Supplementary Material for:
Modeling spatial processes with unknown extremal dependence class

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1 Proofs

Proof of Proposition 1.

\[ P(X_j > x, X_k > x) = P(W_j > x^{1/(1-\delta)} R^{-\delta/(1-\delta)}, W_k > x^{1/(1-\delta)} R^{-\delta/(1-\delta)}) \]

= \[ P(W_j > S, W_k > S), \]

where \( S = x^{1/(1-\delta)} R^{-\delta/(1-\delta)} \), so that \( S \) has support \((0, x^{1/(1-\delta)})\), and Lebesgue density \( f_S(s) = \frac{1-\delta}{\delta} s^{(1-\delta)/\delta-1} x^{-1/\delta} \) on this interval. Using assumption \((3)\), we have

\[ P(W_j > S, W_k > S) = 1 - \delta x^{-1/\delta} \int_0^1 s^{(1-\delta)/\delta-1} \, ds + \frac{1-\delta}{\delta} x^{-1/\delta} \int_1^{x^{1/(1-\delta)}} L_W(s) s^{(1-\delta)/\delta-1/\eta_W-1} \, ds \]

= \[ x^{-1/\delta} + \frac{1-\delta}{\delta} x^{-1/\delta} \int_1^{x^{1/(1-\delta)}} L_W(s) s^{(1-\delta)/\delta-1/\eta_W-1} \, ds. \]

Consider the behavior of \( \int_1^{x^{1/(1-\delta)}} L_W(s) s^{(1-\delta)/\delta-1/\eta_W-1} \, ds \), which is convergent since we have a well defined probability. We will apply Karamata’s Theorem (Resnick, 2007, Theorem 2.1) and so distinguish between the cases when the index of regular variation is \( \geq -1 \). The notation \( g \in RV_\rho \) denotes that a function \( g \) is regularly varying at infinity with index \( \rho \in \mathbb{R} \).

Case 1: \( (1-\delta)/\delta - 1/\eta_W - 1 \geq -1 \) i.e., \( \eta_W \geq \delta/(1-\delta) \). By Karamata’s Theorem \( \int_1^x L(s) s^\theta \, ds \in RV_{\theta+1} \) when \( \theta \geq -1 \). Thus

\[ \int_1^{x^{1/(1-\delta)}} L_W(s) s^{(1-\delta)/\delta-1/\eta_W-1} \, ds = \tilde{L}(x)x^{1/\delta-1/\{\eta_W(1-\delta)\}}, \]
where \( \tilde{L} \) is a new SV function, using also a result on composition of regularly varying functions (Resnick, 2007, Prop. 2.6 (iv)). Overall in Case 1 we thus have

\[
P(X_j > x, X_k > x) = L(x) x^{-1/\{\eta W(1-\delta)\}},
\]

for some slowly varying function \( L \), noting that terms of order \( x^{-1/\delta} \) are absorbed into \( L \) when \( \eta W > \delta/(1-\delta) \).

**Case 2:** \( (1-\delta)/\delta - 1/\eta W - 1 < -1 \) i.e., \( \eta W < \delta/(1-\delta) \). By Karamata’s Theorem

\[
\int_x^\infty L(s)s^\theta ds \in RV_{\theta+1} \text{ when } \theta < -1.
\]

We have

\[
\int_1^{x^{1/(1-\delta)}} L_W(s)s^{(1-\delta)/\delta-1/\eta W-1} ds = \int_1^\infty L_W(s)s^{(1-\delta)/\delta-1/\eta W-1} ds - \int_{x^{1/(1-\delta)}}^\infty L_W(s)s^{(1-\delta)/\delta-1/\eta W-1} ds,
\]

and so the second term on the right-hand side the is regularly varying of index \( 1/\delta - 1/\{\eta W(1-\delta)\} \). The first term on the right-hand side of expression (1) is established by noting that

\[
E\{\min(W_j, W_k)^{(1-\delta)/\delta}\} = \int_0^\infty P\{\min(W_j, W_k)^{(1-\delta)/\delta} > t\} dt
\]

\[
= 1 + \int_1^\infty L_W(t^{\delta/(1-\delta)}) t^{-\delta/\{\eta W(1-\delta)\}} dt = 1 + \frac{1-\delta}{\delta} \int_1^\infty L_W(s)^{(1-\delta)/\delta-1/\eta W-1} ds.
\]

