ON SOME SMOOTHENING EFFECTS OF THE TRANSITION SEMIGROUP OF A LÉVY PROCESS

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Abstract. Let \((P_t)\) be the transition semigroup of a Lévy process \(L\) taking values in a Hilbert space \(H\). Let \(\nu\) be the Lévy measure of \(L\). It is shown that for any bounded and measurable function \(f\),
\[
\int_H |P_t f(x + y) - P_t f(x)|^2 \nu(dy) \leq \frac{1}{t} P_t f^2(x) \quad \text{for all } t > 0, x \in H.
\]
As \(\nu\) can be infinite this formula establishes some smoothening effect of the semigroup \((P_t)\). In the paper some applications of the formula will be presented as well.

1. Introduction

Let \((X_t)\) be the solution to an SDE on a Hilbert space \(H\) driven by a non-degenerate Wiener process \(W\). Let
\[
P_t f(x) = E( f(X_t) | X_0 = x), \quad f \in B_b(H), \quad t \geq 0,
\]
be the corresponding transition semigroup defined on the space of bounded measurable functions \(B_b(H)\). Then the following Bismut–Elworthy–Li formula holds (see [4] or [8])
\[
\langle \nabla P_t f(x), h \rangle_H = \frac{1}{t} E \left( f(X_t) \int_0^t K(s; h) dW_s | X_0 = x \right),
\]
where \(K(s, h)\) is an adapted stochastic processes independent of \(f\). This formula implies the strong Feller property of \((P_t)\), and therefore is very useful for studying its ergodic properties. For other applications we refers readers to e.g. [14, 2, 5].

In this paper, we prove that the transition semigroup \((P_t)\) of a Lévy process \(L\) taking values in a Hilbert space \(H\) exhibits some smoothening effects, namely \(P_t\) transforms \(B_b(H)\) into the intersection of domains of some non-local operators (see Theorem 3.1 for more details). The proof of Theorem 3.1 is very simple and follows [4]. Next, see Corollary 3.2, we will show that
\[
(1.1) \quad \int_H |P_t f(x + y) - P_t f(x)|^2 \nu(dy) \leq \frac{1}{t} P_t f^2(x) \quad \text{for } f \in B_b(H), \quad t > 0, \quad x \in H.
\]
Above \(\nu\) is the Lévy measure of \(L\). Note that if \(\nu\) is infinite, then for any open ball \(B_\varepsilon(0)\) with the center at 0 and radius \(\varepsilon > 0\) one has \(\nu(B_\varepsilon(0)) = +\infty\). Therefore (1.1) means that for any \(x \in H, \ P_t f(x + y), \ y \in B_\varepsilon(0), \) is in a certain sense close to \(P_t f(x)\).

As applications of our general result, we obtain a short time estimate for the semigroup generated by fractional Laplacian and a ‘fractional gradient’ estimate for the perturbed...
stable type stochastic systems considered in [3]. Finally using generalized Campanato’s theorem, we calculate modulus of continuity of the transition semigroups of ‘log $\alpha$-stable’ processes.

The paper is organized as follows: the next section includes some preliminary facts on Lévy processes. The main general results are formulated in Section 3. The last three sections are devoted to applications. In the appendix we prove the generalized Campanato theorem of harmonic analysis.

2. Preliminary facts on Lévy processes

We shall recall here some preliminary facts on Lévy processes (for details see e.g. [1, 9, 10]). Let $(L_t)_{t \geq 0}$ be an $H$-valued Lévy process. It is well known that there is a vector $m \in H$, a symmetric positive definite trace class operator $Q: H \mapsto H$, and a Borel measure $\nu$ on $H$ satisfying

\begin{equation}
\nu(\{0\}) = 0, \quad \int_H 1 \wedge |y|^2_H \nu(dy) < +\infty,
\end{equation}

such that

\[ E e^{i(x,L_t)H} = e^{-t\psi(x)}, \quad x \in H, \]

where the so-called Lévy symbol $\psi$ of $(L_t)$ is given by

\[ \psi(x) = i\langle x, m \rangle_H + \frac{1}{2}(Qx, x)_H + \int_H \left[ e^{i(x,y)H} - 1 - i(x,y)H 1_{(|y| \leq 1)} \right] \nu(dy). \]

We call $\nu$ the Lévy measure of $L$ and $(m, Q, \nu)$ the generating triplet of $L$.

The Poisson random measure associated with $(L_t)$ is defined by

\[ N(t, \Gamma) := \sum_{s \in (0,t]} 1_{\Gamma}(L_s - L_{s-}), \quad \Gamma \in B(H), \quad t > 0, \]

and the compensated Poisson random measure is given by

\[ \tilde{N}(t, \Gamma) = N(t, \Gamma) - t\nu(\Gamma). \]

By the Lévy–Khinchin decomposition (cf. [1, p.108, Theorem 2.4.16] or [9, p. 53, Theorem 4.23]), one has

\[ L_t = mt + W_Q(t) + \int_{\{0 < |x|_H \leq 1\}} x\tilde{N}(dt, dx) + \int_{\{|x|_H > 1\}} xN(dt, dx), \quad t \geq 0, \]

where $W_Q$ is a Wiener process in $H$ with covariance operator $Q$.

