EXPLICIT APPROXIMATIONS OF OPTION PRICES VIA MALLIAVIN CALCULUS IN A GENERAL STOCHASTIC VOLATILITY FRAMEWORK

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ABSTRACT. We obtain an explicit approximation formula for European put option prices within a general stochastic volatility model with time-dependent parameters. Our methodology involves writing the put option price as an expectation of a Black-Scholes formula, reparameterising the volatility process and then performing a number of expansions. The bulk of the work is due to computing a number of expectations induced by the expansion procedure explicitly, which we achieve by appealing to techniques from Malliavin calculus. We obtain the explicit representation of the form of the error generated by the expansion procedure, and we provide sufficient ingredients in order to obtain a meaningful bound. Under the assumption of piecewise-constant parameters, our approximation formulas become closed-form, and moreover we are able to establish a fast calibration scheme. Furthermore, we perform a numerical sensitivity analysis to investigate the quality of our approximation formula in the so-called Stochastic Verhulst model, and show that the errors are well within the acceptable range for application purposes.

Keywords: Stochastic volatility model, Closed-form expansion, Closed-form approximation, Malliavin calculus, Stochastic Verhulst, Stochastic Logistic, XGBM

1. INTRODUCTION

In this article, we consider the European put option pricing problem in a general stochastic volatility model with time-dependent parameters, namely, where the volatility process satisfies the SDE $dV_t = \alpha(t, V_t)dt + \beta(t, V_t)dB_t$ where drift and diffusion coefficients satisfy some regularity properties. The main contributions of this article are an explicit second-order approximation for the price of a European put option, an explicit form for the error induced in the approximation, as well as a fast calibration scheme. Furthermore, the approximation formulas are written in terms of certain iterated integral operators, which, under the assumption of piecewise-constant parameters, are closed-form, yielding a closed-form approximation to the European put option price. As we work in a general framework, we provide sufficient conditions

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regarding existence of a risk-neutral measure in a general inhomogeneous stochastic volatility model, namely Theorem B.1, which is an extension of Theorem 2.4(i) in [23]. We obtain sufficient conditions required in order to obtain a meaningful bound on the error term induced by the approximation procedure, these ingredients essentially being moments involved with the volatility process. Lastly, we perform a numerical sensitivity analysis in the so-called Stochastic Verhulst model in order to assess our approximation procedure in application. Our approximation methodology involves appealing to the mixing solution methodology (see [17, 31, 38]), which reduces the dimensionality of the pricing problem, small vol-of-vol expansion techniques, as well as Malliavin calculus machinery, in the lines of [5, 20].

It is well known that implied volatility is heavily dependent on the strike and maturity of European option contracts. This phenomenon is called the volatility smile, a feature the seminal Black-Scholes model fails to address due to the assumption of a constant volatility [7]. In response, a number of frameworks have been proposed with the intention of accurately modelling the smile effect. In particular, stochastic volatility models are one of a number of classes of models which were developed in order to achieve this. In a stochastic volatility model, the volatility (or variance) process is modelled as a stochastic process itself, possibly correlated with the spot. Empirical evidence has demonstrated that stochastic volatility models are significantly more realistic than models with deterministic volatility. However, with this added complexity comes a cost, as it is usually not possible to compute the prices of even the simplest contracts in a closed-form manner.

Affine stochastic volatility models are a subclass of stochastic volatility models which possess a certain amount of tractability. Specifically, affine models are those in which the characteristic function of the log-spot can be computed explicitly\(^1\), see for example the Heston and Schöbel-Zhu models [16, 32]. As a consequence, in such models, it can be shown that it is possible to express European option prices in a quasi closed-form fashion. However, non-affine models have been shown to be substantially more realistic than their affine counterparts. This empirical evidence has been presented in a number of studies, see for example [9, 13, 18]. For this reason, recently there has been a significant push in the industry towards favouring non-affine models. When one studies a non-affine model, this usually means that prices cannot be represented in any sort of convenient way. Consequently, numerical procedures such as PDE and Monte-Carlo methods have been substantially developed in the literature, see [33, 40].

Closed-form approximations are an alternative methodology for option pricing, where the option price is approximated by a closed-form expression. The purpose of obtaining a closed-form approximation is to achieve a ‘best of both worlds’ scenario; one can utilise a realistic and sophisticated model, yet still have a means to obtain prices of option prices rapidly. Moreover, since transform methods are usually not utilised, time-dependent parameters can often be handled well. One motivation for quick option pricing formulas is calibration, where the option price must be computed several times within an optimisation procedure.

Over the past few decades, closed-form approximation results have been extensively studied in the literature. For example, [24] derive a general closed-form expression for the price of an

\(^1\)More precisely, an affine model is one where the log of the characteristic function of the log-spot is an affine function.
option via a PDE approach, as well as its corresponding implied volatility. A similar approach was then utilised in [25] in order to describe the implied volatility on options on bonds in a general affine term structure model, and later on this was extended to handle options on forward rates and quadratic term structure models [26]. [15] use singular perturbation techniques to obtain an explicit approximation for the option price and implied volatility in their SABR model. [1] show that from the mixing solution, one can approximate the put option price by decomposing it into a sum of two terms, one being completely correlation independent and the other dependent on correlation. However, neither terms are explicit. Furthermore, [3, 4] show that under the assumption of small correlation, an expansion can be performed with respect to the mixing solution, where the resulting expectations can be computed using Malliavin calculus techniques. Similarly, in the case of the time-dependent Heston model, [5] consider the mixing solution and expand around vol-of-vol, performing a combination of Taylor expansions and techniques. Similarly, in the case of the time-dependent Heston model, [5] consider the mixing solution and expand around vol-of-vol, performing a combination of Taylor expansions and computing the resulting terms via Malliavin calculus techniques. [20] adapts the methodology of [5] to the Inverse-Gamma model.

Stochastic volatility models usually either model the volatility directly, or indirectly via the variance process. A critical assumption is that volatility or variance has some sort of mean reversion behaviour, and this is supported by empirical evidence, see for example [14]. Specifically, for modelling the variance, a large class of one-factor stochastic volatility models is given by

\[
\begin{align*}
\text{d}S_t &= S_t((r_t^d - r_t^f)\text{d}t + \sqrt{V_t}\text{d}W_t), \quad S_0, \\
\text{d}V_t &= \kappa_t(\theta_t V_t^\mu - \bar{V}_t^\mu)\text{d}t + \lambda_t V_t^\mu\text{d}B_t, \quad V_0 = v_0, \\
\text{d}(W, B)_t &= \rho_t\text{d}t,
\end{align*}
\]

whereas for modelling the volatility, this class is of the form

\[
\begin{align*}
\text{d}S_t &= S_t((r_t^d - r_t^f)\text{d}t + \tilde{V}_t\text{d}W_t), \quad S_0, \\
\text{d}V_t &= \kappa_t(\theta_t V_t^\tilde{\mu} - \bar{V}_t^\tilde{\mu})\text{d}t + \lambda_t V_t^\tilde{\mu}\text{d}B_t, \quad V_0 = v_0, \\
\text{d}(W, B)_t &= \rho_t\text{d}t,
\end{align*}
\]

for some $\tilde{\mu}, \tilde{\mu}$ and $\mu \in \mathbb{R}$. Some popular models in the literature include:

| Model               | Variance/Volatility | Dynamics of $V$                                                                 | $\tilde{\mu}$ | $\bar{\mu}$ | $\mu$ |
|---------------------|---------------------|-------------------------------------------------------------------------------|----------------|-------------|-------|
| Heston [16]         | Variance            | \(\text{d}V_t = \kappa_t(\theta_t - V_t)\text{d}t + \lambda_t \sqrt{V_t}\text{d}B_t\) | 0              | 1           | 1/2   |
| Schöbel and Zhu [32]| Volatility          | \(\text{d}V_t = \kappa_t(\theta_t - V_t)\text{d}t + \lambda_t \text{d}B_t\)    | 0              | 1           | 0     |
| GARCH [27, 38]      | Variance            | \(\text{d}V_t = \kappa_t(\theta_t - V_t)\text{d}t + \lambda_t V_t \text{d}B_t\) | 0              | 1           | 1     |
| Inverse Gamma [20]  | Volatility          | \(\text{d}V_t = \kappa_t(\theta_t - V_t)\text{d}t + \lambda_t V_t \text{d}B_t\) | 0              | 1           | 1     |
| 3/2 Model [21]      | Variance            | \(\text{d}V_t = \kappa_t(\theta_t V_t^2 - \bar{V}_t^2)\text{d}t + \lambda_t V_t^{3/2} \text{d}B_t\) | 1              | 2           | 3/2   |
| Verhulst [8, 22]    | Volatility          | \(\text{d}V_t = \kappa_t(\theta_t V_t^2 - \bar{V}_t^2)\text{d}t + \lambda_t V_t \text{d}B_t\) | 1              | 2           | 1     |

\[^{2}\text{There exist other classes of stochastic volatility models. For example, the exponential Ornstein-Uhlenbeck model [37] is not included in either of these classes.}\]
In this article, we will obtain an explicit second-order expression for the price of a European put option in the following general model:\(^3\)

\[
\begin{align*}
\text{d}S_t &= (r_d^t - r_f^t)S_t \text{d}t + V_t S_t \text{d}W_t, \quad S_0, \\
\text{d}V_t &= \alpha(t, V_t) \text{d}t + \beta(t, V_t) \text{d}B_t, \quad V_0 = v_0, \\
\text{d}\langle W, B \rangle_t &= \rho_t \text{d}t. \\
\end{align*}
\] (1.1)

where \(W\) and \(B\) are Brownian motions with deterministic, time-dependent instantaneous correlation \((\rho_t)_{0 \leq t \leq T}\), defined on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})\). Here \(T\) is a finite time horizon where \((r_d^t)_{0 \leq t \leq T}\) and \((r_f^t)_{0 \leq t \leq T}\) are the deterministic, time-dependent domestic and foreign interest rates respectively. In addition, \(\rho_t \in [-1, 1]\) for any \(t \in [0, T]\). Furthermore, \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the filtration generated by \((W, B)\) which satisfies the usual assumptions. In the following, \(\mathbb{E}(\cdot)\) (sometimes \(\mathbb{E}[\cdot]\) or simply \(\mathbb{E}\)) denotes the expectation under \(\mathbb{Q}\), where \(\mathbb{Q}\) is a domestic risk-neutral measure which we assume to be chosen, see Theorem B.1 for sufficient conditions on existence on \(\mathbb{Q}\).

The price of a put option with log-strike \(k\) is given by

\[
\text{Put}_G = e^{-\int_0^T r_d^t \text{d}t} \mathbb{E} \left( e^k - S_T \right)_+. 
\]

The methodology utilised in this article has been previously implemented for the subsequent models:

- For the Heston model

\[
\begin{align*}
\text{d}S_t &= S_t((r_d^t - r_f^t) \text{d}t + \sqrt{V_t} \text{d}W_t), \\
\text{d}V_t &= \kappa_t(\theta_t - V_t) \text{d}t + \lambda_t \sqrt{V_t} \text{d}B_t, \\
\text{d}\langle W, B \rangle_t &= \rho_t \text{d}t, \\
\end{align*}
\]

this has been studied by [5].

- For the Inverse-Gamma (IGa) model

\[
\begin{align*}
\text{d}S_t &= S_t((r_d^t - r_f^t) \text{d}t + V_t \text{d}W_t), \\
\text{d}V_t &= \kappa_t(\theta_t - V_t) \text{d}t + \lambda_t V_t \text{d}B_t, \\
\text{d}\langle W, B \rangle_t &= \rho_t \text{d}t, \\
\end{align*}
\]

this has been tackled by [20].

The purpose of this article is to extend the methodology utilised in these aforementioned papers to that of pricing in the general framework eq. (1.1), meaning that the volatility process possesses an arbitrary drift and diffusion coefficient which satisfy some regularity conditions (given in Assumption A and Assumption B), as well as developing an associated fast calibration scheme. The sections are organised as follows:

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\(^3\)Our formulation is for FX market purposes, but can be adapted to equity and fixed income markets easily.
Section 2 details some preliminary calculations. First, we reparameterise the volatility process in terms of a small perturbation parameter, obtaining the process \( \left( V_t^{(\varepsilon)} \right) \). Then, we rewrite the expression for the price of a put option by utilising the mixing solution.

In Section 3 we implement our expansion procedure. Namely, we combine a Taylor expansion of a Black-Scholes formula with a small vol-of-vol expansion of the function \( \varepsilon \mapsto V_t^{(\varepsilon)} \) and its variants. This gives a second-order approximation to the price of a put option.

Section 4 is dedicated to the explicit calculation of terms induced by our expansion procedure from Section 3. In particular, we utilise Malliavin calculus techniques in order to reduce the corresponding terms down into expressions which are in terms of certain iterated integral operators.

In Section 5 we present the explicit form for the error in our expansion methodology. In particular, we provide sufficient ingredients in order to obtain a meaningful bound on the error term.

Section 6 is dedicated to the study of the iterated integral operators in terms of which our approximation formulas are expressed, albeit under the assumption of piecewise-constant parameters. We deduce that under this assumption, our approximation formulas are closed-form. Moreover, this allows us to establish a fast calibration scheme.

Section 7 is dedicated to a numerical sensitivity analysis of our put option approximation formula in the Stochastic Verhulst model.

Remark 1.1. In this article, the phrase ‘explicit expression’ will be used to refer to expressions that can be represented mathematically with an equation that does not involve the subject. The word ‘closed-form’ will be used to describe an explicit expression which only involves elementary functions, and does not involve complicated infinite sums or complicated integrals. For example, let \( g \) be a function with no known closed-form anti-derivative and let

\[
\begin{align*}
  f_1(x) &= \int_0^x g(u)du, \\
  f_2(x) &= e^{\sin(x)}, \\
  f_3(x) &= \int_0^x g(f_3(u))du.
\end{align*}
\]

Then eq. (1.2) is explicit, eq. (1.3) is closed-form and eq. (1.4) is neither.

2. Preliminaries

In order to ensure well-posedness of \( V \) in eq. (1.1), we will require certain assumptions on the regularity of its drift and diffusion coefficients. Furthermore, we will require certain restrictions of the drift and diffusion coefficients when the expansion procedure demands it. In anticipation of this, for the rest of this article, we will enforce the following assumptions on the regularity of the drift and diffusion coefficients of \( V \) in eq. (1.1).
Assumption A. For \( t \in [0, T] \):

(A1) \( \alpha \) is Lipschitz continuous in \( x \), uniformly in \( t \).

(A2) \( \beta \) is Hölder continuous of order \( \geq 1/2 \) in \( x \), uniformly in \( t \).

(A3) There exists a solution of \( V \).

(A4) \( \alpha \) and \( \beta \) satisfy the growth bounds \( x\alpha(t, x) \leq K(1 + |x|^2) \) and \( |\beta(t, x)|^2 \leq K(1 + |x|^2) \) uniformly in \( t \), where \( K \) is a positive constant.

Assumption B. The following properties hold:

(B1) The second derivative \( \alpha_{xx} \) exists and is continuous a.e. in \( x \) and \( t \in [0, T] \).

(B2) The first derivative \( \beta_x \) exists and is continuous a.e. in \( x \) and \( t \in [0, T] \).

The purpose of Assumption A is to guarantee the existence of a pathwise unique strong solution to \( V \) in eq. (1.1), see the Yamada-Watanabe theorem [39], as well as to guarantee that solutions do not explode in finite time. We will comment on the purpose of Assumption B in full detail in Remark 3.2. Clearly (B2) implies (A1); nonetheless we include (A1) for the purpose of clarity.

2.1. Mixing solution. Denote the price of a put option on \( S \) in the general model eq. (1.1) by \( \text{Put}_G \). Namely,

\[
\text{Put}_G = e^{-\int_0^T r_u dt} \mathbb{E}(e^k - S_T)_+.
\]

Let \( X_t := \ln S_t \) denote the log-spot. Now perturb \( X \) in the following way: for \( \varepsilon \in [0, 1] \),

\[
\begin{align*}
\frac{dX_t^{(\varepsilon)}}{dt} &= \left( r_t - f_t - \frac{1}{2} (V_t^{(\varepsilon)})^2 \right) dt + V_t^{(\varepsilon)} dB_t, \quad X_0^{(\varepsilon)} = \ln S_0 =: x_0, \\
\frac{dV_t^{(\varepsilon)}}{dt} &= \alpha(t, V_t^{(\varepsilon)}) dt + \varepsilon \beta(t, V_t^{(\varepsilon)}) dB_t, \quad V_0^{(\varepsilon)} = v_0,
\end{align*}
\]

(2.1)

We can recover the original diffusion from eq. (2.1) as \((S, V) = (\exp(X^{(1)}), V^{(1)})\).

Denote the filtration generated by \( B \) as \( (\mathcal{F}_t^B)_{0 \leq t \leq T} \) and let \( \tilde{X}_t^{(\varepsilon)} := X_t^{(\varepsilon)} - \int_0^t (r_u^{(\varepsilon)} - r_u) du \). By writing \( W_t = \int_0^t \rho_u dW_u + \int_0^t \sqrt{1 - \rho_u^2} dZ_u \), where \( Z \) is a Brownian motion independent of \( B \), it can be seen that

\[
\tilde{X}_T^{(\varepsilon)} | \mathcal{F}_T^B \overset{d}{=} \mathcal{N}(\tilde{\mu}_\varepsilon(T), \tilde{\sigma}_\varepsilon^2(T)),
\]

with

\[
\begin{align*}
\tilde{\mu}_\varepsilon(T) &:= x_0 - \int_0^T \frac{1}{2} (V_t^{(\varepsilon)})^2 dt + \int_0^T \rho_t V_t^{(\varepsilon)} dB_t, \\
\tilde{\sigma}_\varepsilon^2(T) &:= \int_0^T (1 - \rho_t^2)(V_t^{(\varepsilon)})^2 dt.
\end{align*}
\]
Let
\[ g(\varepsilon) := e^{-\int_0^T r_u du} E(e^{k - e^{X_T(\varepsilon)}})_. \]
Then \( g(1) \) is the price of a put option in the general model eq. (1.1). That is, \( g(1) = \text{Put}_G \).

**Proposition 2.1.** The function \( g \) can be expressed as
\[ g(\varepsilon) = E\left(e^{-\int_0^T r_u du} E[(e^{k - e^{X_T(\varepsilon)}}) + \mathcal{F}_T]\right) = E\left[P_{BS}(\hat{\mu}_\varepsilon(T) + \frac{1}{2}\hat{\sigma}_\varepsilon^2(T), \hat{\sigma}_\varepsilon^2(T))\right], \]
where explicitly
\[ \hat{\mu}_\varepsilon(T) + \frac{1}{2}\hat{\sigma}_\varepsilon^2(T) = x_0 - \int_0^T \frac{1}{2}\rho_t^2 (V_t(\varepsilon))^2 dt + \int_0^T \rho_t V_t(\varepsilon) dB_t, \]
\[ \hat{\sigma}_\varepsilon^2(T) = \int_0^T (1 - \rho_t^2) (V_t(\varepsilon))^2 dt, \]
and
\[ P_{BS}(x, y) := e^{-\int_0^T r_t dt} N(-d_{ln}^l) - e^x e^{-\int_0^T r_t dt} N(-d_{ln}^u), \]
\[ d_{ln}^l := d_{ln}^l(x, y) := \frac{x - k + \int_0^T (r_t^d - r_t^f) dt}{\sqrt{\gamma}} \pm \frac{1}{2}\sqrt{\gamma}, \]
where \( N(\cdot) \) denotes the standard normal distribution function.

**Proof.** This result is a consequence of the mixing solution methodology. A derivation can be found in Appendix A. \(\square\)

### 3. Expansion procedure

In this section, we detail our expansion procedure. The notation is similar to that in [5], however there are some differences. The main idea is to approximate the function \( g \) up to second-order through a number of Taylor expansions. The expansion procedure can be briefly summarised by two main steps.

1. First, we expand the function \( P_{BS} \) up to second-order. This step is given in Section 3.2.

2. Then, we expand the functions \( \varepsilon \mapsto V_t(\varepsilon) \) and \( \varepsilon \mapsto (V_t(\varepsilon))^2 \) up to second-order. This step is given in Section 3.1 and Section 3.3.

