ON THE VANISHING OF NEGATIVE HOMOTOPY K-THEORY

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ABSTRACT. We show that the homotopy invariant algebraic K-theory of Weibel vanishes below the negative of the Krull dimension of a noetherian scheme. This gives evidence for a conjecture of Weibel about vanishing of negative algebraic K-groups.

To Chuck Weibel on the occasion of his 65th birthday.

1. Introduction

The aim of this note is to prove the following theorem. For a scheme $X$ and $i \in \mathbb{Z}$ we consider homotopy K-theory $KH_i(X)$ as defined in [11, Sec. IV.12].

**Theorem 1.** Let $X$ be a noetherian scheme of finite Krull dimension $d$. Then $KH_i(X) = 0$ for $i < -d$.

Weibel conjectured the analogue of this theorem with the K-Theory of Bass–Thomason–Trobaugh in place of homotopy K-theory, originally formulated as a question in [9, Qu. 2.9]. In fact Theorem 1 is a special case of Weibel’s original conjecture, as can be seen using the spectral sequence (2).

**Corollary 2.** Let $X$ be a noetherian scheme of finite Krull dimension $d$ and let $p$ be a prime nilpotent on $X$. Then $K_i(X) \otimes \mathbb{Z}[1/p] = 0$ for $i < -d$.

Note that under the conditions of Corollary 2 we have $KH_i(X) \otimes \mathbb{Z}[1/p] \cong K_i(X) \otimes \mathbb{Z}[1/p]$ according to a result of Weibel, see [6, Thm. 9.6], where $K$ denotes the Bass–Thomason–Trobaugh K-theory.

Corollary 2 has been shown in [4] for $X$ quasi-excellent using the alteration theorem of Gabber–de Jong. Our proof is more elementary as instead of weak resolution of singularities we use *platification par éclatement* [5, Thm. 5.2.2].

2. Some reductions

**Proposition 3.** Let $X$ be a noetherian scheme of finite Krull dimension $d$. If $KH_i(O_{X,x}) = 0$ for all $x \in X$ and $i < -\dim(O_{X,x})$ then $KH_i(X) = 0$ for $i < -d$.

In the proof of Proposition 3 we need the following classical result of Grothendieck.

**Lemma 4.** Let $r \geq 0$ be an integer. Let $F$ be a Zariski sheaf on the noetherian scheme $X$. Assume that $F_x = 0$ for all points $x \in X$ with $\dim \{x\} > r$. Then $H^i(X, F) = 0$ for $i > r$.

**Proof.** Let $J = \coprod_{U \subseteq X} F(U)$, where $U$ runs through all open subsets of $X$, and let $I$ be the set of finite subsets of $I$. For $\alpha \in I$ let $F_\alpha$ be the abelian subsheaf of $F$ locally generated by the sections in $\alpha$. Then

$$F = \colim_{\alpha \in I} F_\alpha$$

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as a filtered colimit. Since each local section of $\mathcal{F}$ is supported on a closed subscheme of dimension at most $r$ there are closed immersions $i_\alpha : X_\alpha \to X$ and abelian sheaves $\mathcal{G}_\alpha$ on $X_\alpha$ such that $\dim X_\alpha \leq r$ and such that $i_\alpha_*(\mathcal{G}_\alpha) \cong \mathcal{F}_\alpha$. Then

$$H^i(X, \mathcal{F}) \cong \colim_{\alpha \in I} H^i(X, \mathcal{F}_\alpha) = \colim_{\alpha \in I} H^i(X_\alpha, \mathcal{G}_\alpha)$$

for $i > r$. Here (1) is due to [3] Prop. III.2.9 and (2) is due to [3] Thm. III.2.7.

Proof of Proposition 5. Consider the convergent Zariski-descent spectral sequence, analogous to [3] Thm. 10.3;

$$E_2^{p,q} = H^p(X, K\mathcal{H}_{-q,X}) \Rightarrow KH_{-p-q}(X),$$

where $K\mathcal{H}_{i,X}$ is the Zariski sheaf on $X$ associated with $KH_i$. For $i < -d$ and for $-p - q = i$ let $\mathcal{F}$ be $K\mathcal{H}_{-q,X}$ and let $r$ be $d - q$. Then under the conditions of Proposition 5 we get $\mathcal{F}_x = K\mathcal{H}_{-q}(\mathcal{O}_{X,x}) = 0$ for all $x \in X$ with $\dim \{x\} > r$ since

$$\dim \mathcal{O}_{X,x} \leq \dim X - \dim \{x\} < \dim X - r = q.$$

So by Lemma 4 we deduce $E_2^{p,q} = H^p(X, \mathcal{F}) = 0$ for all $-p - q = i < -d$ and therefore also $KH_i(X) = 0$.

