Abstract

Let Π be a translation invariant point process on the complex plane \( \mathbb{C} \) and let \( \mathcal{D} \subset \mathbb{C} \) be a bounded open set whose boundary has zero Lebesgue measure. We ask what does the point configuration \( \Pi_{\text{out}} \) obtained by taking the points of \( \Pi \) outside \( \mathcal{D} \) tell us about the point configuration \( \Pi_{\text{in}} \) of \( \Pi \) inside \( \mathcal{D} \)? We show that for the Ginibre ensemble, \( \Pi_{\text{out}} \) determines the number of points in \( \Pi_{\text{in}} \). For the translation-invariant zero process of a planar Gaussian Analytic Function, we show that \( \Pi_{\text{out}} \) determines the number as well as the centre of mass of the points in \( \Pi_{\text{in}} \). Further, in both models we prove that the outside says “nothing more” about the inside, in the sense that the conditional distribution of the inside points, given the outside, is mutually absolutely continuous with respect to the Lebesgue measure on its supporting submanifold.
1 Introduction

A point process Π on C is a random locally finite point configuration on the two dimensional Euclidean plane. The probability distribution of the point process Π is a probability measure P[Π] on the Polish space of locally finite point sets on the plane. Point processes on the plane have been studied extensively. For a lucid exposition on general point processes one can look at [DV97]. The group of translations of C acts in a natural way on the space of locally finite point configurations on C. Namely, the translation by z, denoted by T_z, takes the point configuration Λ to the configuration T_z(Λ) := \{x + z | x \in Λ\}. A point process Π is said to be translation invariant if Π and T_z(Π) have the same distribution for all \( z \in \mathbb{C} \). In this work we will focus primarily on translation invariant point processes on C whose intensities are absolutely continuous with respect to the Lebesgue measure. We will consider simple point processes, namely, those in which no two points are at the same location. A simple point process can also be looked upon as a random discrete measure \([Π] = \sum_{z \in Π} δ_z\).

Let \( D \) be a bounded open set in C whose boundary has zero Lebesgue measure. Let \( S \) denote the Polish space of locally finite point configurations on C. The decomposition \( D = D \cup D^c \) induces a factorization \( S = S_{in} \times S_{out} \), where \( S_{in} \) and \( S_{out} \) are respectively the spaces of finite point configurations on \( D \) and locally finite point configurations on \( D^c \). This immediately leads to the natural decomposition \( \Upsilon = (\Upsilon_{in}, \Upsilon_{out}) \) for any \( \Upsilon \in S \), and consequently a decomposition of the point process Π as Π = (Π_{in}, Π{out}).

In this paper, we ask the following question: if we know the configuration Π_{out}, what can we conclude about Π_{in}? We consider with regard to this question the main natural examples of translation invariant point processes on the plane, and provide a complete answer in each case.

The (homogeneous) Poisson point process is the most canonical example of a translation-invariant point process on the plane. Its crucial property is that the point configurations in two disjoint measurable sets are independent of each other. Therefore, our question is a triviality for the Poisson process: the points outside \( D \) do not provide any information about the points inside \( D \).

The two natural examples of translation invariant point processes in the plane that have non-trivial spatial correlations are the Ginibre ensemble and the zeroes of the planar Gaussian Analytic Function. We refer the reader to [HKPV10] for a detailed study of these ensembles. The Ginibre ensemble was introduced in the physics literature by Ginibre [Gin65] as a model based on non-Hermitian random matrices. Like the Poisson process, it is translation-invariant and ergodic under rigid motions of the plane. In fact, it is a determinantal point process with the determinantal kernel \( K(z, w) = \sum_{j=0}^{\infty} (zw)^j/j! \) and the background measure \( e^{-|z|^2} dL(z) \). Here \( L \) denotes the Lebesgue measure on C. Krishnapur [Kr06] has shown that the Ginibre ensemble is the unique determinantal point process on C that is translation invariant, has a sesqui-holomorphic kernel (i.e., the determinantal kernel is holomorphic in the first variable and anti-holomorphic in the second), and is normalized to have unit intensity.

The standard planar Gaussian Analytic Function (abbreviated henceforth as GAF) is the random entire function defined by the series development \( f(z) = \sum_{k=0}^{\infty} \frac{ξ_k}{\sqrt{k!}} z^k \) where \( ξ_k \)'s are i.i.d. standard complex Gaussians. We are interested in the point configuration on C given by the zeroes of this
GAF. The GAF zero process is translation invariant and ergodic, and exhibits local repulsion. It has been studied intensively by several authors including Nazarov, Sodin, Tsirelson, and others (see, e.g., [FH99], [STs1-04], [STs2-06], [STs3-05], [NSV07], [NS10]). Sodin and Tsirelson [STs1-04] have shown that in the class of Gaussian power series, the standard planar GAF is the only one to have a translation invariant zero-set (up to scaling and multiplication by a deterministic entire function with no zeroes).

For further details on these models, we refer the reader to Section 3.

For a pair of random variables \((X,Y)\) which has a joint distribution on a product of Polish spaces \(S_1 \times S_2\), we can define the regular conditional distribution \(\gamma\) of \(Y\) given \(X\) by the family of probability measures \(\gamma(s_1, \cdot)\) parametrized by the elements \(s_1 \in S_1\) such that for any Borel sets \(A \subset S_1\) and \(B \subset S_2\) we have

\[
P(X \in A, Y \in B) = \int_A \gamma(s_1, B) dP[X](s_1)
\]

where \(P[X]\) denotes the marginal distribution of \(X\). For details on regular conditional distributions, see, e.g., [Pa00] or [Bil95].

Recall that \(S_\text{in}\) and \(S_\text{out}\) are Polish spaces. Hence, by abstract nonsense, there exists a regular conditional distribution \(\bar{\rho}\) of \(\Pi_\text{in}\) given \(\Pi_\text{out}\). Clearly, \(\bar{\rho}\) can be seen as the distribution of a point process on \(D\) which depends on \(\Upsilon_\text{out}\).

Let \(\zeta\) be the vector (of variable length) whose co-ordinates are the points of \(\Pi_\text{in}\) taken in uniform random order. We will denote the conditional distribution of \(\zeta\) given \(\Pi_\text{out}\) by \(\rho\). Formally, it is a family of probability measures \(\rho(\Upsilon_\text{out}, \cdot)\) on \(\bigcup_{m=0}^{\infty} D^m\) parametrized by \(\Upsilon_\text{out} \in S_\text{out}\).

For a vector \(\alpha\), we denote by \(\Delta(\alpha)\) the Vandermonde determinant generated by the co-ordinates of \(\alpha\). Note that \(|\Delta(\alpha)|\) is invariant under permutations of the co-ordinates of \(\alpha\).

For two measures \(\mu_1\) and \(\mu_2\) defined on the same measure space \(\Omega\) with \(\mu_1 \ll \mu_2\) (meaning \(\mu_1\) is absolutely continuous with respect to \(\mu_2\)), we will denote by \(\frac{d\mu_1}{d\mu_2}(\omega)\) the Radon Nikodym derivative of \(\mu_1\) with respect to \(\mu_2\) evaluated at \(\omega \in \Omega\). Similarly, by \(\mu_1 \equiv \mu_2\) we mean that the measures \(\mu_1\) and \(\mu_2\) are mutually absolutely continuous, which implies that both \(\mu_1 \ll \mu_2\) and \(\mu_2 \ll \mu_1\) are simultaneously true.

In the case of the Ginibre ensemble, we prove that a.s. the points outside \(D\) determine the number of points inside \(D\), and “nothing more”.

In Theorems 1.1-1.4 we denote the Ginibre ensemble by \(G\) and the GAF zero ensemble by \(F\). As before, \(D\) is a bounded open set in \(\mathbb{C}\) whose boundary has zero Lebesgue measure.

**Theorem 1.1.** For the Ginibre ensemble, there is a measurable function \(N : S_\text{out} \to \mathbb{N} \cup \{0\}\) such that a.s.

\[
\text{Number of points in } G_\text{in} = N(G_\text{out})\,.
\]

Since a.s. the length of \(\zeta\) equals \(N(G_\text{out})\), we can as well assume that each measure \(\rho(\Upsilon_\text{out}, \cdot)\) is supported on \(D^{N(\Upsilon_\text{out})}\).

**Theorem 1.2.** For the Ginibre ensemble, \(\mathbb{P}[G_\text{out}]\)-a.s. the measure \(\rho(\Upsilon_\text{out}, \cdot)\) and the Lebesgue measure \(L\) on \(D^{N(G_\text{out})}\) are mutually absolutely continuous.
In the case of the GAF zero process, we prove that the points outside \( D \) determine the number as well as the centre of mass (or equivalently, the sum) of the points inside \( D \), and “nothing more”.

**Theorem 1.3.** For the GAF zero ensemble,

(i) There is a measurable function \( N : S_{\text{out}} \to \mathbb{N} \cup \{0\} \) such that a.s.

\[
\text{Number of points in } F_{\text{in}} = N(F_{\text{out}}).
\]

(ii) There is a measurable function \( S : S_{\text{out}} \to \mathbb{C} \) such that a.s.

\[
\text{Sum of the points in } F_{\text{in}} = S(F_{\text{out}}).
\]

Define the set

\[
\Sigma_{S(F_{\text{out}})} := \{ \underline{\zeta} \in D^{N(F_{\text{out})}} : \sum_{j=1}^{N(F_{\text{out}})} \zeta_j = S(F_{\text{out}}) \}
\]

where \( \underline{\zeta} = (\zeta_1, \ldots, \zeta_{N(F_{\text{out}})}) \).

Since a.s. the length of \( \underline{\zeta} \) equals \( N(F_{\text{out}}) \), we can as well assume that each measure \( \rho(\Upsilon_{\text{out}}, \cdot) \) gives us the distribution of a random vector in \( D^{N(\Upsilon_{\text{out}})} \) supported on \( \Sigma_{S(\Upsilon_{\text{out}})} \).

**Theorem 1.4.** For the GAF zero ensemble, \( \mathbb{P}[F_{\text{out}}] \)-a.s. the measure \( \rho(F_{\text{out}}, \cdot) \) and the Lebesgue measure \( \mathcal{L}_\Sigma \) on \( \Sigma_{S(F_{\text{out}})} \) are mutually absolutely continuous.

There is a simple relationship between \( \rho(\Pi_{\text{out}}, \cdot) \) and \( \varrho(\Pi_{\text{out}}, \cdot) \). Consider the natural map \( \phi \) from \( \bigcup_{m=0}^{\infty} D^m \) to \( S_m \) which makes a point configuration \( \Upsilon_{\text{in}} \) of size \( m \) from a vector \( \underline{\zeta} \) in \( D^m \) by forgetting the order of the co-ordinates of \( \underline{\zeta} \). It is easy to see that \( \mathbb{P}[\Pi_{\text{out}}] \)-a.s. \( \phi_* \rho(\Pi_{\text{out}}, \cdot) = \varrho(\Pi_{\text{out}}, \cdot) \).

The central question of this paper is partially motivated by the concepts of insertion and deletion tolerance of a point process. Consider a point process \( \Pi \). Fix a bounded open set \( D \), and add to \( \Pi \) a random point uniformly distributed in \( D \). Let us call this perturbed point process \( \Pi_D \). The point process \( \Pi \) is said to be **insertion tolerant** if \( \mathbb{P}[\Pi_D] \ll \mathbb{P}[\Pi] \) for all bounded open sets \( D \). **Deletion tolerance** is similarly defined by deleting a point inside \( D \) (if one exists) uniformly at random. Insertion and deletion tolerance have been investigated by Burton and Keane ([BK89]) in the context of percolation, by Holroyd and Peres ([HP05]) for studying invariant allocation on the plane, and by Heicklen and Lyons ([HLy03]) in the setting of random spanning forests. They have also been studied as topics of their own importance by Holroyd and Soo ([HS10]).

Notice that the question whether \( \mathbb{P}[\Pi_D] \ll \mathbb{P}[\Pi] \) for a given \( D \) can be phrased in terms of the conditional distribution \( \varrho \), and the same holds for the main questions addressed in this paper. Therefore, in more general terms, we are interested in the support and the regularity properties of \( \varrho \).

For a finite point process of fixed size \( n \), the phenomenon of outside points determining the number of inside points is a triviality. However, the behaviour of infinite point processes can be quite different, even when they arise as distributional limits of finite point processes. For example, let us take the disk of area \( n \) centred at the origin and consider the point process \( \Pi_n \) given by \( n \) uniform points inside it. Let \( D \) be the unit disk centred at the origin. The process \( \Pi_n \) clearly has the property that the point configuration outside \( D \) determines the number of points inside \( D \). However, the distributional limit of \( \Pi_n \)-s, as \( n \to \infty \), is the Poisson point process, which has no such
property. Hence, the reason behind this phenomenon to occur in the case of the Ginibre ensemble or the Gaussian zeroes is fundamentally different, and is connected with the spatial correlation properties of the corresponding ensembles.

In addition to answering our central question mentioned in the beginning, Theorems 1.1-1.4 also provide information on the relative strength of spatial correlations in the Ginibre and the GAF zero ensembles. While a simple visual inspection suffices to (heuristically) distinguish a sample of the Poisson process from that of either the Ginibre or the GAF zero process (of the same intensity), the latter two are hard to set apart between themselves. It is therefore an interesting question to devise mathematical statistics that distinguish them. The qualitative idea is that the spatial correlation is much stronger in the GAF zero process than in the Ginibre ensemble. There can be several possible approaches to quantify this heuristic observation. One important feature to look at, for instance, is the rate of decay of the hole probabilities. However, it turns out that both the Ginibre ensemble and the Gaussian zero process behave similarly in this respect. For more details, one can refer to [HKPV10]. Our results clearly demonstrate that the GAF zeroes have much greater spatial dependence, in the sense that the point configuration in the exterior of a open set dictates much more about the one in its interior.

In [STs1-04], Sodin and Tsirelson compared the GAF zero process (CAZP in their terminology) with various models of perturbed lattices. They noticed that the lattice process

\[ \{ \sqrt{3\pi}(k + il) + c e^{2\pi im/3} \eta_{k,l} : k, l \in \mathbb{Z}, m = 0, 1, 2 \}, \]

(where \( \eta_{k,l} \) are i.i.d. standard complex Gaussians, \( c \in (0, \infty) \) is a parameter and \( i \) denotes the imaginary unit) achieves “asymptotic similarity” with the GAF zero process (in the sense that the variances of scaled linear statistics have similar asymptotic behaviour to those for the GAF zeroes). Sodin and Tsirelson further observed that the above perturbed lattice model satisfied two conservation laws: one pertaining to the “mass” and another pertaining to the “centre of mass”. They predicted similar conservation laws for the GAF zero process, although the sense in which such laws would hold was left open to interpretation. Theorem 1.3 establishes two conservation laws, one of which preserves the “mass” (i.e., the number of points), and the other one preserves the “centre of mass”. Moreover, Theorem 1.4 says that these are the only conservation laws for the GAF zero process. We further note that among the perturbed lattice models in [STs1-04], the one that achieves “asymptotic similarity” with the Ginibre ensemble is the process

\[ \{ \sqrt{\pi}(k + il) + c \eta_{k,l} : k, l \in \mathbb{Z} \} \]

where \( \eta_{k,l} \) and \( c \) are as before. In this model, we have one conserved quantity (namely, the “mass”). In Theorems 1.1 and 1.2, we obtain a conservation law for the “mass” (i.e., the number of points) in the Ginibre ensemble, and further, show that there are no other conserved quantities.

En route proving the main theorems mentioned above, we obtain results that are interesting in their own right. For example we prove that the harmonic sum \( \left( \sum_{z \in \Pi} \frac{1}{z} \right) \), for the Ginibre ensemble as well as for the GAF zero process, is a.s. finite (in a precise technical sense specified in Propositions 1.6 and 4.13 and the remarks thereafter. In fact, we show that this sum has a finite first moment for both processes. It is not hard to see that the corresponding sum for the Poisson process does not converge in any reasonable sense. Even for the Ginibre or the GAF zero ensembles, the corresponding sum does not converge absolutely. The underlying reason for the conditional convergence is the mutual cancellation arising from the higher degree of symmetry exhibited by a typical point configuration in the Ginibre or the GAF zero process. This is yet
another manifestation of the fact that the Gaussian zeros or the Ginibre eigenvalues exhibit a much more regular arrangement (which indicates greater rigidity) than, say, the Poisson process.

In a more precise sense, we can define the finite sums \( \alpha_k(n) = (\sum_{z \in \Pi} 1/z^k) \) when \( \Pi = G_n \) or \( F_n \). Our results, as in Proposition 7.9 and Proposition 9.16, establish that both for the Ginibre and the GAF zeroes, these sums converge in probability as \( n \to \infty \). The limit \( \alpha_k \) can be justifiably taken to be an analogue of the sum \( (\sum_{z \in \Pi} 1/z^k) \) for the respective limiting process \( G \) or \( F \).

We also prove a reconstruction theorem for the planar GAF from its zeroes, which essentially says that the GAF zeroes determine a.s. the GAF itself, up to a factor of modulus 1. In what follows, \( \alpha_k \) will denote the random variable introduced above for GAF zeroes, \( P_k \) will be the \( k \)-th Newton polynomial (for details, see Section 11.1). Define \( a_k = P_k(\alpha_1, \cdots, \alpha_k) \) and

\[
\chi = \lim_{k \to \infty} k^{1/2} \left( \sum_{j=0}^{k-1} |P_j(\alpha_1, \cdots, \alpha_j)|^2 \right)^{-1/2}
\]

(the existence of the limit will be proved in the course of proving Theorem 1.5). We state the reconstruction theorem as:

**Theorem 1.5.** Consider the random analytic function \( g(z) = \sum_{k=0}^{\infty} \chi a_k z^k \), which is measurable with respect to the GAF zeroes. There is a random variable \( \zeta \) with uniform distribution on \( S^1 \) and independent of the GAF zeroes, such that a.s. we have \( f(z) = \zeta g(z) \).

It is interesting to compare Theorem 1.5 with the Weierstrass Factorization Theorem (see, e.g., [Rud87] Chapter 15) for reconstructing analytic functions from their zeroes. In the case of the Weierstrass factorization, the key problem is that there is a (random) analytic function (with no zeroes) that occurs as a factor in front of the canonical product formed from the Gaussian zeroes, and a priori there is no concrete information about this function. E.g., it can, in principle, depend on the Gaussian zeroes. However, in Theorem 1.5 we are able to give a concrete description of the factor \( \zeta \) and also the precise dependence of \( g \) on GAF zeroes.

The description in Theorem 1.5 is optimal in the sense that the factor \( \zeta \) cannot be done away with. This can be seen from the fact that if \( \theta \) is a random variable that is uniform in \( S^1 \) and independent of the \( \xi_i \)-s, then the random analytic functions \( \theta f \) and \( f \) are both distributed as planar GAF-s but have the same zeroes; hence from the zeroes of \( f \) we can hope to recover the coefficients of \( f \) only up to such a factor \( \theta \).

Theorem 1.5 can be compared with Theorem 6 in [PV05], where a similar result is established for the zeroes of the Gaussian analytic function on the hyperbolic plane. However, such a result for the planar case is not known, and our approach here is distinct from [PV05], relying crucially on the estimates we obtain in Section 9.2.

We view the conservation laws as the “rigidity” properties of the respective point processes. The absolute continuity (with respect to the Lebesgue measure on the conserved submanifold) of the conditional distribution of the vector of inside points can be viewed as “tolerance”. The heuristic is that due to such mutual absolute continuity, the inside points can form (almost) any configuration on this conserved submanifold.
2 Plan of the paper

In Section 3, we begin with a detailed description of the models we study, and also provide an abstract framework in which other models having similar characteristics can be investigated. Further, we show in Section 4 that for proving the main Theorems 1.1-1.4 it suffices to establish them in the case where $D$ is a disk.

In order to study rigidity phenomena, we devise a unified approach in Section 5, where Theorem 5.1 gives general criteria for a function (of the inside points) to be rigid with respect to a point process. We complete the proofs of Theorems 1.1 and 1.3 by establishing the relevant criteria for the Ginibre and the GAF zero processes.

In Section 6, we study tolerance properties in the general setup introduced in Section 3.3. Theorem 6.2 lays down conditions under which certain tolerance behaviour of a point process can be established. Proving Theorems 1.2 and 1.4 therefore, amounts to showing that the relevant conditions hold for our models. However, unlike the rigidity phenomena, this requires substantially more work, and is carried out in two stages for each process. First, we obtain some estimates for the point processes $G_n$ and $F_n$, which are finite approximations to $G$ and $F$ respectively (see Sections 3.1 and 3.2 for definitions). For the Ginibre ensemble, this is done in Section 7, and for the GAF zeroes this is done in Section 9. Finally, we apply these estimates to deduce that the relevant conditions for tolerance behaviour hold for our models; this is carried out for the Ginibre ensemble in Section 8 and for the GAF zeroes in Section 10.

3 Setup and Models

3.1 The Ginibre Ensemble

Let us consider an $n \times n$ matrix $X_n, n \geq 1$ whose entries are i.i.d. standard complex Gaussians. The vector of its eigenvalues, in uniform random order, has the joint density (with respect to the Lebesgue measure on $\mathbb{C}^n$) given by

$$p(z_1, \ldots, z_n) = \frac{1}{\pi^n \prod_{k=1}^n k!} e^{-\sum_{k=1}^n |z_k|^2} \prod_{i<j} |z_i - z_j|^2.$$

Recall that a determinantal point process on the Euclidean space $\mathbb{R}^d$ with kernel $K$ and background measure $\mu$ is a point process on $\mathbb{R}^d$ whose $k$-point intensity functions with respect to the measure $\mu^\otimes k$ are given by

$$\rho_k(x_1, \ldots, x_k) = \det\left[ (K(x_i, x_j))\right]_{i,j=1}^k.$$

Typically, $K$ has to be such that the integral operator defined by $K$ is a non-negative trace class contraction mapping $L^2(\mu)$ to itself. For a detailed study of determinantal point processes, we refer the reader to [HKPV10] or [Sos00]. A simple calculation involving Vandermonde determinants shows that the eigenvalues of $X_n$ (considered as a random point configuration) form a determinantal point process on $\mathbb{C}$. Its kernel is given by $K_n(z, w) = \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!}$ with respect to the background measure $d\gamma(z) = \frac{1}{\pi} e^{-|z|^2} d\mathcal{L}(z)$ where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{C}$. This point process is the Ginibre ensemble (of dimension $n$), which we will denote by $G_n$. As $n \to \infty$, these point
processes converge, in distribution, to a determinantal point process given by the kernel $K(z, w) = e^{z \bar{w}} = \sum_{k=0}^{\infty} \frac{(zw)^k}{k!}$ with respect to the same background measure $\gamma$. This limiting point process is the infinite Ginibre ensemble $\mathcal{G}$. It is known that $\mathcal{G}$ is ergodic under the natural action of the translations of the plane.

