EXTENSIONS OF I-REVERSIBLE RINGS

VIVEK BHABANI LAMA, SUHAS B N, SUSOBHAN MAZUMDAR AND RAISA DSOUZA

Abstract. A ring $R$ is said to be i-reversible if for every $a, b \in R$, $ab$ is a non-zero idempotent implies $ba$ is an idempotent. It is known that the rings $M_n(R)$ and $T_n(R)$ (the ring of all upper triangular matrices over $R$) are not i-reversible for $n \geq 3$. In this article, we provide a non-trivial i-reversible subring of $M_n(R)$ when $n \geq 3$ and $R$ has only trivial idempotents. We further provide a maximal i-reversible subring of $T_n(R)$ for each $n \geq 3$, if $R$ is a field. We then give conditions for i-reversibility of Trivial, Dorroh and Nagata extensions. Finally, we give some independent sufficient conditions for i-reversibility of polynomial rings, and more generally, of skew polynomial rings.

1. Introduction

A ring $R$ is said to be reversible if for every $a, b \in R$, $ab = 0$ then $ba = 0$. In 1999, Cohn [1] formally introduced the notion of a reversible ring. Anderson and Camillo [2], studied reversible rings under the nomenclature of $ZC_2$, while Krempa and Niewieczerzal [3], called them $C_0$ rings. It is easy to see that a ring $R$ is reversible if and only if for every $a, b \in R$, $ab$ is an idempotent implies $ba$ is an idempotent. In 2020, Anjana Khurana and Dinesh Khurana [4], studied a new class of rings where $ab$ is a non-zero idempotent implies $ba$ is an idempotent and called them i-reversible rings. It was shown that the class of reversible rings is a proper subclass of the class of i-reversible rings. Jung et.al [5] studied the same class of rings under the terminology of quasi-reversible rings.

A complete characterization of i-reversibility of matrix rings over commutative rings was established in [4, Theorem 4.3]. It was proved that for a commutative ring $R$, the matrix ring $M_n(R)$ is i-reversible if and only if $n = 2$ and $R$ has only trivial idempotents. It was also shown in [4, Corollary 3.2] that the upper triangular matrix ring $T_n(R)$ is not i-reversible if $n > 2$. One may ask if there exists at least a non-trivial i-reversible subring of $T_n(R)$. We prove that the ring $D_n(R) = \{(a_{ij}) \in T_n(R) | a_{11} = a_{22} = \cdots = a_{nn}\}$ is such an example for any $n$, provided $R$ has only trivial idempotents (see Theorem 3.1 and Remark 3.2). We also prove that for $n \geq 5$, if $S$ is a subring of $T_n(R)$ such that $D_n(R) \subseteq S$ then $S$ necessarily has a non-trivial idempotent and $S$ is not i-reversible (see

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Theorem 3.9 and Theorem 3.7). Further, when \( n \geq 5 \) and \( R \) is a field, we prove that \( D_n(R) \) is a maximal i-reversible subring of \( T_n(R) \) (see Theorem 3.10). Finally, when \( n = 3 \) or 4 and \( R \) is a field, we prove that \( D_n(R) \) is not a maximal i-reversible subring of \( T_n(R) \) (see Examples 3.11 and 3.12). We also provide maximal i-reversible subrings of \( T_n(R) \) strictly containing \( D_n(R) \) in these two cases (see Theorems 3.14 and 3.16).

In 2003, Kim and Lee [6] show that the polynomial ring over a reversible ring need not be reversible. They show however that if \( R \) is also Armendariz then \( R[x] \) is reversible. In the same article, they provide some sufficient conditions for reversibility of Dorroh, Nagata and Trivial extensions of a ring. In this paper, we discuss similar questions for i-reversibility. Analogous to [6], we show that if \( R \) is i-reversible and Armendariz then \( R[x] \) is i-reversible (see Theorem 5.7). We also provide another sufficient condition for the i-reversibility of \( R[x] \) (see Theorem 5.4). Further, we prove that for Dorroh extensions, under suitable conditions, the reversibility and i-reversibility coincide (see Theorem 4.2). Furthermore, we prove if \( R \) has only trivial idempotents then the Trivial and Nagata extensions of \( R \) over itself are i-reversible (see Theorems 2.6 and 4.6). We also provide a necessary and sufficient condition for the Trivial extension of \( R \) to be i-reversible when \( R \) is abelian and not reversible (see Theorem 2.11).

An interesting generalization of polynomial rings is the skew polynomial rings. The reversibility of skew polynomial rings has been studied in [7, 8]. We provide two independent sufficient conditions for the i-reversibility of skew polynomial rings (see Theorems 6.4 and 6.10).

Throughout the course of the paper, all rings are considered to be associative rings with unity (unless specified otherwise).

2. Trivial Extensions.

In this section, we show that if \( R \) has only trivial idempotents then the trivial extension of \( R \) over itself is i-reversible. We also provide examples to show that this condition is not necessary. However, if \( R \) is abelian and not reversible, it turns out that this condition is necessary as well.

**Definition 2.1.** Let \( R \) be a ring and \( M \) be a bimodule of \( R \). The trivial extension of \( R \) by \( M \) is the ring \( T(R, M) = R \bigoplus M \) with component wise addition and the following multiplication:

\[
(r_1, m_1) \cdot (r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).
\]
The ring $T(R, M)$ is isomorphic to the ring of all matrices of the form $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$, where $r \in R$ and $m \in M$ with usual matrix operations. For now, we focus on the case $M = R$.

**Remark 2.2.** The following statements are easy to observe,

1. If $T(R, R)$ is i-reversible then $R$ is i-reversible.
2. If $R$ is commutative then $T(R, R)$ is commutative (hence i-reversible).
3. If $R$ is reduced then $T(R, R)$ is reversible [6, Proposition 1.6] (hence i-reversible).

At this stage, we ask whether $T(R, R)$ is i-reversible when $R$ is i-reversible. In view of the above remark this answer is trivial if $R$ is commutative or reduced. Hence, it is natural to ask the same question when $R$ is neither commutative nor reduced.

We now provide two examples of non-commutative non-reduced rings whose trivial extensions are not i-reversible.

**Example 2.3.** Let $H$ be the ring of quaternions, $S = T(H, H)$ and $R = T_2(S)$. We have, $e = \begin{bmatrix} E_{11} & 0 \\ 0 & E_{11} \end{bmatrix}$, where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a non-trivial idempotent. Suppose $T(R, R)$ is i-reversible. Then, the corner ring $e T(R, R) e$ is reversible (see [4, Proposition 2.1(4)]).

