Completeness of the ZX-calculus for Pure Qubit
Clifford+T Quantum Mechanics

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Abstract

Recently, we gave a complete axiomatisation of the ZX-calculus for the overall pure qubit quantum mechanics. Based on this result, here we also obtain a complete axiomatisation of the ZX-calculus for the Clifford+T quantum mechanics by restricting the ring of complex numbers to its subring corresponding to the Clifford+T fragment resting on the completeness theorem of the ZW-calculus for arbitrary commutative ring. In contrast to the first complete axiomatisation of the ZX-calculus for the Clifford+T fragment, we have two new generators as features rather than novelties: the triangle can be employed as an essential component to construct a Toffoli gate in a very simple form, while the \( \lambda \) box can be slightly extended to a generalised phase so that the generalised supplementarity (cyclotomic supplementarity) is naturally seen as a special case of the generalised spider rule.

1 Introduction

Clifford+T qubit quantum mechanics (QM) is an approximatively universal fragment of QM, which has been widely used in quantum computing. In contrast to the traditional way of matrix calculation, the ZX-calculus introduced by Coecke and Duncan [1] is a diagrammatical axiomatisation of quantum computing. One of the main open problems of the ZX-calculus is to give a complete axiomatisation for the Clifford+T QM [8]. After the first completeness result on this fragment—the completeness of the ZX-calculus for single qubit Clifford+T QM [3], there finally comes a completion of the ZX-Calculus for the whole Clifford+T QM [4], which contributes a solution to the above mentioned open problem.

Further to the complete axiomatisation of the ZX-Calculus for the Clifford+T fragment QM, we have given a complete axiomatisation of the ZX-calculus for the overall pure qubit QM [6]. In this paper, we first simplify the rule of addition (AD) and show that some rules can be derived from other rules in [6]. Then we obtain a complete axiomatisation of the ZX-calculus for the Clifford+T quantum mechanics by restricting the ring of complex numbers to its subring \( \mathbb{Z}[i, \sqrt{2}] \) based on the completeness theorem of the ZW-calculus for arbitrary commutative ring [7]. In comparison to the completeness proof in [6], a modification of the interpretation from the ZW-calculus to the ZX-calculus is made here.
The main difference between the two complete axiomatisations of the ZX-Calculus for the Clifford+T fragment QM shown in this paper and that presented in [4] is as follows:

1. Although the number of rules (which is 30) listed here is much more than that of [4] (which is 13), the number of nodes in each non-scalar diagram of the extended part of rules (non-stabilizer part) is at most 8 in this paper, in contrast to a maximum of 17 in [4].

2. Following [6], we have still introduced two more generators—the triangle and the λ box—in this paper, while there are only green nodes and red nodes as generators in [4]. Our new generators are features rather than novelties: the triangle can be employed as an essential component to construct a Toffoli gate in a very simple form, while the λ box can be slightly extended to a generalised phase so that the generalised supplementarity (also called cyclotomic supplementarity, with supplementarity as a special case) [5] is naturally seen as a special case of the generalised spider rule. These features are explained in detail in section 4.

3. The translation from the ZX-calculus to the ZW-calculus in our paper is more direct.

2 Universal completion of the ZX-calculus

In the following we list the rules of the ZX-calculus for the overall pure qubit QM as shown in [6].
Figure 1: Standard ZX-calculus rules, where $\alpha, \beta \in [0, 2\pi)$. 
Figure 2: Extended ZX-calculus rules for triangle, where \( \lambda \geq 0, \alpha \in [0, 2\pi) \).
Figure 3: Extended ZX-calculus rules for $\lambda$ and addition, where $\lambda, \lambda_1, \lambda_2 \geq 0, \alpha, \beta, \gamma \in [0, 2\pi)$; in (AD), $\lambda e^{i\gamma} = \lambda_1 e^{i\beta} + \lambda_2 e^{i\alpha}$. 
Note that the upside-down versions of all the above listed rules still hold, thus will be used without being clearly stated.

