Preinjective subfactors of preinjective Kronecker modules

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Abstract

Using a representation theoretical approach we give an explicit numerical characterization in terms of Kronecker invariants of the subfactor relation between two preinjective (and dually preprojective) Kronecker modules, describing explicitly a so called linking module as well. Preinjective (preprojective) Kronecker modules correspond to matrix pencils having only minimal indices for columns (respectively for rows). Thus our result gives a solution to the subpencil problem in these cases (including the completion), moreover the required computations are straightforward and can be carried out easily (both for checking the subpencil relation and constructing the completion pencils based on the linking module). We showcase our method by carrying out the computations on an explicit example.

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1 Introduction

Let $K : \begin{array}{c} \alpha \\ \beta \end{array}$ be the Kronecker quiver, i.e. the quiver having two vertices and two parallel arrows and $k$ an arbitrary field. The path algebra of the Kronecker quiver is the Kronecker algebra and we will denote it by $kK$. A finite dimensional right module over the Kronecker algebra is called a Kronecker module. We denote by mod-$kK$ the category of finite dimensional right modules over the Kronecker algebra. For a module $M \in \text{mod-}kK$, $[M]$ will denote the isomorphism class of $M$ and $tM := M \oplus \cdots \oplus M$ ($t$-times).

A (finite dimensional) $k$-linear representation of the quiver $K$ is a quadruple $M = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$ where $V_1, V_2$ are finite dimensional $k$-vector spaces (corresponding to the vertices) and $\varphi_\alpha, \varphi_\beta : V_2 \rightarrow V_1$ are $k$-linear maps (corresponding to the arrows). Thus a $k$-linear representation of $K$ associates vector spaces to the vertices and compatible $k$-linear functions (or equivalently, matrices) to the arrows. A morphism between two representations $M = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$ and $M' = (V'_1, V'_2; \varphi'_\alpha, \varphi'_\beta)$ is a pair of linear maps $(f_1, f_2)$, where $f_1 : V_1 \rightarrow V'_1$, $f_2 : V_2 \rightarrow V'_2$ and $f_1 \varphi_\alpha = \varphi'_\alpha f_2$, $f_1 \varphi_\beta = \varphi'_\beta f_2$. Let us denote by $\text{rep-}kK$ the category of finite dimensional $k$-representations of the Kronecker quiver. There is a well-known equivalence of categories between mod-$kK$ and rep-$kK$, so that every Kronecker module can be identified with a representation of $K$.

The indecomposable Kronecker modules are of three types: preprojective, preinjective and regular (see the details in the next section). A Kronecker module is preinjective (preprojective) if all its indecomposable components are preinjective (preprojective).

It is easy to see that Kronecker modules are uniquely determined up to isomorphism by two sequences of nonnegative integers and some partitions (see the details in the next section). These numerical invariants are called the Kronecker invariants of the module.

Recall that a module $M'$ is a subfactor of $M$ if there exists a module $L$ with a monomorphism $L \rightarrow M$ and an epimorphism $L \rightarrow M'$ (or equivalently with an epimorphism $M \rightarrow L$ and a monomorphism $M' \rightarrow L$). We will call $L$ a linking module.
We have given in [19] a numerical criterion in terms of Kronecker invariants for the existence of a monomorphism between two preinjective Kronecker modules (and dually for the existence of an epimorphism between two preprojective modules). The criterion is very simple, it is in fact a weighted dominance relation between the invariants. Using this criterion one can also obtain the numerical description of an epimorphism between two preinjective Kronecker modules (and dually of a monomorphism between two preprojective modules) (see [20]). The approach used to obtain the results above is representation theoretical, the methods are homological combined with the knowledge on the category mod-\(kK\).

Note that a different criterion was given by Han Yang in [12], working over an algebraically closed field. He uses calculation of ranks of matrices over polynomial rings and a so called generalization and specialization approach. These matrices appear in the representations and in the morphisms between representations.

In this paper we give a simple explicit numerical characterization in terms of Kronecker invariants of the subfactor relation between two preinjective (and dually preprojective) Kronecker modules and also describe an explicit linking module.

Regarding the Kronecker modules as representations it is obvious that a Kronecker module corresponds to a pair of matrices of the same dimension, thus defining a matrix pencil. Some papers dealing with this connection are the following: [6, 5, 12, 18].

Recall that a matrix pencil over a field \(k\) is a matrix \(A + \lambda B\) where \(A, B\) are matrices over \(k\) of the same size and \(\lambda\) is an indeterminate. Two pencils \(A + \lambda B, A' + \lambda B'\) are strictly equivalent, denoted by \(A + \lambda B \sim A' + \lambda B'\), if and only if there exists invertible, constant (\(\lambda\) independent) matrices \(P, Q\) such that \(P(A' + \lambda B')Q = A + \lambda B\).

Every matrix pencil is strictly equivalent to a canonical diagonal form, described by the classical Kronecker invariants, namely the minimal indices for columns, the minimal indices for rows, the finite elementary divisors and the infinite elementary divisors (see [11] for all the details and [18] for a worked out example).

A pencil \(A' + \lambda B'\) is called subpencil of \(A + \lambda B\) if and only if there are pencils \(A_{12} + \lambda B_{12}, A_{21} + \lambda B_{21}, A_{22} + \lambda B_{22}\) such that
\[
A + \lambda B \sim \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix}.
\]

In this case we also say that the subpencil can be completed to the bigger pencil. We speak about row completion when \(A_{12}, B_{12}, A_{22}, B_{22}\) are zero matrices and about column completion when \(A_{21}, B_{21}, A_{22}, B_{22}\) are zero.

There is an unsolved challenge in pencil theory with lots of applications in control theory (problems related to pole placement, non-regular feedback, dynamic feedback etc. may be formulated in terms of matrix pencils, for details see [13]).

**Challenge:** If \(A + \lambda B, A' + \lambda B'\) are pencils over \(\mathbb{C}\), find a necessary and sufficient condition in terms of their classical Kronecker invariants for \(A' + \lambda B'\) to be a subpencil of \(A + \lambda B\). Also construct the completion pencils \(A_{12} + \lambda B_{12}, A_{21} + \lambda B_{21}, A_{22} + \lambda B_{22}\). A particular case of the challenge above is when we limit ourselves to column or row completions.

