The absence of the Kerr black hole
in the Hořava-Lifshitz gravity

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Abstract

We show that the Kerr metric does not exist as a fully rotating black hole solution to the modified
Hořava-Lifshitz (HL) gravity with $\Lambda_V = 0$ and $\lambda = 1$ case. We perform it by showing that the
Kerr metric does not satisfy full equations derived from the modified HL gravity.

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1. INTRODUCTION

Hořava has proposed a renormalizable theory of modified gravity at a Lifshitz point \[^1,2\], which may be regarded as a UV complete candidate for general relativity. At short distance of the UV scale, the Hořava-Lifshitz (HL) gravity describes interacting non-relativistic gravitons and is supposed to be power counting renormalizable in four dimensions. However, we would like to stress that it belongs to a Lorentz-violating gravity theory even though the Lorentz-symmetry is expected hopefully to be recovered in the IR limit.

On the other hand, its black hole solutions has been intensively investigated in \[^4,19\]. Concerning spherically symmetric solutions, Lü-Mei-Pope (LMP) have obtained the black hole solution with dynamical parameter \(\lambda\) \[^4\] and topological black holes were found in \[^5\]. Its thermodynamics were studied in \[^10\], but there remain unclear issues in obtaining the ADM mass and entropy because their asymptotic spacetimes is Lifshitz \[^8\]. On the other hand, Kehagias and Sfetsos (KS) have found the \(\lambda = 1\) black hole solution in asymptotically flat spacetimes considering the modified HL gravity \[^9\]. Its thermodynamics was investigated in Ref.\[^12,17\]. Park has obtained a \(\lambda = 1\) black hole solution with \(\omega\) and \(\Lambda_W\) \[^14\].

It is very interesting to find a fully rotating black hole solution in the HL gravity since the HL gravity is being considered as a promising modified gravity which violates the Lorentz-symmetry. However, it is a formidable task to find a fully rotating solution because equations of motion to be solved are very complicated. Fortunately, slowly rotating black holes based on the KS \[^20\] and LMP \[^21\] solutions were found in the HL gravity. Here “slowly rotating black hole” means that one considers up to linear order of rotation parameter \(a = J/M\) in the metric functions, equations of motion, and thermodynamic quantities. This implies that the case of \(a \ll 1\) is valid for slowly rotating black hole. We mention that the slowly rotating Kerr black hole could be recovered from the slowly rotating KS black hole solutions in the limit of \(\omega \to \infty\). In this case, the role of \(\omega\) is neglected and thus, the HL gravity reduces to general relativity.

The above case is similar to the parity-violating Chern-Simons (CS) modified gravity \[^22\]. Since the parity-violation may imply the breaking of Lorentz symmetry, the CS modified gravity may belong to the Lorentz-violating theory. It is well known that the CS term could not yield the Kerr metric as a fully rotating black hole solution since the Pontryagin constraint \(*RR = 0\) required by the Bianchi identity is not satisfied. Up to now, the slowly
rotating Kerr black hole is known to be the only solution to the CS modified gravity [23].

Recently, the Penrose process on rotational energy extraction of the black hole in the HL gravity was studied by considering that the Kerr solution is a truly rotating solution to the HL gravity [24]. This approach might be incorrect because the Kerr solution is not yet proved to be a fully rotating solution to the modified HL gravity.

In this work, we wish to show that the Kerr metric does not exist as a fully rotating solution to the modified HL gravity because Lorentz-violating higher order terms are present. Our strategy is to prove the non-existence by plugging the Kerr metric directly to full equations derived from the modified HL gravity. In achieving the non-existence of the Kerr black hole, the order of rotation parameter $a$ plays an important role in classifying full equations. This implies that three full equations are classified according to the order of $a$ and then, we check whether or not each equation in $a$-order is satisfied. This approach is inspired by finding the slowly rotating black hole.

2. HL GRAVITY

In this section, we review briefly the modified HL gravity including a soft-violation term. The ADM formalism implies that the four-dimensional metric of general relativity is parameterized as

$$ds_4^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \quad (1)$$

where the lapse $N$, shift $N^i$, and three-dimensional space metric $g_{ij}$ are all functions of $t$ and $x^i$. The ADM decomposition of the Einstein-Hilbert action with a cosmological constant $\Lambda$ is given by

$$S_{EH} = \frac{1}{16\pi G} \int d^4 x \sqrt{g} N \left( K_{ij} K^{ij} - K^2 + R - 2\Lambda \right), \quad (2)$$

where $G$ is Newton’s constant and extrinsic curvature $K_{ij}$ is defined by

$$K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right). \quad (3)$$

Here a over-dot denotes a derivative with respect to $t$.

The action of the modified HL theory including a soft-violation term $\mu^4 R$ is given by

$$S = \int dt d^3 x \left( \mathcal{L}_0 + \mathcal{L}_1 \right), \quad (4)$$
\[ \mathcal{L}_0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3 \Lambda_W^2)}{8(1 - 3\lambda)} + \mu^4 R \right\}, \tag{5} \]
\[ \mathcal{L}_1 = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2W^4} \left( C_{ij} - \frac{\mu W^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu W^2}{2} R^{ij} \right) \right\}, \tag{6} \]
where \( \lambda, \kappa, \mu, \) and \( W \) are constant parameters to represent the modified HL gravity and \( \Lambda_W \) is a negative cosmological constant. Here, \( \mu^4 \) can be expressed in terms of \( \omega \) as
\[ \mu^4 = \frac{\kappa^2 \mu^2 \omega}{8(3\lambda - 1)}, \quad \omega = \frac{8(3\lambda - 1) \mu^2}{\kappa^2} \tag{7} \]
and \( C_{ij} \) is the Cotton tensor defined by
\[ C^{ij} = \varepsilon^{ik\ell} \nabla_k \left( R^i_{\ell} - \frac{1}{4} R \delta^i_{\ell} \right). \tag{8} \]
Since we wish to find a non-spherical solution of the black hole, we have to know full equations of motion to the action (4), which are composed of three equations. The equation from variation of the lapse function \( N \) is given by
\[ \frac{2}{\kappa^2} (K_{ij} K^{ij} - K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W - \omega) R - 3 \Lambda_W^2}{8(1 - 3\lambda)} + \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2W^4} Z_{ij} Z^{ij} = 0 \tag{9} \]
with
\[ Z_{ij} \equiv C_{ij} - \frac{\mu W^2}{2} R_{ij}. \tag{10} \]
We will focus on this lapse equation (9) for testing the Kerr metric as the solution to the HL gravity. The variation \( \delta N^i \) implies an equation
\[ \nabla_k (K^{k\ell} - \lambda K g^{k\ell}) = 0. \tag{11} \]
Equation of motion from variation of \( \delta g^{ij} \) is complicated to be
\[ \frac{2}{\kappa^2} E_{ij}^{(1)} - \frac{2\lambda}{\kappa^2} E_{ij}^{(2)} + \frac{\kappa^2 \mu^2 (\Lambda_W - \omega)}{8(1 - 3\lambda)} E_{ij}^{(3)} + \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} E_{ij}^{(4)} - \frac{\mu \kappa^2}{4W^2} E_{ij}^{(5)} - \frac{\kappa^2}{2W^4} E_{ij}^{(6)} = 0, \tag{12} \]
where
\[ E_{ij}^{(1)} = N_i \nabla_k K^k_j + N_j \nabla_k K^k_i - K^k_i \nabla_j N_k - K^k_j \nabla_i N_k - N^k \nabla_k K_{ij} - 2N K_{ik} K^k_j - \frac{1}{2} N K^{kl} K_{ik} g_{lj} + N K K_{ij} + K_{ij}, \]
\[ E_{ij}^{(2)} = \frac{1}{2} N K^2 g_{ij} + N_i \partial_j K + N_j \partial_i K - N^k (\partial_i K) g_{kj} + K g_{ij}, \]
\[ E_{ij}^{(3)} = N \left( R_{ij} - \frac{1}{2} R g_{ij} + \frac{3}{2} (\Lambda_W - \omega) g_{ij} \right) - (\nabla_i \nabla_j - g_{ij} \nabla_k \nabla^k) N, \]
\[ E_{ij}^{(4)} = N R \left( 2R_{ij} - \frac{1}{2} R g_{ij} \right) - 2(\nabla_i \nabla_j - g_{ij} \nabla_k \nabla^k) (N R), \]
\[ E_{ij}^{(5)} = \nabla_k \left[ \nabla_j (N Z^k_i) + \nabla_i (N Z^k_j) \right] - \nabla_k \nabla^k (NZ_{ij}) - \nabla_k \nabla_{\ell} (NZ^k_{\ell}) g_{ij}, \]
\[ E_{ij}^{(6)} = \nabla_k \left[ \nabla_j (NZ^k_i) + \nabla_i (N Z^k_j) \right] - \nabla_k \nabla^k (NZ_{ij}) - \nabla_k \nabla_{\ell} (NZ^k_{\ell}) g_{ij}, \]
\begin{align*}
E_{ij}^{(e)} &= - \frac{1}{2} NZ_{\ell k} Z^{\mu \ell} g_{ij} + 2NZ_{ik} Z_{j}^{k} - N(Z_{ik} C_{j}^{k} + Z_{jk} C_{i}^{k}) + NZ_{\ell k} C^{\mu \ell} g_{ij} \\
&\quad - \frac{1}{2} \nabla_k [N \epsilon^{\mu \nu \kappa \ell}] Z_{\nu \kappa} + \frac{1}{2} R^{n \ell} \nabla_n [N \epsilon^{\mu \nu \kappa \ell}] Z_{\nu \kappa} g_{ij} \\
&\quad - \frac{1}{2} \nabla_n [NZ_{m}^{\mu \nu \kappa \ell}(g_{ki} R_{j\ell} + g_{kj} R_{i\ell})] - \frac{1}{2} \nabla_n \nabla^n \nabla_k [N \epsilon^{\mu \nu \kappa \ell} (Z_{mi} g_{kj} + Z_{mj} g_{ki})] \\
&\quad + \frac{1}{2} \nabla_n \nabla_i \nabla_k (NZ_{m}^{\mu \nu \kappa \ell} g_{ij}) - \nabla_j \nabla_k (NZ_{m}^{\mu \nu \kappa \ell} g_{ij}) \\
&\quad - \nabla_n \nabla_i \nabla_k (NZ_{m}^{\mu \nu \kappa \ell} g_{ij}).
\end{align*}

We note that in deriving these equations, we have relaxed both the projectability and detailed-balance conditions since the lapse function $N$ depends on the spatial coordinate $x^i$ as well as the soft-violation term of $\mu^4 R$ is included.

Hereafter, we will focus on the case of $\Lambda^W = 0$, $\lambda = 1$ and $\omega = 16 \mu^2 / \kappa^2$, providing asymptotically flat spacetimes. In this case, we may have the Minkowski background as vacuum solution with \[9\]
\begin{equation}
\frac{c^2}{\kappa^2} = \frac{\mu^4}{2}, \quad \frac{G}{32\pi c} = \frac{\kappa^2}{\mu^4}, \quad A = 0. \quad (13)
\end{equation}

3. KEHAGIAS-SFETTSOS AND ITS SLOWLY ROTATING SOLUTIONS

In this section, we investigate how the KS and its slowly rotating black holes are derived in asymptotically flat spacetimes. First of all, we would like to find a static solution, the KS solution by considering a spherically symmetric line element
\begin{equation}
ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (14)
\end{equation}

In this case, we find that $K_{ij} = 0$ and $C_{ij} = 0$. Equation for the lapse function $N$ can be read as
\begin{equation}
\mu^4 \left[ R - \frac{2}{\omega} \left( R_{ij}^2 - \frac{3}{8} R^2 \right) \right] = 0. \quad (15)
\end{equation}

This leads to the first-order equation for $f$ as
\begin{equation}
\frac{(f - 1)^2}{r^2} - \frac{2(f - 1)f'}{r} - 2\omega (1 - f - rf') = 0. \quad (16)
\end{equation}
The equation (11) from the shift function $\delta N^i$ is trivially satisfied. The equation from $\delta g^{ij}$ reduces to
\[
\frac{\mu^2 \kappa^2}{16} \left[ \omega E^{(3)}_{ij} + \frac{3}{4} E^{(4)}_{ij} \right] - \frac{\mu \kappa^2}{4W^2} \left[ E^{(5)}_{ij} + \frac{2}{\mu W^2} E^{(6)}_{ij} \right] = 0. \tag{17}
\]

The $(rr)$-component of the above equation becomes
\[
- \frac{\mu^2 \kappa^2}{32} \frac{1}{r^4 f} \left[ N' 4rf \left( -1 + f - \omega^2 r^2 \right) - N \left( 1 + f^2 - 2\omega r^2 - 2f + 2\omega r^2 f \right) \right] = 0, \tag{18}
\]
which leads to
\[
\frac{N'}{N} = \frac{1 + f^2 - 2\omega r^2 - 2f + 2\omega r^2 f}{4rf \left( -1 + f - \omega^2 r^2 \right)}. \tag{19}
\]

