Unit Ball Graphs on Geodesic Spaces

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Consider finitely many points in a geodesic space. If the distance of two points is less than a fixed threshold, then we regard these two points as “near”. Connecting near points with edges, we obtain a simple graph on the points, which is called a unit ball graph. If the space is the real line, then it is known as a unit interval graph. Unit ball graphs on a geodesic space describe geometric characteristics of the space in terms of graphs. In this article, we show that chordality and (claw, net)-freeness, which are combinatorial conditions, force the spaces to be $\mathbb{R}$-trees and connected 1-dimensional manifolds respectively, and vice versa. As a corollary, we prove that the collection of unit ball graphs essentially characterizes the real line and the unit circle.

Keywords: geodesic space, $\mathbb{R}$-tree, real tree, 0-hyperbolic space, unit ball graph, unit interval graph, chordal graph, strongly chordal graph, (claw, net)-free graph, Hamiltonian hereditary graph

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1 Introduction

Let $(X, d)$ be a metric space. Consider finitely many points $x_1, \ldots, x_n$ in $X$ and fix a threshold $\delta > 0$. If $d(x_i, x_j) \leq \delta$, we regard $x_i$ and $x_j$ as “near”. We can construct a simple graph on the set $\{x_1, \ldots, x_n\}$ with edges between near points. It might be expected that we could obtain some information about $X$ from graphs constructed in such a way. However, it seems difficult to study metric spaces with unit ball graphs without any other assumptions. For example, let $(X, d)$ be a metric space defined by

$$X := \left\{ (\cos \theta, \sin \theta) \in \mathbb{R}^2 \middle| \frac{\pi}{6} \leq \theta \leq \frac{11\pi}{6} \right\}$$

and $d$ is the restriction of the Euclidean metric in $\mathbb{R}^2$. Take points $x = (\cos \frac{\pi}{6}, \sin \frac{\pi}{6})$, $y = (\cos \frac{11\pi}{6}, \sin \frac{11\pi}{6})$, and $z = (-1, 0)$ and suppose that $\delta = 1$ (see Figure 1). Then $x$ is “near” to $y$ but not to $z$, which seems counterintuitive. It is natural to regard $x$ and $y$ as the “furthest” points in $X$ and $z$ is the midpoint between $x$ and $y$. If we define a metric on $X$ by arc length, then it fits our intuition. Thus, from now on, our interest focus on geodesic spaces defined as follows.
Definition 1.1. Let \((X, d)\) be a metric space and \(x, y \in X\). A geodesic from \(x\) to \(y\) is a distance-preserving map \(\gamma\) from a closed interval \([0, d(x, y)] \subseteq \mathbb{R}\) to \(X\) with \(\gamma(0) = x\) and \(\gamma(d(x, y)) = y\). Its image is said to be a geodesic segment with endpoints \(x\) and \(y\). We say that \((X, d)\) is a geodesic space if every two points are joined by a geodesic. Note that a geodesic segment between two points is not necessarily unique. We will write a geodesic segment whose endpoints are \(x\) and \(y\) as \([x, y]\). If there exists a unique geodesic segment for every pair of points, then we say that \((X, d)\) is uniquely geodesic.

Example 1.2. The \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) and its convex subsets are geodesic spaces. The \(n\)-dimensional sphere \(S^n\) with the great-circle metric is a geodesic space. The vector space \(\mathbb{R}^n\) with \(L_p\)-norm \(|*|_p\) \((1 \leq p \leq \infty)\) is also a geodesic space.

Next we formulate the graphs in which we are interested.

Definition 1.3. Let \((X, d)\) be a (geodesic) metric space. A simple graph \(G = (V_G, E_G)\) is said to be a unit ball graph on \((X, d)\) if there exist a threshold \(\delta > 0\) and a map \(\rho\), called a realization, from the vertex set \(V_G\) to \(X\) such that \(\{u, v\} \in E_G\) if and only if \(d(\rho(u), \rho(v)) \leq \delta\). Let \(\text{UBG}(X, d)\) denote the collection of the unit ball graphs on \((X, d)\). When there is no confusion with the metric, we may write it as \(\text{UBG}(X)\).

Remark 1.4. In this article, the term “graph” refers an undirected simple graph on finite vertices. We frequently identify the vertices of a unit ball graph with the realized points in the space.

Remark 1.5. A unit ball graph is the intersection graph of finitely many closed balls of the same size in a geodesic space. If we scale the metric, then the graph can be the intersection graph of unit balls, that is, balls of radius 1. When we consider the Euclidean spaces, we may always assume that a unit ball graph is the intersection graph of finitely many unit balls.

Let \(H\) be a graph. A graph is said to be \(H\)-free if it has no subgraph isomorphic to \(H\). A graph is called chordal if it is \(C_n\)-free for all \(n \geq 4\), where \(C_n\) denotes the cycle graph on \(n\) vertices. Note that a graph is chordal if and only if every cycle in it of length four or more has a chord, which is an edge connecting non-consecutive vertices of the cycle. For an integer \(n \geq 3\), the (complete) \(n\)-sun (or a trampoline) is a graph on \(2n\) vertices \(\{v_i \mid i \in \mathbb{Z}/2n\mathbb{Z}\}\) such that the even-indexed vertices induce a complete graph, the odd-indexed vertices form an independent set, and an odd-indexed vertex \(v_i\) is adjacent to an even-indexed vertex \(v_j\) if and only if \(i - j = \pm 1\) (See Figure 2 for example). A graph
A graph is called \textbf{sun-free} if it is \emph{n}-sun-free for all \( n \geq 3 \). A graph is called \textbf{strongly chordal} if it is chordal and sun-free. Farber \cite{9} investigated strongly chordal graphs and gave some characterizations. The definition above is one of such characterizations.

