PRIMORDIAL BLACK HOLE FORMATION IN THE MATTER-DOMINATED PHASE OF THE UNIVERSE

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ABSTRACT

We investigate primordial black hole formation in the matter-dominated phase of the universe, where nonspherical effects in gravitational collapse play a crucial role. This is in contrast to the black hole formation in a radiation-dominated era. We apply the Zel’dovich approximation, Thorne’s hoop conjecture, and Doroshkevich’s probability distribution and subsequently derive the production probability \( \beta_0 \) of primordial black holes. The numerical result obtained is applicable even if the density fluctuation \( \sigma \) at horizon entry is of the order of unity. For \( \sigma < 1 \), we find a semi-analytic formula \( \beta_0 \simeq 0.055560 \), which is comparable to the Khlopov–Polnarev formula.

We find that the production probability in the matter-dominated era is much larger than that in the radiation-dominated era for \( \sigma \lesssim 0.05 \), while they are comparable with each other for \( \sigma \gtrsim 0.05 \). We also discuss how \( \sigma \) can be written in terms of primordial curvature perturbations.

Key words: black hole physics – early universe – stars: black holes

1. INTRODUCTION

Primordial black holes are becoming a very important area of study at the intersection of cosmology, astrophysics, high-energy physics, and gravitation. See Carr (2003) and Khlopov (2010) for recent reviews. The abundance of primordial black holes is severely constrained observationally (Carr 1975; Carr et al. 2010, 2016) and this fact has rich implications on the early universe and other relevant fields of physics. Furthermore, recently, LIGO has reported gravitational wave observations of GW150914 (Abbott et al. 2016), and it has been argued (Sasaki et al. 2016) that a binary system of primordial black holes can be a source of gravitational waves of this event. The theoretical prediction of the abundance of primordial black holes based on the physical theory of black hole formation is a key issue for theoretical studies.

Khlopov and Polnarev (Khlopov & Polnarev 1980; Polnarev & Khlopov 1982) pioneered primordial black hole formation in the matter-dominated era of the universe. They argued that if stable superheavy particles predicted in the grand unified theories dominate the universe, the pressure of the matter field can be effectively neglected and the production of primordial black holes is significantly enhanced. More recently, Alabidi & Kohri (2009) and Alabidi et al. (2012, 2013) showed that in the so-called hill-top-type inflation scenario, density perturbations of large amplitude can arise on small scales and lead to an enhanced formation of primordial black holes in an effectively matter-dominated phase of the universe before the reheating phase.

The formation of primordial black holes was conventionally studied in the radiation-dominated era until recently. In this case, the threshold \( \delta_{c*} \) of the amplitude of the density perturbation in the comoving slicing at horizon entry is determined by the Jeans criterion. The production probability \( \beta_0 \) of primordial black holes is given by \( \beta_0 \sim \sqrt{2/\pi} (\sigma/\delta_{c*}) \exp(-\delta_{c*}^2/2\sigma^2) \), where \( \sigma \) is the standard deviation of the density perturbations at the relevant mass scale at horizon entry. The results of numerical relativity simulations in spherical symmetry give the threshold \( \delta_{c*} \approx 0.10 \) (Shibata & Sasaki 1999; Musco et al. 2005; Polnarev & Musco 2007; Musco et al. 2009; Musco & Miller 2013; Harada et al. 2015). Since the threshold value is of the order of unity, gravitational instability leads to the formation of a black hole shortly after the horizon entry.

For the equation of state \( p = \rho c^2 \), the analytic formula for the threshold gives \( \delta_{c*} \approx (3(1+w)/(5+3w)) \sin^2(\pi \sqrt{w/(1+3w)}) \) (Harada et al. 2013) showing a good agreement with the numerical results for \( 0.01 \lesssim w \lesssim 0.6 \) (Musco & Miller 2013). In the limit of \( w \to 0 \), we have \( \delta_{c*} \to 0 \), i.e., the region that is only slightly overdense would necessarily collapse to a black hole. This argument clearly overestimates \( \beta_0 \) because it neglects nonspherical effects. For a density perturbation of small amplitude to collapse to a black hole, it must shrink to a radius much smaller than that at the maximum expansion so that a deviation from spherical symmetry can significantly grow. This instability generally leads to a “pancake” collapse (Lin et al. 1965; Zel’dovich 1970). Khlopov & Polnarev (1980) and Polnarev & Khlopov (1982) discussed that a nonspherical effect significantly suppresses primordial black hole formation and obtained a compact analytic formula for \( \beta_0 \) under the assumption of the small density perturbations. Nonspherical effects on primordial black holes have also recently been discussed by Harada & Jhingan (2016) and Kühnel & Sandstad (2016).

The purpose of the current paper is to estimate \( \beta_0 \) even for a large fluctuation of perturbations and a large deviation from spherical symmetry based on a physical argument and also to reproduce the formula by Khlopov and Polnarev in some sense by adopting the approximation of small fluctuation. We apply the Zel’dovich approximation (Zel’dovich 1970), which is a well-established analytic approximation to describe the nonlinear evolution of density perturbations. See White (2014) and references therein for its validity, application, and limitation. See also Russ et al. (1996) for its general relativistic generalization. We adopt the hoop conjecture proposed by...
Thorne (1972, pp. 231–258) for the formation of a black hole horizon, which has not yet been proven for general situations but is shown to hold even for highly distorted horizons (Yoshino 2008). See also Malec & Xie (2015) for a recent proof for a special case. However, see East et al. (2016) for its possible violation in the presence of a negative energy density. As for the probability distribution of nonspherical perturbations, we adopt Doroshkevich’s (Doroshkevich 1970), which was derived under a least number of natural assumptions.

