MULTIPlicITIES IN ULTRARELATIVISTIC PROTON-(ANTI)PROTON COLLISIONS AND NEGATIVE BINOMIAL DISTRIBUTION FITS

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Likelihood ratio tests are performed for the hypothesis that charged-particle multiplicities measured in proton-(anti)proton collisions at $\sqrt{s} = 0.9$ and 2.36 TeV are distributed according to the negative binomial form. Results indicate that the hypothesis should be rejected in all cases of ALICE-LHC measurements in the limited pseudo-rapidity windows, whereas should be accepted in the corresponding cases of UA5 data. Possible explanations of that and of the disagreement with the least-squares fitting method are given.

Keywords: Likelihood ratio test; negative binomial distribution; charged-particle multiplicity.

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1. Introduction

The UA5 Collaboration noticed for the first time that charged-particle multiplicity distributions measured in high energy proton-(anti)proton collisions in limited intervals of pseudo-rapidity have the negative binomial form\(^{(1)}\). In the present paper this observation will be verified for the collisions at $\sqrt{s} = 0.9$ and 2.36 TeV performed by UA5\(^{(2)}\) and ALICE Collaborations\(^{(3)}\). Only non-single diffractive (NSD) events will be considered because such a case was analyzed with this respect by both Collaborations. In fact, the author investigated ALICE inelastic events also (including the case of $\sqrt{s} = 7$ TeV\(^{(4)}\)), but all fits were entirely unacceptable.

The Negative Binomial Distribution (NBD) is defined as

\[
P(n; p, k) = \frac{k(k+1)(k+2)...(k+n-1)}{n!} (1-p)^n p^k ,
\]

where $n = 0, 1, 2, ..., 0 \leq p \leq 1$ and $k$ is a positive real number. In the application to high energy physics $n$ has the meaning of the number of charged particles detected.
in an event. The expected value $\bar{n}$ and variance $V(n)$ are expressed as:

$$\bar{n} = \frac{k(1 - p)}{p}, \quad V(n) = \frac{k(1 - p)}{p^2}.$$ (2)

In this analysis the hypothesis that the charged-particle multiplicities measured in high energy $p - p$ collisions are distributed according to the NBD is verified with the use of the maximum likelihood method (ML) and the likelihood ratio test. More details of this approach can be found in Refs. 5–7.

There are two crucial reasons for this approach:

(i) The fitted quantity is a probability distribution function (p.d.f.), so the most natural way is to use the ML method, where the likelihood function is constructed directly from the tested p.d.f.. But more important is that because of Wilks’s theorem (see Appendix B) one can easily define a statistic, the distribution of which converges to a $\chi^2$ distribution as the number of measurements goes to infinity. Thus for the large sample the goodness-of-fit can be expressed as a $p$-value computed with the corresponding $\chi^2$ distribution.

(ii) The most commonly used method, the least-squares method (LS) (called also $\chi^2$ minimization), has the disadvantage of providing only the qualitative measure of the significance of the fit, in general. Only if observables are represented by Gaussian random variables with known variances, the conclusion about the goodness-of-fit equivalent to that mentioned in the point (i) can be derived.

It is worth noting that the ML method with binned data and Poisson fluctuations within a bin was already applied to fitting multiplicity distributions to the NBD but at much lower energies (E-802 Collaboration 8).

2. The maximum likelihood method

The number of charged particles $N_{ch}$ is assumed to be a random variable with the p.d.f. given by Eq. (1). Each event is treated as an independent observation of $N_{ch}$ and a set of a given class of events is a sample. For $N$ events in the class there are $N$ measurements of $N_{ch}$, say $\mathbf{X} = \{X_1, X_2, ..., X_N\}$. Some of these measurements can be equal, i.e. $X_i = X_j$ for $i \neq j$ can happen. The whole population consists of all possible events with the measurements of 0, 1, 2,... charged particles and by definition is infinite.

For the class of events one can defined the likelihood function

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*Here, these quantities are distinguished from the experimentally measured the average charged particle multiplicity $\langle N_{ch} \rangle$ and the variance $\sigma^2$.*

*b Precisely, because of the energy conservation the number of produced charged particles is limited but the number of collisions is not.*
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\[ L(X \mid p, k) = \prod_{j=1}^{N} P(X_j; p, k) , \]

where $P(X_j; p, k)$ is the NBD, Eq. (1).

