Categorical and Algebraic Aspects of the Intuitionistic Modal Logic \( \text{IEL}^- \) and its predicate extensions

Daniel Rogozin\(^1,2\)

\(^1\)Lomonosov Moscow State University
\(^2\)Serokell OÜ

Abstract

The system of intuitionistic modal logic \( \text{IEL}^- \) was proposed by S. Artemov and T. Protopopescu as the intuitionistic version of belief logic \([3]\). We construct the modal lambda calculus which is Curry-Howard isomorphic to \( \text{IEL}^- \) as the type-theoretical representation of applicative computation widely known in functional programming. We also provide a categorical interpretation of this modal lambda calculus considering coalgebras associated with a monoidal functor on a cartesian closed category. Finally, we study Heyting algebras and locales with corresponding operators. Such operators are used in point-free topology as well. We study complete semantics à la Kripke-Joyal for predicate extensions of \( \text{IEL}^- \) and related logics using Dedekind-MacNeille completions and modal cover systems introduced by Goldblatt \([31]\). The paper extends the conference paper published in the LFCS’20 volume \([59]\).

Keywords— Intuitionistic modal logic, Modal type theory, Functional programming, Locales, Prenucleus, Cover systems

1 Introduction

1.1 Intuitionistic modal logic and Heyting algebras with operators

Intuitionistic modal logic study extensions of intuitionistic logic with modal operators. One may consider such extensions from two directions. The first perspective is a consideration of intuitionistic modal logic as a branch of modal logic. Here, intuitionistic modalities might be interpreted as a constructive necessity, provability in Heyting arithmetics, intuitionistic knowledge, and so on. The second perspective is the modal type theory providing a more computational interpretation of intuitionistic modalities. Each value in an arbitrary computation is annotated with the relevant data type and modalised type might be one of them.

The first perspective arises to Prior who introduced the system called \( \text{MIPC} \) \([57]\) to investigate modal counterparts of intuitionistic predicate monadic logic. The relation between intuitionistic modalities and quantifiers was developed by Bull \([15]\) and by Ono \([53]\). Monadic Heyting algebras were studied comprehensively by Bezhanishvili as well, see, for instance, \([8]\).

Fischer-Servi provided an intuitionistic analogue of the minimal normal modal logic containing \( \Box \) and \( \Diamond \) as mutually inexpressible connectives \([60]\).

Williamson \([67]\) discussed the question of intuitionistic epistemic modalities considering the problem of an intuitionist knowledge in means of the capability of verification. Artemov and Protopopescu developed this direction further, see \([3]\) and \([58]\).

We also emphasise briefly the direction related to Heyting algebras with operators. Heyting algebras with Fischer-Servi modal operators have a topological duality piggybacked on Esakia’s results \([22]\). This
duality that provides the characterisation of general descriptive frames for extensions of intuitionistic modal logic containing the Fischer-Servi system, see the paper by Palmigiano [54]. Macnab examined the class of Heyting algebras nuclei [49]. We discuss nuclei closely in Section 5. Here we merely claim that logic of Heyting algebras with nucleus and their predicate extensions was investigated by Bezhanishvili and Ghilardi [9]; Goldblatt [28] [31]; Fairtlough, Mendler, and Walton [24] [26].

We refer the reader to this paper by Wolter and Zakharyaschev [69], the paper by Bošić and Došen [14], and the monograph by Simpson [62]. These works contain the underlying results in model-theoretic aspects of intuitionistic modal logic.

1.2 Modalities from a computational perspective

The second perspective we emphasised is related to intuitionistic modalities in a computational landscape. The Curry-Howard correspondence provides bridges between intuitionistic proofs and programs understood in a type-theoretical sense [52] [63].

Modal lambda calculi often correspond to certain intuitionistic modal logics, see the papers by Artemov [2]; Bierman and de Paiva [13]; Davies and Pfenning [18]; Fairtlough and Mendler [25], etc.

De Paiva and Ritter [19] provided a categorical framework in semantics for those type systems. One may study modal operators within Homotopy Type Theory, see the recent paper by Rijke, Shulman, and Spitters [64].

One may find a proof of concept for modal types in functional programming. Let us observe a sort of computation called monadic. A monad is a concept in functional programming implemented in the functional language called Haskell. Moggi examined such monads type-theoretically [51]. Very informally, a monad is a method of structuring a computation as a linearly connected chain of actions within such types as the list or the input/output (IO). Such sequences are often called a pipeline in which one passes a value from an external world and yield a result after the series of actions. There is a way to consider computational monads logically within intuitionistic modal logic.

Functional programming languages such as Haskell, Idris or Purescript have specific type classes for computation within an environment. By computational context (or, environment), we mean some, roughly speaking, type-level map $f$, where $f$ is a “function” from $\ast$ to $\ast$: such a type-level map takes a simple type which has kind $\ast$ and yields another simple type of kind $\ast$. For a more detailed description of the type system with kinds implemented in Haskell see [63].

Here, the underlying type class is Functor which has the following formal definition:

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

Functor provides a generalisation of higher-order functions as map. map merely yields an image of a list by a given function. Let us take a look at its implementation:

```haskell
map :: (a -> b) -> [a] -> [b]
map f [] = []
map f (x:xs) = f x : (map f xs)
```

The first line claims that map is a two-argument function. The arguments of map are a unary function of type $a \rightarrow b$ and a list of elements belonging to $a$. The result of map is a list of $b$. This line of the piece of code is the so-called type-signature. Type-signature describes the behaviour of the function in terms of types of input and output.

The next two lines describe the recursive implementation of map. At first, we tell that an image of the empty list is empty. This part is the termination condition of a recursion. After that, we consider the case with a non-empty list. A non-empty list is a list obtained by adding an element to the top of the list. Suppose one has a list $xs$ and $x$ is an element of type $a$. In the case of non-empty list $x : xs$, one needs to call map recursively on the tail $xs$. We also apply a given function $f$ to the head $x$. Finally, we add $f x$ to the top of the list $map f xs$ which is an image of the tail $xs$.

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1In Haskell, type class is a general interface for some special group of data types.
The list data type is one of the functor instances. Generally, Functor provides a uniform method to carry unary functions through parametrised types. In other words, the notion of a functor in functional programming is a counterpart of the category-theoretic one.

One may extend a functor to the so-called monad which is a functional programming counterpart of Kleisli triples. In Haskell-like languages, one also has the type class called Monad, a type class of an abstract data type of action in some computational environment. Here we define the Monad type class as follows:

```haskell
class Functor m => Monad m where
  return :: a -> m a
  (>>=) :: m a -> (a -> m b) -> m b
```

Monad is a type class that extends Functor with two methods called return and (>>=) (a monadic bind).

Monads present a uniform technique for miscellaneous computations such as computation with a mutable state, many-valued computation, side effect input-output computation, etc. All those kinds of computation have an arrangement in the same fashion as pipelines. Historically, monads were implemented in Haskell to process side-effects that arise in the input/output world. The advantage of a monad is an ability to isolate side-effects within a monad remaining the relevant code purely functional. That is, one has a tool to describe a sequence of actions, where the result of each step depends on the previous ones somehow. In other words, one has so-called monadic binding by which such a sequence of actions with dependencies performs.

Monadic metalanguage is the modal lambda calculus that describes a computation within an abstract monad [51]. From a proof-theoretical point of view, this modal extension of the simply-typed lambda calculus is Curry-Howard isomorphic to lax logic, the logic of Heyting algebras with a nucleus operator we discussed earlier. The typing rules for modalities of this metalanguage correspond to the return and the monadic bind methods.

Let us take a look at the example of a monad. There is a parametrised data type Maybe in Haskell. The main application of Maybe is making a partial function total:

```haskell
data Maybe a = Nothing | Just a
```

The data type consists of two constructors. Suppose we deal with some computation that might terminate with a failure. Nothing is a flag that claims this failure arose. The second constructor Just stores some value of a, a successful result of a considered computation.

For example, one needs to extract the first element of a list. There might be an error if a given array is empty. This problem could be solved with the Maybe data type:

```haskell
safeHead :: [a] -> Maybe a
safeHead [] = Nothing
safeHead (x:xs) = Just x
```

The Maybe instance of Monad is the following one:

```haskell
instance Monad Maybe where
  return = Just
  (Just x) >>= f = f x
  Nothing >>= f = Nothing
```

Here, the return method merely embeds any value of a into the type Maybe a. The implementation of a monadic bind for Maybe is also quite simple. Suppose one has a function f of type a -> Maybe b and some value x of type Maybe a. Here we match on x. If x is Nothing, then the monadic bind yields Nothing. Otherwise, we extract the value of type a and apply a given function.

The monad interface for Maybe allows one to perform sequences of actions, where some values might be undefined. If all values are well defined on each step, then the result of an execution is a term of the form Just n. Otherwise, if something went wrong and we have no required value somewhere, then the computation halts with Nothing. The other examples of Monad instances have more or less the same explanation since the monadic interface was proposed for effects processing.
Let us discuss the Applicative class. Paterson and McBride proposed this class to describe effectful programming in an applicative style [50]. One may consider the Applicative type class as an intermediate one between Functor and Monad. See this paper to have a more precise understanding of the connection between applicative functors and monads [46].