### 3.2 The GAF zero process

Let $\{\xi_k\}_{k=0}^{\infty}$ be a sequence of i.i.d. standard complex Gaussians. Define

$$f_n(z) = \sum_{k=0}^{n} \xi_k \frac{z^k}{\sqrt{k!}} \quad \text{(for } n \geq 0\text{)}, \quad f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}.$$

These are Gaussian processes on $\mathbb{C}$ with covariance kernels given by $K_n(z, w) = \sum_{k=0}^{n} \frac{(zw)^k}{k!}$ and $K(z, w) = \sum_{k=0}^{\infty} \frac{(zw)^k}{k!} = e^{z \bar{w}}$ respectively. A.s. $f_n$ and $f$ are entire functions and the functions $f_n$ converge to $f$ (in the sense of the uniform convergence of holomorphic functions on compact sets).

It is not hard to see (e.g., via Rouche’s theorem) that this implies that the corresponding point processes of zeroes, denoted by $\mathcal{F}_n$, converge a.s. to the zero process $\mathcal{F}$ of the GAF (in the sense of locally finite point configurations converging on compact sets). It is known that $\mathcal{F}$ is ergodic under the natural action of the translations of the plane.

### 3.3 The General Setup

Fix a Euclidean space $\mathcal{E}$ equipped with a non-negative regular Borel measure $\mu$. Let $\mathcal{S}$ denote the Polish space of countable locally finite point configurations on $\mathcal{E}$. Endow $\mathcal{S}$ with its canonical topology, namely the topology of convergence on compact sets (which gives $\mathcal{S}$ a canonical Borel $\sigma$-algebra). Fix a bounded open set $\mathcal{D} \subset \mathcal{E}$ with $\mu(\partial \mathcal{D}) = 0$. Corresponding to the decomposition $\mathcal{E} = \mathcal{D} \cup \mathcal{D}^c$, we have $\mathcal{S} = \mathcal{S}_{\text{in}} \times \mathcal{S}_{\text{out}}$, where $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ denote the spaces of finite point configurations on $\mathcal{D}$ and locally finite point configurations on $\mathcal{D}^c$ respectively.

Let $\Xi$ be a measure space equipped with a probability measure $\mathbb{P}$. For a random variable $Z : \Xi \to \mathcal{X}$ (where $\mathcal{X}$ is a Polish space), we define the push forward $Z_*\mathbb{P}$ of the measure $\mathbb{P}$ by $Z_*\mathbb{P}(A) = \mathbb{P}(Z^{-1}(A))$ where $A$ is a Borel set in $\mathcal{X}$. Also, for a point process $Z' : \Xi \to \mathcal{S}$, we can define point processes $Z'_{\text{in}} : \Xi \to \mathcal{S}_{\text{in}}$ and $Z'_{\text{out}} : \Xi \to \mathcal{S}_{\text{out}}$ by restricting the random configuration $Z'$ to $\mathcal{D}$ and $\mathcal{D}^c$ respectively.

Let $X, X^n : \Xi \to \mathcal{S}$ be random variables such that $\mathbb{P}$-a.s., we have $X^n \to X$ (in the topology of $\mathcal{S}$). We demand that the point processes $X, X^n$ have their first intensity measures absolutely continuous with respect to $\mu$. We can identify $X_{\text{in}}$ (by taking the points in uniform random order) with the random vector $\xi$ which lives in $\bigcup_{m=0}^{\infty} \mathcal{D}^m$. The analogous quantity for $X^n$ will be denoted by $\xi^n$.

For our models we can take $\mathcal{E}$ to be $\mathbb{C}$, $\mu$ to be the Lebesgue measure, and $\mathcal{D}$ to be a bounded open set whose boundary has zero Lebesgue measure.

In the case of the Ginibre ensemble, we can define the processes $\mathcal{G}_n$ and $\mathcal{G}$ on the same underlying probability space so that a.s. we have $\mathcal{G}_n \subset \mathcal{G}_{n+1} \subset \mathcal{G}$ for all $n \geq 1$. For reference, see [Go10]. We take $(\Xi, \mathbb{P})$ to be this underlying probability space, $X^n = \mathcal{G}_n$ and $X = \mathcal{G}$. 

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In the case of the Gaussian zero process, we take \((\Xi, \mathbb{P})\) to be a measure space on which we have countably many standard complex Gaussian random variables denoted by \(\{\xi_k\}_{k=0}^\infty\). Then \(X^n\) is the zero set of the polynomial \(f_n(z) = \sum_{k=0}^{n} \xi_k \frac{z^k}{\sqrt{k!}}\), and \(X\) is the zero set of the entire function \(f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}\). The fact that \(X^n \to X\) \(\mathbb{P}\)-a.s. follows from Rouche’s theorem.

4 Reduction from a general \(\mathcal{D}\) to a disk

In this Section we intend to prove that to obtain Theorems 1.1-1.4, it suffices to consider the case where \(\mathcal{D}\) is an open disk centred at the origin. We will demonstrate the proof for Theorems 1.3 and 1.4; the arguments for Theorems 1.1 and 1.2 are on similar lines.

Let \(\mathcal{D}\) be a bounded open set in \(\mathbb{C}\) whose boundary has zero Lebesgue measure. By translation invariance of the Ginibre ensemble, we take the origin to be in the interior of \(\mathcal{D}\). Let \(\mathcal{D}_0\) be a disk (centred at the origin) which contains \(\overline{\mathcal{D}}\) in its interior (where \(\overline{\mathcal{D}}\) is the closure of \(\mathcal{D}\)).

Suppose we know the point configuration \(\mathcal{F}_{\text{out}}\) to be equal to \(\mathcal{Y}_{\text{out}}\). Further, suppose that we can show that the point configuration \(\mathcal{Y}_{\text{out}}^{\mathcal{D}_0}\) outside \(\mathcal{D}_0\) determines the number \(N_0\) and the sum \(S_0\) of the points inside \(\mathcal{D}_0\) a.s. Since we also know the number and the sum of the points inside \(\mathcal{D}\), we can determine the number \(N\) as well as the sum \(S\) of the points in \(\mathcal{D}\). This proves the rigidity theorem for the GAF zero ensemble (Theorem 1.3) for a general \(\mathcal{D}\).

Now suppose we have the tolerance Theorem 1.4 for a disk. To obtain Theorem 1.4 for \(\mathcal{D}\), we appeal to the tolerance Theorem 1.4 for the disk \(\mathcal{D}_0\). Define

\[
\Sigma := \left\{ (\lambda_1, \ldots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = S, \lambda_j \in \mathcal{D} \right\}
\]

and

\[
\Sigma_0 := \left\{ (\lambda_1, \ldots, \lambda_{N_0}) : \sum_{j=1}^{N_0} \lambda_j = S_0, \lambda_j \in \mathcal{D}_0 \right\}.
\]

The conditional distribution of the vector of points inside \(\mathcal{D}_0\), given \(\mathcal{Y}_{\text{out}}^{\mathcal{D}_0}\), lives on \(\Sigma_0\), in fact it has a density \(f_0\) which is positive a.e. with respect to Lebesgue measure on \(\Sigma_0\). Let there be \(k\) points in \(\mathcal{D}_0 \setminus \mathcal{D}\) and let their sum be \(s\), clearly we have \(N = N_0 - k\) and \(S = S_0 - s\). We parametrize \(\Sigma\) by the last \(N - 1\) co-ordinates. Note that the set \(U := \{(\lambda_2, \ldots, \lambda_N) : (S - \sum_{j=2}^{N} \lambda_j, \lambda_2, \ldots, \lambda_N) \in \Sigma\}\) is an open subset of \(\mathcal{D}^{N-1}\). Further, we define the set \(V := \{(\lambda_1, \ldots, \lambda_k) : \lambda_i \in \mathcal{D}_0 \setminus \mathcal{D}, \sum_{i=1}^{k} \lambda_i = s\}\).

Let the points in \(\mathcal{D}_0 \setminus \mathcal{D}\), taken in uniform random order, form the vector \(\mathbf{z} = (z_1, \ldots, z_k)\). Then we can condition the vector of points in \(\mathcal{D}_0\) to have its last \(k\) co-ordinates equal to \(\mathbf{z}\), to obtain the following formula for the conditional density of the vector of points in \(\mathcal{D}\) at \((\zeta_1, \ldots, \zeta_N)\) \(\in \Sigma\) (with respect to the Lebesgue measure on \(\Sigma\)):

\[
f(\zeta_1, \zeta_2, \ldots, \zeta_N) = \frac{f_0(\zeta_1, \zeta_2, \ldots, \zeta_N, z_1, \ldots, z_k)}{\int_U f_0(s - (\sum_{j=2}^{N} w_j), w_2, \ldots, w_N, z_1, \ldots, z_k) dw_2 \cdots dw_N}. \tag{1}
\]

It is clear that for a.e. \(\mathbf{z} \in V\), we have \(f\) is strictly positive a.e. with respect to Lebesgue measure on \(\Sigma\), because the same is true of \(f_0\) on \(\Sigma_0\).
5 Rigidity Phenomena

We begin by giving a precise definition of rigidity. Recall the general setup in Section 3.3.

**Definition 1.** A measurable function $f_{\text{in}} : S_{\text{in}} \to \mathbb{C}$ is said to be rigid with respect to the point process $X$ on $S$ if there is a measurable function $f_{\text{out}} : S_{\text{out}} \to \mathbb{C}$ such that a.s. we have $f_{\text{in}}(X_{\text{in}}) = f_{\text{out}}(X_{\text{out}})$.

In this section, we prove that the number of points in $D$ in the case of the Ginibre ensemble and the number as well as the sum of the points in $D$ for the GAF zero process are rigid. In fact, we will state some general conditions that ensure such rigid behaviour, and then show that the Ginibre and the GAF satisfy the relevant conditions.

We will use linear statistics of point processes as the main tool that will enable us to obtain the rigidity results.

**Definition 2.** Let $\varphi$ be a compactly supported continuous function on $\mathbb{C}$. The linear statistic corresponding to $\varphi$ is the random variable $\int \varphi d[\pi] = \sum_{z \in \pi} \varphi(z)$.

By a $C^k_c$ function on a Euclidean space $\mathcal{E}$ we denote the space of compactly supported $C^k$ functions on $\mathcal{E}$.

We can now state:

**Theorem 5.1.** Let $\pi$ be a point process on $\mathbb{C}$ whose first intensity is absolutely continuous with respect to the Lebesgue measure, and let $D$ be a bounded open set whose boundary has zero Lebesgue measure. Let $\varphi$ be a continuous function on $\mathbb{C}$. Suppose for any $1 > \varepsilon > 0$, we have a $C^2_c$ function $\Phi^\varepsilon$ such that $\Phi^\varepsilon = \varphi$ on $D$, and $\text{Var} \left( \int_{\mathbb{C}} \Phi^\varepsilon d[\pi] \right) < \varepsilon$. Then $\int_D \varphi d[\pi]$ is rigid with respect to $\pi$.

**Proof.** Consider the sequence of $C^2_c$ functions $\Phi^{2^{-n}}, n \geq 1$. Note that $\mathbb{E} \left[ \int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] \right] = \int_{\mathbb{C}} \Phi^{2^{-n}} \rho_1 d\mathcal{L}$ where $\rho_1(z)$ is the one point intensity function of $\pi$. By Chebyshev's inequality, it is clear that

$$\mathbb{P} \left( \left| \int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] - \mathbb{E} \left[ \int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] \right] \right| > 2^{-n/4} \right) \leq 2^{-n/2}.$$

The Borel Cantelli lemma implies that with probability 1, as $n \to \infty$ we have

$$\left| \int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] - \mathbb{E} \left[ \int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] \right] \right| \to 0.$$

But

$$\int_{\mathbb{C}} \Phi^{2^{-n}} d[\pi] = \int_D \Phi^{2^{-n}} d[\pi] + \int_{D^c} \Phi^{2^{-n}} d[\pi].$$

Thus we have, as $n \to \infty$

$$\left| \int_D \Phi^{2^{-n}} d[\pi] + \int_{D^c} \Phi^{2^{-n}} d[\pi] - \int_{\mathbb{C}} \Phi^{2^{-n}} \rho_1 d\mathcal{L} \right| \to 0. \quad (2)$$
If we know π_{\text{out}} , we can compute \( \int_{\mathcal{D}} \Phi^{2-n} d[\pi] \) exactly, also \( \rho_1 \) is known explicitly; in case of a translation invariant point process \( \pi \) it is, in fact, a constant \( c(\pi) \). Hence, from the limit in (2), a.s. we can obtain \( \int_{\mathcal{D}} \Phi^{2-n} d[\pi] = \int_{\mathcal{D}} \varphi d[\pi] \) as the limit

\[
\lim_{n \to \infty} \int_{\mathcal{C}} \left( \Phi^{2-n} \rho_1 d\mathcal{L} - \int_{\mathcal{D}} \Phi^{2-n} d[\pi] \right).
\]

We now use Theorem 5.1 to establish Theorems 1.1 and 1.3.

**Proof of Theorems 1.1 and 1.3** We have already seen in Section 4 that it suffices to take \( \mathcal{D} \) to be a disk. By translation invariance of \( \mathcal{F} \) and \( \mathcal{G} \), we can assume that \( \mathcal{D} \) is centred at the origin. We intend to construct functions \( \Phi^\varepsilon \) as in Theorem 5.1.

Let \( r_0 = \text{Radius (D)} \). Fix \( \varepsilon > 0 \).

We begin with a continuous function \( \tilde{\Psi} \) on \( \mathbb{R}_+ \cup \{0\} \) such that \( \tilde{\Psi}(r) = 1 \) for \( 0 \leq r \leq r_0 \), \( \tilde{\Psi}'(r) = -\varepsilon/r \) and \( \tilde{\Psi}''(r) = \varepsilon/r^2 \) for \( r_0 \leq r \leq r_0 \exp(1/\varepsilon) \), and \( \tilde{\Psi}(r) = 0 \) for \( r \geq r_0 \exp(1/\varepsilon) \). This can be obtained, e.g., by solving the relevant boundary value problem between \( r_0 \) and \( r_0 \exp(1/\varepsilon) \), and extending to \( \mathbb{R}_+ \cup \{0\} \) in the obvious manner. We then smooth the function \( \tilde{\Psi} \) at \( r_0 \) and \( r_0 \exp(1/\varepsilon) \) such that the resulting function \( \Psi_1 \) is \( C^2 \) on the positive reals, and satisfies \( |\Psi_1'(r)| \leq \varepsilon/r \) and \( |\Psi_1''(r)| \leq \varepsilon/r^2 \) for all \( r > 0 \). Finally, we define the radial \( C^2_c \) function \( \Psi \) on \( \mathbb{C} \) as \( \Psi(z) = \Psi_1(|z|) \).

For \( \mathcal{G} \), we know (see [RV07] Theorem 11) that there exists a constant \( C_1 > 0 \) such that for every radial \( C^2_c \) function \( \Psi \) we have

\[
\text{Var} \left( \int_{\mathcal{C}} \Psi d[\mathcal{G}] \right) \leq C_1 \int_{\mathcal{C}} \|\nabla \Psi(z)\|_2^2 d\mathcal{L}(z).
\]

But from the definition of \( \Psi \) it is clear that \( \int_{\mathcal{C}} \|\nabla \Psi(z)\|_2^2 d\mathcal{L}(z) \leq C_2 \varepsilon \). We apply Theorem 5.1 with \( \varphi \equiv 1 \), and choose \( \Phi^{C_1C_2\varepsilon} = \Psi \) as defined above; recall that \( \Psi \equiv 1 \) on \( \mathcal{D} \).

For \( \mathcal{F} \), we know (see [NS11] Theorem 1.1) that there exists a constant \( C_3 > 0 \) such that every \( C^2_c \) function \( \varphi \) satisfies

\[
\text{Var} \left( \int_{\mathcal{C}} \varphi d[\mathcal{F}] \right) \leq C_3 \int_{\mathcal{C}} \|\Delta \varphi(z)\|_2^2 d\mathcal{L}(z).
\]

For the rigidity of the number of points of \( \mathcal{F} \) in \( \mathcal{D} \) we make exactly the same choice as we did for \( \mathcal{G} \), and note that \( \int_{\mathcal{C}} \|\Delta \Psi(z)\|_2^2 d\mathcal{L}(z) \leq C_4 \varepsilon^2 \). For the rigidity of the sum of points of \( \mathcal{F} \) in \( \mathcal{D} \) we intend to apply Theorem 5.1 with \( \varphi \equiv 1 \) and \( \vartheta \equiv \Psi \) as before. Observe that \( \Delta \theta(z) = 4\frac{\partial \Psi}{\partial z}(z) + z \Delta \Psi(z) \). Using this, for \( \varepsilon < 1 \), we get \( \int_{\mathcal{C}} |\Delta(z \Psi(z))|^2 d\mathcal{L}(z) \leq C_5 \varepsilon \).

It remains to note that for \( z \in \mathcal{D} \) we have \( \Psi(z) = 1 \) and \( \theta(z) = z \).

6 Tolerance: Limits of Conditional Measures

The tolerance properties are established for both models by obtaining explicit bounds on conditional probability measures for finite approximations (finite matrices in case of Ginibre and polynomials.
for GAF) and then passing to the limit. In this Section, we state and prove some general conditions (in the context of the abstract setup considered in Section 3.3), which will enable us to make the transition from the finite ensembles to the infinite one.

We start with the following general proposition:

**Proposition 6.1.** Let \( \Gamma \) be second countable topological space. Let \( \Sigma \) be a countable basis of open sets in \( \Gamma \) and let \( \mathfrak{A} := \{ \bigcup_{i=1}^{\infty} \sigma_i : \sigma_i \in \Sigma, k \geq 1 \} \). Let \( c > 0 \). To verify that two non-negative regular Borel measures \( \mu_1 \) and \( \mu_2 \) on \( \Gamma \) satisfy \( \mu_1(B) \leq c \mu_2(B) \) for all Borel sets \( B \) in \( \Gamma \), it suffices to verify the inequality for all sets in \( \mathfrak{A} \).

**Proof.** Any open set \( U \subset \Gamma \) is a countable union \( \bigcup_{i=1}^{\infty} \sigma_i, \sigma_i \in \Sigma \) because the sets in \( \Sigma \) form a basis for the topology on \( \Gamma \). If we have \( \mu_1(\bigcup_{i=1}^{\infty} \sigma_i) \leq c \mu_2(\bigcup_{i=1}^{\infty} \sigma_i) \), then we can let \( n \to \infty \) to obtain \( \mu_1(U) \leq c \mu_2(U) \). Once we have the inequality for all open sets \( U \), we can extend it to all Borel sets, because for a regular Borel measure \( \mu \) and for any Borel set \( B \subset \Gamma \), we have \( \mu(B) = \inf_{B \subset U} \mu(U) \) where the infimum is taken over all open sets \( U \) containing \( B \).

In this Section, we will work in the setup of Section 3.3 specifying \( \mathcal{D} \) to be an open ball, and requiring that the first intensity of our point process \( X \) is absolutely continuous with respect to the Lebesgue measure on \( \mathcal{E} \). We further assume that \( X \) exhibits rigidity of the number of points. In other words, there is a measurable function \( N : \mathcal{S}_{\text{out}} \to \mathbb{N} \cup \{0\} \) such that a.s. we have

\[
\text{Number of points in } X_{\text{in}} = N(X_{\text{out}}).
\]

In such a situation, we can identify \( X_{\text{in}} \) (by taking the points in uniform random order) with a random vector \( \zeta \) taking values in \( \mathcal{D}^N(X_{\text{out}}) \). Studying the conditional distribution \( \rho(X_{\text{out}}, \cdot) \) of \( X_{\text{in}} \) given \( X_{\text{out}} \) is then the same as studying the conditional distribution of this random vector given \( X_{\text{out}} \). We will denote the latter distribution by \( \rho(X_{\text{out}}, \cdot) \). Note that it is supported on \( \mathcal{D}^N(X_{\text{out}}) \) (see Section 4 for details).

For \( m > 0 \), let \( \mathcal{W}^m_{\text{in}} \) denote the countable basis for the topology on \( \mathcal{D}^m \) formed by open balls contained in \( \mathcal{D}^m \) and having rational centres and rational radii. We define the collection of sets \( \mathcal{W}^m := \{ \bigcup_{i=1}^{k} A_i : A_i \in \mathcal{W}^m_{\text{in}}, k \geq 1 \} \).

Fix an integer \( n \geq 0 \), a closed annulus \( B \subset \mathcal{D} \) whose centre is at the origin and which has a rational inradius and a rational outradius, and a collection of \( n \) disjoint open balls \( B_i \) with rational radii and centres having rational co-ordinates such that \( \{ B_i \cap \mathcal{D} \}_{i=1}^{n} \subset B \). Let \( \Phi(n, B, B_1, \cdots, B_n) \) be the Borel subset of \( \mathcal{S}_{\text{out}} \) defined as follows:

\[
\Phi(n, B, B_1, \cdots, B_n) = \{ \Upsilon \in \mathcal{S}_{\text{out}} : |\Upsilon \cap B| = n, |\Upsilon \cap B_i| = 1 \}.
\]

Then the countable collection \( \Sigma_{\text{out}} = \{ \Phi(n, B, B_1, \cdots, B_n) : n, B, B_i \text{ as above} \} \) is a basis for the topology of \( \mathcal{S}_{\text{out}} \). Define the collection of sets \( B := \{ \bigcup_{i=1}^{k} \Phi_i : \Phi_i \in \Sigma_{\text{out}}, k \geq 1 \} \).