However, $e T(R, R) e \cong T(S, S)$, which is not reversible (see [6, Example 1.7]). This is a contradiction. Therefore, $T(R, R)$ is not i-reversible.

**Example 2.4.** [15, Example 1.10] Let $R = T(H \times H, H \times H)$. Since $H \times H$ is a reduced, $R$ is reversible (see [6, Proposition 1.6]).

Let $\alpha = \begin{bmatrix} (1, 0) & (0, i) \\ (0, 0) & (1, 0) \\ (0, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix}$ and $\beta = \begin{bmatrix} (1, 0) & (0, 1) \\ (0, 0) & (1, 0) \\ (0, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix} \begin{bmatrix} (0, k) & (0, 0) \\ (0, 0) & (0, k) \end{bmatrix}$.

It is clear that $\alpha \beta$ is a non-zero idempotent. However, $\beta \alpha$ is not an idempotent. Therefore, $T(R, R)$ is not i-reversible.

**Remark 2.5.** In Example 2.3, the ring $R$ has a non-trivial idempotent $e$ which is not central. Whereas in Example 2.4, the ring $R$ has non-trivial idempotents and is also abelian.

It thus becomes natural to impose the condition that $R$ has only trivial idempotents in which case, $R$ is also i-reversible and abelian.

**Theorem 2.6.** Let $R$ be a ring. If $R$ has only trivial idempotents then $T(R, R)$ has only trivial idempotents and hence is i-reversible.
Proof. Since $R$ has only trivial idempotents, $R$ is abelian. We first show that if $R$ is abelian, every idempotent of $T(R, R)$ is of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where $a$ is an idempotent in $R$. Let $\alpha = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ be an idempotent in $T(R, R)$. A straightforward computation of $\alpha^2 = \alpha$ yields $a^2 = a$ and $ab + ba = b$. Thus, $a$ is an idempotent in $R$. Since $R$ is an abelian ring with idempotent $a$, we get $(2a - 1)b = 0$. Now, $(2a - 1)^2 = 4a^2 - 4a + 1 = 1$, which shows $2a - 1$ is a unit. Therefore, $b = 0$. Hence $\alpha = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where $a$ is an idempotent in $R$. However, since $R$ has only trivial idempotents the only choices for $a$ are 0 and 1. Therefore, $T(R, R)$ is a ring with only trivial idempotents and hence is i-reversible. □

In view of the above theorem, we may ask, does the i-reversibility of $T(R, R)$ ensure that $R$ has only trivial idempotents? The following example answers this question in the negative.

**Example 2.7.** Let $S$ be a reduced ring with only trivial idempotents and $R = S \oplus S$. Then $R$ is a reduced ring with $(1, 0)$ and $(0, 1)$ as the only non-trivial idempotents. Since $R$ is reduced, $T(R, R)$ is reversible (see [6, Proposition 1.6]). Hence, $T(R, R)$ is i-reversible.

**Corollary 2.8.** Let $S$ be a ring and $R = T(S, S)$. If $S$ has only trivial idempotents then $T(R, R)$ is i-reversible.

**Proof.** Firstly, since $S$ has only trivial idempotents, $S$ is abelian. This implies that every idempotent of $R = T(S, S)$ is of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. However, since $S$ has only trivial idempotents, the only choices for $a$ are $a = 0$ or $a = 1$. Thus, $R$ has only trivial idempotents. Therefore, $T(R, R)$ is i-reversible. □

**Theorem 2.9.** Let $R$ be a ring with a non trivial central idempotent $e$. If $T(R, R)$ is i-reversible then $R$ is reversible.

**Proof.** Let $T(R, R)$ be i-reversible. Then $R$ is also i-reversible. Since $e$ is a non-trivial central idempotent in $R$, we can conclude that $R$ is reversible (see [4, Proposition 2.1.(4)]). □

**Corollary 2.10.** Let $R$ be an abelian ring. If $T(R, R)$ is i-reversible then $R$ is either reversible or $R$ has only trivial idempotents.

**Proof.** The proof is clear in view of Theorem 2.9 and the fact that $R$ is abelian. □
We characterize the i-reversibility of the trivial extension of rings which are abelian but not reversible.

**Theorem 2.11.** Let $R$ be an abelian ring which is not reversible. The ring $T(R, R)$ is i-reversible if and only if $R$ has only trivial idempotents.

*Proof.* If $T(R, R)$ is i-reversible, then since $R$ is not reversible, by the previous corollary, $R$ has only trivial idempotents. For the converse, see Theorem 2.6. □

3. Some Subrings of Triangular Matrix Rings.

In [4], A.Khurana and D.Khurana show that for a ring $R$, $M_n(R)$ is not i-reversible for all $n > 2$. In this section, we show that $D_n(R)$ is an i-reversible subring of $M_n(R)$ when $n \geq 3$ and $R$ has only trivial idempotents. Further, we prove that $D_n(R)$ is a maximal i-reversible subring of $T_n(R)$, when $R$ is a field and $n \geq 5$. Also, for the cases $n = 3$ and 4, we show that $D_n(R)$ is not a maximal i-reversible subring of $T_n(R)$ when $R$ is a field. Finally, in these two cases, we provide maximal i-reversible subrings of $T_n(R)$ strictly containing $D_n(R)$.

**Theorem 3.1.** Let $R$ be a ring and $n \geq 3$ be a positive integer. The ring $D_n(R)$ is i-reversible if and only if $R$ has only trivial idempotents.

*Proof.* Let $D_n(R)$ be i-reversible. If possible, let $R$ have a non-trivial idempotent $e$. Let $\alpha$ be the diagonal matrix with diagonal entries as $e$. Since $\alpha$ is a non trivial idempotent and $D_n(R)$ is i-reversible, the corner ring $\alpha D_n(R)\alpha$ is reversible (see [4, Proposition 2.1.(2)]). This is a contradiction since it is clear that $\alpha D_n(R)\alpha$ is not reversible (for example, $\alpha E_{12}\alpha \cdot \alpha E_{2n}\alpha \neq 0$ and $\alpha E_{2n}\alpha \cdot \alpha E_{12}\alpha = 0$, where $E_{ij}$ is a matrix with $ij^{th}$ entry 1 and other entries 0).