Let $(\mathcal{F}_t)$ be the filtration generated by $(L_t)$, and let us denote by $L^2_{\text{loc}}$ the space of all predictable stochastic process $\psi$ satisfying

\[ E \int_0^t \int_H |\psi(s,y)|^2_H \nu(dy) ds < \infty \quad \text{for } t > 0. \]

Then for any $\psi \in L^2_{\text{loc}}$ the stochastic integral $\int_0^t \int_H \psi(s,y)\tilde{N}(ds, dy)$ is a well-defined square integrable and mean zero martingale. Moreover, the following Itô isometry holds (see e.g. [1, p. 200] or [9, Section 8.7])

\begin{equation}
\mathbb{E} \left[ \int_0^t \int_H \psi(s,y)\tilde{N}(ds, dy) \int_0^t \int_H \varphi(s,y)\tilde{N}(ds, dy) \right] = \mathbb{E} \int_0^t \int_H \psi(s,y)\varphi(s,y)\nu(dy) ds, \quad \psi, \varphi \in L^2_{\text{loc}}.
\end{equation}
Let \( L = (L_t) \) be a Lévy process with a generating triplet \((m, Q, \nu)\). Consider the Markov family

\[ L^x_t = x + L_t, \quad t \geq 0, \ x \in H. \tag{2.3} \]

Its transition semigroup \((P_t)\) is given as follows

\[ P_t f(x) = \mathbb{E} f(L^x_t), \quad t \geq 0, \ f \in B_b(H). \tag{2.4} \]

Observe that \((P_t)\) is \(C_0\) on the space \(UC_b(H)\) of uniformly continuous bounded functions on \(H\), see [9, p. 80]. Moreover, the domain of its generator \(\mathcal{L}\) contains the space \(UC^2_b(H)\), and

\[
\mathcal{L} f(x) = \langle Df(x), m \rangle_H + \frac{1}{2} \text{Trace } Q D^2 f(x) + \int_H (f(x+y) - f(x) - \mathbf{1}_{\{|y|_H < 1\}} \langle Df(x), y \rangle_H) \nu(dy), \quad f \in UC^2_b(H), \ t \geq 0, \ x \in H.
\]

3. Main results

Let \( L \) be a Lévy process on \( H \) with the generation triplet \((m, Q, \nu)\). Let \((L^x_t), t \geq 0, \ x \in H\) be the Markov family given by (2.3), and let \((P_t)\) given by (2.4) be the transition semigroup of \((L^x_t)\).

Given \( q \in L^2(H, \mathcal{B}(H), \nu) \), define

\[
\mathcal{D}_q := \left\{ f \in B_b(H) : \sup_{x \in H} \int_H |f(x+y) - f(x)| |q(y)| \nu(dy) < \infty \right\}.
\]

Next, let

\[
A_q f(x) := \int_H \left[ f(x+y) - f(x) \right] q(y) \nu(dy) \quad \text{for } f \in \mathcal{D}_q, \ x \in H.
\]

Taking into account (2.1), we see that \( C^1_b(H) \subset \mathcal{D}_q \), \( A_q \) is a bounded linear operator from \( C^1_b(H) \) into \( B_b(H) \) and

\[
\|A_q f\|_{\infty} := \sup_{x \in H} |A_q f(x)| \leq 2\|f\|_{\infty} \left( \int_{|y|_H \geq 1} \nu(dy) \right)^{\frac{1}{2}} \left( \int_{|y|_H \geq 1} q^2(y) \nu(dy) \right)^{\frac{1}{2}} + \|Df\|_{\infty} \left( \int_{\{|y|_H < 1\}} |y|_H^2 \nu(dy) \right)^{\frac{1}{2}} \left( \int_{\{|y|_H < 1\}} q^2(y) \nu(dy) \right)^{\frac{1}{2}} < \infty.
\]

**Theorem 3.1.** Let \( q \in L^2(H, \mathcal{B}(H), \nu) \). Then for all \( t > 0 \) and \( f \in B_b(H) \), \( P_t f \in \mathcal{D}_q \) and

\[
|A_q P_t f(x)|^2 \leq \frac{1}{t} P_t f^2(x) \int_H q^2(y) \nu(dy) \quad \text{for all } x \in H.
\]
Thus, by the Hölder inequality and Itô isometry we obtain

\[
\begin{align*}
L_t f(L_t^x) - P_t f(x) &= - \int_0^t \mathcal{L} P_{t-s} f(L_s^x) ds + \int_0^t P_{t-s} \mathcal{L} f(L_s^x) ds \\
&\quad + \int_0^t \int_H [P_{t-s} f(L_s^x + y) - P_{t-s} f(L_s^x)] \tilde{N}(dy, ds) \\
&\quad + \int_0^t \langle DP_{t-s} f(L_s^x), dW_Q(s) \rangle_H \\
&\quad = \int_0^t \int_H [P_{t-s} f(x + y) - P_{t-s} f(x)] \tilde{N}(dy, ds) \\
&\quad + \int_0^t \langle DP_{t-s} f(L_s^x), dW_Q(s) \rangle_H,
\end{align*}
\]

