Optimal relocation strategies for spatially mobile consumers

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Abstract

We develop a model of the behaviour of a dynamically optimizing economic agent who makes consumption-saving and spatial relocation decisions. We formulate an existence result for the model, derive the necessary conditions for optimality and study the behaviour of the economic agent, focusing on the case of a wage distribution with a single maximum.

Keywords: consumption decisions, spatial relocation, optimal control

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1 Introduction

The emergence of the literature on “new economic geography” in the 1990s has rekindled the interest in the spatial aspects of economics. The new generation of models makes heavy use of the standard economics toolkit and analyzes a number of issues from a dynamic perspective or from the perspective of optimizing agents. Interestingly, however, spatial models adopting the perspective of dynamically optimizing consumers remain in relative minority, despite the fact that they are standard fare in mainstream economic research. The models in [1], [2] and [3] are notable exceptions in this respect.

The present work develops a model that studies the behaviour of a dynamically optimizing economic agent who makes two types of interrelated choices: consumption-saving decisions and spatial relocation (migration) decisions. Unlike the constructs in [1], [2] and [3], the consumer in our model has a finite lifetime and a bequest incentive at the end of his life. This departure from classical Ramsey-type models enables richer global dynamics by allowing agents to inherit their ancestors’ savings in a setup akin to that of overlapping

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generations models. It also offers the additional option of introducing heterogeneous agents whose economy-wide behaviour can be obtained through an explicit aggregation rule.

A second important difference with the above papers is that our consumer saves in nominal assets. This partly depends on our choice to center the model around the behaviour of (potentially different) individuals as opposed to that of a representative agent. More importantly, however, the nominal savings feature reflects our belief that pecuniary considerations play an important role in the choice of where to work and how much to consume.

Formally, we cast the model in the form of a continuous-time optimal control problem with a finite planning horizon. The assumptions of the model, while fairly standard in economics, create several mathematical challenges in the present setup. First, they preclude the direct application of the existence theorems for optimal control problems known to the authors. This requires an alternative approach to proving the existence of solutions of the model. In particular, unlike traditional existence proofs in the spirit of Theorem 4, §4.2 in [8], we prove an existence result that dispenses with convexity assumptions on the set of generalized speeds for the optimal control problem. Also, the functional forms employed in the model do not allow one to directly apply Pontryagin’s maximum principle, since the transversality condition for one of the state variables is not defined at the point 0. To be able to use the maximum principle, we prove that for an optimal control-trajectory pair the terminal value of the particular state variable is strictly positive. Finally, economic considerations point to the fact that only a subset of the possible values for the other state variable in the model are of real interest. One way to take care of that issue is to constrain the values of this state variable to lie in a certain set – the interval [0, 1] in our case – for each point in time. However, instead of using an explicit state constraint, which would complicate the use of the maximum principle, we introduce an additional correcting mechanism by suitably defining the wage distribution function \( w(x) \) outside the interval [0, 1]. We claim this mechanism does not influence the other characteristics of the model, while being sufficient to ensure that the optimal state variable never leaves the set in question, and we prove that indeed this is the case.

The rest of the paper is organized as follows. Section 2 introduces the model and the assumptions we make. Section 3 proves the existence of a solution to the model under the above assumptions. Section 4 applies Pontryagin’s maximum principle to obtain necessary conditions for optimality. Section 5 describes some convenient transformations of the system of necessary conditions and comments on the existence of solutions to this system. The analysis in section 6 characterizes the asymptotic behaviour of terminal assets for different sets of model parameters. This establishes facts that are useful for the study of relocation choices in section 7. The results in this section are also of independent economic interest as they shed light on the impact of intra- and intertemporal preferences on the saving decisions of an individual with a sufficiently long planning horizon. Finally, section 7 tackles the question of relocation behaviour in the basic case of a wage distribution having a single maximum (single-peaked wage distribution). The results obtained for this case are intuitive, if unsurprising: in most cases a consumer with a sufficiently long lifespan relocates
toward the wage maximum. While the single-peak case offers easily predictable results, we consider it useful as a testing ground for the model before applying it to more interesting situations. Indeed, preliminary results by the authors on the case of a double-peaked wage distribution suggest that a host of complex situations, including multiple solutions and bifurcations, can arise.

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2 The model

We employ a continuous-time model that deals with the case of a consumer who, given an initial location in space $x_0$ and asset level $a_0$, supplies inelastically a unit of labour in exchange for a location-dependent wage $w(x(t))$, and chooses consumption $c(t)$ and spatial location $x(t)$ over time. The consumer has a finite lifetime $T$ at the end of which a bequest in the form of assets is left. This bequest provides utility to the consumer. More precisely, for $\rho$, $r$, $\eta$, $\xi$, and $p$ – positive constants, and $\theta \in (0, 1)$, we look at the optimal control problem

$$
\begin{align*}
&\max_{c(t), z(t) \in \Delta} J(c(t), z(t)) := \int_0^T e^{-\rho t} \left( \frac{c(t)^{1-\theta}}{1-\theta} - \eta z^2(t) \right) dt + e^{-\rho T} a(T)^{1-\theta} \\
&\text{subject to} \\
&\dot{a}(t) = ra(t) + w(x(t)) - pc(t) - \xi z^2(t), \\
&\dot{x}(t) = z(t), \\
&a(0) = a_0 \geq 0, \\
&x(0) = x_0 \in [0, 1],
\end{align*}
$$

where $a(t)$, $x(t)$ are the state variables, assumed to be absolutely continuous, and $c(t)$, $z(t)$ are the control variables. The set of admissible controls $\Delta$ consists of all pairs of functions $(c(t), z(t))$ which are measurable in $[0, T]$ and satisfy the conditions

$$
\begin{align*}
&0 \leq c(t) \leq C, \\
&|z(t)| \leq Z, \\
&a(T) \geq 0.
\end{align*}
$$
The constants $C$ and $Z$ are such that

\[(2.7)\]
\[
C^\theta > \max \left(1, \mu^{\frac{1}{2}}\right) \left(\frac{a_0 + T \max_x |w(x)|}{p^{1-e^{-rT}}_r}\right),
\]

\[(2.8)\]
\[
Z > \frac{T \max_x |w'(x)| e^{rT}}{2\xi},
\]

where $\mu := \max_{t,t_0 \in [0,T]} e^{(r-\rho)(t-t_0)} > 0$.

**Remark.** The bounds we impose on the admissible controls through equations (2.4) and (2.5) are convenient from a technical viewpoint when proving the existence theorem in section 3. Conditions (2.7) and (2.8) ensure that these constraints are never binding. However, considerations of general nature – both economic and physical – make such constraints appealing.

In the above model $\rho > 0$ is a time discount parameter and $\theta \in (0,1)$ is the utility function parameter. The control $c(t)$ represents physical units of consumption and the control $z(t)$ governs the speed of relocation in space. We assume that relocation in space brings about two type of consequences. First, relocation causes subjective disutility associated with the fact that there is habit formation with respect to the place one occupies. Second, changing one’s location is associated with monetary relocation costs that have to be paid out of one’s income or stock of assets. As a baseline case we choose to capture these phenomena by means of the speed of movement in space $\dot{x}(t)$ or, equivalently, $z(t)$, transformed through a quadratic function. The manner in which spatial relocation affects the consumer’s utility and wealth can vary widely, however, therefore other functional forms are certainly admissible. The parameters $\eta, \xi \geq 0$ multiplying this function measure the subjective disutility from changing one’s location in space and the relocation costs in monetary terms, respectively. The parameters $p > 0$ and $r > 0$ stand for the price of a unit of consumption and the interest rate, respectively.

The nonnegativity condition is imposed on terminal assets $a(T)$ both to have a well-defined objective functional and to capture the intuitive observation that, with a known lifetime, a debtor is unlikely to be allowed to leave behind outstanding liabilities to creditors. The condition $a(T) \geq 0$ also sheds light on the nonnegativity restriction for $a_0$, since in an environment where no debts are allowed at the end of one’s lifetime, no debtor position can be inherited at birth.

For the purposes of our analysis we look at the basic case where economic space is represented by the real line. We are interested in only a subset of it, the interval $[0, 1]$. This is modelled by taking the initial location $x_0 \in [0, 1]$ and requiring the location-dependent wage, which is positive in $(0, 1)$, to be negative outside $[0, 1]$ and to satisfy additional assumptions. Namely, we have $w(x) > 0$, $x \in (0,1)$ and $w(x) < 0$, $x \notin [0,1]$, as well as $w'(x) > 0$, $x \in (−\infty, 0]$ and $w'(x) < 0$, $x \in [1, \infty)$. Later in the paper we formally verify the intuitive claim that an optimal trajectory for $x(t)$ will never leave the interval $[0,1]$ under the above conditions. We also assume that $w(x) \in C^2(\mathbb{R}^1)$ and $w(x)$ is bounded, i.e.
\[
\max_{x \in \mathbb{R}} |w(x)| < +\infty. \text{ We impose additional requirements on } w(x) \text{ to derive some of the results in section 7.}
\]

3 Existence of solutions

Next, we investigate the issue of existence of a solution to the model. The proof requires two intermediate results, shown as lemmas below.

