EXTENSIONS OF THE REDUCED GROUP $C^*$-ALGEBRA OF A FREE PRODUCT OF AMENABLE GROUPS

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Abstract. We prove that the unitary equivalence classes of extensions of $C^*_r(G)$ by any $\sigma$-unital stable $C^*$-algebra, taken modulo extensions which split via an asymptotic homomorphism, form a group which can be calculated from the universal coefficient theorem of $KK$-theory when $G$ is a free product of a countable collection of countable amenable groups.

1. Introduction

The stock of examples of $C^*$-algebras for which the semi-group of extensions by the compact operators is not a group is still growing. The latest newcomers consist of a series of reduced free products of nuclear $C^*$-algebras, cf. [HLSW]. This stresses the necessity of finding a way to handle the many extensions without inverses. In joint work with Vladimir Manuilov the second-named author has proposed a way to amend the definition of the semi-group of extensions of a $C^*$-algebra by a stable $C^*$-algebra in such a way that nothing is changed in the case of nuclear algebras where the usual theory already works perfectly, and such that at least some of the extensions which fail to have inverses in the usual sense become invertible in the new, slightly weaker sense. This new semi-group grew out of investigations of the relation between the $E$-theory of Connes and Higson and the theory of $C^*$-extensions, [MT1]. The change consists merely in trivializing not only the split extensions, but also the asymptotically split extensions; those for which there is an asymptotic homomorphism consisting of right inverses for the quotient map, cf. [MT1]. When an extension can be made asymptotically split by addition of another extension we say that the extension is semi-invertible, and the resulting group of semi-invertible extensions, taken modulo asymptotically split extensions, is an abelian group with a close connection to the $E$-theory of Connes and Higson, [CH]. In some, but not all cases where the usual semi-group of extensions is not a group the alternative definition does give a group; i.e. all extensions are semi-invertible, cf. [MT1], [Th1], [MT3]. Specifically, in [MT1] this was shown to be the case when the quotient is a suspended $C^*$-algebra and in [Th1] when the quotient is the reduced group $C^*$-algebra $C^*_r(F_n)$ of a free group with finitely many generators, and the ideal is the $C^*$-algebra $K$ of compact operators. This gave the first example of a unital $C^*$-algebra for which all extensions by the compact operators are semi-invertible, but not all invertible; by the result of Haagerup and Thorbjørnsen, [HT], there are non-invertible extensions of $C^*_r(F_n)$ by $K$ when $n \geq 2$. The purpose of the present note is to show that the situation in [Th1] is not exceptional at all. This is done by showing that all extensions of a reduced group $C^*$-algebra $C^*_r(G)$ by any stable $\sigma$-unital $C^*$-algebra is semi-invertible when $G$ is the free product of a countable collection of countable amenable groups.

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product of a countable collection of discrete countable and amenable groups. The basic idea of the proof is identical to that employed in [Th1]. The crucial improvement over the argument from [Th1] is that the explicitly given homotopy of representations of \([C]\) is replaced by results of Dadarlat and Eilers from [DE]. The pairing in the first variable of the usual extension group \(\text{Ext}^{-1}\) with \(KK\)-theory and Cuntz’ results on \(K\)-amenability from [C] remain key ingredients.

In [Th1] the inverse of an extension, modulo asymptotically split extensions, could be taken to be invertible in the usual sense, i.e. to admit a completely positive contractive splitting. This turns out to be possible also in the more general situation considered here, and as a consequence it follows that the obvious map from the usual \(KK\)-theory to the group of all extensions, taken modulo asymptotically split extensions, is surjective. By combining results of Cuntz, Tu and Thomsen it follows that \(C^*_r(G)\) satisfies the universal coefficient theorem of Rosenberg and Schochet, and from this it follows easily that the map is also injective. Hence the group of extensions of \(C^*_r(G)\) by \(B\), taken modulo the asymptotically split extensions, can be calculated from \(K\)-theory by use of the UCT.

2. The results

Theorem 2.1. Let \((G_i)_{i \in \mathbb{N}}\) be a countable collection of discrete countable amenable groups and let \(G = \ast_i G_i\) be their free product. Let \(B\) be a stable \(\sigma\)-unital \(C^*\)-algebra. For every extension \(\varphi : C^*_r(G) \to Q(B)\) there is an invertible extension \(\varphi' : C^*_r(G) \to Q(B)\) such that \(\varphi \oplus \varphi'\) is asymptotically split.

