Late-Time Mild Inflation

— a possible solution of dilemma: cosmic age and the Hubble parameter —

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Abstract

We explore the cosmological model in which a late-time mild inflation is realized after the star formation epoch. Non-vanishing curvature coupling of a classical boson field yields this mild inflation without a cosmological constant. Accordingly the lifetime of the present Universe is remarkably increased in our model. Thus we show that the present observed high value of the Hubble parameter $H_0 \approx 70-80 \text{km/sec/Mpc}$ is compatible with the age of the oldest stars 14Gyr without introducing the cosmological constant or the open Universe model. Moreover in our model, the local Hubble parameter becomes larger than the global one. Thus we show that the present observed local Hubble parameter measured by using the Cepheid variables is compatible with the global Hubble parameter measured by using the Sunyaev-Zeldovich effect. Furthermore we examine several aspects of our model: a) The energy conditions in our model are violated. We examine the consequences of these violations. b) There is a natural evolution of the effective gravitational “constant” in high redshift region. This yields drastic change of the stellar luminosity through the constructive equations of a star. We point out that a distant galaxy becomes much dimmer by this effect. c) This varying Gravitational “constant” affects the cosmic expansion speed and the nucleosynthesis process in the early Universe. We point our that this effect constrains the parameters of our model though the fine tuning is always possible.
1 Introduction

Recent many cosmological observations based on the method of Cepheid variables have consistently revealed the value of the present expansion rate of the Universe as $H_0 = 70 - 80 \text{km/sec/Mpc}$ [1][2][3]. These values are appreciably higher than what theoretical cosmologists expected before, $H_0 \approx 50$, mainly from the age analysis of the Universe. Actually the most simple cosmological model with no cosmological constant and no spatial curvature, $H_0 = 70 - 80 \text{km/sec/Mpc}$ infers the age of the Universe as $8.3 - 9.4 \text{Gyr}$. This is an apparent conflict with the age of the oldest stars in the globular clusters, $14 \text{Gyr}$ [4]. Stars should be younger than the Universe! This is the age problem of the Universe.

The problem is not restricted to this. If we look into the cosmological observations on the expansion rate in detail, we realize, besides the age problem, that there is an apparent discrepancy. The local ($z \approx 0.001 - 0.004$) Hubble parameter is consistently larger than the global ($z \approx 0.17 - 0.18$) one. The local observations [1][3][5] are mainly based on the luminosity-periodicity relation of Cepheid variable stars. They find Cepheid variable stars in M81, M100 and NGC4571, nearby galaxies of redshift $z \approx 0.001 - 0.004$. The estimated Hubble parameter is $H_0 = 80 - 90 \text{km/sec/Mpc}$. On the other hand the global observations [6][7] are mainly based on the Sunyaev-Zeldovich effect [7]. They observe distant clusters of galaxies Abel2218 and Abel665 with redshift $z \approx 0.17 - 0.18$ and measured the temperature distortion induced by the inverse Compton scattering of 3K cosmic background radiation by hot gas around the cluster. The estimated Hubble parameter is $H_0 = 50 \text{km/sec/Mpc}$. If these observations are true and the local expansion rate is actually larger than the global one, then we have to reconsider the present standard homogeneous Universe model. Let us call this the local-global $H_0$ discrepancy problem.

Well known solutions for the first age problem are a) to consider an open Universe (low density $\rho$) or b) to introduce the positive cosmological constant ($\Lambda$). This is manifest in the space-space component of the Einstein equation for the scale factor $a(t)$ in the homogeneous and isotropic matter dominated universe,

$$\frac{\ddot{a}}{a} = -g(\rho + 3P) + \frac{\Lambda}{3}c^2,$$

where $g \equiv 4\pi G/3$. In the case a), the maximum lifetime of the Universe is $t_0 = H_0^{-1}$. In the case b), the maximum lifetime is infinite if we fine tune the parameters: $\Lambda \to \Lambda_{cr} \equiv (4/9)H_0^2a_0^2\Omega_0$ where $\Omega \equiv \rho/\rho_{cr}$, $\rho_{cr} \equiv H_0^2/(2g)$. This model is called as the Lemaître Universe [9][10].

For the second local-global discrepancy problem, there are individual extensions of the above solutions a) and b). For the solution a), the authors of Ref. [1][2][3] consider an inhomogeneous cosmology in which the local Universe is underdense and the local expansion rate is higher than the global average. Authors of [4] try to constrain the maximum age of the inhomogeneous Universe based on the argument of Ref. [5]. It would be difficult to obtain an appropriate shape and configuration of the void around us naturally; it would require the anthropological principle. On the other hand for the

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1 Of course we have to wait for much decisive observations in the future. There may be any unknown systematic bias effects in the Sunyaev-Zeldovich method. There appear further complication if we consider another class of measurements of $H_0$ based on the supernovae of type I at maximum B light. It systematically shows the low value for the expansion rate $H_0 = 50 \text{km/sec/Mpc}$ independent of the distance $0.004 < z < 0.2$ [8]. In this paper we do not consider these complications.
solution b), the expansion rate does reduce in past. However the reduction rate seems to be very small. The appreciable reduction of $H_0$ is expected only when $z \geq 1$.

We would like to improve unsatisfactory points of the previous models and would like to propose a new model. We now consider the model which satisfies the following requirements:

1. The Universe has a long lifetime which is compatible with the age of the oldest stars (age problem).

2. Local ($z \approx 0.001-0.004$) Hubble parameter is larger than the global ($z \approx 0.17-0.18$) one: $H_0(\text{local}) > H_0(\text{global})$ (local-global discrepancy problem).

