Local uniqueness for the Dirichlet-to-Neumann map via the two-plane transform

Allan Greenleaf∗
Department of Mathematics
University of Rochester
Rochester, NY 14627

Gunther Uhlmann†
Department of Mathematics
University of Washington
Seattle, WA 98195

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Abstract
We consider the Cauchy data associated to the Schrödinger equation with a potential on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. We show that the integral of the potential over a two-plane $\Pi$ is determined by the Cauchy data of certain exponentially growing solutions on any open subset $U \subset \partial \Omega$ which contains $\Pi \cap \partial \Omega$.

0 Introduction

For $\Omega$ a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $\partial \Omega$, and real-valued $q(x) \in L^\infty(\Omega)$, let

\begin{equation}
\Lambda_q : H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega)
\end{equation}

be the Dirichlet-to-Neumann map associated with the operator $\Delta + q$ on $\Omega$, which is defined if $\lambda = 0$ is not a Dirichlet eigenvalue for $\Delta + q$ on $\Omega$. More generally, one may consider the set of Cauchy data of solutions of

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\((\Delta + q(x))v = 0\), which is defined even if \(\lambda = 0\) is a Dirichlet eigenvalue. Set
\[
\mathcal{CD}_q = \left\{(v|_{\partial \Omega}, \frac{\partial v}{\partial n}|_{\partial \Omega}) \in H^{1/2}(\partial \Omega) \times H^{-1/2}(\partial \Omega) : v \in H^1(\Omega), (\Delta + q)v = 0\right\},
\]
which is a subspace of \(H^{1/2} \times H^{-1/2}\); if \(\Lambda_q\) is defined, then \(\mathcal{CD}_q\) is simply the graph of \(\Lambda_q\).

This paper is concerned with the problem of obtaining partial knowledge of \(q(x)\) from partial knowledge of \(\mathcal{CD}_q\), namely its restriction to certain “small” open subsets of the boundary. The approach taken here is to use concentrated, exponentially growing, approximate solutions to relate \(\mathcal{CD}_q\) on an open set \(U \subset \partial \Omega\) to the two-plane transform of the potential \(q(x)\) on two-planes whose intersections with \(\partial \Omega\) are contained in \(U\).

Let \(M_{2,n}\) denote the \((3n - 6)\)-dimensional Grassmannian of all affine two-planes \(\Pi \subset \mathbb{R}^n\), and
\[
R_{2,n}f(\Pi) = \int_{\Pi} f(y)d\lambda_{\Pi}(y), f \in L^2(\mathbb{R}^n),
\]
denote the two-plane transform on \(\mathbb{R}^n\). Here, \(d\lambda_{\Pi}\) is two-dimensional Lebesgue measure on \(\Pi \in M_{2,n}\), which can be defined by
\[
\langle f, d\lambda_{\Pi} \rangle = \lim_{\epsilon \to 0} \frac{1}{|B^{n-2}(0; \epsilon)|} \int_{\{\text{dist}(x, \Pi) < \epsilon\}} f(x)dx.
\]
(Note that for \(n = 3\), \(R_{2,3}\) is just the usual Radon transform on \(\mathbb{R}^3\).) We will also need the variant of \(d\lambda_{\Pi}\) defined relative to \(\Omega\):
\[
\langle f, d\lambda_{\Pi}^\Omega \rangle = \lim_{\epsilon \to 0} \frac{1}{|B^{n-2}(0; \epsilon)|} \int_{\Omega \cap \{\text{dist}(x, \Pi) < \epsilon\}} f(x)dx,
\]
which gives rise to a two-plane transform relative to \(\Omega\),
\[
R_{2,n}^\Omega f(\Pi) = \int_{\Pi} f(x)d\lambda_{\Pi}^\Omega(x).
\]
Note that if \(\partial \Omega\) is \(C^1\) and \(\Pi \cap \partial \Omega\) transversally, then \(\langle d\lambda_{\Pi}^\Omega, f \rangle = \langle d\lambda_{\Pi}, f \cdot \chi_\Omega \rangle\) and \(R_{2,n}^\Omega f(\Pi) = R_{2,n}(f \cdot \chi_\Omega)(\Pi)\).

For each choice of an orthonormal basis for \(\Pi_0\), the translate of \(\Pi\) passing through the origin, as well as other arbitrary choices made below, we will construct a family, \(\mathcal{F}_q = \{v_z(x) : z \in \mathbb{C}, |z| \geq C\}\), of exponentially growing
solutions of \((\Delta + q(x))v = 0\), concentrated near \(\Pi\). Using these families, we formulate

**Definition**  (i) If \(U \subset \partial \Omega\) is open, \(\mathcal{CD}_{q_1}\) and \(\mathcal{CD}_{q_2}\) are equal on \(U\) relative to \(\mathcal{F}\) at \(z \in \mathbb{C}\) if the solutions in \(\mathcal{F}_{q_1}\) and \(\mathcal{F}_{q_2}\) corresponding to opposite exponential growths, \(v_z^{(1)}\) and \(v_z^{(2)}\), have the same Cauchy data on \(U\):

\[
(v_z^{(1)}|_U, \frac{\partial v_z^{(1)}}{\partial n}|_U) = (v_z^{(2)}|_U, \frac{\partial v_z^{(2)}}{\partial n}|_U).
\]

(ii) \(\mathcal{CD}_{q_1}\) and \(\mathcal{CD}_{q_2}\) are equal on \(U\) for a sequence of exponentially growing solutions if \(\mathcal{CD}_{q_1}\) and \(\mathcal{CD}_{q_2}\) are equal on \(U\) relative to \(\mathcal{F}\) at \(z = z_j\) for some sequence \(\{z_j\}_1^\infty \subset \mathbb{C}\) with \(|z_j| \to \infty\).

We may now state the main result proved here. For each \(\Pi \in M_{2,n}\), let \(\gamma_{\Pi} = \Pi \cap \partial \Omega \subset \partial \Omega\), and let \(H^s(\Omega)\) denote the standard Sobolev space of distributions with \(s\) derivatives in \(L^2(\Omega)\).

**Theorem 1** Let \(n \geq 3\). Assume \(\partial \Omega\) is Lipschitz and potentials \(q_1(x)\) and \(q_2(x)\) are in \(H^s(\Omega)\), for some \(s > \frac{n}{2}\). Let \(\Pi \in M_{2,n}\) and \(\mathcal{F}_{q_1}\) and \(\mathcal{F}_{q_2}\) be families of exponentially growing solutions associated to \(q_1\) and \(q_2\). If, for some fixed neighborhood \(U_{\Pi}\) of \(\gamma_{\Pi}\) in \(\partial \Omega\), \(\mathcal{CD}_{q_1}\) and \(\mathcal{CD}_{q_2}\) are equal on \(U_{\Pi}\) for a sequence of exponentially growing solutions, then

\[
R_{2,n}(q_1 - q_2)(\Pi) = 0,
\]

i.e., \(\int q_1(y)d\lambda^\Omega_{\Pi}(y) = \int q_2(y)d\lambda^\Omega_{\Pi}(y)\).

If \(\mathcal{CD}_{q_1}\) and \(\mathcal{CD}_{q_2}\) equal on all of \(\partial \Omega\) relative to \(\mathcal{F}\), then this implies that \(R_{2,n}(q_1 - q_2)\chi(\Omega)(\Pi) = 0, \forall \Pi \in M_{2,n}\), which by the uniqueness theorem for \(R_{2,n}\) yields that \(q_1 - q_2 \equiv 0\) on \(\Omega\), providing a variant of the global uniqueness theorem for the Dirichlet–to–Neumann map \([SU87a]\). (We note that our technique is limited to three or more dimensions and says nothing in the case \(n = 2\) \([N96]\).) However, one is also able to obtain local uniqueness results by replacing the uniqueness theorem for the two-plane transform with Helgason’s support theorem \([H80, Cor. 2.8]\): if \(C \subset \mathbb{R}^n\) is a closed, convex set and \(f(x)\) a function\(^1\) such that \(R_{2,n}f(\Pi) = 0\) for all \(\Pi\) disjoint from \(C\), then \(\text{supp}(f) \subset C\). We then immediately obtain the following two results.

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\(^1\)The support and uniqueness theorems are usually stated under the assumption that \(f(x)\) is continuous, of rapid decay in the case of the support theorem, but the proofs in \([H80]\) are easily seen to extend to the case where \(f(x) = q(x)\chi_\Omega(x)\) with \(\Omega \subset \mathbb{R}^n\) bounded, \(q \in C(\Omega)\).
Theorem 2 Suppose $\partial \Omega$ and potentials $q_1, q_2$ are as in Thm. 1., and $C \subset \Omega$ is a closed, convex set. If, for all $\Pi \in M_{2,n}$ such that $\Pi \cap C = \emptyset$, there is some neighborhood $U_\Pi$ of $\gamma_\Pi$ on which $\mathcal{C}D_{q_1}$ and $\mathcal{C}D_{q_2}$ are equal for some sequence of exponentially growing solutions, then $\text{supp}(q_1 - q_2) \subseteq C$, i.e., $q_1 = q_2$ on $\Omega \setminus C$.

