Realizing non-Abelian gauge potentials in optical square lattices: an application to atomic Chern insulators

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Abstract

We describe a scheme to engineer non-Abelian gauge potentials on a square optical lattice using laser-induced transitions. We emphasize the case of two-electron atoms, where the electronic ground state $g$ is laser-coupled to a metastable state $e$ within a state-dependent optical lattice. In this scheme, the alternating pattern of lattice sites hosting $g$ and $e$ states depicts a chequerboard structure, allowing for laser-assisted tunnelling along both spatial directions. In this configuration, the nuclear spin of the atoms can be viewed as a ‘flavour’ quantum number undergoing non-Abelian tunnelling along nearest-neighbour links. We show that this technique can be useful to simulate the equivalent of the Haldane quantum Hall model using cold atoms trapped in square optical lattices, offering an interesting route to realize Chern insulators. The emblematic Haldane model is particularly suited to investigate the physics of topological insulators, but requires, in its original form, complex hopping terms beyond nearest-neighbouring sites. In general, this drawback inhibits a direct realization with cold atoms, using standard laser-induced tunnelling techniques. We demonstrate that a simple mapping allows us to express this model in terms of matrix hopping operators that are defined on a standard square lattice. This mapping is investigated for two models that lead to anomalous quantum Hall phases. We discuss the practical implementation of such models, exploiting laser-induced tunnelling methods applied to the chequerboard optical lattice.

(Some figures may appear in colour only in the online journal)

1. Introduction

The discovery of the quantum Hall (QH) effect, in two-dimensional (2D) electronic systems subjected to large magnetic fields, revealed the existence of novel quantum phases: the topological insulating states [1, 2]. Such phases are insulating in the bulk but exhibit current-carrying modes on the edge of the sample. These chiral propagating modes, whose energies are located within the bulk gaps, are protected by topology. They can be formally related to the existence of non-trivial topological invariants—Chern numbers—associated with the band structure [3, 4]. The existence of edge modes is guaranteed as long as the topology of the band structure does not change, irrespective of the details of the microscopic Hamiltonian. In the early age of QH physics, it was generally thought that this phenomenon could only emerge in 2D systems featuring Landau levels, namely in samples necessarily subjected to magnetic fields. Haldane showed in 1988 that this was not a necessary condition [5]. To this aim, he introduced a simple lattice model exhibiting the QH effect without Landau levels, thereafter referred to as the anomalous QH effect. In his seminal work [5], Haldane showed that QH phases were rooted in the breaking of time-reversal symmetry, which potentially increased the possibilities to access this
The original Haldane model [5] is defined on a honeycomb lattice with real nearest-neighbour (NN) and complex next-nearest-neighbour (NNN) hoppings; see figure 1(a). The NN hopping alone leads to a gapless energy spectrum featuring two Dirac cones familiar from graphene studies [28]. The complex NNN hopping, represented by red and blue links in figure 1(a), opens a single gap at half-filling (i.e. around $E = 0$). This bulk energy gap is non-trivial [5]: when the Fermi energy $E_F \approx 0$ lies in this gap, a single edge mode is populated and carries current around the system [4], leading to a quantized Hall conductivity, $\sigma_{xy} = e^2/h = \nu = \pm 1$. From a topological point of view, the lowest energy bulk band $E(k) = 0$ is associated with a non-vanishing Chern number $\nu = \pm 1$, which guarantees the presence of the robust and unique edge mode within the bulk gap [3, 4, 29]. The Chern number $\nu$ can be directly evaluated from the effective Dirac equations associated with the two Dirac cones [5, 28]:

$$\nu = \frac{1}{2} (\text{sign}(M_{K^+}) - \text{sign}(M_{K^-})), \quad (1)$$

where $M_{K^\pm}$ are the effective masses related to the two independent Dirac points $K^\pm$.

Realizing the Haldane model with cold atoms trapped in an optical lattice is attractive, as it would provide a simple playground to investigate the physics of topological (Chern) insulators. Such a proposal was described in [24, 25], where laser-induced tunneling [30] was considered to couple two state-dependent triangular optical lattices. The simplicity of the Haldane model relies on the fact that it is a two-band model, featuring a unique energy bulk gap and hosting a single topological edge mode ($\nu = \pm 1$). This minimal topologically ordered system is to be compared, for instance, to the emblematic Hofstadter model [31], i.e. a tight-binding model for an electron in a uniform magnetic field moving on a square lattice, which is characterized by a multi-band energy spectrum and hosts (infinitely many) QH phases with arbitrary $\nu \in \mathbb{Z}$ [32]. In this sense, the two-band Haldane model is mathematically easier to handle than the Hofstadter model [25], and it thus constitutes a good basis for studying...
the effects of interactions, in particular, in view of realizing fractional Chern insulators [13–16, 33] and topological Mott insulators [27] with cold atoms. Finally, the Haldane model constitutes the building blocks for the Kane–Mele model [7], and therefore, its optical-lattice realization opens a possible route for the observation of the quantum spin Hall effect with cold atoms [34–37].

In this work, one exploits the fact that the honeycomb Haldane model can be mapped into a two-component square lattice with NN and NNN matrix hoppings. We demonstrate that the NNN hoppings, which are inconvenient for an optical-lattice implementation, can be simply omitted: they do not contribute to the appearance of a non-trivial topological phase. This simple observation makes the realization of the Haldane model feasible with laser-assisted tunnelling on a square lattice, at the price of an increased complexity: the resulting model involves non-Abelian gauge potentials [38–40]. The models stemming from this non-Abelian framework are versatile, and thus, they can be generalized to simulate an assortment of topological and Dirac-like systems [34, 41, 42].