The values $\hat{p}$ and $\hat{k}$ for which $L(X \mid p, k)$ has its maximum are the maximum likelihood estimates of parameters $p$ and $k$. This is equivalent to the maximization of the log-likelihood function

\[ \ln L(X \mid p, k) = \sum_{j=1}^{N} \ln P(X_j; p, k) . \] (4)

Thus the values $\hat{p}$ and $\hat{k}$ are the solutions of the equations:

\[ \frac{\partial}{\partial p} \ln L(X \mid p, k) = \sum_{j=1}^{N} \frac{\partial}{\partial p} \ln P(X_j; p, k) = 0 , \]

\[ \frac{\partial}{\partial k} \ln L(X \mid p, k) = \sum_{j=1}^{N} \frac{\partial}{\partial k} \ln P(X_j; p, k) = 0 . \] (5)

It can be proven that one of the necessary conditions for the existence of the maximum is (see Appendix A for details):

\[ \bar{n} = \langle N_{ch} \rangle , \] (6)

i.e. the distribution average has to be equal to the experimental average.

3. Likelihood ratio test

Let divide the sample defined in Sect. 2 into $m$ bins characterized by $Y_i$ - the number of measured charged particles and $n_i$ - the number of entries in the $i$th bin, $N = \sum_{i=1}^{m} n_i$ (details of the theoretical framework of this Section can be found in Refs. 5–7). Then the expectation value of the number of events in the $i$th bin can be written as

\[ \nu_i(\nu_{tot}, p, k) = \nu_{tot} \cdot P(Y_i; p, k) , \] (7)

where $\nu_{tot}$ is the expected number of all events in the sample, $\nu_{tot} = \sum_{i=1}^{m} \nu_i$. This is because one can treat the number of events in the sample $N$ also as a random variable with its own distribution - Poisson one. Generally, the whole histogram can

\[ \text{Now } Y_i \neq Y_j \text{ for } i \neq j \text{ and } i, j = 1, 2, ..., m. \]
be treated as one measurement of \( m \)-dimensional random vector \( \mathbf{n} = (n_1, \ldots, n_m) \) which has a multinomial distribution, so the joint p.d.f. for the measurement of \( N \) and \( \mathbf{n} \) can be converted to the form [57]:

\[
f(\mathbf{n}; \nu_1, \ldots, \nu_m) = \prod_{i=1}^{m} \frac{\nu_i^{n_i}}{n_i!} \exp(-\nu_i).
\]

(8)

Since now \( f(\mathbf{n}; \nu_1, \ldots, \nu_m) \) is the p.d.f. for one measurement, \( f \) is also the likelihood function

\[
L(\mathbf{n} \mid \nu_1, \ldots, \nu_m) = f(\mathbf{n}; \nu_1, \ldots, \nu_m).
\]

(9)

With the use of Eq. (7) the corresponding likelihood function can be written as

\[
L(\mathbf{n} \mid \nu_{\text{tot}}, p, k) = L(\mathbf{n} \mid \nu_1(\nu_{\text{tot}}, p, k), \ldots, \nu_m(\nu_{\text{tot}}, p, k)).
\]

(10)

Then the likelihood ratio is defined as

\[
\lambda = \frac{L(\mathbf{n} \mid \hat{\nu}_{\text{tot}}, \hat{p}, \hat{k})}{L(\mathbf{n} \mid \tilde{\nu}_1, \ldots, \tilde{\nu}_m)} = \frac{L(\mathbf{n} \mid \nu_{\text{tot}}, p, k)}{L(\mathbf{n} \mid n_1, \ldots, n_m)}.
\]

(11)

where \( \hat{\nu}_{\text{tot}}, \hat{p} \) and \( \hat{k} \) are the ML estimates of \( \nu_{\text{tot}}, p \) and \( k \) with the likelihood function given by Eq. (10) and \( \tilde{\nu}_i = n_i, i = 1, 2, \ldots, m \) are the ML estimates of \( \nu_i \) treated as free parameters. Note that since the denominator in Eq. (11) does not depend on parameters, the log-ratio defined as

\[
\ln \lambda(\nu_{\text{tot}}, p, k) = \ln \frac{L(\mathbf{n} \mid \nu_{\text{tot}}, p, k)}{L(\mathbf{n} \mid n_1, \ldots, n_m)}
\]

\[
= -\sum_{i=1}^{m} \left( n_i \ln \frac{n_i}{\nu_i} + \nu_i - n_i \right)
\]

\[
= -\nu_{\text{tot}} + N - \sum_{i=1}^{m} n_i \ln \frac{n_i}{\nu_i},
\]

(12)

where \( \nu_i \) are expressed by Eq. (7), can be used to find the ML estimates of \( \nu_{\text{tot}}, p \) and \( k \). Further, the statistic given by

\[
\chi^2 = -2 \ln \lambda = 2 \sum_{i=1}^{m} \left( n_i \ln \frac{n_i}{\tilde{\nu}_i} + \tilde{\nu}_i - n_i \right)
\]

(13)

approaches the \( \chi^2 \) distribution asymptotically, i.e. as the number of measurements, here the number of events \( N \), goes to infinity (the consequence of the Wilks’s theorem, see Appendix B). The values \( \hat{\nu}_i \) are the estimates of \( \nu_i \) given by
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\[ \hat{\nu}_i = \hat{\nu}_{tot} \cdot P(Y_i; \hat{p}, \hat{k}) \]  

