On the Achievable Rate Region of the $K$-Receiver Broadcast Channels via Exhaustive Message Splitting
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Abstract
This paper focuses on $K$-receiver discrete-time memoryless broadcast channels (DM-BCs) with private messages, where the transmitter wishes to convey $K$ private messages to $K$ receivers respectively. A general inner bound on the capacity region is proposed based on an exhaustive message splitting and a $K$-level modified Marton’s coding. The key idea is to split every message into $\sum_{j=1}^{K} \binom{K}{j}$ submessages corresponding to a set of users who are assigned to recover them, and then send these submessages through codewords that are jointly typical with each other. To ensure the joint typicality among all transmitted codewords, a sufficient condition on the subcodebooks sizes is derived through a newly establishing hierarchical covering lemma, which extends the $2$-level multivariate covering lemma to the $K$-level case including $(2^K - 1)$ random variables with more intricate dependence. As the number of auxiliary random variables and rate constraints both increase linearly with $(2^K - 1)$, the standard Fourier-Motzkin elimination procedure becomes infeasible when $K$ is large. To tackle this problem, we obtain the final form of achievable rate region with a special observation of disjoint unions of sets that constitute the power set of $\{1, \ldots, K\}$. The proposed achievable rate region allows arbitrary input probability mass functions (pmfs) and improves over all previously known ones for $K$-receiver ($K \geq 3$) BCs whose input pmfs should satisfy certain Markov chain(s).

Index Terms
Broadcast channel, achievable rate region, superposition coding, Marton’s coding, covering lemma

I. INTRODUCTION
The 2-receiver discrete-time memoryless broadcast channels (DM-BCs) are first introduced by Cover [3], who proposed the prestigious superposition coding that outperforms the traditional time-division strategy. However, superposition coding is optimal only for certain categories of broadcast channels such as degraded, less noisy and more capable BCs [4]. The best known inner bound on the capacity region of DM-BCs is achieved by Marton’s coding with message splitting [2]. The key idea is to split each source message into common and private parts, where the common part is encoded into a cloud-center codeword, and two private parts are encoded into two separate codewords. To enlarge the achievable rate region, the submitted codewords are jointly typical which is guaranteed by a sufficient condition on the sizes of subcodebooks established by the covering lemma [1].

For the general $K$-receiver DM-BCs, most previous work mainly focused on superposition coding and message splitting (or merging) [5–7] where they require certain Markov chains for auxiliary random variables (RVs). Especially, [7] gives a general inner bound based on superposition coding and rate-splitting using notions from order theory and lattices wherein each receiver decodes its intended message (common or private) along with the partial interference designated to it through rate-splitting. The Marton’s coding with $2$-level superposition coding structure (consisting of one cloud-center codebook and $K$ satellite codebooks) can be easily constructed and analyzed by the multivariate covering lemma and packing lemma [1].

In this paper, we consider $K$-receiver DM-BCs including only private messages. To inherit the characteristics of superposition coding and combine them with Marton’s coding, a general inner bound is proposed based on an exhaustive message splitting and a $K$-level modified Marton’s coding. More specifically, every message is split into $\sum_{j=1}^{K} \binom{K}{j}$ submessages, with each corresponding to a set $S$ belonging to the power set of $\{1, \ldots, K\}$. The submessages respecting to $S$ are encoded into an exclusive codeword and will be decoded by receiver $j$ if $j \in S$. To obtain a potentially larger region, we enlarge the subcodebooks sizes and use a $K$-level Marton’s coding to send all codewords that are jointly typical with each other.

Note that there are mainly two challenges on establishing the final form of achievable rate region. The first one is how to derive rate conditions such that all transmitted codewords are jointly typical. The solution is related to the covering lemma. Unfortunately, the known multivariate covering lemma only deals with sequences which are generated conditionally independently [1], while in our scheme there are $(2^K - 1)$ RVs and the sequences are generated under more intricate dependence. We solve this problem by dividing $(2^K - 1)$ RVs into $K$ levels and process them hierarchically, leading to a new lemma called hierarchical covering lemma. The second challenge lies on how to apply Fourier-Motzkin elimination procedure to obtain the final form of the inner bound. As the number of auxiliary RVs and rate conditions increase linearly with $(2^K - 1)$, it’s infeasible using the standard Fourier-Motzkin elimination procedure, in particular when $K$ is large. To tackle this problem, we aggregate the rate of submessages based on a special observation of disjoint unions of sets that constitute the power set of $\{1, \ldots, K\}$, and finally establish the final form of achievable rate region.
we denote them all by a generic function $X_k$ of receiver capacity region.

Each $M$ transmitter sends the channel input $x$ tends to zero as the blocklength $n$.

Let $\{X_{1}, \ldots, X_{K}\}$ be a tuple of $2^K$ RVs where $I_i \in \mathcal{P}(K)$ for $i = 1, \ldots, 2^K$. Given a set $I = \{I_1, I_2, \ldots, I_{|I|}\}$ with $I_1 < I_2 < \cdots < I_{|I|}$ in dictionary order (e.g., $\{1\}, \{2\}, \{1, 2\}$ is in dictionary order), the subtuple of RVs with indices from $I$ is denoted by $X(I) \triangleq (X_{I_1}, \ldots, X_{I_{|I|}})$, and the corresponding realizations by $x(I) \triangleq (x_{i_1}, \ldots, x_{i_{|I|}})$. Similarly, given $2^K$ random vectors $(X^n_{1}, \ldots, X^n_{2^K})$, let $X^n(I) = (X^n_{I_I} : I_I \in I)$. Let $\bigcup_{I_i \in I} \mathcal{S}_{I_i}$ be a disjoint union where $\mathcal{S}_{I_i} \subseteq \mathcal{P}(K)$ and $\mathcal{S}_{I_i} \cap \mathcal{S}_{I_k} = \emptyset$ if $j \neq k$.

For positive integers $k$ and $j$, we define $x^n_{k} \triangleq (x_{k,1}, \ldots, x_{k,j})$, and $X^n_{k} \triangleq (X_{k,1}, \ldots, X_{k,j})$.

We use $\delta(\epsilon) > 0$ to denote a function of $\epsilon$ that tends to zero as $\epsilon \to 0$. When there are multiple functions $\delta_1(\epsilon), \ldots, \delta_k(\epsilon)$, we denote them all by a generic function $\delta(\epsilon)$ with the understanding that $\delta(\epsilon) = \max \{\delta_1(\epsilon), \ldots, \delta_k(\epsilon)\}$.

Let an all-one column vector $v(1, \ldots, 1)$ with a specified dimension be denoted by $I$. Albeit with an abuse of notation, we use a string of elements in a singleton to represent the set with only one element, e.g., $R_{123} = R_{\{123\}}$, $M_{123} = M_{\{123\}}$, $A_i(1) = A_i(\{1\})$, $B_k(123) = B_k(\{123\})$, etc.

II. CHANNEL MODEL

Consider a $K$-receiver DM-BC with only private messages depicted in Fig. 1. The setup is characterized by a input alphabet $\mathcal{X}$, $K$ output alphabets $\mathcal{Y}_k : k \in \mathcal{K}$, and a collection of channel transition pms $p(y_1, \ldots, y_K|x)$. At time $i \in [1 : n]$, the transmitter sends the channel input $x_i \in \mathcal{X}$, receiver $k \in \mathcal{K}$ observes the output $y_{k,i} \in \mathcal{Y}_k$.

The goal of the communication is that the transmitter convey private messages $M_k$ to receiver $k$, for $k \in \mathcal{K}$, respectively. Each $M_k$ is independently and uniformly distributed over the set $\mathcal{M}_k \triangleq [1 : 2^{R_k}]$, where $R_k$ denotes the communication rate of receiver $k$.

