BOUNDS ON THE TORSION SUBGROUPS OF NÉRON–SEVERI GROUPS

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Abstract. Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety defined by homogeneous polynomials of degree \( \leq d \). We give explicit upper bounds on the order of the torsion subgroup \((\text{NS}_X)_{\text{tor}}\) of the Néron–Severi group of \( X \). The bounds are derived from an explicit upper bound on the number of irreducible components of either the Hilbert scheme \( \text{Hilb}_Q X \) or the scheme \( \text{CDiv}_n X \) parametrizing the effective Cartier divisors of degree \( n \) on \( X \). We also give an upper bound on the number of generators of \((\text{NS}_X)[\ell^\infty]\) uniform as \( \ell \neq \text{char} k \) varies.

1. Introduction

The Néron–Severi group \( \text{NS}_X \) of a smooth proper variety \( X \) over a field \( k \) is the group of divisors modulo algebraic equivalence. If \( k \) is algebraically closed, \( \text{NS}_X \) also equals the group of connected components of the Picard scheme of \( X \). Néron [17, p. 145, Théorème 2] and Severi [20] proved that \( \text{NS}_X \) is finitely generated. The aim of this paper is to give an explicit upper bound on the order of \((\text{NS}_X)_{\text{tor}}\). To the best of the author’s knowledge, this is the first explicit bound on the order of \((\text{NS}_X)_{\text{tor}}\).

**Theorem 5.13.** Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety defined by homogeneous polynomials of degree \( \leq d \). Then

\[
\#(\text{NS}_X)_{\text{tor}} \leq 2^{d^2 + 2r \log_2 r}.
\]

For any prime number \( \ell \neq \text{char} k \), there is a natural isomorphism \((\text{NS}_X)[\ell^\infty] \simeq H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}}[21, 2.2]\). Thus, Theorem 5.13 implies the corollary below.

**Corollary 5.15.** Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety defined by homogeneous polynomials of degree \( \leq d \). Let \( \ell \neq \text{char} k \) be a prime number. Then

\[
\prod_{\ell \neq \text{char} k} \#H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}} \leq 2^{d^2 + 2r \log_2 r}.
\]

We also give a uniform upper bound on the number of generators of \((\text{NS}_X)[\ell^\infty]\) as described below. This bound and a sketch of its proof were suggested by János Kollár after seeing an earlier draft of our paper containing only Theorem 5.13.

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**Theorem 6.3.** Let $X \hookrightarrow \mathbb{P}^r$ be a smooth connected projective variety of degree $d$ over $k$. Let $p = \text{char } k$, and

$$N = \begin{cases} (\text{NS } X)_{\text{tor}}, & \text{if } p = 0 \\ (\text{NS } X)_{\text{tor}}/(\text{NS } X)[p^\infty], & \text{if } p > 0. \end{cases}$$

Then $N$ is generated by less than or equal to $(d - 1)(d - 2)$ elements.

**Corollary 6.4.** Let $X \hookrightarrow \mathbb{P}^r$ be a smooth connected projective variety of degree $d$ over $k$. Let $\ell \neq \text{char } k$ be a prime number. Then $H^2_{\text{et}}(X, \mathbb{Z}_\ell)_{\text{tor}}$ is generated by less than or equal to $(d - 1)(d - 2)$ elements.

The torsion-free quotient $(\text{NS } X)/(\text{NS } X)_{\text{tor}}$ is the group of divisors modulo numerical equivalence. Its rank is bounded by the second Betti number of $X$. Katz found an upper bound on the sum of all Betti numbers of $X$ [10, Theorem 1]; this gives a rough bound on the rank of $\text{NS } X$.

The torsion subgroup $(\text{NS } X)_{\text{tor}}$ is the group of numerically zero divisors modulo algebraic equivalence. This is a birational invariant [1, p. 177]. Recently, Poonen, Testa and van Luijk gave an algorithm to compute it.\footnote{They also found an algorithm computing the torsion free quotient assuming the Tate conjecture.}

The algorithm is based on their theorem that $(\text{NS } X)_{\text{tor}}$ injects into the set of connected components of $\text{Hilb}_Q X$ for some polynomial $Q$ [19, Lemma 8.29]. The order of $(\text{NS } X)_{\text{tor}}$ will be bounded by finding an explicit $Q$ and bounding the number of connected components of $\text{Hilb}_Q X$. We will show that an upper bound on the number of connected components of $\text{CDiv}_n X$ for an integer $n$ depending on $Q$ also gives an upper bound on the order of $(\text{NS } X)_{\text{tor}}$.

Section 3 shows that $Q$ may be taken to be the Hilbert polynomial of $mH$, where $H$ is a hyperplane section of $X$ and $m$ is an explicit integer. Section 4 bounds the number of irreducible components of $\text{Hilb}_Q X$ by using its embedding in a Grassmannian. Section 5 bounds the number of irreducible component of $\text{CDiv}_n X$ by using Kollár’s technique [13, Exercise I.3.28]. Finally, Section 6 gives a uniform upper bound on the number of generators of $(\text{NS } X)[\ell^\infty]$.

From now on, the base field $k$ is assumed to be algebraically closed, since a base change makes the Néron–Severi group only larger [19, Proposition 6.1]. However, no assumption is made on the characteristic of $k$.

2. Notation

Given a scheme $X$ over $k$, let $\text{conn}(X)$ and $\text{irr}(X)$ be the set of connected components of $X$ and the set of irreducible components of $X$, respective. Let $X_{\text{red}}$ be the reduced closed subscheme associated to $X$. If $X$ is smooth and proper, then $\text{NS } X$ denote the Néron–Severi group of $X$. Let $H^i_{\text{et}}(X, \mathcal{F})$ be the $i$-th étale cohomology group of $X$ corresponding to an étale sheaf $\mathcal{F}$. If $k = \mathbb{C}$, then let $X^\text{an}$ be the analytic space of $X$. If $M$ is a topological manifold, then $H^i_{\text{sing}}(M, \mathbb{Z})$ (resp. $H^i_{\text{sing}}(M, \mathbb{Z})$) denotes the $i$-th singular cohomology (resp. homology) group with integer coefficients.

A projective variety is a closed subscheme of $\mathbb{P}^r = \text{Proj } k[x_0, \ldots , x_r]$ for some $r$. Suppose that $X \hookrightarrow \mathbb{P}^r$ is a projective variety. Let $I_X \subset k[x_0, \ldots , x_r]$ be the saturated ideal defining $X$. Given $f_0, \ldots , f_{t-1} \in k[x_0, \ldots , x_r]$, let $V_X(f_0, \ldots , f_{t-1})$ be the subscheme of $X$ defined
by the ideal \( (f_0, \ldots, f_{t-1}) + I_X \). Let \( \mathcal{O}_X, \mathcal{I}_X, \Omega_X \) and \( \omega_X \) be the sheaf of regular functions, the ideal sheaf, the sheaf of differentials and the canonical sheaf of \( X \), respectively.