Han Yang was the first to give a representation theoretical modular approach to the matrix subpencil problem. He proved that the subpencil notion corresponds to the subfactor notion on modular level. Also the Kronecker invariants of a module correspond to the classical Kronecker invariants of the associated pencil.

Preinjective Kronecker modules correspond to matrix pencils having only minimal indices for columns (see details in Section 2). This means that our criterion for monomorphisms between preinjectives in the particular case when the factor is of the form \(tI_0\) (where \(I_0\) is the injective simple module) coincides with the criteria of Baragaña-Zaballa [3] and Mondič [14] (see also [4, 8] as a particular case) for completing by columns a pencil to another one, both of them having only minimal indices for columns. One should note that the existence of a monomorphism in general between two preinjective modules does not have a natural correspondence in pencil theory.

Having in mind all above one can see that our result on the subfactor relation between two preinjective Kronecker gives a solution to the subpencil problem in case both pencils have only minimal indices for columns. The numerical criterion is simple and the explicitly described linking module corresponds in fact to a pencil which is obtained from the smaller one by column completion, the bigger pencil being a row completion of the linking one. Using the linking pencil one can also easily construct the completion pencils. We showcase our result by carrying out the computations on an explicit example.

Dodig and Stosić describe in [10] a numerical criterion in terms of Kronecker invariants for a pencil having only minimal indices for columns to be a subpencil of a general one. Later Dodig gives in [9] a
2 The category of Kronecker modules

In this section we give some details on the category of Kronecker modules (regarded as representations). The interested reader should consult seminal works such as \[11, 17, 2, 16\] for further details, proofs and explanations.

For a module \( M \in \text{mod-}kK \). For two modules \( M, M' \in \text{mod-}kK \) will denote by \( M' \hookrightarrow M \) the fact that there is a monomorphism \( M' \to M \) and by \( M \to M' \) the fact that there is an epimorphism \( M \to M' \). Thus, \( M' \) is a subfactor of \( M \) if there exists a linking module \( L \) such that \( M' \hookrightarrow L \to M \) (or equivalently \( M' \leftarrow L \to M \)).

The simple Kronecker modules (up to isomorphism) are

\[ S_1 : k \cong 0 \quad \text{and} \quad S_2 : 0 \cong k. \]

For a Kronecker module \( M \) we denote by \( \dim M \) its dimension. The dimension of \( M \) is a vector \( \dim M = (\dim S_i(M), \dim S_i(M)) \), where \( \dim S_i(M) \) is the number of factors isomorphic with the simple module \( S_i \) in a composition series of \( M \), \( i = 1, 2 \). Regarded as a representation, \( M : V_1 \cong V_2 \), we have that

\[ \dim M = (\dim_k V_1, \dim_k V_2). \]

The defect of \( M \in \text{mod-}kK \) with \( \dim M = (a, b) \) is defined in the Kronecker case as \( \partial M = b - a \).

An indecomposable module \( M \in \text{mod-}kK \) is a member in one of the following three families: preprojectives, regulars and preinjectives. In what follows we give some details on these families.

The preprojective indecomposable Kronecker modules are determined up to isomorphism by their dimension vector. For \( n \in \mathbb{N} \) we will denote by \( P_n \) the indecomposable preprojective module of dimension \( (n + 1, n) \). So \( P_0 \) and \( P_1 \) are the projective indecomposable modules \( (P_0 = S_1 \text{ being simple}) \). It is known that (up to isomorphism) \( P_n = (k^{n+1}, k^n; f, g) \), where choosing the canonical basis in \( k^n \) and \( k^{n+1} \), the matrix of \( f : k^n \to k^{n+1} \) (respectively of \( g : k^n \to k^{n+1} \)) is \( \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \) (respectively \( \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix} \)). Thus in this case

\[ P_n : k^{n+1} \begin{pmatrix} I_n \\ 0 \end{pmatrix} k^n, \]

where \( I_n \) is the \( n \times n \) identity matrix. We have for the defect \( \partial P_n = -1 \).

We define a preprojective Kronecker module \( P \) as being a direct sum of indecomposable preprojective modules: \( P = P_{a_1} \oplus P_{a_2} \oplus \cdots \oplus P_{a_i} \), where we use the convention that \( a_1 \leq a_2 \leq \cdots \leq a_i \).

The preinjective indecomposable Kronecker modules are also determined up to isomorphism by their dimension vector. For \( n \in \mathbb{N} \) we will denote by \( I_n \) the indecomposable preinjective module of dimension \( (n, n + 1) \). So \( I_0 \) and \( I_1 \) are the injective indecomposable modules \( (P_0 = S_1 \text{ being simple}) \). It is known that (up to isomorphism) \( I_n = (k^n, k^{n+1}; f, g) \), where choosing the canonical basis in \( k^{n+1} \) and \( k^n \), the matrix of \( f : k^{n+1} \to k^n \) (respectively of \( g : k^{n+1} \to k^n \)) is \( \begin{pmatrix} 0 \\ I_n \end{pmatrix} \) (respectively \( \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \)). Thus in this case

\[ I_n : k^n \begin{pmatrix} 0 \\ I_n \end{pmatrix} k^{n+1}, \]

and we have for the defect \( \partial I_n = 1 \).

We define a preinjective Kronecker module \( I \) as being a direct sum of indecomposable preinjective modules: \( I = I_{a_1} \oplus I_{a_2} \oplus \cdots \oplus I_{a_i} \), where we use the convention that \( a_1 \geq a_2 \geq \cdots \geq a_i \).

The regular indecomposable Kronecker modules are those indecomposable modules \( M \in \text{mod-}kK \) which are neither preprojective nor preinjective. We describe here shortly only the case when the base field is algebraically closed. If \( k = \bar{k} \) is algebraically closed, then the regular indecomposables are

\[ R_\mu(n) = k^n \begin{pmatrix} j_{\mu, n} \\ I_n \end{pmatrix} k^n \quad \text{for } k \in \bar{k} \quad \text{and} \quad R_\infty(n) = k^n \begin{pmatrix} I_n \\ j_{\infty, n} \end{pmatrix} k^n, \]
where $J_{p,n}$ is the $n \times n$ Jordan block with eigenvalue $\mu$. The dimension of a regular indecomposable will be $\dim R_p(n) = (n, n)$ and we have for the defect $\partial R_p(n) = 0$, where $p \in \mathbb{k} \cup \{\infty\}$.