The following proposition is immediate in case the scheme $X$ has a desingularization. However, we avoid any assumption on the existence of resolution of singularities by using Raynaud–Gruson’s platification par éclatement instead.

Proposition 5. Let $X$ be a reduced scheme which is quasi-projective over a noetherian ring. Let $f : Y \to X$ a smooth and quasi-projective morphism. Let $k > 0$ be an integer and let $\xi \in K_{-k}(Y)$. There exists a birational projective morphism $p : X' \to X$ such that $\bar{p}^*(\xi) = 0 \in K_{-k}(Y')$ where $\bar{p} : Y' \to Y$ is the pull-back of $p$ along $f$.

Proof. By Bass’s definition of negative $K$-theory [11] Sec. III.4] the group $K_{-k}(Y)$ for $k > 0$ is a quotient of $K_0(Y \times \mathbb{G}^\times_m)$, where $\mathbb{G}^\times_m = \mathbb{A}^1 \setminus \{0\}$. Elements of this $K_0$-group coming from $K_0(Y \times \mathbb{A}^k)$ vanish in $K_{-k}(Y)$.

Without loss of generality $\xi$ is represented by a vector bundle $V$ on $Y \times \mathbb{G}_m^\times$. We can extend $V$ to a coherent sheaf $\check{V}$ on $Y \times \mathbb{A}^k$, see [2] Sec. I.9.4]. Choose an open dense subscheme $U \subseteq X$ such that $\check{V}$ is flat over $U$. This is possible as $X$ is reduced [2] Thm. IV.11.1.1]. By platification par éclatement [5] Thm. 5.2.2 there is a projective birational morphism $p : X' \to X$ which is an isomorphism over $U$ and such that the strict transform $\bar{V}' = p^*(\check{V})$ as a coherent sheaf on $Y' \times \mathbb{A}^k$ is flat over $X'$, here $Y' = X' \times_X Y$.

Recall that the strict transform $p^*(\bar{V})$ is defined as the image of $\check{p}^*(\check{V}) \to j_*j^*\check{p}^*(\check{V})$, where $j : f^{-1}(U) \times \mathbb{A}^k \to Y' \times \mathbb{A}^k$ is the canonical open immersion and where $\check{p}$ denotes the induced morphism $Y' \times \mathbb{A}^k \to Y \times \mathbb{A}^k$.

Note that $\bar{V}'|_{Y' \times \mathbb{G}^\times_m}$ is isomorphic to the usual pull-back of the sheaf $V$, as the latter is flat over $X$.

Lemma 6. $\bar{V}'$ has finite Tor-dimension as an $\mathcal{O}_{Y' \times \mathbb{A}^k}$-sheaf.

Lemma 6 implies by [11] Prop. II.8.3.1] that $\bar{V}'$ induces an element of $K_0(Y' \times \mathbb{A}^k)$ whose restriction to $Y' \times \mathbb{G}^\times_m$ represents $\bar{p}^*(\xi) \in K_{-k}(Y')$ via the Bass construction explained above. As any such element in negative $K$-theory vanishes we have proved Proposition 5.

Proof of Lemma 6. For a noetherian scheme $Z$ we denote by $D(Z)$ the derived category of $\mathcal{O}_Z$-modules whose cohomology sheaves are quasi-coherent and by $D^b(Z)$ the triangulated subcategory of bounded complexes with coherent cohomology sheaves.

Proof of Lemma 6. For a noetherian scheme $Z$ we denote by $D(Z)$ the derived category of $\mathcal{O}_Z$-modules whose cohomology sheaves are quasi-coherent and by $D^b(Z)$ the triangulated subcategory of bounded complexes with coherent cohomology sheaves.
Let $y \in Y' \times \mathbb{A}^k$ be a point with image $x \in X'$. Let $i_y : y \to Y' \times \mathbb{A}^k$ be the natural map, $i_x : F_x \to Y' \times \mathbb{A}^k$ the inclusion of the fiber $F_x$ of $Y' \times \mathbb{A}^k \to X'$ over $x$ and let $i_y^* : y \to F_x$ be the canonical morphism. By [10] Prop. 4.4.11 we must show that $L_i^*(\bar{V}') \in D(y)$ lies in $D^b(y)$. As $\bar{V}'$ is flat over $X'$ we have $L_i^*(\bar{V}') = i_x^*(\bar{V}') \in D^b(F_x)$. As $F_x$ is a regular scheme, $L(i_x^*)^*$ maps $D^b(F_x)$ to $D^b(\mathbb{A}^k)$ [10] Thm. 4.4.16, so $L(i_x^*)^*(\bar{V}') = (L(i_y^*)^* \circ L_i^*)^*(\bar{V}')$ lies in $D^b(y)$.

