Hilbert spaces of states of
\(\mathcal{PT}\)–symmetric harmonic oscillator
near exceptional points

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Abstract

Although the Stone theorem requires that in a physical Hilbert space \(\mathcal{H}\) the time-evolution of a quantum system is unitary if and only if the corresponding Hamiltonian \(H\) is self-adjoint in \(\mathcal{H}\), an equivalent, recently popular picture of the same evolution may be constructed in another, manifestly unphysical Hilbert space \(\mathcal{K}\) in which \(H\) is non-Hermitian but \(\mathcal{PT}\)–symmetric. Unfortunately, one only rarely succeeds in circumventing the key technical obstacle which lies in the necessary ultimate reconstruction of all of the eligible physical Hilbert spaces of states \(\mathcal{H}\) in which \(H\) is self-adjoint. We show that, how, and why such a reconstruction becomes feasible for a spiked harmonic oscillator in a phenomenologically most interesting vicinity of its phase-transition exceptional points.
1 Introduction

Ordinary differential Schrödinger equation

\[ \left( -\frac{d^2}{dr^2} + \frac{G}{r^2} + r^2 \right) \varphi(r) = E \varphi(r) \]  

(1)
is usually used to describe the radial motion of a particle in a \( D \)-dimensional harmonic oscillator well \( V(\vec{r}) = |\vec{r}|^2 \). In multiple textbooks \cite{1} the authors also add a comment that the quantum system remains stable even with attractive, classically unstable spikes in \( V(\vec{r}) = |\vec{r}|^2 - g/|r|^2 \) where \( g < 1/4 \), i.e., in (1), when \( G > -1/4 \) is negative.

In 1999, in a way inspired by Bender and Boettcher \cite{2} we showed, in Ref. \cite{3}, that under the same constraint \( G > -1/4 \) the model remains stable even when it ceases to be Hermitian. We proved, in particular, that the complex shift of the line of coordinates

\[ r \rightarrow r(x) = x - ic \in \mathbb{C}, \quad c > 0, \quad x \in (-\infty, \infty) \]  

(2)
in the same ordinary differential equation (1) still keeps the energy spectrum real, discrete and bounded from below. The regularization of the centrifugal-type singularity in the resulting non-Hermitian but \( \mathcal{PT} \)-symmetric (i.e., parity-times-time-reversal-symmetric \cite{4, 5}) Hamiltonian

\[ H^{(\alpha)} = -\frac{d^2}{dx^2} + (x - ic)^2 + \frac{G}{(x - ic)^2}, \quad \alpha = \sqrt{G + 1/4} > 0, \quad c > 0, \quad x \in (-\infty, \infty) \]  

(3)
made the model eligible as an unusual but still exactly solvable example in supersymmetric quantum mechanics \cite{6}.

Mathematically, our Hamiltonian operator \( H^{(\alpha)} \) is defined in Hilbert space \( \mathcal{K} \) of square-integrable functions of the new real variable \( x \), \( \mathcal{K} = L^2(\mathbb{R}) \). In Ref. \cite{3} it has been shown that in spite of the manifest non-Hermiticity of Hamitonian \( \mathcal{K} \) in \( \mathcal{K} \), its spectrum is all real and defined, in terms of two quantum numbers, by compact formula

\[ E = E_n^{(Q)} = 4n + 2 - 2Q\alpha, \quad Q = \pm 1, \quad n = 0, 1, 2, \ldots \]  

(4)
As functions of the coupling \( G \) of the regularized centrifugal-like spike these eigenvalues are sampled in Fig. 1.

In our present paper a long missing constructive probabilistic interpretation of such an exactly solvable quantum model will be presented in a restriction to the most interesting dynamical regimes which are not too far from the instants of phase transitions called exceptional points (EPs).

