In this work, we investigate the characteristics of the spin-singlet state $\eta_b$ of the bottomonia family via the radiative decays of $\Upsilon(nS) \rightarrow \eta_b + \gamma$. The theoretical estimation of the decay widths is carried out in terms of the light-front quark model (LFQM). Recently CLEO and BaBar collaborations have measured $\mathcal{B}(\Upsilon(3S) \rightarrow \gamma\eta_b)$ and the mass of $\eta_b$. In terms of the data we fix the concerned input parameters in our calculations of $\Upsilon(nS) \rightarrow \eta_b + \gamma$. A special attention is paid on the transition of $\Upsilon(5S) \rightarrow \eta_b + \gamma$. The BELLE data showed that the width of $\Upsilon(5S) \rightarrow \Upsilon(2S,1S) + \pi \pi$ is two orders larger than that of $\Upsilon(4S) \rightarrow \Upsilon(2S,1S) + \pi \pi$, thus some theoretical explanations have been proposed. Among them, it is suggested the inelastic final state interaction (IFSI) $\Upsilon(5S) \rightarrow B\bar{B} \rightarrow \Upsilon(1S) + \pi \pi$ may be a natural one. If so, a similar mechanism also applies to $\Upsilon(5S) \rightarrow B^{(*)}\bar{B}^{(*)} \rightarrow \eta_b + \gamma$, the precise measurement would serve as a good test whether $\Upsilon(5S)$ possess exotic components.

Our calculation in the LFQM indicates that the rate of the direct process $\Upsilon(5S) \rightarrow \eta_b + \gamma$ is not anomalous compared to $\Upsilon(mS) \rightarrow \eta_b + \gamma (m = 1,2,3,4)$, thus if the IFSI does apply, the rate of $\Upsilon(5S) \rightarrow \eta_b + \gamma$ should be larger than the others by orders.

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More recently the CLEO Collaboration confirmed the observation of $\eta_b$ using the database of 6 million $\Upsilon(3S)$ decays and assuming $\Gamma(\eta_b) \approx 10$ MeV, they obtained $\mathcal{B}(\Upsilon(3S) \rightarrow \gamma\eta_b) = (7.1 \pm 1.8 \pm 1.1) \times 10^{-4}$, $M_{\eta_b} = 9391.8 \pm 6.6 \pm 2.0$ MeV and the hyperfine splitting $\Delta M = 68.5 \pm 6.6 \pm 2.0$ MeV, whereas using the database with 9 million $\Upsilon(2S)$ decays they obtained $\mathcal{B}(\Upsilon(2S) \rightarrow \gamma\eta_b) < 8.4 \times 10^{-4}$ at 90% confidence level.

It is noted that the data of the two Collaborations are in accordance on $M_{\eta_b}$ but the central values of $\mathcal{B}(\Upsilon(3S) \rightarrow \gamma\eta_b)$ are different. However, if the experimental errors are taken into account, the difference is still within one standard deviation.

Some theoretical work is devoted to account the experimental results.

In Ref. the authors studied these radiative decays and estimated $\mathcal{B}(\Upsilon(3S) \rightarrow \eta_b + \gamma) = 4 \times 10^{-4}$, $\mathcal{B}(\Upsilon(2S) \rightarrow \eta_b + \gamma) = 1.5 \times 10^{-4}$ and $\mathcal{B}(\Upsilon(1S) \rightarrow \eta_b + \gamma) = 1.1 \times 10^{-4}$ with the mass $M_{\eta_b} = 9.400$ GeV. Their results about $M_{\eta_b}$ and $\mathcal{B}(\Upsilon(3S) \rightarrow \eta_b + \gamma)$ are close to the data. The authors of Ref. systematically investigated the magnetic dipole transition $\Upsilon \rightarrow P\gamma$ in the light-front quark model (LFQM). In the QCD-motivated effective Hamiltonian there are several free parameters, i.e., the quark mass and the in the wavefunction (the notation of $\beta$ was given in the aforementioned literatures) which are fixed by the variational principle, then $\mathcal{B}(\Upsilon(1S) \rightarrow \eta_b + \gamma)$ of $\eta_b$ as the Babar collaboration published its 10-sigma discovery of the $\eta_b$ in 2008 whereas in 2009 the CLEO collaboration released a re-analysis of their data and reported a 4-sigma confirmation of the Babar discovery.
was calculated and the central value is $8.4 \times 10^{-4}$. 2

