A Forward Propagation Algorithm for Online Optimization of Nonlinear Stochastic Differential Equations

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Abstract

Optimizing over the stationary distribution of stochastic differential equations (SDEs) is computationally challenging. [26] proposed a new forward propagation algorithm for the online optimization of SDEs. The algorithm solves an SDE, derived using forward differentiation, which provides a stochastic estimate for the gradient. The algorithm continuously updates the SDE model’s parameters and the gradient estimate simultaneously. This paper studies the convergence of the forward propagation algorithm for nonlinear dissipative SDEs. We leverage the ergodicity of this class of nonlinear SDEs to characterize the convergence rate of the transition semi-group and its derivatives. Then, we prove bounds on the solution of a Poisson partial differential equation (PDE) for the expected time integral of the algorithm’s stochastic fluctuations around the direction of steepest descent. We then re-write the algorithm using the PDE solution, which allows us to characterize the parameter evolution around the direction of steepest descent. Our main result is a convergence theorem for the forward propagation algorithm for nonlinear dissipative SDEs.

1 Introduction

Optimizing over the stationary distribution of a stochastic process is a challenging mathematical and computational problem. Consider a parameterized process $X_{t}^{\theta, x} \in \mathbb{R}^{d}$ which satisfies the stochastic differential equation (SDE):

$$
\begin{align}
    dX_{t}^{\theta, x} &= \mu(X_{t}^{\theta, x}, \theta)dt + \sigma(X_{t}^{\theta, x}, \theta)dW_{t}, \\
    X_{0}^{\theta, x} &= x,
\end{align}
$$

where $\theta \in \mathbb{R}^{\ell}$ and $W_{t}$ is a $d$-dimensional standard Brownian motion. Suppose $X_{t}^{\theta, x}$ is ergodic (to be concretely specified later in the paper) with the stationary distribution $\pi_{\theta}$. Our goal is to select the parameters $\theta$ which minimize the objective function

$$
J(\theta) = \sum_{n=1}^{N} \left( E_{Y \sim \pi_{\theta}}[f_{n}(Y)] - \beta_{n} \right)^{2},
$$

where $f_{n}$ are known functions and $\beta_{n}$ are the target quantities.

Optimizing over the stationary distribution $\pi_{\theta}$ of the parameterized process (1.1) is challenging. For stochastic differential equations (SDEs), the standard approach is to solve a forward Kolmogorov partial differential equation (PDE) and its adjoint PDE at each optimization iteration. At each iteration, a gradient descent step is taken. If the SDE is high-dimensional, this method is computationally expensive or even intractable due to the curse-of-dimensionality. An alternative ad hoc optimization method is to simulate a trajectory of the SDE for a long time interval $[0, T]$ at each optimization iteration and then calculate a gradient by chain rule. The SDE must be re-simulated from scratch at each iteration and the calculated...
gradient is an approximation (with error) since $T$ is finite. Consequently, the method is computationally expensive due to constant re-simulation and furthermore has error. See [26] for a detailed description of existing methods for optimizing the class of models (1.1).

In [26], a new online algorithm was developed to optimize over the stationary distribution of SDEs such as (1.1). The online algorithm simultaneously simulates (1.1) while continuously updating the parameter $\theta_t$ using a stochastic estimate for the gradient $\nabla_\theta J(\theta_t)$. The stochastic estimate for the gradient $\nabla_\theta J(\theta_t)$ is based upon a forward propagation SDE for the gradient of $X_t^\theta$ with respect to $\theta$. [26] rigorously proves convergence of the online forward propagation algorithm for a class of linear SDEs. Numerical experiments demonstrate that the forward propagation algorithm also converges for nonlinear SDEs. In this new paper, we rigorously prove convergence of the forward propagation algorithm for a class of nonlinear SDEs.

For notational convenience (and without loss of generality), we will set $N = 1$ and $\beta_1 = \beta$ in (1.2). The online forward propagation algorithm for optimizing (1.2) is:

$$
\begin{align*}
\frac{d\theta_t}{dt} &= -2\alpha_t \left( f(\tilde{X}_t) - \beta \right) \left( \nabla f(X_t) \tilde{X}_t \right)^T, \\
\frac{d\tilde{X}_t}{d\theta_t} &= \left( \mu_x(X_t, \theta_t) \tilde{X}_t + \mu_\theta(X_t, \theta_t) \right) dt + \left( \sigma_x(X_t, \theta_t) \tilde{X}_t + \sigma_\theta(X_t, \theta_t) \right) dW_t,
\end{align*}
$$

where $W_t$ and $\tilde{W}_t$ are independent Brownian motions and $\alpha_t$ is the learning rate, $\mu_x = \frac{\partial \mu}{\partial x}$, $\mu_\theta = \frac{\partial \mu}{\partial \theta}$, $\sigma_x = \frac{\partial \sigma}{\partial x}$, and $\sigma_\theta = \frac{\partial \sigma}{\partial \theta}$. The learning rate must be chosen such that $\int_0^\infty \alpha_t ds = \infty$ and $\int_0^\infty \alpha_t^2 ds < \infty$. (An example is $\alpha_t = \frac{C}{1 + t}$.) $\tilde{X}_t$ estimates the derivative of $X_t$ with respect to $\theta_t$. The parameter $\theta_t$ is continuously updated using $(f(\tilde{X}_t) - \beta) \nabla f(X_t) \tilde{X}_t$ as a stochastic estimate for $\nabla_\theta J(\theta_t)$.

To better understand the algorithm (1.3), let us re-write the gradient of the objective function using the ergodicity of $X_t^\theta$:

$$
\begin{align*}
\nabla_\theta J(\theta) &= 2 \left( \mathbb{E}_{\sigma_\theta} f(Y) - \beta \right) \nabla_\theta \mathbb{E}_{\sigma_\theta} f(Y) \\
&\approx 2 \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T f(X_t^\theta) dt - \beta \right) \times \nabla_\theta \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T f(X_t^\theta) dt \right).
\end{align*}
$$

Define $\tilde{X}_t^\theta = \nabla_\theta X_t^\theta$, which is the solution of the following SDE:

$$
\begin{align*}
\frac{d\tilde{X}_t^\theta}{dt} &= \left( \mu_x(X_t^\theta, \theta) \tilde{X}_t^\theta + \mu_\theta(X_t^\theta, \theta) \right) dt + \left( \sigma_x(X_t^\theta, \theta) \tilde{X}_t^\theta + \sigma_\theta(X_t^\theta, \theta) \right) dW_t.
\end{align*}
$$

$\tilde{X}_t$ and $\tilde{X}_t^\theta$ satisfy the same equations, except $\theta$ is a fixed constant for $\tilde{X}_t^\theta$ while $\theta_t$ is updated continuously in time for $\tilde{X}_t$. If the derivative and the limit in (1.4) can be interchanged, the gradient can be expressed as

$$
\nabla_\theta J(\theta) = 2 \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T f(X_t^\theta) dt - \beta \right) \times \lim_{T \to \infty} \frac{1}{T} \int_0^T \nabla f(X_t^\theta) \tilde{X}_t^\theta dt,
$$

which suggests a natural stochastic estimate for $\nabla_\theta J(\theta_t)$. Specifically, the forward propagation algorithm (1.3) uses

$$
G(\theta_t) := 2 \left( f(\tilde{X}_t) - \beta \right) \nabla f(X_t) \tilde{X}_t
$$

as a stochastic estimate for $\nabla_\theta J(\theta_t)$. It is expected that $G(\theta_t)$ asymptotically converges to an unbiased estimate for the direction of steepest descent $\nabla_\theta J(\theta_t)$.

For large $t$, we expect that $\mathbb{E} \left[ f(\tilde{X}_t) - \beta \right] \approx \mathbb{E}_{\sigma_\theta} \left[ f(Y) - \beta \right]$ and $\mathbb{E} \left[ \nabla f(X_t) \tilde{X}_t \right] \approx \nabla_\theta \left( \mathbb{E}_{\sigma_\theta} \left[ f(X) - \beta \right] \right)$ since $\theta_t$ is changing very slowly as $t$ becomes large due to $\lim_{t \to \infty} \alpha_t = 0$. Furthermore, since $\tilde{X}_t$ and $X_t$ vary driven by independent Brownian motions, we expect that $\mathbb{E} \left[ 2 \left( f(\tilde{X}_t) - \beta \right) \nabla f(X_t) \tilde{X}_t \right] \approx \nabla_\theta J(\theta_t)$ for large $t$ due to $\tilde{X}_t$ and $(X_t, \tilde{X}_t)$ being close to conditionally independent since $\theta_t$ will be changing very slowly.

\footnote{In this paper’s notation, the Jacobian matrix of a vector value function $f : x \in \mathbb{R}^n \to \mathbb{R}^m$ is an $m \times n$ matrix.}
for large $t$. Thus, we expect that the stochastic sample $G(\theta_t) = 2 (f(\bar{X}_t) - \beta) \nabla f(X_t) \bar{X}_t$ will provide an asymptotically unbiased estimate for the direction of steepest descent $\nabla_\theta J(\theta_t)$ and $\theta_t$ will converge to a local minimum of the objective function $J(\theta)$.

The evolution of the parameters $\theta_t$ in (1.3) can be analyzed by decomposing the dynamics into a gradient descent term and fluctuation terms:

$$
\frac{d\theta_t}{dt} = -2\alpha_t (f(\bar{X}_t) - \beta) \left( \nabla f(X_t) \bar{X}_t \right)^\top
- 2\alpha_t (\mathbb{E}_{\pi_{\theta_t}} f(Y) - \beta) \left( \nabla f(X_t) \bar{X}_t - \nabla_\theta \mathbb{E}_{Y \sim \pi_{\theta_t}} f(Y) \right)^\top
- 2\alpha_t \left( f(\bar{X}_t) - \mathbb{E}_{\pi_{\theta_t}} f(Y) \right) \left( \nabla f(X_t) \bar{X}_t \right)^\top .
$$

(1.8)

We remark here that the transpose in (1.8) is due to $\nabla f$ being a row vector. In [26], convergence of the algorithm (1.3) was proven for linear SDEs.

The main goal of this paper is to rigorously prove the convergence of algorithm (1.3) for a class of nonlinear SDEs (1.1). The mathematical approach uses a Poisson partial differential equation (PDE), such as in [11, 12], to rewrite the fluctuation terms in terms of the solution of the PDEs. The fluctuation terms can be appropriately bounded by proving bounds on the solution to the PDEs. We leverage recent methods from [17] to characterize the convergence rate of the transition semi-group for (1.1) and its derivatives with respect to the initial condition $x$ and the parameter $\theta$, which combined with the moment stability for (1.1), allow us to prove there exists appropriately bounded solutions to the PDE. Once the fluctuation terms have been bounded, using the moment stability for the coupled system (1.3), we can prove convergence of the forward propagation algorithm using the cycle of stopping times argument [1, 20, 22].

### 1.1 Contributions of this Paper

We rigorously prove the convergence of the algorithm (1.3) when $\mu, \sigma$ satisfy the standard dissipative condition and their derivatives are uniformly bounded. Unlike in the traditional stochastic gradient descent algorithm, (1.3) is a fully coupled system and thus the data is not i.i.d. (i.e., $X_t$ is correlated with $X_s$ for $s \neq t$) and, for a finite time $t$, the stochastic update direction $G(\theta_t)$ is not an unbiased estimate of $\nabla_\theta J(\theta_t)$.

Thus one needs to carefully analyze the fluctuations of the stochastic update direction $G(\theta_t)$ around $\nabla_\theta J(\theta_t)$ and prove the stochastic fluctuations vanish in an appropriate way as $t \to \infty$.

Bounds on the fluctuations are challenging to obtain due to the online nature of the algorithm. The stationary distribution $\pi_{\theta_t}$ will continuously change as the parameters $\theta_t$ evolve. Unlike in [26], which studies linear SDEs whose probability density can be characterized via a closed-form formula, in this paper we study nonlinear SDEs and thus more complex calculations are required. The dissipativity of the drift and diffusion terms in (1.1) and the uniform boundedness for their derivatives lead to an exponential convergence rate for the transition semi-group of (1.1) and its derivatives with respect to the initial point $x$ and the parameter $\theta$. A Poisson PDE is constructed for the infinitesimal generator of a certain SDE system: the original nonlinear SDE and an SDE for its derivative. We prove there exists a solution to this Poisson PDE and, furthermore, the solution satisfies a polynomial bound. Then, we are able to analyze the parameter fluctuations in the online forward algorithm using the solution to the Poisson PDE. The fluctuations are re-written in terms of the solution to the Poisson PDEs using Ito’s formula, the bounds for the PDE solutions are subsequently applied, and finally we can show asymptotically that the fluctuations appropriately vanish. Our main theorem proves for nonlinear dissipative SDEs that:

$$
\lim_{t \to \infty} \left| \nabla_\theta J(\theta_t) \right|^2 \overset{\text{a.s.}}{=} 0.
$$

(1.9)
1.2 Literature Review

Recent articles such as [9, 19, 20, 21, 23, 25, 26] have studied continuous-time stochastic gradient descent. Our paper has several important differences as compared to [9, 19, 20, 21, 23]. These previous papers estimate the parameter \( \theta \) for the SDE \( X^\theta_t \) from observations of \( X^\theta_T \) where \( \theta^* \) is the true parameter. In this paper, our goal is to select \( \theta \) such that the stationary distribution of \( X^\theta_t \) matches certain target statistics. Therefore, unlike the previous papers, we are directly optimizing over the stationary distribution of \( X^\theta_t \). Examples of this optimization problem can be found in ergodic stochastic control, bayesian statistics, and reinforcement learning [6, 7, 24].

The analysis in this paper is also related to the literature on multi-scale models and their averaging principle, which arises naturally in many applications in material sciences, chemistry, fluid dynamics, biology, ecology, and climate dynamics (see [15, 27, 28]). There exist two components in multi-scale models, where one evolves much faster than the other. Existing averaging results for multi-scale SDEs can be found in [2, 3, 4, 8, 10, 17, 18]. In these articles, Poisson equation techniques play an important role in analyzing the fluctuations of the fast SDE around the more slowly changing SDE, which has some similarity to the fluctuations of our stochastic gradient estimate \( G(\theta_t) \) in (1.7) around the true gradient \( \nabla J(\theta_1) \). However, in this paper we study a completely new multi-scale system for a novel application: an online, stochastic algorithm for optimizing over the stationary distribution of an SDE.

The presence of the \( X \) process in (1.3) makes the mathematical analysis challenging as the \( X \) term introduces correlation across times, and this correlation does not disappear as time tends to infinity. In order for the algorithm to converge, the fluctuation terms in (1.8) need to decay sufficiently rapidly; we will prove this using the exponential ergodicity of the transition semi-group of \( X^{\theta,x}_t \) and its derivatives with respect to the initial \( x \) and the parameter \( \theta \). [17, 18] use the Poisson equation technique to study the strong convergence rate of the slow component in the multi-scale models to its averaged equation. However, the theoretical results from [11, 12, 17, 18, 20, 21] do not apply to the multi-scale SDE system nor the corresponding Poisson PDE considered in this paper since the diffusion term in our PDE is not uniformly elliptic. This is a direct result of the process \( \tilde{X}_t \) in (1.3), which shares the same Brownian motion with the process \( X_t \). Consequently, we must analyze a new class of Poisson PDEs as in [26]. Under the dissipative condition for nonlinear SDEs, we prove there exists a solution to this new class of Poisson PDEs which satisfies a polynomial bound. The bound on the solution is crucial for analyzing the fluctuations of the parameter evolution in the algorithm (1.3).