We will denote by \( \Omega^m \) the event that \( |X_{\text{in}}| = m \), and by \( \Omega^m_n \) we will denote the event \( |X^m_{\text{in}}| = m \).

**Definition 3.** Let \( p \) and \( q \) be indices (which take values in potentially infinite abstract sets), and \( \alpha(p, q) \) and \( \beta(p, q) \) be non-negative functions of these indices. We write \( \alpha(p, q) \asymp_q \beta(p, q) \) if there exist positive numbers \( k_1(q), k_2(q) \) such that

\[
k_1(q)\alpha(p, q) \leq \beta(p, q) \leq k_2(q)\alpha(p, q) \text{ for all } p, q.
\]

The main point is that \( k_1, k_2 \) in the above inequalities are uniform in \( p \), that is, all the indices in question other than \( q \).
We will also use the notation introduced in Section 3.3. We will define an “exhausting” sequence of events as:

**Definition 4.** A sequence of events \( \{ \Omega(j) \}_{j \geq 1} \) is said to exhaust another event \( \Omega \) if \( \Omega(j) \subset \Omega(j + 1) \subset \Omega \) for all \( j \) and \( \mathbb{P}(\Omega \setminus \Omega(j)) \to 0 \) as \( j \to \infty \).

Let \( \mathcal{M}(\mathcal{D}^m) \) denote the space of all probability measures on \( \mathcal{D}^m \). For two random variables \( U \) and \( V \) defined on the same probability space, we say that \( U \) is measurable with respect to \( V \) if \( U \) is measurable with respect to the sigma algebra generated by \( V \). Finally, recall the definition of \( A^m \) and \( B \) from the beginning of this section.

Now we are ready to state the following important technical reduction:

**Theorem 6.2.** Let \( m \geq 0 \) be such that \( \mathbb{P}(\Omega^m) > 0 \). Suppose that:

(a) There is a map \( \nu : \mathcal{S}_{\text{out}} \to \mathcal{M}(\mathcal{D}^m) \) such that for each Borel set \( A \subset \mathcal{D}^m \), the \( \nu(\cdot, A) \) is a measurable real valued function.

(b) For each fixed \( j \) we have a sequence \( \{ n_k \}_{k \geq 1} \) (which might depend on \( j \)) and corresponding events \( \Omega_{n_k}(j) \) such that:

(i) \( \Omega_{n_k}(j) \subset \Omega^m \).

(ii) \( \Omega(j) := \lim_{k \to \infty} \Omega_{n_k}(j) \) exhaust \( \Omega^m \) as \( j \uparrow \infty \).

(iii) For all \( A \in \mathcal{A}^m \) and \( B \in \mathcal{B} \) we have

\[
\mathbb{P}[(X_{n_k}^m \in A) \cap (X_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] \asymp_j \int_{(X_{\text{out}}^{n_k})^{-1}(B) \cap \Omega_{n_k}(j)} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi) + \vartheta(k; j, A, B). \tag{3}
\]

where \( \lim_{k \to \infty} \vartheta(k; j, A, B) = 0 \) for each fixed \( j, A \) and \( B \).

Then a.s. on the event \( \Omega^m \) we have

\[
\rho(X_{\text{out}}, \cdot) \equiv \nu(X_{\text{out}}, \cdot) \tag{4}
\]

We defer the proof of Theorem 6.2 to Section 12.

We conclude this section with the following simple observation:

**Remark 6.1.** If we have Theorem 6.2 for all \( m \geq 0 \) then we can conclude that (4) holds a.e. \( \xi \in \Xi \).

### 7 Tolerance of the Ginibre Ensemble for a disk

In this section we obtain several estimates necessary to prove Theorem 1.2 in the case where \( \mathcal{D} \) is a disk.

**Remark 7.1.** By the translation invariance of the Ginibre ensemble, we can take \( \mathcal{D} \) to be centred at the origin.

**Notation 1.** For a set \( \Lambda \subset \mathbb{C} \) and \( R > 0 \), we denote \( R \cdot \Lambda = \{ Rz : z \in \Lambda \} \).
7.1 Matrix Approximations

Let \( \delta \in (0,1) \). We consider the event \( \Omega_{m,\delta}^{n} \) which entails that \( \mathcal{G}_{n} \) has exactly \( m \) points inside \( \mathcal{D} \), and there is a \( \delta \) separation between \( \partial \mathcal{D} \) and \( (\mathcal{G}_{n})_{\text{out}} \); the analogous event for \( \mathcal{G} \) will be denoted by \( \Omega_{m,\delta}^{n} \). Notice that \( \Omega_{m,\delta}^{n} \) has positive probability (which is bounded away from 0 uniformly in \( \delta \)), so by the convergence of \( \mathcal{G}_{n} \)-s to \( \mathcal{G} \), we obtain that \( \Omega_{m,\delta}^{n} \)-s have positive probability that is uniformly bounded away from 0 for large enough \( n \). We denote the points of \( \mathcal{G}_{n} \) inside \( \mathcal{D} \) (in uniform random order) by \( \zeta = (\zeta_{1}, \ldots, \zeta_{m}) \) and those outside \( \mathcal{D} \) (in uniform random order) by \( \omega = (\omega_{1}, \omega_{2}, \ldots, \omega_{n-m}) \). Following the notation introduced in Section 1, for a vector \( (\zeta, \omega) \) as above, we denote \( \Upsilon_{m} = \{\zeta_{i}\}_{i=1}^{m} \), \( \Upsilon_{\text{out}} = \{\omega_{j}\}_{j=1}^{n-m} \), and \( \Upsilon = \Upsilon_{\text{in}} \cup \Upsilon_{\text{out}} \). For a vector \( \gamma = (\gamma_{1}, \ldots, \gamma_{N}) \) of \( N \) points in \( \mathbb{C} \), we denote by \( \Delta(\gamma) \) the Vandermonde determinant \( \prod_{i<j}(\gamma_{i} - \gamma_{j}) \). For two vectors \( \gamma_{1}, \gamma_{2} \) we set \( \Delta(\gamma_{1}, \gamma_{2}) = \Delta(\gamma) \) where \( \gamma = (\gamma_{1}, \gamma_{2}) \).

Then the conditional distribution \( \rho(\Upsilon_{\text{out}}, \zeta) \) of \( \zeta \) given \( \Upsilon_{\text{out}} \) has the density

\[
\rho_{\omega}^{n}(\zeta) = C(\omega) |\Delta(\zeta, \omega)|^2 \exp \left( -\sum_{k=1}^{m} |\zeta_{k}|^2 \right)
\]

with respect to the Lebesgue measure on \( \mathcal{D}^{m} \), where \( C(\omega) \) is the normalizing constant (which depends on \( \omega \)). Let \( (\zeta, \omega) \) and \( (\zeta', \omega') \) correspond to two configurations such that the event \( \Omega_{m,\delta}^{n} \) occurs in both cases. Then the ratio of the conditional densities at these two points is given by

\[
\frac{\rho_{\omega}^{n}(\zeta')}{\rho_{\omega}^{n}(\zeta)} = \frac{|\Delta(\zeta', \omega)|^2 \exp(-\sum_{k=1}^{m} |\zeta'_{k}|^2)}{|\Delta(\zeta, \omega)|^2 \exp(-\sum_{k=1}^{m} |\zeta_{k}|^2)}.
\]

Clearly, \( \exp \left( -\sum_{k=1}^{m} |\zeta'_{k}|^2 \right)/\exp \left( -\sum_{k=1}^{m} |\zeta_{k}|^2 \right) \) is bounded above and below by constants which are functions of \( m \) and \( \mathcal{D} \).

To study the ratio of the Vandermonde determinants, we define

\[
\Gamma(\zeta, \omega) = \prod_{1 \leq i \leq m, 1 \leq j \leq n-m} (\zeta_{i} - \omega_{j}).
\]

Then we have

\[
\frac{|\Delta(\zeta', \omega)|^2}{|\Delta(\zeta, \omega)|^2} = \frac{|\Delta(\zeta', \omega)|^2}{|\Delta(\zeta, \omega)|^2} \frac{|\Gamma(\zeta', \omega)|^2}{|\Gamma(\zeta, \omega)|^2}.
\]

In order to bound \( |\Gamma(\zeta', \omega)|^2 \) from above and below uniformly in \( \zeta, \zeta' \in \mathcal{D}^{m} \), it suffices to bound \( |\Gamma(\zeta, \omega)|/|\Gamma(\zeta, \omega)| \) from above and below uniformly in \( \zeta \in \mathcal{D}^{m} \). Here \( \mathbf{0} \) is the vector of all 0-s in \( \mathcal{D}^{m} \). We observe that

\[
\frac{|\Gamma(\zeta, \omega)|}{|\Gamma(\mathbf{0}, \omega)|} = \prod_{i=1}^{m} \left( \prod_{j=1}^{n-m} \left| \frac{\zeta_{i} - \omega_{j}}{\omega_{j}} \right| \right).
\]

Since \( m \) is fixed, it suffices to bound \( \prod_{j=1}^{n-m} \left| \frac{\zeta_{0} - \omega_{j}}{\omega_{j}} \right| \) from above and below uniformly in \( \zeta_{0} \in \mathcal{D} \). To this end we prove
Proposition 7.1. Let $\zeta_0 \in \mathcal{D}$ and $\omega_1, \cdots, \omega_{n-m}$ be such that $|\omega_j| > r + \delta$ for all $j$ where $r$ is the radius of $\mathcal{D}$. Then we have

$$\left| \log \left( \prod_{j=1}^{n-m} \left( \frac{\zeta_0 - \omega_j}{\omega_j} \right) \right) \right| \leq K_1(\mathcal{D}) \left| \sum_{j=1}^{n-m} \frac{1}{\omega_j} \right| + K_2(\mathcal{D}) \left| \sum_{j=1}^{n-m} \frac{1}{\omega_j^2} \right| + K_3(\mathcal{D}, \delta) \left( \sum_{j=1}^{n-m} \frac{1}{|\omega_j|^3} \right).$$

Here $K_1(\mathcal{D})$, $K_2(\mathcal{D})$ and $K_3(\mathcal{D}, \delta)$ are constants depending on $\mathcal{D}$ and $\delta$.

Proof. We begin with $\log \left| \frac{\zeta_0 - \omega_j}{\omega_j} \right| = \log \left| 1 - \frac{\omega_j}{\zeta_0} \right|$. Due to the $\delta$-separation between $\partial \mathcal{D}$ and $\omega$, the ratio $\theta_j := \frac{\omega_j}{\zeta_0}$ satisfies $|\theta_j| \leq \frac{r}{r+\delta} < 1$. Let $\log$ be the branch of the complex logarithm given by the power series development $\log(1 - z) = -\left( \sum_{k=1}^{\infty} \frac{z^k}{k} \right)$ for $|z| < 1$. Then we have

$$\log \left| 1 - \theta_j \right| = \Re \log(1 - \theta_j) = -\Re(\theta_j) - \frac{1}{2} \Re(\theta_j^2) + h(\theta_j)$$

where $|h(\theta_j)| \leq K_3'(\mathcal{D}, \delta)|\theta_j|^3$, where $K_3'$ is a constant depending on $\mathcal{D}$ and $\delta$. Hence,

$$\left| \log \left( \prod_{j=1}^{n-m} \frac{\zeta_0 - \omega_j}{\omega_j} \right) \right| = \left| \sum_{j=1}^{n-m} \log \left( \frac{\zeta_0 - \omega_j}{\omega_j} \right) \right| = -\Re \left( \sum_{j=1}^{n-m} \theta_j \right) - \frac{1}{2} \Re \left( \sum_{j=1}^{n-m} \theta_j^2 \right) + \sum_{j=1}^{n-m} h(\theta_j).$$

Recall that $\theta_j = \frac{\omega_j}{\zeta_0}$ and $|\zeta_0| \leq r$ and $|h(\theta_j)| \leq K_3'(\mathcal{D}, \delta)|\theta_j|^3$. The triangle inequality applied to the above gives us the statement of the Proposition with $K_1(\mathcal{D}) = r$, $K_2(\mathcal{D}) = \frac{1}{2} r^2$ and $K_3(\mathcal{D}, \delta) = K_3'(\mathcal{D}, \delta)r^3$.

As a result, we have

Proposition 7.2. On $\Omega_n^{m, \delta}$, we have a constant $K(\mathcal{D}, \delta) > 0$ such that

$$\exp \left( -4mK(\mathcal{D}, \delta)X_n \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2} \leq p_{\mathcal{D}}(\zeta') \leq p_{\mathcal{D}}(\zeta) \leq \exp \left( 4mK(\mathcal{D}, \delta)X_n \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2},$$

where $X_n = \left| \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega} \right| + \left| \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega^2} \right| + \left( \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega|^3} \right)$ and $\mathbb{E}[X_n] \leq c_1(\mathcal{D}, m) < \infty$.

Proof. Clearly, it suffices to bound $|\Delta(\zeta', \omega)|^2/|\Delta(\zeta, \omega)|^2$ from above and below.

From Proposition 7.1 on $\Omega_n^{m, \delta}$ we have $\log \left( \prod_{j=1}^{n-m} \frac{\zeta_0 - \omega_j}{\omega_j} \right) \leq K(\mathcal{D}, \delta)X_n$ for any $\zeta_0 \in \mathcal{D}$, where $K(\mathcal{D}, \delta) = \max\{K_1, K_2, K_3\}$. Considering this for each $\zeta_i$, $i = 1, \cdots, m$, exponentiating and taking product over $i = 1, \cdots, m$, we get

$$\exp \left( -mK(\mathcal{D}, \delta)X_n \right) \leq \frac{\Gamma(\zeta, \omega)}{|\Gamma(0, \omega)|} \leq \exp \left( mK(\mathcal{D}, \delta)X_n \right).$$

The same estimate holds for $\zeta'$. Since $\frac{\Gamma(\zeta', \omega)}{|\Gamma(\zeta', \omega)|} = \frac{\Gamma(\zeta', \omega)}{|\Gamma(0, \omega)|} \frac{|\Gamma(\zeta, \omega)|}{|\Gamma(\zeta, \omega)|}$, we have

$$\exp \left( -2mK(\mathcal{D}, \delta)X_n \right) \leq \frac{\Gamma(\zeta', \omega)}{|\Gamma(\zeta', \omega)|} \leq \exp \left( 2mK(\mathcal{D}, \delta)X_n \right).$$
In view of (6) and (7), this leads to the desired bound $\frac{\rho_2^{\mu}(\zeta)}{\rho_2^{\mu}(\zeta)}$. In Section 7.2 we will see that each of the three random sums defining $X_n$ has finite expectation. Moreover, those expectations are uniformly bounded by quantities depending only on $D$ and $m$. This yields the statement $\mathbb{E}[X_n] \leq c_1(D, m) < \infty$.

\textbf{Corollary 7.3.} Given $M > 0$, we can replace $X_n$ in Proposition 7.2 by a uniform bound $M$ except on an event of probability less than $c_1(m, D)/M$.

### 7.2 Estimates for Inverse Powers

Our aim in this section is to estimate the sums of inverse powers of the points in $G$ and $G_n$ outside a disk containing the origin. To this end, we first discuss certain estimates on the variance of linear statistics, which are uniform in $n$. Let $B(0; r)$ denote the disk of radius $r$ centred at the origin.

\textbf{Proposition 7.4.} Let $\varphi$ be a compactly supported Lipschitz function, supported inside the disk $B(0; r)$ with Lipschitz constant $\kappa(\varphi)$. Let $\varphi_R(z) := \varphi(z/R)$. Then $\text{Var} \left( \int \varphi_R(z) d[G_n](z) \right) \leq C(\varphi)$, where $C(\varphi)$ is a constant that is independent of $n$. The same conclusion holds for $G$ in place of $G_n$.

To prove Proposition 7.4, we will make use of a general fact about determinantal point processes:

\textbf{Lemma 7.5.} Let $\Pi$ be a determinantal point process with Hermitian kernel $K$. Let $K$ be a reproducing kernel with respect to its background measure $\gamma$, which means $K(x, y) = \int K(x, z)K(z, y) d\gamma(z)$ for all $x, y$. Let $\varphi, \psi$ be compactly supported continuous functions.\newline\text{Cov} \left[ \int \varphi d[\Pi], \int \psi d[\Pi] \right] = \frac{1}{2} \iint (\varphi(z) - \varphi(w))(\psi(z) - \psi(w))|K(z, w)|^2 d\gamma(z) d\gamma(w).

\textbf{Proof.} Expanding the two point correlation in its determinantal formula gives the covariance as

$$\int \varphi(z)\psi(z)K(z, z) d\gamma(z) - \int \int \varphi(z)\psi(w)|K(z, w)|^2 d\gamma(z) d\gamma(w).$$

Using $K(z, z) = \int K(z, w)K(w, z) d\gamma(w)$ and $K(z, w) = \overline{K(w, z)}$, elementary calculations give us the final result.

\textbf{Proof of Proposition 7.4.} We give the proof when $r = 1$, from here the general case is obtained by scaling. In what follows we deal with $G_n$, the result for $G$ follows, for instance, from taking limits as $n \to \infty$ for the result for $G_n$.

Using lemma 7.5 we have

$$\text{Var} \left( \int \varphi_R(z) d[G_n](z) \right) = \frac{1}{2} \int \int |\varphi_R(z) - \varphi_R(w)|^2|K_n(z, w)|^2 d\gamma(z) d\gamma(w)$$

where $\gamma$ is the standard complex Gaussian measure. Now,

$$|\varphi_R(z) - \varphi_R(w)|^2 = |\varphi(z/R) - \varphi(w/R)|^2 \leq \frac{1}{R^2} \kappa(\varphi)^2 |z - w|^2.$$

Therefore, it suffices to bound the integral $\int_{A(R)} |z - w|^2|K_n(z, w)|^2 d\gamma(z) d\gamma(w)$ on the set

$$A(R) := \{(z, w) : |z| \wedge |w| \leq R\}.$$
because outside $A(R)$, we have $\varphi_R(z) = \varphi_R(w) = 0$. We begin with

$$
\int_{A(R)} |z - w|^2 |K_n(z, w)|^2 \, d\gamma(z) \, d\gamma(w) \leq \int_{A_1(R)} |z - w|^2 |K_n(z, w)|^2 \, d\gamma(z) \, d\gamma(w)
$$

$$
+ \int_{A_2(R)} |z - w|^2 |K_n(z, w)|^2 \, d\gamma(z) \, d\gamma(w)
$$

where

$$
A_1(R) = \{|z| \leq 2R, |w| \leq 2R\} \quad \text{and} \quad A_2(R) = \{|z| \leq R, |w| \geq 2R\} \cup \{|w| \leq R, |z| \geq 2R\}.
$$

We first address the case of $A_2(R)$. By symmetry, it suffices to bound the integral over the region $\{|z| \leq R, |w| \geq 2R\}$. In this region, $||z| - |w|| \geq R$, and $|K_n(z, w)|^2 e^{-|z|^2 - |w|^2} \leq e^{-||z| - |w||^2}$. The integral is bounded from above by

$$
\frac{1}{\pi} \int_{|z| \leq R} \left( \int_{|w| \geq 2R} (|z| + |w|)^2 e^{-||z| - |w||^2} \, d\mathcal{L}(w) \right) \, d\mathcal{L}(z).
$$

It is not hard to see that the inner integral is $O(R^2 e^{-R^2})$. Integrating over $|z| \leq R$ gives another factor of $R^2$, so the total contribution is $o(1)$ as $R \to \infty$.

For the integral over $A_1(R)$, we proceed as

$$
\int |z - w|^2 |K_n(z, w)|^2 \, d\gamma(z)
$$

$$
= \int \left( |z|^2 - z\bar{w} - \bar{z}w + |w|^2 \right) \left( \sum_{k=0}^n (z\bar{w})^k / k! \right) \left( \sum_{k=0}^n (\bar{z}w)^k / k! \right) e^{-|z|^2 - |w|^2} \, d\mathcal{L}(z) \, d\mathcal{L}(w).
$$

Now, we integrate the $|z - w|^2$ part term by term. Due to radial symmetry, only some specific terms from $|K_n(z, w)|^2$ contribute. For example, when we integrate the $|z|^2$ term in $|z - w|^2$, only the term $\frac{(z\bar{w})^k (\bar{z}w)^k}{k! / k!}$, $0 \leq k \leq n$, terms in the expanded expression for $|K_n(z, w)|^2$ contribute. When we integrate $z\bar{w}$, only the term $\frac{(z\bar{w})^k (\bar{z}w)^{k+1}}{k! / (k+1)!}$, $0 \leq k \leq n - 1$, terms provide non-zero contributions.

Due to symmetry between $z$ and $w$, it is enough to bound the contribution from $(|z|^2 - z\bar{w})$ by $O(R^2)$.

**$|z|^2$ term:**

Let us denote $|z|^2$ by $x$ and $|w|^2$ by $y$. Then the contribution coming from

$$
\frac{(z\bar{w})^j (\bar{z}w)^j}{j! / j!}
$$

as discussed above can be written as

$$
\int_0^{4R^2} \int_0^{4R^2} \frac{x^{j+1}}{j!} e^{-x} \frac{y^j}{j!} e^{-y} \, dx \, dy.
$$

So, the total contribution due to all such terms, ranging from $k = 0, \cdots, n$ is

$$
\sum_{j=0}^n \left( \int_0^{4R^2} \frac{x^{j+1}}{j!} e^{-x} \, dx \right) \left( \int_0^{4R^2} \frac{y^j}{j!} e^{-y} \, dy \right).
$$

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**z̅w term:**

As above, the contribution coming from the function $\phi(z)$ is defined as

\[
\frac{(z\overline{w})^k (\overline{z}w)^{k+1}}{k! (k + 1)!}\]

is given by

\[
\int_0^{4R^2} \int_0^{4R^2} x^{j+1} \frac{e^{-x}}{j!} y^{j+1} \frac{e^{-y}}{(j+1)!} dx dy.
\]

Therefore the total contribution from $0 \leq k \leq n - 1$ is

\[
\sum_{j=0}^{n-1} \left( \int_0^{4R^2} x^{j+1} \frac{e^{-x}}{j!} dx \right) \left( \int_0^{4R^2} y^{j+1} \frac{e^{-y}}{(j+1)!} dy \right).
\]

We interpret $\frac{x^j}{j!} e^{-x} dx$ as a gamma density, the corresponding random variable being denoted by $\Gamma_{j+1}$.

