A Structure Theorem for Leibniz Homology

Jerry M. Lodder

Mathematical Sciences, Dept. 3MB
Box 30001
New Mexico State University
Las Cruces NM, 88003, U.S.A.
e-mail: jlodder@nmsu.edu

Abstract. Presented is a structure theorem for the Leibniz homology, $HL_*$, of an Abelian extension of a simple real Lie algebra $g$. As applications, results are stated for affine extensions of the classical Lie algebras $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}_n(\mathbb{R})$, and $\mathfrak{sp}_n(\mathbb{R})$. Furthermore, $HL_*(\mathfrak{h})$ is calculated when $\mathfrak{h}$ is the Lie algebra of the Poincaré group as well as the Lie algebra of the affine Lorentz group. The general theorem identifies all of these in terms of $g$-invariants.

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1 Introduction

For a (semi-)simple Lie algebra $g$ over $\mathbb{R}$, the Milnor-Moore theorem identifies the Lie algebra homology, $H^\text{Lie}_*(g; \mathbb{R})$, as a graded exterior algebra on the primitive elements of $H^\text{Lie}_*(g; \mathbb{R})$, i.e.,

$$H^\text{Lie}_*(g) \simeq \Lambda^*(\text{Prim}(H_*(g))),$$

where the coefficients are understood to be in the field $\mathbb{R}$. The algebra structure on $H^\text{Lie}_*(g)$ can be deduced from the exterior product of $g$-invariant cycles on the chain level, and agrees with the corresponding Pontrajagin
product induced from the Lie group \([4]\). For Leibniz homology, however, we have \(HL_n(g) = 0, n \geq 1\), for \(g\) simple with \(R\) coefficients \([8]\). Now, let \(g\) be a simple real Lie algebra, \(h\) an extension of \(g\) by an Abelian ideal \(I\):

\[
0 \longrightarrow I \longrightarrow h \longrightarrow g \longrightarrow 0
\]

Then \(g\) acts on \(I\), and this extends to a \(g\) action on \(\Lambda^*(I)\) by derivations. For any \(g\)-module \(M\), let

\[
M^g = \{m \in M \mid [g, m] = 0 \quad \forall g \in g\}
\]

denote the submodule of \(g\)-invariants. Under a mild hypothesis, we prove that

\[
HL_*(h) \simeq [\Lambda^*(I)]^g \otimes T(K_*)
\]

where

\[
K_* = \text{Ker}(H^{Lie}_*(I; h)^g \to H^{Lie}_{*+1}(h))
\]

and \(T(K_*) = \sum_{n \geq 0} K_*^{\otimes n}\) is the tensor algebra over \(R\). Above, \(H^{Lie}_*(I; h)\) denotes the Lie algebra homology of \(I\) with coefficients in \(h\), and the map \(H^{Lie}_*(I; h) \to H^{Lie}_{*+1}(h)\) is the composition

\[
H^{Lie}_*(I; h) \xrightarrow{j_*} H^{Lie}_*(h; h) \xrightarrow{\pi_*} H^{Lie}_{*+1}(h),
\]

where \(j_*\) is induced by the inclusion of Lie algebras \(j : I \hookrightarrow h\), and \(\pi_*\) is induced by the projection of chain complexes

\[
\pi : h \otimes h^{\wedge n} \to h^{\wedge (n+1)}
\]

\[
\pi(h_0 \otimes h_1 \wedge h_1 \wedge \ldots \wedge h_n) = h_0 \wedge h_1 \wedge h_1 \wedge \ldots \wedge h_n.
\]

Of course, \(K_*\) is computed from the module of \(g\)-invariants, beginning with \(H^{Lie}_*(I; h)^g\) as indicated above.

The main theorem is easily applied when \(g\) is a classical Lie algebra and \(h\) is an affine extension of \(g\). In the final section we state the results for \(g\) being \(\mathfrak{sl}_n(R)\), \(\mathfrak{so}_n(R)\), \(n\) odd or even, and \(I = R^n\), whereby \(g\) acts on \(I\) via matrix multiplication on vectors, which is often called the standard representation. For the (special) orthogonal Lie algebra, \(\mathfrak{so}_n(R)\), the general theorem agrees with calculations of Biyogmam \([1]\). When \(g = \mathfrak{sp}_n(R)\) and \(I = R^{2n}\), we recover the author's previous result \([2]\). Additionally, \(HL_*(h)\) is computed when \(h\) is the Lie algebra of the Poincaré group and the Lie algebra of the affine Lorentz group.
2 Preliminaries on Lie Algebra Homology

For any Lie algebra \( g \) over a ring \( k \), the Lie algebra homology of \( g \), written \( H^\text{Lie}_*(g; k) \), is the homology of the chain complex \( \Lambda^*(g) \), namely

\[
egin{array}{ccccccc}
0 & \leftarrow & g & \leftarrow & g \wedge 2 & \leftarrow & \cdots & \leftarrow & g \wedge (n-1) & \leftarrow & g \wedge n & \leftarrow & \cdots,
\end{array}
\]

where

\[
d(g_1 \wedge g_2 \wedge \cdots \wedge g_n) = \sum_{1 \leq i<j \leq n} (-1)^j (g_1 \wedge \cdots \wedge g_{i-1} \wedge [g_i; g_j] \wedge g_{i+1} \wedge \cdots \wedge g_n).
\]

In this paper \( H^\text{Lie}_*(g) \) denotes homology with real coefficients, where \( k = \mathbb{R} \). Lie algebra homology with coefficients in the adjoint representation, \( H^\text{Lie}_*(g; g) \), is the homology of the chain complex \( g \otimes \Lambda^*(g) \), i.e.,

\[
egin{array}{ccccccc}
g & \leftarrow & g \otimes g & \leftarrow & g \otimes g \wedge 2 & \leftarrow & \cdots & \leftarrow & g \otimes g \wedge (n-1) & \leftarrow & g \otimes g \wedge n & \leftarrow & \cdots,
\end{array}
\]

where

\[
d(g_1 \otimes g_2 \wedge g_3 \wedge \cdots \wedge g_{n+1}) = \sum_{i=2}^{n+1} (-1)^i ([g_1, g_i] \otimes g_2 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_{n+1}) + \sum_{2 \leq i<j \leq n+1} (-1)^j (g_1 \otimes g_2 \wedge \cdots \wedge g_i \wedge g_{i+1} \wedge g_j \wedge \cdots \wedge g_{n+1} \wedge \cdots \wedge g_{n+1}).
\]

The canonical projection \( \pi : g \otimes \Lambda^*(g) \to \Lambda^{*+1}(g) \) given by \( g \otimes g \wedge n \to g \wedge (n+1) \) is a map of chain complexes, and induces a \( k \)-linear map on homology

\[
\pi_* : H^\text{Lie}_n(g; g) \to H^\text{Lie}_{n+1}(g; k).
\]

Let \( HR_n(g) \) denote the homology of the complex

\[
CR_n(g) = (\text{Ker } \pi)_n[1] = \text{Ker } [g \otimes g \wedge (n+1) \to g \wedge (n+2)], \quad n \geq 0.
\]

