HÖLDER REGULARITY OF GENERIC MANIFOLD

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Dedicated to Professor Józef Siciak for his 85th birthday

Abstract: In this paper we study Hölder continuity of the pluricomplex Green function with logarithmic growth at infinity of a smooth generic submanifold of $\mathbb{C}^n$. In particular we prove that the pluricomplex Green function of any $C^2$-smooth generic compact submanifold of $\mathbb{C}^n$ (without boundary) is Lipschitz continuous in $\mathbb{C}^n$.

Key words: Generic manifold, attached analytic discs, plurisubharmonic Green function, pluripolar sets, pluriregular sets, Hölder continuity.

AMS Classification: 32U05, 32U15, 32U35, 32E30, 32V40

1. Introduction and statement of the main result

Real $m-$planes $\Pi \subset \mathbb{C}^n$, $\dim_R \Pi = m$, $m \in \mathbb{N}^+$, which are not contained in any proper complex subspace of $\mathbb{C}^n$ are important in complex analysis and pluripotential theory. The $\mathbb{C}$–hull of such plane $\Pi$ is equal to all $\mathbb{C}^n$ i.e. $\Pi + J\Pi = \mathbb{C}^n$ ($J$ is the standard complex structure on $\mathbb{C}^n$) and any non empty open subset of $\Pi$ is non pluripolar in $\mathbb{C}^n$. Such planes are called generic (real) subspaces of $\mathbb{C}^n$. Correspondingly, a real smooth submanifold $M \subset \mathbb{C}^n$ is said to be generic if for each $z \in M$, its real tangent space $T_z M$ is a generic subspace of $\mathbb{C}^n$ i.e. $T_z M + JT_z M = \mathbb{C}^n$. Such submanifold has real dimension $m \geq n$. The case of minimal dimension $\dim M = n$ is the most relevant for our concern. In this case for each $z \in M$, the tangent space $T_z M$ does not contain any complex line i.e. $T_z M \cap JT_z M = \{0\}$ and $M$ is said to be totally real.

Observe that any smooth Jordan curve in $\mathbb{C}$ is totally real, hence any product of $n$ smooth Jordan curves in $\mathbb{C}$ is a smooth compact totally real submanifold of dimension $n$ in $\mathbb{C}^n$. Moreover the class of smooth compact totally real submanifolds of dimension $n$ in $\mathbb{C}^n$ is stable under small $C^2$-perturbations.

Generic submanifolds of $\mathbb{C}^n$ play an important role in Complex Analysis and Pluripotential Theory (see [Pi74], [KC76], [Sa76], [C92], [EW10], [SZ12]).

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In our previous paper [SZ12], we used the method of attached analytic discs to investigate non plurithinness of generic submanifolds of $\mathbb{C}^n$. We proved in [SZ12], that subsets of full measure in a generic $C^2$-smooth submanifold are non-plurithin at any point.

Here we continue our investigations concerning the pluripotential properties (pluripolarity, pluriregularity) of generic submanifolds in $\mathbb{C}^n$ by studying Hölder continuity of their pluricomplex Green functions.

All these properties can be expressed in terms of the pluricomplex Green function defined as follows.

Given a (bounded) subset $E \subset \mathbb{C}^n$, we define its pluricomplex Green function as follows:

$$V_E(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0\},$$

where $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of psh functions $u$ in $\mathbb{C}^n$ with logarithmic growth at infinity i.e. $\sup\{u(z) - \log^+(z) : z \in \mathbb{C}^n\} < +\infty$ (see [Sic62], [Sic81], [Za76], [Kl]).

Our main result is the following.

**Main theorem.** Let $M \subset \mathbb{C}^n$ be a $C^2$-smooth generic compact submanifold without boundary. Then its pluricomplex Green function $V_M$ is Lipschitz continuous in $\mathbb{C}^n$.

This theorem is concerned with compact submanifolds without boundary. In Section 3, we will consider the more general case of a $C^2$-smooth generic submanifold and prove that its extremal function is Lipschitz near each of its compact subsets (see Theorem 5.1). In the last section we consider the case of a compact $C^2$-smooth generic submanifold with boundary and discuss the Hölder continuity property of its pluricomplex Green function.

From Lipschitz continuity or more generally the Hölder continuity of the pluricomplex Green function $V_E^*(z)$ of a compact set $E \subset \mathbb{C}^n$, it follows that the compact set $E$ satisfies the following Markov’s inequality: there exists positive constants $A, r > 0$ such that

$$\|\nabla P(z)\|_E \leq Ad^r\|P(z)\|_E, \quad z \in \mathbb{C}^n$$

for any polynomial $P$ of degree $d$.

This inequality plays an important role in approximation theory, gives sharp inequalities for polynomials and is useful for constructing continuous extension operators for smooth functions from subsets of $\mathbb{R}^n$ to $\mathbb{C}^n$ (see [PP86], [Ze93]). On the other hand, Complex Dynamic gives a lot of examples of compact subsets for which the pluricomplex Green function $V_E^*(z)$ is Hölder continuous (see [FS92, K95, Ks97]).

