NUMERICAL SOLUTION OF AN OBSTACLE PROBLEM WITH INTERVAL COEFFICIENTS

SONG WANG
Department of Mathematics & Statistics
Curtin University, GPO Box U1987, Perth WA 6845, Australia

(Communicated by Kok Lay Teo)

Abstract. In this work we propose a novel numerical method for a finite-dimensional optimization problem arising from the discretization of an infinite-dimensional constrained optimization problem, called an obstacle problem, with interval coefficients. In this method, the two different ways of characterizing the optimal solutions, i.e., minimizing the mid-point and one end-point (the worst-case scenario) or the mid-point and the width of the objective interval, are formulated as a single constrained multi-objective minimization problem and the KKT conditions of the optimization problem defining the Pareto optimal solution to the multi-objective problem are of the form of a Linear Complementarity Problem (LCP) which is shown to have a unique solution. The LCP is approximated by a non-linear equation using an interior penalty approach. We prove that the penalty equation is uniquely solvable and its solution converges to that of LCP as the penalty constant approaches to zero. Numerical results are presented to demonstrate the usefulness of the numerical method proposed.

1. Introduction. Obstacle problems appear in many areas such as engineering, physics and financial engineering (see, for example, [2, 19, 11, 9, 7]). Such a problem is often called a free boundary problem and can be formulated as a variational inequality or complementarity problem. Mathematically, an infinite-dimensional obstacle problem can be formulated as the following problem.

\[ \inf_{u \in \mathcal{H}} \mathcal{F}(u), \quad (1) \]

subject to \( u \leq q, \quad (2) \)

where \( \mathcal{F} \) is a nonlinear differential operator, \( \mathcal{H} \) is a functional space, usually a Sobolev space containing all feasible solutions to this problem, and \( q \) is a known function so that \( \{ v \in \mathcal{H} : v \leq q \} \) is non-empty. A more rigorous description of the above problem is beyond the scope of this work and we refer the interested reader to [19, 11, 40] for more details. In this work, we assume that \( \mathcal{F} \) is a quadratic functional of the partial derivatives of \( u \). Often the given coefficient functions in \( \mathcal{F} \) involve uncertain parameters whose exact values are unknown.

The infinite-dimensional obstacle problem (1)–(2) can hardly be solved analytically and thus a numerical method such as a finite difference or finite volume
scheme is normally used to approximate (1)–(2) by a finite-dimensional obstacle or
constrained optimization problem (see, for example, [11, 12, 21, 42]). Numerical
solution of the resulting large-scale discreted and finite-dimensional optimization
problems has been discussed in the open literature such as [11, 12, 25, 26, 42, 27, 23].
While these existing methods provide some effective and efficient tools for solving
obstacle problems without uncertainties, how to find a solution to a large-scale
finite-dimensional obstacle problem effectively in the presence of parameter uncer-
tainties still remains a challenge.

An uncertain parameter can be characterized by a random variable or stochastic
process if its statistical properties are known. However, in practice, statistical prop-
erties for an uncertain parameter can hardly be available. Instead, the numerical
range of the parameter can easily determined. In this case, the uncertain parameter
is normally represented by an interval or fuzzy number. In this work we develop a
computational approach for solving the following discretized form of (1)–(2) when
the parameters/coefficients defining the problems are interval numbers:

\[
\min_{x \in \mathbb{R}^n} \quad F(x, p) = \frac{1}{2} x^\top A(p)x - b^\top (p)x, \\
\text{subject to} \quad x \leq c,
\]

where \(A\) is an \(n \times n\) positive-definite and symmetric matrix for a positive integer \(n\), \(b\) and \(c\) are known vectors in \(\mathbb{R}^n\), and \(p = [p_L, p_R]\) is a vector-valued interval number with \(p_L, p_R \in \mathbb{R}^m\) for a positive integer \(m\) satisfying \(p_L \leq p_R\) elementwisely. Clearly,
an optimal policy can be determined using existing methods such as those in [8, 34,
24, 39, 15, 16, 36] when \(p\) is a fixed constant. i.e., \(p_L = p_R\). However, in this paper,
we are concerned with the case that \(p\) is a vector-valued interval number. There
are some numerical methods for interval and fuzzy computing and optimization
problems in the open literature such as those in [17, 3, 14, 43, 1, 29, 31, 20]. However,
most of these methods have been developed for general interval/fuzzy optimization
problems and do not take into consideration of special properties the coefficient of
(3)–(4) possess such as that the system matrix \(A\) may be positive-definite, sparse
\(M\)-matrix. Also, they are not normally designed for large-scale problems. In what
following we shall establish a technique for determining an optimal solution to (3)–
(4) which minimizes the mid-point and the width or the mid- and right-points of
the objective function \(F\). The rest of this paper is organized as follows.

In the next section, we will introduce some existing operations and order relations
for interval numbers. In Section 3, we will establish the mathematical problem for
determining optimal solutions to (3)–(4). We will also derive the KKT conditions
for the Pareto optimal solutions to the problem, which are of the form of a Linear
Complementarity Problem (LCP). In Section 4 we will apply a penalty method
to the resulting LCP and show the unique solvability of the penalty equation and
the convergence of the approximate solution to the exact one. Section 5 contains
some numerical experimental results which demonstrate that our proposed approach
provides an effective and efficient computational tool for determining the optimal
solutions of the obstacle problem in the presence of system parameter uncertainties.

