L}_L. \quad (31)$$

If we particularize to the $\bar{D}\Lambda_{c1}$ molecule, we will end up with

$$V_{C3}(\bar{D}\Lambda_{c1}) = E_a,$$

(32)

which is a really simple potential, where the coupling $E_a$
is unknown.

D. Partial-Wave Projection

For the partial-wave projection of the contact-range poten-
tials (and the OPE potential later on), we will use the
spectroscopic notation $2S+1L_J$ to denote a state with
spin $S$, orbital angular momentum $L$ and total angular
momentum $J$. For the pentaquarks states we are consid-
ering — $P_c$, $P'_c$, $P''_c$ — the relevant partial waves are

$$P_c\left(\frac{1}{2}\right)^- : 2S_{1/2}(\bar{D}\Sigma_c),$$

(33)

$$P'_c\left(\frac{1}{2}\right)^+ : 2S_{1/2}(\bar{D}\Lambda_{c1}) - 2P_{1/2}(\bar{D}^*\Sigma_c) - 4P_{1/2}(\bar{D}^*\Sigma_c),$$

(34)

$$P''_c\left(\frac{1}{2}\right)^+ : 2S_{1/2}(\bar{D}^*\Sigma_c) - 2P_{1/2}(\bar{D}\Lambda_{c1}),$$

(35)

$$P''_c\left(\frac{3}{2}\right)^- : 4S_{3/2}(\bar{D}^*\Sigma_c) - 2P_{3/2}(\bar{D}\Lambda_{c1}),$$

(36)

where we indicate the relevant meson-baryon channels
within parentheses.

E. Momentum-Space Representation

For the momentum-space representation, we simply
project the relevant contact-range potential into the par-
tial waves of interest. For the $P_c$ ($\bar{D}^*\Sigma_c$) pentaquark we simply have

$$\langle p'|V(P_c)|p \rangle = C_a.$$  \hspace{1cm} (37)

Next, for the two $P'_c$ configurations ($J = \frac{1}{2}, \frac{3}{2}$) we have

$$\langle p'|V(P'_c, \frac{1}{2})|p \rangle = \left( \begin{array}{c} C_a - \frac{1}{\sqrt{3}} C_b \frac{2}{\sqrt{3}} \frac{2D_b}{3} p' \frac{2}{\sqrt{3}} \frac{2D_b}{3} p \end{array} \right), \quad (38)$$

$$\langle p'|V(P'_c, \frac{3}{2})|p \rangle = \left( \begin{array}{c} C_a + \frac{1}{\sqrt{3}} C_b - \frac{1}{\sqrt{3}} \frac{2D_b}{3} p' \frac{2}{\sqrt{3}} \frac{2D_b}{3} p \end{array} \right). \quad (39)$$

Finally for the $P''_c$ ($\bar{D}\Lambda_{c1}$) pentaquark we have

$$\langle p'|V(P''_c)|p \rangle = \left( \begin{array}{c} E_a - \frac{2}{\sqrt{3}} \frac{2D_b}{3} p' - \sqrt{\frac{2}{3}} \frac{2D_b}{3} p \end{array} \right),$$

(40)

which can be simplified to a two-channel form if we take
into account that the two P-wave $\bar{D}^*\Sigma_c$ components can adopt the configuration

$$\frac{2}{\sqrt{6}} \left| \bar{D}^*\Sigma_c(2P_{1/2}) \right| + \frac{1}{\sqrt{3}} \left| \bar{D}^*\Sigma_c(4P_{1/2}) \right|,$$

(41)

which maximizes the strength of the transition potential and we end up with

$$\langle p'|V(P''_c)|p \rangle = \left( \begin{array}{c} E_a - \sqrt{\frac{2}{3}} \frac{2D_b}{3} p' \frac{2}{\sqrt{3}} \frac{2D_b}{3} p \end{array} \right).$$

(42)

Notice that this simplification is only possible for the
pionless theory at LO: if we include pion-exchanges or
other effects we will have to revert to the original three-
channel representation.

F. Coordinate Space Representation

We obtain the r-space contact-range potential from Fourier-transforming the p-space one

$$V(\vec{r}) = \int \frac{d^3 \vec{q}}{(2\pi)^3} V(\vec{q}) e^{-i \vec{q} \cdot \vec{r}},$$

(43)

which in the case of the $V_{C1}$ and $V_{C3}$ potentials leads to

$$V_{C1}(\vec{r}) = (C_a + C_b \vec{S}_L \cdot \vec{S}_{L2}) \delta^{(3)}(\vec{r}),$$

(44)

$$V_{C3}(\vec{r}) = (E_a + E_b \vec{S}_L \cdot \vec{L}_{L2}) \delta^{(3)}(\vec{r}).$$

(45)
For the $V_{C2}$ potential, which contains one unit of orbital angular momentum, the transformation is a bit more involved, resulting in

$$V_{C2}(1 \rightarrow 2) = \left[ + i D_a \hat{\sigma}_{L1} \cdot \hat{\nabla} - D_b \hat{\nabla} \cdot \left( \hat{\sigma}_{L1} \times \hat{J}_{L2} \right) \right] \delta^{(3)}(\vec{r}),$$

(46)

which can be further simplified by rewriting

$$\hat{\nabla} \delta^{(3)}(\vec{r}) = \hat{r} \partial_r \delta^{(3)}(\vec{r}),$$

(47)

leading to

$$V_{C2}(1 \rightarrow 2) = \left[ + i D_a \hat{\sigma}_{L1} \cdot \hat{r} - D_b \hat{r} \cdot \left( \hat{\sigma}_{L1} \times \hat{J}_{L2} \right) \right] \partial_r \delta^{(3)}(\vec{r}).$$

(48)

This last expression is particularly useful because the partial wave projection of the $\hat{\sigma}_{L1} \cdot \hat{r}$ and $\hat{r} \cdot \left( \hat{\sigma}_{L1} \times \hat{J}_{L2} \right)$ is identical to their p-space versions. Finally we redefine $V_{C2}(1 \rightarrow 2)$ by a phase to end up with a purely real potential:

$$V_{C2}(1 \rightarrow 2) \rightarrow -i V_{C2}(1 \rightarrow 2).$$

(49)