There is a resulting long exact sequence

\[
\cdots \xrightarrow{\delta^\text{Lie}_{n+1}} HR_{n-1}(g) \xrightarrow{\delta^\text{Lie}_n} H^\text{Lie}_n(g; g) \xrightarrow{\delta^\text{Lie}_{n+1}} H^\text{Lie}_{n+1}(g) \xrightarrow{\delta^\text{Lie}_n} \cdots
\]

\[
\cdots \xrightarrow{\delta^\text{Lie}_0} HR_0(g) \xrightarrow{\delta^\text{Lie}_1} H^\text{Lie}_1(g; g) \xrightarrow{\delta^\text{Lie}_2} H^\text{Lie}_2(g) \xrightarrow{\delta^\text{Lie}_1} \cdots
\]

\[
0 \xrightarrow{\delta^\text{Lie}_0} H^\text{Lie}_0(g; g) \xrightarrow{\delta^\text{Lie}_1} H^\text{Lie}_1(g) \xrightarrow{\delta^\text{Lie}_2} 0.
\]
Now let \( g \) be a simple real Lie algebra and \( h \) an extension of \( g \) by an Abelian ideal \( I \). There is a short exact sequence of real Lie algebras

\[
0 \longrightarrow I \xrightarrow{j} h \xrightarrow{\rho} g \longrightarrow 0,
\]

where \( j : I \to h \) is an inclusion of Lie algebras, and \( \rho : h \to h/I \cong g \) is a projection of Lie algebras. For \( g \in g \) and \( a \in I \), the action of \( g \) on \( I \) can be described as

\[
[g, a] = j^{-1}([h, j(a))],
\]

where \( h \in h \) is any element with \( \rho(h) = g \). The action is well-defined.

Conversely, given any representation \( I \) of \( g \), such as the standard representation of a classical real Lie algebra, then \( h \) can be constructed as the linear span of all elements in \( g \) with all elements in \( I \). Here \( I \) is considered as an Abelian Lie algebra with \( [a, b] = 0 \) for all \( a, b \in I \). Thus, in \( h \), we have

\[
[g_1 + a, g_2 + b] = [g_1, g_2] + [g_1, b] - [g_2, a]
\]

for \( g_1, g_2 \in g \).

**Lemma 2.1.** Let \( g \) be a simple Lie algebra over \( \mathbb{R} \), and let

\[
0 \longrightarrow I \xrightarrow{j} h \xrightarrow{\rho} g \longrightarrow 0,
\]

be an Abelian extension of \( g \). There are natural vector space isomorphisms

\[
H_*^{Lie}(h) \cong [\Lambda^*(I)]^h \otimes H_*^{Lie}(g) \quad (2.1)
\]

\[
H_*^{Lie}(h; h) \cong [H_*^{Lie}(I; h)]^h \otimes H_*^{Lie}(g). \quad (2.2)
\]

**Proof.** Apply the homological version of the Hochschild-Serre spectral sequence to the subalgebra \( g \) of \( h \) \([7]\). Then

\[
H_*^{Lie}(h) \cong H_*^{Lie}(I)^h \otimes H_*^{Lie}(g).
\]

Since \( I \) is Abelian, \( H_*^{Lie}(I)^h = [\Lambda^*(I)]^h \), and isomorphism (2.1) follows. The spectral sequence yields isomorphism (2.2) directly. Note that, since \( I \) acts trivially on \( H_*^{Lie}(I; h) \), we have

\[
H_*^{Lie}(I; h)^h = H_*^{Lie}(I; h)^h
\]

as well, yielding

\[
H_*^{Lie}(h; h) \cong [H_*^{Lie}(I; h)]^h \otimes H_*^{Lie}(g).
\]

Compare with Hochschild and Serre \([3]\). \( \square \)
The natural inclusion \( \mathfrak{g} \hookrightarrow \mathfrak{h} \) of Lie algebras leads to a map of long exact sequences

\[
\delta^{\text{Lie}} : H^r_n(\mathfrak{g}) \rightarrow H^r_{n-1}(\mathfrak{g}; \mathfrak{g}) \rightarrow H^r_n(\mathfrak{g})
\]

where \( \delta^{\text{Lie}} \) is the connecting homomorphism. For \( \mathfrak{g} \) simple, \( H^1_{n-1}(\mathfrak{g}; \mathfrak{g}) = 0 \), \( n \geq 1 \) [2], and

\[
\delta^{\text{Lie}} : H^r_n(\mathfrak{g}) \rightarrow H^{r-1}_{n-3}(\mathfrak{g})
\]

is an isomorphism for \( n \geq 3 \). Note that \( H^1_{n-1}(\mathfrak{g}; \mathfrak{g}) \simeq 0 \) and \( H^2_{n-1}(\mathfrak{g}) \simeq 0 \). The inclusion \( j : I \hookrightarrow \mathfrak{h} \) is \( \mathfrak{g} \)-equivariant and induces an endomorphism

\[
H^r_n(I; \mathfrak{h})^g \xrightarrow{j_*} H^r_n(\mathfrak{h}; \mathfrak{h})^g = H^r_n(\mathfrak{h}; \mathfrak{h}).
\]

Recall that every element of \( H^r_n(\mathfrak{h}; \mathfrak{h}) \) can in fact be represented by a \( \mathfrak{g} \)-invariant cycle at the chain level. Additionally, all elements of \( H^r_n(I; \mathfrak{h})^g \) can be be represented by \( \mathfrak{g} \)-invariant cycles, although in general \( H^r_n(I; \mathfrak{h})^g \) is not isomorphic to \( H^r_n(I; \mathfrak{h}) \). Let \( K_n \) be the kernel of the composition

\[
\pi_* \circ j_* : H^r_n(I; \mathfrak{h})^g \xrightarrow{j_*} H^r_n(\mathfrak{h}; \mathfrak{h}) \xrightarrow{\pi_*} H^r_{n+1}(\mathfrak{h}),
\]

\[
K_n = \text{Ker} [H^r_n(I; \mathfrak{h})^g \rightarrow H^r_{n+1}(\mathfrak{h})], \quad n \geq 0.
\]

**Theorem 2.2.** With \( I, \mathfrak{h} \) and \( \mathfrak{g} \) as in Lemma (2.1), we have

\[
HR^n(\mathfrak{h}) \simeq \delta^{\text{Lie}}[H^r_{n+3}(\mathfrak{g})] \oplus \sum_{i=0}^{n+1} K_{n+1-i} \otimes H^r_{i}(\mathfrak{g}), \quad n \geq 0.
\]

**Proof.** The proof follows from the long exact sequence relating \( HR^r(\mathfrak{h}) \), \( H^r_{n+1}(\mathfrak{h}; \mathfrak{h}) \) and \( H^r_{n+2}(\mathfrak{h}) \) together with a specific knowledge of the generators of the latter two homology groups gleaned from Lemma (2.1).