The encoder maps the messages $(M_1, M_2, \ldots, M_K)$ to a sequence $x_i \in \mathcal{X}$:

$$X_i = f^{(n)}(M_1, M_2, \ldots, M_K),$$  \hspace{1cm} (1)

and receiver $k \in \mathcal{K}$ uses channel outputs $y^n_k$ to estimate $\hat{M}_k$ as a guess of messages $M_k$:

$$\hat{M}_k = g_k^{(n)}(Y^n_k).$$  \hspace{1cm} (2)

A rate region $(R_k : k \in \mathcal{K})$ is called achievable if for every blocklength $n$, there exists an encoding function $f^{(n)}$ and $K$ decoding functions $g_1^{(n)}, \ldots, g_K^{(n)}$ such that the error probability

$$P_e^{(n)} = \Pr\{\hat{M}_k \neq M_k, \exists k \in \mathcal{K}\}$$  \hspace{1cm} (3)

tends to zero as the blocklength $n$ tends to infinity. The closure of the set of achievable rate tuple $(R_k : k \in \mathcal{K})$ is called the capacity region.
III. PRELIMINARY

In this section, we present decomposition of sets which will be used to in Section IV and V.
Given a set \( S \in \mathcal{P}(K) \), \( k \in S \) and \( l, l' \in K \) with \( l \leq l' \), define
\[
\mathcal{A}^{l,l'}(S) \triangleq \{ S' \in \mathcal{P}(K) : S \subseteq S' \text{ and } |S'| = l, l+1, \ldots, l' \},
\]
\[
\mathcal{B}^{l,l'}_k(S) \triangleq \{ S' \subseteq S : k \in S', |S'| = l, l+1, \ldots, l' \}.
\]

To simplify notations, we let
\[
\mathcal{A}^l = \mathcal{A}^{l,l}(\emptyset), \quad \mathcal{A}^{1:K} = \mathcal{A}^{1:K}(\emptyset),
\]
\[
\mathcal{A} = \mathcal{A}^{1:K}(\emptyset), \quad \mathcal{B}_k(S) = \mathcal{B}^{1:|S|}_k(S), \quad \mathcal{B}_l(S) = \mathcal{B}^{1:l}_l(S).
\]

Next we show that
\[
\bigcup_{k \in T} \mathcal{A}(k) = \bigcup_{i=0}^{\left| T \right| - 1} \mathcal{B}_\pi(i),
\]
where for some \( \pi \in \Pi_T \)
\[
\mathcal{B}_\pi(0) = \mathcal{B}_{\pi(i+1)}(K), \quad \mathcal{B}_\pi(i) = \mathcal{B}_{\pi(i+1)}(K \setminus \{ \pi(1), \ldots, \pi(i) \}).
\]

Note that \( \bigcup_{k \in T} \mathcal{A}(k) \) denotes the subset of \( \mathcal{P}(K) \) whose elements contains at least one index belonging to \( T \), i.e.,
\[
\bigcup_{k \in T} \mathcal{A}(k) = \{ S \in \mathcal{P}(K) : T \cap S \neq \emptyset, |S| = 1, \ldots, K \},
\]
and \( \mathcal{B}_\pi(i) \) contains all sets which include \( \pi(i+1) \), and are subsets of \( K \setminus \{ \pi(1), \ldots, \pi(i) \} \). Also, we can have the following relation:
\[
\mathcal{B}_\pi(i) = \mathcal{A}(\{ \pi(i+1) \}) \setminus \bigcup_{k=1}^{\left| T \right| - 1} \mathcal{A}(\{ \pi(i+1) \pi(k) \}).
\]

Eq. (4) describes how we decompose \( \bigcup_{k \in T} \mathcal{A}(k) \) into disjoint sets \( \mathcal{B}_\pi(i) \), for \( i \in \{ 0 : (\left| T \right| - 1) \} \).
For example, consider the case \( K = 3, T = \{ 1, 2, 3 \} \) and \( \pi = (2, 1, 3) \), we have
\[
\mathcal{A}^{1:2}(\{ 2 \}) = \{ \{ 2 \}, \{ 1, 2 \}, \{ 2, 3 \} \},
\]
\[
\bigcup_{k \in T} \mathcal{A}(k) = \{ \{ 1 \}, \{ 2, 3 \}, \{ 1, 2 \}, \{ 1, 3 \}, \{ 2, 3 \}, \{ 1, 2, 3 \} \},
\]
\[
\mathcal{B}_\pi^1(\{ 1, 2, 3 \}) = \{ \{ 1 \}, \{ 1, 2 \}, \{ 1, 3 \} \}, \quad \mathcal{B}_\pi(0) = \{ \{ 2 \}, \{ 1, 2 \}, \{ 2, 3 \}, \{ 1, 2, 3 \} \},
\]
\[
\mathcal{B}_\pi(i)_{i=1} = \{ \{ 1 \}, \{ 1, 3 \} \}, \quad \mathcal{B}_\pi(i)_{i=2} = \{ 3 \}.
\]

We can easily find that (4) is satisfied. Fig. 2 is given for illustration. To simplify the notation, an arbitrary collection of sets, e.g., \( \{ 1, 12, 123 \} \), is recognized as \( \{ \{ 1 \}, \{ 1, 2 \}, \{ 1, 2, 3 \} \} \)
With the definitions above, we can also obtain the following decomposition:
\[
\bigcup_{k \in T} \mathcal{A}^l(k) = \bigcup_{i=0}^{\left| T \right| - 1} \mathcal{B}_\pi^l(i), \quad \bigcup_{k \in T} \mathcal{A}(k) = \sum_{i=1}^{K} \bigcup_{i=0}^{\left| T \right| - 1} \mathcal{B}_\pi^l(i),
\]
where \( \mathcal{B}_\pi^l(i) = \mathcal{B}_{\pi(i+1)}(K \setminus \{ \pi(m) \}_{m=1}^i) \). Here \( \bigcup_{k \in T} \mathcal{A}^l(k) \) denotes the subset of \( \mathcal{P}(K) \) whose each has cardinality \( l \) and contains at least one index belonging to \( T \), i.e.,
\[
\bigcup_{i \in T} \mathcal{A}^l(i) = \{ S \in \mathcal{P}(K) : T \cap S \neq \emptyset, |S| = l \},
\]
and \( \mathcal{B}_\pi^l(i) \) contains all the sets with cardinalities \( l \), containing \( \pi(i+1) \), and being contained in \( K \setminus \{ \pi(m) \}_{m=1}^i \).
Using the notations presented in Section III, a rate region
Theorem 1. The idea is that every time we pick a collection of sets descended from a set in the layer \(l\), the number of unpicked sets in layer \(l - 1\) is one, where layer \(l\) contains all subsets of \(\mathcal{K}\) with cardinality \(l\). Therefore, \(\bigcup_{k \in \{1, 2, 3\}} \mathcal{A}(k) = \mathbb{B}_1(123) \bigcup \mathbb{B}_2(23) \bigcup \mathbb{B}_3(3)\).

IV. MAIN RESULTS

Theorem 1. Using the notations presented in Section III a rate region \(\{R_k : k \in \mathcal{K}\}\) is achievable for the DM-BC \(p(y_1, \ldots, y_K|x)\) if for all \(T \subseteq \mathcal{K}\) and \(\pi \in \Pi_T\),

\[
\sum_{k \in T} R_k \leq \sum_{i=0}^{\left|T\right|-1} \left( \sum_{S \in \mathcal{B}_\pi(i)} H(U_{S_i} | U(\mathcal{A}(S) \setminus S')) - H(U(\mathcal{B}_\pi(i)) | Y_{\pi(i+1)} U(\mathcal{A}(\pi(i+1)) \setminus \mathcal{B}_\pi(i))) \right)
+ \sum_{l=1}^{K-1} \left( H(U(\bigcup_{i=0}^{\left|T\right|-1} \mathcal{B}_\pi(i)) | U(\mathcal{A}(l+1)) \setminus \mathcal{B}_\pi(i)))
- \sum_{S \in \bigcup_{i=0}^{\left|T\right|-1} \mathcal{B}_\pi(i)} H(U_{S} | U(\mathcal{A}(S) \setminus S)) \right),
\]

for some pmf \(p(U(\mathcal{A}))\) and a function \(X = f(U(\mathcal{A}))\).

Proof. The achievable scheme is based on an exhaustive message splitting and a modified Marton’s coding. More specifically, each message \(M_k\), for \(k \in \mathcal{K}\), is split into \(\sum_{j=1}^{K} M_k \triangleq (M_{k, S} : S \subseteq \mathcal{K}, |S| = 1, 2, \ldots, K)\) with \(M_{k, S} \in \emptyset\) if \(k \notin S\). The submessages \(M_S = (M_{k, S} : k \in \mathcal{K})\) are encoded into a codeword \(u_S^\mathcal{A} \sim p(u_S|u(\mathcal{A}(S) \setminus S))\) and will be decoded by receiver \(j\) if \(j \in S\). Then, a modified Marton’s coding is applied to send \((2^K - 1)\) codewords \((u_S^\mathcal{A} : S \subseteq \mathcal{K}, |S| = 1, \ldots, K)\) that are jointly typical with each other. See more detailed proof in Section V

To ensure arbitrary input pmfs \(p(U(\mathcal{A}))\) in the inner bound above, we need to establish a sufficient condition on the sizes of subcodebooks, which is related to the covering lemma. The multivariate covering lemma in [1] only consider sequences generated under simple dependence, e.g., \((u_1^n, \ldots, u_K^n) \sim p(u_1|u_0)p(u_2|u_0) \ldots p(u_K|u_0)\) given a sequence \(u_0^n \sim P_{U_0^n}\). In our scheme there are \((2^K - 1)\) RVs, represented by \((U_{S} : S \in \mathbb{P}(\mathcal{K}), S \neq \emptyset)\), and each symbol of codeword \(U_S^n\) is generated conditionally independent according to \(p(u_S|u(\mathcal{A}(S) \setminus S))\). At first glance it seems overwhelming to derived the sufficient condition that all transmitted codewords are jointly typical. However, by dividing all subcodebooks into \(K\) levels and processing them recursively in a hierarchical manner, we obtain the following new lemma:

