Limiting Behavior of Asteroid Obliquity and Spin Using a Semi-analytic Thermal Model of the YORP Effect

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Abstract

The Yarkovsky–O’Keefe–Radzievskii–Paddack (YORP) effect governs the spin evolution of small asteroids. The axial component of YORP, which alters the rotation rate of the asteroid, is mostly independent of its thermal inertia, while the obliquity component is very sensitive to the thermal model of the asteroid. Here, we develop a semi-analytic theory for the obliquity component of YORP. We integrate an approximate thermal model over the surface of an asteroid, and find an analytic expression for the obliquity component in terms of two YORP coefficients. This approach allows us to investigate the overall evolution of asteroid rotation state, and to generalize the results previously obtained in the case of zero thermal inertia. The proposed theory also explains how a nonzero obliquity component of YORP originates even for a symmetric asteroid, due to its finite thermal inertia. In many cases, this causes equatorial planes of asteroids to align with their orbital planes. The studied nontrivial behavior of YORP as a function of thermal model allows for a new kind of rotational equilibria, which can have important evolutionary consequences for asteroids.

Unified Astronomy Thesaurus concepts: Asteroid dynamics (2210); Asteroid rotation (2211)

1. Introduction

Rotation of kilometer-sized asteroids is governed by the Yarkovsky–O’Keefe–Radzievskii–Paddack (YORP) effect (Rubincam 2000; Vokrouhlický et al. 2015). It is a torque caused by scattering and re-emission of light by the asteroid’s surface, which can change both the asteroid’s rotation rate $\omega$ and the obliquity $\varepsilon$. The part of the torque changing the rotation rate is called the axial component $T_A$, whereas the part affecting obliquity is called the obliquity component $T_\varepsilon$. Characterizing the overall evolution of the rotation state of an asteroid under the combined action of these two components is one of the most fundamental tasks of the YORP theory.

This task has already been solved in our previous paper, Golubov & Scheeres (2019), in a simplified case of zero thermal inertia. Under this constraining assumption, the evolution of asteroids has been simulated numerically, as well as studied analytically in the most typical case. It was shown that most of the asteroids, when starting their evolution from a slow rotation rate, gradually increase it to a certain limit, and if not disrupted by the centrifugal forces in the process, return back to very slow rotation. Inclusion of the tangential YORP into the model of Golubov & Krugly (2012) can qualitatively alter this typical evolution and bring in the possibility of YORP equilibria.

Still, as long as the tangential YORP is disregarded and only the normal YORP is considered, the axial component is indeed, to a very high accuracy, independent of the thermal model. This fundamental fact about YORP was demonstrated in the simulations by Breiter et al. (2010) and later theoretically proven under more general assumptions by Golubov et al. (2016). It allows $T_A$ to be studied in the limit of zero thermal inertia, as it is done in the model by Golubov & Scheeres (2019).

On the other hand, the obliquity component $T_\varepsilon$ is highly sensitive to the thermal inertia. As it has been shown in the numeric simulations by Čapek & Vokrouhlický (2004), $T_\varepsilon$ can be dramatically altered and even flip sign with the change of the heat conductivity. Those authors conducted a deep analysis of the YORP torques, but did not extend their formalism to study the overall asteroid evolution.

A more advanced evolutionary study was later performed by Scheeres & Mirrahimi (2008), although in a simplified model. Their approach avoided rigorous solution of the heat equation by introducing a fixed time lag between the absorption and emission of energy. This allowed the authors to characterize the rotational dynamics—and in particular, to find the possibility of stable equilibria.

Here, we generalize the results of Golubov & Scheeres (2019) for the case of nonzero thermal inertia, basing our approach on the formalism of Golubov et al. (2016). This makes our theory more precise than that of Scheeres & Mirrahimi (2008), and allows us to go farther into analysis of the asteroid than Čapek & Vokrouhlický (2004).

In Section 2, we combine the analytic and numeric approaches to simplify the problem and to reduce all the information about the asteroid shape to two YORP coefficients. Section 3 studies the asteroid evolution in terms of the YORP coefficients, as well as the stable YORP equilibria that can arise on the evolutionary tracks of some asteroids.

