Optimal Binary Search Trees with Near Minimal Height

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Abstract. Suppose we have \( n \) keys, \( n \) access probabilities for the keys, and \( n + 1 \) access probabilities for the gaps between the keys. Let \( h_{\text{min}}(n) \) be the minimal height of a binary search tree for \( n \) keys. We consider the problem to construct an optimal binary search tree with near minimal height, i.e. with height \( h \leq h_{\text{min}}(n) + \Delta \) for some fixed \( \Delta \). It is shown, that for any fixed \( \Delta \) optimal binary search trees with near minimal height can be constructed in time \( O(n^2) \). This is as fast as in the unrestricted case.

So far, the best known algorithms for the construction of height-restricted optimal binary search trees have running time \( O(Ln^2) \), whereby \( L \) is the maximal permitted height. Compared to these algorithms our algorithm is at least faster by a factor of \( \log_2 n \), because \( L \) is lower bounded by \( \log_2 n \).

1 Introduction

Suppose we have \( n \) keys, \( n \) access probabilities for the keys, and \( n + 1 \) access probabilities for the gaps between the keys. The problem to construct a binary search tree for these \( n \) keys that minimizes the expected access time is known as the \textit{optimal binary search tree problem}. Knuth presented in [6] a well-known dynamic programming algorithm that solves this problem in \( O(n^2) \) time.

Apart from the original problem, the construction of optimal binary search trees whose heights are restricted has been considered in the literature. By the height restriction the maximum number of comparisons during a search can be bounded. Thus, an optimal height restricted binary search tree performs well in both the worst and the average case. Itai [5] and Wessner [10] independently discovered construction algorithms for height restricted binary search trees. Their algorithms have running time \( O(Ln^2) \), where \( L \) is the maximal permitted height.

Let \( h_{\text{min}}(n) = \lceil \log_2(n+1) \rceil \) be the \textit{minimal height of a binary search tree for \( n \) keys}. In this paper, we show that for any fixed \( \Delta \) an optimal binary search tree with height \( h \leq h_{\text{min}}(n) + \Delta \) can be constructed in time \( O(n^2) \). This improves the results from Itai and Wessner [5, 10]. Because \( L \geq \lceil \log_2(n+1) \rceil \), the algorithms of Itai and Wessner have running time \( O(n^2 \log n) \) if we use them to construct optimal search trees with height \( h \leq h_{\text{min}}(n) + \Delta \).
Gagie [2, 3] presents a $O(n)$ time algorithm for the restructuring of optimal binary search trees. His algorithm restructures an existing optimal binary search in such a way that the resulting tree has nearly optimal height and cost. In contrast to Gagie’s algorithm our algorithm always selects the best binary search tree from the set of all trees with restricted height.

Other interesting facts about optimal binary search trees can be found in the article of Nagaraj [7]. This article gives a comprehensive survey about optimal binary search trees.

All algorithms for the construction of optimal binary search trees, whether height restricted or not, are based on dynamic programming. They all use step by step construction of larger trees from smaller subtrees. Instead of step by step construction from smaller subtrees we use a decision model where the keys are placed by a sequential decision process in such a way into the tree, that the costs become optimal. This approach is adopted from the construction algorithm for optimal B-trees [1].

The rest of the paper is structured in the following way: in Section 2 a formal description of the problem is given. In Section 3 we present our approach: the decision model is explained and the attached dynamic program is formulated. Section 4 states the solution algorithm and gives the complexity results. Section 5 summarizes the results.

2 The Problem

Now we give the problem formulation. We have $n$ keys $k_1 < k_2 \ldots < k_n$ and $2n + 1$ probabilities $\alpha_0, \beta_1, \alpha_1, \beta_2, \ldots, \beta_n, \alpha_n$.

$\beta_i$ are the key weights and $\alpha_j$ are the gap weights. $\beta_i$ is the probability that key $k_i$ is requested, and $\alpha_j$ is the probability, that a search is made for a key $d$ with $k_j < d < k_{j+1}$. We assume that we have artificial keys $k_0 = -\infty$ and $k_{n+1} = \infty$.