Overall in Case 2 we thus have

\[
P(X_j > x, X_k > x) = E\{\min(W_j, W_k)^{(1-\delta)/\delta}\} x^{-1/\delta} - L(x) x^{-1/\{\eta W(1-\delta)\}}
\]

\[
= E\{\min(W_j, W_k)^{(1-\delta)/\delta}\} x^{-1/\delta} \{1 + o(1)\},
\]

since \( \eta W(1-\delta) < \delta \).

**Proof of Corollary 1.** Since \( X \) has common margins and upper endpoint infinity, the extremal dependence class is determined by the limit

\[
\chi = \lim_{x \to \infty} \frac{P(X_j > x, X_k > x)}{P(X_j > x)}.
\]

**If \( \delta > 1/2 \):** Then \( P(X_j > x) \sim \frac{\delta}{2\delta - 1} x^{-1/\delta} \). We have \( \delta/(1-\delta) > 1 \) so we must be in Case 2, and

\[
\chi = E\{\min(W_j, W_k)^{(1-\delta)/\delta}\} \frac{2\delta - 1}{\delta} > 0,
\]

with expression (10) following as

\[
E\{W_j^{(1-\delta)/\delta}\} = \int_1^\infty w^{(1-\delta)/\delta-2} dw = \frac{\delta}{2\delta - 1}.
\]
If $\delta = 1/2$: Consider the relation (8), and note that since $P(X_j > x)^{-1}$ is regularly varying with limit infinity, then the composition $L_X[P(X_j > x)^{-1}] =: L^*(x)$ is slowly varying at infinity; cf. Resnick (2007, Prop. 2.6(iv)). For $\delta = 1/2$, we have

$$P(X_j > x) = x^{-2\{2\log(x) + 1\}}, \quad P(X_j > x, X_k > x) = E\{\min(W_j, W_k)\}x^{-2\{1 + o(1)\}},$$

since $\eta_W < 1$ by assumption, which puts us in Case 2. We thus have

$$P(X_j > x, X_k > x) = L^*(x)P(X_j > x),$$

so that $\eta_X = 1$ and

$$L^*(x) \sim \frac{E\{\min(W_j, W_k)\}}{\{2\log(x) + 1\}} \to 0, \quad x \to \infty.$$

If $\delta < 1/2$ Then $P(X_j > x) \sim \frac{1-\delta}{1-2\delta}x^{-1/(1-\delta)}$. If $\eta_W \geq \delta/(1-\delta)$ then we are in Case 1 and the survivor function is $L(x)x^{-1/\eta_W(1-\delta)}$. Otherwise if $\eta_W < \delta/(1-\delta)$ we are in Case 2 and the survivor function decays like $x^{-1/\delta}$. In both cases this leads to $\chi = 0$ with coefficient of tail dependence,

$$\eta_X = \begin{cases} \delta/(1-\delta) & \text{if } \eta_W < \delta/(1-\delta) \\ \eta_W & \text{otherwise.} \end{cases}$$

During the proofs of Propositions 2 and 3, we will need results on the quantile function $q(t) := F_X^{-1}\{1 - 1/t\}$, which we give in the following Lemma.

