Confinement and Renormalization

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Abstract

The haaron gas description is reviewed for the QCD vacuum. The role of non-renormalizable operators is emphasised in the mechanism which generates the string tension. Additional examples are mentioned where certain non-renormalizable operators of the bare lagrangian turn out to be important at finite energy scale.

1 Introduction

The confinement of quarks is a rather mysterious phenomenon of high energy physics which challenges our understanding of Quantum Field Theory and provides an ever reviving source of inspiration. One may distinguish two different confinement mechanisms. The soft one, which is responsible for the
screening of the color charge of an isolated quark by the creation of mesons in a manner reminiscent of the sparkling of the supercritical vacuum of QED around a highly charged ion. This latter occurs when a single electron level dives into the Dirac sea of negative energy states. The electron of a virtual $e^+e^-$ pair fills up this hole and the positron escapes to the infinity. The semiclassical condition for this to happen is $Z\alpha_{QED} \approx 1$. For hadrons $Z = 2$ or 3 so one needs a non-perturbative gluonic effect, the anti-screening, to raise the running coupling constant to $\alpha_{QCD} \approx 1$ at the confinement radius.

This confinement mechanism is called soft because it involves energies at the range of the quark rest mass.

It is widely believed that the gluonic vacuum not only amplifies the coupling constant at long distances according to the scenario above but produces another phenomenon, the hard confinement mechanism. This is the emergence of the linear string tension which leads to the separation independent force between static quark charges. It is a hard mechanism because the energy stored in the flux tube is large for well separated charges.

The two mechanisms compete and it is very difficult to disentangle them in the real world which contains the virtual quark-anti quark vacuum polarizations. Only the linear Regge trajectory provides circumstantial evidences for the string picture. The more convincing proof of the hard confining mechanism comes from lattice QCD.

It is clear that both mechanisms are needed to understand the problem of quark confinement. But it is the anti-screening and the emergence of the string tension in the gluonic vacuum which should be clarified first because it is simpler. The soft confining mechanism can only be discussed in the presence of such a non-perturbative medium. Our attention will be limited on the simpler, hard mechanism in the rest of these lectures.

I think that the well defined and clean environment of the numerical simulations of lattice QCD at finite temperature where the deconfinement phase transition can be analyzed is the ideal testing ground for our ideas about confinement. It is mainly due to the result of such numerical studies that the appropriate kinematical framework and the dynamical content of the phase transition can be identified. It was found that the center symmetry which expresses the invariance of the gluonic system under the fundamental group transformations is responsible for the string tension. The dynamical breakdown of this symmetry by the kinetic energy at short time processes links the non-perturbative vacuum with the asymptotically free short
distance phenomena and explains some rather unusual features of the high
temperature phase [4].

The invariance under the center or the fundamental group transforma-
tions might be called "top-secret" symmetry after Coleman's beautiful Erice
lecture [5] because it refers not only to the unobservable gauge transforma-
tions but among them to those which are represented trivially even on the
gauge field. Only the quark field transforms non-trivially under these trans-
f ormations. These transformations are mysterious because it is difficult to
isolate them in the continuum, can be broken dynamically without loosing the
global gauge invariance and are not protected by Ward-identities. Another
characteristic feature of this symmetry is that it requires nontrivial measure
in the functional integral which is invisible in dimensional regularization.
Furthermore the typical configurations in the symmetrical realization of the
path integral contain Dirac delta-type singularities.

I attempt to reconcile these unusual, sometime confusing features of the
hard confinement mechanism and perturbative QCD in these lectures. We
are far from the complete quantitative solution and its outline, the haaron-gas
vacuum [6], can be given only. The starting point is the gauge invariance, the
claim that the hard confinement mechanism can be seen only when the gauge
invariance is guaranteed exactly. In fact, a non-controlled gauge dependent
component of the vacuum state represents a background charge which can
screen the well separated test quarks as the virtual quark-anti quark pair
polarizations do it in the complete vacuum. The ordinary gauge invariance
is easier to guarantee by satisfying the Ward-identities. The question of the
center symmetry is more subtle because it is not protected by simple identities
and can completely be missed in the usual dimensional regularization. Thus
we shall make sure that the center symmetry is present not only formally
but dynamically in the low energy effective theory of the vacuum where the
confining forces can be identified. An interesting question we find in devel-
oping this effective theory, namely the role of non-renormalizable operators
will be discussed in some details in the second part of the lectures.

2 Effective theory of confinement

This Section is devoted to the isolation and the characterization of the mecha-
nism which is responsible for the linear string tension and the chiral sym-
metry breaking in the gluonic vacuum. The center symmetry of the vacuum and its consequence is discussed in the first part. A global symmetry does not give a detailed enough picture of the dynamics so we need a local effective theory for the order parameter. This is introduced in the second part.

2.1 Center symmetry

2.1.1 Functional Schrödinger representation

The center symmetry is simplest to understand in the functional Schrödinger representation. The canonical coordinate is the gluon field, \( A(x) = g A^a(x) \frac{\lambda_a}{2i} \), and the corresponding momentum is the electric field, \( E(x) = E^a(x) \frac{\lambda_a}{2ig} \), where \( E^a(x) = \frac{1}{i} \frac{\delta}{\delta A^a(x)} \). The time component of the gluon field is eliminated by the choice of the temporal gauge, \( A_0(x) = 0 \) and the hamiltonian is of the form

\[
H = -2\text{tr} \int d^3x \left[ \frac{g^2}{2} E^2(x) + \frac{1}{2g^2} B^2(x) \right],
\]

where

\[
B_i \epsilon_{ijk} = \partial_j A_k - \partial_k A_j + [A_j, A_k].
\]

The gauge fixing and the dynamics are invariant under static gauge transformations,

\[
A_\mu(x) \rightarrow A_\mu^\omega(x) = \omega(x)(\partial_\mu + A_\mu)\omega^\dagger(x),
\]

where \( \omega(x) = \omega(x) \in SU(3) \). We are interested in the gauge invariant vacuum sector where the propagator satisfies

\[
< A_f | e^{-iH} | A_i >_0 = < A_f^\omega | e^{-iH} | A_i >_0.
\]

The usual way to select the gauge invariant contributions from the general propagator \( < A_f | e^{-iH} | A_i > \) is to insert a projection operator into the vacuum sector,

\[
< A_f | e^{-iH} | A_i >_0 = < A_f | \mathcal{P}_0 e^{-iH} | A_i > = \int D[\omega(x)] < A_f^\omega | e^{-iH} | A_i >,
\]

where \( \omega = e^{i\alpha^a\lambda^a} \) and

\[
\mathcal{P}_0 = \int D[H[\alpha(x)] e^{i \int d^3x \alpha^a(x) \mathcal{D}E^a(x)}.
\]
The exponent contains the generator of the gauge transformations,

$$\text{DE} = \partial E + [A, E]$$

(7)

and the integral variable appears as a static temporal component of the gauge field when the path integral representation is worked out for (5), \(\alpha^a(x) = tgA_0^a(x)\),

$$\langle A_f | e^{-itH} | A_i \rangle_0 = \int D_H [tgA_0(x)] \int D[A(x, t)] e^{-S_{YM}},$$

(8)

with

$$S_{YM} = \frac{1}{2g^2} \text{tr} \int dx (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^2.$$}

(9)

When the projection operator \(P_0\) is inserted at each time slice, \(t_n = na\), then the corresponding integral variable becomes the time dependent temporal component of the gauge field, \(\alpha^b(x) = agA_0^b(x)\),

$$\langle A | e^{-itH} | A' \rangle_0 = \int D_H [agA_0(x)] \int D[A(x, t)] e^{-S_L}.$$}

(10)

We shall see below that the action is given in lattice regularization is this case.

Global gauge transformations act as the basis transformations on the gauge field,

$$A(x) \rightarrow \omega A(x) \omega^\dagger.$$}

(11)

The center of the gauge group consists of those elements which commute with the whole group, it is \(Z_N = \{e^{i2\pi\ell/N}, \ell = 1, \ldots, N\}\) for \(SU(N)\). Since the center element commute with the generators as well they leave the gauge field invariant in (11). In \(SU(2)\) gauge theory where the center is \(Z_2 = \{1, -1\}\) and (11) expresses the well known fact that rotations by \(2\pi\) leave the vectors invariant. The relation between the center and the fundamental group can be seen by noting that the space of global gauge transformations, \(SU(N)/Z_N\), is \(N\)-fold connected. The center symmetry is the invariance of the propagator \(\langle A_f | e^{-itH} | A_i \rangle_0\) under global center transformations, (4), with \(\omega(x) = e^{i\theta/2}\).

It is easy to construct an order parameter for the center symmetry. The starting point is to note that the gauge invariance of the path integral (10).
is violated at the initial and the final time slices. In fact, the boundary conditions

\[
\begin{align*}
A(x,0) &= A_i(x) \\
A(x,t) &= A_f(x)
\end{align*}
\]  

(12)
do not allow to perform gauge transformations on the time boundaries. The remaining symmetries at the initial and the final time slice are the periodic gauge transformations with period length \( t \) in time since the Hamiltonian is gauge invariant, \([H, DE(x)] = 0\). Thus we have no Ward-identities for non-periodic gauge transformations.