Note that \( H^r_n(I; \mathfrak{h})^g \) contains \( \Lambda^{n+1}(I)^g \) as a direct summand, induced by a \( \mathfrak{g} \)-equivariant chain map

\[
\zeta : \Lambda^{n+1}(I) \rightarrow \mathfrak{h} \otimes I^{\wedge n}
\]

\[
\zeta(a_0 \wedge a_1 \wedge \ldots \wedge a_n) = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i a_i \otimes a_0 \wedge a_1 \wedge \ldots \hat{a}_i \ldots \wedge a_n.
\]
where \( a_i \in I \). Then

\[
\zeta_* : H_{n+1}^{\text{Lie}}(I)^\theta = \Lambda^{n+1}(I)^\theta \to H_n^{\text{Lie}}(I; h)^\theta
\]

is an inclusion, since the composition

\[
\pi \circ \zeta : \Lambda^{n+1}(I) \to h \otimes I^{\wedge n} \to h^{\wedge(n+1)}
\]

is the identity on \( \Lambda^{n+1}(I) \). Let \( \bar{\Lambda}^*(I)^\theta = \sum_{k \geq 1} \Lambda^k(I)^\theta \). Thus, the morphism

\[
\pi_* : H_*^{\text{Lie}}(h; h) \to H_{*+1}^{\text{Lie}}(h)
\]

induces a surjection

\[
H_*^{\text{Lie}}(h; h) \to \bar{\Lambda}^*(I)^\theta \otimes H_*^{\text{Lie}}(g)
\]

with kernel

\[
\sum_{n \geq 0} \sum_{i=0}^{n+1} K_{n+1-i} \otimes H_i^{\text{Lie}}(g).
\]

There is also an inclusion \( i_* : H_n^{\text{Lie}}(g) \to H_n^{\text{Lie}}(h) \), and it follows that \( H_{n+3}^{\text{Lie}}(g) \) maps isomorphically to \( i_* \circ \delta^{\text{Lie}}[H_{n+3}^{\text{Lie}}(g)] \) in the commutative square

\[
\begin{array}{ccc}
H_{n+3}^{\text{Lie}}(g) & \xrightarrow{\delta^{\text{Lie}}} & H_{n}^{\text{Lie}}(g) \\
\downarrow i_* & & \downarrow i_* \\
H_{n+3}^{\text{Lie}}(h) & \xrightarrow{\delta^{\text{Lie}}} & H_{n}^{\text{Lie}}(h)
\end{array}
\]

Recall that \( H_*^{\text{Lie}}(g; g) = 0, * \geq 0 \), for \( g \) simple.

\[\square\]

### 3 Leibniz Homology

Returning to the general setting of any Lie algebra \( g \) over a ring \( k \), we recall that the Leibniz homology \( [5] \) of \( g \), written \( HL_*(g; k) \), is the homology of the chain complex \( T(g) : \)

\[
k \leftarrow^0 g \leftarrow^1 g \otimes g \leftarrow^2 \ldots \leftarrow^d g \otimes (n-1) \leftarrow \ldots,
\]
where
\[
d(g_1, g_2, \ldots, g_n) = \sum_{1 \leq i < j \leq n} (-1)^j (g_1, g_2, \ldots, g_{i-1}, [g_i; g_j], g_{i+1}, \ldots, g_n),
\]
and \((g_1, g_2, \ldots, g_n)\) denotes the element \(g_1 \otimes g_2 \otimes \cdots \otimes g_n \in g^\otimes n\).

The canonical projection \(\pi' : g^\otimes n \to g^\wedge n, n \geq 0\), is a map of chain complexes, \(T(g) \to \Lambda^*(g)\), and induces a \(k\)-linear map on homology
\[
HL_*(g; k) \to H^\text{Lie}_*(g; k).
\]

Letting
\[
(Ker \pi')_n[2] = \text{Ker}\ [g^\otimes(n+2) \to g^\wedge(n+2)], \quad n \geq 0,
\]
Pirashvili \[9\] defines the relative theory \(H^\text{rel}(g)\) as the homology of the complex
\[
C^\text{rel}_n(g) = (\text{Ker} \pi')_n[2],
\]
and studies the resulting long exact sequence relating Lie and Leibniz homology:
\[
\begin{align*}
\cdots & \longrightarrow H^\text{rel}_{n-2}(g) \longrightarrow HL_n(g) \longrightarrow H^\text{Lie}_n(g) \longrightarrow H^\text{rel}_{n-1}(g) \longrightarrow \\
\cdots & \longrightarrow H^\text{rel}_0(g) \longrightarrow HL_2(g) \longrightarrow H^\text{Lie}_2(g) \longrightarrow 0 \\
0 & \longrightarrow HL_1(g) \longrightarrow H^\text{Lie}_1(g) \longrightarrow 0 \\
0 & \longrightarrow HL_0(g) \longrightarrow H^\text{Lie}_0(g) \longrightarrow 0.
\end{align*}
\]

The projection \(\pi' : g^\otimes(n+1) \to g^\wedge(n+1)\) can be factored as the composition of projections
\[
g^\otimes(n+1) \longrightarrow g \otimes g^\wedge n \longrightarrow g^\wedge(n+1),
\]
which leads to a natural map between exact sequences
\[
\begin{align*}
H^\text{rel}_{n-1}(g) & \longrightarrow HL_{n+1}(g) \longrightarrow H^\text{Lie}_{n+1}(g) \stackrel{\delta}{\longrightarrow} H^\text{rel}_{n-2}(g) \\
HR_{n-1}(g) & \longrightarrow H^\text{Lie}_n(g; g) \longrightarrow H^\text{Lie}_{n+1}(g) \stackrel{\delta\text{Lie}}{\longrightarrow} HR_{n-2}(g)
\end{align*}
\]
A key technique in the calculation of Leibniz homology is the Pirashvili spectral sequence [9], which converges to the relative groups $H^\text{rel}_*$. Consider the filtration of

$$C^\text{rel}_n(g) = \text{Ker}(g^{\otimes(n+2)} \to g^{\wedge(n+2)}), \quad n \geq 0,$$

given by

$$F^k_m(g) = g^{\otimes k} \otimes \text{Ker}(g^{\otimes(m+2)} \to g^{\wedge(m+2)}), \quad m \geq 0, \quad k \geq 0.$$

Then $F^*_m$ is a subcomplex of $F^*_m$, and

$$E^0_{m,k} = F^k_m / F^k_{m-1} \approx g^k \otimes \text{Ker}(g \otimes g^{\wedge(m+1)} \to g^{\wedge(m+2)}) = g^k \otimes CR_m(g).$$

From [9], we have

$$E^2_{m,k} \approx H L_k(g) \otimes H R_m(g), \quad m \geq 0, \quad k \geq 0.$$

**Lemma 3.1.** Let $0 \to I \to h \to g \to 0$ be an Abelian extension of a simple real Lie algebra $g$. Then there is a natural injection

$$\epsilon_* : H^\text{Lie}_*(I; h)^g \to H L_{*+1}(h)$$

induced by a $g$-equivariant chain map

$$\epsilon_n : h \otimes \Lambda^n(I) \to h^{\otimes(n+1)}, \quad n \geq 0.$$

**Proof.** For $b \in h$, $a_i \in I$, $i = 1, 2, \ldots n$, define

$$\epsilon_n(b \otimes a_1 \wedge a_2 \wedge \ldots \wedge a_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) b \otimes a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \ldots \otimes a_{\sigma(n)}.$$

