Compactification of the moduli spaces of vortices and coupled vortices

By Gang Tian at MIT and Baozhong Yang* at Stanford

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Abstract

Vortices and coupled vortices arise from Yang-Mills-Higgs theories and can be viewed as generalizations or analogues to Yang-Mills connections and, in particular, Hermitian-Yang-Mills connections. We proved an analytic compactification of the moduli spaces of vortices and coupled vortices on hermitian vector bundles over compact Kähler manifolds. In doing so we introduced the concept of ideal coupled vortices and characterized the singularities of ideal coupled vortices as well as Hermitian-Yang-Mills connections.

1 Introduction

In the theory of holomorphic vector bundles on compact Kähler manifolds, one remarkable result is the theorem of Donaldson-Uhlenbeck-Yau on the Hitchin-Kobayashi correspondence between the stability of a holomorphic vector bundle and the existence of an irreducible Hermitian-Einstein metric on the bundle. This enables us to identify the moduli space of stable holomorphic structures on a complex vector bundle \( E \) with the moduli space of irreducible unitary Hermitian-Yang-Mills connections (HYM connections) with respect to any fixed Hermitian metric \( h \) on \( E \). Uhlenbeck’s removable singularity theorem and compactness theorems (\cite{22}, \cite{23}) for Yang-Mills connections on four manifolds gave a compactification of the moduli space of HYM connections on algebraic surfaces and more generally, of (anti-)self-dual connections any Riemannian four manifolds. With Uhlenbeck’s and Taubes’ work as the analytical basis, Donaldson defined his polynomial invariants for four manifolds via this compactified moduli space. Through Donaldson’s and other people’s work, this idea has been able to provide many deep and surprising results on the differential topology of four manifolds. Generalizations of the compactness theorem for Yang-Mills connections to higher dimensional manifolds have been given in \cite{16} and \cite{20}. One of the main results of \cite{20} is a detailed description of the analytic compactification of the moduli space of HYM connections on higher dimensional Kähler manifolds.

Looking back at their origins in physics, Yang-Mills theories are important special cases of the more general Yang-Mills-Higgs theories in physical models which describe interactions of fields and particles. In a Yang-Mills-Higgs theory people usually study connections coupled with sections of certain vector bundles, which are usually called the Higgs objects, and

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an action which contains, in addition to the curvature term, the interaction terms between
the connections and the Higgs objects. Yang-Mills-Higgs (YMH) connections are the critical
points for such an action functional. Well-studied YMH theories include Ginzburg-Landau
vortices on \( \mathbb{R}^2 \) and monopoles on \( \mathbb{R}^3 \) (See [13] for an introduction and some fundamental
results. [4] contains more recent developments.) For more examples of Yang-Mills-Higgs
theories, we refer the reader to the survey article [5]. The interactions of geometric analy-
sis, geometric topology, algebraic geometry and mathematical physics in the study of YMH
theories make this subject very intriguing.

There is a natural generalization of the vortex equations on \( \mathbb{R}^2 \) to equations on general
compact Kähler manifolds. Let \((E, h)\) be a smooth hermitian vector bundle on a compact
Kähler manifold \((M, \omega)\). Consider an integrable connection \(A\) and a section \(\phi \in \Gamma(E)\), the
Higgs field. The following equations are called the vortex equations,

\[
\begin{align*}
\bar{\partial}_A \phi &= 0, \\
\Lambda F_A - \frac{i}{2} \phi \circ \phi^* + \frac{i}{2} \tau I_E &= 0.
\end{align*}
\]

(1.1) \hspace{2cm} (1.2)

where \(\tau\) is a real parameter. We shall call a solution \((A, \phi)\) to (1.1) and (1.2) a \(\tau\)-vortex
or simply a vortex. A \(\tau\)-vortex is the absolute minimum point for the following YMH
functional

\[
\text{YMH}_\tau(A, \phi) = \int_M |F_A|^2 + |d_A \phi|^2 + \frac{1}{4} |\phi \circ \phi^* - \tau I_E|^2.
\]

A holomorphic pair \((\mathcal{E}, \phi)\) consists of a holomorphic vector bundle \(\mathcal{E}\) and a holomorphic
section \(\phi\) of \(\mathcal{E}\). In [3], stable holomorphic pairs were defined and a Hitchin-Kobayashi
type correspondence between stability of holomorphic pairs and the existence of irreducible
vortices on them were established.

In this paper we are interested in describing the compactification of the moduli space of
vortices and stable pairs. We shall use the results on the convergence and compactness of
pure Yang-Mills connections in [20] and the removable singularity theorems for Hermitian-
Einstein metrics in [8]. A key observation we used in our proofs is a result from O. García-
Prada [10]. This result identifies the vortex equations on bundles over a Kähler manifold
\(M\) with the dimensional reduction of the HYM equations under an \(SU(2)\) action on certain
associated bundles on the manifold \(M \times \mathbb{C}P^1\) (see Section 2 for details.) This correspondence
allows us to apply results on HYM connections to study vortices. In fact, this idea of
dimensional reduction has been explored by E. Witten et al.

A generalization of the vortex equations, the coupled vortex equations, were introduced
by García-Prada in [10]. These equations turn out to be very natural for the setup of the
dimensional reduction mentioned above. Let \((E_1, h_1)\) and \((E_2, h_2)\) be Hermitian vector
bundles on a compact Kähler manifold \((M, \omega)\). Consider integrable connections \(A_i\) on
\((E_i, h_i)\) \((i = 1, 2)\) and a section \(\phi\) of \(\text{Hom}(E_2, E_1)\). The equations we shall consider are

\[
\begin{align*}
\bar{\partial}_{A_1} \otimes \bar{A}_2^* \phi &= 0, \\
\Lambda F_{A_1} - \frac{i}{2} \phi \circ \phi^* + \frac{i}{2} \tau I_{E_1} &= 0, \\
\Lambda F_{A_2} + \frac{i}{2} \phi^* \circ \phi + \frac{i}{2} \tau' I_{E_2} &= 0,
\end{align*}
\]

(1.3) \hspace{2cm} (1.4) \hspace{2cm} (1.5)
where $A_1 \otimes A_2^*$ is the induced connection on $E_1 \otimes E_2^*$ and $\tau$ and $\tau'$ are real parameters. We see that $\tau$ and $\tau'$ are related by the Chern-Weil formula and hence there is only one independent parameter $\tau$. The equations (1.3), (1.4) and (1.5) are called the coupled $(\tau)$-vortex equations and solutions $(A_1, A_2, \phi)$ of them are called coupled $(\tau)$-vortices on $(E_1, E_2)$. The coupled vortex equations are also dimensional reductions of HYM equations on $M \times \mathbb{C}P^1$ (see Section 2 for details.)

We define $(A_1, A_2, \phi, S, C)$ as an ideal coupled $\tau$-vortex on hermitian vector bundles $(E_1, h_1)$ and $(E_2, h_2)$ on $M$ if the singularity set $S$ is a closed subset of $M$ of finite $H^{n-4}$ Hausdorff measure, the triple $(A_1, A_2, \phi)$ is smooth and satisfies the coupled vortex equations on $M \setminus S$, and $C$ is a holomorphic chain of codimension 2 on $M$. We can then define the moduli space $IV_\tau$ of ideal coupled vortices and a natural weak topology on it (see Section 4.)

Our first main result is

**Theorem 1.1 (Theorem 4.1)** The moduli space $IV_\tau$ is compact.

This compactness theorem easily implies the following compactification theorems.

**Theorem 1.2 (Theorem 4.3)** The moduli space $V_\tau$ of coupled $\tau$-vortices on hermitian vector bundles $(E_1, E_2)$ admits a compactification $\bar{V}_\tau$ which is naturally embedded in $IV_\tau$.

**Theorem 1.3 (Theorem 4.5)** The moduli space $V_\tau$ of $\tau$-vortices on a hermitian bundle $E$ over a compact Kähler manifold $(M, \omega)$ admits a compactification in the space of ideal coupled $\tau$-vortices on $E$ and $L$, where $L$ is the trivial line bundle with the product metric on $M$.

We remark here that the compactification and blowup phenomena of $\tau$-vortices appears to be clear only when we embed them into a space of coupled vortices (see Section 2 and 4 for details.) Using the Hitchin-Kobayashi type correspondences for vortices and coupled vortices, these theorems also give the compactification of the moduli space of corresponding holomorphic objects, i.e., stable pairs and triples (see Theorem 4.3 and Theorem 4.6).

Along with the proof of Theorem 4.1, we obtained the following removable singularity theorems for admissible HYM connections (see Section 3 for definition) and ideal coupled vortices, in which the (non-removable or essential) singularities of HYM connections and vortices are characterized precisely. Their proofs are based on Bando and Siu [8] and the recent work of Tao and Tian [21].

**Theorem 1.4** Let $(A, S)$ be an admissible HYM connection on a hermitian vector bundle $E$ over $M^m$. Then there exists $\varepsilon_1 = \varepsilon_1(m) > 0$ such that if we let

$$S_0 = \{ x \in S | \lim_{r \to 0} r^{4-2m} \int_{B_r(x)} |F_A|^2 dv \geq \varepsilon_1 \},$$

then $S_0$ is an analytic subvariety of $M$ of codimension $\geq 3$. The holomorphic bundle $E|_{M \setminus S}$ extends to a reflexive sheaf $E$ over $M$ and $S_0$ is the singularity set of $E$, i.e., the set where $E$ fails to be locally free. After a suitable smooth gauge transformation, the connection $A$ can be extended to be a smooth connection on $M \setminus S_0$. 

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Theorem 1.5 Let \((A_1, A_2, \phi, S, 0)\) be an ideal coupled vortex on hermitian vector bundles \(E_1\) and \(E_2\) over \(M\). There exists \(\varepsilon_2 = \varepsilon_2(m, \tau) > 0\) such that if we define
\[
S_0 = \{ x \in S \mid \lim_{r \to 0} r^{4-2m} \int_{B_r(x)} e_\tau(A_1, A_2, \phi) dv \geq \varepsilon_2 \},
\]
where \(e_\tau(\cdot, \cdot, \cdot)\) is the YM-H action density, then \(S_0\) is an analytic subvariety of \(M\) of codimension \(\geq 3\). The holomorphic bundles \(E_j|\overline{M\setminus S}\) for \(j = 1, 2\) extend to reflexive sheaves \(\mathcal{E}_j\) and \(S_0\) is the union of the singularity sets of \(\mathcal{E}_1\) and \(\mathcal{E}_2\). After a suitable gauge transformation, the triple \((A_1, A_2, \phi)\) can be extended smoothly over \(M \setminus S_0\).

