LOCAL CENTRAL LIMIT THEOREM FOR GRADIENT FIELD MODELS

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Abstract. We consider the gradient field model in $[-N, N]^2 \cap \mathbb{Z}^2$ with a uniformly convex interaction potential. Naddaf-Spencer [22] and Miller [20] proved that the macroscopic averages of linear statistics of the field converge to a continuum Gaussian free field. In this paper we prove the distribution of $\phi(0)/\sqrt{\log N}$ converges uniformly in $\mathbb{R}$ to a Gaussian density, with a Berry-Esseen type bound. This implies the distribution of $\phi(0)$ is sufficiently 'Gaussian like' between $[-\sqrt{\log N}, \sqrt{\log N}]$.

1. Introduction

In this paper we study a two dimensional gradient interface field with a nearest neighbor potential $V$. Explicitly, let $Q_N := [-N, N]^2 \cap \mathbb{Z}^2$ and let the boundary $\partial Q_N$ consist of the vertices in $Q_N$ that are connected to $\mathbb{Z}^2 \setminus Q_N$ by an edge. The gradient field on $Q_N$ with zero boundary condition is a random field denoted by $\phi_{Q_N, 0}$, whose distribution is given by the Gibbs measure

$$d\mu_N = Z_N^{-1} \exp\left[ -\sum_{v \in Q_N} \sum_{i=1}^2 V(\nabla_i \phi(v)) \right] \prod_{v \in \partial Q_N} \delta_0(\phi(v)),$$

where $\nabla_i \phi(v) = \phi(v + e_i) - \phi(v)$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and we set $\phi(v) = 0$ for all $v \in \mathbb{Z}^2 \setminus Q_N$. Here $Z_N$ is the normalizing constant ensuring that $\mu_N$ is a probability measure, i.e. $\mu_N(\mathbb{R}^{[Q_N]}) = 1$. We denote expectation and variance with respect to $\mu_N$ by $\langle \cdot \rangle_{\mu_N}$ and $\text{var}_{\mu_N}$, respectively.

We assume the interaction potential $V: \mathbb{R} \to \mathbb{R}$ in (1.1) satisfies the following:

(i) Symmetry: for every $t \in \mathbb{R}$, we have $V(t) = V(-t)$.

(ii) Uniform convexity: for every $t \in \mathbb{R}$, we have $0 < \lambda \leq V''(t) \leq \Lambda < \infty$.

(iii) Regularity: $V \in C^{2,1}(\mathbb{R})$. In other words, $V''$ is Lipschitz continuous with Lipschitz constant $L$.

The Gibbs measure (1.1) was introduced in the 1970s by Brascamp, Lieb and Lebowitz [11], in the name of anharmonic crystals. Since then, numerous efforts have been made to study the large-scale (macroscopic) statistical behavior of the field $\nabla \phi$. Notable progress was made by Naddaf and Spencer [22], who studied the infinite volume limit of the Gibbs measure (1.1) (the infinite volume Gibbs states were rigorously characterized by Funaki and Spohn [15]), and proved a central limit for (rescaled) linear functions of $\nabla \phi$. More precisely, they consider, for $R \geq 1$, the random variable

$$F_R(\nabla \phi) := R^{-d/2} \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^d f_i \left( \frac{x}{R} \right) (\phi(x + e_i) - \phi(x)),$$
where for each \( i \in \{1, \ldots, d\} \), take \( f_i : \mathbb{R}^d \to \mathbb{R} \) to be a compactly supported, smooth, deterministic function, and proved that the random variable \( F_R \) converges in law to a normal random variable. Later, the result of [22] has been generalized to dynamical settings [16] and also to finite volume measures such that the boundary condition has at most poly-logarithmic fluctuations [20]. We also mention the related work [14] which characterizes the Wulff shape and the large deviation principle for macroscopic profiles of \( \nabla \phi \), and results that are related to the extension of the Gibbs measure (1.1) to non-uniformly convex settings [25, 9, 1, 21, 18, 6, 7], the study of the maximum of the random field (1.1) [8, 26], and to spin models which is related to a Gibbs measure with convex interaction by a duality transform [13].

A natural question arises is whether the Gaussian limit established in [22] holds on much smaller scales, given that the microscopic interaction is not Gaussian. Under an additional assumption on the ellipticity contrast for \( V \), namely \( \Lambda < 2\lambda \), Conlon and Spencer proved a stronger result [12], which states that for the infinite volume gradient Gibbs measure \( \mu \),

\[
\left| \log \left( \exp \left( t(\phi(0) - \phi(x)) \right) \right) \right| \leq \frac{C t^3 \sup_{x \in \mathbb{R}} |V''(x)|}{\varphi}.
\]

The result suggests that under these assumptions of \( V \), the pointwise distribution of \( \phi(x) - \phi(0) \) is close to a Gaussian, and one has to move to a large deviation regime to see non-Gaussian tails.

In this paper we are able to bring down the scale to \( R = 1 \), and prove a local central limit theorem of the gradient Gibbs measure (1.1), under the assumptions on the potential \( V \) given at the beginning of the paper.

**Theorem 1.1.** Let \( \phi \) be sampled from the Gibbs measure (1.1). Assume the potential \( V(\cdot) \) satisfies the conditions (i) -(iii). Then the density function \( g_N \) of \( \phi(0)/\sqrt{\log N} \) converges uniformly in \( \mathbb{R} \) to \( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \). Moreover, there exists \( C < \infty \) such that \( |g_N(x) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}| \leq \frac{C}{\sqrt{\log N}} \).

We remark here the same proof (with a slightly modification of the multiscale argument in Section 5) also gives the local CLT at other locations inside the bulk. Namely, for any \( x_N \in Q_N \) such that the graph distance between \( x_N \) and \( \partial Q_N \) tends to infinity, the density of \( \frac{\phi(x_N)}{\sqrt{\log \text{dist}(x_N, \partial Q_N)}} \) converges uniformly in \( \mathbb{R} \) to the same Gaussian limit. The same proof also works in the infinite volume setting, gives the local CLT for \( \frac{\phi(0) - \phi(x)}{\log \log |x|} \) as \( |x| \to \infty \).

Notice that if \( \phi(0) \) can be written as \( X_1 + \cdots X_{\log N} \), where \( X_i \) are i.i.d random variables, then the Berry-Esseen Theorem gives \( |g_N(x) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}| \leq \frac{C \text{var}(X_1)^2}{\sqrt{\log N}} \). We will see in Section 5 that the analogues of \( X_i \) are increments of harmonic averages of \( \phi \), which are not independent but have certain decoupling properties (thanks to [20]), and Theorem 1.1 gives a Berry-Esseen type estimate for the density function. An immediate consequence of Theorem 1.1 is that the distribution of \( \phi(0) \) is sufficiently spreadout in \( [-\sqrt{\log N}, \sqrt{\log N}] \). Indeed, given any \( a \in [-\sqrt{\log N}, \sqrt{\log N}] \), we apply Theorem 1.1 to obtain

\[
\int_{\frac{-a+1}{\sqrt{\log N}}}^{\frac{a+1}{\sqrt{\log N}}} g_N(x) \, dx - \int_{\frac{-a+1}{\sqrt{\log N}}}^{\frac{a+1}{\sqrt{\log N}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx = O\left( \frac{1}{\log N} \right).
\]
Thus
\[ \mathbb{P}(\phi(0) \in [a, a+1]) = \frac{1}{\sqrt{2\pi \log N}} e^{- \frac{1}{2g} \frac{a^2}{\log N} + O(\frac{1}{\log N})} \]

Why do we expect a local CLT at scale \( O(1) \)? The argument to prove a macroscopic CLT in [22] was based on a beautiful observation that the scaling limit can be derived from an elliptic homogenization problem via the Helffer-Sjöstrand representation [17]. With Armstrong [5], we extend and quantify the homogenization argument by Naddaf and Spencer, based on the quantitative theory for homogenization developed by Armstrong, Kuusi and Mourrat [3, 4]. In particular, we obtained the convergence of the Hessian of the surface tension with an algebraic rate, resolve an open question posed by Funaki and Spohn [15] regarding the \( C^2 \) regularity of surface tension, and the fluctuation-dissipation conjecture of [16]. Following the approach of [5] and [4], we are able to obtain a quantitative homogenization of the Helffer-Sjöstrand PDE, thus estimate the covariance structure of \( \mu_N \), as \( N \) gets large, with a high precision (we also refer to [2] for some further applications of the quantitative homogenization ideas to the \( \nabla \phi \)-model). To obtain a Gaussian limit of \( \phi(0)/\sqrt{\log N} \), we apply the harmonic approximation result by Miller [20], which enables us to write \( \phi(0) \) as the sum of \( \log N \) increments, with certain decoupling properties. These are the two main ingredients behind the proof of Theorem 1.1.

We remark here that the key estimate for proving Theorem 1.1 is the characteristic function asymptotics for \( \left( \exp(i\frac{t}{\sqrt{\log N}} \phi(0)) \right)_{\mu_N} \), given in Lemma 2.1. If one has the convergence stated in Lemma 2.1 for \( t \in \mathbb{R} \), without quantifying the rate of convergence (namely, a CLT for \( \phi(0)/\sqrt{\log N} \)), then it implies a local CLT without any rate of convergence. The rate of convergence for the local CLT depends on the convergence rate and the validity of the \( t \) region such that Lemma 2.1 holds. In this paper we present a proof with a convergence rate using quantitative homogenization, and explain briefly in the next section how it can be simplified if one only aims for a qualitative local CLT.

The proof of Theorem 1.1 presented in this paper implies the asymptotics for the characteristic function for the distribution of \( \phi(0) \), namely there exists some \( g = g(V) > 0 \), such that

\[ (1.3) \quad \langle \exp(i s \phi(0)) \rangle_{\mu_N} \approx e^{- \frac{s^2}{2} g \log N}, \]

as long as \( s = O((\log N)^{-\frac{1}{4}}) \). Using the same major ingredients, but with a more elaborate multiscale argument, we may improve the (1.3) so that it holds for \( s \) within a small neighborhood of the origin, with radius independent of \( N \). It would be very interesting to extend the characteristic function estimates (1.3) beyond \( |s| = 1 \). For this, we include an open question posed by Tom Spencer.

**Open Question:** Consider the lattice dipole gas model, which is a special case of the gradient Gibbs measure (1.1) with \( V(x) = \frac{x^2}{2} + z \cos x \), with \( |z| < 1 \). Prove that the leading term in the asymptotic expansion of the characteristic function

\[ (1.4) \quad \langle \exp(i \pi (\phi(0) - \phi(x))) \rangle_{\mu_{\text{dipole}}} \approx e^{- \frac{s^2}{2} g \log|x|}, \]
for some $g > 0$. Estimates of type (1.4) plays an important role in the study of the (lattice) Coulomb gas and of the Coulomb gas representation of the low temperature Abelian spin models.

We summarize the characteristic function estimates needed for proving Theorem 1.1 in the next section, and introduce some notations in Section 3. In Section 4 we derive a central limit theorem for the linear statistics of $\nabla \phi$ with an algebraic rate of convergence, which quantifies the result of [22], following the argument of [5] and [4]. In Section 5 we recall the harmonic approximation result in [20], and use a multiscale argument to obtain the precise characteristic function asymptotic of $\phi(0)$. Finally, we gave an upper bound for the large $s$ characteristic function in Section 6, based on a Mermin-Wagner type argument, and finishes the proof of Theorem 1.1.

2. Estimates for characteristic functions

Theorem 1.1 follows from the quantitative estimates of the characteristic function below. The first lemma gives the precise estimate for the characteristic function $e^{is\phi(0)}_{\mu_N}$ for $s = o((\log N)^{-\frac{1}{4}})$.

**Lemma 2.1.** There exists $g = g(V) > 0$, such that for $N$ sufficiently large and $t = o((\log N)^{\frac{1}{2}})$, we have

$$
\left\{ \exp \left( i \frac{t}{\sqrt{\log N}} \phi(0) \right) \right\}_{\mu_N} = e^{-\frac{t^2}{2} g} \left( 1 + O \left( \frac{t^2}{(\log N)^{\frac{1}{2}}} \right) \right)
$$

**Remark 2.2.** Lemma 2.1 quantifies the CLT for the pointwise field $\phi(0)/\sqrt{\log N}$, namely for $t \in \mathbb{R}$,

$$
\left\{ \exp \left( i \frac{t}{\sqrt{\log N}} \phi(0) \right) \right\}_{\mu_N} = e^{-\frac{t^2}{2} g} + o_N(1).
$$

The pointwise CLT (2.2) can be proved by combining the CLT for the macroscopic average of the field established in [22, 20], and the multiscale argument presented in [8] or in Section 5 of this paper.

We also need (non-optimal) decay estimate of the characteristic function for large $s$, summarized in the two lemmas below.

**Lemma 2.3.** Let $g = g(V) > 0$ be the same constant as in Lemma 2.1. There exists $\varepsilon = \varepsilon(V) > 0$, such that for $|s| < \varepsilon$, we have for $N$ sufficiently large,

$$
\left\{ \exp \left( i s \phi(0) \right) \right\}_{\mu_N} \leq 2 e^{-\frac{s^2}{2} g \log N}
$$

**Remark 2.4.** If one only aims for the qualitative local CLT, then it suffices to prove a weaker estimate, that there exist $c_1 > 0$ and $\varepsilon = \varepsilon(V) > 0$, such that for $|s| < \varepsilon$, we have for $N$ sufficiently large,

$$
\left\{ \exp \left( i s \phi(0) \right) \right\}_{\mu_N} \leq 2 e^{-c_1 s^2 \log N}.
$$

As will be explained in Section 5, the proof of (2.4) is simpler, and the quantitative CLT presented in Section 4 would not be needed.
Lemma 2.5. There exists $\varepsilon_1 > 0$ and $C < \infty$, such that for $s \in \mathbb{R}$, we have

$$\left| (\exp(is\phi(0)))_{\mu_N} \right| \leq \min\{1 - \varepsilon_1, \frac{C}{s^2} \log N \}$$

Before proving these lemmas, we now explain how they imply Theorem 1.1.