Let $b_i$ be the level resp. the depth of the $i$-th internal node where key $k_i$ is stored, and let $a_j$ be the level of the external node for the gap between $k_j$ and $k_{j+1}$. The root is on level 0. For a binary search tree $T$ we define the weighted path length $\text{wpl}(T)$ by

$$\text{wpl}(T) := \sum_{i=1}^{n} \beta_i (b_i + 1) + \sum_{j=0}^{n} \alpha_j a_j$$

The weighted path length is the expected number of node visits resp. comparisons in a search.

The height $h(T)$ of a tree $T$ is defined as the level of the deepest external node. The minimal height $h_{\min}(n)$ of a binary search tree for $n$ keys is then given by

$$h_{\min}(n) = \lceil \log_2(n + 1) \rceil$$

We want to construct search trees whose heights are nearly minimal. Let $\Delta \geq 0$ be some fixed value. The problem is to find a binary search tree $T$ that
minimizes the weighted path length \(wpl(T)\) subject to the constraint \(h(T) \leq h_{\text{min}}(n) + \Delta\). Such a tree is denoted as an \textit{optimal binary search tree with near minimal height}.

3 Dynamic Programming Model

We model the process of constructing an optimal binary search tree with near minimal height as a decision problem with \(n\) stages. For every key \(k_i\) we have to decide, on which level this key should be placed. Whether placing on some level is feasible, depends on the former decisions for the keys \(k_1\) to \(k_{i-1}\), which define a certain state in the decision process. Then placing the key \(k_i\) on any level results in an increasing weighted path length and a new state. The amount of increasing as well as the new state depend on our decision.

Using this approach, the optimal tree is the result of a sequence of optimal decisions starting in a unique initial state. This leads to a dynamic program \(DP\) of the form \(DP = (S_\nu, A_\nu, D_\nu, T_\nu, c_\nu, C_{n+1})\), where \(n\) is the number of the stages of \(DP\), \(S_\nu\) is the state set of stage \(\nu\), \(1 \leq \nu \leq n+1\), and \(A_\nu\) is the decision set of stage \(\nu\), \(1 \leq \nu \leq n\). The sets \(D_\nu \subseteq S_\nu \times A_\nu\) define the feasible decisions for the states of stage \(\nu\). It holds: \((s, a) \in D_\nu\), if and only if \(a\) is feasible in state \(s\) on stage \(\nu\). The set \(D_\nu(s) := \{a \in A_\nu|(s, a) \in D_\nu\}\) contains all feasible decisions for state \(s\) on stage \(\nu\). \(T_\nu : D_\nu \rightarrow S_{\nu+1}\) is the transition function. Making decision \(a\) in state \(s\) at stage \(\nu\) results in state \(T_\nu(s, a)\) at stage \(\nu + 1\). \(c_\nu : D_\nu \rightarrow \mathbb{R}\) is the cost function of stage \(\nu\). \(c_\nu(s, a)\) gives the costs that arise if we decide to make decision \(a\) in state \(s\) on stage \(\nu\). \(C_{n+1} : S_{n+1} \rightarrow \mathbb{R}\) is the terminal cost function. \(C_{n+1}(s)\) gives the costs that arise if our final state is \(s\).

Now we have to define the components of the dynamic program in such a way that the decision process models the construction of a binary search tree with restricted height. First we give the definition of the states. For motivation take a look at Figure [1]. Suppose we have \(h_{\text{max}} := h_{\text{min}}(n) + \Delta = 3\), that means we can place the keys on levels from 0 to 2.

For a correct placing of a key in the partial tree only the rightmost path fragments from the actual root to the node that contains the largest key is relevant. Due to this fact we can represent a state \(s \in S_\nu\) by a binary vector with \(h_{\text{max}}\) components. We number the vector components from 0 to \(h_{\text{max}} - 1\). Vector component \(s_i\) is related to level \(i\).

\[
s = \begin{pmatrix}
  s_0 \\
  \vdots \\
  s_{h_{\text{max}}-1}
\end{pmatrix}
\quad \text{with } s_i \in \{0, 1\}.
\]

Each vector component \(s_i\) determines, whether the level \(i\) in the rightmost path is occupied. More formally, vector component \(s_i\) is 1 if and only if the largest key on level \(i\) is greater than any key on the levels from 0 to \(i - 1\). For instance
Fig. 1. Tree states in the construction process

the state $s$ resulting from tree (a) in Figure 1 is represented by

$$s = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and the state $s'$ resulting from tree (c) by

$$s' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Observe, that different trees may have the same associated states. For instance the trees (a) and (b) of Figure 1 are both represented by the same state.