The contribution due to the $|z|^2$ term is $\sum_{j=0}^{n} \mathbb{E}[\Gamma_{j+1} 1(\Gamma_{j+1} \leq 4R^2)] \mathbb{P}[\Gamma_{j+1} \leq 4R^2]$ and that due to the $z\overline{w}$ term is $\sum_{j=0}^{n-1} \mathbb{E}[\Gamma_{j+1} 1(\Gamma_{j+1} \leq 4R^2)] \mathbb{P}[\Gamma_{j+1} \leq 4R^2]$.

The difference between the above two terms can be written as:

\[
\mathbb{E}[\Gamma_{n+1} 1(\Gamma_{n+1} \leq 4R^2)] \mathbb{P}[\Gamma_{n+1} \leq 4R^2] + \sum_{j=1}^{n} \mathbb{E}[\Gamma_j 1(\Gamma_j \leq 4R^2)] \left( \mathbb{P}[\Gamma_j \leq 4R^2] - \mathbb{P}[\Gamma_{j+1} \leq 4R^2] \right). \tag{8}
\]

All the expectations in the above are $\leq 4R^2$, and $\mathbb{P}[\Gamma_j \leq 4R^2] \geq \mathbb{P}[\Gamma_{j+1} \leq 4R^2]$ because $\Gamma_{j+1}$ stochastically dominates $\Gamma_j$. Therefore the absolute value of (8), by triangle inequality, is $\leq 4R^2 \mathbb{P}[\Gamma_1 \leq 4R^2] \leq 4R^2$. Combining all of these, we see that $\text{Var} \left( \int \varphi_R(z) d[G_n](z) \right)$ is bounded by $(1/R^2) \kappa(\phi)^2 8R^2 = C(\phi)$.

Let $r_0 = \text{radius} (\mathcal{D})$. Let $\varphi$ be a non-negative radial $C^\infty$ function supported on $[r_0, 3r_0]$ such that $\varphi = 1$ on $[r_0 + \frac{r}{2}, 2r_0]$ and $\varphi(r_0 + r) = 1 - \varphi(2r_0 + 2r)$, for $0 \leq r \leq \frac{1}{2}r_0$. In other words, $\varphi$ is a test function supported on the annulus between $r_0$ and $3r_0$ and its “ascent” to 1 is twice as fast as its “descent”. Let $\bar{\varphi}$ be another radial test function with the same support as $\varphi$, satisfying $\bar{\varphi}(r_0 + xr_0) = 1$ for $0 \leq x \leq \frac{1}{2}$ and $\bar{\varphi} = \varphi$ otherwise. Recall that for a test function $\psi$ and $L > 0$ we denote by $\psi_L$ the scaled function $\psi_L(z) = \psi(z/L)$.

**Proposition 7.6.** Let $r_0$ be the radius of $\mathcal{D}$. Let $\varphi$ and $\bar{\varphi}$ be defined as above.

(i) The random variables

\[
S_l(n) := \int \frac{\bar{\varphi}(z)}{z^l} d[G_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_2(z)}{z^l} d[G_n](z) = \sum_{\omega \in G_n \cap \mathbb{D}^c} \frac{1}{\omega^l} \quad (\text{for } l \geq 1)
\]

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and
\[ \tilde{S}_l(n) := \int \frac{\tilde{\varphi}(z)}{|z|^l} \, d[G_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2j}(z)}{|z|^l} \, d[G_n](z) = \sum_{\omega \in G_n \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \quad (\text{for } l \geq 3) \]

have finite first moments which, for every fixed $l$, are bounded above uniformly in $n$.

(ii) There exists $k_0 = k_0(\varphi) \geq 1$, uniform in $n$ and $l$, such that for $k \geq k_0$ the “tails” of $S_l(n)$ and $\tilde{S}_l(n)$ beyond the disk $2^k \cdot \mathcal{D}$, given by
\[ \tau^n_l(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2j}(z)}{|z|^l} \, d[G_n](z) \quad (\text{for } l \geq 1) \quad \text{and} \quad \tilde{\tau}^n_l(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2j}(z)}{|z|^l} \, d[G_n](z) \quad (\text{for } l \geq 3) \]
satisfy the estimates
\[ \mathbb{E} \left[ \tau^n_l(2^k) \right] \leq C_1(\varphi, l)/2^{kl} \quad \text{and} \quad \mathbb{E} \left[ \tilde{\tau}^n_l(2^k) \right] \leq C_2(\varphi, l)/2^{k(l-2)}. \]

All of the above remain true when $G_n$ is replaced by $\mathcal{G}$, for which we use the notations $S_l$ and $\tilde{S}_l$ to denote the quantities corresponding to $S_l(n)$ and $\tilde{S}_l(n)$.

**Remark 7.2.** For $\mathcal{G}$, by the sum \( \sum_{\omega \in \mathcal{G} \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \) we denote the quantity
\[ S_l = \int \frac{\tilde{\varphi}(z)}{z^l} \, d[\mathcal{G}](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2j}(z)}{z^l} \, d[\mathcal{G}](z) \]
due to the obvious analogy with $G_n$, where the corresponding sum $S_l(n)$ is indeed equal to \( \sum_{\omega \in G_n \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \) with its usual meaning.

**Proof.** Observe that the functions $\tilde{\varphi}$ and $\varphi_{2j}$ for $j \geq 1$ form a partition of unity on $\mathcal{D}^c$, hence we have the identities appearing in part (i).

Fix $n, l \geq 1$. Set $\psi_k = \int \frac{\varphi_{2k}(z)}{z^l} \, d[G_n](z)$ for $k \geq 1$, and $\psi_0 = \int \frac{\tilde{\varphi}(z)}{z^l} \, d[G_n](z)$. When $l \geq 3$ we also define $\gamma_k = \int \frac{\varphi_{2k}(z)}{z^l} \, d[G_n](z)$ for $k \geq 1$, and $\gamma_0 = \int \frac{\tilde{\varphi}(z)}{z^l} \, d[G_n](z)$. Let $\Psi_k$ and $\Gamma_k$ denote the analogous quantities defined with respect to $\mathcal{G}$ instead of $G_n$.

We begin with the observation that for $k \geq 1$ we have $\mathbb{E}[\psi_k] = 0$. This implies that
\[ \mathbb{E}[|\psi_k|^2] \leq \mathbb{E}[|\psi_k|^2]^{1/2} = \sqrt{\text{Var}[\psi_k]} \]
We then apply Proposition 7.4 to the function $\varphi(z)/z^l$ and $R = 2^k$ to obtain $\mathbb{E}|\psi_k| \leq C(\varphi, l)/(2^k)^l$. We also note that
\[ \mathbb{E}[|\psi_0|] \leq \int_{\mathcal{D}^c} \frac{K_n(z, z)e^{-|z|^2}}{|z|^l} \, d\mathcal{L}(z) \leq \int_{\mathcal{D}^c} \frac{1}{|z|^l} \, d\mathcal{L}(z) = c(l). \]
This implies that for $l \geq 1$
\[ \mathbb{E}[|S_l(n)|] \leq \sum_{k=0}^{\infty} \mathbb{E}[|\psi_k|] < \infty. \]
The desired bound for $\mathbb{E}[\tilde{S}_l(n)]$ follows from a direct computation of the expectation using the first intensity, and noting that the first intensity of $G_n$ (with respect to Lebesgue measure) is $K_n(z,z)e^{-|z|^2} \leq 1$ for all $z$.

The estimates for $\tau$ and $\tilde{\tau}$ follow by using the above argument for the sums $\sum_{j=k}^{\infty} \psi_j$ and $\sum_{j=k}^{\infty} \gamma_j$.

**Corollary 7.7.** For $R = 2^k$ for $k \geq k_0$ (as in Proposition 7.6), we have $\mathbb{P}(|\tau_i^n(R)| > R^{-l/2}) \leq c_1(\varphi,l)R^{-l/2}$ and $\mathbb{P}(|\tilde{\tau}_i^n(R)| > R^{-(l-2)/2}) \leq c_2(\varphi,l)R^{-(l-2)/2}$, and these estimates remain true when $G_n$ is replaced with $\mathcal{G}$.

**Proof.** We use the estimates on the expectation of $|\tau_i^n(R)|$ and $|\tilde{\tau}_i^n(R)|$ from Proposition 7.6 and apply Markov’s inequality.

With notations as above, we have

**Proposition 7.8.** For each $l \geq 1$ we have $S_l(n) \rightarrow S_l$ in probability, and for each $l \geq 3$ we have $\tilde{S}_l(n) \rightarrow \tilde{S}_l$ in probability, and hence we have such convergence a.s. along some subsequence, simultaneously for all $l$.

**Proof.** Fix $\delta > 0$. Given $\varepsilon > 0$, we choose $R = 2^k$ large enough such that $c_1 R^{-l/2} < \varepsilon/4$ (for $l \geq 1$) and $c_2 R^{-(l-2)/2} < \varepsilon/4$ for $l \geq 3$ (as in Corollary 7.7), as well as $R^{-l/2} < \delta/4$ for $l = 1, 2$ and $R^{-(l-2)/2} < \delta/4$ for $l \geq 3$. By definition, we have $S_l(n) = \sum_{j=0}^{k} \psi_{2j,l} + \tau_l(2^k)$. On the disk of radius $R$, we have $G_n \rightarrow \mathcal{G}$ a.s. Now choose $n$ large enough so that we have $| \sum_{j=0}^{k} \psi_{2j,l} - \sum_{j=0}^{k} \psi_{2j,l} | < \delta/2$ except on an event of probability $\varepsilon/2$. By choice of $R$, we have $|\tau_l^n(2^k)| < \delta/2$ and $|\tau_l(2^k)| < \delta/2$ except on an event of probability $< \varepsilon/2$. Combining all these, we have $\mathbb{P}(|S_l(n) - S_l| > \delta) \leq \varepsilon$, proving that $S_l(n) \rightarrow S_l$ in probability.

For each $l$, given any sequence we can find a subsequence along which this convergence is a.s. A diagonal argument now gives us a subsequence for which a.s. convergence holds simultaneously for all $l$.

The argument for $\tilde{S}_l$ is similar.

Define $S_k(D, n) = \sum_{z \in G_n} 1/z^k$ and $S_k(D) = \sum_{z \in \mathcal{G}} 1/z^k$. Set $\alpha_k(n) = S_k(D, n) + S_k(n)$ and $\alpha_k = S_k(D) + S_k$. Observe that $\alpha_k(n) = \sum_{z \in G_n} 1/z^k$. Then we have

**Proposition 7.9.** For each $k$, $\alpha_k(n) \rightarrow \alpha_k$ in probability as $n \rightarrow \infty$. Hence, there is a subsequence such that $\alpha_k(n) \rightarrow \alpha_k$ a.s. when $n \rightarrow \infty$ along this subsequence, simultaneously for all $k$.

**Proof.** Since the finite point configurations given by $G_n|_D \rightarrow \mathcal{G}|_D$ a.s. and a.s. there is no point at the origin, therefore $S_k(D, n) \rightarrow S_k(D)$ a.s. This, combined with Proposition 7.8 gives us the desired result.
8 Limiting Procedure for the Ginibre Ensemble

The aim of this section is to use the estimates derived in Section 7 to verify the conditions laid out in Theorem 6.2, so that the limiting procedure outlined in Section 6 can be executed. This will lead us to a proof of Theorem 1.2 for a disk $D$. We have already seen in Section 4 that this implies Theorem 1.2 for general $D$.

**Proof of Theorem 1.2 for a disk.** We will appeal to Theorem 6.2 with $D$ an open disk. We already know from Theorem 1.1 that the number of points in $D$ is rigid. In terms of the notation used in Section 6 we set $X = G$ and $X^n = G_n$.

Fix an integer $m \geq 0$. Consider the event that $m$ points of $G$ inside $D$. The result in Theorem 1.2 is trivial for $m = 0$. Hence, we focus on the case $m > 0$. For $\omega \in S_{\text{out}}$, our candidate for $\nu(\omega, \cdot)$ (refer to Theorem 6.2) is the probability measure $Z^{-1}|\Delta(\zeta')|^2d\mathcal{L}(\zeta)$ on $D^m$, where $Z$ is the normalizing constant. Notice that $\nu$ is a constant when considered as a function mapping $S_{\text{out}}$ to $\mathcal{M}(D^m)$. Since a.s. $\nu(X_{\text{out}}, \cdot)$ is mutually absolutely continuous with respect to the Lebesgue measure on $D^m$, Theorem 6.2 would imply that the same holds true for the conditional distribution $\rho(X_{\text{out}}, \cdot)$ of the points inside $D$ (treated as a vector in the uniform random order).

Now we construct, for each $j$, the sequences $\{n_k(j)\}_{k \geq 1}$ and the events $\Omega_{n_k}(j)$. We proceed as follows. From Proposition 7.3 we get a subsequence $n_k$ such that a.s. $S_1(n_k) \to S_1$, $S_2(n_k) \to S_2$, and $S_3(n_k) \to S_3$. This is going to be our subsequence $n_k(j)$ for all $j$.

Let $M_j \uparrow \infty$ be a sequence of positive numbers such that none of them is an atom of the distributions of $|S_1|$, $|S_2|$ and $|\tilde{S}_3|$.

We will first define the events $\Omega_{n_k}(j)$ by the following conditions:

- (i) There are exactly $m$ points of $G_{n_k}$ inside $D$.
- (ii) There is a $1/M_j$ separation between $\partial D$ and the points of $G_{n_k}$ outside $D$, that is, there is no point of $G_{n_k}$ in the open disk given by the $1/M_j$-thickening of $D$.
- (iii) $|S_1^{n_k}| < M_j, |S_2^{n_k}| < M_j, |\tilde{S}_3^{n_k}| < M_j$.

Clearly, each $\Omega_{n_k}(j)$ is measurable with respect to $G_{n_k}$. On the event $\Omega_{n_k}$ (refer Theorem 6.2), the points in $(G_n)_{\text{out}}$, considered in uniform random order, yield a vector $\omega$ in $(D^c)^{n-m}$. Denoting the conditional distribution of $\zeta$ given $\omega$ to be $\rho_{\omega}^\zeta(\zeta')$ we recall from Proposition 7.2 that

$$\exp \left( -4mK(D, \delta)X_n \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2} \leq \rho_{\omega}^\zeta(\zeta') \leq \exp \left( 4mK(D, \delta)X_n \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2}.$$ 

Therefore, the bounds on $S_1(n_k), S_2(n_k)$ and $\tilde{S}_3(n_k)$ as in condition (iii) above imply that on $\Omega_{n_k}(j)$ we have, with $\delta = 1/M_j$,

$$\exp \left( -12mK(D, \delta)M_j \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2} \leq \rho_{\omega}^\zeta(\zeta') \leq \exp \left( 12mK(D, \delta)M_j \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2}.$$ 

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Let us consider the inequality on the right hand side, namely,

\[
\rho_2^n(\zeta') \leq \exp \left( 12mK(D, \delta)M_j \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2}.
\]

Cross multiplying, we get

\[
\rho_2^n(\zeta')|\Delta(\zeta)|^2 \leq \rho_2^n(\zeta)|\Delta(\zeta')|^2 \exp \left( 12mK(D, \delta)M_j \right).
\]

We now integrate the above inequalities, first with respect to the Lebesgue measure in the variable \( \zeta' \in A \), then with respect to the Lebesgue measure in the variable \( \zeta \in D^n \) and finally with respect to the distribution of \( \omega \) on the event

\[
\{ \omega \in (D^c)^{n-k-m} : (X_{\text{out}}^{n_k})^{-1} \{ \omega_1, \cdots, \omega_{n-k-m} \} \in (X_{\text{out}}^{n_k})^{-1}(B \cap \Omega_{n_k}(j)) \}.
\]

We can carry out the same procedure with the inequality on the left hand side. It can be seen that together they give us condition (3).

Proof. Let \( \Omega(j) := \lim_{k \to \infty} \Omega_{n_k}(j) \) exhausts \( \Omega^m \), and that \( \mathbb{P}(\Omega(j) \Delta \Omega_{n_k}(j)) \to 0 \) as \( k \to \infty \) for each fixed \( j \).

Since the \( M_j \)'s are increasing, we automatically have \( \Omega(j) \subset \Omega(j+1) \). Moreover, recall that \( \Omega^m \) is the event that there are \( m \) points of \( G \) inside \( D \). Each \( \Omega_{n_k}(j) \) satisfies this condition for \( G_{n_k} \). And finally, \( G_{n_k} \to G \) a.s. on \( D \). There three facts together imply that \( \Omega(j) \subset \Omega^m \) for each \( j \). It only remains to check that \( \mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0 \) as \( j \to \infty \). This is the goal of the next proposition, which will complete the proof of Theorem 1.1 for a disk. \( \blacksquare \)

**Proposition 8.1.** Let \( \Omega(j) \) be as defined above. Then there exists \( \Omega^{\text{corr}}(j) \) measurable with respect to \( X_{\text{out}} \) such that \( \mathbb{P}(\Omega(j) \Delta \Omega^{\text{corr}}(j)) = 0 \). Further, \( \mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0 \) as \( j \to \infty \).

**Proof.** Let \( \varepsilon > 0 \). We will demonstrate an event \( A_\varepsilon \) (which depends on \( j \)) such that \( A_\varepsilon \) is measurable with respect to \( X_{\text{out}} \) and there is a bad set \( \Omega_{\text{bad}}^\varepsilon \) of probability \( < \varepsilon \) such that \( \Omega(j) \setminus \Omega_{\text{bad}}^\varepsilon = A_\varepsilon \setminus \Omega_{\text{bad}}^\varepsilon \). As a result, \( \mathbb{P}(\Omega(j) \Delta A_\varepsilon) < \varepsilon \). Then \( \Omega^{\text{corr}}(j) = \lim_{k \to \infty} A_{2^-}\varepsilon \) will give us the desired event. It is easy to check that \( \mathbb{P}(\Omega(j) \Delta \Omega^{\text{corr}}(j)) = 0 \).

Let \( \delta = \delta(\varepsilon) < 1 \) be a small number, depending on \( \varepsilon \), to be chosen later.

We define the event \( A_\varepsilon \) by the following conditions:

- (i) \( N(\mathcal{G}_{\text{out}}) = m \), where \( N \) is as in Theorem 1.1.
- (ii) There is a \( \frac{1}{M_j} \) separation between \( \partial D \) and the points \( \omega \) of \( \mathcal{G} \) outside \( D \).
- (iii) \(|S_1| < M_j - \delta, |S_2| < M_j - \delta, |\bar{S}_3| < M_j - \delta|.

It is clear from the definition of \( A_\varepsilon \) that it is measurable with respect to \( X_{\text{out}} \).

By Proposition 7.8, \( S_i(n_k) \to S_i, i = 1, 2 \) and \( \bar{S}_3(n_k) \to \bar{S}_3 \) a.s. along our chosen subsequence. By Egorov’s Theorem, there is a bad event \( \Omega^1 \) of probability \( < \varepsilon/4 \) such that outside \( \Omega^1 \), this convergence is uniform. Therefore, on \( (\Omega^1)^c \) there is a large \( k_0 \) such that \(|S_i(n_k) - S_i| < \delta, i = 1, 2, 3\).
and $|\widetilde{S}_i(n_k) - \widetilde{S}_i| < \delta$ for all $k \geq k_0$. By a.s. convergence of $G_{n_k}$-s to $G$ on the compact set $2 \cdot \overline{D}$, we have a small event $\Omega^2$ of probability $< \varepsilon/4$ outside which conditions (i) and (ii) in the definition of $\Omega_{n_k}(j)$ are true or false simultaneously for all $G_{n_k}, k \geq k_1$, for some integer $k_1$. By the coupling of $G_{n_k}$-s and $G$ (which entails that $G_n \subset G_{n+1} \subset G$ for all $n$), this would imply that the conditions (i) and (ii) in the definition of $A_\varepsilon$ are also true or false respectively.

Since $M_j$ is not an atom of the distribution of $S_1, S_2$ or $\widetilde{S}_3$, there is an event $\Omega^3$ (of probability $< \varepsilon/2$) outside which each $S_i$ or $\widetilde{S}_i$ is either $> M_j + 2\delta$ or $< M_j - 2\delta$ ($\delta < 1$ is chosen based on $\varepsilon$ so that this condition is satisfied).

Define $\Omega_{bad} = \Omega^1 \cup \Omega^2 \cup \Omega^3$; clearly $P(\Omega_{bad}) < \varepsilon$. We note that on $\Omega_{bad}$, the conditions (i) and (ii) in the definition of $\Omega_{n_k}$ and the conditions (i) and (ii) in the definition of $A_\varepsilon$ are simultaneously true or false (for all $k$ large enough). This is because of the a.s. convergence $G_n \to G$ on the compact set $2 \cdot \overline{D}$. On $\Omega(j) \setminus \Omega_{bad}$, we have $|S_i(n_k)| < M_j$ for all large enough $k$, hence $|S_i(n) - S_i| < \delta$ implies $|S_i| < M_j + \delta$. But we are on $(\Omega^3)^c$, so $|S_i| \in (M_j - 2\delta, M_j + 2\delta)^c$, hence $|S_i| < M_j + \delta$ implies $|S_i| < M_j - 2\delta$, hence we are inside $A_\varepsilon$. Conversely, on $A_\varepsilon \setminus \Omega_{bad}$, we have $|S_i| < M_j - \delta$. But $|S_i(n_k) - S_i| < \delta$ for all large enough $k$. This implies $|S_i(n_k)| < M_j$ for all large enough $k$ which means we are on $\Omega(j)$. Hence $\Omega(j) \setminus \Omega_{bad} = A_\varepsilon \setminus \Omega_{bad}$. As a result, we have $\Omega(j) \Delta A_\varepsilon \subset \Omega_{bad}$, and $P(\Omega(j) \Delta A_\varepsilon) < \varepsilon$.