We then combine both these expansions in order to obtain a second-order approximation for the put option price, which is given in Theorem 3.1. However, this approximation is still in terms of expectations. In Theorem 4.1, we will obtain the explicit second-order approximation in terms of iterated integral operators (defined in Definition 4.1), which will be convenient for our fast calibration scheme in Section 6.

**Remark 3.1.** Let \((t, \varepsilon) \mapsto \xi_t^{(\varepsilon)}\) be a \(C([0, T] \times [0, 1]; \mathbb{R})\) function smooth in \( \varepsilon \). Denote by \( \xi_{i,t}^{(\varepsilon)} := \frac{\partial^{i} \xi_t^{(\varepsilon)}}{\partial \varepsilon^i} \) its \( i \)-th derivative in \( \varepsilon \), and let \( \xi_{i,t} := \xi_{i,t}^{(\varepsilon)}|_{\varepsilon=0} \). Then by a second-order Taylor
expansion around $\varepsilon = 0$, we have the representation

$$\xi_t^{(\varepsilon)} = \xi_{0,t} + \xi_{1,t} + \frac{1}{2}\xi_{2,t} + \Theta_{2,t}^{(\varepsilon)}(\xi)$$

where $\Theta$ is the second-order error term given by Taylor’s theorem. Specifically, for $i \geq 0$,

$$\Theta_{i,t}^{(\varepsilon)}(\xi) := \int_0^\varepsilon \frac{1}{i!}(\varepsilon - u)^i\xi_{i+1,t}^{}du.$$

### 3.1. Expanding processes $\varepsilon \mapsto V_t^{(\varepsilon)}$ and $\varepsilon \mapsto \left(V_t^{(\varepsilon)}\right)^2$

Using the notation from Remark 3.1, we can now represent the functions $\varepsilon \mapsto V_t^{(\varepsilon)}$ and $\varepsilon \mapsto \left(V_t^{(\varepsilon)}\right)^2$ via a Taylor expansion around $\varepsilon = 0$ to second-order.

$$V_t^{(\varepsilon)} = v_{0,t} + \varepsilon V_{1,t} + \frac{1}{2}\varepsilon^2 V_{2,t} + \Theta_2^{(\varepsilon)}(V),$$

$$\left(V_t^{(\varepsilon)}\right)^2 = v_{0,t}^2 + 2\varepsilon v_{0,t} V_{1,t} + \varepsilon^2 \left(V_{1,t}^2 + v_{0,t} V_{2,t}\right) + \Theta_2^{(\varepsilon)}(V^2),$$

where $v_{0,t} := V_{0,t}$.

**Lemma 3.1.** The processes $(V_{1,t})$ and $(V_{2,t})$ satisfy the SDEs

$$dV_{1,t} = \alpha_x(t, v_{0,t}) V_{1,t} dt + \beta(t, v_{0,t}) dB_t, \quad V_{1,0} = 0, \quad (3.2)$$

$$dV_{2,t} = \left(\alpha_{xx}(t, v_{0,t})(V_{1,t})^2 + \alpha_x(t, v_{0,t}) V_{2,t}\right) dt + 2\beta_x(t, v_{0,t}) V_{1,t} dB_t, \quad V_{2,0} = 0, \quad (3.3)$$

with explicit solutions

$$V_{1,t} = e^{\int_0^t \alpha_x(z, v_{0,z})dz} \int_0^t \beta(s, v_{0,s})e^{-\int_0^s \alpha_x(z, v_{0,z})dz} dB_s, \quad (3.4)$$

$$V_{2,t} = e^{\int_0^t \alpha_x(z, v_{0,z})dz} \left\{ \int_0^t \alpha_{xx}(s, v_{0,s})(V_{1,s})^2 e^{-\int_0^s \alpha_x(z, v_{0,z})dz} ds + \int_0^t 2\beta_x(s, v_{0,s}) V_{1,s} e^{-\int_0^s \alpha_x(z, v_{0,z})dz} dB_s \right\}. \quad (3.5)$$

**Proof.** We give a sketch of the proof for $(V_{1,t})$. First, we write

$$dV_{1,t}^{(\varepsilon)} = d\left(\partial_x V_{1,t}^{(\varepsilon)}\right) = \partial_x dV_{1,t}^{(\varepsilon)}.$$ 

The SDE for $V_{1,t}^{(\varepsilon)}$ is given in eq. (2.1). By differentiating, we obtain

$$dV_{1,t}^{(\varepsilon)} = \alpha_x(t, V_{t}^{(\varepsilon)}) V_{1,t}^{(\varepsilon)} dt + \left[\varepsilon \beta_x(t, V_{t}^{(\varepsilon)}) V_{1,t}^{(\varepsilon)} + \beta(t, V_{t}^{(\varepsilon)})\right] dB_t, \quad V_{1,0}^{(\varepsilon)} = 0.$$ 

Letting $\varepsilon = 0$ yields the SDE eq. (3.2). Since the SDE is linear, it can be solved explicitly. This gives the result eq. (3.4). The calculations for $(V_{2,t})$ are similar. \(\square\)

**Remark 3.2.** The purpose of Assumption B is now evident. For Lemma 3.1 to be valid, it is clear that we will require the existence of $\alpha_{xx}$ and $\beta_x$ as well as their continuity a.e. in $x$ and $t \in [0,T]$. This is assumed via (B1) and (B2) respectively in Assumption B.
3.2. Expanding $P_{BS}$. Let
\[
\tilde{P}_T^{(e)} := x_0 - \int_0^T \frac{1}{2} \rho_t^2 \left( V_t^{(e)} \right)^2 dt + \int_0^T \rho_t V_t^{(e)} dB_t,
\]
\[
\tilde{Q}_T^{(e)} := \int_0^T (1 - \rho_t^2) \left( V_t^{(e)} \right)^2 dt.
\]
Immediately we have $\tilde{P}_T^{(e)} = \hat{\mu}(T) + \frac{1}{2} \hat{\sigma}^2(T)$ and $\tilde{Q}_T^{(e)} = \hat{\sigma}^2(T)$. Hence from Proposition 2.1
\[
g(\varepsilon) = \mathbb{E} \left( P_{BS} \left( \tilde{P}_T^{(e)}, \tilde{Q}_T^{(e)} \right) \right). \tag{3.6}
\]
Additionally, introduce the functions
\[
P_T^{(e)} := \tilde{P}_T^{(e)} - \tilde{P}_T^{(0)}
\]
\[
= \int_0^T \rho_t (V_t^{(e)} - v_{0,t}) dB_t - \frac{1}{2} \int_0^T \rho_t^2 \left( (V_t^{(e)})^2 - v_{0,t}^2 \right) dt,
\]
\[
Q_T^{(e)} := \tilde{Q}_T^{(e)} - \tilde{Q}_T^{(0)}
\]
\[
= \int_0^T (1 - \rho_t^2) \left( (V_t^{(e)})^2 - v_{0,t}^2 \right) dt,
\]
and the short-hand
\[
\tilde{P}_{BS} := P_{BS} \left( \tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)} \right),
\]
\[
\frac{\partial^{i+j} \tilde{P}_{BS}}{\partial x^i \partial y^j} := \frac{\partial^{i+j} P_{BS} \left( \tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)} \right)}{\partial x^i \partial y^j}.
\]
As $g(1)$ corresponds to the price of a put option, we are interested in approximating the function $P_{BS}$ at $\left( \tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right)$.

**Proposition 3.1.** By a second-order Taylor expansion, the expression $P_{BS} \left( \tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right)$ can be approximated to second-order as
\[
P_{BS} \left( \tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right) \approx \tilde{P}_{BS} + \left( \partial_x \tilde{P}_{BS} \right) P_T^{(1)} + \left( \partial_y \tilde{P}_{BS} \right) Q_T^{(1)}
\]
\[
+ \frac{1}{2} \left( \partial_{xx} \tilde{P}_{BS} \right) (P_T^{(1)})^2 + \frac{1}{2} \left( \partial_{yy} \tilde{P}_{BS} \right) (Q_T^{(1)})^2 + \left( \partial_{xy} \tilde{P}_{BS} \right) P_T^{(1)} Q_T^{(1)}.
\]

**Proof.** Simply expand $P_{BS}$ around the point
\[
\left( \tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)} \right) = \left( x_0 - \int_0^T \frac{1}{2} \rho_t^2 v_{0,t}^2 dt + \int_0^T \rho_t v_{0,t} dB_t, \int_0^T (1 - \rho_t^2) v_{0,t}^2 dt \right)
\]
and evaluate at $\left( \tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right)$. \hfill \Box

3.3. Expanding functions $\varepsilon \mapsto P_T^{(e)}$, $\varepsilon \mapsto Q_T^{(e)}$ and its variants. The next step in our expansion procedure is to approximate the functions $\varepsilon \mapsto P_T^{(e)}, \varepsilon \mapsto \left( P_T^{(e)} \right)^2, \varepsilon \mapsto Q_T^{(e)}, \varepsilon \mapsto \left( Q_T^{(e)} \right)^2$ and $\varepsilon \mapsto P_T^{(e)} Q_T^{(e)}$. By Remark 3.1 we can write
\[
P_T^{(e)} = P_{0,T} + \varepsilon P_{1,T} + \frac{1}{2} \varepsilon^2 P_{2,T} + \Theta_{2,T}^{(e)}(P),
\]
\[
\left( P_T^{(e)} \right)^2 = P_{0,T}^2 + 2\varepsilon P_{0,T} P_{1,T} + \varepsilon^2 \left( P_{1,T}^2 + P_{0,T} P_{2,T} \right) + \Theta_{2,T}^{(e)}(P^2), \tag{3.7}
\]
First, notice that by their definitions, \[ Q_T^{(c)} = Q_{0,T} + \varepsilon Q_{1,T} + \frac{1}{2} \varepsilon^2 Q_{2,T} + \Theta_1^{(c)}(Q), \]
\[ (Q_T^{(c)})^2 = Q_{0,T}^2 + 2\varepsilon Q_{0,T}Q_{1,T} + \varepsilon^2 (Q_{1,T}^2 + Q_{0,T}Q_{2,T}) + \Theta_2^{(c)}(Q^2), \] (3.8)

and

\[ P_T^{(c)}Q_T^{(c)} = P_{0,T}Q_{0,T} + \varepsilon (Q_{0,T}P_{1,T} + P_{0,T}Q_{1,T}) \]
\[ + \frac{1}{2} \varepsilon^2 (Q_{0,T}P_{2,T} + P_{0,T}Q_{2,T} + 3(Q_{1,T}P_{2,T} + P_{1,T}Q_{2,T})) + \Theta_2^{(c)}(PQ). \] (3.9)

This results in the following lemma.

**Lemma 3.2.** Equations (3.7) to (3.9) can be rewritten as

\[ P_T^{(c)} = \varepsilon P_{1,T} + \frac{1}{2} \varepsilon^2 P_{2,T} + \Theta_1^{(c)}(P), \]
\[ (P_T^{(c)})^2 = \varepsilon^2 P_{1,T}^2 + \Theta_2^{(c)}(P^2), \] (3.10)

\[ Q_T^{(c)} = \varepsilon Q_{1,T} + \frac{1}{2} \varepsilon^2 Q_{2,T} + \Theta_1^{(c)}(Q), \]
\[ (Q_T^{(c)})^2 = \varepsilon^2 Q_{1,T}^2 \Theta_2^{(c)}(Q^2), \] (3.11)

and

\[ P_T^{(c)}Q_T^{(c)} = \varepsilon^2 P_{1,T}Q_{1,T} + \Theta_2^{(c)}(PQ), \] (3.12)

respectively, where

\[ P_{1,T} = \int_0^T \rho_t V_{1,t} dB_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} dt, \]
\[ P_{2,T} = \int_0^T \rho_t V_{2,t} dB_t - \int_0^T \rho_t^2 \left( V_{1,t}^2 + v_{0,t} V_{2,t} \right) dt, \]
\[ Q_{1,T} = 2 \int_0^T (1 - \rho_t^2) v_{0,t} V_{1,t} dt, \]
\[ Q_{2,T} = 2 \int_0^T (1 - \rho_t^2) \left( V_{1,t}^2 + v_{0,t} V_{2,t} \right) dt. \]

**Proof.** First, notice that by their definitions, \( P_{0,T} = P_T^{(0)} = \frac{\hat{P}(0)}{T} - \hat{P}_T^{(0)} = 0 \), and similarly \( Q_{0,T} = 0 \). We will show how to obtain the form of \( P_{1,T} \), the rest being similar. By definition

\[ P_T^{(c)} = \partial_\varepsilon \left( P_T^{(c)} \right) = \partial_\varepsilon \left( x_0 - \int_0^T \frac{1}{2} \rho_t^2 \left( V_t^{(c)} \right)^2 dt + \int_0^T \rho_t V_t^{(c)} dB_t \right) \]
\[ = \int_0^T \rho_t V_t^{(c)} dB_t - \int_0^T \rho_t^2 \left( V_t^{(c)} \right)^2 dt. \]

By putting \( \varepsilon = 0 \) we obtain \( P_{1,T} \), namely

\[ P_{1,T} = \int_0^T \rho_t V_{1,t} dB_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} dt. \]

□
\textbf{Theorem 3.1} (Second-order put option price approximation). Denote by \( \text{Put}^{(2)}_G \) the second-order approximation to the price of a put option in the general model eq. (1.1). Then

\[
\text{Put}^{(2)}_G = \mathbb{E} \hat{P}_{BS}
\]

\[
(C_x :=) \quad + \mathbb{E} \partial_x \hat{P}_{BS} \left( \int_0^T \rho_t \left( V_{1,t} + \frac{1}{2} V_{2,t} \right) dB_t \right)
\]

\[
- \frac{1}{2} \int_0^T \rho_t^2 \left( 2v_{0,t} V_{1,t} + \left( V_{1,t}^2 + v_{0,t} V_{2,t} \right) \right) dt
\]

\[
(C_y :=) \quad + \mathbb{E} \partial_y \hat{P}_{BS} \left( \int_0^T (1 - \rho_t^2)(2v_{0,t} V_{1,t} + \left( V_{1,t}^2 + v_{0,t} V_{2,t} \right) ) dt \right)
\]

\[
(C_{xx} :=) \quad + \frac{1}{2} \mathbb{E} \partial_{xx} \hat{P}_{BS} \left( \int_0^T \rho_t V_{1,t} dB_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} dt \right)^2
\]

\[
(C_{yy} :=) \quad + \frac{1}{2} \mathbb{E} \partial_{yy} \hat{P}_{BS} \left( \int_0^T (1 - \rho_t^2)(2v_{0,t} V_{1,t}) dt \right)^2
\]

\[
(C_{xy} :=) \quad + \mathbb{E} \partial_{xy} \hat{P}_{BS} \left( \int_0^T \rho_t V_{1,t} dB_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} dt \right) \cdot \left( \int_0^T (1 - \rho_t^2)(2v_{0,t} V_{1,t}) dt \right)
\]

Additionally, \( \text{Put}_G = \text{Put}^{(2)}_G + \mathbb{E} (\mathcal{E}) \), where \( \mathcal{E} \) denotes the error in the expansion.

\textit{Proof.} From Proposition 3.1, consider the two-dimensional Taylor expansion of \( P_{BS} \) around

\[
\left( \hat{P}^{(0)}_T, \hat{Q}^{(0)}_T \right)
\]

evaluated at

\[
\left( \hat{P}^{(1)}_T, \hat{Q}^{(1)}_T \right).
\]

Then, substitute in the second-order expressions of \( P^{(1)}_T, \left( P^{(1)}_T \right)^2, Q^{(1)}_T, \left( Q^{(1)}_T \right)^2 \) and \( P^{(1)}_T Q^{(1)}_T \) from Lemma 3.2. As this is a second-order expression, the remainder terms \( \Theta \) are neglected. Taking expectation yields \( \text{Put}^{(2)}_G \).

\textbf{Remark 3.3.} The explicit expression for \( \mathcal{E} \) and the analysis of it is left for Section 5.

4. Explicit price

The goal now is to express the terms \( C_x, C_y, C_{xx}, C_{yy}, C_{xy} \) in terms of the following integral operators.

\textbf{Definition 4.1} (Integral operator). Define the following integral operator

\[
\omega^{(k,l)}_{t,T} := \int_t^T \int_0^u \lambda_x \int_0^z \mu_x d\omega_x d\mu_x du.
\]

In addition, we define the \( n \)-fold iterated integral operator through the following recurrence.

\[
\omega^{(k(n),l(n)),(k(n-1),l(n-1)),...,k(l))}_{t,T} (k(n),l(n)),(k(n-1),l(n-1)),...,k(l)) \quad := \omega^{(k(n),l(n)),(k(n-1),l(n-1)),...,k(l))}_{t,T} (k(n),l(n)),(k(n-1),l(n-1)),...,k(l)) \quad n \in \mathbb{N}.
\]
Assumption C. $\beta(t, x) = \lambda t x^\mu$ for $\mu \in [1/2, 1]$, where $\lambda$ is bounded over $[0, T]$.

We will comment on the reasoning behind Assumption C in Remark E.1.

The main result of this article is the following theorem.

**Theorem 4.1** (Explicit second-order put option price). Enforcing Assumption C, the explicit second-order price of a put option in the general model eq. (1.1) is given by

$$
\text{Put}^{(2)}_{G} = P_{BS}\left(x_0, \int_0^T v_{0,t}^2 dt\right) + 2\omega_{0,T}^{-1}(\alpha_x,\alpha_x) \partial_{xy} P_{BS}\left(x_0, \int_0^T v_{0,t}^2 dt\right)
$$

where the partial derivatives of $P_{BS}$ are given in Appendix D.

**Proof.** The proof is given in Appendix E. □

**Remark 4.1.** In Theorem 4.1, we enforce Assumption C. This means we can obtain the second-order pricing formula for different models by choosing a specific $\alpha(t, x)$ that adheres to Assumption A and Assumption B, as well as a $\mu \in [1/2, 1]$, and then appeal to Theorem 4.1. For instance, if we choose $\alpha(t, x) = \kappa t (\theta t - x)$, then this drift satisfies Assumption A and Assumption B. By choosing some $\mu \in [1/2, 1]$, we will obtain the explicit second-order price of a put option where the volatility obeys the dynamics

$$
\text{d}V_t = \kappa t (\theta - V_t)dt + \lambda t V_t^\mu dB_t, \quad V_0 = v_0.
$$

In particular, to obtain the explicit second-order put option price in the Inverse-Gamma model, choose $\alpha(t, x) = \kappa t (\theta t - x)$ and $\mu = 1$, so that $\alpha_x(t, x) = -\kappa t$ and $\alpha_{xx}(t, x) = 0$. Indeed, this
gives the desired result for the second-order put option price in the Inverse-Gamma model as seen in [20].

**Remark 4.2.** Currently the second-order approximation $\text{Put}_G^{(2)}$ from Theorem 4.1 is expressed in terms of iterated integral operators and partial derivatives of $P_{BS}$. When parameters are assumed to be piecewise-constant, then these iterated integral operators can be expressed in a closed-form manner, which we prove in Section 6.

5. **Error analysis**

This section is dedicated to the explicit representation and analysis of the error induced by our expansion procedure in Section 3. The section is divided into two parts.

1. Section 5.1 is devoted to the explicit representation of the error term induced by the expansion procedure.

2. Section 5.2 details how one would approach bounding the error term induced by the expansion procedure in terms of the remainder terms generated by the approximation of the underlying volatility/variance process.

In this section we will make extensive use of the following notation:

- $L^p := L^p(\Omega, \mathcal{F}, \mathbb{Q})$ denotes the vector space of random variables (identified $\mathbb{Q}$ a.s.) with finite $L^p$ norm, given by $\| \cdot \|_p = [E|\cdot|^p]^{1/p}$.

- For an $n$-tuple $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then $|\alpha| = \sum_{i=1}^n \alpha_i$ denotes its 1-norm.

5.1. **Explicit expression for error.** Recall from Theorem 3.1 that the price of a put option in the general model eq. (1.1) was $\text{Put}_G = \text{Put}_G^{(2)} + E(\mathcal{E})$, where $\text{Put}_G^{(2)}$ is the second-order closed-form price. As our expansion methodology was contingent on the use of Taylor polynomials, the term $\mathcal{E}$ evidently appears due to the truncation of Taylor series. To represent $\mathcal{E}$ we will need explicit expressions for the error terms. These are given by Taylor’s theorem, which we will present here to fix notation. We only consider the results up to second-order.