(3.1)

where the last equality is because \( \mathcal{L} P_t f = P_t \mathcal{L} f \) for \( f \in UC^2_b(H) \). Multiplying the both sides of (3.1) by 

\[
\int_0^t \int_H q(y) \tilde{N}(dy, ds)
\]

and taking into account (2.2), we further get

\[
\begin{align*}
\mathbb{E} \left[ f(L_t^x) \int_0^t \int_H q(y) \tilde{N}(dy, ds) \right] &= \mathbb{E} \int_0^t \int_H [P_{t-s} f(L_s^x + y) - P_{t-s} f(L_s^x)] q(y) \nu(dy) ds \\
&= \int_0^t \int_H [P_s P_{t-s} f(x + y) - P_s P_{t-s} f(x)] q(y) \nu(dy) ds \\
&= t \int_H [P_t f(x + y) - P_t f(x)] q(y) \nu(dy) = t A_q P_t f(x).
\end{align*}
\]

Thus, by the Hölder inequality and Itô isometry we obtain

\[
\begin{align*}
t |A_q P_t f(x)| &\leq \left( \mathbb{E} f^2(L_t^x) \right)^{1/2} \left( \int_0^t \int_H q^2(y) \nu(dy) ds \right)^{1/2} \\
&\leq (P_t f^2(x))^{1/2} t^{1/2} \left( \int_H q^2(y) \nu(dy) \right)^{1/2}.
\end{align*}
\]

Thus the desired estimate holds for any \( f \in UC^2_b(H) \). Assume that \( f \in B_b(H) \). Let \( x \in H \). Then there is a sequence \((f_n) \subset UC^2_b(H)\) such that

\[
\lim_{n \to \infty} P_t f_n^2(x) = P_t f^2(x),
\]

and

\[
\lim_{n \to \infty} P_t f_n(x + y) = P_t f(x + y) \quad \text{for } \nu \text{ almost all } y.
\]

Consequently, the desired estimate for \( f \) follows from the Fatou lemma.

\[ \square \]

**Corollary 3.2.** For arbitrary \( f \in B_b(H) \) we have

\[
\int_H |P_t f(x + y) - P_t f(x)|^2 \nu(dy) \leq \frac{1}{t} P_t f^2(x), \quad x \in H, \ t > 0.
\]

**Proof.** Since

\[
\int_H |P_t f(x + y) - P_t f(x)|^2 \nu(dy) = \sup \left\{ |A_q P_t f(x)|^2, \quad q : \int_H q^2(y) \nu(du) \leq 1 \right\}
\]

the estimate follows from Theorem 3.1.

\[ \square \]
Given $f \in B_b(H)$ we define the difference operator $\nabla^n_{y_1, \ldots, y_n} f(x)$, $x, y_1, \ldots, y_n \in H$ putting
\[
\nabla_y f(x) = f(x + y) - f(x),
\]
\[
\nabla^{n+1}_{y_1, \ldots, y_{n+1}} f(x) = \nabla_{y_{n+1}} \left( \nabla^n_{y_1, \ldots, y_n} f \right)(x).
\]

**Corollary 3.3.** For any $f \in B_b(H)$ and $n \in \mathbb{N}$,
\[
\sup_{x \in H} \int_H \cdots \int_H |\nabla_{y_1, \ldots, y_n} (P_t f)(x)|^2 \nu(dy_1) \cdots \nu(dy_n) \leq \left( \frac{n}{1 - \alpha} \right)^n \|f\|_{\infty}^2.
\]

**Proof.** It is enough to show the estimate for $f \in UC_b^2(H)$. Let $q_1, \ldots, q_n \in L^2(H, \mathcal{B}(H), \nu)$. We claim that the operators $P_t$, $P_{q_1}, \ldots, P_{q_n}$ commute. Indeed, by the Feynman–Kac representation of $P_t$, the Fubini theorem and the fact $X_t(x) + y = X_t(x + y)$, for all $f \in UC_b^2(\mathbb{R}^d)$ we have
\[
P_t A_q f(x) = \mathbb{E} A_q f(X_t(x)) = \mathbb{E} \int_H (f(X_t(x) + y) - f(X_t(x))) q(y) \nu(dy)
\]
\[
= \int_H [\mathbb{E} f(X_t(x) + y) - \mathbb{E} f(X_t(x))] q(y) \nu(dy)
\]
\[
= \int_H [P_t f(x + y) - P_t f(x)] q(y) \nu(dy)
\]
\[
= A_q P_t f(x).
\]
Thus by Theorem 3.1,
\[
\|A_{q_1} \cdots A_{q_n} P_t f\|_{\infty}^2 = \sup_{x \in H} |A_{q_1} \cdots A_{q_n} P_t f(x)|^2
\]
\[
= \sup_{x \in H} \left| (A_{q_1} P_{t/n}) \cdots (A_{q_n} P_{t/n}) f(x) \right|^2
\]
\[
\leq \frac{n}{t} \left\| (A_{q_1} P_{t/n}) \cdots (A_{q_n} P_{t/n}) f \right\|_{\infty}^2.
\]

