Spatial Item Factor Analysis With Application to Mapping Food Insecurity

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September 12, 2018

Abstract:

Item factor analysis is widely used for studying the relationship between a latent construct and a set of observed variables. One of the main assumptions of this method is that the latent construct or factor is independent between subjects, which might not be adequate in certain contexts. In the study of food insecurity, for example, this is likely not true due to a close relationship with socio-economic characteristics, that are spatially structured. In order to capture these effects, we propose an extension of item factor analysis to the spatial domain that is able to predict the latent factors at unobserved spatial locations. We develop a Bayesian sampling scheme for providing inference and illustrate the explanatory strength of our model by application to a study of the latent construct ‘food insecurity’ in a remote urban centre in the Brazilian Amazon. We use our method to map the dimensions of food insecurity in this area and identify the most severely affected areas. Our methods are implemented in an R package, spifa, available from Github.

Keywords: Continuous spatial variation, Factor analysis, Gaussian processes, Item Factor Analysis, Kriging, Model-based geostatistics, Multivariate regression, Spatial prediction

1. Introduction

This paper concerns the analysis of geo-referenced survey data in which there is interest in understanding a set of spatially-varying latent constructs. A latent construct is a complex attribute or property that can be described by a number of characteristics, sometimes elicited through responses to survey questions for example. They are not
rigidly defined, rather the characteristics suggest the construct and may be debated and revised as time progresses. Latent constructs are very widely used across many areas of scientific research; in psychological research for instance, an example of a latent construct would be extroversion. This characteristic is not directly measurable for an individual (unlike age for example), but it can be measured through questionnaires such as the Keirsey Temperament Sorter (Briggs Myers and Myers, 1980, Keirsey, 1998). The idea is that the construct, extroversion, can be indirectly measured through responses to a subset of questions designed to elicit social behaviour and preferences. The collective response to these questions, created for example using a summative operation (in the case of binary data), is used to infer the degree of extroversion, as opposed to introversion, in a person.

Using the language of Item Response Theory (IRT), the individual questions in a survey (or test) are referred to as items, see Hambleton and Swaminathan (1989) for a detailed review. The responses to these items measure different concrete characteristics, known as observable variables. To continue the extroversion example above, item 15 from the Keirsey Temperament Sorter is “At a party, do you (a) interact with many, even strangers or (b) interact with a few friends?” and the observable variable in this case might be ‘interaction preferences in social situations’. Item response theory is a family of statistical models used to relate responses to items to the latent construct(s). These models assume the latent construct or ability (degree of extroversion in this case) is defined on a continuum. This allows us, for instance to score each individual’s ability; to identify which items have the greatest capacity to discriminate between individuals of differing abilities (i.e. how well each item identifies the trait of extroversion in individuals); or to identify the difficulty associated to each item – more ‘difficult’ items in this context would tend to be endorsed by more extroverted individuals, but less often by less extroverted individuals (De Ayala, 2013).

Item response theory has been widely applied in many areas of research. In psychometrics, for example, it has been used to measure the theory of mind ability (Shryane et al., 2008), emotional intelligence (Fiori et al., 2014), self-esteem (Gray-Little et al., 1997). In health and medicine, it is used to determine the health status of patients using self-reported outcomes (Edelen and Reeve, 2007), to measure individual scores of child developmental status (Drachler et al., 2007) and to assess achievement and evaluation of clinical performance (Downing, 2003). In mental health research, it has been used to study disorders like psychopathy (Laurens et al., 2012), alcohol use (Saha et al., 2006) and depression (Sharp et al., 2006). In e-learning, item response theory has been used to develop personalized intelligent tutoring systems that match learner ability and difficulty level (Chen and Duh, 2008). In computerized adaptive testing, it is used in tests like GMAT, GRE or TOEFL to dynamically select the most appropriate items for examinees according to individual abilities (Chen et al., 2006). In marketing, it has been used to measure customer relationship satisfaction (Funk and Rogge, 2007) and to measure extreme response styles (ERS) (de Jong et al., 2008). In criminology, it is applied to the analysis of the causes of crime and deviance using self-reporting measures of delinquency (Osgood et al., 2002) and to measure self-control (Piquero et al., 2000).
Our motivating application concerns the assessment of household food insecurity which is mediated through a family’s ability to access food and also through the supply of food potentially available. Both factors are relevant in the context of our study located in Ipixuna, a remote urban centre in the Brazilian Amazon. Food insecurity was measured using responses to a modified version of the questionnaire proposed by the United States Department of Agriculture (Carlson et al., 1999; National Research Council, 2006). Food insecurity in these remote and roadless urban centres, accessible only by boat or plane, is partly affected by seasonal variation in river levels. During particularly dry months it may be difficult for cargo boats to access the city and in very wet months there are risks of large-scale flooding - disease, loss of home and income. But there are other factors at play too: community, governmental and non-governmental support can bolster a family’s food resources in difficult times (Garrett and Ruel, 1999; Battersby, 2011). As is the case with cities in the West, neighbourhoods with certain characteristics tend to cluster together: it is for exactly this reason that in this paper we propose to extend traditional IRT models to accommodate spatial structure, among other attributes detailed below.

One of the main limitations of classical IRT models is that they assume that the latent construct is unidimensional: this assumption may not be adequate for more complex latent constructs. For example, the items developed to study food insecurity capture a number of different concepts including: (i) the perception of reduction in the quality or quantity of food, (ii) an actual reduction in quality of food, (iii) an actual reduction in quantity of food, and (iv) a reduction in the quantity or quality of food for children in the household. Hence, the construct food insecurity has more than one dimension, and might also depend on characteristics of the population under study, Froelich and Jensen (2002) for example found a further dimension associated with the protection of children from hunger.

In this context, where unidimensional models are not appropriate, researchers have developed Multidimensional Item Response Theory (MIRT) or Item Factor Analysis (IFA), both approaches being conceptually similar (Bock et al., 1988; Chalmers, 2012). These models extend the concept of standard multivariate factor analysis so it can be applied to binary or ordinal data and allows us to study the interaction between multiple items and a multi-dimensional latent construct. Although item factor analysis addresses the problem of uni-dimensionality, there are other limitations of this approach that we seek to address in the present paper.

Firstly, IFA assumes the latent construct of a particular subject to be independent of any other subject. In our subsequent example of food insecurity, this seems inadequate given that households near to each other are more likely to share similar socio-economic conditions and environmental exposures and thus a similar risk of food insecurity. This observation also applies to the analysis of latent constructs in other disciplines where spatial correlation is naturally expected, an example would be socio-economic status itself. Connected to this, an item factor analysis model incorporating spatial random effects would allow us to map the latent factors at unobserved locations, which can be (and is in our case) of scientific interest. With respect to our own and other similar application(s), a complete map of the latent factors over the area under study will
improve our understanding of the construct and help to better inform the decision-making process.

Secondly, IFA only relates items to the latent construct, but not to possible covariates that could help explain why certain individuals might have particularly high or low values of the latent construct. For example, our previous research in this area suggests socio-economic and environmental variables play an important role in determining food insecurity (Parry et al., 2017). In our case, therefore, understanding the relationship between the items, the latent construct and the covariates is highly desirable.

The above summarises our motivation for developing an extension to IFA which we here denominate spatial item factor analysis. Our hierarchical framework allows the latent construct to be split into multiple latent factors, the number and composition of which are determined by initial exploratory analyses. These latent factors are explained by observed covariates and also by spatially-correlated random effects. The relationship between the latent factors and the item responses, in the case of binary outcomes, is mediated through a set of auxiliary variables which handle the conversion between continuous to discrete data forms. We present an efficient Metropolis-within-Gibbs sampling strategy for Bayesian inference with our model.

The structure of the paper is as follows. Details of our proposed model for spatial item factor analysis is presented in Section 2. Bayesian inference for our model through Markov chain Monte Carlo methods is explained in Section 4. Spatial prediction for the latent construct is developed in Section 5. Then we detail application of the model to predicting food insecurity in Section 6. Finally, this paper concludes with a discussion of the advantages, disadvantages and possible extensions to our model in Section 7.

2. Spatial Item Factor Analysis

In this section we develop a modelling framework for spatial item factor analysis. We first introduce classical item factor analysis in Section 2.1, then in Section 2.3 we introduce our new methods. Solutions to identifiability issues in our model are discussed in Section 2.4. We then introduce the matrix form of the auxiliary variables of our model in Section 2.7. We conclude this section with the specification of the likelihood function in Section 2.8.

2.1. Item Factor Analysis

Item factor analysis can be seen as an extension of factor analysis for binary or ordinal data. In the present article, we concentrate on binary outcomes and discuss extensions of the proposed framework to a mix of continuous, binary and ordinal items in the Discussion (Section 7) and in Appendix F.

We begin by considering the response variable $Y_{ij}$ for item $j = 1, 2, \ldots, q$ from subject $i = 1, 2, \ldots, n$ as a binarization around zero of a continuous but unobservable
process $Z_{ij}$, explained by $m$ latent factors $\theta_{i1}, \ldots, \theta_{im}$,

$$Z_{ij} = c_j + \sum_{k=1}^{m} a_{jk} \theta_{ki} + \epsilon_{ij},$$

where $\epsilon_{ij} \sim \mathcal{N}(0, 1)$ and $\{c_j\}$ are intercept parameters that take into model the difficulty of items. High positive (negative) values for $c_j$ increase (reduce) the probability of endorsing $j$-th item, which is why they are also referred to as easiness parameters (Chalmers, 2015). The slopes $\{a_{jk}\}$, commonly called discrimination parameters, indicate how well the $j$-th item can discriminate the $k$-th ability between the subjects under study. If $a_{jk} = 0$, the $k$-th latent factor does not explain the variability of the $j$-th response item, in other words this item does not help to discriminate the $k$-th latent ability between the subjects. In our paper, we also use this parameterisation i.e. using intercepts and slopes. Further details on this model including inference via the expectation-maximization algorithm can be found in Bock et al. (1988).

As well as estimating the easiness and discrimination parameters, interest may also lie in making inferences for the latent factors $\theta_{ki}$, this allows us to differentiate individuals with high or low levels of the construct under study. A practical application of this is in the area of ideal point estimates, where the objective is to estimate the ideological position of a political legislator in order to predict whether they will vote in favour of a particular motion, see Bafumi et al. (2005) for example.

### 2.2. Exploratory and Confirmatory Item Factor analysis

The model defined by Equation 1 is not identifiable due to different types of aliasing, as explained in Section 2.4. We can make the model identifiable by placing restrictions on some parameters. The way this is done yields two different approaches.

We obtain an exploratory item factor analysis if the restrictions are imposed only to make the inference possible, i.e. the restrictions are not related to the construct and data under analysis. In this case, estimates can be rotated under the preference of the researcher.

We obtain a confirmatory item factor analysis if the restrictions are established in a semi-formal manner: the researcher uses their own (or expert) knowledge about the latent construct to establish the structure of an appropriate model (Cai, 2010b). In a confirmatory item factor analysis, the restrictions are designed with a particular study and context in mind, while in exploratory item factor analysis, the restrictions are generally imposed and are not problem-specific. Where experts cannot agree on a particular structure for the model, the option to use measures of model fit (e.g. WAIC or DIC for a Bayesian analysis) is still possible, as is model averaging.

### 2.3. Extension to the Spatial Domain

In our application, we are interested in estimating the easiness and discrimination parameters in order to understand the relationship between the underlying latent factors with the response variables. We also want to be able to predict the latent factors not
only in places where the observations were taken, but also in locations where we have no observations. Our data were costly, difficult and time-consuming to collect, thus our method for predicting food insecurity at new locations is an important step for identifying particularly vulnerable areas that could be targeted for intervention. Since our method can also be used to map and predict the different dimensions of food insecurity, this information could be used to tailor specific interventions to specific regions. This is our motivation for the development of spatial item factor analysis.

The extension of item factor analysis to the spatial domain can be achieved by including a spatial process in the predictor in Equation 1. Frichot et al. (2012), for example, proposed such a model by including a spatially correlated error term $\epsilon_{ij}$. This extension tries to correct the principal components by modelling the residual spatial variation. Our proposed method, spatial item factor analysis allows the latent factors $\theta_{ki}$ to be spatially correlated because the nature of the particular construct we are studying suggests they should be treated in this way. For example, we expect there to be spatial patterns in food insecurity scores across a municipality due to the relationship with socio-economic and environmental variables.

We model the binary response variables as a discrete-state stochastic processes $\{Y_j(s) : s \in D\}$ where $D \subset \mathbb{R}^2$ and the notation $Y_j(s)$ is the response to item $j$ at spatial location $s$. The response variables take values 0 or 1 according to the value assumed by an auxiliary spatial stochastic process $\{Z_j(s) : s \in D\}$:

$$Y_j(s) = \begin{cases} 1 & \text{if } Z_j(s) > 0 \\ 0 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (2)

Conditional on $Z_j(s)$, the values assumed by $Y_j(s)$ are deterministic. We model the auxiliary process as follows:

$$Z_j(s) = c_j + a_j^T \theta(s) + \epsilon_j(s), \quad \epsilon_j(s) \sim \mathcal{N}(0, 1),$$  \hspace{1cm} (3)

where $c_j$ and $a_j$ are respectively the easiness and discrimination parameters. The latent factors, $\theta(s)$, are defined as a continuous-space multivariate stochastic process of dimension $m$. Note that this process is the only source of spatial correlation in $Z_j(s)$ and $Y_j(s)$: if the spatial variation is removed from $\theta_j(s)$, then the model reduces to a simple item factor analysis.

The different assumptions that one can make with respect to $a_j$ and $\theta(s)$ generate different types of models. For example, under the assumption that $\theta(s)$ and $\theta(s')$ are uncorrelated with the further assumption that $\theta(s) \sim \mathcal{N}(0, I_m)$, this generates an exploratory item factor analysis (Cai, 2010a). Alternatively, restrictions on $a_j$ lead to a confirmatory item factor analysis (Cai, 2010b). The reasons why we include these assumptions and restrictions will be explained in Section 2.4: the concern is identifiability and our spatial item factor analysis model requires specific choices here.