2. YORP Coefficients

2.1. Problem Setting

Let us start with the equations of motion of the asteroid, which describe the evolution of its rotation rate $\omega$ and obliquity $\varepsilon$ as a function of time $t'$ (Rubincam 2000):

$$\frac{d\omega}{dt'} = T_\varepsilon,$$  

(1)
where \( I_{z} \) is the asteroid’s moment of inertia, while \( T_{z} \) and \( T_{e} \) are the axial and obliquity components of the YORP torque, acting on the asteroid.

It is convenient to nondimensionalize the problem in the following manner. Let the mean volumetric radius of the asteroid be \( R \) and its density \( \rho \). Then we can introduce the dimensionless moment of inertia \( i_{z} \) by the following equation:

\[
i_{z} = \frac{I_{z}}{\rho R^{5}}.
\]

(3)

The dimensionless YORP torques are introduced as

\[
\tau_{z} = \frac{cT_{z}}{\Phi R^{3}},
\]

(4)

\[
\tau_{e} = \frac{cT_{e}}{\Phi R^{3}}.
\]

(5)

Here, \( c \) is the speed of light and \( \Phi \) is the effective solar constant.

Next, it is convenient to introduce the dimensionless thermal parameter \( \theta \),

\[
\theta = \frac{(C\rho\omega\kappa)^{1/2}}{(e\sigma)^{1/4}(1-A)^{3/4}R^{3/4}}.
\]

(6)

Here, \( A \) is the albedo, \( e \) is the thermal emissivity, \( \sigma \) is the Stefan–Boltzmann constant, \( \kappa \) is the heat conductivity of the material constituting the asteroid surface, and \( C \) is its specific heat capacity. This thermal parameter characterizes the relative importance of thermal inertia of the surface: for \( \theta \ll 1 \), the surface almost instantly adjusts its temperature to the illumination, while for \( \theta \gg 1 \), the surface temperature remains almost constant throughout the rotation period.

The dimensionless time is introduced as \( t = t'/t_{0} \), where

\[
t_{0} = \frac{\sqrt{e\sigma(1-A)^{3/4}R^{2}c}}{C\kappa}.
\]

(7)

The value of \( t_{0} \) characterizes the order of magnitude of the YORP evolution timescale for the thermal parameter \( \theta \sim 1 \) and for the maximal possible strength of YORP \( \tau \sim 1 \). For other values of \( \theta \) and \( \tau \), the timescale would change in direct proportion to \( \theta^{2}\tau^{-1} \).

After all these changes of notation are applied, Equations (1) and (2) assume the following form:

\[
i_{z} \frac{d\theta^{2}}{dt} = \tau_{z},
\]

(8)

\[
i_{z} \frac{\theta^{2}d\psi}{dt} = \tau_{e}.
\]

(9)

2.2. Obliquity Component in Terms of the YORP Coefficients

The dimensionless YORP torques \( \tau_{z} \) and \( \tau_{e} \) can be expressed as integrals over the surface of the asteroid, containing dimensionless pressures \( p_{\text{sin}}^{r} \) and \( p_{\text{cos}}^{r} \). These pressures are some known functions of the thermal parameter \( \theta \), obliquity \( \varepsilon \), and the latitude of the point on the asteroid surface \( \psi \). (See Golubov et al. 2016 for derivations or Appendix A for a summary).

To compute \( p_{\text{sin}}^{r}(\psi, \varepsilon, \theta) \) and \( p_{\text{cos}}^{r}(\psi, \varepsilon, \theta) \), we use an algorithm similar to Breiter et al. (2010). We simulate a one-dimensional heat conductivity in a semi-space under the surface by decomposing the temperature into a Fourier series, expressing the boundary condition as a set of nonlinear equations for the Fourier coefficients, and iteratively finding its solution. Then, the \( p_{\text{sin}}^{r} \) and \( p_{\text{cos}}^{r} \) are expressed in terms of the obtained Fourier series. The results are in agreement with Golubov et al. (2016), who evaluated the same functions, using finite difference method to solve the heat conductivity equation. Still, the method applied here works about three orders of magnitude faster, while providing the same accuracy.

