SOME RESULTS ON MULTIPLICITY-FREE SPACES

LEONARDO BILIOTTI

Abstract. Let \((M, \omega)\) be a connected symplectic manifold on which a connected Lie group \(G\) acts properly and in a Hamiltonian fashion with moment map \(\mu : M \to g^*\). Our purpose is investigate multiplicity-free actions, giving criteria to decide a multiplicity freeness of the action. As an application we give the complete classification of multiplicity-free actions of compact Lie groups acting isometrically and in a Hamiltonian fashion on Hermitian symmetric spaces of noncompact type. Successively we make a connection between multiplicity-free actions on \(M\) and multiplicity-free actions on the symplectic reduction and on the symplectic cut, which allow us to give new examples of multiplicity-free actions.

1. Introduction

Let \((M, \omega)\) be a connected symplectic manifold with a proper and Hamiltonian action of a connected Lie group \(G\) and let \(\mu : M \to g^*\) be a corresponding moment map. In 1984 Guillemin and Sternberg [10], motivated by geometric quantization, introduced the notion of multiplicity-free space when the ring of the \(G\)-invariant functions on \(M\) is commutative with respect to the Poisson-bracket. The manifold \(M\) is called \(G\) multiplicity-free space and the \(G\)-action is called multiplicity-free. The term multiplicity-free comes from the representation theory of Lie groups.

A unitary representation of a Lie group \(G\) on a Hilbert space \(H\) is said to be multiplicity-free if each irreducible representation of \(G\) occurs with multiplicity zero or one in \(H\). The relationships between the two definitions comes via the theory of geometric quantization. The condition that a unitary representation of \(G\) on \(H\) be multiplicity-free is equivalent to the condition that the ring of bounded operators on \(H\) be commutative.

In this paper we investigate multiplicity-free actions, which we also may call coisotropic actions, on a symplectic manifold \(M\), imposing that \(G\) acts properly and in a Hamiltonian fashion on \(M\) and a technical condition, which is needed for applying the symplectic slice, see [2] and [23], and the symplectic stratification of the reduced space given in [2], [23], which is the following.

For every \(\alpha \in g^*\), \(g^*\) is the dual of the Lie algebra of \(G\), \(G\alpha\) is a locally closed coadjoint orbit of \(G\). Observe that the condition of a coadjoint orbit being locally closed is automatic for reductive groups and for their semidirect products with vector spaces. There exists an example of a solvable group due to Mautner [28] p.512, with non-locally closed coadjoint orbits.

One of our purpose is to give Equivalence theorem for multiplicity-free action, which shall allow us to prove that the complete classification of compact Lie groups acting multiplicity-free on irreducible Hermitian symmetric spaces of noncompact type follows from one given in a compact case.

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We also prove a reduction principle for multiplicity-free actions and we make a connection between multiplicity-free actions on $M$ and multiplicity-free actions on the symplectic reduction and on the symplectic cut, mainly in a Kähler geometry. As an application we give new examples of multiplicity-free actions on compact Kähler manifolds which are not Hermitian symmetric spaces.

2. Hamiltonian viewpoint

Let $M$ be a connected differential manifold equipped with a non-degenerate closed 2–form $\omega$. The pair $(M, \omega)$ is called symplectic manifold. Here we consider a finite-dimensional connected Lie group acting smoothly and properly on $M$ so that $g^*\omega = \omega$ for all $g \in G$, i.e. $G$ acts as a group of canonical or symplectic diffeomorphism. For $f, g \in C^\infty(M)$, define $\{f, g\} = \omega(X_f, X_g)$, where $X_f$ and $X_g$ are the vector fields which is uniquely defined by $df = i_{X_f}\omega$ and $dg = i_{X_g}\omega$. It follows that $(C^\infty(M), \{ , \})$ is a Poisson algebra.

The $G$-action is called Hamiltonian, and we said that $G$ acts in a Hamiltonian fashion, if there exists a map

$$\mu : M \longrightarrow g^*,$$

which is called moment map, satisfying:

1. for each $X \in g$ let
   - $\mu^X : M \longrightarrow \mathbb{R}$, $\mu^X(p) = \mu(p)(X)$, the component of $\mu$ along $X$,
   - $X^\#$ be the vector field on $M$ generated by the one parameter subgroup $\{ \exp(tX) : t \in \mathbb{R} \} \subseteq G$.

Then

$$d\mu^X = i_{X^\#}\omega$$

i.e. $\mu^X$ is a Hamiltonian function for the vector field $X^\#$.

2. $\mu$ is $G$–equivariant, i.e. $\mu(gp) = Ad^*\omega(g)(\mu(p))$, where $Ad^*$ is the coadjoint representation on $g^*$.

Let $x \in M$ and $d\mu_x : T_x M \longrightarrow T_{\mu(x)} g^*$ be the differential of $\mu$ at $x$. Then

$$Ker d\mu_x = (T_x G(x))^{\perp_\omega} := \{ v \in T_x M : \omega(v, w) = 0, \forall w \in T_x G(x) \}.$$ 

If we restrict $\mu$ to a $G$–orbit $Gx$, then we have the homogeneous fibration $\mu : Gx \longrightarrow Ad^*(G)\mu(x)$ and the restriction of the ambient symplectic form $\omega$ on the orbit $Gx$ is the pullback by the moment map $\mu$ of the symplectic form on the coadjoint orbit through $\mu(x)$

$$\omega_{|Gx} = \mu^* (\omega|_{Ad^*(G)\mu(x)})_{|Gx}$$

see [2] p. 211, where $\omega_{G\mu(x)}$ is the Kirillov-Konstant-Souriau (KKS) symplectic form on the coadjoint orbit of $\mu(x)$ in $g^*$. This implies the following result.

