Stability Analysis for a Class of Non-Weakly Reversible Chemical Reaction Networks

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Abstract: We consider ordinary differential equations (ODEs) that describe the time evolution of the concentrations of species in chemical reaction networks (CRNs). In order to analyze the convergence of solutions to the ODEs, the chemical reaction network theory has established an important theorem called Deficiency Zero Theorem (DZT). This theorem provides a sufficient condition for any solution to the ODEs to converge to an equilibrium point, based only on the graph structures of the CRNs and the algebraic properties of ODEs. In the present paper, we consider a class of non-weakly reversible chemical reaction networks, to which the DZT cannot be applied since one of the conditions, weak reversibility, is not satisfied. In order to make up for the failure of this important condition, by decomposing the network into weakly reversible sub-networks and applying the DZT to them, we show any solution to the ODEs for our class of networks with positive initial values converges to an equilibrium point on the boundary of the positive orthant.

Key Words: chemical reaction networks, weak reversibility, ordinary differential equations, stability, equilibrium point.

1. Introduction

It is widely known that solutions to ordinary differential equations (ODEs) that describe the time evolution of the concentrations of species in chemical reaction networks (CRNs) can be oscillatory or behave chaotically. For most of the equations, however, the solution converges to an equilibrium point that corresponds to a dynamical equilibrium for the chemical reaction network [1].

Although it is very important in theoretical chemistry to determine whether the dynamics of a CRN converges to a state of dynamical equilibrium or not, no general method to determine that has been established.

In order to formulate and analyze the differential equations describing CRNs, M. Feinberg and his colleagues [1]–[4] have developed the chemical reaction network theory. This theory established the Deficiency Zero Theorem (DZT), which provides a sufficient condition for the ODEs describing the dynamics of a chemical reaction network to have an asymptotically stable equilibrium point in a positive stoichiometric compatibility class, based only on their graph structures of the CRNs and the algebraic properties of the ODEs. The conditions of the theorem do not depend on the size of the system and the values of system parameters.

However, there are many CRNs that do not satisfy the two conditions of the DZT, weak reversibility and zero deficiency, and hence the positive solution to the ODE that describes the dynamics of such a CRN cannot be proved to converge to an equilibrium point based on the DZT.

Although some methods have been proposed for analyzing the convergence to a positive equilibrium point of a solution to ODEs for a class of CRNs that are weakly reversible but with non-zero deficiencies [5],[6], no methods have been established to analyze the dynamics of non-weakly reversible CRNs with zero deficiency, as far as we know.

In the recent work [7], we showed that any solution to the ODE that describes the dynamics of CRNs in a class of non-weakly reversible networks with positive initial values converges to an equilibrium point. The proof given in the paper was, however, in an outline form for a specific example network in our class.

In the present paper, we shall give a general and rigorous proof for the convergence of a solution to an equilibrium point for our class.

The outline of this paper is as follows. In Section 2, we briefly introduce the chemical reaction network theory, which plays an important role throughout this paper. In Section 3, first we provide a class of chemical reaction networks that do not satisfy the weak reversibility, one of the conditions of the Deficiency Zero Theorem. Next, we state two theorems with respect to convergence of a solution to the ODE describing the non-weakly reversible networks in our class. In Section 4, we give proofs of these theorems. In Section 5, we apply our results to an example network. Finally, we give a conclusion of this paper in Section 6.

2. Chemical Reaction Network Theory

In this section, we briefly introduce the chemical reaction network theory [1]–[3].

A chemical reaction network (CRN) in the sense of Feinberg is mathematically defined by a triplet (S,C,R), where the elements S, C and R are defined by

- S: the set of n species in the network, which are denoted by X_1, X_2, ..., X_n,
- C: the set of all complexes y in the network,
- R: the set of all chemical reactions y → y' in the network.

Here a complex is a linear combination of species, y = y_1X_1 + ... + y_nX_n, with coefficients of non-negative integers, and a chemical reaction y → y' denotes that a
complex $y'$ is produced from $y$. We may associate $y$ with the vector of these coefficients, $(y_1, y_2, \ldots, y_m)^T$, and $y$ and $y'$ are called a reactant and a product, respectively.

Next, we denote by $x_i$ the molar concentration of the species $X_i \in S$ and define the non-negative vector of concentrations by $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n$. Then, the time-evolution of $x$ is given by the following ordinary differential equation:

$$\frac{dx(t)}{dt} = \sum_{y \rightarrow y' \in \mathcal{R}} K_{y \rightarrow y'} (x(t))(y' - y), \quad \forall t \geq 0,$$

where a function $K_{y \rightarrow y'} : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$, which is called a kinetics of the chemical reaction $y \rightarrow y'$, is a function of class $C^1$. In particular, the kinetics given by

$$K_{y \rightarrow y'} (x) = k_{y \rightarrow y'} x^i_1 x^2_2 \cdots x^n_n, \quad \forall x \in \mathbb{R}_{\geq 0}^n,$$

is called the mass action kinetics.

In [3],[4], it has been proved that the ODE (1) is non-negative (resp. positive), that is, any solution $x(t)$ to (1) with an initial value $x_0 \in \mathbb{R}_{\geq 0}^n$ (resp. $x_0 \in \mathbb{R}_{>0}^n$, which is the positive orthant of $\mathbb{R}^n$) remains in $\mathbb{R}_{\geq 0}^n$ (resp. $\mathbb{R}_{>0}^n$) for all $t \geq 0$.

Next, we define a subspace $H$ of $\mathbb{R}^n$, which is called the stoichiometric subspace of a chemical reaction network $(S, C, \mathcal{R})$, by

$$H := \text{span}[y' - y | y \rightarrow y' \in \mathcal{R}],$$

and a stoichiometric compatibility class of $(S, C, \mathcal{R})$ by the set $c + H := [c + h | h \in H]$ for $c \in \mathbb{R}^n$, i.e. the subset of $\mathbb{R}^n$ obtained by translating $H$ by $c$. Moreover, we refer to the set $(c + H) \cap \mathbb{R}_{\geq 0}^n$, $c \in \mathbb{R}_{\geq 0}^n$, as a positive stoichiometric compatibility class of the network $(S, C, \mathcal{R})$.

By integrating (1), for all $t \geq 0$ we have

$$x(t) - x_0 = \sum_{y \rightarrow y' \in \mathcal{R}} \left( \int_0^t K_{y \rightarrow y'} (x(s)) \, ds \right) (y' - y),$$

and hence the solution $x$ to the ODE (1) with the initial value $x_0$ remains in $x_0 + H$ for all $t \geq 0$. Together with a positivity of the solution, we see that the solution remains in the positive stoichiometric compatibility class $(x_0 + H) \cap \mathbb{R}_{\geq 0}^n$ for all $t \geq 0$.

Hereafter we abbreviate a stoichiometric subspace, a stoichiometric compatibility class and a positive stoichiometric compatibility class to an SS, an SCC and a PSCC, respectively.