In section 2, we first describe an extension of the scheme proposed in [38] allowing one to realize such non-Abelian gauge potentials. In section 3, we show how these synthetic gauge potentials can be tailored in order to reproduce the gauge potentials. The models stemming from this non-Abelian framework are versatile, and thus, they can be generalized to simulate an assortment of topological and Dirac-like systems [34, 41, 42].

In section 2, we first describe an extension of the scheme proposed in [38] allowing one to realize such non-Abelian gauge potentials. In section 3, we show how these synthetic gauge potentials can be tailored in order to reproduce the original Haldane model on a square lattice. We also briefly comment on the detection of relevant signatures, based on available experimental probes. In section 4, we show that similar considerations can be applied to the so-called π-flux model [13, 39, 43], which leads to identical physics with a simpler setup.

2. Non-Abelian gauge potentials for ultracold atoms in optical lattices

We start by outlining an experimental scheme generating effective non-Abelian gauge fields for ultracold fermionic atoms on a lattice. The scheme is a generalization of the ones proposed by Osterloh et al. [38], building on the earlier proposal by Jaksch and Zoller to realize effective magnetic fields [30]. Both proposals [30, 38] are designed for alkali atoms and rely on using multilevel atoms in a spin-dependent optical lattice, i.e. a periodic trapping potential made from several different sublattices trapping different internal states. A set of laser beams drive Raman transitions that change the internal state, thereby inducing tunnelling from a given sublattice to another. Crucially, the laser-assisted tunnelling process is characterized by a complex matrix element, with an argument determined by the laser phase. A suitable configuration of laser beams then leads to a non-vanishing geometrical phase for an atom tunnelling around a closed trajectory, which realizes an effective magnetic field.

Implementing this scheme experimentally with fermionic alkali atoms is unfortunately plagued by spontaneous emission: the spin-dependent lattice and Raman laser frequencies need to be set relatively close to resonance, leading to heating and losses hardly compatible with quantum gas experiments [44]. Moreover, such spin-dependent lattices are sensitive to the magnetic moment of the atoms, leaving the protocol vulnerable to stray magnetic fields. These issues can be avoided by using alkaline earth atoms (or atoms with a similar level structure, such as ytterbium). We outline below how the proposals must be modified to use these atoms. We first start by recalling the scheme proposed in [44] to realize an effective magnetic field—or equivalently an Abelian gauge structure with U(1) symmetry—and we explicitly apply it to a chequerboard geometry; see figure 2(a). We then explain in a second step how this can be extended to non-Abelian configurations in the spirit of Osterloh et al. [38]. For simplicity, we consider an atom with spin 1/2 and generations of SU(2) gauge potentials. In principle, the method can be extended to other gauge groups but this typically requires many more lasers than the simpler U(1) or SU(2) schemes we outline here. As a result, the experimental implementation becomes significantly more involved.

2.1. Spin-dependent optical lattices, laser-induced tunnelling and the Peierls phase

Two-electron atoms (such as alkaline earth atoms or ytterbium) feature two peripheral electrons that form a singlet1 and the Peierls phase

considerations can be applied to the so-called π-flux model [13, 39, 43], which leads to identical physics with a simpler setup.
tight-binding approximation. We then obtain the effective $g - e$ hopping matrix element between two neighbouring sites, located at $r_g = r_j$ and $r_e = r_j + \delta_{jk}$:

$$J_{jk} = \frac{\hbar}{2} \int w_e^* (r - r_j) w_g (r - r_j) e^{i\mathbf{q \cdot \delta_{jk}}} \mathbf{d}^2 r,$$

where $\Omega$ and $\mathbf{q}$ are the laser Rabi frequency and wavevector, respectively, and $w_g$ and $w_e$ denote the Wannier functions associated with each sublattice. For the chequerboard geometry, the link vectors are given by $\delta_{jk} = r_j - r_j = e_{x'y'}$. It is convenient to rewrite the tunnelling elements as

$$J_{jk} = J_{\text{eff}} e^{i\mathbf{q \cdot \delta_{jk}/2}} = J_{\text{eff}} e^{i\phi_{jk}},$$

$$J_{\text{eff}} = \frac{\hbar}{\Omega} \left( \int w_e^* (r - \delta_{jk}/2) w_g (r + \delta_{jk}/2) e^{i\mathbf{q \cdot \delta_{jk}}} \mathbf{d}^2 r \right),$$

in which case the magnitude of the tunnelling $J_{\text{eff}} = (J_{\text{eff}})^*$ is uniform over the whole lattice. We note that other choices are acceptable for the phases $\phi_{jk}$, as long as the new phases $\phi_{jk}$ satisfy the gauge-transformation relation $\phi_{jk} = \phi_{jk} + \chi (r_j) - \chi (r_j)$, where the function $\chi (r)$ is defined uniformly over the whole lattice. The magnitude of the tunnelling matrix element $J_{\text{eff}}$ is controlled by the Rabi frequency and the overlap integral between the Wannier functions $w_g$ and $w_e$ associated with each sublattice, which have to be calculated numerically [30]. In principle, the laser is also able to induce tunnelling to NNNS of the composite lattice, and beyond. In practice, the matrix elements are exponentially suppressed compared to the ones describing NN tunnelling, and negligible for realistic experimental configurations. This sets a constraint on the class of models that can be realized in this way, as we will discuss later in this paper.