More recently an important result of C.T. Dinh, V. A. Nguyen and N. Sibony shows that the Monge-Ampère measure of a Hölder continuous plurisubharmonic function is a moderate measure (see [DNS]). In particular the equilibrium Monge-Ampère measure $\mu_E := (dd^cV_E)^n$ of a compact subset whose pluricomplex Green function $V_E^*(z)$ is Hölder continuous is
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a moderate measure, which means that it satisfies the following uniform version of Skoda’s integrability theorem: for any compact family $\mathcal{U}$ of psh functions in a neighborhood of a given ball $B \subset \mathbb{C}^n$, there exists $\varepsilon > 0$ and a constant $C > 0$ such that

$$\int_B e^{-\varepsilon u} d\mu_E \leq C, \forall u \in \mathcal{U}.$$  

From this property, it follows that the equilibrium measure $\mu_E$ is ”well dominated” by the Monge-Ampère capacity (see [Ze01]), in the sense that for any given ball $B \subset \mathbb{C}^n$, there is a constant $A > 0$ such that for any Borel set $S \subset B$,

$$\mu_E(S) \leq A \exp \left(-A \text{cap}_B(S)^{-1/n}\right),$$

where $\text{cap}_B(S)$ is the Monge-Ampère capacity ([BT82]).

This property turns out to play an important role in the theory of complex Monge-Ampère equations as was discovered by S. Koldziej (see [Ko98], [Ce98], [GZ07]).

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2. Definitions and preliminaries

Let us recall the following definitions

**Definition 2.1.** 1. We say that a subset $P \subset \mathbb{C}^n$ is pluripolar if there is a plurisubharmonic (psh) function $u : u \not\equiv -\infty$ but $u|_P \equiv -\infty$.

2. We say that $E$ is pluriregular if its pluricomplex Green function satisfies $V_E^*(z)|_E \equiv 0$ i.e. $V_E = V^*_E$ on $\mathbb{C}^n$.

Observe that any pluriregular set is non-pluripolar. It is well-known that if $E$ in non-pluripolar then $V_E^* \in \mathcal{L}(\mathbb{C}^n)$. Moreover if $E$ is pluriregular compact set then $V_E = V^*_E$ is continuous in $\mathbb{C}^n$ (see [Sic81]).

On the other hand, we know from ([BT82]) that if $E$ is non-pluripolar then the locally bounded psh function $V_E^*$ satisfies the following complex Monge-Ampère equation

$$(dd^c V_E)^n = 0, \text{ on } \mathbb{C}^n \setminus \overline{E},$$
which means that the equilibrium measure of $E$ defined as
\[ \mu_E := (dd^c V^*_E)^n \]
is a Borel measure supported in the closed set $\overline{E}$.

Here we will introduce the following important notion.

**Definition 2.2.** We say that a set $E$ is $\Lambda_\alpha$-pluriregular, $\alpha > 0$, if for every compact $K \subset E$ there exist a constant $A = A_K > 0$ and a neighborhood $O = O_K$ of $K$ such that
\[ V^*_E(z) \leq A d^\alpha(z, K), \quad \forall z \in O, \]
where $d$ is the Euclidean distance in $\mathbb{C}^n$.

Roughly speaking, this definition means that the pluricomplex Green function $V^*_E$ of the set $E$ is Hölder continuous near any compact subset $K \subset E$. The following observation, which is essentially due to Z. Blocki, shows that if the set itself $E$ is compact, the definition means that its pluricomplex Green function is Hölder continuous (see [Sic97]).

**Lemma 2.3.** If $E \subset \mathbb{C}^n$ is a $\Lambda_\alpha$-pluriregular compact set then its pluricomplex Green function $V_E$ is Hölder continuous of order $\alpha$ globally in $\mathbb{C}^n$ i.e. for any $z, w \in \mathbb{C}^n$, we have
\[ |V_E(z) - V_E(w)| \leq A |z - w|^\alpha. \]

**Proof.** Observe that $V^*_E$ has a logarithmic growth at infinity. Therefore if $E$ is a $\Lambda_\alpha$-pluriregular compact set then its pluricomplex Green function $V_E$ satisfies (2.1) for all $z \in \mathbb{C}^n$ i.e. for some constant $A > 0$ we have
\[ V_E(z) \leq A d^\alpha(z, E), \quad \forall z \in \mathbb{C}^n. \]

To prove that $V_E$ is Hölder continuous of order $\alpha$ globally in $\mathbb{C}^n$, fix $h \in \mathbb{C}^n$ such that $|h| < \delta$ and observe, that for any $z \in E$, $d(z + h, E) \leq \delta^\alpha$, which implies by the Hölder condition (2.2) that for any $z \in E$, $V_E(z + h) \leq A\delta^\alpha$. Therefore the function defined by $u(z) := V_E(z + h) - A\delta^\alpha$ is a plurisubharmonic function such that $u \in \mathcal{L}(\mathbb{C}^n)$ and $u \leq 0$ on $E$. By the definition of $V_E$, we conclude that $u \leq V_E(z)$ for any $z \in \mathbb{C}^n$, which implies that $V_E$ is Hölder continuous.

\[ \square \]

3. **Analytic discs attached to generic manifolds**

3.1. **Construction of attached analytic discs.** Let $\mathbb{U} := \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$ be the open unit disc and $\mathbb{T} := \partial \mathbb{U}$ the unit circle. An analytic disc of $\mathbb{C}^n$ is a continuous function $f : \mathbb{U} \rightarrow \mathbb{C}^n$, which is holomorphic on $\mathbb{U}$. Let $M \subset \mathbb{C}^n$ be a given subset of $\mathbb{C}^n$ and $\gamma \subset \overline{\mathbb{U}}$ a given connected subset of the closed disc $\overline{\mathbb{U}}$. We say that the analytic disc $f$ is attached to $M$ along $\gamma$ if $f(\gamma) \subset M$.