This paper is organized as follows. In Section 2, we apply the Zel’dovich approximation to the nonlinear evolution of the density perturbations and obtain the criterion of the black hole formation based on Thorne’s hoop conjecture. In Section 3, we introduce the probability distribution for nonspherical perturbations by Doroshkevich and derive an integral expression for the production probability of primordial black holes without assuming the small fluctuation approximation. Moreover, we obtain a semi-analytic formula under the small fluctuation approximation. In Section 4, we discuss our results followed by conclusions in Section 5. We keep both the gravitational constant $G$ and the speed of light $c$ throughout this paper.

2. NONSPHERICAL COLLAPSE OF THE DENSITY PERTURBATIONS

2.1. Zel’dovich Approximation

We begin with the Zel’dovich approximation (Zel’dovich 1970):

$$r_i = a(t) q_i + b(t) p_i(q),$$

(1)

where $a(t) > 0$, $q_i$, and $p_i(q)$ $(i = 1, 2, 3)$ are the scale factor, Lagrangian coordinates, and deviation vector, respectively. The function $b(t)$ denotes a linearly growing mode in the matter-dominated phase of the universe in the framework of the Newtonian cosmology. The scale factor $a(t)$ satisfies the Friedmann equation for a spatially flat universe

$$H^2 = \frac{8\pi}{3} G \bar{\rho},$$

(2)

where $H = \dot{a}/a$ and $\bar{\rho} = \bar{\rho}(t)$ is the density of the Friedmann universe. The conservation law implies $\bar{\rho} \propto 1/a^3$. Although $b(t)$ is a linearly growing mode, the Zel’dovich approximation implies the extrapolation of Equation (1) beyond the linear regime to the nonlinear regime up until a caustic occurs at $q_i$.

We can calculate the deformation tensor $D_{ik}$ such that

$$D_{ik} = \frac{\partial r_i}{\partial q_k} = a(t) \delta_{ik} + b(t) \frac{\partial p_i}{\partial q_k},$$

(3)

The matrix $\partial p_i/\partial q_k$ defines a set of fundamental axes and we can choose $q_i$ so that

$$\frac{\partial p_i}{\partial q_k} = \text{diag}(-\alpha, -\beta, -\gamma),$$

(4)

where $\alpha$, $\beta$, and $\gamma$ are functions of $q_i$ and, hence,

$$D_{ik} = \text{diag}(a - \alpha b, a - \beta b, a - \gamma b).$$

(5)

The mass contained within the Lagrangian volume is conserved, i.e.,

$$dm = \rho d^3r = \bar{\rho} a^3 d^3q \quad \text{or} \quad m = \int \rho d^3r = \bar{\rho} a^3 \int d^3q.$$ 

(6)

Therefore, the density $\rho$ is calculated through the determinant of $D_{ik}$, so that

$$\rho = \frac{a^3}{(a - \alpha b)(a - \beta b)(a - \gamma b)} \bar{\rho}.$$ 

(7)

For convenience, we define a density perturbation in the linear regime

$$\delta_L := \left( \frac{\rho - \bar{\rho}}{\bar{\rho}} \right) = (\alpha + \beta + \gamma) \frac{b}{a}.$$ 

(8)

Note that if we take $b > 0$, we find that $\delta_L > 0$ if and only if $\alpha + \beta + \gamma > 0$. Since the density perturbation in the linear regime grows as the scale factor, i.e., $\delta_L \propto a$, we find

$$b \propto a^2.$$ 

(9)

2.2. Pancake Collapse

Due to the continuity of $\alpha$, $\beta$, and $\gamma$, we can locally take the coordinates $q_i$ such that

$$\begin{cases}
  r_1 = (a - \alpha b) q_1 \\
  r_2 = (a - \beta b) q_2 \\
  r_3 = (a - \gamma b) q_3 \\
\end{cases}$$

(10)

We assume that $\alpha$, $\beta$, and $\gamma$ are constant over the scale in which we are interested. We also assume $\infty > \alpha \geq \beta \geq \gamma > -\infty$ without loss of generality. We fix the scale of the Lagrangian coordinate radius and, hence, consider a ball of radius $q_i$ which gives the comoving scale of the perturbation. We assume that the perturbation will collapse at least along one of the three axes so that $\alpha > 0$. We do not assume that a deviation from spherical symmetry is small.

We define three important moments, the horizon entry time $t = t_h$, maximum expansion time $t = t_f$, and collapse time $t = t_c$.

At the horizon entry time $t = t_h$, the unperturbed physical radius of the mass is equal to the Hubble radius so that

$$a(t_h) q = c H^{-1}(t_h).$$

(11)

Using Equations (2) and (11), we can calculate the mass contained within the radius $q$ to

$$m = \frac{4\pi}{3} \bar{\rho}(t_h) a^3(t_h) q^3 = \frac{c^3}{2GH(t_h)}$$

(12)

and, hence, from Equation (11), we find

$$a(t_h) q = r_g.$$ 

(13)

where $r_g = 2GM/c^2$ is the gravitational radius of the mass $m$. From Equation (8), we can get the following relation between $b(t_f)$ and $\delta_L(t_f)$:

$$\delta_L(t_f) = (\alpha + \beta + \gamma) \frac{b(t_f)}{a}.$$ 

(14)