\[
4. \text{ Application 1: Short time behaviour of the semigroup}
\]

We shall study the fractional gradient estimate of $\alpha$-stable and truncated $\alpha$-stable processes. In this and next sections $H = \mathbb{R}^d$. The norm on $\mathbb{R}^d$ will be denoted by $| \cdot |$.

**Theorem 4.1.** Let the Lévy measure $\nu$ be of the form
\[
\nu(dx) = \frac{1}{|x|^{d+\alpha}} 1_{\{|x| < K\}} dx,
\]
where $\alpha \in (0, 2]$ and $K \in (0, \infty)$. Then for any $\beta \in (\alpha/2, \alpha)$ we have
\[
\|(-\Delta)^{\alpha/2} P_t f\|_{\infty} \leq C(1 + t^{-1/2})\|f\|_{\infty}, \quad \forall t > 0, \ f \in B_b(\mathbb{R}^d),
\]
where $C = C_{\alpha, \beta}$ only depends on $\alpha$ and $\beta$.

**Proof.** Without any loss of generality, we may assume that $K = 1$. Choose $q$ such that
\[
q(y) = |y|^\beta \quad \forall |y| \leq 1, \quad \int_{\{|y| > 1\}} q^2(y) \nu(dy) < \infty.
\]
It is easy to see that
\[
\int_{\{|y| \leq 1\}} q^2(y) \nu(dy) = \int_{\{|y| \leq 1\}} \frac{|y|^{-d-2+2\beta}}{5} \leq C_{\alpha, \beta},
\]
\[
\int_{\{|y| > 1\}} q^2(y) \nu(dy) = C_{\alpha, \beta}.
\]
where $C_{\alpha, \beta} > 0$ depends on $\alpha, \beta$. Therefore, $q \in L^2(\nu) := L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \nu)$.

Observe that
\[
A_q f(x) = \int_{\{|y| \leq 1\}} \frac{f(x + y) - f(x)}{|y|^\alpha + d - \beta} dy + \int_{\{|y| > 1\}} [f(x + y) - f(x)] q(y) \nu(dy)
\]
and
\[
\left| (-\Delta)^{\alpha/2} f(x) \right| \leq \int_{\{|y| > 1\}} \frac{f(x + y) - f(x)}{|y|^\alpha + d - \beta} dy \leq \int_{\{|y| > 1\}} \frac{f(x + y) - f(x)}{|y|^\alpha + d - \beta} dy + \int_{\{|y| \leq 1\}} |f(x + y) - f(x)| q(y) \nu(dy).
\]

It is easy to see that
\[
\int_{\{|y| > 1\}} \frac{f(x + y) - f(x)}{|y|^\alpha + d - \beta} dy \leq c_{\alpha, \beta} \| f \|_\infty,
\]
where $c_{\alpha, \beta} > 0$ depends on $\alpha, \beta$, and that
\[
\left| \int_{\{|y| \leq 1\}} \frac{f(x + y) - f(x)}{|y|^\alpha + d - \beta} dy \right| \leq |A_q f(x)| + \left| \int_{\{|y| > 1\}} [f(x + y) - f(x)] q(y) \nu(dy) \right| \\
\leq t^{-1/2} \| f \|_{L^2(\nu)} \| q \|_{L^2(\nu)} + 2 \| f \|_\infty \| q \|_{L^2(\nu)},
\]
where the last inequality follows from Theorem 3.1 and Hölder’s inequality. Collecting the previous inequalities, we get the desired one. \hfill \Box

5. Application 2: Estimate for a perturbed dynamics studied in [3]

Let $X_t(x)$ be the value at $t$ of the solution to the following stochastic differential equation
\[
(5.1) \quad dX_t = b(X_{t-})dY_t + dZ_t, \quad X_0 = x,
\]
where $b \in C^2_b(\mathbb{R}^d)$ and $Z_t, Y_t$ are both symmetric stable processes with the parameters $\alpha, \beta \in (0, 2)$. Our result below is also true if $b$ is a bounded measurable function, to avoid the complicated differentiability issue and stress the idea, we assume $b \in UC_b^2(\mathbb{R}^d)$.