**Theorem 5.1** (Taylor’s theorem for $g : \mathbb{R}^2 \to \mathbb{R}$). Let $A \subseteq \mathbb{R}^2$, $B \subseteq \mathbb{R}$ and let $g : A \to B$ be a $C^3$ function in a closed ball about the point $(a, b) \in A$. Then Taylor’s theorem states that

\[
g(x, y) = g(a, b) + g_x(a, b)(x - a) + g_y(a, b)(y - b) + \frac{1}{2}g_{xx}(a, b)(x - a)^2 + \frac{1}{2}g_{yy}(a, b)(y - b)^2 + g_{xy}(a, b)(x - a)(y - b) + R(x, y),
\]

where

\[
R(x, y) = \sum_{|\alpha| = 3} \frac{|\alpha|}{\alpha_1!\alpha_2!} E_\alpha(x, y)(x - a)^{\alpha_1}(y - b)^{\alpha_2},
\]

\[
E_\alpha(x, y) = \int_0^1 (1 - u)^2 \frac{\partial^3}{\partial x^{\alpha_1} \partial y^{\alpha_2}} g(a + u(x - a), b + u(y - b))du.
\]
Now recall from Section 3 the functions

\[ \tilde{P}_T^{(e)} = x_0 - \int_0^T \frac{1}{2} \rho_t^2 \left( V_t^{(e)} \right)^2 \, dt + \int_0^T \rho_t V_t^{(e)} \, dB_t, \]

\[ \tilde{Q}_T^{(e)} = \int_0^T (1 - \rho_t^2) \left( V_t^{(e)} \right)^2 \, dt \]

and

\[ P_T^{(e)} := \tilde{P}_T^{(e)} - \tilde{P}_T^{(0)} \]
\[ = \int_0^T \rho_t (V_t^{(e)} - v_{0,t}) \, dB_t - \frac{1}{2} \int_0^T \rho_t^2 \left( \left( V_t^{(e)} \right)^2 - v_{0,t}^2 \right) \, dt, \]

\[ Q_T^{(e)} := \tilde{Q}_T^{(e)} - \tilde{Q}_T^{(0)} \]
\[ = \int_0^T (1 - \rho_t^2) \left( \left( V_t^{(e)} \right)^2 - v_{0,t}^2 \right) \, dt. \]

Furthermore, recall the short hand

\[ \tilde{P}_{BS} \equiv P_{BS} \left( \tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)} \right), \]

\[ \partial^{i+j} \tilde{P}_{BS} = \frac{\partial^{i+j} P_{BS}}{\partial x^i \partial y^j}, \]

**Theorem 5.2** (Explicit error term). The error term \( \mathcal{E} \) in Theorem 3.1 induced from the expansion procedure can be decomposed as

\[ \mathcal{E} = \mathcal{E}_P + \mathcal{E}_V \]

where

\[ \mathcal{E}_P = \sum_{|\alpha| = 1} \frac{|\alpha|}{\alpha_1! \alpha_2!} E_\alpha \left( \tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right) \left( P_T^{(1)} \right)^{\alpha_1} \left( Q_T^{(1)} \right)^{\alpha_2}, \]

\[ E_\alpha \left( \tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right) = \int_{0}^{1} (1 - u)^2 \frac{\partial^2 P_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \left( (1 - u) \tilde{P}_T(0) + u \tilde{P}_T(1), (1 - u) \tilde{Q}_T(0) + u \tilde{Q}_T(1) \right) \, du, \]

and

\[ \mathcal{E}_V = \sum_{|\alpha| = 1} \frac{\partial \tilde{P}_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \Theta_{2,7}^{(1)} (P^{\alpha_1} Q^{\alpha_2}) + \frac{1}{2} \sum_{|\alpha| = 2} \frac{|\alpha|}{\alpha_1! \alpha_2!} \frac{\partial^2 \tilde{P}_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \Theta_{2,7}^{(1)} (P^{\alpha_1} Q^{\alpha_2}), \]

where \( \Theta_{2,7}^{(1)} (P^0 Q^x) \) := \( \Theta_{2,7}^{(1)} (Q^x) \) and \( \Theta_{2,7}^{(1)} (P^y Q^0) := \Theta_{2,7}^{(1)} (P^y) \), for \( x, y \in \{1, 2\} \). Here, \( \mathcal{E}_P \) corresponds to the error in the approximation of the function \( P_{BS} \), and \( \mathcal{E}_V \) corresponds to the error in the approximation of the functions \( \varepsilon \mapsto V_t^{(e)} \) and \( \varepsilon \mapsto \left( V_t^{(e)} \right)^2 \).

**Proof.** The decomposition \( \mathcal{E} = \mathcal{E}_P + \mathcal{E}_V \) is a clear consequence of Taylor’s theorem. The next two subsections are dedicated to representing \( \mathcal{E}_P \) and \( \mathcal{E}_V \) explicitly.

### 5.1.1. Explicit \( \mathcal{E}_P \)

First we will derive \( \mathcal{E}_P \) explicitly, the error term corresponding to the second-order approximation of \( P_{BS} \). In our expansion procedure, we expand \( P_{BS} \) up to second-order around the point

\[ \left( \tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)} \right) = \left( x_0 - \int_0^T \frac{1}{2} \rho_t^2 u_{0,t}^2 \, dt + \int_0^T \rho_t v_{0,t} \, dB_t, \int_0^T (1 - \rho_t^2) u_{0,t}^2 \, dt \right). \]
and evaluate at \( \left( \tilde{P}^{(1)}_T, \tilde{Q}^{(1)}_T \right) \). Thus in the Taylor expansion of \( P_{BS} \), the terms will be of the form

\[
\frac{\partial^{\mid \alpha \mid} P_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \left( P^{(1)}_T \right)^{\alpha_1} \left( Q^{(1)}_T \right)^{\alpha_2}
\]

for \( \mid \alpha \mid = 0, 1, 2 \). By Theorem 5.1 (Taylor’s theorem) we can write the second-order Taylor polynomial of \( P_{BS} \left( P^{(1)}_T, Q^{(1)}_T \right) \) with error term as

\[
P_{BS} \left( \tilde{P}^{(1)}_T, \tilde{Q}^{(1)}_T \right) = \int_0^1 (1-u)^2 \frac{\partial^3 P_{BS}}{\partial x\partial y^2} \left( \tilde{P}_T(0) + uP_T(1), \tilde{Q}_T(0) + uQ_T(1) \right) du
\]

Taking expectation gives PutG. Thus the explicit form for the error term \( \mathcal{E}_P \) is

\[
\mathcal{E}_P = \sum_{\mid \alpha \mid = 3} \frac{\mid \alpha \mid}{\alpha_1!\alpha_2!} E_\alpha \left( P^{(1)}_T, Q^{(1)}_T \right) \left( P^{(1)}_T \right)^{\alpha_1} \left( Q^{(1)}_T \right)^{\alpha_2}.
\]

5.1.2. Explicit \( \mathcal{E}_V \). Now we derive \( \mathcal{E}_V \) explicitly, the error corresponding to the second-order approximation of the functions \( \varepsilon \mapsto V^{(\varepsilon)}_1 \) and \( \varepsilon \mapsto \left( V^{(\varepsilon)}_1 \right)^2 \). Recall from Lemma 3.2, since \( P_{0,T} = \tilde{P}_T^{(0)} - \tilde{P}_T^{(0)} = 0 \) and similarly \( Q_{0,T} = 0 \), we could write

\[
P^{(\varepsilon)}_T = P_{1,T} + \frac{1}{2} \varepsilon^2 P_{2,T} + \Theta^{(\varepsilon)}_{2,T}(P),
\]

\[
(\varepsilon^2 P_{1,T} + \Theta^{(\varepsilon)}_{2,T}(P^2),
\]

\[
Q^{(\varepsilon)}_T = \varepsilon Q_{1,T} + \frac{1}{2} \varepsilon^2 Q_{2,T} + \Theta^{(\varepsilon)}_{2,T}(Q),
\]

\[
(\varepsilon^2 Q_{1,T} + \Theta^{(\varepsilon)}_{2,T}(Q^2),
\]

and

\[
P^{(\varepsilon)}_T Q^{(\varepsilon)}_T = P_{1,T} Q_{1,T} + \Theta^{(\varepsilon)}_{2,T}(PQ),
\]

where

\[
P_{1,T} = \int_0^T \rho_1 V_{1,t} dB_t - \int_0^T \rho_1^2 v_{0,t} V_{1,t} dt,
\]

\[
P_{2,T} = \int_0^T \rho_1 V_{2,t} dB_t - \int_0^T \rho_1^2 \left( V_{1,t}^2 + v_{0,t} V_{2,t} \right) dt,
\]

\[
Q_{1,T} = 2 \int_0^T (1 - \rho_1^2) v_{0,t} V_{1,t} dt,
\]

\[
Q_{2,T} = 2 \int_0^T (1 - \rho_1^2) \left( V_{1,t}^2 + v_{0,t} V_{2,t} \right) dt.
\]
The idea then is to approximate the functions $\varepsilon \mapsto P_T^{(\varepsilon)}$, $\varepsilon \mapsto Q_T^{(\varepsilon)}$ and their variants by their second-order expansions. For example, in the expansion of $P_{BS}$ in eq. (5.1) if we focus on the term corresponding to the first derivative of $P_{BS}$ in its second argument, we have

\[
(\partial_y \tilde{P}_{BS}) Q_T^{(1)} = (\partial_y \tilde{P}_{BS}) (Q_{1,T} + \frac{1}{2} Q_{2,T} + \Theta_{2,T}^{(1)} (Q)) = (\partial_y \tilde{P}_{BS}) (Q_{1,T} + \frac{1}{2} Q_{2,T}) + (\partial_y \tilde{P}_{BS}) (\Theta_{2,T}^{(1)} (Q)).
\]

For the term corresponding to the second derivative of $P_{BS}$ in its second argument, we would have

\[
\frac{1}{2} (\partial_{yy} \tilde{P}_{BS}) (Q_T^{(1)})^2 = \frac{1}{2} (\partial_{yy} \tilde{P}_{BS}) (Q_{2,T} + \Theta_{2,T}^{(1)} (Q^2)) = \frac{1}{2} (\partial_{yy} \tilde{P}_{BS}) (Q_{2,T}) + (\partial_{yy} \tilde{P}_{BS}) (\Theta_{2,T}^{(1)} (Q^2)).
\]

Following this pattern, we can see that the error term $\mathcal{E}_V$ can be written explicitly as

\[
\mathcal{E}_V = \sum_{|\alpha|=1} \frac{\partial \tilde{P}_{BS}}{\partial x^{\alpha_1} y^{\alpha_2}} \Theta_{2,T}^{(1)} (P^{\alpha_1} Q^{\alpha_2}) + \frac{1}{2} \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha_1 \alpha_2} \frac{\partial^2 \tilde{P}_{BS}}{\partial x^{\alpha_1} y^{\alpha_2}} \Theta_{2,T}^{(1)} (P^{\alpha_1} Q^{\alpha_2}).
\]

As the second-order price of a put option is the expectation of our expansion, the goal is to bound $\mathcal{E}$ in $L^1$ for a specific volatility process $V$.

### 5.2. Bounding error term

Our objective is to appeal to the explicit representation of the error term $\mathcal{E}$ as seen in Theorem 5.2 and bound it in $L^1$ under the general model eq. (1.1). As we are in a general framework, the bound obtained will be expressed in terms of expectations of functionals of the variance process. In order to obtain an $L^1$ bound on the error term $\mathcal{E}$, it is sufficient to obtain ingredients given in the following proposition.

**Proposition 5.1.** In order to obtain an $L^1$ bound on the error term $\mathcal{E}$, it is sufficient to obtain:

1. Bounds on $\|\Theta_{2,T}^{(1)} (P^{\alpha_1} Q^{\alpha_2})\|$, where $|\alpha| = 1, 2$.

2. Bounds on $\|P_T^{(1)}\|_p$ and $\|Q_T^{(1)}\|_p$ for $p \geq 2$.

The purpose of the next part of this section is to validate Proposition 5.1.

**Lemma 5.1.** Define

\[
\tilde{P}_{BS}(x, y) := Ke^{-\int_0^T r_t dt} \mathcal{N}(-d_-) - xe^{-\int_0^T r_t dt} \mathcal{N}(-d_+),
\]

\[
d_{\pm}(x, y) := d_{\pm} := \frac{\ln(x/K) + \int_0^T (r_t - r^T) dt}{\sqrt{y}} \pm \frac{1}{2} \sqrt{y}.
\]

Consider the third-order partial derivatives of $\tilde{P}_{BS}$, $\frac{\partial^3 \tilde{P}_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $\alpha_1 + \alpha_2 = 3$ as well as the linear functions $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_+$ such that $h_1(u) = u(d_1 - c_1) + c_1$ and $h_2(u) = u(d_2 - c_2) + c_2$. 
Assume there exists no point \( a \in (0, 1) \) such that
\[
\lim_{u \to a} \frac{\ln(h_1(u)/K) + \int_0^T (r_i^d - r_i^f)dt}{\sqrt{h_2(u)}} = 0 \quad \text{and} \quad \lim_{u \to a} h_2(u) = 0.
\]
Then there exists functions \( M_\alpha \) bounded on \( \mathbb{R}^2_+ \) such that
\[
\sup_{u \in (0, 1)} \left| \frac{\partial^3 P_{\text{BS}}}{\partial x^0 \partial y^0} (h_1(u), h_2(u)) \right| = M_\alpha(T, K).
\]
Furthermore, the behaviour of \( M_\alpha \) for fixed \( K \) and \( T \) is characterised by the functions \( \zeta \) and \( \eta \) respectively, where
\[
\zeta(T) = \hat{A} e^{-\int_0^T r_i^f dt} e^{-E_2 r_2^2(T)} e^{-E_1 \hat{r}(T)} \sum_{i=0}^n c_i \hat{r}^i(T),
\]
with \( \hat{r}(T) := \int_0^T (r_i^d - r_i^f)dt \) and \( E_2 > 0 \), \( E_1 \in \mathbb{R} \), \( \hat{A} \in \mathbb{R} \), \( n \in \mathbb{N} \) and \( c_0, \ldots, c_n \) are constants, and
\[
\eta(K) = \hat{A} K^{-D_2 \ln(K) + D_1} \sum_{i=0}^N C_i (-1)^i \ln^i(K),
\]
with \( D_2 > 0 \), \( D_1 \in \mathbb{R} \), \( \hat{A} \in \mathbb{R} \), \( N \in \mathbb{N} \) and \( C_0, \ldots, C_N \) are constants.

**Proof.** See Lemma 5.2 in [11]. \( \square \)

**Lemma 5.2.** Consider the third-order partial derivatives of \( P_{\text{BS}} \), \( \frac{\partial^3 P_{\text{BS}}}{\partial x^0 \partial y^2} \), where \( \alpha_1 + \alpha_2 = 3 \) as well as the linear functions \( h_1, h_2 : [0, 1] \to \mathbb{R}_+ \) such that \( h_1(u) = u(d_1 - c_1) + c_1 \) and \( h_2(u) = u(d_2 - c_2) + c_2 \). Assume there exists no point \( a \in (0, 1) \) such that
\[
\lim_{u \to a} \frac{h_1(u) - k + \int_0^T (r_i^d - r_i^f)dt}{\sqrt{h_2(u)}} = 0 \quad \text{and} \quad \lim_{u \to a} h_2(u) = 0. \tag{5.2}
\]
Then there exists functions \( B_\alpha \) bounded on \( \mathbb{R}_+ \times \mathbb{R} \) such that
\[
\sup_{u \in (0, 1)} \left| \frac{\partial^3 P_{\text{BS}}}{\partial x^0 \partial y^2} (h_1(u), h_2(u)) \right| = B_\alpha(T, k).
\]
Furthermore, the behaviour of \( B_\alpha \) for fixed \( k \) and \( T \) is characterised by the functions \( \zeta \) and \( \nu \) respectively, where
\[
\zeta(T) = \hat{A} e^{-\int_0^T r_i^f dt} e^{-E_2 r_2^2(T)} e^{-E_1 \hat{r}(T)} \sum_{i=0}^n c_i \hat{r}^i(T),
\]
with \( \hat{r}(T) := \int_0^T (r_i^d - r_i^f)dt \) and \( E_2 > 0 \), \( E_1 \in \mathbb{R} \), \( \hat{A} \in \mathbb{R} \), \( n \in \mathbb{N} \) and \( c_0, \ldots, c_n \) are constants, and
\[
\nu(k) = \hat{A} e^{-D_2 k^2 + D_1 k} \sum_{i=0}^N C_i (-1)^i k^i,
\]
with \( D_2 > 0 \), \( D_1 \in \mathbb{R} \), \( \hat{A} \in \mathbb{R} \), \( N \in \mathbb{N} \) and \( C_0, \ldots, C_N \) are constants.

**Proof.** Lemma 5.2 is very similar to Lemma 5.1, where the latter is the equivalent lemma for the function \( P_{\text{BS}} \). In fact, we will show that Lemma 5.1 implies Lemma 5.2. In the following, we will repeatedly denote by \( F \) or \( G \) to be an arbitrary polynomial of some degree, as well as \( A \) to be an arbitrary constant. That is, they may be different on each use.
First, as a function of $x$ and $y$, notice from Appendix D that the third-order partial derivatives \( \frac{\partial^3 P_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \), where $\alpha_1 + \alpha_2 = 3$ can be written as

\[
A e^x \phi(d_+^{ln}) G(d_-, d_+^{ln}, \sqrt{y}), \quad m \in \mathbb{N}
\]  

(5.3)

except for when $\alpha = (3, 0)$, in which case the partial derivative can be written as

\[
A e^x \phi(d_{+}^{ln}) G(d_{+}, d_{-}^{ln}, \sqrt{y}) + A e^x \phi(d_{+}^{ln})(N(d_{+}^{ln}) - 1), \quad m \in \mathbb{N}.
\]  

(5.4)

Similarly, it can seen that as a function of $x$ and $y$, the third-order partial derivatives of $P_{\text{BS}}$ can be written as

\[
A \frac{\phi(d_{+}^{ln})}{x^{m/2}} F(d_{+}, d_{-}, \sqrt{y}), \quad n \in \mathbb{Z}, m \in \mathbb{N}.
\]  

(5.5)

Recall

\[
d_{+} = d_{+}(x, y) = \frac{\ln(x/K) + \int_0^{T} (r_{t}^{d} - r_{t}^{f}) dt}{\sqrt{y}} + \frac{1}{2} \sqrt{y},
\]

\[
d_{-}^{ln} = d_{-}^{ln}(x, y) = \frac{x - k + \int_0^{T} (r_{t}^{d} - r_{t}^{f}) dt}{\sqrt{y}} + \frac{1}{2} \sqrt{y},
\]

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

Let us consider the cases for which $\alpha \neq (3, 0)$. Without loss of generality, set $k = \ln(K)$. Notice that $d_{+}^{ln}(x, y) = d_{+}(e^x, y)$. Take $n = -1$ in eq. (5.5). Roughly speaking, we will say that two functions $f$ and $g$ are ‘of the same form’ if they are equal up to constant values. Furthermore, we will denote this relation by $f \sim g$. Then comparing eq. (5.3) and eq. (5.5), the form of the partial derivatives of $P_{\text{BS}}$ are the same as the partial derivatives of $P_{\text{BS}}$ composed with the function $e^x$ in its first argument. Specifically, we can write

\[
\frac{\partial^3 P_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(x, y) \sim \frac{\partial^3 P_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(e^x, y).
\]

Now, consider arbitrary functions $f, b : \mathbb{R} \to \mathbb{R}$ such that

\[
\sup_{x \in \mathbb{R}} |f(x)| = L < \infty.
\]

Then it is true that

\[
\sup_{x \in \mathbb{R}} |f(b(x))| = \tilde{L} \leq L < \infty.
\]

Thus

\[
\sup_{u \in (0, 1)} \left| \frac{\partial^3 P_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(h_1(u), h_2(u)) \right| \sim \sup_{u \in (0, 1)} \left| \frac{\partial^3 P_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(e^{h_1(u)}(h_1(u), h_2(u))) \right|.
\]