Proof of Theorem 1.1 with no rate. To prove Theorem 1.1, arguing as the classical local CLT and write

$$\Psi_N(t) := \left( \exp \left( it\phi(0)/\sqrt{\log N} \right) \right)_{\mu_N}$$

Then by inversion theorem,

$$|g_N(x) - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}| = \left| \int_{-\infty}^{\infty} \frac{1}{2\pi} \left( \Psi_N(t) e^{itx} - e^{-\frac{1}{2}g^2 t^2} e^{itx} \right) dt \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \Psi_N(t) - e^{-\frac{1}{2}g^2 t^2} \right| dt$$

We claim the right side above goes to zero by split the integral into three parts:
- For $|t| \leq a$, we apply (2.2) to conclude that $\Psi_N(t)$ goes to $e^{-\frac{1}{2}g^2 t^2}$ uniformly for $t \in [-a, a]$.

Thus

$$\int_{-a}^{a} \left| \Psi_N(t) - e^{-\frac{1}{2}g^2 t^2} \right| dt \to 0$$

as $N \to \infty$.
- For $a \leq |t| \leq \varepsilon \sqrt{\log N}$, we apply (2.4), which yields

$$\Psi_N(t) \leq Ce^{-c_1 t^2}$$

Thus

$$\int_{a \leq |t| \leq \varepsilon \sqrt{\log N}} \left| \Psi_N(t) - e^{-\frac{1}{2}g^2 t^2} \right| dt \leq 2C \int_{a}^{\infty} e^{-c_1 t^2} dt \to 0$$

if we take $a \to \infty$.
- For $|t| \geq \varepsilon \sqrt{\log N}$, we use the bound (2.5) which implies (let $s = t/\sqrt{\log N}$)

$$\int_{|t| \geq \varepsilon \sqrt{\log N}} \left| \Psi_N(t) \right| dt \leq \sqrt{\log N} \int_{|s| \geq 1} \left| \Psi_N(s) \right| ds \leq \sqrt{\log N} \left( \int_{C \geq |s| \geq 1} (1 - \varepsilon_1) \log N ds + \int_{|s| = C} \left( \frac{C}{s^2} \right) \log N ds \right)$$

which goes to 0 as $N \to \infty$. And we conclude Theorem 1.1.

Quantitative proof of Theorem 1.1. To quantify the rate of convergence for the local CLT, take $a_N = \sqrt{\frac{2}{g^2} \log \log N}$ in the proof above.
- For $|t| \leq a_N$, we apply Lemma 2.1 to obtain a rate of convergence that

$$\int_{-a_N}^{a_N} \left| \Psi_N(t) - e^{-\frac{1}{2}g^2 t^2} \right| dt \leq \int_{-a_N}^{a_N} Ce^{-\frac{t^2}{2}} \frac{t^2}{(\log N)^{1/2}} dt \leq \frac{C}{(\log N)^{1/2}}$$

for $N$ sufficiently large.
- For $a_N \leq |t| \leq \varepsilon \sqrt{\log N}$, we apply Lemma 2.3, which yields

$$\Psi_N(t) \leq 2e^{-\frac{1}{2}g^2 t^2}$$

Thus

$$\int_{a_N \leq |t| \leq \varepsilon \sqrt{\log N}} \left| \Psi_N(t) - e^{-\frac{1}{2}g^2 t^2} \right| dt \leq 3 \int_{a_N}^{\infty} e^{-\frac{1}{2}g^2 t^2} dt \leq \frac{C}{(\log N)^{1/2}}$$
for $N$ sufficiently large.

Combine with the estimates for $|t| \geq \sqrt{\varepsilon \log N}$ in the qualitative proof above, we conclude $|g_N(x) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2x^2}}| = O((\log N)^{-\frac{1}{2}})$. \hfill \square

Lemma 2.1 and 2.3 will be proved in Section 5.3, whereas Lemma 2.5 will be proved in Section 6.

3. Preliminaries and Notation

Given a set $U \subseteq Q_\mathbb{R}$, we let $\mathcal{E}(U)$ denote the set of directed edges on $U$ and $U^0$ the interior of $U$. Define $\Omega_0(U)$ to be the set of functions $\phi : U \to \mathbb{R}$ such that $\phi = 0$ on $\partial U$. Given $e = (x, y) \in \mathcal{E}(U)$ and $\phi \in \mathbb{R}^U$, we define $\nabla \phi(e) := \phi(y) - \phi(x)$. The formal adjoint $\nabla^*$ of $\nabla$, which is the discrete version of the negative of the divergence operator, is defined for functions $g : \mathcal{E}(U) \to \mathbb{R}$ by

$$\nabla^* g(x) := \sum_{e \in \mathcal{E}(U)} g(e), \quad x \in U^0. \tag{3.1}$$

The average of a function $f : U \to \mathbb{R}$ on $U$ is denoted as $(f)_U := \frac{1}{|U|} \sum_{x \in U} f(x)$. We define, for each $x \in U$, the basis element $\omega _x \in \Omega_0(U)$ by

$$\omega _x(y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

and the differential operator $\partial_x$ by

$$\partial_x u(\phi) := \lim_{h \to 0} \frac{1}{h} (u(\phi + h\omega _x) - u(\phi)). \tag{3.2}$$

Define $L^p(\mu)$ to be the set of measurable functions $u : \Omega_0(U) \to \mathbb{R}$ such that

$$\|u\|_{L^p(\mu)} := \left( \int_{\Omega} |u(\phi)|^p \, d\mu(\phi) \right)^{\frac{1}{p}} < +\infty.$$ 

We define $H^1(\mu)$ to be

$$\|u\|_{H^1(\mu)} := \left( \|u\|_{L^2(\mu)}^2 + \sum_{x \in U^0} \|\partial_x u\|_{L^2(\mu)}^2 \right)^{\frac{1}{2}}.$$ 

We let $H^{-1}(\mu)$ denote the dual space of $H^1(\mu)$, that is, the closure of $C^\infty(\Omega_0(U))$ functions under the norm

$$\|w\|_{H^{-1}(\mu)} := \sup \left\{ \int_{\Omega} u(\phi) w(\phi) \, d\mu(\phi) : u \in H^1(\mu), \|u\|_{H^1(\mu)} \leq 1 \right\}.$$ 

We define the space $L^2(U, \mu) = L^2(U; L^2(\mu))$ to be the set of measurable functions $u : U \times \Omega_0(U) \to \mathbb{R}$ with respect to the norm

$$\|u\|_{L^2(U, \mu)} := \left( \sum_{x \in U} \|u(x, \cdot\|_{L^2(\mu)}^2 \right)^{\frac{1}{2}}.$$ 

We also define $H^1(U, \mu)$ by the norm

$$\|u\|_{H^1(U, \mu)} := \left( \sum_{x \in U} \|u(x, \cdot\|_{H^1(\mu)}^2 + \sum_{e \in \mathcal{E}(U)} \|\nabla u(e, \cdot\|_{L^2(\mu)}^2 \right)^{\frac{1}{2}}.$$
The subset \( H_0^1(U, \mu) \subseteq H^1(U, \mu) \) consists of those functions \( u \in H^1(U, \mu) \) which satisfy \( u(x, \phi) = 0 \) for every \( \partial U \times \Omega_0(U) \).

We define \( H^{-1}(U, \mu) \) to be the dual space of \( H_0^1(U, \mu) \). That is, \( H^{-1}(U, \mu) \) is the closure of smooth functions with respect to the norm

\[
\| w \|_{H^{-1}(U, \mu)} := \sup \left\{ \sum_{x \in U} \int_{\Omega_0(U)} u(x, \phi) w(x, \phi) \, d\mu(\phi) : u \in H_0^1(U, \mu), \, \| u \|_{H^1(U, \mu)} \leq 1 \right\}.
\]

It is sometimes convenient to work with the volume-normalized versions of the \( L^2 \) and Sobolev norms, defined by

\[
\| u \|_{L^2(U, \mu)} := \left( \frac{1}{|U|} \sum_{x \in U} \| u(x, \cdot) \|_{L^2(\mu)}^2 \right)^{\frac{1}{2}},
\]

\[
\| u \|_{H^1(U, \mu)} := \left( \frac{1}{|U|} \sum_{x \in U} \| u(x, \cdot) \|^2_{H^1(\mu)} + \frac{1}{|U|} \sum_{x \in \Omega(U)} \| \nabla u(e, \cdot) \|^2_{L^2(\mu)} \right)^{\frac{1}{2}},
\]

\[
\| w \|_{H^{-1}(U, \mu)} := \sup \left\{ \frac{1}{|U|} \sum_{x \in U} \int_{\Omega_0(U)} u(x, \phi) w(x, \phi) \, d\mu(\phi) : u \in H_0^1(U, \mu), \, \| u \|_{H^1(U, \mu)} \leq 1 \right\}.
\]

We notice that the formal adjoint of \( \partial_x \) with respect to \( \mu_N \), which we denote as \( \partial^*_x \), is given by

\[
\partial^*_x w := -\partial_x w + \sum_{y \sim x} V'(\phi(y) - \phi(x) - \xi \cdot (y - x)) w(\phi).
\]

This can be easily checked by the identity for all \( u, v \in H^1(\mu_N) \) that

\[
((\partial_x u)v)_{\mu_N} = (u(\partial^*_x v))_{\mu_N}.
\]

We also have the commutator identity

\[
[\partial_x, \partial^*_y] = -\mathbf{1}_{\{x-y\}} V''(\phi(y) - \phi(x) - \xi \cdot (y - x)) + \mathbf{1}_{\{x=y\}} \sum_{e \ni x} V''(\nabla \phi(e) - \nabla \ell_{\xi}(e)).
\]

Define the Witten Laplacian \( \mathcal{L}_{\mu_N} \) as

\[
\mathcal{L}_{\mu_N} F = -\sum_{x \in Q_N} \partial_x^* \partial_x F,
\]

for every cube \( Q \subseteq \mathbb{Z}^d \) and \( u, v \in H^1(Q, \mu) \), we define

\[
\mathcal{B}_{\mu,Q}[u, v] := \frac{1}{|Q|} \sum_{y \in Q_N} \sum_{x \in Q^o} (\partial_y u(x, \cdot), \partial_y v(x, \cdot))_{\mu} + \frac{1}{|Q|} \sum_{e \in \partial(Q)} (\nabla u(e, \cdot) V''(e, \cdot) \nabla v(e, \cdot))_{\mu}
\]

and

\[
\mathcal{E}_{\mu,Q,f}[u] := \frac{1}{2} \mathcal{B}_{\mu,Q}[u, u] - \frac{1}{|Q|} \sum_{e \in \partial(Q)} (f(e, \phi) \nabla u(e, \cdot))_{\mu}.
\]

For \( D \subseteq Q_R \), and \( f : \partial D \to \mathbb{R} \), define the \( \nabla \phi \) measure on \( D \) with Dirichlet boundary condition \( f \) by

\[
d\mu^f_D = Z_D^{-1} \exp \left\{ -\sum_{i=1}^2 \sum_{x \in D} V(\nabla_i \phi(x)) \right\} \prod_{x \in \partial D, \phi(x) - f(x)} \delta_0(\phi(x) - f(x)).
\]
Here $Z_D$ is the normalizing constant ensuring that $\mu^f_D$ is a probability measure. We denote expectation and variance with respect to $\mu^f_D$ by $\mathbb{E}^{D,f}$ and $\text{var}_{D,f}$, respectively.

We finally present the Brascamp-Lieb inequality \cite{B, L}, which states that the variance of observables with respect to a log-concave measure is dominated by that of a Gaussian measure. We denote the Green function for the discrete Laplacian with zero Dirichlet boundary conditions in $Q_L$ by $G_{Q_L}(x,y)$.

**Proposition 3.1** (Brascamp-Lieb inequality for $\mu_L$). For every $F \in H^1(\mu_L)$,

\begin{equation}
\text{var}_{\mu_L}[F] \leq \frac{1}{\lambda} \sum_{x,y \in Q_L^2} G_{Q_L}(x,y) \big( (\partial_x F)(\partial_y F) \big)_{\mu_L}.
\end{equation}

For every $f \in \mathbb{R}^{Q_L}$, we have

\begin{equation}
\log \left( \exp \left( t \sum_{y \in Q_L} \phi(y) f(y) \right) \right)_{\mu_L} \leq \frac{t^2}{2\lambda} \sum_{x,y \in Q_L} G_{Q_L}(x,y) f(x) f(y)
\end{equation}

We sometimes denote by $\mu^f_{G,D}$ the finite volume Gaussian measure in $D$ (i.e., the special case of (3.5) with $V(x) = \frac{1}{2}x^2$). We denote the corresponding expectation and variance by $\mathbb{E}^{G,D,f}$ and $\text{var}_{G,D,f}$ respectively. When $f = 0$ we will omit its appearance on the superscripts.

4. **Quantitative convergence of the variance**

A main ingredient for the refined estimate of $\Psi_N$, defined in (2.6) is the following convergence of the variance of the linear statistics of $\nabla \phi$, with an algebraic rate.

