Generalizing Uniform Distribution Using the Quantile Function

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Abstract. A New Distribution in this paper was derived. The generalization of the uniform distribution using the quantile function of the T-X family depends on the reliability of the exponential distribution, then the quantile function reliability function. We studied the properties (moment s, some reliability analysis, moment generating function, quantile and median, order statistics, entropy). Then estimate the parameters using the maximum likelihood method.

Keywords: Uniform Distribution, Quantile Function, T-X System, Moments, Parameters, statistics.

1. Introduction

Over the past few decades, many classical distributions have been commonly used to model data in many fields such as engineering, technology, climate, medical, biological sciences, demography, economics, and insurance. However, there is a growing need for expanded types of these distributions in many fields applied, such as life analysis and insurance. Therefore several methods for generating new families from distributions have been studied. Some attempts have been made to define new families of probability distributions that extend families of established distributions and at the same time provide great versatility in modeling data in practice. In many realistic cases, classical distributions do not provide adequate score points for real data. Therefore several generators have been proposed in the statistical literature that use one or more parameters to construct novel distributions. Some generators which are well known. Some respected generators are Marshall-Olkin generated family (MO-G) Marshall and Olkin (1997), the beta-G by Eugene, Lee, and Famoye (2002) and Jones (2004), Kumaraswamy-G (Kw-G for short) Cordeiro and de Castro (2011), McDonald-G (Mc-G) by Alexander, Cordeiro, Ortega, and Sarabia (2012), gamma-G (type 1) by Zografos and Balakrishnan (2009), gamma-G (type 2) by Ristić and Balakrishnan (2012), gamma-G (type 3) by Torabi and Hedesh (2012), log-gamma-G by Amini, MirMostafaei, and Ahmadi (2012), logistic-G by Tahir, Cordeiro, Alzaatreh, Mansoor, and Zubair (2015a), exponentiated generalized-G by Cordeiro, Ortega, and da Cunha (2013), Transformed-Transformer (T-X) by Alzaatreh, Lee, and Famoye (2013), exponentiated (T-X) by Alzaghal, Famoye, and Lee (2013), Weibull-G by Bourguignon, Silva, and Cordeiro (2014), Exponentiated half logistic generated family by Cordeiro, Alizadeh, and Ortega (2014a), Lomax-G by Cordeiro, Ortega,
Popovic, and Pescim (2014b), Kumaraswamy Odd log-logistic-G by Alizadeh, Emadi, Doostparast, Cordeiro, Ortega, and Pescim (2015b), Kumaraswamy Marshall-Olkin by Alizadeh, Tahir, Cordeiro, Mansoor, Zubair, and Hamedani (2015c), Beta Marshall-Olkin by Alizadeh, Cordeiro, De Brito, and Demetrio (2015a), Type 1 Half-Logistic family of distributions by Cordeiro, Alizadeh, and Diniz Marinho (2015) and Odd generalized exponential-G by Tahir, Cordeiro, Alizadeh, Mansoor, Zubair, and Hamedani (2015b).

This paper is organized as follows: In 2, In this section we present a probability density function and a cumulative density function with their respective properties. : In 3, As for this section, we present some reliability analysis, including (risk, reverse risk), the rate function and the cumulative risk function for GURQD. : In 4, In this section we presented the moments: In 5, In this section we have introduced the Moment Generation function : In 6, we have introduced Quantile and Median : In 7, we have introduced Order Statistics: In 8, we have introduced Entropy : In 9, we have introduced Estimation

2. Generalizing Uniform Distribution Using the Quantile Function, GURQD

Since \( R(x) = \frac{1}{1-F(x)} \) as head of \( F(x) \), \( R(x) \) has \( u(0,1) \) distribution then
\[ x = R^{-1}(1 - F(x)) \]
is quantile function reliability function.

if \( G(x) = G\left(\frac{Q(F(x))}{b-a}\right) \) where \( x \sim U(a,b) \) then \( F(x) = \frac{x-a}{b-a} \)

\[ x = Q\left(R_y(x)\right) \]
where \( R_y(x) \) is reliability of exponential distribution such that
\[ R_y(x) = e^{-\lambda x} I_{(0,\infty)}(x) \]
then \( x = \ln \frac{1}{\lambda} \) where \( R = R_y \) random variable.

