AN INTEGRAL FORMULA FOR RIEMANNIAN
G–STRUCTURES WITH APPLICATIONS TO ALMOST
HERMITIAN AND ALMOST CONTACT STRUCTURES

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Abstract. For a Riemannian $G$–structure, we compute the divergence of the
vector field induced by the intrinsic torsion. Applying the Stokes theorem, we
obtain the integral formula on a closed oriented Riemannian manifold, which
we interpret in certain cases. We focus on almost hermitian and almost contact
metric structures.

1. Introduction

Equipping a manifold $M$ with a Riemannian metric $g$ is equivalent to reduc-
tion of a frame bundle $L(M)$ to orthogonal frame bundle $O(M)$, i.e. to action of a
structure group $O(n)$. Assuming moreover that $M$ is oriented we can consider the
bundle $SO(M)$ of oriented orthonormal frames. Existence of additional geometric
structure can be considered as a reduction of a structure group $SO(n)$ to a cer-
tain subgroup $G$. For example, almost hermitian structure gives $U(\frac{n}{2})$–structure,
almost contact metric structure $U(\frac{n-1}{2}) \times 1$–structure, etc.

If $\nabla$ is a Levi–Civita connection of $(M, g)$ we may measure the defect of $\nabla$ to be
a $G$–connection. This leads to the notion of an intrinsic torsion. If this $(1, 2)$–
tensor vanishes (in such case we say that a $G$–structure is integrable) then $\nabla$ is
a $G$–connection, which implies that the holonomy group is contained in $G$. We
may classify non–integrable geometries by finding the decomposition of the space
of all possible intrinsic torsions into irreducible $G$–modules. This approach was
initiated by Gray and Hervella for $U(\frac{n}{2})$–structures [10] and later considered for
other structures by many authors [5, 11, 12, 7, 6]. Each so called Gray–Hervella
class, gives some restrictions on the curvature.

One possible approach to curvature restrictions on compact $G$–structures can
be achieved by obtaining integral formulas relating considered objects. This has
been firstly done, in a general case, by Bor and Hernandez Lamone da [3]. They
uses Bochner–type formula for forms being stabilizers of each considered subgroup
in $SO(n)$. They obtained integral formulas for $G = U(\frac{n}{2}), SU(\frac{n}{2}), G_2$ and Spin$_7$
and continued this approach for $Sp(n)Sp(1)$ in [4]. The case $G = U(\frac{n-1}{2}) \times 1$ has
been done later in [8].

In this article, we show how mentioned formulas can be obtained in a different
way. The nice feature of our approach is that the main integral formula (3.11) is
stated in a general case of any $G$–structure for $G \subset SO(n)$. This is achieved by

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considering, so called, characteristic vector field induced by the intrinsic torsion and calculating its divergence. For some Gray–Hervella classes the characteristic vector field vanishes, and then we get point–wise formula relating an intrinsic torsion with a curvature. Moreover, our integral formula, which can be reformulated in such a way that it is equivalent to formulas by Bor and Hernandez Lamoneda [5], gives a priori different information that the ones in [5].

We concentrate on almost hermitian and almost contact metric structures.

In the way described above we recover many well known relations. Let us state some of the consequences of the main integral formula (the objects used in these statements will be defined in appropriate sections):

1. Let \( M \) be a \( G \)–structure such that the orthogonal complement \( g^\perp \) of the Lie algebra \( g \) of Lie group \( G \) in \( \mathfrak{so}(n) \) satisfies \( [g^\perp, g^\perp] \subset g \). Assume that the characteristic vector field vanishes. Then we have the following point–wise formula
\[
\frac{1}{2} s_{g^\perp} = |\xi^\text{alt}|^2 - |\xi^\text{sym}|^2,
\]
where \( s_{g^\perp} \) is a \( g^\perp \)–component of the scalar curvature and \( \xi^\text{alt}, \xi^\text{sym} \) are skew–symmetric and symmetric parts of the intrinsic torsion \( \xi \). In particular, if the intrinsic torsion is totally skew–symmetric then
\[
s_{g^\perp} = 2|\xi|^2 \geq 0.
\]
2. Assume \((M, g, J)\) is closed hermitian manifold of Gray–Hervella type \( W_4 \). Then \( \int_M s - s^* > 0 \), where \( s \) is a scalar curvature and \( s^* \) is \(*\)–scalar curvature.
3. On closed \( SU(n) \)–structure of type \( W_1 \oplus W_2 \oplus W_5 \) we have \( \int_M s = 5 \int_M s^* \).
4. If closed almost contact metric structure is of type \( C_5 \oplus \ldots \oplus C_{10} \), then \( \int_M s - s^* = \int_M \text{Ric}(\zeta, \zeta) \).

2. INTRINSIC TORSION

Let \((M, g)\) be an oriented Riemannian manifold. Denote by \( SO(M) \) the bundle of oriented frames over \( M \). Let \( \nabla \) be the Levi-Civita connection of \( g \) and let \( \omega \) be the induced connection form. Let \( G \subset SO(n) \), where \( n = \dim M \), be a Lie subgroups such that on the level of Lie algebras we have the following decomposition
\[
\mathfrak{so}(n) = g \oplus g^\perp, \quad \text{ad}(G)g^\perp \subset g^\perp,
\]
where the orthogonal complement is taken with respect to the Killing form. Then \( \omega \) decomposes as
\[
\omega = \omega_g \oplus \omega_{g^\perp}.
\]
The component \( \omega_g \) is a connection form in the \( G \)–reduction \( P \subset SO(M) \), if such exists, and therefore defines a Riemannian connection \( \nabla^G \) on \( M \). The difference
\[
\xi_X Y = \nabla^G_X Y - \nabla_X Y, \quad X, Y \in TM,
\]
defines a \((1, 2)\)–tensor called the intrinsic torsion of a \( G \)–structure. \( \xi \) satisfies some skew–symmetry conditions by the fact that \( \xi_X \in g^\perp(TM) \subset \mathfrak{so}(TM) \) where \( g^\perp(TM) \) is the associated bundle of the form \( P \times_{\text{ad}(G)} g^\perp \). In particular,
\[
g(\xi_X Y, Z) = -g(Y, \xi_X Z), \quad X, Y, Z \in TM.
\]
By a definition, the intrinsic torsion measures defect of the Levi–Civita connection to be a $G$–connection. In particular, if $\xi$ vanishes, then the holonomy of $\nabla$ is contained in $G$. Study of the intrinsic torsion and its decomposition into irreducible summands was initiated by Gray and Hervella in the case of $G = U(\mathbb{H})$ [10]. Since then, other possible cases, mainly coming from the Berger classification of non–symmetric irreducible holonomy groups, has been considered (see, for example, [5 11 12 7 6]).