Now we show that the addition rule (AD) in Figure 3 can be simplified. First, by the symmetry of \( \lambda_1 \) and the rule (TR10), we have

\[
\lambda_1 \beta = \lambda_2 \alpha = \lambda_1 \alpha = \lambda_2 \beta
\]

Therefore,

\[
\lambda_1 \beta = \lambda_2 \alpha = \lambda_1 \alpha = \lambda_2 \beta
\]

As a consequence, we have the following commutativity of addition:

\[
\lambda_1 \beta = \lambda_2 \alpha = \lambda_1 \alpha = \lambda_2 \beta
\]

Next we prove that some rules in Figure 2 are derivable.

**Lemma 2.1** The rules (TR4), (TR10), and (TR11) can be derived from other rules.
**Proof:** For the derivation of (TR4), we have

\[
\pi = \pi = \pi = \pi = (4)
\]

where we used (TR3) for the second equality, (TR6) for the third equality, and (TR2) for the last equality.

For the derivation of (TR10), we have

\[
\pi = \pi = \pi = (5)
\]

where we used (TR12) for the second equality, (TR3) for the third equality.

For the derivation of (TR11), we have

\[
\pi = \pi = \pi = \pi = \pi = \pi = (6)
\]

where we used (TR12) for the fourth equality and (TR1) several times. □

If we add a new rule (TR10’) as shown in Figure 5 then the rule TR (5) is also derivable. In fact,
where for the third equality we used the following diagrammatic reasoning via rules (TR1), (K2), (TR9) and (TR10'): 

3 ZX-calculus for Clifford+T quantum mechanics

The ZX-calculus for Clifford+T quantum mechanics is a compact closed category $\mathcal{C}$. The objects of $\mathcal{C}$ are natural numbers: 0, 1, 2, · · · ; the tensor of objects is just addition of numbers: $m \otimes n = m + n$. The morphisms of $\mathcal{C}$ are diagrams of the ZX-calculus. A general diagram $D : k \to l$ with $k$ inputs and $l$ outputs is generated by:
where \( m, n \in \mathbb{N} \), \( \alpha \in \{ \frac{k\pi}{4} | k = 0, 1, 2, 3, 4, 5, 6, 7 \} \), and \( e \) represents an empty diagram.

The composition of morphisms is the same as that of the ZX-calculus for overall qubit QM. Due to the angles in the diagrams being multiples of \( \frac{\pi}{4} \), we call the ZX-calculus generated by the above generators \( \frac{\pi}{4} \)-fragment ZX-calculus.

**Proposition 3.1** The \( \frac{\pi}{4} \)-fragment of the ZX-calculus exactly corresponds to the matrices over the ring \( \mathbb{Z}[i, \frac{1}{\sqrt{2}}] \).

**Proof:** First note that \( \mathbb{Z}[i, \frac{1}{\sqrt{2}}] = \mathbb{Z}[\frac{1}{2}, e^{i\pi/4}] \). It is also clear that each generator of the \( \frac{\pi}{4} \)-fragment ZX-calculus corresponds to a matrix over the ring \( \mathbb{Z}[i, \frac{1}{\sqrt{2}}] \), thus each diagram of the \( \frac{\pi}{4} \)-fragment ZX-calculus must correspond to a matrix over the ring \( \mathbb{Z}[i, \frac{1}{\sqrt{2}}] \).

Conversely, each matrix over the ring \( \mathbb{Z}[i, \frac{1}{\sqrt{2}}] \) can be represented by a normal form in the ZW-calculus with phases belong to the same ring \( \mathbb{Z}[\frac{1}{2}, e^{i\pi/4}] \) [7], hence can be represented by a diagram of the \( \frac{\pi}{4} \)-fragment ZX-calculus via the translation from ZW to ZX as described in [6]. \( \square \)

As was done for the universal ZX-calculus, we extend the language with two new generators— the triangle and the \( \lambda \) box, which will be shown to be representable in the \( \frac{\pi}{4} \)-fragment ZX-calculus (see lemma 3.3):

\[ L : 1 \to 1 \]
\[ T : 1 \to 1 \]

where \( 0 \leq \lambda \in \mathbb{Z}[\frac{1}{4}] \).