A module $R \in \text{mod-}kK$ will be called a regular Kronecker module if it is a direct sum of regular indecomposables. If $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ is a partition, then we use the notation $R_p(\mu) = R_p(\mu_1) \oplus R_p(\mu_2) \oplus \cdots \oplus R_p(\mu_m)$.

The category $\text{mod-}kK$ is a is a Krull-Schmidt category, meaning that every module $M \in \text{mod-}kK$ has a unique decomposition

$$M = (P_{c_1} \oplus \cdots \oplus P_{c_n}) \oplus (\oplus_{p \in \mathbb{P}} R_p(\mu(p))) \oplus (I_{d_1} \oplus \cdots \oplus I_{d_m}),$$

where

- $(c_1, \ldots, c_n)$ is a finite increasing sequence of non-negative integers;
- $\mu(p) = (\mu_1, \ldots, \mu_1)$ is a nonzero partition for finitely many $p \in \mathbb{k} \cup \{\infty\}$;
- $(d_1, \ldots, d_m)$ is a finite decreasing sequence of non-negative integers.

The integer sequences $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_m)$ together with the partitions $\mu(p)$ corresponding to every $p \in \mathbb{k} \cup \{\infty\}$ are called the Kronecker invariants of the module $M$. Hence Kronecker invariants determine a module $M \in \text{mod-}kK$ up to isomorphism.

The following lemmas are well-known:

**Lemma 1.** If there is a short exact sequence $0 \to M' \to M \to M'' \to 0$ of Kronecker modules, then $\dim M = \dim M' + \dim M''$ and $\partial M = \partial M' + \partial M''$.

**Lemma 2.** Preinjectives (respectively preprojectives) are extension closed. This means that in a short exact sequence of the form $0 \to Y \to M \to Y' \to 0$ with $Y, Y'$ preinjectives (preprojectives) $M$ must be also preinjective (preprojective).

The category of Kronecker modules has been extensively studied because the Kronecker algebra is a very important example of a tame hereditary algebra. Moreover, the category has also a geometric interpretation, since it is derived equivalent with the category $\text{Coh}(\mathbb{P}^1(\mathbb{k}))$ of coherent sheaves on the projective line (see [19]).

### 3 Morphisms and short exact sequences

In what follows we compile a few of our recent results on morphisms and short exact sequences of Kronecker modules required for the proofs in the next section. We emphasize that the theorems stated here are valid independently of the underlying field $k$ (as shown in [19]).

We present now the numerical criteria for the existence of a monomorphism $f : I' \to I$ where $I, I'$ are preinjectives. The proof relies on homological algebra and the knowledge on the category $\text{mod-}kK$.

**Theorem 3 ([19]).** Suppose $d_1 \geq \cdots \geq d_n > 0$ and $c_1 \geq \cdots \geq c_m > 0$ are integers. We have a monomorphism

$$f : I_{d_1} \oplus \cdots \oplus I_{d_n} \oplus I_0 \to I_{c_1} \oplus \cdots \oplus I_{c_m} \oplus cI_0$$

if and only if $d \leq c$ and $d_1 + \cdots + d_n \leq \sum_{c_j \leq d} c_j$ for $i = \frac{1}{n}$ (the empty sum being 0).

**Remark 4.** Using the notation $I' = (a_0I_0) \oplus \cdots \oplus (a_nI_n) \oplus \cdots$, $I = (b_0I_0) \oplus \cdots \oplus (b_nI_n) \oplus \cdots$ we have a monomorphism $f : I' \to I$ if and only if

$$a_0 \leq b_0$$
$$a_1 \leq b_1$$
$$a_1 + 2a_2 \leq b_1 + 2b_2$$
$$\cdots$$
$$a_1 + 2a_2 + \cdots + na_n \leq b_1 + 2b_2 + \cdots + nb_n$$

So one can see that in the preinjective case “a kind of” weighted dominance describes the numerical criteria for the embedding (it is well-known that dominance ordering plays a crucial role in partition combinatorics).
One should also note (using Lemma 1) that if $a_m = 0$ for all $m > n$, then we have an exact sequence of the form $0 \to I' \to I \to \beta I_0 \to 0$ (with $\beta$ arbitrary) if and only if $b_m = 0$ for all $m > n$ and

\[
\begin{align*}
  a_0 &\leq b_0 \\
  a_1 &\leq b_1 \\
  a_1 + 2a_2 &\leq b_1 + 2b_2 \\
  &\vdots \\
  a_1 + 2a_2 + \ldots + na_n &= b_1 + 2b_2 + \ldots + nb_n.
\end{align*}
\]

Remark 5. Theorem 3 can be easily dualized for preprojectives.

We are going to state some results on short exact sequences in terms of extension monoid products, so let us introduce this notion shortly. For $d \in \mathbb{N}^2$ let $M_d = \{[M]|M \in \text{mod-}kK, \text{dim } M = d\}$ be the set of isomorphism classes of Kronecker modules of dimension $d$. Following Reineke in [13], for subsets $A \subset M_d$, $B \subset M_e$, we define

\[
A * B = \{[Y] \in M_{d+e} | \exists 0 \to N \to Y \to M \to 0 \text{ exact for some } [M] \in A, [N] \in B\}.
\]

So the product $A * B$ is the set of isoclasses of all extensions of modules $M$ with $[M] \in A$ by modules $N$ with $[N] \in B$. This is in fact Reineke’s extension monoid product using isomorphism classes of modules instead of modules. It is important to know (see [15]) that the product above is associative, i.e. for $A \subset M_d$, $B \subset M_e$, $A \subset M_f$, we have $(A * B) * C = A * (B * C)$. Also $\{[0]\} * A = A * \{[0]\} = A$. We will call the operation “*” simply the extension monoid product.