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$\Delta M$ coming from different experimental measurement and theoretical work.}
\end{figure}

It is also noted that the mass of $m_{h_0}$ = 9.657 (or 9.295) GeV presented in Ref.\cite{14} deviates from the data ($m_{h_0} = 9391.8 \pm 6.6 \pm 2.0$ MeV) and the fitted $\beta$ values are different for singlet and triplet \cite{18}. Here we are going to take an alternative way to fix the values of $\beta$.

Since experimentally, $m_{h_0}$ is determined by $B(\Upsilon(nS) \rightarrow h_0 + \gamma)$ and a study on the radiative decays can offer us much information about the characteristics of $h_0$, one should carefully investigate the transition within a more reliable theoretical framework. That is the aim of the present work, namely we will use the LFQM to evaluate the hadronic matrix element which is governed by the non-perturbative QCD. The method is proven to be successful for calculating the transition rates of the processes where light hadrons exist in the final states. In this work, we first fix $\beta_{nS}$'s for $\Upsilon(nS)$ in terms of their decay constants. Then, using the data of $B(\Upsilon(3S) \rightarrow h_0 + \gamma)$ and $m_{h_0}$, we determine $\beta_{h_0}$. With the parameters being fixed, we are able to estimate the rates $B(\Upsilon(1S) \rightarrow h_0 + \gamma)$, $B(\Upsilon(2S) \rightarrow h_0 + \gamma)$, $B(\Upsilon(4S) \rightarrow h_0 + \gamma)$ and $B(\Upsilon(5S) \rightarrow h_0 + \gamma)$. Since $B(\Upsilon(1S) \rightarrow h_0 + \gamma)$ is sensitive to $\Delta M$, the measurement of $B(\Upsilon(1S) \rightarrow h_0 + \gamma)$ would be helpful for accurately determining the mass of $h_0$.

Recently the transition rates of $\Upsilon(5S) \rightarrow \Upsilon(1S), 2S) + \pi \pi$ were measured by the BELLE Collaboration \cite{21} and it was found that the widths exceed by more than two orders of magnitude the previously measured partial widths between lower $\Upsilon$ resonances. The authors \cite{22} suggested that the re-scattering processes of $\Upsilon(5S) \rightarrow B^{(*)}\bar{B}^{(*)} \rightarrow \Upsilon(mS) + \sigma/f_0(980) \rightarrow \Upsilon(mS) + \pi \pi$ make substantial contributions to the observable rate of the dipion transition of $\Upsilon(5S)$ because its mass exceeds the production threshold of $B^{(*)}\bar{B}^{(*)}$, so that the intermediate bosons $B^{(*)}$ and $\bar{B}^{(*)}$ are on their mass-shell. They apply the same mechanism to study the transition of $\Upsilon(5S) \rightarrow \Upsilon(1S) + \eta$ and find that the re-scattering processes would enhance its width by almost two orders of magnitude. There are different interpretations for the anomalous enhancement of the $\Upsilon(5S)$ decays that the measured resonance $\Upsilon(10870)$ is a mixture of $b\bar{b}$ bound state in the $5S$ state with a hybrid $b\bar{b}g$ or a tetraquark $b\bar{b}gg$ \cite{24}.

