On signs of Hecke eigenvalues of Siegel eigenforms

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Abstract
In this article, we distinguish Siegel cuspidal eigenforms of degree two on the full symplectic group from the signs of their Hecke eigenvalues. To establish our theorem, we obtain a result towards simultaneous sign changes of eigenvalues of two Siegel eigenforms. In the course of the proof, we also prove that the Satake $p$-parameters of two different Siegel eigenforms are distinct for a set of primes $p$ of density 1. The main ingredient to prove the latter result is the theory of Galois representations attached to Siegel eigenforms.

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1 | INTRODUCTION

One of the fundamental and interesting problems in the theory of automorphic forms is whether the set of Hecke eigenvalues determine the eigenform under consideration? This question is well studied for elliptic modular eigenforms and there are many results available in the literature. But in the case of Siegel cuspidal eigenforms, this was a long-standing unanswered problem and only recently in 2018, Schmidt [22] has given an affirmative answer for normalized eigenvalues. To make our statements more concrete, let us introduce some notation. Let $S_k(\Gamma_2)$ be
the space of Siegel cusp forms of weight \( k \) for the symplectic group \( \Gamma_2 := \text{Sp}_4(\mathbb{Z}) \). For an eigenform \( F \in S_k(\Gamma_2) \), we denote the \( n \)th eigenvalue by \( \lambda_F(n) \) and the \( n \)th normalized eigenvalue by \( \lambda_F^*(n) := n^{3/2-k} \lambda_F(n) \). Then Schmidt [22] has proved that if \( F \in S_{k_1}(\Gamma_2) \) and \( G \in S_{k_2}(\Gamma_2) \) are eigenforms such that for all but finitely many primes \( p \)

\[
\lambda_F(p) = \lambda_G(p) \quad \text{and} \quad \lambda_F(p^2) = \lambda_G(p^2),
\]

then \( k_1 = k_2 \) and \( F \) is a constant multiple of \( G \). In the literature, this kind of result is known as the multiplicity one theorem. The result of Schmidt has been improved significantly by the authors of this paper in [12] by proving a strong multiplicity one result which essentially shows that any set of eigenvalues (normalized or non-normalized) at primes \( p \) of positive upper density are sufficient to distinguish the Siegel cuspidal eigenform.

Here is another and more refined question one can ask:

To what extent the signs of the eigenvalues determine an automorphic form uniquely? \( \quad (1) \)

For elliptic modular forms, this problem was first studied by Kowalski et al. [11] by using a very deep result of Ramakrishnan stating that the Rankin–Selberg convolution \( L \)-function associated to two elliptic newforms is the \( L \)-function of some \( \text{GL}_4 \)-form. Soon after, Matomäki [16] refined some of the results of [11]. Loosely speaking, she has proved that if the signs of \( p \)th Hecke eigenvalues of two non-CM elliptic newforms agree on a set of primes \( p \) of analytic density greater than \( 19/25 \), then the two forms are the same.

In this article, we investigate the question (1) in the case of Siegel cuspidal eigenforms. Let us first recall some facts about the signs of their Hecke eigenvalues to put things in order. If \( F \) is a Siegel eigenform with normalized eigenvalues \( \lambda_F(n) \), then we know that \( \lambda_F(n) \) are real numbers for all \( n \geq 1 \). By the works of Breulmann [3] and Kohnen [10], one knows that \( F \) is a Saito–Kurokawa lift if and only if \( \lambda_F(n) > 0 \) for all \( n \geq 1 \). Therefore one should discuss about the signs of \( \lambda_F(n) \) only if \( F \) is not a Saito–Kurokawa lift. Let us denote the subspace of non-Saito-Kurokawa lifts inside \( S_k(\Gamma_2) \) by \( S_k^+(\Gamma_2) \). By using representation theoretic techniques and a weaker result than the generalized Ramanujan conjecture, Pitale and Schmidt [19] have shown that for any Siegel eigenform \( F \in S_k^+(\Gamma_2) \), there are infinitely many primes \( p \) such that the sequence \( \{\lambda_F(p^r)\}_{r \geq 0} \) has infinitely many sign changes. This result has been strengthened by Das and Kohnen [4, Theorem 1.1] by using a completely different method. They have proved that for any positive integer \( j \) with \( 4 \nmid j \), there exists a set of primes of natural density 1 such that for any prime \( p \) in that set, the sequence \( \{\lambda_F(p^{jr})\}_{r \geq 0} \) has infinitely many sign changes.

Next we ask that if given any two distinct non-Saito-Kurokawa Siegel eigenforms \( F \) and \( G \), there exists at least one integer \( m \geq 1 \) such that the sign of \( \lambda_F(m) \) is different from the sign of \( \lambda_G(m) \). Unlike the case of elliptic modular forms, there is no unconditional result known till now about such simultaneous sign change. We first prove the following result on the simultaneous sign change of eigenvalues of two distinct Siegel eigenforms.

**Theorem 1.1.** Let \( F \in S_{k_1}^+(\Gamma_2) \) and \( G \in S_{k_2}^+(\Gamma_2) \) be Siegel eigenforms with normalized eigenvalues \( \lambda_F(n) \) and \( \lambda_G(n) \), respectively. We assume that they are not a constant multiple of each other. Then there exists a set of primes of density 1 such that for any prime \( p \) in that set, the sequence \( \{\lambda_F(p^r)\lambda_G(p^r)\}_{r \geq 0} \) has infinitely many sign changes.