From now on, we use a shorthand notation $(m, n)$ for the site located at $x = md/2$, $y = nd/2$, where $m, n$ are even for state $g$ and odd for state $e$, and $d$ is the lattice spacing for each sublattice; see figure 2(a). Also, we choose $\mathbf{q} = q e_y$ for definiteness. The effective $g - e$ hopping matrix elements between a site located at $r_g = (m, n)$ and its four NNNS are then given by

$$J_{ge} = J_{\text{eff}} e^{i\mathbf{q \cdot (n-1/2)}},$$

$$J_{ge} = J_{\text{eff}} e^{i\mathbf{q \cdot (n+1/2)}},$$

and the reversed paths are given by $J_{eg} = (J_{ge})^*$. Here, the phase factor has been written as $2\pi\alpha = qd$, where $\alpha$ can be controlled from zero to a value larger than one by modifying the orientation of the coupling laser.

Let us note that the chequerboard lattice considered here potentially allows us to induce the tunnelling and the Peierls phases along both spatial directions (in contrast with the ‘column’ geometry previously considered in [30, 44, 48]). This central ingredient will be largely exploited after the present proposal; see section 2.3.

2.2. Realizing Abelian gauge potentials: non-zero flux on the chequerboard lattice

First of all, we note that a particle making a loop around a unit cell of the chequerboard lattice with tunnelling elements (4) does not acquire any phase factor. This is a consequence of unitarity, which imposes that the laser coupling matrix elements for $g \rightarrow e$ transitions (see equation (4)) and its reversed $e \rightarrow g$ counterpart are complex conjugates. Because of the alternating pattern of $g \rightarrow e$ and $e \rightarrow g$ links (see figure 2(a)), the sign of the tunnelling phases alternates when moving in a given direction, which indeed leads to the cancellation of the overall phase picked by a particle hopping around a unit cell of the chequerboard. Starting from the site...
(m, n), and using equation (5), one obtains that the phase acquired along the closed path (m, n) → (m + 1, n + 1) → (m + 2, n) → (m + 1, n − 1) → (m, n) is

$$2\pi \Phi_{\square} = \pi \alpha (n + 1/2) - \pi \alpha (n - 1/2) + \pi \alpha (n - 1/2) - \pi \alpha (n + 1/2) = 0,$$

which indicates that the effective ‘magnetic’ flux \( \Phi_{\square} \) penetrating each plaquette is zero. This trivial flux configuration is in contrast with the ‘column’ geometry discussed in [30, 44, 48], where a staggered effective magnetic flux is generated, and it results from the higher symmetry of the chequerboard lattice. This can be cured by an additional superlattice with double period 2d, generating a potential of the form

$$V_{\text{SL},g}^{\text{ex}}(x) = W_{e,g} \cos^2(\pi x/2d + \varphi),$$

which acts, \textit{a priori}, differently on the g and e atoms. A suitable choice of the relative phase \( \varphi \), which we take to satisfy the relation \( \varphi = -\psi_g/W_e \), leads to on-site energies

$$E_g - V_0 + W_g/2 + \Delta V, \quad \text{for } x/d = 4m,$$
$$E_e + W_e/2 + \Delta V, \quad \text{for } x/d = 4m + 1,$$
$$E_g - V_0 + W_g/2 - \Delta V, \quad \text{for } x/d = 4m + 2,$$
$$E_e + W_e/2 - \Delta V, \quad \text{for } x/d = 4m + 3.$$

Here \( E_{e,g} \) denote the internal energies in free space and \( \Delta V = \frac{-W_g - W_e}{2\sqrt{W_g^2 + W_e^2}} \). In this potential landscape, the resonance frequencies for transitions linking neighbouring sites become non-degenerate:

$$\omega_1 = \omega_0 + \delta V/\hbar \quad \text{for } (4m, 4n) \rightarrow (4m + 1, 4n + 1),$$
$$\omega_2 = \omega_1 + 2\Delta V/\hbar \quad \text{for } (4m + 1, 4n + 1) \rightarrow (4m + 2, 4n),$$
$$\omega_3 = \omega_1 \quad \text{for } (4m + 2, 4n) \rightarrow (4m + 3, 4n + 1),$$
$$\omega_4 = \omega_1 - 2\Delta V/\hbar \quad \text{for } (4m + 3, 4n + 1) \rightarrow (4m + 4, 4n),$$

with \( \hbar \omega_0 = E_e - E_g \) being the bare transition frequency and \( \delta V = V_0 + (W_e - W_g)/2 \).

To the state-dependent lattice, one thus applies three coupling lasers propagating along y. The laser at frequency \( \omega_1 \) is chosen with a wavevector \( q = \mathbf{q}_e \), and the lasers at frequencies \( \omega_1 \pm 2\Delta V \) with the opposite wavevector \( q' = -\mathbf{q}_e \). If we neglect off-resonant transitions, the alteration of the wavevectors compensates the alteration of the sign of the tunnelling phases, thereby leading to a non-zero ‘magnetic’ flux per plaquette that is uniform across the lattice. Starting from the site (m, n), one obtains that the phase acquired along the closed path (m, n) → (m + 1, n + 1) → (m + 2, n) → (m + 1, n − 1) → (m, n) is

$$2\pi \Phi_{\square} = \pi \alpha (n + 1/2) + \pi \alpha (n + 1/2) - \pi \alpha (n - 1/2) - \pi \alpha (n - 1/2) = 2\pi \alpha,$$

where the parameter \( \alpha = qd/2\pi \) is interpreted as the uniform synthetic magnetic flux (in units of the flux quantum) penetrating each plaquette. Thus, this scheme realizes the Hofstadter model [31], generalizing the proposals [30, 44] to the case of the chequerboard geometry.