Given a coherent sheaf \( \mathcal{F} \) on \( X \), let \( \text{HP}_{\mathcal{F}} \) be the Hilbert polynomial of \( \mathcal{F} \), and \( \Gamma(\mathcal{F}) \) be the global section of \( \mathcal{F} \). Given a graded module \( M \) over \( k \), let \( \text{HP}_M \) be the Hilbert polynomial of \( M \), and \( M_t \) be the degree \( t \) part of \( M \). Take an effective divisor \( D \) on \( X \). Let \( \text{HP}_D \) be the Hilbert polynomial of \( D \) as a subscheme of \( \mathbb{P}^r \). Then \( \mathcal{O}(-D) \subset \mathcal{O}_X \) is the ideal sheaf corresponding to \( D \), and

\[
\text{HP}_D = \text{HP}_{\mathcal{O}_X} - \text{HP}_{\mathcal{O}(-D)}.
\]

Let \( \text{Hilb} \, X \) be the Hilbert scheme of \( X \). Given a polynomial \( Q(t) \), let \( \text{Hilb}_Q \, X \) be the Hilbert scheme of \( X \) parametrizing closed subschemes of \( X \) with Hilbert polynomial \( Q \). Let \( \text{Chow}_{\delta,n} \, X \) be the Chow variety of dimension \( \delta \) and degree \( n \) algebraic cycles on \( X \). Let \( \text{CDiv} \, X \) be the scheme parametrizing the effective Cartier divisors on \( X \), and let \( \text{CDiv}_{n} \, X \) be the open and closed subscheme of \( \text{CDiv} \, X \) corresponding to the divisors of degree \( n \). Let \( \text{Pic} \, X \) be the Picard scheme of \( X \). Let \( \text{Alb} \, X \) be the Albanese variety of \( X \). Given a vector space \( V \) and nonnegative number \( t \), let \( \text{Gr}(t,V) \) be the Grassmannian parametrizing \( t \)-dimensional subspaces of \( V \). Let \( \text{Gr}(t,n) = \text{Gr}(t,k^n) \).

Given a set \( S \), let \( \#S \) be the number of elements in \( S \). Give a group \( A \), let \( A_{\text{ab}} \) be the abelianization of \( A \), and \( A^{(\ell)} = \varprojlim A_{\text{ab}}/\ell^n A_{\text{ab}} \) be the maximal pro-\( \ell \) abelian quotient of \( A \). If \( A \) is abelian, let \( A_{\text{tor}} \), \( A[n] \) and \( A[\ell^\infty] \) be the set of torsion elements, \( n \)-torsion elements and \( \ell \)-power torsion elements, respectively. If \( A \) is a finite abelian group, then \( A^* \) is the Pontryagin dual of \( A \).

### 3. Numerical Conditions

Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety defined by polynomials of degree \( \leq d \). Let \( K \) and \( H \) be a canonical divisor and a hyperplane section of \( X \), respectively. The goal of this section is to give an explicit \( m \) such that

\[
\# \text{conn}(\text{Hilb}_{\text{HP},mH} \, X) \geq \#(\text{NS} \, X)_{\text{tor}}.
\]

Poonen, Testa and van Luijk proved that an order of a Néron–Severi group is bounded by a number of connected components of a certain Hilbert scheme. The theorem and the proof below is a reformulation of their work in [19, Section 8.4].

**Theorem 3.1** (Poonen, Testa and van Luijk). *Let \( F \) be a divisor on \( X \) and \( Q = \text{HP}_{\text{O}_X} - \text{HP}_{\text{O}(-F)} \). If \( \text{O}_X(F + D) \) has a global section for every numerically zero divisor \( D \), then*

\[
\# \text{conn}(\text{Hilb}_Q \, X \cap \text{CDiv} \, X) \geq \#(\text{NS} \, X)_{\text{tor}}.
\]

**Proof.** Recall that \( \text{CDiv} \, X \) is an open and closed subscheme of \( \text{Hilb} \, X \) [13, Exercise I.3.28], and there is a natural proper morphism \( \pi : \text{CDiv} \, X \to \text{Pic} \, X \) sending a divisor to the corresponding class [3, p. 214].

Let \( \text{Pic}^c \, X \) be the finite union of connected components of \( \text{Pic} \, X \) parametrizing the divisors numerically equivalent to \( F \). Since the Hilbert polynomial of a divisor is a numerical invariant, \( Q \) is the Hilbert polynomial of each divisor corresponding to a closed point of \( \pi^{-1}(\text{Pic}^c \, X) \). Thus, \( \pi^{-1}(\text{Pic}^c \, X) \subset \text{Hilb}_Q \, X \). Since \( \text{Pic}^c \, X \) is open and closed in \( \text{Pic} \, X \), and \( \text{CDiv} \, X \) is open and closed in \( \text{Hilb} \, X \), the scheme \( \pi^{-1}(\text{Pic}^c \, X) \) is open and closed in
Hilb\(Q\) \(X \cap \text{CDiv} \ X\). Thus,
\[
\# \text{conn}(\text{Hilb}\(Q\) \(X \cap \text{CDiv} \ X\)) \geq \# \text{conn}(\pi^{-1}(\text{Pic}^{c} \ X)).
\]
Since \(F + D\) is linearly equivalent to an effective divisor for every numerically zero divisor \(D\), the morphism \(\pi\) restricts to a surjection \(\pi^{-1}(\text{Pic}^{c} \ X) \to \text{Pic}^{c} \ X\). Hence,
\[
\# \text{conn}(\pi^{-1}(\text{Pic}^{c} \ X)) \geq \# \text{conn}(\text{Pic}^{c} \ X) = (\text{NS}^{\text{tor}} \ X).
\]
The authors of [19] chose \(F = K + (\dim \ X + 2)H\) because of the following partial result towards Fujita’s conjecture.

**Theorem 3.2** (Keeler [11, Theorem 1.1]). Let \(L\) be an ample divisor on \(X\). Then
(a) \(\mathcal{O}_{X}(K + (\dim \ X)H + L)\) is generated by global sections, and
(b) \(\mathcal{O}_{X}(K + (\dim \ X + 1)H + L)\) is very ample.

However, computing the Hilbert polynomial of \(\mathcal{O}(-K)\) is somehow difficult. Therefore, we will show that \(F = ((d - 1) \cdot \text{codim} \ X)H\) is another choice.

**Lemma 3.3.** Let \(Y\) be a nonempty smooth closed subscheme of an affine space \(\mathbb{A}^{r} = \text{Spec} \ k[x_{0}, \cdots, x_{r-1}]\). Suppose that the ideal \(I\) defining \(Y\) is generated by polynomials of degree \(\leq d\). Let \(c = \text{codim} \ Y\). Then there are polynomials \(f_{0}, \cdots, f_{c-1} \in I\) such that
(a) \(\deg f_{i} = d\) for all \(i\), and
(b) \(f_{0} \land \cdots \land f_{c-1}\) represents a nonzero element in \(\land^{c}(I/I^{2})\).

