An anisotropic standing wave braneworld and associated Sturm–Liouville problem

Merab Gogberashvili 1,2, Alfredo Herrera-Aguilar 3,4 and Dagoberto Malagón-Morejón 3

1 Andronikashvili Institute of Physics, 6 Tamarashvili St., Tbilisi 0177, Georgia
2 Javakhishvili State University, 3 Chavchavadze Ave., Tbilisi 0128, Georgia
3 Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, CP 58040, Morelia, Michoacán, Mexico
4 Centro de Estudios en Física y Matemáticas Básicas y Aplicadas, Universidad Autónoma de Chiapas, Calle 4a Oriente Norte 1428, Tuxtla Gutiérrez, Chiapas, Mexico

E-mail: gogber@gmail.com, alfredo.herrera.aguilar@gmail.com and malagon@ifm.umich.mx

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Abstract
We present a consistent derivation of the recently proposed 5D anisotropic standing wave braneworld generated by gravity coupled to a phantom-like scalar field. We explicitly solve the corresponding junction conditions, a fact that enables us to give a physical interpretation to the anisotropic energy–momentum tensor components of the brane. So matter on the brane represents an oscillating fluid which emits anisotropic waves into the bulk. We also analyze the Sturm–Liouville problem associated with the correct localization condition of the transverse to the brane metric and scalar fields. It is shown that this condition restricts the physically meaningful space of solutions for the localization of the fluctuations of the model.

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1. Introduction
Since the early proposals of braneworld models involving large extra dimensions and 4D delta-function sources with both positive and negative tensions [1, 2], there has been a lot of activity in this area with the aim of solving several open questions in modern physics (see [3, 4] for reviews). Most of these models were realized as time-independent field configurations. However, since then, mostly within cosmological approaches, several braneworlds have appeared that assumed time-dependent metrics and fields in an attempt to address several open problems in astrophysics and cosmology [5–12]. In general, these braneworld models have a series of remarkable features: they develop a novel geometrical mechanism of dimensional reduction based on a curved extra dimension, provide a realization of the AdS/CFT correspondence to lowest order, take into account the self-gravity on the brane through its tension, can trap various matter fields (except for gauge bosons) inside the brane,
recast the hierarchy problem into a higher dimensional viewpoint and also lead to cosmological backgrounds with consistent dynamics that recover general relativity results under suitable restrictions on their parameters. All these facts provide a complex but interesting interplay between gravity, particle physics and geometry [4].

However, there are still several open relevant questions which deserve attention: construction of the simplest realistic solution for an astrophysical black hole on the brane and study of its physical properties like staticity and Hawking radiation, development of realistic approximation schemes and numerical codes to study cosmological perturbations on all scales, computation of the CMB anisotropies and large-scale structure to confront their predictions with high-precision observations [4].

One of the main drawbacks of the delta-function braneworld models from the gravitational viewpoint is related to their singularities at the positions of the branes, a fact that gets worse in models with more than six dimensions due to the stronger self-gravity of the brane. In an attempt to heal this problem, several authors smooth out the brane tensions [13] or introduced single [14] or several scalars [15], and phantom [16] or tachyon [17] fields as sources (for a recent review, see [18]), and even considered modified gravity braneworlds [19]. However, scalar field thick brane configurations with 4D Poincaré symmetry also develop naked singularities at the boundaries of the bulk manifold if a mass gap is required to be present in the graviton spectrum of Kaluza–Klein fluctuations (see [14, 20], for instance). An alternative model without such a drawback is provided by a recently proposed de Sitter braneworld model purely generated by 4D and 5D cosmological constants [21].

In this sense, it is important to consider new generalizations that attempt to make more realistic the original braneworld models, or explore other aspects of higher dimensional gravity which are not probed or approached by these simple models.

In this paper, we shall consider the braneworld generated by 5D anisotropic standing gravitational waves coupled to a phantom-like scalar field in the bulk as was recently proposed in [22]. It turns out that standing wave configurations can provide a natural alternative mechanism for localizing 4D gravity as well as for trapping matter fields (for scalar and fermion fields, see [23] and [24], respectively), including gauge bosons [25], which usually are not localized on thin braneworlds. A peculiarity of this anisotropic braneworld scenario is that it possesses non-stationary metric coefficients in the 4D part of the line element. We further impose $Z_2$-symmetry along the extra dimension, a fact that gives rise to the need of introducing also an anisotropic brane energy–momentum tensor for the self-consistency of the model. When the corresponding junction conditions are solved, this fact allows us to give a physical interpretation to the components of the anisotropic energy–momentum tensor of the brane. Then the localization of transverse scalar and metric fields is treated through the associated Sturm–Liouville method that restricts the physically meaningful parameter space of the solution. Some final remarks are given at the end of the paper.

2. The model

In this section, we briefly recall the 5D standing wave braneworld model proposed in [22], which is generated by gravity coupled to a non-self-interacting scalar phantom-like field [16, 18] which depends on time and propagates in the bulk and is given by the action

$$S_b = \int d^5 x \sqrt{g} \left[ \frac{1}{16\pi G_5} (R - 2\Lambda_5) + \frac{1}{2} (\nabla \phi)^2 \right],$$

(1)
where $G_5$ and $\Lambda_5$ are the 5D Newton and cosmological constants, respectively. It is worth noting that in order to avoid the well-known problems of stability which occur with ghost fields, the phantom-like scalar field does not couple to ordinary matter in the model.

