On the stability of homogeneous Einstein manifolds II

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Abstract
For any $G$-invariant metric on a compact homogeneous space $M = G/K$, we give a formula for the Lichnerowicz Laplacian restricted to the space of all $G$-invariant symmetric 2-tensors in terms of the structural constants of $G/K$. As an application, we compute the $G$-invariant spectrum of the Lichnerowicz Laplacian for all the Einstein metrics on most generalized Wallach spaces and any flag manifold with $b_2(M) = 1$. This allows to deduce the $G$-stability and critical point types of each of such Einstein metrics as a critical point of the scalar curvature functional.

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1 | INTRODUCTION

Back in 1915, Hilbert considered in [23] the simplest curvature functional on the space $\mathcal{M}_1$ of all unit volume Riemannian metrics on a given compact manifold $M$, that is, the total scalar curvature

$$\tilde{S}_c(g) := \int_M S_c(g) \, d\text{vol}_g,$$

(1)

and proved that their critical points are precisely the metrics of constant Ricci curvature, that is, $\text{Rc}(g) = \rho g$ for some $\rho \in \mathbb{R}$, so-called Einstein metrics (see [8, 4.21]). A long time later, in the 1970s, Berger [7] and Koiso [26] independently discovered that if one considers the restriction of $\tilde{S}_c$ to the space $C_1$ of all unit volume constant scalar curvature metrics, then the coindex and nullity of any Einstein metric are both finite (modulo trivial variations). A fundamental problem is therefore to determine whether a given Einstein metric $g$ is stable (or linearly stable), that is, coindex and nullity indeed vanish, or equivalently, the Hessian of $\tilde{S}_c$ is negative definite on the tangent space $T_gC_1$ (modulo trivial variations). Any stable Einstein metric is in particular a local maximum of $\tilde{S}_c|C_1$ and it is rigid, in the sense that every Einstein metric sufficiently close to $g$ is isometric to $g$ up to scaling. It was proved by Koiso that the non-degeneracy of the Hessian is actually enough to get rigidity (see [27, Proposition 3.3; 8, 12.66]).

Stability seems to be a very strong property, which is in addition hard to be established. Indeed, the only known examples so far of stable Einstein metrics (or local maxima of $\tilde{S}_c|C_1$) with positive scalar curvature are given by some irreducible symmetric spaces. The space $T_gC_1$ coincides, modulo trivial variations, with $\mathcal{T}_g = \text{Ker} \delta_g \cap \text{Ker} \text{tr}_g$, the space of divergence-free (or transversal) and traceless symmetric 2-tensors, so-called TT-tensors, and if $\text{Rc}(g) = \rho g$, then for any $T \in \mathcal{T}_g$ the Hessian is given by

$$\tilde{S}_c''(T, T) = \frac{1}{2}((2\rho \text{id} - \Delta_L)T, T)_g,$$

(2)

where $\Delta_L$ is the Lichnerowicz Laplacian of $g$ (see [8, 4.64]). Thus $g$ is stable if and only if $2\rho < \lambda_L$, where $\lambda_L$ denotes the smallest eigenvalue of $\Delta_L|_{TT_g}$ (cf. [13, §4] and [38, §1]). The number $\lambda_L$ is usually hard to compute or even estimate, it is known for only few metrics. Note that $\tilde{S}_c''|_{TT_g}$ is non-degenerate if and only if $2\rho \notin \text{Spec}(\Delta_L|_{TT_g})$.

We assume that $M$ is a very strong property in this paper. After fixing a transitive action of a compact Lie group $G$ on $M$, we study the stability of $G$-invariant Einstein metrics, which are precisely the critical points of $S_c : \mathcal{M}_1^G \longrightarrow \mathbb{R}$, where $\mathcal{M}_1^G$ is the finite-dimensional manifold of all unit volume $G$-invariant metrics on $M$. The $G$-action provides a presentation $M = G/K$ of $M$ as a homogeneous
space, where \( K \subset G \) is the isotropy subgroup at some origin point \( o \in M \). It is proved in [30, §3.4] that

\[
T_g \mathcal{M}^G_1 = T_g N_G(K)^* g \oplus T \mathcal{T}_g^G,
\]

where \( N_G(K) \subset \text{Diff}(M) \) is the normalizer of \( K \) and \( T \mathcal{T}_g^G \) is the space of \( G \)-invariant TT-tensors. An Einstein metric \( g \in \mathcal{M}^G_1 \) is therefore called \( G \)-stable when \( \text{Sc}'_g|_{T \mathcal{T}_g^G} < 0 \), or equivalently, \( 2\rho < \lambda^G_L \), where \( \lambda^G_L \) is the smallest eigenvalue of \( \Delta_L|_{T \mathcal{T}_g^G} \). Note that

\[
\lambda_L \leq \lambda^G_L.
\]

One of the aims of [30] and this paper is to show that this ‘algebraic’ upper bound for \( \lambda_L \) is in most cases enough to establish the instability of a homogeneous Einstein metric.

Any \( G \)-stable Einstein metric is in particular a local maximum of \( \text{Sc} : \mathcal{M}^G_1 \to \mathbb{R} \), and if \( \text{Sc}'_g|_{T \mathcal{T}_g^G} \) is non-degenerate, then \( g \) is \( G \)-rigid, in the sense that \( g \) is isolated in the moduli space of \( G \)-invariant Einstein metrics on \( M \) up to pull-back by \( N_G(K) \) and scaling. \( G \)-stability is of course a necessary condition for the classical stability described above, and it is also extremely rare. As far as we know, besides irreducible symmetric metrics and the special case when any subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{k} \) is of the form \( \mathfrak{k} \oplus \mathfrak{a} \) with \([\mathfrak{a}, \mathfrak{a}] = 0 \) (for example, if \( K \) is a maximal subgroup of \( G \), see [9, 39]), the only known \( G \)-stable Einstein metrics with \( \dim \mathcal{M}^G_1 > 1 \) are:

- the standard metric on \( SU(2) \), \( E_7/\text{SO}(8) \) and \( E_8/\text{Spin}(8) \times \text{Spin}(8) \), discovered in [1] (see also [15, Remark 2.3]);
- and the unique Kähler–Einstein metric on each of the 13 flag manifolds with second Betti number \( b_2(M) \) equal to 1, which can be deduced from [6, Theorem 3.1] (here \( G \) is always an exceptional simple Lie group).

It is unknown whether these \( G \)-stable Einstein metrics are stable, as well as whether they realize the Yamabe invariant of \( M \) (that is, whether \( \text{Sc}(g) \) is the supreme among all Yamabe metrics of \( M \), which are those with the smallest scalar curvature in its unit volume conformal class).

**Remark 1.1.** After the first version of the present paper was uploaded to arXiv, many new examples of \( G \)-stable standard metrics have been found in [31] and the \( G \)-stable Einstein metric on \( E_7/\text{SO}(8) \) has been proved to be stable in [36].

On the other hand, if \( g \) is \( G \)-unstable, that is, \( \lambda^G_L < 2\rho \) (or \( \text{Sc}''_g(T, T) > 0 \) for some \( T \in T \mathcal{T}_g^G \)), then \( g \) is also dynamically unstable for the normalized Ricci flow, in the sense that there is an ancient solution emerging from \( g \) (see [13, 28]), and \( g \) does not realize the Yamabe invariant of \( M \) (see [11, Theorem 5.1]). Excepting the ones described in the two items above, all the homogeneous Einstein metrics studied in this paper were found to be \( G \)-unstable. Indeed, \( G \)-instability is considered a generic property of a compact homogeneous Einstein manifold by the experts.

It follows from (2) that the spectrum of the Lichnerowicz Laplacian \( \Delta_L \) restricted to \( T \mathcal{T}_g^G \) plays a crucial role in the study of \( G \)-stability and \( G \)-non-degeneracy. Consider any reductive decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \) of \( M = G/K \) and all the corresponding usual identifications \( T_o M \equiv \mathfrak{p} \), \( S^2(M)^G \equiv \text{sym}(\mathfrak{p})^K \), etc. According to [30, Lemma 4.7], for any \( g \in \mathcal{M}^G \),

\[
\Delta_L T = \langle L_p A \cdot, \cdot \rangle, \quad \forall T \in T \mathcal{T}_g^G, \quad T = \langle A \cdot, \cdot \rangle \in S^2(M)^G, \quad A \in \text{sym}(\mathfrak{p})^K,
\]
where \( L_p = L_p(g) : \text{sym}(p) \rightarrow \text{sym}(p) \) is the self-adjoint operator defined by

\[
\langle L_p A, B \rangle = \frac{1}{2} \langle \partial(A) \mu_p, \partial(B) \mu_p \rangle + 2 \text{tr} M_{\mu_p} AB, \quad \forall A, B \in \text{sym}(p),
\]

and \( \langle \cdot, \cdot \rangle \) is determined by \( g \). Here \( \mu_p := \text{pr}_p \circ \mu \mid_{p \times p} : p \times p \rightarrow p \) and \( M : \Lambda^2 p^* \otimes p \rightarrow \text{sym}(p) \) is the moment map for the usual representation \( \theta \) of \( \mathfrak{gl}(p) \) on \( \Lambda^2 p^* \otimes p \) (see §2.2). Thus \( \lambda_L^G \) (denoted by \( \lambda_p \) in subsequent sections) is the smallest eigenvalue of the operator \( L_p \) restricted to \( \text{sym}(p)_K \) (modulo trivial variations). This was applied to the naturally reductive case in [30], where formula (3) considerably simplifies.

In this paper, we give a formula for the Lichnerowicz Laplacian \( L_p \) in the general case, in terms of the well-known structural constants functions \([ijk]\) of \( G/K \) relative to a \( Q \)-orthogonal decomposition \( p = p_1 \oplus \cdots \oplus p_r \) in \( \text{Ad}(K) \)-invariant subspaces, where \( Q \) is any bi-invariant inner product on \( g \) (see (14)). Assume that the subspaces functions \( p_k \) are all \( \text{Ad}(K) \)-irreducible and pairwise inequivalent, which implies that \( \{ \frac{1}{\sqrt{d_1}} I_1, \ldots, \frac{1}{\sqrt{d_r}} I_r \} \) is an orthonormal basis of \( \text{sym}(p)^K \), where \( d_k := \dim p_k \) and \( I_k \mid_p := \delta_{ki} I \).

**Theorem 1.2** (See Theorem 3.1). For any metric \( g = x_1 Q \mid_{p_1} + \cdots + x_r Q \mid_{p_r} \in \mathcal{M}_G \), the entries of the matrix of \( L_p \) with respect to the above basis are given by

\[
[L_p]_{kk} = \frac{1}{d_k} \sum_{i,j \neq k} \frac{x_k[ijk]}{x_i x_j}, \quad [L_p]_{km} = \frac{1}{\sqrt{d_k d_m}} \sum_i \frac{x_i^2 - x_k^2 - x_m^2}{x_i x_k x_m}[ikm].
\]

The formula is also useful beyond the multiplicity-free case, see Remarks 3.3 and 3.4. Theorem 1.2 allows the systematic computation of the spectrum of \( L_p \) for any homogeneous Einstein metric available in the literature, provided that the numbers \( d_k \) and \( [ijk] \) are known for the homogeneous space. This is usually the case since Einstein equations are often written in terms of these numbers (see §2.5).

The rest of the paper is explorative in nature. In §4, we use the formula in Theorem 1.2 to compute the spectrum of the Lichnerowicz Laplacian \( L_p \) for all Einstein metrics on any generalized Wallach space \( G/K \) with \( G \) simple (5 infinite families and 10 exceptional examples; each space admits between 2 and 4 Einstein metrics), except \( SO^+ (k + l + m) / SO^+(k) \times SO^+(l) \times SO^+(m) \) and \( Sp^+(k + l + m) / Sp^+(k) \times Sp^+(l) \times Sp^+(m) \), where \( k, l, m \) are pairwise different. We also apply Theorem 1.2 in §5 to do the same for all Einstein metrics on flag manifolds with \( b_2(M) = 1 \) (13 exceptional examples, each one admitting between three and six Einstein metrics). The \( G \)-stability and critical point types of each of these metrics so obtained are collected in Tables 2, 6, 8 and Tables 12–15, respectively. The most noteworthy finds of our exploration are:

- the only local maxima of \( Sc \mid_{\mathcal{M}_G} \) are the standard metrics on \( SU(2), E_7/SO(8) \) and \( E_8/\text{Spin}(8) \times \text{Spin}(8) \), and the Kähler metrics on the flag manifolds with \( b_2(M) = 1 \). However, the Kähler–Einstein metrics on flag manifolds with \( b_2(M) = 2 \) and three isotropy summands are never local maxima (cf. [12]);
- any standard Einstein metric on a generalized Wallach space (three infinite families and seven isolated examples) is either a local maximum or a local minimum of \( Sc \mid_{\mathcal{M}_G} \). The value of the volume normalized scalar curvature \( Sc_N \) at the standard metric is always strictly less than for all the other Einstein metrics, except precisely in the cases when it is a local maximum;
a generalized Wallach space admits a local minimum if and only if it admits four Einstein metrics;
• for any of the three Kähler–Einstein metrics on any of the generalized Wallach spaces which are also flag manifolds (necessarily with \( b_2(M) = 2 \)), the kernel of the Lichnerowicz Laplacian \( \Delta_L |_{TT^G} \) is non-trivial. By [32], this implies that there exists a symmetric 2-tensor \( T \) as close to \( \text{Rc}(g) \) as you like such that either the existence or the (local) uniqueness of a solution to the prescribed Ricci curvature problem \( \text{Rc}(g') = cT \) fail (see §4.4). Indeed, for each of these Kähler–Einstein metrics \( g_{KE} \), we provide a pairwise non-homothetic curve \( g_t, \; t \in [1, \infty) \) of \( G \)-invariant metrics (\( \text{Sc}(g_t) \) is strictly increasing) having the same volume as \( g_1 = g_{KE} \) and such that \( \text{Rc}(g_t) = \text{Rc}(g_{KE}) \) for all \( t \geq 1 \). This only happens for these metrics, all the other ones are Ricci locally invertible;
• all the Einstein metrics studied are non-degenerate critical points of \( \text{Sc}|_{\mathcal{M}_1^G} \) and in particular \( G \)-rigid. The \( G \)-rigidity was previously known, it follows from the finiteness (up to scaling) of the set of Einstein metrics on each of these homogeneous spaces (see [3, 33]);
• any non-Kähler Einstein metric on a flag manifold with \( b_2(M) = 1 \) is a saddle point of coindex 1, except for one of the metrics on each of the three spaces \( E_8 / SU(10) \times SU(3) \times SU(1) \), \( E_8 / SU(5) \times SU(4) \times U(1) \) and \( E_8 / SU(5) \times SU(3) \times SU(2) \times U(1) \), which has coindex 2.