Conversely, let $R$ have only trivial idempotents. Let $P$ be an idempotent in $D_n(R)$ with $p$ as the diagonal entries. Since $P^2 = P$, a simple computation yields $p^2 = p$. However, since $R$ has only trivial idempotents $p = 0$ or $p = 1$. If $p = 1$ then $P$ is invertible (since $P = I + N$ for some nilpotent $N$). Also, since $P^2 = P$ we have $P = I$. If $p = 0$ then $P$ is a nilpotent (since $P^n = 0$). Since $P$ is also an idempotent, we arrive at $P = 0$. Therefore $D_n(R)$ has only trivial idempotents and hence is i-reversible. □

**Remark 3.2.** If $n = 2$, then $D_n(R) = T(R, R)$. This is i-reversible when $R$ has only trivial idempotents as discussed in Theorem 2.6.
Corollary 3.3. Let \( R \) be a ring and \( n \) be a positive integer. If \( R \) has only trivial idempotents then the ring \( V_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{ij} = a_{i(j+1)} \forall i, j = 1, 2, \ldots, n-1\} \) is \( i \)-reversible.

Proof. The result is trivial for \( n = 1 \). For \( n = 2 \), we have \( V_n(R) = T(R, R) \). Therefore, by Theorem 2.6, \( V_n(R) \) is \( i \)-reversible in this case. For \( n \geq 3 \), \( V_n(R) \) is a subring of \( D_n(R) \). Since \( R \) has only trivial idempotents, by Theorem 3.1, \( D_n(R) \) is \( i \)-reversible. So \( V_n(R) \) is \( i \)-reversible. \( \square \)

Remark 3.4. The converse of the above corollary is not true. For instance, if \( R \) is a reduced ring containing a non-trivial idempotent, then by [6, Theorem 2.5], we have \( R[x]/(x^n) \) is reversible and hence, \( i \)-reversible. It is not hard to see that for each positive integer \( n \), \( V_n(R) \cong R[x]/(x^n) \).

Theorem 3.5. Let \( R \) be a ring and let \( n \) be a positive integer greater than or equal to 3. The ring \( T(D_n(R), D_n(R)) \) is \( i \)-reversible if and only if \( R \) has only trivial idempotents.

Proof. Suppose \( R \) has only trivial idempotents then \( D_n(R) \) is abelian (see [9, Corollary 3.5]). However, \( D_n(R) \) is not reversible (see [6, Example 1.3] and [6, Example 1.5]). Therefore by Theorem 2.11, \( T(D_n(R), D_n(R)) \) is \( i \)-reversible. Conversely, suppose \( R \) has a non-trivial idempotent. Then by Theorem 3.1, \( D_n(R) \) is not \( i \)-reversible, and so, \( T(D_n(R), D_n(R)) \) is not \( i \)-reversible. \( \square \)

We now ask the following question.

Question 3.6. Given any positive integer \( n \geq 3 \) and any ring \( R \) with only trivial idempotents, can one say \( D_n(R) \) is a maximal \( i \)-reversible subring of \( T_n(R) \)?

We answer this question in the affirmative if \( n \geq 5 \) and in the negative if \( n = 3 \) or \( n = 4 \), when \( R \) is a field. Furthermore, for the cases \( n = 3, 4 \) and a field \( R \), we provide a suitable maximal \( i \)-reversible subring of \( T_n(R) \) that strictly contains \( D_n(R) \).

We first prove the following result which is valid for any ring \( R \).

Theorem 3.7. Let \( R \) be any ring and \( n \geq 5 \). Suppose \( S \) is a ring such that \( D_n(R) \subset S \subset T_n(R) \) and \( S \) has a non-trivial idempotent. Then \( S \) is not \( i \)-reversible.

Proof. If \( R \) has a non-trivial idempotent, then by Theorem 3.1, \( D_n(R) \) itself is not \( i \)-reversible and so \( S \) is not \( i \)-reversible. So it is enough to prove the result when \( R \) is a ring with only trivial idempotents. Let \( A = [a_{ij}] \) be a non-trivial idempotent in \( S \). Let \( A' = A - \text{diag}(A) \), where \( \text{diag}(A) \) is the diagonal matrix with diagonal entries \( a_{ii} \). The
matrix $A' \in D_n(R)$, and so $A' \in S$. Since $S$ is a ring, we have diag$(A) \in S$. The fact that $A$ is an idempotent in $S$ implies each $a_{ii} = 0$ or $1$. Moreover, since $A$ is a non-trivial idempotent, there exists positive integers $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $A - A' = E_{i_1i_1} + E_{i_2i_2} + \cdots + E_{i_ki_k} \neq I$. As $n \geq 5$, we claim that there is at least one non-trivial idempotent of the form $E_{i_1i_1} + E_{i_2i_2} + \cdots + E_{i_ki_k} \in S$, for which $k \geq 3$. If $k = 1$, then $E_{i_1i_1} \in S$, and in this case, $I - E_{i_1i_1} \in S$ will be the required idempotent. If $k = 2$, then $E_{i_1i_1} + E_{i_2i_2} \in S$, and in this case, $I - (E_{i_1i_1} + E_{i_2i_2}) \in S$ will be the required idempotent.

Now consider $\alpha = E_{i_1i_1} + E_{i_2i_2} + \cdots + E_{i_ki_k} \in S$, where $k \geq 3$. Then $\alpha \cdot E_{i_1i_2} \cdot \alpha = E_{i_1i_2}$ and $\alpha \cdot E_{i_2i_k} \cdot \alpha = E_{i_1i_k}$. It is clear that $E_{i_2i_k} \cdot E_{i_1i_2} = 0$ and $E_{i_1i_2} \cdot E_{i_2i_k} = E_{i_1i_k}$. Hence $\alpha S \alpha$ is not reversible. Therefore by [4, Proposition 2.1.(2)], $S$ is not $i$-reversible. \qed

**Remark 3.8.** When $n = 4$, using the same argument as in the above theorem, one can easily see that if $S$ contains an idempotent of the form $E_{i_1i_1} + E_{i_2i_2} + \cdots + E_{i_ki_k}$ where $k = 1$ or $k = 3$, then $S$ is not $i$-reversible.

**Theorem 3.9.** Let $F$ be a field and $n \geq 3$. Let $S$ be a subring of $T_n(F)$ that strictly contains $D_n(F)$. Then $S$ contains a non-trivial idempotent.

