ON HOFFMAN’S CONJECTURAL IDENTITY

MINORU HIROSE, NOBUO SATO

Abstract. In this paper, we shall prove the equality
\[ \zeta(3, \{2\}^n, 1, 2) = \zeta(\{2\}^{n+3}) + 2\zeta(3, 3, \{2\}^n) \]
conjectured by Hoffman using certain identities among iterated integrals on \( \mathbb{P}^1 \setminus \{0, 1, \infty, z\} \).

1. Introduction

Multiple zeta values, or MZVs in short, are real numbers defined by
\[ \zeta(k_1, \ldots, k_d) = \sum_{0 < n_1 < \cdots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}. \]

In February 2000, Hoffman proposed the following conjectural formula on his homepage [5]:
\[ \zeta(3, \{2\}^n, 1, 2) = \zeta(\{2\}^{n+3}) + 2\zeta(3, 3, \{2\}^n) \quad (n \in \mathbb{Z}_{\geq 0}). \]

Here, we used the simplified notations, e.g. \( \zeta(3, \{2\}^n, 1, 2) = \zeta(3, 2, \ldots, 2, 1, 2) \). In [5], Hoffman also mentioned that J. Vermaeren had checked his conjecture up to \( n = 8 \) by using the MZV data mine presented in [1]. In [2] or [3], Charlton proved the existence of a rational number \( q_n \) such that
\[ \zeta(3, \{2\}^n, 1, 2) = q_n \zeta(\{2\}^{n+3}) + 2\zeta(3, 3, \{2\}^n) \quad (n \in \mathbb{Z}_{\geq 0}) \]
by calculating Brown’s operators \( D_r \)’s. However, the method based on Brown’s operators does not seem to provide a way to determine the value of \( q_n \). Thus, in this paper we give a proof of Hoffman’s conjecture by using a completely different approach based on the iterated integrals on \( \mathbb{P}^1 \setminus \{0, 1, \infty, z\} \). More generally, we shall prove the following theorem.

Theorem 1. For \( m, s \in \mathbb{Z}_{\geq 1} \) and \( n \in \mathbb{Z}_{\geq 0} \),
\[ \zeta(\{2\}^{m-1}, 3, \{2\}^n, 1, \{2\}^s) = \zeta(\{2\}^{n+m+s+1}) + \zeta(\{2\}^{s-1}, 3, \{2\}^{m-1}, 3, \{2\}^n) + \zeta(\{2\}^{m-1}, 3, \{2\}^{s-1}, 3, \{2\}^n). \]

This theorem together with the duality relation gives a special case of Charlton’s generalized cyclic insertion conjecture (see [2, Section 2] or [3]). In particular, the case \( m = s = 1 \) of the theorem gives Hoffman’s conjecture.

In Section 2 we briefly review the iterated integrals on \( \mathbb{P}^1 \setminus \{0, 1, \infty, z\} \) which was investigated in [4] by Iwaki, Tasaka and the authors. In Section 3 we give a proof of Theorem 1 using a lemma in Section 2.

The authors thank Michael E. Hoffman for kindly letting us know when the conjecture was formulated.
2. A REVIEW OF ITERATED INTEGRALS ON $\mathbb{P}^1 \setminus \{0, 1, \infty, z\}$

In this section, we review the differential formula for the iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty, z\}$ given in \[1\]. Let $\mathcal{A} := \mathbb{Q}\langle e_0, e_1, e_z \rangle$ denote the non-commutative polynomial algebra generated by three indeterminates $e_0, e_1$ and $e_z$ over $\mathbb{Q}$. Let $\mathcal{A}^1$ denote the subalgebra of $\mathcal{A}$ defined by

$$\mathcal{A}^1 := \mathbb{Q} + \mathbb{Q} e_z + e_1 \mathcal{A} e_0 + e_z \mathcal{A} e_0 + e_1 \mathcal{A} e_z + e_z \mathcal{A} e_z.$$  

We assume that $z \in \mathbb{C} \setminus [0, 1]$ and define a linear function $L$ on $\mathcal{A}^1$, by assigning a word $w = e_{a_1} \cdots e_{a_m}$ in $e_0, e_1$ and $e_z$, the iterated integral

$$I(0; a_1, \ldots, a_m; 1) = \int \prod_{i=1}^{m} \frac{dt_i}{t_i - a_i}.$$  

Here, the path of integration is taken as a line segment between 0 and 1. Note that, for $w \in \mathcal{A}^1$, $L(w)$ defines a holomorphic function of $z$ on $\mathbb{C} \setminus [0, 1]$. The MZVs are, then, expressed as

