Solving the problem of analytical design of the controller for a stationary discrete system with a small step

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Abstract. In this paper, we study the problem of designing a linear controller for a stationary discrete system with a small step. By the method of integral manifolds, the discrete optimal control problem with a small step is subdivided into two subproblems, the solutions of which are found independently of each other. Algorithms for analytical solutions of subproblems are based on the second Lyapunov method. As a result, a linear controller was designed for a controlled stationary discrete system with a small step, consisting of two subcontrollers, which separately regulate the movements of a system of lower dimension. The results obtained can be used to design an optimal linear digital controller.

1. Introduction

One of the modern methods for designing optimal control systems, which finds wide practical application, is the method for constructing solutions to problems of synthesis of a linear controller. A detailed analysis of works devoted to the study of this problem is given in [1], among which one can single out the works [2,3], which considered the problem of designing a controller for a discrete control system with a small step.

This work is a continuation of the studies of these works. In the work the discrete optimal control problem with a small step is subdivided into two subproblems, the solutions of which are found independently of each other using the method of integral manifolds [4].

Algorithms for approximate solutions of subproblems are based on the second Lyapunov method [5].

1.1. Formulation of the problem

Consider the optimal control problem

\[ J = \sum_{k=0}^{\infty} [y'(kT) \mathbf{L}y(kT) + u'(kT)u(kT)] \rightarrow \min \]  

under constraints

\[ y(t + T) = A y(t) + B u(t), \quad y(0) = y_0, \]  

where \( y = \begin{pmatrix} x \\ z \end{pmatrix}, x, z \in \mathbb{R}^n, L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, L_1 > 0, L_2 > 0 \) are symmetric constant matrices, \( u = u(t) - r \)-dimensional control vector, \( t = kT, \ k = 0, 1, \cdots \infty, \ 0 < T \leq 1, \)

\( A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \)

and \( A_i (i = 1, 4) - (n \times n), B_1, B_2 - (n \times r) \) are constant matrices, the prime denotes matrix transposition.

We require the fulfillment of the following conditions:

I. Matrices \( A_i (i = 1, 4) \) are matrices of simple structure and they have no zero eigenvalue
\[ \lambda_i \quad (i = 1, \ldots, n). \]

II. All eigenvalues \( \lambda_i \) of matrix \( A_i \) satisfy the inequalities:
\[ |\lambda_i| < q_0 < 1, \quad \lambda_i + \lambda_j \neq 0, \quad i \geq 1, \quad j \leq n. \]

III. System (2) is completely controllable and observable [5].

When solving problems of controlling objects from various fields of science and technology, difficulties arise due to the high dimension of the models and the presence of several time scales. In this regard, it becomes necessary to separate the state variables in optimal control problems.

Separation of the state variables of the system (2) is carried out using the replacement [6]:
\[
\begin{align*}
    x &= \bar{x} - N\bar{z}, \\
    z &= \bar{z} + Hx.
\end{align*}
\]

Under conditions I, II and taking into account (3), system (2) can be replaced by an equivalent system:
\[
\begin{align*}
    \dot{\bar{x}}(t + T) &= \bar{A}_1\bar{x}(t) + \bar{B}_2 u(t), \\
    \dot{\bar{z}}(t + T) &= \bar{A}_4\bar{z}(t) + \bar{B}_2 u(t),
\end{align*}
\]

where
\[
\begin{align*}
    \bar{A}_1 &= A_1 + A_2H, \quad \bar{A}_4 = A_4 - HA_2, \quad \bar{B}_1 = B_1 + NB_2, \quad \bar{B}_2 = -HB_1 + B_2.
\end{align*}
\]

Here the matrices \( H \) and \( N \) satisfy the matrix Riccati and Lyapunov equations, respectively:
\[
\begin{align*}
    -HA_1 + A_4H - H A_2 H + A_3 &= 0, \\
    -\bar{A}_1 N + N \bar{A}_4 + A_2 &= 0.
\end{align*}
\]