**Proof.** Let \(R = k[x_{0}, \cdots, x_{c-1}]/I\). Then \(I/I^{2}\) is a locally free \(R\)-module of rank \(\text{codim} \ Y\) by [8, Theorem 8.17]. Take any prime ideal \(p \subset R\). Then \((I/I^{2})_{p}\) is a free \(R_{p}\)-module. Thus, there are \(p_{0}, \cdots, p_{c-1} \in I\) and \(q_{0}, \cdots, q_{c-1} \in R \setminus p\) such that
\[
\frac{p_{0}}{q_{0}} \land \frac{p_{1}}{q_{1}} \land \cdots \land \frac{p_{c-1}}{q_{c-1}}
\]
represents a nonzero element of \(\land^{c}(I/I^{2})_{p}\). Hence,
\[
(2) \quad p_{0} \land p_{1} \land \cdots \land p_{c-1}
\]
represents a nonzero element of \(\land^{c}(I/I^{2})\).

Let \(g_{0}, \cdots, g_{c-1}\) be polynomials of degree \(\leq d\) which generate \(I\). Then each \(p_{i}\) can be written as a \(R\)-linear combination of \(g_{i}\)’s. If we expand (2), at least one term should be nonzero in \(\land^{c}(I/I^{2})\). Therefore, we may assume that
\[
g_{0} \land g_{1} \land \cdots \land g_{c-1}
\]
represents a nonzero element of \(\land^{c}(I/I^{2})\). Let \(\ell \notin I\) be a polynomial of degree 1, and let \(f_{i} = \ell^{d - \deg g_{i}}g_{i}\). Then
\[
f_{0} \land f_{1} \land \cdots \land f_{c-1}
\]
represents a nonzero element of \(\land^{c}(I/I^{2})\) and \(\deg f_{i} = d\) for every \(i\). \(\square\)

**Lemma 3.4.** The sheaf \(\mathcal{O}_{X}(-K + (d \cdot \text{codim} \ X - r - 1)H)\) has a global section.
Proof. Let \( c = \text{codim} X \). Since \( X \) is smooth, there is an exact sequence
\[
0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega_{\mathbb{P}^r} \otimes \mathcal{O}_X \to \Omega_X \to 0,
\]
and \( \mathcal{I}_X/\mathcal{I}_X^2 \) is locally free of rank \( c \) [8, Theorem 8.17]. Taking the highest exterior power gives
\[
\omega_{\mathbb{P}^r}|_X \simeq \bigwedge^c (\mathcal{I}_X/\mathcal{I}_X^2) \otimes \omega_X
\]
\[
\omega_X^{-1}(-r - 1) \simeq \bigwedge^c (\mathcal{I}_X/\mathcal{I}_X^2).
\]
Let \( U_i \subset X \) be the affine open set given by \( x_i \neq 0 \). Then there exist polynomials \( f_0, \ldots, f_{c-1} \) of degree \( d \) such that
\[
f_0(x_0/x_i, \ldots, x_r/x_i) \wedge \cdots \wedge f_{c-1}(x_0/x_i, \ldots, x_r/x_i)
\]
represents a nonzero section of \( \bigwedge^c (\mathcal{I}_X/\mathcal{I}_X^2)|_{U_i} \) by Lemma 3.3. Take another \( U_j \subset X \) given by \( x_j \neq 0 \). Then two sections
\[
x_i^d f_0(x_0/x_i, \ldots, x_r/x_i) \wedge \cdots \wedge x_{i}^d f_{c-1}(x_0/x_i, \ldots, x_r/x_i)
\]
and
\[
x_j^d f_0(x_0/x_j, \ldots, x_r/x_j) \wedge \cdots \wedge x_{j}^d f_{c-1}(x_0/x_j, \ldots, x_r/x_j)
\]
give same restrictions in \( \bigwedge^c (\mathcal{I}_X/\mathcal{I}_X^2)|_{U_i \cap U_j} \). Because \( i \) and \( j \) are arbitrary, the sections above extend to a global section of
\[
\omega_X^{-1}(d \cdot \text{codim} X - r - 1) \simeq \bigwedge^c (\mathcal{I}_X/\mathcal{I}_X^2(d)). \tag*{\qed}
\]

Lemma 3.5. Let \( D \) be a divisor on \( X \) numerically equivalent to \( 0 \).\(^2\) Then
(a) \( \mathcal{O}_X(D + ((d - 1) \text{codim} X)H) \) is generated by global sections, and
(b) \( \mathcal{O}_X(D + ((d - 1) \text{codim} X + 1)H) \) is very ample.

Proof. The divisor \( D + H \) is ample, since ampleness is a numerical property. Then \( K + (\dim X)H + (D + H) \) is generated by global section by Theorem 3.2. Thus, Lemma 3.4 implies that
\[
(K + (\dim X)H + (D + H)) + (-K + (d \cdot \text{codim} X - r - 1)H)
\]
\[
= D + ((d - 1) \text{codim} X)H
\]
is generated by global sections. Similarly, \( D + ((d - 1) \text{codim} X + 1)H \) is very ample. \( \Box \)

Theorem 3.6. Let \( m = (d - 1) \text{codim} X \). Then
\[
\# \text{conn}(\text{Hilb}_{\mathbb{P}^r,m} X) \geq \#(\text{NS}_X)_{\text{tor}}.
\]

Proof. By Lemma 3.5(a), we may apply Theorem 3.1 to \( F = mH \). \( \Box \)

\(^2\)The condition ‘numerically equivalent to 0’ can be replaced by ‘numerically effective’ due to Kleiman’s criterion of ampleness [12, Chapter IV §2 Theorem 2].
4. Irreducible Components of Hilbert Schemes

The aim of this section is to give an explicit upper bound on \( \#(\text{NS}_{\text{tor}} X) \) for a smooth projective variety \( X \). Theorem 3.6 implies that it suffices to give an upper bound on the number of connected components of some Hilbert scheme. Recall the definition of Castelnuovo–Mumford regularity and Gotzmann numbers.

**Definition 4.1.** A coherent sheaf \( \mathcal{F} \) over \( \mathbb{P}^r \) is \( m \)-regular if and only if
\[
H^i(\mathbb{P}^r, \mathcal{F}(m - i)) = 0
\]
for every integer \( i > 0 \). The smallest such \( m \) is called the Castelnuovo–Mumford regularity of \( \mathcal{F} \).

**Definition 4.2.** Let \( P \) be the Hilbert polynomial of some ideal \( I \subset k[x_0, \cdots, x_r] \). The Gotzmann number \( \varphi(P) \) of \( P \) is defined as
\[
\varphi(P) = \inf \{ m \mid \mathcal{I}_Z \text{ is } m\text{-regular for every closed subvariety } Z \subset \mathbb{P}^r \text{ with Hilbert polynomial } P \}.
\]

Hilbert schemes can be explicitly described as a closed subscheme of a Grassmannian, by Gotzmann [5].