At the maximum expansion time $t = t_f$, the mass is about to shrink along the $r_1$ axis. Since $r_1(t_f) = 0$ or $a(t_f) = \alpha b(t_f)$, we find

$$\frac{b}{a}(t_f) = \frac{1}{2\alpha}.$$ 

(15)
where we have used Equation (9). The axis \( r_f := r_1(t_f) \) can be calculated to
\[
  r_f = [a(t_f) - \alpha b(t_f)]q = \frac{1}{2}a(t_f)q. \tag{16}
\]

We denote the ratio of \( r_g \) to \( r_f \) with \( \chi \), which is given by
\[
  \chi := \frac{r_g}{r_f} = 2\frac{a(t_f)}{a(t_g)} = 4\frac{\delta_L(t_f)}{\alpha + \beta + \gamma}. \tag{17}
\]

This clearly shows that a perturbation of small amplitude must considerably shrink until it collapses to a black hole. It is also instructive to see the ratio of the radius of the mass to the Hubble radius at the time of maximum expansion as follows.
\[
  \frac{r_f}{cH^{-1}(t_f)} = \left(\frac{1}{2}\right)^{3/2} \chi^{1/2} = \left(\frac{1}{2}\right)^{3/2} \left[\frac{4\alpha}{\alpha + \beta + \gamma}\delta_L(t_f)\right]^{1/2}, \tag{18}
\]

where we have used \( a \propto t^{2/3} \). This means that if \( \chi \ll 1 \) or \( \delta_L(t_f) \ll 1 \), the density perturbation grows to the order of unity only after the radius of the mass becomes much smaller than the Hubble radius. This justifies the use of the Zel’dovich approximation based on the Newtonian gravity in the present setting.

At the collapse \( t = t_c \), \( r_1(t_c) = 0 \) so that
\[
  \frac{b}{a}(t_c) = \frac{1}{\alpha}. \tag{19}
\]

Then, the mass becomes a “pancake” or a two-dimensional ellipse with the seminor and semimajor axes given by
\[
  r_2(t_c) = [a(t_c) - \beta b(t_c)]q \quad \text{and} \quad r_3(t_c) = [a(t_c) - \gamma b(t_c)]q, \tag{20}
\]

respectively. From Equations (9), (15), and (19), we find
\[
  a(t_c)q = 2a(t_f)q = 4r_f. \tag{21}
\]

Using Equations (19) and (20), we find
\[
  r_2(t_c) = 4\left(1 - \frac{\beta}{\alpha}\right)r_f \quad \text{and} \quad r_3(t_c) = 4\left(1 - \frac{\gamma}{\alpha}\right)r_f. \tag{22}
\]

2.3. Black Hole Formation Criterion

The hoop conjecture (Thorne 1972; Misner et al. 1973) states that black holes with horizons form when and only when a mass \( M \) gets compactified into a region whose circumference in every direction is approximately smaller than \( 4\pi GM/c^2 \). The hoop \( C \) of a region is defined as the maximum of its circumferences in all directions. For the pancake, it is given by the circumference of the ellipse. Since the eccentricity of the pancake is given by
\[
  e^2 = 1 - \left(\frac{r_2(t_c)}{r_3(t_c)}\right)^2 = 1 - \left(\frac{\alpha - \beta}{\alpha - \gamma}\right)^2, \tag{23}
\]

the hoop is calculated to
\[
  C = 16\left(1 - \frac{\gamma}{\alpha}\right)E\left[1 - \left(\frac{\alpha - \beta}{\alpha - \gamma}\right)^2\right]r_f, \tag{24}
\]

where \( E(e) \) is the complete elliptic integral of the second kind. Note that \( E(e) \) is a monotonically decreasing function of \( e \in [0, 1] \), where \( E(0) = \pi/2 \) in the circular limit and \( E(1) = 1 \) in the eccentric limit.

According to the hoop conjecture, the condition for black hole formation is given by \( C \leq 2\pi r_g \). Thus, we find the criterion
\[
  s(\alpha, \beta, \gamma) \leq \chi, \tag{25}
\]

Equivalently, we can rewrite the criterion in the following form.
\[
  h(\alpha, \beta, \gamma) \leq 1, \tag{26}
\]

where we define \( h(\alpha, \beta, \gamma) = C/(2\pi r_g) \) according to Yoshino (2008). Using Equation (17), we can calculate the ratio to
\[
  h(\alpha, \beta, \gamma) = \frac{2}{\pi} \left(\frac{\alpha}{\alpha^2}E\left[1 - \left(\frac{\alpha - \beta}{\alpha - \gamma}\right)^2\right] \right), \tag{27}
\]

where we have fixed the normalization of \( b \) so that
\[
  \frac{b}{a}(t) = \frac{a(t)}{a(t_c)}. \tag{28}
\]

The set of Equations (26) and (27) is written only in terms of \( \alpha, \beta, \) and \( \gamma \) and, hence, is the most suitable set of expressions describing the present situation. If the above criterion is not satisfied, a sheet-like caustic occurs at \( t = t_c \) and matter particles should cross each other. These pancakes will then undergo violent relaxation by bouncing several times and eventually get virialized with a large velocity dispersion. The radius of such a virialized object is approximately half the radius of maximum expansion \( r_f \). The object may later shrink and even collapse to a strongly bound object by radiating its energy through some form of radiation. This process happens in a timescale much larger than the Hubble time.