1.3 Organization of Paper

The paper is organized into three main sections. In Section 2, we present the assumptions and the main theorem. Section 3 rigorously proves the convergence of the online forward propagation algorithm for nonlinear dissipative SDEs.

2 Main Results

We will study the convergence of the algorithm (1.3) for a class of nonlinear SDEs satisfying the following conditions.

A1. (Condition on \( \mu \) and \( \sigma \)) There exist constants \( C, \beta > 0 \) such that the following conditions hold for all \( x_1, x_2 \in \mathbb{R}^d, \theta_1, \theta_2 \in \mathbb{R}^\ell \):

- Lipschitz continuity:
  \[
  |\mu(x_1, \theta_1) - \mu(x_2, \theta_2)| + |\sigma(x_1, \theta_1) - \sigma(x_2, \theta_2)| \leq C (|x_1 - x_2| + |\theta_1 - \theta_2|).
  \]  
  (2.1)

- Dissipativity:
  \[
  \langle \mu(x_1, \theta) - \mu(x_2, \theta), x_1 - x_2 \rangle + \frac{\beta}{2} |\sigma(x_1, \theta) - \sigma(x_2, \theta)|^2 \leq -\beta |x_1 - x_2|^2,
  \]  
  (2.2)

where \( \langle a, b \rangle := b^\top a \).
• Uniformly bounded with respect to $\theta$ at $x = 0:
\sup_{\theta \in \mathbb{R}^l} \max\{|\mu(0, \theta)|, |\sigma(0, \theta)|\} \leq C. \quad (2.3)

A2. (Conditions on first-order partial derivatives) The first-order partial derivatives $\nabla_x \mu(x, \theta), \nabla_\theta \mu(x, \theta), \nabla_x \sigma(x, \theta)$, and $\nabla_\theta \sigma(x, \theta)$ exist for any $(x, \theta) \in \mathbb{R}^d \times \mathbb{R}^l$. For any $x_1, x_2 \in \mathbb{R}^d$,
\begin{align*}
&\sup_{\theta \in \mathbb{R}^l} |\nabla_x \mu(x_1, \theta) - \nabla_x \mu(x_2, \theta)| \leq C|x_1 - x_2|, \quad (2.4) \\
&\sup_{\theta \in \mathbb{R}^l} |\nabla_\theta \mu(x_1, \theta) - \nabla_\theta \mu(x_2, \theta)| \leq C|x_1 - x_2|, \quad (2.5) \\
&\sup_{\theta \in \mathbb{R}^l} |\nabla_x \sigma(x_1, \theta) - \nabla_x \sigma(x_2, \theta)| \leq C|x_1 - x_2|, \quad (2.6) \\
&\sup_{\theta \in \mathbb{R}^l} |\nabla_\theta \sigma(x_1, \theta) - \nabla_\theta \sigma(x_2, \theta)| \leq C|x_1 - x_2|. \quad (2.7)
\end{align*}

A3. (Conditions on higher-order partial derivatives) The second-order partial derivatives $\nabla_x^2 \mu(x, \theta), \nabla_\theta^2 \mu(x, \theta), \nabla_x \nabla_\theta \mu(x, \theta)$, and $\nabla_x^2 \nabla_\theta \mu(x, \theta)$ exist for any $(x, \theta) \in \mathbb{R}^d \times \mathbb{R}^l$. For any $x_1, x_2 \in \mathbb{R}^d$,
\begin{align*}
&\sup_{\theta \in \mathbb{R}^l} |\nabla_x^2 \mu(x_1, \theta) - \nabla_x^2 \mu(x_2, \theta)| \leq C|x_1 - x_2|, \quad (2.8) \\
&\sup_{\theta \in \mathbb{R}^l} |\nabla_\theta^2 \mu(x_1, \theta) - \nabla_\theta^2 \mu(x_2, \theta)| \leq C|x_1 - x_2|, \quad (2.9) \\
&\sup_{\theta \in \mathbb{R}^l} |\nabla_x \nabla_\theta \mu(x_1, \theta) - \nabla_x \nabla_\theta \mu(x_2, \theta)| \leq C|x_1 - x_2|, \quad (2.10) \\
&\sup_{\theta \in \mathbb{R}^l} |\nabla_x^2 \nabla_\theta \mu(x_1, \theta) - \nabla_x^2 \nabla_\theta \mu(x_2, \theta)| \leq C|x_1 - x_2|. \quad (2.11)
\end{align*}

Furthermore, if $\mu$ is replaced by $\sigma$, the properties (2.8) - (2.11) also hold, and
\begin{align*}
&\sup_{(x, \theta) \in \mathbb{R}^d \times \mathbb{R}^l} \max\{|\nabla_x^2 \mu(x, \theta)|, |\nabla_\theta^2 \mu(x, \theta)|, |\nabla_x \nabla_\theta \mu(x, \theta)|, |\nabla_x^2 \nabla_\theta \mu(x, \theta)|\} \leq C, \\
&\sup_{(x, \theta) \in \mathbb{R}^d \times \mathbb{R}^l} \max\{|\nabla_x^2 \sigma(x, \theta)|, |\nabla_\theta^2 \sigma(x, \theta)|, |\nabla_x \nabla_\theta \sigma(x, \theta)|, |\nabla_x^2 \nabla_\theta \sigma(x, \theta)|\} \leq C. \quad (2.12)
\end{align*}

A4. The function $f$ in the objective function is continuously differentiable and has uniformly bounded derivatives, i.e. there exists a constant $C$ such that
\begin{equation}
|\nabla^i f(x)| \leq C, \quad \forall x \in \mathbb{R}^d, \ i = 1, 2, 3. \quad (2.13)
\end{equation}

A5. The learning rate $\alpha_t$ satisfies $\int_0^\infty \alpha_t dt = \infty$, $\int_0^\infty \alpha_t^2 dt < \infty$, $\int_0^\infty |\alpha_t'| ds < \infty$, and there is a $p > 0$ such that $\lim_{t \to \infty} \alpha_t^{-p+2p} = 0$.

Remark 2.1. Our assumptions are standard in the mathematical literature which studies ergodic SDEs. We briefly comment on these assumptions before presenting our main theoretical result.

• A sufficient condition for the dissipative SDE (1.1) to satisfy Assumptions A1-A3 is that the first-, second-, and third-order derivatives of $\mu, \sigma$ are uniformly bounded and for any $x, y \in \mathbb{R}^d, \theta \in \mathbb{R}^l$
\begin{equation}
y^\top \nabla_x \mu(x, \theta)y \leq -C|y|^2, \quad |\sigma(x, \theta) - \sigma(y, \theta)| \leq L|x - y|, \quad (2.14)
\end{equation}

where $C, L > 0$ are constants and $\frac{\alpha}{4}L^2 < C$. A classic example is the Langevin Equation, where the drift term is the gradient of some convex potential. That is $\mu(x, \theta) = -\nabla V(x, \theta)$ with $V(x, \theta)$ being convex with respect to $x$. See [13, 14] for a detailed discussion.

• Conditions (2.1) and (2.3) imply that there exists a constant $C > 0$ such that, for any $x \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^l$,
\begin{equation}
|\mu(x, \theta)| + |\sigma(x, \theta)| \leq C (1 + |x|). \quad (2.15)
\end{equation}
• Conditions (2.2) and (2.3) imply that there exists a constant $C > 0$ such that, for any $x \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^\ell$,

$$2\langle \mu(x, \theta), x \rangle + 7|\sigma(x, \theta)|^2 \leq -\beta|x|^2 + C.$$  \hspace{1cm} (2.16)

The derivation of the above inequality is:

\[
2\langle \mu(x, \theta), x \rangle + 7|\sigma(x, \theta)|^2 \\
= 2\langle \mu(x, \theta) - \mu(0, \theta), x \rangle + 2\langle \mu(0, \theta), x \rangle + 7|\sigma(x, \theta) - \sigma(0, \theta) + \sigma(0, \theta)|^2 \\
\leq -2\beta|x|^2 + 2\langle \mu(0, \theta), x \rangle + 7|\sigma(0, \theta)|^2 + 14|\sigma(x, \theta) - \sigma(0, \theta)| \cdot |\sigma(0, \theta)| \\
\leq -\beta|x|^2 + C \left( |\mu(0, \theta)|^2 + |\sigma(0, \theta)|^2 \right),
\]

where step (a) uses the inequality (2.2) and step (b) use Young’s inequality and the inequality (2.1).

Condition (2.2) is used to prove the solution of dynamic (1.3) has uniformly bounded fourth moment.

• Assumption (A1) guarantees (see Theorem 4.3.9 of [16]) that there exists a unique invariant measure $\pi_\theta$ for (1.1) such that

$$\int_{\mathbb{R}^d} |x| \pi_\theta(dx) \leq C < \infty.$$  \hspace{1cm} (2.18)

Under these assumptions, we are able to prove the convergence of the online forward algorithm (1.3).

**Theorem 2.2.** Under Assumptions (A1) - (A5) and for the SDE system (1.3), we have

$$\lim_{t \to \infty} \|\nabla \theta J(\theta_t)\|_{\mathbb{R}^d} = 0.$$  \hspace{1cm} (2.19)

### 3 Proof

The SDE system (1.3) has a unique strong solution.\(^2\) In equation (1.8), we decomposed the evolution of $\theta_t$ into the direction of steepest descent $-\alpha_t \nabla \theta J(\theta_t)$ and two fluctuation terms. Define the fluctuation terms as

$$Z_1^t = (E_{\pi_{\theta_t}} f(Y) - \beta) \left( \nabla f(X_t) \bar{X}_t - \nabla \theta E_{\pi_{\theta_t}} f(Y) \right) \top,$$

$$Z_2^t = \left( f(\bar{X}_t) - E_{\pi_{\theta_t}} f(Y) \right) \left( \nabla f(X_t) \bar{X}_t \right) \top.$$  \hspace{1cm} (3.1)

As in [20], we will study a cycle of stopping times to control the time periods where $|\nabla \theta J(\theta_t)|$ is close to zero and away from zero. Let us select an arbitrary constant $\kappa > 0$ and also define $\mu = \mu(\kappa) > 0$ (to be chosen later). Then, set $\sigma_0 = 0$ and define the cycles of random times

$$0 = \sigma_0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq \ldots,$$

where the stopping times are defined as

$$\tau_n = \inf \{ t > \sigma_{n-1} : |\nabla \theta J(\theta_t)| \geq \kappa \}$$

$$\sigma_n = \sup \left\{ t \geq \tau_n : \frac{|\nabla \theta J(\theta_s)|}{2} \leq |\nabla \theta J(\theta_s)| \leq 2 |\nabla \theta J(\theta_{\tau_n})| \right\}$$

for all $s \in [\tau_n, t]$ and $\int_{\tau_n}^{\tau_{n+1}} \alpha_s ds \leq \mu.$  \hspace{1cm} (3.2)

We define the random time intervals $J_n = [\sigma_{n-1}, \tau_n]$ and $I_n = [\tau_n, \sigma_n]$. We introduce the constant $\eta > 0$ which will be chosen to be sufficiently small later. In order to prove convergence, we will have to show that the fluctuation terms become small as $t \to \infty$. In particular, the following integral of the fluctuation term will be crucial to the convergence analysis:

$$\Delta_{\tau_n, \sigma_n + \eta}^i := \int_{\tau_n}^{\sigma_n + \eta} \alpha_s Z_s^i ds, \quad i = 1, 2.$$  \hspace{1cm} (3.3)

We will begin our analysis by first presenting several lemmas regarding Lipschitz continuity, moment bounds, and ergodicity. The proofs are the same as in [17] and thus we omit them.

\(^2\)Existence and uniqueness can be proven using the standard method of a contraction map; see Theorem 1.2 of [5] for details.
Lemma 3.1 (Lipschitz continuity). For any $t > 0$, $x_i \in \mathbb{R}^d$, and $\theta_i \in \mathbb{R}^\ell$, we have

$$E \left| X_{t}^{\theta_i,x_i} - X_{t}^{\theta_2,x_2} \right|^2 \leq e^{-\beta t} |x_1 - x_2|^2 + C |\theta_1 - \theta_2|^2. \tag{3.4}$$

A proof can be found in Lemma 3.6 of [17].

Lemma 3.2 (Ergodicity). For any $t \geq 0$, $x \in \mathbb{R}^d$, and $\theta \in \mathbb{R}^\ell$,

$$\left| E f(X_t^{\theta,x}) - E_{\pi_\theta} f(Y) \right| \leq C e^{-\frac{\beta^2 t}{2}} (1 + |x|). \tag{3.5}$$

A proof can be found in Proposition 3.7 of [17].

Lemma 3.3 (Moment Bound). There exists a constant $C$ such that

$$E \left| X_{t}^{\theta,x} \right|^2 \leq C (1 + e^{-\beta t} |x|^2), \quad \forall x \in \mathbb{R}^d, t \geq 0. \tag{3.6}$$

Proof. Using Itô’s formula to $\left| X_{t}^{\theta,x} \right|^2$, we have

$$\frac{d}{dt} E \left| X_{t}^{\theta,x} \right|^2 = E \left[ 2 \left( \mu(X_t^{\theta,x}, \theta), X_t^{\theta,x} \right) + \left| \sigma(X_t^{\theta,x}, \theta) \right|^2 \right]$$

$$\leq E \left[ 2 \left( \mu(X_t^{\theta,x}, \theta) - \mu(0, \theta), X_t^{\theta,x} \right) + 2 \left| \sigma(X_t^{\theta,x}, \theta) - \sigma(0, \theta) \right|^2 \right]$$

$$\leq -2\beta E \left| X_t^{\theta,x} \right|^2 + \left( \beta E \left| X_t^{\theta,x} \right|^2 + \frac{1}{\beta} |\mu(0, \theta)|^2 \right) + 2 |\sigma(0, \theta)|^2$$

$$\leq -\beta E \left| X_t^{\theta,x} \right|^2 + C, \tag{3.7}$$

where step (a) uses the dissipativity assumption (2.2) and Young’s inequality and step (b) uses the bound (2.3). Therefore, using a comparison principle for ODEs,

$$E \left| X_{t}^{\theta,x} \right|^2 \leq e^{-\beta t} |x|^2 + C. \tag{3.8}$$

□

Using similar calculations as in Proposition 4.1 of [17], several ergodicity results for $X_t^{\theta}$ can be proven.

Proposition 3.4. Under Assumptions (A1) - (A4), we have the following ergodic bounds:

(i) There exists a constant $C$ such that for any $\theta \in \mathbb{R}^\ell$, $x \in \mathbb{R}^d$, and $t > 0$,

$$\left| \nabla_\theta E f(X_t^{\theta,x}) - \nabla_\theta E_{\pi_\theta} f(Y) \right| \leq C e^{-\beta t} (1 + |x|), \quad i = 0, 1, 2. \tag{3.9}$$

(ii) There exists a constant $C > 0$ such that for any $\theta \in \mathbb{R}^\ell$ and $i = 0, 1, 2$,

$$\left| \nabla_\theta E_{\pi_\theta} f(Y) \right| \leq C. \tag{3.10}$$

(iii) There exists constants $C, \gamma > 0$ such that for any $\theta \in \mathbb{R}^\ell$, $x \in \mathbb{R}^d$, and $t > 0$,

$$\left| \nabla_i \nabla_j E f(X_t^{\theta,x}) \right| \leq C e^{-\gamma t}, \quad i = 0, 1, \quad j = 1, 2. \tag{3.11}$$
The proof method for Proposition 3.4 is the same as in Proposition 4.1 of [17], although we need the convergence result for higher-order derivatives in (3.11). For completeness, we provide the detailed proof for all orders of the derivatives in Appendix A.