\[ G(x) = G\left(Q\left(R_y(x)\right)\right) = G\left(R\right) \]

such that \( Q\left(R_y(x)\right) \) is quantile function reliability function.

\[ G(x) = \ln \frac{1}{\lambda a} - b \]
then \( g(x) = \frac{-1}{\lambda(b-a)} \)

so you find it is impossible to use this substation unless using the following:

\[ R(x) = P(X > x) = \int_0^x f(x) \, dx \]

\[ R_y(x) = \int_{L_R}^{U_R} \frac{dx}{b-a}, \quad L_R < R_y < U_R \]
when \( R = 0 = L_R \), \( R = 1 = U_R \)

\[ R_y(x) = \frac{b}{b-a} \int_{R}^{\infty} \frac{dx}{b-a} \]
\[ = \frac{b - lnR^\frac{1}{\lambda}}{b-a} - \frac{b - lnR^\frac{1}{\lambda}}{(b-a)} - \frac{\lambda b + lnR^\frac{1}{\lambda}}{(b-a)} \]

A random variable R is indicated to be generalizing uniform distribution using the quantile function, GURQD if its pdf is in the form of

\[ w_{GURQD}(R) = \frac{1}{\lambda(b-a)R}, \quad \theta R < \theta R < \theta R, \quad \alpha > b > 0, \lambda > 0 \quad \ldots (1) \]

Which is a pdf as it can be proved as follows , a probability density function?
1) It is easy to check \( w_{GURQD}(R) \geq 0 \)

2) The integral of the function in (1) must equal to 1

\[
\int_{e^{-\lambda a}}^{e^{-\lambda b}} w_{GURQD}(R) \, dR = \int_{e^{-\lambda a}}^{e^{-\lambda b}} \frac{1}{\lambda(b-a)R} \, dR
\]

\[
= \frac{1}{\lambda(b-a)} \left[ e^{-\lambda a} \frac{1}{R} \right]_{e^{-\lambda b}}^{e^{-\lambda a}}
\]

\[
= \frac{1}{\lambda(b-a)} \left[ \ln e^{-\lambda a} - \ln e^{-\lambda b} \right]
\]

\[
= \frac{1}{\lambda(b-a)} \left[ \ln e^{-\lambda a} - \ln e^{-\lambda b} \right] = 1
\]

The limit for the pdf as follows:

\[
w_{GURQD}(R) = \frac{1}{\lambda(b-a)R} \cdot e^{-\lambda b} < R < e^{-\lambda a}
\]

1. \[ \lim_{R \to e^{-\lambda a}} w_{GURQD}(R) = \lim_{R \to e^{-\lambda a}} \frac{1}{\lambda(b-a)R} = \frac{1}{\lambda(b-a)} \lim_{R \to e^{-\lambda a}} \frac{1}{R} \]

\[
= \frac{1}{\lambda(b-a)} e^{-\lambda b}
\]

2. \[ \lim_{R \to e^{-\lambda a}} w_{GURQD}(R) = \lim_{R \to e^{-\lambda a}} \frac{1}{\lambda(b-a)R} = \frac{1}{\lambda(b-a)} \lim_{R \to e^{-\lambda a}} \frac{1}{R} \]

\[
= \frac{1}{\lambda(b-a)} e^{-\lambda a}
\]
Figure 1. This graph showed pdf of $W_{GURQD}(R)$ with the parameters $a=1.5$; $b=(1.7,1.9,2)$, $\lambda=(0.8,1.2,2.8)$.

