The asymptotic comparison of random walks on topological abelian groups

Tobias Fritz

Abstract. We study the asymptotic behaviour of random walks on topological abelian groups $G$. Our main result is a sufficient condition for one random walk to dominate another in the stochastic order induced by any suitably large positive cone $G_+ \subseteq G$, assuming that both walks have Radon distributions and compactly supported steps. We explain in which sense our sufficient condition is very close to a necessary one. Our result is due to Aubrun and Nechita in the one-dimensional case, but new already for $\mathbb{R}^n$ with $n > 1$. It is a direct application of a recently proven theorem of real algebra, namely a Vergleichsstellensatz for preordered semirings.

We then use our result to derive a formula for the rate at which the probabilities of a random walk decay relative to those of another, again for walks on $G$ with compactly supported Radon steps. This can be seen as a relative version of Cramér’s large deviation theorem, since it specializes to the latter in the case where one walk is deterministic.

Contents

1. Introduction 2
2. Radon measures on Hausdorff spaces 4
3. Preordered topological abelian groups 4
4. The preordered semialgebra of measures 6
5. Asymptotic comparison of random walks 12
6. The normalized cumulant-generating function 16
7. A uniform large deviation result and Cramér’s theorem 19
References 23

2010 Mathematics Subject Classification. Primary: 60G50, 60E15; Secondary: 60F10, 06F25, 16Y60.

Acknowledgements. We thank Richard Künig, Rostislav Matveev, Luciano Pomatto, Matteo Smerlak, Arleta Szkoła, Omer Tamuz and Péter Vrana for useful discussions and feedback, as well as David Handelman and Terence Tao for discussion on MathOverflow. Part of this work has been conducted while the author was with the Max Planck Institute for Mathematics in the Sciences and later with the Perimeter Institute for Theoretical Physics, both of which we thank for their outstanding research environments.
1. Introduction

Probability theory offers many classical results on the asymptotic behaviour of random walks, including the strong and weak laws of large numbers, the central limit theorem, and Cramér’s large deviation theorem. In this paper, we will prove a theorem on the asymptotic comparison of two random walks; instead of analyzing only one random walk at a time, it turns out to be interesting and beneficial to compare two random walks, so that some understanding of one can be leveraged to gain information about the other. As we will see, this type of result can be interesting and nontrivial already when one of the two walks is deterministic.

While our main result applies quite generally to walks on topological abelian groups, we state it now for easier readability for the case of random walks on \( \mathbb{R} \); this one-dimensional case has been proven previously by Aubrun and Nechita [2, Section 2].

1.1. Theorem ([2]). Let random variables \( X \) and \( Y \) be real-valued and bounded, and let \((X_i)_{i \in \mathbb{N}}\) and \((Y_i)_{i \in \mathbb{N}}\) be i.i.d. copies. Consider the following conditions:

(i) There is a random variable \( Z \), independent of \( X \) and \( Y \), such that
\[
P[X + Z \geq c] \leq P[Y + Z \geq c] \quad \forall c \in \mathbb{R}.
\] (1.1)

(ii) For \( n \geq 1 \),
\[
P \left[ \sum_{i=1}^{n} X_i \geq c \right] \leq P \left[ \sum_{i=1}^{n} Y_i \geq c \right] \quad \forall c \in \mathbb{R}.
\] (1.2)

(iii) With \( < \) standing for \( < \) or \( \leq \), and similarly for \( > \), the following hold:
\[
E[e^{tX}] < E[e^{tY}], \quad E[e^{-tX}] > E[e^{-tY}] \quad \forall t \in \mathbb{R}_{>0}
\]
\[
\max X < \max Y, \quad \min X > \min Y,
\]
\[
E[X] < E[Y].
\]

Then (i) or (ii) for some \( n \geq 1 \) implies that (iii) holds with non-strict inequalities. Conversely if (iii) holds with strict inequalities, then (i) and (ii) for all \( n \gg 1 \) follow.

Our main result is a generalization of this theorem from the one-dimensional case to the stochastic preorder on topological abelian groups equipped with a suitably large positive cone. In particular, this is new already for the case of \( \mathbb{R}^n \) with \( n > 1 \). We will state and prove this as Theorem 5.6.

The implication from (i) or (ii) to (iii) with non-strict inequalities should not be surprising and is easy to see upon applying each of the relevant summary statistics that appear in (iii). The difficult direction is the converse implication formulated in the final sentence. While this was proven by Aubrun and Nechita in the case of Theorem 1.1 by conventional

---

\(^1\) A minor difference is that Aubrun and Nechita allow \( X \) to be unbounded on the left and \( Y \) to be unbounded on the right. A simple truncation argument can be used to obtain this generalized version from the current Theorem 1.1.
large deviation methods, our proof of the more general Theorem 5.6 proceeds by showing that it is an instance of our recent Vergleichsstellensatz for preordered semirings [7, Theorem 8.6]. In other words, our proof is by reduction to a purely algebraic result.

In Theorem 7.1, we then apply this main result to derive a formula for how the tail probabilities of one random walk decay relative to those of another random walk. This applies to random walks with compactly supported Radon steps on topological vector spaces equipped with a suitably large positive cone. In the $\mathbb{R}$-valued case, this takes the following form.

1.2. Theorem. Let all random variables be real-valued and bounded. Then for random variables $X$ and $Y$ and i.i.d. copies $(X_i)_{i \in \mathbb{N}}$ and $(Y_i)_{i \in \mathbb{N}}$, we have
\[
\sup_{\varepsilon > 0} \limsup_{n \to \infty} \sup_{c \in \mathbb{R}} \frac{1}{n} \log \frac{P\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \geq c + \varepsilon \right]}{P\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i \geq c - \varepsilon \right]} = \sup_{t \geq 0} \log \frac{E[e^{tX}]}{E[e^{tY}]},
\]
where this equation holds in two versions, with $\lim\sup_{n \to \infty}$ standing for $\lim\inf_{n \to \infty}$ or $\lim\sup_{n \to \infty}$.

Taking $X$ to be a fixed number recovers a version of Cramér’s large deviation theorem (for bounded $Y$). We show this in Corollary 7.4, again for random variables taking values in topological vector spaces. It is instructive to see how the Legendre–Fenchel transform of the cumulant-generating function of $Y$ arises from the right-hand side of (1.3) upon taking $X$ to be deterministic.

Summary. We now briefly summarize the content of the individual sections of this paper.

- We briefly present measure-theoretic preliminaries in Section 2, relevant in particular to the infinite-dimensional case.
- We discuss preordered topological abelian groups in Section 3, including the introduction of an order unit condition relevant to our main results (Definition 3.3).
- We consider preordered semialgebras in Section 4 and restate a simplified version of our [7, Theorem 8.6] as Theorem 4.14. This is the algebraic result from which our present results will follow as a special case. We also introduce the relevant version of the test spectrum of a preordered semialgebra in Definition 4.15. Moreover, we also explain how compactly supported Radon measures on a preordered topological abelian group (with an order unit) form a preordered semialgebra of the relevant kind, with multiplication given by the convolution of measures.
- Section 5 then states and proves our main result as Theorem 5.6, switching from measure-theoretical terminology and notation to random variables language.
- Section 6 investigates the test spectrum for the preordered semialgebra of measures under convolution further, and explains in particular how the summary statistics of (iii) relate to cumulant-generating functions. We also sketch how our normalization convention for the cumulant-generating function makes it behave like a family of weighted averages.
Finally, Section 7 uses the results from the previous two sections to derive the general version of Theorem 1.2, and derives a version of Cramér’s large deviation theorem as Corollary 7.4.

2. Radon measures on Hausdorff spaces

We start with some measure-theoretic preliminaries, which are relevant mainly for getting maximum mileage out of our methods by treating infinite-dimensional situations jointly with the finite-dimensional case. Readers who are only interested in random walks in finite dimensions, where our results are still new and nontrivial, can safely skip this preliminary section.

We write $\text{Haus}$ for the category of Hausdorff spaces and continuous maps. For $A \in \text{Haus}$, we denote by $\mathcal{M}(A)$ the set of finite (unsigned) Radon measures on $A$, i.e. the set of finite Borel measures that are inner regular. The regularity of $\mu \in \mathcal{M}(A)$ guarantees that there is a largest closed set of full measure, called the support $\text{supp}(\mu)$. Note that $\mathcal{M}(A)$ contains all finitely supported finite measures. If $f : A \to B$ is continuous, then pushforward of measures defines a map $\mathcal{M}(f) : \mathcal{M}(A) \to \mathcal{M}(B)$ \cite[Section I.5]{16}. We thereby obtain a functor $\mathcal{M} : \text{Haus} \to \text{Set}$.