More explicitly the conclusion is that there is an extension \(\varphi'\), a completely positive contraction \(\psi : C^*_r(G) \to M(B)\) and an asymptotic *-homomorphism \(\pi = (\pi_t)_{t \in [1,\infty)} : C^*_r(G) \to M(B)\), in the sense of Connes and Higson, cf. [CH], such that \(\varphi' = q_B \circ \psi\) and \(\varphi \oplus \varphi' = q_B \circ \pi_t\) for all \(t \in [1,\infty)\), where \(q_B : M(B) \to Q(B)\) is the quotient map.

If only one of the \(G_i\)'s in Theorem 2.1 is non-trivial or if \(G = \mathbb{Z}_2 \ast \mathbb{Z}_2\), the conclusion of the theorem is trivial and can be improved because \(G\) is then amenable. It seems very plausible that such cases are exceptional; indeed it follows from [HLSW] that there is a non-invertible extension of \(C^*_r(G)\) by the compact operators whenever \(G\) is the free product of finitely many non-trivial groups each of which is either abelian or finite and \(G \neq \mathbb{Z}_2 \ast \mathbb{Z}_2\).

As in [Th1] we will prove Theorem 2.1 by use of results from [MT2]. Recall that two extensions \(\varphi, \varphi' : A \to Q(B)\) are strongly homotopic when there is a *-homomorphism \(A \to C[0,1] \otimes Q(B)\) giving us \(\varphi\) when we evaluate at 0 and \(\varphi'\) when we evaluate at 1. By Lemma 4.3 of [MT2] it suffices then to establish the following

Theorem 2.2. Let \((G_i)_{i \in \mathbb{N}}\) be a countable collection of discrete countable amenable groups and let \(G = \ast_i G_i\) be their free product. Let \(B\) be a stable \(\sigma\)-unital \(C^*\)-algebra. For every extension \(\varphi : C^*_r(G) \to Q(B)\) there is an invertible extension \(\varphi' : C^*_r(G) \to Q(B)\) such that \(\varphi \oplus \varphi'\) is strongly homotopic to a split extension.

For the proof of Theorem 2.2 we will need the following notion.
Definition 2.3. Let $A$ be a $C^*$-algebra and $\varphi, \psi$ be $*$-representations of $A$ on some Hilbert spaces. Then $\varphi$ is weakly contained in $\psi$ if $\ker \psi \subseteq \ker \varphi$.

If $\sigma, \pi$ are unitary representations of a locally compact group then $\sigma$ is weakly contained in $\pi$ if and only if the representation of the full group $C^*$-algebra corresponding to $\sigma$ is weakly contained in the representation corresponding to $\pi$. An equivalent definition of weak containment in this case, is that every positive definite function associated to $\sigma$ can be approximated uniformly on compact subsets by finite sums of positive definite functions associated to $\pi$. See sections 3.4 and 18.1 of [Th1] for details.

A proof of the following lemma can be found in [BHV].

Lemma 2.4. Let $\sigma, \pi$ be unitary representations of a locally compact group. Assume that $\sigma$ is weakly contained in the left regular representation $\lambda$. It follows that $\sigma \otimes \pi$ is weakly contained in $\lambda$.

For any discrete group $G$ we denote in the following the canonical surjective $*$-homomorphism $C^*(G) \to C_r^*(G)$ from the full to the reduced group $C^*$-algebra by $\mu$.

Besides the main results of [C] we shall also need the following technical lemma concerning Cuntz’ K-amenability. See [C] for the proof.

Lemma 2.5. Let $H$ be an infinite-dimensional separable Hilbert space and let $G$ be a countable discrete K-amenable group. Then there exist $*$-homomorphisms $\sigma, \sigma_0 : C_r^*(G) \to B(H)$ such that $\sigma \mu, h_t \oplus \sigma_0 \circ \mu : C^*(G) \to B(H)$ are unital, $\sigma \mu(a)-(h_t \oplus \sigma_0 \circ \mu)(a) \in \mathbb{K}$ for all $a \in C^*(G)$, and $[\sigma \circ \mu, h_t \oplus \sigma_0 \circ \mu] = 0$ in $KK(C^*(G), \mathbb{C})$, where $h_t : C^*(G) \to \mathbb{C} \subseteq B(H)$ is the $*$-homomorphism going with the trivial one-dimensional representation $t$ of $G$.