Thus, we would like to further test the mechanism in the radiative decays of $\Upsilon(5S)$. In the radiative decay of $\Upsilon(5S) \rightarrow h_0 + \gamma$, the re-scattering processes $\Upsilon(5S) \rightarrow B^{(*)}\bar{B}^{(*)} \rightarrow \Upsilon(mS) + \gamma$ also exist and one only needs to replace the effective vertex of $B^{(*)}\bar{B}^{(*)}\Upsilon(1S)$ and $B^{(*)}\bar{B}^{(*)}\eta$ by the electromagnetic vertex $B^{(*)}\bar{B}^{(*)}\gamma$ and $B^{(*)}\bar{B}^{(*)}\eta_h$ respectively in the diagrams given in Ref.\cite{22}. Thus one can expect that the corresponding mechanism should enhance the ratio of $\Upsilon(5S) \rightarrow h_0 + \gamma$. Our calculations show that the theoretical estimation on the enhancement factor strongly depends on the parameter $g_{B^{(*)}\bar{B}^{(*)}\eta_h}$ (see below for more details). The future measurements on the radiative decays of $\Upsilon(10870)$ can help to determine if it is the $\Upsilon(5S)$ state as long as the rate of $\Upsilon(5S) \rightarrow h_0 + \gamma$ is obviously larger than that of lower resonances of the family, otherwise other mechanisms may be more favored. Anyhow, the radiative decays would provide a decisive probe for the re-scattering mechanism.

This paper is organized as follows: after the introduction, in section II we present our calculations of the form factors for $\nu \rightarrow P\gamma$ in the LFQM and the corresponding numerical results. In the section III we study the possible re-scattering effects on $\Upsilon(5S) \rightarrow h_0 + \gamma$. The section IV is devoted to our conclusion and discussion.

II. $\Upsilon(nS) \rightarrow h_0 + \gamma$ IN THE LFQM

A. Description of $\Upsilon(nS) \rightarrow h_0 + \gamma$ in the LFQM

The Feynman diagrams describing $\Upsilon(nS) \rightarrow h_0 + \gamma$ are presented in Fig.\cite{23}. In this work, we calculate the transition rate of the radiative decays $\Upsilon(nS) \rightarrow h_0 + \gamma$ in the LFQM.

The transition amplitude of $\Upsilon(nS) \rightarrow h_0 + \gamma$ can be expressed in terms of the form factor $F_{\Upsilon(nS)\rightarrow h_0}(q^2)$ which is defined as \cite{16,19}

$$\langle h_0(P')|J_{\mu n}|\Upsilon(P, h)\rangle = ie^{\mu\nu\rho\sigma}e_{\nu}(P, h)q_{\rho}F_{\Upsilon(nS)\rightarrow h_0}(q^2),$$

(1)

where $P$ and $P'$ are the four-momenta of $\Upsilon(nS)$ and $h_0$, $q = P - P'$ is the four-momentum of the emitted photon and $e_{\nu}(P, h)$ denotes the polarization vector of $\Upsilon(nS)$.
with helicity h. For applying the LFQM, we first let the photon be virtual, i.e. leave its mass-shell \( q^2 = 0 \) into the un-physical region of \( q^2 < 0 \). Then \( F_{\Upsilon(nS)\to\eta_b}(q^2) \) can be obtained in the \( q^2 = 0 \) frame with \( q^2 = q^+ q^- - q_\perp^2 = -q_\perp^2 < 0 \). Then we just analytically extrapolate \( F_{\Upsilon(nS)\to\eta_b}(q_\perp^2) \) from the space-like region to the time-like region \( (q^2 \geq 0) \). By taking the limit \( q^2 \to 0 \), one obtains \( F_{\Upsilon(nS)\to\eta_b}(q^2 = 0) \).

By means of the light front quark model, one can obtain the expression of form factor \( F_{\Upsilon(nS)\to\eta_b}(q^2) \) in parallel, we can calculate the constant in the LFQM

\[
f_V = \frac{\sqrt{N_c}}{4\pi^3 M} \int dx \int d^2 k_\perp \frac{\varphi}{\sqrt{2x(1-x)M_0}} \left[ xM_0^2 - m_1(m_1 - m_2) - \frac{m_1 + m_2}{M_0 + m_1 + m_2} k_\perp^2 \right],
\]
equating the two results. \( \beta_{\Upsilon(nS)} \) is determined (see Table II). Then, we use the formula presented in Section II to calculate \( B(\Upsilon(3S) \to \eta_b + \gamma) \) and compare it with the central value of the experimental data \( B(\Upsilon(3S) \to \eta_b + \gamma) = (7.1 \pm 1.8 \pm 1.1) \times 10^{-4} \) or \( B(\Upsilon(3S) \to \eta_b + \gamma) = (4.8 \pm 0.5 \pm 0.6) \times 10^{-4} \) to fit \( \beta_b \) and the corresponding values are presented in Table III.