**Theorem 4.1** (Quantitative convergence of variance). Fix $R \in [1,\infty)$. Let $\phi$ be sampled from the finite volume Gibbs measure $\mu_R$ with zero boundary condition (1.1). Let $f_R : Q_R \to \mathbb{R}$ and $f \in L^2([0,1]^2)$ be such that there exists $\alpha > 0$, so that $\| \nabla^* \cdot f_R(\overline{r}) - f(\cdot) \|_{L^\infty([0,1]^2)} \leq R^{-\alpha}$. Define the random variable

$$
\Phi_R(f) := R^{-d/2} \sum_{e \in \mathcal{E}(Q_R)} \nabla \phi(e) \cdot f_R(e).
$$

Then there exists $g = g_{V,f} > 0$, $\beta = \beta(d, \alpha, \lambda, \Lambda) \in \left(0, \frac{1}{2}\right]$ and $C_0(d, \lambda, \Lambda) < \infty$ such that, for every $R \in [1,\infty)$,

$$
|\text{var}_{\mu_R}[\Phi_R(f)] - g| \leq C_0 R^{-\beta} \| \nabla^* \cdot f_R \|_{L^\infty}.
$$

**Remark 4.2.** The central limit theorem for the $\nabla \phi$ model, i.e., the convergence of $\Phi_R$ in distribution to a normal random variable, was established in \cite{B, F, L}, without quantifying the rate of convergence.

**Remark 4.3.** It will be clear from Theorem 4.4 below that $g$ can be explicitly written as

$$
g = \int_{[0,1]^2 \times [0,1]^2} f(x)(\nabla^* \cdot \overline{a} \nabla)^{-1}(x,y)f(y) \, dx \, dy,
$$

for some positive definite matrix $\overline{a} = \overline{a}(V)$.

Theorem 4.1 follows from homogenization of an elliptic PDE based on the convergence results of \cite{B}, as we explain below. The starting observation is the variational
characterization of the variance, known as the Helffer-Sjöstrand representation (see [22, 5]), which gives

\begin{equation}
\text{var}_{\mu_R}[R^{-d/2} \sum_{x \in Q_R} \nabla \phi(x) \cdot f_R(x)] = -2 \inf_{w \in H_0^1(Q_R, \mu_R)} E_{\mu_R, Q_R, f_R}[w],
\end{equation}

where we let $E_{\mu, U, f}[-]$ denote the energy functional

$$
E_{\mu, U, f}[w] := \frac{1}{2} \sum_{y \in Q} \sum_{x \in U} \langle (\partial_y w(x, \cdot))^2 \rangle_{\mu} + \frac{1}{2} \sum_{x \in U} \langle V''(x)(\nabla w(x, \cdot))^2 \rangle_{\mu}
- \sum_x \langle f(x, \cdot) w(x, \cdot) \rangle_{\mu}.
$$

The minimizer of (4.1) can be written as $R^{-\frac{d}{2}} u_R$, where $u_R$ solves the Helffer-Sjöstrand PDE

\begin{equation}
\begin{cases}
- \mathcal{L}_\mu u_R + \nabla^* V'' \nabla u_R = \nabla^* f_R & \text{in } Q_R \times \Omega_0(Q_R) \\
u_R = 0 & \text{on } \partial Q_R \times \Omega_0(Q_R),
\end{cases}
\end{equation}

and by testing (4.2) with $u_R$ and integration by parts, we may rewrite the energy functional $E_{\mu_R, Q_R, f_R}[u_R]$, and thus (4.1) as [22, 5]

$$
\text{var}_{\mu_R}[R^{-d/2} \sum_{x \in Q_R} \nabla \phi(x) \cdot f_R(x)] = \sum_{x \in Q_R} R^{-d} (f_R(x) \nabla u_R(x))_{\mu_R}
$$

Therefore Theorem 4.1 follows from the quantitative homogenization of the Helffer-Sjöstrand equation (4.2), presented below.

**Theorem 4.4.** Suppose that $f_R, f$ satisfy the conditions in Theorem 4.1, and let $a$ be the diagonal matrix with $a(e, e) = V''(\nabla \phi(e))$, where $\nabla \phi$ is sampled from the Gibbs measure $\mu_R$ (1.1). Let $u_R, \alpha$ denote respectively the solution to the equations:

\begin{equation}
\begin{cases}
- \mathcal{L}_\mu u_R + \nabla^* a \nabla u_R = \nabla^* f_R & \text{in } Q_R \times \Omega_0(Q_R) \\
u_R = 0 & \text{on } \partial Q_R \times \Omega_0(Q_R),
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
- \nabla \cdot \alpha \nabla u = f & \text{in } [0, 1]^2 \\
u = 0 & \text{on } \partial([0, 1]^2).
\end{cases}
\end{equation}

Then there exists $\beta = \beta(d, \lambda, \Lambda) > 0$, such that

\begin{equation}
\|u_R(R \cdot) - u(\cdot)\|_{L^2(\frac{1}{R} Q_{R, \mu_R})} + \|\nabla u_R(R \cdot) - \nabla u(\cdot)\|_{H^{-1}(\frac{1}{R} Q_{R, \mu_R})} \leq C R^{-\alpha} \|\nabla^* \cdot f_R\|_{L^\infty}.
\end{equation}

Applying Theorem 4.4 and rescale the domain by $R$, we have

$$
\text{var}_{\mu_R}[R^{-d/2} \sum_{x \in Q_R} \nabla \phi(x) \cdot f_R(x)] - \sum_{x \in Q_R} R^{-d} f_R(x) \nabla u \left( \frac{x}{R} \right)
\leq \|\nabla^* \cdot f_R\|_{L^\infty} \|u_R - u \left( \frac{\cdot}{R} \right)\|_{L^2(\frac{1}{R} Q_{R, \mu_R})} \leq C R^{-\beta} \|\nabla^* \cdot f_R\|_{L^\infty}
$$

Moreover, the limit

$$
g := \lim_{R \to \infty} \sum_{x \in Q_R} R^{-d} f_R(x) \nabla u \left( \frac{x}{R} \right) = \int_{[0, 1]^2} f(x) u(x) \, dx = \int_{[0, 1]^2} f(x) \nabla \cdot (\nabla^* \alpha \nabla)^{-1} \nabla \cdot f(y) \, dy dx y$$
exists, by the convergence of Riemann sum to integral, with a rate of convergence \(O(R^{-\alpha})\). Combining these estimates we conclude Theorem 4.1.

4.1. Finite-volume energy quantities. In this section we recall the energy quantities and their quantitative convergence results established in [5]. As can be seen from the variational characterization, the convergence of the energy quantities will play an essential role in the proof of Theorem 4.1. Define the subadditive energy quantity

\[
\nu_R(Q_R, f, p) := \inf_{w \in \mathcal{H}_1^0(Q_R, \mu_{R,p})} E_{\mu_{R,p}}[w],
\]

where \(\mu_{R,p}\) denotes the finite volume Gibbs measure in \(Q_R\) with an affine boundary condition \(\ell_p(x) = p \cdot x\). In what follows we consider \(f = 0\) and simply write it as \(\nu_R(Q_R, p)\). As was explained in (4.1), the minimizer of \(\nu_R(Q_R, p)\), which we denote as \(v(\cdot, Q_R, p)\), solves the Helffer-Sjöstrand equation with an affine boundary condition:

\[
\begin{aligned}
& (-\mathcal{L}_p + \nabla^* a \nabla) v(\cdot, Q_R, p) = 0 \quad \text{in } Q_R^c \times \Omega_0(Q_R), \\
& v(\cdot, Q_R, p) - \ell_p = 0 \quad \text{on } \partial Q_R \times \Omega_0(Q_R).
\end{aligned}
\]

(4.6)

We recall the fact that \(\nu_R\) are actually quadratic polynomials for all \(R \geq 1\), and one may compute the first and second variations of their defining optimization problems. The following lemma is [5, Lemma 5.2].

**Lemma 4.5** (Basic properties of \(\nu_R\)). Fix a cube \(Q \subseteq Q_R\). The quantities \(\nu_R(Q, p)\) and its optimizing functions \(v(\cdot, Q, p)\) satisfy

- Quadratic representation. There exist symmetric matrices \(\bar{a}(Q) \in \mathbb{R}^{d \times d}\), such that

\[
\nu_R(Q, p) = \frac{1}{2} p \cdot \bar{a}(Q)p + \nu_R(Q, 0) \quad \forall p \in \mathbb{R}^d.
\]

(4.7)

where the matrix \(\bar{a}(Q)\) can be characterized such that for all \(p, p' \in \mathbb{R}^d\),

\[
p' \cdot \bar{a}(Q)p = B_{\mu_{R,p},Q}[\ell_{p'}, v(\cdot, Q, p)].
\]

(4.8)

- First variation. The optimizing functions are characterized as follows: \(v(\cdot, Q, p)\) is the unique element of \(\ell_p + H_0^1(Q, \mu_{R,p})\) satisfying

\[
B_{\mu_{R,p},Q}[v(\cdot, Q, p), w] = 0, \quad \forall w \in H_0^1(Q, \mu_{R,p});
\]

(4.9)

- Second variation. For every \(w \in \ell_p + H_0^1(Q, \mu_{R,p})\),

\[
E_{\mu_{R,p},Q}[w] - |Q| \nu_R(Q, p) = \frac{1}{2} B_{\mu_{R,p},Q}[v(\cdot, Q, p) - w, v(\cdot, Q, p) - w]
\]

(4.10)

As \(R \to \infty\), the subadditive quantity \(\nu_R\) is proved to converge with an algebraic rate of convergence. Define, for some positive definite matrix \(\bar{a} = \bar{a}(V)\) and \(\bar{c}\),

\[
\overline{v}(p) = \frac{1}{2} p \cdot \bar{a}p - \bar{c}.
\]

(4.11)

We have

**Proposition 4.6** (Proposition 6.9 of [5]). There exist \(\beta(d, \lambda, \Lambda) \in \left(0, \frac{1}{2}\right]\) and \(C(d, \lambda, \Lambda) < \infty\) such that, for every \(L \in \mathbb{N}\) with \(L \leq R\), we have

\[
|\nu_R(Q_L, p) - \overline{v}(p)| \leq CL^{-\beta}.
\]

(4.12)

Combine with the quadratic representation (4.11), this implies
Corollary 4.7. There exist $\beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}]$ and $C(d, \lambda, \Lambda) < \infty$ such that, for every $R \in \mathbb{N}$,
\begin{equation}
|\bar{a}(Q_R) - \bar{a}| \leq CR^{-\beta}
\end{equation}

4.2. Estimates on finite-volume correctors. A direct consequence of Proposition 4.6 and the quadratic response (4.10) implies the quantitative convergence of the solution to the Dirichlet problem (4.6) to the affine function.

Lemma 4.8. There exist $\beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}]$ and $C(d, \lambda, \Lambda) < \infty$ such that, for every $R \in \mathbb{N}$,
\begin{equation}
\frac{1}{R} \|v(\cdot, Q_R, p) - \ell_p\|_{L^2(Q_{2R}, \mu_R)} \leq CR^{-\beta}.
\end{equation}

An application of the multiscale Poincaré inequality (Proposition A.1) implies the quantitative convergence of the fluxes along the geometric scales $R = 3^n$. We define for every $m \in \mathbb{N}$,
\begin{equation}
\square_m := [-3^m, 3^m]^d \cap \mathbb{Z}^d.
\end{equation}

We also define, for $m \leq n$, $Z_m := 3^m \mathbb{Z}^d \cap \square_n$, so that $\{y + \square_m : y \in Z_m\}$ is a partition of $\square_n$. The next lemma shows the spatial average of the flux is concentrated around its mean.

Lemma 4.9. There exist $\beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}]$ and $C(d, \lambda, \Lambda) < \infty$ such that, for every $m \in \mathbb{N}$,
\begin{equation}
\text{var}_{\mu_{\square_m}} [(a\nabla v(\cdot, \square_m, p))_{\square_m}] \leq C 3^{-m\beta}.
\end{equation}

Proof. We define a localized solution $v_{\text{loc}}$, such that if $x \in \square_m$ is contained in $z + \square_{m/3}$ for some $z \in Z_{m/3}$, set $v_{\text{loc}}(x) = v(x, z + \square_{m/3}, p)$. We notice that
\[
\text{var}_{\mu_{\square_m}} [(a\nabla v(\cdot, \square_m, p))_{\square_m}] \leq \text{var}_{\mu_{\square_m}} [(a\nabla v_{\text{loc}})_{\square_m}] + C 3^{-m\beta/3}.
\]

Indeed, it follows from the second variation and triangle inequality that
\[
\left| \text{var}_{\mu_{\square_m}} [(a\nabla v(\cdot, \square_m, p))_{\square_m}] - \text{var}_{\mu_{\square_m}} [(a\nabla v_{\text{loc}})_{\square_m}] \right| \\
\leq 2 \|a\nabla v(\cdot, \square_m, p) - a\nabla v_{\text{loc}}\|_{L^2(\square_m, \mu_{\square_m})} \leq C \left( \nu_{3m}(\square_{m/3}, p) - \nu_{3m}(\square_m, p) \right),
\]
and this is bounded by $3^{-m\beta/3}$ by the quantitative convergence of energy (Proposition 4.6).

