Determining Feynman integrals with only input from linear algebra

Zhi-Feng Liu\textsuperscript{1} and Yan-Qing Ma\textsuperscript{1,2}\textsuperscript{†}

\textsuperscript{1}School of Physics, Peking University, Beijing 100871, China
\textsuperscript{2}Center for High Energy Physics, Peking University, Beijing 100871, China

(Dated: November 4, 2022)

We find that all Feynman integrals (FIs), having any number of loops, can be completely determined once linear relations between FIs are provided. Therefore, FIs computation is conceptually changed to a linear algebraic problem. Examples up to 5 loops are given to verify this observation. As a byproduct, we get a powerful method to calculate perturbative corrections in quantum field theory.

Introduction. — Feynman integrals (FIs) encode key information of quantum field theories. Study of FIs is important both for exploring mysteries of quantum field theories and for phenomenological application of them. Integrating over some variables is found to be a necessary step to determine FIs in all known systematic methods. This seems to be a reasonable phenomenon, as FIs themselves are defined by integrating over loop momenta. However, because it is usually hard to perform integration in a systematic and efficient way, is it possible to totally bypass integration in determining FIs?

Systematic methods to compute FIs on the market can be divided into direct methods and indirect methods. Direct methods include sector decomposition\textsuperscript{[1–13]}, Mellin-Barnes representation\textsuperscript{[8–13]}, loop-tree duality\textsuperscript{[14–23]}, and so on, where one computes FIs by directly performing integration over some variables. Indirect methods compute FIs indirectly by solving corresponding equations, which include difference equations\textsuperscript{[24–27]} and differential equations\textsuperscript{[28–40]}. To uniquely determine the solution, boundary information are needed in these indirect methods. Unfortunately, the only known systematic way to obtain boundary information is to use direct methods to calculate them. Therefore, integration is still necessary in these indirect methods.

The auxiliary mass flow (AMF) method\textsuperscript{[38–40]} is a kind of differential equations method, which computes FIs by setting up and solving differential equations with respect to an auxiliary mass term $\eta$ (called $\eta$-DEs). The virtue of AMF is that its boundary conditions at $\eta \to \infty$ are simply vacuum bubble integrals, which can be more easily calculated by using other methods\textsuperscript{[41–50]}.

The observation in this Letter is following. Boundary information for AMF, which can always be casted to single-mass vacuum FIs, can be related to propagator integrals (p-integrals) with one less loops. Then, p-integrals can again be calculated by using the AMF method, with input of new boundary information having one less loops. By using this strategy iteratively, we eventually do not need any input for boundary information in the AMF framework. It is thus surprising to find that integration is totally bypassed in determining FIs.

As a result of our observation, FIs can be completely determined once linear relations between FIs are provided, which are used to decompose all FIs to a small set of bases, called master integrals (MIs), and to set up $\eta$-DEs of these MIs. We note that numerically solving ordinary differential equations (like $\eta$-DEs) is a well solved mathematical problem\textsuperscript{[51]}. Therefore, the problem of integrating over loop momenta is now conceptually changed to a linear algebraic problem of exploring the linear space of FIs.

In the rest of the Letter, we first review the AMF method and emphasize its input. We then describe our method to compute boundary conditions within the AMF framework, without any unknown information. Some examples are in order to verify this method. Finally, we propose a powerful way to calculate perturbative corrections within dimensional regularization.

Before continuing, let us first give a brief introduction to FIs. A family of FIs are defined by the following integrals with various values of $\vec{\nu}$,

\begin{equation}
I_{\vec{\nu}} = \int \left( \prod_{i=1}^{L} \int_{D^2} d^{D} \ell_{i} \right) \frac{D_{K+1}^{-\nu_{K+1}} \cdots D_{N}^{-\nu_{N}}}{D_{1}^{-\nu_{1}} \cdots D_{K}^{-\nu_{K}}},
\end{equation}

where $L$ is the number of loops, $\ell_{i}$ are loop momenta, $D$ is the dimensionality of $\ell_{i}$, $D_{1}, \ldots, D_{K}$ are inverse propagators with $\nu_{1}, \ldots, \nu_{K}$ being integers, and $D_{K+1}, \ldots, D_{N}$ are irreducible scalar products introduced for completeness with $\nu_{K+1}, \ldots, \nu_{N}$ being nonpositive integers. It was proved that a family of FIs form a finite-dimensional linear space\textsuperscript{[52]}. That is, any FI in a give family can be decomposed into a linear combination of MIs, which is a finite set of bases of the linear space formed by the family of FIs. Coefficients in this decomposition are rational functions of all natural variables, like $D$, Mandelstam variables, masses, and the $\eta$ introduced in AMF. Information of the linear space are completely encoded in these decompositions, or linear relations between FIs. Decomposition of FIs is usually realized by integration-by-parts (IBP) reduction, which have been extensively studied\textsuperscript{[24, 53–73]}. Having IBP reduction relations, we then only need to study MIs.