A chemical reaction network $(S, C, \mathcal{R})$ can be regarded as a directed graph. Actually, the nodes of the graph are the complexes and a directed edge of the graph is given by a reaction from a complex $y$ to a complex $y'$ if and only if $y \rightarrow y'$ is in $\mathcal{R}$. Each connected component of the directed graph is called a linkage class of the network. When each linkage class is a strongly connected subgraph, the chemical reaction network $(S, C, \mathcal{R})$ is called weakly reversible. In other words, a network $(S, C, \mathcal{R})$ is weakly reversible if and only if for any reaction $y \rightarrow y'$ in $\mathcal{R}$ there exist complexes $y_1, y_2, \ldots, y_k$ such that all of the reactions $y' \rightarrow y_1, y_1 \rightarrow y_2, \ldots, y_k \rightarrow y$ are in $\mathcal{R}$.

At the end of this section, we define an index, which is called the deficiency of $(S, C, \mathcal{R})$, as follows.

**Definition 2.1** The deficiency, a non-negative integer denoted by $\delta$, of a chemical reaction network $(S, C, \mathcal{R})$ is defined by:

$$\delta := m - \ell - \text{dim}(H),$$

where $m = |\mathcal{C}|$, the number of complexes, and $\ell$ is the number of the linkage classes [1],[3].

For a chemical reaction network with the zero deficiency, the following theorem, which is called the Deficiency Zero Theorem, has been proved by M. Feinberg in [1],[3].

**Theorem 2.1** (The Deficiency Zero Theorem) Suppose that a chemical reaction network $(S, C, \mathcal{R})$ has zero deficiency. Then, the following statements hold:

(i) If the network $(S, C, \mathcal{R})$ is not weakly reversible, then the ODE (1) for $(S, C, \mathcal{R})$ admits neither a positive equilibrium nor a periodic orbit containing a point in $\mathbb{R}_{>0}^n$. (ii) If the network $(S, C, \mathcal{R})$ is weakly reversible, then within each PSCC, the ODE (1) for $(S, C, \mathcal{R})$ with the mass action kinetics (2) has a unique equilibrium point, which is locally asymptotically stable relative to the PSCC.

3. **A Class of Non-Weakly Reversible Networks**

We shall deal with a class of chemical reaction networks as shown in Fig. 1, which do not satisfy the weak reversibility, one of the conditions of the Deficiency Zero Theorem.

A network in this class can be decomposed into $N + 1$ sub-networks $(S_i, C_i, \mathcal{R}_i)$ and for each sub-network $(S_i, C_i, \mathcal{R}_i) (i = 2, \ldots, N + 1)$, there are $N$ reactions with reactants in $(S_i, C_i, \mathcal{R}_i)$ and products in $(S_i, C_i, \mathcal{R}_i)$ as denoted by thick arrows in Fig. 1. Obviously, this network is not weakly reversible since these reactions are in one direction.

A chemical reaction network in Fig. 1 can be mathematically described by the following network $(S', C', \mathcal{R}')$:

$$S' := \bigcup_{i=1}^{N+1} S_i, \quad C' := \bigcup_{i=1}^{N+1} C_i,$n

$$\mathcal{R}' := \left( \bigcup_{i=1}^{N+1} \mathcal{R}_i \right) \cup \left( \bigcup_{i=2}^{N+1} \{ y^{(i)}(p_i) \rightarrow y^{(i)}(q_i) \} \right),$$

where $1 \leq p_1 < p_2 < \ldots < p_{N+1} \leq m := |C|, 1 \leq q_1 \leq m := |C|$ (i = 2, ..., $N + 1$) and $S_i, C_i$ and $\mathcal{R}_i$ (i = 1, 2, ..., $N + 1$) satisfy the following three conditions:

1. $S_i = \{ X_1^{(i)}, \ldots, X_{n_i}^{(i)} \}$ and $S_i \cap S_j = \emptyset, (i \neq j)$.
2. $C_i = \{ y^{(i)}(1), y^{(i)}(2), \ldots, y^{(i)}(m_i) \}$, where $y^{(i)}(j) \in \mathbb{Z}_{\geq 0}^{n_i} (j = 1, \ldots, m_i)$ and $C_i$ does not contain a zero complex, a zero vector in $\mathbb{Z}_{\geq 0}^{n_i}$ Moreover for all $y^{(i)}(k) \in C_i, k \neq \ell, \text{supp}(y^{(i)}) \cap \text{supp}(y^{(i)}(j)) = \emptyset$. Here, $\text{supp}(x)$ for $x \in \mathbb{R}^n$ is a subset of species $S$ such that $x_i \in \text{supp}(x)$ if and only if $x_i \neq 0$.
3. Each $(S_i, C_i, \mathcal{R}_i)$ consists of a single linkage class and is weakly reversible.

The condition 1 means that each of species in one sub-network is not in any other. The condition 2 means that species
composing one complex are not contained in any other complex.

For simplicity, we denote by \( (P') \) and \( (P) \) the initial value problem of the ODE (1) with the mass action kinetics (2) for the network \((S', C', R')\) and the sub-network \((S_i, C_i, R_i)\) \((i = 1, \ldots, N + 1)\), respectively. We denote by \( x_{j}^{(i)} \) \((j = 1, \ldots, n_i, i = 1, \ldots, N + 1)\) the molar concentration of species \(X_j^{(i)}\) \((j = 1, \ldots, n_i, i = 1, \ldots, N + 1)\), and define the vector of concentrations of species in the subnetwork \((S_i, C_i, R_i)\) \(i = 1, 2, \ldots, N + 1\) by \( x^{(i)} := (x_j^{(i)})_{j = 1}^{n_i} \in \mathbb{R}^{n_i} \), \((i = 1, \ldots, N + 1)\) and put \( x := (x^{(1)}, \ldots, x^{(N+1)}) \in \mathbb{R}^{N+1}\), where \( n := n_1 + \cdots + n_{N+1}\).

The main results of this paper are in the following two theorems, the proofs of which will be given in Section 4.

**Theorem 3.1** The problem \((P')\) admits neither a positive equilibrium nor a periodic orbit containing a point in \(\mathbb{R}^n_{>0}\).

**Theorem 3.2** Any positive solution \( x(t) \) to \((P') \) with an initial value \( x(0) \in \mathbb{R}^n_{>0}\) converges to an equilibrium point on the boundary of the PSCC containing \( x(0)\), that is, some concentrations of species in chemical reaction network \((S', C', R')\) converge to zero and all others converge to some positive values.

By the definition of weak reversibility and the fact widely known in the graph theory, we see that any non-weakly reversible CRN consists of strongly connected linkage classes at least two of which are linked in one direction. In other words, any non-weakly reversible CRN (any directed graph) can be decomposed into weakly reversible sub-networks (strongly connected components). However, it is hard to tackle such general problems, hence in the present paper we choose one of the simplest single linkage non-weakly reversible network in Fig. 1 as a target of study, although our stability analysis is potentially applicable to more general non-weakly reversible CRNs, which will be dealt with in future work.