2. Preliminaries. Let \(a = [a_L, a_R]\) be an interval number, where \(a_L, a_R \in \mathbb{R}\)
satisfying \(a_L \leq a_R\). Introduce \(a_C = (a_L + a_R)/2\) and \(a_W = (a_R - a_L)/2\). Clearly,
\(a_C\) and \(a_W\) are respectively the center and width of the interval \(a\). Using the center
and width, we can alternatively represent \(a\) by \(a = \langle a_C, a_W \rangle\).
Let $I = \{[a_L, a_R] : a_L \leq a_R, a_L, a_R \in \mathbb{R}\}$ be the set of intervals. On $I$, we define the following operations.

**Definition 2.1.** Let $* \in \{+,-,\cdot,\div\}$ be a binary operation on $\mathbb{R}$. If $a = [a_L, a_R]$ and $b = [b_L, b_R]$ are two interval numbers in $I$, then we define
$$a * b := \{s * t : s \in a, t \in b\}$$
and when $*$ becomes $\div$, it is required that $0 \notin b$.

This definition of arithmetic of intervals has been discussed and used in many existing works such as [17, 3, 13, 14, 5, 30]. This definition of arithmetic operations can also be used for the representation $a = \langle a_C, a_W \rangle$. For example,
$$\langle a_C, a_W \rangle \pm \langle b_C, b_W \rangle = [a_L, a_R] \pm [b_L, b_R] = [a_L \pm b_L, a_R \pm b_R] = \langle a_C \pm b_C, a_W \pm b_W \rangle.$$
Similarly,
$$[a_L, a_R] \cdot [b_L, b_R] = [\min\{a_L b_L, a_R b_L, a_L b_R, a_R b_R\}, \max\{a_L b_L, a_R b_L, a_L b_R, a_R b_R\}] = (a_C, a_W) \cdot (b_C, b_W),$$
$$[a_L, a_R] / [b_L, b_R] = [\min\{a_L / b_L, a_R / b_L, a_L / b_R, a_R / b_R\}, \max\{a_L / b_L, a_R / b_L, a_L / b_R, a_R / b_R\}] = (a_C, a_W) / (b_C, b_W).$$

For a constant $\lambda$, we have
$$\lambda [a_L, a_R] = \begin{cases} [\lambda a_L, \lambda a_R] & \lambda \geq 0, \\ [\lambda a_R, \lambda a_L] & \lambda < 0. \end{cases}$$

We now introduce some order relationships on $I$ which are important for identifying optimal solutions in a minimization problem with interval coefficients.

**Definition 2.2.** Given two intervals $a = \langle a_C, a_W \rangle \in I$ and $b = \langle b_C, b_W \rangle \in I$, we define interval orders $\preceq_{CW}$ and $\succeq_{CW}$ for them as follows.
(a) $a \preceq_{CW} b$ iff $a_C \leq b_C$ and $a_W \leq b_W$,
(b) $a \succeq_{CW} b$ iff $a \preceq_{CW} b$ and $a \neq b$.

The orders defined above are partial orders on $I$. These orders represent the preferences in decision making. More specifically, in a minimization problem with interval coefficients, $a \preceq_{CW} b$ means $a$ is preferable than $b$, because $a$ has smaller mean and variance (width of uncertainty) than $b$.

A decision maker may prefer to minimize the mean and the upper bound of the uncertainty (or the worse-case scenario). For this case, we introduce orders $\preceq_{CR}$ and $\succeq_{CR}$ as follows.

**Definition 2.3.** Given two intervals $a = (a_L, a_R) \in I$ and $b = (b_L, b_R) \in I$, we define interval orders $\preceq_{CR}$ and $\succeq_{CR}$ for them as follows.
(c) $a \preceq_{CR} b$ iff $a_C \leq b_C$ and $a_R \leq b_R$, and
(d) $a \succeq_{CR} b$ iff $a \preceq_{CR} b$ and $a \neq b$.

We comment that in the case of solving maximization problems, the order relationships defined above need modification. For a more detailed discussion we refer to [17]. Other order definitions such as those in [13] have also been used for handling optimization problems with interval coefficients [5].
3. The multi-objective interval obstacle problem. We now consider the solution of (3)--(4) based on the preference relationships defined in Definitions 2.2 and 2.3. Without loss of generality, we assume that $c = 0$ in (4). The case that $c \neq 0$ can be transformed into this one by a simple translation transformation $y = x - c$.

It is clear that, under the preference measure $\preceq_{CW}$, to determine an optimal solution to (3)--(4) when $p \in I^m$, we are to find an $x^* \in \mathbb{R}^n$ such that $F(x^*, a) \preceq_{CW} F(x, a)$ for $x \in \mathbb{R}^n$, $x \leq 0$. Similarly, we can define the optimal solution to (3)--(4) using the order relationship $\preceq_{CR}$.

For any $x \leq 0$, we let
\begin{align}
    p_* &= \arg \min_{s \in p} F(x, s) = \arg \min_{s \in p} \left( \frac{1}{2} x^\top A(s)x - b^\top (s)x \right), \quad (5) \\
    p^* &= \arg \max_{s \in p} F(x, s) = \arg \max_{s \in p} \left( \frac{1}{2} x^\top A(s)x - b^\top (s)x \right). \quad (6)
\end{align}

Note that $p_*, p^* \in \mathbb{R}^m \times \mathbb{R}^n$, since the optimal choice of each component of $p$ may depend on $n$, as can be seen in the numerical experiments section. As we will show later using several examples, for many practical obstacle problems governed by a differential operator, $p_*$ and $p^*$ can be found exactly in the corresponding discrete obstacle problems, though methods of determining for more general quadratic problems are available [29, 28, 6].

