Error bounds in normal approximation for the squared-length of total spin in the mean field classical $N$-vector models

Lê Văn Thành †  Nguyen Ngoc Tu ‡§

Abstract

This paper gives the Kolmogorov and Wasserstein bounds in normal approximation for the squared-length of total spin in the mean field classical $N$-vector models. The Kolmogorov bound is new while the Wasserstein bound improves a result obtained recently by Kirkpatrick and Nawaz [Journal of Statistical Physics, 165 (2016), no. 6, 1114–1140]. The proof is based on Stein’s method for exchangeable pairs.

Keywords: Stein’s method; Kolmogorov distance; Wasserstein distance; mean-field model.

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1 Introduction and main result

Let $N \geq 2$ be an integer, and let $S^{N-1}$ denote the unit sphere in $\mathbb{R}^N$. In this paper, we consider the mean-field classical $N$-vector spin models, where each spin $\sigma_i$ is in $S^{N-1}$, at a complete graph vertex $i$ among $n$ vertices ([5, Chapter 9]). The state space is $\Omega_n = (S^{N-1})^n$ with product measure $P_n = \mu \times \cdots \times \mu$, where $\mu$ is the uniform probability measure on $S^{N-1}$. In the absence of an external field, each spin configuration $\sigma = (\sigma_1, \ldots, \sigma_n)$ in the state space $\Omega_n$ has a Hamiltonian defined by

$$H_n(\sigma) = -\frac{1}{2n} \sum_{i=1}^n \sum_{j=1}^n \langle \sigma_i, \sigma_j \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^N$. Let $\beta > 0$ be the inverse temperature. The Gibbs measure with Hamiltonian $H_n$ is the probability measure $P_{n,\beta}$ on $\Omega_n$ with density function:

$$dP_{n,\beta}(\sigma) = \frac{1}{Z_{n,\beta}} \exp(-\beta H_n(\sigma)) dP_n(\sigma),$$

where $Z_{n,\beta}$ is the normalizing constant.

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†Department of Mathematics, Vinh University, Nghe An 42118, Vietnam. E-mail: levt@vinhuni.edu.vn
‡Department of Applied Sciences, HCMC University of Technology and Education, Ho Chi Minh City, Vietnam.
§Department of Mathematics and Computer Science, University of Science, Viet Nam National University Ho Chi Minh City, Ho Chi Minh City, Vietnam.
E-mail: tunn@hcmute.edu.vn
where $Z_{n, \beta}$ is the partition function: $Z_{n, \beta} = \int_{\Omega_n} \exp(-\beta H_n(\sigma)) \, dP_n(\sigma)$. This model is also called the mean field $O(N)$ model. It reduces to the $XY$ model, the Heisenberg model and the Toy model when $N = 2, 3, 4$, respectively (see, e.g., [5, p. 412]).

Before proceeding, we introduce the following notations. Throughout this paper, $Z$ is a standard normal random variable, and $\Phi(z)$ is the probability distribution function of $Z$. For a real-valued function $f$, we write $\|f\| = \sup_x |f(x)|$. The symbol $C$ denotes a positive constant which depends only on the inverse temperature $\beta$, and its value may be different for each appearance. For two random variables $X$ and $Y$, the Wasserstein distance $d_W$ and the Kolmogorov distance $d_K$ between $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are as follows:

$$d_W(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{\|b\| \leq 1} |E h(X) - E h(Y)|,$$

and

$$d_K(\mathcal{L}(X), \mathcal{L}(Y)) = \sup_{z \in \mathbb{R}} |P(X \leq z) - P(Y \leq z)|.$$

In the Heisenberg model ($N = 3$), Kirkpatrick and Meckes [6] established large deviation, normal approximation results for total spin $S_n = \sum_{i=1}^n \sigma_i$ in the non-critical phase ($\beta \neq 3$), and a non-normal approximation result in the critical phase ($\beta = 3$). The results in [6] are generalized by Kirkpatrick and Nawaz [7] to the mean field $N$-vector models with $N \geq 2$.

Let $I_x$ denote the modified Bessel function of the first kind (see, e.g., [2, p. 713]) and

$$f(x) = \frac{I_{n/2}(x)}{I_{n-1/2}(x)}, \quad x > 0. \quad (1.1)$$

By Lemma A.2 in the Appendix, we have

$$\left(\frac{f(x)}{x}\right)' < 0 \text{ for all } x > 0. \quad (1.2)$$

We also have

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \frac{1}{N} \text{ and } \lim_{x \to \infty} \frac{f(x)}{x} = 0. \quad (1.3)$$