Therefore it suffices to estimate $\text{var}_{\mu_{\square_m}} [(a\nabla v_{\text{loc}})_{\square_m}]$, which we do by using the spectral gap (namely, the Brascamp-Lieb inequality) and that $v_{\text{loc}}$ is a localized solution, to derive a correlation decay for $a\nabla v_{\text{loc}}$. For simplicity, denote by $a\nabla v_{\text{loc}} := (a\nabla v_{\text{loc}})_{\square_m}$. Applying the Brascamp-Lieb inequality (Proposition 3.1) then yields
\begin{equation}
\text{var}_{\mu_{\square_m}} [(a\nabla v_{\text{loc}})_{\square_m}] \leq \frac{1}{\lambda} \sum_{x, y \in \square_m} G_{\square_m}(x, y) \left( (\partial_x a\nabla v_{\text{loc}})(\partial_y a\nabla v_{\text{loc}}) \right)_{\mu_{\square_m}}
\end{equation}

Notice that by definition (3.2)
\[
\partial_x a\nabla v_{\text{loc}} = \lim_{h \to 0} \frac{\sum_{e \in x} a_e(\phi + h\omega_x) - a_e(\phi)}{h} a\nabla v_{\text{loc}}(\phi + h\omega_x) - a\nabla v_{\text{loc}}(\phi)
\]

It follows from the regularity assumption that $a_e = V''$ is uniformly Lipshitz, and since $a_e$ is a function of $\nabla \phi$, we may write $\sum_{e \in x} a_e(\phi + h\omega_x) - a_e(\phi) = \nabla \phi \cdot b^h$, where $b^h$ is
bounded by a constant independent of \( h \). We claim that there exists \( C(d, \lambda, \Lambda) < \infty \), such that

\[
\left\| \frac{a \nabla v_{\text{loc}}(\phi + h \omega_x) - a \nabla v_{\text{loc}}(\phi)}{\sum_{e \in x} a_e(\phi + h \omega_x) - a_e(\phi)} \right\|_{L^2(\mu_{\partial m})} \leq C 3^{-4m/3}.
\]

This implies, by (4.17), and the estimate that \( |\nabla_x \nabla_y \cdot G_{\square m}(x, y)| \leq \frac{C}{\max \{1, |x - y|\}^T} \), that

\[
\text{var}_{\mu_{\square m}} \left[ (a \nabla v_{\text{loc}})_{\square m} \right] \leq \frac{1}{\lambda} \sum_{x, y \in \square_m} \left| b(x) \nabla_x G_{\square m}(x, y) \nabla_y b(y) \right| \left\| \partial_x a \nabla v_{\text{loc}} \right\|_{L^2(\mu_{\square m})} \left\| \partial_y a \nabla v_{\text{loc}} \right\|_{L^2(\mu_{\square m})} \leq C 3^{-8m/3} \sum_{x, y \in \square_m} \frac{C}{\max \{1, |x - y|\}^T} \leq C m 3^{-2m/3}
\]

Thus we conclude the lemma. To prove (4.18), notice that

\[
\left\| \frac{a \nabla v_{\text{loc}}(\phi + h \omega_x) - a \nabla v_{\text{loc}}(\phi)}{\sum_{e \in x} a_e(\phi + h \omega_x) - a_e(\phi)} \right\|_{L^2(\mu_{\square m})} \leq 2 \left\| \frac{a(\phi + h \omega_x)(\nabla v_{\text{loc}}(\phi + h \omega_x) - \nabla v_{\text{loc}}(\phi))}{\sum_{e \in x} a_e(\phi + h \omega_x) - a_e(\phi)} \right\|_{L^2(\mu_{\square m})} + 2 \left\| \frac{((a(\phi + h \omega_x) - a(\phi)) \nabla v_{\text{loc}}(\phi))}{\sum_{e \in x} a_e(\phi + h \omega_x) - a_e(\phi)} \right\|_{L^2(\mu_{\square m})}
\]

And since

\[
((a(\phi + h \omega_x) - a(\phi)) \nabla v_{\text{loc}}(\phi)) \square_m = 3^{-2m} \sum_{e \in x} (a_e(\phi + h \omega_x) - a_e(\phi)) \nabla v_{\text{loc}}(e)
\]

Therefore

\[
\left\| \frac{((a(\phi + h \omega_x) - a(\phi)) \nabla v_{\text{loc}}(\phi))}{\sum_{e \in x} a_e(\phi + h \omega_x) - a_e(\phi)} \right\|_{L^2(\mu_{\square m})} \leq C 3^{-2m} \sum_{e \in x} \left\| \nabla v_{\text{loc}}(e) \right\|_{L^2(\mu_{\square m})}
\]

where the right side is bounded by

\[
C 3^{-2m} \sum_{e \in E(\square_m/3)} \left\| \nabla v_{\text{loc}}(e) \right\|_{L^2(\mu_{\square m})} \leq C 3^{-2m} 3^{2m/3} \nu_{\square m}(\square_m/3, p) \leq C 3^{-4m/3}
\]

To estimate the other term, we claim that for the unique \( z \in \mathcal{Z}_{m/3} \) such that \( x \in z + \square_{m/3} \),

\[
\left\| \frac{\nabla v_{\text{loc}}(\phi + h \omega_x) - \nabla v_{\text{loc}}}{\sum_{e \in x} a_e(\phi + h \omega_x) - a_e(\phi)} \right\|_{L^2(z + \square_{m/3}, \mu_{\square m})} \leq C 3^{2m/3}
\]

Indeed, since \( \tilde{v}_{\text{loc}} := v_{\text{loc}}(\phi + h \omega_x) \) is a solution to the Dirichlet problem:

\[
\begin{cases}
-\mathcal{L}_\mu + \nabla^a(a(\phi + h \omega_x) \nabla) \tilde{v}_{\text{loc}} = 0 & \text{in } z + \square_{m/3} \times \Omega_0(z + \square_{m/3}), \\
\tilde{v}_{\text{loc}} - \ell_p = 0 & \text{on } \partial(z + \square_{m/3} \times \Omega_0(z + \square_{m/3}),
\end{cases}
\]
Testing the equation of \( \tilde{v}_{loc} \) and \( v_{loc} \) and subtract them, we obtain

\[
\left\{ \sum_{e \in \mathcal{E}(z+\Omega_{m/3})} a_e(\phi) \nabla v_{loc}(e) \nabla (v_{loc}(e) - \tilde{v}_{loc}(e)) \right\}_{\mu_{\Omega m}} \\
\leq \left\{ \sum_{e \in \mathcal{E}(z+\Omega_{m/3})} a_e(\phi + h\omega_x) \nabla \tilde{v}_{loc}(e) \nabla (v_{loc}(e) - \tilde{v}_{loc}(e)) \right\}_{\mu_{\Omega m}}
\]

Therefore

\[
\| \nabla \tilde{v}_{loc} - \nabla v_{loc} \|_{L^2(z+\Omega_{m/3},\mu_{\Omega m})}^2 \leq \left\{ \sum_{e \in \mathcal{E}} a_e(\phi + h\omega_x) - a_e(\phi) \right\}_{\mu_{\Omega m}} \nabla v_{loc}(e) \nabla (v_{loc}(e) - \tilde{v}_{loc}(e)) \right\}_{\mu_{\Omega m}}
\]

thus by Cauchy-Schwarz

\[
\left\| \nabla \tilde{v}_{loc} - \nabla v_{loc} \right\|_{L^2(z+\Omega_{m/3},\mu_{\Omega m})}^2 \leq \sum_{e \in \mathcal{E}} \| \nabla v_{loc}(e) \|_{L^2(\mu_{\Omega m})} \leq C3^{2m/3} \nu_{3n}(\square_{m/3}, p)
\]

this is (4.20). Therefore

\[
(4.22) \quad \left\| \frac{(a(\phi + h\omega_x))(\nabla v_{loc}(\phi + h\omega_x) - \nabla v_{loc}(\phi))}{\sum_{e \in \mathcal{E}} a_e(\phi + h\omega_x) - a_e(\phi)} \right\|_{L^2(\mu_{\Omega m})} \leq C3^{-2m} \left\| \nabla v_{loc}(\phi + h\omega_x) - \nabla v_{loc} \right\|_{L^2(z+\Omega_{m/3},\mu_{\Omega m})} \leq C3^{-4m/3}
\]

Combine the estimates (4.19) and (4.22) above we conclude (4.18), and thus finish the proof of the lemma.

We are ready to prove the convergence of the fluxes with an algebraic rate.

**Lemma 4.10.** There exist \( \beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}] \) and \( C(d, \lambda, \Lambda) < \infty \) such that, for every \( n \in \mathbb{N} \),

\[
(4.23) \quad 3^{-2n} \| a \nabla v(\cdot, \square_n, p) - \bar{a}p \|_{H^{-1}(\square_n, \mu_{\Omega m})} \leq C3^{-n\beta}.
\]

**Proof.** We first notice, that by the representation of \( \bar{a}(Q) \) in (4.8), and the definition of \( B \) norm in (3.4), for all \( m \geq 1 \) we may write the spatial average of the flux as

\[
(4.24) \quad \big( (a \nabla v(\cdot, \square_m, p) )_{\square_m} \big)_{\mu_{\Omega m}} = \bar{a}(\square_m) p.
\]

Recall that for \( m \leq n, \ Z_m := 3^m \mathbb{Z}^d \cap \square_n \). We apply the multiscale Poincaré inequality (Proposition A.1) to obtain

\[
(4.25) \quad 3^{-2n} \| a \nabla v(\cdot, \square_n, p) - \bar{a}p \|_{H^{-1}(\square_n, \mu_{\Omega m})} \leq C3^{-n} \| a \nabla v(\cdot, \square_n, p) - \bar{a}p \|_{L^2(\square_n, \mu_{\Omega m})} + C \sum_{m=0}^{n-1} 3^{m-n} \left( \left( \frac{1}{|Z_m|} \sum_{y \in Z_m} \left( a \nabla v(\cdot, \square_n, p) - \bar{a}p \right)_{y+\square_m} \right)^2 \right)^{1/2}_{\mu_{\Omega m}}.
\]

The first term on the right side above is bounded by

\[
C3^{-n} \left( \| \nabla v(\cdot, \square_n, p) \|_{L^2(\square_n, \mu_{\Omega m})} + |p|^2 \right) \leq C'3^{-n}
\]
For the second term, triangle inequality implies that
\[
\frac{1}{|Z_m|} \sum_{y \in Z_m} \left| (a \nabla v(\cdot, \Box_n, p) - \bar{a}p)_{y+\Box_m} \right|^2 \\
\leq \frac{1}{|Z_m|} \sum_{y \in Z_m} \left| \left( (a \nabla v(\cdot, y + \Box_m, p))_{\mu_y + \Box_m} - \bar{a}p \right)_{y+\Box_m} \right|^2 \\
+ \frac{1}{|Z_m|} \sum_{y \in Z_m} \left| \left( (a \nabla v(\cdot, y + \Box_m, p))_{\mu_y + \Box_m} - a \nabla v(\cdot, y + \Box_m, p) \right)_{y+\Box_m} \right|^2 \\
+ \frac{1}{|Z_m|} \sum_{y \in Z_m} \left| (a \nabla v(\cdot, \Box_n, p) - a \nabla v(\cdot, y + \Box_m, p))_{y+\Box_m} \right|^2
\]

By (4.24) and Proposition 4.6, we have
\[
\left| \left( (a \nabla v(\cdot, y + \Box_m, p))_{\mu_y + \Box_m} - \bar{a}p \right)_{y+\Box_m} \right|^2 \leq |\bar{a}(\Box_m) - \bar{a}|^2 |p|^2 \leq C \left( |p| + |q| + K_0 \right)^2 3^{-2\beta m}
\]

By the second variation, we have
\[
\left( \frac{1}{|Z_m|} \sum_{y \in Z_m} \left| (a \nabla v(\cdot, \Box_n, p) - a \nabla v(\cdot, y + \Box_m, p))_{y+\Box_m} \right|^2 \right)^{\mu_0} \leq C \nu_\beta(\Box_m, p) - \nu_\beta(\Box_n, p) \leq C 3^{-m\beta}
\]
where the last inequality follows from the quantitative convergence of $\nu_\beta$ (Proposition 4.6) and triangle inequality.

We also apply the variance estimate Lemma 4.9 to conclude that for every $y \in Z_m$,
\[
\left| \left( (a \nabla v(\cdot, y + \Box_m, p))_{\mu_y + \Box_m} - a \nabla v(\cdot, y + \Box_m, p) \right)_{y+\Box_m} \right|^2 \leq C 3^{-m\beta}
\]
Combining (4.26), (4.27) and (4.28), we conclude there exists $\beta = \beta(d, \lambda, \Lambda) > 0$ and $C = C(d, \lambda, \Lambda) < \infty$, such that
\[
\frac{1}{|Z_m|} \sum_{y \in Z_m} \left| (a \nabla v(\cdot, \Box_n, p) - \bar{a}p)_{y+\Box_m} \right|^2 \leq C 3^{-m\beta}
\]

Substitute the above estimates into (4.25), and summing over $m$, we conclude the Lemma. □

4.3. **Proof of Theorem 4.4.** In the previous subsection, we established the convergence of the solution to a Dirichlet problem with an affine boundary condition, with an algebraic rate of convergence. The equation (4.2) we would like to homogenize is more general, but if we localize it on a mesoscale, the boundary condition becomes approximately affine. In this section we prove Theorem 4.4 by estimating the homogenization error in terms of the error in the convergence of the correctors and fluxes defined in the previous subsection. The proof goes through a standard, deterministic argument known as the two-scale expansions, that follows closely along the argument of [3, 4].

Given $R \geq 1$, let $m = \inf\{k \in \mathbb{N} : 3^k \geq R\}$. We may view the solution $u_R$ (and respectively, $u$) to the the equation (4.2) (resp. (4.4)) as elements in a slightly larger domain $\Box_m \times \Omega_0(\Box_m)$ (resp. in $U := \frac{1}{R}\Box_m$). Specifically, we set the value of $u_R$ to be 0 and $v(\cdot, \Box_n, p)$ (defined in (4.6)) to be $\ell_p$ outside $Q_R$, and set $u = 0$ outside $[0, 1]^2$. 

