A CONSTRUCTION OF LEVEL 1 IRREDUCIBLE MODULES FOR $U_q(\widehat{sp}_4)$ USING LEVEL 2 INTERTWINERS FOR $U_q(\widehat{sl}_2)$

BORIS FEIGIN, JIN HONG, AND TETSUJI MIWA

Abstract. We bosonize certain components of level $\ell$ $U_q(\widehat{sl}_2)$-intertwiners of ($\ell + 1$)-dimensions. For $\ell = 2$, these intertwiners, after certain modification by bosonic vertex operators, are added to the algebra $U_q(\widehat{sl}_2)$ at level 2 to construct all irreducible highest weight representations of level 1 for the quantum affine algebra $U_q(\widehat{sp}_4)$.

1. Introduction

The aim of this paper is to construct the level 1 irreducible highest weight representations of $U_q(\widehat{sp}_4)$. We start with the level 2 irreducible highest weight representations of $U_q(\widehat{sl}_2)$. We construct a representation space of $U_q(\widehat{sp}_4)$ as an infinite sum of representation spaces of $U_q(\widehat{sl}_2)$ tensored by bosonic Fock spaces.

We consider two sets of bosonic oscillators. The first one is a part of the Drinfeld generators of $U_q(\widehat{sl}_2)$ and the second one is added by hand to extend the representation space as mentioned above.

On the extended space thus constructed, we define actions of the Drinfeld generators of $U_q(\widehat{sp}_4)$. For this purpose we use the action of $U_q(\widehat{sl}_2)$ as a part. We also exploit the level 2 intertwiners of dimensions 3 in the sense of [4, 6]. There are two types of such intertwiners, type I and II. One of three components in each case can be written as a vertex operator in terms of bosonic oscillators in $U_q(\widehat{sl}_2)$. We modify these operators by multiplying simple vertex operators constructed from the second set of bosonic oscillators, and add the modified vertex operators to the action.

The construction for the affine Lie algebra case, i.e., $q = 1$, was mentioned in [3]. Our construction is a $q$ deformation of their construction. However, there are two technical differences. First, in the $q$-deformed situation, the type I and II intertwiners are not the same. In our construction we need both in a proper combination. Second, in the $q$-deformed situation, we construct only Drinfeld generators instead of constructing all the $\widehat{sp}_4$ currents as in the affine Lie algebra case. The cost is to prove Drinfeld $q$-Serre relations.

In [3], bosonizations of level 1 representations of $U_q(\widehat{sp}_{2n})$ were constructed. Our construction is different from theirs since we have constructed irreducible representations.

In Section 2, we prepare basic definitions for $U_q(\widehat{sl}_2)$ and $U_q(\widehat{sp}_4)$. In Section 3, we construct special components of the type I and II intertwiners for $U_q(\widehat{sl}_2)$ at level $\ell$, in general. In Section 4, we combine these to construct level 1 irreducible representations of $U_q(\widehat{sp}_4)$.

2. Quantum affine algebras and intertwiners

Basic notations used in this paper is given in this section. We shall deal with two quantum affine algebras, $U_q(\widehat{sl}_2)$ and $U_q(\widehat{sp}_4)$, in this paper. The quantum affine
algebras $U_q$ of type $A_1^{(1)}$ and $C_2^{(1)}$ are generated by the elements
\[ e_i, f_i, t_i^{\pm 1}, q^{\pm d}, \]
with $i = 0, 1$ for type $A_1^{(1)}$ and $i = 0, 1, 2$ for type $C_2^{(1)}$. We shall not write down their defining relations in terms of these generators. The subalgebras of $U_q$ generated by elements above, excluding the elements $q^{\pm d}$, will be denoted by $U'_q$. Hopf algebra structure for both of these algebras is given as follows.

\[ \Delta(e_i) = e_i \otimes t_i + 1 \otimes e_i, \]
\[ \Delta(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i, \]
\[ \Delta(t_i) = t_i \otimes t_i, \]
\[ \Delta(q^{d}) = q^{d} \otimes q^{d}. \]

The canonical central element for $\widehat{\mathfrak{sl}}_2$ is given by $c = h_0 + h_1$ and that of $\widehat{\mathfrak{osp}}_4$ is given by $c = h_0 + h_1 + h_2$.

### 2.1. Intertwiners for $U'_q(\widehat{\mathfrak{sl}}_2)$

We now state the Drinfel’d realization for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. The generators are given by

\[ x^{\pm}(k), a(l), K^{\pm 1}, \gamma^{\pm \frac{1}{2}}, q^{\pm d}. \]

Here, the indices run over $k \in \mathbb{Z}$ and $l \in \mathbb{Z}^\times$. In terms of the generating function

\[ X^{\pm}(z) = \sum_{k \in \mathbb{Z}} x^{\pm}(k) z^{-k-1}, \]

the defining relations are given by

\begin{align*}
(2.1) & \quad [\gamma^{\pm \frac{1}{2}}, u] = 0 \quad \text{for all } u \in U_q(\widehat{\mathfrak{sl}}_2), \\
(2.2) & \quad [a(k), a(l)] = \delta_{k+l,0} \frac{[2k]}{k} \gamma^k - \gamma^{-k}, \\
(2.3) & \quad K a(k) K^{-1} = a(k), \quad K X^{\pm}(z) K^{-1} = q^{\pm 2} X^{\pm}(z), \\
(2.4) & \quad [a(k), X^{\pm}(z)] = \pm \frac{[2k]}{k} \gamma^{\pm |k|/2} z^k X^{\pm}(z), \\
(2.5) & \quad (z - q^{\pm 2} w) X^{\pm}(z) X^{\pm}(w) + (w - q^{\pm 2} z) X^{\pm}(w) X^{\pm}(z) = 0, \\
& \quad [X^{+}(z), X^{-}(w)] \\
(2.6) & \quad = K \exp \left\{ (q - q^{-1}) \sum_{k=-\infty}^{\infty} a(k) \gamma^{k/2} z^{-k} \right\} \frac{\delta(z/w)}{(q - q^{-1}) z w} - K^{-1} \exp \left\{ -(q - q^{-1}) \sum_{k=-\infty}^{\infty} a(-k) \gamma^{-k/2} z^k \right\} \frac{\delta(z/w)}{(q - q^{-1}) z w}, \\
(2.7) & \quad q^d K q^{-d} = K, \quad q^d x^{\pm}(k) q^{-d} = q^k x^{\pm}(k), \quad q^d a(k) q^{-d} = q^k a(k). \]

Here, the notation $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$ is a formal infinite sum. Identification between the two presentations of $U_q(\widehat{\mathfrak{sl}}_2)$ is given in \( \square \).
The algebra $U_q'(\widehat{\mathfrak{sl}_2})$, the subalgebra of $U_q(\widehat{\mathfrak{sl}_2})$ without $q^d$, admits a finite dimensional representation.

\[ V_2^{(t)} = \bigoplus_{j=0}^{\ell} C(q)v_j^{(t)}. \]

The action of $U_q'(\widehat{\mathfrak{sl}_2})$ on $V_2^{(t)}$ is given by

\[
\begin{align*}
\ell v_j^{(t)} &= [j+1]v_{j+1}^{(t)}, \\
e_1 v_j^{(t)} &= [\ell-j+1]v_{j-1}^{(t)}, \\
t_1 v_j^{(t)} &= q^{\ell-2j}v_j^{(t)}, \\
f_0 v_j^{(t)} &= z^{-1}[\ell-j+1]v_{j-1}^{(t)}, \\
e_0 v_j^{(t)} &= z[j+1]v_{j+1}^{(t)}, \\
t_0 v_j^{(t)} &= q^{2j-\ell}v_j^{(t)}.
\end{align*}
\]

Given a dominant integral weight $\lambda$ of level $\ell$, i.e., satisfying $\lambda(h_0 + h_1) = \ell$, we let

\[
\Phi^{(t)}(z) : V_2^{(t)} \otimes V(\lambda) \rightarrow V(\sigma\lambda), \quad \text{(type-I)}
\]

\[
\Psi^{(t)}(z) : V(\lambda) \otimes V_2^{(t)} \rightarrow V(\sigma\lambda), \quad \text{(type-II)}
\]

denote the intertwiners. Here, the map $\sigma$ permutes the fundamental weights $\Lambda_0$ and $\Lambda_1$ of $U_q'(\widehat{\mathfrak{sl}_2})$. Note that these intertwiners are unique up to scalar multiple.