Under the assumption in eq. (5.2), and then using Lemma 5.1, this supremum will not blow up. Clearly, $\sup_{u \in (0, 1)} \frac{\partial^3 P_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(e^{h_1(u)}(h_1(u), h_2(u)))$ is a function of $T$ and $K$. By substituting $k = \ln(K)$ in the result of Lemma 5.1, we obtain the form of $\zeta$ and $\nu$. 
Now for the case of $\alpha = (3, 0)$, we have that
\[
\frac{\partial^3 P_{BS}}{\partial x^3}(x, y) \lesssim \frac{\partial^3 \text{Put}_{BS}}{\partial x^3}(e^x, y) + A e^x \phi(d^n_{+}) (N(d^n_{+}) - 1).
\]

Now
\[
|H(x, y)| = |e^x \phi(d^n_{+}) (N(d^n_{+}) - 1)| \leq e^x \phi(d^n_{+}).
\]

Thus
\[
\sup_{x \in \mathbb{R}} |H(x, y)| = \sup_{x \in \mathbb{R}} |e^x \phi(d^n_{+}) (N(d^n_{+}) - 1)| \leq \sup_{x \in \mathbb{R}} e^x \phi(d^n_{+}) < \infty
\]
and also
\[
\sup_{y \in \mathbb{R}_+} |H(x, y)| = \sup_{y \in \mathbb{R}_+} |\phi(d^n_{+}) (N(d^n_{+}) - 1)| \leq \sup_{y \in \mathbb{R}_+} \phi(d^n_{+}) < \infty.
\]

Hence
\[
\sup_{u \in (0, 1)} H(h_1(u), h_2(u)) \leq \sup_{u \in (0, 1)} e^{h_1(u)} \phi(d^n_{+}(h_1(u), h_2(u))) = \hat{m}(T, k),
\]
where $\hat{m}$ is a bounded function on $\mathbb{R}_+ \times \mathbb{R}$. By direct computation, it is clear that for fixed $T$ the form of $\hat{m}$ is given by
\[
A e^{-\hat{D}_2 k^2} e^{\hat{D}_1 k},
\]
where $\hat{D}_2 > 0$ and $\hat{D}_1 \in \mathbb{R}$. For fixed $k$, it is given by
\[
A e^{-\hat{E}_2 \hat{r}^2(T)} e^{\hat{E}_1 \hat{r}(T)},
\]
where $\hat{E}_2 > 0$ and $\hat{E}_1 \in \mathbb{R}$. Thus
\[
\sup_{u \in (0, 1)} \left| \frac{\partial^3 P_{BS}}{\partial x^3}(h_1(u), h_2(u)) \right| \lesssim \sup_{u \in (0, 1)} \left| \frac{\partial^3 \text{Put}_{BS}}{\partial x^3}(e^{h_1(u)}, h_2(u)) + A H(h_1(u), h_2(u)) \right|
\]
\[
\leq \sup_{u \in (0, 1)} \left| \frac{\partial^3 \text{Put}_{BS}}{\partial x^3}(e^{h_1(u)}, h_2(u)) \right| + A \sup_{u \in (0, 1)} |H(h_1(u), h_2(u))|
\]
\[
\lesssim B_{(3, 0)}(T, k) + A \hat{m}(T, k).
\]

But the form of $\hat{m}$ is exactly that of $B_\alpha$ without the polynomial expression. Thus, the sum of them is again of the form of $B_\alpha$.

\[\hfill \Box\hfill\]

5.2.1. **Bounding $\mathcal{E}_\mathcal{V}$.** We first consider bounding the term $\mathcal{E}_\mathcal{V}$ from Theorem 5.2 in $L^1$. The terms of interest to bound are
\[
\frac{\partial^{(\alpha)}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \Theta_{2,T}^{(1)}(P^{\alpha_1} Q^{\alpha_2}), \quad |\alpha| = 1, 2.
\]

Now the second argument of $\hat{P}_{BS}$ is $Q_{T}^{(0)}$, which is strictly positive. By considering the linear function $u \mapsto (1 - u)Q_{T}^{(0)} + uQ_{T}^{(0)}$, then by Lemma 5.2 this implies $\frac{\partial^{(\alpha)}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \Theta_{2,T}^{(1)} \leq B_\alpha(T, k)$. Thus
\[
\left\| \frac{\partial^{(\alpha)}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \Theta_{2,T}^{(1)}(P^{\alpha_1} Q^{\alpha_2}) \right\| \leq B_\alpha(T, k) \left\| \Theta_{2,T}^{(1)}(P^{\alpha_1} Q^{\alpha_2}) \right\|_2.
\]

(5.6)
Equation (5.6) suggests that obtaining an $L^1$ bound on the remainder term $\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2})$ for $|\alpha| = 1, 2$ is sufficient. This validates (1) in Proposition 5.1.

5.2.2. **Bounding $E_P$.** The terms of interest are

$$E_\alpha \left( \hat{P}_T^{(1)}, \hat{Q}_T^{(1)} \right) \left( P_T^{(1)} \right)^{\alpha_1} \left( Q_T^{(1)} \right)^{\alpha_2}, \quad |\alpha| = 3.$$  

We now define

$$J(u) = (1-u)\hat{P}_T(0) + u\hat{P}_T(1),$$  
$$K(u) = (1-u)\hat{Q}_T(0) + u\hat{Q}_T(1),$$

so that

$$\frac{\partial^3 P_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (J(u), K(u)) = \frac{\partial^3 P_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \left( (1-u)\hat{P}_T(0) + u\hat{P}_T(1), (1-u)\hat{Q}_T(0) + u\hat{Q}_T(1) \right).$$

**Proposition 5.2.** There exists functions $B_\alpha$ with $\alpha_1 + \alpha_2 = 3$ as in Lemma 5.2 where the constants in the definitions of $\zeta$ and $\nu$ are possibly replaced with random variables such that

$$\sup_{u \in (0,1)} \left| \frac{\partial^3 P_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (J(u), K(u)) \right| \leq B_\alpha(T, k) \quad \text{Q.a.s.}$$

**Proof.** Since $J$ and $K$ are linear functions, then from Lemma 5.2, this claim is immediately true if we can show that $K$ is strictly positive Q.a.s. Recall

$$K(u) = (1-u) \left( \int_0^T (1-\rho_t^2)v_{0,t}^2 dt \right) + u \int_0^T (1-\rho_t^2)V_t^2 dt.$$  

$K$ corresponds to the linear interpolation of $\int_0^T (1-\rho_t^2)v_{0,t}^2 dt$ and $\int_0^T (1-\rho_t^2)V_t^2 dt$. It is clear $\sup_{t \in [0,T]} (1-\rho_t^2) > 0$. As $V^2$ corresponds to the variance process, this is always chosen to be a non-negative process such that the set $\{ t \in [0,T] : V_t^2 > 0 \}$ has non-zero Lebesgue measure. Thus these integrals are strictly positive and hence $K$ is strictly positive Q.a.s. \hfill $\Box$

By Proposition 5.2

$$\left| E_\alpha \left( \hat{P}_T^{(1)}, \hat{Q}_T^{(1)} \right) \right| = \left| \int_0^1 (1-u)^2 \frac{\partial^{|\alpha|} P_{BS}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (J(u), K(u)) \, du \right| \leq \frac{1}{3} B_\alpha(T, k).$$

Thus

$$\left\| E_\alpha \left( \hat{P}_T^{(1)}, \hat{Q}_T^{(1)} \right) \left( P_T^{(1)} \right)^{\alpha_1} \left( Q_T^{(1)} \right)^{\alpha_2} \right\| \leq \frac{1}{3} \left\| B_\alpha(T, k) \right\| \left\| \left( P_T^{(1)} \right)^{\alpha_1} \right\|_4 \left\| \left( Q_T^{(1)} \right)^{\alpha_2} \right\|_4. \quad (5.7)$$

Looking at the second and third term on the RHS of eq. (5.7), it is clear one of our objectives is to bound $P_T^{(1)}$ and $Q_T^{(1)}$ in $L^p$ for $p \geq 2$. This validates (2) in Proposition 5.1.

**Lemma 5.3.** The terms from Proposition 5.1 can be bounded if the following quantities can be bounded:

1. $\left\| \Theta_{0,t}^{(1)}(V) \right\|_p$ and $\left\| \Theta_{0,t}^{(1)}(V^2) \right\|_p$ for $p \geq 2$.
2. $\left\| \Theta_{1,t}^{(1)}(V) \right\|_p$ and $\left\| \Theta_{1,t}^{(1)}(V^2) \right\|_p$ for $p \geq 2$.  


(3) \( \|\Theta_{2,t}^{(1)}(V)\|_p \) and \( \|\Theta_{2,t}^{(1)}(V^2)\|_p \) for \( p \geq 2 \).

**Proof.** We will make extensive use of the following form of Jensen’s inequality:

\[
\left( \int_0^T |f(u)|du \right)^p \leq T^{p-1} \int_0^T |f(u)|^p du, \quad p \geq 1. \tag{5.8}
\]

For the rest of this proof, assume that \( p \geq 2 \). We will denote by \( C_p \) and \( D_p \) generic constants that solely depend on \( p \). They may be different on each use. Notice

\[
P_T^{(1)} = \int_0^T \rho_t \Theta_{0,t}^{(1)}(V) dB_t - \frac{1}{2} \int_0^T \rho_t^2 \Theta_{0,t}^{(1)}(V^2) dt,
Q_T^{(1)} = \int_0^T (1 - \rho_t^2) \Theta_{0,t}^{(1)}(V^2) dt.
\]

Applying the Minkowski and Burkholder-Davis-Gundy inequalities, as well as Jensen’s inequality eq. (5.8), we obtain

\[
\|P_T^{(1)}\|_p \leq C_p T^{1 - \frac{1}{p}} \left( \int_0^T \rho_t^p \|\Theta_{0,t}^{(1)}(V)\|^p dt \right)^{1/p} + \frac{1}{2} D_p T^{1 - \frac{1}{p}} \left( \int_0^T \rho_t^2 p \|\Theta_{0,t}^{(1)}(V^2)\|^p dt \right)^{1/p}
\]

and

\[
\|Q_T^{(1)}\|_p \leq C_p T^{1 - \frac{1}{p}} \left( \int_0^T (1 - \rho_t^2)^p \|\Theta_{0,t}^{(1)}(V^2)\|^p dt \right)^{1/p}.
\]

Now also \( (V_{t}^{(1)})^2 = (v_{0,t} + \Theta_{0,t}^{(1)}(V))^2 = v_{0,t}^2 + 2 v_{0,t} \Theta_{0,t}^{(1)}(V) + (\Theta_{0,t}^{(1)}(V))^2 \), so that

\[
\Theta_{0,t}^{(1)}(V^2) = 2 v_{0,t} \Theta_{0,t}^{(1)}(V) + (\Theta_{0,t}^{(1)}(V))^2.
\]

This suggests that finding an \( L^p \) bound on the remainder term \( \Theta_{0,t}^{(1)}(V) \) is sufficient in order to bound \( P_T^{(1)} \) and \( Q_T^{(1)} \) in \( L^p \). This validates (1).

We can write the following remainder terms of \( P \) and \( Q \)

\[
\Theta_{2,t}^{(1)}(P) = \int_0^T \rho_t \Theta_{2,t}^{(1)}(V) dB_t - \frac{1}{2} \int_0^T \rho_t^2 \Theta_{2,t}^{(1)}(V^2) dt,
\]

\[
\Theta_{2,t}^{(1)}(Q) = \int_0^T (1 - \rho_t^2) \Theta_{2,t}^{(1)}(V^2) dt,
\]

\[
\Theta_{2,t}^{(1)}(P^2) = (P_T^{(1)})^2 - P_{2,t}^2 = (P_T^{(1)} - P_{1,t})(P_T^{(1)} + P_{1,t})
\]

\[
= \left( \int_0^T \rho_t \Theta_{1,t}^{(1)}(V) dB_t - \frac{1}{2} \int_0^T \rho_t^2 \Theta_{1,t}^{(1)}(V^2) dt \right)
\]

\[
\cdot \left( \int_0^T \rho_t (2 \Theta_{0,t}^{(1)}(V) - \Theta_{1,t}^{(1)}(V)) dB_t - \frac{1}{2} \int_0^T \rho_t^2 (2 \Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2)) dt \right) \tag{5.9}
\]

\[
\Theta_{2,t}^{(1)}(Q^2) = (Q_T^{(1)})^2 - Q_{2,t}^2 = (Q_T^{(1)} - Q_{1,t})(Q_T^{(1)} + Q_{1,t})
\]

\[
= \left( \int_0^T (1 - \rho_t^2) \Theta_{1,t}^{(1)}(V^2) dt \right) \left( \int_0^T (1 - \rho_t^2) \left[ 2 \Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2) \right] dt \right).
\]
Furthermore, notice
\[
\begin{align*}
\Theta_{1,t}^{(1)}(V^2) &= \Theta_{0,t}^{(1)}(V^2) + 2\nu_0 t \left( \Theta_{1,t}^{(1)}(V) - \Theta_{0,t}^{(1)}(V) \right), \\
\Theta_{2,t}^{(1)}(V^2) &= \Theta_{0,t}^{(1)}(V^2) - 2\nu_0 t \left( \Theta_{1,t}^{(1)}(V) - \Theta_{0,t}^{(1)}(V) \right) - \left( \Theta_{0,t}^{(1)}(V) - \Theta_{1,t}^{(1)}(V) \right)^2.
\end{align*}
\]
Then, by application of the Minkowski, Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities, it is sufficient to obtain \( L^p \) bounds on \( \Theta_{1,t}^{(1)}(V) \) and \( \Theta_{2,t}^{(1)}(V) \) in order to obtain \( L^p \) bounds on the remainders of \( P \) and \( Q \) from eq. (5.9). For the cross remainder term, we have
\[
\|\Theta_{2,T}^{(1)}(PQ)\|_p \leq \|P_{1,T}^{(1)}\|_2 \|Q_{1,T}^{(1)}\|_2 + \|P_{1,T}^{(1)}\|_2 \|Q_{1,T}^{(1)}\|_2.
\]
We just need to check how to obtain \( L^p \) bounds on \( P_{1,T}^{(1)} \) and \( Q_{1,T}^{(1)} \). Notice
\[
\begin{align*}
\|P_{1,T}^{(1)}\|_p &\leq C_p T^{1/p} \left( \int_0^T \rho_t^2 \|\Theta_{0,t}^{(1)}(V) - \Theta_{1,t}^{(1)}(V)\|_p dt \right)^{1/p} \\
&\quad + \frac{1}{2} D_p T^{-1/2} \left( \int_0^T \rho_t^2 \|\Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2)\|_p dt \right)^{1/p}
\end{align*}
\]
and
\[
\|Q_{1,T}^{(1)}\|_p \leq D_p T^{1/2} \left( \int_0^T (1 - \rho_t^2) \|\Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2)\|_p dt \right)^{1/p}.
\]
Again, all we need to obtain \( L^p \) bounds on the cross remainder term are \( L^p \) bounds on \( \Theta_{1,t}^{(1)}(V) \) and \( \Theta_{2,t}^{(1)}(V) \). This validates (2) and (3).

6. Fast calibration procedure

In this section, we present a fast calibration scheme in the Stochastic Verhulst model with time-dependent parameters, here on in referred to as the Verhulst model.\(^4\) Specifically, the dynamics of the spot \( S \) with volatility \( V \) are given by
\[
\begin{align*}
\text{d}S_t &= (r_t - r_t^2) S_t \text{d}t + V_t S_t \text{d}W_t, \quad S_0, \\
\text{d}V_t &= \kappa_t (\theta_t - V_t) \text{d}t + \lambda_t \text{d}B_t, \quad V_0 = v_0, \\
\text{d}(W, B)_t &= \rho_t \text{d}t.
\end{align*}
\]
The deterministic, time-dependent parameters \((\kappa_t)_{0 \leq t \leq T}, (\theta_t)_{0 \leq t \leq T}\) and \((\lambda_t)_{0 \leq t \leq T}\) are all assumed to be positive for all \( t \in [0, T] \) and bounded. By Proposition B.1, a risk-neutral measure exists if \( \rho_t \lambda_t - \kappa_t < 0 \) for all \( t \in [0, T] \).

Remark 6.1 (Stochastic Verhulst model heuristics). The process \( V \) from eq. (6.1) we call the Stochastic Verhulst process, here on in referred to as the Verhulst process. This process is reminiscent of the deterministic Verhulst/Logistic model which most famously arises in population growth models.\(^5\) The Verhulst process behaves intuitively in the following way. Focusing on the drift term of the volatility \( V \) in eq. (6.1), specifically \( \kappa(\theta - V) \), we notice that there is

\(^4\)Commonly the Stochastic Verhulst model also goes by the Stochastic Logistic model, see [8]. It is also referred to as the XGBM model, short for ‘Extended Geometric Brownian Motion’, see [22].

\(^5\)The deterministic Verhulst/Logistic model was first introduced by Verhulst in 1838 [34, 35, 36], then rediscovered and revived by Pearl and Reed in 1920 [29, 30].
a quadratic term. The interpretation here is that $V$ mean reverts to level $\theta$ at a speed of $\kappa V$. That is, the mean reversion speed of $V$ depends on $V$ itself, and is thus stochastic. Contrasting this with the regular linear type mean reversion drift coefficients, namely $\kappa(\theta - V)$, we have that the mean reversion level is still $\theta$, however the mean reversion speed is $\kappa$, and is not directly influenced by $V$. For an in depth discussion of the Verhulst model for option pricing, we refer the reader to [22].

Notice that the Verhulst process $V$ from eq. (6.1) does not satisfy (A1), as its drift coefficient is only locally Lipschitz continuous. However, this is not a problem as its diffusion coefficient is Lipschitz continuous. Hence, we can appeal to the usual Itô style results on existence and uniqueness for solutions to SDEs.

**Proposition 6.1.** Suppose $Y$ solves the SDE

$$dY_t = a_t(b_t - Y_t)dt + c_t Y_t dB_t, \quad Y_0 = y_0 > 0,$$

where $(a_t)_{0 \leq t \leq T}$, $(b_t)_{0 \leq t \leq T}$ and $(c_t)_{0 \leq t \leq T}$ are strictly positive and bounded on $[0, T]$. Then the explicit pathwise unique strong solution is given by

$$Y_t = F_t \left( y_0^{-1} + \int_0^t a_u F_u du \right)^{-1},$$

$$F_t = \exp \left( \int_0^t \left( a_u b_u - \frac{1}{2} c_u^2 \right) du + \int_0^t c_u dB_u \right).$$

**Proof.** Both the drift and diffusion coefficients in the SDE eq. (6.2) are locally Lipschitz, uniformly in $t \in [0, T]$. Clearly the diffusion coefficient obeys the linear growth condition, $(c_t x)^2 \leq K(1 + |x|^2)$ uniformly in $t$, for some constant $K > 0$. In addition, we have that $x[a_t(b_t - x)] \leq K(1 + |x|^2)$ uniformly in $t \in [0, T]$, and thus any potential of explosion in finite time is mitigated (this somewhat non-standard restriction on the growth of the drift is given in [19] Section 4.5, page 135). It remains to be seen that the solution is indeed given by eq. (6.3). Utilising Itô’s formula with $f(x) = x^{-1}$ yields a linear SDE, which results in the explicit solution, namely eq. (6.3). Clearly this solution remains strictly positive in finite time. \qed

The following is the explicit second-order put option price in the Verhulst model, which is a corollary of Theorem 4.1.
Corollary 6.1 (Verhulst model explicit second-order put option price). Under the Verhulst model eq. (6.1), the explicit second-order price of a put option is given by

\[
\text{Put}^{(2)}_{\text{Verhulst}} = P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + 2\omega_{0,T}^{(-\kappa \theta - 2\kappa v_0, \rho \lambda v_0^2), (\kappa \theta - 2\kappa v_0, 0, \rho \lambda v_0^2)} \partial_{xy} P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + \omega_{0,T}^{(-2\kappa \theta - 2\kappa v_0, \kappa^2 v_0^2), (2\kappa \theta - 2\kappa v_0, 1)} \partial_y P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + 2\omega_{0,T}^{(-\kappa \theta - 2\kappa v_0, \rho \lambda v_0^2), (\kappa \theta - 2\kappa v_0, 0, \rho \lambda v_0^2), (2\kappa \theta - 2\kappa v_0, 1)} \partial_{xxy} P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + \omega_{0,T}^{(-2\kappa \theta - 2\kappa v_0, \kappa^2 v_0^2), (\kappa \theta - 2\kappa v_0, -2\kappa, \kappa \theta - 2\kappa v_0, v_0)} \partial_{y} P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + \left\{ 2\omega_{0,T}^{(-\kappa \theta - 2\kappa v_0, \rho \lambda v_0^2), (0, \rho \lambda v_0, (\kappa \theta - 2\kappa v_0, v_0))} \partial_{xxy} P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + 2\omega_{0,T}^{(-\kappa \theta - 2\kappa v_0, \rho \lambda v_0^2), (0, \rho \lambda v_0, (\kappa \theta - 2\kappa v_0, v_0))} \partial_{xxy} P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + 4\omega_{0,T}^{(-\kappa \theta - 2\kappa v_0, \kappa^2 v_0^2), (\kappa \theta - 2\kappa v_0, v_0), (\kappa \theta - 2\kappa v_0, v_0)} \partial_{y} P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) + 2 \left( \omega_{0,T}^{(-\kappa \theta - 2\kappa v_0, \rho \lambda v_0^2), (\kappa \theta - 2\kappa v_0, v_0)} \right)^2 \partial_{xxyy} P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) \right\}.
\]

For convenience we restate the integral operator from Definition 4.1,

\[
\omega^{(k,l)}_{t,T} = \int_t^T I_u e\int_u^T k_s dz du, \tag{6.4}
\]

and its \(n\)-fold iterated extension

\[
\omega^{(k^{(n)}, l^{(n)}), (k^{(n-1)}, l^{(n-1)}), \ldots, (k^{(1)}, l^{(1)})}_{t,T}^{(k^{(n)}, l^{(n)}, \ldots, (k^{(1)}, l^{(1)})}, n \in \mathbb{N}. \tag{6.5}
\]

The rest of this section is devoted to obtaining a fast calibration scheme in the Verhulst model. To do this, we recognise that the approximation of the put option price from Corollary 6.1 is expressed in terms of iterated integral operators eq. (6.4) and eq. (6.5). Our goal is to show that when parameters are assumed to piecewise-constant, these iterated integral operators

- are closed-form, and

- obey a convenient recursive property.