The restriction of the space of gauge transformations to the periodic ones increases the family of gauge independent variables [7]. The observables generated in this manner correspond to the Polyakov line,

\[
\Omega(x) = Pe^{\int_0^t dt' A_0(x,t')},
\]  

(13)
which is the path ordered exponential along the straight line connecting identical three-space points of the initial and the final time slices. Its eigenvalues are invariant under the gauge transformations which are allowed by the boundary conditions, \([12]\). The gauge invariant eigenvalues, \( \lambda_a \), may serve as an order parameter because they transform multiplicatively under center transformations,

\[
\lambda_a \to e^{i\frac{2\pi}{N} \ell} \lambda_a.
\]  

(14)

Since the eigenvalues are not distinguishable our order parameter will be their sum,

\[
\text{tr} \Omega(x) \to e^{i\frac{2\pi}{N} \ell} \text{tr} \Omega(x).
\]  

(15)

It detects the center transformations which are not displayed by the initial and final gluon fields, \([12]\).

The insertion of the projection operator \( \mathcal{P}_0 \) into the amplitude \([1] \) makes the distribution of the order parameter formally center symmetrical. But the dynamical preservation of the center symmetry becomes energetically less favorable for short time processes. This can be seen by the inspection of the effective action for the Polyakov line \([3], [4]\),

\[
S_{\text{eff}}[\Omega] = -\ln \langle A_f^\Omega | e^{-itH} | A_i \rangle.
\]  

(16)

It is formally center symmetrical but the potential barrier between the center symmetrical minima diverges as \( t \to 0 \) due to the large kinetic energy needed for the finite global rotation of the gauge field in short time \([10]\).
2.1.2 Dimensional v.s. lattice regularization

The Haar measure is defined by its invariance under group multiplication,

\[
\int d_H \omega f(\omega) = \int d_H \omega f(\omega'). \tag{17}
\]

This property is required in proving the gauge invariance of \( (5) \),

\[
\langle A'\mid e^{-itH}\mid A \rangle_0 = \int D_H[\omega] \langle A'\mid e^{-itH}\mid A \rangle = \int D_H[\omega] \langle A'\mid e^{-itH}\mid A \rangle = \langle A \mid e^{-itH}\mid A' \rangle_0 . \tag{18}
\]

The invariant measure in the path integral \( (8) \) can be taken into account perturbatively. For this end we write the gauge transformation as \( \omega = vhv^\dagger \) where \( h \) is diagonal, \( h_{jk} = \delta_{jk} e^{iu_j} \). The Haar measure reads as

\[
d_H \omega = dv d^N \rho \sum_{n=\infty} \delta(\sum_j u_j - 2\pi n) \prod_{j<k} \sin^2 \left( \frac{u_j - u_k}{2} \right) . \tag{19}
\]

For the sake of simplicity we shall restrict ourselves to \( SU(2) \) gauge theory. Then \( \omega = e^{u\hat{n}\sigma^j/2i} \) with \( \hat{n}^2 = 1 \) and

\[
d_H \omega = d\hat{n} d\rho \frac{\sin^2 u}{2} = d^3 u \frac{1}{u^2} \sin^2 \frac{u}{2} , \tag{20}
\]

where \( d\hat{n} \) is the uniform integration over \( S^2 \). The Haar measure is a rotational invariant deformation of the flat integration measure, \( d^3u \).

We now return to the path integral where the projection operator is inserted at each time slice and the argument of the Haar measure is \( agA^b_0(x) \),

\[
D_H[agA^b_0] = D[\hat{n}] D[u] e^{\frac{u}{2} \int d^4x \ln \sin^2 au(x)/2} . \tag{21}
\]

Another form is where the Cartesian coordinate system is kept,

\[
D_H[agA^b_0] = D[A^b_0] e^{\frac{1}{a^2} \int d^4x \frac{\ln^2}{a^2} \sin^2 au(x)/2} = D[A^b_0] e^{\int d^4x (-\frac{u^2(x)}{12a^2} + \frac{u^4(x)}{96} + O(a^2))} , \tag{22}
\]

with \( u^2 = a^2 g^2 A^b_0 A^b_0 \) is better suited for perturbation expansion. The UV divergent factor, \( \frac{1}{a^2} \), is needed in order to make the exponent dimensionless.
The nontrivial integral measure can be taken into account by an additional potential for the fields in local regularization schemes where the elementary volume corresponding to a gauge degree of freedom can be identified.

The vertices introduced by the integral measure pose a new problem. In our effort to implement the center symmetry we introduced a term $O(u^2)$ which breaks the gauge invariance of the theory. The remedy for this problem actually comes from another $O(u^2)$ piece in the action. As was mentioned above the measure term can be treated consistently in lattice regularization only. But the gauge invariance requires the use of the link variables $U_\mu(x) = e^{i A_\mu(x)}$ instead of the gauge field, $A_\mu(x)$. The expansion of the plaquette in $A_\mu(x)$ gives rise another quadratic piece which cancels the measure contributions and restores the gauge invariance at one-loop level.

Similar cancellations between the lattice vertices continue to occur at higher order. A formal proof of renormalizability by induction in the order of the loop expansion can be given by the help of the Ward-identities \[11\]. There are two kinds of genuine lattice vertices which have no analogy in dimensionally regulated perturbative QCD. One, like the measure term, is proportional to a negative power of the lattice spacing and UV divergent. The other type which is suppressed by a positive power of the lattice spacing. Neither of these vertices is problematical as far as the overall degree of divergence is concerned. In fact, simple power counting shows that there are no new types of overall divergences because the dimension of these new vertices is supplied by the cut-off itself. To show this consider an observable computed in perturbation expansion where we have one coupling constant only for simplicity,

\[ O = \sum_{n=0}^{\infty} g^n I_n, \]

where $I_n$ stands for a loop integral. Comparing the mass dimensions of both sides we get $[O] = n[g] + [I_n]$. Since the overall degree of divergence of the integral is $\omega(I) = [I]$ we have

\[ \omega(I_n) = [O] - n[g]. \]

This relation shows that the renormalizable coupling constants must have nonnegative dimension except when their negative dimension is provided by the cut-off itself. The difficult part of the proof is to show that the overlapping divergences are removed as well and the counterterms can be chosen to be
gauge invariant. The final result is that the contributions of the diverging lattice vertices cancel in each finite order of the loop expansion at high energy. Furthermore, the order of the loop integration and the removal of the cut-off, \( a \rightarrow 0 \), can be exchanged since the properly regulated theory contains finite, uniformly convergent loop integrals. It is worthwhile noting that this holds only for theories without anomalies \([12], [14]\). Thus the contributions of the lattice vertices cancel and the asymptotically free perturbation expansion based on the usual three and four gluon vertices is recovered.

Dimensional regularization seems to be superior to the lattice regularization because it skips from the very beginning those vertices whose contributions are ultimately canceled. In fact, the suppressed lattice vertices are dropped because only \( \frac{1}{\epsilon} \) or \( \epsilon \)-independent pieces show up in the analytical regularization. The diverging vertices are absent as well because

\[
\frac{1}{a^4} = \int \frac{d^4p}{(2\pi)^4} = \frac{\Omega_4}{(2\pi)^4} \int dp \, p^3
\]  

(25)

and the right hand side is set to zero. Are the complications of the lattice regularization unnecessary? They are certainly unimportant at high energy and in any finite order of the loop expansion. But the situation changes when resummation or non-perturbative approximation is sought. The periodic Haar measure is essential to achieve the vanishing of the Polyakov line and confinement at low temperature \([9]\). For any truncation of the Taylor expansion for the periodic potential in (21) misses the periodicity and with it the confining forces. The perturbative cancellation of the genuine lattice vertices is not enough to expel them from the non-perturbative solution. But even if we had a non-perturbative argument for the possibility of renormalizing the theory in a gauge invariant manner it would not be enough to justify the neglecting of the genuine lattice vertices at finite energy where the asymptotic scaling laws do not apply.

2.1.3 Singular configurations

Center symmetry implies that the eigenvalues of the Polyakov line, (13), are distributed equally around the \( N \)-th roots of 1. The local gauge invariance of (10) allows us to set the integral variables \( e^{agA_0(x,t)} \) to the unit matrix everywhere in the space-time except an arbitrarily chosen equal time hypersurface
where obviously we have
\[ e^{agA_0(x,t_0)} = \omega(x)\Omega(x)\omega^\dagger(x). \] (26)

Whenever an eigenvalue of the Polyakov line is close to a nontrivial \( N \)-th root of 1 we must have \( agA_0 = O(a^0) \). Thus the fraction \( \frac{N-1}{N} \) of the field configurations contain a Dirac delta type singularity, \( A_0 = O(1/ag) \) in the continuum limit. In this manner the apparently harmless observation that the three vectors remains invariant under rotation by 2\( \pi \) brings badly singular configurations in the renormalized path integral. Since the projection operator \( P_0 \) is inserted at each time slice in (10) the configurations display the singularities in the fully gauge invariant path integral, too. The naive continuum expression for kinetic energy in the Yang-Mills lagrangian,
\[ -\text{tr}(\partial_0 A + DA_0)^2 \]
receives the term \( DA_0 \) from the infinitesimal gauge transformations performed between the consecutive time slices, \( \delta A = aDA_0 \). This expression is not invariant any more when the gauge transformations are in a finite, cut-off independent distance from the identity. The more careful derivation of the path integral expression which takes such a singular configurations into account yields the usual lattice regulated kinetic energy in terms of the link variable \( e^{agA_0(x,t)} \). The Lorentz invariant extension of the lagrangian and the path integral leads unambiguously to Wilson’s lattice gauge theory.