The set $S_\nu$ is defined to be the set of all vectors that are possible after the assignment of $\nu - 1$ keys. The initial state set $S_1$ consists of a single state:

$$S_1 := \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$
A decision is characterized by the level on which a key is placed. So we define \( A = A_\nu = \{0, \ldots, h_{\text{max}} - 1\} \). Making decision \( a \) means that the corresponding key is placed on level \( a \). For instance, the tree (a) in Figure 1 is constructed by the decision sequence \( DS = (1, 0, 2, 1) \).

![Decision Example](image)

**Fig. 2.** Feasible and infeasible decision

Let \( s = (s_0, \ldots, s_{h_{\text{max}}-1}) \) be a state. A feasible decision \( a \) for state \( s \) has to fulfill the following conditions:

(i) We can place keys only on unoccupied levels:
\[
s_a = 0
\]

(ii) If a key is placed above some path fragment, this path fragment has to be the deepest path fragment and the key has to be placed directly above this path fragment:
\[
\not\exists i, j : a < i < j \text{ and } s_i = 0 \text{ and } s_j = 1
\]

Condition (i) is obvious. Figure 2 demonstrates condition (ii). The next key \( k_4 \) has to be placed on level 2, because \( k_3 \) becomes the left son of \( k_4 \). If we place \( k_4 \) on level 1, the left son would not be on the next deeper level.

So we can define
\[
D_\nu := \{(s, a) | s \in S_\nu, a \text{ fulfills (i) and (ii)}\}
\]

Observe that the feasible decisions of a state \( s \) are independent of the stage \( \nu \). So we define
\[
D(s) := \{a \in A | a \text{ fulfills (i) to (ii)}\}
\]
as the set of feasible decisions for state \( s \). For every binary search tree (with near minimal height) there exists a unique feasible decision sequence that constructs...
the tree. As an example see the decision sequence to construct tree (a) of Figure 1 (see above). Using this definition each feasible decision sequence leads to trees that are valid binary search trees with the exception of the rightmost path. Trees with invalid rightmost path on stage $n + 1$ are filtered by the terminal cost function $C_{n+1}$ (see below).

![Diagram of trees](image)

**Fig. 3.** Example for a transition

Making a decision $a$ has two effects. First, the level $a$ of the rightmost path becomes occupied and second, the levels from $a + 1$ to $h_{\text{max}} - 1$ become unoccupied. So the definition for the transition function is:

$$T(s, a) := T_\nu(s, a) = \begin{pmatrix} s_0 \\ \vdots \\ s_{a-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Figure 3 shows an example for a single transition. The following state and decision sequence shows the transitions from the initial state to the right tree of Figure 3:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a=1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{a=0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{a=2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{a=1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

If we have a state $s \in S_\nu$, we can deduce from $s$ the preceding decision, i.e. the decision on stage $\nu - 1$ that induced $s$. Take a look at the transition function $T(s, a)$: the largest $i$ with $s_i = 1$ defines this preceding decision.

$$\text{precdec}(s) := \begin{cases} 0 & \text{if } s_0 = \cdots = s_{h_{\text{max}}-1} = 0 \\ \max\{0 \leq i \leq h_{\text{max}}-1 | s_i = 1\} & \text{otherwise} \end{cases}$$
Our cost function $c_\nu(s, a)$ has to consider two aspects: the level of key $k_\nu$ and the level of the gap $(k_{\nu-1}, k_\nu)$. The first is simple: the level of key $k_\nu$ is determined by the decision $a$. With the following Lemma, we are able the determine the level of the gap $(k_{\nu-1}, k_\nu)$.