We next prove that a solution exists to a Poisson equation for the fluctuation terms and, furthermore, that the solution satisfies certain polynomial bounds. We first introduce the process $\tilde{X}_t^{\theta,x,\tilde{x}}$, which satisfies the SDE:

\[
\begin{aligned}
&d\tilde{X}_t^{\theta,x,\tilde{x}} = \left[ \nabla_x \mu(X_t^{\theta,x}, \theta) \tilde{X}_t^{\theta,x,\tilde{x}} + \nabla_\theta \mu(X_t^{\theta,x}, \theta) \right] dt + \left[ \nabla_x \sigma(X_t^{\theta,x}, \theta) \tilde{X}_t^{\theta,x,\tilde{x}} + \nabla_\theta \sigma(X_t^{\theta,x}, \theta) \right] dW_t, \\
&\tilde{X}_0^{\theta,x,\tilde{x}} = \tilde{x},
\end{aligned}
\]

where the Brownian is the same as in (1.1). It should be noted that $\tilde{X}_t^{\theta,x,0} = \nabla_\theta X_t^{\theta,x}$ almost surely.

**Lemma 3.5.** Define the error function

\[
G^1(x, \tilde{x}, \theta) = (\mathbb{E}_{\pi_\theta} f(Y) - \beta) (\nabla f(x) \tilde{x} - \nabla_\theta \mathbb{E}_{\pi_\theta} f(Y))^\top
\]

and the function

\[
v^1(x, \tilde{x}, \theta) = -\int_0^\infty \mathbb{E} G^1(X_t^{\theta,x}, \tilde{X}_t^{\theta,x,\tilde{x}}, \theta) dt.
\]

Let $L_{x,\tilde{x}}^\theta$ denote the infinitesimal generator of the process $(X_t^{\theta,x}, \tilde{X}_t^{\theta,x,\tilde{x}})$, i.e. for any test function $\varphi$

\[
L_{x,\tilde{x}}^\theta \varphi(x, \tilde{x}) = L_{x,\tilde{x}}^\theta \varphi(x, \tilde{x}) + \sum_{k=1}^\ell L_{x,\tilde{x}}^\theta \varphi(x, \tilde{x})
\]

\[
+ \sum_{j=1}^\ell \left( \nabla_{\tilde{x}^j} \nabla_x \varphi(x, \tilde{x}) \sigma(x, \theta) \left( \nabla_x \sigma(x, \theta) \tilde{x}^j + \frac{\partial \sigma(x, \theta)}{\partial \theta_j} \right) \right)
\]

\[
+ \sum_{j<k} \left( \nabla_{\tilde{x}^j} \nabla_{\tilde{x}^k} \varphi(x, \tilde{x}) \left( \nabla_x \sigma(x, \theta) \tilde{x}^j + \frac{\partial \sigma(x, \theta)}{\partial \theta_j} \right) \nabla_x \sigma(x, \theta) \tilde{x}^k + \frac{\partial \sigma(x, \theta)}{\partial \theta_k} \right)
\]

where $\tilde{x}^k$ for $k \in \{1, \ldots, \ell\}$ is the $k$-th column of $\tilde{x}$.

Then, under Assumptions (A1) - (A4), $v^1(x, \tilde{x}, \theta)$ is the classical solution of the Poisson equation

\[
L_{x,\tilde{x}}^\theta u(x, \tilde{x}, \theta) = G^1(x, \tilde{x}, \theta),
\]

where $u = (u_1, \ldots, u_\ell)^\top \in \mathbb{R}_\ell$ is a vector, $L_{x,\tilde{x}}^\theta u(x, \tilde{x}, \theta) = (L_{x,\tilde{x}}^\theta u_1(x, \tilde{x}, \theta), \ldots, L_{x,\tilde{x}}^\theta u_\ell(x, \tilde{x}, \theta))^\top$. Furthermore, the solution $v^1$ satisfies the bound

\[
|v^1(x, \tilde{x}, \theta)| + |
abla \theta v^1(x, \tilde{x}, \theta)| + |\nabla_x v^1(x, \tilde{x}, \theta)| + |\nabla_{\tilde{x}} v^1(x, \tilde{x}, \theta)| \leq C (1 + |x| + |\tilde{x}|),
\]

where $C > 0$ is a constant which does not depend upon $(x, \tilde{x}, \theta)$.

**Proof.** We begin by proving that the integral (3.14) is finite. We divide (3.14) into two terms:

\[
v^1(x, \tilde{x}, \theta) = (\mathbb{E}_{\pi_\theta} f(Y) - \beta) \int_0^\infty \left( \nabla_\theta \mathbb{E}_{\pi_\theta} f(Y) - \mathbb{E} \nabla f(X_t^{\theta,x}) \tilde{X}_t^{\theta,x,\tilde{x}} \right) dt
\]

\[
= (\mathbb{E}_{\pi_\theta} f(Y) - \beta) \left[ \int_0^\infty \left( \nabla_\theta \mathbb{E}_{\pi_\theta} f(Y) - \nabla_\theta \mathbb{E} f(X_t^{\theta,x}) \right)^\top dt + \int_0^\infty \left( \nabla_\theta f(X_t^{\theta,x}) - \nabla_\theta f(X_t^{\theta,x}) \tilde{X}_t^{\theta,x,\tilde{x}} \right)^\top dt \right]
\]

\[
=: v^{1.1}(x, \theta) + v^{1.2}(x, \tilde{x}, \theta).
\]
We first bound $v^{1,1}(x, \theta)$. Following the method in Lemma 3.3 of [26], we have by Proposition 3.4 and the dominated convergence theorem (DCT) that:

\[
|v^{1,1}(x, \theta)| \leq C \int_0^\infty \left| \nabla_\theta \mathbb{E}_{x, \theta} f(Y) - \nabla_\theta \mathbb{E} f(X^{\theta, X}_t) \right| dt \leq C(1 + |x|),
\]

\[
|\nabla_\theta v^{1,1}(x, \theta)| \leq C \int_0^\infty \left| \nabla_\theta \mathbb{E}_{x, \theta} f(Y) - \nabla_\theta \mathbb{E} f(X^{\theta, X}_t) \right| + C \int_0^\infty \left| \nabla_\theta^2 \mathbb{E}_{x, \theta} f(Y) - \nabla_\theta^2 \mathbb{E} f(X^{\theta, X}_t) \right| \leq C(1 + |x|),
\]

\[
|\nabla_x v^{1,1}(x, \theta)| \leq C \int_0^\infty \left| \nabla_x \mathbb{E}_{x, \theta} f(Y) - \nabla_x \mathbb{E} f(X^{\theta, X}_t) \right| dt \leq C, \quad i = 1, 2.
\]

(3.18)

For $v^{1,2}(x, \tilde{x}, \theta)$, define

\[ Z_t = \tilde{X}^{\theta, x, \tilde{x}_1}_t - \tilde{X}^{\theta, x, \tilde{x}_2}_t. \]

We can derive a differential inequality for $Z_i^{k, k}$, the $k$-th column of $Z_t$, using the inequality (A.9):

\[
\frac{d}{dt} \mathbb{E} \left| Z_i^{k, k} \right|^2 \leq \mathbb{E} \left[ 2 \left( \nabla_x \mathbb{E}(X^{\theta, x}_t, \theta) Z_i^{k, k} \right) + \left| \nabla_x \mathbb{E}(X^{\theta, x}_t, \theta) Z_i^{k, k} \right| \right] \leq -\beta \mathbb{E} \left| Z_i^{k, k} \right|^2,
\]

(3.19)

where step (a) is by using Itô’s formula to $\left| Z_i^{k, k} \right|^2$. Therefore, we can prove the exponential decay:

\[
\mathbb{E} \left| \tilde{X}^{\theta, x, \tilde{x}_1}_t - \tilde{X}^{\theta, x, \tilde{x}_2}_t \right|^2 \leq C e^{-\beta t} \left| \tilde{x}_1 - \tilde{x}_2 \right|^2,
\]

\[
\mathbb{E} \left| \nabla_x \tilde{X}^{\theta, x, \tilde{x}}_t \right|^2 \leq C e^{-\beta t}.
\]

(3.20)

Let $\tilde{X}^{\theta, x, \tilde{x}, \cdots, k}_t$ denote the $k$-th column of the matrix $\tilde{X}^{\theta, x, \tilde{x}}_t$ and for any $m \in \{1, \cdots, d\}$, $n \in \{1, \cdots, \ell\}$, we know

\[
d \frac{\partial \tilde{X}^{\theta, x, \tilde{x}, \cdots, k}_t}{\partial \tilde{x}^{m, n}} = \nabla_x \mathbb{E}(X^{\theta, x}_t, \theta) \frac{\partial \tilde{X}^{\theta, x, \tilde{x}, \cdots, k}_t}{\partial \tilde{x}^{m, n}} dt + \nabla_x \mathbb{E}(X^{\theta, x}_t, \theta) \frac{\partial \tilde{X}^{\theta, x, \tilde{x}, \cdots, k}_t}{\partial \tilde{x}^{m, n}} dW_t,
\]

(3.21)

where $\tilde{x}^{m, n}$ denotes the $(m, n)$ element of the matrix $\tilde{x}$. Let

\[
\tilde{Z}_1 = \frac{\partial \tilde{X}^{\theta, x, \tilde{x}_1, \cdots, k}_t}{\partial \tilde{x}^{m, n}} - \frac{\partial \tilde{X}^{\theta, x, \tilde{x}_2, \cdots, k}_t}{\partial \tilde{x}^{m, n}}, \quad \tilde{Z}_2 = \frac{\partial \tilde{X}^{\theta, x, \tilde{x}_1, \cdots, k}_t}{\partial \tilde{x}^{m, n}} - \frac{\partial \tilde{X}^{\theta, x, \tilde{x}_2, \cdots, k}_t}{\partial \tilde{x}^{m, n}}.
\]

Note that $\tilde{Z}_1$ satisfies the SDE

\[
d \tilde{Z}_1 = \nabla_x \mathbb{E}(X^{\theta, x}_t, \theta) \tilde{Z}_1 dt + \nabla_x \mathbb{E}(X^{\theta, x}_t, \theta) \tilde{Z}_1 dW_t
\]

(3.22)

Similar to (3.19), we can get

\[
\frac{d}{dt} \mathbb{E} \left| \tilde{Z}_1 \right|^2 \leq -\beta \mathbb{E} \left| \tilde{Z}_1 \right|^2
\]

(3.23)

which derives

\[
\mathbb{E} \left| \nabla_x \tilde{X}^{\theta, x, \tilde{x}_1}_t - \nabla_x \tilde{X}^{\theta, x, \tilde{x}_2}_t \right|^2 \leq C e^{-\beta t} \left| \tilde{x}_1 - \tilde{x}_2 \right|^2,
\]

\[
\mathbb{E} \left| \nabla_x^2 \tilde{X}^{\theta, x, \tilde{x}}_t \right|^2 \leq C e^{-\beta t}.
\]

(3.24)

Then for $\tilde{Z}_2$

\[
\frac{d}{dt} \mathbb{E} \left| \tilde{Z}_2 \right|^2 \leq -\beta \mathbb{E} \left| \tilde{Z}_2 \right|^2
\]

(3.25)
and as in (A.15)

\[
\frac{d}{dt} \mathbb{E} |\tilde{Z}_t|^2 = \mathbb{E} \left[ 2 \left( \nabla_x \mu(X_t^{\theta, x_1}, \theta) \frac{\partial \tilde{X}_t^{\theta, x_1, \tilde{\xi}, k}}{\partial \tilde{\xi}_m} - \nabla_x \mu(X_t^{\theta, x_2}, \theta) \frac{\partial \tilde{X}_t^{\theta, x_2, \tilde{\xi}, k}}{\partial \tilde{\xi}_m} \right), \tilde{Z}_t^2 \right] + \mathbb{E} \left[ \nabla_x \sigma(X_t^{\theta, x_1}, \theta) \frac{\partial \tilde{X}_t^{\theta, x_1, \tilde{\xi}, k}}{\partial \tilde{\xi}_m} - \nabla_x \sigma(X_t^{\theta, x_2}, \theta) \frac{\partial \tilde{X}_t^{\theta, x_2, \tilde{\xi}, k}}{\partial \tilde{\xi}_m} \right]^2 \right] \\
\leq \mathbb{E} \left[ 2 \left( \nabla_x \mu(X_t^{\theta, x_1}, \theta) \tilde{Z}_t^1 + \tilde{Z}_t^2 \right) \right] + 2 \left[ \nabla_x \sigma(X_t^{\theta, x_1}, \theta) \tilde{Z}_t^1 \right]^2 + \beta \mathbb{E} |\tilde{Z}_t^1|^2 + CE |X_t^{\theta, x_1} - X_t^{\theta, x_2}|^2 \\
\leq - \beta \mathbb{E} |\tilde{Z}_t^1|^2 + C e^{-\beta t} |x_1 - x_2|^2,
\]

which derives

\[
\mathbb{E} \left[ \nabla_x \tilde{X}_t^{\theta, x_1, \tilde{\xi}} - \nabla_x \tilde{X}_t^{\theta, x_2, \tilde{\xi}} \right]^2 \leq C e^{-\beta t} |x_1 - x_2|^2,
\]

\[
\mathbb{E} \left| \nabla_x \nabla_x \tilde{X}_t^{\theta, x_1, \tilde{\xi}} \right|^2 \leq C e^{-\beta t}.
\]

Combining (3.20), (3.24) and (3.36) we can establish bounds on \( v^{1,2}(x, \tilde{x}, \theta) \).

\[
|v^{1,2}(x, \tilde{x}, \theta)| \leq C \int_0^\infty \mathbb{E} \left| \nabla f(X_t^{\theta, x}) \left( \tilde{X}_t^{\theta, x, \tilde{\xi}} - \tilde{X}_t^{\theta, x, 0} \right) \right| dt \leq \int_0^\infty C e^{-\beta t} |\tilde{x}| dt \leq C |\tilde{x}|
\]

\[
|\nabla \tilde{v}^{1,2}(x, \tilde{x}, \theta)| \leq C \int_0^\infty \mathbb{E} \left| \nabla \tilde{X}_t^{\theta, x, \tilde{\xi}} \right| dt \leq \int_0^\infty C e^{-\beta t} dt \leq C, \quad i = 1, 2
\]

\[
|\nabla_x \nabla \tilde{v}^{1,2}(x, \tilde{x}, \theta)| \leq C \int_0^\infty \mathbb{E} \left[ \left| \nabla_x^2 f(X_t^{\theta, x}) \nabla_x \tilde{X}_t^{\theta, x, \tilde{\xi}} \right| \right] dt + C \int_0^\infty \mathbb{E} \left| \nabla_x \nabla_{\tilde{x}} \tilde{X}_t^{\theta, x, \tilde{\xi}} \right| dt \leq C
\]

where in step (a) we use the fact \( \nabla \theta X_t^{\theta, x, a, q} = X_t^{\theta, x, 0} \).