(14)

and if one assumes that \( \nu_{tot} \) does not depend on \( p \) and \( k \) then \( \hat{\nu}_{tot} = N \). For such a case

\[ \sum_{i=1}^{m} \hat{\nu}_i = \sum_{i=1}^{m} n_i \]  

(15)

and Eq. (13) becomes

\[ \chi^2(\hat{p}, \hat{k}) = -2 \ln \lambda = 2 \sum_{i=1}^{m} n_i \ln \frac{n_i}{\hat{\nu}_i} \]  

(16)

Also then one can just put \( \nu_{tot} = N \) and Eq. (12) can be rewritten as

\[ \ln \lambda(p, k) \]

\[ = N \ln N - \sum_{i=1}^{m} n_i \ln n_i + \sum_{i=1}^{m} n_i \ln P(Y_i; p, k) \]

\[ = -\sum_{i=1}^{m} n_i \ln \frac{n_i}{N} + N \sum_{i=1}^{m} \frac{n_i}{N} \ln P(Y_i; p, k) \]

\[ = -N \sum_{i=1}^{m} P_i^{ex} \ln P_i^{ex} + N \sum_{i=1}^{m} P_i^{ex} \ln P_i(Y_i; p, k) \]

(17)

with the term depending on \( p \) and \( k \) the same as Eq. (A.4) and \( P_i^{ex} = n_i/N \). Therefore conclusions of Appendix A holds here, i.e. the necessary conditions for the existence of the maximum is \( \bar{n} = \langle N_{ch} \rangle \), Eq. (A.10) is the equation which determines \( \hat{k} \) and \( \hat{p} \) is obtained with the help of Eq. (A.9). Note that the maximum of \( \ln \lambda \) is the minimum of \( \chi^2 = -2 \ln \lambda \), so from Eqs. (16) and (17) one arrives at

\[ \chi^2_{min} = -2 N \sum_{i=1}^{m} P_i^{ex} \ln \frac{P(Y_i; \hat{p}, \hat{k})}{P_i^{ex}} \]  

(18)

In fact, the method just described assumes that the sum of \( P_i(p, k) \equiv P(Y_i; p, k) \) over all bins equals 1. But only the infinite sum of \( P(n; p, k) \) is 1. However the measured values of \( Y_m \) are big enough (of the order of 20 at least for all considered cases) so in the vicinity of \( \hat{p} \) and \( \hat{k} \) the sum of \( P(n; p, k) \) up to \( n = Y_m \) equals 1 approximately (see the seventh column in Table 1). Nevertheless, to calculate \( \chi^2_{min} \), Eq. (18), \( P(Y_i; \hat{p}, \hat{k}) \) were normalized appropriately and these results are listed in the fifth column of Tables 1-3. Another way to treat this problem is to create arbitrary the \( (m+1) \)st bin for all possible \( n > Y_m \) and with \( P_{m+1}^{ex} = 0 \). Bins with \( P_i^{ex} = 0 \) \((n_i = 0 \) equivalently\) do not contribute in Eq. (18) (see Ref. [5]). In practice, it means
that \( \chi^2_{\text{min}} \) would be calculated also from Eq. (18) but without the normalization. It has turned out that that way leads to much greater values of \( \chi^2_{\text{min}} \).

4. Results and discussion

The method described in Sections 2 and 3 requires that all bins in a given data set have the width equal to 1, so as the experimental probability \( P_i^\text{ex} \) to measure a signal in the \( i \)th bin was equivalent to the probability of the measurement of \((i - 1)\) charged particles (the first bin is the bin of 0 charged particles detected). This is fulfilled for all bins of the considered data sets except the ends of their tails. In these tails the measured values of \( P_i^\text{ex} \) have been uniformly distributed over the bin range so as the method could be applied directly. If the bin width is not significantly greater than 1 then this approximation should not change substantially the value of \( \chi^2_{\text{min}} \) given by Eq. (18) because in the most cases \( P_i^\text{ex} \) at tails are two orders smaller than in the main part of distributions. Also errors in tails are bigger, in the range \( 10 - 50\% \), increasing with \( i \).

Since the test statistic \(-2 \ln \lambda\) has a \( \chi^2 \) distribution approximately in the large sample limit, it can be used as a test of the goodness-of-fit. The result of the test is given by the so-called \( p \)-value which is the probability of obtaining the value of the statistic, Eq. (13), equal to or greater then the value just obtained by the ML method for the present data set, when repeating the whole experiment many times:

\[
p = P(\chi^2 \geq \chi^2_{\text{min}}; n_d) = \int_{\chi^2_{\text{min}}}^{\infty} f(z; n_d)dz ,
\]

where \( f(z; n_d) \) is the \( \chi^2 \) p.d.f. and \( n_d \) the number of degrees of freedom, \( n_d = m - 2 \) here.