3. We do not consider a dilute Universe. We fix $\Omega \equiv \rho/\rho_{cr} = 1$ (flat Universe).

4. As in the Lemaître Universe model, we allow some amount of fine tuning for solving the age problem.

We easily realize from Eq.(1) that large negative pressure in some epoch tends to increase the acceleration ($\ddot{a}$) of the scale factor and extends the cosmic lifetime in the similar way as the small density $\rho$ and the positive cosmological constant $\Lambda$. This is similar to the idea of cosmological inflation in the early Universe [16][17]. However for our purpose to solve the age problem, the inflation must be in the late-time of the Universe definitely after the star formation epoch. Moreover the original scenario of inflation in the early stage does not contribute to the cosmic age because the scenario has too high energy scale such as $10^{15}$GeV.

Therefore we explore the late-time inflationary model after the star formation epoch and study the observational consequences of this model. Apparently this late-time inflation should be mild in the sense that the total expansion of the Universe during this inflation should be small compared with that in the original inflationary model $\approx e^{60}$. In order to yield this late-time inflation, we introduce a hypothetical scalar field with an ultra-light mass and a strong curvature coupling. This type of model has been widely used so far in various contexts in cosmology [18][19], and [21]. We mainly follow these phenomenological arguments in this paper\textsuperscript{2}. Our late-time inflationary model turns out to solve the local-global $H_0$ discrepancy problem as well as the age problem. This is because our model is highly non-linear due to the curvature coupling and the cosmic expansion is not monotonic. The mass of the scalar filed yields oscillatory cosmic expansion rate which can appreciably change within the redshift interval 0.01.

Originally Fakir et al. In Ref.[19] pointed out the possibility of inflation in the model of scalar field with negative\textsuperscript{3} curvature coupling $\xi$ with an appropriate potential. They studied the primordial density fluctuations in their inflationary model. Here we would like to apply the similar model to the late-time cosmology after the star formation epoch. The negative signature of $\xi$ is essential for the inflation and the elongation of the cosmic age. This will be explained in detail in subsequent sections. Therefore, for example, our previous model [21] where we took positive $\xi$ did not show any inflation nor elongation of the cosmic age.

\textsuperscript{2} However in some stage of future researches this kind of scalar field should be identified. See the discussions in the last section.

\textsuperscript{3}Their $\xi$ is minus of our $\xi$. 

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Moreover in our model the effective gravitational constant $G_{\text{eff}}$ changes in time because the scalar field directly couples with the scalar curvature. This may cause the intrinsic change of the stellar luminosity which is known to be very sensitive to the strength of gravity. Actually in the purely inflating phase in our model, the effective gravitational constant becomes half of the present value and the stellar luminosity becomes about six magnitude dimmer than the normal stars now.

The construction of this paper is as follows: In Section 2, we introduce a model of scalar field with a curvature coupling and show the existence of the mild inflation phase. In Section 3, observational properties of this mild Inflation model is explored. We show the cosmic age elongation and the non-linear effect that the local expansion rate becomes higher than the global one. In Section 4, we examine our mild inflationary scenario from wider point of view. We first examine the energy conditions and show that they are violated. Implications of this fact are given. Then we point out that the effective gravitational constant reduces toward past and the possible reduction of the intrinsic luminosity of stars. We also mention the effect of varying $G_{\text{eff}}$ on the nucleosynthesis. In the last Section 5, we summarize our present work and discuss on our future directions of research.

## 2 The Model of Mild Inflation

Let us begin with the following action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} (m^2 - \xi R) \phi^2 - \frac{R}{16\pi G} + \mathcal{L}_m \right]. \quad (2)$$

The scalar field $\phi$ with mass $m$ couples with the scalar curvature $R$ with strength $\xi$ which is taken to be negative. The last term on the right hand side of the above equation $\mathcal{L}_m$ represents the ordinary matter whose energy density is given by $\rho_m$. We assume the isotropic and homogeneous Universe, the Friedman Universe, with no spatial curvature. Then the line element is given by

$$ds^2 = dt^2 - a^2(t)[d\chi^2 + \chi^2 d\Omega^2]. \quad (3)$$

In this Universe, the spatial gradient term of the scalar field tends to reduce in the course of cosmic expansion. Therefore we only consider the spatially uniform configuration for the scalar field from the beginning. The energy-momentum tensor is defined as $T_{\mu\nu} \equiv 2(-\det g_{\mu\nu})^{-1/2} \delta S/\delta g^{\mu\nu}$ and the corresponding density and pressure are given by $\text{diag} T_{\mu\nu} = (\rho, p, p, p)$ with

$$\rho = \frac{1}{2} (\dot{\phi}^2 + m^2 \phi^2) + 3 \xi \phi^2 (H^2 + \frac{k}{a^2}) + 6H \xi \phi \dot{\phi} + \rho_m, \quad p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 + \xi \phi^2 (-3H^2 - \frac{k}{a^2} - 2\dot{H}) - 4H \xi \phi \dot{\phi} - \xi (\phi^2)^{\ddot{\cdot}}, \quad (4)$$

where $k$ is the curvature constant which we eventually set 0. $H$ is the Hubble parameter $H \equiv \dot{a}/a$. Then the equation of motion for the scalar field becomes

$$\ddot{\phi} + 3H \dot{\phi} + (m^2 + 6\xi (\dot{H} + 2H^2 + ka^{-2})) \phi = 0. \quad (5)$$
Time-time component of the Einstein equation becomes
\[ H^2 + ka^{-2} = g(\dot{\phi}^2 + m^2 \phi^2 + 6\xi \phi^2 (H^2 + ka^{-2}) + 12\xi H \dot{\phi} \dot{\phi} + 2\rho_m), \tag{6} \]
and the space-space component becomes
\[ \dot{H}(6\xi \phi^2 (1 - 6\xi) - g^{-1}) = 3\dot{\phi}^2 + 6\xi(4H\dot{\phi} - \dot{\phi}^2 + (m^2 + 12\xi H^2)\phi^2) \]
\[ + ka^{-2}(6\xi \phi^2 + \frac{1}{2}g^{-1}) + 3\rho_m, \tag{7} \]
where \( g = 4\pi G/3. \)