From our numeric simulations, it follows that a reasonable approximation to the functions \( p_{\text{sin}}^{r} \) and \( p_{\text{cos}}^{r} \) is given by the equations

\[
p_{\text{sin}}^{r}(\psi, \varepsilon, \theta) = p_{\text{sin}}^{r}(\theta) \sin 2\varepsilon \sin 2\psi,
\]

\[
p_{\text{cos}}^{r}(\psi, \varepsilon, \theta) = p_{\text{cos}}^{r}(\theta) \sin 2\varepsilon \sin 2\psi.
\]

(10)

These equations basically express the principal nonvanishing terms of the Fourier decomposition of \( p_{\text{sin}}^{r} \) and \( p_{\text{cos}}^{r} \) as a function of \( \psi \) and \( \varepsilon \). Additional arguments for the validity of this approximation are given in Appendix A. The numerically determined \( p_{\text{sin}}^{r}(\theta) \) and \( p_{\text{cos}}^{r}(\theta) \) are plotted in Figure 1.

From Section 3 of Golubov et al. (2016), we know the asymptotics of \( p_{\text{sin}}^{r} \) and \( p_{\text{cos}}^{r} \): \( p_{\text{sin}}^{r} \propto p_{\text{cos}}^{r} \propto \theta^{-1} \) for \( \theta \to \infty \), and \( p_{\text{sin}}^{r} \propto \theta^{0}, p_{\text{cos}}^{r} \propto \theta^{1} \) for \( \theta \to 0 \). We fit the numeric solution for \( p_{\text{sin}}^{r} \) and \( p_{\text{cos}}^{r} \) by the analytic expressions that have the correct
asymptotic behavior (Figure 1)

\[ p_{\sin}(\theta) = \frac{\tilde{p}_s}{\theta + \theta_0} \]
\[ p_{\cos}(\theta) = -\frac{\tilde{p}_c}{(\theta + \theta_0)^2}. \]  

(11)

The best-fit coefficients are about \( \tilde{p}_s = 0.098 \), \( \tilde{p}_c = 0.082 \), and \( \theta_0 = 2.5 \).

With the aid of Equations (10) and (11), the expression for the obliquity component of YORP transforms into

\[ \tau_c = \sin 2c (AC_{\sin} p_{\sin}(0) + (1 - A)C_{\sin} p_{\sin}(\theta) + (1 - A)C_{\cos} p_{\cos}(\theta)). \]  

(12)

The coefficients \( C_{\sin} \) and \( C_{\cos} \) are expressed as integrals over the asteroid surface (see Appendix B). Therefore, all the information about the asteroid shape needed to compute the YORP evolution is contained in these two coefficients. The coefficient \( C_{\sin} \) is proportional to \( C_{\epsilon} \) from Golubov & Scheeres (2019) and differs from it by the factor \( p_{\sin}(0) \), whereas \( C_{\cos} \) is a new concept that was absent in the case of zero thermal inertia.

The first term in Equation (12) proportional to the albedo \( A \) expresses the contribution to YORP from the light scattered by the asteroid. The zero argument of \( p_{\sin} \) arises from the immediacy of light scattering, which is equivalent to no thermal inertia. The following two terms correspond to the re-emitted light, and thus are proportional to the absorption fraction \( 1 - A \).

2.3. Investigation of the YORP Coefficients

To compute the YORP coefficients \( C_{\sin} \) and \( C_{\cos} \) and study the asteroid evolution, we take a sample of 5716 photometric shape models from DAMIT4, Đurech et al. (2010), 29 radar shape models5, and 4 in situ models of asteroids Eros, Itokawa, Bennu, and Ryugu.6 If several photometric or radar shape models of the same asteroid were present in the database, we processed them all independently, whereas for the in situ models, we used the ones with 196,608 facets. We assumed that the \( z \)-axis of the shape model was the rotation axis of the asteroid, in some cases implied a non-principal axis rotation.