The results of the analysis are presented in Table 1. Note that for UA5 cases two possibilities of the corrected number of events are listed. This is because only the measured number of events, 6839, is given in Ref. 2. However, the fits have been done to the corrected distributions, so also the corrected number of events should be put into Eq. (18). The number have been estimated in the following way: in Fig.4 of Ref. 2 the mean of the observed distribution versus the corrected (true) number of particles is plotted, the curve is a straight line roughly with the tangent equal to \( \sim 0.8 \), so one can guess that the efficiency is also about 80\%. Just to check how results are stable with respect to a change in the number of events, the case with 70\% efficiency has been also calculated. As one can see, for all ALICE cases the hypothesis in question should be rejected, whereas for the listed UA5 cases should be accepted. But it was claimed that charged-particle multiplicities measured in the limited pseudo-rapidity windows by the ALICE Collaboration are distributed according to the NBD \(^{3,4,12}\). However that conclusion was the result of the \( \chi^2 \) minimization (the LS method). Therefore it seems to be reasonable to check what are the values of the LS \( \chi^2 \) function at the ML estimators listed in the third and
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Table 1. Results of fitting multiplicity distributions for the NSD events measured in p − \bar{p} (UA5)\textsuperscript{2} and p − p (ALICE)\textsuperscript{3} collisions. The ALICE numbers of events are from Ref. 9. The distributions have been modified in the tails so as all bins have the width 1, see the text for explanations.

| Experiment \( \sqrt{s} \) | N  | \( \hat{k} \) | \( \hat{\rho} \) | \( \chi^2/n_d \) | p-value [%] | \( \sum p_i(\hat{\rho}, \hat{k}) \) quadrature sum | \( \chi^2/\sqrt{s} \) with errors: sum statistical only |
|--------------------------|----|-------------|-------------|----------------|-------------|--------------------------------|-----------------|
| UA5 \( \sqrt{s} = 0.9 \text{ TeV} \) \( |\eta| < 0.5 \) | 8550.0 | 1.5574 | 0.3012 | 0.339 | 99.97 | 0.99996 | 0.375 | na | na |
| UA5 \( \sqrt{s} = 0.9 \text{ TeV} \) \( |\eta| < 0.5 \) | 10000.0 | 1.5574 | 0.3012 | 0.396 | 99.87 | 0.99996 | 0.375 | na | na |
| ALICE \( \sqrt{s} = 0.9 \text{ TeV} \) \( |\eta| < 0.5 \) | 149663.16 | 1.3764 | 0.2767 | 14.155 | 0 | 0.99960 | 1.116 | 0.576 | 3.089 |
| ALICE \( \sqrt{s} = 0.9 \text{ TeV} \) \( |\eta| < 1.0 \) | 128476.45 | 1.4316 | 0.1625 | 37.761 | 0 | 0.99865 | 1.886 | 1.034 | 11.51 |
| ALICE \( \sqrt{s} = 0.9 \text{ TeV} \) \( |\eta| < 1.3 \) | 60142.77 | 1.4955 | 0.1332 | 22.051 | 0 | 0.99876 | 2.993 | 1.671 | 15.31 |
| UA5 \( \sqrt{s} = 0.9 \text{ TeV} \) \( |\eta| < 1.5 \) | 8550.0 | 1.7987 | 0.1385 | 0.812 | 87.81 | 0.99991 | 0.487 | na | na |
| UA5 \( \sqrt{s} = 0.9 \text{ TeV} \) \( |\eta| < 1.5 \) | 10000.0 | 1.7987 | 0.1385 | 0.950 | 59.99 | 0.99991 | 0.487 | na | na |
| ALICE \( \sqrt{s} = 2.36 \text{ TeV} \) \( |\eta| < 0.5 \) | 38970.79 | 1.1778 | 0.2084 | 6.266 | 0 | 0.99930 | 0.888 | 0.501 | 3.592 |
| ALICE \( \sqrt{s} = 2.36 \text{ TeV} \) \( |\eta| < 1.0 \) | 37883.99 | 1.2139 | 0.1180 | 17.416 | 0 | 0.99726 | 2.209 | 1.312 | 17.73 |
| ALICE \( \sqrt{s} = 2.36 \text{ TeV} \) \( |\eta| < 1.3 \) | 22189.40 | 1.2123 | 0.0927 | 15.561 | 0 | 0.99644 | 4.0557 | 2.4537 | 34.40 |

For the sample described in Sect. 3 one can define the LS \( \chi^2 \) function as:
Table 2. Results of testing the NBD for the original data sets of the NSD events measured in $p - \bar{p}$ (UA5) and $p - p$ (ALICE) collisions. The ALICE numbers of events are from Ref. 9. The values of $\hat{k}$ and $\hat{\rho}$ are taken from Table 1.