Since $[a_i, a_j] = 0$ for $a_i, a_j \in I$, it follows that

$$d_{\text{Lieb}} \circ \epsilon_n = \epsilon_{n-1} \circ d_{\text{Lie}}.$$

Also, $\epsilon_n$ is $g$-equivariant, since $g$ acts by derivations on both $h \otimes \Lambda^n(I)$ and $h^{\otimes(n+1)}$. Thus, there is an induced map

$$\epsilon_* : H^\text{Lie}_*(I; h)^g \to H L_{*+1}(h)^g = H L_{*+1}(h).$$

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The composition
\[ \pi' \circ \epsilon_n : \mathfrak{h} \otimes \Lambda^n(I) \to \mathfrak{h}^{\otimes (n+1)} \to \mathfrak{h} \otimes \mathfrak{h}^\wedge n \]
is the identity on \( \mathfrak{h} \otimes \Lambda^n(I) \). Since \( H^\text{Lie}_* (\mathfrak{h}; \mathfrak{h}) \) contains \( H^\text{Lie}_* (I; \mathfrak{h})^g \) as a direct summand via \( H^\text{Lie}_* (I; \mathfrak{h})^g \otimes H^\text{Lie}_0 (\mathfrak{g}) \) (see Lemma (2.1)), it follows that \( (\pi' \circ \epsilon)_* \) and \( \epsilon_* \) are injective.

**Lemma 3.2.** With \( I, \mathfrak{h}, \mathfrak{g} \) as in Lemma (3.1), there is a vector space splitting (that is a splitting of trivial \( \mathfrak{g} \)-modules)
\[ H^\text{Lie}_n (I; \mathfrak{h})^g \simeq [\Lambda^{n+1}(I)]^g \oplus K_n, \]
where
\[ K_n = \text{Ker}[H^\text{Lie}_n (I; \mathfrak{h})^g \to H^\text{Lie}_{n+1} (\mathfrak{h})]. \]

**Proof.** The proof begins with the \( \mathfrak{g} \)-equivariant chain map
\[ \zeta : \Lambda^{n+1}(I) \to \mathfrak{h} \otimes I^\wedge n \]
constructed in Theorem (2.2). Recall that \( K_n \) is defined as the kernel of the composition
\[ H^\text{Lie}_n (I; \mathfrak{h})^g \to H^\text{Lie}_n (\mathfrak{h}; \mathfrak{h}) \to H^\text{Lie}_{n+1} (\mathfrak{h}). \]
Note that \( H^\text{Lie}_n (\mathfrak{h}; \mathfrak{h}) \) contains \( H^\text{Lie}_n (I; \mathfrak{h})^g \) as a summand from Lemma (2.1). \( \square \)

To begin the calculation of the differentials in the Pirashvili spectral sequence converging to \( H^\text{rel}_* (\mathfrak{h}) \), first consider the spectral sequence converging to \( H^\text{rel}_* (\mathfrak{g}) \), where \( \mathfrak{g} \) is simple. We have \( H^\text{Lie}_n (\mathfrak{g}; \mathfrak{g}) = 0 \) for \( n \geq 0 \) from [2] and \( H\mathcal{L}_n (\mathfrak{g}) = 0 \) for \( n \geq 1 \) from [3]. It follows that \( \delta^\text{Lie} : H^\text{Lie}_{n+3} (\mathfrak{g}) \to H^\text{rel}_n (\mathfrak{g}) \) and \( \delta : H^\text{Lie}_{n+3} (\mathfrak{g}) \to H^\text{rel}_n (\mathfrak{g}), n \geq 0, \) are isomorphisms in the square
\[ \begin{array}{ccc}
H^\text{Lie}_{n+3} (\mathfrak{g}) & \xrightarrow{\delta} & H^\text{rel}_n (\mathfrak{g}) \\
\downarrow 1 & & \downarrow 1 \\
H^\text{Lie}_{n+3} (\mathfrak{g}) & \xrightarrow{\delta^\text{Lie}} & H^\text{rel}_n (\mathfrak{g})
\end{array} \]

**Lemma 3.3.** In the Pirashvili spectral sequence converging to \( H^\text{rel}_* (\mathfrak{g}) \) for a simple real Lie algebra \( \mathfrak{g} \), all higher differentials
\[ d^r : E^r_{m,k} \to E^r_{m-r,k+r-1}, \quad r \geq 2, \]
are zero.
Proof. Since \( E^2_{m,k} \simeq \text{HL}_k(\mathfrak{g}) \otimes \text{HR}_m(\mathfrak{g}) \), and \( \text{HL}_k(\mathfrak{g}) = 0 \) for \( k \geq 1 \), it is enough to consider

\[ E^2_{m,0} \simeq \mathbf{R} \otimes \text{HR}_m(\mathfrak{g}) \simeq \delta^\text{Lie}[\text{HL}^\text{Lie}_{m+3}(\mathfrak{g})], \]

for which \( d^r[E^r_{m,0}] = 0, \ r \geq 2. \)

By naturality of the Pirashvili spectral sequence,

\[ \mathcal{F}^*_m(\mathfrak{g}) \hookrightarrow \mathcal{F}^*_m(\mathfrak{h}). \]

Thus, in the spectral sequence converging to \( H^\text{rel}_*(\mathfrak{h}) \), we also have

\[ d^r[\delta^\text{Lie}[\text{HL}^\text{Lie}_{m+3}(\mathfrak{g})]] = 0, \ r \geq 2. \]

Due to the recursive nature of \( H^\text{rel}_*(\mathfrak{h}) \) with \( E^2_{m,k} \simeq \text{HL}_k(\mathfrak{h}) \otimes \text{HR}_m(\mathfrak{h}) \), we calculate \( \text{HL}_n(\mathfrak{h}) \) by induction on \( n \). We have immediately

\[ \text{HL}_0(\mathfrak{h}) \simeq \mathbf{R} \]
\[ \text{HL}_1(\mathfrak{h}) \simeq \text{HL}^\text{Lie}_1(\mathfrak{h}) \simeq I^\mathfrak{g} \simeq \text{HL}_0(I; \mathfrak{h})^\mathfrak{h} \]
\[ \text{HL}_2(\mathfrak{h}) \simeq \text{HL}^\text{Lie}_1(I; \mathfrak{h}) \simeq \text{HL}^\text{Lie}_1(I; \mathfrak{h})^\mathfrak{h}. \]

Elements in

\[ \text{HL}_1(\mathfrak{h}) \otimes \text{HR}_0(\mathfrak{h}) + \text{HL}_0(\mathfrak{h}) \otimes \text{HR}_1(\mathfrak{h}) \]

determine \( \text{HL}_1^\text{rel}(\mathfrak{h}) \), which then maps to \( \text{HL}_3(\mathfrak{h}) \), etc. We now construct the differentials in the Pirashvili spectral sequence.

