Stability of axisymmetric CMC surfaces as steady states for the evolution by surface diffusion

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Abstract. The surface diffusion equation is one of the geometric evolution laws and was first derived by Mullins [6] to model the development of surface grooves at the grain boundaries of a heated polycrystal. It is known that the surface diffusion equation is obtained as the $H^{-1}$-gradient flow of the area functional for the evolving surfaces. Owing to this variational structure, we expect that the constant mean curvature surfaces (CMC surfaces) are steady states for this equation. In this paper, the criteria of the stability of the axisymmetric CMC surfaces will be studied with the help of the linearized stability analysis of the surface diffusion equation.

1. Introduction

Let $\Gamma_t \subset \mathbb{R}^3$ be an evolving surface. The evolution of $\Gamma_t$ is governed by the geometric evolution law

$$V = -\Delta_{\Gamma_t} H \text{ on } \Gamma_t. \tag{1}$$

Here $V$ and $H$ are the normal velocity and the mean curvature of $\Gamma_t$, respectively, and $\Delta_{\Gamma_t}$ is the Laplace-Beltrami operator on $\Gamma_t$. The geometric evolution equation (1) is called the surface diffusion equation, which was first derived by Mullins [6] to model the development of surface grooves at the grain boundaries of a heated polycrystal. Taylor and Cahn [8] showed that the surface diffusion equation (1) is the $H^{-1}$-gradient flow of the area functional for $\Gamma_t$, so that (1) has a variational structure that the area of $\Gamma_t$ decreases in time $t$ whereas the volume of the domain enclosed by $\Gamma_t$ is preserved. This variational structure indicates that the constant mean curvature surfaces (CMC surfaces) are the steady states for (1).

In this paper, we consider the following problem. For $\phi_{\pm} : \mathbb{R}_+ \to \mathbb{R}$, set

$$\Pi_{\pm} = \{(\phi_{\pm}(|\eta|), \eta)^T \mid \eta \in \mathbb{R}^2\}, \quad \Omega = \{(x, \eta)^T \mid \phi_-(|\eta|) \leq x \leq \phi_+(|\eta|), \eta \in \mathbb{R}^2\}. \tag{2}$$

Note that $\partial\Omega = \Pi_- or \Pi_+$. Let us assume that $\Gamma_t \subset \Omega$ and the evolution of $\Gamma_t$ is governed by

$$ V = -\Delta_{\Gamma_t} H \text{ on } \Gamma_t, $$

$$(N_{\Gamma_t}, N_{\Pi_{\pm}})^{\mathbb{R}^3} = \cos \theta_{\pm} \text{ on } \Gamma_t \cap \Pi_{\pm}, $$

$$(\nabla_{\Gamma_t} H, \nu_{\pm})^{\mathbb{R}^3} = 0 \text{ on } \Gamma_t \cap \Pi_{\pm}, $$

$$\Gamma_{t|t=0} = \Gamma_0. \tag{2}$$
Here, $N_t$ and $N_{t_2}$ are the outer unit normals to $\Gamma_t$ and $\Pi_2$ ($= \partial \Omega$), respectively, and $\nu_\pm$ are the outer unit co-normals to $\partial \Omega$ on $\Gamma_t \cap \Pi_\pm$. The problem (2) are obtained as the $H^{-1}$-gradient flow of the capillary energy

$$\text{Area}[\Gamma_t] + \mu_+ \text{Area}[\Sigma_{t,+}] + \mu_- \text{Area}[\Sigma_{t,-}],$$

where $\Sigma_{t,\pm}$ are the part of $\Pi_\pm$ with the boundary $\partial \Sigma_{t,\pm} = \Gamma_t \cap \Pi_\pm$. Note that contact angles $\theta_\pm$ are given by $\cos \theta_\pm = \mu_\pm$ (see [1]).

Our aim is to derive the criteria of the stability of the axisymmetric CMC surfaces, namely, the Delaunay surfaces, which are the steady states of (2), with the help of the linearized stability analysis to (2).

