η-ININVARIANT AND A PROBLEM OF BÉRARD-BERGERY ON THE
EXISTENCE OF CLOSED GEODESICS

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Abstract. We use the η-invariant of Atiyah-Patodi-Singer to compute the Eells-Kuiper invariant for the Eells-Kuiper quaternionic projective plane. By combining with a known result of Béard-Bergery, it shows that every Eells-Kuiper quaternionic projective plane carries a Riemannian metric such that all geodesics passing through a certain point are simply closed and of the same length.

1. Introduction

The η-invariant introduced by Atiyah-Patodi-Singer [APS], as well as its various ramifications, has played important roles in many problems in geometry and topology. In this short paper, we use the η-invariant to compute the Eells-Kuiper invariant for the Eells-Kuiper quaternionic projective plane. By combining with a known result of Béard-Bergery, it shows that every Eells-Kuiper quaternionic projective plane carries a Riemannian metric such that all geodesics passing through a certain point are simply closed and of the same length.

To be more precise, let p be a point in a closed manifold M. Let g be a Riemannian metric on M. The Riemannian structure (M, g) is called an SCP Riemannian structure if all geodesics issued from p are simply closed (periodic) geodesics with the same length. We refer to the classic book [Be] for a systematic account of the SCP structures.

It is clear that there are SCP Riemannian structures on the compact symmetric spaces of rank one (briefed in [Be] as CROSS), namely the unit spheres, the real projective spaces, the complex projective spaces, the quaternionic projective spaces and the Cayley projective plane, endowed with the corresponding canonical metrics. Moreover, a fundamental result of Bott [Bo] states that any smooth manifold carrying an SCP structure should have the same integral cohomology ring as that of a CROSS. On the other hand, there are manifolds verifying the above cohomological condition but not diffeomorphic to any CROSS. For typical examples, we mention the (exotic) homotopy spheres and the Eells-Kuiper (exotic) quaternionic projective planes.

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In 1975, Bérard-Bergery [BB] discovered an $SC^p$ structure on an exotic sphere of dimension 10. He then raised the natural question: is there any (exotic) Eells-Kuiper quaternionic projective plane carrying an $SC^p$ structure? The same question was also posed explicitly by Besse in the classic book [Be, pp. 143]. Moreover, it is pointed out in [Be, pp. 143] that a positive answer to the above question would also give a positive nontrivial example to the following open question: whether a Blaschke manifold at a point would carry an $SC^p$ Riemannian structure?

The purpose of this article is to provide a positive answer to the above two questions concerning the Eells-Kuiper quaternionic projective planes.

Before going on, we describe the Eells-Kuiper quaternionic projective planes as follows, starting with the standard construction of Milnor [Mi1].

For any pair of integers $(h, j)$, let $\xi_{h,j}$ be the $S^3$-bundle over $S^4$ determined by the characteristic map $f_{h,j} : S^3 \to SO(4)$ with $f_{h,j}(u)v = u^hv^j$ for $u \in S^3, v \in R^4$, where we identify $R^4$ with the space of quaternions. It is shown in [Mi1] that when $h + j = 1$, the total space of the above sphere bundle is homeomorphic to the unit sphere $S^7$.

From now on, we denote by $M_h$ this total space corresponding to $(h, j) = (h, 1 - h)$, and denote by $N_h$ the associated disk bundle.

**Remark 1.1.** When $h = 0$ or 1, $M_h$ is just the unit 7-sphere and the sphere bundle is just the Hopf fibration (corresponding to the left or right multiplications of the quaternions, respectively). On the other hand, $M_2$ is the exotic sphere generating the group $\Theta(7)$ (the set of the orientation preserving diffeomorphism classes of 7-dimensional oriented homotopy spheres), which is isomorphic to the cyclic group $Z_{28}$.