If $f : \overline{\mathbb{U}} \rightarrow \mathbb{C}^n$ is an analytic disc and $F$ is a holomorphic function on a neighborhood $D$ of $f(\overline{\mathbb{U}})$, then $F \circ f$ is a holomorphic function on the unit
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If $u$ is a plurisubharmonic function on $D$, then $u \circ f$ is a subharmonic function on $\mathbb{U}$. Therefore analytic discs enable us to reduce multidimensional complex problems to corresponding one dimensional complex problems.

In the proof of our theorem, we need a smooth family of analytic disks. We will use Bishop’s equation for construct such a family (see [B65], [Pi74]). Let $M$ be a totally real submanifold of dimension $n$ given locally by the following equation

$$M := \{ z = x + y \in B \times \mathbb{R}^n : y = h(x) \},$$

where $B \subset \mathbb{R}^n$ is a ball of center $0$ and $h : B \to \mathbb{R}^n$ a smooth map, such that $h(0) = 0$ and $Dh(0) = 0$.

Let $v(\tau) : T \to \mathbb{R}^+ a C^\infty$ function on the unit circle $T$ such that $v|\{e^{i\theta} : \theta \in (0, \pi)\} = 0$ and $v|\{e^{i\theta} : \theta \in (\pi, 2\pi)\} > 0$.

Assume that there exists a continuous mapping $X : T \to \mathbb{R}^n$ which is a solution of the following Bishop equation

$$(3.1) \quad X(\tau) = c - \Im(h \circ X + tv)(\tau), \quad \tau \in T,$$

where $(c, t) \in Q = Q_c \times Q_t \subset \mathbb{R}^n \times \mathbb{R}^n$ is a fixed parameter and $\Im$ is the harmonic conjugate operator defined by the Schwarz integral formula

$$(3.2) \quad \Im(X)(\zeta) = \frac{1}{2\pi} \int_T X(\tau) \Im \frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta} d\tau, \quad \zeta = re^{i\theta},$$

normalized by the condition

$$\Im X(0) = 0.$$

We will consider the unique harmonic extension $X(\zeta)$ of the mapping $X$ to the unit disk $\mathbb{U}$. Then the following mapping

$$(3.3) \quad \Phi(c, t, \zeta) := X(c, t, \zeta) + i[h^*(c, t, \zeta) + tv(\zeta)] = c + i\{h^*(c, t, \zeta) + tv(\zeta) + i\Im[h^*(c, t, \zeta) + tv(\zeta)]\}$$

provides a family of analytic disks $\Phi(c, t, \zeta) : \mathbb{U} \to \mathbb{C}^n$ such that

$$(3.4) \quad \forall (c, t) \in Q, \forall \tau \in \gamma, \Phi(c, t, \tau) \in M.$$

Here $X(c, t, \zeta), h^*(c, t, \zeta)$ and $v(\zeta)$ are harmonic extensions of $X(c, t, \tau)$, $h \circ X(c, t, \tau)$ and $v(\tau)$ to the unit disk $\mathbb{U}$ respectively.

We need a smooth family of disks $\Phi(c, t, \zeta)$. Many constructions of analytic discs attached to generic manifolds along a part the circle have been given by many different authors, depending on the smoothness properties of the manifold (see [Pi74], [75], [Sa76], [C92]). The most general and sharp result was proved by B.Coupet [C92]:

**Theorem** ([C92]). Let $p > 2n + 1, q \geq 1$ be integers and $h \in C^q(B)$. Then there exist a constant $\delta_0 > 0$ independing on $h$ and $p$ such, that for
arbitrary $C^q$-smooth mapping $k(c, t, \tau) : \mathbb{R}^{2n+1} \to \mathbb{R}^n$, with compact support and $\| k \|_{W^{q,p}} \leq \delta_0$, the equation

\begin{equation}
(3.5) \quad u = -\Im(h \circ u) + k
\end{equation}

has a unique solution $u \in W^{q,p}(T \times \mathbb{R}^{2n})$.

Moreover, the harmonic extensions of $u$ and $h \circ u$ to the unit disk $U$ belong to $C^q(U \times \mathbb{R}^{2n})$.

Let now $h \in C^q(B)$. Observe that Bishop’s equation (3.1) is a particular case of the equation (3.5). Therefore, from the theorem of Coupet and Sobolev’s embedding theorem $W^{q,p} \subset C^{q-1}$, it follows that for a small enough neighborhood $Q \ni 0$, $(c, t) \in Q$, the Bishop equation (3.1) has unique solution $X(\tau, c, t) : X, h \circ X \in C^{q-1}(U \times Q) \cap C^q(U \times Q)$. Note that the operator $\Im : W^{q,p} \to W^{q,p}$ is continuous.