**Proposition 2.1.** A $G$-orbit $Gx$ is a symplectic submanifold of $M$ if and only if the moment map restricted to $Gx$ into $G\mu(x)$, $\mu|_{Gx} : Gx \longrightarrow G\mu(x)$, is a covering map. In particular if $G$ is a compact Lie group then $Gx = G\mu(x)$ and $\mu|_{Gx} : Gx \longrightarrow G\mu(x)$ is a diffeomorphism onto.

**Proof.** The first affirmation follows immediately from [2]. If $G$ is compact, coadjoint orbits are of the form $G/C(T)$, where $C(T)$ is the centralizer of the torus $T$. In particular such orbits are simply connected, form which one may deduce $G_x = G_{\mu(x)}$. \hfill \Box
3. Multiplicity-free spaces

Let \((M, \omega)\) be a connected symplectic manifold and let \(G\) be a connected Lie group acting properly and as a group of symplectic diffeomorphism on \(M\).

**Definition 3.1.** The \(G\)-action is called **multiplicity-free**, \(M\) is called a \(G\) multiplicity-free space, if the space of invariant function \(C^\infty(M)^G\) is a commutative Lie algebra with respect the Poisson bracket.

By the definition follows that if \(K \subset G\) and \(M\) is a \(K\) multiplicity-free space then \(M\) is a \(G\) multiplicity-free space as well.

The multiplicity-free actions are also called **coisotropic actions**. This is justified by the following discussion.

**Definition 3.2.** A submanifold \(N\) of a symplectic manifold \((M, \omega)\) is said to be **coisotropic** if and only if for every \(x \in N\), \((T_xN)^{\perp \omega} \subset T_xN\). In particular a \(G\)-action on \(M\) is called coisotropic if and only if there exists an open dense subset \(U \subset M\) with \(Gx\) coisotropic for every \(x \in U\).

**Lemma 3.3.** \(M\) is a \(G\) multiplicity-free space if and only the \(G\)-action on \(M\) is coisotropic.

**Proof.** First of all we note the following easy fact: if \(f \in C^\infty(M)^G\) then for every \(\xi \in \mathfrak{g}\) we have \(\{f, f_\xi\} = 0\), where \(f_\xi\) is defined by \(f_\xi(x) = \mu(x)(\xi)\).

Assume that a generic orbit \(Gx\) is coisotropic.

Let \(f, g \in C^\infty(M)^G\). Since \(\{f, f_\xi\} = \{g, f_\xi\} = 0\) we have \(X_f, X_g \in (T_xG)^{\perp \omega} \subset T_xGx\), since \(Gx\) is coisotropic, for every \(x \in U\). Hence

\[
\{f, g\}(x) = \omega(X_f, X_g) = 0, \ \forall x \in U,
\]

which implies \(\{f, g\} = 0\), since \(U\) is an open dense subset.

Vice-versa, let \(x \in M\) be a regular point. By the slice-theorem there are functions \(f_1, \ldots, f_k \in C^\infty(M)^G\) with \(df_1 \wedge \ldots \wedge df_k \neq 0\) in some neighborhood \(W\) of \(Gx\) and

\[
Gx = \{y \in W : f_1(y) = \ldots = f_k(y) = 0\}.
\]

From \(\{f_i, f_j\} = 0\) one may deduce that \(X_{f_i} \in T_xGx\). Therefore, since \(X_{f_i} \in (T_xG)^{\perp \omega}\), \(i = 1, \ldots, k\) and \(X_{f_1}, \ldots, X_{f_k}\) are independent in \(W\), we have that \(Gx\) is a coisotropic submanifold.

**Remark 3.4.** Our proof is essentially one given in [13]. However in [13] the authors assumed that \(G\) is a compact Lie group, their proof works also when the \(G\)-action is a proper action.

It is standard that given a \(G\)-orbit \(Gx = G/G_x\), study the slice representation, i.e. the linear representation of \(G_x\) induced from the \(G\)-action on \(T_xM/T_xGx\). In [13] p. 274, as a consequence of the arguments used in the Restriction Lemma, it was proved that given a complex orbit \(Gp = G/G_p\) then \(G\) acts coisotropically on \(M\) if and only if \(G_p\) acts coisotropically on the slice, whenever \(M\) is a compact Kähler manifold and \(G\) is a subgroup of its full isometric group. Here, we give the same result in symplectic context.

**Proposition 3.5.** Let \((M, \omega)\) be a symplectic manifold and let \(G\) be a Lie group which acts properly and in a Hamiltonian fashion on \(M\) with moment map \(\mu : M \rightarrow \mathfrak{g}^*\). Let \(Gx\) be a symplectic orbit. If \(M\) is a \(G\) multiplicity-free space then the slice representation is a multiplicity-free representation. Moreover, if \(G\) is compact the vice-versa holds as well.
Proof. It follows from symplectic slice, see \cite{2,13,23}.