There are some possible directions for further studies. Studies of the resolution of singularities of stable sheaves might give us a clearer picture of the boundary points of the compactified moduli spaces. One could also define the concepts of stability and semi-stability for reflexive sheaf pairs or sheaf triples and establish the corresponding Hitchin-Kobayashi type correspondences for them. The topology of the compactified moduli space and its relation with the underlying Kähler manifold and with the parameter \(\tau\) might also be interesting for studies. We hope to address some of these issues in a future paper.

In Section 2, we gave an introduction to the vortex equations and coupled vortex equations on compact Kähler manifolds and reviewed some relevant results, mainly from the references \([2, 10, 11]\). In Section 3, we collected some known results on Hermitian-Yang-Mills connections. In Section 4, we stated and proved our compactification theorems. Finally we proved the removable singularity theorems, Theorem 1.4 and Theorem 1.5, in Section 5.

2 Vortices, coupled vortices and dimensional reductions

Assume that \((M, \omega)\) is an \(m\)-dimensional compact Kähler manifold and \((E, h)\) is a hermitian vector bundle on \(M\). Let \(\mathcal{A}\) denote the set of all unitary connections on \((E, h)\). We define
\[
\mathcal{A}^{1,1} = \{ A \in \mathcal{A} : F_A^{0,2} = 0 \},
\]
the set of integrable connections on \(E\). Consider the following equations for a pair \((A, \phi)\) \(\in \mathcal{A}^{1,1} \times \Gamma(E)\).

\[
\begin{align*}
\bar{\partial}_A \phi &= 0, \\
\Lambda F_A - \frac{i}{2} \phi \otimes \phi^* + \frac{i}{2} \tau I_E &= 0.
\end{align*}
\]

where \(\tau\) is a real parameter, \(\phi^* \in \Gamma(E^*)\) is the dual of \(\phi\) with respect to the the metric \(h\) and \(\Lambda\) is the contraction with the Kähler form \(\omega\). In local coordinates, if \(\{dz^i, d\bar{z}^\bar{i}\}\) is a basis of \(T^*M\) and \(\omega = g_{ij}dz^i d\bar{z}^\bar{j}\), \((g^{ij}) = (g_{ij})^{-1}\), then
\[
\Lambda(f_{ij}dz^i d\bar{z}^\bar{j}) = g^{\bar{i}\bar{j}} f_{ij}.
\]

The equations (2.1) and (2.2) are called the vortex equations. Since (2.1) simply means that \(\phi\) is a holomorphic section of \(E\) with respect to the holomorphic structure defined by
\[\bar{\partial}_A,\ \text{we sometimes call \((2.2)\) the vortex equation. Taking trace of \((2.2)\) and integrating, we see that only for } \tau \text{ such that } \]
\[\mu(E) = \frac{\deg(E)}{\text{rank}(E)} \leq \frac{\tau \text{Vol}(M)}{4\pi},\]
the vortex equation can have solutions. When the equality in above holds, a vortex is given by a Hermitian-Yang-Mills connection \(A\) and \(\phi = 0\).

We define a functional for a pair \((A, \phi) \in \mathcal{A} \times \Gamma(E),\)
\[(2.3) \quad \text{YM}_{\tau}(A, \phi) = \int_M |F_A|^2 + |d_A \phi|^2 + \frac{1}{4} |\phi \otimes \phi^* - \tau I_E|^2\]
For \((A, \phi) \in \mathcal{A}^{1,1} \times \Gamma(E),\) we may compute directly that (see [2] for example),
\[\text{YM}_{\tau}(A, \phi) = \int_M 2|\bar{\partial}_A \phi|^2 + |\Lambda F_A - \frac{i}{2} \phi \otimes \phi^* + \frac{i}{2} \tau I_E|^2\]
\[+ \tau \int_M \text{Tr}(i\Lambda F_A) + \int_M \text{Tr}(F_A^2) \wedge \frac{\omega^{m-2}}{(m-2)!}\]
\[= \int_M 2|\bar{\partial}_A \phi|^2 + |\Lambda F_A - \frac{i}{2} \phi \otimes \phi^* + \frac{i}{2} \tau I_E|^2 + 2\pi \tau \deg(E) - 8\pi^2 \text{Ch}_2(E).\]

Hence the minimum of \(\text{YM}_{\tau}\) is the topological quantity \(2\pi \tau \deg(E) - 8\pi^2 \text{Ch}_2(E).\) This minimum is achieved if and only if \((A, \phi)\) satisfies the vortex equations.

There is another equivalent viewpoint of the vortex equations. We fix a holomorphic structure on a complex vector bundle \(E\) and denote the obtained holomorphic bundle by \(\mathcal{E}.\) Now we allow the hermitian metric \(h\) to vary. For each \(h\) there exists a unique unitary connection which is compatible with the holomorphic structure of \(\mathcal{E}.\) We denote this connection associated to \(h\) by \(A_h\) and its curvature by \(F_h.\) Assume that \(\phi\) is a holomorphic section of \(E.\) The following equation is called the vortex equation for the hermitian metric \(h,\)
\[(2.4) \quad \Lambda F_h - \frac{i}{2} \phi \otimes \phi^* + \frac{i}{2} \tau I_E = 0.\]
It is a standard result that the above two points of views, i.e., fixing the metric to consider special unitary connections and fixing the holomorphic structure to consider special metrics, are equivalent. We will sketch the idea of this equivalence here (see Chap. VII, §1 of [15] for details). We fix a holomorphic bundle \(\mathcal{E}\) and a holomorphic section \(\phi\) here. Suppose that \(\tilde{h}\) is a hermitian metric satisfying \((2.4).\) For any hermitian metric \(h\) on \(E,\) there exists \(g \in \mathcal{G}^C\) in the complex linear gauge group such that \(\tilde{h}(s, t) = h(gs, gt), \forall s, t \in \Gamma(E).\) Suppose that \(A\) and \(\tilde{A}\) are the connections associated to \(h\) and \(\tilde{h}\) respectively. Define \(A'\) by
\[d_{A'} = g \tilde{\partial}_{Ag}^{-1} + (g^*)^{-1} \partial_{Ag^*} = gd_{\tilde{A}}g^{-1}.\]
Then \(A'\) is a unitary connection with respect to \(h,\) and \(F_{A'} = gF_{\tilde{A}}g^{-1},\) hence \(A'\) solves the vortex equation \((2.2)\) with respect to the metric \(h.\)

Let \(\mathcal{E}\) be a rank \(r\) holomorphic vector bundle over \((M, \omega),\) and let \(\phi\) be a holomorphic section of \(\mathcal{E}\) and \(\tau\) be a real parameter. For background on coherent sheaves and stability, we refer the reader to Chapter V of [15].
Definition. The pair \((E, \phi)\) is said to be \(\tau\)-stable if the following conditions are satisfied,

1. \(\mu(E') < \hat{\tau} = \tau \Vol(M)/(4\pi)\) for every coherent subsheaf \(E' \subset E\) with \(\text{rank } E' > 0\).
2. \(\mu(E/E') > \hat{\tau}\) for every coherent subsheaf \(E' \subset E\) with \(0 < \text{rank } E' < r\) and \(\phi \in H^0(M, E')\).

We have the following Hitchin-Kobayashi type theorem from Theorem 2.1.6 and Theorem 3.1.1 of [2].

Theorem 2.1 If there exists a hermitian metric \(h\) on \(E\) which satisfies the vortex equation (2.4), then the bundle splits as \(E = E_\phi \oplus E'\) into a direct sum of holomorphic vector bundles, with \(E_\phi\) containing \(\phi\), such that \((E_\phi, \phi)\) is \(\tau\)-stable and the remaining summands (which together comprise \(E'\)) are all stable and each of slope \(\hat{\tau} = \tau \Vol(M)/(4\pi)\).

Conversely, if \((E, \phi)\) is \(\tau\)-stable, then there exists a hermitian metric \(h\) on \(E\) which is a solution of the vortex equation (2.4).

We define the moduli space of \(\tau\)-vortices on a hermitian vector bundle \((E, h)\) by

\[ V_\tau = \{(A, \phi) \text{ vortices on } (E, h)\}/G, \]

where \(G\) is the unitary gauge transformation group on \((E, h)\) and \(g(A, \phi) = (g(A), g \circ \phi)\) for any \(g \in G\). We define the moduli space of \((\tau\text{-})\)stable (holomorphic) pairs by

\[ M_\tau = \{[(E, \phi)] : (E, \phi) \text{ } \tau\text{-stable pairs with underlying bundle } E\}, \]

where \([\cdot]\) means the isomorphism class of holomorphic bundles with sections. Theorem 2.1 implies that there is an injection \(M_\tau \rightarrow V_\tau\). This fails to be a bijection whenever there is the reducible phenomenon described in the theorem. We let the exception set of values for \(\tau\) be

\[ T = \{\tau | \hat{\tau} = \mu(E') \text{ for } E' \text{ a subbundle of } E\}. \]

If \(\tau\) is not in \(T\), then \(V_\tau = M_\tau\). This identification of the moduli space of vortices with the moduli space of stable pairs is completely analogous to the Hitchin-Kobayashi correspondence.

As we have said in the introduction, vortex equations can be interpreted as the dimensional reduction of the Hermitian-Yang-Mills equation on a certain bundle over \(M \times S^2\). To better describe and explore this dimensional reduction, it is natural to introduce a generalization of the vortex equations.

We assume again that \((M, \omega)\) is an \(m\)-dimensional compact Kähler manifold and \((E_1, h_1), (E_2, h_2)\) are \(C^\infty\) complex vector bundles over \(M\) with hermitian metrics. Let \(A_1^{1,1} \times A_2^{1,1} \times \Gamma(\text{Hom}(E_2, E_1))\), we consider the following equations:

\begin{align*}
\bar{\partial}_{A_1^{1,1} \times A_2^{1,1}} \phi &= 0, \\
\Lambda F_{A_1} - i \frac{\phi \circ \phi^*}{2} + i \frac{\tau I_{E_1}}{2} &= 0, \\
\Lambda F_{A_2} + i \frac{\phi^* \circ \phi}{2} + i \frac{\tau' I_{E_2}}{2} &= 0.
\end{align*}
where \( A_1 \otimes A_2^* \) is the induced connection on \( E_1 \otimes E_2^* \), \( \phi^* \in \Gamma(\text{Hom}(E_1, E_2)) \) is the adjoint of \( \phi \) with respect to \( h_1 \) and \( h_2 \), and \( \tau \) and \( \tau' \) are real parameters. By the Chern-Weil theory, \( \tau \) and \( \tau' \) must satisfy the following relation

\[
(2.9) \quad \tau \text{ rank } E_1 + \tau' \text{ rank } E_2 = 4\pi \frac{\deg E_1 + \deg E_2}{\text{Vol}(M)},
\]

so that there is only one independent parameter \( \tau \).

