Reinforcement Learning with Trajectory Feedback

Yonathan Efroni*, Nadav Merlis* and Shie Mannor
Technion Institute of Technology, Israel

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Abstract

The computational model of reinforcement learning is based upon the ability to query a score of every visited state-action pair, i.e., to observe a per state-action reward signal. However, in practice, it is often the case such a score is not readily available to the algorithm designer. In this work, we relax this assumption and require a weaker form of feedback, which we refer to as trajectory feedback. Instead of observing the reward from every visited state-action pair, we assume we only receive a score that represents the quality of the whole trajectory observed by the agent. We study natural extensions of reinforcement learning algorithms to this setting, based on least-squares estimation of the unknown reward, for both the known and unknown transition model cases, and study the performance of these algorithms by analyzing the regret. For cases where the transition model is unknown, we offer a hybrid optimistic-Thompson Sampling approach that results in a computationally efficient algorithm.

1 Introduction

The field of Reinforcement Learning (RL) tackles the problem of learning how to act optimally in an unknown dynamical environment. Recently, RL witnessed remarkable empirical success (e.g., Mnih et al., 2015, Levine et al., 2016, Silver et al., 2017). However, there are still some matters that hinder its use in practice. One of them, we claim, is the type of feedback an RL agent is assumed to observe. Specifically, in the standard RL formulation, an agent acts in an unknown environment and receives feedback on its actions in the form of a state-action dependent reward signal. Although such an interaction model seems undemanding at first sight, in many interesting problems, such reward feedback cannot be realized. In practice, and specifically in non-simulated environments, it is hardly ever the case an algorithm can query a state-action reward function from every visited state-action pair since such a query can be very costly.

Motivating example: Consider the important challenge of autonomous car driving. Would we want to deploy an RL algorithm for this setting, we would need a reward signal from every visited state-action pair. Obtaining such data is expected to be very costly since it requires scoring each state-action pair with a real number. For example, if a human is involved in giving the feedback, he or she might refuse to supply with such a feedback due to the Sisyphean nature of this task.

Rather than circumventing this problem by deploying heuristics (e.g., by hand-engineering a reward signal), in this work, we relax the feedback mechanism to a more practical one and study RL algorithms in the relaxed setting. Specifically, we study a setting we refer to as RL with trajectory feedback. In RL with trajectory feedback, the agent does not have access to a state-action reward. Instead, it receives the sum of the rewards on the visited trajectory as well as the identity of visited state-action pairs in the trajectory. E.g., for autonomous car driving, we only require feedback on the score of a trajectory, instead of the score of each individual state-action pair. Indeed, this form of feedback is much weaker than the standard RL feedback and is expected to be more common in practical scenarios.

We start by defining our setting and specifying the interaction model of RL with trajectory feedback (Section 2). In Section 3, we introduce a natural least-squares estimator with which the true reward function can be learned based on the trajectory feedback. Building on the least-squares estimator, we study

*Equal contribution, alphabetical order
algorithms that explicitly trade-off exploration and exploitation. We start by considering the case the model is known, however, the reward function needs to be learned. By generalizing the analysis of standard linear bandit algorithms (OFUL [Abbasi-Yadkori et al., 2011] and Thompson-Sampling (TS) for linear bandits [Agrawal and Goyal 2013], we establish performance guarantees for this setup in sections 3 and 5.1. Although the OFUL-based algorithm gives better performance than the TS-based algorithm, its update rule is computationally intractable as it requires solving a convex maximization problem. Thus, in Section 5.2 we generalize the TS-based algorithm to the case both the reward and the transition model are unknown. To this end, we learn the reward by a TS approach and learn the transition model by an optimistic approach. The combination of the two approaches yields a computationally efficient algorithm, which requires solving an empirical MDP in each round. We establish a regret guarantee which scales as $\sqrt{K}$ where $K$ is the number of episodes.

2 Notations and Definitions

We consider finite-horizon MDPs with time-independent dynamics. A finite-horizon MDP is defined by the tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, R, P, H)$, where $\mathcal{S}$ and $\mathcal{A}$ are the state and action spaces with cardinalities $S$ and $A$, respectively. The immediate reward for taking an action $a$ at state $s$ is a random variable $R(s, a) \in [0, 1]$ with expectation $\mathbb{E}[R(s, a)] = r(s, a)$. The transition probability is $P(s' | s, a)$, the probability of transitioning to state $s'$ upon taking action $a$ at state $s$. The initial state in each episode is arbitrarily chosen and $H \in \mathbb{N}$ is the horizon, i.e., the number of time-steps in each episode. We define $[N] \triangleq \{1, \ldots, N\}$, for all $N \in \mathbb{N}$, and throughout the paper use $h \in [H]$ and $k \in [K]$ to denote time-step inside an episode and the index of an episode, respectively.

A deterministic policy $\pi : \mathcal{S} \times [H] \rightarrow \mathcal{A}$ is a mapping from states and time-step indices to actions. We denote by $a_h \triangleq \pi(s_h, h)$, the action taken at time $h$ at state $s_h$ according to a policy $\pi$. The quality of a policy $\pi$ from state $s$ at time $h$ is measured by its value function, which is defined as

$$V_h^\pi(s) \triangleq \mathbb{E} \left[ \sum_{h'=h}^H r(s_{h'}, \pi(s_{h'}, h')) \mid s_h = s \right],$$

where the expectation is over the environment randomness. An optimal policy maximizes this value for all states $s$ and time-steps $h$ simultaneously, and the corresponding optimal value is denoted by $V_h^*(s) \triangleq \max_{\pi} V_h^\pi(s)$, for all $h \in [H]$. We can also reformulate the optimization problem by using the occupancy measure [e.g., Puterman 1994; Altman 1999]. The occupancy measure $q^\pi$ of a policy $\pi$ is defined as the distribution over state-actions generated by executing the policy $\pi$ in the finite-horizon MDP $\mathcal{M}$ with a transition kernel $p$ [e.g., Zimin and Neu, 2013]:

$$q_h^\pi(s, a; p) \triangleq \mathbb{E}\left[\mathbb{I}(s_h = s, a_h = a) \mid s_1 = s_1, p, \pi\right] = \Pr\{s_h = s, a_h = a \mid s_1 = s_1, p, \pi\}.$$

For ease of notation, we define the matrix notation $q^\pi(p) \in \mathbb{R}^{HSA}$ where its $(s, a, h)$ element is given by $q_h^\pi(s, a; p)$. Furthermore, let the average occupancy measure be $d_\pi(p) \in \mathbb{R}^{SA}$ such that $d_\pi(s, a; p) \triangleq \sum_{h=1}^H q_h^\pi(s, a; p)$. For ease of notation, when working with the transition kernel of the ‘real’ model $p = P$, we write $q^\pi = q^\pi(P)$ and $d_\pi = d_\pi(P)$.

This definition implies the following relation:

$$V_1^\pi(s_1; p, r) = \sum_{s, a} r(s, a)q_1^\pi(s, a; p) = \sum_{s, a} d_\pi(s, a; p)r(s, a) = d_\pi(p)^T r,$$

where $V_1^\pi(s_1; p, r)$ is the value of an MDP whose reward function is $r$ and its transition kernel is $p$.

Interaction Model of Reinforcement Learning with Trajectory Feedback. We not define the interaction model of an RL agent which receives a trajectory feedback, the model which we analyze in this work. We consider an agent that repeatedly interacts with an MDP in a sequence of episodes $[K]$. The performance of the agent is measured by its regret, defined as $\text{Regret}(K) \triangleq \sum_{k=1}^K (V_1^*(s_k) - V_1^\pi_k(s_k))$. We denote by $s_h^k$ and $a_h^k$ for the state and the action taken at the $h^{th}$ time-step of the $k^{th}$ episode. At each
episode $k \in [K]$, the agent only observes the cumulative reward experienced while following its policy $\pi_k$ and the identity of the visited state-action pairs, i.e.,

$$\hat{V}_k(s^k_1) = \sum_{h=1}^H R(s^k_h, a^k_h), \text{ and, } \{(s^k_h, a^k_h)\}_{h=1}^H.$$  

This comes in contrast to the standard RL setting, in which the agent observes the reward per visited state-action pair, $\{R(s^k_h, a^k_h)\}_{h=1}^H$. Thus, RL with trajectory feedback receives weaker feedback from the environment on the quality of its actions. Obviously, a standard RL agent can calculate by the feedback it received $\hat{V}_k(s^k_1)$, but one cannot generally reconstruct $\{(R(s^k_h, a^k_h))_{h=1}^H$ by accessing only $\hat{V}_k(s^k_1)$.

Next, we define the filtration $F_k$ that includes all events (states, actions, and rewards) until the end of the $k^{th}$ episode, as well as the initial state of the episode $k + 1$. We denote by $T = KH$, the total number of time-steps (samples). Moreover, we denote by $n_k(s, a)$, the number of times that the agent has visited a state-action pair $(s, a)$, and by $\hat{X}_k$, the empirical average of a random variable $X$. Both quantities are based on experience gathered until the end of the $k^{th}$ episode and are $F_k$ measurable. Furthermore, throughout the paper, the policy $\pi_k$ is only determined by past experience; therefore, it is $F_{k-1}$-measurable.

\section*{Notations.}

We use $\tilde{O}(X)$ to refer to a quantity that depends on $X$ up to poly-log expression of a quantity at most polynomial in $S, A, T, K, H$, and $\frac{1}{\lambda}$. Furthermore, the notation $O(X)$ refers to a quantity that depends on $X$ up to constant multiplicative factor. We use $X \vee Y \overset{\text{def}}{=} \max\{X, Y\}$, and denote $I_n$ as the identity matrix in dimension $n$. Finally, for any positive definite matrix $M \in \mathbb{R}^{n \times n}$ and any vector $x \in \mathbb{R}^n$, we define $\|x\|_M = \sqrt{x^T M x}$.

\section{From Trajectory Feedback to Least-Squares Estimation}

In this section, we examine an intuitive way for estimating the true reward function $r$, given only the cumulative rewards on each of the past trajectories and the identities of visited state-actions. Specifically, we estimate $r$ via a Least-Squares (LS) estimation. Consider past data in the form of (2). To make the connection of this form of feedback to LS estimation more apparent, let us rewrite (2) as follows,

$$\hat{V}_k(s^k_1) = \hat{q}_k^T R,$$

where $\hat{q}_k \in \mathbb{R}^{SAH}$ is the empirical state-action visitation vector given by $\hat{q}_k(s, a, h) \overset{\text{def}}{=} \mathbb{1}(s = s^k_h, a = a^k_h) \in [0, 1]$, and $R \in \mathbb{R}^{SAH}$ is the noisy version of the true reward function, namely $R(s, a, h) = R(s_h, a_h)$. Indeed, since the identity of visited state-action pairs is given to us, we can compute $\hat{q}_k$ using our data. Furthermore, observe that

$$\mathbb{E}[\hat{q}_k^T R|\hat{q}_k] = \sum_{s, a, h} \hat{q}_k(s, a, h)r(s, a) = \sum_{s, a} \left(\sum_{h=1}^H \hat{q}_k(s, a, h)\right) r(s, a) \overset{\text{def}}{=} \hat{d}_k^T r,$$

where the first equality holds since we assume the rewards are i.i.d. and drawn in the beginning of each episode and in the last inequality we defined the empirical state-action frequency vector $\hat{d}_k(s, a) = \sum_{h=1}^H \hat{q}_k(s, a, h)$. Alternatively, we can write $\hat{d}_k(s, a) = \sum_{h=1}^H \mathbb{1}(s = s^k_h, a = a^k_h) \in [0, H]$. This observation makes it apparent that we can think of our data as noisy samples of $\hat{d}_k^T r$, from which it is natural to estimate the reward $r$ by a (regularized) LS estimator, i.e., $\hat{r}_k \in \arg \min_r \left\{ \sum_{l=1}^k ((\hat{d}^l, r) - \hat{V}_l)^2 + \lambda I_{SA} \right\}$, for some $\lambda > 0$. This estimator can also be given in the following closed form solution,

$$\hat{r}_k = (D_k^T D_k + \lambda I_{SA})^{-1} Y_k,$$

where $D_k \in \mathbb{R}^{k \times SA}$ is a matrix with $\{\hat{d}_k^T\}$ in its rows, $Y_k = \sum_{s=1}^k \hat{d}_s \hat{V}_s$ and, thus, $Y_k \in \mathbb{R}^{SA}$.
Algorithm 1 OFUL for RL with Trajectory Feedback and Known Model

Require: $\delta \in (0, 1)$, $\lambda = H$, $l_k = \sqrt{\frac{1}{4}SAH \log \left( \frac{1 + kH^2/\lambda}{\delta/10} \right)} + \sqrt{\lambda SA}$

Initialize: $A_0 = \lambda I_{SA}$

for $k = 1, \ldots, K$ do
  Calculate $\hat{r}_{k-1}$ via LS estimation \[4\]
  Solve $\pi_k \in \arg \max_{\pi} \left( d_T^T \hat{r}_{k-1} + l_{k-1} \|d_\pi\|_{A_{k-1}}^{-1} \right)$
  Interact with the true MDP by $\pi_k$, receive $V_k$ and $\{(s_h^k, a_h^k)\}_{h=1}^H$
  Update $A_k = A_{k-1} + \hat{d}_k d_T^T$ and $Y_k = Y_{k-1} + \hat{d}_k V_k$
end for

A needed property of the estimator $\hat{r}_k$ is for it to be ‘concentrated’ around the true reward $r$. By properly defining the filtration, and observing that $\hat{V}_k = \sqrt{H/4}$ sub-Gaussian given $\hat{d}_k$ (as a sum of $H$ independent variables in $[0, 1]$), it is easy to establish a uniform concentration bound via Theorem 2 of \[Abbasi-Yadkori et al., 2011\] (for completeness, we provide the proof in Appendix \[H\]).

Proposition 1 (Concentration of Reward). Let $A_k \overset{\text{def}}{=} D_T^T D_k + \lambda I_{SA}$ for some $\lambda > 0$. For any $\delta \in (0, 1)$, with probability greater than $1 - \delta/10$ uniformly for all $k \geq 0$, it holds that

$$
\|r - \hat{r}_k\|_{A_k} \leq \sqrt{\frac{1}{4}SAH \log \left( \frac{1 + kH^2/\lambda}{\delta/10} \right)} + \sqrt{\lambda SA} \triangleq l_k.
$$

Relation to Linear Bandits. Assume that the transition kernel $P$ is known while the reward $r$ is unknown. Equation \[H\] together with the fact the set of the average occupancy measures is a convex set \[Altman, 1999\] establishes that RL with trajectory feedback can be understood as an instance of linear-bandits. I.e., it is equivalent to the problem of minimizing the regret $\text{Regret}(K) = \sum_k \max_{d \in \mathcal{K}(P)} d^T r - d_{\pi_k}^T r$ when the feedback is a noisy version of $d_{\pi_k}^T r$ since $\mathbb{E}[\hat{V}_k \mid F_{k-1}] = d_{\pi_k}^T r$.

However, in this and following sections, we make use of $\hat{d}_k$, and not the actual ‘action’ that was taken, $d_{\pi_k}$. First, in RL, this view is much more computationally efficient, since $d_{\pi_k} \in \mathbb{R}^{SA}$ is not necessarily a sparse vector, whereas, $\hat{d}_k$ has at most $H$ non-zero entries. Thus, the update of $A_k$ and $A_{k-1}^{-1}$ can be done more efficiently when using $\hat{d}_k$ instead of $d_{\pi_k}$, e.g., by Sherman-Morrison formula \[Bartlett, 1951\].