$$\zeta(k_1, \ldots, k_d) = (-1)^d L(e_1 e_0^{k_1-1} \cdots e_1 e_0^{k_d-1}).$$  

Let $\partial_z,0$ and $\partial_z,1$ be linear operators on $\mathcal{A}^1$ defined by

$$\partial_z,b(e_{a_1} \cdots e_{a_m}) := \sum_{i=1}^{m} \left( \delta_{\{a_i,a_{i+1}\},\{z,b\}} - \delta_{\{a_{i-1},a_i\},\{z,b\}} \right) e_{a_1} \cdots \hat{e}_{a_i} \cdots e_{a_m}$$  

for $b \in \{0,1\}$. Here, we set $a_0 = 0$, $a_{n+1} = 1$ and $\delta_{\{a,b\},\{c,d\}}$ denotes the Kronecker delta, i.e.

$$\delta_{\{a,b\},\{c,d\}} = \begin{cases} 1 & \text{if } \{a,b\} = \{c,d\} \\ 0 & \text{if } \{a,b\} \neq \{c,d\}. \end{cases}$$  

Note that $\partial_z,0$ and $\partial_z,1$ reduce the word length by 1. The following is a fundamental property of $\partial_z,0$ and $\partial_z,1$.

**Proposition 2** (\[1\]). For $w \in \mathcal{A}^1$,

$$\frac{d}{dz} L(w) = \frac{1}{z} L(\partial_z,0 w) + \frac{1}{z - 1} L(\partial_z,1 w).$$  

Let $\mathcal{I} := e_z \mathcal{A} \subset \mathcal{A}$ be the two-sided ideal generated by $e_z$. The following lemma is useful.

**Lemma 3.** If $w \in \mathcal{I} \cap \mathcal{A}^1$ and $L(\partial_z,0 w) = L(\partial_z,1 w) = 0$, then, $L(w) = 0$.

**Proof.** Since $L(\partial_z,0 w) = L(\partial_z,1 w) = 0$, $\frac{d}{dz} L(w) = 0$ by Proposition 2 and so $L(w)$ is a constant. Since $w \in \mathcal{I} \cap \mathcal{A}^1$, $\lim_{z \to \infty} (L(w)) = 0$, which means the constant is zero. \[\square\]

3. A PROOF OF HOFFMAN’S CONJECTURE

In this section, we prove Theorem \[1\]. Fix $s \in \mathbb{Z}_{>0}$. We set

$$A_n = (e_1 e_0)^n, \quad B_n = (e_1 e_0)^n e_1, \quad A_n = (e_0 e_1)^n, \quad B_n = (e_0 e_1)^n e_0.$$
and

\[ F_{ee}(m, n) := A_s \overline{B}_m e_z A_n + A_m e_z (A_s + \overline{A}_s) e_z A_n + A_m e_z B_n A_s - B_{s+m} e_z A_n - A_m e_z \overline{B}_{s+n} \]

\[ F_{oe}(m, n) := A_s \overline{A}_{m+1} e_z A_n + B_m e_z (A_s + \overline{A}_s) e_z A_n + B_m e_z B_n A_s - A_{s+m+1} e_z A_n - B_m e_z \overline{B}_{s+n} \]

\[ F_{eo}(m, n) := A_s \overline{B}_m e_z \overline{B}_n + A_m e_z (A_s + \overline{A}_s) e_z \overline{B}_n + A_m e_z \overline{A}_{n+1} A_s - B_{s+m} e_z \overline{B}_n - A_m e_z A_{s+n+1} \]

\[ F_{oo}(m, n) := A_s \overline{A}_{m+1} e_z \overline{B}_n + B_m e_z (A_s + \overline{A}_s) e_z \overline{B}_n + B_m e_z \overline{A}_{n+1} A_s - A_{s+m+1} e_z \overline{B}_n - B_m e_z A_{s+n+1} \]

for \( m, n \in \mathbb{Z}_{\geq 0} \). Additionally, we define

\[ F_{ee}(m, n) = F_{oe}(m, n) = F_{eo}(m, n) = F_{oo}(m, n) = 0 \]

if \( m < 0 \) or \( n < 0 \).

