LIE ALGEBROIDS AND POISSON-NIJENHUIS STRUCTURES

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Abstract

Poisson-Nijenhuis structures for an arbitrary Lie algebroid are defined and studied by means of complete lifts of tensor fields.

1. INTRODUCTION

In our previous paper [4], certain definitions and constructions of graded Lie brackets and lifts of tensor fields over a manifold were generalized to arbitrary Lie algebroids. Since Poisson-Nijenhuis structures seem to fit very well to the Lie algebroid language and, as it was recently shown by Kosmann-Schwarzbach in [5], they give examples of Lie bialgebroid structures in the sense of Mackenzie and Xu [9], we would like to present in this note a Lie algebroid approach to Poisson-Nijenhuis structures.

We start with the definition of a pseudo-Lie algebroid structure on a vector bundle \( E \), as a slight generalization of the notion of a Lie algebroid and we show that such structures are determined by special tensor fields \( \Lambda \) on the dual bundle \( E^* \).

Then, we define the complete lift \( d^\Lambda_T(P) \), which reduces to the classical tangent lift \( d_T \) in the case of the tangent bundle \( E = TM \). We prove that, when we start from \( P \in \Gamma(M, \wedge^2 E) \), the complete lift \( d^\Lambda_T(P) \) corresponds to a bracket on sections of \( E^* \), which, in the classical case, is the Fuchssteiner-Koszul bracket on 1-forms. In the case of a Lie algebroid over a single point, \( d^\Lambda_T(P) \) is closely related to the modified Yang-Baxter equation. Deforming the Lie algebroid bracket by a \((1,1)\) tensor \( N \), we find the corresponding bivector field \( \Lambda_N \) on \( E^* \). Assuming some compatibility conditions for \( N \) and \( P \), we can define a Poisson-Nijenhuis structure for a Lie algebroid which provides a whole list of Lie bialgebroid structures.

This unified approach to Poisson and Nijenhuis structures, including the classical case as well, as the case of a real Lie algebra, makes possible to understand common aspects of the theory, which were previously seen separately for different models.

2. TANGENT LIFTS FOR PRE-LIE ALGEBROIDS

Let \( M \) be a manifold and let \( \tau: E \to M \) be a vector bundle. By \( \Phi(\tau) \) we denote the graded exterior algebra generated by sections of \( \tau \): \( \Phi(\tau) = \bigoplus_{k \in \mathbb{Z}} \Phi^k(\tau) \), where \( \Phi^k(\tau) = \Gamma(M, \wedge^k E) \) for \( k \geq 0 \) and \( \Phi^k(\tau) = \{0\} \) for \( k < 0 \). Elements of \( \Phi^0(\tau) \) are functions on \( M \), i.e., sections of the bundle \( \wedge^0 E = M \times \mathbb{R} \). Similarly, by \( \otimes(\tau) \) we denote the tensor algebra \( \otimes(\tau) = \bigoplus_{k \in \mathbb{Z}} \otimes^k(\tau) \), where \( \otimes^k(\tau) = \Gamma(M, \otimes^k_M E) \). The

¹Supported by KBN, grant No 2 PO3A 074 10
dual vector bundle we denote by \( \pi: E^* \to M \). For the tangent bundle \( \tau_M : TM \to M, \Phi(\tau_M) \) is the exterior algebra of multivector fields, and for the cotangent bundle \( \varpi_M : T^* M \to M \), we get \( \Phi(\varpi_M) \), the exterior algebra of differential forms on \( M \).

The cotangent bundle is endowed with the canonical symplectic form \( \omega_M \) and the corresponding canonical Poisson tensor \( \Lambda_M \).

**Definition 2.1** A pseudo-Lie algebroid structure on a vector bundle \( \tau : E \to M \) is a bracket (bilinear operation) \([\cdot, \cdot]\) on the space \( \Phi^1(\tau) = \Gamma(M, E) \) of sections of \( \tau \) and vector bundle morphisms \( \alpha_1, \alpha_r : E \to TM \) (called the left- and right-anchor, respectively), such that

\[
[fX, gY] = f\alpha_1(X)(g)Y - g\alpha_r(Y)(f)X + fg[X, Y] \quad (2.1)
\]

for all \( X, Y \in \Gamma(E) \) and \( f, g \in C^\infty(M) \).

A pseudo-Lie algebroid, with a skew-symmetric bracket \([\cdot, \cdot]\) (in this case the left and right anchors coincide), is called a pre-Lie algebroid.

A pre-Lie algebroid is called a Lie algebroid if the bracket \([\cdot, \cdot]\) satisfies the Jacobi identity, i.e., if it provides \( \Phi^1(\tau) \) with a Lie algebra structure.

In the following, we establish a correspondence between pseudo-Lie algebroid structures on \( E \) and 2-contravariant tensor fields on the bundle manifold \( E^* \) of the dual vector bundle \( \pi : E^* \to M \). Let \( X \in \Phi^1(\tau) \). We define a function \( \iota_{E^*}X \) on \( E^* \) by the formula

\[
E^* \ni a \mapsto \iota_{E^*}X(a) = \langle X(\pi(a)), a \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the canonical pairing between \( E \) and \( E^* \).

Let \( \Lambda \in \Gamma(E, TE \otimes E \otimes E) \) be a 2-contravariant tensor field on \( E \). We say that \( \Lambda \) is linear if, for each pair \((\mu, \nu)\) of sections of \( \pi \), the function \( \Lambda(\mu, \nu)(a) \), defined on \( E \), is linear.

For each 2-contravariant tensor \( \Lambda \), we define a bracket \( \{\cdot, \cdot\}_\Lambda \) of functions by the formula

\[
\{f, g\}_\Lambda = \Lambda(df, dg).
\]

**Theorem 2.1** For every pseudo-Lie algebroid structure on \( \tau : E \to M \), with the bracket \([\cdot, \cdot]\) and anchors \( \alpha_1, \alpha_r \), there is a unique 2-contravariant linear tensor field \( \Lambda \) on \( E^* \) such that

\[
\iota_{E^*}X = \{\iota_{E^*}X, \iota_{E^*}Y\}_\Lambda \quad (2.2)
\]

and

\[
\pi^* (\alpha_1(X)(f)) = \{\iota_{E^*}X, \pi^* f\}_\Lambda, \quad \pi^* (\alpha_r(X)(f)) = \{\pi^* f, \iota_{E^*}X\}_\Lambda, \quad (2.3)
\]

for all \( X, Y \in \Phi^1(\tau) \) and \( f \in C^\infty(M) \).

Conversely, every 2-contravariant linear tensor field \( \Lambda \) on \( E^* \) defines a pseudo-Lie algebroid on \( E \) by the formulae \(2.2\) and \(2.3\).