Therefore, for a $C^2$-smooth generic submanifold $M \subset \mathbb{C}^n$, we obtain a smooth family of disks (3.3), attached to $M$, such that

$$\|\Im X\|_1 \leq A \|X\|_1, \|\Im h \circ X\|_1 \leq A \|h \circ X\|_1,$$

where $A$ is a constant and $\|\cdot\|_1$ is the $C^1$-norm in $\tau \in T$.

3.2. Harmonic measure of boundary set of the unit disk. For arbitrary $\gamma \subset T$ we put $\mathcal{H}(\gamma, U)$– class of functions

$$\{u(\zeta) : u \in \text{sh}(U) \cap C(U), u|_U < 0, u|_\gamma \leq -1\},$$

and set

$$\omega(\zeta, \gamma, U) = \sup\{u(z) : u \in \mathcal{H}\}, \zeta \in U.$$

Then (negative of) the upper semi-continuous regularisation $\omega^*(\zeta, \gamma, U)$ is called the harmonic measure of $\gamma$ with respect to $U$ at the point $\zeta$. The function $\omega^*$ is the unique solution of the Dirichlet problem:

$$\Delta \omega^* = 0, \omega^*|_T = -\chi_\gamma,$$

where $\chi_\gamma$ is the characteristic function of $\gamma$. By Poisson formula

$$\omega^*(\zeta, \gamma, U) = -\frac{1}{2\pi} \int_T \chi_\gamma(\tau) \Re \frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta} d\tau, \quad \zeta = re^{i\theta}.$$

For $\gamma = \{e^{i\varphi} : 0 \leq \varphi \leq \pi\}$ the harmonic measure $\omega^*$ can be expressed as follows.

\begin{equation}
(3.6) \quad \omega^*(\zeta, \gamma, U) = \frac{1}{\pi} \arg \left( \frac{1}{1 + \zeta} \right).
\end{equation}

Let us define the sector at the point $1 = e^{i0} \in \mathbb{U}$ as follows

$$\Omega_{0,\alpha} = \bigcup\{l \cap \mathbb{U} : l \ni 1, \pi/2 \leq \arg l \leq \pi/2 + \alpha\},$$

where $l$ stands for a real line passing through the point 1 and $0 \leq \alpha \leq \pi/2$ is fixed. The sector $\Omega_{0,\alpha}$ at the point $e^{i\alpha} \in \mathbb{U}$ can define in the same way. From (3.6) it follows clearly that
\[ \omega^*(\zeta, \gamma, U) \leq -1 + \alpha/\pi, \quad \forall \zeta \in \Omega_{0, \alpha}. \]

A sector \( \Omega_{a, \alpha} \) at the point \( e^{ia} \) is said to be admissible if \( \Omega_{a, \alpha} \cap \partial U \subset \gamma \). From the last fact, we deduce the following statement.

**Lemma 3.1.** Let \( \gamma = \text{arc}[e^{ia}, e^{ib}] \subset \mathbb{T}, \) \( 0 \leq a < b \leq 2\pi, \) be an arbitrary arc on \( \mathbb{T}, \) and let \( \Omega_{a, \alpha} \), be an admissible sector at the point \( e^{ia}. \) Then \( \omega^*(\zeta, \gamma, U) \) is \( \Lambda_1 \)-continuous in \( U \cup \mathbb{T} \setminus \{ e^{ia}, e^{ib} \}, \) \( \omega^*|_{\gamma^0} \equiv -1, \omega^*|_{\mathbb{T}\setminus\gamma} \equiv 0 \) and \( \omega^* \) satisfies (3.7) in \( \Omega_{a, \alpha}. \)

Here \( \gamma^0 \) denote the interior of the arc \( \gamma. \) We note that if \( \gamma_0 \subset\subset \gamma \) in an arc with non empty interior, then there exist \( \alpha = \alpha(\gamma_0, \gamma) > 0 \) such that \( \Omega_{\tau, \alpha} \) is admissible for every \( \tau \in \gamma_0. \)

4. **Transversality of attached discs to a generic manifold.**

It is clear that the family of analytic discs constructed above

\[ \Phi(c, t, \zeta) = X(c, t, \zeta) + i(h^*(c, t, \zeta) + tv(\zeta)), \quad (c, t) \in Q, \quad \tau \in \partial \bar{U}. \]

for \( c, t \in Q = Q_c \times Q_t, \zeta \in \bar{U}, \) satisfies the following properties:

(4.1) \[ X(c, t, \tau) = c - \Im(h \circ X(c, t, \tau) + tv(\tau)), \quad (c, t) \in Q, \quad \tau \in \partial \bar{U}. \]

(4.2) \[ h^*(c, t, \tau) = h \circ X(c, t, \tau), \quad (c, t) \in Q, \quad \tau \in \partial \bar{U}. \]

(4.3) \[ X(c, 0, \zeta) \equiv c, h^*(c, 0, \zeta) \equiv h(c), \quad \Phi(c, 0, \zeta) \equiv c + ih(c) \in M, \quad c \in Q_c. \]

(4.4) \[ X(c, t, 0) = \frac{1}{2\pi} \int_{T} X(c, t, \tau) d\tau \equiv c, \quad (c, t) \in Q. \]

(4.5) \[ \|X\| \leq O(\|c\| + \|t\|), \quad \|D_{\gamma}X\| \leq O(\|t\|). \]

Here and below \( \|\cdot\| \) is Euclidean norm.

