Fully discrete Kirchhoff formulas with CQ–BEM

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Abstract

In this paper we propose and analyze a fully discrete method for a direct boundary integral formulation of the scattering of a transient acoustic wave by a sound-soft obstacle. The method uses Galerkin-BEM in the space variables and three different choices of time-stepping strategies based on Convolution Quadrature. The numerical analysis of the method is carried out directly in the time domain, not reverting to Laplace transform techniques.

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1 Introduction

In this paper we propose and analyze a fully discrete method for the direct boundary integral formulation of the Dirichlet problem for the causal acoustic wave equation, exterior to a domain with Lipschitz boundary in $\mathbb{R}^d$ ($d = 2$ or $3$). The method arises from using a general Galerkin semidiscretization-in-space and multistep-based Convolution Quadrature (CQ) in time. From the point of view of the numerical method, this paper extends work in [12] and [3]. A survey of recent results for CQ-BEM discretization of a wide variety transient problems can be found in [7].

Analytical literature on time-domain integral equations has typically been focused on integral equations of the first kind arising from indirect formulations. The origin of these techniques was based on Galerkin-in-time methods [1], while CQ techniques were developed only about one decade later. The present paper uses a direct formulation, leading to an integral equation of the first kind similar to those treated in [11, 12, 3]. The main differences lie in the fact that data appear under the action of a retarded integral
operator (which will have to be discretized as well) and that the unknown on the boundary is a quantity of physical interest.

In this paper we propose the development of a systematic analysis of CQ-BEM taking care of all aspects of discretization: (a) data interpolation on the computational grid, (b) Galerkin semidiscretization-in-space of the associated retarded integral equation, (c) discretization in time (using CQ) of the integral operators in both sides of the equation, (d) discretization in time of the postprocessed potentials (acting on the data and unknown of the integral equation) to obtain the scattered wave field in exterior points. The main difference with the traditional black-box analysis proposed by Lubich [12] is in the fact that we propose to do most of the analysis directly in the time domain. Original work in the analysis of CQ-BEM dealt only with the simplest retarded boundary integral equations, that are coercive in the Laplace domain. Coercivity is inherited by the Galerkin semidiscrete-in-space problem, but some properties of the fully discrete problem (including postprocessing of the solution to obtain the associated potentials and treatment of data that appear under the action of retarded integral operators) have to be investigated in a more direct fashion [10]. More recently, some estimates in the time-domain [8, 16] have expanded the analytical toolbox that can be used to prove error estimates for full discretization of retarded boundary integral equations. It has to be noted that most of the literature that is relevant for this analysis had been carried out using Laplace transforms—the paper [15] seems to be a lone exception. The passage through the Laplace domain makes for a relatively streamlined analysis that can be expanded to a wide variety of problems [10, 6] but is likely to yield less sharp results than a direct analysis in the time domain. As announced, in this paper we will develop the more recent technology of time-domain estimates to show properties of the Galerkin semidiscrete-in-space problem (these are pertinent for other kind of time-discretization methods, using Galerkin schemes) and of the full discretization of the problem. In particular, we will obtain a proof of convergence of the trapezoidal rule CQ method that is not directly reachable in the Laplace domain. The tools for this analysis are varied but not complicated: (a) identification of the weak convolutional retarded integral equations and layer potentials with strong solutions of problems in finite domains for finite time intervals, (b) interpretation of Galerkin semidiscretization-in-space with exotic transmission problems following [10, 16], (c) use of the well understood theory of $C_0$-groups of isometries [14] to obtain estimates for the resulting dynamical systems [8], (d) understanding of the process of CQ time-discretization as a direct discretization of the exotic transmission problems in the time-domain (the essence of this idea is already present but not exploited in [12]) and (e) application of standard techniques for numerical analysis of the wave equation in bounded domains to work out the analysis of the fully discrete method.

Foreword. Elementary properties of basic Sobolev spaces $H^1(\Omega)$ and $H^{\pm 1/2}(\Gamma)$, the trace operator and the weak normal derivative, will be used without further reference. The pertinent results can be found in any advanced textbook of elliptic PDE. The monograph [13] contains all of them, as well as some results about steady-state layer potentials and integral operators that will be similar to the ones we will be developing in this paper (and that are used to prove background results that will be explicitly mentioned as they are used). While possible, we will make an effort in clarifying the source of constants in
estimates. Once this is not practical any more, we will use the convention of admitting $C > 0$ to be a constant independent of the associated discretization parameters ($h$ and $\kappa$ in this paper). Vector-valued distributions appear in the background of the theory of retarded layer potentials and integral operators. Their use has been outsourced to some preliminary papers [10, 8, 16] that relate strong and weak solutions of the wave equation. Here we will only employ the basic idea of a causal distribution with values on a space $X$ as a sequentially bounded map $D(\mathbb{R}) \rightarrow X$ that vanishes when applied to elements of $D((-\infty, 0))$. The concepts of differentiation and Laplace transform are then identical to those of scalar distributions.

2 An integral formulation of the scattering problem

Let $\Omega^-$ be a bounded open set in $\mathbb{R}^d$ (with $d = 2$ or 3) with Lipschitz boundary $\Gamma$. We admit the possibility that $\Omega^-$ is not connected, but we demand the complementary domain $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}$ to be connected. The problem of scattering of an incident wave by a sound-soft obstacle can be written by means of the Initial Boundary Value Problem

\begin{equation}
\begin{aligned}
    u_{tt} &= \Delta u \quad \text{in } \Omega^+, \forall t > 0, \\
    u + u^{\text{inc}} &= 0 \quad \text{on } \Gamma, \forall t > 0, \\
    u(\cdot, 0) &= u_t(\cdot, 0) = 0 \quad \text{on } \Gamma.
\end{aligned}
\end{equation}

The incident wave $u^{\text{inc}}$ is a known function. For the model equation to be meaningful we have to assume that $u^{\text{inc}}(\cdot, 0) \equiv u_t^{\text{inc}}(\cdot, 0) \equiv 0$ in a neighborhood of $\Gamma$. The unknown in (2.1) is the scattered wave field, while the total wave is $u + u^{\text{inc}}$. There is no need to impose a radiation condition at infinity since causality of the wave equation takes care of the fact that the support of the solution of (2.1), for any given $t$, is compact.

For all purposes (expository and analytic), it is convenient to understand functions of the space and time variables as functions of $t$ with values on a space of function. Therefore, instead of considering $u = u(x, t)$ as a function in $\Omega^+ \times [0, \infty)$, we will consider $u = u(t)$, where $u(t) \in H^1(\Omega^+)$ for all $t$. It will also be convenient to refer to causal functions as functions $\xi : \mathbb{R} \rightarrow X$ (here $X$ is any Hilbert space) such that $\xi(t) = 0$ for all $t < 0$. The concept of causality can be easily extended to distributions with values in the space $X$.

An integral representation of the solution of (2.1) starts by taking the value of the incident wave on $\Gamma$ for positive values of the time variable. If $\gamma^\pm : H^1(\Omega^\pm) \rightarrow H^{1/2}(\Gamma)$ are the trace operators on $\Gamma$, we consider the causal function

\begin{equation}
\varphi : \mathbb{R} \rightarrow H^{1/2}(\Gamma) \quad \text{such that} \quad \varphi(t) = \gamma^+ u^{\text{inc}}(\cdot, t) \quad \forall t > 0.
\end{equation}

Note that the required regularity of the incident wave for this process to be meaningful is local $H^1$ behavior in a neighborhood of $\Gamma$ and that $u^{\text{inc}}$ can have singularities away from the scattering boundary (this is the case for waves originated by acoustic sources).

Consider now the single and double layer retarded acoustic potentials. Their strong expressions in the three dimensional case (valid for smooth-in-space densities written as functions of the space and time variables) are

\begin{equation}
(S \ast \lambda)(x, t) := \int_{\Gamma} \frac{\lambda(y, t - |x - y|)}{4\pi|x - y|} d\Gamma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma,
\end{equation}
and
\[
(D \ast \varphi)(x, t) := \int_{\Gamma} \nabla_y \left( \frac{\varphi(z, t - |x - y|)}{4\pi|x-y|} \right) \bigg|_{z=y} \cdot \nu(y) \, d\Gamma(y), \quad x \in \mathbb{R}^d \setminus \Gamma,
\]
respectively. In (2.4) the vector \(\nu(y)\) denotes the unit outward pointing normal vector at the point \(y \in \Gamma\). The notation for the layer potentials in (2.3) and (2.4) uses the convolutional symbol to emphasize the fact that these are time-convolution operators (see [9] for a rigorous introduction of these operators in the sense of distributions), since we will take advantage of this convolutional structure for the discretization in the time variable.

By using direct arguments in the time domain [9] or employing Laplace transforms [1, 2], it is possible to prove that if \(\lambda\) is a causal distribution with values in the space \(H^{-1/2}(\Gamma)\), then \(S \ast \lambda\) is a causal distribution with values in the space
\[
H^1_\Delta(\mathbb{R}^d \setminus \Gamma) := \{ u \in H^1(\mathbb{R}^d \setminus \Gamma) : \Delta u \in L^2(\mathbb{R}^d \setminus \Gamma) \},
\]
satisfying
\[
\Delta(S \ast \lambda) = \frac{d^2}{dt^2}(S \ast \lambda) \quad \text{in} \quad \mathbb{R}^d \setminus \Gamma
\]
and
\[
\left[ \gamma(S \ast \lambda) \right] := \gamma^+(S \ast \lambda) - \gamma^+(D \ast \lambda) = 0, \quad \left[ \partial_\nu(S \ast \lambda) \right] := \partial^+_\nu(S \ast \lambda) - \partial^-_\nu(S \ast \lambda) = \lambda. \quad (2.7)
\]
Note that the Laplace operator (in the sense of distributions in \(\mathbb{R}^d \setminus \Gamma\)) and the exterior and interior normal derivatives are well defined in the space \(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)\). Expressions (2.6) and (2.7) can be understood as equalities of causal distributions with values in \(L^2(\mathbb{R}^d \setminus \Gamma)\), \(H^{1/2}(\Gamma)\) and \(H^{-1/2}(\Gamma)\) respectively. The second derivative in (2.6) is defined in the sense of vector valued distributions. The first of the jump relations (2.7) allows us to define the retarded integral operator
\[
V \ast \lambda := \gamma^+(S \ast \lambda) = \gamma^-(S \ast \lambda),
\]
whose integral expression in the three dimensional case (for smooth enough densities) coincides with that of \(S \ast \lambda\), with \(x \in \Gamma\) now.