The pseudo-Lie algebroid structure on \( E \) is a pre-Lie algebroid structure (resp. a Lie algebroid structure) if and only if the tensor \( \Lambda \) is skew-symmetric (resp. \( \Lambda \) is a Poisson tensor).

**Proof.** We shall use local coordinates. Let \((x^a)\) be a local coordinate system on \( M \) and let \( e_1, \ldots, e_n \) be a basis of local sections of \( E \). We denote by \( e^1, \ldots, e^n \) the dual basis of local sections of \( E^* \) and by \((x^a, y^i)\) (resp. \((x^a, \xi_i)\)) the corresponding coordinate system on \( E \) (resp. \( E^* \)), i.e., \( \iota_{E^*}e_i = \xi_i \) and \( \iota_{E^*}x^a = y^i \).

It is easy to see that every linear 2-contravariant tensor \( \Lambda \) on \( E^* \) is of the form

\[
\Lambda = c^k_{ij} \xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \delta_i^a \partial_{\xi_i} \otimes \partial_{x^a} - \sigma_i^a \partial_{x^a} \otimes \partial_{\xi_i}, \quad (2.4)
\]

where \( c^k_{ij}, \delta_i^a \) and \( \sigma_i^a \) are functions of \( x^a \). The correspondence between \( \Lambda \) and a pseudo-Lie algebroid structure is given by the formulae

\[
\begin{align}
[e_i, e_j] &= [e_i, e_j]^\Lambda = c^k_{ij} e_k \\
\alpha_i^L(e_i) &= \delta_i^a \partial_{x^a} \\
\alpha_i^R(e_i) &= \sigma_i^a \partial_{x^a}
\end{align} \quad (2.5)
\]
Theorem 2.2 Let $\tau_i : E_i \to M$, $i = 1, 2$, be vector bundles over $M$ and let $\Psi : E_1 \to E_2$ be a vector bundle morphism over the identity on $M$. Let $\Lambda_i$ be a linear, 2-contravariant tensor on $E_1^*$, $i = 1, 2$. Then
$$[\Psi(X), \Psi(Y)]^{\Lambda_2} = \Psi([X, Y]^{\Lambda_1})$$
if and only if $\Lambda_2$ and $\Lambda_1$ are $\Psi^*$-related, where $\Psi^* : E_2^* \to E_1^*$ is the dual morphism.

Proof. The equality $[\Psi(X), \Psi(Y)]^{\Lambda_2} = \Psi([X, Y]^{\Lambda_1})$ is equivalent to the equality
$$\{(\iota_{E_1^*} X) \circ \Psi^*, (\iota_{E_1^*} Y) \circ \Psi^*\}_{\Lambda_2} = \{\iota_{E_1^*} \Psi(X), \iota_{E_1^*} \Psi(Y)\}_{\Lambda_1} = \{\iota_{E_1^*} X, \iota_{E_1^*} Y\}_{\Lambda_1} \circ \Psi^*. \tag{2.6}$$

Since the exterior derivatives of functions $\iota_{E_1^*} X$ generate $T^* E_1^*$ over an open-dense subset ($E_1^*$ minus the zero section), the equality (2.6) holds if and only if the tensors $\Lambda_1, \Lambda_2$ are $\Psi^*$-related.

To the end of this section we assume that $\Lambda$ is skew-symmetric, i.e., we consider pre-Lie algebroid structures only.

In this case, the bracket $[,]^{\Lambda}$, related to $\Lambda$, defined on sections of $\tau$ can be extended in a standard way (cf. [3, 4]) to the graded bracket on $\Phi(\tau)$. We refer to this bracket as the Schouten-Nijenhuis bracket and we denote it also by $[,]^{\Lambda}$.

Moreover, we can define the ‘exterior derivative’ $d^{\Lambda}$ on $\Phi(\pi)$ and the Lie derivative $L^\Lambda_X : \Phi(\pi) \to \Phi(\pi)$ along a section $X \in \Gamma(M, E)$. Also the Nijenhuis-Richardson bracket and the Frölicher-Nijenhuis bracket can be defined on $\Phi^* \pi (\pi) = \odot_{n \in \mathbb{Z}} \Phi^* \pi (\pi)$, where $\Phi^* \pi (\pi) = \Gamma(M, E \otimes \wedge^n E^*)$. The definitions of these objects are analogous to the definitions in the classical case (cf. [3]).

The bracket $[,]^{\Lambda}$ is a Lie bracket (or, equivalently, $(d^{\Lambda})^2 = 0$) if and only if $\Lambda$ defines a Lie algebroid structure, i.e., if and only if $\Lambda$ is a Poisson tensor. In this case, all classical formulae of differential geometry, like $L^\Lambda_X \circ i_Y - i_Y L^\Lambda_X = i_{[X,Y]}^\Lambda$ etc., remain valid. We should also mention the vertical tangent lift
$$v_\tau : \Gamma(M, \otimes_M^k E) \to \Gamma(E, \otimes^E_\tau T E)$$
given, in local coordinates, by
$$v_\tau(f(x)e_{i_1} \otimes \cdots \otimes e_{i_k}) = f(x)\partial_{y_{i_1}} \otimes \cdots \otimes \partial_{y_{i_k}}.$$ 
In particular, $v_\tau(X \otimes Y) = v_\tau(X) \otimes v_\tau(Y)$ (3). In the case of the tangent bundle, $E = TM$, the vertical lift was denoted by $v_T$ in (3). An analog of the complete tangent lift $d_T$, studied for the tangent bundle in [3], can be defined as follows.

