Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard

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Abstract

Can the vertices of a graph $G$ be partitioned into $A \cup B$, so that $G[A]$ is a line-graph and $G[B]$ is a forest? Can $G$ be partitioned into a planar graph and a perfect graph? The NP-completeness of these problems are just special cases of our result: if $P$ and $Q$ are additive induced-hereditary graph properties, then $(P, Q)$-colouring is NP-hard, with the sole exception of graph 2-colouring (the case where both $P$ and $Q$ are the set $\mathcal{O}$ of finite edgeless graphs). Moreover, $(P, Q)$-colouring is NP-complete iff $P$- and $Q$-recognition are both in NP. This proves a conjecture of Kratochvíl and Schiermeyer.

Kratochvíl and Schiermeyer conjectured in [18] that for any additive hereditary graph properties $P$ and $Q$, recognising graphs in $P \circ Q$ is NP-hard, with the obvious exception of bipartite graphs (the case where both $P$ and $Q$ are the set $\mathcal{O}$ of finite edgeless graphs). They settled the case where $Q = \mathcal{O}$, and it was natural to extend the conjecture to $induced$-hereditary properties. Berger’s result [3] that reducible additive induced-hereditary properties have infinitely many minimal forbidden subgraphs provided support for the extended conjecture.

We prove the extension of the Kratochvíl-Schiermeyer conjecture in this paper. Problems such as the following (for an arbitrary graph $G$) are therefore NP-complete. Can $V(G)$ be partitioned into $A \cup B$, so that $G[A]$ is a
line-graph and $G[B]$ is a forest? Can $G$ be partitioned into a planar graph and a perfect graph? For fixed $k, \ell, m$, can $G$ be partitioned into a $k$-degenerate subgraph, a subgraph of maximum degree $\ell$, and an $m$-edge-colourable subgraph?

Garey et al. [14, 21] essentially showed $(\mathcal{O}, \{\text{forests}\})$-colouring to be NP-complete, while Brandstädt et al. [4, Thm. 3] proved the case $(\mathcal{O}, \{P_4, C_4\}-\text{free graphs})$.

Let $\mathcal{P}$ be a property and let $\mathcal{P}^k$ be the product of $\mathcal{P}$ with itself, $k$ times. Brown and Corneil [6, 8] showed that $\mathcal{P}^k$-recognition is NP-hard when $\mathcal{P}$ is the set of perfect graphs and $k \geq 2$, while Hakimi and Schmeichel [16] did the case $\{\text{forests}\}^2$. There was particular interest in $G$-free $k$-colouring (where $\mathcal{P}$ has just one forbidden induced-subgraph $G$). When $G = K_2$ we get graph colouring, one of the best known NP-complete problems, while subchromatic number [2, 13] (partitioning into subgraphs whose components are all cliques) is the case $G = P_3$. Brown [7] proved the case where $G$ is 2-connected, and Achlioptas [1] showed NP-completeness for all $G$. In fact, Achlioptas’ proof settles the case $\mathcal{R}^k$ for any irreducible additive induced-hereditary $\mathcal{R}$.

1 Preliminaries

We consider only simple finite graphs. We write $G \leq H$ when $G$ is an induced subgraph of $H$. We identify a graph property with the set of graphs that have that property. A property $\mathcal{P}$ is additive, or (induced-)hereditary, if it is closed under taking vertex-disjoint unions, or (induced-)subgraphs. The properties we consider contain the null graph $K_0$ and at least one, but not all (finite simple non-null) graphs.

A $(\mathcal{P}, \mathcal{Q})$-colouring of $G$ is a partition of $V(G)$ into red and blue vertices, such that the red vertices induce a subgraph $G_\mathcal{P} \in \mathcal{P}$, and the blue vertices induce a subgraph $G_\mathcal{Q} \in \mathcal{Q}$. The product of $\mathcal{P}$ and $\mathcal{Q}$ is $\mathcal{P} \circ \mathcal{Q}$, the set of $(\mathcal{P}, \mathcal{Q})$-colourable graphs. We use $(\mathcal{P}, \mathcal{Q})$-colouring, $(\mathcal{P}, \mathcal{Q})$-partition and $(\mathcal{P} \circ \mathcal{Q})$-recognition interchangeably.