To show that $P(\Omega^n \setminus \Omega(j)) \to 0$ as $j \to \infty$, we define events $A'(j) \subset A_\varepsilon$ above by replacing $\delta$ by 1 in the condition (iii) in the definition of $A_\varepsilon$. Clearly, $A'(j) \subset A_\varepsilon$ for each $0 < \varepsilon < 1$, and therefore $A'(j) \subset \lim_{n \to \infty} A_{2-n} = \Omega(j)^{corr}$. It is also clear that $P(\Omega^n \setminus A'(j)) \to 0$ as $j \to \infty$, because $S_1, S_2$ and $S_3$ are well defined random variables (with no mass at $\infty$). These two facts imply that $P(\Omega^n \setminus \Omega(j)) \to 0$ as $j \to \infty$.

This completes the proof of the translation tolerance of the Ginibre ensemble in the case of $D$ being a disk.

9 Tolerance of the GAF zeros for a disk

In this section we obtain the estimates necessary to prove Theorem 1.2 in the case where $D$ is a disk. By translation invariance of $F$, we can take $D$ to be centred at the origin.

9.1 Polynomial Approximations

We focus on the event $\Omega_n^{m,\delta}$ which entails that $f_n$ has exactly $m$ zeroes inside $D$, and there is a $\delta$ separation between $\partial D$ and the outside zeroes. The corresponding event for the GAF zero process has positive probability, so by the distributional convergence $F_n \to F$, we have that $\Omega_n^{m,\delta}$ has positive probability (which is bounded away from 0 as $n \to \infty$).

Let us denote the zeroes of $f_n$ (in uniform random order) inside $D$ by $\zeta = (\zeta_1, \cdots, \zeta_m)$ and those outside $D$ by $\omega = (\omega_1, \omega_2, \cdots, \omega_{n-m})$. Let $s = \sum_{j=1}^{m} \zeta_j$, and $\Sigma_s := \{ (\zeta_1, \cdots, \zeta_m) \in D^m : \sum_{j=1}^{m} \zeta_j = s \}$
Then the conditional density $\rho^n_{\omega,s}(\zeta)$ of $\zeta$ given $\omega,s$ is of the form (see, e.g., [FH99]):

$$
\rho^n_{\omega,s}(\zeta) = C(\omega,s) \frac{|\Delta(\zeta_1, \ldots, \zeta_m, \omega_1, \ldots, \omega_{n-m})|^2}{\left( \sum_{k=0}^n \frac{|\sigma_k(\zeta,\omega)|}{\sqrt{(n)_k}} \right)^{2n+1}} \tag{9}
$$

where $C(\omega,s)$ is the normalizing factor, and for a vector $v = (v_1, \ldots, v_N)$ we define

$$
\sigma_k(v) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} v_{i_1} \cdots v_{i_k}
$$

and for two vectors $\underline{u}$ and $\underline{v}, \sigma_k(\underline{u},\underline{v})$ is defined to be $\sigma_k(\underline{v})$ where the vector $\underline{u}$ is obtained by concatenating the vectors $\underline{u}$ and $\underline{v}$.

Throughout this Subsection 9.1, the zeroes will be those of $f_n$ with $n$ fixed.

Let $(\zeta, \omega)$ and $(\zeta', \omega)$ be two vectors of points (under $\mathcal{F}_n$). Then the ratio of the conditional densities at these two vectors is given by

$$
\frac{\rho^n_{\omega,s}(\zeta')}{\rho^n_{\omega,s}(\zeta)} = \frac{|\Delta(\zeta', \omega)|^2}{|\Delta(\zeta, \omega)|^2} \frac{\left( \sum_{k=0}^n \frac{|\sigma_k(\zeta,\omega)|}{\sqrt{(n)_k}} \right)^{2n+1}}{\left( \sum_{k=0}^n \frac{|\sigma_k(\zeta',\omega)|}{\sqrt{(n)_k}} \right)^{2n+1}} \tag{10}
$$

The expression (10) has two distinct components: the ratio of two Vandermonde determinants and the ratio of certain expressions involving the elementary symmetric functions of the zeroes. We will consider these two components in two separate sections.

### 9.1.1 Ratio of Vandermondes

Here we consider the quantity $|\Delta(\zeta', \omega)|^2/|\Delta(\zeta, \omega)|^2$. We proceed exactly as in the case of the Ginibre ensemble. We refer the reader to section 7.1. The estimates here are valid for all pairs $(\zeta, \zeta') \in \mathcal{D}^m \times \mathcal{D}^m$.

**Proposition 9.1.** On $\Omega^{m,\delta}_n$ there are quantities $K(\mathcal{D}, \delta) > 0$ and $X_n(\omega) > 0$ such that for any $(\zeta, \zeta') \in \mathcal{D}^m \times \mathcal{D}^m$ we have

$$
\exp \left( -2mK(\mathcal{D}, \delta)X_n(\omega) \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2} \leq \frac{|\Delta(\zeta', \omega)|^2}{|\Delta(\zeta, \omega)|^2} \leq \exp \left( 2mK(\mathcal{D}, \delta)X_n(\omega) \right) \frac{|\Delta(\zeta')|^2}{|\Delta(\zeta)|^2}
$$

where $X_n(\omega) = \sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j} + \sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j^2} + \left( \sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega_j|^3} \right)$ and $\mathbb{E}[X_n(\omega)] \leq c_1(\mathcal{D}, m) < \infty$.

**Remark 9.1.** The estimates on $\sum \frac{1}{\omega_j}, \sum \frac{1}{\omega_j^2}$ and $\left( \sum \frac{1}{|\omega_j|^3} \right)$ which are necessary for Proposition 9.1 are proved in section 7.2.

**Corollary 9.2.** Given $M > 0$, we can replace the $X_n(\omega)$ in Proposition 9.1 by a uniform bound $M$ except on an event of probability less than $c(m, \mathcal{D})/M$.  

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9.1.2 Ratio of Symmetric Functions

In this section we will restrict \( \zeta \) and \( \zeta' \) to lie in the same constant-sum hyperplane. Let \( s = \sum_{i=1}^{m} \zeta_i \) and
\[
\Sigma_s := \{ \zeta \in D^m : \sum_{i=1}^{m} \zeta_i = s \}.
\]

Let \( D(\zeta, \omega) = \left( \sum_{k=0}^{n} \left| \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k}}} \right|^2 \right) \).

We want to bound the ratio \( (D(\zeta', \omega)/D(\zeta, \omega))^{n+1} \) from above and below. Our main goal is:

**Proposition 9.3.** Given \( M > 0 \) large enough, \( \exists n_0 \) such that for all \( n \geq n_0 \) the following is true: with probability \( \geq 1 - C/M \) we have, on \( \Omega_{n, \delta} \),
\[
e^{-2K(m,D)M \log M} \leq \left( \frac{D(\zeta', \omega)}{D(\zeta, \omega)} \right)^{n+1} \leq e^{2K(m,D)M \log M}
\]
for all \( \zeta' \in \Sigma_s, \) where \( s = \sum_{i=1}^{m} \zeta_i \) and \( (\zeta, \omega) \) is randomly generated from \( F_n \).

We will first prove several auxiliary propositions.

We begin with the observation
\[
\sigma_k(\zeta, \omega) = \sum_{i=0}^{m} \sigma_i(\zeta)\sigma_{k-i}(\omega), \quad (11)
\]

Note that \( \sigma_0(\zeta) = 1, \sigma_1(\zeta) = s = \sigma_1(\zeta') \) and \( |\sigma_i(\zeta)| < \binom{m}{i} r_0^i \) for all \( i \leq m \), where \( r_0 \) is the radius of \( D \). Since both \( \zeta \) and \( \zeta' \in \Sigma_s \), we have
\[
\sigma_k(\zeta', \omega) = \sigma_k(\zeta, \omega) + \sum_{i=2}^{m} |\sigma_i(\zeta') - \sigma_i(\zeta)|\sigma_{k-i}(\omega), \quad (12)
\]

Taking modulus squared on both sides, we have
\[
|\sigma_k(\zeta', \omega)|^2 = |\sigma_k(\zeta, \omega)|^2 + \sum_{i=2}^{m} |\sigma_i(\zeta') - \sigma_i(\zeta)|^2|\sigma_{k-i}(\omega)|^2 + 2\sum_{i=2}^{m} 2\Re \left( (\sigma_i(\zeta') - \sigma_i(\zeta))\overline{\sigma_k(\zeta, \omega)}\sigma_{k-i}(\omega) \right)
\]
\[
+ \sum_{i,j=2}^{m} \sum_{i \neq j} 2\Re \left( (\sigma_i(\zeta') - \sigma_i(\zeta))(\sigma_j(\zeta') - \sigma_j(\zeta))\overline{\sigma_k(\zeta, \omega)}\sigma_{k-j}(\omega) \right).
\]

Summing the above over \( k = 0, \ldots, n \), we obtain
\[
\sum_{k=0}^{n} |\sigma_k(\zeta', \omega)|^2 = \sum_{k=0}^{n} |\sigma_k(\zeta, \omega)|^2 + \sum_{i=2}^{m} |\sigma_i(\zeta') - \sigma_i(\zeta)|^2 \left( \sum_{k=0}^{n} |\sigma_{k-i}(\omega)|^2 \right)
\]
\[
+ \sum_{i=2}^{m} 2\Re \left( (\sigma_i(\zeta') - \sigma_i(\zeta))\left( \sum_{k=0}^{n} \overline{\sigma_k(\zeta, \omega)}\sigma_{k-i}(\omega) \right) \right)
\]
\[
+ \sum_{i,j=2}^{m} \sum_{i \neq j} 2\Re \left( (\sigma_i(\zeta') - \sigma_i(\zeta))(\sigma_j(\zeta') - \sigma_j(\zeta))\left( \sum_{k=0}^{n} \overline{\sigma_{k-i}(\omega)}\sigma_{k-j}(\omega) \right) \right).
\]

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Dividing throughout by \( \left( \sqrt{\frac{n}{k}} k! \right)^2 \) and applying triangle inequality we get

\[
D(\zeta', \omega) - A(\zeta, \zeta', \omega) \leq D(\zeta', \omega) \leq D(\zeta, \omega) + A(\zeta, \zeta', \omega)
\]

(13)

where we recall the fact that \( D(\zeta, \omega) = \left( \sum_{k=0}^{n} \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k} k!}} \right)^2 \) and \( A(\zeta, \zeta', \omega) = \)

\[
\sum_{i=2}^{m} |\sigma_i(\zeta') - \sigma_i(\zeta)|^2 \left( \sum_{k=0}^{n} \left| \frac{\sigma_{k-i}(\omega)}{\sqrt{\binom{n}{k} k!}} \right|^2 \right) + \sum_{i=2}^{m} 2|\sigma_i(\zeta') - \sigma_i(\zeta)| \left| \sum_{k=0}^{n} \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k} k!}} \frac{\sigma_{k-i}(\omega)}{\sqrt{\binom{n}{k} k!}} \right|
\]

\[
+ \sum_{i \neq j=2}^{m} 2|\sigma_i(\zeta') - \sigma_i(\zeta)| |\sigma_j(\zeta') - \sigma_j(\zeta)| \left| \sum_{k=0}^{n} \frac{\sigma_{k-i}(\omega)}{\sqrt{\binom{n}{k} k!}} \frac{\sigma_{k-j}(\omega)}{\sqrt{\binom{n}{k} k!}} \right|
\]

Divide throughout by \( D(\zeta, \omega) \) in the above inequalities, and observe that

(a) \( i \) and \( j \) vary between 2 and \( m \) (which is fixed and finite),

(b) \( |\sigma_i(\zeta') - \sigma_i(\zeta)| \) is bounded for each \( i \) as discussed immediately before (12).

In view of these facts, we conclude that in order to bound \( (D(\zeta', \omega)/D(\zeta, \omega))^n \) between two constants, it suffices to show that the quantities

\[
\left| \sum_{k=0}^{n} \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k} k!}} \frac{\sigma_{k-i}(\omega)}{\sqrt{\binom{n}{k} k!}} \right| / \left( \sum_{k=0}^{n} \left| \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k} k!}} \right|^2 \right)
\]

(14)

and

\[
\left| \sum_{k=0}^{n} \frac{\sigma_{k-i}(\omega)}{\sqrt{\binom{n}{k} k!}} \frac{\sigma_{k-j}(\omega)}{\sqrt{\binom{n}{k} k!}} \right| / \left( \sum_{k=0}^{n} \left| \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k} k!}} \right|^2 \right)
\]

(15)

for \( m \geq i, j \geq 2 \) are bounded above by \( M/n \) with high probability (depending on \( M \)) where \( M \geq 0 \) is a constant.

Observe that \( \sigma_k(\zeta, \omega)/\sqrt{\binom{n}{k} k!} = \xi_{n-k}/\xi_n \), where the \( \xi_j \)-s are standard complex Gaussians. On the other hand, we do not have good control over \( \sigma_k(\omega) \). To do that, the idea is to obtain a convenient expansion of \( \sigma_k(\omega) \) in terms of \( \sigma_i(\zeta, \omega) \). To this end, we formulate:

**Proposition 9.4.** On the event \( \Omega^m_\delta \) we have, for \( 0 \leq k \leq n - m \),

\[
\sigma_k(\omega) = \sigma_k(\zeta, \omega) + \sum_{r=1}^{k} g_r \sigma_{k-r}(\zeta, \omega)
\]

where a.s. the random variables \( g_r \) are \( O(KD_m) \) as \( r \rightarrow \infty \), for a deterministic quantity \( K(D, m) \) and the constant in \( O \) being deterministic and uniform in \( n \) and \( \delta \).

**Proof.** We begin with the observation that

\[
\sigma_k(\zeta, \omega) = \sum_{r=0}^{m} \sigma_r(\zeta) \sigma_{k-r}(\omega).
\]

(16)
From this, we have
\[ \sigma_k(\omega) = \sigma_k(\zeta, \omega) - \sum_{r=1}^{m} \sigma_r(\zeta) \sigma_{k-r}(\omega). \] (17)

Proceeding inductively in (17) we can similarly expand each of the lower order \( \sigma_{k-r}(\omega) \) in terms of \( \sigma_j(\zeta, \omega) \) and obtain an expansion of \( \sigma_k(\omega) \) in terms of \( \sigma_j(\zeta, \omega), j = 1, \cdots, k \):

\[ \sigma_k(\omega) = \sigma_k(\zeta, \omega) + \sum_{r=1}^{k} g_r \sigma_{k-r}(\zeta, \omega). \] (18)

The coefficient of \( \sigma_k(\zeta, \omega) \) is 1 and the rest of the coefficients are polynomials in \( \sigma_j(\zeta), j = 1, \cdots, m \). Due to the inductive structure, it is easy to see that \( f_r \)-s satisfy a recurrence relation:

\[
\begin{pmatrix}
  g_i \\
g_{i-1} \\
  \vdots \\
g_{i-m+1}
\end{pmatrix} = 
\begin{pmatrix}
  -\sigma_1(\zeta) & -\sigma_2(\zeta) & \cdots & -\sigma_m(\zeta) \\
  1 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
  g_{i-1} \\
g_{i-2} \\
  \vdots \\
g_{i-m}
\end{pmatrix}
\] (19)

with the boundary conditions \( g_i = -\sigma_i(\zeta) \) for \( i = 0, \cdots, m - 1 \) and \( g_i = 0 \) for \( i < 0 \).

Denoting the matrix in (19) above by \( A \), we observe that its eigenvalues are precisely the negatives of the inside zeroes \( \zeta_1, \cdots, \zeta_m \). Due to the recursive structure in (19), we note that \( g_k \) is an element of \( A^k \) applied to the vector \( (-\sigma_{m-1}(\zeta), \cdots, -\sigma_0(\zeta)) \). This implies that \( g_k \) is a linear combination of the entries of \( A^k \) (with the coefficients in the linear combination being independent of \( k \), but depending on \( D \) and \( m \)).

From Proposition 9.5 we that if the eigenvalues of a matrix \( A \) have modulus < \( \rho' \), then the entries of the matrix \( A^k \) are \( o(\rho'^k) \). Now, the eigenvalues of \( A \) are \( \zeta_i \)-s and for each \( i \) we have \( |\zeta_i| \leq \text{radius}(D) = K \). Combining the last two paragraphs, we have \( g_k = O((K(D) + 1)^k) \). Crucially, the constant in the \( O \) above is uniform in \( n \) and \( \delta \) (although it can depend on \( m \)).

We complete this discussion with the following result on the growth of matrix entries:

**Proposition 9.5.** Let \( A \) be a \( d \times d \) matrix. Let \( \rho' > \rho(A) := \max\{|\lambda| : \lambda \text{ an eigenvalue of } A\} \). Then \( (A^r)_{ij} = o(\rho'^r) \) as \( r \to \infty \), for all \( i, j \).

**Proof.** This follows from the well known result (Gelfand’s spectral radius theorem): \( \rho(A) = \lim_{r \to \infty} \|A^r\|^{1/r} \) where \( \| \cdot \| \) denotes the Frobenius norm of the matrix, and the fact that \( \sup_{1 \leq i,j \leq d} |B_{ij}| \leq c(d)\|B\| \) for any \( d \times d \) matrix \( B \).
**Notation:** We denote the falling factorial (of order $k$) of an integer $x$ by

$$(x)_k := x(x-1) \cdots (x-k+1).$$

For $k = 0$ we set $(x)_0 = 1$.

Switching variables to $l = n - k$, we can rewrite the expansion in Proposition 9.4 as

$$\frac{\sigma_{n-l}(\zeta, \omega)}{\sqrt{\binom{n}{n-l}(n-l)!}} = \frac{\xi_l}{\xi_n} + \sum_{r=1}^{n-l} f_r \frac{\xi_{l+r}}{\xi_n} \frac{1}{\sqrt{(l+r)_r}}$$

for $l \geq m$. Define for any integer $l \geq 1$

$$\eta_l^{(n)} = \sum_{r=1}^{n-l} f_r \xi_{l+r} \frac{1}{\sqrt{(l+r)_r}}$$

As a result, for $l \geq m$ we have

$$\frac{\sigma_{n-l}(\omega)}{\sqrt{\binom{n}{n-l}(n-l)!}} = \frac{1}{\xi_n} \left[ \xi_l + \eta_l^{(n)} \right].$$

**Proposition 9.6.** Let $\eta_l^{(n)}$ be as in (20) and $\gamma = \frac{1}{8}$. Then $\exists$ positive random variables $\eta_l$ (independent of $n$) such that a.s. $|\eta_l^{(n)}| \leq \eta_l$, and for fixed $l_0 \in \mathbb{N}$ and large enough $M > 0$ we have

(i) $\mathbb{P} \left[ \eta_l > \frac{M}{l_0} \right. \text{ for some } l \geq 1 \left. \right] \leq e^{-c_1 M^2}$

(ii) $\mathbb{P} \left[ \eta_l > \frac{M}{l_0} \right. \text{ for some } l \geq l_0 \left. \right] \leq e^{-c_2 M^2 l_0^{1/4}}$

where $c_1, c_2$ are constants that depend on the domain $D$ and on $m$.

**Proof.** We begin by recalling from Proposition 9.4 that a.s. $|g_r| \leq K^r$ for some $K = K(D,m)$. Moreover, $\sqrt{(l+p)} \geq \sqrt{2l^{p/4} p^{1/4}}$. Hence,

$$|\eta_l^{(n)}| \leq \sum_{r=1}^{n-l} \frac{K^r}{2^{r/2} l^{r/4} (r!)^{1/4}} |\xi_{l+r}|.$$

Let $B$ be such that $\sup_{r \geq 1} \frac{K^r}{2^{r/2} l^{r/4} (r!)^{1/8}} \leq B$. Then $|\eta_l^{(n)}| \leq \sum_{r=1}^{n-l} \frac{B |\xi_{l+r}|}{l^{r/4} (r!)^{1/8}}$.

Define $\eta_l = \sum_{r=1}^{\infty} \frac{B |\xi_{l+r}|}{l^{r/4} (r!)^{1/8}}$. Clearly, $|\eta_l^{(n)}| \leq \eta_l$. Now, $|\xi_{l+r}|^2$ is an exponential random variable with mean 1, we have

$$\mathbb{P} \left( |\xi_{l+r}| \geq \frac{M l^{p/8} (r!)^{1/16}}{B} \right) \leq \exp \left( -\frac{M^2 l^{p/4} (r!)^{1/8}}{B^{2}} \right).$$

(22)
Moreover,
\[
\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \exp \left( -\frac{M^2 l^r/4 (r!)^{1/8}}{B^2} \right) \leq C_1 \exp \left( -\frac{M^2}{B^2} \right).
\]

So we can have \(|\xi_{l+r}| \leq \frac{M^{r/8} (r!)^{1/16}}{B} \) for all \(l \geq 1, r \geq 0\), except on an event of probability \(\leq C_1 \exp \left( -\frac{M^2}{B^2} \right)\). The last expression can be made arbitrarily small by fixing \(M\) to be large enough. Note that for large enough \(M\), we have \(C \exp \left( -\frac{M^2}{B^2} \right) \leq \exp \left( -cM^2 \right)\). On the complement of this small event, we have for all \(l \geq 1\)
\[
\eta_l \leq \sum_{r=1}^{\infty} \frac{M}{l^r (r!)^{1/16}} \leq \sum_{r=1}^{\infty} \frac{M}{l^r / (r!)^{1/16}} \leq C_1 \frac{M}{l^{1/8}}.
\]

We then absorb this factor of \(C\) into \(M\) by changing the constant \(c_1\) appearing in the exponent of the tail estimate (i) we want to prove.

For proving (ii) we proceed exactly as in the case of (i) above and sum (22) over \(r \geq 1\) and \(l \geq l_0\).