Theorem 2.3 Let $\Lambda$ be a linear bivector field on $E^*$, which defines a pre-Lie algebroid structure on a vector bundle $\tau : E \to M$. Then, there exists a unique $v_\tau$-derivation of order 0
$$d^{\Lambda}_\tau : \otimes (\tau) \to \otimes (\tau_E),$$
which satisfies
$$d^{\Lambda}_\tau (f) = \iota_E d^{\Lambda} f \quad \text{for } f \in C^\infty(M), \tag{2.7}$$
and
$$\iota_{T^\tau E} (d^{\Lambda}_\tau X) \circ R = \iota_{T^\tau E} ([\Lambda, \iota_{E^*} X]) \quad \text{for } X \in \Phi(\tau), \tag{2.8}$$
where $[,]$ is the Schouten bracket of multivector fields on $E$ and $R : T^E E^* \to T^E E$ is the canonical isomorphism of double vector bundles(see [3, [4]]). Moreover, $d^{\Lambda}_\tau$ is a homomorphism of the Schouten-Nijenhuis brackets:
$$d^{\Lambda}_\tau ([X, Y]^\Lambda) = [d^{\Lambda}_\tau X, d^{\Lambda}_\tau Y], \tag{2.9}$$
if and only if $\Lambda$ is a Poisson tensor.
SKETCH OF THE PROOF. Let us take $X \in \Phi^1(\tau)$. The Hamiltonian vector field $G^\Lambda(X) = -[\Lambda, \iota_E X]$ is linear with respect to the tangent vector bundle structure $T \tau : TM \to TM$ \(\square\). Hence, the function $\iota_{T^*E} [\Lambda, \iota_E X]$ is linear with respect to both vector bundle structures on $T^*E$: over $E$ and over $E^*$. It follows that there exists a unique (linear) vector field $d^1_X$ on $E$, such that $\iota_{T^*E} G^\Lambda(X) = -(\iota_{T^*E} d^1_X) \circ R$. We have the formula
\[
d^1_X(fX) = d^1_X(f) v_\tau(X) + v_\tau(f) d^1_X(X)
\]
and, consequently, we can extend $d^1_X$ to a $v_\tau$-derivation on $\otimes(\tau)$. Finally, since $R$ is an anti-Poisson isomorphism, $d^1_X$ is a homomorphism of Schouten-Nijenhuis bracket if and only if
\[
[G^\Lambda(X), G^\Lambda(Y)] = G^\Lambda([X, Y]^\Lambda)
\]
for all $X, Y \in \Phi^1(\tau)$, or, equivalently, if and only if $\Lambda$ is a Poisson tensor. \hfill \blacksquare

Remark. Let us define a mapping
\[
\mathcal{J}_E : \Phi^1(\pi) \to \Phi^1(\tau_E)
\]
by
\[
\mathcal{J}_E(\mu \otimes X) = -\iota_E(\mu) \cdot v_\tau(X).
\]
It has been shown in \cite{4} that $\mathcal{J}_E$ is a homomorphism of the Nijenhuis-Richardson bracket into the Schouten bracket. We have also a mapping
\[
G^\Lambda : \Phi^1(\pi) \to \Phi^1(\tau_E^*) : K \mapsto G^\Lambda(K) = [\Lambda, \mathcal{J}_E^*(K)],
\]
which is, in the case of a Lie algebroid structure, a homomorphism of the Frölicher-Nijenhuis bracket $[\cdot, \cdot]_{\mathcal{F}_N}$, associated to $\Lambda$, into the Schouten bracket \(\square\). The bracket $[\cdot, \cdot]_{\mathcal{F}_N}$ is given by the formula
\[
[X, Y]_\Lambda = [\mu \wedge \nu \otimes [X, Y]^\Lambda + \mu \wedge \mathcal{L}_X^X \nu \otimes Y + \mathcal{L}_Y^X \mu \wedge \nu \otimes X + (-1)^{\mu}(d^\Lambda \mu \wedge \nu \otimes Y + i_Y \mu \wedge d^\Lambda \nu \otimes X).
\] \hfill (2.10)

In local coordinates
\[
\Lambda = \frac{1}{2} c^i_{jkl} \partial_x^i \partial_x^j \partial_x^k.
\] \hfill (2.11)

Then,
\[
d^\Lambda_X(f) = \frac{\partial f}{\partial x^a} \delta^a_j y^j
\] \hfill (2.12)

and
\[
d^\Lambda_X(X^i e_i) = X^i \delta^a_j \partial_x^a + (X^i c^k_{ji} + \frac{\partial X^k}{\partial x^a} \delta^a_j) y^i \partial_y^j.
\] \hfill (2.13)

It follows that, for $P = \frac{1}{2} P^{ij}_{a} e_i \wedge e_j$, we have
\[
d^\Lambda_X(P) = P^{ij} \delta^a_j \partial_y^i \wedge \partial_x^a + (P^{ij} c^k_{ji} + \frac{1}{2} \frac{\partial P^{ij}}{\partial x^a} \delta^a_j) y^i \partial_y^j \wedge \partial_y^j.
\] \hfill (2.14)

Remark. For an arbitrary pseudo-Lie algebroid, we can define the right and the left complete lifts with the use of the right and the left Hamiltonian vector fields instead of $[\Lambda, \iota_E X]$.

The following theorem describes the complete lifts in terms of Lie derivatives and contractions.

**Theorem 2.4** Given a vector bundle $\tau : E \to M$ and a linear bivector field $\Lambda$ on $E^*$, we have
\[(a) \quad v_\tau(X)(\iota_{E\mu}) = v_\tau(\iota_{ix} \mu) = \tau^b(X, \mu),
\]
\[(b) \quad d^1_X(\iota_{E\mu}) = \iota_{E}(\mathcal{L}_X^\Lambda\mu),
\]
where $X \in \Phi^1(\tau)$ and $\mu \in \Phi^1(\pi)$. 
Poisson structure $\Lambda$ on $g$. Kirillov-Souriau tensor in (linear) coordinate system $\xi$ where we used Theorem 2.4. Now, we have

\[ \alpha = \text{Theorem 2.5} \]

Proof. The part (a) has been proved in $[4]$, Theorem 15 c). The part (b) follows from the following sequence of identities:

\[
\begin{align*}
\pi^*_E(d^\Lambda_L(X)(\iota_E\mu)) \circ \mathcal{R} &= \{ \iota_{T^*E}(d^\Lambda_LX), \pi^*_E(\iota_E\mu) \}_\Lambda \circ \mathcal{R} \\
&= \{ \iota_{T^*E}(\iota_E\mu), \iota_{T^*E}([\Lambda, \iota_E\mu]) \}_\Lambda \circ \mathcal{R}, \\
&= \iota_{T^*E}([\iota_E\mu, [\Lambda, \iota_E\mu]]) = \iota_{T^*E}[\mathcal{G}^L(X), \iota_E\mu] \\
&= \iota_{T^*E}([\iota_E\mu, \mathcal{L}^\Lambda(\mu)]) = \pi^*_E(\iota_E(\mathcal{L}^\Lambda_{\mu})) \circ \mathcal{R},
\end{align*}
\]

where we used the equalities $[\mathcal{G}^L(X), \iota_E\mu] = \iota_E(\mathcal{L}^\Lambda_{\mu})$ (see $[4]$, Theorem 15 e)) and $\iota_{T^*E} \iota_E\mu = \pi^*_E(\iota_E(\mathcal{L}^\Lambda_{\mu})) \circ \mathcal{R}$. 