The normalized Ricci flow on \( \mathcal{M}_1^G \) is precisely the gradient flow of \( \text{Sc} \), whose dynamical behavior is mostly governed by the \( G \)-stability types of their fixed points, the \( G \)-invariant Einstein metrics. The Ricci flow on generalized Wallach spaces was studied in [1], where the authors prove that, generically, there are only three possible dynamical behaviors. Our approach provides an alternative proof for that. On flag manifolds with \( b_2(M) = 1 \), the dimension of the unstable manifold for each fixed point at the infinity is given in [6, Theorem 3.1]. It can be shown that such dimension is precisely one more than the coindex of the corresponding critical point. Our computations of the coindex agree with these results.

Remark 1.3 (cf. [13, Remark 1.6]). As proposed in [19], higher dimensional versions of the spherically symmetric Schwarzschild black hole can be constructed by replacing the 2-sphere with other compact Einstein manifolds. The instability condition for the \((n + 2)\)-dimensional black hole obtained is given by

\[
\lambda_L < \frac{\rho}{n - 1} \left( 4 - \frac{(n-5)^2}{4} \right) = \rho \left( \frac{9-n}{4} \right),
\]

where \( \lambda_L \) and \( \rho \) correspond to the Einstein manifold \( M^n \). It is also proved in [19] that this instability criterion is identical to that for the instability of the Freund–Rubin compactification \( AdS_m \times M^n \) in the context of the AdS/CFT correspondence. Remarkably, among all the homogeneous Einstein manifolds studied in this paper, in spite of \( \lambda_L^G \) is in many cases negative (recall that \( \lambda_L \leq \lambda_L^G \)), the Einstein–Kähler metric on the full flag manifold \( M^6 = SU(3)/S(U(1)^3) \) (which is not the standard metric) is the only one satisfying the instability condition above with \( \lambda_L \) replaced by \( \lambda_L^G \). Since \( \lambda_L^G = 0 \) for this particular metric (see Table 2, line \( W_2 \), \( k = 1 \)), it follows that \( \lambda_L \leq 0 \), providing an 8-dimensional generalized black hole which is unstable. Topologically, \( M^6 \) is the projective holomorphic tangent bundle of \( CP^2 \) (see [8, Example 13.80 and Chapter 8]).
Remark 1.4. The arXiv version of this paper contains an appendix including the Maple codes used in the computations made in § 4 and § 5.

2 | PRELIMINARIES

Let $M$ be a compact connected differentiable manifold. We assume that $M$ is homogeneous and fix an almost-effective transitive action of a compact Lie group $G$ on $M$. The $G$-action determines a presentation $M = G/K$ of $M$ as a homogeneous space, where $K \subset G$ is the isotropy subgroup at some point $o \in M$. Let $M^G$ denote the finite-dimensional manifold of all $G$-invariant Riemannian metrics on $M$.

It is well known that $g \in M^G_1$ is Einstein, that is, $Rc(g) = \rho g$ for some $\rho \in \mathbb{R}$ (which is necessarily positive if $G$ is non-abelian), if and only if $g$ is a critical point of the scalar curvature functional $Sc : M^G_1 \rightarrow \mathbb{R}$, where $M^G_1 \subset M^G$ is the codimension one submanifold of all unit volume metrics. We study in this paper the stability of $G$-invariant Einstein metrics on $M$ as critical points of $Sc|_{M^G_1}$, which is encoded in the signature of the second derivative or Hessian $Sc''$. We refer to [30] for a more detailed treatment.

2.1 | $G$-stability

The tangent space of $M^G_1$ at $g$ is given by the subspace of $S^2(M)^G$ of $g$-traceless elements, where $S^2(M)^G$ is the space of all $G$-invariant symmetric 2-tensors. It follows from [30, Corollary 3.12] that

$$T_g M^G_1 = T_g \text{Aut}(G/K) \cdot g \oplus T T^G_g,$$

where $\text{Aut}(G/K) \subset \text{Diff}(M)$ is the Lie group of automorphisms of $G$ taking $K$ onto $K$, acting by pullback on $M^G$ (that is, $T_g \text{Aut}(G/K) \cdot g$ is the space of trivial variations of $g$), and $T T^G_g := (\text{Ker} \delta_g \cap \text{Ker tr}_g)^G$ is the space of $G$-invariant TT-tensors. Note that since $G$ is compact, $T_g \text{Aut}(G/K) \cdot g = T_g N_G(K) \cdot g$, where $N_G(K)$ is the normalizer of $K$ in $G$ and so $T_g N_G(K) \cdot g$ is the tangent space of the $G$-equivariant isometry class of $g$ (see [30, §3.3]).

Definition 2.1. An Einstein metric $g \in M^G_1$ is said to be:

1. $G$-stable: $Sc''|_{T T^G_g} < 0$ (in particular, $g$ is a local maximum of $Sc|_{M^G_1}$);
2. $G$-unstable: $Sc''(T, T) > 0$ for some $T \in T T^G_g$ (g is a saddle point, unless $Sc''|_{T T^G_g} > 0$, in which case $g$ is a local minimum of $Sc|_{M^G_1}$). The coindex is the dimension of the maximal subspace of $T T^G_g$ on which $Sc''$ is positive definite;
3. $G$-non-degenerate: $Sc''|_{T T^G_g}$ is non-degenerate (thus $g$ is $G$-rigid, in the sense that it is an isolated critical point up to the $\text{Aut}(G/K)$-action), and otherwise, $G$-degenerate.

Recall that without assuming $G$-invariance, the classical stability of $g$ means that $Sc''$ is negative definite on $T T^G_g$, the infinite-dimensional vector space of all unit volume constant scalar curvature (non-trivial) variations of $g$. 

It is worth noticing that the space of trivial variations vanishes, that is, $T_g \mathcal{M}_G^1 = T_{T_g G}^G$, under any of the following conditions: $g$ is naturally reductive with respect to $G$; the isotropy representation does not contain the trivial representation; $G$ is semisimple and the isotropy representation is multiplicity-free (that is, any two different $\text{Ad}(K)$-invariant irreducible subspaces are inequivalent as $\text{Ad}(K)$-representations), for example, when $\text{rk}(G) = \text{rk}(K)$ (see [30, §3.3]).

### 2.2 Ricci curvature

Given a compact homogeneous space $M = G/K$ as above, we consider any reductive decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$, giving rise to the usual identification $T_o M \equiv \mathfrak{p}$, where $g$ and $\mathfrak{k}$ are the Lie algebras of $G$ and $K$, respectively. Let $\mu$ denote the Lie bracket of $g$. We extend $\langle \cdot , \cdot \rangle := g_o$ in the usual way to inner products on $\mathfrak{gl}(\mathfrak{p})$ and $\Lambda^2 \mathfrak{p}^* \otimes \mathfrak{p}$, respectively:

$$\langle A, B \rangle := \text{tr} AB^t, \quad \langle \lambda, \lambda \rangle := \sum |\text{ad}_\lambda X_i|^2 = \sum |\lambda(X_i, X_j)|^2,$$

where $\{X_i\}$ is any orthonormal basis of $\mathfrak{p}$ relative to $\langle \cdot , \cdot \rangle$. If

$$\mu_\mathfrak{p} := \text{pr}_\mathfrak{p} \circ \mu|_{\mathfrak{p} \times \mathfrak{p}} : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p},$$

where $\text{pr}_\mathfrak{p} : g \rightarrow \mathfrak{p}$ is the projection on $\mathfrak{p}$ relative to $g = \mathfrak{k} \oplus \mathfrak{p}$, then the Ricci operator $\text{Ric}(g)$ of the metric $g$ is given by

$$\text{Ric}(g) = M_{\mu_\mathfrak{p}} - \frac{1}{2} B_{\mu},$$

where $\langle B_{\mu} \cdot , \cdot \rangle := B_g |_{\mathfrak{p} \times \mathfrak{p}}$, $B_g$ denotes the Killing form of the Lie algebra $g$ and

$$\langle M_{\mu_\mathfrak{p}}, A \rangle := \frac{1}{4} \langle \vartheta(A)\mu_\mathfrak{p}, \mu_\mathfrak{p} \rangle, \quad \forall A \in \mathfrak{gl}(\mathfrak{p}).$$

Here $\vartheta$ is the representation of $\mathfrak{gl}(\mathfrak{p})$ given by

$$\vartheta(A)\lambda := A\lambda(\cdot , \cdot) - \lambda(A\cdot , \cdot) - \lambda(\cdot , A\cdot), \quad \forall A \in \mathfrak{gl}(\mathfrak{p}), \quad \lambda \in \Lambda^2 \mathfrak{p}^* \otimes \mathfrak{p}.$$

The function $M : \Lambda^2 \mathfrak{p}^* \otimes \mathfrak{p} \rightarrow \text{sym}(\mathfrak{p})$ is therefore the moment map from geometric invariant theory (see, for example, [10] and the references therein) for the representation $\vartheta$ of $\mathfrak{gl}(\mathfrak{p})$. Alternatively, the moment map can be defined by

$$\langle M_{\mu_\mathfrak{p}} X, X \rangle = -\frac{1}{2} \sum \langle \mu_\mathfrak{p}(X, X_i), X_j \rangle^2 + \frac{1}{4} \sum \langle \mu_\mathfrak{p}(X_i, X_j), X \rangle^2, \quad \forall X \in \mathfrak{p}.$$

It follows from (7) and (8) that

$$\text{Sc}(g) = -\frac{1}{4} |\mu_\mathfrak{p}|^2 - \frac{1}{2} \text{tr} B_{\mu}.$$

We refer to [29] for more details on this formula for the Ricci curvature from the moving bracket approach point of view.
2.3 Lichnerowicz Laplacian

It is well known that the second variation of the total scalar curvature at any Einstein metric \( g \) on \( M \), say with \( \operatorname{Rc}(g) = \rho g \), is given on \( \mathcal{T}T_g \) by

\[
Sc''_g = \frac{1}{2} \langle (2\rho \operatorname{id} - \Delta_L)^2 \cdot, \cdot \rangle_g,
\]

where \( \Delta_L \) is the Lichnerowicz Laplacian of \( g \) (see [8, 4.64]). Consider the self-adjoint operator

\[
L_p = L_p(g) : \text{sym}(p) \to \text{sym}(p), \quad \text{sym}(p) := \{ A : p \to p : A^t = A \},
\]

defined by

\[
\langle L_p A, B \rangle = \frac{1}{2} \langle \theta(A)_\mu, \theta(B)_\mu \rangle + 2 \operatorname{tr} M_{\mu p} AB. \quad \forall A, B \in \text{sym}(p).
\]

It is proved in [30, Lemma 4.7] that this is precisely the operator defined by \( \Delta_L \) under the usual identifications, that is,

\[
\Delta_L T = \langle L_p A \cdot, \cdot \rangle, \quad \forall T \in \mathcal{T}T^G_g, \quad T = \langle A \cdot, \cdot \rangle \in S^2(M)^G, \quad A \in \text{sym}(p)^K,
\]

where

\[
\text{sym}(p)^K := \{ A \in \text{sym}(p) : [\operatorname{Ad}(K), A] = 0 \} \equiv S^2(M)^G.
\]

According to (11), the \( G \)-stability type of \( g \) is therefore determined by how is the constant \( 2\rho \) suited relative to the spectrum of the Lichnerowicz Laplacian \( L_p \). If \( \lambda_p = \lambda_p(g) \) and \( \lambda_p^{\text{max}} = \lambda_p^{\text{max}}(g) \) denote, respectively, the minimum and maximum eigenvalue of \( L_p = L_p(g) \) restricted to the subspace \( \mathcal{T}T^G_g \), then an Einstein metric \( g \in \mathcal{M}^G_1 \), say \( \operatorname{Rc}(g) = \rho g \), is:

- \( G \)-stable if and only if \( 2\rho < \lambda_p \);
- \( G \)-unstable if and only if \( \lambda_p < 2\rho \);
- \( G \)-non-degenerate if and only if \( 2\rho \notin \text{Spec}(L_p \mid_{\mathcal{T}T^G_g}) \);
- a local minimum of \( \text{Sc} \mid_{\mathcal{M}^G_1} \) if \( \lambda_p^{\text{max}} < 2\rho \).

We note that on trivial variations (see (4)), \( L_p \mid_{\mathcal{T}g \operatorname{Aut}(G/K) \cdot g} = 2\rho \operatorname{id} \) (see [30, (30)]). Recall that \( \lambda_p \) was denoted by \( \lambda_L^G \) in §1.

