On graphs admitting two disjoint maximum independent sets

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Abstract

An independent set \( S \) is maximal if it is not a proper subset of an independent set, while \( S \) is maximum if it has a maximum size. The problem of whether a graph has a pair of disjoint maximal independent sets was introduced by Berge in early 70’s. The class of graphs for which every induced subgraph admits two disjoint maximal independent sets was characterized in (Schaudt, 2015). It is known that deciding whether a graph has two disjoint maximal independent sets is a \( \mathbf{NP} \)-complete problem (Henning et al., 2009).

In this paper, we are focused on finding conditions ensuring the existence of two disjoint maximum independent sets.

Keywords: maximum independent set, shedding vertex, König-Egerváry graph, unicyclic graph, well-covered graph, corona of graphs.

1 Introduction

Throughout this paper \( G = (V, E) \) is a finite, undirected, loopless graph without multiple edges, with vertex set \( V = V(G) \) of cardinality \( |V(G)| = n(G) \), and edge set \( E = E(G) \) of size \( |E(G)| = m(G) \). If \( X \subset V \), then \( G[X] \) is the graph of \( G \) induced by \( X \). By \( G - U \) we mean the subgraph \( G[V - U] \), if \( U \subset V(G) \). We also denote by \( G - F \) the subgraph of \( G \) obtained by deleting the edges of \( F \), for \( F \subset E(G) \), and we write shortly \( G - e \), whenever \( F = \{e\} \).

The neighborhood \( N(v) \) of \( v \in V(G) \) is the set \( \{w : w \in V(G) \text{ and } vw \in E(G)\} \), while the closed neighborhood \( N[v] \) of \( v \) is the set \( N(v) \cup \{v\} \). Let \( \text{deg}(v) = |N(v)| \). If \( \text{deg}(v) = 1 \), then \( v \) is a leaf, and \( \text{Leaf}(G) \) is the set containing all the leaves.

The neighborhood \( N(A) \) of \( A \subset V(G) \) is \( \{v \in V(G) : N(v) \cap A \neq \emptyset\} \), and \( N[A] = N(A) \cup A \). We may also use \( N_G(v), N_G[v], N_G(A) \) and \( N_G[A] \), when referring to neighborhoods in a graph \( G \).

\( C_n \), \( K_n \), \( P_n \), \( K_{p,q} \) denote respectively, the cycle on \( n \geq 3 \) vertices, the complete graph on \( n \geq 1 \) vertices, the path on \( n \geq 1 \) vertices, and the complete bipartite graph on \( p + q \)
vertices, where \( p, q \geq 1 \).

A matching is a set \( M \) of pairwise non-incident edges of \( G \), and by \( V(M) \) we mean the vertices covered by \( M \). If \( V(M) = V(G) \), then \( M \) is a perfect matching. The size of a largest matching is denoted by \( \mu(G) \). If every vertex of a set \( A \) is an endpoint of an edge \( e \in M \), while the other endpoint of \( e \) belongs to some set \( B \), disjoint from \( A \), we say that \( M \) is a matching from \( A \) into \( B \), or \( A \) is matched into \( B \) by \( M \). In other words, \( M \) may be interpreted as an injection from the set \( A \) into the set \( B \).

The disjoint union \( G_1 \cup G_2 \) of the graphs \( G_1 \) and \( G_2 \) with \( V(G_1) \cap V(G_2) = \emptyset \) is the graph having \( V(G_1) \cup V(G_2) \) and \( E(G_1) \cup E(G_2) \) as a vertex set and an edge set, respectively. In particular, \( qG \) denotes the disjoint union of \( q \geq 2 \) copies of the graph \( G \).

A set \( S \subseteq V(G) \) is independent if no two vertices from \( S \) are adjacent, and by \( \text{Ind}(G) \) we mean the family of all the independent sets of \( G \). An independent set \( A \) is maximal if \( A \cup \{v\} \) is not independent, for every \( v \in V(G) - A \). An independent set of maximum size is a maximum independent set of \( G \), and \( \alpha(G) = \max\{|S| : S \in \text{Ind}(G)\} \).