Using a set of rules describing the extension monoid products of Kronecker modules in various cases we have proved in [20] the following property for the extension monoid product of a preinjective and a preprojective Kronecker module:

**Theorem 6.** Let $q > n > 0$, $d_1 \geq \cdots \geq d_q \geq 0$, $c_1 \geq \cdots \geq c_{q-n} \geq 0$ and $0 \leq a_1 \leq \cdots \leq a_n$ be non-negative integers. Then $[I_{c_1} \oplus \cdots \oplus I_{c_{q-n}}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[P_{a_1} \oplus \cdots \oplus P_{a_n}]\}$ if and only if $[I_{d_1+a_1+1} \oplus \cdots \oplus I_{d_q+a_n+1}] \in \{[I_{c_1-a_1} \oplus \cdots \oplus I_{c_{q-n}+a_n+1} \oplus I_0]\} * \{[I_{c_1-a_1+1} \oplus \cdots \oplus I_{c_{q-n}+a_n+1}]\}$, or equivalently there is a short exact sequence

\[
0 \to P_{a_1} \oplus \cdots \oplus P_{a_n} \to I_{c_1} \oplus \cdots \oplus I_{c_{q-n}} \to I_{d_1} \oplus \cdots \oplus I_{d_q} \to 0
\]

if and only if there is a short exact sequence

\[
0 \to I_{d_1+a_1+1} \oplus \cdots \oplus I_{d_q+a_n+1} \oplus I_{d_1+a_1+1} \oplus \cdots \oplus I_{d_q+a_n+1} \to I_{c_1-a_1} \oplus \cdots \oplus I_{c_{q-n}+a_n+1} \oplus I_0 \to 0.
\]

We will use in the proof of our main theorem the following corollary of Theorem 6 obtained by applying the theorem in the special case when kernel in the first short exact sequence is of the form $P_0 \oplus \cdots \oplus P_0$:

**Corollary 7.** Let $q > \alpha > 0$, $d_1 \geq \cdots \geq d_q \geq 0$ and $c_1 \geq \cdots \geq c_{q-n} \geq 0$ be non-negative integers. Then $[I_{c_1} \oplus \cdots \oplus I_{c_{q-n}}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[\alpha P_0]\}$ if and only if $[I_{d_1+1} \oplus \cdots \oplus I_{d_q+1}] \in \{[\alpha I_0]\} * \{[I_{c_1+1} \oplus \cdots \oplus I_{c_{q-n}+1}]\}$, or equivalently there is a short exact sequence

\[
0 \to \alpha P_0 \to I_{c_1} \oplus \cdots \oplus I_{c_{q-n}} \to I_{d_1} \oplus \cdots \oplus I_{d_q} \to 0
\]

if and only if there is a short exact sequence

\[
0 \to I_{c_1+1} \oplus \cdots \oplus I_{c_{q-n}+1} \to I_{d_1+1} \oplus \cdots \oplus I_{d_q+1} \to \alpha I_0 \to 0.
\]

In what follows we will work out how to construct a monomorphism (or an epimorphism) $f : I' \to I$ between two preinjective Kronecker modules, once we have determined using Theorem 3 that $I'$ embeds in $I$ or using Theorem 6 that $I'$ projects on $I$. With $d_1 \geq \cdots \geq d_n > 0$ and $c_1 \geq \cdots \geq c_m > 0$, let $I' = I_{d_1} \oplus \cdots \oplus I_{d_q} \oplus dI_0$ and $I = I_{c_1} \oplus \cdots \oplus I_{c_n} \oplus cI_0$. We also use $p = \sum_{i=1}^{m} c_i$, hence $\dim I' = (p, p + n + d)$ and $\dim I = (s, s + m + c)$.

Identifying the modules with their representations it is clear that we have a monomorphism (or an epimorphism) $f : I' \to I$ if and only if there exists a pair of full-rank matrices $(G, H)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
  k^p & \xrightarrow{A} & k^{p+n+d} \\
  G \downarrow & & \downarrow H \\
  k^s & \xrightarrow{B} & k^{s+m+c}
\end{array}
\]
On the diagram above we have the following matrices: \( A, A' \in M_{p,p+n+d}(k) \), \( B, B' \in M_{s,s+m+c}(k) \), \( G \in M_{s,p}(k) \), \( H \in M_{s+m+c,p+n+d}(k) \), where

\[
A = \begin{pmatrix} A_1 & \cdots & A_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} A'_1 & \cdots & A'_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}
\]

with \( A_i = (I_{d_i} \ 0) \), \( A_i' = (0 \ 1_{d_i}) \), \( i = 1, \ldots, n \)

\[
B = \begin{pmatrix} B_1 & \cdots & B_m \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} B'_1 & \cdots & B'_m \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}
\]

with \( B_j = (I_{c_j} \ 0) \), \( B_j' = (0 \ 1_{c_j}) \), \( j = 1, \ldots, m \).

\[
G = \begin{pmatrix} G_{11} & \cdots & G_{1n} \\ \vdots & \ddots & \vdots \\ G_{m1} & \cdots & G_{mn} \end{pmatrix}, \quad H = \begin{pmatrix} H_{00} & \cdots & H_{01} \\ \vdots & \ddots & \vdots \\ H_{m0} & \cdots & H_{mm} \end{pmatrix}
\]

with the blocks \( G_{ij} \in M_{c_i,d_j}(k) \), \( H_{00} \in M_{c_c,d_d}(k) \), \( H_{i0} \in M_{c_i,d_j+1}(k) \), \( H_{0j} \in M_{c_c,d_j+1}(k) \), \( H_{ij} \in M_{c_i+1,d_j+1}(k) \) for \( i = 1, \ldots, n \), \( j = 1, \ldots, m \). Commutativity of the diagram means the matrices \( G \) and \( H \) have to satisfy the following equalities: \( BH = GA \) and \( B'H' = GA' \). Writing out these equations using block-wise multiplication we immediately get that \( H_{i0} = 0_{c_i,d_j+1} \), while there are no restrictions in choosing \( H_{00} \in M_{c_c,d_d}(k) \) and \( H_{i0} \in M_{c_i+1,d_j}(k) \).