For $A, B \in \text{Haus}$, their product space $A \times B$ is again in $\text{Haus}$. Applying the functoriality to the product projections $A \times B \to A$ and $A \times B \to B$ produces the two components of the marginalization map

$$\Delta_{A,B} : \mathcal{M}(A \times B) \longrightarrow \mathcal{M}(A) \times \mathcal{M}(B).$$

(2.1)

In the other direction, the formation of product measures \cite[p. 63]{16} induces a product map

$$\nabla_{A,B} : \mathcal{M}(A) \times \mathcal{M}(B) \longrightarrow \mathcal{M}(A \times B),$$

(2.2)

where the support of the product measure on $A \times B$ is exactly the product of the respective supports in $A$ and $B$. Both $\Delta_{A,B}$ and $\nabla_{A,B}$ are natural in $A$ and $B$ in the sense of category theory. We refer to \cite{9} for a general theory of these structures and the equations they satisfy.

The compactly supported finite Radon measures form a subset $\mathcal{M}_c(A) \subseteq \mathcal{M}(A)$. It is easy to see that the marginalization and product maps above restrict to corresponding maps on $\mathcal{M}_c$, namely for any $A, B \in \text{Haus}$, we have

Marginalization map: $\Delta_{A,B} : \mathcal{M}_c(A \times B) \longrightarrow \mathcal{M}_c(A) \times \mathcal{M}_c(B)$, \hspace{1cm} (2.3)

Product map: $\nabla_{A,B} : \mathcal{M}_c(A) \times \mathcal{M}_c(B) \longrightarrow \mathcal{M}_c(A \times B)$, \hspace{1cm} (2.4)

3. Preordered topological abelian groups

In this section, we state the relevant definitions for preordered topological abelian groups. For us, a topological group is a group $G$ with a Hausdorff topology such that both the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are continuous. Throughout, we work with topological abelian groups using additive notation.
3.1. Definition. A topological abelian group $G$ is **preordered** if it comes equipped with a positive cone, which is a distinguished closed subset $G_+ \subseteq G$ with

$$G_+ + G_+ \subseteq G_+, \quad 0 \in G_+.$$ 

We refer to [12] for further background on preordered abelian groups in the purely algebraic context.

The paradigmatic examples that we have in mind at this point are $G = \mathbb{R}^d$ with $G_+$ any closed convex cone, or more generally topological vector space equipped with a closed convex cone [1]. In the latter case, we will say that $G$ is a **preordered topological vector space**. However, if $G$ is a topological vector space, then the positive cone $G_+$ is not automatically closed under positive scalar multiplication: taking $G = \mathbb{R}$ and

$$G_+ := \{0\} \cup [1, \infty)$$

still produces a preordered topological abelian group in our sense, but not a preordered topological vector space. The following example is even more peculiar.

3.2. Remark. For a preordered topological abelian group $G$, the set $G_+ - G_+$ is automatically a subgroup of $G$, but it need not be open or closed.

For example, consider the topological abelian group $G = \mathbb{R}$ with positive cone $G_+$ given by zero together with all rationals $p/q$ satisfying $q > 0$ and $p/q \geq \log q$. A short computation shows that this set is indeed closed under addition. It is topologically closed since it contains only finitely many points in every bounded interval. However, we clearly have $G_+ - G_+ = \mathbb{Q}$.

For $a, b \in G$, we write $a \leq b$ as usual if $b - a \in G_+$, defining a preorder relation on $G$ which is translation-invariant. In terms of this, we also have the **order interval**

$$[a, b] := \{x \in G \mid a \leq x \leq b\} = (a + G_+) \cap (b - G_+),$$

which is clearly closed. For a subset $S \subseteq G$, we also write

$$\downarrow S := \{x \in G \mid \exists s \in S, \ x \leq s\} = S - G_+$$

for the **downset** generated by $S$, and similarly $\uparrow S$ for the **upset** $\uparrow S := S + G_+$. A set $S$ is **downward closed** if it is equal to its own downset; and similarly $S$ is **upwards closed** if it is its own upset.

The following definition is standard, at least in the purely algebraic setting [12, p. 4].

3.3. Definition. Let $G$ be a preordered topological abelian group. Then an **order unit** is an element $u \in G_+$ such that:

(a) For every $x \in G$ there is $k \in \mathbb{N}$ with $x \leq ku$.

(b) The order interval $[-u, +u]$ is a neighbourhood of $0 \in G$.

In particular, if $G_+$ has an order unit, then $G = G_+ - G_+$.

3.4. Remark. We comment on the relation between these two conditions. For preordered topological vector spaces, it is well-known that (b) implies (a) [1, Lemma 2.5]. But this is
not true for preordered topological abelian groups in general. An almost trivial example is 
\( G = \mathbb{Z} \) and \( G_+ = \{0\} \), for which \( u = 0 \) satisfies \( (b) \) but not \( (a) \).

The other direction already fails for preordered topological vector spaces. For example, consider \( C([0,1]) \) equipped with the weak-* topology and preordered with respect to the usual closed convex cone containing the nonnegative functions. Then the constant function \( u := 1 \) satisfies \( (a) \), but the order interval \([-1,+1]\) in \( C([0,1]) \) is not a neighbourhood of zero.

3.5. Example. If \( G \) is finite, then every submonoid \( G_+ \subseteq G \) is a positive cone. Since every element is torsion, the positive cones are then exactly the subgroups. It follows that \( G_+ \) has an order unit if and only if \( G = G_+ \), in which case every element is an order unit.

3.6. Example. For \( G = \mathbb{Z}^d \), a positive cone \( G_+ \) has an order unit if and only if \( G = G_+ - G_+ \). Indeed if this condition holds, then we can write the standard basis vectors as \( e_i = x_i - y_i \) for \( x_i, y_i \in G_+ \) for all \( i = 1, \ldots, d \). Hence \( u := x_1 + \ldots + x_d \) is an order unit.

3.7. Example. For \( G = \mathbb{R}^d \) as a topological vector space, consider any closed convex cone \( G_+ \subseteq \mathbb{R}^d \). Then an element of \( G_+ \) is an order unit if and only if it is a topologically interior point.

3.8. Lemma. Let \( u \in G_+ \) be an order unit. Then for every compact \( C \subseteq G \) there is \( k \in \mathbb{N} \) with

\[
C \subseteq \downarrow\{ku\} \cap \uparrow\{-ku\}.
\]

Proof. It is enough to prove \( C \subseteq \downarrow\{ku\} \) for some \( k \), since then \( C \subseteq \uparrow\{-ku\} \) for some \( k \) follows by symmetry. For every \( x \in C \) we have \( k_x \in \mathbb{N} \) with \( x \leq k_x u \). But then also \( x' \leq (k_x + 1)u \) for every \( x' \in [x - u, x + u] \). Since this order interval is a neighbourhood of \( x \), the compactness implies that there are finitely many \( x_1, \ldots, x_n \in C \) such that \( C \subseteq \bigcup_i [x_i - u, x_i + u] \). With \( k := \max_{i=1,...,n} k_{x_i} \), the claim \( C \subseteq \downarrow\{ku\} \) now follows.

\[\Box\]

4. The preordered semialgebra of measures

Convolution. If \( G \) is a topological abelian group, then the multiplication \( G \times G \to G \) induces the convolution of measures map defined as the composition

\[
\mathcal{M}(G) \times \mathcal{M}(G) \longrightarrow \mathcal{M}(G \times G) \longrightarrow \mathcal{M}(G),
\]

where the first map is an instance of \( (2.2) \) and the second one is by functoriality of \( \mathcal{M} \) applied to the multiplication map. More explicitly, the integral of a measurable function \( f : G \to \mathbb{R} \) against the convolution \( \mu \ast \nu \) of two measures \( \mu, \nu \in \mathcal{M}(G) \) is given by \([5, 444]\),

\[
\int f d(\mu \ast \nu) = \int \int f(gh) d\mu(g) d\nu(h).
\]

This convolution operation turns \( \mathcal{M}(G) \) into a commutative monoid with neutral element \( \delta_0 \). A convenient way of proving the relevant associativity and commutativity properties is to use the corresponding associativity and commutativity properties of the formation of product measures \( (2.2) \), which amount to the fact that \( \mathcal{M} \) is a lax symmetric monoidal
functor. The traditional computational proof using the explicit formula for convolution [4, Section 2.5] is the same in spirit.

By Equation (2.4), it follows that $\mathcal{M}_c(G)$ is closed under convolution, and therefore becomes a submonoid of $\mathcal{M}_c(G)$. For $x, y \in G$, we have $\delta_x * \delta_y = \delta_{x+y}$, which makes the inclusion

$$ G \rightarrow \mathcal{M}_c(G), \quad x \mapsto \delta_x $$

(4.3)

into a homomorphism of commutative monoids.

Recall also that if $X$ and $Y$ are independent $G$-valued random variables with distributions $\mu$ and $\nu$, then $\mu * \nu$ is the distribution of the $G$-valued variable $X + Y$.