Our proof of Theorem 2.2 uses the notion of absorbing and unitally absorbing $*$-homomorphisms. We refer to [Th2] for the definition and the proof that they exist in the generality required in the argument. Furthermore, we shall need the following lemma which is a unital version of Lemma 2.2 in [Th3]. The proof is the same.

Lemma 2.6. Let $A$ be a separable unital $C^*$-algebra, $D \subseteq A$ a unital nuclear $C^*$-subalgebra and $B$ a stable $\sigma$-unital $C^*$-algebra. Let $\pi : A \to M(B)$ be a unitaly absorbing $*$-homomorphism. It follows that $\pi|_D : D \to M(B)$ is unitally absorbing.

The next lemma gives us the appropriate substitute for the homotopy of representations of $\mathbb{F}_n$ which was a crucial tool in [Th1].

Lemma 2.7. Let $(G_i)_{i \in \mathbb{N}}$ be a countable collection of discrete countable amenable groups and let $G = \ast_i G_i$ be their free product. Let $\mu : C^*(G) \to C_r^*(G)$ be the canonical surjection and let $h_t : C^*(G) \to \mathbb{C}$ be the character corresponding to the trivial one-dimensional representation of $G$. There is then a separable infinite-dimensional Hilbert space $H$, unital $*$-homomorphisms $\sigma, \sigma_0 : C_r^*(G) \to B(H)$ and a path $\nu_s : C^*(G) \to B(H)$, $s \in [0, 1]$, of unital $*$-homomorphism such that

a) $\nu_0 = \sigma \circ \mu$;
b) $\nu_1 = h_t \oplus \sigma_0 \circ \mu$;
c) $\nu_s(a) - \nu_0(a) \in \mathbb{K}$, $a \in C^*(G)$, $s \in [0, 1]$;
d) $s \mapsto \nu_s(a)$ is continuous for all $a \in C^*(G)$.
Proof. Being amenable $G_i$ has the Haagerup Property. See the discussion in 1.2.6 of [CCJJV]. It follows then from Propositions 6.1.1 and 6.2.3 of [CCJJV] that $G$ has the Haagerup Property. Since the Haagerup Property implies K-amenability by [Dp] we conclude that $G$ is K-amenable. We can therefore pick $*$-homomorphisms $\sigma, \sigma_0 : C^*_r(G) \to B(H)$ as in Lemma 2.7. By adding the same unital and injective $*$-homomorphism to $\sigma$ and $\sigma_0$ we can arrange that both $\sigma$ and $\sigma_0$ are injective and have no non-zero compact operator in their range. Since $\mu|_{C^*_r(G_i)} : C^*(G_i) \to C^*_r(G_i)$ is injective it follows then that $\sigma \circ \mu|_{C^*_r(G_i)}$ and $(h_t \circ \sigma_0 \circ \mu)|_{C^*_r(G_i)}$ are admissible in the sense of Section 3 of [DE] for each $i$. Thus Theorem 3.12 of [DE] applies to show that there is a norm-continuous path $u^i_s, s \in [1, \infty)$, of unitaries in $1 + \mathbb{K}$ such that
\[
\lim_{s \to \infty} \| \sigma \circ \mu|_{C^*_r(G_i)}(a) - u^i_s(h_t \circ \sigma_0 \circ \mu)|_{C^*_r(G_i)}(a)u^*_s \| = 0
\]
for all $a \in C^*(G_i)$ and
\[
\sigma \circ \mu|_{C^*_r(G_i)}(a) - u^i_s(h_t \circ \sigma_0 \circ \mu)|_{C^*_r(G_i)}(a)u^*_s \in \mathbb{K}
\]
for all $a \in C^*(G_i)$ and all $s \in [1, \infty)$. Since the unitary group of $1 + \mathbb{K}$ is connected in norm there are therefore norm-continuous paths of unital $*$-homomorphisms $\nu^j_s : C^*(G_j) \to B(H)$, $s \in [0, 1]$, $j \in \mathbb{N}$, such that
\begin{enumerate}[a)]
\item $\nu^j_0 = \sigma \circ \mu|_{C^*_r(G_j)}$;
\item $\nu^j_1 = \sigma_0 \circ \mu|_{C^*_r(G_j)} \oplus h_t|_{C^*(G_j)}$;
\item $\nu^j_s(a) = \nu^j_0(a) \in \mathbb{K}, a \in C^*(G_j), s \in [0, 1],$
\end{enumerate}
for each $j$. The universal property of the free product construction gives us then a path of unital $*$-homomorphisms $\nu^i_s : C^*(G) \to B(H), s \in [0, 1]$, with the stated properties, a)-d).