On the other hand, from (16) we have $f'$ as
\[
f' = \frac{1 + f^2 - 2\omega r^2 - 2f + 2\omega r^2 f}{2r \left( -1 + f - \omega^2 r^2 \right)}. \tag{20}
\]
Substituting this into (19), we obtain
\[
\frac{N'}{N} = \frac{1}{2} \frac{f'}{f}, \tag{21}
\]
which implies
\[
N^2 = f. \tag{22}
\]
Hence, the KS solution to Eq. (16) is given by
\[
f_{KS} = N^2_{KS} = 1 + \omega r^2 \left( 1 - \sqrt{1 + \frac{4M}{\omega r^3}} \right). \tag{23}
\]

In the limit of $\omega \to \infty$ (equivalently, the decoupling limit of higher curvature terms), it reduces to the Schwarzschild form of
\[
f_{Sch}(r) = 1 - \frac{2M}{r}. \tag{24}
\]

Now we consider an axisymmetric metric for finding a slowly rotating KS black hole
\[
ds_{sr}^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta \left[ d\phi^2 - 2aN^\phi(r) dtd\phi \right]. \tag{25}
\]
Then, the extrinsic curvature tensor is found to be
\[
K_{ij} = \begin{pmatrix}
0 & 0 & -\frac{1}{2} \frac{r^2 a \sin^2 \theta}{\sqrt{f(r)}} \frac{dN^\phi(r)}{dr} \\
0 & 0 & 0 \\
-\frac{1}{2} \frac{r^2 a \sin^2 \theta}{\sqrt{f(r)}} \frac{dN^\phi(r)}{dr} & 0 & 0
\end{pmatrix}, \tag{26}
\]
where \( K_{r\phi} = K_{\phi r} \neq 0 \) are linear order of \( a \), but \( K = 0 \). We note here that all components of Cotton tensor still vanish \( (C_{ij} = 0) \), since it is constructed from the metric \( g_{ij} \) which does not include the rotation parameter \( a \). This implies that if one wishes to find a fully rotating black hole, all higher order terms of \( a \) must be included. Using (26), Eq. (11) reduces to

\[
\nabla_k K^{k\ell} = 0 \rightarrow \text{diag}\left[0, 0, \frac{a\sqrt{f(r)}}{2r} \left(r \frac{d^2 N^\phi(r)}{dr^2} + 4 \frac{dN^\phi(r)}{dr}\right)\right] = 0,
\]

which has a solution with two unknown constants \( C_1 \) and \( C_2 \)

\[
N^\phi(r) = C_1 + \frac{C_2}{r^3}.
\]

For later convenience, we choose the shift function to be

\[
N^\phi(r) = \frac{2M}{r^3}
\]

with \( C_1 = 0 \) and \( C_2 = 2M \). In this case, one has the \( g_{t\phi} \)-component

\[
g_{t\phi} = -ar^2 N^\phi(r) \sin^2 \theta = -\frac{2J}{r} \sin^2 \theta.
\]

Plugging these into Eq. (9) leads to

\[
\mu^4 \left[R + \frac{2}{\omega} \left(R_{ij}^2 - \frac{3}{8} R^2\right)\right] = -\frac{2}{\kappa^2} \left(K_{ij} K^{ij} - K^2\right)
\]

which takes the form

\[
\frac{(f-1)^2}{r^2} - \frac{2(f-1)f'}{r} - 2\omega (1 - f - rf') = \frac{32a^2 \omega M^2 \sin^2 \theta}{\kappa^4 \mu^4 r^4} \simeq 0.
\]

Here, we take the right-hand side to be zero effectively because it is second order of \( a \). Then, the solution is given by the KS-type in Eq. (23).

Consequently, the slowly rotating KS black hole solution is given by

\[
ds_{sr\text{KS}}^2 = -f_{\text{KS}}(r) dt^2 + \frac{dr^2}{f_{\text{KS}}(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta \left(d\phi^2 - \frac{4J}{r^3} dt d\phi\right).
\]

In the limit of \( \omega \rightarrow \infty \), it leads to the slowly rotating Kerr black hole as

\[
ds_{sr\text{Kerr}}^2 = -f_{\text{Sch}}(r) dt^2 + \frac{dr^2}{f_{\text{Sch}}(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta \left(d\phi^2 - \frac{4J}{r^3} dt d\phi\right).
\]

In this sense, we wish to clarify that the slowly rotating Kerr black hole is not the solution to the HL gravity but the Einstein gravity, on the contrary to Ref. [25].
Finally, we explain why the slowly rotating solution is naturally obtained for the HL gravity by examining the order of rotation parameter $a$ in equations of motion. The axisymmetric metric ansatz (25) was implemented by one-component shift vector of $N^\phi$. Hence, extrinsic curvature $K_{ij}$ has off-diagonal components as shown in (26). This implies that equation (31) obtained from $\delta N$ remains unchanged when adding a rotating parameter term to the spherically symmetric case. This is confirmed by showing that $K = 0$ and $K_{ij} = 0$ identically for spherically symmetric case, while $K = 0$ but $K_{ij}K^{ij} = O(a^2)$ for slowly rotating case. Effectively, Eq. (31) is the same equation for both two cases. A shift vector $N^\phi$ could be determined by Eq. (27). We emphasize that Eq. (12) remains unchanged at linear order of $a$. It is clear that $E^{(1)}_{ij} = 0$ for spherically symmetric case and $E^{(1)}_{ij} = O(a^2)$ for slowly rotating case, which is effectively taken to be zero. $E^{(2)}_{ij} = 0$ for both two cases. All other $E^{(r)}_{ij}$ for $r = 3, \cdots, 6$ remain unchanged, since they contain terms derived from $g_{ij}$ which does not carry $a$, and thus, $C_{ij} = 0$ by rotation symmetry in three-dimensional Euclidean space.

4. KERR METRIC IS NOT A ROTATING SOLUTION TO THE HL GRAVITY

In this section, we wish to check explicitly whether or not the Kerr metric is a solution to the HL gravity. For this purpose, we introduce the Kerr line-element written in Boyer-Lindquist coordinates as

$$ds^2_{\text{Kerr}} = -\frac{\rho^2 \Delta_r}{\Sigma^2} dt^2 + \frac{\rho^2}{\Delta_r} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} (d\phi - \xi dt)^2,$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$
$$\Delta_r = (r^2 + a^2) - 2Mr,$$
$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta_r,$$
$$\xi = \frac{2Mar}{\Sigma^2}. \quad (36)$$

From this metric we identify the lapse $N^2$, shift $N^\phi$ and three-dimensional metric $g_{ij}$ with, respectively,

$$N^2(r, \theta) = \frac{\rho^2 \Delta_r}{\Sigma^2}, \quad N^\phi = \xi = \frac{2Mar}{\Sigma^2}, \quad g_{rr} = \frac{\rho^2}{\Delta_r}, \quad g_{\theta\theta} = \rho^2, \quad g_{\phi\phi} = \frac{\Sigma^2 \sin^2 \theta}{\rho^2}. \quad (37)$$
In the limit of \( a \to 0 \), the Kerr spacetimes \((35)\) reduces to the Schwarzschild spacetimes: \((34)\) with \( J = 0 \).

Using Eq.\((35)\), the extrinsic curvature tensor is computed to be

\[
K_{ij} = \begin{pmatrix} 0 & 0 & K_{r\phi} \\ 0 & 0 & K_{\theta\phi} \\ K_{r\phi} & K_{\theta\phi} & 0 \end{pmatrix},
\]

where

\[
K_{r\phi} = \frac{Ma \sin^2 \theta [3r^4 + r^2a^2 + a^2 (r^2 + a^2) \cos^2 \theta]}{K_d} K_f,
\]

\[
K_{\theta\phi} = -\frac{2Mr \sin^3 \theta \cos \theta (r^2 + a^2 - 2Mr)}{K_d} K_f,
\]

with

\[
K_d = a^4 \cos^4 \theta (r^2 + a^2 - 2Mr) + 2ra^2 \cos^2 \theta (r^3 - Mr^2 + ra^2 + Ma^2)
+ r^3 (r^3 + ra^2 + 2Ma^2),
\]

\[
K_f^2 = \frac{r^4 + r^2a^2 + 2Mr a^2 \sin^2 \theta + a^2 (r^2 + a^2) \cos^2 \theta}{(r^2 + a^2 \cos^2 \theta) (r^2 + a^2 - 2Mr)}.
\]

The Ricci tensor takes the form

\[
R_{ij} = \begin{pmatrix} R_{rr} & R_{r\theta} & 0 \\ R_{r\theta} & R_{\theta\theta} & 0 \\ 0 & 0 & R_{\phi\phi} \end{pmatrix},
\]

whose explicit form is given in Appendix A. The lowest order of \( R_{ij} \) in \( a \) is

\[
R^{(0)}_{rr} = \frac{2M}{r^3} \frac{1}{1 - \frac{2M}{r}}, \quad R^{(0)}_{r\theta} = -\frac{M}{r}, \quad R^{(0)}_{\phi\phi} = -\frac{M}{r} \sin^2 \theta,
\]

\[
R^{(2)}_{r\theta} = -\frac{9Ma^2 \sin \theta \cos \theta}{r^4}.
\]

The Ricci scalar is given by

\[
R^{(2)} = -\frac{18M^2 a^2 \sin^2 \theta}{r^6}.
\]

The Cotton tensor has

\[
C_{ij} = \begin{pmatrix} 0 & 0 & C_{r\phi} \\ 0 & 0 & C_{\theta\phi} \\ C_{r\phi} & C_{\theta\phi} & 0 \end{pmatrix},
\]

9
where
\begin{align}
C_{r\phi} &= -2M^2a^2 \cos \theta \frac{F_{r\phi} N_{r\phi}}{D_{r\phi}}, \\
C_{\theta\phi} &= 2M^2a^2 \sin \theta \frac{F_{r\phi} N_{\theta\phi}}{D_{\theta\phi}}.
\end{align}

The explicit forms of $F_{r\phi}, N_{r\phi}, D_{r\phi}, N_{\theta\phi},$ and $D_{\theta\phi}$ are given in Appendix B. The lowest-order term of $C_{ij}$ in $a$ takes the form
\begin{align}
C_{r\phi}^{(2)} &= -\frac{2M^2a^2 \cos \theta}{r^8}, \quad C_{\theta\phi}^{(2)} = -\frac{2M^2a^2 \sin \theta}{r^9}.
\end{align}

The Kerr metric satisfies Eq. (11) with $\lambda = 1$ and $\Lambda = 0$ because it does not contain any HL gravity parameters, as in the Einstein gravity.

However, the Kerr metric does not satisfy Eq. (9) since there exist higher-order curvature terms. In order to see it explicitly, we rewrite Eq. (9) as
\begin{equation}
\frac{2}{\kappa^2} (K_{ij} K^{ij} - K^2 + R) + \left(\mu^4 - \frac{2}{\kappa^2}\right) R + \frac{3\kappa^2 \mu^2}{64} R^2 - \frac{\kappa^2}{2W^4} Z_{ij} Z^{ij} = 0.
\end{equation}

We note that the first term is from the Einstein-Hilbert action and thus, it vanishes for the Kerr metric. The second term is zero when choosing $c^2 = 1(\mu^4 = 2/\kappa^2)$. The last two higher-order terms survive as
\begin{equation}
-\frac{\kappa^3}{2^{7/2}} \left[R_{ij}^2 - \frac{3}{8} R^2\right] - \frac{\kappa^4}{4W^4} C_{ij}^2 = 0,
\end{equation}

because $R \neq 0$, $R_{ij} \neq 0$, $C_{ij} \neq 0$, but $C_{ij} R^{ij} = 0$ for the Kerr solution. In order for (51) to be satisfied, each term should vanish because each term has different power of $\kappa$ and $W$, being considered as independent parameters. Using $\kappa^3 = 2^{9/2}/\omega$, $\omega$ and $W$ are also regarded as two independent parameters. The explicit form of $R^2, R^2_{ij},$ and $C^2_{ij}$ are given in Appendix C upto $a^4$-order. Eq. (51) is split into according to the order of $a$

- $a^0$-order:
\begin{equation}
-\frac{\kappa^3}{2^{7/2}} \left[\frac{6M^2}{r^6}\right] - \frac{\kappa^4}{4W^4} [0] \neq 0,
\end{equation}

- $a^2$-order:
\begin{equation}
-\frac{\kappa^3}{2^{7/2}} \left[\frac{27M^2 \sin^2 \theta}{4r^6} + \frac{18M^2}{r^8} \left(1 - 5 \cos^2 \theta - \frac{3M \sin \theta}{r}\right)\right] - \frac{\kappa^4}{4W^4} [0] \neq 0,
\end{equation}
• $a^4$-order:

$$-rac{\kappa^3}{2^{7/2}} \left[ -\frac{9M^2 \sin^2 \theta}{4r^8} \left( 4 + 13 \cos^2 \theta + \frac{12M \sin^2 \theta}{r} \right) 
+ \frac{6M^2}{r^{10}} \left\{ \frac{81M^2 \sin^4 \theta}{r^2} + \frac{M \sin^2 \theta}{r} \left( 82 \cos^2 \theta - 17 \right) + 81 \cos^4 \theta - 18 \cos^2 \theta - 17 \right\} \right]
- \frac{\kappa^4}{4W^4} \left[ \frac{3528M^4 \sin^2 \theta}{r^{16}} \left( 1 - 2M \sin^2 \theta \right) \right] \neq 0,$$

which shows clearly that the Kerr metric is not a solution to the modified HL gravity. Even at the zeroth order of $a$, the equation \((52)\) is not satisfied, which means that the Schwarzschild metric is not the solution to the modified HL gravity even though it is the solution to the Einstein gravity. When all higher-order terms are turned off ($\kappa \to 0$, $\omega \to \infty$), we find either the Kerr solution or Schwarzschild solution as in the Einstein gravity.

