Some Quantum Estimates Of Hermite-Hadamard Inequalities For \( \varphi \)-Convex Functions*

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Abstract

In this paper, we develop some quantum estimates of Hermite–Hadamard type inequalities for \( \varphi \)-convex functions. In some special cases, these quantum estimates reduce to known results.

1 Introduction

In recent years, the topic of quantum calculus has attracted the attention of several scholars. Quantum calculus stands as a connection between mathematics and physics. It has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities. In quantum analysis, we obtain \( q \)-analogues of mathematical objects that can be recaptured as \( q \to 1^- \). In recent years, many researchers have shown their interest in studying and investigating quantum calculus. Quantum analysis is also very helpful in numerous fields and has large applications in various areas of pure and applied sciences. For some recent developments in quantum calculus, interested readers are referred to [18–20,22].

Theory of inequalities and theory of convex functions are closely related to each other, thus a rich literature on inequalities. One of the most extensively studied inequality in the literature is the Hermite-Hadamard inequality: if \( \mathcal{F} : I \subset \mathbb{R} \to \mathbb{R} \) is a convex function defined on the interval \( I \) of real numbers and \( \omega_1, \omega_2 \in I \) with \( \omega_1 < \omega_2 \), then

\[
\mathcal{F}\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x)dx \leq \frac{\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{2}.
\]

This famous result of Hermite and Hadamard can be considered as necessary and sufficient condition for a function to be convex. Various improvements, generalizations, and variants of this inequality can be found in the papers [1–17,25,27].

Inspired by the works of Liu and Zhuang [23] and Liu, Zhuang and Park [24], we aim to develop some quantum estimates of Hermite-Hadamard type for \( q \)-differentiable \( \varphi \)-convex functions.

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2 Preliminaries

In this section, we recall some previously known concepts and basic results. Let \( \mathcal{I} \) be an interval in real line \( \mathbb{R} \) and \( \varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a bifunction.

**Definition 1** ([21]) A function \( \mathfrak{F} : \mathcal{I} \to \mathbb{R} \) is called convex with respect to \( \varphi \) (briefly \( \varphi \)-convex), if

\[
\mathfrak{F}(t\omega_1 + (1-t)\omega_2) \leq t\mathfrak{F}(\omega_1) + t\varphi(\mathfrak{F}(\omega_1), \mathfrak{F}(\omega_2))
\]

for all \( \omega_1, \omega_2 \in \mathcal{I} \) and \( t \in [0,1] \).

**Remark 1** If we set \( \varphi(A,B) = A - B \) in the above definition, then we recover the classical definition of convex function.

Clearly, any convex function is \( \varphi \)-convex. Furthermore, there exists \( \varphi \)-convex functions which are not convex. For example, we consider \( \mathfrak{F} : \mathbb{R} \to \mathbb{R} \) as

\[
\mathfrak{F}(z) = \begin{cases} 
-2, & \text{if } z \geq 0, \\
-1, & \text{if } z < 0,
\end{cases}
\]

and \( \varphi : [-\infty,0] \times [-\infty,0] \to \mathbb{R} \) as

\[
\varphi(A,B) = \begin{cases} 
A, & \text{if } B = 0, \\
B, & \text{if } A = 0, \\
-A - B, & \text{if } A < 0, B < 0.
\end{cases}
\]

Then, it is not hard to check that \( \mathfrak{F} \) is \( \varphi \)-convex. Also, it is obvious that \( \mathfrak{F} \) is not a convex function. On the other hand, let \( \mathcal{I} = [\omega_1, \omega_2] \subseteq \mathbb{R} \) be an interval and \( 0 < q < 1 \) be a constant.

**Definition 2** ([26]) Assume \( \mathfrak{F} : \mathcal{I} \to \mathbb{R} \) is a continuous function and let \( x \in \mathcal{I} \). Then \( q \)-derivative on \( \mathcal{I} \) of function \( \mathfrak{F} \) at \( x \) is defined as

\[
\omega_1 D_q \mathfrak{F}(x) = \frac{\mathfrak{F}(x) - \mathfrak{F}(qx + (1-q)\omega_1)}{(1-q)(x-\omega_1)}, \quad x \neq \omega_1 \quad \omega_1 D_q \mathfrak{F}(\omega_1) = \lim_{x \to \omega_1} \omega_1 D_q \mathfrak{F}(x).
\]

We say that \( \mathfrak{F} \) is \( q \)-differentiable on \( \mathcal{I} \) provided \( \omega_1 D_q \mathfrak{F}(x) \) exists for all \( x \in \mathcal{I} \). Note that if \( \omega_1 = 0 \) in (2), then \( 0 D_q \mathfrak{F} = D_q \mathfrak{F} \), where \( D_q \) is the well-known \( q \)-derivative of the function \( \mathfrak{F}(x) \) defined by

\[
D_q \mathfrak{F}(x) = \frac{\mathfrak{F}(x) - \mathfrak{F}(qx)}{(1-q)x}.
\]

**Definition 3** ([26]) Let \( \mathfrak{F} : \mathcal{I} \to \mathbb{R} \) be a continuous function. We define the second-order \( q \)-derivative on interval \( \mathcal{I} \), which is denoted as \( \omega_1 D_q^2 \mathfrak{F} \), provided \( \omega_1 D_q \mathfrak{F} \) is \( q \)-differentiable on \( \mathcal{I} \) with \( \omega_1 D_q^2 \mathfrak{F} = \omega_1 D_q (\omega_1 D_q \mathfrak{F}) : \mathcal{I} \to \mathbb{R} \). Similarly, we define higher order \( q \)-derivative on \( \mathcal{I} \), \( \omega_1 D_q^n : \mathcal{I} \to \mathbb{R} \).

**Definition 4** ([26]) Let \( \mathfrak{F} : \mathcal{I} \subset \mathbb{R} \to \mathbb{R} \) be a continuous function. Then \( q \)-integral on \( \mathcal{I} \) is defined by

\[
\int_{\omega_1}^{x} \mathfrak{F}(t) \omega_1 d_q t = (1-q)(x-\omega_1) \sum_{i=0}^{\infty} q^i \mathfrak{F}(q^i x + (1-q^i)\omega_1)
\]

for \( x \in \mathcal{I} \). Note that if \( \omega_1 = 0 \), then we have the classical \( q \)-integral, which is defined by

\[
\int_{0}^{x} \mathfrak{F}(t) d_q t = (1-q)x \sum_{i=0}^{\infty} q^i \mathfrak{F}(q^i x)
\]

for \( x \in [0, \infty) \).
Some Quantum Estimates of Hermite-Hadamard Inequalities

Theorem 1 ([26]) Assume $\mathcal{F}, \mathcal{G} : \mathcal{I} \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in \mathcal{I}$,

$$
\int_{\omega_1}^x [\mathcal{F}(t) + \mathcal{G}(t)] \omega_1 d_q t = \int_{\omega_1}^x \mathcal{F}(t) \omega_1 d_q t + \int_{\omega_1}^x \mathcal{G}(t) \omega_1 d_q t;
$$

$$
\int_{\omega_1}^x (\alpha \mathcal{F})(t) \omega_1 d_q t = \alpha \int_{\omega_1}^x \mathcal{F}(t) \omega_1 d_q t.
$$

Also, we introduce the $q$-analogues of $\omega_1$ and $(x - \omega_1)^n$ and the definition of $q$-Beta function.