3. AN INTEGRAL FORMULA

Let $(M, g)$ be an oriented Riemannian manifold with the Levi-Civita connection $\nabla$. Assume $M$ is a $G$–structure, with $G \subset SO(n)$ such that (2.1) holds and let $\xi$ be the associated intrinsic torsion. Define a vector field $\chi = \chi^G$ by

$$\chi = \sum_i \xi_i e_i,$$

where $(e_i)$ is any orthonormal basis. We call $\chi$ the characteristic vector field of a $G$–structure $M$. Notice that if $\xi$ is skew–symmetric with respect to $X$ and $Y$ then $\chi$ vanishes. This is the case, for example, for nearly Kähler manifolds (see the following sections). Additionally,

$$g(\chi, X) = -\sum_i g(e_i, \xi_i X) = \text{div} X - \text{div}^G X.$$

Thus, vanishing of the characteristic vector field is equivalent to the fact that divergences with respect to $\nabla$ and $\nabla^G$ coincide.

In this section we compute the divergence of $\chi$ with respect to $\nabla$. First, let us recall well–known curvature identities involving the intrinsic torsion:

$$R(X, Y)_g = R^G(X, Y) + [\xi_X, \xi_Y]_g,$$

$$R(X, Y)_{g^\perp} = -(\nabla_X \xi)_Y + (\nabla_Y \xi)_X - 2[\xi_X, \xi_Y] + [\xi_X, \xi_Y]_{g^\perp},$$

where $R$ and $R^G$ are the curvature tensors of $\nabla$ and $\nabla^G$, respectively. We use the following convention for the curvature $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Thus

$$R^G(X, Y) = R(X, Y) + (\nabla_X \xi)_Y - (\nabla_Y \xi)_X + [\xi_X, \xi_Y].$$

Denoting the scalar curvatures of $\nabla$ and $\nabla^G$ by $s$ and $s^G$, respectively, by the above identity we get

$$s^G = s + \sum_{i,j} g((\nabla_{e_i} \xi)_{e_j} e_j, e_i) - \sum_{i,j} g((\nabla_{e_j} \xi)_{e_i} e_j, e_i) + \sum_{i,j} g([\xi_{e_i}, \xi_{e_j}]_{e_j} e_j, e_i).$$

Notice that $(\nabla_X \xi)_Y$ is skew–symmetric, since $\xi_X$ is skew–symmetric,

$$g((\nabla_X \xi)_Y Z, W) = g(\nabla_X \xi_Y Z, W) - g(\xi_Y \nabla_X Z, W) - g(\xi_Y Z, \nabla_X W) + g(Z, \xi_Y \nabla_X W) + (\nabla_X Z, \xi_Y W)$$

$$= -g(Z, \nabla_X \xi_Y W) + g(Z, \xi_Y \nabla_X W) + g(Z, \xi_Y \nabla_X W) - g(Z, \nabla_X \xi_Y W).$$
Thus the first and second sum on the right hand side of (3.4) are opposite. Moreover,

\[
\text{div} \chi = \sum_{i,j} g(\nabla e_i \xi_j e_j, e_i) = \sum_{i,j} g((\nabla e_i \xi) e_j, e_i),
\]

since

\[
\sum_{i,j} (g(\xi_{e_i e_j} e_j, e_i) + g(\xi_j \nabla e_i e_j, e_i)) = \sum_{i,j,k} g(\nabla e_i e_j, e_k)g(\xi e_k e_j, e_i) + \sum_{i,j} g(\xi_j \nabla e_i e_j, e_i)
\]

\[
= - \sum_{i,j,k} g(e_j, \nabla e_i e_k)g(\xi e_k e_j, e_i) + \sum_{i,j} g(\xi_j \nabla e_i e_j, e_i)
\]

\[= 0\]

Let us compute the last term in (3.4),

\[
\sum_{i,j} g([\xi_{e_i}, \xi_j] e_j, e_i) = \sum_{i,j} g(\xi_{e_i} \xi_j e_j, e_i) - g(\xi_j \xi e_i e_j, e_i)
\]

\[
= -|\chi|^2 + \sum_{i,j} g(\xi_j e_i, \xi e_j).
\]

Put

\[
\xi^\text{alt} X^Y = \frac{1}{2}(\xi X^Y - \xi Y^X) \quad \text{and} \quad \xi^\text{sym} X^Y = \frac{1}{2}(\xi X^Y + \xi Y^X).
\]

Then

\[
\sum_{i,j} g(\xi_j e_i, \xi e_j) = |\xi^\text{sym}|^2 - |\xi^\text{alt}|^2.
\]

Substituting (3.5), (3.6) and (3.8) into (3.4) we get the following fact.

**Proposition 3.1.** On an oriented $G$–structure $M$ such that (2.1) holds we have

\[
2\text{div} \chi = s^G - s + |\chi|^2 + |\xi^\text{alt}|^2 - |\xi^\text{sym}|^2.
\]

We will improve above divergence formula a little bit, by getting rid of the component $s^G$ replacing it by $g^\perp$–component of $s$ and some additional term, which vanishes in many cases. Namely, by (3.3), (3.6) and (3.8) we have

\[
s_g = s^G + \sum_{i,j} g([\xi_{e_i}, \xi_j] e_j, e_i) - \sum_{i,j} g([\xi_{e_i}, \xi_j] g^\perp e_j, e_i)
\]

\[
= s^G - |\chi|^2 + |\xi^\text{sym}|^2 - \sum_{i,j} g([\xi_{e_i}, \xi_j] g^\perp e_j, e_i).
\]

Denote the last component in the above formula by $s^\text{alt}_{g^\perp}$, i.e.

\[
s^\text{alt}_{g^\perp} = \sum_{i,j} g([\xi_{e_i}, \xi_j] g^\perp e_j, e_i).
\]

Since $s = s_g + s^\text{alt}_{g^\perp}$, (3.9) can be rewritten in the form contained in the proposition below.
Proposition 3.2. On an oriented $G$–structure $M$ such that (2.1) holds we have

\[(3.10) \quad \text{div} \chi = 1/2 s_{g^\perp} - 1/2 s_{g^\perp} + |\chi|^2 + |\xi_{\text{alt}}|^2 - |\xi_{\text{sym}}|^2.\]

If $M$ is, additionally, closed, then the following integral formula holds

\[(3.11) \quad 1/2 \int_M s_{g^\perp} - s_{g^\perp} = \int_M |\chi|^2 + |\xi_{\text{alt}}|^2 - |\xi_{\text{sym}}|^2.\]

Remark 3.3. Notice that elements $|\chi|^2$, $|\xi_{\text{alt}}|^2$, $|\xi_{\text{sym}}|^2$ are quadratic invariants of the representation of $SO(n)$ in the space of $(1,2)$–tensors with the symmetries of the intrinsic torsion, i.e. the space $T^*M \otimes \mathfrak{so}(TM)$. This implies that $|\xi|^2$ and $|\xi_{\text{alt}}|^2 - |\xi_{\text{sym}}|^2$ are also quadratic invariants. Thus, for an irreducible submodule $U$ of the representation $T^*M \otimes \mathfrak{so}(TM)$, since the space of its quadratic invariants in one dimensional $[2]$, then the number $E_U = |\chi^U|^2 + |\xi^U_{\text{alt}}|^2 - |\xi^U_{\text{sym}}|^2$ is a constant multiple of $|\xi^U|^2$. Here $\xi^U$ denotes the $U$–component of $\xi$ with respect to decomposition into irreducible summands. This approach is also valid for any irreducible module $G$–module in the space of possible intrinsic torsions. This kind of approach, was used in [3] to get integral formulas for many $G$–structures.