We define the components of the intertwiners by

\[
\Phi_j^{(t)}(z) |v\rangle = \Phi^{(t)}(z) v_j^{(t)} \otimes |v\rangle,
\]

\[
\Psi_j^{(t)}(z) |v\rangle = \Psi^{(t)}(z) |v\rangle \otimes v_j^{(t)}.
\]

2.2. Drinfeld realization for $U_q(\widehat{\mathfrak{sp}_4})$. Let $A = (a_{i,j})_{i,j \in I}$ be the Cartan matrix of type $C_2^{(1)}$. Here, the index set $I = \{0, 1, 2\}$. We set $q_1 = q$, $q_2 = q^2$ and let $[n]$ denote the $q$-integer which uses $q_i$ in place of $q$.

The Drinfeld generators for $U_q(\widehat{\mathfrak{sp}_4})$ will be denoted by

\[ x_{\pm}^{\pm}(k), a_i(l), K_i^{\pm 1}, \gamma^{\pm \frac{1}{2}}, q^{\pm d}. \]

Here, the indices run over $i \in I = \{1, 2\}$, $k \in \mathbb{Z}$, and $l \in \mathbb{Z}^\times$. In terms of the generating function

\[ X_{\pm}^{\pm}(z) = \sum_{k \in \mathbb{Z}} x_{\pm}^{\pm}(k) z^{-k-1}, \]

the defining relations are given by

\[
\begin{align*}
[\gamma^{\pm \frac{1}{2}}, u] &= 0 \quad \text{for all } u \in U_q(\widehat{\mathfrak{sp}_4}), \\
K_i K_j &= K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
[a_i(k), a_j(l)] &= \delta_{k+l,0} \frac{[a_{i,j,k}|k]}{k} \gamma^k - \gamma^{-k} q_j - q_j^{-1}, \\
K_j a_i(k) K_j^{-1} &= a_i(k), \quad K_j X_{\pm}^{\pm}(z) K_j^{-1} = q_{\pm a_i} X_{\pm}^{\pm}(z), \\
q^d K_i q^{-d} &= K_i, \quad q^d x_\pm^\pm(k) q^{-d} = q^d x_\pm^\pm(k), \quad q^d a_i(k) q^{-d} = q^d a_i(k), \\
[a_i(k), X_{\pm}^{\pm}(z)] &= \pm \frac{[a_{i,j,k}|k]}{k} \gamma^{|k|/2} z^k X_{\pm}^{\pm}(z), \\
(z - q_i^{\pm a_i-j} w) X_{\pm}^{\pm}(z) X_{\pm}^{\pm}(w) + (w - q_i^{\pm a_i-i} z) X_{\pm}^{\pm}(w) X_{\pm}^{\pm}(z) &= 0,
\end{align*}
\]
Here, we have used the formal notation $S$ from \[8\] and the right hand side denotes elements from this paper.

\[
\begin{align*}
[X^+_s(z), X^-_s(w)] &= \frac{\delta_{ij}}{q_i - q_j}, \\
(2.15) &\times \left( K_i \exp \left\{ \left( q_i - q_j^{-1} \right) \sum_{k=1}^{\infty} a_i(k) \gamma^{k/2} z^{-k} \right\} \frac{\delta(z/\gamma w)}{zw} \\
&- K_i^{-1} \exp \left\{ - \left( q_i - q_j^{-1} \right) \sum_{k=1}^{\infty} a_i(-k) \gamma^{k/2} z^k \right\} \frac{\delta(\gamma z/w)}{zw} \right),
\end{align*}
\]

\[
(3.5) Sym_{z_1, z_2, z_3} \left( \begin{array}{c} X^+_2(w)X^+_1(z_1)X^+_1(z_2)X^+_1(z_3) \\
-3[1]X^+_1(z_1)X^+_2(w)X^+_1(z_2)X^+_1(z_3) \\
+3[1]X^+_1(z_1)X^+_2(z_2)X^+_1(z_2)X^+_1(z_3) \\
- X^+_1(z_1)X^+_1(z_2)X^+_1(z_2)X^+_1(w) \end{array} \right) = 0,
\]

\[
(3.7) Sym_{z_1, z_2} \left( \begin{array}{c} X^+_2(w)X^+_1(z_1)X^+_1(z_2) \\
-2[2]X^+_2(z_1)X^+_1(w)X^+_1(z_2) \\
+X^+_1(z_1)X^+_2(z_2)X^+_1(w) \end{array} \right) = 0.
\]

We remark that identification of this realization with the Drinfeld realization given in \[8\] may be done by mapping

\[
q^{2} \leftrightarrow q,
\]

\[
a_{1,k} \leftrightarrow \frac{1}{[2]} a_1(k),
\]

\[
a_{2,k} \leftrightarrow a_2(k),
\]

and mapping all other elements trivially. Here, the left hand side denotes elements from \[8\] and the right hand side denotes elements from this paper.

3. Bosonization of Intertwiners

We start this section by first recalling some facts from \[9\]. Define endomorphisms $S_i$ ($i = 0, 1$) acting on integrable $U_q(\hat{sl}_2)$-modules by

\[
(3.1) \quad S_i = \exp_{q^{-1}}(q^{-1} e_i t_i^{-1}) \exp_{q^{-1}}(-f_i) \exp_{q^{-1}}(q e_i t_i) q^{h_i(h_i+1)/2}.
\]

Here, we have used the formal notation

\[
\exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{[n]_q!} x^n.
\]

We remark that $\exp_q(x) \exp_{q^{-1}}(-x) = 1$ so that $S_i$ is invertible.

**Proposition 3.1** \[9\]. The endomorphism $S_i$ satisfies the following relations.

\[
(3.2) \quad S_i e_i S_i^{-1} = -f_i t_i,
\]

\[
(3.3) \quad S_i f_i S_i^{-1} = -t_i^{-1} e_i,
\]

\[
(3.4) \quad S_i t_i S_i^{-1} = t_i^{-1},
\]

\[
(3.5) \quad S_i e_j S_i^{-1} = \frac{1}{[2]} (q^{-2} e_j e_i^2 - q^{-1}[2] e_i e_j + e_i^2 e_j),
\]

\[
(3.6) \quad S_i f_j S_i^{-1} = \frac{1}{[2]} (q^2 f_i^2 f_j - q[2] f_i f_j f_i + f_j f_i^2),
\]

\[
(3.7) \quad S_i t_j S_i^{-1} = t_j t_i^2.
\]

*Here, we only take $i \neq j$.\]
3.1. The operator $D^\frac{1}{2}$. We now set
\begin{equation}
D^\frac{1}{2} = S_0 t_0^{-1} \sigma.
\end{equation}
Recall that $\sigma$ permutes the fundamental weights $\Lambda_0$ and $\Lambda_1$. Here, the map
\begin{equation*}
\sigma : V(\lambda) \rightarrow V(\sigma \lambda)
\end{equation*}
sends $v_\lambda$ to $v_{\sigma \lambda}$ and satisfies $\sigma f_i = f_1 - i \sigma$. Hence the operator $D^\frac{1}{2}$ is a map from
$V(\lambda)$ to $V(\sigma \lambda)$.

**Proposition 3.2.** On the Drinfel’d generators, the operator $D^\frac{1}{2}$ has the following properties.
\begin{align}
D^\frac{1}{2} a(n) D^{-\frac{1}{2}} &= a(n), \\
D^\frac{1}{2} K D^{-\frac{1}{2}} &= \gamma^{-1} K, \\
D^\frac{1}{2} X^\pm(z) D^{-\frac{1}{2}} &= -z^\mp 1 X^\pm(z).
\end{align}

**Proof.** Equation (3.10) is immediate from (3.4).
Let us show the last one. The two special cases
\begin{align}
D^\frac{1}{2} x^+(0) D^{-\frac{1}{2}} &= -x^+(1), \\
D^\frac{1}{2} x^-(0) D^{-\frac{1}{2}} &= -x^-(1)
\end{align}
may be obtained from (3.2) and (3.3). From the defining relations for $U_q(\widehat{sl}_2)$, we may write
\begin{equation}
x^-(2) = -\gamma^{-\frac{1}{2}} [a(1), x^-(1)] = -\gamma^{-\frac{1}{2}} [e_1 e_0 - q^2 e_0 e_1, e_0 t_0^{-1}].
\end{equation}
This looks quite similar to the right hand side of (3.5). Indeed, if we compare this with the outcome of applying Proposition 3.1 to
$D^\frac{1}{2} X^-(1) D^{-\frac{1}{2}}$, we obtain
\begin{equation}
D^\frac{1}{2} x^-(1) D^{-\frac{1}{2}} = -x^-(2).
\end{equation}
Equations (3.12) and (3.14) may then be applied to
\begin{equation*}
[x^+(0), x^-(1)] = \gamma^{-\frac{1}{2}} Ka(1)
\end{equation*}
to show
\begin{equation}
D^\frac{1}{2} a(1) D^{-\frac{1}{2}} = a(1).
\end{equation}
With this, starting from (3.12) and (3.13), we may recursively show (3.11) true for all negative powers of $z$. Positive powers of (3.11) may similarly be shown after obtaining $D^\frac{1}{2} a(-1) D^{-\frac{1}{2}} = a(-1)$.