Here is the precise definition of Applicative:

```haskell
class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

The main aim of an applicative functor is a generalisation the action of a functor for functions having an arbitrary arity, for instance:

```haskell
liftA2 :: Applicative f =
  (a -> b -> c) -> f a -> f b -> f c
liftA2 f x y = ((pure f) <*> x) <*> y
```

`liftA2` is a version of `fmap` for arbitrary two-argument function. It is clear that one may implement `liftA3`, `liftA4`, and `liftAn` for each `n < ω`. In the case of lists, `liftA2` passes a two-argument function, two lists, and yields the list obtained by applying to every possible pair the first element of which is an element of the first list and the second element belongs to the second list.

In this paper, we consider applicative computation type-theoretically. The modal axioms of IEL⁺ and types of the Applicative methods in Haskell-like languages are quite similar. We investigate the relationship between intuitionistic epistemic logic IEL⁺ and applicative computation providing the modal lambda calculus Curry-Howard isomorphic to IEL⁺.

This calculus consists of the rules for simply-typed lambda-calculus extended via the special modal rules. We assume that the proposed type system axiomatises applicative computation. We provide a proof-theoretical view of this sort of computation in functional programming and prove such metatheoretical properties as strong normalisation and confluence. The initial idea to consider applicative functors type-theoretically belongs to Krishnaswami [41]. We are going to develop his ideas considering the IEL⁺ from a computational perspective. Litak et. al. [47] made an observation that the logic IEL⁺ might be treated as a logic of an applicative functor as well.

In further sections, we study semantical questions of IEL⁺ and related logics. We study categorical semantics for the provided modal lambda calculus and cover semantics for quantified versions of intuitionistic modal logic with IEL⁺-like modalities.

2 The intutionistic modal logic IEL⁺

Intuitionistic modal logic IEL⁺ was proposed by S. Artemov and T. Protopopescu [3]. According to the authors, IEL⁺ represents beliefs agreed with BHK-semantics of intuitionistic logic. IEL⁺ is a weaker version of the system IEL that represents knowledge as provably consistent intuitionistic belief. This logic consists of the following axioms and derivation rules:

**Definition 1.** Intuitionistic epistemic logic IEL⁺:

1. \((φ → (ψ → θ)) → ((φ → ψ) → (φ → θ))\)
2. \(φ → (ψ → φ)\)
3. \(φ → (ψ → (φ ∧ ψ))\)
4. \(φ_1 ∧ φ_2 → φ_i, i = 1, 2\)

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²John Connor (the City College of New York) also connected the intuitionistic epistemic logic IEL⁺ with propositional truncation in Homotopy Type Theory. Those results were presented at the category theory seminar, the CUNY Graduate Centre. At the moment, there is only a video of that talk on YouTube.
5. \((\varphi \rightarrow \theta) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \lor \psi \rightarrow \theta))\)
6. \(\varphi_i \rightarrow \varphi_1 \lor \varphi_2, \ i = 1, 2\)
7. \(\bot \rightarrow \varphi\)
8. \(\bigcirc(\varphi \rightarrow \psi) \rightarrow (\bigcirc \varphi \rightarrow \bigcirc \psi)\)
9. \(\varphi \rightarrow \bigcirc \psi\)
10. From \(\varphi \rightarrow \psi\) and \(\varphi\) infer \(\psi\) (Modus ponens).

The \(\varphi \rightarrow \bigcirc \psi\) axiom is also called co-reflection. One may consider this axiom as the principle connecting intuitionistic truth and intuitionistic knowledge. From a Kripkean point, \(\text{IEL}^-\) is the logic of all frames \(\langle W, \leq, E \rangle\), where \(\langle W, R \rangle\) is a partial order and \(E\) is a binary “knowledge” relation, a subrelation of \(\leq\). The relation \(E\) obeys the following conditions:

1. \(E(w) \subseteq \uparrow w\) for each \(w \in W\).
2. \(E(u) \subseteq E(w)\), if \(w Ru\).

A model for \(\text{IEL}^-\) is a quadruple \(\mathcal{M} = \langle W, \leq, E, \gamma \rangle\), an extended intuitionistic Kripke model with the additional forcing relation for modal formulas defined via the relation \(E\). The \(\bigcirc\) connective has the “necessity” semantics:

\[\mathcal{M}, x \models \bigcirc \varphi \iff \forall y \in E(x) \mathcal{M}, y \models \varphi.\]

\(\text{IEL}\), the full epistemic intuitionistic logic, extends \(\text{IEL}^-\) as \(\text{IEL} = \text{IEL}^- \odot \bigcirc \varphi \rightarrow \neg \neg \varphi\). This additional axiom is often called the intuitionistic reflection principle. An \(\text{IEL}^-\)-frame is an \(\text{IEL}^-\) frame with the condition \(E(u) \neq \varnothing\) for each \(u \in W\). One has the following theorem proved by Artemov and Protopopescu \([3]\) by the canonical model on prime theories:

**Theorem 1.**

Let \(L \in \{\text{IEL}^-, \text{IEL}\}\), then \(\text{Log} (\text{Frames}(L)) = L\).

V. Krupski and A. Yatmanov investigated proof-theoretical and algorithmic aspects of the logic \(\text{IEL}\).

In this paper \([42]\), they provided the sequent calculus for \(\text{IEL}\) and proved that the derivability problem of this calculus is PSPACE-complete. \(\text{IEL}^-\) is decidable as well since this logic has the finite model property, see the paper by Wolter and Zakharyaschev \([68]\).

For further purposes, we define the natural deduction calculus for \(\text{IEL}^-\) that we call \(\text{NIEL}^-\). For simplicity, we restrict our language to \(\land, \lor, \rightarrow\) and \(\bigcirc\).

**Definition 2.** The natural deduction calculus \(\text{NIEL}^-\) for \(\text{IEL}^-\) is an extension of the intuitionistic natural deduction calculus with the additional inference rules for modality:

\[
\begin{align*}
\Gamma, \varphi & \vdash \varphi & \text{ax} \\
\Gamma, \varphi & \vdash \psi & \rightarrow_l \\
\Gamma & \vdash \varphi & \land_l \\
\Gamma & \vdash \bigcirc \varphi & \bigcirc_l
\end{align*}
\]

The first modal rule allows one to derive co-reflection and its consequences. The second modal rule is a counterpart of \(\bigcirc_l\) rule in natural deduction calculus for constructive \(\text{K} \) (see \([10]\)). We will denote \(\Gamma \vdash \bigcirc \varphi_1, \ldots, \varphi_n \vdash \psi\) and \(\psi_1, \ldots, \psi_n \vdash \psi\) as \(\Gamma \vdash \bigcirc \psi\) and \(\bigcirc \psi \vdash \psi\) respectively for brevity.

It is straightforward to check that the second modal rule is equivalent to the \(\text{K} \bigcirc\)-rule.
Let us show that one may translate \( \text{NIEL}^- \) into \( \text{IEL}^- \) as follows:

**Lemma 1.** \( \Gamma \vdash_{\text{NIEL}^-} \varphi \Rightarrow \text{IEL}_{\land,\land,\land} \vdash \bigwedge \Gamma \rightarrow \varphi \).

**Proof.** Induction on the derivation. Let us consider the modal cases.

1. If \( \Gamma \vdash_{\text{NIEL}^-} \varphi \), then \( \text{IEL}_{\land,\land,\land} \vdash \bigwedge \Gamma \rightarrow \land \varphi \).

   \begin{align*}
   (1) & \quad \bigwedge \Gamma \rightarrow \varphi & \text{assumption} \\
   (2) & \quad \varphi \rightarrow \land \varphi & \text{co-reflection} \\
   (3) & \quad \bigwedge \Gamma \rightarrow ((\varphi \rightarrow \land \varphi) \rightarrow (\bigwedge \Gamma \rightarrow \land \varphi)) & \text{IPC theorem} \\
   (4) & \quad \varphi \rightarrow \land \varphi \rightarrow (\bigwedge \Gamma \rightarrow \land \varphi) & \text{from (1), (3) and MP} \\
   (5) & \quad \bigwedge \Gamma \rightarrow \land \varphi & \text{from (2), (4) and MP}
   \end{align*}

2. If \( \Gamma \vdash_{\text{NIEL}^-} \land \varphi \) and \( \bar{A} \vdash \psi \), then \( \text{IEL}_{\land,\land,\land} \vdash \bigwedge \Gamma \rightarrow \land \psi \).

   \begin{align*}
   (1) & \quad \bigwedge \Gamma \rightarrow \land \varphi_1, \ldots, \land \varphi_n & \text{assumption} \\
   (2) & \quad \bigwedge \Gamma \rightarrow \land_{i=1}^n \varphi_i & \text{IEL\textsuperscript{-} theorem} \\
   (3) & \quad \land_{i=1}^n \varphi_i \rightarrow \land \land_{i=1}^n \varphi_i & \text{IEL\textsuperscript{-} theorem} \\
   (4) & \quad \land \bigwedge_{i=1}^n \varphi_i & \text{from (2), (3) and transitivity} \\
   (5) & \quad \land_{i=1}^n \varphi_i \rightarrow \psi & \text{assumption} \\
   (6) & \quad (\land_{i=1}^n \varphi_i \rightarrow \psi) \rightarrow \land (\land_{i=1}^n \varphi_i \rightarrow \psi) & \text{co-reflection} \\
   (7) & \quad \land_{i=1}^n \varphi_i \rightarrow \psi & \text{from (5), (6) and MP} \\
   (8) & \quad \land_{i=1}^n \varphi_i \rightarrow \land \psi & \text{from (7) and normality} \\
   (9) & \quad \land \bigwedge_{i=1}^n \varphi_i \rightarrow \land \psi & \text{from (4), (8) and transitivity}
   \end{align*}

\[ \square \]

**Lemma 2.** If \( \text{IEL}_{\land,\land,\land} \vdash A \), then \( \text{NIEL}^- \vdash A \).

**Proof.** By straightforward derivation of modal axioms in \( \text{NIEL}^- \). We will consider those derivations via terms below. \[ \square \]

One may enrich the observed natural deduction calculus with the well-known inference rules for disjunction and bottom and prove the same lemmas as above. We build further the typed lambda-calculus based on the \( \text{NIEL}^- \) by proof-assignment in the inference rules.

### 3 Modal Lambda Calculus based on the \( \text{IEL}^- \) logic

Let us define terms and types.

**Definition 3.** The set of terms:

Let \( \mathcal{V} = \{x, y, z, \ldots\} \) be the set of variables, the following grammar generates the set \( \Lambda_{\land} \) of terms:

\[