The Einstein equations for action (1) read

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_5 T_{\mu\nu} - \Lambda_5 g_{\mu\nu}, $$

where Greek indices run from 0 to 5, labeling the 5D spacetime, and $T_{\mu\nu}$ is the energy–momentum tensor for the scalar field,

$$ T_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi. $$

(2)

Following [22], we use the anisotropic metric ansatz

$$ ds^2 = e^{2A(r)} (dt^2 - e^{-2u(t,r)} dx^2 - e^{-2u(t,r)} dy^2 - e^{-2u(t,r)} dz^2) - dr^2, $$

where Latin indices stand for the 4D spacetime coordinates ($x^0 = t, x^i$ with $i = 1, 2, 3$ are the spatial coordinates $x, y$ and $z$, respectively) and the extra dimension is denoted as $x^5 = r$. This metric generalizes straightforwardly the thin brane metric ansatz [2], where $A(r) \sim |r|$, which is recovered in the limit when $u(t,r)$ vanishes. The warp factor here $A(r)$ is an arbitrary function of the extra coordinate $r$ and, in principle, may model a thick brane configuration in the spirit of [14].

This braneworld constitutes a generalization of the thin brane model with the peculiarity that now the brane possesses anisotropic oscillations on it, which send a wave into the bulk (as in [26]), i.e. the brane is warped along the spatial coordinates $x, y$ and $z$ through the factor $e^{-2u(t,r)}$, depending on the time $t$ and the extra coordinate $r$. Several anisotropic braneworld models have previously been considered in the literature [6–11] when addressing relevant cosmological issues such as anisotropy dissipation during inflation [8], braneworld isotropization with the aid of magnetic fields [10] and localization of test particles [11]. Moreover, as a general feature it has been established that anisotropic metrics on the brane prevent the bulk from being static [9, 10].

The phantom-like scalar field $\phi(t, r)$ obeys the Klein–Gordon equation on the background spacetime given by (4),

$$ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = \Box \phi = e^{-2A(r)} \ddot{\phi} - \dot{\phi}^2 - 4A' \phi' = 0, $$

where the overdots mean the time derivatives and the primes stand for the derivatives with respect to the extra coordinate $r$.

We further write the Einstein equations in the form

$$ R_{\mu\nu} = -\partial_\mu \sigma \partial_\nu \sigma + \frac{1}{2} g_{\mu\nu} \Lambda_5, $$

where the gravitational constant has been absorbed in the definition of the scalar field:

$$ \sigma = \sqrt{8\pi G_5 \phi}, $$

and we have used the reduced form of the energy–momentum tensor of the phantom-like scalar field,

$$ T_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = -\nabla_\mu \sigma \nabla_\nu \sigma, $$

and $T \equiv g^{\mu\nu} T_{\mu\nu}.$

It turns out that from the 05-component of the Einstein equations (6), it follows that the fields $\sigma$ and $u$ are related by [22]

$$ \sigma(t, r) = \sqrt{\frac{2}{3}} u(t, r). $$

(9)
up to an additive constant that can be absorbed into a redefinition of the 4D spatial coordinates in (4).

On the other hand, by combining the 22- and 33-components of the Einstein equations (6), and comparing the result to the corresponding 55-component, one obtains a quite strong restriction on the function \( A(r) \),

\[
A'' = 0,
\]
whose solution is linear in \( r \) but corresponds to divergent warp factors. Another interesting fact that follows from the same analysis is that this condition also implies a sort of fine tuning, where the 5D cosmological constant is related to \( A(r) \) as follows:

\[
\Lambda_5 = 6A'^2.
\]

It is worth noting that relation (9) remains valid even if we include a self-interaction potential for the scalar field.

In order to obtain an asymptotically vanishing warp factor and to study a more interesting braneworld with the above \( A'' = 0 \) behavior on the bulk, we modify the initial model by imposing \( Z_2 \)-symmetry along the extra dimension and by introducing a thin brane at \( r = 0 \), which, for consistency with (4), must be supported by an anisotropic energy–momentum tensor. This implies that the warp factor of metric (4) adopts the known form, similar to [2]

\[
dx^2 = e^{2\alpha(r)} \left( dr^2 - e^\sigma \, dx^2 - e^\nu \, dy^2 - e^{-2\alpha} \, dz^2 \right) - dr^2,
\]

where \( \alpha \) is an arbitrary constant. The non-zero components of the Ricci tensor for this metric read

\[
\begin{align*}
R_{tt} &= e^{2\alpha(r)} \left[ -\frac{1}{2} e^{-2\alpha(r)} u'^2 + 4\alpha^2 + 2\alpha' \right], \\
R_{xx} &= R_{yy} = e^{2\alpha(r)} \left[ \frac{1}{2} e^{-2\alpha(r)} u'^2 - 4\alpha^2 - 2ae(r)u' - 2\alpha(r) + \frac{1}{2} u'^2 \right], \\
R_{zz} &= e^{2\alpha(r)} \left[ - e^{-2\alpha(r)} u^2 - 4\alpha^2 + 4ae(r)u' - 2\alpha(r) + u'^2 \right], \\
R_{rr} &= -\frac{3}{2} u'^2 - 4\alpha^2 - 8\alpha' \delta(r), \\
R_{\tau \tau} &= -\frac{3}{2} u'^2,
\end{align*}
\]

where \( \epsilon(r) \) is the sign function. The presence of terms proportional to \( \delta(r) \) in the Ricci tensor, together with the anisotropy of the metric, forces us to define an anisotropic energy–momentum tensor on the brane \( \tau^\mu_\nu \) with different stresses along different directions. The simplest way to accomplish this is by proposing

\[
\tau^\mu_\nu = \delta(r) \text{ diag}[\lambda_0, \lambda_1, \lambda_2, \lambda_3, 0], \quad \lambda_1 = \lambda_2.
\]