**Lemma 1.** Let $\text{klevel}(k_\nu)$ denote the level of key $k_\nu$ and let $\text{glevel}(k_{\nu-1}, k_\nu)$ denote the level of the gap $(k_{\nu-1}, k_\nu)$. Then we have

$$\text{glevel}(k_{\nu-1}, k_\nu) = 1 + \max\{\text{klevel}(k_{\nu-1}), \text{klevel}(k_\nu)\}$$

**Proof.** Adjacent keys cannot be on the same level. So we have either $\text{klevel}(k_{\nu-1}) < \text{klevel}(k_\nu)$ or $\text{klevel}(k_{\nu-1}) > \text{klevel}(k_\nu)$.

In the case of $\text{klevel}(k_{\nu-1}) < \text{klevel}(k_\nu)$, the key $k_\nu$ is in the right subtree of key $k_{\nu-1}$ and the gap $(k_{\nu-1}, k_\nu)$ is the left son of the node that contains $k_\nu$. In the other case the key $k_{\nu-1}$ is in the left subtree of key $k_\nu$ and the gap $(k_{\nu-1}, k_\nu)$ is the right son of the node that contains $k_{\nu-1}$. In both cases the equation of Lemma 1 is valid.

The cost functions $c_\nu(s, a)$ are defined by:

$$c_\nu(s, a) := (1 + \max\{\text{precdec}(s), a\}) \cdot \alpha_{\nu-1} + (a + 1) \cdot \beta_\nu$$

This definition utilizes Lemma 1: $\text{klevel}(k_{\nu-1})$ is equivalent to $\text{precdec}(s)$ and $\text{klevel}(k_\nu)$ to the decision $a$.

The terminal costs $C_{n+1}$ model whether our final state fulfills the tree conditions. In particular, we have to check whether the right most path contains unoccupied levels above occupied levels. For instance, tree (c) of Figure is not a valid search tree because level 1 is not occupied but level 2 is. We have:

$$C_{n+1}(s) = \begin{cases} (1 + \text{precdec}(s)) \cdot \alpha_n & \text{if } s_0 = 1 \text{ and } \exists i < j : s_i = 0 \land s_j = 1 \\ \infty & \text{otherwise} \end{cases}$$

To check whether there exists an unoccupied level we use an adaption of condition (ii) of the feasible decision set $D(s)$. If the root level is occupied and there exists no unoccupied level above an occupied level the terminal costs consist of the access probability $\alpha_n$ of the last gap multiplied by the level of key $k_n$ plus 1.

Now the definition of the dynamic program $DP$ is complete. Using this definition the optimization problem is

$$F := \sum_{\nu=1}^n c_\nu(s_\nu, a_\nu) + C_{n+1}(s_{n+1}) \to \min$$

subject to:

$$s_1 = (0 \cdots 0) \quad a_\nu \in D(s_\nu), 1 \leq \nu \leq n \quad s_{\nu+1} = T(s_\nu, a_\nu), 1 \leq \nu \leq n$$

The value $F$ of the objective function yields the minimum weighted path length and the tree is given by the optimal sequence $(a_1, \ldots, a_n)$ of feasible decisions.
4 Algorithm and Complexity

For the solution of this optimization problem we use a common dynamic programming algorithm, cf. [8].

Algorithm 1.

1. /* Initialization */
2. forall \( s \in S_{n+1} \)
3. \( V_0(s) \leftarrow C_{n+1}(s) \)
4. /* Backward Computation */
5. for \( \nu \leftarrow n \) downto 1 do
6. forall \( s \in S_{\nu} \) do
7. \( V_\nu(s) \leftarrow \infty \)
8. \( \pi_\nu(s) \leftarrow \text{undefined} \)
9. forall \( a \in D(s) \) do
10. if \( c_\nu(s, a) + V_{\nu+1}(T(s, a)) < V_\nu(s) \) then
11. \( V_\nu(s) \leftarrow c_\nu(s, a) + V_{\nu+1}(T(s, a)) \)
12. \( \pi_\nu(s) \leftarrow a \)
13. /* Forward Computation */
14. \( s \leftarrow (0 \cdots 0) \)
15. \( F \leftarrow V_1(s) \)
16. for \( \nu \leftarrow 1 \) to \( n \) do
17. \( a_\nu \leftarrow \pi_\nu(s) \)
18. \( s \leftarrow T(s, a_\nu) \)

\( V_\nu(s) \) is the value function which represents the minimal costs to reach a terminal state from state \( s \) on stage \( \nu \). In line (1) and (2) we initialize the value function with the terminal costs. \( \pi_\nu(s) \) represents the optimal decision for state \( s \) on stage \( \nu \). The value function \( V_\nu(s) \) and the optimal decision \( \pi_\nu(s) \) is determined by the Bellman equation

\[
V_\nu(s) = \min_{a \in D(s)} \{ c_\nu(s, a) + V_{\nu+1}(T(s, a)) \}
\]

which is solved for all states on all stages in lines (4) to (11).