\[ \text{2.3. Realizing non-Abelian gauge potentials} \]

This proposal can be generalized to non-Abelian gauge potentials when \( g \) and \( e \) represent manifolds of degenerate states (see figures 2(a) and (b)). Let us consider a 2 × 2 system with states \( |g_{1/2}, g_{1/2} \rangle \) and \( |g_{-1/2}, e_{1/2} \rangle \), as realized in the fermionic \({\text{Yb}}\) with nuclear spin 1/2. In a moderate magnetic field (a few tens of Gauss), the Zeeman effect allows us to distinguish between \( \sigma^z \) transitions \( g_{\pm 1/2} \rightarrow e_{\mp 1/2} \) that change the spin projection, from \( \sigma_{x,y} \) transitions \( g_{\pm 1/2} \rightarrow e_{\mp 1/2} \) that conserve it (see figure 2(b)). This allows us to correlate tunnelling in a spatial direction with rotations in internal states and state-dependent tunnelling phases. Laser-assisted tunnelling along \( x' \) or \( y' \) is now described by 2 × 2 matrices \( \hat{U}_{x'} \) and \( \hat{U}_{y'} \) acting on a two-component spinor. If it is possible to arrange such that \( \hat{U}_{x'}, \hat{U}_{y'} \neq 0 \), this mimics a non-Abelian gauge potential \( \mathbf{A} \) through \( \hat{U} = e^{\mathbf{i} \mathbf{A} \cdot \mathbf{d}} \).

In order to realize \( \hat{U}_{x'} \neq \hat{U}_{y'} \), one must distinguish between the tunnellings directed along \( e_{x'} \) and \( e_{y'} \). Therefore, this requires two superlattice potentials with period 2d along the \( x- \) and \( y- \)axes, with different heights \( \Delta V_{x'} \) and \( \Delta V_{y'} \). This generates nine different transition frequencies differing by \( 0, \Delta V_{x'}, \pm \Delta V_{x'} \pm \Delta V_{y'}, 2\Delta V_{x'}, 2\Delta V_{y'}, \) which can be grouped into two families. One family corresponds to transitions along \( e_{x'} \), and another one corresponds to transitions along \( e_{y'} \).

We first consider the transitions along \( e_{x'} \). Let us choose the coupling lasers propagating along \( z \) (thus inducing tunnelling with no relative phase for motion in the \( x-y \) plane), with a linear polarization \( \sigma^z \), and tuned on resonance in the absence of magnetic field. Such lasers will induce a tunnelling matrix \( \propto \hat{\sigma}_x \). A second set of lasers is added, with polarization \( \sigma^{x'} \), on resonance with transitions along \( e_{y'} \). This configuration generates a spin–orbit coupling of the form

$$\hat{U}_{x'} \propto \hat{\sigma}_x, \quad \hat{U}_{y'} \propto e^{\mathbf{i} \phi_1} \hat{\sigma}_y,$$

where \( \phi_1 \) is the relative phase of the \( \sigma^{x'} \)-polarized laser with respect to the \( \sigma^z \)-polarized one.

More complicated tunnelling matrices with complex position-dependent elements can be generated using a different laser configuration. Two specific configurations, leading to the analogue of the Haldane model [5], will be discussed below. In principle, arbitrary tunnelling matrices can be generated in this way. However, the increase in complexity makes it experimentally very challenging. The configuration described above, for instance, generates no less than nine different frequencies. Those frequencies are typically in the 10 kHz range, much less than the absolute optical frequency \( \omega_0 \) corresponding to the bare atomic transition. As a result, they can be generated from a single laser using acoustic-optical or electro-optical modulators. This represents nevertheless a significant technological challenge for designing arbitrary tunnelling matrices.

5 Technically, one should also verify that the tunnelling matrices \( \hat{U}_{x',y'} \) correspond to non-trivial Wilson loops; see [40].
3. The Haldane model on the square lattice

3.1. Mapping the Haldane model to a non-Abelian model on a square lattice

The idea that we expose in this section is to propose and analyse a mapping from the original Haldane model, involving complex tunnelling phases on a honeycomb lattice [5], into a spin-1/2 model defined on the square lattice; see figures 1(a) and (b). This is possible in two dimensions since any lattice with \( N \) inequivalent sites within its unit cell can be formally labelled using (a) two spatial coordinates \((m, n)\) describing the location of the unit cell and (b) an additional ‘pseudo-spin’ index \( \tau = 1, \ldots, N \) labelling the inequivalent sites. Since the honeycomb lattice features two inequivalent sites, denoted \( A \) and \( B \), it can be mapped into a square lattice hosting spin-1/2 objects.

Our starting point is to describe the honeycomb lattice using the notations illustrated in figure 1(a). In these notations [43], one uses discrete coordinates \((m, n)\) to label the unit cells of the honeycomb, where \( m, n \) are integers. Here each ‘site’ \((m, n)\) hosts the wavefunctions \( \psi_A(m, n) \) and \( \psi_B(m, n) \), associated with the two inequivalent sites of the honeycomb. In other words, the honeycomb lattice can be interpreted as a square lattice with a pseudo-spin 1/2 structure. Thus, the tight-binding Hamiltonian describing NN hopping between \( A \) and \( B \) neighbouring sites can be mapped into an equivalent operator, which involves hopping between opposite pseudo-spin components, which are located at neighbouring sites of a square lattice; see figure 1(b). This mapping smoothly deforms the original Brillouin zone and energy bands but does not affect the physical properties, such as bulk gaps and topological phases, which are discussed in the following.