**Theorem 2.5** If $P \in \Phi^2(\tau)$, then $d_P(P)$ defines a pre-Lie algebroid structure on $E^*$ with the bracket

\[
[\mu, \nu]^{d_P(P)} = E^\mu - E^\nu - d_P(\mu, \nu),
\]

where $P_\mu = \iota_\mu P$, and the anchor is given by

\[ \alpha^{d_P(P)}(\mu) = \alpha^L(P_\mu). \]

Proof. It is sufficient to consider $P = X \wedge Y, X, Y \in \Phi^1(\tau)$. Let us denote $d_P(X)$ and $d_P(Y)$ by $\bar{X}$ and $\bar{Y}$, $\iota_E(\mu)$ by $\bar{Y}$, $\iota_E(\mu)$ by $\bar{X}$ and $\bar{Y}, \mu^L(\mu) = d^L(\mu)$ by $\bar{X}$ and $\bar{d}$. Then, we have

\[
\begin{align*}
\{ \iota_E\mu, \iota_E\nu \}^{d_P(P)} &= X(\iota_E\mu)Y(\iota_E\nu) - X(\iota_E\nu)Y(\iota_E\mu) - \bar{X}(\iota_E\mu)Y(\iota_E\nu) + \bar{X}(\iota_E\nu)Y(\iota_E\mu) \\
&= \iota_E(Y(\mu)\mathcal{L}_X(\nu) - Y(\mu)\mathcal{L}_X(\nu) - (X, \nu)\mathcal{L}_Y(\mu) + (X, \mu)\mathcal{L}_Y(\nu)) \\
&= \iota_E(\mathcal{L}_P(\mu) - \mathcal{L}_P(\nu) - d(\mu, \nu)).
\end{align*}
\]

where we used Theorem 2.4. Now, we have

\[
[\mu, \nu]^{d_P(P)} = E^\mu - E^\nu - d_P(\mu, \nu) = f[\mu, \nu]^{d_P(P)} + (E_P\mu)\nu,
\]

so that $\alpha^{d_P(P)}(\mu) = E^\mu(f) = \alpha^L(P_\mu)(f)$. 

In the case of the canonical Poisson structure on the tangent bundle $\tau_M: TM \to M$, associated with the canonical Poisson tensor $\Lambda^L$ on $T^*M$, our definition of $d_P(P)$ gives the standard tangent complete lift $d_P$. Moreover, the bracket $\mathcal{F}^L$ of 1-forms is the bracket introduced independently in $[2, 8, 10]$ and corresponding to the lift $d_P$ (cf. $[2, 8, 10]$).

**Example.** Let us consider a Lie algebroid over a point, i.e., a real Lie algebra $\mathfrak{g}$ with a basis $e_1, \ldots, e_m$, and its dual space $\mathfrak{g}^*$ with the dual basis $e^1, \ldots, e^m$. We have also the corresponding linear coordinate system $\xi_1, \ldots, \xi_m$ on $\mathfrak{g}^*$ and the coordinate system $y^1, \ldots, y^m$ on $\mathfrak{g}$. The linear Poisson structure $\Lambda$ on $\mathfrak{g}^*$, associated with the Lie bracket $[,]^L$ on $\mathfrak{g}$, is the well-known Kostant-Kirillov-Souriau tensor

\[
\Lambda = \frac{1}{2}c^k_{ij}\xi_k \wedge \partial_{\xi_i}.
\]

Here $c^k_{ij}$ are the structure constants with respect to the chosen basis. The exterior derivative $d^L: \Lambda \mathfrak{g}^* \to \Lambda \mathfrak{g}^*$ is the dual mapping to the Lie bracket:

\[
d^L(\mu)(X, Y) = (\mu, [Y, X])^L,
\]

i.e., $d^L$ is the Chevalley cohomology operator. For $X \in \Phi^1(\tau) = \mathfrak{g}$, the tangent complete lift $d_P(x)$ is the fundamental vector field of the adjoint representation, corresponding to $x$:

\[
d_P(e_i) = c^k_{ij}y^j \partial_{y^k}.
\]
3. NIJENHUIS TENSORS AND POISSON-NIJENHUIS STRUCTURES FOR LIE ALGEBROIDS

Let a vector bundle \( \tau: E \to M \) be given a pseudo-Lie algebroid structure, associated with a tensor field \( \Lambda \) on \( E^* \), and let \( N: E \to E \) be a vector bundle morphism over the identity. We can represent \( N \), as well as its dual \( N^* \), by a tensor field \( N \in \Phi^1(\tau) \). This tensor field defines operations in \( \Phi^1(\tau) \) and \( \Phi^1(\pi) \), which we denote by the same symbol \( i_N \). If \( N = X_i \otimes \mu^i \), \( X_i \in \Phi^1(\tau) \), \( \mu^i \in \Phi^1(\pi) \), the operation \( i_N \) is given by the formulae

\[
i_N X = (X, \mu^i)X_i \quad \text{and} \quad i_N \mu = (X, \mu^i)\mu^i,
\]

where \( X \in \Phi^1(\tau), \mu \in \Phi^1(\pi) \). In the notation of \( \mathbb{3} \), \( i_N X = NX \) and \( i_N \mu = t_N \mu \). It is obvious that we can extend \( i_N \) to a derivation of the tensor algebra, putting

\[
i_N (A \otimes B) = (i_N A) \otimes B + A \otimes (i_N B).
\]

Using \( N \), we can deform the bracket \([ , ]\) to a bracket \([ , ]\) on \( \Phi^1(\tau) \) by the formula

\[
[X, Y] = [NX, Y] + [X, NY] - N[X, Y].
\]

**Theorem 3.1** The deformed bracket \( [ , ] \) defines on \( E \) a pseudo-Lie algebroid structure, with the anchors \( (\alpha^N)_L = \alpha^L \circ N \) and \( (\alpha^N)_R = \alpha^R \circ N \). The associated tensor field is given by

\[
\Lambda_N = \mathcal{L}_{\tau^E^*}(N) \Lambda,
\]

where \( \mathcal{L}_{\tau^E^*}(N) \) is the standard Lie derivative along the vector field \( \tau^E^* (N) \).

If \( \Lambda \) is skew-symmetric, then \( \Lambda_N \) is also skew-symmetric and the Schouten-Nijenhuis bracket, induced by \( \Lambda_N \), can be written in the form, similar to \( \mathbb{3} \),

\[
[X, Y] = [NX, Y] + [X, NY] - N[X, Y]
\]

for \( X, Y \in \Phi(\tau) \). Moreover,

\[
d\Lambda_N = i_N \circ d\Lambda - d\Lambda \circ i_N.
\]

The proof is based on the following Lemma.

**Lemma 3.1** For \( X \in \Phi^1(\tau) \), we have

\[
\iota_{E^*}(NX) = -\mathcal{L}_{\tau^E^*}(N)(\iota_{E^*})
\]

and

\[
\mathcal{L}_{\tau^E^*}(N) \nu_\pi(\mu) = \nu_\pi(i_N \mu)
\]

for \( X \in \Phi^1(\tau), \mu \in \Phi^1(\pi) \).