The analysis of \( \nabla \tilde{v}^{1,2} \) for \( i = 1, 2 \) and \( \nabla \tilde{v}^{1,2} \) is similar to the calculations for \( v^{1,1} \). Define

\[
\tilde{Z}_t = \nabla_x \tilde{X}_t^{\theta, x, \tilde{\xi}} = \nabla_x \tilde{X}_t^{\theta, x, \tilde{\xi}}.
\]

\( \tilde{Z}_t \) satisfies the SDE:

\[
d\tilde{Z}_t = \left( \langle \nabla_x^2 \mu(X_t^{\theta, x}, \theta) \nabla_{\theta \theta} X_t^{\theta, x}, Z_t \rangle + \nabla_x \nabla_{\theta \theta} \mu(X_t^{\theta, x}, \theta) Z_t + \nabla_x \mu(X_t^{\theta, x}, \theta) \tilde{Z}_t \right) dt,
\]

\[
+ \left( \langle \nabla_x^2 \sigma(X_t^{\theta, x}, \theta) \nabla_{\theta \theta} X_t^{\theta, x}, Z_t \rangle + \nabla_x \nabla_{\theta \theta} \sigma(X_t^{\theta, x}, \theta) Z_t + \nabla_x \sigma(X_t^{\theta, x}, \theta) \tilde{Z}_t \right) dW_t,
\]

where \( \langle , \rangle \) in the equation above is defined as:

\[
\langle \nabla_x^2 \mu(X_t^{\theta, x}, \theta) \nabla_{\theta \theta} X_t^{\theta, x}, Z_t \rangle \rangle_{m,p,q} = \left. \nabla_x^2 \mu_m(X_t^{\theta, x}, \theta) \frac{\partial X_t^{\theta, x}}{\partial \theta_q}, Z_t^{p} \right),
\]

\[
\langle \nabla_x^2 \sigma(X_t^{\theta, x}, \theta) \nabla_{\theta \theta} X_t^{\theta, x}, Z_t \rangle \rangle_{m,p,q} = \left. \nabla_x^2 \sigma_m(X_t^{\theta, x}, \theta) \frac{\partial X_t^{\theta, x}}{\partial \theta_q}, Z_t^{p} \right).
\]

Thus by the same calculations as in (A.15) and the uniform bounds for the derivatives of \( \mu, \sigma \), we can derive the differential inequality:

\[
\frac{d}{dt} \mathbb{E} |\tilde{Z}_t|^2 \leq - \beta \mathbb{E} |\tilde{Z}_t|^2 + C \mathbb{E} |Z_t|^2.
\]

Using an integrating factor, we have

\[
\frac{d}{dt} \left( e^{\beta t} \mathbb{E} |\tilde{Z}_t|^2 \right) \leq C e^{\beta t} \mathbb{E} |Z_t|^2,
\]
which combined with (3.20) yields
\[ 
\mathbb{E}|\nabla_\theta \tilde{X}_t^{\theta,x,\tilde{x}_1} - \nabla_\theta \tilde{X}_t^{\theta,x,\tilde{x}_2}|^2 = \mathbb{E} |\tilde{Z}_t|^2 \leq e^{-\beta t} |\tilde{x}_1 - \tilde{x}_2|^2 + e^{-\beta t} \int_0^t e^{\beta s} \mathbb{E} |Z_s|^2 \, ds 
\]
\[ 
\leq Ce^{-\frac{2\beta}{3}t} |\tilde{x}_1 - \tilde{x}_2|^2. 
\] (3.33)

Consequently,
\[ 
|\nabla_\theta v^{1,2}(x, \tilde{x}, \theta)| \leq C|\tilde{x}| \cdot |\nabla_\theta \mathbb{E}_x f(Y) + C \int_0^\infty \mathbb{E} \left| \nabla f(X_t^{\theta,x}) \left( \nabla_\theta \tilde{X}_t^{\theta,x,\tilde{x}} - \nabla_\theta \tilde{X}_t^{\theta,x,0} \right) \right| dt 
\]
\[ 
+ \int_0^\infty \mathbb{E} \left| \left( \nabla^2 f(X_t^{\theta,x}) \nabla_\theta X_t^{\theta,x} \tilde{X}_t^{\theta,x,\tilde{x}} - \tilde{X}_t^{\theta,x,0} \right) \right| dt 
\]
\[ 
\stackrel{(a)}{\leq} C|\tilde{x}| + \int_0^\infty Ce^{-\frac{2\beta}{3}t} |\tilde{x}| dt + \int_0^\infty Ce^{-\frac{2\beta}{3}t} |\tilde{x}| dt 
\]
\[ 
\leq C|\tilde{x}|, 
\] (3.34)

where in step (a) we use the Cauchy-Schwarz inequality, (A.11), (3.20) and (3.33).

Finally, for the derivatives with respect to \( x \), define
\[ 
\dot{Z}_t = \nabla_x \tilde{X}_t^{\theta,x,\tilde{x}_1} - \nabla_x \tilde{X}_t^{\theta,x,\tilde{x}_2} 
\]
and as in (3.29) and (3.30) it satisfies the SDE
\[ 
\dot{Z}_t = \left( \left( \nabla^2_{x_2} \mu(X_t^{\theta,x}, \tilde{x}) \nabla_x X_t^{\theta,x}, Z_t \right) + \nabla \mu(X_t^{\theta,x}, \theta) \dot{Z}_t \right) dt + \left( \left( \nabla^2_{x_2} \sigma(X_t^{\theta,x}, \tilde{x}) \nabla_x X_t^{\theta,x}, Z_t \right) + \nabla \sigma(X_t^{\theta,x}, \theta) \dot{Z}_t \right) dW_t, 
\] (3.35)

Similarly, we can derive the differential inequality
\[ 
\frac{d}{dt} \mathbb{E} |\dot{Z}_t|^2 \leq -\beta \mathbb{E} |\dot{Z}_t|^2 + C \mathbb{E} |Z_t|^2. 
\]

Consequently,
\[ 
\mathbb{E} |\nabla_x \tilde{X}_t^{\theta,x,\tilde{x}_1} - \nabla_x \tilde{X}_t^{\theta,x,\tilde{x}_2}|^2 \leq Ce^{-\frac{2\beta}{3}t} |\tilde{x}_1 - \tilde{x}_2|^2. 
\] (3.36)

Due to Lemma 3.1,
\[ 
\mathbb{E} |X_t^{\theta,x_1} - X_t^{\theta,x_2}|^2 \leq e^{-\beta t} |x_1 - x_2|^2, 
\] (3.37)

which, combined with the dominated convergence theorem, yields
\[ 
\mathbb{E} \left| \nabla_x X_t^{\theta,x} \right|^2 \leq e^{-\beta t}. 
\] (3.38)

Therefore,
\[ 
|\nabla_x v^{1,2}(x, \tilde{x}, \theta)| \leq C \int_0^\infty \mathbb{E} \left| \nabla f(X_t^{\theta,x}) \left( \nabla_x \tilde{X}_t^{\theta,x,\tilde{x}} - \nabla_x \tilde{X}_t^{\theta,x,0} \right) \right| dt + C \int_0^\infty \mathbb{E} \left| \left( \nabla^2 f(X_t^{\theta,x}) \nabla_x X_t^{\theta,x} \tilde{X}_t^{\theta,x,\tilde{x}} - \tilde{X}_t^{\theta,x,0} \right) \right| dt 
\]
\[ 
\stackrel{(a)}{\leq} \int_0^\infty Ce^{-\beta t} |\tilde{x}| dt + \int_0^\infty Ce^{-\frac{2\beta}{3}t} |\tilde{x}| dt 
\]
\[ 
\leq C|\tilde{x}|, 
\] (3.39)

where step (a) is by Cauchy-Schwarz inequality, (3.38), (3.20) and (3.36). The bound for \( \nabla_x^2 v^{1,2} \) follows from exactly the same method. Combining the bounds for \( v^{1,1} \) and \( v^{1,2} \) proves the desired bound (3.16). Using the same calculations as in Lemma 3.3 of [26], we can show that \( v^1 \) is the classical solution of PDE (3.15) and thus the proof is completed. \( \square \)
We will also need bounds on the moments of \(X_t\) and \(\tilde{X}_t\) in order to analyze the fluctuation term \(\Delta_{t_n, \sigma_{n+\eta}}^t\).

**Lemma 3.6.** There exists a constant \(C > 0\) such that the processes \(X_t, \tilde{X}_t\) in (1.3) satisfy

\[
\mathbb{E}_x[X_t^4] \leq C (1 + |x|^8), \quad \mathbb{E}_{x, \tilde{x}}[\tilde{X}_t^4] \leq C (1 + |	ilde{x}|^8),
\]

where \(\mathbb{E}_{x, \tilde{x}}\) is the conditional expectation given that \(X_0 = x\) and \(\tilde{X}_0 = \tilde{x}\). Furthermore, we have the bounds

\[
\mathbb{E}_x \left( \sup_{0 \leq t' \leq t} |X_{t'}|^4 \right) = O(\sqrt{t}) \quad \text{as} \ t \to \infty, \quad \text{(3.41)}
\]

\[
\mathbb{E}_{x, \tilde{x}} \left( \sup_{0 \leq t' \leq t} |\tilde{X}_{t'}|^4 \right) = O(\sqrt{t}) \quad \text{as} \ t \to \infty. \quad \text{(3.42)}
\]

**Proof.** By Itô’s formula, for any \(m \geq 1\) we have

\[
d|X_t|^{2m} = 2m|X_t|^{2m-2} \langle \mu(X_t, \theta_t), \, X_t \rangle dt + 2m|X_t|^{2m-2} \langle \sigma(X_t, \theta_t), \, X_t \rangle dW_t + m|X_t|^{2m-2} \cdot |\sigma(X_t, \theta_t)|^2 dt + 2m(m - 1)|X_t|^{2m-4} \cdot |\langle X_t, \sigma(X_t, \theta_t) \rangle|^2. \quad \text{(3.43)}
\]

We use induction to prove the bound for the 8-th moment. First let \(m = 1\) in (3.43), by the same proof as in Lemma 3.3, we have

\[
\frac{d}{dt} \mathbb{E}_x |X_t|^2 \leq -\beta \mathbb{E}_x |X_t|^2 + \mathbb{E}_x \left( \frac{1}{\beta} |\mu(0, \theta_t)|^2 + 2 |\sigma(0, \theta_t)|^2 \right) \leq -\beta \mathbb{E}_x |X_t|^2 + C, \quad \text{(3.44)}
\]

which yields the bound for the second moment

\[
\mathbb{E}_x |X_t|^2 \leq C (1 + |x|^2). \quad \text{(3.45)}
\]

For \(k \in \{1, 2, \ldots, \ell\}\), let \(\tilde{X}_t^{i,k}\) denote the \(k\)-th column of \(\tilde{X}_t\).\(\tilde{X}_t^{i,k}\) satisfies the following SDE:

\[
d|\tilde{X}_t^{i,k}|^2 = 2 \left( \nabla_x \mu(X_t, \theta_t) \tilde{X}_t^{i,k} + \frac{\partial \mu(X_t, \theta_t)}{\partial \theta_k} \tilde{X}_t^{i,k} \right) dt + \left( |\nabla_x \sigma(X_t, \theta_t) \tilde{X}_t^{i,k} + \frac{\partial \sigma(X_t, \theta_t)}{\partial \theta_k} \tilde{X}_t^{i,k}|^2 \right) dt \quad \text{(3.46)}
\]

Similar to (A.10), we can derive the differential inequality

\[
\frac{d}{dt} \mathbb{E}_{x, \tilde{x}} |\tilde{X}_t^{i,k}|^2 \leq -\beta \mathbb{E}_{x, \tilde{x}} |\tilde{X}_t^{i,k}|^2 + \mathbb{E}_{x, \tilde{x}} \left( \frac{1}{\beta} \left| \frac{\partial \mu(X_t, \theta_t)}{\partial \theta_k} \right|^2 + 2 \left| \frac{\partial \sigma(X_t, \theta_t)}{\partial \theta_k} \right|^2 \right) \leq -\beta \mathbb{E}_{x, \tilde{x}} |\tilde{X}_t^{i,k}|^2 + C. \quad \text{(3.47)}
\]

Therefore,

\[
\mathbb{E}_{x, \tilde{x}} |\tilde{X}_t| \leq C (1 + |	ilde{x}|^2). \quad \text{(3.48)}
\]

Now let \(m = 2\) in (3.43) and use the bound (2.16),

\[
\frac{d}{dt} \mathbb{E}_x |X_t|^4 = 4 \mathbb{E}_x \left( |X_t|^2 (\mu(X_t, \theta_t), \, X_t) \right) dt + \mathbb{E}_x \left( 2 |X_t|^3 |\sigma(X_t, \theta_t)|^2 + 4 |\langle \sigma(X_t, \theta_t), \, X_t \rangle|^2 \right) \leq \mathbb{E}_x \left( |X_t|^2 (4|\mu(X_t, \theta_t), \, X_t| + 6|\sigma(X_t, \theta_t)|^2 \right) \leq -\beta \mathbb{E}_x |X_t|^4 + C \mathbb{E}_x |X_t|^2, \quad \text{(3.49)}
\]

which together with (3.45) and Gronwall’s inequality prove the bound for fourth moment of \(X_t\). Similarly, as in (2.17)

\[
\frac{d}{dt} \mathbb{E}_{x, \tilde{x}} |\tilde{X}_t^{i,k}|^4 \leq \mathbb{E}_{x, \tilde{x}} \left( |\tilde{X}_t^{i,k}|^2 \left( 4 \left| \nabla_x \mu(X_t, \theta_t) \tilde{X}_t^{i,k} + \frac{\partial \mu(X_t, \theta_t)}{\partial \theta_k} \tilde{X}_t^{i,k} \right) \right) + 6 \left| \nabla_x \sigma(X_t, \theta_t) \tilde{X}_t^{i,k} + \frac{\partial \sigma(X_t, \theta_t)}{\partial \theta_k} \tilde{X}_t^{i,k} \right|^2 \right) \leq -\beta \mathbb{E}_{x, \tilde{x}} |\tilde{X}_t^{i,k}|^4 + C \mathbb{E}_{x, \tilde{x}} |\tilde{X}_t^{i,k}|^2, \quad \text{(3.50)}
\]
which together with (3.48) derives the estimate for $\hat{X}_t$ in (3.40). By induction, we can prove the bound for the sixth and eighth moments of $(X_t, \hat{X}_t)$ in (3.40).

Finally, as in (3.43) and use (2.17), we have

\[
|X_t|^8 = |x|^8 + 8 \int_0^t |X_s|^6 \langle \mu(X_s, \theta_s), \ X_s \rangle \, ds + 8 \int_0^t |X_s|^6 \langle \sigma(X_s, \theta_s), \ X_s \rangle \, dW_s \\
+ 24 \int_0^t |X_s|^4 \cdot |\langle \sigma(X_s, \theta_s), \ X_s \rangle|^2 \, ds + 4 \int_0^t |X_s|^6 \cdot |\sigma(X_s, \theta_s)|^2 \, ds
\]

\[
\leq -4\beta \int_0^t |X_s|^8 \, ds + C \int_0^t |X_s|^6 \, ds + 8 \int_0^t |X_s|^6 \langle \sigma(X_s, \theta_s), \ X_s \rangle dW_s,
\]

which together with the Burkholder-Davis-Gundy inequality and (3.45) derive that there exists a constant $C$ such that

\[
E_x \sup_{0 \leq t' \leq t} |X_{t'}|^8 \leq |x|^8 + ct + CE_x \left( \int_0^t |X_s|^4 \cdot |\sigma(X_s, \theta_s)|^2 \, ds \right) \frac{1}{2}
\]

\[
\leq |x|^8 + ct + CE_x \left( \sup_{0 \leq t' \leq t} |X_{t'}|^8 \cdot \int_0^t |X_s|^6 \cdot |\sigma(X_s, \theta_s)|^2 \, ds \right) \frac{1}{2}
\]

\[
\leq |x|^8 + ct + \frac{1}{2} E_x \sup_{0 \leq t' \leq t} |X_{t'}|^8 + C \int_0^t E_x \left[ |X_s|^6 + |X_s|^8 \right] \, ds,
\]

where step (a) is by Young’s inequality. Thus, combining (3.40) and (3.52) we obtain

\[
E_x \left( \sup_{0 \leq t' \leq t} |X_{t'}|^8 \right) = O(t) \quad \text{as } t \to \infty,
\]

which derives (3.41). Similarly for (3.42), using Itô’s formula for $|\hat{X}_t|^8$ and the Burkholder-Davis-Gundy inequality,

\[
E_{x, \bar{x}} \sup_{0 \leq t' \leq t} |\hat{X}_{t'}|^8 \leq |\bar{x}|^8 + ct + CE_{x, \bar{x}} \left( \int_0^t |\bar{X}_s|^4 \cdot \left| \nabla_x \sigma(X_s, \theta_s) \bar{X}_s + \nabla \theta \sigma(X_s, \theta_s) \right|^2 \, ds \right) \frac{1}{2}
\]

\[
\leq |\bar{x}|^8 + ct + \frac{1}{2} E_{x, \bar{x}} \sup_{0 \leq t' \leq t} |\bar{X}_{t'}|^8 + C \int_0^t E_{x, \bar{x}} \left[ |\bar{X}_s|^6 + |\bar{X}_s|^8 \right] \, ds,
\]

which together with (3.40) derive (3.42).

\[\square\]

We can now bound the first fluctuation term $\Delta^1_{x_n \sigma_n \eta}$ in (3.3) using the estimates from Lemma 3.5 and Lemma 3.6.