In the case $\beta > N$, from (1.2) and (1.3), there is a unique strictly positive solution $b$ to the equation

$$x - \beta f(x) = 0. \quad (1.4)$$

Based on their large deviations, Kirkpatrick and Nawaz [7] argued that in the case $\beta > N$, there exists $\varepsilon > 0$ such that

$$P \left( \frac{\beta|S_n|}{n} - b \geq x \right) \leq e^{-C_n x^2}$$

for all $0 \leq x \leq \varepsilon$, where $S_n = \sum_{i=1}^n \sigma_i$ is total spin. It means that $|S_n|$ is close to $bn/\beta$ with high probability. On the other hand, all points on the hypersphere of radius $bn/\beta$ will have equal probability due to symmetry. Based on these facts, they considered the fluctuations of the squared-length of total spin:

$$W_n := \sqrt{n} \left( \frac{\beta^2}{n^2 b^2} |S_n|^2 - 1 \right), \quad (1.5)$$

where $S_n = \sum_{j=1}^n \sigma_j$. Let

$$B^2 = \frac{4 \beta^2}{(1 - \beta f(b)) b^2} \left[ 1 - \frac{(N - 1)}{b^2} \right]. \quad (1.6)$$
Kirkpatrick and Nawaz [7] proved that when $\beta > N$, the bounded-Lipschitz distance between $W_n/B$ and $Z$ is bounded by $C(\log n/n)^{1/4}$. Their proof is based on Stein’s method for exchangeable pairs (see Stein [10]). Recall that a random vector $(W, W')$ is called an exchangeable pair if $(W, W')$ and $(W', W)$ have the same distribution. Kirkpatrick and Nawaz [7] construct an exchangeable pair as follows. Let $W_n$ be as in (1.5) and let $\sigma' = \{\sigma'_1, \ldots, \sigma'_n\}$, where for each $i$ fixed, $\sigma'_i$ is an independent copy of $\sigma$ given $\{\sigma_j, j \neq i\}$, i.e., given $\{\sigma_j, j \neq i\}$, $\sigma'_i$ and $\sigma_i$ have the same distribution and $\sigma'_i$ is conditionally independent of $\sigma_i$ (see, e.g., [4, p. 964]). Let $I$ be a random index independent of all others and uniformly distributed over $\{1, \ldots, n\}$, and let
\[
W'_n = \sqrt{n} \left( \frac{\beta^2}{n^2b^2} |S'_n|^2 - 1 \right),
\]
where $S'_n = \sum_{j=1}^n \sigma_j - \sigma_I + \sigma'_I$. Then $(W_n, W'_n)$ is an exchangeable pair (see Kirkpatrick and Nawaz [7, p. 1124], Kirkpatrick and Meckes [6, p. 66]).

The bound $C(\log n/n)^{1/4}$ obtained by Kirkpatrick and Nawaz [7] is not sharp. The aim of this paper is to give the Kolmogorov and Wasserstein distances between $W_n/B$ and $Z$ with optimal rate $Cn^{-1/2}$.

The main result is the following theorem. We recall that, throughout this paper, $C$ is a positive constant which depends only on $\beta$, and its value may be different for each appearance.

**Theorem 1.1.** Let $\beta > N$ and $f$ be as in (1.1). Let $b$ be the unique strictly positive solution to the equation $x - \beta f(x) = 0$ and $B^2$ as in (1.6). For $W_n$ as defined in (1.5), we have
\[
\sup_{\|b'\| \leq 1} |Eh(W_n/B) - Eh(Z)| \leq Cn^{-1/2},
\]
and
\[
\sup_{z \in \mathbb{R}} |P(W_n/B \leq z) - \Phi(z)| \leq Cn^{-1/2}.
\]

The Wasserstein bound in Theorem 1.1 will be a consequence of the following proposition, a version of Stein’s method for exchangeable pairs. It is a special case of Theorem 2.4 of Eichelsbacher and Löwe [4] or Theorem 13.1 in [3].

**Proposition 1.2.** Let $(W, W')$ be an exchangeable pair and $\Delta = W - W'$. If $E(\Delta|W) = \lambda(W + R)$ for some random variable $R$ and $0 < \lambda < 1$, then
\[
\sup_{\|b'\| \leq 1} |Eh(W) - Eh(Z)| \leq \sqrt{2/\pi}E \left[ 1 - \frac{1}{2\lambda} E(\Delta^2|W) \right] + \frac{1}{2\lambda} E|\Delta|^3 + 2E|R|.
\]

The Kolmogorov distance is more commonly used in probability and statistics, and is usually more difficult to handle than the Wasserstein distance. Recently, Shao and Zhang [9] proved a very general theorem. Their result is as follows.

**Proposition 1.3.** Let $(W, W')$ be an exchangeable pair and $\Delta = W - W'$. Let $\Delta^* := \Delta^*(W, W')$ be any random variable satisfying $\Delta^*(W, W') = \Delta^*(W', W)$ and $\Delta^* \geq |\Delta|$. If $E(\Delta|W) = \lambda(W + R)$ for some random variable $R$ and $0 < \lambda < 1$, then
\[
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq E \left[ 1 - \frac{1}{2\lambda} E(\Delta^2|W) \right] + \frac{1}{\lambda} E|E(\Delta^*|W)| + E|R|.
\]

Shao and Zhang [9] applied their bound in Proposition 1.3 to get optimal bound in many problems, including a bound of $O(n^{-1/2})$ for the Kolmogorov distance in normal approximation of total spin in the Heisenberg model. We note that if $|\Delta| \leq a$, then the following result is an immediate corollary of Proposition 1.3. In this case, the bound is much simpler than that of Proposition 1.3.
Corollary 1.4. If $|\Delta| \leq a$, then

$$
sup_{z \in \mathbb{R}}|P(W \leq z) - \Phi(z)| \leq E \left| 1 - \frac{1}{2\lambda} E(\Delta^2 |W|) + (E|W| + 1)a + E|R| \right|. \tag{1.10}$$

Proof. In Proposition 1.3, let $\Delta^* = a$, then

$$
E \left| E(\Delta \Delta^* |W|) \right| = aE \left| E(\Delta |W|) \right| \leq a\lambda (E|W| + E|R|). \tag{1.11}
$$

If $E|R| \geq 1$, then (1.10) is trivial. If $E|R| < 1$, then (1.10) follows immediately from (1.11) and Proposition 1.3.