$$\chi^2_{LS}(p, k) = \sum_{i=1}^{m} \frac{(P_{ex} - P(Y_i; p, k))^2}{err_i^2},$$

where $err_i$ is the uncertainty of the $i$th measurement. Here this function is not

| Experiment $\sqrt{s}$ | N      | $\hat{k}$ | $\hat{\rho}$ | $\chi^2/n_d$ | p-value [\%] | $\chi^2_{LS}/n_d$ with errors: |
|----------------------|--------|-----------|--------------|-------------|--------------|-------------------------------|
|                      |        |           |              |            |              | quadrature sum statistical |
|                      |        |           |              |            |              | sum only ~ $\sqrt{n_d}$ |
| UA5 $\sqrt{s} = 0.9$ TeV $| \eta| < 0.5$ | 8550.0  | 1.5574     | 0.3012      | 0.211       | 99.998 | 0.072 | na | na | 0.203 |
|                      | 10000.0| 1.5574    | 0.3012      | 0.247       | 99.991 | 0.072 | na | na | 0.237 |
| ALICE $\sqrt{s} = 0.9$ TeV $| \eta| < 0.5$ | 149663.16 | 1.3764   | 0.2767      | 14.498     | 0 | 0.728 | 0.381 | 2.458 | 15.107 |
|                      | 128476.45 | 1.4316 | 0.1625      | 36.855     | 0 | 1.718 | 0.948 | 11.010 | 38.017 |
|                      | 60142.77 | 1.4955 | 0.1332      | 24.323     | 0 | 2.213 | 1.767 | 15.201 | 25.771 |
| UA5 $\sqrt{s} = 0.9$ TeV $| \eta| < 1.5$ | 8550.0  | 1.7987     | 0.1385      | 1.099      | 28.94 | 0.362 | na | na | 1.14  |
|                      | 10000.0 | 1.7987    | 0.1385      | 1.286      | 8.06 | 0.362 | na | na | 1.33  |
| ALICE $\sqrt{s} = 2.36$ TeV $| \eta| < 0.5$ | 38970.79 | 1.1778   | 0.2084      | 7.030      | 0 | 0.761 | 0.428 | 3.805 | 7.465 |
|                      | 37883.99 | 1.2139 | 0.1180      | 18.535     | 0 | 2.288 | 1.362 | 18.802 | 20.282 |
|                      | 22189.40 | 1.2123 | 0.0927      | 18.233     | 0 | 4.245 | 2.599 | 39.647 | 19.980 |
minimized with respect to \( p \) and \( k \) as in the LS method but is calculated at ML estimates of \( p \) and \( k \), i.e. at \( \hat{p} \) and \( \hat{k} \). One can see from the eight and ninth columns of Table\([4]\) that \( \chi^2_{LS}/n_d \) values are significant for the ALICE narrowest pseudo-rapidity windows, what agrees with the results of Ref.\([12]\)

Since the determination of \( \hat{k} \) and \( \hat{p} \) has been done for the distributions modified in their tails, as it has been just explained, one should check what values of \( \chi^2 \) and \( \chi^2_{LS} \) are at \( \hat{k} \) and \( \hat{p} \) for the original data sets. It means that if the \( i \)th bin width is greater than 1, instead of \( P(Y_i; \hat{p}, \hat{k}) \) in Eq.\((15)\) the appropriate sum \( \sum P(n; \hat{p}, \hat{k}) \) over \( n \in \text{bin } i \) is taken. The results of the check are presented in Table\([2]\). Qualitatively the results are the same as in Table\([1]\) only slight differences in numbers can be noticed except the UA5 cases (for |\( \eta \)| < 0.5 \( \chi^2 \) has decreased more than 2 times, but the change is in the good direction). This is because the maximal width of a tail bin is 2 for all ALICE cases, but is 8 and 17 for UA5 windows |\( \eta \)| < 0.5 and |\( \eta \)| < 1.5, respectively. Of course, the assumption of the uniform distribution inside a wider bin causes greater discrepancies. Nevertheless, the results of the test for both UA5 cases are positive even if \((\hat{k}, \hat{p})\) is not the maximum of the exact likelihood function (in fact, values of \( \hat{k} \) are the same as those obtained by UA5 Collaboration in Ref.\([2]\)). This is guaranteed by the Wilks’s theorem (see Appendix B), which allows for the test of a single point in the parameter space. Then the tested point might not be the best estimate of the true value but the hypothesis in question becomes the hypothesis only about a particular distribution (a simple hypothesis). This is also the reason why \( n_d = m \) in Table\([2]\). In terms of rigorous statistics single points are tested in there.