The stochastic preorder. Suppose now that $G$ is a preordered topological abelian group with positive cone $G_+$. We now extend the resulting preorder on $G$ to a preorder on $\mathcal{M}(G)$. We refer to Strassen [17, Theorem 11], Edwards [3, Theorem 7.1] and Kellerer [14, Proposition 3.12] for more general definitions and proofs of the following equivalence, which crucially rely on the assumption that the measures involved are Radon, but neither use the group structure nor Hausdorffness of $G$.

4.1. Proposition. For $\mu, \nu \in \mathcal{M}(G)$ with $\mu(G) = \nu(G)$, the following are equivalent:

(a) $\mu(C) \leq \nu(C)$ for every closed upset $C \subseteq G$.

(b) $\mu(U) \leq \mu(U)$ for every open upset $U \subseteq G$.

(c) For every monotone and lower semi-continuous function $f : G \rightarrow \mathbb{R}$, we have

$$ \int f \, d\mu \leq \int f \, d\nu. $$

(d) There is $\lambda \in \mathcal{M}(G \times G)$ with marginals $\mu$ and $\nu$, and such that $\lambda$ is supported on the preorder relation $\{(x, y) \mid x \leq y\} \subseteq G \times G$.

4.2. Definition. The stochastic preorder is the relation on $\mathcal{M}(G)$ defined by these equivalent conditions.

We also write $\mu \leq \nu$ to denote this relation for $\mu, \nu \in \mathcal{M}(G)$. By definition, $\mu \leq \nu$ can hold only if the normalizations are the same, $\mu(G) = \nu(G)$. Intuitively, $\mu \leq \nu$ means that $\nu$ can be obtained from $\mu$ by merely moving mass upwards in the preorder. Note that $x \leq y$ in $G$ is equivalent to $\delta_x \leq \delta_y$ in $\mathcal{M}(G)$.

Either of the first three equivalent conditions obviously shows that $\leq$ is a preorder relation, i.e. is reflexive and transitive. If the preorder on $G$ is antisymmetric, or equivalently if $G_+ \cap (-G_+) = \{0\}$, then it is known that the stochastic preorder is antisymmetric too [8]. The most well-known instance of the stochastic preorder is for $G = \mathbb{R}$ and $G_+ = \mathbb{R}_+$, in which case it is also called the usual stochastic order or first-order stochastic dominance. In this case, we have $\mu \leq \nu$ if and only if

$$ \mu([c, \infty)) \leq \nu([c, \infty)) \quad \forall c \in \mathbb{R}, $$

(4.4)

since in this case the closed and upward closed sets are exactly the $[c, \infty)$. This system of inequalities can be understood intuitively upon thinking of $\mu$ and $\nu$ as return distributions
of a financial asset: then this condition states that the return distribution described by $\nu$ is unambiguously (non-strictly) preferable over the one given by $\mu$ [13].

We now turn to a consideration relating the stochastic preorder with supports.

4.3. Lemma. Suppose that $\mu, \nu \in M(G)$ with $\mu(G) = \nu(G)$ are such that $x \leq y$ for all $x \in \text{supp}(\mu)$ and $y \in \text{supp}(\nu)$. Then $\mu \leq \nu$.

Proof. We assume $\mu(G) = \nu(G) = 1$ without loss of generality. Then this follows e.g. from condition (d) of Proposition 4.1 upon taking $\lambda$ to be the product measure $\mu \otimes \nu$. □

Preordered semialgebra structure. $M(G)$ carries both an additive commutative monoid structure given by addition of measures, as well as the commutative monoid structure given by convolution (4.1), which distributes over the addition. This makes $M(G)$ into a commutative semiring. While we refer to the literature for the full definitions [11], it may help to note that a semiring is like a ring, except in that additive inverses generally do not exist. As usual, we denote its operations by $+$ and $\cdot$ and the corresponding neutral elements by 0 and 1. Since all of the semirings considered in this paper are commutative, we no longer mention the commutativity assumption explicitly, but it will be assumed throughout.

Thus $M_c(G)$ is a semiring and $M(G)$ is a subsemiring. Both semirings also carry an additional scalar multiplication by nonnegative reals, as per the following definition.

4.4. Definition. A semialgebra $S$ is a semiring together with a semiring homomorphism $\mathbb{R}_+ \to S$,

$$\mathbb{R}_+ \times S \longrightarrow S$$

which is additive\(^2\) in each argument and satisfies $1x = x$ as well as $(sx)(xy) = (rs)(xy)$ for all $r, s \in \mathbb{R}_+$ and $x, y \in S$.

Since we will not consider scalar multiplication by any other semiring than $\mathbb{R}_+$, we leave out mention of $\mathbb{R}_+$ in the term “semialgebra”. Here are the two most important examples that are relevant for the theory of preordered semirings itself [7].

4.5. Example. (a) $\mathbb{R}_+$ itself, with its usual algebraic structure, is an $\mathbb{R}_+$-semialgebra.

(b) The tropical reals $\mathbb{T}_+$ are the $\mathbb{R}_+$-semialgebra given by the semiring\(^3\)

$$\left(\mathbb{R}_+, \max, \cdot\right),$$

meaning that addition is formation of the maximum with neutral element 0, while multiplication is as usual. As for scalar multiplication, we put $rx := x$ for all $r \in \mathbb{R}_{>0}$ as well as $0x := 0$.

\(^2\)By additivity, we mean both that binary addition and nullary addition, i.e. the neutral element 0, are preserved.

\(^3\)The tropical reals are usually defined as a different but isomorphic semiring, namely $(\mathbb{R} \cup \{-\infty\}, \max, +)$, where the logarithm and exponential implement an isomorphism between this definition and ours. The multiplicative version that we use turns out to be more convenient for our purposes, in particular Definition 4.15.
Our main object of study will be $M_c(G)$, considered as a semialgebra and together with the stochastic preorder. What type of algebraic structure do we get if we also take the stochastic preorder into account? This preorder interacts well with the algebraic structure by making all algebraic operations monotone, in the following sense.

4.6. Definition. A preordered semiring $S$ is a semiring together with a preorder relation $\leq$ such that for all $a, x, y \in S$,
\[ x \leq y \implies a + x \leq a + y, \quad ax \leq ay. \]

A preordered semialgebra is a semialgebra which is preordered as a semiring.

Note that the scalar multiplication $x \mapsto rx$ for $r \in \mathbb{R}^+$ is automatically monotone, since $x \leq y$ implies that $(r1)x \leq (r1)y$. We present some examples, starting with our primary object of study, and then presenting examples which will also play an important role in our results.

4.7. Example. Let $G$ be a preordered topological abelian group. Then $M_c(G)$ is a preordered semialgebra with respect to addition, scalar multiplication and convolution of measures as algebraic operations, and with respect to the stochastic preorder as preorder.

4.8. Example. If $S$ is a preordered semialgebra, then we write $S^{\text{op}}$ for the same semialgebra but with the opposite preorder, meaning that $x \leq y$ holds in $S^{\text{op}}$ if and only if $x \geq y$ holds in $S$. Clearly $S^{\text{op}}$ is again a preordered semialgebra.

4.9. Example. Consider the semialgebras $\mathbb{R}_+$ and $\mathbb{T}\mathbb{R}_+$ from Example 4.5. Both of these are preordered semialgebras with respect to the usual order on the real numbers. $\mathbb{R}_+^{\text{op}}$ and $\mathbb{T}\mathbb{R}_+^{\text{op}}$ are the same preordered semialgebras carrying the opposite of the usual order.

The following growth condition from [7, Definition 4.44] will be key for us.

4.10. Definition ([6, Definition 3.28]). Let $S$ be a preordered semiring. An element $v \in S$ with $v \geq 1$ is power universal if for every $x \leq y$ in $S$, there is $k \in \mathbb{N}$ such that
\[
    y \leq v^k x. \tag{4.5}
\]

The following shows that this indeed applies in the case of interest to us. We continue assuming that $G$ is a preordered topological abelian group and prove two auxiliary statements about the preordered semialgebra $M_c(G)$.

4.11. Lemma. Let $u \in G_+$ be an order unit (Definition 3.3). Then $v := \delta_u$ is power universal in $M_c(G)$.

**Proof.** If $\mu, \nu \in M_c(G)$ satisfy $\mu \leq \nu$, then in particular $\mu(G) = \nu(G)$. We can thus assume that both $\mu$ and $\nu$ are probability measures without loss of generality. With $\mu_-$ denoting the pushforward of $\mu$ along the inversion map $G \to G$, it is then enough to show that there is $k \in \mathbb{N}$ with
\[
    \nu \leq \delta_{ku}, \quad \mu_- \leq \delta_{ku},
\]
since then also $\delta_{-ku} \leq \mu$ and therefore
\[
    \nu \leq \delta_{2ku} * \delta_{-ku} \leq \delta_{2ku} * \mu = v^{2k} * \mu.
\]
Since the argument for \( \mu_\leq \leq \delta_ku \) is the same, we only need to show that there is \( k \) with \( \nu \leq \delta_ku \). But since \( \text{supp}(\nu) \) is compact by assumption, this follows from Lemma 3.8 and Lemma 4.3.