Let $\mathcal{P}$ be an additive induced-hereditary property. Then $\mathcal{P}$ is reducible if it is the product of two additive induced-hereditary properties; otherwise it is irreducible. It is true, though by no means obvious, that if $\mathcal{P}$ is the product of any two properties, then it is also the product of two additive induced-hereditary properties [11].
The set of minimal forbidden induced-subgraphs for $\mathcal{P}$ is $\mathcal{F}(\mathcal{P}) := \{H \notin \mathcal{P} \mid \forall G < H, G \in \mathcal{P}\}$. Note that $\mathcal{F}(\mathcal{O}) = \{K_2\}$, while all other properties have forbidden subgraphs with at least 3 vertices. $\mathcal{P}$ is additive if every graph in $\mathcal{F}(\mathcal{P})$ is connected. Every hereditary property is induced-hereditary, and the product of additive (induced-hereditary) properties is additive (induced-hereditary).

A graph $H$ is strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable if there is exactly one ordered partition $(V_1, \ldots, V_n)$ of $V(H)$ such that for all $i$, $H[V_i] \in \mathcal{P}_i$. More precisely, suppose $V(H) = U_1 \cup \cdots \cup U_n$, where $H[U_i] \in \mathcal{P}_i$ for all $i$. Then

(a) there is a permutation $\phi$ of $\{1, \ldots, n\}$ such that $V_i = U_{\phi(i)}$;
(b) if $i, j$ are in the same cycle of $\phi$, then $\mathcal{P}_i = \mathcal{P}_j$.

When the $\mathcal{P}_i$'s are additive induced-hereditary and irreducible, Mihók [20] gave a construction that can easily be adapted (cf. [10, Thm. 5.3], [11], [5]) to give a strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph $H$ with $V_n \neq \emptyset$. We use $H$ to show that $A \circ B$-recognition is at least as hard as $A$-recognition, when $A$ and $B$ are additive induced-hereditary properties (the result is not true for all properties, e.g., $B := \{G \mid |V(G)| \geq 10\}$).

1. **Theorem.** Let $A$ and $B$ be additive induced-hereditary properties. Then there is a polynomial-time transformation from the $A$-recognition problem to the $(A \circ B)$-recognition problem.

**Proof:** It is clearly enough to prove this when $B$ is irreducible. For any graph $G$ we will construct (in time linear in $|V(G)|$) a graph $G'$ such that $G \in A$ if and only if $G' \in A \circ B$.

Let $A = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{n-1}$, $B = \mathcal{P}_n$, where the $\mathcal{P}_i$'s are irreducible additive induced-hereditary properties. Let $H$ be a fixed strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph, with partition $(V_1, \ldots, V_n)$, such that $V_n \neq \emptyset$. Let $v_H$ be some fixed vertex in $V_1$.

For any graph $G$, we construct $G'$ by taking a copy of $G$ and a copy of $H$, and making every vertex of $G$ adjacent to every vertex of $N(v_H) \cap V_n$. By additivity of $A$, if $G$ is in $A$, then $G'$ is in $A \circ B$.

Conversely, if $G' \in A \circ B = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, then it has an ordered partition $(W_1, \ldots, W_n)$ with $W_i \in \mathcal{P}_i$ for each $i$. Since the $\mathcal{P}_i$'s are induced-hereditary, $G'[W_i] \in \mathcal{P}_i$ implies $G'[W_i \cap V(H)] \in \mathcal{P}_i$. Then

$^{1}$Up to some permutation of the subscripts as in (a), (b).
Suppose some $w \in V(G)$ is in $W_n$. Now $(V_1 \setminus \{v_H\}, V_2, \ldots, V_{n-1}, V_n \cup \{w\})$ is a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $(H-v_H)+w \cong H$. Then $(V_1 \setminus \{v_H\}, V_2, \ldots, V_{n-1}, V_n \cup \{v_H\})$ is a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $H$ that is different from $(V_1, \ldots, V_n)$ (since $V_n \neq \emptyset$), a contradiction.