There exists a neighborhood of the orbit $Gx$ which is $G$-equivariantly symplectomorphic to a neighborhood of the zero section of the symplectic manifold $(Y = G \times_{G_x} (\mathfrak{g}_\beta / \mathfrak{g}_x)^* \oplus V), \tau)$, see \cite{2,23} for an explicit description of the symplectic form $\tau$; with a $G$-moment map $\mu$ given by

$$\mu([g, m, v]) = Ad(g)(\beta + j(m) + i(\mu_V(v))),$$

where $\beta = \mu(x)$, $i : \mathfrak{g}_x^* \rightarrow \mathfrak{g}^*$ is the transpose of the projection $p : \mathfrak{g} \rightarrow \mathfrak{g}_x$, $j : (\mathfrak{g}_\beta / \mathfrak{g}_x)^* \rightarrow \mathfrak{g}_x^*$ ($\mathfrak{g}_x^*$ is the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$) is defined by a choice of a $G_x$-equivariant splitting and finally $\mu_V$ is the moment map of the $G_x$-action on the symplectic subspace $V$ of $(T_x Gx, \omega(x))$. Note that $V$ is isomorphic to the quotient $((T_x Gx)^\perp \omega / ((T_x Gx)^\perp \cap T_x Gx))$. In the sequel we denote by $\omega_V = \omega(x)|_V$.

Since $Gx$ is symplectic, $(Y = G \times_{G_x} V, \tau)$ and the moment map $\mu$ becomes

$$\mu([g, m]) = Ad^*(g)(\beta + i(\mu_V(m))),$$

Assume that $M$ is a $G$-multiplicity-free space and let $y = [e, m] \in Y$ be such that $Gy$ is coisotropic.

Let $Y \in ker(\mu_V)_m$. Noting $d\mu_{[e,m]}(0,Y) = d(\mu_V)_m(Y) = 0$, which implies, from \cite{11}, $Y \in (T_y Gy)^\perp \subset T_y Gy$. This claims $Y \in T_y Gy \cap V = T_m Gx m$, i.e the slice representation is multiplicity-free.

Assume that $G$ is a compact Lie group. It is well known

$$\text{cohom}(G, M) = \text{cohom}(G_x, V),$$

which follows from the classical slice theorem for proper actions \cite{20}, and $rk(G) = rk(G_x)$, since $Gx$ is a symplectic manifold, where cohom$(G, M)$ denotes the codimension of a principal orbit and for every compact group $K$, $rk(K)$ denotes the rank, namely the dimension of the maximal torus. If $G_x$ acts multiplicity-free on $V$ then $\text{cohom}(G_x, V) = rk(G_x) - rk(G_{princ})$, see \cite{13}, where $G_{princ}$ is the principal isotropy subgroup of the action, which implies that

$$\text{cohom}(G, M) = rk(G) - rk(G_{princ}).$$

Therefore, from Theorem 3 \cite{13} p. 269, we get $G$ acts multiplicity-free on $M$. \hfill \Box

Corollary 3.6. Let $M$ be an irreducible Hermitian irreducible symmetric space of non compact type. Let $G$ be a compact group which acts in a Hamiltonian fashion on $M$. Then $G$ acts multiplicity-free on $M$ if and only if it acts multiplicity-free on $M^*$, the corresponding irreducible Hermitian symmetric space of compact type.

Proof. Since $G$ is compact it has a fixed point $x \in M$, from a Theorem of Cartan, see \cite{12}. Hence $G$ acts multiplicity-free on $M$ if and only if $G$ acts multiplicity-free on the tangent space at $x$ if and only if it acts multiplicity free on $M^*$.

\textbf{Remark 3.7.} Corollary 3.6 classifies completely the compact Lie groups acting in a Hamiltonian fashion and coisotropically on the irreducible Hermitian symmetric spaces of noncompact type, due the results proved in \cite{3,6,21}.

4. \textbf{Equivalence Theorems for multiplicity-free action}

From now on we assume that $(M, \omega)$ is a connected symplectic manifold acted on properly and in a Hamiltonian fashion by a connected Lie group $G$. We denote by $\mu : M \rightarrow \mathfrak{g}^*$ the corresponding moment map for the $G$-action on $M$. 
Let $\alpha \in g^*$. We define the corresponding reduced space

$$M_\alpha = \mu^{-1}(G\alpha)/G,$$

to be the topological quotient of the subset $\mu^{-1}(G\alpha)$ of $M$ by the action of $G$. It is well known, see [1], [2], [23], that the reduced space $M_\alpha$ is a union of symplectic manifolds and it can be endowed with a Poisson structure which arise from Poisson structure on the orbits space. Hence $M_\alpha$ is a symplectic stratified space and the manifolds which decompose $M_\alpha$ are called pieces.

Here we analyze the case when $G = G_1 \times G_2$, where $G_i$, $i = 1, 2$ are closed connected subgroup of $G$. We assume also that the coadjoint orbits of $G, G_1$ and $G_2$ are locally closed spaces.

Obviously $g^* = g_1^* \oplus g_2^*$ and the moment map $\mu = \mu_1 + \mu_2$, where $\mu_i$ is the corresponding moment map for the $G_i$-action on $M$, $i = 1, 2$. Since $\mu$ is $G$-equivariant, we have that $\mu_1$ is invariant under $G_2$ and $\mu_2$ is invariant under $G_1$.