We now rescale the problem and study the Dirichlet problem in a fixed domain with mesh size goes to zero. Denote by $\varepsilon = R^{-1}$ and for $i = 1, \ldots, d$ define the finite volume corrector

$$\chi^\varepsilon_{e_i}(\cdot) := u(\cdot, \Box_{m}, e_i) - \ell_{e_i}. $$

For $u$ that solves (4.4), we construct the modified two scale expansion

$$w^\varepsilon(x) = u(x) + \varepsilon \nabla u(x) \cdot \chi^\varepsilon(\frac{x}{\varepsilon}), \quad \forall x \in U$$

**Step 1.** Substitute $w^\varepsilon$ into the Eq. (4.2) to obtain there exist $\beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}]$ and $\mathcal{C}(d, \lambda, \Lambda) < \infty$ such that,

$$\|\nabla^* \cdot (a \nabla w^\varepsilon) - \mathcal{L}_{\mu} w^\varepsilon - \nabla \cdot f^\varepsilon\|_{H^{-1}(U, \mu_R)} \leq \mathcal{C}\|f\|_{L^\infty(U)} \varepsilon^\beta.$$  

We begin by computing

$$\nabla w^\varepsilon = \nabla u + \nabla u(\cdot) \cdot \nabla \chi^\varepsilon + \varepsilon \sum_{j=1}^{d} \chi^\varepsilon_{e_j}(\frac{\cdot}{\varepsilon}) \nabla \nabla_j u$$

$$= \sum_{j=1}^{d} \nabla_j u \nabla v(\frac{\cdot}{\varepsilon}, \Box_{m}, e_j) + \varepsilon \sum_{j=1}^{d} \chi^\varepsilon_{e_j}(\frac{\cdot}{\varepsilon}) \nabla \nabla_j u$$

where we used the definition of $\chi^\varepsilon_{e_j}$ in the last line. This yields

$$\nabla^* a \nabla w^\varepsilon = \sum_{j=1}^{d} \nabla_j u \nabla v(\frac{\cdot}{\varepsilon}, \Box_{m}, e_j) + \sum_{j=1}^{d} \nabla^* \nabla_j u \left( a \nabla v(\frac{\cdot}{\varepsilon}, \Box_{m}, e_j) \right) + \varepsilon \nabla^* a \left( \sum_{j=1}^{d} \chi^\varepsilon_{e_j}(\frac{\cdot}{\varepsilon}) \nabla \nabla_j u \right)$$

We also obtain, using that $u(x)$ has no $\phi$-dependence,

$$\mathcal{L}_{\mu} w^\varepsilon = \sum_{j=1}^{d} \nabla_j u \cdot \mathcal{L}_{\mu} v(\frac{\cdot}{\varepsilon}, \Box_{m}, e_j)$$

Therefore

$$\nabla^* a \nabla w^\varepsilon - \mathcal{L}_{\mu} w^\varepsilon = \sum_{j=1}^{d} \nabla_j u \left( \nabla^* \nabla v(\frac{\cdot}{\varepsilon}, \Box_{m}, e_j) - \mathcal{L}_{\mu} v(\frac{\cdot}{\varepsilon}, \Box_{m}, e_j) \right)$$

$$+ \sum_{j=1}^{d} \nabla^* \nabla_j u \left( a \nabla v(\frac{\cdot}{\varepsilon}, \Box_{m}, e_j) - \bar{a} e_j \right) + \sum_{j=1}^{d} \nabla^* \nabla_j u \cdot \bar{a} e_j + \varepsilon \nabla^* a \left( \sum_{j=1}^{d} \chi^\varepsilon_{e_j}(\frac{\cdot}{\varepsilon}) \nabla \nabla_j u \right)$$

The first term on the right side above vanishes since $v$ solves the equation (4.6). The third term is

$$\sum_{j=1}^{d} \nabla^* \nabla_j u \cdot \bar{a} e_j = \nabla^* \bar{a} \nabla u = f$$

To estimate the rest two terms, notice that $u$ solves the constant coefficient equation (4.4), and therefore

$$\|\nabla^k \nabla_j u\|_{L^\infty(U)} \leq \left\| \int_{[0,1]^2} \nabla^{k+1} \nabla_j G_{\pi}(x, y) f(y) \, dy \right\|_{L^\infty([0,1]^2)} \leq C\|f\|_{L^\infty([0,1]^2)},$$

where $G_{\pi}(x, y)$ is the Green’s function for the operator $\nabla^* \bar{a} \nabla$ in $[0, 1]^2$ with zero boundary condition.
We apply the quantitative convergence of the corrector (Lemma 4.8) and the fluxes (Lemma 4.10) in the rescaled setting, which implies there there exist $\beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}]$ and $C(d, \lambda, \Lambda) < \infty$ such that,

$$
\sum_{j=1}^{d} \left\| \nabla^{+} \cdot \nabla_{j} u \left( a \nabla v(\cdot, e_{j}) - \bar{a}e_{j} \right) \right\|_{H^{-1}(U, \mu_{R})}^{2} \leq C \left\| \nabla_{j} u \right\|_{L^{\infty}}^{2} \sum_{j=1}^{d} \left\| a \nabla v(\cdot, e_{j}) - \bar{a}e_{j} \right\|_{H^{-1}(U, \mu_{R})}^{2} \leq C \left\| f \right\|_{L^{\infty}([0,1]^{2})} \varepsilon^{\beta}
$$

and

$$
\left\| \varepsilon \nabla^{+} \cdot a \left( \sum_{j=1}^{d} \chi_{e_{j}}(\cdot) \nabla_{j} u \right) \right\|_{L^{2}(U, \mu_{R})} \leq C \left\| \nabla_{j} u \right\|_{L^{\infty}} \sum_{j=1}^{d} \left\| \chi_{e_{j}} \right\|_{L^{2}(Q_{R}, \mu_{R})} \leq C \left\| f \right\|_{L^{\infty}([0,1]^{2})} \varepsilon^{\beta}
$$

Thus we conclude

$$
\left\| \nabla^{+} \cdot (a \nabla w^{\varepsilon}) + \mathcal{L}_{\mu} w^{\varepsilon} - f \right\|_{H^{-1}(U, \mu_{R})} \leq C \varepsilon^{\beta}.
$$

And Step 1 follows from the fact that $\left\| \nabla^{+} \cdot f^{\varepsilon}(\cdot) - f(\cdot) \right\|_{L^{\infty}([0,1]^{2})} \leq \varepsilon^{\alpha}$.

**Step 2** We deduce that there exist $\beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}]$ and $C(d, \lambda, \Lambda) < \infty$ such that,

$$
\left\| u^{\varepsilon} - w^{\varepsilon} \right\|_{H^{-1}(U, \mu_{R})} \leq C \left\| f \right\|_{L^{\infty}(U)} \varepsilon^{\beta}.
$$

Since $u$ solves (4.2), we use the coercivity of the $B_{\mu,Q}$ norm defined in (3.4) to obtain

$$
\left\| u^{\varepsilon} - w^{\varepsilon} \right\|_{H^{-1}(U, \mu_{R})}^{2} \leq CB_{\mu_{R}, Q_{R}}[u^{\varepsilon} - w^{\varepsilon}, u^{\varepsilon} - w^{\varepsilon}]
$$

Testing (4.2) by $u^{\varepsilon} - w^{\varepsilon}$ then implies

$$
B_{\mu_{R}, Q_{R}}[u^{\varepsilon} - w^{\varepsilon}, u^{\varepsilon}] = \frac{1}{|Q_{R}|} \sum_{e \in \mathcal{E}(e_{Q_{R}})} \left\{ \left( \nabla u^{\varepsilon}(\cdot, \cdot) - \nabla w^{\varepsilon}(\cdot, \cdot) f^{\varepsilon}(\cdot) \right) \right\}_{\mu_{R}}.
$$

Therefore by the definition of $B_{\mu,Q}$ and Poincaré inequality,

$$
\left\| \nabla u^{\varepsilon} - \nabla w^{\varepsilon} \right\|_{L^{2}(U, \mu_{R})}^{2} \leq \frac{1}{|Q_{R}|} \sum_{e \in \mathcal{E}(e_{Q_{R}})} \left\{ \left( \nabla u^{\varepsilon}(\cdot, \cdot) - \nabla w^{\varepsilon}(\cdot, \cdot) f^{\varepsilon}(\cdot) \right) \right\}_{\mu_{R}} - CB_{\mu_{R}, Q_{R}}[w^{\varepsilon}, u^{\varepsilon} - w^{\varepsilon}]
$$

\begin{align*}
&\leq C \left\| u^{\varepsilon} - w^{\varepsilon} \right\|_{H^{1}(e_{Q_{R}}, \mu_{R})} \left\| \nabla^{+} \cdot (a \nabla w^{\varepsilon}) - \mathcal{L}_{\mu} w^{\varepsilon} - \nabla^{+} \cdot f^{\varepsilon} \right\|_{H^{-1}(e_{Q_{R}}, \mu_{R})} \\
&\leq C \left\| \nabla u^{\varepsilon} - \nabla w^{\varepsilon} \right\|_{L^{2}(e_{Q_{R}}, \mu_{R})} \left\| \nabla^{+} \cdot (a \nabla w^{\varepsilon}) - \mathcal{L}_{\mu} w^{\varepsilon} - \nabla^{+} \cdot f^{\varepsilon} \right\|_{H^{-1}(U, \mu_{R})}
\end{align*}

Absorbing $\left\| \nabla u^{\varepsilon} - \nabla w^{\varepsilon} \right\|_{L^{2}(e_{Q_{R}}, \mu_{R})}$ to the left side and combining with Step 1 we conclude Step 2.

**Step 3** We show that there exist $\beta(d, \lambda, \Lambda) \in (0, \frac{1}{2}]$ and $C(d, \lambda, \Lambda) < \infty$ such that,

$$
\left\| w^{\varepsilon} - u \right\|_{L^{2}(e_{Q_{R}}, \mu_{R})} + \left\| \nabla u^{\varepsilon} - \nabla u \right\|_{H^{-1}(U, \mu_{R})} \leq C \varepsilon^{\beta}.
$$

When $f^{\varepsilon}$ is a constant in $Q_{R}$, this just follows from the quantitative convergence of the corrector (Lemma 4.8). Here we simply compute the derivative of the second term in the two-scale expansion.

Indeed, we have

$$
\left\| \nabla u^{\varepsilon} - \nabla u \right\|_{H^{-1}(U, \mu_{R})} \leq C \left\{ \sum_{j=1}^{d} \chi_{e_{j}}(\cdot) \nabla_{j} u \right\}_{L^{2}(U, \mu_{R})} \leq C \left\| \nabla_{j} u \right\|_{L^{\infty}([0,1]^{2})} \sum_{j=1}^{d} \varepsilon \chi_{e_{j}}(\cdot) \left\| \nabla_{j} u \right\|_{L^{2}(e_{Q_{R}}, \mu_{R})}
$$
Applying Lemma 4.8 we conclude
\[ \| \nabla w^\varepsilon - \nabla u \|_{H^{-1}(U,\mu_R)} \leq C\varepsilon^\beta, \]
Since \( w^\varepsilon - u \in H_0^1(U,\mu_R) \), we apply Lemma A.2 to conclude
\[ \| w^\varepsilon - u \|_{L^2(\varepsilon Q_R,\mu_R)} \leq C \| \nabla w^\varepsilon - \nabla u \|_{H^{-1}(U,\mu_R)} \leq C\varepsilon^\beta. \]

**Step 4** Finally, we conclude by combining Steps 2 and 3 which yields
\[ \| \nabla u^\varepsilon - \nabla u \|_{H^{-1}(U,\mu_R)} \leq \| \nabla u^\varepsilon - \nabla w^\varepsilon \|_{H^{-1}(U,\mu_R)} + \| \nabla w^\varepsilon - \nabla u \|_{H^{-1}(U,\mu_R)} \leq C(1 + \| f \|_{L^\infty(U)}) \varepsilon^\beta. \]
Finally, the bound for \( \| u^\varepsilon - u \|_{L^2(\varepsilon Q_R,\mu_R)} \) follows from the above \( H^{-1} \) bound for the gradients and the fact that since \( u^\varepsilon - u \in H_0^1(U,\mu_R) \), we have by Lemma A.2
\[ \| u^\varepsilon - u \|_{L^2(\varepsilon Q_R,\mu_R)} \leq C \| \nabla u^\varepsilon - \nabla u \|_{H^{-1}(U,\mu_R)}. \]

5. Decoupling of the \( \nabla \phi \)-field

5.1. **Approximate harmonic coupling.** For the discrete Gaussian Free Field(GFF), there is a nice orthogonal decomposition. More precisely, the conditioned field inside the domain is the discrete harmonic extension of the boundary value to the whole domain plus an independent copy of a zero boundary discrete GFF.

While this exact decomposition does not carry over to general gradient Gibbs measures, the next result due to Jason Miller, see [20], provides an approximate version.

**Theorem 5.1** ([20]). Let \( D \subseteq \mathbb{Z}^2 \) be a simply connected domain of diameter \( R \), and denote \( D^r = \{ x \in D : \text{dist}(x,\partial D) > r \} \). Suppose that \( f : \partial D \to \mathbb{R} \) satisfies \( \max_{x \in \partial D} | f(x) | \leq 2 |\log R|^2 \). Let \( \phi \) be sampled from the Gibbs measure (3.5) on \( D \) with zero boundary condition, and let \( \phi^f \) be sampled from Gibbs measure on \( D \) with boundary condition \( f \). Then there exist constants \( c, \gamma, \delta' \in (0,1) \), that only depend on \( V \), so that if \( r > cR^\gamma \), then the following holds. There exists a coupling \( (\phi,\phi^f) \), such that if \( \tilde{\phi} : D^r \to \mathbb{R} \) is discrete harmonic with \( \tilde{\phi}|_{\partial D^r} = \phi^f - \phi|_{\partial D^r} \), then
\[ \mathbb{P}(\phi^f = \phi + \tilde{\phi} \text{ in } D^r) \geq 1 - cR^{-\delta'}. \]

Here and in the sequel of the paper, for a set \( A \subseteq \mathbb{Z}^2 \) and a point \( x \in \mathbb{Z}^2 \), we use \( \text{dist}(x,A) \) to denote the (lattice) distance from \( x \) to \( A \). Since the above theorem requires that the boundary condition \( f \) is not too large, we introduce the “good” event
\[ \mathcal{M} = \left\{ \phi : \max_{v \in \partial D} | \phi(v) | < (\log R)^2 \right\}, \]
which is typical since the Brascamp-Lieb inequality implies that \( \max_{v \in \partial D} | \phi(v) | \leq O(\log R) \) with high probability. Indeed, by applying the exponential Brascamp-Lieb inequality (3.7) and a union bound we immediately obtain

**Lemma 5.2.** There is some \( c_1 > 0 \), such that \( \mathbb{P}^{D,\emptyset}(\mathcal{M}^c) \leq \exp\left( -c_1 (\log R)^3 \right) \).