The initial conditions of system (4), (5) take the form:
\[
\begin{align*}
    \bar{x}(0) &= \bar{x}_0, \\
    \bar{z}(0) &= \bar{z}_0,
\end{align*}
\]

where
\[
\begin{align*}
    \bar{x}_0 &= x_0 + N\bar{z}_0, \\
    \bar{z}_0 &= z_0 - Hx_0.
\end{align*}
\]

1.1.1. The solution of the problem. It is known that under conditions II, systems (4) and (5) have integral manifolds [4]:
\[
\begin{align*}
    \theta_\nu &= \{ (\bar{x}, H\bar{x}) \mid \bar{x} \in \mathbb{R}^n, z = H\bar{x}, z \in \mathbb{R}^n \}, \\
    \theta_\sigma &= \{ ((-N\bar{z}), \bar{z}) \mid \bar{x} = -N\bar{z} \in \mathbb{R}^n, \quad z \in \mathbb{R}^n \}.
\end{align*}
\]

Let us define the following vectors from the columns of the matrix \( B_1 \) and \( B_2 \):
\[
\begin{align*}
    b_j^{(1)} &= \begin{pmatrix} b_{11}^{(1)} & b_{21}^{(1)} & \cdots & b_{ni}^{(1)} \end{pmatrix}', \\
    b_j^{(2)} &= \begin{pmatrix} b_{11}^{(2)} & b_{21}^{(2)} & \cdots & b_{ni}^{(2)} \end{pmatrix}',
\end{align*}
\]

\( j = 1, \ldots, n, \quad i = 1, r \), where \( b_j^{(1)} \), \( b_j^{(2)} \) are matrix elements \( B_1 \) and \( B_2 \).

Let the following conditions be met
\[
\left( \begin{array}{c} b_j^{(1)}' \\ Hb_j^{(1)} \end{array} \right) \in \theta_\nu, \quad \left( \begin{array}{c} (-N\bar{b}_j^{(2)})' \\ \bar{b}_j^{(2)} \end{array} \right) \in \theta_\sigma,
\]

where
\[
\begin{align*}
    \bar{b}_j^{(1)} &= b_j^{(1)} + NB_j^{(2)}, \quad b_j^{(2)} = b_j^{(2)} - Hb_j^{(1)}, \quad \bar{b}_j^{(1)} = \begin{pmatrix} \bar{b}_{11}^{(1)} & \bar{b}_{21}^{(1)} & \cdots & \bar{b}_{ni}^{(1)} \end{pmatrix}', \\
    \bar{b}_j^{(2)} &= \begin{pmatrix} \bar{b}_{11}^{(2)} & \bar{b}_{21}^{(2)} & \cdots & \bar{b}_{ni}^{(2)} \end{pmatrix}'.
\end{align*}
\]

Then the original problem is reduced to independent synthesis of controllers in systems (4), (5). Moreover, the control \( u(t) \) can be defined in the form
\[
\begin{align*}
    u(t) &= \begin{pmatrix} -\bar{B}_1 K_\nu \bar{x}(t), \bar{x} \in \mathbb{R}^n \\ -\bar{B}_2 K_\sigma \bar{z}(t), \bar{z} \in \mathbb{R}^n \end{pmatrix},
\end{align*}
\]

where \( K_\nu, \quad K_\sigma = (n \times n) \) gain matrices to be defined.
Now we represent problem (1), (2) in the form of the following two subproblems, solutions of which are found independently of each other:

$$J_y = \sum_{k=0}^{\infty} [u'(kT)u(kT) + \ddot{x}'(kT)L_y \ddot{x}(kT)] \rightarrow \min,$$

where $$L_y = L_1 + H' L_2 H.$$  

$$J_\sigma = \sum_{k=0}^{\infty} [u'(kT)u(kT) + \ddot{z}'(kT)L_\sigma \ddot{z}(kT)] \rightarrow \min,$$

where $$L_\sigma = N' L_1 N + (E_n - N H N') L_2 (E_n - N H).$$

When constructing solutions to problem (14) - (17), we use the Lyapunov method [7]. For these problems, the Lyapunov functions can be represented as:

$$F(\ddot{x}, t) = \ddot{x}'(t) P \ddot{x}(t), \quad G(\ddot{z}, t) = \ddot{z}'(t) Q \ddot{z}(t),$$

where $$P$$ and $$Q$$ - positive definite matrices.