We have an immediate consequence of the formula (3.10).

Corollary 3.4. Assume $M$ is an oriented $G$–structure. If (2.1) holds, $[g^\perp, g^\perp] \subset g$ and the characteristic vector field vanishes, then

\[1/2 s_{g^\perp} = |\xi_{\text{alt}}|^2 - |\xi_{\text{sym}}|^2.\]

In particular, if the intrinsic torsion is totally skew–symmetric, then

\[s_{g^\perp} = 2|\xi|^2 \geq 0\]

with the equality if and only if a $G$–structure $M$ is integrable (i.e. $\xi = 0$).

The consequences of the integral formula will be presented in the following section for certain choices of $G$.

4. APPLICATIONS TO CERTAIN RIEMANNIAN $G$–STRUCTURES

In this section we rewrite formulae (3.10) and (3.11) for certain $G$–structures. We also give some applications of these relations. We will show that obtained formulas are consistent with the Bochner type formulae obtained, using representation theory, in [3].
4.1. Almost product structures. We show that the divergence and integral formulae obtained in the previous section agree with the Walczak formulas \cite{13}. Since this integral formula has found many applications, we will only concentrate on deriving it from (3.11) and state its one corollary, which will be needed later.

Let \((M, g)\) be an oriented Riemannian manifold, with tho complementary orthogonal oriented distributions \(D\) and \(D^\perp\), i.e. \(TM = D \oplus D^\perp\). Thus the bundle of oriented frames \(SO(M)\) has a reduction to a subgroup \(SO(m) \times SO(n-m) \subset SO(n)\), where \(m = \dim D\). On the level of Lie algebras

\[
\mathfrak{so}(n) = (\mathfrak{so}(m) \oplus \mathfrak{so}(n-m)) \oplus \mathfrak{m},
\]

where

\[
\mathfrak{m} = (\mathfrak{so}(m) \oplus \mathfrak{so}(n-m))^\perp = \left\{ \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \right\}.
\]

Let \(\nabla\) be the Levi–Civita connection of \(g\). Since the orthogonal projection is just a restriction to non–diagonal blocks, it follows that the intrinsic torsion equals

\[
\xi_XY = -(\nabla_XY^\top)^\perp - (\nabla_XY^\perp)^\top,
\]

where \(Y^\top\) and \(Y^\perp\) denotes the components of \(Y\) in \(D\) and \(D^\perp\). Notice that \(\xi\) is made of shape operators and fundamental forms of distributions \(D\) and \(D^\perp\).

Recall, that the second fundamental form, for example, of \(D\) is a \((1,2)\)–symmetric tensor \(B = B_D\) of the form

\[
B(X,Y) = \frac{1}{2}(\nabla_X Y + \nabla_Y X)^\perp, \quad X,Y \in D.
\]

Additionally, we will use integrability tensor \(T = T^D\) being just

\[
T(X,Y) = \frac{1}{2}[X,Y]^\perp.
\]

Notice that \(B(X,Y) + T(X,Y) = (\nabla_X Y)^\perp\), hence \(B\) and \(T\) are symmetrization and alternation of (a minus of) a part of the intrinsic torsion reduced to \(D\).

For an orthonormal basis \((e_i)\) adapted to decomposition \(D \oplus D^\perp\), denote by \(e_A\) components of \((e_i)\) in \(D\) and by \(e_\alpha\) components of \((e_i)\) in \(D^\perp\). The characteristic vector field \(\chi\) equals

\[
\chi = -\sum_A (\nabla_{e_A} e_A)^\perp - \sum_\alpha (e_\alpha e_\alpha)^\top = -H - H^\perp,
\]

where \(H\) and \(H^\perp\) are mean curvature vectors of \(D\) and \(D^\perp\) respectively. Since

\[
\sum_{A,\alpha} |(\nabla_{e_\alpha} e_A)^\perp|^2 = \sum_{A,\alpha,\beta} g(\nabla_{e_\alpha} e_A, e_\beta)^2 = \sum_{A,\alpha,\beta} g(\nabla_{e_\alpha} e_\beta, e_A)^2 = \sum_{\alpha,\beta} |(\nabla_{e_\alpha} e_\beta)^\top|^2
\]

and analogously interchanging \(e_\alpha\) with \(e_A\), then

\[
|\xi^{\text{alt}}|^2 = |T|^2 + |T^\perp|^2 + \frac{1}{4} \sum_{A,B} |(\nabla_{e_A} e_B)^\perp|^2 + \frac{1}{4} \sum_{\alpha,\beta} |(\nabla_{e_\alpha} e_\beta)^\top|^2
\]

and

\[
|\xi^{\text{sym}}|^2 = |B|^2 + |B^\perp|^2 + \frac{1}{4} \sum_{\alpha,\beta} |(\nabla_{e_\alpha} e_\beta)^\top|^2
\].
Denoting by $s_{\text{mix}}$ so called mixed scalar curvature,

$$s_{\text{mix}} = \sum_{A,\alpha} g(R(e_A, e_\alpha)e_\alpha, e_A)$$

we get

$$s_m = \sum_{i,j} g(R(e_i, e_j) e_j, e_i) = 2 \sum_{A,\alpha} g(R(e_A, e_\alpha)e_\alpha, e_A) = 2s_{\text{mix}}.$$ 

Putting all facts together (3.10) implies Walczak formula [14]

$$-\text{div}(H + H^\perp) = -s_{\text{mix}} + |H|^2 + |H^\perp|^2 + |T|^2 - |B|^2 - |B^\perp|^2.$$ 

Assuming $M$ is closed, the following Walczak integral formula holds [14]

$$\int_M s_{\text{mix}} = \int_M |H|^2 + |H^\perp|^2 + |T|^2 - |B|^2 - |B^\perp|^2.$$ 

This formula has found many applications. Let us only state one of its consequences for $D$ of codimension 1, since it will be used in one of forthcoming subsections. In this case, clearly, $T^\perp = 0$ and $B^\perp = H^\perp$. Denoting the unit positively oriented vector field orthogonal to $D$ by $\zeta$, we have $H = - (\text{div}\zeta)\zeta$. Moreover,

$$s_{\text{mix}} = \sum_A g(R(e_A, \zeta)\zeta, e_A) = \text{Ric}(\zeta, \zeta).$$