**Lemma 3.4.** In the Pirashvili spectral sequence converging to \( H^\text{rel}_*(\mathfrak{h}) \), the differential

\[ d^r[E^0_{m,0}] = d^r[\text{HL}_0(\mathfrak{h}) \otimes \text{HR}_m(\mathfrak{h})] = d^r[\mathbf{R} \otimes \text{HR}_m(\mathfrak{h})] \]

is given by

\[ d^r[\delta^\text{Lie}(\text{HL}^\text{Lie}_{m+3}(\mathfrak{g}))] = 0, \ r \geq 2, \]
\[ d^r[K_{m+1}] = 0, \ r \geq 2, \]
\[ d^{m+3-p} : K_{m+1-p} \otimes \text{HL}^\text{Lie}_p(\mathfrak{h}) \to K_{m+1-p} \otimes \delta^\text{Lie}[\text{HL}^\text{Lie}_p(\mathfrak{g})], \text{ where} \]
\[ K_{m+1-p} \otimes \delta^\text{Lie}[\text{HL}^\text{Lie}_p(\mathfrak{g})] \subseteq \text{HL}_{m+2-p}(\mathfrak{h}) \otimes \text{HR}_{p-3}(\mathfrak{h}) \subseteq E^2_{p-3,m+2-p}, \]
\[ d^{m+3-p}(\omega_1 \otimes \omega_2) = \omega_1 \otimes \delta(\omega_2). \]
Proof. Since elements in $K_{m+1}$ are represented by cycles in $H^\text{Lie}_{m+1}(h; I)^g$ that map via $\epsilon_*$ to cycles in $HL_{m+2}(h)$, it follows $d^r[K_{m+1}] = 0$, $r \geq 2$.

Let $[\omega_2] \in H^\text{Lie}_p(g)$. Consider

$$\omega_2 = \sum_{i_1, i_2, \ldots, i_p} g_{i_1} \wedge g_{i_2} \wedge \ldots \wedge g_{i_p} \in g^\otimes p.$$ 

Now, using the homological algebra of the long exact sequence relating Leibniz and Lie-algebra homology,

$$\tilde{\omega}_2 = \sum_{i_1, i_2, \ldots, i_p} g_{i_1} \otimes g_{i_2} \otimes \ldots \otimes g_{i_p} \in g^\otimes p$$ 

is a chain with $\pi'(\tilde{\omega}_2) = \omega_2$ and $d(\tilde{\omega}_2) \neq 0$ in the Leibniz complex. Moreover,

$$d(\tilde{\omega}_2) \in \text{Ker}(g^\otimes (p-1) \to g^\wedge (p-1))$$ 

and $d(\tilde{\omega}_2)$ represents the class $\delta(\omega_2)$ in $H^\text{rel}_{p-3}(g) \simeq HR_{p-3}(g)$. Since $w_1 \in K_{m+1-p}$ is $g$-invariant, $[\omega_1, g] = 0$ $\forall g \in g$. Also, $\epsilon(\omega_1)$ is a $g$-invariant cycle representing $\omega_1$ in $T(g)$. Thus,

$$d^{m+3-p}(\omega_1 \otimes \omega_2) = \omega_1 \otimes d(\tilde{\omega}_2) = \omega_1 \otimes \delta(\omega_2).$$

By a similar argument to that in Lemma (3.4), we have that

$$d^r[\Lambda^*(I)^g \otimes \delta^\text{Lie}(H^{\text{Lie}}_{s+3}(g))] = 0, \quad r \geq 2,$$

where

$$\Lambda^k(I)^g \otimes \delta^\text{Lie}(H^{\text{Lie}}_{m+3}(g)) \subseteq HL_k(h) \otimes HR_m(h) \subseteq E^2_{m,k}.$$ 

Thus, the elements in $M = \Lambda^*(I)^g \otimes \delta^\text{Lie}(H^{\text{Lie}}_{s+3}(g))$ represent absolute cycles in $H^\text{rel}_s(h)$. We claim that

$$M \subseteq \text{Ker}(H^\text{rel}_s(h) \to HL_{s+2}(h)),$$

which follows from:

**Lemma 3.5.** The boundary map

$$\delta : H^\text{Lie}_{s+3}(h) \to H^\text{rel}_s(h)$$

satisfies

$$\delta[\Lambda^k(I)^g \otimes H^{\text{Lie}}_{m+3}(g)] = \Lambda^k(I)^g \otimes \delta^\text{Lie}[H^{\text{Lie}}_{m+3}(g)],$$

for $k \geq 0$, $m \geq 0$. 

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Proof. The proof follows from the long exact sequence relating Lie and Leibniz homology by choosing representatives for the Lie classes \( \Lambda^k(I)^g \otimes H_{m+3}^{\text{Lie}}(g) \) at the chain level. Also, note that for \( g \) simple, \( H^1_{\text{Lie}}(g) = 0 \) and \( H^2_{\text{Lie}}(g) = 0 \).

Now let

\[
\omega_1 \otimes (\omega_2 \otimes \omega_3) \in \Lambda^k(I)^g \otimes (K_{m+1-p} \otimes H_p^{\text{Lie}}(g)) \subseteq E^2_{m,k}.
\]

Then, using the \( g \)-invariance of elements in \( \Lambda^* (I)^g \) and \( K_* \), as well as the \( h \)-invariance of \( \Lambda^* (I)^g \), we have

\[
d^{m+3-p} (\omega_1 \otimes \omega_2 \otimes \omega_3) = (\omega_1 \otimes \omega_2) \otimes \delta(\omega_3) \in
(\Lambda^k(I)^g \otimes K_{m+1-p}) \otimes \delta^{\text{Lie}}[H_p^{\text{Lie}}(g)]
\subseteq HL_{k+m+2-p}(h) \otimes HR_{p-3}(g) \subseteq E^2_{p-3,k+m+2-p}.
\]

For the remainder of this section, we suppose:

**Hypothesis A:** Every element of \( K_n \) has a \( h \)-invariant representative in \( HL_{n+1}(h) \) at the chain level.

Since \( I \) acts trivially on \( H^1_{\text{Lie}}(I; h) \), we have \( H^1_{\text{Lie}}(I; h)^g = H^1_{\text{Lie}}(I; h)^h \), and Hypothesis A is reasonable. The end of this section offers a canonical construction of \( h \)-invariants.