**Proof.** Let $A \in S \setminus D_n(F)$ and $B = \text{diag}(A)$ be the corresponding diagonal matrix. By the same arguments as in Theorem 3.7, $B \in S$. Let $b_1, b_2, \cdots, b_k$ be the distinct diagonal entries of $B$. This implies that the minimal polynomial of $B$ is of degree $k$. Therefore, the set $\{I, B, B^2, \cdots, B^{k-1}\}$ is a linearly independent subset of $S$. Now, for each $m$ such that $1 \leq m \leq k$, let $E^{(m)} = [e^{(m)}_{ij}]$ be the diagonal matrix with $e^{(m)}_{ii} = \begin{cases} 1, & \text{if } B_{ii} = b_m \\ 0, & \text{if } B_{ii} \neq b_m \end{cases}$, where $B_{ii}$ is the $ii^{th}$ entry of $B$. It is clear that the set $\{E^{(1)}, E^{(2)}, \cdots, E^{(k)}\}$ is linearly independent. Let $S' = \text{span}\{E^{(1)}, E^{(2)}, \cdots, E^{(k)}\}$. It is not hard to see that the set $\{I, B, B^2, \cdots, B^{k-1}\} \subseteq S'$, and so, $\text{span}\{I, B, B^2, \cdots, B^{k-1}\} \subseteq S'$. As both span$\{I, B, B^2, \cdots, B^{k-1}\}$ and $S'$ are of dimension $k$ as vector spaces over $F$, we can conclude that span$\{I, B, B^2, \cdots, B^{k-1}\} = S'$. This implies $S'$ is a subset of $S$. Clearly each $E^{(m)} \in S$ is a non-trivial idempotent. \qed

**Theorem 3.10.** Let $F$ be a field and $n \geq 5$. Then $D_n(F)$ is a maximal $i$-reversible subring of $T_n(F)$.

**Proof.** By Theorem 3.1, $D_n(F)$ is $i$-reversible. Let $S$ be a subring of $T_n(F)$ such that $D_n(F) \not\subset S$. We have to show that $S$ is not $i$-reversible. By Theorem 3.7, it is enough to
show that $S$ contains a non-trivial idempotent. But this is true from Theorem 3.9. This completes the proof.

The following examples illustrate that Theorem 3.7 is not true if $n = 3$ or $4$.

**Example 3.11.** Let $F$ be a field and $S_3(F) = \left\{ [a_{ij}] \in T_3(F) \mid a_{11} = a_{22} \right\}$. It is clear that $S_3(F)$ is a subring of $T_3(F)$ such that $D_3(F) \subseteq S_3(F)$. It is also clear that $S_3(F)$ has non-trivial idempotents (for example $E_{11} + E_{22}$). We claim that $S_3(F)$ is i-reversible. Suppose $E = [e_{ij}]$ is a non-trivial idempotent in $S_3(F)$. This implies either $e_{11} = 1$ and $e_{33} = 0$ or $e_{11} = 0$ and $e_{33} = 1$. Further, the fact that $E^2 = E$ will imply $e_{12} = 0$. This will mean, $E$ is either of the form,

$$E_1 = \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad E_2 = \begin{pmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{23} \\ 0 & 1 & 0 \end{pmatrix}.$$

For $A, B \in S_3(F)$, suppose $AB$ is an idempotent of the form $E_1$, then the product of the first $2 \times 2$ blocks of $A$ and $B$ is identity and and $a_{33}b_{33} = 0$. This implies that the product of the first $2 \times 2$ blocks of $B$ and $A$ is identity and and $b_{33}a_{33} = 0$. So, $BA$ is again of the form $E_1$. Hence, $BA$ is an idempotent in $S_3(F)$. Again if $A$ and $B$ are such that $AB$ is an idempotent of the form $E_2$ then, the product of the first $2 \times 2$ blocks of $A$ and $B$ is the zero matrix and and $a_{33}b_{33} = 1$. Now since $F$ is reduced, by [6, Proposition 1.6], $T(F, F)$ is reversible. So, the product of the first $2 \times 2$ blocks of $B$ and $A$ is the zero matrix. Also $a_{33}b_{33} = 1$ implies $b_{33}a_{33} = 1$. Therefore, $BA$ is a matrix of the form $E_2$. Hence $BA$ is an idempotent in this case as well. This proves that $S_3(F)$ is i-reversible.

**Example 3.12.** Let $F$ be a field and $S_4(F) = \left\{ [a_{ij}] \in T_4(F) \mid a_{11} = a_{22} \text{ and } a_{33} = a_{44} \right\}$. It is clear that $S_4(F)$ is a subring of $T_4(F)$ such that $D_4(F) \nsubseteq S_4(F)$, and that $S_4(F)$ has non-trivial idempotents (for example $E_{11} + E_{22}$). We claim that $S_4(F)$ is i-reversible. Suppose $E = [e_{ij}]$ is a non-trivial idempotent in $S_4(F)$. This implies either $e_{11} = 1$ and $e_{33} = 0$ or $e_{11} = 0$ and $e_{33} = 1$. Further, the fact that $E^2 = E$ will imply $e_{12} = 0$ and $e_{34} = 0$. Therefore, $E$ is either of the form

$$E_1 = \begin{pmatrix} 1 & 0 & e_{13} & e_{14} \\ 0 & 1 & e_{23} & e_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad E_2 = \begin{pmatrix} 0 & 0 & e_{13} & e_{14} \\ 0 & 0 & e_{23} & e_{24} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

A similar argument as in the previous example will now show that $S_4(F)$ is i-reversible.
Remark 3.13. The above examples stand valid even if the base ring is any reduced ring with trivial idempotents instead of a field. However, we need the base ring to be a field in order to establish the following results.

Theorem 3.14. Let $F$ be a field. Then $S_3(F)$ is a maximal i-reversible subring of $T_3(F)$.

Proof. Firstly, it is clear from Example 3.11 that $S_3(F)$ is an i-reversible subring of $T_3(F)$.

Now suppose $S$ is any subring of $T_3(F)$ that strictly contains $S_3(F)$. Then since $S$ is also a vector subspace of $T_3(F)$ and $\dim_F(T_3(F)) = \dim_F(S_3(F)) + 1$, we can conclude that $S = T_3(F)$. Therefore, $S_3(F)$ is a maximal i-reversible subring of $T_3(F)$. □

Remark 3.15. From the above proof, we can actually conclude that $S_3(F)$ is a maximal subring of $T_3(F)$.

Theorem 3.16. Let $F$ be a field. Then $S_4(F)$ is a maximal i-reversible subring of $T_4(F)$.