2 Physics behind \( \mathcal{PT} \)-symmetric harmonic oscillator

At the time of the publication of Ref. \cite{3} in 1999, a consistent physical interpretation of similar Schrödinger equations has not been available yet (see, e.g., its most recent reviews in \cite{7, 8, 9}). Only
Figure 1: $G$–dependence of spectrum of $\mathcal{PT}$–symmetric harmonic oscillator (...). Vertical lines mark the exceptional-point values of coupling $G^{(EP)}_\alpha = \alpha^2 - 1/4$ at $\alpha = (EP) = 0, 1, 2, 3$ and $4$.

step by step it has been clarified that the new variable $x$ does not carry the physical meaning of an observable of a particle position [10]. The intuitively appealing concept of $\mathcal{PT}$–symmetry was not yet re-assigned its traditional mathematical meaning of Krein-space-based $\mathcal{P}$–pseudo-Hermiticity [11, 12, 13]. Last but not least, time was still needed for the reformulation of quantum mechanics in which even non-Hermitian Hamiltonians with real spectra may generate unitary evolution.

Before an ultimate formulation of these ideas (see, e.g., [4, 12]) it was necessary to rediscover several older results by mathematicians [14] and physicists [15, 16].

2.1 Three Hilbert space formulation of quantum mechanics [17]

During the process of understanding of the Bender’s and Boettcher’s conjectures [2] it appeared necessary to replace, first of all, the mathematically friendly Hilbert space $\mathcal{K}$ by its unitarity-compatible alternative (we will denote it by dedicated symbol $\mathcal{H}$ in what follows). These two Hilbert spaces differ just by na amendment of the inner product $(\cdot, \cdot)$ [16]. In the standard Dirac’s bra-ket notation we may write

$$ (\psi_a, \psi_b)_\mathcal{K} = \langle \psi_a | \psi_b \rangle, \quad (\psi_a, \psi_b)_\mathcal{H} = \langle \psi_a | \Theta | \psi_b \rangle. $$

(5)
Here, the ad hoc, Hilbert-space-metric operator $\Theta = \Theta^\dagger > 0$ must be such that the evolution generated by $H$ becomes unitary in $\mathcal{H}$ [18,19]. This means that the metric must be, by construction, Hamiltonian-dependent, i.e., such that

$$H^\dagger \Theta = \Theta H.$$  \hfill (6)

Next, once we factorize the metric

$$\Theta = \Omega^\dagger \Omega$$  \hfill (7)

we reveal that the operator $\Omega$ maps the ket-vector elements $|\psi\rangle$ of $\mathcal{K}$ (or, equivalently, of $\mathcal{H}$) on the new, “curly” kets which span another, third Hilbert space denoted as $\mathcal{L}$,

$$|\psi\rangle = \Omega |\psi\rangle, \quad |\psi\rangle \in \mathcal{H}, \quad |\psi\rangle \in \mathcal{L}.$$  \hfill (8)

This construction implies the equivalence between the following two inner products,

$$(\psi_a, \psi_b)_\mathcal{H} = (\psi_a, \psi_b)_\mathcal{L}.$$  \hfill (9)

We may conclude that the quantum system in question may be represented by the Hamiltonian $H$ acting in Hilbert space $\mathcal{H}$ or, equivalently, by the Hamiltonian

$$h = \Omega H \Omega^{-1}$$  \hfill (10)

defined in Hilbert space $\mathcal{L}$. In this framework we may say that $h$ is self-adjoint in $\mathcal{L}$ while $H$ is self-adjoint in $\mathcal{H}$ [19]. The third, manifestly unphysical Hilbert space $\mathcal{K}$ is just a mathematically preferred auxiliary space in which the calculations are all performed – this is the reason why $H$ is often (and misleadingly) called non-Hermitian.

In an application of the three-Hilbert-space picture to our harmonic oscillator Hamiltonian we may also observe that it is non-Hermitian in the manifestly unphysical Hilbert space $\mathcal{K} = L^2(\mathbb{R})$. Obviously, unless we specify the physical Hilbert space $\mathcal{H}$ [i.e., the metric $\Theta$], the description of the system remains unfinished, leaving the information about physics incomplete. The necessity of the completion (i.e., of the specification of metric $\Theta$) follows from the necessity of the standard probabilistic interpretation of the model. In applications, such a requirement reflects the weakest point of the whole theory. In fact, for a long time it remained unnoticed that the present spiked harmonic oscillator model offers one of the rare opportunities of its consequent and complete implementation.