If \(\varphi\) is a causal distribution with values in \(H^{1/2}(\Gamma)\), then \(D \ast \varphi\) is a causal distribution with values in \(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)\) satisfying
\[
\Delta(D \ast \varphi) = \frac{d^2}{dt^2}(D \ast \varphi) \quad \text{in} \quad \mathbb{R}^d \setminus \Gamma
\]
and
\[
\left[ \gamma(D \ast \varphi) \right] = -\varphi, \quad \left[ \partial_\nu(D \ast \varphi) \right] = 0. \quad (2.10)
\]
We then define the retarded boundary integral operator
\[
K \ast \xi = \frac{1}{2} \gamma^+(D \ast \xi) + \frac{1}{2} \gamma^-(D \ast \xi).
\]
An integral expression for this operator in the three dimensional case coincides with that of the layer operator \(D \ast \xi\) (see (2.4)). Any causal \(H^1_\Delta(\mathbb{R}^d \setminus \Gamma)\)-valued tempered distribution \(u\) such that
\[
\ddot{u} = \Delta u \quad \text{in} \quad \mathbb{R}^d \setminus \Gamma
\]
(with equality as \(L^2(\mathbb{R}^d \setminus \Gamma)\)-valued distributions and with the usual notation \(\ddot{u} = \frac{d^2 u}{dt^2}\)) can be represented with Kirchhoff’s formula

\[ u = S \ast [\partial_\nu u] - D \ast [\gamma u]. \]

Therefore, if we consider a solution of (2.1) as a causal tempered \(H^1(\Omega^+)^\text{-valued distribution that is extended by zero to } \Omega^-\) and denote \(\varphi\) as in (2.2), we can write

\[ u = -S \ast \lambda - D \ast \varphi \]  

(2.12)

where \(\lambda := \partial_\nu u\). Using the definitions of the boundary operators in (2.8) and (2.11) as well as the first of the jump relations (2.10), it follows that \(\lambda\) satisfies the following equation

\[ V \ast \lambda = \frac{1}{2} \varphi - K \ast \varphi. \]  

(2.13)

The analysis of [1] includes a proof of the unique solvability of the operator equation in (2.13) and a Sobolev estimate for the solution of \(V \ast \xi = \varphi\). Also [8, Theorem 6.2] contains an estimate of the solution operator for equation (2.13) and its postprocessing (2.12).

3 Discretization

We start by assuming that the data function \(\varphi\) has been approximated. This is the usual approach of the engineering literature (see the exposition of a very similar family of methods for elastic waves in [17] for instance) and will be for us a motive to studying the propagation of errors in data, a study that will be needed for analysis of the fully discrete schemes. We therefore assume that a causal function \(\varphi_h : \mathbb{R} \to H^{1/2}(\Gamma)\) is given as an approximation to \(\varphi\).

3.1 Semidiscretization in space

We consider a discrete space \(X_h \subset L^\infty(\Gamma)\) and substitute (2.13) by the search of a causal function \(\lambda_h : \mathbb{R} \to X_h\) such that

\[ \langle \mu_h, V \ast \lambda_h \rangle_\Gamma = \langle \mu_h, \frac{1}{2} \varphi_h - K \ast \varphi_h \rangle_\Gamma \]  

(3.1)

\(\forall \mu_h \in X_h\).

Here and in the sequel the angled brackets denote the \(H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)\) duality product. The solution of (3.1) is then used for the discrete representation formula

\[ u_h := -S \ast \lambda_h - D \ast \varphi_h. \]  

(3.2)

For the sake of clarity, let us write down the system (3.1) in the three dimensional case, when data have been approximated by a function \(\varphi_h : \mathbb{R} \to Y_h \subset W^{1,\infty}(\Gamma)\), where \(Y_h\) is finite dimensional. Let \(\{N_1, \ldots, N_J\}\) and \(\{M_1, \ldots, M_K\}\) be respective bases of \(X_h\) and \(Y_h\). Data and unknown can then be represented by their coefficients:

\[ \varphi_h(t) = \sum_{k=1}^K \varphi_k(t)M_k, \quad \lambda_h(t) = \sum_{j=1}^J \lambda_j(t)N_j. \]  

(3.3)
This is equivalent to substituting the $Y_h$- and $X_h$-valued functions by a finite set of casual scalar functions. Problem (3.1) is equivalent to the system

$$
\sum_{j=1}^{J} \int_{\Gamma} \frac{N_i(x)N_j(y)}{4\pi|x-y|}\lambda_j(t - |x-y|)d\Gamma(x)d\Gamma(y) = \frac{1}{2} \sum_{k=1}^{K}\left(\int_{\Gamma} N_i(x)M_k(x)d\Gamma(x)\right)\varphi_k(t) + \sum_{k=1}^{K} \int_{\Gamma} M_{i,k}(x,y)\left(\frac{\varphi_k(t - |x-y|)}{|x-y|} + \dot{\varphi}_k(t - |x-y|)\right)d\Gamma(x)d\Gamma(y) \tag{3.4}
$$

(for $i = 1, \ldots, J$), where

$$
M_{i,k}(x,y) := \frac{(x-y) \cdot \nu(y)}{4\pi|x-y|^2}N_i(x)M_k(y).
$$

The following spaces will be relevant in the sequel:

$$
\mathcal{W}^k_0(\mathbb{R}; X) := \{\rho \in C^{k-1}(\mathbb{R}; X) : \text{supp } \rho \subset [0, \infty), \rho^{(k)} \in L^1_{\text{loc}}(\mathbb{R}; X)\},
$$

$$
C^k_0(\mathbb{R}; X) := \{\rho \in C^k(\mathbb{R}; X) : \text{supp } \rho \subset [0, \infty)\}.
$$

**Theorem 3.1.** Let $\varphi \in \mathcal{W}^1_0(\mathbb{R}; H^{1/2}(\Gamma))$, assume that the solution of (2.13) satisfies $\lambda \in \mathcal{W}^2_0(\mathbb{R}; H^{-1/2}(\Gamma))$ and let $u$ be given by (2.12). Assume also that $\varphi_h \in \mathcal{W}^1_0(\mathbb{R}; Y_h)$. Then the semidiscrete equation (3.1) has a unique solution $\lambda_h$. Let finally $u_h$ be given by (3.2). Then, for all $t \geq 0$,}

$$
\|(\lambda - \lambda_h)(t)\|_{-1/2,\Gamma} \leq C(1 + t) \left(\sum_{\ell=0}^{2} \int_{0}^{t} \|((\lambda^{(\ell)} - \Pi_h\lambda^{(\ell)})(\tau)\|_{-1/2,\Gamma}d\tau \right. \\
+ \sum_{\ell=0}^{4} \left. \|(\varphi^{(\ell)} - \varphi_h^{(\ell)})(\tau)\|_{1/2,\Gamma}d\tau\right) \tag{3.5}
$$

$$
\|(u - u_h)(t)\|_{1,\mathbb{R}^d} \leq C(1 + t) \left(\sum_{\ell=0}^{2} \int_{0}^{t} \|((\lambda^{(\ell)} - \Pi_h\lambda^{(\ell)})(\tau)\|_{-1/2,\Gamma}d\tau \right. \\
+ \sum_{\ell=0}^{2} \left. \|(\varphi^{(\ell)} - \varphi_h^{(\ell)})(\tau)\|_{1/2,\Gamma}d\tau\right), \tag{3.6}
$$

where $\Pi_h : H^{-1/2}(\Gamma) \rightarrow X_h$ is the orthogonal projection onto $X_h$.

Note that by [3] Theorem 6.2, if $\varphi \in \mathcal{W}^1_0(\mathbb{R}; H^{1/2}(\Gamma))$, then $\lambda \in C^0_0(\mathbb{R}; H^{-1/2}(\Gamma))$. Time regularity of the solution is then guaranteed by the sufficient (but not necessary) condition $\varphi \in \mathcal{W}^1_0(\mathbb{R}; H^{1/2}(\Gamma))$. The proof of Theorem 3.1 will be given at the end of Section 5. If data are not discretized the second group of terms in the error estimates of Theorem 3.1 is not needed.
3.2 Full discretization

In a final step, we substitute the four time convolutions that appear in (3.1) and (3.2) with a discrete convolution based on one of the applicable Convolution Quadrature methods. For a fixed time-step \( \kappa > 0 \), the CQ method applied to the discretization of (3.1) and (3.2) produces (in theory) casual functions \( \lambda^\kappa : \mathbb{R} \rightarrow X_h \) and \( u^\kappa : \mathbb{R} \rightarrow H^1_\Lambda (\mathbb{R}^d \setminus \Gamma) \). In practice, these functions are evaluated in equally spaced time steps \( t_n := n \kappa \) and only these values of the functions are obtained. To obtain values at other times, the method has to be run again, starting at \( t_0 := -\varepsilon \kappa \) for instance. Therefore, even if the theory of CQ deals with functions of continuous time, in practice the solutions can be understood as functions of discrete time, i.e., sequences.

Let us briefly explain what the CQ discretization of (3.4) consists of. First of all, we consider the complex matrix valued functions \( V^h(s) \in \mathbb{R}^{J \times J} \) and \( K^h(s) \in \mathbb{R}^{J \times K} \) with elements

\[
V^h(s)_{i,j} := \int_{\Gamma} \int_{\Gamma} N_i(x)N_j(y) \frac{e^{-s|x-y|}}{4\pi|x-y|} d\Gamma(x) d\Gamma(y) \tag{3.7}
\]

\[
K^h(s)_{i,j} := \int_{\Gamma} \int_{\Gamma} \Phi_{i,k}(x,y) (1+s|x-y|) \frac{e^{-s|x-y|}}{|x-y|} d\Gamma(x) d\Gamma(y) \tag{3.8}
\]

and the matrix with elements

\[
I_{i,k} := \int_{\Gamma} N_i(x)M_k(x) d\Gamma(x).
\]

Note that \( V^h(s) \) and \( K^h(s) \) are the Laplace transforms of the operators that appear in (3.4). We then construct the Taylor expansions

\[
V^h(\kappa^{-1}\delta(\zeta)) = \sum_{n=0}^{\infty} V^h[n] \zeta^n \quad K^h(\kappa^{-1}\delta(\zeta)) = \sum_{n=0}^{\infty} K^h[n] \zeta^n, \tag{3.9}
\]

where \( \delta \) is one of the following functions

\[
\delta(\zeta) := \begin{cases} 1 - \zeta, & \text{(backward Euler method)} \\ \frac{3}{2} - 2\zeta + \frac{1}{2}\zeta^2, & \text{(BDF2)} \\ 2 \frac{1-\zeta}{1+\zeta}, & \text{(trapezoidal rule)}. \end{cases}
\]

Data discretization consists of the construction of vectors \( \varphi^h[n] := (\varphi_k(n \kappa))_{k=1}^{K} \) for \( n \geq 0 \) (recall (3.3)). The unknowns are vectors \( \lambda^h[n] \in \mathbb{R}^J \) satisfying

\[
\sum_{m=0}^{n} V^h[m] \lambda^h[n-m] = \frac{1}{2} I_h \varphi^h[n] + \sum_{m=0}^{n} K^h[m] \varphi^h[n-m], \quad n \geq 0. \tag{3.10}
\]

We can thus associate

\[
\lambda^h[n] = (\lambda_1[n], \ldots, \lambda_J[n]) \quad \mapsto \quad \lambda^h[n] := \sum_{j=1}^{J} \lambda_j[n] N_j \in X_h
\]
to obtain a fully discrete approximation of \( \lambda_h(t_n) \approx \lambda(t_n) \).

