RATIONAL CURVES ON A SMOOTH HERMITIAN SURFACE II

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Abstract. In characteristic \( p > 0 \) and for \( q \) a power of \( p \), we compute the number of nonplanar rational curves of arbitrary degrees on a smooth Hermitian surface of degree \( q + 1 \) in the case where they have a parametrization given by polynomials with at most 4 terms. It is shown that a smooth Hermitian cubic surface contains infinitely many rational curves of degree 3 and 6. On the other hand, for all other cases the numbers of curves are finite and they are exactly determined. Further such rational curves are given explicitly up to projective isomorphism and their smoothness are checked.

1. Introduction

Let \( q \) be a power of a prime \( p \) and let \( k \) be an algebraic closure of the finite field \( \mathbb{F}_q \). We denote by a column vector \( \mathbf{x} = (x_0, x_1, x_2, x_3) \) a point in the \( k \)-projective space \( \mathbb{P}^3 \). For a nonzero 4-by-4 matrix \( A \) with entries in \( k \), a \( k \)-Hermitian surface of degree \( q + 1 \) is defined by

\[
X_A := \{ \mathbf{x} \in \mathbb{P}^3 \mid \mathbf{x}^tA\mathbf{x}^{(q)} = 0 \},
\]

where \( \mathbf{x}^{(q)} = (x_0^q, x_1^q, x_2^q, x_3^q) \). The map \( x \mapsto x^q \) is an involutive automorphism on \( \mathbb{F}_{q^2} \), which is an analog of the complex conjugate \( z \mapsto \bar{z} \) on the complex numbers. If \( \mathbf{t}A = A^{(q)} \) then \( X_A \) is defined over \( \mathbb{F}_{q^2} \) and is simply called a Hermitian surface. Especially, when \( A \) is equal to the identity matrix \( I \), the associated surface \( X_I \) is the Fermat surface of degree \( q + 1 \). One can easily show that \( X_A \) is smooth if and only if \( A \) is invertible.

Lines on a Hermitian surface, more generally, linear spaces on a Hermitian hypersurface have been extensively studied in various papers, for example \( \cite{6, 2, 7, 4, 5} \). In contrast, there are few results known about rational curves of degree \( > 1 \) on a Hermitian surface.

Let \( A \) be an invertible 4-by-4 matrix with entries in \( k \) and let \( R_d \) be the set of nonplanar rational curves of degree \( d \) contained in \( X_A \). In \( \cite{4} \) we studied on \( R_d \) for \( d \leq q + 1 \). The result may be summarized as follows:

The set \( R_d \) is empty for \( d < q + 1 \). In addition, no conics are also contained in \( X_A \). On the other hand, when \( d = q + 1 \) the set \( R_d \) is not empty and all curves in \( R_d \) are projectively isomorphic over \( k \) to an explicitly given curve, and furthermore the number of curves in \( R_d \) is exactly \( q^d(q^3 + 1)(q^2 - 1) \).

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In this paper, we extend this result to the case $d \geq q+2$ under the assumption that the curves in $R_d$ can be parametrized by polynomials with at most 4 terms. We refer to the curves satisfying such condition as “tetranomial curves” in this paper. By definition, for a nonplanar tetranomial curve of degree $d$ with the locus $t(x_0, x_1, x_2, x_3)$, there are an invertible 4-by-4 matrix $F$ with entries in $k$ and integers $i, j$ with $1 \leq i < j \leq d-1$ such that

$t(x_0, x_1, x_2, x_3) = F \cdot (s^{d-i}t^i, s^{d-j}t^j, t^d),$

where $s, t$ are homogeneous coordinates of $\mathbb{P}^1$. We denote by $C(d, i, j)$ a curve for when $F = I$ in the above expression. Our purpose is to prove the following theorem:

**Theorem 1.1.** Let $T_d$ be the set of nonplanar tetranomial curves of degree $d$ on a smooth $k$-Hermitian surface. If $T_d$ is non-empty then $T_d$ is equal to one of the following three cases:

C1) $d = q + 1$,

$$|T_d| = \begin{cases} \infty & \text{for } q = 2 \\ q^4(q^3 + 1)(q^2 - 1) & \text{for } q \geq 3, \end{cases}$$

all elements of $T_d$ are projectively isomorphic over $k$ to $C(q+1, 1, q)$.