We call a triple \( (A_1, A_2, \phi) \) a coupled \((\tau)-\)vortex if it satisfies equations (2.6), (2.7) and (2.8). Coupled vortices are the absolute minima of the following Yang-Mills-Higgs type functional on \( A_1^{1,1} \times A_2^{1,1} \times \Omega^0(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \),

\[
\text{YMHH}_r(A_1, A_2, \phi) = \int_M |F_{A_1}|^2 + |F_{A_2}|^2 + |d_{A_1 \otimes A_2^*} \phi|^2 + \frac{1}{4} |\phi \circ \phi^* - \tau I_{E_1}|^2 + \frac{1}{4} |\phi^* \circ \phi + \tau' I_{E_2}|^2 dv.
\]

We denote the integrand above by \( e_r(A_1, A_2, \phi) \) and call it the YMHH action density for the triple \( (A_1, A_2, \phi) \).

Next we describe the dimensional reduction in the setting of triples (for details and proofs see §3 of [10]). Let \( p : M \times S^2 \to M \), \( q : M \times S^2 \to M \) be the natural projections. Assume again that \( (E_1, h_1) \) and \( (E_2, h_2) \) are hermitian vector bundles on \( M \), and \( H^{\otimes 2} \) is the degree 2 line bundle with the standard metric \( h' \) (up to a constant) on \( \mathbb{C}P^1 = S^2 \). We shall consider the bundle

\[
F = F_1 \oplus F_2 = p^* E_1 \oplus (p^* E_2 \otimes q^* H^{\otimes 2})
\]

on \( M \times S^2 \) with the induced metric \( h = p^* h_1 \oplus p^* h_2 \otimes q^* h'_2 \). Consider the left \( SU(2) \) action on \( M \times S^2 \) which is trivial on \( M \) and standard on \( S^2 \) (i.e., coming from the Hopf fibration \( SU(2) \to S^2 \) and the product structure of \( SU(2) \)). There is a natural lift of this action to the bundle \( F \), which we shall describe below.

Note that the total space of \( p^* E_1 \) is \( E_1 \times S^2 \), and the action of \( SU(2) \) on \( p^* E_1 \) is defined to be trivial on \( E_1 \) and standard on \( S^2 \), similarly we define the action of \( SU(2) \) on \( p^* E_2 \). Recall that \( H^{\otimes 2} = SU(2) \times \rho_2 \mathbb{C} \), where we regard via the Hopf fibration \( SU(2) \) as a principle \( S^1 \) bundle over \( S^2 \) and \( \rho_2 \) is the representation \( S^1 \to S^1 = U(1) \) given by \( \rho_2 : e^{i\alpha} \to e^{i2\alpha} \). This gives a natural action of \( SU(2) \) on \( H^{\otimes 2} \) on the left. Since \( q^* H^{\otimes 2} = M \times H^{\otimes 2} \), we require the action of \( SU(2) \) on \( q^* H^{\otimes 2} \) to be trivial on \( M \) and as above on \( H^{\otimes 2} \).

Any unitary connection on \((F, h)\) is then of the form

\[
(2.10) \quad d_A = \begin{pmatrix} d_{\tilde{A}_1} & \beta \\ -\beta^* & d_{\tilde{A}_2} \end{pmatrix},
\]

where \( \tilde{A}_1, \tilde{A}_2 \) are connections on \((F_1, \tilde{h}_1)\) and \((F_2, \tilde{h}_2)\) respectively, and \( \beta \in \Omega^1(M \times S^2, \text{Hom}(F_2, F_1)) \). It can be shown (as in Prop. 3.5 of [10]) that for any \( SU(2) \)-invariant connection \( A \) on \((F, h)\), we have

\[
(2.11) \quad \tilde{A}_1 = p^* A_1, \tilde{A}_2 = p^* A_2 \otimes q^* A_2',
\]
where \( A_1 \) and \( A_2 \) are connections on \((E_1, h_1)\) and \((E_2, h_2)\) respectively, and \( A'_2 \) is the unique \( SU(2) \)-invariant connection on \((H^\otimes 2, h'_2)\). We also have

\[
\beta = p^* \phi \otimes q^* \alpha, \tag{2.12}
\]

where \( \phi \in \Omega^0(M, E_1 \otimes E_2^*), \) and \( \alpha \) is the unique \( SU(2) \)-invariant element of \( \Omega^1(S^2, H^\otimes -2) \), up to a constant factor. The proofs of these facts exploit the \( SU(2) \)-invariance of the objects and study the restriction to fibers to show that certain components have to vanish (for details see the proof of Prop. 3.5 in [10]).

Let \( A_1 \) and \( A_2 \) be the spaces of unitary connections on \((E_1, h_1)\) and \((E_2, h_2)\) respectively and \( A_{SU(2)} \) be the space of \( SU(2) \)-invariant unitary connections on \((F, h)\). Then after fixing the choice of the one form \( \alpha \), we have a one-to-one correspondence between

\[
A_{SU(2)} \leftrightarrow A_1 \times A_2 \times \Gamma(Hom(E_2, E_1)) \tag{2.13}
\]

given by \( A \leftrightarrow (A_1, A_2, \phi) \) as in (2.10), (2.11) and (2.12). If we restrict (2.13) to integrable connections, there is a one-to-one correspondence

\[
A^{1,1}_{SU(2)} \leftrightarrow A^{1,1}_1 \times A^{1,1}_2 \times \Gamma(Hom(E_2, E_1)) \tag{2.14}
\]

We denote by \( G_{SU(2)} \) the \( SU(2) \)-invariant gauge transformation group on \( F \) and \( G_i \) the unitary gauge transformation groups on \( E_i \) \((i = 1, 2)\). Using a similar argument to the above we can write every \( g \in G_{SU(2)} \) as

\[
g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},
\]

for \( g_1 \in G_1 \) and \( g_2 \in G_2 \). Hence there is a one-to-one correspondence of the gauge groups,

\[
G_{SU(2)} \leftrightarrow G_1 \times G_2. \tag{2.15}
\]

It is then clear that there is a one-to-one correspondence between the configuration spaces,

\[
A_{SU(2)} / G_{SU(2)} \overset{1:1}{\leftrightarrow} A_1 \times A_2 \times \Gamma(Hom(E_2, E_1)) / G_1 \times G_2,
\]

where \((g_1, g_2)(A_1, A_2, \phi) = (g_1(A_1), g_2(A_2), g_1 \circ \phi \circ g_2^{-1})\) is the gauge action of \( G_1 \times G_2 \) on triples.

From now on, we shall fix a Kähler metric

\[
\omega_{\sigma} = \omega \oplus \sigma \omega_{CP^1}
\]

on \( M \times S^2 \), where \( \omega_{CP^1} \) is the standard Fubini-Study metric on \( CP^1 \) so that \( \int_{CP^1} \omega_{CP^1} = 1 \), and \( \sigma \) is given by the following formula,

\[
\sigma = \frac{2r_1 \text{Vol}(M)}{(r_1 + r_2)\hat{\tau} - \text{deg} E_1 - \text{deg} E_2}, \tag{2.16}
\]

where \( r_1 = \text{rank} E_1 \) and \( r_2 = \text{rank} E_2 \) and \( \hat{\tau} = \tau \text{Vol}(M)/(4\pi) \). In what follows, we shall fix the choice of \( \alpha \) such that

\[
\alpha \wedge \alpha^* = \frac{i}{2} \sigma \omega_{CP^1}. \tag{2.17}
\]

8
Define the Yang-Mills functional for a connection $A$ on the hermitian vector bundle $F$ with respect to $\omega_\sigma$ by

$$\text{YM}_\sigma(A) = \int_{M \times S^2} \left| F_A \right|^2 \frac{\omega_\sigma^{m+1}}{(m+1)!},$$

where $| \cdot |_\sigma$ is the norm induced by the hermitian metric on $F$ and the Kähler metric $\omega_\sigma$ on $M \times S^2$.

If $A \in \mathcal{A}^{SU(2)}$ corresponds to $(A_1, A_2, \phi) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \Gamma(\text{Hom}(E_2, E_1))$ in (2.13), then we have the following lemma relating the energy densities of $\text{YMH}_\tau$ and $\text{YM}_\sigma$.

**Lemma 2.1**

(2.18) $$|F_A|_\sigma^2 = e_\tau(A_1, A_2, \phi) + c(\tau),$$

where $c(\tau)$ is a constant depending only on $\tau$, $\text{Vol}(M)$ and the degrees of $E_1$ and $E_2$.

**Proof.** Via the expression (2.10), we compute that

$$F_A = \begin{pmatrix} p^*F_{A_1} - \beta \wedge \beta^* & d_{A_1 \otimes A_2} \beta \\ -(d_{A_1 \otimes A_2} \beta^*)^* & p^*F_{A_2} - 4\pi i\omega_{CP^1} + \beta^* \wedge \beta \end{pmatrix}.$$  

We also note that with the convention of (2.17),

$$\beta \wedge \beta^* = \frac{i}{2}p^*(\phi \phi^*)\omega_{CP^1},$$

$$\beta^* \wedge \beta = -\frac{i}{2}p^*(\phi^* \phi)\omega_{CP^1}.$$  

These together give us (for simplicity we omit the $p^*$'s in the following equation),

$$|F_A|_\sigma^2 = |F_{A_1}|^2 + \frac{1}{4} |\phi \phi^*|^2 + |F_{A_2}|^2 + \frac{1}{4} |\phi^* \phi| - \frac{8\pi}{\sigma} |E_2|^2$$

$$+ 2|\alpha|^2|d_{A_1 \otimes A_2} \phi|^2 + 2|d_{A_2} \alpha|^2|\phi|^2$$

$$= e_\tau(A_1, A_2, \phi) + \left(\frac{16\pi^2}{\sigma^2} r_2^2 - \frac{1}{4} (\sigma^2 r_1^2 + \tau^2 r_2^2)\right).$$

In the above we used the fact that $d_{A_2} \alpha = 0$, $|\alpha|^2_\sigma = |\alpha \wedge \alpha^*| = 1/2$, and the identity $\frac{4\pi}{\sigma} = \tau - \tau'$. \hfill \Box

As a corollary of Lemma 2.1, we have the following identity,

(2.19) $$\text{YM}_\sigma(A) = \sigma \text{YMH}_\tau(A_1, A_2, \phi) + C(\tau),$$

where $C(\tau)$ is a geometrical constant depending on $\tau$.

By direct computation it can be shown (see §3 of [10]) that under the correspondance (2.13), a connection $A$ on $M \times S^2$ is a Hermitian-Yang-Mills connection with respect to $\omega_\sigma$ if and only if $(A_1, A_2, \phi)$ is a coupled $\tau$-vortex, where $\sigma$ is related to $\tau$ by (2.16).

Assume that rank $E_2 = 1$, we can define a concept of stability for holomorphic triples $(E_1, E_2, \phi)$, where $E_1$ and $E_2$ are holomorphic bundles with underlying topological bundle $E_1$ and $E_2$ respectively and $\phi$ is a holomorphic section of $\text{Hom}(E_2, E_1)$.  