\[
\Box
\]

Lemma 2.8. In the setting of Theorem 2.1 it holds that every extension $\varphi : C^*(G) \to Q(B)$ of $C^*(G)$ by $B$ is invertible. If $\varphi$ is unital, it is invertible in the semi-group of unitary equivalence classes of unitary extensions, modulo the unital split extensions.

Proof. Assume first that $\varphi$ is unital. For each $i \in \mathbb{N}$ the $C^*$-algebra $C^*_r(G_i) = C^*(G_i)$ is nuclear and hence the unital extensions $\varphi_i = \varphi|_{C^*_r(G_i)} : C^*(G_i) \to Q(B)$ are all invertible. There are therefore unitary extensions $\psi_i : C^*(G_i) \to Q(B)$ and $*$-homomorphisms $\pi_i : C^*(G_i) \to M(B)$ such that $\varphi_i \oplus \psi_i = q_B \circ \pi_i$, $i \in \mathbb{N}$. Let $\omega_i : C^*(G_i) \to \mathbb{C}$ denote the $*$-homomorphism corresponding to the trivial unitary representation of $G_i$. By replacing $\pi_i$ with $\pi_i + \omega_i\pi_i(1)^{\perp}$ we may assume that $\pi_i$ is unital. The universal property of the free product gives us a unital extension $\psi = \star_i \psi_i : C^*(G) \to Q(B)$ and a unital $*$-homomorphism $\pi = \star_i \pi_i : C^*(G) \to M(B)$. Since $\varphi \oplus \psi = q_B \circ \pi$, this completes the proof of the unital case.

Now let $\varphi$ be a general extension. Again consider $\varphi_i = \varphi|_{C^*_r(G_i)} : C^*(G_i) \to Q(B)$. Then $\varphi_i(1) = \varphi_j(1) = p$ for all $i, j \in \mathbb{N}$, so the extensions $\tilde{\varphi_i} := \varphi_i + \omega_ip^{\perp}$ are all unital. As above we get $\psi : C^*(G) \to Q(B)$ and a unital $*$-homomorphism $\pi : C^*(G) \to M(B)$ such that $(\star_i \tilde{\varphi_i}) \oplus \psi = q_B \circ \pi$. Since $\star_i \tilde{\varphi_i}$ and $\varphi \oplus (\star_i \omega_ip^{\perp})$ are equal in $\text{Ext}(C^*(G), B)$ this completes the proof. \[\Box\]
Proof of Theorem 2.2. In order to control the image of the unit for the extensions we consider, we need a result of Skandalis which we first describe. Note that the unital inclusion \(i : \mathbb{C} \to C^*(G)\) has a left-inverse \(h_t : C^*(G) \to \mathbb{C}\) given by the trivial one-dimensional representation \(t\). Therefore the map

\[
i^* : \text{Ext}^{-1}(C^*(G), SB) \to \text{Ext}^{-1}(\mathbb{C}, SB) = K_0(B)
\]

is surjective. We put this into the six-term exact sequence of Skandalis, 10.11 in [S], whose proof can be found in [MT4]. Using the notation from [MT4] we obtain the following commuting diagram with exact rows:

\[
\begin{array}{cccc}
0 & \text{Ext}^{-1}_{\text{unital}}(C^*(G), B) & \text{Ext}^{-1}(C^*(G), B) & K_0(Q(B)) \\
& \mu^* & \mu^* & \\
\text{Ext}^{-1}_{\text{unital}}(C^*_i(G), B) & \text{Ext}^{-1}(C^*_i(G), B) & K_0(Q(B)) & \\
\end{array}
\]

(2.1)

\(G\) is \(K\)-amenable as observed in the proof of Lemma 2.7. By [C] this implies that \(\mu^* : \text{Ext}^{-1}(C^*_i(G), B) \to \text{Ext}^{-1}(C^*(G), B)\) is an isomorphism.