The singular structure of the configurations in the path integral is the rule rather than an exception. In fact, consider the path integral for a free massless particle in \( D \) dimension,
\[ \prod_x \int d\phi(x)e^{-\frac{1}{2}aD^{-2}\sum(\phi(x+\mu) - \phi(x))^2}. \] (27)

The typical configuration is where each contribution to the kinetic energy is \( O(a^0) \),
\[ \phi(x+\mu) - \phi(x) = O(a^{1-D/2}). \] (28)

The trajectories are non-differentiable for \( D = 1 \), in Quantum Mechanics, \[ 13, 14 \], have finite discontinuities in two dimensions, \[ 15 \], and develop Dirac delta type power singularities for \( D > 2 \). The Fourier transform of the one-loop momentum space UV divergences yields the same result. Due to this singular structure the topological concepts introduced on the tree level may not survive the renormalization procedure \[ 16 \].
2.2 Effective theory

2.2.1 Sine-Gordon model

Our goal is to construct a local effective theory which comprises the low energy effects of the center symmetrical fluctuations of the vacuum [6]. In the lattice regulated theory we find the usual three and four gluon vertices, infinitely many genuine lattice vertices and the lattice propagator. The lattice vertices cancel against each other perturbatively in the UV region. We suspect that the periodicity of the integral measure is crucial at low energies where the cancellation does not hold any more. Thus we shall ignore all but the integral measure vertices for $A_0$.

The construction of the effective theory for SU(2) gauge theory is as follows: We start in the gauge where $A_0$ is diagonal, $A_0^a(x) = \delta^{a3} u(x)$ and a U(1) local gauge symmetry is left only. Then we eliminate the off diagonal gauge field components which are charged with respect to the diagonal U(1) gauge group. The resulting effective theory is a nonlinear U(1) gauge theory. So far this is the usual procedure for setting up the Abelian confinement scenario, [17], except that the Polyakov line is used to single out the Abelian subgroup. Then we choose Feynman gauge and eliminate the space-like component of the gauge field. We arrive at an effective theory for $u(x)$ where the interactions come from the effective vertices generated by the elimination procedure and the measure term.

Is the measure term is the same in the effective theory as in the original bare lagrangian? Recall that the boundary condition in time for the trajectories in the path integral (10) restricts the gauge invariance for periodic gauge transformations. Thus the integral measure for the eigenvalues of the Polyakov line is not protected by gauge symmetry and gets renormalized. But in the same time the center symmetry remains present and requires that the integral measure be periodic. So the integral measure for the diagonal component of the local $A_0(x, t)$ must get renormalized as well. In fact, the eigenvalues of the Polyakov line are given by the field $u(x)$,

$$\lambda_a(x) = e^{\pm ig \int du(x,t)},$$

and its integral measure would not be renormalized unless the integral measure for $u(x)$ is renormalized.

In the spirit of the gradient expansion we ignore all derivative coupling
and truncate the effective theory onto the lagrangian

$$L = \frac{1}{2} (\partial_\mu u)^2 - V(u)$$  \hspace{1cm} (30)$$

where the local potential is periodic,

$$V(u + 2\pi/\kappa) = V(u),$$  \hspace{1cm} (31)

$$V(u) = 2 \sum_{m>0} v_m \cos mk_u.$$  \hspace{1cm} (32)

The contribution $m = 0$ is eliminated by the condition $\int du V(u) = 0$. This four dimensional sine-Gordon type model is our effective theory. It will be shown that despite the absence of the other interaction it reproduces the salient non-perturbative features of the vacuum.

### 2.2.2 Haaron gas

Consider first a simplified version of the effective theory with one Fourier mode only,

$$L = \frac{1}{2} (\partial_\mu u)^2 - 2\lambda \cos \kappa u.$$  \hspace{1cm} (33)

The generator functional defined as

$$Z[J] = \int D[u] e^{-\int dx [\frac{1}{2} (\partial_\mu u(x))^2 - \lambda (e^{i\kappa u(x)} + e^{-i\kappa u(x)}) + ig J(x) u(x)]}$$  \hspace{1cm} (34)

what is expanded in $\lambda$,

$$Z[J] = \sum_{n=0}^\infty \frac{\lambda^n}{n!} \prod_{j=1}^n \left( \int dx_j \sum_{m(j) = \pm 1} \right) I_{mm} I_{JJ} I_{mJ},$$  \hspace{1cm} (35)

where

$$I_{mm} = e^{-\frac{g^2}{2} \sum_{k,j=1}^n m(j)G(x_j - x_k)m(k)}$$

$$I_{JJ} = e^{-\frac{g^2}{2} \int dx dx' G(x-y)J(x)J(y)}$$

$$I_{mJ} = e^{-g\kappa \sum_{j=1}^n m(j) \int dx G(x-y)J(y)}$$  \hspace{1cm} (36)
and

\[ G(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ipx}}{p^2} = \frac{1}{4\pi^2x^2}. \]  

(37)

The interpretation of these expressions is quite straightforward: The sine-Gordon model is equivalent with the grand canonical ensemble of point particles with charge \( \pm \kappa \) and fugacity \( \lambda \) interacting via the Coulomb potential (37). The logarithm of the factors \( I_{mm}, I_{JJ}, I_{mJ} \) contains the energies due to the self interaction of the Coulomb particles, the source and the energy of the source in the Coulomb fields of the particles, respectively.

We now return to the effective theory (30) where the repetition the steps shown above gives

\[
Z[J] = \int D[u] e^{-\int d^4 x \left( \frac{1}{4} (\partial_\mu u(x))^2 + igJ(x)u(x) \right) + \frac{1}{2} \sum_{m>0} \sum_{m'=0} v_m \cos m\kappa u} \\
= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} (\int d^4 x_j \sum_{m(j)=-\infty}^{\infty} v_{m(j)}) I_{mm} I_{JJ} I_{mJ}. \]  

(38)

The charge is quantized in the units of \( \kappa \) and the Fourier coefficients turn into the fugacities. The Coulomb particles are called haarons because they represent the effects of the Haar measure.

The haaron-haaron interactions polarize the gas and introduce the Thomas-Fermi screening. The screening mass square is the curvature of the effective potential of the field \( u(x) \) at the vacuum expectation value. The manifest center symmetry, the invariance of the theory with respect to the shift \( u(x) \rightarrow u(x) + 2\pi / \kappa \) allows the constant effective potential only. This is in agreement with earlier numerical finding, namely that the eigenvalues of the Polyakov line decouple from the dynamics in the low temperature phase of QCD and its distribution is given solely by the integration measure without the influence of the action [18]. The absence of the screening in a Coulomb gas is attributed to the negative fugacities. Some of the Fourier coefficients, \( v_m \), can be negative indicating negative probability in the classical Coulomb gas picture of the sine-Gordon model. This makes the cancellation between the screening effects of different charges possible. In other words, the self energy computed in the framework of the perturbation expansion may be vanishing for appropriately chosen coupling constants, \( v_m \). The dynamical issue of confinement is the assumption of the center symmetry, the absence of screening in the effective theory. By the help of the massless propagator,
the confining forces and the chiral symmetry breaking can be obtained in a partial resummation of the perturbation expansion.

After having excluded the generation of the screening mass the polarization of the Coulomb gas becomes less important and the approximation \( I_{mn} = 1 \), the neglect of the Coulomb interactions between the particles is more reasonable. The remaining contributions can be resummed in \( (38) \) yielding

\[
Z[J] = e^{-\frac{g^2}{\pi} \int d^4x d^4y J(x) G(x-y) J(y) - \int d^4x V(i g \int d^4y G(x-y) J(y))} = e^{-\int d^4x \left( \frac{g^2}{2} J(x)^2 + V(i U(x)) \right)}.
\] (39)

The exponent of the second equation gives the energy of the source system as the sum of the perturbative current-current interaction energy and the potential energy in terms of the Coulomb potential created by the source,

\[
U(x) = g \int d^4r G(x-y) J(y).
\] (40)

The only unusual detail is the factor \( i \) in the argument of the potential \( V \) whose role is to remove the periodicity after the elimination of the quantum fluctuations. It is worthwhile noting that the Wick rotation into real time is nontrivial and the factor \( i \) remains present.

### 2.2.3 Static charges

The energy \( E \) of a test charge can be read off in the long time limit as \( tE = -\ln Z \) by choosing

\[
J(x) = \int ds \frac{dy^0}{ds} \delta^{(4)}(x - y(s)),
\] (41)

where \( y^\mu(s) \) is the world line of the charge. We take a static charge, \( y^\mu(s) = (s, x_0) \), and use \( (39) \),

\[
E_{np} = \int d^3x V \left( \frac{ig\kappa}{4\pi|x - x_0|} \right)
\] (42)

for the non-perturbative second term in the exponent of \( (39) \).

