Abstract

In this paper we describe the Hopf algebras on planar binary trees used to renormalize the Feynman propagators of quantum electrodynamics, and the coaction which describes the renormalization procedure. Both structures are related to some semi-direct coproduct of Hopf algebras.

1 Introduction

Planar binary trees exhibit surprisingly rich structures, often related to analogue ones on (non binary) rooted trees. For instance, in the last decades several Hopf algebras on families of trees have been discovered in different frameworks: by R.Grossman and R.G. Larson [1] in connection to the Butcher group introduced by J.C. Butcher [6] to solve differential equations; by D. Kreimer [4] to describe the renormalization of perturbative quantum field theory; by J.-L. Loday and M. Ronco [19] in the framework of dendriform algebras; by L. Foissy [10] as a noncommutative extension of the Butcher-Kreimer algebra on rooted trees. In particular, the relationship between some of these Hopf algebras have been studied by F. Painate [24], by Foissy [10] and by R. Holtkamp [13]. Moreover, these Hopf algebras present a universal property which was first described by A. Connes and Kreimer in [7], and then used by I. Moerdijk in [22] to introduce a large class of Hopf algebras.

Planar binary trees were also used in [1] to solve perturbatively the system of functional differential equations satisfied by the 2-points correlation functions of quantum electrodynamics (QED). QED is the quantum field theory which describes the dynamics of interacting electrons and photons. The interaction between these two particles is usually represented by Feynman diagrams. In this context, each tree corresponds to a finite sum of appropriate Feynman diagrams and the explicit relations are given in [2]. Following the classical Feynman rules, cf. for instance [15] or [25], we consider two Feynman amplitudes associated to each tree, one for the electron propagator and one for the photon propagator. These amplitudes are in general sums of divergent integrals which need to be renormalized, and in [2] we give a perturbative solution in term of trees of the equations satisfied by the renormalized propagators of QED.

In principle, the renormalization procedure can be described directly on the propagators as the action of a group, called the renormalization group. In practice, the elements of this group are known only through computations made on Feynman diagrams: the so-called Forest Formula [18] describes the relation between the perturbative coefficients of the propagators before and after the renormalization. In [14], D. Kreimer and A. Connes discovered that the operations involved in this formula define the structure of a commutative Hopf algebra on the set of Feynman diagrams labeled by some indices. This result shows that the labeled Feynman diagrams are the natural local coordinates of quantum field theory, and that the renormalization group can be recovered as the set of characters of this Hopf algebra. Therefore, the renormalization procedure is known if we can construct the characters from the data we
know of the quantum field theory: the Feynman amplitudes and the counterterm maps imposed by the physical renormalization prescription.

In scalar field theory, the amplitudes and the counterterms are scalar maps which preserve the junction of Feynman graphs, that is, they are characters of the Connes-Kreimer Hopf algebra given in [8]. The relationship between the renormalization group and the local coordinates Hopf algebra is then the classical Tannaka-Krein duality which holds between each affine algebraic group and its coordinate ring.

In vector or spinor valued field theories (such as QED), the propagators are matrices, hence the Feynman amplitudes and the counterterms are maps which take value in a non-commutative ring. Because of Feynman rules, they still respect the product between Feynman graphs, but they can not be usual characters of a commutative coordinate ring. In this case, the Connes-Kreimer commutative Hopf algebra dual to the renormalization group of course still exists, but it does not help us to recover the group through the physical data, because neither the amplitudes nor their matrix elements are characters of this algebra. The alternative approach is to look for a suitable algebra whose matrix-valued characters are the known Feynman amplitudes and counterterm maps. It might not exist, or not be a Hopf algebra. In fact, these “non-commutative characters” do not satisfy any known duality principle, for two reasons. Given a group $G = \text{Hom}_{\text{Alg}}(\mathcal{H}, A)$ through a set of algebra homomorphisms between two non-commutative algebras, the group law on $G$ induces a coproduct on $\mathcal{H}$ which, in general, takes value in the free product $\mathcal{H} \star \mathcal{H}$, therefore $\mathcal{H}$ is not necessarily a Hopf algebra in the usual sense. Dually, if $\mathcal{H}$ is a non-commutative Hopf algebra, then the set $\text{Hom}_{\text{Alg}}(\mathcal{H}, A)$ is a groupoid, in general, but not necessarily a group.

In this paper, we show that there exists a non-commutative Hopf algebra which represents the renormalization group of QED, and an associated coaction on the algebra dual to the propagators which describes the renormalization in local coordinates. In our case, we expand the QED propagators as asymptotic series over the planar binary trees instead of the Feynman diagrams, therefore our local non-commutative coordinates are the trees. The coproducts and coactions are forced by the relationship between the bare and the renormalized propagators found in [2]. The resulting Hopf algebras of renormalization for photons and electrons look very different, both for the algebra and the coalgebra structures. However, they can be interpreted as semidirect coproducts of similar Hopf algebras, and thus directly related to a standard form of the renormalization group.

The paper is organized as follows. In the second section we recall the algebraic tools needed to present a non-commutative version of the renormalization group, which is a semidirect product of two groups, and of the renormalization action. The main tools come from the semidirect or smash coproduct of Hopf algebras, introduced by R. Molnar in [23].

In the third, fourth and fifth sections, we define the non-commutative Hopf algebras which correspond to the electron and photon propagators; the Hopf algebra which corresponds to the renormalization of the coupling constant of QED; and finally the renormalization Hopf algebras and the renormalization coactions for the electron and for the photon propagators. To describe these structures we only need some grafting and pruning operations on trees. The choice of such operations, which looks apparently arbitrary, is in fact forced by the combinatorial operations on the Feynman graphs related to the trees, cf. [2]. It is then even more surprising how the basic operations on trees turn out to be deeply related to those used by J.-L. Loday in his *Arithmetree*, [17].

The main application of these Hopf algebras, namely the renormalization of QED propagators, is recalled in the last section.

Notations. We suppose that all vector spaces and algebras are defined over the field $\mathbb{C}$ of complex numbers, but this choice is not necessary. For any set $X$, we denote by $\mathbb{C}X$ the vector space spanned by $X$, by $\mathbb{C}\langle X \rangle$ the tensor algebra on $X$ (noncommutative polynomials), and by $\mathbb{C}[X]$ the symmetric algebra on $X$ (commutative polynomials).

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1 We consider Feynman graphs with one external leg removed. The junction is then the same as the disjoint union.

2 We consider here the usual propagators multiplied by the inverse of the free propagator.
2 Renormalization group and semidirect coproduct of Hopf algebras

The renormalization of quantum fields can be formalized as an action of the renormalization group on the set of propagators associated to the quantum fields: a bare propagator $D$ is turned into a renormalized propagator $\bar{D} = D \cdot \eta$ by the action of an element $\eta$ of the renormalization group. In perturbative quantum field theory, all these groups and sets are made of formal series in the powers of the coupling constant, which is the fine structure constant $\alpha$ in QED (i.e. the square of the electric charge divided by $4\pi$). Among such series, two basic operations are possibly allowed and determine the algebraic part of the renormalization procedure: the (pointwise) product and the composition, or substitution.

The set of propagators is in fact a group $G_p$ of series with the pointwise product, since their constant term is invertible. The composition, instead, is the natural operation in the group $G_c$ which renormalizes the coupling constant. Such series have zero constant term, and invertible first term. By substitution, the group $G_c$ also acts on $G_p$ from the right, and the action $(f, \varphi) \mapsto f^\varphi$ is associative with respects to the composition in $G_c$, i.e.

$$f^{(\varphi \psi)} = (f^\varphi)^\psi,$$

and commutes with the product in $G_p$, in the sense that

$$(fg)^\varphi = (f^\varphi)(g^\varphi).$$

Then, in QED we can distinguish two renormalization groups, one for the electron and one for the photon propagators.