One considers the Haldane model, which features both NN and NNN hoppings on the honeycomb lattice (figure 1(a)). The single-particle Schrödinger equations, satisfied by the wavefunctions \( \psi_A(m, n) \) and \( \psi_B(m, n) \), take a very simple form in terms of the notations introduced above, and read

\[
E \psi_A(m, n) = \psi_A(m, n) + \psi_B(m - 1, n) + \psi_B(m, n - 1)
+ i \lambda \left( \psi_A(m + 1, n) - \alpha \psi_A(m + 1, n - 1) + \psi_A(m, n - 1) - \psi_A(m - 1, n) + \alpha \psi_A(m - 1, n + 1) \right),
\]

\[
E \psi_B(m, n) = \psi_A(m, n) + \psi_A(m + 1, n) + \psi_A(m, n + 1)
+ i \lambda \left( \psi_B(m + 1, n) - \psi_B(m, n + 1) + \alpha \psi_B(m + 1, n - 1) - \psi_B(m - 1, n) + \psi_B(m, n - 1) - \alpha \psi_B(m - 1, n + 1) \right),
\]

where the tunnelling amplitude \( J \) to NNs of the honeycomb lattice is set to \( J = -1 \), and thus defines our unit of energy. Here, \( \lambda \) is the amplitude of the NNN hoppings on the honeycomb lattice and it corresponds to the \( t_2 \) parameter of the original Haldane model (or equivalently to the intrinsic spin–orbit coupling strength in the spin-1/2 Kane–Mele model [7]). We remind that the complex NNN hoppings, which are responsible for the opening of the non-trivial topological bulk gap in this model, introduce a chirality in the system; see figure 1(a). Note that we have included an additional parameter \( \alpha \in [0, 1] \)—for reasons that will appear clear after the mapping to a square lattice—with \( \alpha = 1 \) in the original Haldane model.

Setting \( \Psi(m, n) = (\psi_A(m, n), \psi_B(m, n)) \), one can rewrite equation (10) as

\[
E \Psi(m, n) = \hat{F} \Psi(m, n) + \hat{U}_x \Psi(m + 1, n) + \hat{U}_y \Psi(m, n + 1)
+ \hat{D} \Psi(m + 1, n - 1) + \hat{U}_y^\dagger \Psi(m - 1, n),
\]

(11)

where \( \hat{U}_x = \hat{\sigma}_+ + i \lambda \hat{\sigma}_z, \hat{U}_y = \hat{\sigma}_- - i \lambda \hat{\sigma}_z, \hat{F} = \hat{\sigma}_x \) and \( \hat{D} = -i \alpha \lambda \hat{\sigma}_x = -i \hat{d}_{\text{diag}} \hat{\sigma}_x \). Here, we have introduced the Pauli matrices \( \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \) and \( \hat{\sigma}_- = (\hat{\sigma}_z - i \hat{\sigma}_y)/2 \). Equation (11) describes a non-interacting two-component system, evolving on a square lattice; see figure 1(b). The obtained model is characterized by non-Abelian hopping operators \( \hat{U}_{x,y} \) that act along the links, but also by an on-site spin-mixing term \( \hat{F} \) and diagonal hoppings \( \hat{D} \) with amplitude \( \hat{d}_{\text{diag}} = \alpha \lambda \); see figure 1(b). To keep with the notations used in the preceding section, we denote by \( x', y' \) the principal axes of the square lattice.

Realizing non-Abelian hoppings \( \hat{U}_{x',y'} \) between the NNs of a square optical lattice is a possible but difficult task, as discussed above. The most important issue is the presence of additional diagonal matrix hoppings \( \hat{D} \), which will be exponentially small in a cold-atom realization compared to \( \hat{U}_{x,y} \) (see the discussion in the preceding section). Consequently, a natural question arises: how important is this diagonal hopping \( \hat{D} \) for the obtention of the topological phase? Would the non-trivial bulk gap and corresponding edge states survive in the limit \( \alpha \to 0? \) We answer this question by investigating the fate of the energy spectrum and topological order (Chern number and edge states) as the Haldane model described by equation (11) is reduced to the simpler model with \( \hat{D} = 0 \).

In momentum space, the Hamiltonian describing our system reads

\[
H(k_x, k_y) = \mathbf{d}(k) \cdot \hat{\sigma},
\]

(12)

where \( \hat{\sigma} \) is a vector of Pauli matrices and the vector \( \mathbf{d}(k) \) is given by

\[
\mathbf{d}(k) = \begin{pmatrix}
\cos(k_y) + \cos(k_y) + 1
\sin(k_y) + \sin(k_y)
2\lambda[\sin(k_y) - \sin(k_y)] + 2\alpha\lambda\sin(k_y) - k_y
\end{pmatrix}.
\]

(13)

In the limit \( \lambda = 0 \), the energy spectrum is gapless and describes a semi-metal at half-filling. The two branches are

\[
E_{\pm}(k; \lambda = 0) = \pm \sqrt{3} / 2 (\cos k_y + \cos k_y + \cos(k_y - k_y)),
\]