Thus no vertex of $G$ is in $W_n$, and so $G \leq G'[W_1 \cup \cdots \cup W_{n-1}] \in \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{n-1} = A$, and $G \in A$ as required. □

We will prove the main result by transforming $p$-in-$r$-SAT to $(\mathcal{P}, \mathcal{Q})$-colouring, where $p$ and $r$ are fixed integers depending on $\mathcal{P}$ and $\mathcal{Q}$. Schaefer [23] showed $p$-in-$r$-SAT to be NP-complete, even for formulae with all literals unnegated, for any fixed $p$ and $r$, so long as $1 \leq p < r$ and $r \geq 3$. We restate it as:

$p$-in-$r$-COLOURING

Instance: an $r$-uniform hypergraph.

Problem: is there a set of vertices $U$ such that, for each hyper-edge $e$, $|U \cap e| = p$?

2 NP-hardness

2. Theorem. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive induced-hereditary properties, $\mathcal{P} \circ \mathcal{Q} \neq \mathcal{O}^2$. Then $(\mathcal{P} \circ \mathcal{Q})$-recognition is NP-hard. Moreover, it is NP-complete iff $\mathcal{P}$- and $\mathcal{Q}$-recognition are both in NP.

Proof: We will prove the first part; the second part then follows by Theorem [11]. Also by Theorem [11] (and by the well-known NP-hardness of recognising $\mathcal{O}^3$ [14]), we need only consider the case where $\mathcal{P}$ and $\mathcal{Q}$ are irreducible. By Theorem [11] there is a strongly uniquely $(\mathcal{P}, \mathcal{Q})$-colourable graph $G_{\mathcal{P}, \mathcal{Q}}$ that we use to “force” vertices to be in $\mathcal{P}$ or $\mathcal{Q}$.

More formally, let the unique partition be $V(G_{\mathcal{P}, \mathcal{Q}}) = U_\mathcal{P} \cup U_\mathcal{Q}$. Choose $p \in U_\mathcal{P}$. If $G_{\mathcal{P}, \mathcal{Q}} \leq H$, and $v \notin V(G_{\mathcal{P}, \mathcal{Q}})$ satisfies $N(v) \cap U_\mathcal{Q} = N(p) \cap U_\mathcal{Q}$, then in any $(\mathcal{P}, \mathcal{Q})$-colouring of $H$, $v$ must be in the $\mathcal{P}$-part2; otherwise, in

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2To be precise, we mean that $v$ is coloured the same as $p$: if $\mathcal{P} = \mathcal{Q}$ then a $(\mathcal{P}, \mathcal{Q})$-colouring is also a $(\mathcal{Q}, \mathcal{P})$-colouring, but we adopt the convention that the $\mathcal{P}$-part is the part containing $p$. 

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we could transfer \( p \) over to the \( Q \) part, giving us a different \((P, Q)\)-colouring. Similarly we choose \( q \in U_Q \), whose neighbours we use to force vertices to be in \( Q \). \( G_{P, Q} \) is our first gadget.

An end-block of a graph \( G \) is a block of \( G \) that contains at most one cut-vertex of \( G \); if \( G \) has no cut-vertices, then \( G \) is itself an end-block. Let \( B_P \) be an end-block of \( F_P \in \mathcal{F}(P) \), chosen to have the least number of vertices among all the end-blocks of all the graphs in \( \mathcal{F}(P) \) (see Figure 1). Because \( P \) is additive and non-trivial, \( F_P \) is connected and has at least two vertices, so \( B_P \) has \( k \geq 2 \) vertices. The point to note is that, if \( H \) is a graph in \( P \), then adding an end-block with fewer than \( k \) vertices produces another graph in \( P \).

Let \( y_P \) be the unique cut-vertex contained in \( B_P \) (if \( B_P = F_P \), pick \( y_P \) arbitrarily), and let \( x_P \) be a vertex of \( B_P \) adjacent to \( y_P \). Let \( F'_P \) be the graph obtained by adding an extra copy of \( B_P \) (incident to the same cut-vertex \( y_P \)), and let \( x'_P \) be a vertex in this new copy that is adjacent to \( y_P \).