Let \( T = \{0 = T_0, T_1, \ldots, T_{N-1}, T_N = T\} \), where \( T_i < T_{i+1} \) be a collection of maturity dates on \([0, T]\), with \( \Delta T_i := T_{i+1} - T_i \) and \( \Delta T_0 = 1 \). When the dummy functions are piecewise-constant, that is, \( l^{(n)}_t = l^{(n)}_i \) on \( t \in [T_i, T_{i+1}) \) and similarly for \( k^{(n)} \), we can recursively calculate the
integral operators eq. (6.4) and eq. (6.5). Consider the ODE for \((v_{0,t})\) in the Verhulst model eq. (6.1),
\[
dv_{0,t} = \kappa_t (\theta_t - v_{0,t}) v_{0,t} dt, \quad v_{0,0} = v_0.
\]
It is true that an explicit solution exists for this ODE eq. (6.6), namely
\[
v_{0,t} = e^{\int_t^0 \kappa_s \theta_s dz} \left( \frac{v_{0,s}}{1 + v_{0,s} \int_t^0 \kappa_u e^{\int_u^0 \kappa_v \theta_v dz} du} \right).
\]
However, the solution is a quotient, and unfortunately we cannot utilise it in order to make the following recursive formulas simpler, unlike in [20]. Instead we will compute values of \((v_{0,t})\) on a grid in order to well approximate any integrals involving them. However, it will not be wise to compute \((v_{0,t})\) over the grid \(\mathcal{T}\), as in practice it is quite coarse. Instead we compute \(v_{0,t}\) on the grid \(\hat{T} = \{0, \hat{T}_1, \ldots, \hat{T}_{N-1}, T\}\), where \(\hat{T}_i < \hat{T}_{i+1}\) such that \(\hat{T} \supseteq \mathcal{T}\). Similarly define \(\Delta \hat{T}_i := \hat{T}_{i+1} - \hat{T}_i\) with \(\Delta \hat{T}_0 = 1\). Hence we have two grids, \(\mathcal{T}\) which contains the maturity dates and \(\hat{T}\) which contains \(\mathcal{T}\) and is where \((v_{0,t})\) is computed over. Define
\[
e^{(k^{(n)}, \ldots, k^{(1)})}_{t, \hat{T}_{i+1}} := e^{\int_t^0 \sum_{j=1}^n k_j^{(j)} dz},
\]
\[
e^{(h^{(n)}, \ldots, h^{(1)})}_{v, \hat{T}_{i+1}} := e^{\int_0^v \sum_{j=1}^n h_j^{(j)} dz},
\]
\[
\varphi^{(k, h, p)}_{t, \hat{T}_{i+1}} := \int_t^{T_{i+1}} \gamma_t^p (u) e^{\int_t^u k_z + h_z v_{0,z} dz} du,
\]
where \(\gamma_t(u) := (u - T_i)/\Delta T_i\) and \(p \in \mathbb{N} \cup \{0\}\). In addition, define the \(n\)-fold extension of \(\varphi^{(\cdot, \cdot, \cdot)}\),
\[
\varphi^{(k^{(n)}, h^{(n)}, p_n), \ldots, (k^{(1)}, h^{(1)}, p_1)}_{t, \hat{T}_{i+1}} := \int_t^{T_{i+1}} \gamma_t^{p_n} (u) e^{\int_t^u k_z^{(n)} + h_z^{(n)} v_{0,z} dz}
\]
\[
\cdot \varphi^{(k^{(n-1)}, h^{(n-1)}, p_{n-1}), \ldots, (k^{(2)}, h^{(2)}, p_2), (k^{(1)}, h^{(1)}, p_1)}_{a, \hat{T}_{i+1}} du,
\]
where \(p_n \in \mathbb{N} \cup \{0\}\).

We now assume that the dummy functions are piecewise-constant on \(\mathcal{T}\). However, since \((v_{0,t})\) is computed over the finer grid \(\hat{T}\), to make this recursion simpler, we will assume that we are working on the finer grid \(\hat{T}\) rather than \(\mathcal{T}\), so that \(v_{0,t}\) can be approximated by \(v_{0,\hat{T}_i}\) over \([\hat{T}_i, \hat{T}_{i+1}]\). Moreover, since the dummy functions are piecewise-constant on \(\mathcal{T}\), then there exists an equivalent parameterisation on \(\hat{T}\). For example, let \(k_i\) be the constant value of \(k\) on \([\hat{T}_i, \hat{T}_{i+1}]\). Then there exists \(\hat{T}_i, \hat{T}_{i+1}, \ldots, \hat{T}_j\) such that \(\hat{T}_i = T_i\) and \(\hat{T}_j = T_{i+1}\). Then let \(\hat{k}_m := k_i\) for \(m = i, \ldots, j\). Thus, without loss of generality, we can assume that we are working on \(\hat{T}\) and we will suppress the tilde from now on. With the assumption that the dummy functions are
piecewise-constant, we can obtain the integral operator at time $T_{i+1}$ expressed by terms at $T_i$.

\[
\begin{align*}
&\omega_{0,T_{i+1}}^{(k^{(1)}+h^{(1)})v_0,...,(l^{(1)})v_0^2)} \\
&= \omega_{0,T_i}^{(k^{(1)}+h^{(1)})v_0,...,(l^{(1)})v_0^2)} + (k^{(2)}+h^{(2)})v_0,...,(l^{(2)})v_0^2),\omega_{0,T_{i+1}}^{(k^{(1)}+h^{(1)})v_0,...,(l^{(1)})v_0^2)}
\end{align*}
\]

\[
\begin{align*}
&\omega_{0,T_{i+1}}^{(k^{(2)}+h^{(2)})v_0,...,(l^{(2)})v_0^2)} \\
&= \omega_{0,T_i}^{(k^{(2)}+h^{(2)})v_0,...,(l^{(2)})v_0^2)} + (k^{(3)}+h^{(3)})v_0,...,(l^{(3)})v_0^2),\omega_{0,T_{i+1}}^{(k^{(2)}+h^{(2)})v_0,...,(l^{(2)})v_0^2)}
\end{align*}
\]

\[
\begin{align*}
&\omega_{0,T_{i+1}}^{(k^{(3)}+h^{(3)})v_0,...,(l^{(3)})v_0^2)} \\
&= \omega_{0,T_i}^{(k^{(3)}+h^{(3)})v_0,...,(l^{(3)})v_0^2)} + (k^{(4)}+h^{(4)})v_0,...,(l^{(4)})v_0^2),\omega_{0,T_{i+1}}^{(k^{(3)}+h^{(3)})v_0,...,(l^{(3)})v_0^2)}
\end{align*}
\]

The only terms here that are not closed-form are the functions $e_i^{(\cdots),e_{\nu,T}}$ and $\varphi_{t,T_i+1}^{(\cdot,\cdot)}$.

For $t \in (T_i, T_{i+1}]$, we can derive the following:

\[
\begin{align*}
&\epsilon_{t}^{(k^{(n)},...,k^{(1)})} = \epsilon_{t,T_i}^{(k^{(n)},...,k^{(1)})} e^{\Delta T_{T_i}(t)} \sum_{j=1}^{k^{(j)}} \epsilon_{t}^{(j)} = \epsilon_{t}^{(1)} \sum_{j=1}^{k^{(j)}} e^{\Delta T_{T_i}(t)} \sum_{j=1}^{k^{(j)}} \epsilon_{t}^{(j)} \\
&\epsilon_{\nu,t}^{(k^{(n)},...,h^{(1)})} = \epsilon_{T_i}^{(k^{(n)},...,h^{(1)})} e^{\Delta T_{T_i}(t)} \sum_{j=1}^{h^{(j)}} \epsilon_{\nu,t}^{(j)} = \epsilon_{\nu,t}^{(1)} \sum_{j=1}^{h^{(j)}} e^{\Delta T_{T_i}(t)} \sum_{j=1}^{h^{(j)}} \epsilon_{\nu,t}^{(j)}
\end{align*}
\]

where $\epsilon_0^{(k^{(n)},...,k^{(1)})} = 1$ and $\epsilon_0^{(h^{(n)},...,h^{(1)})} = 1$.

Let $\hat{k}_i := k_i + h_i v_{0,T_i}$ and $\tilde{k}_i^{(n)} := k_i^{(n)} + h_i^{(n)} v_{0,T_i}$. Then

\[
\begin{align*}
&\tilde{k}_i^{(k^{(h,p)})} = \left\{ \begin{array}{ll} \\
\frac{1}{\hat{k}_i} e^{\hat{k}_i \Delta T_{T_i} - \hat{k}_i \Delta T_{T_{i+1}}(t) - \frac{p}{\hat{k}_i} e^{\hat{k}_i \Delta T_{T_{i+1}}(t)}} - \frac{p}{\hat{k}_i} e^{\hat{k}_i \Delta T_{T_{i+1}}(t)} & \hat{k}_i \neq 0, p \geq 1,
\end{array} \right. \\
&\tilde{k}_i^{(0)} = 0, p = 0,
\end{align*}
\]

\[
\begin{align*}
\tilde{k}_i = \frac{1}{\hat{k}_i} \Delta T_{T_i} \left( 1 - \gamma_i^{p+1}(t) \right)
\end{align*}
\]
In addition, for \( n \geq 2 \),
\[
\varphi_{t,T_{i+1}}^{(k^{(n)},h^{(n)},p_n),...,(k^{(1)},h^{(1)},p_1)} = \frac{1}{P_n} \left( \varphi_{t,T_{i+1}}^{(k^{(n)}+k^{(n-1)},h^{(n)}+h^{(n-1)},p_n+p_n-1),(k^{(n-2)},h^{(n-2)},p_{n-2}),...,(k^{(1)},h^{(1)},p_1)} - \gamma_{p_n}^n(t) e^{\Delta T_i \gamma_i(t)} \varphi_{t,T_{i+1}}^{(k^{(n-1)},h^{(n-1)},p_{n-1}),...,(k^{(1)},h^{(1)},p_1)} \right),
\]
\[
\tilde{k}^{(n)}_i \neq 0, p_n \geq 1,
\]
\[
\frac{1}{P_n} \left( \varphi_{t,T_{i+1}}^{(k^{(n)}+k^{(n-1)},h^{(n)}+h^{(n-1)},p_n-1),(k^{(n-2)},h^{(n-2)},p_{n-2}),...,(k^{(1)},h^{(1)},p_1)} - \gamma_{p_n+1}^n(t) e^{\Delta T_i \gamma_i(t)} \varphi_{t,T_{i+1}}^{(k^{(n-1)},h^{(n-1)},p_{n-1}),...,(k^{(1)},h^{(1)},p_1)} \right),
\]
\[
\tilde{k}^{(n)}_i = 0, p_n \geq 0.
\]

Remark 6.2 (Fast calibration scheme). To implement our fast calibration scheme, one executes
the following algorithm. Let \( \mu_t = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)}) \) be an arbitrary set of parameters
and denote by \( \omega_t \) an arbitrary integral operator.

- Calibrate \( \mu \) over \([0,T_1]\) to obtain \( \mu_0 \). This involves computing \( \omega_{T_1} \).

- Calibrate \( \mu \) over \([T_1,T_2]\) to obtain \( \mu_1 \). This involves computing \( \omega_{T_2} \) which is in terms of \( \omega_{T_1} \),
the latter already being computed in the previous step.

- Repeat until time \( T_N \).

7. Numerical tests and sensitivity analysis

Remark 7.1. The language and methodology in this section is not dissimilar to that of [11].
This is because the numerical sensitivity analysis methodology carried out here is essentially
the same, and the main difference is that we utilise our approximation method established in
this article rather than the ones found in [11].

In this section, we will numerically investigate the accuracy of our closed-form approximation
formula in the Stochastic Verhulst model (Corollary 6.1) by considering the sensitivity of our
approximation formula whilst varying one parameter at a time, with the others all being fixed.
Namely, for an arbitrary set of piecewise-constant parameters \((\kappa_t, \theta_t, \lambda_t, \rho_t) \equiv (\kappa, \theta, \lambda, \rho) =: \mu, \)
we vary only one of \( \kappa, \theta, \lambda, \rho \) at a time and keep the rest fixed. Then, we will compute the
difference (signed error) in implied volatilities via our approximation formula as well as the
Monte-Carlo for maturity times \( T \in \{1/12, 3/12, 6/12, 1\} \equiv \{1M, 3M, 6M, 1Y\} \) and strikes corresponding to Put 10, 25, and ATM deltas. Thus, the signed error of the implied volatility for a given parameter set \( \mu \), maturity \( T \), and strike \( K \) is
\[
\text{Error}(\mu, T, K) = \sigma_{\text{IM-Approx}}(\mu, T, K) - \sigma_{\text{IM-Monte}}(\mu, T, K).
\]
The Monte-Carlo simulation is implemented by taking advantage of a variation of the mixing solution methodology from Appendix A. Using this relationship for Monte-Carlo simulation, there is no need to simulate $S$, one need only simulate $V$. This reduces the runtime as well as standard error of the procedure.

We will employ the usual Euler-Maruyama method to simulate the variance process. For all our simulations, we use 2,000,000 Monte Carlo paths, and 24 time steps per day, where a year is comprised of 252 trading days. This is to reduce the Monte-Carlo and discretisation errors sufficiently well.

**Remark 7.2.** The code utilised to obtain the numerical results in this section is available at GitHub [12]. In particular, what is provided is:

- A routine which computes our closed-form approximation of put option prices for the Stochastic Verhulst model with piecewise-constant parameter inputs.
- A routine which implements the Monte-Carlo simulation via the mixing solution methodology for the pricing of put option prices in the Stochastic Verhulst model with piecewise-constant parameter inputs.
- A routine which compares the accuracy and runtimes of the aforementioned methods.

### 7.1. Stochastic Verhulst model sensitivity analysis.

**Remark 7.3.** A piecewise-constant parameter which is piecewise-constant over $n$ intervals will be called an $n$-piece parameter or a parameter with $n$ pieces.

We start from a “safe” parameter set, given by:

\[
\begin{array}{cccc}
S_0 & v_0 & r^d & r^f \\
100 & 18\% & 0.02 & 0 \\
\end{array}
\]

with

| $T$ | $\kappa$ | $\theta$ | $\lambda$ | $\rho$ |
|-----|---------|---------|----------|-------|
| 1M  | 5.00   | 1.70\% | 0.414    | -0.391|
| 3M  | 5.00   | 1.70\% | 0.414    | -0.391|
| 6M  | 5.00   | 1.70\% | 0.414    | -0.391|
| 1Y  | 5.00   | 1.70\% | 0.414    | -0.391|

However, we would like to consider piecewise-constant parameters, as our closed-form approximation method is designed in order to take advantage of piecewise-constant parameter inputs. To do so, we will make our parameters piecewise-constant over three time intervals, the length of the first, second and third time interval having proportions 1/4, 1/4, 1/2 of the
maturity time $T$ respectively. For example, if $T = 1/12$ then a 3-piece parameter is piecewise-constant on the time intervals $[0, 1/48), [1/48, 2/48), [2/48, 4/48)$. To choose the “safe” values of the 3-piece parameters, we will simply (albeit somewhat artificially) perturb the value of each safe parameter over each time interval, yielding the following table of “safe” 3-piece parameter sets:

| $T$ | Piece | Proportion | $\kappa$ | $\theta$ | $\lambda$ | $\rho$ | $r^d$ | $r^f$ |
|-----|-------|------------|---------|---------|---------|-------|-------|-------|
| 1M  | 1     | 1/4        | 4.80    | 1.70%   | 0.394   | -0.371 | 1%    | 0     |
|     | 2     | 1/4        | 5.20    | 2.10%   | 0.434   | -0.411 | 3%    | 0     |
|     | 3     | 1/2        | 5.00    | 1.90%   | 0.414   | -0.391 | 2%    | 0     |
| 3M  | 1     | 1/4        | 4.80    | 1.70%   | 0.394   | -0.371 | 1%    | 0     |
|     | 2     | 1/4        | 5.20    | 2.10%   | 0.434   | -0.411 | 3%    | 0     |
|     | 3     | 1/2        | 5.00    | 1.90%   | 0.414   | -0.391 | 2%    | 0     |
| 6M  | 1     | 1/4        | 4.80    | 1.70%   | 0.394   | -0.371 | 1%    | 0     |
|     | 2     | 1/4        | 5.20    | 2.10%   | 0.434   | -0.411 | 3%    | 0     |
|     | 3     | 1/2        | 5.00    | 1.90%   | 0.414   | -0.391 | 2%    | 0     |
| 1Y  | 1     | 1/4        | 4.80    | 1.70%   | 0.394   | -0.371 | 1%    | 0     |
|     | 2     | 1/4        | 5.20    | 2.10%   | 0.434   | -0.411 | 3%    | 0     |
|     | 3     | 1/2        | 5.00    | 1.90%   | 0.414   | -0.391 | 2%    | 0     |

In our numerical analysis, we vary one of the 3-piece parameters ($\kappa, \theta, \lambda, \rho$) with the rest fixed, and then compute implied volatilities via both the closed-form approximation formula as well as the Monte-Carlo method as described above. Specifically, we select a 3-piece parameter from ($\kappa, \theta, \lambda, \rho$), and start at 40% of its safe 3-piece parameter value, then increase the value of each piece in increments of 20%, all the way up to 160%, whilst keeping the other three 3-piece parameters fixed at their safe value. We then repeat this process with each of the other three 3-piece parameters. The relevant tables are Table 7.1, Table 7.2, Table 7.3, and Table 7.4, for the analysis of $\kappa$, $\theta$, $\lambda$, and $\rho$ respectively. Note that the $\rho$ table values are reversed, as $\rho$ is negative, and thus 160% of the safe 3-piece $\rho$ is approximately $(-0.594, -0.658, -0.626)$. This ensures that the parameter values are increasing for all tables.
Table 7.1. Signed error of implied volatilities in basis points, computed in the Stochastic Verhulst model with $\kappa$ varying from 40% to 160% of its “safe” 3-piece value.