### 4. The Proofs of Theorems 3.1 and 3.2

First, we consider the initial value problem \((P_i)\) of the subnetwork \((S_i, C_i, R_i)\) \((i = 1, 2, \ldots, N + 1)\).

From the conditions 2 and 3 of the network \((S, C, R)\) and Theorem 2.1 (ii), we have the following theorem immediately.

**Theorem 4.1** The problem \((P_i)\) \((i = 1, 2, \ldots, N + 1)\) has a unique equilibrium point within each PSCC, and this equilibrium point is locally asymptotically stable relative to the PSCC.

**Proof:** We see from the conditions 2 and 3 of the network \((S, C, R)\) that the SS, \(H_i\), is spanned by \(m_i - 1\) linearly independent vectors as

\[
H_i = \text{span}\left\{ y^{(i)}(2) - y^{(i)}(1), \ldots, y^{(i)}(m_i) - y^{(i)}(1) \right\}, \quad (6)
\]

which implies that \(\dim(H_i) = m_i - 1\).

Since we see from the condition 3 that the number of linkage classes \(\ell_j\) of the network is unity, the deficiency \(\delta_j\) of the network is \(\delta_j = |C_j| - \ell_j - \dim(H_j) = m_j - 1 - (m_i - 1) = 0\).

Besides, from the condition 3 the network \((S, C, R)\) is weakly reversible, and hence Theorem 2.1 (ii) can be applied to each network \((S_i, C_i, R_i)\). The proof is complete. \(\square\)

We shall show that any positive solution to the problem \((P_i)\) \((i = 1, \ldots, N + 1)\) is bounded globally in time. In order to do this, we start by defining conservativity of a chemical reaction network, giving a lemma [8] and proving some properties are equivalent to the conservativity.

**Definition 4.1** Consider a chemical reaction network \((S, C, R)\) with \(n\) species \(S = \{X_1, \ldots, X_n\}\). Let \(H\) be the SS of \((S, C, R)\). The network \((S, C, R)\) is said to be conservative if there exists a vector \(a \in \mathbb{R}^n_{>0}\) such that \(a \cdot h = 0\) for all \(h \in H\).

**Lemma 4.1** (Stiemke [8]) For \(u_1, \ldots, u_m \in \mathbb{R}^n\), exactly one of the following two conditions holds:

1. There exists \(c \in \mathbb{R}^n\) such that

\[
\left( \sum_{i=1}^m c_i u_i \right)_j \leq 0, \quad j = 1, \ldots, n,
\]

and at least one of the inequalities is strict.

2. There is a \(w \in \mathbb{R}^n_{>0}\) such that \(w \cdot u_i = 0\) for each \(i = 1, \ldots, m\).

**Theorem 4.2** For any CRN \((S, C, R)\), the following three conditions are equivalent to each other:

1. The CRN \((S, C, R)\) is conservative.

2. Any non-zero vector in \(H\) consists of both positive and negative components.

3. There are \(n - s\) linearly independent vectors of positive integers orthogonal to \(H\). Here \(s = \dim(H)\).

**Proof for 1 \(\Rightarrow\) 2** Suppose that the condition 1 holds. Then by the definition of conservativity there exists a vector \(a \in \mathbb{R}^n_{>0}\) such that \(a \cdot h = 0\) for all \(h \in H\). Now we assume that 2 does not hold. Then there exists a non-zero vector \(h \in H \cap \mathbb{R}^n_{>0}\). Thus we see \(a \cdot h > 0\), which is in contradiction with the conservativity.

**Proof for 2 \(\Rightarrow\) 1** Suppose that the condition 2 holds. Letting \(|h_1, \ldots, h_s|\) be a basis for \(H\) we see that the linear combination \(\sum_{i=1}^s c_i h_i\) has both positive and negative components for all non-zero \(c \in \mathbb{R}^s\). Therefore by virtue of Lemma 4.1, we see that there exists a positive vector \(a \in \mathbb{R}^n_{>0}\) such that \(a \cdot h_i = 0\) \((1 \leq \forall i \leq s)\). This implies \(a \cdot h = 0\) for all \(h \in H\) because \(|h_1, \ldots, h_s|\) is a basis of \(H\). This completes the proof.

**Proof for 1 \(\Leftrightarrow\) 3** It is trivial that 3 implies 1, hence we shall prove that 1 implies 3. Suppose that 1 holds. Then there exists a vector \(b \in \mathbb{R}^n_{>0}\) that is orthogonal to \(H\). We can also find a set of linearly independent positive vectors \(|b_1, \ldots, b_{n-s}|\) each vector in which is orthogonal to the \(s\)-dimensional subspace \(H\). By the definition of an SS, we have linearly independent integer-valued vectors \(|h_1, \ldots, h_s| \in \mathbb{Z}^n_{>0}\) such that \(H = \text{span}(h_1, \ldots, h_s)\). Let \(A := |h_1, \ldots, h_s| \in \mathbb{Z}^{n-s}\). As shown above, there exist linearly independent positive-valued vectors \(|b_1, \ldots, b_{n-s}|\) such that \(Ab_i = 0\) for all \(i = 1, \ldots, n-s\).

The solution space for the simultaneous linear equations \(Ax = 0\) with integer coefficient matrix \(A\) is spanned by a set of linearly independent vectors of rational numbers, \(|q_1, \ldots, q_{n-s}|\). Since rational numbers are dense in the set of real numbers, we can find a set of linearly independent positive-valued rational vectors orthogonal to \(H\) by slightly changing the above positive-valued vectors \(|b_1, \ldots, b_{n-s}|\). Besides, by multiplying this vector by a sufficiently large integer we obtain \(n - s\) linearly independent vectors of positive integers orthogonal to \(H\).

This completes the proof. \(\square\)

We see that each chemical reaction network \((S_i, C_i, R_i)\), \(i = 1, 2, \ldots, N + 1\) is conservative.
Lemma 4.2  Each chemical reaction network \((S_i, C_i, R_i), i = 1, 2, \ldots, N + 1\) is conservative. Furthermore, for the SS of \((S_i, C_i, R_i), H_i\), there exist \(n_i - \dim(H_i)\) positive integer vectors \(a^{(0)}(k) = (a^{(0)}_i(k), \ldots, a^{(0)}_{m}(k)) \in \mathbb{Z}^+_{\geq 0}, k = 1, \ldots, n_i - \dim(H_i)\) satisfying \(a^{(0)}(k) \cdot h = 0\) for any \(h \in H_i\).