Dependence between \( C_{\sin} \) and \( C_{\cos} \) is studied in Figure 2. One can see that \( C_{\cos} \) is positive in the predominant majority of cases, whereas \( C_{\sin} \) has equal probabilities of being positive or negative. The symmetric distribution of the points in the plot implies absence of correlation between \( C_{\sin} \) and \( C_{\cos} \). On the other hand, there is a strong correlation between the asteroid pole flattening and \( C_{\cos} \) as revealed by the color coding. For example, the two overlapping, red, open squares that are the lowest points in the plot correspond to two models of asteroid 4179 Toutatis. This asteroid experiences tumbling, and the \( z \)-axis of its shape model is oriented in such an unnatural way that \((a + b)/2c \approx 0.4\) is much less than unity, once again confirming the mentioned correlation.

\begin{itemize}
  \item DAMIT: https://astro.troja.mff.cuni.cz/projects/damit/.
  \item Asteroid Radar Research. Asteroid Shape Models https://echo.jpl.nasa.gov/asteroids/shapes/shapes.html.
  \item PDS Small Bodies Node. Shape Models of Asteroids, Comets, and Satellites, https://sbn.psi.edu/pds/shape-models/.
\end{itemize}
the obliquity. According to Equation (9), this corresponds to \( \tau_c < 0 \). One can then look at Equation (12), note that \( C_{\sin} = 0 \) for ellipsoidal asteroid (see Appendix B for the proof), and conclude that \( C_{\cos}P_{\cos} < 0 \). As \( P_{\cos} \) is always negative (see Figure 1), \( C_{\cos} \) has to be positive, just as it can be seen in Figure 2.

Moreover, the YORP torque is zero for spherical asteroids, where the light pressure forces have zero lever arm, and it rises for more flattened asteroids as the lever arm increases, which agrees with the monotonic growth of \( C_{\cos} \) as a function of flattening in Figure 3. Naturally, this effect also vanishes for very fast and very slow rotators, as they do not have a significant temperature differences between the evening and the morning sides. This agrees with the asymptotic behavior of \( P_{\cos} \), which vanishes in the limits \( \theta \to 0 \) and \( \theta \to \infty \).

It is important to note that \( C_{\cos} \) is nonzero even for a perfect ellipsoid, while \( C_{\sin} \) is only produced by its asymmetry, which is usually slight. This can explain why, in most of the cases, \( C_{\cos} \) is about an order of magnitude greater than \( C_{\sin} \).

3. YORP Evolution and Equilibria

3.1. Overall YORP Evolution

From Golubov & Scheeres (2019), we know an approximate expression for \( \tau_c \), which in our present notations looks like

\[
\tau_c = \frac{C_{\sin}P_k}{\alpha \theta_0} \cos(2\varepsilon + \beta),
\]

(13)

where the coefficients \( \alpha \approx 0.72 \) and \( \beta \approx 0.33 \). When Equations (12) and (13) are substituted into Equations (8) and (9), a full set of evolutionary equations is obtained. It describes \( \theta \) and \( \varepsilon \) as functions of time \( t \).

In this section, we will investigate the typical solutions of these equations. As \( C_{\cos} > 0 \) in the majority of cases (see Figure 2), this is what we assume henceforth. On the other hand, the sign of \( C_{\sin} \) seems to be positive and negative with equal probabilities, thus we consider both cases.

In the case \( C_{\sin} < 0 \), the topology of the solution is the same as in Golubov & Scheeres (2019). The asteroids start at small rotation rates, accelerate their rotation, and then slow it down, if not disrupted by the centrifugal forces on the way. The most important quantitative differences from Golubov & Scheeres (2019) occur at \( \theta \sim 1 \), where the large \( C_{\cos} \) term causes obliquity to evolve much faster than the rotation rate.

In the opposite case of \( C_{\sin} > 0 \), a qualitatively new behavior can arise, as it is shown in Figure 5. At \( \theta \ll 1 \) and \( \theta \gg 1 \), the contribution from \( C_{\sin} \) dominates in Equation (12), causing \( \varepsilon \) to increase. On the other hand, at \( \theta \sim 1 \), the major contribution to Equation (12) can arise from \( C_{\cos} \), leading to \( \varepsilon \ll 0 \). These areas with different signs of \( \tau_c \) are separated by the curves where \( \tau_c = 0 \), which in our approximation are straight. They are shown in Figure 5 with blue dashed lines. Additionally, Equation (13) turns into zero at \( \varepsilon \approx 55^\circ \), which is shown in the figure with a green dashed line. These lines split the phase plane into parts where \( \theta \) and \( \varepsilon \) change in different directions, shown with colored arrows in Figure 5. On the boundaries, two equilibrium points originate. The lower one, at \( (\theta_0^+, \varepsilon_0) \), is an unstable saddle point. The upper equilibrium, at \( (\theta_0^- \varepsilon_0) \), is a focal point, whose stability requires additional investigation. We postpone the discussion of their stability until Section 3.3, and first discuss under which circumstances and in which rotation states such equilibria occur.