Given an intermediate subgroup \( K \subset H \subset N_G(K) \), if \( \text{sym}(p)^H \) is the subspace of \( \text{sym}(p)^K \) of those maps which are \( \operatorname{Ad}(H) \)-invariant, then

\[
L_p \mid_{\text{sym}(p)^H} \subset \text{sym}(p)^H,
\]

for any \( g \in \mathcal{M}^{G,H} := \mathcal{M}^G \cap \text{sym}(p)^H \), the submanifold of \( G \)-invariant metrics which are in addition \( \operatorname{Ad}(H) \)-invariant (see [30, §4.4]). Note that since \( \lambda_p \leq \text{Spec}(L_p \mid_{\mathcal{T}T^G_{\text{sym}(p)^H}}) \), if the minimum eigenvalue of \( L_p \mid_{\mathcal{T}T^G_{\text{sym}(p)^H}} \) is \( \leq 2\rho \), then \( g \) is \( G \)-unstable, and that \( g \) is \( G \)-degenerate if \( 2\rho \in \text{Spec}(L_p \mid_{\mathcal{T}T^G_{\text{sym}(p)^H}}) \).
Remark 2.2. It may be the case that $\mathcal{M}^G = \text{Aut}(G/K) \cdot \mathcal{M}^{G,H}$, that is, any $G$-invariant metric is isometric to an $\text{Ad}(H)$-invariant one. A typical example of this behavior is the Stiefel manifold $\text{SO}(k+2)/\text{SO}(k)$, $k \geq 3$ (see [25, §4] and Examples 3.7 and 3.8). Since the function $\text{Sc}$ therefore takes the same values on $\mathcal{M}^{G,H}_1$ than on $\mathcal{M}^G_1$, the $G$-stability and critical point types of an Einstein metric $g \in \mathcal{M}^{G,H}_1$ as a critical point of $\text{Sc}|_{\mathcal{M}^{G,H}_1}$ and $\text{Sc}|_{\mathcal{M}^G_1}$ coincide, and so such types can be obtained from the spectrum of $L_p |_{\mathfrak{T}^G_R \text{sym}(\mathfrak{p})^H}$.

2.4 | Structural constants

We fix a bi-invariant inner product $Q$ on $\mathfrak{g}$ and consider the $Q$-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and as a background metric, $Q|_\mathfrak{p} \in \mathcal{M}^G$. For each $Q$-orthogonal decomposition,
\[ \mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r, \]
(13)
in $\text{Ad}(K)$-invariant subspaces (not necessarily $\text{Ad}(K)$-irreducible), we denote the corresponding structural constants by
\[ [ijk] := \sum_{\alpha,\beta,\gamma} Q([e^i_\alpha, e^j_\beta], e^k_\gamma)^2, \]
(14)
where $\{e^i_\alpha\}$, $\{e^j_\beta\}$ and $\{e^k_\gamma\}$ are $Q$-orthonormal basis of $\mathfrak{p}_i$, $\mathfrak{p}_j$ and $\mathfrak{p}_k$, respectively. Note that $[ijk]$ is invariant under any permutation of $ijk$ and it vanishes if and only if $[\mathfrak{p}_i, \mathfrak{p}_j] \perp \mathfrak{p}_k$. These numbers have been strongly used in the literature since McKenzie Wang and Wolfgang Ziller introduced them in their pioneering article [39].

For each $g \in \mathcal{M}^G$, there exists at least one decomposition as in (13) which is also $g$-orthogonal and
\[ g = x_1 Q|_{\mathfrak{p}_1} + \cdots + x_r Q|_{\mathfrak{p}_r}, \quad x_i > 0. \]
(15)
The metric $g$ will often be denoted by $(x_1, \ldots, x_r)$. We note that
\[ |\mu_p|^2 = \sum_{i,j,k} \frac{x_k}{x_i x_j} [ijk], \]
(16)
as $\{\frac{1}{\sqrt{x_i}} e^i_\alpha\}$ is a $g$-orthonormal basis of $\mathfrak{p}_i$ for all $i$.

2.5 | Einstein equations

Recall from (6) that the Ricci operator of $g$ is given by $\text{Ric}(g) = M_{\mu_p} - \frac{1}{2} B_{\mu}$. If we assume that, for each $k$, $M_{\mu_p}|_{\mathfrak{p}_k} = m_k I_{\mathfrak{p}_k}$ for some $m_k \in \mathbb{R}$ ($I_{\mathfrak{s}}$ denotes the identity map on $\mathfrak{s}$ for any vector space $\mathfrak{s}$), then it follows from (9) that
\[ m_k = -\frac{1}{2d_k} \sum_{i,j} \frac{x_j}{x_i x_k} [ijk] + \frac{1}{4d_k} \sum_{i,j} \frac{x_k}{x_i x_j} [ijk], \quad \forall k = 1, \ldots, r, \]
(17)
where $d_k := \dim \mathfrak{p}_k$ and $g = (x_1, \ldots, x_r)$. Let us also assume that $-B_g|\mathfrak{p}_k \times \mathfrak{p}_k = b_k Q$, $b_k \in \mathbb{R}$, which implies $B_g|\mathfrak{p}_k = -\frac{b_k}{x_k} I_{\mathfrak{p}_k}$. Note that $b_k \geq 0$, where equality holds if and only if $\mathfrak{p}_k \subset \mathfrak{z}(\mathfrak{g})$, and that if $G$ is semisimple and $Q = -B_g$, then $b_k = 1$ for all $k$.

We therefore obtain that

$$\text{Ric}(g)|_{\mathfrak{p}_k} = \rho_k I_{\mathfrak{p}_k}, \quad \rho_k = \frac{b_k}{2x_k} + m_k,$$

where each number $\rho_k$ can be written in the following ways:

\begin{align*}
\rho_k &= \frac{b_k}{2x_k} - \frac{1}{2d_k} \sum_{i,j} \frac{x_j}{x_i} [ijk] + \frac{1}{4d_k} \sum_{i,j} \frac{x_k}{x_i} [ijk], \\
&= \frac{b_k}{2x_k} - \frac{1}{4d_k} \sum_{i,j} \left( \frac{x_i}{x_j} + \frac{x_j}{x_i} - \frac{x_k}{x_i} \right) [ijk], \\
&= \frac{b_k}{2x_k} - \frac{1}{4d_k} \sum_{i,j} \frac{x_i^2 + x_j^2 - x_k^2}{x_i x_j x_k} [ijk].
\end{align*}

(18)

The Einstein equations therefore become \text{Ric}(g) = \rho I$ if and only if

$$\rho_k = \rho, \quad \forall k = 1, \ldots, r \quad \text{and} \quad \langle \text{Ric}(g) p_i, p_j \rangle = 0, \quad \forall i \neq j.$$

The two assumptions we have made above (namely, $M_{\mathfrak{p}_k}|_{\mathfrak{p}_k} = m_k I_{\mathfrak{p}_k}$ and $-B_g|_{\mathfrak{p}_k} = b_k Q|_{\mathfrak{p}_k}$) hold, for instance, in the case when the functions $\mathfrak{p}_k$ are all $\text{Ad}(K)$-irreducible, or more in general, if they are all $\text{Ad}(H)$-invariant and irreducible for some intermediate subgroup $K \subset H \subset N_G(K)$. If in addition the functions $\mathfrak{p}_k$ are pairwise inequivalent as $K$-representations (respectively, $H$-representations), then any $G$-invariant (respectively, $(G \times H)$-invariant) metric is of the form (15) and the right-hand side Einstein equations are automatically fulfilled.

These intermediate subgroups $K \subset H \subset N_G(K)$ can be used to search for Einstein metrics in the difficult case when $G/K$ is not multiplicity-free (see [37] and references therein).

Example 2.3. The isotropy representation $\mathfrak{p}$ of $\text{SO}(7)/\text{SO}(2)$ is highly non-multiplicity free, it decomposes in ten 1-dimensional and five 2-dimensional irreducible $\text{SO}(2)$-subrepresentations. However, for the subgroup $H = \text{SO}(4) \times \text{SO}(2)$ of the normalizer of $\text{SO}(2)$, $\mathfrak{p}$ only admits four inequivalent $\text{Ad}(H)$-irreducible subspaces of dimensions 2, 4, 6 and 8, respectively.

2.6 Scalar curvature

It follows from (10) and (16) (or alternatively, from (18)) that the scalar curvature of a metric $g = (x_1, \ldots, x_r)$ is given by

$$\text{Sc}(g) = \frac{1}{2} \sum_k \frac{b_k d_k}{x_k} - \frac{1}{4} \sum_{i,j,k} \frac{x_k}{x_i x_j} [ijk].$$

(19)
We note that this formula is assuming nothing about $\mathfrak{M}_p$, and if we set $Q := -B_8$ in the case when $G$ is semisimple, then $b_k = 1$ for all $k$ and the formula is valid for any $\text{Ad}(K)$-invariant decomposition as in (13).

A useful homothety invariant of $g$ is the volume normalized scalar curvature defined by

$$\text{Sc}_N(g) := (\det_Q g)^{\frac{1}{n}} \text{Sc}(g) = (x_1^{d_1} \ldots x_r^{d_r})^{\frac{1}{n}} \text{Sc}(g). \quad (20)$$

Note that $g \in \mathcal{M}^G$ is Einstein if and only if $g$ is a critical point of $\text{Sc}_N : \mathcal{M}^G \to \mathbb{R}$.

If $G/K$ is not multiplicity-free, then it is sometimes useful to use instead formula (10) for the scalar curvature to find Einstein metrics (see [25]).

### 3 A FORMULA FOR $L_p$ IN TERMS OF STRUCTURAL CONSTANTS

Let $g$ be a $G$-invariant metric on a connected homogeneous space $M = G/K$ with $G$ compact and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ a reductive decomposition as in § 2.4. Recall that $L_p : \text{sym}^{\mathfrak{p}}(\mathfrak{h}) \subset \text{sym}^{\mathfrak{p}}(\mathfrak{h})$ for any intermediate subgroup $K \subset H \subset N_G(K)$, so

$$\text{Spec}(L_p |_{\text{sym}^{\mathfrak{p}}(\mathfrak{h})}) \subset \text{Spec}(L_p |_{\text{sym}^{\mathfrak{p}}(\mathfrak{k})}).$$

In the case when $\mathfrak{p}$ is $\text{Ad}(H)$-multiplicity-free, we consider the orthonormal basis of $\text{sym}^{\mathfrak{p}}(\mathfrak{h})$ given by

$$\left\{ \frac{1}{\sqrt{d_1}} I_1, \ldots, \frac{1}{\sqrt{d_r}} I_r \right\}, \quad (21)$$

where $I_k$ is the block map $[0, \ldots, 0, \mathfrak{p}_k, 0, \ldots, 0]$ of $\mathfrak{p}$.

**Theorem 3.1.** Assume that the subspaces functions $\mathfrak{p}_k$ are all $\text{Ad}(H)$-invariant, $\text{Ad}(H)$-irreducible and pairwise inequivalent for some intermediate subgroup $K \subset H \subset N_G(K)$. Then, for any metric $g = (x_1, \ldots, x_r)$, the entries of the matrix of the Lichnerowicz Laplacian $L_p$ with respect to the orthonormal basis (21) of $\text{sym}^{\mathfrak{p}}(\mathfrak{h})$ are given by

$$[L_p]_{kk} = \frac{1}{d_k} \sum_{i,j \neq k} \frac{x_k}{x_i x_j} [ijk] + \frac{1}{d_k} \sum_{i \neq k} \frac{x_i}{x_k} [ikk], \quad \forall k = 1, \ldots, r,$$

and

$$[L_p]_{km} = \frac{1}{\sqrt{d_k d_m}} \sum_{i} \frac{x_i^2 - x_k^2 - x_m^2}{x_i x_k x_m} [ikm], \quad \forall k \neq m.$$

**Remark 3.2.** In the case when $H = K$, that is, $G/K$ is multiplicity-free, this is indeed the matrix of $L_p : \text{sym}^{\mathfrak{p}}(\mathfrak{K}) \to \text{sym}^{\mathfrak{p}}(\mathfrak{K})$.

**Remark 3.3.** The spectrum of the symmetric $r \times r$ matrix given in the theorem, restricted to the hyperplane $\{(a_1, \ldots, a_r) : \sum \sqrt{d_i} a_i = 0\}$ (which corresponds to $\text{sym}_0(\mathfrak{p})^H$), is therefore contained
in $[\lambda_p, \lambda_p^{max}]$ (see § 2.3), providing estimates which are very useful in the study of the $G$-stability of an Einstein metric $g$.

**Remark 3.4.** If the decomposition $p = p_1 \oplus \cdots \oplus p_r$ is only assumed to be $\text{Ad}(K)$-invariant and we call $L$ the symmetric $r \times r$ matrix defined as in the above theorem using the corresponding structural constants, then $L$ is a principal submatrix of the matrix of $L_p |_{\text{sym}(p)^x}$ and so the spectrum of $L$ restricted to the hyperplane $\sum \sqrt{d_i} a_i = 0$ (intersected with $\mathcal{T} \mathcal{T}_g^G$ if necessary) lies in $[\lambda_p, \lambda_p^{max}]$. In particular, the $G$-instability of an Einstein metric $g$ (say $\text{Rc}(g) = \rho g$) follows as soon as some eigenvalue of $L$ is less than $2\rho$.

**Proof.** Recall from § 2.2 that $g$ defines inner products on $\text{sym}(p)$ and $\Lambda^2 p^* \otimes p$ both denoted by $\langle \cdot, \cdot \rangle$. According to the definition of the operator $L_p$ given in (12),

$$\langle L_p I_k, I_m \rangle = \frac{1}{2} \langle \theta(I_m) \theta(I_k) \mu_p, \mu_p \rangle + 2m_k \langle I_k, I_m \rangle, \quad \forall k, m. \quad (22)$$

If we write $\mu_p : p \times p \to p$ (see (5)) as

$$\mu_p = \sum_{i,j,l} \mu^l_{ij}, \quad \mu^l_{ij} : p_i \times p_j \to p_l,$$

then following properties can be easily checked.

- $\mu^l_{ij}(X, Y) = -\mu^l_{ji}(Y, X)$ for all $X \in p_i, Y \in p_j$.
- $\mu^l_{ij} \perp \mu^l_{i'j'}$ for any $(i, j, l) \neq (i', j', l')$.
- $|\mu^l_{ij}|^2 = \frac{x_i x_j}{x_l x_l} [ijl]$; in particular, $\sum_{i,j,k} \frac{x_k}{x_i x_j} [ijk] = |\mu_p|^2$.

