The effect of spherical aberration on singularities and spectral changes of focused beams

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New Journal of Physics 8 (2006) 93
Received 21 February 2006
Published 9 June 2006
Online at http://www.njp.org/
doi:10.1088/1367-2630/8/6/093

Abstract. The singular transformation and spectral behaviour of spatially fully coherent polychromatic plane beams focused by an aperture lens with spherical aberration (SA) is studied. The expressions of the light intensity and the spectrum distributions in the focal region of a plane beam focused by a lens with SA are derived. It is shown that, with the SA, one axial singularity will split into two new singularities. At the off-axial positions, new singularities appear and split with increasing SA. The phase at the positions of the singularities is indeterminate and the spectrum is split into two lines with equal height. In addition, the numerical calculation shows that the spectrum is red-shifted at the side of the singularities close to the focus and blue-shifted at the side far away from the focus.

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1. Introduction

In recent years, a great deal of attention has been paid to the structure of wave fields in the neighbourhood of points where the field amplitude has zero value [1]–[3]. At such points the phase of the fields is singular. It has been known that in the neighbourhood of the singular points, the wave fields exhibit a rather complex structure, for example, dislocations and optical vortices. Studies of phenomena associated with phase singularities have gradually developed into a new branch of physical optics, sometimes referred to as singular optics [1].

Most of the publications concerned with singular optics deal with monochromatic waves. More recently, singular optics has been extended to polychromatic cases by Professor Wolf and his co-workers [4, 5], in which they have showed that drastic spectral changes occur in the vicinity of intensity zeros in the focal region of polychromatic, spatially coherent, converging spherical waves, that is, the spectrum is red-shifted at some points, blue-shifted at others, and is split into two lines elsewhere. Similar phenomena of drastic spectral changes were also found in Young’s double-slit interference experiments etc [6]–[10]. These drastic spectral changes in the vicinity of singular points are shown to have close relation to spectral switches, which were found several years ago [11, 12], and the relation between the drastic spectral change in the vicinity of singular points and the spectral switches were studied in [13]. Otherwise, the influence of spherical aberration (SA) on the intensity distribution and phase singularities has generated considerable interests in recent years [14, 15].

In this paper, we investigate the singular transformation and spectral behaviour of spatially fully coherent polychromatic plane beams focused by an aperture lens with SA. The light intensity and the spectrum distributions in the focal region of a plane beam focused by a lens with SA are investigated in detail. It is demonstrated that the spectrum is split into two lines with equal height at the positions of the singularities. There are singularities at the geometrical focal plane and at the optical axis when a plane beam is focused by an ideal aperture lens without SA. But with the SA, the singularities at the geometrical focal plane will shift and the singularities at the optical axis will split. The numerical calculation shows that the spectrum is red-shifted at the side of the singularities close to the geometrical focus and blue-shifted at the side far away from the geometrical focus.

2. Simulation model

Suppose that a spatially fully coherent polychromatic plane beams $E(r', \omega)$ which is written as $E(r', \omega) = A(\omega)$, is incident upon an aperture lens of full width $2a$ and focal length $f$ with SA at the $z = 0$ plane shown in figure 1. Therefore the field emerging from the lens with SA $\Phi_R$, at a typical point $P$, is expressed as

$$E_0(P) = E_0(r', \omega) = A(\omega)\exp\left[\frac{ik}{\Phi_R - \frac{r'^2}{2f}}\right], \quad (1)$$

where $r'$ is the radial coordinate and $\omega$ is angular frequency.
Figure 1. Geometry for focusing plane beams by a lens with SA. $\Sigma$ is the wave front behind the lens; $\Sigma^*$ is the spherical reference sphere centred at $z = f$.