The above energy–momentum tensor can be interpreted as certain ‘anisotropic effective fluid’ which constitutes a mixture of a vacuum fluid (which is characterized just by a brane tension) and an anisotropic matter fluid (see [4] and references therein for a complete decomposition of the energy–momentum tensor induced on the brane).

The parameter \( \lambda_0 \) is the energy density of the ‘effective fluid’ and \( \lambda_m (m = 1, 2, 3) \) are the stresses along the \( m \) directions (see below). In general, these quantities depend only on time. A similar energy–momentum tensor was implemented in the context of Eötvös branewrldes with variable (time-dependent) tensions in analogy with fluid membranes in [12].

By taking into account the above considerations, the field equations (6) change as follows:

\[
R_{\mu \nu} = -\partial_\mu \sigma \partial_\nu \sigma + \frac{2}{3} g_{\mu \nu} \Lambda_5 + 8\pi G_5 \ddot{\tau}_{\mu \nu},
\]

where the reduced energy–momentum tensor,

\[
\ddot{\tau}_{\mu \nu} = \tau_{\mu \nu} - \frac{1}{3} \delta_{\mu \nu} \tau, \quad \tau = \delta(r) \text{ diag} [\lambda_0, \lambda_1, \lambda_2, \lambda_3, 0].
\]
corresponds to the matter content on the brane and takes the form
\[ \bar{\tau}_{\mu \nu} = \frac{1}{3} \delta(r) \text{diag}(2\lambda_0 - 2\lambda_1 - \lambda_3, (\lambda_0 - \lambda_1 + \lambda_3) e^\omega, (\lambda_0 - \lambda_1 + \lambda_3) e^\omega, (\lambda_0 - 2\lambda_1 - 2\lambda_3) e^{-2a}, (\lambda_0 + 2\lambda_1 + \lambda_3)) \]  
\tag{17}

By making use of (12), (15) and (17), one can show that relation (9) between the scalar field and the metric function is satisfied as well. Relation (11) computed with the aid of metric (12) adopts the following form:
\[ \Lambda_5 = 6a^2. \]  
\tag{18}

Therefore, the system of Einstein and field equations reduces to a single ordinary differential equation,
\[ e^{-2a(r)} \bar{u}'' - u'' - 4ae(r)u' = 0. \]  
\tag{19}

In addition to the latter, the existence of the thin brane in our setup gives rise to a relationship between the jump of the first derivative of the function \( u \) at \( r = 0 \) (denoted by \( [u'] \)), the stresses \( \lambda_m \) and the parameter \( a \). A way to derive these relationships consists in integrating (15) with respect to \( r \) in the range \([ -\epsilon, \epsilon ]\) and then making \( \epsilon \) tend to zero, yielding the following junction conditions for the stresses \( \lambda_m \):
\[ a = \frac{4\pi G_5}{3} (2\lambda_0 - 2\lambda_1 - \lambda_3), \]
\[ [u'] = \frac{16\pi G_5}{3} (-\lambda_0 + \lambda_1 - \lambda_3) - 4a, \]
\[ [u'] = \frac{8\pi G_5}{3} (\lambda_0 + 2\lambda_1 - 2\lambda_3) + 2a, \]
\[ a = -\frac{\pi G_5}{3} (\lambda_0 + 2\lambda_1 + \lambda_3). \]  
\tag{20}

Although the above system seems to be overdetermined since we have four equations for just three variables \( \lambda_0, \lambda_1 \) and \( \lambda_3 \), it is possible to show that only three equations are independent.

We then construct a standing wave solution to (19) by implementing the ansatz
\[ u(t, r) = C \sin(\omega t) f(r), \]  
\tag{21}

where \( C \) and \( \omega \) are some constants. In [22], it was argued that a standing wave configuration in the bulk would in principle provide an alternative mechanism for localizing gravity and binding matter fields (through an anisotropic force which drives the matter fields/particles toward the nodes), including light (see [23–25] for recent results).

The equation for the radial function \( f(r) \) from (21) reads
\[ f'' + 4ae(r)f' + \omega^2 e^{-2a|r|} f = 0. \]  
\tag{22}

The general solution of this equation adopts the following form:
\[ f(r) = e^{-2a|r|} \left[ A J_2 \left( \frac{\omega}{|a|} e^{-|a|r} \right) + B Y_2 \left( \frac{\omega}{|a|} e^{-|a|r} \right) \right], \]  
\tag{23}

where \( A \) and \( B \) are the arbitrary constants and \( J_2 \) and \( Y_2 \) are the second-order Bessel functions of the first and second kind, respectively. It is worth noting that although the \( Y_2 \) Bessel function is singular as \( r \to \infty \) for \( a > 0 \), the whole function (23) vanishes in this limit since its behavior is dominated by the \( e^{-2a|r|} \) factor.