Lastly, we show that Eq. \((12)\) is not satisfied by Kerr metric in the order-by-order of $a$. The explicit form of $E_{ij}^{(k)}$ is shown in Appendix D. Eq. \((12)\) can be split into

• $a^0$-order, $(rr)$ component:

$$\frac{3\kappa}{2^{3/2}} \left[ \frac{M^2}{r^6 \sqrt{1 - 2M/r}} \right] \neq 0,$$

• $a^0$-order, $(\theta\theta)$ component:

$$-\frac{3\kappa}{2^{3/2}} \left[ \frac{M^2}{r^4 \sqrt{1 - 2M/r}} \right] \neq 0,$$

• $a^0$-order, $(\phi\phi)$ component:

$$-\frac{3\kappa}{2^{3/2}} \left[ \frac{M^2 \sin^2 \theta}{r^4 \sqrt{1 - 2M/r}} \right] \neq 0,$$

• $a^2$-order, $(rr)$ component:

$$-\frac{3\kappa}{2^{5/2}} \left[ \frac{M^2}{r^8 \left( 1 - 2M/r \right)^{3/2}} \left\{ 12 - 28 \cos^2 \theta - \frac{M}{r} \left( 13 - 46 \cos^2 \theta \right) - \frac{M^2}{r^2} \left( 20 \sin^2 \theta \right) \right\} \right]
+ \frac{357\kappa^2}{W^4} \left[ \frac{M^3}{r^{11} \sqrt{1 - 2M/r}} \left( 1 - 3 \cos^2 \theta \right) \right] \neq 0,$$
• $a^2$-order, $(r\theta)$ component:
\[
\frac{3\kappa}{2^{5/2}} \left[ \frac{M^2}{r^8 \sqrt{1 - \frac{2M}{r}}} \left( 61 - 136 \frac{M}{r} \right) \right] - \frac{21\kappa^2}{2W^4} \left[ \frac{M^3}{r^{10} \sqrt{1 - \frac{2M}{r}}} (197 - 403 \cos^2 \theta) \right] \neq 0,
\]
(59)

• $a^2$-order, $(r\phi)$ component:
\[
\frac{9\kappa^{3/2}}{2^{11/4}W^2} \left[ \frac{M^2 \cos \theta \sin^2 \theta}{r^9} \left( 228 - 533 \frac{M}{r} \right) \right] \neq 0,
\]
(60)

• $a^2$-order, $(\theta\theta)$ component:
\[
- \frac{3\kappa}{2^{7/2}} \left[ \frac{M^2}{r^6 \sqrt{1 - \frac{2M}{r}}} \left\{ -81 + 73 \cos^2 \theta + \frac{M}{r} (354 - 336 \cos^2 \theta) - \frac{M^2}{r^2} (380 \sin^2 \theta) \right\} \right] \\
- \frac{21\kappa^2}{W^4} \left[ \frac{M^3}{r^9 \sqrt{1 - \frac{2M}{r}}} \left\{ -15 - 2 \cos^2 \theta + \frac{M}{r} (106 - 72 \cos^2 \theta) - \frac{M^2}{r^2} (152 \sin^2 \theta) \right\} \right] \\
\neq 0,
\]
(61)

• $a^2$-order, $(\theta\phi)$ component:
\[
\frac{3 \cdot 3\kappa^{3/2}}{4 \cdot 2^{7/4}W^2} \left[ \frac{M^2 \sin^3 \theta}{r^9} \left( 513 - 2796 \frac{M}{r} + 3625 \frac{M^2}{r^2} \right) \right] \neq 0,
\]
(62)

• $a^2$-order, $(\phi\phi)$ component:
\[
\frac{3\kappa}{2^{5/2}} \left[ \frac{M^2 \sin^2 \theta}{r^6 \sqrt{1 - \frac{2M}{r}}} \left\{ -81 + 89 \cos^2 \theta + \frac{M}{r} (404 - 422 \cos^2 \theta) - \frac{M^2}{r^2} (488 \sin^2 \theta) \right\} \right] \\
+ \frac{21\kappa^2}{W^4} \left[ \frac{M^3 \sin^2 \theta}{r^9 \sqrt{1 - \frac{2M}{r}}} \left\{ -32 + 49 \cos^2 \theta + \frac{M}{r} (140 - 174 \cos^2 \theta) - \frac{M^2}{r^2} (152 \sin^2 \theta) \right\} \right] \\
\neq 0,
\]
(63)
• $a^4$-order, $(rr)$ component:

\[
\begin{align*}
\frac{3\kappa}{2^{9/2}} & \left[ \frac{M^2}{r^{10}} \left(1 - \frac{2M}{r}\right)^{5/2} \right] \left\{ -64 + 1232 \cos^2 \theta - 1672 \cos \theta 
\right. \\
& \left. + \frac{M}{r} \left( 612 - 6364 \cos^2 \theta + 7708 \cos^4 \theta \right) - \frac{M^2}{r^2} \left( 1029 - 9406 \cos^2 \theta + 10267 \cos^4 \theta \right) 
\right. \\
& \left. - \frac{M^3}{r^3} \left( 868 + 1208 \cos^2 \theta - 2076 \cos^4 \theta \right) + \frac{M^4}{r^4} \left( 2004 \sin^4 \theta \right) \right\} \\
\end{align*}
\]

\[
\begin{align*}
-3 \frac{\kappa^2}{r^{13}} & \left[ \frac{M^3}{(1 - \frac{2M}{r})^{5/2}} \right] \left\{ 679 + 3303 \cos^2 \theta - 6898 \cos^4 \theta 
\right. \\
& \left. - \frac{M}{r} \left( 5443 + 13482 \cos^2 \theta - 30351 \cos^4 \theta \right) + \frac{M^2}{r^2} \left( 19826 + 1650 \cos^2 \theta - 32664 \cos^4 \theta \right) 
\right. \\
& \left. - \frac{M^3}{r^3} \left( 35240 - 48060 \cos^2 \theta + 12820 \cos^4 \theta \right) + \frac{M^4}{r^4} \left( 23856 \sin^4 \theta \right) \right\} \neq 0, \quad (64)
\end{align*}
\]

• $a^4$-order, $(r\phi)$ component:

\[
\begin{align*}
\frac{3\kappa}{2^{5/2}} & \left[ \frac{M^2 \cos \theta \sin \theta}{r^9} \right] \left(1 - \frac{2M}{r}\right)^{3/2} \left\{ 140 - 616 \cos^2 \theta - \frac{M}{r} \left( 1002 - 3091 \cos^2 \theta \right) 
\right. \\
& \left. + \frac{M^2}{r^2} \left( 2598 - 4858 \cos^2 \theta \right) - \frac{M^3}{r^3} \left( 2280 \sin^2 \theta \right) \right\} \\
- \frac{3 \kappa^2}{2 W^4} & \left[ \frac{M^3 \cos \theta \sin \theta}{r^{12}} \right] \left(1 - \frac{2M}{r}\right)^{3/2} \left\{ -1552 - 18464 \cos^2 \theta + \frac{M}{r} \left( 1513 + 80510 \cos^2 \theta \right) 
\right. \\
& \left. + \frac{M^2}{r^2} \left( 17292 - 101207 \cos^2 \theta \right) - \frac{M^3}{r^3} \left( 28086 \sin^2 \theta \right) \right\} \\
& \left. + \frac{M^4 \sin^2 \theta}{r^{14}} \right\} \neq 0, \quad (65)
\end{align*}
\]

• $a^4$-order, $(r\theta)$ component:

\[
\begin{align*}
\frac{-3 \kappa^{3/2}}{2^{11/4} W^2} & \left[ \frac{M^2 \cos \theta \sin^2 \theta}{r^{10}} \right] \left(1 - \frac{2M}{r}\right) \left\{ -1914 + 8010 \cos^2 \theta + \frac{M}{r} \left( 12825 - 29888 \cos^2 \theta \right) 
\right. \\
& \left. - \frac{M^2}{r^2} \left( 17424 \sin^2 \theta \right) \right\} \\
- \frac{1764 \kappa^2}{W^4} & \left[ \frac{M^4 \sin^2 \theta}{r^{14}} \right] \sqrt{1 - \frac{2M}{r} \sin^2 \theta} \left( 1 - \frac{2M}{r} \sin^2 \theta \right) \neq 0, \quad (66)
\end{align*}
\]
\[ \frac{3\kappa}{2^{9/2}} \left[ \frac{M^2}{r^8 (1 - \frac{2M}{r})^{3/2}} \left\{ -192 - 2904 \cos^2 \theta + 2992 \cos^4 \theta \right. \right. \\
- \frac{M}{r^3} (352 - 22540 \cos^2 \theta + 21732 \cos^4 \theta) + \frac{M^2}{r^2} (10487 - 68870 \cos^2 \theta + 57891 \cos^4 \theta) \\
- \frac{M^3}{r^3} (31756 + 98280 \cos^2 \theta - 66524 \cos^4 \theta) + \frac{M^4}{r^4} (27468 \sin^4 \theta) \left. \right\} \right] \\
+ \frac{3\kappa^2}{W^4} \left[ \frac{M^3}{r^{11} (1 - \frac{2M}{r})^{3/2}} \left\{ 501 + 798 \cos^2 \theta - 2638 \cos^4 \theta \right. \right. \\
- \frac{M}{r^3} (5443 - 7103 \cos^2 \theta - 3805 \cos^4 \theta) + \frac{M^2}{r^2} (24092 - 54816 \cos^2 \theta + 25130 \cos^4 \theta) \\
- \frac{M^3}{r^3} (47848 - 10962 \cos^2 \theta + 61764 \cos^4 \theta) + \frac{M^4}{r^4} (34776 \sin^4 \theta) \left. \right\} \right] \\
+ \frac{M^4}{r^4} \left( -588 \sin^2 \theta + \frac{M}{r} (2352 - 3528 \cos^2 \theta + 1176 \cos^4 \theta) \\
+ \frac{M^2}{r^2} (2352 \sin^4 \theta) \right) \right\} \right] \neq 0, \tag{67} \]

\[ \frac{3\kappa}{2^{5/2}} \left[ \frac{M^2 \sin^3 \theta}{r^9} \left\{ -144 - 10680 \cos^2 \theta - \frac{M}{r} (5771 - 72572 \cos^2 \theta) \right. \right. \\
+ \frac{M^2}{r^2} (44773 - 140725 \cos^2 \theta) - \frac{M^3}{r^3} (73174 \sin^2 \theta) \left. \right\} \right] \\
- \frac{1764\kappa^2}{W^4} \left[ \frac{M^4 \sin^2 \theta}{r^{14} \sqrt{1 - \frac{2M}{r} \sin^2 \theta}} \left( 1 - \frac{2M}{r} \sin^2 \theta \right) \right] \right\} \right] \neq 0, \tag{68} \]

- \( a^4 \)-order, \((\theta \theta)\) component:
\[ a^4 \text{-order, } (\phi\phi) \text{ component:} \]

\[
-\frac{3\kappa}{2^{9/2}} \left[ \frac{M^2 \sin^2 \theta}{r^8 \left( 1 - \frac{2M}{r} \right)^{3/2}} \left\{ 48 - 2412 \cos^2 \theta + 2468 \cos^4 \theta \right. \right.
\]

\[ -\frac{M}{r} \left( 1628 - 21852 \cos^2 \theta + 20680 \cos^4 \theta \right) \]

\[ + \frac{M^2}{r^2} \left( 12277 - 71578 \cos^2 \theta + 57793 \cos^4 \theta \right) \]

\[ -\frac{M^3}{r^3} \left( 31812 - 101800 \cos^2 \theta + 69988 \cos^4 \theta \right) + \frac{M^4}{r^4} \left( 26756 \sin^4 \theta \right) \right] \right] \}

\[ -\frac{3}{2^{3/2}} \left[ \frac{M^3 \sin^2 \theta}{r^{11} \left( 1 - \frac{2M}{r} \right)^{3/2}} \left\{ 329 - 3429 \cos^2 \theta + 4603 \cos^4 \theta \right. \right.
\]

\[ -\frac{M}{r} \left( 1502 - 23791 \cos^2 \theta + 27764 \cos^4 \theta \right) \]

\[ + \frac{M^2}{r^2} \left( 6198 - 58678 \cos^2 \theta + 58074 \cos^4 \theta \right) \]

\[ -\frac{M^3}{r^3} \left( 12216 - 58640 \cos^2 \theta + 46424 \cos^4 \theta \right) + \frac{M^4}{r^4} \left( 9016 \sin^4 \theta \right) \right] \}

\[ -\frac{M^4 \sin^2 \theta}{r^{14} \left( 1 - \frac{2M}{r} \right)^{3/2}} \left\{ 588 - \frac{M}{r} \left( 2352 + 1178 \cos^2 \theta \right) + \frac{M^2}{r^2} \left( 2352 \sin^4 \theta \right) \right\} \right]

\[ \neq 0. \quad (69) \]