There are two kinds of rules for the morphisms of \( 
\): the structure rules for \( 
\) as an compact closed category, as well as standard rewriting rules listed in Figure 4 and our extended rules listed in Figure 5 and Figure 6.

Note that all the diagrams should be read from top to bottom.
Figure 4: Rules for the $\frac{\pi}{4}$-fragment ZX-calculus, where $\alpha, \beta \in \{\frac{k\pi}{4} | k = 0, 1, \ldots, 7\}$. 
Figure 5: Extended $\frac{\pi}{4}$-fragment ZX-calculus rules for triangle, where $0 \leq \lambda \in \mathbb{Z}[\frac{1}{2}]$, $\alpha \in \{\frac{k\pi}{4} | k = 0, 1, \cdots, 7\}$. 
Figure 6: Extended $\frac{\pi}{4}$-fragment ZX-calculus rules for $\lambda$ and addition, where $0 \leq \lambda, \lambda_1, \lambda_2 \in \mathbb{Z}[\frac{1}{4}], \alpha, \beta, \gamma \in \{\frac{k\pi}{4} | k = 0, 1, \cdots, 7\}$; in (AD), $\lambda e^{i\gamma} = \lambda_1 e^{i\alpha} + \lambda_2 e^{i\beta}$. 
Note that the upside-down versions of all the above listed rules still hold. Since now we focus on the $\frac{\pi}{4}$-fragment ZX-calculus, the empty rule (IV) in Figure 3 is changed to the form of rule (IV’) in Figure 6. However, we still have the following useful property.

**Lemma 3.2** The frequently used empty rule can be derived from the $\frac{\pi}{4}$-fragment ZX-calculus:

$$
\begin{array}{c}
\text{Lemma 3.2} \\
\text{The frequently used empty rule can be derived from the } \frac{\pi}{4} \text{-fragment } \text{ZX-calculus:}
\end{array}
$$

$$
\begin{array}{c}
\text{(6)} \\
\text{Proof:}
\end{array}
$$

$$
\begin{array}{c}
\text{Lemma 3.3} \\
\text{The triangle } \bigtriangleup \text{ and the lambda box } \lambda \text{ are expressible in the } \frac{\pi}{4} \text{-fragment } \\
\text{ZX-calculus.}
\end{array}
$$

**Proof:** The triangle $\bigtriangleup$ has been represented in the $\frac{\pi}{4}$-fragment ZX-calculus in [2][4], we give the decomposition form according to [2] as follows:

$$
\begin{array}{c}
\text{Lemma 3.3} \\
\text{The triangle } \bigtriangleup \text{ and the lambda box } \lambda \text{ are expressible in the } \frac{\pi}{4} \text{-fragment } \\
\text{ZX-calculus.}
\end{array}
$$

$$
\begin{array}{c}
\text{Proof:}
\end{array}
$$

$$
\begin{array}{c}
\text{Lemma 3.3} \\
\text{The triangle } \bigtriangleup \text{ has been represented in the } \frac{\pi}{4} \text{-fragment ZX-calculus in } \\
\text{[2][4], we give the decomposition form according to [2] as follows:}
\end{array}
$$

$$
\begin{array}{c}
\text{(7)} \\
\text{Proof:}
\end{array}
$$

Now we deal with the lambda box. First we can write $\lambda$ as a sum of its integer part and remainder part: $\lambda = [\lambda] + \{\lambda\}$, where $[\lambda]$ is a non-negative integer and $0 \leq \{
\[ \lambda < 1. \text{ Since } \lambda \in \mathbb{Z}[\frac{1}{2}], \{\lambda\} \text{ can be uniquely written as a binary expansion of the form } a_1 \frac{1}{2} + \cdots + a_s \frac{1}{2^s} \text{, where } a_i \in \{0, 1\}, i = 1, \cdots, s. \text{ For the integer part } [\lambda], \text{ the corresponding } \lambda \text{ box has been represented in the } \frac{1}{2} \text{-fragment ZX-calculus during the universal completion of the ZX-calculus. For the remainder part } \{\lambda\}, \text{ it is sufficient to express the } \lambda \text{ box for } \lambda = \frac{1}{2} \text{ in terms of triangle and } Z, X \text{ phases, since one can apply the addition rule (AD) by recursion. Actually, we have}

\[ \lambda = \ldots = \lambda = \lambda. \]

Therefore,

\[ \lambda = \lambda. \]

The diagrams in the ZX-calculus for Clifford+T QM have the same standard interpretation as that for the whole qubit QM.