Then, the first differences of function (18) can be written in the form:

$$\Delta F(\ddot{x}, t) = F(\ddot{x}(t + T)) - F(\ddot{x}(t)), \quad \Delta G(\ddot{z}, t) = G(\ddot{z}(t + T)) - G(\ddot{z}(t)).$$

Consider problem (14), (15). According to the Lyapunov method, the first difference of the Lyapunov function must be negative definite. Combining the condition of negative definiteness of the first difference of the Lyapunov function (19) with functional (14), we put

$$\ddot{x}'(t + T) P \ddot{x}(t + T) - \ddot{x}'(t) P \ddot{x}(t) = -[u'(t) u(t) + \ddot{x}'(t) L_y \ddot{x}(t)].$$

Taking into account (13), (15), from (21) we have

$$\ddot{x}'(t) [\dddot{A}_1 P \dddot{A}_1 - \dddot{A}_1 P \dddot{B}_1 \dddot{B}_1 K_y - K_y' \dddot{B}_1 \dddot{B}_1 P \dddot{A}_1 + K_y' \dddot{B}_1 \dddot{B}_1 P \dddot{B}_1 \dddot{B}_1 K_y - P] \dddot{x}(t) =$$

$$= -\dddot{x}'(t) [K_y' \dddot{B}_1 \dddot{B}_1 K_y + L_y] \dddot{x}(t).$$

For any $$\dddot{x}(t)$$ from (22) we obtain

$$P = \dddot{A}_1 P \dddot{A}_1 - \dddot{A}_1 P \dddot{B}_1 \dddot{B}_1 K_y - K_y' \dddot{B}_1 \dddot{B}_1 P \dddot{A}_1 + K_y' \dddot{B}_1 \dddot{B}_1 P \dddot{B}_1 \dddot{B}_1 K_y,$$

$$K_y' \dddot{B}_1 \dddot{B}_1 K_y + L_y = 0.$$ 

Similarly, for problem (16), (17) from the condition

$$\ddot{z}'(t + T) Q \ddot{z}(t + T) - \ddot{z}'(t) Q \ddot{z}(t) = -[u'(t) u(t) + \ddot{z}'(t) L_\sigma \ddot{z}(t)]$$

taking into account (13), (17) from (25) we obtain

$$\ddot{z}'(t) [\dddot{A}_4 Q \dddot{A}_4 - \dddot{A}_4 Q \dddot{B}_2 \dddot{B}_2 K_\sigma - K_\sigma' \dddot{B}_2 \dddot{B}_2 Q \dddot{A}_4 + K_\sigma' \dddot{B}_2 \dddot{B}_2 Q \dddot{B}_2 \dddot{B}_2 K_\sigma - Q] \dddot{z}(t) =$$

$$= -\ddot{z}'(t) [K_\sigma' \dddot{B}_2 \dddot{B}_2 K_\sigma + L_\sigma] \dddot{z}(t).$$

For any $$\dddot{z}(t)$$ from (26) we have

$$Q = \dddot{A}_4 Q \dddot{A}_4 - \dddot{A}_4 Q \dddot{B}_2 \dddot{B}_2 K_\sigma - K_\sigma' \dddot{B}_2 \dddot{B}_2 Q \dddot{A}_4 + K_\sigma' \dddot{B}_2 \dddot{B}_2 Q \dddot{B}_2 \dddot{B}_2 K_\sigma,$$

$$K_\sigma' \dddot{B}_2 \dddot{B}_2 K_\sigma + L_\sigma = 0.$$ 

Theorem. Let conditions II, III, (12), (21) and (25) be satisfied. Then in the integral manifolds (11) there is a unique control function (13) for system (15) and (17), which minimizes functionals (14), (16), respectively, and their minimum values have the form:

$$\min f_y = \dddot{x}'(0) P \dddot{x}(0).$$
Now let us prove the uniqueness of the solution to problem (14), (15). Let the optimal solution to the problem be the function:

\[ u_\nu(t) = -\bar{B}_1'K_\nu \bar{x}(t), \quad u_\sigma(t) = -\bar{B}_2'K_\sigma \bar{z}(t). \]