Theorem 3 Suppose $\partial \Omega$ is $C^2$ and strictly convex, and potentials $q_1, q_2$ are as in Thm. 1. If, for some $r > 0$, $\mathcal{C}D_{q_1}$ and $\mathcal{C}D_{q_2}$ are equal on $B$ for some sequence of exponentially growing solutions for all surface balls $B = B^n(x_0; r) \cap \partial \Omega \subset \partial \Omega$, then

$$\text{dist}(\text{supp}(q_1 - q_2), \partial \Omega) \geq Cr^2,$$

i.e., $q_1 = q_2$ on the tubular neighborhood $\{x \in \overline{\Omega} : \text{dist}(x, \partial \Omega) \leq Cr^2\}$ of $\partial \Omega$ in $\Omega$.

Remark

The conclusions of Thms. 2 and 3 can be strengthened by combining them with a result in Isakov [Is]. Namely, if either $C \subset \subset \Omega$ in Thm. 2, or the assumption of Thm. 3 holds for some $r > 0$, we can conclude from Thm. 2 or 3 that $\text{supp}(q_1 - q_2) \subset \subset \Omega$. By Ex. 5.7.4 in [Is], based on a technique of Kohn and Vogelius [KV85], this, together with the condition that $\Lambda_{q_1} = \Lambda_{q_2}$ on some open set $\mathcal{U} \subset \partial \Omega$, implies that $q_1 \equiv q_2$ everywhere on $\Omega$. We are indebted to Adrian Nachman for pointing this out to us.

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1 Approximate solutions

To prove Thm. 4, we first construct exponentially growing approximate solutions for $(\Delta + q)v = 0$. As considered in [Is, SU86, SU87], let

$$Q = \{\rho \in \mathbb{C}^n : \rho \cdot \rho = 0\}$$

be the (complex) characteristic variety of $\Delta$. Each $\rho \in Q$ can be written as $\rho = \frac{\rho}{|\rho|} = \frac{1}{\sqrt{2}}(\omega_R + i\omega_I) \in \mathbb{R} \cdot (S^{n-1} + iS^{n-1})$, with $\omega_R \cdot \omega_I = 0$. For $\rho \in Q$, let $\Delta_\rho = \Delta + 2\rho \cdot \nabla$. Then

$$\Delta_\rho + q(x) = e^{-\rho \cdot x}(\Delta + q(x))e^{\rho \cdot x},$$

(1.1)
so that, with \( v(x) = e^{\rho \cdot x} u(x) \),
\[
(1.2) \quad (\Delta_\rho + q(x)) u(x) = w(x) \iff (\Delta + q(x)) v(x) = e^{\rho \cdot x} w(x)
\]
and, in particular, \( (\Delta_\rho + q(x)) u(x) = 0 \iff (\Delta + q(x)) v(x) = 0 \).

Now, given a potential \( q(x) \) and a two-plane \( \Pi \in M_{2,n} \), we will construct an approximate solution \( u_{app} \) to \( (\Delta_\rho + q) u = 0 \), supported near \( \Pi \):

**Theorem 4**  Let \( \Omega \) be Lipschitz and \( q(x) \in H^s(\Omega) \) for some \( s > \frac{n}{2} \). Then, for any \( 0 < \beta < \frac{1}{2} \) fixed, the following holds: \( \exists \, \epsilon > 0 \) such that, for any \( \rho = \frac{1}{\sqrt{\epsilon}} |\rho| (\omega_R + i\omega_I) \in \mathcal{Q} \) and any two-plane \( \Pi \) parallel to \( \Pi_0 = \text{span}\{\omega_R, \omega_I\} \), we can find an approximate solution \( u_{app} = u_{app}(x, \rho, \Pi) \) to \( (\Delta_\rho + q(x)) u = 0 \) satisfying
\[
(1.5) \quad \| u_{app} \|_{L^2(\mathbb{R}^n)} \leq C, \quad \| u_{app} \|_{L^2(\Omega)} \simeq |\lambda_\Pi^\Omega(\Pi \cap \Omega)|^{\frac{1}{2}} \text{ as } |\rho| \to \infty
\]
\[
(1.6) \quad \text{supp}(u_{app}) \subset \left\{ x \in \mathbb{R}^n : \text{dist}(x, \Pi) \leq \frac{2}{|\rho|^\beta} \right\}
\]
and
\[
(1.7) \quad \| (\Delta_\rho + q) u_{app} \|_{L^2(\mathbb{R}^n)} \leq \frac{C}{|\rho|^\epsilon}.
\]

Furthermore, for any two such solutions, \( u_{app}^{(1)}, u_{app}^{(2)} \), associated with possibly different potentials \( q_1(x), q_2(x) \) and with \( \rho_1 \in \mathcal{Q}, \rho_2 = e^{-i\theta} \rho_1 \) or \( \rho_2 = e^{i\theta} \rho_1 \in \mathcal{Q} \),
\[
(1.8) \quad u_{app}^{(1)}(\cdot, \rho_1, \Pi) u_{app}^{(2)}(\cdot, \rho_2, \Pi) \to d\lambda_\Pi^\Omega \text{ weakly as } |\rho_1| \to \infty.
\]

In fact, as will be seen below, \( u_{app} = u_0 + u_1 \) with \( u_0 \) depending only on \( \Pi \) and \( |\rho| \) and satisfying (1.5).

Now, we may apply the results of [SU86, SU87a] (see also [Ha96]) to find a solution \( u_2 \) of
\[
(\Delta_\rho + q) u_2 = - (\Delta_\rho + q) u_{app} \in L^2_{\text{comp}}(\mathbb{R}^n),
\]
uniformly in \( H_t^1 \) and with a gain of \( |\rho|^{-1} \) in \( L_t^2 \), as long as \( |\rho| \geq C \) with \( C \) depending only on \( \|q\|_{\infty} \) and \( \text{diam}(\Omega) \). Here, \( H_t^s \) and \( L_t^2 \) the weighted versions of these spaces, as in [SU87a], for some fixed \(-1 < t < 0\). By these results and (1.7),
\[
\| u_2 \|_{H_t^1(\mathbb{R}^n)} \leq c \| (\Delta_\rho + q) u_{app} \|_{L^2_{t+1}(\mathbb{R}^n)} \leq c |\rho|^{-\epsilon}, \quad \| u_2 \|_{L_t^2} \leq C |\rho|^{-1-\epsilon}.
\]
(The statements in \[SU86,SU87a\] are for \(q \in C^\infty\), but the proofs are easily seen to hold if \(q \in H^s(\Omega)\) with \(s > \frac{n}{2}\). Also, the weights will be irrelevant since we will be working on \(\Omega\).) Thus, \(u = u_{app} + u_2 = u_0 + u_1 + u_2\) is an exact solution of \((\Delta _\rho + q)u = 0\) on \(\mathbb{R}^n\), satisfying
\[
\|u - u_0\|_{L^2} \leq c|\rho|^{-\epsilon} \quad \text{and} \quad \|u_2\|_{H^s} \leq |\rho|^{s-1-\epsilon}, \forall 0 \leq s \leq 1.
\]