As pointed out in [22], along with solution (23), we impose the ghost-like field to be unobservable on the position of the thin brane, since it oscillates together with the gravitational field (see (9)). A way to accomplish this consists in setting a boundary condition that nullifies
the ghost-like field at the position of the brane \( r = 0 \), a condition that quantizes the oscillation frequency of the standing wave:

\[
\frac{\omega}{|a|} = X_n, 
\]

(24)

where \( X_n \) is the \( n \)th zero of the second-order Bessel function \( J_2 \) or \( Y_2 \) depending on whether one takes \( A = 0 \) or \( B = 0 \) in (23), since the zeros of \( J_2 \) and \( Y_2 \) do not coincide. For an increasing \((a > 0)\)/decreasing \((a < 0)\) warp factor, the ghost-like field \( \sigma(t, r) \) and the metric function \( u(t, r) \) vanish at a finite/infinite number of points \( r_m \) along the extra dimension in the bulk (nodes), forming 4D spacetime ‘islands’ at these nodes, where the matter particles are assumed to be bound.

The induced 4D Einstein spacetime is locally AdS4 (in the vicinity of the positions of the island universes \( r_m \) along the extra dimension) if

\[
4R_{ab} = \alpha(r_m)q_{ab}, \quad a, b = 0, 1, 2, 3, \tag{25}
\]

where \( 4R_{ab} \) and \( q_{ab} \) are the induced 4D Ricci and metric tensors, respectively, and \( \alpha(r_m) > 0 \) is a constant depending on the position of the island \( r_m \).

By computing the 4D induced metric on the brane islands, we can write down the components of the 4D induced Ricci tensor as functions of the induced metric tensor and its first two derivatives, i.e. as functions of \( u(t, r) \) and its first two partial derivatives with respect to time and the extra coordinate. By looking for a proportionality between the induced metric and Ricci tensors on the island universes, i.e. when \( u(t, r_m) = 0 \) (however, the derivatives of \( u \) at \( r_m \) do not vanish), we see that the AdS4 effective character of the 4D island universes can be reached only in two distinct cases:

(i) by restricting the amplitude of the anisotropic oscillations to be very small, more precisely when \( AC \ll 1 \), for the islands which are located at finite \( r_m \);

(ii) asymptotically along the fifth dimension \((r_m \to \infty)\).

Thus, even when at the \( r_m \) points where the island universes are located we have \( u(t, r_m) = 0 \) and metric (12) reduces to the standard 5D thin braneworld metric of [2], whereas setup (1) simplifies to an action describing 5D gravity with a cosmological constant, plus a delta-function brane as in [2], leading to the known Randall–Sundrum braneworld which is an AdS5 slice, in general, the metric in the islands (i.e. for each particular node \( r_m \)) is not locally AdS4, but just in the above referred two cases.

Returning to the problem of junction conditions (20), if we use (21) and (23) to solve them, we obtain

\[
\lambda_0 = -\frac{3a}{4\pi G_5}, \\
\lambda_1 = \lambda_2 = \frac{AC\omega \epsilon(a) \sin(\omega t) - 3a}{16\pi G_5} (J_3 - J_1) \big|_{\omega/|a|} - \frac{3a}{4\pi G_5}, \\
\lambda_3 = \frac{AC\omega \epsilon(a) \sin(\omega t) - 3a}{8\pi G_5} (J_3 - J_1) \big|_{\omega/|a|}, \\
[u'] = \frac{16\pi G_5}{3}(\lambda_1 - \lambda_3) = AC\omega \epsilon(a) \sin(\omega t) (J_3 - J_1) \big|_{\omega/|a|}. \tag{26}
\]

From these relations, it can be seen that \( \lambda_0 = \rho \) is constant, and it can be interpreted as the energy density of the ‘effective fluid’ as mentioned above, while the stresses oscillate in the \( x, y \) and \( z \) directions with the same frequency as the function \( u \) does, as it was expected since it is precisely along these directions that the metric coefficients do oscillate in the bulk. On the other hand, the spatial components of the energy–momentum tensor on the brane
can be consistently interpreted following a systematic covariant analysis developed from the viewpoint of a brane-bound observer [27, 28].

Let us decompose the tension components (26) as follows:

\[
\lambda_i = -p + \pi_i (i = 1, 2, 3),
\]

where the quantity \( p \) represents a constant isotropic pressure and the \( \pi_i \) are the components of the anisotropic stress that oscillate along the spatial direction \( x' \):

\[
p = \frac{3a}{4\pi G_5},
\]

\[
\pi_1 = \pi_2 = \frac{AC_0 \epsilon(a) \sin(\omega t)}{16\pi G_5} (J_3 - J_1) |\omega/|a| |
\]

\[
\pi_3 = -2\pi_1 = -\frac{AC_0 \epsilon(a) \sin(\omega t)}{8\pi G_5} (J_3 - J_1) |\omega/|a| |.
\]

By taking into account that in the solution for the junctions conditions (26) the parameter \( \omega/|a| \) is a zero of \( J_2 \), but not of \( J_1 \) and \( J_3 \), in general the anisotropic stresses are different from zero. Furthermore, the stresses along the directions \( x \) and \( y \) are equal, whereas the stress in the \( z \)-direction \( \pi_3 \) is twice in amplitude and possesses an opposite phase with respect to \( \pi_1 \). This is a direct consequence of the symmetry of the metric under the exchange \( x \leftrightarrow y \), on the one hand, and the differences in amplitude and phase in the argument of the exponential metric coefficients, on the other hand.