Lemma 1 (Hierarchical Covering Lemma). Let \(U(\mathcal{A}) \sim p(u(\mathcal{A}))\) and \(\epsilon_K < \epsilon_{K-1} \cdots < \epsilon_1\). Let \(U_K^n \sim p(u_K^n)\) be a random sequence with \(\lim_{n \to \infty} P\{U_K^n \in \mathcal{T}_{\epsilon_K}^{(n)}\} = 1\). For each \(l \in [1 : K - 1]\) and \(S \in \mathbb{A}_l\), let \(U_S^n(m_S), m_S \in [1 : 2^{nrS}]\), be pairwise conditionally independent sequences, each distributed according to \(\prod_{i=1}^{l} P_{U_S^n|U(\mathcal{A}(S) \setminus S)}(u_{S,i}|u_{S',i} : S' \in (\mathcal{A}(S) \setminus S))\). Assume that \((U_S^n(m_S) : m_S \in [1 : 2^{nrS}], S \in \mathbb{A}_l)\) are mutually conditionally independent given \((U_S^n(m_S) : S \in (\mathcal{A}(S) \setminus S))\). Then there exists \(\delta(\epsilon_1)\) that tends to zero as \(\epsilon_1 \to 0\) such that

\[
\lim_{n \to \infty} P\{U_S^n(m_S) : S \in \mathbb{A}_1^{(K-1)}, U_K^n \notin \mathcal{T}_{\epsilon_1}^{(n)}\}
\]

for all \((m_S : S \in \mathbb{A}_1^{(K-1)}) \in \prod_{S \in \mathbb{A}_1^{(K-1)}} [1 : 2^{nr_S}]\).
if
\[
\sum_{S \in J_i} r_S > \sum_{S \in J_i} H\left( U_S \mid U(\mathbb{A}(S) \setminus S) \right) - H\left( U(J_i) \mid U(\mathbb{A}^{(l+1):K}) \right) + \delta(\epsilon),
\]
(10)
for all $J_i \subseteq \mathbb{A}$ and $l \in [1 : (K - 1)]$.

**Proof.** See the proof in Appendix A.

**Lemma 2.** Consider the inequality condition in hierarchical covering lemma:
\[
\sum_{S \in J_i} r_S > \sum_{S \in J_i} H\left( U_S \mid U(\mathbb{A}(S) \setminus S) \right) - H\left( U(J_i) \mid U(\mathbb{A}^{(l+1):K}) \right) + \delta(\epsilon).
\]
If the set $J_i$ is split into $N \in [1 : |J_i|]$ disjoint pieces $\{J'_1, J'_2, \ldots, J'_N\}$ satisfying
\[
J_i = \bigcup_{i=1}^{N} J'_i, \text{ with } J'_i \cap J'_j = \emptyset, \text{ if } i \neq j,
\]
then for all $i \in [N]$, we have
\[
\sum_{S \in J'_i} r_S > \sum_{S \in J'_i} H\left( U_S \mid U(\mathbb{A}(S) \setminus S) \right) - H\left( U(J'_i) \mid U(\mathbb{A}^{(l+1):K}) \right) + \delta_{J'_i}(\epsilon).
\]
With simple justification, the sum of lower bounds for $\sum_{S \in J_i} r_S, \forall i \in [N]$ is not greater than the lower bound for $\sum_{S \in J_i} r_S$, which means combinations of split inequalities will not induce a smaller region bounded by the overall inequalities that simultaneously restrain all variables.

**Proof.** It's equivalent to prove that
\[
\sum_{S \in J_i} H\left( U_S \mid U(\mathbb{A}(S) \setminus S) \right) - H\left( U(J_i) \mid U(\mathbb{A}^{(l+1):K}) \right) =^a \sum_{i=1}^{N} \sum_{S \in J'_i} \left( H\left( U_S \mid U(\mathbb{A}(S) \setminus S) \right) - H\left( U(J'_i) \mid U(\mathbb{A}^{(l+1):K}), U(J'_1), \ldots, U(J'_{i-1}) \right) \right)
\]
\[
\geq^b \sum_{i=1}^{N} \left( \sum_{S \in J'_i} H\left( U_S \mid U(\mathbb{A}(S) \setminus S) \right) - H\left( U(J'_i) \mid U(\mathbb{A}^{(l+1):K}) \right) \right),
\]
where (a) follows by the chain rule of entropy and (11); (b) holds because conditioning reduces entropy.

**Corollary 1.** Define
\[
I_{12} := \min_{i \in \{1,2\}} \{ I(U_{12}U_{123}; Y_i) \},
I_{13} := \min_{i \in \{1,3\}} \{ I(U_{13}U_{123}; Y_i) \},
I_{23} := \min_{i \in \{2,3\}} \{ I(U_{23}U_{123}; Y_i) \},
\]
and
\[
\Delta := I(U_2; U_{13}U_{12}U_{23}U_{123}) + I(U_3; U_{12}U_{13}U_{23}U_{123}) + I(U_1; U_2U_{12}U_{13}U_{23}U_{123}) + I(U_1; U_3U_{12}U_{13}U_{23}U_{123}) + I(U_1; U_2U_{12}U_{13}U_{23}U_{123}) + I(U_2; U_3U_{12}U_{13}U_{23}U_{123}).
\]
A rate region \((R_1, R_2, R_3)\) is achievable for 3-receiver DM-BC \(p(y_1, y_2, y_3|x)\) if
\[
\begin{align*}
R_1 &< I(U_1U_{12}U_{13}U_{123}; Y_1) - I(U_1; U_{23}U_{12}U_{13}U_{123}), \\
R_2 &< I(U_2U_{12}U_{23}U_{123}; Y_2) - I(U_2; U_{13}U_{12}U_{13}U_{123}), \\
R_3 &< I(U_3U_{13}U_{23}U_{123}; Y_3) - I(U_3; U_{12}U_{12}U_{13}U_{123}), \\
R_1 + R_3 &< I(U_1U_{12}Y_1 | U_{13}U_{12}U_{13}U_{123}) + I(U_2U_{23}; Y_2 | U_{12}U_{12}U_{13}U_{123}) - I(U_2; U_{12}U_{12}U_{13}U_{123}) - I(U_1; U_{13}U_{12}U_{13}U_{123}) - I(U_1; U_{13}U_{12}U_{13}U_{123}), \\
R_2 + R_3 &< I(U_2U_{12}Y_2 | U_{12}U_{12}U_{13}U_{123}) + I(U_3U_{13}; Y_3 | U_{12}U_{12}U_{13}U_{123}) - I(U_2; U_{12}U_{12}U_{13}U_{123}) - I(U_3; U_{12}U_{12}U_{13}U_{123}) - I(U_1; U_{13}U_{12}U_{13}U_{123}), \\
R_1 + R_2 + R_3 &< I(U_1U_{12}Y_1 | U_{12}U_{12}U_{13}U_{123}) + I(U_2U_{12}; Y_2 | U_{12}U_{12}U_{13}U_{123}) + I(U_3U_{13}; Y_3 | U_{12}U_{12}U_{13}U_{123}) - I(U_2; U_{12}U_{12}U_{13}U_{123}) - I(U_3; U_{12}U_{12}U_{13}U_{123}) - I(U_1; U_{13}U_{12}U_{13}U_{123}) - \Delta, \\
R_1 + R_2 + R_3 &< I(U_1U_{12}Y_1 | U_{13}U_{12}U_{13}U_{123}) + I(U_2U_{12}; Y_2 | U_{12}U_{12}U_{13}U_{123}) + I(U_3U_{13}; Y_3 | U_{12}U_{12}U_{13}U_{123}) - I(U_2; U_{12}U_{12}U_{13}U_{123}) - I(U_3; U_{12}U_{12}U_{13}U_{123}) - I(U_1; U_{13}U_{12}U_{13}U_{123}) - \Delta, \\
R_1 + R_2 + R_3 &< I(U_1U_{12}Y_1 | U_{13}U_{12}U_{13}U_{123}) + I(U_2U_{12}; Y_2 | U_{12}U_{12}U_{13}U_{123}) + I(U_3U_{13}; Y_3 | U_{12}U_{12}U_{13}U_{123}) - I(U_2; U_{12}U_{12}U_{13}U_{123}) - I(U_3; U_{12}U_{12}U_{13}U_{123}) - I(U_1; U_{13}U_{12}U_{13}U_{123}) - \Delta,
\end{align*}
\]
for some pmf \(p(u_1u_2u_3u_{12}u_{13}u_{23}u_{123})\) and a function \(x = f(u_1u_2u_3u_{12}u_{13}u_{23}u_{123})\).