| $\kappa$ | 40% | 60% | 80% | 100% | 120% | 140% | 160% |
|----------|-----|-----|-----|------|------|------|------|
| ATM      |     |     |     |      |      |      |      |
| 1M       | -0.72 | -0.41 | -0.09 | 0.22 | 0.52 | 0.81 | 1.10 |
| 3M       | 3.70  | 4.57 | 5.39 | 6.16 | 6.89 | 7.59 | 8.25 |
| 6M       | 6.54  | 7.93 | 9.19 | 10.34 | 11.41 | 12.40 | 13.32 |
| 1Y       | 12.65 | 14.35 | 15.88 | 17.28 | 18.56 | 19.75 | 20.85 |
| Put 25   |     |     |     |      |      |      |      |
| 1M       | -0.21 | 0.12 | 0.43 | 0.73 | 1.03 | 1.32 | 1.60 |
| 3M       | 3.09  | 3.93 | 4.71 | 5.45 | 6.15 | 6.81 | 7.43 |
| 6M       | 5.93  | 7.19 | 8.33 | 9.36 | 10.31 | 11.18 | 12.00 |
| 1Y       | 12.37 | 13.58 | 14.68 | 15.69 | 16.63 | 17.51 | 18.34 |
| Put 10   |     |     |     |      |      |      |      |
| 1M       | -0.54 | -0.20 | 0.10 | 0.40 | 0.69 | 0.98 | 1.26 |
| 3M       | 1.31  | 2.15 | 2.92 | 3.64 | 4.32 | 4.96 | 5.57 |
| 6M       | 3.30  | 4.51 | 5.57 | 6.51 | 7.37 | 8.15 | 8.88 |
| 1Y       | 9.39  | 10.16 | 10.83 | 11.42 | 11.98 | 12.50 | 13.00 |

Table 7.2. Signed error of implied volatilities in basis points, computed in the Stochastic Verhulst model with $\theta$ varying from 40% to 160% of its “safe” 3-piece value.

| $\theta$ | 40% | 60% | 80% | 100% | 120% | 140% | 160% |
|----------|-----|-----|-----|------|------|------|------|
| ATM      |     |     |     |      |      |      |      |
| 1M       | 0.35 | 0.30 | 0.26 | 0.22 | 0.17 | 0.13 | 0.09 |
| 3M       | 6.56 | 6.42 | 6.29 | 6.16 | 6.03 | 5.89 | 5.76 |
| 6M       | 11.02 | 10.79 | 10.57 | 10.34 | 10.12 | 9.89 | 9.66 |
| 1Y       | 18.35 | 17.99 | 17.64 | 17.28 | 16.93 | 16.57 | 16.21 |
| Put 25   |     |     |     |      |      |      |      |
| 1M       | 0.84 | 0.81 | 0.77 | 0.73 | 0.69 | 0.65 | 0.61 |
| 3M       | 5.82 | 5.70 | 5.57 | 5.45 | 5.32 | 5.20 | 5.07 |
| 6M       | 9.97 | 9.77 | 9.56 | 9.36 | 9.15 | 8.95 | 8.74 |
| 1Y       | 16.58 | 16.29 | 15.99 | 15.69 | 15.39 | 15.09 | 14.79 |
| Put 10   |     |     |     |      |      |      |      |
| 1M       | 0.50 | 0.48 | 0.44 | 0.40 | 0.36 | 0.33 | 0.29 |
| 3M       | 3.98 | 3.87 | 3.76 | 3.64 | 3.53 | 3.42 | 3.30 |
| 6M       | 6.99 | 6.83 | 6.67 | 6.51 | 6.35 | 6.19 | 6.03 |
| 1Y       | 11.89 | 11.73 | 11.58 | 11.42 | 11.27 | 11.11 | 10.95 |
Table 7.3. Signed error of implied volatilities in basis points, computed in the Stochastic Verhulst model with $\lambda$ varying from 40% to 160% of its “safe” 3-piece value.

| $\lambda$ | 40%  | 60%  | 80%  | 100% | 120% | 140% | 160% |
|-----------|------|------|------|------|------|------|------|
| ATM       |      |      |      |      |      |      |      |
| 1M        | -0.28| -0.17| 0.00 | 0.22 | 0.48 | 0.80 | 1.18 |
| 3M        | 4.82 | 5.13 | 5.58 | 6.16 | 6.88 | 7.75 | 8.77 |
| 6M        | 8.22 | 8.71 | 9.41 | 10.34| 11.51| 12.93| 14.63|
| 1Y        | 14.12| 14.86| 15.91| 17.28| 18.99| 21.07| 23.55|
| Put 25    |      |      |      |      |      |      |      |
| 1M        | 0.44 | 0.49 | 0.59 | 0.73 | 0.93 | 1.18 | 1.50 |
| 3M        | 4.62 | 4.74 | 5.01 | 5.45 | 6.08 | 6.92 | 8.01 |
| 6M        | 7.88 | 8.07 | 8.55 | 9.36 | 10.57| 12.25| 14.45|
| 1Y        | 13.19| 13.49| 14.28| 15.69| 17.82| 20.78| 24.67|
| Put 10    |      |      |      |      |      |      |      |
| 1M        | 0.65 | 0.56 | 0.47 | 0.40 | 0.36 | 0.36 | 0.41 |
| 3M        | 4.17 | 3.89 | 3.69 | 3.64 | 3.80 | 4.22 | 4.96 |
| 6M        | 7.03 | 6.55 | 6.33 | 6.51 | 7.24 | 8.64 | 10.82|
| 1Y        | 11.61| 10.86| 10.70| 11.42| 13.27| 16.44| 21.12|

Table 7.4. Signed error of implied volatilities in basis points, computed in the Verhulst model with $\rho$ varying from 160% to 40% of its “safe” 3-piece value.

| $\rho$  | 160% | 140% | 120% | 100% | 80%  | 60%  | 40%  |
|---------|------|------|------|------|------|------|------|
| ATM     |      |      |      |      |      |      |      |
| 1M      | 0.63 | 0.44 | 0.31 | 0.22 | 0.17 | 0.18 | 0.25 |
| 3M      | 8.30 | 7.46 | 6.75 | 6.16 | 5.68 | 5.32 | 5.07 |
| 6M      | 13.29| 12.13| 11.15| 10.34| 9.70 | 9.23 | 8.92 |
| 1Y      | 21.43| 19.82| 18.44| 17.28| 16.34| 15.61| 15.09|
| Put 25  |      |      |      |      |      |      |      |
| 1M      | 0.95 | 0.86 | 0.78 | 0.73 | 0.71 | 0.71 | 0.75 |
| 3M      | 6.55 | 6.13 | 5.76 | 5.45 | 5.20 | 5.01 | 4.88 |
| 6M      | 10.55| 10.10| 9.70 | 9.36 | 9.07 | 8.85 | 8.70 |
| 1Y      | 17.18| 16.64| 16.14| 15.69| 15.29| 14.94| 14.67|
| Put 10  |      |      |      |      |      |      |      |
| 1M      | -0.18| 0.04 | 0.22 | 0.40 | 0.56 | 0.71 | 0.84 |
| 3M      | 2.51 | 2.93 | 3.30 | 3.64 | 3.94 | 4.18 | 4.38 |
| 6M      | 4.25 | 5.10 | 5.85 | 6.51 | 7.08 | 7.55 | 7.91 |
| 1Y      | 8.84 | 9.76 | 10.63| 11.42| 12.14| 12.75| 13.24|

The sensitivity analysis fares well, with errors in implied volatility being less than 15bp for reasonable parameter values (namely the 100% column). For more unreasonable parameter values the errors grow but do not exceed 25bp. This is consistent with our expectations, and indeed such errors are acceptable for use within applications.

As one would expect, the errors do grow as maturity increases. However, it seems that something obtuse occurs with when considering out of the money Put options. We would expect the approximation methodology to break down, however it seems that the errors fare better for out of the money Put options. A more thorough investigation into the error analysis would be required in order to better understand this phenomenon.
8. Conclusion

We have provided a second-order approximation for the price of a put option in a general stochastic volatility framework with general drift and power-type diffusion coefficients that satisfy some regularity conditions, as well as an associated fast calibration scheme. Moreover, when parameters are assumed to be piecewise-constant, our approximation formula is closed-form. In addition, this assumption allows us to devise a fast calibration scheme by exploiting recursive properties of the iterated integral operators in terms of which our approximation formulas are expressed. We establish the explicit form of the error term induced by the expansion, and we determine sufficient ingredients for obtaining a meaningful bound on this error term, these ingredients essentially being higher order moments pertaining to the volatility process. We perform a numerical sensitivity analysis for the approximation formula in the Stochastic Verhulst model, and show that the error is small, behaves as we expect with respect to parameter changes, and is within an acceptable range for application purposes. It is also worth noting that we also derive sufficient conditions regarding existence of a risk-neutral measure in a general inhomogeneous stochastic volatility model, albeit this is not the main focus of the article. It is of our opinion that our general second-order approximation formula will be incredibly useful for practitioners since: obtaining the pricing formula for different stochastic volatility models only requires basic differentiation, the formula is essentially instantaneous to compute, the fast calibration scheme can be used to calibrate models rapidly, and the error is sufficiently low for the purposes of application.

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Appendix A. Mixing solution

In this appendix, we present a derivation of the result referred to as the mixing solution by [17]. This result is crucial for the expansion methodology implemented in Section 3. Hull and White first established the expression for the case of independent Brownian motions \( W \) and \( B \). Later on, this was extended to the correlated Brownian motions case, see [31, 38].

**Theorem A.1** (Mixing solution). Under a chosen domestic risk-neutral measure \( Q \), suppose that the spot \( S \) with volatility \( V \) are given as the solution to the general model eq. (1.1). Define \( X \) as the log-spot and \( k \) the log-strike. Namely, \( X_t = \ln S_t \) and \( k = \ln K \). Then

\[
\text{Put}_G = e^{-\int_0^T r_t \, dt} \mathbb{E}(e^k - e^{X_T})_+ = \mathbb{E} \left( e^{-\int_0^T r_t \, dt} \mathbb{E}[(e^k - e^{X_T})_+ \mid \mathcal{F}_T] \right) = \mathbb{E} \left[ P_{BS} \left( x_0 - \int_0^T \frac{1}{2} \rho_t^2 V_t^2 \, dt + \int_0^T \rho_t V_t dB_t, \int_0^T V_t^2 (1 - \rho_t^2) \, dt \right) \right],
\]

where \( P_{BS} \) is given in eq. (2.2).
Proof. By writing the driving Brownian motion of the spot as \( W_t = \int_0^t \mu_t dB_u + \int_0^t \sqrt{1 - \rho_t^2} dZ_u \), where \( Z \) is a Brownian motion under \( Q \) which is independent of \( B \), this yields the pathwise unique strong solution of \( X \) as
\[
X_T = x_0 + \int_0^T (r^d_t - r^f_t - \frac{1}{2} V^2_t) dt + \int_0^T \rho_t V_t dB_t + \int_0^T V_t \sqrt{1 - \rho_t^2} dZ_t.
\]
First, notice that \( V \) is adapted to the filtration \( (\mathcal{F}^B_t)_{0 \leq t \leq T} \). Thus, it is evident that \( X_T|_{\mathcal{F}^B_T} \) will have a normal distribution. Namely,
\[
X_T|_{\mathcal{F}^B_T} \sim \mathcal{N}(\hat{\mu}(T), \hat{\sigma}^2(T)),
\]
where
\[
\begin{align*}
\hat{\mu}(T) &:= x_0 + \int_0^T \left( r^d_t - r^f_t \right) dt - \frac{1}{2} \int_0^T V^2_t dt + \int_0^T \rho_t V_t dB_t, \\
\hat{\sigma}^2(T) &:= \int_0^T V^2_t (1 - \rho_t^2) dt.
\end{align*}
\]
Also, let \( \hat{\mu}(T) := \mu(T) - \int_0^T (r^d_t - r^f_t) dt \). Hence the calculation of \( e^{-\int_0^T r^d_t dt} E[(e^k - e^{X_T})_+|\mathcal{F}^B_T] \) will result in a Black-Scholes-like formula.
\[
e^{-\int_0^T r^d_t dt} E[(e^k - e^{X_T})_+|\mathcal{F}^B_T] = e^k e^{-\int_0^T r^d_t dt} \mathcal{N}\left( \frac{k - \hat{\mu}(T)}{\hat{\sigma}(T)} \right) - e^{-\int_0^T r^d_t dt} \mathcal{E}(r^d(T) + \frac{1}{2} \hat{\sigma}^2(T)) \mathcal{N}\left( \frac{k - \hat{\mu}(T) - \frac{1}{2} \hat{\sigma}^2(T)}{\hat{\sigma}(T)} \right)
\]
\[
= e^k e^{-\int_0^T r^d_t dt} \mathcal{N}\left( \frac{k - \hat{\mu}(T)}{\hat{\sigma}(T)} - \frac{1}{2} \hat{\sigma}^2(T) + \frac{1}{2} \hat{\sigma}(T) \right) - e^{\hat{\mu}(T) + \frac{1}{2} \hat{\sigma}^2(T)} e^{-\int_0^T r^d_t dt} \mathcal{N}\left( \frac{k - \hat{\mu}(T) - \frac{1}{2} \hat{\sigma}^2(T)}{\hat{\sigma}(T)} - \frac{1}{2} \hat{\sigma}(T) \right)
\]
\[
= e^k e^{-\int_0^T r^d_t dt} \mathcal{N}\left( \frac{k - (\hat{\mu}(T) + \frac{1}{2} \hat{\sigma}^2(T)) - \int_0^T (r^d_t - r^f_t) dt}{\hat{\sigma}(T)} + \frac{1}{2} \hat{\sigma}(T) \right) - e^{\hat{\mu}(T) + \frac{1}{2} \hat{\sigma}^2(T)} e^{-\int_0^T r^d_t dt} \mathcal{N}\left( \frac{k - (\hat{\mu}(T) + \frac{1}{2} \hat{\sigma}^2(T)) - \int_0^T (r^d_t - r^f_t) dt}{\hat{\sigma}(T)} - \frac{1}{2} \hat{\sigma}(T) \right).
\]
It is now immediate that \( e^{-\int_0^T r^d_t dt} E[(e^k - e^{X_T})_+|\mathcal{F}^B_T] = \mathcal{P}_{BS} (\hat{\mu}(T) + \frac{1}{2} \hat{\sigma}^2(T), \hat{\sigma}^2(T)) \).

**Appendix B. Existence of a domestic risk-neutral measure**

In this appendix, we consider the existence of a domestic risk-neutral measure in the Verhulst model. To elaborate, first consider the general model eq. (1.1) under the probability measure \( Q \). Let \( F_t := S_t e^{\int_0^t r^d_u du} / e^{\int_0^t r^d_u du} \) be the foreign bank account denominated in units of domestic currency and discounted in units of domestic currency. If \( F_t \) is a martingale under \( Q \), then \( Q \) is indeed a domestic risk-neutral measure. However, this is not always guaranteed. The issue pertaining to the existence of a risk-neutral measure in stochastic volatility models has been identified and studied originally by [23] as well as [2]. Both sets of authors provide conditions for when a (time-homogeneous) stochastic volatility model is legally specified under a risk-neutral measure. A comprehensive survey on this issue is given in [6]. However, to our knowledge, the case of time-inhomogeneous stochastic volatility models (that is, with time-dependent parameters) has not been studied in the literature. Thus, we aim to extend these
results to time-inhomogeneous stochastic volatility models. First, notice that the general model eq. (1.1) can be reexpressed as

\[ dF_t = F_t V_t dW_t, \quad F_0 = S_0, \]
\[ dV_t = \alpha(t, V_t) dt + \beta(t, V_t) dB_t, \quad V_0 = v_0, \]
\[ d\langle W, B \rangle_t = \rho_t dt, \]

(B.1)

where we stress that \( W \) and \( B \) are Brownian motions with deterministic, time-dependent instantaneous correlation \( \rho_t \) for \( 0 \leq t \leq T \), defined on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})\).

The question of whether or not eq. (1.1) is truly specified under a domestic risk-neutral measure is equivalent to the question of whether or not \((F_t)\) is a martingale under \(\mathbb{Q}\).

**Theorem B.1.** Let Assumption A hold. Suppose the following condition is true:

\[ \sup_{t \in [0,T]} \limsup_{x \to \infty} \frac{\rho t \beta(t, x) + \alpha(t, x)}{x} < \infty. \]

(B.2)

Then \((F_t)\) is a martingale.

**Proof.** We essentially follow in a very similar vein to Theorem 2.4 (i) in [23]. Let \( \tau_n := \inf\{t \geq 0 : V_t > n\} \) be the first time \( V \) crosses the level \( n \). Clearly \( \tau_n \uparrow \infty \) \(\mathbb{Q}\) a.s. Define \( F^n_t := F_{t \wedge \tau_n} \).

It is well known that \((F^n_t)\) and \((F_t)\) possess the pathwise unique strong solutions

\[ F^n_t = F_0 \exp \left( \int_0^t V_u 1_{\{u \leq \tau_n\}} dW_u - \frac{1}{2} \int_0^t V_u^2 1_{\{u \leq \tau_n\}} du \right) \]

and

\[ F_t = F_0 \exp \left( \int_0^t V_u dW_u - \frac{1}{2} \int_0^t V_u^2 du \right) \]

respectively. Now \((F^n_t)\) is a martingale for each \( n \in \mathbb{N} \). Furthermore, \((F_t)\) is a non-negative local-martingale. Thus, by Fatou’s lemma, \((F_t)\) is a non-negative supermartingale. Utilising the condition eq. (B.2), this implies that there exists some constant \( M > 0 \) such that

\[ \rho t \beta(t, x) + \alpha(t, x) \leq M(1 + x), \]

(B.3)

for all \( x \geq 0 \), uniformly in \( t \leq T \). Now if we show that

\[ \sup_n E(F^n_t \log(F^n_t)) < \infty \]

then by the Vallée Poussin theorem, \((F^n_t)_n\) is uniformly integrable and thus \( E(F_t) = F_0 \) for each \( t \leq T \). This, combined with the fact that \((F_t)\) is a non-negative supermartingale, will ensure \((F_t)\) is a martingale. Now,

\[ F^n_t \log(F^n_t) = F^n_t \left( \log(F_0) + \int_0^t V_u 1_{\{u \leq \tau_n\}} dW_u - \frac{1}{2} \int_0^t V_u^2 1_{\{u \leq \tau_n\}} du \right). \]