Second, and much more importantly, this view allows us to generalize (in Section \[5\]) the algorithm to the case where the transition model $P$ is unknown. When the transition model is not known and estimated via $P$, there is an error in identifying the action, in the view of linear bandits, since $d_{\pi}(P) \neq d_{\pi}(\hat{P})$. This error, had we used the ‘naive’ linear bandit approach, would result in errors in the matrix $A_k$. Since our estimator uses the empirical average state-action frequency, $\hat{d}_k$, the fact the model is unknown does not distort the reward estimation.

4 OFUL for RL with Trajectory Feedback and Known Model

Given the concentration of the estimated reward in Proposition \[H\], it is natural to follow the optimism in the face of uncertainty approach, as used in the OFUL algorithm \[Abbasi-Yadkori et al., 2011\] for linear bandits. We adapt this approach to the RL with trajectory feedback as depicted in Algorithm \[I\]. On each episode, we need to find a policy that maximizes the value $V^\pi(s_1;P,\hat{r}_{k-1}) = d_{\pi}^T \hat{r}_{k-1}$, combined with a ‘confidence’ term $\|d_\pi\|_{A_{k-1}^{-1}}$ that properly encourages the policy $\pi_k$ to be exploratory.

The next theorem establishes the performance guarantee of Algorithm \[I\]

Theorem 2 (OFUL for RL with Trajectory Feedback and Known Model). For any $\delta \in (0, 1)$, it holds with probability greater than $1 - \delta$ that for all $K > 0$,

$$
\text{Regret}(K) \leq O \left( SAH \sqrt{K} \log \left( \frac{KH}{\delta} \right) \right).
$$

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$$
\text{Regret}(K) \leq O \left( SAH \sqrt{K} \log \left( \frac{KH}{\delta} \right) \right).
$$
Algorithm 2 TS for RL with Trajectory Feedback and Known Model

Require: $\delta \in (0, 1)$, $\lambda = H$, $l_k = \frac{1}{4}SAH \log \left( \frac{1+kH^2/\lambda}{\delta/10} \right) + \sqrt{\lambda S \delta} / 2$, $v_k = \sqrt{9SAH} \log \frac{kH^2}{\delta/10}$

Initialize: $A_0 = \lambda I_{S,A}$.

for $k = 1, ..., K$ do

Calculate $\hat{r}_{k-1}$ via LS estimation

Draw noise $\xi_k \sim \mathcal{N}(0, v_k^2 A_{k-1}^{-1})$ and define $\hat{r}_k = \hat{r}_{k-1} + \xi_k$

Solve an MDP with perturbed empirical reward $\pi_k \in \arg \max_\pi d(P)^T \hat{r}_k$

Interact with the true MDP by $\pi_k$, receive $\hat{V}_k$ and $(s_k^k, d_k^k)_{h=1}^H$

Update $A_k = A_{k-1} + \hat{d}_k \hat{d}_k^T$ and $Y_k = Y_{k-1} + \hat{d}_k \hat{V}_k$.

end for

The proof follows the techniques in [Abbasi-Yadkori et al., 2011] while explicitly dealing with the fact the ‘action’ that the agent takes at the $k$th episode, $d_{\pi_k}$, is not the action used in the LS estimator, $\hat{d}_k$. We give a brief proof sketch that highlights how to deal with this discrepancy relatively to previous analysis (see Appendix C for the full proof).

Proof Sketch. By the optimism of the update rule, following [Abbasi-Yadkori et al., 2011], it is possible to show that $V_1^*(s_k^k) \leq d_k^T \hat{r}_{k-1} + l_{k-1} \|d_{\pi_k}\|_{A_{k-1}^{-1}}$, for any $k > 0$. Thus, we only need to bound the on-policy prediction error given as follows,

$$\text{Regret}(K) = \sum_{k=1}^K (V_1^* - V_{1,1}^* \leq \sum_{k=1}^K (d_{\pi_k}(P)^T \hat{r}_{k-1} + l_{k-1} \|d_{\pi_k}(P)\|_{A_{k-1}^{-1}} - V_1^*) \leq 2l_K \sum_{k=1}^K \|d_{\pi_k}(P)\|_{A_{k-1}^{-1}},$$

where the last inequality can be derived using Proposition 1. At this point, we would like to apply the elliptical potential lemma [Abbasi-Yadkori et al., 2011], by which $\sum_{k=1}^K \|x_k\|_{A_{k-1}^{-1}} \leq \tilde{O}(\sqrt{AK})$ when $A_{k-1} = \sum_{s=1}^{k-1} x_s^x x_s^T + \lambda I_d$ and $x_s \in \mathbb{R}^d$. To do so, observe that $d_{\pi_k} = \mathbb{E}[d_k|F_{k-1}]$. Thus, applying Jensen’s inequality, which is valid by the convexity of norm, we get

$$\mathbb{E}[\|d_k\|_{A_{k-1}^{-1}} | F_{k-1}] \leq 2l_K \sum_{k=1}^K \mathbb{E}[\|d_k\|_{A_{k-1}^{-1}} | F_{k-1}] = 2l_K \sum_{k=1}^K \|\delta_k\|_{A_{k-1}^{-1}} + 2l_K \sum_{k=1}^K \Delta_k(d_k),$$

where $\Delta_k(d_k) = \mathbb{E}[\|\delta_k\|_{A_{k-1}^{-1}} | F_{k-1}] - \|\delta_k\|_{A_{k-1}^{-1}}$ is a martingale difference noise. Also, notice that $\|\delta_k\|_{A_{k-1}^{-1}} \in [0, H/\sqrt{\lambda}]$ by norm equivalences and the form of $A_{k-1}$ and $\delta_k \in \mathbb{R}^{SA}$. We can now apply the elliptical potential lemma and bound the first term of (5) by $\tilde{O}(l_K \sqrt{SAK})$. The second term of (6) is a martingale difference sequence, thus it can be bounded by applying Azuma-Hoeffding’s inequality by $\tilde{O}(l_K \sqrt{H^2SAK})$ with high probability. Combining all the above concludes the proof.

Although Algorithm 1 provides a natural solution to the problem, it results in a major computational disadvantage. The optimization problem needed to be solved in each iteration is a convex maximization problem (known to generally be NP-hard) [Atamtürk and Gómez, 2017]. Furthermore, since $\|d_\pi\|_{A_{k-1}^{-1}}$ is non-linear in $d_\pi$, it restricts us from solving this problem by means of Dynamic Programming. In the next section, we follow a different route and formulate a Thompson Sampling based algorithm, with computational complexity that amounts to solving an MDP on each episode.
5 Thompson Sampling for Trajectory Based RL

The OFUL-based algorithm for RL with trajectory feedback, analyzed in the previous section, was shown to give good performance in terms of regret. However, implementing the algorithm seems to be computationally hard. Instead of following the OFUL-based approach, in this section, we analyze a Thompson Sampling (TS) approach for RL with trajectory feedback.

We start by studying the performance of Algorithm 2 which assumes access to the transition model (as in Section 4). Then, we study Algorithm 3 which generalizes the latter method to the case where the transition model is unknown. In this generalization, we use an optimistic-based approach to learn the transition model, and a TS-based approach to learn the reward. The combination of optimism and TS results in a tractable algorithm in which every iteration amounts to solving an empirical MDP (which can be done by Dynamic Programming). The reward estimator in both Algorithm 2 and Algorithm 3 is the same LS estimator (4) used for the OFUL-like algorithm.

5.1 TS for Trajectory Based RL and Known Model

For general action sets, it is known that OFUL [Abbasi-Yadkori et al., 2011] results in a computationally intractable update rule. One popular approach to mitigate the computational burden is to resort to TS for linear bandits [Agrawal and Goyal, 2013]. Then, the update rule amounts to solving a linear optimization problem over the action set. Yet, the reduced computational complexity of TS comes with a cost in the sample complexity. Specifically, for linear bandit problem in dimension $d$ OFUL achieves $\tilde{O}(d\sqrt{T})$ whereas TS achieves $\tilde{O}(d^{3/2}/\sqrt{T})$ [Agrawal and Goyal, 2013, Abeille et al., 2017].

Algorithm 2 can be understood as a TS variant of Algorithm 1. Unlike the common TS algorithm for linear bandits, Algorithm 2 uses the LS estimator in Section 3, i.e., the one which uses the empirical state-action distributions $\hat{d}_k$, instead of the true ‘action’ $d_\pi_k$. Extending techniques from [Agrawal and Goyal, 2013, Russo, 2019] and the proof of Theorem 2 we obtain the following performance guarantee for Algorithm 2 (see Appendix D for the full proof).

**Theorem 3** (TS for RL with Trajectory Feedback and Known Model). For any $\delta \in (0, 1)$, it holds with probability greater than $1 - \delta$ that for all $K > 0$,

$$
\text{Regret}(K) \leq O\left((SA)^{3/2}H\sqrt{K\log(K)}\log\left(\frac{KH}{\delta}\right) + SAH\sqrt{\log\left(\frac{KH^2}{\delta}\right)}\right).
$$

See that Theorem 3 establishes a regret guarantee of $d^{3/2}/\sqrt{K}$ since the dimension of the specific linear bandit problem is $d = SA$ (see (1)). This is the type of regret we would expect from a TS algorithm [Agrawal and Goyal, 2013, Abeille et al., 2017]. It is an interesting question whether this bound can be improved due to the structure of the problem.

5.2 UCBVI-TS for Trajectory Based RL

In previous sections, we devised algorithms for RL with trajectory feedback, assuming access to the true transition model, and that only the reward function is needed to be learned. In this section, we relax this assumption and study the setting in which the transition model is also unknown.

This setting, in which the transition model is unknown, highlights the importance of the LS estimator (4), which uses the empirical state-action frequency $d_k$, instead of $d_\pi_k$. I.e., when the transition model is not given, we do not have access to $d_\pi_k$. Yet, $d_k$ depends only on the observed sequence of state-action pairs in the $k^{th}$ episode, and, thus, it does not depend on whether or not the algorithm has access to the true model. For this reason, the LS estimator (4) is much more amenable to use in RL with trajectory feedback when the transition model is not given and needed to be estimated.

Algorithm 4, which we refer as UCBVI-TS, uses a combined TS and optimistic approach for RL with trajectory feedback. At each episode, the algorithm perturbs the LS estimation of the reward $\tilde{r}_{k-1}$ by a random Gaussian noise $\xi_k$, similarly to Algorithm 2. Furthermore, to encourage the agent to learn the
unknown transition model, UCBVI-TS (Upper Confidence Value Iteration and Thompson Sampling) adds to the reward estimation the bonus $b_{k-1}^{pw} \in \mathbb{R}^{SA}$ where

$$b_{k-1}^{pw}(s, a) \simeq \frac{H}{\sqrt{n_{k-1}(s, a) \vee 1}}$$

up to logarithmic factors (similarly to [Azar et al., 2017]). Then, it simply solves the empirical MDP defined by the plug-in transition model $\hat{P}_{k-1}$ and the reward function $\hat{r}_{k-1} = \hat{r}_{k-1} + \xi_k + b_{k-1}^{pw}$. The next result establishes the performance guarantee for UCBVI-TS with trajectory feedback (see proof in Appendix E.3).

**Theorem 4 (UCBVI-TS Performance Guarantee).** For any $\delta \in (0, 1)$, it holds with probability greater than $1 - \delta$ that for all $K > 0$,

$$\text{Regret}(K) \leq O\left( SH(SA + H) \sqrt{AHK \log K} \log \left( \frac{SAHK}{\delta} \right)^{\frac{3}{2}} \right) + O\left( H^2 \sqrt{S(SA + H)^2} \log \left( \frac{SAHK}{\delta} \right)^{\frac{3}{2}} \sqrt{\log K} \right)$$

thus, discarding logarithmic factors and constants and assuming $SA \geq H$, $\text{Regret}(K) \leq \tilde{O}\left( S^2A^{3/2}H^2\sqrt{K} \right)$.

6 Discussion and Conclusions

In this work we formulated the framework of RL with trajectory feedback and studied different RL algorithms in the presence of such feedback. Indeed, in practical scenarios, such feedback is more reasonable to have, as it is weaker feedback relatively to the standard RL one. For this reason, we believe studying it and understanding the gaps between the trajectory feedback RL and standard RL is of importance. The central result of this work is a hybrid optimistic-TS based RL algorithm with a provably bounded $\sqrt{K}$ regret that can be applied when both the reward and transition model are unknown and, thus, needed to be learned. Importantly, the suggested algorithm is computationally tractable, as it requires to solve an empirical MDPs and not a convex maximization problems.

Regret minimization for standard RL has been extensively studied. Previous algorithms for this scenario can be roughly divided into optimistic algorithms [Jaksch et al., 2010, Azar et al., 2017, Jin et al., 2018, Dann et al., 2019, Zanette and Brunskill, 2019, Simeonov and Jamieson, 2018, Efroni et al., 2019] and Thompson-Sampling (or Posterior-Sampling) based algorithms [Osband et al., 2013, Gopalan and Mannor, 2015, Osband and Van Roy, 2017, Russo, 2019]. Nonetheless, and to the best of our knowledge, we are the first to present a hybrid approach that utilizes both concepts in the same algorithm. Specifically, we combine the optimistic confidence-intervals of UCBVI [Azar et al., 2017] alongside linear TS for the reward [Agrawal and Goyal, 2013] and also take advantage of analysis tools for posterior sampling in RL [Russo, 2019].
In the presence of trajectory-feedback, our algorithms make use of concepts from linear bandits to learn the reward. Specifically, we use both OFUL [Abbasi-Yadkori et al., 2011] and linear TS [Agrawal and Goyal, 2013, Abeille et al., 2017], whose regret bounds for \( d \)-dimension problems after \( K \) time-steps with 1-subgaussian noise are \( \tilde{O}\left(d\sqrt{K}\right) \) and \( \tilde{O}\left(d^2\sqrt{K}\right) \), respectively. These bounds directly affect the performance in the RL setting, but the adaptation of OFUL leads to a computationally-intractable algorithm. In addition, when there are at most \( N \) contexts, it is possible to achieve a regret bound of \( \tilde{O}\left(\sqrt{dK\log N}\right) \) [Chu et al., 2011]; however, the number of deterministic policies, which are the number of ‘contexts’ for RL with trajectory-feedback, is exponential in \( S \), namely, \( A^{SH} \). Therefore, such approaches will lead to similar guarantees to OFUL and will also be computationally intractable.

In terms of regret bounds, the minimax regret in the standard RL setting is \( \tilde{O}\left(\sqrt{SAHT}\right) \) [Osband and Van Roy, 2016, Azar et al., 2017]. For linear bandits with \( \sqrt{H} \)-subgaussian noise, the minimax performance bounds with are \( \tilde{O}\left(d\sqrt{HK}\right) \) [Dani et al., 2008]. Specifically, in RL we set \( d = SA \), which leads to \( \tilde{O}\left(SA\sqrt{HK}\right) \). Nonetheless, for RL with trajectory feedback and known model, the context space is the average occupancy measures \( d_\pi \), which is heavily-structured, so it is an open question whether the minimax regret bound remains \( \tilde{O}\left(SA\sqrt{HK}\right) \) for RL with trajectory feedback, when the transition model is known, or whether it can be improved. Finally, when the model is unknown, our algorithm enjoys a regret of \( \tilde{O}\left(S^2A^{3/2}H^{3/2}\sqrt{K}\right) \) when \( H \leq SA \). A factor of \( \sqrt{SA} \) is a direct result of the TS-approach, that was required to make to algorithm tractable, and an additional \( \sqrt{S} \) only appears when the model is unknown. Moreover, extending OFUL to the case of unknown model and following a similar analysis to Theorem 4 would still yield this extra \( \sqrt{S} \) factor (and would result in computationally hard algorithm), in comparison to the case where we know the model. It is an open question whether this additional factor can also be improved.