As was noted previously, the higher-order terms violate the Lorentz symmetry. Hence, it is conjectured that the absence of Kerr solution is related closely to the violation of the Lorentz-symmetry in the HL gravity. In order to explore a connection between Kerr solution and Lorentz-symmetry, we introduce one known-example. It is well known that the CS modified gravity could not provide the Kerr metric as a fully rotating black hole solution since it cannot satisfy the Pontryagin constraint. The action for the CS modified gravity is given by

\[
S_{EH} + S_{CS} + S_m = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{1}{4} \vartheta \ast RR + \kappa^2 \mathcal{L}_m \right), \quad (70)
\]

where \( \vartheta \) is a CS parameter and thus, \( v_\mu = \partial_\mu \vartheta \) plays a role of the embedding vector. \( \mathcal{L}_m \) represents the matter whose energy-momentum tensor is \( T^{\mu\nu} \). The equation of motion CS modified gravity is given by

\[
G^{\mu\nu} + C^{\mu\nu} = \kappa^2 T^{\mu\nu}, \quad (71)
\]

where \( C^{\mu\nu} \) is the Cotton tensor defined as

\[
C^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial S_{CS}}{\partial g_{\mu\nu}}. \quad (72)
\]
Requiring the Bianchi identity and using the energy-momentum conservation, we have the condition
\[ \nabla^\mu C_{\mu\nu} = 0. \]  
(73)

Considering the relation
\[ \nabla^\mu C_{\mu\nu} = \frac{v_\nu}{8} \ast RR, \]  
(74)

the Pontryagin constraint
\[ \mathcal{P} = \ast RR = 0, \]  
(75)

should be satisfied for any solution to the CS modified gravity. However, for the Kerr solution (35), one has \( \nabla^\mu C_{\mu\nu} = 0 \), but \( \mathcal{P} \) is not zero as
\[ \mathcal{P} = \ast RR = \frac{96aM^2r}{\rho_{12}} \cos \theta \left( r^2 - 3a^2 \cos^2 \theta \right) \left( 3r^2 - a^2 \cos^2 \theta \right). \]  
(76)

In the limit of \( a \to 0 \), the Schwarzschild solution is recovered with \( \mathcal{P} = 0 \). Therefore, for any finite \( a \), the Pontryagin term is non-vanishing and thus, the Kerr spacetime cannot be a solution to the CS modified gravity equation.

5. DISCUSSIONS

We have shown that the Kerr metric is not a solution of the modified HL gravity with \( \lambda = 1 \) and \( \Lambda_W = 0 \). This was performed by checking whether or not three equations are satisfied for the Kerr metric, according to the \( a \)-order. It is shown that the lapse equation (9) and the metric equation (12) are not satisfied for the Kerr metric even at the zeroth order of rotating parameter \( a \). This implies that the Schwarzschild metric is not a solution to the modified HL gravity because in the limit of \( a \to 0 \), the Kerr metric reduces to the Schwarzschild metric. The dissatisfaction comes from the presence of higher-order curvature terms in the modified HL gravity, which may enable us to carry the power-counting renormalizability out.

The only allowable rotating solution to the modified HL gravity is the slowly rotating KS solution which includes the effect \( (\omega) \) of higher-order curvature terms. We mention again that the slowly rotating Kerr metric is not the solution to the modified HL gravity but the Einstein gravity.
In conclusion, the absence of a fully rotating solution in the modified HL gravity provides another dark-side, in addition to the strong coupling issue because astrophysical black holes may be considered as the Kerr black hole [26].

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**Appendix A: Ricci tensor and scalar**

We show the explicit form of $R_{ij}$ and $R$ for the Kerr metric. Ricci tensor is given by

$$
R_{ij} = \begin{pmatrix}
R_{rr} & R_{r\theta} & 0 \\
R_{r\theta} & R_{\theta\theta} & 0 \\
0 & 0 & R_{\phi\phi}
\end{pmatrix},
$$

(A1)

where

$$
R_{rr} = -M \frac{Q_{rr}}{P_{rr}},
$$

(A2)

$$
Q_{rr} = 2 r^{11} - 16 r^7 a^4 \cos^2 \theta - 22 r^3 a^8 \cos^4 \theta + 3 r a^{10} \cos^6 \theta + 3 r^5 a^6 \cos^6 \theta
+ 8 r^7 a^4 + 7 r^9 a^2 + 3 r^5 a^6 + 6 r^3 a^8 \cos^6 \theta + 7 M r^8 a^2 \cos^2 \theta + 30 M r^2 a^8 \cos^4 \theta
+ 16 M r^6 a^4 \cos^4 \theta - 14 M r^2 a^8 \cos^6 \theta - 16 M r^2 a^8 \cos^2 \theta + 12 M^2 r^2 a^6 \cos^6 \theta
- 11 M r^4 a^6 \cos^6 \theta - 24 M^2 r^3 a^6 \cos^4 \theta + 35 M r^4 a^6 \cos^4 \theta - 4 M^2 r^5 a^4 \cos^4 \theta
- 29 M r^4 a^6 \cos^2 \theta + 12 M^2 r^3 a^6 \cos^2 \theta - 18 M r^6 a^4 \cos^2 \theta + 8 M^2 r^5 a^4 \cos^2 \theta
- 7 M r^8 a^2 + 2 M r^6 a^4 - 4 M^2 r^5 a^4 + 5 M r^4 a^6 - 4 r^7 a^4 \cos^4 \theta - 17 r^5 a^6 \cos^2 \theta
- 5 r^9 a^2 \cos^2 \theta - 17 r^5 a^6 \cos^4 \theta - 9 r a^{10} \cos^4 \theta - 6 r^3 a^8 \cos^2 \theta
+ M a^{10} \cos^6 \theta - M a^{10} \cos^4 \theta,
$$

(A3)
\[ P_{rr} = -r^{14} \left( 1 - \frac{2M}{r} \right) - r^8 a^6 + 8 M^2 r^{10} a^2 + 8 M^3 r^7 a^4 - 3 r^2 a^{12} \cos^8 \theta \]
\[ -r^6 a^8 \cos^8 \theta - 4 r^2 a^{12} \cos^6 \theta - 12 r^{10} a^4 \cos^2 \theta \]
\[ -12 r^8 a^6 \cos^2 \theta - 18 r^6 a^6 \cos^4 \theta - 18 r^6 a^8 \cos^4 \theta - 12 r^6 a^8 \cos^6 \theta \]
\[ -12 a^{10} (\cos (\theta))^6 r^4 - 4 (\cos (\theta))^2 a^8 r^6 - 6 (\cos (\theta))^4 a^{10} r^4 - 4 a^2 (\cos (\theta))^2 r^{12} \]
\[ -6 r^{10} a^4 \cos^4 \theta - 4 r^8 a^6 \cos^6 \theta - 3 r^4 a^{10} \cos^8 \theta - 12 M r^7 a^6 \cos^2 \theta \]
\[ +12 M r^3 a^{10} \cos^6 \theta - 12 M r^5 a^8 \cos^2 \theta - 12 M r^3 a^{10} \cos^4 \theta + 36 M r^5 a^8 \cos^6 \theta \]
\[ +24 M^2 r^8 a^4 \cos^2 \theta + 12 M r^{11} a^2 \cos^2 \theta + 36 M^2 r^4 a^8 \cos^4 \theta + 12 M r^9 a^4 \cos^2 \theta \]
\[ +36 M^2 r^7 a^6 \cos^2 \theta - 16 M^3 r^3 a^8 \cos^6 \theta + 8 M^3 r^3 a^8 \cos^8 \theta + 16 M^3 r^5 a^6 \cos^6 \theta \]
\[ -16 M^3 r^7 a^4 \cos^2 \theta + 24 M^2 r^6 a^6 \cos^2 \theta - 8 M^2 r^{10} a^2 \cos^2 \theta + 20 M r^7 a^6 \cos^6 \theta \]
\[ -16 M^2 r^4 a^8 \cos^6 \theta - 8 M^2 r^4 a^8 \cos^2 \theta - 4 M^2 r^2 a^{10} \cos^4 \theta + 24 M r^9 a^4 \cos^4 \theta \]
\[ +12 M^2 r^6 a^6 \cos^4 \theta + 16 M^2 r^2 a^{10} \cos^6 \theta - 32 M^2 r^6 a^6 \cos^6 \theta + 8 M^3 r^3 a^8 \cos^4 \theta \]
\[ +16 M^3 r^5 a^6 \cos^2 \theta - 28 M^2 r^8 a^4 \cos^4 \theta - 32 M^3 r^5 a^6 \cos^4 \theta + 8 M^3 r^7 a^4 \cos^4 \theta \]
\[ -4 M r^{12} \cos^6 \theta + 6 M r^{12} \cos^8 \theta + 12 M r^3 a^{10} \cos^8 \theta - 12 M r^3 a^{10} \cos^8 \theta \]
\[ +6 M r^5 a^8 \cos^8 \theta - 12 M^2 r^4 a^8 \cos^8 \theta - 6 M r^9 a^4 + 4 M^2 r^8 a^4 - a^{14} \cos^8 \theta \]
\[ -4 M^2 r^6 a^6 - 4 M r^7 a^6 - 3 r^{10} a^4 - 3 r^{12} a^2. \quad (A4) \]

\[ R_{r\theta} = -Ma^2 \sin \theta \cos \theta \frac{Q_{r\theta}}{P_{r\theta}}, \quad (A5) \]

\[ Q_{r\theta} = 9 r^8 + 4 Mr a^6 \cos^4 \theta + 8 Mr^3 a^4 \cos^4 \theta - 3 r^4 a^4 \cos^4 \theta - 6 r^2 a^6 \cos^4 \theta \]
\[ -3 a^8 \cos^4 \theta - 12 Mr^5 a^2 \cos^2 \theta + 6 r^6 a^2 \cos^2 \theta - 4 Mr a^6 \cos^2 \theta \]
\[ +6 r^2 a^6 \cos^2 \theta + 12 r^4 a^4 \cos^2 \theta - 24 Mr^3 a^4 \cos^2 \theta + 18 r^6 a^2 \]
\[ +16 a^4 Mr^3 + 12 a^2 Mr^5 + 9 a^4 r^4, \quad (A6) \]
\[
P_{r\theta} = r^{12} + 4 M^2 r^2 a^8 \cos^4 \theta + 8 M^2 r^4 a^6 \cos^2 \theta + r^8 a^4 + 2 r^{10} a^2 + 4 M r^7 a^4 + 4 M^2 r^6 a^4 + 4 M r^9 a^2 - 4 M r^3 a^8 \cos^8 \theta - 4 M r^5 a^{10} \cos^8 \theta + 4 M^2 r^2 a^8 \cos^8 \theta
\]

\[
-8 M r^3 a^8 \cos^6 \theta + 4 M r a^{10} \cos^6 \theta - 8 M^2 r^2 a^8 \cos^8 \theta - 12 M r^5 a^6 \cos^6 \theta
\]

\[
+8 M^2 r^4 a^6 \cos^6 \theta + 12 M r^3 a^8 \cos^4 \theta - 16 M^2 r^4 a^6 \cos^4 \theta - 12 M r^7 a^4 \cos^4 \theta
\]

\[
+4 M^2 r^6 a^4 \cos^4 \theta + 8 M r^7 a^4 \cos^2 \theta - 8 M^2 r^6 a^4 \cos^2 \theta + 12 M r^5 a^6 \cos^2 \theta
\]

\[
-4 M r^9 a^2 \cos^2 \theta + r^4 a^8 \cos^8 \theta + 2 r^2 a^{10} \cos^8 \theta
\]

\[
+8 r^4 a^8 \cos^6 \theta + 4 r^2 a^{10} \cos^6 \theta + 4 r^6 a^6 \cos^6 \theta + 12 r^6 a^6 \cos^4 \theta
\]

\[
+6 r^4 a^8 \cos^4 \theta + 6 r^8 a^4 \cos^4 \theta + 4 r^6 a^6 \cos^2 \theta + 8 r^8 a^4 \cos^2 \theta
\]

\[
+4 r^{10} a^2 \cos^2 \theta + a^{12} \cos^8 \theta,
\]