In a similar manner, we can impose a particular structure on the latent factor $\theta(s)$ in order to create our spatial item factor model. Since one of our interests is in predicting the latent factors at unobserved locations $s^*$; we define the structure of $\theta(s)$ through a set of spatial covariates $x(s) = (x_1(s), \ldots, x_p(s))$. In this way the model allows us to
understand why certain individuals have high or low scores. The inclusion of covariates in factor analyses leads to multiple indicators, multiple causes models (MIMIC) in the literature on structural equation modelling (SEM), see Tekwe et al. (2014) for example. We include a latent spatial stochastic process \( \{w(s) : s \in D\} \) into our model for \( \theta(s) \), defining the \( m \)-dimensional latent factor as:

\[
\theta(s) = B^\top x(s) + w(s) + v(s),
\]

where \( B \) is an \( p \times m \) matrix of slopes associating a set of covariates \( x(s) \) with the latent factor \( \theta(s) \) and \( v(s) \) as defined below. Note that we will eventually assume that the covariates have been standardised, see Section 2.4 for further details. We define \( w(s) = \{w_k(s)\}_{k=1}^m \) to be a set of zero-mean, independent, stationary and isotropic Gaussian processes with variance \( \sigma_k^2 \) and correlation function \( \rho_k(u) \) at distance \( u \),

\[
w_k(s) \sim \text{GP}(0, \sigma_k^2, \rho_k(u)), \quad k = 1, \ldots, m.
\]

We use vector notation to denote \( w(s) \) because later, in Section 2.5, we discuss further extensions to the structure of \( w(s) \), such as allowing correlation between the \( w_k(s) \) and thus at the outset we wish to think of this as a multivariate Gaussian process (MGP).

Finally, the \( m \)-dimensional random vector \( v(s) \) is the remaining uncertainty in the latent factors that is neither explained by the covariates nor by \( w(s) \). We assume \( v(s) \) is a zero-mean multivariate normal distribution with covariance matrix \( \Sigma_v \),

\[
v(s) \sim \mathcal{N}(0, \Sigma_v).
\]

Equation 4 has the same structure as a multivariate geostatistical model. However, in our case, the dependent variable, \( \theta(s) \), is a low-dimensional latent process instead of a high-dimensional observed process as in Gelfand et al. (2004). A similar structure including fixed and random effects is also discussed in Chalmers (2015), but the author does not attempt to model unexplained spatial variation. In addition, the author mainly focuses on including covariates at the item level, whereas our emphasis is on the inclusion of covariates at the subject level which will then allow us to make predictions about individuals at unobserved locations.

Substituting the structure of the latent factors \( \theta(s) \) into Equation 3 results in

\[
Z_j(s) = c_j + a_j^\top [B^\top x(s) + w(s) + v(s)] + \epsilon_j(s).
\]

We note that if \( a_j \) were known, then Equation 7 would be a multivariate geostatistical model. The main challenges in our proposed model come from the inclusion of the interaction between the latent variables with the (unknown) slopes, \( a_j \).

In theory, the proposed model could be used in both exploratory and confirmatory factor analysis. However, we suggest using the model for confirmatory factor analysis in which there is no rotation of the latent factors - in this way, the correlation parameters are directly interpretable. If on the other hand, the latent factors have been rotated, as in exploratory analysis, interpreting the correlation parameters is then more difficult.
The relationship between covariates $x(s)$, latent factors $\theta(s)$, auxiliary latent variables $Z(s)$ and response variables $Y(s)$ can be seen more clearly through an example of spatial confirmatory factor analysis, as shown in Figure 1. This figure shows a directed graph with twelve items $Y_j(s)$, or response variables, four latent factors $\theta_k(s)$, four Gaussian processes $w_k(s)$ and six covariates $x_l(s)$. We have introduced $\eta_k$ as a linear combination of the covariates in order to have a more clear visualization of the model. In this example some of the coefficients, $a_j$, are set to zero so that each factor is only explained by a subset of items. It can be seen that the 12-dimensional response vector $Y(s)$ is reduced to a 4-dimensional space of factors $\theta(s)$. These factors allowed to be correlated with each other and also spatial correlation is permitted within factors. At the top of the figure, it is shown how covariates $x(s)$ are used to predict the latent factors $\theta(s)$.

2.4. Identifiability and restrictions

The model presented above is subject to the same identifiability problems as those found in factor analysis and structural equation modelling. Identifiability issues arise when different sets of parameters lead to the same likelihood in a structured way - this leads to symmetry in the posterior (or objective function) i.e. there are multiple modes. In our model, these identifiability issues could be due to additive, scaling, rotational or reflection aliasing, which will be discussed in detail below.

Additive aliasing occurs when the item difficulties $c_j$ and the product $a_j^T \theta_j$ have free means. Under this situation a constant value could be added and subtracted to each term respectively and the probability density function will be unchanged. Similarly, if $a_j^T$ is multiplied by a constant and $\theta_j$ divided by the same constant, then the probability density function is constant leading to scaling aliasing.
In order to address the issues of scaling and additive aliasing in classical item factor analysis as in Equation 1 it is common to assume that $\mathbf{\theta}_{ik} \sim \mathcal{N}(0, 1)$ \cite{Bafumi2005}. A generalisation of this would be to assume $\mathbf{\theta}_i \sim \mathcal{N}(0, \Sigma_{\theta})$, whence the previous solution is obtained by setting that $\Sigma_{\theta} = I$ for an exploratory factor analysis or by setting $\text{diag}(\Sigma_{\theta}) = 1$ for a confirmatory factor analysis.

The spatial item factor model presented in Section 2.3 does not suffer from additive aliasing because we are already assuming the processes $\mathbf{w}(s)$ and $\mathbf{v}(s)$ are zero-mean. As mentioned above, we assume that the covariates included in Equation 4 are standardised, which leads to the latent factor $\mathbf{\theta}(s)$ having mean zero.

However, our model does suffer from scaling aliasing, so we are required to restrict the variances of the latent factors $\mathbf{\theta}(s)$, this is complicated by the presence of covariates. One simple way of achieving the required restriction is by fixing the variance of one of the terms inside the structure of the latent factors in Equation 4, see Appendix A.1 for details. This is usually applied to the multivariate error term, $\mathbf{v}(s)$, as in \cite{Tekwe2014}. It is sufficient to fix a diagonal matrix $\mathbf{D}$, which contains the marginal standard deviations of $\mathbf{v}(s)$, $\text{diag}(\mathbf{D}) = (\sigma_{v1}, \ldots, \sigma_{vm})^\top$, such that

$$\Sigma_v = \mathbf{D} \mathbf{R}_v \mathbf{D},$$

where $\mathbf{R}_v$ is a correlation matrix. The usual restrictions applied in exploratory or confirmatory item factor analysis are equivalent to setting $\mathbf{D} = I$. If the model includes both covariates and Gaussian processes and we are conducting an exploratory item factor analysis, then this method does not work well because the marginal variances of the latent factors $\mathbf{\theta}(s)$ might become too big and consequently, the discrimination parameters would have to be close to zero and become unidentifiable; this is due to the multi-modal shape of the likelihood function. For confirmatory item factor analysis the condition $\mathbf{D} = I$ is sufficient to eliminate issues of scaling aliasing, see Appendix A.1.

Although the restrictions imposed modify the interpretation of the discrimination parameters $\mathbf{a}_j$ because the latent factors are on different scales, they are only necessary in order to make the inference possible. Therefore, a scaling transformation can applied post-estimation in order to recover the correct interpretation of the discrimination parameters:

$$Z_j(s) = c_j + \mathbf{a}^\top \mathbf{Q} \mathbf{Q}^{-1} \mathbf{\theta}(s) + \epsilon_j(s),$$

where $\mathbf{Q}$ is a diagonal matrix of the standard deviations for $\mathbf{\theta}(s)$. This transformation leads to a new vector of latent abilities $\mathbf{Q}^{-1} \mathbf{\theta}(s)$ with unit variances and discrimination parameters $\mathbf{a}^\top \mathbf{Q}$ with the usual interpretation as in item factor analysis, see Section 4.4 for further details.

Returning to classical item factor analysis, the other two types of aliasing, rotational and reflection, are due to the fact that linear transformations of the slope parameters $\mathbf{a}_j^* = \mathbf{a}_j^\top \mathbf{\Lambda}^{-1}$ and of the latent factors $\mathbf{\theta}_i^* = \mathbf{\Lambda} \mathbf{\theta}_i$ lead to the same probability density function of the original parameters $\mathbf{a}_j$ and $\mathbf{\theta}_i$ given that $\mathbf{\Lambda}^{-1} \mathbf{\Lambda} = I$ \cite{Erosheva2011}. In exploratory factor analysis $\mathbf{\Lambda}$ is an orthogonal matrix because it
is assumed $\Sigma_\theta = I$; it can be shown that this implies $\Lambda \Lambda^\top = I$. In the case of rotational aliasing the matrix of the linear transformation has $m(m - 1)/2$ degrees of freedom. Hence, $m(m - 1)/2$ restrictions can be applied to eliminate this type of aliasing. The usual criteria is to set $(a_1, \ldots, a_g)^\top$ to be a lower triangular matrix (Geweke and Zhou, 1996). For reflection aliasing, there are $2^m$ orthogonal matrices $\Lambda$ obtained by simultaneously changing the signs of $a_j$ and $\theta_i$. In this case, identifiability can be ensured by setting the diagonal elements of $A = (a_1, \ldots, a_j)^\top$ to be positive (Geweke and Zhou, 1996).

For spatial exploratory item factor analysis the above restrictions on the discrimination parameters, or similar, are necessary. For confirmatory factor analysis it is sufficient to fix $m(m - 1)/2$ entries (usually the value chosen is zero) of $A$ and also set as positive (or negative) one element from each column of $A$; the former addresses rotation aliasing, and the latter reflection aliasing. More generally, a set of restrictions can be induced through a linear association between the constrained parameters $a^*_j$ and the free parameters $a_j$,

$$a^*_j = u_j + L_j a_j,$$

where the vector $u_j$ are the values that are to be fixed, while the matrix $L_j$ indicates which elements of the free-parameter $a_j$ are to be activated (Cai, 2010b). In the example below, the third and fourth elements of the parameter vector are set to 0 and 1 respectively.

$$\begin{pmatrix} a^*_{j1} \\ a^*_{j2} \\ a^*_{j3} \\ a^*_{j4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{j1} \\ a_{j2} \\ a_{j3} \\ a_{j4} \end{pmatrix} = \begin{pmatrix} a_{j1} \\ a_{j2} \\ 0 \\ 1 \end{pmatrix}$$

In practice, achieving the required positivity (or negativity) constraints is accomplished through the appropriate specification of the marginal prior distributions, see Section 4.2 for details.

### 2.5. Allowing Further Flexibility on the Multivariate Spatial Structure

In the discussion above, we proposed using a set of independent Gaussian processes in the structure of the latent factors $\theta(s)$. However, it can be the case that some of the factors $\theta_k(s)$ are not spatially correlated, or that some of the unexplained variation in two or more factors may have a common (spatially-correlated) component. In this situation it will be desirable, respectively, to include spatial structure on only a subset of the factors, or to share the spatial structure across several factors.

In a similar way to how restrictions were imposed on the discrimination parameters, we can use an $m \times g$ transformation matrix $T$ to convert $g$ independent standard Gaussian processes in $w(s)$ into an $m$-dimensional multivariate Gaussian process, $w^*(s)$:

$$w^*(s) = Tw(s).$$

An example is given in Equation 13, where after transforming, $w_1(s)$ is common to the first and second factor and the second factor has an additional spatial structure,
namely $w_2(s)$; $w_3(s)$ features in the third factor, and the last factor does not include any Gaussian process i.e. it is not spatially structured.

\[
\begin{pmatrix}
w_1^*(s) \\
w_2^*(s) \\
w_3^*(s) \\
w_4^*(s)
\end{pmatrix}
= 
\begin{pmatrix}
t_{11} & 0 & 0 \\
t_{21} & t_{22} & 0 \\
0 & 0 & t_{33}
\end{pmatrix}
\begin{pmatrix}
w_1(s) \\
w_2(s) \\
w_3(s)
\end{pmatrix}
= 
\begin{pmatrix}
t_{11}w_1(s) \\
t_{21}w_1(s) + t_{22}w_2(s) \\
t_{33}w_3(s)
\end{pmatrix}
\]  \quad (13)

Notice that the variance of $w^*(s)$ is controlled by $T$. Using this stochastic process in Equation 4, we re-define the $m$-dimensional latent factor of our model as:

\[
\theta(s) = \mathbf{B}^\top \mathbf{x}(s) + w^*(s) + \mathbf{v}(s).
\]  \quad (14)

The methods described in the section are closely connected to multivariate geostatistical models of coregionalization [Gelfand et al. 2004, Fanshawe and Diggle 2012]. The main difference is that here we are using it as a way for the user to control the nature of interrelationships between factors (which would obviously change according to the problem and data under study), rather than allowing free reign estimating all the elements of this matrix. There is a sense in which the restrictions imposed can be thought of as prior specification. Provided the ‘correct’ overall structure of $T$ has been chosen, such restrictions are also beneficial; in particular if $m > g$ then inference becomes more tractable – both in terms of computation, and subsequently interpretation.

In the absence of expert opinion (but preferably in the presence of it), we suggest using an exploratory item factor analysis before applying our model in order to evaluate these characteristics and decide on the structure of the multivariate spatial correlation defined through $T$.