Definition 5 ([22]) For any real number $\omega_1$,

$$
[\omega_1] = \frac{1 - q^{\omega_1}}{1 - q}
$$

is called the $q$-analogue of $\omega_1$. In particular, for $i \in \mathbb{Z}^+$, we denote

$$
[i] = \frac{1 - q^i}{1 - q} = q^{i-1} + \cdots + q + 1.
$$

Definition 6 ([22]) If $i$ is an integer, the $q$-analogue of $(x - \omega_1)^i$ is the polynomial

$$(x - \omega_1)^i_q = \begin{cases} 1, & \text{if } i = 0, \\ (x - \omega_1)(x - q\omega_1) \cdots (x - q^{i-1}\omega_1), & \text{if } i \geq 1 \end{cases}$$

Definition 7 ([22]) For any $t, s > 0$,

$$
\beta_q(t, s) = \int_0^1 x^{t-1} (1 - qx)^{s-1} d_q x
$$

(3)

is called the $q$-Beta function. Note that

$$
\beta_q(t, 1) = \int_0^1 x^{t-1} d_q x = \frac{1}{[t]},
$$

(4)

where $[t]$ is the $q$-analogue of $t$.

At last, we present the following lemmas [23] that will be used in this paper.

Lemma 1 (a) Let $\mathcal{F}(x) = 1$. Then we have

$$
\int_0^1 x d_q x = (1 - q) \sum_{i=0}^{\infty} q^i = 1.
$$

(b) If $\mathcal{F}(x) = x$ for $x \in [0, 1]$, then we have

$$
\int_0^1 x^2 d_q x = (1 - q) \sum_{i=0}^{\infty} q^{2i} = \frac{1}{1 + q}.
$$

(c) Let $\mathcal{F}(x) = x^2$ for $x \in [0, 1]$. Then we have

$$
\int_0^1 x^2 d_q x = (1 - q) \sum_{i=0}^{\infty} q^{3i} = \frac{1}{1 + q + q^2}.
$$
Lemma 2  (a) Let $\mathcal{F}(x) = 1 - qx$ for $x \in [0, 1]$, where $0 < q < 1$ is a constant. Then we have
\[
\int_0^1 (1 - qx) \, d_q x = \int_0^1 x \, d_q x - q \int_0^1 x^2 \, d_q x = \frac{1}{1 + q}.
\]
(b) Let $\mathcal{F}(x) = x(1 - qx)$ for $x \in [0, 1]$, where $0 < q < 1$ is a constant. Then we have
\[
\int_0^1 x(1 - qx) \, d_q x = \int_0^1 x \, d_q x - q \int_0^1 x^2 \, d_q x = \frac{1}{(1 + q)(1 + q + q^2)}.
\]
(c) If $\mathcal{F}(x) = x^2(1 - qx)$ for $x \in [0, 1]$, where $0 < q < 1$ is a constant, then we have
\[
\int_0^1 x^2(1 - qx) \, d_q x = \int_0^1 x^2 \, d_q x - q \int_0^1 x^3 \, d_q x = \frac{1}{(1 + q + q^2)(1 + q^2 + q^3)}.
\]

3 Hermite-Hadamard Inequalities for $\varphi$-Convex Functions

We need the following auxiliary result, which will be useful in proving our main results. This result was also proved by Liu and Zhuang [23].

Lemma 3 Let $\mathcal{F} : I = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice $q$-differentiable function on $I^0$ with $\omega_1 D_q^2 \mathcal{F}$ continuous and integrable on $\mathcal{F}$, where $0 < q < 1$. Then the following identity holds:
\[
\frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \, d_q x = \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \int_0^1 t(1 - qt) \, D_q^2 \mathcal{F}((1 - t)\omega_1 + t\omega_2) \, d_q t. \tag{5}
\]

The next theorem gives some estimates for the left-hand side of the result of (5) through $\varphi$-convex functions.

Theorem 2 Let $\mathcal{F} : I = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice $q$-differentiable function on $I^0$ with $\omega_1 D_q^2 \mathcal{F}$ continuous and integrable on $\mathcal{F}$, where $0 < q < 1$. If $|\omega_1 D_q^2 \mathcal{F}|^m$ is $\varphi$-convex on $[\omega_1, \omega_2]$ for $m \geq 1$, then the following inequality holds:
\[
\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \, d_q x \right| \leq \frac{q^2(\omega_2 - \omega_1)^2}{(1 + q)^2 - \frac{1}{m}} \left( K_1 |\omega_1 D_q^2 \mathcal{F}(\omega_1)|^m + K_2 \varphi \left( |\omega_1 D_q^2 \mathcal{F}(\omega_2)|^m, |\omega_1 D_q^2 \mathcal{F}(\omega_1)|^m \right) \right)^{\frac{1}{m}}, \tag{6}
\]
where
\[
K_1 = \int_0^1 t(1 - qt)^m \, d_q t = (1 - q) \sum_{i=0}^{\infty} q^i (1 - q^{i+1})^m
\]
and
\[
K_2 = \int_0^1 t^2(1 - qt)^m \, d_q t = (1 - q) \sum_{i=0}^{\infty} q^{3i} (1 - q^{i+1})^m.
\]
Proof. Using Lemma 3, Hölder inequality and the fact that $|\omega_i D_q^2 \tilde{\mathcal{F}}|^m$ is a $\varphi$-convex function, we have

$$
\left| \frac{q \tilde{\mathcal{F}}(\omega_1) + \tilde{\mathcal{F}}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \tilde{\mathcal{F}}(x) \omega_1^m dx \right|

\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \int_0^1 t(1 - qt) \omega_1 D_q^2 \tilde{\mathcal{F}}((1 - t) \omega_1 + t \omega_2) \omega_1^m dt

\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 t(1 - qt) |\omega_1 D_q^2 \tilde{\mathcal{F}}((1 - t) \omega_1 + t \omega_2)|^m \omega_1^m dt \right)^{\frac{1}{m}}

\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 t(1 - qt) |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_1)|^m \omega_1^m dt \right)^{\frac{1}{m}}

+ \varphi \left( |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_2)|^m, |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_1)|^m \right) \int_0^1 t^2(1 - qt) \omega_1^m dt \right)^{\frac{1}{m}}.

Now, applying Lemma 1(b), we have

$$
\left| \frac{q \tilde{\mathcal{F}}(\omega_1) + \tilde{\mathcal{F}}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \tilde{\mathcal{F}}(x) \omega_1^m dx \right|

\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \frac{1}{1 + q} \right)^{1 - \frac{m}{q}} \left( |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_1)|^m K_1 + \varphi \left( |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_2)|^m, |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_1)|^m \right) K_2 \right)^{\frac{1}{m}}

= \frac{q^2 (\omega_2 - \omega_1)^2}{(1 + q)^2 - \frac{m}{q}} \left( K_1 |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_1)|^m + K_2 \varphi \left( |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_2)|^m, |\omega_1 D_q^2 \tilde{\mathcal{F}}(\omega_1)|^m \right) \right)^{\frac{1}{m}}.