Notice \( F^n_t / F_0 \) is a Radon-Nikodym derivative which defines a measure \( \hat{\mathbb{Q}} \). And by Girsanov’s theorem, \( \hat{W}_t = W_t - \int_0^t V_u 1_{\{u \leq \tau_n\}} du \) is a Brownian motion under \( \hat{\mathbb{Q}} \). Denote the expectation
under \( \hat{Q} \) by \( \hat{E} \). Then
\[
\mathbb{E}(F^m \log(F^m)) = F_0 \log(F_0) + \mathbb{E} \left( F^m \left[ \int_0^T V_t \mathbf{1}_{\{t \leq \tau_n\}} dW_t - \frac{1}{2} \int_0^T V_t^2 \mathbf{1}_{\{t \leq \tau_n\}} dt \right] \right)
\]
\[
= F_0 \log(F_0) + F_0 \hat{E} \left( \int_0^T V_t \mathbf{1}_{\{t \leq \tau_n\}} d\hat{W}_t + \frac{1}{2} \int_0^T V_t^2 \mathbf{1}_{\{t \leq \tau_n\}} dt \right)
\]
\[
= F_0 \log(F_0) + \frac{F_0}{2} \int_0^T \hat{E}(V_t^2 \mathbf{1}_{\{t \leq \tau_n\}}) dt
\]
\[
\leq F_0 \log(F_0) + \frac{F_0}{2} \int_0^T \hat{E}(V_t^2) dt.
\]
So it suffices to determine a bound on \( \hat{E}(V_t^2) \) for \( t \leq T \). Now write \( B_t = \int_0^t \rho_u dW_u + \int_0^t \sqrt{1 - \rho_u^2} dZ_u \) where \( Z \) is a \( \hat{Q} \) Brownian motion independent of \( W \). Then
\[
B_t = \int_0^t \rho_u \mathbf{1}_{\{u \leq \tau_n\}} du + \int_0^t \rho_u d\hat{W}_u + \int_0^t \sqrt{1 - \rho_u^2} dZ_u.
\]
Hence,
\[
dV_t = \left[ \rho V_t \beta(t, V_t) V_t \mathbf{1}_{\{t \leq \tau_n\}} + \alpha(t, V_t) \right] dt + \beta(t, V_t) d\hat{B}_t
\]
where \( \hat{B}_t = \int_0^t \rho_u d\hat{W}_u + \int_0^t \sqrt{1 - \rho_u^2} dZ_u \) is a Brownian motion under \( \hat{Q} \). Utilising eq. (B.3) yields
\[
dV_t \leq M(1 + V_t) dt + \beta(t, V_t) d\hat{B}_t.
\]
Then for the second moment,
\[
\hat{E}(V_t^2) \leq \hat{E} \left( v_0 + M \int_0^t (1 + V_u) du + \int_0^t \beta(u, V_u) d\hat{B}_u \right)^2
\]
\[
\leq 4v_0^2 + 4M^2 \hat{E} \left( \int_0^t (1 + V_u) du \right)^2 + 2 \hat{E} \left( \int_0^t \beta(u, V_u) d\hat{B}_u \right)^2
\]
\[
\leq 4v_0^2 + 4M^2 \int_0^t \hat{E}(1 + V_u)^2 du + 2 \int_0^t \hat{E}(\beta(u, V_u))^2 du
\]
\[
\leq 4v_0^2 + 4M^2 \int_0^t \hat{E} \left[ (2 + V_u^2) + \frac{K}{2M^2} (1 + V_u^2) \right] du,
\]
where we have used that \( (a + b)^2 \leq 2a^2 + 2b^2 \), \( \left( \int_0^t f(u) du \right)^2 \leq \int_0^t f^2(u) du \) (Jensen’s inequality) and (A3) in Assumption A. Define \( m_{2,t} := \hat{E}(V_t^2) \). Then (after redefining constants), this suggests we study the integral inequality
\[
m_{2,t} \leq c_0(1 + t) + c \int_0^t m_{2,u} du.
\]
Utilising Gronwall’s inequality, we obtain
\[
m_{2,t} \leq c_0(1 + t)e^{ct}.
\]
\[\square\]

**Proposition B.1.** Consider the Verhulst model eq. (6.1) specified under the probability measure \( \bar{Q} \). Then \( \bar{Q} \) is a domestic risk-neutral measure if
\[
\rho_t \lambda_t - \kappa_t \leq 0
\]
holds for all \( t \in [0, T] \).
Proof. We utilise Theorem B.1 with \( \alpha(t, x) = \kappa_t(\theta_t - x) \) and \( \beta(t, x) = \lambda_t x \). Then

\[
\frac{\rho_t \lambda_t x^2 + \kappa_t(\theta_t - x) x}{x} = x(\rho_t \lambda_t - \kappa_t) + \kappa_t \theta_t.
\]

Clearly eq. (B.2) is satisfied if and only if \( \rho_t \lambda_t - \kappa_t \leq 0 \) for all \( t \in [0, T] \). So in this case, \( \mathbb{Q} \) is a domestic risk-neutral measure. \( \square \)

**Appendix C. Malliavin calculus machinery**

In the following appendix we give a short excerpt on Malliavin calculus. This is predominantly to fix notation. We point the reader towards [28] for a complete and accessible source on Malliavin calculus.

The underlying framework of Malliavin calculus involves a so-called isonormal Gaussian process \( W \). Specifically, \( W = \{ W(h) : h \in H \} \) is a zero-mean Gaussian process induced by an underlying real, separable Hilbert space \( H \) such that \( \mathbb{E}(W(h)W(g)) = \langle h, g \rangle_H \). We need only make use of Malliavin calculus when the underlying Hilbert space is

\[
H = L^2([0, T]) \equiv L^2([0, T], \mathcal{B}([0, T]), \lambda^*),
\]

where \( \lambda^* \) is the one-dimensional Lebesgue measure. Thus the inner product on \( H \) is

\[
\langle h, g \rangle_H = \int_0^T h_t g_t \lambda^*(dt) = \int_0^T h_t g_t dt.
\]

Our Gaussian process \( \tilde{W} \) will be explicitly given as \( \tilde{W}(h) := \int_0^T h_t d\tilde{B}_t \), where \( \tilde{B} \) is a Brownian motion with natural filtration \( (\mathcal{F}_t^\tilde{B})_{0 \leq t \leq T} \) and \( h \in L^2([0, T]) \). By use of the zero-mean and Itô isometry properties of the Itô integral, it can be seen that such a Hilbert space \( H \) and Gaussian process \( \tilde{W} \) satisfy the framework for Malliavin calculus.

**Definition C.1** (Malliavin derivative). Let

\[
S_n := \left\{ F = f \left( \int_0^T h_{1,t} d\tilde{B}_t, \ldots, \int_0^T h_{n,t} d\tilde{B}_t \right) : f \in C_p^\infty(\mathbb{R}^n; \mathbb{R}), h_i, \in H \right\}
\]

and \( S := \bigcup_{n \geq 1} S_n \). Here \( C_p^\infty(\mathbb{R}^n; \mathbb{R}) \) is the space of smooth Borel measurable functions \( f : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) which have at most polynomial growth. Thus, the elements of \( S_n \) are random variables. For \( F \in S_n \), the Malliavin derivative \( D \) is an unbounded operator from \( S_n \subseteq L^p(\Omega) \rightarrow L^p([0, T] \times \Omega) \) for \( p \geq 1 \) and is given by

\[
D_t F := \sum_{i=1}^n \partial_i f \left( \int_0^T h_{1,u} d\tilde{B}_u, \ldots, \int_0^T h_{n,u} d\tilde{B}_u \right) h_{i,t}.
\]

It can be shown that the Malliavin derivative \( D \) is a closable operator on \( L^p(\Omega) \) into \( L^p([0, T] \times \Omega) \). We denote the closed extension of it again by \( D \) and moreover, its domain is denoted by \( \mathbb{D}^{1,p} \). Another way to think about the domain \( \mathbb{D}^{1,p} \) is as the completion of \( S \) with respect to the seminorm

\[
\| F \|_{1,p} := \left( \mathbb{E}|F|^p + \mathbb{E} \left[ \int_0^T (D_t F)^p dt \right] \right)^{1/p}
\]

where \( F \in S \) and \( p \geq 1 \).
The Malliavin derivative satisfies a duality relationship. In the case of adapted processes, the duality relationship reads as follows.

**Proposition C.1** (Malliavin duality relationship). Let $G \in \mathcal{D}^{1,2}$ and $\alpha \in L^2([0,T] \times \Omega)$ such that $\alpha$ is adapted to the filtration $(\mathcal{F}^B_t)_{0 \leq t \leq T}$. Then

$$
E \left( \int_0^t \alpha_s (D_s G) ds \right) = E \left( G \int_0^t \alpha_s d\tilde{B}_s \right)
$$

for any $t < T$.

**Proof.** See [28].

**Lemma C.1** (Malliavin integration by parts). Let $\tilde{T} \leq T$ and $\hat{T} \leq T$. Also, let $\alpha \in L^2([0,T] \times \Omega)$ such that $\alpha$ is adapted to $(\mathcal{F}^B_t)_{0 \leq t \leq T}$. Then,

$$
E \left[ l \left( \int_0^\tilde{T} h_u d\tilde{B}_u \right) \left( \int_0^{\hat{T}} \alpha_u d\tilde{B}_u \right) \right] = E \left[ l' \left( \int_0^\hat{T} h_u d\tilde{B}_u \right) \left( \int_0^{\tilde{T} \wedge \hat{T}} h_u \alpha_u du \right) \right].
$$

In particular, for $\tilde{T} = T$ and $\hat{T} = t < T$,

$$
E \left[ l \left( \int_0^T h_u d\tilde{B}_u \right) \left( \int_0^t \alpha_u d\tilde{B}_u \right) \right] = E \left[ l' \left( \int_0^T h_u d\tilde{B}_u \right) \left( \int_0^t h_u \alpha_u du \right) \right].
$$

**Proof.** Let $G = l \left( \int_0^T h_u d\tilde{B}_u \right)$. Then $G \in \mathcal{S}_1 \subseteq \mathcal{D}^{1,2}$ and $D_t G = l' \left( \int_0^T h_u d\tilde{B}_u \right) h_t 1_{\{t \leq \tilde{T}\}}$. The result follows by a consequence of Proposition C.1.

**Appendix D. $P_{BS}$ partial derivatives**

This appendix contains some partial derivatives for the Black-Scholes put option formula $P_{BS}$ (see eq. (2.2)). One can think of these partial derivatives as being analogous to the Black-Scholes Greeks. However, these are slightly different as our Black-Scholes formulas are parameterised with respect to log-spot and integrated variance rather than spot and volatility respectively.

**D.1. First-order $P_{BS}$**

\[
\frac{\partial_x P_{BS}}{P_{BS}} = e^x e^{-\int_0^T r_u du} \left( N\left(d_+ \right) - 1 \right), \\
\frac{\partial_y P_{BS}}{P_{BS}} = \frac{e^x e^{-\int_0^T r_u du} \phi(d_+)}{2\sqrt{y}}.
\]
D.2. Second-order $P_{BS}$

\[ \partial_{xx} P_{BS} = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{\sqrt{y}} + \partial_x P_{BS} \]

\[ = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{\sqrt{y}} + e^x e^{-\int_0^T \rho_0 \phi \left( N(d^1_n) - 1 \right)} , \]

\[ \partial_{xy} P_{BS} = (-1) \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{2y} , \]

\[ \partial_{yy} P_{BS} = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{4y^{3/2}} (d^1_n - d^1_n - 1) . \]

D.3. Third-order $P_{BS}$

\[ \partial_{xxx} P_{BS} = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{y} (\sqrt{y} - d^1_n) + \partial_{xx} P_{BS} \]

\[ = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{y} (2\sqrt{y} - d^1_n) + e^x e^{-\int_0^T \rho_0 \phi \left( N(d^1_n) - 1 \right)} , \]

\[ \partial_{xxy} P_{BS} = (-1) \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{2y^{3/2}} \left( d^1_n - d^1_n + (1 - d^1_n d^1_n \right) , \]

\[ \partial_{xyy} P_{BS} = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{4y^2} \left( (2d^1_n - \sqrt{y}) + (1 - d^1_n d^1_n ) \left( d^1_n - \sqrt{y} \right) , \right) \]

\[ \partial_{yyy} P_{BS} = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{8y^{5/2}} \left( (d^1_n d^1_n - 1)^2 - (d^1_n + d^1_n)^2 + 2 \right) . \]

D.4. Fourth-order $P_{BS}$

\[ \partial_{xxxx} P_{BS} = (-1) \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{y^{3/2}} (1 - (d^1_n - \sqrt{y})^2) + \partial_{xx} P_{BS} \]

\[ = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{y^{3/2}} \left[ (d^1_n - \sqrt{y})^2 + 2y - d^1_n \sqrt{y} - 1 \right] + e^x e^{-\int_0^T \rho_0 \phi \left( N(d^1_n) - 1 \right)} , \]

\[ \partial_{xxyy} P_{BS} = \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{2y^2} \left[ (\sqrt{y} - d^1_n)(d^1_n d^1_n - 2) + (\sqrt{y} + d^1_n) - d^1_n y - \sqrt{y} (1 - d^1_n d^1_n ) \right] , \]

\[ \partial_{xyxy} P_{BS} = (-1) \frac{e^x e^{-\int_0^T \rho_0 \phi(d^1_n) \, du}}{2y^{5/2}} \left[ 3d^1_n d^1_n + \frac{1}{2} (d^1_n - d^1_n)^2 d^1_n - \frac{1}{2} (d^1_n - d^1_n)^2 \right] \]

\[ + \frac{1}{2} y - \frac{1}{2} \sqrt{y} \left( 2d^1_n + d^1_n - \frac{3}{2} \right) . \]
\[ \partial_{xyyy} P_{BS} = e^x e^{-\int_0^T r_u du} \frac{\phi(d^n)}{8y^{7/2}} \left[ 2y^{3/2}(d^n_{-1})^2 - 1 - (d^n_{-1} + d^n_{+1}) \right] \]
\[ + \sqrt{y}(\sqrt{y} - d^n_{-1}) \left( (d^n_{-1} + d^n_{+1})^2 - 2 \right), \]
\[ \partial_{yyyy} P_{BS} = e^x e^{-\int_0^T r_u du} \frac{\phi(d^n)}{8y^{7/2}} \left( \frac{1}{2} (d^n_{-1} - 1)^2 (d^n_{-1} - 5) - (d^n_{-1} - 1)(d^n_{-1} + d^n_{+1}) \right) \]
\[ - \frac{1}{2} (d^n_{-1} + d^n_{+1})^2 (d^n_{-1} - 7) + (d^n_{-1} - 1) \].

**APPENDIX E. PROOF OF THEOREM 4.1**

In this appendix, we provide the proof of Theorem 4.1. In order to do so, we will utilise results from Malliavin calculus extensively. A short treatment of Malliavin calculus is presented in Appendix C. In addition to Malliavin calculus machinery, we will require the following ingredients.

**Proposition E.1** (P prized derivative relationship).
\[ \partial_y P_{BS}(x, y) = \frac{1}{2} \left( \partial_{xx} P_{BS}(x, y) - \partial_x P_{BS}(x, y) \right). \]

**Proof.** A simple application of differentiation yields the result.\hfill \Box

In addition, we will make extensive use of the stochastic integration by parts formula, which we will list here for convenience.

**Proposition E.2** (Stochastic integration by parts). Let \( X \) and \( Y \) be semimartingales with respect to a filtration \( \mathcal{F}_t \). Then we have
\[ X_T Y_T = \int_0^T X_t dY_t + \int_0^T Y_t dX_t + \int_0^T d\langle X, Y \rangle_t, \]
given that the above Itô integrals exist. In particular, if \( X_t = \int_0^t x_u d\tilde{X}_u \) and \( Y_t = \int_0^t y_u d\tilde{Y}_u \), where \( \tilde{X} \) and \( \tilde{Y} \) are semimartingales and \( x \) and \( y \) are stochastic processes adapted to the underlying filtration \( (\mathcal{F}_t) \) such that \( X \) and \( Y \) exist, then the stochastic integration by parts formula reads as
\[ \int_0^T x_t d\tilde{X}_t \int_0^T y_t d\tilde{Y}_t = \int_0^T \left( \int_0^t x_u d\tilde{X}_u \right) y_t d\tilde{Y}_t + \int_0^T \left( \int_0^t y_u d\tilde{Y}_u \right) x_t d\tilde{X}_t + \int_0^T x_t y_t d\langle \tilde{X}, \tilde{Y} \rangle_t. \]

**Lemma E.1.** Let \( Z \) be a semimartingale such that \( Z_0 = 0 \) and let \( f \) be a Lebesgue integrable deterministic function. Then
\[ \int_0^T f_t Z_t dt = \int_0^T \omega_{t,T}^{Z}(0, f) dZ_t. \]

**Proof.** A simple application of Proposition E.2 (stochastic integration by parts) gives the desired result.\hfill \Box

Now we can proceed with the proof of Theorem 4.1. The proof is quite long and arduous, and hence will be broken up over a number of subsections, each more or less dedicated to the
calculation of either \( \mathbb{E}\tilde{P}_\text{BS}, C_x, C_y, C_{xx}, C_{yy} \) or \( C_{xy} \). For the time being we will not yet enforce Assumption C.

E.1. \( \mathbb{E}\tilde{P}_\text{BS} \). Notice that \( \mathbb{E}\tilde{P}_\text{BS} = g(0) = \mathbb{E}(e^k - e^{\chi(t)})_+ \). Since the perturbed volatility process \( V_t^{(\varepsilon)} \) is deterministic when \( \varepsilon = 0 \), then \( g(0) \) will just be a Black-Scholes formula. Thus we have

\[
\mathbb{E}\tilde{P}_\text{BS} = P_\text{BS} \left( x_0, \int_0^T \sigma^2_v \, dt \right).
\]

E.2. \( C_x \). Using Lemma C.1 (Malliavin integration by parts),

\[
\mathbb{E}\partial_x \tilde{P}_\text{BS} \int_0^T \rho_t \left( V_{1,t} + \frac{1}{2} V_{2,t} \right) \, dB_t = \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \int_0^T \rho^2_v \, \left( V_{1,t} + \frac{1}{2} V_{2,t} \right) \, dt.
\]

Furthermore, using Proposition E.1 (\( P_\text{BS} \) partial derivative relationship),

\[
\mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \int_0^T \rho^2_v \, \left( V_{1,t} + \frac{1}{2} V_{2,t} \right) \, dt = \mathbb{E}(2\partial_y + \partial_x) \tilde{P}_\text{BS} \int_0^T \rho^2_v \, \left( V_{1,t} + \frac{1}{2} V_{2,t} \right) \, dt.
\]

Thus

\[
C_x = 2\mathbb{E}\partial_y \tilde{P}_\text{BS} \int_0^T \rho^2_v \, \left( V_{1,t} + \frac{1}{2} V_{2,t} \right) \, dt - \frac{1}{2} \mathbb{E}\partial_x \tilde{P}_\text{BS} \int_0^T \rho^2_v \, V^2_{1,t} \, dt.
\]

E.3. \( C_{xx} \). For \( C_{xx} \) we first use Proposition E.2 (stochastic integration by parts) to reduce this expression.

\[
C_{xx} = \frac{1}{2} \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \rho_t V_{1,t} \, dB_t - \int_0^T \rho^2_v \, V_{1,t} \, dt \right)^2
\]

\[
= \frac{1}{2} \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \rho_t V_{1,t} \, dB_t \right)^2 - \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \rho^2_v \, V_{1,t} \, dt \right) \left( \int_0^T \rho_t V_{1,t} \, dB_t \right)
\]

\[
+ \frac{1}{2} \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \rho^2_v \, V_{1,t} \, dt \right)^2
\]

\[
= \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \left( \int_0^t \rho^2_v \, V_{1,s} \, ds \right) \rho^2_v \, V_{1,t} \, dt \right)
\]

\[
- \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \left( \int_0^t \rho^2_v \, V_{1,s} \, ds \right) \rho_t V_{1,t} \, dB_t \right) + \int_0^T \left( \int_0^t \rho_v \, V_{1,s} \, ds \right) \rho^2_v \, V_{1,t} \, dt
\]

\[
+ \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \left( \int_0^t \rho_v \, V_{1,s} \, ds \right) \rho_t V_{1,t} \, dB_t \right) + \frac{1}{2} \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \rho^2_v \, V^2_{1,t} \, dt \right)
\]

\[
= \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \left( \int_0^t \rho_v \, V_{1,s} \, ds \right) \left( \rho_t V_{1,t} \, dB_t - \rho^2_v \, V_{1,t} \, dt \right) \right)
\]

\[
+ \frac{1}{2} \mathbb{E}\partial_{xx} \tilde{P}_\text{BS} \left( \int_0^T \rho^2_v \, V^2_{1,t} \, dt \right).
\]
Using Lemma C.1 (Malliavin integration by parts),
\[
C_{xx} = -\mathbb{E} \partial_{xx} \tilde{P}_{BS} \left( \int_0^T \left\{ \left( \int_0^t \rho_s V_{1,s} dB_s - \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) \rho^2_t v_{0,t} V_{1,t} dt \right\} \right)
+ \mathbb{E} \partial_{xx} \tilde{P}_{BS} \left( \int_0^T \left( \int_0^t \rho_s V_{1,s} dB_s - \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) \rho^2_t v_{0,t} V_{1,t} dt \right) + \frac{1}{2} \mathbb{E} \partial_{xx} \tilde{P}_{BS} \left( \int_0^T \rho^2_t V_{1,t} dt \right).
\]
Furthermore, using Proposition C.1 (\(P_{BS}\) partial derivative relationship),
\[
C_{xx} = 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \left( \int_0^t \rho_s V_{1,s} dB_s - \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) \rho^2_t v_{0,t} V_{1,t} dt \right)
+ \frac{1}{2} \mathbb{E} \partial_{xx} \tilde{P}_{BS} \left( \int_0^T \rho^2_t V_{1,t} dt \right) .
\]
Adding the terms \(C_x, C_{xx}\) and \(C_y\) yields
\[
C_x + C_{xx} + C_y = \mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T 2v_{0,t} V_{1,t} + V_{1,t}^2 v_{0,t} V_{2,t} dt \right)
+ 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \left( \int_0^t \rho_s V_{1,s} dB_s - \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) \rho^2_t v_{0,t} V_{1,t} dt \right).
\]