2.1 The QED renormalization groups. The electron renormalization group is the semidirect product $G^c \rtimes G^p$ made of pairs $(\varphi, f)$ in the direct product $G^c \times G^p$ with group law

$$(\varphi, f) \cdot_K (\psi, g) := (\varphi \psi, f^\varphi g).$$

The renormalization procedure is the right action of $G^c \rtimes G^p$ on $G^p$ obtained by embedding $G^p$ in $G^c \rtimes G^p$ through $f \mapsto (1_c, f)$, applying the semidirect product law in $G^c \rtimes G^p$ and then projecting onto the $G^p$ component, that is,

$$f \cdot_\sigma (\varphi, f) := f^\varphi f.$$

In this model, the $G^c$ component of $G^c \rtimes G^p$ represents the renormalization of the fine structure constant, while the $G^p$ component represents the inverse $Z_\alpha^{-1}$ of the electron renormalization factor. In the analogue situation for the photon renormalization, it was proved by J.C. Ward [27] that the fine structure constant is renormalized exactly by the inverse $Z_\alpha^{-1}$ of the photon renormalization factor. In other words, we identify the series of type $G^c$ and $G^p$ by multiplying or dividing by the fine structure constant. In our model, this identification consists of a map $s : G^c \longrightarrow G^p$ which is a 1-cocycle of $G^c$ with values in $G^p$, that is,

$$s(\psi)[s(\varphi \psi)]^{-1}[s(\varphi)\psi] = 1_p \quad \text{for all } \varphi, \psi \in G^c.$$

The photon renormalization group is then the group $G^c$ itself, and the renormalization procedure simply becomes the action on $G^p$ of $G^c$ identified with the subgroup $G^c \rtimes \{s(G^c)\}$ of $G^c \rtimes G^p$,

$$f \cdot_\sigma \varphi := f^\varphi s(\varphi).$$

2.2 Feynman amplitudes and characters. The Feynman bare and renormalized amplitudes $U$, $R$, and the counterterm maps $C$ are all the data which allow to reconstruct the bare and the renormalized propagators and the elements of the renormalization group, starting from the appropriate set of Feynman diagrams (or trees in our case).

Moreover, by definition of the Feynman rules, they preserve the natural product which joins together two (amputated) Feynman diagrams.
For scalar field theories, these maps take scalar values, so they can be recognized as characters of the coordinate rings of the groups involved. In fact, since $G^c$ and $G^p$ are two affine groups, the semidirect product $G^c \ltimes G^p$ is also affine. Let us denote by $\mathbb{C}(G^c)$, $\mathbb{C}(G^p)$ and $\mathbb{C}(G^c \ltimes G^p)$ their coordinate rings. They are commutative Hopf algebras, in perfect duality with the original groups. More precisely, the groups can be reconstructed as the sets

$$G^c \cong \text{Hom}_{\text{Alg}}(\mathbb{C}(G^c), \mathbb{C}), \quad G^p \cong \text{Hom}_{\text{Alg}}(\mathbb{C}(G^p), \mathbb{C}), \quad G^c \ltimes G^p \cong \text{Hom}_{\text{Alg}}(\mathbb{C}(G^c \ltimes G^p), \mathbb{C}),$$

of characters of the coordinate rings, which are the algebra homomorphisms from the rings to the field of scalars, endowed with the convolution products.

For QED, the maps $U$, $R$ and $C$ do not anymore have scalar values. For single particle Green functions, the maps $U$ and $R$ take value in the ring of $4 \times 4$ complex matrices, and $C$ is defined by the value of $U$ and $R$ at some fixed momentum, through the renormalization conditions. Therefore, $C$ is a priori a matrix. In the case of QED in flat space time, Lorentz invariance implies that $C$ is a scalar multiplied by a fixed matrix. However, for applications in curved space time, or to treat several fermions at once, it is interesting to allow matrix-valued counterterms. Therefore, in general, the maps $U$, $R$ and $C$ are not anymore characters of the coordinate rings of the groups. A duality, if it exists, should then be searched between the groups $G^p$, $G^c$, $G^c \ltimes G^p$ and some algebras $\mathcal{H}(G^p)$, $\mathcal{H}(G^c)$, $\mathcal{H}(G^c \ltimes G^p)$ such that

$$U, R \in \text{Hom}_{\text{Alg}}(\mathcal{H}(G^p), \mathcal{A}), \quad C \in \text{Hom}_{\text{Alg}}(\mathcal{H}(G^c \ltimes G^p), \mathcal{A}),$$

where $\mathcal{A}$ is the non-commutative ring where $U$, $R$ and $C$ take values. This leads us to consider some non-commutative versions of the coordinate rings.

### 2.3 The Hopf algebra of a semidirect product of groups.
Let us recall how to construct the coordinate ring $\mathbb{C}(G^c \ltimes G^p)$ and the coactions on $\mathbb{C}(G^p)$. Denote by $\Delta^c : \mathbb{C}(G^c) \longrightarrow \mathbb{C}(G^c) \otimes \mathbb{C}(G^c)$ and $\Delta^p : \mathbb{C}(G^p) \longrightarrow \mathbb{C}(G^p) \otimes \mathbb{C}(G^p)$ the coproducts dual to the group laws of $G^c$ and $G^p$, in the sense that

$$\langle \varphi, a \rangle = \langle \varphi \otimes \psi, \Delta^c a \rangle \quad \text{and} \quad \langle f g, b \rangle = \langle f \otimes g, \Delta^p b \rangle,$$

where $\langle , \rangle : G^c \times \mathbb{C}(G^c) \longrightarrow \mathbb{C}$ is the evaluation map $\langle \varphi, a \rangle = a(\varphi)$. Also, denote by $\delta : \mathbb{C}(G^p) \longrightarrow \mathbb{C}(G^p) \otimes \mathbb{C}(G^c)$ the coaction dual to the action of $G^c$ on $G^p$,

$$\langle h^\varphi, b \rangle = \langle h \otimes \varphi, \delta(b) \rangle.$$

The map $\delta$ is coassociative with respect to $\Delta^c$ and commutes with $\Delta^p$.

Then, the coordinate ring $\mathbb{C}(G^c \ltimes G^p)$ is the tensor product algebra $\mathbb{C}(G^c) \otimes \mathbb{C}(G^p)$, endowed with the coproduct $\Delta^\times$ dual to the group law $\cdot^\times$, i.e.

$$\langle (\varphi, f) \cdot^\times (\psi, g), a \otimes b \rangle = \langle \varphi \otimes f \otimes \psi \otimes g, \Delta^\times(a \otimes b) \rangle.$$

Explicitly, $\Delta^\times$ is the algebra morphism given by

$$\Delta^\times(a \otimes b) = \Delta^c(a) \cdot [\delta \otimes \text{Id}] \Delta^p(b),$$

where we omit the symbol of the componentwise product in the algebra $\mathbb{C}(G^c) \otimes \mathbb{C}(G^p)$ between the image of $\Delta^c$ in $\mathbb{C}(G^c) \otimes 1 \otimes \mathbb{C}(G^c) \otimes 1$, and the image of $(\delta \otimes \text{Id}) \Delta^p$ in $1 \otimes \mathbb{C}(G^p) \otimes \mathbb{C}(G^c) \otimes \mathbb{C}(G^p)$.

Moreover, the action of $G^c \ltimes G^p$ on $G^p$ induces a dual coaction $\delta^\times : \mathbb{C}(G^p) \longrightarrow \mathbb{C}(G^p) \otimes \mathbb{C}(G^c \ltimes G^p)$, which is simply the second component of the coproduct, i.e.

$$\delta^\times(b) = (\delta \otimes \text{Id}) \Delta^p(b).$$

Similarly, if we denote by $\sigma : \mathbb{C}(G^p) \longrightarrow \mathbb{C}(G^c)$ the linear map dual to the 1-cocycle $s : G^c \longrightarrow G^p$, then $\sigma$ satisfies the identity

$$(m_{21}^2 \otimes m_{135}^1)(\text{Id} \otimes \Delta^c \otimes \text{Id} \otimes \text{Id})(\sigma \otimes \sigma \otimes \sigma \otimes \text{Id})(\text{Id} \otimes \delta \otimes \text{Id} \otimes \text{Id}) \cdot \Delta^p)^2 = \Delta^c \cdot \epsilon^c \epsilon^p,$$
where \( m_{i,j,k} \) denotes the multiplication in \( \mathbb{C}(G^c) \) applied to positions \((i, j, k)\), \( S^p \) is the antipode of the Hopf algebra \( \mathbb{C}(G^p) \), \( (\Delta^p)^2 = (\Delta^p \otimes \text{Id})\Delta^p = (\text{Id} \otimes \Delta^p)\Delta^p \), \( \epsilon^c : \mathbb{C} \rightarrow \mathbb{C}(G^c) \) is the unit of \( \mathbb{C}(G^c) \), and finally \( \epsilon^p : \mathbb{C}(G^p) \rightarrow \mathbb{C} \) is the counit of \( \mathbb{C}(G^p) \). This condition is equivalent to require that

\[
\Delta^c \sigma = (\sigma \otimes \text{Id})(\text{Id} \otimes m^c)(\delta \otimes \sigma)\Delta^p.
\]

The coaction of \( \mathbb{C}(G^c) \) on \( \mathbb{C}(G^p) \) dual to the action \( \ast \sigma \) is then the algebra morphism \( \delta^\sigma : \mathbb{C}(G^p) \rightarrow \mathbb{C}(G^p) \otimes \mathbb{C}(G^c) \) given by

\[
\delta^\sigma(b) = (\text{Id} \otimes \epsilon^p)(\delta \otimes \sigma)\Delta^p(b).
\]

### 2.4 The semidirect coproduct of Hopf algebras.

The formulas of section 2.3 make sense for all Hopf algebras, even not commutative ones. The generalisation to arbitrary Hopf algebras has been studied by R. Molnar [23], B. Lin [10], D. Radford [24], S. Majid [21] and others.