From the above definitions we see that the lower and upper bound of the interval $F(x, p)$ are respectively $F_L(x, p) = F(x, p_*)$ and $F_R(x, p) = F(x, p^*)$ for any $x \leq 0$. Therefore, we define the following optimization problem so that its optimal solutions are ‘preferred’ solutions to the interval obstacle problem (3)--(4) (with $c = 0$).

\[
    \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} (F_L(x, p) + F_R(x, p)) , (1 - \theta)F_R(x, p) - \theta F_L(x, p) \right\}, \quad (7)
\]

subject to $x \leq 0$, \quad (8)

where $\theta = 0$ or $1/2$. When $\theta = 1/2$, a solution to (7) provides an optimal solution to (3)--(4) with respect to the order $\preceq_{CW}$, while when $\theta = 0$, it is an optimal solution to (3)--(4) with respect to $\preceq_{CR}$. Therefore, we have combined the two preference measures into (7).

Note that (7)--(8) is a multi-objective optimization problem. In what follows we shall find Pareto optimal solutions to this problem [35]. To achieve this, we consider the following optimization problem:

\[
    \min_{x \in \mathbb{R}^n} \frac{\lambda}{2} [F_L(x, p) + F_R(x, p)] + (1 - \lambda) [ (1 - \theta) F_R(x, p) - \theta F_L(x, p) ]
\]

\[
    = \left( \frac{\lambda}{2} - (1 - \lambda) \theta \right) F_L(x, p) + \left( \frac{\lambda}{2} + (1 - \lambda)(1 - \theta) \right) F_R(x, p),
\]

subject to $x \leq 0$.

where $\lambda \in [0, 1]$ is a parameter. The cost function in (9) is a weighted sum of the two objectives in (7). Using the definitions of $p_*$ and $p^*$, we rewritten the above optimization problem as

\[
    \min_{x \in \mathbb{R}^n} \left( \frac{\lambda}{2} - (1 - \lambda) \theta \right) F(x, p_*) + \left( \frac{\lambda}{2} + (1 - \lambda)(1 - \theta) \right) F(x, p^*), \quad (9)
\]

subject to $x \leq 0$, \quad (10)

where $F$ is defined in (3).
For \( x \in \mathbb{R}^n \), we introduce two matrices \( B \) and \( d \) given by

\[
B = \begin{bmatrix}
\frac{\lambda}{2} - (1 - \lambda)\theta \\
\frac{\lambda}{2} + (1 - \lambda)(1 - \theta)
\end{bmatrix} A(p_*) + \begin{bmatrix}
\frac{\lambda}{2} + (1 - \lambda)\theta \\
\frac{\lambda}{2} + (1 - \lambda)(1 - \theta)
\end{bmatrix} A(p^*),
\]

\[
d = \begin{bmatrix}
\frac{\lambda}{2} - (1 - \lambda)\theta \\
\frac{\lambda}{2} + (1 - \lambda)(1 - \theta)
\end{bmatrix} b(p_*) + \begin{bmatrix}
\frac{\lambda}{2} + (1 - \lambda)\theta \\
\frac{\lambda}{2} + (1 - \lambda)(1 - \theta)
\end{bmatrix} b(p^*).\]

Using \( B \) and \( d \) and the definition of \( F \), it is easy to see that

\[
\nabla_x \left[ \left( \frac{\lambda}{2} - (1 - \lambda)\theta \right) F(x, p_*) + \left( \frac{\lambda}{2} + (1 - \lambda)(1 - \theta) \right) F(x, p^*) \right] = Bx - d,
\]

and thus the KKT conditions for (9)–(10), after eliminating the multiplier, are given in the following Linear Complementarity Problem (LCP).

**Problem.** Find \( x \in \mathbb{R}^n \) satisfying

\[
Bx - d \leq 0, \quad (11)
\]

\[
x \leq 0, \quad (12)
\]

\[
x^\top (Bx - d) = 0 \quad (13)
\]

for \( \lambda \in [0, 1] \) and \( \theta \in \{0, 1/2\} \).

As mentioned before, \( A(p) \) in (3) is usually a positive-definite symmetric matrix if an appropriate numerical scheme such as one of those in [22, 42, 27, 23] is used for the discretization of (1). Thus, there exists a positive constant \( \alpha \) such that

\[
z^\top A(p)z \geq \alpha \|z\|^2_2, \quad \forall z \in \mathbb{R}^n,
\]

where \( p \) is the vector-valued interval number introduced before. The properties of the matrix \( B \) are given in the following theorem.

**Theorem 3.1.** The matrix \( B \) has the following properties.

- When \( \theta = 0 \), \( B \) is positive-definite, i.e.,

\[
z^\top Bz \geq \alpha \|z\|^2_2
\]

for \( z \in \mathbb{R}^n \) and \( \lambda \in [0, 1] \), where \( \alpha > 0 \) is the constant defined in (14).

- When \( \theta = 1/2 \), \( B \) is non-singular and satisfies

\[
z^\top Bz \geq \alpha \lambda \|z\|^2_2 + (b(p^*) - b(p_*))^\top z
\]

for \( z \in \mathbb{R}^n \) and \( \lambda \in (0, 1] \).