According to the Collins formula [16], we obtain the optical field of the point Q at the $z$ plane

$$E(Q) = E(r, \varphi, z, \omega)$$

$$= -\frac{ik}{2\pi B} \exp(ikz) \int_0^a \int_0^{2\pi} E_0(P) \exp \left\{ \frac{ik}{2B} \left[ Ar'^2 + Dr'^2 - 2r' \cdot r \cos(\varphi' - \varphi) \right] \right\} r' \, dr' \, d\varphi',$$

(2)

where $k = 2\pi/\lambda = \omega/c$ is the wavenumber associated with angular frequency $\omega$, the integral is performed over the portion of the reference surface. The positions of P and Q can be denoted as $(r', \varphi', 0)$ and $(r, \varphi, z)$ respectively, in cylindrical coordinates. A, B and D are elements of the transfer matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

(3)

The radius of the lens is assumed to be $a$. Equation (2) is the expression for the optical field of a plane beam focused by a lens with SA $\Phi_R$. For simplicity, we discuss only primary SA, which is given by

$$\Phi_R = C_4 r'^4,$$

(4)

where $C_4$ is the SA coefficient.

The spectral density function $S(r, z, \omega)$ of the light at point Q can be given by

$$S(r, z, \omega) = E^*(r, \varphi, z, \omega) E(r, \varphi, z, \omega),$$

(5)

where the asterisk denotes the complex conjugate.

On substituting equations (2)–(4) into equation (5) and after some algebra, we obtain

$$S(\rho, z, \omega) = S^{(0)}(\omega)M(\rho, z, \omega)$$

(6)
with
\[ S^{(0)}(\omega) = A^*(\omega)A(\omega) \] (7)
and
\[ M(\rho, z, \omega) = \frac{\pi^2 \omega^2}{(z/f)^2 \omega_0^2} \left[ \left( \int_0^N J_0 \left( \frac{2\pi\rho \omega}{\omega_0 z/f} \sqrt{xN} \right) \cos \left\{ \frac{2\pi\omega}{\omega_0} \left[ \frac{C_4 a^4}{\lambda_0} \left( \frac{x}{N} \right)^2 + \frac{1 - z/f}{2z/f} x \right] \right) dx \right)^2 
+ \left( \int_0^N J_0 \left( \frac{2\pi\rho \omega}{\omega_0 z/f} \sqrt{xN} \right) \sin \left\{ \frac{2\pi\omega}{\omega_0} \left[ \frac{C_4 a^4}{\lambda_0} \left( \frac{x}{N} \right)^2 + \frac{1 - z/f}{2z/f} x \right] \right) dx \right)^2 \right], \] (8)
where
\[ \rho = \frac{r}{a} \] (relative polar radius),
\[ \rho = \frac{r^2}{\lambda_0 f} \] (normalized polar radius),
\[ N = \frac{a^2}{\lambda_0 f} \] (Fresnel number).

\( \omega_0 \) and \( \lambda_0 \) are the central frequency and central wavelength of the original spectrum \( S^{(0)}(\omega) \) respectively. For definiteness, assume that the original spectrum \( S^{(0)}(\omega) \) has a Gaussian shape centred at frequency \( \omega_0 \)
\[ S^{(0)}(\omega) = S_0 \exp \left\{-\frac{(\omega - \omega_0)^2}{2\sigma_0^2} \right\}, \] (12)
where \( S_0 \) is a positive constant and \( \sigma_0 \) is the bandwidth.

\( M(\rho, z, \omega) \) is called the spectral modifier of the plane beam. \( J_0(\rho) \) is the zero-order Bessel function of the first kind. Equation (6) provides an analytical expression for the spectrum of spatially coherent plane beams focused by an aperture lens with SA. It follows from (6) that the spectrum is equal to the original spectrum \( S^{(0)}(\omega) \) and spectral modifier \( M(\rho, z, \omega) \), and the latter depends on the SA coefficient \( C_4 \), axial distance \( z \) and relative polar position \( \rho \).

The substitution from (12) into (6) yields
\[ S(\rho, z, \omega) = S_0 \exp \left\{-\frac{(\omega - \omega_0)^2}{2\sigma_0^2} \right\} M(\rho, z, \omega). \] (13)
Assume that the incident wave is monochromatic; in this case the intensity at the point \( Q(\rho, z) \) is
\[ I(Q) = S(\rho, z, \omega_0) = S_0 M(\rho, z, \omega_0). \] (14)
In figure 2(a), we give singularity (intensity) distributions in the focal region of $z, \rho$ with SA coefficient $C_4 = 0 \text{ m}^{-3}$ for $N = 100$, $\omega_0 = 3 \times 10^{15} \text{ s}^{-1}$ and $a = 1 \times 10^{-3} \text{ m}$. The normalized intensity of the central points in the dark rings is zero. Therefore, these points are singularities. As shown in figure 2(a) with no SA for $0.95 \leq z/f \leq 1.05$ and $0 \leq \rho \leq 0.02$, there are six singularities (actually, there are three dark rings) at the geometrical focal plane and four singularities (A, B, C, D) at the $z$-axis.