It is shown by Eells-Kuiper [EK2] that the homotopy sphere $M_h$ is diffeomorphic to $S^7$ if and only if the following congruence holds for $h$,

$$\frac{h(h - 1)}{56} \equiv 0 \mod Z.$$  \hspace{1cm} (1.1)

From now on, we assume that $h$ satisfies (1.1). Then there is a diffeomorphism $\sigma : M_h \to S^7$. Let $X_{h,\sigma}$ denote the 8 dimensional closed smooth manifold constructed from $N_h$ by attaching the unit disk $D^8$ by the diffeomorphism $\sigma : \partial(N_h) = M_h \to \partial(D^8) = S^7$. This is what we call an Eells-Kuiper quaternionic projective plane, first constructed in [EK1]. We remark that when $h = 0$ or 1, and $\sigma = \text{id}$, $X_{h,\sigma}$ is just the standard quaternionic projective plane $HP^2$. We also mention a deep result due to Kramer and Stolz [KraS] which asserts that the diffeomorphism type of the resulting manifold $X_{h,\sigma}$ does not depend on the choice of the diffeomorphism $\sigma : M_h \to S^7$.

\footnote{Cf. [Be, 5.37 on page 135] for a definition.} \footnote{Indeed, Eells and Kuiper showed in [EK1] that the $X_{h,\sigma}$’s are the only 8 dimensional closed smooth manifolds admitting a Morse function with 3 critical points.
Let $\tau_h$ be the canonical involution on $M_h$ obtained by the fiberwise antipodal involution on $S^3$. By [BB, Theorem 1] and the above result of Kramer-Stolz, to prove that $X_{h,\sigma}$ carries an $SC^p$ Riemannian structure, one only needs to show that there is a diffeomorphism $\sigma': M_h \to S^7$ such that $\tau \sigma' = \sigma' \tau_h$, where $\tau$ is the standard antipodal involution of $S^7$. Equivalently, one needs only to show that the quotient manifold $M_h/\tau_h$ is diffeomorphic to $\mathbb{R}P^7$. This is the content of the following main result of this paper.

**Theorem 1.1.** The involution $\tau_h$ on $M_h \cong S^7$ is equivalent to the standard antipodal involution on $S^7$. In other words, $M_h/\tau_h$ is diffeomorphic to $\mathbb{R}P^7$.

**Corollary 1.1.** Every Eells-Kuiper quaternionic projective plane admits an $SC^p$ Riemannian structure.

**Remark 1.2.** Since there is infinitely many Eells-Kuiper quaternionic projective planes not diffeomorphic to each other, the above Corollary actually shows that there is an infinite family of pairwise non-diffeomorphic manifolds $M$ with the cohomology ring of $H\mathbb{P}^2$ such that each $M$ admits an $SC^p$ Riemannian structure.

The rest of this article is organized as follows. In Section 2, we reduce the proof of Theorem 1.1 to a problem of computing the Eells-Kuiper $\mu$ invariant introduced in [EK2]. In Section 3, we recall the results of Donnelly [D1] and Kreck-Stolz [KreS] (cf. [G]) which use $\eta$-invariants to express the $\mu$-invariant, and then carry out the required computation of the involved $\eta$ invariant.

2. **Theorem 1.1 and the Eells-Kuiper $\mu$ invariant**

As was indicated in [BB, pp. 240], by results of Mayer [Ma], there could only be two possibilities for $M_h/\tau_h$. That is, it is diffeomorphic either to $\mathbb{R}P^7$ or to the connected sum $\mathbb{R}P^7\#14M_2$, where $14M_2$ is the connected sum $M_2\#\cdots\#M_2$ of 14 copies of $M_2$.

On the other hand, Milnor [Mi2] showed that the Eells-Kuiper $\mu$ invariant of $\mathbb{R}P^7$ and $\mathbb{R}P^7\#14M_2$ takes different values. Thus, in order to prove Theorem 1.1, one need only to show that the $\mu$ invariant of $M_h/\tau_h$ is different from that of $\mathbb{R}P^7\#14M_2$.

For completeness, we recall the definition of the Eells-Kuiper $\mu$ invariant in our situation. Let $M$ be a 7 dimensional closed oriented spin manifold such that the 4-th cohomology group $H^4(M; \mathbb{R})$ vanishes. If $M$ bounds a compact oriented spin manifold $N$, then the first Pontrjagin class $p_1(N) \in H^4(N, M; \mathbb{Q})$ is well-defined.