**Theorem 1.1** [2] In a graph \( G \), an independent set \( S \) is maximum if and only if every independent set disjoint from \( S \) can be matched into \( S \).

Let \( \text{core}(G) = \bigcap\{S : S \in \Omega(G)\} \), where \( \Omega(G) \) denotes the family of all maximum independent sets [14].

**Theorem 1.2** [10] A connected bipartite graph \( G \) has a perfect matching if and only if \( \text{core}(G) = \emptyset \).

If \( \alpha(G) + \mu(G) = n(G) \), then \( G \) is a König-Egerváry graph [8, 28]. It is known that every bipartite graph is a König-Egerváry graph as well.

Let \( v \in V(G) \). If for every independent set \( S \) of \( G - N[v] \), there exists some \( u \in N(v) \) such that \( S \cup \{u\} \) is independent, then \( v \) is a shedding vertex of \( G \) [30]. Clearly, no isolated vertex may be a shedding vertex. On the other hand, every vertex of degree \( n(G) - 1 \) is a shedding vertex. Let \( \text{Shed}(G) \) denote the set of all shedding vertices. For instance, \( \text{Shed}(K_1) = \emptyset \), while \( \text{Shed}(K_n) = V(K_n) \) for every \( n \geq 2 \).

A vertex \( v \) of a graph \( G \) is simplicial if the induced subgraph of \( G \) on the set \( N[v] \) is a complete graph and this complete graph is called a simplex of \( G \). Clearly, every leaf is a simplicial vertex. Let \( \text{Simp}(G) \) denote the set of all simplicial vertices.

**Proposition 1.3** [30] If \( v \in \text{Simp}(G) \), then \( N(v) \subseteq \text{Shed}(G) \).

A graph \( G \) is said to be simplicial if every vertex of \( G \) belongs to a simplex of \( G \). By Proposition [12], if every simplex of a simplicial graph \( G \) contains two simplicial vertices at least, then \( \text{Shed}(G) = V(G) \). The converse is not necessarily true. For instance, \( C_5 \) has no simplicial vertex, while \( \text{Shed}(C_5) = V(C_5) \).

A vertex \( v \in V(G) \) is codominated if there is another vertex \( u \in V(G) \) such that \( N[u] \subseteq N[v] \). In such a case, we say that \( v \) is codominated by \( u \). For instance, consider the graphs \( G_1 \) and \( G_2 \) from Figure 11, \( x, z \in \text{Shed}(G_1) \), and both vertices are codominated, while \( w \in \text{Shed}(G_2) \) and \( w \) is not codominated.
Lemma 1.4 \[3\] Every codominated vertex is a shedding vertex as well. Moreover, in a bipartite graph, each shedding vertex is also a codominated vertex, and if \(x\) is codominated by \(y\), then \(y\) is a leaf.

Theorem 1.5 \[4\] If \(v \in \text{Shed}(G)\), then one of the following hold:

(i) there exists \(u \in N(v)\), such that \(N[u] \subseteq N[v]\), i.e., \(v\) is a codominated vertex;
(ii) \(v\) belongs to some \(5\)-cycle.

A graph is well-covered if all its maximal independent sets are also maximum \[25\]. If \(G\) is well-covered, without isolated vertices, and \(n(G) = 2\alpha(G)\), then \(G\) is a very well-covered graph \[8\]. The only well-covered cycles are \(C_3, C_4, C_5\) and \(C_7\), while \(C_4\) is the unique very well-covered cycle.

Theorem 1.6 \[19\] \(G\) is very well-covered if and only if \(G\) is a well-covered König-Egerváry graph.

Let \(H = \{H_v : v \in V(G)\}\) be a family of graphs indexed by the vertex set of a graph \(G\). The corona \(G \circ H\) of \(G\) and \(H\) is the disjoint union of \(G\) and \(H_v, v \in V(G)\), with additional edges joining each vertex \(v \in V(G)\) to all the vertices of \(H_v\). If \(H_v = H\) for every \(v \in V(G)\), then we denote \(G \circ H\) instead of \(G \circ H\) \[11\]. It is known that \(G \circ H\) is well-covered if and only if each \(H_v, v \in V(G)\), is a complete graph \[29\].