2) optimal controls \( u_\nu(t) \) and \( u_\sigma(t) \) provide asymptotic stability of closed systems:

\[ \bar{x}(t + T) = (\bar{A}_1 - \bar{B}_1\bar{B}_1'K_\nu + L_\nu + \bar{A}_1'P\bar{A}_1 - \bar{A}_1'P\bar{B}_1\bar{B}_1'K_\nu - K_\nu'\bar{B}_1\bar{A}_1P + K_\nu'\bar{B}_1\bar{B}_1'P\bar{B}_1\bar{B}_1'K_\nu - L_\nu)x(0), \]

Comparing the right-hand sides of (36) and (37), we obtain

\[ \sum_{k=0}^{\infty} \Delta F[\bar{x}, kT] = \sum_{k=0}^{\infty} \bar{x}'(kT)P\bar{x}(N) - \bar{x}'(0)P\bar{x}(0). \]  

Using (13) and (15), from (35) we have

\[ \sum_{k=0}^{\infty} \Delta F[\bar{x}, kT] = \sum_{k=0}^{\infty} \bar{x}'(kT)[K_\nu'\bar{B}_1\bar{B}_1'K_\nu + L_\nu + \bar{A}_1'PA_1 - \bar{A}_1'PB_1\bar{B}_1'K_\nu - K_\nu'\bar{B}_1\bar{A}_1P + K_\nu'\bar{B}_1\bar{B}_1'P\bar{B}_1\bar{B}_1'K_\nu - L_\nu]x(0). \]

Under conditions (24), we obtain

\[ \lim_{N \to \infty} \bar{x}'(N)P\bar{x}(N) = 0. \]

and using (23) from (38) we obtain (29).

Consider the first difference of the function \( \Delta F(\bar{x}, t) \) from (19)

\[ \Delta F(\bar{x}, t) = F[\bar{x}(t + T)] - F[\bar{x}(t)] = \bar{x}'(t + T)P\bar{x}(t + T) - \bar{x}'(t)P\bar{x}(t). \]

Substitution \( \bar{x}(t + T) \) from (33) to (40) gives:

\[ \Delta F(\bar{x}, t) = \bar{x}'(t)[\bar{A}_1 - \bar{B}_1\bar{B}_1'K_\nu]P(\bar{A}_1 - \bar{B}_1\bar{B}_1'K_\nu)\bar{x}(t) - \bar{x}'(t)P\bar{x}(t) = \]

\[ = \bar{x}'(t)\left[\bar{A}_1'PA_1 - \bar{A}_1'PB_1\bar{B}_1'K_\nu - K_\nu'\bar{B}_1\bar{A}_1P + K_\nu'\bar{B}_1\bar{B}_1'P\bar{B}_1\bar{B}_1'K_\nu - L_\nu\right]\bar{x}(t). \]

We transform the last equality

\[ \Delta F(\bar{x}, t) = \bar{x}'(t)[\bar{A}_1'PA_1 - (\bar{A}_1' - K_\nu\bar{B}_1\bar{B}_1')P\bar{B}_1\bar{B}_1'K_\nu - K_\nu'\bar{B}_1\bar{A}_1P + K_\nu'\bar{B}_1\bar{B}_1'P\bar{B}_1\bar{B}_1'K_\nu - L_\nu]\bar{x}(t). \]

Since \( P \) is a positive definite matrix, expression on the right side of the relation \( \Delta F(\bar{x}, t) \) is negative definite and in accordance with the Lyapunov stability theorem, system (33) is asymptotically stable.