Eq. (5.1) in more general setting has been studied by Chen and Wang ([3]) and has a unique weak solution. Let
\[
P_t f(x) = \mathbb{E} f(X_t(x)), \quad \forall f \in B_b(\mathbb{R}^d),
\]
be the corresponding transition semigroup. The semigroup is $C_0$ on the space $UC_b(\mathbb{R}^d)$. Let $\mathcal{L}$ be the generator of $(P_t)$ considered on $UC_b(\mathbb{R}^d)$. It is well known that $UC_b^2(\mathbb{R}^d) \subset \text{Dom}(\mathcal{L})$, and that for all $f \in UC_b^2(\mathbb{R}^d)$,
\[
\mathcal{L} f = -(-\Delta)^{\alpha/2} f - |b(x)|^\beta (-\Delta)^{\beta/2} f,
\]
and the following backward Kolmogorov equation holds
\[
(5.2) \quad \partial_t P_t f = \mathcal{L} P_t f.
\]
We shall use (4.1) to show some properties of the associated backward Kolmogorov equation.

**Theorem 5.1.** If $\beta \in (0, \alpha/2)$, then there exists a $t_0 \in (0, 1)$ depending on $\alpha, \beta$ and $\|b\|_\infty$, such that for any $f \in UC_b(\mathbb{R}^d)$,
\[
\|(-\Delta)^{\beta/2} P_t f\|_\infty \leq C_1 t^{-1/2} \| f \|_\infty, \quad t \leq t_0,
\]
\[
\|(-\Delta)^{\beta/2} P_t f\|_\infty \leq C_2 \| f \|_\infty, \quad t > t_0,
\]
where $C_1, C_2$ depend on $\alpha, \beta$ and $t_0$.

**Proof.** Since $UC^2_b(\mathbb{R}^d)$ is dense in $UC_b(\mathbb{R}^d)$ and $(-\Delta)^{\beta/2}$ is closable, it suffices to show (5.3) for $f \in UC^2_b(\mathbb{R}^d)$.

For any $f \in UC^2_b(\mathbb{R}^d)$, define $P^0_t f(x) = \mathbb{E}[f(Z_t + x)]$. It satisfies
\begin{equation}
(\partial_t P^0_t f = -(-\Delta)^{\alpha/2} P^0_t f.
\end{equation}
$(P^0_t)_{t \geq 0}$ can be extended to a Markov $C_0$-semigroup on $UC_b(\mathbb{R}^d)$. Thanks to (5.2) and (5.4), using the classical Duhamel principle we obtain
\begin{equation}
P_t f(x) = P^0_t f(x) - \int_0^t P^0_s \beta((\Delta)^{\beta/2} P_s f(x))ds.
\end{equation}

Since $b \in UC^2_b(\mathbb{R}^d)$ and $f \in UC^2_b(\mathbb{R}^d)$, $P_t f \in UC^2_b(\mathbb{R}^d)$ and $P^0_t f \in UC^2_b(\mathbb{R}^d)$ both hold. By (4.1), we have
\begin{equation}
\|(-\Delta)^{\beta/2} P_t^0 f\|_{\infty} \leq C(1 + t^{-1/2})\|f\|_{\infty} \leq Ct^{-1/2} \|f\|_{\infty}, \quad \forall t < 1,
\end{equation}
which, together with (5.5), yields
\begin{align*}
\|(-\Delta)^{\beta/2} P_t f\|_{\infty} &\leq Ct^{-1/2} \|f\|_{\infty} + C \int_0^t (t - s)^{-1/2} \|b\|_{\infty} \|(-\Delta)^{\beta/2} P_s f\|_{\infty} ds \\
&\leq Ct^{-1/2} \|f\|_{\infty} + C \int_0^t (t - s)^{-1/2} \|b\|_{\infty} \|(-\Delta)^{\beta/2} P_s f\|_{\infty} ds \\
&= Ct^{-1/2} \|f\|_{\infty} + C \int_0^t (t - s)^{-1/2} \|b\|^2 \|b\|_{\infty} \|b\|_{\infty} \|(-\Delta)^{\beta/2} P_s f\|_{\infty} ds.
\end{align*}

Define
\begin{equation*}
L_T := \sup_{0 \leq t \leq T} t^{1/2} \|(-\Delta)^{\beta/2} P_t f\|_{\infty}
\end{equation*}
with $T > 0$ to be chosen later. From the previous inequality we have
\begin{equation}
L_T \leq C \|f\|_{\infty} + CT^{1/2} \sup_{0 \leq t \leq T} \int_0^t s^{-1/2} (t - s)^{-1/2} ds L_T, \\
= C \|f\|_{\infty} + CB(3/2, 3/2) T^{1/4} \|b\|_{\infty} L_T,
\end{equation}
where $B$ is the beta function. Choose a $t_0 \in (0, 1)$ (depending on $\alpha, \beta$, and $\|b\|_{\infty}$) such that $CB(3/2, 3/2)t_0^{1/2} \|b\|_{\infty} < \frac{1}{2}$, we obtain
\begin{equation*}
L_{t_0} \leq 2C \|f\|_{\infty}.
\end{equation*}

This immediately implies the first estimate in the theorem.