**Theorem 4.3** (Gotzmann). Let \( P \) be the Hilbert polynomial of some ideal \( I \subset k[x_0, \cdots, x_r] \). Assume that \( t \geq \varphi(P) \). Then
\[
\iota_t : \text{Hilb}^{(t+r)-P(t)} \mathbb{P}^r \to \text{Gr}(P(t), k[x_0, \cdots, x_r]_t)
\]
\[
[Y] \mapsto \Gamma(\mathcal{I}_Y(t))
\]
gives a well-defined closed immersion. Moreover, the image is the collection of linear spaces \( T \subset k[x_0, \cdots, x_r]_t \) such that
(a) \( \dim (x_0 T + \cdots + x_r T) \leq P(t + 1) \).

**Proof.** See [5, Section 3]. \( \square \)

Let \( X \hookrightarrow \mathbb{P}^r \) be a projective variety and \( Q \) be a polynomial. Then there is the natural closed embedding
\[
\text{Hilb}_Q X \hookrightarrow \text{Hilb}_Q \mathbb{P}^r.
\]

**Theorem 4.4.** Use the notation in Theorem 4.3. Let \( X \hookrightarrow \mathbb{P}^r \) be a projective variety defined by polynomials of degree \( \leq d \). Assume that \( t \geq \max\{ \varphi(P), d \} \). Then the image of
\[
\text{Hilb}^{(t+r)-P(t)} X
\]
under \( \iota_t \) is the collection of linear spaces \( T \subset k[x_0, \cdots, x_r]_t \) such that
(a) \( \dim (x_0 T + \cdots + x_r T) \leq P(t + 1) \) and
(b) \( \Gamma(\mathcal{I}_X(t)) \subset T \).

**Proof.** See the proof of [19, Lemma 8.23] \( \square \)

Therefore, an upper bound on Gotzmann numbers will give an explicit construction of a Hilbert scheme. Such a bound is given by Hoa [9, Theorem 6.4(i)].
Theorem 4.5 (Hoa). Let $I \subset k[x_0, \cdots, x_r]$ be an nonzero ideal generated by homogeneous polynomials of degree at most $d \geq 2$. Let $a$ be the Krull dimension of $k[x_0, \cdots, x_r]/I$. Then

$$\varphi(\text{HP}_I) \leq \left(\frac{3}{2}d^{r+1-a} + d\right)^{a^2-1}. $$

Once a Hilbert scheme is explicitly constructed, we can bound the number of the irreducible components by the lemma below.

Lemma 4.6. If $X \hookrightarrow \mathbb{A}^r$ is an affine scheme defined by polynomials of degree $\leq d$, then

$$\# \text{ irr}(X) \leq d^r. $$

Proof. This is a special case of the Andreotti-Bălzout inequality [4, Lemma 1.28]. \hfill $\square$

The Grassmannian $\text{Gr}(k, n)$ is covered by open sets isomorphic to $\mathbb{A}^{k(n-k)}$. The conditions in Theorem 4.4 can be translated into explicit equations in such an affine space, giving the lemma below.

Lemma 4.7. Let $V$ and $W$ be vector spaces. Let $n = \dim V$. Let $\varphi_i : V \to W$ be a linear map for each $i = 0, \cdots, r$. Let $U \subset V$ be a subspace. Let $X$ be the collection of $T \in \text{Gr}(q, V)$ satisfying

(a) $\dim(\varphi_0(T) + \cdots + \varphi_r(T)) \leq p$, and

(b) $U \subset T$.

Then $X$ is a closed subscheme of $\text{Gr}(q, V)$, and with $n = \dim V$,

$$\# \text{ conn}(X) \leq \max\{p + 1, q + 1\}^{q(n-q)}. $$

Proof. Given a subspace $S \subset V$ of dimension $n - q$, there is an open set

$$U_S = \{T \in \text{Gr}(q, V) | U \cap T = \{0\}\}. $$

Choose a basis of $V$ such that $S$ is spanned by the first $n - q$ entries. Then every $T \in U_S$ is uniquely represented as a column space of a block matrix

$$N_T = \begin{pmatrix} X_{n-q,q} \\ I_q \end{pmatrix}, $$

where $X_{n-q,q}$ is an $(n-q) \times q$ matrix, and $I_q$ is the $q \times q$ identity matrix. The entries of $X_{n-q,q}$ can be regarded as indeterminates, giving an isomorphism $U_S \simeq \mathbb{A}^{(n-q)q}$. Let $\varphi_{i}^{\otimes q}(N_T)$ be the $(\dim W) \times q$ matrix obtained by applying $\varphi_i$ to every column of $N_T$. Then

$$\dim(\varphi_0(T) + \cdots + \varphi_r(T)) \leq p $$

if and only if every $(p + 1) \times (p + 1)$ minor of the block matrix

$$\begin{pmatrix} \varphi_0^{\otimes k}(X_{n-q,q}) & \varphi_1^{\otimes k}(X_{n-q,q}) & \cdots & \varphi_r^{\otimes k}(X_{n-q,q}) \end{pmatrix} $$

is zero.

Let $M_U$ be a matrix whose columns form a basis of $U$. Then $U \subset T$ if and only if every $(q + 1) \times (q + 1)$ minor of

$$\begin{pmatrix} X_{n-q,q} \\ I_q \end{pmatrix} M_U $$

is zero.
is zero. Thus, \( \mathbf{X} \cap U_S \) is defined by the minors in \( U_S \), meaning that \( \mathbf{X} \) is a closed subscheme of \( \text{Gr}(q,V) \).

Now, take one point from each irreducible component of \( \mathbf{X} \), and let them be \( T_0, \cdots, T_{\ell-1} \subset V \). Then \((n-k)\)-dimensional subspace \( S \subset V \) can be chosen such that

\[
S \cap \left( \bigcup_{i=0}^\ell T_i \right) = \{0\},
\]

implying that every irreducible component intersects with \( U_S \). Since \( \mathbf{X} \cap U_S \) is defined by polynomials of degree \( \leq \max\{p+1, q+1\} \), Lemma 4.6 implies

\[
\# \text{irr}(\mathbf{X}) = \# \text{irr}(\mathbf{X} \cap U_S) \leq \max\{p+1, q+1\}^q(n-q). \tag*{□}
\]

**Lemma 4.8.** Let \( X \hookrightarrow \mathbb{P}^r \) be a projective variety defined by polynomials of degree \( \leq d \). Let \( P \) be the Hilbert polynomial of an ideal. If \( t \geq \max\{\varphi(P), d, 8r\} \) and \( r \geq 2 \), then

\[
\# \text{irr} \left( \text{Hilb}_{r^t-P(t)} X \right) \leq t^{r^2r}.
\]