**Theorem 3.6.** Let \( 0 \to I \to h \to g \to 0 \) be an Abelian extension of a simple real Lie algebra \( g \). Then, under Hypothesis A, we have

\[
HL_* (h) \simeq \Lambda^* (I)^g \otimes T(K_*),
\]

where \( T(K_*) = \sum_{n \geq 0} K_*^{\otimes n} \) denotes the tensor algebra, and

\[
K_n = \text{Ker}[H^1_{\text{Lie}}(I; h)^g \to H^1_{\text{Lie}}(g)].
\]

**Proof.** It follows by induction on \( \ell \) that for certain \( r \)

\[
d^r [K_*^{\otimes \ell} \otimes H^1_{\text{Lie}}(g)] \to K_*^{\otimes \ell} \otimes \delta^{\text{Lie}}[H^1_{\text{Lie}}(g)]
\]

is given by

\[
d^r (\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_\ell \otimes v) = \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_\ell \otimes \delta(v),
\]

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where $\omega_i \in K_*$ and $v \in H_*^{\text{Lie}}(g)$. By a similar induction argument, we have

$$d''[\Lambda^*(I^g) \otimes K_*^{\otimes \ell} \otimes H_*^{\text{Lie}}(g)] \to \Lambda^*(I^g) \otimes K_*^{\otimes \ell} \otimes \delta^{\text{Lie}}[H_*^{\text{Lie}}(g)]$$

is given by

$$d''(u \otimes \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_\ell \otimes v) = u \otimes \omega_1 \otimes \ldots \otimes \omega_\ell \otimes \delta(v),$$

where $u \in \Lambda^*(I^g)$, $\omega_i \in K_*$ and $v \in H_*^{\text{Lie}}(g)$. The only absolute cycles in the Pirashvili spectral sequence are elements of

$$\Lambda^*(I^g) \otimes K_*^{\otimes \ell},$$

which are not in $\text{Im}\, \delta : H_{*+3}^{\text{Lie}}(h) \to H_*^{\text{rel}}(h)$. By induction on $\ell$,

$$H_{\ell}^{\text{rel}}(h) \simeq \Lambda^*(I^g) \otimes T(K_*).$$

We now study elements in $K_n$ and outline certain canonical constructions to produce $h$-invariants. Recall that $H_*^{\text{Lie}}(I; h)^g$ is the homology of

$$h^g \leftarrow (h \otimes I)^g \leftarrow (h \otimes I^2)^g \leftarrow \ldots \leftarrow (h \otimes I^n)^g \leftarrow (h \otimes I^{n+1})^g.$$

Note that as $g$-modules, $h \simeq g \oplus I$ and

$$h \otimes I^{\otimes n} \simeq (g \otimes I^{\otimes n}) \oplus (I \otimes I^{\otimes n}).$$

Thus, $(h \otimes I^{\otimes n})^g \simeq (g \otimes I^{\otimes n})^g \oplus (I \otimes I^{\otimes n})^g$. Any element in $K_n$ having a representative in $(I \otimes I^{\otimes n})^g$ is necessarily an $h$-invariant at the chain level, since $I$ acts trivially on $(I \otimes I^{\otimes n})^g$. Of course, all elements of $(I \otimes I^{\otimes n})^g$ are cycles.

**Lemma 3.7.** Any non-zero element in $K_n$ having a representative in

$$(g \otimes I^{\otimes n})^g$$

is determined by an injective map of $g$-modules, $\alpha : g \to I^{\otimes n}$, where $g$ acts on itself via the adjoint action.
Proof. Let $B : \mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}^* = \text{Hom}_\mathbb{R}(\mathfrak{g}, \mathbb{R})$ be the isomorphism from a simple Lie algebra to its dual induced by the Killing form. Then the composition
\[
\begin{array}{c}
g \otimes I^{\wedge n} \xrightarrow{B \otimes 1} \mathfrak{g}^* \otimes I^{\wedge n} \xrightarrow{\simeq} \text{Hom}_\mathbb{R}(\mathfrak{g}, I^{\wedge n})
\end{array}
\]
is $\mathfrak{g}$-equivariant, and induces an isomorphism
\[
(g \otimes I^{\wedge n})^\mathfrak{g} \xrightarrow{\simeq} \text{Hom}_\mathfrak{g}(\mathfrak{g}, I^{\wedge n}).
\]
Since $\mathfrak{g}$ is simple, a non-zero map of $\mathfrak{g}$-modules $\alpha : \mathfrak{g} \to I^{\wedge n}$ has no kernel, and $\mathfrak{g} \simeq \text{Im}(\alpha)$. 

Consider the special case where $I \simeq \mathfrak{g}$ as $\mathfrak{g}$-modules, although $I$ remains an Abelian Lie algebra. Let $\alpha : \mathfrak{g} \to I$ be a $\mathfrak{g}$-module isomorphism and let $B^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ be the inverse of $B : \mathfrak{g} \to \mathfrak{g}^*$ in the proof of Lemma (3.7). For a vector space basis $\{b_i\}_{i=1}^n$ of $\mathfrak{g}$, let $\{b_i^*\}_{i=1}^n$ denote the dual basis.

Lemma 3.8. With $\alpha : \mathfrak{g} \simeq I$ as above, the balanced tensor
\[
\omega = \sum_{i=1}^n B^{-1}(b_i^*) \otimes \alpha(b_i) + \alpha(B^{-1}(b_i^*)) \otimes b_i \in \mathfrak{h} \otimes \mathfrak{h}
\]
is $\mathfrak{h}$-invariant.

Proof. By construction,
\[
\sum_{i=1}^n B^{-1}(b_i^*) \otimes \alpha(b_i) \in g \otimes I \hookrightarrow \mathfrak{h} \otimes \mathfrak{h}
\]
is $\mathfrak{g}$-invariant. Since $\alpha$ is an isomorphism of $\mathfrak{g}$-modules, it follows that
\[
\sum_{i=1}^n \alpha(B^{-1}(b_i^*)) \otimes b_i \in I \otimes \mathfrak{g} \hookrightarrow \mathfrak{h} \otimes \mathfrak{h}
\]
is also a $\mathfrak{g}$-invariant. Now, let $a \in I$. There is some $g_0 \in \mathfrak{g}$ with $\alpha(g_0) = a$. 

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Thus,
\[
[\omega, a] = [\omega, \alpha(g_0)] \\
= \sum_{i=1}^{n} [B^{-1}(b_i^*) \otimes \alpha(b_i), \alpha(g_0)] + [\alpha(B^{-1}(b_i^*)) \otimes b_i, \alpha(g_0)] \\
= \sum_{i=1}^{n} \alpha([B^{-1}(b_i^*), g_0]) \otimes \alpha(b_i) + \alpha(B^{-1}(b_i^*)) \otimes \alpha([b_i, g_0]) \\
= \sum_{i=1}^{n} (\alpha \otimes \alpha) ([B^{-1}(b_i^*) \otimes b_i, g_0]) = 0.
\]

Since \(\alpha : g \to I\) is an isomorphism of \(g\)-modules, it follows that if
\[
\sum_{i=1}^{n} B(b_i^*) \otimes \alpha(b_i) \in g \otimes I
\]
is a \(g\)-invariant, then \(\sum_{i=1}^{n} B(b_i^*) \otimes b_i \in g \otimes g\) is also a \(g\)-invariant. \(\square\)