This set of equations admits a special solution if the parameter \( \xi \) is negative, \( k = 0, \) and \( \rho_m = 0: \)
\[ \dot{\phi} = 0, \quad \dot{H} = 0, \]
\[ H_s^2 = \frac{m^2}{(12|\xi|)}, \quad \phi_s^2 = \frac{1}{6|\xi|g}. \tag{8} \]

This is nothing but the de Sitter space-time the exponentially inflating Universe. In this solution, the effective mass of the scalar field \( (m^2 + 6\xi(\dot{H} + 2H^2 + ka^{-2})) \) vanishes and the expansion rate \( H \) is frozen. Therefore the evolution of the scale factor becomes convex \( \ddot{a}/a = H_s^2 > 0 \) and the age of the universe is infinitely elongated. It should be emphasized that this solution is possible only for negative curvature coupling \( \xi < 0 \). If positive, the age of the universe would have been reduced.

This solution should be compared with the extreme Lemaître universe, the Einstein static universe, in which the scale factor is frozen. Also in this case, the age of the Universe is infinitely elongated. However this Lemaître universe is unstable; any small amount of perturbation forces the Universe collapse or re-expand forever. Similar but much milder instability exists in our model. A small deviation from this \( H \)-frozen solution is shown to be unstable. Let us consider small deviations \( x \) and \( y: \)
\[ \phi = \phi_s(1 + x), \quad H = H_s(1 + y). \tag{9} \]

Then the linearized equations of motion for the deviations are
\[ \begin{pmatrix} x \\ y \end{pmatrix} ' = \begin{pmatrix} \frac{2}{1+3|\xi|} & -2 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{10} \]
where the prime denotes the derivative with respect to the time measured by the frozen Hubble parameter \( H_s t. \) The above matrix has one positive eigenvalue around 1 and one negative eigenvalue around \( -4 \) for \( |\xi| \gg 1. \) Therefore the \( H \)-frozen solution is unstable and the inflation eventually terminates. However the scalar field is globally oscillating due to the finite mass \( m \) and the field configuration returns closely to the original configuration. Then the Universe reenters the inflationary phase. After some amount of inflation the configuration deviates from the \( H \)-frozen phase again. This behavior repeats several times during the cosmic expansion. Accordingly the Universe shows piece-wise inflationary expansion repeating the \( H \)-frozen phase. This is the essence of the mild inflationary model.

\[ ^4 \text{Another choice } m^2 < 0, \text{ } \xi > 0 \text{ also admits this solution though unrealistic.} \]
In order to have a rough idea for the mild inflation, we numerically solve the above set of equations. A typical example is shown in Fig. 1 and Fig. 2.

Fig. 1.

We took our parameters as follows:

\[ m = 100H_0, \quad k = 0, \quad \xi = -80, \quad \Omega_{\text{matter}} = 0.01, \]  

(11)

The above choice of mass \( m \approx 10^{-31} \text{eV} \) yields several H-frozen phases until present time. If the mass were much larger than the above choice, then the piece-wise inflation would become less prominent and the energy density starts to behave as the ordinary dust. The above choice of curvature coupling \( \xi \) yields appropriate strength in the oscillation of the Hubble expansion rate. If \( \xi \) were much smaller than the above choice, then the inflation strength would become weaker. Spatially flatness, \( k = 0 \), is assumed for simplicity and the density parameter of matter \( \Omega_m \) is set to be the lowest possible value from typical observations. In this paper, we would like to propose a new type of cosmology and would like to show qualitative characteristics of the model. Therefore we will not try the best fit of our model with observations. In this sense the above choice of parameters are somewhat tentative though we will not change the choice in this paper. We set the “final” conditions at the present time \( t_0 \) as

\[ \phi(t_0) = 0.0023\sqrt{4\pi G/3}. \]  

(12)

This value is also somewhat arbitrary. Setting the value \( \phi(t_0) \) and the present value of the expansion rate \( H_0 \) automatically fixes the value \( \dot{\phi}(t_0) \) through the constraint equation Eq.(6). We notice in Fig. 1 that the piece-wise inflation is realized in the scale factor. The oscillatory nature of the scalar field becomes manifest in Fig. 2. The scale factor vanishes at \( t = -1.45802H_0^{-1} \), with which we identify the point of big bang.

Several comments on this H-frozen phase are in order. 1) The condition \( \xi < 0 \) is inevitable for the existence of this phase. As we can see in the numerical calculation in Fig. 1 that the curve of the scale factor vs. the cosmic time becomes piece-wise convex downward for \( \xi < 0 \). On the other hand if \( \xi > 0 \), the graph would become piece-wise concave. The former convex case the cosmic age is increased and the latter concave case it is reduced. 2) This type of inflation induced by the negative curvature coupling was studied in the paper [19] and mainly applied to the density perturbations in the early Universe. We are applying the similar model to the directly observable universe. In Ref. [19], the existence of the self coupling of the scalar field and very large curvature

\footnote{5 If we increase \( \Omega_m \), the inflationary period and the age of the Universe would be reduced.}

\footnote{6 As we will see in later in Fig. 3, the value 0.0023 for \( \phi(t_0) \) is chosen in favor of longer age of the Universe.}
3 Observational properties of the Mild Inflation

We now study the observational properties of the mild inflationary model of Universe.