Finally, \(F_q = \{ v_z : |z| \geq C \} = \{ e^{\rho \cdot x}u(x, \Pi, \rho) : \rho = Re(z)\omega_R + iIm(z)\omega_I, |z| \geq C \}\) is the associated family of exponentially growing solutions used in the statements of the theorems. To prove Thm. 1, we assume that \(q_1, q_2\) and \(\Pi \in M_{2,n}\), \(U_\Pi \subset \partial \Omega\) are as in its statement. We will make use of a variant of Alessandrini’s identity [A]. For \(j = 1, 2\), let \(v^{(j)}_{\rho_j}\) be the exact solution to \((\Delta + q_j)v = 0\) constructed above, so that \(v^{(j)}_{\rho_j}(x) = e^{\rho_j \cdot x}u^{(j)}(x, \Pi, \rho_j)\), with \(u^{(j)} = u_{app}^{(j)} + u_2^{(j)}\). Taking \(\rho_1 = \rho, \rho_2 = -\rho\), consider the quantity
\[
I = \int_{\partial \Omega \setminus U_\Pi} \partial u^{(1)}_\rho \cdot \frac{\partial v^{(2)}_{-\rho}}{\partial n} - v^{(1)}_\rho \cdot \frac{\partial v^{(2)}_{-\rho}}{\partial n} e^{d\sigma}.
\]
Under the assumption that \(v^{(1)}_\rho\) and \(v^{(2)}_{-\rho}\) have the same Cauchy data on \(U_\Pi\), \(I\) is equal to the integral of the same expression over \(\partial \Omega \setminus U_\Pi\). Observing that
\[
\frac{\partial v^{(1)}_{\rho}}{\partial n} = e^{\rho \cdot x}(\frac{\partial }{\partial n} + (\rho \cdot n(x)))u^{(1)} \quad \text{and} \quad \frac{\partial v^{(2)}_{\rho}}{\partial n} = e^{-\rho \cdot x}(\frac{\partial }{\partial n} - (\rho \cdot n(x)))u^{(2)},
\]
we see that the exponentials cancel and the integrand of \(I\) is
\[
= \frac{\partial u^{(1)}_\rho}{\partial n} \cdot u^{(2)}_\rho - u^{(1)}_\rho \cdot \frac{\partial u^{(2)}_\rho}{\partial n} + 2(\rho \cdot n(x))u^{(1)} u^{(2)}.
\]
Since (1.6) implies that \(\text{supp}(u_{app}^{(j)}|\partial \Omega), \text{supp}(\frac{\partial u_{app}^{(j)}}{\partial n}|\partial \Omega) \subset U_\Pi\) for \(|\rho|\) sufficiently large, we have that
\[
I = \int_{\partial \Omega \setminus U_\Pi} \frac{\partial u^{(1)}_\rho}{\partial n} \cdot u^{(2)}_\rho - u^{(1)}_\rho \cdot \frac{\partial u^{(2)}_\rho}{\partial n} + 2(\rho \cdot n(x))u^{(1)} u^{(2)} d\sigma.
\]
We estimate
\[
\left| \int_{\partial \Omega \setminus U_0} \frac{\partial u_2^{(1)}}{\partial n} \cdot u_2^{(2)} d\sigma \right| \leq \left\| \frac{\partial u_2^{(1)}}{\partial n} \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} \cdot \left\| u_2^{(2)} \right\|_{H^{\frac{1}{2}}(\partial \Omega)}
\]
\[
\leq \left\| u_2^{(1)} \right\|_{H^{1}(\Omega)} \cdot \left\| u_2^{(2)} \right\|_{H^{\frac{1}{2}}(\partial \Omega)}
\]
\[
\leq C \left\| u_2^{(1)} \right\|_{H^1(\Omega)} \cdot \left\| u_2^{(2)} \right\|_{H^1(\Omega)} \leq C \left\| u_2^{(1)} \right\|_{H^1(\Omega)} \cdot \left\| u_2^{(2)} \right\|_{H^1(\Omega)}
\]

and similarly for the second term. Now note that \( |\rho \cdot n(x)| \leq c|\rho| \) since \( \partial \Omega \) is Lipschitz, and
\[
\left\| u_2^{(j)} \right\|_{L^2(\partial \Omega)} \leq \left\| u_2^{(j)} \right\|_{H^\sigma(\partial \Omega)} \leq c_\sigma \left\| u_2^{(j)} \right\|_{H^{\sigma+\frac{1}{2}}(\Omega)} \leq C ' |\rho|^{\sigma - \frac{1}{2} - \epsilon}
\]

for any \( \sigma > 0 \), and thus the third term is dominated by \( (C ')^2 |\rho| \cdot |\rho|^{2\sigma - 1 - 2\epsilon} \to 0 \) as \( |\rho| \to 0 \) if we choose \( 0 < \sigma < \epsilon \).

On the other hand,
\[
I = \int_{\partial \Omega} \frac{\partial v^{(1)}}{\partial n} \cdot v^{(2)} - v^{(1)} \cdot \frac{\partial v^{(2)}}{\partial n} d\sigma
\]
\[
= \int_{\Omega} \Delta (v^{(1)}) \cdot v^{(2)} - v^{(1)} \cdot \Delta (v^{(2)}) dx \text{ by Green’s Thm.}
\]
\[
= \int_{\Omega} (-q_1 v^{(1)}) \cdot v^{(2)} - v^{(1)} \cdot (-q_2 v^{(2)}) dx
\]
\[
= \int_{\Omega} (q_2 - q_1) v^{(1)} v^{(2)} dx = \int_{\Omega} (q_2 - q_1) u^{(1)} u^{(2)} dx
\]

since the exponentials cancel. As \( u^{(1)} \cdot u^{(2)} = (u^{(1)}_{app} + u^{(2)}_{app}) \cdot (u^{(1)}_{app} + u^{(2)}_{app}) \) and the leading term \( u^{(1)}_{app} u^{(2)}_{app} \to d\lambda_0^\Omega \) weakly as \( |\rho| \to \infty \) by (1.8), while the remaining terms \( \to 0 \) since \( \| u^{(j)}_{app} \|_{L^2(\Omega)} \leq C \) by (1.5) and \( \| u^{(j)}_{2} \|_{L^2(\Omega)} \leq c |\rho|^{-1-\epsilon} \), we conclude that \( I \to R^2_{p,n} (q_2 - q_1)(\Pi) \) as \( |\rho| \to \infty \), finishing the proof of Thm. 1.

Now, to start the proof of Thm. 2 we may use the rotation invariance of \( \Delta \) and the invariance of \( Q \) under \( S^1 = \{ e^{i\theta} \} \), and note that it suffices to treat the case\(^2 \) \( \rho = |\rho|(\vec{e}_1^p + i\vec{e}_2^p) \), where \( \{ \vec{e}_1, \ldots, \vec{e}_n \} \) is the standard orthonormal
\(^2\)Of course, the length of this element of \( Q \) is \( \sqrt{2}|\rho| \), but this is irrelevant for the proofs, and denoting the length of \( |\rho|(\vec{e}_1 + i\vec{e}_2) \) by \( |\rho| \) is notationally convenient.
basis for $\mathbb{R}^n$. Write $x \in \mathbb{R}^n$ as $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ and similarly $\xi = (\xi', \xi'')$.

If $\Pi \in M_{2,n}$ is parallel to $\text{span}\{\omega_R, \omega_I\} = \text{span}\{e_1, e_2\} = \mathbb{R}^2 \times \{0\}$, then $\Pi = \text{span}\{\tilde{e}_1, \tilde{e}_2\} + (0, x_0'')$ for some $x_0'' \in \mathbb{R}^{n-2}$. Given $|\rho| > 1$ and $x_0'' \in \mathbb{R}^{n-2}$, we will define an approximate solution $u(x, \rho, \Pi)$ to $(\Delta_\rho + q(x))u = 0$ on $\mathbb{R}^n$, of the form $u(x, \rho, \Pi) = u_0(x, \rho, \Pi) + u_1(x, \rho, \Pi)$.

For notational convenience, we will usually suppress the dependence on $\rho$ and $\Pi$ and simply write $u(x) = u_0(x) + u_1(x)$. We will use various cutoff functions $\chi_j$; for $j$ even or odd, $\chi_j$ will always denote a function of $x'$ or $x''$, respectively. Also, $\Pi = \text{span}(a, r)$ and $S^{m-1}(a, r)$ will denote the closed ball and sphere of radius $r$ centered at a point $a \in \mathbb{R}^m$.

To define $u_0$, first fix $\chi_0 \in C^\infty_c(\mathbb{R}^2)$ with $\chi_0 \equiv 1$ on $B^2(0; R)$ for any $R > \sup\{|x'| : (x', x'') \in \Omega, x'' \in \mathbb{R}^{n-2}\}$; let $C_0 = ||\chi_0||_{L^2(\mathbb{R}^2)}$. Secondly, let $\psi_1 \in C^\infty_c(\mathbb{R}^{n-2})$ be radial, nonnegative, supported in the unit ball, and satisfy

$$\int_{\mathbb{R}^{n-2}} (\psi_1(x''))^2dx'' = 1.$$

Now, for $\beta > 0$ to be fixed later, we let $\delta$ be the small parameter $\delta = |\rho|^{-\beta}$ and define

$$\chi_1(x'') = \delta^{-\frac{n-2}{2}} \psi_1 \left( \frac{x'' - x_0''}{\delta} \right),$$

so that

$$(1.9) \quad ||\chi_1||_{L^2(\mathbb{R}^{n-2})} = ||\psi_1||_{L^2(\mathbb{R}^{n-2})} = 1, \forall \delta > 0.$$

Set $u_0(x) = u_0(x', x'') = \chi_0(x') \chi_1(x'')$; then $u_0$ is real, $||u_0||_{L^2(\mathbb{R}^n)} = C_0$ and $||u_0||_{L^2(\Omega)} \to [\lambda_\Pi(\Pi \cap \Omega)]^{\frac{1}{2}}$ as $\delta \to 0^+$, i.e., as $|\rho| \to \infty$. Note also that $||u_0||_{H^s} \leq c\delta^{-1} = c|\rho|^\beta$, so that $||u_0||_{H^s} \leq c|\rho|^s\delta^s$ for $0 \leq s \leq 1$. Since $\Delta_\rho = \Delta + 2\rho \cdot \nabla = \Delta + 2|\rho|(\tilde{e}_1 + i\tilde{e}_2) \cdot \nabla = \Delta + 4|\rho|\bar{\delta}_{x'}$ and $\rho \perp \mathbb{R}^{n-2}$,

$$(\Delta_\rho + q(x))u_0 = (\Delta \chi_0) \cdot \chi_1 + 2(\nabla \chi_0) \cdot (\nabla \chi_1) + \chi_0(\Delta \chi_1) + 2(\rho \cdot \nabla)(\chi_0) \chi_1 + 2 \chi_0(\rho \cdot \nabla)(\chi_1) + q \chi_0 \chi_1$$

$$= \chi_0(x')(\Delta_{x''} + q)(\chi_1)(x'')$$

on $B^2(0; R) \times \mathbb{R}^{n-2}$, the first and fourth terms after the first equality vanishing because $(\rho \cdot \nabla)(\chi_0) = 2\bar{\delta}_{x'} \chi_0 \equiv 0$ on $B^2(0; R)$, and the second and fifth equalling zero because $\nabla \chi_1 \perp \mathbb{R}^2$.  