Following [EK2 (11)], we define $\mu(M) \in \mathbb{R}/\mathbb{Z}$ by

$$\mu(M) \equiv \frac{p_1^2(N)}{2^7 \times 7} - \frac{\text{Sign}(N)}{2^6 \times 7} \mod \mathbb{Z},$$

(2.1)

By the above diffeomorphism type result, it is clear that $M_h/\tau_h$ verifies this condition.
where $p_1^2(N)$ denotes the corresponding Pontrjagin number and Sign$(N)$ is the Signature of $N$.

Now set $M = M_h$, $N = N_h$. Let $x \in H^4(S^4; \mathbb{Z})$ be the generator. By [Mi1], one has

$$e(\xi_{h,1-h}) = x, \quad p_1(\xi_{h,1-h}) = \pm 2(2h - 1)x,$$

where $e(\xi_{h,1-h})$ and $p_1(\xi_{h,1-h})$ are the Euler class and the first Pontrjagin class of the sphere bundle $\xi_{h,1-h}$ respectively. Also by [Mi1], one has

$$\text{Sign}(N_h) = 1.$$  

(2.2) and (2.3), one deduces as in [Mi1] and [EK2] that

$$p_1^2(N_h) - 2 \frac{\text{Sign}(N_h)}{2^9 \times 7} = h(h - 1) - \frac{1}{56},$$

which is an integer in view of the assumption (1.1).

Recall that by [Mi2], one has $\mu(RP^7) = \pm \frac{1}{32}$ while $\mu(RP^7 \# 14M_2) = \pm \frac{1}{32} + \frac{1}{2}$. Thus, in order to prove Theorem 1.1, one need only to prove the following result.

**Theorem 2.1.** The following identity holds for any integer $h$ verifying (1.1),

$$\mu(M_h/\tau_h) \equiv \pm \frac{1}{32} \pmod{\mathbb{Z}}.$$  

(2.5)

Theorem 2.1 will be proved in Section 3

### 3. A PROOF OF THEOREM 2.1

In this section, we compute $\mu(M_h/\tau_h)$. The obvious difficulty is that one does not find easily an 8 dimensional spin manifold with boundary $M_h/\tau_h$. Instead, we will make use of an intrinsic formula for the $\mu$ invariant, which is given by Donnelly [Di] and Kreck-Stolz [KreS] (cf. the survey paper of Goette [G]).

Indeed, for any 7 dimensional closed oriented spin manifold $M$ with $H^4(M; \mathbb{R}) = 0$, let $g^{TM}$ be a Riemannian metric on $TM$. Let $\nabla^{TM}$ be the associated Levi-Civita connection. Let $p_1(TM, \nabla^{TM})$ be the corresponding first Pontrjagin form (cf. [Z, Section 1.6.2]). Then there is a 3-form $\tilde{p}_1(TM, \nabla^{TM})$ on $M$ such that

$$d\tilde{p}_1(TM, \nabla^{TM}) = p_1(TM, \nabla^{TM}).$$

(3.1)

Let $D_M$ (resp. $B_M$) be the Dirac (resp. Signature) operator associated to $g^{TM}$. Let $\eta(D_M), \eta(B_M)$ be the Atiyah-Patodi-Singer $\eta$ invariant of $D_M$, $B_M$ (cf. [APS]). Let

$$\eta(D_M) = \frac{1}{2}(\dim(\ker D_M) + \eta(D_M))$$

be the corresponding reduced $\eta$-invariant.
By [DI] and [KreS] (cf. [G] pp. 424), the $\mu$ invariant defined in (2.1) can be represented by

$$
\mu(M) \equiv \eta(D_M) + \frac{\eta(B_M)}{2^5 \times 7} - \frac{1}{2^7 \times 7} \int_M p_1(TM, \nabla^{TM}) \wedge \hat{\rho}_1(TM, \nabla^{TM}) \mod \mathbb{Z}. \tag{3.2}
$$