A graph in \( \text{UBG}(\mathbb{R}) \) is called a \textbf{unit interval graph} also known as an \textbf{indifference graph}, which is a very important object in combinatorics. There are several linear-time algorithms for recognizing unit interval graphs. Also, they are characterized by forbidden induced subgraphs as explained below.

\textbf{Theorem 1.6} (Wegner \cite{14}, Roberts \cite{12}). \emph{A graph is unit interval if and only if it is chordal and \((\text{claw, net, 3-sun})\)-free} (see Figure 3).

A graph in \( \text{UBG}(S^1) \) is called a \textbf{unit circular-arc graph}, which is one of a natural generalization of unit interval graphs and this class is also well investigated (see \cite{10}, for example). Tucker \cite{13, Theorem 3.1 and Theorem 4.3} characterized unit circular-arc graphs in terms of forbidden induced subgraphs.

A graph in \( \text{UBG}(\mathbb{R}^2) \) is called a \textbf{unit disk graph}. Breu and Kirkpatrick \cite{4} showed that the recognition of unit disk graphs is NP-hard. Therefore it seems very difficult to characterize unit disk graphs in terms of forbidden induced subgraphs. Recently, Atminas and Zamaraev \cite{2} discovered infinitely many minimal non-unit disk graphs.

A graph in \( \text{UBG}(\mathbb{R}^2, || \cdot ||_{\infty}) = \text{UBG}(\mathbb{R}^2, || \cdot ||_1) \) is called a \textbf{unit square graph}. Breu \cite[Corollary 3.46.2]{3} proved that the recognition of unit square graphs is also NP-hard. Neuen \cite{11} proved that the graph isomorphism problem for unit square graphs can be solved in polynomial time and investigated a lot of properties of unit square graphs. For instance, Neuen showed that every unit square graph is \((K_{1,5}, K_{2,3}, \overline{3K_2})\)-free (see \cite{11}).
Notice that the collection of unit ball graphs can distinguish these four geodesic spaces (see Table 4). However, we can find easily non-isometric geodesic spaces whose unit ball graphs coincide. For example, UBG([0,1]) = UBG(R). More generally, when X is a convex subset of R^n with non-empty interior, we have that UBG(X) = UBG(R^n). In Section 2, we will give a sufficient condition for the coincidence of the collections of unit ball graphs (Corollary 2.5).

The main contribution of this article is to characterize geodesic spaces whose unit ball graphs are chordal and geodesic spaces whose unit ball graphs are (claw, net)-free. In order to state the results, we define R-trees and tripods. There are several equivalent definitions for R-trees. Here we give one of them (see Section 3, for other conditions and details).

**Definition 1.7.** A geodesic space is said to be an R-tree (or a real tree) if it is uniquely arc-connected, that is, every pair of points in it is joined by a unique arc.

**Definition 1.8.** A subset Y of a geodesic space X is said to be a tripod (see Figure 5) if there exist four distinct points x_1, x_2, x_3, y ∈ Y and geodesic segments [x_i, y] (i = 1, 2, 3) such that

\[ Y = [x_1, y] \cup [x_2, y] \cup [x_3, y] \text{ and } [x_i, y] \cap [x_j, y] = \{y\} \text{ if } i \neq j. \]

The main results are as follows.

**Theorem 1.9.** Let (X, d) be a geodesic space. Then the following are equivalent:

1. Every unit ball graph on X is strongly chordal.
2. Every unit ball graph on X is chordal.
3. X is an R-tree.
Theorem 1.10. Let \((X, d)\) be a geodesic space. Then the following are equivalent:

1. Every unit ball graph on \(X\) is (claw, net)-free.
2. \(X\) has no tripod.
3. \(X\) is homeomorphic to a manifold of dimension at most 1, that is, \(X\) is similar to \(S^1\) or isometric to an interval, that is, a convex subset of \(\mathbb{R}\).

Remark 1.11. According to [8], every (claw, net)-free graph has a Hamiltonian path. Hence we can deduce that a graph is (claw, net)-free if and only if it is a Hamiltonian-hereditary graph, that is, every induced connected component of it has a Hamiltonian path as stated in [7]. Clearly, unit ball graphs on intervals and \(S^1\) are Hamiltonian-hereditary graphs. Theorem 1.10 asserts that spaces whose unit ball graphs are Hamiltonian-hereditary graphs are exactly intervals and \(S^1\).

Theorem 1.9, 1.10, and Table 4 lead to the following corollaries.

Corollary 1.12. Let \((X, d)\) be a geodesic space. Then the following are equivalent:

1. \(\text{UBG}(X) = \text{UBG}(\mathbb{R})\).
2. \(X\) is isometric to an interval which is not the single-point space.

Corollary 1.13. Let \((X, d)\) be a geodesic space. Then the following are equivalent:

1. \(\text{UBG}(X) = \text{UBG}(S^1)\).
2. \(X\) is similar to \(S^1\).