This result is typical inasmuch as the argument of the potential \( V(u) \) is small in the IR since the Coulomb field created by the source approaches
zero at the infinity for localized source. Thus the infrared contribution to the energy is controlled by the behavior of the potential around zero, \( V(u) = V(0) + \frac{u^2}{2} V''(0) + O(u^4) \). Hence we have

\[
E_{np} = -V''(0) \left( \frac{g \kappa}{4\pi} \right)^2 \int d^3x \frac{1}{x^2} + \text{const.} \tag{43}
\]

The curvature of the bare potential should be negative, \( V''(0) < 0 \), in order to arrive at flat effective potential in the IR. The linear infrared divergence of the integral indicates the absence of localized charged among the asymptotic states of the effective theory. Note that the Thomas-Fermi screening would regulate this integral and localized charges would be allowed. Thus one expects the effective theory to undergo a phase transition and develop screening at the deconfining phase transition.

Similar computation allows us to derive the static potential between test charges. To this end we take

\[
J(x) = \int ds [\delta^{(4)}(x - y(s)) - \delta^{(4)}(x - y(s) - L)], \tag{44}
\]

with \( L = (0, L) \). The non-perturbative part of (39) gives

\[
E_{np} = -\int d^3 x V \left( \frac{ig\kappa}{4\pi|x|} - \frac{ig\kappa}{4\pi|x - L|} \right) = -V''(0) \frac{g^2}{8\pi} |L| + \sum_{n=0}^{\infty} c_n |L|^{-n}. \tag{45}
\]

The leading infrared part is a linearly rising potential with the string tension

\[
\sigma = -V''(0) \frac{g^2}{8\pi}, \tag{46}
\]

and the coefficients in the sub-leading pieces, \( c_n \), are ultraviolet divergent.

### 2.2.4 Nambu-Jona-Lasinio model

It is straightforward to include quark fields into the effective theory. The resulting lagrangian is

\[
L = \frac{1}{2} (\partial_\mu u)^2 - V(u) - \bar{\psi}[i\not\!\partial - g\gamma_0 \sigma^3 u]\psi. \tag{47}
\]
It is not Lorentz or gauge invariant since the diagonal component of $A_0$ are kept only after making approximations. The elimination of the field $u(x)$ in the free hadron gas approximation gives (39) except the external source is replaced by $J(x) = \bar{\psi}\gamma_0\sigma^3\psi$,

$$S = \int d^4x \left\{ -\bar{\psi}\not{\partial}\psi + V(i g \int d^4y G(x-y)J(y)) \right\}$$

$$-\frac{g^2}{2} \int d^4x d^4y J(x) G(x-y) J(y)$$

$$= \int d^4x \left\{ -\bar{\psi}\not{\partial}\psi + V(iU(x)) - \frac{g}{2} \int d^4x J(x)U(x) \right\},$$  \quad (48)

where (40) is used to express the Coulomb field of the dynamical source $J$. The second term in the first equation gives an infinite series of non-local vertices. The $O(J^2)$ action is

$$S = -\int d^4x \bar{\psi}\not{\partial}\psi$$

$$+ \int d^4x d^4y J(x) \left\{ -\frac{g^2}{2} G(x-y) + \frac{4\pi\sigma}{g^2} G_2(x-y) \right\} J(y),$$  \quad (49)

where

$$G_2(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^4}.$$  \quad (50)

Notice that the static potential generated by $G_2(x)$ is linear with string tension $\sigma$. There have been several studies of the NJL model with such kind of interaction and the self consistent gap equation approximation shows the dynamical breakdown of the chiral symmetry induced by the strong repulsion \cite{19}.

It is illuminating to eliminate the quark field degrees of freedom in favor of a composite meson field. By the help of the identity

$$\int D[\Phi] e^{-\frac{i}{2} \int dx dy \Phi(x)K^{-1}(x,y)\Phi(y) + \int dx A(x)\Phi(x)} = e^{\int dx dy A(x)K(x,y)A(y)}$$  \quad (51)

we find

$$S = -\text{tr} \ln[i\not{\partial} + \gamma_0\sigma^3\Phi] + \frac{1}{2g^2} \int dx \Phi \frac{\partial^4}{\partial^2 \Phi}.$$  \quad (52)

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Notice the strong IR dependence of the wavefunction renormalization constant
\[
Z(p^2) = \frac{p^2}{\frac{4\pi\sigma}{g^2} - p^2}.
\]  
(53)

Since the restoring force for the small fluctuations of the meson field is weak in the IR there are strong interactions between the low energy mesons and the quark-anti quark vacuum polarizations.

It is worthwhile noting that the Euclidean mesonic effective theory is defined only below the Landau pole \( p^2 < \frac{4\pi}{g^2} \sigma \) since beyond this limit the Hubbard-Stratanovich transformation (51) requires imaginary meson field.

2.2.5 Haaron gas and localization

There is a simple physical picture behind the haaron gas description of the QCD vacuum. Consider the quenched multi-quark Green function in the free haaron gas approximation,
\[
<0| T[\bar{\psi}(y_1) \cdots \bar{\psi}(y_n)\psi(z_1) \cdots \psi(z_n)] |0> = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \left( \int d^4 x_j \sum_{m(j)=-\infty}^{\infty} v_{m(j)} \right) \nonumber
\]
\[<\bar{\psi}(y_1) \cdots \bar{\psi}(y_n)\psi(z_1) \cdots \psi(z_n)>_U, \]  
(54)

where the factor in the last line is the free fermion Green function on the imaginary background field
\[
U(x) = \frac{i\kappa}{4\pi^2} \sum_{j=1}^{n} \frac{m(j)}{(x - x_j)^2},
\]  
(55)
e.g.
\[
<\bar{\psi}(y)\psi(z)>_U = \left[ \frac{1}{[i\partial - g\gamma_0\sigma^3U]} \right](y, z).
\]  
(56)

The expression (54) is reminiscent of the fermionic Green functions in the dilute instanton gas approximation. But our result is not a semiclassical contribution, the haaron gas may be dense and is not an extremum of the Yang-Mills action. Furthermore there are no zero modes and result is infrared.
finite. For the dilute haaron gas the fermion propagator can be factorized and the zero mode dominance yields the condensate

\[ \langle 0 | \bar{\psi} \psi | 0 \rangle = -V(0) \text{tr} \int d^4x < \bar{\psi}(x) \psi(x) >_U \]

(57)

where propagator in the right hand side is evaluated on a single haaron background. Similarly to the case of the instantons there are localized zero modes for the Dirac operator which make the integral finite and generate the chiral symmetry breaking.

The hard confinement mechanism is similar to localization observed in strongly correlated electron systems, [20], except that it takes place in the space-time rather than the three-space. We have seen that the IR divergence of the single quark energy makes the quark propagator vanishing. It is easy to reproduce this result by the localization scenario. In fact, consider an isolated quark which is sent through the vacuum. The oscillation due to the phase shift which is generated by the long range haaron potential cancels the propagator.

In order to understand the propagation of a quark-anti quark pair we have to take into account the correlation between the haarons. The coupling constants \( v_m \) provide the scale for the finite haaron density in the vacuum and the average haaron distance appears as a correlation length of the quenched Coulomb potential generated by the haarons. Let us follow a very simple way of keeping track of this correlation: The haaron potential is considered completely correlated or uncorrelated for separations which are less or more than the correlation length, respectively. The propagator of a meson can be written as the sum over the world lines of the quark-anti quark pairs. The phase shifts of the quark-anti quark state cancel so long as the separation of the quark-anti quark pair is smaller than the correlation length. When the distance between the pair increases beyond the correlation length then the statistically uncorrelated potential creates a non-vanishing phase shift which in turn suppresses the contribution after the haaron gas averaging. Thus the quark-anti quark pair tends to stick together and the confinement radius is the correlation length of the haaron potential. The similarity between this scenario and the realization of confinement in the stochastic vacuum [21] is remarkable.
2.2.6 Double role of $V(u)$

The periodic potential $V(u)$ has two rather different roles in our approximate solution. On the one hand, it controls the large amplitude fluctuations of the field $u(x)$ in the effective lagrangian, $\langle \Phi \rangle$. The order of magnitude of the center symmetrical fluctuations is $\frac{1}{\kappa}$ and the behavior of $V(u)$ in this range, in particular the periodicity, is important. On the other hand, the same potential appears after the partial resummation in the free haaron gas. Its typical argument is purely imaginary and small in absolute magnitude as far as the IR physics is concerned because the Coulomb potential approaches zero for large distances. The approximation $V(u) \approx V(0) + V''(0)u^2/2$ around the origin is sufficient to express the string tension and the chiral condensate. After the elimination of the fluctuations the non-fluctuating argument of the potential is small and the IR physics is governed by the behavior of the potential around zero.

One finds that the periodicity of the potential and the manifest center symmetry has two consequences in the effective theory: In the UV it keeps the fluctuations of the field $u(x)$ large and thereby it creates the non-perturbative environment. In the IR regime it protects against mass generation. This latter appears as the discrete analog of the chiral symmetry. What is furthermore interesting is that a discrete symmetry is responsible for the massless behavior.