Furthermore, because FIs containing linear propagators can be determined by FIs containing only quadratic
nonzero contributions only come from integration regions $\sim \infty$ their MIs) are eventually determined.\[10\]

The auxiliary mass flow method. — To determine $I_\sigma$ defined in Eq. (1), in the AMF method one introduces an auxiliary family of integrals defined by

$$\tilde{I}_\sigma(\eta) = \int \left( \prod_{i=1}^{L} \frac{d^D \ell_i}{i \pi^{D/2}} \right) \frac{\tilde{D}_{K+1}^{\nu_{K+1}} \cdots \tilde{D}_{N}^{\nu_{N}}}{\tilde{D}_{1}^{\nu_{1}} \cdots \tilde{D}_{K}^{\nu_{K}}}. \tag{2}$$

Without loss of generality, we assume $\nu_1 > 0$ and $D_1 = \ell_1^2 - m^2 + i0^+$ where $m$ can be zero. We can then choose the propagator mode $[40]$ to set $\tilde{D}_1 = D_1$ for $i > 1$ and modify the mass term for $i = 1$ by

$$\tilde{D}_1 = \ell_1^2 - m^2 - \eta. \tag{3}$$

Original $I_\sigma$ can be obtained by taking $\eta \to i0^-$,

$$I_\sigma = \lim_{\eta \to i0^-} \tilde{I}_\sigma(\eta). \tag{4}$$

Let us denote MIs of the auxiliary family by $\tilde{J}_\eta$, and denote its dimension by $n$. Using IBP reduction, $\frac{\partial}{\partial \eta} \tilde{J}_\eta$ can be again expressed as linear combinations of $\tilde{J}_\eta$, which results in a system of closed $\eta$-DEs,

$$\frac{\partial}{\partial \eta} \tilde{J}_\eta = A(\eta) \tilde{J}_\eta, \tag{5}$$

where $A(\eta)$ is an $n \times n$ matrix with entries rationally depending on $\eta$. Supposing that we already have boundary conditions in hand, we can solve the $\eta$-DEs numerically [38, 51] to obtain $\tilde{J}_\eta$ and thus their limit $\tilde{J}(i0^-)$. As $I_\sigma(\eta)$ can be expressed as linear combinations of $\tilde{J}_\eta$ using IBP reduction, all original FIs $I_\sigma$ (and certainly also their MIs) are eventually determined.

An advantage of AMF is that boundary conditions at $\eta \to \infty$ can be systematically calculated. In this limit, nonzero contributions only come from integration regions where linear combinations of loop momenta are either of $\mathcal{O} (\sqrt{\eta})$ or $\mathcal{O} (1)$ [73, 76]. In each of these limited number of regions, a general propagator can be expressed as

$$\frac{1}{(\ell_L + \ell_S + p)^2 - m^2 - \kappa \eta},$$

where $\ell_L$ is the $\mathcal{O} (\sqrt{\eta})$ part of loop momenta, $\ell_S$ is the $\mathcal{O} (1)$ part of loop momenta, $p$ is a linear combination of external momenta, $m$ is the mass, and $\kappa = 0$ or 1. Then, if $\ell_L \neq 0$ or $\kappa \neq 0$, we can simplify the propagator by

$$\frac{1}{(\ell_L + \ell_S + p)^2 - m^2 - \kappa \eta} \sim \frac{1}{\ell_L^2 - \kappa \eta}. \tag{6}$$

Otherwise, the propagator is unchanged. After the above simplification, the resulted new FIs at boundary are either single-mass vacuum FIs or simpler FIs comparing with the original FIs. For the later cases, we can compute them again using AMF, which needs even simpler FIs as input for boundary conditions.

By using AMF iteratively, to determine any $L$-loop FI, we eventually only need single-mass vacuum FIs no more than $L$ loops as additional input besides IBP reductions. Diagrams of some typical single-mass vacuum FIs are shown in Fig. 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Some typical Feynman diagrams of single-mass vacuum FIs up to 5 loops, where solid lines denote massive propagators, dotted lines denote massless propagators.}
\end{figure}

Determine single-mass vacuum Feynman integrals. — Now let us assume that $I_\sigma$ defined in Eq. (1) are single-mass vacuum FIs, with $D_1 = \ell_1^2 - m^2 + i0^+$ as the only massive propagator and $\nu_1 > 0$. Without loss of generality, we set $m^2 = 1$ in the rest of this Letter.