The organization of this article is as follows. In Section 2, we study basic properties of unit ball graphs. In Section 3, we review the theory of \(\mathbb{R}\)-trees and prove Theorem 1.9. In Section 4, we will prove Theorem 1.10.

2 Basic properties

First of all, we begin with the following proposition, which is easy to prove.

Proposition 2.1. The class of unit ball graphs on a metric space \((X, d)\) is hereditary. Namely, if \(G \in \text{UBG}(X)\), then every induced subgraph of \(G\) belongs to \(\text{UBG}(X)\).

Therefore every class \(\text{UBG}(X)\) has a characterization in terms of forbidden induced subgraphs. However, as mentioned in the introduction, it is difficult to characterize \(\text{UBG}(X)\) in general. Next, we treat the most trivial case, that is, the single-point space \(\{\ast\}\).

Proposition 2.2. The following statements hold true:

1. \(\text{UBG}(\{\ast\})\) consists of complete graphs, or equivalently \(2K_1\)-free graphs.
(2) Let \((X, d)\) be a metric space. Then \(\text{UBG}(X) = \text{UBG}(*\}) \) if and only if \(X = \{\ast\}\).

\textbf{Proof.} The assertion \((\ast)\) is trivial. To show \((\ast')\), take two distinct points \(x, y \in X\). Then the unit ball graph on \(\{x, y\}\) with threshold \(d(x, y)/3\) is \(2K_1\). Thus the assertion holds. \(\square\)

By definition, a unit ball graph is the intersection graph of finitely many \textit{closed} balls of the same size. Next we show that we may use \textit{open} balls instead of closed ones.

\textbf{Proposition 2.3.} A simple graph \(G = (V_G, E_G)\) is a unit ball graph on a metric space \((X, d)\) if and only if there exist \(\delta > 0\) and a map \(\rho : V_G \to X\) such that \(\{u, v\} \in E_G\) if and only if \(d(\rho(u), \rho(v)) < \delta\).

\textbf{Proof.} Assume that \(G\) is a unit ball graph on \(X\). Then, by definition, there exist \(\delta > 0\) and a map \(\rho : V_G \to X\) such that \(\{u, v\} \in E_G\) if and only if \(d(\rho(u), \rho(v)) \leq \delta\). Let \(\delta' = \min \{d(\rho(u), \rho(v))| \{u, v\} \notin E_G\}\). Then we have \(\{u, v\} \in E_G\) if and only if \(d(\rho(u), \rho(v)) < \delta'\). The converse can be proven in a similar way. \(\square\)

Next, we give a sufficient condition for inclusion of the classes of unit ball graphs.

\textbf{Proposition 2.4.} Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Suppose that every finite subset in \(X\) is similarly embedded into \(Y\). Namely, assume that, for any finite subset \(S\), there exist \(r > 0\) and a map \(f : S \to Y\) such that \(d_Y(f(a), f(b)) = rd_X(a, b)\) for any \(a, b \in S\). Then \(\text{UBG}(X) \subseteq \text{UBG}(Y)\).

\textbf{Proof.} Let \(G \in \text{UBG}(X)\) with a realization \(\rho\) and a threshold \(\delta\). Since \(\rho(V_G)\) is finite, by the assumption, there exist \(r > 0\) and \(f : \rho(V_G) \to Y\) such that \(d_Y(f(\rho(u)), f(\rho(v))) = rd_X(\rho(u), \rho(v))\) for any \(u, v \in V_G\). Hence we conclude that \(G \in \text{UBG}(Y)\) with a realization \(f \circ \rho\) and a threshold \(\delta/r\). \(\square\)

\textbf{Corollary 2.5.} Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Suppose that every finite subset in \(X\) is similarly embedded into \(Y\), and vice versa. Then \(\text{UBG}(X) = \text{UBG}(Y)\).

From Proposition 2.4 we have that \(\text{UBG}(\mathbb{R}^m) \subseteq \text{UBG}(\mathbb{R}^n)\) whenever \(m \leq n\). Therefore, intuitively, higher dimensional spaces could have more unit ball graphs. However, the converse is not true in general as follows.

\textbf{Proposition 2.6.} There exists a geodesic space \(X\) such that its Lebesgue covering dimension is \(1\) and \(\text{UBG}(X)\) consists of all graphs.

\textbf{Proof.} Let \(G\) be a connected graph and \(X_G\) the geodesic space obtained by replacing the edges of \(G\) with a copy of the unit interval \([0, 1]\) (so \(X_G\) is the underlying space of a \(1\)-dimensional simplicial complex). Clearly, \(G \in \text{UBG}(X_G)\). Choose a vertex of \(G\) and connect a geodesic segment of length \(1\) to the corresponding point of \(X_G\). Let \(X\) be a geodesic space obtained by gluing the other endpoints with respect to each connected graph and its countably many copies. Then every graph belongs to \(\text{UBG}(X)\). Obviously, the Lebesgue covering dimension of \(X\) is \(1\). \(\square\)

It is not clear whether the converse of Corollary 2.5 holds true or not. However, for geodesic spaces \(\mathbb{R}, S^1\), and \(\{\ast\}\), it is true by Corollary 1.12, 1.13, and Proposition 2.2 (2).
3 Chordal graphs and $\mathbb{R}$-trees

In this section, we will give a proof of Theorem 1.9. Note that the implication $(1) \Rightarrow (2)$ follows by definition. As mentioned before, an $\mathbb{R}$-tree is a geodesic space in which every pair of points is joined by a unique arc, that is, the image of a topological embedding of a closed interval. Note that every arc in an $\mathbb{R}$-tree is a (unique) geodesic segment and hence an $\mathbb{R}$-tree is uniquely geodesic. Obviously, the real line $\mathbb{R}$ and intervals are $\mathbb{R}$-trees. The underlying space of a 1-dimensional connected acyclic simplicial complex is also an $\mathbb{R}$-tree.