\Lambda_{\land} ::= \mathcal{V} | (\lambda \mathcal{V} \Lambda_{\land}) | (\Lambda_{\land} \Lambda_{\land}) | ((\Lambda_{\land} \Lambda_{\land})) | (\pi_1 \Lambda_{\land}) | (\pi_2 \Lambda_{\land}) | (\text{pure} \ \Lambda_{\land}) | (\text{let} \ \Lambda_{\land}^* = \Lambda_{\land}^* \text{ in } \Lambda_{\land})
\]
Definition 4. The set of types:
Let $\mathbb{T} = \{p_0, p_1, \ldots \}$ be the set of atomic types, the set $\mathbb{T}_0$ of types is generated by the grammar:

$$\mathbb{T}_0 ::= \mathbb{T} \cup (\mathbb{T}_0 \rightarrow \mathbb{T}_0) \cup (\mathbb{T}_0 \times \mathbb{T}_0) \cup ((\mathbb{T}_0))$$

A context has the standard definition \[52\] [63] as a sequence of type declarations $\Gamma = \{x_0 : \varphi_1, \ldots, x_n : \varphi_n, \ldots \}$. Here $x_i$ is a variable and $\varphi_i$ is a type for each $i \leq n < \omega$.

Definition 5. The modal lambda calculus $\lambda_{\text{IEL}}^\varnothing$:

\[
\begin{align*}
\Gamma, x : \varphi &\vdash x : \varphi & \text{ax} \\
\Gamma \vdash \lambda x. M : \varphi &\implies \Gamma \vdash \lambda x. M : \varphi & \rightarrow_i \\
\Gamma \vdash M : \varphi &\implies \Gamma \vdash N : \psi & \times_i \\
\Gamma \vdash M : \varphi &\implies \Gamma \vdash N : \psi & \rightarrow_e \\
\Gamma \vdash \text{pure} M : \text{pure} \varnothing &\implies \Gamma \vdash \text{pure} M : \text{pure} \varnothing & \rightarrow_t \\
\end{align*}
\]

$\Gamma \vdash \overline{M} : \text{pure} \varnothing$ is a short form for the sequence $\Gamma \vdash M_1 : \text{pure} \varnothing, \ldots, \Gamma \vdash M_n : \text{pure} \varnothing$ and $\overline{M} : \text{pure} \varnothing \vdash N : \psi$ is a short form for $x_1 : \varphi_1, \ldots, x_n : \varphi_n \vdash N : B$. We use this short form instead of $\text{let} \ C x_1, \ldots, x_n = \overline{M_1}, \ldots, \overline{M_n} \text{ in } N$. The $\rightarrow_t$-typing rule is the same as $\rightarrow$-introduction in monadic metalanguage \[55\]. $\rightarrow_t$ injects an object of type $A$ into $\emptyset$. According to this rule, it is clear that the type constructor $\text{pure}$ reflects the method $\text{pure}$ in the Applicative class.

The rule let$\emptyset$ is similar to the $\rightarrow$-rule in typed lambda calculus for intuitionistic normal modal logic $\text{IK}$, see \[38\]. Informally, one may read $\text{let} \ C \overline{M} \text{ in } N$ as a simultaneous local binding in $N$, where each free variable of a term $N$ should be binded with term of modalised type from $\overline{M}$. In other words, we modalise all free variables of a term $N$ and ‘substitute’ them to the terms belonging to the sequence $\overline{M}$.

As a matter of fact, our calculus extends the typed lambda calculus for $\text{IK}$ with $\rightarrow_t$-rule with the co-reflection rule allowing one to modalise any type of an arbitrary context.

Here are some examples:

\[
\begin{align*}
\Gamma \vdash x : \varphi \vdash x : \varphi &\implies \Gamma \vdash \text{pure} x : \text{pure} \varnothing & \rightarrow_t \\
\Gamma \vdash f : \overline{\text{pure} (\varphi \rightarrow \psi)} \vdash f : \overline{\text{pure} (\varphi \rightarrow \psi)} &\implies \Gamma \vdash g : \varphi \rightarrow \psi \vdash g : \varphi \rightarrow \psi & \rightarrow_e \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash f : \overline{\text{pure} (\varphi \rightarrow \psi)}, x : \overline{\text{pure} \varnothing} &\vdash \text{let} \ C g, y = f, x \text{ in } gy : \overline{\text{pure} \psi}, g : \varphi \rightarrow \psi \vdash g : \varphi \rightarrow \psi & \rightarrow_e \\
\Gamma \vdash \lambda f. \lambda x. \text{let} \ C g, y = f, x \text{ in } gy : \overline{\text{pure} \psi} &\implies \Gamma \vdash \lambda f. \lambda x. \text{let} \ C g, y = f, x \text{ in } gy : \overline{\text{pure} \psi} & \rightarrow_t \\
\end{align*}
\]
Here we provided the derivations for modal axioms of $\text{IEL}^-$. In fact, we proved Lemma 2 using proof-assignment.

Now we define free variables and substitutions:

**Definition 6.** The set $\text{FV}(M)$ of free variables for a term $M$:

1. $\text{FV}(x) = \{x\}$.
2. $\text{FV}(\lambda x.M) = \text{FV}(M) \setminus \{x\}$.
3. $\text{FV}(MN) = \text{FV}(M) \cup \text{FV}(N)$.
4. $\text{FV}(\langle M, N \rangle) = \text{FV}(M) \cup \text{FV}(N)$.
5. $\text{FV}(\pi_i M) = \text{FV}(M)$, $i = 1, 2$.
6. $\text{FV}(\text{pure } M) = \text{FV}(M)$.
7. $\text{FV} \left( \text{let } \circ \overline{\sigma} = \overline{M} \text{ in } N \right) = \cup_{i=1}^{n} \text{FV}(M)$, where $n = |\overline{M}|$.

**Definition 7.** Substitution:

1. $x[x := N] = N$, $x[y := N] = x$.
2. $(MN)[x := N] = M[x := N]N[x := N]$.
3. $(\lambda x.M)[y := N] = \lambda x.M[y := N]'$ $y \in \text{FV}(M)$.
4. $(M, N)[x := P] = (M[x := P], N[x := P])$.
5. $(\pi_i M)[x := P] = \pi_i(M[x := P])$, $i = 1, 2$.
6. $(\text{pure } M)[x := P] = \text{pure } (M[x := P])$.
7. $(\text{let } \circ \overline{\sigma} = \overline{M} \text{ in } N)[y := P] = \text{let } \circ \overline{\sigma} = (\overline{M}[y := P]) \text{ in } N$.

Substitutions and free variables for terms of the kind $\text{let } \circ \overline{\sigma} = \overline{M} \text{ in } N$ are defined similarly to [33]. That is, we do not take into account free variables of $N$ because those variables occur in the list $\overline{\sigma}$ and are eliminated by the assignment $\overline{\sigma}$.

The reduction rules are the following ones:

**Definition 8.** $\beta$-reduction rules for $\lambda_{\text{IEL}^-}$:

1. $(\lambda x.M)N \rightarrow_{\beta} M[x := N]$.
2. $\pi_1(M, N) \rightarrow_{\beta} M$.
3. $\pi_2(M, N) \rightarrow_{\beta} N$.
4. $\text{let } \circ \overline{\sigma}, y, \overline{\tau} = \overline{M}, \text{let } \circ \overline{\sigma} = \overline{N} \text{ in } Q, \overline{\sigma} \text{ in } R \rightarrow_{\beta}$
   let $\circ \overline{\sigma}, \overline{\tau}, \overline{\sigma} = \overline{M}, \overline{N}, \overline{Q} \text{ in } R[y := Q]$.
5. $\text{let } \circ \overline{\sigma} = \text{pure } \overline{M} \text{ in } N \rightarrow_{\beta} \text{pure } N[\overline{\sigma} := \overline{M}]$.
6. $\text{let } \circ \overline{\sigma} = \text{ in } M \rightarrow_{\beta} \text{pure } M$, where $\overline{\sigma}$ is an empty sequence of terms.

If $M$ reduces to $N$ by one of these rules, then we write $M \rightarrow_{r} N$. A multistep reduction $\rightarrow_{r}$ is a reflexive transitive closure of $\rightarrow_{r}$. $\rightarrow_{r}$ is a symmetric closure of $\rightarrow_{r}$. Now we formulate the standard lemmas.

**Proposition 1.** The generation lemma for $\circ_1$.

Let $\Gamma \vdash \text{pure } M : \circ \varphi$, then $\Gamma \vdash M : \varphi$.

**Proof.** Straightforwardly. 

**Lemma 3.** Basic lemmas.

1. If $\Gamma \vdash M : \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : \varphi$. 

2. If $\Gamma \vdash M : \varphi$, then $\Delta \vdash M : \varphi$, where $\Delta = \{x : \psi \mid (x : \psi) \in \Gamma \land x \in \text{FV}(M)\}$.

3. If $\Gamma, x : \varphi \vdash M : \phi$ and $\Gamma \vdash N : \varphi$, then $\Gamma \vdash M[x := N] : \psi$.

Proof.

The items 1-2 are proved by induction on $\Gamma \vdash M : \varphi$. The third item is shown by induction on the derivation of $\Gamma \vdash N : \psi$. $\square$

Theorem 2. Subject reduction.

If $\Gamma \vdash M : \varphi$ and $M \rightarrow_r N$, then $\Gamma \vdash N : \varphi$.

Proof. Induction on the derivation $\Gamma \vdash M : \varphi$ and on the generation of $\rightarrow_r$. The general statement follows from transitivity of $\rightarrow_r$, Proposition $1$ and Lemma $3$. $\square$

Theorem 3. $\rightarrow_{\beta}$ is strongly normalising.

Proof. Follows from Theorem $2$ below, so far as reduction in the monadic metalanguage is strongly normalising $[7]$ and $\lambda^{\text{IEL}}$ is sound with respect to the monadic metalanguage. $\square$

Theorem 4. $\rightarrow_r$ is confluent.

Proof. By Newman’s lemma $[63]$, if a given relation is strongly normalising and locally confluent, then this relation is confluent. It is sufficient to show that a multistep reduction is locally confluent.

Lemma 4. If $M \rightarrow_r N$ and $M \rightarrow_r Q$, then there exists some term $P$, such that $N \rightarrow_r P$ and $Q \rightarrow_r P$.

Proof. Let us consider the following critical pairs and show that they are joinable:

1. 

$\begin{align*}
\text{let } & \circ x = (\text{let } \circ y = \text{pures } N \text{ in } P) \text{ in } M \\
& \downarrow \beta \\
\text{let } & \circ y = \text{pure } N \text{ in } M[x := P] \\
\text{let } & x = \text{pure } P[y := N] \text{ in } M
\end{align*}$

$\begin{align*}
\text{let } & \circ y = \text{pure } N \text{ in } M[x := P] \rightarrow_{\beta} \\
& \text{pure } M[x := P][y := N] \\
\text{let } & x = \text{pure } P[y := N] \text{ in } M \\
\text{Since } & x \notin y \\
& \text{pure } M[x := P][y := N]
\end{align*}$

2. 

$\begin{align*}
\text{let } & \circ x = (\text{let } \_ = \_ \text{ in } N) \text{ in } M \\
& \downarrow \beta \\
\text{let } & \_ = \_ \text{ in } M[x := N] \\
\text{let } & x = \text{pure } N \text{ in } M
\end{align*}$

$\begin{align*}
\text{let } & \_ = \_ \text{ in } M[x := N] \rightarrow_{\beta} \text{let } \circ (M[x := N]) \\
\text{let } & x = \text{pure } N \text{ in } M \rightarrow_{\beta} \text{pure } (M[x := N])
\end{align*}$

$\square$
One may consider four critical pairs analysed in the confluence proof for the lambda-calculus based on the intuitionistic normal modal logic IK [38]. Those pairs are joinable in our calculus as well.

3.1 Relation with the monadic metalanguage