For $S_n = \sum_{i=1}^{n} \sigma_i$, and for $W_n$ and $W_n'$ respectively defined in (1.5) and (1.7), we have

$$
|\Delta| = |W_n - W_n'| = \frac{\beta^2}{n^{3/2}b^2} \left| |S_n|^2 - |S_n'|^2 \right| \leq \frac{4\beta^2}{n^{1/2}b^2},
$$

since $|S_n| + |S_n'| \leq 2n$ and $|S_n| - |S_n'| \leq |\sigma_I - \sigma_I'| \leq 2$. Therefore, we will apply Corollary 1.4 to obtain the Kolmogorov bound in Theorem 1.1.

2 Proof of the main result

The proof of Theorem 1.1 depends on Kirkpatrick and Nawaz’s finding [7]. Applying Proposition 1.2 and Corollary 1.4, Theorem 1.1 follows from the following proposition.

Proposition 2.1. Let $\beta > N$, and let $f$ be as in (1.1), $b$ the unique strictly positive solution to the equation $x - \beta f(x) = 0$. Let $W_n$ and $W_n'$ be as in (1.5) and (1.7), respectively. Then the following statements hold:

(i) $|W_n - W_n'| \leq 4\beta^2 b^{-2} n^{-1/2}$ and $EW_n^2 \leq C$,

(ii) $E(W_n - W_n'|W_n) = \lambda(W_n + R)$, where $\lambda = \frac{1 - \beta f'(b)}{n}$ and $R$ is a random variable satisfying $E|R| \leq Cn^{-1/2}$,

(iii) $E\left| \frac{1}{2\lambda} E((W_n - W_n')^2|W_n)) - B^2 \right| \leq Cn^{-1/2}$, where $B^2$ is defined in (1.6).

Remark 2.2. Kirkpatrick and Nawaz’s [7] used their large deviation result for total spin $S_n$ to prove that $EW_n^2 \leq C \log n$. Intuitively, we see that this bound would be improved to $EW_n^2 \leq C$ since $W_n$ approximates a normal distribution. By a more careful estimate, we can prove that $E (|S_n|/n - b)^2 \leq C/n$ (see Lemma A.1). This will lead to desired bound $EW_n^2 \leq C$. Kirkpatrick and Nawaz’s [7] also proved that

$$
E\left| \frac{1}{2\lambda} E((W_n - W_n')^2|W_n)) - B^2 \right| \leq C \left( \frac{\log n}{n} \right)^{1/4}.
$$

To get optimal bound of order $n^{-1/2}$ for this term, we use a fine estimate of function $f(x) = I_{\mathbb{Z}}(x)/I_{\mathbb{Z}}(x)$ (Lemma A.2) and a technique developed recently by Shao and Zhang [9, Proof of (5.51)].

Proof of Proposition 2.1. (i) We have

$$
|W_n - W_n'| = \frac{\beta^2}{b^2 n^{1/2}} \left| |S_n|^2 - |S_n'|^2 \right| = \frac{\beta^2}{b^2 n^{1/2}} \left| \langle S_n + S_n', S_n - S_n' \rangle \right|
\leq \frac{2\beta^2 |S_n - S_n'|}{b^2 n^{1/2}} = \frac{2\beta^2 |\sigma_I - \sigma_I'|}{b^2 n^{1/2}} \leq \frac{4\beta^2}{b^2 n^{1/2}}.
$$
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The proof of the first half of (i) is completed. Now, apply Lemma A.1 given in the Appendix, we have

$$E W_n^2 = n E \left( \frac{\beta |S_n|}{n b} + 1 \right)^2 \leq C n E \left( \frac{\beta |S_n|}{n b} - 1 \right)^2 \leq C.$$  

(ii) Kirkpatrick and Nawaz [7, equation (9)] showed that

$$E(W_n - W'_n | W_n) = \frac{2}{\sqrt{n}} W_n + \frac{2}{\sqrt{n}} - \frac{2 \beta}{n^{1/2} b^2} \left( \frac{\beta |S_n|}{n} \right) f \left( \frac{\beta |S_n|}{n} \right) + R_1,$$  

(2.1)

where $R_1$ is a random variable satisfying $E|R_1| \leq C n^{-3/2}$. Set $g(x) = x f(x), x > 0$. By Taylor’s expansion, we have for some positive random variable $\xi$:

$$g \left( \frac{\beta |S_n|}{n} \right) = g(b) + g'(b) \left( \frac{\beta |S_n|}{n} - b \right) + g''(\xi) \frac{\beta |S_n|}{n} - b \right)^2.$$  

(2.2)