In order to illustrate the dependence of our results on \( m_b \), we re-set \( m_b = 5.2 \) GeV, which was adopted by the authors of Ref. 10 and fitted \( \beta_{\Upsilon(nS)} \) and \( \beta_b \) again. Using the new parameter the \( B(\Upsilon(nS) \to \eta_b + \gamma) \) is computed and the result is also listed in Table III. From the Table III we find

1. The predicted branching ratios \( B(\Upsilon(1S) \to \eta_b + \gamma) \) and \( B(\Upsilon(4S) \to \eta_b + \gamma) \) are not sensitive to \( m_b \) and \( \beta \).

2. Numerical results

In Ref. 10 the authors fixed the parameter \( \beta_b = 1.145 \) or 1.803 \(^3\) in the Gaussian wavefunction using the variational method with \( m_b = 5.2 \) GeV. However their predictions on the mass \( m_{\eta_b} = 9.657 \) (or 9.295) GeV and \( m_{\Upsilon(1S)} = 9.691 \) (or 9.558) GeV (the values in the brackets are obtained with a different potential form), which obviously deviate from data \( (m_{\eta_b} = 9300 \pm 20 \pm 20 \) GeV and \( m_{\Upsilon(1S)} = 9460.30 \pm 0.26 \) GeV \(^2\)), thus we are going to fix \( \beta \) in an alternative way.

First we set the b quark mass as \( m_b = 4.64 \) GeV which was used in Ref. 18. Then we extract the decay constant \( f_{\Upsilon(nS)} \) of \( \Upsilon(nS) \) from the data \( \Gamma(\Upsilon(nS) \to e^+e^-) \) with

\[
\Gamma(V \to e^+e^-) = \frac{4\pi \alpha^2}{27 M_V} f_V^2.
\]

equating the two results. \( \beta_{\Upsilon(nS)} \) is determined (see Table II). Then, we use the formula presented in Section II to calculate \( B(\Upsilon(3S) \to \eta_b + \gamma) \) and compare it with the central value of the experimental data \( B(\Upsilon(3S) \to \eta_b + \gamma) = (7.1 \pm 1.8 \pm 1.1) \times 10^{-4} \) or \( B(\Upsilon(3S) \to \eta_b + \gamma) = (4.8 \pm 0.5 \pm 0.6) \times 10^{-4} \) to fit \( \beta_b \) and the corresponding values are presented in Table III. At last with all the parameters we estimate \( B(\Upsilon(1S) \to \gamma\eta_b)), B(\Upsilon(2S) \to \eta_b + \gamma), B(\Upsilon(4S) \to \eta_b + \gamma) \) and \( B(\Upsilon(5S) \to \eta_b + \gamma) \) which are shown in Table III. It is noted that at this step, we only consider the direct decay modes, but for \( \Upsilon(5S) \to \eta_b + \gamma \), the re-scattering effect may play a dominant role as mentioned in the introduction and we will discuss the details in next section.

\[^3\] The authors of Ref. 10 used the harmonic oscillator and linear potential forms for the confinement term in their computations, thus they obtained two different values for the \( \beta \) parameter. Since later in our calculations we do not evaluate the spectra of the concerned hadrons, the concrete value of \( \beta \) does not influence our numerical results.
but $B(\Upsilon(2S) \to \eta_b + \gamma)$ and $B(\Upsilon(5S) \to \eta_b + \gamma)$ slightly change as $m_b$ and $\beta$ vary within certain ranges:

2. Our results about $B(\Upsilon(1S) \to \eta_b + \gamma)$ and $B(\Upsilon(2S) \to \eta_b + \gamma)$ are somehow larger than that given in Ref. [6];

3. If the final state interaction is not taken into account, the branching ratio of $\Upsilon(5S) \to \eta_b + \gamma$ is not anomalous compared to $\Upsilon(mS) \to \eta_b + \gamma$ ($m = 1, 2, 3, 4$). The BELLE Collaboration [26] found the rate of $\Upsilon(5S) \to \Upsilon(1S, 2S) + \pi\pi$ is anomalously large compared to the similar dipion transitions between lower $\Upsilon$ resonances, one has reason to doubt if such anomaly would appear in $\Upsilon(5S) \to \eta_b + \gamma$.