It remains to deal with the first equation. To this end, recall that one of the defining relations (2.6) for $U_q(\widehat{sl}_2)$ may be written as
\begin{equation}
[x^+(k), x^-(l)] = \frac{\gamma_{k-l} \psi_{k+l} - \gamma_{-k-l} \psi_{k+l}}{q - q^{-1}}.
\end{equation}
where
\begin{align*}
\sum_{k=0}^{\infty} \psi_k z^{-k} &= K \exp \left\{ (q - q^{-1}) \sum_{k=1}^{\infty} a(k) z^{-k} \right\}, \\
\sum_{k=0}^{\infty} \varphi_k z^k &= K^{-1} \exp \left\{ -(q - q^{-1}) \sum_{k=1}^{\infty} a(-k) z^k \right\}.
\end{align*}
Applying (3.11) to (3.16), we may show $D^{\pm}a(k)D^{-\frac{1}{2}} = a(k)$ for all nonnegative $k$. Since the $\gamma^{-1}$ term comes from the $K$ inside $\psi_k$, we may start with $D^{\pm}a(1)D^{-\frac{1}{2}} = a(1)$, which we already obtained, to recursively show

$$D^{\pm}a(k)D^{-\frac{1}{2}} = a(k)$$

for all positive $k$. Results for negative $k$ may be obtained similarly. □

3.2. Bosonization of $\Phi_0^{(\ell)}(z)$. We shall follow [3, 4] in realizing the highest component $\Phi_0^{(\ell)}$ of the type-I $U'_q(\widehat{sl}_2)$-intertwiner.

Since all components of the intertwiner act on a level $\ell$ module, we will freely use $\gamma = q^\ell$. Starting from the fact that $\Phi^{(\ell)}(z)$ commutes with the $U'_q(\widehat{sl}_2)$-action, we obtain the following identities.

(3.17) $$[\ell - j + 1] \Phi_{j-1}^{(\ell)}(z) t_1 = e_1 \Phi_j^{(\ell)}(z) - \Phi_j^{(\ell)}(z)e_1,$$

(3.18) $$z [j + 1] \Phi_{j+1}^{(\ell)}(z) t_0 = e_0 \Phi_j^{(\ell)}(z) - \Phi_j^{(\ell)}(z)e_0,$$

(3.19) $$[j + 1] \Phi_{j+1}^{(\ell)}(z) = f_1 \Phi_j^{(\ell)}(z) - q^{(2j-\ell)} \Phi_j^{(\ell)}(z)f_1,$$

(3.20) $$z^{-1}[\ell - j + 1] \Phi_{j-1}^{(\ell)}(z) = f_0 \Phi_j^{(\ell)}(z) - q^{(\ell-2j)} \Phi_j^{(\ell)}(z)f_0,$$

(3.21) $$t_1 \Phi_j^{(\ell)}(z)t_1^{-1} = q^{(\ell-j)} \Phi_j^{(\ell)}(z),$$

(3.22) $$t_0 \Phi_j^{(\ell)}(z)t_0^{-1} = q^{(2j-\ell)} \Phi_j^{(\ell)}(z).$$

From (3.17) and (3.20) we get

(3.23) $$[x^+(0), \Phi_0^{(\ell)}(z)] = 0,$$

(3.24) $$[x^+(-1), \Phi_0^{(\ell)}(z)] = 0.$$

We may combine the equations (3.18) and (3.19) by removing the left hand side. After changing everything into the Drinfeld notation we get

(3.25) $$z \Phi_0^{(\ell)}(z)x^-(0) - \gamma x^-(0) \Phi_0^{(\ell)}(z) = \left(\gamma \Phi_0^{(\ell)}(z)x^-(1) - x^-(1) \Phi_0^{(\ell)}(z)\right).$$

With equations (3.23), (3.24), and (3.25), we may proceed as in [3, pp. 73–74] to obtain the following proposition

**Proposition 3.3.** The component $\Phi_0^{(\ell)}(z)$ of the type-I intertwiner satisfies the following set of equations.

(3.26) $$K \Phi_0^{(\ell)}(z)K^{-1} = \gamma \Phi_0^{(\ell)}(z),$$

(3.27) $$[X^+(w), \Phi_0^{(\ell)}(z)] = 0,$$

(3.28) $$[a(\pm k), \Phi_0^{(\ell)}(z)] = \frac{1}{k} \gamma^{\frac{j}{2}} [k\ell] z^{\pm k} \Phi_0^{(\ell)}(z).$$

This proposition allows us to guess a realization for $\Phi_0^{(\ell)}(z)$.

**Theorem 3.4.** The component $\Phi_0^{(\ell)}(z)$ of the type-I intertwiner may be realized, up to scalar multiple, as follows.

$$\Phi_0^{(\ell)}(z) = \exp \left( \sum_{k=1}^{\infty} \frac{\gamma^{\frac{j}{2}}}{[2k]} a(-k) z^k \right) \exp \left( - \sum_{k=1}^{\infty} \frac{\gamma^{\frac{j}{2}}}{[2k]} a(k) z^{-k} \right) \times D^{\pm} z^h_1.$$

**Proof.** That this definition of $\Phi_0^{(\ell)}(z)$ satisfies the three equations of Proposition 3.3 may be checked through routine calculations.

To prove that this is the correct realization for $\Phi_0^{(\ell)}(z)$, first recall that the group of equations (3.17)–(3.22) is equivalent to the definition of the intertwiner $\Phi^{(\ell)}(z)$. **
So let us define all other components of the intertwiner by (3.19) and show that the defined components satisfy the equations (3.17)–(3.22).

Checking equations (3.21) and (3.22) is trivial. Let us consider (3.17). The $j = 0$ case is immediate from (3.27). So let us assume (3.17) true for $j = s$. Then, using $j = s$ and $j = s - 1$ cases of (3.19), we have

$$
[s + 1](e_1 \Phi^{(f)}_{s+1}(z) - \Phi^{(f)}_{s+1}(z)e_1)
= e_1 f_1 \Phi^{(f)}_{s+1}(z) - q^{2s-\ell}e_1 \Phi^{(f)}_{s+1}(z)f_1 - f_1 \Phi^{(f)}_{s+1}(z)e_1 + q^{2s-\ell} \Phi^{(f)}_{s+1}(z)f_1 e_1
= ([e_1, f_1] \Phi^{(f)}_{s+1}(z) - q^{2s-\ell} \Phi^{(f)}_{s+1}(z)[e_1, f_1])
+ [\ell - s + 1](f_1 \Phi^{(f)}_{s+1}(z)t_1 - q^{2s-\ell} \Phi^{(f)}_{s+1}(z)t_1 f_1)
= [\ell - 2s] \Phi^{(f)}_{s+1}(z)t_1 + [\ell - s + 1][s] \Phi^{(f)}_{s+1}(z)t_1
= [s + 1][\ell - s] \Phi^{(f)}_{s+1}t_1.
$$

This shows the induction step and (3.17) is true for all $j$.

In a very similar way, we may show (3.20) true for all $j$, if we can show (3.18) true for all $j$.

It only remains to show (3.18). We prove it by induction on $j$. The $j = 0$ case of (3.18) is equivalent to the validity of (3.25). And the induction step for (3.18) may be shown using (3.19) as in the previous two induction proofs. So this last step reduces to showing (3.25). Using (2.4) on the explicit formula for $\Phi^{(f)}_{s+1}(z)$, given in the statement of this theorem, we may calculate

$$
(1 - \gamma w/z) \exp \left( - \sum_{k=1}^{\infty} \frac{\gamma^k}{2k} a(k) z^{-k} \right) X^-(w)
= X^-(w) \exp \left( - \sum_{k=1}^{\infty} \frac{\gamma^k}{2k} a(k) z^{-k} \right),
$$

$$
(1 - \gamma z/w) X^-(w) \exp \left( \sum_{k=1}^{\infty} \frac{\gamma^k}{2k} a(-k) z^{-k} \right)
= \exp \left( \sum_{k=1}^{\infty} \frac{\gamma^k}{2k} a(-k) z^{-k} \right) X^-(w).
$$

With this, we can show

$$
(z - \gamma w) \Phi^{(f)}_{s+1}(z) X^-(w) = -(w - \gamma z) X^-(w) \Phi^{(f)}_{s+1}(z).
$$

It leads to

$$
z \left( \Phi^{(f)}_{s+1}(z) X^-(w) - \gamma X^-(w) \Phi^{(f)}_{s+1}(z) \right)
= w \left( \gamma \Phi^{(f)}_{s+1}(z) X^-(w) - X^-(w) \Phi^{(f)}_{s+1}(z) \right),
$$

which is a generalization of (3.25).