For \( i \in \{1 \ldots n\} \) and \( j \in \{1 \ldots m\} \) let us write for the corresponding blocks \( G_{ij} = (g_{ij'})_{c_i \times d_j} \) and \( H_{ij} = (h_{ij'})_{(c_i+1) \times (d_j+1)} \) and expand the equations:

\[
\begin{align*}
B_iH_{ij} &= G_{ij}A_j \\
B'_{i}H_{ij} &= G_{ij}A'_j 
\end{align*}
\]

All the entries in the blocks \( H_{ij} \) and \( G_{ij} \) must satisfy the following relations: \( h_{ij',j'} = h_{i+1,j'+1} = g_{i',j'} \), \( i' = 1, \ldots, c_i \), \( j' = 1, \ldots, d_j \) and consequently \( g_{i',j'} = g_{i'+1,j'+1} \), \( i' = 1, \ldots, c_i \), \( j' = 1, \ldots, d_j \) (i.e. the elements along all top-left to bottom-right diagonals in the blocks \( H_{ij} \) and \( G_{ij} \) are equal). Using \( h_{i+1,j'+1} = g_{i+1,j'+1} = 0 \) for \( i = 1, \ldots, c_i \), \( j = 1, \ldots, d_j \) we can explicitly give the elements of the blocks \( H_{ij} \) and \( G_{ij} \) as:

\[
h_{ij',j'} = \begin{cases} 0 & j' \neq j \in \{0, \ldots, d_j - c_i\}, \ i' = 1, \ldots, c_i + 1, \ j' = 1, \ldots, d_j + 1 \\
\end{cases}
\]

\[
g_{ij',j'} = \begin{cases} 0 & j' - i' \notin \{0, \ldots, d_j - c_i\}, \ i' = 1, \ldots, c_i + 1, \ j' = 1, \ldots, d_j + 1 \\
\end{cases}
\]

where \( c_i \in \mathbb{Z} \). If \( d_j \geq c_i \) and at least one value \( \gamma_{j' - i'} \neq 0 \), then the blocks \( H_{ij} \) and \( G_{ij} \) both have full-rank. With this information in mind (and the fact that the elements of the top-left block \( H_{00} \) may be chosen arbitrarily) it is straightforward to construct the full-rank matrices \( H \) and \( G \).

### 4 Matrix pencils as Kronecker modules

Next we will translate all the terms taken from pencil theory into the language of Kronecker modules (representations). Indeed one can easily see that a matrix pencil \( A + \lambda B \in M_{m,n}(k[\lambda]) \) corresponds to the Kronecker module \( M_{A,B} = (k^m, k^n, f_A, f_B) \), where choosing the canonical basis in \( k^n \) and \( k^m \) the matrix of \( f_A : k^n \to k^m \) (respectively of \( f_B : k^n \to k^m \)) is \( A \) (respectively \( B \)). The strict equivalence \( A + \lambda B \sim A' + \lambda B' \) means the isomorphism of modules \( M_{A,B} \cong M_{A',B'} \). It is also known that a pencil \( A' + \lambda B' \) is a subpencil of \( A + \lambda B \) if and only if the module \( M_{A,B'} \) is a subfactor of \( M_{A,B} \) (see [12]).
It is also clear that we have the following correspondence between the classical Kronecker invariants and the Kronecker invariants (for Kronecker modules) introduced in Section 2; the minimal indices for rows correspond to the integers \((c_1, \ldots, c_n)\) parameterizing the preprojective part, the minimal indices for columns correspond to the integers \((d_1, \ldots, d_m)\) parameterizing the preinjective part, the finite elementary divisors correspond to the nonzero partitions \(\mu^{(p)}, p \in k\) and the infinite elementary divisors to the partition \(\mu^{(\infty)}\) (more precisely the partition \(\mu^{(p)}\) describes the dimensions of the Jordan blocks corresponding to \(p\)).

Based on [12] we easily obtain the following:

**Proposition 8.** \(A' + \lambda B' \in \mathcal{M}_{m', n'}(k[\lambda])\) is a subpencil of \(A + \lambda B \in \mathcal{M}_{m, n}(k[\lambda])\) if and only if \(m \geq m', n \geq n'\) and \([A_{\lambda}, B_{\lambda}] \in \{([n - n']I_0) \ast ([M_{A', B'}]) \ast \{([m - m']P_{0})\}\).  

**Proof.** First note that \([A_{\lambda}, B_{\lambda}] \in \{([n - n']I_0) \ast ([M_{A', B'}]) \ast \{([m - m']P_{0})\}\) if and only if \(\exists L \in \text{mod-}kK\) such that \([L] \in \{([n - n']I_0) \ast ([M_{A', B'}])\) and \([A_{\lambda}, B_{\lambda}] \in \{([L]) \ast \{([m - m']P_{0})\}\). Also, the condition \(m \geq m'\) and \(n \geq n'\) is clear.

“\(\Rightarrow\)” From \([L] \in \{([n - n']I_0) \ast ([M_{A', B'}])\) and \([A_{\lambda}, B_{\lambda}] \in \{([L]) \ast \{([m - m']P_{0})\}\) we deduce the existence of the short exact sequences

\[
0 \to M_{A', B'} \to L \to (n - n')I_0 \to 0
\]

and

\[
0 \to (m - m')P_0 \to M_{A, B} \to L \to 0,
\]

hence there exist an embedding and a projection \(M_{A', B'} \to L = M_{A, B}, \) i.e. \(M_{A', B'}\) a subfactor of \(M_{A, B}\) and the fact that \(A' + \lambda B'\) is a subpencil of \(A + \lambda B\) follows.

“\(\Leftarrow\)” If \(A' + \lambda B' \in \mathcal{M}_{m', n'}(k[\lambda])\) is a subpencil of \(A + \lambda B \in \mathcal{M}_{m, n}(k[\lambda])\), then there exist completion matrices \(A_{12} + \lambda B_{12}, A_{21} + \lambda B_{21}, A_{22} + \lambda B_{22}\) such that

\[
A + \lambda B \sim \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix} = \tilde{A} + \lambda \tilde{B},
\]

that is, the modules \(M_{A, B}\) and \(M_{\tilde{A}, \tilde{B}}\) are isomorphic.