9
**Definition.** Assume that rank\(E_2 = 1\). We define a triple \((E_1, E_2, \phi)\) to be \(\tau\)-stable if the pair \((E_1 \otimes E_2^*, \phi)\) is \(\tau\)-stable.

We have the following theorem (Theorem 4.33 in [10]).

**Theorem 2.2** Let \((A_1, A_2, \phi)\) be a coupled \(\tau\)-vortex and let \(E_1 = (E_1, \bar{\partial}_{A_1})\) and \(E_2 = (E_2, \bar{\partial}_{A_2})\). Then \(E_1\) decomposes as a direct sum \(E_1 = E' \oplus E''\), such that \(\phi \in \text{Hom}(E_2, E')\), \((E', E_2, \phi)\) is \(\tau\)-stable and \(E''\) is a direct sum of stable bundles of the slope \(\hat{\tau} = \tau \text{Vol}(M)/(4\pi)\).

Conversely, if \((E_1, E_2, \phi)\) is \(\tau\)-stable, then for any hermitian metrics \(h_1, h_2\) on \(E_1, E_2\) respectively, there exists a solution \((A_1, A_2, \phi)\) to the coupled \(\tau\)-vortex equations such that \(E_1 = (E_1, \bar{\partial}_{A_1})\) and \(E_2 = (E_2, \bar{\partial}_{A_2})\) and the solution is unique up to unitary gauge transformations.

The following discussion is similar to the case of vortices and stable pairs. We define the moduli space of coupled \(\tau\)-vortices on hermitian bundles \(E_1\) and \(E_2\) by

\[
\mathcal{V}_\tau = \{(A_1, A_2, \phi) \text{ coupled vortices on } (E_1, E_2)\}/\mathcal{G}_1 \times \mathcal{G}_2,
\]

where \(\mathcal{G}_1\) and \(\mathcal{G}_2\) are the unitary gauge transformation groups on \(E_1\) and \(E_2\). If rank \(E_2 = 1\), we can define the moduli space of stable triples by

\[
\mathcal{M}_\tau = \{[(E_1, E_2, \phi)] : (E_1, E_2, \phi) \text{ } \tau\text{-stable triple with underlying bundles } E_1 \text{ and } E_2\},
\]

where \([\cdot]\) means the holomorphic isomorphism class. Theorem 2.2 implies that when rank \(E_2 = 1\), there is an injection \(\mathcal{M}_\tau \hookrightarrow \mathcal{V}_\tau\). This fails to be a bijection when there is the reducible phenomenon described in the theorem. We let

\[
(2.20) \quad \mathcal{T} = \{\tau|\hat{\tau} = \mu(E') \text{ for } E' \text{ a subbundle of } E_1\}.
\]

If \(\tau\) is not in \(\mathcal{T}\), then \(\mathcal{V}_\tau = \mathcal{M}_\tau\).

Next we consider the relation between stable pairs and triples. Let \(L\) be the trivial smooth line bundle over \(M\) and let \(\mathcal{M}_\tau\) be the moduli space of \(\tau\)-stable triples with the underlying smooth bundles being \(E\) and \(L\). Let \(\mathcal{E}\) and \(\mathcal{L}\) be holomorphic structures on \(E\) and \(L\). Since by definition, \((\mathcal{E}, \mathcal{L}, \phi)\) is \(\tau\)-stable if and only if \((\mathcal{E} \otimes \mathcal{L}^*, \phi)\) is stable, we have a map \(\mathcal{M}_\tau \rightarrow M_\tau\) given by

\[
[(\mathcal{E}, \mathcal{L}, \phi)] \mapsto [(\mathcal{E} \otimes \mathcal{L}^*, \phi)].
\]

The group of holomorphic bundles supported by \(L\), \(\text{Pic}^0(M)\), acts on \(M_\tau\) by

\[
(\mathcal{E}, \mathcal{L}, \phi) \mapsto (\mathcal{E} \otimes \mathcal{U}, \mathcal{L} \otimes \mathcal{U}, \phi), \quad \text{for } \mathcal{U} \in \text{Pic}^0(M).
\]

It is then clear that the above map gives an identification

\[
(2.21) \quad M_\tau \cong \mathcal{M}_\tau / \text{Pic}^0(M)
\]

Finally we consider the relation between the vortex equations and the coupled vortex equations. We let \((E_1, h_1) = (E, h)\) of rank \(r\) and \((E_2, h_2) = (L, h_0)\), where \(L\) is the trivial bundle and \(h_0\) is the product metric. Then a coupled \(\tau\)-vortex \((A, A', \phi)\) on \((E, L)\) satisfies

\[
(2.22) \quad \bar{\partial}_A \phi = 0,
\]

\[
(2.23) \quad \Lambda F_A - \frac{i}{2} \phi \circ \phi^* + \frac{i}{2} \tau \Lambda E = 0,
\]

\[
(2.24) \quad \Lambda F_{A'} + \frac{i}{2} \phi^* \circ \phi + \frac{i}{2} \tau' = 0,
\]
where $\tau r + \tau' = 4\pi \deg E/\text{Vol}(M)$. We see that (2.22) and (2.23) are actually the vortex equations on $(E,h)$. Hence $(A,\phi)$ is a vortex on $(E,h)$. On the other hand, if $(A,\phi)$ is a solution to the vortex equations (2.22) and (2.23) on $(E,h)$, then it is easy to check that the linear equation (2.24) admits a solution $A'$ (see for example [11] p.541 - 542). The solution $A'$ is unique if we fix the holomorphic structures on $L$ and require $A'$ to be compatible with the given holomorphic structure. Therefore, if we fix the trivial holomorphic structure on $L$, there is an embedding

\[(2.25)\quad V_{\tau} \hookrightarrow V_{\tau},\]

For each holomorphic structure on $L$, there is a unique solution $A'$. Hence if we take all the holomorphic structures on $L$ into consideration, then there is an identification

\[(2.26)\quad V_{\tau} \cong V_{\tau}/\text{Pic}^0(M).\]

3 Hermitian-Yang-Mills connections

In this section we collected some results on the blow-up phenomena of Yang-Mills connections and in particular, some results on Hermitian-Yang-Mills connections. The main references for this section are [20] and [8].

Assume that $M$ is an $n$-dimensional Riemannian manifold and $E$ is a smooth vector bundle on $M$ with a compact structure group $G$. Let $A$ be the set of all $G$-connections on $E$.

The Yang-Mills functional $\text{YM} : A \to \mathbb{R}$ is defined by

$$\text{YM}(A) = \int_M |F_A|^2 dv.$$ 

A smooth connection $A$ is called a Yang-Mills connection if and only if $A$ is a critical point of the functional $\text{YM}$, or equivalently, $A$ satisfies the Yang-Mills equation

$$d^*_A F_A = 0.$$ 

Definition. 1) $(A,S)$ is called an admissible connection if $S \subset M$, $H^{n-4}(S) = 0$, $A$ is a smooth connection on $M \setminus S$ and $\text{YM}(A) < \infty$. $S$ is called the singular set of $A$.

2) $(A,S)$ is called an admissible Yang-Mills connection if $(A,S)$ is an admissible connection and $d^*_A F_A = 0$ on $M \setminus S$.

Sometimes, when the singular set $S$ is understood, we simply say $A$ is an admissible connection.

By using variations generated by a vector field on $M$, we can derive the following first variation formula for smooth Yang-Mills connections (first derived by Price [17]),

\[(3.1)\quad \int_M |F_A|^2 \text{div} X - 4 \sum_{1 \leq i < j \leq n} \langle F_A(\nabla e_i, X, e_j), F_A(e_i, e_j) \rangle dv = 0.\]

This formula is true for any compactly supported $C^1$ vector field $X$ on $M$. This motivates the following definition.
Definition. An admissible connection \((A, S)\) is said to be stationary if the first variation formula is true for \(A\) with any compactly supported \(C^1\) vector field \(X\) on \(M\).

Assume that \(A\) is a stationary connection. By using a cutoff of the radial vector field \(X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\) in the first variation formula, we can obtain the following important monotonicity formula first shown by Price [17] (also see Tian [20] for a more general version). Let \(\text{injrad}(x)\) denotes the injective radius of \(x \in M\).

**Proposition 3.1** For any \(x \in M\), there exist positive constants \(a = a(x)\) and \(r_x < \text{injrad}(x)\) which only depend on the supremum bound of the curvature of \(M\), such that if \(0 < \sigma < \rho \leq r_x\), then

\[
\rho^{4-n} e^{\rho^2} \int_{B_{\rho}(x)} |F_A|^2 \, dv - \int_{B_{\rho}(x)} |F_A|^2 \, dv \\
\geq 4 \int_{B_{\rho}(x) \setminus B_{\sigma}(x)} r^{4-n} \left| \frac{\partial}{\partial r} F_A \right|^2 \, dv.
\]

If \(M\) is flat, then we may take \(a = 0\).

We have the following a priori pointwise estimate for smooth Yang-Mills connections by Uhlenbeck and also by Nakajima [16]. It can be proven by using the Bochner-Weitzenböck formula and a method similar to Schoen’s method (IX.4.2 in [19]) in proving the a priori pointwise estimate for stationary harmonic maps.

**Proposition 3.2** Assume that \(A\) is a smooth Yang-Mills connection. There exist \(\varepsilon_0 = \varepsilon_0(n) > 0\) and \(C = C(M, n) > 0\) such that for any \(x \in M\), \(\rho < r_x\), if \(\rho^{4-n} \int_{B_{\rho}(x)} |F_A|^2 \, dv \leq \varepsilon_0\), then

\[
\sup_{B_{\frac{3}{2} \rho}(x)} \frac{1}{\rho^2} |F_A|(x) \leq C \left( \int_{B_{\rho}(x)} |F_A|^2 \, dv \right)^{\frac{1}{2}}.
\]

Using the a priori pointwise estimate and monotonicity formula, compactness theorems about Yang-Mills connections were first proven in Uhlenbeck [23] and Nakajima [16]. We quote the following compactness theorem from [20].

**Proposition 3.3** Assume that \(\{(A_i, S_i)\}\) is a sequence of stationary admissible Yang-Mills connections on \(E\), with \(YM(A_i) \leq \Lambda\), where \(\Lambda\) is a constant. Let \(S_{\text{cls}} = \limsup_{i \to \infty} S_i\). Assume also that \(H^{n-4}(S_{\text{cls}}) = 0\). Then there exist a subsequence \(\{A_i\}\), a closed subset \(S_b\) of \(M\) with \(H^{n-4}(S_b \cap K) < \infty\) for any compact subset \(K \subset M\), a nonnegative \(H^{n-4}\)-integrable function \(\Theta\) on \(S_b\), gauge transformations \(\sigma_i \in \Gamma(\text{Aut} P)\) and a smooth Yang-Mills connection on \(M \setminus S_b\), such that the following holds:

1. On any compact set \(K \subset M \setminus (S_b \cup S_{\text{cls}})\), \(\sigma_i(A_i)\) converges to \(A\) in \(C^\infty\) topology.
2. \(|F_{A_j}|^2 \, dv \to |F_A|^2 \, dv + \Theta H^{n-4}|S_b\) weakly as measures on \(U\).