### 2.2 Exceptional points

Besides an expected confirmation of complexification of the whole spectrum of model (3) at negative $\alpha < 0$ (the effect widely known under the nickname of a spontaneous breakdown of
$\mathcal{PT}$-symmetry \cite{2,20}, one of the key results of paper \cite{3} was the observation that at the positive integer values of $\alpha = 1, 2, \ldots$ the energy levels cross but remain real. Due to the exact solvability of the model it was easy to reveal that at all of these values of the parameter the unavoided eigenvalue crossings were accompanied also by the parallelization and degeneracy of the related pairs of eigenvectors. Indeed, for the bound state wave functions expressed in terms of Laguerre polynomials,

$$\varphi(x) = \text{const.} \, (x - ic)^{-Q\alpha + 1/2} e^{-(x - ic)^2/2} \, L_n^{-Q\alpha} \left[ (x - ic)^2 \right], \quad n = 0, 1, \ldots$$

the rigorous proof of the parallelizations was based on the elementary identities like

$$L_n^{-1} \left[ (x - ic)^2 \right] = -(x - ic)^2 L_n \left[ (x - ic)^2 \right]$$

etc.

Using the terminology as introduced by Kato \cite{21} all of the integer values of $\alpha = 0, 1, \ldots$ may be called exceptional points (EPs). At these values, operator $H^{(\alpha)}$ ceases to be diagonalizable. In the context of quantum mechanics this has the following important consequence (see the reasons, e.g., in \cite{12}).

**Lemma 1** \cite{3} Operator \cite{3} may play the role of Hamiltonian of a unitary quantum system only if $\alpha > 0$ and $\alpha \notin \mathbb{Z}$.

| $\alpha = \alpha(G)$ | 0 | 1 | 2 | 3 | 4 |
|-----------------------|---|---|---|---|---|
| $E_n^{(Q))(G)$       |   |   |   |   |   |
| $G = -1/4$           |   |   |   |   |   |
| $G = 3/4$            |   |   |   |   |   |
| $G = 15/4$           |   |   |   |   |   |
| $G = 35/4$           |   |   |   |   |   |
| $G = 63/4$           |   |   |   |   |   |
| $E_0^{(+)} = E_0^{(-)}$ |   |   |   |   |   |
| $E_1^{(+)} = E_1^{(-)}$ |   |   |   |   |   |
| $E_2^{(+)} = E_2^{(-)}$ |   |   |   |   |   |
| $E_3^{(+)} = E_3^{(-)}$ |   |   |   |   |   |
| $E_4^{(+)} = E_4^{(-)}$ |   |   |   |   |   |

Table 1: EP degeneracies
The detailed nature of EP-related degeneracies can vary with our choice of $\alpha^{(EP)} = \alpha^{(EP)}_K = K$ where $K = 0, 1, \ldots$ (see Table 1). At these points the lost possibility of diagonalization of $H^{(\alpha)}$ can only be replaced by its canonical representation,

$$H^{(\alpha)} Q^{(\alpha)} = Q^{(\alpha)} \mathcal{J}^{(\alpha)}, \quad \alpha = \alpha^{(EP)} \in \mathbb{Z}.$$  \hspace{1cm} (12)

An optimal choice of the infinite-dimensional canonical representative $\mathcal{J}^{(\alpha)}$ of the EP limit of the Hamiltonian will be specified below. This choice will enable us to treat the transition-matrix solutions $Q^{(\alpha)}$ of Eq. (12) as a certain degenerate EP analogue of the set of eigenvectors forming an unperturbed basis. In such a perspective our recent experience with the EP-based perturbation theory will find its new application as a tool of making, finally, the consistent and constructive physical interpretation of our non-Hermitian but unitary harmonic oscillator quantum model near its EP singularities complete.

Expectedly, without an additional information about dynamics there will be infinitely many such completions. In the related literature, unfortunately, one rarely finds a sufficiently nontrivial example of such a variability of options. In our present paper such an example is provided.