Using (8), we obtain that

$$\theta(I_k) \mu^l_{ij} = \begin{cases} 0, & k \neq i, j, l, \\ -\mu^l_{kj}, & k = i \neq j, l, \\ -\mu^l_{jk}, & k = j \neq i, l, \\ -2\mu^l_{kk}, & k = i = j \neq l, \\ \mu^l_{ij}, & k = l \neq i, j, \\ 0, & k = l = i \neq j, \\ 0, & k = l = j \neq i, \\ -\mu^l_{kk}, & k = l = i = j, \end{cases}$$
and

\[
\theta(I_m)\theta(I_k)\mu_{ij}^l = \begin{cases} 
\mu_{kj}^l, & m = k = i \neq j, l, \\
\mu_{ik}^l, & m = k = j \neq i, l, \\
4\mu_{kk}^l, & m = k = i = j, \\
\mu_{ij}^k, & m = k = l \neq i, j, \\
\mu_{kk}^k, & m = k = l = i = j, \\
\mu_{km}^l, & k = i \neq m = j \neq l \neq k, \\
-\mu_{kj}^m, & k = i \neq m = l \neq j \neq k, \\
0, & k = i \neq m = j = l, \\
\mu_{mk}^k, & k = j \neq m = i \neq l \neq k, \\
-\mu_{ik}^m, & k = j \neq m = l \neq i \neq k, \\
0, & k = j \neq m = i = l, \\
-2\mu_{kk}^m, & k = i = j \neq m = l, \\
-\mu_{mk}^k, & k = l \neq m = i \neq j \neq k, \\
-\mu_{im}^k, & k = l \neq m = j \neq i \neq k, \\
-2\mu_{mm}^k, & k = l \neq m = i = j. 
\end{cases}
\]

This implies that if we set \([ijk]_g := |\mu_{ij}^k|^2\), then

\[
\langle \theta(I_k)\theta(I_k)\mu_p,\mu_p \rangle = \sum_{j,l\neq k}[kjl]_g + \sum_{i,l\neq k}[kil]_g + 4\sum_{l\neq k}[kkl]_g + \sum_{i,j\neq k}[ijk]_g + [kkk]_g,
\]

\[
= 2\sum_{i,j\neq k}[kij]_g + 4\sum_{i\neq k}[kki]_g + \sum_{i,j\neq k}[ijk]_g + [kkk]_g,
\]

and for \(k \neq m\),

\[
\langle \theta(I_m)\theta(I_k)\mu_p,\mu_p \rangle = \sum_{l\neq k,m}[kml]_g - \sum_{j\neq k,m}[kjm]_g + \sum_{l\neq k,m}[kml]_g - \sum_{i\neq k,m}[kim]_g \\
- 2[kkm]_g - \sum_{j\neq k,m}[mjk]_g - \sum_{i\neq k,m}[mik]_g - 2[mmk]_g \\
= 2\sum_{i\neq k,m}[kmi]_g - 2\sum_{i\neq k,m}[kim]_g \\
- 2[kkm]_g - 2\sum_{i\neq k,m}[mik]_g - 2[mmk]_g.
\]
It now follows from (22) and (17) that in terms of $\{\frac{1}{\sqrt{d_1}} I_1, \ldots, \frac{1}{\sqrt{d_r}} I_r\}$,

$$[L_p]_{kk} = \frac{1}{d_k} \langle L_p I_k, I_k \rangle = \frac{1}{2d_k} \langle \Theta(I_k) \Theta(I_k) \mu_p, \mu_p \rangle + 2m_k$$

$$= \frac{1}{2d_k} \left( 2 \sum_{i,j \neq k} [ki]_g + 4 \sum_{i \neq k} [kki]_g + \sum_{i,j} [ijk]_g + [kkk]_g - 2 \sum_{i,j} [ki]_g + \sum_{i,j} [ijk]_g \right),$$

$$= \frac{1}{2d_k} \left( 2 \sum_{i,j \neq k} [ki]_g + \sum_{i} [kk]_g + 2 \sum_{i,j} [ijk]_g + 2 \sum_{i} [kk]_g - 2 \sum_{i,j} [ki]_g + \sum_{i,j} [ijk]_g \right),$$

$$= \frac{1}{d_k} \left( \sum_{i \neq k} [kk]_g + \sum_{i,j \neq k} [ijk]_g \right),$$

and for $k \neq m$,

$$[L_p]_{km} = \frac{1}{\sqrt{d_k \sqrt{d_m}}} \langle L_p I_m, I_k \rangle = \frac{1}{2 \sqrt{d_k \sqrt{d_m}}} \langle \Theta(I_m) \Theta(I_k) \mu_p, \mu_p \rangle$$

$$= \frac{1}{\sqrt{d_k \sqrt{d_m}}} \left( \sum_{i \neq k, m} [kmi]_g - \sum_{i \neq k, m} [kim]_g - [kkm]_g - \sum_{i \neq k, m} [mik]_g - [mmk]_g \right).$$

Finally, we use that $[ijk]_g = \frac{x_k}{x_i x_j} [ijk]$ to derive the formulas as stated in the theorem, concluding the proof.

**Remark 3.5.** If $g$ is naturally reductive with respect to $G$ and $\mathfrak{p}$, then the numbers $[ijk]_g$ are invariant under any permutation of $ijk$ and so one obtains from the above proof that

$$[L_p]_{kk} = \frac{1}{d_k} \sum_{j \neq k; i} [ijk]_g, \quad \forall k, \quad [L_p]_{km} = -\frac{1}{\sqrt{d_k \sqrt{d_m}}} \sum_i [ikm]_g, \quad \forall k \neq m.$$ 

This provides an alternative proof of [30, Theorem 5.3].

**Example 3.6.** Consider the full flag manifold $\text{SU}(n)/T$, where $T$ is the diagonal maximal torus. The standard reductive decomposition is given by

$$\mathfrak{su}(n) = \mathfrak{t} \oplus \mathfrak{p}_{12} \oplus \mathfrak{p}_{13} \oplus \cdots \oplus \mathfrak{p}_{(n-1)n},$$

where each $\mathfrak{p}_{ij} := \{zE_{ij} - E_{ji} : z \in \mathbb{C}\}$ is 2-dimensional and they are all $\text{Ad}(T)$-irreducible and pairwise inequivalent. Note that $\dim \mathcal{M}^G = \frac{n(n-1)}{2}$. Thus $[\mathfrak{p}_{ij}, \mathfrak{p}_{kl}] = 0$ if $\{i, j\}$ and $\{k, l\}$ are either equal or disjoint, and $[\mathfrak{p}_{ij}, \mathfrak{p}_{il}]$ is non-zero and it is contained in $\mathfrak{p}_{jk}$ for all $j \neq k$. Moreover, since all the non-zero structural constants are equal to $\frac{1}{n}$, it follows from (18) that the standard
or Killing metric $g_B = (1, \ldots, 1)$ (set $Q = -B_{\mathfrak{su}(n)}$) is Einstein with $2\rho = 1 - \frac{1}{2n(n-2)}$. According to Theorem 3.1, for $H = T$, one obtains that $[L_p] = \frac{1}{2n} (2(n-2)I - \text{Adj}(X))$, where $X = J(n, 2, 1)$ is the Johnson graph with parameters $(n, 2, 1)$ (see [20, §1.6]) and $\text{Adj}(X)$ denotes its adjacency matrix. The spectrum of $\text{Adj}(X)$ is therefore given by \{2(n - 2), n - 4, -2\} with multiplicities $1, n - 1, \frac{n(n-3)}{2}$, respectively (see [20, §10.1, §10.2]). This implies that

$$\text{Spec}(L_p) = \{0, \lambda_p, \lambda_{\text{max}}^p\}, \quad \lambda_p = \frac{1}{2}, \quad \lambda_{\text{max}}^p = \frac{n - 1}{n}, \quad \forall n \geq 4,$$

with multiplicities $1, n - 1$ and $\frac{n(n-3)}{2}$, respectively. For $n = 3$, $X$ is the complete graph on three vertices and so the spectrum of $\text{Adj}(X)$ equals \{2, -1\}, with respective multiplicities 1 and 2. Thus $\lambda_p = \lambda_{\text{max}}^p = \frac{1}{2}$ and has multiplicity 2. A straightforward comparison between the values of $2\rho, \lambda_p$ and $\lambda_{\text{max}}^p$ gives the following picture.

- The standard metric $g_B$ on each $\text{SU}(n)/T$ is always $G$-unstable with coindex $n - 1$.
- $g_B$ is a local minimum for $n = 3$, it is $G$-degenerate for $n = 4$ and it is a saddle point for any $n \geq 5$.
- $g_B$ is always $G$-non-degenerate, and in particular $G$-rigid, except on $\text{SU}(4)/T$. We do not know whether $g_B$ is a local minimum for $\text{SU}(4)/T$ or not.

This example is a particular case of [30, §6], where the $G$-stability of the standard metric is studied using the spectrum of $\text{Adj}(X)$ on $\text{SU}(nk)/\text{SU}(U(k)^n), \text{Sp}(nk)/\text{Sp}(k)^n$ and $\text{SO}(nk)/\text{SO}(k)^n$.

**Example 3.7.** The isotropy representation of the Stiefel manifold $\text{SO}(k + 2)/\text{SO}(k)$, $k \geq 3$, decomposes as $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$, where $d_1 = d_2 = k$, $d_3 = 1$ and $\mathfrak{p}_1, \mathfrak{p}_2$ are equivalent $\text{Ad}(K)$-representations of real type. This implies that $\dim \mathcal{M}^G = 4$. It is proved in [25, §4] that this space admits a unique Einstein metric given by $g = (1, 1, \frac{2k}{k+1})$, with $2\rho = \frac{k}{k+1}$, and that for the intermediate subgroup $H := \text{SO}(k) \times \text{SO}(2)$ of the normalizer of $\text{SO}(k)$ one has that $\mathcal{M}^G = \text{Ad}(H) \cdot \mathcal{M}^{G,H}$. Thus we only need to compute the spectrum of $L_p(g)$ restricted to $\text{sym}(\mathfrak{p})^H$ (see Remark 2.2), which is precisely the 3-dimensional space of diagonal $\text{Ad}(H)$-invariant tensors since $\mathfrak{p}_1, \mathfrak{p}_2$ are now inequivalent $\text{Ad}(H)$-representations. It follows from Theorem 3.1 that

$$L_p|_{\text{sym}(\mathfrak{p})^H} = \begin{bmatrix}
\frac{k+1}{2k^2} & \frac{k^2 - 2k - 1}{2k(k+1)} & -\frac{\sqrt{k}}{k+1} \\
\frac{k^2 - 2k - 1}{2k^2(k+1)} & \frac{k+1}{2k} & -\frac{\sqrt{k}}{k+1} \\
-\frac{\sqrt{k}}{k+1} & -\frac{\sqrt{k}}{k+1} & \frac{2k}{k+1}
\end{bmatrix},$$

and it is therefore straightforward to check that $\lambda_p = \frac{2k+1}{k^2(k+1)}$ and $\lambda_{\text{max}}^p = \frac{2k+1}{k+1}$. Since $\lambda_p < 2\rho < \lambda_{\text{max}}^p$, we obtain that $g$ is always a $G$-non-degenerate and $G$-unstable saddle point.

**Example 3.8.** In the case $S^2 \times S^3 = \text{SO}(4)/\text{SO}(2)$, one also has as above that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$, $d_1 = d_2 = 2$, $d_3 = 1$ and $\mathfrak{p}_1, \mathfrak{p}_2$ equivalent, but here $\dim \mathcal{M}^G = 5$ since $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are of complex type. Besides the Einstein metric $g = (1, 1, \frac{4}{3})$, there is another Einstein metric $g_\infty \in \mathcal{M}^G$, which was discovered in [2] and is isometric to the product metric. The metric $g_\infty$ actually
belongs to a one-parameter family of Einstein metrics in $\mathcal{M}^G$ given by the orbit $\text{Aut}(G/K) \cdot g_x$ (note that $\text{Aut}(G/K) = O(2)$). The argument used in Example 3.7 to compute the spectrum of $L_p(g)$ is still valid, giving that $\lambda_p(g) = \frac{5}{12}$ and $\lambda_p^{\text{max}}(g) = \frac{5}{3}$. On the other hand, we have that $\dim T_{g_x} \text{Aut}(G/K) \cdot g_x = 1$ (see (4)) and it is easy to see by using that $g_x$ is the product metric that $g_x$ is a $G$-non-degenerate and $G$-unstable saddle point of coindex 1.

Beyond left-invariant metrics on Lie groups, the above is the only example we know of a continuous family of Einstein metrics in $\mathcal{M}^G_1$, that is, with $\dim T_g \text{Aut}(G/K) \cdot g > 0$.

3.1 Case $r = 2$

For homogeneous spaces with only two isotropy summands, the above formulas considerably simplify. We refer to [18, 22] for a complete classification of these spaces (43 infinite families and 78 isolated examples) together with their Einstein metrics (only 19 isolated examples do not admit any). Note that $\dim \mathcal{M}^G_1 = 1$ in this case (with the only exception of $SO(8)/G_2$), so $\text{Sc} |_{\mathcal{M}^G_1}$ is a one-variable function.

It follows from (18) that the Ricci eigenvalues are

$$\rho_1 = \left(\frac{b_1}{2} - \frac{[111]}{4d_1} - \frac{[122]}{2d_1}\right) \frac{1}{x_1} - \frac{[122] x_2}{2d_1 x_1^2} + \frac{[122] x_1}{4d_1 x_2^2},$$

$$\rho_2 = \left(\frac{b_2}{2} - \frac{[222]}{4d_2} - \frac{[112]}{2d_2}\right) \frac{1}{x_2} - \frac{[122] x_1}{2d_2 x_2^2} + \frac{[112] x_2}{4d_2 x_1^2},$$

and from Theorem 3.1 that

$$[L_p]_{11} = \frac{[122] x_1}{d_1 x_2^2} + \frac{[112] x_2}{d_1 x_1^2}, \quad [L_p]_{22} = \frac{[122] x_2}{d_2 x_1^2} + \frac{[112] x_1}{d_2 x_2^2}.$$

Since $\text{Spec}(L_p) = \{0, \lambda_p\}$, we obtain that

$$\lambda_p = [L_p]_{11} + [L_p]_{22} = \frac{d_1 + d_2}{d_1 d_2} \left(\frac{[122] x_2}{x_1^2} + \frac{[112] x_1}{x_2^2}\right).$$

This can be used to find the $G$-stability and critical point types of any Einstein metric $(x_1, x_2)$ on a homogeneous space $M = G/K$ of this kind, after computing or finding in the literature the structural constants of $G/K$ (see [31] for the case of standard Einstein metrics).