With the previous conventions and the power counting we use (for which $D_a$ is a subleading order effect), we end up with the r-space potentials

$$V(\vec{r}; \vec{P}_c) = C_a \delta^{(3)}(\vec{r}),$$

(50)

$$V(\vec{r}; \vec{P}_c^*; \frac{1}{2}) = \left( \frac{C_a}{\sqrt{3}} - \frac{2 D_b}{3} \partial_r \right) \delta^{(3)}(\vec{r}),$$

(51)

$$V(\vec{r}; \vec{P}_c^*; \frac{3}{2}) = \left( C_a + \frac{2}{3} C_b - \frac{1}{\sqrt{3}} \frac{2 D_b}{3} \partial_r \right) \delta^{(3)}(\vec{r}),$$

(52)

$$V(\vec{r}; \vec{P}_c^*; \vec{L}_a) = \left( \frac{E_a}{\sqrt{3}} - \sqrt{2} \frac{2 D_b}{3} \partial_r \right) \delta^{(3)}(\vec{r}),$$

(53)

where for the $P'_c$ pentaquark we have written the simplified two-channel version of the potential.

### Table I. Matrix elements of the vector operators for the partial waves we are considering in this work.

| Molecule | Partial Waves | $J^P$ | $\hat{\sigma}_{L1} \cdot \hat{r}$ | $\hat{r} \cdot (\hat{\sigma}_{L1} \times \hat{S}_{L2})$ |
|----------|---------------|-------|-----------------------------------|-------------------------------------------------|
| $D\Lambda_{c1} - D^*\Sigma_c$ | $2S_{1/2}^2P_{1/2}^2P_{1/2}$ | $\frac{1}{2}$ | \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} | \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} |
| $D^*\Sigma_c - D\Lambda_{c1}$ | $2S_{1/2}^2P_{1/2}$ | $\frac{1}{2}$ | \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} | \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} |
| $\tilde{D}^*\Sigma_c - D\Lambda_{c1}$ | $4S_{3/2}^2P_{3/2}$ | $\frac{3}{2}$ | \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} | \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} |

### G. Regularization and Renormalization

The contact-range potentials we are using are not well-defined unless we include a regulator to suppress the unphysical high-momentum components of the potential. For the p-space version of the potential this is done with the substitution

$$\langle p'| V_C | p \rangle \rightarrow \langle p'| V_{C\Lambda\Sigma} | p \rangle f\left(\frac{p'}{A}\right) f\left(\frac{P}{A}\right),$$

(54)

with $f(x)$ a regulator function, for which we will choose a Gaussian, $f(x) = e^{-x^2}$. For the r-space version of the potential we will use a delta-shell regulator

$$\delta^{(3)}(\vec{r}) \rightarrow \frac{\delta(r - R_c)}{4\pi R_c^2},$$

(55)

$$\partial_r \delta^{(3)}(\vec{r}) \rightarrow \frac{3 \delta(r - R_c)}{R_c 4\pi R_c^3},$$

(56)

with $R_c$ the coordinate space cutoff, where the $3/R_c$ factor in the derivative of the delta is chosen for its Fourier-transform to be either $p$ or $p'$ in the $R_c \rightarrow 0$ limit after the partial wave projection.

### H. Dynamical Equation

For finding the location of the bound states we have to iterate the r- or p-space potentials that we have obtained within a dynamical equation. For the r-space potential, we will solve the reduced Schrödinger equation

$$-u''(r) + \sum_b 2\mu_b V_{ab}(r) u_b(r) + \frac{L_a(L_a + 1)}{r^2} u_a(r) = -\gamma_a^2 u_a(r),$$

(57)

where $a$, $b$ are indices we use to represent the different channels in the molecules we are considering as detailed in Eqs. [33][36], while $V_{ab}$ is the potential between two channels, see Eqs. [34][35], which is regularized as in Eqs. [32][34] and [35][36]. The reduced mass, angular momentum and wave number of a given channel $a$ are represented by $\mu_a$, $L_a$ and $\gamma_a$. In turn the wave number is
given by \( \gamma_a = \sqrt{2\mu_a(M_{th(a)} - M)} \), with \( M_{th(a)} \) the mass of the two-hadron threshold for channel \( a \) and \( M \) the mass of the molecular pentaquark we are predicting.

For the p-space potential we will solve the Lippmann-Schwinger equation as applied to the pole of the \( \Sigma \)-matrix, that is:

\[
\phi_a(p) = \sum_b \int \frac{d^3q}{(2\pi)^3} \frac{\langle p|V_{ab}|q\rangle}{M_{th(b)} - M - \frac{q^2}{2\mu_b}} \phi_b(q),
\]

where \( a, b \) represent the channel, \( \phi_a \) is the vertex function for channel \( a \) (where the vertex function is related to the residue of the \( \Sigma \)-matrix), \( V_{ab} \) is the potential between two channels, see Eqs. (34)-(10), which is regularized according to Eq. (63), and \( M \) is the mass of the molecular pentaquark, while \( M_{th(b)} \) and \( \mu_a \) are the two-hadron threshold and the reduced mass for a given channel \( a \).

III. HALF-PIONFUL THEORY

The exchange of one pion between the \( \bar{D}^*\Sigma_c \) and \( \bar{D}\Lambda_{c1} \) channels has the particularity that its range is extremely enhanced. The reason is that the pion in the \( \Sigma_c\Lambda_{c1}\pi \) and \( D^*D\pi \) vertices can be emitted or absorbed almost on the mass-shell, resulting in an improved range. Besides, owing to the opposite parity of the \( \Sigma_c \) and \( \Lambda_{c1} \) baryons, the pion exchange in this vertex is in S-wave. In combination with the standard P-wave pion in the vertex involving the charmed mesons, the outcome is that instead of having a central and tensor forces with orbital angular momentum \( L = 0 \) and 2 respectively, we end up with a vector force with \( L = 1 \). The long-range behavior of the vector force is \( 1/r^2 \), i.e. an inverse square-law potential, which can trigger a series of interesting theoretical consequences when the strength of the potential is above a certain critical value \( g_1 \). Yet, as explained in Ref. \[22\], this is probably not the case for the LHCb pentaquarks as hadronic molecules.