To characterize the spectral changes, we introduce the mean frequency, which is defined as

$$\bar{\omega}(\rho) = \frac{\int \omega S(\rho, z, \omega) \, d\omega}{\int S(\rho, z, \omega) \, d\omega}.$$  

When the normalized mean frequency $\bar{\omega}(\rho) > \omega_0$, the spectrum is called blue-shifted, whereas $\bar{\omega}(\rho) < \omega_0$ the spectrum is red-shifted. The mean frequency $\bar{\omega}$ of the spectrum corresponding to the singularity (intensity) distribution in figure 2(a) in the focal region of $z, \rho$ is plotted in figure 2(b), for $C_4 = 0 \text{ m}^{-3}$, $\omega_0 = 3 \times 10^{15} \text{ s}^{-1}$, $\sigma_0 = 0.03 \times 10^{15} \text{ s}^{-1}$ and $N = 100$. The colour is more red or blue as the spectrum is more red-shifted or blue-shifted respectively. It can be seen from the figures that the spectrum is red-shifted at the side of the singularities close to the focus and blue-shifted at the side far away from the focus.

In figure 3(a) with $C_4 = 0.1 \times 10^6 \text{ m}^{-3}$, the six singularities at the geometrical focal plane in the figure 2(a) shift far away from the aperture lens. Compared to the figure 2(a), the axial singularities A and B split into $A_1A_2$ (actually, it is a dark ring) and $B_1B_2$ respectively. Although $B_1B_2$ seems like a short line in figure 3(a), actually, the point B has split into a little dark
Figure 3. (a) Singularity distributions in the focal region of $z, \rho$ with $C_4 = 0.1 \times 10^6 \text{ m}^{-3}$. (b) Colour-coded plot of the mean frequency $\bar{\sigma}$ corresponding to the singularity distribution in figure 3(a). The other calculation parameters are the same as those in figure 2.

Figure 4. (a) Singularity distributions in the focal region of $z, \rho$ with $C_4 = 0.1 \times 10^6 \text{ m}^{-3}$. (b) Colour-coded plot of the mean frequency $\bar{\omega}$ corresponding to the singularity distribution in figure 4(a).

It can be observed clearly in figure 3(b), colour-coded plot of the mean frequency $\bar{\sigma}$ corresponding to the singularity distribution in figure 3(a).

When $C_4$ is equal to $-0.1 \times 10^6 \text{ m}^{-3}$, the six singularities shift towards the aperture lens just as seen in figure 4(a), and the axial singularities C and D split into $C_1C_2$ and $D_1D_2$ respectively. The corresponding mean frequency distribution of the figure 4(a) is plotted in figure 4(b).
Figure 5. The splitting process of the singularity A at intervals of $\pi/4$ with the SA from $0 \text{ m}^{-3}$ to $700 \text{ m}^{-3}$ for $1.020375 \leq z/f$ and $0 \leq \rho \leq 0.0004$. The width of the diagram is about $1.646\lambda_0$.

Figure 6. Normalized spectrum at the positions of the singularities appearing (such as A, E, A_1, A_2, etc) in the focal region of $z$, $\rho$ with the changing of SA, $\omega_0 = 3 \times 10^{15} \text{s}^{-1}$, $\sigma_0 = 0.03 \times 10^{15} \text{s}^{-1}$ and $N = 100$. The dashed curve indicates the original spectrum $S^{(0)}(\omega)$, and the solid curve indicates the spectrum $S(\omega)$ in the diffraction field.

The splitting process of the singularity A with the SA from $0 \text{ m}^{-3}$ to $700 \text{ m}^{-3}$ for $1.020375 \leq z/f \leq 1.02044$ and $0 \leq \rho \leq 0.0004$ is shown in figure 5. It can be seen that with the SA one axial singularity will split into two new singularities (a dark ring) and equiphase lines converge at the singularities, that is, the phase at the singularity is indeterminate. From the process we can see that with the SA $C_4 = 0 \text{ m}^{-3}$, the phase increases the value 0 on following a small circuit around the singularity, that is, there is no topological charge associated with this singularity. However, the topological charge of new singularities appearing with the increase in SA is not zero. The topological charges of the upper new singularity and the lower one are $-1$ and $+1$. We can see that the total topological charge is conserved [15, 17].