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A Nomenclature

\[ l_k = \sqrt{\frac{1}{4} SAH \log \left( \frac{1+kH^2}{\delta} \right) } + \sqrt{\lambda SA} \] 

\[ v_k = \sqrt{\frac{9}{4} SAH \log \frac{kH^2}{\delta/10} } \]

\[ c = 2\sqrt{2\pi e} \]

\[ g_k = l_{k-1} + (2c + 1)v_k \left( \sqrt{SA} + \sqrt{10 \log k} \right) = O \left( \sqrt{SA} \sqrt{H} \log \frac{kH^2}{\delta} \log k \right). \]

\( \hat{r}_k \) - the Least Square estimator based on samples from \( k \) episodes.

\( \hat{d}_k \) - the empirical state-action frequency at the \( k^{th} \) episode.

B The Expected Elliptical Potential Lemma

**Lemma 5 (Expected Elliptical Potential Lemma).** Let \( \{F^d_k\}_{k=1}^\infty \) be a filtration such that for any \( k \) \( F_k \subseteq F^d_k \).

Assume that \( d_{\pi_k} = E \left[ \hat{d}_k \mid F_{k-1}^d \right] \). Then, for all \( \lambda > 0 \), it holds that

\[ \sum_{k=0}^K E \left[ \|d_{\pi_k}\|_{A_{k-1}} \mid F_{k-1} \right] \leq 2 \sqrt{\frac{H^2}{\lambda} K \log \left( \frac{2K}{\delta} \right) } + \sqrt{2 \frac{H^2}{\lambda} KSA \log \left( \lambda + \frac{KH^2}{SA} \right) } \]

\[ = O \left( \sqrt{\frac{H^2}{\lambda} KSA \log \left( \frac{HK}{\delta} \right) } \right), \]

uniformly for all \( K > 0 \), with probability greater than \( 1 - \delta \).

**Proof.** Applying Jensen’s inequality, we get

\[ E \left[ \|d_{\pi_k}\|_{A_{k-1}} \mid F_{k-1} \right] = E \left[ E \left[ \|\hat{d}_k\|_{A_{k-1}} \mid F_{k-1} \right] \right] \]

\[ \leq E \left[ E \left[ \|\hat{d}_k\|_{A_{k-1}} \mid F_{k-1} \right] \right] \]

\[ = E \left[ \|\hat{d}_k\|_{A_{k-1}} \mid F_{k-1} \right] \quad \text{(Jensen’s Inequality. Norm is convex.)} \]

\[ = E \left[ \|\hat{d}_k\|_{A_{k-1}} \mid F_{k-1} \right] \quad \text{(Tower Property)} \]

Adding and subtracting the random variable \( \|\hat{d}_k\|_{A_{k-1}} \) we get

\[ (i) \leq \sum_{k=1}^K E \left[ \|\hat{d}_k\|_{A_{k-1}} \mid F_{k-1} \right] = \sum_{k=1}^K E \left[ \|\hat{d}_k\|_{A_{k-1}} \mid F_{k-1} \right] - \|\hat{d}_k\|_{A_{k-1}} + \sum_{k=1}^K \|\hat{d}_k\|_{A_{k-1}}. \]

It is evident that \( (a) \) is a martingale difference sequence, with elements bounded by

\[ \|\hat{d}_k\|_{A_{k-1}} \leq \frac{1}{\sqrt{\lambda}} \|\hat{d}_k\|_2 \leq \frac{1}{\sqrt{\lambda}} \|\hat{d}_k\|_1 = \frac{H}{\sqrt{\lambda}}. \]

Thus, applying Azuma-Hoeffding’s inequality, for any fixed \( K > 0 \), w.p. at least \( 1 - \delta' \)

\[ (a) \leq \sqrt{2 \frac{H^2}{\lambda} K \log \left( \frac{1}{\delta'} \right) }. \]
Setting $\delta' \leftarrow \delta/(2K^2)$ and taking the union bound on all $K > 0$ we have that

\[(a) \leq \sqrt{2 \frac{H^2}{\lambda} K \log \left( \frac{2K^2}{\delta} \right)} \leq 2 \sqrt{\frac{H^2}{\lambda} K \log \left( \frac{2K}{\delta} \right)}\]

does not hold for some $K > 0$ with probability smaller than $\sum_{K=1}^{\infty} \frac{\delta}{K^2} \leq \delta$. Thus, it holds with probability greater than $1 - \delta$.

Term (b) can be bounded by applying the elliptical potential lemma (Lemma 22) as follows.

\[(b) \leq \sqrt{K} \sqrt{\sum_{k=0}^{K} \lambda \left\| \hat{d}_k \right\|_{A_k^{-1}}^2} \quad \text{(Jensen’s inequality)}\]

\[= \sqrt{\frac{H^2}{\lambda}} \sqrt{K} \sqrt{\sum_{k=0}^{K} \lambda \frac{\left\| \hat{d}_k \right\|_{A_k^{-1}}^2}{H^2}}\]

\[= \sqrt{\frac{H^2}{\lambda}} \sqrt{K} \sqrt{\sum_{k=0}^{K} \min \left( \frac{\lambda}{H^2} \left\| \hat{d}_k \right\|_{A_k^{-1}}^2, 1 \right)} \quad \text{($\| \hat{d}_k \|_{A_k^{-1}} \leq \frac{1}{\lambda} \| \hat{d}_k \| \leq \frac{H^2}{\lambda}$)}\]

\[\leq \sqrt{\frac{H^2}{\lambda}} \sqrt{2KSA \log \left( \lambda + \frac{KH^2}{SA} \right)} \quad \text{(Lemma 22)}\]

Combining the bounds on (a), (b) (and observing that the bound on (b) is greater than the one on (a)), we conclude the proof of the lemma.
C OFUL for RL with Trajectory Feedback and Known Model

We prove the following theorem.

**Theorem 2 (OFUL for RL with Trajectory Feedback and Known Model).** For any $\delta \in (0, 1)$, it holds with probability greater than $1 - \delta$ that for all $K > 0$,

\[
\text{Regret}(K) \leq O\left( SAH \sqrt{K \log \left( \frac{KH}{\delta} \right)} \right).
\]

*Proof.* We define the good event $G$ as the event for all $k > 0$,

\[
\|\hat{r}_k - r\|_{A_k} \leq \sqrt{\frac{1}{4}SAH \log \left( \frac{1 + kH^2/\lambda}{\delta/10} \right)} + \sqrt{\lambda SA} \overset{\text{def}}{=} l_k,
\]

and the event that for all $k > 0$,

\[
\sum_{k=0}^{K} \mathbb{E}\left[\|d_{\pi_k}\|_{A_k^{-1}} | F_{k-1}\right] \leq 2 \sqrt{\frac{H^2}{\lambda} K \log \left( \frac{20K}{\delta} \right)} + \sqrt{\frac{2H^2}{\lambda} KSA \log \left( \lambda + \frac{KH^2}{SA} \right)},
\]

for all $K > 0$.

By Proposition 1 the first event holds with probability greater than $1 - \frac{\delta}{10}$. By Lemma 5 with $F_k^d = F_k$, the second event holds with probability greater than $1 - \frac{\delta}{10}$. Taking the union bound establishes that $\Pr\{G\} \geq 1 - \frac{\delta}{10} - 1 - \delta$.

Now, let $C_k \overset{\text{def}}{=} \{\hat{r} : \|\hat{r} - \hat{r}_k\|_{A_k} \leq l_k\}$. Conditioning on $G$, it holds that $r \in C_k$ for all $k > 0$. Thus,

\[
d_{\pi_k}^T \hat{r}_{k-1} + l_{k-1} \|d_{\pi_k}\|_{A_k^{-1}} = \max_{\pi} \left( d_{\pi}^T \hat{r}_{k-1} + l_{k-1} \|d_{\pi}\|_{A_k^{-1}} \right) = \max_{\pi} \max_{\hat{r} \in C_{k-1}} d_{\pi}^T \hat{r} \geq d_{\pi}^T r,
\]

i.e., the algorithm is optimistic. We initially bound the regret following similar analysis to Abbasi-Yadkori et al. (2011) as follows.

\[
\text{Regret}(K) = \sum_{k=1}^{K} d_{\pi_k}^T r - d_{\pi_k}^T r \\
\leq \sum_{k=1}^{K} d_{\pi_k}^T \hat{r}_{k-1} + l_{k-1} \|d_{\pi_k}\|_{A_k^{-1}} - d_{\pi_k}^T r \quad \overset{\text{Eq. (7)}}{=}
\sum_{k=1}^{K} d_{\pi_k}^T (\hat{r}_{k-1} - r) + l_{k-1} \|d_{\pi_k}\|_{A_k^{-1}}
\leq \sum_{k=1}^{K} \|d_{\pi_k}\|_{A_k^{-1}} \|\hat{r}_{k-1} - r\|_{A_k} + l_{k-1} \|d_{\pi_k}\|_{A_k^{-1}}
\leq 2l_K \sum_{k=1}^{K} \|d_{\pi_k}\|_{A_k^{-1}},
\]

where the last relation holds conditioning on $G$ and that $l_K \geq l_k$ for all $k \leq K$.

Setting $\lambda = H$ and observing that conditioning on the good event it holds that

\[
\sum_{k=0}^{K} \mathbb{E}\left[\|d_{\pi_k}\|_{A_k^{-1}} | F_{k-1}\right] \leq O\left( \sqrt{HKSA \log \left( \frac{HK}{\delta} \right)} \right),
\]

we conclude that

\[
\text{Regret}(K) \leq O\left( SAH \sqrt{K \log \left( \frac{KH}{\delta} \right)} \right).
\]
D Thompson Sampling for RL with Trajectory Feedback and Known Model

In this section, we prove Theorem 3.

D.1 The Good Event

We now specify the good event \( \mathcal{G} \). We establish the performance of our algorithm conditioning on the good event. In the following, we show the good event occurs with high probability. Define the following set of events:

\[
E^r(k) = \left\{ \forall d \in \mathbb{R}^{SA} : |d^T (\hat{r}_k - r)| \leq l_k \right\}
\]

\[
E^d(K) = \left\{ \sum_{k=1}^{K} \mathbb{E} \left[ \|d_{x_k} \|_{\mathbb{A}_k^{-1}} | F_{k-1} \right] \leq 2 \sqrt{\frac{H^2}{\lambda} K \log \left( \frac{20K}{\delta} \right)} + \sqrt{2 \frac{H^2}{\lambda} K S A \log \left( \lambda + \frac{K H^2}{SA} \right)} \right\}
\]

The event \( E^d \) holds uniformly for all \( k \geq 0 \), with probability greater than \( 1 - \frac{\delta}{10} \), by Proposition 7 combined with Cauchy-Schwarz inequality:

\[
d^T (\hat{r}_k - r) = (A_k^{-1/2} d)^T (A_k^{-1/2} (\hat{r}_k - r)) \leq \left\| A_k^{-1/2} d \right\|_2 \left\| A_k^{1/2} (\hat{r}_k - r) \right\|_2 = \|d\|_{\mathbb{A}_k^{-1}} \|\hat{r}_k - r\|_{\mathbb{A}_k} \leq l_k \|d\|_{\mathbb{A}_k^{-1}}.
\]

The proof of the high probability bound is similar to [Agrawal and Goyal, 2013], and the bound on the conditional expectation is an extension that we required for our analysis. The next result shows that by perturbing the LS estimator with Gaussian noise we get an effective optimism with a fixed probability. It follows standard analysis of [Agrawal and Goyal, 2013].

Lemma 6. Let the good event be \( \mathcal{G} = E^r \cup E^d \). Then, \( \Pr(\mathcal{G}) \geq 1 - \frac{\delta}{2} \).

Proof. The event \( E^r \) holds uniformly for all \( k \geq 0 \), with probability greater than \( 1 - \frac{\delta}{10} \), by Proposition 7 combined with Cauchy-Schwarz inequality:

\[
\Pr \left\{ \sum_{k=1}^{K} \mathbb{E} \left[ \|d_{x_k} \|_{\mathbb{A}_k^{-1}} | F_{k-1} \right] \leq 2 \sqrt{\frac{H^2}{\lambda} K \log \left( \frac{20K}{\delta} \right)} + \sqrt{2 \frac{H^2}{\lambda} K S A \log \left( \lambda + \frac{K H^2}{SA} \right)} \right\} \geq 1 - \frac{\delta}{2}.
\]

D.2 Optimism with Fixed Probability

We start by stating three lemmas that will be essential to our analysis, both for known and unknown model. The proof of the lemmas can be found in Appendix 1. The first result analyzes the concentration of the TS noise around zero:

Lemma 7 (Concentration of Thompson Sampling Noise). Let

\[
E^{\xi}(k) = \left\{ \forall d \in \mathbb{R}^{SA} : |d^T \xi_k| \leq v_k \left( \sqrt{SA} + \sqrt{16 \log k} \right) \right\},
\]

where \( \xi_k \sim N(0, v_k A_k^{-1}) \). Then, for any \( k > 0 \) it holds that \( \Pr(E^{\xi}(k) | F_{k-1}) \geq 1 - \frac{\delta}{10} \). Moreover, for any random variable \( X \in \mathbb{R}^{SA} \), it holds that

\[
\mathbb{E} \left[ \|X^T \xi_k| F_{k-1} \right] \leq v_k \left( \sqrt{SA} + \sqrt{16 \log k} \right) \mathbb{E} \left[ \|X\|_{A_k^{-1}} | F_{k-1} \right] + \frac{v_k \sqrt{SA}}{k^2} \sqrt{\mathbb{E} \left[ \|X\|_{A_k^{-1}}^2 | F_{k-1} \right]}.
\]

The proof of the high probability bound is similar to [Agrawal and Goyal, 2013], and the bound on the conditional expectation is an extension that we required for our analysis. The next result shows that by perturbing the LS estimator with Gaussian noise we get an effective optimism with a fixed probability. It follows standard analysis of [Agrawal and Goyal, 2013].

Lemma 8 (Optimism with Fixed Probability). Let \( \hat{r}_k = \xi_k + r_{k-1} \). Assume that \( \lambda \in [1, H] \). Then, for any \( k > 1 \) and any filtration \( F_{k-1} \) such that \( E^r(k-1) \) is true and any model \( P' \) that is \( F_{k-1} \)-measurable, it holds that

\[
\Pr \left\{ d_{\pi^*}(P')^T \hat{r}_k > d_{\pi^*}(P')^T r | F_{k-1} \right\} \geq \frac{1}{c},
\]

where \( c \) is a constant.
for \(c = 2\sqrt{2\pi e}\). Specifically, for any \(\pi_k \in \arg \max_{\pi} d_\pi(P^T r_k)\) it also holds that
\[
\Pr\{d_{\pi_k}(P^T r_k) > d_{\pi_k}(P^T r) \mid F_{k-1}\} \geq \frac{1}{c}.
\]

Notice that since the model is known to the learner, we can fix \(P' = P\), but the general statement will be of use when the model is unknown. Finally, we require the following technical result:

**Lemma 9.** Let \(\xi_k, \xi'_k\) be i.i.d. random variables given \(F_{k-1}\). Also, let \(x_{k-1} \in \mathbb{R}^{SA}\) be some \(F_{k-1}\)-measurable random variable and \(P'\) be an \(F_{k-1}\)-measurable transition kernel. Finally, let \(\tilde{\pi} \in \arg \max_{\pi} d_\pi(P^T(x_{k-1} + \xi_k))\). Then,
\[
\mathbb{E}\left[\left(\tilde{d}(P')^T(x_{k-1} + \xi_k) - \mathbb{E}[\tilde{d}(P')^T(x_{k-1} + \xi_k) \mid F_{k-1}]\right)^+ \mid F_{k-1}\right] \leq \mathbb{E}[|d_\tilde{\pi}(P')\xi_k| + |d_\tilde{\pi}(P')\xi'_k\mid F_{k-1}].
\]

Using these results, we now prove a variation of Lemma 6 from [Russo, 2019]. We find this technique more easily generalizable (relatively to the analysis in e.g., [Agrawal and Goyal, 2013]) to the next section in which we extend this approach to the case a model is not given.