The postprocessing step to compute the approximated scattered field can be explained in a similar way. The \( s \)-domain semidiscrete single and double layer potentials correspond to vector valued functions with domain \( \mathbb{R}^{3} \setminus \Gamma \)

\[
S_h(s)_j := \int_{\Gamma} \frac{e^{-s|\cdot-y|}}{4\pi|\cdot-y|} N_j(y) d\Gamma(y),
\]

\[
D_h(s)_k := \int_{\Gamma} \frac{(\cdot-y) \cdot \nu(y)}{4\pi|\cdot-y|^2} (1+s|\cdot-y|) \frac{e^{-s|\cdot-y|}}{|\cdot-y|} M_k(y) d\Gamma(y).
\]

The CQ method uses the same strategy as in (3.9) to produce sequences of vector valued functions \( S_h^k[n] \) and \( D_h^k[n] \), defined in \( \mathbb{R}^{3} \setminus \Gamma \), and uses them to construct the approximations

\[
u_h^k[n] = -\sum_{m=0}^{n} S_h^k[m] \chi_h^k[n-m] - \sum_{m=0}^{n} D_h^k[m] \varphi_h^k[n-m], \quad n \geq 0.
\]

In the two-dimensional case the expressions for the fully discrete method are very similar, using Hankel functions instead of exponential expressions. For instance,

\[
\nabla_h(s)_{ij} := \frac{1}{4} \int_{\Gamma} \int_{\Gamma} H_{0}^{(1)}(is|x-y|) N_i(x) N_j(y) d\Gamma(x) d\Gamma(y),
\]

\[
\mathbb{K}_h(s)_{ik} := -\frac{s}{4} \int_{\Gamma} \int_{\Gamma} H_{1}^{(1)}(is|x-y|) \frac{(x-y) \cdot \nu(y)}{|x-y|} N_i(x) M_k(y) d\Gamma(x) d\Gamma(y).
\]

**Remark 3.1.** When data are not approximated, the indices \( k \) have to be ignored. Instead of having a matrix \( \mathbb{K}_h(s) \) we have operators \( H^{1/2}(\Gamma) \rightarrow \mathbb{R}^J \) obtained by substituting the basis function \( M_k \) by a general element of \( H^{1/2}(\Gamma) \). In this way, the matrix-vector products \( \mathbb{K}_h^k[m] \varphi_h^k[n-m] \) have to be substituted by the action of operators \( \mathbb{K}^k[m] : H^{1/2}(\Gamma) \rightarrow \mathbb{R}^J \) on elements \( \varphi^k[n-m] = \varphi(t_{n-m}) \in H^{1/2}(\Gamma) \). Similar changes have to be applied to the double layer potential and to the matrix \( \nabla_h \) which is now substituted by an operator \( H^{1/2}(\Gamma) \rightarrow \mathbb{R}^J \) corresponding to testing a function with the basis functions \( N_i \).

The analysis of the difference between the semidiscrete and the fully discrete solution at the different time-steps is carried out separately for the three time-discretization methods. This is done in Section 6.

### 4 The semidiscrete Galerkin projection

Consider a casual smooth function \( \lambda : \mathbb{R} \rightarrow H^{-1/2}(\Gamma) \) and a finite dimensional space \( X_h \subset H^{-1/2}(\Gamma) \). The aim of this section is the analysis of the semidiscrete discretization process looking for a causal function \( \lambda^G_h : \mathbb{R} \rightarrow X_h \) such that

\[
\langle \mu_h, (\mathcal{V} \ast \lambda^G_h)(t) \rangle_\Gamma = \langle \mu_h, (\mathcal{V} \ast \lambda)(t) \rangle_\Gamma \quad \forall \mu_h \in X_h, \quad \forall t
\]

and outputs the potential

\[
u_h^G := S \ast \lambda^G_h.
\]
Using a simple Laplace transform argument and the estimates of [1] (see also [10]), it is easy to prove that (4.1) has at most one continuous causal solution.

Before proceeding to state and prove the main result of this section, we are going to introduce some constants related to the geometry of the problem and associated functional inequalities.

### 4.1 Some inequalities

Let $R > 0$ be such that

$$\overline{\Omega_-} \subset B_0 := B(0; R) = \{x \in \mathbb{R}^d : |x| < R\}$$

and let us consider the balls $B_T := B(0; R + T)$ for $T \geq 0$. Let then $C_T > 0$ be taken so that

$$\|u\|_{B_T} \leq C_T \|\nabla u\|_{B_T}, \quad \forall u \in H^1_0(B_T). \quad (4.3)$$

A simple scaling argument shows that we can take $C_T = C_0(1 + T/R)$. Therefore, the constant $C_T$ grows linearly with $T$. The trace operator on the boundary $\partial B_T$ will be denoted $\gamma_T$.

We will also consider a constant for the following two-sided trace inequality:

$$\|\gamma u\|_{1/2, \Gamma} \leq C_T \|u\|_{1, B_0}, \quad \forall u \in H^1(B_0). \quad (4.4)$$

Next, we consider a one-sided lifting of the trace onto $\Gamma$ in the form of a bounded linear operator $L : H^{1/2}(\Gamma) \rightarrow H^1(\mathbb{R}^d \setminus \Gamma)$ such that

$$L\varphi \equiv 0 \quad \text{in} \quad \Omega_+ \quad \text{and} \quad \gamma^- L\varphi = \varphi \quad \forall \varphi \in H^{1/2}(\Gamma). \quad (4.5)$$

Note that $[\gamma L\varphi] = \varphi$ and that there exists $C_L > 0$ such that

$$\|L\varphi\|_{1, \mathbb{R}^d \setminus \Gamma} = \|L\varphi\|_{1, \Omega_-} \leq C_L \|\varphi\|_{1/2, \Gamma}, \quad \forall \varphi \in H^{1/2}(\Gamma). \quad (4.6)$$

Finally, using the weak definition of the normal derivative, we can fix a constant $C_\nu > 0$ such that

$$\|\partial_\nu u\|_{-1/2, \Gamma} \leq C_\nu \left(\|\Delta u\|^2_{B_0 \setminus \Gamma} + \|\nabla u\|^2_{B_0 \setminus \Gamma}\right)^{1/2}, \quad \forall u \in H^1_\Delta(B_0 \setminus \Gamma). \quad (4.7)$$

### 4.2 Estimates for the Galerkin projection

**Theorem 4.1.** Let $\lambda \in \mathcal{W}_G^0(\mathbb{R}; H^{-1/2}(\Gamma))$. Then the solution of the semidiscrete problem (4.1) and its associated potential (4.2) satisfy

$$\lambda_h^G \in C_0(\mathbb{R}; H^{-1/2}(\Gamma)), \quad u_h^G \in C_0(\mathbb{R}; H^1(\mathbb{R}^d)).$$

Moreover, for all $t \geq 0$,

$$\|u_h^G(t)\|_{1, \mathbb{R}^d} \leq 2C_T \left(\|\lambda(t)\|_{-1/2, \Gamma} + \sqrt{1 + C_T^2} B_2^{-1/2}(\lambda, t)\right), \quad (4.8)$$

$$\|\lambda_h^G(t)\|_{-1/2, \Gamma} \leq (1 + C_T C_\nu) \|\lambda(t)\|_{-1/2, \Gamma} + \sqrt{2} C_T C_\nu B_2^{-1/2}(\lambda, t), \quad (4.9)$$

where

$$B_2^{-1/2}(\lambda, t) := \int_0^t \left(\|\lambda(\tau)\|_{-1/2, \Gamma} + \|\dot{\lambda}(\tau)\|_{-1/2, \Gamma}\right) d\tau.$$
The proof of Theorem 4.1 will occupy the remainder of this section. The proof will never use that $X_h$ is finite dimensional. If we take $X_h = H^{-1/2}(\Gamma)$, (4.8) gives a bound for the single layer acoustic operator that reproves [6, Theorem 3.1].

Theorem 4.1 will be proved for $\lambda \in C_0^2(\mathbb{R}; H^{-1/2}(\Gamma))$. The extension to the general case follows by a simple density argument. Also, we will prove the results in a finite interval $[0, T]$, and bounds (4.8)-(4.9) will only be proved for $t = T$. Since $T$ is arbitrary, this is equivalent to having proved the results for any $t$.

Because of the finite speed of propagation of solutions to the wave equation, it is possible to understand (formally at the beginning) $u_{Gh}^\xi$ as a solution of the following wave propagation problem on a truncated domain with non-standard transmission conditions (see [10, 16])

\begin{align}
  u_{Gh}^\xi(t) &\in H_0^1(B_T) \quad 0 \leq t \leq T, \\
  \ddot{u}_{Gh}^\xi(t) &= \Delta u_{Gh}^\xi(t) \quad 0 \leq t \leq T, \\
  \gamma u_{Gh}^\xi(t) - (\nabla * \lambda)(t) &\in X_h^\circ \quad 0 \leq t \leq T, \\
  [\partial_n u_{Gh}^\xi(t)] &\in X_h \quad 0 \leq t \leq T, \\
  u_{Gh}^\xi(0) = \dot{u}_{Gh}^\xi(0) &= 0.
\end{align}

The set $X_h^\circ$ is the polar set or annihilator of $X_h$, i.e.,

$$X_h^\circ := \{ \rho \in H^{1/2} (\Gamma) : \langle \mu_h, \rho \rangle_\Gamma = 0 \quad \forall \mu_h \in X_h \}. $$

If $u \in H_\Delta^1(B_T \setminus \Gamma)$, the condition $[\partial_n u] \in X_h$ can be rewritten as a set of restrictions

$$\langle [\partial_n u], \rho \rangle_\Gamma = 0 \quad \forall \rho \in X_h^\circ,$$

or equivalently, as

$$(\nabla u, \nabla v)_{B_T \setminus \Gamma} + (\Delta u, v)_{B_T \setminus \Gamma} = 0 \quad \forall v \in V_h^T$$

where

$$V_h^T := \{ v \in H_0^1(B_T) : \gamma v \in X_h^\circ \}. $$

**Proposition 4.2** (Uniqueness). *Problem (4.10) has at most one solution*

$$u_{Gh}^\xi \in C^2([0, T]; L^2(B_T)) \cap C^1([0, T]; H_0^1(B_T)) \cap C([0, T]; H_\Delta^1(B_T \setminus \Gamma)).$$

If this solution exists and $\lambda_h^\xi := [\partial_n u_{Gh}^\xi]$, then $u_h^\xi$ and $\lambda_h^\xi$ coincide with the solution of (4.1)-(4.2) on the interval $[0, T]$.

**Proof.** Uniqueness of solution follows from an elementary energy argument. Careful, but not complicated, use of extension operators from $B_T$ to $\mathbb{R}^d$ and from $[0, \infty)$ to $\mathbb{R}$ in the sense of vector valued distributions, can be used following [6, Sections 4 & 5], to prove the relation between a strong solution of the transmission problem and a weak distributional solution of (4.1)-(4.2).