It is easy to check by Definition 4 that

$$
K_1 = \int_0^1 t(1 - qt)^m dt = (1 - q) \sum_{i=0}^\infty q^{2i}(1 - q^{i+1})^m

$$

and

$$
K_2 = \int_0^1 t^2(1 - qt)^m dt = (1 - q) \sum_{i=0}^\infty q^{3i}(1 - q^{i+1})^m.

$$

Thus, we get (6). \qed

Remark 2 By setting $\varphi(A, B) = A - B$ in Theorem 2, we recapture [23, Theorem 5.1].

Corollary 1 In Theorem 2, if $q \rightarrow 1^-$, then we get

$$
K_1 = \int_0^1 t(1 - t)^m dt = \frac{1}{(m + 1)(m + 2)}, \quad K_2 = \int_0^1 t^2(1 - t)^m dt = \frac{2}{(m + 1)(m + 2)(m + 3)}

$$

and (6) reduces to the following inequality:

$$
\left| \frac{\tilde{\mathcal{F}}(\omega_1) + \tilde{\mathcal{F}}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \tilde{\mathcal{F}}(x)dx \right|

\leq \frac{(\omega_2 - \omega_1)^2}{2^{2 - \frac{m}{q}}} \left( \frac{1}{(m + 1)(m + 2)} |\tilde{\Phi}(\omega_1)|^m + \frac{2}{(m + 1)(m + 2)(m + 3)} \varphi \left( |\tilde{\Phi}'(\omega_2)|^m, |\tilde{\Phi}'(\omega_1)|^m \right) \right)^{\frac{1}{m}}.

$$
Remark 3 If \( \varphi(A, B) = A - B \) and \( q \to 1^- \), then (6) reduces to the following inequality:

\[
\frac{\|J(t, \omega)\|}{2} - \frac{1}{2} \int_{-1}^{1} J(t, \omega) dt \leq \frac{(\omega_2 - \omega_1)^2}{2^2 - \frac{1}{q}} \left( (m+1)|J'(\omega_1)|^m + 2|J'(\omega_2)|^m \right)^{\frac{1}{m}}.
\]

Corollary 2 If \( m \) is a positive integer, then Theorem 2 amounts to:

\[
(1 - qt)^m \leq (1 - qt_q)^m,
\]

and (6) reduces to

\[
\left| \frac{q J(t, \omega) + J(t, \omega)}{1 + q} - \frac{1}{2} \int_{-1}^{1} J(t, \omega) dt \right| \leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} (L) \left( \frac{1}{1 + q} + \frac{\varphi \left( \left| \frac{1}{1 + q} \right| \omega, D_q J(t, \omega) \right) \left| \frac{1}{1 + q} \right| \omega, D_q J(t, \omega) \right) \right)^{\frac{1}{m}},
\]

where

\[ L = \int_{0}^{1} t(1 - qt)^a_0 dt = (1 - q) \sum_{i=0}^{\infty} q^{2i}(1 - q^{i+1})^n. \]

Theorem 3 Let \( \mathcal{J} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{R} \) be a twice \( q \)-differentiable function on \( \mathcal{I} \) and \( \omega, D_q \mathcal{J} \) be continuous and integrable on \( \mathcal{I} \), where \( 0 < q < 1 \). If \( |\omega, D_q \mathcal{J}|^m \) is \( \varphi \)-convex on \( [\omega_1, \omega_2] \), where \( n, m > 1, \frac{1}{n} + \frac{1}{m} = 1 \), then

\[
\left| \frac{q J(t, \omega) + J(t, \omega)}{1 + q} - \frac{1}{2} \int_{-1}^{1} J(t, \omega) dt \right| \leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} (L) \left( \frac{1}{1 + q} + \frac{\varphi \left( \left| \frac{1}{1 + q} \right| \omega, D_q J(t, \omega) \right) \left| \frac{1}{1 + q} \right| \omega, D_q J(t, \omega) \right) \right)^{\frac{1}{m}},
\]

Proof. UsingLemma 3, Hölder inequality and the fact that \( |\omega, D_q \mathcal{J}|^m \) is a \( \varphi \)-convex function, we have
Applying Lemmas 1(b) and 1(c), we have

\[
\left| \frac{q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(\xi) d\xi \right|
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 t(1 - qt)^n \, dt \right) \frac{1}{\beta_q(2, n + 1)} \cdot \left( \left| \frac{\mathfrak{F}(\omega_1)}{1 + q} + \varphi \left( \frac{\left| \mathfrak{F}(\omega_2) \right|^m}{1 + q + q^2} \right) \right| \right) ^{\frac{1}{n}}.
\]

It is easy to check by Definition 4 that

\[
L = \int_0^1 t(1 - qt)^n \, dt = (1 - q) \sum_{i=0}^{\infty} q^i (1 - q^{i+1})^n,
\]

and thus, we get (7).  

Remark 4 By setting \( \varphi(A, B) = A - B \) in Theorem 3, we recapture [23, Theorem 5.2].

Corollary 3 By letting \( q \to 1^- \) in Theorem 3, we get

\[
L = \int_0^1 t(1 - t)^n \, dt = \frac{1}{(n + 1)(n + 2)},
\]

and (7) reduces to the following inequality:

\[
\left| \frac{\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x) \, dx \right|
\leq \frac{(\omega_2 - \omega_1)^2}{2} \left( \frac{1}{(n + 1)(n + 2)} \right) \frac{1}{\beta_q(2, n + 1)} \cdot \left( \left| \frac{\mathfrak{F}(\omega_1)}{1 + q} + \varphi \left( \frac{\left| \mathfrak{F}(\omega_2) \right|^m}{1 + q + q^2} \right) \right| \right) ^{\frac{1}{n}}.
\]

Remark 5 If \( \varphi(A, B) = A - B \) and \( q \to 1^- \), then (7) reduces to the following inequality:

\[
\left| \frac{\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x) \, dx \right|
\leq \frac{(\omega_2 - \omega_1)^2}{2} \left( \frac{1}{(n + 1)(n + 2)} \right) \frac{1}{\beta_q(2, n + 1)} \cdot \left( \left| \frac{\mathfrak{F}(\omega_1)}{1 + q} + \varphi \left( \frac{\left| \mathfrak{F}(\omega_2) \right|^m}{1 + q + q^2} \right) \right| \right) ^{\frac{1}{n}}.
\]

Corollary 4 In Theorem 3, if \( n \) is a positive integer, then

\[
(1 - qt)^n \leq (1 - qt)_{q^n},
\]

and (7) reduces to

\[
\left| \frac{q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(\xi) d\xi \right|
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \beta_q(2, n + 1) \right) \frac{1}{\beta_q(2, n + 1)} \cdot \left( \left| \frac{\mathfrak{F}(\omega_1)}{1 + q} + \varphi \left( \frac{\left| \mathfrak{F}(\omega_2) \right|^m}{1 + q + q^2} \right) \right| \right) ^{\frac{1}{n}}.
\]
**Theorem 4** Let $\mathcal{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice $q$-differentiable function on $\mathcal{I}$ and $\omega, D_q^2 \mathcal{F}$ be continuous and integrable on $\mathcal{I}$, where $0 < q < 1$. If $|\omega, D_q^2 \mathcal{F}|^m$ is $\varphi$-convex on $[\omega_1, \omega_2]$, where $n, m > 1$, $\frac{1}{n} + \frac{1}{m} = 1$, then

$$
\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x)_{\omega_1} \, dx \right|
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \frac{1}{(R)^{\frac{1}{q}}}
\left( \frac{|\omega, D_q^2 \mathcal{F}(\omega_1)|^m + \varphi \left( \frac{|\omega, D_q^2 \mathcal{F}(\omega_2)|^m}{1 + q} \right) \right) \frac{1}{(R)^{\frac{1}{q}}}.
$$

where

$$
R = \int_0^1 t^n(1 - qt)^n \, dq \, dt = (1 - q) \sum_{i=0}^{\infty} (q^i)^{n+1} (1 - q^{i+1})^n.
$$