(13)

and they touch at the two inequivalent Dirac points \( K^+ = (2\pi/3, 4\pi/3), K^- = (4\pi/3, 2\pi/3) \), where \( E_{\pm} = 0 \) (or equivalently, \( d_x + id_y = 0 \)). When \( \lambda \neq 0 \), a gap

\[
\Delta = 2|d_{\tau}(k^+)| = 4|\lambda|\sqrt{3} \left( 1 + \frac{\alpha}{2} \right)
\]

(14)

opens at the Dirac points. It is well established for two-component lattice models that the topology of the ground band
Figure 3. Energy spectrum \( E = E(k_x) \) for \( \alpha = 0.05\pi \): (a) \( \alpha = 1 \), (b) \( \alpha = 0 \) for a cylindrical geometry aligned along \( x' \). This spectrum shows the projected bulk bands \( E_{\pm}(k) \rightarrow E_{\pm}(k_x) \) and also reveals the presence of topological edge states inside the bulk gap [4]. In the standard case \( \alpha = 1 \), the topological bulk gap is \( \Delta = 6\sqrt{3}\lambda \approx 0.5 \). In the absence of the diagonal matrix hopping, \( \hat{t} \), the gap is \( \Delta = 4\sqrt{3}\lambda \approx 0.34 \) but survives. The energy unit is given by the tunnelling amplitude \( J \).

is entirely characterized by the vector \( d \) parameterizing the Hamiltonian: the Chern number of the band \( v \) is identical to the winding number of \( \mathbf{d}(k) \), as \( k \) is varied in the first Brillouin zone [12, 24, 25, 29]. As already presented in equation (1), this winding number \( v \) can be directly related to the effective masses associated with the two Dirac points \( K^{\pm} \). Here, these masses are given by

\[
M^{\pm} = d_z(K^{\pm}) = \pm 2\lambda \sqrt{3} \left( 1 + \frac{\alpha}{2} \right),
\]

as can be deduced by developing the Hamiltonian \( H(k', k_y) \) in the vicinity of the Dirac points \( K^{\pm} \). The effective masses have different signs, which according to equation (1) leads to a non-zero Chern number \( v = \pm 1 \). Moreover, these masses are non-vanishing in the limit \( \alpha \rightarrow 0 \). Therefore, the bulk energy gap opened by the perturbation \( \lambda \), stemming from the complex Haldane NNN hopping, survives for any value of \( \alpha \) (including the extreme value \( \alpha = 0 \)). Since the masses \( M^{\pm} \) preserve their sign all along the transformation \( \alpha = 1 \rightarrow \alpha = 0 \), the Chern number in equation (1) remains constant and non-trivial for any value of \( \alpha \). The latter remark can also be formulated in the following manner: since the bulk gap is preserved during this transformation, the topological Chern number characterizing the bulk bands is unaffected [3]. This result is further illustrated in figure 3, which shows the energy spectrum for the two limiting cases \( \alpha = 0 \) and \( \alpha = 1 \). In figures 3(a) and (b), the spectrum was obtained using a cylindrical geometry aligned along \( x' \): it shows the projected bulk bands \( E_{\pm}(k) \rightarrow E_{\pm}(k_x) \) and the topological edge states inside the bulk gap [4]. This result demonstrates that the diagonal hopping induced by the operator \( \hat{D} = -i\alpha\lambda \hat{\sigma}_z \), in equation (11) does not play any role in the realization of topological insulating phases and can thus be omitted for the sake of experimental feasibility.

To summarize, the physics of the Haldane model can be explored with an optical square lattice described by the simplified tight-binding Hamiltonian

\[
\mathcal{H} = \sum_{m,n} \hat{c}_{m+1,n}^{\dagger} \hat{U}_{x} c_{m,n} + \hat{c}_{m,n+1}^{\dagger} \hat{U}_{x} c_{m,n} + \hat{c}_{m-1,n}^{\dagger} \hat{U}_{x} c_{m,n} + \hat{c}_{m,n-1}^{\dagger} \hat{U}_{x} c_{m,n} + \hat{c}_{m,n}^{\dagger} \hat{F} c_{m,n},
\]

\( \hat{U}_{x} = \hat{\sigma}_+ + i\lambda \hat{\sigma}_z, \quad \hat{U}_{x'} = \hat{\sigma}_- - i\lambda \hat{\sigma}_z, \quad \hat{F} = \hat{\sigma}_x, \)

where the two-component operator \( \hat{c}_{m,n}^{\dagger} \) creates a particle at site \((m, n)\) and where the spin-1/2 structure is implicit. We note that this system is formally similar to the HgMnTe quantum wells model proposed in [12, 17] to realize the anomalous QH effect.

3.2. Implementation

We now discuss how the Hamiltonian (equation (16)) can be implemented using the methods discussed in section 2. We start with the terms \( \hat{U}_{x} = +i\lambda \hat{\sigma}_z \) and \( \hat{U}_{x'} = -i\lambda \hat{\sigma}_z \). First, we consider transitions corresponding to tunnelling along the \( x' \)-direction. For each value of the nuclear spin \((+1/2 \text{ or } -1/2)\), one can apply laser-assisted tunnelling with \( \pi \) polarization. The lasers propagate along \( z \) (thus inducing no tunnelling phase), with a polarization along \( y \) parallel to the direction of an applied magnetic field large enough to split the different transitions. Since the transition frequencies for \( g_{1/2} \rightarrow e_{1/2} \) and \( g_{-1/2} \rightarrow e_{-1/2} \) are different, they must be addressed separately with independent lasers; see figure 2(b).