The ZW-calculus for Clifford+T QM almost remain the same as for the whole qubit QM, except that now \( r \) in the generator \( r \) lies in the ring \( \mathbb{Z}[i, \sqrt{2}] \).

4 Features of the new generators

In this section, we show the features of the two generators—the triangle and the \( \lambda \) box.

The triangle notation was first introduced in [4] as a shortcut for the proof of completeness of the ZX-calculus for Clifford+T QM. Afterwards, it is employed as a generator for a complete axiomatisation of the ZX-calculus for the whole pure qubit QM in [6] and for the Clifford+T fragment in this paper. The purpose to use it as a generator is to make the rewriting rules simple and the translation between the ZX-calculus and the ZW-calculus direct.

Moreover, very recently we find that the triangle can be an essential component for
the construction of a Toffoli gate as shown in the following form:

where the triangle with \(-1\) on the top-left corner is the inverse of the normal triangle.

In contrast to the standard circuit form which realises the Toffoli gate in elementary gates [10], the form of (9) is much more simpler, thus promising for simplifying Clifford + T quantum circuits with the aid of a bunch of ZX-calculus rules involving triangles.

Unexpectedly, we also realise that the denotation of a slash box used in [2] to construct a Toffoli gate is just a triangle (up to a scalar) as shown in (7).

Next we illustrate the feature of the \( \lambda \) box. In [6] and the previous parts of this paper, the \( \lambda \) box is restricted to be parameterised by a non-negative real number. While in [9], it has been generalised to a general green phase of form \( \Box \) with arbitrary complex number as a parameter. Similarly, we have the general red phase \( \Box \) = \( \Box \). Below we give the spider form of general phase which are interpreted in Hilbert spaces:

where \( a \) is an arbitrary complex number. The generalised spider rules and colour
change rule are depicted in the following:

\[
\begin{array}{c}
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\end{array}
\]

\[
\begin{array}{c}
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\end{array}
\]

\[
\begin{array}{c}
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\ldots \\
\end{array}
\]

where \(a, b\) are arbitrary complex numbers.

Now we consider the generalised supplementarity—also called cyclotomic supplementarity, with supplementarity as a special case—which is interpreted as merging \(n\) subdiagrams if the \(n\) phase angles divide the circle uniformly [5]. We give the diagrammatic form of the generalised supplementarity as follows:

\[
\begin{array}{c}
\alpha + \frac{2\pi}{n} \\
\alpha + \frac{4\pi}{n} \\
\vdots \\
\alpha + \frac{2(n-1)\pi}{n} \\
\end{array}
\]

\[
\begin{array}{c}
\text{na + (n - 1)π} \\
\end{array}
\]

where there are \(n\) parallel wires in the diagram at the right-hand side.

Next we show that the generalised supplementarity can be seen as a special form of the generalised spider rule as shown in (11). For simplicity, we ignore scalars in the rest of this section.

First note that by comparing the normal form translated from the ZW-calculus [7], we have

\[
\begin{array}{c}
\text{a} \\
\text{=} \\
\text{=} \\
\end{array}
\]

where \(a \in \mathbb{C}, a \neq 1\).

Especially,

\[
\begin{array}{c}
\text{a} \\
\text{=} \\
\text{=} \\
\end{array}
\]

where \(a \in [0, 2\pi), a \neq \pi\). For \(a = \pi\), we can use the \(\pi\) copy rule directly.
Then

\[
\frac{\alpha}{n} = \frac{\alpha}{n} + \frac{2\pi}{n} \quad \text{for } n \geq 1
\]

\[
\prod_{n \geq 1} \left( 1 - e^{i\left(\alpha + \frac{2\pi}{n}\right)} \right) = 1 - e^{i\alpha}
\]

Note that if \( n \) is odd, then

\[
\frac{1 - e^{i\alpha}}{1 + e^{i\alpha}} = \frac{1 - e^{i\alpha}}{1 + e^{i\alpha}} = e^{i\alpha}
\]

If \( n \) is even, then

\[
\frac{1 - e^{i\alpha}}{1 + e^{i\alpha}} = 1
\]

It is not hard to see that if we consider the parity of \( n \) in the right diagram of (12) with no consideration of scalars, then by Hopf law we get the same result as shown in (16) and (17).