Let us define a massless p-integral

$$\tilde{I}_\sigma(\ell_1^2) = \int \left( \prod_{i=2}^{L} \frac{d^D \ell_i}{i \pi^{D/2}} \right) \frac{D_{K+1}^{\nu_{K+1}} \cdots D_{N}^{\nu_{N}}}{D_1^{\nu_1} \cdots D_K^{\nu_K}}. \tag{7}$$

with $\nu = (\nu_2, \cdots, \nu_N)$, where $\ell_1$ presents as its “external momentum” and $\ell_1^2$ is its only mass scale. Based on dimensional counting, we have

$$\tilde{I}_\sigma(\ell_1^2) = (-\ell_1^2)^{(L-1)D/2 - \nu + \nu_1} \tilde{I}_\sigma(-1), \tag{8}$$

where $\nu = \sum_{i=1}^{N} \nu_i$. The original integral $I_\sigma$ is then factorized to two parts and can be evaluated as

$$I_\sigma = \int \frac{d^D \ell_1}{i \pi^{D/2}} (-\ell_1^2)^{(L-1)D/2 - \nu + \nu_1} \tilde{I}_\sigma(-1) = \frac{\Gamma(\nu - LD/2) \Gamma(LD/2 - \nu + \nu_1)}{(-1)^{\nu_1} \Gamma(\nu_1) \Gamma(D/2)} \tilde{I}_\sigma(-1), \tag{9}$$

which determines a $L$-loop single-mass vacuum FI $I_\sigma$ by a $(L - 1)$-loop massless p-integral $I_\sigma(-1)$. This relation is well-known.

Here comes the key observation: the $(L - 1)$-loop massless p-integral $\tilde{I}_\sigma(-1)$ can be computed via AMF discussed in the last section, which requires single-mass vacuum FIs no more than $(L - 1)$ loops as additional input.
besides IBP reductions. Therefore, we find that, with linear algebra provided by IBP reductions, single-mass vacuum FIs with \( L \) loops are determined by that with less than \( L \) loops. This works iteratively until the boundary at \( L = 1 \). Vacuum FIs with \( L = 1 \) are completely determined by the relation (9) by noticing that the value of 0-loop \( p \)-integral is simply 1.

We eventually arrive at a surprising conclusion that all single-mass vacuum FIs, and therefore all FIs, can be completely determined once linear algebraic relations between different FIs are provided. This conclusion is valid for any number of loops \( L \) and arbitrary dimensionality \( D \).

**Examples.** — To better understand the above observation, let us compute some FIs.

One of the simplest examples is the 2-loop single-mass vacuum integral shown in Fig. 1 (b), defined by

\[
I_{(1,1)} = \int \left( \prod_{i=1}^{2} \frac{d^D \ell_i}{i \pi^{D/2}} \right) \frac{1}{(\ell_1^2 - \eta)(\ell_2 + \ell_2 - \ell_1 + \ell_2)^2},
\]  

(10)

where Feynman prescription \( i0^+ \) for each denominator is suppressed. The relation (9) gives

\[
I_{(1,1)} = \frac{\Gamma(3 - D)\Gamma(D - 2)}{-\Gamma(1)\Gamma(D/2)} \tilde{I}_{(1,1)} (-1),
\]

(11)

with

\[
\tilde{I}_{(1,1)} (-1) = \int \frac{d^D \ell_2}{i \pi^{D/2} \ell_2^2} \frac{1}{(\ell_2 + p)^2},
\]

(12)

where \( p^\mu \) satisfies \( p^2 = -1 \).

To calculate the 1-loop \( p \)-integral \( \tilde{I}_{(1,1)} (-1) \) via the AMF method, we introduce auxiliary integrals

\[
\tilde{I}_{(1,0)} (\eta) = \int \frac{d^D \ell_2}{i \pi^{D/2} \ell_2^2} \frac{1}{\eta},
\]

\[
\tilde{I}_{(1,1)} (\eta) = \int \frac{d^D \ell_2}{i \pi^{D/2} (\ell_2^2 - \eta)(\ell_2^2 + \ell_2 - \eta + \ell_2)^2},
\]

(13)

(14)

which are MIs of the corresponding auxiliary family. Denoting \( \tilde{J} = (\tilde{I}_{(1,0)}, \tilde{I}_{(1,1)}) \), \( \eta \)-DEs can be obtained using IBP reductions,

\[
\frac{\partial}{\partial \eta} \tilde{J}(\eta) = \begin{pmatrix}
\frac{1 - \epsilon}{\eta^2 (1 + \eta)} & 0 \\
\frac{1 - 2 \epsilon}{\eta^2 (1 + \eta)} & \frac{1 - 2 \epsilon}{1 + \eta}
\end{pmatrix} \tilde{J}(\eta).
\]

(15)

As \( \eta \to \infty \), only the integration region \( |\ell_2| \sim {\cal O}(\sqrt{\eta}) \) gives nonzero contribution. Thus we have

\[
\tilde{I}_{(1,0)} (\eta) = \eta^{D/2 - 1} \int \frac{d^D \ell_2}{i \pi^{D/2} \ell_2^2} \frac{1}{\ell_2 - 1} = \eta^{D/2 - 1} (-1) \Gamma(1 - D/2),
\]

(16)

where in the last step the relation (9) has been used, and

\[
\tilde{I}_{(1,1)} (\eta) \eta \to \infty \int \frac{d^D \ell_2}{i \pi^{D/2} (\ell_2^2 - \eta)\ell_2^2} = \eta^{D/2 - 2} \int \frac{d^D \ell_2}{i \pi^{D/2} (\ell_2^2 - 1)\ell_2^2} = \eta^{D/2 - 2} \int \frac{d^D \ell_2}{i \pi^{D/2} \ell_2^2 - 1} = \tilde{I}_{(1,0)} (\eta) \frac{1}{\eta},
\]

(17)

where scaleless integrals are omitted in the third line.