2.6. Auxiliary Variables in the Identifiable Spatial Item Factor Analysis

Using the restricted discrimination parameters $\mathbf{a}^*$ defined in Equation 10 and the new definition of the latent factor $\theta(s)$ in Equation 14, we obtain an identifiable and flexible model for spatial item factor analysis where the auxiliary variables $Z_j(s)$ have the following structure

\[
Z_j(s) = c_j + a_j^\top \theta(s) + \epsilon_j(s) = c_j + a_j^\top [\mathbf{B}^\top \mathbf{x}(s) + w^*(s) + \mathbf{v}(s)] + \epsilon_j(s).
\]  \quad (15)

We are assuming that the structure of the restricted discrimination parameters $a_j^*$ and also the multivariate Gaussian process $w^*(s)$ will be informed by expert opinion through direct involvement of researchers in the area of application and/or through consulting the academic literature in that area.

Doing this not only allows our model to be identifiable, but it also allows us to obtain interpretable latent factors which are practically useful to researchers in the field under consideration.
2.7. Matrix Form of the Auxiliary Variables

Expressing the terms in our model at the individual level as above (and in Equation 15) is convenient for understanding the various components; however, in the sequel, we will use the matrix form of our model in order to define the likelihood function (Section 2.8) and later derive the conditional distributions of the posterior (Section 4).

Before proceeding with the matrix form of our model, we introduce some further notational conventions. Let $\alpha(s)$ be a $q$-variate random variable at spatial location $s$. Then if $s = (s_1, s_2, \ldots, s_n)^T$ is a set of locations, we will define the $q$-vector $\alpha_i = \alpha(s_i) = (\alpha_1(s_i), \ldots, \alpha_q(s_i))^T$ and the $n$-vector $\alpha_{[j]} = \alpha_j(s) = (\alpha_j(s_1), \ldots, \alpha_j(s_n))^T$.

With the above conventions, the collection of auxiliary random variables $Z = (Z_{[1]}^T, \ldots, Z_{[q]}^T)^T$ for $q$ items at $n$ locations can be expressed as

$$Z = (I_q \otimes 1_n)c + (A^* \otimes I_n)\theta + \epsilon$$

(16)

where $I_q$ and $I_n$ are identity matrices of dimension $q$ and $n$ respectively, $1_n$ is a $n$-dimensional vector with all elements equals to one, $c = (c_1, \ldots, c_q)^T$ is a vector arrangement of the easiness parameters, $A^*_{q \times m} = (\alpha_i^*, \ldots, \alpha_q^*)^T$ is a matrix arrangement of the restricted discrimination parameters, $\theta = (\theta_{[l]}^T, \ldots, \theta_{[m]}^T)^T$ and $\epsilon = (\epsilon_{[1]}^T, \ldots, \epsilon_{[q]}^T)^T$ is a $nq$-vector of residual terms.

The vector of latent abilities $\theta$ with respect to Equation 14 can be expressed as

$$\theta = (I_m \otimes X)\beta + (T \otimes I_n)w + v,$$

(17)

where $\beta = \text{vec}(B)$ is a column-vectorization of the multivariate fixed effects, $X_{n \times p} = (x_1, \ldots, x_n)^T$ is the design matrix of the covariates, $w = (w_{[l]}^T, \ldots, w_{[m]}^T)^T$ is the collection of the multivariate Gaussian process and $v = (v_{[l]}^T, \ldots, v_{[m]}^T)^T$ is the collection of the multivariate residual terms. Substituting Equation 17 into Equation 16, we obtain:

$$Z = (I_q \otimes 1_n)c + (A^* \otimes X)\beta + (A^*T \otimes I_n)w + (A^* \otimes I_n)v + \epsilon.$$

(18)

This matrix representation is useful for deriving the multivariate marginal and conditional distributions of $Z$ in the following sections.

Alternatively, the collection of auxiliary variables $Z$ can also be expressed as

$$Z = (I_q \otimes 1_n)c + (I_q \otimes \Theta)a^* + \epsilon$$

$$= (I_q \otimes 1_n)c + (I_q \otimes \Theta)u + (I_q \otimes \Theta)La + \epsilon,$$

(19)

where $\Theta_{n \times m} = (\theta_{[l]}, \ldots, \theta_{[m]})$ is the matrix of latent abilities, $u = (u_{[1]}^T, \ldots, u_{[q]}^T)^T$ are the restrictions defined in Equation 10, $a = (a_1^T, \ldots, a_q^T)^T$ are the free discrimination parameters and $L = \oplus_{j=1}^qL_j$ is the direct sum of the activation matrices defined in Equation 10 (recall these link the free discrimination parameters $a$ with the constrained discrimination parameters $a^*$). We later use Equation 19 in the derivation of the conditional posterior distribution of the discrimination parameters $a$. 

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2.8. Likelihood Function

A challenging aspect of our motivating application, see Section 6, is the fact that not all questions are answered by all households: the items in Section C of the questionnaire only apply to households with children. More generally, it is common to have to deal with missing data (in this case item responses) in statistics, therefore in the present section we begin to introduce notation for observed and missing data; this will be revisited several times in Section 4 and is also connected with prediction.

Let \( s = (s_1, s_2, \ldots, s_n)^\top \) be a set of locations at which data from \( q \) items has been collected. Let the random variable \( Y_{ij} = Y_j(s_i) \) be the \( j \)-th item response at location \( s_i \). Using notation introduced in Section 2.7, let \( Y = (Y_1^\top, \ldots, Y_q^\top)^\top \) be the collection of responses to all items. These can be divided into two groups; the set of observed variables \( Y_{obs} \) and the set of variables that were missing \( Y_{mis} \).

The marginal likelihood function for our spatial item factor analysis model is obtained by integrating the joint density of the observed variables \( Y_{obs} \), the associated auxiliary variables \( Z_{obs} \) and the collection of latent abilities \( \theta = (\theta_1, \ldots, \theta_m)^\top \);

\[
L(c, a, B, T, \phi, R_v) = \int \int Pr (y_{obs} \mid z_{obs}) Pr (z_{obs}, \theta \mid c, a, B, T, \phi, R_v) \, dz_{obs} \, d\theta, \tag{20}
\]

where \( a = (a_1^\top, \ldots, a_q^\top)^\top \) is vector arrangement of all the discrimination parameters and \( \phi = (\phi_1, \ldots, \phi_g)^\top \) is the vector of scale parameters of the \( g \)-dimensional Gaussian process \( w(s) \).

In Equation 20, the structure of our model implies

\[
Pr (y_{obs} \mid z_{obs}) = \prod_{o_{ij}=1} \Pr (y_{ij} \mid z_{ij}), \tag{21}
\]

where \( o_{ij} \) is an indicator variable with value equals to one when the variable \( Y_{ij} \) has been observed (i.e. is not missing) and zero otherwise. We further have:

\[
Pr (z_{obs}, \theta \mid c, a, B, T, \phi, R_v) = \prod_{o_{ij}=1} \Pr (z_{ij} \mid \theta_i, c_j, a_j, B) \Pr (\theta \mid T, \phi, R_v), \tag{22}
\]

and densities on the right hand side are normally distributed.

Note that the definition of the likelihood function through Equation 20, 21 and 22 does not depend on the missing observations. Therefore, if some items were not observed in some of the locations, inference will still be possible provided the missing data are missing at random [Merkle 2011]. Using this likelihood, inference from the model can proceed in a number of ways. Maximum likelihood estimation can be achieved by approximating the likelihood function in Equation 20 using a variety of Monte Carlo methods or via stochastic approximation [Cai 2010b]. However in the present article, we focus on a Bayesian approach as shown in Section 4.

Our likelihood function can also be written using the auxiliary variables associated with both the observed and missing responses:

\[
L(c, a, B, T, \phi, R_v) = \int Pr (y_{obs} \mid z_{obs}) Pr (z \mid c, a, B, T, \phi, R_v) \, dz. \tag{23}
\]
The advantage of this representation is that the joint density of the auxiliary variables \( \Pr(z \mid c, a, B, T, \phi, R_v) \) can be obtained in a straightforward manner using Equation 18. It is normally distributed with mean

\[
\mu_z = (I_q \otimes 1_n)c + (A^* \otimes X)\beta
\]  

(24) and covariance matrix

\[
\Sigma_z = (A^*T \otimes I_n)\Sigma_w(T^TA^* \otimes I_n) + (A^* \otimes I_n)DR_vD(A^* \otimes I_n)
\]  

(25)

where \( \Sigma_w = \oplus_k \Sigma_{w_k} \) is the direct sum of the covariance matrices of the independent Gaussian processes. We prefer this last definition of the likelihood function as it allows us to handle the missing data using data augmentation, see Section 4.3.

3. R Package

Our model is implemented in an open-source R package, spifa, available from Github, https://github.com/ErickChacon/spifa. This package implements the Bayesian inferential method outlined below in full, allowing the user to specify the structure of the multivariate Gaussian processes and prior hyperparameters; model selection is also available through the DIC. The package has functions for summarising model output, for MCMC diagnostics and for the production of predictive maps via sf methods [Pebesma 2018]. The inferential code is written using C++, Rcpp, RcppArmadillo and OpenBLAS to make efficient use of multi-CPU hardware architectures.

4. Bayesian Inference Using Markov Chain Monte Carlo

In this section we describe a Metropolis-within-Gibbs algorithm for Bayesian inference with the spatial item factor analysis model proposed in Section 2. We first present the Bayesian formulation of our model in Section 4.1; then in Section 4.2, we provide details of the prior specifications; lastly, we conclude by explaining the sampling scheme for the parameters and auxiliary variables in Section 4.3.

4.1. Bayesian Spatial Item Factor Analysis Model

As illustrated in Figure 1, we factored the joint likelihood in a natural way into four model hierarchies. The first three hierarchies are: the data level \( \Pr(y_{obs} \mid z_{obs}) \), at the level of the auxiliary variables \( \Pr(z \mid \theta, c, a) \), at the level of the latent factors \( \Pr(\theta \mid \beta, T, \phi, R_v) \). For our Bayesian model, we add an additional level for the parameters \( c, a, \beta, T, \phi \) and \( R_v \). The posterior distribution of the model is

\[
\Pr(z, c, a, \theta, \beta, T, \phi, R_v \mid y_{obs}) \propto \Pr(y_{obs} \mid z_{obs}) \Pr(z \mid \theta, c, a) \Pr(\theta \mid \beta, T, \phi, R_v) \Pr(c) \Pr(a) \Pr(\beta) \Pr(T) \Pr(\phi) \Pr(R_v).
\]  

(26)

This choice of factorisation allows us to take advantage of conjugacy for some parameters and also marginalise terms that may lead to slow convergence/mixing e.g. the multivariate Gaussian process \( w \) and the multivariate residual term \( v \).
4.2. Priors

We assume Gaussian distributions for the easiness, discrimination and fixed effects parameters:

\[ c \sim \mathcal{N}(0, \Sigma_c), \quad a_j \sim \mathcal{N}(\mu_{a_j}, \Sigma_{a_j}), \quad \beta \sim \mathcal{N}(0, \Sigma_\beta), \]  

(27)

where \( \mu_c \) and \( \mu_{a_j} \) are the mean parameters, and \( \Sigma_c, \Sigma_{a_j} \) and \( \Sigma_\beta \) are diagonal covariance matrices.

With respect to the \( m \)-dimensional Gaussian process \( w^*(s) \), we assume that the associated parameters have a log-normal distribution,

\[ \text{vec}^*(T) \sim \mathcal{LN}(\mu_T, \Sigma_T), \quad \phi \sim \mathcal{LN}(\mu_\phi, \Sigma_\phi), \] 

(28)

where \( \text{vec}^*(T) \) is a vector of the non-zero values of \( T \), \( \mu_T \) and \( \mu_\phi \) are the mean parameters and \( \Sigma_T \) and \( \Sigma_\phi \) are diagonal covariance matrices of the log-transformation of the parameters.

Finally, we use an LKJ distribution for the correlation matrix \( R_v \) of the multivariate residual term, which is defined as:

\[ \Pr(\mathbf{R}_v) \propto \det(\mathbf{R}_v)^{\eta-1}. \]  

(29)

Here, \( \eta \) is the shape parameter of the LKJ distribution. If \( \eta = 1 \), the density is uniform; for bigger values \( \eta > 1 \), the mode is a identity matrix; and band diagonal matrices are more likely when \( 0 < \eta < 1 \) \cite{Lewandowski2009}.

4.3. Sampling Scheme

Samples from the posterior distribution (Equation 26) are drawn using blocked Gibbs sampling where possible. In cases where the conditional posterior distribution is not available analytically, we used Metropolis Hastings to update parameters, details below.

Auxiliary variables

Recall from above that we introduced a distinction between the observed variables \( Y_{obs} \) and the set that could not been observed \( Y_{mis} \). In a similar way, we divide the associated auxiliary variables into two groups, \( Z_{obs} \) and \( Z_{mis} \). From Equation 18, the joint vector of auxiliary variables \( Z \) is normally distributed given the easiness parameters \( c \), the discrimination parameters \( a \) and the latent factors \( \theta \):

\[ \Pr(z \mid c, a, \theta) = \mathcal{N}(z \mid (I_q \otimes I_n)c + (A^* \otimes I_n)\theta, I_{nq}). \] 

(30)

In the equation above it can be seen that any two elements of \( Z \) are conditionally independent given \( c, a \) and \( \theta \) because the covariance is the identity matrix. Using the fact that this joint density can also be written as the product of two marginal densities and that \( Y_{obs} \) is conditionally independent of \( Z_{mis} \) given \( Z_{mis} \), as shown in Appendix.
B.1, the conditional posterior distribution for the auxiliary variables $Pr (z | y_{obs}, c, a, \theta)$ is

$$Pr (z_{obs}, z_{mis} | y_{obs}, c, a, \theta) \propto Pr (z_{obs} | c, a, \theta) Pr (z_{mis} | c, a, \theta) Pr (y_{obs} | z_{obs}). \quad (31)$$

Hence, using Equation 21, the conditional posterior distribution for $Z_{obs}$ is

$$Pr (z_{obs} | y_{obs}, c, a, \theta) \propto Pr (z_{obs} | c, a, \theta) \prod_{oij=1} Pr (y_{ij} | z_{ij}), \quad (32)$$

which is a marginal truncated normal distribution obtained from Equation 30. Note that $Pr (y_{ij} | z_{ij}) = I_{z_{ij}>0} I_{z_{ij} \leq 0}$, where $I(.)$ is the indicator function. In a similar way, we obtain that the conditional posterior distribution of the auxiliary variables related to the missing data $Z_{mis}$ as

$$Pr (z_{mis} | y_{obs}, c, a, \theta) \propto Pr (z_{mis} | c, a, \theta), \quad (33)$$

which is a marginal distribution of Equation 30. Hence, the only difference between the posterior of both sets of variables is that it is truncated for the $Z_{obs}$ and unrestricted for $Z_{mis}$.