It is noted that $B(\Upsilon(1S) \to \eta_b + \gamma)$ is sensitive to $m_{\eta_b}$ (or $\Delta M$) since the decay width is proportional to $(\Delta M)^3$, thus as $\Delta M$ is small, i.e., the masses of initial and daughter mesons are close to each other, any small changes of $m_{\eta_b}$ which has not been accurately measured yet [23], can lead to a remarkable difference. In Fig. 3 we display the dependence of $B(\Upsilon(1S) \to \eta_b + \gamma)$ on $\Delta M$. Thus the accurate measurement on $B(\Upsilon(1S) \to \eta_b + \gamma)$ will be a great help to determine the mass of $m_{\eta_b}$.

### III. POSSIBLE RE-SCATTERING EFFECTS INDUCING A LARGE $B(\Upsilon(5S) \to \eta_b + \gamma)$

As aforementioned, the re-scattering of hadrons may remarkably enhance the rate of $\Upsilon(nS)$ ($n > 4$) $\to \eta_b + \gamma$. Similar to the re-scattering effects on in the branching ratios of $\Upsilon(5S) \to \Upsilon(1S) + \pi\pi$ [22] and $\Upsilon(nS)(n \geq 4) \to \Upsilon(1S)\eta$ [22], the transitions $\Upsilon(nS) \to \eta_b + \gamma$ can occur via re-scattering sub-processes with the intermediate states being $B^{(*)}B^{(*)}$ where another $B^{(*)}$ meson is exchanged at the t-channel. The corresponding diagrams are depicted in Fig. 4. Since the other diagrams can be obtained by a charge conjugation transformation of $B^{(*)} \to B^{(*)}$ the contribution of each diagram in Figs. 4 (a-f) should be multiplied by a factor 2.

Now let us calculate the amplitudes of the re-scattering processes which occur at the hadron level. Following Refs. [22,23] we will not account for the contribution from the dispersive parts of the diagrams where $B^{(*)}$ from $\Upsilon(nS)$ are off-shell, but only consider the absorptive parts where $B^{(*)}$ are real particles on their mass-shells. The off-shell effect of the meson exchanged at t-channel is compensated by a monopole form factor which is also reflects the inner structures of the mesons at the effective vertex

$$F(m_i, q^2) = \frac{(\Lambda + m_i)^2 - m^2}{(\Lambda + m_i)^2 - q^2}, \quad (7)$$

where $q$ and $m_i$ are the momentum and mass of the exchanged meson respectively. And the cutoff is set as $\Lambda = 600$ GeV [22, 23].

The absorptive part of the amplitude is read as

$$\text{Abs}_i = \frac{|p_1|}{32\pi^2 m_{\Upsilon(nS)}} \int d\Omega A_i [\Upsilon(nS) \to B^{(*)} \bar{B}^{(*)}] \times C_i [B^{(*)} \bar{B}^{(*)} \to \eta_b + \gamma] \times F(m_i, q^2), \quad (8)$$

with $i = a, b, c, d, e, f$. Here, $d\Omega$ and $p_1$ are the solid angle and linear momentum of the on-shell $B^{(*)}$ in the rest frame of $\Upsilon(nS)$, respectively.

The effective couplings for $\Upsilon BB$, $\Upsilon B^*B$ and $\Upsilon B^*B^*$ we adopt in this work are directly borrowed from Refs. [22, 23], and some discussions will be made in the last
FIG. 3: The dependance of $\mathcal{B}(\Upsilon(1S) \to \gamma \eta_b)$ on $\Delta M$.

FIG. 4: The diagrams for $\Upsilon(nS) \to B^{(*)} B^{(*)} \to \eta_b + \gamma$. Other diagrams can be obtained by a charge conjugation transformation $B^{(*)} \leftrightarrow B^{(*)}$.