Therefore, (4.2) becomes

$$\int \text{Ric}(\zeta, \zeta) = \int_M (\text{div}\zeta)^2 + |T|^2 - |B|^2.$$ 

### 4.2. Almost hermitian structures

Assume $(M, g, J)$ is an oriented Riemannian manifold with an almost complex structure $J$, i.e., $J^2 = -\text{id}_{TM}$, which is hermitian, i.e., $g(JX, JY) = g(X, Y)$ for $X, Y \in TM$. Then $(M, g, J)$ is of even dimension $2n$ and induces $U(n)$–structure. On the level of Lie algebras, we have

$$\mathfrak{so}(2n) = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp,$$

where

$$\mathfrak{u}(n) = \{ A \in \mathfrak{so}(n) \mid AJ = JA \}, \quad \mathfrak{u}(n)^\perp = \{ A \in \mathfrak{so}(n) \mid AJ = -JA \}.$$ 

In particular, $[\mathfrak{u}(n)^\perp, \mathfrak{u}(n)^\perp] \subset \mathfrak{u}(n)$, thus $\mathfrak{s}_{\text{alg}}^{\mathfrak{u}(n)^\perp} = 0$. The orthogonal projection from $\mathfrak{so}(n)$ to $\mathfrak{u}(n)^\perp$ equals $A \mapsto \frac{1}{2} (A + JAJ)$. Thus the $\mathfrak{u}(n)$–component of $R$ is given by

$$R(X, Y)_{\mathfrak{u}(n)} = \frac{1}{2} (R(X, Y) + J \circ R(X, Y) \circ J).$$

Moreover, the intrinsic torsion, being informally the projection of $-\nabla$ to $\mathfrak{u}(n)^\perp$, is given by the formula

$$\xi_X Y = -\frac{1}{2} J(\nabla_X J)Y.$$

Hence, the characteristic vector field $\chi$ is the following

$$\chi = -\frac{1}{2} J(\text{div} J).$$
Let us describe the intrinsic torsion with the use of the Nijenhuis tensor $N$ and the Kähler form $\Omega$. Recall that
\[
N(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = (\nabla_X J) JY - (\nabla_Y J) JX + (\nabla_{JX} J) Y - (\nabla_{JY} J) X
\]
and
\[
\Omega(X, Y) = g(X, JY).
\]
It is a famous theorem by Newlander and Nirenberg that vanishing of the Nijenhuis tensor is equivalent to integrability of $J$, i.e. existence of complex coordinates adapted to $J$. In can be shown \[1\] that
\[
\Omega(X, Y) = 4g(\xi_X Y, Z) + d\Omega(X, JY, Z) - g(N(Y, Z), X).
\]
Unfortunately, this shows that $\xi$ has no particular symmetries and it is hard to give nice interpretations for the symmetrized and skew–symmetrized intrinsic torsion $\xi^{\text{sym}}$ and $\xi^{\text{alt}}$, respectively. Therefore, it is convenient to consider some restrictions or decomposition of the intrinsic torsion. The space of all possible intrinsic torsions is, in this case, $T^*M \otimes \mathfrak{u}(n)^\perp(TM)$. Decomposing this space into irreducible modules with respect to $U(n)$–action, we get so called Gray–Hervella classes \[10\]
\[
T^*M \otimes \mathfrak{u}(n)^\perp(TM) = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4,
\]
where each class can be characterized as follows:

- $\mathcal{W}_1$: $\xi_X Y = - \xi_Y X$, in particular, $\chi = 0$.
- $\mathcal{W}_2$: $g(\xi_X Y, Z) + g(\xi_Z X, Y) + g(\xi_Y Z, X) = 0$. Then $\chi = 0$.
- $\mathcal{W}_3$: $\xi_X Y = \xi_{JX}(JY)$ and $\chi = 0$.
- $\mathcal{W}_4$: $-4\xi_X Y = \theta(Y) X + \theta(JY) JX - g(X, Y) \theta^\sharp - g(X, JY) \theta^\sharp$ for $\theta \in \Gamma(T^*M)$.

It can be shown \[10\] that $\mathcal{W}_1 \oplus \mathcal{W}_2$ and $\mathcal{W}_3 \oplus \mathcal{W}_4$ are described by relations
\[
\xi_{JX} JY = -\xi_X Y \quad \text{and} \quad \xi_{JX} JY = \xi_X Y,
\]
whereas $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ is characterized by vanishing of the characteristic vector field. In such case we have point–wise formula for the $\mathfrak{u}(n)^\perp$–component of the scalar curvature
\[
\frac{1}{2} s_{\mathfrak{u}(n)^\perp} = |\xi^{\text{alt}}|^2 - |\xi^{\text{sym}}|^2.
\]

The left–hand–side has a nice interpretation, which is valid for all Gray–Hervella classes. Define Ricci and $*$–Ricci tensors by
\[
\text{Ric}(X, Y) = \sum_i g(R(X, e_i) e_i, Y), \quad \text{Ric}^*(X, Y) = g(R(X, e_i) Je_i, JY).
\]
These induce, taking the trace, scalar curvatures
\[
s = \sum_i \text{Ric}(e_i, e_i), \quad s^* = \sum_i \text{Ric}^*(e_i, e_i).
\]
Hence

$$s_{u(n)} = \sum_{i,j} g(R(e_i, e_i)_{u(n)}) e_i, e_j$$

(4.10)

$$= \frac{1}{2} \sum_{i,j} g(R(e_j, e_i)_{u(n)}) e_i, e_j + g(JR(e_j)_{e_i} e_i, e_j)$$

$$= \frac{1}{2} (s - s^*).$$

Now, we will discuss the relations between elements in the divergence formula (8.10) in each pure class separately. We will proceed by studying quadratic invariants of the $U(n)$–representation on the space of intrinsic torsions $T^*M \otimes u(n)^{-1}(TM)$ [10]:

$$i_1 = \sum_{i,j,k} \alpha(e_i, e_j, e_k)^2$$

$$i_3 = \sum_{i,j,k} \alpha(e_i, e_j, e_k) \alpha(e_i, e_j, e_k)$$

$$i_4 = \sum_{i,j,k} \alpha(e_i, e_j, e_k) \alpha(e_i, e_j, e_k),$$

where $\alpha(X, Y, Z) = g(\xi_X Y, Z)$. Notice that

$$i_1 = |\xi|^2, \quad i_2 = |\xi_{sym}|^2 - |\xi_{alt}|^2, \quad i_4 = |\chi|^2.$$

It is not hard to see by definitions of each pure class $\mathcal{W}_i$ and conditions (4.8) the following relations contained in the table below hold. In the table $i_j^{(k)}$ denotes invariant $i_j$ considered for class $\mathcal{W}_k$.