**Proposition 3.1** (See [6, Proposition 2.3] and [5], for example). Let $(X, d)$ be a geodesic space. Then the following conditions are equivalent.

1. $X$ is an $\mathbb{R}$-tree
2. $X$ has no subspace homeomorphic to $S^1$.
3. For any $x, y, z \in X$, whenever $[x, y] \cap [y, z] = \{y\}$, the union $[x, y] \cup [y, z]$ is a geodesic segment joining $x$ and $z$.
4. $X$ is a Gromov 0-hyperbolic space, that is, for any geodesic segments $[x, y], [x, z], [y, z]$, we have $[x, z] \subseteq [x, y] \cup [y, z]$.

3.1 Proof of Theorem 1.9 $(2) \Rightarrow (3)$

The following two propositions are required. The first one is very famous.

**Proposition 3.2.** Suppose that $A$ and $B$ be disjoint compact subsets of a metric space $(X, d)$. Then

$$d(A, B) := \inf_{a \in A, b \in B} d(a, b) > 0.$$ 

**Proposition 3.3.** Let $S$ be a connected subset of a topological space and $\mathcal{U}$ a finite open covering of $S$ in which each member intersects with $S$. Then the intersection graph of $\mathcal{U}$ is connected.

**Proof.** Assume that the intersection graph of $\mathcal{U}$ is disconnected. Then there exist non-empty subsets $U_1$ and $U_2$ of $\mathcal{U}$ such that $\mathcal{U} = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, and $U_1 \cup U_2 = \emptyset$ for any $U_i \in \mathcal{U}(i = 1, 2)$. For each $i \in \{1, 2\}$, let $O_i := \bigcup_{U \in \mathcal{U}} U$. Then $S \subseteq O_1 \cup O_2$ and $S \cap O_i \neq \emptyset$ for each $i \in \{1, 2\}$. Therefore $S$ is disconnected. Thus the assertion holds.

**Proof of Theorem 1.9 $(2) \Rightarrow (3)$**. We assume that $X$ is not an $\mathbb{R}$-tree and show that there exists $G \in \text{UBG}(X)$ such that $G$ is non-chordal. By Proposition 3.1, there exists a topological embedding $\phi : S^1 \to X$. We consider $S^1$ as $\mathbb{R}/4\mathbb{Z}$ and put $p_i := \phi(i)$, and $S_i := \phi([i, i+1])$ for $i \in \{1, 2, 3, 4\} \subseteq \mathbb{R}/4\mathbb{Z}$ (see Figure 6). By Proposition
there exists $r > 0$ such that $d(S_1, S_3) > 2r$ and $d(S_2, S_4) > 2r$. Let $U_r(x)$ denote the open ball of radius $r$ with center $x$. Note that $U_r(x) \cap U_r(y) = \emptyset$ whenever $(x, y) \in S_1 \times S_3$ or $(x, y) \in S_2 \times S_4$. For every $i$, $\{ U_r(x) \}_{x \in S_i}$ is an open covering of $S_i$. Since $S_i$ is compact, we have a finite subset $V_i \subseteq S_i$ such that $\{ U_r(x) \}_{x \in V_i}$ is a finite open covering of $S_i$. Adding two points $p_i, p_{i+1}$ if necessary, we may suppose that $p_i, p_{i+1} \in V_i$. Let $G$ be the intersection graph of $\{ U_r(x) \}_{x \in V_1 \cup \cdots \cup V_4}$ and we will identify the vertices of $G$ with the corresponding points. We will show that $G$ is not chordal. By Proposition 3.3, each induced subgraph $G[V_i]$ is connected. Take a shortest path $\pi_i$ from $p_i$ to $p_{i+1}$ in $G[V_i]$. Since $p_i \in S_{i-1}$ and $p_{i+1} \in S_{i+1}$, they are not adjacent. Therefore the length of the path $\pi_i$ is at least two and every intermediate vertex of $\pi_i$ is an interior point of $S_i$. Connecting the paths $\pi_1, \ldots, \pi_4$, we have a cycle $C = (V_C, E_C)$ satisfying the following conditions.

(i) $V_C \cap V_i \neq \emptyset$ and $C[V_C \cap V_i]$ is a chordless path for each $i$.

(ii) $C$ has a vertex corresponding to an interior point of $S_i$ for each $i$.