The following geometric transversality property will be crucial for the proof of our main theorem.

**Lemma 4.1.** Let \( \gamma_0 \subset\subset \gamma \) an arc with non empty interior. Then for small enough \( Q \) the attached disks \( \Phi(c, t, \zeta) \), \( t \neq 0, \) for \( \zeta \to \tau \in \gamma_0 \) meet \( M \) transversally.

**Proof.** For the normal derivative \( D_{n_{\gamma}} \) at the points \( \tau \in \gamma_0 \) we have

\[ \text{Im}D_{n_{\gamma}} \Phi(c, t, \tau) = D_{n_{\gamma}} h^*(c, t, \tau) + tD_{n_{\gamma}} v(\tau) \]

and

\[ \left| \text{Im}D_{n_{\gamma}} \Phi(c, t, \tau) \right| \geq \| t \| \| b - O(\varepsilon) \| \| t \| = \| t \| (b - O(\varepsilon)), \]
where
\[ b := \inf_{\gamma_0} \left| D_n u(\tau) \right| > 0 \]
and \( \varepsilon = \sup \{ \| c \| + \| t \| : c \in Q_c, t \in Q_t \} \).

It follows, that for \( O(\varepsilon) < \frac{b}{2} \)
\[
(4.6) \quad \left| \text{Im} D_n \Phi(c, t, \tau) \right| \geq \| t \| \frac{b}{2} \quad \forall \tau \in \gamma_0,
\]
i.e. the disks \( \Phi(c, t, \zeta) \) meet \( M \) for \( \zeta \to \tau \in \gamma_0 \) transversally. \( \square \)

**Corollary 4.2.** Let \( Q' = \{ \| t \| = \sigma \} \subset Q_t \), where \( \sigma > 0 \). Then there exist a neighborhood \( \Omega' \supset \gamma_0 \) and a constant \( C > 0 \) such that
\[
(4.7) \quad \begin{align*}
d_C(\zeta, \gamma_0) &\leq Cd_{C^n}[\Phi(c, t, \zeta), M] \\
d_{C^n}[\Phi(c, t, \zeta), \Phi(c, t, \gamma_0)] &\leq Cd_{C^n}[\Phi(c, t, \zeta), M],
\end{align*}
\]
\( \forall \zeta \in \Omega = \overline{U} \cap \Omega' \), \( t \in Q' \), \( c \in \overline{Q}_c \). Here \( d_C \) and \( d_{C^n} \) are Euclidean distances on \( C \) and \( C^n \), respectively.

**Proof.** The statement clearly follows from (4.6), because for every fixed \( t^0 \in Q', c^0 \in \overline{Q}_c \) we can write (4.7), which then will be true in some neighborhoods \( B_c \ni c^0, B_t \ni t^0 \). \( \square \)

**Lemma 4.3.** For every \( \Omega' \supset \gamma_0 \) and for every \( Q'_t = \{ \| t \| = \sigma \} \subset Q_t \), \( \sigma > 0 \) small enough the closed set \( W = \{ \Phi(c, t, \zeta) \in C^n : c \in \overline{Q}_c, t \in Q'_t, \zeta \in \Omega = \overline{U} \cap \Omega' \} \) contains the point \( 0 \in M \) in its interior in \( C^n \). i.e. \( 0 \in W \).

**Proof.** By (4.3) \( X(c, t, \zeta) \equiv c \) if \( t = 0 \). Since \( X \) is smooth, then for small enough fixed \( t^0 \) and for arbitrary fixed \( \tau^0 \in \gamma_0 \) the image \( X(c, t^0, \tau^0) : c \in Q_c \) contains \( 0 \in \mathbb{R}^n \). It follows, that \( 0 \in W \). Moreover, \( W \neq \emptyset \) and if for some \( \| t^0 \| \leq \sigma \), \( \zeta^0 \in \overline{U} \)
\[
(4.8) \quad x(c, t^0, \zeta^0) \in \frac{1}{2}Q_c, \ \text{then} \ c \in Q_c.
\]

Now we assume by contradiction that \( 0 \in \partial W \). Then \( C^n \setminus W \) is open and contains 0 on its boundary. It is clear, that near 0 there exists a point \( p^0 = (x^0, y^0) \in \partial W \setminus M \) such, that \( x^0 \in \frac{1}{2}Q_c \) and \( p^0 = \Phi(c^0, t^0, \zeta^0) \) for some \( c^0 \in \overline{Q}_c \), \( \| t^0 \| = \sigma \), \( \zeta^0 \in \Omega' \cap \overline{U} \).

For simplicity we may assume that \( t^0 = (0, ..., 0, \sigma) \) and set \( 'c = (c_1, ..., c_{n-1}), 't = (t_1, ..., t_{n-1}) \). From (4.8) it follows also, that \( c \in Q_c \).