Now, we begin by writing the pion-exchange Lagrangian for the \( \Sigma_c \) to \( \Lambda_{c1} \) transition in the heavy superfield notation:

\[
\mathcal{L}_{HH\pi} = \frac{g_1}{\sqrt{2f_\pi}} \text{Tr} \left[ H_Q^\dagger \tau_a \vec{\sigma} \cdot \vec{\nabla} \pi_a H_Q \right],
\]

\[
\mathcal{L}_{RS\pi} = \frac{h_2}{f_\pi} \tilde{R}_Q t_a \partial_\mu \pi_a \cdot \vec{S}_Q + C.C.,
\]

which are obtained from the non-relativistic limits of the Lagrangians of Refs. \[29\], \[32\]. The light-quark notation version happens to be trivial

\[
\mathcal{L}_{qLqL\pi} = \frac{g_1}{\sqrt{2f_\pi}} q_L^\dagger \tau_a \vec{\sigma} L \cdot \vec{\nabla} \pi_a q_L,
\]

\[
\mathcal{L}_{cLcL\pi} = \frac{h_2}{f_\pi} v^\dagger_L t_a \partial_\mu \pi_a a_L + C.C.
\]

From the previous Lagrangians we can derive the OPE potential in momentum space, which reads as follows

\[
V_{\text{OPE}}(\vec{q}, 1 \to 2) = \frac{g_1 h_2}{\sqrt{2f_\pi}} \vec{r}_1 \cdot \vec{r}_2 \frac{\vec{q} \cdot \vec{r}_1 \cdot \vec{r}_2 \cdot \vec{q}}{\vec{q}^2 \mu^2},
\]

where we are indicating that this is the transition potential in the \( \bar{D}(\Sigma^*) \to \bar{D}(\Lambda^*_c) \) direction. The operator \( \vec{r}_1 \cdot \vec{r}_2 = \sqrt{2} \) for total isospin \( I = \frac{1}{2} \) and 0 otherwise. The equivalent expression in coordinate space can be obtained by Fourier-transforming the previous expression, where in addition we include a phase to follow the convention of having a purely real transition potential

\[
V_{\text{OPE}}(\vec{r}, 1 \to 2) = \frac{g_1 h_2}{4\pi \sqrt{2f_\pi}} \frac{e^{-\mu_\tau}}{\mu_\tau} \left( 1 + \frac{1}{\mu_\tau} \right)
\]

with \( W_E \) defined as

\[
W_E(\vec{r}) = \frac{g_1 h_2}{4\pi} \mu_\tau \frac{1}{\sqrt{2f_\pi}} \frac{e^{-\mu_\tau}}{\mu_\tau} \left( 1 + \frac{1}{\mu_\tau} \right)
\]

For the couplings we have taken \( g_1 = 0.59 \) (as deduced from the \( D^* \to D\pi \) and \( D^*_s \to D_s\pi \) decays \[33\], \[34\]), \( h_2 = 0.63 \) (from the analysis of Ref. \[35\], where \( h_2 \) is extracted from \( \Gamma(\Lambda_{c1} \to \Sigma_c\pi) \) as measured by CDF \[36\], \( f_\pi = 130 \) MeV and \( \omega_\pi \approx (m(\Lambda_{c1}) - m(\Sigma_c)) \approx (m(D^*) - m(D)) \approx m_\pi \), with \( m_\pi = 138 \) MeV. Finally \( \mu_\pi = \sqrt{m_\pi^2 - \omega_\pi^2} \approx 0 \), a value we will further discuss in the following lines.

A. Infrared regularization

In the \( \mu_\pi \to 0 \) limit, which is close to the physical situation we are dealing with and will probably represent a good approximation of it, the previous OPE potential becomes a \( 1/r^2 \) infinite-range potential. In particular the p-space potential reads

\[
V_{\text{OPE}}(\vec{q}, 1 \to 2) = \frac{g_1 h_2}{\sqrt{2f_\pi}} \frac{\vec{r}_1 \cdot \vec{r}_2 \omega_\pi \vec{r}_1 \cdot \vec{q}}{\vec{q}^2},
\]

while for the r-space potential we can take this approximation into account within the function \( W_E \)

\[
W_E(r) = \frac{g_1 h_2}{4\pi} \mu_\tau \frac{1}{\sqrt{2f_\pi}} \frac{1}{r^2}
\]

Of course this is merely an approximation. What is actually happening is that the modulus of the effective pion mass \( |\mu_\pi| \) will be in general considerably smaller than the pion mass \( m_\pi \) (or any other hadronic scale for that matter). We have \( |\mu_\pi| \sim (10 - 20) \) MeV, its concrete value depending on the specific particle channel under consideration. In a few particle channels \( \mu_\pi \) is purely imaginary, indicating the possibility of decay into the \( DS_\pi \) channel, and in others it is real. A detailed treatment of these difference is however outside the scope of the present manuscript.

Here we will opt for the much easier treatment we were describing above, that is, to assume that \( \mu_\pi = 0 \). For
taking into account that the range of the OPE potential is actually not infinite we will include an infrared cutoff. For the partial-wave projection of OPE in momentum space, we will introduce an infrared cutoff $\Lambda_{IR}$ in the following way

$$\langle p'|V_{OPE}|p \rangle \rightarrow \langle p'|V_{OPE}|p \rangle \theta(|q_- - \Lambda_{IR}|) \theta(|q_+ - \Lambda_{IR}|),$$

(68)

with $q_- = p - p'$ and $q_+ = p + p'$, with the infrared cutoff chosen within the cutoff window $\Lambda_{IR} = (10 - 20)$ MeV, which corresponds with the size of the modulus of the effective pion mass. In coordinate space the inclusion of the infrared cutoff $R_{IR}$ will be considerably simpler

$$V_{OPE}(\vec{r}) \rightarrow V_{OPE}(\vec{r}) \theta(R_{IR} - r),$$

(69)

where we will take $R_{IR} = (10 - 20)$ fm.