5 Interpretations from ZX-calculus to ZW-calculus and back forth

As for the Clifford+T QM, the interpretation \([\_]_{XW}\) from ZX-calculus to ZW-calculus remains the same:

\[
\left[ \cdots \right]_{XW} = \cdots, \quad \left[ \cdots \right]_{XW} = \cdots
\]

\[
\left[ \cdots \right]_{XW} = \cdots, \quad \left[ \cdots \right]_{XW} = \cdots
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\left[ \cdots \right]_{XW} = \cdots, \quad \left[ \cdots \right]_{XW} = \cdots
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\[
\left[ \cdots \right]_{XW} = \cdots, \quad \left[ \cdots \right]_{XW} = \cdots
\]
\[ [D_1 \otimes D_2]_{\text{wx}} = [D_1]_{\text{wx}} \otimes [D_2]_{\text{wx}}, \quad [D_1 \circ D_2]_{\text{wx}} = [D_1]_{\text{wx}} \circ [D_2]_{\text{wx}}, \]

where \( 0 \leq \lambda \in \mathbb{Z}[\frac{1}{2}], \alpha \in \{ \frac{k\pi}{4} | k = 0, 1, \ldots, 7 \} \).

**Lemma 5.1** Suppose \( D \) is an arbitrary diagram in ZX-calculus. Then \( [[D]_{\text{wx}}] = [D] \).

The proof is easy.

On the other hand, the interpretation \([\cdot]_{\text{wx}}\) from ZW-calculus to ZX-calculus for Clifford+T QM is almost the same as the case of the overall qubit QM except for the \( r \)-phase part:

\[
\begin{align*}
[[\cdots]]_{\text{wx}} &= \cdots, & [[\cdot]]_{\text{wx}} &= \cdot, & [[\hugelbracket\cdot\hugerrbracket]]_{\text{wx}} &= \hugelbracket\cdot\hugerrbracket, \\
[[\hugelbracket\cdot\hugerrbracket\cdots]]_{\text{wx}} &= \hugelbracket\cdot\hugerrbracket\cdots, & [[\cdot]]_{\text{wx}} &= \cdot, \\
[[\hugelbracket\cdot\hugerrbracket]]_{\text{wx}} &= \hugelbracket\cdot\hugerrbracket, & [[\hugelbracket\cdot\hugerrbracket\cdots]]_{\text{wx}} &= \hugelbracket\cdot\hugerrbracket\cdots. 
\end{align*}
\]

where \( r = a_0 + a_1e^{i\frac{\pi}{4}} + a_2e^{i\frac{3\pi}{4}} + a_3e^{i\pi}, \quad a_j \in \mathbb{Z}[\frac{1}{2}], \quad j = 0, 1, 2, 3 \). Note that the representation of \( a_j \) box is described in Lemma 3.3.

**Lemma 5.2** Suppose \( D \) is an arbitrary diagram in ZW-calculus. Then \( [[[D]_{\text{wx}}] = [D] \).
The proof is easy.

**Lemma 5.3** Suppose $D$ is an arbitrary diagram in ZX-calculus. Then $ZX \vdash \llbracket D \rrbracket_{XW} = D$.

**Proof:** By the construction of $\llbracket : XW \rrbracket$ and $\llbracket : WX \rrbracket$, we only need to prove for $D$ as a generator of ZX-calculus. The first six generators in ZX-calculus are the same as the first six generators in ZW-calculus, so we just need to check for the last four generators in ZX-calculus, i.e., the green phase gate, the Hadamard gate, the $\lambda$ box and the triangle.