The following finite approximation result will be a crucial stepping stone for the proof of our main result presented in the next section.

4.12. Lemma. For \( u \in G \) an order unit, the order interval
\[
[\mu * \delta_{-2u}, \mu * \delta_{+2u}]
\]
contains a finitely supported measure for every \( \mu \in M_c(G) \).

Proof. We assume \( \mu(G) = 1 \) without loss of generality. Since \( \text{supp}(\mu) \) is compact by assumption and \( [x - u, x + u] \) is a neighbourhood of \( x \) for every \( x \in G \), we have finitely many \( x_1, \ldots, x_n \in \text{supp}(\mu) \) such that
\[
\text{supp}(\mu) \subseteq \bigcup_{i=1}^n [x_i - u, x_i + u],
\]
as in the proof of Lemma 3.8. The sets \( \text{supp}(\mu) \cap [x_i - u, x_i + u] \) generate a finite Boolean algebra of measurable sets with atoms \( B_1, \ldots, B_m \subseteq \text{supp}(\mu) \). Upon choosing arbitrary points \( y_j \in B_j \), we define
\[
\nu := \sum_{j=1}^m \mu(B_j) \delta_{y_j}.
\]
We then argue that \( \nu \leq \mu * \delta_{2u} \); the other claimed inequality works analogously. Indeed consider the measure on \( G \times G \) given by
\[
\lambda := \sum_{j=1}^m \delta_{y_j} \otimes (\mu|_{B_j} * \delta_{2u}).
\]
Its two marginals are \( \nu \) and \( \mu * \delta_{2u} \), respectively, so it is enough to prove that \( \lambda \) is supported on the relation \( \leq \). But this is because of \( B_j \subseteq [x_i - u, x_i + u] \) for some \( i \), which implies the relevant \( y_j \in \downarrow(B_j + 2u) \).

We now turn to stating a slightly simplified version of our recent Vergleichstellensatz [7, Theorem 8.6]. We will instantiate it on \( M_c(G) \) in the next section, which gives our main result. We need one more definition first.

4.13. Definition. If \( S \) and \( T \) are semialgebras, then a semialgebra homomorphism from \( S \) to \( T \) is a map \( \phi : S \to T \) which preserves addition, multiplication, and scalar multiplication: for all \( x, y \in S \) and \( r \in \mathbb{R}_+ \),
\[
\phi(x + y) = \phi(x) + \phi(y), \quad \phi(xy) = \phi(x)\phi(y), \quad \phi(rx) = r\phi(x),
\]
as well as the neutral elements, \( \phi(0) = 0 \) and \( \phi(1) = 1 \).

Now here is [7, Theorem 8.6], specialized to the case of preordered semialgebras and to the case where the power universal element \( v \) is invertible; both of these assumptions result in some small simplifications.
4.14. Theorem. Let \( S \) be a preordered semialgebra with a power universal element \( v \in S \) that is multiplicatively invertible, and suppose that \( S \) comes equipped with a surjective homomorphism \( \| \cdot \| : S \to \mathbb{R}_+ \) such that
\[
a \leq b \implies \|a\| = \|b\| \implies a \sim b,
\]
where \( \sim \) denotes the equivalence relation generated by \( \leq \).

For nonzero \( x, y \in S \) with \( \|x\| = \|y\| = 1 \), suppose that the following hold:

(a) For every monotone semialgebra homomorphism \( \phi : S \to \mathbb{K} \) with trivial kernel and \( \mathbb{K} \in \{ \mathbb{R}_+, \mathbb{R}_+^{\text{op}}, \mathbb{T}_{\text{R}_+}, \mathbb{T}_{\text{R}_+}^{\text{op}} \} \),
\[
\phi(x) < \phi(y),
\]

apart from \( \phi = \| \cdot \| \) itself.

(b) For every \( \mathbb{R}_+ \)-linear monotone map \( D : S \to \mathbb{R} \) which satisfies the Leibniz rule
\[
D(ab) = D(a)\|b\| + \|a\| D(b)
\]
as well as \( D(v) = 1 \), we have
\[
D(x) < D(y).
\]

Then also the following hold:

(c) There is nonzero \( a \in S \) with \( \|a\| = 1 \) and
\[
ax \leq ay.
\]

(d) We have
\[
x^n \leq y^n \quad \forall n \gg 1.
\]

Conversely, if (c) or (d) holds for some \( n \geq 1 \), then (a) and (b) follow with non-strict inequality.

Here, the final sentence is very simple to prove, and follows upon simply applying the relevant maps \( \phi \) and \( D \) to either assumed inequality and cancelling the resulting term involving \( a \) or the \( n \)-th power. The implication from (a)+(b) to (c)+(d) is the actual result, which is much deeper and requires a substantial theory development \cite{7}. Note that it is close to being a converse to the final sentence, in the sense that its assumptions amount to strict inequalities in the \( \phi \) and \( D \), while the final sentence concludes non-strict inequalities but is otherwise essentially the same.

In order to apply the theorem in a concrete case, it is necessary to characterize first the inequalities required by conditions (a) and (b). In addition, it is also helpful to have an idea of what type of structure is formed by this family of inequalities in general. This is what we turn to now.

There are five types of relevant inequalities corresponding to various types of monotone maps out of \( S \). In \cite{7}, we have introduced the following terminology for talking about these types.

- Monotone homomorphisms \( \phi : S \to \mathbb{T}_{\text{R}_+} \) are \textbf{max-tropical}.
- Monotone homomorphisms \( \phi : S \to \mathbb{R}_+ \) are \textbf{max-temperate}.
Monotone $\mathbb{R}_+\text{-linear derivations } D : S \rightarrow \mathbb{R}$ (maps satisfying the Leibniz rule above) are **arctic**.

- Monotone homomorphisms $\phi : S \rightarrow \mathbb{R}_+^\text{op}$ are **min-temperate**.
- Monotone homomorphisms $\phi : S \rightarrow \mathbb{T}\mathbb{R}_+^\text{op}$ are **min-tropical**.

It is useful to consider all of these together as defining a family of inequalities parametrized by a suitable topological space. This is the space which we denoted $\text{TSper}(S)$ in [7, Section 8]. We recall its definition here, assuming that $S$ is as in Theorem 4.14.

**4.15. Definition.** The test spectrum $\text{TSper}(S)$ is defined as the disjoint union

\[
\text{TSper}(S) := \{\text{monotone homs } \phi : S \rightarrow \mathbb{R}_+ \text{ or } \phi : S \rightarrow \mathbb{R}_+^\text{op} \} \setminus \{\| \cdot \|\} \\
\cup \{\text{monotone homs } \phi : S \rightarrow \mathbb{T}\mathbb{R}_+^\text{op} \text{ with } \phi(v) = e\} \\
\cup \{\text{monotone homs } \phi : S \rightarrow \mathbb{T}\mathbb{R}_+^\text{op} \text{ with } \phi(v) = e^{-1}\} \\
\cup \{\text{monotone } \mathbb{R}_+\text{-linear derivations } D : S \rightarrow \mathbb{R} \text{ with } D(v) = 1\}.
\]

and carrying the coarsest topology which makes the **logarithmic evaluation maps**

\[
\text{lev}_x(\phi) := \frac{\log \phi(x)}{\log \phi(u)}, \quad \text{lev}_x(D) := D(x)
\]

continuous for all $x \in S$ with $\|x\| = 1$.

Note that there is a minor difference relative to [7, Definitions 8.3 and 8.4], namely that we now define logarithmic evaluation maps $\text{lev}_x$ rather than logarithmic comparison maps, which in terms of the above maps are defined for $x, y \in S$ with $\|x\| = \|y\| = 1$ as

\[
\text{lc}_{x,y} := \text{lev}_y - \text{lev}_x.
\]

It is sufficient to consider the case $\|x\| = \|y\| = 1$ rather than the weaker $\|x\| = \|y\|$ as in [7, Definition 8.4], since we are only dealing with the semialgebra case, and $x$ and $y$ can be normalized by scalar multiplication. Then $\text{lc}_{x,y}$ is continuous if both $\text{lev}_x$ and $\text{lev}_y$ are; and conversely, $\text{lev}_x = \text{lc}_{u,x} - 1$ shows that the $\text{lev}_x$ are continuous as soon as the $\text{lc}_{x,y}$ are.

The conditions (a) and (b) of Theorem 4.14 can now be conveniently summarized as saying that

\[
\text{lev}_x < \text{lev}_y,
\]

where the strict inequality must be pointwise strict on $\text{TSper}(S)$. We also recall [7, Proposition 8.5].