**Table 1. Quadratic invariants for Gray–Hervella classes $\mathcal{W}_k$**

| $\mathcal{W}_1$ | $i_1^{(1)} = 0$, | $i_3^{(1)} = 0$, | $i_1^{(1)} = -i_1^{(1)}$, | $i_3^{(1)} = -i_3^{(1)}$ |
| $\mathcal{W}_2$ | $i_4^{(2)} = 0$, | $i_3^{(2)} = 0$, | $i_2^{(2)} = i_2^{(2)}$, | $i_2^{(2)} = i_2^{(2)}$ |
| $\mathcal{W}_3$ | $i_4^{(3)} = 0$, | $i_3^{(3)} = 0$, | $i_2^{(3)} = i_2^{(3)}$, | $i_2^{(3)} = i_2^{(3)}$ |
| $\mathcal{W}_4$ | $i_4^{(4)} = \frac{1}{4}(n - 1)i_1^{(4)}$, | $i_3^{(4)} = i_3^{(4)}$, | $i_2^{(4)} = i_2^{(4)}$, | $i_2^{(4)} = i_2^{(4)}$ |

Decompose $\xi$ and $\chi$ with respect to the Gray–Hervella classes as follows

$$\xi = \xi^1 + \xi^2 + \xi^3 + \xi^4, \quad \chi = \chi^1 + \chi^2 + \chi^3 + \chi^4$$

end let

$$E_k = |\chi^k|^2 + |\xi^k_{alt}|^2 - |\xi^k_{sym}|^2 = -i_2^{(k)} + i_4^{(k)}.$$

It can be shown that

$$i_j = \sum_k i_j^{(k)}, \quad j = 1, 2, 3, 4.$$

Thus, by above considerations (see also Remark 3.3), we have

$$E_1 = |\xi|^2, \quad E_2 = -\frac{1}{2} |\xi|^2, \quad E_3 = 0, \quad E_4 = \frac{1}{2}(n - 1)|\xi|^2.$$

Hence

$$\text{div} \chi = |\xi|^2 - \frac{1}{2} |\xi|^2 + \frac{n - 1}{2} |\xi|^2 - \frac{1}{4} (s - s^*).$$

(4.11)
which implies the integral formula by Bor and Hernandez Lamoneda [3] (assuming \( M \) is closed)
\[
\int_{M} (2|\xi|^2 - |\xi^2|^2 + (n-1)|\xi^4|) \, d\text{vol}_M = \frac{1}{2} \int_{M} s - s^* \, d\text{vol}_M.
\]

Let us list two applications: one which was not stated in [3] but, although not directly, can be found [13] and the second one being reformulation of Corollary 3.4.

**Proposition 4.1.** Assume \((M, g, J)\) is of type \(\mathcal{W}_1\), where \( M \) is closed. Then
\[
\int_{M} s - s^* = 8(n-1)^2 \int_{M} |\theta|^2.
\]
In particular, if \( \int_{M} s = \int_{M} s^* \), then \((M, g, J)\) is Kähler.

**Proposition 4.2.** Assume \((M, g, J)\) is a nearly–Kähler manifolds, i.e. of type \(\mathcal{W}_1\). Then
\[
s - s^* = 4|\xi|^2 = |\nabla J|^2.
\]

**Proof.** Follows directly by (4.10) and by Corollary 3.4. \(\square\)

**Example 4.3.** Consider a six sphere \( S^6 \) with a natural Riemannian metric \( g \) induced from \(\mathbb{R}^7 \) and an almost complex \( J \) structure induced from cross product on \(\mathbb{R}^7 \). It is well known that \( J \) is \( g \)-orthogonal and nearly–Kähler \((\mathcal{W}_1)\). It can be shown that \( s = 30 \), \( s^* = 6 \) and \( |\nabla J| = 24 \) [9]. This justifies Proposition 4.2.

### 4.3. Special almost hermitian structures

Assume \((M, g, J)\) is an almost hermitian manifold equipped with a complex volume form \( \Psi = \psi_+ + i\psi_- \) such that \( \langle \Psi, \Psi \rangle_C = 1 \), where the inner product is a natural extension of an inner product for real valued forms. This structure defines reduction of a structure group to special unitary group \( SU(n) \), hence a \( SU(n) \)--structure. On the level of Lie algebras we have
\[
\mathfrak{so}(2n) = \mathfrak{su}(n) \oplus (\mathfrak{u}(n)^\perp \oplus \mathbb{R}),
\]
since \( \mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathbb{R} \). For an element \( A \in \mathfrak{u}(n) \) let
\[
A = \begin{pmatrix}
A_0 & -A_1 \\
A_1 & A_0
\end{pmatrix} \in \mathfrak{so}(2n).
\]
Then \( A \in \mathfrak{u}(n) \) if and only if \( A \in \mathfrak{u}(n) \) and \( \text{tr}A_1 = 0 \). Notice that
\[
\text{tr}A_1 = \frac{1}{2} \sum_i g(Ae_i, Je_i) = -\frac{1}{2} \text{tr}(AJ).
\]
Thus, the orthogonal projection from \( \mathfrak{so}(2n) \) to \( \mathfrak{su}(n)^\perp = \mathfrak{u}(n)^\perp \oplus \mathbb{R} \) equals
\[
A \mapsto \frac{1}{2}(A + JAJ) - \frac{1}{2n} (\text{tr}A) J.
\]
The intrinsic torsion \( \xi \) equals \( \xi = \xi^{U(n)} + \eta \), where \( \xi^{U(n)} \) is the intrinsic torsion of related \( U(n) \)--structure and
\[
\eta_{XY} = -\frac{1}{2n} \sum_i g(\xi X Je_i, e_i)JY.
\]
Denote the one form on the right–hand–side of the above formula evaluated on \( JX \) not \( X \) also by \( \eta \), so \( \eta_X Y = \eta(JX)JY \). This convention will appear to be useful. Denote the class in the space of all possible intrinsic torsions induced by \( \eta \) by \( \mathcal{W}_5 \). Thus we have a decomposition
\[
\xi = \xi^1 + \xi^2 + \xi^3 + \xi^4 + \eta \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5.
\]

**Remark 4.4.** As it was noticed in [3] module \( \mathcal{W}_5 \) is not in general orthogonal to other modules \( \mathcal{W}_i \), \( i = 1, 2, 3, 4 \). This follows from the fact that \( \mathcal{W}_5 \equiv T^*M \equiv \mathcal{W}_4 \), thus there are isomorphic modules or, in other words, the multiplicity of \( \mathcal{W}_5 \) or \( \mathcal{W}_4 \) is 2. This also implies that these two mentioned modules are not orthogonal but they are orthogonal to remaining ones. We will proceed in a different way than in [3].