Let \(\varphi : C^*_i(G) \to Q(B)\) be a unital extension. Let \(\pi_1 : C^*_i(G) \to Q(B)\) be a unitaly absorbing split extension (whose existence is guaranteed by [Th2]) and set \(\varphi' = \varphi \oplus \pi_1\). It follows from Lemma 2.8 and a diagram chase in (2.1) that there is an invertible unital extension \(\varphi'' : C^*_i(G) \to Q(B)\) such that

\[
[\varphi' \circ \mu \oplus \varphi'' \circ \mu] = 0
\]

(2.2)

in \(\text{Ext}^{-1}_{\text{unital}}(C^*(G), B)\). Since \(C^*(G_i)\) is nuclear \(\mu|_{C^*_i(G_i)} : C^*(G_i) \to C^*_i(G_i), i \in \mathbb{N}\), is an \(*\)-isomorphism and it follows from Lemma 2.6 that \(\pi_1|_{C^*_i(G_i)} : C^*_i(G_i) \to Q(B)\) is unitally absorbing for each \(i \in \mathbb{N}\). Hence \(\pi_1 \circ \mu|_{C^*_i(G_i)} : C^*(G_i) \to Q(B)\) is a unitaly absorbing split extension. It follows therefore from (2.2) that \((\varphi' \circ \mu \oplus \varphi'' \circ \mu)|_{C^*_i(G_i)}\) is a unitaly split extension for each \(i\). As in the proof of Lemma 2.8 this implies that \(\varphi' \circ \mu \oplus \varphi'' \circ \mu\) is unitally split. There is therefore a unitary representation \(R : G \to M(B)\) such that

\[
q_B \circ h_R = \varphi' \circ \mu \oplus \varphi'' \circ \mu,
\]

(2.3)

where \(h_R : C^*(G) \to M(B)\) is the \(*\)-homomorphism defined by \(R\).

Consider the homotopy \(\nu_s\) from Lemma 2.7. Let \(\nu'_s : G \to B(H)\) be the unitary representation defined by \(\nu_s\) so that \(\nu_s = h_{\nu'_s}\). It follows from the property a) of Lemma 2.7 that \(\nu'_0\) is weakly contained in the left-regular representation of \(G\) and from b) that \(\nu'_1\) is a direct sum \(t \oplus \lambda_0\) where \(\lambda_0\) is a representation of \(G\) which is weakly contained in the left-regular representation of \(G\). Consider the unitary representations

\[
R \otimes \nu'_s : G \to M(B \otimes B(H)) \subseteq M(B \otimes K), \quad s \in [0, 1].
\]

Then \(q_{B \otimes K} \circ h_{R \otimes \nu'_s} : C^*(G) \to Q(B \otimes K), \quad s \in [0, 1]\), is a norm-continuous path of extensions. Note that

\[
q_{B \otimes K} \circ h_{R \otimes \nu'_0} = q_{B \otimes K} \circ h_{R \otimes t} \oplus q_{B \otimes K} \circ h_{R \otimes \lambda_0} = (\varphi' \oplus \varphi'') \circ \mu \oplus \mu \circ q_{B \otimes K} \circ h_{R \otimes \lambda_0}.
\]

Since \(R \otimes \nu'_0\) and \(R \otimes \lambda_0\) are weakly contained in the left-regular representation of \(G\) by Lemma 2.4 it follows from an argument almost identical with one used in [Th1] that each \(q_{B \otimes K} \circ h_{R \otimes \nu'_s}\) factors through \(C^*_i(G)\) and gives us a strong homotopy connecting the split extension \(q_{B \otimes K} \circ h_{R \otimes \nu'_0} : C^*_i(G) \to Q(B \otimes K)\) to the direct sum \(\varphi' \oplus \varphi'' \oplus q_{B \otimes K} \circ h_{R \otimes \lambda_0}\).
For completeness we include the argument: Let $s \in [0,1]$ and $x = \sum_j c_j g_j \in \mathbb{C}G$, where $c_j \in \mathbb{C}$ and $g_j \in G$. Then

$$h_{R \otimes \nu'}(x) = \sum_j c_j R(g_j) \otimes \nu_0(g_j) + \sum_j c_j R(g_j) \otimes \Delta(g_j), \quad (2.4)$$

where $\Delta(g_j) = \nu'_s(g_j) - \nu_0(g_j)$. Note that $\Delta(g_j) \in \mathbb{K}$ by $c)$. Since $\nu'_0$ is weakly contained in the left regular representation we can use Lemma 2.4 to conclude that $\left\| \sum_j c_j R(g_j) \otimes \nu'_0(g_j) \right\| \leq \| x \|_{C^*_r(G)}$ and hence

$$\left\| q_{B \otimes \mathbb{K}} \left( \sum_j c_j R(g_j) \otimes \nu'_0(g_j) \right) \right\| \leq \| x \|_{C^*_r(G)}.$$