Let \( H^c \) and \( H^p \) be two Hopf algebras with multiplications \( m^c, m^p \) and coproducts \( \Delta^c, \Delta^p \). Suppose that \( H^c \) coacts on \( H^p \) from the right, and that the coaction \( \delta : H^p \rightarrow H^p \otimes H^c \) satisfies

\[
\begin{align*}
(\delta \otimes \text{Id})\delta &= (\text{Id} \otimes \Delta^c)\delta \\
(\Delta^p \otimes \text{Id})\delta &= m^p_{23}(\delta \otimes \delta)\Delta^p,
\end{align*}
\]

where \( m^p_{23} \) multiplies what is in the position 2 by what is in the position 4 and puts it in the position 3. Then, the semidirect or smash coproduct \( H^c \ltimes H^p \) is the tensor algebra \( H^c \otimes H^p \) endowed with the coproduct

\[
\Delta^\ast(a \otimes b) := \Delta^c(a) \[(\delta \otimes \text{Id})\Delta^p(b)\], \quad a \in H^c, b \in H^p,
\]

and the counit \( \epsilon^\ast(a \otimes b) := \epsilon_1(a)\epsilon_2(b) \).

Molnar proved in [23] that \( H^c \ltimes H^p \) is a coalgebra. In particular, it follows that the map \( \delta^\ast : H^p \rightarrow H^p \otimes (H^c \ltimes H^p) \) given by \( \delta^\ast(b) = (\delta \otimes \text{Id})\Delta^p(b) \) is a coaction, i.e. it is coassociative with respect to \( \Delta^\ast \).

He also proved that \( H^c \ltimes H^p \) is a bialgebra if \( H^c \) is commutative. In this case it is also a Hopf algebra, with antipode

\[
S^\ast(a \otimes b) := S^c(a) \left[\tau(\text{Id} \otimes S^c)\delta(S^p b)\right] = \tau(S^p \otimes S^c)(\text{Id} \otimes m^c)(\delta \otimes \text{Id})(b \otimes a).
\]

Moreover, in this case the coaction \( \delta^\ast \) is also an algebra morphism.

### 2.8 Lemma.

Let \( H^c \) and \( H^p \) be two Hopf algebras such that \( H^c \) coacts on \( H^p \) as above (\( H^c \) is not necessarily commutative). Suppose that there exists a map \( \sigma : H^p \rightarrow H^c \) with the property that if \( \delta^\sigma : H^p \rightarrow H^p \otimes H^c \) is the map defined by

\[
\delta^\sigma := m^p_{23}(\delta \otimes \sigma)\Delta^p
\]

then \( \sigma \) interwines \( \delta^\sigma \) and \( \Delta^c \), i.e.

\[
\Delta^c \sigma = (\sigma \otimes \text{Id})\delta^\sigma.
\]

Then \( \delta^\sigma \) is coassociative with respect to \( \Delta^c \).

**Proof.** Let us adopt the following Sweedler conventions:

\[
\Delta^c(a) = \sum a_{(1)} \otimes a_{(2)}, \quad \Delta^p(b) = \sum b_{(1)} \otimes b_{(2)}, \quad \delta(b) = \sum b_{(l)} \otimes b_{(r)}.
\]
Then for any \( b \in \mathcal{H}^p \) we have

\[
(\delta^\sigma \otimes \text{Id})\delta^\sigma(b) = \sum \delta^\sigma(b_{(1)1}) \otimes b_{(1)2}\sigma(b_{(2)}) = \sum b_{(1)11} \otimes b_{(1)12}\sigma(b_{(2)}) = \sum b_{(1)11r} \otimes b_{(1)12r}\sigma(b_{(2)}) = \sum b_{(1)11r} \otimes b_{(1)12r}\sigma(b_{(2)})
\]

where the equality (1) follows from (2.9) applied to \( b_{(1)} \), the equality (2) follows from (2.3) applied to \( b_{(1)1} \), the equality (3) follows from the coassociativity of \( \Delta^p \) applied to \( b \), and the equality (4) follows from (2.9) applied to \( b_{(2)} \).

\[
\delta^\sigma \otimes \text{Id})\delta^\sigma(b) = \sum \delta^\sigma(b_{(1)1}) \otimes b_{(1)2}\sigma(b_{(2)}) = \sum b_{(1)11} \otimes b_{(1)12}\sigma(b_{(2)}) = \sum b_{(1)11r} \otimes b_{(1)12r}\sigma(b_{(2)}) = \sum b_{(1)11r} \otimes b_{(1)12r}\sigma(b_{(2)}) = \sum b_{(1)11r} \otimes b_{(1)12r}\sigma(b_{(2)})
\]

3 Propagators Hopf algebras on trees

3.1 Planar binary trees. By planar binary tree we mean a connected and oriented planar graph with no cycle, such that each internal vertex has one incoming and two outgoing edges. The incoming and outgoing edges of a tree are called respectively the root and the leaves. Such trees are naturally graded by the number of internal vertices, that we call the order. We denote by \(|t|\) the order of a tree \( t \). Up to continuous transformations of the plane which fix the root and the leaves, there are \( c_n = \frac{(2n)!}{n!(n+1)!} \) trees with order \( n \). We denote by \( Y_n \) the set of trees \( t \) with \(|t| = n\), and by \( Y = \bigcup_{n \geq 0} Y_n \) the set of all planar binary trees. Here are the sets of trees with order 0, 1, 2 and 3:

\[
Y_0 = \{ | \}, \\
Y_1 = \{ \U, \U \}, \\
Y_2 = \{ \U \U, \U \U, \U \U, \U \U \}, \\
Y_3 = \{ \U \U \U, \U \U \U, \U \U \U, \U \U \U \}
\]

Let \( \vee : Y_n \times Y_m \rightarrow Y_{n+m+1} \) denote the map which grafts two trees on a new root, for instance,

\[
\U \vee \U = \U \U, \\
\U \vee | = \U \U.
\]

Then, each tree \( t \neq | \) is the grafting \( t = t^l \vee t^r \) of two uniquely determined trees \( t^l, t^r \) with smaller order.

3.2 The products over and under. Following the notations of J.-L. Loday and M. Ronco in [17], [20], we call over and under the graded products \( /, \backslash : Y_n \times Y_m \rightarrow Y_{n+m} \) defined by the recurrence relations

\[
t/s := (t/s^l) \vee s^r \text{ for } s = s^l \vee s^r, \\
t/| := t,
\]

and similarly

\[
t\backslash s := t^l \vee (t^r \backslash s) \text{ for } t = t^l \vee t^r, \\
| \backslash s := s.
\]
These operations graft one tree on the other one according to the rules \( t/s = t\backslash s \) and \( t\backslash s = t^s \). For instance,

\[
\begin{align*}
\Upsilon / \Upsilon &= \Upsilon, & \Upsilon / \Upsilon &= \Upsilon, \\
\Upsilon \backslash \Upsilon &= \Upsilon, & \Upsilon \backslash \Upsilon &= \Upsilon.
\end{align*}
\]

Both products are clearly associative (non-commutative), and for both the root tree \( | \) is a unit. Moreover, any tree \( t = t^l \lor t^r \) can be decomposed as \( t = t^l \lor (| \lor t^r) \) or as \( t = (t^l \lor |) \lor t^r \). Hence, the trees of the form \( | \lor t =: V(t) \), for any \( t \in Y \), form a system of generators of \((Y,/)\), and similarly the trees of the form \( t \lor | \) form a system of generators of \((Y,\backslash)\).

### 3.3 The pruning coalgebras.

Identify \( CY \) with its linear dual \( CY^* \), and consider the coproducts \( \Delta^p_\gamma, \Delta^p_e : CY \rightarrow CY \otimes CY \) dual of the products / and \( \backslash \) respectively,

\[
\Delta^p_\gamma(t) = \sum_{t=t_1t_2} t_1 \otimes t_2,
\]

\[
\Delta^p_e(t) = \sum_{t=t_1 \backslash t_2} t_1 \otimes t_2.
\]

Of course, \( \Delta^p_\gamma \) and \( \Delta^p_e \) are graded coassociative operations, and together with the counit \( \epsilon \) dual to the unit \( | \), defined as \( \epsilon(|) = 1 \) and \( \epsilon(t) = 0 \) if \( t \neq | \), they define on \( CY \) two different structures of graded coalgebra.