We are now ready to prove bounds on the quantities in (13) and (15). To this end, we first express them in terms of \(\xi\)-s, using \(\sigma_{n-l}(\xi, \omega) / \sqrt{(n-l)!} (n-l)! = \xi_l / \xi \) and (21). For \(2 \leq i, j \leq m\) these expressions, (after clearing out \(|\xi_{l+1}|^2\) from numerator and denominator) reduce to:

\[
\left| \sum_{l=0}^{n-i-l} \frac{\xi_{l+i} + \eta_{l+i}^{(n)}}{\sqrt{(l+i)} \sqrt{(l+j)}} \right| \left/ \left( \sum_{l=0}^{n} |\xi_l|^2 \right) \right.
\]
(23)

and

\[
\left| \sum_{l=0}^{n-i-l-j} \frac{\xi_{l+i} + \eta_{l+i}^{(n)} \xi_{l+j} + \eta_{l+j}^{(n)}}{\sqrt{(l+i)} \sqrt{(l+j)}} \right| \left/ \left( \sum_{l=0}^{n} |\xi_l|^2 \right) \right.
\]
(24)

Define
\[
E_n = \sum_{l=0}^{n} |\xi_l|^2, \quad L_{ij}^{(n)} = \sum_{l=0}^{n-i-n-j} \frac{\xi_{l+i} \xi_{l+j}}{\sqrt{(l+i)} \sqrt{(l+j)}} ,
\]
\[
M_{ij}^{(n)} = \sum_{l=0}^{n-i-n-j} \frac{|\xi_{l+i}| \eta_{l+j}^{(n)}}{\sqrt{(l+i)} \sqrt{(l+j)}} , \quad N_{ij}^{(n)} = \sum_{l=0}^{n-i-n-j} \frac{\eta_{l+i} \eta_{l+j}^{(n)}}{\sqrt{(l+i)} \sqrt{(l+j)}} .
\]

For \(i = 0\) the product \((l+i)\) in the denominator is replaced by \(1\). Expanding out the products and applying the triangle inequality in (23) and (24), we observe that to upper bound the expressions in (23) and (24) it suffices to upper bound the following quantities for \(2 \leq i, j \leq m\) (remember that \(|\eta_l^{(n)}| \leq \eta_l\) for all \(l\)):

\[
\left( \frac{L_{0j}^{(n)}}{E_n} + \frac{M_{0j}^{(n)}}{E_n} \right) , \quad \left( \frac{L_{ij}^{(n)}}{E_n} + \frac{M_{ij}^{(n)}}{E_n} + \frac{M_{ji}^{(n)}}{E_n} + \frac{N_{ij}^{(n)}}{E_n} \right)
\]
(25)
Let
\[ Y_n = \sum_{i=2}^{m} L_{0i}^{(n)} + \sum_{i=2}^{m} M^{(n)}_{0i} + \sum_{i,j \geq 2}^{m} L_{ij}^{(n)} + \sum_{i,j \geq 2}^{m} M_{ij}^{(n)} + \sum_{i,j \geq 2}^{m} N^{(n)}_{ij}. \]

Recall (13) and recall that for fixed \( m \) and \( D \), we have that \( \sigma_i(\zeta) \) are uniformly bounded. Putting all these together, we deduce that: for some constant \( K(m, D) \) we have
\[
1 - K(m, D) \frac{Y_n}{E_n} \leq \frac{D(\zeta', \omega)}{D(\zeta, \omega)} \leq 1 + K(m, D) \frac{Y_n}{E_n}. \quad (26)
\]
Regarding \( Y_n \) and \( E_n \), we have the following estimates:

**Proposition 9.7.** Given \( M > 0 \) we have:
(i) \( \mathbb{P}[Y_n \geq M \log M] \leq c(m, D)/M \),
(ii) Given \( M > 0 \) there exists \( n_0 \) such that for \( n \geq n_0 \) we have \( \mathbb{P}[\frac{n}{M} \leq |E_n| \leq 2n] \geq 1 - \frac{1}{M} \).

**Proof.** Part (i):
We will show that whenever \( X = L_{ij}^{(n)}, X = M_{ij}^{(n)} \) or \( X = N_{ij}^{(n)} \), we have that \( X \) satisfies \( \mathbb{P}[|X| > M \log M] < c'(m, D)/M \). This would imply \( \mathbb{P}[Y_n > M \log M] < c_1(m, D)/M \) albeit for a different constant \( c(m, D) \). This is easily seen as follows. The number of summands in the definition of \( Y_n \) is a polynomial in \( m \), let us call it \( p(m) \). Now, for each summand \( X \) of \( Y_n \), we have
\[
\mathbb{P}\left[|X| > \frac{M \log M}{p(m)}\right] \leq \mathbb{P}\left[|X| > \frac{M}{p(m)} \log \frac{M}{p(m)}\right] \leq c'(m, D)p(m)/M.
\]
If \( Y_n > M \log M \) and there are \( p(m) \) summands, at least one of the summands must be \( > M \log M/p(m) \). By a union bound over the summands, this implies \( \mathbb{P}[Y_n > M \log M] \leq c'(m, D)p(m)^2/M \).

Setting \( c(m, D) = c'(m, D)p(m)^2 \) as the new constant, this would give us part (i) of the proposition.

It remains to show that each of the summands in \( Y_n \) satisfies \( \mathbb{P}[|X| > M \log M] < c(m, D)/M \) with the quantity \( c(m, D) \) being uniform in \( n \). The understanding is that \( c \) will depend on the particular summand, and we will take a maximum over the \( p(m) \) summands in order to obtain \( c'(m, D) \) as in the previous paragraph. We will take up the cases of \( L, M \) and \( N \) separately.

**Estimating \( L_{ij}^{(n)}, i \vee j \geq 2 \):**
We have
\[
\mathbb{E}[|L_{ij}^{(n)}|^2] = \sum_{l=0}^{n-i \wedge n-j} \frac{1}{(l+i)(l+j)} \leq \sum_{l=0}^{\infty} \frac{1}{(l+i)(l+j)} = c(i, j) \leq c < \infty
\]
(since either \( i \) or \( j \geq 2 \)). An application of Chebyshev’s inequality proves the desired tail bound on \( L_{ij}^{(n)} \).

**Estimating \( M_{ij}^{(n)}, j \geq 2 \):**
By Proposition 9.7 we can assume (dropping an event $A$ of probability $\leq e^{-c(\log M)^2}$) that $\max_{l \geq 1} \eta_l \leq \log M/1^{1/8}$. Applying triangle inequality to the definition of $M_{ij}^{(n)}$ and using the upper bound on $\eta_{l+j}$ we have

$$E \left[ |M_{ij}^{(n)}| 1_{A^c}\right] \leq \log M E \left[ \sum_{l=0}^{n-i \wedge n-j} \frac{|\xi_{l+i}|}{(l+j)^{1/8}(l+i)(l+j)} \right] \leq m(i, j) \log M \leq c \log M < \infty$$

where $m(i, j) = \sum_{l=0}^{\infty} \frac{E[|\xi|]}{(l+j)^{1/8}(l+i)(l+j)} \leq c < \infty$, and $\xi$ is a standard complex Gaussian. In the last step, we have used the fact that $j \geq 2$ in order to upper bound $m(i, j)$ uniformly in $i$ and $j$. Applying Markov inequality gives us the desired tail estimate $P(|M_{ij}^{(n)}| 1_{A^c} > M \log M) \leq c/M$. This, along with $P(A) < e^{-c(\log M)^2}$ completes the proof.

Estimating $N_{ij}^{(n)}$, $i \wedge j \geq 2$:

We consider the event $A$ as in the case of $M_{ij}^{(n)}$. On $A^c$, we use $\eta_{l+i} \eta_{l+j} \leq (\log M)^2 < M \log M$ (for large $M$), and the rest is bounded above by

$$\sum_{l=0}^{n-i \wedge n-j} \frac{1}{l^{1/4}(l+i)(l+j)} \leq \sum_{l=0}^{\infty} \frac{1}{(l+i)^{1/8}(l+j)^{1/8}(l+i)(l+j)} = c(i, j) \leq c < \infty.$$

This, along with $P[A] < e^{-c(\log M)^2}$ establishes the desired tail bound on $N_{ij}^{(n)}$.

All these arguments complete the proof of part (i) of Proposition 9.7.

Part(ii):

We simply observe that $|\xi|^2$-s are i.i.d. exponentials and by the strong law of large numbers, $E_n/n \to 1$ a.s. From this, the statement of part (ii) follows.

Define

$$L_{ij} = \sum_{l=0}^{\infty} \frac{\xi_{l+i}\xi_{l+j}}{\sqrt{(l+i)(l+j)}}, \quad \tau(L_{ij}^{(n)}) = L_{ij} - L_{ij}^{(n)}, (i \wedge j \geq 2)$$

$$M_{ij} = \sum_{l=0}^{\infty} \frac{|\xi_{l+i}|\eta_{l+j}}{\sqrt{(l+i)(l+j)}}, \quad \tau(M_{ij}^{(n)}) = M_{ij} - M_{ij}^{(n)}, (j \geq 2)$$

$$N_{ij} = \sum_{l=0}^{\infty} \frac{\eta_{l+i}\eta_{l+j}}{\sqrt{(l+i)(l+j)}}, \quad \tau(N_{ij}^{(n)}) = N_{ij} - N_{ij}^{(n)}, (i \wedge j \geq 2).$$

For $i = 0$ the product $(l+i)_i$ in the denominator is replaced by 1. The above random variables have finite first moments, as can be seen by arguing on similar lines to the estimates in Proposition 9.7. Similar arguments with first moments of $\tau(L_{ij}^{(n)})$, $\tau(M_{ij}^{(n)})$, $\tau(N_{ij}^{(n)})$ also show that:

**Proposition 9.8.** As $n \to \infty$, we have each of the random variables $\tau(L_{ij}^{(n)})$, $\tau(M_{ij}^{(n)})$, $\tau(N_{ij}^{(n)})$, as defined above, converge to 0 in $L^1$, and hence, in probability.
We omit the proof to avoid repetitiveness.

Now we are ready to prove Proposition 9.3.

Proof. Proof of Proposition 9.3
We refer back to equation (26). By virtue of Proposition 9.7 we have, dropping an event of probability $\leq \frac{c}{M}$, we have $|Y_n/E_n| \leq 2M \log M/n$ (for $n \geq n_0$, where $n_0$ is large enough such that Proposition 9.7 part(ii) holds). Raising this to the $(n + 1)$th power we obtain the desired result. All constants are absorbed in $K(m, D)$. The randomness is clearly in $(\zeta, \omega)$. ■

For the following corollary we recall the definition of $\Sigma_s$ from the beginning of Section 9.1.2.

Corollary 9.9. Given $M > 0$ large enough, $\exists n_0$ such that for all $n \geq n_0$, we have, on $\Omega_n^{m,\delta}$ (except on an event of probability $\leq C/M$)

$$e^{-4K(m, D)M \log M} \leq \left( \frac{D(\zeta'', \omega)}{D(\zeta', \omega)} \right)^{n+1} \leq e^{4K(m, D)M \log M}$$

for all $(\zeta'', \zeta') \in \Sigma_s \times \Sigma_s$ where $(\zeta, \omega)$ is picked randomly from the distribution $\mathbb{P}[F_n]$ and $s = \sum_{i=1}^m \zeta_i$.

Proof. We reduce to random $s$ from a random $\zeta$ by taking the ratio

$$\frac{D(\zeta'', \omega)}{D(\zeta', \omega)} = \frac{D(\zeta'', \omega)}{D(\zeta', \omega)} \frac{D(\zeta', \omega)}{D(\zeta, \omega)}$$

for $(\zeta', \zeta'') \in \Sigma_s \times \Sigma_s$ where $s = \sum_{i=1}^m \zeta_i$, and applying the bounds in Proposition 9.3. ■

9.1.3 Estimate for Ratio of Conditional Densities

Proposition 9.10. There exist constants $K(m, D, \delta)$ such that given $M > 0$ large enough, we have for $n \geq n_0(m, M, D)$ the following inequalities hold on $\Omega_n^{m,\delta}$, except for an event of probability $\leq c(m, D)/M$:

$$\exp \left( -K(m, D, \delta)M \log M \right) \frac{\Delta(\zeta'')^2}{\Delta(\zeta)^2} \leq \frac{\rho_{\omega,s}(\zeta'')}{\rho_{\omega,s}(\zeta')} \leq \exp \left( K(m, D, \delta)M \log M \right) \frac{\Delta(\zeta'')^2}{\Delta(\zeta')}$$

uniformly for all $(\zeta', \zeta'') \in \Sigma_s \times \Sigma_s$, where $(s, \omega)$ corresponds to a point configuration picked randomly from $\mathbb{P}[F_n]$.

Proof. We simply put together the estimates for the ratios of the Vandermondes as well as the symmetric functions and subsume all relevant constants in $c(m, D)$ and $K(m, D, \delta)$. ■
9.2 Estimates for Inverse Powers of Zeroes

In this section we prove estimates on the (smoothed) sum of inverse powers of GAF zeroes.

Proposition 9.11. Let $\Phi$ be a $C^\infty$ radial function supported on the annulus between $r_0$ and $3r_0$. Let $\Phi_R = \Phi(z/R)$. We have

\[
(i) \mathbb{E} \left[ \int \frac{\Phi_R(z)}{|z|^l} d[\mathcal{F}_n](z) \right] \leq C(r_0, \Phi, l)/R^l
\]

\[
(ii) \mathbb{E} \left[ \int \frac{\Phi_R(z)}{|z|^l} d[\mathcal{F}_n](z) \right] \leq C_1(r_0, \Phi, l)/R^{l-2}.
\]

The same is true for $\mathcal{F}$ in place of $\mathcal{F}_n$. The constants $C(r_0, \Phi, l)$ and $C_1(r_0, \Phi, l)$ are uniform in $n$.

Proof. Part (i): We begin with

\[
\int \frac{1}{z^l} \Phi_R(z) d[\mathcal{F}_n](z) = \int \frac{1}{z^l} \Phi_R(z) \Delta \log(|f_n(z)|) d\mathcal{L}(z).
\]

Now, $\log(\sqrt{\mathbb{E}(|f_n(z)|^2)})$ is a radial function, and Laplacian of a radial function is also radial. Hence,

\[
\int \frac{1}{z^l} \Phi_R(z) \Delta \log(\sqrt{\mathbb{E}(|f_n(z)|^2)}) d\mathcal{L}(z) = 0
\]

because for $l \geq 1$, $1/z^l$ when integrated against a radial function is always 0. Let $\hat{f}_n(z) = f_n(z)/\sqrt{\mathbb{E}(|f_n(z)|^2)}$. Then the above argument implies

\[
\int \frac{1}{z^l} \Phi_R(z) d[\mathcal{F}_n](z) = \int \frac{1}{z^l} \Phi_R(z) \Delta \log(|\hat{f}_n(z)|) d\mathcal{L}(z). \tag{27}
\]

Integrating by parts on the right hand side of (27) we have

\[
\mathbb{E} \left[ \int \frac{1}{z^l} \Phi_R(z) d[\mathcal{F}_n](z) \right] \leq \int \left| \Delta \left( \frac{1}{z^l} \Phi_R(z) \right) \right| \mathbb{E} \left[ |\log(|\hat{f}_n(z)|)| \right] d\mathcal{L}(z). \tag{28}
\]

Now, the integrand is non-zero only for $Rr_0 \leq |z| \leq 3Rr_0$. Hence it is easy to see that $\left| \Delta \left( \frac{1}{z^l} \Phi_R(z) \right) \right| \leq C(r_0, \Phi, l)/R^{l+2}$. Further, $\mathbb{E} \left[ |\log(|\hat{f}_n(z)|)| \right]$ is a constant because $\hat{f}_n$ is $N_{\mathbb{C}}(0, 1)$.

Finally, $\int_{3Rr_0, D \setminus Rr_0, D} d\mathcal{L}(z) = 8\pi r_0^2 R^2$, where $D$ is the unit disk.

Putting all these together, we have

\[
\int \left| \Delta \left( \frac{1}{z^l} \Phi_R(z) \right) \right| \mathbb{E} \left[ |\log(|\hat{f}_n(z)|)| \right] d\mathcal{L}(z) \leq C(r_0, \Phi, l)/R^l
\]

as desired (all constants are subsumed in $C(r_0, \Phi, l)$).

Since $\Phi$ is a radial function on $\mathbb{C}$, therefore there exists a function $\tilde{\Phi}$ on $\mathbb{R}_{\geq 0}$ such that $\Phi(z) = \tilde{\Phi}(|z|)$.
• Part (ii): We have, with \( r = |z| \),
\[
\mathbb{E} \left[ \int \frac{1}{|z|^l} \Phi_R(z) d\mathcal{F}_n(z) \right] = c \int \frac{1}{r^l} \Phi_R(r) \Delta \log(\sqrt{K_n(z,z)}) r dr
\]
where the integral on the right hand side is over the non-negative reals. Integrating by parts,
\[
\int \frac{1}{r^l} \Phi_R(r) \Delta \log(\sqrt{K_n(z,z)}) r dr \leq \int \left| \Delta \left( \frac{1}{r^l} \Phi_R(r) \right) \right| \log(\sqrt{K_n(z,z)}) r dr.
\]
But \( \log(\sqrt{K_n(z,z)}) \leq \log(\sqrt{K(z,z)}) = \frac{1}{2} r^2 \leq \frac{9}{2} r_0^2 R^2 \) and \( |\Delta \Phi_R(z)| \leq C(r_0, \Phi, l)/R^{l+2} \), hence
\[
\int \left| \Delta \left( \frac{1}{r^l} \Phi_R(r) \right) \right| \log(\sqrt{K_n(z,z)}) r dr \leq C_1(r_0, \Phi, l)/R^{l-2}
\]
for another constant \( C_1 \) (which is clearly uniform in \( n \)).

**Remark 9.2.** The constant \( C(r_0, \Phi, l) \) above is of the form \( p(l) \left( \frac{1}{r_0} \right)^l C(\Phi) \) where \( p \) is a fixed polynomial (of degree 2), and \( C(\Phi) \) is a constant that depends only on \( \Phi \). Both \( p \) and \( C(\Phi) \) are uniform in \( n \). Similarly, \( C_1(r_0, \Phi, l) \) is of the form \( p_1(l) \left( \frac{1}{r_0} \right)^{l-2} C_1(\Phi) \); where \( p_1 \) is another degree 2 polynomial and \( C_1(\Phi) \) is uniform in \( n \).

Let \( r_0 = \text{radius} (D) \). Let \( \varphi \) be a non-negative radial \( C_c^\infty \) function supported on \( [r_0, 3r_0] \) such that \( \varphi = 1 \) on \( [r_0 + \frac{r_0}{2}, 2r_0] \) and \( \varphi(r_0 + r) = 1 - \varphi(2r_0 + 2r) \), for \( 0 \leq r \leq \frac{1}{2} r_0 \). In other words, \( \varphi \) is a test function supported on the annulus between \( r_0 \) and \( 3r_0 \) and its “ascent” to 1 is twice as fast as its “descent”.

Let \( \varphi \) be another radial test function with the same support as \( \varphi \), and \( \tilde{\varphi}(r_0 + x r_0) = 1 \) for \( 0 \leq x \leq \frac{1}{2} \) and \( \tilde{\varphi} = \varphi \) otherwise.

**Proposition 9.12.**
\[
\mathbb{E} \left[ \left| \int \frac{\varphi(z)}{|z|^l} d\mathcal{F}_n(z) \right| \right] \leq \mathbb{E} \left[ \left| \int \frac{\tilde{\varphi}(z)}{|z|^l} d\mathcal{F}_n(z) \right| \right] \leq c(r_0, \varphi, l)
\]
where \( c(r_0, \varphi, l) \) is uniform in \( n \), and the same result remains true when \( \mathcal{F}_n \) is replaced by \( \mathcal{F} \).

**Proof.** For fixed \( l \), let \( a(n) = \int \frac{\tilde{\varphi}(z)}{|z|^l} d\mathcal{F}_n(z) \) and \( b(n) = \int \frac{\varphi(z)}{|z|^l} d\mathcal{F}_n(z) \).

We have, \( \mathbb{E}[|a(n)|] \leq \mathbb{E}[b(n)] = C \int \frac{\tilde{\varphi}(z)}{|z|^l} \Delta \log(K_n(z,z)) dL(z) \) for some constant \( C \). Using the uniform convergence of the continuous functions \( \Delta \log(K_n(z,z)) \to \Delta \log(K(z,z)) < \infty \) on the (compact) support of \( \tilde{\varphi} \), we deduce that \( \mathbb{E}[b(n)]\)'s are uniformly bounded by constants that depend on \( r, \tilde{\varphi} \) and \( l \). It is obvious from the above argument that the same holds true for \( \mathcal{F} \) instead of \( \mathcal{F}_n \).

**Proposition 9.13.** Let \( r_0 \) be the radius of \( D \). Let \( \varphi \) and \( \tilde{\varphi} \) be defined as above.

(i) The random variables
\[
S_l(n) := \int \frac{\tilde{\varphi}(z)}{|z|^l} d\mathcal{F}_n(z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2j}(z)}{|z|^l} d\mathcal{F}_n(z) = \sum_{\omega \in \mathcal{F}_n \cap D^c} \frac{1}{\omega^l} \quad (\text{for } l \geq 1)
\]
and
\[ \tilde{S}_l(n) := \int \frac{\tilde{\varphi}(z)}{|z|^l} d[F_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2j}(z)}{|z|^l} d[F_n](z) = \sum_{\omega \in F_n \cap D^n} \frac{1}{|\omega|^l} \quad (\text{for } l \geq 3) \]

have finite first moments which, for every fixed \( l \), are bounded above uniformly in \( n \).