For the second estimate, taking $P_{t_0/2} f$ rather than $f$ as the initial data, by the same procedure as above we have
\begin{align*}
\|(-\Delta)^{\beta/2} P_t f\|_{\infty} &\leq C_1 \left(t - \frac{t_0}{2}\right)^{-\frac{1}{2}} \|P_{t_0/2} f\|_{\infty} \\
&\leq C_1 \left(t - \frac{t_0}{2}\right)^{-\frac{1}{2}} \|f\|_{\infty}, \quad \forall t \in (t_0/2, 3t_0/2).
\end{align*}
Therefore
\begin{align*}
\|(-\Delta)^{\beta/2} P_t f\|_{\infty} &\leq C_1 \left(\frac{t_0}{2}\right)^{-\frac{1}{2}} \|f\|_{\infty}, \quad \forall t \in (t_0, 3t_0/2).
\end{align*}
Now taking $P_{t_0}f$ as the initial data, we obtain
\[ \|(-\Delta)^{\beta/2}P_{t}f\|_\infty \leq C_1 \left( \frac{t_0}{2} \right)^{-\frac{\beta}{2}} \|f\|_\infty, \quad \forall t \in (3t_0/2, 2t_0). \]
Iterating the above argument, we finally get
\[ \|(-\Delta)^{\beta/2}P_{t}f\|_\infty \leq C_1 \left( \frac{t_0}{2} \right)^{-\frac{\beta}{2}} \|f\|_\infty, \quad \forall t \geq t_0, \]
which is the desired second estimate.

\[ \square \]

6. Application 3: modulus of continuity of transition semigroup of log-\(\alpha\)-stable processes

When the Lévy measure \(\nu\) on \(\mathbb{R}^d\) satisfies
\[ (6.1) \quad \int_{\{|x|<r\}} \nu(dx) = \infty, \quad \forall r > 0, \]
the process \((L_t)_{t\geq 0}\) has infinite small jumps in any time interval \([t, t+\delta]\). Then (see e.g. [10]), the law \(\Sigma(L_t)\) of \(L_t\) is continuous but not necessarily absolutely continuous with respect to Lebesgue measure. However, if additionally \(\nu\) is absolutely continuous then \(\Sigma(L_t)\) is absolutely continuous. Let us also recall, see e.g. [7], that the law \(\Sigma(L_t)\) is absolutely continuous if and only if for the corresponding semigroup satisfies \(P_t: B_c(\mathbb{R}^d) \to C_b(\mathbb{R}^d)\). Thus the absolute continuity is equivalent to strong Feller property.

Below we provide some estimates for the moduli of continuity of the transition semigroup which is beyond the scope of \(\alpha\)-stable type process. Namely, assume that the Lévy measure \(\nu\) is absolutely continuous with respect to Lebesgue measure on the ball \(B_1(0) = \{x: |x| < 1\}\) and
\[ (6.2) \quad \frac{\nu(dx)}{dx} \geq \frac{|\log_2 |x||^{2\alpha}}{|x|^d}, \quad x \in B_1(0), \]
where \(\alpha \in (1, \infty)\) is a constant.

**Theorem 6.1.** Let \((L_t)_{t\geq 0}\) be a Lévy process with Lévy measure \(\nu\) satisfying (6.2). Then \((L_t)_{t\geq 0}\) is strong Feller. Moreover, there exists an \(r_0 > 0\) such that
\[ (6.3) \quad |P_t f(x) - P_t f(y)| \leq \frac{C}{|\log_2 |x-y||^{\alpha-1}}, \quad |x-y| \leq r_0, \]
where \(C\) depends on \(\alpha, d, t\) and \(r_0\).

Let \(\Omega \subset \mathbb{R}^d\) be an open set. Given a function \(f: \Omega \to \mathbb{R}, \quad x \in \Omega \) and \(r > 0\) such that the ball \(B_r(x) \subset \Omega\), define
\[ \bar{f}_{x,r} := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy. \]
To show the theorem, we need to use the following lemma, which is a generalized Campanato theorem. The proof of the lemma is postponed to the appendix.

**Lemma 6.2.** Let \(\Omega \subset \mathbb{R}^d\) be open and bounded, and let \(f: \Omega \to \mathbb{R}\) be a bounded function. Assume that there are constants \(C > 0\) and \(\alpha > 1\) such that
\[ (6.4) \quad \int_{B_r(x)} |f(y) - \bar{f}_{x,r}|^2dy \leq C \frac{r^d}{|\log_2 r|^{2\alpha}}, \]
where \(\bar{f}_{x,r}\) is the average of \(f\) over the ball \(B_r(x)\) centered at \(x\).
for any ball $B_r(x) \subset \Omega$. Then $f$ is uniformly continuous and there exists an $r_0 > 0$ such that for any open $\hat{\Omega} \subset \Omega$ with $\operatorname{diam}(\hat{\Omega}) < r_0$ and $\operatorname{dist}(\hat{\Omega}, \partial \hat{\Omega}) > r_0$, we have

\begin{equation}
|f(x) - f(y)| \log_2 |x - y|^{\alpha - 1} \leq \hat{C},
\end{equation}

where $\hat{C}$ depends on $C$, $\operatorname{dist}(\hat{\Omega}, \partial \hat{\Omega}), d, \alpha$ and $r_0$.

