Non-regular Potentials and Sources for Static Axisymmetric Electrovacs

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Abstract
In this lecture a new formalism for constructing electromagnetic surface sources for static axisymmetric electrovacs is presented. The electrostatic and magnetostatic sources are derived from the discontinuities of the scalar potentials. This formalism allows the inclusion of two kinds of dipole sources: Sheets of dipoles and the dipole moment of a distribution of monopoles. It is a generalization of a previous formalism in order to cope with asymptotically monopolar electric fields.

1 Introduction
The problem of finding compact physically reasonable material sources that could be matched to an appropriate asymptotically flat vacuum spacetime is a task of great relevance in general relativity. Unfortunately, very few exact solutions are known for the stationary axisymmetric case (cfr. [1] for a recent review).

Instead of considering volume sources this lecture will be devoted to the calculation of electromagnetic sources for static axisymmetric asymptotically flat electrovacs. The electric charge distribution is easily obtained from the integration of the Maxwell equations [2], but electric and magnetic moment distributions cannot be calculated in that way. In order to achieve that goal a generalization of the approach followed in [3], [4] will be attempted. In these references magnetic and electric sources for static electrovacs were constructed thanks to the introduction of an asymptotically cartesian coordinate \( Z \), provided that the electric field was not monopolar. A way to circumvent that difficulty will be shown here. This will also allow to calculate the contribution of the charge density to the electric dipole.
In the next section the Green identity will be used to rederive the classical expression for the dipole surface density that arises from a discontinuous scalar potential. This will be helpful to understand its generalization to curved spacetimes in section 3. An example of an application of this formalism is presented in section 4. The results will be discussed at the end.

2 Non-relativistic dipole surface sources

Let us consider a non-relativistic physical vector field, $E$, (electric or magnetic) which can be obtained by differentiation of a scalar potential, $V$, ($E = -dV$) that fulfills the flat-spacetime Laplace equation. From the classical theory of potential [5] it is known that, if the field is discontinuous across a surface $S$, (and therefore the normal derivative of $V$ is discontinuous) then on crossing $S$ a layer of monopole charge is encountered and the surface density thereof, $\sigma_1$, is given by:

$$\sigma_1 = \frac{1}{4\pi} [E \cdot n] = -\frac{1}{4\pi} \left[ \frac{dV}{dn} \right]$$

where $n$ is the outer unitary normal to $S$ and a square bracket denotes the difference $([a] = a^+ - a^-)$ between the values of a quantity on the outer $(a^+)$ and inner $(a^-)$ sides of $S$.

If, besides, not only the field but also the potential is discontinuous on $S$, then the dipole density on $S$ can be constructed in the following way: Since both the cartesian coordinate $z$ and the scalar potential $V$ satisfy the Laplace equation out of $S$, then the Green identity is valid on $\mathbb{R}^3_+ \cup \mathbb{R}^3_-$, that is, the euclidean space outside and inside the surface $S$:

$$0 = \int_{\mathbb{R}^3_+ \cup \mathbb{R}^3_-} d^3x (V \Delta z - z \Delta V) = \int_{S^+(\infty) \cup S^-} dS \left( V \frac{dz}{dn} - z \frac{dV}{dn} \right) - \int_{S^+} dS \left( V \frac{dz}{dn} - z \frac{dV}{dn} \right)$$

since the boundary of $\mathbb{R}^3_\pm$ consists of the sphere at infinity and $S$ and the boundary of $\mathbb{R}^3_\pm$ is just $S$.

Taking into account that the asymptotic behaviour of $V$ is known from its expansion in Legendre polynomials and inverse powers of the spherical radius, the integral at infinity can be performed and the other terms can be identified as the dipole surface density $\sigma_2$ on $S$:

$$\sigma_2 = \frac{1}{4\pi} \left\{ n \cdot u_z [V] - z \left[ \frac{dV}{dn} \right] \right\}$$

The first term in this expression arises as the contribution of a sheet of dipoles [5] whilst the second one is the moment density of the $\sigma_1$ distribution.
3 Relativistic thin shells

In this section a way of generalizing the expression for the dipole density to Maxwell fields in curved spacetimes will be introduced. The metric for the static axially symmetric electromagnetic-gravitational system can be written in Weyl coordinates:

\[ ds^2 = -e^{2U} \, dt^2 + e^{-2U} \{ e^{2k} (d\rho^2 +dz^2) + \rho^2 \, d\phi^2 \} \] (4)

where \( U, k \) are functions of \( \rho \) and \( z \).

The scalar potential \( V \) for either the electric or the magnetic field satisfies the following equation, that can be derived from Maxwell’s vacuum equations only, even if the electromagnetic stress tensor is not the source of the gravitational field:

\[ \frac{1}{\sqrt{g}} \partial_\mu \left\{ \sqrt{g} e^{-U} \, g^{\mu\nu} \partial_\nu V \right\} = 0 \] (5)

where the metric \( g \) is the one induced by (4) on each of the hypersurfaces \( t = \text{const.} \).

Hence the results will be valid also for Maxwell fields in the geometry defined by (4).