In all ALICE cases \( \chi^2 \) values listed in the fifth column of Table\([2]\) are only slightly smaller than corresponding ones from Table\([1]\). For |\( \eta \)| < 0.5 the decrease is about 2\%, for other cases is less than 0.1\%. Also \( \chi^2/n_d \) values are much greater than 1. Therefore it is reasonable to recognize \( \hat{k} \) and \( \hat{p} \) determined for modified data sets as a good approximations of the ML estimators. Thus the hypothesis about the NBD should be rejected on the basis of obtained values of \( \chi^2/n_d \) and \( p \)-values.

One can also compare \( \chi^2/n_d \) with \( \chi^2_{LS}/n_d \) calculated for the original data sets and the same \( \hat{k} \) and \( \hat{p} \). The results are listed in four last columns of Table\([2]\) for various treatment of errors. Note that for UA5 conclusions from both statistics are exactly the same. In the ALICE both cases of the window |\( \eta \)| < 0.5, \( \chi^2_{LS}/n_d < 1 \) is acceptable for errors expressed as the quadrature sum of statistical and systematical components and is smaller than the corresponding values in Table\([1]\). In other ALICE cases \( \chi^2_{LS}/n_d \) is substantially greater than 1 for the same treatment of errors. This is in the full agreement with the results of Ref.\([12]\). One can also check what \( \chi^2_{LS}/n_d \) is if only statistical errors are taken into account. The results are listed in the next to last column of Table\([2]\). For all ALICE cases the values are much greater than 1. This means that acceptable \( \chi^2_{LS}/n_d \) was obtained only because of significant systematic errors of ALICE measurements. The word ”significant” is subjective, here means ”significant with respect to the sample size”, not to the value of \( P_{ex} \).

The crucial question is now why the conclusions from \( \chi^2 \) and \( \chi^2_{LS} \) test statistics
Table 3. Results of testing the NBD for the original data sets of the NSD events measured in $p - \bar{p}$ (UA5) and $p - p$ (ALICE) collisions at $\sqrt{s} = 0.9$ TeV. The values of $\hat{k}$ and $\hat{p}$ are taken from Table 1 but the ALICE numbers of events have been changed arbitrarily to the UA5 number of events.

| Experiment $\sqrt{s}$ | N  | $\hat{k}$   | $\hat{p}$   | $\chi^2/n_d$ | p-value [%] | $\chi^2_{LS}/n_d$ with errors: |
|-----------------------|----|-------------|-------------|--------------|-------------|---------------------------------|
|                       |    |             |             | $\chi^2$    |             |                                 |
|                       |    |             |             | $(n_d)$      |             |                                 |
| UA5 $\sqrt{s} = 0.9$ TeV | 8550.0 | 1.5574 | 0.3012 | 0.211 | 99.998 | 0.072 | na | na | 0.203 |
| $|\eta|<0.5$           |    | $\pm0.0365$ | $\pm0.005$ | 4.859 |             |                                 |
| ALICE $\sqrt{s} = 0.9$ TeV | 8550.0 | 1.3764 | 0.2767 | 0.828 | 70.37 | 0.728 | 0.381 | 2.458 | 0.863 |
| $|\eta|<0.5$           |    | $\pm0.0318$ | $\pm0.0051$ | 19.88 |             |                                 |
| ALICE $\sqrt{s} = 0.9$ TeV | 8550.0 | 1.4316 | 0.1625 | 2.453 | 5 $\cdot$ 10$^{-5}$ | 1.718 | 0.948 | 11.010 | 2.530 |
| $|\eta|<1.0$          |    | $\pm0.0272$ | $\pm0.0029$ | 103.01 |             |                                 |
| ALICE $\sqrt{s} = 0.9$ TeV | 8550.0 | 1.4955 | 0.1332 | 3.458 | 7 $\cdot$ 10$^{-13}$ | 2.213 | 1.276 | 15.201 | 3.664 |
| $|\eta|<1.3$          |    | $\pm0.0271$ | $\pm0.0024$ | 165.97 |             |                                 |
| ALICE $\sqrt{s} = 0.9$ TeV | 8550.0 | 1.7987 | 0.1385 | 1.999 | 28.94 | 0.362 | na | na | 1.14 |
| $|\eta|<1.5$          |    | $\pm0.0319$ | $\pm0.0024$ | 57.16 |             |                                 |

The values are the same for UA5 data but entirely opposite for ALICE measurements. The main difference between both statistics is that $\chi^2$ depends explicitly on the number of events but $\chi^2_{LS}$ does not. On opposite, $\chi^2$ does not depend on the actual errors but $\chi^2_{LS}$ does. In fact, $\chi^2$ statistic implicitly assumes errors of the type $\sqrt{n_i}$, what is the straightforward result of the form of the likelihood function, Eqs. (8) and (9), namely the product of Poisson distributions. This is revealed when one compare $\chi^2/n_d$ and $\chi^2_{LS}/n_d$ with errors $\sim \sqrt{n_i}$ (the fifth and last column in Table 2). The values are practically the same.