**Proof.** Using Lemma 3, Hölder inequality and the fact that $|\omega, D_q^2 \mathcal{F}|^m$ is a $\varphi$-convex function, we have

$$
\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x)_{\omega_1} \, dx \right|
= \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \int_0^1 t(1 - qt) \omega_1 D_q^2 \mathcal{F}((1 - t)\omega_1 + t \omega_2) \, dt \, dq
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 t^n(1 - qt)^n \, dq \, dt \right)^{\frac{1}{q}} \left( \int_0^1 |\omega, D_q^2 \mathcal{F}((1 - t)\omega_1 + t \omega_2)|^m \, dq \, dt \right)^{\frac{1}{m}}
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 t^n(1 - qt)^n \, dq \, dt \right)^{\frac{1}{q}} \left( |\omega, D_q^2 \mathcal{F}(\omega_1)|^m \int_0^1 t \, dq \, dt \right)^{\frac{1}{m}}
$$

Employing Lemmas 1(a) and 1(b), we obtain

$$
\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x)_{\omega_1} \, dx \right|
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \frac{1}{(R)^{\frac{1}{q}}}
\left( |\omega, D_q^2 \mathcal{F}(\omega_1)|^m + \varphi \left( \frac{|\omega, D_q^2 \mathcal{F}(\omega_2)|^m}{1 + q} \right) \right) \frac{1}{(R)^{\frac{1}{q}}}.
$$

It is easy to check by Definition 4 that

$$
R = \int_0^1 t^n(1 - qt)^n \, dq \, dt = (1 - q) \sum_{i=0}^{\infty} (q^i)^{n+1} (1 - q^{i+1})^n,
$$

and thus, we get (8). $$\blacksquare$$

**Remark 6** By setting $\varphi(A, B) = A - B$ in Theorem 4, we recapture [23, Theorem 5.4].

**Corollary 5** In Theorem 4, if $q \rightarrow 1$, then we have

$$
R = \int_0^1 t^n(1 - t)^n \, dt = \beta(n + 1, n + 1).
$$
Using the properties of Beta function, that is, \( \beta(x, x) = 2^{1-2x} \beta(\frac{1}{2}, x) \) and \( \beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \), we obtain that

\[
\beta(n+1, n+1) = 2^{1-2(n+1)} \beta\left(\frac{1}{2}, n+1\right) = 2^{-2n-1} \frac{\Gamma(\frac{1}{2}) \Gamma(n+1)}{\Gamma(\frac{3}{2} + n)},
\]

where \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \) and \( \Gamma(t) \) is Gamma function:

\[
\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx, \quad t > 0.
\]

Thus, inequality (8) reduces to the following inequality:

\[
\left| \frac{\mathcal{H}(\omega_1) + \mathcal{H}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{H}(x) dx \right| \leq \frac{(\omega_2 - \omega_1)^2}{8} \left( \frac{\sqrt{\pi}}{2} \right)^{\frac{m}{2n}} \left( \Gamma(n+1) \right)^{\frac{m}{2}} \left( \frac{\mathcal{H}''(\omega_1)}{2} + \mathcal{H}''(\omega_2) \right)^{\frac{1}{m}}.
\]

**Remark 7** If \( \varphi(A, B) = A - B \) and \( q \to 1^- \), then (8) reduces to the following inequality:

\[
\left| \frac{\mathcal{H}(\omega_1) + \mathcal{H}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{H}(x) dx \right| \leq \frac{(\omega_2 - \omega_1)^2}{8} \left( \frac{\sqrt{\pi}}{2} \right)^{\frac{m}{2n}} \left( \Gamma(n+1) \right)^{\frac{m}{2}} \left( \frac{\mathcal{H}''(\omega_1)}{2} + \mathcal{H}''(\omega_2) \right)^{\frac{1}{m}}.
\]

**Corollary 6** In Theorem 4, if \( n \) is a positive integer, \( n > 1 \), then

\[
(1 - qt)^n \leq (1 - qt)^\frac{n}{q},
\]

and (8) reduces to

\[
\left| \frac{q\mathcal{H}(\omega_1) + \mathcal{H}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{H}(x) dx \right| \leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \frac{1}{\beta_q(n+1, n+1)} \right)^{\frac{1}{n}} \left( \left| \omega_1 D_q^2 \mathcal{H}(\omega_1) \right|^m + \varphi\left( \left| \omega_1 D_q^2 \mathcal{H}(\omega_2) \right|^m \right) \right)^{\frac{1}{m}}.
\]

**Theorem 5** Let \( \mathcal{H} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R} \) be a twice \( q \)-differentiable function on \( \mathcal{I} \) with \( \omega_1 D_q^2 \mathcal{H} \) continuous and integrable on \( \mathcal{I} \) where \( 0 < q < 1 \). If \( \left| \omega_1 D_q^2 \mathcal{H} \right|^m \) is \( \varphi \)-convex on \( [\omega_1, \omega_2] \) where \( n, m > 1 \), \( \frac{1}{n} + \frac{1}{m} = 1 \), then

\[
\left| \frac{q\mathcal{H}(\omega_1) + \mathcal{H}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{H}(x) dx \right| \leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \frac{1}{\beta_q(n+1)} \right)^{\frac{1}{n}} \left( \left| W_1 \omega_1 D_q^2 \mathcal{H}(\omega_1) \right|^m + W_2 \varphi\left( \left| \omega_1 D_q^2 \mathcal{H}(\omega_2) \right|^m \right) \right)^{\frac{1}{m}}, \quad (9)
\]

where

\[
W_1 = \int_0^1 (1 - qt)^m \, dt = (1 - q) \sum_{i=0}^{\infty} q^i (1 - q^{i+1})^m
\]
and

\[ W_2 = \int_0^1 t(1 - qt)^m \, dt = (1-q)\sum_{i=0}^{\infty} q^i(1-q^{i+1})^m. \]