Suppose that $C_0$ is a minimal cycle satisfying these conditions. By the condition (ii), the length of $C_0$ is at least four. Let $i \in \{1, 2, 3, 4\}$. From (i), there exists no chord between two vertices in $V_{C_0} \cap V_i$. By minimality of $C_0$, there exists no chord connecting $V_{C_0} \cap V_i$ and $V_{C_0} \cap V_{i+1}$. By choice of $r$, there exists no chord joining $V_{C_0} \cap V_i$ and $V_{C_0} \cap V_{i+2}$. Thus $C_0$ is a chordless cycle of length at least four and hence $G$ is a non-chordal unit ball graph on $X$.

\[\square\]

3.2 Proof of Theorem 1.9 (3) $\Rightarrow$ (1)

Proposition 3.4. Let $(X, d)$ be a geodesic space. Then the following conditions hold.

(1) Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Suppose that $\bigcap_{i=1}^n [x_i, y_i] \neq \emptyset$. Then $\sum_{i=1}^n d(x_i, y_i) \geq \sum_{i=1}^n d(x_i, y_{\sigma(i)})$ for any permutation $\sigma$. 

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(2) Let $G \in \text{UBG}(X)$. Suppose that $\{x_i, y_i\} \in E_G$ for each $i \in \{1, \ldots, n\}$ and there exists a permutation $\sigma$ such that $\{x_i, y_{\sigma(i)}\} \notin E_G$ for any $i$. Then $\bigcap_{i=1}^{n} [x_i, y_i] = \emptyset$.

Proof. Take a point $p \in \bigcap_{i=1}^{n} [x_i, y_i]$. Then we have

$$
\sum_{i=1}^{n} d(x_i, y_i) = \sum_{i=1}^{n} (d(x_i, p) + d(p, y_i)) = \sum_{i=1}^{n} d(x_i, p) + \sum_{i=1}^{n} d(p, y_i)
$$

$$
= \sum_{i=1}^{n} d(x_i, p) + \sum_{i=1}^{n} d(p, y_{\sigma(i)}) = \sum_{i=1}^{n} (d(x_i, p) + d(p, y_{\sigma(i)})) \geq \sum_{i=1}^{n} d(x_i, y_{\sigma(i)}),
$$

where we apply the triangle inequality.

By the assumption, there exists a threshold $\delta > 0$ such that $d(x_i, y_i) \leq \delta$ and $d(x_i, y_{\sigma(i)}) > \delta$ for any $i \in \{1, \ldots, n\}$. Assume that $\bigcap_{i=1}^{n} [x_i, y_i] \neq \emptyset$. Then by the assertion (1) we have

$$
n\delta \geq \sum_{i=1}^{n} d(x_i, y_i) \geq \sum_{i=1}^{n} d(x_i, y_{\sigma(i)}) > n\delta.
$$

This contradiction proves the assertion. \hfill \Box

**Proposition 3.5** (Alperin-Bass Proposition 2.17, Chiswell Lemma 1.9(b)). Let $A$ and $B$ be non-empty closed connected subsets of an $\mathbb{R}$-tree such that $A \cap B = \emptyset$. Then there exist unique points $a \in A$ and $b \in B$ such that $[a, b] \cap A = \{a\}$ and $[a, b] \cap B = \{b\}$.

**Definition 3.6.** We call $[a, b]$ in Proposition 3.5 the bridge between $A$ and $B$. Let $Y$ be a non-empty closed connected subset of an $\mathbb{R}$-tree and $x$ a point. Call the other endpoint $y$ of the bridge $[x, y]$ between $\{x\}$ and $Y$ the closest point in $Y$ to $x$ when $x \notin Y$. If $x \in Y$, define the closest point as $x$ itself.

**Lemma 3.7.** Suppose that the $n$-cycle $C_n$ is a unit ball graph of a geodesic space $X$. Let $\{x_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ be the vertex set of $C_n$ with $\{x_i, x_{i+1}\} \in E_{C_n}$ ($i \in \mathbb{Z}/n\mathbb{Z}$). Suppose that $\{x_i, x_{i+1}\}, \{x_j, x_{j+1}\}$ are non-adjacent edges. Then $[x_i, x_{i+1}] \cap [x_j, x_{j+1}] = \emptyset$.

Proof. Since $\{x_i, x_{i+1}\}, \{x_j, x_{j+1}\} \in E_{C_n}$ and $\{x_i, x_j\}, \{x_{i+1}, x_{j+1}\} \notin E_{C_n}$, we have immediately $[x_i, x_{i+1}] \cap [x_j, x_{j+1}] = \emptyset$ by Proposition 3.3 (2). \hfill \Box

**Lemma 3.8.** Suppose that a geodesic space $X$ admits an $n$-sun as a unit ball graph. Let $\{x_i\}_{i \in \mathbb{Z}/2n\mathbb{Z}}$ be the vertex set such that even-indexed vertices induce a clique, odd-indexed vertices are independent, and two consecutive vertices form an edge. Then the following assertions hold.

(1) Suppose that $\{x_i, x_{i+1}\}$ and $\{x_j, x_{j+1}\}$ are non-adjacent edges with $j \neq i \pm 2$. Then $[x_i, x_{i+1}] \cap [x_j, x_{j+1}] = \emptyset$.