Let $\alpha = \alpha_1 + \alpha_2$. The $G_1$-action on the pieces of the reduced space $M_{\alpha_2} = \mu_2^{-1}(G_2\alpha_2)/G_2$ is symplectic. These moment maps on the pieces fit together to form an application

$$\mu_{1,2} : M_{\alpha_2} \rightarrow g_1^*,$$

such that $\mu_{1,2} = \mu_1 \circ \pi_2$, where $\pi_2$ is the projection on $M_{\alpha_2}$. Clearly $G_2$ acts on the reduced space $M_{\alpha_1} = \mu_1^{-1}(G_1\alpha_1)/G_1$ with moment map

$$\mu_{2,1} : M_{\alpha_1} \rightarrow g_2^*$$

such that $\mu_{2,1} = \mu_2 \circ \pi_1$, where $\pi_1$ is the projection on $M_{\alpha_1}$.

We introduce the notion of multiplicity-free space for the reduced space. Indeed, we say that the $G_1$-action on $M_{\alpha_2}$ is multiplicity-free if the ring of $G_1$-invariant functions of $M_{\alpha_2}$, [1], is a commutative Poisson algebra.

We may also define the reduced space with respect the $G_2$-action on $M_{\alpha_1}$.

Now, we shall give a criterion for a $G$-action to be a multiplicity-free action. We begin with the following lemma.

**Lemma 4.1.** Let $(M, \omega)$ be a symplectic manifold and let $G = G_1 \times G_2$ be a connected Lie group which acts in a Hamiltonian fashion on $M$. Let $\alpha_1 \in g_1^*$. Then the $G_2$-action on $M_{\alpha_1}$ is multiplicity-free if and only if for every $\alpha_2 \in g_2^*$ the reduced space $(M_{\alpha_1})_{\alpha_2}$ are points $\mu_{2,1} : M_{\alpha_1} \rightarrow g_2^*$ be the corresponding moment map of the $G_2$-action on the reduced space. We recall that the smooth function on the reduced spaces are defined by

$$C^\infty(M_{\alpha_1}) = C^\infty(M)^{G_1}|_{\mu_1^{-1}(G_1\alpha_1)}$$

and the reduced space is a locally finite union of symplectic manifolds (symplectic pieces) which are the following manifolds.

Let $H$ be a subgroup of $G$. The set $M^{(H)}$ of orbit of type $H$, i.e. the set of points which orbits are isomorphic to $G/H$, is a submanifold of $M$ [20]. The set $(\mu_1^{-1}(G_1\alpha_1) \cap M^{(H)})$ is a submanifold of constant rank and the quotient

$$(M_{\alpha_1})^{(H)} = (\mu_1^{-1}(G_1\alpha_1) \cap M^{(H)})/G_1,$$
is a symplectic manifold which inclusion \((M_{\alpha_1})^{(H)} \hookrightarrow M_{\alpha_1}\) is a Poisson map \(\mathbb{2}\).

The \(G_2\)-action preserves \((M_{\alpha_1})^{(H)}\), and the following topological space
\[
((M_{\alpha_1})^{(H)} \cap \mu_{2,1}^{-1}(G_2\alpha_2))/G_2 = \bigcup_{i \in I} S_i
\]
is a stratified symplectic manifold which restrictions map
\[
r^H_{\alpha_2} : C^\infty((M_{\alpha_1})^{(H)} \cap \mu_{2,1}^{-1}(G_1\alpha_1))^{G_2} \to C^\infty(S_i)
\]
are Poisson and surjective. Therefore, if \(C^\infty(M_{\alpha_1})^{G_2}\) is abelian, the algebra \(C^\infty(S_i), \ i \in I\) must be abelian, and \(S_i\) must be discrete and therefore a points.

Vice-versa, if all reduced spaces are points then \(r^H_{\alpha_2}([f, g]) = 0\) for all \(\alpha_2 \in g_2^*\), and every \(H\) subgroup of \(G_1\), so that \([f, g] = 0\).

**Theorem 4.2.** Let \((M, \omega)\) be a symplectic manifold with a proper and Hamiltonian action of a connected Lie group \(G = G_1 \times G_2\). Assume also that \(G_i, \ i = 1, 2\) are closed connected Lie group and the coadjoint orbits of \(G, G_1\) and \(G_2\) are locally closed. Hence \(M\) is a \(G\) multiplicity-free space if and only if for every \(\alpha = \alpha_1 + \alpha_2 \in g_1^*\) \(M_{\alpha_1}\) is a \(G_1\) multiplicity-free space if and only if \(M_{\alpha_2}\) is a \(G_2\) multiplicity-free space.

**Proof.** It follows immediately from the above result. Indeed, it is easy to check that \(M_{\alpha} = \mu^{-1}(G\alpha)/G\) is homeomorphic to \((M_{\alpha_1})_{\alpha_2}\) or equivalently is homeomorphic to \((M_{\alpha_2})_{\alpha_1}\); the homeomorphism is given by the natural application
\[
(M_{\alpha_1})_{\alpha_2} \to M_{\alpha}, \quad [[x]] \to [x]
\]
which preserves the symplectic pieces, concluding the prove.