The monadic metalanguage is the modal lambda-calculus based on the categorical semantics of computation proposed by Moggi [51]. As we mentioned above, the monadic metalanguage might be considered as the type-theoretical representation of computation with an abstract data type of action. In fact, the monadic metalanguage is a type-theoretical formulation for monadic computation implemented in Haskell. We show that \( \lambda_{\text{IEL}} \) is sound with respect to the monadic metalanguage.

**Definition 9.** The monadic metalanguage extends the simply-typed lambda calculus with the additional typing rules:

\[
\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash \text{val } M : \nabla \varphi \; \nabla I} \quad \frac{\Gamma \vdash M : \nabla \varphi, \Gamma, x : \varphi \vdash N : \nabla \psi}{\Gamma \vdash \text{let } \varphi \; \text{val } x = M \; \text{in } N : \nabla \psi \; \text{let}_{\varphi}}
\]

The reduction rules are the following ones (in addition to the standard rule for abstraction and application):

1. \( \text{let } \text{val } x = \text{val } M \; \text{in } N \rightarrow_{\beta} N[x := M] \)
2. \( \text{let } \text{val } x = (\text{let } \text{val } y = N \; \text{in } P) \; \text{in } M \rightarrow_{\beta} \text{let } \text{val } y = N \; \text{in } (\text{let } \text{val } x = P \; \text{in } M) \)
3. \( \text{let } \text{val } x = M \; \text{in } x \rightarrow_{\eta} M \)

Let us define the translation \( . \) from \( \lambda_{\text{IEL}} \) to the monadic metalanguage:

1. \( .p_i \) = \( p_i \), where \( p_i \) is atomic
2. \( .\varphi \rightarrow .\psi \) = \( .\varphi \rightarrow .\psi \)
3. \( .\Box \varphi \) = \( \nabla .\varphi \)
4. \( .\text{pure } M \) = \( \text{val } .M \)
5. \( .\text{let } .\varphi \; \text{in } N \rightarrow .\text{let } .\varphi \; \text{in } N \)

where \( \text{let } .\varphi \; \text{in } N \) denotes \( \text{let } \text{val } x_1 = M_1 \; \text{in } (\ldots \; \text{in } (\text{let } \text{val } x_n = M_n \; \text{in } N) \ldots) \)

It is clear that, if \( \Gamma = \{x_1 : \varphi_1, \ldots, x_n : \varphi_n\} \) is a context, then \( \Gamma' = \{x_1 : .\varphi_1, \ldots, x_n : .\varphi_n\} \). Let us denote \( \vdash_{\text{IEL}} \) as the derivability relation in \( \lambda_{\text{IEL}} \) in order to distinguish from the monadic metalanguage derivability.

**Lemma 5.**

If \( \Gamma \vdash_{\text{IEL}} M : A \), then \( \Gamma' \vdash .M : .A \) in the monadic metalanguage.

**Proof.** By induction on \( \Gamma \vdash_{\text{IEL}} M : A \). One may prove the cases of \( \Box \) and \( \text{let}_\Box \) as follows:

\[
\begin{align*}
\Gamma' & \vdash .M : .A \\
\Gamma' \vdash \text{val } .M : \nabla .A \\
\varphi & \vdash .\Box \; \text{in } N : .B \\
\Gamma' & \vdash .\Box \; \text{in } .N : .B \\
\Gamma' & \vdash \text{let } .\varphi \; \text{val } .M : .N' : \nabla .B
\end{align*}
\]
Now one may formulate the following lemma:

**Lemma 6.**

1. \( \textit{'}M[x := N] \textit{'} = \textit{'}M'[x := \textit{'}N'] \)
2. \( M \xrightarrow{\tau} N \Rightarrow \textit{'}M' \xrightarrow{\beta} \textit{'}N' \)

**Proof.**

1. Induction on the structure of \( M \).
2. By the induction on \( N \):

   (a) For simplicity, we consider the case with only one variable in \texttt{let} \( \circ \) local binding, that can be easily extended to an arbitrary number of variables in local binding:
   
   \[
   \text{let } \circ x = (\text{let } \circ \varphi = \textit{'}N \text{ in } P) \text{ in } M' = \text{let val } x = (\text{let val } \varphi = \textit{'}N' \text{ in } \text{val } P') \text{ in } \text{val } M' \rightarrow_{\beta} \text{let val } \varphi = \textit{'}N' \text{ in } \text{val } M'[x := \textit{'}P'] = \text{let } \circ \varphi = \textit{'}N \text{ in } \text{val } M[x := P']
   \]

   (b) \( \text{let } \circ \varphi = \textit{'}pure \textit{'}N \text{ in } M' = \text{let val } \varphi = \textit{'}N \text{ in } \text{val } M' \rightarrow_{\beta} \text{val } M'[\varphi := \textit{'}N'] = \text{let val } \varphi = \textit{'}pure \textit{'}M[\varphi := \textit{'}N'] \)

   (c) \( \text{let } \circ x = M \text{ in } x' = \text{let val } x = \textit{'}M' \text{ in } \text{val } x \rightarrow_{\eta} \textit{'}M' \)

**Theorem 5.**

IEL\(^{-}\) is sound with respect to the monadic metalanguage.

**Proof.** Follows from the lemmas above.

---

### 3.2 Categorical semantics

In this subsection, we provide categorical semantics for the modal lambda calculus proposed above considering the co-reflection principle coalgebraically. Here we need a bit of category theory. We recall the required definitions first. For the abstract definitions of category, functor, natural transformation see the book by Goldblatt [32] or the book by MacLane and Moerdijk [48]. We piggyback the construction used in the proof of the completeness for simply-typed lambda-calculus, see [1] and [44] to have comprehensive details.

**Definition 10.** A category \( \mathcal{C} \) is called cartesian closed if this category has products \( A \times B \), exponentials \( B^A \) and a terminal object \( 1 \) satisfying the universal product and exponentiation properties.

Following to Bellin et. al. [6] and Kakutani [38] [39], we interpret a modal operator as a monoidal endofunctor on a cartesian closed category. Monoidal endofunctors are introduced as morphisms of those categories that respect monoidal structure, products and a terminal object in our case. Here we refer to the work by Eilenberg and Kelly for precise details [21]. We define a monoidal endofunctor on a cartesian closed category as an underlying notion.

**Definition 11.** Let \( \mathcal{C} \) be a cartesian closed category and \( F : \mathcal{C} \rightarrow \mathcal{C} \) an endofunctor, \( F \) is called monoidal if there exists a natural transformation \( m \) consisting of components \( m_{A,B} : FA \times FB \rightarrow F(A \times B) \) and a natural transformation \( \eta : 1 \rightarrow F1 \) such that the well-known diagrams commute (MacLane pentagon and triangle identity).
The abstract definition of a coalgebra is the following one:3

**Definition 12.** Let C be a category and \( F : C \to C \) an endofunctor. If \( A \in \text{Ob}(C) \), then an \( F \)-coalgebra is a pair \( \langle A, \alpha \rangle \), where \( \alpha \in \text{Hom}_C(A, FA) \). An \( F \)-coalgebra homomorphism from \( \langle A, \alpha \rangle \) to \( \langle A', \beta \rangle \) is a map \( f \in \text{Hom}_C(A, A') \) such that the following square commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow{f} & & \downarrow{\beta \circ Ff} \\
B & \xrightarrow{\beta} & FB
\end{array}
\]

Given a natural transformation \( \alpha : \text{Id}_C \to F \), one may associate an \( F \)-coalgebra \( \langle A, \alpha_A \rangle \) for each \( A \in \text{Ob}(C) \). Homomorphisms of such coalgebras are defined by naturality.

**Definition 13.** Let \( C \) be a cartesian closed category, \( F : C \to C \) a monoidal functor on \( C \), and \( \alpha : \text{Id}_C \to F \) a natural transformation. An IEL-\(-\)category is a pair \( \langle C, F, \alpha \rangle \) such that the following coherence conditions hold:

1. \( u = \alpha_1 \), where \( \alpha_1 \)
2. \( m_{A,B} \circ (\alpha_A \times \alpha_B) = \alpha_{A \times B} \), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\alpha_{A \times B}} & FA \times FB \\
\downarrow{m_{A,B}} & & \downarrow{m_{A,B}} \\
F(A \times B) & \xrightarrow{\alpha_{A \times B}} & FA \times FB
\end{array}
\]

The following construction describes the standard construction of typed lambda-calculus semantics. First of all, let us define semantic brackets \( [\_] \), a semantic translation from \( \lambda_{\text{IEL}} \) to the IEL-\(-\)category \( \langle C, F, \alpha \rangle \). Suppose one has an assignment \( \hat{\_} \) that maps every primitive type to some object of \( C \). Such semantic brackets \( [\_] \) have the following inductive definition:

1. \( [i] := \hat{i} \)
2. \( [\varphi \to \psi] := [\varphi]^\psi \)
3. \( [\varphi \times \psi] := [\varphi] \times [\psi] \)
4. \( [\bigcirc \varphi] = F([\varphi]) \)

We extend this interpretation for contexts by induction too:

1. \( [\_] = I \), where \( I \) is a terminal object of a given CCC
2. \( [\Gamma, x : \varphi] = [\Gamma] \times [\varphi] \)

Typing rules have the interpretation as follows. We understand typing assignments \( \Gamma \vdash M : A \) as arrows of the form \( [\Gamma \vdash M : \varphi] = [M] : [\Gamma] \to [\varphi] \):

\[
\begin{array}{c}
\pi_2 : [\Gamma] \times [\varphi] \to [\varphi] \\
\Lambda([M]) : [\Gamma] \to [\varphi]^{[\Gamma]} \\
\pi_1 : [\Gamma] \times [\varphi] \to [\varphi] \\
\eta_\varphi : [\Gamma] \to [\varphi] \\
\end{array}
\]

3Coalgebraic techniques are widely used in logic and computer science as well, see [12] [13] [69].
\[
\begin{align*}
\langle M_1, \ldots, M_n \rangle : \Gamma & \rightarrow \frac{}{\mathbb{F}[\phi]} \\
\langle N \rangle : \Gamma & \rightarrow \frac{\bigwedge_{i=1}^n \phi_i}{\psi}
\end{align*}
\]

The let \( \Box \)-rule has the interpretation similar to \( \Box \)-rule in term calculus for intuitionistic \( K \) \[3\]. The semantic brackets respect substitution and reduction:

**Lemma 7.**

1. \( \langle M[x_1 := M_1, \ldots, x_n := M_n] \rangle \equiv \langle M \rangle \circ \langle M_1, \ldots, M_n \rangle \).
2. If \( \Gamma \vdash M : A \) and \( M \rightarrow N \), then \( \langle \Gamma \upharpoonright M : A \rangle \equiv \langle \Gamma \upharpoonright N : A \rangle \).