(ii) There exists \( k_0 = k_0(\varphi) \geq 1 \), uniform in \( n \) and \( l \), such that for \( k \geq k_0 \) the "tails" of \( S_l(n) \) and \( \tilde{S}_l(n) \) beyond the disk \( 2^k \cdot D \), given by
\[
\tilde{\tau}_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2j}(z)}{|z|^l} d[F_n](z) \quad (\text{for } l \geq 1) \quad \text{and} \quad \tau_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2j}(z)}{|z|^l} d[F_n](z) \quad (\text{for } l \geq 3)
\]
satisfy the estimates
\[
\mathbb{E} \left[ \left| \tau_l^n(2^k) \right| \right] \leq C_1(\varphi, l)/2^{kl/2} \quad \text{and} \quad \mathbb{E} \left[ \left| \tilde{\tau}_l^n(2^k) \right| \right] \leq C_2(\varphi, l)/2^{k(l-2)/2}.
\]

All of the above remain true when \( F_n \) is replaced by \( F \), for which we use the notations \( S_l \) and \( \tilde{S}_l \) to denote the quantities corresponding to \( S_l(n) \) and \( \tilde{S}_l(n) \).

**Remark 9.3.** For \( F \), by the sum \( \left( \sum_{\omega \in F \cap D^n} \frac{1}{|\omega|^l} \right) \) we denote the quantity \( S_l = \int \frac{\tilde{\varphi}(z)}{z^l} d[F](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2j}(z)}{z^l} d[F](z) \) due to the obvious analogy with \( F_n \), where the corresponding sum \( S_l(n) \) is indeed equal to \( \left( \sum_{\omega \in F_n \cap D^n} \frac{1}{|\omega|^l} \right) \) with its usual meaning.

**Proof.** The functions \( \tilde{\varphi} \) and \( \varphi_{2j} \), \( j \geq 1 \) form a partition of unity on \( D^n \), hence the identities appearing in part (i). That the left hand side in part (i) has finite expectation can be seen from corollary 9.12 and Proposition 9.12 it is uniformly bounded in \( n \) because so are \( C(r_0, \varphi, l) \) in corollary ?? and \( c(r_0, \varphi_0) \) in Proposition 9.12.

Fix \( n, l \geq 1 \). Set \( \psi_k = \int \frac{\varphi_{2j}(z)}{z^l} d[F_n](z) \) for \( k \geq 1 \), and \( \psi_0 = \int \frac{\tilde{\varphi}(z)}{z^l} d[F_n](z) \). When \( l \geq 3 \) we also define \( \gamma_l = \int \frac{\varphi_{2j}(z)}{z^l} d[F_n](z) \) for \( k \geq 1 \), and \( \gamma_0 = \int \frac{\tilde{\varphi}(z)}{z^l} d[F_n](z) \). Let \( \Psi_k \) and \( \Gamma_k \) denote the analogous quantities defined with respect to \( F \) instead of \( F_n \).

We begin with the observation that for \( k \geq 1 \) we have \( \mathbb{E}[\psi_k] = 0 \). This implies that
\[
\mathbb{E}[|\psi_k|] \leq \left( \mathbb{E}[|\psi_k|^2] \right)^{1/2} = \sqrt{\text{Var}[\psi_k]}
\]
We then apply Proposition 9.11 part (i) to the function \( \Phi = \varphi \) and \( R = 2^k \) to obtain
\[
\mathbb{E}[|\psi_k|] \leq C(r_0, \varphi, l)/(2^k)^l. \tag{29}
\]
We also observe that \( \mathbb{E}[|\psi_0|] \leq \mathbb{E}[|\gamma_0|] \) which, for fixed \( l \), is uniformly bounded above in \( n \), using Proposition 9.12.
The above arguments imply that for $l \geq 1$
\[ \mathbb{E}[|S_l(n)|] \leq \sum_{k=0}^{\infty} \mathbb{E}[|\psi_k|] < \infty. \]

The results for $\tilde{S}_l(n)$ are similar, utilizing part (ii) of Proposition 9.11.

To obtain $k_0$ as in part (ii), we recall Remark 9.2 and find $k_0$ such that $p(l) \left( \frac{1}{r_0} \right)^l 2^{-kl/2}C(\varphi) \leq 1/2$ for all $k \geq k_0$. Such a $k_0$ can be obtained as follows: first fix $k$ and let $l \to \infty$ in $p(l) \left( \frac{1}{r_0} \right)^l 2^{-kl/2}C(\varphi)$; if $k$ is large enough, this will $\to 0$. Fix such a $k$, then for large enough $l \geq l_0$ and this $k$, we have $p(l) \left( \frac{1}{r_0} \right)^l 2^{-kl/2}C(\varphi) \leq 1/2$. Note that if we increase $k$ further, the inequality will still remain true for all $l \geq l_0$. To take care of the first $l_0$ terms, we simply need to pick a $k_0$ much larger, so that $p(l) \left( \frac{1}{r_0} \right)^l 2^{-kl/2}C(\varphi) \leq 1/2$ holds for all $l \geq 1$. This $k_0$ will clearly be independent of $n$, because so is $C(\varphi)$, and by choice it is independent of $l$. To estimate $\mathbb{E}[|\tau_l^n(2^k)|]$, we now simply sum (29) for $k \geq k_0$.

The result for $\tilde{\tau}_l^n(2^k)$ follows from a similar argument.

The same arguments yield the corresponding results when $F_n$ is replaced by $F$.

**Corollary 9.14.** For $R = 2^k, k \geq k_0$ as in Proposition 9.13 We have $\mathbb{P}[|\tau_l^n(R)| > R^{-l/4}] \leq R^{-l/4}$ and $\mathbb{P}[|\tilde{\tau}_l^n(R)| > R^{-(l-2)/4}] \leq R^{-(l-2)/4}$, and these estimates remain true when $f_n$ is replaced with $f$.

**Proof.** We use the estimates on the expectation on $|\tau_l^n(R)|$ and $|\tilde{\tau}_l^n(R)|$ from Proposition 9.13 and apply Markov’s inequality.

With notations as above, we have

**Proposition 9.15.** For each $l \geq 1$ we have $S_l(n) \to S_l$ in probability, and for each $l \geq 3$ we have $\tilde{S}_l^n \to \tilde{S}_l$ in probability, and hence we have such convergence a.s. along some subsequence, simultaneously for all $l$.

**Proof.** We argue on similar lines to the proof of Proposition 7.8.

Define $S_k(D, n) = \sum_{z \in F_n} 1/z^k$ and $S_k(D) = \sum_{z \in F} 1/z^k$. Set $\alpha_k(n) = S_k(D, n) + S_k(n)$ and $\alpha_k = S_k(D) + S_k$. Observe that $\alpha_k(n) = \sum_{z \in F_n} 1/z^k$. Then we have:

**Proposition 9.16.** For each $k$, $\alpha_k(n) \to \alpha_k$ in probability as $n \to \infty$. Hence, there is a subsequence such that $\alpha_k(n) \to \alpha_k$ a.s. when $n \to \infty$ along this subsequence, simultaneously for all $k$.

**Proof.** Since the finite point configurations given by $F_n|_D \to F|_D$ a.s. and a.s. there is no point at the origin, therefore $S_k(D, n) \to S_k(D)$ a.s. This, combined with Proposition 9.15 gives us the desired result.
10 Limiting procedure for GAF zeroes

In this section, we use the estimates for $\mathcal{F}_n$ to prove Theorem 1.4 for a disk $D$ centred at the origin. We know from Section 4 that this is sufficient in order to obtain Theorems 1.3 and 1.4 in the general case. We will work in the framework of Section 6. More specifically, we will show that the conditions for Theorem 6.2 are satisfied, which will give us the desired conclusion. We will introduce the definitions and check all the conditions here except the fact $\Omega(j)$ exhausts $\Omega^m$. This last criterion will be verified in the subsequent Proposition 10.1.

In terms of the notation used in Section 6 we have $X^n = \mathcal{F}_n$ and $X = \mathcal{F}$.

**Proof of Theorem 1.4 for a disk.** We will invoke Theorem 6.2. We will define the relevant quantities (as in the statement of Theorem 6.2) and verify that they satisfy the required conditions for that theorem to apply.

Following the notation in Theorem 6.2 we begin with $\Omega^m$, $m \geq 0$. The cases $m = 0$ and $m = 1$ are trivial, so we focus on the case $m \geq 2$.

Our candidate for $\nu(X_{\text{out}}(\xi), \cdot)$ (refer to Theorem 6.2) is the probability measure $Z^{-1}|\Delta(\bar{\zeta})|^2 d\mathcal{L}_\zeta(\bar{\zeta})$ on $\Sigma_S(X_{\text{out}}(\xi))$, where $\Delta(\bar{\zeta})$ is the Vandermonde determinant formed by the coordinates of $\bar{\zeta}$, $\mathcal{L}_\zeta$ is the Lebesgue measure on $\Sigma_S(X_{\text{out}}(\xi))$ and $Z$ is the normalizing factor. Here we recall the definition of $S(X_{\text{out}}(\xi))$ from Theorem 1.3 and the definition of $\Sigma_S(X_{\text{out}}(\xi))$ from Section 9.1.2. Note that as soon as we define $\nu(X_{\text{out}}, \cdot)$ which maps $\xi$ to $\mathcal{M}(\mathcal{D}^m)$, this automatically induces a map from $\mathcal{S}_{\text{out}}$ to $\mathcal{M}(\mathcal{D}^m)$ which satisfies the required measurability properties.

To find the sequence $n_k$ (which will be the same for every $j$ in our case), we proceed as follows. Let $N_g(K)$ denote the number of zeroes of the function $g$ in a set $K$, $\Gamma$ denote the closed annulus of thickness 1 around $D$, and in the next statement let $Z$ be any of the variables $L, M$ or $N$ as in Proposition 9.7 (and the immediately preceding discussion) with $p = 0$ or 2 $\leq p \leq m$ and $2 \leq q \leq m$. We have the probabilities of each of the following events converging to 0 as $n \to \infty$: $\{ |S_j(n) - S_j| > 1 \}$ for $j = 1, 2, 3$, $\{|N_f(\mathcal{D}) \neq N_{f_n}(\mathcal{D})|\}$, $\{|N_f(\Gamma) \neq N_{f_n}(\Gamma)|\}$, $\{|\tau(Z_{pq}^{(n)})| > 1\}$ and $\left( \frac{n}{2} \right)^3 \leq \frac{1}{2} \left( \sum_{j=0}^{n} \right)$. Call the union of these events $\mathcal{B}_n$. For a given $k \geq 1$, let $n'_k$ be such that $\mathbb{P}(\mathcal{B}_n) < 2^{-k}$ for all $n \geq n'_k$. From Proposition 9.15 we have a sequence such that $S_j(n) \to S_j$ ($j = 1, 2$) and $S_3(n) \to S_3$ a.s. as $n \to \infty$ along that sequence. For a given $k \geq 1$, we define $n_k$ to be the least integer in that sequence which is $\geq n'_k$.

Fix a sequence of positive real numbers $M_j \uparrow \infty$.

On the event $\Omega^m_{n_k}$ (which entails that $f_{n_k}$ has $m$ zeroes inside $D$), let $\zeta(n_k)$ and $\omega(n_k)$ respectively denote the vector of inside and outside zeroes of $f_{n_k}$ (taken in uniform random order). Let $s(n_k)$ denote the sum of the inside zeroes: $s(n_k) := \sum_{j=1}^{m} \zeta(n_k)_j$, where $\zeta(n_k)_j$ are the co-ordinates of the vector $\zeta(n_k)$. By $\Sigma(s(n_k))$ we will denote the (random) set $\{ \zeta' \in \mathcal{D}^m : \sum_{j=1}^{m} \zeta'_j = s(n_k) \}$. Also recall the notation $\rho^{n_k}(\zeta)$ to be the conditional density (with respect to the Lebesgue measure on $\Sigma_s$) of the inside zeroes (at $\zeta \in \mathcal{D}^m$) given the vector of outside zeroes to be $\omega$ and the sum of the inside zeroes to be $s$.

We define our event $\Omega_{n_k}(j)$ by the following conditions on the zero set $(\zeta(n_k), \omega(n_k))$ of $f_{n_k}$:
1. There are exactly $m$ zeroes of $f_{nk}$ in $\mathcal{D}$

2. There are no zeroes of $f_{nk}$ in the closed annulus of thickness $1/M_j$ around $\mathcal{D}$.

3. $|S_1(n_k)| \leq M_j, |S_2(n_k)| \leq M_j, |S(n_k)| \leq M_j$.

4. There exists $\xi' \in \Sigma_{s(n_k)}$ such that $(\xi', \varpi(n_k))$ satisfies (\varpi here is an abbreviation for $\varpi(n_k)$):

   \[
   \begin{align*}
   (a) & \quad \left| \sum_{r=0}^{nk} \frac{\sigma_r(\xi', \varpi) \sigma_{r-i}(\varpi)}{\sqrt{(n)^{r!}} / \sqrt{(n)^{r!}}} \right| \left( \sum_{r=0}^{nk} \frac{\sigma_r(\xi', \varpi)}{\sqrt{(n)^{r!}}} \right)^2 \leq M_j/n_k \quad \text{for each } 2 \leq i \leq m. \\
   (b) & \quad \left| \sum_{r=0}^{nk} \frac{\sigma_r(\xi', \varpi) \sigma_{r-j}(\varpi)}{\sqrt{(n)^{r!}} / \sqrt{(n)^{r!}}} \right| \left( \sum_{r=0}^{nk} \frac{\sigma_r(\xi', \varpi)}{\sqrt{(n)^{r!}}} \right)^2 \leq M_j/n_k \quad \text{for all } 2 \leq i, j \leq m. 
   \end{align*}
   \]

Clearly, $\Omega_{nk}(j)$ depends only on the quantities $s(n_k)$ and $\varpi(n_k)$. In particular, for a vector $\varpi$ of outside zeroes of $f_{nk}$, if there exists $\xi \in \Sigma_s$ such that $(\xi, \varpi)$ satisfies the conditions demanded in the definition of $\Omega_{nk}(j)$, then all $(\xi', \varpi)$ (with $\xi' \in \Sigma_{s}$) satisfies those conditions. From Proposition 9.1, 14, 15 and the discussion therein, it is clear that on the event $\Omega_{nk}(j)$ we have

\[
m(j)|\Delta(\xi)|^2 \leq \rho_{\omega, s}(\xi) \leq M(j)|\Delta(\xi)|^2
\]

for positive quantities $M(j)$ and $m(j)$ (that depend on $M_j$).

To obtain (31), we introduce some further notations. On the event $\Omega_{nk}(j)$, let $\gamma_{nk}(\varpi; s)$ denote the conditional probability measure on the sum $s$ of inside zeroes given the vector of outside zeroes of $f_{nk}$ to be $\varpi$. Further, let $\mathcal{L}_s$ denote the Lebesgue measure on the set $\Sigma_s$.

For any $A \in \mathfrak{A}^m$ and $B \in \mathcal{B}$, we can write

\[
\mathbb{P}[(X_{in} \in A) \cap (X_{out}^{nk} \in B) \cap \Omega_{nk}(j)] = \int_{B \cap \Omega_{nk}(j)} \left( \int_{A \cap \Sigma_s} \rho_{\omega, s}^{nk}(\xi) d\mathcal{L}_s(\xi) \right) d\gamma_{nk}(\varpi; s) d\mathbb{P} X_{out}^{nk} (\varpi). \tag{31}
\]

Setting $h(s, A) := \left( \int_{A \cap \Sigma_s} |\Delta(\xi)|^2 d\mathcal{L}_s(\xi) \right) / \left( \int_{\Sigma_s} |\Delta(\xi)|^2 d\mathcal{L}_s(\xi) \right)$, we get from (31) that the right hand side of (31) is

\[
\int_{B \cap \Omega_{nk}(j)} \left( \int_{A \cap \Sigma_s} \rho_{\omega, s}^{nk}(\xi) d\mathcal{L}_s(\xi) \right) d\gamma_{nk}(\varpi; s) d\mathbb{P} X_{out}^{nk} (\varpi) \prec_j \int_{B \cap \Omega_{nk}(j)} h(s, A) d\gamma_{nk}(\varpi; s) d\mathbb{P} X_{out}^{nk} (\varpi). \tag{32}
\]

The last expression can be written as $\mathbb{E} \left[ h(s(n_k), A) 1[X_{out}^{nk} \in B] 1[\Omega_{nk}(j)] \right]$, where $1[E]$ is the indicator function of the event $E$. Note that for fixed $A$, the function $h(s, A)$ is continuous in $s$, because $A \in \mathfrak{A}^m$, and $0 \leq h(s, A) \leq 1$. Since, as $k \to \infty$, we have $s(n_k) \to S(X_{out})$ a.s., therefore using the Dominated Convergence Theorem we get

\[
\mathbb{E} \left[ h(s(n_k), A) 1[X_{out}^{nk} \in B] 1[\Omega_{nk}(j)] \right] = \mathbb{E} \left[ h(S(X_{out}), A) 1[X_{out}^{nk} \in B] 1[\Omega_{nk}(j)] \right] + o_k(1), \tag{33}
\]

where $o_k(1)$ is a quantity which $\to 0$ as $k \to \infty$, for fixed $A$ and $B$ (actually, in this particular case, it can be easily seen that the convergence is uniform in $B \in \mathcal{B}$). Also, as $k \to \infty$, we have $1[X_{out}^{nk} \in B] \to 1[X_{out} \in B]$ since $B$ is a compact set. Since $0 \leq h(s, A) \leq 1$ a.s., this implies that

\[
\mathbb{E} \left[ h(S(X_{out}), A) 1[X_{out}^{nk} \in B] 1[\Omega_{nk}(j)] \right] = \mathbb{E} \left[ h(S(X_{out}), A) 1[X_{out} \in B] 1[\Omega_{nk}(j)] \right] + o_k(1). \tag{34}
\]
Observe that for \( \xi \in \Xi \), we have \( h(S(X_{\text{out}}(\xi)), A) = \nu(X_{\text{out}}(\xi), A) \). Therefore, putting (31)-(34) together, we obtain (3). The only condition in Theorem 6.2 that remains to be verified is the fact that \( \Omega(j) \)-s exhaust \( \Omega^m \), which will be taken up in Proposition 10.1.

By Theorem 6.2, this proves that a.s. we have \( \rho(X_{\text{out}}, \cdot) \equiv \nu(X_{\text{out}}, \cdot) \). Since a.s. \( \nu(X_{\text{out}}, \cdot) \equiv \mathcal{L}_\Sigma \), we have \( \rho(X_{\text{out}}, \cdot) \equiv \mathcal{L}_\Sigma \), as desired.

We end this section with a proof that \( \Omega(j) \)-s exhaust \( \Omega^m \):

**Proposition 10.1.** With definitions as above, \( \Omega(j) := \lim_{k \to \infty} \Omega_{nk}(j) \) exhausts \( \Omega^m \) as \( j \to \infty \).

**Proof.** We begin by showing that \( \Omega(j) \subset \Omega^m \) for each \( j \). Due to the convergence of \( X^{nk} \to X \) on compact sets, we have \( \lim_{k \to \infty} \Omega_{nk}^m = \Omega^m \). Since \( \Omega_{nk}(j) \subset \Omega_{nk}^m \) for each \( k \), therefore \( \Omega(j) \subset \Omega^m \) for each \( j \). Since \( M_j < M_{j+1} \), it is also clear that \( \Omega_{nk}(j) \subset \Omega_{nk}(j+1) \) for each \( k \). Hence \( \Omega(j) \subset \Omega(j+1) \).

To show that \( \mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0 \) as \( j \to \infty \), for each \( j \) we first define the event \( \Omega_{nk}^1(j) \) by demanding that the zeroes \( (\zeta, \omega) \) of \( f_{nk} \) satisfy the following conditions:

1. There are exactly \( m \) zeroes of \( f_{nk} \) in \( \mathcal{D} \)
2. There are no zeroes of \( f_{nk} \) in the closed annulus of thickness \( 1/M_j \) around \( \mathcal{D} \).
3. \( |S_1(nk)| \leq M_j, |S_2(nk)| \leq M_j, |S_3(nk)| \leq M_j \).
4. \( \left| \sum_{k=0}^{n} \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k}}} \sigma_{k-i}(\omega) \right| / \left( \sum_{k=0}^{n} \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k}}} \right)^2 \leq M_j/nk \) for each \( 2 \leq i \leq m \).
5. \( \left| \sum_{k=0}^{n} \frac{\sigma_{k-i}(\omega)}{\sqrt{\binom{n}{k}}} \sigma_{k-j}(\omega) \right| / \left( \sum_{k=0}^{n} \frac{\sigma_k(\zeta, \omega)}{\sqrt{\binom{n}{k}}} \right)^2 \leq M_j/nk \) for all \( 2 \leq i, j \leq m \).

Clearly, \( \Omega_{nk}^1(j) \subset \Omega_{nk}(j) \) and hence \( \lim_{k \to \infty} \Omega_{nk}^1(j) = \Omega^1(j) \subset \Omega(j) \).

Next, we define the event \( \Omega_{nk}^2(j) \) by the following conditions (refer to Proposition 9.7 and the discussion immediately preceding it to recall the notations):

1. There are exactly \( m \) zeroes of \( f_{nk} \) in \( \mathcal{D} \)
2. There are no zeroes of \( f_{nk} \) in the closed annulus of thickness \( 1/M_j \) around \( \mathcal{D} \).
3. \( |S_1(nk)| \leq M_j, |S_2(nk)| \leq M_j, |S_3(nk)| \leq M_j \).
4. Each of \( |L_{0q}^{nk}|, |M_{0q}^{nk}|, |L_{pq}^{nk}|, |M_{pq}^{nk}|, |N_{pq}^{nk}| \) is \( \leq M_j/8 \) for all \( 2 \leq p, q \leq m \).
5. \( E_{nk}/nk \geq 1/2 \).

It is clear from (25) that \( \Omega_{nk}^2(j) \subset \Omega_{nk}^1(j) \).

Finally, we define an event \( \Omega^3(j) \) by the conditions:
1. There are exactly \( m \) zeroes of \( f \) in \( D \).

2. There are no zeroes of \( f \) in the closed annulus of thickness \( 1/M_j \) around \( D \).

3. \(|S_1| \leq M_j, |S_2| \leq M_j - 1, |S_3| \leq M_j - 1|.

4. Each of \(|L_0q|, |M_0q|, |L_{pq}|, |M_{pq}|, |N_{pq}|\) is \( \leq \frac{M_j}{8} - 1 \) for all \( 2 \leq p, q \leq m \).