**Theorem 4.2** is not difficult to prove. However, from Theorem 4.2 we may deduce some interesting facts.

**Proposition 4.3.** Let \(N\) be closed \(G\)-invariant symplectic submanifold of \(M\). If \(M\) is a \(G\) multiplicity-free space then so is \(N\).

**Proof.** \(G\) acts on \(N\) in a Hamiltonian fashion with moment map \(\overline{\mu} : N \to g^*, \overline{\mu}(x) = \mu(x)\), which is the restriction of \(\mu\) on \(N\). In particular, for every \(\alpha \in g^*\) the reduced space \(N_\alpha \subset M_\alpha\), which implies that the topological space \(N_\alpha\) are points.

**Corollary 4.4.** If \(G\) is a compact Lie group acting multiplicity-free on \(M\) then
\[
M^G := \{x \in M : Gx = x\},
\]
must be a finite set.

Another interesting application of Theorem 4.2 is the following result.

**Proposition 4.5.** Let \((M, \omega)\) be a symplectic manifold with a Hamiltonian circle action. Let \(K\) be a connected Lie group which acts properly and in a Hamiltonian fashion on \(M\). Assume also that the \(K\)-action commutes with the circle action. If \(M\) is a \(K\) multiplicity-free space then so is the \(K\)-action induced on any symplectic cut.

**Proof.** We briefly describe the symplectic cut, see \(\mathbb{17}\) and \(\mathbb{7}\) for more details.

Let \((M, \omega)\) be a symplectic manifold with a Hamiltonian action of a one-dimensional torus \(T^1\) with moment map \(\phi : M \to \mathbb{R}\). We consider the symplectic manifold \(N = M \times \mathbb{C}\), equipped with the symplectic form \(\omega - \frac{i}{2}dz \wedge d\overline{z}\). \(T^1\) acts on \(N\) with its product action and
this action is a Hamiltonian action with moment map \( \psi(p, z) = \phi(p) + |z|^2 \). The reduced space \( M^\lambda = \psi^{-1}(\lambda)/S^1 \), \( \lambda \in \mathbb{R} \) is the symplectic cut.

The \( K \)-action on \( M \times \mathbb{C} \) is given by \( k(m, z) = (km, z) \). Since the \( K \)-action commutes with the \( T^1 \)-action, it induces a Hamiltonian action on the symplectic cut with moment map \( \mathcal{F}([x, z]) = \mu(x) \) where \( \mu \) is the moment map of the \( K \)-action on \( M \). Note that \( K \times T^1 \)-action is multiplicity-free on \( M \times \mathbb{C} \) if and only if the \( K \)-action is on \( M \). Therefore, if \( K \) acts multiplicity-free on \( M \), from Theorem 4.7, \( K \) acts multiplicity-free on the symplectic cut.

Let \( H \) be a compact subgroup of \( G \) and let \( N(H) \) be its normalizer in \( G \). It is well-known that the Lie group \( L = N(H)/H \) acts freely and properly on the submanifold \( M^H := \{ x \in M : G_x = H \} \). Moreover, since \( T_x M^H = (T_x M)^H \), \( M^H \) is symplectic.

In [2], it was proved that \( L \) acts in a Hamiltonian fashion on \( M^H \), the dual of the Lie algebra of \( L \) is naturally isomorphic to the subspace \( (\mathfrak{h})^H \) of the \( H \)-fixed vectors in the annihilator of \( \mathfrak{h} = \text{Lie}(H) \) in \( \mathfrak{g}^* \). Furthermore, given \( \alpha = \mu(x) \), where \( x \in M^H \), then
\[
G\mu^{-1}(\alpha) \cap M^H / G \cong (M^H)_{\alpha_o},
\]
\( \cong \) means symplectically diffeomorphic, where \( \alpha_o = \pi(\alpha) \) and \( \pi : (\mathfrak{h})^H \to l^* \) be the natural projection. This proves that if \( M \) is a \( G \) multiplicity-free space then, from Theorem 4.7, \( M^H \) is a \( L \) multiplicity-free space. Hence, we have the following result.

**Proposition 4.6.** Let \( H \) be a compact subgroup of \( G \). If \( G \) acts coisotropically on \( M \) then \( L \) acts coisotropically on \( M^H \).

Next, we claim the reduction principle for a multiplicity-free action.

**Proposition 4.7.** (Reduction principle) Let \( G \) be a connected Lie group acting properly on a connected symplectic manifold \( M \). Let \( H \) be a principal isotropy for the \( G \)-action. Then \( G \) acts coisotropically on \( M \) if and only if \( L = N(H)/H \) acts coisotropically on \( M^H \).

**Proof.** Since the \( G \)-action is proper and preserves \( \omega \), there exists a \( G \)-invariant almost complex structure \( J \), i.e. \( J : TM \to TM \) be such that \( J^2 = -\text{Id} \), adapted to \( \omega \), i.e. \( \omega(J\cdot, J\cdot) = \omega(\cdot, \cdot) \) and \( g = \omega(\cdot, J\cdot) \) is a Riemannian metric, see [2].

Now let \( H \) be a principal isotropy and let \( L = N(H)/H \). It is well-known that
\[
M^H \cong N(H)/H \times (T_x G x)^{\perp}\,
\]
see [20], which implies \( T_x L x = (T_x G x)^H \).