**Lemma 3.7.** Under Assumptions A1 - A5, for any fixed $\eta > 0$,

\[
|\Delta^1_{x_n \sigma_n \eta}| \to 0 \text{ as } n \to \infty, \quad \text{a.s.}
\]

**Proof.** We will express $\Delta^1_{x_n \sigma_n \eta}$ in terms of the Poisson equation in Lemma 3.5 and then prove it vanishes as $n$ becomes large. Consider the function

\[
G^1(x, \bar{x}, \theta) = (\mathcal{E}_{\sigma} f(Y) - \beta) (\nabla f(x) \bar{x} - \nabla \theta \mathcal{E}_{\sigma} f(Y)) \top.
\]

By Lemma 3.5, the Poisson equation $L^0_{\sigma} u(x, \bar{x}, \theta) = G^1(x, \bar{x}, \theta)$ will have a unique smooth solution $u^1(x, \bar{x}, \theta)$ that grows at most linearly in $(x, \bar{x})$. Let us apply Itô’s formula to the function

\[
u^1(t, x, \bar{x}, \theta) := \alpha t v^1(x, \bar{x}, \theta) \in \mathbb{R}^\ell,
\]

\[
\sum_{i=1}^\ell \nu^1_i(t, x, \bar{x}, \theta) = \frac{1}{2} \sum_{i=1}^\ell \left( \frac{\partial^2 v^1}{\partial x_i \partial \bar{x}_j} \right) (\partial x_i \partial \bar{x}_j + \partial x_i \partial \theta \sum_{s \neq i} \partial x_s \partial \bar{x}_j) ds.
\]

\[
\frac{1}{2} \sum_{i=1}^\ell \left( \frac{\partial^2 v^1}{\partial x_i \partial \bar{x}_j} \right) (\partial x_i \partial \bar{x}_j + \partial x_i \partial \theta \sum_{s \neq i} \partial x_s \partial \bar{x}_j) ds.
\]

\[
\frac{1}{2} \sum_{i=1}^\ell \left( \frac{\partial^2 v^1}{\partial x_i \partial \bar{x}_j} \right) (\partial x_i \partial \bar{x}_j + \partial x_i \partial \theta \sum_{s \neq i} \partial x_s \partial \bar{x}_j) ds.
\]
evaluated on the stochastic process \((X_t, \tilde{X}_t, \theta_t)\). Recall that \(u_i\) denotes the \(i\)-th element of \(u\) and \(\tilde{X}_i^{k}\) be the \(k\)-th column of the matrix \(X_t\) for \(i, k \in \{1, 2, \cdots, \ell\}\). Then,

\[
u_i^1 \left( \sigma, X_\sigma, \tilde{X}_\sigma, \theta_\sigma \right) = u_i^1 \left( \tau, X_\tau, \tilde{X}_\tau, \theta_\tau \right) + \int_\tau^\sigma \partial_\sigma u_i^1 \left( s, X_s, \tilde{X}_s, \theta_s \right) ds + \int_\tau^\sigma L_{x,x}^\sigma u_i^1 \left( s, X_s, \tilde{X}_s, \theta_s \right) ds + \int_\tau^\sigma \nabla_\theta u_i^1 \left( s, X_s, \tilde{X}_s, \theta_s \right) d\theta_s + \int_\tau^\sigma \nabla_x u_i^1 \left( s, X_s, \tilde{X}_s, \theta_s \right) \sigma(X_s, \theta_s) dW_s
\]

\[
+ \sum_{k=1}^\ell \int_\tau^\sigma \nabla_x \kappa u_i^1 \left( s, X_s, \tilde{X}_s, \theta_s \right) \left( \nabla_x \sigma(X_s, \theta_s) \tilde{X}_s^\kappa + \frac{\partial \sigma(X_s, \theta_s)}{\partial \theta_k} \right) dW_s.
\]

Rearranging the previous equation, we obtain the representation

\[
\Delta_1^{\tau_n, \alpha_n + \eta} = \int_\tau^{\tau_n} \alpha_s G^1(X_s, \tilde{X}_s, \theta_s) ds = \int_\tau^{\tau_n} L_{x,x}^\sigma u_i^1 \left( s, X_s, \tilde{X}_s, \theta_s \right) ds
\]

\[
= \alpha_{\sigma_n + \eta} v^1 \left( X_{\sigma_n + \eta}, \tilde{X}_{\sigma_n + \eta}, \theta_{\sigma_n + \eta} \right) - \alpha_{\tau_n} v^1 \left( X_{\tau_n}, \tilde{X}_{\tau_n}, \theta_{\tau_n} \right) - \int_\tau^{\tau_n} \alpha_s v^1 \left( X_s, \tilde{X}_s, \theta_s \right) ds
\]

\[
+ \int_\tau^{\tau_n} 2\alpha_s^2 \nabla \theta v^1 \left( X_s, \tilde{X}_s, \theta_s \right) \left( f(\tilde{X}_s) - \beta \right) \left( \nabla f(X_s) \tilde{X}_s \right)^\top ds
\]

\[
- \int_\tau^{\tau_n} \alpha_s v^1 \left( X_s, \tilde{X}_s, \theta_s \right) \sigma(X_s, \theta_s) dW_s
\]

\[
- \sum_{k=1}^\ell \int_\tau^{\tau_n} \alpha_s \nabla \kappa v^1 \left( X_s, \tilde{X}_s, \theta_s \right) \left( \nabla_x \sigma(X_s, \theta_s) \tilde{X}_s^\kappa + \frac{\partial \sigma(X_s, \theta_s)}{\partial \theta_k} \right) dW_s.
\]

The next step is to treat each term on the right hand side of (3.56) separately. For this purpose, let us first set

\[
J_{1,1}^1 = \alpha_t \sup_{s \in [0, t]} \left| v^1 \left( X_s, \tilde{X}_s, \theta_s \right) \right|.
\]

By (3.16) and Lemma 3.6, there exists a constant \(C\) such that

\[
E \left| J_{1,1}^1 \right|^2 \leq C \alpha_t^2 \left[ \sup_{s \in [0, t]} |X_s|^2 + \sup_{s \in [0, t]} |\tilde{X}_s|^2 \right]
\]

\[
= C \alpha_t^2 \left[ 1 + \sqrt{t} \left( E \sup_{s \in [0, t]} |X_s|^2 + E \sup_{s \in [0, t]} |\tilde{X}_s|^2 \right)^\frac{1}{2} \right]
\]

\[
\leq C \alpha_t^2 \sqrt{t}.
\]

Let \(p > 0\) be the constant in Assumption A5 such that \(\lim_{t \to \infty} \alpha_t^{2t^{1/2+2p}} = 0\) and for any \(\delta \in (0, p)\) define the event \(A_{t, \delta} = \{ J_{1,1}^1 \geq t^{\delta-p} \} \). Then we have for \(t\) large enough such that \(\alpha_t^{2t^{1/2+2p}} \leq 1\)

\[
P \left( A_{t, \delta} \right) \leq E \left| J_{1,1}^1 \right|^2 \leq C \alpha_t^{2t^{1/2+2p}} \leq C \frac{1}{t^{2\delta}}.
\]

The latter implies that

\[
\sum_{m \in \mathbb{N}} P \left( A_{2^m, \delta} \right) < \infty.
\]

Therefore, by the Borel-Cantelli lemma we have that for every \(\delta \in (0, p)\) there is a finite positive random variable \(d(\omega)\) and some \(m_0 < \infty\) such that for every \(m \geq m_0\) one has

\[
J_{2,m}^{1,1} \leq \frac{d(\omega)}{2^{m(p-\delta)}}.
\]
Thus, for \( t \in [2^m, 2^{m+1}) \) and \( m \geq m_0 \) one has for some finite constant \( C < \infty \)

\[
J_{t,1}^{1,1} \leq C \alpha_{2^{m+1}} \sup_{s \in (0, 2^m+1]} \left| v^1 \left( X_s, \tilde{X}_s, \theta_s \right) \right| \leq C \frac{d(\omega)}{2^{(m+1)(p-\delta)}} \leq C \frac{d(\omega)}{\rho - \delta},
\]

which proves that for \( t \geq 2^{m_0} \) with probability one

\[
J_{t,1}^{1,1} \leq C \frac{d(\omega)}{\rho - \delta} \to 0, \text{ as } t \to \infty. \quad (3.59)
\]

Next we consider the term

\[
J_{t,0}^{1,2} = \int_0^t \left| \alpha' v^1 \left( X_s, \tilde{X}_s, \theta_s \right) - 2\alpha^2 \nabla \theta v^1 \left( X_s, \tilde{X}_s, \theta_s \right) \left( f(\tilde{X}_s) - \beta \right) \left( \nabla f(X_s)\tilde{X}_s \right)^\top \right| ds.
\]

Noting that by the same approach for \( X_t \) in Lemma 3.6, we can prove that there exists a constant \( C > 0 \) such that

\[
E_x |\tilde{X}_t|^4 \leq C (1 + |\tilde{x}|^4), \quad E_x \left( \sup_{0 \leq t' \leq t} |\tilde{X}_{t'}|^2 \right) = O(\sqrt{t}) \quad \text{as } t \to \infty. \quad (3.60)
\]

Thus

\[
\sup_{t>0} E \left| J_{t,0}^{1,2} \right|^{(a)} \leq C \int_0^\infty \left( |\alpha'\sigma| + \alpha^2 \right) \left( 1 + E |X_s|^4 + E |\tilde{X}_s|^4 + E |\tilde{X}_s|^2 \right) ds
\]

\[
\leq C \int_0^\infty (|\alpha'| + \alpha_2^2) ds
\]

\[
\leq C,
\]

where step \( (a) \) is by Assumption A4 and (3.16) and in step \( (b) \) we use (3.40). Thus there is a finite random variable \( J_{\infty,0}^{1,2} \) such that

\[
J_{t,0}^{1,2} \to J_{\infty,0}^{1,2} \text{ as } t \to \infty \text{ with probability one.} \quad (3.61)
\]

The last term we need to consider is the martingale term

\[
J_{t,0}^{1,3} = \int_0^t \alpha \nabla v^1 \left( X_s, \tilde{X}_s, \theta_s \right) \sigma(X_s, \theta_s) dW_s
\]

\[
+ \sum_{k=1}^t \int_0^t \alpha \nabla \sigma \left( X_s, \tilde{X}_s, \theta_s \right) \left( \nabla_x \sigma(X_s, \theta_s) \tilde{X}_{s,k}^{\top} + \frac{\partial \sigma(X_s, \theta_s)}{\partial \theta_k} \right) dW_s.
\]

By Doob’s inequality, Assumption A5, (3.16), (3.40), and using calculations similar to the ones for the term \( J_{t,0}^{1,2} \), we can show that for some finite constant \( C < \infty \),

\[
\sup_{t>0} E \left| J_{t,0}^{1,3} \right|^2 \leq C \int_0^\infty \alpha_s^2 \left( 1 + E |X_t|^4 + E |\tilde{X}_t|^4 \right) ds < \infty
\]

Thus, by Doob’s martingale convergence theorem there is a square integrable random variable \( J_{\infty,0}^{1,3} \) such that

\[
J_{t,0}^{1,3} \to J_{\infty,0}^{1,3} \text{ as } t \to \infty \text{ both almost surely and in } L^2. \quad (3.62)
\]

Let us now return to (3.56). Using the terms \( J_{t,1}^{1,1}, J_{t,2}^{1,2}, \) and \( J_{t,0}^{1,3} \) we can write

\[
|A_{\tau_n, \sigma_n + \eta}^1| \leq J_{\sigma_n + \eta}^{1,1} + J_{\sigma_n + \eta, \tau_n}^{1,2} + J_{\sigma_n + \eta, \tau_n}^{1,3},
\]

which together with (3.59), (3.61), and (3.62) prove the statement of the Lemma. \( \square \)

We will next prove a similar convergence result for \( \Delta_{\tau_n, \sigma_n + \eta}^2 \). We must first prove an extension of Lemma 3.5 for the Poisson equation.
Lemma 3.8. Define the error function

\[ G^2(x, \tilde{x}, \bar{x}, \theta) = [f(\bar{x}) - \mathbf{E}_{\pi_{\theta}} f(Y)] (\nabla f(x) \tilde{x})^T. \]  

(3.63)

Under Assumptions (A1) - (A4), the function

\[ v^2(x, \tilde{x}, \bar{x}, \theta) = - \int_0^\infty \mathbb{E}G^2(X^\theta_t, \tilde{X}^\theta_{t,x}, \bar{X}^\theta_{t,x}, \theta) dt \]  

(3.64)

is the classical solution of the Poisson equation

\[ \mathcal{L}^\theta_{x, \tilde{x}, \bar{x}} u(x, \tilde{x}, \bar{x}, \theta) = G^2(x, \tilde{x}, \bar{x}, \theta), \]  

(3.65)

where \( \mathcal{L}^\theta_{x, \tilde{x}, \bar{x}} \) is generator of the process \( (X^\theta, \tilde{X}^\theta_{t,x}, \bar{X}^\theta_{t,x}) \), i.e. for any test function \( \varphi \)

\[ \mathcal{L}^\theta_{x, \tilde{x}, \bar{x}} \varphi(x, \tilde{x}, \bar{x}, \theta) = \mathcal{L}^\theta_{x, \tilde{x}, \bar{x}} \varphi(x, \tilde{x}, \bar{x}) + \mathcal{L}^\theta_{x, \tilde{x}, \bar{x}} \varphi(x, \tilde{x}, \bar{x}). \]  

(3.66)

Furthermore, this solution satisfies the bound

\[ |v^2(x, \tilde{x}, \bar{x}, \theta)| + |\nabla v^2(x, \tilde{x}, \bar{x}, \theta)| + |\nabla \theta v^2(x, \tilde{x}, \bar{x}, \theta)| + |\nabla \bar{x} v^2(x, \tilde{x}, \bar{x}, \theta)| + |\nabla \bar{\theta} v^2(x, \tilde{x}, \bar{x}, \theta)| \leq C (1 + |\bar{x}|) (1 + |\tilde{x}|), \]  

(3.67)

where \( C \) is a constant independent of \( (x, \tilde{x}, \bar{x}, \theta) \).