2.2.7 Center symmetry and instantons

The perturbative ordered vacuum is based on the configuration $A_\mu = 0$ and the small fluctuations around it. This gives $\frac{1}{V} \int d^3 x \text{tr} \Omega(x) \approx 2$ in $SU(2)$ theory which indicates that the transition amplitudes are not center symmetrical. What modes are responsible for the restoration of the center symmetrical vacuum, $\frac{1}{V} \int d^3 x \text{tr} \Omega(x) = 0$? It is reasonable to expect that localized configurations which have large entropy and interpolate between the center related minima of the effective potential for the order parameter will reach this goal for sufficiently large values of the time parameter $t$ of the matrix element of the time evolution operator $\langle \Omega \rangle$.

The shape of such an interpolating configuration can be found as the solution of the Yang-Mills equations of motion. This turns out to be an instanton. The reason is that the map of the three-space into the gauge
group is \( S_3 \to S_3 \) when the space is spherically compactified. The Polyakov line configurations are labeled by the Pontryagin index, the winding number. But this is just the topological charge, \([22]\). Thus the Polyakov line of an instanton winds around the \( SU(2) \) group space as the space coordinates moves around the whole three-space. The asymptotical value for large coordinates is \( \Omega \approx 1 \) and in order to cover the whole group space it has to take the value \( \Omega = -1 \) somewhere. Due to the rotational symmetry of the solution this happens at the center of the instanton. In this manner \( \text{tr}\Omega(x) \) is a spherically symmetrical localized function which interpolates between \( \pm 2 \). The gas of such "domains" restores the center symmetry. Note that the size of the instantons which might play role in the center symmetry restoration must be around the confinement radius.

3 Universality and condensates

The picture of the confining gluonic vacuum outlined above raises more questions than answers. The justification of the emphasis put on the center symmetry comes from the numerical experiences in lattice gauge theory, namely from the observation that the fate of the center symmetry is related to the existence of the string tension. But the important IR effects are produced by the Haar measure vertices of the path integral, by the vertices which are labeled as non-renormalizable or irrelevant according to the renormalization group. These vertices are set to zero in dimensional regularization. Does that mean that the usual class of renormalizable field theories is not sufficient to parametrize all possible physics of a given set of particles and there might be different QCDs? We are not in the position to answer affirmatively this question which goes beyond the perturbation expansion. Instead, my goal will be to indicate only a gap in the usual argument which might make possible to describe the non-perturbative IR effects more systematically and in the same time to find new continuum field theories.

There is a well known example where the usual power counting might miss a relevant operator, the strong coupling QED \([23]\). The point is that the anomalous dimensions should be included into the usual power counting argument in classifying the operators of the theory. The extrapolation of the perturbative anomalous dimension of the electron field gives dimension four for the four fermion operator when \( e \approx 1 \). Thus we have the possibility for a
new relevant coupling constant in strong coupling QED.

We shall give two other examples for the generation of new relevant coupling constants. They might be more realistic because one of them refers to asymptotically free theories and the other is the Higgs sector of the Standard Model. In both cases the apparent violation of the universality and the emergence of new parameters are related to condensates.

3.1 Localized saddle points

Our example for non-perturbatively generated relevant coupling constants in asymptotically free models is based on the higher order derivative terms in the lagrangian. If localized saddle points, coherent states, appears in the theory then the higher order derivatives may deform them substantially. Depending on the sign of the derivatives the saddle points may shrink to the cut-off size and saturate the path integral with cut-off effects at each length scale. What is interesting is that this may happen despite the weakness of the coupling constant at the cut-off scale because the derivatives of the saddle point are sufficiently large.

3.1.1 Bare vs. renormalized expansion

The starting point is the difference between the bare and the renormalized saddle point expansion \[24\]. The well defined path integral is given only for the bare, regulated theory,

\[
\int D[A_\mu] e^{-S_B} = \int D[A_\mu] e^{-S_0 - S_i - S_{CT}}
\]

\[
= \int D[A_\mu] e^{-S_0} \left\{ 1 - S_i - S_{CT} + \cdots \right\}, \tag{58}
\]

where the small parameter of the perturbation expansion is \(g_B\). The bare action is the sum of the renormalized one, \(S_R = S_0 + S_i\), and the counterterms, \(S_{CT}\). The new expansion parameter, \(g_R\), is obtained in the renormalized perturbation expansion after taking into account the cancellations in the Taylor expansion of the second equation. But note that this procedure is formal in the sense that there is no well defined path integral with the small expansion parameter \(g_R\). In such a manner the reliability and the applicability of the perturbation expansion should be investigated in the framework of the bare
rather than the renormalized series since the expansion is done before the cancellations take place. In fact, the phase transitions provide evidences that the renormalizable running coupling constants can not even characterize the theory in a unique manner.

The difference between these two expansion schemes is more pronounced for theories with dimensionless coupling constants only, such as the $SU(2)$ Yang-Mills theory. Here the renormalized saddle point expansion is based on the strategy that the saddle points are selected by $S_R$ and the counterterms are taken into account on the higher loop level only. The tree level saddle points are the instantons whose action is independent of the instanton size due to the scale invariance of $S_R$. The higher loop contributions break this scale invariance in a manner that large instantons are preferred and the non-interacting instanton gas picture becomes inconsistent. The bare saddle point expansion brakes the scale invariance on the tree level already since a scale parameter, the cut-off, appears explicitly in the regulated action. Such a tree level violation of the scale invariance is governed by the dimensional parameters of the bare action. In order to explore the possibilities of the breakdown of the scale invariance we introduce additional dimensional coupling constants in the bare theory. The coupling constants with positive mass dimension are superrenormalizable and influence the distribution of the large, i.e. cut-off independent saddle points. They are excluded in Yang-Mills theory by symmetry. The coupling constants with negative mass dimension are non-renormalizable and govern the distribution of the ultraviolet saddle points around the scale of the cut-off.

The regulators are represented by irrelevant operators because they are supposed to supress the fluctuations in the UV regime only. In this manner different regulators give different distribution for the small instantons. The instantons whose size parameter is in the vicinity of the cut-off and whose dynamics is influenced by the regulators will be called mini-instantons. The usual Wilson-type single plaquette action decreases monotonically with the size of the instantons. The too low action of the mini-instantons leads to divergent topological susceptibility and nonunique topological charge in lattice regularization. The usual strategy to cope with this problem is to construct topological charge operator or improved action which cuts out or supresses the mini-instantons, alas topological defects. This is certainly justified when we are to establish a lattice regulated Yang-Mills model as close as possible to the continuum perturbation expansion. But our goal is different
in the search of the confinement mechanism when the results of the dimensional regulated renormalized saddle point expansion can not always be used. Instead, we take the bare theory seriously, on a non-perturbative manner at each length scale and try to trace downs the modifications of the dynamics due to the mini-instantons, if exist, even in the IR regime.

### 3.1.2 Mini-instantons

We start with the bare theory which involves some higher dimensional terms, \[ L_B = -\frac{1}{4g^2} F^a_{\mu\nu} \left( \delta^{ab} + \frac{c_2}{\Lambda^2} D^{2ab} + \frac{c_4}{\Lambda^4} D^{4ab} \right) F^{a\mu\nu}. \] The higher order pieces usually come from a heavy particle exchange in the energy range \( \Lambda \). The coupling constants \( c_\alpha \) are irrelevant according to the power counting. The simplest way to see this is to consider an observable obtained in the perturbation expansion, \[ <O> = \mu \left[ O \right] \mu = \mu \left[ O \right] \sum_{jkl} g^j \left( \frac{c_2 \mu^2}{\Lambda^2} \right)^k \left( \frac{c_4 \mu^4}{\Lambda^4} \right)^l I_{jkl} \left( \frac{\mu}{\Lambda} \right). \] The characteristic scale, \( \mu \), of the observable is used to give its dimension and each insertion of a vertex with the coupling constant \( \frac{c_\alpha}{\Lambda^\alpha} \) brings the factor \( \mu^\alpha \) by dimensional reasons. So long as the theory is infrared finite and the limit \( \mu/\Lambda \rightarrow 0 \) is convergent the \( c_\alpha \) dependence drops in the renormalized observables. The power counting gives another important result, which has already been mentioned above in connection with the renormalizability of lattice gauge theory. Namely, the coupling constants \( c_\alpha \) do not harm renormalizability, they are actually a variant of the Pauli-Villars regulators.

The power counting argument is no longer valid if there are localized saddle points in the theory. In that case the dependence in the coupling constants is not necessarily polynomial and the suppressing factor \( \left( \frac{\mu}{\Lambda} \right)^\alpha \) may be missing. As an example take \( Z_1/Z_0 \), the ratio of the partition function of the one and the zero instanton sector. In order to find the saddle point consider a one parameter family of the instanton configurations labeled by a scale parameter \( \rho \). The action of the instanton is \[ S_B(\rho) = -\frac{8\pi^2}{4g^2} \left( 1 - \frac{\tilde{c}_2}{(\Lambda\rho)^2} + \frac{\tilde{c}_4}{(\Lambda\rho)^4} \right). \]
where $\tilde{c}_\alpha/c_\alpha$, $\alpha = 2, 4$ are positive, $\rho$ and $g$ independent constants. The stable mini-instanton size, $\tilde{\rho}$, is obtained by solving $(\tilde{\rho} \Lambda)^2 = 2 \tilde{c}_4/\tilde{c}_2$,

$$S_B(\tilde{\rho}) = \frac{8\pi^2}{g^2} \left(1 - \frac{c_2^2}{4\tilde{c}_4}\right).$$

(62)

The one loop approximation yields

$$\frac{Z_1}{Z_0} = C g^p V \Lambda^4 e^{-S_B(\tilde{\rho})},$$

(63)

where $V$ is the four volume $[27]$. Observe the absence of the perturbative suppressing factor $1/(\Lambda \tilde{\rho})^2$ in (62) and (63). What is not suppressed here is a cut-off contribution because $1/(\Lambda \tilde{\rho})^2 = O(\Lambda^0)$.