**Proof.** Let \( N = X_i \otimes \mu^i \), \( X_i \in \Phi^1(\tau) \) and \( \mu^i \in \Phi^1(\pi) \). We have

\[
\iota_{E^*}(NX) = \iota_{E^*}((X, \mu^i)X_i) = \pi^*((X, \mu^i))\iota_{E^*}(X_i)
\]

\[
= \nu_\pi((X, \mu^i))\iota_{E^*}(X_i).
\]

On the other hand,

\[
\mathcal{L}_{\tau^E^*}(N)(\iota_{E^*}X) = (\iota_{E^*}(X_i)\nu_\pi(\mu^i))\iota_{E^*}X = \iota_{E^*}(X_i)\nu_\pi(\mu^i)\iota_{E^*}X
\]

\[
= \iota_{E^*}(X_i)\nu_\pi((X, \mu^i)),
\]

according to Theorem 15 c) in \( \mathbb{3} \).
Similarly,
\[ [\mathcal{J}_{E^*}(N), v_\pi(\mu)] = [\iota_{E^*}(X_i), v_\pi(\mu_i), v_\pi(\mu)] \]
\[ = -v_\pi(\mu_i) \wedge [\iota_{E^*}(X_i), v_\pi(\mu)], \]
since the vertical vector fields commute. Following Theorem 15 c) in \[4\], we get
\[ [\iota_{E^*}(X_i), v_\pi(\mu)] = -v_\pi(i_X, \mu) \]
and, consequently,
\[ [\mathcal{J}_{E^*}(N), v_\pi(\mu)] = v_\pi(\mu_i \wedge i_X, \mu) = v_\pi(i_N \mu). \]

**Proof of Theorem 3.1.** Using Lemma and properties of the Lie derivative, we get
\[ \iota_{E^*}([X, Y]^A) = \iota_{E^*}([X, Y]^A + [X, NY]^A - N[X, Y]^A) \]
\[ = \{-\mathcal{L}_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}X), \iota_{E^*}Y\}_A + \{\iota_{E^*}X, \mathcal{L}_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}Y)\}_A + \mathcal{L}_{\mathcal{J}_{E^*}(N)}\{\iota_{E^*}X, \iota_{E^*}Y\}_A \]
\[ = \{\iota_{E^*}X, \iota_{E^*}Y\}_E \mathcal{J}_{E^*} \mathcal{N} \Lambda. \]
The general form of the corresponding Schouten bracket follows inductively from the Leibniz rule for the Schouten bracket \([,] \Lambda\) and from \[3.2\].

In order to prove \[3.4\], we, again, use Lemma and \[\text{Theorem } 15 \text{ d:} \]
\[ v_\pi(d^{A_N} \mu) = [\Lambda, v_\pi \mu] = [[\mathcal{J}_{E^*} N, A], v_\pi \mu] \]
\[ = [\mathcal{J}_{E^*} N, [A, v_\pi \mu]] - [A, [\mathcal{J}_{E^*} N, v_\pi \mu]] \]
\[ = [\mathcal{J}_{E^*} N, v_\pi(d\Lambda_N)] - [A, v_\pi(i_N \mu)] = v_\pi(i_N d\Lambda - d\Lambda i_N \mu). \]

In local coordinates, we have
\[ N = N^l e_i \otimes e^j, \]
\[ \Lambda = c_i^j \xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \delta_i^j \partial_{\xi_i} \otimes \partial_{x^a} - \sigma_i^a \partial_{x^a} \otimes \partial_{\xi_i}, \]
\[ \mathcal{J}_{E^*} N = N^l_k \xi_i \partial_{\xi_k}, \]
and
\[ \Lambda_N = \left( c_i^j N^l + c_i^k N^l_j - c_i^j N^l_k + \delta_i^j \frac{\partial N^l_k}{\partial x^a} - \sigma_i^a \frac{\partial N^l_k}{\partial x^a} \right) \xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} \]
\[ + N^l_k \delta_i^j \partial_{\xi_i} \otimes \partial_{x^a} - N^l_k \sigma_i^a \partial_{x^a} \otimes \partial_{\xi_i}. \]

**Theorem 3.2** For \( X \in \otimes(\tau) \) and skew-symmetric \( \Lambda \), we have
\[ d^{\Lambda_N}(X) = d^{\Lambda}(i_N X) - \mathcal{L}_{\mathcal{J}_{E}(N)} d^{\Lambda}(X). \]

**Proof.** Since \( d^{\Lambda_N} \) and \( d^\Lambda \) are \( v_\tau \)-derivations of order 0 on \( \otimes(\tau) \) and \( \mathcal{L}_{\mathcal{J}_{E}(N)} v_\tau(X) = 0 \) \( (\mathcal{J}_{E}(N) \) is vertical), it is enough to consider the case \( X \in \Phi^1(\tau) \). For such \( X \)
\[ \iota_{T^*E} \left( d_T^{\Lambda_N}(X) \right) \circ R = \iota_{T^*E}[\Lambda, \iota_{E^*} X] = \iota_{T^*E}[\mathcal{L}_{\mathcal{J}_{E^*}(N)} \Lambda, \iota_{E^*} X] \]
\[ = \iota_{T^*E} \left( \mathcal{L}_{\mathcal{J}_{E^*}(N)}[\Lambda, \iota_{E^*} X] - \iota_{T^*E}[\Lambda, \mathcal{L}_{\mathcal{J}_{E^*}(N)}(\iota_{E^*} X)] \right). \]

Since \( \mathcal{L}_{\mathcal{J}_{E^*}(N)} = -\iota_{E^*}(NX) \) \[3.3\], then
\[ -\iota_{T^*E}[\Lambda, \mathcal{L}_{\mathcal{J}_{E^*}(N)}(\iota_{E^*} X)] = \iota_{T^*E}[\Lambda, \iota_{E^*}(NX)] = \iota_{E^*} \left( d_T^{\Lambda}(NX) \right) \circ R. \]
On the other hand,
\[ \iota_{T^*E^*}(\mathcal{L}_{\mathcal{J}_E^*(N)}[\Lambda, \iota_{E^*X}]) = \{\iota_{T^*E^*}, (\mathcal{J}_E^*(N)), \iota_{T^*E^*}[\Lambda, \iota_{E^*X}]\}_{\Lambda^*} \]
\[ = \{\iota_{T^*E^*}(\mathcal{J}_E^N) \circ \mathcal{R}, \iota_{T^*E^*}(d\delta(X)) \circ \mathcal{R}\}_{\Lambda^*} \]
\[ = -\{\iota_{T^*E^*}(\mathcal{J}_E^N), \iota_{T^*E^*}(d\delta(X))\}_{\Lambda^*} \circ \mathcal{R} \]
\[ = -\iota_{T^*E^*}(\mathcal{L}_{\mathcal{J}_E^N}(d\delta(X))), \]
where we used the equality \(\iota_{T^*E^*}(\mathcal{J}_E^N) = \iota_{T^*E^*}(\mathcal{J}_E^*(N))\). Since \(\mathcal{R}\) is an isomorphism and \(\iota_{T^*E^*}\) is injective, the theorem follows.