**Proof.** Using Lemma 3, Hölder inequality and the fact that \(|\omega_1 D^m_q \mathfrak{F}|^m\) is a \(\varphi\)-convex function, we have

\[
\left| q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2) - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)\omega_1 \, dx \right| \\
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \int_0^1 t(1 - qt) \omega_2 \mathfrak{F}((1-t)\omega_1 + t\omega_2) \, dt \right) \\
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \int_0^1 t(1 - qt) \omega_2 \mathfrak{F}((1-t)\omega_1 + t\omega_2) \, dt \right) \\
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \int_0^1 t^n \omega_2 \mathfrak{F}((1-t)\omega_1 + t\omega_2) \, dt \right)^{\frac{1}{n}} \\
+ \varphi \left( |\omega_1 D^m_q \mathfrak{F}(\omega_2)|^m, |\omega_1 D^m_q \mathfrak{F}(\omega_1)|^m \right) \int_0^1 t(1 - qt)^m \, dt \right)^{\frac{1}{m}},
\]

and applying (4) in Definition 7, we have

\[
\left| q\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2) - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)\omega_1 \, dx \right| \\
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \frac{1}{[n+1]} \right)^{\frac{1}{n}} \left( |\omega_1 D^m_q \mathfrak{F}(\omega_1)|^m \right)^{\frac{1}{n}} \left( \int_0^1 (1 - qt)^m \, dt \right)^{\frac{1}{m}} \\
+ \varphi \left( |\omega_1 D^m_q \mathfrak{F}(\omega_2)|^m, |\omega_1 D^m_q \mathfrak{F}(\omega_1)|^m \right) \int_0^1 t(1 - qt)^m \, dt \right)^{\frac{1}{m}} \\
= \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \frac{1}{[n+1]} \right)^{\frac{1}{n}} \left( \left( W_1 |\omega_1 D^m_q \mathfrak{F}(\omega_1)|^m + W_2 \varphi \left( |\omega_1 D^m_q \mathfrak{F}(\omega_2)|^m, |\omega_1 D^m_q \mathfrak{F}(\omega_1)|^m \right) \right) \right)^{\frac{1}{m}}.
\]

It is easy to check by Definition 4 that

\[ W_1 = \int_0^1 (1 - qt)^m \, dt = (1-q)\sum_{i=0}^{\infty} q^i(1-q^{i+1})^m \]

and

\[ W_2 = \int_0^1 t(1 - qt)^m \, dt = (1-q)\sum_{i=0}^{\infty} q^i(1-q^{i+1})^m, \]

thus, we get (9).  

**Remark 8** By setting \(\varphi(A, B) = A - B\) in Theorem 5, we recapture [23, Theorem 5.5].

**Corollary 7** In Theorem 5, if \(q \to 1\), then we have

\[ W_1 = \int_0^1 (1-t)^m \, dt = \frac{1}{m+1}, \quad W_2 = \int_0^1 t(1-t)^m \, dt = \frac{1}{(m+1)(m+2)} , \]
and (9) reduces to the following inequality:

$$
\left| \frac{\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) dx \right| 
\leq \frac{(\omega_2 - \omega_1)^2}{2} \left( \frac{1}{n + 1} \right)^{\frac{1}{m}} \left( \frac{(m + 1) |\mathcal{F}''(\omega_1)|^m + \varphi (|\mathcal{F}''(\omega_2)|^m)}{(m + 1)(m + 2)} \right)^{\frac{1}{m}}.
$$

(10)

**Remark 9** If $\varphi(A, B) = A - B$ and $q \to 1$, then (9) reduces to the following inequality:

$$
\left| \frac{\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) dx \right| 
\leq \frac{(\omega_2 - \omega_1)^2}{2} \left( \frac{1}{n + 1} \right)^{\frac{1}{m}} \left( \frac{(m + 1) |\mathcal{F}''(\omega_1)|^m + |\mathcal{F}''(\omega_2)|^m}{(m + 1)(m + 2)} \right)^{\frac{1}{m}}.
$$

(11)

**Corollary 8** In Theorem 5, if $m$ is a positive integer, $m > 1$, then

$$(1 - qt)^m \leq (1 - qt)^m_q,$$

and (9) reduces to

$$
\left| q\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2) \right| - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) dq \leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \frac{1}{n + 1} \right)^{\frac{1}{m}} \left( \beta_q(1, m + 1) |\omega_q D_q^2 \mathcal{F} (\omega_1)|^m 
\right.
\left. + \beta_q(2, m + 1) \varphi (|\omega_q D_q^2 \mathcal{F} (\omega_2)|^m, |\omega_q D_q^2 \mathcal{F} (\omega_1)|^m) \right)^{\frac{1}{m}}.
$$

Theorem 6 Let $\mathcal{F} : I = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice $q$-differentiable function on $I$ with $\omega_q D_q^2 \mathcal{F}$ be continuous and integrable on $I$ where $0 < q < 1$. If $|\omega_q D_q^2 \mathcal{F}|^m$ is $\varphi$-convex on $[\omega_1, \omega_2]$ where $n, m > 1$, $\frac{1}{n} + \frac{1}{m} = 1$, then

$$
\left| q\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2) \right| - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) dq \leq \frac{q^2(\omega_2 - \omega_1)^2}{1 + q} \left( \frac{1}{n + 1} \right)^{\frac{1}{m}} \left( \beta_q(1, m + 1) |\omega_q D_q^2 \mathcal{F} (\omega_1)|^m 
\right.
\left. + \beta_q(2, m + 1) \varphi (|\omega_q D_q^2 \mathcal{F} (\omega_2)|^m, |\omega_q D_q^2 \mathcal{F} (\omega_1)|^m) \right)^{\frac{1}{m}}.
$$

(12)

where

$$
M = \int_0^1 (1 - qt)^n dq = (1 - q) \sum_{i=0}^{\infty} q^i(1 - q^{i+1})^n.
$$
Proof. Using Lemma 3, Hölder inequality and the fact that \(|\omega_1 D_q^{2} \mathfrak{F})^m| is a ϕ-convex function, we have

\[
|q \mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)| - \frac{1}{1 + q} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x) d_q x = \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \int_0^1 t (1 - qt) \omega_1 D_q^{2} \mathfrak{F}((1 - t)\omega_1 + t\omega_2) dt \\
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 (1 - qt)^n d_q t \right)^\frac{1}{n} \left( \int_0^1 t^m |\omega_1 D_q^{2} \mathfrak{F}((1 - t)\omega_1 + t\omega_2)|^m d_q t \right)^\frac{1}{m+1} \\
= \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} (M)^\frac{1}{n} \left( \frac{|\omega_1 D_q^{2} \mathfrak{F}(\omega_1)|^m}{m+1} + \frac{\varphi (|\omega_1 D_q^{2} \mathfrak{F}(\omega_2)|^m, |\omega_1 D_q^{2} \mathfrak{F}(\omega_1)|^m)}{m+2} \right)^\frac{1}{m+1}.
\]

and applying (4) in Definition 7, we have

\[
\left| q \mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2) \right| - \frac{1}{1 + q} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x) d_q x \\
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 (1 - qt)^n d_q t \right)^\frac{1}{n} \left( \int_0^1 t^m |\omega_1 D_q^{2} \mathfrak{F}(\omega_1)|^m d_q t \right)^\frac{1}{m+1} \\
= \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} (M)^\frac{1}{n} \left( \frac{|\omega_1 D_q^{2} \mathfrak{F}(\omega_1)|^m}{m+1} + \frac{\varphi (|\omega_1 D_q^{2} \mathfrak{F}(\omega_2)|^m, |\omega_1 D_q^{2} \mathfrak{F}(\omega_1)|^m)}{m+2} \right)^\frac{1}{m+1}.
\]

It is easy to check by Definition 4 that

\[
M = \int_0^1 (1 - qt)^n d_q t = (1 - q) \sum_{i=0}^{\infty} q^i (1 - q^{i+1})^n,
\]

thus, we get (10). □

Remark 10 By setting \(\varphi(A, B) = A - B\) in Theorem 6, we recapture [23, Theorem 5.6].

Corollary 9 In Theorem 6, if \(q \to 1\), then we have

\[
M = \int_0^1 (1 - t)^n dt = \frac{1}{n+1}
\]

and (12) reduces to (10) in Corollary 7.