**Proof.** The result directly comes from Theorem 1 with \(K = 3\). For example, for \(\mathcal{T} = \{1, 2\}\) and \(\pi = (1, 2)\), in (9), then we have \(A^K = \{123\}\), \(A^{2, K} = \{12, 13, 23, 123\}\), and,
\[
\mathcal{B}_+(0) = \mathbb{B}_1(123) = \{123, 12, 13, 1\},
\]
\[
\mathcal{B}_+(1) = \mathbb{B}_2(23) = \{23, 2\},
\]
which satisfy the decomposition \(\bigcup_{i \in \{1, 2\}} A^i(i) = \{1, 12, 13, 23, 123\} = \mathcal{B}_+(0) \sqcup \mathcal{B}_+(1)\). With \(\pi = (1, 2)\) and \(\pi = (2, 1)\), we obtain upper bound for \(R_1 + R_2\) in (12d). The remaining rate constraints are acquired similarly.

**Remark 1** (Comparison with superposition coding and Marton’s coding). Our achievable region generalizes that introduced by standard Marton’s coding and superposition coding, i.e., both of them are special cases of our general-form intended coding. Furthermore, the rate region in Theorem 1 contains the regions resulted from the two aforementioned coding schemes.
- Our coding degenerates into Marton’s coding by setting \(U_S = \text{const}\), \(\forall S \in A^{2, (K-1)}\), which indicates our rate region contains that derived by Marton’s coding;
- Since superposition coding is optimal for degraded/less noisy/more capable DM-BC, which is barely guaranteed in many cases, i.e., without knowing the concrete relation according to the Markov chain, our rate region contains that derived from superposition coding.

In the future work, we will evaluate our rate region for some specific DM-BCs to show that our coding scheme strictly improves previously known inner bounds.

V. ACHIEVABLE CODING SCHEME FOR THEOREM 1

We first present our scheme for 3-receiver DM-BC as an illustration, and then extend it to general \(K\)-receiver DM-BC model for \(K \geq 2\).
To simplify notations, we denote \(C_S(\mathbf{m}_S)\) by \(C_S(\mathbf{m})\) with cognition of the subscript of \(\mathbf{m}\) from \(C_S\), similarly for \(U_S^g(\mathbf{m}_S, l_S)\) denoted by \(U_S^g(\mathbf{m}, l)\) and its realization \(u_S^g(\mathbf{m}_S, l_S)\) denoted by \(u_S^g(\mathbf{m}, l)\).
A. Coding scheme for 3-receiver DM-BCs

1) Rate splitting: Divide $M_j \in [1 : 2^R_j]$, $j \in \{1, 2, 3\}$, into four independent messages ($M_{j,S} \in [1 : 2^{R_j \cdot s}]$; $S \in A(j)$). Hence, $R_j = \sum_{S \in A(j)} R_{j,S}$. More precisely

$$
M_1 = (M_{1,123}, M_{1,12}, M_{1,13}, M_{1,1}), \\
M_2 = (M_{2,123}, M_{2,12}, M_{2,23}, M_{2,2}), \\
M_3 = (M_{3,123}, M_{3,23}, M_{3,13}, M_{3,3}), \\
R_1 = R_{1,123} + R_{1,12} + R_{1,13} + R_{1,1}, \\
R_2 = R_{2,123} + R_{2,12} + R_{2,23} + R_{2,2}, \\
R_3 = R_{3,123} + R_{3,23} + R_{3,13} + R_{3,3}.
$$

For convenience, let $\mathbf{R}_{123} \triangleq R_{1,123} + R_{2,123} + R_{3,123}$, $\mathbf{R}_{12} \triangleq R_{1,12} + R_{2,12} + R_{3,12}$, $\mathbf{R}_{13} \triangleq R_{1,13} + R_{3,13}$, $\mathbf{R}_{23} \triangleq R_{2,23} + R_{3,23}$, $\mathbf{R}_1 \triangleq R_{1,1}$, $\mathbf{R}_2 \triangleq R_{2,2}$, $\mathbf{R}_3 \triangleq R_{3,3}$ and $\mathbf{M}_{123} \triangleq (M_{1,123}, M_{1,23}, M_{1,32}, M_{1,3}, M_{1,2}), \mathbf{M}_{12} \triangleq (M_{1,12}, M_{1,2}, M_{1,2}), \mathbf{M}_{23} \triangleq (M_{2,23}, M_{2,32}, M_{2,3}, M_{2,2}), \mathbf{M}_1 \triangleq (M_{1,1}, M_{1,3}).$

2) Codebook generation: Fix a pmf $p(u_{123}, u_{12}, u_{23}, u_{13}, u_1, u_2, u_3)$, Randomly and independently generate $2^{nR_{123}}$ sequences $u^n_{123}(\mathbf{m})$ each according to $\prod_{i=1}^3 p(u_{123,i}(u_{123,i}))$. For each $\mathbf{m}_{12} \in \prod_{i=1}^3 [1 : 2^{nR_{12}}]$, generate a subcodebook $C_{12}(\mathbf{m})$ consisting of $2^{n(R_{12} - R_{12})}$ independent generated sequences $u^n_{12}(\mathbf{m}, l_2) \in \prod_{i=1}^3 [1 : 2^{nR_{12},i}]$, each according to $\prod_{i=1}^3 p(u_{12,i}(u_{12,i}, u_{123,i}))$. In the same way, for each $\mathbf{m}_{23}$ and $\mathbf{m}_{13}$, generate subcodebooks $C_{23}(\mathbf{m})$ and $C_{13}(\mathbf{m})$, respectively. For each $\mathbf{m}_i \in \prod_{i=1}^3 [1 : 2^{nR_{1,i}}]$, generate a subcodebook $C_i(\mathbf{m})$ consisting of $2^{n(R_{i} - R_{i})}$ independent generated sequences $u^n_{i}(\mathbf{m}, l_i) \in \prod_{i=1}^3 [1 : 2^{nR_{1,i},i}]$, each according to $\prod_{i=1}^3 p(u_{i,i}(u_{i,i}, u_{123,i}))$. For $\mathbf{m}_i$ and $\mathbf{m}_j$, generate corresponding subcodebooks similarly.

3) Encoding: For each $\mathbf{m}_S$, $S \in \{1, 2, 3, 12, 13, 23\}$, find an index tuple $(l_1, l_2, l_3, l_{12}, l_{13}, l_{23})$ such that $u^n_S(\mathbf{m}, l) \in C_S(\mathbf{m}), S \in \{1, 2, 3, 12, 13, 23\}$, and

$$
(u^n_S(\mathbf{m}, l) : S \in \{1, 2, 3, 12, 13, 23\}) \in T^n_e.
$$

If more than one such index tuple can be found, choose an arbitrary one among those. If no such tuple exists, just choose $(l_1, l_2, l_3, l_{12}, l_{13}, l_{23}) = (1, 1, 1, 1, 1, 1)$. Then the transmitter generates $x^n(\mathbf{m}_{123}, \mathbf{m}_{12}, \mathbf{m}_{13}, \mathbf{m}_{23}, \mathbf{m}_{11,1}, \mathbf{m}_{22,2}, \mathbf{m}_{33,3})$ as

$$
x_i = x(u_{123,i}(\mathbf{m}), u_{12,i}(\mathbf{m}, l), u_{23,i}(\mathbf{m}, l), u_{13,i}(\mathbf{m}, l) , u_{1,i}(\mathbf{m}, l), u_{2,i}(\mathbf{m}, l)),$$

for $i = 1, \ldots, n$.

4) Decoding: Decoder $j = 1$ declares that $(\hat{m}_{1,123}, \hat{m}_{1,12}, \hat{m}_{1,13}, \hat{m}_{1,1})$ is sent if it is the unique message such that

$$
(u^n_{123}(\mathbf{m}), u^n_S(\mathbf{m}, l) : S \in \{1, 2, 13\}, y^n) \in T^n_e.
$$

Similarly, receiver $j = 2, 3$ uses joint typicality decoding to find the unique message $(\hat{m}_{2,123}, \hat{m}_{2,12}, \hat{m}_{2,23}, \hat{m}_{2,2})$ and $(\hat{m}_{3,123}, \hat{m}_{3,13}, \hat{m}_{3,23}, \hat{m}_{3,3})$, respectively.