4 Applications

We compute the Leibniz homology for extensions of the classical Lie algebras \(\mathfrak{sl}_n(\mathbb{R})\), \(\mathfrak{so}_n(\mathbb{R})\), and \(\mathfrak{sp}_n(\mathbb{R})\). Additionally, \(HL_*\) is calculated for the Lie algebra of the Poincaré group \(\mathbb{R}^4 \rtimes SL_2(\mathbb{C})\) and the Lie algebra of the affine Lorentz group \(\mathbb{R}^4 \rtimes SO(3,1)\). To describe a common setting for these examples, let \(g\) be a (semi-)simple real Lie algebra, and consider \(g \subseteq \mathfrak{gl}_n(\mathbb{R})\). Then \(g\) acts on \(I = \mathbb{R}^n\) via matrix multiplication on vectors in \(\mathbb{R}^n\), which is often called the standard representation. Consider
\[
\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}
\]
as a vector space basis for \(\mathbb{R}^n\). Then the elementary matrix with 1 in row \(i\), column \(j\), and 0s everywhere else becomes \(x_i \frac{\partial}{\partial x^j}\). In the sequel, \(h\) denotes the real Lie algebra formed via the extension
\[
0 \longrightarrow I \longrightarrow h \longrightarrow g \longrightarrow 0.
\]
Also, the element
\[
\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \in I^\wedge n
\]
is the volume form, and often occurs as a \(g\)-invariant.
Corollary 4.1. Let \( g \) be a simple Lie algebra and \( I \) an Abelian Lie algebra, both over \( \mathbb{R} \). If \( h \simeq g \oplus I \) as Lie algebras, i.e., \( h \) is reductive, then
\[
HL_*(h) \simeq \Lambda^*(I) \otimes T(K_*),
\]
\[
K_* = \text{Ker}(I \otimes \Lambda^*(I) \to \Lambda^{*+1}(I))
\]

Proof. Since \( g \) acts trivially on \( I \), it follow that
\[
[\Lambda^*(I)]^g = \Lambda^*(I),
\]
\[
H^\text{Lie}_*(h) \simeq H^\text{Lie}_*(g) \otimes \Lambda^*(I),
\]
\[
H^\text{Lie}_*(I; h)^g \simeq I \otimes \Lambda^*(I).
\]
Thus, \( K_n = \text{Ker}(I \otimes \Lambda^n(I) \to \Lambda^{n+1}(I)) \).

By way of comparison, from [6], under the hypotheses of Corollary (4.1), we have
\[
HL_*(h) \simeq HL_*(g) \simeq HL_*(I) \simeq T(I).
\]
Thus, as vector spaces, \( \Lambda^*(I) \otimes T(K_*) \simeq T(I) \). For tensors of degree two, the above isomorphism becomes
\[
\Lambda^2(I) \oplus S^2(I) \simeq I^\otimes 2,
\]
where \( S^2(I) \) denotes the second symmetric power of \( I \).

Corollary 4.2. For \( g = \mathfrak{sl}_n(\mathbb{R}) \) and \( I = \mathbb{R}^n \) the standard representation of \( \mathfrak{sl}_n(\mathbb{R}) \), we have
\[
HL_*(h) \simeq [\Lambda^*(I)]^{\mathfrak{sl}_n(\mathbb{R})} = \langle \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \rangle.
\]

Proof. In this case there are no non-trivial \( \mathfrak{sl}_n(\mathbb{R}) \)-module maps from \( \mathfrak{sl}_n(\mathbb{R}) \) to \( I^\otimes k \). Using Lemma (3.7), we have
\[
H^\text{Lie}_*(I; h)^{\mathfrak{sl}_n(\mathbb{R})} \simeq H^\text{Lie}_*(I; I)^{\mathfrak{sl}_n(\mathbb{R})} \simeq [I \otimes \Lambda^*(I)]^{\mathfrak{sl}_n(\mathbb{R})}
\]
\[
= \langle \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \rangle.
\]
Also,
\[
H^\text{Lie}_*(h) \simeq H^\text{Lie}_*(\mathfrak{sl}_n(\mathbb{R})) \otimes [\Lambda^*(I)]^{\mathfrak{sl}_n(\mathbb{R})}
\]
\[
\simeq H^\text{Lie}_*(\mathfrak{sl}_n(\mathbb{R})) \otimes \langle \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \rangle.
\]
Thus, \( T(K_*) = \sum_{m \geq 0} K_*^{\otimes m} = \mathbb{R} \), and the corollary follows from Theorem (3.6). \qed
Corollary 4.3. Consider $g = \mathfrak{sl}_2(\mathbb{C})$ as a real Lie algebra with real vector space basis:

\[
\begin{align*}
v_1 &= x_1 \frac{\partial}{\partial x^1} + x_2 \frac{\partial}{\partial x^2} - x_3 \frac{\partial}{\partial x^3} - x_4 \frac{\partial}{\partial x^4}, \\
v_2 &= x_1 \frac{\partial}{\partial x^2} - x_2 \frac{\partial}{\partial x^1} - x_3 \frac{\partial}{\partial x^4} + x_4 \frac{\partial}{\partial x^3}, \\
v_3 &= x_1 \frac{\partial}{\partial x^3} + x_2 \frac{\partial}{\partial x^4}, \\
v_4 &= x_1 \frac{\partial}{\partial x^4} - x_2 \frac{\partial}{\partial x^3}, \\
v_5 &= x_3 \frac{\partial}{\partial x^1} + x_4 \frac{\partial}{\partial x^2}, \\
v_6 &= x_3 \frac{\partial}{\partial x^2} - x_4 \frac{\partial}{\partial x^1}.
\end{align*}
\]

For $I = \mathbb{R}^4$ the standard representation of $\mathfrak{sl}_2(\mathbb{C}) \subseteq \mathfrak{sl}_4(\mathbb{R})$, we have

\[HL_*(\mathfrak{h}) \simeq [\Lambda^* (I)]^{\mathfrak{sl}_2(\mathbb{C})}\]

Proof. Again, there are no non-trivial $\mathfrak{sl}_2(\mathbb{C})$-module maps from $\mathfrak{sl}_2(\mathbb{C})$ to $I^\wedge k$. Thus, $T(K_*) = \mathbb{R}$ in this case as well. The reader may check that $[\Lambda^2 (I)]^{\mathfrak{sl}_2(\mathbb{C})}$ has a real vector space basis given by the two elements:

\[
\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4}, \quad \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}.
\]

Furthermore, $[\Lambda^4 (I)]^{\mathfrak{sl}_2(\mathbb{C})}$ is a one-dimensional (real) vector space on the volume element

\[
\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}.
\]

For $k = 1, 3$, we have $[\Lambda^k (I)]^{\mathfrak{sl}_2(\mathbb{C})} = 0$. \( \square \)

Corollary 4.4. \[1\] Let $g = \mathfrak{so}_n(\mathbb{R})$, $n \geq 3$, and

\[
\alpha_{ij} = x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i} \in \mathfrak{so}_n(\mathbb{R}), \quad 1 \leq i < j \leq n.
\]

For $I = \mathbb{R}^n$ the standard representation of $\mathfrak{so}_n(\mathbb{R})$, we have

\[HL_*(\mathfrak{h}) \simeq [\Lambda^* (I)]^{\mathfrak{so}_n(\mathbb{R})} \otimes T(W),\]

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where $W$ is the one-dimensional vector space with $\mathfrak{h}$-invariant basis element

$$\omega = \sum_{\sigma \in \text{Sh}_{2, n-2}} \text{sgn}(\sigma) \alpha_{\sigma(1)} \alpha_{\sigma(2)} \otimes \epsilon \left( \frac{\partial}{\partial x^{\sigma(3)}} \wedge \frac{\partial}{\partial x^{\sigma(4)}} \wedge \ldots \wedge \frac{\partial}{\partial x^{\sigma(n)}} \right)$$

$$+ (-1)^{n+1} \sum_{\sigma \in \text{Sh}_{n-2, 2}} \text{sgn}(\sigma) \epsilon \left( \frac{\partial}{\partial x^{\sigma(1)}} \wedge \frac{\partial}{\partial x^{\sigma(2)}} \wedge \ldots \wedge \frac{\partial}{\partial x^{\sigma(n-2)}} \right) \otimes \alpha_{\sigma(n-1)} \alpha_{\sigma(n)},$$

and $\epsilon : I^{\wedge(n-2)} \to I^{\otimes(n-2)} \hookrightarrow \mathfrak{h}^{\otimes(n-2)}$ is the skew-symmetrization map. Above, $\text{Sh}_{p, q}$ denotes the set of $p, q$ shuffles in the symmetric group $S_{p+q}$.