We consider the transformation
\[
(4.9) \quad S('c', 't', \zeta) = \Phi('c', c^0_n, 't', t^0_n, \zeta) : 'Q \times \overline{U} \longrightarrow C^n,
\]
where \( 'Q := \{ z \in Q : c_n = c^0_n, t_n = t^0_n \} \subset \mathbb{R}^{2n-2} \).
Then \( S('c', 't^0, \zeta^0) = p^0 \) and its Jacobian is given by
\[
J('c', 't', \zeta) = J(c, t, \zeta)|_{c_n = c^0_n, t_n = t^0_n},
\]
where

\[
\begin{aligned}
&\begin{bmatrix}
\frac{\partial X_1}{\partial c_1} & \cdots & \frac{\partial X_{n-1}}{\partial c_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial X_1}{\partial c_{n-1}} & \cdots & \frac{\partial X_{n-1}}{\partial c_{n-1}} \\
\end{bmatrix} \quad \begin{bmatrix}
\frac{\partial Y_1}{\partial c_1} & \cdots & \frac{\partial Y_{n-1}}{\partial c_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial Y_1}{\partial c_{n-1}} & \cdots & \frac{\partial Y_{n-1}}{\partial c_{n-1}} \\
\end{bmatrix} \\
&\quad \begin{bmatrix}
\frac{\partial X_1}{\partial c_n} \\
\vdots \\
\frac{\partial X_1}{\partial c_{n-1}} \\
\end{bmatrix} \quad \begin{bmatrix}
\frac{\partial Y_1}{\partial c_n} \\
\vdots \\
\frac{\partial Y_1}{\partial c_{n-1}} \\
\end{bmatrix} \\
&\quad \begin{bmatrix}
\frac{\partial X_1}{\partial c^m} \\
\vdots \\
\frac{\partial X_1}{\partial c^{m-1}} \\
\end{bmatrix} \quad \begin{bmatrix}
\frac{\partial Y_1}{\partial c^m} \\
\vdots \\
\frac{\partial Y_1}{\partial c^{m-1}} \\
\end{bmatrix} \\
&\end{aligned}
\]

Here \( \zeta = \zeta' + i \zeta'' \) and \( Y_k(c, t, \zeta) = h^* \circ X(c, t, \zeta) + t_k v(\zeta), \ k = 1, \ldots, n. \)

The determinant \( J \) is composed by \( 9 \) block matrices \( D_{ij}, i, j = 1, 2, 3. \)

We will show that \( J((c^0, t^0, \zeta^0)) \neq 0, \) which will imply that the operator \( S \) is a local diffeomorphism in a neighborhood of the point \( (c^0, t^0, \zeta^0) \).

Indeed, by (4.3) \( X(c, 0, \zeta) \equiv c, \ h^*(c, 0, \zeta) \equiv h(c) \) and then\n
\[
\begin{vmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{vmatrix}_{(c, 0, \zeta)} = D_{11} \cdot D_{22} = v^{n-1}(\zeta)
\]

and

(4.10) \[
\begin{vmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{vmatrix}_{(c, t, \zeta)} = v^{n-1}(\zeta) + O(\varepsilon),
\]

where we recall that \( \varepsilon = \text{sup} \{ \| c \| + \| t \| : c \in Q_c, t \in Q_t \} \). Note also that

\[
D_{33} = \left| \frac{\partial X_n}{\partial c^m} \right| \left| \frac{\partial Y_n}{\partial c^m} \right| = \left| \frac{d}{d\zeta} (X_n + iY_n) \right|^2.
\]

Now consider the right hand side near the arc \( \gamma \). It is clear that for every \( s > 0 \), there is an open set \( \hat{\Omega} \supset \gamma_0 \) such that

(4.11) \[
\left| \frac{d}{d\zeta} (X_n + iY_n)(c, t, \zeta) \right|^2 \geq |D_\tau X_n(c, t, \tau)|^2 - s, \forall \zeta \in U \cap \hat{\Omega}, \tau \in \gamma_0.
\]

We calculate \( D_\tau X(c, t, \tau) \), for \( (c, t) \in Q, \ \tau \in T, \)

(4.12) \[
D_\tau X(c, t, \tau) = -D_\tau \Im h \circ X(c, t, \tau) - tD_\tau \Im v(\tau).
\]

Since, \( D_\tau \Im v(\tau) = D_n \Im v(\tau) \), where \( D_n \) is the normal derivative \( \vec{n} \), then (4.12) implies\n
\[
D_\tau X(c, t, \tau) + tD_n \Im v(\tau) = \Im D_{\tau h} \circ X(c, t, \tau).
\]
For $k$-coordinate of vector $X(\tau) = X(c, t, \tau)$ we have

\begin{equation}
\left\| D_{\tau}X_k(c, t, \tau) + t_k D_{\tau}^n v(\tau) \right\| = \|SD_{\tau}h_k \circ X(c, t, \tau)\| \leq \\text{const} \|D_{\tau}h_k \circ X(c, t, \tau)\| \leq O(\varepsilon) \|D_{\tau}X(c, t, \tau)\|.
\end{equation}

Therefore,

\begin{equation}
\left| t_k D_{\tau}^n v(\tau) - O(\varepsilon) \|t\| \right| \leq \|D_{\tau}X_k(c, t, \tau)\| \leq \|t_k D_{\tau}^n v(\tau) + O(\varepsilon) \|t\| , \ 1 \leq k \leq n , \ \tau \in T.
\end{equation}

The second part of (4.14) implies

\begin{equation}
\|D_{\tau}X(c, t, \tau)\| \leq C \|t\| , \ (c, t, \tau) \in Q \times T , \ C - \text{constant}
\end{equation}

As in Lemma 4.1 if $b = \inf_{\gamma_0} \left| D_{\tau}^n v(\tau) \right| > 0$ and $O(\varepsilon) < \frac{b}{2}$, then the first part of (4.14) implies

\begin{equation}
\|D_{\tau}X_k(c, t, \tau)\| \geq \|t_k\| - \|t\|b/2,
\end{equation}

for $\tau \in \gamma, 1 \leq k \leq n$.