Proof We show that the chemical reaction network \((S_i, C_i, R_i)\) satisfies the condition 2 of Theorem 4.2, that is, any non-zero vector in \(H_i\) consists of both positive and negative components. For all non-zero vectors \(h \in H_i\), there exist \(a_2, \ldots, a_{m_0} \in \mathbb{R}\) such that

\[ h = \sum_{l=1}^{m_0} a_l(y^{(0)}(l) - y^{(0)}(1)) + \sum_{l=1}^{m_0} a_l(y^{(0)}(l) - y^{(0)}(1)), \]

where index sets \(I_l\) and \(J_l\) are defined by

\[ I_l := \{l \in [2, m_0] | a_l > 0\}, \]
\[ J_l := \{l \in [2, m_0] | a_l < 0\}. \]

In the case where \(I_l \neq \emptyset\) and \(J_l \neq \emptyset\), we see from the condition 3 of the network \((S_i, C_i, R_i)\) that \(h_i > 0\) for all \(s\) such that \(X^{(0)}_s \in \text{supp}(y^{(0)}(j))\) for \(j \in I_l\) and \(h_i < 0\) for all \(s\) such that \(X^{(0)}_s \in \text{supp}(y^{(0)}(j))\) for \(j \in J_l\). In the case where \(I_l \neq \emptyset\) and \(J_l = \emptyset\), we also see from the condition 3 that \(h_i > 0\) for all \(s\) such that \(X^{(0)}_s \in \text{supp}(y^{(0)}(j))\) for \(j \in I_l\) and \(h_i < 0\) for all \(s\) such that \(X^{(0)}_s \in \text{supp}(y^{(0)}(j))\). Hence, the network \((S_i, C_i, R_i)\) satisfies the condition 2 of Theorem 4.2. Therefore, the lemma follows from the equivalence of the conditions 1, 2 and 3. \(\square\)

Using this lemma, we can show the following theorem.

Theorem 4.3  Any positive solution \(x^{(0)}(t)\) to the problem \((P_i), i = 1, 2, \ldots, N + 1\) is bounded globally in time, that is,

\[ \sup_{t \geq 0} \|x^{(0)}(t)\| < \infty, \quad (7) \]

Proof Since the network \((S_i, C_i, R_i)\) is conservative, we see from Theorem 4.2 that there exist \(n_i - \dim(H_i)\) linearly independent positive integer vectors \(a^{(0)}(k) = (a^{(0)}_1(k), \ldots, a^{(0)}_{m}(k))^T \in \mathbb{Z}^+_{\geq 0}, k = 1, \ldots, n_i - \dim(H_i)\) satisfying \(a^{(0)}(k) \cdot h = 0\) for all \(h \in H_i\).

Now, we define linear functions \(T^{(0)}_k : \mathbb{R}^n \rightarrow \mathbb{R}, k = 1, \ldots, n_i - \dim(H_i)\)

\[ T^{(0)}_k(x^{(0)}) := \sum_{j=1}^{n_i} a^{(0)}_{j}(k) x^{(0)}_j, \quad k = 1, \ldots, n_i - \dim(H_i). \quad (8) \]

By considering the time derivative \(\dot{T}^{(0)}_k\) of these functions along the positive solution \(x^{(0)}(t)\) to \((P_i)\), we easily see that \(T^{(0)}_k(x^{(0)}(t)) = T^{(0)}_k(x^{(0)}(0)), k = 1, \ldots, n_i - \dim(H_i)\) for all \(t \geq 0\).

Hence, by putting

\[ M(x^{(0)})(0) := \max_{1 \leq t \leq n_i - \dim(H_i)} T^{(0)}_k(x^{(0)}(0)), \]

we see that \(0 \leq x^{(0)}_j(t) \leq M(x^{(0)}(0)), j = 1, \ldots, n_i\) for all \(t \geq 0\), which implies that \(\limsup_{t \rightarrow \infty} \|x^{(0)}(t)\| < \infty. \quad \square\)

Moreover, it is easy to see that the linear functions \(T^{(0)}_k\) characterize the PSCC \((c + H_i) \cap \mathbb{R}^n_0\) for any \(c \in \mathbb{R}^n_0\) as

\[ P^{(0)}(c) := \{ x^{(0)} \in \mathbb{R}^n_0 | T^{(0)}_k(x^{(0)}) = a^{(0)}_j(k) \cdot c, k = 1, \ldots, n_i - \dim(H_i) \}. \quad (9) \]

We shall show that the unique equilibrium point within each PSCC of the problem \((P_j), j = 1, 2, \ldots, N + 1\), which has been shown to exist in Theorem 4.1, is globally asymptotically stable relative to the PSCC. First we give the definitions of a semi-locking set and semi-conservativity [2].

Definition 4.2  For a network \((S, C, R)\), a non-empty subset \(W\) of \(S\) is called a semi-locking set if \(W \cap \text{supp}(y) \neq \emptyset\) for any reaction \(y \rightarrow y' \in \mathbb{R}\) such that \(W \cap \text{supp}(y') \neq \emptyset. \) When \(W\) does not contain any other semi-locking set, it is called a minimal semi-locking set.

Definition 4.3  For a network \((S, C, R)\), let \(W\) be a non-empty subset of \(S\) and \(H\) be the SS of \((S, C, R)\). The network \((S, C, R)\) is said to be semi-conservative with respect to \(W\) if there exists a vector \(a \in \mathbb{R}^n_0\) satisfying \(\text{supp}(a) = W\) and \(a \cdot h = 0\) for all \(h \in H\).

The following lemma plays an important role in showing globally asymptotic stability of the unique equilibrium point within each PSCC of the problem \((P_j), j = 1, 2, \ldots, N + 1\).

Lemma 4.3  The network \((S_i, C_i, R_i), i = 1, \ldots, N + 1\) is semi-conservative with respect to each minimal semi-locking set.

Proof Let \(W\) be a minimal semi-locking set. Then we see from the conditions 2 and 3 of the network \((S_i, C_i, R_i)\) that every minimal semi-locking set consists of \(m\) species each of which appears in exactly one complex. That is, there exist \(X^{(0)}_h \in \text{supp}(y^{(0)}(k)), k = 1, \ldots, m_i\) such that \(W = (X^{(0)}_1, \ldots, X^{(0)}_m) \subset S_i\), and \(X^{(0)}_h \neq X^{(0)}_k, k \neq l. \)

Now we denote by \(c^{(0)}_{jl}\) the \(j\)-th component of \(y^{(0)}(k), k = 1, \ldots, m_i\) and define \(a^{(0)}(W) = (a^{(0)}_1, \ldots, a^{(0)}_m)^T \in \mathbb{R}^n_0\) by

\[ a^{(0)}_{jl} := \begin{cases} \prod_{k=1, k \neq l}^{m_i} c^{(0)}_{jk} > 0, & j = 1 \text{ to } m_i, \text{ otherwise.} \\ 0, \end{cases} \quad (10) \]

Then, from the condition 2 we have

\[ a^{(0)}(W) \cdot (y^{(0)}(l) - y^{(0)}(1)) = m \sum_{k=1}^{m_i} c^{(0)}_{jk} - m \sum_{k=1}^{m_i} c^{(0)}_{j}\]
\[ = 0, l = 2, \ldots, m_i \quad \text{for all } h \in H_i. \quad \square \]

Here we introduce the following two lemmas [2] to show globally asymptotic stability of the unique equilibrium point within each PSCC of the problem \((P_j), j = 1, 2, \ldots, N + 1\).