Recall that the girth of a graph \(G\) is the length of a shortest cycle contained in \(G\), and it is defined as the infinity for every forest.

Theorem 1.7 (i) \[10\] Let \(G\) be a connected graph of girth \(\geq 6\), which is isomorphic to neither \(C_7\) nor \(K_1\). Then \(G\) is well-covered if and only if \(G = H \circ K_1\) for some graph \(H\).

(ii) \[18\] Let \(G\) be a connected graph of girth \(\geq 5\). Then \(G\) is very well-covered if and only if \(G = H \circ K_1\) for some graph \(H\).

The graph \(P_3\) has two disjoint maximal independent sets, while \(C_4\) has even two disjoint maximum independent sets. On the other hand, the graph \(C_5 \circ K_1\) has no pair of disjoint maximal independent sets. The graphs from Figure 1 have pairs of disjoint
maximal (non-maximum) independent sets, while the graphs from Figure 2 have pairs of disjoint maximum independent sets.

The research on the graphs admitting two disjoint maximal independent sets has its roots in [1, 23]. Further, this topic was studied in [5, 7, 9, 13, 24, 26]. A constructive characterization of trees that have two disjoint maximal independent sets of minimum size may be found in [12].

By definition, for well-covered graphs to find out two disjoint maximum independent sets is the same as to detect two maximum independent sets.

Theorem 1.8 [26] Let $G$ be a well-covered graph without isolated vertices. If $G$ does not contain $C_{2k+1} \cup K_1$ as an induced subgraph for $k \geq 1$, then $G$ has two disjoint maximum independent sets.

The same problem in line graphs is about two disjoint maximum matchings.

Theorem 1.9 [16] A bipartite graph has two disjoint perfect matchings if and only if it has a partition of its vertex set comprising of a family of simple cycles.

The most well-known subclass of graphs with two disjoint maximum independent sets is the family of $W_2$-graphs. Recall that a graph $G$ belongs to $W_2$ if every two pairwise disjoint independent sets are included in two pairwise disjoint maximum independent sets [22, 27].

In this paper, we concentrate on graphs admitting two disjoint maximum independent sets.

2 General graphs

Theorem 2.1 For every graph $G$, the following assertions are equivalent:

(i) $G$ has two disjoint maximum independent sets;

(ii) there exists a maximum independent set $S$ such that $\alpha(G - S) = \alpha(G)$;

(iii) there exists a matching $M$ of size $\alpha(G)$ such that $G[V(M)]$ is a bipartite graph;

(iv) $G$ has an induced bipartite subgraph of order $2\alpha(G)$;

(v) there exists a set $A \subset V(G)$ such that $G - A$ is a bipartite graph having a perfect matching of size $\alpha(G)$.

Proof. (i) $\Leftrightarrow$ (ii) It is clear.

(i) $\Rightarrow$ (iii) Assume that $S_1$, $S_2$ are two disjoint maximum independent sets in $G$. Then $G[S_1 \cup S_2]$ is a bipartite subgraph in $G$, having, by Theorem 1.1, a perfect matching $M$ of size $\alpha(G)$. Clearly, $G[V(M)] = G[S_1 \cup S_2]$.

(iii) $\Rightarrow$ (i) Suppose that there exists a matching $M$ of size $\alpha(G)$ such that $G[V(M)]$ is a bipartite graph. Consequently, the bipartition $\{A, B\}$ of $G[V(M)]$ provides two disjoint maximum independent sets, namely $A$ and $B$.

(iii) $\Rightarrow$ (iv) It is clear.

(iv) $\Rightarrow$ (iii) Let $H = (A, B, U)$ be an induced bipartite subgraph of order $2\alpha(G)$. Hence, $|A| = |B| = \alpha(G)$, since $A$ and $B$ are independent sets in $G$. Moreover, Theorem 1.1 ensures a perfect matching of size $\alpha(G)$ in $H$.

(iii) $\Leftrightarrow$ (v) It is clear. ■
Corollary 2.2  If $G$ has two disjoint maximum independent sets, then $\mu(G) \geq \alpha(G)$.  