To prove the above result, Theorem 3.1 plays a vital role which is the most technical part of this paper. It states that the Satake \( p \)-parameters of any two Siegel eigenforms \( F \) and \( G \), as in
Theorem 1.1, are distinct for a set of primes $p$ of density 1. To establish this result, we make use of certain tools from the theory of Galois representations attached to Siegel eigenforms.

Remark 1.2. We want to emphasize that the ideas used in the proof of Theorem 1.1 can be adopted to prove a similar result for some other automorphic forms. For example, one can easily prove such simultaneous sign change result for eigenvalues of two non-CM elliptic newforms such that neither of them is a Galois-conjugate to a twist of the other, which generalizes [9, Theorem 3]. In the elliptic case, we need to use [15, Theorem 3.2.2] to prove an analogous result to Theorem 3.1.

Now we state our next result which answers question (1) for Siegel modular forms.

**Theorem 1.3.** Let $F \in S^{1}_{k_1}(\Gamma_2)$ and $G \in S^{1}_{k_2}(\Gamma_2)$ be Siegel eigenforms with normalized eigenvalues $\lambda_F(n)$ and $\lambda_G(n)$, respectively. If
\[
\liminf_{x \to \infty} \frac{\# \{ n \leq x : \text{sign}(\lambda_F(n)) \neq \text{sign}(\lambda_G(n)) \}}{x} = 0,
\]
then $k_1 = k_2$ and $F$ is a scalar multiple of $G$.

The main ingredient in the proof of Theorem 1.3 is the existence of a prime power $p^t_0$ for which $\lambda_F(p^t_0) \lambda_G(p^t_0) < 0$, governed by Theorem 1.1. We want to emphasize that if one could manage to get a prime $p_0$ such that $\lambda_F(p_0) \lambda_G(p_0) < 0$, then Theorem 1.3 can be refined to distinguish $F$ and $G$ by the signs of their eigenvalues indexed by squarefree positive integers. Moreover, it would be of interest to study a similar problem for prime indices.

**Applications**

We now use the above results to obtain similar results for the Fourier coefficients of a Siegel eigenform. In particular, under certain assumptions, it is possible to characterize Siegel eigenforms in terms of signs of its Fourier coefficients supported on matrices of the form $nT_0$, where $T_0$ is a certain fixed symmetric, half-integral, positive definite $2 \times 2$ matrix and $n$ varies over a set of integers of positive lower density.

Let $F \in S_k(\Gamma_2)$ be a Siegel eigenform with eigenvalues $\mu_F(n)$ and Fourier coefficients $A_F(T)$. Suppose $T_0$ is a symmetric, half-integral, positive-definite matrix such that $-\det(2T_0) = -D_0$ is a fundamental discriminant and $\mathbb{Q}(\sqrt{-D_0})$ has class number 1. Then [1, Theorem 2.4.1] simplifies to give the following relation:
\[
\sum_{n \geq 1} \frac{A_F(nT_0)}{n^s} = A_F(T_0) \sum_{n \geq 1} \frac{\mu_F(n)}{n^s},
\]
in some right half plane $\text{Re}(s) \gg 1$. In particular, if $A_F(T_0) \neq 0$, then
\[
\frac{A_F(nT_0)}{A_F(T_0)} = \mu_F(n), \quad \text{for all } n \geq 1.
\]
Since the sign of $\mu_F(n)$ and $\lambda_F(n)$ are the same, by using the above relation together with Theorems 1.1 and 1.3, we have the following result.
Corollary 1.4. Let $F \in S_{k_1}^{\perp} (\Gamma_2)$ and $G \in S_{k_2}^{\perp} (\Gamma_2)$ be Siegel eigenforms with Fourier coefficients $A_F(T)$ and $A_G(T)$, respectively. Suppose there exists a $T_0$ such that $-\det(2T_0) = -D_0$ is fundamental, $\mathbb{Q}(\sqrt{-D_0})$ has class number 1 and $A_F(T_0) = A_G(T_0) = 1$. Then we have the following.

1. If $F$ and $G$ are not scalar multiples of each other, then there exists a set of primes of density 1 such that for each prime $p$ in this set, the sequence $\{A_F(p^rT_0)A_G(p^rT_0)\}_{r \geq 0}$ has infinitely many sign changes.

2. If

$$\liminf_{x \to \infty} \frac{\#\{n \leq x : \text{sign}(A_F(nT_0)) \neq \text{sign}(A_G(nT_0))\}}{x} = 0,$$

then $k_1 = k_2$ and $F$ is a scalar multiple of $G$.

Structure of the paper
In § 2, we recall Siegel modular forms and some properties of the spinor $L$-functions. We also review some standard results about the Galois representations attached to Siegel eigenforms. These representations play a vital role in proving the distinctness of Satake parameters of two Siegel eigenforms in § 3. Also in the same section, we show that the degree of a certain polynomial, appearing in the Euler product of a certain Dirichlet series, is at most 14. Finally, by using the properties of $\mathfrak{B}$-free numbers and the results obtained in § 3, we prove Theorems 1.1 and 1.3 in §§ 4 and 5, respectively.

2 | PREREQUISITES

In this section, we recall some basic properties of Siegel modular forms, associated spinor $L$-functions and Galois representations which are based primarily on [1] and [18].