**Proof.** There are two non-trivial $\mathfrak{so}_n(\mathbb{R})$-module maps $\mathfrak{so}_n(\mathbb{R}) \to I^{\wedge k}$ to consider

$$\rho_1 : \mathfrak{so}_n(\mathbb{R}) \to I^{\wedge 2}, \quad \rho_2 : \mathfrak{so}_n(\mathbb{R}) \to I^{\wedge(n-2)},$$

given by

$$\rho_1(\alpha_{ij}) = \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

$$\rho_2(\alpha_{ij}) = \text{sgn}(\tau) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^j} \ldots \wedge \frac{\partial}{\partial x^n},$$

where $\tau$ is the permutation sending

$$1, 2, \ldots, i, \ldots, j, \ldots, n \quad \text{to} \quad i, j, 1, 2, \ldots, n.$$

Now, $\sum_{i<j} \alpha_{ij} \otimes \epsilon \left( \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \right)$ is not a cycle in the Leibniz complex, while

$$\lambda = \sum_{\sigma \in \text{Sh}_{2, n-2}} \text{sgn}(\sigma) \alpha_{\sigma(1)} \alpha_{\sigma(2)} \otimes \epsilon \left( \frac{\partial}{\partial x^{\sigma(3)}} \wedge \frac{\partial}{\partial x^{\sigma(4)}} \wedge \ldots \wedge \frac{\partial}{\partial x^{\sigma(n)}} \right)$$

is a cycle in $\mathfrak{h}^{\otimes(n-1)}$, and $\omega$ above is a homologous $\mathfrak{h}$-invariant cycle. For $\sigma \in S_n$, let

$$a(\sigma) = \sum_{i=1}^{n-2} \alpha_{\sigma(i)} \alpha_{\sigma(n-1)} \otimes \alpha_{\sigma(i)} \alpha_{\sigma(n)},$$

$$\gamma = \sum_{\sigma \in \text{Sh}_{n-2, 2}} \text{sgn}(\sigma) \epsilon \left( \frac{\partial}{\partial x^{\sigma(1)}} \wedge \frac{\partial}{\partial x^{\sigma(2)}} \wedge \ldots \wedge \frac{\partial}{\partial x^{\sigma(n-2)}} \right) \otimes a(\sigma).$$
Then in the Leibniz complex,
\[ d(\gamma) = (n - 2)\beta, \]
\[ \beta = (-1)^{n+1} \sum_{\sigma \in \text{Sh}_{n-2,2}} \text{sgn}(\sigma) \epsilon \left( \frac{\partial}{\partial x^{\sigma(1)}} \wedge \frac{\partial}{\partial x^{\sigma(2)}} \wedge \ldots \wedge \frac{\partial}{\partial x^{\sigma(n-2)}} \right) \otimes \alpha_{\sigma(n-1)\sigma(n)}. \]

Thus, \( \omega \) and \( \lambda \) are homologous in \( HL_* \). From [1],
\[ [\Lambda^*(I)]^{so_4(R)} = \langle \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \rangle. \]

**Corollary 4.5.** Let \( g = so(3, 1) \) and
\[ \alpha_{ij} = x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i} \in so(3, 1), \quad 1 \leq i < j \leq 3, \]
\[ \beta_{ij} = x_i \frac{\partial}{\partial x^j} + x_j \frac{\partial}{\partial x^i} \in so(3, 1), \quad i = 1, 2, 3, j = 4. \]

For \( I = R^4 \) the standard representation of \( so(3, 1) \), we have
\[ HL_*(h) \simeq [\Lambda^*(I)]^{so(3, 1)} \otimes T(W), \]
where \( W \) is the one-dimensional vector space with \( h \)-invariant basis element
\[ \omega = \alpha_{12} \otimes \left( \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4} \right) + \alpha_{13} \otimes \left( \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4} \right) + \alpha_{23} \otimes \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} \right) \]
\[ + \beta_{14} \otimes \left( \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \right) - \beta_{24} \otimes \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} \right) + \beta_{34} \otimes \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \right) \]
\[ - \left( \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4} \right) \otimes \alpha_{12} + \left( \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4} \right) \otimes \alpha_{13} - \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} \right) \otimes \alpha_{23} \]
\[ - \left( \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \right) \otimes \beta_{14} + \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} \right) \otimes \beta_{24} - \left( \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \right) \otimes \beta_{34}. \]

**Proof.** The proof follows from identifying \( so(3, 1) \) module maps \( \rho : so(3, 1) \rightarrow I^k \) and constructing \( h \)-invariants via balanced tensors. Note that the Killing form to establish \( g \simeq g^* \) is different for \( so_4(R) \) and \( so(3, 1) \). \( \square \)

**Corollary 4.6.** Let \( g = sp_n(R) \) be the real symplectic Lie algebra with vector space basis given by the families:
\( x_k \frac{\partial}{\partial y}, \quad k = 1, 2, 3, \ldots, n, \)

\( y_k \frac{\partial}{\partial x}, \quad k = 1, 2, 3, \ldots, n, \)

\( x_i \frac{\partial}{\partial y} + x_j \frac{\partial}{\partial y}, \quad 1 \leq i < j \leq n, \)

\( y_i \frac{\partial}{\partial x} + y_j \frac{\partial}{\partial x}, \quad 1 \leq i < j \leq n, \)

\( y_j \frac{\partial}{\partial y} - x_i \frac{\partial}{\partial y}, \quad i = 1, 2, 3, \ldots, n, \quad j = 1, 2, 3, \ldots, n. \)

Let \( I = \mathbb{R}^{2n} \) have basis

\[
\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^n}.
\]

Then

\[
HL_*(\mathfrak{h}) \cong [\Lambda^*(I)]^{sp_n} = \Lambda^*(\omega_n),
\]

where \( \omega_n = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i}. \)

**Proof.** Since there are no non-trivial \( sp_n(\mathbb{R}) \)-module maps \( sp_n(\mathbb{R}) \to I^\wedge k \), we have \( T(K_*) = \mathbb{R} \). The algebra of symplectic invariants \( [\Lambda^*(I)]^{sp_n} \) is identified in another paper [7]. \( \square \)

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