Let us compare the isotropic and the anisotropic parts of (14). In order to do that, we shall consider the following physically interesting limits.

(a) The first case is \( \omega/|a| \approx 1 \), and depending on the amplitudes \( A \) and \( C \) in the expressions for the components \( \lambda_i \), the dominant term can be the oscillatory or the constant one (\( 3a/4\pi G_5 \)), or both terms can be of the same order.

(1a) If \( \omega/|a| \approx 1 \) and \( AC \ll 1 \), the constant term dominates and \( \lambda_i \approx \lambda_0 = -3a/4\pi G_5 \). Then the oscillatory terms in \( \lambda_i \) can be neglected compared to the contribution given by the constant terms and the thin brane becomes ‘isotropic’, recovering the standard thin brane case [2], where the state equation is \( p = -\rho \). Thus, the energy–momentum tensor on the brane only has a relevant contribution coming from the tension of the brane.

(2a) If \( \omega/|a| \approx 1 \) and \( AC \gg 1 \), the oscillatory terms are dominant in \( \lambda_i \), and \( \max(\lambda_i) \gg \lambda_0 \), implying that the energy–momentum tensor of the thin brane describes a kind of exotic matter given by the following expression:

\[
\tau^\mu_\nu \approx \delta(r) \text{diag}[\lambda_0, \lambda_1 + \lambda_0, \lambda_2 + \lambda_0, \lambda_3 + \lambda_0, 0], \quad \lambda_1 = \lambda_2,
\]

where

\[
\lambda_1 = \frac{AC_0 \epsilon(a) \sin(\omega t)}{16\pi G_5} (J_3 - J_1) |\omega/|a| |
\]

\[
\lambda_3 = \frac{AC_0 \epsilon(a) \sin(\omega t)}{8\pi G_5} (J_3 - J_1) |\omega/|a| |.
\]

In this case, the matter is highly anisotropic because the amplitude of the stationary wave is large.

(3a) If \( \omega/|a| \approx 1 \) and \( AC \approx 1 \), the oscillatory and constant terms are of the same order and \( \lambda_i \sim \lambda_0 \sim a \). Due to this fact, in this case there is no way to highlight an interesting physical situation, in contrast to cases (1a) and (2a).
The second interesting limit takes place when the stationary wave that propagates in the bulk oscillates in the high-frequency regime \((\omega/|a| \gg 1)\); in this case, we have again three situations.

(1b) If \(\omega/|a| \gg 1\) and \(AC \ll 1\) with \(ACX_n \to C_1\), where \(C_1\) is a finite constant, we have the same result as in case (3a). Thus, the oscillatory and constant terms are of the same order.

(2b) If \(\omega/|a| \gg 1\) and \(AC \approx 1\), we have the same result as in case (2a): the matter fluid is highly anisotropic. But in contrast to (2a), in this case it is because the frequency of the stationary wave is large; therefore, it contributes to the growth of the anisotropic part of the energy–momentum tensor.

(3b) If \(\omega/|a| \gg 1\) and \(AC \gg 1\), we have the same result as in (2a): an oscillatory energy–momentum tensor of the thin brane which describes a kind of exotic matter. In this case, this fact takes place because the frequency and amplitude of the stationary wave are large.

Note that since \(\omega/|a|\) is a zero of \(J_2\), the quantity \(\omega/|a| > 1\) and the third case \(\omega/|a| \ll 1\) is excluded from our analysis.

In summary, the two most relevant limits are

(I) when the matter on the brane hardly oscillates due to the small amplitude of the anisotropies compared to the brane’s tension, leading to a quasi-isotropic brane. This limit mimics the Randall–Sundrum model;

(II) when the matter on the brane is highly anisotropic and a stationary wave propagates through the bulk. In this case, we are far from the Randall–Sundrum limit.

It would be interesting to study a braneworld isotropization mechanism for our brane universe similar to those proposed in \([8, 10]\). In the latter model, the brane anisotropic energy leaks into the bulk as it evolves, a phenomenon that can also be interpreted from a completely 4D point of view in the framework of the AdS/CFT correspondence as particle production in the CFT, where energy is drawn from the anisotropy to fuel the process, leading to an isotropic brane over time. In principle, this study could be carried out for our anisotropic braneworld model since we have completely solved the 5D Einstein equations, a necessary requirement for the computation of the projection of the Weyl tensor into the brane.

### 3. Localization of transverse scalar and metric fields

In the original paper \([22]\), it was shown that near the nodes \(r_m\), where \(u(r, t) \approx 0\), metric (12) adopts the usual thin brane form of \([2]\) and one recovers localized 4D gravity on the island universes in the usual way through a massless mode solution for the equation that governs the dynamics of transverse traceless metric fluctuations.