We will use repeatedly the following consequence of Theorem 5.1. It applies to functions \( \rho \) such that the integral of \( \rho \) against a harmonic function is always zero.
Lemma 5.3 ([8], Lemma 2.7). There exists constants $\delta = \delta(V), \gamma = \gamma(V) > 0$ and $C < \infty$, such that for any simply connected $D \subseteq \mathbb{Z}^2$ of diameter $R$, any $r > R^\gamma$ and any $\rho: D \to \mathbb{R}$ supported on $D^r$ that satisfies $\sum_{x \in D^r} \rho(x) f(x) = 0$ for all functions $f$ harmonic in $D^r$, and $\sum_{y \in D} |\rho(y)| < \infty$, we have for $R$ large enough,

$$
\left| E^{D,f} \left[ \exp \left( \sum_{x \in D} \rho(x) \phi^f(x) \right) 1_M \right] - E^{D,0} \left[ \exp \left( \sum_{x \in D} \rho(x) \phi(x) \right) 1_M \right] \right| \\
\leq 2 \exp \left( C \var_{G,D} \left( \sum_{x \in D} \rho(x) \phi(x) \right) \right) R^{-\delta}.
$$

We will apply Lemma 5.3 to the increment of harmonic averages of the field in $D$, defined in the section below. The increments have finite variances and thus changing the boundary to zero only gives an error of order $R^{-\delta}$.

5.2. Harmonic averages. We will apply Theorem 5.1 to study the harmonic average of the $\nabla \phi$ field. Given $B \subseteq \mathbb{Z}^2$, $v \in B$ and $y \in \partial B$, we denote by $a_B(v, \cdot)$ the harmonic measure on $\partial B$ seen from $v$. In other words, let $S^v$ denote the simple random walk starting at $x$, and $\tau_{\partial B} = \inf \{t > 0 : S(t) \in \partial B \}$, we have

$$
a_B(x, y) = \mathbb{P} \left( S^v[\tau_{\partial B}] = y \right).$$

Given $v \in \mathbb{Z}^2$ and $R > 0$, let $B_R(v) = \{ y \in \mathbb{Z}^2 : |v_1 - y_1|^2 + |v_2 - y_2|^2 < R^2 \}$. When $v = 0$ we simply write $B_R(v)$ as $B_R$. Define the circle average of the field with radius $R$ at $v$ by

$$
C_R(v, \phi) = \sum_{y \in \partial B_R(v)} a_{B_R(v)}(v, y) \phi(y).
$$

We introduce the geometric scales in order to carry out the multiscale argument to prove Lemma 2.1 and 2.3. Let $\gamma = \gamma(V)$ be the constant in Theorem 5.1, define the sequence of numbers $\{r_k\}_{k=1}^\infty$, $\{r_{k,\cdot}\}_{k=0}^\infty$ and $\{r_{k,-}\}_{k=0}^\infty$ by

$$
\begin{align*}
r_k &= e^{-k} N, \\
r_{k,\cdot} &= 1 + r_{k,-} \gamma r_k, \\
r_{k,-} &= 1 - r_{k,-} \gamma r_k.
\end{align*}
$$

We also define

$$
\begin{align*}
X_{r_{k,\cdot}}(v) &= \sum_{r=(1+\frac{1}{2}\gamma r_{k,-})r_{k,\cdot}}^{(1+\frac{1}{2}\gamma r_{k,-})r_{k,\cdot}} \left( \frac{1}{2} r_{k,-} \right)^{-1} C_{r}(v, \phi), \\
X_{r_{k,-}}(v) &= \sum_{r=(1-\frac{1}{2}\gamma r_{k,-})r_{k,-}}^{(1-\frac{1}{2}\gamma r_{k,-})r_{k,-}} \left( \frac{1}{2} r_{k,-} \right)^{-1} C_{r}(v, \phi), \\
X_r(v) &= \sum_{r=(1-\frac{1}{2}\gamma r_{k,-})r_k}^{(1+\frac{1}{2}\gamma r_{k,-})r_k} \left( \frac{1}{2} r_{k,-} \right)^{-1} C_{r}(v, \phi),
\end{align*}
$$

where $C_r$ is defined in (5.1).

We also set the increment process $A_k := X_{r_{k+1}} - X_{r_{k-}}$ and the boundary layer error $E_k := X_r - X_{r_{k-}}$. 
Notice that the harmonic average process $X$ defined above is slightly different from the one in [8], Section 3. Namely, the width of the harmonic average is thinner (of the order $r_k^{1-\gamma}$ instead of $r_k$). The increment of the harmonic average process, $A_k$, is crucial to the multiscale argument below. We may write

\[(5.3)\]

$$A_k(v, \phi) = \left( (1 + \frac{1}{2} 1^{1-\gamma})_{r_k} \right) \left( \frac{1}{2} r_{k+1}^{1-\gamma} \right) \left( \sum_{r=\frac{1}{2} r_{k+1}}^{1} \right) \left( \sum_{r=\frac{1}{2} r_{k-1}}^{1} \right) \sum_{y \in \partial B_r(v)} a_{B_r(v)}(v, y) \phi(y).$$

This can be written as $\sum_{y \in B_{r_k}(v)} \rho_{r_k}(v, y) \phi(y)$, where we define

\[(5.4)\]

$$\rho_{r_k}(v, y) = \left( (\frac{1}{2} r_{k+1}^{1-\gamma}) \left( \sum_{r=\frac{1}{2} r_{k+1}}^{1} \right) \left( \sum_{r=\frac{1}{2} r_{k-1}}^{1} \right) \right) a_{B_{r_k}(v)}(v, y).$$

Here we omit the dependence of $r$ in the definition of $\rho_{r_k}$.

We now state two important properties of the increment process $A_k$. They are consequences of Theorem 5.1, and the proof are essentially the same as Lemma 3.1 and 3.2 of [8].

**Lemma 5.4.** For any $k \in \mathbb{N}$, and any discrete harmonic function $h$ in $B_{r_k}$, we have $\sum_{y \in B_{r_k}(v)} \rho_r(v, y) h(y) = 0$.

**Proof.** Denote $B_{r_k}$ by $D$. Suppose $h$ takes the boundary value $h|_{\partial D} = H$. We conclude the proof by showing for any $r \geq 1$

$$\sum_{y \in \partial B_r(v)} a_{B_r(v)}(v, y) h(y) = h(v).$$

Indeed, since $h$ is harmonic,

$$h(y) = \sum_{z \in \partial D} a_D(y, z) H(z).$$

Using the fact that

$$\sum_{y \in \partial B_r(v)} a_{B_r(v)}(v, y) a_D(y, z) = a_D(v, z),$$

we obtain

$$\sum_{y \in \partial B_r(v)} a_{B_r(v)}(v, y) h(y) = \sum_{z \in \partial D} a_D(v, z) H(z) = h(v).$$

\[\square\]

The following lemma is a consequence of Theorem 5.1 and the lemma above.

**Lemma 5.5.** Suppose the same conditions in Theorem 5.1 holds. Let $\delta$ be the constant from Theorem 5.1. Let $\phi^f$ be sampled from gradient field (3.5) on $B_{r_k}$ with boundary condition $f$, and $\phi^0$ be sampled from the zero boundary gradient field on $B_{r_k}$. Then, on an event with probability $1 - O(r_k^{-\delta})$, we have

$$A_k(v, \phi^f) = A_k(v, \phi^0).$$
5.3. **Proof of Lemma 2.1 and Lemma 2.3.** A key ingredient for the characteristic function estimates (Lemma 2.1 and Lemma 2.3) is the decoupling estimates Proposition 5.6 below.

For \( r > 0 \) denote by \( \mathbb{P}^{r,0} \) the law of the Ginzburg-Landau field in \( B_r(v) \) with zero boundary condition (and denote by \( \mathbb{E}^{r,0} \) the corresponding expectation). The basic building block of Lemma 2.1 and 2.3 is the following.

**Proposition 5.6.** There exists \( \delta = \delta(V) > 0 \) and \( C = C(d, \lambda, \Lambda) < \infty \), such that for all \( |s|^2 < \min\{\delta \log N - k - 1, \frac{1}{2}\} \) and \( k \leq N - \sqrt{\log N} \),

\[
(5.5) \quad \log (\exp(isX_{r_k}))_{\mu_N} = \sum_{j=1}^{k} \log \mathbb{E}^{r,j,0} [\exp(isA_j)] + \log (\exp(isX_{r_0}))_{\mu_N} + O\left( \sum_{j=1}^{k} e^{C_2^2 k r^{-\delta}} \right).
\]

**Proof.** We gave an inductive proof, by running an induction jointly with (5.5) and (5.6)

\[
(5.6) \quad \langle \exp(isX_{r_k}) \rangle_{\mu_N} \geq e^{-C s^2 k}
\]

for some \( C = C(d, \lambda, \Lambda) < \infty \). For the base case \( k = 0 \), (5.5) trivially holds. To see (5.6) holds for \( k = 0 \), we Taylor expand the exponential and apply the Brascamp-Lieb inequality to obtain

\[
\langle \exp(isX_{r_k}) \rangle_{\mu_N} \geq 1 - \frac{s^2}{2} \var_{\mu_N} X_{r_0} \geq 1 - C s^2 \var_{\mu_G, Q_N} X_{r_0} \geq e^{-C s^2}
\]

To keep iterating, we denote \( \mathcal{M}_k := \{ \max_{x \in B_k} |\phi(x)| \leq (\log r_k)^2 \} \). By Lemma 5.7, \( \mathbb{P}^{N,0}[\mathcal{M}^c_k] \leq N^{-c_1} \). On the event \( \mathcal{M}_k \), we may write

\[
(5.7) \quad \langle \exp(isX_{r_k}) \1_{\mathcal{M}_k} \rangle_{\mu_N} = \langle \exp(isX_{r_{k-1}}) \mathbb{E}[\exp(isA_{k-1}) \exp(is\mathcal{E}_{k-1}) | \mathcal{F}_{k-1}] \1_{\mathcal{M}_k} \rangle_{\mu_N}
= \langle \exp(isX_{r_{k-1}}) \mathbb{E}[\exp(isA_{k-1}) | \mathcal{F}_{k-1}] \1_{\mathcal{M}_k} \rangle_{\mu_N}
+ \langle \exp(isX_{r_{k-1}}) \exp(isA_{k-1}) (\exp(is\mathcal{E}_{k-1}) - 1) \1_{\mathcal{M}_k} \rangle_{\mu_N}
\]

We conclude using Lemma 5.3 that

\[
\mathbb{E}[\exp(isA_{k-1}) | \mathcal{F}_{k-1}] = \mathbb{E}^{r_{k-1},0}[\exp(isA_{k-1})] + R_{k-1}
\]

such that with probability \( 1 - \mathbb{P}^{N,0}[\mathcal{M}^c_k] \), there is some \( C < \infty \) and \( \delta_1 > 0 \) such that \( |R_{k-1}| \leq 2 \mathbb{E}^{r_{k-1},0}[\exp(isA_{k-1})] r_{k-1}^{-\delta_1} \leq C r_{k-1}^{-\delta_1} \), and on an event with probability \( \mathbb{P}^{N,0}[\mathcal{M}^c_k] \), \( |R_{k-1}| \) is bounded by 1. This implies, in particular, \( \langle |R_{k-1}|^2 \rangle_{\mu_N} \leq C r_{k-1}^{-\delta_1} \). For the last term in (5.7), applying the Brascamp-Lieb inequality yields

\[
\langle |\exp(is\mathcal{E}_{k-1}) - 1| \rangle_{\mu_N} \leq |s| \langle |\mathcal{E}_{k-1}| \rangle_{\mu_N} \leq \left( 2 \mathbb{E}^{r_{k-1},0}[\exp(isA_{k-1})] + O(r_{k-1}^{-\delta_1}) \right) \leq C r_{k-1}^{-\gamma/2}.
\]

The two estimates above implies that for every \( k \in \mathbb{N} \), we have

\[
(5.8) \quad \langle \exp(isX_{r_k}) \1_{\mathcal{M}_k} \rangle_{\mu_N} = \langle \exp(isX_{r_{k-1}}) \1_{\mathcal{M}_k} \rangle_{\mu_N} \mathbb{E}^{r_{k-1},0}[\exp(isA_{k-1})] + O(r_{k-1}^{-\delta_1/2}) + O(r_{k-1}^{-\gamma/2}).
\]

Denote \( \min\{\delta_1/2, \gamma/2\} \) as \( \delta \). We now apply the induction hypothesis and the fact that

\[
\mathbb{E}^{r_{k-1},0}[\exp(isA_{k-1})] \geq 1 - \frac{s^2}{2} \var_{\mathcal{D}_{r_{k-1}}} A_{k-1} \geq 1 - C s^2 \var_{\mathcal{D}_{r_{k-1}}} A_{k-1} \geq e^{-C s^2}
\]
to obtain for some $C < \infty$,

$$(\exp(isX_{r_{k}}) 1_{M_{k}})_{\mu_{N}} \geq e^{-Cs^{2}(k+1)} + O(r_{k-1}^{-\delta})$$

therefore (5.6) follows from the fact that $e^{-Cs^{2}(k+1)} > r_{k-1}^{-\delta/2}$, which is implied by the condition $|s|^{2} < \min\left(\frac{\ln N - k - 1}{C_{k}}, \frac{1}{2}\right)$, and $\mathbb{P}^{N,0}[\mathcal{M}_{k}^{c}] \leq N^{-c_{1}}$. Finally, by taking the logarithm of (5.8) we obtain (5.5) for the case $k = 1$.

\[\square\]

**Lemma 5.7.** There exists $c_{1} > 0$, such that for all $k \leq \log N - \sqrt{\log N}$, $\mathbb{P}^{N,0}[\mathcal{M}_{k}^{c}] \leq N^{-c_{1}}$.