**Proof.** From the definition of \( B \), we have, for any \( z \in \mathbb{R}^n \),

\[
z^\top Bz = \begin{bmatrix}
\frac{\lambda}{2} - (1 - \lambda)\theta \\
\frac{\lambda}{2} + (1 - \lambda)(1 - \theta)
\end{bmatrix} z^\top A(p_*)z + \begin{bmatrix}
\frac{\lambda}{2} + (1 - \lambda)\theta \\
\frac{\lambda}{2} + (1 - \lambda)(1 - \theta)
\end{bmatrix} z^\top A(p^*)z
\]

\[
= \begin{cases}
\frac{\lambda}{2} \sum A(p_*)z + \left(1 - \frac{\lambda}{2}\right) z^\top A(p^*)z, & \theta = 0, \\
\frac{\lambda}{2} \sum A(p_*)z + \frac{1}{2} \left(z^\top A(p^*)z - z^\top A(p_*)z\right) & \theta = \frac{1}{2},
\end{cases}
\]

**Case 1.** \( \theta = 0 \). Since \( \frac{\lambda}{2} \geq 0 \) and \( 1 - \frac{\lambda}{2} > 0 \), from (14) and (17) we have

\[
z^\top Bz \geq \frac{\lambda}{2} \alpha \|z\|^2_2 + \left(1 - \frac{\lambda}{2}\right) \alpha \|z\|^2_2 = \alpha \|z\|^2_2.
\]

Thus, we have proved (15).
Case 2. \( \theta = 1/2 \). From (17) and (14) we have

\[
\begin{align*}
  z^\top Bz &= \lambda z^\top A(p_\ast)z + \left(\frac{1}{2} z^\top A(p^\ast)z - b(p^\ast)^\top z\right) - \left(\frac{1}{2} z^\top A(p_\ast)z - b(p_\ast)^\top z\right) \\
  &= \lambda z^\top A(p_\ast)z + (F(z, p^\ast) - F(z, p_\ast)) + (b(p^\ast) - b(p_\ast))^\top z \\
  &\geq \lambda \alpha \|z\|^2 + (b(p^\ast) - b(p_\ast))^\top z,
\end{align*}
\]

since \( F(z, p^\ast) - F(z, p_\ast) \geq 0 \) by (5) and (6). This is (16), and thus we have proved this theorem.

We now show that \( B \) is non-singular when \( \lambda \in (0, 1] \) and \( \theta = 1/2 \) by contradiction. Suppose \( B \) is singular. Then there exists a \( z_c \in \mathbb{R}^n \), \( z_c \neq 0 \) such that

\[
Bz_c = 0.
\]

Left-multiplying both sides of the above equation by \( z_c^\top \) and using (16) we have

\[
0 = z_c^\top Bz_c \geq \lambda \alpha \|z_c\|^2 + (b(p^\ast) - b(p_\ast))^\top z_c.
\]

Similarly, since \( B(-z_c) = 0 \) also holds, using the same argument as given above we have

\[
0 \geq \lambda \alpha \|z_c\|^2 + (b(p^\ast) - b(p_\ast))^\top (-z_c).
\]

Adding up the above two inequalities gives \( \lambda \alpha \|z_c\|^2 \leq 0 \) and thus \( z_c = 0 \), contradicting the assumption that \( z_c \neq 0 \). Therefore, \( B \) is non-singular when \( \theta = 1/2 \) and \( \lambda > 0 \).

Let \( K = \{z \in \mathbb{R}^n : z \leq 0\} \). The it is easy to see that \( K \) is a convex set. We consider the following variational inequality problem:

**Problem.** Find \( x \in K \), such that for all \( z \in K \),

\[
(z - x)^\top (Bx - d) \geq 0,
\]

(18)

**Theorem 3.2.** A vector \( x \) is a solution to Problem 3 if and only if it is a solution to Problem 3.

The results in the above theorem is well-known and its proof can be found in various existing works such as [8, 16]. The unique solvability of Problem 3 is given in the following theorem.

**Theorem 3.3.** Problem 3 has a unique solution when \( \theta = 0 \), \( \lambda \in [0, 1] \), or when \( \theta = 1/2 \), \( \lambda \in (0, 1] \).

**Proof.** Note that when \( \theta = 0 \), from (15) we see that \( B \) is symmetric and positive-definite, and Problem 3 has a unique solution by [8, Theorem 2.3.3].

When \( \theta = 1/2 \) and \( \lambda \in (0, 1] \), from (16) we have, for any \( z \in \mathbb{R}^n \),

\[
\frac{z^\top Bz}{\|z\|^2} \geq \frac{1}{\|z\|^2} \left[ \alpha \lambda \|z\|^2 + (b(p^\ast) - b(p_\ast))^\top z \right] \\
= \alpha \lambda \|z\|^2 + (b(p^\ast) - b(p_\ast))^\top \frac{z}{\|z\|^2} \\
\geq \alpha \lambda \|z\|^2 - \|b(p^\ast) - b(p_\ast)\|_2.
\]

From this inequality we have \( \lim_{\|z\|_2 \to \infty} \frac{z^\top Bz}{\|z\|^2} = \infty \) for \( \lambda \in (0, 1] \). Thus, the mapping \( B \) is coercive, and so Problem 3 has at least one solution by [8, Theorem 2.3.3].
We now show that the solution is unique when $\theta = 1/2$ and $\lambda \in (0, 1)$. Suppose both $x$ and $y$ are solutions to Problem 3. Then, $x$ and $y$ satisfy (18), i.e.,

$$(z - x)^\top (Bx - d) \geq 0, \quad \text{and} \quad (z - y)^\top (By - d) \geq 0$$

for any $z \in \mathcal{K}$. For any real number $\rho > 0$. Replacing $z$ in the above inequality with $y$ and $x$ respectively, we have

$$(y - x)^\top (Bx - d) \geq 0, \quad \text{and} \quad (x - y)^\top (By - d) \geq 0.$$ 