5) Analysis of the probability of error: Assume without loss of generality that

$$(\mathbf{M}_{123}, \mathbf{M}_{12}, \mathbf{M}_{13}, \mathbf{M}_{23}, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3) = (1, 1, 1, 1, 1, 1, 1)
$$

is sent and let $L_{12}, L_{13}, L_{23}, L_{1}, L_{2}, L_{3}$ be the index tuple of selected sequences

$$(U^n_{123}(1), U^n_{12}(1, L), U^n_{13}(1, L), U^n_{23}(1, L), U^n_1(L), U^n_2(L), U^n_3(L)) \\
\in C_{12}(1) \times C_{13}(1) \times C_{23}(1) \times C_{1}(1) \times C_{2}(1) \times C_{3}(1).$$

We note that the subcodebook $C_S(\mathbf{m})$ consists of $2^{n(R_S - R_{S})}$ i.i.d. $U^n_S(\mathbf{m}, l)$ sequences, $\forall S \in \{1, 2, 3, 12, 13, 23\}$. By the hierarchical covering lemma (with $r_S = R_S - R_{S}$), we obtain a set of constraints for $S \in \{12, 13, 23\}$.

$$
r_12 + r_{13} > I(U_{12}; U_{13})/U_{123}, \\
r_12 + r_23 > I(U_{12}; U_{23})/U_{123}, \\
r_{13} + r_23 > I(U_{13}; U_{23})/U_{123}, \\
r_12 + r_{13} + r_23 > I(U_{12}; U_{23})/U_{123} + I(U_{13}; U_{23}) + I(U_{12}; U_{13})/U_{123}, U_{23}).
$$

(14a) (14b) (14c) (14d)
Again, using hierarchical covering lemma for $S \in \{1, 2, 3\}$, we attain the region for $r_1, r_2, r_3$:

\begin{align}
    r_1 &> I(U_1; U_{23}|U_{123}U_{12}U_{13}), \\
    r_2 &> I(U_2; U_{13}|U_{123}U_{12}U_{23}), \\
    r_3 &> I(U_3; U_{12}|U_{123}U_{13}U_{23}), \\
    r_1 + r_2 &> I(U_1; U_2|U_{123}U_{12}U_{13}U_{23}) \\
    &+ I(U_1; U_{23}|U_{123}U_{12}U_{13}) + I(U_2; U_{13}|U_{123}U_{23}U_{12}), \\
    r_1 + r_3 &> I(U_1; U_3|U_{123}U_{12}U_{13}U_{23}) \\
    &+ I(U_1; U_{23}|U_{123}U_{12}U_{13}) + I(U_3; U_{12}|U_{123}U_{13}U_{23}), \\
    r_2 + r_3 &> I(U_2; U_3|U_{123}U_{12}U_{13}U_{23}) \\
    &+ I(U_2; U_{13}|U_{123}U_{12}U_{23}) + I(U_3; U_{12}|U_{123}U_{13}U_{23}), \\
    r_1 + r_2 + r_3 &> I(U_1; U_2|U_{123}U_{12}U_{13}U_{23}) \\
    &+ I(U_1; U_3|U_{123}U_{12}U_{13}U_{23}) + I(U_2; U_{13}|U_{123}U_{23}U_{12}), \\
    r_1 + r_2 + r_3 + r_4 &> I(U_1; U_2; U_3|U_{123}U_{12}U_{13}U_{23}). \\
\end{align}

By the symmetry of decoders, we first consider the average probability of error for decoder 1. To analyze it, the Table I lists all possible pmfs of message tuple $(U^0_{123}(m), U^0_{122}(m', t), U^0_{123}(m', t'), U^0_{123}(m, t'), Y^n_t)$.

| $m_{123}$ | $(m_{12}, t'_{12})$ | $(m_{13}, t'_{13})$ | $m_{1,1}$ | Joint pmf |
|----------|------------------|------------------|--------|---------|
| $1_{123}$ | $(1_{12}, t_{12})$ | $(1_{13}, t_{13})$ | 1     | $p^* p(y^n_1 | u^n_{123} u^n_{12} u^n_{13})$ |
| $1_{123}$ | $(1_{12}, t_{12})$ | $(1_{13}, t_{13})$ | *     | $p^* p(y^n_1 | u^n_{123} u^n_{12} u^n_{13})$ |
| $1_{123}$ | $(1_{12}, t_{12})$ | *                 | *     | $p^* p(y^n_1 | u^n_{123} u^n_{12})$ |
| $1_{123}$ | *                 | $(1_{13}, t_{13})$ | *     | $p^* p(y^n_1 | u^n_{123} u^n_{13})$ |
| $1_{123}$ | *                 | *                 | *     | $p^* p(y^n_1 | u^n_{123})$ |

* denote the message or index $m_s \neq l_s, l'_s, s \in \{123, 12, 13, 1\}$.

And $p^*$ denote joint pmf $p(u^n_{123}, u^n_{12}, u^n_{13}, u^n_1)$.

**TABLE I: Table of joint pmfs of all possible error messages**

By the conditional typicality lemma, the first term $P(\mathcal{E}_{1,0})$ tends to zero as $n \to \infty$ since that $(w^n_S(m_S, l_S) : S \in \{1, 2, 3, 12, 13, 23\}) \in \mathcal{T}^{(n)}_e$ and $\epsilon' < \epsilon$.

Considering the second term $P(\mathcal{E}_{1,1})$, note that $U^n_{12}(l) \equiv \prod_{i=1}^{n} P_{U_i}(u_{1,i})$ is independent of $Y^n_1$ for every $U_i(l) \notin C_1(1)$, then the variables $U^n_{12}(*)$ is independent of $Y^n_1$ on condition of $(U^n_{123}(1), U^n_{123}(1, l), U^n_{123}(1, 1, l))$. By the packing lemma, $P(\mathcal{E}_{1,1})$ tends to zero as $n \to \infty$ if

$$\hat{R}_1 < I(U_1; Y_1|U_{123}U_{12}U_{13}).$$
Similarly, $P(\mathcal{E}_{1,1,2}), P(\mathcal{E}_{1,1,3}), P(\mathcal{E}_{1,1,2,3})$ and $P(\mathcal{E}_{1,1,2,3})$ tend to zero as $n \to \infty$ if
\[
\tilde{R}_{12} + \tilde{R}_1 < I(U_{12}U_1; Y_1 | U_{123}U_{13}) + I(U_{12}; U_{13} | U_{123}), \\
\tilde{R}_{13} + \tilde{R}_1 < I(U_{13}U_1; Y_1 | U_{123}U_{12}) + I(U_{12}; U_{13} | U_{123}), \\
\tilde{R}_{12} + \tilde{R}_{13} + \tilde{R}_1 < I(U_{12}U_{13}U_1; Y_1 | U_{123}) + I(U_{12}; U_{13} | U_{121}).
\]

For receiver 2 and 3, the corresponding inequalities can be derived in a similar way.

From the lower bound of linear combinations of $R_S$ and the upper bound of linear combinations of $\tilde{R}_S$ about receiver 1, 2 and 3, eliminating $\tilde{R}_{123}, \tilde{R}_{12}, \tilde{R}_{13}, \tilde{R}_{23}$ and $R_1, R_2, R_3$ by our elimination procedure, an example given in Figure 2 yields the characterization (12a).

**B. Coding scheme for $K \geq 2$**

For each $k \in \mathcal{K}$, split message $M_k \in [1: 2^{nR_k}]$ into $\sum_{i=1}^{K} \binom{K}{i}$ submessages $M_k \triangleq (M_{k,S} : S \subseteq \mathcal{K}, |S| = 1, 2, \ldots, K)$ with $M_{k,S} \in [1: 2^{R_{k,S}}]$ and $R_{k,S} = 0$ if $k \notin S$. Similarly to $K = 3$ case, we use the following notations
\[
\hat{R}_S \triangleq \sum_{k=1}^{K} \tilde{R}_{k,S}, \quad R_S \triangleq \sum_{k=1}^{K} R_{k,S}, \quad r_S \triangleq \hat{R}_S - R_S,
\]
and $M_S \triangleq (M_{1,S}, \ldots, M_{K,S})$.