**Proof of Theorem 6.1.** Let $f \in B_b(\mathbb{R}^d)$ and set $g(x) = P_t f(x)$. For any $t > 0$, we have

\begin{equation}
\int_{B_r(x)} |g(y) - \bar{g}_{x,r}|^2 \, dy = \int_{\{|y-x| \leq r\}} \frac{1}{|B_r(x)|} \left| \int_{\{|z-x| \leq r\}} (g(y) - g(z)) \, dz \right|^2 \, dy \leq \int_{\{|y-x| \leq r\}} \frac{1}{|B_r(x)|} \left| \int_{\{|z-x| \leq r\}} |g(y) - g(z)|^2 \, dz \, dy \right| \leq \int_{\{|y-x| \leq r\}} \frac{1}{|B_r(x)|} \left| \int_{\{|z-x| \leq 2r\}} |g(y) - g(z)|^2 \, dz \, dy \right|
\end{equation}

where the last inequality follows from the inclusion

$$\{ |y - x| \leq r \} \cap \{ |z - x| \leq r \} \subset \{ |y - x| \leq r \} \cap \{ |y - z| \leq 2r \}.$$

From (6.6) we further get

\begin{align*}
\int_{B_r(x)} |g(y) - \bar{g}_{x,r}|^2 \, dy & \leq \int_{\{|y-x| \leq r\}} \frac{1}{|B_r(x)|} \int_{\{|z-x| \leq 2r\}} |g(y) - g(z)|^2 \frac{|z - y|^d}{|\log_2 |y - z||^{2\alpha}} \frac{|\log_2 |z - y||^{2\alpha}}{|z - y|^d} \, dz \, dy \\
& = \int_{\{|y-x| \leq r\}} \frac{1}{|B_r(x)|} \int_{\{|z-x| \leq 2r\}} |g(y + z) - g(y)|^2 \frac{|z|^d}{|\log_2 |z||^{2\alpha}} \frac{|\log_2 |z||^{2\alpha}}{|z|^d} \, dz \, dy \\
& \leq \int_{\{|y-x| \leq r\}} \frac{1}{|B_r(x)|} \int_{\{|z-x| \leq 2r\}} |g(y + z) - g(y)|^2 \frac{|z|^d}{|\log_2 |z||^{2\alpha}} \, \nu(\text{d}z) \, dy
\end{align*}

Since $\frac{r^d}{|\log_2 r|^{2\alpha}}$ is decreasing as $r < r_0/2$ for small $r_0 > 0$, the above inequality further gives

\begin{align*}
\int_{B_r(x)} |g(y) - \bar{g}_{x,r}|^2 \, dy & \leq C \frac{r^d}{|\log_2 r|^{2\alpha}} \frac{1}{|B_r(x)|} \int_{\{|y-x| \leq r\}} \int_{\{|z-x| \leq 2r\}} |g(y + z) - g(y)|^2 \, \nu(\text{d}z) \, dy \\
& \leq C \frac{r^d}{|\log_2 r|^{2\alpha}} \sup_{y \in B_r(x)} \int_{\mathbb{R}^d} \int_{\{|z-x| \leq 2r\}} |g(y + z) - g(y)|^2 \, \nu(\text{d}z) \\
& \leq C \frac{r^d}{|\log_2 r|^{2\alpha}} \int_{\mathbb{R}^d} \int_{\{|z-x| \leq 2r\}} |g(y + z) - g(y)|^2 \, \nu(\text{d}z) \\
& \leq C \frac{r^d}{|\log_2 r|^{2\alpha}} t^{-1} \|f\|_\infty^2,
\end{align*}

where the last inequality follows from Corollary 3.2. Combining the estimate above and Lemma 6.2 we obtain the desired conclusion. \qed
7. Appendix: Proof of Lemma 6.2