**4.16. Proposition.** The test spectrum $\text{TSper}(S)$ is a compact Hausdorff space.

5. Asymptotic comparison of random walks

In order to apply Theorem 4.14 to $\mathcal{M}_c(G)$, we thus still need to determine the relevant test spectrum. Throughout this section, $G$ is again a preordered topological abelian group with positive cone $G_+$ and order unit $u \in G_+$ in the sense of Definition 3.3. We write

\[
G_+^* := \{\text{monotone group homomorphisms } G \rightarrow \mathbb{R}\},
\]
and denote the application of \( t \in G^*_+ \) to a group element \( x \in G \) usually by \( \langle t, x \rangle := t(x) \).

5.1. **Lemma.** Every \( t \in G^*_+ \) is continuous as a map \( t : G \to \mathbb{R} \).

While this can be seen as a consequence of the purely algebraic \[12, Proposition 7.18\], we offer an independent proof.

**Proof.** We assume \( t(u) = 1 \) without loss of generality. It is enough to show that \( t \) is continuous at zero, or equivalently that for every \( n \in \mathbb{N}_{>0} \) the set \( t^{-1}([-\frac{1}{n}, \frac{1}{n}]) \) is a neighbourhood of zero. But this is the case because the map \( G \rightarrow G, x \mapsto nx \) is continuous, and hence the set
\[
\{ x \in G \mid u \leq nx \leq u \} \subseteq t^{-1} \left( [-\frac{1}{n}, \frac{1}{n}] \right)
\]
is a neighbourhood of zero. \( \square \)

We now introduce the moment-generating function for \( G \), and will then use it to characterize the five parts of the test spectrum. We start with the max-temperate part.

5.2. **Lemma.** The monotone semialgebra homomorphisms \( \phi : \mathcal{M}_c(G) \to \mathbb{R}_+ \) are precisely the maps of the form
\[
\mu \mapsto \int e^{\langle t, x \rangle} d\mu(x),
\]for some \( t \in G^*_+ \).

**Proof.** Every such map is clearly \( \mathbb{R}_+ \)-linear by linearity of the integral in the measure. The multiplicativity follows by the formula (4.2) for the integral of a function against a convolution,
\[
\int e^{\langle t, x \rangle} d(\mu * \nu)(x) = \int \int e^{\langle t, y+z \rangle} d\mu(y) d\nu(z) = \left( \int e^{\langle t, y \rangle} d\mu(y) \right) \left( \int e^{\langle t, z \rangle} d\nu(z) \right).
\]
Monotonicity in \( \mu \) holds because the integrand \( x \mapsto e^{\langle t, x \rangle} \) is a monotone and lower semicontinuous function, where the latter is a consequence of Lemma 5.1, and therefore Proposition 4.1 applies.

For the converse, let \( \phi : \mathcal{M}_c(G) \to \mathbb{R}_+ \) be a monotone homomorphism. Restricting \( \phi \) along the inclusion homomorphism \( G \to \mathcal{M}_c(G) \) from (4.3) shows that
\[
t(x) := \log \phi(\delta_x)
\]defines an element \( t \in G^*_+ \). The formula (5.1) then holds by definition for all delta measures, and \( \mathbb{R}_+ \)-linearity implies that it therefore also holds for all finitely supported measures. But then together with monotonicity, Lemma 4.12 shows that the value \( \phi(\mu) \) for any \( \mu \) differs from (5.1) by a factor of at most \( \phi(2u) \). Applying this statement to a power \( \mu^n \) and taking \( n \to \infty \) proves that \( \phi(\mu) \) actually coincides with (5.1). \( \square \)
We need two more similar results, one for monotone homomorphisms to the tropical reals and one for monotone derivations, corresponding to the max-tropical and the arctic parts of the real spectrum, respectively.

5.3. Lemma. The monotone semialgebra homomorphisms \( \phi : \mathcal{M}_c(G) \to \mathbb{TR}_+ \) are precisely the maps of the form

\[
\mu \mapsto \exp \left( \max_{x \in \text{supp}(\mu)} \langle t, x \rangle \right)
\]

for some \( t \in G_+^* \).

Proof. Since the support of the sum of two measures is exactly the union of the supports, this map indeed takes addition to maximization. It also takes the zero measure to \( e^{-\infty} = 0 \), and a positive scalar multiple \( r\mu \) to the same number as any measure \( \mu \) itself. Since the support of a convolution is exactly the Minkowski sum of its supports, which becomes multiplication upon exponentiation, it follows that (5.2) indeed defines a semialgebra homomorphism. Monotonicity follows from the characterization of the stochastic preorder of Proposition 4.1(d) in terms of a joint distribution \( \lambda \) supported on the relation \( \leq \); considering the support of \( \lambda \) as a subset of \( G \times G \) shows that if \( \mu \leq \nu \) and \( x \in \text{supp}(\mu) \), then there must be \( y \in \text{supp}(\nu) \) with \( x \leq y \).

Conversely, suppose that \( \phi : \mathcal{M}_c(G) \to \mathbb{TR}_+ \) is a monotone semialgebra homomorphism. Then define an element \( t \in G_+^* \) by \( t(x) := \log \phi(\delta_x) \). The assumption that \( \phi \) preserves multiplication shows that \( t(x + y) = t(x) + t(y) \), while monotonicity of \( t \) is obvious. Hence indeed \( t \in G_+^* \). The \( \mathbb{R}_+ \)-linearity of \( \phi \) then implies that \( \phi \) coincides with (5.2) for all finitely supported \( \mu \). For general \( \mu \), we again use Lemma 4.12, which shows that \( \phi(\mu) \) differs from the value of (5.2) by a factor of at most \( \phi(2u) \). Applying this statement to the powers \( \mu^*n \) and taking \( n \to \infty \) proves the claim. \( \square \)

5.4. Remark. Replacing \( G_+ \) by \( -G_+ \) in Lemma 5.2 and Lemma 5.3 shows that the monotone homomorphisms \( \mathcal{M}_c(G) \to \mathbb{R}_+^\text{op} \) and \( \mathcal{M}_c(G) \to \mathbb{TR}_+^\text{op} \) are also of the specified form, but with \( -t \) in place of \( t \). The previous two lemmas thus also characterize the min-temperate and min-tropical parts of the real spectrum.

5.5. Lemma. The maps \( D : \mathcal{M}_c(G) \to \mathbb{R} \) which are monotone, \( \mathbb{R}_+ \)-linear and satisfy the Leibniz rule

\[
D(\mu \ast \nu) = D(\mu)\nu(G) + \mu(G)D(\nu)
\]

are precisely the maps of the form

\[
\mu \mapsto \int \langle t, x \rangle d\mu(x)
\]

for \( t \in G_+^* \).

Proof. The \( \mathbb{R}_+ \)-linearity is obvious, and monotonicity holds by monotonicity and continuity of \( t \) itself. The Leibniz rule follows again by (4.2) and the additivity of \( t \),

\[
\int t(x) d(\mu \ast \nu)(x) = \iint t(y + z) d\mu(y) d\nu(z)
\]
\begin{align*}
= \left( \int t(y) d\mu(y) \right) \nu(G) + \mu(G) \left( \int t(z) d\nu(z) \right).
\end{align*}

Conversely, suppose that \( D : \mathcal{M}_c(G) \to \mathbb{R} \) has the assumed properties, and put \( t(x) := D(\delta_x) \). Then the assumed monotonicity of \( D \) and the Leibniz rule show that \( t \in G_+^* \). It follows then by \( \mathbb{R}_+ \)-linearity that \( D \) coincides with (5.3) on the finitely supported \( \mu \). To show this for all \( \mu \), we consider \( \mu(G) = 1 \) without loss of generality and again apply Lemma 4.12. Together with the Leibniz rule, this shows that \( D(\mu) \) for arbitrary \( \mu \) differs from (5.3) by at most \( D(2u) \). Applying this statement to a power \( \mu^n \) then proves the claim. \( \square \)

Theorem 4.14 then instantiates to the following main result, which we formulate directly in probabilistic terms using random variables. In the following, all inequalities between \( G \)-valued random variables refer to the preorder on \( G \) induced by the positive cone \( G_+ \), and are to be interpreted as holding almost surely. Let us also say that a random variable is Radon if its distribution is a Radon measure.

5.6. Theorem. Let \( G \) be a topological abelian group, preordered with respect to a positive cone \( G_+ \subseteq G \) having an order unit \( u \in G_+ \). Let all random variables be \( G \)-valued, compactly supported and Radon.