The coproducts \( \Delta^p_\gamma \) and \( \Delta^p_e \) break all the branches of a tree which are respectively on the left and on the right of the root, and places them on the same side. It is useful to give a recursive definition of these coproducts: for any \( t, s \in Y \) we have

\[
\Delta^p_\gamma(|) = | \otimes |,
\]

\[
\Delta^p_\gamma(t \lor s) = t \lor s \otimes | + \sum_{\Delta^p_\gamma t} t_{(1)} \otimes t_{(2)} \lor s,
\]

and similarly

\[
\Delta^p_e(|) = | \otimes |,
\]

\[
\Delta^p_e(t \lor s) = | \otimes t \lor s + \sum_{\Delta^p_e s} t \lor s_{(1)} \otimes s_{(2)},
\]

where we use the standard Sweedler notation \( \Delta^p_\gamma(t) = \sum t_{(1)} \otimes t_{(2)} \) and \( \Delta^p_e(s) = \sum s_{(1)} \otimes s_{(2)} \). The pruning operator of \( [1] \) is the reduced coproduct \( P(t) = \Delta^p_e(t) - t \otimes | - | \otimes t \).

### 3.6 The photon and electron propagator Hopf algebras.

If we extend the pruning coproducts \( \Delta^p_\gamma \) and \( \Delta^p_e \) multiplicatively on tensor products of trees, and we set the root tree \( | \) as unit, we obtain two different Hopf algebras \( H^\gamma \) and \( H^e \), which are neither commutative nor cocommutative. Therefore we set \( H^\gamma, H^e := C(Y)/(1 - |) \) as the free associative algebras on the set of trees where we identify the formal unit \( 1 \) with the root tree \( | \), and we consider \( H^\gamma \) with the Hopf structure induced by \( \Delta^p_\gamma \), and \( H^e \) with the Hopf structure induced by \( \Delta^p_e \). For notational convenience, we omit the tensor product symbols.

Beside the natural grading coming from the tensor powers, on a tensor product of trees we can define a total order as the sum of the orders of the trees,

\[
|t_1 \ldots t_k| = |t_1| + \cdots + |t_k|.
\]
Then the algebras $\mathcal{H}_n^\gamma$ and $\mathcal{H}_n^e$ are graded connected Hopf algebras, with homogeneous components

$$
\mathcal{H}_n^\gamma, \mathcal{H}_n^e = \bigoplus_{n_1 + \cdots + n_k = n} \mathbb{C}Y_{n_1} \otimes \cdots \otimes \mathbb{C}Y_{n_k}.
$$

In particular, the electron pruning antipode $S_p^e$ is the graded algebra anti-morphism automatically defined on generators by the recursive formula

$$
S_p^e(1) = 1 \quad \text{and} \quad S_p^e(t) = -t - \sum_{P(t)} S_p^e(t(1))t(2) = -t - \sum_{P(t)} t(1)S_p^e(t(2)).
$$

Since $S_p^e$ plays an explicit role in the renormalization of the electron propagator, we give a few examples:

$$
S_p^e(\gamma) = -\gamma,
S_p^e(\gamma^2) = -\gamma + \gamma^2,
S_p^e(\gamma^3) = -\gamma + \gamma^2 \gamma + \gamma \gamma^2.
$$

Notice that the coproduct $\Delta_p^e$ is neither commutative nor cocommutative, and $S_p^e \circ S_p^e \neq \text{Id.}$

4 Charge Hopf algebra on trees

4.1 The charge algebra. Let $\mathcal{H}^\gamma := \mathbb{C}[V(t), t \in \gamma]$ be the polynomial algebra generated by all trees of the form $V(t) = | \vee t$. Since each tree $t \in \gamma$ can be uniquely decomposed as $t = t_l/V(t_r)$, the map $V(t) \mapsto V(t)$ and $1 \mapsto |$ is an algebra isomorphism from $\mathcal{H}^\gamma$ to the abelianization of $(\mathbb{C}Y,/)$. Under the inverse of this isomorphism, the natural homogeneous component $\mathbb{C}Y_n$ of degree $n$ in $\mathbb{C}Y$ corresponds to the subspace $\mathcal{H}_n^\gamma = \bigoplus_{n_1 \leq \cdots \leq n_k} \mathbb{C}V(Y_{n_1}) \otimes \cdots \otimes \mathbb{C}V(Y_{n_k})$ of total degree $n = n_1 + \cdots + n_k + k$ in $\mathcal{H}^\gamma$.

From now on, we identify $\mathcal{H}^\gamma$ with $(\mathbb{C}Y,/)_{ab}$, and represent the unit 1 as the root tree $|$. 

4.2 The charge Hopf algebra. Define a coproduct $\Delta^\gamma : \mathcal{H}^\gamma \rightarrow \mathcal{H}^\gamma \otimes \mathcal{H}^\gamma$ and a coaction $\delta : \mathcal{H}^\gamma \rightarrow \mathcal{H}^\gamma \otimes \mathcal{H}^\gamma$ as the two linear operators satisfying the following recursive relations:

$$
\Delta^\gamma | = | \otimes |,
\Delta^\gamma V(t) = | \otimes V(t) + \delta V(t),
\Delta^\gamma (t \vee s) = \Delta^\gamma t/\Delta^\gamma (V(s));
$$

and

$$
\delta | = | \otimes |,
\delta V(t) = (V \otimes \text{Id})\delta(t),
\delta (t \vee s) = \Delta^\gamma t/\delta (V(s)).
$$

For instance, the coproduct on small generator trees yields

$$
\Delta^\gamma \gamma = \gamma \otimes | + | \otimes \gamma,
\Delta^\gamma \gamma^2 = \gamma^2 \otimes | + | \otimes \gamma^2,
\Delta^\gamma \gamma^3 = \gamma^3 \otimes | + \gamma^2 \otimes \gamma + | \otimes \gamma^3,
\Delta^\gamma \gamma^4 = \gamma^4 \otimes | + \gamma^3 \otimes \gamma + \gamma^2 \otimes \gamma + | \otimes \gamma^4.
$$
Similarly, the coaction on small generator trees yields

\[
\begin{align*}
\delta \, \cdot &= \cdot \otimes \cdot, \\
\delta \, \cdot &= \cdot \otimes \cdot, \\
\delta \, \cdot &= \cdot \otimes \cdot + \cdot \otimes \cdot, \\
\delta \, \cdot &= \cdot \otimes \cdot.
\end{align*}
\]

Let \( \epsilon : H^\alpha \longrightarrow \mathbb{C} \) be the linear map which sends all the trees to 0 except the root \(|\) which is sent to 1.

4.3 Theorem. The algebra \( H^\alpha \) is a graded connected commutative Hopf algebra. Moreover, \( \delta \) is a right \( \Delta^\alpha \)-coaction, that is

\[
(\delta \otimes \text{Id})\delta = (\text{Id} \otimes \Delta^\alpha)\delta.
\]

Proof. We first observe that the coproduct preserves the grading of \( H^\alpha \), that is

\[
\Delta^\alpha(H^\alpha_n) \subset \bigoplus_{p+q=n} H^\alpha_p \otimes H^\alpha_q.
\]

Since \( H^\alpha_0 \) is spanned by a single tree \(|\), the graded algebra \( H^\alpha \) is connected.

By recursion arguments, it is then easy to see that the only terms of total degree \((n, 0)\) and \((0, n)\), in the image of \( \Delta^\alpha \), consist of the primitive part \( t \otimes 1 \) and \( 1 \otimes t \) for any tree \( t \). Then, the map \( \epsilon \) is a counit for \( \Delta^\alpha \), and the antipode \( S^\gamma : H^\alpha \longrightarrow H^\alpha \) is the graded algebra isomorphism automatically defined on the generators by the recursive formula

\[
S^\gamma(t) = -t - \sum_{\Delta^\alpha(t)} S^\gamma(t_{(1)})/t_{(2)},
\]

where \( \Delta^\alpha(t) = \Delta^\alpha(t) - t \otimes 1 - 1 \otimes t \) is the reduced coproduct.