Proof. The proof is exactly the same as in Lemma 3.5 except for the presence of the dimension \( \bar{x} \) and \( \mathcal{L}_\bar{x} \).

Since \( X^\theta_t \) and \( \tilde{X}^\theta_t \) are i.i.d., the bounds from Proposition 3.4 are also true for \( \tilde{X}_t \). We first show that the integral (3.64) is finite. Note that

\[ v^2(x, \tilde{x}, \bar{x}, \theta) = \int_0^\infty \mathbb{E} \left[ \left( \mathbf{E}_{\pi_{\theta}} f(Y) - f(\tilde{X}^\theta_t) \right) \cdot (\nabla f(x) \tilde{x})^T \right] dt \]

(3.68)

where step (a) is due to the independence of \( \tilde{X}^\theta_t \) and \( (X^\theta, \tilde{X}^\theta_{t,x}) \). As in (3.46) and (3.47), we can prove

\[ \mathbb{E} \left| \tilde{X}^\theta_{t,x} \right|^2 \leq C (1 + |\tilde{x}|^2) \]

and thus by Assumption A4

\[ \mathbb{E} \left| \nabla f(X^\theta_t) \tilde{X}^\theta_{t,x} \right| \leq C \mathbb{E} \left| \tilde{X}^\theta_{t,x} \right| \leq C (1 + |\tilde{x}|), \]  

(3.69)

which together with Proposition 3.4 yields

\[ |v^2(x, \tilde{x}, \bar{x}, \theta)| \leq C (1 + |\tilde{x}|) \cdot \int_0^\infty \left| \mathbb{E} f(\tilde{X}^\theta_t) - \mathbf{E}_{\pi_{\theta}} f(Y) \right| dt \leq C (1 + |\tilde{x}|) (1 + |\tilde{x}|). \]  

(3.70)

We next show that \( v^2(x, \tilde{x}, \bar{x}, \theta) \) is differentiable with respect to \( (x, \tilde{x}, \bar{x}, \theta) \). Similar to Lemma 3.5, we first change the order of differentiation and integration and show the corresponding integral exists. Then, we apply DCT to prove that the differentiation and integration can be interchanged. For the ergodic process \( \tilde{X}^\theta_t \), by Proposition 3.4 and (3.69), we have the following bound for \( i = 1, 2 \):

\[ |\nabla^i_x v^2(x, \tilde{x}, \bar{x}, \theta)| \leq \int_0^\infty \left| \nabla^i_x \mathbb{E} f(\tilde{X}^\theta_t) \cdot \mathbb{E} \left[ \nabla f(X^\theta_t) \tilde{X}^\theta_{t,x} \right] \right| dt \leq C (1 + |\tilde{x}|). \]  

(3.71)

Note that

\[ \nabla x X^\theta_t = \tilde{X}^\theta_{t,x} \]  

(3.72)
and thus, as in the proof of Proposition 3.4 and Lemma 3.5, it is easy to prove the bounds

\[
\sup_{\theta \in \mathbb{R}^l, x \in \mathbb{R}^d} \left| \nabla_x^i \hat{X}_t^{\theta, x, \hat{x}} \right|^2 \leq C e^{-\beta t}, \quad \sup_{\theta \in \mathbb{R}^l, x \in \mathbb{R}^d} \left| \nabla_x^i \hat{X}_t^{\theta, x, \hat{x}} \right|^2 \leq C e^{-\beta t}, \quad i = 1, 2, 
\]

(3.73)

which derives

\[
\sum_{i=1}^2 \left| \int_{-4}^2 \left[ \nabla_x^i \mathbb{E} \left[ \nabla f(X_t^{\theta, x}) \hat{X}_t^{\theta, x, \hat{x}} \right] \right] + \left[ \nabla_x^i \mathbb{E} \left[ \nabla f(X_t^{\theta, x}) \hat{X}_t^{\theta, x, \hat{x}} \right] \right] \right| \leq C \left( 1 + \left| \bar{x} \right| \right),
\]

(3.74)

Therefore, for \( i = 1, 2 \),

\[
\left| \nabla_x^i u^2(x, \hat{x}, \bar{x}, \theta) \right| \leq \int_0^\infty \left| \mathbb{E} \pi_x f(Y) - \mathbb{E} f(X_t^{\theta, x}) \right| + \left| \nabla_x^i \mathbb{E} \left[ \nabla f(X_t^{\theta, x}) \hat{X}_t^{\theta, x, \hat{x}} \right] \right| \leq C \left( 1 + \left| \bar{x} \right| \right),
\]

(3.75)

and

\[
\left| \nabla_x^i \nabla_x^i u^2(x, \hat{x}, \bar{x}, \theta) \right| = \int_0^\infty \nabla_x \left( \left| \mathbb{E} \pi_x f(Y) - \mathbb{E} f(X_t^{\theta, x}) \right| + \left| \nabla_x^i \mathbb{E} \left[ \nabla f(X_t^{\theta, x}) \hat{X}_t^{\theta, x, \hat{x}} \right] \right| \right) \right| dt \leq C \left( 1 + \left| \bar{x} \right| \right) .
\]

(3.76)

Finally,

\[
\left| \nabla_x \nabla_x^i u^2(x, \hat{x}, \bar{x}, \theta) \right| \leq \int_0^\infty \left| \mathbb{E} \pi_x f(Y) - \mathbb{E} f(X_t^{\theta, x}) \right| + \left| \nabla_x \nabla_x^i \mathbb{E} \left[ \nabla f(X_t^{\theta, x}) \hat{X}_t^{\theta, x, \hat{x}} \right] \right| \right| dt \leq C \left( 1 + \left| \bar{x} \right| \right).
\]

(3.77)

By the same calculations as in Lemma 3.3 of [26], it can be shown that \( u^2 \) is the classical solution of PDE (3.65) and the bound (3.67) holds. \( \square \)

Now we can bound the second fluctuation term \( Z_t^2 \). The proof is exactly the same as in Lemma 3.7.

**Lemma 3.9.** Under Assumptions (A1) - (A5), for any fixed \( \eta > 0 \),

\[
\left| \Delta_{:, \sigma, n+\eta}^2 \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{a.s.}
\]

(3.78)

**Proof.** Consider the function

\[
G^2(x, \hat{x}, \bar{x}, \theta) = \left[ f(x) - \mathbb{E} \pi_x f(Y) \right] \left( \nabla f(x) \hat{x} \right) ^T .
\]

(3.79)

Let \( u_t^2 \) be the solution of (3.65) in Lemma 3.8. We apply Itô formula to the function \( u^2(t, x, \hat{x}, \bar{x}, \theta) = \alpha_t u^2(x, \hat{x}, \bar{x}, \theta) \) evaluated on the stochastic process \( (X_t, \hat{X}_t, \bar{X}_t, \theta_t) \) and get for any \( i \in \{ 1, 2, \cdots, \ell \} \)

\[
\begin{aligned}
&u_t^2 \left( \sigma, X_\sigma, \hat{X}_\sigma, \bar{X}_\sigma, \theta_\sigma \right) - u_t^2 \left( \tau, X_\tau, \hat{X}_\tau, \bar{X}_\tau, \theta_\tau \right) \\
= &\int_\tau^\sigma \partial_\sigma u_t^2 \left( s, X_s, \hat{X}_s, \bar{X}_s, \theta_s \right) ds + \int_\tau^\sigma \mathcal{L}_x u_t^2 \left( s, X_s, \hat{X}_s, \bar{X}_s, \theta_s \right) ds + \int_\tau^\sigma \mathcal{L}_x^2 u_t^2 \left( s, X_s, \hat{X}_s, \bar{X}_s, \theta_s \right) ds \\
+ &\int_\tau^\sigma \nabla \theta u_t^2 \left( s, X_s, \hat{X}_s, \bar{X}_s, \theta_s \right) ds + \int_\tau^\sigma \nabla_x u_t^2 \left( s, X_s, \hat{X}_s, \bar{X}_s, \theta_s \right) dW_s + \int_\tau^\sigma \nabla_x^2 u_t^2 \left( s, X_s, \hat{X}_s, \bar{X}_s, \theta_s \right) dW_s \\
+ &\sum_{k=1}^\ell \int_\tau^\sigma \nabla_{x,k} u_t^2 \left( s, X_s, \hat{X}_s, \bar{X}_s, \theta_s \right) \left( \nabla_x \sigma(X_s, \theta_s) \hat{X}_s^k + \frac{\partial_x \sigma(X_s, \theta_s)}{\partial \theta_k} \right) dW_s.
\end{aligned}
\]

(3.80)
Rearranging the previous equation, we obtain the representation

$$
\Delta^2_{\tau_n, \sigma_n + \eta} = \int_{\tau_n}^{\sigma_n + \eta} \alpha s G^2(X_s, \tilde{X}_s, X_s, \theta_s) ds - \int_{\tau_n}^{\sigma_n + \eta} \mathcal{L}_{\sigma_n + \eta} v^2(s, X_s, \tilde{X}_s, X_s, \theta_s) ds
$$

Thus for the first set

$$
\therefore \alpha_{\sigma_n + \eta} v^2 \left( X_{\sigma_n + \eta}, \tilde{X}_{\sigma_n + \eta}, X_{\sigma_n + \eta}, \theta_{\sigma_n + \eta} \right) - \alpha_{\sigma_n + \eta} v^2 \left( X_{\tau_n}, \tilde{X}_{\tau_n}, X_{\tau_n}, \theta_{\tau_n} \right) - \int_{\tau_n}^{\sigma_n + \eta} \alpha_s v^2 \left( X_s, \tilde{X}_s, X_s, \theta_s \right) ds
$$

Next we consider the term

$$
\sum_{\tau_n} \alpha_s v^2 \left( X_s, \tilde{X}_s, X_s, \theta_s \right) dW + \int_{\tau_n}^{\sigma_n + \eta} 2\alpha_s \nabla \theta v^2 \left( X_s, \tilde{X}_s, X_s, \theta_s \right) (f(\tilde{X}_s) - \beta) \left( \nabla f(X_s) \tilde{X}_s \right)^T ds
$$

The next step is to treat each term on the right hand side of (3.81) separately. For this purpose, let us first set

$$
J_t^{2,1} = \alpha_t \sup_{s \in [0, t]} \left| v^2 \left( X_s, \tilde{X}_s, X_s, \theta_s \right) \right|.
$$

Combining Lemma 3.6, (3.67) and (3.60), we know that there exists a constant $C$ such that

$$
\mathbb{E} \left[ J_t^{2,1} \right]^2 \leq C \alpha_t^2 \mathbb{E} \left[ 1 + \sup_{s \in [0, t]} \left| X_s \right|^4 + \sup_{s \in [0, t]} \left| \tilde{X}_s \right|^4 \right]
$$

$$
\leq C \alpha_t^2 \left[ 1 + \frac{\mathbb{E} \sup_{s \in [0, t]} \left| X_s \right|^4 + \mathbb{E} \sup_{s \in [0, t]} \left| \tilde{X}_s \right|^4}{\sqrt{t}} \right]
$$

$$
\leq C \alpha_t^2 \sqrt{t}.
$$

Let $p > 0$ be the constant in Assumption A5 such that $\lim_{t \to \infty} \alpha_t^2 t^{1/2 + 2p} = 0$ and for any $\delta \in (0, p)$ define the event $A_{t, \delta} = \left\{ J_t^{2,1} \geq \varepsilon^{\delta - p} \right\}$. Then we have for $t$ large enough such that $\alpha_t^2 t^{1/2 + 2p} \leq 1$ and

$$
\mathbb{P} \left( A_{t, \delta} \right) \leq \frac{\mathbb{E} \left[ J_t^{2,1} \right]^2}{t^{2(\delta - p)}} \leq C \alpha_t^2 t^{1/2 + 2p} \leq C \frac{1}{t^{2\delta}}.
$$

The latter implies that

$$
\sum_{m \in \mathbb{N}} \mathbb{P} \left( A_{2m, \delta} \right) < \infty.
$$

Therefore, by the Borel-Cantelli lemma we have that for every $\delta \in (0, p)$ there is a finite positive random variable $d(\omega)$ and some $m_0 < \infty$ such that for every $n \geq m_0$ one has

$$
J_{2m}^{2,1} \leq \frac{d(\omega)}{2^{m(p - \delta)}}.
$$

Thus for $t \in [2^m, 2^{m+1})$ and $m \geq m_0$ one has for some finite constant $C < \infty$

$$
J_t^{2,1} \leq C \alpha_{2m+1} \sup_{s \in (0, 2^{m+1})} \left| v^2 \left( X_s, \tilde{X}_s, X_s, \theta_s \right) \right| \leq C \frac{d(\omega)}{2^{(m+1)(p - \delta)}} \leq C \frac{d(\omega)}{t^{2p - \delta}}.
$$

which derives that for $t \geq 2^{m_0}$ we have with probability one

$$
J_t^{2,1} \leq C \frac{d(\omega)}{t^{p - \delta}} \to 0, \text{ as } t \to \infty.
$$

(3.84)

Next we consider the term

$$
J_t^{2,2} = \int_{0}^{t} \alpha_s v^2 \left( X_s, \tilde{X}_s, X_s, \theta_s \right) - 2\alpha_s^2 \nabla \theta v^2 \left( X_s, \tilde{X}_s, X_s, \theta_s \right) (f(\tilde{X}_s) - \beta) \left( \nabla f(X_s) \tilde{X}_s \right)^T ds
$$
and thus we see that there exists a constant $0 < C < \infty$ such that
\[
\sup_{t > 0} \mathbb{E} \left| J_{t, 0}^{2, 2} \right| \leq C \int_0^\infty \left( |a_s' + a_s|^2 \right) \left( 1 + \mathbb{E} |\bar{X}_s|^4 + \mathbb{E} |\bar{X}_s|^4 \right) ds \leq C,
\]
where in step (a) we use (3.67) and in step (b) we use Lemma 3.6 and (3.60). Thus we know there is a finite random variable $J_{t, 0}^{2, 2}$ such that
\[
J_{t, 0}^{2, 2} \to J_{\infty, 0}^{2, 2} \text{ as } t \to \infty \text{ with probability one.} \tag{3.85}
\]
The last term we need to consider is the martingale term
\[
J_{t, 0}^{2, 3} = \int_0^t \alpha_s \nabla_x v^2 \left( X_s, \bar{X}_s, \bar{X}_s, \theta_s \right) dW_s + \int_0^t \alpha_s \nabla_x v^2 \left( X_s, \bar{X}_s, \bar{X}_s, \theta_s \right) d\bar{W}_s + \sum_{k=1}^t \int_0^\infty \alpha_s \nabla_x v^2 \left( X_s, \bar{X}_s, \bar{X}_s, \theta_s \right) \left( \nabla_x \sigma(X_s, \theta_s) \bar{X}_s \bar{v} \right) + \left( \partial \sigma(X_s, \theta_s) \bar{X}_s \right) dW_s.
\]

Notice that Doob’s inequality and the bounds of (3.67) (using calculations similar to the ones for the term $J_{t, 0}^{2, 2}$) give us that for some finite constant $K < \infty$, we have
\[
\sup_{t > 0} \mathbb{E} \left| J_{t, 0}^{2, 3} \right|^2 \leq K \int_0^\infty \alpha_s^2 ds < \infty.
\]
Thus, by Doob’s martingale convergence theorem there is a square integrable random variable $J_{\infty, 0}^{(3)}$ such that
\[
J_{t, 0}^{2, 3} \to J_{\infty, 0}^{2, 3} \text{ as } t \to \infty \text{ both almost surely and in } L^2. \tag{3.86}
\]

Let us now go back to (3.81). Using the terms $J_{t}^{2, 1}$, $J_{t, 0}^{2, 2}$ and $J_{t, 0}^{2, 3}$ we can write
\[
|\Delta_{\tau_n, \sigma_n + \eta}^2| \leq |J_{\sigma_n + \eta}^{2, 1} + J_{\sigma_n + \eta, \tau_n}^{2, 2} + J_{\sigma_n + \eta, \tau_n}^{2, 3}|
\]
which together with (3.84), (3.85) and (3.86) prove the statement of the Lemma.