3 Physical Hilbert space of oscillator near the spontaneous breakdown of $\mathcal{PT}$–symmetry ($\alpha^{(EP)} = 0$)

Let us initiate our analysis of oscillator (3) in the dynamical regime of the smallest positive parameters $\alpha$. Only in the next section we will make the analysis complete by extending it to all of the EP neighborhoods of $\alpha \approx K$ with $K = 1, 2, \ldots$.

Near the lowermost EP limit $\alpha \to 0^+$ an inspection of Fig. 1 reveals that the full, infinite-dimensional Hilbert space may be decomposed into a sequence of two-dimensional subspaces $\mathcal{K}^{(\alpha)}_{(2)}$,

$$\mathcal{K} = \bigoplus_{n=0}^{\infty} \mathcal{K}^{(\alpha)}_{(2)}.$$  \hspace{1cm} (13)

The vanishing $\alpha$ loss of the diagonalizability of $H^{(\alpha)}$ may be best reflected by the choice of the canonical representation matrix $\mathcal{J}^{(0)}$ of Eq. (12) in the following block-diagonal-matrix form of a direct sum of Jordan matrices,

$$\mathcal{J}^{(0)} = J^{[2]}(2) \bigoplus J^{[2]}(6) \bigoplus J^{[2]}(10) \bigoplus \ldots , \quad J^{[2]}(E) = \begin{pmatrix} E & 1 \\ 0 & E \end{pmatrix}.$$  \hspace{1cm} (14)

Having specified this matrix we have to solve Eq. (12) yielding the infinite-dimensional transition matrix. The columns of this matrix may then play the role of an unperturbed basis in $\mathcal{K}$.  

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Such a construction generates, finally, a simplified isospectral zero-order representation of our Hamiltonian,

\[ S^{(0)}(\alpha) = [Q^{(0)}]^{-1} H^{(0)} Q^{(0)} = \mathcal{J}^{(0)} + \text{corrections}, \quad 0 < \alpha \ll 1. \]  

(15)

In other words this means that our Hamiltonian will have the infinite-dimensional block-diagonal matrix structure,

\[
S^{(0)}(\alpha) = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & \ldots \\
0 & 2 & 0 & 0 & 0 & \ldots \\
0 & 0 & 6 & 1 & 0 & \ldots \\
0 & 0 & 0 & 6 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix} + \text{corrections}
\]

where the blocks are the two-by-two Jordan matrices. It is now necessary to specify the first-order correction term in (15).

In the dynamical regime of small and positive \( \alpha \), i.e., close to the leftmost EP instability at \( \alpha_0^{(EP)} = 0 \) we have to describe the energies as functions of the coupling constant \( G \). Indeed, once we set \( G = G^{(EP)} + \xi \) and once we rewrite formula (4) for energies as a function of \( \xi \), we notice the qualitative difference between the left and right vicinities of \( \xi = 0 \). As long as only the right, real-energy vicinity with \( \xi > 0 \) in \( E_n^{(\pm)} = 4n + 2 \pm 2\sqrt{\xi} \) is of our present interest, we know that at its EP boundary the whole spectrum degenerates pairwise, \( \lim_{\alpha \to 0} E_n^{(\pm)} \to 4n + 2, \ n = 0, 1, \ldots \).

Our recent experience with corrections to non-diagonalizable matrices \[22\] warns us against a naive expectation that the correction term in (15) should be of order \( O(\alpha) \). An independent version of the same warning came also from Ref. \[23\] and/or from inspection of Fig. 1. We found that the dominant, leading-order correction appearing in Eq. (15) may be written in an apparently counterintuitive but still remarkably elementary explicit form,

\[ S^{(0)}(\alpha) = \mathcal{J}^{(0)} + \xi \mathcal{V}^{(0)} + \text{higher order corrections}, \quad \xi = \mathcal{O}(\alpha^2) \]

(16)

with elementary block-diagonal matrix of perturbations

\[
\mathcal{V}^{(0)} = [J[2](0)]^T \bigoplus [J[2](0)]^T \bigoplus [J[2](0)]^T \bigoplus \ldots
\]

(17)

where, the superscript \( T \) marks the matrix transposition.