By solving the \( \eta \)-DEs (15) together with boundary conditions at \( \eta \to \infty \) in Eqs. (10) and (17), \( \tilde{I}_{(1,1)} (-1) = \tilde{I}_{(1,1)} (i0^-) \) is determined. We thus obtain the desired FI \( I_{(1,1)} \) using the relation (11).

Clearly, the same procedure can be used to compute any FI. Let us give the result of another example shown in Fig. 1 (c), which is one of the most complicated 5-loop single-mass vacuum FIs. Following the above described procedure, we can compute all MIs in this family to very high precision, with only input of IBP reductions. The result of the corner integral with 10-digit precision is given by

\[
-2.073855510 \epsilon^{-2} - 7.812755312 \epsilon^{-1} - 17.25882864 + 717.6808845 \epsilon + 8190.876448 \epsilon^2 + 78840.29598 \epsilon^3 + 566649.1116 \epsilon^4 + 3901713.802 \epsilon^5 + 23702384.71 \epsilon^6,
\]

(18)

where we have set \( D = 4 - 2 \epsilon \) with only 9 orders in \( \epsilon \) expansion are shown, although more orders and digits can be easily obtained. The first seven terms of the expansion agree with that obtained in Ref. [18], and other terms are new.

**A new method to calculate perturbative corrections.** — An important feature of our strategy is that the FIs we calculate can have arbitrary dimensionality. This on the one hand makes our strategy applicable for a general theory, e.g., nonrelativistic theory with dimensionality equals 3. And on the other hand, by sampling different dimensionality around a fixed value, say 4 - 2 \( \epsilon \) with some small values of \( \epsilon \), we can fit the Laurent expansion with respect to \( \epsilon \) to any desired order, which is actually the way we obtain the results in Eq. (18).

If we apply the above strategy directly to physical processes, we arrive at a new and powerful method to calculate perturbative corrections. Let us explain this by an example: the next-to-next-to-leading order (NNLO) QCD correction to top-antitop quark pair fully inclusive production cross section in lepton colliders \( e^+ e^- \to \gamma^* \to t\bar{t} + X \), which have been previously calculated in Refs. [24, 25]. In our method, we calculate bare cross section (before renormalization) with a numerical value of \( \epsilon \), and then renormalize it in the standard \( \overline{\text{MS}} \) scheme, with the same value of \( \epsilon \). To show a numerical result, we choose center-of-mass energy \( s \), renormalization scale \( \mu \) and top quark mass as \( \mu = \sqrt{s} = 1 \) and \( m_t^2 = 1/8 \). We ignore contributions from internal top quark loops and that
from photon interacting with other five type of quarks, because these contributions are very small. Then, if we set $\epsilon = 0.001$, the NNLO correction gives
\[ \sigma^{\text{NNLO}}_{0.0011} / (\alpha_s^2) = 9.261823090, \tag{19} \]
where only 10 digits are shown. Because cross section is a physical quantity that is free of divergence, $\sigma^{\text{NNLO}}$ can give an estimation of total cross section up to $\mathcal{O}(\epsilon)$ error. Now let us calculate the cross section with another value $\epsilon = 0.0011$, which gives
\[ \sigma^{\text{NNLO}}_{0.0011} / (\alpha_s^2) = 9.262629688. \tag{20} \]

The fact that $\sigma^{\text{NNLO}}_{0.001}$ and $\sigma^{\text{NNLO}}_{0.0011}$ has a relative difference at $\mathcal{O}(1/1000)$ level confirms two things. First, $\sigma^{\text{NNLO}}_\epsilon$ calculated here is free of $1/\epsilon^n$ divergence, or else the difference should be at $\mathcal{O}(1)$ level. Second, $\sigma^{\text{NNLO}}_\epsilon = \sigma^{\text{NNLO}}_{\epsilon_1} + \mathcal{O}(\epsilon_1 - \epsilon_2)$ is justified. Therefore, we can fit a linear function of $\epsilon$ by combining values of $\sigma^{\text{NNLO}}_{0.001}$ and $\sigma^{\text{NNLO}}_{0.0011}$ to provide a better estimation of $\sigma^{\text{NNLO}}_\epsilon$,
\[ \sigma^{\text{NNLO}}_0 / (\alpha_s^2) \approx 9.2537 + \mathcal{O}(\epsilon^2), \tag{21} \]
which becomes closer to the exact result $9.253454354$. By calculating each one more value of $\epsilon$, we can further improve the estimation with uncertainty suppressed by one higher order in $\epsilon$.