Proof. Let $S_1, S_2$ are two disjoint maximum independent sets in $G$. By Theorem 1.1 there exists some matching from $S_1$ into $S_2$. Since $|S_1| = |S_2| = \alpha(G)$, we infer that $\mu(G) \geq \alpha(G)$.

Theorem 2.3  Let $S \subseteq \text{Ind}(G)$ and $S \subseteq \text{Shed}(G)$, then  
(i) the number of independent sets of size $|S|$ in $G$ is greater or equal to $2^{|S|}$;  
(ii) there exist some maximal independent set $U$ disjoint from $S$, and a matching from $S$ into $U$.

Proof. Let $A = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subseteq S$. Now, we are constructing an independent set $I_A = (S - A) \cup B_A$ such that $S \cap B_A = \emptyset$ and $|A| = |B_A|$.

We start by taking a vertex $y_{i_1} \in N_G(x_{i_1})$, such that $I_{i_1} = (S - A) \cup \{y_{i_1}\}$ is independent. This is possible, because $x_{i_1}$ is a shedding vertex.

Further, let us consider the vertex $x_{i_j}$ for each $2 \leq j \leq k$. Then, there exists some $y_{i_j} \in N_G(x_{i_j})$, such that $y_{i_j}$ is not adjacent to any vertex of $I_{i_{j-1}}$, since $x_{i_j}$ is shedding and $I_{i_{j-1}}$ is independent. Hence, the set $I_{i_j} = I_{i_{j-1}} \cup \{y_{i_j}\}$ is independent.

Finally, $I_A = I_{i_k} = (S - A) \cup B_A$, where $B_A = \{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\}$. Clearly, $S \cap B_A = \emptyset$, and $\{x_{i_1}y_{i_1}, x_{i_2}y_{i_2}, \ldots, x_{i_k}y_{i_k}\}$ is a matching from $A$ into $B_A \subseteq I_A$.

Suppose $A_1, A_2 \subseteq S$. If $A_1 \neq A_2$, then $I_{A_1} = (S - A_1) \cup B_{A_1} \neq (S - A_2) \cup B_{A_2} = I_{A_2}$, since $S - A_1 \neq S - A_2$. In other words, every subset of $S$ produces an independent set of the same size, and all these sets are different. Thus the graph $G$ has $2^{|S|}$ independent sets of cardinality $|S|$, at least.

If $A = S$, then $I_S$ is disjoint from $S$. To complete the proof, one has just to enlarge $I_S$ to a maximal independent set, say $U$. The sets $U$ and $S$ are disjoint, since there is a matching from $S$ into $I_S \subseteq U$.

Corollary 2.4  If $G$ has a maximal independent set $S$ such that $S \subseteq \text{Shed}(G)$, then there exists a maximal independent set $U$ disjoint from $S$ such that $|S| \leq |U|$.

Corollary 2.5  If $G$ has a maximum independent set $S$ such that $S \subseteq \text{Shed}(G)$, then $|\Omega(G)| \geq 2^{\alpha(G)}$, while some $I \in \Omega(G)$ is disjoint from $S$.

The friendship graph $F_q = K_1 \circ qK_2, q \geq 2$ shows that $2^{\alpha(G)}$ is a tight lower bound for $|\Omega(G)|$ in graphs with a maximum independent set consisting of only shedding vertices.

Combining Theorem 2.3 and Corollary 2.2 we deduce the following.

Corollary 2.6  If $G$ has a maximum independent set $S \subseteq \text{Shed}(G)$, then $\mu(G) \geq \alpha(G)$.

Notice that each graph from Figure 2 has a maximum independent set containing only shedding vertices, and hence, by Theorem 2.3, each one has two disjoint maximum independent sets.

Corollary 2.7  If $p \geq 2$, then $G \circ K_p$ has two disjoint maximum independent sets.

Proof. Since $p \geq 2$, Proposition 1.3 implies that Shed$(G) = V(G)$. Further, the conclusion follows according to Corollary 2.5.

It is worth mentioning that if $G$ has a pair of disjoint maximum independent sets, it may have Shed$(G) = \emptyset$; e.g., $G = K_{n,n}$ for $n \geq 2$.  