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To define the second term in the approximate solution, \( u_1(x) \), we make use of a truncated form of the Faddeev Green function, \( G_\rho \), and an associated projection operator. The operator \( \Delta_\rho \) has, for \( \rho \in Q \), (full) symbol

\[
\sigma(\xi) = -[|\xi|^2 - 2|\rho|\omega_I \cdot \xi] + i2|\rho|(|\omega_R \cdot \xi|),
\]

and so for \( \frac{\rho}{|\rho|} = e_1 + ie_2 \), we have

\[
\sigma(\xi) = -[|\xi| - |\rho|e_2|^2 - |\rho|^2] + i(2|\rho|\xi_1),
\]

which has (full) characteristic variety

\[
\Sigma_\rho = \{ \xi \in \mathbb{R}^n : \xi_1 = 0, |\xi| - |\rho|e_2 = |\rho| \}
= \{0\} \times S^{n-2}((|\rho|, 0, \ldots, 0); |\rho|) \subset \mathbb{R}_{\xi_1} \times \mathbb{R}_{\xi_2, \xi''}.
\]

The Faddeev Green function is then defined by

\[
G_\rho = \left( -\sigma(\xi)^{-1} \right) \in S'(\mathbb{R}^n).
\]

We now introduce, for an \( \epsilon_0 > 0 \) to be fixed later, a tubular neighborhood of \( \Sigma_\rho \),

\[
T_\rho = \{ \xi : \text{dist}(\xi, \Sigma_\rho) < |\rho|^{-\frac{1}{2} - \epsilon_0} \},
\]

as well as its complement, \( T_\rho^C \), and let \( \chi_{T_\rho}, \chi_{T_\rho^C} \) be their characteristic functions. Define a projection operator, \( P_\rho \), and a truncated Green function, \( \tilde{G}_\rho \), by

\[
\tilde{P}_\rho f(\xi) = \chi_{T_\rho}(\xi) \cdot \hat{f}(\xi) \quad \text{and}
\]
\[
(\tilde{G}_\rho f)^\wedge(\xi) = \chi_{T_\rho^C}(\xi) \cdot [-\sigma(\xi)]^{-1} \hat{f}(\xi)
\]

for \( f \in S(\mathbb{R}^n) \). Note that \( \Delta_\rho \tilde{G}_\rho = I - P_\rho \).

Choose a \( \psi_3 \in C_0^\infty(\mathbb{R}^n) \), supported in \( B^{n-2}(0; 2) \), radial and with \( \psi_3 \equiv 1 \) on \( \text{supp}(\psi_1) \), and set \( \chi_3(x'') = \psi_3\left(\frac{x'' - x'_0}{\delta}\right) \). We now define the second term, \( u_1(x, \rho, \Pi) \) in the approximate solution by

\[
u_1(x) = -\chi_3(x'')\tilde{G}_\rho((\Delta_\rho + q(x))u_0(x))
\]

and set \( u(x) = u_0(x) + u_1(x) \). Then \( u_1 \) (as well as \( u_0 \)) is supported in \( \{ x : \text{dist}(x, \Pi) \leq 2\delta \} \), yielding (1.6). We will see below that \( \|u_1\|_{L^2(\Omega)} \leq C|\rho|^{-\epsilon} \) as \( |\rho| \to \infty \), so that (1.5) holds as well, so that the first part of (1.9) holds.
as well. To start the proof of (1.7), note that

\[(\Delta_\rho + q)(u_0 + u_1) = (\Delta_\rho + q)u_0 - (\Delta_\rho + q)\chi_3 \tilde{G}_\rho((\Delta_\rho + q)u_0)\]

\[= (\Delta_\rho + q)u_0 - \chi_3(\Delta_\rho + q)\tilde{G}_\rho((\Delta_\rho + q)u_0)\]

\[- |\Delta_\rho + q, \chi_3| \tilde{G}_\rho((\Delta_\rho + q)u_0)\]

\[= (\Delta_\rho + q)u_0 - \chi_3(I - P_\rho)(\Delta_\rho + q)u_0 - \chi_3 q \tilde{G}_\rho(\Delta_\rho + q)u_0\]

\[\quad - 2(\nabla \chi_3 \cdot \nabla x') \tilde{G}_\rho(\Delta_\rho + q)u_0 - (\Delta_{x''} \chi_3) \tilde{G}_\rho(\Delta_\rho + q)u_0\]

\[= \chi_3 P_\rho(\Delta_\rho + q)u_0\]

\[- |q \chi_3 + 2(\nabla \chi_3 \cdot \nabla x') - (\Delta_{x''} \chi_3)] \tilde{G}_\rho(\Delta_\rho + q)u_0\]

on \(\Omega\), since \(\chi_3 \equiv 1\) on \(\text{supp}(\chi_1)\). Now, since \(q_1\chi_3 \in L^\infty\), \(|\nabla \chi_3| \leq C \delta^{-1} = c|\rho|^\beta\)
and \(|\Delta_{x''} \chi_3| \leq C \delta^{-2} = c|\rho|^{2\beta}\), (1.7) will follow if we can show that for some \(\epsilon > 0\),

\[(1.16) \quad \|P_\rho(\Delta_\rho + q)u_0\|_{L^2(\Omega)} \leq C|\rho|^{-\epsilon},\]

\[(1.17) \quad \|D''|\tilde{G}_\rho(\Delta_\rho + q)u_0\|_{L^2(\Omega)} \leq C|\rho|^{-\beta - \epsilon},\]

\[(1.18) \quad \|\tilde{G}_\rho(\Delta_\rho + q)u_0\|_{L^2(\Omega)} \leq C|\rho|^{-2\beta - \epsilon},\]

with \(C\) independent of \(|\rho| > 1\). Before proceeding to prove these, we note that for any \(u^{(1)}, u^{(2)}\) constructed in this way for the same two-plane \(\Pi\),

\[u^{(1)}_0(x)u^{(2)}_0(x) = \chi_0^2(x')\delta^{-(n-2)}\psi_1^2 \left( \frac{x'' - x''_0}{\delta} \right) \to d\lambda_\Omega^\Pi \text{ in } \Omega\]

as \(\delta \to 0\) by (1.11), while \(u^{(1)}_1 u^{(2)}_0 + u^{(1)}_0 u^{(2)}_1 + u^{(1)}_1 u^{(2)}_1 \to 0\) in \(L^2(\Omega)\), yielding (1.8). Thus, we are reduced to establishing (1.17–1.19).

### 2 \(L^2\) estimates

We will first prove (1.17)–(1.19) under the simplifying assumption that \(q_1, q_2 \in C^{\alpha-1+\sigma}(\Omega)\) for some \(\sigma > 0\), turning to the Sobolev space case in Section 3. Start by noting that the desired estimates (1.17)–(1.19) cannot be simply obtained from operator norms; for example, \(\|P_\rho\|_{L^2 \to L^2} = 1\) for all \(\rho\). One needs to make use of the special structure of \((\Delta_\rho + q)u_0\); we first deal with \(\Delta_\rho u_0\), leaving \(q(x) \cdot u_0\) for the end. So, we will show that \(\|P_\rho \Delta_\rho u_0\|_{L^2} \leq C|\rho|^{-\epsilon}\), etc. Since \(\nabla \chi_0 \cdot \nabla \chi_1 \equiv 0\),

\[(2.1) \quad \Delta_\rho u_0 = \chi_0 \Delta_{x''} \chi_1 + (\Delta_{x'} + 4|\rho| \overline{\partial_{x'}})(\chi_0) \cdot \chi_1.\]
The second term is supported on \( \Omega^c \), but \( P_\rho \) and \( \tilde{G}_\rho \) are nonlocal operators and we need to control the contribution from this term. However, because \( \Delta_{x'}(\chi_0) \) is a fixed, \( \delta \)-independent element of \( C_0^\infty(\mathbb{R}^2) \), this can be handled in the same way as the \( q(x) \cdot u_0 \) terms of (1.17–1.19), which will be dealt with later. The contribution from \( 4|\rho|\overline{\mathcal{G}}\chi_0 \cdot \chi_1 \) will be handled at the end.