A more careful analysis of the fluctuation determinant shows that the mini-instantons dominate (63) for $1 - \tilde{c}_2^2/4\tilde{c}_4 < X < 1$, where $X$ is a given positive constant. When this inequality is satisfied then one of the following two possibilities is realized: (i) $c_\alpha$ are relevant or (ii) $c_\alpha$ are irrelevant but the beta function for $g$ is non-universal. In fact, let us assume that $c_\alpha$ are irrelevant. Then the cut-off independence of (63) gives the beta function

$$\beta_g = \Lambda \frac{dg}{d\Lambda} = -\frac{g^3}{4\pi^2(1 - \tilde{c}_2^2/4\tilde{c}_4)} < -\frac{g^3}{4\pi^2X}.$$ 

(64)

In either case the coupling constants $c_\alpha$ modify the physical content of the theory at length scales which are independent of the cut-off. In other words, the saturation of the path integral by the mini-instantons which are close to the cut-off makes the cut-off effects present and changes the dynamics at finite length scales.

### 3.2 Multiple fixed points

Another example for the unusual relevant parameters is when there are several fixed points in a theory and a given operator is relevant at one fixed point and irrelevant at another one. The appearance of multiple fixed points is typical in Particle Physics where different interactions are found to be dominant at different energy scales.
3.2.1 Theory of Everything

Let us imagine the renormalized trajectory of the Theory of Everything (TOE). This theory includes all physics, at any length scale, by definition. In order to understand its features it is useful to summarize the connection between the language of the renormalization group when applied in Statistical Physics and in Quantum Field Theory.

The UV fixed point is where the correlation length is infinite. It corresponds to the infinite value of the cut-off, to the renormalized theories. The applicability of the linearized version of the blocking relations gives the scaling regimes. In these regions we have a classification of the coupling constants. Those coupling constants which decrease or increase when we move towards the IR regime are called irrelevant or relevant, respectively. The irrelevant coupling constants increase as we move into the opposite, UV, direction. This is what we always do in the renormalization of a Quantum Field Theory. Thus the irrelevant operators prevent us from going ”back” to the UV fixed point and from removing the cut-off. Hence they are non-renormalizable.

Finally, universality states that the physics at the length scale which corresponds to the IR end of the scaling regime is insensitive for the choice of the irrelevant coupling constants. This translates into the usual rule of model building in Particle Physics, the use of renormalizable theories only. The non-renormalizable theories were first excluded due to their uncontrollable UV behavior. By some tricks, such as the use of the cut-off to suppress the non-renormalizable coupling constants mentioned above may help to eliminate the divergences from the theory but the predictive power and the simplicity are lost. A simpler argument to ignore non-renormalizable coupling constants is offered by the renormalization group: They become small anyhow at the scale of the observations so we might as well set all of them zero from the very beginning, at the cut-off.

We now return to the renormalized trajectory of the TOE. At its asym-
totically high energy scaling regime, in the vicinity of the UV fixed point the trajectory is governed by the relevant coupling constants. But it is the lesson of the Wilson-Kadanoff blocking that all coupling constants what might be generated later should be included into the theory from the very beginning. Since the TOE describes the physics down to the classical regime all effective coupling constants must appear in the theory. In this manner the coupling constant space includes GUT, Standard Model, Nuclear, Condensed Matter, Solid State and Atomic Physics parameters, too. Say a QCD quark-gluon vertex appears as a multi-particle composite vertex on the level of the TOE. We must include all composite vertices in the action what might be needed to characterize the physics at lower energies.

Suppose that we increase the energy of the measurement from few GeV. The renormalized trajectory converges to the UV fixed point of QCD only in the model computations. In the real world the electro-weak interactions which are represented by the non-renormalizable current-current vertices well below the Standard Model scale start to deflect the trajectory form the QCD UV fixed point. As we further increase the energy the local effective vertices explode at the threshold of the weak vector bosons. From now on we see the scaling laws of the Standard Model. The renormalized trajectory stays in the vicinity of the UV fixed point of the Standard Model (the triviality problem of the Higgs sector is now ignored for simplicity) so long as the effective vertices generated by the exchanges of a superheavy vector boson are weak. By repeting this argument at each intermediate fixed point we find that the renormalized trajectory of the TOE visits several fixed points, those of the GUT, Standard Model, QCD, QED, Condensed Matter, Solid State and Atomic Physics among others, as we move towards the IR. Finally there is an IR fixed point. There are further fixed points in the coupling constant space which can be approached by the renormalized trajectory when the environmental parameters, such as the temperature or chemical potentials are properly tuned.

Since the same operator algebra is classified in the vicinity of each fixed point it may happen that a given operator turns out to be relevant at some fixed point and irrelevant at others. It is certainly right that the renormalized trajectory is influenced only by the relevant operators of a fixed point in the scaling regime of the fixed point in question. Thus we have islands of universality around each fixed point. But a coupling constant which happens to be irrelevant at one fixed point may turn out to important at another fixed
point. Thus it is a quite involved question that what operators are relevant at a certain energy range because the answer depends not only on the energy range considered but all scaling regime between the energy range in question and the true UV fixed point. We shall see below the possibility that other fixed points toward the IR direction may influence the result, too.

3.3 From the superconductor to the Standard Model

To simplify the mixing of different fixed points consider the strong and the electromagnetic interactions for electrons, muons and nucleons at finite baryon density. This gives a well defined UV scaling regime until the Landau pole of QED. For the appropriate choice of the chemical potential we may have another scaling regime at lower energy which is related to Condensed Matter or Solid State Physics. The two scaling regimes define two classification schemes for the operators. Let us call an operator relevant or irrelevant at a fixed point if it is contained in the relevant scaling operator set of the fixed point or not, respectively. Each operator belongs to one of the classes (rel,rel), (rel,irr), (irr,rel) or (irr,irr). Here the first and the second property refers to the behavior of the operator in the UV or the IR scaling regime.

The electron mass, $m_e$, is of the type (rel,rel) since it is a renormalizable parameter of the QED lagrangian and influences the phase transitions at lower energies. On the contrary, the muon mass, $m_\mu$, is (rel,irr) because the effects of the muons are shielded by those of the electrons at energies below $m_\mu$.

The interesting class is the (irr,rel). A coupling constant of that type drops as we lower the observational energy from the UV cut-off and becomes undetectable. But it starts to grow as we arrive at the IR scaling regime and may play important role there and at lower energies. This is a "hidden parameter" in the sense that it must be specified at the microscopical scales but its effects appear at much longer length scales only. It represents an elementary interaction which is not detectable on the level of the elementary constituents and becomes important only when certain long range structure such as the solid state lattice is formed. An example of this type of operator is an effective four fermion contact term which incorporates the weak attractive forces between the electrons of the solid due to the phonon exchange. This non-renormalizable term is responsible for the superconducting ground state whose influence becomes dominant at very low energy. This four fermion contact term generated by the phonons is the analogy of the contribution
According to the strategy of the Wilson-Kadanoff blocking all operators which are generated at lower energy scales should be included in the hamiltonian from the very beginning. The appearance of non-renormalizable terms should not be a serious problem except for the UV fixed point of the TOE. In fact, since all lower energy theory is effective only the scaling laws inferred in the vicinity of a fixed point change as we go up in energy and the apparent explosion of the non-renormalizable coupling constants gives rise to stability in the inter-fixed point region. One is left with wild speculations only concerning the TOE. A possible scenario to include all coupling constant is to require the UV finiteness of this ultimate theory.

The crucial question whose clarification requires the detailed quantitative analysis is whether the initial value for the coupling constants of the type (irr,rel) effect the infrared behavior of the theory. In other words, whether the dynamical growth of these coupling constants as we pass the crossover in lowering the cut-off between the two fixed points is modified by the initial conditions taken at the UV side. In case of QED this is the question whether the Fermi constant or other effective vertex of similar structure generated at the level of the Standard Model or beyond influences the supercurrent density in solids.

We find an aspect of the condensate formation in the example above which is different from those observed in regard to the mini-instantons. The four fermion interaction becomes important at energies which are well below the characteristic mass scale, \( m_e \). What kind of degrees of freedom correspond to this new operator? One part of the answer is trivial. Namely, the degrees of freedom which are responsible of this vertex belong to the coherent state of photons with nonzero momentum, the solid state lattice. The excitations of this condensate, the phonons, generate the effective vertex. The other part of the question, the degrees of freedom influenced by this vertex is less obvious. They come from a condensate again, the boose condensate of the Cooper pairs. Thus we have two condensates below \( m_e \) and their dynamics are related to a non-renormalizable operator in the lagrangian. Other field theoretical models show similar behavior [29].

Historically, the discovery of the superconducting state of matter was a total surprise. The understanding of Quantum Field Theory was not sufficient at that time to predict such phenomena by a theory which was well tested in few particle process only. The renormalization group provides us
the language and the framework to study problems like this and the systematic search of relevant but non-renormalizable operators can reveal such surprises.