Notice (refer to Proposition 9.8) that for each of the random variables \( Z^{(n_k)}(pq) \) in condition 4 defining \( \Omega_2^{n_k}(j) \) and \( Z_{pq} \) in condition 4 defining \( \Omega^2(j) \) (where \( Z = L, M, N \)) we have

\[
|Z_{pq}^{(n_k)}| \leq |Z_{pq}| + |\tau(Z_{pq}^{(n_k)})|.
\]

(35)

Consider the event \( F_{n_k} \) where any one of the following conditions hold:

1. There are \( m \) zeroes of \( f \) in \( D \) but this is not true for \( f_{n_k} \).

2. There are no zeroes of \( f \) in the closed annulus of thickness \( 1/M_j \) around \( D \), but this is not true for \( f_{n_k} \).

3. \(|S_1 - S_1(n_k)| > 1\) or \(|S_2 - S_2(n_k)| > 1\) or \(|S_3 - S_3(n_k)| > 1|.

4. \(|\tau(Z_{pq}^{(n_k)})| > 1\) for any of the random variables appearing in condition 4 of \( \Omega_3^{n_k}(j) \).

5. \( E_{n_k}/n_k < 1/2 \).

It is easy to see (using (35)) that \( \Omega^3(j) \setminus F_{n_k} \subset \Omega_3^{n_k}(j) \subset \Omega_{n_k}(j) \). However, by our choice of the sequence \( n_k \), we have \( \mathbb{P}(F_{n_k}) < 2^{-k} \), hence by Borel Cantelli lemma, \( \lim_{k \to \infty} (\Omega^3(j) \setminus F_{n_k}) = \Omega^3(j) \). Hence, \( \Omega^3(j) \subset \Omega(j) = \lim_{k \to \infty} \Omega_{n_k}(j) \).

However, each random variable used in defining \( \Omega^3(j) \) does not put any mass at \( \infty \), and the thickness of the annulus around \( D \) in condition 2 in its definition goes to 0 as \( j \to \infty \). Hence, the probability that each of the conditions 2-4 defining \( \Omega^3(j) \) holds goes to 1 as \( j \to \infty \). Finally, condition 1 is just the definition of \( \Omega^m \). Hence, as \( j \to \infty \), we have \( \mathbb{P}(\Omega^m \setminus \Omega^3(j)) \to 0 \). But \( \Omega^3(j) \subset \Omega(j) \), hence \( \mathbb{P}(\Omega^m \setminus \Omega(j)) \to 0 \) as \( j \to \infty \), as desired.

\[ \blacksquare \]

11 Reconstruction of GAF from Zeroes and Vieta’s formula

In this section we prove the reconstruction Theorem 1.5 for the planar GAF. En route, we establish an analogue of Vieta’s formula for the planar GAF.
11.1 Vieta’s formula for the planar GAF

It is an elementary fact that for a polynomial

\[ p(z) = \sum_{j=0}^{N} a_j z^j \]

whose roots are \( \{z_j\}_{j=1}^{N} \), we have, for any \( 1 \leq k \leq N \),

\[ a_{N-k}/a_N = \sum_{i_1 < i_2 < \cdots < i_k} z_{i_1} \cdots z_{i_k}. \]  \hspace{1cm} (36)

When \( a_0 \neq 0 \), we equivalently have

\[ a_k/a_N = \sum_{i_1 < i_2 < \cdots < i_k} \frac{1}{z_{i_1} \cdots z_{i_k}}. \]  \hspace{1cm} (37)

This kind of result is broadly referred to as Vieta’s formula. For entire function, such results do not hold in general; for example it may not be possible to provide any reasonable interpretation to the function of the zeroes appearing on the right hand side of (37).

The quantity on the right hand side of (36) is the elementary symmetric function of order \( k \) in the variables \( z_1, \cdots, z_N \), denoted by \( e_k(z_1, \cdots, z_N) \). If we introduce the power sum \( \beta_k = \sum_{j=1}^{N} z_j^k \), then it is known that for each \( k \) we have

\[ e_k(z_1, \cdots, z_N) = P_k(\beta_1, \cdots, \beta_k) \]

where \( P_k \) is a homogeneous symmetric polynomial of degree \( k \) in the variables \( \beta_1, \cdots, \beta_k \). The polynomial \( P_k \) is called the Newton polynomial of degree \( k \), and it is known that the coefficients of \( P_k \) depend only on \( k \) and do not depend on \( n \) (refer [Sta99], chapter 7).

**Proposition 11.1.** For the planar GAF zero process, we have, a.s. \( \frac{\xi_k}{\xi_0} = P_k(\alpha_1, \alpha_2, \cdots, \alpha_k) \) for each \( k \geq 1 \).

**Proof.** We begin by observing that under the natural coupling of the planar GAF \( f \) and its approximating polynomials \( f_n \), we have by the Vieta’s formula for each \( f_n \):

\[ \frac{\xi_k}{\xi_0} = P_k(\alpha_1(n), \cdots, \alpha_k(n)) \]  \hspace{1cm} (38)

where \( P_k \) is, as before, the Newton polynomial of degree \( k \). Clearly, \( P_k \) is continuous in the input variables. We also know, from Proposition 9.15 that, as \( n \to \infty \) (possibly along some appropriately chosen subsequence), \( \alpha_k(n) \to \alpha_k \) a.s., simultaneously for all \( k \geq 1 \). Taking this limit in (38) and using the continuity of \( P_k \), we get a.s.

\[ \frac{\xi_k}{\xi_0} = P_k(\alpha_1, \cdots, \alpha_k). \]  \hspace{1cm} (39)
11.2 Proof of Theorem 1.5

Since $\xi_0$ is a complex Gaussian, therefore a.s. $|\xi_0| \neq 0$. For $|\xi_0| \neq 0$, we can write

$$f(z) = \frac{\xi_0}{|\xi_0|} |\xi_0| \left( 1 + \frac{\xi_1}{\xi_0} z + \frac{\xi_2}{\xi_0} z^2 + \cdots + \frac{\xi_k}{\xi_0} z^k + \cdots \right)$$  (40)

From Proposition 11.1 we have that for each $k$ the random variable $\frac{\xi_k}{\xi_0}$ is measurable with respect to $F$. From the strong law of large numbers, a.s. we have

$$|\xi_0|^2 + \ldots + |\xi_{k-1}|^2 \to 1.$$  

Therefore

$$|\xi_0| = \lim_{k \to \infty} k^{1/2} \left( \frac{1}{k} \sum_{j=0}^{k-1} \frac{|\xi_j|^2}{|\xi_0|^2} \right)^{-1/2} = \chi.$$  

Since $k^{1/2} \left( \sum_{j=0}^{k-1} \frac{|\xi_j|^2}{|\xi_0|^2} \right)^{-1/2}$ is measurable with respect to $F$ for each $k$, therefore $\chi = |\xi_0|$ is also measurable with respect to $F$.

Let us define the random variable $\zeta = \xi_0/|\xi_0|$ (it is set to be equal to 0 when $|\xi_0| = 0$) and the random function $g$ as in the statement can clearly be written as

$$g(z) = |\xi_0| \left( 1 + \frac{\xi_1}{\xi_0} z + \frac{\xi_2}{\xi_0} z^2 + \cdots + \frac{\xi_k}{\xi_0} z^k + \cdots \right).$$

Then (40) can be re-written as

$$f(z) = \zeta g(z)$$  (41)

almost surely, and $g$ is measurable with respect to $F$ and $\zeta$ is distributed uniformly on $S^1$.

Therefore, all that remains to complete the proof is to show that $\zeta$ and $F$ are independent. For this, note that for $\theta \in S^1$ the random function

$$|\xi_0| + \sum_{j \geq 1} \frac{\theta \xi_j}{\sqrt{j}} z^j$$

has the same distribution (irrespective of $\theta$). This is because since the $\xi_i$s are complex Gaussians with mean 0 and variance 1, the vectors $(|\xi_0|, \theta \xi_j)_{j \geq 1}$ and $(|\xi_0|, \xi_j)_{j \geq 1}$ have the same distribution for each fixed $\theta \in S^1$.

Now note that

$$g(z) = |\xi_0| + \sum_{j \geq 1} \frac{\zeta \xi_j}{\sqrt{j}} z^j.$$  

Therefore, the distribution of $g(z)$ given $\zeta$ does not depend on the value of $\zeta$. Hence, the random function $g$ and $\zeta$ are independent. But, $g$ and $f$ have the same zero set a.s. Hence, $\zeta$ and $F$ are independent random variables. This completes the proof.
12 Proof of Theorem 6.2

**The main proof.** We claim that it suffices to show that for every \( j_0 \), we have positive real numbers \( M(j_0) \) and \( m(j_0) \) such that for any \( A \in \mathfrak{A}^m \) and any Borel set \( B \) in \( S_{\text{out}} \)

\[
\mathbb{P} ((X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0)) \leq M(j_0) \int_{X_{\text{out}}^{-1}(B)} \nu(\xi, A)d\mathbb{P}(\xi) \tag{42}
\]

and

\[
m(j_0) \int_{X_{\text{out}}^{-1}(B) \cap \Omega(j_0)} \nu(\xi, A)d\mathbb{P}(\xi) \leq \mathbb{P} (X_{\text{in}} \in A \cap (X_{\text{out}} \in B)) . \tag{43}
\]

Once we have (42), we can invoke Proposition 12.1. Setting \( \mu_1(\xi, \cdot) = \rho(X_{\text{out}}(\xi), \cdot), \mu_2(\xi, \cdot) = \nu(X_{\text{out}}(\xi), \cdot), a(j_0) = 1 \) and \( b(j_0) = M(j_0) \) in (15) we get \( \rho(X_{\text{out}}, \cdot) \ll \nu(X_{\text{out}}, \cdot) \) a.s.

If we have (43), we can again appeal to Proposition 12.1. Setting \( \mu_1(\xi, \cdot) = \nu(X_{\text{out}}(\xi), \cdot), \mu_2(\xi, \cdot) = \rho(X_{\text{out}}(\xi), \cdot), a(j_0) = m(j_0) \) and \( b(j_0) = 1 \) in (15) we get \( \nu(X_{\text{out}}, \cdot) \ll \rho(X_{\text{out}}, \cdot) \) a.s.

The last two paragraphs together imply that \( \rho(X_{\text{out}}, \cdot) \equiv \nu(X_{\text{out}}, \cdot) \) a.s., as desired.

To establish (42) and (43), we begin with a fixed \( j_0 \), a set \( A \in \mathfrak{A}^m \) and a Borel set \( B \) in \( S_{\text{out}} \). We will invoke Proposition 12.2 in the following manner. Let the multiplicative constants appearing in the \( \approx \) relation in (4) with \( j = j_0 \) be \( m(j_0) \) and \( M(j_0) \) respectively, with \( m(j_0) \leq M(j_0) \). In Proposition 12.2 we set \( h_1(\xi) = 1, h_2(\xi) = \nu(\xi, A), U_1 = A, U_2 = \mathcal{D}^m, V = B, a(j_0) = 1, b(j_0) = M(j_0) \) to obtain (42). On the other hand, setting \( h_1(\xi) = \nu(\xi, A), h_2(\xi) = 1, U_1 = \mathcal{D}^m, U_2 = A, V = B, a(j_0) = m(j_0), b(j_0) = 1 \) in Proposition 12.2 we obtain (43).

This completes the proof of theorem 6.2.

We end this section with Propositions 12.1 and 12.2 used in the above proof.

**Proposition 12.1.** Let \( \mu_1 \) and \( \mu_2 \) be two functions mapping \( \Xi \rightarrow \mathcal{M}(\mathcal{D}^m) \) such that for each Borel set \( A \subset \mathcal{D}^m \), the function \( \xi \rightarrow \mu_j(\xi, A) \) is measurable, \( j = 1, 2 \). Suppose \( \mu_1 \) and \( \mu_2 \) satisfy, for each positive integer \( j_0 \), \( A \in \mathfrak{A}^m \) and Borel set \( B \subset \mathcal{D}^m \)

\[
a(j_0) \int_{X_{\text{out}}^{-1}(B) \cap \Omega(j_0)} \mu_1(\xi, A)d\mathbb{P}(\xi) \leq b(j_0) \int_{X_{\text{out}}^{-1}(B)} \mu_2(\xi, A)d\mathbb{P}(\xi) \tag{44}
\]

for some positive numbers \( a(j_0) \) and \( b(j_0) \). Then a.s. we have

\[
\mathbb{E} [\mu_1(\xi, \cdot)|X_{\text{out}}(\xi)] \ll \mathbb{E} [\mu_2(\xi, \cdot)|X_{\text{out}}(\xi)]. \tag{45}
\]

**Proof.** (44) implies that for each \( A \in \mathfrak{A}^m \) a.s. in \( X_{\text{out}} \) we have

\[
a(j_0)\mathbb{E} [\mu_1(\xi, A)|X_{\text{out}}(\xi), \Omega(j_0) \text{ occurs }] \mathbb{P}(\Omega(j_0) \text{ occurs } |X_{\text{out}}(\xi)) \leq b(j_0)\mathbb{E}[\mu_2(\xi, A)|X_{\text{out}}(\xi)]. \tag{46}
\]

For almost every configuration \( \omega \in S_{\text{out}} \) (with respect to the measure \( \mathbb{P}_{X_{\text{out}}} \)) we have (46) for all \( A \in \mathfrak{A}^m \), and therefore by the regularity of the Borel measures on the two sides, (46) extends to all
Borel sets $A \subset \mathcal{D}^m$ (see Proposition 6.1). Now, for $\omega \in \mathcal{S}_{\text{out}}$, suppose that $A \subset \mathcal{D}^m$ is a Borel set such that $\mathbb{E}[\mu_2(\xi, A)|X_{\text{out}}(\xi) = \omega] = 0$. Then (46) implies that

$$\mathbb{E}[\mu_1(\xi, A)|X_{\text{out}}(\xi) = \omega, \Omega(j_0) \text{ occurs}] \mathbb{P}(\Omega(j_0) \text{ occurs} | X_{\text{out}}(\xi) = \omega) = 0, \quad (47)$$

for each $j_0$. But $\Omega(j_0)$ exhausts $\Omega^m$, hence

$$\mathbb{E}[\mu_1(\xi, A)|X_{\text{out}}(\xi) = \omega, \Omega(j_0) \text{ occurs}] \mathbb{P}(\Omega(j_0) \text{ occurs} | X_{\text{out}}(\xi) = \omega) \rightarrow 0$$
as $j_0 \uparrow \infty$. By letting $j_0 \uparrow \infty$ in (47), we obtain the fact that a.s. on $\Omega^m$ we have $\mathbb{E}[\mu_2(\xi, A)|X_{\text{out}}(\xi)] = 0$ implies $\mathbb{E}[\mu_1(\xi, A)|X_{\text{out}}(\xi)] = 0$. In other words, a.s. on $\Omega^m$, we have (45).

**Proposition 12.2.** Suppose we have measurable functions $h_1$ and $h_2$ mapping $\Xi \rightarrow [0, 1]$, and measurable sets $U_1, U_2 \in \mathcal{A}^m$. Define $\mathbb{P}_{h_i}$ to be the finite non-negative measure on $\Xi$ given by $d\mathbb{P}_{h_i}(\xi) = h_i(\xi)d\mathbb{P}(\xi)$, $i = 1, 2$. Suppose there are positive numbers $a(j_0)$ and $b(j_0)$ such that the following inequality holds for all $\tilde{V} \in \mathcal{B}$:

$$a(j_0)\mathbb{P}_{h_1}[(X_{\text{in}}^{n_k} \in U_1) \cap (X_{\text{out}}^{n_k} \in \tilde{V}) \cap \Omega_{n_k}(j_0)] \leq b(j_0)\mathbb{P}_{h_2}[(X_{\text{in}}^{n_k} \in U_2) \cap (X_{\text{out}}^{n_k} \in \tilde{V}) \cap \Omega_{n_k}(j_0)] + \vartheta(k) \quad (48)$$

where $\vartheta(k) = \vartheta(k; j_0, U_1, U_2, \tilde{V}) \rightarrow 0$ as $k \rightarrow \infty$ for fixed $j_0, U_1, U_2, \tilde{V}$. Then for all Borel sets $V$ in $\mathcal{S}_{\text{out}}$, we have

$$a(j_0)\mathbb{P}_{h_1}[(X_{\text{in}} \in U_1) \cap (X_{\text{out}} \in V) \cap \Omega(j_0)] \leq b(j_0)\mathbb{P}_{h_2}[(X_{\text{in}} \in U_2) \cap (X_{\text{out}} \in V)] \quad (49)$$

**Proof.** In what follows, we will denote by $\mathbb{P}_h$ the non-negative finite measure on $\Xi$ obtained by setting $d\mathbb{P}_h(\xi) = h(\xi)d\mathbb{P}(\xi)$ where $h : \Xi \rightarrow [0, 1]$ is a measurable function. We note that for any event $E$, we have $0 \leq \mathbb{P}_h(E) \leq \mathbb{P}(E)$. Fix a $U \in \mathcal{A}^m$.

For any Borel set $V$ in $\mathcal{S}_{\text{out}}$, given $\varepsilon > 0$ we can find a $V_{\varepsilon} \in \mathcal{B}$ such that

$$\mathbb{P}(X_{\text{out}}^{-1}(V) \Delta X_{\text{out}}^{-1}(V_{\varepsilon})) < \varepsilon.$$

This can be seen by considering the push forward probability measure $(X_{\text{out}}), \mathbb{P}$ on $\mathcal{S}_{\text{out}}$. The aim of this reduction is to exploit the fact that as $k \rightarrow \infty$, we have $1_{V_{\varepsilon}}(X_{\text{out}}^{n_k}) \rightarrow 1_{V_{\varepsilon}}(X_{\text{out}})$ a.s.

We start with $\mathbb{P}_h[(X_{\text{in}}^{n_k} \in U) \cap (X_{\text{out}}^{n_k} \in V_{\varepsilon}) \cap \Omega_{n_k}(j_0)]$. This is equal to

$$\mathbb{P}_h[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V_{\varepsilon}) \cap \Omega_{n_k}(j_0)] + o_k(1; V_{\varepsilon})$$

where $o_k(1; V_{\varepsilon})$ stands for a quantity that tends to 0 as $k \rightarrow \infty$ for fixed $V_{\varepsilon}$. This step uses the fact that $1[X_{\text{out}}^{n_k} \in V_{\varepsilon}] \rightarrow 1[X_{\text{out}} \in V_{\varepsilon}]$ and $1[X_{\text{in}}^{n_k} \in U] \rightarrow 1[X_{\text{in}} \in U]$ a.s. The expression in the last display above equals

$$\mathbb{P}_h[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0)] + o_{\varepsilon}(1) + o_k(1; V_{\varepsilon})$$

where $o_{\varepsilon}(1)$ denotes a quantity that tends to 0 uniformly in $k$ as $\varepsilon \rightarrow 0$. 

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We upper bound the probability in the last display simply by \( P_h \left[ (X_{in} \in U) \cap (X_{out} \in B) \right] \). Putting all these together, we have

\[
P_h[(X_{in}^{nk} \in U) \cap (X_{out}^{nk} \in V_{\varepsilon}) \cap \Omega_{nk}(j_0)] \leq P_h \left[ (X_{in} \in U) \cap (X_{out} \in V) \right] + o_{\varepsilon}(1) + o_k(1; V_{\varepsilon}) \tag{50}
\]

To obtain a comparable lower bound, we need to work more. We begin with

\[
P_h \left[ (X_{in} \in U) \cap (X_{out} \in V) \cap \Omega_{nk}(j_0) \right] \geq P_h \left[ (X_{in} \in U) \cap (X_{out} \in V) \cap \Omega_{nk}(j_0) \cap \Omega(j_0) \right].
\]

Observe that \( \Omega(j_0) = \lim_{k \to \infty} \Omega_{nk}(j_0) \) and \( P \left( \lim_{k \to \infty} \Omega_{nk}(j_0) \Delta \left( \bigcap_{l \geq k} \Omega_{nl}(j_0) \right) \right) = o_k(1) \) where \( o_k(1) \) denotes a quantity \( \to 0 \) as \( k \to \infty \), (for a fixed \( j_0 \)) uniformly in all the other quantities.

Hence we have

\[
P_h \left[ (X_{in} \in U) \cap (X_{out} \in V) \cap \Omega_{nk}(j_0) \cap \Omega(j_0) \right]
= P_h \left[ (X_{in} \in U) \cap (X_{out} \in V) \cap \Omega_{nk}(j_0) \cap \left( \bigcap_{l \geq k} \Omega_{nl}(j_0) \right) \right] + o_k(1).
\]

But \( \Omega_{nk}(j_0) \cap \left( \bigcap_{l \geq k} \Omega_{nl}(j_0) \right) = \left( \bigcap_{l \geq k} \Omega_{nl}(j_0) \right) \) and hence the probability in the last display equals

\[
P_h \left[ (X_{in} \in U) \cap (X_{out} \in V) \cap \Omega(j_0) \right] + o_k(1).
\]

The arguments above result in

\[
P_h \left[ (X_{in}^{nk} \in U) \cap (X_{out}^{nk} \in V_{\varepsilon}) \cap \Omega_{nk}(j_0) \right] \geq P_h \left[ (X_{in} \in U) \cap (X_{out} \in V) \cap \Omega(j_0) \right] + o_k(1; V_{\varepsilon}) + o_{\varepsilon}(1) + o_k(1). \tag{51}
\]

Now, we wish to prove (49). We appeal to (50) with \( h = h_2, U = U_2 \) and to (51) with \( h = h_1, U = U_1 \) to obtain, using (18) (applied with \( V = V_{\varepsilon} \)),

\[
a(j_0)P_{h_1} \left[ (X_{in} \in U_1) \cap (X_{out} \in V) \cap \Omega(j_0) \right] \leq b(j_0)P_{h_2} \left[ (X_{in} \in U_2) \cap (X_{out} \in V) \right] + o_{\varepsilon}(1) + o_k(1; V_{\varepsilon}) + o_k(1) + \vartheta(k). \tag{52}
\]

We first keep all the other quantities fixed and let \( k \to \infty \), after that we let \( \varepsilon \to 0 \) to obtain (49), as desired.

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