Remark 11 If \(\varphi(A, B) = A - B\) and \(q \to 1\), then (12) reduces to (11) in Remark 9.

Corollary 10 In Theorem 6, if \(n\) is a positive integer, \(n > 1\), then

\[
(1 - qt)^n \leq (1 - qt)^{\frac{n}{2}}
\]

and (12) reduces to

\[
\left| q \mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2) \right| - \frac{1}{1 + q} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x) d_q x \\
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} (\beta_q(1, n+1))^\frac{1}{n} \left( \frac{|\omega_1 D_q^{2} \mathfrak{F}(\omega_1)|^m}{m+1} + \frac{\varphi (|\omega_1 D_q^{2} \mathfrak{F}(\omega_2)|^m, |\omega_1 D_q^{2} \mathfrak{F}(\omega_1)|^m)}{m+2} \right)^\frac{1}{m+1}.
\]
Theorem 7 Let $\mathcal{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice $q$-differentiable function on $\mathcal{I}$ with $\omega, D_q^2 \mathcal{F}$ being continuous and integrable on $\mathcal{I}$ where $0 < q < 1$. If $|\omega, D_q^2 \mathcal{F}|^m$ is $\varphi$-convex on $[\omega_1, \omega_2]$ for $m \geq 1$, then
\[
\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) d_q x \right| 
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( V_1 |\omega, D_q^2 \mathcal{F}(\omega_1)|^m + V_2 \varphi \left( |\omega, D_q^2 \mathcal{F}(\omega_2)|^m, |\omega, D_q^2 \mathcal{F}(\omega_1)|^m \right) \right)^{\frac{1}{m}},
\]
where
\[
V_1 = \int_0^1 t (1 - qt)^m d_q t = (1 - q) \sum_{i=0}^{\infty} (q^i)^{m+1} (1 - q^{i+1})^m
\]
and
\[
V_2 = \int_0^1 t^{m+1} (1 - qt)^m d_q t = (1 - q) \sum_{i=0}^{\infty} (q^i)^{m+2} (1 - q^{i+1})^m.
\]
Proof. Using Lemma 3, Hölder inequality and the fact that $|\omega, D_q^2 \mathcal{F}|^m$ is a $\varphi$-convex function, we have
\[
\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) d_q x \right| 
= \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \int_0^1 t (1 - qt) |\omega, D_q^2 \mathcal{F}((1 - t)\omega_1 + t\omega_2)| d_q t 
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \int_0^1 t (1 - qt) |\omega, D_q^2 \mathcal{F}((1 - t)\omega_1 + t\omega_2)| d_q t 
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 d_q t \right)^{1 - \frac{1}{m}} \left( \int_0^1 t (1 - qt)^m |\omega, D_q^2 \mathcal{F}((1 - t)\omega_1 + t\omega_2)|^m d_q t \right)^{\frac{1}{m}} 
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_0^1 d_q t \right)^{1 - \frac{1}{m}} \left( |\omega, D_q^2 \mathcal{F}(\omega_1)|^m \right) \left( \int_0^1 t (1 - qt)^m d_q t \right)^{\frac{1}{m}} 
+ \varphi \left( |\omega, D_q^2 \mathcal{F}(\omega_2)|^m, |\omega, D_q^2 \mathcal{F}(\omega_1)|^m \right) \left( \int_0^1 t^{m+1} (1 - qt)^m d_q t \right)^{\frac{1}{m}},
\]
and applying Lemma 1(a), we have
\[
\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) d_q x \right| 
\leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( |\omega, D_q^2 \mathcal{F}(\omega_1)|^m \right) \int_0^1 t (1 - qt)^m d_q t 
+ \varphi \left( |\omega, D_q^2 \mathcal{F}(\omega_2)|^m, |\omega, D_q^2 \mathcal{F}(\omega_1)|^m \right) \int_0^1 t^{m+1} (1 - qt)^m d_q t \right)^{\frac{1}{m}} 
= \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( V_1 |\omega, D_q^2 \mathcal{F}(\omega_1)|^m + V_2 \varphi \left( |\omega, D_q^2 \mathcal{F}(\omega_2)|^m, |\omega, D_q^2 \mathcal{F}(\omega_1)|^m \right) \right)^{\frac{1}{m}}.
\]
It is easy to check by Definition 4 that
\[
V_1 = \int_0^1 t (1 - qt)^m d_q t = (1 - q) \sum_{i=0}^{\infty} (q^i)^{m+1} (1 - q^{i+1})^m
\]
and
\[
V_2 = \int_0^1 t^{m+1} (1 - qt)^m d_q t = (1 - q) \sum_{i=0}^{\infty} (q^i)^{m+2} (1 - q^{i+1})^m.
\]
thus, we get (13). □

**Remark 12** By setting \( \varphi(A, B) = A - B \) in Theorem 7, we recapture \([23, \text{Theorem 5.3}]\).

**Corollary 11** In Theorem 7, if \( q \to 1 \), then we have

\[
V_1 = \int_0^1 t^m(1-t)^m \, dt = \beta(m+1, m+1), \quad V_2 = \int_0^1 t^{m+1}(1-t)^m \, dt = \beta(m+2, m+1),
\]

and (13) reduces to the following inequality:

\[
\left| \frac{\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \, dx \right| \\
\leq \frac{(\omega_2 - \omega_1)^2}{2} \left( \beta(m+1, m+1) |\mathcal{F}'''(\omega_1)|^m + \beta(m+2, m+1) \varphi \left( |\mathcal{F}''(\omega_2)|^m, |\mathcal{F}'''(\omega_1)|^m \right) \right)^{\frac{1}{m}}.
\]

**Remark 13** If \( \varphi(A, B) = A - B \) and \( q \to 1 \), then (13) reduces to the following inequality:

\[
\left| \frac{\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \, dx \right| \\
\leq \frac{(\omega_2 - \omega_1)^2}{2} \left( \beta(m+1, m+2) |\mathcal{F}''(\omega_1)|^m + \beta(m+2, m+1) |\mathcal{F}'''(\omega_2)|^m \right)^{\frac{1}{m}}.
\]

**Corollary 12** In Theorem 7, if \( m \) is a positive integer, \( m > 1 \), then

\[
(1 - qt)^m \leq (1 - qt)_q^m,
\]

and (13) reduces to

\[
\left| \frac{q\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \, dx \right| \\
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \beta_q(m+1, m+1) \left| \omega_1 D_q^2 \mathcal{F}(\omega_1) \right|^m + \beta_q(m+2, m+1) \varphi \left( \left| \omega_1 D_q^2 \mathcal{F}(\omega_2) \right|^m, \left| \omega_1 D_q^2 \mathcal{F}(\omega_1) \right|^m \right) \right)^{\frac{1}{m}}.
\]

**Theorem 8** Let \( \mathcal{F} : I = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R} \) be a twice-q-differentiable function on \( I^0 \) with \( \omega_1 D_q^2 \mathcal{F} \) continuous and integrable on \( I \) where \( 0 < q < 1 \). If \( \omega_1 D_q^2 \mathcal{F} \) is \( \varphi \)-convex on \([\omega_1, \omega_2]\) for \( m \geq 1 \), then

\[
\left| \frac{q\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \, dx \right| \\
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \frac{1}{1+q} \right)^{1-\frac{1}{m}} \left( \beta_q(m+1, 2) \left| \omega_1 D_q^2 \mathcal{F}(\omega_1) \right|^m + \beta_q(m+2, 2) \varphi \left( \left| \omega_1 D_q^2 \mathcal{F}(\omega_2) \right|^m, \left| \omega_1 D_q^2 \mathcal{F}(\omega_1) \right|^m \right) \right)^{\frac{1}{m}}.
\]
Thus, we get (14).