**Proof.** Theorem 4.4 and Lemma 4.7 implies that

\[
\# \text{irr} \left( \text{Hilb}_{r^t-P(t)} X \right) \leq \max \{ P(t+1)+1, P(t)+1 \}^{P(t)(t^{r^t})-P(t)}
\]

\[
\leq \left( \binom{t+1+r}{r} + 1 \right) \left( \frac{(t+1+r)^2}{r!} \right).
\]

Since \( r \geq 2 \) and \( t \geq 8r \),

\[
\binom{t+1+r}{r} = \frac{(t+r+1) \cdots (t+2)}{r!}
\]

\[
\leq \frac{(t+r+1)^r}{2^{r-1}}
\]

\[
\leq \left( \frac{10t}{8} \right)^r \frac{1}{2^{r-1}}
\]

\[
< t^r.
\]

Therefore,

\[
\# \text{irr} \left( \text{Hilb}_{r^t-P(t)} X \right) \leq (t^r)^2 = t^{r^2}. \tag*{□}
\]

Now, we are ready to give an upper bound:

**Theorem 4.9.** Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety defined by homogeneous polynomials of degree \( \leq d \). Then

\[
(3) \quad \#(\text{NS} \ X)_{\text{tor}} \leq 2^{dr+3\log_2 r}.
\]

**Proof.** If \( X \) is a curve or a projective space, then \((\text{NS} \ X)_{\text{tor}} = 0 \). Thus, we may assume that \( r \geq 3, d \geq 2 \) and \( \text{codim} \ X \geq 1 \). Moreover, \( X \) may be assumed to be not contained in any hyperplane.
Let $I$ be the ideal defining $X$. Let $H$ be a hyperplane section of $X$ cut by $x_r = 0$. Let $m = (d-1) \operatorname{codim} X \geq 1$. Then $I + (x^m)$ is the ideal defining $mH$ where $m = (d-1) \operatorname{codim} X$. Let $t = (2rd)^{(r+1)2^{r-2}}$ and $a = \dim X$. Then $t \geq d$ and $t \geq 8r$. Moreover,

$$t = \left(\frac{3}{2}(rd)^{r+1-a} + \frac{1}{2}(rd)^{r+1-a}\right) \cdot 2^{2^{r-2}}$$

$$\geq \left(\frac{3}{2}(rd)^{r+1-a} + rd\right) \cdot 2^{a-1}$$

$$\geq \varphi(\mathcal{H} \mathcal{P}_{mH}) \text{ (by Theorem 4.5)}.$$

Thus, Theorem 3.6 and Lemma 4.8 implies that

$$\#(\text{NS}_X)_{\text{tor}} \leq \# \text{conn}(\mathcal{H} \mathcal{P}_{mH}X) \leq \# \text{irr}(\mathcal{H} \mathcal{P}_{mH}X) \leq t^{rt^{2r}}.$$

Notice that

$$\log_2 \log_d \log_2 \left(t^{rt^{2r}}\right) = \log_2 \log_d \left(t^{2r}r \log_2 t\right)$$

$$\leq \log_2 \log_d \left(t^{2r}2^{r-1}(r+1)r^2d\right)$$

(since $\log_2(2rd) \leq 2rd$)

$$\leq \log_2 \left(2r \log_d t + r + \log_2(r+1)r^2\right)$$

(since $d \geq 2$)

$$\leq \log_2 \left(2^{r-1}r(r+1)(\log_2 r + 2) + r + \log_2(r+1)r^2\right)$$

$$\leq r + 3 \log_2 r.$$

As a result,

$$\#(\text{NS}_X)_{\text{tor}} \leq 2^{2^{r+3} \log_2 r}.$$

□

5. Irreducible Components of Chow Varieties

In this section, $X \hookrightarrow \mathbb{P}^r$ is a smooth projective variety defined by homogeneous polynomials of degree $\leq d$. The goal of this section is to give a better upper bound on the order of $(\text{NS}_X)_{\text{tor}}$. János Kollár pointed out that the bound may be also derived form an upper bound on the number of connected components of $\text{Chow}_{\delta,n}X$. Our goal only requires a bound for $\text{CDiv}_nX$ instead of Chow varieties of arbitrary dimensions.

Theorem 5.1. Let $n = (d-1) \operatorname{codim} X \cdot \deg X$. Then

$$\# \text{conn}(\text{CDiv}_nX) \geq \#(\text{NS}_X)_{\text{tor}}.$$

Proof. Let $H \subset X$ be a hyperplane section, $m = (d-1) \operatorname{codim} X$, and $Q = \mathcal{H} \mathcal{P}_{mH}$. If $D$ is a closed subscheme with Hilbert polynomial $Q$, then $\deg D = \deg mH = n$. Hence,

$$\mathcal{H} \mathcal{P}_QX \cap \text{CDiv} X \subset \text{CDiv}_nX.$$

Since $\mathcal{H} \mathcal{P}_QX$, $\text{CDiv} X$ and $\text{CDiv}_nX$ are open and closed in $\mathcal{H} \mathcal{P}X$, Lemma 3.5 and Theorem 3.1 implies that

$$\# \text{conn}(\text{CDiv}_nX) \geq \# \text{conn}(\mathcal{H} \mathcal{P}_QX \cap \text{CDiv} X) \geq \#(\text{NS}_X)_{\text{tor}}.$$

□
In [13, Exercise I.3.28], Kollár gives an explicit upper bound on \( \# \text{irr}(\text{Chow}_{\delta,n} \mathbb{P}^r) \) and an outline of the proof. Moreover, [13, Exercise I.3.28.13] suggests an exercise to find an explicit upper bound on \( \# \text{irr}(\text{Chow}_{\delta,n} X) \), and the proof can be found in [7, Section 2]. However, the proof works only if \( \text{char} \, k = 0 \), and it does not give a bound in a closed form.

Therefore, we will give another complete proof, but most of the proof up to Lemma 5.11 is just a modification of Kollár’s technique and [7, Section 2]. Moreover, we will only bound \( \# \text{irr}(\text{CDiv}_n X) \), because this restriction avoids the bad behavior of Chow varieties in positive characteristic, simplifies the proof and slightly improves the result.

**Lemma 5.2.** Let \( D \subset X \) be a nonzero effective divisor on \( X \) of degree \( n \). Then there are \( f \) and \( g \) in \( \Gamma(X, \mathcal{I}_D(n)) \) such that \( D \) is the largest effective divisor contained in \( V_X(f,g) \).

**Proof.** Take any point \( x \in X \) and a generic linear projection \( \rho_0 : \mathbb{P}^r \to \mathbb{P}^{\dim X} \). Then \( \rho_0(D) \) is a hypersurface and \( \rho_0|_x \) is étale at \( x \). Let \( f_0 \) be a homogeneous polynomial of degree \( n \) defining \( \rho_0(D) \), and \( f \in \Gamma(X, \mathcal{I}_D(n)) \) be its pullback. Then

\[
V_X(f) = D \cup E_0 \cup E_1 \cup \cdots \cup E_{t-1}
\]

for some irreducible closed subschemes \( E_i \subset X \) not set theoretically contained in \( D \).