Let us begin by describing all objects contained in the divergence formula (3.10). We have
\[
\chi = \chi^{U(n)} + \sum_i \eta(\xi_i)\xi_i = \chi^{U(n)} + \eta^\sharp
\]
and
\[
\sum_{i,j} g(\xi_i e_j, \xi_j e_i) = \sum_{i,j} g(\xi_i^{U(n)} e_j, \xi_j^{U(n)} e_i) + 2 \sum_{i,j} g(\xi_i^{U(n)} e_j, \eta(\xi_j)\xi_i) = \sum_{i,j} g(\xi_i^{U(n)} e_j, \xi_j^{U(n)} e_i) - 2\eta(\chi^{U(n)}) + |\eta^\sharp|^2.
\]

Thus
\[
\text{div} \chi = -\frac{1}{2} s_{\mathfrak{su}(n)^\perp} + \frac{1}{2} s_{\mathfrak{su}(n)^\perp}^\text{alt} + |\chi|^2 - \sum_{i,j} g(\xi_i e_j, \xi_j e_i)
\]
\[
= -\frac{1}{2} s_{\mathfrak{su}(n)^\perp} + \frac{1}{2} s_{\mathfrak{su}(n)^\perp}^\text{alt} + |\chi^{U(n)}|^2 + 2\eta(\chi^{U(n)}) + |\eta^\sharp|^2
\]
\[- \sum_{i,j} g(\xi_i^{U(n)} e_j, \xi_j^{U(n)} e_i) + 2\eta(\chi^{U(n)}) - |\eta^\sharp|^2
\]
\[=
\text{div} \chi^{U(n)} + \frac{1}{2} s_{\mathfrak{su}(n)^\perp}^\text{alt} - \frac{1}{2} s_{\mathfrak{su}(n)^\perp} + \frac{1}{2} s_{\mathfrak{su}(n)^\perp}^\text{alt} + 4\eta(\chi^{U(n)}).
\]

Hence
\[
(4.12) \quad \text{div} \eta^\sharp = -\frac{1}{2} s_\mathbb{R} + \frac{1}{2} s_{\mathfrak{su}(n)^\perp}^\text{alt} + 4\eta(\chi^{U(n)}),
\]

where we split \( s_{\mathfrak{su}(n)^\perp} \) into \( s_{\mathfrak{u}(n)^\perp} \) and \( s_\mathbb{R} \) with respect to the decomposition \( \mathfrak{su}(n)^\perp = \mathfrak{u}(n)^\perp \oplus \mathbb{R} \). Applying the Stokes theorem, assuming \( M \) is closed we get the following integral formula.

**Proposition 4.5.** On a closed \( SU(n) \)–structure \( (M, g, J) \) with the \( \mathcal{W}_5 \)-component induced by the 1–form \( \eta \) we have the following integral formula
\[
(4.13) \quad 8 \int_M \eta(\chi^{U(n)}) \, d\text{vol}_M = \int_M s_\mathbb{R} - s_{\mathfrak{su}(n)^\perp}^\text{alt} \, d\text{vol}_M.
\]
In particular, if \((M, g, J)\) is of Gray–Hervella class \(W_1 \oplus W_2 \oplus W_3\) treated as \(U(n)\)–structure, then \(\int_M s_R = \int_M s_{\text{alt}}^{\text{su}(n)^\perp}\).

The values of \(s_R\) and \(s_{\text{alt}}^{\text{su}(n)^\perp}\) can be computed explicitly. Namely,

\[
s_R = \sum_{i,j} g(R(e_i, e_j)R e_j, e_i) \\
= -\frac{1}{2n} \sum_{i,j} \text{tr}(R(e_i, e_j)J)g(J e_j, e_i) \\
= -\frac{1}{2n} \sum_{i,j} g(R(J e_j, e_j)J e_i, e_i) \\
= \frac{1}{2n} \sum_{i,j} (g(R(e_i, J e_j)e_j, e_i) + g(R(e_j, J e_i)J e_j, e_i)) \\
= \frac{1}{n} s^*.
\]

To compute \(s_{\text{alt}}^{\text{su}(n)^\perp}\), it is convenient to determine the component \([\text{su}(n)^\perp, \text{su}(n)^\perp]_{\text{su}(n)^\perp}\).

For \(A = A_0 + \lambda J\) and \(B = B_0 + \mu J\), where \(A_0, B_0 \in \mathfrak{u}(n)^\perp\), by the relation 

\[
[A, B]_{\text{su}(n)^\perp} = [A_0, B_0]_{\mathbb{R}} + \lambda [J, B_0] + \mu [A_0, J] \\
= -\frac{1}{n} \text{tr}(A_0 B_0 J) J + 2(\mu A_0 - \lambda B_0) J.
\]

Hence,

\[
s_{\text{alt}}^{\text{su}(n)^\perp} = \sum_{i,j} \left(-\frac{1}{n} \text{tr}(\xi_{e_i} \xi_{e_j} J)g(J e_j, e_i) + 2g(\eta(J e_j)\xi_{e_i} J e_j - \eta(J e_i)\xi_{e_j} J e_j, e_i) \right) \\
= -\frac{1}{n} \sum_{i,j} g(\xi_{J e_j} \xi_{e_j} J e_j, e_i) + 2 \sum_{i,j} (\eta(J e_j)g(\xi_{e_i} J e_j, e_i) - \eta(J e_i)g(\xi_{e_j} J e_j, e_i)) \\
= \frac{1}{n} \sum_{i,j} g(\xi_{e_j} J e_j, \xi_{e_j} J e_j) + 4\eta(\chi^{U(n)}) \\
= -\frac{1}{n} \sum_{i,j} g(\xi_{e_j} e_i, \xi_{e_j} J e_i) - 4\eta(\chi^{U(n)}).
\]

Denote by \(\xi^{U(n),12}\) and \(\xi^{U(n),34}\) the \(W_1 \oplus W_2\) and \(W_3 \oplus W_4\) components of \(\xi^{U(n)}\).

By (4.8), we get

\[
\sum_{i,j} g(\xi_{e_j} e_i, \xi_{e_j} J e_i) = -|\xi^{U(n),12}|^2 + |\xi^{U(n),34}|^2.
\]

We have proved the following corollary.

**Corollary 4.6.** On a closed \(SU(n)\)–structure \((M, g, J)\) with the \(W_5\)-component induced by the 1–form \(\eta\) we have the following integral formula

\[
\int_M s^* = \int_M (|\xi^{U(n),12}|^2 - |\xi^{U(n),34}|^2) + 4n \int_M \eta(\chi^{U(n)}).
\]
In particular, if $\xi^{U(n)} \in W_1 \oplus W_2 \oplus W_3 \oplus W_5$, then
\[ \int_M s^* = \int_M |\xi^{U(n),1}|^2 + |\xi^{U(n),2}|^2 - |\xi^{U(n),3}|^2. \]