Lemma 3.1 Let the functions \( x_i, i = 1, 2, \ldots, \) and \( \bar{x} \) be defined on \([0, T]\) and take values in the interval \([a, b]\). Let \( x_i \) tend uniformly to \( \bar{x} \) as \( i \to \infty \) (denoted by \( x_i \Rightarrow \bar{x} \)) and \( w \in C^0[a, b] \). Then, in \([0, T]\), as \( m \to \infty \) we have

i) \( \frac{1}{m} \sum_{i=1}^{m} x_i \Rightarrow \bar{x}, \)

ii) \( w(x_m) \Rightarrow w(\bar{x}), \)

iii) \( w(\frac{1}{m} \sum_{i=1}^{m} x_i) \Rightarrow w(\bar{x}). \)

Proof. The proof directly replicates the standard proofs of counterpart results on numerical sequences. ■

Lemma 3.2 (The Banach-Saks Theorem) Let \( \{v_n\}_{n=1}^{\infty} \) be a sequence of elements in a Hilbert space \( H \) which are bounded in norm: \( \|v_n\| \leq K = \text{const}, \forall n \in \mathbb{N}. \) Then, there exist a subsequence \( \{v_{n_k}\}_{k=1}^{\infty} \) and an element \( v \in H \) such that

\[
\left\| \frac{v_{n_1} + \cdots + v_{n_s}}{s} - v \right\| \to 0 \text{ as } s \to \infty.
\]

Proof. See, for example, [5, pp.78-81]. ■

Theorem 3.3 Under the assumptions stated in section 2, there exists a solution \((c(t), z(t)) \in \Delta \) of problem (2.1)-(2.3).

Proof. We start by noting that the set of admissible controls \( \Delta \) is nonempty. To see this, choose controls \( c(t) \equiv c_0 = \text{const} \) and \( z(t) \equiv 0 \). Then, any \( c_0 \in (0, w(x_0)/p] \) will ensure that \( a(T) \geq 0 \).

Next, observe that \((c(t), z(t)) \in \Delta \) implies \( c(t), z(t) \in L_\infty[0, T] \) and

\[
(3.1) \quad 0 \leq a(T) \leq \text{const} = e^{rT}(a_0 + T \max_{x \in \mathbb{R}} |w(x)|).
\]

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(We note that (2.6) implies the following bounds,
\[
p∥c(t)∥_{L^1[0,T]}, ξ∥z(t)∥^2_{L^2[0,T]} ≤ e^{rT} \left( a_0 + T \max_{x} |w(x)| \right),
\]
which do not depend on the constants C and Z.)

Through an application of Hölder’s inequality one verifies that \( \int_0^T c(t)^{1−θ}e^{-ρt} dt \leq const(T)∥c(t)∥^θ_{L^1} \).

Thus, for \((c(t), z(t)) ∈ Δ\), the objective functional (2.1) is bounded. Consequently, \( J_0 := \sup_{(c(t), z(t)) ∈ Δ} J(c(t), z(t)) < ∞ \). Then we can choose a sequence of controls \( \{ (c_k(t), z_k(t)) \} ⊂ Δ \) such that \( J(c_k(t), z_k(t)) → J_0 \).

Let \( a_k(t) \) and \( x_k(t) \) be the state variables corresponding to the controls \((c_k(t), z_k(t))\). It is easy to verify that the functions \( a_k(t) \) and \( x_k(t) \) form a uniformly bounded and equicontinuous set. Then, by the Arzelà-Ascoli theorem (see, e.g., [8], Ch.4), there exists a subsequence \((a_{k_q}(t), x_{k_q}(t)) \) ⇒ \((\bar{a}(t), \bar{x}(t))\).

Then, if \( c_{k_q}(t) \) and \( z_{k_q}(t) \) are the controls corresponding to \((a_{k_q}(t), x_{k_q}(t))\), by Lemma 3.2 we can in turn choose subsequences \( c_{k_q}(t) \) and \( z_{k_q}(t) \) whose arithmetic means tend in \( L^2[0,T] \) norm to some elements in \( L^2[0,T] \), denoted \( \bar{c}(t) \) and \( \bar{z}(t) \), respectively. However, we do not claim that \( \bar{a}(t) \) and \( \bar{x}(t) \) correspond to \( \bar{c}(t) \) and \( \bar{z}(t) \). For brevity we introduce the notation \( c_q(t) := c_{k_q}(t) \), \( z_q(t) := z_{k_q}(t) \) etc., as well as \( \bar{c}_m(t) := \frac{1}{m}\sum_{q=1}^m c_q(t) \) and \( \bar{z}_m(t) := \frac{1}{m}\sum_{q=1}^m z_q(t) \).

Then, we have established that: (1) \((a_q(t), x_q(t)) \) ⇒ \((\bar{a}(t), \bar{x}(t))\) as \( q → ∞ \) and (2) \( \bar{c}_m(t) \rightrightarrows \bar{c}(t) \), \( \bar{z}_m(t) \rightrightarrows \bar{z}(t) \) as \( m → ∞ \).

Recall that \( a_q(t) \) and \( x_q(t) \) correspond to \( c_q(t) \) and \( z_q(t) \) as solutions to the respective differential equations (2.2) and (2.3).

So far, it is not clear whether \( \bar{c}_m(t) \) and \( \bar{z}_m(t) \) are admissible. It is immediately seen that they satisfy (2.4) and (2.5) but the corresponding a(T) may fail to satisfy (2.6). However, we can show that the controls \( \bar{c}(t) \) and \( \bar{z}(t) \) are admissible.

To prove the last claim, note first that according to [72] Ch.7, §2.5, Prop.4 we can choose a subsequence of \( \{ \bar{c}_m(t), \bar{z}_m(t) \} \) that converges a.e. to \((\bar{c}(t), \bar{z}(t))\) and, after passing to the limit, we obtain that \( \bar{c}(t) \) and \( \bar{z}(t) \) satisfy (2.4) and (2.5).

It remains to show that \( \bar{a}(T) = e^{rT} \left[ a_0 + \int_0^T [w(\bar{x}(t)) - p\bar{c}(t) - ξ \bar{z}^2(t)] e^{-rt} dt \right] ≥ 0 \),

where \( \bar{x}(t) = x_0 + \int_0^t \bar{z}(τ)dτ \).

Consider
\[
\bar{a}_m(T) = e^{rT} \left[ a_0 + \int_0^T [w(\bar{x}_m(t)) - p\bar{c}_m(t) - ξ \bar{z}_m^2(t)] e^{-rt} dt \right], \tag{3.2}
\]

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with $\tilde{x}_m(t) = x_0 + \int_0^t \tilde{z}_m(\tau)d\tau = \frac{1}{m}\sum_{q=1}^m \left(x_0 + \int_0^t z_q(\tau)d\tau\right) = \frac{1}{m}\sum_{q=1}^m x_q(t)$. Adding and subtracting $\frac{1}{m}\sum_{q=1}^m w(x_q(t))$, and applying Jensen’s inequality to the term $\tilde{z}_m^2(t)$, we obtain

\[
\tilde{a}_m(T) \geq e^{rT} \int_0^T \left[w(\tilde{x}_m(t)) - \frac{1}{m} \sum_{q=1}^m w(x_q(t))\right] e^{-rt} dt +
\]

\[
\frac{1}{m} \sum_{q=1}^m e^{rT} \left[a_0 + \int_0^T [w(x_q(t)) - pc_q(t) - \xi z_q^2(t)] e^{-rt} dt\right] \geq
\]

\[
e^{rT} \int_0^T \left[w(\tilde{x}_m(t)) - \frac{1}{m} \sum_{q=1}^m w(x_q(t))\right] e^{-rt} dt \tag{3.3}
\]

By Lemma 3.1 both integrands inside the square brackets in the last line of (3.3) tend uniformly to $w(\tilde{x}(t))$, so that the integral tends to zero. Thus, if $\lim_{m \to \infty} \tilde{a}_m(T)$ exists, we have $\lim_{m \to \infty} \tilde{a}_m(T) \geq 0$.

We proceed to check that $\lim_{m \to \infty} \tilde{a}_{m_j}(T) = \tilde{a}(T)$ for a suitable subsequence $\tilde{a}_{m_j}(T)$. We know that $\frac{1}{m} \sum_{q=1}^m x_q(t) = x_0 + \int_0^t \frac{1}{m} \sum_{q=1}^m z_q(\tau)d\tau$. Since $\frac{1}{m} \sum_{q=1}^m x_q(t) \to \tilde{x}(t)$ and, additionally, it is easy to verify by applying Hölder’s inequality that $\int_0^t \tilde{z}_m(\tau)d\tau \to \int_0^t \tilde{z}(\tau)d\tau$ when $\tilde{z}_m(t) \to \tilde{z}(t)$, we obtain $\tilde{x}(t) = x_0 + \int_0^t \tilde{z}(\tau)d\tau = \tilde{\tilde{x}}(t)$.

As $\tilde{c}_m(t) \to \tilde{c}(t)$ and $\tilde{z}_m(t) \to \tilde{z}(t)$, there exist subsequences $\tilde{c}_{m_j}(t)$ and $\tilde{z}_{m_j}(t)$ such that $\tilde{c}_{m_j}(t) \to \tilde{c}(t)$ and $\tilde{z}_{m_j}(t) \to \tilde{z}(t)$. To simplify notation, we refer to the new subsequences as $\tilde{c}_j(t)$ and $\tilde{z}_j(t)$. Since the function $z^2$ is bounded on $[-Z, Z]$, by Lebesgue’s dominated convergence theorem $\int_0^T \xi \tilde{z}_j^2(t)e^{-rt} dt \to \int_0^T \xi \tilde{z}^2(t)e^{-rt} dt$. It can also be verified that $\int_0^T \tilde{c}_j(t)e^{-rt} dt \to \int_0^T \tilde{c}(t)e^{-rt} dt$. Lastly, we know that $\int_0^T w(\tilde{x}_j(t))e^{-rt} dt \to \int_0^T w(\tilde{x}(t))e^{-rt} dt$ as $w(\tilde{x}_j(t)) \to w(\tilde{x}(t))$. Consequently, the limit of (3.2) as $m_j \to \infty$ exists and is equal to $\tilde{a}(T)$, so that $\tilde{a}(T) \geq 0$. This shows that $\tilde{c}(t)$ and $\tilde{z}(t)$ are admissible.