We choose a phase of \( \pm\pi/2 \) for the two lasers driving the two \( \pi \) transitions for nuclear spins \( \pm 1/2 \). This gives a tunnelling matrix proportional to

\[
\hat{U}_{x} \propto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\hat{\sigma}_z.
\]

Another set of lasers can be used to generate tunnelling operators along \( e_{x'} \), also driving \( \pi \) transitions for nuclear spins \( \pm 1/2 \) with phases \( \mp\pi/2 \). This generates a tunnelling matrix

\[
\hat{U}_{x'} \propto -i\hat{\sigma}_z.
\]

Next, we consider additional coupling lasers propagating along \( +e_{y} \), with \( \sigma^z \) polarization, thus generating a tunnelling matrix \( \propto \hat{\sigma}_z \). For instance, the tunnelling matrix element for the transition linking site \( r_{x} = (m, n) \) to \( r_{x} = r_{y} + e_{y} = (m + 1, n) \) is then

\[
\langle m + 1, n | \hat{U}_{x'} | m, n \rangle \propto e^{i\phi_{m,n}} \hat{\sigma}_- = e^{i\pi n \alpha} \hat{\sigma}_-, \]

where we used equation (3), and where \( n \) designates the \( y \) coordinate of the site \((m, n)\), which is here labelled according to the \( x' - y' \) axis system; see figure 2. We note that each phase \( \phi_{m,n} \) is driven by a specific laser, corresponding to the different transition frequencies offered by the superlattice potential. By adjusting the relative phases of these lasers, one can annihilate the undesired phases \( \phi_{m,n} \propto 0 \) uniformly on the lattice, and thus realize constant tunnelling operators \( \hat{U}_{x',y} \propto \hat{\sigma}_z \) as in equation (16). Combining the lasers generating the \( \pi \) and \( \sigma^z \) transitions, one obtains the required NN tunnelling terms in equation (16). Finally, the on-site term \( \hat{F} \propto \hat{\sigma}_x \) can be generated by an additional radio frequency field resonant at the Larmor frequency, acting uniformly on all sites.\(^6\)

Although the scheme seems directly feasible, it is rather complex due to the large number of transition frequencies involved. In section 4, we present an equivalent model that leads to a simpler implementation.

\(^6\) Note that with our choice of phases, the phase of this radio frequency field plays the role of the phase reference, to which the phases of the lasers inducing tunnelling process can be locked.
4. The $\pi$-flux model

The physics of the honeycomb Haldane model can be alternatively studied by considering non-trivial hoppings on the $\pi$-flux model [13, 39, 43]. The latter is illustrated in figure 5(a) and is characterized by tunnelling amplitudes $\pm J$ along alternate columns. In the absence of additional hoppings, this system exhibits a gapless spectrum with two Dirac cones, and it is therefore physically equivalent to the honeycomb lattice [43]. In order to open a non-trivial topological gap, with the Chern number $v = \pm 1$, complex diagonal hoppings are required; see the purple arrows in figure 5(a). As for the original honeycomb model discussed in the previous section, these chiral and complex NNN hoppings highly diminish the feasibility of this Haldane-like model with cold atoms trapped in optical lattices. Note that the $\pi$-flux model without NNN hoppings is equivalent to the Hofstadter model with the magnetic flux set to $\Phi = 1/2$ [44, 58].

In this section, we follow the same strategy as before: we consider the two-component wavefunction $\psi(m, n) = (\psi_A(m, n), \psi_B(m, n))$, which describes particles on alternate columns; see the blue (A) and red (B) sites in figure 5(a). In these notations, the single-particle Schrödinger equation satisfied by $\psi(m, n)$ in this model then takes the form

$$E\psi(m, n) = \hat{F}\psi(m, n) + \hat{U}_x\psi(m + 1, n) + \hat{U}_y\psi(m, n + 1) + \hat{D}_1\psi(m + 1, n + 1) + \hat{D}_2\psi(m + 1, n - 1) + \hat{U}_x\psi(m - 1, n - 1) + \hat{U}_y\psi(m, n - 1) + \hat{D}_1\psi(m - 1, n - 1) + \hat{D}_2\psi(m - 1, n + 1),$$

where $\hat{U}_x = \hat{\sigma}_x, \hat{U}_y = -\hat{\sigma}_x - i\hat{\lambda}\hat{\sigma}_y, \hat{F} = \hat{\sigma}_z$, and $\hat{D}_{1,2} = \pm i\alpha\lambda\hat{\sigma}_z$. Therefore, the model is translated into a non-Abelian square lattice, with direct tunnelling operators $\hat{U}_{x,y}$, an on-site term $\hat{F}$ and two diagonal hopping matrices $\hat{D}_{1,2}$; see figure 5(b).

The $\pi$-flux model can be analysed in the same way as in the preceding section, with identical conclusions. In the limit $\lambda = 0$, the energy spectrum is given by

$$E_{\pm}(\lambda = 0) = \pm \sqrt{2 + 2\cos k_x + 2\cos^2 k_y},$$

with two independent Dirac points at $K^+ = (\pi, \pi/2)$ and $K^- = (\pi, 3\pi/2)$. In the vicinity of the Dirac points, the effective masses are given by

$$M^\pm = \pm 2\lambda(1 + \alpha).$$

As a result, the Chern number $v = \pm 1$ remains non-zero even without diagonal hoppings ($\alpha = 0$), as found previously.