In local coordinates, for \(\Lambda\) as in \[\[\text{3.1}\]\], we have
\[ d^\Lambda_N(X^i e_i) = X^i N^k_i \delta^a_k \partial_x^a + \]
\[ + \left( X^i (N^k_j c^a_{ki} + N^k_i c^a_{jk}) - \delta^a_i \frac{\partial N^a}{\partial x^a} - \delta^a_i \frac{\partial N^a}{\partial x^a} N^k_j \delta^a_k \right) y_j y^g. \quad (3.8) \]

**Remark.** If we treat the Schouten brackets \(B^\Lambda = [\cdot, \cdot]^\Lambda\) and \(B^\Lambda_N = [\cdot, \cdot]^\Lambda_N\) as bilinear operators on \(\Phi(\tau)\), then formula \[\[\text{3.2}\]\] means
\[ B^\Lambda_N = [i_N, B^\Lambda]_{N-R}, \quad (3.9) \]
where \([\cdot, \cdot]_{N-R}\) is the Nijenhuis-Richardson bracket of multilinear graded operators of a graded space in the sense of \[\[\text{3.4}\]\]. Similarly, \[\[\text{3.4}\]\] means that
\[ d^\Lambda_{\iota_X} = [i_N, d^\Lambda_{\iota_X}]_{N-R}. \quad (3.10) \]

This interpretation will be used later, together with the Jacobi identity for the \([\cdot, \cdot]_{N-R}\).

**Definition 3.1** A tensor \(N \in \Gamma(M, E \otimes E^*)\) is called a Nijenhuis tensor for \(\Lambda\) (or, for a Lie algebroid structure defined by \(\Lambda\)), if the Nijenhuis torsion
\[ T^\Lambda_N(X, Y) = 1\frac{d}{\delta X^a}[N[X, Y]]^\Lambda_N - [NX, NY]^\Lambda \]
vanishes for all \(X, Y \in \Gamma(E)\).

The classical version of the following is well known (cf. \[\[\text{3.1}\]\]).

**Theorem 3.3**

(a) \(N\) is a Nijenhuis tensor for \(\Lambda\) if and only if \(\Lambda\) and \(\Lambda_N = \mathcal{L}_{\mathcal{J}_E^*(N)}\Lambda\) are \(N^*\)-related.

(b) The Nijenhuis torsion corresponds to the Frölicher-Nijenhuis bracket:
\[ T^\Lambda_N(X, Y) = \frac{1}{2}[N, N]^\Lambda_{F-N}(X, Y). \]

(c) \[ [B^\Lambda, i_N]_{R-N}, i_N]_{N-R} = 2T^\Lambda_N + [B^\Lambda, i_N^2]_{N-R}, \]
where \((X_i \otimes \mu^i)^2 = (X_i, \mu^i) \mu^i \otimes X_j, \]

(d) If \(N\) is a Nijenhuis tensor, then \(\Lambda_N\) is a Poisson tensor.

**Proof.**

(a) Since \(\Lambda_N = \mathcal{L}_{\mathcal{J}_E^*(N)}\Lambda\) induces the deformed bracket \(B^\Lambda_N = [\cdot, \cdot]^\Lambda_N\), this part follows from Theorem \[\[\text{2.2}\]\].

(b) Let \(N = X_i \otimes \mu^i\), then
\[ [NX, NY]^\Lambda = (X, \mu^i) (Y, \mu^j)[X_i, X_j]^\Lambda + (X, \mu^i) \mathcal{L}_X^\Lambda((Y, \mu^j)) X_j - (Y, \mu^j) \mathcal{L}_Y^\Lambda((X, \mu^i)) X_i \]
and

\[
N[X,Y]_N^A = N ([X,\mu^i][X_i,Y]^A - L_X^B ((X,\mu^i))X_i + (Y,\mu^i)[X,X_j]
\]
\[
+ L_X^B ((Y,\mu^i))X_j - \langle [X,Y]^A,\mu^i \rangle X_i
\]
\[
= \langle X,\mu^i \rangle ([X_i,Y]^A,\mu^i)X_j - L_X^B ((X,\mu^i))X_i
\]
\[
+ (Y,\mu^i) ([X,X_j]^A,\mu^i)X_i + L_X^B ((Y,\mu^i))X_j - \langle [X,Y]^A,\mu^i \rangle (X_i,\mu^i)X_j. \quad (3.12)
\]

Hence, using properties of Lie derivatives, we get

\[
T_N^A (X,Y) = \langle X,\mu^i \rangle (Y,\mu^i) [X_i, X_j]^A + \langle X,\mu^i \rangle (Y,\mu^i) L_X^B (X_i,\mu^i)X_j
\]
\[
- (Y,\mu^i) (X,\mu^i) L_X^B (X_i,\mu^i)X_j + d\mu^i (X,Y) (X_i,\mu^i)X_j
\]
\[
= \left( \frac{1}{2} \mu^i \otimes [X_i, X_j]^A + \mu_i \wedge L_X^B \mu^i \otimes X_j + d\mu^i \otimes i_X, \mu^j \otimes X_j \right) (X,Y)
\]
\[
= \frac{1}{2} [N, N]^A_{P-N} (X,Y).
\]

(c)

\[
[[B^A, i_N]_{N-R}, i_N]_{N-R} (X,Y) =
\]
\[
= ([N^2 X, Y]^A + [N X, N Y]^A - N [N X, Y]^A) + ([N X, N Y]^A + [X, N^2 Y]^A - N [X, N Y]^A)
\]
\[
- (N [N X, Y]^A + N [N X, Y]^A - N^2 [X,Y]^A)
\]
\[
= 2 ([N X, N Y]^A - N ([N X, Y]^A + [X, N Y]^A - N [X, Y]^A))
\]
\[
+ [N^2 X, Y]^A + [X, N^2 Y]^A - N^2 [X,Y]^A = 2T_N^A (X,Y) + [B^A, i_N]_{N-R} (X,Y). \quad (3.13)
\]

(d) The Schouten bracket induced by \( \Lambda_X \) is given by \([i_N, B^A]_{N-R}\) and it is known from general theory \(\ref{fn:3.3}\), that it defines a graded Lie algebra structure if and only if its Nijenhuis-Richardson square vanishes. Using the graded Jacobi identity for \([\,,\,]_{N-R} \), we get

\[
[[i_N, B^A]_{N-R}, [i_N, B^A]_{N-R}]_{N-R} = [[i_N, B^A]_{N-R}, i_N]_{N-R} - B^A]_{N-R}
\]
\[
= -2 [T_N^A, B^A]_{N-R} + [[i_N, B^A]_{N-R}, B^A]_{N-R} = 0,
\]

since \(T_N^A = 0\) and \([B^A, B^A]_{N-R} = 0\) (\([\,,\,]^A\) is a Lie bracket) implies that \((a a_{B^A}^{-R})^2 = 0\).