3. PRODUCTION PROBABILITY OF PRIMORDIAL BLACK HOLES

3.1. Doroshkevich’s Probability Distribution

The probability distribution of \( \alpha, \beta, \) and \( \gamma \) is given by Doroshkevich (1970) as
\[
  w(\alpha, \beta, \gamma) d\alpha d\beta d\gamma = \frac{27}{8\sqrt{5} \pi \sigma_3^3} \exp \left[-\frac{3}{5\sigma_3^3}(\alpha^2 + \beta^2 + \gamma^2) \right] \left[\frac{1}{2}(\alpha \beta + \beta \gamma + \gamma \alpha)\right] \cdot (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) d\alpha d\beta d\gamma, \tag{29}
\]

where \( \infty > \alpha \geq \beta \geq \gamma > -\infty \) is assumed and \( \sigma_3 \) is a positive constant. The above distribution is derived by assuming that each of the independent components of the deformation tensor has a Gaussian distribution. We should note that the probability for \( \alpha, \beta, \) and \( \gamma \) to take values that are very close to each other is suppressed because of the factor of \((-\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)\).
It would be more comprehensive to rewrite the above in the following form.

$$w(\alpha, \beta, \gamma) d\beta d\gamma$$

$$= -\frac{27}{8\sqrt{5}\pi\sigma_3^5} \exp \left[ -\frac{1}{10\sigma_3^4} (\alpha + \beta + \gamma)^2 \right]$$

$$- \frac{1}{4\sigma_3^2} \left[ (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \right]$$

$$\cdot (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) d\beta d\gamma. \quad (30)$$

Introducing new variables $x$, $y$, and $z$ by

$$x = \frac{\alpha + \beta + \gamma}{3},$$

$$y = \frac{(\alpha - \beta) - (\beta - \gamma)}{4},$$

and

$$z = \frac{\alpha - \gamma}{2}, \quad (31)$$

respectively, we find

$$w(\alpha, \beta, \gamma) d\beta d\gamma = \tilde{w}(x, y, z) dxdydz$$

$$= -\frac{27}{\sqrt{5}\pi\sigma_3^5} (2y - z)(2y + z) x \exp \left[ -\frac{9}{10} \left( \frac{x}{\sigma_3} \right)^2 \right]$$

$$- 2 \left( \frac{y}{\sigma_3} \right)^2 - \frac{3}{2} \left( \frac{z}{\sigma_3} \right)^2 \right] dxdydz, \quad (32)$$

where $-\infty < x < \infty$, $-\infty < y < \infty$, and $2|y| \leq z < \infty$. Using Equations (62), (64), and (65), we can explicitly show

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\alpha} dy \int_{-\infty}^{0} d\gamma w(\alpha, \beta, \gamma)$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{2|y|}^{\infty} d\gamma \tilde{w}(x, y, z) = 1. \quad (33)$$

3.2. Integral Expression and Numerical Integration

We define $\sigma$ as the standard deviation of $\delta_L(t_0)$, which is given by

$$\sigma^2 = \delta_L^2(t_0) = (\alpha + \beta + \gamma)^3 \left( \frac{h}{a} \right)^2 \left( t_0 \right) = 5\sigma_3^2, \quad (34)$$

where we have used Equations (28) and (29). The production probability of primordial black holes is given by

$$\beta_0 = \int_0^\infty da \int_0^\alpha d\beta \int_0^\beta d\gamma \theta(1 - h(\alpha, \beta, \gamma))$$

$$\times w(\alpha, \beta, \gamma), \quad (35)$$

where we have used Equation (26). Note that we still allow both a large deviation from spherical symmetry and a large amplitude for density perturbations. Note also that, in principle, one can add a condition $\alpha + \beta + \gamma > 0$, i.e., $\delta_L > 0$ for black hole formation. Although this gives another factor $\theta(\alpha + \beta + \gamma)$ in the integrand in Equation (35), it generally results in very little change in the numerical value of $\beta_0$ and hence we proceed without this condition. We have numerically implemented the triple integration to obtain $\beta_0$ and present the result in Figure 1. We can see that $\beta_0$ is a monotonically increasing function of $\sigma$. For small $\sigma$, $\beta_0$ tends to be proportional to $\sigma^5$ and is best fit by

$$\beta_0 \simeq 0.056\sigma^5. \quad (36)$$

As $\sigma$ increases beyond 0.01, $\beta_0$ becomes significantly larger than the above power-law formula.

3.3. Analytic Estimate for $\sigma \ll 1$

We can show that one of the three integrals in expression (35) can be analytically implemented. If the constraint $\sigma \ll 1$ is imposed, we can obtain a semi-analytic estimate of $\beta_0$ by a single numerical integration. We here present the semi-analytic formula

$$\beta_0 \simeq 0.05556\sigma^5. \quad (37)$$

This confirms the best-fit curve (36) for small $\sigma$. In the same framework, we can analytically derive lower and upper bounds on $\beta_0$ so that

$$0.01338\sigma^5 \lesssim \beta_0 \lesssim 0.1280\sigma^5. \quad (38)$$

Since the derivation of the above results is rather technical, we postpone to describe it in the Appendix B. Instead, we will give a comprehensive argument below, which provides us with a physical picture of black hole formation as well as the analytic lower and upper bounds on $\beta_0$. Since $E(e)$ is a monotonically decreasing function of $e \in [0, 1]$, Equation (27) implies

$$h(\alpha, \beta, \gamma) \geq \frac{2}{\pi} \frac{\alpha - \gamma}{\alpha^2}. \quad (39)$$