E.4. \(C_{xy}\). For \(C_{xy}\) we use Proposition E.2 (stochastic integration by parts) to obtain
\[
\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \rho_t V_{1,t} dB_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} dt \right) \left( \int_0^T \left( 1 - \rho_t^2 \right) (2v_{0,t} V_{1,t}) dt \right)
= 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \left( 1 - \rho_s^2 \right) v_{0,s} V_{1,s} ds \right) \left( \rho_t V_{1,t} dB_t - \rho_t^2 v_{0,t} V_{1,t} dt \right)
+ 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \left( 1 - \rho_s^2 \right) v_{0,s} V_{1,s} ds \right) \left( 1 - \rho_t^2 \right) v_{0,t} V_{1,t} dt
= 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \left( 1 - \rho_s^2 \right) v_{0,s} V_{1,s} ds \right) \left( \rho_t V_{1,t} dB_t - \rho_t^2 v_{0,t} V_{1,t} dt \right)
- 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \rho_s^2 v_{0,s} V_{1,s} ds \right) v_{0,t} V_{1,t} dt
+ 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \rho_s V_{1,s} dB_s \right) v_{0,t} V_{1,t} dt
- 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T \rho_s V_{1,s} dB_s - \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) \rho_t^2 v_{0,t} V_{1,t} dt .
\]
Furthermore, using Proposition C.1 (Malliavin duality relationship),
\[
\hat{C}_{xy} := 2\mathbb{E} \partial_{xy} \tilde{P}_{BS} \left( \int_0^T v_{0,t} V_{1,t} \left( \int_0^t \rho_s V_{1,s} dB_s \right) dt \right) = 2 \int_0^T \mathbb{E} \partial_{xy} \tilde{P}_{BS} v_{0,t} V_{1,t} \left( \int_0^t \rho_s V_{1,s} dB_s \right) dt
\]
Using the definition of the Malliavin derivative, we obtain

\[ D_s(\partial_{xy} \bar{P}_{BS} v_0, V_{1,\cdot}) = \partial_{xy} \bar{P}_{BS} v_0, t \rho_s v_0, s 1_{\{s \leq t\}} + \partial_{xy} \bar{P}_{BS} D_s(\rho_s v_0, s 1_{\{s \leq t\}}) \]

\[ = \partial_{xy} \bar{P}_{BS} v_0, t \rho_s v_0, s 1_{\{s \leq t\}} + \partial_{xy} \bar{P}_{BS} v_0, t \rho_s v_0, s 1_{\{s \leq t\}} \]

\[ + \partial_{xy} \bar{P}_{BS} v_0, t \left( e^{\int_0^t \alpha_x(u, v_0, u) du} \beta(s, v_0, s) e^{-\int_0^t \alpha_x(z, v_0, z) dz} 1_{\{s \leq t\}} \right), \]

where we have used the explicit form for \( V_{1,\cdot} \) from eq. (3.4). Thus using Proposition C.1 (Malliavin duality relationship),

\[ 2 \int_0^T E \left( \int_0^t \rho_s V_{1,s} D_s(\partial_{xy} \bar{P}_{BS} v_0, V_{1,\cdot}) ds \right) dt \]

\[ = 2 \int_0^T E \partial_{xy} \bar{P}_{BS} \left( \int_0^t \rho_s^2 v_0, s \rho_s v_0, s ds \right) v_0, t V_{1,\cdot} dt \]

\[ + 2 \int_0^T E \partial_{xy} \bar{P}_{BS} \left( \int_0^t \rho_s v_0, s \beta(s, v_0, s)e^{-\int_0^t \alpha_x(z, v_0, z) dz} ds \right) v_0, t e^{\int_0^t \alpha_x(z, v_0, z) dz} dt \]

\[ = 2 E \partial_{xy} \bar{P}_{BS} \int_0^T \left( \int_0^t \rho_s^2 v_0, s \rho_s v_0, s ds \right) v_0, t V_{1,\cdot} dt \]

\[ + 2 E \partial_y \bar{P}_{BS} \int_0^T e^{\int_0^t \alpha_x(z, v_0, z) dz} v_0, t \left( \int_0^t v_0, s e^{\int_0^t \alpha_x(z, v_0, z) dz} dB_s \right) dt. \]

We will now enforce Assumption C.

**Remark E.1.** The purpose of Assumption C is:

1. \( \beta(t, x) = \lambda_t x^\mu \) for \( \mu \geq 1/2 \) is Hölder continuous of order \( \geq 1/2 \) in \( x \) uniformly in \( t \in [0, T] \), and \( \beta_x(t, x) = \lambda_t x^{\mu-1} \) is continuous a.e. in \( x \) and \( t \in [0, T] \). Thus, Assumption A and Assumption B are satisfied.

2. Such a diffusion coefficient is common in application, see for example SABR model [15] and CEV model [10].

Truthfully, we could leave \( \beta \) as an arbitrary diffusion coefficient that solely obeys the conditions in Assumption A and Assumption B. However, for the purposes of application and also for our fast calibration scheme in Section 6, it will be more insightful to have this form for \( \beta \). For the interested reader, all the following calculations still remain valid solely under Assumption A and Assumption B.

Due to Assumption C, we can rewrite \( V_{1,\cdot} \) and \( V_{2,\cdot} \) from Lemma 3.1 as

\[ V_{1,\cdot} = e^{\int_0^t \alpha_x(z, v_0, z) dz} \int_0^t \lambda_s v_0, s e^{-\int_0^t \alpha_x(z, v_0, z) dz} dB_s, \quad (E.1) \]

\[ V_{2,\cdot} = e^{\int_0^t \alpha_x(z, v_0, z) dz} \left\{ \int_0^t \alpha_x(s, v_0, s)(V_{1,s})^2 e^{-\int_0^t \alpha_x(z, v_0, z) dz} ds \right. \]

\[ + \left. \int_0^t 2 \mu \lambda_s v_0, s^{-1} V_{1,s} e^{-\int_0^t \alpha_x(z, v_0, z) dz} dB_s \right\}. \quad (E.2) \]
Then we obtain
\[ C_{xy} = 2E\partial_{xy} \tilde{P}_{BS} \int_0^T \left( \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) v_{0,t} V_{1,t} dt \]
\[ + 2E\partial_y \tilde{P}_{BS} \int_0^T v_{0,t} e^{\int_0^t \alpha_{x}(z,v_{0,s}) ds} \left( \int_0^t \lambda_s v_{0,s}^{\mu-1} V_{1,s} e^{-\int_0^s \alpha_{x}(z,v_{0,s}) ds} dB_s \right) dt. \]

Hence

\[ C_{xy} = 2E\partial_{xy} \tilde{P}_{BS} \left( \int_0^T \left( \int_0^t (1 - \rho_s^2) v_{0,s} V_{1,s} ds \right) \left( \rho_t V_{1,t} dB_t - \rho_t^2 v_{0,t} V_{1,t} dt \right) \right) \]
\[ - 2E\partial_{xy} \tilde{P}_{BS} \int_0^T \left( \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) v_{0,t} V_{1,t} dt \]
\[ + 2E\partial_{xy} \hat{P}_{BS} \int_0^T v_{0,t} e^{\int_0^t \alpha_{x}(z,v_{0,s}) ds} \left( \int_0^t \lambda_s v_{0,s}^{\mu-1} V_{1,s} e^{-\int_0^s \alpha_{x}(z,v_{0,s}) ds} dB_s \right) dt \]
\[ - 2E\partial_{xy} \hat{P}_{BS} \left( \int_0^T \left( \int_0^t \rho_s V_{1,s} dB_s - \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) \rho_t v_{0,t} V_{1,t} dt \right). \]

E.5. \( C_{yy} \). \( C_{yy} \) is given by Proposition E.2 (stochastic integration by parts) as

\[ 4E\partial_{yy} \hat{P}_{BS} \left( \int_0^T \left\{ \int_0^t (1 - \rho_s^2) v_{0,s} V_{1,s} ds \right\} (1 - \rho_t^2) v_{0,t} V_{1,t} dt \right). \]

E.6. Adding \( C_x, C_y, C_{xx}, C_{xy} \) and \( C_{yy} \). Now we add up all the terms after manipulation from Appendices E.1 to E.5:

\[ (C_x + C_y + C_{xx}) + C_{xy} + C_{yy} \]
\[ = E\partial_{xy} \hat{P}_{BS} \left( \int_0^T 2v_{0,t} V_{1,t} + V_{1,t}^2 + v_{0,t} V_{2,t} dt \right) \]
\[ + 2E\partial_{xy} \hat{P}_{BS} \int_0^T v_{0,t} e^{\int_0^t \alpha_{x}(z,v_{0,s}) ds} \left( \int_0^t \lambda_s v_{0,s}^{\mu-1} V_{1,s} e^{-\int_0^s \alpha_{x}(z,v_{0,s}) ds} dB_s \right) dt \]
\[ + 2E\partial_{xy} \hat{P}_{BS} \left( \int_0^T \left( \int_0^t (1 - \rho_s^2) v_{0,s} V_{1,s} ds \right) \left( \rho_t V_{1,t} dB_t - \rho_t^2 v_{0,t} V_{1,t} dt \right) \right) \]
\[ - 2E\partial_{xy} \hat{P}_{BS} \int_0^T \left( \int_0^t \rho_s^2 v_{0,s} V_{1,s} ds \right) v_{0,t} V_{1,t} dt \]
\[ + 2E\partial_{xy} \hat{P}_{BS} \int_0^T \left( \int_0^t \rho_s v_{0,s} V_{1,s} ds \right) v_{0,t} V_{1,t} dt \]
\[ + 4E\partial_{yy} \hat{P}_{BS} \left( \int_0^T \left\{ \int_0^t (1 - \rho_s^2) v_{0,s} V_{1,s} ds \right\} (1 - \rho_t^2) v_{0,t} V_{1,t} dt \right). \]
Then
\[ C_x + C_y + C_{xx} + C_{xy} + C_{yy} \]
\[ = \mathbb{E} \partial_y \tilde{P}_{BS} \left( \int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t}dt \right) \]
\[ + 2\mathbb{E} \partial_y \tilde{P}_{BS} \int_0^T v_{0,t}e^{\int_0^t \alpha_s(z_v,v_s)dz_s} \left( \int_0^t \lambda_s v_{0,s}^{-1}V_{1,s}e^{-\int_0^s \alpha_x(z_v,v_s)dz}dB_s \right) dt \]
\[ + 2\mathbb{E} \partial_x \tilde{P}_{BS} \left( \int_0^T \left( \int_0^t (1 - \rho_s^2)v_{0,s}V_{1,s}ds \right) \left( \rho_t V_{1,t}dB_t - \rho_t^2 v_{0,t}V_{1,t}dt \right) \right) \]
\[ + 4\mathbb{E} \partial_y \tilde{P}_{BS} \int_0^T \left( \int_0^t \rho_s^2 v_{0,s}V_{1,s}ds \right) v_{0,t}V_{1,t}dt \]
\[ + 4\mathbb{E} \partial_y \tilde{P}_{BS} \left( \int_0^T \left\{ \int_0^t (1 - \rho_s^2)v_{0,s}V_{1,s}ds \right\} (1 - \rho_t^2)v_{0,t}V_{1,t}dt \right) , \]
where we have used Proposition E.1 (\( P_{BS} \) partial derivative relationship), then Proposition C.1 (Malliavin duality relationship) plus Proposition E.1 (\( P_{BS} \) partial derivative relationship) for the first and second equalities respectively. Combining and then simplifying the preceding \( \partial_{yy} \) terms yields
\[ C_x + C_y + C_{xx} + C_{xy} + C_{yy} \]
\[ = \mathbb{E} \partial_y \tilde{P}_{BS} \left( \int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t}dt \right) \]
\[ + 2\mathbb{E} \partial_y \tilde{P}_{BS} \int_0^T v_{0,t}e^{\int_0^t \alpha_s(z_v,v_s)dz_s} \left( \int_0^t \lambda_s v_{0,s}^{-1}V_{1,s}e^{-\int_0^s \alpha_x(z_v,v_s)dz}dB_s \right) dt \]
\[ + 4\mathbb{E} \partial_y \tilde{P}_{BS} \left( \int_0^T \left( \int_0^t \rho_s^2 v_{0,s}V_{1,s}ds \right) v_{0,t}V_{1,t}dt \right) . \]

Lastly, notice by Proposition E.2 (stochastic integration by parts) that
\[ 2 \left( \int_0^T \left( \int_0^t v_{0,s}V_{1,s}ds \right) v_{0,t}V_{1,t}dt \right) = \left( \int_0^T v_{0,t}V_{1,t}dt \right)^2 . \]

**Proposition E.3.** In view of the calculations in Appendices E.1 to E.6, and under Assumption C, we obtain the simpler form of the second-order approximation Put\(_G^{(2)}\) from Theorem 3.1.
as

\[
\text{Put}^{(2)}_G = P_{BS} \left( x_0, \int_0^T v_{0,t}^2 dt \right) \\
+ \mathbb{E} \partial_y \tilde{P}_{BS} \left( \int_0^T \rho v_{0,t} \xi V_{1,t} + V_{1,t}^2 + v_{0,t} V_{2,t} dt \right) \\
+ 2 \mathbb{E} \partial_y \tilde{P}_{BS} \left( \int_0^T v_{0,t} e^{\int_0^t \alpha_x(z,v_{0,s}) dz} \left( \int_0^t \lambda_s v_{0,s}^{-1} V_{1,s} e^{-\int_0^t \alpha_x(z,v_{0,s}) dz} dB_s \right) dt \right) \\
+ 2 \mathbb{E} \partial_y y \tilde{P}_{BS} \left( \int_0^T v_{0,t} V_{1,t} dt \right)^2.
\]

(E.3)

The last step is to reduce these remaining expectations in Proposition E.3 down by eliminating the stochastic processes \((V_{1,t})\) and \((V_{2,t})\). To do so, we will require one more lemma, which is a consequence of Lemma E.1, and the new forms of \(V_{1,t}\) and \(V_{2,t}\) under Assumption C, namely eqs. (E.1) and (E.2).

**Lemma E.2.** The following equalities hold:

\[
\mathbb{E} \left( l \left( \int_0^T \rho v_{0,t} dB_t \right) \int_0^T \xi V_{1,t} dt \right) = \omega_{0,T} (\alpha_x, \rho \lambda v_0^{\mu + 1}, \alpha_x, \xi) \mathbb{E} \left( l^{(1)} \left( \int_0^T \rho v_{0,t} dB_t \right) \right),
\]

(E.4)

\[
\mathbb{E} \left( l \left( \int_0^T \rho v_{0,t} dB_t \right) \int_0^T \xi V_{1,t}^2 dt \right) = 2 \omega_{0,T} (\alpha_x, \rho \lambda v_0^{\mu + 1}, \alpha_x, \alpha_x, \xi) \mathbb{E} \left( l^{(2)} \left( \int_0^T \rho v_{0,t} dB_t \right) \right)
\]

(E.5)

\[
\mathbb{E} \left( l \left( \int_0^T \rho v_{0,t} dB_t \right) \int_0^T \xi V_{2,t} dt \right) = \omega_{0,T} (\alpha_x, \rho \lambda v_0^{\mu + 1}, \alpha_x, \alpha_x, \xi) \mathbb{E} \left( l \left( \int_0^T \rho v_{0,t} dB_t \right) \right)
\]

\[
+ 2 \omega_{0,T} (\alpha_x, \rho \lambda v_0^{\mu + 1}, \alpha_x, \alpha_x, \xi) \mathbb{E} \left( l^{(2)} \left( \int_0^T \rho v_{0,t} dB_t \right) \right)
\]

(E.6)

\[
\mathbb{E} \left( l \left( \int_0^T \rho v_{0,t} dB_t \right) \int_0^T \xi V_{1,t} dt \right)^2 = 2 \omega_{0,T} (\alpha_x, \rho \lambda v_0^{\mu + 1}, \alpha_x, \xi, \xi) \mathbb{E} \left( l \left( \int_0^T \rho v_{0,t} dB_t \right) \right)
\]

\[
+ \left( \omega_{0,T} (\alpha_x, \rho \lambda v_0^{\mu + 1}, \alpha_x, \xi) \right)^2 \mathbb{E} \left( l^{(2)} \left( \int_0^T \rho v_{0,t} dB_t \right) \right).
\]

(E.7)

Here we write \(\alpha_x := \alpha_x (\cdot, v_{0, \cdot})\) and \(\alpha_{xx} := \alpha_{xx} (\cdot, v_{0, \cdot})\) for readability.

**Proof.** We will only show how to obtain eq. (E.4). Equations (E.5) to (E.7) can be obtained in a similar way. First, we replace \(V_{1,t}\) with its explicit form from eq. (E.1). Thus we can write
the left hand side of eq. (E.4) as
\[
\mathbb{E} \left( \int_0^T \rho_t v_0, t dB_t \right) \int_0^T \xi_t e^{\int_0^t \alpha_x(z,v_0, \cdot) dz} \left( \int_0^t \lambda_s v_0^\mu_s e^{-\int_0^s \alpha_x(z,v_0, \cdot) dz} dB_s \right) dt. 
\]
Using Lemma E.1 with \( f_t = \xi_t e^{\int_0^t \alpha_x(z,v_0, \cdot) dz} \) and \( Z_t = \int_0^t \lambda_s v_0^\mu_s e^{-\int_0^s \alpha_x(z,v_0, \cdot) dz} dB_s \), we get
\[
\mathbb{E} \left( \int_0^T \rho_t v_0, t dB_t \right) \int_0^T \xi_t e^{\int_0^t \alpha_x(z,v_0, \cdot) dz} \left( \int_0^t \lambda_s v_0^\mu_s e^{-\int_0^s \alpha_x(z,v_0, \cdot) dz} dB_s \right) dt)
= \mathbb{E} \left( \int_0^T \rho_t v_0, t dB_t \right) \int_0^T \omega_{l,T}^\alpha e^{-\int_0^t \alpha_x(z,v_0, \cdot) dz} dB_t. 
\]
Lastly, appealing to the Malliavin integration by parts Lemma C.1 we obtain
\[
\mathbb{E} \left( \int_0^T \rho_t v_0, t dB_t \right) \int_0^T \omega_{l,T}^\alpha \rho_t \lambda_t v_0^\mu e^{-\int_0^t \alpha_x(z,v_0, \cdot) dz} dB_t 
= \mathbb{E} \left( \int_0^T \rho_t v_0, t dB_t \right) \int_0^T \omega_{l,T}^\alpha \rho_t \lambda_t v_0^\mu e^{-\int_0^t \alpha_x(z,v_0, \cdot) dz} dB_t 
= \mathbb{E} \left( \int_0^T \rho_t v_0, t dB_t \right). 
\]
In addition, to obtain eq. (E.7), notice the following integral property holds:
\[
\left( \omega_{l,T}^{(k(2),j(2)),(k(1),j(1))} \right)^2 = 2\omega_{l,T}^{(k(2),j(2)),(k(1),j(1))} + 4\omega_{l,T}^{(k(2),j(2)),(k(1),j(1))}. 
\]
With Lemma E.2 in our arsenal, we can eliminate the processes \( V_{1,t} \) and \( V_{2,t} \) from Proposition E.3. Using Lemma C.1 (Malliavin integration by parts) and eq. (E.4) yields
\[
2\mathbb{E} \partial_y \hat{P}_{BS} \left( \int_0^T \rho_t v_0, t dB_t \right) \int_0^T \xi_t e^{\int_0^t \alpha_x(z,v_0, \cdot) dz} \left( \int_0^t \lambda_s v_0^\mu_s e^{-\int_0^s \alpha_x(z,v_0, \cdot) dz} dB_s \right) dt 
= 2\mathbb{E} \partial_y \hat{P}_{BS}. 
\]
Finally, using Lemma E.2 on the rest of the terms in a similar way, we obtain the explicit second-order price given in Theorem 4.1.