After the backward computation terminates, the \( \pi_\nu \) define an optimal policy. To get the optimal decision sequence we apply the \( \pi_\nu \) in a forward computation (line (13) to (17)) beginning with our initial state. As a result the \( a_\nu \) represent the decision sequence to build an optimal tree and the value of \( F \) is the weighted path length of the optimal tree.

With the decision sequence \( DS = (a_1, \ldots, a_n) \) that defines the optimal binary search tree we are able to build the corresponding tree in linear time, as for each key \( k_\nu \) the level where \( k_\nu \) has to be placed is given by the decision \( a_\nu \).

Example 1. Suppose we have keys \( k_1, \ldots, k_4 \) with access probabilities \( \beta_1 = \frac{3}{16}, \beta_2 = \frac{1}{16}, \beta_3 = \frac{1}{4}, \beta_4 = \frac{1}{2} \) and \( \alpha_0 = \cdots = \alpha_4 = 0 \). Let \( \Delta = 0 \), that means we have to construct a tree of height \( \lceil \log_2(5) \rceil = 3 \).
Figure 4 shows the search graph for this problem. The number adjacent to an arc represents the cost $c_{\nu}(s,a)$ of the corresponding transition. The terminal costs $C_5(s)$ are shown right beside the states for the state sets $S_1$ to $S_4$. Observe, that the value function $V_\nu(s)$ is shown right beside the states for the state sets $S_1$ to $S_4$. Observe, that the value function of state $(1,11) \in S_4$ yields $\infty$ because of an empty decision set.

The best decision sequence $DS = (1,2,0,1)$ is given by the bold arcs. Its overall cost is $\frac{25}{16}$, that means the corresponding optimal binary search tree has a weighted path length of $\frac{25}{16}$. Figure 5 shows the corresponding tree.

Our complexity results are based on bounds for the cardinality of the state sets $S_\nu$ and the decision sets $D_\nu$.

**Theorem 1.** For all state sets $S_\nu (\nu = 1, \ldots, n + 1)$ we have:

$$|S_\nu| \leq 2^{4+1}(n + 1)$$

**Proof.** Let $h_{max}(n) := h_{min}(n) + \Delta$ and $S := \{0,1\}^{h_{max}(n)}$. With these definitions we get

$$|S_\nu| \leq |S| = 2^{h_{max}(n)} = 2^{h_{min}(n)+\Delta}$$

Using $h_{min}(n) = \lceil \log_2(n + 1) \rceil$ we get

$$|S| \leq 2^{\lceil \log_2(n+1) \rceil + \Delta}$$

$$\leq 2^{4+1} \cdot 2^{\log_2(n+1)}$$

$$= 2^{4+1} \cdot (n + 1)$$
Corollary 1. For any fixed $\Delta$ the cardinality of the state sets $S_\nu$ is bounded by $O(n)$.

Theorem 2. For all feasible decision sets $D_\nu (\nu = 1, \ldots , n + 1)$ we have:

$$|D_\nu| \leq 2^{\Delta + 2}(n + 1)$$

Proof. Let $h_{\text{max}}(n) := h_{\text{min}}(n) + \Delta$, $S := \{0, 1\}^{h_{\text{max}}(n)}$ and $D := \{(s,a) | s \in S, a \text{ is feasible for } s\}$. With these definitions we get $|D_\nu| \leq |D|$ for all $\nu = 1, \ldots , n$.