Lemma 4.4  Consider the ODE (1) for \((S, C, R)\). Let \(W \subset S\) be a non-empty subset. If there exists an initial value \(x(0) \in \mathbb{R}^n_0\) such that \(\omega_{\text{stable}}(x(0)) \cap L_W \neq \emptyset, \) then \(W\) is a semi-locking set. Here \(L_W\) is a subset of \(\mathbb{R}^n\) defined by \(L_W := \{ x \in \mathbb{R}^n | x_i = 0 \Leftrightarrow X_i \in W \}. \)
**Lemma 4.5** Suppose the ODE $\dot{\varphi} = f(\varphi)$ with a $C^1$ function $f$ defined on an open set $\Omega \subset \mathbb{R}^n$ has a unique equilibrium point $\bar{x} \in \Omega$ and there exists a globally defined Lyapunov function $V$ satisfying the following conditions:

1. $V(\varphi) \geq 0$, and the equality holds if and only if $\varphi = \bar{x}$.
2. $\frac{d}{dt} V(\varphi(t)) \leq 0$, and the equality holds if and only if $\varphi(t) = \bar{x}$ for all $t \geq 0$.
3. $V(\varphi) \to \infty$ as $\|\varphi\|_{\mathbb{R}^n} \to \infty$.

Then it holds that for any solution $\varphi(t)$, either $\varphi(t) \to \bar{x}$ or $\varphi(t) \to \partial \Omega$ in $\mathbb{R}^n$ as $t \to \infty$, where $\partial \Omega$ is the boundary of $\Omega$.

By using Lemmas 4.3, 4.4 and 4.5, we have the following theorem.

**Theorem 4.4** The problem $(P_i) (i = 1, 2, \ldots, N + 1)$ has a unique equilibrium point within each PSCC, and this equilibrium point is globally asymptotically stable relative to the PSCC.

**Proof:** From Theorem 4.1, $(P_1)$ has a unique equilibrium point within each PSCC, and this equilibrium point is locally asymptotically stable relative to the PSCC. Moreover, we see from Theorem 4.3 that any positive solution to $(P_1)$ with an initial value $x(0) \in \mathbb{R}^n_{>0}$ is bounded, hence its omega-limit set $\omega(x(0))$ is non-empty.

The function $V: \mathbb{R}^n_{>0} \to \mathbb{R} \geq 0$ defined by

$$V(x) := \sum_{i=1}^{n} (x_i \ln x_i - \ln x_i^*) - 1 + \bar{x}_i,$$

where $x_i$ is the unique equilibrium point on the PSCC $P_i(x(0))$, satisfies (1)–(3) of Lemma 4.5, and hence we see from Lemma 4.5 that a positive solution $x(t)$ to $(P_1)$ satisfies either $x(t) \to \bar{x}$ or $x(t) \to \partial \Omega$ as $t \to \infty$.

We assume that $x(t) \to \partial \Omega$ as $t \to \infty$, that is, $\omega(x(0)) \subseteq \partial \Omega = \emptyset$. Since the boundary $\partial \Omega$ is a union of the sets $L_w$ for some $W \in S$, $\omega(x(0)) \cap L_w = \emptyset$ for some $W$ holds. Thus, there exists $w_i \in \omega(x(0)) \cap L_w$, and we see from Lemma 4.4 that $w_i$ is a semi-locking set.

For a minimal semi-locking set $W \subset W$, since from Lemma 4.3 the network $(S_i, C_i, R_i)$ is semi-conservative with respect to $W$, there exists an $a(W) \in \mathbb{R}^n_{>0}$ satisfying $\text{supp} (a(W)) = W$ and $a(W) \cdot h = 0$ for all $h \in H_i$. Now, we define a linear function $T_{a(W)} : \mathbb{R}^n \to \mathbb{R}$ by

$$T_{a(W)}(y) := \sum_{j=1}^{n} a_j(W) y_j.$$

By considering the time derivative $\dot{T}_{a(W)}$ of these functions along the positive solution $x(t)$ to $(P_1)$, we easily see that $\dot{T}_{a(W)}(y) = T_{a(W)}(y(0)) > 0$ for all $t \geq 0$. Hence, the continuity of $\dot{T}_{a(W)}(y(t))$ with respect to $t$, we see $\dot{T}_{a(W)}(y(t)) > 0$, which is in contradiction with $w_i \in L_w$. Therefore it holds that $x(t) \to \bar{x}$. The proof of this theorem is complete.

Now, we shall give the proof of Theorem 3.1.

**Proof of Theorem 3.1:** We first prove

$$y^{(i)}(q_i) - y^{(i)}(p_i) \notin \text{span} \left( y' - y, y \to y' \in \bigcup_{j=1}^{N+1} R_j \right)$$

for all $i = 2, 3, \ldots, N + 1$.

Since $\text{supp}(y^{(i)}(k)) \cap \text{supp}(y^{(i)}(l)) = \emptyset$, $k \neq l$, and hence $y^{(i)}(q_i) - y^{(i)}(p_i) \in \text{span} \left( y' - y, y \to y' \in R_i \cup R_j \right)$. (14)

Besides, for $\text{supp}(y^{(i)}(q_i)) \cap \text{supp}(y^{(i)}(p_i)) = \emptyset$, we see $y^{(i)}(q_i)$ is in span $[y' - y \mid y \to y' \in R_i]$, i.e. in $H_i$. However, from the proof of Lemma 4.2, any non-zero vector in $H_i$ consists of both positive and negative components, while $y^{(i)}(q_i)$ is non-zero and non-negative. Hence, (13) holds for all $i = 2, 3, \ldots, N + 1$.

From the proof of Theorem 4.1, the deficiency $\delta_i$ of each network $(S_i, C_i, R_i)$, $i = 1, 2, \ldots, N + 1$ is given by $\delta_i = m_i - 1 - s_i = 0$, which implies $s_i = m_i - 1$ where $s_i = \dim(H_i)$.

Defining the number of complexes, the numbers of linkage classes and the SS of $(S', C', R')$ by $m', p'$ and $H'$ respectively, we easily see from the conditions 2 and 3, and (14) that $m' = \Sigma_{i=1}^{N+1} m_i$, $p' = 1$ and $s' = \text{dim}(H') = \Sigma_{i=1}^{N+1} s_i + N$, and hence we have $\delta' = m' - p' - s' = \Sigma_{i=1}^{N+1} m_i - 1 - (\Sigma_{i=1}^{N+1} s_i + N) = \Sigma_{i=1}^{N+1} m_i - 1 - s_i + N - 1 = N - 0$.

Therefore, the deficiency of the network $(S', C', R')$ is zero. Besides, it is clear that the network is not weakly reversible, hence this theorem follows from Theorem 2.1 (i). □

Next, we shall give the proof of Theorem 3.2.

First, by using (8) in the proof of Theorem 4.3 we show in the following two theorems that any positive solution to the problem $(P')$ of the network $(S', C', R')$ is bounded globally in time.