Consider the following short exact sequences, where we have identified the module \(M_{\tilde{A}, \tilde{B}}\) with \(M_{A, B}\):

![Diagram](image-url)

As it can be seen from these two short exact sequences, the module \(L\) may be constructed such that \([L] \in \{([n - n']I_0) \ast ([M_{A', B'}])\) and \([A_{\lambda}, B_{\lambda}] \in \{([L]) \ast \{([m - m']P_{0})\}\), so \([A_{\lambda}, B_{\lambda}] \in \{([n - n']I_0) \ast \{([m - m']P_{0})\}\) follows immediately.
5 Complete solution to the subpencil problem in a particular case

Let us consider matrix pencils $A + \lambda B$, $A' + \lambda B'$ over $k$, having only minimal indices for columns among their classical Kronecker invariants. In this case, $A + \lambda B \sim \text{diag}(L_{\varepsilon_1}, \ldots, L_{\varepsilon_p})$ and $A' + \lambda B' \sim \text{diag}(L'_{\varepsilon'_1}, \ldots, L'_{\varepsilon'_q})$, where $0 \leq \varepsilon_1 \leq \cdots \leq \varepsilon_p$ and $0 \leq \varepsilon'_1 \leq \cdots \leq \varepsilon'_q$ are the minimal indices for columns and

$$L_\varepsilon := \begin{pmatrix} \lambda & 1 \\ & \ddots \\ & & \lambda \\ & & & 1 \end{pmatrix} \in \mathcal{M}_{\varepsilon, \varepsilon+1}(k[\lambda])$$

are the corresponding blocks on the diagonal (for further details see [11]). Hence, as explained before, one may identify the pencil $A + \lambda B$ with the preinjective module $M_{A,B} = I = I_{\varepsilon_1} \oplus \cdots \oplus I_{\varepsilon_1} \in \text{mod}-kK$ and the pencil $A' + \lambda B'$ with the preinjective module $M_{A',B'} = I' = I'_{\varepsilon'_1} \oplus \cdots \oplus I'_{\varepsilon'_1} \in \text{mod}-kK$. Using this identification, we have that $A' + \lambda B'$ is a subpencil of $A + \lambda B$ if and only if $I'$ is a subfactor of $I$, that is if and only if there exists a Kronecker module $L \in \text{mod}-kK$ such that $I' \hookrightarrow L \hookrightarrow I$. We have the following theorem (where $[x]$ denotes the integer part of $x$):

**Theorem 9.** If $I' \subset a_n I_n \oplus \cdots \oplus a_0 I_0$ and $I = c_n I_n \oplus \cdots \oplus c_0 I_0$ are preinjective Kronecker modules, then $I'$ is a subfactor of $I$ (i.e. $\exists L$ such that $I' \subset L \subset I$) if and only if

$$b_1 \leq \frac{1}{2} \left( \sum_{i=1}^n (i+1)c_i - \sum_{i=2}^n (i+1)b_i \right) \quad \text{and} \quad b_0 \geq a_0,$$

where the sequence $b_n, \ldots, b_0$ is defined by the following (decreasing) recursion:

$$b_t = \begin{cases} \min(a_n, c_n) & t = n \\ \min \left( \sum_{i=1}^n ia_i - \sum_{i=t+1}^n ib_i, \sum_{i=1}^n (i+1)c_i - \sum_{i=t+1}^n (i+1)b_i \right) & 2 \leq t < n \\ \frac{1}{2} \sum_{i=1}^n ia_i - \frac{1}{2} \sum_{i=2}^n ib_i & t = 1 \\ \frac{1}{2} \sum_{i=0}^n (i+1)c_i - \frac{1}{2} \sum_{i=1}^n (i+1)b_i & t = 0 \end{cases}$$

Moreover, in this case the values $b_n, \ldots, b_0$ are non-negative and one of the linking modules is $L = b_n I_n \oplus \cdots \oplus b_0 I_0$. Note also that in pencil language $L$ is obtained from $I'$ by column completion and $I$ is obtained from $L$ by row completion.

**Proof.** First we show that $b_n, \ldots, b_1 \geq 0$. For $b_n = \min(a_n, c_n)$, the inequality $b_n \geq 0$ holds. Suppose $b_t \geq 0$ holds for all $t$, where $l \leq t \leq n$, $l \in \{2, \ldots, n\}$. We are going to show that $b_{t-1} \geq 0$ is also true. Since

$$b_t = \min \left( \sum_{i=1}^n ia_i - \sum_{i=t+1}^n ib_i, \sum_{i=1}^n (i+1)c_i - \sum_{i=t+1}^n (i+1)b_i \right),$$

follows that $0 \leq b_t \leq \sum_{i=1}^n ia_i - \sum_{i=t+1}^n ib_i$. So $\sum_{i=t-1}^n ia_i - \sum_{i=t+1}^n ib_i = (t-1)a_{t-1} + \sum_{i=t}^n ia_i - \sum_{i=t+1}^n ib_i - ta_{t-1} \geq 0$. In the same way $\sum_{i=t-1}^n (i+1)c_i - \sum_{i=t+1}^n (i+1)b_i = ta_{t-1} + \sum_{i=t}^n (i+1)c_i - \sum_{i=t+1}^n (i+1)b_i - (t+1)b_t \geq tc_{t-1} \geq 0$, so $b_{t-1} \geq 0$.

From Proposition 8 we know that $I'$ is a subfactor of $I$ if and only if $[I] \in \{[\beta I_0] \ast \{I'\} \ast \{\alpha P_0\}\}$ for some $\alpha, \beta \in N$. The extension monoid product is associative, so $[I] \in \{[\beta I_0] \ast \{I'\}\}$ if and only if $[I] \in \{[\beta L] \ast \{I'\}\}$ for some $L \in \text{mod}-kK$, where $[L] \in \{[\beta I_0] \ast \{I'\}\}$. Since $[L] \in \{[\beta I_0] \ast \{I'\}\}$ and $I'$ is preinjective, it results from Lemma 2 that $L \in \text{mod}-kK$ must also be preinjective.

Let us use now the multiplicative notation for $L$ as well, that is, let use suppose $L = \cdots \oplus u_n I_n \oplus \cdots \oplus u_0 I_0$ (we have no requirement for $u_n$ to be non-zero).