Actually the effect of this infrared cutoff is only important if the strength of the $1/r^2$ potential is equal or larger to the critical triggering a geometric spectrum. This does not happen for any of the pentaquarks we are considering, at least with the currently known values of the couplings $g_1$ and $g_2$. However in the $P_c^*$ pentaquark the strength is not far away to that critical value [22], indicating that in this case the results will have a larger dependence on the infrared cutoff.

### B. Partial-wave projection

The partial-wave projection of the OPE potential is trivial for its coordinate space representation: owing to its clear separation into a radial and angular piece – Eq. [64] – it merely requires to consider the partial wave projection of the vector operator $\vec{\sigma}_{L_1} \cdot \vec{r}$, which we already showed in Table [II].

For the momentum-space representation of the potential the partial-wave projection is a bit more complex, yet it can be written as

$$\langle p'(S' L', p')|V|p(S L)\rangle = \frac{g_1 h_2}{\sqrt{2 f^2}} \vec{r}_1 \cdot \vec{r}_2 \omega_{\pi}$$

$$\times \langle S' L', p'|\vec{q}|S L \rangle$$

$$\times \langle p'(L')|\frac{1}{|q|} |p(L)\rangle,$$

(70)

where the matrix elements of the vector operator are again to be found in Table [II] to which we have to add the partial wave projection of the $1/|q|$ potential:

$$\langle p'(1)|\frac{1}{|q|}|p(0)\rangle = \frac{2\pi}{p'} \left[ 1 + \frac{p'^2 - p^2}{2pp'} \log\left(\frac{p + p'}{p - p'}\right) \right],$$

$$\langle p'(0)|\frac{1}{|q|}|p(1)\rangle = \frac{2\pi}{p} \left[ 1 + \frac{p^2 - p'^2}{2pp'} \log\left(\frac{p + p'}{p - p'}\right) \right].$$

(71)

Of course, we still supplement the previous expressions with the infrared cutoff of Eq. [68].

### IV. THE PENTAQUARK TRIO REVISITED

In this Section we consider the description of the $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$ pentaquarks within the EFTs proposed in this work. We will begin by reviewing their standard molecular interpretations as $\bar{D}\Sigma$ and $\bar{D}^*\Sigma$ bound states and then we will move to the novel molecular interpretation in which the $\bar{D}\Lambda_{c1}$ channel is included as an explicit degree of freedom for the $P_c(4440)$ and $P_c(4457)$ pentaquarks. The prediction of a $\bar{D}\Lambda_{c1}$ bound state is contingent on an unknown coupling constant, $E_0$. For dealing with this issue we will consider two different estimations of the value of this coupling and the predictions they will entail.

#### A. The standard molecular interpretation

We begin by reviewing the standard molecular interpretation of Ref. [5], in which the pentaquarks were considered to be $\bar{D}\Sigma$ and $\bar{D}^*\Sigma$ molecules (without any $\bar{D}\Lambda_{c1}$ component) described by a pionless EFT. This pionless EFT is equivalent to using the $V_{c1}$ contact-range potential of Eq. [15], which contains two independent couplings $C_a$ and $C_b$. The original procedure [5] for determining these two couplings was as follows:

(i) use the $P_c(4440)$ and $P_c(4457)$ as $\bar{D}^*\Sigma$ molecules to determine the $C_a$ and $C_b$ couplings;

(ii) postdict the $P_c(4312)$ as a $\bar{D}\Sigma$ molecule and compare with its experimental location.

For convenience we will modify the previous procedure in this manuscript:

(i) use the $P_c(4312)$ and $P_c(4457)$ as $\bar{D}\Sigma$ and $\bar{D}^*\Sigma$ molecules to determine the $C_a$ and $C_b$ couplings;

(ii) postdict the $P_c(4440)$ as a $\bar{D}^*\Sigma$ molecule and compare with its experimental location.

This choice guarantees that the prediction of the pentaquark trio remain all below their respective meson-baryon thresholds: the later inclusion of the $\bar{D}\Lambda_{c1}$ channel can in a few instances move the $P_c(4457)$ a bit above the threshold for hard cutoffs if we fit the couplings as in Ref. [5].

Now for the $\bar{D}^*\Sigma$ molecules there are two spin configurations, $J = \frac{1}{2}$ and $\frac{3}{2}$, but we do not know which one corresponds with each of the pentaquarks. As a consequence we consider two scenarios, $A_0$ and $B_0$:

(a) in scenario $A_0$ the $P_c(4440)$ has $J = \frac{1}{2}$, while the spin of the $P_c(4457)$ is $J = \frac{3}{2}$,

(b) in scenario $B_0$ the $P_c(4440)$ has $J = \frac{3}{2}$, while the spin of the $P_c(4457)$ is $J = \frac{1}{2}$,
TABLE II. The contact-range couplings $C_a$, $C_b$ and $E_a$ when the $D^*\Sigma_c$ and $D\Lambda_c$ channels do not couple. $C_a$ and $C_b$ are obtained from the condition of reproducing the mass of the $P_c(4312)$ and $P_c(4457)$ as molecular pentaquarks in p- and r-space (as indicated by type of cutoff: $\Lambda$ and $R_c$). Scenario $\Lambda$ (and its variants) corresponds to considering that the spin-parities of the $P_c(4440)$ and $P_c(4457)$ are $J^P = \frac{1}{2}^-$ and $\frac{3}{2}^-$, respectively, while scenario B corresponds to the opposite identification. $E_a^{\text{crit}}$ is the critical value of the $E_a$ coupling required for the uncoupled $D\Lambda_{c1}$ system to bind.