Adding up both sides of the above inequalities gives

$$(y - x)^\top (Bx - d) + (x - y)^\top (By - d) = (y - x)^\top B(x - y) \geq 0.$$ 

Thus for any positive real number $\rho > 0$, we have

$$(\rho y - \rho x)^\top B(\rho x - \rho y) \geq 0,$$

or

$$(\rho y - \rho x)^\top B(\rho y - \rho x) \leq 0.$$ 

Using (16), we have from the above inequality

$$(b(p^*) - b(p_*))^\top (\rho y - \rho x) + \alpha \lambda \rho^2 \|y - x\|^2 \leq 0,$$

from which we have

$$\alpha \lambda \rho^2 \|y - x\|^2 \leq -(b(p^*) - b(p_*))^\top (\rho y - \rho x) \leq \rho \|b(p^*) - b(p_*)\|_2 \|y - x\|_2.$$ 

Therefore, we have

$$\|y - x\|_2 \leq \frac{\|b(p^*) - b(p_*)\|_2}{\alpha \lambda \rho}.$$ 

Note that $\rho > 0$ is arbitrary. This implies that the above inequality holds only when $\|y - x\|_2 = 0$. Thus we have proved the theorem.

We comment that from Theorem 3.1 we see that when $\theta = 0$, (7)–(8) is to minimize the mean (mid-point) and the worse case-scenario (right end-point) of the interval/uncertain cost function. In this case, (9)–(10) is uniquely solvable for any $\lambda \in [0, 1]$. When $\theta = 1/2$, (7)–(8) is to minimize the mean and the width of cost function. In this case, (9)–(10) is uniquely solvable only when $\lambda \neq 0$. This is understandable as the problem of minimizing the width of the interval cost only may have more than one solution. As a matter of fact, this case has the trivial optimal solution $x = 0$.

We also comment that in [13], the authors propose a new order relation which is to compare the mid-points of two given intervals. If the mid-points are equal to each other, then compare their widths. It is easy to see that the interval optimization problem with respect to this order is equivalent to (9)–(10) with $\lambda = 1$ and $\theta = 1/2$, as minimizing the mean yields a unique solution.

4. **The interior penalty method for (11)–(13).** The problem (11)–(13) is a large-scale Linear Complementarity Problem (LCP). Existing methods for solving general LCPs can be found in [8]. In recent years, various penalty methods have been developed for solving large-scale LCPs arising from discretization of an obstacle problem in both classical and financial engineering [32, 40, 39, 44, 36, 37, 4]. In this work, we use the interior penalty method proposed in [41, 38] to solve (11)–(13). This method is based on interior point techniques for constrained optimization problems (see, for example, [10, 18]).
We introduce the element-by-element matrix (Hadamard) division operation ./, used in Matlab, defined as $y./z = (y_1/z_1, \ldots, y_n/z_n)$ for $y, z \in \mathbb{R}^n$ or $y./z = (y/z_1, \ldots, y/z_n)$ if $y$ is a scalar and $z \in \mathbb{R}^n$. It is proposed in [41, 38] to approximate (11)–(13) by the following nonlinear equation:

$$Bx_\mu - \mu ./ x_\mu = d,$$

where $\mu > 0$ is the penalty constant. More specifically, we use the negative solution $x_\mu$ to (19) to approximate the solution $x$ to Problem 3 and expect $x_\mu \to x$ as $\mu \to +0$. In the rest of this section we will prove that (19) has a unique solution satisfying $x_\mu < 0$ and the solution converges to that of (11)–(13) at the rate of $\mathcal{O}(\sqrt{\mu})$ as $\mu \to +0$. For brevity, we only present a proof to these results for $\theta = 1/2$, as the proof for $\theta = 0$ can be regarded as special cases of setting $(b(p^*) - b(p_*))^\top z = 0$ and $\lambda = \alpha$ in (16). We start this discussion with the following lemma.

**Lemma 4.1.** For any given $\mu > 0$, let $x_\mu$ be a solution to (19) satisfying $x_\mu < 0$. Then there exists a positive constant $M$, independent of $\mu$ and $z_\mu$, such that

$$||x_\mu||_2 \leq M,$$

when $\theta = 0$ or $\theta = 1/2$ and $\lambda > 0$.

**Proof.** We only prove this result for $\theta = 1/2$. Left-multiplying (19) by $x_\mu^\top$ gives

$$x_\mu^\top Bx_\mu - \mu = x_\mu^\top d.$$

Using (16) we have from the above equation,

$$\alpha \lambda ||x_\mu||_2^2 + (b(p^*) - b(p_*))^\top x_\mu \leq x_\mu^\top Bx_\mu = x_\mu^\top d + \mu,$$

from which we obtain

$$\alpha \lambda ||x_\mu||_2^2 \leq \mu + [d - (b(p^*) - b(p_*))]^\top x_\mu \leq \mu + C||x_\mu||_2 \leq \frac{C^2}{2\alpha \lambda} + \frac{\alpha \lambda ||x_\mu||_2^2}{2} + \mu,$$

where $C$ denotes a positive constant, independent of $\mu$ and $z_\mu$. In the above we used the Young’s inequality. It is easy to see that (20) follows from the above inequality.

We are ready to show that (19) has a unique solution.

**Theorem 4.2.** For any $\mu > 0$, (19) has a unique solution $x_\mu < 0$ when $\theta = 0$ or $\theta = 1/2$ and $\lambda > 0$.