(2) When $\{x_i, x_{i+1}\}, \{x_j, x_{j+1}\}$, and $\{x_k, x_{k+1}\}$ are non-adjacent edges, we have that $[x_i, x_{i+1}] \cap [x_j, x_{j+1}] \cap [x_k, x_{k+1}] = \emptyset$. 9
Therefore we conclude that every unit ball graph on the n-sun is chordal and sun-free. Assume that there exists n ≥ 4 such that \( C_n \in \text{UBG}(X) \). Let \( \{x_i\}_{i \in \mathbb{Z}/n\mathbb{Z}} \) be the vertices of \( C_n \) with \( \{x_i, x_{i+1}\} \in E_{C_n} \). Then \( A := \{x_1, x_2\} \) and \( B := \bigcup_{i=3}^{n-1} \{x_i, x_{i+1}\} \) are disjoint closed connected subsets by Lemma 3.7. By Proposition 3.5 both of the segments \([x_1, x_n]\) and \([x_2, x_3]\) contain the bridge between \( A \) and \( B \). In particular, we have \([x_1, x_n] \cap [x_2, x_3] \neq \emptyset\), which contradicts to Lemma 3.7. Thus we conclude that every unit ball graph on \( X \) is chordal.

Next, suppose that an n-sun is a unit ball graph on \( X \). Let \( \{x_i\}_{i \in \mathbb{Z}/2n\mathbb{Z}} \) be the vertex set of the n-sun with the conditions mentioned in Lemma 3.8. Let \( A := \{x_1, x_2\} \) and \( B := \bigcup_{i=3}^{n-2} \{x_i, x_{i+1}\} \). By Lemma 3.8 (1), we have \( A \) and \( B \) are disjoint closed connected subsets of \( X \). Let \([a, b]\) be the bridge between \( A \) and \( B \) and set \( Y := A \cup [a, b] \cup B \). Let \( y_3 \) and \( y_{2n} \) be the closest points in \( Y \) to \( x_3 \) and \( x_{2n} \).

We will show that \( y_3, y_{2n} \in A \cup B \). Assume that \( y_3 \notin A \cup B \). Then we have \( y_3 \in [a, b] \subseteq [x_1, x_{2n-1}] \subseteq [x_{2n}, x_1] \cup [x_{2n-1}, x_{2n}] \) by Proposition 3.3 and 3.1. Furthermore, by Proposition 3.5 again, we have \( y_3 \in [x_3, x_4] \cap [x_2, x_3] \). Therefore \( y_3 \in ([x_{2n}, x_1] \cap [x_3, x_4]) \cup ([x_{2n-1}, x_{2n}] \cap [x_2, x_3]) \). However, by Lemma 3.8 (1), we have \([x_{2n}, x_1] \cap [x_3, x_4] = [x_{2n-1}, x_{2n}] \cap [x_2, x_3] = \emptyset\), a contradiction. Hence \( y_3 \in A \cup B \). We can show that \( y_{2n} \in A \cup B \) in a similar way.

Assume that \( y_3 \in A \) and \( y_{2n} \in B \). Then \([x_3, x_4] \cap [x_{2n}, x_1] \supseteq [a, b]\), which contradicts to Lemma 3.8 (1). The condition \( y_3 \in B \) and \( y_{2n} \in A \) also leads to a contradiction. If \( y_3, y_{2n} \in A \), then \([x_3, x_4] \cap [x_{2n}, x_{2n-1}] \supseteq [a, b]\). Therefore \( A \cap [x_3, x_4] \cap [x_{2n}, x_{2n-1}] \supseteq a \). This is a contradiction to Lemma 3.8 (2). Finally assume that \( y_3, y_{2n} \in B \). Then we have \( B \cap [x_3, x_2] \cap [x_{2n}, x_1] \ni b \), which is again a contradiction to Lemma 3.8 (2).

Therefore we conclude that every unit ball graph on \( X \) is sun-free. Thus the proof has been completed.

4 (claw.net)-free graphs and 1-dimensional manifolds

In this section, we will prove Theorem 1.10. As mentioned Remark 1.11, it is well known that the implication (3) ⇒ (1) holds true.
Lemma 4.1. Let $G$ be a graph on vertex set $\{a_i\}_{i=0}^l \cup \{b_i\}_{i=0}^m \cup \{c_i\}_{i=0}^n$ with positive integers $l, m, n$ satisfying the following conditions.

(i) $\{a_i\}_{i=0}^l \cup \{b_i\}_{i=0}^m \cup \{c_i\}_{i=0}^n$ induce chordless paths.

(ii) $a_0, b_0, a_l$ are leaves.

(iii) $\{a_l, b_m, c_n\}$ induces a triangle.

Then $G$ has an induced subgraph isomorphic to a claw or a net.

Proof. We proceed by induction of the number of the vertices of $G$. The initial case where $l = m = n = 1$ is trivial since $G$ itself is a net. Hence we may assume that $l \geq 2$ by symmetry.

If the neighborhood of $a_1$ coincides with $\{a_0, a_2\}$, then $G \setminus \{a_0\}$ satisfies the assumptions. Therefore, by the induction hypothesis, $G \setminus \{a_0\}$ and hence $G$ has a claw or a net. Without loss of generality, we may assume that there exists a minimal integer $i$ such that $\{a_i, b_i\}$ is an edge of $G$. By the assumption (ii) we have $1 \leq i \leq m$.