**Lemma 10** (Consequences of Optimism). Assume that \(\lambda \in [1, H]\). Also, conditioned on \(F_{k-1}\), let \(\xi_k\) be an independent copy of \(\xi_k\). Then, for any \(k > 1\) and any filtration \(F_{k-1}\) such that \(E^k(\cdot)\) occurs, it holds that
\[
\mathbb{E}[d_{\pi_k}^T r - d_{\pi_k}^T r \mid F_{k-1}] \leq (2c + 1)v_k \left(\sqrt{SA} + \sqrt{16 \log k}\right) \mathbb{E}[|d_{\pi_k}|_{A_{k-1}^{-1}} \mid F_{k-1}] + \left(2c + 1\right)v_k H \sqrt{SA} \sqrt{k} + \mathbb{E}[d_{\pi_k}^T \hat{r}_{k-1} - d_{\pi_k}^T r \mid F_{k-1}].
\]

**Proof.** We use the following decomposition,
\[
\mathbb{E}[d_{\pi_k}^T r - d_{\pi_k}^T r \mid F_{k-1}] = \mathbb{E}[d_{\pi_k}^T r - d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}] + \mathbb{E}[d_{\pi_k}^T \tilde{r}_k - d_{\pi_k}^T r \mid F_{k-1}]. \tag{9}
\]

We start by bounding (i) and show that
\[
(i) \leq c \mathbb{E}\left[\left(d_{\pi_k}^T \tilde{r}_k - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}]\right)^+ \mid F_{k-1}\right]. \tag{10}
\]

If \((i) = d_{\pi_k}^T r - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}] < 0\) the inequality trivially holds. Otherwise, let \(a \equiv d_{\pi_k}^T r - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}] \geq 0\). Then,
\[
\mathbb{E}\left[\left(d_{\pi_k}^T \tilde{r}_k - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}]\right)^+ \mid F_{k-1}\right]
\geq a \cdot \Pr(d_{\pi_k}^T \tilde{r}_k - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}] \geq a \mid F_{k-1})
\]
\[
= (d_{\pi_k}^T r - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}]) \cdot \Pr(d_{\pi_k}^T \tilde{r}_k \geq d_{\pi_k}^T r \mid F_{k-1})
\]
\[
\geq (d_{\pi_k}^T r - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}]) \cdot \frac{1}{c}, \tag{Lemma 8}
\]

which implies that \((i) \leq c \mathbb{E}\left[\left(d_{\pi_k}^T \tilde{r}_k - \mathbb{E}[d_{\pi_k}^T \tilde{r}_k \mid F_{k-1}]\right)^+ \mid F_{k-1}\right]\). Next, we apply Lemma 9 with \(x_{k-1} = \tilde{r}_{k-1}\) and \(P' = P\) (and, therefore, \(\tilde{\pi} = \pi_k\)), which yields
\[
(i) \leq c \left(\mathbb{E}[|d_{\pi_k}\xi_k| \mid F_{k-1}] + \mathbb{E}[|d_{\pi_k}\xi'_k\mid F_{k-1}]\right).
\]

Substituting the bound on \((i)\) in (9) and using the definition \(\tilde{r}_k = \tilde{r}_{k-1} + \xi_k\) we get
\[
\mathbb{E}[d_{\pi_k}^T r - d_{\pi_k}^T r \mid F_{k-1}] \leq c \left(\mathbb{E}[|d_{\pi_k}\xi_k| \mid F_{k-1}] + \mathbb{E}[|d_{\pi_k}\xi'_k\mid F_{k-1}] + \mathbb{E}[d_{\pi_k}(\tilde{r}_{k-1} + \xi_k) - d_{\pi_k}^T r \mid F_{k-1}]\right)
\leq (c + 1)\mathbb{E}[|d_{\pi_k}\xi_k| \mid F_{k-1}] + c\mathbb{E}[|d_{\pi_k}\xi'_k\mid F_{k-1}] + \mathbb{E}[d_{\pi_k}^T \tilde{r}_{k-1} - d_{\pi_k}^T r \mid F_{k-1}]].
\]

Finally, we apply Lemma 7 on the two noise terms, with \(X = d_{\pi_k}\). To this end, notice that
\[
||d_{\pi_k}|_{A_{k-1}^{-1}} \leq \frac{|d_{\pi_k}|_2}{\sqrt{\lambda}} \leq \frac{|d_{\pi_k}|}{\sqrt{\lambda}} = \frac{H}{\sqrt{\lambda}}.
\]

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Then, applying the lemma yields:
\[
\mathbb{E}[d_{\pi_k}^T r - d_{\pi_k}^T r | F_{k-1}] \leq (2c + 1)v_k \left( \sqrt{SA} + \sqrt{16 \log k} \right) \mathbb{E}\left[ \|d_{\pi_k} r\|_{\Lambda_{A_k}^{-1}} | F_{k-1} \right] + \frac{(2c + 1)v_k \sqrt{SA}}{\sqrt{A_k^2}} + \mathbb{E}[d_{\pi_k}^T f_{k-1} - d_{\pi_k}^T r | F_{k-1}].
\]

D.3 Proof of Theorem 3

We are now ready to prove the following result.

**Theorem 3** (TS for RL with Trajectory Feedback and Known Model). For any \( \delta \in (0, 1) \), it holds with probability greater than \( 1 - \delta \) that for all \( K > 0 \),
\[
\text{Regret}(K) \leq O (SA)^{3/2} H \sqrt{K \log(K)} \log\left( \frac{KH}{\delta} \right) + SAH \sqrt{\log\left( \frac{KH^2}{\delta} \right)}.
\]

**Proof.** We start by conditioning on the good event, which occurs with probability greater than \( 1 - \frac{\delta}{2} \). Conditioned on the good event,
\[
\text{Regret}(K) = \sum_{k=1}^{K} d_{\pi_k}^T r - d_{\pi_k}^T r
\]
\[
= \sum_{k=1}^{K} \mathbb{E}\left[ (d_{\pi_k}^T r - d_{\pi_k}^T r) | F_{k-1} \right] + \sum_{k=1}^{K} d_{\pi_k}^T r - d_{\pi_k}^T r - \mathbb{E}\left[ (d_{\pi_k}^T r - d_{\pi_k}^T r) | F_{k-1} \right]
\]
\[
= \sum_{k=1}^{K} \mathbb{E}\left[ (d_{\pi_k}^T r - d_{\pi_k}^T r) | F_{k-1} \right] + \sum_{k=1}^{K} \mathbb{E}\left[ d_{\pi_k}^T r | F_{k-1} \right] - d_{\pi_k}^T r
\]

The first term \((i)\) is bounded in Lemma 11 by
\[
(i) \leq O \left( g_k \sqrt{HSAK \log\left( \frac{KH}{\delta} \right)} + SAH \sqrt{\log\left( \frac{KH^2}{\delta} \right)} \right),
\]
where \( g_k = l_{k-1} + v_k \left( \sqrt{SA} + \sqrt{16 \log k} \right) \). The second term, \((ii)\), is a martingale difference sequence with random variables bounded in \([0, H]\). Applying Azuma-Hoeffding’s inequality for probabilities \( \frac{H}{2} \) and taking the union bound on all \( K > 0 \), we get the following relation holds with probability greater than \( 1 - \frac{\delta}{2} \).
\[
(ii) \leq O \left( H \sqrt{K \log\left( \frac{K}{\delta} \right)} \right).
\]

Bounding \((12)\) by the bounds in \((13)\) and \((14)\) concludes the proof since
\[
g_k \leq O \left( SA \sqrt{H \log(K) \log\left( \frac{KH}{\delta} \right)} \right).
\]
The bound holds for all \( K > 0 \) w.p. at least \( 1 - \frac{\delta}{2} \), using the union bound on the good event (w.p. \( 1 - \delta/2 \)) and Equation \((13)\) (each w.p. \( 1 - \delta/2 \)).

**Lemma 11** (Conditional Gap Bound). Conditioning on the good event and setting \( \lambda = H \), the following bound holds.
\[
\sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}^T r - d_{\pi_k}^T r | F_{k-1}] \leq O \left( g_k \sqrt{HSAK \log\left( \frac{KH}{\delta} \right)} + SAH \sqrt{\log\left( \frac{KH^2}{\delta} \right)} \right),
\]
where \( g_k = l_k + (2c + 1)v_k \left( \sqrt{SA} + \sqrt{16 \log k} \right) \).
Proof. Conditioning on the good event, and by Lemma 10 with $\lambda = H$, we get

$$
\sum_{k=1}^{K} \mathbb{E}[(d_{r_k}^T r - d_{r_k}^T r)|F_{k-1}]
\leq \sum_{k=1}^{K} \left( (2c+1)v_k \left( \sqrt{SA} \right) \right) \mathbb{E} \left[ \|d_{\pi_k}\|_{A_{k-1}^{-1}} |F_{k-1}\right] + \frac{(2c+1)v_k \sqrt{SAH}}{k^2} + \mathbb{E} \left[ d_{r_k}^T \hat{r}_{k-1} - d_{r_k}^T r |F_{k-1}\right] 
\leq (2c+1)v_K \left( \sqrt{SA} + \sqrt{16 \log K} \right) \sum_{k=1}^{K} \mathbb{E} \left[ \|d_{\pi_k}\|_{A_{k-1}^{-1}} |F_{k-1}\right] + \sum_{k=1}^{K} \mathbb{E} \left[ \|d_{\pi_k}\|_{A_{k-1}^{-1}} \|r - \hat{r}_{k-1}\|_{A_{k-1}} |F_{k-1}\right]
+ 2(2c+1)v_K \sqrt{SAH}
\leq (2c+1)v_K \left( \sqrt{SA} + \sqrt{16 \log K} \right) l_{k-1} \sum_{k=1}^{K} \mathbb{E} \left[ \|d_{\pi_k}\|_{A_{k-1}^{-1}} |F_{k-1}\right] + 2(2c+1)v_K \sqrt{SAH}
= g_K \sum_{k=1}^{K} \mathbb{E} \left[ \|d_{\pi_k}\|_{A_{k-1}^{-1}} |F_{k-1}\right] + 2(2c+1)v_K \sqrt{SAH}
\leq g_K \left( 2 \sqrt{HK \log \frac{20K}{\delta}} + \sqrt{2HKSA \log \left( \lambda + \frac{KH^2}{SA} \right)} \right) + 2(2c+1)v_K \sqrt{SAH}
= O \left( g_K \sqrt{HSAK \log \left( \frac{KH}{\delta} \right)} + SAH \sqrt{\log \left( \frac{KH^2}{\delta} \right)} \right).
$$

(1) is since $v_k \leq v_K$ for any $k \leq K$ and $\sum_{k=1}^{K} \frac{1}{k^2} \leq 2$. Next, (2) holds under the good event (and specifically, $E^r$), and since $l_{k-1} \leq l_{K-1}$ for any $k \leq K$, where the last relation holds conditioning on $G$ (and, specifically, $E^r$) and using $l_K \geq l_k$ for any $k \in [K]$.  Finally, (3) holds by $E^d$ with $\lambda = H$ and substitution of $v_K$. \qed

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E Thompson Sampling for RL with Trajectory Feedback and Unknown Model

E.1 The Good Event

We now specify the good event $G$. We establish the performance of our algorithm conditioning on the good event. In the following, we show the good event occurs with probability greater than $1 - \delta/2$. Define the following set of events

\[ E^r(k) = \{ \forall d \in \mathbb{R}^{S,A} : |d^T(\hat{r}_k - r)| \leq l_k \|d\|_{A_k^{-1}} \} \]

\[ E^p(k) = \{ \forall s \in S, a \in A : \|P(\cdot|s, a) - \tilde{P}_k(\cdot|s, a)\|_1 \leq \sqrt{\frac{4S\log \frac{40SAHk^3}{\delta}}{n_{k-1}(s, a) \vee 1}} \} \]

\[ E^{pv}(k) = \{ \forall s, a, h : \left| (\tilde{P}_k(\cdot|s, a) - P(\cdot|s, a))^T V_{h+1} \right| \geq \sqrt{\frac{H^2 \log \frac{40SAH^2k^3}{\delta}}{2(n_k(s, a) \vee 1)}} = t_k^{pv}(s, a) \} \]

\[ E^N(K) = \{ \sum_{k=1}^{K} E^p \left[ \sum_{h=1}^{H} \frac{1}{n_{k-1}(s_h, a_h^{k})} \left| F_{k-1} \right| \right] \leq 16H^2 \log \left( \frac{20K^2}{\delta} \right) + 4SAH + 2\sqrt{2} \sqrt{SAHK \log HK} \} \]

\[ E^d(K) = \{ \sum_{k=0}^{K} E \left[ \|d_{\pi_k}\|_{A_{k}^{-1}} \left| F_{k-1} \right| \right] \leq 2 \sqrt{\frac{H^2}{\lambda} K \log \left( \frac{20K}{\delta} \right)} + 2 \sqrt{\frac{H^2}{\lambda} KSA \log \left( \frac{\lambda + KH^2}{SA} \right)} \}. \]

Furthermore, we define the events in which all the former events hold uniformly, namely, $E^r = \cap_{k \geq 0} E^r(k)$, $E^p = \cap_{k \geq 0} E^p(k)$, $E^{pv} = \cap_{k \geq 0} E^{pv}(k)$, $E^N = \cap_{K \geq 1} E^N(K)$ and $E^d = \cap_{K \geq 1} E^d(K)$.

**Lemma 12.** Let the good event be $G = E^r \cap E^p \cap E^{pv} \cap E^N \cap E^d$. Then, $\Pr\{G\} \geq 1 - \delta/2$.

**Proof.** We analyze the probability each of the events does not hold and upper bound this probability by $\delta/10$. Taking the union bound then concludes the proof. The claim that $\Pr\{E^r(k) | F_{k-1}\} \geq 1 - \frac{1}{k}$ for any $k > 0$ is proved in Lemma [7] (proved in Agrawal and Goyal 2013 as well).