Therefore, the formation criterion $h \lesssim 1$ implies

$$\alpha - \gamma \lesssim \frac{\pi}{2} \alpha^2. \quad (40)$$
As we can see in Equation (32), the probability that \( x, y, \) or \( z \) take a value much larger than \( \sigma = \sigma / \sqrt{5} \) is exponentially suppressed. Then, for \( \sigma \ll 1 \), it is most probable that \( (\alpha + \beta + \gamma) = O(\sigma), (\alpha - \gamma) = O(\sigma), \) and \( (\alpha - \beta) - (\beta - \gamma) = O(\sigma) \). This means \( \alpha = O(\sigma), \beta = O(\sigma), \) and \( \gamma = O(\sigma) \). Therefore, Equation (40) immediately implies \( \alpha - \gamma = O(\sigma^2) \), and, hence, \( \alpha - \beta = (\sigma^2) \) and \( \beta - \gamma = O(\sigma^2) \), since \( \alpha \geq \beta \geq \gamma \). Then, \( \alpha, \beta, \) and \( \gamma \) are equal to each other to \( O(\sigma) \). Therefore, the collapse is predominantly nearly spherically symmetric to \( O(\sigma) \). In this case, \( \alpha > 0 \) implies \( x > 0 \) to \( O(\sigma) \).

Although the collapse is nearly spherical, it does not mean that the pancake is nearly circular but the size of the pancake is \( \sim \sigma f_0 \). The elliptic integral in Equation (27) for \( h(\alpha, \beta, \gamma) \) makes it very difficult to directly integrate Equation (35). Here, we estimate the integral by using the inequality \( E(0) \geq E(e) \geq E(1) \) for \( 0 \leq e \leq 1 \). In the circular limit \( e = 0 \), we can approximate \( h \) as

\[
\beta_0 \simeq \int_0^\infty dx \int_0^{x^2} dy \int_{2|y|}^{y^2} dz \bar{w}(x, y, z)
\]

\[
\simeq \frac{7}{2^4 \cdot 3^6 \sqrt{10\pi}} \sigma^5 \simeq 0.01338 \sigma^5,
\]

where we have used the approximation \( x^2 \ll \sigma \) and Equation (63). In the eccentric limit \( e = 1 \) of the pancake, \( \beta_0 \) can be similarly given by

\[
\beta_0 \simeq \int_0^\infty dx \int_0^{x^2} dy \int_{2|y|}^{x^2} dz \bar{w}(x, y, z)
\]

\[
\simeq \frac{7}{2^4 \cdot 3^6 \sqrt{10\pi}} \left( \frac{\pi}{2} \right)^5 \sigma^5 \simeq 0.1280 \sigma^5.
\]

Clearly, the circular and eccentric limits, (44) and (45), correspond to the lower and upper bounds, respectively. The semi-analytic formula (37) implies that highly eccentric pancakes give a significant contribution to the probability of black hole formation.

Note that there is a nonvanishing probability where \( \alpha, \beta, \) or \( \gamma \) takes a value of the order greater than \( \sigma \) and the resulting collapse is highly nonspherical. The semi-analytic formula (37) does not include this probability. We should recall that the full expression (35) includes all such cases. As seen in Figure 1, for \( \sigma \lesssim 0.01 \), the semi-analytic formula (37) agrees with the numerical result very well. On the other hand, for \( \sigma \gtrsim 0.01 \), the numerical result is larger than the semi-analytic formula (37). This suggests that a highly nonspherical collapse results in black hole formation with a significant probability for \( \sigma \gtrsim 0.01 \).

For \( \sigma \ll 1 \), as the probability of black hole formation is dominated by near-spherical collapse, we can neglect gravitational radiation in the course of collapse because the energy gravitationally radiated away from the mass is suppressed by a factor of \( \sigma^4 \). On the other hand, in the violent relaxation phase on and after the first caustic, gravitational radiation can be significantly large. In the presence of a large velocity dispersion after the first caustic, it is highly nontrivial whether such gravitational radiation significantly affects the picture of the virialization. For the moment, we adopt the scenario where the mass once gets virialized through the violent relaxation and it may collapse to a black hole in a timescale much larger than the Hubble time. This scenario must be tested by numerical simulations and it is also very interesting to estimate gravitational radiation in the nonspherical formation of primordial black holes and virialized objects.

4. DISCUSSION

4.1. Comparison with the Khlopov–Polnarev Formula

It is important to compare the present result with the previous result in the literature. The Khlopov–Polnarev criterion for primordial black hole formation is given by (Khlopov & Polnarev 1980; Polnarev & Khlopov 1982; Khlopov 2010)

\[
\bar{s} \lesssim \chi,
\]

where they put

\[
\bar{s} = \max(|\alpha - \beta|, |\beta - \gamma|, |\gamma - \alpha|) \quad \text{and} \quad \chi = \frac{r_g}{r_f} \approx b_3(t_f).
\]

From the above, they derived the probability for primordial black hole formation as

\[
\beta_0 = \frac{27}{8 \sqrt{5} \pi} \int_0^\infty d\sigma e^{-\pi \sigma^2} \int_{0-\alpha}^\infty (\alpha - \beta) d\beta \times \int_{0-\alpha}^\beta (\alpha - \gamma)(\beta - \gamma) d\gamma \simeq 0.02 \chi^5.
\]

Let us compare our new estimate with the Khlopov–Polnarev one. We should first note that the new estimate is written in terms of the density fluctuation \( \sigma \), while the Khlopov–Polnarev one is written in terms of \( \chi \), which is dealt with as if it were a definitive value in Khlopov & Polnarev (1980) and Polnarev & Khlopov (1982). In spite of this critical conceptual difference, it is very curious that the new formula (37) for \( \sigma \lesssim 1 \) agrees with the Khlopov–Polnarev one (48) only within a factor of three if we simply identify \( \chi \) with \( \sigma \). However, we would like to emphasize that the present analysis can deal with a large deviation from spherical symmetry and a large fluctuation \( \sigma \) by adopting the hoop conjecture as the black hole formation criterion.