Take \( e_i \in E_i \setminus D \) for each \( i \), and let \( \rho_1 : \mathbb{P}^r \to \mathbb{P}^{\dim X} \) be another generic linear projection. Then \( e_i \notin \rho_1(D) \) for every \( i \). Let \( g_0 \) be a homogeneous polynomial of degree \( n \) defining \( \rho_1(D) \), and \( g \in \Gamma(X, \mathcal{I}_D(n)) \) be its pullback. Then \( V_X(f,g) \) contains \( D \) but not \( E_i \) for all \( i \). Thus, \( D \) is the largest divisor contained in \( V_X(f,g) \). \( \square \)

**Definition 5.3.** Let

\[
N_n = \binom{n + r}{n} - 1.
\]

Then \( \mathbb{P}^{N_n} \) parameterizes nonzero homogeneous polynomials of degree \( n \) with \( r + 1 \) variables up to constant factors. Let

\[
S_n = \left\{ (f,g) \in (\mathbb{P}^{N_n})^2 \mid \text{codim}_X(V_X(f,g)) \leq 1 \right\} \quad \text{and} \quad T_n = \left\{ ((f,g), [D]) \in S_n \times \text{CDiv} \, X \mid D \subset V_X(f,g) \right\}.
\]

Let \( p : T_n \to S_n \) and \( q : T_n \to \text{CDiv} \, X \) be the natural projections.

**Lemma 5.4.** Let \( X \hookrightarrow (\mathbb{P}^r)^2 \) be a closed subscheme defined by bihomogeneous polynomials of total degree \( \leq d \). Then

\[
\# \text{irr}(X) \leq d^{2r}.
\]

**Proof.** Let \( L_0 \) and \( L_1 \) be generic hyperplanes of \( \mathbb{P}^r \). Then \( L_0 \times \mathbb{P}^r \) and \( \mathbb{P}^r \times L_1 \) do not contain any irreducible component of \( X \). Let

\[
U = (\mathbb{P}^r \setminus L_0) \times (\mathbb{P}^r \setminus L_1) \cong \mathbb{A}^{2r}.
\]

Then

\[
\# \text{irr}(X) = \# \text{irr} \,(X \cap U),
\]

and \( X \cap U \hookrightarrow \mathbb{A}^{2r} \) is defined by polynomials of degree \( \leq d \). Consequently, Lemma 4.6 proves the inequality. \( \square \)
**Lemma 5.5.** Let \( n \) be a positive integer. Then \( S_n \) is closed in \((\mathbb{P}^N)^2\) and
\[
\# \text{irr}(S_n) \leq \left(\frac{2\max\{n,d\} + (r-1)d}{r}\right)^{2(n+r)-2}.
\]

**Proof.** Take \((f,g) \in (\mathbb{P}^N)^2\) as in Definition 5.3. Then \(\text{codim}_X(V_X(f,g)) = 1\), if and only if the intersection of \(V_X(f,g)\) with \( t = \dim X - 1 \) number of generic hyperplane sections is nonempty. Let \( h_i = \sum_{j=0}^{r-1} \xi_{i,j} x_j \) be a generic hyperplane section for each \( i \), where \( \xi_{i,j} \) are indeterminates. Take a base extension to \( k(\{\xi_{i,j}\}_{i,j})\). Then
\[
\text{codim}_X(V_X(f,g)) = 1 \\
\iff V_X(f, g, h_0, \ldots, h_{t-1}) \neq \emptyset \\
\iff (x_0, \ldots, x_r)^{2n,n} \in (f,g,h_0,\ldots,h_{t-1}) + I_X \\
\text{(by [13, Corollary I.7.4.4.3])} \\
\iff \text{rank} \left( (f,g,h_0,\ldots,h_{t-1}) + I_X \right) < \left(\frac{2\max\{n,d\} + (r-1)d}{r}\right).
\]
The last condition can be translated into bihomogeneous polynomials in the coefficients of \( f \) and \( g \) of total degree \( \left(\frac{2\max\{n,d\} + (r-1)d}{r}\right) \). Lemma 5.4 proves the inequality. \(\square\)

**Lemma 5.6.** The set \( T_n \) is a closed subset of \( S_n \times \text{CDiv } X \).

**Proof.** Since \( \text{CDiv } X \) is open and closed in \( \text{Hilb } X \), it suffices to show that
\[
T^Q_n = \{((f,g), [Z]) \in S_n \times \text{Hilb}_Q X \mid Z \subset V_X(f,g)\}
\]
is closed in \( S_n \times \text{Hilb}_Q X \) for every Hilbert polynomial \( Q \). Recall that Theorem 4.3 gives a closed embedding
\[
u_t: \text{Hilb}_Q X \to \text{Gr}(P(t), k[x_0,\ldots,x_r]_t) \\
[Z] \mapsto \Gamma(\mathcal{I}_Z(t))
\]
for some polynomial \( P \) and every large \( t \). This gives a closed embedding
\[
(S_n \times \text{Hilb}_Q X) \hookrightarrow (\mathbb{P}^N \times \text{Gr}(P(t), k[x_0,\ldots,x_r]_t)).
\]
We may assume that \( t \geq n \). Notice that \( Z \subset V(f,g) \) if and only if the saturation of \((f,g)\) is contained in the saturated ideal defining \( Z \). Thus,
\[
Z \subset V(f,g) \iff (f,g)_t \subset \Gamma(\mathcal{I}_Z(t)) \\
\iff \dim (\Gamma(\mathcal{I}_Z(t)) + (f,g)_t) \leq P(t).
\]
Note that \( \mathbb{P}^N \times \text{Gr}(P(t), k[x_0,\ldots,x_r]_t) \) is covered by the standard affine open spaces. In such affine open spaces, the last condition is expressed as \((P(t) + 1) \times (P(t) + 1)\) minors of some matrix. Consequently, \( T^Q_n \) is identified with a closed subset of \( \mathbb{P}^N \times \text{Gr}(P(t), k[x_0,\ldots,x_r]_t) \). \(\square\)

**Definition 5.7.** Let \( \text{PDiv}_n X \) be the union of the irreducible components of \( \text{CDiv}_n X \) which contains at least one closed point corresponding to a reduced and irreducible divisor.

**Lemma 5.8.** Let \( F \) be an irreducible component of \( \text{PDiv}_n X \). Then there is a unique irreducible component \( E \) of \( T_n \) such that \( q(E) = F \).
Lemma 5.12. That this map is injective. □

E where Lemma 5.8 and Lemma 5.10 define a map

Proof. Lemma 5.11.

Let

Proof. S

Lemma 5.10.