Here we shall consider the localization of small perturbations of a real massless scalar field defined by the action

\[
S_{\varphi} = -\frac{1}{2} \int \sqrt{g} \, d^4x \, dr \, g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi. \tag{30}
\]

We consider a massless scalar field since it is important to obtain an oscillating solution in our model; otherwise there will be no standing wave since the solution for the scalar field \(\varphi\) will be dissipative (as in the case of including a mass term, for instance). It is necessary to mention that unlike \([22]\), we study the localization of scalar field in the system formed by the thin brane located in \(r = 0\) and all the island universes. This will be seen below from the fact that
the norm of fluctuations is defined on the whole domain of the extra dimension \( r \); thus, if it is finite for the entire domain, it will also be finite for single islands or a set of island universes.

The scalar field equation corresponding to action (30) is
\[
\ddot{\psi} - e^{-a} (\varphi_{xx} + \varphi_{yy}) - e^{2a} \varphi_{zz} - e^{2a(r)} [4ae(r) \varphi' + \varphi''] = 0,
\]
(31)
where the subscripts \( xx, yy \) and \( zz \) denote second differentiation with respect to \( x, y \) and \( z \), respectively. Let us propose the following ansatz for the scalar field:
\[
\varphi = L(r, t) P_1(x) P_2(y) P_3(z),
\]
(32)
which transforms the previous equation into
\[
\frac{\ddot{L}}{L} - e^{-a} \left( \frac{1}{P_1}\frac{d^2P_1}{dx^2} + \frac{1}{P_2}\frac{d^2P_2}{dy^2} - \frac{e^{2a}}{P_3}\frac{d^2P_3}{dz^2} - e^{2a(r)} [4ae(r) L' + L''] \right) = 0.
\]
(33)
Let us also set
\[
\frac{d^2P_i}{dx_i^2} = -k_i^2 P_i \quad (i = 1, 2, 3).
\]
(34)
The parameters \( k_i \) can be physically regarded as the momenta along the \( i \) direction of the brane without considering the anisotropic character of the background metric.

By taking into account (34), we obtain
\[
\ddot{L} - e^{2a(r)} [4ae(r)L' + L''] = -\left( (k_1^2 + k_2^2 - k_3^2) e^{-a} + k_3^2 \right) L.
\]
(35)
In the limit \( u \to 0 \), this equation has the well-known solution
\[
L \sim e^{\kappa y}, \quad k_0^2 - k_i^2 = 0,
\]
(36)
corresponding to the 4D massless scalar mode, which can be localized on the thin brane located at \( r = 0 \) by the warp factor \( e^{2a(r)} \) if \( a < 0 \) (see, for example, [29]).

In this paper, we consider a simple case of transverse fluctuations to the island universes, i.e. when \( k_i \approx 0 \). In this case, equation (35) takes the form
\[
\ddot{L} - e^{2a(r)} [4ae(r)L' + L''] = 0.
\]
(37)
By proposing an oscillatory ansatz similar to (21)
\[
L(r, t) = K \sin(\Omega t) g(r),
\]
(38)
where \( \Omega \) and \( K \) are some constants, we obtain
\[
g'' + 4ae(r)g' + \Omega^2 e^{-2a(r)} g = 0,
\]
(39)
which is the same equation as (22) for the metric function \( f(r) \), and thus possesses the solution
\[
g(r) = e^{-2a(r)} \left[ D J_2 \left( \frac{\Omega}{|a|} e^{-a(r)} \right) + E Y_2 \left( \frac{\Omega}{|a|} e^{-a(r)} \right) \right],
\]
(40)
where \( D \) and \( E \) are the integration constants. In contrast with the frequency of the ghost-like field, \( \omega \), the parameter \( \Omega \) in (40) is not quantized since in general \( \Omega \neq \omega \).

In order to analyze the proper way of normalization of \( g(r) \), we rewrite (39) as
\[
- (e^{2a(r)} g')' = \Omega^2 e^{2a(r)} g.
\]
(41)
Now we recall the Sturm–Liouville method associating with the equation,
\[
- [p(r)y']' + q(r)y = \lambda s(r)y,
\]
(42)
of a variable \( y(r) \) the norm
\[
||y||^2 = \int_b^c |y(r)|^2 s(r) \, dr,
\]
(43)
where $\lambda$ is an eigenvalue parameter, while $b$ and $c$ are the arbitrary real constants. By comparing (42) with (41), we see that $y(r) = g(r)$, $q(r) = 0$, $s(r) = e^{2ar}$, $\lambda = \Omega^2$, $p(r) = e^{4ar}$ and the correct norm for (39) is given by

$$
S_{\phi} \sim \int |g(r)|^2 e^{2ar} dr,
$$

(44)

where $b = 0$ and $c = \infty$. Thus, $g(r)$ belongs to the Hilbert space $H$, consisting of all such functions for which (44) is finite [30].

Another way of looking at this problem consists in considering the action for the scalar field (30) and taking into account (4) and the expression for $\sqrt{g}$, in order to show that its nontrivial part (the kinetic term) is

$$
S_{\phi} \sim \int d^4 x e^{2ar} L^2 + \cdots,
$$

(45)

where the dots denote the 5D contribution of the scalar field. If we further make use of relation (38), then the normalization condition along the extra coordinate reads

$$
S_{\phi} \sim \int dr |g(r)|^2 e^{2ar},
$$

(46)

which precisely coincides with relation (44). Thus, if we want to have a localized transverse 5D scalar field on the brane (and due to relation (9), the gravitational waves $u$ as well) the integral over $r$ in (46) must be finite.