So, for the time being, we are interested in estimating \( \|P_\rho(\chi_0(x')\Delta_{x''}\chi_1(x''))\|_{L^2} \), etc. Now, \( \Delta_{x''}\chi_1(x'') = \delta^{-2}\chi_5(x'') \), where \( \chi_5(x'') = \delta^{-\frac{n-2}{2}}\psi_5(\frac{x''-x_0''}{\delta}) \) is associated with the radial function \( \psi_5 = \Delta_{x''}\psi_1 \) as \( \chi_1 \) is associated with \( \psi_1 \). Note for future use that \( \tilde{\psi}_5 \) vanishes to second order at 0. Of course, \( \chi_0 \in C_0^\infty \Rightarrow \hat{\chi}_0 \in \mathcal{S}(\mathbb{R}^n) \), but looking ahead to estimating the terms involving \( q(x) \cdot u_0(x) \), we will now prove the analogues of (1.17–1.19) where \( P_\rho \) and \( \tilde{G}_\rho \) act on \( \chi_2(x')\Delta\chi_1(x'') \), under the weaker assumption that \( \chi_2 \) is radial and satisfies the uniform decay estimate

\[
(2.2)_\alpha \quad |\hat{\chi}_2(\xi)| \leq C(1 + |\xi|)^{-\alpha}
\]

for some \( \alpha > 0 \).

Now, by (1.14) and Plancherel,

\[
\|P_\rho(\chi_2\Delta\chi_1)\|_{L^2(\Omega)} \leq \|(P_\rho(\chi_2\Delta\chi_1))^\wedge\|_{L^2(\mathbb{R}^n)} = \|\delta^{-2}|\hat{\chi}_2(\xi')|\delta^{\frac{n-2}{2}}|\hat{\psi}_5(\delta\xi'')|\|_{L^2(T_\rho)}.
\]

The characteristic variety \( \Sigma_\rho \), of which \( T_\rho \) is a tubular neighborhood, passes through the origin, and we may represent \( \Sigma_\rho \) near \( O \) as a graph over the \( \xi'' \)-plane: \( \Sigma_\rho = \Sigma^s_\rho \cup \Sigma^n_\rho \cup \Sigma^e_\rho \), with

\[
(2.3) \quad \Sigma^s_\rho = \left\{ \xi_1 = 0, \xi_2 = |\rho| - (|\rho|^2 - |\xi''|^2)^{\frac{1}{2}}, |\xi''| \leq \frac{|\rho|}{2} \right\}
\]

\[
\simeq \left\{ \xi_1 = 0, \xi_2 = \frac{|\xi''|}{2|\rho|}, |\xi''| \leq \frac{|\rho|}{2} \right\}
\]

a neighborhood of the south pole \( O \),

\[
(2.4) \quad \Sigma^n_\rho = \left\{ \xi_1 = 0, \xi_2 = |\rho| + (|\rho|^2 - |\xi''|^2)^{\frac{1}{2}}, |\xi''| \leq \frac{|\rho|}{2} \right\}
\]

\[
\simeq \left\{ \xi_1 = 0, \xi_2 = 2|\rho| - \frac{|\xi''|^2}{2|\rho|}, |\xi''| \leq \frac{|\rho|}{2} \right\}
\]

a neighborhood of the north pole \((0, 2|\rho|, 0, \ldots, 0)\), and \( \Sigma^e_\rho \) a neighborhood of the equator \( \{ \xi \in \Sigma_\rho : \xi_2 = |\rho| \} \). We have a corresponding decomposition
\[ T_\rho = T_\rho^s \cup T_\rho^n \cup T_\rho^e, \] where, e.g.,

\[ (2.5) \quad T_\rho^s \simeq \left\{ (\xi', \xi'') : \xi' \in B^2 \left( \left( 0, \frac{|\xi''|^2}{2|\rho|} \right); |\rho|^{-\frac{1}{2} - \epsilon_0} \right), |\xi''| \leq \frac{|\rho|}{2} \right\}. \]

Recalling that \( \chi_2 \) and \( \psi_3 \) are radial, so are \( \tilde{\chi}_2 \) and \( \tilde{\psi}_3 \), and by abuse of notation we consider these as functions of one variable satisfying \((2.2)_\alpha\) and rapidly decreasing, respectively. Thus, using polar coordinates in \( \xi'' \),

\[ \| \chi_2 \Delta \chi_1 \|^2_{L^2(T_\rho^s)} \simeq \int_0^{\frac{|\rho|}{2}} \int_{B^2 \left( \left( \frac{r}{|\rho|}, \frac{r^2}{2|\rho|^2} \right); |\rho|^{-\frac{1}{2} - \epsilon_0} \right)} |\tilde{\chi}_2(\xi')|^2 d\xi' \delta^{\frac{n}{2} - 6} |\tilde{\psi}_5(\delta r)|^2 r^{n-3} dr \]

\[ \simeq \int_0^{\frac{|\rho|}{2}} \int_{B^2 \left( (0,0); |\rho|^{-\frac{1}{2} - \epsilon_0} \right)} |\tilde{\chi}_2|^2 d\xi' \delta^{\frac{n}{2} - 6} |\tilde{\psi}_5(\delta r)|^2 r^{n-2} \frac{dr}{r} \]

\[ + \int_0^{\frac{|\rho|}{2}} \left| \tilde{\chi}_2 \left( \frac{r^2}{2|\rho|^2} \right) \right|^2 \left| \tilde{\psi}_5(r) \right|^2 r^{n-2} \frac{dr}{r}. \]

Since we will be taking \( \delta = |\rho|^{-\beta} \) with \( \beta < \frac{1}{4} \), if we choose \( 0 < \epsilon_0 < 2\left( \frac{1}{4} - \beta \right) \), then the quantity \( |\rho|^{-\frac{1}{2} \delta} \to \infty \) as \( |\rho| \to \infty \) and so

\[ (2.7) \quad \| \chi_2 \Delta \chi_1 \|^2_{L^2(T_\rho^s)} \leq c \frac{\delta^{\frac{1}{2} - n}}{|\rho|^{1 + 2\epsilon_0}} \left( \int_0^{\frac{|\rho|}{2}} \left| \tilde{\psi}_5(r) \right|^2 r^{n-2} \frac{dr}{r} \right) \]

\[ + \int_0^{\frac{|\rho|}{2}} \left| \tilde{\chi}_2 \left( \frac{r^2}{2|\rho|^2} \right) \right|^2 \left| \tilde{\psi}_5(r) \right|^2 r^{n-2} \frac{dr}{r} \]

\[ \leq c \left( \delta^4 |\rho|^{-1} \right)^{-1}, \]

which is \( \leq c |\rho|^{-2\epsilon} \) with \( \epsilon = \frac{1}{2} \left( 1 - 4 \beta \right) > 0 \).

The other contributions to \( \| P_\rho \chi_2 \Delta \chi_1 \|^2_{L^2} \), coming from \( T_\rho^n \) and \( T_\rho^e \) are handled similarly and are even smaller due to the decrease of \( \tilde{\chi}_2 \) and \( \tilde{\psi}_5 \).

We next turn to estimating \( \| D'' |\tilde{G}_\rho \Delta u_0 \|^2_{L^2} \); by the remark above, we may concentrate on the \( \chi_2 \Delta \chi_1 \) term of \( \Delta u_0 \). Then

\[ (2.8) \quad \| D'' |\tilde{G}_\rho (\chi_2 \Delta \chi_1) \|^2_{L^2(\Omega)} \leq \| \xi''(\sigma(\xi))^{-1}(\chi_2 \Delta \chi_1)^\wedge(\xi) \|^2_{L^2(T_\rho^e)}. \]

We may cover \( T_\rho^C \) by \( T_\rho^{C,s} \cup T_\rho^{C,n} \cup T_\rho^{C,e} \cup T_\rho^{C,\infty} \), where

\[ (2.9) \quad T_\rho^{C,s} = \left\{ \xi : \xi' \in B^2 \left( \left( 0, \frac{|\xi''|^2}{2|\rho|} \right); |\rho|^{-\frac{1}{2} - \epsilon_0} \right) \cap B^2 \left( \left( 0, 2|\rho| - \frac{|\xi''|^2}{2|\rho|} \right); \frac{1}{4} |\rho| \right) \right\}, \]

\[ \left| \xi'' \right| \leq \frac{|\rho|}{2} \right\}, \]
$T^C_ρ$ is defined similarly,

\[(2.10) \quad T^C_ρ = \left\{ \xi : \frac{|ρ|}{4} < ξ_2 < \frac{7|ρ|}{4}, |ρ|^{-\frac{1}{2}} < \text{dist}(ξ, Σ_ρ) < |ρ|, |ξ''| < 2|ρ| \right\} \]

and

\[(2.11) \quad T^C_ρ^∞ = \left\{ ξ : |ξ| ≥ 3|ρ|, |ξ''| ≥ \frac{3}{2}|ρ| \right\}. \]