Component containing S

Since R

Thus, we can take W

Because

is unique. □

Lemma 5.9. Let F and E be as in Lemma 5.8. Then there is (f, g) ∈ Sn such that E is the only irreducible component of Tn satisfying (f, g) ∈ p(E).

Proof. Let W ⊂ CDivn be the complement of the image of the proper morphism

Then W parametrizes the reduced and irreducible divisors of degree n on X. Notice that W ∩ F ≠ ∅, because F ⊂ PDivn X. The uniqueness of E implies that there is a dense open set U ⊂ F such that q−1(U) does not intersect with any irreducible component of Tn other than E. Thus, we can take [D] ∈ W ∩ U. Then D is a reduced and irreducible divisor, since [D] ∈ W. Lemma 5.2 implies that there is (f, g) ∈ Sn such that

p−1((f, g)) = \{((f, g), [D])\}.

Then E is the only irreducible component of Tn containing ((f, g), [D]), because [D] ∈ U. □

Lemma 5.10. Let F and E be as in Lemma 5.8. Then p(E) is an irreducible component of Sn.

Proof. Let

Rn = q−1(CDiv1 X ∪ CDiv2 X ∪ ⋯ ∪ CDivn deg X X) ⊂ Tn.

Since Rn is proper, the restriction p|Rn : Rn → Sn is also proper. Moreover, p|Rn is surjective, because deg Vx(f, g) ≤ n deg X for every (f, g) ∈ Sn. Thus, every irreducible component of Sn is the image of an irreducible component of Rn under p|Rn. Let B ⊂ Sn be the irreducible component containing p(E). Then Lemma 5.9 implies that p(E) = B. □

Lemma 5.11. If n is a positive integer, then

# irr(PDivn X) ≤ # irr(Sn).

Proof. Lemma 5.8 and Lemma 5.10 define a map

\[ \text{irr}(\text{PDiv}_n X) \rightarrow \text{irr}(S_n) \]

\[ F \mapsto P(E), \]

where E is determined by F as in Lemma 5.8. Then Lemma 5.8 and Lemma 5.9 implies that this map is injective. □

Lemma 5.12. Let n be a positive integer. Then

\[ # \text{irr}(\text{CDiv}_n X) \leq 2^n \left( 2 \max\{n, d\} + (r - 1)d \right)^{2(n + r) - 2}. \]
Proof. Notice that
\[
\prod_{n_0 + \cdots + n_{t-1} = n} \text{PDiv}_{n_0} X \times \cdots \times \text{PDiv}_{n_{t-1}} X \longrightarrow \text{CDiv}_n X
\]

([D_0], \cdots, [D_{t-1}]) \mapsto [D_0 + \cdots + D_{t-1}].

is surjective, where the disjoint union runs over all integer partitions of \(n\). Thus, the left-hand side has more irreducible components. Take any integer partition \(n_0 + \cdots + n_{t-1} = n\). Then by Lemma 5.5 and Lemma 5.11,

\[
\# \text{irr}(\text{PDiv}_{n_0} X \times \cdots \times \text{PDiv}_{n_{t-1}} X) = \# \text{irr}(\text{PDiv}_{n_0} X) \times \cdots \times \# \text{irr}(\text{PDiv}_{n_{t-1}} X)
\]

\[
\leq \left(2 \max\{n_0, d\} + (r - 1)d\right)^{2^{(n_0 + r)^2} - 2} \times \cdots \times \left(2 \max\{n_{t-1}, d\} + (r - 1)d\right)^{2^{(n_{t-1} + r)^2} - 2}
\]

\[
\leq \left(2 \max\{n, d\} + (r - 1)d\right)^{2^{(n + r)^2} + \cdots + 2^{(n_{t-1} + r)^2} - 2}.
\]

Let
\[
D(x) = 2\left(x + \frac{r}{d}\right) - 2.
\]

Then \(D(0) = 0\) and \(D\) is concave above in \([0, \infty)\). Therefore,

\[
D(n_0) + D(n_1) + \cdots + D(n_{t-1}) \leq D(n).
\]

Furthermore, the number of integer partitions of \(n\) is less than or equal to \(2^n\). This proves the inequality. \(\square\)

Now, we are ready to give a new upper bound:

**Theorem 5.13.** Let \(X \hookrightarrow \mathbb{P}^r\) be a smooth projective variety defined by homogeneous polynomials of degree \(\leq d\). Then

(4) \[
\#(\text{NS}_X)_{\text{tor}} \leq 2^{d^2 + 2r \log_2 r}.
\]

Proof. If \(X\) is a curve or a projective space, then \((\text{NS}_X)_{\text{tor}} = 0\). Thus, we may assume that \(r \geq 3, d \geq 2, \deg X \geq 2\) and \(r - 2 \geq \text{codim} X \geq 1\). Let \(n = (d - 1) \text{codim} X \cdot \deg X\). Then

\[
d \leq n \leq (r - 2)(d - 1)d^{r-2}.
\]

Theorem 5.1 and Lemma 5.12 implies that

\[
\#(\text{NS}_X)_{\text{tor}} \leq \text{conn}(\text{CDiv}_n X) \leq \text{irr}(\text{CDiv}_n X) \leq 2^n \left(2n + (r - 1)d\right)^{2^{(n + r)^2} - 2}.
\]
Since
\[
\log_2 \left( \frac{2n + (r - 1)d}{r} \right) \leq \log_2 (2n + (r - 1)d)^r \\
\leq r \log_2 (rd)^r \\
\leq r (\log_2 r + r \log_2 d) \\
\leq r^2 (1 + \log_2 d) \\
\leq r^2 d
\]
and
\[
2 \binom{n + r}{r} \leq \frac{2(n + r)^r}{r!} \\
\leq \frac{(rd^{r-1})^r}{3},
\]
we have
\[
\log_d \log_2 (\#(NSX)_{tor}) \leq \log_d \log_2 \left( 2^n \left( \frac{2n + (r - 1)d}{r} \right)^{2(n+r)} \right) \\
\leq \log_d \left( n + r^2 d \frac{(rd^{r-1})^r}{3} \right) \\
\leq \log_d \left( r^{r+2} d^{r(r-1)+1} \right) \\
\leq (r + 2) \log_2 r + r(r - 1) + 1 \\
\leq 2r \log_2 r + r^2.
\]

We now give an application to the torsion subgroups of second cohomology groups.

**Lemma 5.14.** Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety of degree \( d \). Let \( \ell \neq \text{char } k \) be a prime number. The embedding \((NSX) \otimes \mathbb{Z}_\ell \hookrightarrow H^2_{\text{ét}}(X, \mathbb{Z}_\ell) \) [15, Remark V.3.29] induces by the Kummer sequence restricts to an isomorphism.
\[
(\text{NSX})[\ell^{\infty}] \simeq H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}}.
\]

**Proof.** See [21, 2.2]. \( \square \)

Therefore, the bound of Theorem 5.13 implies the corollaries below.