The diagrams for $\Upsilon(nS) \to B^{(*)} B^{(*)} \to \eta_b + \gamma$. Other diagrams can be obtained by a charge conjugation transformation $B^{(*)} \leftrightarrow B^{(*)}$.

\[ g_{\gamma BB} = 2.5 \]
\[ g_{\gamma B^* B} = 1.4 \pm 0.3 \]
\[ g_{\gamma B^* B} = 2.5 \pm 0.4. \] (10)

Following the strategy of Ref.23, we obtain

\[ \mathcal{L}_{\gamma BB} = g_{\gamma BB} A_\mu (\partial^\mu B \gamma^\dag - B \partial^\mu B^\dag), \] (11a)
\[ \mathcal{L}_{\gamma B^* B} = \frac{g_{\gamma B^* B}}{m_{B^*}} e^{\mu \nu \alpha \beta} \partial_\mu \partial_\nu \partial_\alpha \partial_\beta \times (B^\dag \alpha \beta B^B - B \partial_\alpha \partial_\beta B^\dag), \] (11b)
\[ \mathcal{L}_{\gamma B^* B} = g_{\gamma B^* B} (A^\mu B^\nu \gamma_\alpha \beta \partial_\mu B^\nu \partial_\alpha \partial_\beta B^\dag), \] (11c)
\[ \mathcal{L}_{B^* B \eta_b} = ig_{B^* B \eta_b} B^\dag_\mu \partial_\mu \eta_b B, \] (11d)
\[ \mathcal{L}_{B^* B \eta_b} = \frac{g_{B^* B \eta_b}}{m_B} e^{\mu \nu \alpha \beta} \partial_\mu B^\dag_\nu \partial_\alpha \partial_\beta \eta_b. \] (11e)

With the heavy quark spin symmetry22, we have

\[ g_{\gamma BB} = g_{\gamma B^* B} = g_{\gamma B^* B}, \]
\[ g_{\eta_b B^* B} = g_{\eta_b B^* B} = g_{\eta_b(1S) BB} \] (12)

In terms of the theoretically evaluated value of $\Gamma(B^{*+} \to B^{*+} \gamma) = 0.40 \pm 0.03$ keV we can fix $g_{B^* B} \approx 3.5$. The coupling $g_{\eta_b B^* B}$ should be at the order of $O(1)$, but may vary within a reasonable range. If we choose $g_{\eta_b B^* B} \approx g_{\eta_b B^* B} \approx 1$ the contributions of the diagrams (a), (c), (d), (f) in Fig. 4 to $B(\Upsilon(5S) \to \gamma \eta_b)$ are approximately a few of $10^{-7}$ GeV and that of the diagrams (b), (e) are slightly smaller than the direct transition. If the interference between the contributions of the direct decay and that through the re-scattering is constructive, the total width would be about a few times larger, but if it is destructive, the width would be very suppressed. However, the coupling $g_{\gamma(1S) BB}$ may be as large as 15 as was estimated in Refs.22-23. If we use this value for the coupling constants $g_{\eta_b B^* B}$, $g_{\eta_b B^* B}$, the contributions of the diagrams in Fig. 4 would enhance the total width by more than two orders. In principle, the diagram (a) in Fig. 4 can contribute to $\Upsilon(4S) \to \eta_b + \gamma$, but the mass of $\Upsilon(4S)$ is just above the threshold of $B^* B$, so that the phase space would greatly suppress the contribution of the diagram.