3. **Proof of Theorem 1**

In the proof of Theorem 1 we can, using Proposition 3, restrict to schemes $X$ which are quasi-projective over noetherian rings. For such $X$ we argue inductively on the dimension $d = \dim(X)$. We may assume that $X$ is reduced as $KH_i(X) = KH_i(X_{red})$, use [8] Thm. 2.3 and Zariski-descent. The case $d = 0$ of Theorem 1 is shown in [8] Prop. 3.1.

Let $d > 0$ and assume Theorem 1 for all schemes of Krull dimension less than $d$ which are quasi-projective over a noetherian ring. Let $\Delta^p$ be the usual cosimplicial scheme defined in degree $p$ by $\Delta^p = \text{Spec}(\mathbb{Z}[T_0, \ldots, T_p]/(\sum T_j - 1))$. There is a right half-plane spectral sequence

$$E^1_{p,q}(X) = KH_p(X \times \Delta^p) \Rightarrow KH_{p+q}(X),$$

functorial in $X$, see [7] Prop. 5.17. This is the Bousfield–Kan spectral sequence arising from the simplicial spectrum $K(X \times \Delta^*)$ whose homotopy colimit is $KH(X)$ by definition. For each $p + q$ there is a filtration

$$0 = F_{-1}(X) \subseteq F_0(X) \subseteq F_1(X) \subseteq \ldots \cup_{p=0}^{\infty} F_p(X) = KH_{p+q}(X)$$

with $F_p(X)/F_{p-1}(X) \cong E^\infty_{p,q}(X)$.

Let $i < -d$. In order to conclude that $KH_i(X) = 0$, we show inductively on $p \geq 0$ that the group $F_p(X)$ in the filtration vanishes for all $X$ as above with $\dim(X) \leq d$ at once. Fix a scheme $X$ of Krull dimension $d$ which is quasi-projective over a noetherian ring and let $\gamma \in F_p(X)$ be an element. We have $F_p(X) \cong E^\infty_{p,q}(X)$ by the induction hypothesis on $p$. As $E^\infty_{p,q}(X)$ is a quotient of $E^1_{p,q}(X)$, the element $\gamma$ lifts to a class $\xi \in KH_q(X \times \Delta^p)$. Note that $q < -d < 0$.

By Proposition 3 applied to the morphism $Y = X \times \Delta^p \to X$, we find a projective birational morphism $p: X' \to X$ such that $p^*(\xi) = 0 \in KH_q(Y')$, here $Y' = X' \times_X Y$. We choose a nowhere dense closed subscheme $Z \to X$ such that $p$ is an isomorphism outside $Z$ and obtain a cdh-distinguished square

$$
\begin{array}{ccc}
Z' & \longrightarrow & X' \\
\downarrow & & \downarrow p \\
Z & \longrightarrow & X.
\end{array}
$$

As homotopy $K$-Theory satisfies cdh-descent by [11] Thm. 3.9, we get a long exact sequence

$$\cdots \to KH_{i+1}(Z') \to KH_i(X) \to KH_i(Z) \oplus KH_i(X') \to \cdots.$$  

The groups $KH_{i+1}(Z')$ and $KH_i(Z)$ vanish by the induction hypothesis on $d$ as $\dim(Z')$, $\dim(Z) < d$, so $KH_i(X) \to KH_i(X')$ is injective (recall that $i < -d$). Hence, it suffices to show that $\gamma = 0$. Since $KH_i(X') \leq \dim(X')$, we have $F_p(X') \cong E^\infty_{p,q}(X')$ by the induction hypothesis. The morphism $p^* : KH_i(X) \to KH_i(X')$ restricts to a morphism $F_p(X) \to F_p(X')$ which is compatible with $\tilde{p}^* : KH_q(Y) \to KH_q(Y')$. Since $\tilde{p}^*(\xi) = 0 \in KH_q(Y')$, we conclude that $p^*(\gamma) = 0$, so $\gamma = 0$. Hence we obtain $F_p(X) = 0$. 


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