Set $V = \frac{\beta |S_n|}{n b} + 1$, we have $1 \leq V \leq C$ and

$$V \left( \frac{\beta |S_n|}{n b} - 1 \right) = \frac{b W_n}{\sqrt{n}} + \frac{b W_n}{\sqrt{n}} \left( \frac{1}{2} - \frac{1}{V} \right)$$

$$= \frac{b W_n}{\sqrt{n}} - \frac{b W_n}{2 \sqrt{n V}} \left( \frac{\beta |S_n|}{n} - 1 \right) = \frac{b W_n}{2 \sqrt{n V}} - \frac{b W_n}{2 \sqrt{n V}} \left( \frac{\beta |S_n|}{n} - 1 \right) = \frac{2}{\sqrt{n}} W_n - \frac{2}{\sqrt{n V}} \left( \frac{\beta |S_n|}{n} - 1 \right).$$  

(2.3)

Combining (2.1)-(2.3) and noting that $b = \beta f(b)$, we have

$$E(W_n - W'_n | W_n)$$

$$= \frac{2 W_n}{n} + \frac{2}{\sqrt{n}} + R_1 - \frac{2 \beta}{n^{1/2} b^2} \left( g(b) + g'(b) \left( \frac{b W_n}{2 \sqrt{n V}} - \frac{b W_n}{2 n V^2} + \frac{g''(\xi) b^2 W_n^2}{2 n V^2} \right) \right)$$

$$= \frac{2 W_n}{n} + \frac{2}{\sqrt{n}} + R_1 - \frac{2 \beta}{n^{1/2} b^2} \left( \frac{b^2}{\beta} + \left( \frac{b}{\beta} + b f'(b) \right) \left( \frac{b W_n}{2 \sqrt{n V}} - \frac{b W_n}{2 n V^2} + \frac{g''(\xi) b^2 W_n^2}{2 n V^2} \right) \right)$$

$$= \frac{2 W_n}{n} + \frac{2}{\sqrt{n}} + R_1 - \frac{1 - \beta f'(b)}{n} (W_n + R),$$

where

$$R = \frac{\beta W_n}{n^{3/2} V^2} \left( \frac{1}{\beta} + \frac{f'(b)}{n} - g''(\xi) \right).$$

By Lemma A.2 (ii), we have $|g''(\xi)| < 6$. Since $V \geq 1$, $E W_n^2 \leq C$ and $E|R_1| \leq C n^{-3/2}$, we conclude that $E|R| \leq C n^{-1/2}$. The proof of (ii) is completed.

(iii) Denote $I_d$ is the $n \times n$ identity matrix and set $\sigma^{(i)} = S_n - \sigma_i, b_i = \beta |\sigma^{(i)}| / n, r_i = \sigma^{(i)} / |\sigma^{(i)}|$. From Kirkpatrick and Nawaz [7, Equations (11) and (12)], we have

$$E((W_n - W'_n)^2 | \sigma) = 2 \lambda B^2 + \frac{4 \beta^4}{n^4 b^4} \sum_{i=1}^{n} \left( 1 - \frac{N - 1}{\beta} \right)^2 \left( |\sigma^{(i)}| - \frac{(n - 1)^2 b^2}{\beta^2} \right)$$

$$- \frac{8 \beta^3}{n^3 b^3} \sum_{i=1}^{n} \left( |\sigma^{(i)}| \sigma_{i, \sigma^{(i)}} - \frac{n^2 b^3}{\beta^4} \right)$$

$$+ \frac{4 \beta^4}{n^4 b^4} \sum_{i=1}^{n} \left( \sigma_{i, \sigma^{(i)}}^2 - \left( 1 - \frac{N - 1}{\beta} \right) \frac{(n - 1)^2 b^2}{\beta^2} \right)$$

$$+ \frac{4 \beta^4}{n^4 b^4} \sum_{i=1}^{n} \sum_{j,k \neq i} \sigma_j^T R_i \sigma_k,$$
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where

$$R'_i = \left( \frac{f(b_i)}{b_i} - \frac{1}{\beta} \right) I_d - \left( \frac{N f(b_i)}{b_i} - \frac{N}{\beta} \right) P_i - \left( f(b_i) - \frac{b}{\beta} \right) \left( r_i \sigma_i^T + \sigma_i r_i^T \right),$$

and $P_i$ is orthogonal projection onto $r_i$. Therefore,

$$\frac{1}{2\lambda} E((W_n - W'_n)^2 | \sigma)) - B^2 = \frac{2\beta^4}{n^3 b^4 (1 - \beta f'(b))} \left( R_2 - \frac{2b}{\beta} R_3 + R_4 + R_5 \right), \quad (2.4)$$

where

$$R_2 = \sum_{i=1}^{n} \left( 1 - \frac{N - 1}{\beta} \right) \left( |\sigma^{(i)}|^2 - \frac{(n - 1)^2 b^2}{\beta^2} \right),$$

$$R_3 = \sum_{i=1}^{n} \left( |\sigma^{(i)}| |\sigma_i, \sigma^{(i)}| - \frac{n^2 b^3}{\beta^3} \right),$$

$$R_4 = \sum_{i=1}^{n} \left( |\sigma_i, \sigma^{(i)}|^2 - \left( 1 - \frac{N - 1}{\beta} \right) \frac{(n - 1)^2 b^2}{\beta^2} \right),$$

$$R_5 = \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sigma_j^T R'_i \sigma_k.$$