Now let us prove the uniqueness of the solution to problem (14), (15). Let the optimal solution to the problem be the function

\[ u_\nu(t) = V_\nu(t) - \bar{B}_1'K_\nu \bar{x}(t), \quad t = kT. \]

Then problem (14), (15) takes the form:

\[ J_\nu = \sum_{k=0}^{\infty} \left[ \left( V_\nu(kT) - \bar{B}_1'K_\nu \bar{x}(kT) \right)'(V_\nu(kT) - \bar{B}_1'K_\nu \bar{x}(kT)) + \bar{x}'(kT)L_\nu \bar{x}(kT) \right] \to \min, \]
\[ \tilde{x}(t + T) = (\tilde{A}_1 - \tilde{B}_1 \tilde{B}_1^T K_\nu) \tilde{x}(t) + \tilde{B}_1 V_\nu(t). \]  

(43)

Transforming the expressions for the sums from (42), taking into account (24), we obtain

\[ J_\nu = \sum_{k=0}^{\infty} [V_\nu(kT)\tilde{V}_\nu(kT) - \tilde{V}_\nu(kT)\tilde{B}_1^T K_\nu \tilde{x}(kT) - \tilde{x}'(kT)K_\nu \tilde{B}_1 V_\nu(kT)]. \]  

(44)

Obviously the best choice in (44)

\[ V_\nu(t) = 0. \]

Similarly, for problem (16), (17) we have:

\[ \sum_{k=0}^{\infty} \Delta G[\tilde{z}, kT] = \sum_{k=0}^{\infty} \left\{ G[\tilde{z}(k+1)] - G[\tilde{z}(k)] \right\}, \]

\[ \sum_{k=0}^{\infty} \Delta G[\tilde{z}, kT] = \tilde{z}'(N)Q\tilde{z}(N) - \tilde{z}'(0)Q\tilde{z}(0). \]

(45)

(46)

Comparing the right-hand sides of (46) and (47), we obtain

\[ \sum_{k=0}^{\infty} \tilde{z}'(kT)\left[ K_\nu^T \tilde{B}_1^T K_\sigma + L_\sigma + \tilde{A}_4 Q \tilde{A}_4 - \tilde{A}_4 Q \tilde{B}_2 \tilde{B}_2^T K_\sigma - \tilde{K}_\nu \tilde{B}_2 P \tilde{A}_4 + K_\nu^T \tilde{B}_2^T Q \tilde{B}_2 \tilde{B}_2^T K_\sigma - Q \right] \tilde{z}(kT) - \tilde{z}'(N)Q\tilde{z}(N) + \tilde{z}'(0)Q\tilde{z}(0) = 0. \]

(47)

(48)

Adding the left-hand side of equality (48) to functional (16), we obtain

\[ J = \sum_{k=0}^{\infty} \tilde{z}'(kT)\left[ K_\nu^T \tilde{B}_1^T K_\sigma + L_\sigma \right] \tilde{z}(kT) + \sum_{k=0}^{\infty} \tilde{z}'(kT)\left[ K_\nu^T \tilde{B}_1^T K_\sigma + L_\sigma + \tilde{A}_4 Q \tilde{A}_4 - \tilde{A}_4 Q \tilde{B}_2 \tilde{B}_2^T K_\sigma - \tilde{K}_\nu \tilde{B}_2 P \tilde{A}_4 + K_\nu^T \tilde{B}_2^T Q \tilde{B}_2 \tilde{B}_2^T K_\sigma - Q \right] \tilde{z}(kT) - \tilde{z}'(N)Q\tilde{z}(N) + \tilde{z}'(0)Q\tilde{z}(0) = 0. \]

(49)

Under conditions (28),

\[ \lim_{N \to \infty} \tilde{z}'(N)Q\tilde{z}(N) = 0 \]

(50)

and using (27) from (49) we obtain (30), as required.

The uniqueness of the solution to problem (16), (17) and the asymptotic stability of system (34) are proved similarly to problem (14), (15) and system (33).

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