One has the lower bounds on $σ$,

\[(2.12) \quad |σ(ξ)| ≥ \begin{cases} C|ρ|\text{dist}(ξ, Σ_ρ), & |ξ| ≤ 3|ρ| \\ C|ξ|^2, & |ξ| ≥ 3|ρ| \end{cases} \]

with $C$ (as always) uniform in $|ρ|$. The first inequality in (2.12) follows from noting that $\frac{1}{2}∇σ(ξ) = (ξ - |ρ|e_2) + i(|ρ|e_1)$, so that $|∇σ(ξ)| = 2\sqrt{2}|ρ|$ on $Σ_ρ$, while the second follows from $\text{Re}(σ(ξ)) = \text{dist}(ξ, |ρ|e_2)^2 - |ρ|^2$. Using the first estimate in (2.12), we can then dominate the contribution to the right side of (2.8) from the region $T^C_ρ$ by

\[(2.13) \quad δ^{n-6} \delta^{-\frac{α}{2}} \int_{|ξ''| \leq \frac{7}{4}} \int_{B^2((0, \frac{5|ρ|^2}{4|ρ|})^2):|ρ|^{-\frac{1}{2}} - \hat{σ}} \frac{C |ξ''|^2}{2|ρ|} |ξ''|^2 dξ'dξ''dξ'' \]

The inner integral is the convolution

\[|ρ|^{-2} \left( |\tilde{χ}_2|^2 \ast_{\mathbb{R}^2} \chi\left\{ |ξ'| ≥ \frac{|ξ'|^{-\frac{1}{2}} - \hat{σ}}{2|ρ|} \right\} \right)_{|ξ''| = \frac{|ξ''|^2}{2|ρ|}}. \]

An elementary calculation shows that, for $\tilde{χ}_2$ satisfying (2.2) for some $0 < α < 1$, and any $0 < a < 1$,

\[(2.14) \quad |\tilde{χ}_2|^2 \ast_{\mathbb{R}^2} \frac{\chi\{ |ξ'| ≥ a \}}{|ξ'|^2} ≤ \begin{cases} C_1(1 + \log(a^{-1})), & |ξ'| ≤ 1 \\ C_2|ξ'|^{-2} + C_3|ξ'|^{-2α} \log \left( \frac{|ξ'|}{a} \right), & |ξ'| ≥ 1, \end{cases} \]

so that, taking $a = |ρ|^{-\frac{1}{2}} - \hat{σ} = \frac{|ξ''|^2}{2|ρ|}$, the inner integral in (2.13) is

\[\leq \begin{cases} C_1|ρ|^{-2} \log |ρ|, & 0 < |ξ''| ≤ \sqrt{2}|ρ|^{\frac{1}{2}} \\ C_2|ξ''|^{-4} + C_3|ρ|^{2α-2}|ξ''|^{-4α} \log \left( \frac{|ξ''|^2}{2|ρ|^{\frac{1}{2}} - \hat{σ}} \right), & \sqrt{2}|ρ|^{\frac{1}{2}} ≤ |ξ''| ≤ \frac{|ρ|}{2}. \end{cases} \]
Employing polar coordinates in $\xi''$ and rescaling by $\delta$, we see that (2.13) is

\[
\leq C_1 \delta^{-6} |\rho|^{-2} \log |\rho| \int_{0}^{\sqrt{2}|\rho|^{\frac{1}{2} \delta}} |\hat{\psi}_5(r)|^2 r^n dr + C_2 \delta^{-2} \int_{\sqrt{2}|\rho|^{\frac{1}{2} \delta}}^{\frac{|\rho|}{2} \delta} |\hat{\psi}_5(r)|^2 r^{n-4} dr + C_3 \delta^{4a-4} |\rho|^{2a-2} \log |\rho| \int_{\sqrt{2}|\rho|^{\frac{1}{2} \delta}}^{\frac{|\rho|}{2} \delta} |\hat{\psi}_5(r)|^2 r^{n-2-4a} dr.
\]

With $\delta = |\rho|^{-\beta}$, $\beta < \frac{1}{4}$, $|\rho|^{\frac{1}{2} \delta} \to \infty$ as $|\rho| \to \infty$, and thus we estimate this for any $N > 0$ (using the rapid decay of $\hat{\psi}_5$) by

\[
C_1 |\rho|^{6\beta-2} \log |\rho| + C_2 \delta^{-2} (|\rho|^{\frac{1}{2} \delta})^{-N} + C_3 |\rho|^{(4-4a)\beta+2a-2} \log |\rho|(|\rho|^{\frac{1}{2} \delta})^{-N},
\]

the first term of which will be less than the desired $|\rho|^{-2\beta-2\epsilon}$, for any $\alpha > 0$, if $\beta < \frac{1}{4}$ and $\epsilon = \frac{1}{2}(1-4\beta)$; the second and third terms are rapidly decaying simply because $\beta < \frac{1}{2}$.

Moving ahead for the moment to (1.18), the contribution to $\|\tilde{G}_\rho \chi_2 \Delta \chi_1\|_{L^2}^2$ (which we want $\leq C|\rho|^{-4\beta-2\epsilon}$) from $T_{\rho}^{C,n}$ is handled in the same fashion, the only differences being the absence of the multiplier $|D^n| = |\xi''|$ on the left and the improved gain we are demanding on the right. Taking these into account, we need to control

\[
(2.15) \quad C_1 \delta^{-4} |\rho|^{-2} \log |\rho| \int_{0}^{\sqrt{2}|\rho|^{\frac{1}{2} \delta}} |\hat{\psi}_5(r)|^2 r^{n-2} dr + C_2 \delta^{-2} \int_{\sqrt{2}|\rho|^{\frac{1}{2} \delta}}^{\frac{|\rho|}{2} \delta} |\hat{\psi}_5(r)|^2 r^{n-6} dr + C_3 \delta^{4a-2} |\rho|^{2a-2} \log |\rho| \int_{\sqrt{2}|\rho|^{\frac{1}{2} \delta}}^{\frac{|\rho|}{2} \delta} |\hat{\psi}_5(r)|^2 r^{n-4-4a} dr
\]

\[
\leq C_1 \delta^{-4} |\rho|^{-2} \log |\rho| + C_2 (|\rho|^{\frac{1}{2} \delta})^{-N} + C_N \delta^{4a-2} |\rho|^{2a-2} \log |\rho|(|\rho|^{\frac{1}{2} \delta})^{-N},
\]

and this is $\leq C|\rho|^{-4\beta-2\epsilon}$ provided $\beta < \frac{1}{4}$, $\epsilon < \frac{1}{2}(1-4\beta)$ and $N$ is sufficiently large.

The contributions to (1.18) from $T_{\rho}^{C,n}$ and $T_{\rho}^{C,e}$ are handled similarly. To treat the contribution from $T_{\rho}^{C,\infty}$, we use the second estimate in (2.12) and
calculate (for (1.18))

\[(2.16) \quad \| \xi'' \| (\sigma(\xi))^{-1}(\chi_2 \Delta \chi_1 \chi)(\xi) \|_{L^2(T^\infty_\rho)}^2 \]

\[\leq C \int_{|\xi| \geq 3|\rho|} \delta^{-6} |\widehat{\chi_2(\xi')}|^2 |\widehat{\psi_5(\delta \xi'')}|^2 (\xi''')^2 d\xi' d\xi''\]

\[\leq C \left( \int_{|\xi''| \leq |\rho|} \delta^{-6}|\rho|^{-2\alpha - 2} |\widehat{\psi_5(\delta \xi'')}|^2 |\xi''|^2 d\xi'' + \int_{|\xi''| \geq |\rho|} \delta^{-6}|\rho|^{-2\alpha} |\widehat{\psi_5(\delta \xi'')}|^2 |\xi''|^{-2\alpha} d\xi'' \right)\]

\[= C \left( \delta^{-6}|\rho|^{-2\alpha - 2} \int_0^{|\rho|} |\widehat{\psi_5(r)}|^2 r^n \frac{dr}{r} + \delta^{2\alpha - 4} \int_{|\rho|}^{\infty} |\widehat{\psi_5(r)}|^2 r^{n - 2\alpha - 2} \frac{dr}{r} \right)\]

\[\leq C(\delta^{-6}|\rho|^{-2\alpha - 2} + \delta^{2\alpha - 4}(|\rho|^{-N})) \quad \forall \ N > 0,\]

which, for \( \delta = |\rho|^{-\beta} \) and \( N \) large is \( \leq C|\rho|^{-2\beta - 2\epsilon} \) provided \( \beta < \frac{1}{4} \) and \( \epsilon < \alpha + 1 - 4\beta \). A similar analysis holds for the \( T^\infty_\rho \) contribution to (1.19).