**Proof.**

1. By simple induction on \( M \). Let us check only the modal cases.

\[
\begin{align*}
\Gamma \vdash \text{pure} \ M & : \Box \phi \equiv \Gamma \vdash \text{pure} \ (M[x := \bar{M}]) : \Box \phi \\
\eta_{\Box A} \circ \langle M[x := \bar{M}] \rangle & = \alpha_{\Box A} \circ \langle M \rangle \circ \langle M_1, \ldots, M_n \rangle \\
(\alpha_{\Box A} \circ \langle M \rangle) \circ \langle M_1, \ldots, M_n \rangle & = \Gamma \vdash \text{pure} \ M_1 : \Box \phi \circ \langle M_1, \ldots, M_n \rangle
\end{align*}
\]

2. The cases with \( \beta \)-reductions for let \( \Box \) are shown in [35]. Those cases are similar to our ones. Let us consider the cases with the pure terms that immediately follow from the coherence conditions of an \( \text{IEL} \)-category and the previous item of this lemma.

(a) \( \Gamma \vdash \text{let} \ \bar{x} = \bar{M} : \Box \phi \equiv \Gamma \vdash \text{let} \ \bar{x} = \bar{M} : \Box \phi \\
\Gamma \vdash \text{let} \ \bar{x} = \bar{M} : \Box \phi & \equiv \Gamma \vdash \text{pure} \ M : \Box \phi.
\]

(b) \( \text{let} \ \bar{x} = \bar{M} = \text{pure} \ M : \Box \phi \\
\text{let} \ \bar{x} = \bar{M} : \Box \phi & \equiv \text{let} \ \bar{x} = \bar{M} : \Box \phi.
\]

The following soundness theorem follows from the lemma above and the whole construction:

**Theorem 6.** **Soundness**

Let \( \Gamma \vdash \phi \) and \( M = N \), then \( \langle \Gamma \upharpoonright M : \phi \rangle \equiv \langle \Gamma \upharpoonright N : \phi \rangle \).

The completeness theorem is proved via the syntactic model. We will consider term model for the simply-typed lambda-calculus with \( \times \) and \( \rightarrow \) standardly described in \[16,17\].

Let us define a binary relation on lambda-terms \( \sim_{\phi, \psi} \subseteq (\mathcal{V} \times \Lambda_o)^2 \) as:

\[
(x, M) \sim_{\phi, \psi} (y, N) \iff x : \phi \vdash M : \psi & y : \phi \vdash N : \psi & M =_\cdot N[y := x]
\]

We will denote equivalence class as \( [x, M]_{\phi, \psi} = \{ y, N \mid (x, M) \sim_{\phi, \psi} (y, N) \} \) (we will drop indices below). Let us recall the definition of the category \( \mathcal{C}(\lambda) \), a model structure for the simply-typed lambda calculus.

The category \( \mathcal{C}(\lambda) \) has the class of objects defined as \( \text{Ob}_{\mathcal{C}(\lambda)} = \{ \phi \mid \phi \in \mathcal{P} \} \cup \{ \mathbb{I} \} \). For \( \hat{\phi}, \hat{\psi} \in \text{Ob}_{\mathcal{C}(\lambda)} \), the set of morphisms has the form \( \text{Hom}_{\mathcal{C}(\lambda)}(\hat{\phi}, \hat{\psi}) = \{ (x, M) \mid x : \phi \vdash M : \psi \} \). Let \( [x, M] \in \text{Hom}_{\mathcal{C}(\lambda)}(\hat{\phi}, \hat{\psi}) \) and \( [y, N] \in \text{Hom}_{\mathcal{C}(\lambda)}(\hat{\psi}, \hat{\theta}) \), then \( [y, N] \circ [x, M] = [x, N[y := M]] \). Identity morphisms are \( \text{id}_y = [x, x] \).
The category $\mathcal{C}(\lambda)$ is cartesian closed since $\mathbb{1}$ is a terminal object such that $\text{Hom}_{\mathcal{C}(\lambda)}(\mathbb{1}, \varphi) = \{\varphi\}$. Canonical projections are defined as $[x, \pi, x] = \text{Hom}_{\mathcal{C}(\lambda)}(\varphi_1 \times \varphi_2, \varphi)$ for $i = 1, 2$. The evaluation arrow is a morphism $\text{ev}_{\varphi, \psi} = [x, (\pi_1 x)(\pi_2 x)] \in \text{Hom}_{\mathcal{C}(\lambda)}(\varphi \times \psi, \psi)$.

Let us define a map $F : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda)$, such that for all $[x, M] \in \text{Hom}_{\mathcal{C}(\lambda)}(\varphi, \psi)$, $[x, M] = [y, \text{let } x = y \text{ in } M] \in \text{Hom}_{\mathcal{C}(\lambda)}(\varphi \times \psi, \psi)$. The following functoriality condition might be easily checked with the reduction rules:

1. $\mathcal{C}(\lambda) \circ [g \circ f] = \mathcal{C}(\lambda) \circ [g] \circ [f]$.
2. $\mathcal{C}(\lambda) \circ [id_A] = id_{\mathcal{C}(\lambda)}$.

We define the following maps. $\eta : \text{Id}_{\mathcal{C}(\lambda)} \to F$ such that for each $\varphi \in \text{Ob}_{\mathcal{C}(\lambda)}$ one has $\eta_\varphi = [x, \text{pure } x] \in \text{Hom}_{\mathcal{C}(\lambda)}(\varphi, \varphi \times \varphi)$. We express a monoidal transformation as $m_{\varphi, \psi} : \text{F}(\varphi \times \psi) \to \text{F}(\varphi \times \psi)$ such that one has $m_{\varphi, \psi} = [p, \text{let } x, y = \pi_1 p, \pi_2 p \text{ in } (x, y)] \in \text{Hom}_{\mathcal{C}(\lambda)}(\varphi \times \psi)$.

Lemma 8.

1. $F(f) \circ \alpha_f = \alpha_f \circ f$.
2. $(m_{\varphi, \psi}) \circ (\alpha_x \times \alpha_y) = \alpha_x \times \alpha_y$.
3. $u_1 = \eta_\varphi$.

Proof.

1. $\eta_{\varphi} \circ f = [y, \text{pure } y] \circ [x, M] = [x, \text{pure } y[y := M]] = [x, \text{pure } M]$.

From the other hand, one has:

$\text{C}(\lambda) \circ [\eta_{\varphi}] = [z, \text{let } x = z \text{ in } M] \circ [x, \text{pure } x] = [x, \text{let } x = z \text{ in } M[z := \text{pure } x]] = [x, \text{let } x = \text{pure } x \text{ in } M] = [x, \text{pure } M[x := x]] = [x, \text{pure } M]$.

2. $m_{A, B} \circ (\eta_A \times \eta_B) = \eta_{A \times B}$.

$\text{C}(\lambda) \circ [\eta_{A \times B}] = [q, \text{let } x, y = \pi_1 q, \pi_2 q \text{ in } (x, y)] \circ [p, \text{pure } (\pi_1 p), \text{pure } (\pi_2 p)] = [p, \text{let } x, y = \pi_1 (\text{pure } \pi_1 p), \pi_2 (\text{pure } \pi_2 p) \text{ in } (x, y)] = [p, \text{let } x, y = \text{pure } (\pi_1 p), \text{pure } (\pi_2 p) \text{ in } (x, y)] = [p, \text{pure } (\pi_1 p), \text{pure } (\pi_2 p) \text{ in } (x, y)] = [p, \text{pure } (\pi_1 p, \pi_2 p) \text{ in } (x, y)] = [p, \text{pure } (\pi_1 p, \pi_2 p)] = [p, \text{pure } p] = \eta_{A \times B}$.

3. Immediately.

The previous results imply completeness.

Lemma 9. $\langle \mathcal{C}(\lambda), \mathcal{C}(\lambda) \rangle$ is an IEL category.

4 Prenuclear algebras and their representation

4.1 The background on locales, nuclei, and localic cover systems

A Heyting algebra is a bounded distributive lattice $\mathcal{H} = \langle H, \wedge, \vee, \top, \bot \rangle$ with the binary operation $\Rightarrow$ such that the following equivalence hold:

$$a \wedge b \leq c \text{ if and only if } a \leq b \Rightarrow c$$
Recall that a **locale** is a complete lattice $\mathcal{L} = \langle L, \wedge, \vee \rangle$ satisfying the infinite distributive law:

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\} \text{ for each } B \subseteq L.$$  

The notion of a locale coincides with the notion of a complete Heyting algebra since an implication might uniquely defined for each $a, b \in L$ as

$$a \Rightarrow b = \bigvee \{ c \in L \mid a \wedge c \leq b \}$$

Here we note that the categories of complete Heyting algebras and locales are not the same since their classes of morphisms are different. We do not take into consideration these categories, so we assume that locale and complete Heyting algebra are synonymous terms.

A locale is a central object in point-free topology, where a locale is a lattice-theoretic counterpart of a topological space. The aim of this discipline is to study point-set topology concerning topological spaces only with the structure of their topologies as lattices of opens without mentioning points. For the further discussion see [30] [37] [18] [50]. In usual point-set topology, we are often interested in subspaces. In point-free topology, subspaces are characterised via operators on a locale called nuclei. A nucleus on a Heyting algebra is a multiplicative closure operator or a completion operator according to the Dragalin’s terminology [20].

**Definition 14.** A nucleus on a Heyting algebra $\mathcal{H}$ is a monotone map $j : \mathcal{L} \to \mathcal{L}$ such that

1. $a \leq ja$
2. $ja = jja$
3. $j(a \wedge b) = ja \wedge jb$

One may consider a nucleus operator as a lattice-theoretic analogue of a Lawvere-Tierney topology that generalises the notion of a Grothendieck topology on a presheaf topos. In its turn, Lawvere-Tierney topology provides a modal operator often called a geometric modality [45]. Here, one may read $j\varphi$ as “it is locally the case that $\varphi$”. The logic of Heyting algebras with a nucleus operator was studied by Goldblatt from Kripkean and topos-theoretic perspectives, see [28] and [32] as well.

It is also well-known that the set of fixpoints of a nucleus on a Heyting algebra is a Heyting subalgebra. From a point-free topological view, nuclei characterise sublocales [50]. Those operators play a tremendous role in a locale representation as well. In this monograph [20], Dragalin showed that any complete Heyting algebra is isomorphic to the locale of fixpoints of a nucleus operator on the algebra of up-sets. Moreover, any spatial locale (the lattice of open sets) is isomorphic to the complete Heyting algebra of fixpoints of a nucleus operator generated by a suitable Dragalin frame. We recall that a Dragalin frame is a structure that generalises both Kripke and Beth semantics of intuitionistic logic. Bezhanishvili and Holliday strengthened this result for arbitrary complete Heyting algebras, see [10] and [11] as well.

Goldblatt provided the alternative way of a locale representation [31] [33] with cover systems. Perhaps, Dragalin frames and Goldblatt cover systems may be connected to each other somehow, but it seems that the relationship between them is not investigated yet.