In this method, we do not need to manipulate a Laurent expansion of $\epsilon$ during the intermediate stage of calculation, and thus the computational time can be usually reduced by several times. This improvement of efficiency is very important for cutting-edge problems. Actually, using this method we have successfully calculated the above mentioned $t\bar{t}$ production to next-to-next-to-leading order for the first time, which will be presented elsewhere [80].

Summary and outlook. — By combining the recently proposed AMF method and the relation Eq. (4), we find that all FIs, with any number of loops and arbitrary dimensionality, can be completely determined once linear relations between FIs are provided. This interesting observation conceptually changes FIs computation to an algebra problem. This observation has been explicitly verified by some examples up to 5 loops.

For phenomenological purpose, many general FIs need to be calculated. The mainstream method on the market to compute FIs can be divided into two steps. In the first step one reduces all FIs to MIS and in the second step one calculates these MIS. Both of the two steps are found to be very difficult for current cutting-edge problems. With our strategy, IBP reduction becomes the only obstacle for FIs calculation. Our strategy has been implemented in the package AMFlow [81], which can fully automatically calculate general FIs, with any number of loops, to high precision, as far as IBP reduction is successful. These features make our method unique comparing with other methods of FIs computation on the market.

Because FIs with any dimensionality can now be calculated, we can calculate physical processes directly with a given small value of $\epsilon$, the dimensional regulator. In this way, we can significantly improve the efficiency of perturbative calculation. Furthermore, our method is applicable for a general theory, like nonrelativistic theory with dimensionality equals 3.

Acknowledgments. — We thank X. Chen, X. Liu, X. Li, X. Guan and W.H. Wu for many useful communications and discussions. The work is supported in part by the National Natural Science Foundation of China (Grants No. 11875071, No. 11975029), the National Key Research and Development Program of China under Contracts No. 2020YFA0406400, and the High-performance Computing Platform of Peking University.

---

* xiangshui@pku.edu.cn
* yqma@pku.edu.cn

[1] K. Hepp, Proof of the Bogolyubov-Parasiuk theorem on renormalization, Commun. Math. Phys. 2 (1966) 301–326 [InSPIRE].
[2] M. Roth and A. Denner, High-energy approximation of one loop Feynman integrals, Nucl. Phys. B 479 (1996) 495–514 [hep-ph/9605420 InSPIRE].
[3] T. Binoth and G. Heinrich, An automated algorithm to compute infrared divergent multiloop integrals, Nucl. Phys. B585 (2000) 741–759 [hep-ph/0004013 InSPIRE].
[4] G. Heinrich, Sector Decomposition, Int. J. Mod. Phys. A23 (2008) 1457–1486 [arXiv:0803.4177 InSPIRE].
[5] A. V. Smirnov, FIESTA4: Optimized Feynman integral calculations with GPU support, Comput. Phys. Commun. 204 (2016) 189–199 [arXiv:1511.03614 InSPIRE].
[6] S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, and T. Zirke, SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop, Comput. Phys. Commun. 196 (2015) 470–491 [arXiv:1502.06595 InSPIRE].
[7] S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk, and T. Zirke, pySecDec: a toolbox for the numerical evaluation of multi-scale integrals, Comput. Phys. Commun. 222 (2018) 313–326 [arXiv:1703.09692 InSPIRE].
[8] E. E. Boos and A. I. Davydychev, A Method of evaluating massive Feynman integrals, Theor. Math. Phys. 89 (1991) 1052–1063 [InSPIRE, Teor. Mat. Fiz.89,56(1991)].
[9] V. A. Smirnov, Analytical result for dimensionally regularized massless on shell double box, Phys. Lett. B460 (1999) 397–401 [hep-ph/9905323 InSPIRE].
[10] J. B. Tausk, Nonplanar massless two loop Feynman diagrams with four on-shell legs, Phys. Lett. B 469 (1999) 225–234 [hep-ph/9909506 InSPIRE].
I. Dubovyk, A. Freitas, J. Gluza, K. Grzanka, S. Laporta, High precision calculation of multiloop Q. Song and A. Freitas, Space-time dimensionality D as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D. Nucl. Phys. B830 (2010) 474–492

[11] M. Czakon, Automated analytic continuation of Mellin-Barnes integrals, Comput. Phys. Commun. 175 (2006) 559–571 [hep-ph/0511200] InSPIRE.

[12] A. V. Smirnov and V. A. Smirnov, On the Resolution of Singularities of Multiple Mellin-Barnes Integrals, Eur. Phys. J. C 62 (2009) 445–449 [arXiv:0901.0386] InSPIRE.

[13] J. Gluza, K. Kajda, and T. Riemann, AMBRE: A Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals, Comput. Phys. Commun. 177 (2007) 879–893 [arXiv:0704.3170] InSPIRE.

[14] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo, and J.-C. Winter, From loops to trees by-passing Feynman’s theorem, JHEP 09 (2008) 065 [arXiv:0804.2423] InSPIRE.