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4.3 An underlying initial value problem

Associated to the space $V_h^T$ given in (4.12), we consider the space

$$D_h^T := \{ v \in V_h^T : \Delta v \in L^2(B_T \setminus \Gamma), [\partial_n v] \in X_h \}$$

(both definitions coincide by (4.11)). In the frame of the triple $D_h^T \subset V_h^T \subset L^2(B_T)$ we can consider the unbounded operator $\Delta : D_h^T \subset L^2(B_T) \rightarrow L^2(B_T)$ (the distributional Laplacian in $B_T \setminus \Gamma$)) and the initial value problem

$$\dot{v}(t) = \Delta v(t) + f(t) \quad \forall t \geq 0, \quad v(0) = \dot{v}(0) = 0. \quad (4.15)$$

Strong solutions of this problem will be those with values in $D_h^T$. Following [8, Section 8 and Appendix], we can also consider weak solutions of (4.15). In order to do that, we start in the Gelfand triple $V_h^T \subset L^2(B_T) \cong L^2(B_T)' \subset (V_h^T)'$, we extend $\Delta$ to an unbounded operator $\Delta : V_h^T \subset (V_h^T)' \rightarrow (V_h^T)'$ and we only look for solutions with values in $V_h^T$. In this case, the additional boundary condition that appears in the definition of $D_h^T$ and the extended definition of the Laplace operator are part of the same expression: namely, the differential equation (4.15) is to be understood as

$$(\dot{v}(t), w)_{(V_h^T)' \times V_h^T} + (\nabla v(t), \nabla w)_{B_T} = (f(t), w)_{B_T} \quad \forall w \in V_h^T. \quad (4.16)$$

The following two results follow from [8, Appendix].

**Proposition 4.3** (Strong solutions). If $f : [0, \infty) \rightarrow V_h^T$ is continuous, then there exists $v : [0, \infty) \rightarrow D_h^T$ such that

$$v \in C^2([0, \infty); L^2(B_T)) \cap C^1([0, \infty); H^1_0(B_T)) \cap C([0, \infty); H^1_a(B_T \setminus \Gamma)), \quad (4.17)$$

satisfying (4.15) and the bounds

$$\|v(t)\|_{B_T} \leq C_T \int_0^t \|f(\tau)\|_{B_T} d\tau,$$

$$\|\nabla v(t)\|_{B_T} \leq \int_0^t \|f(\tau)\|_{B_T} d\tau,$$

$$\|\Delta v(t)\|_{B_T \setminus \Gamma} \leq \int_0^t \|\nabla f(\tau)\|_{B_T} d\tau.$$

**Proposition 4.4** (Weak solutions). If $f : [0, \infty) \rightarrow L^2(B_T)$ is continuous, then there exists $v : [0, \infty) \rightarrow V_h^T$ such that

$$v \in C^2([0, \infty); (V_h^T)') \cap C^1([0, \infty); L^2(B_T)) \cap C([0, \infty); H^1_0(B_T)),$$

satisfying the weak form of (4.15) (see (4.16)) and the bounds

$$C_T^{-1} \|v(t)\|_{B_T} \leq \|\nabla v(t)\|_{B_T} \leq \int_0^t \|f(\tau)\|_{B_T} d\tau.$$

Moreover, $\bar{v}(t) := \int_0^t v(\tau) d\tau$ is a continuous $D_h^T$-valued function.
4.4 A decomposition of \( u_h^G \)

As explained in Section 4.2, we are going to deal with \( \lambda \in \mathcal{C}^2([0, T]; H^{-1/2}(\Gamma)) \) such that \( \lambda(0) = \dot{\lambda}(0) = 0 \) and obtain bounds in the interval \([0, T]\) as well. The proof of this result follows from the decomposition

\[
u_h^G = \mathcal{S} \ast \lambda + u_h^0 + v_h^0, \quad (4.18)
\]

where \( u_h^0 : [0, \infty) \to V_h^T \) solves the steady-state transmission problems (for all \( t \geq 0 \))

\[
u_h^0(t) \in V_h^T, \quad -\Delta u_h^0(t) + u_h^0(t) = 0 \quad \text{in} \quad B_T \setminus \Gamma, \quad [\partial_n u_h^0(t)] + \lambda(t) \in X_h, \quad (4.19)
\]

and \( v_h^0 : [0, \infty) \to D_h^T \) is a solution of the evolution problem

\[
\begin{align*}
\ddot{v}_h^0(t) &= \Delta v_h^0(t) + u_h^0(t) - \dot{u}_h^0(t) = \Delta v_h^0(t) + \Delta u_h^0(t) - \dot{u}_h^0(t) & \quad t \geq 0, \\
v_h^0(0) &= \dot{v}_h^0(0) = 0.
\end{align*}
\]

We start by analyzing the three terms in (4.18) one by one. Note that we still need to show that the decomposition (4.18) holds true, that is, that the sum of the three functions in the right hand side of (4.18) is \( u_h^G \).

1. By Theorem 3.1 and Proposition 4.2 in [8], it follows that

\[
\mathcal{S} \ast \lambda \in \mathcal{C}^2([0, T]; L^2(B_T)) \cap \mathcal{C}^1([0, T]; H_0^1(B_T)) \cap \mathcal{C}([0, T]; H_\Delta^1(B_T \setminus \Gamma)),
\]

that for all \( t \in [0, T] \),

\[
\begin{align*}
\frac{d^2}{dt^2}(\mathcal{S} \ast \lambda)(t) &= \Delta(\mathcal{S} \ast \lambda)(t), \\
[\partial_n(\mathcal{S} \ast \lambda)(t)] &= \lambda(t), \\
(\mathcal{S} \ast \lambda)(0) &= \frac{d}{dt}(\mathcal{S} \ast \lambda)(0) = 0,
\end{align*}
\]

and

\[
\|(\mathcal{S} \ast \lambda)(t)\|_{1, B_T} \leq C_T \left( \|\lambda(t)\|_{-1/2, \Gamma} + \sqrt{1 + C_T^2 B_2^{-1/2}(\lambda, t)} \right) \quad 0 \leq t \leq T. \quad (4.23)
\]

2. We next analyze the behavior of \( u_h^0 \). The variational formulation of (4.19) is

\[
u_h^0(t) \in V_h^T, \quad (\nabla u_h^0(t), \nabla v)_{B_T} + (u_h^0(t), v)_{B_T} = -\langle \lambda(t), \gamma v \rangle_{\Gamma} \quad \forall v \in V_h^T.
\]

This is a well posed problem, that depends on \( t \), only because data depend on \( t \). In particular,

\[
u_h^0 \in \mathcal{C}^2([0, T]; H_0^1(B_T) \cap H_\Delta^1(B_T \setminus \Gamma)), \quad u_h^0(0) = \dot{u}_h^0(0) = 0.
\]

The variational formulation (4.24) and the trace inequality (4.6) show that

\[
\|u_h^0(t)\|_{1, B_T} \leq C_T \|\lambda(t)\|_{-1/2, \Gamma}. \quad (4.26)
\]

On the other hand, since \( \Delta u_h^0(t) = u_h^0(t) \), then (4.7) implies that

\[
\|\partial_n u_h^0(t)\|_{-1/2, \Gamma} \leq C_\nu \|u_h^0(t)\|_{1, B_T} \leq C_\nu C_T \|\lambda(t)\|_{-1/2, \Gamma}. \quad (4.27)
\]

Differentiating (4.23) twice with respect to \( t \), it also follows that

\[
\|\ddot{u}_h^0(t)\|_{1, B_T} \leq C_T \|\dot{\lambda}(t)\|_{-1/2, \Gamma}. \quad (4.28)
\]
3. We can apply Proposition 4.3 to the solution of (4.20), taking \( f := u_h^0 - \bar{u}_h^0 \). It then follows that

\[
\psi_h \in C^2([0, \infty); L^2(B_T)) \cap C^1([0, \infty); H^1_0(B_T)) \cap C([0, \infty); H^1_\Delta(B_T \setminus \Gamma)).
\]

(4.29)

We also obtain the bounds

\[
\|v_h^0(t)\|_{B_T} \leq C_t \int_0^t \|\bar{u}_h^0(\tau) - \bar{u}_h^0(\tau)\|_{B_T} d\tau \leq C_T C_1 B_2^{-1/2}(\lambda, t)
\]

(4.30)

(we have used (4.26) and (4.28) in the last step),

\[
\|\nabla v_h^0(t)\|_{B_T} \leq C_T B_2^{-1/2}(\lambda, t)
\]

(4.31)

and

\[
\|\Delta v_h^0(t)\|_{B_T \backslash \Gamma} \leq C_T B_2^{-1/2}(\lambda, t).
\]

(4.32)

From the bound for the jump of the normal derivative (4.7) and (4.31)-(4.32), it follows that

\[
\|\partial_\nu v_h^0(t)\|_{-1/2, \Gamma} \leq \sqrt{\bar{C}_1} C_\nu B_2^{-1/2}(\lambda, t).
\]

(4.33)

4.5 Proof of Theorem 4.1

Let now \( u_h^G \) be defined by (4.18). To prove that \( u_h^G \) satisfies (4.13), we just have to use (4.21), (4.25) and (4.29). To prove that \( u_h^G \) satisfies problem (4.10), we add the equations that are satisfied by the three components of the sum, namely (4.22), (4.19) and (4.20). The uniqueness result of Proposition 4.2 shows then that the function defined by (4.18) can be identified with the solution of (4.10) in the interval \([0, T]\).

By (4.23), (4.26), (4.30) and (4.31), it follows that

\[
\|u_h^G(t)\|_{1, B_T} \leq 2C_T \|\lambda(t)\|_{-1/2, \Gamma} + C_T \left(\sqrt{1 + C_t^2} + \sqrt{1 + C_r^2}\right) B_2^{-1/2}(\lambda, t).
\]

(4.34)

Taking \( t = T \) and using that \( T \) is arbitrary, (4.8) follows.

Since \( \lambda_h^G(t) = [\partial_\nu u_h^G(t)] = \lambda(t) + [\partial_\nu u_h^0(t)] + [\partial_\nu v_h^0(t)], \) and \([\partial_\nu \cdot] : H^1_\Delta(B_T \setminus \Gamma) \to H^{-1/2}(\Gamma)\) is bounded, then (4.25) and (4.29) imply that \( \lambda_h^G \in \mathcal{C}([0, T], H^{-1/2}(\Gamma)) \). The uniqueness argument of Proposition 4.2 then proves that the solution of (4.1) is a causal continuous \( H^{-1/2}(\Gamma) \)-valued function. Finally, inequalities (4.27) and (4.33) prove (4.9).