It should be pointed out here that the finiteness of the above-obtained norm (44) is exactly the same for the phantom-like scalar field ($\phi$, or $\sigma$), which indeed vanishes at the island branes. Therefore, the latter field will also be localized if expression (46) constitutes a finite quantity.

By considering the change of variable $v = \Omega e^{-ar}/|a|$ for the case $a > 0$, we have the following integrating limits: $v_1 = \Omega/|a|$ and $v_2 = 0$. Alternatively, for $a < 0$, we obtain $v_1 = \Omega/|a|$, while $v_2 = \infty$.

In the language of this new coordinate, integral (46) adopts the form

$$
S_{\phi} \sim \int v |D|^2 |J_2(v)|^2 + D\bar{E}J_2(v)\bar{Y}_2(v) + \bar{D}E\bar{J}_2(v)Y_2(v) + |E|^2 |Y_2(v)|^2 \, dv,
$$

(47)

where $\bar{X}$ denotes the complex conjugate of $X$. It is easy to get the behavior of the integral (47) by direct computation. It turns out that only in the case $E = 0$ and $a > 0$, this integral is finite. Therefore, the massless scalar field $\phi$ is localized on the brane at $r = 0$, as well as in all the island branes of the model since in the latter case, one should just change the integration limits $v_1$ and $v_2$, obtaining a finite value for each island universe or for their set.

It should be mentioned that if one compares our result to the result obtained in [22, 29], it seems that they are in an apparent contradiction, because the scalar field is localized for $a > 0$ in our case, while in these papers this field is localized for $a < 0$. However, the point here is that we are investigating the transverse modes and not the modes along the brane. Also in our work $\Omega \neq 0$ in ansatz (38); otherwise $L \equiv 0$ and the scalar field $\phi$ would be trivial. A way of avoiding this trouble consists in extending ansatz (38) to contain a nonzero constant phase $\alpha \neq 2\pi n$ with $n \in \mathbb{Z}$ in the argument of the sinus function, i.e.

$$
L = K \sin (\Omega t + \alpha) g(r).
$$

(48)

In this case, the $\Omega = 0$ value corresponds to the lowest frequency of oscillation of the nontrivial mode of the scalar field $\phi$, which indeed adopts a constant value along the fifth dimension (as well as $g$). Then this field is localized for $a < 0$ like in [22, 29] according to the normalization condition (44).
Now let us roughly analyze the gravitational sector from this localizing point of view. The Ricci scalar for metric (12) reads
\[ R_5 = -\frac{3}{2} e^{-2\alpha(r)} u'^2 + \frac{3}{2} u'^2 + 20a^2 + 16\delta(r). \] (49)

Thus, the gravitational 5D action is
\[ S_g = \int \sqrt{g} \, dx^5 R_5 \sim -\frac{3}{2} \int dx^5 e^{2\alpha(r)} u'^2 + \ldots, \] (50)
where the dots denote 5D contributions. Due to relation (9) between the phantom-like scalar field \( \sigma \) and the gravitational field \( u \), it turns out (as expected) that the localization properties of these fields are similar. Thus, the gravitational field \( u \) is also localized if we choose \( E = 0 \) and \( a > 0 \) in solution (40) under ansatz (38).

4. Final remarks

In this paper, we presented a consistent derivation of the 5D anisotropic standing wave braneworld proposed in [22] by initially assuming a \( Z_2 \)-symmetric factor and introducing a simple anisotropic energy–momentum tensor with different stresses along different spacetime directions on the 3-brane. We also derived and explicitly solved the corresponding junction conditions for the braneworld model and obtained analytical expressions for the (oscillating) stresses of the 3-brane along the 4D spatial directions.

By carefully looking at the energy–momentum tensor of the 3-brane proposed in (14), we can infer from (26) that it corresponds to an anisotropic effective fluid which is a mixture of a vacuum fluid characterized by its tension and an anisotropic oscillating matter fluid. We can also conclude that the 4D matter corresponding to this source has an exotic nature because it violates the weak energy condition for increasing warp factors and satisfies it for decreasing warp factors \( (a < 0) \). It is worth mentioning here that this condition is also violated in the thin brane models [2]. Besides, it also violates the dominant energy condition since it is an oscillating fluid which emits anisotropic waves into the bulk with different amplitudes (which change sign over time and eventually disappear) and phases along the directions \( x, y \) and \( z \). Nevertheless, one can look for an alternative mechanism that could lead to a more physical picture and eventually heal this drawback related to the anisotropy of the metric ansatz (12), like proposing a more involved energy–momentum tensor or add some matter fields on the 3-brane. An interesting situation takes place when the amplitude of the anisotropies is small with respect to the tension on the brane (quasi-isotropic limit), since in this case the braneworld is effectively isotropic. Moreover, within the framework of the model presented here, it is possible to study the braneworld isotropization in which anisotropies dissipate via inflation [8] or leakage of thermal graviton radiation into the bulk [10], a relevant phenomenon from the brane cosmological viewpoint that deserves more attention.