The cdf of generalizing uniform distribution using the quantile function, GURQD

$W_{GURQD}(R) = \int_{1-e^{-\lambda a}}^{R} w_{GURQD}(R) dR$

$= \int_{1-e^{-\lambda a}}^{R} b e^{-\lambda b} \frac{1}{\lambda(b-a)R} dR$

$= \frac{1}{\lambda(b-a)} \int_{1}^{R} \frac{1}{R} dR$

$= \frac{1}{\lambda(b-a)} \ln R \mid_{1}^{R} e^{\lambda b}$

$= \frac{1}{\lambda(b-a)} \left( \ln R - \ln e^{-\lambda b} \right)$

$W_{GURQD}(R) = \frac{1}{\lambda(b-a)} \left( \ln R + \lambda b \right)$  ...(2)
3. Some Reliabilities Analysis:

The reliability is discussed such as (reverse hazard, hazard) rate function and cumulative hazard function for the GURQD.

Subsequently, the probability for a system survives beyond a specified time is known as reliability function or (survivor, survival) function and it is set out in the following function.

\[
R_{GURQD} (R) = 1 - W_{GURQD} (R)
\]

\[
= 1 - \frac{1}{\lambda (b - a)} \left( \ln R + \lambda \cdot \frac{b}{b - a} \right)
\]

\[
= 1 - \frac{\ln R + \lambda \cdot \frac{b}{b - a}}{\lambda (b - a)} \quad \ldots (3)
\]

Note that \( R_{GURQD} (R) + W_{GURQD} (R) = 1 \)

The hazard function is also defined as hazard rate or failure rate and it is given by

\[
h (R) = \frac{W_{GURQD} (R)}{1 - W_{GURQD} (R)}
\]

Figure 2. This graph showed cdf of \( W_{GURQD} (R) \) with the parameters \( \lambda = 4.5 \); 
\( b = (1.3, 1.8, 1.9) \) \( a = (0.9, 1.5, 1.8) \).
\[ = \frac{1}{R \left[ \lambda (b-a) + \left( \ln R + \lambda b \right) \right]} \quad \text{...}(4) \]

and the reverse hazard function for the GURQD, is given by

\[
h_r (R) = \frac{w_{\text{GURQD}}(R)}{w_{\text{GURQD}}(R)} \]
\[ = \frac{1}{R \left( \ln R + \lambda b \right)} \quad \text{...}(5) \]

and the cumulative hazard function for the GURQD, is given by

\[
H_R (R) = -\log \left( 1 + \frac{1}{\lambda (b-a)} \left( \ln R + \lambda b \right) \right) \]

### 4. Moments

Let R denotes the random variable of GURQD, then the r-th order moment \(E \left( R^r \right)\) of GURQD can be obtained as

\[
E \left( R^r \right) = \int_{e^{-\lambda b}}^{e^{-\lambda a}} \frac{w_{\text{GURQD}}(u)}{R^r} \, du \\
= \int_{e^{-\lambda b}}^{e^{-\lambda a}} \frac{R^r}{\lambda (b-a) R} \, dR \\
= \frac{1}{\lambda (b-a)} \left[ e^{-\lambda a} - e^{-\lambda b} \right] R^{r-1} \, dR \\
= \frac{1}{\lambda (b-a)} \left[ e^{-\lambda a} - e^{-\lambda b} \right] R^r \\
= \frac{1}{\lambda (b-a)} \left[ \frac{(e^{-\lambda a})^r}{r} - \frac{(e^{-\lambda b})^r}{r} \right] \]

Thus, the \(r^{th}\) order moment of GURQD, is obtained

\[
E \left( R^r \right) = -\frac{1}{\lambda r (b-a)} \left[ e^{-\lambda r a} - e^{-\lambda r b} \right] \]

Put \(r = 1, 2, 3, 4\) we will obtain the first four moments
Mean \( = \mu_1 = \frac{1}{\lambda} \left[ e^{-\lambda a} - e^{-\lambda b} \right] \)

\( \mu_2 = \frac{1}{2\lambda (b-a)} \left[ e^{-2\lambda a} - e^{-2\lambda b} \right] \)

\( \mu_3 = \frac{1}{3\lambda (b-a)} \left[ e^{-3\lambda a} - e^{-3\lambda b} \right] \)

\( \mu_4 = \frac{1}{4\lambda (b-a)} \left[ e^{-4\lambda a} - e^{-4\lambda b} \right] \)