The $\pi$-flux QH model can thus be explored by realizing the tight-binding Hamiltonian

$$\mathcal{H} = \sum_{m,n} c_{m-1,n}^\dagger \hat{U}_x c_{m,n} + c_{m,n+1}^\dagger \hat{U}_y c_{m,n} + c_{m-1,n}^\dagger \hat{U}_x^{\dagger} c_{m,n} + c_{m,n+1}^\dagger \hat{U}_y^{\dagger} c_{m,n} + c_{m,n}^\dagger \hat{F} c_{m,n},$$

$$\hat{U}_x = \hat{\sigma}_x, \quad \hat{U}_y = -\hat{\sigma}_x - i\hat{\lambda}\hat{\sigma}_y, \quad \hat{F} = \hat{\sigma}_z.$$ (22)

In fact, this is the simplest scheme, exploiting non-Abelian hopping operators on a square lattice, which leads to a topological (Chern) insulating phase. Indeed, the hopping operators $\hat{U}_{x,y}$ should necessarily contain a term proportional to a Pauli matrix, $\hat{U}_{x,y} \propto \hat{\sigma}_{x,y}$, in order to induce the Dirac
Figure 5. (a) The \( \pi \)-flux model on the square lattice. The unit cells of the \( \pi \)-flux lattice, with inequivalent sites \( A, B \), are labelled by the coordinates \( (m, n) \in \mathbb{Z} \). Two inequivalent sites \( A, B \) belonging to the same unit cell are connected by a full black line. Standard NN hopping operators are characterized by the tunnelling factors \( \pm J \). The complex NNN hopping, with the tunnelling factor \( i\lambda \), are represented by purple dotted arrows (the hoppings with the opposite factor, \(-i\lambda\), correspond to the reversed paths). The chirality introduced by the NNN terms potentially results in anomalous QH phases. (b) The same \( \pi \)-flux model translated into a non-Abelian square lattice, with matrix hopping operators \( U_{x,y} \). The ‘undesired’ diagonal hoppings \( D_{1,2} \) are depicted by red dotted arrows and disappear in the limit \( \alpha = 0 \) (cf the text). Note that when \( \alpha = 0 \), this model reduces to the non-Abelian optical lattice illustrated in figure 1(c). The modified \( \pi \)-flux model, corresponding to \( \alpha = 0 \), is represented in the appendix.

In conclusion, we have described an experimental scheme to generate non-Abelian gauge potentials for cold atoms in a square optical lattice. The scheme generalizes the proposals of [30, 38] originally formulated for alkali atoms, to two-electron atoms (alkaline earth atoms, ytterbium, erbium, etc), and requires additional superlattice potentials to generate finite flux per elementary cell of the lattice [44]. We have shown that such arrangements can be used to generate topological phases with the non-zero Chern number. We have detailed two examples of microscopic models exhibiting such phases, the first one based on an explicit mapping from the Haldane honeycomb lattice to a multicomponent model defined on the square lattice with non-Abelian tunnelling operators, and the second one built directly on the square lattice (the so-called \( \pi \)-flux model). The latter model is slightly simpler, but its experimental implementation would still represent a considerable challenge due to the relatively large number of laser frequencies that need to be controlled precisely. As explained before, the different transition frequencies can be generated from a single laser using frequency modulators, a standard tool in optoelectronics. As a result, we believe that the experimental implementation is challenging but possible. Besides the models described here, generating non-Abelian gauge fields on a lattice can simulate many different systems, including lattice-gauge-theory models [38] and spin–orbit coupled gases [7], with complete freedom over the choice of the microscopic Hamiltonian [34, 41, 42]. This is in contrast with Raman-coupled bulk systems, for example, where the form of the spin–orbit coupling is constrained [59]. We also stress that although the laser-coupled honeycomb lattice described in [24, 25] should be easier to implement experimentally, it does not offer the versatility of the non-Abelian optical lattices discussed in this work. As a result, the system described in this paper has the potential to realize other kinds of topological insulating phases, such as, for instance, the \( \mathbb{Z}_2 \) topological insulators characterized by a topological invariant \( \nu_{\mathbb{Z}_2} = \pm 1 \) and exhibiting the quantum spin Hall effect [7, 34–37, 60–63].
Figure A1. (a) The modified Haldane model obtained by setting $\alpha = 0$ in equations (10) and (11). Here, we only represent the NNN hopping terms involving the two inequivalent lattice sites $A, B$ of the central unit cell $(m, n)$. The ‘missing’ NNN hopping terms are represented by red and blue dotted lines. (b) The modified $\pi$-flux model obtained by setting $\alpha = \pi$ in equation (19). These pictures are to be compared with the original Haldane and $\pi$-flux models illustrated in figures 1(a) and 5(a), respectively.

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Appendix. The non-Abelian optical lattice and the modified Haldane/$\pi$-flux models

In section 3, we considered a modified version of the Haldane model that was obtained by taking the limit $\alpha \to 0$, namely by neglecting the NNN hopping terms that led to non-zero diagonal hopping operators $\hat{D}$ in the non-Abelian model described by equation (11). This modification was motivated by the fact that the model with $\hat{D} \to 0$ could be realized in square optical lattices with non-Abelian hopping operators acting along NN links. To be complete, we represent in figure A1(a) the NNN hopping terms of the original Haldane model that have been omitted in the transformation leading to the cold-atom model described by Hamiltonian (16). We represent in figure A1(b) the modified $\pi$-flux model, leading to the Hamiltonian (22).

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