Assume that $i < m$. If $\{a_1, b_{i+1}\}$ is not an edge of $G$, then the four vertices $a_1, b_{i-1}, b_i, b_{i+1}$ form a claw by the minimality of $i$ and the assumption (ii). Now suppose that $\{a_1, b_{i+1}\}$ is an edge. Note that $\{a_1, b_i, b_{i+1}\}$ induces a triangle. Take a shortest path from $b_{i+1}$ to $c_0$ in the induced subgraph on $\{b_{i+1}, \ldots, b_m, c_n, \ldots, c_0\}$. This path together with two paths on $\{a_0, a_1\}$ and $\{b_0, \ldots, b_i\}$ induce a subgraph of $G$ satisfying the assumptions. Making use of the induction hypothesis, we have that $G$ has a claw or a net. Hence we may assume that $i = m$.

If $\{a_1, c_n\}$ is an edge of $G$, then the graph $G \setminus \{a_2, \ldots, a_l\}$ satisfies the assumptions with the triangle $\{a_1, b_m, c_n\}$. Therefore we may assume that $\{a_1, c_n\}$ is not an edge of $G$. If $\{b_{m-1}, c_n\}$ is an edge of $G$, then the paths on $\{a_0, a_1, b_m\}, \{b_0, \ldots, b_{m-1}\}, \{c_0, \ldots, c_n\}$ induce a subgraph satisfying the assumptions with the triangle $\{b_m, b_{m-1}, c_n\}$. Hence we may assume that $\{b_{m-1}, c_n\}$ is not an edge of $G$. Then the induced subgraph on $\{a_1, b_{m-1}, b_m, c_n\}$ is a claw. Thus the assertion holds true.

Proof of Theorem 1.10 (1) ⇒ (2). Assume that there exist four points $x_1, x_2, x_3, y \in X$ forming a tripod with center $y$. By Proposition 3.2 there exists $r > 0$ such that

$$d(\{x_i\}, [x_{i+1}, y] \cup [x_{i+2}, y]) > 2r$$

for every $i \in \mathbb{Z}/3\mathbb{Z}$.

Let $\gamma$ be a geodesic from $x_1$ to $y$ and $l$ the greatest integer less than or equal to $d(x_1, y)$. Define a sequence $\{a_i\}_{i=0}^l$ by $a_i := \gamma(ir)$. Note that $U_r(a_i) \cap U_r(a_j) \neq \emptyset$ if and only if $|i - j| \leq 1$. Define sequences $\{b_i\}_{i=0}^m$ and $\{c_i\}_{i=0}^n$ with respect to $x_2$ and $x_3$ in a similar way. By the choice of $r$, we have $U_r(a_0) \cap U_r(b_i) = \emptyset$, $U_r(a_0) \cap U_r(c_i) = \emptyset$, and so on. Moreover we have $U_r(a_l) \cap U_r(b_m) \cap U_r(c_n) \ni y$. Therefore the intersection graph of open balls of radius $r$ with center points in $\{a_i\}_{i=0}^l \cup \{b_i\}_{i=0}^m \cup \{c_i\}_{i=0}^n$ satisfies the assumptions of Lemma 4.1 and hence it has a claw or a net as an induced subgraph.
4.2 Proof of Theorem 1.10 (2) ⇒ (1)

Lemma 4.2. Let $(X, d)$ be a geodesic space with no tripods. Then the following conditions hold.

(1) Suppose that $x, y, z \in X$ satisfy $[x, z] \cap [y, z] \supseteq \{z\}$. Then $x \in [y, z]$ or $y \in [x, z]$. In particular, if $X$ is uniquely geodesic, then $[x, z] \subseteq [y, z]$ or $[y, z] \subseteq [x, z]$.

(2) Let $x, y, z$ be distinct points with $y \in [x, z]$. Then a geodesic segment between $x$ and $y$ is unique and hence it is a subsegment of $[x, z]$.

Proof. (1) Assume that $x \notin [y, z]$ and $y \notin [x, z]$. We will prove that $X$ has a tripod. Let $E := [x, z] \cap [y, z]$ and $\gamma$ the geodesic corresponding to $[x, z]$. Since $E$ is compact, there exists a point $q \in E$ such that $d(x, q) = \min_{p \in E} d(x, p)$. Note that $q \neq x, y, z$ since $x, y \notin E$ and $E \supseteq \{z\}$. Take a geodesic segments $[x, q]$ from $[x, z]$ and two segments $[y, q], [z, q]$ from $[y, z]$ (Note that a geodesic segment joining two points is not necessarily unique). Then we may conclude that $[x, q], [y, q]$ and $[z, q]$ form a tripod by the choice of $q$.

(2) Let $[x, y], [y, z]$ denote the geodesic segments in $[x, z]$ and assume that there exists another geodesic segment $[x, y']$ between $x$ and $y$. If $[x, y] \subseteq [x, y']$, then $[x, y] = [x, y']$, which is a contradiction. Hence there exists $p \in [x, y] \setminus [x, y']$. Let $[p, z]$ be the geodesic segment in $[x, z]$ and $[x, z]$ the geodesic segment obtained by connecting $[x, y']$ and $[y, z]$. Then $[p, z] \cap [x, z] \supseteq [y, z] \supseteq \{z\}$. Hence, by (1), we have $p \in [x, z]'$ or $x \in [p, z]$. Each case yields a contradiction. □

Proof of Theorem 1.10 (2) ⇒ (3). First we assume that $X$ is an $\mathbb{R}$-tree and construct a distance-preserving map from $X$ to $\mathbb{R}$. We may assume that $X$ is not the single-point space. Take two distinct points $q_+, q_- \in X$ and let $q_0$ be the midpoint between $q_+$ and $q_-.$