**The event $E^r$.** The event holds uniformly for all $k \geq 0$, with probability greater than $1 - \frac{\delta}{10}$, by Proposition 4 combined with Cauchy-Schwarz inequality:

\[ d^T(\hat{r}_k - r) = \left( A_k^{-1/2} d \right)^T \left( A_k^{-1/2} (\hat{r}_k - r) \right) \leq \|A_k^{-1/2} d\|_2 \|A_k^{-1/2} (\hat{r}_k - r)\|_2 = \|d\|_{A_k^{-1}} \|\hat{r}_k - r\|_{A_k} \leq l_k \|d\|_{A_k^{-1}}. \]

**The event $E^p$.** Note that the event trivially holds if $n_k(s, a) = 0$, so we assume w.l.o.g. that $n_k(s, a) \geq 1$. Next, for any fixed $k$, we apply the following concentration inequality for L1-norm [Weissman et al. 2003]:

**Lemma 13.** Let $X_1, \ldots, X_m$ be i.i.d random values over $\{1, \ldots, a\}$ such that $\Pr\{X_n = i\} = p_i$, and define $\hat{p}_m(i) = \frac{1}{m} \sum_{n=1}^{m} 1(X_n = i)$. Then, for all $\delta, \epsilon \in (0, 1)$,

\[ \Pr\left\{ \|\hat{p}_m - p\|_1 \geq \sqrt{\frac{2a \log \frac{2}{\delta}}{m}} \right\} \leq \epsilon, \]

Next, we fix $\delta' = \frac{\delta}{20SAH^2}$. Then, noting that $n_k(s, a) \leq kH$ and taking the union bounds over all possible values of $s, a, n_k(s, a)$, we get that $\Pr\{E^p(k)\} \geq 1 - \frac{\delta}{2000}$. Finally, taking the union bound over all possible values of $k$ and recalling that $\sum_{k=1}^{\infty} \frac{1}{k} \leq 2$, we get that $\Pr\{E^p\} \geq 1 - \delta/10$, which concludes the proof.
The event $E^p$. Notice that the event trivially holds if $n_k(s, a) = 0$, since $V_{k+1}^*(s') \in [0, H]$ and $P, \bar{P}_k$ are in the $S$-dimensional simplex for any $s, a$; thus, and w.l.o.g., we assume that $n_k(s, a) \geq 1$. Next, we fix $s, a, h$ and $\delta' \in (0, 1)$ and let $s'_1, \ldots, s'_m$ be $m$ i.i.d samples from $P\{\cdot|s, a\}$. Finally, we define $X_i = V_{k+1}^*(s'_i) \in [0, H]$ for $i \in [m]$. Specifically, notice that $E[X_i] = P\{\cdot|s, a\}^T V_{k+1}^*$ and we can write $\bar{P}_k \{\cdot|s, a\}^T V_{k+1}^* = \frac{1}{m_k(s, a)} \sum_{i=1}^{m_k(s, a)} X_i$. Then, by Hoeffding’s inequality, w.p. $1 - \delta'$
\[
\Pr\left\{ \frac{1}{m} \sum_{i=1}^{m} X_i \geq \sqrt{\frac{H^2}{2} \log \frac{2}{\delta}} \right\} \leq \delta'.
\]
Next, notice that for any $k, n_k(s, a) \in [Hk]$. Fixing $m = n_k(s, a)$ and taking the union bound over all possible values yields w.p. $1 - \delta'$
\[
\Pr\left\{ \left| \bar{P}_k \{\cdot|s, a\}^T V_{k+1}^* \right| \geq \sqrt{\frac{H^2}{2} \log \frac{2Hk}{\delta'}} \right\} \leq \delta'.
\]
Finally, choosing $\delta' = \frac{\delta}{20SAHK^2}$ and taking the union bound over all possible values of $s, a, h$ and $k > 0$ leads us to $\Pr\{E^p\} \geq 1 - \frac{\delta}{10}$.

The event $E^N$. For any fixed $K$, it holds with probability greater than $1 - \delta'$ that
\[
\sum_{k=1}^{K} \mathbb{E} \left[ \sum_{h=1}^{H} \frac{1}{n_{k-1}(s_h^k, a_h^k) \vee 1} | F_{k-1} \right] \leq 16H^2 \log \left( \frac{1}{\delta} \right) + 4SAH + 2\sqrt{2SAHK} \log HK.
\]
by Lemma [19] Setting $\delta' = \frac{\delta}{20SAHK^2}$ and applying the union bound we get that
\[
\sum_{k=1}^{K} \mathbb{E} \left[ \sum_{h=1}^{H} \frac{1}{n_{k-1}(s_h^k, a_h^k) \vee 1} | F_{k-1} \right] \leq 16H^2 \log \left( \frac{20H^2}{\delta} \right) + 4SAH + 2\sqrt{2SAHK} \log HK,
\]
for all $K > 0$ w.p. at least $1 - \delta/10$.

The event $E^d$. This event holds with probability greater than $1 - \delta/10$ by Lemma [K] with $\lambda = H$.

\[\square\]

E.2 Optimism with Fixed Probability

We remind the reader the $\bar{r}_k^h \overset{\text{def}}{=} \bar{r}_{k-1} + b_k^p + \xi_k$ where $\bar{r}_{k-1}$ is the LS-estimator, $b_k^p$ is the bonus at the beginning of the $k^{th}$ episode, and $\xi_k$ is the randomly drawn noise at the $k^{th}$ episode. The following result which establishes an optimism with fixed probability holds.

Lemma 14 (Optimism with Fixed Probability). Assume that $\lambda \in [1, H]$. Then, for any $k > 1$ and any filtration $F_{k-1}$ such that $E^N(k-1)$ and $E^p(k-1)$ are true,
\[
\Pr\{d_{\pi}(\bar{P})^T \bar{r}_k > d_{\pi}^* r | F_{k-1}\} \geq \frac{1}{c}
\]
for $c = 2\sqrt{2\pi e}$.

Proof. First, by definition of the update rule, we have
\[
d_{\pi}(\bar{P}_{k-1})^T \bar{r}_k - d_{\pi}^* r = \max_{\pi} d_{\pi}(\bar{P}_{k-1})^T \bar{r}_k - d_{\pi}^* r \geq d_{\pi}(\bar{P}_{k-1})^T \bar{r}_k - d_{\pi}^* r.
\]
Next, we establish that
\[
d_{\pi}(\bar{P}_{k-1})^T \bar{r}_k - d_{\pi}^* r \geq d_{\pi}(\bar{P}_{k-1})^T \bar{r}_k - d_{\pi}^* r.
\]
where \( \tilde{r}_k = \hat{r}_{k-1} + \xi_k \). By the value difference lemma (Lemma 17, and given \( F_{k-1} \), it holds that

\[
d_{\pi^*}(\bar{P}_{k-1})^T \hat{r}_k - d_{\pi^*}^T r
\]

\[
= \mathbb{E} \left[ \sum_{h=1}^{H} \tilde{p}_k^h - r(s_h, a_h) + (\bar{P}_{k-1} \cdot s_h, a_h) - P(\cdot | s_h, a_h))^T V_{k+1}^* | s_1, \bar{P}_{k-1}, \pi^* \right] \tag{Lemma 17}
\]

\[
= \mathbb{E} \left[ \sum_{h=1}^{H} (\tilde{r}_k - r)(s_h, a_h) + \bar{b}_{k-1}^p(s_h, a_h) + (\bar{P}_{k-1} \cdot s_h, a_h) - P(\cdot | s_h, a_h))^T V_{k+1}^* | s_1, \bar{P}_{k-1}, \pi^* \right] \tag{E^{\pi^*}(k-1) holds in \( F_{k-1} \)}
\]

\[
= \mathbb{E} \left[ \sum_{h=1}^{H} (\tilde{r}_k - r) + (s_h, a_h) | s_1, \bar{P}_{k-1}, \pi^* \right] = d_{\pi^*}(\bar{P}_{k-1})^T \tilde{r}_k - d_{\pi^*}(\bar{P}_{k-1})^T r,
\]

which establishes (10). Finally, since \( E^*(k-1) \) is true under \( F_{k-1} \) and \( \bar{P}_{k-1} \) is \( F_{k-1} \)-measurable, we can apply Lemma 8 which leads to the desired result:

\[
\Pr\{d_{\pi^*}(\bar{P}_{k-1})^T \hat{r}_k > d_{\pi^*}^T r | F_{k-1} \} = \Pr\{d_{\pi^*}(\bar{P}_{k-1})^T \hat{r}_k - d_{\pi^*}^T r > 0 | F_{k-1} \}
\]

\[
\geq \Pr\{d_{\pi^*}(\bar{P}_{k-1})^T \tilde{r}_k - d_{\pi^*}^T r > 0 | F_{k-1} \} \tag{By (15)}
\]

\[
\geq \Pr\{d_{\pi^*}(\bar{P}_{k-1})^T \tilde{r}_k - d_{\pi^*}(\bar{P}_{k-1})^T r > 0 | F_{k-1} \} \tag{By (16)}
\]

\[
\geq \frac{1}{c}. \tag{Lemma 8}
\]

Next, we generalize Lemma 10 from the case a model is known to the case the model is unknown (this lemma is a variation of Lemma 6 from Russo 2019).

**Lemma 15.** Assume that \( \lambda \in [1, H] \). Then, for any \( k > 1 \) and any filtration \( F_{k-1} \) such that \( E^*(k-1) \) and \( E^{\pi^*}(k-1) \) are true, it holds that

\[
\mathbb{E}[d_{\pi^*}^T r - d_{\pi^*}^T r | F_{k-1}] \leq (2c + 1) v_k \left( \sqrt{SA} + \sqrt{16 \log k} \right) \mathbb{E}\left[ ||d_{\pi^*}(\bar{P}_{k-1})||_{A_{k-1}} | F_{k-1} \right] + \frac{(2c + 1) v_k H \sqrt{SA}}{\sqrt{\lambda k^2}}
\]

\[
+ \mathbb{E}\left[ d_{\pi^*}(\bar{P}_{k-1})^T \tilde{r}_k - d_{\pi^*}^T r | F_{k-1} \right].
\]

where \( \tilde{r}_k = \hat{r}_k + \bar{b}_{k-1}^p \).

**Proof.** We use the following decomposition,

\[
\mathbb{E}[d_{\pi^*}^T r - d_{\pi^*}^T r | F_{k-1}] = \mathbb{E}[d_{\pi^*}^T r - d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k] + \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k - d_{\pi^*}^T r | F_{k-1}]. \tag{17}
\]

We start by bounding (i) and show that

\[
(i) \leq c \mathbb{E}\left[ (d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k - \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k | F_{k-1}])^+ | F_{k-1} \right]. \tag{18}
\]

If (i) \( d_{\pi^*}^T r - \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k | F_{k-1}] < 0 \) the inequality trivially holds. Let \( a \equiv d_{\pi^*}^T r - \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k | F_{k-1}] \geq 0 \). Then,

\[
\mathbb{E}\left[ (d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k - \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k | F_{k-1}])^+ | F_{k-1} \right]
\]

\[
\geq a \Pr(d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k - \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k | F_{k-1}] \geq a | F_{k-1}) \tag{Markov’s inequality}
\]

\[
= (d_{\pi^*}^T r - \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k | F_{k-1}]) \Pr(d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k \geq d_{\pi^*}^T r | F_{k-1})
\]

\[
\geq (d_{\pi^*}^T r - \mathbb{E}[d_{\pi^*}(\bar{P}_{k-1})^T \bar{r}_k | F_{k-1}]) \frac{1}{c} \tag{Lemma 14}
\]

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which implies that $(i) \leq c \mathbb{E}\left[(d_{\pi_k}(\hat{P}_{k-1})^T \tilde{r}_k^b - \mathbb{E}[d_{\pi_k}(\hat{P}_{k-1})^T \tilde{r}_k^b|F_{k-1}] + |F_{k-1}\right]$. Next, we apply Lemma 7 with \( x_{k-1} = \tilde{r}_{k-1}^b \) and \( P' = \hat{P}_{k-1} \) (and, therefore, \( \hat{\pi} = \pi_k \)), which yields

\[
(i) \leq c(\mathbb{E}[d_{\pi_k}(\hat{P}_{k-1})^T \xi_k|F_{k-1}] + \mathbb{E}[d_{\pi_k}(\hat{P}_{k-1})^T \xi_k||F_{k-1}])
\]

Substituting the bound on $(i)$ in (17) and using the definition \( \tilde{r}_k^b = r_{k-1}^b + \xi_k \) we get

\[
\mathbb{E}[d_{\pi_k}(\hat{P}_{k-1})^T r_{k-1}^b|F_{k-1}] = c(\mathbb{E}[d_{\pi_k}(\hat{P}_{k-1})^T \xi_k|F_{k-1}] + \mathbb{E}[d_{\pi_k}(\hat{P}_{k-1})^T \xi_k||F_{k-1}])
\]

Finally, we apply Lemma 7 on the two noise terms, with \( X = d_{\pi_k} \). To this end, notice that

\[
\|d_{\pi_k}(\hat{P}_{k-1})\|_{A^{-1}_{k-1}} \leq \frac{\|d_{\pi_k}(\hat{P}_{k-1})\|_2}{\sqrt{\lambda}} \leq \frac{\|d_{\pi_k}(\hat{P}_{k-1})\|_1}{\sqrt{\lambda}} = H \sqrt{\lambda}
\]

Then, applying the lemma yields:

\[
\mathbb{E}[d_{\pi_k}^T r_{k-1}^b - d_{\pi_k}^T r_{k-1}^b|F_{k-1}] \leq (2c + 1) v_k \left( \sqrt{SA + \sqrt{16\log k}} \right) \mathbb{E}\left[\|d_{\pi_k}(\hat{P}_{k-1})\|_{A^{-1}_{k-1}}|F_{k-1}\right] + \frac{(2c + 1)v_k H \sqrt{SA}}{\sqrt{\lambda k^2}} + \mathbb{E}[d_{\pi_k}^T r_{k-1}^b - d_{\pi_k}^T r_{k-1}^b|F_{k-1}].
\]

**E.3 Proof of Theorem 4**

We are now ready to prove the theorem.

**Theorem 4 (UCBVI-TS Performance Guarantee).** For any \( \delta \in (0, 1) \), it holds with probability greater than \( 1 - \delta \) that for all \( K > 0 \),

\[
\text{Regret}(K) \leq O\left( SH(SA + H) \sqrt{AHK \log K \log \left( \frac{SAH}{\delta} \right)^{\frac{3}{2}}} \right)
\]

\[
+ O\left( H^2 \sqrt{S(SA + H)^2 \log \left( \frac{SAH}{\delta} \right)^2} \sqrt{\log K} \right)
\]

thus, discarding logarithmic factors and constants and assuming \( SA \geq H \), \( \text{Regret}(K) \leq \tilde{O}\left( S^2 A^{3/2} H^2 \sqrt{K} \right) \).