5
3 König-Egerváry graphs

Theorem 3.1 \(G\) is a König-Egerváry graph with two disjoint maximum independent sets if and only if \(G\) is a bipartite graph having a perfect matching.

Proof. Let \(S_1, S_2 \in \Omega(G)\) and \(S_1 \cap S_2 = \emptyset\). Since \(G\) is a König-Egerváry graph and \(S_1 \subseteq V(G) - S_2\), we get

\[
\alpha(G) = |S_1| \leq |V(G) - S_2| = |V(G)| - \alpha(G) = \mu(G) \leq \alpha(G).
\]

It follows that \(S_1 = V(G) - S_2\) and \(\mu(G) = \alpha(G)\). Hence, \(G = (S_1, S_2, E(G))\) is a bipartite graph with a perfect matching.

The converse is evident.

Corollary 3.2 If \(G\) is a very well-covered graph having two disjoint maximum independent sets, then \(G\) is a bipartite graph with a perfect matching.

Proof. By Theorem 1.6, \(G\) is a König-Egerváry graph. Further, according to Theorem 3.1, \(G\) is a bipartite graph with a perfect matching.

The converse of Corollary 3.2 is not true; e.g., \(G = C_6\). It is worth mentioning that \(C_5\) is well-covered, non-bipartite, and has some pairs of disjoint maximum independent sets.

Corollary 3.3 The corona \(H \circ K_1\) has two disjoint maximum independent sets if and only if \(H\) is a bipartite graph.

Proof. By Corollary 3.2, \(H \circ K_1\) must be bipartite, because it is a very well-covered graph with two disjoint maximum independent sets. Hence, \(H\) itself must be bipartite as a subgraph of \(H \circ K_1\).

Conversely, \(H \circ K_1\) is a bipartite graph, because \(H\) is bipartite. Clearly, \(H \circ K_1\) has a perfect matching. Therefore, \(H \circ K_1\) has two disjoint maximum independent sets, in accordance with Theorem 3.1.

Evidently, \(C_5\) and \(C_7\) are well-covered and they both have disjoint maximum independent sets.

Corollary 3.4 Let \(G\) be a well-covered graph of girth \(\geq 6\) with \(K_1 \neq G \neq C_7\), or \(G\) be a very well-covered graph of girth \(\geq 5\). Then \(G\) has two disjoint maximum independent sets if and only if \(G\) is bipartite.

Proof. According to Theorem 1.1(i), (ii), \(G\) must be under the form \(G = H \circ K_1\). Now, the result follows by Corollary 3.3.
Clearly, the graphs $G_1, G_2$ from Figure 3 are well-covered. The graph $G_1$ has no pair of disjoint maximum independent sets, while $G_2$ has such pairs. The graph $G_3$ from Figure 3 is not even well-covered, but it has some pairs of disjoint maximum independent sets.

It is known that $G$ is a König-Egerváry graph if and only if every maximum matching matches $V(G) - S$ into $S$, for each $S \in \Omega(G)$ [21].

**Theorem 3.5** If $G$ is a König-Egerváry graph, then $|\Omega(G)| \leq 2^{\alpha(G)}$. Moreover, the equality $|\Omega(G)| = 2^{\alpha(G)}$ holds if and only if $G = \alpha(G) K_2$.

**Proof.** Let $S \in \Omega(G)$ and $M_G$ be a maximum matching of $G$. Then $|V(G) - S| = |M_G| = \mu(G)$ and each maximum matching of $G$ matches $V(G) - S$ into $S$, since $G$ is a König-Egerváry graph. Thus every maximum independent set different from $S$ must contain vertices belonging to $V(G) - S$. Let us define a graph $H$ as follows: $V(H) = S \cup (V(G) - S) \cup A$, where $A$ is comprised of $|S| - |V(G) - S|$ new vertices, while $E(H) = M$, where $M$ is a perfect matching that matches $S$ into $(V(G) - S) \cup A$ and $M_G \subseteq M$. In the other words, $H = |S| K_2 = \alpha(G) K_2$. Then $\Omega(G) \subseteq \Omega(H)$, and hence, we infer that $|\Omega(G)| \leq |\Omega(H)| = 2^{\alpha(G)}$, as required.