1) **Codebook generation:** Fix a pmf $p(u(S))$ and function $x(u(S))$ and let $\hat{R}_S \geq R_S, \forall S \in \mathcal{A}$. Randomly and independently generate $2^{nR_S}$ sequences $u^n_S(m, k) \in \prod_{i=1}^{n} \{1 : 2^{nR_{i,k}}\}$ for all $S \in \mathcal{A}^{1:(K-1)}$, construct a subcodebook $\mathcal{C}_S(m)$ consisting of $2^{(R_S - R_H)}$ i.i.d. generated sequences $u^n_S(m, j), (m_S,j_S) \in \prod_{i=1}^{K} \{1 : 2^{nR_{i,S}}\} \times \{1 : 2^{n(R_S - R_H)}\}$, each according to $\prod_{i=1}^{n} p_{u(k)}(u_{k,i})$, find an index tuple $(j_S : S \in \mathcal{A})$ such that, $l \in [1 : (K-1)], u^n_S(m, j) \in \mathcal{C}_S(m)$ and
\[
(u^n_S(m, j) : S \in \mathcal{A}), u^n_S(m) \in \mathcal{T}^{(n)}_{e}
\]
If there is more than one such tuple, pick an arbitrary one among them. If no such tuple exists, pick $(j_S : S \in \mathcal{A}) = 1$. Then generate $x^n(m_S : S \in \mathcal{A})$ with
\[
x_i = x(u_{k,i}(m), u_{S,i}(m, j) : \forall S \in \mathcal{A}),
\]
for $i \in [n]$.

To send the message tuple $(m_1, \ldots, m_K) = (m_S : S \in \mathcal{A})$, transmit $x^n(m_S : S \in \mathcal{A})$.

3) **Decoding:** Let $\epsilon > \epsilon^*$. Decoder $i, i \in \mathcal{K}$, declares that $m_i$ is sent if it is the unique message such that
\[
(y^n_i, u^n_S(m, j) : S \in \mathcal{A}^{1:(K-1)}(i), u^n_S(m) \in \mathcal{T}^{(n)}_{e}
\]
for some $(u^n_S(m, j) : S \in \mathcal{A}^{1:(K-1)}(i), u^n_S(m) \in \prod_{S \in \mathcal{A}^{1:(K-1)}} C_S(1) \times C_k(1))$.

4) **Analysis of the probability of error:** Assume without loss of generality that $(M_1, \ldots, M_K) = (M_S : \forall S \in \mathcal{A} = (1, \ldots, 1)$ and let $(J_S : S \in \mathcal{A}^{1:(K-1)})$ be the index tuple of the chosen sequences $(U_S : S \in \mathcal{A}^{1:(K-1)}) \in \prod_{S \in \mathcal{A}^{1:(K-1)}} C_S(1)$. Then decoder $i$ makes an error only if one or more of the following events occur:
\[
\begin{align*}
\mathcal{E}_0 &= \{ (U^n_S(m, j) : S \in \mathcal{A}^{1:(K-1)}, U^n_S(m)) \\
&\notin \mathcal{T}^{(n)}_{e} \cup (U^n_S : S \in \mathcal{A}) \in \prod_{S \in \mathcal{A}} C_S(1), \forall l \in [1 : (K-1)] \}, \\
\mathcal{E}_{i,1} &= \{ (Y^n_i, U^n_S(m, j) : S \in \mathcal{A}^{1:(K-1)}(i), U^n_S(m)) \notin \mathcal{T}^{(n)}_{e} \}, \\
\mathcal{E}_{i,2} &= \{ (Y^n_i, U^n_S(m, j) : S \in \mathcal{A}^{1:(K-1)}(i), U^n_S(m)) \in \mathcal{T}^{(n)}_{e} \text{ for some } (U^n_S(m, j) : S \in \mathcal{A}^{1:(K-1)}(i)) \notin \prod_{S \in \mathcal{A}^{1:(K-1)}} C_S(1) \}. 
\end{align*}
\]
Therefore, the probability of error for decoder $i$ is upper bounded as

$$P(\mathcal{E}_i) \leq P(\mathcal{E}_0) + P(\mathcal{E}_0 \cap \mathcal{E}_{i,1}) + P(\mathcal{E}_{i,2}).$$

To bound $P(\mathcal{E}_0)$, we utilize the hierarchical covering lemma proposed before. Noticing that sequences among the subcodebooks $\mathcal{C}_S(m)$, $S \in \mathcal{A}$, $i \in [1 : (K - 1)]$ are mutually conditionally independent, the usage of hierarchical covering lemma is straightforward with $r_S = R_S - R_S$. Hence, $P(\mathcal{E}_0)$ tends to zero as $n \to \infty$ if (10) holds for all $l \in [1 : (K - 1)]$ and $\mathcal{J}^l \in \mathcal{A}^l$.

To bound $P(\mathcal{E}_0 \cap \mathcal{E}_{i,1})$, by the conditional typicality lemma (11) it tends to zero as $n \to \infty$.

To bound $P(\mathcal{E}_{i,2})$, noticing that $U^n_{\mathcal{S}}(M_i, j) \sim \prod_{i=1}^n u_i(u_i(S), S) u_i(S') | S' \in \mathcal{A}(S) \setminus S)$ and $Y^n_{\mathcal{I}}$ are independent of $U^n_{\mathcal{S}}(M_i, j) \notin \mathcal{C}_S(1)$, $\forall S \in \mathcal{A}^{1 : (K - 1)}(i)$, as well as $U_{\mathcal{S}}(M_i) \notin \mathcal{C}_S(1)$. Furthermore, if $\forall S \in \mathcal{A}(i)$, $U^n_{\mathcal{S}} \notin \mathcal{C}_S(1)$, then by the conditional coding distribution, for all $S' \in \mathcal{B}_i(S)$, we have $U_{\mathcal{S}}(M_i, j) \notin \mathcal{C}_S(1)$. Thus, $\forall i \in \mathcal{K}$, $\forall J \subseteq \mathcal{A}(i)$ by identifying $\bigcup_{\mathcal{S} \in J} \mathcal{B}_i(S)$, we obtain inequalities with the help of packing lemma:

$$\sum_{S \in \bigcup_{\mathcal{S} \in J} \mathcal{B}_i(S)} \tilde{R}_S < \sum_{S' \in \bigcup_{\mathcal{S} \in J} \mathcal{B}_i(S')} H(U_{S'} | U(A(S') \setminus S')) - H(U(\bigcup_{\mathcal{S} \in J} \mathcal{B}_i(S))) \delta(e).$$

(17)

5) Eliminating $(\tilde{R}_{k,S}, r_S : k \in \mathcal{K}, S \in \mathcal{A})$: Due to massive numbers of $\tilde{R}_{k,S}$ and $r_S$, using standard Fourier-Motzkin elimination to obtain the achievable rate constraint is disastrous. We first present observations which help the elimination, and then find all valid constraints for $\sum_{k \in \mathcal{T}} R_k$ for all $\mathcal{T} \subseteq \mathcal{K}$.

Observation: Let $\mathcal{S}(i, J) \triangleq \bigcup_{\mathcal{S} \in J} \mathcal{B}_i(S)$, and denote the right hand term of (17) by $I_{\mathcal{S}(i, J)}$, thus (17) can be rewritten as below.

$$\sum_{S \in \mathcal{S}(i, J)} \tilde{R}_S < I_{\mathcal{S}(i, J)}.$$  

(18)

a) If there exists $i', i''$ and $J', J''$ such that

$$\mathcal{S}(i, J) = \mathcal{S}(i', J') \cup \mathcal{S}(i'', J''),$$

then

$$I_{\mathcal{S}(i, J)} \leq I_{\mathcal{S}(i', J')} + I_{\mathcal{S}(i'', J'')}.$$  

(19a)

b) If there exists $i'$ and $J'$ such that $\mathcal{S}(i, J) \subseteq \mathcal{S}(i', J')$, then

$$I_{\mathcal{S}(i, J)} \leq I_{\mathcal{S}(i', J')}.$$  

(19b)

From the rate-splitting procedure, we have $R_i = \sum_{S \in \mathcal{A}(i)} R_{k,S}$, for all $i \in \mathcal{K}$. Thus in order to find rate constraints for $\sum_{k \in \mathcal{T}} R_k$, we must find all valid constraints for $\sum_{k \in \mathcal{T}} \sum_{S \in \mathcal{A}(k)} R_{k,S}$. By definitions in (16), we have

$$\sum_{k \in \mathcal{T}} (R_{k}) \leq \sum_{S \in \bigcup_{k \in \mathcal{A}(k)} \mathcal{A}(k)} \tilde{R}_S = \sum_{S \in \bigcup_{k \in \mathcal{T}} \mathcal{A}(k)} \tilde{R}_S - \sum_{S \in \bigcup_{k \in \mathcal{T}} \mathcal{A}(k)} r_S.$$  

(21)

where (a) is an equation if and only if $R_{k,S} = 0$ for all $k \notin \mathcal{T}$.

Consider a permutation $\pi \in \Pi_\mathcal{T}$. For easy reading, rewrite the decomposition (4):

$$\bigcup_{k \in \mathcal{T}} \mathcal{A}(k) = \bigcup_{i=0}^{\mathcal{T} - 1} \mathcal{B}_\pi(i).$$  

(22)

For $R_{\pi(1)}$, it must involve $R_{\pi(1), S} : S \in \mathcal{A}(\pi(1)).$ Since $\mathcal{A}(\pi(1)) = \mathcal{B}_\pi(0) = \mathcal{S}(\pi(1), K)$ is the largest set with every element containing $\pi(1)$, by observation in (19) and (20), we have the only valid rate constraint $\sum_{S \in \mathcal{A}(\pi(1))} \tilde{R}_S \leq I_{\mathcal{S}(\pi(1))} = I_{\mathcal{S}(0)},$ to support $R_{\pi(1)}$.