3.2. Existence of YORP Equilibria

To find the equilibria, we substitute Equation (11) into Equation (12), equate its right-hand side to zero, and solve for
The roots of the resulting quadratic equation are

$$\theta^\pm_0 = \frac{\theta_0}{2 \bar{p}_c C_{\sin}} (\bar{p}_c (1 - A))$$

$$C_{\cos} - \bar{p}_c (1 + A) C_{\sin}$$

$$\pm \sqrt{-4 \bar{p}_c^2 A C_{\sin}^2 + (\bar{p}_c (1 - A) C_{\cos} - \bar{p}_c (1 + A) C_{\sin})^2}.$$  

(14)

The equilibria exist when the expression under the square root is positive, which is equivalent to the following condition:

$$\frac{C_{\cos}}{C_{\sin}} > \frac{1 + \sqrt{A \bar{p}_c^2}}{1 - \sqrt{A \bar{p}_c^2}}.$$  

(15)

The areas where this condition is met are shown in shades of gray in Figure 2. We see that the majority of points with $C_{\sin} > 0$ satisfy this condition. This result comes naturally from the fact that for most of the asteroids $C_{\cos} \gg |C_{\sin}|$ (see Figure 2).

Equation (14) is illustrated in Figure 6, which shows the equilibrium points $\theta_1$ and $\theta_2$ as functions of $C_{\cos}/C_{\sin}$ for several fixed values of albedo $A$. As the ratio $C_{\cos}/C_{\sin}$ increases, a bifurcation occurs, in which two equilibria originate.

It is easy to understand why YORP equilibria always appear in pairs. In the absence of tangential YORP, $\tau_z$ is independent of $\theta$. Thus, the condition $\tau_z = 0$ prescribes the value of $\varepsilon$. With $\varepsilon$ being fixed, the second condition for the equilibrium $\tau_z = 0$ turns into an equation only for $\theta$. Applying the asymptotics for $\bar{p}_{\sin}$ and $\bar{p}_{\cos}$ to Equation (12), one can see that $\tau_z(0^\circ) = \tau_z(180^\circ) = A$. Therefore, $\tau_z$ has the same sign at 0 and $\infty$. Due to the continuity of the function $\tau_z(\theta)$, it implies that the total number of sign reversals (i.e., equilibria) is even. This fact was omitted by Čapek & Vokrouhlický (2004), who unphysically assumed $A = 0$, and thus for many asteroids observed only one sign reversal.

The rotation periods $P_{1,2}$ corresponding to the equilibrium thermal parameters $\theta_{1,2}$ are depicted in Figure 7. The horizontal axis shows different heliocentric distances. The three colors show different values of the ratio $C_{\cos}/C_{\sin}$, corresponding to the 10th, 50th, and 90th percentile of the distribution of studied DAMIT asteroids. Two thermal inertias $\Gamma$ are shown with different types of lines, corresponding to the lower and upper boundary of their typical values (Hanuš et al. 2018). Two albedos $A = 0.05$ and $A = 0.25$ were chosen to represent low- and high-albedo asteroids (Masiero et al. 2011) and depicted with thin and thick lines correspondingly.

The plotted lines show where $\tau_z$ changes sign, and small arrows point to where the sign is negative. To put it in a simple way, one can assume that in the direction of the arrows, $C_{\cos}$ dominates over $C_{\sin}$, and the result is the relatively fast alignment of the asteroid equatorial planes with their orbital planes, i.e., $\varepsilon = 0^\circ$ or $180^\circ$. In the direction opposite to the arrows, $C_{\sin}$ dominates over $C_{\cos}$, resulting in small probabilities of $\varepsilon = 0^\circ/180^\circ$ and $\varepsilon = 90^\circ$ and an evolution similar to the one described for the low thermal inertia limit by Golubov & Scheeres (2019).

One can see that, depending on the values of parameters, the evolutionary regime can be very different. For most of the high thermal inertia, near-Earth asteroids (NEAs) $C_{\cos}$ is indeed dominant, whereas in the main asteroid belt, this is true for only ~50% of the bodies. Among low thermal inertia asteroids $C_{\cos}$ dominates for ~50% of the NEAs, but loses to $C_{\sin}$ for the overwhelming majority of the main belt asteroids. We must conclude that the widely acknowledged alignment of asteroid equatorial and orbital planes due to YORP is only partially true and does not describe the entire asteroid population.