3.3. **Bosonization of $\Psi^{(f)}_{s+1}(z)$**. Realization for the lowest component $\Psi^{(f)}_{s+1}(z)$ of the type-II intertwiner will be given in this subsection. Since all the steps are as in the previous subsection, we shall be very brief.

(3.29) \[ [\ell - j + 1] \Psi^{(f)}_{j-1}(z) = e_1 \Psi^{(f)}_{j}(z) - q^{(\ell-2j)} \Psi^{(f)}_{j} e_1, \]

(3.30) \[ z [j + 1] \Psi^{(f)}_{j+1}(z) = e_0 \Psi^{(f)}_{j}(z) - q^{(2j-\ell)} \Psi^{(f)}_{j} e_0, \]

(3.31) \[ [j + 1] \Psi^{(f)}_{j+1}(z) t_1^{-1} = f_1 \Psi^{(f)}_{j}(z) - \Psi^{(f)}_{j}(z) f_1, \]
Proposition 3.5. The component \( \Psi_k^{(\ell)}(z) \) of the type-II intertwiner satisfies the following set of equations.

\[
\begin{align*}
(3.35) & \quad K \Psi_k^{(\ell)}(z) K^{-1} = \gamma^{-1} \Psi_k^{(\ell)}(z), \\
(3.36) & \quad [X^-(w), \Psi_k^{(\ell)}(z)] = 0, \\
(3.37) & \quad [a(\pm k), \Psi_k^{(\ell)}(z)] = -\frac{1}{k} \gamma^{-k} [k \ell] (q^2 z)^{k \ell} \Psi_k^{(\ell)}(z).
\end{align*}
\]

Theorem 3.6. The component \( \Psi_k^{(\ell)}(z) \) of the type-II intertwiner may be realized, up to scalar multiple, as follows.

\[
\Psi_k^{(\ell)}(z) = \exp \left( -\sum_{k=1}^{\infty} \frac{\gamma^{-k}}{2k} a(-k)(q^2 z)^k \right) \exp \left( \sum_{k=1}^{\infty} \frac{\gamma^{-k}}{2k} a(k)(q^2 z)^{-k} \right) \times D^{-\frac{\ell}{2}} (q^2 z)^{-\frac{\ell}{2}h_1}.
\]

3.4. Relations for vertex operators. With the explicit bosonizations obtained in the previous sections, we may write down relations for these vertex operators.

Lemma 3.7. We have the following commutation relation between vertex operators and the grading operator.

\[
\begin{align*}
q^d \Phi_0^{(2)}(z) q^{-d} &= q^{\frac{1}{2} \lambda(h_0) - 1} \Phi_0^{(2)}(q^{-1} z), \\
q^d \Psi_2^{(2)}(z) q^{-d} &= q^{-\frac{1}{2} \lambda(h_1)} \Psi_2^{(2)}(q^{-1} z),
\end{align*}
\]

as operators from \( V(\lambda) \) to \( V(\sigma \lambda) \).

Proof. We shall deal with just the first one. Second one may be done similarly.

Using the explicit realization of \( \Phi_0^{(2)}(z) \), we may reduce this proof to showing

\[q^d D^\frac{1}{2} z^\frac{k}{2} h_1 q^{-d} = q^{\frac{1}{2} \lambda(h_0) - 1} D^\frac{1}{2} (q^{-1} z)^{\frac{1}{2} h_1}.\]

To show this, we first follow the next set of equalities.

\[
\begin{align*}
q^d D^\frac{1}{2} z^\frac{k}{2} h_1 q^{-d} &= q^d S_0 t_0^{-1} \sigma z^\frac{k}{2} h_1 q^{-d} \\
&= q^d q^{-d} S_1 t_1^{-1} z^\frac{k}{2} h_1 \\
&= q^d \sigma q^{-d} D^\frac{1}{2} z^\frac{k}{2} h_1.
\end{align*}
\]

Now, we can show that, as operators on an irreducible highest weight module of highest weight \( \mu \),

\[q^d \sigma q^{-d} = q^{\frac{1}{2} \mu(h_1)} q^{-\frac{1}{2} h_1}.\]

It can be done by checking the action of both sides on a weight vector of weight \( \mu = (\alpha_0 + y \alpha_1) \). Recalling that \( D^\frac{1}{2} \) is a map from \( V(\lambda) \) to \( V(\sigma \lambda) \), we can continue as follows.

\[
\begin{align*}
q^d D^\frac{1}{2} z^\frac{k}{2} h_1 q^{-d} &= q^{\frac{1}{2} \sigma \lambda(h_1)} q^{-\frac{1}{2} h_1} D^\frac{1}{2} z^\frac{k}{2} h_1, \\
&= q^{\frac{1}{2} \lambda(h_0) - 1} D^\frac{1}{2} (q^{-1} z)^{\frac{1}{2} h_1} \\
&= q^{\frac{1}{2} \lambda(h_0) - 1} D^\frac{1}{2} (q^{-1} z)^{\frac{1}{2} h_1}.
\end{align*}
\]

The proof is complete. \( \square \)
Lemma 3.9. We have the following identities concerning normal orderings of level ℓ operators to the algebra \( U \), now on, we shall assume \( \ell = 2 \) case of this lemma.

For example, we have
\[
\begin{align*}
: a(k) a(l) : & = \begin{cases} 
a(k) a(l) & \text{if } k < 0, \\
(a(l) a(k) & \text{if } k > 0,
\end{cases} \\
: D^{\frac{1}{2}} h_1 : & = : h_1 D^{\frac{1}{2}} : = D^{\frac{1}{2}} h_1.
\end{align*}
\]

For example, we have
\[
\begin{align*}
z^{\frac{1}{2}} h_1 D^{\frac{1}{2}} & = z : z^{\frac{1}{2}} h_1 D^{\frac{1}{2}} :.
\end{align*}
\]

We also use the formal notation
\[
(x;p) = \prod_{k=0}^{\infty} (1 - xp^k).
\]

Lemma 3.8. We have the following identities concerning normal orderings.

\[
\begin{align*}
\Phi_{0}^{(\ell)}(z) \Phi_{0}^{(\ell)}(w) & = z^{\frac{1}{2}} (q^{2\ell} w/z; q^4)_{\infty} : \Phi_{0}^{(\ell)}(z) \Phi_{0}^{(\ell)}(w) :, \\
\Phi_{0}^{(\ell)}(z) \Psi_{\ell}^{(\ell)}(w) & = \left( \frac{1}{z} \right)^{\frac{1}{2}} (q^{4\ell} w/z; q^4)_{\infty} : \Phi_{0}^{(\ell)}(z) \Psi_{\ell}^{(\ell)}(w) :, \\
\Psi_{\ell}^{(\ell)}(z) \Phi_{0}^{(\ell)}(w) & = \left( \frac{1}{q^2 z} \right)^{\frac{1}{2}} (q^{4\ell} w/z; q^4)_{\infty} : \Psi_{\ell}^{(\ell)}(z) \Phi_{0}^{(\ell)}(w) :, \\
\Psi_{\ell}^{(\ell)}(z) \Psi_{\ell}^{(\ell)}(w) & = (q^{2\ell} z^{\frac{1}{2}} (q^{2\ell} w/z; q^4)_{\infty} : \Psi_{\ell}^{(\ell)}(z) \Psi_{\ell}^{(\ell)}(w) :.
\end{align*}
\]

Proof. This may be easily verified by applying the following standard formula.
\[
\exp(A) \exp(B) = \exp([A, B]) \exp(B) \exp(A), \quad \text{when } [A, B] \text{ is a scalar.}
\]

In simplifying the outcome, the identity
\[
\sum_{k=1}^{\infty} \frac{[k]}{k} z^k = \log \frac{(q^{2\ell} z; q^4)_{\infty}}{(q^{2\ell} z; q^4)_{\infty}}
\]
will be useful.

\[\blacksquare\]

For later use, we write down the \( \ell = 2 \) case of this lemma.