3.1 age of the present Universe

Apparently the existence of the inflationary phase extends the age of the Universe. This is because the curve of the scale factor vs. cosmic time becomes convex downward as we see in Fig. 1. However actually, the late-time inflation cannot continue forever. If it were the case, the cosmic matter baryonic density would be thoroughly diluted away; manifest conflict with the observations. According to the recent observations the baryonic matter should occupy one to ten percent of the total matter density of the Universe. This fact most severely constrains the total amount of inflation. Baryonic matter density would be diluted, in the pure inflationary phase, by the factor \((a_f/a_i)^3\) where \(a_f\) and \(a_i\) are the scale factors at the initial stage and final stage of the inflation phase, respectively.

Our late-time inflation is not a pure inflation but piece-wise inflation. Therefore at each inflationary phase, some amount of the baryonic matter density is diluted away and the total accumulation of the loss determines the present baryonic matter density. Stronger inflation much elongates cosmic age but too much inflation makes the Universe empty. We need compromise and took \(\Omega_{\text{matter}} = 0.01\) in our numerical calculations.

In our numerical calculations in the previous section, we set the parameters as Eq. (11) Eq.(12). We obtained the present age of the Universe \(t_0 = -1.45802H_0^{-1}\) by identifying the big bang at the point when the scale factor vanishes (Fig.1). Then the age of the present Universe is read as 17.86Gyr if we adopt \(H_0 = 80\text{km/sec/Mpc}\). This is not a very special value. The age of the Universe is naturally elongated. We numerically checked this by varying the “final” condition \(\phi_0\) fixing all the other parameters. We show the result in Fig. 3.

According to Fig. 3, we obtain the cosmic age larger than 17Gyr for the parameter range \(\phi(t_0)/\sqrt{4\pi G/3} = 0.0015 - 0.006\). This variation of \(\phi_0\) is equivalent to the variation of the identification of the oscillation phase at the present time. If the phase is chosen so that the present expansion rate becomes local maximum, then it corresponds to the
local maximum of the cosmic age. Therefore each peak in Fig. 3 corresponds to the local maximum of the present expansion rate. The total number of oscillations or the total number of H-frozen phases is different from peak to peak in Fig. 3. At the peak \( \phi_0 = 0.0023 \) (which we have chosen before in the previous section), the number of H-frozen phase becomes maximum. The number of H-frozen phase for each peak is denoted in Fig. 3. Note that larger number of H-frozen phase yields larger cosmic age.

### 3.2 local and global Hubble parameters

Now we turn our attention to the second problem the local-global discrepancy of \( H_0 \). In our model, the expansion rate of the Universe is oscillating around the globally reducing component. Therefore, depending on the phase of the oscillation, the Hubble parameter averagingly smaller in the past than the value at present. It’s typical reduction time scale is given by the oscillation period. For example in the previous numerical calculations, the change of the Hubble parameter in the small redshift region is shown in Fig. 4.

The dashed line in this graph represents the expansion rate \( H \equiv \dot{a}(t)/a(t) \) versus redshift \( z \). However this is the bare Hubble parameter and is different from the observable Hubble parameter \( H_{\text{obs}} \). The observable Hubble parameter is usually obtained either from a) the distance measured through the luminosity of a star or b) the distance measured from the angular diameter of an X-ray cluster. These distances are translated into the coordinate distance \( \chi \) in Eq.(8). The coordinate distance \( \chi_1 \) to an object which emits light at the cosmic time \( t_1 \) and shows redshift \( z \) is simply given by integrating \( d\chi \) in Eq.(8) along the light path \( (ds^2 = 0) \) aided by Eq.(6). In the simplest standard cosmology \( \Omega = 1, \Lambda = 0 \), it becomes

\[
\chi_1 = \int_{a_1}^{a_0} \frac{dt}{a(t)} = \frac{2}{H_0a_0}(1 - (1 + z)^{-1/2}). \tag{13}
\]

Therefore the observable Hubble parameter is obtained from this relation as

\[
H_0 = \frac{2}{\chi_1a_0}(1 - (1 + z)^{-1/2}) \equiv H_{\text{obs}}. \tag{14}
\]

This observable Hubble parameter is plotted in Fig. 4 with the solid line. This observable parameter behaves much milder than the bare Hubble parameter because it is an integral of the inverse of the scale factor which is already an integral of the oscillating quantity \( H \). It is obvious from Fig. 4 that \( H_{\text{obs}} \) rapidly reduces in past and stays almost constant with low value. Already at \( z = 0.04 \) the Hubble parameter \( H_{\text{obs}} \) reduces almost 60 percent of the present value. This is the characteristic of the recent observations on the Hubble parameter mentioned in the introduction. Typical observed values \( H_{\text{obs}} \approx 70 - 80 \text{km/sec/Mpc} \) at \( z \approx 0.001 - 0.004 \) and \( H_{\text{obs}} \approx 50 \text{km/sec/Mpc} \) at \( z \approx 0.17 - 0.18 \) are qualitatively consistent with our results of numerical calculations.