The lines with downward-pointing arrows correspond to potentially stable equilibria, similar in kind to the filled red circle in Figure 5. Such equilibria are physically feasible only if they result in realistic rotation periods, hours to days, to avoid both tumbling or rotational disruption.

Typical ranges of such periods for NEAs and main-belt asteroids (MBAs) are marked in the plot. It can be seen from the plot that NEAs can have equilibria with realistic periods if they have high thermal inertias $\Gamma$ and the most probable $C_{\cos}/C_{\sin}$ ratios, or if they have low $\Gamma$ and high $C_{\cos}/C_{\sin}$. MBAs are expected to have realistic periods if they have high $\Gamma$ and above-average $C_{\cos}/C_{\sin}$ ratios.
The lower parts of both panels (\(\theta \ll 1\)) show the geometry of phase trajectories described by Golubov & Scheeres (2019). The upper parts of the two panels are essentially the same, but with the factor of 1/4 slower evolution in obliquity.

In the middle part of the upper panel (\(\theta \sim 1\)), the evolution is also similar to Golubov & Scheeres (2019) but even faster due to the contribution from the \(C_{\cos}\) term. Therefore, for negative \(C_{\cos}/C_{\sin}\), the entire phase portrait has a trivial topology.

On the other hand, in the middle of the lower panel (\(C_{\cos}/C_{\sin} > 0\)) two equilibrium points can appear, with a complex geometry of phase trajectories around them. The lower equilibrium is an unstable saddle point. The trajectories around the upper equilibrium are closed in our model. Thus, it is neither a stable focus nor an unstable focus, but rather a neutral center. However, this result is model-dependent and breaks if higher-order terms are taken into account.

Therefore, the factorization of Equation (16) creates both an opportunity for analytic solution and an obstacle for stability analysis. In a more realistic theory, more Fourier terms should be taken in Equation (12), thus breaking the factorization. This analytic approach will be further explored in our future article, while now we limit ourselves to a simple illustration of stability in one individual case.

For this purpose, we created a program that simulates dynamical evolution of an asteroid. It solves Equations (8) and (9) with the right-hand sides precisely computed for a given asteroid shape. The thermal model for \(\tau_{eps}\) uses the Fourier algorithm described in Section 2.2. Sample results of this simulation are shown in Figure 9.

One can see a stable focal point. The asteroid starts its evolution far away from it, performs a number of oscillations with a decreasing amplitude, and eventually converges to the equilibrium. From Figure 9, it can be seen that the attraction basin around the focal point is large. Each asteroid whose shape permits such a stable equilibrium has a substantial probability of acquiring the initial conditions within this attraction basin, e.g., as a result of collision. Then, the asteroid sinks to the equilibrium and resides there until the next collision or other major perturbation.

### 3.3. Stability of YORP Equilibria

The physical significance of the focal equilibrium point depends on its stability. Unfortunately, our analytic model does not allow us to conclude whether the focus is stable or not. The full set of evolutionary equations is obtained by substituting Equations (13) and (12) into Equations (8) and (9) correspondingly:

\[
\frac{d\theta^2}{dt} = \frac{1}{i_\omega} \alpha \frac{C_{\sin} p_{\sin}}{\cos (2\varepsilon) + \beta},
\]

\[
\frac{d\varepsilon}{dt} = \frac{1}{\theta^2} (AC_{\sin} p_{\sin} (0) + (1 - A)C_{\sin} p_{\sin} (\theta) + (1 - A)C_{\cos} p_{\cos} (\theta)).
\]

The right-hand sides of the equations are factorized. They are products of functions depending on either \(\varepsilon\) or \(\theta\). We divide the first equation by the second one and separate the variables. The integral of the resulting expression gives an implicit solution of the system Equation (16),

\[
\frac{1}{4\alpha} (\beta \ln \tan \varepsilon + \ln \sin 2\varepsilon) - \ln \theta + (1 - A) \ln (\theta + \theta_0) - \frac{(1 - A)C_{\cos} p_{\cos} (\theta)}{C_{\sin} p_{\sin} (\theta + \theta_0)} = \text{Const.}
\]