2.1 | Siegel modular forms

The real symplectic unimodular group of degree 2 is defined by

$$\text{Sp}_4(\mathbb{R}) = \{ M \in \text{GL}_4(\mathbb{R}) : MJM^t = J \},$$

where $J = \left( \begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix} \right)$, $M^t$ denotes the transpose matrix of the matrix $M$, $0_2$ is the $2 \times 2$ zero matrix and $I_2$ is the $2 \times 2$ identity matrix. Let $\Gamma_2 := \text{Sp}_4(\mathbb{Z})$ be the subgroup of $\text{Sp}_4(\mathbb{R})$ consisting of matrices with integer entries. For a positive integer $k$, we denote the space of Siegel modular forms (respectively, cusp forms) of weight $k$ on the group $\Gamma_2$ by $M_k(\Gamma_2)$ (respectively, $S_k(\Gamma_2)$). There is an algebra of Hecke operators acting on the space $M_k(\Gamma_2)$ which preserves $S_k(\Gamma_2)$. A Siegel modular form in $S_k(\Gamma_2)$ is called a Siegel eigenform if it is a common eigenvector of all the Hecke operators.

If $k$ is an even integer, the Saito–Kurokawa conjecture asserts the existence of a lifting of any modular form of weight $2k - 2$, level 1 to a Siegel modular form of degree 2 of weight $k$ on $\Gamma_2$. This conjecture has been proved now due to the work of many mathematicians. The image of this lifting is a special subspace of $M_k(\Gamma_2)$ named Maass space. Moreover, this Saito-Kurokawa lifting
is an isomorphism from the modular forms space onto the Maass space mapping Eisenstein series to Eisenstein series, cusp forms to cusp forms, and eigenforms to eigenforms.

Note that the space $S_k(\Gamma_2)$ is a Hilbert space under the Petersson inner product given by [Equation (1.1.16)]. We denote by $S_k^\perp(\Gamma_2)$ the subspace which is the orthogonal complement to the Maass space in $S_k(\Gamma_2)$.

### 2.2 Satake $p$-parameters and Spinor $L$-functions

Fix a Siegel eigenform $F \in S_k(\Gamma_2)$ with $n$th Hecke eigenvalue $\mu_F(n)$ and $n$th normalized Hecke eigenvalue $\lambda_F(n) := \frac{\mu_F(n)}{n^{k-3/2}}$. For any prime $p$, let $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ be the Satake $p$-parameters of $F$. Then we know that $\alpha_{0,p}^2 \alpha_{1,p} \alpha_{2,p} = p^{2k-3}$. We now define the complex numbers

$$\beta_{1,p} := \frac{\alpha_{0,p}}{p^{k-3/2}}, \quad \beta_{2,p} := \frac{\alpha_{0,p} \alpha_{1,p}}{p^{k-3/2}}, \quad \beta_{3,p} := \frac{\alpha_{0,p} \alpha_{2,p}}{p^{k-3/2}}, \quad \beta_{4,p} := \frac{\alpha_{0,p} \alpha_{1,p} \alpha_{2,p}}{p^{k-3/2}} \quad (3)$$

and by abuse of notation, we call them the (normalized) Satake $p$-parameters of $F$. It is clear that $\beta_{1,p} = \beta_{4,p}^{-1}$ and $\beta_{2,p} = \beta_{3,p}^{-1}$.

The spinor $L$-function attached to $F$ is defined by

$$L(s, F, spin) = \prod_{p \text{ prime}} L_p(s, F, spin),$$

where

$$L_p(s, F, spin) = \prod_{1 \leq i \leq 4} (1 - \beta_{i,p} p^{-s})^{-1}. \quad (4)$$

Indeed, one can obtain that (see, for example [18, Theorem 3.11])

$$L_p(s, F, spin)^{-1} = p^{-4s} - \lambda_F(p)p^{-3s} + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{-1})p^{-2s} - \lambda_F(p)p^{-s} + 1.$$  

Remark 2.1. The above equation shows that the functions $p^{k-3/2} \beta_{i,p}$ are the roots of the $p$th Hecke polynomial (cf. (9)). In other words,

$$\prod_{1 \leq i \leq 4} (X - p^{k-3/2} \beta_{i,p}) = X^4 - \mu_F(p)X^3 + (\mu_F(p)^2 - \mu_F(p^2) - p^{2k-4})X^2 - \mu_F(p)p^{2k-3}X + p^{4k-6}.$$  

This fact will play a crucial role in the proof of Theorem 3.1.

### 2.3 Bounds of eigenvalues

We now assume that the eigenform $F \in S_k^\perp(\Gamma_2)$. Then the generalized Ramanujan conjecture proved by Weissauer [24] asserts that for any prime $p$, we have

$$|\beta_{i,p}| = 1 \quad \text{for all } 1 \leq i \leq 4. \quad (5)$$
However, the above assertion is not true if $F$ is a Saito–Kurokawa lift. Writing $L(s, F, \text{spin}) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$ as a Dirichlet series, one deduces from (5) that

$$|a_F(n)| \leq d_4(n) \quad \text{for all} \quad n \geq 1,$$

where $d_4(n)$ is the number of ways of writing $n$ as a product of four positive integers. Furthermore, it easily follows from [1, Theorem 1.3.2] that

$$\sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} = \zeta(2s + 1)L(s, F, \text{spin}) = \zeta(2s + 1)^{-1}\sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

from which we conclude that for any integer $n \geq 1$, we have

$$\lambda_F(n) = \sum_{d^2 \mid n} \frac{\mu(d)}{d} a_F(n/d^2),$$

where $\mu$ is the Möbius function. Using the bound (6), we see that $|\lambda_F(n)| \leq \sum_{d^2 \mid n} \frac{d_4(n/d^2)}{d}$ and therefore for any $\varepsilon > 0$, we have