For $R_2$, noting that $|\sigma^{(i)}| - S_n \leq 1$, then by Lemma A.1, we have

$$\left( E \left| \frac{\beta |\sigma^{(i)}|}{n} - b \right| \right)^2 \leq E \left| \frac{\beta |\sigma^{(i)}|}{n} - b \right|^2 \leq E \left| \frac{\beta S_n}{n} - b \right|^2 + \frac{C}{n^2} \leq \frac{C}{n}. \quad (2.5)$$

Thus,

$$E|R_2| \leq C \sum_{i=1}^{n} E \left| |\sigma^{(i)}|^2 - \frac{(n - 1)^2 b^2}{\beta^2} \right|$$

$$\leq C n^2 \sum_{i=1}^{n} \left( E \left| \frac{\beta |\sigma^{(i)}|^2}{n^2} - b^2 \right| + \frac{(2n - 1) b^2}{n^2} \right) \quad (2.6)$$

$$\leq C n^2 \left( \sum_{i=1}^{n} E \left| \frac{\beta |\sigma^{(i)}|}{n} - b \right| + C \right) \leq C n^{5/2}.$$

For $R_3$, we have

$$E|R_3| = E \sum_{i=1}^{n} \left| S_n |\sigma_i, S_n \rangle - \frac{n^2 b^3}{\beta^2} + |\sigma^{(i)}| |\sigma_i, \sigma^{(i)}\rangle - | S_n |\sigma_i, S_n \rangle \right|$$

$$\leq E \left| S_n^3 - \frac{n^3 b^3}{\beta^2} \right| + E \left| \sum_{i=1}^{n} |\sigma^{(i)}| |\sigma_i, \sigma^{(i)}\rangle - | S_n |\sigma_i, S_n \rangle \right|$$

$$\leq C n^2 E \left| S_n \right| - \frac{nb}{\beta} + E \left| \sum_{i=1}^{n} \left( |\sigma^{(i)}| - | S_n \rangle \right) \langle \sigma_i, \sigma^{(i)}\rangle - | S_n |\sigma_i, \sigma_n \rangle \right| \quad (2.7)$$

$$\leq C n^3 E \left| \frac{\beta |S_n|}{n} - b \right| + E \left| \sum_{i=1}^{n} \left( |\sigma_i, \sigma^{(i)}\rangle | + | S_n \rangle \right) \right|$$

$$\leq C n^3 E \left| \frac{\beta |S_n|}{n} - b \right| + C n^2 \leq C n^{5/2}.$$
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To bound $E|R_5|$, we note that

\[
\begin{align*}
\sum_{i=1}^n \sum_{j, k \neq i} \sigma_j^T R_i \sigma_k &= \sum_{i=1}^n \sum_{j, k \neq i} \left( \frac{f(b_i)}{b_i} - \frac{1}{\beta} \right) \langle \sigma_j, \sigma_k \rangle - \left( f(b_i) - \frac{b}{\beta} \right) \sigma_j^T \left( r_i \sigma_i^T + \sigma_i r_i^T \right) \sigma_k \\
&= \sum_{i=1}^n \left[ \left( \frac{f(b_i)}{b_i} - \frac{1}{\beta} \right) |\sigma^{(i)}|^2 - 2 \left( f(b_i) - \frac{b}{\beta} \right) |\sigma^{(i)}| \langle \sigma^{(i)}, \sigma_i \rangle \right] \\
&\quad - \sum_{i=1}^n \left( \frac{N f(b_i)}{b_i} - \frac{N}{\beta} \right) \sum_{j, k \neq i} \text{Trace}(\sigma_k \sigma_j^T r_i r_i^T) \\
&= \sum_{i=1}^n \left[ \left( \frac{f(b_i)}{b_i} - \frac{1}{\beta} \right) |\sigma^{(i)}|^2 - 2 \left( f(b_i) - \frac{b}{\beta} \right) |\sigma^{(i)}| \langle \sigma^{(i)}, \sigma_i \rangle \right] \\
&\quad - \sum_{i=1}^n \left( \frac{N f(b_i)}{b_i} - \frac{N}{\beta} \right) \langle \sigma^{(i)}, r_i \rangle^2 \\
&= \sum_{i=1}^n (1 - N) \left( \frac{f(b_i)}{b_i} - \frac{1}{\beta} \right) |\sigma^{(i)}|^2 - 2 \sum_{i=1}^n \left( f(b_i) - \frac{b}{\beta} \right) |\sigma^{(i)}| \langle \sigma^{(i)}, \sigma_i \rangle \\
&\quad := R_{51} - 2R_{52}.
\end{align*}
\]

Since $1/\beta = f(b)/b$ and $b_i = \beta |\sigma^{(i)}|/n$, we have

\[
E|R_{51}| = E \left| \sum_{i=1}^n (1 - N) \left( \frac{f(b_i)}{b_i} - \frac{f(b_i)}{b} \right) |\sigma^{(i)}|^2 \right| \\
\leq C n^2 \sum_{i=1}^n E|b_i - b| \quad \text{(by Lemma A.2 (iii) and the fact that $|\sigma^{(i)}| \leq n$)} \tag{2.8}
\]

\[
\leq C n^2 \sum_{i=1}^n \left( |\beta| S_n - \frac{b}{n} \left( |\sigma^{(i)}| - |S_n| \right) \right) \\
\leq C n^{5/2} \quad \text{(by (2.5) and the fact that $||\sigma^{(i)}| - |S_n|| \leq 1$)}.
\]