For the phase gate, we have

\[
\llbracket \begin{array}{c}
\text{\rotatebox{90}{\(\alpha\)}} \\
\text{\rotatebox{-90}{\(\alpha\)}}
\end{array} \rrbracket_{XW} = \llbracket e^{i \alpha} \rrbracket_{WX}
\]

For the Hadamard gate, we have

\[
\llbracket H \rrbracket_{XW} = \llbracket \sqrt{2} - 2 \rrbracket_{WX}
\]

For the $\lambda$ box and the triangle, we have

\[
\llbracket \llbracket \frac{\sqrt{3}}{2} \rrbracket \rrbracket_{XW} = \llbracket \frac{\sqrt{3}}{2} \rrbracket_{WX}
\]
Here we used $\sqrt{2-2} = -1 + \frac{1}{2}e^{i\pi/4} - \frac{1}{2}e^{i3\pi/4} = -1 + \frac{1}{2}e^{i\pi/4} + \frac{1}{2}e^{i5\pi/4}$. Finally, it is easy to check that

$$
\begin{bmatrix} 1 \end{bmatrix}_{xw} w_x = \begin{bmatrix} 0 \end{bmatrix}_{xw} w_x = \begin{bmatrix} 0 \end{bmatrix}_{xw} w_x
$$

□
6 Completeness

**Proposition 6.1** If $\forall ZW \vdash D_1 = D_2$, then $\forall ZX \vdash [D_1]_{WX} = [D_2]_{WX}$.

**Proof:** Since the derivation of equalities in $ZW$ and $ZX$ is made by rewriting rules, we need only to prove that $ZX \vdash [D_1]_{WX} = [D_2]_{WX}$ where $D_1 = D_2$ is a rewriting rule of $ZW$-calculus. Most proofs of this proposition have been done in the case of universal completion of the $ZX$-calculus [6], we only need to check for the last 5 rules $\text{rng}_c^e$, $\text{rng}_c^e$, $\text{nat}_c^e$, $\text{nat}_c^e$, $\text{ph}'$, which involve white phases in the $ZW$-calculus for Clifford+T QM. The rules $\text{nat}_c^e$ and $\text{ph}'$ are easy to check, we just deal with the rules $\text{rng}_c^e$, $\text{rng}_c^e$ and $\text{nat}_c^e$ in the appendix. □

**Theorem 6.2** The $ZX$-calculus is complete for Clifford+T QM: if $[D_1] = [D_2]$, then $ZX \vdash D_1 = D_2$.

**Proof:** Suppose $D_1, D_2 \in ZX$ and $[D_1] = [D_2]$. Then by lemma 5.1 $[[D_1]_{WX}] = [[D_2]_{WX}]$. Thus by the completeness of $ZW$-calculus in any commutative ring [7], $ZW \vdash [D_2]_{WX} = [D_2]_{WX}$. Now by proposition 6.1 $ZX \vdash [[D_1]_{WX}]_{WX} = [[[D_2]_{WX}]_{WX}$. Finally, by lemma 5.5 $ZX \vdash D_1 = D_2$. □

7 Conclusion and further work

In this paper, we give a complete axiomatisation of the $ZX$-calculus for the Clifford+T QM based on our complete axiomatisation for the overall pure qubit QM [6] and the completeness theorem of the $ZW$-calculus [7]. We also show the features of our new generators in contrast to the complete axiomatisation for the Clifford+T QM shown in [4].

A natural thing to do next would be applying the rules of this paper to the simplification of Clifford+T quantum circuits.

It is also interesting to incorporate the rules obtained here in the automated graph rewriting system Quantomatic [11].