The main consequence of these formulae is that in every two-dimensional subspace \( \mathcal{K}^{[2]}_{(n)} \) we have a block-diagonalized leading-order Hamiltonian

\[ S^{(0)}(\alpha) \approx S_0^{(0)}(\alpha) = H^{[2]}_{(0)}(\xi) \bigoplus H^{[2]}_{(1)}(\xi) \bigoplus \ldots \]

(18)
where
\[
H_{(n)}^{[2]}(\xi) = J_{(n)}^{[2]}(E^{(+)}_n) + \xi \left[ J_{(n)}^{[2]}(0) \right]^T = \begin{pmatrix} E^{(+)}_n & 1 \\ \xi & E^{(+)}_n \end{pmatrix}, \quad E^{(+)}_n = E^{(+)}_n|_{\alpha=0} = 4n + 2. \quad (19)
\]

For the latter submatrices we can solve the related time-independent Schrödinger equations in closed form,
\[
H_{(n)}^{[2]}(\xi) \begin{pmatrix} 1 \\ \eta_{\pm} \end{pmatrix} = \left( E^{(+)}_n + \eta_{\pm} \right) \begin{pmatrix} 1 \\ \eta_{\pm} \end{pmatrix}, \quad \eta_{\pm} = \pm \sqrt{\xi}. \quad (20)
\]

This has the following consequence.

**Lemma 2** For approximate two by two matrix Hamiltonians (19) the unfolding energies are real if and only if the small parameter \(\xi\) is non-negative, \(\xi \geq 0\).

At non-negative \(\xi\) we have \(\xi = \alpha^2\) and \(\eta_{\pm} = \pm \alpha\) in (20). The approximate Hamiltonian (19) may be then made Hermitian along the lines outlined in subsection 2.1. Via a mere redefinition of inner products (5) our unphysical but mathematically optimal Hilbert space \(\mathcal{H}_{(n)}^{[2]}\) is converted into its correct physical alternative \(\mathcal{H}_{(n)}^{[2]}\).

**Lemma 3** Metric operators \(\Theta\) making Hamiltonian (19) Hermitian (in \(\mathcal{H}_{(n)}^{[2]}\)) read
\[
\Theta = \Theta_{(n)}^{[2]}(\alpha, b_n) = \begin{pmatrix} \alpha & b_n \\ b_n & 1/\alpha \end{pmatrix} \quad (21)
\]
and form a one-parametric family numbered by a real variable \(b_n\) such that \(|b_n| < 1\).

**Proof.** The Hermiticity of matrix \(H_{(n)}^{[2]}(\xi)\) in the physical Hilbert space \(\mathcal{H}_{(n)}^{[2]}\) means that this matrix satisfies condition (6) of subsection 2.1. This condition (written in \(\mathcal{K}_{(n)}^{[2]}\)) may be perceived as a set of linear equations for the matrix elements of the unknown matrix \(\Theta\). This matrix must be Hermitian and positive definite [16]. Under these constraints, an easy algebra leads to the result.

**Theorem 4** At small \(\alpha\) the infinite-dimensional matrix Hamiltonian \(S_{(0)}^{(0)}(\alpha)\) of Eq. (18) becomes Hermitian in the ad hoc physical Hilbert space
\[
\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{(n)}^{[2]} \quad (22)
\]
whenever we introduce, in \(\mathcal{K} = \bigoplus_{n} \mathcal{K}_{(n)}^{[2]}\), one of the amended, nontrivial inner-product metrics
\[
\Theta = \bigoplus_{n=0}^{\infty} \Theta_{(n)}^{[2]}(\alpha, b_n). \quad (23)
\]

The optional sequence of parameters \(b_n \in (-1, 1)\) with \(n = 0, 1, \ldots\) is arbitrary.
Proof. The infinite-dimensional matrix Hamiltonian (16) must be shown compatible with the Dieudonné’s Hermiticity condition (3) of subsection 2.1 below, but this follows from the block-diagonality of the participating infinite-dimensional matrices, and from Lemma 3.