**Theorem 4.5** For any positive solution $x(t)$ to $(P')$, $x^{(i)}(t)$ is bounded globally in time, that is,

$$\sup_{t \geq 0} \|x^{(i)}(t)\| < +\infty.$$  

**Proof:** By taking the time derivative of the functions $y^{(i)}(k), k = 1, \ldots, n_i = \dim(H_i)$ given by (8) along the positive solution $x(t)$ to $(P')$, we have

$$\frac{d}{dt} \sum_{j=1}^{N+1} a_j^{(i)}(k) \frac{d}{dt} x^{(i)}(j) = \sum_{j=1}^{N+1} a_j^{(i)}(k) x^{(i)}(j)(p_i) \leq \sum_{i=2}^{N+1} x^{(i)}(j)(p_i) \geq 0.$$  

Hence from the positivity of the solution to $(P')$ we have $\sum_{i=2}^{N+1} x^{(i)}(j)(t) \leq \sum_{i=2}^{N+1} x^{(i)}(0)$ for all $t \geq 0$ and putting $M^{(i)}(x(0)) := \max_{1 \leq i \leq n_i = \dim(H_i)} \sum_{j=1}^{N+1} a_j^{(i)}(k)$, we see that for all $t \geq 0$, $0 \leq x^{(i)}(t) \leq M^{(i)}(x(0)), j = 1, \ldots, n_i$ which implies $\sup_{t \geq 0} \|x^{(i)}(t)\| < +\infty$. □

**Lemma 4.6** The following limit exists:

$$\lim_{t \to \infty} \int_{0}^{t} \sum_{i=1}^{n_i} x^{(i)}(p_i)(s)ds, i = 2, 3, \ldots, N + 1.$$  

(17)
Proof: Integrating (16) from 0 to t, we have
\[ T_k^{(i)}(x^{(i)}(0)) - T_k^{(i)}(x^{(i)}(t)) = \sum_{j=2}^{N+1} k_j^{(i)}(\gamma_{1}^{(i)}) \sum_{j=1}^{n} a_j^{(i)}(k) y_j^{(i)}(q_j) \int_0^t \prod_{i=1}^{n} x_i^{(j)}(p_i)(s)ds. \]
Hence from the positivity of the solution to \((P')\) we have
\[ \frac{T_k^{(i)}(x^{(i)}(0))}{k_j^{(i)}(\gamma_{1}^{(i)}) \sum_{j=1}^{n} a_j^{(i)}(k) y_j^{(i)}(q_j)} \geq \int_0^t \prod_{i=1}^{n} x_i^{(j)}(p_i)(s)ds, \forall t \geq 0, \ i = 2, \ldots, N + 1. \]
Since the above integral is an increasing function of \(t\), (17) holds. \(\square\)

Theorem 4.6 For any positive solution \(x(t)\) to \((P')\), \(x^{(i)}(t), i = 2, \ldots, N + 1\) is bounded globally in time, that is,
\[ \sup_{t \geq 0} \|x^{(i)}(t)\| < +\infty. \] (18)

Proof: In the same way as the proof of Theorem 4.5, we first take the time derivative of the functions \(T_k^{(i)}, k = 1, \ldots, n_i - \dim(H_i)\) given by (8) along the positive solution \(x(t)\) to \((P')\). Then for all \(i = 2, \ldots, N + 1\) we have
\[ \frac{d}{dt} T_k^{(i)}(x^{(i)}(t)) = \sum_{j=1}^{n_i} a_j^{(i)}(k) \frac{d}{dt} x_j^{(i)}(t) = k_j^{(i)}(\gamma_{1}^{(i)}) \sum_{j=1}^{n_i} a_j^{(i)}(k) y_j^{(i)}(q_j) \prod_{i=1}^{n_i} x_i^{(j)}(p_i)(t). \] (19)
Hence, from Lemma 4.6 we have
\[ T_k^{(i)}(x^{(i)}(t)) \leq T_k^{(i)}(x^{(i)}(0)) + k_j^{(i)}(\gamma_{1}^{(i)}) \sum_{j=1}^{n_i} a_j^{(i)}(k) y_j^{(i)}(q_j) \int_0^t \prod_{i=1}^{n_i} x_i^{(j)}(p_i)(s)ds =: T_k^{(i)}(x^{(i)}(0)), \forall t \geq 0, \] (20)
and putting
\[ M^{(i)}(x^{(0)}(0)) := \max_{1 \leq j \leq n_i - \dim(H_i)} T_k^{(i)}(x^{(i)}(0)), \]
we see that for all \(t \geq 0, 0 \leq x^{(i)}(t) \leq M^{(i)}(x^{(0)}(0)), i = 1, \ldots, n_i\) which implies \(\sup_{t \geq 0} \|x^{(i)}(t)\| < +\infty. \) \(\square\)

By integrating (19) from 0 to \(t\) and applying Lemma 4.6, we have the following theorem.

Theorem 4.7 For any positive solution \(x(t)\) to \((P')\), we have
\[ \lim_{t \to \infty} T_k^{(i)}(x^{(i)}(0)) = \lim_{t \to \infty} T_k^{(i)}(x^{(i)}(t)) =: T_k^{(i)}(x^{(i)}(0)) \]
\[ = T_k^{(i)}(x^{(i)}(0)), \quad k = 1, \ldots, n_i - \dim(H_i), \quad i = 2, \ldots, N + 1. \]

On the basis of the boundedness, Lemma 4.6 and Theorems 4.5 and 4.6, we show the convergence of the solution to \((P')\).

First, in order to show the convergence of \(x^{(i)}\), we give the following lemma.

Lemma 4.7 We consider the positive solution \(x(t) \in C^1([0, \infty); \mathbb{R}^n_{>0})\) to the following ODE:
\[ \frac{d}{dt} x_i(t) = b_i f(x(t)), \quad i = 1, \ldots, n, \]
\[ x(0) = x_0 \in \mathbb{R}^n_{>0}, \] (21)
where \(f \in C(\mathbb{R}^n; \mathbb{R})\) and \(b_i > 0, i = 1, \ldots, n\). If the solution satisfies
\[ \lim_{t \to \infty} \prod_{i=1}^{n} x_i^k(t) = 0, \] (23)
then there exists \(k \in \{1, \ldots, n\}\) such that \(\lim_{t \to \infty} x_k(t) = 0\), and the limit, \(\lim_{t \to \infty} x_k(t)\), exists for any other \(i \in \{1, \ldots, n\}\).

Proof: Putting \(a_i := \prod_{k=1, k \neq i}^{n} b_k, i = 1, \ldots, n\), we have
\[ a_i \dot{x}_i(t) - a_j \dot{x}_j(t) = \left( \prod_{k=1, k \neq i}^{n} b_k \right) - \left( \prod_{k=1, k \neq j}^{n} b_k \right) f(x(t)) = 0, \]
\[ 1 \leq i, j \leq n, i \neq j, \forall t \geq 0, \] (24)
which implies that \(a_i x_i(t) - a_j x_j(t) = a_i x_i(0) - a_j x_j(0)\) for all \(t \geq 0\).