First we prove by induction that the operator \( \delta \) defines a left \( \Delta^\alpha \)-coaction of \( H^\alpha \) on itself. It is true on \(|\). Suppose that it is true for all the trees up to order \( n \), and let \( V(t) \) has order \( n+1 \). Then

\[
(\delta \otimes \text{Id})\delta(V(t)) = (\delta \circ V \otimes \text{Id})\delta(t) = (V \otimes \text{Id} \otimes \text{Id}) (\delta \otimes \text{Id}) \delta(t) = (V \otimes \Delta^\alpha) \delta(t)
\]

Now let \( s \lor t = s/V(t) \) has order \( n+1 \), with \( s \neq |\). Then both \( s \) and \( V(t) \) have order smaller or equal to \( n \). Let us fix the Sweedler notations

\[
\Delta^\alpha(s) = \sum s_{(1)} \otimes s_{(2)}, \quad \delta(t) = \sum t_{(t) \otimes t_{(r)}}.
\]

On one side we have

\[
(\delta \otimes \text{Id})\delta(s/V(t)) = (\delta \otimes \text{Id})[\Delta^\alpha(s)/\delta V(t)] = (\delta \otimes \text{Id}) \sum_{\delta t, \Delta^\alpha s} s_{(1)}/V(t_{(1)}) \otimes s_{(2)}/t_{(r)}
\]

and on the other side we have

\[
(\text{Id} \otimes \Delta^\alpha)\delta(s/V(t)) = [(\text{Id} \otimes \Delta^\alpha)\Delta^\alpha(s)]/[[(\delta \otimes \text{Id})\delta V(t)]].
\]
so the equality holds by inductive hypothesis.

Now we prove by induction that the operator \( \Delta^\alpha \) is coassociative, that is \((\text{Id} \otimes \Delta^\alpha) \Delta^\alpha = (\Delta^\alpha \otimes \text{Id}) \Delta^\alpha\), using the fact that \( \delta \) is a coaction. Since \( \Delta^\alpha \) is multiplicative, we only need to prove it on the generators \( V(t) \). It is true on \( t = 1 \). Suppose that \( \Delta^\alpha \) is coassociative on all the trees with order up to \( n \), and let \( V(t) \) be a generator with order \( n + 1 \). Then by definition of \( \Delta^\alpha \) we have on one side

\[
(\text{Id} \otimes \Delta^\alpha)\Delta^\alpha V(t) = | \otimes \Delta^\alpha V(t) + (\text{Id} \otimes \Delta^\alpha)\delta(V(t))
= | \otimes | \otimes V(t) + | \otimes \delta(V(t)) + (\text{Id} \otimes \Delta^\alpha)\delta(V(t)),
\]

and on the other side

\[
(\Delta^\alpha \otimes \text{Id})\Delta^\alpha V(t) = \Delta^\alpha(1) \otimes V(t) + (\Delta^\alpha \otimes \text{Id})\delta(V(t))
= | \otimes | \otimes V(t) + (\Delta^\alpha \circ V \otimes \text{Id})\delta(t)
= | \otimes | \otimes V(t) + (\text{Id} \otimes V \otimes \text{Id})(| \otimes \delta(t)) + (\delta \circ V \otimes \text{Id})\delta(t)
= | \otimes | \otimes V(t) + | \otimes \delta(V(t)) + (\delta \circ \text{Id})(V \otimes \delta(t))
= | \otimes | \otimes V(t) + | \otimes \delta(V(t)) + (\delta \circ \text{Id})\delta(V(t)).
\]

Then, the two sides are equal because \((\text{Id} \otimes \Delta^\alpha)\delta(V(t)) = (\delta \circ \text{Id})\delta(V(t))\). \(\square\)

4.5 The non-commutative charge Hopf algebra. Let \( \widetilde{\mathcal{H}}^\alpha := \mathbb{C}[V(t), t \in \mathbf{Y}] \) be the algebra of non-commutative polynomials on the trees of the form \( V(t) \). Then the charge algebra \( \mathcal{H}^\alpha \) is the abelian quotient of \( \widetilde{\mathcal{H}}^\alpha \). Moreover, the isomorphism \( \mathcal{H}^\alpha \cong (\mathbf{CY},/)\) of \( \mathbf{H} \) can be lifted to an isomorphism \( \widetilde{\mathcal{H}}^\alpha \cong (\mathbf{CY},/) \). Therefore, the formulas employed in \( \mathbf{H} \) to define a coproduct \( \Delta^\alpha \) and a coaction \( \delta \) on \( \mathcal{H}^\alpha \) can be adopted to define some lifted maps \( \Delta^\alpha \) and \( \delta \) from \( \mathcal{H}^\alpha \) to \( \mathcal{H}^\alpha \otimes \mathcal{H}^\alpha \). These lifted maps are defined as the original ones on the generators, and no ambiguity comes from a product of generator trees if we require \( \Delta^\alpha \) and \( \delta \) to be algebra morphisms.

4.6 Theorem. The algebra \( \widetilde{\mathcal{H}}^\alpha \) is a graded connected Hopf algebra, which is neither commutative nor cocommutative.

Proof. We can repeat the proof of \( \mathbf{4.3} \), since we never used the commutativity of the product in \( \mathcal{H}^\alpha \). \(\square\)

5 QED Hopf algebra and coactions on trees

5.1 The electron and photon coactions. Since \( \widetilde{\mathcal{H}}^\alpha \cong \mathbf{CY} \) as a vector space, the coaction \( \widetilde{\delta} \) on \( \widetilde{\mathcal{H}}^\alpha \) given in \( \mathbf{4.3} \) can be seen as a linear map \( \delta : \mathbf{CY} \rightarrow \mathbf{CY} \otimes \mathbf{CY} \). Since \( \mathbf{CY} \) is the set of generators of the algebras \( \mathcal{H}^\gamma \) and \( \mathcal{H}^e \), and \( \delta(|1|) = | \otimes | \) , we can extend \( \delta \) to two maps \( \delta^\gamma : \mathcal{H}^\gamma \rightarrow \mathcal{H}^\gamma \otimes \mathcal{H}^\alpha \) and \( \delta^e : \mathcal{H}^e \rightarrow \mathcal{H}^e \otimes \mathcal{H}^\alpha \) defined as \( \delta \) on the generators (single trees), extended multiplicatively on tensor products,

\[
\delta^\gamma(t_1 \cdots t_n) = \delta^\alpha(t_1 \cdots t_n) := \widetilde{\delta}(t_1) \cdots \widetilde{\delta}(t_n),
\]

and finally passed to the quotient \( \mathcal{H}^\alpha \rightarrow \mathcal{H}^\gamma \). Explicitly, \( \delta^\gamma \) and \( \delta^e \) can be recursively defined as

\[
\delta^\gamma(t \vee s) = \sum_{\Delta^\alpha t, \delta^\gamma s} t(1) \vee s(\gamma) \otimes t(2)/s(\alpha), \tag{5.2}
\]

\[
\delta^e(t \vee s) = \sum_{\Delta^\alpha t, \delta^e s} t(1) \vee s(e) \otimes t(2)/s(\alpha), \tag{5.3}
\]

where we use the Sweedler notations

\[
\delta^\gamma s = \sum s(\gamma) \otimes s(\alpha), \quad \delta^e s = \sum s(e) \otimes s(\alpha).
\]
5.4 Lemma. The maps $\delta^\gamma$ and $\delta^e$ are right $\Delta^\alpha$-coactions, i.e. they satisfy (2.3), and they commute respectively with $\Delta^\alpha_s$ and $\Delta^\alpha_t$, i.e. they satisfy (2.4).

Proof. Since the proof is exactly the same in the two cases, we do it explicitly only for $\delta^e$.

The map $\delta^e$ is a right $\Delta^\alpha$-coaction, because we already proved that the identity

$$(\delta^e \otimes \text{Id})\delta^e = (\text{Id} \otimes \Delta^\alpha)\delta^e$$

holds on single trees, and on a product $t_1 \cdots t_n$, it follows from the fact that

$$(\delta^e \otimes \text{Id})\delta^e(t_1 \cdots t_n) = [(\delta^e \otimes \text{Id})\delta^e(t_1)] \cdots [(\delta^e \otimes \text{Id})\delta^e(t_n)]$$

and similarly

$$(\text{Id} \otimes \Delta^\alpha)\delta^e(t_1 \cdots t_n) = [(\text{Id} \otimes \Delta^\alpha)\delta^e(t_1)] \cdots [(\text{Id} \otimes \Delta^\alpha)\delta^e(t_n)].$$

Let us prove that $\delta^e$ commutes with $\Delta^\alpha_p$, i.e. that

$$(\Delta^\alpha_p \otimes \text{Id})\delta^e = m^3_{24}(\delta^e \otimes \delta^e)\Delta^\alpha_p,$$

where $m^3_{24}$ is the commutative multiplication in $H^\alpha$ with the notations of (2.4).