**Latent Factors**

The conditional posterior distribution of the latent abilities is

$$Pr (\theta | z, c, a, \beta) \propto Pr (z | c, a, \theta) Pr (\theta | \beta, T, \phi, R_n), \quad (34)$$

where the joint density of the auxiliary variables $Pr (z | c, a, \theta)$ is a Gaussian distribution, given in Equation 30, and the density of the latent factors, as defined in Equation 17, is also a Gaussian distribution,

$$Pr (\theta | \beta, T, \phi, R_n) = N(\theta | (I_m \otimes X)\beta, (T \otimes I_n)\Sigma_w (T^\top \otimes I_n) + DR_n D \otimes I_n), \quad (35)$$

where $\Sigma_w = \oplus_{k=1}^g \Sigma_{w_k}$. Hence, the conditional posterior $Pr (\theta | z, c, a, \beta)$ is defined by the product of two normal densities that leads to a normal density with covariance matrix

$$\Sigma_{\theta | z} = (A^* \otimes I_n) (A^* \otimes I_n) + (T \otimes I_n) \Sigma_w (T^\top \otimes I_n) D R_n D \otimes I_n)^{-1}, \quad (36)$$

and mean

$$\mu_{\theta | z} = \Sigma_{\theta | z} (A^* \otimes I_n) (z - (I_q \otimes I_n)c) + \Sigma_{\theta | z} (T \otimes I_n) \Sigma_w (T^\top \otimes I_n) D R_n D \otimes I_n)^{-1} (I_m \otimes X)\beta. \quad (37)$$
Fixed effects

For the multivariate fixed effects $\beta$, the conditional posterior

$$
\Pr (\beta \mid y_{\text{obs}}, z, c, a) \propto \Pr (\theta \mid \beta, T, \phi, R_v) \Pr (\beta)
$$

is given by the product of two normal densities obtained from Equation 17 and 27,

$$
\mathcal{N}(\theta \mid (I_m \otimes X)\beta, (T \otimes I_n)\Sigma_w(T^T \otimes I_n) + R \otimes I_n)\mathcal{N}(\beta \mid 0, \Sigma_\beta),
$$

that also leads to a multivariate normal distribution with covariance matrix

$$
\Sigma_{\beta_j} = ((I_m \otimes X)^T ((T \otimes I_n)\Sigma_w(T^T \otimes I_n) + R \otimes I_n)^{-1} (I_n \otimes X) + \Sigma_\beta^{-1})^{-1},
$$

and mean

$$
\mu_{\beta_j} = \Sigma_{\beta_j} (I_m \otimes X^T)((T \otimes I_n)\Sigma_w(T^T \otimes I_n) + R \otimes I_n)^{-1} \theta.
$$

Easiness parameters

The conditional posterior distribution of the easiness parameters $c$,

$$
\Pr (c \mid y, z, a, \theta) \propto \Pr (z \mid \theta, c, a) \Pr (c),
$$

is also the product of two normal densities obtained from Equation 30 and 27,

$$
\Pr (c \mid y, z, a, \theta) \propto \mathcal{N}(z \mid (I_q \otimes 1_n)c + (I_q \otimes 1_n)\theta, I_{nq})\mathcal{N}(c \mid 0, \Sigma_c),
$$

leading to a multivariate normal density with covariance matrix

$$
\Sigma_{c_j} = ((I_q \otimes 1_n)^T(I_q \otimes 1_n) + \Sigma_c^{-1})^{-1} = (\text{diag}(\Sigma_c)^{-1} + n)^{-1},
$$

and mean

$$
\mu_{c_j} = \Sigma_{c_j} (I_q \otimes 1_n^T)(z - (A^* \otimes I_n)\theta).
$$

Discrimination parameters

Due to the structure of our hierarchical model in Section 4.1, the conditional posterior distribution of the discrimination parameters,

$$
\Pr (a \mid y, z, c, \theta) \propto \Pr (z \mid \theta, c, a) \Pr (a),
$$

is determined by the product of two Gaussian densities obtained from Equation 30 and 27,

$$
\mathcal{N}(z \mid (I_q \otimes 1_n)c + (I_q \otimes \Theta^*)u + (I_q \otimes \Theta^*)La, I_n)\mathcal{N}(a \mid \mu_a, \Sigma_a),
$$

which, similar to previous parameters, leads to a Gaussian density with covariance matrix

$$
\Sigma_{a_j} = (L^T(I_q \otimes \Theta^*\Theta^*)L + \Sigma_a^{-1})^{-1},
$$

and mean

$$
\mu_{a_j} = \Sigma_{a_j} L^T(I_q \otimes \Theta^*\Theta^*)L(z - (I_q \otimes 1_n)c - (I_q \otimes \Theta^*)u) + \Sigma_{a_j} \Sigma_a^{-1} \mu_a.
$$
Covariance parameters

Unlike the previous parameters, the parameters of the multivariate Gaussian process $w^*(s)$ and the multivariate residual term $v(s)$ can not be directly sampled from their conditional posterior density as they are not available analytically. However, this density can be defined up to a constant of proportionality,

$$\Pr(\text{vec}^*(T), \phi, R_v | \theta, \beta) \propto \Pr(\theta | \beta, T, \phi, R_v) \Pr(T) \Pr(\phi) \Pr(R_v). \quad (50)$$

In order to obtain an MCMC chain that mixes over the real line, we work with $\log(\phi)$ instead of $\phi$ and $\log(\text{vec}^*(T))$ instead of $\text{vec}^*(T)$. For the correlation $R_v$, we use canonical partial correlation, transforming to a set of free parameters $\nu \in \mathbb{R}^{m(m-1)/2}$, see Lewandowski et al. (2009) for further details.

We use an adaptive random-walk Metropolis Hastings algorithm to sample from this part of the posterior distribution. The covariance matrix of the proposal, is adapted to reach a fixed acceptance probability (e.g. 0.234). More specifically, we implemented algorithm 4 proposed in Andrieu and Thoms (2008) using a deterministic adaptive sequence $\gamma_i = C/i^\alpha$ for $\alpha \in ((1+\lambda)^{-1}, 1]$, where $\lambda > 0$. In the tests we have run and in our food insecurity application, this algorithm and choice of parameters performs well (see details below for our choice of $C$ and $\alpha$).

4.4. Scaling Samples for Interpretation

In Section 2.4, we saw how restricting the standard deviations of the multivariate residual term $v(s)$ is necessary to make our model identifiable (Equation 8). However, we can not ensure that the latent factors will be on the same scale, which leads to a loss of interpretation of the discrimination parameters $a_j$. As proposed in the same section, after the samples of the MCMC have obtained, we can transform the parameters in order to obtain latent factors with expected variance equal to 1 to solve this problem. We can then obtain the matrix $Q$ of Equation 8 by filling the diagonal elements with the expected variances of the samples of the latent factors $\theta(s)$. We then make the following transformations

$$a_j \leftarrow Q a_j, \quad \theta_i \leftarrow Q^{-1} \theta_i, \quad B \leftarrow Q^{-1} B, \quad T \leftarrow Q^{-1} T, \quad D \leftarrow Q^{-1} D; \quad (51)$$

the correct interpretation of the parameters is then recovered.

4.5. Model Selection Using the Deviance Information Criterion

Bayesian model selection for the spatial item factor analysis can be done by using any of the information criteria normally applied in Bayesian modelling; here we focus on the deviance information criterion (DIC) proposed by Spiegelhalter et al. (2002). A Bayesian version of the Akaike information criterion, the DIC encapsulates the trade-off between goodness of fit and model complexity. This complexity, measured through the effective number of parameters, is determined by the difference between the mean of the deviance and the deviance of the mean,

$$p_D = D(\alpha) - D(\bar{\alpha}). \quad (52)$$
The deviance in our case is given by
\[ D(\alpha) = -2 \log\{Pr(y | \alpha)\} + 2 \log\{Pr(y | \mu(\alpha) = y)\}, \] (53)
where \(Pr(y | \mu(\alpha) = y)\) is the likelihood associated with a saturated model. The DIC can then be calculated as:
\[ DIC = \bar{D(\alpha)} + p_D, \] (54)
modes with a lower DIC are preferred.

In order to be able to calculate this quantity for our model, we require the density function of the responses \(Y\) given all the parameters of the model, which is expressed as
\[ \log(Pr(y | \alpha)) = \sum_{o_{ij}=1} (y_{ij} \log(\Phi(c_j + a_j^T \theta_j)) + (1 - y_{ij}) \log(1 - \Phi(c_j + a_j^T \theta_j))). \] (55)
where \(o_{ij}\) is a binary variable taking value equal to one when the variable \(Y_{ij}\) has been observed and zero otherwise.

5. Prediction of Latent Factors

In this section, our interest is on the spatial prediction of the latent factors \(\tilde{\theta}\) at a set of locations that we have not observed data, \(\tilde{s}\). As is customary, we obtain the predictive distribution by integrating out the parameters of the model from the joint density of \(\tilde{\theta}\) and the parameters,
\[ Pr(\tilde{\theta} | y, X, \tilde{X}) = \int Pr(\tilde{\theta} | \theta, B, \sigma^2, \phi, R_v, y, X, \tilde{X}) Pr(\theta, B, \sigma^2, \phi, R_v | y, X) d\theta dB d\sigma^2 d\phi dR_v. \]

Note that a vectorized version of Equation 4 can be expressed as
\[ \tilde{\theta} = (I_m \otimes \tilde{X}) \beta + (T \otimes I_n) \tilde{w} + \tilde{v}. \] (56)
Under these expressions, it can be shown that \(\theta\) and \(\tilde{\theta}\) are normally distributed with parameters
\[ E[\theta] = (I_m \otimes X) \beta, \quad V[\theta] = (T \otimes I_n) V[w] (T^T \otimes I_n) + V[v] \] (57)
\[ E[\tilde{\theta}] = (I_m \otimes \tilde{X}) \beta, \quad V[\tilde{\theta}] = (T \otimes I_n) V[\tilde{w}] (T^T \otimes I_n) + V[\tilde{v}]. \] (58)
Furthermore, the cross-covariance can be obtained as
\[ \text{Cov} \left[ \tilde{\theta}, \theta \right] = \text{Cov} \left[ (T \otimes I_n) \tilde{w} + \tilde{v}, (T \otimes I_n) w + v \right] = (T \otimes I_n) \text{Cov} \left[ \tilde{w}, w \right] (T^T \otimes I_n), \] (59)
where $\text{Cov} \begin{pmatrix} \tilde{w}, w \end{pmatrix}$ is a block diagonal matrix as both $\tilde{w}$ and $w$ are multivariate independent Gaussian process, see Section 2.3. Hence, the conditional distribution of $\theta$ is $\text{Pr}(\tilde{\theta} | \theta, B, \sigma^2, \phi, R_v, y, X, \tilde{X})$, a normal distribution with mean and variance

\begin{align*}
E \left[ \tilde{\theta} | \theta \right] &= E \left[ \tilde{\theta} \right] + \text{Cov} \left[ \tilde{\theta}, \theta \right] V \left[ \theta \right]^{-1} (\theta - E \left[ \theta \right]) \quad (60) \\
V \left[ \tilde{\theta} | \theta \right] &= V \left[ \tilde{\theta} \right] - \text{Cov} \left[ \tilde{\theta}, \theta \right] V \left[ \theta \right]^{-1} \text{Cov} \left[ \theta, \tilde{\theta} \right] . \quad (61)
\end{align*}

Predictions are obtained by generating $\tilde{\theta}$ from this conditional distribution for a set of samples $\theta, B, \sigma^2, \phi, R_v$ obtained from the joint posterior via MCMC.

6. Case of Study: Predicting Food Insecurity in an Urban Centre in Brazilian Amazonia

In this section, we detail results from our motivating application: modelling and prediction of food insecurity in a remote urban centre, Ipixuna, in the Amazonas state, Brazil.

“Food security [is] a situation that exists when all people, at all times, have physical, social and economic access to sufficient, safe and nutritious food that meets their dietary needs and food preferences for an active and healthy life” (FAO 2003, pp 313). The Food and Agriculture Organisation of the United Nations estimate that in 2016, there were 815,000,000 undernourished people in the world (http://www.fao.org/state-of-food-security-nutrition) and it is not a coincidence that the 2030 Agenda for Sustainable Development incorporated as a main goal to end hunger and prevent all forms of malnutrition by 2030 (http://www.fao.org/3/a-I7695e.pdf). The majority of undernourished people live in developing countries and in regions where there is high socioeconomic and environmental vulnerability. Brazilian Amazonia is one such region. Food insecurity describes the opposite situation, in which individual or household access to sufficient, safe and nutritious food is not a guarantee (National Research Council 2006). For policy makers, understanding the level of food insecurity in a region is crucial in the planning of interventions designed to foster development and improve the quality of life for these populations. Therefore, being able to understand the spatial structure of food insecurity and to be able to map (i.e. predict) it is highly relevant for both fundamental science and policy makers alike. Mapping intra-urban inequalities in health in developing countries has been identified as a research priority; “The power of maps is often overlooked: maps that show politicians how their specific (disaggregated) area is faring in terms of health are often powerful prompts for action” (Harpham 2009).