Lemma 3.9. We have the following identities concerning normal orderings of level 2 vertex operators.

\[
\begin{align*}
\Phi_{0}^{(2)}(z) \Phi_{0}^{(2)}(w) & = z (1 - q^2 w/z) : \Phi_{0}^{(2)}(z) \Phi_{0}^{(2)}(w) :, \\
\Phi_{0}^{(2)}(z) \Psi_{2}^{(2)}(w) & = \frac{1}{z (1 - q^2 w/z)} : \Phi_{0}^{(2)}(z) \Psi_{2}^{(2)}(w) :, \\
\Psi_{2}^{(2)}(z) \Phi_{0}^{(2)}(w) & = \frac{1}{q^2 z (1 - w/q^2 z)} : \Psi_{2}^{(2)}(z) \Phi_{0}^{(2)}(w) :, \\
\Psi_{2}^{(2)}(z) \Psi_{2}^{(2)}(w) & = q^2 z (1 - w/q^2 z) : \Psi_{2}^{(2)}(z) \Psi_{2}^{(2)}(w) :.
\end{align*}
\]

4. Extended actions of vertex operators (\( \ell = 2 \))

In this section, we define an algebra \( U' \). This is done by adding two vertex operators to the algebra \( U'_q(\hat{\mathfrak{s}l}_2) \) after tensoring by some correction terms. From now on, we shall assume \( \ell = 2 \). This is equivalent to saying \( \gamma = q^2 \) in \( U_q(\hat{\mathfrak{s}l}_2) \).
4.1. **Heisenberg algebra.** Let us define a Heisenberg algebra $\mathcal{B}'$. It is to be generated by the element $b(k)$ with $k \in \mathbb{Z}$. Defining relations are given below.

\begin{align}
[b(k), b(l)] &= \delta_{k+l,0} \frac{1}{k}(q^{2k} - 1 + q^{-2k}) \quad (k \neq 0), \\
[b(0), b(k)] &= 0.
\end{align}

We may consider the representation $\mathcal{V}_p$, over $\mathcal{B}'$, which contains the vacuum vector $v(p)$ satisfying

\begin{align*}
b(k)v(p) &= 0 \quad \text{for } k > 0, \\
b(0)v(p) &= pv(p).
\end{align*}

The operator $T^r$ acts from $\mathcal{V}_p$ to $\mathcal{V}_{p+r}$. It commutes with all $b(k)$ ($k \neq 0$) and sends $v(p) \mapsto v(p + r)$. We may trivially check

\begin{equation}
[b(0), T^r] = r.
\end{equation}

We define two operators acting on $\bigoplus_{p \in \mathbb{Z}} \mathcal{V}_p$.

\begin{align*}
\Omega_0(z) &= \exp \left( \sum_{k=1}^{\infty} b(-k) q^k z^k \right) \exp \left( - \sum_{k=1}^{\infty} b(k) q^k z^{-k} \right) T^1 z^{b(0)}, \\
\Omega_2(z) &= \exp \left( - \sum_{k=1}^{\infty} b(-k) q^{-k} z^k \right) \exp \left( \sum_{k=1}^{\infty} b(k) q^{-k} z^{-k} \right) T^{-1} z^{-b(0)}.
\end{align*}

**Normal ordering** in the algebra $\mathcal{B}'$ is defined by

\begin{align*}
: b(k)b(l) : &= \begin{cases} 
 b(k)b(l) & \text{if } k < 0, \\
 b(l)b(k) & \text{if } k > 0,
\end{cases} \\
: T^r b(0) : &= : b(0) T^r : = T^r b(0).
\end{align*}

**Lemma 4.1.** Normal ordering of products of the operators $\Omega_0$ and $\Omega_2$ are as follows.

\begin{align}
\Omega_0(z)\Omega_0(w) &= z \frac{(1 - w/z)(1 - q^4 w/z)}{1 - q^2 w/z} : \Omega_0(z)\Omega_0(w) :, \\
\Omega_0(z)\Omega_2(w) &= \frac{1}{z} \frac{1 - w/z}{1 - w/q^2 z} : \Omega_0(z)\Omega_2(w) :, \\
\Omega_2(z)\Omega_0(w) &= \frac{1}{z} \frac{1 - w/z}{z(1 - w/z)(1 - q^2 w/z)} : \Omega_2(z)\Omega_0(w) :, \\
\Omega_2(z)\Omega_2(w) &= z \frac{(1 - w/z)(1 - w/q^4 z)}{1 - w/q^2 z} : \Omega_2(z)\Omega_2(w) :.
\end{align}

4.2. **The algebra $U'$.** We finally set

\begin{align}
Y^+(z) &= \Psi^{(2)}_2(q^{-2} z) \otimes \Omega_2(z), \\
Y^-(z) &= \Phi^{(2)}_0(z) \otimes \Omega_0(z).
\end{align}

We write

\begin{equation}
Y^\pm(z) = \sum_{k \in \mathbb{Z}} y^\pm(k) z^{-k-1}.
\end{equation}
They are operators acting on the following spaces, or on their direct sums.
\[
\mathcal{V}(0) = \left(V(2\Lambda_0) \otimes \bigoplus_{p \in \mathbb{Z}} \mathcal{V}_p\right) \oplus \left(V(2\Lambda_1) \otimes \bigoplus_{p \in \mathbb{Z}} \mathcal{V}_{p+1}\right)
\]
\[
\mathcal{V}(1) = \left(V(\Lambda_0 + \Lambda_1) \otimes \bigoplus_{p \in \mathbb{Z}} \mathcal{V}_{p+1}\right)
\]
\[
\mathcal{V}(2) = \left(V(2\Lambda_0) \otimes \bigoplus_{p \in \mathbb{Z}} \mathcal{V}_{p+1}\right) \oplus \left(V(2\Lambda_1) \otimes \bigoplus_{p \in \mathbb{Z}} \mathcal{V}_p\right)
\]
Note that each \(\mathcal{V} = \mathcal{V}(j)\) \((j = 0, 1, 2)\) may be seen both as a \(U'_q(\widehat{\mathfrak{sl}_2})\)-module and as a \(B'\)-module.

**Definition 4.2.** The algebra \(U' = U'(\widehat{\mathfrak{sl}_2})\) is defined to be the subalgebra of \(\text{End}(\mathcal{V}(j))\) generated by all elements of the quantum affine algebra \(U'_q(\widehat{\mathfrak{sl}_2})\), all elements of the Heisenberg algebra \(B'\), and all coefficients of the modified vertex operators \(Y^{\pm}(z)\).

**Lemma 4.3.** For each \(p \in \mathbb{Z}\), the following equations are true up to nonzero scalar multiple.

\[
v_{2\Lambda_1} \otimes v(p+1) = y^-(p+1) (v_{2\Lambda_0} \otimes v(p))
\]
\[
v_{\Lambda_0+\Lambda_1} \otimes v(p+\frac{1}{2}) = x^- (p) y^-(p+\frac{1}{2}) (v_{\Lambda_0+\Lambda_1} \otimes v(p))
\]
\[
v_{2\Lambda_0} \otimes v(p+1) = (x^-(p+1))^2 y^-(p+2) (v_{2\Lambda_1} \otimes v(p))
\]
\[
v_{2\Lambda_1} \otimes v(p-1) = (x^-(p))^2 y^+(p) (v_{2\Lambda_0} \otimes v(p))
\]
\[
v_{\Lambda_0+\Lambda_1} \otimes v(p+\frac{1}{2}) = x^- (p) y^+(p) (v_{\Lambda_0+\Lambda_1} \otimes v(p+\frac{1}{2}))
\]
\[
v_{2\Lambda_0} \otimes v(p-1) = y^+(p) (v_{2\Lambda_1} \otimes v(p)).
\]

**Proof.** We shall prove just the second one. Other cases are similar.

We may calculate
\[
y^-(p+1) (v_{\Lambda_0+\Lambda_1} \otimes v(p-\frac{1}{2})) = Dz^2 v_{\Lambda_0+\Lambda_1} \otimes v(p+\frac{1}{2}).
\]
We claim that
\[
Dz^2 v_{\Lambda_0+\Lambda_1} \in V(\Lambda_0 + \Lambda_1)_{\Lambda_0+\Lambda_1-\alpha_0}.
\]
Note that the space on the right is of dimension 1, so that, if our claim is correct, \(Dz^2 v_{\Lambda_0+\Lambda_1}\) is a nonzero scalar multiple of \(x^+(p) v_{\Lambda_0+\Lambda_1}\). Also note that
\[
x^+(p) x^-(-1) v_{\Lambda_0+\Lambda_1} = v_{\Lambda_0+\Lambda_1}.
\]
So our claim is equivalent to the statement given in this Lemma.

The claim is proved by computing the weight of \(Dz^2 v_{\Lambda_0+\Lambda_1}\), using equation (3.10) and Lemma 3.7.

Recall that \(V(\lambda)\) is irreducible under the action of \(U'_q(\widehat{\mathfrak{sl}_2})\) and that each \(V_p\) is also irreducible under the action of \(B'\). So each of the tensored spaces \(V(\lambda) \otimes \mathcal{V}_p\) is irreducible under the action of \(U'_q(\widehat{\mathfrak{sl}_2}) \cup B'\). Now, Lemma 4.3 shows that \(Y^{\pm}(z)\) links these irreducible tensored spaces. Thus we have obtained the following proposition.