Here, we discuss an inhomogeneity effect on primordial black hole formation. Khlopov and Polnarev (Khlopov & Polnarev 1980; Polnarev & Khlopov 1982) argued that if the central concentration within the overdense region is sufficiently high, a caustic, or a shell-focusing singularity in modern terminology, occurs at the center of the mass before a black hole horizon is formed. They assumed that the equation of state then changes to that of radiation \( p = \rho c^2 / 3 \) due to the rise of the density in the central region and an arising pressure gradient would prevent the collapse from being a black hole. If the interaction between particles is not sufficiently strong, they assumed that particles escaping from the central region would also prevent black hole formation. According to the above
argument, they put another factor $\chi^{3/2}$ to the probability based on the analysis of the Lemaître–Tolman–Bondi dust solution. They finally obtained $\beta_0 \simeq 0.02\chi^{13/2}$. This can be recast in the form $\beta_0 \sim \sigma^{13/2}$ in terms of $\sigma$ in our formulation. Although it is hard to generally exclude such a scenario, we only point out here that such an effect, if any, is highly dependent on the matter model. It is also likely that, even if pressure arises in the central region, its gradient just slows down the collapse of the central region and eventually a black hole horizon forms as the surrounding layers fall down and accumulate on the central region. In the current paper, we focus on the nonspherical effect and simply neglect this model-dependent inhomogeneity effect.

### 4.2. Comparison with the Production Rate in the Radiation-Dominated Era

In the presence of relativistic pressure $p = w/\rho c^2$, where $w = 1/3$ for radiation, the criterion of primordial black hole formation is predominantly determined by the Jeans scale argument. Following Harada et al. (2013, 2015), let us describe the criterion by using the density perturbation $\delta$ at horizon entry in the comoving slicing in the long-wavelength limit, where $\delta$ is defined as

$$\delta := \lim_{\epsilon \to 0} e^{-2\delta}$$

and $\epsilon = ck/(aH)$. Note that $\delta$ is the density perturbation averaged over the comoving coordinate scale $k^{-1}$ and is proportional to $\epsilon^2$ for a fixed $k$. By using the threshold value $\delta_c$, the criterion for black hole formation is given by $\delta \gtrsim \delta_c$, where

$$\delta_c \simeq \frac{3}{2w + 5} \sin^2 \left( \frac{\pi \sqrt{w}}{1 + 3w} \right),$$

(50)

while the maximum value for $\delta$ is given by

$$\delta_{\text{max}} \simeq \frac{3}{2w + 5},$$

(51)

if we neglect a relatively minor dependence on the density profile. Although the above formula is based on spherical symmetry, nonspherical effects are expected to be subdominant for the relativistic pressure $w = O(1)$, because $\delta_c = O(1)$ and there is little time left for nonsphericity to sufficiently grow, even if it grows, until a black hole horizon forms. If the density perturbation obeys a Gaussian distribution, the production rate of primordial black holes is given by (Carr 1975; Harada et al. 2013)

$$\beta_0 \simeq \int_{\delta_c}^{\delta_{\text{max}}} \frac{d\delta}{2\sqrt{2\pi} \sigma} \exp \left( -\frac{\delta^2}{2\sigma^2} \right) = \text{erfc} \left( \frac{\delta_c}{\sqrt{2}\sigma} \right) - \text{erfc} \left( \frac{\delta_{\text{max}}}{\sqrt{2}\sigma} \right) \approx \frac{\sqrt{2}}{\pi} \frac{\sigma}{\delta_c} \exp \left( -\frac{\delta_c^2}{2\sigma^2} \right),$$

(52)

where $\sigma^2 \equiv \langle \delta^2 \rangle$, the factor two comes from the Press–Schechter argument, $\text{erfc}(x)$ is the complementary error function and the tightest expression is valid only for $(\delta_{\text{max}} - \delta_c)/\sigma \gg 1$ and $\delta_c/\sigma \gg 1$. We plot the second left expression in Equation (52) as a function of $\sigma$ in the radiation-dominated era in Figure 1 with the green line, where $w = 1/3$, $\delta_c \simeq 0.4135$ and $\delta_{\text{max}} = 2/3$ are chosen.

We can see that the production probability in the radiation-dominated era is smaller than that in the matter-dominated era for $\sigma \lesssim 0.05$. This can be understood as an effect of the relativistic pressure, which suppresses the collapse of the overdense region to a black hole. On the other hand, for $\sigma \gtrsim 0.05$, the graph seems to show that the production probability in the matter-dominated era is smaller than that in the radiation-dominated era. Although we should be cautioned that nonspherical effects in the radiation-dominated era are simply neglected in Equation (52), we can conclude that the production of primordial black holes in the matter-dominated era can be suppressed by the nonspherical effect so strongly that the production rate can be as small as or smaller than that in the radiation-dominated era for $\sigma \gtrsim 0.05$. To clarify this issue, it will be important to investigate nonspherical effects on primordial black hole formation in the presence of relativistic pressure. It should also be noted that as indicated by Kopp et al. (2011), there is a serious problem in the conventional assumption of the Gaussian distribution for the density perturbation because of the presence of its maximum value and its double-valued nature, in particular for large amplitude of fluctuation. It is also important to investigate this issue further.