Next we show that $[x, q_0] \supseteq [q_-, q_0], x \in [q_-, q_+], \text{ or } [q_0, q_+] \subseteq [q_0, x]$ for each $x \in X$. In order to prove this statement, consider geodesic segments $[q_-, q_+]$ and $[q_-, x]$. If $[q_-, q_+] \cap [q_-, x] = \{q_-\}$, then $[x, q_+] = [x, q_-] \cup [q_-, q_+] \supseteq [q_-, q_+]$ by Proposition 3.1 and hence $[x, q_0] \supseteq [q_-, q_0]$. When $[q_-, q_+] \cap [q_-, x] \supseteq \{q_+\}$, we have $x \in [q_-, q_+]$ or $[q_-, q_+] \subseteq [q_-, x]$ by Lemma 1.2. The latter condition implies $[q_0, q_+] \subseteq [q_0, x]$.

Define subspaces $X_+$ and $X_-$ by

$$
X_+ := \{x \in X \mid d(x, q_-) > d(x, q_+)\},
$$

$$
X_- := \{x \in X \mid d(x, q_-) < d(x, q_+)\}.
$$

Note that for every $x \in X \setminus \{q_0\}$ we have that $x \in X_+$ if and only if $x \in [q_0, q_+]$ or $[q_0, q_+] \subseteq [q_0, x]$, and also we have that $x \in X_-$ if and only if $x \in [q_-, q_0]$ or $[q_-, q_0] \supseteq [q_-, q_0]$.

Suppose that $x \in X \setminus (X_+ \cup X_-)$, that is, $d(x, q_-) = d(x, q_+).$ Then $x \in [q_-, q_+]$ and hence $x = q_0$. Thus we obtain the decomposition $X = X_- \cup \{q_0\} \cup X_+$. Define a map
\( \phi : X \rightarrow \mathbb{R} \) by

\[
\phi(x) := \begin{cases} 
0 & \text{if } x = q_0, \\
d(q_0, x) & \text{if } x \in X_+, \\
-d(q_0, x) & \text{if } x \in X_-.
\end{cases}
\]

Now we will show that \( \phi \) preserves the distance. We will treat only the case where \( x, y \in X_+ \) since the other cases is clear or similar and show that one of \([q_0, x]\) and \([q_0, y]\) contains the other. If \( x \in [q_0, q_+] \) or \( y \in [q_0, q_+] \), then it is clear. Thus we may assume that \([q_0, q_+] \subseteq [q_0, x] \) and \([q_0, q_+] \subseteq [q_0, y] \). By Lemma 4.2 (1) we have \([q_0, y] \subseteq [q_0, x] \) or \([q_0, y] \supseteq [q_0, x] \). Therefore we may conclude that \( d(x, y) = |d(q_0, x) - d(q_0, y)| = |\phi(x) - \phi(y)| \). Hence \( X \) is isometric to an interval.

Second we assume that \( X \) is not an \( \mathbb{R} \)-tree and show that \( X \) is similar to \( S^1 \). By Proposition 3.1 there exist distinct three points \( x, y, z \in X \) and segments \( [x, y], [x, z], [y, z] \) such that \( [x, y] \cap [y, z] = \{y\} \) and \( [x, z] \neq [x, y] \cup [y, z] \). We show that \( [x, z] \cap [y, z] = \{z\} \) and \( [x, y] \cap [x, z] = \{x\} \). To prove the former, assume \( [x, z] \cap [y, z] \supseteq \{z\} \). Lemma 1.2 (1) asserts that \( x \in [y, z] \) or \( y \in [x, z] \). If \( x \in [y, z] \), then \( x \in [x, y] \cap [y, z] = \{y\} \). Hence \( x = y \), which is a contradiction. When \( y \in [x, z] \), apply Lemma 1.2 (2), we obtain \( [x, y] \) and \( [y, z] \) are subsegments of \([x, z]\) and hence \( [x, z] = [x, y] \cup [y, z] \), which is again a contradiction. The latter case can be proved by symmetry.

If \( X = [x, y] \cup [x, z] \cup [y, z] \), then it is easy to prove that \( X \) is similar to \( S^1 \). Assume that there exists a point \( p \in X \setminus [x, y] \cup [x, z] \cup [y, z] \). Take a geodesic segment \([p, z]\) and consider it together with the geodesic segments \([x, z], [y, z] \). Since \( X \) has no tripod, we have \([x, z] \cap [p, z] \supseteq \{z\} \) or \([y, z] \cap [p, z] \supseteq \{z\} \). Without loss of generality we may assume that the latter condition holds. Since \( p \notin [y, z] \), we have \( y \in [p, z] \) by Lemma 1.2 (1). The segments \([y, z]\) is a subsegment of \([p, z]\) by Lemma 1.2 (2). Take the subsegment \([p, y] \subseteq [p, z] \). Note that \([p, y] \cap [y, z] = \{y\} \). We may deduce that \( x \in [p, y] \) in a similar way. Moreover, take the subsegment \([p, x] \subseteq [p, y] \) and we may show that \( z \in [p, x] \), which is a contradiction. Therefore we can conclude that \( X = [x, y] \cup [x, z] \cup [y, z] \) and it is similar to \( S^1 \).

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