**Acknowledgement**

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**References**

[1] B. Coecke, R. Duncan (2011), Interacting quantum observables: Categorical algebra and diagrammatics. New Journal of Physics 13, p. 043016.
Appendix

Proposition 7.1 (ZW rule nat$n$)

\[
ZX + \left[ \begin{array}{c}
\rotatebox{90}{$\bigcirc$
\hspace{10pt}r
\hspace{10pt}r
\hspace{10pt}r}
\end{array} \right]_{wx} = \left[ \begin{array}{c}
\rotatebox{90}{$\bigcirc$
\hspace{10pt}r
\hspace{10pt}r}
\end{array} \right]_{wx},
\]

where \( r = a_0 + a_1 e^{i\frac{\pi}{4}} + a_2 e^{i\frac{3\pi}{4}} + a_3 e^{i\frac{5\pi}{4}}, a_j \in \mathbb{Z}[\frac{1}{4}], \) \( j = 0, 1, 2, 3. \)
Proof: Let $c_k = a_k e^{i\mu_k}$, $k = 0, 1, 2, 3$. Then

Thus

Note that all the ZW rules we applied here have been proved to be true as well under the interpretation $\llbracket \cdot \rrbracket_{WX}$. □

Proposition 7.2 (ZW rule rmg∗)
\[ ZX + \left[ \begin{array}{c} r \\ s \end{array} \right]_{WX} = \left[ \begin{array}{c} r + s \end{array} \right]_{WX} , \quad (21) \]

where \( r = a_0 + a_1 e^{i\frac{\pi}{2}} + a_2 e^{i\frac{3\pi}{2}} + a_3 e^{i\frac{5\pi}{2}} \), \( s = b_0 + b_1 e^{i\frac{\pi}{2}} + b_2 e^{i\frac{3\pi}{2}} + b_3 e^{i\frac{5\pi}{2}} \), \( a_j, b_j \in \mathbb{Z}[\frac{1}{2}] \), \( j = 0, 1, 2, 3 \).

**Proof:** Let \( c_k = a_k e^{i\frac{k\pi}{4}} \), \( d_k = b_k e^{i\frac{k\pi}{4}} \), \( k = 0, 1, 2, 3 \). Then we have

\[ n_{q_2} \left[ \begin{array}{c} c_0 \\ c_1 \\ c_2 \\ c_3 \\ d_0 \\ d_1 \\ d_2 \\ d_3 \end{array} \right]_{WX} = (AD) \left[ \begin{array}{c} c_0 + d_0 \\ c_1 + d_1 \\ c_2 + d_2 \\ c_3 + d_3 \end{array} \right]_{WX} \]

\[ \left[ \begin{array}{c} r \\ s \end{array} \right]_{WX} \]

Note that all the ZW rules we applied here have been proved to be true as well under the interpretation \([\cdot]_{WX}\).

\[ \square \]

**Proposition 7.3 (ZW rule \( rq_{q_2} \))**

\[ ZX + \left[ \begin{array}{c} s \\ r \end{array} \right]_{WX} = \left[ \begin{array}{c} rs \end{array} \right]_{WX} , \quad (23) \]

where \( r = a_0 + a_1 e^{i\frac{\pi}{2}} + a_2 e^{i\frac{3\pi}{2}} + a_3 e^{i\frac{5\pi}{2}} \), \( s = b_0 + b_1 e^{i\frac{\pi}{2}} + b_2 e^{i\frac{3\pi}{2}} + b_3 e^{i\frac{5\pi}{2}} \), \( a_j, b_j \in \mathbb{Z}[\frac{1}{2}] \), \( j = 0, 1, 2, 3 \).
Proof: Let $c_k = a_k e^{j \varphi_k}$, $d_k = b_k e^{j \varphi_k}$, $k = 0, 1, 2, 3$. Then by (19) we have

\[
\begin{bmatrix}
    & s \\
    r &
\end{bmatrix}_{WX} = \begin{bmatrix}
    c_0 & c_1 & c_2 & c_3 \\
\end{bmatrix}_{WX}, \quad \begin{bmatrix}
    & s \\
    r &
\end{bmatrix}_{WX} = \begin{bmatrix}
    d_0 & d_1 & d_2 & d_3 \\
\end{bmatrix}_{WX}
\]

Therefore,

\[
\begin{bmatrix}
    & s \\
    r &
\end{bmatrix}_{WX} = \begin{bmatrix}
    d_0, d_1, d_2, d_3 \\
\end{bmatrix}_{WX}
\]
Note that all the ZW rules we applied here have been proved to be true as well under the interpretation $[\cdot]_{WX}$. □