Our aim will be to cope with compact sources, therefore only asymptotically flat metrics will be considered:

\[ ds^2 \simeq -(1 - \frac{2m}{r}) \, dt^2 + (1 + \frac{2m}{r})(dr^2 + (r^2 + \alpha r)(d\theta^2 + \sin^2 \theta \, d\phi^2)) \] (6)

in terms of the total mass \( m \) and a constant \( \alpha \) in a coordinate patch described by \( \{ t, \phi, r, \theta \} \). On the other hand, the scalar potential will be required to have the following expansion, where \( Q \) is the total monopole charge, \( d \) is the total dipole moment and \( \beta \) is another constant:

\[ V = \frac{Q}{r} + \frac{d \cos \theta}{r^2} + \frac{\beta}{r^2} + O(r^{-3}) \] (7)

If the source for the field is located on a surface \( S \), then equation (5) can be integrated on the regions \( V^+ \) and \( V^- \) in which the hypersurfaces \( t = \text{const.} \) are split by \( S \). Since the integrand is a total derivative, the integral can be reduced to a surface integral on \( S \) and another on the sphere at infinity and therefore it yields the expression for the monopole charge density:

\[ \sigma_1 = -\frac{1}{4\pi} e^{-U} \left[ \frac{dV}{dn} \right] \] (8)

which is the formula that was obtained by Israel in [2], but written in terms of the potential instead of the field. The only difference with the non-relativistic situation is the appearance of a metric factor.
After the fashion of [3], [6], [7], an asymptotically cartesian function $Z$ will be introduced and will be taken to satisfy the same differential equation as $V$:

$$\frac{1}{\sqrt{g}} \partial_\mu \left\{ \sqrt{g} e^{-U} g^{\mu\nu} \partial_\nu Z \right\} = 0 \quad (9)$$

Hence a Green identity can be used to reduce the following integral on the two regions $V^+_3$ and $V^-_3$:

$$0 = \int_{\partial V^+_3 \cup \partial V^-_3} dS e^{-U} \left( Z \frac{dV}{dn} - V \frac{dZ}{dn} \right) +\int_{V^+_3 \cup V^-_3} \sqrt{g} \frac{1}{\sqrt{g}} \left\{ Z \partial_\mu (\sqrt{g} e^{-U} g^{\mu\nu} \partial_\nu V) - V \partial_\mu (\sqrt{g} e^{-U} g^{\mu\nu} \partial_\nu Z) \right\} dx^1 dx^2 dx^3 = \int_{\partial V^+_3 \cup \partial V^-_3} dS e^{-U} \left( Z \frac{dV}{dn} - V \frac{dZ}{dn} \right) \quad (10)$$

Assuming that $V$ may be discontinuous on $S$ and calculating the integral at infinity with the information given by the asymptotic expansions, the following expression is obtained:

$$0 = -4\pi d +\int_{S} dS e^{-U} \left[ V \frac{dZ}{dn} - Z \frac{dV}{dn} \right] \quad (11)$$

This equation yields the expression for the dipole density on $S$, as it happened in the non-relativistic case, in terms of the discontinuities of $V$:

$$\sigma_2 = \frac{1}{4\pi} e^{-U} \left[ V \frac{dZ}{dn} - Z \frac{dV}{dn} \right] \quad (12)$$

This formula is again similar to the classical one (3), except for a metric factor. As it happened then, there is a contribution of a sheet of dipoles and also of the distribution of monopoles. This last term was not considered in [3].

4 An example: Bonnor’s magnetic dipole

As an example of how this formalism works, the magnetic source for Bonnor’s magnetic dipole [8] will be calculated. This solution is the Bonnor transform of the Kerr metric [9]:

$$ds^2 = -(1 - \frac{2mr}{r^2 - a^2 \cos^2 \theta})^2 dt^2 + (1 - \frac{2mr}{r^2 - a^2 \cos^2 \theta})^{-2} \{(r^2 - a^2 - 2mr) \sin^2 \theta d\phi^2 + (r^2 - a^2 \cos^2 \theta - 2mr)^4 \{(\theta^2 + \frac{dt^2}{r^2 - 2mr - a^2}) \} \quad (13)$$
\[ V = \frac{2am\cos\theta}{r^2 - a^2\cos^2\theta} \]  

(14)

which describes the field around a magnetic dipole of mass equal to \( 2m \) and magnetic moment \( 2am \).

As it was done in [3] the radial coordinate will be taken to be nonnegative. Events on the surface \( r = 0 \) with collatitude \( \theta \) will be identified with those with \( \pi - \theta \) and therefore the range of \( \theta \) will be restricted to \( [0, \pi/2) \) to avoid double-counting them. This means that the magnetic potential is discontinuous on \( r = 0 \):

\[ [V] = -\frac{4m}{a\cos\theta} \]  

(15)

Since the solution satisfies the required asymptotic behaviours, only a \( Z \) function satisfying (9) is needed for constructing the magnetic source:

\[ Z = (r - 3m)\cos\theta - \frac{2a^2m\cos^3\theta}{r^2 - a^2\cos^2\theta} \]  

(16)

Applying (8) and (12) to this solution, the expressions for the magnetic density are obtained:

\[ \sigma_2 = m \frac{a^2\cos^2\theta - m^2\sin^2\theta)^{3/2}}{\pi a^4 \cos^4\theta} \]  

(17)

Obviously there is no monopole density. Integrating \( \sigma_2 \) on the surface \( r = 0 \), the correct result for the magnetic dipole moment is obtained:

\[ d = \int_S \sigma_2 \, dS = \int_0^{\pi/2} \int_0^{2\pi} \, d\theta \, (ma \sin\theta) = 2ma \]  

(18)

This is the same result that was obtained in [3] since there is no contribution from monopole sheets, as was to be expected.

5 Discussion

It has been shown in this lecture a new method for constructing electromagnetic sources for static axially symmetric spacetimes. With this new approach the inclusion of asymptotically monopolar electric fields has been achieved. This was not possible in a previous formalism [3] because vector potentials were used and therefore Dirac string singularities would be present if monopoles were included. Also the influence of the charge distribution on the dipole density has been considered. A further generalization to stationary nonstatic axisymmetric spacetimes [10] is in preparation.
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