Proof. It is clear from Example 3.12 that $S_4(F)$ is i-reversible. Let $S$ be a subring of $T_4(F)$ such that $S$ strictly contains $S_4(F)$. Let $A \in S \setminus S_4(F)$ and $B = \text{diag}(A)$. Let $b_{ii}$’s denote the diagonal entries of $B$. We now consider the following cases.

Case I: If $B$ has 3 or 4 distinct diagonal entries, then by the same arguments as in the proof of Theorem 3.9 and Remark 3.8, $S$ is not i-reversible.

Case II: If $B$ is such that $b_{11} = b_{22} = b_{33} = b$ and $b_{44} = a$, then clearly $B' = B - aI \in S$.

This implies $(b-a)^{-1}I \cdot B' = E_{11} + E_{22} + E_{33} \in S$. So, by Remark 3.8, $S$ is not i-reversible.

A similar argument can be applied if $B$ is such that $b_{11} = b_{22} = b_{44} = b$ and $b_{33} = a$, $b_{11} = b_{33} = b_{44} = b$ and $b_{22} = a$ or $b_{22} = b_{33} = b_{44} = b$ and $b_{11} = a$.

Case III: If $B$ is such that $b_{11} = b_{33} = a$ and $b_{22} = b_{44} = b$, then by similar arguments as in the previous case, we can conclude that $E_{11} + E_{33} \in S$. As $S$ contains $S_4(F)$, $E_{11} + E_{22} \in S$. So, $E_{11} = (E_{11} + E_{33}) \cdot (E_{11} + E_{22}) \in S$. Therefore, by Remark 3.8, $S$ is not i-reversible.

Case IV: If $B$ is such that $b_{11} = b_{44} = a$ and $b_{22} = b_{33} = b$, then by similar arguments as in the previous cases, we can conclude that $E_{11} + E_{44} \in S$. This implies $E_{11} \in S$. Again by Remark 3.8, $S$ is not i-reversible. □

4. Dorroh Extensions and Nagata Extensions.

In this section, we discuss the i-reversibility of Dorroh Extensions and Nagata Extensions.
**Definition 4.1.** Let \( R \) be an algebra over a commutative ring \( S \). The Dorroh extension of \( R \) by \( S \) is the ring \( R \times S \) with the following operations:

\[
(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)
\]

\[
(r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2).
\]

The Dorroh extension of \( R \) by \( S \) is a ring with unity \((0, 1)\).

**Theorem 4.2.** Let \( S \) be a commutative ring and \( R \) be an algebra (not necessarily unital) over \( S \) containing a non-zero central idempotent \( e \). Let \( D \) be the Dorroh extension of \( R \) by \( S \). Then \( D \) is i-reversible if and only if \( D \) is reversible.

**Proof.** Let \( D \) be i-reversible. Clearly, \((e, 0)\) is a non-trivial idempotent in \( D \). Now, \((r, s)(e, 0) = (re + se, 0) = (e, 0)(r, s)\), which makes \((e, 0)\) a central idempotent. Therefore, \( D \) is reversible (see [4, Proposition 2.1.(4)]). The reverse implication is clear. \( \square \)

The Dorroh extension is usually used to embed a non-unital ring into a unital ring. However, Alwis studied the Dorroh extension of \( Z \) over \( Z \) (see [14]). Cannon and Neuerburg [16] generalized these results to the Dorroh extension of \( R \) over \( Z \) where \( R \) is any unital ring (and hence a algebra over \( Z \)). More recently, Alhribat et al. further generalized these results to Dorroh extensions of a unital algebra \( R \) over a unital ring \( S \) (see [13] for details). The following theorem gives a necessary and sufficient condition for the i-reversibility of the Dorroh extension of a unital algebra \( R \) over a commutative unital ring \( S \).

**Theorem 4.3.** Let \( R \) be a unital algebra over a commutative unital ring \( S \) and \( D \) be the Dorroh extension of \( R \) by \( S \). The ring \( D \) is i-reversible if and only if \( R \) is reversible.

**Proof.** Since \( S \) is a commutative ring with unity and \( R \) is a unital algebra over \( S \), by [13 Proposition 1.3], we have \( D \cong S \times R \). We know that \( S \times R \) is i-reversible if and only if \( S \) and \( R \) are reversible. As \( S \) is commutative, we further have \( S \times R \) is i-reversible if and only if \( R \) is reversible.

**Remark 4.4.** Theorem [4,3] can be thought of as a particular case of Theorem 4.2 where \( e \) is the unity in \( R \). Therefore, the condition that \( R \) and \( S \) are reversible in Theorem 4.3 is also a necessary and sufficient condition for the reversibility of \( D \). \( \square \)
Definition 4.5. Let $R$ be a commutative ring, $M$ be an $R$-module and $\sigma$ be an endomorphism of $R$. Give $R \oplus M$ a (possibly non-commutative) ring structure with component-wise addition and multiplication defined by,

$$(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, \sigma(r_1) m_2 + r_2 m_1)$$

where $r_i \in R$ and $m_i \in M$. This extension is called the Nagata extension of $R$ by $M$ and $\sigma$.

Theorem 4.6. Let $R$ be a commutative ring and $N$ be the Nagata Extension of $R$ by $M$ and $\sigma$. If $R$ has only trivial idempotents then $N$ is i-reversible.

Proof. Let $\alpha \beta$ be an idempotent, where $\alpha = (a_1, b_1), \beta = (a_2, b_2) \in N$. We have,

$$\alpha \beta = (a_1 a_2, \sigma(a_1) b_2 + a_2 b_1)$$

$$(\alpha \beta)^2 = (a_1 a_2 a_1 a_2, \sigma(a_1 a_2)(\sigma(a_1) b_2 + a_2 b_1) + a_1 a_2 (\sigma(a_1) b_2 + a_2 b_1))$$

$$\beta \alpha = (a_2 a_1, \sigma(a_2) b_1 + a_1 b_2)$$

$$(\beta \alpha)^2 = (a_2 a_1 a_2 a_1, \sigma(a_2 a_1)(\sigma(a_2) b_1 + a_1 b_2) + a_2 a_1 (\sigma(a_2) b_1 + a_1 b_2))$$