Now consider the double covering $M_h \to M_h/\tau_h$. We fix a spin structure on $M_h/\tau_h$ and lift everything from $M_h/\tau_h$ to $M_h$. We get that

$$
\mu(M_h/\tau_h) \equiv \eta(P_h D_{M_h}) + \frac{\eta(P_h B_{M_h})}{2^5 \times 7} - \frac{1}{2^7 \times 7} \int_{M_h} p_1(TM_h, \nabla^{TM_h}) \wedge \hat{\rho}_1(TM_h, \nabla^{TM_h}) \mod \mathbb{Z}, \tag{3.3}
$$

where $P_h = \frac{1}{2}(1 + \tau_h)$ is the canonical projection. Here $\tau_h$ denotes the lifted actions on the corresponding vector bundles.

Indeed, recall that $M_h$ is a fiber bundle over $S^4$ with fiber $S^3$. It is the boundary of the unit disk bundle $N_h$ over $S^4$, while $\tau_h$ is the canonical involution which maps on each fiber by mapping a point to its antipodal. This involution extends canonically to an involution on $N_h$ which we still denote by $\tau_h$. Clearly, the fixed point set of $\tau_h$ on $N_h$ is $S^4$, the image of the zero section of the disk bundle.

Let $g^{TN_h}$ be a $\tau_h$-invariant Riemannian metric on $TN_h$ such that it restricts to $g^{TM_h}$ on $\partial N_h = M_h$ and is of product structure near $M_h$ (the existence of such a metric is clear). Let $\nabla^{TN_h}$ be the associated Levi-Civita connection.

By dimensional reason we see that we are in the situation of even type in the sense of [AB] Proposition 8.46. Thus there exists a $\tau_h$-equivariant spin structure on $N_h$, such that it induces a $\tau_h$-equivariant spin structure on $M_h$, which equals the one lifted from the spin structure given on $M_h/\tau_h$. In particular, $\tau_h$ lifts to an action on the associated spinor bundle $S(TN_h) = S_+(TN_h) \oplus S_-(TN_h)$ associated to $(TN_h, g^{TN_h})$, preserving the corresponding $\mathbb{Z}_2$-grading. It induces an action on $S(TM_h) = S_+(TM_h)|_{M_h}$. Moreover, the lifted $\tau_h$-action commutes with the Dirac operator $D_{TN_h} : \Gamma(S(TM_h)) \to \Gamma(S(TM_h))$, and thus also commutes with the induced Dirac operator $D_{TM_h} : \Gamma(S(TM_h)) \to \Gamma(S(TM_h))$, which in turn determines a Dirac operator on $M_h/\tau_h$ on which one can apply (3.2) and (3.3).

Let $D_{N_h, +} : \Gamma(S_+(TN_h)) \to \Gamma(S_-(TN_h))$ be the natural restriction of $D_{N_h}$. By the Atiyah-Patodi-Singer index theorem [APS] and its equivariant extension by Donnelly [D2], one finds

$$
\eta(P_h D_{M_h}) \equiv \frac{1}{2} \int_{N_h} \widehat{\alpha}(TN_h, \nabla^{TN_h}) + \frac{1}{2} \int_{S^4} A_1 \mod \mathbb{Z}, \tag{3.4}
$$

where the mod $\mathbb{Z}$ term comes from the Atiyah-Patodi-Singer type index $\text{ind}_{APS}(P_h D_{N_h, +})$, $\widehat{\alpha}(TN_h, \nabla^{TN_h})$ is the Hirzebruch $\widehat{\alpha}$-form associated to $\nabla^{TN_h}$ (cf. [Z] Section 1.6.3) and $A_1$ is the canonical contribution on the fixed point set (which by the local index theory is the same as the usual fixed point set contribution appearing in the equivariant Atiyah-Singer index theorem for compact group actions on closed manifolds).
Similarly,
\[ \eta(P_h B_{M_h}) = \frac{1}{2} \int_{N_h} L(TN_h, \nabla^{TN_h}) + \frac{1}{2} \int_{S^4} A_2 - \frac{1}{2} \text{Sign}(N_h) - \frac{1}{2} \text{Sign}(N_h, \tau_h), \] (3.5)
where \(L(TN_h, \nabla^{TN_h})\) is the Hirzebruch L-form associated to \(\nabla^{TN_h}\) (cf. [Z] Section 1.6.3), \(A_2\) is the canonical contribution on the fixed point set and \(\text{Sign}(N_h, \tau_h)\) is the notation for the equivariant Signature with respect to \(\tau_h\).