(A7)

\[
R_{\theta\theta} = -M r \frac{Q_{\theta\theta}}{P_{\theta\theta}}, \quad \text{(A8)}
\]

\[
Q_{\theta\theta} = r^{10} - 22 M r^3 a^6 \cos^2 \theta + 34 M r a^8 \cos^4 \theta + 16 M^2 r^2 a^6 \cos^6 \theta - 32 M^2 r^2 a^6 \cos^4 \theta + 16 M^2 r^4 a^4 \cos^2 \theta + 16 M^2 r^2 a^6 \cos^2 \theta - 14 M r a^8 \cos^2 \theta - 8 M^2 r^4 a^4 \cos^4 \theta
\]

\[
-8 M^2 r^4 a^4 + 6 M r^3 a^6 + 5 r^8 a^2 - 2 M r^7 a^2 + 4 M r^5 a^4
\]

\[
-4 r^8 a^2 \cos^2 \theta + r^6 a^4 \cos^4 \theta + 6 r^4 a^6 \cos^6 \theta + 7 r^6 a^4 + 3 r^4 a^6
\]

\[
-6 M r^5 a^4 \cos^2 \theta + 36 M r^3 a^6 \cos^4 \theta + 2 M r^5 a^4 \cos^4 \theta - 20 M r^3 a^6 \cos^6 \theta + 2 M r^7 a^2 \cos^2 \theta - 20 M r a^8 \cos^6 \theta + 6 a^{10} \cos^6 \theta - 9 a^{10} \cos^4 \theta
\]

\[
-14 r^6 a^4 \cos^2 \theta - 7 r^4 a^6 \cos^4 \theta + 12 r^2 a^8 \cos^6 \theta - 6 r^2 a^8 \cos^2 \theta
\]

\[
-16 r^4 a^6 \cos^2 \theta - 17 r^2 a^8 \cos^4 \theta,
\]

(A9)

\[
P_{\theta\theta} = r^{12} + 4 M^2 r^2 a^8 \cos^4 \theta + 8 M^2 r^4 a^6 \cos^2 \theta + r^8 a^4 + 2 r^{10} a^2 + 4 M r^7 a^4 + 4 M^2 r^6 a^4 + 4 M r^9 a^2 - 4 M r^3 a^8 \cos^8 \theta - 4 M r a^{10} \cos^8 \theta
\]

\[
+4 M^2 r^2 a^8 \cos^8 \theta - 8 M r^3 a^8 \cos^6 \theta + 4 M r a^{10} \cos^6 \theta - 8 M^2 r^2 a^8 \cos^6 \theta
\]

\[
-12 M r^5 a^6 \cos^6 \theta + 8 M^2 r^4 a^6 \cos^6 \theta + 12 M^2 r^3 a^8 \cos^4 \theta - 16 M^2 r^4 a^6 \cos^4 \theta
\]

\[
-12 M r^7 a^4 \cos^4 \theta + 4 M^2 r^6 a^4 \cos^4 \theta + 8 M r^7 a^4 \cos^2 \theta - 8 M^2 r^6 a^4 \cos^2 \theta
\]

\[
+12 M r^5 a^6 \cos^2 \theta - 4 M r a^8 \cos^8 \theta + r^4 a^8 \cos^8 \theta + 2 r^2 a^{10} \cos^8 \theta
\]

\[
+8 r^4 a^8 \cos^6 \theta + 4 r^2 a^{10} \cos^6 \theta + 4 r^6 a^6 \cos^6 \theta + 12 r^6 a^6 \cos^4 \theta
\]

\[
+6 r^4 a^8 \cos^4 \theta + 6 r^8 a^4 \cos^4 \theta + 4 r^6 a^6 \cos^2 \theta + 8 r^8 a^4 \cos^2 \theta
\]

\[
+4 r^{10} a^2 \cos^2 \theta + a^{12} \cos^8 \theta,
\]

(A10)
\[ R_{\phi\phi} = -M \sin^2 \theta \frac{Q_{\phi\phi}}{P_{\phi\phi}}, \]  
(A11)

\[ Q_{\phi\phi} = -r^{11} + 2 r^7 a^4 \cos^2 \theta + 5 r^3 a^8 \cos^4 \theta + 3 r a^{10} \cos^6 \theta + 3 r^5 a^6 \cos^6 \theta 
- r^7 a^4 - 2 r^9 a^2 + 6 r^3 a^8 \cos^6 \theta + 13 M r^8 a^2 \cos^2 \theta 
+ 20 M r^2 a^8 \cos^4 \theta - 2 M r^6 a^4 \cos^4 \theta - 18 M r^2 a^8 \cos^6 \theta - 2 M r^2 a^8 \cos^2 \theta 
+ 20 M^2 r^3 a^6 \cos^6 \theta - 15 M r^4 a^6 \cos^4 \theta - 40 M^2 r^3 a^6 \cos^2 \theta + 7 M r^4 a^6 \cos^4 \theta 
- 4 M^2 r^5 a^4 \cos^4 \theta + 9 M r^4 a^6 \cos^2 \theta + 20 M^2 r^3 a^6 \cos^2 \theta + 12 M r^6 a^4 \cos^2 \theta 
+ 8 M^2 r^5 a^4 \cos^2 \theta - 13 M r^8 a^2 - 10 M r^6 a^4 - 4 M^2 r^5 a^4 - M r^4 a^6 
+ 5 r^7 a^4 \cos^4 \theta + r^5 a^6 \cos^2 \theta + r^9 a^2 \cos^2 \theta + 10 r^5 a^6 \cos^4 \theta 
+ M a^{10} \cos^6 \theta - M a^{10} \cos^4 \theta, \]  
(A12)

\[ P_{\phi\phi} = -r^{12} + 2 M r a^{10} \cos^2 \theta - r^{10} a^2 - 2 M r^9 a^2 + 8 M r^3 a^8 \cos^8 \theta 
- 2 M r a^{10} \cos^8 \theta - 8 M r^3 a^8 \cos^6 \theta + 12 M r^5 a^6 \cos^6 \theta + 8 M r^7 a^4 \cos^4 \theta 
- 8 M r^7 a^4 \cos^2 \theta + 2 M r^9 a^2 \cos^2 \theta - 5 r^4 a^8 \cos^8 \theta - 5 r^2 a^{10} \cos^8 \theta 
- 10 r^4 a^8 \cos^6 \theta - 10 r^6 a^4 \cos^6 \theta - 10 r^6 a^6 \cos^4 \theta - 10 r^8 a^4 \cos^4 \theta 
\cos^4 \theta - 5 r^8 a^4 \cos^2 \theta - 5 r^{10} a^2 \cos^2 \theta - 12 M r^5 a^6 \cos^4 \theta 
- a^{12} \cos^10 \theta - r^2 a^{10} \cos^10 \theta. \]  
(A13)

Ricci scalar is given by

\[ R = -2 M^2 a^2 \sin^2 \theta \frac{Q_R}{P_R}, \]  
(A14)

\[ Q_R = 9 r^8 - 3 r^4 a^4 \cos^2 \theta + a^8 \cos^4 \theta - 6 r^2 a^6 \cos^4 \theta + 8 M r^3 a^4 \cos^4 \theta 
- 8 M r^3 a^4 \cos^2 \theta + 2 r^2 a^6 \cos^2 \theta + 6 r^6 a^2 \cos^2 \theta + a^4 r^4 + 6 r^6 a^2, \]  
(A15)
\[ P_R = r^{14} + 24 Mr^5 a^8 \cos^4 \theta - 4 Mr^3 a^{10} \cos^6 \theta - 4 M r^2 a^{12} \cos^8 \theta + 4 M^2 r^2 a^{10} \cos^8 \theta \\
+ 5 r^2 a^{12} \cos^8 \theta + 5 r^6 a^8 \cos^8 \theta + a^{14} \cos^{10} \theta \\
+ 2 r^2 a^{12} \cos^{10} \theta + r^4 a^{10} \cos^{10} \theta + 4 Mr^{11} a^2 + 10 r^{10} a^4 \cos^2 \theta + 5 r^8 a^6 \cos^2 \theta \\
+ 20 r^8 a^6 \cos^4 \theta + 10 r^6 a^8 \cos^4 \theta + 20 r^6 a^8 \cos^6 \theta + 10 r^4 a^{10} \cos^6 \theta \\
+ 5 r^{12} a^2 \cos^2 \theta + 10 r^{10} a^4 \cos^2 \theta + 10 a^6 \cos^6 \theta r^8 + 10 r^4 a^{10} \cos^8 \theta \\
+ 16 Mr^7 a^6 \cos^2 \theta + 16 Mr^3 a^{10} \cos^6 \theta - 8 Mr^5 a^8 \cos^6 \theta - 8 M^2 r^8 a^4 \cos^2 \theta \\
- 4 Mr^{11} a^2 \cos^2 \theta + 12 M^2 r^4 a^8 \cos^4 \theta + 12 M r^9 a^4 \cos^2 \theta + 8 Mr^7 a^6 \cos^4 \theta \\
+ 16 a^6 M^2 r^6 \cos^2 \theta - 24 Mr^7 a^6 \cos^6 \theta - 24 M^2 r^4 a^8 \cos^6 \theta - 16 Mr^9 a^4 \cos^4 \theta \\
- 24 M^2 r^6 a^6 \cos^4 \theta + 4 M^2 r^2 a^{10} \cos^6 \theta + 12 M^2 r^6 a^6 \cos^6 \theta + 4 M^2 r^8 a^4 \cos^4 \theta \\
+ 4 M r^3 a^{10} \cos^8 \theta - 12 M r^3 a^{10} \cos^8 \theta - 8 M^2 r^2 a^{10} \cos^8 \theta - 16 M r^5 a^8 \cos^8 \theta \\
+ 12 M^2 r^4 a^8 \cos^8 \theta + 4 M r^9 a^4 + 4 M^2 r^8 a^4 + r^{10} a^4 + 2 r^{12} a^2. \quad (A16) \]
Appendix B: Cotton tensor

We show the explicit form of $C_{r\phi}$ and $C_{\theta\phi}$ for the Kerr metric. $F_{r\phi}$, $N_{r\theta}$, $D_{r\theta}$, $N_{\theta\phi}$, and $D_{23}$ appeared in Eqs. (47) and (48) are given by

\[ F_{r\phi}^2 = \frac{r^2 + a^2 - 2Mr}{(r^2 + a^2 \cos^2 \theta) (r^4 + r^2a^2 + 2Mr^2 \sin^2 \theta + r^2a^2 \cos^2 \theta + a^4 \cos^2 \theta)}, \tag{B1} \]