By an application of Lebesgue’s dominated convergence theorem to the respective terms in (2.1), we get $\lim_{j \to \infty} J(\tilde{c}_j(t), \tilde{z}_j(t)) = J(\tilde{c}(t), \tilde{z}(t))$.

Define $\rho_{m_j}(T) := e^{rT} \int_0^T \left[w(\tilde{x}_{m_j}(t)) - \frac{1}{m_j} \sum_{q=1}^{m_j} w(x_q(t))\right] e^{-rt} dt$. Obviously, $\tilde{a}_{m_j}(T) = \tilde{a}_{m_j}(T) - \rho_{m_j}(T)$ also tends to $\tilde{a}(T)$ and $\tilde{a}_{m_j}(T) \geq \frac{1}{m_j} \sum_{q=1}^{m_j} a_q(T)$, where $a_q(T)$ corresponds to $(c_q(t), z_q(t))$. Then, indexing by $j$ instead of $m_j$ to simplify notation, we get

\[
J_0 \geq J(\tilde{c}(t), \tilde{z}(t)) = \lim_{j \to \infty} \left\{ \int_0^T \left[\tilde{c}_j(t)^{1-\theta} - \eta \tilde{z}_j^2(t)\right] e^{-rt} dt + e^{-rT} \tilde{a}_j^{1-\theta}(T) \right\} \geq
\]
\[
\lim_{j \to \infty} \left\{ \frac{1}{j} \sum_{i=1}^{j} \left[ \int_{0}^{T} \left[ \frac{c_i^{1-\theta}(t)}{1-\theta} - \eta z_i^{2}(t) \right] e^{-\rho t} dt + e^{-\rho T} a_i^{1-\theta}(T) \right] \right\} = \\
\lim_{j \to \infty} \left\{ \frac{1}{j} \sum_{i=1}^{j} J(c_i(t), z_i(t)) \right\} = J_0,
\]
where the inequality is a consequence of the fact that the functions \( \sigma \mapsto \sigma^{1-\theta} \) and \( z \mapsto (-z^2) \) are concave and we can apply Jensen’s inequality. This shows that the admissible pair \((\bar{c}(t), \bar{z}(t))\) is optimal, as required. ■

4 Necessary conditions for optimality

In this section we turn to the derivation of a set of necessary conditions for optimality on the basis of Pontryagin’s maximum principle. To apply the maximum principle, however, we need to ensure that the terminal utility from assets \( e^{-\rho T} a(T)^{1-\theta}/(1-\theta) \) is well-behaved at least for the optimal value of terminal assets. To this end, we prove the following

Theorem 4.1 For the optimal controls \((c(t), z(t))\) the terminal value of assets \( a(T) \) is strictly positive for any \( T > 0 \).

Proof. Let us assume that there is a time \( T_0 > 0 \) for which \( a(T_0) = 0 \).

Step 1. We first verify that it is impossible to have \( c(t) \equiv 0 \). Assuming that \( c(t) \equiv 0 \), together with \( a(T_0) = 0 \), yields the objective functional

\[
J(0, z(t)) = -\eta \int_{0}^{T_0} z^2(t)e^{-\rho t} dt \leq 0.
\]

If one of the following two conditions is valid:

1. \( a_0 > 0 \) and \( x_0 \in [0, 1] \); 
2. \( a_0 = 0 \) and \( x_0 \in (0, 1) \),

then we can choose the admissible pair \( \bar{z}(t) \equiv 0 \) (so that \( x(t) \equiv x_0 \)) and \( \bar{c}(t) \equiv c_0 = \text{const} > 0 \), where \( c_0 \) is such that

\[
a_0 + \int_{0}^{T_0} [w(x_0) - pc_0] e^{-\rho t} dt = 0.
\]
The last condition is equivalent to
\[ a_0 + T_0 w(x_0) \frac{e^{-rT_0} - 1}{-r} = pc_0 \frac{e^{-rT_0} - 1}{-r} \]
and therefore \( c_0 > 0 \). Then
\[ J(\bar{c}(t), \bar{z}(t)) = \int_0^{T_0} \frac{c_0^{1-\theta}}{1-\theta} e^{-\rho t} dt > 0, \]
contradicting the optimality of \((c(t), z(t))\).

The case \( a_0 = 0 \) and \( x_0 = 0 \) or 1 is pathological in the sense that the consumer has neither current income \((w(0)=w(1)=0)\), nor initial wealth. Economically, it is implausible to expect that such a consumer will manage to obtain a loan. From a purely formal point of view, however, the consumer could get a loan and finance his relocation even in this case. Moreover, he will be able to attain positive consumption levels.

To illustrate the above claim, suppose that \( x_0 = 0, a_0 = 0 \) and the consumer spends all the income left after paying the relocation costs. Fix \( \varepsilon_0 > 0 \) in such a way that \( w'(x) \geq w'(0)/2 > 0 \) for \( x \in [0, \varepsilon_0] \). Let the relocation strategy be given by the control \( \bar{z}(t) = \varepsilon \sin \frac{\pi}{T} t, \varepsilon > 0 \). Then consumption is given by \( \bar{c}(t) = w(\bar{x}(t)) - \xi \bar{z}^2(t) \), where \( \bar{x}(t) \) is
\[ \bar{x}(t) = \varepsilon \int_0^t \sin \left( \frac{\pi}{T} \tau \right) d\tau = \frac{\varepsilon T}{\pi} \left( 1 - \cos \frac{\pi t}{T} \right) = \frac{2\varepsilon T}{\pi} \sin^2 \frac{\pi t}{2T}. \]

Then, \( \bar{x}(T) = \frac{2T\varepsilon}{\pi} < \varepsilon_0 \) for \( \varepsilon \) sufficiently small. Notice that
\[ w(\bar{x}(t)) = w(\bar{x}(t)) - w(0) = w'(x^*(t))\bar{x}(t) \geq \frac{w'(0)}{2} \bar{x}(t), \]
for some \( x^*(t) \in (0, \bar{x}(t)) \). Consequently, we obtain
\[ w(\bar{x}(t)) - \xi \bar{z}^2(t) \geq \varepsilon \left[ \frac{w'(0)}{2} \frac{2T}{\pi} \sin^2 \frac{\pi t}{2T} - \varepsilon \xi \sin^2 \frac{\pi t}{T} \right] = \varepsilon \sin^2 \frac{\pi t}{2T} \left[ \frac{Tw'(0)}{\pi} - 4\varepsilon \xi \cos^2 \frac{\pi t}{2T} \right]. \]

Consumption will be positive if
\[ g(t) := \frac{Tw'(0)}{\pi} - 4\varepsilon \xi \cos^2 \frac{\pi t}{2T} > 0 \text{ for } t \in [0, T]. \]

For \( \varepsilon \) small \( g(0) = Tw'(0)/\pi - 4\xi \varepsilon > 0 \). Also,
\[ g'(t) = 4\varepsilon \xi \frac{\pi}{2T} \cos \frac{\pi t}{2T} \sin \frac{\pi t}{2T} = 2\varepsilon \xi \frac{\pi}{T} \sin \frac{\pi t}{T} \geq 0 \text{ for } t \in [0, T]. \]

Thus, \( g(t) \geq g(0) > 0 \), as required.
Remark. It is easy to see that in the above example we can take $\tilde{z}(t)$ to be any smooth function that is positive on $(0, T)$, zero for $t = 0, T$ and $\dot{\tilde{z}}(0) > 0$.

Step 2. Since $c(t) \neq 0$, there exists a set $A \subset [0, T]$, $\text{meas } A > 0$, such that

$$\text{essinf}_{t \in A} c(t) > \varepsilon_1 > 0.$$ 

Let us take the control pair $(\tilde{c}(t), \tilde{z}(t))$ with $\tilde{c}(t) := c(t) - \varepsilon \chi_A(t)$ and $\tilde{z}(t) := z(t)$, where $\chi_A(t)$ is the indicator function of the set $A$ and $\varepsilon \in (0, \varepsilon_1)$. These controls are admissible if we have terminal assets $\tilde{a}(T_0) > 0$, $\forall \varepsilon \in (0, \varepsilon_1)$. To verify the last claim, we take

$$\tilde{a}(T_0) = e^{\rho T_0} \left[ a_0 + \int_0^{T_0} w(x(s)) - p(c(s) - \varepsilon \chi_A(s)) - \xi z^2(s) e^{-\rho s} ds \right] = e^{\rho T_0} \int_A p e^{-\rho s} ds = \varepsilon C_1,$$

where $C_1 := e^{\rho T_0} \int_A e^{-\rho s} ds > 0$.

An application of Taylor’s formula yields

$$\frac{\bar{c}(t)^{1-\theta}}{1-\theta} = \frac{c(t)^{1-\theta}}{1-\theta} + (-\varepsilon \chi_A(t))c(t)^{-\theta} + (-\varepsilon \chi_A(t))^2 \frac{\theta}{2} c^*(t)^{-\theta-1},$$

where $c^*(t) = \alpha(t) \bar{c}(t) + (1 - \alpha(t))c(t)$, $\alpha(t) \in (0, 1)$ or $c^*(t) = c(t) - \varepsilon \chi_A(t) \alpha(t)$. Note also that for $t \in A$ we have $0 < c(t) - \varepsilon_1 \leq c^*(t) \leq c(t)$, so that $(c(t) - \varepsilon_1)^{-\theta-1} \geq c^*(t)^{-\theta-1} \geq c(t)^{-\theta-1}$.