$$|\lambda_F(n)| \ll n^{\varepsilon}. \quad (8)$$

### 2.4 Galois representations attached to Siegel modular forms

Let $F \in S_k(\Gamma_2)$ be a Siegel eigenform with $n$th eigenvalue $\mu_F(n)$. Suppose $E$ denotes the number field generated by $\mu_F(n)$, $n \geq 1$. It follows from the works of Taylor, Laumon and Weissauer that one can attach a family of Galois representations to $F$. More precisely, if $\lambda$ is a prime in $E$ above a rational prime $\ell$ and $E_{\lambda}$ denotes the completion of $E$ at $\lambda$, then there exists a continuous semisimple Galois representation $\rho_{F,\lambda} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(E_{\lambda})$ such that $\rho_{F,\lambda}$ is unramified outside $\ell$. Moreover, the characteristic polynomial of the Frobenius $\text{Frob}_p$ at $p \neq \ell$ is

$$X^4 - \mu_F(p)X^3 + (\mu_F(p^2) - \mu_F(p^2 - p^{2k-4}))X^2 - p^{2k-3}\mu_F(p)X + p^{4k-6}. \quad (9)$$

Furthermore, for all but finitely many primes $\ell$, the representation $\rho_{F,\lambda}$ is valued in $\text{GSp}_4(E_{\lambda})$ (see, for example, [14, Corollary 2.2]) and hence we can view $\rho_{F,\lambda}$ (after conjugating this representation) as a representation valued in $\text{GSp}_4(O_{E_{\lambda}})$, where $O_{E_{\lambda}}$ is the ring of integers of $E_{\lambda}$. Therefore, the maximal possible image for $\rho_{F,\lambda}$ is the group

$$A_{\lambda} = \{ \gamma \in \text{GSp}_4(O_{E_{\lambda}}) : \text{sim}(\gamma) \in (\mathbb{Z}^\times)^{2k-3} \},$$

where $\text{sim}(\gamma)$ denotes the similitude of $\gamma$. Let $F$ be a non-Saito-Kurokawa lift. Dieulefait [6, Theorem 4.2] has proved that if $\ell$ is large enough and splits completely in $E$, then the image of $\rho_{F,\lambda}$ is
A_1. In a recent paper [14], the precise image of the Galois representations attached to Siegel eigenforms of higher level has been studied. Since F is of level 1, it cannot have any (non-trivial) inner twists. This can be easily deduced from the fact that if (σ, χ) is an inner twist of a Siegel eigenform of level N and nebentype ε, then we have χ^2 = σ(ε) · ε^(-1) and hence the only prime divisors of the conductor of χ are the primes dividing N (see [14, Section 2.1]). So, it follows from [14, Theorem 1.4 (ii)]) that for any sufficiently large prime ℓ, the image of ρ_F, λ is A_1.

For our purpose, we need the information about the images of the product Galois representations which we recall from [13]. Let F and G be Siegel eigenforms and let E be the number field generated by all the Hecke eigenvalues of F and G. Let λ be a prime in E above a rational prime ℓ. Let ρ_{F, λ} and ρ_{G, λ} be the λ-adic Galois representations attached to F and G. From the above discussion, we may assume that, for a large prime ℓ, both of these representations are valued in GSp_4(Ω_{E, λ}). Consider the product Galois representation

$$\rho_{F, \lambda} \times \rho_{G, \lambda} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GSp}_4(\mathbb{Q}_{E, \lambda} \times \text{GSp}_4(\mathbb{Q}_{E, \lambda})$$

defined by

$$\sigma \mapsto (\rho_{F, \lambda}(\sigma), \rho_{G, \lambda}(\sigma)).$$

Then we have the following result.

**Theorem 2.2** [13]. Let F ∈ S_{k_1} \Gamma_2 and G ∈ S_{k_2} \Gamma_2 be Siegel eigenforms such that F is not a constant multiple of G. Then for all but finitely many primes ℓ, the image of ρ_{F, λ} × ρ_{G, λ} is

$$\{(γ_1, γ_2) ∈ \text{GSp}_4(\mathbb{Q}_{E, \lambda}) \times \text{GSp}_4(\mathbb{Q}_{E, \lambda}) : \exists v ∈ \mathbb{Z}_ℓ^\times, \text{sim}(γ_i) = v^{2k_i - 3}, 1 ≤ i ≤ 2 \}.$$

### 2.5 | Algebraic Chebotarev density theorem

We now recall the following result, a special case of a result of Rajan [20], giving an algebraic formulation of the Chebotarev density theorem for the density of primes satisfying an algebraic conjugacy condition. This plays a crucial role in proving Theorem 3.1.

**Theorem 2.3** [20, Theorem 3]. Let K be a finite extension of \mathbb{Q}_ℓ and let G be an algebraic group over K. Let \mathcal{X} be a subscheme of G defined over K that is stable under the adjoint action of G. Suppose that

$$R : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow G(K)$$

is a Galois representation which is unramified outside a finite set of primes. Let H denote the Zariski closure of Im(R) (the image of R) in G(K), with identity connected component H° and component group Φ = H/H°. For each φ ∈ Φ, assume that H^φ denotes its corresponding connected component. Let

$$\Psi = \{φ ∈ Φ : H^φ ⊂ \mathcal{X} \}.$$ 

Then the set of primes p with R(Frob_p) ∈ \mathcal{X}(K) ∩ Im(R) has density \frac{|\Psi|}{|Φ|}.
2.6 \hspace{1em} \textit{B-free numbers}

The notion of \textit{B-free numbers} was first introduced by Erdös in [7]. These numbers are a certain generalization of squarefree numbers.