5 The semidiscrete Galerkin solver

In this section we study how Galerkin semidiscretization depends on data. Our starting point is a causal function \( \varphi : \mathbb{R} \to H^{1/2}(\Gamma) \). We then consider the function \( \lambda_h^\varphi : \mathbb{R} \to X_h \) such that

\[
\langle \mu_h, (\mathcal{V} \ast \lambda_h^\varphi)(t) \rangle_\Gamma = \langle \mu_h, \frac{1}{2} \varphi(t) - (\mathcal{K} \ast \varphi)(t) \rangle_\Gamma \quad \forall \mu_h \in X_h
\]

(5.1)

and the associated exterior solution

\[
u_h^\varphi := -\mathcal{S} \ast \lambda_h^\varphi - \mathcal{D} \ast \varphi.
\]

(5.2)
Using a simple Laplace transform argument and the estimates of [11] (see also [10]), it is easy to prove that (5.1) has at most one continuous causal solution. Moreover, uniqueness can be also established for weaker solutions, where for instance, $\lambda^\varphi_h$ is the distributional derivative of a continuous causal $X_h$-valued function.

### 5.1 Estimates for the Galerkin solver

**Theorem 5.1.** Let $\varphi \in W^d_0(\mathbb{R}; H^{1/2}(\Gamma))$. Then the solution of the semidiscrete problem (5.1) and its associated potential (5.2) satisfy

$$
\lambda^\varphi_h \in C(\mathbb{R}; H^{-1/2}(\Gamma)), \quad u^\varphi_h \in C^1(\mathbb{R}; H^1(\mathbb{R}^d \setminus \Gamma)).
$$

Moreover, for all $t \geq 0$,

$$
\|u^\varphi_h(t)\|_{1,\mathbb{R}^d\setminus\Gamma} \leq C_L \left( \|\varphi(t)\|_{1/2,\Gamma} + \sqrt{1 + C^2_f} B^1_2(\varphi, t) \right), \quad (5.3)
$$

$$
\|\lambda^\varphi_h(t)\|_{-1,\mathbb{R}^d\setminus\Gamma} \leq \sqrt{2} C^\varphi C_L \left( 4 \|\varphi(t)\|_{1/2,\Gamma} + 2 \|\varphi(t)\|_{1/2,\Gamma} + B^1_4(\varphi, t) \right), \quad (5.4)
$$

where

$$
B^1_2(\varphi, t) := \int_0^t \left( \|\varphi(\tau)\|_{1/2,\Gamma} + \|\varphi(\tau)\|_{1/2,\Gamma} \right) d\tau,
$$

$$
B^1_4(\varphi, t) := 4 B^1_2(\varphi, t) + B^1_2(\varphi, t).
$$

The proof of this result follows partially the steps of the proof of Theorem 4.1. The analysis will be more involved because of the occurrence of a non-homogeneous essential transmission condition (see (5.5d) below), that cannot be easily lifted with a continuous potential. Like in Section 4, we will prove the estimates on a fixed time interval $[0, T]$, taking advantage of the fact that finite speed of propagation will allow us to impose a homogeneous Dirichlet boundary condition in $\partial B_T$. Again, we will only assume that $\varphi \in C^4([0, T]; H^{1/2}(\Gamma))$ with $\varphi^{(k)}(0) = 0$ for $k \leq 3$. The result for general $\varphi$ can be extended with a density argument.

One of the keys towards the proof of the result lies in the fact if $u^\varphi_h$ is smooth enough, then $u^\varphi_h$ is a strong solution of the following problem:

$$
u^\varphi_h(t) \in H^1(B_T \setminus \Gamma) \quad 0 \leq t \leq T, \quad (5.5a)
$$

$$
\gamma_T u^\varphi_h(t) = 0 \quad 0 \leq t \leq T, \quad (5.5b)
$$

$$
\ddot{u}^\varphi_h(t) = \Delta u^\varphi_h(t) \quad 0 \leq t \leq T, \quad (5.5c)
$$

$$
[\gamma u^\varphi_h(t)] = \varphi(t) \quad 0 \leq t \leq T, \quad (5.5d)
$$

$$
\gamma^+ u^\varphi_h(t) \in X^\varphi_h \quad 0 \leq t \leq T, \quad (5.5e)
$$

$$
[\partial_\nu u^\varphi_h(t)] \in X_h \quad 0 \leq t \leq T, \quad (5.5f)
$$

$$
u^\varphi_h(0) = \dot{u}^\varphi_h(0) = 0. \quad (5.5g)
$$

At the same time, we can combine (5.5c) and (5.5f), multiply by $w \in V^T_h$ and integrate, to obtain a weaker form of the differential equation and the natural boundary condition

$$
(\ddot{u}^\varphi_h(t), w)_{B_T} + (\nabla u^\varphi_h(t), \nabla w)_{B_T \setminus \Gamma} = 0 \quad \forall w \in V^T_h, \quad 0 \leq t \leq T. \quad (5.6)
$$
If we now define

\[ w_h^\varphi(t) := \int_0^t u_h^\varphi(\tau) \, d\tau, \quad (5.7) \]

it also follows that

\[ (\dot{w}_h^\varphi(t), w)_{B_T} + (\nabla w_h^\varphi(t), \nabla w)_{B_T \setminus \Gamma} = 0 \quad \forall w \in V_h^T, \quad 0 \leq t \leq T. \quad (5.8) \]

The following double uniqueness result will help us recognize \( u_h^\varphi \) in the two decompositions that will be given below.

**Proposition 5.2 (Uniqueness).** There exists at most one

\[ u_h^\varphi \in C^1([0, T]; L^2(B_T)) \cap C([0, T]; H^1(B_T \setminus \Gamma)) \]

satisfying the essential boundary and transmission conditions \((5.5b), (5.5d), (5.5e)\), and such that \( w_h^\varphi \) defined with \((5.7)\) satisfies \((5.8)\). If such a solution exists and, in addition, \( w_h^\varphi \in C([0, T]; H^1(B_T \setminus \Gamma)) \), then \( \lambda_h^\varphi = -\frac{d}{dt} \left[ \partial_t w_h^\varphi \right] \) (with differentiation in the sense of \( H^{-1/2}(\Gamma) \)-valued distributions) solves \((5.1)\) and \( u_h^\varphi \) satisfies \((5.2)\).

**Proof.** Let \( u_h^{\varphi=0} \) be a solution of \((5.5b), (5.5d)\) with \( \varphi = 0 \), \((5.5e)\) and \((5.5g)\) such that the corresponding \( w_h^{\varphi=0} \) satisfies \((5.8)\). Then \( u_h^{\varphi=0} : [0, T] \to V_h^T \) is continuous and we have enough regularity to test \((5.8)\) with \( u_h^{\varphi=0}(t) = \dot{w}_h^{\varphi=0}(t) \in V_h^T \) and prove that

\[ \frac{d}{dt} \left( \| \dot{w}_h^{\varphi=0}(t) \|_{B_T}^2 + \| \nabla \dot{w}_h^{\varphi=0}(t) \|_{B_T \setminus \Gamma}^2 \right) = 0. \]

The proof of uniqueness of solution is now straightforward. The final statement follows from the same arguments developed in [8, Section 5].

**Corollary 5.3.** There exists at most one

\[ u_h^\varphi \in C^2([0, T]; L^2(B_T)) \cap C^1([0, T]; H^1(B_T \setminus \Gamma)) \]

that solves \((5.5)\). Moreover, if such a solution exists, then \( \lambda_h^\varphi := -[\partial_t u_h^\varphi] \) solves \((5.1)\) and \( u_h^\varphi \) satisfies \((5.2)\).

### 5.2 A first decomposition of \( u_h^\varphi \)

For the arguments of this section, we only need \( \varphi \in C^2([0, T]; H^{1/2}(\Gamma)) \) with \( \varphi(0) = \varphi(0) = 0 \). In a first step, we formally decompose

\[ u_h^\varphi = u_h^0 + v_h^0, \quad (5.9) \]

where \( u_h^0 : [0, T] \to H^1(B_T \setminus \Gamma) \) is the solution of the variational problems (for \( 0 \leq t \leq T \))

\[ u_h^0(t) \in H^1(B_T \setminus \Gamma), \]

\[ \gamma_T u_h^0(t) = 0, \quad [\gamma u_h^0(t)] = \varphi(t), \quad \gamma^+ u_h^0(t) \in X_h^0, \quad (5.10) \]

\[ (\nabla u_h^0(t), \nabla v)_{B_T \setminus \Gamma} + (u_h^0(t), v)_{B_T} = 0 \quad \forall v \in V_h^T, \]
and \( v^0_h : [0, T] \rightarrow V^T_h \) is a solution of
\[
(\ddot{v}^0_h(t), w)(v^T_h \times v^T_h) + (\nabla v^0_h(t), \nabla w)_{B_T}
= (u^0_h(t) - \ddot{u}_h^0(t), w)_{B_T} \quad \forall w \in V^T_h, \quad 0 \leq t \leq T,
\]
\( v^0_h(0) = \dot{v}_h^0(0) = 0. \)

1. Problem (5.10) has a unique solution by a simple coercivity argument. Since dependence on \( t \) happens only through the non-homogeneous essential transmission condition (compare with (4.24), where the condition was natural), it is simple to prove that
\[
u \text{ in (5.10), we can prove that}
\]
and
\[
u \in C^2([0, T]; H^1(B_T \setminus \Gamma)), \quad u^0_h(0) = \dot{u}_h^0(0) = 0.
\]

Using the lifting operator \( L \) given in (4.5)-(4.6) and taking \( u^0_h(t) - L\varphi(t) \in V^T_h \) as a test function in (5.10), we can prove that
\[
\|u^0_h(t)\|_{1,B_T \setminus \Gamma} \leq C_L\|\varphi(t)\|_{1/2,\Gamma}.
\]

Taking second derivatives with respect to time in (5.10) and using same kind of argument, we prove
\[
\|\ddot{u}_h^0(t)\|_{1,B_T \setminus \Gamma} \leq C_L\|\ddot{\varphi}(t)\|_{1/2,\Gamma}.
\]

2. Proposition 4.4 can now be invoked to prove that the evolution problem (5.11) has a unique solution
\[
v^0_h \in C^1([0, T]; H^1_0(B_T)) \cap C([0, T]; L^2(B_T)),
\]
satisfying the bounds
\[
C_T^{-1}\|v^0_h(t)\|_{B_T} \leq \|\nabla v^0_h(t)\|_{B_T} \leq \int_0^t \|\dot{u}_h^0(\tau) - \ddot{u}_h^0(\tau)\|_{B_T} d\tau \leq C_LB_{2/1}^1(\varphi, t),
\]
where we have used (5.13)-(5.14) in the last inequality.