Ipixuna, shown in Figure 2, is a ‘jungle town’ located on the banks of the River Juruá; it is unconnected to the Brazilian road network, and is several thousand kilometers of upstream boat travel from the Amazonas state capital, Manaus. Being remote and ‘roadless’, Ipixuna exhibits very high social vulnerability and it is also prone to extreme hydro-climatic events such as floods and droughts, which pose a serious risk of harm to the local population (Parry et al 2018).
6.1. Data description

Our data were collected from 200 randomly sampled households in August 2015 (low-water dry season) and also in March 2016 (high water rainy season). The spatial distribution of these samples can be seen in Figure 2; these points have been jittered for privacy reasons: they just give a general sense of where samples were taken from. The interview responses to our questionnaire elicited information on household food insecurity over the previous month. Focussing on household food insecurity, rather than on individual food insecurity, is relevant in our setting because we wish to capture social inequalities which affect co-habitant groups of individuals.

The questionnaire contained items initially validated by the United States Department of Agriculture and additional items that are relevant in the context of Brazilian Amazonia. To be more precise, our survey instrument was based on a validated version of the above specific to Brazil: ‘The Brazilian Household Food Insecurity Scale’ Segall–Corrêa et al. (2014). We adapted this scale for the Amazon region, to include 5 additional questions designed to capture last-resort coping strategies, which our earlier pilot work showed to be indicative of severe food insecurity. This pilot work conducted in Autazes, another remote urban centre in the Amazonas state, used qualitative research including focus groups and semi-structured interviews in order to better understand regional responses/adaptations to food insecurity. In addition to the two months of data-collection in Ipixuna, we returned in May 2017 to hold a workshop with diverse citizen stakeholders. We presented and discussed exploratory results and also conducted site visits to neighbourhoods identified as particularly vulnerable. These interactions were highly beneficial in interpreting the results of our spatial models below.

In total the questionnaire contained 18 items relating to food insecurity, see Table 1 and Appendix D. Items in Section A of our questionnaire referred to the household as a whole, those in Section B referred to adults only, Section C concerned children and...
Section D included items related to the regional context of our study. The regionally-specific questions in section D were designed to measure similar aspects as contained in the general scale, but measured through common coping strategies employed in this locality.

The items with higher endorsement probability were numbers 15, 3, 1, 18, and 2, see Table 1. In the present context, endorsement simply means ‘answering with an affirmative’. This indicates that it is common that Ipixuna citizens obtain credit for eating, eat few food types, are worried that food will end, reduce meat or fish consumption, or run out of food. Of the 200 surveyed households, 25 did not have children and this led to missing data on the 6 items of associated with food insecurity in children, see Section D in Table 1.

Table 1: Summary of the food insecurity items: i) the number of missing values (#NA) and the proportion of endorsement (π) are shown for the descriptive analysis, ii) the posterior median of the discrimination parameters {\(\hat{A}_1, \hat{A}_2, \hat{A}_3\}) are shown for the confirmatory factor analysis (CIFA), and iii) the posterior median of the discrimination and easiness parameters {\(\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{c}\}) are shown for the spatial item factor analysis (SPIFA).

Full details about the sections and complete questions can be seen in the Appendix D.

| Item | Section | Question | Descriptive | CIFA | SPIFA |
|------|---------|----------|-------------|------|-------|
|      |         |          | #NA \(\pi\) \(\hat{A}_1\) \(\hat{A}_2\) \(\hat{A}_3\) \(\hat{A}_1\) \(\hat{A}_2\) \(\hat{A}_3\) \(\hat{c}\) |          |      |       |
| 1    | A       | worry that food ends | 0 0.56 1.62 - 1.79 - 0.44 |      |       |
| 2    | A       | run out of food     | 0 0.52 1.49 - - 1.87 0.21 |      |       |
| 3    | A       | ate few food types  | 0 0.64 1.68 - - 1.83 - 1.47 |      |       |
| 4    | A       | skip a meal         | 0 0.30 1.48 - 1.01 1.91 - 1.00 -0.62 |      |       |
| 5    | A       | ate less than required | 0 0.41 0.88 - 1.77 1.39 - 1.50 -0.07 |      |       |
| 6    | A       | hungry but did not eat | 0 0.24 1.26 - 1.52 1.83 - 1.51 -1.44 |      |       |
| 7    | A       | one meal per day    | 0 0.26 1.57 - - 1.82 - -0.53 |      |       |
| 8    | B       | ate few food types  | 25 0.49 1.69 - - 1.90 - -0.60 |      |       |
| 9    | B       | ate less than required | 25 0.31 1.89 - - 2.24 - -0.34 |      |       |
| 10   | B       | decreases food quantity | 25 0.36 2.16 - - 2.51 - -0.03 |      |       |
| 11   | B       | skip a meal         | 25 0.23 2.01 - - 2.54 - -1.06 |      |       |
| 12   | B       | hungry but did not eat | 25 0.20 2.11 - - 2.56 - -1.32 |      |       |
| 13   | B       | one meal per day    | 25 0.18 1.95 - - 2.45 - -1.52 |      |       |
| 14   | C       | food just with farinha | 0 0.17 0.34 1.24 0.63 1.28 -1.60 |      |       |
| 15   | C       | credit for eating   | 0 0.68 0.72 - - 0.79 - 0.62 |      |       |
| 16   | C       | borrowed food       | 0 0.14 - 1.42 - - 1.61 -1.89 |      |       |
| 17   | C       | meal at neighbors   | 0 0.17 - 0.97 - - 1.01 -1.24 |      |       |
| 18   | C       | reduced meat or fish | 0 0.54 1.28 - - 1.43 - -0.76 |      |       |

The table values are not provided directly in the text but are implied in the context and can be inferred from the description and the structure of the table.
leads to: the first factor being explained by items 3–14 and 18; the second, by items 1 and 15–17; and the third by items 2, 4–6 and 14. To perform Bayesian inference, we used standard normal priors for the easiness parameters $c_j$; standard normal priors for the discrimination parameters with exception of $\{A_{11,1}, A_{13,1}, A_{16,2}, A_{14,3}\}$ for which we used normal priors with mean $\mu = 1$ and standard deviation $\sigma = 0.45$; and an LKJ prior distribution with hyper-parameter $\eta = 1.5$ for the correlation matrix of the latent factors. The adaptive MCMC scheme had parameters $C = 0.7$ and $\alpha = 0.8$ with target acceptance probability of 0.234. We ran the Metropolis-within-Gibbs algorithm for 100,000 iterations discarding the first 50,000 iterations and storing 1 in 10 of the remaining iterations. Convergence was assessed visually; stationarity was observed from around the iteration 10,000 of the burn-in period.

The posterior median of the discrimination parameters $\{\hat{A}_{1}, \hat{A}_{2}, \hat{A}_{3}\}$ of the CIFA model is shown in Table 1. These values show that questions related to reduction of quality and quantity of food in the diet of children, items 10–12, are the top three most important items for the first factor. The second factor includes three items relating to Amazonian coping strategies (15–17) and one concerning anxiety (1). Note that using credit (15), borrowing food (16) or relying on neighbours for meals (17) are likely sources of anxiety in their own right. Finally, the third factor is related mainly to the reduction in quantity of food (2 and 4–6) and one item associated with substitution of normal foods with only toasted manioc flour, a staple carbohydrate in low-income households (14).

In order to evaluate the spatial correlation in the obtained factors, we use the empirical variogram: see Figure 3. This exploratory tool for determining the extent and form of (spatial) correlation is defined as a function of the distance $u$,

$$\hat{\gamma}(u) = \frac{1}{2} \hat{E} \left[ (w(s) - w(s + u))^2 \right].$$

The initial increasing behaviour of the variogram, mainly, observed in the first and third factors is evidence for spatial correlation in these dimensions of food insecurity.
6.3. Spatial confirmatory item factor analysis

We placed the same restrictions on the discrimination parameters for the spatial item factor as we did for the confirmatory item factor analysis. Based on the empirical variograms shown in Figure 3, we proposed three models; model 1 includes a Gaussian process in the first latent factor (SPIFA I), model 2 includes Gaussian processes in the first and third factor (SPIFA II), and model 3 includes Gaussian processes in the three factors (SPIFA III). We used the exponential correlation function to model the spatial structure of each of the Gaussian processes in our model. Spatial predictors were not included in our model because these are insufficiently finely resolved in our study area. For instance, there are only 8 census sectors (from the 2010 demographic census by the Brazilian Institute for Geography and Statistics (IBGE)) covering Ipixuna - in the future, we are planning a larger scale analysis in which spatial predictors will be included; our software package is already able to handle this case.

We used the same prior specifications as in the CIFA model for the easiness parameters $c_j$, discrimination parameters $A_{jk}$ and correlation matrix $R_v$. In addition, we used log-normal priors $\mathcal{LN}(\log(160), 0.3)$, $\mathcal{LN}(\log(80), 0.3)$ and $\mathcal{LN}(\log(80), 0.3)$ for the scale parameters $\{\phi_1, \phi_2, \phi_3\}$ of the Gaussian processes in factor 1, 2 and 3 respectively; and the log-normal prior distribution $\mathcal{LN}(\log(0.4), 0.4)$ for all the free elements of $T$. The adaptive MCMC scheme had parameters $C = 0.7$ and $\alpha = 0.8$ with target acceptance probability of 0.234. We ran the Metropolis-within-Gibbs algorithm for 300,000 iterations discarding the first 150,000 iterations and storing 1 in every 150 iterations. Convergence was again assessed visually with stationarity occurring around the iteration 40,000 of the burnin period. Usually, mixing is slower for the elements of $T$ and the scale parameter $\phi$ of the Gaussian processes.

Figure 3: Empirical variogram $\hat{\gamma}(u)$ for each latent factor: the points represent the empirical values and the lines the smoothed version of the empirical variogram.
We compared these three models using the Deviance Information Criterion (DIC), see Table 2. We can see that the classical confirmatory model (CIFA) has lowest effective number of parameters (328.14); this model has independent random effects only. In contrast, the spatial models include both independent and spatial random effects as explained in Section 2. The DIC for the three spatial models is lower than that for CIFA, hence by this measure, it is statistically advantageous in terms of model fit to allow the factors to be spatially correlated. Of the three spatial models, SPIFA III, the model including Gaussian processes in all three factors, has the best performance in terms of DIC (2195.529). Hence in the remainder of this section, we focus on the results from this model. The trace-plots of for the elements of $T$ and the scale parameters $\phi$ of the Gaussian processes for our selected model can be seen on Figure 10 and 11 respectively. Additional (representative) trace-plots for random selected parameters can be seen in Appendix E.

**Table 2:** Deviance Information Criteria (DIC) for the Proposed Models: without spatial correlation (CIFA), with spatial correlation in factor 1 (SPIFA I), with spatial correlation in factor 1 and 3 (SPIFA II) and with spatial correlation in all factors (SPIFA III).

| Model    | Posterior Mean Deviance | Effective Number of Parameters | DIC     |
|----------|-------------------------|-------------------------------|---------|
| CIFA     | 1894.228                | 328.1377                      | 2222.365|
| SPIFA I  | 1865.85                 | 334.228                       | 2200.078|
| SPIFA II | 1862.156                | 339.2756                      | 2201.432|
| SPIFA III| 1856.354                | 339.1752                      | 2195.529|

The posterior medians of the discrimination parameters $\{\hat{A}_1, \hat{A}_2, \hat{A}_3\}$ for the selected model (SPIFA III) are shown in Table 1 under the column of SPIFA. We can see that the median of the obtained parameters have a broadly a similar structure as for the CIFA model, so their interpretation is as discussed in the previous section; notice that most of the discrimination parameters are higher for the SPIFA model. The last column of Table 1 shows the posterior median of the easiness parameters $\hat{c}$; note the items with high easiness are those most frequently answered with an affirmative (‘endorsed’). This column shows that eating few food types (item 3), obtaining credit for eating (item 15) and worrying that food will end (item 1) are the most common behaviours in the population of Ipixuna. Borrowing food (item 16), eating food just with farinha (item 14), having children with one meal per day (item 13) and feeling hungry but do not eat (item 6 and 12) are less common.
Figure 4: Median of the predicted latent factors of food insecurity.

Figure 4 shows the posterior median of each of the three factors over our study area. The left plot shows that the first factor has a strong spatial structure; the respective posterior median of the standard deviation and scale parameters of the associated Gaussian process are $\hat{T}_{1,1} = 0.465$ and $\hat{\phi}_1 = 214$ meters. Examining the middle plot, for the second factor, it can be seen that the spatial structure is not as strong as the first factor. The respective parameters of the associated Gaussian process have posterior medians $\hat{T}_{2,2} = 0.205$ and $\hat{\phi}_2 = 83.6$. The right hand plot, referring to the third factor, shows moderate spatial structure with similar median posterior estimates as the second factor: $\hat{T}_{3,3} = 0.287$ and $\hat{\phi}_3 = 78.8$.