**Proposition 4.4.** Each \(\mathcal{V}(j)\) \((j = 0, 1, 2)\) is irreducible under the action of \(U'\).

For later use, we state some calculation results that may be obtained from lemmas concerning normal ordering, appearing in previous sections.
We shall write

\[ a(k)x^\pm(l) : = a(k)x^\pm(l) \] if \( k < 0 \),
\[ x^\pm(l)a(k) : = x^\pm(l)a(k) \] if \( k > 0 \),
\[ D^\pm x^\pm(l) : = x^\pm(l)D^\pm, \]
\[ h_1 x^\pm(l) : = x^\pm(l)h_1 : = x^\pm(l)h_1. \]

(4.10) \[ X^+(z) Y^+(w) = \frac{1}{1 - w/q^2 z} : X^+(z) Y^+(w) : . \]
(4.11) \[ Y^+(z) X^+(w) = \frac{w}{z} \frac{1}{1 - w/q^2 z} : Y^+(z) X^+(w) : . \]
(4.12) \[ X^-(z) Y^-(w) = \frac{1}{1 - q^2 w/z} : X^-(z) Y^-(w) : . \]
(4.13) \[ Y^-(z) X^-(w) = \frac{w}{z} \frac{1}{1 - q^2 w/z} : Y^-(z) X^-(w) : . \]
(4.14) \[ Y^+(z) Y^+(w) = (z - q^4 w)(z - w) : Y^+(z) Y^+(w) : . \]
(4.15) \[ Y^-(z) Y^-(w) = (z - w)(z - q^4 w) : Y^-(z) Y^-(w) : . \]
(4.16) \[ Y^+(z) Y^+(w) = \frac{1}{z^2} \frac{1}{(1 - q^2 w/z)(1 - w/q^2 z)} : Y^+(z) Y^+(w) : . \]
(4.17) \[ Y^-(z) Y^+(w) = \frac{1}{z^2} \frac{1}{(1 - q^2 w/z)(1 - w/q^2 z)} : Y^-(z) Y^+(w) : . \]
(4.18) \[ [a(k), Y^\pm(z)] = \frac{[2k]}{k} q^{\pm|k|} z^k Y^\pm(z). \]
(4.19) \[ [b(k), Y^\pm(z)] = \frac{(q^{2k} - 1 + q^{-2k})}{|k|} q^{\pm|k|} z^k Y^\pm(z). \]

\[ X^\pm(w) Y^\pm(z_1) Y^\pm(z_2) \]
(4.20) \[ = \frac{1}{1 - q^{\mp 2 z_1/w}} \frac{1}{1 - q^{\mp 2 z_2/w}} \times (z_1 - q^{\pm 4 z_2})(z_1 - z_2) : X^\pm(w) Y^\pm(z_1) Y^\pm(z_2) : . \]
\[ Y^\pm(z_1) X^\pm(w) Y^\pm(z_2) \]
(4.21) \[ = - \frac{w}{z_1} \frac{1}{1 - q^{\mp 2 w/z_1}} \frac{1}{1 - q^{\mp 2 z_2/w}} \times (z_1 - q^{\pm 4 z_2})(z_1 - z_2) : Y^\pm(z_1) X^\pm(w) Y^\pm(z_2) : . \]
\[ Y^\pm(z_1) Y^\pm(z_2) X^\pm(w) \]
(4.22) \[ = \frac{w^2}{z_1 z_2} \frac{1}{1 - q^{\mp 2 w/z_1}} \frac{1}{1 - q^{\mp 2 w/z_2}} \times (z_1 - q^{\pm 4 z_2})(z_1 - z_2) : Y^\pm(z_1) Y^\pm(z_2) X^\pm(w) : . \]

5. S懦射从到U"'

Let us restrict \( U_q'(\mathfrak{osp}_4) \) to level 1 representations and denote it by \( (U_q'(\mathfrak{osp}_4))_1 \). Recall from 3, 4, and 5 that under the identification between the two presentations of \( U_q'(\mathfrak{osp}_4) \), we have

\[ t_0 \leftrightarrow \gamma(K_1^2 K_2)^{-1}, \]

(5.1)
so that

\begin{equation}
q^{2c} = t_0 t_1 t_2 \leftrightarrow \gamma,
\end{equation}

where \( c = h_0 + h_1 + h_2 \) is the canonical central element. This shows that restricting to level 1 is equivalent to setting \( \gamma = q^2 \) in \( U_q(\widehat{\mathfrak{sl}}_4) \). This section is devoted to giving a surjection from the quantum affine algebra \( (U'_q(\widehat{\mathfrak{sp}}_4))_1 \) to \( U' \). This would imply that the irreducible representation \( V \) constructed for \( U'' \) is also an irreducible representation for \( U'_q(\widehat{\mathfrak{sp}}_4) \).

**Theorem 5.1.** The following map defines a surjection from \( (U'_q(\widehat{\mathfrak{sp}}_4))_1 \) to \( U' \).

\[
X^\pm_1(z) \mapsto X^\pm(z), \quad X^\pm_2(z) \mapsto Y^\pm(z),
\]

\[
a_1(\pm k) \mapsto a(\pm k), \quad a_2(\pm k) \mapsto \frac{1}{[2]}(a(\pm k) + [2] b(\pm k)),
\]

\[
K_1 \mapsto K, \quad K_2 \mapsto (q^{2b(0)} K)^{-1}.
\]

The rest of this section is devoted to proving this theorem. Since the surjectivity is obvious, it suffices to show that the image under this map of every defining relation for \( U'_q(\widehat{\mathfrak{sp}}_4) \), given in Subsection 5.2, is also satisfied in \( U' \).

The \( i = j = 1 \) cases of these relations are immediate from the defining relations of \( U_q(\mathfrak{sl}_2) \), so we shall not mention them below. The first few equations are easy to check, so let us start with (2.13).

**Equation (2.13):** The \( i = 1, j = 2 \) case follows from (4.18). To check the \( i = 2, j = 1 \) case, we refer to (2.4) and notice that \( -\frac{2k}{|2|} = [-k]_2 \). Last of all, to check the \( i = j = 2 \) case, by use of (4.18) and (4.19), it suffice to calculate

\[
-\frac{1}{[2]} \left( \pm \frac{[2]k}{k} q^{\mp |k|} z^k \pm \frac{[2]k}{|k|} (q^{2k} - 1 + q^{-2k}) q^{\mp |k|} z^k \right)
\]

\[
= \pm \frac{1}{[2]} \frac{[2]k}{k} (q^{2k} + q^{-2k}) q^{\mp |k|} z^k
\]

\[
= \pm \frac{[2]k}{k} q^{\mp |k|} z^k.
\]

**Equation (2.14):** This follows from the equations (4.10)-(4.15). For example, from (4.14), one can show

\[
(z - q^4 w) Y^+(z) Y^+(w) + (w - q^4 z) Y^+(w) Y^+(z) = 0.
\]

**Equation (2.15):** The \( i = 1, j = 2 \) case follows from (3.27). Likewise, the \( i = 2, j = 1 \) case follows from (3.36). To check the remaining \( i = j = 2 \) case, we refer to (4.16) and (4.17). We may calculate

\[
\frac{1}{z^2} \frac{1}{(1 - q^2 w/z)(1 - w/q^2 z)} = \frac{1}{w^2} \frac{1}{(1 - q^2 z/w)(1 - z/q^2 w)}
\]

\[
= \delta(z/q^2 w) \frac{1}{w^2} \frac{w/q^2 z}{1 - w/q^2 z} + \delta(q^2 z/w) \frac{1}{w^2} \frac{q^2 w/z}{1 - q^2 w/z}
\]

\[
= \frac{1}{q^2 - q^{-2}} \left( \delta(z/q^2 w) - \delta(q^2 z/w) \right) \frac{1}{w}
\]

and the rest follows if we carefully write down : \( Y^+(z) Y^-(w) : \).