5.3 A second decomposition of \( u^\varphi_h \)

An alternative decomposition to (5.9) is needed to bound the density \( \lambda^T_h \). From this moment on, we need \( \varphi \in C^4_0([0, T]; H^{1/2}(\Gamma)) \). We now write
\[
u^\varphi = u^1_h + v^1_h,
\]
where \( u^1_h : [0, T] \rightarrow H^1(B_T \setminus \Gamma) \) is the solution of the variational problems (for \( 0 \leq t \leq T \))
\[
u^\varphi \in C^1([0, T]; H^1_0(B_T)), \quad \nu^\varphi \in C([0, T]; L^2(B_T)),
\]
and \( v^1_h : [0, T] \rightarrow D^T_h \) is a solution of
\[
\ddot{v}^1_h(t) = \Delta v^1_h(t) + f(t), \quad 0 \leq t \leq T,
\]
\[v^1_h(0) = \dot{v}_h^1(0) = 0,
\]
where \( f := u^1_h - \ddot{u}_h^1 + L\varphi - L\ddot{\varphi} = \Delta u^1_h - \ddot{u}_h^1 : [0, T] \rightarrow V^T_h \) is continuous.
1. It is clear that \( u_h^1 \in C^2([0, T]; H^1_\Delta(B_T \backslash \Gamma)) \) and \( u_h^1(0) = \hat{u}_h^1(0) = 0 \). Also, using

\[ v = u_h^1(t) - L\varphi(t) \in V_h^T \]

as test function in (5.18), we can prove that

\[
\|u_h^1(t)\|_{1, BT \backslash \Gamma} \leq \|L\varphi(t)\|_{1, BT \backslash \Gamma} + \|u_h^1(t) - L\varphi(t)\|_{1, BT \backslash \Gamma} \\
\leq C_L(\|\varphi(t)\|_{1/2, \Gamma} + \|\dot{\varphi}(t)\|_{1/2, \Gamma})
\]

and therefore

\[
\|\Delta u_h^1(t)\|_{BT \backslash \Gamma} \leq C_L(\|\varphi(t)\|_{1/2, \Gamma} + 2\|\dot{\varphi}(t)\|_{1/2, \Gamma})
\]

2. Regularity of the solution of (5.19) is given by (4.17) in Proposition 4.3. Since (5.20) can be differentiated twice with respect to time, we can bound

\[
\|
abla f(t)\|_{BT} \leq \|u_h^1(t)\|_{1, BT \backslash \Gamma} + \|u_h^1(t)\|_{1, BT \backslash \Gamma} + C_L(\|\varphi(t)\|_{1/2, \Gamma} + \|\dot{\varphi}(t)\|_{1/2, \Gamma}) \\
\leq C_L(4\|\varphi(t)\|_{1/2, \Gamma} + 5\|\dot{\varphi}(t)\|_{1/2, \Gamma} + \|\varphi(\cdot)(\cdot)\|_{1/2, \Gamma})
\]

and thus

\[
\|\Delta u_h^1(t)\|_{BT \backslash \Gamma} \leq \int_0^t \|
abla f(\tau)\|_{BT} d\tau \leq C_L B_4(\varphi, t) \quad 0 \leq t \leq T.
\]

### 5.4 Proof of Theorem 5.1

We first define \( u_h^\varphi \) with (5.9), where \( u_h^0 \) solves (5.10) and \( u_h^0 \) solves (5.11). By (5.10), (5.11), (5.12), and (5.15), it follows that \( u_h^\varphi \) is the only weak solution of (5.5) in the sense of Proposition 5.2 and it can be thus identified with the solution of (5.1)-(5.2). As a direct consequence of (5.13) and (5.16), it follows that

\[
\|u_h^\varphi(t)\|_{1, BT \backslash \Gamma} \leq C_L(\|\varphi(t)\|_{1/2, \Gamma} + \sqrt{1 + C_T^2 B_2^{1/2}(\varphi, t)}), \quad 0 \leq t \leq T,
\]

and

\[
\|
abla u_h^\varphi(t)\|_{BT} \leq C_L(\|\varphi(t)\|_{1/2, \Gamma} + B_2^{1/2}(\varphi, t)), \quad 0 \leq t \leq T.
\]

We next define \( u_h^\varphi = u_h^1 + v_h^1 \), where \( u_h^1 \) satisfies (5.18) and \( v_h^1 \) satisfies (5.19). It is clear that \( u_h^\varphi \) is a strong solution of (4.3) and therefore (Corollary 5.3) coincides with the solution of (5.1)-(5.2). Since \( u_h^\varphi \in C([0, T]; H^1_\Delta(B_T \backslash \Gamma)) \), then \( \lambda_h^\varphi = -\|\partial_\nu u_h^\varphi\| \in C([0, T]; H^{-1/2}(\Gamma)) \). By (5.20) and (5.21) it follows that

\[
\|\Delta u_h^\varphi(t)\|_{BT \backslash \Gamma} \leq C_L(4\|\varphi(t)\|_{1/2, \Gamma} + 2\|\varphi(t)\|_{1/2, \Gamma} + B_4^{1/2}(\varphi, t)) \quad 0 \leq t \leq T.
\]

This inequality, (5.24), and (4.17) prove finally that

\[
\|\lambda_h^\varphi(t)\|_{-1/2, \Gamma} \leq \sqrt{C_L} C_\nu(4\|\varphi(t)\|_{1/2, \Gamma} + 2\|\varphi(t)\|_{1/2, \Gamma} + B_4^{1/2}(\varphi, t)) \quad 0 \leq t \leq T.
\]
5.5 Proof of Theorem 3.1

We use the notation (4.1) for the Galerkin projection and (5.1) for the Galerkin solver. Since, \((\Pi_h \lambda)^G = \Pi_h \lambda\), we can bound
\[
\|\lambda - \lambda_h(t)\|_{-1/2,\Gamma} \leq \|\lambda - \Pi_h \lambda\|_{-1/2,\Gamma} + \|\lambda - \Pi_h \lambda\|_h + \|\lambda^{\phi^h} - \phi^h\|_{-1/2,\Gamma}.
\]
The bound (3.5) follows then from Theorems 4.1 and 5.1. Similarly we write
\[
\|u - u_h(t)\|_{1,R^d} \leq \|S^* (\lambda - \Pi_h \lambda)(t)\|_{1,R^d} + \|S^* (\lambda - \Pi_h \lambda)^G_h(t)\|_{1,R^d} + \|u^{\phi^h} - \phi^h\|_{1,R^d}
\]
and use [8, Theorem 3.1], Theorem 4.1 and Theorem 5.1 to prove (3.6).

6 Analysis of time discretization

6.1 A passage to the Laplace domain

For any value \(s \in \mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re} s > 0\}\), we consider the fundamental solution of the differential operator \(\Delta - s^2\), namely,
\[
\Phi(x, y; s) := \begin{cases} \frac{i}{4}H_0^{(1)}(is|x-y|), & \text{for } d = 2, \\ e^{-s|x-y|}/(4\pi|x-y|), & \text{for } d = 3. \end{cases}
\]
(The function \(H_0^{(1)}\) is the Hankel function of the first kind and order zero.) We also consider the single and double layer potentials
\[
S(s)\lambda := \int_{\Gamma} \Phi(\cdot, y; s)\lambda(y)d\Gamma(y) : \mathbb{R}^d \setminus \Gamma \to \mathbb{C},
\]
\[
D(s)\varphi := \int_{\Gamma} \partial_{\nu(y)}\Phi(\cdot, y; s)\varphi(y)d\Gamma(y) : \mathbb{R}^d \setminus \Gamma \to \mathbb{C}
\]
and the associated integral operators
\[
V(s) := \gamma^+ S(s) = \gamma^- S(s) \quad K(s) := \frac{1}{2}\gamma^+ D(s) + \frac{1}{2}\gamma^- D(s).
\]
Consider then the Laplace transforms of the semidiscrete data \(\Phi_h := \mathcal{L}\{\varphi_h\}\) and of the semidiscrete solutions \(\Lambda_h := \mathcal{L}\{\lambda_h\}\) and \(U_h := \mathcal{L}\{u_h\}\). For \(z \in \mathbb{C}_+\) and \(G \in H^{1/2}(\Gamma)\), we consider the uniquely solvable transmission problem looking for \(V \in H^1(\mathbb{R}^d \setminus \Gamma)\) such that
\[
z^2 V - \Delta V = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma \quad \text{(6.1a)}
\]
\[
[\gamma V] = G, \quad \text{(6.1b)}
\]
\[
\gamma^- V \in X^0_h, \quad \text{(6.1c)}
\]
\[
[\partial_{\nu} V] \in X_h. \quad \text{(6.1d)}
\]

**Proposition 6.1.** For all \(s \in \mathbb{C}_+\), \(U_h(s)\) is the unique solution of the transmission problem (6.1) with \(z = s\) and \(G = \Phi_h(s)\). Moreover \(\Lambda_h(s) = -[\partial_{\nu} U_h(s)]\).
Proof. Note that for all \( G = \Phi \) and \( \lambda^\kappa \) with \( u_h^\kappa(n) = [u_h^\kappa]_n \) and \( \lambda_h^\kappa(n) = [\lambda^\kappa]_n \). Let then \( \Lambda_h^\kappa := \mathcal{L}\{\lambda^\kappa\} \) and \( U_h^\kappa := \mathcal{L}\{u_h^\kappa\} \).

**Proposition 6.2.** For all \( s \in \mathbb{C}_+ \), \( U_h^\kappa(s) \) is the unique solution of the transmission problem (6.1) with \( G = \Phi_h(s) \). Moreover \( \Lambda^\kappa_h(s) = -[\partial_u U_h^\kappa(s)] \).

Proof. By construction \( \Lambda^\kappa_h(s) \in X_h \) and

\[
V(s\kappa)\Lambda^\kappa_h(s) - \frac{1}{2}\Phi_h(s) + K(s\kappa)\Phi_h(s) \in X_h^0, \quad U_h^\kappa(s) = -S(s\kappa)\Lambda^\kappa_h(s) - D(s\kappa)\Phi_h(s).
\]

The result is then straightforward.

Note that each occurrence of \( s\kappa \) in (6.3) corresponds to the discretization of a convolution process.