To find out what is the reason for the above-mentioned disagreement the calculations of Table 2 have been repeated for ALICE measurements at $\sqrt{s} = 0.9$ TeV but with the arbitrary assumption that all cases have the same number of events as UA5 ones. The results are listed in Table 3. One can see that now there is full agreement between $\chi^2$ and $\chi^2_{LS}$ test statistic results for all ALICE cases. This means that the accuracy with which experimental distributions approximate the NBD has not increased in ALICE data even though the sample sizes are one order greater. But the accuracy should increase with the sample size because if the hypothesis is true the postulated form of distribution is exact for the whole population. So with the growing number of events, the experimental distribution should be closer to the postulated one. This is also seen in the form of $\chi^2_{min}$, Eq. (13), where the linear
dependence on $N$ is explicit. To keep $\chi^2_{\text{min}}$ at least constant when $N$ (the sample size) is growing the relative differences between $P(Y_i)$ and $P_{\text{ex}}$ have to decrease.

5. Conclusions

The main conclusion is that the hypothesis of the NBD of charged-particle multiplicities measured by the ALICE Collaboration in proton-proton collisions at $\sqrt{s} = 0.9$ and 2.36 TeV should be rejected for all pseudo-rapidity window classes. This is the result of likelihood ratio tests performed for the corresponding data samples. The significant systematic errors are the reasons for acceptable values of the least squares test statistic for the narrowest pseudo-rapidity window measurements.

The second conclusion is that the size of "proper" errors (i.e. not too big and not too small, both extremes cause the false inference from $\chi^2_{\text{LS}}/n_d$ values) is somehow related to the sample size. Here, for instance, errors of the type $\sqrt{n_i}$ could be "a frame of reference" as it has been revealed from the results gathered in Tables 2 and 3. This is connected with the meaning of the formulation of a hypothesis. If the hypothesis is true, it means that the form of a distribution postulated by this hypothesis is exact for the whole population. Thus for the very large samples (as in all ALICE cases) the measured distribution should be very close to that postulated. The performed analysis has shown that the ALICE experimental errors are much bigger than the acceptable discrepancies (acceptable for these sample sizes). Therefore $\chi^2$ and $\chi^2_{\text{LS}}$ test statistics give the opposite answers in the narrowest pseudo-rapidity windows of the ALICE measurements. For the UA5 sample sizes, which are much smaller than the ALICE ones, the experimental errors have turned out to be of the order of acceptable discrepancies, so both test statistics give the same answer.

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Appendix A.

The sample defined in Sect. 2 can be divided into $m$ bins with the different value of measured $N_{\text{ch}}$ in each bin. Let $n_i$ be the number of events in the $i$th bin, i.e. events with the same measured value of $N_{\text{ch}}$, say $Y_i$. Then the number of events in the sample equals

$$N = \sum_{i=1}^{m} n_i .$$

(A.1)
Dividing by \( N \) one can obtain the condition for experimental probabilities (frequencies) \( P_{i}^{ex} \):

\[
1 = \sum_{i=1}^{m} \frac{n_{i}}{N} = \sum_{i=1}^{m} P_{i}^{ex} .
\] (A.2)

Now the likelihood function, Eq. (3), can be rewritten as

\[
L(X \mid p, k) = \prod_{j=1}^{N} P(X_{j}; p, k) = \prod_{i=1}^{m} P(Y_{i}; p, k)^{n_{i}}
\]

\[
= L(Y \mid p, k) = \left[ \prod_{i=1}^{m} P(Y_{i}; p, k)^{n_{i}} \right]^{N} = \left[ \prod_{i=1}^{m} P(Y_{i}; p, k)^{P_{i}^{ex}} \right]^{N},
\] (A.3)

and the corresponding log-likelihood function reads

\[
\ln L(Y \mid p, k) = N \sum_{i=1}^{m} P_{i}^{ex} \ln P(Y_{i}; p, k) .
\] (A.4)