Also the variance is calculated by

\[
\text{var}(R) = \text{E}(R^2) - \left( \text{E}(R) \right)^2
\]

\[
= \frac{1}{2\lambda (b-a)} \left[ e^{-2\lambda a} - e^{-2\lambda b} \right] - \left( \frac{1}{\lambda (b-a)} \left[ e^{-\lambda a} - e^{-\lambda b} \right] \right)^2
\]

\[
\text{var}(R) = \mu_2 - (\mu_1)^2 \quad \ldots (6)
\]

Standard Deviation \( \sigma = \sqrt{\text{var}(R)} \)

\[
= \sqrt{\frac{1}{2\lambda (b-a)} \left[ e^{-2\lambda a} - e^{-2\lambda b} \right] - \left( \frac{1}{\lambda (b-a)} \left[ e^{-\lambda a} - e^{-\lambda b} \right] \right)^2}
\]

\[
= \sqrt{\mu_2 - (\mu_1)^2} \quad \ldots (7)
\]

Coefficient of Variation

\[
(CV) = \frac{\sigma}{\mu_1} = \frac{\frac{1}{2\lambda (b-a)} \left[ e^{-2\lambda a} - e^{-2\lambda b} \right] - \left( \frac{1}{\lambda (b-a)} \left[ e^{-\lambda a} - e^{-\lambda b} \right] \right)^2}{\sqrt{\mu_2 - (\mu_1)^2}} \cdot \frac{\mu_2 - (\mu_1)^2}{\mu_1} \quad \ldots (8)
\]

Such that \( \mu_1 \) First moment and \( \mu_2 \) The second moment.

Coefficient of variation \( (\gamma) = \frac{\sigma}{\mu} \)
coefficient of skewness \( K_a = \frac{\overline{x} - M_o}{\sigma} = \frac{\mu_1 - e^{-\lambda(a+b)}}{\sqrt{\mu_2 - (\mu_1)^2}} \) \( \mu_1 \) ... (9)

coefficient of kurtosis

\[
L = \frac{M_4}{(\sigma)^4} = \frac{\mu_4}{(\mu_1)^4} = \frac{\mu_4}{\left[ \frac{\mu_2 - (\mu_1)^2}{\mu_1} \right]^2} = \frac{\mu_4}{\left[ \frac{\mu_2 - (\mu_1)^2}{\mu_1} \right]^2} \] ... (10)

5. Moment Generating Function

the MGF of GURQN, was detected by

\[
M(t) = E(e^{tR}) = \left[ e^{-\lambda a} e^{\lambda b} e^{tR} \right]_{GURQD} (R) dR
\]

\[
= \left[ \frac{1}{e^{\lambda b}} \right] e^{-\lambda a} e^{tR} \frac{1}{l} dR
\]

From Taylor series of \( e^{tR} \) get on \( e^{tR} = \sum_{i=0}^{\infty} \frac{(tR)^i}{i!} \) then

\[
= \frac{1}{e^{\lambda b}} \left[ e^{-\lambda a} \sum_{i=0}^{\infty} \frac{(tR)^i}{i!} R^{-1} \right] dR
\]

\[
= \frac{1}{e^{\lambda b}} \left[ e^{-\lambda a} \sum_{i=0}^{\infty} \frac{t^i (R)^i}{i!} R^{-1} \right] dR
\]

\[
= \frac{e^{-\lambda a}}{\lambda(b-a)} \left[ \sum_{i=0}^{\infty} \frac{t^i (R)^i}{i!} \right] e^{-\lambda b} R^{-1} dR
\]

\[
= \frac{e^{-\lambda a}}{\lambda(b-a)} \left[ \sum_{i=0}^{\infty} \frac{t^i (R)^i}{i!} \right] e^{-\lambda b} R^{-1} dR
\]
The $r$-th moment about the mean of a r.v. $R$ is called Central moments, stated by $\mu_r$, is the expected value of $(R - \mu_R)^r$ Simbolized