Example 3.9. If $M = G/K$ is a flag manifold (that is, $G$ simple and $K$ the centralizer of a torus in $G$, see § 5) with two isotropy summands (see, for example, [6, Table 1] for a complete list, consisting of three classical families and ten exceptional examples), then it can be assumed that the only non-zero structural constant is $[112] = \frac{d_1 d_2}{d_1 + 4d_2}$ (set $Q := -B_g$) and so,

$$\rho_1 = \frac{1}{2x_1} - \frac{[112] x_2}{2d_1 x_1^2}, \quad \rho_2 = \frac{1}{2x_2} - \frac{[112] (2x_1^2 - x_2^2)}{4d_2 x_1^2 x_2}, \quad \lambda_p = \frac{(d_1 + d_2) x_2}{(d_1 + 4d_2) x_1^2}.$$
Table 1  Generalized Wallach spaces with $G$ simple (see [14, 34]). In case $W_1$, the triple $(k, 2, 2)$ (and its permutations) must be avoided, and $(k, 1, 1)$ (that is, the Stiefel manifold $SO(k + 2)/SO(k)$) are the only instances in which $G/K$ is not multiplicity-free. The spaces $W_2, W_3$ and $W_4$ are also flag manifolds (see §5).

| $W_i$ | $g/f$ | $a_1 = \frac{[123]}{d_1}$ | $a_2 = \frac{[123]}{d_2}$ | $a_3 = \frac{[123]}{d_3}$ | $[123]$ |
|-------|-------|------------------|------------------|------------------|--------|
| $W_1$ | $\mathfrak{a} \oplus (k + l + m)$ | $k$ | $\frac{l}{2(k + l + m - 2)}$ | $\frac{m}{2(k + l + m - 2)}$ | $\frac{k l m}{2(k + l + m - 2)}$ |
| $W_2$ | $\mathfrak{a} \oplus (k + l + m)$ | $\mathfrak{a} \oplus (l + m)$ | $\frac{m}{2(k + l + m)}$ | $\frac{m}{2(k + l + m)}$ | $\frac{k l m}{m}$ |
| $W_3$ | $\mathfrak{a} \oplus (k + l + m)$ | $\mathfrak{a} \oplus (k) \oplus \mathfrak{a} \oplus (m)$ | $\frac{k}{2(k + l + m)}$ | $\frac{l}{2(k + l + m)}$ | $\frac{m}{2(k + l + m)}$ |
| $W_4$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{l - 1}{4}$ | $\frac{m - 1}{4l}$ | $\frac{m - 1}{4l}$ | $\frac{l + 1}{4l}$ |
| $W_5$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{l - 2}{4(l - 1)}$ | $\frac{m - 2}{4(l - 1)}$ | $\frac{m - 2}{4(l - 1)}$ | $\frac{l - 2}{2}$ |
| $W_6$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $W_7$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $W_8$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{2}{9}$ | $\frac{2}{9}$ | $\frac{2}{9}$ | $\frac{2}{9}$ |
| $W_9$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{2}{9}$ | $\frac{2}{9}$ | $\frac{2}{9}$ | $\frac{2}{9}$ |
| $W_{10}$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{2}{9}$ | $\frac{2}{9}$ | $\frac{2}{9}$ | $\frac{2}{9}$ |
| $W_{11}$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{5}{18}$ | $\frac{5}{18}$ | $\frac{5}{18}$ | $\frac{5}{18}$ |
| $W_{12}$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{5}{18}$ | $\frac{5}{18}$ | $\frac{5}{18}$ | $\frac{5}{18}$ |
| $W_{13}$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{4}{15}$ | $\frac{4}{15}$ | $\frac{4}{15}$ | $\frac{4}{15}$ |
| $W_{14}$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{5}{18}$ | $\frac{5}{18}$ | $\frac{5}{18}$ | $\frac{5}{18}$ |
| $W_{15}$ | $\mathfrak{a} \oplus (k + l + m)$ | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ |

It is easy to see that there are exactly two solutions to $\rho_1 = \rho_2$ up to scaling, the Kähler–Einstein metric $g_0 = (1, 2)$ and a non-Kähler Einstein metric given by $g_1 = (1, \frac{4d_2}{d_1 + 2d_2})$. On the other hand, a straightforward inspection gives that $g_0$ is always $G$-stable (that is, $2\rho < \lambda_p$) and $g_1$ is always $G$-unstable (that is, $\lambda_p < 2\rho$). In particular, $g_0$ is a local maximum of $\text{Sc}_{\mathfrak{a}} | M^*_G$ and $g_1$ a local minimum for any of these spaces (cf. [5, Theorem 1]).

4  GENERALIZED WALLACH SPACES

A connected homogeneous space $M = G/K$ with $G$ compact semisimple is called a generalized Wallach space if the $B_\mathfrak{a}$-orthogonal reductive decomposition $g = \mathfrak{f} \oplus p$ satisfies that $p = p_1 \oplus p_2 \oplus p_3$ for some $\text{Ad}(K)$-irreducible subspaces functions $p_i$ such that $[p_i, p_j] \subset \mathfrak{f}$ for all $i = 1, 2, 3$. Equivalently, the only non-zero structural constant is $[123]$. The generalized Wallach spaces with $G$ simple are listed in Table 1, together with the numbers $[123]$ and $a_i := \frac{[123]}{d_i}$, where $d_i := \text{dim} p_i$ (see [14, 34]). For a table with the dimensions functions $d_i$, see [15, Table 1].

In what follows, as an application of the formula given in Theorem 3.1, we compute the spectrum of the Lichnerowicz Laplacian $L_p$ for each $G$-invariant Einstein metric on most of these spaces (see Tables 2, 6 and 8). In this way, we obtain the $G$-stability type of all these Einstein
TABLE 2  Einstein metrics $g_1$, $g_2$, $g_3$ on generalized Wallach spaces with $a_1 = a_2 = a_3 = b$ (see [33]). The metrics functions $g_i$ all satisfy that $\lambda_p < 2\rho < \lambda_{\text{max}}^p$ and so they are $G$-unstable and $G$-non-degenerate saddle points with coindex 1. All the standard metrics functions $g_0$ which are $G$-unstable are local minima of coindex 2 since $\lambda_p = \lambda_{\text{max}}^p < 2\rho$. Note that $g_1$, $g_2$, $g_3$ are all Kähler metrics in the flag cases $W_2$, $W_5$ and $W_7$, which are precisely the cases when $\lambda_p = 0$

| $W_i$ | $g/t$ | $b$ | $\frac{1-2b}{2b}$ | $g$ | $\lambda_p$ | $\lambda_{\text{max}}^p$ | $2\rho$ | Type |
|-------|-------|-----|----------------|-----|-----------|---------------|-------|------|
| $W_1$ | $\mathfrak{so}(3)$ | $\frac{1}{2}$ | $g_0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $G$-stab. |
| $W_2$ | $\mathfrak{so}(3k)$ | $k$ | $\frac{3k}{2(3k-2)}$ | $g_0$ | $\frac{3k}{2(3k-2)}$ | $\frac{3k}{2(3k-2)}$ | $\frac{5k-4}{2(3k-2)}$ | $G$-unst. |
| $W_3$ | $\mathfrak{sp}(3k)$ | $k$ | $\frac{3k}{2(3k+1)}$ | $g_0$ | $\frac{3k}{2(3k+1)}$ | $\frac{3k}{2(3k+1)}$ | $\frac{5k+2}{2(3k+1)}$ | $G$-unst. |
| $W_4$ | $\mathfrak{sp}(8)$ | $\frac{3}{2}$ | $g_0$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ | $G$-unst. |
| $W_5$ | $\mathfrak{sp}(8)$ | $\frac{5}{2}$ | $g_0$ | $\frac{5}{2}$ | $\frac{5}{2}$ | $\frac{5}{2}$ | $G$-unst. |
| $W_6$ | $\mathfrak{sp}(8)$ | $\frac{7}{2}$ | $g_0$ | $\frac{7}{2}$ | $\frac{7}{2}$ | $\frac{7}{2}$ | $G$-unst. |
| $W_7$ | $\mathfrak{sp}(8)$ | $\frac{9}{2}$ | $g_0$ | $\frac{9}{2}$ | $\frac{9}{2}$ | $\frac{9}{2}$ | $G$-unst. |
| $W_8$ | $\mathfrak{sp}(8)$ | $\frac{11}{2}$ | $g_0$ | $\frac{11}{2}$ | $\frac{11}{2}$ | $\frac{11}{2}$ | $G$-unst. |
| $W_9$ | $\mathfrak{sp}(8)$ | $\frac{13}{2}$ | $g_0$ | $\frac{13}{2}$ | $\frac{13}{2}$ | $\frac{13}{2}$ | $G$-unst. |
| $W_{10}$ | $\mathfrak{sp}(8)$ | $\frac{15}{2}$ | $g_0$ | $\frac{15}{2}$ | $\frac{15}{2}$ | $\frac{15}{2}$ | $G$-unst. |
| $W_{11}$ | $\mathfrak{sp}(8)$ | $\frac{17}{2}$ | $g_0$ | $\frac{17}{2}$ | $\frac{17}{2}$ | $\frac{17}{2}$ | $G$-unst. |
| $W_{12}$ | $\mathfrak{sp}(8)$ | $\frac{19}{2}$ | $g_0$ | $\frac{19}{2}$ | $\frac{19}{2}$ | $\frac{19}{2}$ | $G$-unst. |
| $W_{13}$ | $\mathfrak{sp}(8)$ | $\frac{21}{2}$ | $g_0$ | $\frac{21}{2}$ | $\frac{21}{2}$ | $\frac{21}{2}$ | $G$-unst. |
| $W_{14}$ | $\mathfrak{sp}(8)$ | $\frac{23}{2}$ | $g_0$ | $\frac{23}{2}$ | $\frac{23}{2}$ | $\frac{23}{2}$ | $G$-unst. |
| $W_{15}$ | $\mathfrak{sp}(8)$ | $\frac{25}{2}$ | $g_0$ | $\frac{25}{2}$ | $\frac{25}{2}$ | $\frac{25}{2}$ | $G$-unst. |

metrics, as well as what type of critical points of $\text{Sc} |_{M_i^G}$ are. It was proved in [1, Theorem 7] that, generically, there are three possible behaviors depending on the region the triple $(a_1, a_2, a_3)$ attached to $G/K$ belongs. The regions are described in [15, Table 2] (the case 5 is incorrectly assigned to region $O_3$, it is actually in $O_1$). It is easy to check that the spaces $W_1$ with $l = m$ are in $O_1$. There is no contradiction between our results and the behaviors described in these papers.

Note that $\dim M_i^G = 2$ for any generalized Wallach space (except $SO(k + 2)/SO(k)$, see Example 3.7) and so

$$\text{Spec}(L_p(g)) = \{0, \lambda_p, \lambda_{\text{max}}^p\}$$
for any Einstein metric \( g = (x_1, x_2, x_3) \in \mathcal{M}^G \). We will always set \( Q = -B \) as a background metric. Using that \([123]\) is the only non-zero structural constant, it easily follows from Theorem 3.1 that

\[
L_p(g) = \begin{bmatrix}
\frac{2x_1}{x_2x_3} & \frac{x^2_1 - x^2_2 - x^2_3}{x_1x_2x_3} & \frac{x^2_1 - x^2_2 - x^2_3}{x_1x_2x_3} \\
\frac{x^2 - x^2_2 - x^2_3}{x_1x_2x_3} & \frac{2x_2}{x_1x_3} & \frac{x^2_1 - x^2_2 - x^2_3}{x_1x_2x_3} \\
\frac{x^2 - x^2_1 - x^2_3}{x_1x_2x_3} & \frac{x^2 - x^2_1 - x^2_3}{x_1x_2x_3} & \frac{2x_3}{x_1x_2}
\end{bmatrix}.
\]

(23)

### 4.1 Case \( d_1 = d_2 = d_3 \)

It is proved in \([33, \text{Theorem 3}]\) that \( G/K (\neq \text{SO}(3)) \) admits in this case four \( G \)-invariant Einstein metrics up to scaling, the standard metric \( g_B = (1, 1, 1) \) and

\[
g_1 = \left( \frac{1 - 2b}{2b}, 1, 1 \right), \quad g_2 = \left( 1, \frac{1 - 2b}{2b}, 1 \right), \quad g_3 = \left( 1, 1, \frac{1 - 2b}{2b} \right),
\]

where \( b := a_1 = a_2 = a_3 = \frac{[123]}{d_i} \). Note that \( b \leq \frac{1}{2} \) and equality holds if and only if \( g/\mathfrak{k} = \mathfrak{so}(3) \). It follows from (18) that for \( g_B \), \( 2\rho = 1 - b \) and by (23),

\[
L_p(g_B) = \begin{bmatrix}
2b & -b & -b \\
-b & 2b & -b \\
-b & -b & 2b
\end{bmatrix}, \quad \text{so} \quad \lambda_p = \lambda_p^{\max} = 3b.
\]

On the other hand, for each \( g_i \) we have that \( 2\rho = \frac{1 + 2b}{2} \),

\[
L_p(g_i) = \begin{bmatrix}
(1 - 2b) & -\frac{1}{2}(1 - 2b) & -\frac{1}{2}(1 - 2b) \\
-\frac{1}{2}(1 - 2b) & \frac{4b^2}{1 - 2b} & -\frac{4b^2 + 4b - 1}{2(1 - 2b)} \\
-\frac{1}{2}(1 - 2b) & \frac{4b^2 + 4b - 1}{2(1 - 2b)} & 4b^2 - 1
\end{bmatrix},
\]

and thus it is straightforward to check that

\[
\text{Spec}(L_p(g_i)) = \left\{ 0, \frac{12b^2 + 4b - 1}{2(1 - 2b)}, \frac{3(1 - 2b)}{2} \right\}, \quad \forall i = 1, 2, 3.
\]

All these data together with the resulting types have been collected in Table 2. We furthermore give in Table 3 the value of the volume normalized scalar curvature \( S_{cN} \) (see (20)) for these metrics.