By a direct computation, one has
\[ \frac{1}{2} \int_{N_h} \hat{A}(TN_h, \nabla^{TN_h}) + \frac{1}{2^{6} \times 7} \left( \int_{N_h} L(TN_h, \nabla^{TN_h}) - \text{Sign}(N_h) \right) \]
\[ - \frac{1}{2^{8} \times 7} \int_{M_h} p_1(TM_h, \nabla^{TM_h}) \wedge \hat{p}_1(TM_h, \nabla^{TM_h}) = \frac{p_1^2(N_h)}{2^{8} \times 7} - \frac{\text{Sign}(N_h)}{2^6 \times 7}. \] (3.6)

From (2.4) and (3.3)-(3.6), we find that
\[ \mu(M_h/\tau_h) \equiv \frac{h(h-1)}{112} + \frac{1}{2} \int_{S^4} A_1 + \frac{1}{2^{6} \times 7} \int_{S^4} A_2 - \frac{\text{Sign}(N_h, \tau_h)}{2^{6} \times 7} \mod \mathbb{Z}. \] (3.7)

Now let \(W_h\) denote the normal bundle in \(N_h\) to the submanifold \(S^4\), the fixed point set of \(\tau_h\). It is clear that \(\tau_h\) acts on \(W_h\) by multiplication by \(-1\).

By (2.2) and [LM] pp. 267, one finds
\[ \int_{S^4} A_1 = \pm \frac{1}{32} \int_{S^4} p_1(W_h) = \pm \frac{(2h-1)}{16}. \] (3.8)

Similarly, by [LM] pp. 265 and (2.2), one has
\[ \int_{S^4} A_2 = \int_{S^4} c(W_h) = 1. \] (3.9)

On the other hand, since \(S^4\) is the fixed point set of \(\tau_h\), \(\tau_h\) preserves \(x \in H^4(S^4; \mathbb{Z})\). Thus one has
\[ \text{Sign}(N_h, \tau_h) = 1. \] (3.10)

From (3.7)-(3.10), one gets
\[ \mu(M_h/\tau_h) \equiv \frac{h(h-1)}{112} \pm \frac{2h-1}{32} \mod \mathbb{Z}. \] (3.11)

We now claim that under the condition (1.1), (2.5) follows from (3.11).

Indeed, under the assumption (1.1), one has \(h \equiv 0, 1, 8, 49 \mod 56\mathbb{Z}\). Thus we only need to do the case by case checking as follows, where by “≡” we mean that the congruence is mod \(\mathbb{Z}\).

(i) For \(h = 56k\), then \(\frac{h(h-1)}{112} \equiv \frac{k}{2}\), while \(\frac{2h-1}{32} \equiv -\frac{1}{32} + \frac{k}{2}\).
(ii) For \(h = 56k + 1\), then \(\frac{h(h-1)}{112} \equiv \frac{k}{2}\), while \(\frac{2h-1}{32} \equiv \frac{1}{32} + \frac{k}{2}\).
(iii) For \(h = 56k + 8\), one has \(\frac{h(h-1)}{112} \equiv \frac{1}{2} + \frac{k}{2}\), while \(\frac{2h-1}{32} \equiv -\frac{1}{32} + \frac{1}{2} + \frac{k}{2}\).
(iv) For \(h = 56k + 49\), one has \(\frac{h(h-1)}{112} \equiv \frac{k}{2}\), while \(\frac{2h-1}{32} \equiv \frac{1}{32} + \frac{k}{2}\).
Combining (i)-(iv) with (3.11), we always have (2.5).

The proof of Theorem 2.1, as well as of Theorem 1.1 and Corollary 1.1 is complete.

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