On single trees, we prove it by induction. It is true for the root tree $1$, so let us suppose that the equality holds for all trees up to order $n$, and let $t \vee s$ has order $n + 1$. Then, on the left hand side we have

$$(\Delta^\alpha_p \otimes \text{Id})\delta^e(t \vee s) = \sum_{\Delta^\alpha t, \delta^e s} \Delta^\alpha_p(t(1) \vee s(c)) \otimes t(2)/s(\alpha)$$

$$= \sum_{\Delta^\alpha t, \delta^e s} | \otimes t(1) \vee s(c) \otimes t(2)/s(\alpha) + \sum_{\Delta^\alpha t, \delta^e s} t(1) \vee s(c) \otimes s(e(2) \otimes t(2))/s(\alpha),$$

while on the right hand side we have

$$m^3_{24}(\delta^e \otimes \delta^e)\Delta^\alpha_p(t \vee s) = m^3_{24}(\delta^e \otimes \delta^e) \left[ | \otimes t \vee s + \sum_{\Delta^\alpha t} t \vee s(1) \otimes s(2) \right]$$

$$= m^3_{24} \left( \sum_{\Delta^\alpha t, \delta^e s} | \otimes t(1) \vee s(1) \otimes t(2)/s(2) + \sum_{\Delta^\alpha s} \delta^e(t \vee s(1)) \otimes \delta^e(s(2)) \right)$$

$$= \sum_{\Delta^\alpha t, \delta^e s} | \otimes t(1) \vee s(c) \otimes t(2)/s(\alpha) + \sum_{\Delta^\alpha t, \delta^e s} t(1) \vee s(1c) \otimes s(1c) \otimes t(2)/s(1c)/s(2a).$$

Then the two sides coincide, because for the tree $s$ we know that

$$\sum_{\delta^e s} s(c(1)) \otimes s(c(2)) \otimes s(\alpha) = \sum_{\Delta^\alpha s, \delta^e s(1c), \delta^e s(2a)} s(1c) \otimes s(2c) \otimes s(1c)/s(2a).$$

Finally, we prove that the equality hold on a tensor product $ts \in H^e$. On one side we have

$$(\Delta^\alpha_p \otimes \text{Id})\delta^e(ts) = (\Delta^\alpha_p \otimes \text{Id}) \left[ \sum_{\delta^e t, \delta^e s} t(c) \otimes t(\alpha)/s(\alpha) \right]$$

$$= \sum_{\delta^e t, \delta^e s} t(c) \otimes t(c) s(c) \otimes t(\alpha)/s(\alpha)$$

$$= [(\Delta^\alpha_p \otimes \text{Id})\delta^e(t)] \cdot [(\Delta^\alpha_p \otimes \text{Id})\delta^e(s)].$$
On the other side we have

\[ m_{24}^3(\delta^e \otimes \delta^e)\Delta_P^e(ts) = m_{24}^3(\delta^e \otimes \delta^e) \left[ \sum_{\Delta^e t, \Delta^e s} t(1)s(1) \otimes t(2)s(2) \right] \]

\[ = m_{24}^3 \left[ \sum_{\delta^e t(1), \delta^e s(1), \delta^e t(2), \delta^e s(2)} t(1)s(1)c / t(1a)s(1a)c / t(2a)s(2a)c / s(2a) \right] \]

which is equal to

\[ [m_{24}^3(\delta^e \otimes \delta^e)\Delta_P^e(t)] \ [m_{24}^3(\delta^e \otimes \delta^e)\Delta_P^e(s)] \]

because / is commutative in \( \mathcal{H}^a \). Then the two sides coincide by inductive hypothesis. \( \square \)

5.5 The QED Hopf algebra. By Mohan’s result of [23], the smash coproduct \( \mathcal{H}^\text{QED} := \mathcal{H}^\alpha \ltimes \mathcal{H}^e \), as defined in [24], is then a graded connected Hopf algebra, which is neither commutative nor cocommutative. The grading is given by the sum of the orders of all the trees appearing in a monomial.

The coproduct \( \Delta^\text{QED} : \mathcal{H}^\text{QED} \rightarrow \mathcal{H}^\text{QED} \otimes \mathcal{H}^\text{QED} \) is explicitly given by

\[ \Delta^\text{QED}(t \otimes s_1 \ldots s_n) := \Delta^a(t) \ [(\delta^e \otimes \text{Id})\Delta_P^e(s_1 \ldots s_n)]. \]

5.6 The electron renormalization coaction. As in (2.4), we can then define a coaction of \( \mathcal{H}^\text{QED} \) on \( \mathcal{H}^e \), as the map \( \Delta^e : \mathcal{H}^e \rightarrow \mathcal{H}^e \otimes \mathcal{H}^\text{QED} \) given by

\[ \Delta^e(s_1 \ldots s_n) := (\delta^e \otimes \text{Id})\Delta_P^e(s_1 \ldots s_n). \]

For instance,

\[ \Delta^e | = | \otimes | \otimes | \]
\[ \Delta^e \gamma = \gamma \otimes | \otimes | + | \otimes | \otimes \gamma \]
\[ \Delta^e \gamma \gamma = \gamma \otimes | \otimes | + \gamma \otimes \gamma \otimes | + | \otimes | \otimes \gamma \]
\[ \Delta^e \gamma = \gamma \otimes | \otimes | + | \otimes | \otimes \gamma \]
\[ \Delta^e \gamma = \gamma \otimes | \otimes | + \gamma \otimes \gamma \otimes | + | \otimes | \otimes \gamma \]
\[ \Delta^e \gamma = \gamma \otimes | \otimes | + \gamma \otimes \gamma \otimes | + | \otimes | \otimes \gamma \]
\[ \Delta^e \gamma = \gamma \otimes | \otimes | + \gamma \otimes \gamma \otimes | + | \otimes | \otimes \gamma \]

5.7 Lemma. The coaction \( \Delta^e \) of \( \mathcal{H}^\text{QED} \) on \( \mathcal{H}^e \) can be defined recursively as

\[ \Delta^e | = | \otimes | \otimes |, \]
\[ \Delta^e(t \forall s) = | \otimes | \otimes t \forall s + \sum_{\Delta^e(t), \Delta^e(s)} t(1) \forall s(1) \otimes t(2)/s(2) \otimes s(3), \]
\[ \Delta^e(s_1 \ldots s_n) = \Delta^e(s_1) \cdots \Delta^e(s_n), \]

where we adopt a Sweedler’s notation \( \Delta^e s = \sum s(1) \otimes s(2) \otimes s(3) \).
Proof. Since the maps $\Delta^e$ and $\delta^e$ are algebra morphisms, we only need to show it on the generators. We show it by induction on the order of trees. It is true for the root tree $| \cdot |$. Suppose that the equality holds for all trees up to order $n$, and let $t \vee s$ have order $n + 1$. Then, in particular for $s$, we know that

$$\Delta^e s = \sum s_{(1)} \otimes s_{(2)} \otimes s_{(3)} = \sum_{\Delta^p(s) \sigma(s_{(2)})} s_{(1\alpha)} \otimes s_{(1\alpha)} \otimes s_{(2)}.$$ 

So, applying the definition of $\Delta^e$ on $t \vee s$ and using the recursive definition (3.3) for $\Delta^p$ and (5.2) for $\delta^e$, we have

$$\Delta^e (t \vee s) = (\delta^e \otimes \text{Id}) \Delta^p (t \vee s) = | \otimes | \otimes t \vee s + \sum_{\Delta^p s} t_{(1)} \vee s_{(1)} \otimes t_{(2)}/s_{(1\alpha)} \otimes s_{(2)}$$

$$= | \otimes | \otimes t \vee s + \sum_{\Delta^o(s) \delta^e(s_{(2)})} t_{(1)} \vee s_{(1)} \otimes t_{(2)}/s_{(2)} \otimes s_{(3)}.$$ 

\[\square\]

5.8 The photon renormalization coaction. Exactly as in (5.3), the semi-direct coproduct $H^o \ltimes H^\gamma$ is a graded connected Hopf algebra, with twisted coproduct

$$t \otimes s_1 \ldots s_n \mapsto \Delta^o(t) \cdot \sum_{\Delta^p(s) \delta^o(s_{(2)})} s_{(1\alpha)} \otimes s_{(1\alpha)} \otimes s_{(2)},$$

which coacts on $H^\gamma$ from the right, with coaction given by the restriction of the coproduct to the subspace $H^o$, as in (2.4) and (5.6).