Examining the obtained maps of food insecurity for the first factor, we can see there are lower levels of food insecurity around the center of the study area and more severe food insecurity around the locations A (71.695° W, 7.045° S), B (71.69° W, 7.045° S), C (71.698° W, 7.052° S) and D (71.685° W, 7.06° S). In this city, location C refers to the flood-prone, politically marginalized and poor neighbourhood of Turrufão. Housing is on stilts, and there is no sanitation and poor provision of public services. Point A refers to the flood-prone, poorest part of another marginalized neighbourhood, Multirão Novo. These households are also located at the edge of a large stream called Igarapé Turrufão. The area B is a relatively new neighbourhood, Morro dos Encanados, which is poor and prone to surface flooding from rainfall. Area D is at the edge of the River Jurú and is highly flood-prone. It is also a relatively new and very poor neighbourhood called Bairro da Várzea. Hence, the common characteristic among these locations is that they are poor, marginalized and mostly flood-prone neighbourhoods on the peri-urban fringe. Most of the heads of households in these neighbourhoods are rural-urban migrants (often relatively recent), and many of their livelihoods are still based in rural areas. These relatively large areas are capturing indications of relatively severe food insecurity, yet without apparent anxiety and a distinct absence of some coping strategies: borrowing food, eating in other households or accessing credit.
With respect to the map of the second factor, we can identify higher levels of food insecurity around location E (71.69° W, 7.048° S), F (71.686° W, 7.048° S) and G (71.698° W, 7.058° S). Location E covers a large and older area of the town, covering proportions of two neighbourhoods: Bairro do Cemitério and Multirão Velho. They are not flood prone and not so marginalized and poor, though certainly not wealthy. It is plausible that this factor captures more moderate food insecurity and coping strategies associated with higher levels of horizontal social capital and access to credit. Area F is the larger part of Morro dos Encanados (see above). Area G is another flood-prone peri-urban neighbourhood on the other side of the River Jurua, by the name of Bairro da Ressaca.

In the map of the third factor of food insecurity, we can see areas of severe food insecurity around H (71.693° W, 7.045° S), I (71.687° W, 7.057° S) and J (71.688° W, 7.056° S). Area H covers the border between two poor, peri-urban neighbourhoods: Bairro da Liberdade and Multirão Novo. Point I is an area of Morro dos Encanados. Area J is the beginning of the peri-urban, flood-prone region and the poor area, Bairro da Várzea.

While spatial plots of the posterior median tell us where food insecurity is high and low on average, we ideally also need to take into account the spatial sampling design, since we will be better able to estimate food insecurity where we have more data points. One such measure are exceedance probabilities: the posterior probability that the factor exceeds a given threshold; this takes into account both the mean and the variance of the factor at each location.

In figure Figure 5, we show the probability that the latent factor is greater than zero in order to identify areas over and below average. It so happens that in the present case, the pattern of high and low food insecure areas remain similar with respect to the maps of the median for each factor.

![Figure 5: Exceedance probabilities Pr (θk(s) > 0) of the latent factors of food insecurity.](image-url)
Identifying these areas of high (and also low) food insecurity is of relevance for future research in this area, for example: exploring the social and environmental (e.g. household flood risk due to elevation) determinants of vulnerability to food insecurity. Understanding the spatial-variation of food insecurity at local (e.g. neighbourhood or street) scales will also allow us to continue our dialogue with local government and other stakeholders around which are the priority areas for intervention and what type(s) of intervention should be deployed in order to reduce the risk of food insecurity.

7. Discussion

In this work we have developed a new extension of item factor analysis to the spatial domain, where the latent factors are allowed to be spatially correlated. Our model allows for the inclusion of predictors to help explain the variability of the factors. These developments allow us to make prediction of the latent factors at unobserved locations as shown in our case of study of food insecurity in the Brazilian Amazon. We solved the issues of identifiability and interpretability by employing a similar strategy as for confirmatory item factor analysis in order to obtain an identifiable model, and by standardizing the resulting factors after inference. Our model has been successfully implemented in an open source R package.

Since item factor analysis is used across such a wide range of scientific disciplines, we believe that our model and method of inference will be important for generating and investigating many new hypotheses. For instance, it could be used to model socio-economic status.

Computationally, our model is more efficient compared to a model where the spatial structure is used at the level of the response variables. By including spatial structure at the level of the factors, we reduce the computational cost from $O(q^3n^3)$ to $O(m^3n^3)$ where the number of items ($q$) is usually much greater than the number of latent factors ($m$). However, computational expense remains a limitation of our proposed model, which would still be intractable for large datasets $O(m^3n^3)$. On the other hand, our model does not require a spatial dataset, and the cost for a general (parametric) covariance structure for the Gaussian processes would remain as $O(m^3n^3)$.

For larger datasets, we can reduce the computational burden by using alternatives to the Gaussian process. For example, we could use spatial basis functions (Fahrmeir et al. 2004), nearest neighbour Gaussian processes (Datta et al. 2016) or stochastic partial differential equations (Lindgren et al. 2011) to reduce the cost. This is not so obvious because some of the nice properties of these processes can be lost when working with multivariate models.

Our model can be extended to the spatio-temporal domain, though again with increased computational expense, depending on the chosen parameterisation of the spatio-temporal correlation. A more complex extension of our model would allow the use of binary, ordinal and continuous items and would also allow predictors to be related in a non-linear way to the latent factors. These extensions would allow us to answer more complex research questions and would also improve prediction of the latent factors, see Appendix F. Extensions to other distributional assumptions (e.g. heavier
tailed densities) are also possible if one desires to trade the convenience conjugacy for realism; the Gaussian model fitted our particular dataset well.

Acknowledgements

This research was funded by a Future Research Leaders Fellowship to LP (ES/K010018/1), the Newton Fund/FAPEAM (ES/M011542/1), Brazil’s CNPq (CsF PVE 313742/2013-8) and CAPES-ProAmazonia (Projeto 3322-2013), and the European Commission Horizon 2020 RISE programme (Project 691053 - ODYSSEA). Erick Chacon’s studentship was funded by the Faculty of Health and Medicine, Lancaster University.

References

Andrieu, C. and Thoms, J. (2008). A tutorial on adaptive MCMC. *Statistics and Computing*, 18(4):343–373.

Bafumi, J., Gelman, A., Park, D. K., and Kaplan, N. (2005). Practical issues in implementing and understanding Bayesian ideal point estimation. *Political Analysis*, 13(2):171–187.

Battersby, J. (2011). Urban food insecurity in Cape Town, South Africa: An alternative approach to food access. *Development Southern Africa*, 28(4):545–561.

Bock, R. D., Gibbons, R., and Muraki, E. (1988). Full-Information Item Factor Analysis. *Applied Psychological Measurement*, 12(3):261–280.

Briggs Myers, I. and Myers, P. B. (1980). *Gifts Differing: Understanding Personality Type*. Davies-Black Pub.

Cai, L. (2010a). High-dimensional Exploratory Item Factor Analysis by A Metropolis-Hastings Robbins-Monro Algorithm. *Psychometrika*, 75(1):33–57.

Cai, L. (2010b). Metropolis-Hastings Robbins-Monro Algorithm for Confirmatory Item Factor Analysis. *Journal of Educational and Behavioral Statistics*, 35(3):307–335.

Carlson, S. J., Andrews, M. S., and Bickel, G. W. (1999). Measuring food insecurity and hunger in the United States: development of a national benchmark measure and prevalence estimates. *The Journal of Nutrition*, 129(2S Suppl):510S–516S.

Chalmers, R. P. (2012). mirt: A Multidimensional Item Response Theory Package for the R Environment. *Journal of Statistical Software*, 48(6).

Chalmers, R. P. (2015). Extended Mixed-Effects Item Response Models With the MH-RM Algorithm. *Journal of Educational Measurement*, 52(2):200–222.

Chen, C. M. and Duh, L. J. (2008). Personalized web-based tutoring system based on fuzzy item response theory. *Expert Systems with Applications*, 34(4):2298–2315.
Chen, C. M., Liu, C. Y., and Chang, M. H. (2006). Personalized curriculum sequencing utilizing modified item response theory for web-based instruction. *Expert Systems with Applications*, 30(2):378–396.

Datta, A., Banerjee, S., Finley, A. O., and Gelfand, A. E. (2016). Hierarchical Nearest-Neighbor Gaussian Process Models for Large Geostatistical Datasets. *Journal of the American Statistical Association*, 111(514):800–812.

De Ayala, R. J. (2013). *The theory and practice of item response theory*. Guilford Publications.

de Jong, M. G., Steenkamp, J.-B. E., Fox, J.-P., and Baumgartner, H. (2008). Using Item Response Theory to Measure Extreme Response Style in Marketing Research: A Global Investigation. *Journal of Marketing Research (JMR)*, 45(1):104–115.

Downing, S. M. (2003). Item response theory: applications of modern test theory in medical education. *Medical Education*, 37(8):739–745.

Drachler, M. L., Marshall, T., Carlos, J., and Leite, D. C. (2007). A continuous-scale measure of child development for population-based epidemiological surveys: a preliminary study using Item Response Theory for the Denver Test. *Pediatric and Perinatal Epidemiology*, 21:138–53.

Edelen, M. O. and Reeve, B. B. (2007). Applying item response theory (IRT) modeling to questionnaire development, evaluation, and refinement. *Quality of Life Research*, 16(SUPPL. 1):5–18.

Erosheva, E. A. and Curtis, S. M. (2011). Dealing with rotational invariance in bayesian confirmatory factor analysis. *Department of Statistics, University of Washington, Seattle, Washington, USA*.

Fahrmeir, L., Kneib, T., and Lang, S. (2004). Penalized Structured Additive Regression for Space-Time Data: A Bayesian Perspective. *Statistica Sinica*, 14:731–761.

Fanshawe, T. R. and Diggle, P. J. (2012). Bivariate geostatistical modelling: A review and an application to spatial variation in radon concentrations. *Environmental and Ecological Statistics*, 19(2):139–160.

FAO (2003). Food and agriculture organization of the united nations. trade reforms and food security: Conceptualizing the linkages. Technical report, Food and Agriculture Organization of the United Nations.

Fiori, M., Antonietti, J.-P., Mikolajczak, M., Luminet, O., Hansenne, M., and Rossier, J. (2014). What Is the Ability Emotional Intelligence Test (MSCEIT) Good for? An Evaluation Using Item Response Theory. *PLoS ONE*, 9(6):e98827.

Frichot, E., Schoville, S., Bouchard, G., and François, O. (2012). Correcting principal component maps for effects of spatial autocorrelation in population genetic data. *Frontiers in Genetics*, 3(NOV):1–9.
Froelich, A. G. and Jensen, H. H. (2002). Dimensionality of the USDA food security index. *Unpublished Manuscript.*

Funk, J. L. and Rogge, R. D. (2007). Testing the Ruler With Item Response Theory: Increasing Precision of Measurement for Relationship Satisfaction With the Couples Satisfaction Index. *Journal of Family Psychology, 21*(4):572–583.

Garrett, J. L. and Ruel, M. T. (1999). Are determinants of rural and urban food security and nutritional status different? Some insights from Mozambique. *World Development,* 27(11):1955–1975.

Gelfand, A. E., Schmidt, A. M., Banerjee, S., and Sirmans, C. F. (2004). Nonstationary multivariate process modeling through spatially varying coregionalization. *Test,* 13(2):263–312.

Geweke, J. and Zhou, G. (1996). Measuring the Pricing Error of the Arbitrage Pricing Theory. *Review of Financial Studies,* 9(2):557–587.

Gray-Little, B., Williams, V. S. L., and Hancock, T. D. (1997). An Item Response Theory Analysis of the Rosenberg Self-Esteem Scale. *Personality and Social Psychology Bulletin,* 23(5):443–451.

Hambleton, R. and Swaminathan, H. (1989). *Item Response Theory: Principles and Applications.* Evaluation in education and human services. Kluwer-Nijhoff Pub.

Harpham, T. (2009). Urban health in developing countries: what do we know and where do we go? *Health & place,* 15(1):107–116.

Holmes, C. C. and Held, L. (2006). Bayesian auxiliary variable models for binary and multinomial regression. *Bayesian Analysis,* 1(1 A):145–168.

Keirsey, D. (1998). *Please Understand Me 2.* Prometheus Nemesis.

Laurens, K. R., Hobbs, M. J., Sunderland, M., Green, M. J., and Mould, G. L. (2012). Psychotic-like experiences in a community sample of 8000 children aged 9 to 11 years: An item response theory analysis. *Psychological Medicine,* 42(7):1495–1506.

Lewandowski, D., Kurowicka, D., and Joe, H. (2009). Generating random correlation matrices based on vines and extended onion method. *Journal of Multivariate Analysis,* 100(9):1989–2001.

Lindgren, F., Rue, H., and Lindström, J. (2011). An explicit link between gaussian fields and gaussian markov random fields: The stochastic partial differential equation approach. *Journal of the Royal Statistical Society. Series B: Statistical Methodology,* 73(4):423–498.

Merkle, E. C. (2011). A Comparison of Imputation Methods for Bayesian Factor Analysis Models. *Journal of Educational and Behavioral Statistics,* 36(2):257–276.
National Research Council (2006). Item Response Theory and Food Insecurity. In Wunderlich, G. S. and Norwood, J. L., editors, Food Insecurity and Hunger in the United States: An Assessment of the Measure, chapter 5. The National Academies Press, Washington, DC.

Osgood, D. W., McMorris, B. J., and Potenza, M. T. (2002). Analyzing multiple-item measures of crime and deviance I: Item response theory scaling. Journal of Quantitative Criminology, 18(3):267–296.

Parry, L., Davies, G., Almeida, O., Frausin, G., de Moraé’s, A., Rivero, S., Filizola, N., and Torres, P. (2017). Social Vulnerability to Climatic Shocks Is Shaped by Urban Accessibility. Annals of the American Association of Geographers, 4452(October):1–19.

Parry, L. T. W., Davies, G., Almeida, O., Frausin Bustamante, G. G., de Moraé’s, A., Rivero, S., Filizola, N., and Torres, P. (2018). Social vulnerability to climatic shocks is shaped by urban accessibility. Annals of the Association of American Geographers, 108(1):125–143.

Pebesma, E. (2018). sf: Simple Features for R. R package version 0.6-3.

Piquero, A. R., MacIntosh, R., and Hickman, M. (2000). Does Self-Control Affect Survey Response? Applying Exploratory, Confirmatory, and Item Response Theory Analysis To Grasmick Et Al.’S Self-Control Scale. Criminology, 38(3):897–930.