**Proof.** We start be conditioning on the good event, which occurs with probability greater than \( 1 - \delta \) (Lemma 12). Conditioning on the good event,

\[
\text{Regret}(K) = \sum_{k=1}^{K} d_{\pi_k}^T r_{k-1}^b - d_{\pi_k}^T r_{k-1}^b
\]

\[
= \sum_{k=1}^{K} \mathbb{E}\left[(d_{\pi_k}^T r_{k-1}^b - d_{\pi_k}^T r_{k-1}^b)|F_{k-1}\right] + \sum_{k=1}^{K} d_{\pi_k}^T r_{k-1}^b - d_{\pi_k}^T r_{k-1}^b - \mathbb{E}\left[(d_{\pi_k}^T r_{k-1}^b - d_{\pi_k}^T r_{k-1}^b)|F_{k-1}\right]
\]

\[
= \sum_{k=1}^{K} \mathbb{E}\left[(d_{\pi_k}^T r_{k-1}^b - d_{\pi_k}^T r_{k-1}^b)|F_{k-1}\right] + \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}^T r_{k-1}^b|F_{k-1}] - d_{\pi_k}^T r_{k-1}^b.
\]
The first term \((i)\) is bounded in Lemma 16 by

\[
(i) \leq O\left( H^2 \sqrt{S(SA + H)^2} \log\left( \frac{SAHK}{\delta} \right)^2 \sqrt{\log K} + SH(SA + H) \sqrt{AHK} \log K \log \left( \frac{SAHK}{\delta} \right)^{\frac{3}{2}} \right). \tag{20}
\]

The second term \((ii)\) is a sum over bounded martingale difference terms, and can be bounded exactly as in the proof of Theorem 3, using Azuma-Hoeffding inequality (see Section D). Specifically, we with probability greater than \(1 - \frac{\delta}{2}\), uniformly for all \(K > 0\), we have

\[
(ii) \leq O\left( H \sqrt{K \log \left( \frac{K}{\delta} \right)} \right),
\]

Combining the bounds on \((i)\) and \((ii)\) concludes the proof. \(\square\)

**Lemma 16 (Conditional Gap Bound Unknown Model).** Conditioning on the good event and setting \(\lambda = H\), the following bound holds.

\[
\sum_{k=1}^{K} \mathbb{E}\left[ (d_{\pi_k}^T r - d_{\bar{\pi}_k}^T r) | F_{k-1} \right] \leq O\left( H^2 \sqrt{S(SA + H)^2} \log\left( \frac{SAHK}{\delta} \right)^2 \sqrt{\log K} + SH(SA + H) \sqrt{AHK} \log K \log \left( \frac{SAHK}{\delta} \right)^{\frac{3}{2}} \right)
\]

**Proof.** We start by following similar steps to the proof of Theorem 3 when the transition model is known. Conditioning on the good event and by Lemma 15 with \(\lambda = H\), we get

\[
\sum_{k=1}^{K} \mathbb{E}\left[ (d_{\pi_k}^T r - d_{\bar{\pi}_k}^T r) | F_{k-1} \right] \\
\leq \sum_{k=1}^{K} (2c + 1)v_k \left( \sqrt{SA} + \sqrt{16 \log K} \right) \mathbb{E}\left[ ||d_{\pi_k}(\bar{P}_{k-1})||_{A_{k-1}^{-1}} | F_{k-1} \right] + \sum_{k=1}^{K} \frac{(2c + 1)v_k \sqrt{SAH}}{k^2} \\
+ \sum_{k=1}^{K} \mathbb{E}\left[ d_{\pi_k}(\bar{P}_{k-1})^T \hat{r}_{k-1}^b - d_{\bar{\pi}_k}^T r | F_{k-1} \right] \\
\overset{(1)}{\leq} (2c + 1)v_K \left( \sqrt{SA} + \sqrt{16 \log K} \right) \sum_{k=1}^{K} \mathbb{E}\left[ ||d_{\pi_k}(\bar{P}_{k-1})||_{A_{k-1}^{-1}} | F_{k-1} \right] + 2(2c + 1)v_K \sqrt{SAH} \\
+ \sum_{k=1}^{K} \mathbb{E}\left[ d_{\pi_k}(\bar{P}_{k-1})^T \hat{r}_{k-1}^b - d_{\bar{\pi}_k}^T r | F_{k-1} \right] \tag{21}
\]

where the last inequality is since \(v_k \leq v_K\) for any \(k \leq K\) and \(\sum_{k=1}^{K} \frac{1}{k^2} \leq 2\). Next, we further decompose the
The second term of (22) can be bounded by

\[ \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}(\tilde{P}_{k-1})^T \tilde{r}_{k-1} - d_{\pi_k}^T r|F_{k-1}] \]

\[ = \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}(\tilde{P}_{k-1})^T (\tilde{r}_{k-1} - r)|F_{k-1}] + \sum_{k=1}^{K} \mathbb{E}\left[ (d_{\pi_k} - d_{\pi_k}(\tilde{P}_{k-1}))^T r|F_{k-1} \right] + \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}(\tilde{P}_{k-1})^T b_{k-1}^{pv}] \]

\[ \leq \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}(\tilde{P}_{k-1})\|_{A_{k-1}} \|\tilde{r}_{k-1} - r\|_{A_{k-1}} |F_{k-1}\right] + \sum_{k=1}^{K} \mathbb{E}[\|d_{\pi_k} - d_{\pi_k}(\tilde{P}_{k-1})\|_1 r\|_{\infty} |F_{k-1}|] \]

\[ + \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}(\tilde{P}_{k-1})^T b_{k-1}^{pv}] \]

(Cauchy-Schwarz)

where the last inequality holds under \( E^* \), since \( l_{k-1} \leq l_{K-1} \) for all \( k \leq K \) and \( ||r||_{\infty} \leq 1 \). Substituting back into (21) and defining \( g_k = l_k + (2c+1)\nu_k \left( \sqrt{SAH} + \sqrt{16 \log k} \right) \), we get

\[ \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}^T r - d_{\pi_k}^T \tilde{r}|F_{k-1}] \leq gK \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}(\tilde{P}_{k-1})\|_{A_{k-1}} |F_{k-1}\right] + \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}(\tilde{P}_{k-1})^T b_{k-1}^{pv}] \]

\[ + \sum_{k=1}^{K} \mathbb{E}[\|d_{\pi_k} - d_{\pi_k}(\tilde{P}_{k-1})\|_1 |F_{k-1}|] + 2(2c+1)\nu_K \sqrt{SAH} \]

(22)

Before we continue, we focus on simplifying the first two terms of (22). The first term of (22) can be bounded by

\[ \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}(\tilde{P}_{k-1})\|_{A_{k-1}} |F_{k-1}\right] \leq \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}\|_{A_{k-1}} |F_{k-1}\right] + \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}(\tilde{P}_{k-1}) - d_{\pi_k}\|_{A_{k-1}} |F_{k-1}\right] \]

\[ \leq \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}\|_{A_{k-1}} |F_{k-1}\right] + \frac{1}{\sqrt{H}} \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}(\tilde{P}_{k-1}) - d_{\pi_k}\|_1 |F_{k-1}\right] \]

where in the last inequality, we used the fact that \( \lambda = H \), and therefore, for any \( x \in \mathbb{R}^{SA} \),

\[ ||x||_{A_{k-1}} \leq \frac{1}{\sqrt{\lambda}} ||x||_2 = \frac{1}{\sqrt{H}} ||x||_2 \leq \frac{1}{\sqrt{H}} ||x||_1 \]

The second term of (22) can be bounded by

\[ \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k}(\tilde{P}_{k-1})^T b_{k-1}^{pv} |F_{k-1}] \]

\[ = \sum_{k=1}^{K} \mathbb{E}\left[ (d_{\pi_k}(\tilde{P}_{k-1}) - d_{\pi_k}) b_{k-1}^{pv} |F_{k-1}\right] + \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k} b_{k-1}^{pv} |F_{k-1}] \]

\[ \leq \sum_{k=1}^{K} \mathbb{E}\left[ \|d_{\pi_k}(\tilde{P}_{k-1}) - d_{\pi_k}\| b_{k-1}^{pv} \|_{\infty} |F_{k-1}\right] + \sum_{k=1}^{K} \mathbb{E}[d_{\pi_k} b_{k-1}^{pv} |F_{k-1}] \]

(Hölder’s inequality)

\[ \leq H \sqrt{\frac{1}{2} \log \left( \frac{40SAH K}{\delta} \right)} \sum_{k=1}^{K} \mathbb{E}[\|d_{\pi_k}(\tilde{P}_{k-1}) - d_{\pi_k}\|_1 |F_{k-1}\]

\[ + H \sqrt{\frac{1}{2} \log \left( \frac{40SAH K}{\delta} \right)} \sum_{k=1}^{K} \mathbb{E}\left[ \sum_{k=1}^{H} \frac{1}{\sqrt{n_{k-1}(s_{k-1}^k, a_{k}^k)} \vee 1} |F_{k-1}| \right] \].

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where we bounded $\|h_{k-1}^{\pi_{k}}\|_{\infty} \leq H \sqrt{\frac{3}{4} \log \left(\frac{40SAHK}{\delta}\right)}$.

Finally, substituting both these terms back into (22) yields

$$
\sum_{k=1}^{K} \mathbb{E}\left[(d_{k}^{T} - d_{\pi_{k}}^{T}) \mathbb{1}(E^{k}(k) | F_{k-1})\right]
$$

$$
\leq g_{K} \sum_{k=1}^{K} \mathbb{E}\left[\|d_{\pi_{k}}\|_{A^{-1}_{k-1}} | F_{k-1}\right] + \left(\frac{g_{K}}{\sqrt{H}} + H \sqrt{\frac{1}{2} \log \left(\frac{40SAHK}{\delta}\right)} + 1 \right) \sum_{k=1}^{K} \mathbb{E}\left[\|d_{\pi_{k}}(\tilde{P}_{k-1}) - d_{\pi_{k}}\|_{1} | F_{k-1}\right]
$$

$$
+ H \sqrt{\frac{1}{2} \log \left(\frac{40SAHK}{\delta}\right)} \sum_{k=1}^{K} \mathbb{E}\left[\sum_{h=1}^{H} \frac{1}{\sqrt{n_{k-1}(s_{h}^{k}, a_{h}^{k}) \vee 1}} | F_{k-1}\right] + 2(2c + 1)v_{K} \sqrt{SAH}. \quad (23)
$$

**Bounding (a):** Conditioning on the good event, and, specifically, when $E^{d}$ holds, term (a) is bounded by

$$
(a) \leq O \left(\frac{g_{K}}{\sqrt{H}} \mathbb{E}_{\pi}^{d} \left|HKS\log \left(\frac{Hk}{\delta}\right)\right|\right) \leq O \left((SA)^{3/2}H \sqrt{K \log(K) \log \left(\frac{KH}{\delta}\right)}\right)
$$

where the second relation holds since $g_{K} \leq O \left(SA \sqrt{H \log(K) \log \left(\frac{KH}{\delta}\right)}\right)$. Observe that this term also appeared in the case the transition model is given (see Section D).

**Bounding (b):** Term (b) originates from the fact the transition model is not the true one and, thus, needs to be learned. Specifically, we use the following bound:

$$
\sum_{k=1}^{K} \mathbb{E}\left[\|d_{\pi_{k}}(\tilde{P}_{k-1}) - d_{\pi_{k}}\|_{1} | F_{k-1}\right]
$$

$$
\leq H \sum_{k=1}^{K} \mathbb{E}\left[\sum_{h=1}^{H} P(\cdot | s_{h}^{k}, a_{h}^{k}) - \tilde{P}_{k-1}(\cdot | s_{h}^{k}, a_{h}^{k}) \|_{1} | F_{k-1}\right] \quad \text{(Lemma 18)}
$$

$$
\leq O \left(H \sqrt{S \log \left(\frac{SAHK}{\delta}\right)} \sum_{k=1}^{K} \mathbb{E}\left[\sum_{h=1}^{H} \frac{1}{\sqrt{n_{k-1}(s_{h}^{k}, a_{h}^{k}) \vee 1}} | F_{k-1}\right] \right) \quad \text{(The Event $E^{p}$ holds)}
$$

Then, noticing that

$$
\frac{g_{K}}{\sqrt{H}} + H \sqrt{\frac{1}{2} \log \left(\frac{40SAHK}{\delta}\right)} + 1 = O \left((SA + H) \sqrt{\log K \log \left(\frac{SAHK}{\delta}\right)}\right),
$$

we get

$$
(b) \leq O \left(H^{2} \sqrt{S(SA + H)^{2} \log \left(\frac{SAHK}{\delta}\right)} + SH \sqrt{AHK \log \left(\frac{SAHK}{\delta}\right)}\right).
$$

**Bounding (c):** Term (c) measures the on policy value of the bonus on the estimates MDP. We bound this term using $E^{N}$ by

$$
(c) \leq O \left(H^{2} \left(SA + H \log \left(\frac{K}{\delta}\right)\right) \sqrt{\log \left(\frac{SAHK}{\delta}\right)} + \sqrt{SAHK \log HK \log \left(\frac{SAHK}{\delta}\right)}\right).
$$

Importantly, notice that (a) and (c) are negligible, when compared to (b). Then, substituting the bounds back into (23) concludes the proof. \qed
F Value Difference Lemmas

Lemma 17 (Value difference lemma, e.g., Dann et al. 2017, Lemma E.15). Consider two MDPs $M = (S,A,P,r,H)$ and $M' = (S,A,P',r',H)$. For any policy $\pi$ and any $s,h$ the following relation holds:

$$V^\pi_h(s; M) - V^\pi_h(s; M')$$

$$= \mathbb{E} \left[ \sum_{h'=h}^H (r_h'(s_h',a_h') - r_h(s_h',a_h')) + (P - P')(\cdot | s_h',a_h')^\top V^\pi_{h+1}(\cdot; M') | s_h = s, \pi, P) \right].$$

The following lemma, to the best of our knowledge, is a new result. It bound the $L_1$ norm of the occupancy measure of a fixed policy for different models, $\|q^\pi(P_1) - q^\pi(P_2)\|_1$, by the on trajectory difference of models. Using this result, as oppose to previous techniques (e.g., Jin and Lue (2019)), simplifies the analysis by much. Furthermore, it highlights an interesting relation between the occupancy measure and the difference between models which was not observed in previous literature.