Similarly,

\[
E|R_{52}| = E \left| \sum_{i=1}^n (1 - N) \left( f(b_i) - f(b) \right) |\sigma^{(i)}| \langle \sigma^{(i)}, \sigma_i \rangle \right| \\
\leq C n^2 \sum_{i=1}^n E|b_i - b| \quad \text{(by Lemma A.2 (i) and the fact that $|\sigma^{(i)}| \leq n$)} \tag{2.9}
\]

\[
\leq C n^{5/2}.
\]

Combining (2.8) and (2.9), we have

\[
E|R_5| \leq C n^{5/2}. \tag{2.10}
\]

Bounding $E|R_4|$ is the most difficult part. Here we follow a technique developed by Shao and Zhang [9, Proof of (5.51)]. Set

\[
a = \left( 1 - \frac{N - 1}{\beta} \right) \frac{(n - 1)^2 \beta^2}{\beta^2}, \quad a^{(1,2)} = S_n - \sigma_1 - \sigma_2, \quad V_1 = \langle \sigma_1, a^{(1,2)} \rangle^2, \quad V_2 = \langle \sigma_2, a^{(1,2)} \rangle^2,
\]

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we have
\[ |\langle \sigma_1, \sigma^{(1)} \rangle^2 - V_1| \leq Cn, \quad |\langle \sigma_1, \sigma^{(2)} \rangle^2 - V_2| \leq Cn. \]

It follows that
\[
ER_4^2 = nE \left( |\langle \sigma_1, \sigma^{(1)} \rangle^2 - a| \right)^2 - n(n-1)E \left( |\langle \sigma_1, \sigma^{(1)} \rangle^2 - a| \right) \left( |\langle \sigma_2, \sigma^{(2)} \rangle^2 - V_2 + V_2 - a| \right) \\
\leq Cn^5 + n(n-1) \left| E \left( |\langle \sigma_1, \sigma^{(1)} \rangle^2 - V_1 + V_1 - a| \right) \left( |\langle \sigma_2, \sigma^{(2)} \rangle^2 - V_2 + V_2 - a| \right) \right| \\
\leq Cn^5 + n(n-1) \left| E(V_1 - a)(V_2 - a) \right| \\
\leq Cn^5 + n(n-1) \left| E(V_1 - E(V_1|(\sigma_j)_{j>2})) \left( V_2 - E(V_2|(\sigma_j)_{j>2}) \right) \right| \\
+ n(n-1) \left| E \left( E(V_1|(\sigma_j)_{j>2}) - a \right) \left( E(V_2|(\sigma_j)_{j>2}) - a \right) \right| \\
:= Cn^5 + n(n-1)(|R_{41}| + |R_{42}|).
\]

Define a probability density function
\[
p_{12}(x, y) = \frac{1}{Z_{12}} \exp \left( \frac{\beta}{n} \langle x + y, \sigma^{(1,2)} \rangle \right), \quad x, y \in S^{N-1},
\]
where $Z_{12}^2$ is the normalizing constant. Let $(\xi_1, \xi_2) \sim p_{12}(x, y)$ given $(\sigma_j)_{j>2}$, and for $i = 1, 2$
\[
\tilde{V}_i = E \left( |\langle \xi_i, \sigma^{(1,2)} \rangle^2 |(\sigma_j)_{j>2} \right).
\]

Similar to Shao and Zhang [9, pages 97, 98], we can show that
\[
R_{41} = E \left( |\langle \xi_1, \sigma^{(1,2)} \rangle^2 - \tilde{V}_1 \right) \left( |\langle \xi_i, \sigma^{(1,2)} \rangle^2 - \tilde{V}_2 \right) + H_1,
\]
and
\[
R_{42} = E \left( \tilde{V}_1 - a \right) \left( \tilde{V}_2 - a \right) + H_2,
\]
where $|H_1| \leq Cn^3$ and $|H_2| \leq Cn^3$. Let
\[
b_{12} = \frac{\beta|\langle \sigma^{(1,2)} \rangle|^2}{n}.
\]

By Lemma A.3 and the definition of $a$, we have
\[
\left| \tilde{V}_1 - a \right| = \left| \left( 1 - \frac{(N-1)f(b_{12})}{b_{12}} \right) \left( \sigma^{(1,2)} \right)^2 - \left( 1 - \frac{N-1}{\beta} \right) \left( \frac{(n-1)^2b^2}{\beta^2} \right) \right| \\
\leq Cn^2 \left( \frac{\beta^2|S_n|^2}{n^2} - b \right) + Cn.
\]

Using similar estimate for $|\tilde{V}_2 - a|$, then we have
\[
E \left| \left( \tilde{V}_1 - a \right) \left( \tilde{V}_2 - a \right) \right| \leq C \left( n^4E \left| \frac{\beta|S_n|}{n} - b \right|^2 + n^3E \left| \frac{\beta|S_n|}{n} - b \right| + n^2 \right) (2.17)
\]
\[
\leq Cn^3 \quad \text{(by Lemma A.1)}.
\]
Note that given \((\sigma_j)_{j>2}, \xi_1\) and \(\xi_2\) are conditionally independent. It implies that
\[
E \left( (\xi_i, \sigma^{(1,2)})^2 - \tilde{V}_1 \right) \left( (\xi_i, \sigma^{(1,2)})^2 - \tilde{V}_2 \right) = 0. \tag{2.18}
\]
Combining (2.11)-(2.18), we have \(ER_4^2 \leq Cn^5\), and so
\[
E|R_4| \leq Cn^{5/2}. \tag{2.19}
\]
Combining (2.4), (2.6), (2.7), (2.10) and (2.19), we have
\[
E \left| \frac{1}{2\lambda} E((W_n - W_n')^2|W_n) - B \right| \leq Cn^{-1/2}.
\]
The proposition is proved.