| Scenario $\Lambda$ (GeV) | $C_a$ (fm$^2$) | $C_b$ (fm$^2$) | $E_a^{\text{crit}}$ (fm$^2$) |
|--------------------------|----------------|----------------|-----------------|
| $A_0$                    | 0.5            | -2.17          | +0.55           | -1.13           |
| $A_0$                    | 1.0            | -0.80          | +0.13           | -0.57           |
| $B_0$                    | 0.5            | -2.17          | -0.27           | -1.13           |
| $B_0$                    | 1.0            | -0.80          | -0.07           | -0.57           |

where we use the subscript “zero” to indicate that this is the base case in which the $D\Lambda_{c1}$ channel is not included. Then we postdict the location of the $P_c(4440)$ in each scenario, resulting in

$$M_{A_0} = 4440.1 \ (4434.5) \ \text{MeV},$$

$$M_{B_0} = 4449.6 \ (4447.6) \ \text{MeV},$$

for the p-space Gaussian regulator with $\Lambda = 0.5 \ (1.0) \ \text{GeV}$ and

$$M_{A_0} = 4438.9 \ (4433.8) \ \text{MeV},$$

$$M_{B_0} = 4449.3 \ (4447.5) \ \text{MeV},$$

for the r-space delta-shell regulator with $R = 1.0 \ (0.5) \ \text{fm}$. These numbers are to be compared with the experimental value $M = 4440.3 \pm 1.3^{+4.1}_{-4.8} \ \text{MeV}$, which indicates that scenario $A_0$ is marginally preferred over scenario $B_0$ (particularly for softer cutoffs). This coincides with the conclusions of the previous pionless EFT of Ref. [2].

**B. The novel molecular interpretation**

Now we explore the novel molecular interpretation we propose, in which the $P_c(4312)$ is a $D\Sigma_c$ molecule while the $P_c(4440)$ and $P_c(4457)$ are $D^*\Sigma_c$-$D\Lambda_{c1}$ molecules. The contact-range piece of the potential for the pionless and half-pionful EFTs is given by Eqs. [47, 39], which contain three independent coupling constants ($C_a$, $C_b$ and $D_b$). Finally we conjecture the existence of a $D\Lambda_{c1}$ S-wave molecule, which we call the $P_c^*$ and for which the contact-range piece of the potential is given by Eq. [10], which includes a new coupling ($E_a$).

Of these four couplings, we can determine three of them — $C_a$, $C_b$ and $D_b$ — from the masses of the three pentaquarks. The procedure we will follow is:

(i) use the $P_c(4312)$ as a $D\Sigma_c$ molecule to determine the $C_a$ coupling,

(ii) use the $P_c(4440)$ and $P_c(4457)$ as $D^*\Sigma_c$-$D\Lambda_{c1}$ molecules to determine the $C_b$ and $D_b$ couplings,

(iii) if there is no solution for the previous procedure, we will set $D_b = 0$ and, as in the uncoupled-channel case, we will determine $C_b$ from the condition of reproducing the $P_c(4457)$ pole,

(iii) finally determine for which values of $E_a$ the $P_c^*$ (the conjectured S-wave $D\Lambda_{c1}$ molecule) binds and compare these values with expectations from naive dimensional analysis (NDA). As in the standard molecular interpretation, we have two possible scenarios which we now call $A_1$ and $B_1$, where $A_1$ ($B_1$) corresponds to the $P_c(4457)$ being a $J = \frac{1}{2}^-$ ($\frac{3}{2}^-$) molecule. We will further subdivide the scenario $A_1$ ($B_1$) into a pionless and a half-pionful version, which we will denote $A_1^\pi$ ($B_1^\pi$) and $A_1^\Sigma$ ($B_1^\Sigma$), respectively. It happens that the couplings can be compared with NDA, in particular $D_b$ and $E_a$: the $D_b$ comparison can provide an indirect estimation of the likelihood of scenarios $A_1$ and $B_1$, while $E_a$ will provide the binding likelihood of the $P_c^*$ pentaquark.

To illustrate this idea, we can consider the pionless p-space calculation, which for $\Lambda = 0.5 \ \text{GeV}$ in scenario $A_1^\pi$ and $B_1^\pi$ gives

$$C_a = -2.17 \ \text{fm}^2,$$

$$C_b = +0.55 \ \text{fm}^2 \ (A_1^\pi),$$

$$D_b = +0.00 \ \text{fm}^2 \ (A_1^\pi),$$

$$C_b = -0.85 \ \text{fm}^2 \ (B_1^\pi),$$

$$D_b = +0.99 \ \text{fm}^2 \ (B_1^\pi).$$

This translates into the following condition for the $P_c^*$ to bind

$$E_a \leq -1.13 \ \text{fm}^2 \ (A_1^\pi),$$

$$E_a \leq +0.04 \ \text{fm}^2 \ (B_1^\pi),$$

which in scenario $A_1^\pi$ requires the coupling $E_a$ to be attractive, while scenario $B_1^\pi$ will lead to binding even for a slightly repulsive coupling. For the calculation we have used the following values for the masses of the hadrons involved: $m(D) = 1867.22 \ \text{MeV}$, $m(D^*) = 2008.61 \ \text{MeV}$, $m(\Sigma_c) = 2453.54 \ \text{MeV}$, $m(\Lambda_{c1}) = 2592.25 \ \text{MeV}$, which are the isospin averages of the PDG values [37].

A complete list of the couplings can be consulted in Table [III] for the different EFTs and regulators considered in
From this we see that $\Lambda$ which is to be expected for the coupling of a two-body system that binds $^{38}$, while $C_b$ and $D_b$ are closer to natural, though this depends on the cutoff (particularly for $D_b$, see Table III). In addition we can appreciate that in scenario A1 the binding of the $P'_c$ pentaquark is possible but not particularly probable, as the size of the coupling $E_a$ that is required to bind is larger than the NDA expectation. In contrast, in scenario B1 the coupling $E_a$ required to bind falls well within what is expected from NDA. Thus in this second case binding seems to be much more likely.

Regarding the $P'_c$ pentaquark, we can deduce its probable mass from the NDA estimation of the $E_a$ coupling, provided this coupling is attractive:

$$E_a^{\text{NDA}} \simeq -\frac{4\pi}{M^2}. \quad (85)$$

Within scenario B1, this estimation of the coupling consistently generates a shallow $P'_c$ close to the $\Delta A_{c1}$ threshold, where the concrete predictions can be consulted in Table IV. Of course the question is whether it is sensible to assume that the $E_a$ coupling is attractive. We will examine the validity of this assumption in the next few lines.