We examine that framework closely. First of all, let us recall some relevant notions. Let $\langle P, \leq \rangle$ be a poset. A subset $A \subseteq P$ is called upwardly closed, if $x \in A$ and $x \leq y$ implies $y \in A$. For $A \subseteq P$, $\uparrow A = \{ x \in P \mid \exists y \in A \ y \leq x \}$. If $x \in P$, then the **cone** at $x$ is an up-set $\uparrow x = \{ x \}$. A subset $Y \subseteq P$ refines a subset $X \subseteq P$ if $Y \subseteq \uparrow X$. By $\text{Up}(P, \leq)$ we will mean the poset (in fact, the locale) of all upwardly closed subsets of a partial order $\langle P, \leq \rangle$. It is also clear that the set of all upwardly closed sets forms a locale.

Here we consider triples $S = \langle P, \leq, \triangleright \rangle$, where $\langle P, \leq \rangle$ is a poset and $\triangleright$ is a binary relation between $P$ and $\text{P}(P)$. Given $x \in P$ and $C \subseteq P$, then we say that $x$ is **covered** by $C$ ($C$ is an $x$-cover), if $x \triangleright C \ (C \triangleright x)$.

---

4There is a third synonym for locales and complete Heyting algebras called frame, but we already used this term in means of Kripke semantics.

5Such topology is often called pointless, but we find the point-free topology term more appropriate.

6This note is based on the recent conversation between Prof. Valentin Shehtman and the author.
Cover systems were presented to study local truth that comes from a topological and topos-theoretic intuition. A statement is locally true concerning some object as topological space or an open subset if this object has an open cover in each member of which the statement is true. For instance, such a statement might be local equality of continuous maps, see [28] and also [29]. An abstract cover system has the following definition:

**Definition 15.** A triple \( S = \langle P, \leq, \triangleright \rangle \) as above is called cover system, if the following axioms hold for \( x \in P \):

1. *(Existence)* There exists an \( x \)-cover \( C \) such that \( C \sqsubseteq \uparrow x \)
2. *(Transitivity)* Let \( x \triangleright C \) and for each \( y \in C \) \( y \triangleright C_y \), then \( x \triangleright \bigcup_{y \in C} C_y \)
3. *(Refinement)* If \( x \leq y \), then any \( x \)-cover might be refined to a \( y \)-cover. That is, \( C \triangleright x \) implies that there exists an \( y \)-cover \( C' \) such that \( C' \sqsubseteq C \)

Let \( S \) be a cover system, let us define an operator \( j : \mathcal{P}(P) \to \mathcal{P}(P) \) as

\[
j X = \{ x \in P \mid \exists C x \triangleright C \sqsubseteq X \}
\]

If \( x \in j X \) is called a local member of \( X \). A subset \( X \subseteq P \) is called localised if \( j X \subseteq X \). A localised up-set is called a proposition. \( \text{Prop}(S) \) is the set of all propositions of a cover system. Goldblatt showed that such an operator is a closure operator on a locale of all up-sets [31] that follows from the axioms of a cover system. According to that, a subset \( X \) is a proposition iff \( X = \uparrow X = j X \).

**Definition 16.** A cover system is called localic, if the following axiom holds:

Every \( x \)-cover can be refined to an \( x \)-cover that is included in \( \uparrow x \).

That is, \( x \triangleright C \) implies that there exists \( x \triangleright C' \) such that \( C' \sqsubseteq \uparrow C \) and \( C' \sqsubseteq \uparrow x \).

This localic axiom makes that \( j \)-operator a nucleus. That is, if \( S = \langle P, \leq, \triangleright \rangle \) is a localic cover system, then \( \text{Prop}(S) \) is a sublocale of \( \text{Up}(P, \leq) \) since the set of fixpoints of nucleus is a sublocale of \( \text{Up}(P, \leq) \). Here we strengthen the fourth axiom of a localic cover system as:

Every \( x \)-cover is included in \( \uparrow x \).

Such a local cover system is called a strictly localic cover system. The stronger fourth axiom is built in such generalisation of open-covers systems as Grothendieck topology and cover schemes [5] [18].

The representation theorem for an abritrary locale is the following one [31]:

**Theorem 7.** Let \( L \) be a locale, then there exists a strictly localic cover system \( S \) such that \( L \cong \text{Prop}(S) \).

**Proof.** We provide a proof sketch in order to remain the paper self-contained.

Given a locale \( L = \langle L, \sqcup, \sqcap \rangle \). Let us define \( S_L = \langle L, \leq, \triangleright \rangle \) such that \( x \leq y \) iff \( y \leq x \) and \( x \triangleright C \) iff \( x = \sqcup C \) in \( L \). Then \( S_L \) is a localic cover system. The strictness follows from the fact that if \( x \triangleright C \), that is, \( x = \sqcup C \), then \( C \sqsubseteq \{ x \} = \{ y \mid y \leq x \} \). Every cone \( \{ x \} = \uparrow x \) is localised, thus, \( \{ x \} \) is a proposition. It is not to so difficult to see that an arbitrary proposition of \( S_L \) is a downset of \( \leq \).

An isomorphism itself is established with the map \( x \mapsto \{ x \} \).

As a consequence, one has a uniform embedding for arbitrary Heyting algebras as follows:

**Theorem 8.** Every Heyting algebra is isomorphic to a subalgebra of propositions of a suitable strictly localic cover system.

**Proof.** Every Heyting algebra has a Dedekind-Macneille completion \( H \hookrightarrow H' \), where \( H' \) is a locale, see [31]. But \( H' \) is isomorphic to the locale of propositions of a strictly localic cover system \( S_{H'} \).
Strictly localic cover systems provide alternative model structures for intuitionistic predicate logic. Let \( S = \langle P, \leq, \triangleright \rangle \) be a strictly localic cover system and let \( D \) be a non-empty set, a domain of individuals. Let \( V \) be a valuation function that maps each \( k \)-ary predicate letter \( P \) to \( V(P) : D^k \to \text{Prop}(S) \). To interpret variables, we use \( D \)-assignments that have the form of infinite sequences \( \sigma = \langle \sigma_0, \sigma_1, \ldots, \sigma_n, \ldots \rangle \), where \( \sigma_i \in D \) for each \( i < \omega \). A \( D \)-assignment maps each variable \( x_i \) to the corresponding \( \sigma_i \). Given an assignment \( \sigma \) and \( d \in D \), then \( \sigma(d/n) \) is a \( D \)-assignment obtained from \( \sigma \) replacing \( \sigma_i \) with \( d \).

By IPL-model we will mean a structure \( \mathcal{M} = \langle S, D, V \rangle \), where \( S \) is a strictly localic cover system, \( D \) is a domain of individuals, and \( V \) is a \( D \)-valuation. Given a \( D \)-assignment and \( x \in S \), the truth relation \( \mathcal{G}, x, \sigma \models \varphi \) is defined inductively:

1. \( \mathcal{M}, x, \sigma \models P(x_{n_1}, \ldots, x_{n_k}) \) if \( x \in V(P)(\sigma_{n_1}, \ldots, \sigma_{n_k}) \).
2. \( \mathcal{M}, x, \sigma \models \varphi \land \psi \) if \( \mathcal{M}, x, \sigma \models \varphi \) and \( \mathcal{M}, x, \sigma \models \psi \).
3. \( \mathcal{M}, x, \sigma \models \forall x \varphi \) if there exists an \( x \)-cover \( C \) such that for each \( y \in C \mathcal{M}, x, \sigma \models \varphi \) or \( \mathcal{M}, x, \sigma \not\models \psi \).
4. \( \mathcal{M}, x, \sigma \models \varphi \rightarrow \psi \) if for all \( y \in \uparrow x \), if \( \mathcal{M}, y, \sigma \models \varphi \) implies \( \mathcal{M}, y, \sigma \models \psi \).
5. \( \mathcal{M}, x, \sigma \models \exists x \varphi \) if for each \( d \in D \), \( \mathcal{M}, x, \sigma(d/n) \models \varphi \).
6. \( \mathcal{M}, x, \sigma \models \exists x \varphi \) if there exist an \( x \)-cover \( C \) and \( d \in D \) such that for each \( y \in C \) one has \( \mathcal{M}, y, \sigma(d/n) \models \varphi \).

Given a formula \( \varphi \), one may associate a truth set \( \| \varphi \|_S^\mathcal{M} \) defined in means of locale operations on \( \text{Prop}(S) \):

1. \( \| P(x_{n_1}, \ldots, x_{n_k}) \|_\sigma^\mathcal{M} = V(P)(\sigma_{n_1}, \ldots, \sigma_{n_k}) \)
2. \( \| \varphi \land \psi \|_\sigma^\mathcal{M} = \| \varphi \|_\sigma^\mathcal{M} \land \| \psi \|_\sigma^\mathcal{M} \)
3. \( \| \varphi \lor \psi \|_\sigma^\mathcal{M} = j(\| \varphi \|_\sigma^\mathcal{M} \lor \| \psi \|_\sigma^\mathcal{M} ) \)
4. \( \| \varphi \rightarrow \psi \|_\sigma^\mathcal{M} = \| \varphi \|_\sigma^\mathcal{M} \Rightarrow \| \psi \|_\sigma^\mathcal{M} \)
5. \( \| \forall x \varphi \|_\sigma^\mathcal{M} = \bigwedge_{d \in D} \| \varphi \|_{\sigma(d/n)}^\mathcal{M} \)
6. \( \| \exists x \varphi \|_\sigma^\mathcal{M} = j(\bigvee_{d \in D} \| \varphi \|_{\sigma(d/n)}^\mathcal{M} ) \)

where \( j \) is an associated nucleus on the locale of \( S \)-propositions.

Thus, one has the completeness theorem\(^7\).

**Theorem 9.** Intuitionistic first-order logic is sound and complete with respect to IPL-models.

We define modal cover systems. Suppose one has a localic cover system \( S = \langle S, \leq, \triangleright \rangle \). We seek to extend \( S \) with a binary relation \( R \) on \( S \) that yields an operator on \( \text{Prop}(S) \):

\[ \langle R \rangle A = \{ x \in S \mid \exists y \in A x Ry \} = R^{-1}(A) \]

**Definition 17.** A quadruple \( \mathcal{M} = \langle S, \leq, \triangleright, R \rangle \) is called a modal cover system, if a triple \( \langle S, \leq, \triangleright \rangle \) is a strictly localic cover system and the following conditions hold:

1. (Confluence) If \( x \leq y \) and \( x R z \), then there exists \( w \) such that \( y R w \) and \( z \leq w \).
2. (Modal localisation) If there exists \( C \) such that \( x \triangleright C \subseteq \langle R \rangle A \), then there exists \( y \in R(x) \) with a \( y \)-cover included in \( X \).