\[ N_{r\phi} = -21r^{14} - 30Mr^{11}a^2 - 128Mr^9a^4 - 94Mr^7a^6 - 4Mr^5a^8 - 12M^2r^8a^4 - 84r^{12}a^2 \]
\[ + 3r^6a^8 - 4r^6a^8 \cos^2 \theta - 3a^{14} \cos^8 \theta + 2a^{14} \cos^8 \theta - 52r^6M^2a^6 - 104r^{10}a^4 \]
\[ - 44r^8a^6 - 10r^4a^{10} \cos^8 \theta - 6a^{12}r^2 \cos^8 \theta + 102r^8a^6 \cos^4 \theta + 50r^8a^6 \cos^4 \theta \]
\[ + 2r^4a^{10} \cos^4 \theta + 53r^{10}a^4 \cos^4 \theta - 9r^2a^{12} \cos^4 \theta - 70r^{10}a^4 \cos^2 \theta - 124r^8a^6 \cos^2 \theta \]
\[ - 66r^6a^8 \cos^2 \theta - 3r^{12}a^2 \cos^2 \theta - 9r^4a^{10} \cos^2 \theta + 78r^6a^8 \cos^2 \theta + 76r^4a^{10} \cos^2 \theta \]
\[ + 26r^2a^{12} \cos^6 \theta + 31r^8a^6 \cos^6 \theta - 98Mr^9a^4 \cos^4 \theta + 34Mr^7a^6 \cos^4 \theta \]
\[ + 190Mr^5a^8 \cos^4 \theta + 30Mr^{11}a^2 \cos^2 \theta + 226Mr^9a^4 \cos^2 \theta + 170Mr^7a^6 \cos^2 \theta \]
\[ - 110Mr^7a^6 \cos^6 \theta - 202Mr^5a^8 \cos^6 \theta - 114Mr^3a^{10} \cos^6 \theta + 102Mr^3a^{10} \cos^4 \theta \]
\[ - 2Mr^5a^8 \cos^2 \theta + 10Mr^{12} \cos^6 \theta - 12M^2r^8a^4 \cos^4 \theta - 244M^2r^6a^6 \cos^4 \theta \]
\[ - 180M^2r^4a^8 \cos^4 \theta + 24M^2r^8a^4 \cos^2 \theta + 200M^2r^6a^6 \cos^2 \theta + 96M^2r^6a^6 \cos^6 \theta \]
\[ + 120M^2r^4a^8 \cos^6 \theta - 8M^2r^2a^{10} \cos^6 \theta + 18Mr^5a^8 \cos^8 \theta + 20Mr^3a^{10} \cos^8 \theta \]
\[ + 6Mr^{12} \cos^8 \theta - 20M^2r^4a^8 \cos^8 \theta + 4M^2r^2a^{10} \cos^8 \theta + 4M^2r^2a^{10} \cos^4 \theta \]
\[ - 4Mr^{12} \cos^4 \theta - 8Mr^3a^{10} \cos^2 \theta + 80M^2r^4a^8 \cos^2 \theta, \tag{B2} \]
\[ D_{r\phi} = -r^{20} - 3 r^{18} a^{2} - 3 r^{16} a^{4} - r^{14} a^{6} - a^{20} \cos^{14} \theta - 7 r^{18} a^{2} \cos^{2} \theta - 105 r^{8} a^{12} \cos^{8} \theta \\
-35 r^{14} a^{6} \cos^{6} \theta - 21 r^{16} a^{4} \cos^{4} - 63 r^{14} a^{6} \cos^{4} \theta - 21 r^{10} a^{10} \cos^{4} \theta - 105 r^{12} a^{8} \cos^{6} \theta \\
-35 r^{8} a^{12} \cos^{6} \theta - 105 r^{10} a^{10} \cos^{8} \theta - 35 r^{6} a^{14} \cos^{8} \theta - 35 r^{12} a^{8} \cos^{8} \theta \\
-63 r^{6} a^{14} \cos^{10} \theta - 21 r^{10} a^{10} \cos^{10} \theta - 21 r^{14} a^{6} \cos^{2} \theta - 63 r^{12} a^{8} \cos^{4} \theta \\
-105 r^{10} a^{10} \cos^{6} \theta - 63 r^{8} a^{12} \cos^{10} \theta - 21 r^{4} a^{16} \cos^{10} \theta - 21 r^{16} a^{4} \cos^{2} \theta \\
-7 r^{12} a^{8} \cos^{2} \theta - 6 M r^{17} a^{2} - 12 M r^{15} a^{4} - 6 M r^{13} a^{6} - 12 M^{2} r^{14} a^{4} - 12 M^{2} r^{12} a^{6} \\
-8 M^{3} r^{11} a^{6} - r^{6} a^{14} \cos^{14} \theta - 3 r^{2} a^{18} \cos^{14} \theta - 3 r^{4} a^{16} \cos^{14} \theta - 7 r^{8} a^{12} \cos^{12} \theta \\
-21 r^{6} a^{14} \cos^{12} \theta - 21 r^{4} a^{16} \cos^{12} \theta - 7 r^{2} a^{18} \cos^{12} \theta + 6 M r^{5} a^{14} \cos^{14} \theta \\
+8 M^{3} r^{3} a^{14} \cos^{14} \theta + 12 M r^{3} a^{16} \cos^{14} \theta - 12 M^{2} r^{2} a^{16} \cos^{14} \theta + 6 M r a^{18} \cos^{18} \theta \\
-12 M^{2} r^{4} a^{14} \cos^{14} \theta + 36 M r^{7} a^{12} \cos^{12} \theta + 66 M r^{5} a^{14} \cos^{12} \theta - 60 M^{2} r^{6} a^{12} \cos^{12} \theta \\
+24 M r^{3} a^{16} \cos^{12} \theta - 36 r^{4} a^{14} M^{2} \cos^{12} \theta + 32 r^{5} a^{12} M^{3} \cos^{12} \theta - 6 M r a^{18} \cos^{12} \theta \\
+24 M^{2} r^{2} a^{16} \cos^{12} \theta - 24 M^{3} r^{3} a^{14} \cos^{12} \theta + 90 M r^{9} a^{10} \cos^{10} \theta + 144 M r^{7} a^{12} \cos^{10} \theta \\
-120 M^{2} r^{8} a^{10} \cos^{10} \theta + 18 M r^{5} a^{14} \cos^{10} \theta + 48 M^{3} r^{7} a^{10} \cos^{10} \theta - 36 M r^{3} a^{16} \cos^{10} \theta \\
+108 M^{2} r^{4} a^{14} \cos^{10} \theta - 96 M r^{5} a^{12} \cos^{10} \theta - 12 M^{2} r^{2} a^{16} \cos^{10} \theta + 24 M^{3} r^{3} a^{14} \cos^{10} \theta \\
+120 M r^{11} a^{8} \cos^{8} \theta + 150 M r^{9} a^{10} \cos^{8} \theta - 120 M^{2} r^{10} a^{8} \cos^{8} \theta - 60 M r^{7} a^{12} \cos^{8} \theta \\
+120 M r^{12} a^{10} \cos^{8} \theta - 90 M r^{5} a^{14} \cos^{8} \theta + 180 M^{2} r^{6} a^{12} \cos^{8} \theta + 32 M^{3} r^{9} a^{8} \cos^{8} \theta \\
-144 r^{7} a^{10} M^{3} \cos^{8} \theta - 60 M^{2} r^{4} a^{14} \cos^{8} \theta + 96 M^{3} r^{5} a^{12} \cos^{8} \theta - 8 M^{3} r^{3} a^{14} \cos^{8} \theta \\
+90 M r^{13} a^{6} \cos^{6} \theta + 60 M r^{11} a^{8} \cos^{6} \theta - 150 M r^{9} a^{10} \cos^{6} \theta - 60 M^{2} r^{12} a^{6} \cos^{6} \theta \\
+180 M^{2} r^{10} a^{8} \cos^{6} \theta - 120 M r^{7} a^{12} \cos^{6} \theta + 120 M^{2} r^{8} a^{10} \cos^{6} \theta - 120 M^{2} r^{6} a^{12} \cos^{6} \theta \\
+8 M^{3} r^{11} a^{6} \cos^{6} \theta - 96 M^{3} r^{9} a^{8} \cos^{6} \theta + 144 r^{7} a^{10} M^{3} \cos^{6} \theta - 32 M^{3} r^{5} a^{12} \cos^{6} \theta \\
+36 M r^{15} a^{4} \cos^{4} \theta - 18 M r^{13} a^{6} \cos^{4} \theta - 144 M r^{11} a^{8} \cos^{4} \theta - 90 M r^{9} a^{10} \cos^{4} \theta \\
-12 M r^{14} a^{4} \cos^{4} \theta + 108 M^{2} r^{12} a^{6} \cos^{4} \theta - 120 M^{2} r^{8} a^{10} \cos^{4} \theta - 24 M^{3} r^{11} a^{6} \cos^{4} \theta \\
+96 M^{3} r^{9} a^{8} \cos^{4} \theta - 48 M^{3} r^{7} a^{10} \cos^{4} \theta + 6 M r^{17} a^{2} \cos^{2} \theta - 24 M r^{15} a^{4} \cos^{2} \theta \\
-66 M r^{13} a^{6} \cos^{2} \theta - 36 M r^{11} a^{8} \cos^{2} \theta + 24 M^{2} r^{14} a^{4} \cos^{2} \theta - 36 M^{2} r^{12} a^{6} \cos^{2} \theta \\
-60 M^{2} r^{10} a^{8} \cos^{2} \theta + 24 M^{3} r^{11} a^{6} \cos^{2} \theta - 32 M^{3} r^{9} a^{8} \cos^{2} \theta, \quad (B3) \]
\[ N_{\theta \phi} = 21 r^{13} + 12 M^2 r^7 a^4 \cos^4 \theta + 7 M r^2 a^{10} \cos^4 \theta - 113 M r^4 a^8 \cos^4 \theta - 27 M r^6 a^6 \cos^4 \theta \\
+ 117 M r^8 a^4 \cos^2 \theta - 96 M^2 r^5 a^6 \cos^2 \theta - 24 M^2 r^7 a^4 \cos^2 \theta + 11 M r^4 a^8 \cos^2 \theta \\
- 99 M r^6 a^6 \cos^2 \theta - 147 M r^8 a^4 \cos^2 \theta - 21 M r^{10} a^2 \cos^2 \theta - 22 M r^2 a^{10} \cos^8 \theta \\
- 17 M r^4 a^8 \cos^8 \theta + 20 M^2 r^3 a^8 \cos^8 \theta - 40 M^2 r^3 a^8 \cos^6 \theta - 96 M^2 r^5 a^6 \cos^6 \theta \\
+ 15 M r^2 a^{10} \cos^6 \theta + 119 M r^4 a^8 \cos^6 \theta + 121 M r^6 a^6 \cos^6 \theta + 20 M^2 r^3 a^8 \cos^4 \theta \\
+ 192 M^2 r^5 a^6 \cos^4 \theta + 8 r^3 a^{10} \cos^8 \theta + 4 r^5 a^8 \cos^8 \theta - M a^{12} \cos^8 \theta \\
+ M a^{12} \cos^6 \theta + 3 r a^{12} \cos^6 \theta - 21 r^3 a^{10} \cos^6 \theta - 71 r^5 a^8 \cos^6 \theta \\
- 31 r^7 a^6 \cos^6 \theta + 9 r^3 a^{10} \cos^4 \theta - 47 r^5 a^8 \cos^4 \theta - 133 r^7 a^6 \cos^4 \theta \\
- 53 r^9 a^4 \cos^4 \theta + 9 r^5 a^8 \cos^2 \theta + r^7 a^6 \cos^2 \theta - 21 r^9 a^4 \cos^2 \theta \\
+ 12 M^2 r^7 a^4 + 21 M r^{10} a^2 + 30 M r^8 a^4 + 5 M r^6 a^6 + 3 r^7 a^6 + 3 r^{11} a^2 \cos^2 \theta \\
+ 8 r a^{12} \cos^8 \theta + 33 r^{11} a^2 + 19 r^9 a^4, \quad (B4) \]
\[ D_{\theta \phi} = -r^{20} - 3 r^{18} a^2 - 3 r^{16} a^4 - r^{14} a^6 - a^{20} \cos^{14} \theta - 7 r^{18} a^2 \cos^2 \theta - 105 r^8 a^{12} \cos^8 \theta - 35 r^{14} a^6 \cos^6 \theta - 21 r^{16} a^4 \cos^4 \theta - 63 r^{14} a^6 \cos^8 \theta - 21 r^{10} a^{10} \cos^4 \theta - 105 r^{12} a^8 \cos^6 \theta - 35 r^{10} a^{10} \cos^8 \theta - 63 r^{12} a^8 \cos^8 \theta - 35 r^{10} a^{10} \cos^6 \theta - 35 r^{12} a^8 \cos^4 \theta - 21 r^{10} a^{10} \cos^{12} \theta - 21 r^{14} a^6 \cos^2 \theta - 21 r^{16} a^4 \cos^2 \theta - 6 r^{17} a^2 - 12 r^{15} a^4 - 6 r^{13} a^6 - 12 M^2 r^{14} a^4 - 12 M^2 r^{12} a^6 - 8 M^3 r^{11} a^6 - r^6 a^{14} \cos^{14} \theta - 3 r^2 a^{18} \cos^{14} \theta - 3 r^4 a^{16} \cos^{14} \theta - 7 r^8 a^{12} \cos^{12} \theta - 21 r^6 a^{14} \cos^{12} \theta - 21 r^4 a^{16} \cos^{12} \theta - 7 r^2 a^{18} \cos^{12} \theta + 6 M r^5 a^{14} \cos^{14} \theta + 8 M^3 r^3 a^{14} \cos^{14} \theta + 12 M r^3 a^{16} \cos^{14} \theta - 12 M^2 r^2 a^{16} \cos^{14} \theta + 6 M r^5 a^{14} \cos^{14} \theta - 12 M^2 r^4 a^{14} \cos^{14} \theta + 36 M r^7 a^{12} \cos^{12} \theta + 66 M r^5 a^{14} \cos^{12} \theta - 60 M^2 r^6 a^{12} \cos^{12} \theta + 24 M r^3 a^{16} \cos^{12} \theta - 36 M^2 r^4 a^{14} \cos^{12} \theta + 32 M^3 r^5 a^{12} \cos^{12} \theta - 6 M r^5 a^{12} \cos^{12} \theta + 24 M^2 r^2 a^{16} \cos^{12} \theta - 24 M^3 r^3 a^{14} \cos^{12} \theta + 90 M r^9 a^{10} \cos^{10} \theta + 144 M r^7 a^{12} \cos^{10} \theta - 120 M^2 r^8 a^{10} \cos^{10} \theta + 18 M r^5 a^{14} \cos^{10} \theta + 48 M^3 r^7 a^{10} \cos^{10} \theta - 36 M r^3 a^{16} \cos^{10} \theta + 108 M^2 r^4 a^{14} \cos^{10} \theta - 96 M^3 r^5 a^{12} \cos^{10} \theta - 12 M^2 r^2 a^{16} \cos^{10} \theta + 24 M^3 r^3 a^{14} \cos^{10} \theta + 120 M r^{11} a^8 \cos^8 \theta + 150 M r^9 a^{10} \cos^8 \theta - 120 M^2 r^8 a^6 \cos^8 \theta - 60 M r^7 a^{12} \cos^8 \theta + 120 M^2 r^8 a^6 \cos^8 \theta - 90 M r^5 a^{14} \cos^8 \theta + 180 M^2 r^6 a^{12} \cos^8 \theta + 32 M^3 r^9 a^8 \cos^8 \theta - 144 M^3 r^7 a^{10} \cos^8 \theta - 60 M^2 r^4 a^{14} \cos^8 \theta + 96 M^3 r^5 a^{12} \cos^8 \theta - 8 M^3 r^3 a^{14} \cos^8 \theta + 90 M r^{13} a^6 \cos^6 \theta + 60 M r^{11} a^8 \cos^6 \theta - 150 M r^9 a^{10} \cos^6 \theta - 60 M^2 r^{12} a^6 \cos^6 \theta + 180 M^2 r^{10} a^8 \cos^6 \theta - 120 M r^7 a^{12} \cos^6 \theta + 120 M^2 r^8 a^6 \cos^6 \theta - 120 M^2 r^6 a^{12} \cos^6 \theta + 8 M^3 r^{11} a^6 \cos^6 \theta - 96 M^3 r^9 a^8 \cos^6 \theta + 144 M^3 r^7 a^{10} \cos^6 \theta - 32 M^3 r^9 a^8 \cos^6 \theta + 36 M r^{15} a^4 \cos^4 \theta - 18 M r^{13} a^6 \cos^4 \theta - 144 M r^{11} a^8 \cos^4 \theta - 90 M r^9 a^{10} \cos^4 \theta - 12 M^2 r^{14} a^4 \cos^4 \theta + 108 M^2 r^{12} a^6 \cos^4 \theta - 120 M^2 r^8 a^{10} \cos^4 \theta - 24 M^3 r^9 a^8 \cos^4 \theta + 96 M^3 r^7 a^{10} \cos^4 \theta - 48 M^3 r^7 a^{10} \cos^4 \theta + 6 M r^{17} a^2 \cos^2 \theta - 24 M r^{15} a^4 \cos^2 \theta - 66 M r^{13} a^6 \cos^2 \theta - 36 M r^{11} a^8 \cos^2 \theta + 24 M^2 r^{14} a^4 \cos^2 \theta - 36 M^2 r^{12} a^6 \cos^2 \theta - 60 M^2 r^{10} a^8 \cos^2 \theta + 24 M^3 r^{11} a^6 \cos^2 \theta - 32 M^3 r^9 a^8 \cos^2 \theta. \]
Appendix C: Curvature square terms