Saha, T. D., Chou, S. P., and Grant, B. F. (2006). Toward an alcohol use disorder continuum using item response theory: Results from the National Epidemiologic Survey on Alcohol and Related Conditions. Psychological Medicine, 36(7):931–941.

Segall-Corrêa, A. M., Marin-León, L., nonez, H. M.-Q., and Pérez-Escamilla, R. (2014). Refinement of the Brazilian household food insecurity measurement scale: Recommendation for a 14-item EBIA. Revista de Nutrição, 27(2):241–251.

Sharp, C., Goodyer, I. M., and Croudace, T. J. (2006). The Short Mood and Feelings Questionnaire (SMFQ): A unidimensional item response theory and categorical data factor analysis of self-report ratings from a community sample of 7-through 11-year-old children. Journal of Abnormal Child Psychology, 34(3):379–391.

Shryane, N. M., Corcoran, R., Rowse, G., Moore, R., Cummins, S., Blackwood, N., Howard, R., and Bentall, R. P. (2008). Deception and false belief in paranoia: Modelling Theory of Mind stories. Cognitive Neuropsychiatry, 13(1):8–32.

Spiegelhalter, D. J., Best, N. G., Carlin, B. P., and Van Der Linde, A. (2002). Bayesian measures of model complexity and fit. Journal of the Royal Statistical Society. Series B: Statistical Methodology, 64(4):583–616.

Tekwe, C. D., Carter, R. L., Cullings, H. M., and Carroll, R. J. (2014). Multiple indicators, multiple causes measurement error models. Statistics in Medicine, 33(25):4469–4481.
Appendices

Appendix A: Spatial item factor analysis

A.1. Scaling aliasing

Restricting the variances of the latent abilities to one, \( \text{diag}(\Sigma_\theta) = 1 \), is the same as restricting the variances of the residual term \( v(s) \) because

\[
V[b_k^\top x(s) + w_k(s) + v_k(s)] = 1 \tag{62}
\]

for \( k = 1, \ldots, m \) implies that

\[
V[v_k(s)] = 1 - V[b_k^\top x(s) + w_k(s)] = \sigma^2_{v_k}. \tag{63}
\]

More generally, the constrain \( \text{diag}(\Sigma_\theta) \) is equivalent to set the covariance matrix of \( v(s) \) as \( \Sigma_v = D_1 R_v D_1 \), where \( D_1 \) is a diagonal matrix with elements \( \sigma_{v_k} \). Then the covariance matrix of the latent abilities \( \theta(s) \) is expressed as

\[
V[\theta(s)] = V[B^\top x(s)] + V[w(s)] + D_1 R_v D_1, \tag{64}
\]

the problem with this restriction is that \( \sigma_{v_k} \) need to be known.

Inference can be attained by introducing arbitrary values. Consider the transformation \( D_2 = DD_1^{-1} \), where \( D \) is a diagonal matrix with arbitrary values, then we can define

\[
\tilde{\theta}(s) = a_j^\top D_2 \theta(s) = a_j^\top \theta(s). \tag{65}
\]

Note that under this transformation, the variance of the new latent variable \( \tilde{\theta}(s) = D_2 \theta(s) \) is defined as

\[
V[\tilde{\theta}(s)] = V[B^\top x(s)] + V[w(s)] + D_2 R_v D_2, \tag{66}
\]

where \( B^\top = D_2 B^\top \) and \( \tilde{w}(s) = D_2 w(s) \). It can be seen that defining an arbitrary diagonal matrix \( D \) still allows us to make inference given that the marginal variances of \( \tilde{\theta}(s) \) are still restricted. In this case, the variances are equal to the squared values of the diagonal matrix \( D_2 \), \( \text{diag}(\Sigma_{\tilde{\theta}}) = \text{diag}(D_2^2) \).

If we choose \( D = I \); then \( D_2 = D_1^{-1} \), \( \tilde{\theta}(s) = D_1^{-1} \theta(s) \), \( \text{diag}(\Sigma_{\tilde{\theta}}) = \text{diag}(D_1^{-2}) \) and

\[
V[\tilde{\theta}(s)] = V[B^\top x(s)] + V[w(s)] + R_v. \tag{67}
\]

This transformation allows us to make inference, but the interpretation of the transformed parameters \( \tilde{a}_j \) are not the same as in the classical item factor analysis because the marginal variances of \( \tilde{\theta}(s) \) are not equal to 1, \( \text{diag}(\Sigma_{\tilde{\theta}}) \neq 1 \). To recover the interpretation of the discrimination parameters, we simply compute the standard deviations of \( \tilde{\theta}(s) \) after sampling to obtain the estimated \( Q = \tilde{D}_1^{-1} \), and back-transformed \( a_j = Q \tilde{a}_j \) and \( \theta(s) = Q^{-1} \tilde{\theta}(s) \) as explained in Section 4.4.
Appendix B: Markov chain Monte Carlo scheme sampling

B.1. Posterior of auxiliary variables

We show the details of how to obtain Equation 31 to specify the posterior distribution of the auxiliary variables of our model.

\[
\Pr(z | y_{\text{obs}}, c, a, \theta) = \Pr(z_{\text{obs}}, z_{\text{mis}} | y_{\text{obs}}, c, a, \theta)
\]

\[
\propto \Pr(z_{\text{obs}}, z_{\text{mis}} | c, a, \theta) \Pr(y_{\text{obs}} | z_{\text{obs}}, z_{\text{mis}})
\]

\[
\propto \Pr(z_{\text{obs}} | c, a, \theta) \Pr(z_{\text{mis}} | z_{\text{obs}}, c, a, \theta) \Pr(y_{\text{obs}} | z_{\text{obs}}, z_{\text{mis}})
\]

\[
\propto \Pr(z_{\text{obs}} | c, a, \theta) \Pr(z_{\text{mis}} | c, a, \theta) \Pr(y_{\text{obs}} | z_{\text{obs}}).
\]  

(68)

The last line is obtained because \(Z_{\text{mis}}\) and \(Z_{\text{obs}}\) are conditionally independent given \(\{c, a, \theta\}\) and because \(Y_{\text{obs}}\) is conditionally independent of \(Z_{\text{mis}}\) given \(Z_{\text{obs}}\).

Appendix C: Alternative Sampling Schemes

C.1. Alternative sampling scheme using marginalization

In Section 4.1, we defined the Bayesian model such us the conditional probability \(\Pr(z | \theta, c, a)\) plays a main role to derive the posterior conditional distributions of the associated parameters. This was convenient to obtain the analytical expression of the conditional posterior distributions; however, convergence can be slow due to nested relationship in the updates of the Gibbs sampling. An Alternative approach to achieve faster convergence, in terms of iterations, is to marginalize some parameters such as the nested relationship is reduced.

Considering the definition of the auxiliary variables in Equation 18, we can see that any element from the set \(\{c, \beta, w, v\}\) can be marginalized due to conjugacy Gaussian properties. Let \(\alpha\) be the subset of parameters that we wish to marginalize and \(\gamma\) the subset of remaining parameters which will not be marginalized. Additionally, let \(X_\alpha\) and \(X_\gamma\) be the associated design matrix, and let \(\Sigma_\alpha\) and \(\Sigma_\gamma\) be the associated covariance matrices. Then the auxiliary variables can be expressed as

\[
Z = X_\gamma \gamma + X_\alpha \alpha + \epsilon,
\]  

(69)

where at least one of the design matrices \(X_\beta\) and \(X_\gamma\) will depend of the restricted discrimination parameters \(A^*\). Then, composition sampling, as shown in Holmes and Held (2006), can be used to sample from the posterior distribution of the model using the following equivalence

\[
\Pr(z, \alpha, \gamma, a, T, \phi, R_v | y) = \Pr(z, \gamma, a, T, \phi, R_v | y) \Pr(\alpha | z, \gamma, a, T, \phi, R_v),
\]  

(70)

such as the first term of the right hand side does not depend of the set of parameters \(\alpha\). This way convergence is expected to be faster and the parameters included in \(\alpha\) can be
simulated by \textit{composition} sampling once the convergence of the remaining parameters is ensured.

Sampling from the marginalized parameters $\alpha$ can be done straight away because the conditional distribution given $\gamma$ is a Gaussian density,

$$
\Pr(\alpha \mid z, \gamma, a, T, \phi, R_v) \propto \mathcal{N}(z \mid X_\gamma \gamma + X_\alpha \alpha, I_{ny}) \mathcal{N}(\alpha \mid 0, \Sigma_\alpha),
$$

with mean and covariance:

$$
\Sigma_{\alpha \mid z} = (X_\alpha^\top X_\alpha + \Sigma_\alpha^{-1})^{-1}
$$

$$
\mu_{\alpha \mid z} = \Sigma_{\alpha \mid z} X_\alpha^\top (z - X_\gamma \gamma).
$$

In some cases, computational advantage can be gained considering that

$$
(X_\alpha^\top X_\alpha + \Sigma_\alpha^{-1})^{-1} = \Sigma_\alpha - \Sigma_\alpha X_\alpha^\top (X_\alpha \Sigma_\alpha X_\alpha^\top + I)^{-1} X_\alpha \Sigma_\alpha.
$$

Obtaining posterior samples for $\{z, \gamma, a, T, \phi, R_v\}$ is more complicated, but can be achieved using Metropolis within Gibbs sampling. For this, we should notice that

$$
\Pr(z, \gamma, a, T, \phi, R_v \mid y) \propto \Pr(y \mid z) \Pr(z \mid \gamma, a, T, \phi, R_v) \Pr(\gamma) \Pr(\alpha)
$$

$$
\Pr(T) \Pr(\phi) \Pr(R).
$$

Hence, using Equation 69, the conditional posterior for $Z$ is

$$
\Pr(z \mid y, \gamma, a, T, \phi, R_v) \propto \mathcal{N}(z \mid X_\gamma \gamma, X_\alpha \Sigma_\alpha X_\alpha^\top + I_{ny}) \prod_{i,j} \Pr(y_{ij} \mid z_{ij})
$$

which is a truncated multivariate normal distribution. Unfortunately, sampling can not be done directly, but Gibbs sampling can be used taking into advantage that the conditional posterior of the marginalized parameters $\alpha$ given all the auxiliary variables except $Z_k$, $\Pr(\alpha \mid z_{-k}, \gamma, a, T, \phi, R_v)$, is a Normal distribution with covariance and mean:

$$
\Sigma_{\alpha \mid z_{-k}} = \Sigma_{\alpha \mid z} + \frac{\Sigma_{\alpha \mid z} x_{ak} x_{ak}^\top \Sigma_{\alpha \mid z}}{1 - h_{kk}}
$$

$$
\mu_{\alpha \mid z_{-k}} = \mu_{\alpha \mid z} - \frac{\Sigma_{\alpha \mid z} x_{ak}}{1 - h_{kk}} (z_k - x_{\beta k} \beta_k - x_{\alpha k} \mu_{\alpha \mid z}),
$$

where $h_{kk} = x_{\alpha k}^\top \Sigma_{\alpha \mid z} x_{\alpha k}$. Then, we can sample from the leave-one-out marginal predictive densities,

$$
\Pr(z_k \mid z_{-k}, y_k, \gamma, a, T, \phi, R_v) = \int \Pr(z_k \mid \alpha, y_k, \gamma, a) \Pr(\alpha \mid z_{-k}, \gamma, a, T, \phi, R_v) \, d\alpha,
$$

being proportional to

$$
\mathbb{1}_{(z_k > 0)} \mathbb{1}_{(z_k \le 0)} \int \mathcal{N}(z_k \mid x_{\gamma k}^\top \gamma + x_{\alpha k}^\top \alpha, 1) \Pr(\alpha \mid z_{-k}, \gamma, a, T, \phi, R_v) \, d\alpha
$$

35
which are univariate Normal truncated densities,
\[ \propto \mathcal{N} \left( x_{\alpha_k}^T \gamma + x_{\alpha_k}^T \mu_{\alpha|z} - w_k(z_k - x_{\alpha_k}^T \gamma - x_{\alpha_k}^T \mu_{\alpha|z}), 1 + w_k \right) 1_{(z_k > 0)} 1_{(z_k \leq 0)}, \]  
(80)
where \( w_k = h_{kk}/(1-h_{kk}) \). As explained in Holmes and Held (2006), each time a sample \( z_k \) is drawn, the conditional mean \( \mu_{\alpha|z} \) must be updated. Denoting \( S = \Sigma_{\alpha|z} X_{\alpha}^T \), the conditional mean can be expressed as \( \mu_{\alpha|z} = S^{-1} z - S \gamma \), and it can efficiently be updated as
\[
\begin{align*}
\mu_{\alpha|z}^{\text{new}} &= S^{-1} z_i^{\text{new}} - S z_i - S \gamma_i \\
\mu_{\alpha|z}^{\text{old}} &= S^{-1} z_i^{\text{old}} - S z_i - S \gamma_i \\
\end{align*}
(81)
(82)
(83)
(84)

Finally, because we do not get analytically expressions for the conditional distributions of remaining parameters \( \{a, T, \phi, R_v\} \), we can use Metropolis-Hasting or others samplers like Hamiltonian Monte Carlo to obtain draws from them. The posterior is only defined up to a constant of proportionality
\[ \Pr (a, T, \phi, R_v | \gamma, z) \propto \Pr (z | \gamma, T, \phi, R_v) \Pr (a) \Pr (T) \Pr (\phi) \Pr (R_v). \]  
(85)
Note that an adequate transformation will be required to sample these parameters as explained in Section 4.3.6.