Since \((T_x G x)^{\perp}= J((T_x G x)^{\perp})\) and \((T_x G x)^{\perp} \subset (T_x M)^H \), recall that \( H \) acts trivially on the slice due the fact that \( G x \) is a principal orbit, we get that
\[
J((T_x G x)^{\perp}) \subset T_x G x \Leftrightarrow J((T_x G x)^{\perp}) \subset (T_x G x)^H.
\]
Therefore, recall that \((T_x L x)^{\perp} \cap T_x M^H = (T_x G x)^{\perp}\) since \( G x \) is principal, we have that \( G x \) is coisotropic in \( M \) if and only if \( L x \) is in \( M^H \). \( \square \)

We conclude this section giving the Equivariant mapping lemma, see [13], in a symplectic context.

**Proposition 4.8.** Let \( (M, \omega) \) and \( (N, \omega_N) \) be connected symplectic manifolds and \( G \) be a connected Lie group acting on both manifolds properly, and in a Hamiltonian fashion. Let
\[ \phi : M \rightarrow N \] be a smooth surjective \( G \)-equivariant map with \( \text{rank} \phi = \dim N \). Assume that for every \( p \in M \), \( \text{Ker} \phi_p \) is a symplectic subspace and

\[ d\phi_p : ((\text{Ker} \phi)^\perp, \omega) \rightarrow (T_pN, \omega_o) \]

is a symplectomorphism. If \( M \) is a \( G \) multiplicity-free space then so is \( N \).

**Proof.** Let \( f \in C^\infty(N)^G \). The function \( \tilde{f} = f \circ \phi \) is a \( G \)-invariant function of \( M \). Take the vector field \( X_f \) such that \( df = i_{X_f} \omega_o \). By assumption the vector field \( \tilde{X} \in (\text{Ker} \phi)^\perp \) such that \( d\phi(\tilde{X}) = X_f \) is the symplectic gradient of \( \tilde{f} \). Hence, given \( f, g \in C^\infty(N)^G \) there exist \( \tilde{f}, \tilde{g} \in C^\infty(M)^G \) such that \( \{f, g\}(\phi(x)) = \{\tilde{f}, \tilde{g}\}(x) \) which conclude our proof. \( \square \)

5. **multiplicity-free spaces in Kähler geometry**

In the sequel we shall assume that \( M \) is a compact Kähler manifold and \( G \) is a closed subgroup of its full isometry group acting in a Hamiltonian fashion on \( M \). Note that this action is automatically holomorphic by a Theorem of Konstant (see [16] p.242).

In [22] it was introduced the homogeneity rank of \( (G, M) \) as the following integer

\[ \text{homrk}(G, M) = \text{rk}(G) - \text{rk}(G_{\text{princ}}) - \text{cohom}(G, M), \]

where \( G_{\text{princ}} \) is the principal isotropy subgroup of the action, the integer \( \text{cohom}(G, M) \) is the codimension of the principal orbit and, for a compact Lie group \( H \), \( \text{rk}(H) \) denotes the rank, namely the dimension of the maximal torus.

In [13] it was proved that a group \( G \) acts multiplicity-free on \( M \) if and only if the homogeneity rank vanishes.

Our purpose is to make a connection between homogeneity rank of \( (G, M) \) and homogeneity rank of \( (G, M_\lambda) \), where \( M_\lambda \) is the reduced space obtained from a torus action.

Let \( K \) be a semisimple compact Lie subgroup of \( G \) and let \( T^k \) be a \( k \)-dimensional connected torus which centralizes \( K \) in \( G \), i.e. \( T^k \subset C_G(K) \). In the sequel we denote by

\[ \phi : M \rightarrow \mathfrak{k} \oplus \mathfrak{t}_k, \]

where \( \mathfrak{t}_k = \text{Lie}(T^k) \), the moment map of the \( T^k \cdot K \)-action on \( M \) and with \( \mu \), respectively with \( \psi \), the moment map of the \( K \)-action on \( M \), respectively a moment map of the \( T^k \)-action on \( M \).

Let \( \lambda \in \mathfrak{t}_k \) be such that \( T^k \) acts freely on \( \psi^{-1}(\lambda) \). The symplectic reduction

\[ (M_\lambda = \psi^{-1}(\lambda)/T^k, \omega_\lambda), \]

is a symplectic manifold and \( \omega_\lambda \) satisfies

\[ \pi^*(\omega_\lambda) = i^*(\omega), \]

where \( \pi \) is the natural projection \( \psi^{-1}(\lambda) \xrightarrow{\pi} M_\lambda \) and \( i \) is the inclusion \( M_\lambda \hookrightarrow M \). [8], [18]. Since \( K \) commutes with \( T^k \), \( K \) acts on \( M_\lambda \) in a Hamiltonian fashion with moment map

\[ \overline{\mu} : M_\lambda \rightarrow \mathfrak{k}, \overline{\mu}([x]) = \mu(x). \]

Indeed, it is easy to see that \( \overline{\mu} \) is \( K \)-equivariant. Hence the problem is then restricted to verify that for every \( Z \in \mathfrak{k} \) we have \( d\overline{\mu} = i_{\overline{Z}^\#} \omega_\lambda \), where \( \overline{Z}^\# \) is the vector field on \( M_\lambda \) generated by the one parameter subgroup \( \exp(t\overline{Z}) \).
Let $X \in T_pM_\lambda$ and let $\tilde{X} \in T_\mu^{-1}(\lambda)$ such that $\pi_*(\tilde{X}) = X$. Since $\pi_*(Z^\#) = \tilde{Z}^\#$, where $Z^\#$ is the Killing field induced from $Z$ in $M$, it follows
\[
d\psi^Z(X) = d\psi^Z(\tilde{X}) = i_{Z^\#}\omega(\tilde{X}) = \pi^*\omega_\lambda(Z^\#, \tilde{X}) = i_{\tilde{Z}^\#}\omega_\lambda(X),
\]
thus $K$ acts in a Hamiltonian fashion on $M^\lambda$.