Let us compare

$$J(c(t), z(t)) = \int_0^{T_0} \frac{c(t)^{1-\theta}}{1-\theta} e^{-\rho s} dt - \eta \int_0^{T_0} z^2(t) e^{-\rho s} dt$$

and

$$J(\bar{c}(t), \bar{z}(t)) = \int_0^{T_0} \frac{\bar{c}(t)^{1-\theta}}{1-\theta} e^{-\rho s} dt - \eta \int_0^{T_0} \bar{z}^2(t) e^{-\rho s} dt + \frac{\tilde{a}(T_0)^{1-\theta}}{1-\theta} e^{-\rho T_0}$$

$$= J(c(t), z(t)) + \left[ -\varepsilon \int_A c(t)^{-\theta} e^{-\rho s} dt - \frac{\theta \varepsilon^2}{2} \int_A (c(t) - \varepsilon \alpha(t))^{-1-\theta} e^{-\rho s} dt \right] +$$

$$+ \frac{(\varepsilon C_1)^{1-\theta}}{1-\theta} e^{-\rho T_0}.$$ 

We will show that $J(\bar{c}(t), \bar{z}(t)) > J(c(t), z(t))$ for $\varepsilon \in (0, \varepsilon_1)$ sufficiently small. This will be true if we are able to establish that

$$\frac{(\varepsilon C_1)^{1-\theta}}{1-\theta} e^{-\rho T_0} > \varepsilon \int_A c(t)^{-\theta} e^{-\rho s} dt + \frac{\varepsilon^2 \theta}{2} \int_A (c(t) - \varepsilon_1)^{-1-\theta} e^{-\rho s} dt,$$

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where the last integral provides an upper bound on \( \int_A (c(t) - \varepsilon \alpha(t))^{-1-\theta} e^{-\rho t} dt \). Denoting the respective positive constants in the above inequality by \( B_1, B_2 \) and \( B_3 \), we obtain

\[
\varepsilon^{1-\theta} B_1 > \varepsilon B_2 + \varepsilon^2 B_3
\]

or

\[
B_1 > \varepsilon^\theta B_2 + \varepsilon^{1+\theta} B_3,
\]

which is obviously true for \( \varepsilon \in (0, \varepsilon_1) \) sufficiently small. This contradicts the optimality of \((c(t), z(t))\). Thus, \( a(T_0) = 0 \) cannot be true and hence \( a(T_0) > 0 \). \( \Box \)

On the basis of Theorem 4.1 an optimal solution \( \bar{c}(t), \bar{z}(t) \) to problem (2.1)-(2.3) (possibly non-unique) also solves the following problem, where the controls \((c(t), z(t)) \in \Delta_1 \subset \Delta:\)

\[
\max_{c(t), z(t) \in \Delta_1} J(c(t), z(t))
\]

\[
\dot{a}(t) = ra(t) + w(x(t)) - pc(t) - \xi^2(t)
\]

\[
\dot{x}(t) = z(t)
\]

\[
a(0) = a_0 \geq 0,
\]

\[
x(0) = x_0 \in [0, 1],
\]

\[
a(T) \geq \delta > 0,
\]

with \( \delta \) being an appropriate constant, strictly smaller than the optimal value of terminal assets.

To avoid burdensome notation, from now on we do not append additional symbols to the state, costate and control variables in the model when referring to their optimal values. However, we use alternative symbols to denote alternative sets of variables to be compared with the optimal ones.

Taking into account that we do not impose any state constraints on the problem, Theorem 5.2.1 in [4] provides the set of necessary conditions. To derive the latter, we define the Hamiltonian for the problem

\[
H := H(t, a, x, \varphi, \psi, p_1, p_2) =
\]

\[
p_1(ra + w(x) - p\varphi - \xi \psi^2) + p_2\psi + \lambda_0 e^{-\rho t} \left( \frac{\varphi^{1-\theta}}{1-\theta} - \eta \psi^2 \right),
\]

where \( \lambda_0 \in \{0, 1\} \). Then

1) The costate variables \( p_i(t), \ i = 1, 2 \), satisfy a.e. on \((0, T)\)

\[
\dot{p}_1(t) = -rp_1(t),
\]

\[
\dot{p}_2(t) = -p_1(t)w'(x(t)).
\]
2) The function \( \varphi, \psi \mapsto H(t, a(t), x(t), \varphi, \psi, p_1(t), p_2(t)) \) attains its maximum with respect to \( \varphi, \psi \) at the point \((c(t), z(t))\) for almost all \( t \in [0, T] \), where \( \varphi, \psi \) satisfy the constraints on the function values arising from \( \Delta_1 \), i.e. \( \varphi \in [0, C] \) and \( |\psi| \leq Z \):

\[
H(t) := H(t, a(t), x(t), c(t), z(t), p_1(t), p_2(t)) = \max_{\varphi, \psi} H(t, a(t), x(t), \varphi, \psi, p_1(t), p_2(t)).
\]

(4.4)

3a) Since \( a(t) \) and \( x(t) \) are fixed at \( t = 0 \), the values of \( p_i(0) \) are arbitrary, i.e.

\[
p_1(0) = \lambda_1, \quad \lambda_1 \in \mathbb{R},
\]

(4.5)

\[
p_2(0) = \lambda_2, \quad \lambda_2 \in \mathbb{R}.
\]

(4.6)

3b) Since the terminal values \((a(T), x(T))\) of the state variables are at an interior point of the target set

\[
\{(a, x) \in \mathbb{R}^2 | a \geq \delta > 0, x \in \mathbb{R}^1 \},
\]

the corresponding normal cone is trivial and the transversality condition at the right endpoint \( T \) has the form

\[
p_1(T) = \lambda_0 e^{-\rho T} a(T)^{-\theta},
\]

(4.7)

\[
p_2(T) = 0,
\]

(4.8)

(cf. condition 4) in Theorem 5.2.1 and the functional form for \( f(x(b)) \) in §5.2 in [4]).

4) The variables \( p_1(t), p_2(t), \lambda_0 \) are not simultaneously equal to zero.

Below we specify the form of the necessary conditions in greater detail.

According to (4.2) and (4.5) we have

\[
p_1(t) = \lambda_1 e^{-rt}.
\]

(4.9)

**Proposition 4.2** If there exists \( t_0 \in [0, T] \) such that \( c(t_0) \in (0, C) \), then \( c(t) > 0 \) for almost all \( t \in [0, T] \).

**Proof.** If there exists \( t_0 \) with the above properties, then (4.4) implies

\[
\frac{\partial}{\partial \varphi} H(t_0, a(t_0), x(t_0), \varphi, z(t_0), p_1(t_0), p_2(t_0)) \bigg|_{\varphi = c(t_0)} = 0,
\]

i.e.

\[
-p \lambda_1 e^{-rt_0} + \lambda_0 e^{-\rho t_0} c(t_0)^{-\theta} = 0.
\]

(4.10)
If we assume that \( \lambda_0 = 0 \), then \( \lambda_1 = 0 \) and, because of (4.9), one obtains \( p_1(t) \equiv 0 \). This implies that \( p_2(t) \equiv const = \lambda_2 \). Now (4.8) shows that \( \lambda_2 = 0 \), which constitutes a contradiction with condition 4) from the cited theorem in [4]. Therefore, \( \lambda_0 = 1 \).

Assume that there exists \( t_1 \in [0, T] \) for which \( c(t_1) = 0 \). Then, for all sufficiently small \( \varphi > 0 \) we have
\[
H(t_1, a(t_1), x(t_1), \varphi, z(t_1), p_1(t_1), p_2(t_1)) - H(t_1, a(t_1), x(t_1), 0, z(t_1), p_1(t_1), p_2(t_1)) \leq 0,
\]
i.e.
\[
-p_1(t_1)\varphi + \lambda_0 e^{-\rho t_1} \frac{\varphi - \theta}{1 - \theta} \leq 0.
\]
Since \( \lambda_0 > 0 \), for \( \varphi \to 0^+ \) the last inequality leads to a contradiction. This proves the proposition. ■

**Corollary 4.3** If there exists \( t_1 \in [0, T] \) for which \( c(t_1) = 0 \), then \( \lambda_0 = 0 \) and \( c(t) = 0 \) for almost all \( t \in [0, T] \).

**Proof.** The conclusion on \( \lambda_0 \) can be obtained in the same manner as in the proof of Proposition 4.2 by passing to the limit as \( \varphi \to 0^+ \). If we assume the existence of a point \( t_0 \in [0, T] \) for which \( c(t_0) > 0 \), we can proceed as in the proof of the proposition and get \( p_1(t) \equiv p_2(t) \equiv 0 \) and \( \lambda_0 = 0 \), which is impossible. ■

**Proposition 4.4** The optimal consumption cannot be identically zero.

**Proof.** Assume that the controls \( c(t) \equiv 0 \) and \( z(t) \) are optimal. Then
\[
J(0, z(t)) = -\int_0^T \eta z^2(t)e^{-\rho t} dt + e^{-\rho T}a(T)^{1-\theta}.
\]
Take the controls \( \tilde{c}(t) = \varepsilon \) and \( \tilde{z}(t) = z(t) \), where \( \varepsilon > 0 \) is sufficiently small. These controls are admissible, as the respective value of terminal assets is
\[
\tilde{a}(T) = e^{\rho T} \left\{ a_0 + \int_0^T \left[ \theta w(x(t)) - \rho \varepsilon - \xi z^2(t) \right] e^{-\rho t} dt \right\} = a(T) - \varepsilon C_1,
\]
where \( C_1 := e^{\rho T} p \int_0^T e^{-\rho t} dt > 0 \). It is evident that for \( \varepsilon \) sufficiently small we have \( \tilde{a}(T) > \delta \), since \( a(T) > \delta \). It remains to check that for \( \varepsilon \) close to zero we have
\[
J(\varepsilon, z(t)) = \int_0^T \frac{\varepsilon^{1-\theta}}{1-\theta} e^{-\rho t} dt - \eta \int_0^T z^2(t)e^{-\rho t} dt + e^{-\rho T} \frac{a(T) - \varepsilon C_1}{1 - \theta} >
\]
\[
J(0, z(t)) = -\int_0^T \eta z^2(t)e^{-\rho t} dt + e^{-\rho T}a(T)^{1-\theta},
\]
13
which is equivalent to
\[
\varepsilon^{1-\theta} C_2 > \frac{e^{-\rho T}}{1-\theta} \left[ a(T)^{1-\theta} - (a(T) - \varepsilon C_1)^{1-\theta} \right], \quad C_2 := \text{const} > 0.
\]

The last expression is obviously true for all \( \varepsilon \) sufficiently small. ■

**Remark.** So far it is clear that the optimal consumption cannot be identically zero and that if there exists \( t_0 \) such that \( c(t_0) \in (0, C) \), then \( \lambda_0 = 1 \). It remains to check whether we can have \( c(t) = C \) for some \( t \).