We finally consider the errors of time discretization of the semidiscrete-in-space problem

\[
e = e_h^\kappa := u_h - u_h^\kappa, \quad E := U_h - U_h^\kappa = \mathcal{L}\{e\}, \quad \varepsilon = \varepsilon_h^\kappa := \lambda_h - \lambda_h^\kappa.
\]

On time steps, we will be considering the errors

\[
e_n := e(t_n) = u_h(t_n) - u_h^\kappa[n], \quad \varepsilon_n := \varepsilon(t_n) = \lambda_h(t_n) - \lambda_h^\kappa[n].
\]

Applying Propositions 6.1 and 6.2, it follows that for all \( s \in \mathbb{C}_+ \)

\[
E(s) \in D_h := \{ u \in H^1(\mathbb{R}^d) \cap H^1_\lambda(\mathbb{R}^\mathbb{C} \setminus \Gamma) : \gamma u \in X_h^0, [\partial_u u] \in X_h \},
\]

and

\[
s^2\kappa E(s) - \Delta E(s) = (s^2\kappa - s^2)U_h(s) =: \Theta(s).
\]

In the time domain, the function \( \theta = \theta_h^\kappa \), whose Laplace transform is \( \Theta \), corresponds to a consistency error of the time discretization—the approximation of the second derivative by the particular CQ scheme—applied to the semidiscrete-in-space solution. Before we start the analysis of each of the time discretization methods, let us mention the integration-by-parts formula in \( D_h \), which will play an important role in the forthcoming analysis:

\[
(\Delta u, v)_{\mathbb{R}^d \setminus \Gamma} + (\nabla u, \nabla v)_{\mathbb{R}^d} = ([\partial_u u], \gamma v) = 0 \quad \forall u, v \in D_h.
\]

Bounds with respect to data will be given in terms of the following quantities:

\[
B_{\kappa}^{1/2}(\varphi, t) := \sum_{t=3}^k \int_0^t \|\varphi^{(t)}_{\kappa}(\tau)\|_{1/2, \mathbb{R}^d} d\tau \quad (k \geq 5).
\]

The following product (semi)norm

\[
\|(u, v)\| := \left(\|\nabla u\|^2_{\mathbb{R}^d} + \|v\|^2_{\mathbb{R}^d}\right)^{1/2}
\]

will be used to simplify some formulas.
6.2 Analysis of the Backward Euler discretization

Proposition 6.3. For all $n \geq 1$,

$$
\| (e_n, \frac{1}{\kappa}(e_n - e_{n-1})) \| \leq \kappa C t_n (1 + t_n) B^{1/2}_5(\varphi_h, t_n).
$$

(6.7)

Proof. Let $f_n := \frac{1}{\kappa}(e_n - e_{n-1})$. Noticing that for the Backward Euler discretization $s_\kappa = \frac{1}{\kappa}(1 - e^{-s_\kappa})$, the error equation (6.5) can be written as

$$
e_n - e_{n-1} = \kappa f_n, \quad f_n - f_{n-1} = \kappa \Delta e_n + \kappa \theta_n
$$

(6.8)

where we can bound the consistency error as

$$
\| \theta_n \|_{ \mathbb{R}^d } = \left\| \frac{u_h(t_n) - 2u_h(t_n-1) + u_h(t_n-2)}{\kappa^2} - \ddot{u}_h(t_n) \right\|_{ \mathbb{R}^d } \leq \frac{5\kappa}{3} \max_{t_{n-2} \leq t \leq t_n} \| u_h^{(3)}(t) \|_{ \mathbb{R}^d }.
$$

(6.9)

Testing the equations (6.8) with $-\Delta e_n$ and $f_n$ respectively, adding the results, and applying the integration by parts formula (6.6) (note that $e_n \in D_h$ for all $n$, since $e$ takes values in this space by (6.4)), it follows that

$$(\nabla e_n, \nabla e_n)_{\mathbb{R}^d} + (f_n, f_n)_{\mathbb{R}^d} = (\nabla e_n, \nabla e_{n-1})_{\mathbb{R}^d} + (f_n, f_{n-1})_{\mathbb{R}^d} + \kappa (\theta_n, f_n)_{\mathbb{R}^d}$$

and therefore

$$
\| (e_n, f_n) \| \leq \| (e_{n-1}, f_{n-1}) \| + \kappa \| \theta_n \|_{ \mathbb{R}^d } \quad \forall n.
$$

(6.10)

Then, by induction

$$
\| (e_n, f_n) \| \leq \kappa \sum_{j=1}^n \| \theta_j \|_{ \mathbb{R}^d }.
$$

Using now (6.9) and Theorem 5.1 (recall that $\varphi \mapsto u_h^{(3)}$ is a convolution operator and therefore commutes with differentiation), it follows that

$$
\| \theta_j \|_{ \mathbb{R}^d } \leq \frac{5}{3} C_L \kappa \left( \max_{t_{j-2} \leq t \leq t_j} \| \varphi_h^{(3)}(t) \|_{1/2, \Gamma} + \sqrt{1 + C_L^2 B^{1/2}_2(\varphi_h^{(3)}, t_n)} \right),
$$

(6.11)

where we have also used the fact that the constant $C_L$ of (5.3) grows with $t$. Adding the bounds (6.11) for different values of $j$, using the overestimate

$$
\max_{t_{j-2} \leq t \leq t_j} \| \varphi_h^{(3)}(t) \|_{1/2, \Gamma} \leq \max_{0 \leq t \leq t_n} \| \varphi_h^{(3)}(t) \|_{1/2, \Gamma} \leq \int_0^{t_n} \| \varphi_h^{(4)}(t) \|_{1/2, \Gamma} dt,
$$

the result follows. \qed

Theorem 6.4. For all $n \geq 1$

$$
\| e_n \|_{1, \mathbb{R}^d} \leq \kappa t_n (1 + t_n^2) B^{1/2}_5(\varphi_h, t_n)
$$

(6.12)

$$
\| \varepsilon_n \|_{-1/2, \Gamma} \leq \kappa (1 + t_n^2) B^{1/2}_6(\varphi_h, t_n).
$$

(6.13)
Proof. Using the notation of the proof of Proposition \[6.3\] and since \(e_n = e_{n-1} + \kappa f_n\), we can easily bound (using Proposition \[6.3\])

\[
\|e_n\|_{\mathbb{R}^d} \leq \kappa \sum_{j=1}^{n} \|f_j\|_{\mathbb{R}^d} \leq \kappa C t_n (1 + t_n) B_6^{1/2}(\varphi_h, t_n).
\]

This inequality and Proposition \[6.3\] prove \(6.12\).

Note now that \(f_n = f(t_n)\), where \(f = (e - e(\cdot - \kappa))/\kappa\). Therefore, using the second of the equalities \(6.8\), we can bound

\[
\|\Delta e_n\|_{\mathbb{R}^d \setminus \Gamma} = \left\| \frac{1}{\kappa} (f_n - f_{n-1}) - \theta_n \right\|_{\mathbb{R}^d} \leq \max_{t_{n-1} \leq t \leq t_n} \|\dot{f}(t)\|_{\mathbb{R}^d} + \|\theta_n\|_{\mathbb{R}^d}.
\]  

(6.14)

For \(t \in [t_{n-1}, t_n]\), we can construct a mesh with time-step \(\kappa\) that includes \(t\). Then, applying Proposition \[6.1\] with data \(\dot{\varphi}_h\) on this mesh, it follows that

\[
\|\dot{f}(t)\|_{\mathbb{R}^d} \leq \kappa C t(1 + t) B_5^{1/2}(\varphi_h, t) \leq \kappa C t_n(1 + t_n) B_5^{1/2}(\varphi_h, t_n).
\]

This inequality, the bound \(6.11\) for the consistency error and \(6.14\) provide a bound for the Laplacian of the error

\[
\|\Delta e_n\|_{\mathbb{R}^d \setminus \Gamma} \leq \kappa C (1 + t_n^2) B_6^{1/2}(\varphi_h, t_n).
\]

(6.15)

Since by Propositions \[6.1\] and \[6.2\] it follows that \(\left[\partial_\nu e_n\right] = -\varepsilon_n\), the bound \(6.13\) is a direct consequence of \(4.7\), Proposition \[6.3\] and \(6.16\). \(\square\)

### 6.3 Analysis of the BDF2 discretization

**Lemma 6.5.** Given \(G \in L^2(\mathbb{R}^d)\), let \(V \in D_h\) solve

\[
s^2 V - \Delta V = (s^2 - s^2) s^{-3} G \quad \text{in } \mathbb{R}^d \setminus \Gamma.
\]

Then

\[
\|(V, s \kappa V)\| \leq C \kappa^2 \frac{|s|}{\min\{1, \Re s\}} \|G\|_{\mathbb{R}^d} \quad \forall s \in \mathbb{C}_+, \forall \kappa, \forall h.
\]

**Proof.** Consider first a general \(z \in \mathbb{C}_+, H \in L^2(\mathbb{R}^d)\) and \(V \in D_h\) such that \(z^2 V - \Delta V = H\). By \(6.6\), it follows that

\[
z^2 \|V\|_{\mathbb{R}^d}^2 + \|\nabla V\|_{\mathbb{R}^d}^2 = (H, \nabla V)_{\mathbb{R}^d}
\]

and therefore

\[
(\Re z) \|(V, z V)\| \leq \|H\|_{\mathbb{R}^d}.
\]  

(6.16)

Taking now \(H := (s^2 - s^2) s^{-3} G\), \(z = s \kappa\), and noticing that

\[
|s \kappa| \leq C_1 |s|, \quad |s \kappa - s| \leq C_2 \kappa^2 |s|^2, \quad \Re s \kappa \geq C_3 \min\{1, \Re s\}, \quad \forall s \in \mathbb{C}_+, \forall \kappa,
\]

the result follows from \(6.16\). \(\square\)
In the next results we will refer to the operator
\[ \partial_x g := \frac{1}{\kappa} \left( \frac{3}{2} g + 2g(\cdot - \kappa) + \frac{1}{2} g(\cdot - 2\kappa) \right) \quad \mathcal{L}\{\partial_x g\} = s\kappa G(s), \]
which is the discrete derivative associated to the BDF2 method.

**Proposition 6.6.** Let \( f := \partial_x e \) and \( f_n := f(t_n) = \frac{1}{\kappa} \left( \frac{3}{2} e_n - 2e_{n-1} + \frac{1}{2} e_{n-2} \right) \). Then for all \( n \geq 1 \)
\[ \| (e_n, f_n) \| \leq \kappa^2 C t_n (1 + t_n^2) B_{1/2}^2 (\varphi_h, t_n). \]

**Proof.** By (6.5) and Lemma 6.5 it follows that
\[ \| (E(s), F(s)) \| \leq C \kappa^2 \frac{|s|}{\min\{1, \text{Re} s\}} \| s^3 U_h(s) \|_{\mathbb{R}^d} \quad s \in \mathbb{C}_+. \]
Using then [8, Theorem 7.1], it follows that
\[ \| (e(t), f(t)) \| \leq C \kappa^2 t \sum_{\ell=3}^6 \int_0^t \| u_h^{(\ell)}(\tau) \|_{\mathbb{R}^d} d\tau \quad \forall t. \]
(6.17)
Since by Theorem 5.1 we can bound
\[ \int_0^t \| u_h(\tau) \|_{\mathbb{R}^d} d\tau \leq C (1 + t^2) B_{1/2}^2 (\varphi_h, t), \]
(6.18)
the result is a direct consequence of (6.17).