Now we choose an index \(k\) satisfying
\[ a_k x_k(0) = \min_{1 \leq i \leq n} a_i x_i(0). \]
Then since for all \(t \geq 0\)
\[ x_k(t) = a_k x_k(t) + a_k x_k(0) - a_k x_k(0), \quad i = 1, \ldots, n, i \neq k, \] (25)
we have
\[ \prod_{i=1}^{n} x_i^k(t) = x_k^k(t) \prod_{i=1, i \neq k}^{n} \frac{a_i x_i(t) + a_i x_i(0) - a_k x_k(0)}{a_k}, \forall t \geq 0. \]
We assume that \(\lim_{t \to \infty} x_k(t) \neq 0\). Then we see from
\[ a_k x_k(0) - a_k x_k(0) \geq 0 \] that \(\lim_{t \to \infty} x_k(t) > 0\), hence
\[ \lim_{t \to \infty} \prod_{i=1}^{n} x_i^k(t) > 0, \] (26)
which is in contradiction with (23).

Therefore we have \(\lim_{t \to \infty} x_k(t) = 0\). Furthermore, we see from (25) that any other variable \(x_i(t)\) converges to a non-negative value. \(\square\)

From Lemma 4.7, we obtain the following lemma.

Lemma 4.8 For any positive solution \(x(t)\) to \((P')\), there exists \(x^{(i)}_i \in \text{supp}(x^{(i)}(p_i))\) satisfying \(\lim_{t \to \infty} x^{(i)}_i(t) = 0\) for any \(k = 2, \ldots, N + 1\). Moreover, any other concentration of the species contained in \(\text{supp}(x^{(i)}(p_i))\) converges to a non-negative value.

Proof: We easily see from the ODE (1) with mass action kinetics (2) and Theorem 4.5 that for all \(t \geq 0\),
\[ \frac{d}{dt} \left( \prod_{i=1}^{n} x_i^{(j)}^{(i)}(p_i)(t) \right) < +\infty, \quad k = 2, \ldots, N + 1. \] (27)
Hence, by combining (27) with Lemma 4.6 and applying Barbalat’s lemma [9], we have
\[ \lim_{t \to \infty} \prod_{i=1}^{n} x_i^{(j)}^{(i)}(p_i)(t) = 0, \quad k = 2, \ldots, N + 1. \] (28)
Now, for any index $l$ satisfying $X_l^{(1)} \in \text{supp}(y_l^{(1)}(p_k))$, the ODE of $x_{l}^{(1)}$ can be given by $\dot{x}_{l}^{(1)}(t) = y_{l}^{(1)}(p_k)j_{l}(x(t))$ where

\[
    f_{l}^{(1)}(x) := -\sum_{y^{(1)}(p_k) \neq y^{(1)}(p_k)} k_{l}^{(1)}(p_{k} \rightarrow y^{(1)}(p_k)) \times_{l}^{(1)}(p_k) + \sum_{y^{(1)}(p_k) \neq y^{(1)}(p_k)} k_{l}^{(1)}(y^{(1)}(p_k)) x_{l}^{(1)}(t).
\]  

(29)

Hence, from Lemma 4.7 this lemma holds. \hfill \Box

Theorem 4.8 For any positive solution $x(t)$ to $(P')$, there exists a vector $\vec{x}^{(1)} \in \mathbb{R}^{n_0}$ satisfying $(\vec{x}^{(1)})^{T} = 0$ for all $y^{(1)} \in C_1$ such that $x^{(1)}(t) \rightarrow \vec{x}^{(1)}$, $t \rightarrow \infty$.

Proof: From Lemma 4.8, there exists $X_l^{(1)} \in \text{supp}(y_l^{(1)}(p_k))$ for any $k = 2, \ldots, N + 1$ satisfying $\lim_{t \to \infty} x_{l}^{(1)}(t) = 0$. We see from Theorem 4.5 that the derivative $\dot{x}_{l}^{(1)}(t)$ is uniformly continuous with respect to $t$, hence, by applying Barbalat’s lemma we have $\dot{x}_{l}^{(1)}(t) \rightarrow 0$, $t \rightarrow \infty$.

From the proof of Lemma 4.8, we have

\[
    \dot{x}_{l}^{(1)}(t) + y_{l}^{(1)}(p_k) \sum_{y^{(1)}(p_k) \neq y^{(1)}(p_k)} k_{l}^{(1)}(p_{k} \rightarrow y^{(1)}(p_k)) \times_{l}^{(1)}(p_k) = y_{l}^{(1)}(p_k) \sum_{y^{(1)}(p_k) \neq y^{(1)}(p_k)} k_{l}^{(1)}(y^{(1)}(p_k)) x_{l}^{(1)}(t),
\]  

(30)

and

\[
    \lim_{t \to \infty} y_{l}^{(1)}(p_k) \sum_{y^{(1)}(p_k) \neq y^{(1)}(p_k)} k_{l}^{(1)}(y^{(1)}(p_k)) x_{l}^{(1)}(t) = 0,
\]  

(31)

which implies that $\lim_{t \to \infty} x_{l}^{(1)}(t) = 0$ for all $y^{(1)} \in C_1$ satisfying $y^{(1)} \rightarrow y^{(1)}(p_k) \in R_1$.

Hence, we see from Lemma 4.8 that for all $y^{(1)} \in C_1$ satisfying $y^{(1)} \rightarrow y^{(1)}(p_k) \in R_1$ all variables contained in $\text{supp}(y^{(1)})$ converge.

By repeating the above argument and using weak reversibility of $(S_1, C_1, R_1)$, we can show that this theorem holds. \hfill \Box

Finally, we show the convergence of $x^{(i)}$, $i = 2, \ldots, N + 1$, of the solution $x(t)$ to $(P')$ with the initial value $x(0) = x_0 \in \mathbb{R}^{n_0}$, preparing the following two lemmas.

Lemma 4.9 The $\omega$-limit set $\omega(x(0))$ of $(P')$ is a non-empty, compact, and invariant subset of $X \subseteq \mathbb{R}^{n_0}$ defined by

\[
    X := \left\{ x \in \mathbb{R}^{n_0} \left| x^{(i)} = \pi^{(1)}, \sqrt{\sum_{i=1}^{n_0} \left( x^{(i)} - \pi^{(1)} \right)^2} \leq 2 \right| \right. \left( x^{(i)}(t) = 2^{k-1} \sum_{i=1}^{n_0} \left( x^{(i)}(t) - x^{(i)}(0) \right), \right. \left. k = 1, \ldots, t \neq \text{dim}(H), \right. \left. i = 2, \ldots, N + 1 \right\}.
\]  

(32)

where $\pi^{(1)}$ is the vector introduced in Theorem 4.8.

Proof: This follows from the general theory of ODEs and the direct consequence of Theorems 4.6, 4.7 and 4.8, so we omit the proof of this theorem here. \hfill \Box

From the above results, we obtain the following lemma immediately.