**Proof.** Again, we only consider the case when $\theta = 1/2$. Let us first show that (19) has no more than one solution.

Suppose both $x < 0$ and $y < 0$ satisfy (19). Then we have

$$B(x - y) - \mu (1./x - 1./y) = 0.$$

Left-multiplying the above equality by $(x - y)^\top$ and using (16) we have

$$\alpha \lambda ||x - y||_2^2 + (b(p^*) - b(p_*))^\top (x - y) - \mu (x - y)^\top (1./x - 1./y) \leq 0. \quad (21)$$

Similarly, the above analysis is also true if we replace $x - y$ with $y - x$ and $1./x - 1./y$ with $1./y - 1./x$. Thus,

$$\alpha \lambda ||y - x||_2^2 + (b(p^*) - b(p_*))^\top (y - x) - \mu (y - x)^\top (1./y - 1./x) \leq 0. \quad (22)$$
Adding up both sides of (21) and (22) gives

\[2\alpha \lambda \|x - y\|^2_2 - 2\mu(x - y)\top(1./x - 1./y) = 2\alpha \lambda \|y - x\|^2_2 + 2\mu \sum_{i=1}^{n} (x_i - y_i)^2/x_iy_i \leq 0.\]

Since \(x_iy_i > 0\) for any \(i = 1, 2, ..., n\), the above inequality implies \(\|x - y\|^2_2 = 0\). Thus, (19) has at most one solution.

To show that (19) has a solution, we introduce a bounded subset \(S\) of \(K\) defined by \(S = \{x \in \mathbb{R}^n : -\varepsilon^{-1}e < z < -\delta e\}\), where \(e = (1, ..., 1)^\top \in \mathbb{R}^n\) and both \(\varepsilon\) and \(\delta\) are (small) positive constants. Let \(W(x) = Bx - \mu/|x - d|\). Then \(W : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is continuous. Letting \(\partial S\) denote the boundary of \(S\), we show, when \(\varepsilon > 0\) and \(\delta > 0\) are sufficiently small, \(0 \notin W(\partial S)\). We decompose \(S\) into \(S_1\) and \(S_2\), where \(S_1\) contains the boundary points which have at least one component equal to \(-\delta\) and \(S_2\) contains those having at least one component equal to \(-\varepsilon^{-1}\). We consider the following two cases.

**Case 1.** Suppose \(0 \in W(\partial S_1)\). In this case, there must be an \(x \in \partial S_1\) such that \(x_k = -\delta\) and

\[W_k(x) = (Bx)_k + \frac{\mu}{\delta} - d_k = 0, \text{ or } (Bx)_k = -\mu/\delta + d_k\]

for a feasible \(k\). This implies that \(Bx\) is unbounded for this particular solution \(x\) to (19) when \(\delta \to +0\). This clearly contradicts Lemma 4.1. Thus, \(0 \notin W(\partial S_1)\).

**Case 2.** Suppose \(0 \in W(\partial S_2)\). Then, there is an \(x \in S_2\) with \(x_k = -\varepsilon^{-1}\) for a feasible \(k\) such that \(W(x) = 0\). In fact this is not possible when \(\varepsilon\) is sufficiently small, as otherwise, \(x\) will violate (20).

Combining the above two cases we see that when \(\varepsilon > 0\) and \(\delta > 0\) are both sufficiently small, \(0 \notin W(\partial C)\). Moreover, the gradient of \(W\) is \(\nabla W(x) = B + \mu \text{diag}(x_1^{-2}, x_2^{-2}, ..., x_n^{-2})\) for any \(x < 0\). It is easy to show, using the same argument as that for showing the non-singularity of \(B\) in the proof of Theorem 3.1, that \(\nabla W(x)\) is non-singular for any \(x < 0\). Thus, the degree of \(W\), defined as \(\text{sign(det}(\nabla F))\), at any point is non-zero. Therefore, by the Kronecker Theorem, \(W(z) = 0\) has a solution [33, p.161]. This concludes the proof of the theorem.

The rate of convergence for \(x_\mu\) is established in the following theorem.

**Theorem 4.3.** Let \(x^*\) and \(x_\mu < 0\) be solutions to Problem 3 and (19) respectively. Then, \(x^*\) and \(x_\mu\) satisfy

\[\|x^* - x_\mu\|_2 \leq \sqrt{\frac{nm}{C}},\]  \hspace{1cm} (23)

where \(C = \alpha\) when \(\theta = 0\) and \(C = \lambda\) when \(\theta = 1/2\) and \(\lambda > 0\).

**Proof.** We only prove the case that \(\theta = 1/2\) and \(\lambda > 0\). From Theorem 3.2 we see that \(x^*\) satisfies (18). Since \(x_\mu \in \mathcal{K}\), we have

\[(x_\mu - x^*)\top Bx^* \geq (x_\mu - x^*)\top d.\]

Left-multiplying (19) by \((x_\mu - x^*)\top\) gives

\[(x_\mu - x^*)\top Bx_\mu - (x_\mu - x^*)\top(\mu/x_\mu) = (x_\mu - x^*)\top d.\]