We see that at sufficiently small parameters \( \alpha > 0 \), our \( \mathcal{PT} \)-symmetric harmonic-oscillator Hamiltonian (3) defined in auxiliary, unphysical Hilbert space \( \mathcal{K} = L_2(-\infty, \infty) \) of Eq. (13) acquires the status of standard self-adjoint generator of unitary evolution. Nevertheless, different choices of the sequence of parameters \( \{b_n\} \) define phenomenologically non-equivalent quantum systems. In any such a system the observables must be represented by operators \( \Lambda \) which are self-adjoint in the physical Hilbert space \( \mathcal{H} \) of Eq. (22). Even for the block-diagonal subset \( \Lambda = \bigoplus_n \Lambda^{[2]}(n) \) of observables the general form of their admissible submatrices

\[
\Lambda^{[2]}(n) = \begin{pmatrix} u & v \\ y & z \end{pmatrix}
\]

remains \( b_n \)-dependent in general. Indeed, in a parallel to Eq. (6) of subsection 2.1 these submatrices must satisfy the metric-dependent Hermiticity constraint

\[
\left[ \Lambda^{[2]}(n) \right]^\dagger \Theta^{[2]}(n)(\alpha, b_n) = \Theta^{[2]}(n)(\alpha, b_n) \Lambda^{[2]}(n).
\]

Lemma 5 Condition (25) is satisfied if and only if we restrict

\[
y = y(b_n) = \alpha^2 v + \alpha b_n (z - u)
\]

in (24).

In an elementary check, the latter construction of observables reproduces the initial leading-order Hamiltonian at \( v = 1 \) and \( u = z = 0 \). It is also easy to verify that the most popular complementary observable of charge \( [4] \) is obtained at \( v = 1/\alpha \) and \( u = z = b_n = 0 \).

Marginally, let us add that once we reparametrize \( \alpha = \exp t \) and \( b_n = \cos \phi \) in (21) and once we put \( \phi = \mu + \nu \) we may also factorize the metric (cf. Eq. (7)) yielding

\[
\Omega^{[2]}(n) = \begin{pmatrix} p & a \\ a & q \end{pmatrix}, \quad p = e^{t/2} \sin \mu, \quad q = e^{-t/2} \sin \nu, \quad a = e^{t/2} \cos \mu = e^{-t/2} \cos \nu.
\]

On these grounds, whenever needed, we may perform transition to the third Hilbert space \( \mathcal{L} \) using Eq. (8) of subsection 2.1. Redundant as this step may seem to be, the work in the latter space is often recommended in conventional textbooks, mainly for establishing easier contacts with experimentalists (cf., e.g., \([10, 24]\)).
4 Physical Hilbert spaces of oscillators near unavoided level crossings \((\alpha^{(EP)} = 1, 2, \ldots)\)

In Fig. 1 we notice a significant qualitative difference between the leftmost EP at \(\alpha = \alpha_0^{(EP)} = 0\) (to the left of which the spectrum complexifies) and the remaining EP family of \(\alpha^{(EP)}_K = K\) with \(K = 1, 2, \ldots\) (in the respective vicinities of which the spectra remain real). In what follows we intend to show that at \(K \geq 1\) such a qualitative phenomenological difference is also reflected by the related mathematics.

First of all we notice that in the limit \(\alpha \to \alpha^{(EP)}_K\) the \(K\)-plet of the lowermost energy levels remains non-degenerate (ND). In a small vicinity of \(\alpha^{(EP)}_K\) the \(K\)-dimensional Hilbert space \(\mathcal{K}^{[K]}_{ND}\) spanned by the corresponding wave functions may be characterized, for this reason, by the unit-matrix metric of textbooks, \(\Theta^{[K]}_{ND} = I\). Hence, this subspace may be treated as equivalent to its two physical alternatives, \(\mathcal{K}^{[K]}_{ND} \equiv \mathcal{H}^{[K]}_{ND} \equiv \mathcal{H}^{[K]}_{\tilde{N}D}\). For this reason the full Hilbert spaces

\[
\mathcal{K} = \mathcal{K}^{[K]}_{ND} \bigoplus \mathcal{K}^{[2]}_{(0)} \bigoplus \mathcal{K}^{[2]}_{(1)} \bigoplus \ldots, \quad \mathcal{H} = \mathcal{H}^{[K]}_{ND} \bigoplus \mathcal{H}^{[2]}_{(0)} \bigoplus \mathcal{H}^{[2]}_{(1)} \bigoplus \ldots \tag{28}
\]