The explicit form of $R^2$, $R_{ij}^2 = R_{ij}R^{ij}$, and $C_{ij}^2 = C_{ij}C^{ij}$ is calculated up to $a^4$-order.

$R^2$ is given by

$$R^2 = -\frac{18M^2 \sin^2 \theta}{r^6} a^2 + \frac{6M^2 \sin^2 \theta}{r^8} \left( 4 + 13 \cos^2 \theta + \frac{12M \sin^2 \theta}{r} \right) a^4 + O(a^6). \quad \text{(C1)}$$

$R_{ij}R^{ij}$ takes the form

$$R_{ij}R^{ij} = \frac{6M^2}{r^6} + \frac{18M^2}{r^8} \left( 1 - 5 \cos^2 \theta - \frac{3M \sin \theta}{r} \right) a^2 + \frac{6M^2}{r^10} \left[ \frac{81M^2 \sin^4 \theta}{r^2} + \frac{M \sin^2 \theta}{r} (82 \cos^2 \theta - 17) + 81 \cos^4 \theta - 18 \cos^2 \theta - 17 \right] a^4 + O(a^6). \quad \text{(C2)}$$

Finally, $C_{ij}C^{ij}$ is given by

$$C_{ij}C^{ij} = \frac{3528M^4 \sin^2 \theta}{r^{16}} \left( 1 - \frac{2M \sin^2 \theta}{r} \right) a^4 + O(a^6). \quad \text{(C3)}$$

Appendix D: $E_{ij}^{(k)}$ up to $a^4$-order

We show the explicit form $E_{ij}^{(k)}$. All other terms not listed here are zero.

$E_{rr}^{(1)}$ is given by

$$E_{rr}^{(1)} = \frac{9M^2 \sin^2 \theta}{r^6 \sqrt{1 - \frac{2M}{r}}} a^2 - \frac{3M^2 \sin^2 \theta}{r^8 (1 - \frac{2M}{r})^{3/2}} \left[ 7 + 10 \cos^2 \theta + \frac{M}{r} (4 - 35 \cos^2 \theta) - \frac{30M^2 \sin^2 \theta}{r^2} \right] a^4 + O(a^6), \quad \text{(D1)}$$

$$E_{r\theta}^{(1)} = -\frac{12M^2 \sin^3 \theta \cos \theta}{r^7} \sqrt{1 - \frac{2M}{r}} a^4 + O(a^6), \quad \text{(D2)}$$

$$E_{\theta\theta}^{(1)} = -\frac{9M^2 \sin^2 \theta}{r^4} \sqrt{1 - \frac{2M}{r}} a^2 + \frac{3M^2 \sin^2 \theta}{r^6 \sqrt{1 - \frac{2M}{r}}} \left[ 4 + 10 \cos^2 \theta + \frac{M}{r} (4 - 35 \cos^2 \theta) - \frac{30M^2 \sin^2 \theta}{r^2} \right] a^4 + O(a^6), \quad \text{(D3)}$$
Explicitly, we show that $E_{ij}^{(2)} = 0.$

$E_{ij}^{(3)}$ is given by

$$E_{ij}^{(3)} = -\frac{9M^2 \sin^2 \theta a^2}{r^6 \sqrt{1 - \frac{2M}{r}}}$$

$$+ \frac{3M^2 \sin^2 \theta}{r^8 (1 - \frac{2M}{r})^{3/2}} \left[ 7 + 10 \cos^2 \theta + \frac{M}{r} (4 - 35 \cos^2 \theta) - \frac{30M^2 \sin^2 \theta}{r^2} \right] a^4$$

$$+ \mathcal{O}(a^6),$$

(D5)
\(E_{ij}^{(4)}\) is

\[
E_{rr}^{(4)} = -\frac{72 M^2}{r^8 \sqrt{1 - \frac{2M}{r}}} \left( 7 - 9 \cos^2 \theta - \frac{12 M \sin^2 \theta}{r} \right) a^2 \\
+ \frac{6 M^2}{r^{10} \left( 1 - \frac{2M}{r} \right)^{3/2}} \left[ 140 + 684 \cos^2 \theta - 984 \cos^4 \theta \right] a^4 \\
+ \frac{M}{r} \left( 52 - 3780 \cos^2 \theta + 4024 \cos^4 \theta \right) \\
- \frac{M^2}{r^2} \left( 1891 - 7254 \cos^2 \theta + 5363 \cos^4 \theta \right) + \frac{2502 M^3 \sin \theta}{r^3} a^6 \\
+ O(a^6),
\] (D9)

\[
E_{r\theta}^{(4)} = \frac{72 M^2 \sin \theta \cos \theta}{r^7 \sqrt{1 - \frac{2M}{r}}} \left( 7 - \frac{15 M}{r} \right) a^2 \\
+ \frac{24 M^2 \sin \theta \cos \theta}{r^9 \left( 1 - \frac{2M}{r} \right)^{3/2}} \left[ 90 - 246 \cos^2 \theta - \frac{M}{r} \left( 627 - 1298 \cos^2 \theta \right) \right] a^4 + O(a^6),
\] (D10)

\[
E_{\theta\theta}^{(4)} = \frac{72 M^2}{r^6 \sqrt{1 - \frac{2M}{r}}} \left[ 18 - 17 \cos^2 \theta - \frac{M}{r} \left( 81 - 79 \cos^2 \theta \right) + \frac{90 M^2 \sin^2 \theta}{r^2} \right] a^2 \\
- \frac{6 M^2}{r^8 \sqrt{1 - \frac{2M}{r}}} \left[ 224 + 1452 \cos^2 \theta - 1608 \cos^4 \theta \right] a^4 \\
+ \frac{M}{r} \left( 316 - 9888 \cos^2 \theta + 9424 \cos^4 \theta \right) - \frac{M^2}{r^2} \left( 6793 - 24582 \cos^2 \theta + 17789 \cos^4 \theta \right) \\
+ \frac{M^3}{r^3} \left( 10746 - 21492 \cos^2 \theta + 10746 \cos^4 \theta \right) a^4 + O(a^6),
\] (D11)

\[
E_{\phi\phi}^{(4)} = \frac{72 M^2 \sin^2 \theta}{r^6} \sqrt{1 - \frac{2M}{r}} \left( 17 - 16 \cos^2 \theta - \frac{45 M \sin^2 \theta}{r} \right) a^2 \\
- \frac{6 M^2 \sin^2 \theta}{r^8 \sqrt{1 - \frac{2M}{r}}} \left[ 92 + 1416 \cos^2 \theta - 1440 \cos^4 \theta \right] a^4 \\
+ \frac{M}{r} \left( 652 - 9708 \cos^2 \theta \right) - \frac{M^2}{r^2} \left( 6157 - 22830 \cos^2 \theta + 16678 \cos^4 \theta \right) \\
+ \frac{M^3}{r^3} \left( 9234 - 18468 \cos^2 \theta + 9234 \cos^4 \theta \right) a^4 + O(a^6).
\] (D12)
\( E^{(5)}_{ij} \) is given by

\[
E^{(5)}_{rr} = -\frac{4\mu W^2 M^2}{r^6 \sqrt{1 - \frac{2M}{r}}} \left[ -\frac{\mu W^2 M^2}{2r^8 (1 - \frac{2M}{r})^{3/2}} [168 - 229 \cos^2 \theta \\
- \frac{M}{r} (786 - 910 \cos^2 \theta) + \frac{904 M^2 \sin^2 \theta}{r^2} \right] a^2 \\
+ \frac{\mu W^2 M^2}{2r^{10} (1 - \frac{2M}{r})^{5/2}} [364 + 662 \cos^2 \theta - 1338 \cos^4 \theta \\
- \frac{M}{r} (1026 + 8021 \cos^2 \theta - 10221 \cos^4 \theta) \\
- \frac{M^2}{r^2} (4300 - 31809 \cos^2 \theta + 28710 \cos^4 \theta) \\
+ \frac{M^3}{r^3} (17532 - 52536 \cos^2 \theta + 35004 \cos^4 \theta) \\
- \frac{M^4}{r^4} (15228 - 31056 \cos^2 \theta + 15228 \cos^4 \theta) \right] a^4 + \mathcal{O}(a^6),
\]

(D13)

\[
E^{(5)}_{r\theta} = \frac{3\mu W^2 M^2 \sin \theta \cos \theta}{2r^7 \sqrt{1 - \frac{2M}{r}}} \left( 8 - \frac{11M}{r} \right) a^2 \\
+ \frac{3\mu W^2 M^2 \sin \theta \cos \theta}{2r^9 (1 - \frac{2M}{r})^{3/2}} [130 - 178 \cos^2 \theta - \frac{M}{r} (949 - 1103 \cos^2 \theta) \\
+ \frac{M^2}{r^2} (2194 - 2315 \cos^2 \theta) - \frac{1642 M^3 \sin^2 \theta}{r^3} \right] a^4 + \mathcal{O}(a^6),
\]

(D14)

\[
E^{(5)}_{r\phi} = -\frac{252 M^2 \sin^2 \theta \cos \theta}{r^8} \left( 4 - \frac{11M}{r} \right) a^2 \\
- \frac{12 M^2 \sin^2 \theta \cos \theta}{r^{10}} \left[ 234 - 990 \cos^2 \theta - \frac{M}{r} (1569 - 4068 \cos^2 \theta) + \frac{2419 M^2 \sin^2 \theta}{r^2} \right] a^4 \\
+ \mathcal{O}(a^6),
\]