Suppose that \( X \) and \( Y \) are random variables, not jointly distributed, and let \( (X_i)_{i \in \mathbb{N}} \) and \( (Y_i)_{i \in \mathbb{N}} \) be i.i.d. copies. Consider the following conditions:

(i) There is a joint distribution of \( X \) and \( Y \) and a third random variable \( Z \), independent of \( X \) and \( Y \), such that

\[
X + Z \leq Y + Z. \tag{5.4}
\]

(ii) For \( n \geq 1 \), there is a joint distribution of the i.i.d. copies such that

\[
\sum_{i=1}^{n} X_i \leq \sum_{i=1}^{n} Y_i. \tag{5.5}
\]

(iii) With \( < \) standing for \( < \) or \( \leq \), and similarly for \( > \), the following hold for all nonzero \( t \in G_+^* \):

\[
\mathbb{E}[e^{\langle t, X \rangle}] < \mathbb{E}[e^{\langle t, Y \rangle}], \quad \mathbb{E}[e^{-\langle t, X \rangle}] > \mathbb{E}[e^{-\langle t, Y \rangle}], \tag{5.6}
\]

\[
\max\langle t, X \rangle < \max\langle t, Y \rangle, \quad \min\langle t, X \rangle > \min\langle t, Y \rangle, \tag{5.7}
\]

\[
\mathbb{E}[\langle t, X \rangle] < \mathbb{E}[\langle t, Y \rangle]. \tag{5.8}
\]

Then (i) or (ii) for some \( n \geq 1 \) implies that (iii) holds with non-strict inequalities. Conversely if (iii) holds with strict inequalities, then (i) and (ii) for all \( n \gg 1 \) follow.

Note that the inequalities (iii) do indeed not require \( X \) and \( Y \) to be jointly distributed. As with Theorem 4.14, the forward direction is easy to see by applying the respective functions to the assumed inequality. Our main result is the converse direction, and we state the forward direction mainly to indicate that our converse is close to necessary and sufficient: the only difference is in the strictness of the inequalities in (iii).
Proof. This follows upon instantiating Theorem 4.14 on $\mathcal{M}_c(G)$, taking $\|\cdot\| : \mathcal{M}_c(G) \to \mathbb{R}_+$ to be given by the normalization homomorphism $\mu \mapsto \mu(G)$, and translating the statement into random variables language. The power universality of $\delta_u$ holds by Lemma 4.11. The relevant monotone quantities are exactly the specified ones, as per Lemmas 5.2, 5.3 and 5.5 and Remark 5.4, where we have in addition taken the logarithm of those of the form (5.2), and dropped the conventional normalization requirement on the tropical and arctic parts of (5.7) and (5.8).

For $G = \mathbb{R}$ and $G_+ = \mathbb{R}_+$, the above result specializes to the limit theorem of Aubrun and Nechita which we had stated in the introduction as Theorem 1.1, using the fact that in this case, the stochastic preorder is characterized by the given inequalities between cumulative distribution functions per (4.4).

5.7. Remark. It may be worth pointing out that the existence of a $Z$ as in (5.4) and (1.1) varies strongly with whether we require $Z$ to be compactly supported or not. While our result is concerned with the bounded case, another recent result of Pomatto, Strack and Tamuz for the real-valued case [15] shows that such a $Z$ with merely finite first moment exists already as soon as only $\mathbb{E}[X] < \mathbb{E}[Y]$ holds.

6. The normalized cumulant-generating function

The conditions (5.6)–(5.8) may seem a bit unwieldy, and for successful application of Theorem 5.6 it is imperative to understand these conditions better and in particular how they relate. This is what we do in this section, by investigating the structure of the test spectrum as a compact Hausdorff space. In our current context, this space behaves a lot like a projective version of $G_+^*$, so we introduce a more concise notation and denote it by $\hat{P}(G_+)$, indicating its projective nature.

6.1. Definition. Let $G$ be a topological abelian group preordered with respect to a positive cone $G_+ \subseteq G$ having an order unit $u \in G_+$. Then $\hat{P}(G_+)$ is the topological space defined as follows. Its underlying set is the disjoint union of the following five parts:

- **The max-tropical part,** given by
  \[ \{ t \in G_+^* \mid t(u) = 1 \} \]

- **The max-temperate part,** given by
  \[ G_+^* \setminus \{0\} \]

- **The arctic part,** given by
  \[ \{ t \in G_+^* \mid t(u) = 1 \} \]

- **The min-temperate part,** given by
  \[ (-G_+^*) \setminus \{0\} \]

- **The min-tropical part,** given by
  \[ \{ t \in (-G_+^*) \mid t(u) = -1 \} \]
\( \hat{\mathcal{P}}(G_+) \) carries the coarsest topology which makes the logarithmic evaluation maps \( \text{lev}_\mu : \hat{\mathcal{P}}(G_+) \to \mathbb{R} \) defined as

\[
\text{lev}_\mu(t) := \begin{cases} 
\max_{x \in \text{supp}(\mu)} \langle t, x \rangle & \text{if } t \text{ is max-tropical or min-tropical}, \\
\langle t, u \rangle^{-1} \log \int_G e^{\langle t, x \rangle} d\mu(x) & \text{if } t \text{ is max-temperate or min-temperate}, \\
\int_G \langle t, x \rangle d\mu(x) & \text{if } t \text{ is arctic}.
\end{cases}
\]

continuous for all \( \mu \in \mathcal{M}_c(G) \) with \( \mu(G) = 1 \).

**6.2. Remark.** Several comments are in order to make sense of this definition.

(a) We will not introduce separate notation for the five parts of \( \hat{\mathcal{P}}(G_+) \), but distinguish them in words.

(b) For \( t \) in the min-temperate or min-tropical part, it is important to keep in mind that \( \langle t, u \rangle < 0 \).

(c) For nonzero \( t \in G^*_+ \), consider the associated max-temperate point of \( \hat{\mathcal{P}}(G_+) \), and let \( r \in \mathbb{R}_{>0} \) be a scalar. Then \( rt \) again represents a max-temperate point with logarithmic evaluation map given by\(^4\)

\[
\mu \mapsto \frac{\log \int_G e^{r\langle t, x \rangle} d\mu(x)}{r\langle t, u \rangle}.
\]

(6.1)

Assuming that \( t \) is normalized to \( \langle t, u \rangle = 1 \), it also defines a max-tropical and an arctic point of \( \hat{\mathcal{P}}(G_+) \). And indeed the corresponding logarithmic evaluation maps arise from (6.1) as limits in \( r \): taking \( r \to \infty \) recovers the tropical case,

\[
\lim_{r \to \infty} \frac{\log \int_G e^{r\langle t, x \rangle} d\mu(x)}{r\langle t, x \rangle} = \max_{x \in \text{supp}(\mu)} \langle t, x \rangle,
\]

and \( r \to 0 \) recovers the arctic case,

\[
\lim_{r \to 0} \frac{\log \int_G e^{r\langle t, x \rangle} d\mu(x)}{r\langle t, x \rangle} = \int_G \langle t, x \rangle d\mu(x),
\]

both of which follow by an elementary calculation, assuming that \( \mu \) is a probability measure.

(d) In the tropical and arctic parts, we could have replaced the normalization by considering equivalence classes of homomorphisms up to scalar multiplication. However, we follow [7, Section 8] by imposing normalization, and this choice will also be more convenient in the following.

**6.3. Proposition.** \( \hat{\mathcal{P}}(G_+) \) is a compact Hausdorff space.

**Proof.** This is an instance of Proposition 4.16 upon noting that the space \( \text{TSper}(\mathcal{M}_c(G)) \) coincides with our current \( \hat{\mathcal{P}}(G_+) \); but this statement in turn is a consequence of Lemmas 5.2, 5.3 and 5.5 and Remark 5.4.

---

\(^4\) As an interesting aside, this function of \( r \) is constant whenever \( \mu = \delta_x \) for \( x \in G \).
The main use of \( \hat{P}(G_+) \) is that it allows us to summarize the inequalities (5.6)–(5.8) concisely as a pointwise inequality between continuous functions on \( \hat{P}(G_+) \), namely as
\[
\text{lev}_X \prec \text{lev}_Y,
\]
where we omit notational distinction between a random variable and its distribution. In the non-strict case, this inequality is simply pointwise inequality \( \leq \) between continuous functions on \( \hat{P}(G) \), and similarly in the strict case. The compactness in particular can be useful in concrete applications of Theorem 5.6. We will develop one such application in the next section.

6.4. Example. Consider \( G = \mathbb{R} \) and \( G_+ = \mathbb{R}_+ \) with order unit \( u = 1 \). Then we have the following classes of relevant monotone maps, which let us identify \( \hat{P}(\mathbb{R}_+) \) as a topological space with \( \mathbb{R} \cup \{\pm \infty\} \).

- The max-tropical part is \( +\infty \), corresponding to the monotone homomorphism \( M_c(\mathbb{R}) \to \mathbb{T}\mathbb{R}_+ \) given by
  \[
  X \mapsto \exp(\max X).
  \]

- The max-temperate part is \( (0, +\infty) \), corresponding to monotone homomorphisms \( M_c(\mathbb{R}) \to \mathbb{R}_+ \) having the form
  \[
  X \mapsto \mathbb{E}[e^{tX}],
  \]
  for parameter \( t \in (0, \infty) \).