may be, for our present purpose of the construction of its physics-representing amendment, reduced to the respective relevant tilded subspaces

\[
\tilde{\mathcal{K}} = \tilde{\mathcal{K}}^{[2]}_{(0)} \bigoplus \tilde{\mathcal{K}}^{[2]}_{(1)} \bigoplus \tilde{\mathcal{K}}^{[2]}_{(2)} \bigoplus \ldots, \quad \tilde{\mathcal{H}} = \tilde{\mathcal{H}}^{[2]}_{(0)} \bigoplus \tilde{\mathcal{H}}^{[2]}_{(1)} \bigoplus \tilde{\mathcal{H}}^{[2]}_{(2)} \bigoplus \ldots \tag{29}
\]

In the small left and right vicinities of exceptional points \(\alpha^{(EP)} = K \geq 1\) the reduction of attention will also involve the omission, from our considerations, of the trivial, diagonal-matrix sub-Hamiltonian \(H^{[K]}_{ND}\) such that

\[
\left( H^{[K]}_{ND} \right)_{jj} = E^{(+)}_j = -2K + 2 + 4j, \quad j = 0, 1, \ldots, K - 1.
\]

In the only relevant (i.e., in our notation, in the tilded) part \([29]\) of full spaces the rest of the spectrum remains doubly degenerate forming the sequence sampled in Table 1

\[E_n^{(-)} = E_{n+K}^{(+)} = 2K + 2 + 4n, \quad n = 0, 1, \ldots\]

This enables us to establish, in three steps, several \(K > 0\) parallels with the preceding \(K = 0\) results. In the first step we introduce the tilded version of the canonical EP Hamiltonian,

\[
\tilde{\mathcal{J}}^{(K)} = J^{[2]}(2K + 2) \bigoplus J^{[2]}(2K + 6) \bigoplus \ldots \tag{30}
\]

Up to the omission of the first \(K\) non-degenerate levels this is a perfect \(K > 0\) analogue of the \(K = 0\) EP Hamiltonian of Eq. \([14]\). In the second step we define the infinite-dimensional tilded transition matrices \(\tilde{\Phi}^{(K)}\) as solutions of a tilded version of Eq. \([12]\). In the third step, as above, we finally use these transition matrices to define the unperturbed basis (cf. \([22]\)).
In the vicinity of $\alpha^{(EP)}_K = K$, as a result, the tilded $K > 0$ analogue of the simplified Hamiltonian of Eq. (16) is obtained,

$$\tilde{\mathcal{H}}^{(K)}(\alpha) = \left[\tilde{Q}^{(K)}(\alpha)\right]^{-1} \tilde{H}^{(K)}(\alpha) \tilde{Q}^{(K)} = \tilde{J}^{(K)} + \delta^2 \tilde{V}^{(K)} + \text{higher order corrections}.$$  

(31)

The matrix of perturbations itself remains the same as above, $\tilde{V}^{(K)}(\alpha) = V^{(0)}$ [cf. Eq. (17) above]. What is, nevertheless, different is the role of the new small parameter $\delta = \delta(\alpha) = \alpha - K$. One of the reasons is that the unfolded spectrum remains real at both of its signs. Hence, the approximate leading-order tilded Hamiltonian

$$\tilde{\mathcal{H}}^{(K)}_0(\alpha) = \tilde{H}^{(2)}_{(0)}(\delta(\alpha)) \bigoplus \tilde{H}^{(2)}_{(1)}(\delta(\alpha)) \bigoplus \ldots$$

(32)

with

$$\tilde{H}^{(2)}_{(n)}(\delta) = J^{(2)}(E^{(-)}_n) + \delta^2 \left[ J^{(2)}(0) \right]^T = \begin{pmatrix} E^{(-)}_n & 1 \\ \delta^2 & E^{(-)}_n \end{pmatrix}, \quad E^{(-)}_n = E^{(-)}_n|_{\alpha=K} = 2K + 4n + 2$$

(33)

has different spectral properties determined by the related Schrödinger equation

$$\tilde{H}^{(2)}_{(n)}(\delta) \begin{pmatrix} 1 \\ \pm \delta \end{pmatrix} = (E^{(+)}_n \pm \delta) \begin{pmatrix} 1 \\ \pm \delta \end{pmatrix}.$$ 

(34)

Still, many of the consequences remain similar.