**Equation (2.16):** This one is long. Let us just sketch the + part. First, substitute the bosonization of \( Y^+(w) \) in place of \( X^+_2(w) \) and substitute \( X^+(z_i) \) in place of \( X^+_1(z_i) \) in (2.16). If we then send all \( a(k) \) with \( k < 0 \) to the left of all the \( X^+(z_i) \)
and send all other parts of $Y^+(w)$ to the right of all the $X^+(z_i)$, then we are left with showing that
\[
\text{Sym}_{z_1, z_2, z_3} \begin{pmatrix}
A(z_1, z_2, z_3, w) \\
\times \exp(-\sum a_- w^+) \\
\times X^+(z_1) X^+(z_2) X^+(z_3) \\
\times \exp(+\sum a_+ w^-) D^{-\frac{1}{2}}(w)^{-h_+}
\end{pmatrix} = 0,
\]
where
\[
A(z_1, z_2, z_3, w) = \begin{pmatrix}
\frac{z_1 z_2 z_3}{w^3} & 1 & 1 & 1 \\
\frac{z_2 z_3}{w} & 1 & 1 & 1 \\
\frac{z_3}{w} & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix} + [3] \frac{z_2 z_3}{w} \frac{1}{1 - q^2 z_1} \frac{1}{1 - q^2 z_2} \frac{1}{1 - q^2 z_3}
\]
Here, we have written the bosonized parts of $Y^+(w)$ in a simplified manner. Now, if we use the simple relation
\[
\frac{y/x}{1 - y/x} = \delta(x/y) - \frac{1}{1 - x/y},
\]
we may show
\[
A(z_1, z_2, z_3, w) = \frac{z_1 z_2 z_3}{w^3} (B(z_1, z_2, z_3, w) + C(z_1, z_2, z_3, w)),
\]
with
\[
B(z_1, z_2, z_3, w) = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} - [3] q^2 \frac{1}{1 - q^2 z_1} \frac{1}{1 - q^2 z_2} \frac{1}{1 - q^2 z_3}
\]
and
\[
C(z_1, z_2, z_3, w) = \begin{pmatrix}
[3] q^2 \frac{1}{1 - q^2 z_1} \frac{1}{1 - q^2 z_2} \frac{1}{1 - q^2 z_3} \\
+ [3] q^2 \frac{1}{1 - q^2 z_1} \frac{1}{1 - q^2 z_2} \frac{1}{1 - q^2 z_3} \\
- [3] q^4 \frac{1}{1 - q^2 z_1} \frac{1}{1 - q^2 z_2} \frac{1}{1 - q^2 z_3} \\
+ q^6 \frac{1}{1 - q^2 z_1} \frac{1}{1 - q^2 z_2} \frac{1}{1 - q^2 z_3}
\end{pmatrix}.
\]
It suffices to show
\[
(5.3) \quad \text{Sym}_{z_1, z_2, z_3} (B(z_1, z_2, z_3, w) X^+(z_1) X^+(z_2) X^+(z_3)) = 0,
\]
and
\[ \text{Sym}_{z_1, z_2, z_3} \left( C(z_1, z_2, z_3, w) X^+(z_1) X^+(z_2) X^+(z_3) \right) = 0. \]

These may be done through extensive use of relation (2.3). For example, to do the first one, we first unify denominators. There are many cancellation of terms and we have
\[
B(z_1, z_2, z_3, w) = \left( \frac{(q^2 - q^{-2})}{1 - z_1/q^2 w} \left( 1 - z_2/q^4 w \right) \left( 1 - z_3/q^6 w \right) \right) \times \left( \frac{1}{1 - q^2 z_1/w} \left( 1 - q^2 z_2/w \right) \left( 1 - q^2 z_3/w \right) \right).
\]

We throw away all symmetric parts, and are left with showing
\[
\text{Sym}_{z_1, z_2, z_3} \{ (-z_1 + (q^2 + q^4)z_2 - q^6 z_3) X^+(z_1) X^+(z_2) X^+(z_3) \} = 0,
\]
\[
\text{Sym}_{z_1, z_2, z_3} \{ (-z_1 z_2 - q^8 z_3 + (q^2 + q^4)z_1 z_3) X^+(z_1) X^+(z_2) X^+(z_3) \} = 0.
\]
The first symmetrization expands into
\[
(-z_1 + (q^2 + q^4)z_2 - q^6 z_3) X^+(z_1) X^+(z_2) X^+(z_3)
\]
\[
+ (-z_1 + (q^2 + q^4)z_3 - q^6 z_2) X^+(z_2) X^+(z_3) X^+(z_2)
\]
\[
+ (-z_2 + (q^2 + q^4)z_2 - q^6 z_1) X^+(z_2) X^+(z_3) X^+(z_2)
\]
\[
+ (-z_2 + (q^2 + q^4)z_1 - q^6 z_3) X^+(z_1) X^+(z_3) X^+(z_3)
\]
\[
+ (-z_3 + (q^2 + q^4)z_1 - q^6 z_2) X^+(z_1) X^+(z_3) X^+(z_3)
\]
\[
+ (-z_3 + (q^2 + q^4)z_2 - q^6 z_1) X^+(z_1) X^+(z_3) X^+(z_3).
\]

In this long equation, we may single out
\[
(-z_1 + q^2 z_2) X^+(z_1) X^+(z_2) X^+(z_3) + (-z_2 + q^2 z_1) X^+(z_2) X^+(z_1) X^+(z_3) = 0,
\]
\[
(q^4 z_2 - q^6 z_3) X^+(z_1) X^+(z_2) X^+(z_3) + (q^4 z_3 - q^6 z_2) X^+(z_1) X^+(z_3) X^+(z_2) = 0,
\]
\[
(-z_1 + q^2 z_3) X^+(z_1) X^+(z_3) X^+(z_2) + (-z_3 + q^2 z_1) X^+(z_3) X^+(z_1) X^+(z_2) = 0,
\]
and so on. Note that these are all zero, thanks to (2.3). So, equation (5.5) is indeed true. The second symmetrization (5.6) may similarly be shown to be equal to zero, leading to (5.3).

Showing the validity of (5.4) is quite similar, except that we have the added complexity of using
\[
x \delta(x/y) = y \delta(x/y)
\]
at appropriate places.

Equation (2.17) : With equations (1.20)-(1.22) in hand, it suffices to show that the symmetrization of
\[
\left( \frac{1}{1 - q^2 z_1/w} \right) \left( \frac{1}{1 - q^4 z_2/w} \right) \left( \frac{1}{1 - q^2 z_3/w} \right) \times (z_1 - q^{+4} z_2) (z_1 - z_2)
\]

\[
+ \left[ \frac{w}{z_1} \left( \frac{1}{1 - q^2 w/z_1} \right) \left( \frac{1}{1 - q^4 w/z_1} \right) \right] \times (z_1 - q^{+2} z_2) (z_1 - z_2)
\]

\[
+ \frac{w^2}{z_1 z_2} \left( \frac{1}{1 - q^4 w/z_1} \right) \left( \frac{1}{1 - q^4 w/z_2} \right)
\]
with respect to $z_1$ and $z_2$ is equal to zero. It is done through routine calculation. This concludes the proof of Theorem \[5.1\].

6. REDUCIBLE HIGHEST WEIGHT MODULES

We show in this section that the irreducible modules created in Section 4 are actually highest weight modules of $U_q(\mathfrak{sl}_n)$.

Let us use the notation

$$[a, b]_v = ab - v ba.$$  

It is easy to check that

$$[a, [b, c]]_v = [[a, b], c]_{uv/x} + x[b, [a, c]]_{uv/x},$$

for $x \neq 0 (\ell)$. We first prepare a small lemma.

**Lemma 6.1.** As operators acting on the highest weight vector $v_\lambda \in V(\lambda)$,

$$[x^{-}(0), [\Phi_0^{(2)}(z), x^{-}(1)]_{q^{-2}}]_1$$

$$= -q^{-2}z^{-1}D^x f_1^2 \exp \left(\sum_{k=1}^{\infty} \frac{q^k}{2k} a(-k) z^k\right) \hat{z}^{h_1}.$$  

**Proof.** We may calculate

$$[x^{-}(0), [\Phi_0^{(2)}(z), x^{-}(1)]_{q^{-2}}]_1$$

$$= [[x^{-}(0), \Phi_0^{(2)}(z)]_{q^{-2}}, x^{-}(1)]_1 + q^{-2}[\Phi_0^{(2)}(z), [x^{-}(0), x^{-}(1)]_{q^2}]_1.$$  

Notice that the second term is zero because,

$$[x^{-}(0), x^{-}(1)]_{q^2} = [f_1, e_0 t_0^{-1}]_{q^2} = [f_1, e_0] t_0^{-1} = 0. $$

And from (3.25), we have

$$[x^{-}(0), \Phi_0^{(2)}(z)]_{q^{-2}} = q^{-2}z^{-1}[x^{-}(1), \Phi_0^{(2)}(z)]_{q^2}$$

with $x^{-}(1) = e_0 t_0^{-1}$. So, when acting on an extremal vector,

$$[x^{-}(0), [\Phi_0^{(2)}(z), x^{-}(1)]_{q^{-2}}]_1$$

$$= -q^{-2}z^{-1}x^{-}(1)x^{-}(1)[\Phi_0^{(2)}(z)]_1$$

$$= -q^{-2}z^{-1}x^{-}(1)x^{-}(1)$$

$$\times \exp \left(\sum_{k=1}^{\infty} \frac{q^k}{2k} a(-k) z^k\right) \exp \left( -\sum_{k=1}^{\infty} \frac{q^k}{2k} a(k) z^{-k}\right),$$

$$\times D^x \hat{z}^{h_1}$$

$$= -q^{-2}z^{-1}x^{-}(1)x^{-}(1)D^x \exp \left(\sum_{k=1}^{\infty} \frac{q^k}{2k} a(-k) z^k\right) \hat{z}^{h_1},$$

$$= -q^{-2}z^{-1}D^x \hat{z}^{h_1}.$$  

We have used Theorem \[3.4\] and equations (3.9) and (3.13) in the above line of equalities. We’ve also used the fact $a(k) v_\lambda = 0$ for all $k > 0$.