**Lemma 1.** For $\delta > 1/2$, the marginal quantile function $q(t) = F_X^{-1}(1 - 1/t)$ satisfies

$$q(t) = \left(\frac{\delta}{2\delta - 1}\right)^{\delta} t^{\delta} \left[1 - (1-\delta) \left(\frac{\delta}{2\delta - 1}\right)^{(1-2\delta)/(1-\delta)} t^{(1-2\delta)/(1-\delta)} \{1 + o(1)\}\right], \quad t \to \infty.$$

**Proof.** The quantile function is obtained by solving $1 - F_X\{q(t)\} = t^{-1}$, which leads to

$$\frac{\delta}{2\delta - 1} q(t)^{-\delta} \left\{1 - \frac{1-\delta}{\delta} q(t)^{(1-2\delta)/(\delta(1-\delta))}\right\}^{\delta} = t^{-1}$$

and thus

$$q(t) \left\{1 - \frac{1-\delta}{\delta} q(t)^{(1-2\delta)/(\delta(1-\delta))}\right\}^{\delta} = \left(\frac{\delta}{2\delta - 1}\right)^{\delta} t^{\delta}. \tag{3}$$

Since $q(t) \to \infty$ as $t \to \infty$, and $(1-2\delta)/(\delta(1-\delta)) < 0$ for $\delta > 1/2$, expression (3) leads to

$$q(t) = \left(\frac{\delta}{2\delta - 1}\right)^{\delta} t^{\delta} \{1 + o(1)\}, \quad t \to \infty,$$

which can be fed back into (3) to give the result claimed. ✷
Proof of Proposition 2. When \( \delta > 1/2 \), the exponent function \( V(x_1, \ldots, x_d) \) is obtained from the limit
\[
V(x_1, \ldots, x_d) = \lim_{t \to \infty} t(1 - P[X_1 \leq F_X^{-1}(1 - (tx_1)^{-1}), \ldots, X_1 \leq F_X^{-1}(1 - (tx_d)^{-1})]).
\]
Using Lemma 1, we have \( q(tx) = (\frac{\delta}{2\delta - 1})^\delta (tx)^\delta \{1 + o(1)\} \), and so
\[
1 - P\{X_1 \leq q(tx_1), \ldots, X_1 \leq q(tx_d)\} = P\left\{ \max_{j=1, \ldots, d} \frac{X_j}{q(tx_j)} > 1 \right\} = \int_0^1 P\left[ \max_{j=1, \ldots, d} \frac{W_j^{(1-\delta)/\delta}}{x_j(1 + o(1))} > tu \right] du = \frac{1}{t} \int_0^t P\left[ \max_{j=1, \ldots, d} \frac{W_j^{(1-\delta)/\delta}}{x_j(1 + o(1))} > z \right] dz.
\]
For sufficiently large \( t \), an integrable function of the form \( P \left[ K \max_{j=1, \ldots, d} \frac{W_j^{(1-\delta)/\delta}}{x_j(1 + o(1))} > z \right] \), \( 1 < K < \infty \), dominates the integrand over \((0, \infty)\) and thus the above integral tends to
\[
\int_0^\infty P\left\{ \max_{j=1, \ldots, d} \frac{W_j^{(1-\delta)/\delta}}{(\frac{\delta}{2\delta - 1})^\delta x_j} > z \right\} dz = E\left\{ \max_{j=1, \ldots, d} \frac{W_j^{(1-\delta)/\delta}}{x_j} \right\}\left(2\delta - 1\right)^\frac{1}{\delta}
\]
and hence
\[
\lim_{t \to \infty} t[1 - P\{X_1 \leq q(tx_1), \ldots, X_d \leq q(tx_d)\}] = E\left\{ \max_{j=1, \ldots, d} \frac{W_j^{(1-\delta)/\delta}}{x_j} \right\}\left(2\delta - 1\right)^\frac{1}{\delta}
\]
the final line following by equation (2). \( \blacksquare \)