Lemma 18 (Total Variation Occupancy Measure Difference). Let $P_1$ and $P_2$ represent two transition probabilities defined on a common state-action space. Let $\pi$ be a fixed policy and let $q^\pi(P_1), q^\pi(P_2) \in \mathbb{R}^{S^2}$ be the occupancy measure of $\pi$ w.r.t. $P_1$ and $P_2$, respectively, and w.r.t. a common fixed initial state $s_1$. Then,

$$\|q^\pi(P_1) - q^\pi(P_2)\|_1 \leq H \mathbb{E} \left[ \sum_{h=1}^H \|P_1(\cdot | s_h, a_h) - P_2(\cdot | s_h, a_h)\|_1 | s_1, \pi, P_2 \right]$$

Furthermore, for $d_\pi(s; a; p) = \sum_{h=1}^H q^\pi_h(s, a; P)$, it holds that

$$\|d_\pi(P_1) - d_\pi(P_2)\|_1 \leq H \mathbb{E} \left[ \sum_{h=1}^H \|P_1(\cdot | s_h, a_h) - P_2(\cdot | s_h, a_h)\|_1 | s_1, \pi, P_2 \right]$$

Proof. Fix $s, a, h$. Observe that

$$q^\pi_h(s, a; P_1) = \mathbb{E} \left[ \mathbb{1}(s_h = s, a_h = a) | s_1, P_1, \pi \right] = \mathbb{E} \left[ \sum_{h'=1}^H r_h^{s,a,h}(s_h', a_h') | s_1, P_1, \pi \right],$$

where

$$r_h^{s,a,h}(s', a') = \mathbb{1}(s' = s, a' = a, h' = h).$$

Thus, we can perceive $q^\pi_h(s, a; P_1) - q^\pi_h(s, a; P_2)$ as the difference in values w.r.t. to two MDPs $M_1 = (S,A,P_1, r^{s,a,h}, H)$, $M_2 = (S,A,P_2, r^{s,a,h}, H)$ and fixed policy $\pi$. Doing so, we can apply the value difference lemma and get

$$\|q^\pi_h(s, a; P_1) - q^\pi_h(s, a; P_2)\|_1 = \left| \sum_{h'=1}^H \mathbb{E} \left[ (P_1(\cdot | s_h', a_h') - P_2(\cdot | s_h', a_h'))^\top V^\pi_{h'+1}(s, a, P_1) | s_1, \pi, P_2 \right] \right| \leq \sum_{h'=1}^H \mathbb{E} \left[ \sum_{s''} |P_1(s'' | s_h', a_h') - P_2(s'' | s_h', a_h')| V^\pi_{h'+1}(s''; s, a, P_1) | s_1, \pi, P_2 \right].$$

where the inequality is by the triangle inequality and since $V \geq 0$, Summing on both sides on $s, a, h$ and using linearity of expectation we get

$$\sum_{s,a,h} |q^\pi_h(s, a; P_1) - q^\pi_h(s, a; P_2)|$$

$$\leq \sum_{h'=1}^H \mathbb{E} \left[ \sum_{s''} |P_1(s'' | s_h', a_h') - P_2(s'' | s_h', a_h')| V^\pi_{h'+1}(s''; s, a, P_1) | s_1, \pi, P_2 \right],$$

$$\leq \sum_{h'=1}^H \mathbb{E} \left[ \sum_{s''} |P_1(s'' | s_h', a_h') - P_2(s'' | s_h', a_h')| V^\pi_{h'+1}(s''; s, a, P_1) | s_1, \pi, P_2 \right].$$

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Observe that by definition

\[ V_{h'+1}(s''; s, a, h, P_1) = \mathbb{E}[\mathbb{I}(s_h = s, a_h = a) \mid s'', \pi, P_1], \]

for \( h' + 1 \leq h \) and zero otherwise. Using this, we see that for any \( s'' \) and \( h' + 1 \)

\[ \sum_{s, a, h} V_{h'+1}^\pi(s''; s, a, h, P_1) = \mathbb{E} \left[ \sum_{s, a} \sum_{h=h'+1}^H \mathbb{I}(s_h = s, a_h = a) \mid s_{h'+1} = s'', \pi, P_1 \right] \leq H \]

since in a single trajectory a policy can visit at most at \( H \) state-action pairs. Plugging this bound back and observing that

\[ \sum_{s''} |P_1(s'' \mid s_{h'}, a_{h'}) - P_2(s'' \mid s_{h'}, a_{h'})| = \|P_1(\cdot \mid s_{h'}, a_{h'}) - P_2(\cdot \mid s_{h'}, a_{h'})\|_1 \]

concludes the proof of the first inequality. The second inequality is a direct result of the first inequality and the triangle inequality, namely,

\[ \|d_\pi(P_1) - d_\pi(P_2)\|_1 = \sum_{s, a, h} |q_h^\pi(s, a; P_1) - q_h^\pi(s, a; P_2)| \]

\[ \leq \sum_{s, a, h} |q_h^\pi(s, a; P_1) - q_h^\pi(s, a; P_2)| \quad \text{(triangle inequality)} \]

\[ = \|q^\pi(P_1) - q^\pi(P_2)\|_1 \]

\[ \leq H \mathbb{E} \left[ \sum_{h=1}^H \|P_1(\cdot \mid s_h, a_h) - P_2(\cdot \mid s_h, a_h)\|_1 \mid s_1, \pi, P_2 \right]. \]
G Cumulative Visitation Bounds

The following result follows the analysis of Lemma 10 in \cite{Jin2019} and generalizes their result to the stationary MDP setting (by doing so, a factor of $\sqrt{H}$ is shaved off relatively to their bound). This result can also be found in other works such as \cite{Dann2015}. However, we find the analysis somewhat simpler and for this reason we supply it.

**Lemma 19 (Expected Cumulative Visitation Bound).** Let $\{F_k\}_{k=1}^K$ be a filtration. Then, with probability greater than $1 - \delta$ it holds that

$$
\sum_{k=1}^K \mathbb{E} \left[ \sum_{h=1}^H \frac{1}{\sqrt{n_{k-1}(s_h^k, a_h^k)} \lor 1} \mid F_{k-1} \right] \leq 16H^2 \log \left( \frac{1}{\delta} \right) + 4SAH + 2\sqrt{SAHK \log HK}
$$

$$
= O \left( H \left( SA + H \log \left( \frac{1}{\delta} \right) \right) + \sqrt{SAHK \log HK} \right).
$$

**Proof.** We have that

$$
\sum_{k=1}^K \mathbb{E} \left[ \sum_{h=1}^H \frac{1}{\sqrt{n_{k-1}(s_h^k, a_h^k)} \lor 1} \mid F_{k-1} \right] = \sum_{k=1}^K \sum_{s,a} \frac{d_{\pi_k}(s,a)}{\sqrt{n_{k-1}(s,a) \lor 1}}
$$

$$
= \sum_{k=1}^K \sum_{s,a} d_{\pi_k}(s,a) - \sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\} \sum_{k=1}^K \sum_{s,a} \frac{\sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\}}{\sqrt{n_{k-1}(s,a) \lor 1}}.
$$

Observe that $\sum_{s,a} \frac{d_{\pi_k}(s,a) - \sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\}}{\sqrt{n_{k-1}(s,a) \lor 1}} \leq H$. Applying Freedman’s Inequality (Lemma 23) with $\eta = 1/8H$ we get

$$
(i) \leq \frac{1}{8H} \sum_{k=1}^K \mathbb{E} \left[ \left( \sum_{s,a} \frac{d_{\pi_k}(s,a) - \sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\}}{\sqrt{n_{k-1}(s,a) \lor 1}} \right)^2 \mid F_{k-1} \right] + 8H^2 \log \left( \frac{1}{\delta} \right).
$$

Furthermore, we have that

$$
\mathbb{E} \left[ \left( \sum_{s,a} \frac{d_{\pi_k}(s,a) - \sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\}}{\sqrt{n_{k-1}(s,a) \lor 1}} \right)^2 \mid F_{k-1} \right] \leq 2 \mathbb{E} \left[ \left( \sum_{s,a} \frac{\sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\}}{\sqrt{n_{k-1}(s,a) \lor 1}} \right)^2 \mid F_{k-1} \right] + 2 \mathbb{E} \left[ \left( \sum_{s,a} \frac{d_{\pi_k}(s,a)}{\sqrt{n_{k-1}(s,a) \lor 1}} \right)^2 \mid F_{k-1} \right]
$$

$$
\leq 2H \mathbb{E} \left[ \sum_{s,a} \frac{\sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\}}{\sqrt{n_{k-1}(s,a) \lor 1}} \mid F_{k-1} \right] + \sum_{s,a} d_{\pi_k}(s,a) \sqrt{n_{k-1}(s,a) \lor 1}.
$$

(1) is since $\forall a, b, \ (a + b)^2 \leq 2(a^2 + b^2)$. In (2), notice that for both terms, the argument of the square function $f(x) = x^2$ is in $[0, H]$; then, we can bound $x^2 \leq Hx$. Finally, for (3), recall that $d_{\pi_k}(s,a) = \mathbb{E} \left[ \sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\} \mid F_{k-1} \right]$. Plugging this and (25) back into (24), we get

$$
\sum_{k=1}^K \sum_{s,a} \frac{d_{\pi_k}(s,a)}{\sqrt{n_{k-1}(s,a) \lor 1}} \leq \frac{1}{2} \sum_{k=1}^K \sum_{s,a} \frac{d_{\pi_k}(s,a)}{\sqrt{n_{k-1}(s,a) \lor 1}} + 8H^2 \log \left( \frac{1}{\delta} \right) + \sum_{k=1}^K \sum_{s,a} \frac{\sum_{h=1}^H 1\{s_h^k = s, a_h^k = a\}}{\sqrt{n_{k-1}(s,a) \lor 1}},
$$

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and, thus,

\[
\sum_{s,a} \sum_{k=1}^{K} \frac{d_{s_k}(s,a)}{\sqrt{n_{k-1}(s,a) \lor 1}} \leq 16H^2 \log \left( \frac{1}{\delta} \right) + 2 \sum_{s,a} \sum_{k=1}^{K} \frac{\sum_{h=1}^{H} 1(s_h^k = s, a_h^k = a)}{\sqrt{n_{k-1}(s,a) \lor 1}}.
\]  

(26)

To bound the second term in (26) we fix an \(s, a\) pair and bound the following sum.

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1(s_h^k = s, a_h^k = a)}{\sqrt{n_{k-1}(s,a) \lor 1}}
\]

\[
= \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1(n_{k-1}(s,a) \lor 1 < H)}{\sqrt{n_{k-1}(s,a) \lor 1}} + \sum_{h=1}^{H} \frac{1(n_{k-1}(s,a) \lor 1 \geq H)}{\sqrt{n_{k-1}(s,a) \lor 1}}
\]

\[
\leq 2H + \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1(n_{k-1}(s,a)) \lor 1 \geq H)}{\sqrt{n_{k-1}(s,a) \lor 1}}
\]

\[
\leq 2H + \sqrt{KH} \left( \sum_{k=1}^{K} 1(n_{k-1}(s,a) \lor 1 \geq H) \right)
\]

(1)

(2)

(3)

where (1) is since if \(1(s_h^k = s, a_h^k = a) = 1\), then \(n_k(s,a)\) will increase by 1; therefore, both indicators in the first term can be true only \(2H - 1 \leq 2H\) times (the extreme case is when \(n_{k-1}(s,a) = H - 1\) and \(s_h^k = s, a_h^k = a\) for all \(h \in [H]\)). Next, (2) is by Cauchy-Schwartz inequality and (3) holds by Lemma 20. Plugging this bound back into (26) we get

\[
(26) \leq 16H^2 \log \left( \frac{1}{\delta} \right) + 2 \sum_{s,a} \sum_{h=1}^{H} \left( 1(s_h^k = s, a_h^k = a) \right)
\]

\[
\leq 16H^2 \log \left( \frac{1}{\delta} \right) + 4S AH \leq 2 \sqrt{SAH K \log \left( \frac{1}{\delta} \right)}
\]

(Jensen’s Inequality)

\[
= 16H^2 \log \left( \frac{1}{\delta} \right) + 4S AH + 2 \sqrt{SAH K \log \left( \frac{1}{\delta} \right)}
\]

(\(\sum_{s,a} n_k(s,a) = HK\))

where we used the fact that \(\sqrt{\log(x)}\) is concave for \(x \geq 1\). This concludes the proof.

\[
\square
\]

**Lemma 20** (Cumulative Visitation Bound). For any fixed \((s,a)\) pair, it holds that

\[
\sum_{k=1}^{K} 1(n_{k-1}(s,a) \geq H) \frac{\sum_{h=1}^{H} 1(s_h^k = s, a_h^k = a)}{n_{k-1}(s,a) \lor 1} \leq 2 \log(n_k(s,a) \lor 1).
\]
Proof. The following relations hold for any fixed $s, a$ pair.

\begin{align*}
\sum_k \mathbb{I}(n_{k-1}(s, a) \geq H) \sum_{h=1}^{H} \frac{\mathbb{I}(s_h^k = s, a_h^k = a)}{n_{k-1}(s, a) \lor 1} \\
= \sum_k \mathbb{I}(n_{k-1}(s, a) \geq H) \sum_{h=1}^{H} \frac{\mathbb{I}(s_h^k = s, a_h^k = a)}{n_k(s, a)} \frac{n_k(s, a)}{n_{k-1}(s, a)} \\
\leq 2 \sum_k \mathbb{I}(n_{k-1}(s, a) \geq H) \frac{n_k(s, a) - n_{k-1}(s, a)}{n_k(s, a)} \\
\leq 2 \sum_k \mathbb{I}(n_{k-1}(s, a) \geq H) \log \left( \frac{n_k(s, a)}{n_{k-1}(s, a)} \right) \\
\leq \mathbb{I}(n_K(s, a) \geq H) \cdot 2 \log n_K(s, a) - 2 \log(H) \\
\leq 2 \log(n_K(s, a) \lor 1).
\end{align*}
H Useful Results

**Theorem 21** (Abbasi-Yadkori et al. 2011, Theorem 2). Let \( \{F_k\}_{k=0}^\infty \) be a filtration. Let \( \{\eta_k\}_{k=0}^\infty \) be a real-valued stochastic process such that \( \eta_k \) is \( F_k \)-measurable and \( \eta_k \) is conditionally \( \sigma \)-sub-Gaussian for \( \sigma \geq 0 \). Let \( \{x_k\}_{k=0}^\infty \) be an \( \mathbb{R}^d \)-valued stochastic process s.t. \( X_k \) is \( F_{k-1} \)-measurable and \( \|x_k\| \leq L \). Define \( y_k = \langle x_k, w \rangle + \eta_k \) and assume that \( \|w\| \leq R \) and \( \lambda > 0 \). Let

\[
\hat{w}_k = (X_k^T X_k + \lambda I_d)^{-1}X_k^T y_k,
\]

where \( X_k \) is the matrix whose rows are \( x_1^T, \ldots, x_k^T \) and \( Y_k = (y_1, \ldots, y_k)^T \). Then, for any \( \delta > 0 \) with probability at least \( 1 - \delta \) for all, \( t \geq 0 \) \( w \) lies in the set

\[
\left\{ w \in \mathbb{R}^d : \|\hat{w}_k - w\|_{V_k} \leq \sigma \sqrt{d \log \frac{1 + kL^2/\lambda}{\delta}} + \lambda^{1/2} R \right\}.
\]

The previous theorem can be easily extended to our setting, as stated in Proposition 1:

**Proposition 1** (Concentration of Reward). Let \( A_k \overset{\text{def}}{=} D_k^T D_k + \lambda I_{SA} \) for some \( \lambda > 0 \). For any \( \delta \in (0,1) \), with probability greater than \( 1 - \delta/10 \) uniformly for all \( k \geq 0 \), it holds that

\[
\|r - \hat{r}_k\|_{A_k} \leq \sqrt{\frac{1}{4} SAH \log \left( \frac{1 + kH^2/\lambda}{\delta/10} \right)} + \sqrt{\lambda SA} \overset{\text{def}}{=} l_k.
\]

**Proof.** Define the filtration \( \bar{F}_k = \sigma(\hat{d}_{\pi_1}, \ldots, \hat{d}_{\pi_{k+1}}, \eta_1, \ldots, \eta_k) \), where

\[
\eta_k = \sum_{h=1}^H (R(s^k_h, a^k_h) - r(s^k_h, a^k_h)) = \sum_{h=1}^H R(s^k_h, a^k_h) - \hat{a}^T \hat{r} x r.
\]

Specifically, notice that \( \hat{d}_k \in \mathbb{R}^{SA} \) is \( F_{k-1} \)-measurable, \( \eta_k \) is \( F_k \)-measurable and that \( \eta_k \) is \( \sqrt{H/2} \) sub-Gaussian given \( F_{k-1} \), as a (centered) sum of \( H \) conditionally independent random variables bounded in \([0, 1]\). Also note that \( \|r\|_2 \leq \sqrt{SA} \) and \( \|d_k\|_2 \leq H \). Then, for \( A_k = D_k D_k^T + \lambda I \), we can apply Theorem 2 of Abbasi-Yadkori et al. 2011 (also restated in Theorem 21). Specifically, the theorem implies that for any \( \delta' > 0 \) with probability at least \( 1 - \delta' \), it holds that

\[
\forall k \geq 0, \|\hat{r}_k - r\|_{A_k} \leq \sqrt{\frac{1}{4} SAH \log \left( \frac{1 + kH^2/\lambda}{\delta'} \right)} + \sqrt{\lambda SA}
\]

Applying this results for \( \delta' = \frac{\delta}{10} \) concludes the proof.