For a comparison, we also plotted the Hubble parameter in the simplest standard model by the thin solid line. The observed Hubble parameter \( H_{\text{obs}} \) defined above becomes constant in this case.
The reduction time scale is basically determined by the oscillation period of the expansion rate which is controlled by the mass parameter $m$ of the scalar field. The strength of reduction is basically determined from the non-linearity of the set of evolution equations Eqs. (5), (6), and (7) controlled by the parameter $\xi$. Though we do not try to fine tune these parameters to fit to the observations in this paper, these parameters may become important in the near future when many reliable observations are available.

The reduction of $H$ or $H_{\text{obs}}$ toward the past is closely connected with the elongation of the age of the Universe. Suppose we vary the present phase of the oscillation, or equivalently the “final” condition $\phi(t_0)$. If the phase is chosen so that we observe the most decreasing Hubble parameter toward the past, then this is the most favorable situation for the age problem. The age of the Universe in this case becomes a local maximum in the variation of the phase. This is because, in this situation, we are now located in the phase of $H$ local maximum whose value should be fixed by observations. Therefore the global average of the Hubble parameter becomes minimum and the cosmic age becomes local maximum. This specially “lucky” phases correspond to several peaks in Fig. 3. This situation seems to be very similar in appearance to the model of inhomogeneous Universe in Refs. [11][12][13]. However, unlike the inhomogeneous model, we do not have to rely upon the anthropological principle in order to set our location just in the center of a void. In our model, any observer at present time $t_0$ observe the virtual void structure with the observer located just on the center. This is because the change of $H$ in redshift $z$ is caused by a temporal structure in the homogeneous Universe and is not caused by real inhomogeneity.

For the comparison with the simplest standard cosmology, we have shown in Fig. 5. the $m$-$M$ and Log($z$) relation (Hubble diagram), where $m - M$ is the distance modulus the difference of the apparent luminosity and the absolute luminosity of galaxies. We have chosen the present Hubble parameter $H_0 = 80 \text{km/sec/Mpc}$ and therefore

$$m - M = 5 \log_{10}((1 + z)\chi) - 5(\log_{10}(H_0 \text{pc})) - 5$$

$$= \log_{10}((1 + z)\chi) + 42.87.$$  \hspace{1cm} (16)

The thick solid line in this graph represents this relation which is expected in our model.

Fig. 5.

In the simplest standard model with $\Omega = 1$, $\Lambda = 0$, the relation is given by

$$m - M = 5 \log_{10}(2(1 + z)(1 - \frac{1}{\sqrt{1 + z}})) - 5(\log_{10}(H_0 \text{pc})) - 5$$

$$= \log_{10}((1 + z)(1 - \frac{1}{\sqrt{1 + z}})) + 42.87.$$  \hspace{1cm} (17)

The thin solid line in Fig. 5 represents this relation. Since larger distance modulus $m - M$ means smaller $H_0$, it is obvious in Fig. 5 that $H_0(\text{local}) > H_0(\text{global})$ is actually realized.
4 Examination of the model

In this section, we examine other aspects of our late-time inflationary model. Because this model is a new proposal for the solution of cosmological problems, we need to examine it in a wide point of view. We simply examine the natural consequences of our model and do not try to justify our model\textsuperscript{7}. This is because we believe that there are many points to be developed in our model and therefore it is too early to judge the final validity of the model. Among various interesting aspects of this model we picked up three topics: Violation of energy conditions, The luminosity biasing and Nucleosynthesis.

4.1 Hubble parameter and energy condition

The elongation of the cosmic age is closely related with the violation of energy conditions. We first examine these energy conditions in our model.

According to the argument in the reference \cite{14}, the volume expansion rate

\[
H_v \equiv \frac{1}{3 \sqrt{\det \gamma}} \frac{\partial}{\partial t} \sqrt{\det \gamma},
\]

satisfies the inequality

\[
H_v(t_0) t_0 \leq 1 \tag{20}
\]

under the following conditions

1) Einstein’s theory of gravity is correct.

2) Caustic (at which \( H_v = \infty \)) is identified with the Big Bang Singularity.

3) \( H_v(t_0) > 0 \)

4) The strong energy condition holds:

\[
R_{\mu\nu}V^\mu V^\nu \geq 0 \tag{21}
\]

for any timelike vector \( V^\mu \).

The volume expansion rate \( H_v \) reduces to the Hubble constant in the homogeneous and isotropic Universe. The inequality Eq.(20) is essentially exhausted in the Landau-Lifshitz’s textbook \cite{15}:

\[
\frac{\partial}{\partial t} \frac{1}{H_v} \leq 1. \tag{22}
\]

In our late-time inflationary model, the first three conditions 1) 2) 3) hold but the last condition 4). Let us investigate this point. The strong energy condition becomes

\[
\rho + p > 0, \quad \text{and} \quad \rho + 3p > 0 \tag{23}
\]

if we diagonalize the energy momentum tensor. We can examine these conditions using the expression Eq.(4). We numerically calculated \( \rho, \rho + p \) and \( \rho - |p| \) in Fig. 6a ∼ 6c.

\textsuperscript{7}In particular, we do not further fine tune the parameters of the model.
As we can see immediately the strong energy condition is apparently periodically violated. This is the reason why we could obtain the long lifetime of the Universe free from the constraint Eq.(20).