As shown in figure 6, compared to the original spectrum, the observed spectrum at the positions of the singularities appearing (such as point A($z/f = 1.0204082$, $\rho = 0$) and point...
Figure 7. (a) Singularity distributions in the focal region of $z, \rho$ with $C_4 = 0.3 \times 10^6 \text{m}^{-3}$. (b) Colour-coded plot of the mean frequency $\bar{\omega}$ corresponding to the singularity distribution in figure 6(a). The other calculation parameters are the same as those in figure 2.

Figure 8. (a) Singularity distributions in the focal region of $z, \rho$ with $C_4 = 0.35 \times 10^6 \text{m}^{-3}$. (b) Colour-coded plot of the mean frequency $\bar{\omega}$ corresponding to the singularity distribution in figure 7(a).

$E(z/f = 1, \rho = 0.006099)$ when $C_4 = 0$, point $A_1(z/f = 1.0237091, \rho = 0.0017675)$ and point $E(z/f = 1.002905, \rho = 0.0061198)$ when $C_4 = 0.1 \times 10^6 \text{m}^{-3}$, etc is split into two lines with nearly equal height, sometimes this phenomenon is called a spectral switch.

The singularity distributions of the focused plane beam in the focal region of $z, \rho$ with the increasing of $C_4$ and their corresponding mean frequency distributions are given from figures 7–10, where the calculation parameters are $N = 100$, $\omega_0 = 3 \times 10^{15} \text{s}^{-1}$ and
Figure 9. (a) Singularity distributions in the focal region of $z, \rho$ with $C_4 = 0.45 \times 10^6$ m$^{-3}$. (b) Colour-coded plot of the mean frequency $\bar{\omega}$ corresponding to the singularity distribution in figure 8(a).

Figure 10. (a) Singularity distributions in the focal region of $z, \rho$ with $C_4 = 0.55 \times 10^6$ m$^{-3}$. (b) Colour-coded plot of the mean frequency $\bar{\omega}$ corresponding to the singularity distribution in figure 9(a).

As can be seen, with the increasing of the SA, new singularities $H_1H_2$ (actually, it is a ring) appear in figure 7 and split into $H_{11}H_{12}$ and $H_{21}H_{22}$ (there are two rings) in figure 10.

In figure 11, the intensity and mean frequency distribution with the increase in SA from $0$ m$^{-3}$ to $0.58 \times 10^6$ m$^{-3}$ for $0.95 \leq z/f \leq 1.05$ and $0 \leq \rho \leq 0.02$ is shown. The other
Figure 11. The intensity and mean frequency distribution with the increase in SA from 0 \( m^{-3} \) to \( 0.58 \times 10^6 \ m^{-3} \) for \( 0.95 \leq z/f \leq 1.05 \) and \( 0 \leq \rho \leq 0.02 \). The other calculation parameters are the same as those in figure 7.

calculation parameters are the same as those in figure 7. From the figure the changing process of singularities appearing, shifting and vanishing can be seen clearly.

4. Conclusions

In this paper, we have studied the singularity transformation and spectral behaviour of spatially fully coherent polychromatic plane beams focused by an aperture lens with SA both analytically and numerically. It has been shown that the phase at the singularity is indeterminate and with the increase in SA, one axial singularity will split into two new singularities and new off-axial singularities will appear and split again. At the positions of the singularities, the spectrum is split into two lines with equal height. This phenomenon is called the spectral switch. In addition, the numerical calculation shows that the spectrum is red-shifted at the side of the singularities close to the focus and blue-shifted at the side far away from the focus. It has been shown that the singularity position at which the spectral switches occur is dependent on the SA \( C_4 \), axial distance \( z \) and relative polar radius \( \rho \). The potential applications of the spectral switch in optical interconnection and optical communication were shown in [18, 19].

Acknowledgments

This research was supported by the National Natural Science Foundation of China (grant no 60477041) and Fujian Natural Science Foundation of China (grant no A0510018).

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