IV. CONCLUSION

By studying the radiative decay of $\Upsilon(nS) \to \eta_b + \gamma$, we can learn much about the hadronic structure of $\eta_b$. Efforts have been made to explore the spin singlet $\eta_b$ in the Data-book of 2008, $\eta_b$ was still omitted from the summary table 23. In fact, determination of the mass of $\eta_b$ is made via the radiative decays of $\Upsilon(nS) \to \eta_b + \gamma$, and the recent data show that $m_{\eta_b} = 9388.9^{+3.1}_{-2.3}(stat) \pm 2.7(syst)$ by the $\Upsilon(3S)$ data
and $m_{b} = 9394.2^{+4.8}_{-5.5}(\text{stat}) \pm 2.0(\text{syst})$ by the $\Upsilon(2S)$ data \cite{11}. Recently, Penin \cite{28} reviewed the progress for determining the mass of $\eta_b$ and indicated that the accurate theoretical prediction of $m_{b}$ would be a great challenge. Indeed, determining the wavefunction of $\eta_b$ would be even more challenging. In this work, we are not going to obtain the wavefunction or even the mass of $\eta_b$ based on the fundamental theories, such the non-perturbative QCD, but using the radiative transition to testify the phenomenologically determined wavefunction as long as the mass is well measured. We carefully study the transition rates of the radiative decays which would help experimentalists to extract information about $m_{b}$. The transition rate of $\Upsilon(1S) \rightarrow \eta_b + \gamma$ is very sensitive to the mass splitting $\Delta m = m_{\Upsilon(1S)} - m_{\eta_b}$ (see the text above), thus an accurate measurement of the radiative decay may be more useful to learn the spin dependence of the bottomonia.

Recently our experimental colleagues have made some progress. CLEO and BaBar collaborator measure the $\Upsilon(3S) \rightarrow \eta_b + \gamma$ and the mass $m_{\eta_b}$ which offer an opportunity for us to more deeply study $\eta_b$.

Following Ref. \cite{11} we systematically study $\Upsilon(nS) \rightarrow \eta_b + \gamma$ in the LFQM. We take an alternative way which is different from that adopted in Ref. \cite{11} to fix the parameters i.e., namely we use the data to fit $\beta_{\Upsilon(nS)}$ and $\beta_{\eta_b}$ in the wavefunctions. Then using these parameters we estimate $\Upsilon(1S) \rightarrow \eta_b + \gamma$, $\Upsilon(2S) \rightarrow \eta_b + \gamma$, $\Upsilon(4S) \rightarrow \eta_b + \gamma$ and $\Upsilon(5S) \rightarrow \eta_b + \gamma$. Our result indicates $\Upsilon(2S) \rightarrow \eta_b + \gamma$ and $\Upsilon(5S) \rightarrow \eta_b + \gamma$ are more sensitive to the parameters than $\Upsilon(1S) \rightarrow \eta_b + \gamma$ and $\Upsilon(4S) \rightarrow \eta_b + \gamma$.

Since the value of $\Upsilon(1S) \rightarrow \eta_b + \gamma$ is sensitive to $\Delta M$, we show the dependance of $B(\Upsilon(1S) \rightarrow \eta_b + \gamma)$ on $\Delta M$. We hope that our experimental colleagues will conduct accurate measurements in the near future to determine the precise value of $m_{\eta_b}$.

The anomalous largeness of the branching ratio of $\Upsilon(5S) \rightarrow \Upsilon(1S, 2S) + \pi \pi$ motivates a hot surf of theoretical studies \cite{28}. It was suggested that the re-scattering effects may explain the unusual large branching ratio. This mechanism should be tested somewhere else. In Ref. \cite{28} the authors evaluated the effect induced by the mechanism for $\Upsilon(5S) \rightarrow \Upsilon(1S, 2S) + \eta$ and found that the corresponding branching ratio is also greatly enhanced compared to the transition among lower resonances. We suggest to further test the mechanism at the radiative decays where the effective electromagnetic vertex is relatively simple. Our result which is obtained in terms of the LFQM, indicates the branching ratio of $\Upsilon(5S) \rightarrow \eta_b + \gamma$ is not anomalous compared to $\Upsilon(mS) \rightarrow \eta_b + \gamma$ ($m = 1, 2, 3, 4$) as long as the re-scattering is not taken into account. However, there could be a two-order enhancement in magnitude for $\Upsilon(5S) \rightarrow \eta_b + \gamma$ which is induced by the re-scattering effects. Thus measurement of $\Upsilon(5S) \rightarrow \eta_b + \gamma$ would be an ideal probe for the re-scattering mechanism which successfully explains the data of $\Upsilon(5S) \rightarrow \Upsilon(1S, 2S) + \pi \pi$. This is one of the tasks of the LHCb which will be oper-

ating very soon.