However we readily recognize that the weak energy condition and dominant energy condition are also violated as well as strong energy condition. The weak energy condition is the requirement that the energy density is non-negative for any observer. That is,

\[ T_{\mu\nu} V^\mu V^\nu \geq 0 \]  

(24)

for any timelike vector \( V^\mu \). In our case, it reduces to

\[ \rho > 0, \text{ and } \rho + p > 0. \]  

(25)

Our numerical calculation shows that our model violates the second inequality though the first inequality is satisfied. The dominant energy condition is the requirement that the energy flow \( T_{\mu\nu} V^\mu \) is non-spacelike for any timelike vector \( V^\mu \). In our case, this reduces to

\[ \rho > 0, \text{ and } \rho - |p| > 0. \]  

(26)

Our numerical calculation shows that our model violates the second inequality. These violations seem to be a general feature of the model with the scalar field with curvature coupling. Actually even if we take other coupling constants \( \xi \) such as +80 or +10, the weak and dominant energy conditions are violated. Therefore it is clear that this scalar field cannot couple with the ordinary observable matter. Because otherwise we would see the negative energy density and therefore the stability of matter is not guaranteed.\(^8\)

4.2 Effective Gravitational Constant

The curvature coupling of the scalar field in our model yields the temporal change of the effective gravitational constant. From the action Eq.(2), the effective gravitational constant is read as the inverse of the coefficient of the scalar curvature \( R \).

\[ G_{\text{eff}}(t) = \frac{1}{G^{-1} - 8\pi \xi \phi(t)^2}. \]  

(27)

This parameter characterizes the gravitational interaction with time scale shorter than that of the scalar field. The latter is roughly determined by the inverse of the mass \( m^{-1} = 0.01 H_0^{-1} \) as we see from Eq.(31). Because the parameter \( \xi \) is taken to be negative in our model, \( G_{\text{eff}}(t) \) reduces on average if we go back in time. For example, \( G_{\text{eff}}(t) \) in the purely H-frozen phase exactly becomes one half of the present value. Actual temporal change of \( G_{\text{eff}}(t) \) is numerically calculated and shown in Fig. 7.

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\(^8\)If there exists observable matter with negative energy density, then this matter is highly unstable and therefore there will be a violent activity associated with this matter.
Within the redshift one, $G_{\text{eff}}(t)$ changes up to five percent. However, it drastically changes at around redshift two and beyond.

On the other hand, the luminosity of a star is known to be very sensitive to the value of the gravitational constant. Now we examine this effect. The basic macroscopic equations for a spherically symmetric star are the equation of hydrostatic equilibrium, mass continuity, energy continuity, and radiative transport equations:\[23:\]

\[
\frac{dp}{dr} = -\frac{\rho GM(r)}{r^2},
\]

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho,
\]

\[
\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon,
\]

\[
\frac{dT(r)}{dr} = -\frac{3\kappa \rho L}{16\pi acT^3 r^2},
\]

where

\[
\frac{ac}{4} = \frac{\pi^2 ck^4}{60c^3 \hbar^3}
\]

is the Stefan-Boltzmann constant. Here $r$ is the radial distance measured from the stellar center. $\rho$, $L(r)$, and $T(r)$ are the mass density, the luminosity and the temperature at the position $r$, respectively. $M(r)$ is the mass contained within a sphere of radius $r$. These equations must be solved simultaneously with the microscopic equations, state equation, energy generation equation, and the equation for the opacity:

\[
p = \frac{\rho}{\mu m_H} kT,
\]

\[
\epsilon = \epsilon_0 XZ T^n,
\]

\[
\kappa_{\text{bf}} = \kappa_0 Z (1 + X) \rho T^{-3.5},
\]

where $X$ and $Z$ are the hydrogen and metal mass fractions respectively. $\kappa_{\text{bf}}$ is the absorption (opacity) of a photon by a bound electron. They form a complicated set of equations which can be exactly solved only by numerical calculations. Here in order to obtain a rough analytical estimate, we replace the differentiation such as $dp/dr$ by the ratio of averaged variables such as $p/R$. Then we obtain

\[
R = b G^{(2n-15)/(2n-1)} M^{(2n-9)/(2n-1)},
\]

\[
T = \mu m (kb)^{-1} G^{14/(2n-1)} M^{8/(2n-1)},
\]

\[
L = \epsilon_0 XZ \left( \frac{\mu m}{kb} \right)^n G^{-14n/(2n-1)} M^{(10n-1)/(2n-1)},
\]

\[
T_s = (\pi ac)^{-1/4} (\epsilon_0 XZ)^{1/4} \frac{\mu m^{n/4}}{k} b^{-2n+1/4} G^{(5n+15)/(4n-2)} M^{(6n+17)/(8n-4)},
\]

where

\[
b \equiv \frac{3^3 \kappa_0 f_0}{4^4 \pi^2 ac} \frac{2/(2n-1)}{2/(2n-1) - \frac{\mu m}{k} [\mu m]^{(2n-15)/(2n-1)} [ZX(1 + X)]^{2/(2n-1)}}.
\]
For example if the star is on the stage that the CNO-cycle is the dominant energy generation process, then \( n = 15 \) and

\[
\begin{align*}
    b & = \text{const.}[ZX(1 + X)]^{0.069}, \\
    R & = \text{const.}[bG^{0.52}M^{0.72}], \\
    T & = \text{const.}[b^{-1}G^{0.48}M^{0.28}], \\
    L & = \text{const.}[XZb^{-15}G^{7.2}M^{5.1}], \\
    T_s & = \text{const.}[(XZ)^{1/4}b^{-4.3}G^{1.6}M^{0.92}].
\end{align*}
\]  

If it is on the stage that the p-p chain is the dominant energy generation process, then \( n = 4 \) and

\[
\begin{align*}
    b & = \text{const.}[ZX(1 + X)]^{0.29}, \\
    R & = \text{const.}[bG^{-1}M^{-0.14}], \\
    T & = \text{const.}[b^{-1}G^{2}M^{1.14}], \\
    L & = \text{const.}[b^{-4}G^{8}M^{5.57}], \\
    T_s & = \text{const.}[(XZ)^{1/4}b^{-2}G^{2.5}M^{1.46}].
\end{align*}
\]  

In both cases we notice large exponents 7.2 (for \( n = 15 \)) and 8 (for \( n = 4 \)) on \( G \) for the stellar luminosity \( L \).