Proof of Proposition 3. The function
\[
\chi_u = P\{F_X(X_j) > u \mid F_X(X_k) > u\} = \frac{P\{X_j > F_X^{-1}(u), X_k > F_X^{-1}(u)\}}{1 - u}
\]
Lemma 1 gives the behavior of \( F_X^{-1}(u) = q\{(1 - u)^{-1}\} \), whilst the proof of Proposition 1 provides \( P(X_j > x, X_k > x) = E\{\min(W_j, W_k)^{(1-\delta)/\delta}x^{-1/\delta} - L(x)x^{-1/(\eta x(1-\delta))}\} \), giving
\[
\frac{P\{X_j > F_X^{-1}(u), X_k > F_X^{-1}(u)\}}{1 - u} = E\{\min(W_j, W_k)^{(1-\delta)/\delta}\}\frac{F_X^{-1}(u)^{1-\delta}}{1 - u} - L(F_X^{-1}(u))\frac{F_X^{-1}(u)^{1/(\eta x(1-\delta))}}{1 - u} = X\left[1 + \frac{1 - \delta}{\delta}\left(\frac{\delta}{2\delta - 1}\right)^{(1-2\delta)/(1-\delta)}(1 - u)^{(2\delta - 1)/(1-\delta)}\{1 + o(1)\}\right] - L\{(1 - u)^{-1}\}(1 - u)^{\delta/(\eta x(1-\delta))}\{1 + o(1)\};
\]
with constant terms absorbed in to \( L \). Since \((2\delta - 1)/(1 - \delta) < \delta/(\eta_W (1 - \delta)) - 1 \) for \( \eta_W < 1 \), the result follows.

2 Supporting information for Section 3

![Boxplots](image)

Figure 1: Boxplots for the MLEs \( \log(\hat{\lambda}) \) and \( \hat{\nu} \), estimated concurrently with \( \hat{\delta} \) as in Figure 3 of Section 3.2.
3 Supporting information for Section 4

3.1 Bootstrap procedure

To demonstrate that the stationary bootstrap procedure described in §4.1 adequately reproduces the temporal dependence in the extremes, we consider a spatial extension of the extremal index for univariate time series. For a stationary time series \( \{X_t\} \), the extremal index, \( \theta \in [0, 1] \), can be defined as

\[
\theta = \lim_{n \to \infty} P(X_2 \leq u_n, \ldots, X_{p_n} \leq u_n | X_1 > u_n),
\]

where \( p_n = o(n) \) and \( u_n \) is a series such that \( n\{1 - F(u_n)\} \to \tau \in (0, \infty) \). The extremal index describes the degree of temporal clustering of extremes, with \( 1/\theta \) the limiting mean cluster size. A popular estimator for \( \theta \) is the so-called Runs Estimator (Smith and Weissman, 1994). The estimate is formed by taking the reciprocal of the mean cluster size, whereby threshold exceedances are determined to be part of different clusters (the same cluster) if they are separated by a run of at least \( m \) (fewer than \( m \)) consecutive non-exceedances.

In our application we have a time series of spatial processes \( \{X_t(s)\} \), which, as we consider winter months only, may reasonably be deemed stationary. In analogy to the univariate case, we define clusters of spatial threshold exceedances as follows. A realization of the process is deemed to be a “threshold exceedance” if the observation at any site exceeds a given threshold. Clusters are then defined as sequences of threshold exceedances separated by a run of at least \( m \) non-exceedances, and \( \theta \) as the reciprocal mean cluster size. Figure 2 displays a histogram of estimated \( \theta \)s, using a value of \( m = 1 \), from 200 bootstrap samples, along with that from the original dataset of 50 sites temporally thinned to one observation per day. The threshold value used was the 95%-quantile, as in the model fit. The agreement between the original and bootstrap samples indicates that the temporal structure of the extremes is adequately reproduced.

![Figure 2: Estimates of the extremal index from the time series of spatial processes using the stationary bootstrap sampling procedure described in §4.1. The vertical line is the value from the original sample.](image-url)
3.2 Additional model fit diagnostics

Figure 3: Distribution of the number of exceedances, given at least one exceedance of the 95%-quantile threshold. These histograms are based on the data at the 20 sites used for fitting the model (top) or all 50 sites (bottom). Model-based quantities are calculated for our new model (left) and the Gaussian model (right) by simulating $10^5$ values from the fitted dependence models.

References

Resnick, S. I. (2007). *Heavy Tail Phenomena: Probabilistic and Statistical Modeling*. Springer, New York.

Smith, R. L. and Weissman, I. (1994). Estimating the extremal index. *J. Roy. Statist. Soc. B*, pages 515–528.