**Definition 2.4.** Let \( B = \{ b_i : 1 < b_1 < b_2 < \ldots \} \) be a sequence of positive integers such that

\[
\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty \quad \text{and} \quad \gcd(b_i, b_j) = 1 \quad \text{for} \quad i \neq j.
\]

A positive integer \( n \) is said to be \( B \)-free if it is not divisible by \( b_i \) for any \( i \geq 1 \).

Let \( \mathcal{A} \) be the set of all \( B \)-free numbers, then a result of Erdös [7, Theorem 3] states that

\[
\#\{n \leq x : n \in \mathcal{A}\} \sim \delta x \quad \text{as} \quad x \to \infty,
\]

(10)

where \( \delta = \prod_{b_i \in B} (1 - \frac{1}{b_i}) \). Note that if we take \( B \) to be the sequence of squares of all primes, then the set of \( B \)-free numbers is nothing but the set of all squarefree numbers.

3 \hspace{1em} \textbf{TECHNICAL RESULTS}

3.1 \hspace{1em} \textbf{Distinctness of Satake \( p \)-parameters}

We start by recalling a result of Weiss about the distinctness of Satake \( p \)-parameters of a non-Saito-Kurokawa Siegel eigenform. For \( F \in S_{k_1}^\perp (\Gamma_2) \) with Satake \( p \)-parameters \( \{ \beta_{i,p} \}_{1 \leq i \leq 4} \), defined by (3), he proved that there exists a set of primes of density 1 such that for any prime \( p \) in that set, all the four elements \( \beta_{i,p}, 1 \leq i \leq 4 \), are distinct [23, Corollary 5.11]. Using a similar argument, we prove the following result about the distinctness of Satake \( p \)-parameters for two Siegel eigenforms.

**Theorem 3.1.** Let \( F \in S_{k_1}^\perp (\Gamma_2) \) and \( G \in S_{k_2}^\perp (\Gamma_2) \) be Siegel eigenforms which are not a constant multiple of each other. For a prime \( p \), let \( \{ \beta_{i,p} \}_{1 \leq i \leq 4} \) and \( \{ \delta_{i,p} \}_{1 \leq i \leq 4} \) be the Satake \( p \)-parameters of \( F \) and \( G \), respectively. Then there exists a set of primes of density 1 such that for any prime \( p \) in that set, all the eight elements

\[
\{ \beta_{i,p}, \delta_{j,p} : 1 \leq i, j \leq 4 \}
\]

are distinct.

**Proof.** Let \( E \) be the number field generated by the eigenvalues of \( F \) and \( G \). Let \( \lambda \) be a prime in \( E \) above a large rational prime \( \ell \) such that the assertion of Theorem 2.2 is true. In particular, both the representations \( \rho_{F,\lambda} \) and \( \rho_{G,\lambda} \) are valued in \( \text{GSp}_4(\mathcal{O}_{E,\lambda}) \). We now consider the \( \lambda \)-adic product Galois representation

\[
R_\lambda \coloneqq \rho_{F,\lambda} \times \rho_{G,\lambda} : \text{Gal}(\overline{Q}/Q) \longrightarrow \mathcal{G}(\mathcal{O}_{E,\lambda}),
\]
where \( G := G_{Sp,4} \times G_{Sp,4} \) is the algebraic group considered over \( \overline{\mathbb{Q}} \). For any prime \( p \neq \ell \), if

\[
R_\lambda(Frob_p) = (\gamma_1, \gamma_2),
\]

then by Remark 2.1, it is easy to observe that the eigenvalues of \( \text{sim}(\gamma_2)\gamma_2 \) and \( \text{sim}(\gamma_1)\gamma_2 \) are

\[
p^{2(k_1+k_2-3)}\beta_{1,p}^2, \quad p^{2(k_1+k_2-3)}\beta_{2,p}^2, \quad p^{2(k_1+k_2-3)}\beta_{3,p}^2, \quad p^{2(k_1+k_2-3)}\beta_{4,p}^2,
\]

and

\[
p^{2(k_1+k_2-3)}\delta_{1,p}^2, \quad p^{2(k_1+k_2-3)}\delta_{2,p}^2, \quad p^{2(k_1+k_2-3)}\delta_{3,p}^2, \quad p^{2(k_1+k_2-3)}\delta_{4,p}^2,
\]

respectively. Hence in order to complete the proof, it is sufficient to show that for a set of primes \( p \) of density 1, any two elements in the collection of all the eight eigenvalues of \( \text{sim}(\gamma_2)\gamma_2 \) and \( \text{sim}(\gamma_1)\gamma_2 \) are distinct, where \( R_\lambda(Frob_p) = (\gamma_1, \gamma_2) \).

Let \( \mathcal{X} \) be the set of elements \( (\gamma_1, \gamma_2) \) of \( G \) such that at least two eigenvalues are the same among all the eight eigenvalues of \( \text{sim}(\gamma_2)\gamma_2 \) and \( \text{sim}(\gamma_1)\gamma_2 \). Then we need to show that the set of primes \( p \) such that \( R_\lambda(Frob_p) \in \mathcal{X}(E_\lambda) \cap \text{Im}(R_\lambda) \) is of density 0, where \( \mathcal{X}(E_\lambda) \) is the set of \( E_\lambda \)-points of \( \mathcal{X} \). We will prove it by applying Theorem 2.3 in this setting and for that, we first need to check certain required properties of \( \mathcal{X} \) and understand the image of \( R_\lambda \).