**Remarks.** 1) We shall make a little remark about the condition \(H^{n-4}(S_{\text{cls}}) = 0\) here. This condition will be trivially satisfied if all \(A_i\) are smooth YM connections, i.e., all \(S_i\)'s are empty. This condition is necessary for the application of the a priori estimates to extract
a convergent subsequence. We conjecture that the theorem is true without this condition, but the proof will need a very general regularity theorem for stationary admissible YM connections, which has not been proved yet.

2) The density function $\Theta$ is defined by
\[
\Theta(x) = \lim_{\rho \to 0} \liminf_{j \to \infty} \rho^{4-n} \int_{B_\rho(x)} |F_{A_{ij}}|^2 dv.
\]

3) The closed set $S_b$ is given by
\[
S_b = \{ x \in M : \Theta(x) \geq \varepsilon_0 \}. \tag{3.4}
\]
where $\varepsilon_0$ is as in Prop. 3.2.

Define
\[
T = \text{the closure of } \{ x \in S_b | \Theta(x) > 0, \lim_{r \to 0^+} r^{4-n} \int_{B_r(x)} |F_A|^2 dv = 0 \}.
\]

It is easy to show that the measure $\Theta H^{n-4} |S_b$ is equal to $\Theta H^{n-4} |T$. $(T, \Theta)$ is called the blow-up locus of the sequence $\{A_{ij}\}$. The set $T$ may be shown to be rectifiable (see §3.3 of [20]), and hence $(T, \Theta)$ defines a rectifiable varifold. We know very little about the blow-up set of sequences of general Yang-Mills connections without further restrictions.

The most important examples of Yang-Mills connections include self-dual and anti-self-dual connections on four manifolds, and Hermitian-Yang-Mills connections (also called Hermitian-Einstein connections in literature) on Hermitian vector bundles over Kähler manifold. Assume that $(M, \omega)$ is an $m$-dimensional Kähler manifold and $(E, h)$ is a Hermitian vector bundle over $M$. A unitary connection $A$ on $(E, h)$ is a Hermitian-Yang-Mills connection (HYM connections) if it is integrable and
\[
\Lambda F_A = \lambda I_E,
\]
where $\lambda$ is a constant. HYM connections are the absolute minima of the Yang-Mills functional.

One of the main results in [20] (Theorem 4.3.3) is the following characterization of the blow-up locus of a sequence of HYM connections (in [20], a more general class of connections, $\Omega$-anti-self-dual connections are treated).

**Theorem 3.1** Let $(M, \omega)$ be an $m$-dimensional compact Kähler manifold and $(E, h)$ a hermitian vector bundle over $M$. Let $\{A_i\}$ be a sequence of Hermitian-Yang-Mills connections on $E$. Then by passing to a subsequence, $A_i$ converges to an admissible Hermitian-Yang-Mills connection $A$ (in the sense in Prop. 3.3) with the blow-up locus equivalent as a $(2m-4)$ varifold to $(S, \Theta)$, such that $S = \bigcup \alpha S_\alpha$ is a countable union of $(m-2)$ dimensional holomorphic subvarieties $S_\alpha$‘s, and $\frac{1}{\Theta_s} |\Theta_s |S_\alpha$ are positive integers for every $\alpha$. There is the following convergence of measures,
\[
|F_{A_i}|^2 dv \to |F_A|^2 dv + \Theta H^{2m-4} |S.
\]

The following removable singularity/extension theorem is proved in [8] and the proof is based on results in [1], [18].
Because of the finiteness of the YMH energy of the triple \((A_1, A_2, \phi)\), we shall leave the details to the reader as it is completely analogous to (and simpler than) what we shall do about vortices and coupled vortices in the next section.

### 4 Compactification of the moduli spaces of vortices and coupled vortices

Assume as in Section 2 that \((M, \omega)\) is an \(m\)-dimensional compact Kähler manifold, and that \((E_1, h_1)\) and \((E_2, h_2)\) are Hermitian vector bundles on \(M\).

**Definition.** An ideal coupled \((\tau-)\) vortex on the Hermitian bundles \((E_1, h_1)\) and \((E_2, h_2)\) is a quintuple \((A_1, A_2, \phi, S, C)\) such that the following holds:

- \(S\) is a closed subset of finite \(H^{n-4}\) measure of \(M\).
- \(A_1\) and \(A_2\) are smooth integrable connections on \(E_1|_{M\setminus S}\) and \(E_2|_{M\setminus S}\) respectively.
- \(\phi\) is a holomorphic section of \(\text{Hom}(E_2, E_1)|_{M\setminus S}\).
- \(\phi\) satisfies the coupled \(\tau\)-vortex equations \((2.6), (2.7)\) and \((2.8)\) on \(M\setminus S\). \(C = (T, \Theta)\) is a \((2m-4)\)-dimensional current on \(M\). \(\text{supp}\, C = T = \bigcup_\alpha T_\alpha\) is a countable union of \((m-2)\)-dimensional holomorphic subvarieties of \(M\), and \(\frac{1}{8\pi^2} \Theta|_{T_\alpha}\) are positive integers. In other words, \(\frac{1}{8\pi^2} C\) is a holomorphic chain of codimension 2 on \(M\).

We also require the following energy identity to be satisfied:

\[
\text{YMH}_\tau(A_1, A_2, \phi) + \|C\| = \text{YMH}_\tau(A_1, A_2, \phi) + \int_M \Theta dH^{2m-4}\{T = E(\tau)\},
\]

where \(E(\tau)\) is the value of the YMH functional of a smooth coupled \(\tau\)-vortex on \((E_1, h_1)\) and \((E_2, h_2)\), which is a geometric constant depending on \(\tau\).

**Remark.** Assume that \((A_1, A_2, \phi, S, C)\) is an ideal coupled vortex on \((E_1, E_2)\). The integrable connections \(A_1\) and \(A_2\) give holomorphic structures on \(E_1|_{M\setminus S}\) and \(E_2|_{M\setminus S}\). Because of the finiteness of YMH energy of the triple \((A_1, A_2, \phi)\), we see via Theorem 3.2 that these holomorphic bundles on \(M\setminus S\) extend to reflexive sheaves \(\mathcal{E}_1\) and \(\mathcal{E}_2\) over \(M\). Because \(\phi \in \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)|_{M\setminus S}\) and \(\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)\) is reflexive and hence normal, we have that
\[ \phi \text{ extends as a sheaf homomorphism from } E_1 \text{ to } E_2. \] Therefore, an ideal coupled vortex gives rise to two reflexive sheaves \( E_1 \) and \( E_2 \) together with a sheaf homomorphism \( \phi \in \text{Hom}(E_1, E_2) \).

It is not hard to see via Theorem 3.2 that \( A_1 \) and \( A_2 \) extend as smooth connections to the common locally free part of \( E_1 \) and \( E_2 \), which is an open set whose complement is an analytic subvariety of \( M \) of codimension at least 3.

We define an equivalence relation among ideal coupled vortices in the following way. Let \((A_1, A_2, \phi, S, C) \sim (A_1', A_2', \phi', S', C')\) if \( C = C' \) as currents and there exists a closed subset \( S'' \) of \( M \) with finite \( H^{n-4} \) measure such that \((A_1, A_2, \phi)\) and \((A_1', A_2', \phi')\) are gauge equivalent via smooth gauge transformations on \( M \setminus S'' \), i.e., there exist smooth gauge transformations \( g_i \) on \( E_i|_{M \setminus S''} \) for \( i = 1, 2 \) such that \((g_1(A_1), g_2(A_2), g_1 \circ \phi \circ g_2^{-1}) = (A_1', A_2', \phi')\) over \( M \setminus S'' \).

Let the moduli space of ideal coupled vortex \((A_1, A_2, \phi, S, C)\) if there exist smooth gauge transformations \( g_i \) on \( E_i|_{M \setminus S''} \) for \( i = 1, 2 \) such that \((g_1(A_1), g_2(A_2), g_1 \circ \phi \circ g_2^{-1}) = (A_1', A_2', \phi')\) over \( M \setminus S'' \).

Let the moduli space of ideal coupled vortex \((A_1, A_2, \phi, S, C)\) if there exist smooth gauge transformations \( g_i \) on \( E_i|_{M \setminus S''} \) for \( i = 1, 2 \) such that \((g_1(A_1), g_2(A_2), g_1 \circ \phi \circ g_2^{-1}) = (A_1', A_2', \phi')\) over \( M \setminus S'' \).

We define \( IV_\tau = IV_\tau(E_1, E_2) = \{ \text{ideal coupled } \tau\text{-vortices on } (E_1, h_1) \text{ and } (E_2, h_2) \}/\sim \).

We define the weak topology on \( IV_\tau \) via the following notion of weak convergence. We say a sequence of ideal coupled vortices \((A_{i,1}, A_{i,2}, \phi_i, S_i, C_i)\) weakly converges to an ideal coupled vortex \((A_1, A_2, \phi, S, C)\) if there exist smooth gauge transformations \( g_{i,j} \) on \( E_j|_{M \setminus (S_i \cup S)} \) \((j = 1, 2)\) such that

\[
(g_{i,1}(A_{i,1}), g_{i,2}(A_{i,2}), g_{i,1} \circ \phi_i \circ g_{i,2}) \to (A_1, A_2, \phi) \text{ in } C_c^\infty(M \setminus S),
\]

and there is the following convergence of measures,

\[
e_\tau(A_{i,1}, A_{i,2}, \phi) dv + 8\pi^2 C_i \to e_\tau(A_1, A_2, \phi) dv + 8\pi^2 C.
\]

It is easy to see that this weak convergence descends to \( IV_\tau \) to define the weak convergence of equivalence classes of ideal coupled vortices and the limit of a sequence in \( IV_\tau \) is unique. We note that the moduli space of ideal coupled vortex \((A_1, A_2, \phi, S, C)\) if there exist smooth coupled vortex \((A_1, A_2, \phi)\) and thus associates an ideal coupled vortex to it.

With the above definitions made, we have the following compactness theorem.

**Theorem 4.1** Assume that \((E_1, h_1)\) and \((E_2, h_2)\) are hermitian complex vector bundles over a compact Kähler manifold \( M \). Then the moduli space \( IV_\tau \) of ideal coupled vortices on \((E_1, E_2)\) is compact in the weak topology.