We now turn to controlling the \( q(x)u_0(x) \) terms in (1.17)–(1.19), as well as the contributions from the \( \Delta(\chi_0) \cdot \chi_1 \) term in (2.1). Note that since \( q(x) \) is \( C^{n-1+\sigma} \) (for some \( \sigma > 0 \)), \( q(x) \) has an extension (see, e.g., [St70,Ch.6]) to a \( C^{n-1+\sigma} \) function of compact support on \( \mathbb{R}^n \), which we also denote by \( q \). The restriction of \( q \) to any \( \Pi \in M_{2n} \) is still \( C^{n-1+\sigma} \).

Let \( \{D_t : 0 < t < \infty\} \) be the one-parameter group of partial dilations on \( S'(\mathbb{R}^n) \),

\[(D_t f)(\xi', \xi'') = t^{n-2} f(\xi', t\xi''),\]

which, for \( f, g \in L^1 \), satisfy \( \int_{\mathbb{R}^n} D_t f d\xi = \int_{\mathbb{R}^n} f d\xi \) and \( D_t (f * g) = D_t f * D_t g \). Then

\[(2.17) \quad \widehat{\widehat{qu_0}}(\xi) = \widehat{\widehat{q}} \ast \widehat{u_0}(\xi) = D_\delta(D_{\delta^{-1}} \widehat{q}) \ast \delta^{-\frac{n-2}{2}} D_\delta(\widehat{\chi_0}(\xi') \widehat{\psi_1}(\xi'') e^{ix_0^0 \cdot \xi''})\]

\[= D_\delta(D_{\delta^{-1}}(\widehat{q}) \ast \delta^{-\frac{n-2}{2}} \widehat{\chi_0}(\xi') \widehat{\psi_1}(\xi'') e^{ix_0^0 \cdot \xi''}).\]

Now, as \( \delta = |\rho|^{-\beta} \to 0 \), \( D_{\delta^{-1}}(\widehat{q}) = \delta^{-(n-2)} \widehat{q}(\xi', \delta \xi'') \) converges weakly to the singular measure

\[(2.18) \quad Q(\xi') \otimes \delta(\xi'') = Q(\xi')d\xi',\]

\[Q(\xi') \otimes \delta(\xi'') = Q(\xi')d\xi',\]
where \( Q(\xi') = \int_{\mathbb{R}^{n-2}} \tilde{q}(\xi', \xi'') d\xi'' \); note that \( q \in C^{n-1+\gamma} \) implies that the integral defining \( Q \) converges and \( Q \) satisfies (2.2)\(_{1+\gamma}\). Letting \( F(\xi) = \tilde{\chi}_0(\xi') \tilde{\psi}_1(\xi'') e^{ix'' \xi' \cdot \xi''} \), it follows from (2.17) that

\[
(2.19) \tilde{\chi}_0(\xi) = D_\delta(D_{\delta^{-1}}(\tilde{q}) \ast \delta^{-\frac{n-2}{2}} F) = D_\delta((Qd\xi') \ast \delta^{-\frac{n-2}{2}} F) + D_\delta((D_{\delta^{-1}}(\tilde{q} - Qd\xi') \ast \delta^{-\frac{n-2}{2}} F).
\]

If we define \( \tilde{\chi}_4(\xi') = Q \ast \mathbb{R}^2 \tilde{\chi}_0(\xi') \), then \( \tilde{\chi}_4 \) also satisfies condition (2.2)\(_{1+\gamma}\) (and thus (2.2)\(_{\alpha'}\) for \( 0 < \alpha' < 1 \), so that (2.14) can be applied), and the first term in (2.19) is

\[
(2.20) D_\delta((Qd\xi') \ast \delta^{-\frac{n-2}{2}} F) = \tilde{\chi}_4(\xi') \delta^{-\frac{n-2}{2}} \tilde{\psi}_1(\delta\xi'') e^{i\delta x'' \cdot \xi''}.
\]

Thus, the contributions to \( \|P_\delta(qu_0)\|_{L^2} \), \( \|D^n|G_\rho(qu_0)|\|_{L^2} \) and \( \|G_\rho(qu_0)\|_{L^2} \) from the first term in (2.19) may be handled as the main \( \chi_2 \Delta \chi_1 \) term was earlier, with the obvious absence of the factor \( \delta^{-2} \). To control the contributions from the second term in (2.19), we use the elementary

**Lemma 5** Let \( \varphi(x), f(x) \) be functions on \( \mathbb{R}^m \) such that \( \varphi(x), |x|\varphi(x), f(x) \) and \( |\nabla f(x)| \) are in \( L^1(\mathbb{R}^m) \). Then, \( \forall \epsilon > 0 \)

\[
\left| \left( e^{-m} \varphi \left( \frac{x}{\epsilon} \right) - \int_{\mathbb{R}^m} \varphi dy \right) \delta(x) \ast f(x) \right| \leq C_m(\|\varphi\|_{L^1} + \|x|\varphi\|_{L^1}) \cdot \left( \|f\|_{L^\infty(B(0;|x| - 1))} + \|\nabla f\|_{L^\infty(B(x;1))} \right) \cdot \epsilon.
\]

Applying this for \( \epsilon = \delta, \xi' \in \mathbb{R}^2 \) fixed, and using \( F \in \mathcal{S}, \ |\tilde{q}(\xi)| \leq C(1 + |\xi|)^{-(n-1+\gamma)} \), we find that, \( \forall N > 0 \)

\[
(2.21) \|D_{\delta^{-1}}(\tilde{q} - Qd\xi') \ast F(\xi)\| \leq C_N(1 + |\xi'|)^{-\gamma} (1 + |\xi''|)^{-N} \delta.
\]

Hence, the second term in (2.19) is \( \leq C_N \delta^{\frac{2}{7}} (1 + |\xi'|)^{-\gamma} (1 + |\xi''|)^{-N} \) and this allows the contributions to (1.17)–(1.19) to be dealt with as the \( \chi_2 \Delta x'' \chi_1 \) term was before.

Finally, we need to establish the estimates (1.17–1.19) for the \( 4|\rho| \bar{\partial} \chi_0 \) term in (2.1); thus, we need to show

\[
\begin{align*}
(2.22) \|P_\rho(\bar{\partial} \chi_0 \cdot \chi_1)\|_{L^2} & \leq C |\rho|^{-1-\epsilon}, \\
(2.23) \|D^n|G_\rho(\bar{\partial} \chi_0 \cdot \chi_1)|\|_{L^2} & \leq C |\rho|^{-1-\beta-\epsilon}, \text{ and} \\
(2.24) \|G_\rho(\bar{\partial} \chi_0 \cdot \chi_1)\|_{L^2} & \leq C |\rho|^{-1-2\beta-\epsilon},
\end{align*}
\]

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for some $\epsilon > 0$. Using the fact that $\widehat{\partial \chi}_0(\xi')$ is rapidly decreasing and vanishes to first order at $\xi' = 0$, we may replace (2.6) with

$$\|\widehat{\partial \chi}_0\|_{L^2(T^\rho_\rho)}^2 \approx \int_0^{\rho_\rho} \int_{B^2((0,0,0);|\rho|^{-\frac{1}{2} - \epsilon})} |\widehat{\partial \chi}_0(\xi')|^2 d\xi' \delta^{n-2} |\psi_1(\delta r)|^2 r^{n-3} dr$$

$$\leq c_N \left( \begin{array}{c} |\rho|^{-2-4\epsilon} \int_0^{\rho_\rho} |\psi_1|^2 r^{n-2} dr \\ + |\rho|^{-3-2\epsilon} \int_0^{\rho_\rho} |\psi_1|^2 r^{n+2} dr \\ + |\rho|^{-1-2\epsilon + N} \int_0^{\rho_\rho} |\psi_1|^2 r^{n-2-2N} dr \end{array} \right)$$

$$\leq c_N \left( |\rho|^{-2-4\epsilon} + |\rho|^{-3-2\epsilon + 4\beta} |\rho|^{-1-2\epsilon + 2\beta - N} \right)$$

(2.25)

for any $N, N' \geq 0$. As before, the contributions from $T^\rho_\rho$ and $T^e_\rho$ are handled similarly. Since $\epsilon_0 < \frac{1}{2} - 2\beta$, if $N'$ is chosen large enough this yields (2.23) with $\epsilon \leq 2\epsilon_0$, which is weaker than the previously imposed $\epsilon < \frac{1}{2}(1 - 4\beta)$.

The desired estimates (2.23), (2.24) are even easier and hold for any $\beta < \frac{1}{2}$. The contribution to (2.24) from $T^{C,s}_\rho$ is controlled as in (2.13), but with the factor $\delta^{n-2}$ and with the $\widehat{\chi}_2$ in the integrand replaced by $\widehat{\partial \chi}_0$; this is then dominated in the same manner as below (2.14). The $T^{C,s}_\rho$ contribution to (2.25) is estimated as in (2.15), but with the absence of the $\delta^{-4}$. All other contributions are dealt with similarly.