Clearly, \( \mathcal{X} \) is stable under the conjugate action of \( G \) because the eigenvalues do not depend on conjugacy classes. Also, \( \mathcal{X} \) is a closed subscheme of \( G \). To see this, consider the polynomial

\[
f(X) = P(\text{sim}(\gamma_2)\gamma_2)(X) \cdot P(\text{sim}(\gamma_1)\gamma_2)(X),
\]

where \( P(\gamma)(X) \) denotes the characteristic polynomial of \( \gamma \in G_{Sp,4} \). Then \( \mathcal{X} \) is the vanishing set of the discriminant of the polynomial \( f \).

Let \( H := \text{Im}(R_\lambda) \) denote the Zariski closure of \( \text{Im}(R_\lambda) \). From Theorem 2.2, it follows that the algebraic group \( H \) is connected which might be well known to the experts but for the sake of completeness, we have given an idea of its proof in Lemma 3.2.

Summarizing the above discussions, we have:

- a representation \( R_\lambda : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow G(E_\lambda) \) such that the Zariski closure of its image (denoted by \( H \)) is a connected algebraic group; and
- a closed subscheme \( \mathcal{X} \) of \( G \) that is stable under the action of \( G \) by conjugation.

We are now ready to apply Theorem 2.3. Since \( H \) is connected, the identity connected component \( H^0 \) is equal to \( H \) and hence the component group \( \Phi \) is trivial. Also, it is obvious that \( H \varsubsetneq \mathcal{X} \) which gives \( |\Psi| = 0 \). Applying Theorem 2.3, we deduce that the set of primes \( p \) such that \( R_\lambda(Frob_p) \in \mathcal{X}(E_\lambda) \cap \text{Im}(R_\lambda) \) has density 0. This completes the proof.

Finally, we give a proof of the fact that the algebraic group \( H \), defined in the proof of the above theorem, is connected.

**Lemma 3.2.** The algebraic group \( H \) is homeomorphic to \( G_{Sp,4} \times \text{Sp,4} \). In particular, \( H \) is connected.
Proof. From Theorem 2.2, recall that
\[
\text{Im}(R_\lambda) = \left\{ (\gamma_1, \gamma_2) \in \text{GSp}_4(O_{E_\lambda}) \times \text{GSp}_4(O_{E_\lambda}) : \exists \nu \in \mathbb{Z}_\mathbb{F}^\times, \text{sim}(\gamma_1) = \nu^{2k_1-3}, 1 \leq i \leq 2 \right\}.
\]
Let \( A_{\lambda,1} = \{ \gamma \in \text{GSp}_4(O_{E_\lambda}) : \text{sim}(\gamma) \in (\mathbb{Z}_\mathbb{F}^\times)^{2k_1-3} \} \). We now consider the following split short exact sequence:
\[
1 \longrightarrow A_{\lambda,1} \xrightarrow{\xi} \text{Im}(R_\lambda) \longrightarrow \text{Im}(R_\lambda)/\xi(A_{\lambda,1}) \longrightarrow 1.
\]
In the above sequence, for \( \gamma_1 \in A_{\lambda,1} \) with \( \text{sim}(\gamma_1) = \nu^{2k_1-3} \), the injective homomorphism \( \xi \) is defined by
\[
\xi(\gamma_1) = (\gamma_1, \gamma_2), \text{ where } \gamma_2 = J(\nu, k_2) := \begin{pmatrix} v^{2k_2-3}I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}
\]
and the section \( \nu \) is the natural projection. Therefore, we have \( \text{Im}(R_\lambda) \simeq A_{\lambda,1} \times \text{Im}(R_\lambda)/\xi(A_{\lambda,1}) \).

From the proof of [12, Theorem 1.5], we know that \( A_{\lambda,1} \) is Zariski dense in \( \text{GSp}_4 \) and so by using the fact that a product of closed spaces is closed in the product topology and \( H \) is the Zariski closure of \( \text{Im}(R_\lambda) \), we obtain
\[
H \simeq \text{GSp}_4 \times \left( \text{Im}(R_\lambda)/\xi(A_{\lambda,1}) \right).
\]
By [17, Lemma 3], we know that \( \text{Sp}_4(O_{E_\lambda}) \) is Zariski dense in \( \text{Sp}_4 \) and hence to complete the proof it is sufficient to show that
\[
\text{Im}(R_\lambda)/\xi(A_{\lambda,1}) \simeq \text{Sp}_4(O_{E_\lambda}).
\]
Fix a coset \((\gamma_1, \gamma_2)\xi(A_{\lambda,1})\) in the quotient group with \( \text{sim}(\gamma_1) = \nu^{2k_1-3} \) for some \( \nu \in \mathbb{Z}_\mathbb{F}^\times \). Then we can write
\[
(\gamma_1, \gamma_2)\xi(A_{\lambda,1}) = (\gamma_1, \gamma_2)(\gamma_1^{-1}, J(\nu, -k_2))\xi(A_{\lambda,1}) = (I_4, \gamma_2J(\nu, -k_2))\xi(A_{\lambda,1}).
\]
It is now easy to see that the map \((\gamma_1, \gamma_2)\xi(A_{\lambda,1}) \mapsto \gamma_2J(\nu, -k_2)\) is a well-defined injective homomorphism onto \( \text{Sp}_4(O_{E_\lambda}) \), which completes the proof. \( \square \)

3.2 Euler product of the spinor \( L \)-functions

The aim of this section is to show that the degree of the polynomial, appearing in the Euler product of a certain Dirichlet series with coefficients the product of eigenvalues at prime powers for a fixed prime, is at most 14. This result improves on a result of Gun et al. [9, Lemma 15]. Although the result of Gun et al. is sufficient for our purpose, we would like to present the proof of our result...
since it gives a better result by following a completely different but elementary approach compared to [9, Lemma 15]. We follow the idea presented in the proof of [5, Lemma 2.7.13].