**Lemma 22** (Elliptical Potential Lemma, Abbasi-Yadkori et al. 2011, Lemma 11). Let \( \{x_k\}_{k=1}^\infty \) be a sequence in \( \mathbb{R}^d \) and \( V_k = V + \sum_{i=1}^k x_i x_i^T \). Assume \( \|x_k\| \leq L \) for all \( k \). Then,

\[
\sum_{i=1}^k \min \left( \|x_i\|_{V_i^{-1}}^2, 1 \right) \leq 2 \log \frac{\det(V_k)}{\det(V)} \leq 2d \log \left( \frac{\text{trace}(V) + kL^2}{d} \right) - 2 \log \det(V).
\]

Furthermore, if \( \lambda_{\min}(V) \geq \max(1, L^2) \) then

\[
\sum_{i=1}^t \|x_i\|_{V_i^{-1}}^2 \leq 2 \log \frac{\det(V_t)}{\det(V)} \leq 2d \log \frac{\text{trace}(V) + tL^2}{d}.
\]

**Lemma 23** (Beygelzimer et al. 2011, Freedman’s Inequality). Let \( Y_1, \ldots, Y_K \) be a martingale difference sequence w.r.t. a filtration \( \{F_k\}_{k=1}^K \). Assume \( |Y_k| \leq R \) a.s. for all \( k \). Then, for any \( \delta \in (0,1) \) and \( \eta \in [0, 1/R] \), with probability greater than \( 1 - \delta \) it holds that

\[
\sum_{k=1}^K Y_k \leq \sum_{k=1}^K E[Y_k^2 | F_{k-1}] + \frac{R}{\eta} \log \left( \frac{1}{\delta} \right).
\]
I Properties of Thompson Sampling

**Lemma 7** (Concentration of Thompson Sampling Noise). Let

\[
E^\xi(k) = \{d \in \mathbb{R}^{SA} : |d^T \xi_k| \leq v_k \left( \sqrt{SA + \sqrt{16 \log k}} \right) \|d\|_{A^{-1}_{k-1}} \},
\]

where \( \xi_k \sim \mathcal{N}(0, v_k A^{-1}_{k-1}) \). Then, for any \( k > 0 \) it holds that \( \Pr\{E^\xi(k)|F_{k-1}\} \geq 1 - \frac{1}{k^r} \). Moreover, for any random variable \( X \in \mathbb{R}^{SA} \), it holds that

\[
E[|X^T \xi_k||F_{k-1}|] \leq v_k \left( \sqrt{SA + \sqrt{16 \log k}} \right) E[\|X\|_{A^{-1}_{k-1}}|F_{k-1}|] + \frac{v_k \sqrt{SA}}{k^2} \sqrt{E[|X|^2_{A^{-1}_{k-1}}]|F_{k-1}|}.
\]

**Proof.** Let \( d \in \mathbb{R}^{SA} \) and notice that given \( F_{k-1}, A_{k-1} \) is fixed. Thus, we have

\[
|d^T \xi_k| = |d^T A_{k-1}^{-1/2} \cdot A_{k-1}^{1/2} \xi_k| \\
\overset{(1)}{\leq} \|A_{k-1}^{-1/2} d\|_2 \|A_{k-1}^{1/2} \xi_k\|_2 \\
= v_k \|d\|_{A_{k-1}^{-1}} \|A_{k-1}^{1/2} \xi_k\|_2 \\
= v_k \|d\|_{A_{k-1}^{-1}} \|\xi_k\|_2 \tag{27}
\]

(1) is due to the Cauchy-Schwartz inequality. In (2) we defined \( \zeta_k = \frac{1}{v_k} A_{k-1}^{1/2} \xi_k \in \mathbb{R}^{SA} \), which is a vector with independent standard Gaussian components. Specifically, notice that \( \|\xi_k\|_2 \) is chi-distributed. Then, we can apply Lemma 1 of [Laurent and Massart, 2000], which implies w.p. at least \( 1 - \delta \),

\[
\|\xi_k\|_2 \leq \sqrt{SA + 2 \sqrt{SA \log \frac{1}{\delta} + 2 \log \frac{1}{\delta}}} = \left( \sqrt{SA + \sqrt{\log \frac{1}{\delta}}} \right)^2 + \log \frac{1}{\delta} \leq \sqrt{SA + \sqrt{\log \frac{1}{\delta} + \sqrt{\log \frac{1}{\delta}}}}
\]

Taking \( \delta = \frac{1}{k^r} \) and substituting this bound into (27) implies that w.p. at least \( 1 - \frac{1}{k^r} \),

\[
|d^T \xi_k| \leq v_k \|d\|_{A_{k-1}^{-1}} \left( \sqrt{SA + \sqrt{16 \log k}} \right),
\]

and thus \( \Pr\{E^0(k)|F_{k-1}\} \geq 1 - \frac{1}{k^r} \).

To prove the second result of the lemma, we decompose the conditional expectation as follows:

\[
E[|X^T \xi_k||F_{k-1}|] = E[|X^T \xi_k||E^\xi(k)|F_{k-1}] + E\left[|X^T \xi_k| \mathbb{1}(E^\xi(k))|F_{k-1}\right].
\]

For the first term of (28), notice that the bound in \( E^\xi(k) \) holds for any \( d \in \mathbb{R}^{SA} \); therefore, it also holds for any random variable \( X \in \mathbb{R}^{SA} \):

\[
E[|X^T \xi_k| \mathbb{1}(E^\xi(k))|F_{k-1}] \leq v_k \left( \sqrt{SA + \sqrt{16 \log k}} \right) E\left[\|X\|_{A_{k-1}^{-1}} \mathbb{1}(E^\xi(k))|F_{k-1}\right] \\
\leq v_k \left( \sqrt{SA + \sqrt{16 \log k}} \right) E\left[\|X\|_{A_{k-1}^{-1}}|F_{k-1}\right].
\]
For the second term of (28), we apply Cauchy-Schwarz (CS) inequality:
\[
\mathbb{E}\left[|X^T \xi_k| \mathbb{1}\left(E^k(k)\right) |F_{k-1}\right] \overset{(CS)}{\leq} \sqrt{\mathbb{E}\left[|X^T \xi_k|^2 |F_{k-1}\right]} \sqrt{\mathbb{E}\left[\mathbb{1}\left(E^k(k)\right) |F_{k-1}\right]}
\]
\[
= v_k \sqrt{\mathbb{E}\left[|X^T A_{k-1}^{-1/2} \xi_k|^2 |F_{k-1}\right]} \sqrt{\mathbb{P}\left\{E^k(k) |F_{k-1}\right\}}
\]
\[
\overset{(CS)}{\leq} v_k \sqrt{\mathbb{E}\left[\|X\|_k^2 |F_{k-1}\right]} \sqrt{\mathbb{E}\left[\|\xi_k\|_k^2 |F_{k-1}\right]} \sqrt{\mathbb{P}\left\{E^k(k) |F_{k-1}\right\}}
\]
\[
\leq \frac{v_k \sqrt{SA}}{k^2} \sqrt{\mathbb{E}\left[\|X\|_{A_{k-1}}^2 |F_{k-1}\right]}
\]
where the last inequality is since \(\mathbb{P}\left\{E^k(k) |F_{k-1}\right\} \leq \frac{1}{k^2}\) and \(\mathbb{E}\left[\|\xi_k\|_k^2 |F_{k-1}\right] = SA\). The proof is concluded by substituting both results back into (28).

\[\square\]

**Lemma 8 (Optimism with Fixed Probability).** Let \(\hat{r}_k = \xi_k + r_{k-1}\). Assume that \(\lambda \in [1, H]\). Then, for any \(k > 1\) and any filtration \(F_{k-1}\) such that \(E^r(k-1)\) is true and any model \(P'\) that is \(F_{k-1}\)-measurable, it holds that
\[
\mathbb{P}\left\{d_{\pi}(P')^T \hat{r}_k > d_{\pi}(P')^T r | F_{k-1}\right\} \geq \frac{1}{c},
\]
for \(c = 2\sqrt{2\pi e}\). Specifically, for any \(\pi_k \in \arg\max_{\pi} d_{\pi}(P')^T \hat{r}_k\) it also holds that
\[
\mathbb{P}\left\{d_{\pi_k}(P')^T \hat{r}_k > d_{\pi_k}(P')^T r | F_{k-1}\right\} \geq \frac{1}{c}.
\]

**Proof.** First, recall that under \(E^r(k-1)\), for any fixed \(d \in \mathbb{R}^SA\), it holds that \(|d^T \hat{r}_{k-1} - d^T r| \leq l_{k-1} \|d\|_{A_{k-1}}\). Specifically, the relation holds for \(d = d_{\pi}(P')\). Also recall that conditioned on \(F_{k-1}, d_{\pi}(P')^T \hat{r}_k\) is a Gaussian random variable with mean \(d_{\pi}(P')^T \hat{r}_{k-1}\) and standard deviation \(v_k \|d_{\pi}(P')\|_{A_{k-1}}\). Then, we can write
\[
\mathbb{P}\left\{d_{\pi}(P')^T \hat{r}_k > d_{\pi}(P')^T r | F_{k-1}\right\} = \mathbb{P}\left\{\frac{d_{\pi}(P')^T \hat{r}_k - d_{\pi}(P')^T \hat{r}_{k-1}}{v_k \|d_{\pi}(P')\|_{A_{k-1}}} > \frac{d_{\pi}(P')^T r - d_{\pi}(P')^T \hat{r}_{k-1}}{v_k \|d_{\pi}(P')\|_{A_{k-1}}} | F_{k-1}\right\},
\]
Next, we define \(Z_k = \frac{d_{\pi}(P')^T r - d_{\pi}(P')^T \hat{r}_{k-1}}{v_k \|d_{\pi}(P')\|_{A_{k-1}}}\) and bound \(Z_k\) as follows:
\[
Z_k \leq |Z_k| = \left|\frac{d_{\pi}(P')^T r - d_{\pi}(P')^T \hat{r}_{k-1}}{v_k \|d_{\pi}(P')\|_{A_{k-1}}}\right| \overset{(1)}{\leq} \frac{l_k \|d_{\pi}(P')\|_{A_{k-1}}}{v_k \|d_{\pi}(P')\|_{A_{k-1}}} = \frac{\sqrt{4SAH \log \left(\frac{1+kH^2}{\delta/10}\right)}}{\sqrt{9SAH \log \frac{kH^2}{\delta/10}}} \overset{(3)}{\leq} 1
\]
where (1) is by the definition of \(E^r(k-1)\) and by bounding \(l_{k-1} \leq l_k\), (2) is a direct substitution of \(l_k\) and \(v_k\) and (3) holds for any \(\lambda \in [1, H]\) and \(k > 1\). Then, we can write
\[
\mathbb{P}\left\{d_{\pi}(P')^T \hat{r}_k > d_{\pi}(P')^T r | F_{k-1}\right\} = \mathbb{P}\left\{\frac{d_{\pi}(P')^T \hat{r}_k - d_{\pi}(P')^T \hat{r}_{k-1}}{v_k \|d_{\pi}(P')\|_{A_{k-1}}} > Z_k | F_{k-1}\right\}
\]
\[
\geq \mathbb{P}\left\{\frac{d_{\pi}(P')^T \hat{r}_k - d_{\pi}(P')^T \hat{r}_{k-1}}{v_k \|d_{\pi}(P')\|_{A_{k-1}}} > 1 | F_{k-1}\right\}
\]
\[
\geq \frac{1}{2\sqrt{2\pi e}} \overset{\text{Borjesson and Sundberg, 1979}}{=} \frac{1}{c},
\]
where the last inequality is since for \(X \sim \mathcal{N}(0, 1)\) and any \(z > 0\), \(\mathbb{P}\{X > z\} \geq \frac{1}{\sqrt{2\pi} e^{-z^2/2}}\). Finally, notice that \(d_{\pi_k}(P')^T \hat{r}_k = \max_{\pi} d_{\pi}(P')^T \hat{r}_k \geq d_{\pi_k}(P')^T \hat{r}_k\). Thus, the second result of the lemma is a direct consequence of its first result. \[\square\]
Lemma 9. Let $\xi_k, \xi'_k$ be i.i.d. random variables given $F_{k-1}$. Also, let $x_{k-1} \in \mathbb{R}^{SA}$ be some $F_{k-1}$-measurable random variable and $P'$ be an $F_{k-1}$-measurable transition kernel. Finally, let $\tilde{\pi} \in \arg \max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi_k)$. Then,

$$
\mathbb{E} \left[ (d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi_k) - \mathbb{E}[d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi_k)|F_{k-1}])^+ |F_{k-1} \right] \leq \mathbb{E}[d_{\tilde{\pi}}(P')^T\xi_k + d_{\tilde{\pi}}(P')^T\xi'_k |F_{k-1}] 
$$

Proof. First, observe that since $d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi_k) = \max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi_k)$ and $\xi_k, \xi'_k$ are identically distributed, it also holds that $d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi_k) = \max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi'_k)$. Therefore,

$$
\mathbb{E} \left[ (d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi_k) - \mathbb{E}[d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi_k)|F_{k-1}])^+ |F_{k-1} \right] 
= \mathbb{E} \left[ \max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi_k) - \mathbb{E}[\max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi'_k)|F_{k-1}] \right]^+ |F_{k-1} \right], 
$$

(29)

Next, by definition, it holds that

$$
\mathbb{E}[\max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi'_k)|F_{k-1}] \geq \mathbb{E}[d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi'_k)|F_{k-1}] .
$$

Notice that conditioned on $F_{k-1}$, $\xi_k$ and $\xi'_k$ are independent, and that $x_{k-1}$ and $P'$ are $F_{k-1}$-measurable. Thus, we can write

$$
\mathbb{E}[\max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi_k)|F_{k-1}] = \mathbb{E}[\max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi'_k)|F_{k-1}, \xi_k]
$$

$$
\overset{(*)}{=} \mathbb{E}[\max_{\pi'} d_{\pi'}(P')^T(x_{k-1} + \xi'_k)|F_{k-1}, \xi_k, \pi_k]
$$

$$
\geq \mathbb{E}[d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi'_k)|F_{k-1}, \xi_k, \pi_k],
$$

where $(*)$ is since $\pi_k$ is deterministically determined by $F_{k-1}$ and $\xi_k$. Plugging this back into (29) we get

$$
\mathbb{E}[d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi_k) - \mathbb{E}[d_{\tilde{\pi}}(P')^T(x_{k-1} + \xi'_k)|F_{k-1}, \xi_k, \pi_k])^+ |F_{k-1} \right] \leq \mathbb{E}[d_{\tilde{\pi}}(P')^T\xi_k - d_{\tilde{\pi}}(P')^T\xi'_k|F_{k-1}, \xi_k, \pi_k] \quad (\forall x \in \mathbb{R} : (x)^+ \leq |x|)
$$

$$
\leq \mathbb{E}[d_{\tilde{\pi}}(P')^T\xi_k - d_{\tilde{\pi}}(P')^T\xi'_k|F_{k-1}, \xi_k, \pi_k] \quad \text{(Jensen’s Inequality)}
$$

$$
= \mathbb{E}[d_{\tilde{\pi}}(P')^T\xi_k - d_{\tilde{\pi}}(P')^T\xi'_k|F_{k-1}] \quad \text{(Tower Property)}
$$

$$
\leq \mathbb{E}[d_{\tilde{\pi}}(P')^T\xi_k|F_{k-1}] + \mathbb{E}[d_{\tilde{\pi}}(P')^T\xi'_k|F_{k-1}] \quad \text{(Triangle Inequality)}
$$