This equation provides an implicit solution of Equation (16), which is plotted in Figure 8. The two panels show two values of \(C_{\cos}/C_{\sin}\), negative and positive, with the absolute value at the median of distribution of the DAMIT shapes.
This is not a good approximation, as the axial component of emergence from each of them with a new shape, an asteroid can eventually acquire such $C_{\sin}$ and $C_{\cos}$ to be locked in a stable rotation state. It would preserve this stable rotation until an external perturbation (collision, orbital change) either kicks it out or destroys its stability. With its significance for the asteroid evolution, this equilibrium is similar to the ones previously discussed in literature (Golubov & Scheeres 2016; Golubov et al. 2018; Golubov & Scheeres 2019), although caused by different physical mechanisms. It requires neither YORP nor asteroid binarity, and thus in some sense it is the simplest of all types of equilibria.

Stability of these rotation states presents a major theoretical challenge. As in the simplest model, one cannot determine whether the focal point is stable, a more sophisticated theory needs to be developed, accounting for the higher-order Fourier terms in Equations (12) and (13). We will target the analytic and numeric study of these terms in the next article, thus making the simulation of the YORP evolution even more realistic.

The proposed model provides a compromise between accuracy and simplicity. It takes into account the thermal contribution to the YORP effect, but boils down the information about the asteroid shape to a few free parameters, allowing to keep track of their physical meaning and their individual impact on the simulation results. This model is similar to the one by Golubov & Scheeres (2019), and only slightly more complicated, but the inclusion of $C_{\cos}$ and the associated thermal model allows to bring in a great amount of new physics. The latter includes the new kind of stable equilibria and the preferential alignment of the equatorial and orbital planes at thermal parameters of the order of unity. The new insight into the structure of asteroid families, dynamics of asteroid pairs, and MBA-NEA asteroid transport can be obtained by conducting a rigorous simulation of the asteroid evolution, similar to Marzari et al. (2020). It should account for YORP, Yarkovsky, and collisions, and such a simple but realistic model of YORP presents the cornerstone for such modeling of asteroid evolution.

Appendix A

Methods for Computation of the YORP Effect

Here, we briefly review the YORP theory from Golubov et al. (2016). The components of the nondimensional YORP torque are expressed as integrals over the asteroid surface $S$ by the following equations:

$$
\tau_\epsilon = \frac{1}{R^3} \oint_S dS \sin \Delta \cos \eta \cos \psi \frac{p^*}{(\psi, \epsilon)} \, \, (\psi, \epsilon), \quad (A1)
$$

$$
\tau_\psi = -\frac{1}{R^3} \oint_S dS \sin \Delta \cos \eta \sin \psi \times (Ap^*_{\sin} (\psi, \epsilon) + (1 - A)p^*_{\cos} (\psi, \epsilon, \theta)) + \cos \psi \sin \eta - \sin \psi \cos \eta \cos \Delta \times (1 - A)p^*_{\cos} (\psi, \epsilon, \theta). \quad (A2)
$$

The angles $\psi$, $\eta$, and $\Delta$ are defined by the orientation of a surface element on the asteroid, and explained in Figure 10.
Here, $p^s$, $p^\alpha$, $p^\beta$, and $p^\gamma$ are the dimensionless YORP pressures. The former two of them are defined as follows:

$$p^s(\psi, \varepsilon) = \frac{2}{3\pi^2} \int_{-\pi/2}^{\pi/2} d\phi \left[ 1 - (\sin \phi \cos \psi \sin \varepsilon - \sin \psi \cos \varepsilon)^2 \right],$$

$$p^\alpha(\psi, \varepsilon) = \frac{2}{3\pi^2} \int_{-\pi/2}^{\pi/2} d\phi \sin \phi \left[ 1 - (\sin \phi \cos \psi \sin \varepsilon - \sin \psi \cos \varepsilon)^2 \right].$$

The latter two pressures, $p^\beta$ and $p^\gamma$, are defined via the weighted averages of the dimensionless temperature, which is found by the heat conduction equation:

$$p^\beta(\psi, \varepsilon, \theta) = \frac{1}{6\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\psi \frac{\tau^4}{\sin \phi},$$

$$p^\gamma(\psi, \varepsilon, \theta) = \frac{1}{6\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\psi \frac{\tau^4}{\cos \phi}.$$