**Lemma 3.3.** Let $F \in S_{k_1}^\perp (\Gamma_2)$ and $G \in S_{k_2}^\perp (\Gamma_2)$ be Siegel eigenforms with respective $n$th normalized eigenvalues $\lambda_F(n)$ and $\lambda_G(n)$. Let $\{\beta_{i,p}\}_{1 \leq i \leq 4}$ and $\{\delta_{i,p}\}_{1 \leq i \leq 4}$ be the Satake $p$-parameters of $F$ and $G$, respectively. Then for any prime $p$ and any $s$ with $\text{Re}(s) > 0$, we have

$$
\sum_{r=0}^{\infty} \frac{\lambda_F(p^r) \lambda_G(p^r)}{p^{rs}} = g_p(p^{-s}) \prod_{1 \leq i, j \leq 4} (1 - \beta_{i,p} \delta_{j,p} p^{-s})^{-1},
$$

where $g_p(p^{-s})$ is a polynomial in $p^{-s}$ of degree at most 14.

**Proof.** By using (7) and (8), we deduce that for each prime $p$ and $\text{Re}(s) > 0$, we have

$$
\Phi_F(p^{-s}) := \sum_{r=0}^{\infty} \frac{\lambda_F(p^r)}{p^{rs}} = \left(1 - \frac{1}{p^{2s+1}}\right) \prod_{1 \leq i \leq 4} (1 - \beta_{i,p} p^{-s})^{-1},
$$

and

$$
\Phi_G(p^{-s}) := \sum_{r=0}^{\infty} \frac{\lambda_G(p^r)}{p^{rs}} = \left(1 - \frac{1}{p^{2s+1}}\right) \prod_{1 \leq i \leq 4} (1 - \delta_{i,p} p^{-s})^{-1}.
$$

Let $s_0 = \sigma_0 + it_0$ be any complex number such that $\sigma_0 > 0$. Let $C$ be the circle centred at 0 of radius bigger than 1 but strictly less than $2^{s_0}$. Now consider the integral

$$
\frac{1}{2\pi i} \int_{C} \Phi_F(p^{-s_0}z) \Phi_G(1/z) \frac{dz}{z}.
$$

Using the above series form of $\Phi_F$ and $\Phi_G$, we deduce that the above integral (11) is equal to

$$
\sum_{r, r' = 0}^{\infty} \lambda_F(p^r) \lambda_G(p^{r'}) p^{-s_0 r} \frac{1}{2\pi i} \int_{C} z^{-r-r'-1} dz = \sum_{r=0}^{\infty} \lambda_F(p^r) \lambda_G(p^r) p^{-s_0 r}.
$$

By using the Euler product form of $\Phi_F$ and $\Phi_G$, the integral (11) is equal to

$$
\frac{1}{2\pi i} \int_{C} \prod_{1 \leq i \leq 4} \left(1 - \beta_{i,p} z \right) \prod_{1 \leq i \leq 4} \left(1 - \delta_{i,p} z \right) \frac{dz}{z}.
$$

Since $|\beta_{i,p}| = |\delta_{i,p}| = 1$ for $1 \leq i \leq 4$, the singularities of the integrand of the above integral which are inside $C$ are the poles at $z = \delta_{i,p}$ for $1 \leq i \leq 4$. Applying Cauchy’s residue theorem, we deduce that

$$
\frac{1}{2\pi i} \int_{C} \Phi_F(p^{-s_0}z) \Phi_G(1/z) \frac{dz}{z} = g_p(p^{-s_0}) \prod_{1 \leq i, j \leq 4} (1 - \beta_{i,p} \delta_{j,p} p^{-s_0})^{-1},
$$

where $g_p(p^{-s})$ is a polynomial in $p^{-s}$ of degree at most 14.
where \( g_p(p^{-s_0}) \) is a polynomial in \( p^{-s_0} \) of degree at most 14. Therefore, from (12) and (13), we conclude the result. \( \square \)

## 4 | PROOF OF THEOREM 1.1

For a prime \( p \), let \( \{\beta_{i,p}\}_{1 \leq i \leq 4} \) and \( \{\delta_{i,p}\}_{1 \leq i \leq 4} \) be the respective Satake \( p \)-parameters of \( F \) and \( G \), defined by (3). Let \( P \) be the set of primes \( p \) such that any two elements in the set

\[
\{\beta_{i,p}, \delta_{j,p} : 1 \leq i, j \leq 4\}
\]

are distinct. From Theorem 3.1, we know that the set of primes \( P \) is of density 1. To prove Theorem 1.1, we show that for any \( p \in P \), the sequence \( \{\lambda_F(p^r)\lambda_G(p^r)\}_{r \geq 0} \) changes sign infinitely often.