The two \( G \)-stable standard metrics found on \( E_7/\text{SO}(8) \) and \( E_8/(\text{Spin}(8) \times \text{Spin}(8)) \) deserve special attention, note that they are in particular local maxima of \( S_{c\mathcal{M}^G} \). We do not know whether these Einstein metrics are stable in the classical sense (see Remark 1.1). We do not know whether they realize the Yamabe invariant either.
TABLE 3 Volume normalized scalar curvature of Einstein metrics in Table 2. Note that $\text{Sc}_N(g_B) < \text{Sc}_N(g_i)$ always holds, except for $W_{11}$ and $W_{13}$, that is, precisely when the standard metric $g_B$ is a local maximum.

| $W_i$ | $g$ | $(\det g_B)^{\frac{1}{2}} \text{Sc}(g)$ | $W_i$ | $g$ | $(\det g_B)^{\frac{1}{2}} \text{Sc}(g)$ |
|-------|-----|-------------------------------------|-------|-----|-------------------------------------|
| $W_1$ | $g_B$ | $\frac{1}{2}(k = 3)$ | $W_7$ | $g_B$ | $20$ |
|       | $g_i$ | $16 \cdot 2^\frac{1}{2} \approx 20.1587$ |        | $g_i$ | $\frac{12}{5} \approx 70.4$ |
| $W_2$ | $g_B$ | $\frac{3k^2}{2}$ | $W_{11}$ | $g_B$ | $455 \frac{12}{12} \approx 37.9166$ |
|       | $g_i$ | $2^\frac{1}{2}k^2 \approx 2.52k^2$ |        | $g_i$ | $\frac{11}{5}28^\frac{1}{2} \approx 11.1341$ |
| $W_3$ | $g_B$ | $\frac{m(5k+2)}{4(3k+1)}$ | $W_{13}$ | $g_B$ | $\frac{352}{5} \approx 70.4$ |
|       | $g_i$ | $\frac{3k^2(4k+1)}{3k+1} \frac{(2k+1)^{\frac{1}{2}}}{k}$ |        | $g_i$ | $\frac{104}{5} \approx 70.3958$ |
| $W_5$ | $g_B$ | $\frac{15}{2} = 7.5$ | $W_{15}$ | $g_B$ | $32 \frac{5}{5} \approx 10.6666$ |
|       | $g_i$ | $6 \cdot 2^\frac{1}{2} \approx 7.5595$ |        | $g_i$ | $\frac{11}{5}28^\frac{1}{2} \approx 11.1341$ |

4.2 Case $d_1 = d_2 \neq d_3$

We set $b := a_1 = a_2$ and $c := a_3$. According to [33, Theorem 4], under this assumption, $G/K$ admits at most four $G$-invariant Einstein metrics given by:

- $g_{q^\pm} = (1, 1, q^\pm)$, where

$$q^\pm = \frac{1 \pm \sqrt{1 - 4(b + c)(1 - 2c)}}{2(b + c)},$$

- $g_{p^\pm} = (p^\pm, 1, 2b(p^\pm + 1))$, where

$$p^\pm = \frac{-1 + 2b - 8b^2(b + c) \pm \sqrt{(1 - 2b + 8b^2(b + c))^2 - 4(b + c)^2(1 - 4b^2)^2}}{2(b + c)(1 - 4b^2)}.$$

Note that $p^+p^- = 1$ and $1 < p^+$. The radicand is positive if and only if

$$T := 1 - 2(2b + c) + 16b^2(b + c) > 0. \quad (24)$$

It is worth pointing out that the non-negativity of the radicands is a necessary condition for the existence of the corresponding metric, which may or may not hold.

For the family $W_1$ with $l = m$, it is easy to see that

$$T = \frac{-2k^2 + 2(k + m)(m - 2)^2 + 8(m - 1)}{(k + 2m - 2)^3}, \quad (25)$$

and

$$p^\pm = \frac{5m^3 + (9k - 16)m^2 + (20 + 5k^2 - 20k)m + k^3 - 6k^2 - 8 + 12k \pm 2(m + k - 2)\sqrt{D_1(k, m)}}{(k - 2 + m)(k - 2 + 3m)(k + m)}, \quad (26)$$
where
\[ D_1(k, m) := \frac{1}{2} (k - 1 + m)(k + 2m - 2)^2 T, \]
which is positive if and only if \( T > 0 \).
On the other hand, for \( W_3 \) with \( l = m \),
\[
p^{\pm} = \frac{10mk + 5m^3 + 5mk^2 + 9m^2k + (k + 1)^3 + 5m + 8m^2 \pm (2m + k + 1)\sqrt{D_2(k, m)}}{(k + 1 + 3m)(k + 1 + m)(m + k)},
\]
where
\[ D_2(k, m) := (2k + 1 + 2m)((k + 1)^2 + 4mk + 2m^2k + 2m^3 + 4m^2 + 4m), \]
is always positive. All these metrics have been listed in Table 4 and the values of \( S_{cN}(g) \) are given in Table 5.

We compute next the spectrum of \( L_\gamma \) for each of these metrics. For \( q = q^{\pm} \), we have that \( 2\rho = 1 - bq \) by (18), and from (23) we obtain that
\[
L_\gamma(g_q) = \begin{bmatrix}
\frac{2b}{q} & \frac{b(q^2 - 2)}{q} & \frac{-\sqrt{bcq}}{q} \\
\frac{b(q^2 - 2)}{q} & \frac{2b}{q} & \frac{-\sqrt{bcq}}{q} \\
\frac{-\sqrt{bcq}}{q} & \frac{-\sqrt{bcq}}{q} & 2cq
\end{bmatrix}, \quad \text{Spec}(L_\gamma) = \left\{ 0, \frac{b(4 - q^2)}{q}, q(b + 2c) \right\}.
\]

For \( p = p^{\pm} \), one obtains from (18) that \( 2\rho = \frac{(1+p)(1-4b^2)}{2p} \) and it follows from (23) that
\[
L_\gamma(g_p) = \begin{bmatrix}
\frac{p}{p+1} & \frac{4b^2p^2 + 8b^2p + 4b^2 - p^2 - 1}{2p(p+1)} & \frac{-\sqrt{c(4b^2p + p + 4b^2 - 1)}}{2p} \\
\frac{4b^2p^2 + 8b^2p + 4b^2 - p^2 - 1}{2p(p+1)} & \frac{1}{p(p+1)} & \frac{-\sqrt{c(4b^2p - p + 4b^2 + 1)}}{2p} \\
\frac{-\sqrt{c(4b^2p + p + 4b^2 - 1)}}{2p} & \frac{-\sqrt{c(4b^2p - p + 4b^2 + 1)}}{2p} & \frac{4bc(p+1)}{p}
\end{bmatrix}.
\]

It is therefore straightforward to check that the spectrum is given by
\[
\text{Spec}(L_\gamma) = \left\{ 0, \frac{4cb^2(p+1)^2 + b(p^2 + 1) \pm \sqrt{2bD(p)}}{2bp(p+1)} \right\},
\]
where \( D(p) := c_4 p^4 + c_3 p^3 + c_2 p^2 + c_3 p + c_4 \) and
\[
c_4 = (b + c)(8b^3c + 1 - 4b^2 + 8b^4), \quad c_3 = 8b^2(b + c)(4bc + 4b^2 - 1),
\]
\[
c_2 = 96cb^4 - 8b^3 + 48c^2b^3 + 48b^5 - 2c - 8cb^2.
\]
TABLE 4 Einstein metrics on generalized Wallach spaces with $a_1 = a_2 = b \neq a_3 = c$, given by $g_{q^+} = (1, 1, q^2)$ and $g_{q^-} = (p^\pm, 1, 2b(p^\pm + 1))$ (see [33]). Assume that $k \neq m$ everywhere

| $W_i$ | $g$ | $b$ | $c$ | $q^+$ |
|-------|-----|-----|-----|-------|
| $W_1$ | $\frac{d\omega(4)}{d\omega(2)}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{4}{3}$ |
| $W_1$ | $\frac{d\omega(k+2)}{d\omega(k)}$ | $\frac{1}{2k}$ | $\frac{1}{2}$ | $\frac{2k+1}{k+1}$ (see Example 3.7) |
| $W_1$ | $\frac{d\omega(k+2m)}{d\omega(k+2m)}$ | $\frac{m}{2(k+2m)}$ | $\frac{k}{2(k+2m)}$ | $\frac{k+2m+1}{k+m}$ (see (26)) |
| $W_2$ | $\frac{d\omega(k+2m)}{d\omega(k+2m)}$ | $\frac{m}{2(k+2m)}$ | $\frac{k}{2(k+2m)}$ | $q^+ = 2$ |
| $W_3$ | $\frac{d\omega(2l)}{d\omega(1l)}$ | $\frac{l-2}{4(l-1)}$ | $\frac{1}{2(l-1)}$ | $q^+ = 2$ |
| $W_5$ | $\frac{d\omega(4)}{d\omega(4)}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $p^+ = \frac{5}{3}$ |
| $W_6$ | $\frac{d\omega(12)}{d\omega(12)}$ | $\frac{1}{5}$ | $\frac{4}{15}$ | $p^+ = \frac{5}{3}$ |
| $W_{12}$ | $\frac{d\omega(4)}{d\omega(4)}$ | $\frac{1}{4}$ | $\frac{1}{9}$ | $p^+ = \frac{499+9\sqrt{1777}}{392}$ |
| $W_{14}$ | $\frac{d\omega(5)}{d\omega(5)}$ | $\frac{1}{8}$ | $\frac{1}{9}$ | $p^+ = \frac{499+9\sqrt{1777}}{392}$ |

The case $W_1$ with $m = 1$, that is, the Stiefel manifold $SO(k+2)/SO(k)$, $k \geq 2$, was separately worked out in Example 3.7 since it is not multiplicity-free.

Finally, a straightforward inspection of all the above information gives that either $\lambda_p < 2\rho < \lambda_p^{\text{max}}$ or $\lambda_p^{\text{max}} < 2\rho$ in all cases. This produces Table 6, containing the $G$-stability and critical point types of all the metrics in Table 4.

Remark 4.1. For each space in Table 2, the Einstein metrics functions $g_i$ have the same value of $\text{Sc}_N$ (see Table 3) and identical eigenvalues of $L_p$ up to scaling. The same holds for the two Einstein
TABLE 5  Volume normalized scalar curvature of Einstein metrics in Table 4. In cases \( W_1 \) and \( W_3 \), we set \( q := q^\pm \) and \( p := p^\pm \)

| \( W_i \) | \( g \) | \( (\det g)\frac{1}{2^n} \text{Sc}(g) \) |
|---|---|---|
| \( W_1 \) | \( q^\pm \) | \( \frac{m(2k+m)k(4m+k-2)-(m+q)}{4(2m+k-2)}q^m \) |
| \( m \geq 3 \) | \( p^\pm \) | \( \frac{(p+1)m(2k+m)3m^2+4km^2+4k^2+4k+4}{4p(2m+k-2)^2}p^m \) |
| \( W_2 \) | \( q^\pm \) | \( \frac{m(2k+m)(m+k)}{2m+k}q^m \) |
| \( q^- \) | \( \frac{m(2k+m)(m+k)(m+2k)}{2m+k}q^m \) |
| \( p^+ \) | \( \frac{m(2k+m)(m+k)}{2m+k}q^m \) |
| \( p^- \) | \( \frac{m(2k+m)(m+k)}{3m+k}q^m \) |
| \( W_3 \) | \( q^\pm \) | \( \frac{m(2k+m)^2}{2m+k+1}q^m \) |
| \( p^\pm \) | \( \frac{(p+1)m(2k+m)(m+k+2m)}{p^m(2m+k+1)}q^m \) |
| \( W_5 \) | \( q^\pm \) | \( \frac{l(2l+1)}{4}q^\pm \) |
| \( q^- \) | \( \frac{(2l+1)(2l+2)+4}{4l}q^- \) |
| \( p^+ \) | \( \frac{l(l+1)}{4}q^\pm \) |
| \( p^- \) | \( \frac{(3l-4)(2l+1)}{4}q^\pm \) |
| \( W_6 \) | \( p^+ \) | \( \frac{28}{5} (120)^\frac{1}{2} \approx 21.9907 \) |
| \( p^- \) | \( \frac{28}{5} (120)^\frac{1}{2} \approx 21.9907 \) |
| \( W_{12} \) | \( q^\pm \) | \( \approx 69.1037 \) |
| \( q^- \) | \( \approx 68.5187 \) |
| \( W_{14} \) | \( p^+ \) | \( \approx 14.5750 \) |
| \( p^- \) | \( \approx 14.5750 \) |

metrics on \( W_6 \) and \( W_{14} \) (see Table 5). As far as we know, the question of whether these metrics are homothetic (that is, isometric up to scaling) or not remains open, except for few particular cases (Table 7).