2. Delaunay surfaces and the eigenvalue problem

Let $\Gamma_*$ be the Delaunay surfaces and set

$$\Gamma_* = \{(x_*(s), y_*(s) \cos \zeta, y_*(s) \sin \zeta)^T | s \in [0, d], \zeta \in [0, 2\pi]\},$$

where $s$ is the arc-length parameter of a generating curve $(x_*(s), y_*(s))^T$. Furthermore, let $H_*$ be a constant satisfying $H_* \neq 0$ (assuming $H_* < 0$). According to [3, 7, 2], a generating curve $(x_*(s), y_*(s))^T$ of the Delaunay surface with a constant mean curvature $H_*$ is represented by

$$x_*(s) = \int_0^s \frac{1 - B \sin(2H_*(\sigma - \tau))}{\sqrt{1 + B^2 - 2B \sin(2H_*(\sigma - \tau))}} d\sigma,$$

$$y_*(s) = -\frac{1}{2H_*} \sqrt{1 + B^2 - 2B \sin(2H_*(s - \tau))},$$

where $B \in (0, 1)$ and $\tau \in \mathbb{R}$ are constants. Note that $\Gamma_*$ is a cylinder if $B = 0$, an unduloid if $0 < B < 1$, a sphere if $B = 0$ and a nodoid if $B > 1$.

Applying an axisymmetric perturbation $v(s, t)$ from the Delaunay surfaces $\Gamma_*$ and linearizing it, we obtain

$$\left\{ \begin{array}{l}
  v_t = -\frac{1}{2} \Delta_{\Gamma_*} L[v] \quad \text{for } (s, t) \in [0, d] \times [0, T], \\
  \partial_s v \pm (\kappa_{\Pi_\pm} \csc \theta_\pm - \kappa_{\Gamma_*} \cot \theta_\pm) v = 0 \quad \text{for } s = 0, d, \ t \in [0, T], \\
  \partial_s L[v] = 0 \quad \text{for } s = 0, d, \ t \in [0, T],
\end{array} \right. \quad (3)$$

where $L[v] = \Delta_{\Gamma_*} v + |A_*|^2 v$ with

$$\Delta_{\Gamma_*} = \frac{1}{y_*} \left\{ \partial_s (y_* \partial_s) + \frac{1}{y_*} \partial_{\zeta}^2 \right\}, \quad |A_*|^2 = (-x_*'' y_*' + x_*' y_*'')^2 + \left( \frac{x_*'}{y_*} \right)^2,$$

and

$$\kappa_{\Pi_\pm} = \pm \frac{\phi_+(y_*)}{\{1 + (\phi_+(y_*))^2\}^{3/2}}, \quad \kappa_{\Gamma_*} = -x_*'' y_*' + x_*' y_*''.$$

Note that $\kappa_{\Pi_\pm}$ and $\kappa_{\Pi_2}$ are the curvature of $x = -\phi_-(y)$ at $y = y_*(0)$ and $x = \phi_+(y)$ at $y = y_*(d)$, respectively, and $\kappa_{\Gamma_*}$ is the curvature of the generating curve $(x_*(s), y_*(s))^T$. Since $v$ is an axisymmetric perturbation, $v$ is independent of $\zeta$, so that $\Delta_{\Gamma_*} v = (1/y_*) \{\partial_s (y_* \partial_s v)\}$.

Let us consider the eigenvalue problem corresponding to the linearized problem (3):

$$\left\{ \begin{array}{l}
  -\Delta_{\Gamma_*} L[w] = \lambda w \quad \text{for } s \in [0, d], \\
  \partial_s w \pm (\kappa_{\Pi_\pm} \csc \theta_\pm - \kappa_{\Gamma_*} \cot \theta_\pm) w = 0 \quad \text{at } s = 0, d, \\
  \partial_s L[w] = 0 \quad \text{at } s = 0, d.
\end{array} \right. \quad (4)$$
We say that the steady states $\Gamma_s$ is linearly stable under an axisymmetric perturbation if and only if all of eigenvalues of (4) are negative. Now we obtain the following properties for the eigenvalue problem (4):