The first condition is a general requirement for intuitionistic modal logic allowing \( \langle R \rangle A \) to be an up-set whenever \( A \) is. The modal localisation principle claims that \( \text{Prop}(\mathcal{M}) \) is closed under \( \langle R \rangle \).

There is a representation theorem for locales with monotone operators, see [31] to have a proof in detail:

---

\(^7\) Here we note that this conclusion admits generalisations and provides complete semantics for predicate substructural logics, see, e. g., [39].
Theorem 10. Let $\mathcal{L}$ be a locale and $m : \mathcal{L} \to \mathcal{L}$ a monotone map on $\mathcal{L}$, then the algebra $\langle \mathcal{L}, m \rangle$ is isomorphic to the algebra $\langle \text{Prop}(\mathcal{S}_c), \langle R_m \rangle \rangle$.

Proof. As we already know by Theorem 7 $\mathcal{L} = \langle L, \vee, \wedge \rangle$ is isomorphic to the locale $\text{Prop}(\mathcal{S}_c)$. $\mathcal{S}_c = \langle L, \sqsubseteq, \triangleright \rangle$ is a strictly localic cover system, where $x \sqsubseteq y$ iff $y \sqsubseteq x$ and $x \triangleright C$ iff $x = \bigvee C$ in $\mathcal{L}$. We recall that this isomorphism was established with map $x \mapsto (x) = \{y \in L | y \sqsubseteq x \}$. Let us put $x R_m y$ iff $x \sqsubseteq my$. The relation is well-defined and the confluence and modal localisation conditions holds. The key observation is that $(ma) = \langle R_m \rangle(a)$. $\square$

Goldblatt introduced modal cover systems to provide semantics for quantified lax logic and intuitionistic counterparts of modal predicate logics $K$ and $S4$ [31]. In the next subsection, we introduce similar cover systems to provide complete semantics for intuitionistic predicate modal logics with $\text{IEL}^\sim$-like modalities.

4.2 Prenuclei operators

We discuss prenuclei operators, overview their use cases and provide representation for Heyting algebras with such operators via suitable modal localic cover systems. A weaker version of nuclei operators is quite helpful in point-free topology as well.

Definition 18.

Let $\mathcal{H}$ be a Heyting algebra, a prenucleus on $\mathcal{H}$ is an operator monotone $j : \mathcal{H} \to \mathcal{H}$ such that for each $a, b \in \mathcal{H}$:

1. $a \sqsubseteq ja$
2. $ja \wedge b \sqsubseteq j(a \wedge b)$.

A prenucleus is called multiplicative if it distributives over finite infima.

By prenuclear algebra, we will mean a pair $\langle \mathcal{H}, j \rangle$, where $j$ is a prenucleus on $\mathcal{H}$. A prenuclear algebra is localic when its Heyting reduct is a locale. A prenuclear algebra is multiplicative if its prenucleus is. Simmons calls multiplicative prenuclei merely as prenuclei [61], but this term is more spread for operators as defined above, see, e.g. [56]. We introduce the term “multiplicative prenucleus” in order to distinguish all those operators from each other since we are going to consider both of them.

Prenuclei operators have an application in point-free topology in factorising locales considering sublocales as quotients. See the paper by Banaschewski [4] and the monograph by Picado and Pultr [56] for the discussion in detail. We just note that one may generate nucleus from a prenucleus the generated by a sequence of prenuclei parametrised over ordinals.

One may involve multiplicative prenuclei to the study of nuclei lattices. Infima are defined pointwise there. Joins are more awkward to be defined explicitly. Multiplicative prenuclei provide a suitable description of nuclei joins in such locales. Multiplicative prenuclei form a locale as well and they are closed under composition and pointwise directed joins. Thus, one may define joins of nuclei in means of so-called nuclear reflection, an approximation of nucleus via prenuclei. Here we refer the reader to this paper [23], where this aspect has a more comprehensive explanation.

The other aspect of multiplicative prenuclei were studied by Haykazyan and Simmons [55] [61]. They consider the special multiplicative prenucleus. Given a bounded distributive lattice $\mathcal{L}$, one may introduce a preorder $\leq$ defined as follows for each $a, b \in \mathcal{L}$:

$$a \leq b \iff \forall c \in \mathcal{L} a \vee c = \top \Rightarrow b \vee c = \top$$

If this preorder on a locale is agreed with the parent order, then this complete Heyting algebra is called subfit. This preorder also has an associated map $\xi : a \mapsto \bigvee \{b \in \mathcal{L} | b \leq a\}$ as it is observed by Coquand [19]. $\xi$ is a prenucleus on an arbitrary locale as it is shown by Simmons [61], where he studies certain properties of nuclei on the locale of complete Heyting algebra ideals. Moreover, one may associate a certain nucleus with the prenucleus $\xi$ to measure the subfitness of a locale.

Let us define prenuclear cover systems to have a suitable representation for prenuclear algebras.
Definition 19. Let $\mathcal{S} = \langle S, \preceq, \models \rangle$ be a modal cover system, then $\mathcal{S}$ is called prenuclear, if the following two conditions hold:

1. $R$ is reflexive.
2. Let $x, y \in S$ such that $xRy$, then there exists $z \in \uparrow y$ such that $x \preceq z$ and $x \in R(z)$.

One may visualise the second condition with the following diagram:

\[
\begin{array}{c}
\quad x \\
R \downarrow \\
\quad y \\
\end{array}
\]

This lemma claims that a prenuclear cover system is well-defined as follows, the similar statement was proved by Goldblatt for nuclear cover systems $[31]$: 

Lemma 10. Let $\mathcal{S} = \langle P, \subseteq, \models \rangle$ be a prenuclear cover system, then $\langle P \rangle$ is a prenucleus on $\text{Prop}(\mathcal{S})$, that is for each $A, B \in \text{Prop}(\mathcal{S})$:

1. $A \subseteq \langle P \rangle A$
2. $A \cap \langle P \rangle B \subseteq \langle P \rangle (A \cap B)$

Proof. The condition $A \subseteq \langle P \rangle A$ holds according to the standard modal logic argument.

Let us check the second condition. Let $A \cap \langle P \rangle B$, then $x \in A$ and $x \in R(y)$ for some $y \in B$. $xRy$ implies there exists $z \in \uparrow y$ such that $xRz$ and $x \preceq z$. $A$ is an up-set, then $z \in A$, so $z \in A \cap B$, but $xRz$, thus, $x \in \langle P \rangle (A \cap B)$. \qed 

The lemma above allows one to extend the representation of arbitrary modal cover system described in the proof of Theorem 10 to prenuclear ones:

Theorem 11. Every localic prenuclear algebra is isomorphic to the algebra of propositions associated with some modal prenuclear localic cover system.

Proof. Let $\mathcal{L} = \langle L, \lor, \land \rangle$ be a locale and $\mathcal{L} = \langle L, \models \rangle$ a localic prenuclear algebra. Then $\mathcal{S}_L = \langle L, \preceq, \models, R_\models \rangle$ is a modal cover system, where $xR_\models y$ iff $x \preceq y$. Let us ensure that this cover system is prenuclear.

The relation is clearly reflexive, $xR_\models x$ follows from the inflationary condition. The second prenuclear cover system axiom also holds. $xR_\models y$, then $x \preceq y$. Let us put $z = x \land y$, then $xR_\models z$ since $x \preceq x \land y \preceq y(x \land y)$. $y \preceq z$ holds obviously. \qed 

To embed an arbitrary prenuclear algebra, Heyting reduct of which is a non-necessarily complete one, one need to preserve prenuclei under Dedekind-MacNeille completions. First of all, we recall what that completion is. Given a bounded lattice $\mathcal{L}$, a completion of $\mathcal{L}$ is a complete lattice $\overline{\mathcal{L}}$ that contains $\mathcal{L}$ as a sublattice. A completion $\overline{\mathcal{L}}$ is called Dedekind-McNeille if every element of $\overline{\mathcal{L}}$ is both a join and meet of elements of $\mathcal{L}$, that is for each $a \in \overline{\mathcal{L}}$ (see $[17]$ to read more about lattice completions):

\[
a = \lor\{b \in \mathcal{L} \mid a \leq b\} = \land\{b \in \mathcal{L} \mid b \leq a\}.
\]

The class of all Heyting algebras is closed under Dedekind-MacNeille completions: if $\mathcal{H}$ is a Heyting algebra, then $\overline{\mathcal{H}}$ is a locale. An implication in an arbitrary Heyting algebra $\mathcal{H}$ has an extension as follows, where $a, b \in \overline{\mathcal{H}}$:

\[
a \Rightarrow b = \land\{c \rightarrow d \mid a \geq c \in \mathcal{H} \& d \leq b \in \mathcal{H}\}.
\]

Given a lattice $\mathcal{L}$ and $f : \mathcal{L} \rightarrow \mathcal{L}$ a monotone function on this lattice, let us define maps $f^\wedge, f^\vee : \overline{\mathcal{L}} \rightarrow \overline{\mathcal{L}}$ for $a \in \overline{\mathcal{L}}$:

\[
\text{Completions of Heyting algebras are interesting topic itself, we refer the reader to these papers $[24]$ $[33]$ for further discussion.}
\]

19
\[ f^\circ(a) = \bigvee \{ f(x) \mid a \geq x \in \mathcal{L} \} \]
\[ f^*(a) = \bigwedge \{ f(x) \mid a \leq x \in \mathcal{L} \} \]

\( f^\circ \) and \( f^* \) both extend \( f \) and \( f^\circ \leq f^* \) in means of the pointwise order. Generally, neither \( f^\circ \) is multiplicative nor \( f^* \), if \( f \) is. One the other hand, if \( f \) is a multiplicative function on a Heyting algebra, so \( f^\circ \) is, see \([12]\).

**Lemma 11.** Let \( \iota \) be a prenucleus on a Heyting algebra \( \mathcal{H} \), then \( \iota^* \) is a prenucleus on \( \overline{\mathcal{H}} \).

**Proof.** The proof is similar for the analogous statement about nuclei \([31]\). \( \iota \) is inflationary, so \( \iota^* \) is, it is readily checked. Let us check that \( a \land \iota^* b \leq \iota^*(a \land b) \) for each \( a, b \in \overline{\mathcal{H}} \). \( \square \)

One may prove the following representation theorem for Heyting algebra with prenuclei operators combining Theorem \([11]\) and Lemma \([11]\).