### 4.3. Density Perturbation in Terms of Primordial Curvature Perturbation

Here, let us first briefly introduce the relativistic cosmological perturbation theory based on Lyth et al. (2005) and Harada et al. (2015). In the long-wavelength limit $\epsilon \to 0$, the density perturbation $\delta$ can be written in the comoving slicing as (Harada et al. 2015)

$$\delta = -\frac{4(1 + w)}{3w + 5} \frac{c^2}{a^2 H^2} \frac{\Delta \psi}{\psi^5}$$

(53)

for the equation of state $p = w\rho c^2$, where $\Delta$ is the Laplacian in the flat three-space, the curvature variable $\psi$ is defined so that the spatial metric $\gamma_{ij}$ is written in a form $\gamma_{ij} = \psi^4 a^2 \delta_{ij}$ by introducing the conformal spatial metric $\tilde{\gamma}_{ij}$ whose determinant is equal to that of the metric in the flat three-space, and the decaying mode is neglected. In the long-wavelength limit, $\psi$ takes an identical value, whether it is described in the comoving slicing, uniform-density slicing, or constant-mean-curvature slicing, and is also conserved in the above slicings for adiabatic perturbation.

The curvature perturbation $\zeta$ can be defined by

$$e^{-2\zeta} = \psi^4$$

under the uniform-density slicing condition (Lyth et al. 2005). This $\zeta$ is often used for calculating primordial cosmological perturbations generated in inflation. In terms of $\zeta$, Equation (53) is rewritten in the form

$$\delta = -\frac{4(1 + w)}{3w + 5} \frac{c^2}{a^2 H^2} e^{\Delta \zeta} \Delta \psi / \psi^5,$$

(55)

We can calculate $\delta_k$ given by Equation (49) and its variance by $\sigma_k^2 = \langle \delta_k^2 \rangle$, where the comoving wave number $k$ is explicitly indicated. Note that the above argument so far does not require $\zeta$ to be small. If we additionally linearize Equation (55) with respect to $\zeta$, we find

$$\delta_k = 2 \left( \frac{1 + w}{3w + 5} \right) \zeta_k \bigg|_{k = aH(t_c)}$$

(56)
The probability distribution reduces to
\[ \sigma_k^2 = 4 \left(\frac{1 + w}{3w + 5}\right)^2 \langle \zeta_k^2 \rangle_{k=aH(t)} . \]  

(57)

Noting that the Zel’dovich approximation in the matter-dominated phase correctly describes the density perturbation in the linear regime in the Newtonian gravity and this coincides with that in the relativistic perturbation theory in the comoving slicing both at subhorizon and superhorizon scales (Peebles 1980; Hwang et al. 2012), we can adopt an identification
\[ \delta_k(t_i) = \delta(t_i) . \]  

(58)

Therefore, we can identify the density perturbation in the Newtonian cosmology with that in the comoving slicing in the general relativistic perturbation theory in the linear regime. In fact, in the Newtonian gauge with the Newtonian potential \( \phi \), we can find \( \zeta = (5/3) \phi/c^2 \) in the matter-dominated phase of the universe in the linear regime. By linearizing Equation (55), we can recover the Poisson equation both in subhorizon and superhorizon scales as
\[ \frac{1}{a^2} \Delta \phi = 4\pi G \rho \delta . \]  

(59)

In summary, in the radiation-dominated phase, Equation (57) with \( w = 1/3 \) reduces to
\[ \sigma_k^2 = \frac{16}{81} \langle \zeta_k^2 \rangle_{k=aH(t)} , \]  

(60)

where \( \sigma_k \) is the standard deviation of the density perturbations at horizon entry in the linear regime in the comoving slicing in the general relativistic perturbation theory. In the matter-dominated phase, Equation (57) with \( w = 0 \) reduces to
\[ \sigma_k^2 = \frac{4}{25} \langle \zeta_k^2 \rangle_{k=aH(t)} , \]  

(61)

and \( \sigma_k \) can be identified with the standard deviation of the density perturbations at horizon entry in the linear regime in the Newtonian cosmology.

5. CONCLUSION

We have studied primordial black hole formation in the matter-dominated era of the universe. In this epoch, in the absence of relativistic pressure, nonspherical effects play a crucial role and gravitational collapse does not necessarily lead to black hole formation. We have applied the Zel’dovich approximation to the nonlinear evolution of density perturbations in an expanding universe, Thorne’s hoop conjecture for the formation of a black hole horizon, and Doroshkevich’s probability distribution for the eigenvalues of the deformation tensor. We have succeeded in obtaining an integral expression for the probability of black hole formation, which allows for a large fluctuation of density perturbations and a large deviation from spherical symmetry and a large fluctuation of density perturbations. Both the integral expression and semi-analytic formula are applicable for the estimate of abundance of primordial black holes in the matter-dominated era of the universe, such as the first-order phase transition, the ending phase of inflation before reheating and the late-time matter-dominated era following the matter-radiation equality. We have compared the new estimate of the production rate with that in the radiation-dominated phase of the universe and found that the matter dominance strongly enhances primordial black hole formation for small density fluctuation, while it does not for larger density fluctuation. We have also presented a formula that gives the initial density fluctuation in terms of primordial curvature perturbation.