By (3.10), we know that
\[
|\nabla_\theta J(\theta)| = 2 \mathbb{E}_{\sigma_n} f(Y) - \beta \cdot |\nabla_\theta \mathbb{E}_{\sigma_n} f(Y)| \leq C. \tag{3.87}
\]
Therefore, the objective function $J(\theta)$ is Lipschitz continuous with respect to $\theta$. The following lemmas are the same as in [26b] and thus we omit the proofs.

**Lemma 3.10.** Under Assumptions (A1)-(A5), choose $\mu > 0$ in (3.2) such that for the given $\kappa > 0$, one has $3\mu + \frac{\mu}{6\kappa} = \frac{1}{2L_{\mathcal{F}, \mathcal{J}}}$, where $L_{\mathcal{F}, \mathcal{J}}$ is the Lipschitz constant of objective function $J$ in (1.2). Then for $n$ large enough and $\eta > 0$ small enough (potentially random depending on $n$), one has $\int_{\tau_n}^{\sigma_n + \eta} \alpha_s ds > \mu$. In addition we also have $\frac{\mu}{2} \leq \int_{\tau_n}^{\sigma_n} \alpha_s ds \leq \mu$ with probability one.

**Lemma 3.11.** Under Assumptions (A1)-(A5), suppose that there exists an infinite number of intervals $I_n = [\tau_n, \sigma_n]$. Then there is a fixed constant $\gamma_1 = \gamma_1(\kappa) > 0$ such that for $n$ large enough,
\[
J(\theta_{\sigma_n}) - J(\theta_{\tau_n}) \leq -\gamma_1. \tag{3.88}
\]

** Lemma 3.12.** Under Assumptions (A1)-(A5), suppose that there exists an infinite number of intervals $I_n = [\tau_n, \sigma_n]$. Then, there is a fixed constant $\gamma_2 < \gamma_1$ such that for $n$ large enough,
\[
J(\theta_{\tau_n}) - J(\theta_{\sigma_n - 1}) \leq \gamma_2. \tag{3.89}
\]
Proof of Theorem 2.2: Recalling (3.2), we know \( \tau_n \) is the first time \( |\nabla \theta J(\theta_i)| > \kappa \) when \( t > \sigma_{n-1} \). Thus, for any fixed \( \kappa > 0 \), if there are only a finite number of \( \tau_n \), then there is a finite \( T^* \) such that \( |\nabla \theta J(\theta_i)| \leq \kappa \) for \( t \geq T^* \). We now use a “proof by contradiction”. Suppose that there are infinitely many instances of \( \tau_n \). By Lemmas 3.11 and 3.12, we have for sufficiently large \( n \) that

\[
J(\theta_{\sigma_n}) - J(\theta_{\tau_n}) \leq -\gamma_1 \\
J(\theta_{\tau_n}) - J(\theta_{\sigma_n-1}) \leq \gamma_2,
\]

where \( 0 < \gamma_2 < \gamma_1 \). Choose \( N \) large enough so that the above relations hold simultaneously for \( n \geq N \). Then,

\[
J(\theta_{\tau_{m+1}}) - J(\theta_{\tau_n}) = \sum_{n=N}^{m} \left[ J(\theta_{\sigma_n}) - J(\theta_{\tau_n}) + J(\theta_{\tau_{m+1}}) - J(\theta_{\sigma_n}) \right] \\
\leq \sum_{k=N}^{m} (-\gamma_1 + \gamma_2) \\
< (m - N) \times (-\gamma_1 + \gamma_2).
\]

Letting \( m \to \infty \), we observe that \( J(\theta_{\tau_m}) \to -\infty \), which is a contradiction, since by definition \( J(\theta_i) \geq 0 \). Thus, there can be at most finitely many \( \tau_n \). Thus, there exists a finite time \( T \) such that almost surely \( |\nabla \theta J(\theta_i)| < \kappa \) for \( t \geq T \). Since \( \kappa \) is arbitrarily chosen, we have proven that \( |\nabla \theta J(\theta_i)| \to 0 \) as \( t \to \infty \) almost surely.

\[\square\]

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Appendix

A Proof of Proposition 3.4

Proof of (i). (3.9) with \( i = 0 \) holds from Lemma 3.2. And for \( i = 1 \), define

\[
\hat{f}(t,x,\theta) = \mathbf{E} f(X^{\theta,x}_t) \quad \text{and} \quad \hat{f}_{t_0}(t,x,\theta) = \hat{f}(t,x,\theta) - \hat{f}(t+t_0,x,\theta)
\]

and we have

\[
\lim_{t_0 \to \infty} \hat{f}_{t_0}(t,x,\theta) = \mathbf{E} f(X^{\theta,x}_t) - \mathbf{E} \sigma_{t_0} f(Y)
\]

Note that by markov property of \( X^\theta_t \), we have

\[
\hat{f}_{t_0}(t,x,\theta) = \hat{f}(t,x,\theta) - \mathbf{E} f(X^{\theta,x}_{t+t_0}) = \hat{f}(t,x,\theta) - \mathbf{E} \left[ f(X^{\theta,x}_{t+t_0}) \bigg| \mathcal{F}_{t_0} \right] = \hat{f}(t,x,\theta) - \mathbf{E} \left[ \hat{f}(t,X^{\theta,x}_{t_0},\theta) \right]
\]

Then we obtain

\[
\nabla \theta \hat{f}_{t_0}(t,x,\theta) = \nabla \theta \hat{f}(t,x,\theta) - \mathbf{E} \left[ \nabla \theta \hat{f}(t,X^{\theta,x}_{t_0},\theta) \right] - \mathbf{E} \left[ \nabla_x \hat{f}(t,X^{\theta,x}_{t_0},\theta) \nabla \theta X^{\theta,x}_{t_0} \right].
\]

We need the following statement

- For any \( t \geq 0, x \in \mathbb{R}^d, \theta \in \mathbb{R}^\ell \)

\[
|\nabla_x \hat{f}(t,x,\theta)| \leq C e^{-\frac{At}{2}}.
\]
• There exist \( \eta > 0 \) such that for any \( t \geq 0, \theta \in \mathbb{R}^\ell, x_1, x_2 \in \mathbb{R}^d \)
\[
\left| \nabla_\theta \hat{f}(t, x_1, \theta) - \nabla_\theta \hat{f}(t, x_2, \theta) \right| \leq C e^{-\frac{\eta}{2} t} |x_1 - x_2|.
\] (A.4)

For the first statement, we have by Lemma 3.1
\[
\left| \hat{f}(t, x_1, \theta) - \hat{f}(t, x_2, \theta) \right| = \left| \mathbf{E} f(X_t^\theta, x_1) - \mathbf{E} f(X_t^\theta, x_2) \right| \leq C \left| X_t^\theta, x_1 - X_t^\theta, x_2 \right| \leq C e^{-\frac{\eta}{2} t} |x_1 - x_2|,
\]
which implies (A.3). Then for the second statement, assumptions A1 and A2 imply that \( X_t^\theta, x \) is differentiable w.r.t. \( \theta \) and its derivative \( \nabla_\theta X_t^\theta, x \) satisfies
\[
\begin{cases}
\quad \left. d\nabla_\theta X_t^\theta, x \right| = \left[ \nabla_x \mu(X_t^\theta, x, \theta) \nabla_\theta X_t^\theta, x + \nabla_\theta \mu(X_t^\theta, x, \theta) \right] dt + \left[ \nabla_x \sigma(X_t^\theta, x, \theta) \nabla_\theta X_t^\theta, x + \nabla_\theta \sigma(X_t^\theta, x, \theta) \right] dW_t,

\quad \nabla_\theta X_0^\theta, x = 0.
\end{cases}
\] (A.5)

where the SDE in (A.5) can be written explicitly as
\[
d\frac{\partial X_t^\theta, x}{\partial \theta_n} = \left[ \nabla_x \mu_m(X_t^\theta, x, \theta) \frac{\partial X_t^\theta, x}{\partial \theta_n} + \frac{\partial \mu_m(X_t^\theta, x, \theta)}{\partial \theta_n} \right] dt + \sum_{k=1}^d \left[ \nabla_x \sigma_{mk}(X_t^\theta, x, \theta) \frac{\partial X_t^\theta, x}{\partial \theta_n} + \frac{\partial \sigma_{mk}(X_t^\theta, x, \theta)}{\partial \theta_n} \right] dW^k_t
\] (A.6)

for \( m \in \{1, 2, \cdots, d\} \) and \( n \in \{1, 2, \cdots, \ell\} \). Due to the dissipativity of \( \mu \), for any \( x, \ y \) and \( t > 0 \), we have that
\[
\langle \mu(x + ty, \theta) - \mu(x, \theta), ty \rangle + \frac{3}{2} |\sigma(x + ty, \theta) - \sigma(x, \theta)|^2 \leq -\beta |t|^2 y^2.
\] (A.7)

Therefore,
\[
\frac{1}{t^2} \left[ \begin{array}{c}
\int_0^t d\mu(x + sy, \theta)ds, \quad ty \\
\int_0^t d\sigma(x + sy, \theta)ds
\end{array} \right] \geq \left[ \begin{array}{c}
\frac{1}{t} \int_0^t \mu_x(x + sy, \theta) yds, \quad y \\
\frac{1}{t} \int_0^t \sigma_x(x + sy, \theta) yds
\end{array} \right] \geq -\beta |y|^2.
\] (A.8)

Taking the limit \( t \to 0^+ \) yields
\[
\langle \mu_x(x, \theta) y, \ y \rangle + \frac{3}{2} |\sigma_x(x, \theta) y|^2 \leq -\beta |y|^2, \quad \forall x, y \in \mathbb{R}^d.
\] (A.9)

Combining (A.5) and (A.9), for any \( k \in \{1, 2, \cdots, \ell\} \)
\[
\frac{d}{dt} \mathbf{E} \left| \frac{\partial X_t^\theta, x}{\partial \theta_k} \right|^2
\]
\[
= \mathbf{E} \left[ 2 \left\langle \nabla_x \mu(X_t^\theta, x, \theta) \frac{\partial X_t^\theta, x}{\partial \theta_k}, \quad \frac{\partial \mu(X_t^\theta, x, \theta)}{\partial \theta_k}, \quad \frac{\partial X_t^\theta, x}{\partial \theta_k} \right\rangle + \left| \nabla_x \sigma(X_t^\theta, x, \theta) \frac{\partial X_t^\theta, x}{\partial \theta_k} \right|^2 \right] + \mathbf{E} \left[ 2 \left\langle \frac{\partial \mu_m(X_t^\theta, x, \theta)}{\partial \theta_k}, \quad \frac{\partial \mu_m(X_t^\theta, x, \theta)}{\partial \theta_k}, \quad \frac{\partial \sigma_{mk}(X_t^\theta, x, \theta)}{\partial \theta_k} \right\rangle \right] \geq -2\beta \mathbf{E} \left| \frac{\partial X_t^\theta, x}{\partial \theta_k} \right|^2 + \beta \mathbf{E} \left| \frac{\partial X_t^\theta, x}{\partial \theta_k} \right|^2 + \left( \frac{1}{\beta} \mathbf{E} \left| \frac{\partial \mu_m(X_t^\theta, x, \theta)}{\partial \theta_k} \right|^2 + 2 \mathbf{E} \left| \frac{\partial \sigma_{mk}(X_t^\theta, x, \theta)}{\partial \theta_k} \right|^2 \right)
\]
\[
\leq -\beta \mathbf{E} \left| \frac{\partial X_t^\theta, x}{\partial \theta_k} \right|^2 + C.
\] (A.10)
where step (a) is by the uniform boundedness of $\nabla_\theta \mu, \nabla_\theta \sigma$ from (2.1). Thus
\[
\sup_{\theta \in \mathbb{R}^\ell, x \in \mathbb{R}^d} \left| \nabla_\theta X_t^{\theta, x} \right|^2 \leq C, \quad \forall t \geq 0. \tag{A.11}
\]

Then
\[
\begin{align*}
\left| \nabla_\theta \hat{f}(t, x_1, \theta) - \nabla_\theta \hat{f}(t, x_2, \theta) \right| & = \left| \nabla_\theta \hat{E}(X_t^{\theta, x_1}) - \nabla_\theta \hat{E}(X_t^{\theta, x_2}) \right| \\
& = \left| \nabla \hat{f}(X_t^{\theta, x_1}) \nabla \theta X_t^{\theta, x_1} \right| - \left| \nabla \hat{f}(X_t^{\theta, x_2}) \nabla \theta X_t^{\theta, x_2} \right| \\
& \leq \left| \nabla \hat{f}(X_t^{\theta, x_1}) \nabla \theta X_t^{\theta, x_1} \right| - \left| \nabla \hat{f}(X_t^{\theta, x_2}) \nabla \theta X_t^{\theta, x_1} \right| + \left| \nabla \hat{f}(X_t^{\theta, x_2}) \nabla \theta X_t^{\theta, x_2} \right|
\end{align*}
\]
\[
:= I_1 + I_2 \tag{A.12}
\]

For the terms $I_1$, it follows from Assumption A4, Lemma 3.1 and (A.11) that
\[
I_1 \leq C \left\{ \mathbb{E} \left| X_t^{\theta, x_1} - X_t^{\theta, x_2} \right|^2 \right\}^{\frac{1}{2}} \cdot \left( \mathbb{E} \left| \nabla_\theta X_t^{\theta, x_1} \right| \right)^{\frac{1}{2}} \leq C e^{-\alpha t} |x_1 - x_2|. \tag{A.13}
\]

Then for $I_2$, by the same calculation in Lemma 3.1 we have
\[
\mathbb{E} \left| \nabla_\theta X_t^{\theta, x_1} - \nabla_\theta X_t^{\theta, x_2} \right|^2 \leq e^{-\beta t} |x_1 - x_2|^2. \tag{A.14}
\]

Actually, define $Y_t = \nabla_\theta X_t^{\theta, x_1} - \nabla_\theta X_t^{\theta, x_2}$ and Let $Y_t^{::k}$ denote its k-th column, i.e.
\[
Y_t^{::k} := \frac{\partial X_t^{\theta, x_1}}{\partial \theta_k} - \frac{\partial X_t^{\theta, x_2}}{\partial \theta_k}
\]

Then from (A.5) we have for any $k \in \{1, 2, \ldots, \ell\}$
\[
\begin{align*}
\frac{d}{dt} \mathbb{E} \left| Y_t^{::k} \right|^2 & = \mathbb{E} \left[ 2 \left\langle \nabla_x \mu(X_t^{\theta, x_1}, \theta) \frac{\partial X_t^{\theta, x_1}}{\partial \theta_k} + \frac{\partial \mu(X_t^{\theta, x_1}, \theta)}{\theta_k} - \nabla_x \sigma(X_t^{\theta, x_2}, \theta) \frac{\partial X_t^{\theta, x_2}}{\partial \theta_k} - \frac{\partial \sigma(X_t^{\theta, x_2}, \theta)}{\theta_k}, Y_t^{::k} \right\rangle \right] \\
& \quad + \mathbb{E} \left[ \left| \nabla_x \sigma(X_t^{\theta, x_1}, \theta) \frac{\partial X_t^{\theta, x_1}}{\partial \theta_k} + \frac{\partial \sigma(X_t^{\theta, x_1}, \theta)}{\theta_k} - \nabla_x \sigma(X_t^{\theta, x_2}, \theta) \frac{\partial X_t^{\theta, x_2}}{\partial \theta_k} - \frac{\partial \sigma(X_t^{\theta, x_2}, \theta)}{\theta_k} \right|^2 \right] \tag{A.15}
\end{align*}
\]
\[
\leq \mathbb{E} \left[ 2 \left\langle \nabla_x \mu(X_t^{\theta, x_1}, \theta) Y_t^{::k}, Y_t^{::k} \right\rangle + 2 \left| \nabla_x \sigma(X_t^{\theta, x_1}, \theta) Y_t^{::k} \right|^2 \right] + \beta \mathbb{E} |Y_t^{::k}|^2 + C \mathbb{E} \left| X_t^{\theta, x_1} - X_t^{\theta, x_2} \right|^2
\leq - \beta \mathbb{E} |Y_t^{::k}|^2 + C e^{-\beta t} |x_1 - x_2|^2,
\]

which derives (A.14). Noting that from Assumption A4 we have $|\nabla f| \leq C$ and thus
\[
I_2 \leq C \mathbb{E} \left| \nabla_\theta X_t^{\theta, x_1} - \nabla_\theta X_t^{\theta, x_2} \right| \leq C e^{-\frac{\beta}{2} t} |x_1 - x_2|. \tag{A.16}
\]

Therefore, (A.13) and (A.16) imply (A.4).