3.3.1 Wegner-Haughton equation

In order to separate the impact of different scaling regimes on the renormalized trajectory one has to follow the mixing of each operator which might prove to be relevant at some fixed point or important at the crossovers. The differential form of the renormalization group equation, [28], is more suited for this goal because it handles the mixing of infinitely many coupling constants in a very economical manner. We shall derive the equation for a scalar field theory by using sharp momentum space cut-off.

Denote the bare action by \( S_k[\phi] \) where \( k \) is the UV cut-off. We shall obtain \( S_{k-\nabla k}[\phi] \) in the loop expansion,

\[
e^{-S_{k-\nabla k}[\phi]} = \int D[\hat{\phi}] e^{-S_k[\phi + \hat{\phi}]} = e^{-S_k[\phi + \hat{\phi}_0]} - \frac{1}{2} \text{tr} \ln \frac{\delta S[\phi + \hat{\phi}_0]}{\delta \phi} + O(\nabla k/k),
\]

(65)

where the Fourier amplitude of the fields \( \phi(x) \) and \( \hat{\phi}(x) \) is non-vanishing for \( p < k - \nabla k \) and \( k - \nabla k < p < k \), respectively and the saddle point is given by \( \frac{\delta S[\phi + \hat{\phi}_0]}{\delta \phi} = 0 \). Since the n-loop contributions include an n-fold integration over the functional space \( \hat{\phi} \) they are proportional to \( (\delta k/k)^n \) in the absence of massless singularities in the given kinematical region. Thus \( \delta k/k \) appears as a new small parameter and the exact functional differential equation obtained in the limit \( \nabla k \to 0 \) includes the one-loop contribution only.

In order to simplify (65) we use the gradient expansion,

\[
S[\phi] = \sum_{n=0}^{\infty} \int dx U_n(\phi(x), \partial^{2n}),
\]

(66)

where \( U_n \) is a homogeneous function of order \( 2n \) in the derivative. This expansion will be truncated at \( n = 1 \),

\[
S[\phi] = \int dx \frac{Z(\phi)}{2}(\partial \phi)^2 + U(\phi),
\]

(67)

and furthermore the simplification \( Z(\phi) = 1 \) will be used to derive a simple differential equation for the potential \( U \). In order to pick up the local potential from the action we choose a homogeneous background field \( \phi(x) = \Phi \).
The functional differential equation reduces to

\[ e^{-V U_k - \nabla_k} = e^{-V U_k(\Phi)} - \frac{\nabla}{(2\pi)^d} \int \ln(p^2 + V_k''(\Phi)) \]

where the \( V \) stands for the space-time volume and the integration extends over the shell \( k - \nabla k < p < k \). In the limit \( \nabla k \to 0 \) one easily finds

\[ k \partial_k U_k(\Phi) = -\frac{\Omega_d k^d}{2(2\pi)^d} \ln(k^2 + U_k''(\Phi)) \]  

(69)

where \( \Omega_d \) denotes the d-dimensional solid angle. This equation represents the one-loop resummed mixing of the coupling constants of the potential \( U_k(\Phi) = \sum_n g_n \Phi^n \). In fact, the expansion of the logarithm in the second derivative of the potential gives

\[ k \partial_k U_k(\Phi) = -\frac{\Omega_d k^d}{2(2\pi)^d} \sum_n \frac{1}{n} \left( -\frac{U_k''(\Phi)}{k^2 + U_k''(\Phi)} \right)^n \]  

(70)

up to a field independent constant. This is the sum over the Feynman graphs contributions which come from the infinitesimal loop integration volume. The circumstance that the right hand side includes the running potential \( U_k(\Phi) \) rather than the bare one, \( U_\Lambda(\Phi) \), indicates that the contributions of the successive eliminations of the degrees of freedom are piled up during the integration of the differential equation and the solution of the renormalization group equation resummes the perturbation series. The solution of the differential equation interpolates between the bare and the effective potential as \( k \) is lowered from the original cut-off \( \Lambda \) to zero.

### 3.4 IR fixed point

The infrared fixed point is always trivial for theories with mass gap. In fact, as the block size extends beyond the correlation length the evolution of the coupling constants slows down and we find a manifold of stable fixed points. The only relevant coupling constants is the mass since it is divergent in the units of the cut-off. But this controls a quadratic operator so has trivial effects only. In order to find an example where the IR fixed point generates non-renormalizable relevant operators we look for models with massless excitations. The massless one component \( \phi^4 \) model is not appropriate due to
the Coleman-Weinberg mechanism \[30\]. We need a symmetry to keep the mass gap zero in the presence of the interactions. The simplest example is the linear sigma model in the symmetry broken phase. Let us start with the $O(N)$ invariant lagrangian,

$$
L = \frac{1}{2} (\partial_\mu \phi^a)^2 + U(|\phi^a|),
$$

(71)

$a = 1, \cdots, N$, in four dimensions. The renormalization group equation for the potential is of the form \[31\]

$$
k \partial_k U = -\frac{k^4}{16\pi^2} \left( \ln(k^2 + \partial^2 U) + (N - 1) \ln(k^2 + \partial^2_{tr} U) \right),
$$

(72)

where $\partial_t$ and $\partial_{tr}$ denote the derivatives in the along the vacuum expectation value and the transverse directions of the internal space, respectively. The beta function of the n-order massive mode vertex is defined by

$$
\beta_n = k \partial_k \partial^n U_k(\Phi).
$$

(73)

Note that these functions depend explicitly on the scale $k$ and $\Phi$. The strength of the effective interactions for the particlelike excitations are obtained by setting $\Phi^a = \langle \varphi^a(x) \rangle$.

Let us identify the leading IR piece of the beta functions. Since

$$
\beta_1 = -\frac{k^4}{16\pi^2} \left( \frac{\partial^2_t U_k(\Phi)}{k^2 + \partial^2_{tr} U_k(\Phi)} + (N - 1) \frac{\partial_t \partial^2_{tr} U_k(\Phi)}{k^2 + \partial^2_{tr} U_k(\Phi)} \right),
$$

(74)

the most important IR contribution of the higher order beta functions comes from the highest order power of the transverse denominator,

$$
\beta_n = (-1)^n (N - 1) \frac{k^4}{16\pi^2} \left( \frac{\partial_t \partial^2_{tr} U_k(\Phi)}{k^2 + \partial^2_{tr} U_k(\Phi)} \right)^n (1 + O(k^2/\partial_t \partial^2_{tr} U)).
$$

(75)

The dimension of the corresponding coupling constant, $g_n$, is $4 - n$ so the beta function, $\tilde{\beta}_n$, for $\tilde{g}_n = g_n k^{n-4}$ in the IR regime is

$$
\tilde{\beta}_n = (-1)^n \frac{N - 1}{16\pi^2} \left( \frac{\partial_t \partial^2_{tr} U_k(\Phi)}{k + \partial^2_{tr} U_k(\Phi)/k} \right)^n (1 + O(k^2/\partial_t \partial^2_{tr} U)) + (n - 4) \tilde{g}_n.
$$

(76)

According to the Goldstone theorem $\partial^2_{tr} U_{k=0}(\langle \phi \rangle) = 0$. We shall assume that $\partial_t \partial^2_{tr} U_k(\Phi) > 0$ and $\partial^2_{tr} U_k(\langle \phi \rangle) = o(k)$ which is supported by the
simple one-loop solution. The result is that the odd vertices whose beta function is negative correspond to relevant coupling constants. The vertices \( \phi^n \), \( n = 6, 8, \cdots \) which are irrelevant at the UV scaling regime give rise relevant operator(s) at the IR fixed point of the symmetry broken theory. The inclusion of the renormalization of \( Z(\Phi) \) leaves our conclusion unchanged [31].

3.5 Couplings of the Goldstone modes

The IR divergence in the beta functions poses an interesting problem for the chiral perturbation expansion and the Standard Model. It is well known that the effective theory of the Goldstone modes, the nonlinear sigma model is IR finite. This comes about because the Goldstone modes fluctuations have no restoring force by symmetry so they interact with each other via gradient couplings which suppress the IR divergences. This makes the chiral models which include only the Goldstone modes IR finite.