**Proposition 4.** For \( m, n \in \mathbb{Z}_{\geq 0} \), we have

\[ L(F_{ee}(m, n)) = L(F_{oe}(m, n)) = L(F_{eo}(m, n)) = L(F_{oo}(m, n)) = 0. \]

**Proof.** We shall prove the proposition by induction on the word length. For \( w \in A^1 \cap \mathbb{Q}(e_0, e_1) \), we set

\[ \Delta(w) = w - \tau(w), \]

where \( \tau \) is an anti-automorphism defined by \( \tau(e_0) = -e_1 \) and \( \tau(e_1) = -e_0 \). Since

\[ \partial_{z,0} (A_s \overline{B}_m e_z A_n) = A_s \overline{A}_m e_z A_n - A_s \overline{B}_{m+n}, \]

\[ \partial_{z,0} (A_m e_z (A_s + \overline{A}_s) e_z A_n) = \begin{cases} B_{m-1} e_z (A_s + \overline{A}_s) e_z A_n - A_{s+m} e_z A_n - A_m e_z A_{s+n} & \text{for } m > 0, \\
-A_s e_z A_n - e_z A_{s+n} & \text{for } m = 0, \end{cases} \]

\[ \partial_{z,0} (A_m e_z B_n A_s) = \begin{cases} B_{m-1} e_z B_n A_s + \tau(A_s \overline{B}_{m+n}) & \text{for } m > 0, \\
\tau(A_s \overline{B}_{m+n}) & \text{for } m = 0, \end{cases} \]

\[ \partial_{z,0} (-B_{s+m} e_z A_n) = 0, \]

\[ \partial_{z,0} (-A_m e_z \overline{B}_{s+n}) = \begin{cases} -B_{m-1} e_z \overline{B}_{s+n} + A_m e_z A_{s+n} & \text{for } m > 0, \\
e_z A_{s+n} & \text{for } m = 0, \end{cases} \]

we find

\[ \partial_{z,0} F_{ee}(m, n) = F_{oe}(m-1, n) - \Delta(A_s \overline{B}_{m+n}) \]

for \( m, n \in \mathbb{Z}_{\geq 0} \). Similarly,

\[ \partial_{z,1} F_{ee}(m, n) = -F_{eo}(m, n-1) + \Delta(A_s \overline{B}_{m+n}), \]

\[ \partial_{z,0} F_{oe}(m, n) = 0, \]

\[ \partial_{z,1} F_{oe}(m, n) = F_{ee}(m, n) - F_{oo}(m, n-1), \]

\[ \partial_{z,0} F_{eo}(m, n) = -F_{ee}(m, n) + F_{oo}(m-1, n), \]

\[ \partial_{z,1} F_{eo}(m, n) = 0, \]

\[ \partial_{z,0} F_{oo}(m, n) = -F_{oe}(m, n) + \Delta(A_s \overline{B}_{m+n+1}), \]

\[ \partial_{z,1} F_{oo}(m, n) = F_{eo}(m, n) - \Delta(A_s \overline{B}_{m+n+1}). \]
Using Lemma 3 and the duality relation, i.e. \( L(\Delta(w)) = 0 \) for \( w \in \mathcal{A}^1 \cap \mathbb{Q} \langle e_0, e_1 \rangle \), we obtain the proposition.

By Proposition 4 it follows that

\[
0 = \lim_{z \to -0} L(F_{ee}(m, n)) = L(A_s B_m B_n + A_n B_s B_n + A_m A_{n+1} A_s - A_{s+m+n+1})
\]

\[
= (-1)^{s+m+n} \left\{ \zeta(\{2\}^{s-1}, 3, \{2\}^{m-1}, 3, \{2\}^n) + \zeta(\{2\}^{m-1}, 3, \{2\}^{s-1}, 3, \{2\}^n) \\
- \zeta(\{2\}^{m-1}, 3, \{2\}^n, 1, \{2\}^s) + \zeta(\{2\}^{s+m+n+1}) \right\},
\]

which proves Theorem 1.

REFERENCES

[1] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren. The multiple zeta value data mine. Comput. Phys. Comm., 181(3):582–625, 2010.
[2] S. P. Charlton. Identities arising from coproducts on multiple zeta values and multiple polylogarithms. Durham theses, 2016. Available at Durham E-Theses Online: http://etheses.dur.ac.uk/11834/1/
[3] S. P. Charlton. The alternating block decomposition of iterated integrals, and cyclic insertion on multiple zeta values. preprint, 2017. arXiv:1703.03784v1 [math.NT].
[4] M. Hirose, K. Iwaki, N. Sato, and K. Tasaka. Sum/duality formulas for iterated integrals and multiple zeta values. preprint, 2017.
[5] M. E. Hoffman. Multiple Zeta Values. Hoffman’s info page. http://www.usna.edu/Users/math/meh/mult.html

(Minoru Hirose) MULTIPLE ZETA RESEARCH CENTER, KYUSHU UNIVERSITY
E-mail address: m-hirose@math.kyushu-u.ac.jp

(Nobuo Sato) DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY
E-mail address: saton@math.kyoto-u.ac.jp