- The arctic part consists only of \( 0 \in \mathbb{R} \), corresponding to the monotone derivation
  \[
  X \mapsto \mathbb{E}[X].
  \]

- The min-temperate part is \( (-\infty, 0) \), corresponding to monotone homomorphisms \( M_c(\mathbb{R}) \to \mathbb{R}_+^{\text{op}} \) having the form
  \[
  X \mapsto \mathbb{E}[e^{-tX}],
  \]
  for parameter \( t \in (0, \infty) \).

- The min-tropical part is \( -\infty \), corresponding to the monotone homomorphism
  \( M_c(\mathbb{R}) \to \mathbb{T}\mathbb{R}_+^{\text{op}} \) given by
  \[
  X \mapsto \exp(-\min X).
  \]

Applying the definition of the logarithmic evaluation maps then shows that these are simply the functions of the form
\[
\text{lev}_X(t) = \frac{\log \mathbb{E}[e^{tX}]}{t}
\]
for \( t \in \mathbb{R} \setminus \{0\} \), reproducing the correct limits as \( t \to 0 \) and \( t \to \pm \infty \), namely \( \mathbb{E}[X] \) and \( \min X \) and \( \max X \).

This indicates that our logarithmic evaluation maps \( \text{lev}_X \) should be thought of as a "normalized" version of the cumulant-generating function; the only difference with respect to the usual cumulant-generating function \( \log \mathbb{E}[e^{tX}] \) is the normalization in the denominator. Dividing by \( t \) is exactly what makes the cumulant-generating function have the
relevant limiting values at \( t = 0 \) and \( t \to \pm \infty \). It also means that our version of the cumulant-generating function can be thought of as a family of weighted averages: the value \( \text{lev}_X(t) \) is always in the interval \([\min X, \max X]\) and coincides with these values for \( t = \pm \infty \); and for larger \( t \) the averaging attributes higher weight to the right, while being “unbiased” at \( t = 0 \), for which we get the usual expectation value.

6.5. Example. Consider \( \mathbb{R}^2 \) with positive cone given by the positive quadrant \( \mathbb{R}_+^2 \). Then as a topological space, \( \hat{\mathbb{P}}(\mathbb{R}_+^2) \) looks like the union of the positive quadrant and its opposite, together with a projective point at infinity for every half-line through the origin and pointing in the direction of either quadrant, and in addition with the origin blown up to an interval.

7. A uniform large deviation result and Cramér’s theorem

We now formulate a weaker version of Theorem 5.6, one where we relax the properties under consideration such that only the moment-generating function \( t \mapsto \mathbb{E}[e^{\langle t, X \rangle}] \) for \( t \in G^*_+ \) matters. It has turned out to be convenient to phrase this result also only in the more specific case of topological vector spaces.

The advantages of the following result over Theorem 5.6 are that it is somewhat easier to state, that it is closer to traditional large deviation theory, and that it makes more explicit how our results can be thought of as a duality between the asymptotic behaviour of random walks and the moment-generating or cumulant-generating function.

7.1. Theorem. Let \( V \) be a topological vector space preordered with respect to a closed convex cone \( V_+ \subseteq V \) with \( u \in V_+ \) such that \([-u, +u]\) is a neighbourhood of zero. Let all random variables be \( V \)-valued, compactly supported and Radon.

For random variables \( X \) and \( Y \) and i.i.d. copies \((X_i)_{i \in \mathbb{N}}\) and \((Y_i)_{i \in \mathbb{N}}\), we have

\[
\sup_{\varepsilon > 0} \lim_{n \to \infty} \sup_{C} \frac{1}{n} \log \frac{\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} X_i \in C \right]}{\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} Y_i + \varepsilon u \in C \right]} = \sup_{t \in V_+^*} \log \frac{\mathbb{E}\left[e^{\langle t, X \rangle}\right]}{\mathbb{E}\left[e^{\langle t, Y \rangle}\right]},
\]

(7.1)

where \( C \) ranges over all closed upsets in \( V \) and the statement holds in two versions, with \( \lim_{n \to \infty} \) standing for \( \liminf_{n \to \infty} \) or \( \limsup_{n \to \infty} \).

A few comments are in order before we get to the proof. The fraction on the left-hand side is understood to be \( \infty \) if the denominator vanishes and the numerator does not; we similarly stipulate that \( \frac{0}{0} := 0 \), so that those cases for which both vanish do not contribute to \( \sup_C \). The fraction on the left is monotonically nondecreasing as \( \varepsilon \to 0 \), so that the supremum over \( \varepsilon > 0 \) is equivalently a limit \( \varepsilon \to 0 \). Finally, replacing \( C \) by \( C - \varepsilon u \) shows that the \( +\varepsilon u \) term in the denominator on the left-hand side can likewise be replaced by an analogous \( -\varepsilon u \) term in the numerator, since

\[
\sup_{C} \frac{\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} X_i - \varepsilon u \in C \right]}{\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} Y_i \in C \right]} = \sup_{C'} \frac{\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} X_i \in C \right]}{\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} Y_i + \varepsilon u \in C \right]},
\]

Let us now turn to the proof, which consists of a reduction to Theorem 5.6.
Proof. We introduce two auxiliary variables in terms of the given ones, and also depending on additional parameters for which we will choose concrete values below.

- For $p \in (0,1)$ and $k \in \mathbb{N}$, consider the variable $X'$ which coincides with $X$ with probability $p$ and is equal to $-ku$ with probability $1 - p$.

- For $\epsilon > 0$, consider $Y' := Y + \epsilon u$.

We also choose corresponding i.i.d. copies $X'_i$ and $Y'_i$ for $i \in \mathbb{N}$.

These new variables have normalized cumulant-generating functions taking the form, for $t$ in the temperate part of $\mathbb{P}(V_+)$,

\[
\begin{align*}
\text{lev}_{X'}(t) &= \frac{\log \left( p \mathbb{E}[e^{\langle t, X \rangle}] + (1 - p)e^{-k\langle t, u \rangle} \right)}{\langle t, u \rangle}, \\
\text{lev}_{Y'}(t) &= \frac{\log \mathbb{E}[e^{\langle t, Y \rangle}]}{\langle t, u \rangle} + \epsilon.
\end{align*}
\]

We first prove the inequality $\geq$ in the claimed equation (7.1), with $\liminf_{n \to \infty}$ in place of $\lim_{n \to \infty}$. This inequality direction is the easy "forward" direction, conceptually analogous to the easy directions in Theorems 4.14 and 5.6. Since this inequality direction is trivial if the left-hand side is $\infty$, we assume that it is finite. Then consider any value of $p$ such that

\[-\log p > \sup_{\epsilon > 0} \lim_{n \to \infty} \frac{1}{n} \log \frac{\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in C \right]}{\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i + \epsilon u \in C \right]}.
\]

The goal is then to show that $-\log p \geq \log \frac{\mathbb{E}[e^{\langle t, X \rangle}]}{\mathbb{E}[e^{\langle t, Y \rangle}]}$ for any $t \in V_+^*$, or equivalently that $p\mathbb{E}[e^{\langle t, X \rangle}] \leq \mathbb{E}[e^{\langle t, Y \rangle}]$. Indeed for fixed $\epsilon > 0$ and $p$ as above, choose $n$ such that the inequality

\[-\log p \geq \sup_{C} \frac{1}{n} \log \frac{\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in C \right]}{\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i + \epsilon u \in C \right]}
\]

still holds. But then this equivalently means that

\[p^n \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in C \right] \leq \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i' \in C \right] \quad \forall C.
\]

By choosing $k$ large enough and applying Lemma 3.8, the distribution of $X'$ will have the property that $\frac{1}{n} \sum_{i=1}^{n} X'_i$ is below the support of $Y$, and hence also below the support of $\frac{1}{n} \sum_{i=1}^{n} Y'_i$, as soon as just one of the $X'_i$ is equal to $-ku$. This event is complementary to $X'_i = X_i$ for all $i$, which has probability $p^n$. Therefore also

\[\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} X'_i \in C \right] \leq \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y'_i \in C \right] \quad \forall C.
\]

Thus $\text{lev}_{X'} \leq \text{lev}_{Y'}$ on $\hat{\mathbb{P}}(V_+)$ follows from the easy forward direction of Theorem 5.6. But then the desired $p\mathbb{E}[e^{\langle t, X \rangle}] \leq \mathbb{E}[e^{\langle t, Y \rangle}]$ follows from the above formulas (7.2) as $\epsilon \to 0$. 