**Corollary 5.15.** Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety defined by homogeneous polynomials of degree \( \leq d \). Let \( \ell \neq \text{char } k \) be a prime number. Then \( \prod_{\ell \neq \text{char } k} \# H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}} \leq 2^{r^2 + 2r \log_2 r} \).

**Proof.** This follows from Theorem 5.13 and Lemma 5.14. \( \square \)

**Corollary 5.16.** Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth projective variety over \( \mathbb{C} \) defined by homogeneous polynomials of degree \( \leq d \). Then
\[
\# H^2_{\text{sing}}(X^{\text{an}}, \mathbb{Z})_{\text{tor}} \leq 2^{d^{r^2 + 2r \log_2 r}}.
\]
Proof. This follows from Corollary 5.15 and the fact that
\[
\prod_{\ell \text{ is prime}} H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}} \simeq H^2_{\text{sing}}(X^{\text{an}}, \mathbb{Z})_{\text{tor}}. \]
\[ \Box \]

6. The Number of Generators of \((\text{NS } X)_{\text{tor}}\)

In this section, \(X \hookrightarrow \mathbb{P}^r\) is a smooth connected projective variety, and \(\ell\) is a prime number not equal to \(\text{char } k\). The goal is to give a uniform upper bound on the number of generators of \((\text{NS } X)[\ell^\infty]\). Our approach is a simplification of a brief sketch suggested by János Kollár.

Lemma 6.1. Let \(X \hookrightarrow \mathbb{P}^r\) be a smooth connected projective variety of degree \(d\), and \(x_0\) be a geometric point of \(X\). Let \(\ell \neq \text{char } k\) be a prime number. Then
\[
(\text{NS } X)[\ell^\infty]^* \simeq \pi_1^{\text{ét}}(X, x_0)_{\text{tor}}^{\ell}[\ell^\infty].
\]

Proof. The exact sequence in [22, Proposition 69] gives an exact sequence
\[
0 \to (\text{NS } X)[\ell^\infty]^* \to \pi_1^{\text{ét}}(X, x_0)^{(\ell)} \to \pi_1^{\text{ét}}(\text{Alb } X, 0)^{(\ell)} \to 0,
\]
by taking the maximal pro-\(\ell\) abelian quotient. Since \(\pi_1^{\text{ét}}(\text{Alb } X, 0)^{(\ell)}\) is a free \(\mathbb{Z}_\ell\)-module,
\[
(\text{NS } X)[\ell^\infty]^* \simeq \pi_1^{\text{ét}}(X, x_0)^{(\ell)}_{\text{tor}} \simeq \pi_1^{\text{ét}}(X, x_0)^{\text{ab}}[\ell^\infty]. \]
\[ \Box \]

If \(M\) is a topological manifold, the linking form implies that \(H^2_{\text{sing}}(M, \mathbb{Z})_{\text{tor}}^* \simeq H^1_{\text{sing}}(M, \mathbb{Z})_{\text{tor}}^*\). Lemma 5.14 and Lemma 6.1 imply the étale analogy that \(H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}}^* \simeq \pi_1^{\text{ét}}(X, x_0)^{\text{ab}}[\ell^\infty]\).

Lemma 6.2. Let \(X \hookrightarrow \mathbb{P}^r\) be a smooth connected projective variety of degree \(d\). Then \((\text{NS } X)[\ell^\infty]^*\) is generated by less than or equal to \((d - 1)(d - 2)\) elements.

Proof. For a general linear space \(L \subset \mathbb{P}^r\) of dimension \(r - \dim X + 1\), the intersection \(C := X \cap L\) is a connected smooth curve of degree \(d\). Take a geometric point \(x_0 \in C\). Then the natural map
\[
\pi_1^{\text{ét}}(C, x_0) \to \pi_1^{\text{ét}}(X, x_0)
\]
is surjective, by repeatedly applying the Lefschetz hyperplane theorem for étale fundamental groups [6, XII. Corollaire 3.5]. Let \(g\) be the genus of \(C\). Then Lemma 6.1 implies that \((\text{NS } X)[\ell^\infty]^*\) is isomorphic to a subquotient of
\[
\pi_1^{\text{ét}}(C, x_0)^{(\ell)}_{\text{tor}} \simeq \mathbb{Z}_\ell^{2g}. \]
Notice that \(2g \leq (d - 1)(d - 2)\) because \(\deg C = d\). As a result, \((\text{NS } X)[\ell^\infty]^*\) is generated by \(\leq (d - 1)(d - 2)\) elements.
\[ \Box \]

Theorem 6.3. Let \(X \hookrightarrow \mathbb{P}^r\) be a smooth connected projective variety of degree \(d\) over \(k\). Let \(p = \text{char } k\), and
\[
N = \begin{cases} 
(\text{NS } X)_{\text{tor}}, & \text{if } p = 0 \\
(\text{NS } X)_{\text{tor}}/(\text{NS } X)[p^\infty], & \text{if } p > 0.
\end{cases}
\]
Then \(N\) is generated by less than or equal to \((d - 1)(d - 2)\) elements.
Since a product of finitely many finite cyclic groups of coprime orders is again cyclic,
\[ N \simeq \prod_{\ell \neq p} (\text{NS } X)[\ell^\infty] \]
is a product of \( (d-1)(d-2) \) cyclic groups by Lemma 6.2.

Corollary 6.4. Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth connected projective variety of degree \( d \) over \( k \). Let \( \ell \neq \text{char } k \) be a prime number. Then \( H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}} \) is generated by less than or equal to \( (d-1)(d-2) \) elements.

Proof. This follows from Lemma 6.2 and Lemma 5.14.

Corollary 6.5. Let \( X \hookrightarrow \mathbb{P}^r \) be a smooth connected projective variety of degree \( d \) over \( \mathbb{C} \). Then \( \text{H}^2_{\text{sing}}(X^{an}, \mathbb{Z}) \) is generated by less than or equal to \( (d-1)(d-2) \) elements.

Proof. This follows from Theorem 6.3 and the fact that
\[ (\text{NS } X)_{\text{tor}} \simeq \prod_{\ell \text{ is prime}} (\text{NS } X)[\ell^\infty] \simeq \prod_{\ell \text{ is prime}} H^2_{\text{ét}}(X, \mathbb{Z}_\ell)_{\text{tor}} \simeq H^2_{\text{sing}}(X^{an}, \mathbb{Z})_{\text{tor}}. \]

The bounds in this section exclude the case \( \ell = \text{char } k \), because of the bad behavior of the étale cohomology at \( p \). One may try to overcome this by using Nori’s fundamental group scheme [18]. However, the Lefschetz hyperplane theorem for Nori’s fundamental group scheme is no longer true [2, Remark 2.4].

Question 6.6. Can one use another cohomology to prove the analogue of Lemma 6.2 for \( (\text{NS } X)[p^\infty] \) in characteristic \( p \)?

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