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APPENDIX A
FORMULAS OF GAUSSIAN INTEGRALS

We present the following well-known formulas for Gaussian integrals,
\[ \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = \frac{(2n-1)!!}{(2\alpha)^n} \sqrt{\pi / \alpha} , \]  

(62)

\[ \int_{0}^{\infty} x^{2n} e^{-\alpha x^2} dx = \frac{1}{2} \frac{(2n-1)!!}{(2\alpha)^n} \sqrt{\pi / \alpha} , \]  

(63)

\[ \int x^n e^{-\alpha x^2} dx = -\frac{1}{2\alpha} e^{-\alpha x^2} , \]  

(64)

\[ \int x^3 e^{-\alpha x^2} dx = -\frac{1}{2\alpha} (1 + \alpha x^2) e^{-\alpha x^2} , \]  

(65)

\[ \int x^3 e^{-\alpha x^2} dx = -\frac{1}{\alpha^2} (2 + 2\alpha x^2 + \alpha^2 x^4) e^{-\alpha x^2} , \]  

(66)

where \( \alpha > 0 \), \( n \) is a nonnegative integer and we have put \( 0!! = (-1)!! = 1 \).

APPENDIX B
ESTIMATE OF THE INTEGRAL IN \( \beta_0 \)

To estimate \( \beta_0 \), we further change the variables from \((x, y, z)\) given by Equation (31) to \((t, u, z)\) defined as
\[ t = \frac{x}{z} , \quad u = \frac{y}{z} , \quad z = z \]  

(67)

where the domain is given by \(-\infty < t < \infty , -1/2 < u < 1/2 , 0 < z < \infty \). The probability distribution is rewritten in the form
\[ \tilde{w}(t, u, z) dt du dz = \frac{27}{\sqrt{5 \pi \sigma_3^5}} (2u - 1)(2u + 1) z^5 \times \exp[-A(t, u) z^2] dt du dz , \]  

(68)

where
\[ A(t, u) := \frac{9}{10} \left( \frac{t}{\sigma_3} \right)^2 + 2 \left( \frac{u}{\sigma_3} \right)^2 + \frac{3}{2} \left( \frac{1}{\sigma_3} \right)^2 , \]  

(69)
Since
\[ h = \frac{4}{\pi^2} \left( t + \frac{2}{3}u + 1 \right)^2 E \left( \sqrt{1 - \left( u + \frac{1}{2} \right)^2} \right), \] (70)
the criterion \( h < 1 \) can be written in the following form.
\[ z > z_\ast(t, u) := \frac{4}{\pi} \left( t + \frac{2}{3}u + 1 \right)^2 E \left( \sqrt{1 - \left( u + \frac{1}{2} \right)^2} \right). \] (71)

We also find that \( \alpha > 0 \) implies \( t > -1 - (2/3)u \). Therefore, \( \beta_0 \) can be calculated as follows.
\[
\beta_0 = -\frac{27}{5} \int_{-1/2}^{1/2} du (2u - 1)(2u + 1) \times \int_{-1/2}^{1/2} dz e^{-Az^2} \\
= \frac{27}{5} \int_{-1}^{1} du \int_{-1/2}^{1/2} d\tilde{z} e^{-Az^2} \\
\] where we have used Equation (66) and omitted the arguments \( t, \) of \( A \) and \( z_\ast \). Thus, the integration with respect to \( \tilde{z} \) is done.

For \( \sigma \ll 1 \), we can find that the dominant contribution in the \( t \)-integral comes from the region \( t > \sigma^{-1} \) because the contribution from the outside of this region is exponentially suppressed. For this region, when \( t \gg 1 \), we find
\[ A \approx \frac{9}{10} \frac{t}{\sigma^2}, \quad z_\ast \approx \frac{4 E}{\pi t^2}, \quad \text{and} \quad \frac{A z^4}{\sigma^3} \approx \frac{72}{5 \pi^2 \sigma^3} t^2, \] (73)
where we have omitted the argument of \( E \). Since the contribution to the integral with respect to \( t \) in Equation (72) from the interval \([-1 - (2/3)u, 0]\) is negligible, changing the variable from \( t \) to \( w = 1/t \), we obtain
\[
\beta_0 \approx -\frac{8 \sqrt{5}}{27 \pi^2 \sigma^3} \int_{-1/2}^{1/2} du (2u - 1)(2u + 1) \int_0^\infty dw w^4 \\
\times (25 \pi^4 s^3 + 360 \pi^2 \sigma^3 w^2 E^2) \\
+ 2592 w^{-4} \exp \left( -\frac{72 w^2 E^2}{5 \pi^2 \sigma^3} \right) \\
= \frac{7 \cdot 5^{9/2} \pi^9 \sigma^2}{2^{10} \cdot 3^{5/2} \sqrt{10}} \int_{-1/2}^{1/2} du (1 - 2u)(1 + 2u) E^{-5} \\
= \frac{7 \cdot 5^{9/2} \pi^9 \sigma^2}{2^{9} \cdot 3^{6/10}} E^{-5} \sigma^5, \] (74)
where we have used \( \sigma = \sqrt{5} \sigma_3 \) and Equation (63) and defined \( E \) as
\[ E^{-5} = \frac{3}{2} \int_{-1}^{1} du (1 - 2u) \\
(1 + 2u) E^{-5} \left( \sqrt{1 - \left( u + \frac{1}{2} \right)^2} \right). \] (75)
The direct numerical integration with respect to \( u \) gives Equation (37) with \( \tilde{E} \approx 1.182 \), while it is clear that if we replace \( \tilde{E} \) in the last expression in Equation (74) with the circular limit value \( \pi/2 \) and eccentric limit value 1, we obtain the same lower and upper bounds as are given by Equations (44) and (45), respectively.

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