[15] G. Rodrigo, S. Catani, T. Gleisberg, F. Krauss, and J.-C. Winter, From multileg loops to trees (by-passing Feynman’s Tree Theorem), Nucl. Phys. B Proc. Suppl. 183 (2008) 262–267 [arXiv:0807.0651] InSPIRE.

[16] I. Bierenbaum, S. Catani, P. Draggiotis, and G. Rodrigo, A Tree-Loop Duality Relation at Two Loops and Beyond, JHEP 10 (2010) 073 [arXiv:1007.0194] InSPIRE.

[17] I. Bierenbaum, S. Buchta, P. Draggiotis, I. Malamos, and G. Rodrigo, Tree-Loop Duality Relation beyond simple poles, JHEP 03 (2013) 025 [arXiv:1211.5048] InSPIRE.

[18] E. T. Tomboulis, Causality and Unitarity via the Tree-Loop Duality Relation, JHEP 05 (2017) 148 [arXiv:1701.07082] InSPIRE.

[19] R. Runkel, Z. Ször, J. P. Vesea, and S. Weinzierl, Causality and loop-tree duality at higher loops, Phys. Rev. Lett. 122 (2019) 111603 [arXiv:1902.02135] InSPIRE. [Erratum: Phys.Rev.Lett. 123, 059902 (2019)].

[20] Z. Capatti, V. Hirschi, D. Kermanschah, and B. Ruijl, Loop-Tree Duality for Multiloop Numerical Integration, Phys. Rev. Lett. 123 (2019) 151602 [arXiv:1906.06138] InSPIRE.

[21] J. J. Aguilera-Verdugo, F. Driencourt-Mangin, R. J. Hernández-Pinto, J. Plenter, S. Ramirez-Uribe, and T. Torres Bobadilla, Open Loop Amplitudes and Causality to All Orders and Powers from the Tree-Loop Duality, Phys. Rev. Lett. 124 (2020) 211602 [arXiv:2001.03564] InSPIRE.

[22] Q. Song and A. Freitas, On the evaluation of two-loop electroweak box diagrams for e+e− → HZ production, JHEP 04 (2021) 179 [arXiv:2101.00308] InSPIRE.

[23] T. Dubovyk, A. Freitas, J. Gluza, K. Grzanka, M. Hidding, and J. Usovitsch, Evaluation of multiloop multi-scale Feynman integrals for precision physics, arXiv:2201.02576 InSPIRE.

[24] S. Laporta, High precision calculation of multiloop Feynman integrals by difference equations, Int. J. Mod. Phys. A 15 (2000) 5087–5159 [hep-ph/0004033] InSPIRE.

[25] R. N. Lee, Space-time dimensionality D as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D, Nucl. Phys. B830 (2010) 474–492 [arXiv:0911.0252] InSPIRE.

[26] R. N. Lee and V. A. Smirnov, The Dimensional Recurrence and Analyticity Method for Multicomponent Master Integrals: Using Unitarity Cuts to Construct Homogeneous Solutions, JHEP 12 (2012) 104 [arXiv:1209.0339] InSPIRE.

[27] R. N. Lee and K. T. Mingulov, Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method, Comput. Phys. Commun. 203 (2016) 255–267 [arXiv:1507.04256] InSPIRE.

[28] A. V. Kotikov, Differential equations method: New technique for massive Feynman diagrams calculation, Phys. Lett. B254 (1991) 158–164 [InSPIRE].

[29] A. V. Kotikov, Differential equation method: The Calculation of N point Feynman diagrams, Phys. Lett. B 267 (1991) 123–127 [InSPIRE]. [Erratum: Phys.Lett.B 295, 409–409 (1992)].

[30] E. Remiddi, Differential equations for Feynman graph amplitudes, Nuovo Cim. A110 (1997) 1435–1452 [hep-th/9711188] InSPIRE.

[31] T. Gehrmann and E. Remiddi, Differential equations for two loop four point functions, Nucl. Phys. B580 (2000) 485–518 [hep-th/9912329] InSPIRE.

[32] M. Argeri and P. Mastrolia, Feynman Diagrams and Differential Equations, Int. J. Mod. Phys. A 22 (2007) 4375–4436 [arXiv:0707.4037] InSPIRE.

[33] S. Müller-Stach, S. Weinzierl, and R. Zayadeh, Picard-Fuchs equations for Feynman integrals, Commun. Math. Phys. 326 (2014) 237–249 [arXiv:1212.4389] InSPIRE.

[34] J. M. Henn, Multiloop integrals in dimensional regularization made simple, Phys. Rev. Lett. 110 (2013) 251601 [arXiv:1304.1806] InSPIRE.

[35] J. M. Henn, Lectures on differential equations for Feynman integrals, J. Phys. A 48 (2015) 153001 [arXiv:1412.2296] InSPIRE.