**Remark 4.7.** The above integral formula, however formulated in a different way, can be found in [3]. Let us be more precise. In [3] authors state some consequences of their formula for almost hermitian structures with vanishing first Chern class $c_1(M)$. Let us derive the first Chern class in our setting. It is known [9] that the first Chern form $\gamma$ is given by
\[ 8\pi \gamma = -\varphi + 2\psi, \]
where
\[ \varphi(X,Y) = \text{tr}((\nabla_X J)(\nabla_Y J)), \quad \psi(X,Y) = \text{tr}(R(X,Y) \circ J). \]
It is not hard to see, that
\[ \varphi(X,Y) = 4 \sum_i g(\xi_X e_i, \xi_Y e_i), \quad \psi(X,Y) = -2\text{Ric}^*(X, JY). \]
Thus, using the same arguments as before Corollary 4.6, we get
\[ 2\pi \text{tr}^* \gamma = |\xi^{U(n),3}|^2 - |\xi^{U(n),2}|^2 + s^*, \]
where $\text{tr}^* \gamma = \sum_i \gamma(e_i, J e_i)$. Notice that vanishing of the first Chern class, i.e., $\gamma = \text{da}$ for some 1–form, is equivalent to the fact that $\int_M \text{tr}^* \gamma = 0$. Thus by Corollary 4.6, $c_1(M) = 0$ if and only if $\int_M \eta(\chi^{U(n)}) = 0$. Finally, note that by (3.2) we have
\[ \int_M s = 5 \int_M s^*. \]
By Corollary 4.6 we have
\[ \int_M s^* = \int_M |\xi^{U(n)}|^2. \]
It suffices to notice that by (1.10), (4.9), $\frac{1}{4}(s - s^*) = |\xi^{U(n)}|^2$. 

**Corollary 4.8.** Consider an $SU(n)$–structure $(M, g, J)$ which is of type $W_1 \oplus W_5$. Then
\[ \int_M s = 5 \int_M s^*. \]

**Proof.** By Corollary 4.6 we have
\[ \int_M s^* = \int_M |\xi^{U(n)}|^2. \]
It suffices to notice that by (1.10), (4.9), $\frac{1}{4}(s - s^*) = |\xi^{U(n)}|^2$. 

### 4.4. Almost contact metric structures.
Let $(M, g)$ be a $(2n+1)$–dimensional manifold together with a 1–form and its dual unit vector field $\xi$ and $\varphi \in \text{End}(TM)$ such that
\[ \varphi^2 X = -X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \]
Notice that $\varphi$ defines almost complex structure which is $g$–orthogonal on the distribution $\ker \eta$. Thus we get $U(n) \times 1$–structure. On the level of Lie algebras we have
\[ \mathfrak{so}(2n+1) = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp, \]
where $\mathfrak{u}(n)^\perp$ is isomorphic to the space of block matrices of the form
\[ \begin{pmatrix} B & a^t \\ -a & t \end{pmatrix}, \quad B \in \mathfrak{u}(n)^\perp, \quad a \in \mathbb{R}^{2n}, \quad t \in \mathbb{R}. \]
Since here $\zeta = e_{2n+1}$, $\varphi$ is a natural complex structure on $\mathbb{R}^{2n}$ and zero on $\zeta$, it is easy to see that the orthogonal projection from $\mathfrak{so}(2n+1)$ onto $\mathfrak{u}(n)^\perp$ equals

\[
A \mapsto \frac{1}{2}(A + \varphi A \varphi + \zeta^T A \otimes \zeta + \zeta^T \otimes A \zeta).
\]

Rewriting this formula with the use of the one–form $\eta(\equiv \zeta^T)$, the intrinsic torsion satisfies the following relation

\[
(4.16) \quad \xi_X Y = \varphi(\xi_X \varphi Y) + \eta(\xi_X Y) \zeta + \eta(Y) \xi_X \zeta.
\]

This, moreover, implies the formula for the intrinsic torsion

\[
(4.17) \quad \xi_X Y = \frac{1}{2}(\nabla_X \varphi) \varphi Y + \frac{1}{2}(\nabla_X \eta) Y \cdot \zeta - \eta(Y) \nabla_X \zeta.
\]

By (4.14) it follows that

\[
\varphi(\nabla_X \varphi) Y = -(\nabla_X \varphi) \varphi Y + (\nabla_X \eta) Y \cdot \zeta + \eta(Y) \nabla_X \zeta.
\]

Thus, we may write the intrinsic torsion in an alternative way

\[
(4.18) \quad \xi_X Y = \frac{1}{2} \varphi(\nabla_X \varphi) Y + (\nabla_X \eta) Y \cdot \zeta - \frac{1}{2} \eta(Y) \nabla_X \eta.
\]

Hence, the characteristic vector field in this case equals

\[
(4.19) \quad \chi = -\frac{1}{2} \varphi(\text{div} \varphi) + (\text{div} \zeta) \zeta - \frac{1}{2} \nabla \zeta.
\]

The condition $\xi \in T^*M \otimes \mathfrak{u}(n)^\perp(TM)$ is equivalent to the relation (4.16). Decomposing the space $T^*M \otimes \mathfrak{u}(n)^\perp(TM)$ into irreducible $U(n) \times 1$–modules, we get twelve classes $\mathcal{C}_1, \ldots, \mathcal{C}_{12}$. First four are isomorphic to Gray–Hervella classes $\mathcal{W}_1, \ldots, \mathcal{W}_4$. Let us describe these spaces in more detail. Put

\[
\mathcal{D}_1 = \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_4, \quad \mathcal{D}_2 = \mathcal{C}_5 \oplus \ldots \oplus \mathcal{C}_{11}, \quad \mathcal{D}_3 = \mathcal{C}_{12}.
\]

Each of above spaces is characterized as follows [5]:

**Class $\mathcal{D}_1$:** $\xi_X Y = \xi_X \zeta = 0$. Applying formulas (4.17) and (4.18) we obtain $\nabla \zeta = 0$ and hence $\chi = -\frac{1}{2} \varphi(\text{div} \varphi)$, as expected, since in this case, being not very precise, $\xi$ is the intrinsic torsion on the almost hermitian structure $\ker \eta$.

**Class $\mathcal{D}_2$:** $\xi_X Y = \eta(X) \xi_X + \eta(Y) \xi_X + \eta(\xi_X Y) \zeta$. Taking $X = Y = \zeta$ we get $\xi_X \zeta = -\eta(\xi_X \zeta) \zeta$ and $\eta$ to both sides we get $\xi_X \zeta = 0$. Taking $X = Y = e_1$ we obtain

\[
\text{pr}_{\ker \eta} \chi = 0.
\]

Hence, $\chi \in \text{span} \zeta$, so by (4.19), $\chi = (\text{div} \zeta) \zeta$. Moreover, it is easy to see that for $X, Y \in \ker \eta$

\[
\xi_X Y \in \text{span} \zeta, \quad \xi_X Y \in \ker \eta, \quad \xi_X \zeta \in \ker \eta, \quad \xi_X \zeta = 0.
\]