$$\mu_r = E\left[(R - \mu_R)^r\right]$$

$$= \int_{\mu_R - \lambda \mu \beta}^{\mu_R + \lambda \mu \beta} (R - \mu_R)^r w_{\text{GURQD}}(R) dR \quad \text{for } r = 0, 1, 2, \ldots$$

let $\nu = \mu_R$

$$= \int_{\mu_R - \lambda \mu \beta}^{\mu_R + \lambda \mu \beta} (R - \nu)^r w_{\text{GURQD}}(R) dR$$

$$= \frac{1}{\lambda(b-a)} \int_{-\lambda(a-b)}^{\lambda(a-b)} (R - \nu)^r R^{-1} dR$$

$$= \frac{1}{\lambda(b-a)} \sum_{k=0}^{r} (-1)^k \int_{-\lambda(a-b)}^{\lambda(a-b)} \left(\frac{R}{R-k} \right)^r \nu^k R^{-1} dR$$

$$= \frac{1}{\lambda(b-a)} \sum_{k=0}^{r} (-1)^k \nu^k \left(\frac{R}{R-k} \right)^r \int_{-\lambda(a-b)}^{\lambda(a-b)} R^{r-k} dR$$

$$= \frac{1}{\lambda(b-a)} \sum_{k=0}^{r} (-1)^k \nu^k \left(\frac{R}{R-k} \right)^r \left(\frac{e^{-\lambda(a-b)}}{r-k} - \frac{e^{-\lambda b}}{r-k} \right)$$

Then we have the rth central moment of $R$ about $\mu_R$.
6. Quantile and Median:

\[ W_{GURQD} (R) = \frac{1}{\lambda(b - a)} (\ln R + \lambda b) \]

By solving \( W_{GURQD} (R) = q \) for given value, the quantile \( Q(q) \) is obtained

\[ q = \frac{1}{\lambda(b - a)} (\ln Q + \lambda b) \]

\[ Q(q) = \exp \left[ q \left( \frac{\lambda(b - a)}{\lambda(b - a)} \right) - \lambda b \right] \]

Setting \( q = 0.5 \) yields the median of GURQD.

7. Order Statistics O.S.:

Let \( R_1, R_2, \ldots, R_n \) is a simple random sample with distribution function \( W_{GURQD}(R) \) and density function \( w_{GURQD}(R) \). Let the order statistics denote as \( R_{(1:n)} \leq R_{(2:n)} \leq \ldots \leq R_{(n:n)} \) obtained from the sample. The d.f. of \( R_{(i:n)} \), \( 1 \leq i \leq n \) is provided by:

\[ w_{i:n}(R) = \frac{n!}{(i-1)!(n-i)!} \left[ W_{GURQD}(R) \right]^{i-1} (1 - W_{GURQD}(R))^{n-i} w_{GURQD}(R), R > 0 \]

The pdf of the maximum O.S. \( R_n \) is specified by:

\[ w_{R_n}(R) = \frac{n!}{(i-1)!(n-i)!} \left[ W_{GURQD}(R) \right]^{i-1} (1 - W_{GURQD}(R))^{n-i} w_{GURQD}(R), R > 0 \]

and the smallest O.S. \( R_1 \) of the pdf is provided by:

\[ w_{R_1}(R) = \frac{n!}{(i-1)!(n-i)!} \left[ W_{GURQD}(R) \right]^{i-1} (1 - W_{GURQD}(R))^{n-i} w_{GURQD}(R), R > 0 \]

8. Entropy:

Entropy is a measure of a random variable R, variance or uncertainty. The Shannon and Renyi entropies Renyi (1961), Shannon (2001) are two common entropy measures. The entropy of a Shannon variable R with pdf \( w_{GURQD}(R) \) is defined as
Entropy = $E \left( -\log w(R) \right)$

$$= \int e^{-\lambda a} e^{-\lambda b} w_{GURQD}(R) \left[ -\log w(R) \right] dR$$

$$= \int e^{-\lambda a} e^{-\lambda b} w_{GURQD}(R) \left[ -\log \left( \frac{1}{\lambda (b-a) R} \right) \right] dR$$