With respect to both arguments $\psi$ and $\varepsilon$, the function $p^s$ is even, while the functions $p^\alpha$, $p^\beta$, and $p^\gamma$ are odd:

$$p^s(\psi, -\varepsilon) = p^s(\psi, \varepsilon),$$

$$p^\alpha(\psi, -\varepsilon) = -p^\alpha(\psi, \varepsilon),$$

$$p^\beta(\psi, -\varepsilon, \theta) = -p^\beta(\psi, \varepsilon, \theta),$$

$$p^\gamma(\psi, -\varepsilon, \theta) = -p^\gamma(\psi, \varepsilon, \theta).$$

The functions $p^\beta$ and $p^\gamma$ have the following limiting behavior:

$$p^\beta(\psi, 0) = p^\beta(\psi, \infty) = 0,$$

$$p^\gamma(\psi, 0) = p^\gamma(\psi, \infty) = 0.$$

Symmetries of the problem require that all functions $p^\alpha$ and $p^\gamma$ vanish when either $\psi$ or $\varepsilon$ equals either 0 or $\pm \pi/2$. Moreover, the functions are odd with respect to both arguments $\psi$ and $\varepsilon$. This allows us to guess that qualitatively good approximations to $p^\alpha$ and $p^\beta$ can be given by the expressions

$$p^\alpha(\psi, \varepsilon, \theta) = p^\alpha(\psi, \varepsilon) \sin 2\varepsilon \sin 2\psi,$$

$$p^\beta(\psi, \varepsilon, \theta) = p^\beta(\psi, \varepsilon) \sin 2\varepsilon \sin 2\psi.$$
The Taylor decomposition of this expression gives the following approximation valid for \( |a - c| \ll c \):

\[
C'_{\cos} \approx \frac{32\pi}{15} \left( \frac{a}{c} - 1 \right).
\]  

(B7)

For a triaxial ellipsoid, the integral cannot be computed in a closed form, but as a simple estimate, we can substitute \( a \) in the former expressions with the mean equatorial radius \( (a + b)/2 \). Equation (B6) in this approximation is plotted in Figure 11 with a solid black line. It agrees well with the results of simulations for different asteroid shapes. No dependence is seen between \( \cos \) and the asteroid’s surface roughness, which is color-coded.

A simple estimate for \( C''_{\cos} \) can be obtained in the following manner. Let \( \Delta \) be some effective value of \( \Delta \). We assume it to be roughly constant, and take it out of the integral. The remaining integral is evaluated in the same way as Equation (B6), leading to a similar expression:

\[
C''_{\cos} = -\frac{8\pi}{1} \left( 1 - \cos \frac{\Delta}{a} \right) \left( \frac{2}{3} + \frac{1}{c^2} - \frac{a^2}{c^2} - 1 \right) - \frac{a^2}{c^2} \frac{1}{\left( \frac{a^2}{c^2} - 1 \right)^{3/2}} \arctan \left( \frac{a^2}{c^2} - 1 \right).
\]  

(B8)

Again, we use this equation as an approximation for triaxial ellipsoids, by substituting \( (a + b)/2 \) instead of \( a \). This ignores the fact that the difference between \( a \) and \( b \) causes a complicated correlation between \( \Delta \) and the position on the asteroid. The Taylor approximation to Equation (B8) in the case of \( |a - c| \ll c \) gives

\[
C''_{\cos} \approx -\frac{16\pi}{15} (1 - \cos \frac{\Delta}{a}).
\]  

(B9)

This equation is illustrated in Figure 12. We see a good agreement between Equation (B8) and the simulations for the shape models.

In Figure 13, \( C''_{\cos} \) is studied as a function of the ellipticity of the asteroid’s equator, \( e = (a - b)/(a + b) \). The figure shows a negative trend, which can be fitted by a linear dependence. Still, the trend is worse than in Figure 12. This is natural, as high ellipticity of the equator implies a large average deviation \( \Delta \) between the equatorial projections of the radius vector and the normal vector.

The distribution of asteroids from the DAMIT database over \( C_{\cos} \) is shown in Figure 14. We see once again that this coefficient is positive in almost all the cases.

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