This concludes the proof of Thm.11 for the case of potentials in the Hölder class $C^{n-1+\sigma}(\Omega), \sigma > 0$. The restrictions on $\beta$ and $\epsilon$ that we have needed are that $\beta < \frac{1}{2}$ and $\epsilon < \frac{1}{2}(1 - 4\beta)$.  

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3 Remarks

(i) The proof of Thm. 4 needs to be slightly modified if we assume that the potential \( q(x) \) belongs to the Sobolev space \( H^{2+\sigma}(\Omega) \) for some \( \sigma > 0 \). Since \( \partial \Omega \) is Lipschitz, such a \( q(x) \) can, by the Calderón extension theorem, be extended to be in \( H^{2+\sigma}(\mathbb{R}^n) \). Again denoting the extension by \( q \), one has by Cauchy-Schwarz

\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^{n-2}} (1 + |\xi''|)|\hat{q}(\xi', \xi'')|d\xi'' \right)^2 (1 + |\xi'|)^\sigma d\xi' \leq c(\|q\|_{2+\sigma})^2\tag{3.1}
\]

Thus, \( Q \) as in (2.18) belongs to \( L^2(\mathbb{R}^2; (1 + |\xi'|)^\sigma d\xi') \), so that \( \hat{\chi}_4 = Q \ast_{\mathbb{R}^2} \hat{\chi}_o \in L^2(\mathbb{R}^2; (1 + |\xi'|)^\sigma d\xi') \cap L^\infty \). Replacing the uniform decay estimate (2.2) with \( (2.2)_\sigma \) will allow us to handle the first term in (2.19). Furthermore, if for \( \xi' \) fixed, we let \( \phi(\cdot) = \hat{q}(\xi', \cdot) \) in Lemma 5, then \( \phi(\xi'') \) and \( |\xi''|\phi(\xi'') \) are in \( L^1(\mathbb{R}^{n-2}) \) with norms (as functions of \( \xi' \)) in \( L^2(\mathbb{R}^2; (1 + |\xi'|)^\sigma d\xi') \), and so the second term in (2.19) is \( \leq c_N \hat{\chi}_6(\xi')(1 + |\delta\xi''|)^{-N}, \forall N \), with \( \hat{\chi}_6 \) satisfying condition (3.2)\(_\sigma\). So, we are reduced to repeating the analysis of Section 2 with (2.2)\(_\sigma\) replaced by (3.2)\(_\sigma\). The decay of \( \hat{\chi}_2 \) was used in only two places in the argument. In (2.14), under (3.2)\(_\sigma\), we have the same estimate except for the absence of \( |\xi'|^{-2\alpha} \); however, this loss is absorbed into terms rapidly decreasing in \( |\rho|^{\frac{1}{4}}\delta = |\rho|^{\frac{1}{4}}\beta \) where (2.14) is used. On the other hand, in (2.16) we may estimate the inner integral by

\[
\int_{|\xi'| \geq |\rho|} |\hat{\chi}_2(\xi')|^2 \frac{d\xi'}{(|\xi'|^2 + |\xi''|^2)^2} \leq \int_{\mathbb{R}^2} |\hat{\chi}_2|^2 \frac{d\xi'}{(1 + |\xi'|)^\sigma |\xi'|^4} \leq c|\rho|^{-4-\sigma} \text{ if } |\xi''| \leq \rho \tag{3.3}
\]

and

\[
\int_{\mathbb{R}^2} |\hat{\chi}_2(\xi')|^2 \frac{d\xi'}{(|\xi'|^2 + |\xi''|^2)^2} \leq c|\xi''|^{-4} \text{ if } |\xi'| \geq \rho, \tag{3.4}
\]

so that
\[ (3.5) \quad \left\| \xi'' \right\| L^2(T^C, \infty) \leq C \left( \int_{|\xi''| \leq |\rho|} \delta^{n-6} \rho^{-4-\sigma} |\hat{\psi}(\xi'')|^2 |\xi''|^2 d\xi'' + \int_{|\xi''| \geq |\rho|} \delta^{n-6} |\hat{\psi}(\delta \xi'')|^2 |\xi''|^{-2} d\xi'' \right) \]

\[ = C \left( \delta^{-6} |\rho|^{-4-\sigma} \int_0^{|\rho|\delta} |\hat{\psi}(r)|^2 r^n \frac{dr}{r} + \delta^{-2} \int_{|\rho|\delta}^{\infty} |\hat{\psi}(r)|^2 r^{n-4} \frac{dr}{r} \right) \]

\[ \leq C_N (\delta^{-6} |\rho|^{-4-\sigma} + \delta^{-2} (||\rho|\delta|^{-N}) \]

\[ = C_N \left( |\rho|^{6\beta-4-\sigma} + |\rho|^{2\beta} (|\rho|^{\beta-\frac{1}{2}})^N \right), \quad \forall N, \]

which is \( \leq c |\rho|^{-2\beta-\epsilon} \) for \( N \) sufficiently large, since \( \beta < \frac{1}{2} \). The restrictions on \( \beta \) and \( \epsilon \) are as before.

(ii) The construction of the approximate solutions given by Thm. 4 may be generalized by taking \( \chi_0 \) to be an arbitrary analytic function of \( z = x_1 + ix_2 \), defined on a domain \( \Pi \cap \Omega \subset \subset \Omega' \subset \Pi \). Since \( \overline{\partial} \chi_0 = \Delta' \chi_0 \equiv 0 \) on \( \Omega \), the resulting \( u = u_0 + u_1 \) is still an approximate solution in the sense of Thm. 4, except that (1.8) no longer applies. Thus, Thm. 1 can be strengthened to conclude that \( (q_1 - q_2)|_{\Pi} \) is orthogonal in \( L^2(\Pi \cap \Omega, d\lambda) \) to the Bergman space \( A^2(\Pi \cap \Omega) \) of square-integrable holomorphic functions on \( \Pi \cap \Omega \). Furthermore, by repeating the construction using \( \overline{\rho} = \frac{1}{\sqrt{2}} |\rho| (\omega_R - i\omega_I) \), which induces the conjugate complex structure on \( \Pi \), for which the \( \overline{\partial} \) operator equals the \( \partial \) operator induced by \( \rho \), we obtain that \( (q_1 - q_2)|_{\Pi} \) is also orthogonal to the conjugate Bergman space \( A^2(\Pi \cap \Omega) \) of anti-holomorphic functions. (The analogue of this in two dimensions was obtained in \([SU87b]\).) It would be interesting to make further use of this information.

(iii) To obtain variants of Thm. 1 establishing smaller sets of uniqueness in \( \partial \Omega \), it might be useful to use approximate solutions associated to different two-planes. For this, it seems necessary to construct approximate solutions with much thinner supports, i.e., to overcome the restriction \( \beta < \frac{1}{4} \) in Thm. 4. Such an improvement might also be useful in extending the results to \( q_i \in L^\infty \).

References

[A] G. Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal., 27 (1988), 153–172.
[C] A. P. Calderón, On an inverse boundary value problem, in Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro, 1980.

[F] L. D. Faddeev, Growing solutions of the Schrödinger equation, Dokl. Akad. Nauk SSR, 165 (1965), 514–517 (trans. Sov. Phys. Dokl. 10 (1966), 1033.)

[G] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston, 1985.

[H65] S. Helgason, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds, Acta Math. 113 (1965), 153–180.

[H80] S. Helgason, The Radon Transform, Birkhäuser, Boston, 1980.

[Ha96] P. Hähner, A periodic Faddeev–type solution operator, Jour. Diff. Eqns. 128 (1996), 300–308.

[Is] V. Isakov, Inverse Problems for Partial Differential Equations, Springer–Verlag, Berlin, 1998.

[KV84] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements, Comm. Pure Appl. Math. 37 (1984), 113–123.

[KV85] R. Kohn and M. Vogelius, Determining Conductivity by Boundary Measurements II: Interior Results, Comm. Pure Appl. Math. 38 (1985), 643–667.

[N88] A. Nachman, Reconstruction from boundary measurements, Annals of Math. 128 (1988), 531–576.

[N96] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Annals of Math. 143 (1996), 71–96.

[St70] E. M. Stein, Singular Integrals and Differentiability properties of Functions, Princeton Univ. Press, Princeton, 1970.

[SU86] J. Sylvester and G. Uhlmann, A uniqueness theorem for an inverse boundary value problem in electrical prospection, Comm. Pure Appl. Math. 39 (1986), 92–112.
[SU87a] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Annals of Math.* **125** (1987), 153–169.

[SU87b] J. Sylvester and G. Uhlmann, Remarks on an inverse boundary value problem, in *Pseudo-differential operators: proceedings of a conference held in Oberwolfach, February 2-8, 1986*, H. O. Cordes, B. Gramsch and H. Widom., eds., *Lecture notes in mathematics* **1256**, Springer Verlag, Berlin, 1987.

[SU88] J. Sylvester and G. Uhlmann, Inverse boundary value problems at the boundary-continuous dependence, *Comm. Pure Appl. Math.* **41** (1988), 197–221.