The following theorem is, essentially, due to Mackenzie and Xu \(\ref{fn:3.3}\).

**Theorem 3.4** Let \( \Lambda \) be a Poisson tensor on \( E^* \) and let \( P \in \Phi^2(\tau) \). Then

(a) \( d_\Lambda^A (P) \) induces a pre-Lie algebroid structure on \( E^* \), with the bracket and the anchor described in Theorem \(\ref{th:3.2}\). The exterior derivative, induced by \( d_\Lambda^A (P) \), is given by

\[
d^{\Phi_\Lambda} (P) (X) = [P, X]^A. \quad (3.14)
\]

Moreover,

\[
\frac{1}{2} [P, P]^A (\mu, \nu, \gamma) = \langle \vec{P} ([\mu, \nu]^{d_\Lambda^A (P)}) - [P_\mu, P_\nu]^A, \gamma \rangle \quad (3.15)
\]

for all \( \mu, \nu, \gamma \in \Phi^1(\pi) \) and \( P \) is a Poisson tensor for \( \Lambda \) (i.e., \([P, P]^A = 0\)) if and only if \( \Lambda \) and \( d_\Lambda^A (P) \) are \( \vec{P} \)-related, where \( \vec{P} (\mu) = P_\mu = i_{\mu} P \).

(b) If \( P \) is, in addition, a Poisson tensor for \( \Lambda \), then \( d_\Lambda^A (P) \) is a Poisson tensor and Poisson tensors \( \Lambda, d_\Lambda^A (P) \) induce a Lie bialgebroid structure on bundles \( E \) and \( E^* \), i.e.,

\[
d_\Lambda \left( [\mu, \nu]^{d_\Lambda^A (P)} \right) = [d_\Lambda \mu, \nu]^{d_\Lambda^A (P)} + (-1)^{\mu+1} [\mu, d_\Lambda \nu]^{d_\Lambda^A (P)}. \quad (3.16)
\]
PROOF. The proof of (3.13) is completely analogous to the proof in the classical case (see [3]). The remaining part of (a) follows from Theorem 2.2. Part (b) is proved in [3].

Remark. Due to the result of Kosmann-Schwarzbach ([3], 3.16) is equivalent to
\[ [P, [X, Y]^\Lambda]^\Lambda = [[P, X]^\Lambda, Y]^\Lambda + (-1)^2 [X, [P, Y]^\Lambda]^\Lambda, \tag{3.17} \]
which is a special case of the graded Jacobi identity for the bracket \([\cdot, \cdot]^\Lambda\).

The fact that \(d\Lambda^P(P)\) is a Poisson tensor, if \([P, P]^\Lambda = 0\), is a direct consequence of 2.11. The converse to this is not true, in general, as shows the following example.

Example 2. For a Lie algebroid over a point, i.e., for a Lie algebra \(g\) with the bracket \([\cdot, \cdot]^\Lambda\), corresponding to a Kirillov-Kostant-Souriau tensor \(\Lambda\) on \(g^*\), \(P \in \wedge^2 g\) is a Poisson tensor for \(\Lambda\) if and only if \(P\) is an \(r\)-matrix satisfying the classical Yang-Baxter equation \([P, P]^\Lambda = 0\).

On the other hand, \(d\Lambda^P(P)\) is a Poisson tensor if and only if \(d\Lambda^P([P, P]^\Lambda) = 0\) which means, that \(\text{ad}_\xi[P, P]^\Lambda = 0\) for all \(\xi \in g\), i.e., the equation \(d\Lambda^P([P, P]^\Lambda) = 0\) is the modified Yang-Baxter equation.

**Definition 3.2.** Let \(P \in \Phi^2(\tau)\) be a Poisson tensor with respect to a Lie algebroid structure on \(E\), associated to a Poisson tensor \(\Lambda\) on \(E^*\), and let \(N \in \Phi^1(\tau)\) be a Nijenhuis tensor for \(\Lambda\). We call the pair \((P, N)\) a Poisson-Nijenhuis structure for \(\Lambda\) if the following two conditions are satisfied:

1. \(NP = PN^*\), where \(NP(\mu, \nu) = P(\mu, i_N \nu)\) and \(PN^*(\mu, \nu) = P(i_N \mu, \nu)\),
2. \(d\Lambda^N(P) = (d\Lambda^P(P))_N\).

Remark. Since \(NP + PN^* = i_N P\) and, according to Theorems 3.1 and 2.2,
\[ (d\Lambda^P(P))_N = \mathcal{L}_{\mathcal{J}_E(N)} d\Lambda^P(P), \]
\[ d\Lambda^N(P) = d\Lambda^P(i_N P) - \mathcal{L}_{\mathcal{J}_E(N)} d\Lambda^P(P), \]
the condition (2) can be replaced by
\[ (2') \mathcal{L}_{\mathcal{J}_E(N)} d\Lambda^P(P) = (d\Lambda^P(P))_N = d\Lambda^P(NP). \]

**Theorem 3.5** If \((P, N)\) is a Poisson-Nijenhuis structure for \(\Lambda\) then \(NP\) is a Poisson tensor for \(\Lambda\) and we have the following commutative diagram of Poisson mappings between Poisson manifolds.

\[
\begin{array}{ccc}
(E^*, \Lambda) & \xrightarrow{-\mathcal{P}} & (E, d\Lambda^P(P)) \\
\downarrow \mathcal{N}^* & & \downarrow \mathcal{N} \\
(E^*, \mathcal{L}_E(N) \Lambda) & \xrightarrow{-\mathcal{P}} & (E, d\Lambda^P(NP) = (d\Lambda^P(P))_N)
\end{array}
\]

where \(\Lambda_N = \mathcal{L}_{\mathcal{J}_E(N)} \Lambda\) and \((d\Lambda^P(P))_N = \mathcal{L}_{\mathcal{J}_E(N)} d\Lambda^P(P)\). Moreover, every structure from the left-hand side of this diagram constitutes a Lie bialgebroid structure with every right-hand side structure.