Comparing the first component of $\alpha \beta$ and $(\alpha \beta)^2$, we get that $a_1 a_2$ is an idempotent in $R$. Hence, $a_1 a_2 = 0$ or $a_1 a_2 = 1$. If $a_1 a_2 = 0$, we have $\sigma(a_1 a_2) = 0$. This will imply $\alpha \beta = 0$ leaving nothing to prove. If $a_1 a_2 = 1$ and $\sigma(a_1 a_2) = 0$ then $\beta \alpha = (1, \sigma(a_2) b_1 + a_1 b_2) = (\beta \alpha)^2$. So, in this case $N$ is i-reversible. Finally, if $a_1 a_2 = 1$ and $\sigma(a_1 a_2) = 1$ then comparing the second component of $\alpha \beta$ and $(\alpha \beta)^2$, we have $\sigma(a_1) b_2 + a_2 b_1 = 0$. Multiplying throughout by $\sigma(a_2)$, we get $b_2 + \sigma(a_2) a_2 b_1 = 0$. Further multiplying this equation by $a_1$, we get $b_2 a_1 + \sigma(a_2) b_1 = 0$. Therefore, $\beta \alpha = (1, 0) = (\beta \alpha)^2$ and $N$ is i-reversible. \qed

Remark 4.7. The converse of the above theorem is not true. For example if $\sigma = Id_R$, where $R$ is a commutative ring with non-trivial idempotents (say $\mathbb{Z}_6$) then the Nagata extension of $R$ by $R$ and $\sigma$ is just $T(R, R)$. The commutativity of $R$ will imply the commutativity of $T(R, R)$ and hence its i-reversibility. We further provide an example where $\sigma \neq Id_R$.

Example 4.8. Let $D$ be a commutative domain of characteristic zero, and $R = D \oplus D$ with component-wise addition and multiplication. Define $\sigma : R \rightarrow R$ by $\sigma(s, t) = (t, s)$, then $\sigma$ is an automorphism of $R$. Since $D$ is a domain, the only idempotents in $R$ are $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$. Let $N$ denote the Nagata extension of $R$ by $R$ and $\sigma$. Let $\alpha = (r_1, m_1), \beta = (r_2, m_2) \in N$ for some $r_1, m_1, r_2, m_2 \in R$ such that $\alpha \beta$ is a non-zero
idempotent. Now, \((\alpha\beta)^2 = \alpha\beta\) implies
\[
((r_1r_2)^2, \sigma(r_1r_2)(\sigma(r_1)m_2 + r_2m_1) + r_1r_2(\sigma(r_1)m_2 + r_2m_1)) = (r_1r_2, \sigma(r_1)m_2 + r_2m_1)
\]
and therefore, \(r_1r_2\) is an idempotent in \(R\). If \(r_1r_2 = (0,0)\) it gives \((\alpha\beta)^2 = 0\) and hence \(\alpha\beta = 0\). This contradicts the fact that \(\alpha\beta\) is a non-zero idempotent. If \(r_1r_2 = (1,0)\) or \((0,1)\), a direct computation gives \(\beta\alpha = (\beta\alpha)^2\). Finally, if \(r_1r_2 = (1,1)\), letting \(r_1 = (r'_1, \tilde{r}_1), r_2 = (r'_2, \tilde{r}_2), m_1 = (m'_1, \tilde{m}_1)\) and \(m_2 = (m'_2, \tilde{m}_2)\) we get,
\[
\beta\alpha = ((1,1), (\tilde{r}_2m'_1 + r'_1m'_2, r'_2\tilde{m}_1 + \tilde{r}_1\tilde{m}_2))
\]
\[
(\beta\alpha)^2 = ((1,1), 2(\tilde{r}_2m'_1 + r'_1m'_2, r'_2\tilde{m}_1 + \tilde{r}_1\tilde{m}_2)).
\]
However, since \(\alpha\beta\) is an idempotent, comparing the second component of \(\alpha\beta\) and \((\alpha\beta)^2\) gives
\[
\tilde{r}_1m'_2 + r'_2m'_1 = 0. \tag{4.1}
\]
Multiplying (4.1) from the left side with \((\tilde{r}_2r'_1)\) yields
\[
r'_1m'_2 + \tilde{r}_2m'_1 = 0. \tag{4.2}
\]
Again, multiplying (4.1) from the left side with \((r'_2\tilde{r}_1)\) we get
\[
\tilde{r}_1\tilde{m}_2 + r'_2\tilde{m}_1 = 0. \tag{4.3}
\]
Adding (4.2) and (4.3), we get
\[
r'_1m'_2 + \tilde{r}_2m'_1 + \tilde{r}_1\tilde{m}_2 + r'_2\tilde{m}_1 = 0.
\]
Therefore, \((\beta\alpha)^2 = (\beta\alpha)\) and hence \(N\) is i-reversible.

5. I-reversibility of Polynomial and Laurent Polynomial Rings.

In [4, Proposition 2.1(7)] and [5, Example 1.10], the authors provide an example of a reversible ring \(S\) such that \(S[x]\) and \(S[x, x^{-1}]\) are not i-reversible. In this section, we provide an example of a non-reversible, i-reversible ring \(S\) such that \(S[x]\) and \(S[x, x^{-1}]\) are not i-reversible. We further give two independent sufficient conditions for the polynomial ring and Laurent polynomial ring to be i-reversible.

We need the following result from [4],

**Proposition 5.1.** [4, Corollary 3.2] For any ring \(R\) and any \(n > 1\), \(T_n(R)\) is i-reversible if and only if \(n = 2\) and \(R\) is reversible and has only trivial idempotents.

We give an example of a non-reversible, i-reversible ring \(S\) such that \(S[x]\) is not i-reversible.
Example 5.2. In [6, Example 2.1.], Kim and Lee provide a reversible ring $R$ such that $R[x]$ is not reversible. Further, Jung et.al in [5, Example 2.1] show that $R$ has only trivial idempotents. By Proposition 5.1, $S = T_2(R)$ is i-reversible. Clearly, $S$ is not reversible (for example $E_{12}E_{11} = 0$ and $E_{11}E_{12} \neq 0$). Next, we show $S[x]$ is not i-reversible. Since $S[x] \cong \left[ \begin{array}{cc} R[x] & R[x] \\ 0 & R[x] \end{array} \right]$, if $S[x]$ were i-reversible then $R[x]$ would be reversible (by Proposition 5.1), which is a contradiction. Hence, $S[x]$ is not i-reversible. Also, $S[x, x^{-1}]$ is not i-reversible since any subring of an i-reversible ring is i-reversible.