Similarly, we choose \( B_Q \) to be an end-block of \( F_Q \in \mathcal{F}(Q) \), minimal among the end-blocks of graphs in \( \mathcal{F}(Q) \); we add a copy of \( B_Q \), and pick \( x_Q \), \( y_Q \) and \( x'_Q \) as above. We identify \( x_P \) with \( x_Q \), \( y_P \) with \( y_Q \), \( x'_P \) with \( x'_Q \), and label the identified vertices \( x, y, x' \).

Finally, we force all the vertices of \( F'_P \) (except for \( x, y, x' \)) to be in \( P \), and all the vertices of \( F'_Q \) (except for \( x, y, x' \)) to be in \( Q \). That is, we add a copy of \( G_{P, Q} \), and make every vertex of \( F'_P - \{x, y, x'\} \) adjacent to every vertex of \( N(p) \cap U_Q \), and every vertex of \( F'_Q - \{x, y, x'\} \) adjacent to every vertex of \( N(q) \cap U_P \) (cf. Figure 1).

It can be readily checked that the resulting gadget \( R \) (for ‘replicator’) has the following properties:

- In a \((P, Q)\)-colouring of \( R \), if \( x \) is in \( P \), then \( y \) is in \( Q \) and \( x' \) is in \( P \); similarly, if \( x \) is in \( Q \), then \( y \) is in \( P \) and \( x' \) is in \( Q \). So \( x \) and \( x' \) always have the same colour, that is different from that of \( y \). Moreover, there is at least one colouring (in fact, exactly one) in which \( x \) and \( x' \) are in \( P \), and at least one in which both are in \( Q \).

- Identify \( x \) with a vertex \( z \) of some graph \( H \) to obtain \( H_R \); then \((P, Q)\)-colourings of \( H \) and \( R \) that agree on \( x \), together give a \((P, Q)\)-colouring of \( H_R \). We can then similarly identify \( x' \) with some vertex \( z' \) of a graph \( H' \), and attach more copies of \( R \) at \( x \) or \( x' \).
Figure 1: The forbidden graphs $F_P$ and $F_Q$, and the replicator gadget $R$. 
We thus have a gadget that “replicates” the colour of \( x \) on \( x' \), while preserving valid colourings.

Let \( H_P \) be a forbidden subgraph for \( P \) with the least possible number of vertices, say \( p + 1 \); similarly choose \( H_Q \in \mathcal{F}(Q) \) on \( q + 1 \) vertices, where \( q + 1 \) is as small as possible, so any graph on at most \( p \) (resp. \( q \)) vertices is in \( P \) (resp. \( Q \)). Since \( P \) and \( Q \) are not both \( O \), \( p + q \geq 3 \), and so \( p\)-in-\((p + q)\)-COLOURING is NP-complete. We will construct a third gadget to transform this to \((P, Q)\)-colouring.

We start with an independent set \( S \) on \( p + q \) vertices, \( \{x_1, \ldots, x_{p+q}\} \). For every \((p + 1)\)-subset of \( S \), say \( T_j = \{x_1, \ldots, x_{p+1}\} \), add a disjoint copy of \( H_P \) whose vertices are labeled \( x^j_1, \ldots, x^j_{p+1} \). For each \( i = 1, \ldots, p + 1 \), use a new copy \( R_{i,j} \) of \( R \) to ensure that \( x_i \) and \( x^j_i \) are always coloured the same; to do this, identify the vertices \( x \) and \( x' \) of \( R_{i,j} \) with \( x_i \) and \( x^j_i \). For every \((q + 1)\)-subset of \( S \) we add a copy of \( H_Q \) in the same manner. Thus every vertex \( x_i \in S \) will have \( \ell = (p+q-1) + (p+q-1) \) ‘shadow vertices’ \( x^1_i, \ldots, x^\ell_i \) from copies of \( H_P \) and \( H_Q \). Call this gadget \( N \) (for ‘pin cushion’ — the copies of \( H_P \) and \( H_Q \) being stuck into the independent set \( S \) by ‘pins’ or ‘replicators’).