We first establish the following result.

**Proposition 4.5** It is impossible for the optimal \( c(t) \) to satisfy
\[
\text{(4.11)} \quad c(t) \geq C_0 > 0,
\]
where
\[
\text{(4.12)} \quad C_0 > \frac{a_0 + T \max_x |w(x)|}{p^{1-e^{-\rho T}}}. \tag{4.12}
\]

**Proof.** Notice that if condition (4.11) is true, then the inequality \( a(T) \geq \delta \) is violated. Indeed, if (4.11) holds, then
\[
a(T) = e^{rt} \left[ a_0 + \int_0^T \left[ w(x(t)) - p c(t) - \xi z^2(t) \right] e^{-rt} dt \right] \leq e^{rt} \left[ a_0 + \int_0^T \left[ \max_x |w(x)| - p C_0 \right] e^{-rt} dt \right],
\]
which is negative when (4.12) holds. ■

**Proposition 4.6** The number \( \lambda_1 \) is strictly positive.

**Proof.** We know that for the optimal \( c(t) \) it is impossible to have \( c(t) \geq C_0 \) or \( c(t) \equiv 0 \). Consequently, there exists \( t_0 \in [0, T] \) for which \( c(t_0) \in (0, C_0) \). Then
\[
\frac{\partial}{\partial \varphi} H(t_0, a(t_0), x(t_0), \varphi, z(t_0), p_1(t_0), p_2(t_0)) \big|_{\varphi=c(t_0)} = 0,
\]
and hence (4.10) holds. This in turn implies that \( \lambda_0 = 1 \), as well as
\[
p \lambda_1 e^{-rt_0} = e^{-\rho t_0} c(t_0)^{-\theta}.
\]
Therefore, we have \( \lambda_1 > 0 \) and
\[
\text{(4.13)} \quad \lambda_1 = \frac{e^{(r-\rho) t_0} c(t_0)^{-\theta}}{p}.
\]
■
Proposition 4.7 There does not exist $t \in [0, T]$ for which $c(t) = C$.

Proof. Assuming the contrary, by the maximum principle we obtain

$$\frac{H(t, a(t), x(t), \varphi, z(t), p_1(t), p_2(t)) - H(t, a(t), x(t), C, z(t), p_1(t), p_2(t))}{\varphi - C} \geq 0$$

for $\varphi \in (0, C)$ and so for $\varphi \to C - 0$ we get

$$-p\lambda_1 e^{-rt} + e^{-\rho t} C^{-\theta} \geq 0,$$

which implies

$$C^\theta \leq \frac{e^{(r-\rho) t}}{\lambda_1 p} = \frac{e^{(r-\rho) t} c(t_0)^\theta}{e^{(r-\rho) t_0}} \leq e^{(r-\rho)(t-t_0)} C_0^\theta \leq \mu C_0^\theta,$$

where $\mu := \max_{t, t_0 \in [0, T]} e^{(r-\rho)(t-t_0)} > 0$. In other words,

$$C \leq \mu \frac{1}{1} C_0,$$

which is impossible. ■

The results obtained so far allow us to to find an expression for the optimal consumption $c(t)$.

Corollary 4.8 For each $t \in [0, T]$ we have $c(t) \in (0, C)$. The optimal consumption rule has the form

$$c(t) = \left[ \frac{1}{p\lambda_1} \right]^{\frac{1}{\theta}} e^{-\frac{r}{\theta} t} = \frac{1}{p\lambda_1^{\frac{1}{\theta}}} e^{-\frac{r}{\theta} (T-t)} a(T).$$

Before deriving an expression for the optimal relocation control $z(t)$, we note that (4.3) and (4.8) imply

$$p_2(t) = \lambda_1 \int_t^T w'(x(\tau)) e^{-r\tau} d\tau = e^{(r-\rho) T} a(T)^{-\theta} \int_t^T w'(x(\tau)) e^{-r\tau} d\tau.$$

Proposition 4.9 For each $t \in [0, T]$ we have the strict inequality

$$|z(t)| < Z.$$
Proof. Assume, for example, that \( z(t_1) = Z \) for some \( t_1 \in [0, T] \). Then, after passing to the limit in the respective difference quotient, we obtain

\[
-p_1(t_1)\xi 2Z + p_2(t_1) - 2\eta Ze^{-\rho t_1} \geq 0,
\]

so that

\[
p_2(t_1) \geq 2(\xi \lambda_1 e^{-rt_1} + \eta e^{-\rho t_1})Z.
\]

Similarly, the assumption that \( z(t_2) = -Z \) for some \( t_2 \in [0, T] \) leads to

\[
-p_2(t_2) \geq 2(\xi \lambda_1 e^{-rt_2} + \eta e^{-\rho t_2})Z.
\]

In both cases we have \((i = 1 \text{ or } 2)\)

\[
 Z \leq \frac{\pm p_2(t_i)}{2(\xi \lambda_1 e^{-rt_i} + \eta e^{-\rho t_i})} \leq \frac{|p_2(t_i)|}{2(\xi \lambda_1 e^{-rt_i} + \eta e^{-\rho t_i})} \leq \frac{\lambda_1 \int_{t_i}^T w'(x(\tau)) e^{-r \tau} d\tau}{2(\xi \lambda_1 e^{-rt_i} + \eta e^{-\rho t_i})} \leq \max_x |w'(x)| \left| \frac{T - t_i}{2\xi e^{-rt_i}} \right| \leq \frac{T \max_x |w'(x)|}{2\xi} e^{rT},
\]

which is impossible by the definition of \( Z \). ■

Corollary 4.10 For \( t \in [0, T] \) we have for the optimal relocation speed \( z(t) \in (-Z, Z) \) and then

\[
 (4.16) \quad z(t) = \frac{p_2(t)}{2(\xi \lambda_1 e^{-rt} + \eta e^{-\rho t})}.
\]

5 Existence of a solution of the system of necessary conditions

In order to facilitate the study of the differential equations arising from the set of necessary conditions in section 4, it would prove convenient to rewrite the differential system. Theorem 3.3 guarantees the existence of a solution to the problem (2.1)-(2.3) which in turn ensures the existence for each \( T > 0 \) of a solution to the following problem:

\[
 \dot{x}(t) = \frac{y(t)}{F(t)},
 \dot{y}(t) = -w'(x(t))\lambda_1 e^{-rt},
 x(0) = x_0,
 y(T) = 0,
\]

\[
 (5.1)
\]
where \( y(t) := p_2(t) \) and \( F(t) := 2(\xi \lambda_1 e^{-rt} + \eta e^{-\rho t}) \). It follows that there exists a solution to the problem

\[
\begin{align*}
\frac{d}{dt}(F(t)\dot{x}(t)) + w'(x(t))\lambda_1 e^{-rt} &= 0, \\
x(0) &= x_0, \\
\dot{x}(T) &= 0.
\end{align*}
\] (5.2)

The latter fact can also be established without recourse to Theorem 3.3. Following the procedure described in §73 of [9], we construct the respective Green function and transform (5.2) in the form

\[
x(t) = \int_0^T K(t,\tau)\lambda_1 e^{-r\tau} w'(x(\tau))d\tau,
\] (5.3)

where

\[
K(t,\tau) = \begin{cases} 
\int_0^\tau \frac{1}{F(s)}ds, & \tau \in [0,t] \\
\int_t^T \frac{1}{F(s)}ds, & \tau \in [t,T]
\end{cases}
\]

Since the function \( w'(x) \) is bounded and continuous by assumption, a solution to (5.3) exists. This is a consequence of Leray-Schauder index theory (see §2.4 in [10]). Also, the solution to (5.2) may not be unique, as can be seen from simple examples of eigenfunction problems that possess nontrivial solutions.

A solution to (5.1) or (5.2) can be viewed as a particular member of the family of solutions \((x(t,\alpha), y(t,\alpha))\) to the Cauchy problem

\[
\begin{align*}
\dot{x}(t) &= y(t) \\
\dot{y}(t) &= -w'(x(t))\lambda_1 e^{-rt}, \\
x(0) &= x_0, \\
y(0) &= \alpha,
\end{align*}
\] (5.4)

where \( \alpha \) has been chosen appropriately, so that

\[
y(T,\alpha) = 0.
\] (5.5)

The existence of a unique solution to (5.4) on the interval \([0,T]\) for initial data \((x_0,\alpha)\) and each \( T > 0 \) is ensured by Corollary 3.1, chapter 2, in [6].

Since (5.5) is equivalent to \( \dot{x}(T,\alpha) = 0 \), we can integrate the differential equation in (5.2) over \([0,T]\) to arrive at an equivalent form of (5.5):

\[
\alpha = \lambda_1 \int_0^T w'(x(\tau,\alpha))e^{-r\tau}d\tau.
\] (5.6)

It is straightforward to verify the following
Proposition 5.1 The function \( x(t) \equiv x_0 \) is a solution to (5.2) if and only if the point \( x_0 \) is a critical point for \( w(x) \), i.e. \( w'(x_0) = 0 \).