[36] F. Moriello, Generalised power series expansions for the elliptic planar families of Higgs + jet production at two loops, JHEP 01 (2020) 150 [arXiv:1907.13234] InSPIRE.

[37] M. Hidding, DiffEzr, a Mathematica package for computing Feynman integrals in terms of one-dimensional series expansions, Comput. Phys. Commun. 269 (2021) 108125 [arXiv:2006.05510] InSPIRE.

[38] X. Liu, Y.-Q. Ma, and C.-Y. Wang, A Systematic and Efficient Method to Compute Multi-loop Master Integrals, Phys. Lett. B779 (2018) 353–357 [arXiv:1711.09572] InSPIRE.

[39] X. Liu, Y.-Q. Ma, W. Tao, and P. Zhang, Calculation of Feynman loop integration and phase-space integration via auxiliary mass flow, Chin. Phys. C 45 (2021) 013115 [arXiv:2009.07987] InSPIRE.

[40] X. Liu and Y.-Q. Ma, Multiloop corrections for collider processes using auxiliary mass flow, Phys. Rev. D 105 (2022) 015003 [arXiv:2107.01864] InSPIRE.

[41] A. I. Davydychev and J. B. Tausk, Two loop selfenergy diagrams with different masses and the momentum
expansion, *Nucl. Phys. B397* (1993) 123–142 [InSPIRE].

[42] D. J. Broadhurst, Massive three - loop Feynman diagrams reducible to SC** primitives of algebras of the sixth root of unity, *Eur. Phys. J. C8* (1999) 311–333

[43] Y. Schröder and A. Vuorinen, High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles, *JHEP* 06 (2005) 051 [hep-ph/0503209 InSPIRE].

[44] T. Lütke, Fully massive vacuum integrals at 5 loops, PhD thesis, Bielefeld U., 2015 [InSPIRE].

B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, Three-loop massive tadpoles and polylogarithms through weight six, *JHEP* 08 (2017) 024 [arXiv:1705.05136 InSPIRE].

[45] T. Lütke, A. Maier, P. Marquard, and Y. Schroeder, Complete renormalization of QCD at five loops, *JHEP* 03 (2017) 020 [arXiv:1701.07068 InSPIRE].

[46] P. A. Baikov and K. G. Chetyrkin, Four Loop Massless Propagators: An Algebraic Evaluation of All Master Integrals, *Nucl. Phys. B837* (2010) 186–220 [arXiv:1004.1153 InSPIRE].

[47] R. N. Lee, A. V. Smirnov, and V. A. Smirnov, Master Integrals for Four-Loop Massless Propagators up to Transcendental Four-Weight Twelve, *Nucl. Phys. B856* (2012) 95–110 [arXiv:1108.0732 InSPIRE].

[48] A. Georgoudis, V. Goncalves, E. Panzer, and R. Pereira, Five-loop massless propagator integrals, [arXiv:1802.00803 InSPIRE].

[49] A. Georgoudis, V. Goncalves, E. Panzer, R. Pereira, A. V. Smirnov, and V. A. Smirnov, Glue-and-cut at five loops, *JHEP* 09 (2021) 098 [arXiv:2104.08272 InSPIRE].

W. Wason, Asymptotic Expansions for Ordinary Differential Equations, Dover Publications, Inc., 1987

[50] A. V. Smirnov and A. V. Petukhov, The Number of Master Integrals is Finite, *Lett. Math. Phys. 97* (2011) 37–44 [arXiv:1004.4199 InSPIRE].

[51] K. G. Chetyrkin and F. V. Tkachov, Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops, *Nucl. Phys. B319* (1989) 159–204 [InSPIRE].

[52] J. Gluza, K. Kajda, and D. A. Kosower, Towards a Basis for Planar Two-Loop Integrals, *Phys. Rev. D83* (2011) 045012 [arXiv:1009.0472 InSPIRE].

[53] R. M. Schabinger, A New Algorithm For The Generation Of Unitarity-Compatible Integration By Parts Relations, *JHEP* 01 (2012) 077 [arXiv:1111.4220 InSPIRE].

[54] A. von Manteuffel and C. Studerus, Reduze 2 - Distributed Feynman Integral Reduction, [arXiv:1201.4330 InSPIRE].

[55] R. N. Lee, LiteRed 1.4: a powerful tool for reduction of multiloop integrals, *J. Phys. Conf. Ser. 523* (2014) 012059 [arXiv:1310.1145 InSPIRE].

[56] A. von Manteuffel and R. M. Schabinger, A novel approach to integration by parts reduction, *Phys. Lett. B744* (2015) 101–104 [arXiv:1406.4513 InSPIRE].

[57] K. J. Larsen and Y. Zhang, Integration-by-parts reductions from unitarity cuts and algebraic geometry, *Phys. Rev. D93* (2016) 041701 [arXiv:1511.01071 InSPIRE].

[58] T. Peraro, Scattering amplitudes over finite fields and multivariate functional reconstruction, *JHEP* 12 (2016) 030 [arXiv:1608.01902 InSPIRE].