A Appendix

In this Section, we will prove the technical results that used in the proof of Theorem 1.1.

Lemma A.1. We have
\[
E \left| \frac{\beta}{n} - b \right|^2 \leq C.
\]
Proof. By the large deviation for \(S_n/n\) [7, Proposition 2] and the argument in [7, p. 1126], one can prove that there exists \(\varepsilon > 0\) such that
\[
P \left( \left| \frac{\beta}{n} - b \right| \geq x \right) \leq e^{-Cn x^2}
\]
for all \(0 \leq x \leq \varepsilon\). Since \(\left| \frac{\beta}{n} - b \right| \leq C\), it implies that
\[
E \left| \frac{\beta}{n} - b \right|^2 \leq 2 \int_0^\varepsilon xP \left( \left| \frac{\beta}{n} - b \right| > x \right) dx
\]
\[
+ E \left( \left| \frac{\beta}{n} - b \right| I \left( \left| \frac{\beta}{n} - b \right| > \varepsilon \right) \right)
\]
\[
\leq 2 \int_0^\varepsilon xe^{-Cn x^2} dx + CP \left( \left| \frac{\beta}{n} - b \right| > \varepsilon \right)
\]
\[
\leq \frac{C}{n} + C e^{-Cn \varepsilon^2} \leq \frac{C}{n}. \tag*{\Box}
\]

Lemma A.2. Let \(x > 0\) and \(f(x) = \frac{I_{N/2}(x)}{I_{N/2-1}(x)}\). Then the following statements hold:

(i) \(0 < f'(x) < \frac{1}{N-1} \leq 1\).
(ii) \(|(xf(x))''| < 6\).
(iii) \(-5 \leq -\frac{5}{N-1} < \left( \frac{f(x)}{x} \right)' < 0\).

Proof. As was showed in [7, p. 1134], we have
\[
f'(x) = 1 - \frac{N-1}{x} f(x) - f^2(x). \tag{A.1}
\]
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It implies

$$\frac{f(x)}{x} = 1 - \frac{f'(x) - f^2(x)}{N - 1},$$

(A.2)

and

$$f^2(x) = 1 - \frac{N - 1}{x} f(x) - f'(x).$$

(A.3)

Amos [1, p. 243] proved that

$$0 < f'(x) < \frac{f(x)}{x}.$$  

(A.4)

Combining (A.2)-(A.4), we have

$$0 < f'(x) < \frac{f(x)}{x} < \frac{1}{N - 1}, \text{ and } f^2(x) < 1.$$  

(A.5)

Therefore,

$$|(xf(x))''| = |2f'(x) + xf''(x)|
\leq 2 + (N - 1)f'(x) + 2xf(x)f'(x) + \frac{(N - 1)f(x)}{x}
\leq 4 + 2f^2(x) \text{ (by the first half of (A.5))}
\leq 6 \text{ (by the second half of (A.5)).}$$

The proof of (i) and (ii) is completed. For (iii), we have

$$(\frac{f(x)}{x})' = \frac{1}{x} \left( f'(x) - \frac{f(x)}{x} \right).$$

(A.6)

Combining the first half of (A.5) and (A.6), we have $\left( \frac{f(x)}{x} \right)' < 0$. It follows from (A.1), (A.5) and (A.6) that

$$\left( \frac{f(x)}{x} \right)' = \frac{1}{x} \left( 1 - \frac{Nf(x)}{x} \right) - \frac{f^2(x)}{x} > \frac{1}{x} \left( 1 - \frac{Nf(b_{12})}{x} \right) - \frac{1}{N - 1}.$$

(A.7)

Apply Theorem 2 (a) of Näsell [8], we can show that

$$\frac{1}{x} \left( 1 - \frac{Nf(x)}{x} \right) > \frac{-4}{N - 1}.$$  

(A.8)

Combining (A.7) and (A.8), we have $\left( \frac{f(x)}{x} \right)' > \frac{-5}{N - 1}$. The proof of (iii) is completed. 