This sensitive dependence of \( L \) on \( G \) yields a drastic results in our cosmological model. According to the above estimate, the stars in the redshift range from zero to 0.2, \( G_{\text{eff}}(t)/G_0 \) is about 0.98 and the luminosity fluctuate only fifteen percent. On the other hand, the stars in the redshift range from two to five, \( G_{\text{eff}}(t) \) is about 0.8 on average and the luminosity fluctuate about 85 percent. Furthermore if the redshift exceeds about ten, the stellar luminosity reduces more than several hundreds times and actually such stars cannot be observed at all. It would therefore become very difficult to detect, in high redshift region, the ordinary stars and clusters of them shining by the nuclear reactions triggered by the self gravity. If we fined luminous object with high redshift, the emission process of it would be very different from the ordinary stars. Observationally, the most distant galaxy which is confirmed to have its luminosity from the collection of ordinary stars is located at around \( z = 3[24] \). Therefore it may be interesting to examine the possible change of galactic spectrum as a function of redshift, though actually the evolutionary effect makes the spectrum complicated. The above reduction of stellar luminosity at high redshift region may also cause the reduction of the number counts of galaxies in the past.

### 4.3 Nucleosynthesis

We have previously explored mainly the directly observable epoch in the Universe i.e. the redshift less than about ten. However a naive extrapolation of our model into much early epoch may be interesting though there is no guarantee that the dynamics of the scalar field and its coupling to the scalar curvature \( \xi \) are not changed from those at present. We consider the effect of \( G_{\text{eff}}(t) \) on the nucleosynthesis in this subsection.

In the early Universe the energy density is expected to be dominated by radiation and the scalar field contribution to the total energy density becomes less important. However since the scalar field couples with the scalar curvature, it strongly affects the effective
gravitational constant even in the early Universe. Actually the asymptotic solution of Eq.(5) in the radiation dominated era \((H = 1/(2t))\) becomes
\[ \phi = c_1 + c_2 t^{-1/2} \] (51)
with \(c_1, c_2\) some constants. We can safely discard the growing mode toward the past \((c_2 = 0)\) if we assume that the scalar field was finite in the past. Even in this case \(G_{\text{eff}}(t)\) in the epoch of nucleosynthesis might be generally different from the present value. Since the value of \(G_{\text{eff}}(t)\) determines the expansion rate at that time, it strongly affects the dynamics of nucleosynthesis and the primordial abundance of \(^4\text{He}\). According to the arguments in Ref. [25], the constraint is obtained as
\[ \frac{|G_{\text{eff}}(t_{\text{ns}}) - G_0|}{G_0} < 0.4, \] (52)
where \(t_{\text{ns}}\) is the time of nucleosynthesis. Though it is always possible to fine tune the parameter \(c_1\) so that the above constraint is satisfied, it goes beyond the scope of the present paper to argue what guarantees this tuning.

On the other hand we doubt that our scalar field with tiny mass \((10^{-31}\text{eV})\) is the elementary field even at the epoch of nucleosynthesis (energy scale of MeV). If it decomposes into other elementary fields or its curvature coupling vanishes at the epoch of nucleosynthesis then \(G_{\text{eff}}(t_{\text{ns}})\) is the same as \(G_0\) and it does not affect the dynamics of nucleosynthesis.

5 Conclusions and discussions

We first pointed out two basic problems in the present cosmology: a) The age problem and b) the local-global \(H_0\) discrepancy problem. In order to solve these problems we have introduced a cosmological model with scalar field which has negative curvature coupling constant \(\xi\). In this model, we have obtained sufficient amount of elongation of the cosmic age: 17.86Gyr if we adopt \(H_0 = 80\text{km/sec/Mpc}\) for an appropriate choice of parameters. Moreover this model shows approximately forty percent reduction of \(H_{\text{obs}}\) within \(z \approx 0.04\) and this may partially explain the low values of \(H_{\text{obs}}\) measured by the method of Sunyaev-Zeldovich effect.

In a simple open Universe model the cosmic age is also elongated however the Hubble parameter never reduces toward the past. In a model with cosmological constant the cosmic age is also elongated and the Hubble parameter can reduce toward the past however the reduction rate is extremely small (about one percent at redshift 0.04). In a model with inhomogeneous matter distribution, the rapid reduction of the Hubble parameter toward the past may be accounted for but the age problem still remains. Our late-time inflationary scenario will be the unique model which can account for both the rapid reduction of \(H\) and the age problem.

These extreme properties are due to the piece-wise mild inflation realized in our model. For this mechanism, large negative \(\xi\) has been essential. If we would have taken positive \(\xi\), the cosmic age would have been shorten.

Then we have examined couple of consequences of the late-time inflationary scenario. They constrain the parameters and simultaneously indicate interesting developments of our model. Both of them should be continuously studied and will be reported soon.
1. Our model violates all energy conditions, strong, dominant, and weak energy conditions. Therefore our scalar field cannot couple with the ordinary matter.