In a \((P, Q)\)-colouring of \( N \), no \( p + 1 \) vertices of \( S \) can be in \( P \), and no \( q + 1 \) vertices can be in \( Q \), so exactly \( p \) vertices of \( S \) are in \( P \), and exactly \( q \) are in \( Q \). Conversely, suppose that exactly \( p \) vertices of \( S \) are coloured red, and the other \( q \) are blue; colour each vertex \( x^j_i \) the same as \( x_i \), \( 1 \leq i \leq p + q \), \( 1 \leq j \leq \ell \). Then each copy of \( H_P \) has at most \( p \) red and at most \( q \) blue vertices, giving it a valid \((P, Q)\)-colouring. The colouring on the rest of each gadget \( R_{i,j} \) is then forced, and we have a \((P, Q)\)-colouring of all of \( N \).

Now, given a \((p + q)\)-uniform hypergraph \( H \), we stick a copy of \( N \) onto every hyper-edge. The resulting graph is \((P, Q)\)-colourable iff \( H \) has a \( p\)-in-\((p + q)\)-COLOURING.

3 New directions

How far can the main result be extended? Uniquely \((P_1, \ldots, P_n)\)-partitionable graphs exist even in many cases where the \( P_i \)’s are not additive [12]; however, this includes finite \( P_i \)’s, so the existence of uniquely colourable graphs does not guarantee NP-hardness.
It may be useful to restate the result as follows: if the graphs in \( F(P) \) and \( F(Q) \) are all connected, then \((P, Q)\)-colouring is NP-hard. This is also true if the graphs in \( F(P) \) and \( F(Q) \) are all disconnected, since \( G \in P \circ Q \Leftrightarrow \overline{G} \in P \circ Q \), where \( P \) is defined by \( F(P) := \{ H \mid H \in F(P) \} \).

A natural problem to tackle next would be classifying the complexity of \( R^k \)-recognition, where \( R \) has both connected and disconnected minimal forbidden induced-subgraphs. One of the simplest such cases is \( R = (O \cup K) \), where \( K \) is the set of all cliques: \( F(O \cup K) = \{ P_3, \overline{P_3} \} \). Gimbel et al. \cite{15} noted that \( G \in O^k \Leftrightarrow nG \in (O \cup K)^k \) (where \( n = |V(G)| \)); so \((O \cup K)^k\)-recognition is NP-complete for \( k \geq 3 \) (and, in fact, polynomial for \( k = 1, 2 \)).

Another natural problem is \((P, Q)\)-colouring, where all graphs in \( F(P) \) are connected, and all those in \( F(Q) \) are disconnected. In all problems, it may make sense to restrict attention to hereditary properties with finitely many forbidden subgraphs.

Another class of problems often considered in the literature is \((D : P)\)-recognition: given a graph \( G \) in the domain \( D \), is \( G \in P? \) This is just \((D \cap P)\)-recognition; if \( D \) and \( P \) are both additive induced-hereditary, then so is \( D \cap P \), with \( F(D \cap P) = \min \leq (F(D) \cup F(P)) \). We leave it as an open question, for reducible \( P \), to determine when \( D \cap P \) is also reducible; Mihók’s characterisations \cite{19,20} of reducibility may prove useful.

4 Notes and acknowledgements

The most important part of the proof is the ‘replicator’ gadget. Phelps and Rödl \cite[Thm. 6.2]{22} and Brown \cite[Thm. 2.3]{7} used different gadgets to perform similar roles. The forcing technique of Theorem \cite{11} was first used in \cite[Thm. 2]{18} and \cite[Lemma 3]{5}.

Contacts with Lozin were very helpful, as they spurred the author to look at \((K_m\text{-free}, K_n\text{-free})\)-colouring, not knowing it had been settled in \cite{9}. Kratochvíl and Schiermeyer \cite{18} proved a special case of Theorem \cite{2} that covered the case \( m = 2 \); \((K_2\text{-free}, K_n\text{-free})\)-colouring; I started my proof for general \( m \) and \( n \) by adapting theirs, and ended up strengthening and simplifying it considerably.

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