How many feasible decisions exists for a state $s \in S$? Take a look at condition (ii) in the definition of $D(s)$ (see Section 3). If $s_{h_{\text{max}}-1} = 1$ there is at most one feasible decision $a$, which is determined by the highest index $a$ with $s_a = 0$. That means, that half of all the states in $S$ have only one feasible decision. States with $s_{h_{\text{max}}-1} = 0$ and $s_{h_{\text{max}}-2} = 1$, which comprise a quarter of all states in $S$, have at most two decisions. Generalized, $\frac{1}{2^k}|S|$ states of all the states in $S$ have $k$ feasible decisions. We get:

$$|D_\nu| \leq |D|$$
$$\leq 1 \cdot \frac{1}{2}|S| + 2 \cdot \frac{1}{4}|S| + 3 \cdot \frac{1}{8}|S| + \cdots$$
$$\leq \sum_{k=0}^{\infty} \frac{k}{2^k} \cdot |S|$$
$$= \left( \sum_{k=0}^{\infty} \frac{k+1}{2^k} - \sum_{k=0}^{\infty} \frac{1}{2^k} \right) \cdot |S|$$
$$= \left( \frac{1}{1 - \frac{1}{2}^2} - \frac{1}{1 - \frac{1}{2}} \right) \cdot |S|$$
$$= 2 \cdot |S|$$
$$\leq 2^{\Delta + 2}(n + 1)$$

Corollary 2. For any fixed $\Delta$ Algorithm constructs an optimal binary search with height $h \leq h_{\text{min}}(n) + \Delta$ in time $O(n^2)$.

Proof. We have to iterate over the $n$ stages from $n$ down to 1. In doing so, the cardinality of each state set $S_\nu$ and each feasible decision set $D_\nu$ is bounded by $O(n)$ for fixed $\Delta$. All operations can be executed in constant time. It follows, that the overall running time is $O(n^2)$.

5 Summary

We have presented a quadratic time algorithm to compute optimal binary search trees with near minimal height, i.e. with height $h \leq h_{\text{min}}(n) + \Delta$ and fixed $\Delta$. The algorithm was adopted from the construction algorithm for optimal B-tress. The construction process was modeled by a decision oriented dynamic program: In
the model we have to decide key by key, on which level the key should be placed. The tree conditions are represented by additional constraints and a terminal cost function.

It seems to be easy to apply this approach to other kinds of trees. By applying the construction algorithm of [1], it should be possible to construct optimal B-trees with near minimal height and fixed order in quadratic time, too. The construction of unrestricted optimal B-trees needs time $O(n^{2+\frac{1}{\log k}})$. A generalization of the binary tree model to multiway trees of a fixed order should also lead to a quadratic time algorithm in contrast to the cubic time algorithms for the unrestricted case [4, 9]. This means for both cases, that optimal trees with near minimal height can be constructed faster than unrestricted trees. If we consider that optimal trees have typically a low height, the approach of height restriction may lead to fast construction algorithms, which generate optimal trees with high probability.

References

[1] Peter Becker. A new algorithm for the construction of optimal b-trees. In Proceedings of the 4th Scandinavian Workshop on Algorithm Theory (SWAT ’94), pages 49–60, 1994.
[2] Travis Gagie. New ways to construct binary search trees. In Proceedings of the 14th International Symposium on Algorithms and Computation (ISAAC 2003), pages 537–543, 2003.
[3] Travis Gagie. Restructuring binary search trees revisited. Information Processing Letters, 95:418–421, 2005.
[4] L. Gotlieb. Optimal multi-way search trees. SIAM Journal on Computing, 10(3):422–433, 1981.
[5] A. Itai. Optimal alphabetic trees. SIAM Journal on Computing, 5:101–110, 1976.
[6] D. E. Knuth. Optimum binary search trees. Acta Informatica, 1:79–110, 1971.
[7] S. V. Nagaraj. Optimal binary search trees. Theoretical Computer Science, 188:1–44, 1997.
[8] K. Neumann and M. Morlock. Operations Research. Hanser, Munich, 2002.
[9] V. K. Vaishnavi, H. P. Kriegel, and D. Wood. Optimum multiway search trees. Acta Informatica, 14(2):119–133, 1980.
[10] R. L. Wessner. Optimal alphabetic search trees with restricted maximal height. Information Processing Letters, 4:90–94, 1976.