We also showed the correct way of defining the norm for the transverse scalar field \( \varphi \) through the Sturm–Liouville method, a fact that leads to the localization of this field just when \( a > 0 \) and \( E = 0 \) in solution (40). However, it should be pointed out that this localization approach involves all the island brane universes as well as the thin brane located at \( r = 0 \), while the approach taken in [22] considers the gravity localization on each of the island universes and is related to the massless modes of the transverse traceless metric fluctuations of the system, i.e. to the 4D gravitons that live in each 3-brane. Our results seem to agree with recent results obtained in [11], where localization of massive test particles about a thick braneworld with a time-dependent extra dimension arises just for increasing warp factors due to an oscillatory behavior in time-like geodesics. Further work must be done in order to get localization mechanisms of matter fields with decreasing warp factors, since these are useful
in solving the hierarchy problem. We would also like to point out that further investigations toward the localization of three generations of fermion fields in this model, both at the thin brane located at $r = 0$ and at the island universes located at $r_m$, are in progress and will be published elsewhere.

We finally would like to point out that in order to avoid the instabilities related to the ghost-like scalar field of this braneworld, we can appeal to an alternative approach of interpreting the phantom-like scalar field $\phi$ as the geometrical scalar field of a 5D Weyl integrable manifold [31, 32], where a scalar appears through the definition of the covariant derivative of the metric tensor,

$$D_\gamma g_{\alpha\beta} = g_{\alpha\beta} \delta_\gamma \phi.$$  \hspace{1cm} (51)

This is a generalization of the Riemannian geometry, in which the covariant derivative of the metric tensor obeys the metricity condition, i.e. it vanishes

$$D_\gamma g_{\alpha\beta} = 0.$$  \hspace{1cm} (52)

On the other hand, this relation indicates that the Weylian affine connections are not metric compatible since they also involve the scalar field in their definition:

$$\Gamma^\rho_{\mu\nu} = \{\rho_{\mu\nu}\} - \frac{1}{2} \big( \phi_{,\mu} \delta^\rho_{\nu} + \phi_{,\nu} \delta^\rho_{\mu} - g_{\mu\nu} \phi_{,\rho} \big),$$  \hspace{1cm} (53)

where $\{\rho_{\mu\nu}\}$ are the Christoffel symbols. As a consequence, in an integrable Weyl manifold specified by the pair $(g_{\mu\nu}, \phi)$, the non-metricity condition (51) implies that the length of a vector is altered by parallel transport.

The 5D Weyl action can be written as

$$S^W_5 = \int_{M^5_5} d^5x \sqrt{|\hat{g}|} \frac{1}{16\pi G_5} e^{-\frac{1}{2} \tilde{\xi}} \left[ R + 3\tilde{\xi} (\nabla \phi)^2 - 2U(\phi) \right],$$  \hspace{1cm} (54)

where $M^5_5$ is a Weylian integrable manifold, $\tilde{\xi}$ is an arbitrary coupling parameter and $U(\phi)$ is a self-interaction potential for the scalar field $\phi$. From the formalism itself it becomes clear that this action is of pure geometrical nature since the scalar field that couples non-minimally to gravity is precisely the scalar $\phi$ that enters the definition of the affine connections of the Weyl manifold (53) and the non-metricity condition (51) and, thus, cannot be neglected at all in our setup.

By performing the conformal transformation,

$$\hat{g}_{\mu\nu} = e^{\frac{\xi}{2}} g_{\mu\nu},$$  \hspace{1cm} (55)

we map the Weyl action (54) onto the Einstein frame:

$$S^E_5 = \int_{M^5_5} d^5x \sqrt{|\hat{g}|} \frac{1}{16\pi G_5} \left[ \hat{R} + 3\xi (\hat{\nabla} \phi)^2 - 2\Lambda_5 \right],$$  \hspace{1cm} (56)

where all hatted magnitudes and operators are defined in the Riemann manifold, $\xi = \tilde{\xi} - 1$, $\hat{U}(\phi) = e^{-\phi} U(\phi) = \Lambda_5$ and we have set $U(\phi) = \Lambda_5 e^\phi$ in (54) in order to obtain a 5D cosmological constant in (56) as in (1). In this frame, we have a theory which describes 5D gravity minimally coupled to a scalar field plus a cosmological constant; the affine connections become the Christoffel symbols, the metricity condition is recovered and Weyl’s scalar field in (56) imitates a massless scalar field (either an ordinary scalar or a ghost-like scalar depending on the sign of $\xi$ [31, 32]).

In the alternative approach mentioned above, we can start with action (54) which does correspond to a conventional scalar field in Einstein frame’s action (56) under the conformal transformation (55). Then we can compute the corresponding field equations under metric (12) (which will be different from (15) since the affine connections involve the scalar field) and
solve them in the Weyl frame. We further can set $\phi_{\mu \nu} = 0$ on the brane in the solution of the field equations (which also will be different from the solution in the Riemann manifold). Thus, in this way we can recover an effective 4D universe in the Einstein frame at the brane (and possibly at the island universes if our solution possesses the same structure as the braneworld obtained in [22]) completely free of ghost instabilities since the Weyl scalar corresponds to a conventional real scalar field. This scenario still has to be investigated in full detail, a direction which is currently under research.

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