We need the following result from [10],

Theorem 5.3. [10, Theorem 5] For a ring $R$ the following conditions are equivalent,

1. $R$ is abelian.
2. Idempotents of $R$ commute with units of $R$.
3. The set of idempotents of $R[[x]]$ is equal to the set of idempotents of $R$.
4. The set of idempotents of $R[x, x^{-1}]$ is equal to the set of idempotents of $R$.
5. The set of idempotents of $R[x]$ is equal to the set of idempotents of $R$.
6. There exists $n \geq 1$ such that $R[x]$ does not contain idempotents which are polynomials of degree $n$.

Theorem 5.4. If a ring $R$ has only trivial idempotents then $R[x]$, $R[[x]]$ and $R[x, x^{-1}]$ are i-reversible.

Proof. Since, the only idempotents of $R$ are trivial, $R$ is an Abelian ring. Therefore, idempotents of $R[x]$, $R[[x]]$ and $R[x, x^{-1}]$ are exactly idempotents of $R$ (see [10, Theorem 5]). Therefore $R[x]$, $R[[x]]$ and $R[x, x^{-1}]$ are rings with only trivial idempotents and hence i-reversible.

Theorem 5.5. Let $R$ be an abelian ring. Then $R[x]$ is i-reversible if and only if $R[x, x^{-1}]$ is i-reversible.

Proof. The only case which is not obvious is when $R$ has a non-trivial idempotent and $R[x]$ is i-reversible. In this case, $R$ becomes reversible and the proof follows from [6, Lemma 2.2.]

Definition 5.6. A ring $R$ is said to be Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_m x^m$; $g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $i, j$. 
**Theorem 5.7.** Let \( R \) be an Armendariz ring, then the following statements are equivalent:

1. \( R \) is i-reversible.
2. \( R[x] \) is i-reversible.
3. \( R[x, x^{-1}] \) is i-reversible.

**Proof.** If \( R \) is a ring with only trivial idempotents, the result is clear in view of [10, Theorem 5.]. If \( R \) contains a non-trivial idempotent then \( R \) is reversible (see [4, Proposition 2.1.(4)]). The result is evident in view of [6, Proposition 2.3]. \( \square \)

**Example 5.8.** Let \( S \) be a commutative ring with only trivial idempotents. \( R = T_2(S) \) is not Armendariz (see [11, Example 1]). By Proposition 5.1 we have \( R[x] \cong \begin{bmatrix} S[x] & S[x] \\ 0 & S[x] \end{bmatrix} \) is i-reversible.

The following examples show that \( R \) containing only trivial idempotents and \( R \) being Armendariz are independent conditions.

**Example 5.9.** Let \( S \) be a reduced ring with a non-trivial idempotent \( e \) (for example \( \mathbb{Z}_6 \)). Then the ring \( R = T(S, S) \) is an Armendariz ring (see [11, Corollary 4]) with non-trivial idempotents (for example the diagonal matrix with diagonal entries \( e \)).

**Example 5.10.** Let \( D \) be a domain, \( S = T(D, D) \) and \( R = T(S, S) \) which is a ring with only trivial idempotents (see Theorem 2.6). However, \( R \) is not Armendariz (see [11, Example 5]).

### 6. Skew Polynomial Rings

In this section, we ask whether Theorem 5.4 and Theorem 5.7 generalize to the case of skew polynomial rings \((R[x; \sigma])\). We also provide conditions on \( R \) and \( \sigma \) for which \( R[x; \sigma] \) is not i-reversible.

**Definition 6.1.** Let \( R \) be a ring and \( \sigma \) be an endomorphism of \( R \) such that \( \sigma(1) = 1 \). We construct a skew polynomial ring by stipulating that for \( b \in R \), \( xb = \sigma(b)x \) on the set of left polynomials over \( R \) and multiplication between elements in \( R \) defined in the usual sense. The ring of skew polynomials over \( R \) is denoted by \( R[x; \sigma] \).

**Remark 6.2.** The following statements are easy to observe,

1. Skew polynomial rings can be similarly constructed on the set of right polynomials over \( R \) by stipulating \( bx = x\sigma(b) \). In this case, we can observe that \( R[x, \sigma]/(x^2) \) is isomorphic to the Nagata extension of \( R \) by \( R \) and \( \sigma \).
(2) Suppose \( R \) is a ring in which \( \sigma(1) = 1 \) implies \( \sigma = Id_R \) (for example \( \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_n \)), then \( R[x;\sigma] = R[x] \).

We first give a sufficient condition for the i-reversibility of \( R[x;\sigma] \). For that, we need the following Lemma.

**Lemma 6.3.** Let \( R \) be a ring and \( \sigma \) be an endomorphism of \( R \) which fixes a central idempotent \( e \). Then any idempotent in \( R[x;\sigma] \) with constant term \( e \) has to be \( e \) itself.

**Proof.** Let \( p(x) = e + d_1 x + d_2 x^2 + \cdots + d_k x^k \in R[x;\sigma] \) be an idempotent. Since \((p(x))^2 = p(x)\) and \( e \) is a central idempotent fixed by \( \sigma \), comparing the coefficient of \( x \) in \((p(x))^2 \) and \( p(x) \), we get \((2e - 1)d_1 = 0\). Since \( 2e - 1 \) is a unit, we get \( d_1 = 0 \). Similarly, for all \( i \geq 2 \), comparing the coefficients of \( x^i \) recursively gives \( d_2 = d_3 = \cdots = d_k = 0 \). Therefore, \( p(x) = e \). \( \square \)

**Theorem 6.4.** Let \( R \) be a ring with only trivial idempotents then \( R[x;\sigma] \) is i-reversible.

**Proof.** Let \( f(x)g(x) \) be a non-zero idempotent in \( R[x;\sigma] \), where \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \). Then \( a_0 b_0 \) is an idempotent in \( R \). Since \( R \) has only trivial idempotents the choices for \( a_0 b_0 \) are 0 and 1. But by Lemma 6.3 and the fact that \( f(x)g(x) \neq 0 \) we get, \( a_0 b_0 = 1 \) and hence \( f(x)g(x) = 1 \). This proves that \( g(x)f(x) \) is an idempotent making \( R[x;\sigma] \) i-reversible. \( \square \)

We now see conditions under which \( R[x;\sigma] \) is not i-reversible.

**Theorem 6.5.** Let \( R \) be an i-reversible ring with a non-trivial central idempotent \( e \). If \( \sigma \) is a non-injective endomorphism of \( R \) such that \( \sigma(e) = e \) then \( R[x;\sigma] \) is not i-reversible.