It seems surprising at the first moment that the IR finiteness is lost when the massive modes are added to the Goldstone particles. Though the on-shell amplitudes remain IR finite off-shell the IR divergences appear in the massive particle Green functions. These divergences can be found in the one-loop effective potential of the linear sigma model where the heavy particle legs are connected by a massless particle loop. This is just the contribution singled out in (75). This divergence comes from the coupling between the Goldstone and the heavy modes via the potential \( U(\phi^n) \). This coupling contains no derivatives and leads to the IR divergence in the beta function. The absence of the derivatives in this coupling can be understood by noting that the transverse modes of the linear sigma model are not exactly the Goldstone modes but contain the heavy mode at higher order in the fluctuations around the vacuum. Since the massless propagator of Goldstone modes comes from the second order contributions of the action and the mixing between the Goldstone and the heavy modes is of higher order the massive mode interacts by itself by the long range massless propagator. This IR divergence of the effective potential makes the dynamics of the condensate sensitive for the coupling constants which would have been completely unimportant otherwise. Similar strongly coupled long range interactions have been noticed in the framework of our extended Nambu-Jona-Lasinio model, after equation (12).
In regard to the Standard Model this argument raises the possibility that the higher order, non-renormalizable Higgs vertices which are generated by the exchange of the superheavy particles of the GUT scale may have an unusual strong influence on the low energy physics. These vertices are certainly small as we follow the renormalized trajectory around the energy scale of the Higgs mass, \( M_H \), but they may become large in the IR scaling regime. Since we have a differential equation the initial conditions at the UV cut-off determine the effective coupling constants in the IR region. Both the suppression in the UV and the amplification in the IR generate qualitatively similar power dependence around the fixed points according to the linearized blocking equations. So it seems plausible to expect seizable sensitivity of the IR physics on these non-renormalizable bare parameters. If the detailed numerical solution of the renormalization group equation reveals such a sensitivity then new parameters of the Standard Model are found. These parameters become important well below the mass gap. What degrees of freedom do they influence? The answer points again to the condensate since \( \langle \phi(x) \rangle \) is determined by the long range interactions between the heavy and the Goldstone modes. It is reasonable to assume that the number of new parameters is the number of the dynamically generated condensate. There is a difference with superconductivity in the manner the non-renormalizable vertices influence the excitation spectrum. The four fermion interaction leads to the Cooper pair formation whose condensate gives rise a new low energy excitation spectrum. In the case of the Higgs particle \( \langle \phi(x) \rangle \) generates the mass for the fermions and the gauge bosons. Thus the modification of the strength of the condensate feeds back to the whole mass spectrum of the theory.

4 Conclusions

The confinement of quarks has generated different models and led to the development of a number of non-perturbative mechanisms. The present lectures intend to introduce few new pieces to this collection. These are the breakdown of the fundamental group symmetry, the role of the nontrivial integration measure in the path integral, the presence of singular configurations in renormalized theories and the appearance of new parameters by the condensates or the multiple fixed point structure. QCD has a lesson for us to learn in each of these directions. Such a rich and many-sided theory can only
raise our determination to aim at synthesis and arrive at a comprehensive understanding.

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**References**

[1] V. N. Gribov, unpublished.

[2] W. Greiner, B. Muller, J. Rafelski, *Quantum Electrodynamics of Strong Fields*, Springer-Verlag, 1981.

[3] J. Polonyi, in *Quark-Gluon Plasma*, ed. R. Hwa, World Scientific, 1988.

[4] J. Polonyi, *Physics of the Quark-Gluon Plasma, Acta Phys. Hung.* in print.

[5] S. Coleman, *Aspects of Symmetry*, Cambridge University Press, 1985.

[6] K. Johnson, L. Lellouch J. Polonyi, *Nucl. Phys.* B367 675, 1991.

[7] J. Polonyi, in the Proceedings of *Advances in Lattice Gauge Theories* (Tallahassee, FL, 1985).

[8] N. Weiss, *Phys. Rev.* D24 475 1981; *Phys. Rev.* D25 2667; 1982 *Can. J. Phys.* 59 1686, 1981.

[9] J. Polonyi, K. Szlachanyi, *Phys. Lett.* 110B, 395, 1982.

[10] J. Polonyi, in *Hot and Dense Nuclear Matter*, W. Greiner, H. Stöcker A. Gallman edt. Plenum Press, 1994.
[11] B. W. Lee, *Phys. Rev.* D9 933, 1974; H. Kluber-Stern, J. B. Zuber, *Phys. Rev.* D12 467, 482, 3159, 1975; J. Zinn-Justin, in *Trends in Elementary Particle Physics*, Proceedings of the 12th Schools of Theoretical Physics, Karpacz, 1975, Acta Universitatis Wratislaviensis 368; B. W. Lee, in *Methods in Field Theory*, Les Houches 1975, North-Holland/World Scientific; S. D. Joglekar, B. W. Lee, *Ann. Phys.* , 97 160, 1976; J. C. Taylor, *Gauge Theories of Weak Interactions*, Cambridge UNiversity Press, 1976; T. Riesz, *Nucl. Phys.* B318 417, 1989.

[12] T. Riesz, *Comm.Math. Phys.* 117 79, 1988.

[13] L. S. Schulman, *Techniques and Applications of Path Integrations*, Wiley-Interscience, 1981.

[14] J. Polonyi, *Renormalization Group in Quantum Mechanics*, submitted to *Ann. Phys.* .

[15] K. Huang, J. Polonyi, *Int. J. of Mod. Phys.*, A6 409, 1991.

[16] J. Polonyi, *Topology Renormalized*, in the Proceedings of LATT'90, Tallahassee, FL, 1990.

[17] G. ’tHooft, *Nucl. Phys. B190* 455, 1981; *Phys. Scr.* 25 133, 1982.

[18] J. Polonyi, W. Wyld, *Random Field, Gauge invariance and Charges I.-II.* CTP-1452, CTP-1458, 1987, unpublished.

[19] R. Delburgo, M. D. Scadron, *J. Phys.* G5 1621, 1979; J. Finger, J. Mandula, *Nucl. Phys.* B199 168, 1982; A. Amer, A. Le Yaouanc, L. Oliver, O. Péne, J-C. Raynal, *Phys. Rev.* D28 1530, 1983; A. Le Yaouanc, L. Oliver, O. Péne, J-C. Raynal, *Phys. Rev.* D29 1233, 1984; A. Le Yaouanc, L. Oliver, S. Ono, O. Péne, J-C. Raynal, *Phys. Rev.* D31 137, 1985; S. Adler, C. Davis, *Nucl. Phys.* B244 469, 1984; S. Adler, *Progr. Theor. Phys. Suppl.* 86 12, 1986; M. Hirata, *Progr. Theor. Phys.* 77 939, 1987; R. Alkofer, P. A. Amundsen, *Nucl. Phys.* B306 305, 1988; M. Baker, J. S. Ball, F. Zacharaisen *Phys. Rev.* D38 1926, 1988.

[20] P. W. Anderson, *Phys. Rev.* 109 1492, 1958.
[21] M. B. Voloshin, *Nucl. Phys.* B154 365, 1979; H. Leutwyler, *Phys. Lett.* B98 447, 1981; P. Olesen, *Nucl. Phys.* B200 381, 1982; V. N. Baier, Yu. F. Pinelis, *Phys. Lett.* B116 179, 1982; D. Gromes, *Phys. Lett.* B115 482, 1982; G. Cerci, A. Di Giacomo, G. Paffuti, *Z. Phys.* C18 135, 1983; T. I. Belova, Yu. M. Makeenko, M. I. Polikarpov, A. I. Veselov, *Nucl. Phys.* 230 473, 1984; I.I Balitsky, *Nucl. Phys.* B254 166, 1985; H. Campostrini, A. Di Giacomo, S. Olejnik, *Z. Phys.* C31 577, 1986; L. S. Celenza, C. H. Shakin, *Phys. Rev.* D34 1591, 1986; H. D. Dosch, *Phys. Lett.* B156 365, 1987; *Phys. Lett.* B190 177, 1987; V. Marquard, H. G. Dosch, *Phys. Rev.* D35 2238, 1987; P. V. Landshoff O. Nachtmann, *Z. Phys.* C35 405, 1987. Yu. Simonov, *Nucl. Phys.* B307 512, 1988.

[22] P. Woit, *Phys. Rev. Lett.* 51 630, 1983; J. Polonyi, *Phys. Rev.* D29 716, 1984.

[23] P. I. Fomin, V. A. Miransky, *Phys. Rev. Lett.* 64B 166, 1976; V. A. Miransky, *Nowo Cim.* 90A 149, 1985; W. A. Bardeen, C. N. Leung, S. T. Love, *Phys. Rev. Lett.* 56 1230, 1986; C. N. Leung, S. T. Love, W. A. Bardeen, *Nucl. Phys.* B273 649, 1986; *Nucl. Phys.* B323 493, 1989.

[24] V. Branchina, J. Polonyi, *Nucl. Phys.* B433 99, 1995.

[25] G. ’tHooft, *Phys. Rev.* D14 3432, 1976; A. Polyakov, *Nucl. Phys.* B120 429, 1977.

[26] M. Luscher, *Nucl. Phys.* B200 429, 1977.

[27] V. Branchina, J. Polonyi, in preparation.

[28] N. F. Nicoll, T. S. Chang, H. E. Stanley, *Phys. Lett.* A57 7,1976; F. J. Wegner, A. Haughton, *Phys. Rev.* A8 401, 1972; J. Polchinski, *Nucl. Phys.* B231 269, 1984; A. Hasenfratz, P. Hazenfratz, *Nucl. Phys.* B270 685, 1986; A. Margaritis, G. Odor, A. Patkos, *Z. Phys.* C39 109, 1988; C. Wetterich, *Nucl. Phys.* B352 529, 1991; S. B. Liao, J. Polonyi, *Ann. Phys.* 222 122, 1993; T. Morris, *Phys. Lett.* B334 335, 1994.

[29] V. Branchina, J. Fingberg, H. Mohrbach, J. Polonyi, in preparation.
[30] S. Coleman, E. Weinberg, *Phys. Rev.* **D7** 1888, 1973.

[31] S. B. Liao, J. Polonyi, *Phys. Rev.* **D51** 4474, 1995.