Lemma 4.10 For any semi-locking set $W$ of the network $(S_1', C_1', R_1')$ including $W_1 \subset S_1$ such that $W \cap (S_2 \cup \ldots \cup S_{N+1} \neq 0$, it holds that $\omega(x(0)) \cap L_W = \emptyset$. Here, the set $W_1$ is given by

\[
    W_1 := \left\{ x^{(1)} \in S_1 \left| x^{(1)} \right._{\pi^{(1)}} = \emptyset \right. \right. \right\}.
\]  

(33)

Proof: We easily see that for any semi-locking set $W$ of the network $(S_1', C_1', R_1')$ including $W_1$ such that $W \cap (S_2 \cup \ldots \cup S_{N+1} \neq 0$, there exists $i \in [2, \ldots, N + 1]$ such that $W_i \cup W_i \subset W$, where $W_i$ is a minimal semi-locking set of the network $(S_i, C_i, R_i)$.

We assume that $\omega(x(0)) \cap L_W \neq \emptyset$. Since $W_i$ is a minimal semi-locking set of the network $(S_i, C_i, R_i)$, by taking the time derivative of $T_W^{(1)}$ given by (12) along a positive solution $x(t)$ to $(P')$, we have

\[
    \lim_{t \to \infty} T_W^{(1)}(x^{(i)}(t)) = T_W^{(1)}(x^{(i)}(0)) + \sum_{i=1}^{n_0} \int_0^\infty x^{(i)}(s) ds > 0,
\]  

(34)

However, for all $w \in \omega(x(0)) \cap L_W$ we see from the continuity of $T_W^{(1)}(x^{(i)}(t))$ with respect to $t$ that $T_W^{(1)}(x^{(i)}(0)) > 0$, which is in contradiction with $w \in L_W$ and $W_i \subset W$.

Therefore, it holds that $\omega(x(0)) \cap L_W = \emptyset$. \hfill \Box

Now, we define a subset $X'$ of $X$ by

\[
    X' := \left\{ x \in X \left| x^{(i)} \in \mathbb{R}^{n_0}, \sqrt{\sum_{i=1}^{n_0} \left( x^{(i)} - \pi^{(1)} \right)^2} \leq 2 \right| \right. \left( x^{(i)}(t) = 2^{k-1} \sum_{i=1}^{n_0} \left( x^{(i)}(t) - x^{(i)}(0) \right), \right. \left. k = 1, \ldots, t \neq \text{dim}(H), \right. \left. i = 2, \ldots, N + 1 \right\}.
\]  

(35)

We easily see that the set $X'$ is a forward invariant set, that is, any positive solution $x(t)$ to $(P')$ with an initial value $x(0) \in X'$ remains $x(t) \in X'$ for all $t \geq 0$. Also, together with Lemmas 4.4 and 4.10 we see that $\omega(x(0)) \subset X'$.

By using the results above and the following theorem [10], we can prove Theorem 3.2.

Theorem 4.9 Suppose that $C$ is a nonempty, compact and invariant set of a dynamical system generated by a system $\dot{x} = f(x)$ defined on some set $\Omega \subseteq \mathbb{R}^n$ and that $\Omega$ is a globally asymptotically stable equilibrium point. Then $C = \{ \Xi \}$.

The proof of Theorem 3.2: We have $x^{(1)} = \pi^{(1)}$ for all $x \in X'$, which implies that

\[
    \sum_{i=1}^{n_0} x^{(i)}(t) \theta^{(1)}(p_k) = 0, i = 2, \ldots, N + 1.
\]

Hence, we see that solutions to the problem $(P')$ restricted on the set $X'$ are equivalent to those to independent problem of $(P_i)$, $i = 2, \ldots, N + 1$.

Here we see that the conditions $T_k^{(1)}(x^{(i)}) = T_k^{(1)}(x^{(i)}(0))$, $k = 1, \ldots, n_i - \text{dim}(H)$ in (35) determine PSCCs of $(P_i)$, $i = 2, \ldots, N + 1$. Hence we see from Theorem 4.4 that the problem $(P')$ on the set $X'$ has a unique globally asymptotically stable equilibrium point $\Xi := \{ x^{(1)}; x^{(2)}; \ldots; x^{(N+1)} \}' \in X'$ since $X'$ is forward invariant. Here, the vector $\pi^{(1)}(p_k)$ represents a unique globally asymptotically stable equilibrium point on the PSCC of each $(P_i)$, $i = 2, \ldots, N + 1$. Besides, since from Lemmas 4.9 and 4.10 $\omega(x(0))$ of $(P')$ is a non-empty, compact, and invariant subset of $X'$, by applying Theorem 4.9 we have $\omega(x(0)) = \{ \Xi \}$, which completes the proof. \hfill \Box
5. Example

As an application of Theorem 3.2, we consider the CRN [1] given in Fig. 2. This network consists of three sub-networks (S_i, C_i, R_i) (i = 1, 2, 3), where S_i, C_i, and R_i are given by:

S_1 = \{X_1, X_2, X_3\}, S_2 = \{X_4, X_5, X_6\}, S_3 = \{X_7, X_8, X_9\},
C_1 = \{X_1 + X_2, X_3\}, C_2 = \{X_4 + X_5, X_6\}, C_3 = \{X_7, X_8, 2X_9\},
R_1 = \{X_1 + X_2 \to X_3, X_3 \to X_1 + X_2\},
R_2 = \{X_4 + X_5 \to X_6, X_6 \to X_1 + X_3\},
R_3 = \{X_7 \to X_8, X_8 \to 2X_9, 2X_9 \to X_8, 2X_9 \to X_7\}.

Since this network is not weakly reversible, we can not apply Theorem 2.1 (DZT) to it. However, it is easily verified that any positive solution to the system (1) for the CRN shown in Table 1 and the rate constants in Fig. 2.

Theorem 2.1 (DZT) to it. However, it is easily verified that any positive solution to the system (1) for the CRN shown in Table 1 and the rate constants in Fig. 2.

The numerical calculation is executed for 0 ≤ t ≤ 10 on the basis of Eq. (1) with the mass action kinetics (2), using the initial values as shown in Table 1. Consequently, we can confirm that the solution of Eq. (1) converges to an equilibrium point on the boundary of the PSCC.

A numerical example is illustrated in Fig. 3 and Table 1. The numerical calculation is executed for 0 ≤ t ≤ 10 on the basis of Eq. (1) with the mass action kinetics (2), using the initial values shown in Table 1 and the rate constants in Fig. 2.

Actually, we clearly see from Fig. 3 that x_3(t) vanishes and x_6(t) converges to some positive values, besides other variables of concentrations converge to some non-negative values as shown in Table 1. Consequently, we can confirm that the solution of Eq. (1) converges to an equilibrium point on the boundary of the PSCC.

In the present paper, we have discussed a stability analysis for a certain class of non-weakly reversible chemical reaction networks. By decomposing the network into weakly reversible sub-networks and applying the DZT to them, we have proven any solution with positive initial values converges to an equilibrium point on the boundary of the positive orthant.

In future work, we will deal with a more general class of non-weakly reversible chemical reaction networks than that dealt with in this work, and construct an extended Deficiency Zero Theorem.

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