4.3  Case \( d_1, d_2, d_3 \) pairwise different

This case will be treated partially, since the Einstein metrics are not available in the literature for all the spaces. It follows from [14] that in the case \( W_3 \) (with \( k, l, m \) pairwise different), \( G/K \) admits exactly four \( G \)-invariant Einstein metrics up to scaling (see also [33, Theorem 5]):

\[
g_0 = (l + m, k + m, k + l), \quad g_k = (l + m + 2k, k + m, k + l), \quad g_l = (l + m, k + 2l + m, k + l), \quad g_m = (l + m, k + m, k + l + 2m).
\]

For \( g_0 \), one obtains from (18) that

\[
2p = \frac{(k + l)(k + m)(l + m) + 2klm}{(k + l + m)(k + l)(k + m)(l + m)}, \quad (29)
\]
### Table 6

| $W_i$ | $g$          | $\lambda_p$ | $\lambda_{\text{max}}^p$ | $2\phi$ | Type       |
|-------|-------------|-------------|--------------------------|--------|------------|
| $W_1$ | $g_\uparrow$| $\frac{k}{12}$ | $\frac{5}{3}$           | $\frac{2}{3}$ | saddle     |
| $W_1$ | $g_\times$  | $\frac{k+1}{k+1}$ | $\frac{k}{k+1}$         |         | saddle     |
| $W_1$ | $g_\downarrow$ | $\frac{2k+1}{k(k+1)}$ | $\frac{2k+1}{k+1}$ | $\frac{k}{k+1}$ | saddle     |

$k \geq 3, m = 2$

| $W_1$ | $g_\uparrow$ | $\frac{m(4-q^2)}{2(k+2m-2)q}$ | $\frac{q(2k+m)}{2(k+2m-2)}$ | $\frac{2(k+2m-2)-mq}{2(k+2m-2)}$ | saddle     |
| $W_1$ | $g_\downarrow, T > 0$ | $\frac{m(4-q^2)}{2(k+2m-2)q}$ | $\frac{q(2k+m)}{2(k+2m-2)}$ | $\frac{2(k+2m-2)-mq}{2(k+2m-2)}$ | loc.min.    |
| $W_1$ | $g_\downarrow, T < 0$ | $\frac{2(k+2m-2)}{m(4-q^2)}$ | $\frac{2(k+2m-2)}{q(2k+m)}$ | $\frac{2(k+2m-2)-mq}{2(k+2m-2)}$ | saddle     |
| $W_2$ | $g_\uparrow$ | $0$ | $\frac{2k+m}{k+2m}$ | $\frac{k+m}{k+2m}$ | $\frac{k+m}{k+2m}$ | saddle     |
| $W_2$ | $g_\downarrow, k < m$ | $\frac{k}{k+m}$ | $\frac{m(2k+m)}{(k+2m)(k+m)}$ | $\frac{k^2+m^2+3km}{(k+m)(k+2m)}$ | loc.min.    |
| $W_2$ | $g_\downarrow, k > m$ | $\frac{m(2k+m)}{(k+2m)(k+m)}$ | $\frac{q(2k+m)}{2(k+2m-2)}$ | $\frac{m(4-q^2)}{2(k+2m-2)q}$ | loc.min.    |
| $W_2$ | $g_p^+$ | $0$ | $\frac{(k+3m)(k+m)}{(2m+k(k+3m))}$ | $\frac{k+3m}{k+2m}$ | saddle     |
| $W_2$ | $g_p^-$ | $0$ | $\frac{(k+3m)(k+m)}{(2m+k(k+3m))}$ | $\frac{k+3m}{k+2m}$ | saddle     |

and from (23) that

$$\text{Spec}(L_p(g_\theta)) = \left\{ 0, \frac{(k+l)(k+m)(l+m) + 4klm \pm \sqrt{P(k,l,m)}}{2(k+l+m)(k+l)(k+m)(l+m)} \right\},$$

where

$$P(k, l, m) = (k+l)(k+m)(l+m)((k+l)(k+m)(l+m) - 8klm).$$
TABLE 7  Einstein metrics in Table 4 on the spaces $W_6, W_{12}$ and $W_{14}$. They are all $G$-unstable and $G$-non-degenerate saddle points with coindex 1

| $W_i$ | $g$ | $\lambda_p$ | $\lambda_{p,\text{max}}$ | $2\varphi$ | Type |
|-------|-----|-------------|----------------|--------|------|
| $W_6$ | $g_{p^+}$ | $67 - \sqrt{1465} \approx 0.2393$ | $67 + \sqrt{1465} \approx 0.8772$ | $\frac{1}{3}$ | $0.6$ | saddle |
|       | $g_{p^-}$ | $67 - \sqrt{1465} \approx 0.3989$ | $67 + \sqrt{1465} \approx 1.4621$ | 1 | | saddle |
| $W_{12}$ | $g_{q^+}$ | $\frac{9 - \sqrt{218}}{14} \approx 0.2582$ | $\frac{165 + 11 \sqrt{218}}{210} \approx 1.0677$ | $\frac{55 - \sqrt{218}}{70} \approx 0.7087$ | saddle |
|       | $g_{q^-}$ | $\frac{165 - 11 \sqrt{218}}{210} \approx 0.5036$ | $\frac{9 + \sqrt{218}}{14} \approx 1.0275$ | $\frac{55 + \sqrt{218}}{70} \approx 0.8626$ | saddle |
| $W_{14}$ | $g_{p^+}$ | 0.1494 | 0.8657 | $\frac{28(99 + \sqrt{1177})}{9(499 + \sqrt{1177})} \approx 0.5134$ | saddle |
|       | $g_{p^-}$ | 0.3080 | 1.7839 | $\frac{28(99 - \sqrt{1177})}{9(499 - \sqrt{1177})} \approx 1.0579$ | saddle |

For the others metrics on $W_2$, as well as for the cases $W_8$ and $W_{10}$, which both admit exactly two Einstein metrics (see [14]), we proceeded as above and all the information has been collected in Table 8. The volume normalized scalar curvature of the metrics on $W_8$ and $W_{10}$ are given by

$$Sc_N(W_8, g_1) \approx 21.7434, \quad Sc_N(W_8, g_2) \approx 21.5470, \quad Sc_N(W_{10}, g_1) \approx 36.7796, \quad Sc_N(W_{10}, g_2) \approx 37.1468.$$

**Example 4.2.** It is proved in [14, Section 4.1] that $W_4 = SU(2l)/U(l)$, $l \geq 2$, admits exactly two Einstein metrics, given by $g = (1, x_2, x_3)$, where $x_3$ is one of the only two positive roots of the quartic polynomial

$$12l^4 x^4 - (48l - 8)l^3 x^3 + (72l^2 - 36l - 4)l^2 x^2 - (48l^3 - 48l^2 + 4l + 4)lx + 12l^4 - 20l^3 + 7l^2 + 2l - 1,$$
and
\[ x_2 = \frac{2l^2 x_3^2 + 2lx_3 + 1 - l - 2l^2}{2l(2lx_3 - 2l + 1)}. \]

Using, respectively, (23) and (18), it is straightforward to check that both metrics satisfy
\[ \{\lambda_p, \lambda_{p}^{\text{max}}\} = \frac{(l - 1)x_3^2 + (l + 1)x_3^3 \pm \sqrt{Q(l, x_2, x_3)}}{4lx_2x_3}, \quad 2\rho = \frac{1 + 4x_2x_3 - x_2^2 - x_3^2}{4x_2x_3}, \]

where
\[ Q(l, x_2, x_3) = (2l + 4l^2)x_3^4 + (2 - 4l^2 - 4l^2x_2^2 + 2l)x_3^2 - 1 + 2x_2^2 - 2l^2x_2^2 + 4l^2x_2^4 + 4l^2. \]

With the help of a computer, we have checked that \( \lambda_p < 2\rho < \lambda_{p}^{\text{max}} \) holds for any \( 2 \leq l \leq 20 \), so in these cases both metrics are \( G \)-non-degenerate and \( G \)-unstable saddle critical points of \( S^c |_{\mathcal{M}^G_1} \) of coindex 1. This numerical evidence suggests that these metrics are saddle points for any \( l \geq 2 \).

The only cases which have not been considered in this paper all have pairwise different \( d_1, d_2, d_3 \) and correspond to the spaces \( W_1 \) (between two and four Einstein metrics, see [15, Theorem 1.3]) and \( W_3 \) (four Einstein metrics, see [33, Example 4]). This is due to the fact that the Einstein metrics are not explicitly given in these papers, only existence results are provided.

### 4.4 Application to the prescribed Ricci curvature problem

Given a symmetric 2-tensor \( T \) on a differentiable manifold \( M \), asking about the existence and uniqueness (up to scaling) of a Riemannian metric \( g \) and a constant \( c > 0 \) such that \( \text{Rc}(g) = cT \), is called the prescribed Ricci curvature problem (see, for example, [8, Chapter 5]). For a \( G \)-invariant tensor \( T \) and metric \( g \) on a homogeneous space \( M = G/K \), the question is therefore about the image and the injectivity (up to scaling) of the differentiable function \( \text{Rc} : \mathcal{M}^G \rightarrow S^2(M)^G \). A metric \( g \in \mathcal{M}^G \) is called Ricci locally invertible if \( \text{Rc} \) is as invertible as it can be near \( g \), that is, for any \( T \) near \( \text{Rc}(g) \), existence and (local) uniqueness of solutions to the prescribed Ricci curvature problem are guaranteed (see [32] for a more detailed treatment). According to [32, Lemma 6.1] (see also [30, Lemma 4.5]), \( \text{dRc} |_{g} = \frac{1}{2} L_p |_{S^2(M)^G} \) after identifications, so a necessary condition for the Ricci local invertibility of \( g \) is that \( \dim \ker L_p |_{S^2(M)^G} = 1 \) (see [32, Theorem 1.3]). The metrics with \( \lambda_p = 0 \) contained in Tables 2, 6 and 8 are therefore examples of \( G \)-invariant Einstein metrics which are not Ricci locally invertible. These are precisely the Kähler–Einstein metrics on flag manifolds with \( b_2(M) = 2 \). On the other hand, it is easy to check that the remaining Einstein metrics considered in this paper are all Ricci locally invertible.

In what follows, we give examples of curves of \( G \)-invariant metrics with identical Ricci tensors. The metric \( g_{KE} = (1, 1, 2) \) is Kähler–Einstein on each of the following homogeneous spaces,

\[ \text{SO}(2l)/U(1) \times U(l - 1), \quad l \geq 4, \quad E_6/\text{SO}(8) \times U(1) \times U(1), \]

and \( g_{KE} = (l + m, k + m, k + l + 2m) \) is a Kähler–Einstein metric on

\[ \text{SU}(k + l + m)/S(U(k) \times U(l) \times U(m)), \quad k, l, m \geq 1. \]
TABLE 9  Einstein metrics on flag manifolds with $b_r(M) = 1$ and $r = 3$ (see [5]). On the fourth row, the embedding of $SU(6) \times SU(2) \times U(1)$ on $E_7$ is different from the one for the generalized Wallach space $W$.  

| $G/K$ | $g$ | $(x_1, x_2, x_3)$ | $(\det_{g_0}(g) \frac{1}{2} \text{Sc}(g))$ |
|-------|-----|-----------------|---------------------------------|
| $E_6$ | $g_0$ | (1, 2, 3) | $\frac{157}{30} \approx 52.3333$ |
| $E_6$ | $g_1$ | (1, 0.914286, 1.54198) | 66.9159 |
| $E_6$ | $g_2$ | (1, 1.0049, 0.129681) | 65.6151 |
| $SU(6) \times SU(2) \times U(1)$ | $g_0$ | (1, 2, 3) | $\frac{782}{15} \approx 52.1333$ |
| $SU(6) \times SU(2) \times U(1)$ | $g_1$ | (1, 1.071586, 1.253432) | 69.5453 |
| $SU(6) \times SU(2) \times U(1)$ | $g_2$ | (1, 1.056853, 0.473177) | 69.1155 |
| $SU(5) \times SU(3) \times U(1)$ | $g_0$ | (1, 2, 3) | $\frac{317}{18} \approx 17.5556$ |
| $SU(5) \times SU(3) \times U(1)$ | $g_1$ | (1, 0.678535, 1.201221) | 37.3277 |
| $SU(5) \times SU(3) \times U(1)$ | $g_2$ | (1, 1.090568, 0.546044) | 37.3277 |
| $SU(4) \times SU(3) \times U(1)$ | $g_0$ | (1, 2, 3) | $\frac{203}{12} \approx 16.8333$ |
| $SU(4) \times SU(3) \times U(1)$ | $g_1$ | (1, 1.04268, 0.373467) | 22.0134 |
| $SU(4) \times SU(3) \times U(1)$ | $g_2$ | (1, 1.04268, 0.373467) | 22.0134 |
| $SU(3) \times SU(2) \times U(1)$ | $g_0$ | (1, 2, 3) | $\frac{100}{7} \approx 14.2857$ |
| $SU(3) \times SU(2) \times U(1)$ | $g_1$ | (1, 0.678535, 1.201221) | 14.9311 |
| $SU(3) \times SU(2) \times U(1)$ | $g_2$ | (1, 0.678535, 1.201221) | 14.9311 |
| $SU(2) \times U(1)$ | $g_0$ | (1, 2, 3) | $\frac{25}{12} \approx 2.0833$ |
| $SU(2) \times U(1)$ | $g_1$ | (1, 1.67467, 2.05238) | 3.7104 |
| $SU(2) \times U(1)$ | $g_2$ | (1, 1.67467, 2.05238) | 3.4422 |

Note that the last $g_{KE}$ is a multiple of $(1,1,2)$ if $k = l$. In all the above cases, a straightforward computation using (18) gives that if $g = (x_1, x_2, x_3)$, then $Rc(g) = Rc(g_{KE})$ if and only if $x_3 = x_1 + x_2$ (recall that the Ricci tensor in terms of a $-B_g$-orthonormal basis is given by $Rc(g) = (x_1\rho_1, x_2\rho_2, x_3\rho_3)$). This implies that the metrics

$$g_t := \left(t, \frac{1}{2} \left(- t + \left(t^2 + \frac{8}{t}\right)^{1/2}\right), \frac{1}{2} \left(t + \left(t^2 + \frac{8}{t}\right)^{1/2}\right)\right), \quad t > 0,$$

have all the same volume and Ricci tensor as $g_1 = g_{KE} = (1, 1, 2)$. This curve was recently discovered by Pulemotov and Ziller on the full flag manifold $SU(3)/S(U(1)^3)$.