Since the logarithm of the NBD is given by

\[
\ln P(n; p, k) = \sum_{j=1}^{n} \ln (k + j - 1) + n \ln (1 - p) + k \ln p - \ln (n!)
\] (A.5)

the necessary conditions for the existence of the maximum, Eqs. (2), have the following form:

\[
\frac{\partial}{\partial p} \ln L(Y \mid p, k)
\]

\[
= N \sum_{i=1}^{m} P_{i}^{ex} \left[ - Y_{i} \left( \frac{1}{1 - p} + \frac{k}{p} \right) \right]
\]

\[
= N \left[ - \frac{1}{1 - p} \sum_{i=1}^{m} P_{i}^{ex} Y_{i} + \frac{k}{p} \sum_{i=1}^{m} P_{i}^{ex} \right]
\]

\[
= N \left[ - \frac{1}{1 - p} (N_{ch}) + \frac{k}{p} \right] = 0 ,
\] (A.6)

\[
\frac{\partial}{\partial k} \ln L(Y \mid p, k)
\]
Multiplicities in $p-p(\bar{p})$ collisions and NBD

$$= N \sum_{i=1}^{m} P_{i}^{ex} \left[ \sum_{j=1}^{Y_{i}} \frac{1}{k+j-1} + \ln p \right]$$

$$= N \left[ \sum_{i=1}^{m} P_{i}^{ex} \sum_{j=1}^{Y_{i}} \frac{1}{k+j-1} + \ln p \right] = 0 \ , \quad (A.7)$$

where the sum over $j$ is 0 if $Y_{i} = 0$.

From Eqs. (A.6) and (2) one can obtain:

$$\langle N_{ch} \rangle = \frac{k(1-p)}{p} = \bar{n} \ . \quad (A.8)$$

Expressing $p$ as a function of $k$ and $\langle N_{ch} \rangle$

$$\frac{1}{p} = \frac{\langle N_{ch} \rangle}{k} + 1 \ , \quad (A.9)$$

and substituting it to Eq. (A.7) the equation which determines $\hat{k}$ is obtained:

$$\frac{\partial}{\partial k} \ln L(Y | p, k)$$

$$= N \left[ \sum_{i=1}^{m} P_{i}^{ex} \sum_{j=1}^{Y_{i}} \frac{1}{k+j-1} - \ln \left( 1 + \frac{\langle N_{ch} \rangle}{k} \right) \right] = 0 \ .$$

(A.10)

The above equation can be solved numerically. Having obtained $\hat{k}$ and substituting it into Eq. (A.9) $\hat{p}$ is derived.

**Appendix B. Wilks’s Theorem**

Let $X$ be a random variable with p.d.f $f(X, \theta)$, which depends on parameters $\theta = \{\theta_{1}, \theta_{2}, ..., \theta_{d}\} \in \Theta$, where a parameter space $\Theta$ is an open set in $\mathbb{R}^{d}$. For the set of $N$ independent observations of $X$, $X = \{X_{1}, X_{2}, ..., X_{N}\}$, one can defined the likelihood function

$$L(X | \theta) = \prod_{j=1}^{N} f(X_{j}; \theta) \ . \quad (B.1)$$

Now consider $H_{0}$, a $k$-dimensional subset of $\Theta$, $k < d$. Then the maximum likelihood ratio can be defined as

$$\lambda = \frac{\max_{\theta \in H_{0}} L(X | \theta)}{\max_{\theta \in \Theta} L(X | \theta)} \ . \quad (B.2)$$
This is a statistic because it does not depend on parameters $\theta$ no more, in the numerator and the denominator there are likelihood function values at the ML estimators of parameters $\theta$ with respect to sets $H_0$ and $\Theta$, respectively.

The Wilks’s theorem says that under certain regularity conditions if the hypothesis $H_0$ is true (i.e. it is true that $\theta \in H_0$), then the distribution of the statistic $-2 \ln \lambda$ converges to a $\chi^2$ distribution with $d - k$ degrees of freedom as $N \to \infty$. The proof can be found in Ref. 11. Note that $k = 0$ is possible, so one point in the parameter space (one value of the parameter) can be tested as well.

References

1. UA5 Collab. (G. J. Alner et al.), Phys. Lett. B 160, 193 (1985).
2. UA5 Collab. (R. E. Ansorge et al.), Z. Phys. C 43, 357 (1989).
3. ALICE Collab. (K. Aamodt et al.), Eur. J. Phys. C 68, 89 (2010).
4. ALICE Collab. (K. Aamodt et al.), Eur. J. Phys. C 68, 345 (2010).
5. G. Cowan, Statistical data analysis, (Oxford University Press, Oxford, 1998)
6. F. James, Statistical methods in experimental physics, (World Scientific, Singapore, 2006)
7. S. Baker and R. D. Cousins, Nucl. Instrum. Meth. 221, 437 (1984).
8. E-802 Collab. (T. Abbott et al.), Phys. Rev. C 52, 2663 (1995).
9. J. F. Grosse-Oetringhaus, private communication.
10. P. G. Hoel, Introduction to mathematical statistics, 4th edn. (Wiley, New York, 1971)
11. R. M. Dudley, 18.466 Mathematical Statistics, Spring 2003, (Massachusetts Institute of Technology: MIT OpenCourseWare), http://ocw.mit.edu/courses/mathematics/18-466-mathematical-statistics-spring-2003/lecture-notes/
12. T. Mizoguchi and M. Biyajima, Eur. Phys. J. C 70, 1061 (2010).