**Theorem 6.7.** For all \( n \geq 1 \),
\[ \| e_n \|_{1, \mathbb{R}^d} \leq \kappa^2 t_n (1 + t_n^2) B_{1/2}^2 (\varphi_h, t_n), \]  
(6.19)
\[ \| e_n \|_{-1/2, \Gamma} \geq \kappa^2 C (1 + t_n^3) B_{3/2}^2 (\varphi_h, t_n). \]
(6.20)

**Proof.** The proof is very similar to that of Theorem 6.4 Using a simple stability argument for recurrences, we first show that
\[ \| e_n \|_{\mathbb{R}^d} \leq \kappa \sum_{j=1}^n \| f_j \|_{\mathbb{R}^d}. \]
This inequality and Proposition 6.6 prove (6.19). We next use that \( \Delta e_n = (\partial_x f)(t_n) - \theta_n \) (see (6.5) and the definition of \( f \) in Proposition 6.6) to bound
\[ \| \Delta e_n \|_{\mathbb{R}^d \setminus \Gamma} \leq \frac{3}{2} \| f_n - f_{n-1} \|_{\mathbb{R}^d} + \frac{1}{2} \| f_{n-1} - f_{n-2} \|_{\mathbb{R}^d} + \| (\partial_x^2 u_h - \bar{u}_h)(t_n) \|_{\mathbb{R}^d} \]
\[ \leq \frac{3}{2} \max_{t_{n-2} \leq \tau \leq t_n} \| f(\tau) \|_{\mathbb{R}^d} + C \kappa^2 \max_{t_{n-4} \leq \tau \leq t_n} \| u_h^{(4)}(\tau) \|_{\mathbb{R}^d}. \]
Using then (6.17) and (6.18) we obtain a bound
\[ \| \Delta e_n \|_{\mathbb{R}^d \setminus \Gamma} \leq \kappa^2 (1 + t_n^2) B_{3/2}^2 (\varphi_h, t_n) \]
and (6.20) follows from this and Proposition 6.6 using that \( \varepsilon_n = -[\partial_x e_n] \).
6.4 Analysis of the trapezoidal rule discretization

Proposition 6.8. For all \( n \geq 1 \)
\[
\left\| \left( \frac{1}{\kappa}(e_{n+1} + e_n), \frac{1}{\kappa}(e_{n+1} - e_n) \right) \right\| \leq \kappa^2 C t_n^{1/2} B_2^{1/2}(\varphi_h, t_n^{(5)}, t_{n+1}).
\]

Proof. In the case of the trapezoidal rule, the error equation (6.5) can be written as
\[
\frac{1}{\kappa^2}(1 - e^{-sk})^2 E(s) = \frac{1}{2}(1 + e^{-sk})^2 \Delta E(s) + \frac{1}{2}\kappa^2 (1 - e^{-sk})^2 U_h(s) - \frac{1}{2}(1 + e^{-sk})^2 s^2 U_h(s).
\]
In the time domain, this gives
\[
\frac{1}{\kappa^2}(e_{n+1} - 2e_n + e_{n-1}) = \frac{1}{4}\Delta(e_{n+1} + 2e_n + e_{n-1}) + \chi_n,
\]
where
\[
\chi_n := \frac{1}{\kappa^2}(u_h(t_{n+1}) - 2u_h(t_n) + u_h(t_{n-1})) - \frac{1}{2}(\ddot{u}_h(t_{n+1}) + 2\ddot{u}_h(t_n) + \ddot{u}_h(t_{n-1})).
\]
Using the integration by parts formula (6.6) we obtain
\[
\left( \frac{1}{\kappa}(e_{n+1} - e_n) - \frac{1}{\kappa}(e_n - e_{n-1}), v \right)_{\mathbb{R}^d} + \frac{\kappa}{2}\left( \nabla\left( \frac{1}{2}(e_{n+1} + e_n) + \nabla(\frac{1}{2}(e_n + e_{n-1}), \nabla v) \right) = \kappa(\chi_n, v)_{\mathbb{R}^d} \quad \forall v \in D_h.
\]
Testing (6.22) with
\[
v := \frac{1}{\kappa}(e_{n+1} - e_{n-1}) = \frac{1}{\kappa}(e_{n+1} - e_n) + \frac{1}{\kappa}(e_n - e_{n-1}) = \frac{2}{\kappa}\left( \frac{1}{2}(e_{n+1} + e_n) - \frac{1}{2}(e_n + e_{n-1}) \right),
\]
it follows that
\[
\left\| \left( \frac{1}{2}(e_{n+1} + e_n), \frac{1}{\kappa}(e_{n+1} - e_n) \right) \right\|^2 = \left\| \left( \frac{1}{2}(e_{n+1} + e_n), \frac{1}{\kappa}(e_n - e_{n-1}) \right) \right\|^2
\]
\[
+ \kappa \left( \chi_n, \frac{1}{\kappa}(e_{n+1} - e_n) + \frac{1}{\kappa}(e_n - e_{n-1}) \right)_{\mathbb{R}^d}
\]
\[
= \kappa \sum_{j=0}^n \left( \chi_j, \frac{1}{\kappa}(e_{j+1} - e_j) + \frac{1}{\kappa}(e_j - e_{j-1}) \right)_{\mathbb{R}^d}. \tag{6.23}
\]
Consider now the mesh-grid with nodes in the midpoints \( t_j^{j+\frac{1}{2}} := (j + \frac{1}{2})\kappa, \) the piecewise constant function \( \tilde{\chi} \) such that \( \tilde{\chi}(t) = \chi_j \) in \( (t_{j-\frac{1}{2}}, t_{j+\frac{1}{2}}) \), and the continuous piecewise linear functions \( \tilde{f} \) and \( \tilde{e} \) such that
\[
\tilde{f}(t_{j+\frac{1}{2}}) = \frac{1}{\kappa}(f_{j+1} - f_j), \quad \tilde{e}(t_{j+\frac{1}{2}}) = \frac{1}{2}(e_{j+1} + e_j).
\]

We can then write (6.23) as
\[
\left\| \left( \tilde{e}(t_{n+\frac{1}{2}}), \tilde{f}(t_{n+\frac{1}{2}}) \right) \right\|^2 = 2 \int_0^{t_{n+\frac{1}{2}}} (\tilde{\chi}(\tau), \tilde{f}(\tau))_{\mathbb{R}^d} d\tau.
\]

Given \( n \), we choose \( n^* \leq n \) such that \( \left\| \left( \tilde{e}(t_{n^*+\frac{1}{2}}), \tilde{f}(t_{n^*+\frac{1}{2}}) \right) \right\| \) is maximized. Then
\[
\left\| \left( \tilde{e}(t_{n^*+\frac{1}{2}}), \tilde{f}(t_{n^*+\frac{1}{2}}) \right) \right\|^2 \leq 2\left\| \left( \tilde{e}(t_{n^*+\frac{1}{2}}), \tilde{f}(t_{n^*+\frac{1}{2}}) \right) \right\| \int_0^{t_{n^*+\frac{1}{2}}} \| \tilde{\chi}(\tau) \|_{\mathbb{R}^d} d\tau
\]
\[23\]
and therefore (after comparing with the maximum at $t_{n+\frac{1}{2}}$ and overestimating the integral in the right-hand side)

$$\| (\tilde{e}(t_{n+\frac{1}{2}}), \tilde{f}(t_{n+\frac{1}{2}})) \| \leq 2 \int_{0}^{t_{n+\frac{1}{2}}} \| \tilde{\chi}(\tau) \|_{\mathbb{R}^d} d\tau \leq 2t_{n+1} \max_{j \leq n} \| \chi_j \|_{\mathbb{R}^d} \quad \forall n.$$ 

Since

$$\| \chi_n \|_{\mathbb{R}^d} \leq C\kappa^2 \max_{t_{n-1} \leq \tau \leq t_{n+1}} \| u_h^{(4)}(\tau) \|_{\mathbb{R}^d}, \quad (6.24)$$

the result follows by (6.18). \(\square\)

**Theorem 6.9.** For all $n \geq 1$,

$$\| e_n \|_{\mathbb{R}^d} \leq \kappa^2 Ct_n^2 B^{1/2}_1(\varphi_h, t_n), \quad (6.25)$$

$$\| \frac{1}{2}(e_{n+1} + 2e_n + e_{n-1}) \|_{-1/2, \Gamma} \leq \kappa^2 C(1 + t_{n+1}^2) B^{1/2}_1(\varphi_h, t_{n+1}). \quad (6.26)$$

**Proof.** Since

$$e_n = \frac{1}{\kappa} \sum_{j=0}^{n-1} (e_{j+1} - e_j),$$

the first bound follows from Proposition 6.8. By (6.21) we can bound

$$\| \frac{1}{2} \Delta (e_{n+1} + 2e_n + e_{n-1}) \|_{\mathbb{R}^d \setminus \Gamma} \leq C \max_{t_{n-1} \leq \tau \leq t_{n+1}} \| \ddot{e}(\tau) \|_{\mathbb{R}^d} + \| \chi_n \|_{\mathbb{R}^d} \quad (6.27)$$

The first term in the right-hand-side of (6.27) can be bounded using (6.25) applied to $\dot{\varphi}_h$ using a time-grid with time-step $\kappa$ that includes the point where the maximum is attained. The second term of (6.27) is bounded using (6.24). The result follows from the fact that $e_n = -\| \partial_\nu e_n \|$. \(\square\)

**7 Final comments**

In this paper we have given a full analysis of the discretization with Galerkin is space and three particular instances of Convolution Quadrature in time of a direct formulation for the exterior Dirichlet problem for the wave equation. The full error estimates are the result of Theorem 3.1 for the semidiscretization in space process and Theorems 6.4 (backward Euler), 6.7 (BDF2), and 6.9 (trapezoidal rule) for time discretization.

An indirect formulation, i.e.,

$$V * \xi = \varphi \quad u = S * \xi$$

follows from very similar arguments. The Galerkin projection is the same and therefore, the analysis of the semidiscrete in pase system is a particular case of the results in this paper. The Galerkin solver (see Section 5) is however slightly different. While its analysis is not needed for the semidiscretization in space, it is needed for the time discretization.
This analysis is likely to be extremely similar to the one given here. In terms of its Laplace transform, the semidiscrete problem is

\[ s^2 U_h(s) - \Delta U_h(s) = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma \]
\[ \gamma U_h(s) = 0, \]
\[ \gamma U_h(s) - \Phi(s) \in X^\circ_h, \]
\[ [\partial_\nu U_h(s)] \in X_h. \]

with \( \Xi_h(s) = [\partial_\nu U_h(s)] \). This is a very similar problem (same kind of transmission conditions) to problem (6.1) (see also Proposition 6.1). In particular, the error equations to compare semidiscrete and fully discrete solutions (6.4)-(6.5) are the same as in the case of the direct formulation and all the arguments of Section 6 hold, contingent to having proved the estimates of Theorem 5.1 adapted to the new kind of Galerkin solver.

All the arguments that have been used in this paper can be easily extended to the case of the single layer potential for the elastic wave equation in any dimension.

Much of the analysis of Sections 4-6 can be done using estimates in the Laplace domain. That gives a more streamlined way of proving estimates, although they come with either worse constants (for growth in time) or with higher continuity requirements: see [8, Section 7] for a comparison of Laplace domain and time domain techniques applied to estimating layer potentials and integral operators. The analysis of semidiscretization in space using the Laplace domain can be adapted from the techniques developed in [10]. Analysis of convolution quadrature can then be carried out using the very general results of Lubich [12] applied to the semidiscrete operators. It has to be noted, though, that the analysis in [12] does not cover the case of the trapezoidal rule (the reference [3] circumvents this difficulty nevertheless), while the relatively traditional time-domain analysis of Section 6.4 – based on understanding the semidiscrete equations as a transmission problem and, in particular, on the integration by parts formula (6.6) –, is applicable. Similarly, the use of multistage convolution quadrature [4], [5] and variable-step convolution quadrature [11] can be applied using estimates in the Laplace domain and it remains to be seen whether a time-domain analysis is practicable and produces different or improved results.

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