Clearly, $H$ is well-covered. Consequently, every vertex of $H$ is contained in some maximum independent set, and adding an edge to $E(H)$ reduces the number of maximum independent sets.

Suppose $|\Omega(G)| = 2^{\alpha(G)}$. Then $\Omega(G) = \Omega(H)$, since $\Omega(G) \subseteq \Omega(H)$.

First, $V(G) = V(H)$, because, otherwise, if there exists some vertex $x \in V(H) - V(G)$, then each maximum independent set of $H$ containing $x$ does not appear in $\Omega(G)$, in contradiction with $\Omega(G) = \Omega(H)$.

Second, $E(G) = E(H)$, since otherwise, if there is an edge $xy \in E(G) - E(H)$, then each maximum independent set of $H$ containing $\{x, y\}$ does not appear in $\Omega(G)$, in contradiction with $\Omega(G) = \Omega(H)$.

If $G = \alpha(G) K_2$, then clearly, $|\Omega(G)| = 2^{\alpha(G)}$. ■

**Theorem 3.6** For a König-Egerváry graph $G$, the following assertions are equivalent:

(i) there is a maximum independent set included in Shed$(G)$;

(ii) $|\Omega(G)| = 2^{\alpha(G)}$;

(iii) Shed$(G) = V(G)$;

(iv) every maximum independent set is included in Shed$(G)$;

(v) there exist two disjoint maximum independent sets included in Shed$(G)$;

(vi) $G = \alpha(G) K_2$.

**Proof.** (i) $\Rightarrow$ (ii) By Corollary 2.5 and Theorem 3.5

(ii) $\Rightarrow$ (iii) By Theorem 3.5 every vertex of $G$ is a leaf. Hence, Shed$(G) = V(G)$.

(iii) $\Rightarrow$ (iv) Evident.

(iv) $\Rightarrow$ (v) By Corollary 2.5 $G$ has two disjoint maximum independent sets.

(v) $\Rightarrow$ (i) Obvious.

(vi) $\Leftrightarrow$ (ii) By Theorem 3.5 ■
4 Trees

Clearly, a leaf is a shedding vertex if and only if its unique neighbor is a leaf as well. Hence, the only tree \( T \) having \( \text{Shed}(T) = V(T) \) is \( T = K_2 \). Notice that \( \text{Shed}(P_4) = \{ v : \deg(v) = 2 \} \), and no maximal independent set of \( P_4 \) is included in \( \text{Shed}(P_4) \).

**Proposition 4.1** Let \( T \) be a tree, which is not isomorphic to \( K_2 \), and let \( S \) be a maximal independent set. Then \( S \subseteq \text{Shed}(T) \) if and only if \( S = \text{Shed}(T) \).

**Proof.** The assertion is clearly true for \( T = K_1 \), as \( \text{Shed}(K_1) = \emptyset \). Assume that \( T \neq K_1 \).

By Theorem 1.5, it follows that the shedding vertices of a tree are exactly the neighbors of its leaves.

Let \( S \subseteq \text{Shed}(T) \) be a maximal independent set in \( T \). Thus each vertex in \( V(T) - S \) has a neighbor in \( S \).

Assume, to the contrary, that there is some \( v \in \text{Shed}(T) - S \). Hence, there must exist \( vu, vu \in E(T) \) such that \( u \) is a leaf and \( v \in S \). Consequently, we infer that \( x \notin \text{Shed}(T) \), and consequently, \( S \cup \{ x \} \) is an independent set larger than \( S \), in contradiction with the maximality of \( S \). In conclusion, \( S = \text{Shed}(T) \).

The converse is evident. ■

**Corollary 4.2** A tree \( T \) has a maximum independent set consisting of only shedding vertices if and only if \( T = K_2 \).