On one hand we have $[L] \in \{[\beta I_0] \ast \{I'\}\}$ (with $\beta$ arbitrary) if and only if we have the short exact sequence $0 \to I' \to L \to \beta I_0 \to 0$, that is if and only if $u_m = 0$ for all $m > n$ and the following is true...
By extracting the inequalities from the last equality in both systems and coupling them together we get
that
\[
\begin{align*}
  a_0 & \leq u_0 \\
  a_1 & \leq u_1 \\
  a_1 + 2a_2 & \leq u_1 + 2u_2 \\
  & \vdots \\
  a_1 + 2a_2 + \ldots + (n-1)a_{n-1} & \leq u_1 + 2u_2 + \ldots + (n-1)u_{n-1} \\
  a_1 + 2a_2 + \ldots + (n-1)a_{n-1} + n a_n & = u_1 + 2u_2 + \ldots + (n-1)u_{n-1} + nu_n.
\end{align*}
\]
On the other hand, using Corollary we have \([I] \in \{[\beta I_0]\} \ast \{[\alpha P_0]\}\) if and only if \([L^{(1)}] \in \{[\alpha I_0]\} \ast \{[I^{(1)}]\}\), if and only if we have the short exact sequence \(0 \to I^{(1)} \to L^{(1)} \to \alpha I_0 \to 0\), where \(L^{(1)} = u_n I_{n+1} \oplus \cdots \oplus u_1 I_1\) and \(I^{(1)} = c_n I_{n+1} \oplus \cdots \oplus c_0 I_1\). So this condition translates to
\[
\begin{align*}
  c_0 & \leq u_0 \\
  c_0 + 2c_1 & \leq u_0 + 2u_1 \\
  & \vdots \\
  c_0 + 2c_1 + \cdots + nc_{n-1} & \leq u_0 + 2u_1 + \cdots + nu_n \\
  c_0 + 2c_1 + \cdots + nc_{n-1} + (n+1)c_n & = u_0 + 2u_1 + \cdots + nu_n + (n+1)u_n.
\end{align*}
\]
By extracting the inequalities from the last equality in both systems and coupling them together we get that \([I] \in \{[\beta I_0]\} \ast \{[I']\} \ast \{[\alpha P_0]\}\) (with arbitrary \(\alpha\) and \(\beta\)) if and only if there exist non-negative integers \(u_0, u_1, \ldots, u_n\) such that the following system is satisfied:
\[
\begin{align*}
  u_0 & \geq a_0 \\
  nu_n & \leq na_n \\
  (n-1)u_{n-1} + nu_n & \leq (n-1)a_{n-1} + na_n \\
  & \vdots \\
  2u_2 + \ldots + (n-1)u_{n-1} + nu_n & \leq 2a_2 + \ldots + (n-1)a_{n-1} + na_n \\
  u_1 + 2u_2 + \ldots + (n-1)u_{n-1} + nu_n & = a_1 + 2a_2 + \ldots + (n-1)a_{n-1} + na_n \\
  u_0 + 2u_1 + \cdots + nu_{n-1} + (n+1)u_n & = c_0 + 2c_1 + \cdots + nc_{n-1} + (n+1)c_n \\
  2u_1 + \cdots + nu_{n-1} + (n+1)u_n & \leq 2c_1 + \cdots + nc_{n-1} + (n+1)c_n \\
  & \vdots \\
  nu_{n-1} + (n+1)u_n & \leq nc_{n-1} + (n+1)c_n \\
  (n+1)u_n & \leq (n+1)c_n.
\end{align*}
\]
Going further, we can write the system in the following equivalent form:
\[
\begin{align*}
  u_n & \leq \min(a_n, c_n) \\
  u_{n-1} & \leq \min\left(\frac{(n-1)a_{n-1} + na_n - nu_n}{n-1}, \frac{nc_{n-1} + (n+1)c_n - (n+1)u_n}{n}\right) \\
  & \vdots \\
  u_2 & \leq \min\left(\frac{\sum_{i=2}^n ia_i - \sum_{i=3}^n iu_i}{2}, \frac{\sum_{i=2}^n (i+1)c_i - \sum_{i=3}^n (i+1)u_i}{3}\right) \\
  u_1 & = \frac{n}{\sum_{i=1}^n ia_i - \sum_{i=2}^n iu_i} \leq \frac{1}{2} \left(\frac{\sum_{i=1}^n (i+1)c_i - \sum_{i=2}^n (i+1)u_i}{\frac{n}{3}}\right) \\
  u_0 & = \sum_{i=0}^n (i+1)c_i - \sum_{i=1}^n (i+1)u_i \geq a_0
\end{align*}
\]
“\(\Longleftarrow\)" We have seen that the recursive definition of the values \(b_n, \ldots, b_0\) assure \(b_n, \ldots, b_1 \geq 0\). Moreover, if the inequalities
\[
b_1 \leq \frac{1}{2} \left(\sum_{i=1}^n (i+1)c_i - \sum_{i=2}^n (i+1)b_i\right) \quad \text{and} \quad b_0 \geq a_0
\]
are also satisfied, then we can take \((u_0, u_1, \ldots, u_n) = (b_0, b_1, \ldots, b_n)\), which is a non-negative integer solution for the system.

“\(\implies\)” Suppose that there exists \(u_0, u_1, \ldots, u_n \in \mathbb{N}\) such that the previous system is satisfied (note that if \(u_2, \ldots, u_n\) are chosen, then \(u_0\) and \(u_1\) are determined) and consider the definition of the sequence \(b_0, b_1, \ldots, b_n\) from the statement of the theorem. If \((u_2, u_3, \ldots, u_n) \neq (b_2, b_3, \ldots, b_n)\) then let \(t \in \{2, \ldots, n\}\) be the greatest index such that \(u_t \neq b_t\). Then we must have \(u_n = b_n, \ldots, u_{t+1} = b_{t+1}\) and \(u_t < b_t\). Let us denote \(d = b_t - u_t > 0\). We perform the following change of variables: \(u_0' = u_0, \ldots, u_{t-3}' = u_{t-3}\), \(u_{t-2}' = u_{t-2} + d\), \(u_{t-1}' = u_{t-1} - 2d\), \(u_t' = u_t + d\), \(u_{t+1}' = u_{t+1}, \ldots, u_n' = u_n\). Direct calculations show that \(u_0', u_1', \ldots, u_n' \in \mathbb{Z}\) also satisfy the system, moreover \(u_0' = b_0, \ldots, u_t' = b_t\). Repeating this process we find the following integer solution of the system: \(u_0'' = b_0, \ldots, u_n'' = b_2, u_1'', u_t''\). The last two equations in the system guarantee that in fact \(u_t'' = b_t\) and \(u_0'' = b_0\) hence they satisfy the two inequalities from the statement of the theorem.

From the proof above one can see that a possible linking module is \(L = b_nI_n \oplus \cdots \oplus b_0I_0\).