We shall follow [6]. For $0 < r_2 < r_1 < \min\{\text{dist} (\hat{\Omega}, \partial \Omega), 1\}$ and any $x \in \hat{\Omega}$, we have

\[(7.1)\]  

\[
|\tilde{f}_{x,r_1} - \tilde{f}_{x,r_2}| \leq \frac{1}{|B_{r_2}|} \int_{B_{r_2}(x)} |f(y) - \tilde{f}_{x,r_2}| \, dy + \frac{1}{|B_{r_2}|} \int_{B_{r_2}(x)} |f(y) - \tilde{f}_{x,r_1}| \, dy
\]

\[
\leq \sqrt{\frac{1}{|B_{r_2}|} \int_{B_{r_2}(x)} |f(y) - \tilde{f}_{x,r_2}|^2 \, dy} + \sqrt{\frac{1}{|B_{r_2}|} \int_{B_{r_2}(x)} |f(y) - \tilde{f}_{x,r_1}|^2 \, dy}
\]

\[
\leq \sqrt{\frac{1}{|B_{r_2}|} \int_{B_{r_2}(x)} |f(y) - \tilde{f}_{x,r_2}|^2 \, dy} + \sqrt{\frac{|B_{r_1}|}{|B_{r_2}|} \frac{1}{|B_{r_1}|} \int_{B_{r_1}(x)} |f(y) - \tilde{f}_{x,r_1}|^2 \, dy}
\]

\[
\leq C \left[ \frac{1}{|\log_2 r_2|} + \left( \frac{r_1}{r_2} \right)^{d/2} \frac{1}{|\log_2 r_1|} \right]
\]

\[
\leq C \left[ 1 + \left( \frac{r_1}{r_2} \right)^{d/2} \right] \frac{1}{|\log_2 r_1|},
\]

where the last inequality is by the assumption of the lemma. For all $0 < r_n < r_m < \text{dist} (\hat{\Omega}, \partial \Omega)$, define

\[
N := \left[ \log_2 \left( \frac{r_m}{r_n} \right) \right],
\]

without loss of generality we assume $r_m < 1/2$. By (7.1) we have

\[
|\tilde{f}_{x,r_n} - \tilde{f}_{x,r_m}| \leq \sum_{k=1}^{N} |\tilde{f}_{x,2^{-k}r_m} - \tilde{f}_{x,2^{-(k+1)}r_m}| + |\tilde{f}_{x,2^{-N}r_m} - \tilde{f}_{x,r_n}|
\]

\[(7.2)\]

\[
\leq C_d \sum_{k=1}^{N} \frac{1}{|k - 1 - \log_2 r_m|^\alpha} + C_d \frac{1}{|N - \log_2 r_m|^\alpha}
\]

\[
\leq C_{d,\alpha} \frac{1}{|\log_2 r_m|^{\alpha-1}}.
\]

Hence, there exists an $\tilde{f}$ such that

\[
\lim_{r \to 0} \tilde{f}_{x,r} = \tilde{f}(x), \quad \forall x \in \hat{\Omega},
\]

and there exists an $r_0 > 0$ such that as $r < r_0$,

\[
|\tilde{f}_{x,r} - \tilde{f}(x)| \leq C_{d,\alpha} \frac{1}{|\log_2 r|^{\alpha-1}}, \quad \forall x \in \hat{\Omega}.
\]

(7.3)

On the other hand, by the Lebesgue theorem,

\[
\lim_{r \to 0} \tilde{f}_{x,r} = f(x), \quad \text{a.s.} \ x \in \hat{\Omega}.
\]

By (7.3), all the points in $\hat{\Omega}$ are Lebesgue points. Hence, $\tilde{f}_{x,r} \to f(x)$ uniformly for $x \in \hat{\Omega}$ as $r \to 0$ with

\[
(7.4) \quad |\tilde{f}_{x,r} - f(x)| \leq C_{d,\alpha} \frac{1}{|\log_2 r|^{\alpha-1}}, \quad \forall x \in \hat{\Omega}.
\]

Now for $x, y \in \hat{\Omega}$, denote $r = |x - y|$, we have

\[
|\tilde{f}_{y,2r} - \tilde{f}_{x,2r}| \leq |\tilde{f}_{y,2r} - f(z)| + |f(z) - \tilde{f}_{x,2r}|, \quad z \in B_{2r}(x) \cap B_{2r}(y).
\]

(7.5)
Since $B_{2r}(x) \cap B_{2r}(y)$ contains a ball with radius $r$, as $r < r_0/2$,

\[
|\bar{f}_{y,2r} - \bar{f}_{x,2r}| \leq \frac{1}{|B_r|} \int_{B_{2r}(x) \cap B_{2r}(y)} |\bar{f}_{y,2r} - f(z)| + |f(z) - \bar{f}_{x,2r}| dz
\]

\[
\leq \frac{1}{|B_r|} \int_{B_{2r}(y)} |\bar{f}_{y,2r} - f(z)| dz + \frac{1}{|B_r|} \int_{B_{2r}(x)} |f(z) - \bar{f}_{x,2r}| dz
\]

\[
\leq C \frac{1}{|\log_2 r|^\alpha},
\]

where the last inequality is by the assumption of the lemma. Observe that

\[
|f(x) - f(y)| \leq |f(x) - \bar{f}_{x,2r}| + |f(y) - \bar{f}_{y,2r}| + |\bar{f}_{y,2r} - \bar{f}_{x,2r}|.
\]

This, together with (7.6) and (7.4), immediately implies the desired inequality.

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