Let $[p] \in M_\lambda$. It is easy to see that $k[p] = [p]$ if and only if there exists $r(k) \in T^k$ such that $kp = r(k)p$, which is unique since $T^k$ acts freely on $\psi^{-1}(\lambda)$. This means that the following application
\[
K[p] \overset{F}{\longrightarrow} (T^k \cdot K)_p, \quad F(k) = kr(k)^{-1},
\]
is a covering map, due the fact that $K$ is semisimple. Hence
\[
\dim K[p] = \dim(T^k \cdot K)_p - \dim T^k.
\]
Since $M$ is a compact manifold, we may extend the $T^k$-action to a holomorphic action of $(\mathbb{C}^*)^n$ which commutes with the $K$-action. This can be easily deduced from the following easy fact: let $X, Y$ be holomorphic fields. If $[X, Y] = 0$ then $[X, J(Y)] = 0$, since $[X, J(Y)] = J([X, Y]) = 0$, due the fact that $M$ is Kähler. In particular the infinitesimal generatores of the $K$-action commute with ones of the $(\mathbb{C}^*)^n$-action, proving that the two action commute as well.

The set $(\mathbb{C}^*)^n \cdot \psi^{-1}(\lambda)$ is an open subset. Indeed, for every $p \in \psi^{-1}(\lambda)$, denoting with $\mathfrak{z}_p$ the vector subspace of $T_pM$ spanned by the infinitesimal generator of the $T^k$-action on $M$, we have $T_p\psi^{-1}(\lambda) \oplus J(\mathfrak{z}_p) = T_pM$, since $\lambda$ is a regular value, which implies our affirmation. In particular the open subset $(\mathbb{C}^*)^n \cdot \psi^{-1}(\lambda)$ contains regular points. Hence there exists an element
\[
q = \rho_1 \cdots \rho_n \exp(i\theta_1) \cdots \exp(i\theta_n)p = \rho \exp(i\Theta)p \in (\mathbb{C}^*)^n \cdot \psi^{-1}(\lambda),
\]
such that the orbit $(T^k \cdot K)q$ is a principal orbit. Since $K$ commutes with $(\mathbb{C}^*)^n$, we get that $(T^k \cdot K)_p = (T^k \cdot K)_q$ which means that $p$, which lies in $\psi^{-1}(\lambda)$, is a regular point. Therefore, from (4) we deduce that $K[p]$ is a principal orbit and from (3), we get
\[
\text{homrk}(K, M_\lambda) = \text{homrk}(T^k \cdot K, M),
\]
which proves the following result.

**Proposition 5.1.** Let $G$ be a connected compact Lie group acting isometrically and in a Hamiltonian fashion on a compact Kähler manifold $M$. Let $K$ be a compact semisimple Lie group of $G$ which centralizer in $G$ contains a $k$-dimensional connected torus $T^k$. Let $\lambda \in \mathfrak{t}_k$ be such that $T^k$ acts freely on $\psi^{-1}(\lambda)$, where $\psi$ is a moment map of $T^k$-action on $M$. Then the $(T^k \cdot K)$-action is coisotropic on $M$ if and only if the $K$-action is on $M_\lambda = \psi^{-1}(\lambda)/T^k$.

If we consider a one-dimensional torus $T^1$ we may investigate the $K$-action on the Kähler cut $M^\lambda$ obtained from the $T^1$-action. Here we only assume that the $K$-action commutes with the $T^1$-action. It is easy to check that $K[v, z] = K_v$ when $z \neq 0$. Since $\{[v, z] \in M^\lambda : z \neq 0\}$ is an open subset, one may deduce that
\[
\text{homrk}(K, M) = \text{homrk}(K, M^\lambda).
\]
Hence $K$ acts coisotropically on $M$ if and only if $K$ acts on $M^\lambda$ which proves the following result.
Proposition 5.2. Let $G$ be a connected compact Lie group acting isometrically and in a Hamiltonian fashion on a compact Kähler manifold $M$. Let $K$ be a compact Lie group of $G$ whose centralizer in $G$ contains a one-dimensional torus $T^1$. Let $\lambda \in \mathfrak{t}_1$ be such that $T^1_\lambda$ acts freely on $\psi^{-1}(\lambda)$, where $\psi$ is a moment map of $T^1$-action on $M$. Then $K$-action is coisotropic on $M$ if and only if the $K$-action on the Kähler cut $M_\lambda$ is.