By (4.10) and (4.11) it follows that

\begin{equation}
|J(c', t, \zeta)| = |D_{11}||D_{22}| \left| \frac{d}{d\zeta}(x_n + iy_n)(c', t, \zeta) \right|^2 + O(\varepsilon) \geq [v^{n-1}(\zeta) + O(\varepsilon)] \cdot \left[ |t_n b/2|^2 - s \right] + O(\varepsilon),
\end{equation}

for all $(c', t, \zeta) \in Q \times [U \cap \Omega']$, because $\|t^0\| = |t^0_n|$. We can take $\Omega \cap \Omega'$ instead of $\Omega$ and observe that all functions $O(\cdot)$ do not depend on $\zeta$. Therefore if we take $\varepsilon, s$ small enough, then $|J(c', t, \zeta)| > 0$.

Since, the plane $\{t_n = t^0_n\}$ is tangent to the sphere $\|t\| = \sigma$ at the point $t^0$, the Jacobian of the restriction $\tilde{S} = \Phi \left( c', c^0_n, \sqrt{\|t_1\|^2 + \ldots + \|t_{n-1}\|^2}, \zeta \right)$ also is not zero at the point $(c', t^0, \zeta^0)$. In particular, the operator

\begin{equation}
\tilde{S} : U_1 \times U_2 \times U_3 \rightarrow U(p)
\end{equation}

is a homeomorphism, where $U_1 \subset \mathbb{R}^{n-1}$ – a neighborhood of the point $c^0$, $U_2 \subset Q'_t$ – a neighborhood of $t^0 \in Q'_t$ and $U_3 = \{||\zeta - \zeta_0| < \sigma'\} \subset \Omega, \sigma' > 0$, is a neighborhood of $\zeta_0$. It follows, that the open set $U(p) \subset W$, that is contradiction to $p \in \partial W$. \hfill \square

5. PROOF OF THE MAIN THEOREM

First we observe that from the results of Edigarian-Wiegerinck [EW10] and the authors [SZ12], it follows that $M \subset \mathbb{C}^n$ is a pluriregular set. Indeed, it was proved in [SZ12] that a set of full measure in a generic manifold $M$ is not thin. Since the set $P = \{z \in M : V_M^*(z) > 0\}$, where $V_M^*(z)$ is Green function, is pluripolar by Bedford and Taylor ([BT82]), it has zero-measure (see [Sa76, C92]) and then the set $M \setminus P$ is not thin. Therefore $V_M^* \equiv 0$ on $M$, i.e. $M$ is pluriregular. Note that in [EW10] non-thinness of $M \setminus P$ was
proved for $C^1$-smooth manifold $M$ and for a pluripolar set $P \subset M$, which implies that an arbitrary $C^1$-smooth generic manifold is pluriregular.

Our main theorem will be a consequence of the following result, thanks to Lemma 2.3.

**Theorem 5.1.** Any $C^2$-smooth generic submanifold $M \subset \mathbb{C}^n$ is $\Lambda_1$-pluriregular.

**Proof.** We first reduce to the case of a totally real submanifold. Fix a point, say $z^0 = 0 \in M$. Changing holomorphic coordinates in $\mathbb{C}^n$, we can assume that the tangent space $T_0 M$, which by definition does not contain any complex hyperplane, can be written as $T_0 M = \{z = x + iy \in \mathbb{C}^n : y_1 = \cdots = y_{2n-m} = 0\}$.

Hence for a small neighborhood $G = G_1 \times G_2$ of the origin with

$$G_1 = \{(x, y') = (x, y_{2n-m+1}, \ldots, y_n) \in \mathbb{R}^n \times \mathbb{R}^{m-n} : |x| \leq \delta, |y'| < \delta\},$$

$$G_2 = \{y' = (y_1, \ldots, y_{2n-m}) \in \mathbb{R}^{2n-m} : |y'| < \delta\},$$

we can represent $M$ as a graph

$$M \cap G = \{z \in G : y' = h(x, y'')\},$$

where $h$ is $C^2$ smooth mapping from $G_1$ into $G_2$.

Observe that for each small enough $y''_0$ the intersection $M \cap \Pi\{y''_0\}$ of $M$ with the plane $\Pi\{y''_0\} := \{z \in \mathbb{C}^n : y'' = y''_0\}$ is an $n$-dimensional generic manifold. Moreover, since the Green function is monotonic, i.e., $V(z, E_1) \geq V(z, E_2)$ for $E_1 \subset E_2$, it is enough to prove the theorem in the case when $M$ is generic of dimension $n$, hence totally real of dimension $n$.