Subtracting both sides of the above inequality from the corresponding sides of (24) we have

\[(x_\mu - x^*)\top B(x^* - x_\mu) + (x_\mu - x^*)\top(\mu/x_\mu) \geq 0,\]
from which we have
\[
(x_\mu - x^*)^T B(x_\mu - x^*) \leq (x_\mu - x^*)^T (\mu./x_\mu) \leq \mu(n - (x^*)^T/x_\mu) \leq \mu n, \tag{25}
\]
since \(- (x^*)^T/x_\mu \leq 0\). Using (16) we obtain from the above inequality
\[
\alpha \lambda \|x_\mu - x^*\|_2^2 + (b(p^*') - b(p_*)) (x_\mu - x^*) \leq (x_\mu - x^*)^T B(x_\mu - x^*) \leq \mu n.
\]

Following the same procedure for deducing the above inequality we can also show
\[
\lambda \|x^* - x_\mu\|_2^2 + (b(p^*) - b(p_*)) (x^* - x_\mu) \leq -(x^* - x_\mu)^T (\mu./x_\mu) \leq \mu n. \tag{26}
\]

Adding up both sides of (25) and (26) and dividing both sides of the resulting inequality by 2 give
\[
\lambda \|x_\mu - x^*\|_2^2 \leq \mu n.
\]
Taking the square-root on both sides of the above estimate we have (23). \qed

We comment that (19) is a smooth nonlinear system in the interior of \(\mathcal{K}\) and the smooth Newton’s algorithm proposed in [41, 38] can be used for solving it numerically.

5. Numerical experiments. We now present some numerical results to demonstrate the usefulness and effectiveness of the numerical method presented. All numerical experiments have been performed in double precision under Matlab environment.

**Test 1.** Let \(\Omega = (0, 1)^2\). Find \(u\) in a proper functional space with \(u = 0\) on \(\partial \Omega\), the boundary of \(\Omega\), such that
\[
u = \arg \min_{v \leq q} J(v) = \int_{\Omega} \left[ \frac{p_1}{2} (1 + s + t)^p \| \nabla v \|^2 - p_3 g(s, t) v(s, t) \right] d\Omega,
\]
where \(p_1\), \(p_2\) and \(p_3\) are constants, and \(g(s, t)\) and \(q(s, t)\) are given functions.

We partition \(\Omega\) into a uniform mesh with mesh \((N+1) \times (N+1)\) nodes \((x_i, y_j) = (i, j)h\) for \(i, j = 0, 1, \ldots, N\), where \(N\) denotes a positive integer and \(h = 1/N\).

Applying standard finite differences to the above problem yields the following finite-dimensional problem.
\[
\min_{V \leq Q} F(V) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left[ \frac{p_1}{2} (1 + s_i + t_j)^p \left( \frac{(V_{i+1,j} - V_{i,j})^2}{h^2} + \frac{(V_{i,j+1} - V_{i,j})^2}{h^2} \right) \right.
\]
\[
\left. - p_3 g_{i,j} V_{i,j} \right] h^2, \tag{27}
\]
where \(V = \{V_{i,j}\}\) with \(V_{i,j}\) an approximation to \(v(s_i, t_j)\), \(Q = \{Q_{i,j}\}\) with \(Q_{i,j} = q(s_i, t_j)\) and \(g_{i,j} = g(s_i, t_j)\). Note that \(V_{i,j} = 0\) when any of \(i\) or \(j\) equals 1 or \(N\).

If we let \(L = N - 1\) and \(x_{(j-1)N+i} = V_{i,j}\) for \(i, j = 1, 2, \ldots, N-1\), then (27) can be rewritten as
\[
\min_{x \leq c} F(x) = \sum_{k=1}^{n} \frac{p_1}{2} (1 + d_k)^p \left( \frac{(x_{k+1} - x_k)^2}{h^2} + \frac{(x_{k+L} - x_k)^2}{h^2} \right) - p_3 f^T x \right] h^2, \tag{28}
\]
with \(n = (N - 1)^2\). The definitions of \(c\), \(d_k\) and \(f\) in the above expression are self-explanatory.

Now, we assume that \(p_1\), \(p_2\) and \(p_3\) are the interval numbers \(p_1 = [0.9, 1.2]\), \(p_2 = [1.9, 2.1]\) and \(p_3 = [0.8, 1.3]\) and consider the solution of (28). It is easy to
see that the optimal choices for \( p_1, p_2 \) and \( p_3 \) are \( p_{1,*} = 0.9, p_{1}^* = 1.2, p_{2,*} = 1.9, p_2^* = 2.1 \), and
\[
(p_{1})_* = \frac{1 - \text{sign}(f^T x)}{2}(p_3) + \frac{1 + \text{sign}(f^T x)}{2}(p_3),
\]
\[
(p_{3})^* = \frac{1 - \text{sign}(f^T x)}{2}(p_3) + \frac{1 + \text{sign}(f^T x)}{2}(p_3).
\]

It is also easy to verify that the gradient of \( F/h^2 \) is
\[
\frac{1}{h^2} \nabla F(x) = A(p_1, p_2)x - b(p_3),
\]
where \( b_k = p_3 f_k \) and \( A \) is a symmetric penta-diagonal matrix with the non-zeros entries

\[
\begin{align*}
    a_{k,k-L} &= -\frac{p_1}{h^2}(1 + d_{k-L}) p_3, & a_{k,k-1} &= -\frac{p_1}{h^2}(1 + d_{k-1}) p_3, & a_{k,k+1} &= -\frac{p_1}{h^2}(1 + d_k) p_3, \\
    a_{k+L} &= a_{k,k+1}, & a_{k,k} &= -(a_{k,k-L} + a_{k,k-1} + a_{k,k+1} + a_{k,k+L})
\end{align*}
\]

for all feasible \( k \in \{1, 2, ..., n\} \). It is easy to verify that \( A \) is a symmetric and positive-definite. It is also an \( M \)-matrix. We comment that, strictly speaking, the optimal values of \( p_3 \) is a function of \( x \). However, their derivatives are zero almost everywhere and thus the above expression for \( \nabla F \) holds almost everywhere in \( K \).