$$= \int e^{-\lambda a} e^{-\lambda b} \left[ \frac{1}{\lambda (b-a) R} \right] \log \left( \frac{1}{\lambda (b-a) R} \right) dR$$

$$= \frac{1}{\lambda (b-a)} \int e^{-\lambda b} \left[ \frac{1}{R} \right] \log \left( b-a + \log R \right) dR$$

$$= \frac{1}{\lambda (b-a)} \left( \frac{1}{2} \log \frac{b-a + \log e}{2} \right)$$

$$= \frac{1}{\lambda (b-a)} \left( \frac{1}{2} \log \frac{b-a + \log e}{2} \right)$$

$$9. \text{ Estimation:}$$

We will use maximum likelihood method for estimating the parameters of generalizing uniform distribution using the exponential quantile function. Let $R_1, R_2, \ldots, R_n$ indicate the sample size $n$ at random from the GURQD. Then the likelihood function is given by

$$L(R_i; \lambda, b, a) = \prod_{i=1}^{n} w \left( R_i; \lambda, b, a \right)$$

$$= \prod_{i=1}^{n} \frac{1}{\lambda (b-a) R_i}$$

$$\ln L(\lambda, b, a) = \sum_{i=1}^{n} \ln \left( \frac{1}{\lambda (b-a) R_i} \right)$$

$$= -\sum_{i=1}^{n} \left( \ln \lambda + \ln (b-a) + \ln R_i \right)$$
\[-n \ln \lambda - n \ln (b - a) - \sum_{i=1}^{n} \ln R_i\]

\[\frac{\partial}{\partial \lambda} \ln L(\lambda, b, a) = -\frac{n}{\lambda} = 0\]

\[-n = \hat{\lambda} * 0 \Rightarrow n = 0 \quad (Absurd \ result)\]

\[\frac{\partial}{\partial b} \ln L(\lambda, b, a) = \frac{n}{(b - a)} = 0\]

\[-n - (\hat{b} - a)^* 0 \Rightarrow n = 0 \quad (Absurd \ result)\]

\[\frac{\partial}{\partial a} \ln L(\lambda, b, a) = \frac{n}{(b - a)} = 0\]

\[n = (b - a)^* 0 \Rightarrow n = 0 \quad (Absurd \ result)\]

\[L \left( R_i ; \lambda, b, a \right) = \prod_{i=1}^{n} \frac{1}{\lambda (b - a) R_i} = \left( \frac{1}{\lambda (b - a)} \right)^n \sum_{i=1}^{n} R_i^{-1}\]

L is maximum if \( \lambda (b - a) \) is minimum

\[\lambda a \leq R_i \leq \lambda b \Rightarrow \lambda a \leq R(1), \ldots, R(n) \leq \lambda b\]

\[\lambda a - R(1) \text{ and } \lambda b - R(n) \text{ then } a - \frac{R(1)}{\lambda}, b - \frac{R(n)}{\lambda} \text{ and } \lambda - \frac{R(1)}{a}, \lambda - \frac{R(n)}{b}\]

\[\hat{a} = \frac{\min \left\{ R(1), \ldots, R(n) \right\}}{\lambda}, \quad \hat{b} = \frac{\max \left\{ R(1), \ldots, R(n) \right\}}{\lambda} \quad \text{and}\]

\[\hat{\lambda} = \frac{\min \left\{ R(1), \ldots, R(n) \right\}}{b}, \quad \hat{\lambda} = \frac{\max \left\{ R(1), \ldots, R(n) \right\}}{\hat{b}}\]

10. Conclusion

Overall, in this study is considered a new three parametric distribution called as the generalizing uniform distribution using the quantile function, GURQD. As well the distribution has found by the C.D.F. of the T-X family that is given by \( U \left[ Q \left( R_y \left( x \right) \right) \right] \) where U is cdf of uniform distribution, \( R_y \) is cdf of the reliability of exponential distribution and Q is quantile function reliability function. In addition, there are a three parametric distribution with two scale and one shape parameter were studied. As by using certain special functions, the mathematical properties, moments, failure rate, survival function, reverse hazard rate were obtained.
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