The analysis of the solutions of the system of necessary conditions, which is carried out in section 7, provides the dynamics of the behaviour of the economic agent, implied by this model, in the baseline case when the wage distribution has a single maximum point on the interval \([0, 1]\). Prior to that, the next section studies the properties of the function \( T \mapsto a(T) \) as \( T \to \infty \).

6 Dynamics of terminal assets \( a(T) \) for different time horizons

In this section we study the dependence of optimal terminal assets \( a(T) \) on the length of the time horizon \( T \). Although terminal assets is the natural object of study due to the fact that it is easily interpretable in economic terms, the discussion may equally well be framed in terms of the behaviour of \( \lambda_1 \), viewed as a function of the time horizon \( T \) and denoted \( \lambda_1(T) \). This approach is feasible by virtue of the relationship

\[
\lambda_1(T) = e^{(r-\rho)T}a(T)^{-\theta}.
\]

Below we derive upper and lower bounds on \( \lambda_1(T) \), which will be needed in the analysis of section 7. We assume for simplicity that \( x_0 \in (0, 1) \), i.e. \( w(x_0) > 0 \), as well as that \( a_0 > 0 \). Also, in this section we denote by \( C \) different constants that do not depend on \( T \). Since we do not use the bound on the control \( c(t) \) from (2.4) in this section, no confusion can arise from this convention.

6.1 An upper bound on \( \lambda_1(T) \)

The pair \((c(t) \equiv c_0 = \text{const}, z(t) \equiv 0)\) is admissible for \( c_0 \) appropriately chosen. Then \( x(t) \equiv x_0 \) and we set \( c_0 := \frac{w(x_0)}{p} \). In this case terminal assets are

\[
\tilde{a}(T) = e^{rT} \left[ a_0 + \int_0^T [w(x(t)) - pc(t) - \xi^2(t)] e^{-rT} dt \right] = a_0 e^{rT}.
\]

Consequently,

\[
J(c_0, 0) = c_1 (1 - e^{-\rho T}) + c_2 e^{[r(1-\theta)-\rho]T},
\]

where \( c_1 \) and \( c_2 \) are constants that depend on \( c_0 \) and \( a_0 \).

On the other hand, for the optimal controls \((c(t), z(t))\) we have

\[
J(c(t), z(t)) = \int_0^T \left[ a(T) \frac{1}{p^{1/\theta}} e^{\frac{\omega(T)}{1-\theta}(T-t)} \right]^{1-\theta} \frac{e^{-\rho t}}{1-\theta} dt - \eta \int_0^T z^2(t)e^{-\rho t} dt + \frac{a(T)^{1-\theta}}{1-\theta} e^{-\rho T},
\]
which takes different forms depending on whether $\rho - r(1 - \theta)$ is different from zero.

Since $J(c(t), z(t)) \geq J(c_0, 0)$, for $\rho - r(1 - \theta) \neq 0$ we obtain

$$a(T)^{1-\theta} e^{-\rho T} \left[ 1 + \frac{1}{\rho - r(1 - \theta)} \frac{\rho - r(1 - \theta) T - 1}{\rho - r(1 - \theta)} \right] \geq c_1 + c_2 e^{-(\rho - r(1 - \theta)T) - c_1 e^{-\rho T}}.$$

Thus,

$$a(T) \geq \begin{cases} \tilde{c}_1 e^{\rho T} + \tilde{c}_2 e^{r(1 - \theta)T} - \tilde{c}_1 \frac{\rho - r(1 - \theta) T - 1}{\rho - r(1 - \theta)} & \\
1 + \frac{1}{\rho - r(1 - \theta)} \frac{\rho - r(1 - \theta) T - 1}{\rho - r(1 - \theta)} & \\
\end{cases},$$

where $\tilde{c}_1 = (1 - \theta)c_1$, $\tilde{c}_2 = (1 - \theta)c_2$. Using the last expression together with (6.1), we can derive upper bounds on $\lambda_1(T)$.

**Proposition 6.1** Under the assumptions of this section, we have ($\forall T > 0$):

$$\lambda_1(T) \leq \begin{cases} C, & \text{if } \rho - r(1 - \theta) > 0, \\
Ce^{-\rho T}, & \text{if } \rho - r(1 - \theta) < 0, \\
C(1 + T)^{\frac{\theta}{1 - \theta}}, & \text{if } \rho - r(1 - \theta) = 0. \\
\end{cases}$$

and, accordingly,

$$a(T) \geq \begin{cases} Ce^{\frac{\rho}{\rho - r(1 - \theta)} T}, & \text{if } \rho - r(1 - \theta) > 0, \\
Ce^{rT}, & \text{if } \rho - r(1 - \theta) < 0, \\
C e^{\frac{\rho}{\rho - r(1 - \theta)} T}, & \text{if } \rho - r(1 - \theta) = 0. \\
\end{cases}$$

### 6.2 A lower bound on $\lambda_1(T)$

We first look at a particular case of the main problem, for which

$$w(x) \equiv W = \text{const.}$$

Then $\dot{p}_2 \equiv 0$ which, together with the transversality condition $p_2(T) = 0$, yields $p_2(t) \equiv 0$, i.e. $z(t) \equiv 0$ and $x(t) \equiv x_0$.

The optimal consumption rule is $c(t) = \frac{1}{\rho - r)^{\frac{\theta}{\rho}} \tilde{a}(T) e^{\frac{\rho - r}{\rho - r(1 - \theta)T}}$, where $\tilde{a}(T)$ is the optimal terminal value of assets for the problem with condition (6.5). We will calculate $\tilde{a}(T)$ from

$$\tilde{a}(T) = e^{\rho T} \left[ a_0 + \int_0^T \left( W - p^{\frac{\rho}{\rho - r}} \tilde{a}(T) e^{\frac{\rho - r}{\rho - r(1 - \theta)T}} \right) e^{-rt} dt \right].$$
Thus, we find

\[
\bar{a}(T) \leq \begin{cases} 
  Ce^{r\theta T}, & \text{if } \rho - r(1 - \theta) > 0, \\
  Ce^{r T}, & \text{if } \rho - r(1 - \theta) < 0, \\
  Ce^{\rho T}, & \text{if } \rho - r(1 - \theta) = 0.
\end{cases}
\]  

**Proposition 6.2** Let \( w(x) \leq W \), \( \forall x \), and let \( a(T) \) and \( \bar{a}(T) \) be the optimal terminal asset values for the problems with wage distributions \( w(x) \) and \( W \), respectively (all other parameters of the two problems being identical). Then

\[ a(T) \leq \bar{a}(T). \]

**Proof.** Since according to (6.1) optimal consumption for the two problems has the form \( a(T)\Psi(t) \) and \( \bar{a}(T)\Psi(t) \) with one and the same function \( \Psi(t) \), we obtain

\[
[\bar{a}(T) - a(T)] \left( 1 + e^{rT} \int_{0}^{T} p\Psi(t)e^{-rt}dt \right) = e^{rT} \int_{0}^{T} [W - w(x(t))]e^{-rt}dt + e^{rT} \int_{0}^{T} z^2(t)e^{-rt}dt,
\]

where \( x(t) \) and \( z(t) \) refer to the variables in the problem with wage distribution \( w(x) \). This completes the proof. \( \blacksquare \)

From Proposition 6.2 and equations (6.1) and (6.6), we obtain

**Proposition 6.3** Under the assumptions of this section, we have (\( \forall T > 0 \)):

\[
a(T) \leq \begin{cases} 
  Ce^{r\theta T}, & \text{if } \rho - r(1 - \theta) > 0, \\
  Ce^{r T}, & \text{if } \rho - r(1 - \theta) < 0, \\
  Ce^{\rho T}, & \text{if } \rho - r(1 - \theta) = 0.
\end{cases}
\]

and, accordingly,

\[
\lambda_1(T) \geq \begin{cases} 
  C, & \text{if } \rho - r(1 - \theta) > 0, \\
  Ce^{-(\rho-r(1-\theta))T}, & \text{if } \rho - r(1 - \theta) < 0, \\
  C(1 + T)^\theta, & \text{if } \rho - r(1 - \theta) = 0.
\end{cases}
\]

**Remark.** The bounds derived above can be refined in some cases. For instance, the first inequality in (6.7) implies very different behaviour of \( a(T) \) depending on whether \( \rho \in (r(1 - \theta), r) \), \( \rho = r \) or \( \rho > r \).
7 Optimal relocation strategies for single-peaked wage distributions

This section studies the optimal relocation behaviour of the consumer, as described by $x(t)$, in the important case of single-peaked wage distributions on the interval $[0, 1]$. We demonstrate first the validity of the following general claim (under the conditions stated at the end of section 2):

**Proposition 7.1** The optimal trajectory $x(t)$ remains in the interval $[0, 1]$, regardless of the particular form of the wage distribution $w(x)$ in $[0, 1]$.

**Proof.** Notice that since $\dot{x}(t) = p_2(t)/F(t)$, with $F(t)$ defined as in section 5 in view of (4.15) we can write

$$\dot{x}(t) = \frac{\lambda_1}{F(t)} \int_t^T w'(x(\tau))e^{-rt}d\tau = G(t) \int_t^T w'(x(\tau))e^{-rt}d\tau,$$

where $G(t) := \lambda_1/F(t)$.