Thus, by the formulas (4.17) and (4.18) for the intrinsic torsion we get ($X, Y \in \ker \eta$)

\[
\begin{align*}
\xi_X Y &= (\nabla_X \eta) Y \cdot \zeta, \\
\xi_X \zeta &= \frac{1}{2} (\nabla_X \varphi) \varphi Y, \\
\xi_X \zeta &= -\nabla_X \zeta.
\end{align*}
\]
Denoting by $B^n$ and $T^n$ the (symmetric) second fundamental form and integrability tensor of $\text{ker} \eta$, respectively, we have (again $X,Y \in \text{ker} \eta$)

\[
\begin{align*}
\xi^\text{alt}_X Y &= -T^n(X,Y), \\
\xi^\text{sym}_X Y &= -B^n(X,Y), \\
\xi^\text{alt}_X \zeta &= -\frac{1}{2} \nabla_X \zeta - \frac{1}{4} (\nabla_\zeta \varphi) \varphi X, \\
\xi^\text{sym}_X \zeta &= -\frac{1}{2} \nabla_X \zeta + \frac{1}{4} (\nabla_\zeta \varphi) \varphi X.
\end{align*}
\]

Therefore

(4.20) \[ |\chi|^2 + |\xi^\text{alt}|^2 - |\chi^\text{sym}|^2 = (\text{div} \zeta)^2 + |T|^2 - |B|^2 + \eta(\text{div} \varphi(\nabla_\zeta \varphi)). \]

**Class $D_3$:** $\xi_X Y = \eta(X)\eta(Y)\xi_\zeta + \eta(X)\eta(\xi_\zeta Y)\zeta$. Therefore, for $X,Y \in \text{ker} \eta$

\[
\begin{align*}
\xi_X Y &= 0, \\
\xi_X \zeta &= 0, \\
\xi_\zeta \zeta &= 0, \\
\zeta \xi \zeta &= \ker \eta.
\end{align*}
\]

Hence, $\chi = \xi_\zeta \zeta = -\nabla_\zeta \zeta$ and $|\chi|^2 + |\xi^\text{alt}|^2 - |\chi^\text{sym}|^2 = 0$.

We now turn to computations of the scalar curvature components. Since $[\mathfrak{u}(n)\perp, \mathfrak{u}(n)\perp] \subset \mathfrak{u}(n)$ it follows that $s^\text{alt}_{\mathfrak{u}(n)\perp}$ vanishes. For $s_{\mathfrak{u}(n)\perp}$ by (4.15) we have

\[
\begin{align*}
s_{\mathfrak{u}(n)\perp} &= \sum_{i,j} g(R(e_i, e_j)_{\mathfrak{u}(n)\perp} e_j, e_i) \\
&= \frac{1}{2}(s - s^\ast) + \frac{1}{2} \sum_{i,j} (\eta(R(e_i, e_j)e_j) \eta(e_i) + \eta(e_j)g(R(e_i, e_j)\zeta, e_i) \\
&= \frac{1}{2}(s - s^\ast) + \sum_{i,j} g(R(\zeta, e_j)e_j, \zeta) \\
&= \frac{1}{2}(s - s^\ast) + \text{Ric}(\zeta, \zeta).
\end{align*}
\]

where, as in the almost hermitian case, the $\ast$-scalar curvature is defined as follows

\[ s^\ast = \text{tr} \text{Ric}^\ast = \sum_{i,j} g(R(e_i, e_j)\varphi e_j, \varphi e_i). \]

Concluding we have the following result.

**Proposition 4.9.** Let $(M,g,\varphi,\eta,\zeta)$ be an almost contact metric structure of class $D_2$. Then

(4.21) \[ \text{div}((\text{div} \zeta)\zeta) = (\text{div} \zeta)^2 + |T^n|^2 - |B^n|^2 + \eta(\text{div} \varphi(\nabla_\zeta \varphi)) - \frac{1}{4}(s - s^\ast) - \frac{1}{2}\text{Ric}(\zeta, \zeta). \]

If additionally, $M$ is closed, then the following integral formula holds

(4.22) \[ \frac{1}{4} \int_M s - s^\ast - 2\text{Ric}(\zeta, \zeta) = \int_M \eta(\text{div} \varphi(\nabla_\zeta \varphi)). \]

**Proof.** The only explanation is needed for the proof of the integral formula. It follows by (4.21) and integral formula (4.3). \[ \square \]

Let us list some direct consequences of above fact.
Corollary 4.10. If an almost contact metric structure \((M, g, \varphi, \eta, \zeta)\), where \(M\) is closed, is of type \(C_5 \oplus \ldots \oplus C_{10}\), then
\[
\int_M s - s^* = 2 \int_M \text{Ric}(\zeta, \zeta).
\]
Proof. By classification of almost contact metric structures (see [5] Table III) and remark below, if \(\xi \in C_5 \oplus \ldots \oplus C_{10}\), then \(\nabla_\zeta \varphi = 0\). Now, it suffices to apply Proposition 4.9. \(\square\)

Corollary 4.11. If the almost contact metric structure \((M, g, \varphi, \eta, \zeta)\), where \(M\) is closed, is of type \(C_5 \oplus \ldots \oplus C_{10}\) such that \(\int_M s - s^* < 0\), then distribution \(
ker \eta\) is not totally geodesic.
Proof. By Corollary 4.10 and (4.3) \(\int_M |H^\eta|^2 + |T^\eta|^2 - |B^\eta|^2 = \int_M \text{Ric}(\zeta, \zeta) < 0\). Thus the second fundamental form \(B^\eta\) does not vanish. \(\square\)

Remark 4.12. Note that in [5] types of almost contact metric structures, i.e. irreducible modules of \(T^* M \otimes \mathfrak{su}(n)^\perp (TM)\), were classified with respect to \(\alpha(X, Y, Z) = (\nabla_X \Phi)(Y, Z)\). It is well known that this is equivalent to considering the intrinsic torsion as a map \(\beta(X, Y, Z) = g(\xi_X Y, Z)\). The correspondence follows from the fact that \(\nabla_X \Phi = \xi_X \Phi\), since \(\nabla^{U(n) \times SU(1)} \Phi = 0\). This implies a direct relation
\[
4.23 \quad \alpha(X, Y, Z) = \beta(X, Y, \varphi Z) - \beta(X, Z, \varphi Y).
\]
Note that we should be careful with studying irreducible modules \(C_1, \ldots, C_{12}\), since the correspondence \(\alpha \leftrightarrow \beta\) interchanges some of the modules, which is underlined in the table below.

| \(\alpha\) | \(C_1\) | \(C_2\) | \(C_3\) | \(C_4\) | \(C_5\) | \(C_6\) | \(C_7\) | \(C_8\) | \(C_9\) | \(C_{10}\) | \(C_{11}\) | \(C_{12}\) |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(\beta\) | \(C_1\) | \(C_2\) | \(C_3\) | \(C_4\) | \(C_5\) | \(C_6\) | \(C_7\) | \(C_8\) | \(C_9\) | \(C_{10}\) | \(C_{11}\) | \(C_{12}\) |

In an analogous way as for \(U(n)\)–structures, it can be shown, with a little bit more effort, that the integral formula (3.11) in this case is equivalent with the integral formula obtained in [8].

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