**Proof.** Clearly, \( T \neq K_1 \), since \( \text{Shed}(K_1) = \emptyset \).

Assume, on the contrary, that \( T \neq K_2 \) and let \( S \) be a maximum independent set such that \( S \subseteq \text{Shed}(T) \). Since \( \text{Shed}(K_1) = \emptyset \), we infer that \( V(T) \geq 3 \). By Proposition 4.1, we know that \( S = \text{Shed}(T) \).

Clearly, \( \text{Leaf}(T) \) is independent and \( |\text{Leaf}(T)| \geq 2 \). By Theorem 1.5, every vertex of \( S \) has a leaf as a neighbor. Consequently, \( \text{Leaf}(T) \) is a maximum independent set, because \(|\text{Leaf}(T)| \geq |S| = \alpha(T)|. Since all the vertices of \( \text{Leaf}(T) \) are leaves of \( T \), the subgraph \( T - \text{Leaf}(T) \) is a tree containing all the vertices of \( S \). Hence, there exists some \( v \in V(T - \text{Leaf}(T)) - S \), because \( S \) is independent in the tree \( T - \text{Leaf}(T) \) and \(|S| \geq 2 \).

Hence, no neighbor of \( v \) in \( T \) is a leaf. Therefore, \( \text{Leaf}(T) \cup \{ v \} \) is an independent set larger than \( \text{Leaf}(T) \), in contradiction with \( \text{Leaf}(T) \in \Omega(T) \). In conclusion, \( T = K_2 \).

The converse is obvious. ■

Notice that \(|\text{Shed}(K_1)| = 0 = \alpha(K_1) - 1 \), while \(|\text{Shed}(K_2)| = 2 = \alpha(K_2) + 1 \).

**Proposition 4.3** For a tree \( T \neq K_2 \) the following are true:

(i) \(|\text{Shed}(T)| \leq \alpha(T)|; 
(ii) if \( \text{Shed}(T) \) is independent, then \(|\text{Shed}(T)| \leq \alpha(T) - 1 \).

Moreover, both inequalities are tight.

**Proof.** (i) According to Theorem 1.5, \(|\text{Shed}(T)| \leq |\text{Leaf}(T)| \). In addition, \(|\text{Leaf}(T)| \leq \alpha(T) \), since \( \text{Leaf}(T) \) is independent. The series of graphs \( P_n \circ K_1, n \geq 2 \) shows that \( \alpha(T) \) is the tight upper bound for \(|\text{Shed}(T)|\).

(ii) The inequality \(|\text{Shed}(T)| \leq \alpha(T) - 1 \) directly follows from Corollary 4.2.

Let \( p \geq 2 \), and \( T \) be the tree obtained from \( K_{1,p} \) by adding \( p \) vertices, each one joined to a leaf of \( K_{1,p} \). The set \( \text{Shed}(T) \) consists of all the leaves of \( K_{1,p} \), while \( \alpha(T) = p + 1 \).
Hence, $|\text{Shed} (T)| = p = \alpha (T) - 1$. Thus, in this case, $\alpha (T) - 1$ is the tight upper bound for $|\text{Shed} (T)|$. \hfill \blacksquare

5 Unicyclic graphs

A graph $G$ is unicyclic if it is connected and has a unique cycle, which we denote by $C = (V(C), E(C))$. Let $N_1(C) = \{v : v \in V(G) - V(C), N(v) \cap V(C) \neq \emptyset\}$, and $T_x = (V_x, E_x)$ be the maximum subtree of $G - xy$ containing $x$, where $x \in N_1(C), y \in V(C)$.

For instance, every $C_{2k}$ is a unicyclic König-Egerváry graph, while every $C_{2k+1}$ is a unicyclic non-König-Egerváry graph.

**Lemma 5.1** \cite{20} If $G$ is a unicyclic graph, then $n(G) - 1 \leq \alpha(G) + \mu(G) \leq n(G)$.

**Theorem 5.2** \cite{20} Let $G$ be a unicyclic non-König-Egerváry graph. Then the following assertions are true:

(i) $W \in \Omega (T_x)$ if and only if $W = S \cap V (T_x)$ for some $S \in \Omega (G)$;

(ii) core $(G) = \bigcup \{\text{core} (T_x) : x \in N_1(C)\}$.