### C. Can we further pinpoint the location of the $P'_c$ pentaquark?

Regarding $E_a$, it will be useful not only to determine its sign but also its size beyond the NDA estimation we have already used to argue the existence of the $P'_c$ pentaquark. From arguments regarding the saturation of contact-range couplings by light-mesons $^{31}$, the light-meson contributions to $E_a$ can be divided into two components

$$E_a = E_a^S + E_a^V, \quad (86)$$

which correspond to the scalar ($\sigma$) and vector ($\omega$) meson contributions. The scalar and vector contributions

| Scenario $\Lambda$ (GeV) | $E_a^{\text{crit}}$ (fm$^2$) | $E_a^{\text{NDA}}$ (fm$^2$) | $M^{\text{NDA}}(P'_c)$ |
|------------------------|-----------------------------|-----------------------------|---------------------------|
| $B'_1$                 | 0.5  | +0.04 | −0.49 | 4457.0 |
| $B'_2$                 | 1.0  | −0.40 | −0.49 | 4457.9 |
| $B'_3$                 | 0.5  | +0.18 | −0.49 | 4456.3 |
| $B'_4$                 | 1.0  | −0.35 | −0.49 | 4457.0 |
| Scenario $R_c$ (fm) | $E_a^{\text{crit}}$ (fm$^2$) | $E_a^{\text{NDA}}$ (fm$^2$) | $M^{\text{NDA}}(P'_c)$ |
| $B'_1$                 | 0.5  | −0.42 | −0.49 | 4458.1 |
| $B'_2$                 | 1.0  | −0.15 | −0.49 | 4458.2 |
| $B'_3$                 | 0.5  | −0.36 | −0.49 | 4457.3 |
| $B'_4$                 | 1.0  | −0.11 | −0.49 | 4457.7 |

**TABLE IV.** The mass of the $P'_c$ pentaquark as deduced from the NDA estimate of the $E_a$ coupling (assuming it is attractive) in scenario $B_1$, both in the pionless and half-pionful theory. For reference, the $\Delta A_{c1}$ threshold is located at 4459.5 MeV in the isospin-symmetric limit.
are attractive and repulsive ($E^S_0 < 0$ and $E^V_0 > 0$), respectively. At first sight this ambiguous result seems to indicate that we cannot determine the sign of $E_a$, yet this would be premature. As a matter of fact the same situation would arise had we applied this argument to the two-nucleon system, but it happens that the deuteron binds. The reason is that the scalar meson contributions have a longer range than the vector meson ones, leading to net attraction.

This seems to be the case not only in the two-nucleon system, but also in the $D\Lambda_c$ case: according to a recent calculation in the one-boson-exchange model \[41\], the $D\Lambda_c$ system is not far away from binding. In fact, had we adapted the recent one-boson exchange model of Ref. \[42\] (originally intended for the $D^*(\pi)\Sigma_c^*(\pi)$ molecules) to the $D\Lambda_c$ system, the system will not bind, yet its two-body scattering length $a_2$ would probably be unnaturally large

$$a_{2}^{\text{OBE}}(D\Lambda_c) = -24.1^{+20.7}_{-\infty}(+9.5) \text{ fm}, \tag{87}$$

where the errors are computed as in Ref. \[42\] and which are compatible with binding\[3\] (the lower error indicates that the scattering length changes sign, hence the $-\infty$, and that in that case its value would be $+9.1 \text{ fm}$). This reinforces the conclusions derived from Ref. \[41\] for the $D\Lambda_c$ case. That is, we expect $E_a < 0$ and close to the value required to have a shallow bound state in the absence of coupling with the $D^*\Sigma_c$ channel. All this makes the $P_c'$ pentaquark very likely in scenario $B_1$, as we will now show with explicit calculations.

If we now describe the $D\Lambda_c$ two-body system in a pionless EFT, the coupling $E_a$ can be determined from the value of the scattering length that we have already computed within the OBE model, leading to

$$E_a = -1.09^{+0.21}_{-0.18} (-0.55^{+0.06}_{-0.04}) \text{ fm}^2, \tag{88}$$

for $\Lambda = 0.5 (1.0) \text{ GeV}$ if we do the calculations in p-space, or alternatively

$$E_a = -1.10^{+0.21}_{-0.19} (-0.56^{+0.06}_{-0.05}) \text{ fm}^2, \tag{89}$$

for $R_c = 1.0 (0.5) \text{ fm}$ in r-space. As already explained, this extracted value of the coupling is enough as to guarantee binding in scenario $B_1$, both in the pionless and pionful versions. This would lead to a $P_c'$ that is bound by (4 $- 9$) $\text{MeV}$ depending on the case. The predicted locations can be found in Table V where we have only consider scenario $B_1$ (for which binding is more probable). We can appreciate that the predictions are very similar, independently of the cutoff or whether the calculation has been done in r- or p-space. For a more graphical comparison we have included Fig. 4 which shows the dependence of the binding energy on the coupling $E_a$ for the half-pionful theory in momentum space (we have chosen this particular calculation as the representative case, as the other three calculations in scenario $B_1$ would yield pretty similar results). Notice that in Fig. 4 we indicate the most probable values of $E_a$ and the binding energy of the $P_c'$ within a square.

### D. Can scenario $A$ be discarded?

A preliminary examination of the different determinations of the couplings presented in Table III reveals that $D_0 = 0$ in scenario $A$. The reason for this is that in general it is not possible to reproduce the two $D^*\Sigma_c$ pentaquarks in this scenario. This seems counter intuitive at first, but actually there are good reasons for this to be the case, which have to do with coupled-channel dynamics and which we will explain below.

First, we will consider a molecular pentaquark $P_Q$ in the heavy-quark limit, in which the masses of the charmed hadrons diverge and we can ignore the kinetic energy of the hadrons. In this limit the binding energy of a molecular pentaquark is given by

$$B_{P_Q} = -\langle V^S \rangle, \tag{90}$$

where $\langle V^S \rangle$ is the expected value of the S-wave potential and where we have taken the convention that the binding energy $B_{P_Q}$ is a positive number, thus the minus sign in front of $\langle V^S \rangle$. Now we consider the case where the molecular pentaquark contains an additional P-wave component, for which the coupled-channel potential reads

$$V_{P_Q} = \begin{pmatrix} V^S & \lambda V^{SP} \\ \lambda V^{SP} & 0 \end{pmatrix}, \tag{91}$$

and $\lambda = 1.19 \pm 0.03 \text{ GeV}$, while the couplings of the $\Lambda_c$ and $\Sigma_c$ baryons to the $\sigma$ and $\omega$ happen to be identical. Finally $\Lambda_c$ does not couple to the $\rho$, owing to isospin.
This simplifies the S-to-P wave contribution to the binding with \( \Delta P \).