C.2. Marginalizing the Gaussian process and individual random effect

In the spatial item factor analysis, it seems reasonable to desired to marginalized the more high-dimensional terms like the multivariate Gaussian process \( w \) and the multivariate residual term \( v \). In this case, the marginalized parameters are defined as \( \alpha = (w^T, v^T)^T \) with associated design matrix \( X_{\alpha} = (A^T \otimes I_n, A^* \otimes I_n) \). The remaining parameters would be \( \gamma = (c^T, \beta^T)^T \) with design matrix \( X_{\gamma} = (I_q \otimes 1_n, A^* \otimes X) \). The covariance matrix of these collections of parameters are obtained as \( \Sigma_{\alpha} = \text{diag}(\Sigma_w, \Sigma_v) \) and \( \Sigma_{\gamma} = \text{diag}(\Sigma_c, \Sigma_\beta) \). Given these definitions, it can be noticed that the some of the terms required for the sampling are
\[
\begin{align*}
X_{\gamma}^T \gamma &= (I_q \otimes 1_n) c + (A^* \otimes X) \beta \\
X_{\alpha} \Sigma_{\alpha} X_{\alpha}^T &= (A^T \otimes I_n)(\sum_{k=1}^m w_k)(T^T A^* \otimes I_n) + (A^* R_v A^T \otimes I_n), \\
X_{\alpha}^T X_{\alpha} &= \begin{pmatrix} T^T A^* A^T & T^T A^* A^* \\ A^* A^T & A^* A^* \end{pmatrix} \otimes I_n, \\
\end{align*}
(86)
(87)
and also
\[
X_{\alpha}^T (z - X_{\gamma}) = \begin{pmatrix} (T^T A^* \otimes I_n)(z - X_{\gamma}) \\
(A^* \otimes I_n)(z - X_{\gamma}) \end{pmatrix} = \begin{pmatrix} \text{vec}((Z - 1_n c^T - XBA^*)A^T) \\
\text{vec}((Z - 1_n c^T - XBA^*)A^*) \end{pmatrix}. \\
(88)
(89)
\]
As mentioned before, we can take advantage of Equation 74 and additionally reduce the dimension of the computational cost considering that

\[
X_a \Sigma_a X_a^T + I_{nq} = (A^T \otimes I_n) \Sigma_w (T^T A^T \otimes I_n) + (A^T R_v A^T \otimes I_n) + I_{nq} = (A^T \otimes I_n) \Sigma_w (T^T A^T \otimes I_n) + ((A^T R_v A^T + I_q) \otimes I_n),
\]

that the inverse of this is

\[
(X_a \Sigma_a X_a^T + I_{nq})^{-1} = ((A^T \otimes I_n) \Sigma_w (T^T A^T \otimes I_n) + (A^T R_v A^T \otimes I_n) + I_{nq})^{-1} = ((A^T \otimes I_n) \Sigma_w (T^T A^T \otimes I_n) + (A^T R_v A^T + I_q) \otimes I_n)^{-1} = (A^T R_v A^T + I_q)^{-1} \otimes I_n \Sigma_w (T^T A^T (A^T R_v A^T + I_q)^{-1} \otimes I_n)
\]

and that the determinant is

\[
\det(X_a \Sigma_a X_a^T + I_{nq}) = \det((A^T \otimes I_n) \Sigma_w (T^T A^T \otimes I_n) + (A^T R_v A^T + I_q) \otimes I_n) = \det((A^T R_v A^T + I_q) \otimes I_n) \det(((A^T R_v A^T + I_q)^{-1} \otimes I_n) \Sigma_w (T^T A^T \otimes I_n) + I_{nq})
\]

\[
= \det((A^T R_v A^T + I_q) \otimes I_n) \det(((T^T A^T \otimes I_n)((A^T R_v A^T + I_q)^{-1} \otimes I_n) (A^T \otimes I_n) \Sigma_w + I_{nm})
\]

\[
= \det((A^T R_v A^T + I_q) \otimes I_n) \det(((T^T A^T (A^T R_v A^T + I_q)^{-1} \otimes I_n) \Sigma_w + I_{nm})
\]

C.3. Marginalizing all the possible set of parameters

Let \( \alpha = (c^T, \beta^T, w^T, v^T)^T \) denote the collection of model terms that will be marginalized with associated design matrix \( X_a = (I_q \otimes I_n, A^* \otimes X, A^T \otimes I_n, A^* \otimes I_n) \). The covariance matrix of this collection of parameters is obtained as \( \Sigma_{\alpha} = \text{diag}(\Sigma_c, \Sigma_\beta, \Sigma_w, \Sigma_v) \). Then \( \gamma \) would be an empty set and will simply be removed from the expressions shown in Section C.1.

The sampling follows the explanation presented in Section C.1, but it is worth to notice that

\[
X_a^T X_a = \begin{pmatrix}
I_q \otimes 1_n^T 1_n & A^* \otimes 1_n^T X & A^T \otimes 1_n^T 1_n & A^* \otimes 1_n^T 1_n \\
A^T \otimes X^T 1_n & A^* A^* \otimes X^T X & A^T A^T \otimes X^T & A^* A^* \otimes X^T \\
T^T A^T \otimes 1_n & T^T A^T A^* \otimes X & T^T A^T A^T \otimes I_n & T^T A^T A^* \otimes I_n \\
A^T \otimes I_n & A^* A^* \otimes X & A^T A^T \otimes I_n & A^* A^* \otimes I_n
\end{pmatrix}
\]

(94)

\[
X_a^T z = \begin{pmatrix}
(I_q \otimes 1_n^T) z \\
A^T \otimes X^T) \\
T^T A^T \otimes I_n) z \\
A^T \otimes I_n) z
\end{pmatrix} = \begin{pmatrix}
\text{vec}(1_n^T Z) \\
\text{vec}(X^T Z A^*) \\
\text{vec}(Z A^T) \\
\text{vec}(Z A^*)
\end{pmatrix}
\]

(95)
Appendix D: Full Questions Used in Survey

In this section, we give an explanation and translation of the 18 questions used in our ‘Food Insecurity Survey of Road-less Urban Areas and Surrounding Rural Areas in Amazonas (2015-16)’ by researchers from Lancaster University, the Oswaldo Cruz Foundation (Fiocruz) and the Federal Universities of Pará (UFPA) and Amazonas (UFAM).

The following questions were used in interviews with heads of households regarding their perceptions of food insecurity. These questions are based on the Brazilian Food Insecurity Scale (Segall-Corrêa et al., 2014), though modified to reflect a 30 day (rather than 3-month) time period and with slight changes in wording to reflect the fact that cash is not the only means to acquire food in the Amazonian context. Note that the English is a ‘back-translation’ of what was really asked.

Section A:

Portuguese: Nos últimos 30 dias, ou seja, desde o dia _______ (mesmo dia atual) do mês de _______ (1 mês atrás): English: During the past 30 days:

**Question 1:** Portuguese: Vocês, deste domicílio, já tiveram a preocupação de que os alimentos acabassem antes de poderem comprar ou receber mais comida? English: Were you, in this household, worried that you would run out of food before being able to buy or receive more food?

**Question 2:** Portuguese: Os alimentos acabaram antes que vocês tivessem condições para adquirir mais comida? English: Did you run out of food before having the means to acquire more?

**Question 3:** Portuguese: Vocês comeram apenas alguns poucos tipos de alimentos que ainda tinham, porque o dinheiro acabou? English: Did you have to consume just a few types of foods (remaining) because you ran out of money?

Section B:

Portuguese: Agora vou perguntar apenas sobre você e os outros adultos (18 anos ou mais) da sua casa. Alguma de vocês, alguma vez: English: Now I’m going to ask you only about you and other adults (18 years and above) in your household. Did any of you adults:

**Question 4:** Portuguese: Deixou de fazer alguma refeição, porque não havia dinheiro para comprar comida? English: Skip a meal because there was not enough money to buy food?

**Question 5:** Portuguese: Comeu menos do que achou que devia, porque não havia dinheiro para comprar comida? English: Eat less than what you thought you should because there was not enough money to buy food?

**Question 6:** Portuguese: Sentiu fome, mas não comeu porque não havia dinheiro para comprar comida? English: Feel hungry but did not eat because there was not enough
money to buy food?

**Question 7:** Portuguese: Fez apenas uma refeição ao dia ou ficou um dia inteiro sem comer, porque não havia dinheiro para comprar a comida? English: Go without eating for a whole day or just have one meal in a whole day because there was not enough money to buy food?

**Section C:**

Portuguese: Agora vou perguntar apenas sobre os moradores menores de 18 anos da sua casa. Algum deles, alguma vez: English: Now I’m going to ask only about those in the household under 18 years old. Did any of them:

**Question 8:** Portuguese: Comeu apenas algunos poucos tipos de alimentos que ainda tinham, porque o dinheiro acabou? English: Eat only a few types of food that you still had left, because money had run out?

**Question 9:** Portuguese: Não comeu quantidade suficiente de comida porque não havia dinheiro para comprar comida? English: Not eat enough because there was not enough money to buy food?

**Question 10:** Portuguese: Foi diminuída a quantidade de alimentos das refeições de algum morador com menos de 18 anos de idade, porque não havia dinheiro para comprar a comida? English: Reduce the size of meals of your children/adolescents because there was not enough money to buy food?

**Question 11:** Portuguese: Deixou de fazer alguma refeição, porque não havia dinheiro para comprar comida? English: Skip a meal because there was not enough money to buy food?

**Question 12:** Portuguese: Sentiu fome, mas não comeu porque não havia dinheiro para comprar mais comida? English: Were your children/adolescents ever hungry but you just could not buy more food?

**Question 13:** Portuguese: Fez apenas uma refeição ao dia ou ficou sem comer por um dia inteiro, porque não havia dinheiro para comprar comida? English: Did your children go without food for a whole day or just have one meal in a whole day because there was not enough money to buy food?

**Section D – Regionalized food security questions**

Portuguese: Nos últimos 30 dias, ou seja, desde o dia ______ do mês passado, alguma vez, o(a) senhor(a) ou alguém aqui desta casa: English: During the previous 30 days, at some time did you or anyone else in this household:

**Question 14:** Portuguese: Fez alguma refeição apenas com farinha ou chicó porque não tinha outro alimento? English: Had a meal with only toasted manioc flour (or this with water and salt) because there were no other foods?

**Question 15:** Portuguese: Teve que pegar crédito ou comprar fiado na taberna, mercadinho ou vendedor para comprar comida porque não tinha mais dinheiro? English:
Have to borrow money or buy food on credit at a shop because there was no other money?

**Question 16:** Portuguese: Emprestou comida de outra família porque faltou em casa e não tinha dinheiro? English: Borrowed food from another Family because you had none at home and had no money?

**Question 17:** Portuguese: Fez as refeições na casa de vizinhos, amigos ou parentes porque não tinha comida em casa? English: Had meal(s) in the home of neighbours, friends or relatives because there was no food at home?

**Question 18:** Portuguese: Diminuiu a quantidade de carne ou peixe em alguma refeição para economizar? English: Reduce the quantity of meat or fish in a meal in order to economize?

**Appendix E: Traceplots of the Case of Study**

![Traceplots](image)

**Figure 6:** Traceplots of difficulty parameters: only 3 out of 18 were randomly selected to be shown.
Figure 7: Traceplots of discrimination parameters: only 5 out of 22 were randomly selected to be shown.

Figure 8: Traceplots of discrimination parameters: only 5 out of 600 were randomly selected to be shown.
Figure 9: Traceplots of correlation parameters.

Figure 10: Traceplots of unrestricted standard deviations parameters for the multivariate Gaussian process.
Appendix F: Extension to Mixed Outcome Types

In order to deal with binary, ordinal or continuous items, we can extend the spatial item factor analysis by considering $q_1$ ordinal items and $q_2$ continuous items. We do not need to differentiate another set of binary items given they are simply ordinal items with two categories. The $q_1$ ordinal times can be modelled as spatial discrete-valued stochastic processes $\{Y_j(s) : s \in D\}$, where $D \subset \mathbb{R}^2$ and the random variable $Y_j(s)$ can take values $\{0, 1, \ldots, K_j - 1\}$. Notice that $K_j$ represents the number of categories for ordinal item $j = 1, \ldots, q_1$. We assume that the values of the $q_1$ discrete-valued stochastic processes are determined by an auxiliary real-valued stochastic processes $\{Z_{o,j}(s) : s \in D\}$ and thresholds $\gamma_j = (\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{j(K_j-1)})^T$ such as

$$Y_j(s) = k \iff -\gamma_{jk} \leq Z_{o,j}(s) < -\gamma_{j(k+1)}, \text{ for } k = 0, 1, \ldots, (K_j - 1),$$

where $\gamma_{j0} = -\infty$ and $\gamma_{j(K_j)} = \infty$. The $q_2$ continuous items can be modelled as real-valued stochastic processes $\{Z_{c,j}(s) : s \in D\}$ for $j = 1, \ldots, q_2$. Then, we can defined the spatial random vector $Z(s) = (Z_{o,1}(s), \ldots, Z_{o,q_1}(s), Z_{c,1}(s), \ldots, Z_{c,q_2}(s))^T$, a collection of the auxiliary random variables $Z_{o,j}(s)$ associated to the ordinal items and the observable random variables $Z_{c,j}(s)$ associated to the continuous items, and define the factor model at this level such as

$$Z_j(s) = c_j + a_j^T\theta(s) + \epsilon_j(s), \text{ for } j = 1, \ldots, q_1 + q_2$$

where, due to identifiability, $c_j = 0$ for $j = 1, \ldots, q_1$ and where the $m$-dimensional latent factors are modelled including multivariate non-linear effects, $f(x_j(s)) : \mathbb{R} \rightarrow \mathbb{R}^m$,

$$\theta(s) = \sum_{i=1}^p f(x_i(s)) + w^*(s) + v(s).$$
Finally, to make the model identifiable, the error term is defined as

$$
\epsilon_j(s) \sim \begin{cases} 
\mathcal{N}(0, 1) & \text{for } j = 1, \ldots, q_1 \\
\mathcal{N}(0, \sigma_j) & \text{for } j = q_1 + 1, \ldots, q_1 + q_2
\end{cases}
$$