[59] P. Mastrolo and S. Mizera, Feynman Integrals and Intersection Theory, *JHEP* 02 (2019) 139 [arXiv:1810.03818 InSPIRE].

[60] X. Liu and Y.-Q. Ma, Determining arbitrary Feynman integrals by vacuum integrals, *Phys. Rev. D99* (2019) 071501 [arXiv:1801.10523 InSPIRE].

[61] X. Guan, X. Liu, and Y.-Q. Ma, Complete reduction of integrals in two-loop five-light-parton scattering amplitudes, *Phys. Rev. B192* (2011) 101–104 [arXiv:1912.09294 InSPIRE].

[62] J. Klappert and F. Lange, Reconstructing rational functions with FireFly, *Comput. Phys. Commun. 247* (2020) 106051 [arXiv:1904.00099 InSPIRE].

[63] T. Peraro, FiniteFlow: multivariate functional reconstruction using finite fields and dataflow graphs, *JHEP* 07 (2019) 031 [arXiv:1905.08019 InSPIRE].

[64] H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolo, L. Mattiazzi, and S. Mizera, Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers, *JHEP* 05 (2019) 153 [arXiv:1901.11510 InSPIRE].

[65] Y. Wang, Z. Li, and N. Ul Basat, Direct reduction of multiloop multiscale scattering amplitudes, *Phys. Rev. D 101* (2020) 076023 [arXiv:1901.09390 InSPIRE].

[66] A. V. Smirnov and F. S. Chuharev, FIRE6: Feynman Integral REduction with Modular Arithmetic, *Comput. Phys. Commun. 247* (2020) 106877 [arXiv:1901.07808 InSPIRE].

[67] J. Klappert, F. Lange, P. Maierhöfer, and J. Usoschtsch, Integral reduction with Kira 2.0 and finite field methods, *Comput. Phys. Commun. 266* (2021) 108024 [arXiv:2008.06494 InSPIRE].

[68] J. Boehm, M. Wittmann, Z. Wu, Y. Xu, and Y. Zhang, IBP reduction coefficients made simple, *JHEP* 12 (2020) 054 [arXiv:2008.13194 InSPIRE].

[69] N. u Basat, Z. Li, and Y. Wang, Reduction of the planar double-box diagram for single-top production via auxiliary mass flow, *Phys. Rev. D 104* (2021) 056020 [arXiv:2102.08225 InSPIRE].

[70] M. Heller and A. von Manteuffel, MultivariateApart: Generalized partial fractions, *Comput. Phys. Commun. 271* (2022) 108174 [arXiv:2101.08283 InSPIRE].

[71] D. Bendle, J. Boehm, M. Heymann, R. Ma, M. Rahn, L. Ristau, M. Wittmann, Z. Wu, and Y. Zhang, Two-loop five-point integration-by-parts relations in a usable form, [arXiv:2104.06866 InSPIRE].

[72] Z.-F. Liu and Y.-Q. Ma, Automatic computation of Feynman integrals containing linear propagators via auxiliary mass flow, *Phys. Rev. D 105* (2022) 074003 [arXiv:2201.11636 InSPIRE].

[73] M. Beneke and V. A. Smirnov, Asymptotic expansion of Feynman integrals near threshold,
[76] V. A. Smirnov, Problems of the strategy of regions, *Phys. Lett. B* **465** (1999) 226–234 [hep-ph/9907471](http://arxiv.org/abs/hep-ph/9907471) [InSPIRE].

[77] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser, Three loop polarization function and O($\alpha_s^3$) corrections to the production of heavy quarks, *Nucl. Phys. B* **482** (1996) 213–240 [hep-ph/9606230](http://arxiv.org/abs/hep-ph/9606230) [InSPIRE].

[78] J. Gao and H. X. Zhu, Top Quark Forward-Backward Asymmetry in $e^+e^-$ Annihilation at Next-to-Next-to-Leading Order in QCD, *Phys. Rev. Lett.* **113** (2014) 262001 [arXiv:1410.3165](http://arxiv.org/abs/1410.3165) [InSPIRE].

[79] L. Chen, O. Dekkers, D. Heisler, W. Bernreuther, and Z.-G. Si, Top-quark pair production at next-to-next-to-leading order QCD in electron positron collisions, *JHEP* **12** (2016) 098 [arXiv:1610.07897](http://arxiv.org/abs/1610.07897) [InSPIRE].

[80] X. Chen, X. Guan, C.-Q. He, X. Liu, and Y.-Q. Ma, Heavy-quark-pair production at lepton colliders at NNNLO in QCD, [arXiv:2209.14259](http://arxiv.org/abs/2209.14259) [InSPIRE].

[81] X. Liu and Y.-Q. Ma, AMFlow: a Mathematica Package for Feynman integrals computation via Auxiliary Mass Flow, [arXiv:2201.11669](http://arxiv.org/abs/2201.11669) [InSPIRE].