**Theorem 5.3** A unicyclic graph $G$ has two disjoint maximum independent sets if and only if, either $G$ is a bipartite graph with a perfect matching, or there is a vertex $v$ belonging to its unique cycle, such that $G - v$ has a perfect matching.

**Proof.** Let $S_1, S_2 \in \Omega (G)$ be such that $S_1 \cap S_2 = \emptyset$. Clearly, core $(G) = \emptyset$.

Case 1. $G$ is a König-Egerváry graph. Then, by Theorem 5.1, it follows that $G$ must a bipartite graph with a perfect matching.

Case 2. $G$ is not a König-Egerváry graph. Since core $(G) = \emptyset$, Theorem 5.2(ii) implies core $(T_x) = \emptyset$ for every $x \in N_1(C)$. Hence, each $T_x$ has a perfect matching, by Theorem 1.2, and $S_1 \cap V (T_x), S_2 \cap V (T_x) \in \Omega (T_x)$, according to Theorem 5.2(i). Therefore, we get

$$|S_1 \cap V (C)| = |S_2 \cap V (C)| = \frac{|V (C)| - 1}{2}.$$ 

Thus, there is some $v \in V (C)$, such that $v \notin S_1 \cup S_2$. Finally, $G - v$ is a forest with a perfect matching.

Conversely, if $G$ is a bipartite graph with a perfect matching, then its bipartition is comprised of two disjoint maximum independent sets.
Otherwise, there is a vertex \( v \) belonging to its unique cycle, such that \( G - v \) is a forest with a perfect matching. If \( \{ A, B \} \) is a bipartition of the vertex set of \( G - v \), then clearly, \( A \) and \( B \) are disjoint independent sets of \( G \) of size equal to 

\[
\mu(G) = \mu(G-v) = \frac{n(G) - 1}{2}.
\]

Moreover, \( A, B \in \Omega(G) \), because, otherwise, we have 

\[
n(G) - 1 - \mu(G) = \alpha(G) > |A| = \mu(G-v) = \mu(G),
\]

which leads to the following contradiction: \( n(G) - 1 > 2\mu(G) \). 

It is well-known that the matching number of a bipartite graph \( G \) can be computed in \( O(n(G)^{5/2}) \) \([14]\). Thus Theorem 5.3 implies the following.

**Corollary 5.4** One can decide in polynomial time whether a unicyclic graph has two disjoint maximum independent sets.

### 6 Conclusions

Theorem \([7]\) and Corollary 3.4 provide us with a complete description of very well-covered graphs of girth \( \geq 5 \) containing a pair of disjoint maximum independent sets. Corollary 3.2 tells us that the only girth under consideration left is four.

**Problem 6.1** Find a constructive characterization of very well-covered graphs (bipartite well-covered) of girth equal to 4, that have two disjoint maximum independent sets at least.

The same question may be asked about other classes of graphs. Recall that \( G \) is an edge \( \alpha \)-critical graph if \( \alpha(G-e) > \alpha(G) \), for every \( e \in E(G) \). For instance, every odd cycle \( C_{2k+1} \) and its complement are edge \( \alpha \)-critical graphs. Moreover, both \( C_{2k+1} \) and \( \overline{C}_{2k+1} \) have two disjoint maximum independent sets.

**Conjecture 6.2** \([15]\) If \( G \) is an edge \( \alpha \)-critical graph without isolated vertices, then it has two disjoint maximum independent sets.

It is known that the decision problem whether there are two disjoint maximal independent sets in a graph is \( \textit{NP} \)-complete \([13]\).

**Conjecture 6.3** It is \( \textit{NP} \)-complete to recognize (well-covered) graphs with two disjoint maximum independent sets.

The friendship graph \( F_q \) is a non-König-Egerváry graph with exactly \( 2^{\alpha(G)} \) maximum independent sets. Thus Theorem 5.4 motivates the following.

**Problem 6.4** Characterize non-König-Egerváry graphs with \( |\Omega(G)| = 2^{\alpha(G)} \).
7 Acknowledgements

We express our gratitude to Isabel Beckenbach, who suggested a number of remarks that helped us make proofs of some theorems clearer.

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