To handle the second term in (2.4) note that $M(B) \otimes \mathbb{K} / B \otimes \mathbb{K} \simeq Q(B) \otimes \mathbb{K}$ so

$$\left\| q_{B \otimes \mathbb{K}} \left( \sum_j c_j R(g_j) \otimes \Delta(g_j) \right) \right\| = \left\| \sum_j c_j (\varphi' \oplus \varphi'') (g_j) \otimes \Delta(g_j) \right\|_{Q(B) \otimes \mathbb{K}}.$$

Since $\varphi' \oplus \varphi'': C^*_r(G) \to Q(B)$ is injective (because $\varphi'$ contains the unitarily absorbing split extension $\pi_1$) and $(\varphi' \oplus \varphi'') \otimes \text{id}_\mathbb{K}$ isometric,

$$\left\| \sum_j c_j (\varphi' \oplus \varphi'') (g_j) \otimes \Delta(g_j) \right\|_{Q(B) \otimes \mathbb{K}} = \left\| \sum_j c_j \lambda(g_j) \otimes \Delta(g_j) \right\|_{C^*_r(G) \otimes \mathbb{K}}.$$

And

$$\left\| \sum_j c_j \lambda(g_j) \otimes \Delta(g_j) \right\|_{C^*_r(G) \otimes \mathbb{K}} =$$

$$\left\| \sum_j c_j \lambda(g_j) \otimes \nu'_s(g_j) - \sum_j c_j \lambda(g_j) \otimes \nu'_0(g_j) \right\|_{C^*_r(G) \otimes \mathbb{K}} \leq 2 \| x \|_{C^*_r(G)},$$

by Fell’s absorption principle or Lemma 2.4. Inserting these estimates into (2.4) yields the conclusion that

$$\left\| q_{B \otimes \mathbb{K}} \circ h_{R \otimes \nu'}(x) \right\| \leq 3 \| x \|_{C^*_r(G)},$$

proving that $q_{B \otimes \mathbb{K}} \circ h_{R \otimes \nu'}$ factors through $C^*_r(G)$ as claimed.

It remains to reduce the general case of a possibly non-unital extension to the case of a unital extension. Let $\varphi: C^*_r(G) \to Q(B)$ be an arbitrary extension. From Lemma 2.4 and K-amenability we get an invertible extension $\varphi': C^*_r(G) \to Q(B)$ such that $[\varphi \circ \mu \oplus \varphi' \oplus \mu] = 0$ in $\text{Ext}^{-1}(C^*_r(G), B)$. In particular,

$$p = (\varphi \circ \mu \oplus \varphi' \circ \mu)(1)$$

is a projection which represents 0 in $K_0(Q(B))$. Since $K_0(M(B)) = 0$ we see $[1 - p] + [p] = [1] = 0$ in $K_0(Q(B))$ so we find that also $p^\perp = 1 - p$ represents 0 in $K_0(Q(B))$. Since $M_k(Q(B)) \simeq Q(B)$ for all $k$ this implies that

$$p \perp 1 \sim 0 \perp 1 \text{ and } p^\perp \perp 1 \sim 0 \perp 1.$$

in $M_2(Q(B))$, where $\sim$ is Murray-von Neumann equivalence. It follows that
\[ p \oplus 1 \oplus 0 \sim 1 \oplus 0 \oplus 0 \text{ and } p^\perp \oplus 0 \oplus 1 \sim 0 \oplus 1 \oplus 1 \]
in $M_3(Q(B))$. So there is a unitary $w \in M_4(Q(B))$ contained in the connected component of the unit in the unitary group of $M_4(Q(B))$ such that
\[ w(p \oplus 1 \oplus 0 \oplus 0)w^* = 1 \oplus 0 \oplus 0 \oplus 0. \]