Example 5.3. Let $\omega = \sqrt{-1} \sum_{i=1}^{n+1} dz_i \wedge d\bar{z}_i$ be a Kähler form on $\mathbb{C}^{n+1}$. Consider the following $S^1$-action on $(\mathbb{C}^{n+1}, \omega)$:

$$t \in S^1 \mapsto \theta t = \text{multiplication by } e^{it}.$$ 

The action is Hamiltonian with moment map $\mu(z) = |z|^2 + \text{constant}$. If we choose the constant to be $-1$, then $\mu^{-1}(0) = S^{2n+1}$ is the unit sphere on which $S^1$ acts freely and the Kähler reduction $\mu^{-1}(0)/S^1$ is just $\mathbb{P}^n(\mathbb{C}) = \text{SU}(n+1)/\text{S(U(1) \times U(n))}$. Therefore, by Proposition 5.1 a compact connected Lie subgroup $K$ of $\text{SU}(n+1)$ acts multiplicity-free on $\mathbb{P}^n(\mathbb{C})$ if and only if $S^1 \cdot K$ acts multiplicity-free on $\mathbb{C}^n$. Kac [14] and Benson and Ratclif [3] have given the complete classification of linear coisotropic actions, from which one has the full classification of coisotropic actions on $\mathbb{P}^n(\mathbb{C})$.

If we consider the cut of $\mathbb{C}^{n+1}$ at $\lambda > 0$, with respect the above $S^1$-action, we obtain, see [7], $\mathbb{P}^{n+1}(\mathbb{C})$, with $\lambda$ times the Fubini-Study metric. Hence $G \subset \text{SU}(n+1)$ acts coisotropically on $\mathbb{P}^{n+1}(\mathbb{C})$ if and only if it acts coisotropically on $\mathbb{C}^{n+1}$.

6. Multiplicity-free actions on compact non-Hermitian symmetric spaces

Let $T^1$ acting on $\mathbb{P}^n(\mathbb{C})$, as

$$(t, [z_o, \ldots, z_n]) \mapsto [z_o, \ldots, tz_n].$$

This is a Hamiltonian action with moment map

$$\phi([z_o, \ldots, z_n]) = \frac{1}{2} \|z_n\|^2.$$ 

Note that $\phi([0, \ldots, 1])$ is the maximum value of $\phi$ and $\phi^{-1}(\frac{1}{2}) = [0, \ldots, 1]$. Hence (see [7] page 5) if $\lambda = \frac{1}{2} - \epsilon$, $\epsilon \equiv 0$, then the Kähler cut $\mathbb{P}^n(\mathbb{C})^\lambda$ is the blow up of $\mathbb{P}^n(\mathbb{C})$ at $[0, \ldots, 1]$.

Let $T^n$ be a torus acting on $\mathbb{P}^n(\mathbb{C})$ as follows

$$(t_1, \ldots, t_n)([z_o, \ldots, z_n]) = (t_1z_o, t_2z_1, \ldots, t_nz_{n-1}, z_n).$$

This action is Hamiltonian and the principal orbits are Lagrangian; therefore $T^n$ acts coisotropically on $\mathbb{P}^n(\mathbb{C})$. Since $T^n$-action commutes with the above $T^1$-action, from Proposition 5.2 we get $T^n$ acts coisotropically on the blow-up at one point of $\mathbb{P}^n(\mathbb{C})$.

We may generalize the above procedure as follows.

Let $G$ be a connected compact Lie group acting coisotropically on a compact Kähler manifold. It is well-known that, see [12], $\mathfrak{g} = \mathfrak{g}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, and if we denote by $G_{ss}$ the semisimple connected compact Lie group whose Lie algebra is $[\mathfrak{g}, \mathfrak{g}]$, then

$$G = Z(G) \cdot G_{ss}.$$ 

From Proposition 5.1 if $G_{ss}$ acts coisotropically on $M$, then so is the $G_{ss}$-action induced on $M^\lambda$, the reduced space obtained from $T^k \subset Z(G)$.

Let $T^1$ be a one-dimensional torus which lies on $Z(G)$. If $K \subset G$ is a compact Lie group acting coisotropically on $M$ then from Proposition 5.2 $K$ acts coisotropically on the Kähler cut, obtained from the $T^1$-action on $M$. In particular, see [7], if $\lambda_o$ is a maximum for the
moment map of the $T^1$-action then $M^\lambda$, $\lambda = \lambda_o - \epsilon$, $\epsilon \cong 0$, is the blow-up of $M$ along the complex submanifold $\psi^{-1}(\lambda_o)$, where $\psi$ is the corresponding moment map for the $T^1$-action on $M$.

In [5], [6], [21], the complete classification of coisotropic actions on irreducible Hermitian symmetric spaces of compact type is given. Therefore, it is easy to construct examples using the above strategy. For example, the $SU(n)$-action on $SO(2n)/U(n)$ induces a coisotropic action on Kähler cut given by $Z(U(n))$. More generally, if $M = L/P$ is an irreducible Hermitian symmetric space of compact type, then $Z(P)$ is a one-dimensional torus. Since the $P$-action on $M$ is coisotropic, see [5], [6], [21], $P$ acts coisotropically on the Kähler cut with respect the $Z(P)$-action on $M$.

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Dipartimento di Matematica, Università Politecnica delle Marche, Via Brecce Bianche, 60131, Ancona Italy
E-mail address: bilotti@dipmat.univpm.it