2. In our model the effective gravitational constant changes with time. It reduces toward the past and becomes almost half of the present value at the redshift about ten. Because the luminosity of the ordinary star highly depends on the value of the gravitational constant \( L \propto G^{7-8} \), we expect that the detection of the ordinary stars will become much more difficult in such redshift region.

3. Naive extrapolation of the temporal change of the gravitational constant affects the expansion speed of the Universe at the epoch of nucleosynthesis. In order not to destroy the present scenario of nucleosynthesis, we need some amount of fine tuning of the parameters.

In this article we have used the hypothetical scalar field with tiny mass and have not identified the field. There is a possibility that this field is the ordinary photon field with a tiny mass.\(^9\) Experimentally, the upper limit of photon mass is given as \( 6 \times 10^{-16} \text{eV} \) from the measurement of the Jupiter’s magnetic field by Pioneer-10 satellite\([26]\). For example, the Proca field action is given by

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (m^2 - \xi R) B_{\mu} B_{\nu} g^{\mu\nu} - \frac{1}{16\pi G} R \right],
\]

where \( F_{\mu\nu} \equiv \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} \). The energy momentum tensor for this model becomes

\[
T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{4} g_{\mu\nu} F^{\alpha \beta} F_{\alpha \beta} - F_{\alpha \mu} F^{\alpha \nu} + m^2 (B_{\mu} B_{\nu} - \frac{1}{2} g_{\mu\nu} B^2) \\
- \xi (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) B^2 - \xi B_{\mu} B_{\nu} R - \xi g_{\mu \nu} \Box B^2 + \xi (B^2)_{;\mu \nu}.
\]

If we consider the simplest uniform electromagnetic field, the Universe becomes anisotropic and we should consider Bianchi type or Kantowski-Sachs type cosmologies. The full study on this issue will be reported by the present authors in the near future.

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\(^9\)In this case, we also have to reconsider the energy conditions and examine the violation of them.
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Figure Captions

Fig.1: The scale factor vs. the cosmic time in the mild inflationary model. The scale factor is measured in units of $a_0 = a(t_0)$. The horizontal axes is $t - t_0$ measured in units of $H_0^{-1}$. We took our parameters as $m = 100H_0$, $k = 0$, $\xi = -80$, $\Omega_{\text{matter}} = 0.01$ and set the “final” conditions at the present time $t_0$ as $\phi(t_0) = 0.0023$. We notice that the piece-wise inflation is realized in the oscillatory evolution of the Universe. At each short inflationary phase the line becomes convex downward and this convexity elongates the cosmic age. The scale factor vanishes at $t = -1.45802$ where we identify the point of big bang.

Fig.2: The scalar field vs. the cosmic time in the mild inflationary model. The scalar field is measured in units of $\sqrt{g} \equiv \sqrt{4\pi G/3}$. The horizontal axes is $t - t_0$ measured in units of $H_0^{-1}$. This graph shows highly non-linear oscillation caused by strong curvature coupling term.

Fig.3: The cosmic age vs. the “final” condition $\phi_0$. The age is measured in units of $H_0^{-1}$. The scalar field is measured in units of $\sqrt{g} \equiv \sqrt{4\pi G/3}$. The value $\dot{\phi}_0$ is automatically fixed from the constraint equation Eq.(6). This variation of $\phi_0$ is equivalent to the variation of the identification of the present phase in the oscillating evolution. Each peak in the figure corresponds to the local maximum of the present expansion rate. The total number of H-frozen phases is different from peak to peak and is denoted in the figure. At the peak $\phi_0 = 0.0023$, this number becomes maximum. Note that larger number of H-frozen phase yields larger cosmic age.

Fig.4: The Hubble parameter vs. redshift. The vertical axes is normalized by $H_0$. Dashed line in this graph represents the expansion rate $H \equiv \dot{a}(t)/a(t)$ versus redshift $z$. However this is the bare Hubble parameter and is different from the observable Hubble parameter $H_{\text{obs}}$ (solid line) defined in Eq.(hubble). It is obvious from the figure that $H_{\text{obs}}$ rapidly reduces toward the past and stays almost constant with low value. Thin solid line is the bare Hubble parameter $H$ in the standard cosmology with $\Omega = 1$. $H_{\text{obs}}$ in this case becomes constant in $z$.

Fig.5: The Hubble diagram. The thick solid line represents the distance modulus $m-M$ vs. $\log(z)$ in our model. The thin solid line represents the same relation in the standard model with $\Omega = 1$. The thick solid line has the same tendency as the dilute Universe (smaller $\Omega$).

Fig.6: Numerically calculated $\rho$ (6a), $\rho + p$ (6b), and $\rho - |p|$ (6c) vs. cosmic time. The vertical axes is in units of $H_0^2g = 4\pi G_0H_0^2/3$. The horizontal axes is $t - t_0$ measured in units of $H_0^{-1}$. As we can immediately see all the energy conditions are apparently violated periodically. Violations of the weak energy condition and dominant energy condition suggest that our hypothetical scalar field cannot couple with the ordinary matter.

Fig.7: The temporal change of $G_{\text{eff}}(t)$ in the late-time inflationary model. The horizontal axes is the redshift $z$. The vertical axes is normalized by $G_0$. Within the redshift one, $G_{\text{eff}}(t)$ changes up to several percent. However it drastically changes at around redshift two and beyond. Because the luminosity of a star is known to be very
sensitive to the value of the gravitational constant \((L \propto G^{7-8})\), distant stars would become much dimmer in our model.