Substitute $f_1 = x^{-}(0)$ to complete the proof.

We now give a theorem that contains most of the result we have wanted.
Theorem 6.2. For each $j = 0, 1, 2$, the irreducible $U_q^{\ast}(\check{\frak{p}}_4)$-module $V(j)$ is a highest weight module of highest weight weight $\Lambda_j$. The highest weight vectors are given as follows.

$$v_{2\Lambda_0} \otimes v(0) \in \mathcal{V}(0),$$
$$v_{\Lambda_0 + \Lambda_1} \otimes v(-1/2) \in \mathcal{V}(1),$$
$$v_{2\Lambda_0} \otimes v(-1) \in \mathcal{V}(2).$$

Proof. Using Theorem 5.1 and equation (5.3), it is straightforward to check that the weight of each proposed highest weight vector is $\Lambda_j$.

It suffices to show that these are indeed killed by each $e_i$ ($i = 0, 1, 2$). Recall from (4) that,

$$e_0 = q^2 \cdot [x^+_1(0), [x^+_2(0), x^+_1(1)]_{q^{-1}}]_1 K_1^{-2} K_2^{-1},$$
$$e_1 = x^+_1(0),$$
$$e_2 = x^+_2(0),$$

for elements of $(U_q^{\ast}(\check{\frak{p}}_4))^1$. Let us try the last one, $w = v_{2\Lambda_0} \otimes v(-1)$, as an example. Other cases are simpler.

Action of $e_1$ on $w$ is trivially zero. To see the action of $e_2$, we look for the coefficient of $z^{-1}$ in $Y^+(z) \cdot w$. As $b(k)v(-1) = 0$ for all $k > 0$, we have

$$\Omega_2(z)v(-1) = z \exp \left( -\sum_{k=1}^{\infty} b(-k)q^{-k}z^k \right) v(-2).$$

Hence, the smallest power of $z$ with nonzero coefficient appearing from the $\Omega_2(z)$ part of $Y^+(z) \cdot w$ is $z$. Likewise, we have

$$\Psi^{(2)}_2(q^{-2}z)v_{2\Lambda_0} = D^{-\frac{j}{2}} \exp \left( \frac{q}{2k}a(-k)z^k \right) v_{2\Lambda_0}.$$ 

The smallest power possible is $z^0$. Combined, they imply that the smallest power of $z$ appearing in $Y^+(z) \cdot w$ is at least $z^1$. The coefficient of $z^{-1}$ in $Y^+(z) \cdot w$ is zero and hence $e_2w = 0$.

It remains to check the action of $e_0$. Again, we look for the coefficient of $z^{-1}$ in

$$q^2 \cdot [x^-(0), [Y^-(z), x^-(1)]_{q^{-1}}]_1 K_1^{-2} K_2^{-1} \cdot w.$$ 

As before, we may verify that the smallest power of $z$ with nonzero coefficient appearing from the $\Omega_0(z)$ part is $z^{-1}$. Using Lemma 5.1, we see that what remains is

$$-z^{-1}D^{\frac{j}{2}}f_1^2 q \cdot \exp \left( \frac{q^k}{2k}a(-k)z^k \right) z^{\frac{jh_1}{2}} K^{-1} v_{2\Lambda_0}.$$ 

Since $f_1v_{2\Lambda_0} = 0$, the smallest nonzero power of $z$ is (or, at least for the moment, seems to be) $z^0$ with the coefficient

$$-D^{\frac{j}{2}}f_1^2 \frac{q}{[2]}(a(-1)v_{2\Lambda_0}.$$ 

We ignore the insignificant coefficients and follow

$$D^{\frac{j}{2}} f_1^2 a(-1) v_{2\Lambda_0} = D^{\frac{j}{2}}(x^-(0))^2 a(-1) v_{2\Lambda_0},$$
$$= (x^-(1))^2 a(-1) D^{\frac{j}{2}} v_{2\Lambda_0},$$
$$= (x^-(1))^2 a(-1) v_{2\Lambda_1},$$
$$= x^-(1)(a(-1)x^-(1) + q[2]x^-(0)) v_{2\Lambda_1},$$
$$= e_0 f_0^{-1}(a(-1)e_0 f_0^{-1} + q[2]f_1) v_{2\Lambda_1}. $$

\[ = q[2]c_0 t_0^{-1} f_1 v_{2\Lambda_1}, \]
\[ = q^3[2] t_0^{-1} f_1 c_0 v_{2\Lambda_1}, \]
\[ = 0. \]

Hence, the smallest nonzero power of \( z \) was not \( z^0 \), but \( z^1 \). Recalling that the smallest power obtained from \( \Omega(0) \) part was \( z^{-1} \), we conclude that the coefficient of \( z^{-1} \), which we have been looking for, is zero.

We have \( cw = 0 \) and the vector \( w = v_{2\Lambda_0} \otimes v(-1) \) is an extremal vector. This completes the proof.

Now, note that \( U_q^{'}(\hat{\mathfrak{so}}_4) \) is a \( d \)-graded algebra. And we know that the irreducible highest weight modules are quotients of \( U_q^{'}(\hat{\mathfrak{so}}_4) \) by \( d \)-graded submodules. So setting the degree of each highest weight vector to zero determines the action of \( q^d \) uniquely to the rest of the irreducible module.

We have thus reached our goal.

**Theorem 6.3.** \( \mathcal{V}(j) \) is the irreducible highest weight module over \( U_q^{'}(\hat{\mathfrak{so}}_4) \) of highest weight \( \Lambda_j \), for each \( j = 0, 1, 2 \).

We end this paper by writing down the explicit action of \( q^d \) on our new realization of highest weight modules. First, add an element \( q^d \) to the Heisenberg algebra \( \mathcal{B}' \) with the relation

\[ q^d(b(k))q^{-d} = q^k(b(k)) \]

The resulting algebra is denoted by \( \mathcal{B} \). We next make each \( \mathcal{V}_p \) into a \( \mathcal{B} \)-module. It suffices to define the action of \( q^d \) on each \( \mathcal{V}(p) \). This is done differently, depending on the irreducible highest weight module we want to create.

\[ \mathcal{V}(0) : \quad q^d v(p) = \begin{cases} q^{-\frac{d}{2}p^2} v(p) & \text{for } p \in 2\mathbb{Z}, \\ q^{-\frac{d}{2}(p^2+1)} v(p) & \text{for } p \in 2\mathbb{Z} + 1. \end{cases} \]

\[ \mathcal{V}(1) : \quad q^d v(p) = q^{-\frac{d}{2}(p-\frac{1}{2})(p+\frac{3}{2})} v(p) & \text{for } p \in \mathbb{Z} + \frac{1}{2}, \]

\[ \mathcal{V}(2) : \quad q^d v(p) = \begin{cases} q^{-\frac{d}{2}p^2} v(p) & \text{for } p \in 2\mathbb{Z}, \\ q^{-\frac{d}{2}(p^2-1)} v(p) & \text{for } p \in 2\mathbb{Z} + 1. \end{cases} \]

Finally, the action of \( q^d \) on \( \mathcal{V}(j) \) \( (j = 0, 1, 2) \) is then equal to

\[ q^d \otimes q^d \in \text{End} \mathcal{V}(j). \]

Here, \( q^d \) on the left of the tensor sign is the usual \( U_q^{'}(\hat{\mathfrak{so}}_4) \)-action, and \( q^d \) on the right of the tensor sign signifies the action just defined.

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L. D. Landau Institute of Theoretical Physics, Chernogolovka 142432, Russian Federation

Korea Institute for Advanced Study, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea

Department of Mathematics, Kyoto University, Graduate School of Science, Sakyo, Kyoto, Japan

E-mail address: feigin@landau.ac.ru, jinhong@kias.re.kr, tetsuji@kusm.kyoto-u.ac.jp