To solve Test 1 numerically, we choose
\[
g(s, t) = -5(1 + s + t) \left[ (2(1 - 3s^2) - 6s(1 + s + t))(t - t^3) + (2(1 - 3t^2) - 6t(1 + s + t))(s - s^3) \right], \\
q(s, t) = 0.3 + |s - 0.5| + |t - 0.5|.
\]

It is easy to verify, using the calculus of variations, that when omitting the constraint \( v \leq q \) and setting \( p_1 = 1 = p_3 \) and \( p_2 = 2 \), the unconstrained optimization problem has the 1st-order optimality condition \( -\nabla \cdot [(1 + s + t)^2 \nabla v] = g \) which has the analytical solution \( u_{\text{unc}} = 5(s - s^2)(t - t^3) \).

Now, we choose \( N = 100 \) and a sequence of \( \lambda \) given by \( \lambda_i = a + (i - 1)(a - a)/40 \) for \( i = 1, 2, ..., 40, \) where \( a = 0 \) when \( \theta = 0 \) and \( a = 0.01 \) when \( \theta = 0.5 \). We also choose \( \mu = 10^{-10} \) and using the Newton algorithm proposed in [41] to solve (19). The computed optimal center \( F_C \), right end-point \( F_R \) and width \( F_W \) for \( \theta = 0 \) and 0.5 are plotted in Figures 1 and 2 respectively. The Pareto fronts are also depicted in Figures 1(b) and 2(b) respectively which consist of the Pareto optimal solutions corresponding to \( \theta = 0 \) and \( \theta = 0.5 \) respectively. In our computations we have observed that when \( \theta = 0.5 \) and \( \lambda \) is small, our Newton’s algorithm fails to converge in 500 iterations. This is because the system matrix \( B \) in (11) may no long be positive-definite as shown in Theorem 3.1. In fact, when \( \theta = 0.5 \) and \( \lambda = 0, (9)-(10) \) is to minimize the width of the interval objective only and the optimal solution is \( x = 0 \), though \( B \) may no longer be positive-definite, or an \( M \)-matrix. From Figure 2(a) we see that the computational error when \( \lambda < 0.2 \) is noticeable.

**Test 2.** This test is the same as Test 1 with \( p_1 = 1 = p_3 = p_2/2 \). However, we assume that there is an absolute error of up to 0.1 in each \( \frac{1}{2}(1 + d_k)^2 \) and up to 1 in \( f_k \) in (28). More specifically, we replace \( \frac{1}{2}(1 + d_k)^2 \) and \( f_k \) in (28) with the
following interval numbers
\[
d_k' = \left[ \frac{1}{2}(1 + d_k)^2 - 0.1 \times \text{rand}(), \frac{1}{2}(1 + d_k)^2 + 0.1 \times \text{rand}() \right],
\]
\[
f_k' = [f_k - \text{rand}(), f_k + \text{rand}]
\]
for \( k = 1, 2, \ldots, n \), where \( \text{rand}() \) denotes the random number generator in Matlab.

In this case, it is easy to see that optimal values for \( d_k' \) and \( f_k' \) are
\[
(d_k')^* = (d_k)_L, \quad (d_k')^* = (d_k)_R,
\]
\[
(f_k')^* = \frac{1 - \text{sign}(x_k)}{2} (f_k')_R + \frac{1 + \text{sign}(x_k)}{2} (f_k')_L,
\]
\[
(f_k')^* = \frac{1 - \text{sign}(x_k)}{2} (f_k')_L + \frac{1 + \text{sign}(x_k)}{2} (f_k')_R.
\]
This problem is solved on the same mesh as the one in Test 1 and the results for \( \theta = 0 \) and \( \theta = 0.5 \) are depicted in Figures 3 and 4 respectively. From Figure 3 we see that, when \( \theta = 0 \), both the average and maximum costs \( F_C \) and \( F_R \) are almost constant when \( \lambda \) ranges from 0 to 1. However, when \( \theta = 0.5 \), we see from Figure 4 that the average cost \( F_C \) decreases quickly from 0 to a certain value as \( \lambda \) increases, and then \( F_C \) decreases slowly as \( F_W \) (or \( \lambda \)) increases.

Finally, we plot the computed optimal solution \( u \), along with the upper bound \( q \) when \( \theta = 0 \) and \( \lambda = 0.5 \) in Figure 5(a) and the difference between the optimal solutions corresponding to \( \theta = 0 \) and \( \theta = 0.5 \), respectively, in Figure 5(b).

5. Conclusions. In this work we have proposed a novel method for a large-scale constrained optimization problem arising from the discretization of an infinite-dimensional obstacle problem with uncertainty in the form of interval coefficients. The problem was formulated as a multi-objective optimization problem and the KK-T conditions corresponding to the Pareto optimal solutions to the multi-objective optimization problem are in the form of a linear complementarity problem (LCP).
We have shown that the LCP has a unique solution. A interior penalty method has been used for the LCP and the penalized problem is shown to be uniquely solvable. We have also established the rate of convergence of the solution to the penalized problem. Numerical results have been presented to demonstrate the effectiveness and usefulness of our method when used for solving from non-trivial test problems.

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Received July 2018; 1st revision December 2018; Final revision April 2019.

E-mail address: song.wang@curtin.edu.au