**Proof.** The tensors \(\Lambda_N\) and \(d\Lambda^P(P)\) are Poisson. The mappings \(-\mathcal{P}: (E^*, \Lambda) \rightarrow (E, d\Lambda^P(P))\) and \(\mathcal{N}^*: (E, \Lambda) \rightarrow (E^*, \mathcal{L}_E(N) \Lambda)\) are Poisson, in view of Theorems 3.3 and 3.4. The assumption \(NP = PN^*\) implies that the diagram is commutative. To show that the mapping \(-\mathcal{P}: (E^*, \mathcal{L}_E(N) \Lambda) \rightarrow (E, (d\Lambda^P(P))_N)\) is Poisson, it is enough to check that, under the assumption \(NP = PN^*\), the vector fields \(\mathcal{J}_E(N)\) and \(\mathcal{J}_E(N)\) are \(-\mathcal{P}\)-related. One can do it easily. Since \(\Lambda\) and \(d\Lambda^P(P)\) are \(-\mathcal{P}\)-related, also \(\Lambda_N = [\mathcal{J}_E(N), \Lambda]\) and \((d\Lambda^P(P))_N = [\mathcal{J}_E(N), d\Lambda^P(P)]\) are \(-\mathcal{P}\)-related. Hence, the equality \((d\Lambda^P(P))_N = d\Lambda^N(P)\) implies that \(\Lambda\) and \(d\Lambda^N(P)\) are \(-NP\)-related and, according to Theorem 3.4 (a), \(NP\) is a Poisson tensor for \(\Lambda\).

The fact that the mapping \(\mathcal{N}: (E, d\Lambda^P(P)) \rightarrow (E, (d\Lambda^P(P))_N)\) is Poisson follows from the identity
\[
\langle X, [N, N][d\Lambda^P(P)](\alpha, \beta) \rangle = \langle [N, N][d\Lambda^P(P)](X, P\beta), \alpha \rangle \]
\[+ 2\langle X, C^\Lambda(P, N)(i_N \alpha, \beta) \rangle - 2\langle NX, C^\Lambda(P, N)(\alpha, \beta) \rangle, \tag{3.18} \]
where $C^\Lambda(P, N)(\alpha, \beta) = [\alpha, \beta]d^\Lambda_{\{NP\}} - [\alpha, \beta]^\Lambda_{\{NP\}}$. This is a generalization of an analogous identity in [3], with a completely parallel proof.

The pairs $(\Lambda, d^\Lambda_{\{P\}})$ and $(\Lambda, d^\Lambda_{\{NP\}})$ constitute Lie bialgebroids by Theorem [3, b), since $P$ and $NP$ are Poisson tensors for $\Lambda$.

Similarly, $(\Lambda_N, d^\Lambda_{\{NP\}}) = d^\Lambda_{\{NP\}}(P)$ constitute a Lie bialgebroid, since $P$ is a Poisson tensor for $\Lambda_N$ ($\Lambda_N$ and $d^\Lambda_{\{NP\}}$ are $-\tilde{P}$-related).

To show that the pair $(\Lambda_N, d^\Lambda_{\{NP\}}(P))$ forms a Lie bialgebroid, we have to prove that $d^\Lambda_{\{NP\}} = d^\Lambda_{\{NP\}}$ is a derivation of the Schouten bracket $B = [\cdot, \cdot]d^\Lambda_{\{NP\}}$, i.e., $[d^\Lambda_{\{NP\}}, B]_{N-R} = 0$. Since, due to [3, 3],

$$[d^\Lambda_{\{NP\}}, B]_{N-R} = [i_N, d^\Lambda_{\{NP\}}(P)]_{N-R} = [i_N, [d^\Lambda_{\{NP\}}, B]_{N-R}]_{N-R}$$

+ $[d^\Lambda_{\{NP\}}, i_N, B]_{N-R} = [d^\Lambda_{\{NP\}}, B]_{N-R} = 0$, (3.19)

in view of the fact that $[i_N, B]_{N-R}$ is the bracket associated to $(d^\Lambda_{\{NP\}}(P))_{N}$, for which $d^\Lambda$ is a derivation.

\[\square\]

**Remark.** The above diagram is a dualization of a similar diagram in [1].

In the case of the canonical Lie algebroid on $E = TM$, the fact that $((\Lambda_M), d^\Lambda_{\{TP\}})$ constitutes a Lie bialgebroid is equivalent to the fact that $(P, N)$ is a Poisson-Nijenhuis structure, as it was recently shown in [3]. This is due to the formulae

$$A(f, g) = \langle(NP - PN^*)d^\Lambda_{\{NP\}} g, d^\Lambda_{\{NP\}} f\rangle, \quad \text{(3.20)}$$

$$A(d^\Lambda_{\{NP\}} f, g) = C^\Lambda(P, N)(d^\Lambda_{\{NP\}} f, d^\Lambda_{\{NP\}} g) + d^\Lambda A(f, g), \quad \text{(3.21)}$$

where $A = [d^\Lambda_{\{NP\}}, B_{d^\Lambda_{\{NP\}}}]_{N-R}$, and the fact that $A$ satisfies a Leibniz rule and $d^\Lambda f$ generate $T^*M$, for $\Lambda = \Lambda_M$.

In general, this is not true and we can have $((\Lambda)_{\{NP\}}, d^\Lambda_{\{TP\}})$ being a Lie bialgebroid with $(P, N)$ not being Poisson-Nijenhuis structure for $\Lambda$, even if we assume the equality $NP = PN^*$, as shows the following example.

**Example.** As a Lie algebroid over a single point, let us take a Lie algebra $\mathfrak{g}$ spanned by $\xi_1, \xi_2, \xi_3, \xi_4$ with the bracket defined by $\Lambda = \xi_3 \partial_{\xi_1} \wedge \partial_{\xi_2}$. The tensor $P = \partial_{\xi_2} \wedge \partial_{\xi_4}$ is a Poisson tensor with $d^\Lambda_{\{TP\}} P = y_1 \partial_{y_1} \wedge \partial_{y_4}$.

The tensor

$$N = -\xi_1 \otimes y_1 + \sum_{i=2}^4 \xi_i \otimes y_i$$

is a Nijenhuis tensor for $\Lambda$ and $\Lambda_{\{NP\}} = -\Lambda$. Moreover, $NP = PN^* = P$, so that $(\Lambda, d^\Lambda_{\{NP\}})$ constitutes a Lie bialgebroid. In this case, however, $d^\Lambda_{\{NP\}} P = -d^\Lambda_{\{NP\}} P = -d^\Lambda_{\{NP\}} P$ and $(PN)$ is not a Poisson-Nijenhuis structure.

It is easy to see that, as in the classical case, a Poisson-Nijenhuis structure for a Lie algebroid induces a whole hierarchy of compatible Poisson structures and Nijenhuis tensors (see [3]). Since this theory goes quite parallel to the classical case, we will not present details here.

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