Thus we have

\[
\lambda^2 \propto \{1, 4, 6\},
\]

for the \( P_c^+(1/2) \), \( P_c^+(3/2) \) and \( P_c' \) pentaquarks, respectively. The actual effect of the P-wave channel also depends on the inverse of the mass gap, i.e.

\[
\Delta B_{P_Q} \propto -\frac{\lambda^2}{\Delta P},
\]

which implies that the impact of the \( \bar{D}A_{c1} \) channel will be larger in the \( P_c(4457) \) pentaquark than in the \( P_c(4440) \) one \((\Delta P = -2.2 \text{ and } -19.2 \text{ MeV})\) respectively. However, once we take into account the finite mass of the hadrons, the effect of the mass gap on the \( P_c(4457) \) will diminish in relative terms as it will be softened owing to the kinetic energy contributions.

In scenario A the \( P_c(4457) \) receives a large boost to its binding energy: the coupling to the \( \bar{D}A_{c1} \) channel is 4 times stronger than for the \( P_c(4440) \) pentaquark. Besides, the \( P_c(4457) \) is also considerably closer to the \( \bar{D}A_{c1} \) threshold. Thus the \( P_c(4457) \) receives much more attraction from the coupled-channel dynamics than the \( P_c(4440) \), to the point that it is not possible to reproduce the two of them simultaneously. That is, the \( P_c(4457) \) as a \( J = \frac{3}{2} \bar{D}^*\Sigma_c \) molecule is only possible if the coupling \( D_b \) is much smaller than its NDA estimation. In scenario B this does not happen because the \( P_c(4457) \) couples much more weakly to the \( \bar{D}A_{c1} \) channel than in scenario A. But this does not imply that scenario A can be discarded though: we simply do not know the size of the coupling \( D_b \) and it is within the realm of possibilities that its size is considerably smaller than the NDA estimation.

Here it is important to mention that the two theoretical scenarios we have presented (A and B) are but a subset of all the possible scenarios. We have three molecular explanations \((P_c^+(1/2), P_c^+(3/2) \text{ and } P_c')\) for two pentaquarks, which gives a total of six possible scenarios instead of the two we are considering. But with the exception of scenarios A and B, it is not possible to determine the value of the couplings in other cases. For instance, had we assumed that the \( P_c(4440) \) is the \( J = \frac{3}{2} \bar{D}^*\Sigma_c \) molecule and \( P_c(4457) \) the \( J = \frac{3}{2} \bar{D}A_{c1} \) one, i.e. the scenario originally proposed in Ref. [8], we would have ended with three unknown couplings \((C_b, D_b \text{ and } E_a)\) for two pentaquarks. Though this limitation can indeed be overcome by invoking NDA, the resulting analysis is considerably more involved than in scenarios A and B and thus we have decided not to consider them in this work.

3 Notice that we are assuming an attractive S-wave potential \(- \langle V_S \rangle < 0\) – and that we always have \( \langle (V_{SP})^2 \rangle > 0\).
Another factor that we have not taken into account in the present analysis is the effect of the $\bar{D}\Lambda_c$ channel, which lies about 20 MeV above the $D\Lambda_c$ threshold. The $\bar{D}\Lambda_c$ channel can mix with the $J = \frac{3}{2}^-$ $D^*\Sigma_c$ one, inducing a bit of extra attraction in this later case. However, from Eq. (90) and the larger mass gap for the $\bar{D}\Lambda_c$ channel ($\Delta P = -55.0$ MeV versus $-19.2$ MeV for the $D\Lambda_c$ one for scenario B), we expect this effect to be fairly modest.

V. CONCLUSIONS

In this manuscript we have considered the impact of the $D\Lambda_c$ channel for the description of the $P_c(4440)$ and $P_c(4457)$ pentaquarks. Within the molecular picture, the standard interpretation of the $P_c(4440)$ and $P_c(4457)$ states is that they are $D^*\Sigma_c$ bound states. This is motivated by the closeness of the $D^*\Sigma_c$ threshold to the location of the two pentaquark states. But the same is true for the $D\Lambda_c$ threshold, which naturally prompts the question of what is the contribution of this channel to the description of the pentaquarks $[8, 22, 43]$.

For answering this question we have analyzed the inclusion of $D\Lambda_c$ channel from the EFT perspective. We find that the importance of the $D\Lambda_c$ channel depends on which are the quantum numbers of the $P_c(4440)$ and $P_c(4457)$ pentaquarks: in the standard molecular interpretation ($D^*\Sigma_c$) their quantum numbers can be either $J^P = \frac{1}{2}^-$ or $\frac{3}{2}^-$, but we do not know which quantum numbers correspond to which pentaquark. There are two possibilities: that the $P_c(4440)$ and $P_c(4457)$ are respectively the $J^P = \frac{1}{2}^-$ and $\frac{3}{2}^-$ $D^*\Sigma_c$ molecules, or vice versa. The first possibility, which we call scenario A, corresponds to the standard expectation that hadron masses increase with spin. The second possibility, scenario B, represents the opposite pattern, which has recently been conjectured to be a property of hadronic molecules [40].

In scenario A the inclusion of the $D\Lambda_c$ channel is inconsequential for the description of the molecular pentaquarks: the $D\Lambda_c$ can effectively be ignored, as the transition potential between the $D\Lambda_c \rightarrow D^*\Sigma_c$ channels is weak. However this is not the case in scenario B, where the inclusion of the $D\Lambda_c$ channel can potentially have important consequences on the pentaquark spectrum. In this case the coupling between the $D\Lambda_c$ and $D^*\Sigma_c$ channels is strong enough as to facilitate the binding of the $D\Lambda_c$ system in S-wave, as happened in Ref. [8]. Right now there is no experimental determination of the quantum numbers of the pentaquarks, with different theoretical explorations favoring different scenarios. We see a preference towards A in Refs. [8, 45] and towards B in Refs. [7, 9, 44, 45], though other scenarios are possible: for instance in Ref. [8] the $\frac{1}{2}^- D^*\Sigma_c$ pentaquark does not bind. Within the molecular picture there seems to be a tendency for pionless theories to favor A, while theories that include pion exchange effects tend to fall into scenario B.