(D15)
$$E_{\theta \theta}^{(5)} = \frac{7\mu W^2 M^2}{2r^4} \sqrt{1 - \frac{2M}{r}} + \frac{\mu W^2 M^2}{2r^6 \sqrt{1 - \frac{2M}{r}}} \left[ 240 - 248 \cos^2 \theta + \frac{1145M \cos^2 \theta}{r} + \frac{1298M^2 \sin^2 \theta}{r^2} \right] a^2$$

$$- \frac{\mu W^2 M^2}{4r^8 \left(1 - \frac{2M}{r}\right)^{3/2}} \left[ 738 + 2232 \cos^2 \theta - 2928 \cos^4 \theta \right.$$

$$- \frac{M}{r} \left(1236 + 24240 \cos^2 \theta - 25338 \cos^4 \theta\right) - \frac{M^2}{r^2} \left(17672 - 98136 \cos^2 \theta + 80349 \cos^4 \theta\right)$$

$$+ \frac{M^3}{r^3} \left(62664 - 173700 \cos^2 \theta + 111036 \cos^4 \theta\right)$$

$$- \frac{M^4}{r^4} \left(56532 - 113064 \cos^2 \theta + 56532 \cos^4 \theta\right) \left]\right. a^4 + O(a^6), \quad (D16)$$

$$E_{\theta \phi}^{(5)} = - \frac{168 M^2 \sin^3 \theta}{r^7} \left(9 - \frac{48M}{r} + \frac{61M^2}{r^2}\right) a^2$$

$$+ \frac{12M^2 \sin^3 \theta}{r^9} \left[ 24 + 1320 \cos^2 \theta - 1608 \cos^4 \theta + \frac{M}{r} \left(666 - 8862 \cos^2 \theta\right) \right.$$

$$- \frac{M^2}{r^2} \left(5294 - 16858 \cos^2 \theta\right) + \frac{8575M^3 \sin^2 \theta}{r^3} \left]\right. a^4 + O(a^6), \quad (D17)$$

$$E_{\phi \phi}^{(5)} = \frac{7\mu W^2 M^2 \sin^2 \theta}{2r^4} \sqrt{1 - \frac{2M}{r}} + \frac{\mu W^2 M^2 \sin^2 \theta}{2r^6 \sqrt{1 - \frac{2M}{r}}} \left[ 712 - 720 \cos^2 \theta - \frac{M}{r} \left(3426 - 3449 \cos^2 \theta\right) + \frac{4018M^2 \sin^2 \theta}{r^2} \right] a^2$$

$$- \frac{\mu W^2 M^2 \sin^2 \theta}{4r^8 \left(1 - \frac{2M}{r}\right)^{3/2}} \left[ 306 + 10140 \cos^2 \theta - 10404 \cos^4 \theta \right.$$

$$+ \frac{M}{r} \left(4788 - 91072 \cos^2 \theta + 86146 \cos^4 \theta\right)$$

$$- \frac{M^2}{r^2} \left(52752 - 303736 \cos^2 \theta + 250869 \cos^4 \theta\right)$$

$$+ \frac{M^3}{r^3} \left(146184 - 448836 \cos^2 \theta + 302652 \cos^4 \theta\right)$$

$$- \frac{M^4}{r^4} \left(124532 - 249064 \cos^2 \theta + 124532 \cos^4 \theta\right) \left]\right. a^4 + O(a^6). \quad (D18)$$
Finally, $E_{ij}^{(6)}$ is given by

\[
E_{rr}^{(6)} = \frac{5\mu^2 W^4 M^2}{4r^6 \sqrt{1 - \frac{2M}{r}}} \left[ -\frac{1}{4} \frac{\mu^2 W^4 M^2}{r^8 (1 - \frac{2M}{r})^{3/2}} \left\{ -10 + 70 \cos^2 \theta + \frac{M}{r} (123 - 238 \cos^2 \theta) \right\} a^2 \\
\left[ -\frac{M^2}{r^2} (196 \sin^2 \theta) \right] - \frac{1}{4} \frac{M}{r^{11}} \sqrt{1 - \frac{2M}{r}} \left( 2856 - 8568 \cos^2 \theta \right) \right] a^2 \right.
\]

\[
+ \frac{1}{8} \frac{\mu^2 W^4 M^2}{r^{10} (1 - \frac{2M}{r})^{5/2}} \left\{ -2 - 94 \cos^2 \theta + 756 \cos^4 \theta \right\} a^2 \\
+ \frac{M}{r} (108 + 2422 \cos^2 \theta - 5040 \cos^4 \theta) \\
+ \frac{M^2}{r^2} (1166 - 11242 \cos^2 \theta + 12472 \cos^4 \theta) \\
+ \frac{M^3}{r^3} (-5484 + 19080 \cos^2 \theta - 13596 \cos^4 \theta) \\
+ \frac{M^4}{r^4} (5532 - 11064 \cos^2 \theta + 5532 \cos^4 \theta) \right\} \}
\]

\[
+ \frac{1}{8} \frac{M^3}{r^{13} (1 - \frac{2M}{r})^{5/2}} \left\{ 32592 + 158544 \cos^2 \theta - 331104 \cos^4 \theta \right\} a^2 + O(a^6),
\]

(D19)
\[ E_{r\theta}^{(6)} = \left[ -\frac{9}{2} \frac{\mu W^4 M^2 \cos \theta \sin \theta}{r^7} \sqrt{1 - \frac{2M}{r}} \right. \\
\left. + \frac{3}{2} \frac{M^3 \cos \theta \sin \theta}{r^{10}} \left( 2758 - 5642 \frac{M}{r} \right) \right] a^2 \\
\left. \quad + \left[ -\frac{3}{2} \frac{\mu W^4 M^2 \cos \theta \sin \theta}{r^9} \left\{ -28 \cos^2 \theta + \frac{M}{r} \left( -35 + 150 \cos^2 \theta \right) \right\} \\
\left. \quad \quad + \frac{M^2}{r^2} \left( 146 - 264 \cos^2 \theta \right) - \frac{M^3}{r^3} \left( 152 \sin^2 \theta \right) \right\} \\
\left. \quad - \frac{3}{2} \frac{M^3 \cos \theta \sin \theta}{r^{12}} \left\{ 3106 + 36928 \cos^2 \theta - \frac{M}{r} \left( 3026 + 161020 \cos^2 \theta \right) \right\} \\
\left. \quad - \frac{M^2}{r^2} \left( 34584 - 202414 \cos^2 \theta \right) + \frac{M^3}{r^3} \left( 56172 \sin^2 \theta \right) \right\} \\
\left. \quad + \frac{3}{2} \frac{M^4 \sin^2 \theta}{r^{14}} \left\{ -2352 + \frac{M}{r} \left( 9408 - 4704 \cos^2 \theta \right) \right\} - \frac{M^2}{r^2} \left( 9408 \sin^2 \theta \right) \right] a^4 + O(a^6), \quad (D20) \]

\[ E_{r\phi}^{(6)} = -\frac{9}{4} \frac{\mu W^2 M^2 \cos \theta \sin^2 \theta}{r^8} \left( 4 + \frac{83M}{r} \right) a^2 \\
\left. + \frac{3}{4} \frac{\mu W^2 M^2 \cos \theta \sin^2 \theta}{r^{10}} \left\{ -42 + 90 \cos^2 \theta \right\} \\
\left. \quad + \frac{M}{r} \left( 273 + 2656 \cos^2 \theta \right) + \frac{M^2}{r^2} \left( 1928 \sin^2 \theta \right) \right\} a^4 + O(a^6), \quad (D21) \]
\[ E^{(6)}_{\theta \theta} = -\frac{1}{4} \mu^2 W^4 M^2 r^4 \sqrt{1 - \frac{2M}{r}} \]

\[ + \left[ \frac{1}{4} \mu^2 W^4 M^2 r^6 \left\{ 3 + 8 \cos^2 \theta - \frac{M}{r} (3 + 20 \cos^2 \theta) - \frac{M^2}{r^2} (8 \sin^2 \theta) \right\} \right] \]

\[ + \frac{1}{4} \frac{M^3}{r^9} \left\{ -2520 - 336 \cos^2 \theta + \frac{M}{r} (17808 - 12096 \cos^2 \theta) - \frac{M^2}{r^2} (25536 \sin^2 \theta) \right\} \]

\[ + \left[ -\frac{1}{8} \frac{\mu^2 W^4 M^2}{r^8} \left( 1 - \frac{2M}{r} \right)^{3/2} \left\{ -18 - 54 \cos^2 \theta + 180 \cos^4 \theta \right\} \right] \]

\[ + \frac{M}{r} \left( 114 + 486 \cos^2 \theta - 1056 \cos^4 \theta \right) - \frac{M^2}{r^2} \left( 10 + 1830 \cos^2 \theta - 2319 \cos^4 \theta \right) \]

\[ - \frac{M^3}{r^3} \left( 804 + 3168 \cos^2 \theta + 2364 \cos^4 \theta \right) + \frac{M^4}{r^4} \left( 1020 \sin^4 \theta \right) \]

\[ - \frac{1}{8} \frac{M^3}{r^{11}} \left( 1 - \frac{2M}{r} \right)^{3/2} \left\{ 24048 + 38304 \cos^2 \theta - 126624 \cos^4 \theta \right\} \]

\[ - \frac{M}{r} \left( 260784 - 340944 \cos^2 \theta - 182640 \cos^4 \theta \right) \]

\[ + \frac{M^2}{r^2} \left( 1156416 - 2631168 \cos^2 \theta + 1206240 \cos^4 \theta \right) \]

\[ - \frac{M^3}{r^3} \left( 2296704 - 5261376 \cos^2 \theta + 2964672 \cos^4 \theta \right) + \frac{M^4}{r^4} \left( 1669248 \sin^4 \theta \right) \]

\[ - \frac{1}{8} \frac{M^4}{r^{14}} \left( 1 - \frac{2M}{r} \right)^{3/2} \left\{ -28224 \sin^2 \theta + \frac{M}{r} \left( 112896 - 169344 \cos^2 \theta + 56448 \cos^4 \theta \right) \right\} \]

\[ - \frac{M^2}{r^2} \left( 112896 \sin^4 \theta \right) \right\} \right\} a^4 + \mathcal{O}(a^6), \quad \text{(D22)} \]

\[ E^{(6)}_{\theta \phi} = \frac{3}{4} \frac{\mu W^2 M^2 \sin^3 \theta}{r^9 \left( 1 - \frac{2M}{r} \right)} \left( 9 - \frac{108M}{r} + \frac{209M^2}{r^2} \right) a^2 \]

\[ + \left[ \frac{3}{4} \frac{\mu W^2 M^2}{r^9} \sin \theta \left\{ -48 + 168 \cos^2 \theta - 120 \cos^4 \theta \right\} \right] \]

\[ + \frac{M}{r} \left( 443 - 2119 \cos^2 \theta + 1676 \cos^4 \theta \right) \]

\[ - \frac{M^2}{r^2} \left( 2421 - 8282 \cos^2 \theta + 5861 \cos^4 \theta \right) + \frac{M^3}{r^3} \left( 4574 \sin^4 \theta \right) \]

\[ + \frac{3}{4} \frac{M^4 \sin^2 \theta}{r^{14}} \left( 4704 - 9408 \sin^2 \theta \frac{M}{r} \right) \right\} a^4 + \mathcal{O}(a^6), \quad \text{(D23)} \]
\[ E^{(6)}_{\phi\phi} = -\frac{1}{4} \mu^2 W^4 M^2 \sin^2 \theta \sqrt{1 - \frac{2M}{r}} \]

\[ + \left[ - \frac{1}{4} \mu^2 W^4 M^2 \sin^2 \theta \left\{ 10 - 21 \cos^2 \theta - \frac{M}{r} \left( 81 - 104 \cos^2 \theta + \frac{M^2}{r^2} \left( 124 \sin^2 \theta \right) \right) \right\} \]

\[ - \frac{1}{4} M^3 \sin^2 \theta \left\{ -5376 + 8232 \cos^2 \theta + \frac{M}{r} \left( 23520 - 29232 \cos^2 \theta \right) \right\} \]

\[ - \frac{M^2}{r^2} \left( 5536 \sin^2 \theta \right) \right\} a^2 \]

\[ + \left\{ - \frac{1}{4} \mu^2 W^4 M^2 \sin^2 \theta \left\{ -36 + 150 \cos^2 \theta - 222 \cos^4 \theta \right\} \right. \]

\[ + \frac{M}{r} \left( 240 - 1864 \cos^2 \theta + 2080 \cos^4 \theta \right) - \frac{M^2}{r^2} \left( 762 - 6262 \cos^2 \theta + 5979 \cos^4 \theta \right) \]

\[ + \frac{M^3}{r^3} \left( 1500 - 7560 \cos^2 \theta + 6060 \cos^4 \theta \right) - \frac{M^4}{r^4} \left( 1292 \sin^4 \theta \right) \right\} \]

\[ + \frac{1}{8} \frac{M^3 \sin^2 \theta}{r^{11}} \left( 1 - \frac{2M}{r} \right)^{3/2} \left\{ 7920 - 164592 \cos^2 \theta + 220944 \cos^4 \theta \right\} \]

\[ - \frac{M}{r} \left( 72096 - 1141968 \cos^2 \theta + 1332672 \cos^4 \theta \right) \]

\[ + \frac{M^2}{r^2} \left( 297504 - 2816544 \cos^2 \theta + 2787552 \cos^4 \theta \right) \]

\[ - \frac{M^3}{r^3} \left( 586368 - 2814720 \cos^2 \theta + 2228352 \cos^4 \theta \right) + \frac{M^4}{r^4} \left( 432768 \sin^4 \theta \right) \right\} \]

\[ + \frac{1}{8} \frac{M^4 \sin^2 \theta}{r^{14}} \left( 1 - \frac{2M}{r} \right)^{3/2} \left\{ 28224 - \frac{M}{r} \left( 112896 + 56448 \cos^2 \theta \right) \right\} \]

\[ + \frac{M^2}{r^2} \left( 112896 \sin^2 \theta \right) \right\} \right] a^4 + O(a^6), \]  

(D24)

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