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Appendix

The incoming (outgoing) meson in Fig. 2 has the momentum $P^{(i)} = p_{1}^{(i)} + p_{2}$ where $p_{1}^{(i)}$ and $p_{2}$ are the momenta of the off-shell quark and antiquark and

\begin{align*}
  p_{1}^{+} &= x_{1}P^{+}, & p_{2}^{+} &= x_{2}P^{+}, \\
  p_{1\perp} &= x_{1}P_{\perp} + k_{\perp}, & p_{2\perp} &= x_{2}P_{\perp} - k_{\perp}, \\
  p_{1}'^{+} &= x_{1}P_{\perp}' + k_{\perp}', & p_{2}'^{+} &= x_{2}P_{\perp}' + k_{\perp}', \\
  p_{1\perp}' &= x_{1}P_{\perp}' + k_{\perp}', & p_{2\perp}' &= x_{2}P_{\perp}' - k_{\perp}'
\end{align*}

with $x_{1} + x_{2} = 1$, where $x_{i}$ and $k_{\perp}(k_{\perp}')$ are internal variables. $M_{0}$ and $\tilde{M}_{0}$ are defined

\begin{align*}
  M_{0}^{2} &= \frac{k_{\perp}^{2} + m_{2}^{2}}{x_{1}} + \frac{k_{\perp}'^{2} + m_{2}^{2}}{x_{2}}, \\
  \tilde{M}_{0} &= \sqrt{M_{0}^{2} - (m_{1} - m_{2})^{2}}.
\end{align*}

The radial wavefunctions $\phi$ related to $\Upsilon(nS)$ are de-
\begin{align*}
\phi(1S) &= 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{\partial k_z}{\partial x}} \exp \left( - \frac{k_z^2 + k_1^2}{2 \beta^2} \right), \\
\phi(2S) &= 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{\partial k_z}{\partial x}} \exp \left( - \frac{k_z^2 + k_1^2}{2 \beta^2} \right) \\
&\times \left( -3 + 2 \frac{k_z^2 + k_1^2}{\beta^2} \right), \\
\phi(3S) &= 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{\partial k_z}{\partial x}} \exp \left( - \frac{k_z^2 + k_1^2}{2 \beta^2} \right) \\
&\times \left( -15 - 4 \frac{k_z^2 + k_1^2}{\beta^2} + 4 \left( \frac{k_z^2 + k_1^2}{\beta^2} \right)^2 \right), \\
\phi(4S) &= 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{\partial k_z}{\partial x}} \exp \left( - \frac{k_z^2 + k_1^2}{2 \beta^2} \right) \\
&\times \left( -105 - 121 \frac{k_z^2 + k_1^2}{\beta^2} - 84 \left( \frac{k_z^2 + k_1^2}{\beta^2} \right)^2 \right) \\
&\times \left( 12 \sqrt{35} \right), \\
\phi(5S) &= 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{\partial k_z}{\partial x}} \exp \left( - \frac{k_z^2 + k_1^2}{2 \beta^2} \right) \\
&\times \left( 945 - 2520 \frac{k_z^2 + k_1^2}{\beta^2} + 1512 \left( \frac{k_z^2 + k_1^2}{\beta^2} \right)^2 \right) \\
&\times \left( 72 \sqrt{70} \right), \\
&\times \left( -288 \left( \frac{k_z^2 + k_1^2}{\beta^2} \right)^3 + 16 \left( \frac{k_z^2 + k_1^2}{\beta^2} \right)^4 \right),
\end{align*}

with

\begin{align*}
k_z &= \frac{x_2 M_0}{2} - \frac{m_1^2 + k_1^2}{2 x_2 M_0}, \\
\frac{\partial k_z}{\partial x} &= \frac{M_0}{4 x_1 x_2} \left[ 1 - \left( \frac{m_1^2 - m_2^2}{M_0^2} \right)^2 \right].
\end{align*}

More information can be found in Ref. [18].
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