For $R_{\pi(2)}$, since $\sum_{S \in \mathcal{A}(\pi(2))} \tilde{R}_S$ already contains $\sum_{S \in \mathcal{A}(\pi(1), \pi(2))} \tilde{R}_S$, thus we need to find the rate constraints for $\sum_{S \in \mathcal{A}(\pi(2)) \setminus \mathcal{A}(\pi(1, \pi(2)))} \tilde{R}_S$. Since $\mathcal{A}(\pi(2)) \setminus \mathcal{A}(\pi(1), \pi(2))) = \mathcal{B}_\pi(1) = \mathcal{S}(\pi(2), K \setminus \{\pi(1)\})$, and $\mathcal{B}_\pi(1)$ is second largest sets with every element containing $\pi(2)$ and excluding $\pi(1)$, by observation in (19) and (20), we obtain the only effective rate constraints $\sum_{S \in \mathcal{B}_\pi(1)} \tilde{R}_S \leq I_{\mathcal{S}(\pi(1))}$ to support $R_{\pi(2)}$. Following the similar steps and by (21), we find all valid rate constraints under the permutation $\pi$ for all $R_{\pi(k)}, k \in [1 : |\mathcal{T}|]$:

$$\sum_{k \in \mathcal{T}} R_k \leq \sum_{S \in \bigcup_{i=0}^{\mathcal{T} - 1} \mathcal{B}_\pi(i)} I_{\mathcal{S}(\pi(1))} - \sum_{S \in \bigcup_{k \in \mathcal{A}(k)} \mathcal{A}(k)} r_S.$$  

(23)
Thus, formally we have for all $\mathcal{T} \subseteq \mathcal{K}$ and $\pi \in \Pi_{\mathcal{T}}$:

$$\sum_{k \in \mathcal{T}} R_k \leq \sum_{S \in \bigcup_{l=0}^{T^1-1} \mathcal{B}_\pi(i)} IS_{\pi(i)} - \sum_{S \in \bigcup_{k \in \mathcal{T}} \mathcal{A}(k)} r_S \leq \sum_{S \in \bigcup_{l=0}^{T^1-1} \mathcal{B}_\pi(i)} IS_{\pi(i)} - \sum_{l=1}^{K-1} \sum_{S \in \bigcup_{i=0}^{T^1-1} \mathcal{B}^l_s(i)} r_S \leq \sum_{i=0}^{T^1-1} \left( \sum_{S \in \mathcal{B}_\pi(i)} H(U_{\mathcal{S}} \mid U(\mathcal{A}(S') \setminus S')) - H(U(\mathcal{B}_\pi(i)) \mid Y_{\pi(i+1)} U(\mathcal{A}(\pi(i+1)) \setminus \mathcal{B}_\pi(i))) \right) + \sum_{l=1}^{K-1} \left( H(U(\bigcup_{i=0}^{T^1-1} \mathcal{B}_s^l(i)) \mid U(\mathcal{A}(l+1):K)) - \sum_{S \in \bigcup_{i=0}^{T^1-1} \mathcal{B}_s^l(i)} H(U_{\mathcal{S}} \mid U(\mathcal{A}(S) \setminus S)) \right) ,$$

where (a) comes from the disjoint decomposition in [8]; (b) holds by letting $\mathcal{J}^l = \bigcup_{i=0}^{T^1-1} \mathcal{B}_s^l(i)$ in Lemma 1 and by Lemma 2.

VI. CONCLUSION

In this paper, we propose new scheme for $K$-receiver DM-BCs with private messages based on exhaustive message splitting and $K$-level Marton’s coding. The achievable rate region allows arbitrary pmfs, and improves over all previously known ones for $K \geq 3$ case whose input pmfs should satisfy certain Markov chain(s). A hierarchical covering lemma is established which extends the 2-level multivariate covering lemma to $K$-level case.

APPENDIX A

PROOF OF HIERARCHICAL COVERING LEMMA [1]

Let

$$\mathcal{M}_{\mathcal{A}^l} \triangleq \prod_{S \in \mathcal{A}^l} \mathcal{M}_S = \prod_{S \in \mathcal{A}^l} [1 : 2^{nrs}] . \quad (24)$$

Divide the messages and codewords into $K$ levels, where at level $l \in [1 : K]$, there are messages sets $\{\mathcal{M}_S : S \in \mathcal{A}^l\}$, and codewords

$$(U^n_S(m_S) : S \in \mathcal{A}^l, m_S \in \mathcal{M}_{\mathcal{A}^l}) .$$

The proof of hierarchical covering lemma starts at level $l = K - 1$, finding a message tuple $(m_S : S \in \mathcal{A}^{K-1})$ such that $(u^n_S(m_S) : S \in \mathcal{A}^{K-1}, u^n_S) \in \mathcal{T}_{e_{K-1}}^{(n)}$. Then with a descending order of $l = K - 2, K - 1, \ldots, 1$, we find a message tuple $(m_S : S \in \mathcal{A}^l)$ such that $(u^n_S(m_S) : S \in \mathcal{A}^l), u^n_S) \in \mathcal{T}_{e_{l+1}}^{(n)}$. For each $l \in [1 : K]$, we use $\mathcal{S}^l_i$ to denote the element of $\mathcal{A}^l$ for $i = 1, \ldots, |\mathcal{A}^l|$, i.e.,

$$\mathcal{A}^l = \{\mathcal{S}^l_1, \ldots, \mathcal{S}^l_{|\mathcal{A}^l|}\} .$$

We define, for all $l \in [1 : K]$, the random events

$$\mathcal{E}_l = \left\{ (U^n_S(m_S) : S \in \mathcal{A}^{(K-1)}, U^n_S) \notin \mathcal{T}_{e_{l+1}}^{(n)}, \forall (m_S : S \in \mathcal{A}^{(K-1)}) \in \prod_{S \in \mathcal{A}^{(K-1)}} [1 : 2^{nrs}] \right\} ,$$

$$\mathcal{A}_l = \{ (m_{\mathcal{S}^l_1}, \ldots, m_{\mathcal{S}^l_{|\mathcal{A}^l|}}) : (U^n(m_{\mathcal{S}^l_1}) \ldots U^n(m_{\mathcal{S}^l_{|\mathcal{A}^l|}}), u^n(\mathcal{A}^{(l+1):K})) \in \mathcal{T}_{e_{l+1}}^{(n)} \} ,$$

$$\mathcal{Q}_l = \{ u^n(m_{\mathcal{S}^l_1}) \ldots u^n(m_{\mathcal{S}^l_{|\mathcal{A}^l|}}) : j \in ([l+1]:K) \} \in \mathcal{T}_{e_{l+1}}^{(n)} ,$$

where $\mathcal{S}^l_1$ denotes the message tuple of $\mathcal{A}^l$.
where
\[ u^n(\mathbb{A}^{(l+1)\cdot K}) = (u^n(m_{S_1}) \ldots u^n(m_{[a]_l})) : j \in [(l + 1) \cdot (K - 1)], u^n_k \]

Note that \( E_l \) denotes the event that there isn’t any tuple of sequences from level \( l \) to \( K \) such that they are jointly typical. Correspondingly, \( E_l^c \) denotes the event that there exists such tuple of sequences. \( A_l \) denotes the set of messages at level \( l \) such that the corresponding codewords are jointly typical with a given tuple of jointly typical codewords from levels \( l + 1 \) to \( K \). \( Q_l \) denotes a condition that there exists message \( (m_{S_1}, \ldots, m_{[a]_l}) \in \mathcal{M}_{A_l} \) at level \( l + 1, \ldots, K \) such that the corresponding codewords are jointly typical.

Our goal is to find rate conditions such that there exists a tuple of messages \( (m_S : S \in \mathbb{A}^{1 \cdot (K-1)}) \) satisfying \( (U^n_S(m_S)) : S \in \mathbb{A}^{1 \cdot (K-1)}, U^n_S \in T_{\xi_l}^{(n)} \), which is equivalent to \( P\{E_1\} \to 0 \) as \( n \to \infty \).