On the spaces $SO(8)/U(1) \times U(3), E_6/ SO(8) \times U(1) \times U(1)$ and $SU(3k)/ S(U(k)^3)$ (that is, when $d_1 = d_2 = d_3$), the scalar curvature is given by

$$\text{Sc}(g_t) = c \left(-t^2 + 3 \left(t^2 + \frac{8}{t}\right)^{1/2} + \frac{4}{t}\right), \quad \text{for some } c > 0,$$

which is a convex function on $(0, \infty)$ with a global minimum at $t = 1$. The family $\{g_t : 1 \leq t\}$ is therefore pairwise non-homothetic as $\text{Sc}(g_t)$ is strictly increasing on $[1, \infty)$ (note that also
TABLE 10 Einstein metrics on flag manifolds with $b_2(M) = 1$ and $r = 4$ (see [4])

| $G/K$         | $g$        | $(x_1, x_2, x_3, x_4)$ | $\text{(det}_{g_0}^\frac{1}{2} \text{Sc}(g))$ |
|---------------|------------|------------------------|-----------------------------------------------|
| $F_4$ SU(3)×SU(2)×U(1) | $g_0$ | (1, 2, 3, 4) | $\frac{20}{9} \sqrt{24} \approx 14.5995$ |
|               | $g_1$ | (1.2761, 1.9578, 2.3178) | 14.5693 |
|               | $g_2$ | (1.07904, 0.2291, 1.0097) | 14.0370 |
| $E_7$ SU(4)×SU(3)×SU(2)×U(1) | $g_0$ | (1, 2, 3, 4) | $\frac{212}{9} \sqrt{24} \approx 38.0563$ |
|               | $g_1$ | (1.82333, 1.2942, 1.3449) | 37.6284 |
|               | $g_2$ | (1.9912, 0.5783, 1.1312) | 37.3618 |
| $E_8$ SU(7)×SU(2)×U(1) | $g_0$ | (1, 2, 3, 4) | $\frac{637}{15} \sqrt{23} \approx 70.2624$ |
|               | $g_1$ | (1.9133, 1.4136, 1.5196) | 69.6567 |
|               | $g_2$ | (1.9663, 0.4898, 1.0809) | 68.8049 |

TABLE 11 Einstein metrics on flag manifolds with $b_2(M) = 1$ and $r = 5, 6$ (see [16])

| $G/K$         | $g$        | $(x_1, ..., x_r)$ | $\text{(det}_{g_0}^\frac{1}{2} \text{Sc}(g))$ |
|---------------|------------|-----------------|-----------------------------------------------|
| $E_6$ SU(3)×SU(2)×U(1) | $g_0$ | (1, 2, 3, 4, 5) | $\frac{572}{15} \sqrt{3} \approx 69.9307$ |
|               | $g_1$ | (1, 0.599785, 1.08371, 0.901823, 1.22291) | 68.8905 |
|               | $g_2$ | (1, 1.01237, 0.546007, 1.05352, 1.10879) | 68.7023 |
|               | $g_3$ | (1, 1.08294, 1.04088, 0.532615, 1.10351) | 68.7757 |
|               | $g_4$ | (1, 0.720713, 1.02546, 0.475234, 1.07095) | 68.6913 |
|               | $g_5$ | (1, 1.03732, 1.04718, 1.03082, 0.29862) | 68.4798 |
| $E_7$ SU(4)×SU(3)×SU(2)×U(1) | $g_0$ | (1, 2, 3, 4, 5, 6) | $\frac{159}{5} \sqrt{2} \approx 69.0420$ |
|               | $g_1$ | (1, 0.823084, 1.1467, 1.17377, 1.42664, 1.46519) | 68.6856 |
|               | $g_2$ | (1, 0.986536, 0.636844, 1.06853, 1.13323, 0.921127) | 68.4684 |
|               | $g_3$ | (1, 0.90422, 0.778283, 0.927483, 1.03408, 0.359949) | 68.2283 |
|               | $g_4$ | (1, 0.954875, 0.965321, 1.00534, 0.290091, 1.01965) | 67.8054 |

\{g_t : 0 < t ≤ 1\} is pairwise non-homothetic. Since for any $t > 1$, $g_t$ has the same Ricci eigenvalues as $g_s$, where $s := \frac{1}{2}(-t + (t^2 + 8t)^{1/2})$, we do not know whether these pairs of metrics are isometric or not.

For the rest of the Kähler–Einstein metrics which are not Ricci locally invertible, that is, $g_{p±}$ on $SO(2l)/U(1) \times U(l − 1)$, $l ≥ 5$ and $W_2$ with $k = l$, and the three Kähler-Einstein metrics on $W_2$ with pairwise different $k, l, m$, the same behavior occurs and an explicit curve $g_t$ such that $\text{Rc}(g_t) \equiv \text{Rc}(g_1)$ and with the same properties as above can easily be given.

Remark 4.3. After the first version of this paper appeared in arXiv, these examples were generalized in [35].
| $G/K$ | $g$ | $\lambda_p$ | $\lambda_{\text{max}}$ | $2p$ |
|---|---|---|---|---|
| $E_8$ | $g_0$ | $\frac{6 - \sqrt{51}}{30}$ | $\frac{6 + \sqrt{51}}{30}$ | $\frac{19}{30}$ |
| | $g_1$ | 0.478572 | 1.55965 | 0.821452 |
| | $g_2$ | 0.118731 | 1.272251 | 0.829109 |
| $E_7$ | $g_0$ | $\frac{11 - \sqrt{6}}{10}$ | $\frac{11 + \sqrt{6}}{10}$ | $\frac{17}{30}$ |
| | $g_1$ | 0.469601 | 1.311724 | 0.825338 |
| | $g_2$ | 0.352429 | 1.157063 | 0.772758 |
| $E_6$ | $g_0$ | $\frac{7 - \sqrt{2}}{6}$ | $\frac{7 + \sqrt{2}}{6}$ | $\frac{11}{18}$ |
| | $g_1$ | 0.469601 | 1.485343 | 0.816530 |
| | $g_2$ | 0.194860 | 1.232011 | 0.820660 |
| $E_6$ | $g_0$ | $\frac{9 - \sqrt{33}}{24}$ | $\frac{9 + \sqrt{33}}{24}$ | $\frac{7}{12}$ |
| | $g_1$ | 0.465849 | 1.391997 | 0.815861 |
| | $g_2$ | 0.281933 | 1.187358 | 0.801967 |
| $F_4$ | $g_0$ | $\frac{5}{6}$ | $\frac{4}{3}$ | $\frac{5}{9}$ |
| | $g_1$ | 0.469601 | 1.311722 | 0.825338 |
| | $g_2$ | 0.352428 | 1.157064 | 0.772757 |
| $G_2$ | $g_0$ | $\frac{1}{2}$ | $\frac{5}{4}$ | $\frac{5}{12}$ |
| | $g_1$ | 0.413430 | 1.210009 | 0.502068 |
| | $g_2$ | 0.19355 | 2.670881 | 0.970058 |

5 | FLAG MANIFOLDS

A flag manifold (or generalized flag manifold) is a homogeneous space $M = G/K$, where $G$ is a compact semisimple Lie group and $K$ is the centralizer of a torus in $G$. It is called full flag when $K$ is a maximal torus of $G$. Any compact, simply connected and de Rham irreducible homogeneous Kähler manifold is isometric to a $G$-invariant metric on a flag manifold $G/K$ with $G$ simple and simply connected. The isotropy representation of any flag manifold is multiplicity-free. We refer to [3, 6] and references therein for more information on flag manifolds.

In this section, we compute the $G$-stability and critical point types of all Einstein metrics on flag manifolds with $b_2(M) = 1$, where $b_2(M)$ denotes the second Betti number of $M$. As in §4, this will follow from the knowledge of the spectrum of $L_p$, which is computed using Theorem 3.1 and explicitly provided for all Einstein metrics. Our results on $G$-stability agree with [6, Theorem 3.1], where the Ricci flow behavior on flag manifolds with $b_2(M) = 1$ is studied, and also with [21, Theorems 1 and 2], where the case of flag manifolds with three isotropy summands is considered.

The number $r$ of Ad($K$)-irreducible summands of the isotropy representation of these spaces goes from 2 to 6, and each of them admits a unique Kähler–Einstein metric given by
Table 13: Einstein metrics in Table 10. The Kähler metric $g_0$ is always $G$-stable and all the other ones are $G$-non-degenerate and $G$-unstable with coindex as in the last column.

| $G/K$ | $g$  | $\lambda_1$ | $\lambda_2$ | $\lambda_{max}$ | $2\rho$ | $c_x$ |
|-------|------|-------------|-------------|----------------|--------|------|
| $E_6$ | $g_0$ | 1          | 20          | 7             | $\frac{14}{6}$ | 0    |
|       | $g_1$ | 0.4111     | 0.8447      | 1.2064        | 0.5380 | 1    |
|       | $g_2$ | 0.2108     | 1.2583      | 2.1506        | 0.8231 | 1    |
| $E_7$ | $g_0$ | $\frac{11-\sqrt{13}}{12}$ | $\frac{53}{54}$ | $\frac{11+\sqrt{13}}{12}$ | $\frac{4}{9}$ | 0    |
|       | $g_1$ | 0.5001     | 1.0457      | 1.2838        | 0.7173 | 1    |
|       | $g_2$ | 0.4331     | 1.0696      | 1.3332        | 0.7626 | 1    |
| $E_8$ | $g_0$ | $\frac{9}{10} - \frac{\sqrt{13}}{15}$ | $\frac{14}{15}$ | $\frac{9}{10} + \frac{\sqrt{13}}{15}$ | $\frac{11}{30}$ | 0    |
|       | $g_1$ | 0.4826     | 1.0134      | 1.2468        | 0.6781 | 1    |
|       | $g_2$ | 0.3937     | 1.1307      | 1.3923        | 0.7826 | 1    |
| $E_8$ | $g_0$ | $\frac{19}{20} - \frac{\sqrt{309}}{60}$ | $\frac{97}{90}$ | $\frac{19}{20} + \frac{\sqrt{309}}{60}$ | $\frac{7}{15}$ | 0    |
|       | $g_1$ | 0.5039     | 1.1437      | 1.4399        | 0.7970 | 1    |
|       | $g_2$ | 0.1698     | 0.9025      | 1.1115        | 0.7038 | 1    |
|       | $g_3$ | 0.4527     | 1.0347      | 1.3018        | 0.7173 | 1    |
|       | $g_4$ | 0.2157     | 0.6048      | 1.5116        | 0.7173 | 1    |
|       | $g_5$ | 0.2406     | 0.791323    | 1.169060      | 1.320873 | 0.7173 | 2    |

Table 14: Einstein metrics on $E_6/SU(5) \times SU(4) \times U(1)$ (see Table 11). The Kähler metric $g_0$ is $G$-stable and the remaining ones are all $G$-non-degenerate and $G$-unstable.

| $g$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_{max}$ | $2\rho$ | $c_x$ |
|-----|-------------|-------------|-------------|----------------|--------|------|
| $g_0$ | 0.483308 | 0.807779 | 1.002382 | 1.156529 | $\frac{11}{30} \approx 0.36$ | 0    |
| $g_1$ | 0.510214 | 1.027352 | 1.238460 | 1.655126 | 0.757540 | 1    |
| $g_2$ | 0.444857 | 1.112644 | 1.251940 | 1.598781 | 0.73104 | 1    |
| $g_3$ | 0.387659 | 0.850534 | 1.094764 | 1.233562 | 0.678788 | 1    |
| $g_4$ | 0.458617 | 0.613228 | 1.437954 | 1.612703 | 0.773963 | 2    |
| $g_5$ | 0.240674 | 0.791323 | 1.169060 | 1.320873 | 0.674542 | 1    |

$g_0 := (1, 2, \ldots, r)$, with respect to a suitable decomposition $p = p_1 \oplus \cdots \oplus p_r$ in $\text{Ad}(K)$-irreducible subspaces. Note that $\text{dim} \mathcal{M}_1^r = r-1$.

The case when $r = 2$ has been worked out in Example 3.9. Actually any flag manifold with $r = 2$ has $b_2(M) = 1$. On the other hand, the remaining flag manifolds with $r = 3$ all have $b_2(M) = 2$ and are given by the generalized Wallach spaces $W_2$, $W_5$ and $W_7$, already studied in §4.

The structural constants and numerical approximations of all Einstein metrics (other than $g_0$) on flag manifolds with $b_2(M) = 1$ are given in [5] for $r = 3$ (the case $E_7/SU(5) \times SU(3) \times U(1)$ had to be corrected since the right dimensions are $(60,30,10)$, see [17]), in [4] for $r = 4$ and in [16] for $r = 5, 6$. The metrics, including their volume normalized scalar curvature $S_{c_N}$ relative to the Killing metric $g_0 = (1, \ldots, 1)$ (see (20)), are listed in Tables 9–11. It follows that the Einstein metrics on each space are always pairwise non-homothetic. Also note that $S_{c_N}(g_i) > S_{c_N}(g_j)$ for any $i \geq 1$ in all cases.
TABLE 15 Einstein metrics on $E_6/\text{SU}(5) \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ (see Table 11). The Kähler metric $g_i$ is $G$-stable and the other ones are all $G$-unstable

| $g$  | $\lambda_p$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_{\text{max}}$ | $2\rho$ | $\varepsilon x$ |
|------|-------------|-------------|-------------|-------------|-----------------|--------|--------------|
| $g_1$ | 0.373056    | 0.616209    | 0.784713    | 0.907323    | 1.025363        | $\frac{3}{10} = 0.3$ | 0            |
| $g_2$ | 0.560023    | 0.753518    | 0.990611    | 1.177858    | 1.218291        | 0.627866 | 1            |
| $g_3$ | 0.499591    | 0.995988    | 1.060532    | 1.188804    | 1.256011        | 0.697205 | 1            |
| $g_4$ | 0.371562    | 0.622626    | 0.981804    | 1.317906    | 1.460064        | 0.735036 | 2            |

It is straightforward to check that the corresponding numerical approximations for the eigenvalues of the $r \times r$ matrix $L_p$ and the $G$-stability types of these Einstein metrics are given as in Tables 12–15, according to the number $r$. They are all $G$-non-degenerate and Ricci locally invertible (see § 4.4).

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