We now show the other inequality direction with limsup \( n \to \infty \) in place of \( \lim n \to \infty \), which is enough to prove the whole claim. Consider now any value of \( p \) with 

\[- \log p > \sup_{t \in V_+^*} \log \frac{E[e^{\langle t, X \rangle}]}{E[e^{\langle t, Y \rangle}]} ,\]  

(7.3)

or equivalently \( pE[e^{\langle t, X \rangle}] < E[e^{\langle t, Y \rangle}] \) for all \( t \in V_+^* \), now assuming without loss of generality that the right-hand side of (7.1) is finite. Fix \( \varepsilon > 0 \). Then for every \( t \) in the max-temperate part of \( \hat{P}(V_+^*) \), the formulas (7.2) show that there is \( k \gg 1 \) such that \( \text{lev}_{X'}(t) < \text{lev}_{Y'}(t) \). The same strict inequality holds on the max-tropical part because of \( \varepsilon > 0 \). For every point of the min-tropical, min-temperate and arctic parts, the \( +ku \) component of \( X' \) dominates for \( k \to \infty \), and therefore we can again find \( k \gg 1 \) such that \( \text{lev}_{X'} < \text{lev}_{Y'} \) at every such point. By compactness of \( \hat{P}(V_+^*) \), it follows that some fixed \( k \gg 1 \) works for all of these points. Taking all this together, we have \( \text{lev}_{X'} < \text{lev}_{Y'} \) on all of \( \hat{P}(V_+^*) \) for suitable \( k \gg 1 \).

Thus Theorem 5.6 shows that we have, in terms of i.i.d. copies: for all \( n \gg 1 \),

\[
P \left[ \frac{1}{n} \sum_{i=1}^{n} X'_i \in C \right] \leq P \left[ \frac{1}{n} \sum_{i=1}^{n} Y'_i \in C \right] \quad \forall C.
\]

Since \( \sum_{i=1}^{n} X'_i \) coincides with \( \sum_{i=1}^{n} X_i \) with probability at least \( p^n \) and \( \sum_{i=1}^{n} Y'_i = \sum_{i=1}^{n} Y_i + \varepsilon un \), we therefore obtain that for all \( n \gg 1 \),

\[
p^n P \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in C \right] \leq P \left[ \frac{1}{n} \sum_{i=1}^{n} Y_i + \varepsilon u \in C \right] \quad \forall C,
\]

which translates into

\[
\sup_{C} \frac{1}{n} \log \frac{P \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in C \right]}{P \left[ \frac{1}{n} \sum_{i=1}^{n} Y_i + \varepsilon u \in C \right]} \leq - \log p.
\]

Letting \( p \) approach the bound given in (7.3) and noting that \( \varepsilon > 0 \) was arbitrary proves the inequality \( \leq \) in the claimed equation (7.1) with limsup \( n \to \infty \), which is enough. □

For \( V = \mathbb{R} \) and \( V_+ = \mathbb{R} \), our new Theorem 7.1 specializes to Theorem 1.2 from the introduction.

We end this paper by explaining how Theorem 7.1 specializes further to a version of Cramér’s classical large deviation theorem, namely Corollary 7.4 below. Notably, the latter arises by taking one of the two variables in Theorem 7.1 to be deterministic. This justifies thinking of Theorem 7.1, and thereby also of the stronger Theorem 5.6, intuitively as a result on uniform large deviations for random walks, where the “uniform” is measured relative to another random walk. In the following, \( V \) is as in Theorem 7.1.

7.2. Definition. Let \( X \) be a \( V \)-valued compactly supported Radon random variable. Then its rate function \( \Lambda^* : V \to [0, \infty] \) is given by

\[
\Lambda^*(c) := \sup_{t \in V_+^*} \left( \langle t, c \rangle - \log E[e^{\langle t, X \rangle}] \right).
\]
7.3. Lemma. The rate function $\Lambda^*$ is continuous at every $c \in V$ with $\Lambda^*(c) < \infty$.

Proof. Being a pointwise supremum of linear functions, $\Lambda^*$ is convex. In particular the restricted function $\mathbb{R} \to [0, \infty) \setminus \{0\}$ is a one-dimensional convex function and hence continuous at every point at which it is finite. Thus since $\Lambda^*(c) < \infty$ by assumption, for given $\varepsilon > 0$ we in particular have $\delta > 0$ such that $|\Lambda^*(c \pm \delta u) - \Lambda^*(c)| < \varepsilon$.

The claim now follows since $\Lambda^*$ is also monotone (as a supremum of monotone functions) and the order interval $[c - \delta u, c + \delta u]$ is a neighbourhood of $c$. □

7.4. Corollary. Let $V$ be a topological vector space preordered with respect to a closed convex cone $V_+ \subseteq V$ with $u \in V_+$ such that $[-u, +u]$ is a neighbourhood of zero. Let $X$ be a $V$-valued random variable, compactly supported and Radon, and let $(X_i)_{i \in \mathbb{N}}$ be i.i.d. copies. Then for every $c \in V$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \geq c \right] = -\Lambda^*(c).$$

Proof. In order to match this up with Theorem 7.1, we denote the variables that appear in the statement by $Y$ and $Y_i$ instead.

Consider the special case of Theorem 7.1 where $X := c$ is constant. Then we have, trivially,

$$\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} X_i \in C \right] = \begin{cases} 1 & \text{if } c \in C, \\ 0 & \text{if } c \notin C. \end{cases}$$

Therefore the supremum over $C$ in (7.1) is achieved at $C = \uparrow \{c\}$, resulting in

$$\sup_{\varepsilon > 0} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i \geq c - \varepsilon u \right] = \sup_{t \in V_+^*} \log \left( \mathbb{E}[e^{t(c)}] \right) = \Lambda^*(c),$$

or equivalently

$$\inf_{\varepsilon > 0} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i \geq c - \varepsilon u \right] = -\Lambda^*(c).$$

Monotonicity in $\varepsilon$ now shows that, for every $\varepsilon > 0$,

$$-\Lambda^*(c + \varepsilon u) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i \geq c \right] \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} Y_i \geq c \right] \leq -\Lambda^*(c).$$

Thus the claim follows in the limit $\varepsilon \to 0$ by continuity of $\Lambda^*$, Lemma 7.3. □

Note that, at least for locally convex Polish spaces, Corollary 7.4 is a special case of known versions of Cramér’s theorem, such as the one given by Gao [10].


References

[1] Charalambos D. Aliprantis and Rabee Tourky. Cones and duality, volume 84 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2007.
[2] Guillaume Aubrun and Ion Nechita. Stochastic domination for iterated convolutions and catalytic majorization. Ann. Inst. Henri Poincaré Probab. Stat., 45(3):611–625, 2009.
[3] David A. Edwards. On the existence of probability measures with given marginals. Ann. Inst. Fourier (Grenoble), 28(4):53–78, 1978.
[4] Gerald B. Folland. A course in abstract harmonic analysis. Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.
[5] D. H. Fremlin. Measure theory. Vol. 4. Torres Fremlin, Colchester, 2006. Topological measure spaces. Part I, II. Corrected second printing of the 2003 original.
[6] Tobias Fritz. Abstract Vergleichsstellensätze for preordered semifields and semirings I. arXiv:2003.13835.
[7] Tobias Fritz. Abstract Vergleichsstellensätze for preordered semifields and semirings II. arXiv:2112.05949.
[8] Tobias Fritz. Antisymmetry of the stochastical order on all ordered topological spaces. Anal. Geom. Metr. Spaces, 7(1):250–252, 2019. arXiv:1810.06771.
[9] Tobias Fritz and Paolo Perrone. Bimonoidal structure of probability monads. In Proceedings of the Thirty-Fourth Conference on the Mathematical Foundations of Programming Semantics (MFPS XXXIV), volume 341 of Electron. Notes Theor. Comput. Sci., pages 121–149. Elsevier Sci. B. V., Amsterdam, 2018. arXiv:1804.03527.
[10] Fuqing Gao. A note on Cramer’s theorem. In Séminaire de Probabilités, XXXI, volume 1655 of Lecture Notes in Math., pages 77–79. Springer, Berlin, 1997.
[11] Jonathan S. Golan. Semirings and their applications. Kluwer Academic Publishers, Dordrecht, 1999. Updated and expanded version of The theory of semirings, with applications to mathematics and theoretical computer science [Longman Sci. Tech., Harlow, 1992].
[12] Kenneth R. Goodearl. Partially ordered abelian groups with interpolation, volume 20 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1986.
[13] Josef Hadar and William R. Russell. Rules for ordering uncertain prospects. The American Economic Review, 59(1):25–34, 1969.
[14] Hans G. Kellerer. Duality Theorems for Marginal Problems. Zeitschrift für Warscheinlichkeitstheorie und verwandte Gebiete, 67:399–432, 1984.
[15] Luciano Pomatto, Philipp Strack, and Omer Tamuz. Stochastic dominance under independent noise. Journal of Political Economy, to appear. arXiv:1807.06927.
[16] Laurent Schwartz. Radon measures on arbitrary topological spaces and cylindrical measures. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973. Tata Institute of Fundamental Research Studies in Mathematics, No. 6.
[17] Volker Strassen. The existence of probability measures with given marginals. Annals of Mathematical Statistics, 36:423–439, 1965.

Department of Mathematics, University of Innsbruck, Austria

Email address: tobias.fritz@uibk.ac.at