In this case, we show the local Hölder pluriregularity of $M$, using previous results from Section 4. Fix a point $p \in M$, a ball $B(p) = B_x \times B_y \subset \mathbb{C}^n$ centered at the point $p$ such that $M_p := M \cap B(p)$ is the graph of a $C^2$-smooth function. Then by Corollary 4.2, for arbitrary fixed small $\sigma > 0$ there exist a neighborhood $\Omega' \supset \gamma_0$ and a constant $C > 0$, depending on the point $p$, such that the inequalities (4.7) hold. By Lemma 4.3 $O(p) \ni p$, where $O(p) = W^0$ is the interior of the set $W = W(p, \Omega', \sigma)$, constructed in Lemma 4.3.

Fix a point $z^0 \in O(p) \setminus M$ and a disk $\Phi(c, t, \zeta) : \Phi(c^0, t^0, \zeta^0) = z^0$, with $c^0 \in \mathcal{T}$, $t^0 \in Q_t$, $\zeta^0 \in \mathcal{U} \cap \Omega'$. Then the function $V_{M_p} \circ \Phi(c^0, t^0, \zeta) \in sh(\mathcal{U})$ and $V_{M_p} \circ \Phi|\gamma \equiv 0$. Let $C'' = \max_{B(p)} V_{M_p}(z) < \infty$. By the theorem of two constants we have

$$V_{M_p} \circ \Phi(c^0, t^0, \zeta) \leq C''[\omega^*(\zeta, \gamma, \mathcal{U}) + 1], \quad \zeta \in \mathcal{U}. \tag{5.1}$$

Therefore the first part of Lemma 3.1 and (4.7) yields the inequality

$$V_{M_p}(z^0) = V_{M_p} \circ \Phi(c^0, t^0, \zeta^0) \leq C''[\omega^*(\zeta^0, \gamma, \mathcal{U}) + 1] \leq C' C'' d_{\mathcal{C}}(\zeta^0, \gamma_0) \leq C_p d_{\mathcal{C}}(z^0, M_p), \tag{5.2}$$
for all $z^0 \in O(p)$, where $C_p := CC' C''$ depends on the fixed point $p \in M$ and on the corresponding family of analytic discs, attached to $M$ locally, in a neighborhood of $p$.

Now given a compact set $K \subset M$ we can apply the previous estimate to each point of $K$. Then by compactness we can find a finite number of points $p_1, \cdots, p_k$ of $K$, a finite number of balls $B(p_1), \cdots, B(p_k)$ and a finite numbers of open sets $O(p_1), \cdots O(p_k)$ such that

$$V_{M, p}(z) \leq C_p d_{\mathbb{C}^n}(z, M_p),$$

for any $z \in O(p)$ and $p = p_1, \cdots, p_k$. Now observe that $O = \cup_{1 \leq i \leq k} O(p_i)$ is a neighborhood of $K$ and shrinking a little bit the open sets $O_p$ we can assume that for any $p = p_i$ and $z \in O_p$, $d_{\mathbb{C}^n}(z, M_p) \leq d(z, M)$. Since $V_M \leq V_{M, p}$, it follows that $V_M(z) \leq Ad_{\mathbb{C}^n}(z, K)$, for any $z \in O.$

6. Open problems

Let $D \subset M$ a domain with $C^1$-smooth boundary $\partial D$. Lemma 4.2 states that a neighborhood of the generic manifold locally consists in the interior $\tilde{W}$ of the set $W = \{ \Phi(c, t, \zeta) : c \in \overline{Q}, t \in \overline{Q}', \zeta \in \Omega = \overline{U} \cap \overline{U}' \}$. It seems clear, at least intuitively, that if here, instead of $\Omega$, we take its part $\Omega_{a, \alpha}$, $\alpha > 0$ (see Lemma 3.1), then we should see that $\tilde{W}$ contains some wedge

$$\{ z \in \mathbb{C}^n : d_{\mathbb{C}^n}(z, M) < C_\alpha \cdot d_{\mathbb{C}^n}(z, \partial D) \},$$

where $C_\alpha > 0$ is a constant. If this is true then we could prove: arbitrary close $C^1$-domain $\overline{D}$ in $C^2$-smooth generic manifold is pluriregular, i.e. the Green function $V_*(z, \overline{D})$ is continuous in $\mathbb{C}^n$. The proof easily follows by the well-known criteria of pluriregularity (see [Sa80]) and by the following lemma.

**Lemma 6.1.** If $f(\lambda)$ is a $C^1$-smooth function on $[0, 1] \subset \mathbb{R}$, then for every $\varepsilon > 0$ there exist polynom $p(\lambda)$ such that

$$|p(\lambda) - f(\lambda)| \leq \varepsilon \lambda, \ \lambda \in [0, 1].$$

The authors do not know any proof of the following

**Conjecture:** let $D \subset M$ be a bounded domain in $M$ with smooth boundary, then $\overline{D} \subset \mathbb{C}^n$ is $\Lambda_{1/2}$-pluriregular i.e. its pluricomplex Green function $V(z, \overline{D})$ is Hölder continuous of order $1/2$ in $\mathbb{C}^n$.

We note that if $M$ is real analytic generic manifold then the conjecture is true.

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