If scenario B happens to be the one preferred by nature, the prospects for the $J^P = \frac{1}{2}^+$ $D\Lambda_c$ molecule to bind are good: though the fate of this bound state is ultimately contingent on the unknown short-distance details of the interaction, phenomenological arguments indicate a moderate attraction between the $D$ meson and $\Lambda_c$ baryon at short distances. If this is the case and this molecule binds, it might very well be that the $P_c(4457)$ is a double peak, containing both a $D^*\Sigma_c$ and a $D\Lambda_c$ molecule with opposite parities. If scenario A is the one that actually describes the pentaquarks, the $J^P = \frac{1}{2}^-$ pentaquark cannot be discarded either — there is the possibility that it binds even without coupling to the $D^*\Sigma_c$ channel — but is less likely to exist nonetheless.

Yet we stress the exploratory nature of the present manuscript: the EFT framework requires experimental input and a series of assumptions for it to be able to generate predictions. In this regard it would be very welcome to have phenomenological explorations of the $D\Lambda_c$ interaction and the $D\Lambda_c \rightarrow D^*\Sigma_c$ transition.

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Appendix A: Pion- and Rho-like couplings

In this appendix we discuss the possible sources of saturation of the $D_a$ and $D_b$ contact-range couplings that mediate the $D^*\Sigma_c \rightarrow D\Lambda_c$ transition. Regarding the coupling $D_a$, its similarity with the exchange of a pseudoscalar is evident from a direct comparison of from particularizing the contact-range potential of Eq. (25) for $D_b = 0$, that is:

$$V_{\text{C2}(a)}(1 \rightarrow 2) = +D_a \vec{\sigma}_{L1} \cdot \vec{q},$$

(A1)

and the OPE potential of Eq. (53). Saturation of the $D_a$ coupling from a derivative pseudoscalar meson such as the pion will lead to the approximation [31, 40]

$$D_a^{(\pi)} \propto \frac{g_1 f_2}{\sqrt{2} f_{\pi}} \vec{r}_1 \cdot \vec{r}_2 \frac{\omega_\pi}{\mu_\pi^2}.$$  (A2)

However saturation is only known to work if the regularization scale is close to the mass of the exchanged meson [40]. Taking into account that the pion is the lightest meson and that the cutoff range we are using is $\Lambda = (0.5 - 1.0)$ GeV, we do not expect the $D_a$ coupling to receive contributions coming from pion exchange. If we consider the exchange of heavier mesons, there is no clear candidate for the exchange of a pseudoscalar meson in the mass range comprised by our choice of a cutoff.
Hence the decision to consider that \( D_a = 0 \) at lowest order.

For the \( D_b \) coupling the situation is different, because the \( \bar{D}^* \Sigma_c \rightarrow \bar{D} \Lambda_{c1} \) transition can happen via rho-exchange. The relevant Lagrangians read

\[
\mathcal{L}_{\rho L} = g_{\rho L} q_L \tau_a \rho a_0 q_L - \frac{f_{\rho L}}{4M} \epsilon^{ijk} q_L \sigma_{L,k} \gamma_5 (\partial_i \rho_{a j} - \partial_j \rho_{a i}) q_L, \quad (A3)
\]

\[
\mathcal{L}_{\rho R} = \frac{f_{\rho R}}{2M} \bar{q} R_L a_0 L + C.C. \quad (A4)
\]

where \( q_L, a_L \), and \( v_L \) are the light subfields of the \( D^{(*)} \), \( \Sigma^{(*)} \), and \( \Lambda^{(*)} \) charmed hadrons, \( \rho_{a \mu} \) is the rho meson field, with \( \mu \) a Lorentz index (\( i \) is used to indicate \( \mu = 1, 2, 3 \)) and \( a \) and isospin index, \( t_a \) and \( \tau_a \) are isospin matrices, \( g_{\rho L}, f_{\rho L} \), and \( f_{\rho R} \) are coupling constant, and \( M \) is a mass scale for the magnetic and dipole terms (i.e. the piece proportional to \( g_{\rho L} \), \( f_{\rho L} \) respectively). The charge-like term (i.e. the one proportional to \( g_{\rho L} \)) can contribute to \( \bar{D} \Sigma_c \rightarrow \bar{D} \Lambda_{c1} \), \( \bar{D}^* \Sigma_c \rightarrow \bar{D}^* \Lambda_{c1} \) transitions, but not to the \( \bar{D}^{(*)} \Sigma_c \rightarrow \bar{D}^{(*)} \Lambda_{c1} \) one which is of interest for this work. The magnetic and electric dipole terms of these Lagrangian lead to the potential

\[
V_\rho(\bar{q}, 1 \rightarrow 2) = \frac{f_{\rho L}}{2M} \frac{f_{\rho R}}{2M} \bar{q} L \bar{T}_1 \cdot \bar{T}_2 \bar{q} \cdot (\bar{q} L \times \bar{T}_L),
\]

(A5)

where \( \omega_\rho \simeq (m(\Lambda_{c1}) - m(\Sigma_c)) \simeq (m(D^*) - m(D)) \), \( \mu^2 = m_{\rho}^2 - \omega_\rho^2 \) and the rest of the terms have the same meaning as in Eqs. \((25)\) and \((63)\). Finally the saturation of the \( D_b \) coupling by the rho will lead to a value proportional to

\[
\bar{D}_b^{(\rho)} \propto \frac{f_{\rho L}}{2M} \frac{f_{\rho R}}{2M} \bar{q} \bar{L} \cdot \bar{T}_2 \bar{q} \cdot \mu^2.
\]

(A6)

This is why we keep \( D_b \) as a leading-order effect, but consider \( D_a \) to be subleading.

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