**Proof.** Clearly, \( e \) is a non-trivial idempotent in \( R[x;\sigma] \). Also, \( x^k e = \sigma^k(e)x^k = ex^k \) for any \( k \in \mathbb{N} \). Therefore, \( e \) is a non-trivial central idempotent in \( R[x;\sigma] \). Assume \( R[x;\sigma] \) is i-reversible. The fact that \( e \) is a non-trivial central idempotent implies \( R[x;\sigma] \) is reversible (see [4, Proposition 2.1.(4)]). As \( \sigma \) is a non-injective, there exists \( b \neq 0 \in R \) such that \( \sigma(b) = 0 \). Clearly, \( bx \neq 0 \) and \( xb = 0 \), which is a contradiction. Therefore, \( R[x;\sigma] \) is not i-reversible. \( \square \)

We now give two examples to show that the conditions on \( \sigma \) from Theorem 6.5 are not sufficient conditions for the non i-reversibility of \( R[x;\sigma] \).

**Example 6.6.** Let \( R \) be the ring of real sequences with pointwise addition and multiplication as the operations. Let \( \sigma \) be the endomorphism of \( R \) given by \( \sigma(a_1,a_2,a_3,a_4,\cdots) = \cdots \).
(a_2, a_3, a_4, \cdots). Then \( \sigma \) is clearly non-injective and does not fix any non-trivial idempotent in \( R \). Consider the polynomials \( f(x) = (0, 1, 1, 0, 0, \cdots) + (0, 1, 0, 0, \cdots) \) and \( g(x) = (0, 1, 1, 0, 0, \cdots) \) in \( R[x, \sigma] \). Then \( fg \) is a non-zero idempotent but \( gf = f \) is not. Thus \( R[x, \sigma] \) is not i-reversible.

Example 6.7. Let \( S \) be a ring with a non-trivial central idempotent \( e \) and let \( R = S \times S \). Let \( \sigma \) be the endomorphism on \( R \) such that \( \sigma(a, b) = (b, a) \). It is not hard to see that \((e, e)\) is a non trivial central idempotent in \( R[x, \sigma] \). Suppose, \( R[x, \sigma] \) is i-reversible. By [4, Proposition 2.1.(4)], \( R[x, \sigma] \) is reversible. This gives a contradiction since we have \( f = (1, 0)x \) and \( g = (1, 0) \in R[x, \sigma] \) such that \( fg = 0 \) however \( gf \neq 0 \).

The following example shows that unlike polynomial rings, \( R \) being i-reversible and Armendariz is not a sufficient condition for i-reversibility of \( R[x; \sigma] \).

Example 6.8. Let \( S \) be a reduced ring with a non-trivial central idempotent \( e \). Let \( R = T(S, S) \) with the endomorphism \( \sigma \) defined by \( \sigma \left[ \begin{array}{cc} a & b \\ 0 & a \end{array} \right] = \left[ \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right] \). Since \( S \) is reduced, \( R \) is Armendariz (see [11, Corollary 4]) and reversible (see [6, Proposition 1.6]). Using the fact that \( \sigma \) fixes the non-trivial central idempotent \( \left[ \begin{array}{cc} e & 0 \\ 0 & e \end{array} \right] \) and by Theorem 6.5, \( R[x; \sigma] \) is not i-reversible.

We now provide a class of rings \( R \) which may contain non-trivial idempotents yet, \( R[x; \sigma] \) is i-reversible (c.f. Example 6.12).

Definition 6.9. Let \( R \) be a ring with an endomorphism \( \sigma \). \( R \) is called \( \sigma \)-Armendariz if for polynomials \( p(x) = \sum_{i=0}^{m} a_i x^i \) and \( q(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma] \), \( p(x)q(x) = 0 \) implies \( a_i b_j = 0 \) for all \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \).

Theorem 6.10. Let \( R \) be an i-reversible ring. If \( R \) is \( \sigma \)-Armendariz then \( R[x; \sigma] \) is i-reversible.

Proof. Let \( R \) be an i-reversible \( \sigma \)-Armendariz ring. If \( R \) has only trivial idempotents then \( R[x; \sigma] \) is i-reversible (see Theorem 6.4). Let \( R \) contain a non-trivial idempotent \( e \). Now, \( R \) is abelian since \( R \) is \( \sigma \)-Armendariz, (see [12, Lemma 1.1]). Hence \( e \) is a central idempotent and \( R \) is reversible (see [11, Proposition 2.1.(4)]). Finally, \( R \) is a \( \sigma \)-Armendariz and reversible implies \( R[x; \sigma] \) is reversible (see [8, Theorem 5]) and hence i-reversible. \( \Box \)

Definition 6.11. Let \( R \) be a ring with an endomorphism \( \sigma \). Then \( R \) is called \( \sigma \)-rigid if for any \( a \in R \) with \( a \sigma(a) = 0 \), we have \( a = 0 \).
Example 6.12. Let $S$ be an $\alpha$-rigid ring where $\alpha$ is an endomorphism of $S$ and let $R = S \times S$ with component-wise addition and multiplication. Since $S$ is an $\alpha$-rigid ring hence $S$ is reduced (see [17]). Therefore, $R$ is reversible. Consider the endomorphism $\sigma$ of $R$, defined by $\sigma(a, b) = (\alpha(a), \alpha(b))$. In view of [12] Example 1.4(2), we have, $R$ is $\sigma$-rigid and hence $\sigma$-Armendariz (see [12, Lemma 1.3]). It is clear that $R$ contains non-trivial idempotents.

Example 6.13. The ring $R$ in Example 5.10 contains only trivial idempotents. However, $R$ is not $\text{Id}_R$-Armendariz.

Remark 6.14. The above examples (Example 6.12 and Example 6.13) show that $R$ containing only trivial idempotents and $R$ being $\sigma$-Armendariz are independent conditions.

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Department of Mathematics, St. Joseph’s College (Autonomous), Bangalore, India

*Email address*: v.bhabani.lama@gmail.com

Department of Mathematics, St. Joseph’s College (Autonomous), Bangalore, India

*Email address*: suhas.b.n@sjc.ac.in

Department of Mathematics, St. Joseph’s College (Autonomous), Bangalore, India

*Email address*: susobhan@sjc.ac.in

Department of Mathematics, St. Joseph’s College (Autonomous), Bangalore, India

*Email address*: raisadsouza@sjc.ac.in