Lemma A.3. With the notation in the proof of Theorem 1.1, we have

$$\hat{V}_i = |\sigma^{(1,2)}|^2 \left( 1 - \frac{(N - 1)f(b_{12})}{b_{12}} \right), \ i = 1, 2.$$
Proof. Let $A_N = 2\pi N^2/\Gamma(N/2)$ the Lebesgue measure of $S^{N-1}$. It follows from (2.12) that

$$Z_{12}^2 = \int_{S^{N-1}} \int_{S^{N-1}} \exp \left( \frac{\beta}{n} (x + y, \sigma^{(1,2)}) \right) d\mu(x) d\mu(y)$$

$$= \left( \int_{S^{N-1}} \exp \left( \frac{\beta}{n} (x, \sigma^{(1,2)}) \right) d\mu(x) \right)^2$$

$$= \left( \frac{A_{N-1}}{A_N} \int_0^\pi e^{b_{12} \cos \varphi_{N-2}} \sin^{N-2} \varphi_{N-2} d\varphi_{N-2} \right)^2$$

$$= \left( \frac{A_{N-1}}{A_N} \sqrt{\pi \Gamma(N/2 - 1/2)} I_{N/2 - 1}(b_{12}) \right)^2,$$

where we have used formula

$$I_\nu(z) = \frac{1}{\sqrt{\pi} (\nu + 1/2)} \left( \frac{\nu}{2} \right) ^\nu \int_0^\pi \exp(z \cos \theta) \sin^{2\nu} \theta d\theta$$

(see, e.g., Exercise 11.5.4 in [2]) in the last equation. For $i = 1, 2$, we have

$$V_i = \frac{1}{Z_{12}} \int_{S^{N-1}} (\theta, \sigma^{(1,2)})^2 \exp \left[ \frac{\beta}{n} (\theta, \sigma^{(1,2)}) \right] d\mu(\theta)$$

$$= \frac{1}{Z_{12}} \int_{S^{N-1}} |\sigma^{(1,2)}|^2 \left( \int_{S^{N-1}} \exp \left( \frac{\beta |\sigma^{(1,2)}|}{n} \left( \theta, \sigma^{(1,2)} \right) \right) d\mu(\theta) \right)$$

$$= |\sigma^{(1,2)}|^2 \frac{A_{N-1}}{A_N} \int_0^\pi \cos^2 \varphi_{N-2} \sin^{N-2} \varphi_{N-2} e^{b_{12} \cos \varphi_{N-2}} d\varphi_{N-2}$$

$$= |\sigma^{(1,2)}|^2 \frac{A_{N-1}}{A_N} \int_0^\pi e^{b_{12} \cos \varphi_{N-2}} \sin^{N-2} \varphi_{N-2} d\varphi_{N-2}$$

$$- \int_0^\pi e^{b_{12} \cos \varphi_{N-2}} \sin^N \varphi_{N-2} d\varphi_{N-2}$$

$$= \left( 1 - \frac{A_{N-1}}{A_N} \int_0^\pi e^{b_{12} \cos \varphi_{N-2}} \sin^N \varphi_{N-2} d\varphi_{N-2} \right) |\sigma^{(1,2)}|^2$$

$$= \left( 1 - \frac{A_{N-1}}{A_N Z_{12}} \int_0^\pi \frac{e^{b_{12} \cos \varphi_{N-2}} \sin^N \varphi_{N-2}}{(b_{12}/2)^{N/2}} I_{N/2}(b_{12}) \right) |\sigma^{(1,2)}|^2$$

$$= \left( 1 - \frac{(N-1)f(b_{12})}{b_{12}} \right) |\sigma^{(1,2)}|^2.$$

Finally, we would like to note again that Proposition 1.2 is a special case of Theorem 2.4 of Eichelsbacher and Löwe [4] or Theorem 13.1 in [3], but the constants in the bound may be different from those of Theorem 2.4 in [4] or Theorem 13.1 in [3]. Since the proof is short and simple, we will present here.

Proof of the Proposition 1.2. Let $h : \mathbb{R} \to \mathbb{R}$ such that $||h'|| \leq 1$ and $E|h(Z)| < \infty$, and let $f := f_h$ be the unique solution to the Stein’s equation $f'(w) - w f(w) = h(w) - Eh(Z)$. Since $(W, W')$ is an exchangeable pair and $E(W - W'|W) = \lambda(W + R),

$$0 = E(W - W')(f(W) + f(W'))$$

$$= E(W - W')(f(W') - f(W)) + 2Ef(W)(W - W')$$

$$= E(W - W')(f(W') - f(W)) + 2\lambda Ef(W)E(W - W'|W)$$

$$= E\Delta(f(W') - f(W)) + 2\lambda Ef(W) + 2\lambda Ef(W)R.$$
It thus follows that
\[ |Eh(W) - Eh(Z)| = |E(f'(W) - Wf(W))| \]
\[ = \left| E \left( f'(W) + \frac{1}{2\lambda} E\Delta(f(W') - f(W)) + Ef(W)R \right) \right| \]
\[ = \left| E \left( f'(W) \left( 1 - \frac{1}{2\lambda} E\Delta^2|W| \right) + \frac{1}{2\lambda} \Delta (f(W') - f(W) + \Delta f(W')) + f(W)R \right) \right| \]
\[ \leq \|f'| \sigma \left| 1 - \frac{1}{2\lambda} E\Delta^2|W| \right| + \frac{1}{4\lambda} \|f''\| \sigma |\Delta|^3 + \|f| \sigma R| \quad \text{(A.9)} \]

By Lemma 2.4 in [3] we have
\[ \|f\| \leq 2, \quad \|f'\| \leq \sqrt{2/\pi}, \quad \|f''\| \leq 2. \quad \text{(A.10)} \]

The conclusion of the proposition follows from (A.9) and (A.10). □

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