Let $\chi : C^*_r(G) \to Q(B)$ be a unital split extension. It follows that
\[ w((\varphi \circ \mu \oplus \varphi' \circ \mu) \oplus \chi \circ \mu \oplus 0 \oplus 0)w^* = \psi_0 \oplus 0 \oplus 0 \oplus 0 \quad (2.5) \]
for some unital extension $\psi_0 : C^*(G) \to Q(B)$. It follows from (2.5) that $\psi_0$ factors through $C^*_r(G)$, i.e. there is a unital extension $\psi : C^*_r(G) \to Q(B)$ such that $\psi_0 = \psi \circ \mu$. Via an isomorphism $M_4(Q(B)) \simeq M_2(Q(B))$ which leaves the upper lefthand corner unchanged, we see that there is an invertible extension $\varphi'' : C^*_r(G) \to Q(B)$ and a unitary $u$ in the connected component of 1 such that
\[ \text{Ad } u \circ (\varphi \oplus \varphi'') = \psi \oplus 0 \]
as $\ast$-homomorphisms $C^*_r(G) \to Q(B)$. Since $\psi$ is unital the first part of the proof gives us an invertible (unital) extension $\psi' : C^*_u(G) \to Q(B)$ such that $\psi \oplus \psi'$ is strongly homotopic to a split extension. Since
\[ \varphi \oplus \varphi'' \oplus \psi' = \text{Ad}(u^* \oplus 1) \circ (\psi \oplus 0 \oplus \psi'), \]
we conclude that $\varphi \oplus \varphi'' \oplus \psi'$ is strongly homotopic to a split extension. Note that $\varphi'' \oplus \psi'$ is invertible. $\square$

Let $A$ be a separable $C^*$-algebra and $B$ a stable $\sigma$-unital $C^*$-algebra. Following [MT1] we let $\text{Ext}^{-1/2}(A, B)$ denote the group of unitary equivalence classes of semi-invertible extensions of $A$ by $B$. There is then an obvious map
\[ \text{Ext}^{-1}(A, B) \to \text{Ext}^{-1/2}(A, B) \]
which in [Th1] was shown to be an isomorphism when $B = \mathbb{K}$ and $A = C^*_r(\mathbb{F}_n)$. We can now extend this conclusion as follows.

**Theorem 2.9.** Let $(G_i)_{i \in \mathbb{N}}$ be a countable collection of discrete countable amenable groups and let $G = *_{i \in \mathbb{N}} G_i$ be their free product. Let $B$ be a stable $\sigma$-unital $C^*$-algebra. It follows that $C^*_r(G)$ satisfies the UCT and that the natural map $\text{Ext}^{-1}(C^*_r(G), B) \to \text{Ext}^{-1/2}(C^*_r(G), B)$ is an isomorphism.

**Proof.** It follows from Theorem 2.1 that the map $\text{Ext}^{-1}(C^*_r(G), B) \to \text{Ext}^{-1/2}(C^*_r(G), B)$ is surjective. To conclude that the map is also injective note that the six-term exact sequence of $K$-theory arising from an asymptotically split extension has trivial boundary maps and the resulting group extensions are split. Hence the injectivity of the map we consider will follow if we can show that $C^*_r(G)$ satisfies the UCT. Since $G$ is $K$-amenable $C^*_r(G)$ is $KK$-equivalent to $C^*(G)$, cf. [C], so we may as well show that $C^*(G)$ satisfies the UCT. We do this in the following three steps: Since the class of $C^*$-algebras which satisfies the UCT is closed under countable inductive limits we need only show that $C^*(\ast_{i \leq n} G_i)$ satisfies the UCT. Next observe that it follows from [Th3] that an amalgamated free product $A \ast_C B$ of unital separable $C^*$-algebras $A$ and $B$ is $KK$-equivalent to the mapping
cone of the inclusion $C \subseteq A \oplus B$. Thus $A \star_C B$ will satisfy the UCT when $A$ and $B$ do. Since

$$C^* \left( \bigast_{i \leq n} G_i \right) \simeq C^* (G_1) \star_C C^* (G_2) \star_C \cdots \star_C C^* (G_n)$$

we can apply this observation $n - 1$ times to conclude that $C^* \left( \bigast_{i \leq n} G_i \right)$ satisfies the UCT if each $C^* (G_i)$ does. And this follows from [Tu] because $G_i$ is amenable.

□

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