Lemma 6 Metric operators making Hamiltonian (33) Hermitian in $\tilde{\mathcal{H}}^{(2)}_{(n)}$ form a one-parametric family

$$\tilde{\Theta}^{(2)}_{(n)}(\delta, c_n) = \begin{pmatrix} \delta & c_n \\ c_n & 1/\delta \end{pmatrix}$$

(35)

where $\delta = \delta(\alpha) = \alpha - K \neq 0$ is small, and where $-1 < c_n < 1$.

Proof. The construction is analogous to the one described in the proof of Lemma 3.

□

Theorem 7 Tilded Hamiltonian $\tilde{\mathcal{H}}^{(K)}_0(\alpha)$ of Eq. (32) is Hermitian in any tilded physical Hilbert space $\tilde{\mathcal{H}}$ of Eq. (29) characterized by the metric

$$\tilde{\Theta} = \bigoplus_{n=0}^{\infty} \tilde{\Theta}^{(2)}_{(n)}(\delta(\alpha), c_n)$$

(36)

where all of the parameters $c_n \in (-1, 1)$ are variable.

Proof. In comparison with Theorem 4 the only modification of the proof is that now we ignore the low-lying bound-state $K-$plets as controlled by trivial metric $\Theta^{[K]}_{ND} = I$. Thus, at $K > 0$ the proof remains analogous while paying attention just to the “tilded” Hilbert-space subspaces. □
In the light of the closeness of parallels between the $K = 0$ and $K > 0$ EP-related scenarios we leave the last-step $K > 0$ upgrade of the construction of the admissible classes of observables (24) to interested readers. For compensation let us add here that in general, the variable parameters in metric (36) may be chosen $\alpha$—dependent, $c_n = c_n(\alpha)$. Fortunately, in the light of an appropriate upgrade of Lemma 5 it is clear that this would only imply an inessential modification of the physics described by the model.

The only important exception occurs at the $\delta = 0$ EP interface admitting a discontinuous jump in $c_n = c_n(\alpha)$ at $\alpha = K$. From the point of view of quantum physics such a jump (reflecting the “punched”, two-sided nature of the theoretically admissible diagonalizable-Hamiltonian vicinity of all of the EPs at $K > 0$) would have to be interpreted, in the terminology of Ref. [25], as a quantum phase transition of the second kind.

5 Conclusions

In the Carroll’s book about Alice’s adventures [26] one reads that the “Cheshire Cat appears in a tree”, and then he “disappears but his grin remains behind to float on its own in the air” [27]. In a fairly close parallel to the story (and to its time-to-time use in physics [28]) the appearance of the imaginary cubic potential $V(x) = ix^3$ [2, 29] (the Cheshire Cat of quantum theory?) directed the community to the wide acceptance of the parity-times-time-reversal symmetry ($\mathcal{PT}$—symmetry) in quantum mechanics [4, 12] but, while “the grin” (i.e., the inspiring concept of $\mathcal{PT}$—symmetry) is still “floating in the air” [8, 9], the imaginary cubic potential itself has repeatedly been shown to disappear from the scene of physics because “there is no quantum-mechanical Hamiltonian associated with it” [30]. An evidence is now available that many non-Hermitian, Cheshire-Cat—resembling models of dynamics exhibit certain “unexpected wild properties” [31] and that they “are not equivalent to Hermitian models, but that they rather form a separate model class with purely real spectra” [32].

In our present letter we weakened the present-day wave of scepticism by showing that there also exist non-Hermitian models in which, after an appropriate formulation of the theory, even a very wild, quantum-phase-transition-opening behavior can be given an entirely conventional, unitary-evolution interpretation compatible with the dictum of standard textbooks.

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**Author Contribution statement**

The author is the single author of the paper.