**Proof.** By making a union bound and applying the exponential Brascamp-Lieb (3.7) and the Chebyshev inequality, we have for all $t \in \mathbb{R}$,

$$\mathbb{P}^{N,0}[\mathcal{M}_{k}^{c}] \leq \sum_{x \in B_{k}} \mathbb{P}^{N,0}[\phi(x) > (\log r_{k})^{2}] \leq r_{k}^{2}e^{-t(\log r_{k})^{2}}e^{Ct^{2}\log N}.$$ 

Optimize over $t$, and use the fact that $\log r_{k} \geq (\log N)^{1/2}$, we see that there exists $c_{1} > 0$, such that for $N$ sufficiently large,

$$\mathbb{P}^{N,0}[\mathcal{M}_{k}^{c}] \leq r_{k}^{2}e^{-\frac{2c_{1}(\log r_{k})^{2}}{\log N}} \leq N^{-c_{1}}$$

\[\square\]

We also need an algebraic rate convergence of the variance of $A_{k}$ stated below.

**Lemma 5.8.** There exists $g = g(V) > 0$ and $\beta = \beta(d, \lambda, \Lambda) > 0$, such that for all $j = 1, \ldots, \log N$,

$$\mathbb{E}^{r_{j},0}[A_{j}^{2}] = g + O(r_{j}^{-\beta}).$$

**Proof.** Recall from (5.3) that, we may write $A_{j} = \sum_{x \in B_{r_{j}}} \phi(x)\rho_{r_{j}}(x)$, where $\rho_{r_{j}}$ is defined in (5.4). Denote by $G_{r_{j}}(x, \cdot)$ the Dirichlet Green’s function in $B_{r_{j}}$. We may use the integration by parts $\phi(x) = \sum_{x \in E(B_{r_{j}})} \nabla\phi(e)\nabla G_{r_{j}}(x, e)$ to write $A_{j}$ as

$$A_{j} = \sum_{x \in E(B_{r_{j}})} \nabla\phi(e) \sum_{x \in B_{r_{j}}} \nabla G_{r_{j}}(x, e)\rho_{r_{j}}(x)$$

In order to apply Theorem 4.1, define

$$f_{r_{j}}(e) := r_{j} \sum_{x \in B_{r_{j}}} \nabla G_{r_{j}}(x, e)\rho_{r_{j}}(x)$$

Thus $\nabla^{*} f_{r_{j}}(x) = r_{j} \rho_{r_{j}}(x)$ and $A_{j} = r_{j}^{-1} \sum_{x \in E(B_{r_{j}})} \nabla\phi(e) f_{r_{j}}(e)$. Notice that, as $r \to \infty$, the rescaled harmonic measure

$$ra_{B_{r}}(0, \cdot) \to 1/2\pi.$$ 

Thus as $r \to \infty$, $r\rho_{r}$ converges to

$$f(x) := \frac{1}{2\pi r_{j+1}} 1_{|x| = r_{j+1}} - \frac{1}{2\pi r_{j-1}} 1_{|x| = r_{j-1}}$$

and that $\|r\rho_{r} - f\|_{L^{\infty}(r^{-1}B_{r})} = O(r^{-1})$.

Applying Theorem 4.1 using that $\sup_{j \geq 1} r_{j}^{1/2}\rho_{r_{j}} \|_{L^{\infty}} < \infty$, there exists $g = g(V) > 0$ and $\beta = \beta(d, \lambda, \Lambda) > 0$, such that

$$|\mathbb{E}^{r_{j},0}[A_{j}^{2}] - g| \leq Cr_{j}^{-\beta}.$$
We apply Lemma 5.3 and Lemma 5.7 to conclude

\[ (\exp(is\phi(0)))_{\mu_N} = \left\{ E \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) | F_{k_2} \right] \exp(isX_{r_{k_2}}) \exp(isE_{k_2}) \right\}_{\mu_N} \]

We apply Lemma 5.3 and Lemma 5.7 to conclude

\[ E \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) | F_{k_2} \right] = E \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) 1_{M_{k_2}} | F_{k_2} \right] 
+ E \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) 1_{M_{k_2}^c} | F_{k_2} \right] = E_{r_{k_2}^0} \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) \right] + R_{k_2}, \]

where, for some \( \delta = \delta(V) \in (0, \frac{1}{2}) \), \( \left\{ R_{k_2} \right\}_{\mu_N} \leq r_{k_2}^{-\delta} \leq e^{-\delta \log N}. \) Thus

\[ (\exp(is\phi(0)))_{\mu_N} = E_{r_{k_2}^0} \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) \right] \left( \exp(isX_{r_{k_2}}) \right)_{\mu_N} + \left( \exp(isE_{k_2}) - 1 \right) \exp(is\phi(0) - isX_{r_{k_2}^c}) \exp(isX_{r_{k_2}}) \left( \exp(is\phi(0)) \right)_{\mu_N} \]

The first term on the right side gives the main contribution. We claim that there exists \( g = g(V) > 0 \) and \( \beta = \beta(V) > 0 \), such that

\[ (\exp(isX_{r_{k_2}}))_{\mu_N} = e^{-\frac{t^2}{2g}} \left( 1 + O \left( \frac{t^2}{(\log N)^{\frac{1}{2}}} \right) \right) \]

together with the Brascamp-Lieb inequality, which implies

\[ 1 \geq E_{r_{k_2}^0} \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) \right] \geq 1 - \frac{s^2}{2} \mathrm{var}_{\mu_{r_{k_2}^c}} [\phi(0) - X_{r_{k_2}^c}] \geq 1 - \frac{C_{1}s^2}{2} \mathrm{var}_{\mu_{G,B_{r_{k_2}^c}}} [\phi(0) - X_{r_{k_2}^c}] \]
\[ \geq 1 - C_2 \frac{t^2}{(\log N)^{\frac{1}{2}}} \]

Therefore

\[ E_{r_{k_2}^0} \left[ \exp(is\phi(0) - isX_{r_{k_2}^c}) \right] \left( \exp(isX_{r_{k_2}}) \right)_{\mu_N} = e^{-\frac{t^2}{2g}} \left( 1 + O \left( \frac{t^2}{(\log N)^{\frac{1}{2}}} \right) \right) \]

To show (5.11), we apply Proposition 5.6 with \( s = \frac{t}{\sqrt{\log N}} \) (thus \( s^2 \ll \frac{\delta(\log N - k - 1)}{C_k} = O((\log N)^{-3/2}) \)), which yields

\[ \log \left( \exp(isX_{r_{k_2}}) \right)_{\mu_N} = \sum_{j=1}^{k_2} \log E_{r_j}^{0} \left[ \exp(isA_j) \right] + \log \left( \exp(isX_{r_0}) \right)_{\mu_N} + O \left( \sum_{j=1}^{k_2} e^{Cs^2k_{r_{k_2}^{-\delta}}} \right). \]

Note that \( \sum_{j=1}^{k_2} e^{Cs^2k_{r_{k_2}^{-\delta}}} \leq C_{r_{k_2}^{-\delta}} \leq C e^{-\delta \sqrt{\log N}}. \) We also apply the Brascamp-Lieb inequality to see that

\[ \left| E_{r_j}^{0} \left[ \exp(isA_j) \right] - 1 - \frac{s^2}{2} E_{r_j}^{0} \left[ A_j^2 \right] \right| \leq C_s^4 E_{r_j}^{0} \left[ A_j^4 \right] \leq \frac{Ct^4}{(\log N)^2} E_{G,B_{r_j}} \left[ A_j^4 \right] = O \left( \frac{t^4}{(\log N)^2} \right) \]
Thus
\[ \sum_{j=1}^{k_2} \log \mathbb{E}^{r_j,0} \left[ \exp(isA_j) \right] = \sum_{j=1}^{k_2} -\frac{s^2}{2} \mathbb{E}^{r_j,0} \left[ A_j^2 \right] + O\left( \frac{t^4}{\log N} \right) \]
We apply Lemma 5.8 to conclude that there exists \( g = g(V) > 0 \) and \( \beta = \beta(d, \lambda, \Lambda) > 0 \), such that
\[ \mathbb{E}^{r_j,0} \left[ A_j^2 \right] = g + O(r_j^{-\beta}) , \]
this yields
\[ \sum_{j=1}^{k_2} -\frac{s^2}{2} \mathbb{E}^{r_j,0} \left[ A_j^2 \right] = -\frac{t^2 \log N - (\log N)^\frac{1}{2}}{2g} \log N + O\left( e^{-\beta \sqrt{\log N}} \right) , \]
and by the Brascamp-Lieb inequality, which gives \( \log \{ \exp (isX_{k_1}) \} = O\left( \frac{1}{\log N} \right) \), we conclude (5.11).

The other terms on the right side of (5.10) can be estimated by
\[ \left| \left( R_{k_2} \exp(isX_{k_2}) \right) \right| \leq e^{-\delta \sqrt{\log N}} , \]
and
\[ \left\{ \left( \exp(is\mathcal{E}_{k_2}) - 1 \right) \exp(is\phi(0) - isX_{k_2} \cdot) \exp(isX_{k_2}) \right\} = \frac{s^2}{2} \left\{ \mathcal{E}_{k_2}^2 \right\} \leq \frac{C t^2}{2 \log N} \left\{ \mathcal{E}_{k_2}^2 \right\} \leq O\left( \frac{1}{\log N} \right) \]
Substitutes the estimates above into (5.10) we conclude Lemma 2.1.

\[ \square \]

**Proof of Lemma 2.3.** The proof is very similar to Lemma 2.1 and thus we give a sketch here. We apply Proposition 5.6 and stop at \( k_1 := \frac{1}{2} \log N \). By conditioning,

(5.13)
\[ \left\{ \exp(is\phi(0)) \right\} = \left\{ \mathbb{E} \left[ \exp(is\phi(0) - isX_{k_1} \cdot) \big| \mathcal{F}_{k_1} \right] \exp(isX_{k_1}) \exp(is\mathcal{E}_{k_1}) \right\} \]
We then apply Lemma 5.3 and Lemma 5.7 to obtain

(5.14)
\[ \left\{ \exp(is\phi(0)) \right\} = \mathbb{E}^{r_{k_1},0} \left[ \exp(is\phi(0) - isX_{k_1} \cdot) \right] \left\{ \exp(isX_{k_1}) \right\} + \left\{ R_{k_1} \exp(isX_{k_1}) \right\} + \left\{ \left( \exp(is\mathcal{E}_{k_1}) - 1 \right) \exp(is\phi(0) - isX_{k_1} \cdot) \exp(isX_{k_1}) \right\} \]
where, for some \( \delta = \delta(V) \in (0, \frac{1}{2}) \), \( \left\{ R_{k_1} \right\} \leq r_{k_1}^{-\delta} \leq N^{-\delta/2} \). Thus,
\[ \left| \left( R_{k_1} \exp(isX_{k_1}) \right) \right| \leq N^{-\delta/2} , \]
and by the Brascamp-Lieb inequality,
\[ \left\{ \left( \exp(is\mathcal{E}_{k_1}) - 1 \right) \exp(is\phi(0) - isX_{k_1} \cdot) \exp(isX_{k_1}) \right\} \leq \frac{s^2}{2} \left\{ \mathcal{E}_{k_1}^2 \right\} = O\left( N^{-\gamma} \right) \]
Let \( \delta > 0 \) and \( C < \infty \) be the constants from Proposition 5.6, and we take \( \varepsilon \) sufficiently small so that \( \varepsilon^2 \leq \frac{C}{2\varepsilon} \). For the first term on the right side of (5.14),
which is absolutely bounded by \(|(\exp(isX_{r_{k_1}}))_{\mu_N}|\), we apply Proposition 5.6 (with \(|s| < \varepsilon\)) to obtain

\[
\log\left(\exp\left(isX_{r_{k_1}}\right)\right)_{\mu_N} = \sum_{j=1}^{k_1} \log E^{r_j,0}[\exp(isA_j)] + \log(\exp(isX_{r_0}))_{\mu_N} + O\left(\sum_{j=1}^{k_1} e^{Cs^2k_1r_j^{-\delta}}\right).
\]

Since \(s^2 \leq \frac{\delta}{2C}\), we have

\[
\sum_{j=1}^{k_1} e^{Cs^2k_1r_j^{-\delta}} \leq r_{k_1}^{-\delta/4} \leq N^{-\delta/8}.
\]

Notice that, since the distribution of \(A_j\) is symmetric for the zero boundary field,

\[
E^{r_j,0}[\exp(isA_j)] \leq 1 - \frac{s^2}{2} E^{r_j,0}[A_j^2] + \frac{s^4}{24} E^{r_j,0}[A_j^4]
\]

We apply the Brascamp-Lieb inequality to conclude, there exists an absolute constant \(M < \infty\), such that

\[
\max_{j=1,\ldots,k_1} E^{r_j,0}[A_j^4] \leq C \max_{j=1,\ldots,k_1} E^{G,B_{r_j}}[A_j^4] \leq M
\]

On the other hand, applying Lemma 5.8 to conclude that there exists \(g = g(V) > 0\) and \(\beta = \beta(d, \lambda, \Lambda) > 0\), such that for each \(j = 1, \ldots, k_1\),

\[
E^{r_j,0}[A_j^2] = g + O(N^{-\beta}),
\]

Thus, by choosing \(\varepsilon > 0\) such that \(\varepsilon^2 = \min\{\frac{g}{2M}, \frac{\delta}{2C}, \frac{3\gamma}{2g}, \frac{3\delta}{4g}\}\), we see that for all \(|s| < \varepsilon|\)

\[
E^{r_j,0}[\exp(isA_j)] \leq 1 - \frac{s^2}{3} g
\]

Thus for \(N\) sufficiently large,

\[
\left(\exp\left(isX_{r_{k_1}}\right)\right)_{\mu_N} \leq \left(1 + O(N^{-\frac{\delta}{8}})\right) \prod_{j=1}^{k_1} \left(1 - \frac{s^2}{4} g\right) + O(N^{-\frac{\delta}{2}} + N^{-\gamma}) \leq 2e^{-\frac{s^2}{4} g \log N}
\]

Substitutes these estimates into (5.14) we conclude the Lemma.

\[
\square
\]

Remark 5.9. Notice that if we aims for a weaker bound (2.4), then instead of applying 5.8 in the proof above, we only need a uniform lower bound, namely,  \(E^{r_j,0}[A_j^2] \geq c_1\) for some \(c_1 > 0\). This can be proved, for example, by using the Mermin-Wagner argument as we did in the next section.