The probability of \( E_1 \) can be upper bounded as
\[ P\{E_1\} = P\{E_1 \cap E_2^c\} + P\{E_1 \cap E_2\} \leq P\{E_1 | E_2^c\} + P\{E_2\}. \]

With recursively upper bounding \( P\{E_l\} \), we obtain:
\[ P\{E_1\} \leq \sum_{l=1}^{K-1} P\{E_1 | E_{l+1}^c\} + P\{E_K\} = \sum_{l=1}^{K-1} P\{|A_l| = 0\} + P\{E_K\}, \tag{25} \]

where the last equality holds by definitions of \( A_l \) and \( E_l \). Since \( P\{E_K\} \) tends to zero as \( n \to \infty \) by the property of typicality, and \( K \) is a fixed channel parameter, in order to make \( P\{E_1\} \) tend to 0, it’s sufficient to let \( P\{|A_l| = 0\} \) tend 0 as \( n \to \infty \) for all \( l \in [1 : K-1] \).

To analyze \( (25) \), \( P\{|A_l| = 0\} \) is upper bounded by a variation of Chebyshev inequality:
\[ P\{|A_l| = 0\} \leq P\{(|A_l| - E|A_l|)^2 \leq (E|A_l|)^2 \} \leq \frac{\text{Var}(|A_l|)}{(E|A_l|)^2}. \]

Using indicator random variables, \( |A_l| \) can be written as
\[ |A_l| = \sum_{j=1}^{[A_l]} \sum_{m_{S_j} \in \mathcal{M}_{S_j}} \text{I}(m_{S_1}, \ldots, m_{[a]_l}) \tag{26a} \]

where
\[ \text{I}(m_{S_1}, \ldots, m_{[a]_l}) = \begin{cases} 1 & \text{if } Q_l \text{ given } Q_{l+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{26b} \]

for each \( (m_{S_1}, \ldots, m_{[a]_l}) \in \mathcal{M}_{A_l} \).

For all \( J_l \subseteq \mathbb{A}_l \), define
\[ p_{J_l} = P\{ (U^n(m_{S_1}) \ldots U^n(m_{[a]_l})) : u^n(\mathbb{A}^{(l+1)\cdot K}) \in T^{(n)}_{\xi_l}, \]
\[ (U^n(m_{S_1}) \ldots U^n(m_{[a]_l})) : u^n(\mathbb{A}^{(l+1)\cdot K}) \in T^{(n)}_{\xi_l}, \]

with \( m_{S_j} = m_{S_j} \) if \( S_j \in J_l \) and vice versa
\[ |u^n(\mathbb{A}^{(l+1)\cdot K}) \in T^{(n)}_{\xi_{l+1}}\}. \tag{27} \]

From \( (26) \) and definition of \( p_{J_l} \) in \( (27) \), we have
\[ E(|A_l|) = 2^n \sum_{S \in A_l} r_S \cdot p_{A_l}, \]
\[ E(|A_l|^2) = \sum_{m_{S_1}, \ldots, m_{[a]_l}} \sum_{p_{J_l}} p_{J_l}. \]

\[ \leq 2^n \sum_{S \in A_l} r_S \cdot \sum_{J_l \subseteq \mathbb{A}_l} 2^n \sum_{S \in A_l \setminus J_l} r_S \cdot p_{J_l}. \]
From definition of $p_{J'}$ and independence of codewords, we have $(p_{A_i})^2 = p_0$. Hence
\[
\text{Var}(|A_i|) \leq 2^n \sum_{S \in J} r_S \sum_{J' \subseteq (R \setminus \emptyset)} 2^n \sum_{S \in J \setminus J'} r_S \cdot p_{J'}.
\]
Next we compute the upper bound of $p_{J'}$.

\[
p_{J'} = P \left\{ \left( U^n(m_{S_1}) \ldots U^n(m_{S_{|J|}}), u^n(A^{(l+1):K}) \right) \in T_{t_i}^{(n)} \right\}
\]
\[
\quad \cdot P \left\{ \left( U^n(m_{S_1}') \ldots U^n(m_{S_{|J|}}'), u^n(A^{(l+1):K}) \right) \in T_{t_i}^{(n)} \right\}
\]
with $m_{S_j}' = m_{S_j}$ if $S_j \in J$ and vice versa $Q_{l+1}$
\[
= P \left\{ \left( U^n(m_{S_1}) \ldots U^n(m_{S_{|J|}}), u^n(A^{(l+1):K}) \right) \in T_{t_i}^{(n)} \right\}
\]
\[
\cdot P \left\{ \left( U^n(m_{S_1}') \ldots U^n(m_{S_{|J|}}'), u^n(A^{(l+1):K}) \right) \in T_{t_i}^{(n)} \right\}
\]
\[
\cdot \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) - \delta(\epsilon_i) \right)
\]
\[
\leq 2^n(H(U(A') | U(A^{(l+1):K})) + \delta(\epsilon_i))
\]
\[
2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) - \delta(\epsilon_i) \right)
\]
\[
2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
(28)

where (a) follows by properties of joint typicality and the mutually conditional independence property, i.e., given $u^n(A^{(l+1):K})$, $(U^n(m_{S_j}': S \in A^{l \setminus J})$ and $(U^n(m_{S_j}: S \in A^{l \setminus J})$ are conditionally independent. The lower bound of $p_{A_i}$ can be derived as

\[
p_{A_i} = P \left\{ \left( U^n(m_{S_1}) \ldots U^n(m_{S_{|J|}}), u^n(A^{(l+1):K}) \right) \in T_{t_i}^{(n)} \right\}
\]
\[
\left( U^n(m_{S_1}' \ldots U^n(m_{S_{|J|}}'), u^n(A^{(l+1):K}) \right) \in T_{t_i}^{(n)} \right\}
\]
\[
\geq (1 - \epsilon_i) \cdot 2^n(H(U(A') | U(A^{(l+1):K})) - \delta(\epsilon_i))
\]
\[
2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
(29)

where the last equality holds by the properties of jointly typicality. From (28) and (29), we have
\[
\frac{p_{J'}}{(p_{A_i})^2} \leq 2^n(H(U(A') | U(A^{(l+1):K})) + \delta(\epsilon_i))
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\cdot (1 - \epsilon_i)^{-2} \cdot 2^n(H(U(A') | U(A^{(l+1):K})) + \delta(\epsilon_i))
\]
\[
\cdot 2^{-2n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\leq (1 - \epsilon_i)^{-2} \cdot 2^n[H(U(A') | U(A^{(l+1):K})]
\]
\[
\cdot 2^n[H(U(A') | U(A^{(l+1):K})]
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\cdot 2^{n[H(U(A') | U(A^{(l+1):K})]
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\cdot 2^{n[H(U(A') | U(A^{(l+1):K})]
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
\[
\cdot 2^{-n} \left( \sum_{S \subseteq J} H(U_{S_j} | U(A^{(l+1):K})) + \delta(\epsilon_i) \right)
\]
Therefore, we finally obtain the desired inequality

$$\frac{\text{Var}(|A_l|)}{(E(|A_l|))^2} \leq (1 - \epsilon)^{-2} \cdot 2^{-n} \sum_{S \subseteq A^l} r_S \cdot \sum_{J^l \subseteq A^l \setminus \emptyset} \left(2^n \sum_{S \subseteq A^l \setminus J^l} r_S \cdot \frac{P_{J^l}}{(P_{A^l})^2} \right)$$

$$\leq (1 - \epsilon)^{-2} \sum_{J^l \subseteq A^l \setminus \emptyset} \left(2^{-n} \sum_{S \subseteq J^l} r_S \cdot 2^{n \delta(\epsilon)} \cdot 2^n H(U(A^l \setminus J^l) | U(J^l), U(A^{l+1}; K)) - H(U(A^l) | U(A^{l+1}; K)) \right)$$

which tends to zero as $n \to \infty$ if for all $J^l \subseteq A^l$

$$\sum_{S \in J^l} r_S > H(U(A^l \setminus J^l) | U(J^l), U(A^{l+1}; K))$$

$$- H(U(A^l) | U(A^{l+1}; K))$$

$$+ \sum_{S \subseteq A^l} H(U_S | U(A^l \setminus S) \setminus S)$$

$$- \sum_{S \subseteq A^l \setminus J^l} H(U_S | U(A^l \setminus S) \setminus S) + \delta(\epsilon)$$

$$= H(U(A^l \setminus J^l) | U(J^l), U(A^{l+1}; K))$$

$$- H(U(A^l) | U(A^{l+1}; K))$$

$$+ \sum_{S \subseteq J^l} H(U_S | U(A^l \setminus S) \setminus S) + \delta(\epsilon)$$

$$= \sum_{S \subseteq J^l} H(U_S | U(A^l \setminus S) \setminus S)$$

$$- H(U(J^l) | U(A^l | A^{l+1}; K)) + \delta(\epsilon).$$

Therefore, from (25), $P\{E_1\}$ tends to zero as $n \to \infty$ if conditions (30) hold for all $l \in [1 : (K - 1)]$.

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