Assume first that at time $t_1$ the point $x(t)$ leaves the interval $[0, 1]$ to the left (i.e. leaves the interval at $x = 0$) and remains to the left of zero until $t = T$, so that $x(t) < 0$ for $t \in (t_1, T]$. Then, for $t \in [t_1, T]$, $w'(x(t)) > 0$ and consequently $\dot{x}(t) > 0$. This would imply that for some $t_\ast \in (t_1, T)$, $x(T) - x(t_1) = (T - t_1)\dot{x}(t_\ast) > 0$, or $x(T) > x(t_1) = 0$, which contradicts the assumption that $x(t) < 0$ for $t \in (t_1, T]$. Thus, $x(t)$ cannot leave the interval $[0, 1]$ to the left and remain outside it until the end of the planning horizon $T$. A similar argument shows that it is impossible for $x(t)$ to leave the interval $[0, 1]$ to the right and stay there.

Let us now assume that $x(t)$ leaves the interval $[0, 1]$ to the left of zero at time $t_1$ and returns back at time $t_2 > t_1$. Again, for $t \in (t_1, t_2)$ we have $x(t) < 0$ and $w'(x(t)) > 0$, which means that $\int_{t_1}^{t_2} w'(x(t))e^{-rt}dt > 0$. Since $x(t)$ leaves the interval $[0, 1]$ to the left at $t_1$, it must be that $\dot{x}(t_1) \leq 0$. By the same logic, at time $t_2$ we should have $\dot{x}(t_2) \geq 0$. Then one obtains

$$\dot{x}(t_1) = G(t_1) \int_{t_1}^{t_2} w'(x(t))e^{-rt}dt + G(t_1) \int_{t_1}^{t_2} w'(x(t))e^{-rt}dt = G(t_1) \int_{t_1}^{t_2} w'(x(t))e^{-rt}dt + \frac{G(t_1)}{G(t_2)} \int_{t_2}^{T} w'(x(t))e^{-rt}dt$$

$$= G(t_1) \int_{t_1}^{t_2} w'(x(t))e^{-rt}dt + \frac{G(t_1)}{G(t_2)} \dot{x}(t_2) > 0,$$

which contradicts the condition $\dot{x}(t_1) \leq 0$. Hence it is impossible for $x(t)$ to temporarily leave the interval $[0, 1]$ to the left. Naturally, this argument can be applied with obvious
modifications to the hypothesis that $x(t)$ temporarily goes to the right of $x = 1$. Summarizing the above conclusions, we see that the optimal $x(t)$ remains in the interval $[0, 1]$.

We turn next to the main object of study for this section: the case when the wage function has a single peak on the interval $[0, 1]$. The example of the quadratic function $w(x) = x(1 - x)$ in a neighbourhood of the interval $[0, 1]$ may facilitate visualization.

Assume that in this case the initial location $x_0$ lies to the left of the wage peak, i.e. if $x_1 := \arg\max_{x \in [0, 1]} w(x)$, then $0 \leq x_0 < x_1$. For the remainder of this section we will assume that $w'(x) > 0$, $x \in [0, x_1]$ and $w'(x) < 0$, $x \in [x_1, 1]$. In this case, for $T$ sufficiently small, $x(T)$ will remain in a small neighbourhood of $x_0$. However, this means that $w'(x(t)) > 0$ for any $t \in [0, T]$ and therefore $\dot{p}_2(t) < 0$, which implies $p_2(t) > 0$ since $p_2(T) = 0$. As $p_2(t) > 0$, we obtain $\dot{x}(t) > 0$. In words, for sufficiently small planning horizons the consumer unambiguously relocates toward the wage maximum.

If $x(T) \in (x_0, x_1)$, we have, using the notation in section 5, $\dot{y}(t) = -w'(x(t))p_1(t) < 0$. Since $y(T) = 0$, it follows that $y(t) > 0$, $t \in [0, T)$. Then $\dot{x}(t) = \frac{y(t)}{F(t)} > 0$, so that $x(t)$ is monotonically increasing.

We note that there does not exist a solution to the system of necessary conditions for which $x(T) = x_1$. For such a solution we would have $y(T) = 0$ and one could compare this solution of the stationary solution $\bar{x}(t) \equiv x_1$, $\bar{y}(t) \equiv 0$. Then, the uniqueness of the solution to a Cauchy problem (for identical data at $t = T$) shows that the two solutions coincide. This, however, is impossible, since for $t = 0$ the values of the two solutions are different ($x(0) = x_0 \neq \bar{x}(0) = x_1$).

We would like to check whether it is possible for the terminal location $x(T)$ to lie to the right of the wage peak for $x_0 < x_1$. To this end, assume that $x(T) > x_1$ and let $t_1$ be the time when point $x_1$ is reached last, i.e. $x(t_1) = x_1$ and for $t \in (t_1, T)$ we have $x(t) > x_1$. (In other words, $t_1 = \sup\{t \in [0, T]|x(t) = x_1\}$.) Then, by the mean value theorem, $0 < x(T) - x(t_1) = (T - t_1)\frac{\dot{y}(t_1)}{F(t_1)}$ for some $t_* \in (t_1, T)$. However, since $\dot{y}(t) > 0$, $t \in (t_1, T)$, and $y(T) = 0$ imply $y(t_*) < 0$, we obtain $\frac{\dot{y}(t_*)}{F(t_*)} < 0$, which is a contradiction. Thus, $x(T)$ cannot lie to the right of $x_1$.

The systematic study of the relocation behaviour of the economic agent in this case can be reduced to the analysis of the way in which the solutions to the Cauchy problem (5.4) behave for different values of the parameter $\alpha$. Those solutions that satisfy (5.3) are also solutions to the system of necessary conditions (5.1), i.e. extremals. Through this approach we can also obtain information on the number of solution to the problem at hand. Of course, if only one extremal exists, then it is the solution we seek.

We remind the reader that the number $\lambda_1 = \lambda_1(T)$ is fixed, insofar as $T$ is fixed.

**Case I:** $\alpha \leq 0$. It is obvious that for small $t$ we have $y(t) < 0$ since $\dot{y}(t) < 0$. For such $t$ we have

$$x(t) = x_0 + \int_0^t \frac{y(\tau)}{F(\tau)} d\tau < x_0,$$

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i.e. the agent shifts toward $x = 0$. It means that $\dot{y}(t)$ remains negative, so that $y(t) = \alpha + \int_0^t \dot{y}(\tau) d\tau$ also remains negative and $x(t)$ keeps moving to the left. Thus, it is impossible for (5.5) to become true, i.e. there are no extremals among the solutions of (5.4) for $\alpha \leq 0$.

**Case II: $\alpha > 0$.** In this case we have $y(t) > 0$ in a neighbourhood of $t = 0$ and so $x(t)$ moves to the right in the direction of the point $x_1$. However, $\dot{y}(t) = -w'(x(t))p_1(t) < 0$, so that $y(t)$ decreases. If it turns out that $y(T) = 0$ and $x(T) < x_1$, then the respective solution is an extremal. The existence of such an extremal is guaranteed by Theorem 3.3. The latter claim can be established through an alternative approach, which allows us to ascertain the number of extremals.

We introduce the notation $M(\alpha)$ for the right-hand side of (5.6). Since $x(t) < x_1$ for $t \in [0, T]$, we get

$$M(\alpha) \leq \lambda_1 \max_x |w'(x)| \frac{1 - e^{-rt}}{r} =: M_0,$$

i.e. in view of (5.6) the relevant values of $\alpha$ lie in the interval $(0, M_0)$.

It is easy to see that the function

$$g(\alpha) := \alpha - M(\alpha)$$

is continuous on the interval $[0, M_0 + 1]$ and satisfies the inequalities

$$g(0) < 0 < g(M_0 + 1).$$

Consequently, there exists $\alpha > 0$ for which $g(\alpha) = 0$, i.e. which satisfies (5.6).

**Proposition 7.2** For the case of a single-peaked wage distribution $w(x)$ with $w''(x) \leq 0$ in $[0, 1]$, there exists a unique extremal for the system (5.1).

**Proof.** Assume that at least two different extremals exist. They solve the system (5.4) for different positive values $\alpha_1 \neq \alpha_2$, for which $\alpha_i - M(\alpha_i) = 0$, $i = 1, 2$. Then

$$(\alpha_2 - \alpha_1) \left(1 - \frac{d}{d\alpha} M(\alpha)\right) = 0, \quad \alpha^* = \kappa \alpha_1 + (1 - \kappa) \alpha_2, \quad \kappa \in (0, 1).$$

(7.1)

The derivative

$$\frac{d}{d\alpha} \left( \lambda_1 \int_0^T w'(x(t, \alpha)) e^{-rt} dt \right)\bigg|_{\alpha = \alpha^*}$$

has the form

$$\lambda_1 \int_0^T w''(x(t, \alpha^*)) \frac{\partial x(t, \alpha^*)}{\partial \alpha} e^{-rt} dt,$$

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with \( x_\alpha(t, \alpha) := \frac{\partial x(t, \alpha)}{\partial \alpha} \) and \( y_\alpha(t, \alpha) := \frac{\partial y(t, \alpha)}{\partial \alpha} \) satisfying the equations of variation [9, Ch.V, Theorem 3.1]:

\[
\begin{align*}
\dot{x}_\alpha(t, \alpha) &= y_\alpha(t, \alpha), \\
\dot{y}_\alpha(t, \alpha) &= -u''(x(t, \alpha))x_\alpha(t, \alpha)\lambda_1 e^{-rt}, \\
x_\alpha(0, \alpha) &= 0, \\
y_\alpha(0, \alpha) &= 1.
\end{align*}
\]