**Proof.**

Let \((A_{i,1}, A_{i,2}, \phi_i, S_i, C_i)\) be a sequence of coupled \( \tau \)-vortices on hermitian bundles \( E_1 \) and \( E_2 \). We adopt the notations from Section 2. Let \( \sigma \) be defined by \( \tau \) in \((2.16)\). Let \( F = p^*E_1 \oplus p^*E_2 \otimes q^*H^{\otimes 2} \) be the \( SU(2) \)-equivariant Hermitian vector bundle on \( M \times S^2 \).

Each coupled vortex \((A_{i,1}, A_{i,2}, \phi_i)\) corresponds as in \((2.14)\) to an \( SU(2) \)-invariant HYM connection \( \tilde{A}_i \) on bundle \( F \) with respect to the Kähler form \( \omega_\sigma \).

We define the concentration set of the sequence \( \{A_i\} \) as follows (recall definition from \((3.4)\)),

\[
\tilde{S} = \{ x \in M \times S^2 \mid \lim_{r \to 0} \liminf_{i \to \infty} r^{-(2m+2)} \int_{B_r(x, M \times S^2)} |F_{\tilde{A}_i}|^2 dv_\sigma \geq \varepsilon \},
\]

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where $\varepsilon < \varepsilon_0$ is to be determined later, $a$ is a constant only depending on the curvature bound of $M$, and $dv_g$ is volume form of the metric $\omega_g$. Since $\tilde{A}_i$ are $SU(2)$-invariant connections, the curvature density $|F_{\tilde{A}_i}|^2$ are $SU(2)$-invariant functions, hence by the definition $\tilde{S}$ is an $SU(2)$-invariant subset. Let $\tilde{S} = p^{-1}S, S \subset M$. We call $S$ the concentration set of the sequence $(A_{i,1}, A_{i,2}, \phi_i)$ and we shall show that after gauge transformations, a subsequence of $(A_{i,1}, A_{i,2}, \phi_i)$ converges to a coupled vortex $(A_1, A_2, \phi)$ on $M \setminus S$.

We need the following removable singularity theorem. Its proof will be postponed to the next section.

**Theorem 4.2** Assume that $(A_1, A_2, \phi, S)$ is an admissible coupled vortex on trivial vector bundles $E_1$ and $E_2$ over $B_r(0) \subset \mathbb{C}^m$. Assume also that the hermitian metrics on the bundles and the Kähler metric on the ball $B_r(0)$ are comparable with the standard product metrics and Kähler metric by a constant factor $c$. Then there exists $\varepsilon_2 = \varepsilon_2(c, \tau) > 0$, such that if $x \in M, 0 < r < \text{injrad}(x)$, and

\begin{equation}
(4.4)
\int_{B_r(x,M)} r^{4-2m} e(A_1, A_2, \phi)dv < \varepsilon_1,
\end{equation}

then there exist smooth gauge transformations $g_j$ on $E_j|_{B_{r/2}(x)_S}$ for $j = 1, 2$ such that the triple $(g_1(A_1), g_2(A_2), g_1 \circ \phi \circ g_2^{-1})$ extends smoothly over $B_{r/2}(x)$.

We shall also need the following lemma in the proof of Theorem 4.1.

**Lemma 4.1** If $(A_1, A_2, \phi)$ is a coupled $\tau$-vortex, then the following equations hold:

\begin{align}
(4.5) & & d^*_{A_1} F_{A_1} = -2(m-1)J (d_{A_1}(\phi \circ \phi^*)), \\
(4.6) & & d^*_{A_2} F_{A_2} = 2(m-1)J (d_{A_2}(\phi^* \circ \phi)), \\
(4.7) & & \partial_{A_1 \otimes A_2} \phi = 0.
\end{align}

**Proof of Lemma.** Let $\Omega = \omega^{n-2}/(n-2)!$. Then the operator $*\Omega: \Lambda^2_c(M) \to \Lambda^2_c(M)$ has eigenvalues $\pm 1$ and $\Lambda^2_c(M) = \Lambda^2(M) \otimes \mathbb{C}$ decomposes as $\Omega$-self-dual and $\Omega$-anti-self-dual parts. The space of $\Omega$ self-dual 2-forms has the decomposition

\begin{equation}
\Lambda^+_{\mathbb{C}}(M) = \Lambda^0_{\mathbb{C}}(M) \cdot \omega \oplus \Lambda^0_{\mathbb{C}} \oplus \Lambda^2_{\mathbb{C}}. \nonumber
\end{equation}

For a connection $A$ on a complex vector bundle $E$ on $M$, we let $H_A$ be the projection of $F_A$ to the $\Lambda^0_{\mathbb{C}}(M) \cdot \omega$ part in the above decomposition. Then we have

\begin{equation}
H_A = (-2i\Lambda F_A) \cdot \omega. \nonumber
\end{equation}

Now assume $(A_1, A_2, \phi)$ is a coupled $\tau$-vortex. Since $A_1$ and $A_2$ are integrable connections, $F_{A_1}$ and $F_{A_2}$ are $(1,1)$ forms. We have,

\begin{align}
(4.8) & & d^*_{A_1} F_{A_1} = - * d_{A_1} * (F_{A_1} - H_{A_1} + H_{A_1}) \\
& & = - * d_{A_1} (-\Omega \wedge (F_{A_1} - H_{A_1}) + \Omega \wedge H_{A_1}) \\
& & = -2 * (\Omega \wedge d_{A_1} H_{A_1}) \quad \text{(by the Bianchi identity and } d\Omega = 0) \\
& & = -2 * (\Omega \wedge d_{A_1} ((\phi \circ \phi^* - \tau F_{E_1}) \omega)) \quad \text{(by } (2.7)) \\
& & = -2 * (\frac{\omega^{m-1}}{(m-2)!} \wedge d_{A_1} (\phi \circ \phi^*)) \quad \text{(by } d\omega = 0) \\
& & = -2(m-1)J (d_{A_1}(\phi \circ \phi^*)),
\end{align}
where $J$ is the complex structure acting on 1-forms. This is exactly (4.3). Similarly, we have the equation (4.6) for $A_2$. (4.7) is just a copy of the vortex equation (2.6).

Assume that $x \in M \setminus S$ and $\bar{x} \in p^{-1}(x) \subset \tilde{S}$, then by the definition of $S$ and $\tilde{S}$, there exist $r \in (0, \text{dist}(x, S)) \cap (0, \text{injrad}(\bar{x}))$, and $N = N(x) > 0$, such that for $i \geq N$,

$$r^{4-(2m+2)} \int_{B_r(\bar{x}, M \times S^2)} |F_{\tilde{A}}|^2 dv < \varepsilon. \quad (4.8)$$

It follows from Lemma 2.1 that if $r$ satisfies $c(\tau)r^4 \leq \varepsilon$ for a suitable constant $c(\tau)$, then

$$r^{4-2m} \int_{B_r(x, M)} e(A_{i,1}, A_{i,2}, \phi_i) dv < \frac{2\varepsilon}{\sigma} \quad (4.9)$$

If we take $\varepsilon < 2^{2m-5}\sigma\varepsilon_1$ and fix trivialisations of $E_1$ and $E_2$ over $B_r(x, M)$, then by (4.3) and Theorem 1.2, there exist gauge transformations to make the triple $(A_{i,1}, A_{i,2}, \phi_i)$ smooth over $B_0(x)$. We will assume that $(A_{i,1}, A_{i,2}, \phi_i)$ is already in such a smoothing gauge and hence $\tilde{A}$ is also smooth over $B_0(\bar{x}, M \times S^2)$. Since the left hand sides of (4.8) and (4.9) are scaling invariant, we may rescale to assume that $r = 8$. In what follows we shall only consider those $i \geq N$ when we restrict our attention on the ball $B_r(x)$.

Since $\tilde{A}$ is now smooth on $B_1(\bar{x}, \bar{M})$, the pointwise a priori estimate for Yang-Mills connections (Prop. 3.2) and (4.8) imply that there exists a uniform constant $C > 0$ such that

$$\sup_{B_1(\bar{x}, M \times S^2)} |F_{\tilde{A}_i}|^2 \leq C\varepsilon, \quad (4.10)$$

if we assume that $\varepsilon \leq \varepsilon_0$ for $\varepsilon_0$ in Prop. 3.2. By Lemma 2.1 and its proof, we have that $e_r(A_{i,1}, A_{i,2}, \phi_i) = |F_{\tilde{A}_i}|^2 - c(\tau)$ and $|F_{A_{i,1}}|^2 + |F_{A_{i,2}}|^2 + |d_{A_{i,1}} \otimes A_{i,2}^* \phi_i|^2 \leq |F_{\tilde{A}_i}|^2$. Thus we have

$$\sup_{B_1(x, M)} |F_{A_{i,1}}|^2 + |F_{A_{i,2}}|^2 + |d_{A_{i,1}} \otimes A_{i,2}^* \phi_i|^2 \leq C\varepsilon, \quad (4.11)$$

$$\sup_{B_1(x, M)} |\phi_i \circ \phi_i^* - \tau I_{E_1}|^2 + |\phi_i^* \circ \phi_i + \tau^* I_{E_2}|^2 \leq C. \quad (4.12)$$

Fix a local trivialization for $E_1$ and $E_2$ on $B_1(x, M)$. If $\varepsilon$ is sufficiently small, by (4.11), we may apply the existence of Coulomb gauges (Theorem 2.7 in Uhlenbeck [22]) to find Coulomb gauges for $A_{i,1}$ and $A_{i,2}$ on $B_1(x, M)$. In other words, there exist gauge transformations $g_{i,1}$ on $E_1|_{B_1(x)}$ and $g_{i,2}$ on $E_2|_{B_1(x)}$ such that the connections $A'_{i,1} = g_{i,1}(A_{i,1})$ and $A'_{i,2} = g_{i,2}(A_{i,2})$ satisfy

$$d^* A'_{i,j} = 0, \quad \text{on } B_1(x), \quad j = 1, 2, \quad (4.13)$$

$$* A'_{i,j} = 0, \quad \text{on } \partial B_1(x), \quad j = 1, 2, \quad (4.14)$$

$$||A'_{i,j}||_{L^p(B_1(x))} \leq C_p ||F_{A_{i,j}}||_{L^\infty(B_1(x))} \leq C_p \varepsilon, \quad \forall 1 \leq p < \infty, j = 1, 2. \quad (4.15)$$

Let $\phi'_i = g_{i,1} \circ \phi \circ g_{i,2}^{-1}$. The equations (4.5), (4.6), (4.7), (4.11), and (4.12) are gauge equivariant and hence they hold if we replace $(A_{i,1}, A_{i,2}, \phi_i)$ by $(A'_{i,1}, A'_{i,2}, \phi'_i)$. We observe that (4.13) and (4.14), (4.5) and (4.7) form an elliptic system for the triple

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(A'_{i,1}, A'_{i,2}, \phi'_i) over B_1(x). Now (4.11), (4.12) and (4.15) imply that \( \phi'_i \in L^p(B_1(x)) \) for any \( p < \infty \). This together with (4.15) give a starting point to carry out bootstrapping argument and obtain bounds on the supremum norm of all derivatives of \((A'_{i,1}, A'_{i,2}, \phi'_i)\) on \( B_1(x) \). In particular, we see that the triple \((A'_{i,1}, A'_{i,2}, \phi'_i)\) is smooth on \( B_1(x) \) and \((g_{i,1}, g_{i,2})\) are smooth gauge transformations on \( B_1(x) \). The bounds on derivatives of \((A'_{i,1}, A'_{i,2}, \phi'_i)\) are uniform in \( i \), hence by passing to a subsequence, \((A'_{i,1}, A'_{i,2}, \phi'_i)\) converges to a triple \((A'_1, A'_2, \phi')\) in smooth topology on \( \bar{B}_{\bar{x}}(x) \).