In Theorem 8, if $q \to 1$, then
\begin{align*}
\beta(m + 1, 2) &= \int_0^1 t^m(1-t)dt = \frac{1}{(m+1)(m+2)}, \\
\beta(m+2, 2) &= \int_0^1 t^{m+1}(1-t)dt = \frac{1}{(m+2)(m+3)},
\end{align*}

then (14) reduces to
\begin{align*}
&\left| \frac{\mathcal{H}(\omega_1) + \mathcal{H}(\omega_2)}{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{H}(x)dx \right| \\
\leq &\left( \frac{\omega_2 - \omega_1}{2} \right)^2 \left( \frac{2}{(m+1)(m+2)(m+3)} \right)^{\frac{1}{m}} \left( (m+3)|\mathcal{H}''(\omega_1)|^m + (m+1)|\mathcal{H}''(\omega_2)|^m \right)^{\frac{1}{m}}.
\end{align*}

Remark 14 By setting $\varphi(A, B) = A - B$ in Theorem 8, we recapture [23, Theorem 5.9].

Corollary 13 In Theorem 8, if $q \to 1$,
\begin{align*}
\beta(m + 1, 2) &= \int_0^1 t^m(1-t)dt = \frac{1}{(m+1)(m+2)}, \\
\beta(m+2, 2) &= \int_0^1 t^{m+1}(1-t)dt = \frac{1}{(m+2)(m+3)},
\end{align*}

then (14) reduces to
\begin{align*}
&\left| \frac{\mathcal{H}(\omega_1) + \mathcal{H}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{H}(x)dx \right| \\
\leq &\left( \frac{\omega_2 - \omega_1}{4} \right)^2 \left( \frac{2}{(m+1)(m+2)(m+3)} \right)^{\frac{1}{m}} \left( (m+3)|\mathcal{H}''(\omega_1)|^m + (m+1)|\mathcal{H}''(\omega_2)|^m \right)^{\frac{1}{m}}.
\end{align*}

Remark 15 If $\varphi(A, B) = A - B$ and $q \to 1$, then (14) reduces to the following inequality:
\begin{align*}
&\left| \frac{\mathcal{H}(\omega_1) + \mathcal{H}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{H}(x)dx \right| \\
\leq &\left( \frac{\omega_2 - \omega_1}{4} \right)^2 \left( \frac{2}{(m+1)(m+2)(m+3)} \right)^{\frac{1}{m}} \left( 2|\mathcal{H}''(\omega_1)|^m + (m+1)|\mathcal{H}''(\omega_2)|^m \right)^{\frac{1}{m}}.
\end{align*}
Theorem 9 Let $\hat{\mathfrak{F}} : I = [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice $q$-differentiable function on $I$ with $\omega_1 D_q^2 \hat{\mathfrak{F}}$ be continuous and integrable on $I$ where $0 < q < 1$. If $|\omega_1 D_q^2 \hat{\mathfrak{F}}|^m$ is $\varphi$-convex on $[\omega_1, \omega_2]$ where $n, m > 1, \frac{1}{n} + \frac{1}{m} = 1$, then

$$ \left| q \hat{\mathfrak{F}}(\omega_1) + \hat{\mathfrak{F}}(\omega_2) \right| \frac{1}{1 + q} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x \leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_{0}^{1} t^n (1 - qt) d_q t \right)^\frac{1}{m} \left( \int_{0}^{1} (1 - qt) \left| \omega_1 D_q^2 \hat{\mathfrak{F}}((1 - t)\omega_1 + t\omega_2) \right|^m d_q t \right)^\frac{1}{m}, $$

and applying Lemma 2(a) and (b) and the fact that $|\omega_1 D_q^2 \hat{\mathfrak{F}}|^m$ is $\varphi$-convex, we have

$$ \left| q \hat{\mathfrak{F}}(\omega_1) + \hat{\mathfrak{F}}(\omega_2) \right| \frac{1}{1 + q} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x \leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_{0}^{1} t^n (1 - qt) d_q t \right)^\frac{1}{m} \left( \int_{0}^{1} (1 - qt) \left| \omega_1 D_q^2 \hat{\mathfrak{F}}((1 - t)\omega_1 + t\omega_2) \right|^m d_q t \right)^\frac{1}{m}. $$

Proof. Using Lemma 3, Hölder inequality and the fact that $|\omega_1 D_q^2 \hat{\mathfrak{F}}|^m$ is a $\varphi$-convex function, we have

$$ \left| q \hat{\mathfrak{F}}(\omega_1) + \hat{\mathfrak{F}}(\omega_2) \right| \frac{1}{1 + q} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x \leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_{0}^{1} t^n (1 - qt) d_q t \right)^\frac{1}{m} \left( \int_{0}^{1} (1 - qt) \left| \omega_1 D_q^2 \hat{\mathfrak{F}}((1 - t)\omega_1 + t\omega_2) \right|^m d_q t \right)^\frac{1}{m}, $$

and applying Lemma 2(a) and (b) and the fact that $(1 - qt) = (1 - qt)^1_q$, we have

$$ \left| q \hat{\mathfrak{F}}(\omega_1) + \hat{\mathfrak{F}}(\omega_2) \right| \frac{1}{1 + q} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \hat{\mathfrak{F}}(x) d_q x \leq \frac{q^2 (\omega_2 - \omega_1)^2}{1 + q} \left( \int_{0}^{1} t^n (1 - qt) d_q t \right)^\frac{1}{m} \left( \int_{0}^{1} (1 - qt) \left| \omega_1 D_q^2 \hat{\mathfrak{F}}((1 - t)\omega_1 + t\omega_2) \right|^m d_q t \right)^\frac{1}{m}. $$

Hence, we get (15).

Remark 16 By setting $\varphi(A, B) = A - B$ in Theorem 9, we recapture [23, Theorem 5.10].

Corollary 14 In Theorem 9, if $q \rightarrow 1$,

$$ \beta(n + 1, 2) = \int_{0}^{1} t^n (1 - t) dt = \frac{1}{(n + 1)(n + 2)}, $$

and (15) reduces to

$$ \left| \frac{3 \hat{\mathfrak{F}}''(\omega_1)}{2} + \hat{\mathfrak{F}}''(\omega_2) \right| \frac{1}{(n + 1)(n + 2)} \left( \left| \hat{\mathfrak{F}}''(\omega_1) \right|^m + \frac{\varphi \left( \left| \hat{\mathfrak{F}}''(\omega_2) \right|^m, \left| \hat{\mathfrak{F}}''(\omega_1) \right|^m \right)}{2} \right)^\frac{1}{m}. $$
Remark 17 If \( \varphi(A, B) = A - B \) and \( q \to 1 \), then (15) reduces to the following inequality:

\[
\left| \frac{\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| \leq \frac{(\omega_2 - \omega_1)^2}{2(n+1)(n+2)} \left( \frac{2}{n+1} \right)^{\frac{1}{n}}. 
\]