Consequently, \( x_\alpha(t, \alpha^*) \) solves the linear equation

\[
\frac{d}{dt} (F(t)\dot{x}_\alpha(t, \alpha^*)) + u''(x(t, \alpha^*))\lambda_1 e^{-rt}x_\alpha(t, \alpha^*) = 0
\]

for initial data \( x_\alpha(0, \alpha^*) = 0 \) and \( \dot{x}_\alpha(0, \alpha^*) = 1 \). Multiplying by \( x_\alpha(t, \alpha^*) \) and integrating over \((0, t)\), we obtain

\[
F(t)\dot{x}_\alpha(t, \alpha^*)x_\alpha(t, \alpha^*) = \int_0^t F(\tau)\dot{x}_\alpha^2(\tau, \alpha^*)d\tau + \int_0^T (-u''(x(\tau, \alpha^*)))\lambda_1 e^{-r\tau}x_\alpha^2(\tau, \alpha^*)d\tau \geq 0.
\]

Taking into account that \( F(t)\dot{x}_\alpha(t, \alpha^*)x_\alpha(t, \alpha^*) = \frac{4}{\pi}(x_\alpha^2(t, \alpha^*) - F(t)/2) \) we establish that the function \( x_\alpha^2(t, \alpha^*) \) is increasing. In view of the initial conditions, in a small interval \((0, \varepsilon)\) we have \( x_\alpha(t, \alpha^*) > 0 \). This inequality holds for all \( t \in (0, T) \), for otherwise there would exist \( \bar{t} \in (\varepsilon, T) \) for which \( x_\alpha(\bar{t}, \alpha^*) = 0 \). The latter would lead to the contradiction \( 0 < x_\alpha^2(\varepsilon/2, \alpha^*) \leq x_\alpha^2(\bar{t}, \alpha^*) = 0 \). Taking into account that \( u''(x) \leq 0 \), we obtain from (7.1) that \( \alpha_1 = \alpha_2 \), i.e. the two extremals coincide. ■

We augment the above results by investigating the dependence of the final location \( X(T) := x(T; T, x_0) \) of the agent on the length of the time horizon \( T \). Since \( X(T) < x_1, \forall T > 0 \), we have \( l := \sup_{T>0} X(T) \leq x_1 \).

**Proposition 7.3** Under the assumptions of Proposition 7.2, we have the following classification. If \( \rho \geq r \) or \( \rho \in (0, r(1 - \theta)) \), then \( l = x_1 \). If \( \rho \in (r(1 - \theta), r) \), it is possible to have \( l < x_1 \) for appropriate values of the parameters of the problem.

**Proof.** Assume that \( l < x_1 \). For \( \rho \geq r \), we have \( e^{(r - \rho)t} \leq 1 \) and so

\[
X(T) = x_0 + \int_0^T \int_0^T \frac{\lambda_1(T)w'(s)e^{-rs}}{2(\xi_1(T)e^{-rt} + \eta e^{-rt})}d\tau ds \geq x_0 + \frac{\lambda_1(T)w'(l)}{2(\xi_1(T) + \eta)} \int_0^T e^{rt} \left( \int_0^T e^{-rs}d\tau \right) d\tau,
\]

where the integral evaluates to \( \frac{1}{\eta} \left[ T - \frac{1}{r} + e^{-rt} \right] \).

According to the results from section 6, the expression \( \frac{\lambda_1(T)}{2(\xi_1(T) + \eta)} \) does not tend to zero as \( T \to \infty \), so \( \lim_{T \to \infty} X(T) = \infty \), which contradicts the fact that \( X(T) \) is bounded.

For \( \rho < r \), we study three cases according to the behaviour of \( \lambda_1(T) \):
i) $\rho \in (r(1 - \theta), r)$. In this case $\lim_{T \to \infty} \lambda_1(T) = \text{const}$.

ii) $\rho \in (0, r(1 - \theta))$. In this case $\lim_{T \to \infty} \lambda_1(T) = \infty$.

iii) $\rho = r(1 - \theta)$. In this case $C_1(1 + T)^\theta \leq \lambda_1(T)$.

For case i) we have

$$X(T) \leq x_0 + \frac{\lambda_1(T) w'(x_0)}{2\eta} \int_0^T e^{\rho T} \left( \int_0^{\rho T} e^{-rs} ds \right) d\tau \leq x_0 + \frac{\lambda_1(T) w'(x_0)}{2\rho (r - \rho) \eta},$$

after taking into account that the integral evaluates to $\frac{1}{T} \left[ \frac{1}{\tau - \rho} - \frac{r}{(r - \rho)\rho} e^{-(r - \rho)\tau} + \frac{e^{-\tau}}{\rho} \right]$. It is clear that, for instance, for large values of $\eta$ this upper bound on $X(T)$ can be strictly smaller than $x_1$.

In case ii), defining $A := r(1 - \theta) - \rho > 0$, we have from section 6

$$C_1 e^{AT} \leq \lambda_1(T) \leq C_2 e^{AT}$$

with $C_i > 0$, $i = 1, 2$.

Consequently, assuming that $l < x_1$, we have

$$(7.2) \quad X(T) \geq x_0 + \frac{C_1 w'(l)}{2r} \int_0^T e^{AT} \frac{e^{-rt} - e^{-rT}}{\zeta e^{AT} \tau + \eta e^{-\rho \tau}} d\tau,$$

where $\zeta := \xi C_2 > 0$. Denote the integral in (7.2) by $I$. We have

$$I = \int_0^T \frac{1 - e^{rt} e^{-rT}}{\zeta + \eta e^{(A + r\theta)\tau}} d\tau \geq \int_0^A \frac{1 - e^{rt} e^{-rT}}{\zeta + \eta e^{(A + r\theta)\tau}} d\tau.$$

Since $e^{-AT} e^{(A + r\theta)\tau} \leq 1$ for $\tau \in [0, AT/(A + r\theta)]$, we have from the last expression

$$I \geq \int_0^A \frac{1 - e^{rt} e^{-rT}}{\zeta + \eta} d\tau = \frac{A}{(A + r\theta)(\zeta + \eta)} e^{-rT} - \frac{e^{A + r\theta rT} - 1}{r} \zeta + \eta.$$

The last expression tends to infinity as $T \to \infty$, implying that $X(T)$ is unbounded. This contradiction shows that $l = x_1$.

In case iii) the condition from section 6 is equivalent to

$$\frac{1}{\lambda_1(T)} \leq \frac{1}{C_1(1 + T)^\theta}.$$
Assume that $l < x_1$. Then

$$X(T) \geq x_0 + \frac{w'(l)\lambda_1(T)}{2r} \int_0^T \frac{1 - e^{-rT}e^{\tau r}}{\xi \lambda_1(T) + \eta e^{r\tau}} d\tau \geq x_0 + \frac{w'(l)}{2r} \int_0^T \frac{1 - e^{-rT}e^{\tau r}}{\xi + \frac{\eta e^{r\tau}}{C_1(1+T)^\theta}} d\tau =$$

$$x_0 + \frac{w'(l)}{2r} C_1(1+T)^\theta \int_0^T \frac{1 - e^{-rT}e^{\tau r}}{\xi C_1(1+T)^\theta + e^{r\tau}} d\tau.$$

Set $B = B(T) := \frac{\xi C_1(1+T)^\theta}{\eta}$ and introduce the change of variables $\mu = e^{r\tau}$ in the last expression to obtain

$$X(T) \geq x_0 + \text{const}(1+T)^\theta \int_1^{e^{rT}} \frac{1 - e^{-rT}\mu^\theta}{r\theta\mu(\mu + B)} d\mu.$$

This requires us to study the behaviour of two expressions.

First, we have

$$(1 + T)^\theta \int_1^{e^{rT}} \frac{d\mu}{\mu(\mu + B)} = (1 + T)^\theta \frac{1}{B} \left[ \ln \left( 1 + \frac{1}{\text{const}(1+T)^\theta} \right) + \ln \left( 1 + \text{const}(1+T)^\theta \right) \right].$$

When $T \to \infty$, the first logarithm tends to zero and the second one tends to infinity, i.e. the whole expression tends to infinity.

Second, note that

$$0 \leq (1 + T)^\theta e^{-rT} \int_1^{e^{rT}} \frac{\mu^\theta}{\mu(\mu + B)} d\mu \leq \frac{(1 + T)^\theta}{e^{rT}} \int_1^{e^{rT}} \frac{\mu^\theta}{\mu^2} d\mu = \frac{\theta}{1 - \theta} \left[ (1 + T)^\theta - (1 + T)^0 \right].$$

The last expression tends to zero as $T \to \infty$.

Combining the above results, we obtain $X(T) \to \infty$, which contradicts the fact that $X(T)$ is bounded. Thus, in this case $l = x_1$. ■

References

[1] Baldwin, R., “The Core-Periphery Model with Forward-Looking Expectations,” Regional Science and Urban Economics, Vol. 31, pp 21-49, 2001.

[2] Boucekkine, R., C. Camacho and B. Zou, “Bridging the gap between growth theory and economic geography: The spatial Ramsey model,” 2006, mimeo.
[3] Brito, P., “The dynamics of growth and distribution in a spatially heterogeneous world,” Working papers, Department of Economics, ISEG, WP13/2004/DE/UECE, November 2004.

[4] Clarke, F.H., *Optimization and Nonsmooth Analysis*, 1983, John Wiley: New York.

[5] Diestel, J., *Geometry of Banach Spaces: Selected Topics*, 1975, Springer-Verlag: Berlin.

[6] Hartman, P., *Ordinary Differential Equations*, 1964, John Wiley & Sons.

[7] Kolmogorov, A.N., S.V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, 1976, Nauka: Moscow (in Russian).

[8] Lee, E.B., L. Markus, *Foundations of Optimal Control Theory*, 1968, John Wiley: New York.

[9] Lovitt, W.V., *Linear Integral Equations*, 1924, McGraw-Hill Book Co.: New York.

[10] Nirenberg, L., *Topics in Nonlinear Functional Analysis*, 1974, Courant Inst.: NY Univ.