We now fix a prime \( p \in P \) and on the contrary, we assume that the sequence \( \{\lambda_F(p^r)\lambda_G(p^r)\}_{r \geq 0} \) has all but finitely many terms non-negative. Since \( F \) and \( G \) are non-Saito-Kurokawa lifts, by using (8) we see that the Dirichlet series

\[
\sum_{r=0}^{\infty} \frac{\lambda_F(p^r)\lambda_G(p^r)}{p^{rs}} \tag{14}
\]

converges absolutely for \( \text{Re}(s) > 0 \). Also, from Lemma 3.3, we know that

\[
\sum_{r=0}^{\infty} \frac{\lambda_F(p^r)\lambda_G(p^r)}{p^{rs}} = g_p(p^{-s}) \prod_{1 \leq i, j \leq 4} (1 - \beta_{i,p}\delta_{j,p}p^{-s})^{-1}, \tag{15}
\]

where \( g_p(p^{-s}) \) is a polynomial in \( p^{-s} \) of degree at most 14. Since \( |\beta_{i,p}| = |\delta_{j,p}| \) for all \( 1 \leq i, j \leq 4 \), the function

\[
\prod_{1 \leq i, j \leq 4} (1 - \beta_{i,p}\delta_{j,p}p^{-s})^{-1}
\]

has 16 singularities on the line \( \text{Re}(s) = 0 \). On the other hand, as a polynomial in \( p^{-s} \), \( g_p(p^{-s}) \) is of degree at most 14 showing that the function on the right of (15) and hence the Dirichlet series (14) has at least two singularities on the line \( \text{Re}(s) = 0 \). Therefore, the abscissa of absolute convergence of the Dirichlet series (14) is 0. Thus by Landau’s theorem [2, Theorem 11.13] on Dirichlet series with non-negative coefficients, we deduce that the Dirichlet series (14) has a singularity at \( s = 0 \) and thus the series on the right of (15) has a singularity at \( s = 0 \). It follows that \( \prod_{1 \leq i, j \leq 4} (1 - \beta_{i,p}\delta_{j,p}) = 0 \) and hence

\[
\beta_{i,p} = \delta_{j,p}^{-1} \quad \text{for some} \quad i, j.
\]

But we know that \( \delta_{1,p}^{-1} = \delta_{4,p}, \delta_{2,p}^{-1} = \delta_{3,p} \) and hence we arrive at a contradiction because of the choice of our prime \( p \).
5 PROOF OF THEOREM 1.3

Consider the set

\[ \mathcal{F} := \{ n \in \mathbb{N} : \lambda_F(n)\lambda_G(n) < 0 \}. \]

Suppose \( F \) is not a scalar multiple of \( G \). To prove the theorem, we show that

\[ \#\{ n \leq x : n \in \mathcal{F} \} \gg x, \]

that is, there exists an absolute constant \( c > 0 \) such that \( \#\{ n \leq x : n \in \mathcal{F} \} \geq c x \) for all sufficiently large \( x \). By Theorem 1.1, there exists a prime \( p_0 \) and an integer \( t \geq 1 \) such that \( \lambda_F(p_0^t)\lambda_G(p_0^t) < 0 \).

Define

\[ \mathcal{B} := \{ p_0 \} \cup \{ p : p \neq p_0 \text{ and } \lambda_F(p)\lambda_G(p) = 0 \} \cup \{ p^2 : p \neq p_0 \text{ and } \lambda_F(p)\lambda_G(p) \neq 0 \}, \]

and let \( \mathcal{A} \) be the set of all \( \mathcal{B} \)-free numbers. Then any element \( n \) of \( \mathcal{A} \) is a squarefree positive integer satisfying \( \lambda_F(n)\lambda_G(n) \neq 0 \). We now claim that

\[ \sum_{b \in \mathcal{B}} \frac{1}{b} < \infty. \]

Since \( \sum_p \frac{1}{p^2} < \infty \), to prove the above claim it is sufficient to show that \( \sum_{\lambda_F(p)\lambda_G(p)=0} \frac{1}{p} < \infty. \) But

\[ \sum_{\lambda_F(p)\lambda_G(p)=0} \frac{1}{p} \leq \sum_{\lambda_F(p)=0} \frac{1}{p} + \sum_{\lambda_G(p)=0} \frac{1}{p}. \]

We know from [21, Theorem 4] that there exists a \( \delta > 0 \) such that

\[ \#\{ p \leq x : \lambda_F(p) = 0 \} \ll \frac{x}{(\log x)^{1+\delta}}. \]

Using the above result, integration by parts yields

\[ \sum_{p \leq x} \frac{1}{p} = \int_{2}^{x} \frac{1}{u} \left( \sum_{p \leq x} \frac{1}{p} \right) \ll 1 + \int_{2}^{x} \frac{du}{u(\log u)^{1+\delta}} \ll 1. \]

Similarly, we have

\[ \sum_{p \leq x} \frac{1}{p} \ll 1. \]

This proves our claim.
Note that the set \( A \) can be written as a disjoint union of the sets \( A_1 \) and \( A_2 \) defined by

\[
A_1 = \{ n \in A : \lambda_F(n)\lambda_G(n) < 0 \}, \quad A_2 = \{ n \in A : \lambda_F(n)\lambda_G(n) > 0 \}.
\]

If \( n \in A_2 \), then \( p_0^t n \in F \). Therefore \( F \supseteq A_1 \cup p_0^t A_2 \) and hence we have

\[
\#\{ n \leq x : n \in F \} \geq \#\{ n \leq x : n \in A_1 \cup p_0^t A_2 \} \geq \#\{ n \leq x/p_0^t : n \in A \}.
\]

Next by using (10) we deduce that \( \#\{ n \leq x/p_0^t : n \in A \} \sim \delta x/p_0^t \), for a constant \( \delta > 0 \). Therefore for sufficiently large \( x \), we have

\[
\frac{\#\{ n \leq x/p_0^t : n \in A \}}{x} \geq c,
\]

where \( c \) is a positive constant. This in turn implies \( \#\{ n \leq x : n \in F \} \gg x \).

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