**Theorem 12.** Every prenuclear algebra is isomorphic to the algebra of propositions obtained by some prenuclear localic cover system.

We consider the multiplicative case. Lower extensions respect multiplicativity and upper ones preserve inflationarity. We provide the equivalent definition of a multiplicative prenuclear algebra as follows to simplify the issue:

**Proposition 2.** Let \( \mathcal{H} \) be a Heyting algebra and \( j \) a function that preserves finite infima, then for each \( a, b \in \mathcal{H} \) one has \( a \leq ja \) iff \( a \land jb \leq j(a \land b) \).

**Proof.** Both implications are quite simple. One has \( a = a \land \top = a \land j\top \leq j(a \land \top) = ja \). On the other hand, \( a \land jb \leq ja \land jb = j(a \land b) \). \( \square \)

**Lemma 12.** Let \( \mathcal{H} \) be a Heyting algebra and \( \iota \) a multiplicative prenucleus on \( \mathcal{H} \), then \( \iota^\circ \) is a multiplicative nucleus on \( \overline{\mathcal{H}} \).

**Proof.** According to Proposition \([2]\) one may equivalently replace the inflationarity condition to \( j\top = \top \) and \( a \land ib \leq \iota(a \land b) \). In fact, one needs to check that the inequation \( x \land \iota^\circ y \leq \iota^\circ(x \land y) \) holds for each \( x, y \in \overline{\mathcal{H}} \). One has:

\[ x \land \iota^\circ y = \bigvee \{ a \in \mathcal{H} \mid a \leq x \} \land \bigvee \{ b \in \mathcal{H} \mid b \leq y \} = \bigvee \{ a \land cb \mid a \leq x, b \leq y, a, b \in \mathcal{H} \} \leq \bigvee \{ \iota(a \land b) \mid a \leq x, b \leq y, a, b \in \mathcal{H} \} \leq \iota^\circ(x \land y) \]

\( \iota^\circ \) is multiplicative since \( \iota \) is multiplicative. Thus, \( \iota^\circ \) is a multiplicative prenucleus on \( \mathcal{H} \). \( \square \)

Let us define a suitable cover system.

**Definition 20.** Let \( \mathcal{M} = \langle S, \leq, \triangleright, R \rangle \) be a modal cover system, then \( \mathcal{M} \) is called multiplicative prenuclear if the following conditions hold:

1. \( R \) is serial, that is, for each \( x \in S \) there exists \( y \in S \) such that \( xRy \).
2. if \( xRy \) and \( xRz \) then there exists \( w \in x \uparrow \cap y \uparrow \) such that \( xRw \).
3. Let \( x, y \in S \) such that \( xRy \), then there exists \( z \in y \uparrow \) such that \( x \leq z \) and \( x \in R(z) \).

One may consider a multiplicative prenuclear frame as an \( R_\triangleright \)-reduct of a CK-modal cover system \([31]\) with the added principle that corresponds to the second postulate of a prenuclear cover system. Such a cover system describes the logic with modal axioms \( \top \lor \top \lor \top \Rightarrow \top \lor (p \land q) \), and \( p \land q \Rightarrow \top \lor (p \land q) \) plus the \( \lor \)-monotonicity rule. It is not so hard to see that this logic is deducitively equivalent to \( \mathsf{IEL}^\lor \) over intuitionistic logic.

**Lemma 13.** Let \( \mathcal{M} = \langle S, \leq, \triangleright, R \rangle \) be a multiplicative prenuclear cover system, then \( \langle R \rangle \) is a multiplicative prenucleus on \( \text{Prop}(S) \).
Proof. \(xRy\) is clearly serial. The multiplicativity follows from the second postulate of a multiplicative prenuclear cover system. The third equation is proved similarly to Lemma 10.

Theorem 13.

1. Every localic multiplicative prenuclear algebra is representable as a modal locale of the propositions obtained by a suitable modal cover system.
2. Every multiplicative prenuclear algebra is isomorphic to the subalgebra to the algebra of propositions obtained by some multiplicative prenuclear localic cover system.

Proof.

1. The proof is similar to Theorem 11 concerning Lemma 13.
2. Follows from the previous item and Lemma 12.

Finally, we consider IEL-cover systems and corresponding multiplicative prenuclear algebras, where the equation \(j \perp = \perp\) is satisfied. We call such multiplicative prenuclear algebras dense. In particular, \(j \perp = \perp\) implies \(j' \perp = \perp\). Thus, if an operator on a Heyting algebra is a dense multiplicative prenucleus, so its lower Dedekind-MacNeille completion is.

An IEL-cover system is a multiplicative prenuclear system \(S = \langle P, \leq, \triangleright, R \rangle\) such that for each \(x, y \in S\) if \(xRy\) and \(y \triangleright \emptyset\) implies \(x \triangleright \emptyset\). This condition yields \(R/\emptyset = \emptyset\).

Thus, one may immediately extend Theorem 13:

Theorem 14.

1. Every localic dense multiplicative prenuclear algebra is isomorphic to the algebra of propositions of some IEL-cover system.
2. Every dense multiplicative prenuclear algebra is isomorphic to the subalgebra of propositions of some IEL-cover system.

4.3 Completeness theorems

In this subsection, by IEL⁻ we mean the set of formulas defined as the closure of this set

\[
\text{IPC} \oplus \varphi \to \bigcirc \varphi \oplus \varphi \wedge \psi \to \bigcirc (\varphi \wedge \psi)
\]

under the monotonicity rule: from \(\varphi \to \psi\) infer \(\bigcirc \varphi \to \bigcirc \psi\).

Let us define first intuitionistic modal predicate logic QIEL⁻ as an extension of intuitionistic predicate logic with modal axioms that correspond to the conditions of a prenucleus operator. We consider a signature consisting of predicate symbols of an arbitrary arity lacking function symbols and individual constants.

1. IEL⁻-axioms
2. \(\forall x \varphi \to \varphi(t/x)\)
3. \(\varphi(t/x) \to \exists x \varphi\)
4. The inference rules are Modus Ponens, Bernays rules, and \(\bigcirc\)-monotonicity.

Then QIEL⁻ = QIEL⁻ \(\oplus \bigcirc (\varphi \to \psi) \to (\bigcirc \varphi \to \bigcirc \psi)\) and QIEL = QIEL⁻ \(\oplus \neg \bigcirc \perp\), where \(\neg \varphi = \varphi \to \perp\).

In this section we show that the logics above are complete with respect to their suitable cover systems. Let \(L\) be a logic above QIEL⁻, let us define their models. Let \(\mathcal{C}\) be a prenuclear cover system \(\mathcal{M} = \langle S, \leq, \triangleright, R \rangle, V\) a valuation function, and \(D\) a set of individuals, then an \(L\)-cover model is a triple \(\mathfrak{M} = \langle \mathcal{M}, V, D \rangle\). Given a \(D\)-assignment and \(x \in S\), a modal operator has the following semantics:

\[\mathfrak{M}, x, \sigma \models \bigcirc \varphi\text{ if there exists }y \in R(x)\text{ such that }\mathcal{M}, y, \sigma \models \varphi.\]
In contrast to Kripkean semantics of IEL-like logics, we interpret modality in terms of “possibility”. Indeed, one may reformulate the truth condition above in means of an \(\langle R \rangle\)-operator on the locale of propositions:

\[|| \circ \phi ||^M = ||\langle R \rangle \phi ||^M\]

The completeness theorem converts the Lindenbaum-Tarski algebra to a suitable locale with a certain operator via Dedekind-MacNeille completion. After that, we represent this algebra as an algebra of propositions of a localic system by the representation theorem we proved. To be more precise, one has:

**Theorem 15.** Let \(L \in \{IEL_\prec, IEL_\prec, IEL\}\), then \(Q\ell L\) is sound and complete with respect to their cover systems.

**Proof.** Let us consider the \(QIEL_\prec\)-case, the rest two cases are shown in the same fashion via relevant representation and Dedekind-MacNeille completions. Let \(Fm\) be the set of all formulas and \(V\) the set of all variables, then one has an equivalence relation \(\phi \sim \psi \) if and only if \(QIEL_\prec \phi \vdash \psi \) and \(QIEL_\prec \psi \vdash \phi \). Then, one has an ordering on \(Fm/\sim\) defined as \([\phi] \leq [\psi]\) if \(QIEL_\prec \phi \vdash \psi\). The operations on \(Fm/\sim\) are defined as:

- \([\phi \land \psi] = [\phi] \wedge [\psi]\)
- \([\phi \lor \psi] = [\phi] \lor [\psi]\)
- \([\phi \rightarrow \psi] = [\phi] \Rightarrow [\psi]\)
- \([\forall x \phi] = \bigwedge_{x \in V} [\phi]\)
- \([\exists x \phi] = \bigvee_{x \in V} [\phi]\)
- \([\circ \phi] = [\circ [\phi]]\)
- \(T = [\phi], \) where \(IEL_\prec \vdash \phi\)

This algebra is clearly prenuclear, but its Heyting reduct is not necessarily complete. By Lemma 12, one may embed the Lindenbaum-Tarski algebra \(\mathcal{L}_{QIEL_\prec}\) to the prenucleus \(\mathcal{O}^* \) on \(F/\sim\). A localic prenuclear algebra \(\mathcal{L}_{QIEL_\prec}(F/\sim, \mathcal{O}^*)\) is isomorphic to some prenuclear cover system. Thus, by Theorem 11, one has an isomorphism \(f: \langle F/\sim, \mathcal{O}^* \rangle \cong \langle \text{Prop}(\mathcal{S}_{QIEL_\prec}), \langle R \circ \rangle \rangle\), where \(\mathcal{S}_{QIEL_\prec}\) is an obtained prenuclear cover system. Let define a \(QIEL_\prec\) cover model \(\mathcal{M} = \langle \mathcal{S}_{QIEL_\prec}, D, V \rangle\) putting \(D = V\). A valuation \(V\) is defined as \(V(P)(x_{n_1}, \ldots, x_{n_k}) = f([P(x_{n_1}, \ldots, x_{n_k})])\). Here, a \(D\)-assignment \(\sigma\) is merely an identity function. Here, the key observation is \([\phi]^{\mathcal{M}} = f[\phi]\) that might be shown by easy induction on \(\phi\). Then, if \(\phi\) is true in every \(QIEL_\prec\)-model, then \([\phi]^{\mathcal{M}} = T\), thus, \(f[\phi] = T\). Hence, \(QIEL_\prec \vdash \phi\). Thus, \(QIEL_\prec\) is sound and complete with respect models on prenuclear cover systems.

The \(QIEL_\prec\) (QIEL) case follows from the same construction using Theorem 13 (Theorem 14).

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