However it is not the semidirect coproduct $H^o \ltimes H^\gamma$ which describes the renormalization of the photon propagators. As we sketched in (2.1), the photon renormalization Hopf algebra is the charge algebra $H^o$, and the coaction is a semidirect coproduct coaction induced by a 1-cocycle, as in (2.4).

Let $\sigma : H^\gamma \longrightarrow H^o$ be the algebra morphism defined by

$$\sigma(t_1 \ldots t_n) := t_1/\ldots/t_n.$$

Then define $\Delta^\gamma : H^\gamma \longrightarrow H^o \otimes H^o$ as the map

$$\Delta^\gamma := m_{23}^3(\delta^\gamma \otimes \sigma) \Delta^p.$$

Since $\sigma$ is an algebra morphism, $\Delta^\gamma$ is also an algebra morphism.

5.9 Lemma. The map $\Delta^\gamma$ is a right coaction of $H^o$ on $H^\gamma$, i.e. it is coassociative with respect to $\Delta^o$.

Proof. By lemma (2.8), it is sufficient to show that the map $\sigma$ interwines $\Delta^o$ and $\Delta^\gamma$, i.e. for any $t_1 \ldots t_n \in H^\gamma$ we have

$$\Delta^o \sigma(t_1 \ldots t_n) = (\sigma \otimes \text{Id}) \Delta^\gamma(t_1 \ldots t_n).$$

If $n > 1$, the result follows from the fact that all the maps are algebra morphisms. So we only need to check it on a single tree $t$, for which $\sigma(t) = t$. We prove it by induction on the order of the trees. The equality holds for $| \cdot |$, suppose that it holds for a tree $t$, i.e. that

$$\sum_{\Delta^o t} t_{(1)} \otimes t_{(2)} = \sum_{\Delta^p t, \delta^o t_{(1)}} \sigma(t_{(1\gamma)}) \otimes t_{(2)}/t_{(1\alpha)} = \sum_{\Delta^o t, \delta^o t_{(1)}} t_{(1\gamma)} \otimes t_{(2)}/t_{(1\alpha)}.$$ 

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Then for a tree $t \vee s$ with larger order we have:

\[
(\sigma \otimes \text{Id})\Delta^\gamma(t \vee s) = (\sigma \otimes \text{Id})m_{23}^3(\delta^\gamma \otimes \text{Id})\Delta^\gamma_3(t \vee s)
\]

\[
= (\sigma \otimes \text{Id})m_{23}^3 \left[ \delta^\gamma(t \vee s) \otimes 1 + \sum_{\Delta^\gamma_{t,t}^\gamma=1} \delta^\gamma(t_{(1)}) \otimes t_{(2)} \vee s \right]
\]

\[
= (1) \delta^\gamma(t \vee s) + \sum_{\Delta^\gamma_{t,t}^\gamma=1} \sigma(t_{(1)}) \otimes t_{(2)} / (t_{(2)} \vee s)
\]

\[
= \delta^\gamma(t \vee s) + \sum_{\Delta^\gamma_{t,t}^\gamma=1} t_{(1)} \otimes t_{(2)} / V(s) = \Delta^\alpha(t \vee s),
\]

where the equality (1) holds because $\delta^\gamma$ applied to a single tree produces only single tree components on the left hand side, hence $(\sigma \otimes \text{Id})\delta^\gamma(t \vee s) = \delta^\gamma(t \vee s)$. □

Remark that the coaction $\Delta^\gamma$ applied to a single tree also produces only single tree components on the left hand side, hence $(\sigma \otimes \text{Id})\Delta^\gamma(t) = \Delta^\gamma(t)$ for any $t \in CY$. In conclusion, we obtain a very simple expression for the photon renormalization coaction on single trees.

5.10 Corollary. The photon renormalization coaction $\Delta^\gamma$ restricted to the subspace of single trees coincides with the non-commutative charge coproduct $\tilde{\Delta}^\alpha$,

\[
\Delta^\gamma(t) = \tilde{\Delta}^\alpha(t), \quad \text{for any } t \in Y.
\]

Examples of $\Delta^\gamma(t)$ for small order trees can then be constructed directly from (4.2):

\[
\begin{align*}
\Delta^\gamma 1 &= 1 \otimes 1 \\
\Delta^\gamma Y &= Y \otimes 1 + 1 \otimes Y, \\
\Delta^\gamma YY &= YY \otimes 1 + 2 Y \otimes Y + 1 \otimes YY, \\
\Delta^\gamma YY &= Y \otimes 1 + 1 \otimes YY, \\
\Delta^\gamma YY &= YY \otimes 1 + 3 YY \otimes Y + 3 Y \otimes YY + 1 \otimes YY, \\
\Delta^\gamma YY &= YY \otimes 1 + YY \otimes Y + YY \otimes YY + 1 \otimes YY, \\
\Delta^\gamma YY &= YY \otimes 1 + YY \otimes Y + YY \otimes YY + 1 \otimes YY, \\
\Delta^\gamma YY &= YY \otimes 1 + YY \otimes Y + YY \otimes YY + 1 \otimes YY, \\
\Delta^\gamma YY &= YY \otimes 1 + YY \otimes Y + YY \otimes YY + 1 \otimes YY.
\end{align*}
\]

6 Renormalization of tree-expanded QED propagators

Let $\alpha_0$ be the bare fine structure constant (before renormalization). For each momentum vector $q \in C^4$, let $D(\alpha_0; q)$ and $S(\alpha_0; q)$ denote the bare Feynman propagators for the photon and electron fields, as considered in [3]. Following [4], consider the tree-expansions

\[
\begin{align*}
D(\alpha_0; q) &= \sum_{t \in Y} U^\gamma_q(t)\alpha_0^{|t|}, \\
S(\alpha_0; q) &= \sum_{t \in Y} U^\gamma_q(t)\alpha_0^{|t|},
\end{align*}
\]

Remark that $D(\alpha_0; q)$ and $S(\alpha_0; q)$ differ from the usual QED propagators respectively by a factor $D_0(q)^{-1}$ and $S_0(q)^{-1}$, which are the inverse of the free propagators.
that is, the expansions of these propagators as power series of \( \alpha_0 \) with coefficients labeled by planar binary trees. For single particles, the coefficients \( U^q_0(\bigotimes) \) and \( U^c_0(\bigotimes) \) of the root tree represent the free propagators, which, by assumption, are the identity \( 4 \times 4 \) matrix \( I \). For higher order trees, the coefficients \( U^q_0(\bigotimes) \) and \( U^c_0(\bigotimes) \) can be explicitly determined as Feynman amplitudes, since each tree is a finite sum of appropriate Feynman diagrams, cf. [3]. In alternative, they can be determined recursively, as showed in [1], starting from the coefficients of the smaller trees \( t' \) and \( t'' \) such that \( t' \cup t'' = t \). In conclusion, the tree-expansions (6.1,6.2) allow to consider the QED propagators with two algebra morphisms \( U^q_0 \) and \( U^c_0 \), on \( \mathcal{H}^q \) and \( \mathcal{H}^c \) respectively, such that

\[
\langle t, D(\alpha_0; q) \rangle = \langle U^q_0, t \rangle = U^q_0(t),
\]

\[
\langle t, S(\alpha_0; q) \rangle = \langle U^c_0, t \rangle = U^c_0(t).
\]

Moreover, in [3] it was shown that the product of propagators is dual to the pruning coproducts, that is

\[
\langle t, D(\alpha_0; q)D(\alpha_0; q) \rangle = \langle U^q_0 \otimes U^q_0, \Delta^q_0(t) \rangle,
\]

\[
\langle t, S(\alpha_0; q)S(\alpha_0; q) \rangle = \langle U^c_0 \otimes U^c_0, \Delta^c_0(t) \rangle.
\]

Furthermore, let \( \alpha \) be the renormalized fine structure constant, and let \( \bar{D}(\alpha; q) \) and \( \bar{S}(\alpha; q) \) denote the massless renormalized propagators as in [4]. Again, consider the tree-expansions

\[
\bar{D}(\alpha; q) = \sum_{t \in Y} R^q_0(t)\alpha_0^{\left| t \right|}
\]

\[
\bar{S}(\alpha; q) = \sum_{t \in Y} R^c_0(t)\alpha_0^{\left| t \right|}
\]

as power series on \( \alpha_0 \), also starting with the unperturbed coefficient given by \( I \). As before, these expansions determine two algebra morphisms \( R^q_0 \) and \( R^c_0 \) on \( \mathcal{H}^q \) and \( \mathcal{H}^c \) respectively. The aim of renormalization theory is to find their values \( R^q_0(t) \) and \( R^c_0(t) \) on all the trees. In [4], we gave some recursive solutions with respect to the order of the trees. Here we recall how the relationship between all these coefficients can be given in terms of the Hopf algebras and coactions on trees defined in the previous sections.