6. A Mermin-Wagner bound

In this section we prove Lemma 2.5. The upper bound (2.5) is obtained by using the following Lemma.

Lemma 6.1. Let \(X\) be a random variable taking values on a unit circle, and \(f: [0,2\pi] \to [0,1]\) be its density function. Suppose that \(\forall a \in [0,2\pi], \) we have

\[
f\left(a + \frac{1}{2}\right) f\left(a - \frac{1}{2}\right) \geq c f(a)^2
\]

(6.1)
Then the random variable has a bounded density on the circle, and \( f^2 e^{i\theta} f(\theta) d\theta \leq 1 - \varepsilon \). Moreover, if there exist \( t > 1 \) and \( C < \infty \), such that \( \forall a, b \in [0, 2\pi) \),

\[
(6.2) \quad f(a + b) f(a - b) \geq e^{-C/t^2} f(a)^2
\]

then the characteristic function bound can be improved to \( f^2 e^{i\theta} f(\theta) d\theta \leq C/t^r \).

We first prove Lemma 2.5 based on Lemma 6.1. The argument presented below follows closely a Mermin-Wagner type estimate which has been done in e.g., [19, 24, 23]. Denote by \( x_k = (2^k, 0) \) for \( k = 0, \ldots, \log N \). And

\[
(6.3) \quad \mathcal{F}_k := \sigma(\phi(x) - \phi(y) : |x_1| + |x_2| = 2^k \text{ and } |y_1| + |y_2| = 2^k).
\]

In other words, \( \mathcal{F}_k \) specifies all the gradients of the field on the boundary of a diamond of radius \( 2^k \). A key observation is that conditioned on \( \mathcal{F}_k \), the gradients of the field inside the region \( |x_1| + |x_2| < 2^k \) and the gradients in the region \( |x_1| + |x_2| > 2^k \) are independent. By progressively conditioning the gradients on the layers \( \mathcal{F}_k \), we have

\[
(6.4) \quad \langle \exp(is\phi(0)) \rangle_{\mu_N} = \prod_{k=0}^{\log N} \mathbb{E}\left[\exp(is(\phi(x_k) - \phi(x_{k-1})) | (\mathcal{F}_k)_{0 \leq k \leq \log N})\right]_{\mu_N}
\]

Thus (2.5) follows if we can show there exist \( \varepsilon > 0 \) and \( C < \infty \), such that \( \forall k = 1, \ldots, \log N \), and \( \forall t > 1 \),

\[
(6.5) \quad E\left[\exp(it(\phi(x_k) - \phi(x_{k-1})) | (\mathcal{F}_k)_{0 \leq k \leq \log N})\right] \leq \min\{1 - \varepsilon, \frac{C}{t^2}\}.
\]

We show (6.5) using a Mermin-Wagner type argument. Define the deformation \( \tau(x) := \begin{cases} \frac{|x_1| + |x_2|}{2^k}, & 2^{k-1} \leq |x_1| + |x_2| \leq 2^k, \\ 0, & |x_1| + |x_2| \geq 2^k \end{cases} \)

which interpolates between \( b \) and \( 0 \), and does not change the gradients for all \( (\mathcal{F}_k)_{0 \leq k \leq \log N} \). We also notice that a straightforward calculation yields \( \sum_{e \in \mathcal{E}(Q_N)} (\nabla \tau(e))^2 \leq Cb^2 \) for some \( C < \infty \).

Let \( \phi^+ := \phi + \tau \) and \( \phi^- := \phi - \tau \). We see the densities of \( \phi^+ \) and \( \phi^- \) (conditioning on all the gradients \( (\mathcal{F}_k)_{0 \leq k \leq \log N} \)) satisfies

\[
(6.7) \quad g(\phi^+) g(\phi^-) = \frac{1}{Z_N^2} \exp\left(-\sum_{(x,y) \in \mathcal{E}(Q_N)} V(\phi(x) - \phi(y) + \tau(x) - \tau(y))\right) \\
\cdot \exp\left(-\sum_{(x,y) \in \mathcal{E}(Q_N)} V(\phi(x) - \phi(y) - \tau(x) + \tau(y))\right) \\
\geq \frac{1}{Z_N^2} \exp\left(-2 \sum_{(x,y) \in \mathcal{E}(Q_N)} V(\phi(x) - \phi(y)) - \frac{1}{2} \sup_{x \in \mathbb{R}} V''(x) \sum_{e \in \mathcal{E}(Q_N)} (\nabla \tau(e))^2\right) \geq e^{-Cb^2} g(\phi)^2
\]

for some \( c > 0 \).
For any set $E \subseteq \Omega_0(Q_N)$ of field configurations, integrate the above inequality of densities over $E$ and applying the Cauchy-Schwarz inequality, we obtain

$$\mathbb{P}(\phi^+ \in E | (\mathcal{F}_k)_{0 \leq k \leq \log N}) \cdot \mathbb{P}(\phi^- \in E | (\mathcal{F}_k)_{0 \leq k \leq \log N}) \geq e^{-Ct^2} \mathbb{P}(\phi \in E | (\mathcal{F}_k)_{0 \leq k \leq \log N})^2.$$ 

Set $t = 1$ and $b = \frac{1}{2}$. If we denote by $f$ the conditional density of $\exp(i(\phi(x_k) - \phi(x_{k-1}))$ on the unit circle (conditioning on all the gradients $(\mathcal{F}_k)_{0 \leq k \leq \log N}$), then it follows that (6.1) is satisfied (for $t = 1$).

To see that (6.2) holds for $t > 1$, notice the fact that for $t$ large, we may rescale $\phi \to t\phi$, so that the corresponding potential is given by $V(\hat{\gamma})$, which has second derivative of order $t^{-2}$. Thus following the same argument as above, one obtains an improved bound for the conditional densities (for large $t$),

$$\mathbb{P}(\phi^+ \in E | (\mathcal{F}_k)_{0 \leq k \leq \log N}) \cdot \mathbb{P}(\phi^- \in E | (\mathcal{F}_k)_{0 \leq k \leq \log N}) \geq e^{-Ct^2} \mathbb{P}(\phi \in E | (\mathcal{F}_k)_{0 \leq k \leq \log N})^2.$$ 

Therefore by applying Lemma 6.1 we conclude that

$$\mathbb{E}[\exp(it(\phi(x_k) - \phi(x_{k-1})) | (\mathcal{F}_k)_{0 \leq k \leq \log N}] \leq \min\{1 - \varepsilon, \frac{C}{t^2}\}$$

And by the conditioning (6.4) we conclude Lemma 2.5.

Finally we give a proof of Lemma 6.1.

Proof of Lemma 6.1. We focus on the proof of $\int_0^{2\pi} e^{i\theta} f(\theta) d\theta \leq \frac{C}{t^2}$ as the $1 - \varepsilon$ bound is a classical result and its proof can be found in [19, 24, 23]. Using (6.2), we see that the probability density has a ratio bounded by $e^{C/t^2}$ uniformly over the circle.

Thus for all $x \in [0, 2\pi)$ and any interval $I \subseteq [0, 2\pi)$ of length at most $\pi$,

$$\left(\int_I f(\theta) d\theta\right)^2 e^{-\frac{C}{t^2}} \leq \int_I f(\theta + x) f(\theta - x) d\theta \leq \int_I f(\theta + x) e^{\frac{C}{t^2}} f(\theta - x) d\theta\leq e^{\frac{C}{t^2}} \int_{I+x} f(\theta) d\theta \int_{I-x} f(\theta) d\theta,$$

thus

$$\left(\int_I f(\theta) d\theta\right)^2 e^{-\frac{C}{t^2}} \leq \left(\int_{I+x} f(\theta) d\theta\right) \left(\int_{I-x} f(\theta) d\theta\right).$$

In particular, fix $t \in \mathbb{R}$, by taking $I = [(k-1)\pi/m, k\pi/m)$, for some $m \in \mathbb{N}$, and $k = 1, \ldots, 2m$ we have

$$\left(\max_{k=1, \ldots, 2m} \int_{(k-1)\pi/m}^{k\pi/m} f(\theta) d\theta\right)^2 \leq e^{\frac{2C}{t^2}} \left(\min_{k=1, \ldots, 2m} \int_{(k-1)\pi/m}^{k\pi/m} f(\theta) d\theta\right) \left(\max_{k=1, \ldots, 2m} \int_{(k-1)\pi/m}^{k\pi/m} f(\theta) d\theta\right).$$

Thus

$$\left(\max_{k=1, \ldots, 2m} \int_{(k-1)\pi/m}^{k\pi/m} f(\theta) d\theta\right) \leq e^{\frac{2C}{t^2}} \left(\min_{k=1, \ldots, 2m} \int_{(k-1)\pi/m}^{k\pi/m} f(\theta) d\theta\right).$$
Therefore for each $k$, we have

\[
\left| \int_{\pi \frac{k\pi}{m}}^{\pi \frac{(k+1)\pi}{m}} e^{i\theta} f(\theta) d\theta + \int_{\pi \frac{(k+1)\pi}{m}}^{\pi \frac{2(k+1)\pi}{m}} e^{i\theta} f(\theta) d\theta \right| \leq \left| e^{i\frac{k\pi}{m}} \int_{\pi \frac{k\pi}{m}}^{\pi \frac{(k+1)\pi}{m}} f(\theta) d\theta - e^{i\frac{k\pi}{m}} \int_{\pi \frac{(k+1)\pi}{m}}^{\pi \frac{2(k+1)\pi}{m}} f(\theta) d\theta \right| \\
+ \max_{\theta \in \left[\pi \frac{(k+1)\pi}{m}, \pi \frac{2(k+1)\pi}{m}\right]} \left| e^{i\frac{(k+1)\pi}{m}} - e^{i\frac{k\pi}{m}} \left| \int_{\pi \frac{k\pi}{m}}^{\pi \frac{(k+1)\pi}{m}} f(\theta) d\theta + \int_{\pi \frac{(k+1)\pi}{m}}^{\pi \frac{2(k+1)\pi}{m}} f(\theta) d\theta \right| \right|
\]

\[
\leq \left( 1 - e^{-\frac{2C}{l^2}} + O\left( \frac{1}{m} \right) \right) \left| \int_{\pi \frac{k\pi}{m}}^{\pi \frac{(k+1)\pi}{m}} f(\theta) d\theta + \int_{\pi \frac{(k+1)\pi}{m}}^{\pi \frac{2(k+1)\pi}{m}} f(\theta) d\theta \right| + O\left( \frac{1}{m^4l^2} \right)
\]

Summing over $k$ yields

\[
\left| \int_{0}^{2\pi} e^{i\theta} f(\theta) d\theta \right| \leq 1 - e^{-\frac{2C}{l^2}} + O\left( \frac{1}{m} \right)
\]

by taking $m > \frac{1}{e} l^2$ we conclude the lemma. \hfill \Box

**Appendix A. Multiscale Poincare inequality,**

We state a discrete version of the multiscale Poincare inequality, which provides an estimate of the $H^{-1}(\Box_n)$ norm of a function in terms of its spatial averages in triadic subcubes.

**Proposition A.1 ([3, 5]).** (Multiscale Poincare inequality) Let $Z_n = 3^n \mathbb{Z}^2 \cap \Box_m$. Then

\[
\|f\|_{H^{-1}(\Box_m)} \leq C \|f\|_{L^2(\Box_m)} + C \sum_{n=0}^{m-1} \left( \frac{1}{|Z_n|} \sum_{y \in Z_n} |(f)_{y + \Box_n}|^2 \right)^{1/2}.
\]

We also record a lemma here that the $H^{-1}$ norm of $\nabla u$ can bound the $L^2$ oscillation of $u$.

**Lemma A.2.** There exists $C(d) < \infty$ such that for every $m \in \mathbb{N}$ and $u \in H^1(\Box_m, \mu_N)$,

\[
\|u - (u)_{\Box_m}\|_{L^2(\Box_m, \mu_N)} \leq C \|\nabla u\|_{H^{-1}(\Box_m, \mu_N)}
\]

And, for every $u \in H_0^1(\Box_m, \mu_N)$,

\[
\|u\|_{L^2(\Box_m, \mu_N)} \leq C \|\nabla u\|_{H^{-1}(\Box_m, \mu_N)}
\]

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