- $\{\lambda_n\} \subset \mathbb{R}$ (i.e. Linearized operator is self-adjoint in the suitable setting) with $\lambda_1 \geq \lambda_2 \geq \cdots$. This means that if $\lambda_1 < 0$, $\Gamma_s$ are linearly stable.
- The eigenvalues depend continuously on the parameters $\kappa_{\Pi\pm}$, $\kappa_{\Gamma_s}$ and $d$, and are monotone decreasing in $\kappa_{\Pi\pm}$.
- $\lambda_1 \leq 0$ if $d$ is small enough and $\kappa_{\Pi\pm}$ is large enough.
- 0 is an eigenvalue of (4) if and only if the parameters $\kappa_{\Pi\pm}$, $H_s$, $B$, $d$, $\tau$ and $\theta_{\pm}$ fulfill

$$A^w(H_s, B, d, \tau)\kappa_{\Pi\pm} + B^w(H_s, B, d, \tau, \theta_+)\kappa_{\Pi\pm} + B^w(H_s, B, d, \tau, \theta_-)\kappa_{\Pi\pm} + C^w(H_s, B, d, \tau, \theta_{\pm}) = 0.$$  

Here $A^w$, $B^w$ and $C^w$ depend on the configuration of $\Gamma_s$ and satisfy $B^w B^w - A^w C^w \geq 0$.

Let $d_1 > 0$ be the first zero point of $A^w(H_s, B, d, \tau)$ and define the left hand side of (5) as $D(\kappa_{\Pi\pm}, H_s, B, d, \tau, \theta_{\pm})$. Then the properties mentioned in the above imply that if the parameters $\kappa_{\Pi\pm}, H_s, B, d, \tau$ and $\theta_{\pm}$ satisfy

$$D(\kappa_{\Pi\pm}, H_s, B, d, \tau, \theta_{\pm}) > 0, \quad \kappa_{\Pi\pm} > -\frac{B^w(H_s, B, d, \tau, \theta_-)}{A^w(H_s, B, d, \tau)}$$

then $\lambda_1 < 0$, that is, the steady states $\Gamma_s$ are linearly stable under an axisymmetric perturbation.

![Figure 1](image1.png)

**Figure 1.** An example of the configuration drawn by (5) for $0 < d < d_1$ in the $(\kappa_{\Pi\pm}, d, \kappa_{\Pi\pm})$-coordinate space. In this figure, $H_s, B, \tau$ and $\theta_{\pm}$ are given.

If you want to know the details of this section, see [4, 5].

3. Examples

Let us consider, for example, the case of $\theta_- = \pi/4$, $\theta_+ = \pi/3$, $\Pi_- = \{(0, \eta)^T | \eta \in \mathbb{R}^2\}$ and $\Pi_+ = \{(3, \eta)^T | \eta \in \mathbb{R}^2\}$. In Figure 2, we introduce some possible configurations of the Delaunay surfaces. Figure 3 shows the criterion of the linearized stability for the Delaunay surfaces in Figure 2. If the pair of parameters $(\kappa_{\Pi\pm}, \kappa_{\Pi\pm})$ is included in the gray region, the corresponding Delaunay surface is linearly stable under an axisymmetric perturbation. According to Figure 3, for all of the cases, $(\kappa_{\Pi\pm}, \kappa_{\Pi\pm}) = (0, 0)$ is included in the gray region. This implies that all of the Delaunay surfaces in Figure 2 are linearly stable. In Figure 4, we introduce another possible configuration of the unduloids in this setting for $0 < d < d_1$ and the criterion of the linearized stability of its unduloid. In this case, $(\kappa_{\Pi\pm}, \kappa_{\Pi\pm}) = (0, 0)$ is not included in the gray region. That is, the unduloid in Figure 4 is unstable.
Figure 2. The case of $\theta_- = \pi/4$ and $\theta_+ = \pi/3$. (a) A part of the unduloid with $B = 0.75$, $H_\ast \approx -0.50$, $d \approx 3.28$ and $\tau \approx 0.45$. (b) A part of the sphere with $H_\ast \approx -0.40$, $d \approx 3.25$ and $\tau = 0$. (c) A part of the nodoid with $B = 1.25$, $H_\ast \approx -0.36$, $d \approx 3.25$ and $\tau \approx -0.26$.

Figure 3. The criteria of the linearized stability for the configurations in Figure 2. The left is for (a), the middle is for (b) and the right is for (c), respectively.

Figure 4. (Left) A part of the unduloid with $B = 0.75$, $H_\ast \approx -0.71$, $d \approx 3.51$ and $\tau \approx 0.32$ in the case of $\theta_- = \pi/4$ and $\theta_+ = \pi/3$. (Right) The criterion of the linearized stability for this unduloid.

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