We may cover the non-concentration set \( M \setminus S \) by a countable union of balls \( B_{r_\alpha}(x_\alpha) \) such that (4.13) and (4.13) apply with \( r = 8r_\alpha \) and \( x = x_\alpha \). Applying the above analysis to each ball \( B_{8r_\alpha}(x_\alpha) \), by passing to a subsequence, there exist smooth gauge transformations \((g_{i,1,\alpha}, g_{i,2,\alpha})\) on \((E_1, E_2)|_{B_{r_\alpha}(x_\alpha) \cap S_i}\), such that the sequence \((g_{i,1,\alpha}(A_{i,1}), g_{i,2,\alpha}(A_{i,2}), g_{i,1,\alpha} \circ \phi_i \circ g_{i,2,\alpha})^{-1}\) is smooth on \( B_{r_\alpha}(x_\alpha) \) and converges in smooth topology to a triple \((A'_{1,\alpha}, A'_{2,\alpha}, \phi'_\alpha)\) on \( B_{r_\alpha}(x_\alpha) \). We can now use a standard diagonal process of gluing gauges (see for example 4.4.8 in Donaldson and Kronheimer [8]), again passing to a subsequence, to obtain smooth gauge transformations \((g_{i,1}, g_{i,2})\) on \( M \setminus (S_i \cap S) \), such that \((g_{i,1}(A_{i,1}), g_{i,2}(A_{i,2}), g_{i,1} \circ \phi_i \circ g_{i,2}^{-1})\) converges to \((A_1, A_2, \phi)\) in smooth topology on compact subsets of \( M \setminus S \).

Now we recall from the dimensional reduction in Section 2 that there is a one-to-one correspondence between the gauge groups \( \mathcal{G}_1 \times \mathcal{G}_2 \) and \( \mathcal{G}^{SU(2)} \) and a one-to-one correspondence between coupled vortices and \( SU(2) \)-invariant HYM connections. Hence it follows from the above that smooth \( SU(2) \)-invariant gauge transformations, \( \tilde{A}_i \) converges to an \( SU(2) \)-invariant HYM connection \( \tilde{A} \) in smooth topology on compact subsets of \( M \times S^2 \setminus \hat{S} \).

Now Prop. 3.3 on pure Yang-Mills connections implies that there exists a nonnegative density function \( \Theta, H^{2m-4} \) measurable on \( \hat{S} \) such that

\[
|F_{\tilde{A}}|_\sigma^2 dv_{\sigma} \to \mu = |F_{A}|_\sigma^2 dv_{\sigma} + \tilde{\Theta} H^{2m-4}|\hat{S} \quad \text{as measures.}
\]

We notice that since \( |F_{\tilde{A}}|_\sigma^2 dv_{\sigma} \) are \( SU(2) \)-invariant measures on \( M \times S^2 \), the limit measure \( \mu \) must also be \( SU(2) \)-invariant. Recall the definition of the density function \( \tilde{\Theta} \),

\[
\tilde{\Theta}(\tilde{x}) = \lim_{r \to 0} \frac{\mu(B_r(\tilde{x}, M \times S^2))}{r^{2m+2-4}}, \forall \tilde{x} \in \hat{S}_b.
\]

It is clear from the invariance of \( \mu \) that \( \tilde{\Theta} \) is \( SU(2) \)-invariant. Hence \( \tilde{\Theta} = \Theta \circ p \) for a function \( \Theta \) on \( S \).

By Theorem 3.1 on the blow-up of HYM connections, the blowup locus of \( \tilde{A}_i \) is of the form \( \tilde{C} = (\tilde{T}, \tilde{\Theta}) \), where \( \tilde{T} = \text{supp} \tilde{C} \) is a countable union of \((m+1-2)\)-dimensional holomorphic subvarieties. Because the current \( \tilde{C} \) is \( SU(2) \)-invariant, we have \( \tilde{T} = T \times S^2 = (\cup_{\alpha=1}^\infty T_{\alpha}) \times S^2 \), where \( T_{\alpha} \) are \((m-2)\)-dimensional holomorphic subvarieties of \( M \) and the induced function \( \tilde{\Theta} \) satisfies that \((1/8\pi^2)\Theta|_{T_{\alpha}} \) is a constant positive integer for any \( \alpha \). Let \( C' = (T, \Theta) \), then \( \frac{1}{8\pi^2} C' \) is a holomorphic chain by the above. Because of the energy identity (4.1), the currents \( C_i \) are uniformly bounded in the mass norm, hence passing to a subsequence, \( C_i \) converges to a current \( C'' \) as currents. \( \frac{1}{8\pi^2} C'' \) is an integral, positive \((m-2, m-2)\) current, hence by the result of King [12], or Harvey and Shiffman [12], \( \frac{1}{8\pi^2} C'' \) is a holomorphic chain. We define \( C = C' + C'' \), then \((A_1, A_2, \phi, S, C)\) is an ideal coupled vortex and the previous argument shows that a subsequence of \((A_{i,1}, A_{i,2}, \phi_i, S_i, C_i)\) weakly converges to \((A_1, A_2, \phi, S, C)\). \( \square \)
For hermitian vector bundles \((E_1, h_1)\) and \((E_2, h_2)\) on \(M\), as mentioned before, the moduli space \(V_\tau\) of coupled \(\tau\)-vortices naturally embeds into \(IV_\tau\). Hence we have the following compactification theorem as a corollary of Theorem 4.1.

**Theorem 4.3** The moduli space \(V_\tau\) of coupled \(\tau\)-vortices on hermitian bundles \((E_1, h_1)\) and \((E_2, h_2)\) over a compact Kähler manifold \((M^m, \omega)\) has a compactification \(\bar{V}_\tau\) which is embedded in the space of ideal coupled \(\tau\)-vortices on \(M\).

In view of the Hitchin-Kobayashi type correspondence established in Theorem 2.2 and the remark following the definition of ideal coupled vortices, we have the following corollary.

**Theorem 4.4** Assume that \((E_1, h_1)\) and \((E_2, h_2)\) are hermitian complex vector bundles over a compact Kähler manifold \((M, \omega)\) and rank \(E_2 = 1\) and \(\tau \notin \mathcal{T}\) (\(\mathcal{T}\) as defined in (2.20)). The moduli space \(M_\tau\) of stable holomorphic \(\tau\)-triples on \((E_1, h_1)\) and \((E_2, h_2)\) admits a compactification in the moduli space of ideal coupled \(\tau\)-vortices on \((E_1, h_1)\) and \((E_2, h_2)\).

**Remark.** From the remark following the definition of ideal coupled vortices (at the beginning of this section), we know that an ideal coupled vortex \((A_1, A_2, \phi, S, C)\) gives rise to reflexive sheaves \(E_1\) and \(E_2\) and a sheaf homomorphism \(\phi \in \text{Hom}(E_1, E_2)\). We should be able to define suitable notions of stability and semi-stability for such triples \((E_1, E_2, \phi)\) (by algebraic criteria) so that the stability of a triple is equivalent to the existence of a singular coupled vortex \((A_1, A_2, \phi)\) on it. This will be the Hitchin-Kobayashi correspondence for stable triples in the sheaf version. With this work done, Theorem 4.4 could be written in a nicer way, i.e., the compactification of the moduli space of stable bundle triples lies in the moduli space of stable sheaf triples, which will be in the algebraic category.

Now assume that \((E, h)\) is a hermitian vector bundle over \(M\). In order to get a compactification of the moduli space \(V_\tau(E)\) of \(\tau\)-vortices on \((E, h)\), we use the embedding of \(V_\tau(E)\) into \(V_\tau(E, L)\), the moduli space of coupled \(\tau\)-vortices on \(E\) and \(L\), where \(L\) is the trivial smooth line bundle over \(M\) with the product metric \(h_0\). We have the following compactification theorem of the moduli space of vortices.

**Theorem 4.5** The moduli space \(V_\tau\) of \(\tau\)-vortices on a hermitian vector bundle \(E\) over a compact Kähler manifold \((M, \omega)\) admits a compactification in the space of ideal coupled \(\tau\)-vortices on \(E\) and \(L\), where \(L\) is the trivial line bundle with the product metric on \(M\).

**Remarks.** 1) Assume that \((A_1, A_2, \phi, S, C)\) is an ideal coupled \(\tau\)-vortex on \(E\) and \(L\). Then it gives rise to reflexive sheaves \(E_1\) and \(L\). \(L\) is a line bundle because it is a rank 1 reflexive sheaf. The degree of \(L\) is again equal to 0, that of the trivial bundle, because in passing to a weak limit, the topological change happens only on the blowup set, a codimension 2 set, and that doesn’t affect the first chern class; hence \(L\) is topologically trivial. Now we may determine the connection \(A_2\) uniquely in terms of the connection \(A_1\), \(\phi\) and the holomorphic structure \(L\) from the equation (2.8). Thus the ideal coupled vortex essentially can be viewed as an ‘ideal vortex’ \((A_1, \phi, S, C)\) on the bundle \(E_1 \otimes L^*\). And the boundary points of the compactification \(V_\tau\) are actually ‘ideal vortices’. However, we would not use the concept of ‘ideal vortices’ here as it would not make things much simpler.
2) The compactification given in Theorem 4.5 is somewhat indirect since in order to compactify vortices, we turn to ideal coupled vortices. However, we do not know a more direct way of compactifying the space of vortices. Notice in particular the blow-up set of a sequence of vortices \((A_i, \phi_i)\) is not where the YMH energy of \((A_i, \phi_i)\) concentrates. Instead it is the set where the YMH energy of the sequence of coupled vortices \((A_i, A'_i, \phi_i)\) concentrates, where \(A'_i\) is a connection on \(L\) determined by \((A_i, \phi_i)\) from (2.8). In this regard, we may say that the blow-up phenomena of vortices is only clear when we put them in the setting of coupled vortices.