Let \( Z_3(\alpha) \) and \( Z_2(\alpha) \) denote the renormalization factors for the photon and the electron propagators. They satisfy the Dyson formulas

\[
\bar{D}(\alpha; q)Z_3(\alpha) = D(\alpha_0; q)
\]

\[
\bar{S}(\alpha; q)Z_2(\alpha) = S(\alpha_0; q)
\]

and the charge renormalization formula proved by Ward

\[
\alpha_0(\alpha) = \alpha Z_3(\alpha)^{-1}.
\]

As explained in [2], [3], trees represent sums of Feynman diagrams. It is well known that the renormalization factors are expanded only over 1PI Feynman graphs, and this property corresponds to the following expansions over trees:

\[
Z_3(\alpha) = 1 - \sum_{t \in Y} C^q(V(t))\alpha_0^{\left| t \right| + 1},
\]

\[
Z_2(\alpha) = 1 + \sum_{t \neq \bigotimes} C^c(S^p(t))\alpha_0^{\left| t \right|},
\]

where \( V(t) \) are the generators of the algebra \( \mathcal{H}^q \) and \( S^p(t) \) are the elements of the algebra \( \mathcal{H}^c \) transformed under the right pruning antipode defined in (3.6). Once again, these expansions determine two algebra

\[4\] Because \( U^q_0(\bigotimes) = D_0(q)D_0(q)^{-1} = I \) and similarly \( U^c_0(\bigotimes) = S_0(q)S_0(q)^{-1} = I \).

\[5\] The recursive solutions given in [3] are valid in massive renormalization.
morphisms \( C^\gamma \) and \( C^e \) on \( \mathcal{H}^\gamma \) and \( \mathcal{H}^e \) respectively. In the present case they are both scalars, but the formalism based on planar binary trees allows to consider also non-scalar maps. Moreover, the Ward formula for the fine structure constant tells us that the map \( C^\gamma \) is also an algebra morphism on \( \mathcal{H}^\alpha \).

We finally state a results of [2], which shows that the coproduct on trees previously defined encodes the relationship between the amplitudes, before and after the renormalization.

6.5 Theorem. The relation between the coefficients of the expansions (6.1) and (6.3) for the bare and the renormalized photon propagators is

\[
R^\gamma_q(t) = \langle U^\gamma \otimes C^\gamma, \Delta^\gamma(t) \rangle = \sum_{\Delta^\gamma(t)} U^\gamma(t_{(1)}) C^\gamma(t_{(2)}).
\]

The relation between the coefficients of the expansions (6.2) and (6.4) for the bare and the renormalized electron propagators is

\[
R^e_q(t) = \langle \Delta^e \otimes C^\gamma \otimes C^e, \Delta^e(t) \rangle = \sum_{\Delta^e(t)} U^e(t_{(1)}; q) C^\gamma(t_{(2)}) C^e(S^p_t(3)).
\]

It remains to show that the Hopf algebra \( \mathcal{H}^\alpha \) describes the renormalization of the fine structure constant \( \alpha \), i.e. that

\[
\langle (\alpha_1 \circ \alpha_2)(\alpha), t \rangle = \langle \alpha_1 \otimes \alpha_2, \Delta^\alpha t \rangle,
\]

if \( \alpha_2(\alpha) \) and \( \alpha_1(\alpha_2) \) are two successive renormalizations. This is the topic of the paper [3] in preparation, where we define the natural group of composition of series expanded over trees.

7 Conclusions

In [4], Connes and Kreimer relate the renormalization group of the \( \Phi^3 \) theory, based on Feynman graphs, to the group of formal diffeomorphisms on the complex line, and to the Birkhoff decomposition of holomorphic line bundles on the circle. In their case, since the renormalization Hopf algebra is commutative, the Milnor-Moore theorem allows to study the dual renormalization group even without knowing it explicitly. In our case, since the renormalization Hopf algebra of QED propagators is neither commutative nor cocommutative, we need first to introduce its natural dual group, which \textit{a priori} does not necessarily exist. This is the topic of the paper [4] in preparation.

However, the QED Hopf algebra on trees can be directly related to the Hopf algebra dual to the group of formal diffeomorphisms. This comparison can be done simply by summing up all the trees at a given order \( n \), which corresponds to the order \( n \) of interaction for the particle Green functions. The result is a non-commutative version of the Hopf algebra of formal diffeomorphisms, described in [4].

Finally, all these results suggest a natural question which has no answer yet: can the renormalization group of perturbative quantum field theories be realized as an “automorphism group” on some space?

References

[1] Ch. Brouder. \textit{On the trees of quantum fields}, Eur. Phy. J. C, 12 (2000), 535-549.

[2] Ch. Brouder and A. Frabetti. \textit{Renormalization of QED with planar binary trees}, Eur. Phys. J. C 19 (2001), 715-741.

[3] Ch. Brouder and A. Frabetti. \textit{Noncommutative renormalization of massless QED}, hep-th/0011161.

[4] Ch. Brouder and A. Frabetti. \textit{Noncommutative Hopf algebra of formal diffeomorphisms}, in preparation.

[5] Ch. Brouder and A. Frabetti. \textit{Groups of tree-expanded series}, in preparation.

[6] J.C. Butcher. \textit{An algebraic theory of integration methods}, Math. Comput., 26 (1972), 79-106.
[7] A. Connes and D. Kreimer. *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys. 199 n.1 (1998), 203-242.

[8] A. Connes and D. Kreimer. *Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys. 210 n.1 (2000), 249-273.

[9] A. Connes and D. Kreimer. *Renormalization in quantum field theory and the Riemann-Hilbert problem II: the β function, diffeomorphisms and the renormalization group*, Comm. Math. Phys. 216 n.1 (2001), 215-241.

[10] L. Foissy. *Les algèbres de Hopf des arbres enracinés décorés*, Thèse Univ. Reims (2001), [math.QA/0105212](https://arxiv.org/abs/math.QA/0105212).

[11] A. Frabetti. *Simplicial properties of the set of planar binary trees*, J. Alg. Comb. 13 (2001), 41-65.

[12] R. Grossman and R.G. Larson. *Hopf algebraic structure of families of trees*, J. Algebra 126 n.1 (1989), 184-210.

[13] R. Holtkamp. *Comparison of Hopf algebras on trees*, Preprint (2001).

[14] D. Kreimer. *On the Hopf algebra structure of perturbative quantum field theory*, Adv. Th. Math. Phys. 2 (1998), 303-334.

[15] C. Itzykson and J.-B. Zuber. *Quantum Field Theory*, McGraw-Hill, New York, 1980.

[16] B. Lin. *Crossed coproducts of Hopf algebras*, Comm. Alg. 10 n.1 (1982), 1-17.

[17] J.-L. Loday. *Arithmetree*, Preprint [math.CO/0112034](https://arxiv.org/abs/math.CO/0112034).

[18] J.-L. Loday. *Algèbres ayant deux opérations associatives (digèbres)*, C. R. Acad. Sci. Paris 321 (1995), 141-146.

[19] J.-L. Loday and M.O. Ronco. *Hopf algebra of the planar binary trees*, Adv. Math. 139 (1998), 293-309.

[20] J.-L. Loday and M.O. Ronco. *Order structure and the algebra of permutations and planar binary trees*, J. Alg. Comb., to appear.

[21] S. Majid. *Physics for algebraists: noncommutative and noncocommutative Hopf algebras by a bicrossedproduct*, J. Alg. 130 n.1 (1990), 17-64.

[22] I. Moerdijk. *On the Connes-Kreimer construction of Hopf algebras*, Cont. Math. 271 (2001), 311-321.

[23] R.K. Molnar. *Semi-direct products of Hopf algebras*, J. Alg. 47 (1977), 29-51.

[24] F. Panaite. Relating the Connes-Kreimer and the Grassman-Larson Hopf algebras on rooted trees, Lett. Math. Phys. 51 n.3 (2000), 211-219.

[25] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory*, Perseus Books Pub. L.L.C., 1995.

[26] D. Radford. *The structure of Hopf algebras with a projection*, J. Alg. 92 n.2 (1985), 322-347.

[27] J.C. Ward. *An identity in quantum electrodynamics*, Phys. Rev. 78 (1950), 182.