**Remark 10.** Theorem \(\text{[B]}\) does not change if we take preprojective modules instead of preinjectives.

**Example 11.** Consider the following matrix pencils over \(\mathbb{C}\) written in canonical diagonal form and having only minimal indices for columns among their classical Kronecker invariants:

\[
A + \lambda B = \begin{pmatrix}
\lambda & 1 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda \\
\end{pmatrix} \in \mathcal{M}_{10,14}(\mathbb{C}[\lambda])
\]

and

\[
A' + \lambda B' = \begin{pmatrix}
0 & \lambda & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 1 & 0 \\
\end{pmatrix} \in \mathcal{M}_{8,12}(\mathbb{C}[\lambda]).
\]

The pencil \(A + \lambda B\) has \(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 3\), and \(\varepsilon_4 = 1\) as its minimal indices for columns, while in the case of the pencil \(A' + \lambda B'\) these are \(\varepsilon_1' = 0\), \(\varepsilon_2' = 5\), \(\varepsilon_3' = 2\) and \(\varepsilon_4' = 1\). Hence the corresponding modules are \(M_{A,B} = I_3 \oplus I_3 \oplus I_1 \oplus I_4\) and \(M_{A',B'} = I_3 \oplus I_2 \oplus I_1 \oplus I_4\). Written using the multiplicative notation used in Theorem \(\text{[B]}\) \(M_{A',B'} = \bigoplus_{i=0}^5 a_i I_i\) and \(M_{A,B} = \bigoplus_{i=0}^5 c_i I_i\), where \((a_0, a_1, \ldots, a_5) = (1, 1, 1, 0, 0, 1)\) and \((c_0, c_1, \ldots, c_5) = (0, 1, 0, 3, 0, 0)\). We use the recursive formula from the theorem to compute the sequence \((b_0, b_1, \ldots, b_5) = (2, 1, 2, 1, 0, 0)\) and to find out that the inequalities

\[
b_1 \leq \frac{1}{2} \left( \sum_{i=1}^5 (i+1)c_i - \sum_{i=2}^5 (i+1)b_i \right) \quad \text{and} \quad b_0 \geq a_0
\]

are satisfied. So \(A' + \lambda B'\) is a subpencil of \(A + \lambda B\) or equivalently, \(M_{A',B'}\) is a subfactor of \(M_{A,B}\), i.e. \(\exists L\) such that \(M_{A',B'} \hookrightarrow L \twoheadrightarrow M_{A,B}\). Moreover, we can take the linking module \(L\) to be \(L = \bigoplus_{i=0}^5 b_i I_i = I_3 \oplus I_2 \oplus I_1 \oplus I_0 \oplus I_0\). We could use at this point Theorem \(\text{[B]}\) to verify the existence of the embedding \(M_{A',B'} \hookrightarrow L\) and Corollary \(\text{[C]}\) to verify the existence of the projection \(L \twoheadrightarrow M_{A,B}\) with the kernel equal to \(2P_0\). The matrix pencil corresponding to the module \(L\) is

\[
L_1 + \lambda L_2 = \begin{pmatrix}
0 & 0 & \lambda & 1 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 & 0 \\
\lambda & 0 & \lambda & 1 & 0 & 0 \\
0 & \lambda & 0 & \lambda & 1 & 0 \\
\end{pmatrix} \in \mathcal{M}_{8,14}(\mathbb{C}[\lambda]).
\]

Let us construct now the completion matrices \(A_{12} + \lambda B_{12}, A_{21} + \lambda B_{21}, A_{22} + \lambda B_{22}\), i.e. those matrix blocks for which the following equivalence holds:

\[
A + \lambda B \sim \begin{pmatrix}
A' + \lambda B' & A_{12} + \lambda B_{12} \\
A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22}\end{pmatrix}.
\]

Since we have an embedding \(M_{A',B'} \hookrightarrow L\), we must have \(f = (F_1, F_2)\), where \(F_1 \in \mathcal{M}_{14,12}(\mathbb{C})\) and \(F_2 \in \mathcal{M}_{8}(\mathbb{C})\) are full-rank matrices such that \((L_1 + \lambda L_2)F_1 = F_2(A' + \lambda B')\). Also, for the projection
From now on we work along the proof of Proposition 1. in [12]. The matrices $G_1^{-1}F_1$ and $F_2^{-1}G_2$ are also full-rank matrices, so there are non-singular square matrices $C_1$, $C_2$, $D_1$ and $D_2$ such that $G_1^{-1}F_1 = C_1 \begin{pmatrix} 1 & 12 \\ 0 & 0 \end{pmatrix}$ and $F_2^{-1}G_2 = D_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} D_2$, respectively. In our case these matrices are

\[
C_1 = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_2 = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
D_1 = \mathbb{I}_8, \quad \text{and} \quad D_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Using these matrices we can write:

\[
A' + \lambda B' = F^{-1}_2 F_2 (A' + \lambda B') = F^{-1}_2 (L_1 + \lambda L_2) F_1 \\
= F^{-1}_2 (L_1 + \lambda L_2) G_1 G^{-1}_2 F_1 = F^{-1}_2 G_2 (A + \lambda B) G^{-1}_1 F_1 \\
= D_1 (I_8 \ 0) D_2 (A + \lambda B) C_1 \begin{pmatrix} 1_{12} \\ 0 \end{pmatrix} C_2.
\]

So \( D_1^{-1} (A' + \lambda B') C_2^{-1} = (I_8 \ 0) D_2 (A + \lambda B) C_1 \begin{pmatrix} 1_{12} \\ 0 \end{pmatrix} \), hence

\[
A' + \lambda B' = (I_8 \ 0) \begin{pmatrix} D_1 \\ I_2 \end{pmatrix} D_2 (A + \lambda B) C_1 \begin{pmatrix} C_2 \\ I_2 \end{pmatrix} \begin{pmatrix} 1_{12} \\ 0 \end{pmatrix},
\]

where

\[
C' = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Obviously, \( A + \lambda B \sim D_2 (A + \lambda B) C' \), where

\[
D_2 (A + \lambda B) C' = \begin{pmatrix}
0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix},
\]

with the completion pencils

\[
A_{12} + \lambda B_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{21} + \lambda B_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{22} + \lambda B_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Remark 12. The calculations were verified using the computer algebra system Maxima [21].

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