Via the Hitchin-Kobayashi type correspondence given by Theorem 2.1, we have,

**Theorem 4.6** Assume that \(\tau \notin T\) (\(T\) as defined in (2.3)). Then the moduli space \(M_\tau\) of stable \(\tau\)-pairs on a Hermitian complex vector bundle \((E, h)\) over compact a Kähler manifold \((M^m, \omega)\) admits a compactification in the moduli space of ideal coupled vortices on \(E\) and \(L\), where \(L\) is the trivial bundle with the product metric on \(M\).

**Remarks.** 1) Again, like in the remark following Theorem 4.10, we should be able to define an algebraic concept of stable (or semi-stable) pairs \((E, \phi)\) of a coherent reflexive sheaf and a section of it, and to interpretate the compactification in the algebraic category. We hope to clarify these issues in a future paper.

2) When \(M\) is a Kähler surface, let \((A_1, A_2, \phi, C)\) be an ideal coupled vortex on hermitian bundles \((E_1, h_1)\) and \((E_2, h_2)\). Then because reflexive sheaves are locally free on Kähler surfaces, we find that \((A_1, A_2, \phi)\) extends as a smooth coupled vortex on some hermitian bundles \((E'_1, h'_1)\) and \((E'_2, h'_2)\) on \(M\). \(C\) is now given by a finite set of points with multiplicities on \(M\). Hence the compactification is much easier to describe and it is reminiscent of the well-known moduli spaces of self-dual connections on four-manifolds. It would be a natural question to ask whether we can describe the compactified moduli spaces more specifically, and in particular, compute their topological invariants.

3) In order to understand the compactification better, we have the following question. Are there any effective bounds on the measure of the singular set of a reflexive sheaf in terms of topological or analytical quantities?

### 5 Removable singularity theorems

In this section we give the proofs of Theorem 4.4, 4.5 and 4.2. We shall first prove the following \(\varepsilon\)-regularity theorem for HYM connections.

**Theorem 5.1** Assume that \(E\) is a trivial complex vector bundle over \(B_r(0) \subset \mathbb{C}^m\) with the product hermitian metric. Assume also that the hermitian metric on the bundle and the Kähler metric on the ball \(B_r(0)\) are comparable with the standard product metrics and Kähler metric by a constant factor \(c\). Then there exists a constant \(\varepsilon_1(m, c) > 0\), such that if \((A,S)\) is an admissible HYM connection on \(E\), with

\[
r^{4-2m} \int_{B_r(0)} |F_A|^2 dv \leq \varepsilon_1,
\]
then there exists a smooth gauge transformation \( \sigma \) on \( B_{r/2}(0) \setminus S \), such that \( \sigma(A) \) can be extended smoothly over \( B_{r/2}(0) \).

**Proof.** Because \((5.1)\) is scaling invariant, we may rescale and assume that \( r = 1 \). Let \( E_0 \) be the holomorphic bundle over \( B_S(0) \setminus S \) which is topologically \( E \) over \( B_S(0) \setminus S \) and has the holomorphic structure given by \( \partial A \). We first resort to Theorem 3.2 to extend \( E_0 \) to a reflexive sheaf \( \mathcal{E} \) over \( B_S(0) \). Let \( S_1 \) be the singular set of the reflexive sheaf \( \mathcal{E} \), i.e. the subset of \( M \) where \( \mathcal{E} \) is not locally free. It is a standard fact that \( S_1 \) is an analytic subvariety of dimension at most \( m - 3 \). Theorem 3.2 also implies that the hermitian metric \( h \) extends smoothly over the locally free part \( B_S(0) \setminus S_1 \). Hence the connection \( A \), determined by \( h \) and the holomorphic structure, also extends smoothly over \( B_S(0) \setminus S_1 \).

Since \( S_1 \) is stratified by smooth complex submanifolds of \( M \), we shall make induction on the complex dimension of the top strata \( S_0 \) of \( S_1 \) to show that, under the assumption \((5.1)\), \( S_1 = \emptyset \).

If \( S_0 = \emptyset \), then \( S_1 = \emptyset \). Assume that we have shown that if \( \dim S_0 < k \), then \( S_1 = \emptyset \). Assume now that \( \dim S_0 = k \). Take a generic point \( x_0 \in S_0 \). With a suitable choice of coordinates, a neighborhood of \( x_0 \) in \( B(1) \) can be written in the form of \( N = B_s^{m-k} \times B_s \) such that \( x_0 = (0,0) \), and \( S_1 \cap N = S_0 \cap N = \{0\} \times B_s^k \), where \( B_s^k \) stands for the ball of radius \( s \) centered at origin in \( \mathbb{C}^k \). Again after rescaling, we may assume that \( s = 2 \).

We first claim that the vector bundle \( \mathcal{E}|_{N \setminus S_0} = \mathcal{E}|_{(B_s^{m-k} - \{0\}) \times B_s} \) is trivial as a smooth vector bundle over \( N \setminus S_0 \). Let \( F = \mathcal{E}|_{(B_s^{m-k} - \{0\}) \times \{0\}} \) and \( A' = A|_{\partial B_s^{m-k} \times \{0\}} \). If \( \varepsilon_1 < \varepsilon_0 \), we have by the a priori estimates (Prop. 3.3) that

\[
|F_{A'}(y)|^2 \leq |F_A|^2(y) \leq C(m)\varepsilon_1, \quad \forall y \in \partial B_s^{m-k} \times \{0\}
\]

We can then apply the argument of Lemma 2.2 of Uhlenbeck \([22]\) to assert that the bundle \( F \) restricts to a trivial smooth bundle over \( \partial B_1^{m-k} \times \{0\} \), if \( \varepsilon_1 \) is sufficiently small. Let \( \{e_1, \ldots, e_l\} \) be a frame of \( F \) over \( \partial B_1^{m-k} \times \{0\} \) and fix a smooth connection on \( \mathcal{E}|_{N \setminus S_0} \), we may use parallel transport to obtain a smooth trivialization \( \{e_1, \ldots, e_n\} \) of \( \mathcal{E}|_{N \setminus S_0} \) and the claim is proved.

The trivialization of \( \mathcal{E} \) away from the singular set allows us to express \( A \) as a matrix valued 1-form. If \( \varepsilon_1 \) is sufficiently small, the singularity removal theorem of Tao and Tian \([21]\) implies that there exists a smooth gauge transformation \( g \) on \( (B^{m-k}_s - \{0\}) \times B_s^k \) such that \( g(A) \) extends as a smooth HYM connection over a trivial bundle \( H \) over \( B^{m-k}_s \times B_s^k \).

Define the holomorphic structure on \( H \) by \( \tilde{\partial}_g \). Let \( \{f_1, \ldots, f_l\} \) be a frame of holomorphic sections of \( H \) over \( B_s^{m-k} \times B_s^k \). Since \( \tilde{\partial}_g(A) = g \circ \tilde{\partial}_A \circ g^{-1} \), \( \{g^{-1}f_1, \ldots, g^{-1}f_l\} \) gives a holomorphic frame for the reflexive sheaf \( \mathcal{E} \) over \( (B_s^{m-k} - \{0\}) \times B_s^k \). Since \( \mathcal{E} \) is a reflexive coherent sheaf and \( \mathcal{E} \) agrees with the trivial holomorphic bundle away from a singular set of codimension at least 3, it follows that \( \mathcal{E} \) is actually a trivial holomorphic bundle over \( B_s^{m-k} \times B_s^k \). This implies that \( \mathcal{E} \) is locally free at \( x_0 \). It is a contradiction and the claim is established. \( \square \)

**Proof of Theorem 4.3.** We shall use the notations from Section 2. Let \( \hat{A} \) be the \( SU(2) \)-invariant admissible HYM connection on the \( SU(2) \)-invariant bundle \( F = p^*E_1 \oplus (p^*E_2 \otimes
with a smooth Hermitian-Einstein metric $h$ to a reflexive sheaf $F$ locally extends across the singularity to a holomorphic bundle. Since we know from Bando and Siu’s theorem (Theorem 3.2) that $F$ extends uniquely as a reflexive sheaf $\tilde{F}$ over $B_r(0) \times S^2$, it follows that $\tilde{F}$ is a holomorphic bundle over $B_r(0) \times S^2$ with a smooth Hermitian-Einstein metric $h$. $\tilde{F}$ and its metric are $SU(2)$-invariant because so are $F$ and its metric. Let $E_1, E_2$ be the reflexive sheaves extending $E_1, E_2$ over $B_r(0)$ (by Theorem 3.2). Then $\tilde{F}$ has a splitting

$$\tilde{F} = p^*E_1 \oplus (p^*E_2 \otimes q^*H^2),$$

because the splitting is valid on $(B_r(0) \setminus S) \times S^2$ and the sheaves involved are reflexive. With the following lemma, we deduce that $E_1$ and $E_2$ are smooth holomorphic bundles over $B_r(0)$. Since the hermitian metric $h$ on $\tilde{F}$ is $SU(2)$-invariant, it must be of the form

$$h = \begin{pmatrix} \tilde{h}_1 & 0 \\ 0 & \tilde{h}_2 \end{pmatrix},$$

where $\tilde{h}_1 = p^*\tilde{h}_1$, $\tilde{h}_2 = p^*\tilde{h}_2 \otimes q^*h'_2$, and $\tilde{h}_i$ is a metric on $E_i$ and $h'_2$ is the $SU(2)$-invariant metric on $H^2$ (see Prop. 3.2 of [10]). $h_i$ and the holomorphic structures on $E_i$ then give a smooth coupled $\tau$-vortex $(A_1', A_2', \phi')$ on $B_r^2(x, M)$. $(E_i, \tilde{h}_i)|_{B_r^2(0)}$ is isomorphic to $(E_i, h_i)|_{B_r^2(0)}$ as hermitian holomorphic vector bundles because by definition, the former ones are extensions of the latter ones. It follows that $(A_1', A_2', \phi)$ is gauge equivalent to $(A_1, A_2, \phi)$ on $B_r^2(0)$ and gives the desired smooth extension. \□

**Lemma 5.1** Assume that $E_1$ and $E_2$ are two coherent sheaves on a complex manifold $M$ such that $E_1 \oplus E_2 = \tilde{F}$, where $\tilde{F}$ is a holomorphic vector bundle of finite rank over $M$. Then $E_1$ and $E_2$ are locally free.

**Proof of lemma.** The statement is local in nature. Let $x$ be any point of $M$. Because of $(E_1)_x \oplus (E_2)_x = \tilde{F}_x$, we see that $(E_i)_x$ $(i = 1, 2)$ are projective, hence are free. Because the sheaves are coherent, we see that $E_1$ and $E_2$ are free in a neighborhood of $x$. That finishes the proof of the lemma. \□

In the end, we remark that the proofs of Theorem 1.4 and 1.5 follow easily from Theorem 5.1 and 4.2 through a standard covering argument and we shall leave the proof to the reader.

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Department of Mathematics, Massachusetts Institute of Technology, 77 Mass. Ave., Cambridge, MA 02139, USA
email: tian@math.mit.edu

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