Theorem 10 Let \( \mathfrak{F} : \mathcal{I} = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R} \) be a twice \( q \)-differentiable function on \( \mathcal{I} \) with \( \omega_1, D_q^2 \mathfrak{F} \) be continuous and integrable on \( \mathcal{I} \) where \( 0 < q < 1 \). If \( |\omega_1, D_q^2 \mathfrak{F}| \) is \( \varphi \)-convex on \( [\omega_1, \omega_2] \), then

\[
\left| q \mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2) \right|_{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| 
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \left( 1 + q + q^2 + q^3 \right) |\omega_1, D_q^2 \mathfrak{F}(\omega_1)| + (1 + q)\varphi \left( |\omega_1, D_q^2 \mathfrak{F}(\omega_2)|, |\omega_1, D_q^2 \mathfrak{F}(\omega_1)| \right) \right). 
\]

Proof. Using Lemma 3, Hölder inequality and the fact that \( |\omega_1, D_q^2 \mathfrak{F}| \) is a \( \varphi \)-convex function, we have

\[
\left| q \mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2) \right|_{1+q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| 
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \int_0^1 t(1 - q)t \omega_1 D_q^2 \mathfrak{F}((1 - t)\omega_1 + t\omega_2)dt \right) 
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \int_0^1 t(1 - q)t \omega_1 D_q^2 \mathfrak{F}((1 - t)\omega_1 + t\omega_2)dt \right) 
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \int_0^1 t(1 - q)t \omega_1 D_q^2 \mathfrak{F}((1 - t)\omega_1 + t\omega_2)dt \right). 
\]

applying Lemma 2(b) and (c), we have

\[
\left| q \mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2) \right| - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| 
\leq \frac{q^2(\omega_2 - \omega_1)^2}{1+q} \left( \left( 1 + q + q^2 + q^3 \right) |\omega_1, D_q^2 \mathfrak{F}(\omega_1)| + (1 + q)\varphi \left( |\omega_1, D_q^2 \mathfrak{F}(\omega_2)|, |\omega_1, D_q^2 \mathfrak{F}(\omega_1)| \right) \right). 
\]

thus, we get (16). \( \square \)

Remark 18 By setting \( \varphi(A, B) = A - B \) in Theorem 10, we recapture [23, Theorem 5.7].

Corollary 15 In Theorem 10, if \( q \to 1 \), then (16) reduces to the following inequality:

\[
\left| \frac{\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| \leq \frac{(\omega_2 - \omega_1)^2}{24} \left( 2 |\mathfrak{F}''(\omega_1)| + \varphi \left( |\mathfrak{F}''(\omega_2)|, |\mathfrak{F}''(\omega_1)| \right) \right). 
\]

Remark 19 If \( \varphi(A, B) = A - B \) and \( q \to 1 \), then (16) reduces to the following inequality:

\[
\left| \frac{\mathfrak{F}(\omega_1) + \mathfrak{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{F}(x)dx \right| \leq \frac{(\omega_2 - \omega_1)^2}{24} \left( |\mathfrak{F}''(\omega_1)| + |\mathfrak{F}''(\omega_2)| \right). 
\]
Theorem 11 Let $\mathcal{F} : I = [\omega_1, \omega_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice $q$-differentiable function on $I^0$ with $\omega_1 D_q^2 \mathcal{F}$ be continuous and integrable on $I$ where $0 < q < 1$. If $|\omega_1 D_q^2 \mathcal{F}|^m$ is $\varphi$-convex on $[\omega_1, \omega_2]$ for $m \geq 1$, then

$$\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \omega_1 d_q x \right| \leq \frac{q^2 (\omega_2 - \omega_1)^2}{(1 + q)^2 (1 + q + q^2)} \left( \frac{1}{1 + q + q^2 + q^3} \right)^m \left( \frac{1}{1 + q + q^2 + q^3} \right)^m + (1 + q) \varphi \left( \left| \omega_1 D_q^2 \mathcal{F}(\omega_2) \right|^m, \left| \omega_1 D_q^2 \mathcal{F}(\omega_1) \right|^m \right)^m. \quad (17)$$

Proof. Using Lemma 3, Hölder inequality and the fact that $|\omega_1 D_q^2 \mathcal{F}|^m$ is a $\varphi$-convex function, we have

$$\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \omega_1 d_q x \right| = \frac{q^2 (\omega_2 - \omega_1)^2}{(1 + q)^2} \left( \int_0^1 t(1 - qt) \omega_1 D_q^2 \mathcal{F}((1 - t)\omega_1 + t\omega_2) d_q t \right) \leq \frac{q^2 (\omega_2 - \omega_1)^2}{(1 + q)^2} \left( \int_0^1 t(1 - qt) \omega_1 D_q^2 \mathcal{F}((1 - t)\omega_1 + t\omega_2) d_q t \right)^\frac{1}{m} \left( \int_0^1 (1 - qt) \omega_1 D_q^2 \mathcal{F}(\omega_1) d_q t \right)^{\frac{1}{m}} + \varphi \left( \left| \omega_1 D_q^2 \mathcal{F}(\omega_2) \right|^m, \left| \omega_1 D_q^2 \mathcal{F}(\omega_1) \right|^m \right) \int_0^1 (1 - qt) d_q t \right)^\frac{1}{m}$$

and applying Lemma 2(b) and (c), we have

$$\left| \frac{q \mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{1 + q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) \omega_1 d_q x \right| \leq \frac{q^2 (\omega_2 - \omega_1)^2}{(1 + q)^2} \left( \frac{1}{1 + q + q^2} \right)^{\frac{1}{m}} \left( \frac{1}{1 + q + q^2} \right)^{\frac{1}{m}} \left( \frac{1}{1 + q + q^2} \right)^{\frac{1}{m}} + \varphi \left( \left| \omega_1 D_q^2 \mathcal{F}(\omega_2) \right|^m, \left| \omega_1 D_q^2 \mathcal{F}(\omega_1) \right|^m \right) \cdot \frac{1}{1 + q + q^2 + q^3} \left( \frac{1 + q + q^2 + q^3}{1 + q + q^2 + q^3} \right)^{\frac{1}{m}} \frac{1}{1 + q + q^2 + q^3} \left( \frac{1 + q + q^2 + q^3}{1 + q + q^2 + q^3} \right)^{\frac{1}{m}} \frac{1}{1 + q + q^2 + q^3} \left( \frac{1 + q + q^2 + q^3}{1 + q + q^2 + q^3} \right)^{\frac{1}{m}}$$

Thus, we get (17). ■

Remark 20 By setting $\varphi(A, B) = A - B$ in Theorem 11, we recapture [23, Theorem 5.8].

Corollary 16 In Theorem 11, if $q \to 1$, then (17) reduces to the following inequality:

$$\left| \frac{\mathcal{F}(\omega_1) + \mathcal{F}(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathcal{F}(x) dx \right| \leq \frac{(\omega_2 - \omega_1)^2}{12} \left( \frac{2 |\mathcal{F}(\omega_1)|^m + \varphi \left( |\mathcal{F}''(\omega_2)|^m, |\mathcal{F}''(\omega_1)|^m \right) }{2} \right)^{\frac{1}{m}}.$$
4 Conclusion

Quantum calculus has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities. In this paper, develop some quantum estimates of Hermite-Hadamard type inequalities for $\varphi$-convex functions. Theses results in some special cases recapture the known results. We hope that our results may be helpful for further study.

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