Finally, by estimate (A.2), (A.3), (A.4) and (A.11), we have
\[
|\nabla_\theta f_{t_0}(t, x, \theta)| \leq C e^{-\frac{\alpha}{2} t_0} \mathbb{E} |x - X_t^{\theta, x}| + C e^{-\frac{\beta}{2} t} \leq C e^{-\frac{\beta}{3} t}(1 + |x|), \quad \forall t_0 \geq 0. \tag{A.17}
\]

Let $t_0 \to \infty$ and by DCT, we have
\[
\left| \nabla_\theta \mathbb{E} f(X_t^{\theta, x}) - \nabla_\theta \mathbb{E}_\pi f(Y) \right| \leq C e^{-\frac{\beta}{3} t}(1 + |x|). \tag{A.18}
\]
The proof of (3.9) for \( i = 2 \) follows from the same idea. Actually, from (A.2) we know

\[
\left| \nabla_\theta \hat{f}_t(t, x, \theta) \right| \leq \left| \nabla^2_\theta \hat{f}(t, x, \theta) \right| - \mathbb{E} \left[ \nabla_\theta \hat{f}(t, X^\theta_{t_0}, \theta) \right] \\
+ \mathbb{E} \left[ \nabla_\theta \nabla^2_\theta \hat{f}(t, X^\theta_{t_0}, \theta) \nabla_\theta X^\theta_{t_0} \right] \\
+ \mathbb{E} \left[ \nabla^2_\theta \hat{f}(t, X^\theta_{t_0}, \theta) \nabla^2_\theta \nabla_\theta X^\theta_{t_0} \right] \\
+ \mathbb{E} \left[ \nabla \hat{f}(t, X^\theta_{t_0}, \theta) \nabla^2_\theta X^\theta_{t_0} \right] \\
:= I_3 + I_4 + I_5 + I_6.
\]

First for the term \( I_3 \), note that

\[
\nabla_\theta \hat{f}(t, x, \theta) = \nabla_\theta \mathbb{E} f(X^\theta_{t_0}, \theta) = \mathbb{E} \left[ \nabla f(X^\theta_{t_0}) \nabla_\theta X^\theta_{t_0} \right],
\]

and thus

\[
\nabla^2_\theta \hat{f}(t, x, \theta) = \mathbb{E} \left[ \nabla^2 f(X^\theta_{t_0}) \nabla_\theta X^\theta_{t_0}, \nabla \nabla_\theta X^\theta_{t_0} \right] + \mathbb{E} \left[ \nabla f(X^\theta_{t_0}) \nabla^2_\theta \nabla_\theta X^\theta_{t_0} \right],
\]

where \( \nabla f(X^\theta_{t_0}) \nabla^2_\theta f(X^\theta_{t_0}) \) is defined as:

\[
\left[ \nabla f(X^\theta_{t_0}) \nabla^2_\theta f(X^\theta_{t_0}) \right]_{m,n} = \frac{\partial^2 X_{t_0}^{\theta,x}}{\partial \theta_m \partial \theta_n}.
\]

Then for any \( x_1, x_2 \in \mathbb{R}^d \),

\[
\left| \nabla^2_\theta \hat{f}(t, x_1, \theta) - \nabla^2_\theta \hat{f}(t, x_2, \theta) \right| \\
\leq \mathbb{E} \left[ \nabla^2 f(X^\theta_{t_0}) \nabla_\theta X^\theta_{t_0}, \nabla \nabla_\theta X^\theta_{t_0} \right] - \mathbb{E} \left[ \nabla^2 f(X^\theta_{t_0}) \nabla_\theta X^\theta_{t_0}, \nabla \nabla_\theta X^\theta_{t_0} \right] \\
+ \mathbb{E} \left[ \nabla f(X^\theta_{t_0}) \nabla^2_\theta X^\theta_{t_0} \right] - \mathbb{E} \left[ \nabla f(X^\theta_{t_0}) \nabla^2_\theta X^\theta_{t_0} \right] \\
:= I_{3,1} + I_{3,2}.
\]

As in (A.10), by Itô’s formula for any \( k \in \{1, 2, \cdots, \ell\} \)

\[
\frac{d}{dt} \mathbb{E} \left| \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^4 \leq \mathbb{E} \left[ \left| \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2 \right] \left( 4 \left| \nabla_\mu(X^\theta_{t_0}, \theta) \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2 + \left| \nabla_\sigma(X^\theta_{t_0}, \theta) \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2 \right) \\
+ 6 \left| \nabla_\sigma^2(X^\theta_{t_0}, \theta) \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2 \right) \\
\leq \mathbb{E} \left[ \left| \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2 \right] \left( -4\beta \left| \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2 + 2 \left| \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2 + C \left( \left| \frac{\partial \mu(X^\theta_{t_0}, \theta)}{\partial \theta_k} \right|^2 + \left| \frac{\partial \sigma(X^\theta_{t_0}, \theta)}{\partial \theta_k} \right|^2 \right) \right) \\
\leq -2\beta \mathbb{E} \left| \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^4 + C \mathbb{E} \left| \frac{\partial X_{t_0}^{\theta,x}}{\partial \theta_k} \right|^2,
\]

where step (a) use the same calculations as in (2.17). Thus combining (A.11) and (A.22), we have

\[
\sup_{\theta \in \mathbb{R}^d, x \in \mathbb{R}^d} \left| \nabla_\theta X_{t_0}^{\theta,x} \right|^4 \leq C, \ \forall t \geq 0.
\]
Then for $I_{3,1}$, by direct computation, we know there exists $\gamma > 0$ such that

$$I_{3,1} \leq \mathbb{E} \left[ \left( \nabla^2 f(X^\theta_{\mathbf{x}}, x_1) - \nabla^2 f(X^\theta_{\mathbf{x}}, x_2) \right) \cdot \left( \nabla^2 f(X^\theta_{\mathbf{x}}, x_1) - \nabla^2 f(X^\theta_{\mathbf{x}}, x_2) \right) \right] + \mathbb{E} \left[ \left( \nabla^2 f(X^\theta_{\mathbf{x}}, x_1) - \nabla^2 f(X^\theta_{\mathbf{x}}, x_2) \right) \cdot \left( \nabla^2 f(X^\theta_{\mathbf{x}}, x_1) - \nabla^2 f(X^\theta_{\mathbf{x}}, x_2) \right) \right]$$

$$\leq \mathbb{E} \left[ \nabla^2 f(X^\theta_{\mathbf{x}}, x_1) - \nabla^2 f(X^\theta_{\mathbf{x}}, x_2) \right] \cdot \left( \nabla^2 f(X^\theta_{\mathbf{x}}, x_1) - \nabla^2 f(X^\theta_{\mathbf{x}}, x_2) \right) \cdot \left( \nabla^2 f(X^\theta_{\mathbf{x}}, x_1) - \nabla^2 f(X^\theta_{\mathbf{x}}, x_2) \right)$$

$$\leq C \left( E \left| X^\theta_{\mathbf{x}} - X^\theta_{\mathbf{x}} \right|^2 \right)^{\frac{1}{2}} + C \left( E \left| \nabla X^\theta_{\mathbf{x}} \right|^2 \right)^{\frac{1}{2}} \cdot \left( E \left| \nabla X^\theta_{\mathbf{x}} \right|^2 \right)^{\frac{1}{2}} \cdot \left( E \left| \nabla X^\theta_{\mathbf{x}} \right|^2 \right)^{\frac{1}{2}}$$

$$\leq Ce^{-\gamma t} |x_1 - x_2|,$$

(A.24)

where step (a) we use Lemma 3.1, (A.14) and (A.23).

Under the assumptions A1-A3 and using the same calculations as in (A.10) and (A.15), it is easy to prove that

$$\sup_{x \in \mathbb{R}^d, \theta \in \mathbb{R}^\ell} \mathbb{E} \left| \nabla^2 X^\theta_{\mathbf{x}} \right|^2 \leq C, \quad \forall t \geq 0,$$

(A.25)

and there exists $\gamma > 0$ such that

$$\sup_{\theta \in \mathbb{R}^\ell} \mathbb{E} \left| \nabla^2 X^\theta_{\mathbf{x}} - \nabla^2 X^\theta_{\mathbf{x}} \right|^2 \leq Ce^{-\gamma t} |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^d, \ t \geq 0.$$

(A.26)

Thus we have

$$I_{3,2} \leq \left| \mathbb{E} \nabla f (X^\theta_{\mathbf{x}}, x) \left( \nabla^2 X^\theta_{\mathbf{x}} - \nabla^2 X^\theta_{\mathbf{x}} \right) \right| + \mathbb{E} \left( \nabla f (X^\theta_{\mathbf{x}}, x) - \nabla f (X^\theta_{\mathbf{x}}, x) \right) \nabla^2 X^\theta_{\mathbf{x}} \leq Ce^{-\gamma t} |x_1 - x_2|.$$

(A.27)

Combining (A.21), (A.24) and (A.27), we have

$$I_3 \leq Ce^{-\gamma t} \left( \mathbb{E} \left| x - X^\theta_{\ell_0} \right| \right) \leq Ce^{-\gamma t} (1 + |x|).$$

(A.28)

For the term $I_4$, note that

$$\nabla X^\theta_{\mathbf{x}} \nabla \theta f(t, x, \theta) = \mathbb{E} \left( \nabla^2 f (X^\theta_{\mathbf{x}}, x) \nabla X^\theta_{\mathbf{x}} \right) + \mathbb{E} \nabla f (X^\theta_{\mathbf{x}}, x) \nabla X^\theta_{\mathbf{x}}.$$

By Lemma 3.1 and (A.14), we have

$$\sup_{x \in \mathbb{R}^d, \theta \in \mathbb{R}^\ell} \left( \mathbb{E} \left| \nabla X^\theta_{\mathbf{x}} \right|^2 + \mathbb{E} \left| \nabla X^\theta_{\mathbf{x}} \right|^2 \right) \leq Ce^{-\frac{d}{t}}, \quad t \geq 0,$$

which derives that

$$\left| \nabla X^\theta_{\mathbf{x}} \nabla \theta f(t, x, \theta) \right| \leq Ce^{-\frac{d}{t}}.$$

(A.29)

Hence, it is easy to see that

$$I_4 \leq Ce^{-\frac{d}{t}} \mathbb{E} \left| \nabla X^\theta_{\ell_0} \right| \leq Ce^{-\frac{d}{t}}.$$

(A.30)

For the term $I_5$, by a similar argument as for $I_4$, we have

$$\sup_{x \in \mathbb{R}^d, \theta \in \mathbb{R}^\ell} \mathbb{E} \left| \nabla^2 X^\theta_{\mathbf{x}} \right|^2 \leq Ce^{-\beta t}, \quad t \geq 0,$$

and then

$$\left| \nabla^2 f(t, x, \theta) \right| \leq Ce^{-\beta t}.$$ (A.31)

Hence,

$$I_5 \leq Ce^{-\frac{\beta t}{t}}.$$ (A.32)
Finally for $I_6$, from (A.3) and (A.25) we can get

$$I_6 \leq Ce^{-\frac{\alpha t}{2}}.$$  

(A.33)

Hence, combining (A.28), (A.30), (A.32) and (A.33) there exists $\gamma > 0$ such that

$$|\nabla^2_{t,x}f(t,x,\theta)| \leq Ce^{-\gamma t} \leq Ce^{-\gamma t}(1 + |x|), \quad \forall t_0 \geq 0.$$  

(A.34)

Let $t_0 \to \infty$, we have

$$\left|\nabla^2_{t,x}E_f(X^\theta_{t,x}) - \nabla^2_{t,x}E_{\pi_\theta}f(Y)\right| \leq Ce^{-\gamma t}(1 + |x|).$$  

(A.35)

Proof of (ii) and (iii). First note that by Lemma 3.3

$$\left|E_{\pi_\theta}f(Y)\right| = \lim_{t \to \infty} \left|E_f(X^\theta_{t,x})\right| \leq C.$$  

(A.36)

By (A.11), we have

$$\left|\nabla_{\theta}E_f(X^\theta_{t,x})\right| = \left|E\nabla f(X^\theta_{t,x})\nabla_{\theta}X^\theta_{t,x}\right| \leq C,$$  

(A.37)

which together with (3.9) derive

$$\left|\nabla_{\theta}E_{\pi_\theta}f(Y)\right| = \lim_{t \to \infty} \left|\nabla_{\theta}E_f(X^\theta_{t,x})\right| \leq C.$$  

(A.38)

Similarly we can get the bound for $\left|\nabla^2_{t,x}E_f(X^\theta_{t,x})\right|$ and derive (3.10) for $i = 2$.

Then for (3.11), the case for $j = 1$ directly follows from (A.3), (A.4) and the case for $i = 0, j = 2$ follows from (A.31). Finally, it is easy to prove there exists $\gamma > 0$ such that for $i, j \in \{0, 1\}$

$$\sup_{\theta \in \mathbb{R}^t} E\left[\nabla^j_x\nabla^i_{\theta}X^\theta_{x_1} - \nabla^j_x\nabla^i_{\theta}X^\theta_{x_2}\right]^2 \leq Ce^{-\gamma t}|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^d, \quad t \geq 0.$$  

(A.39)

Thus by the same method to estimate (A.21), we can get there exists $\eta > 0$ such that

$$\left|\nabla_x\nabla_{\theta}f(t,x_1,\theta) - \nabla_x\nabla_{\theta}f(t,x_2,\theta)\right|$$

$$\leq E\left[\nabla^2 f(X^\theta_{x_1})\nabla_x X^\theta_{x_1}, \nabla_{\theta}X^\theta_{x_1}\right] - \left(\nabla^2 f(X^\theta_{x_2})\nabla_x X^\theta_{x_2}, \nabla_{\theta}X^\theta_{x_2}\right)\right|$$

$$+ E\left[\nabla f(X^\theta_{x_1})\nabla_{\theta}X^\theta_{x_1} - \nabla f(X^\theta_{x_2})\nabla_{\theta}X^\theta_{x_2}\right]$$

$$\leq Ce^{-\gamma t}|x_1 - x_2|,$$  

(A.40)

which derives the case for $i = 1, j = 2$ and thus the proof is completed.

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