C2) $d = (q+1)q$ for $q$ even,

$$|T_d| = \begin{cases} \infty & \text{for } q = 2 \\ q^6(q^4 - 1)(q^2 - 1) & \text{for } q \geq 3, \end{cases}$$

all elements of $T_d$ are projectively isomorphic over $k$ to $C((q+1)q, q+1, q^2 + 1)$.

C3) $d = (q+1)q/2$ for $q$ odd,

$$|T_d| = \begin{cases} 2q^6(q^4 - 1)(q^2 - 1) & \text{for } q \equiv 1 \mod 4 \\ 4q^6(q^4 - 1)(q^2 - 1) & \text{for } q \equiv 3 \mod 4, \end{cases}$$

all elements of $T_d$ are projectively isomorphic over $k$ to $C((q+1)q/2, (q+1)/2, (q^2 + 1)/2)$.

Here C1 has already proven for $q \geq 3$ in [4]. When $q = 2$, C1 states that a smooth $k$-Hermitian cubic surface contains an infinite number of nonplanar rational curves of degree 3, however, which seems to contradict a result of [4]. Unfortunately, there is actually an oversight in the case $q = 2$ of [4]. We recall that the Fermat surface is exceptionally rational when its degree is 3. Indeed when $q = 2$ non-empty $T_d$ splits into infinitely many $\text{Aut}(X_A)$-orbits, where $\text{Aut}(X_A)$ is the projective automorphism group of a smooth $k$-Hermitian surface $X_A$ (see Remark 3.3). Further we will see that all curves of C2, C3 are singular, although those of C1 are smooth (see Remark 2.3).

**Example 1.2.** The case $q = 2$: There are infinitely many nonplanar rational curves of degree 3 and infinitely many nonplanar tetranomial curves of degree 6 on a smooth $k$-Hermitian cubic surface, and their curves are
projectively isomorphic over \( k \) to \( C(3, 1, 2) \) and \( C(6, 3, 5) \), respectively.

**The case \( q = 3 \):** There are exactly 18144 nonplanar rational curves of degree 4 and 1866240 nonplanar tetranomial curves of degree 6 on a smooth \( k \)-Hermitian quartic surface, and their curves are projectively isomorphic over \( k \) to \( C(4, 1, 3) \) and \( C(6, 2, 5) \), respectively.

**The case \( q = 4 \):** There are exactly 249600 nonplanar rational curves of degree 5 and 15667200 nonplanar tetranomial curves of degree 20 on a smooth \( k \)-Hermitian quintic surface, and their curves are projectively isomorphic over \( k \) to \( C(5, 1, 4) \) and \( C(20, 5, 17) \), respectively.

**The case \( q = 5 \):** There are exactly 1890000 nonplanar rational curves of degree 6 and 468000000 nonplanar tetranomial curves of degree 15 on a smooth \( k \)-Hermitian sextic surface, and their curves are projectively isomorphic over \( k \) to \( C(6, 1, 5) \) and \( C(15, 3, 13) \), respectively.

In the next section, we give a necessary condition for which \( T_d \neq \emptyset \), and determine the curves in \( T_d \) up to \( k \)-projective isomorphisms. In Section 3, we associate the set \( T_d \) modulo \( \text{Aut}(X_A) \) to a certain set consisting of matrices of rank 4 by introducing some notations, and show that there is a one-to-one correspondence between them. Moreover we deduce that the number of \( \text{Aut}(X_A) \)-orbits is exactly 1 if \( q \geq 3 \) otherwise an infinite number. In Section 4, by computing the stabilizer of a curve in \( T_d \) for \( q \geq 3 \), we complete the proof of Theorem 1.1.

2. A CONDITION FOR WHICH \( T_d \neq \emptyset \)

Let \( s, t \) be homogeneous coordinates of \( \mathbb{P}^1 \). We write the vector
\[
\begin{pmatrix} s^d, s^{d-i} t, f_{B}(d, i, j) \end{pmatrix} \in \mathbb{P}^3 \quad \text{with } 1 \leq i < j \leq d - 1
\]
as \( v(d, i, j) \) shortly. Let \( B \) and \( F \) be 4-by-4 invertible matrices with entries in \( k \). We denote by \( f_B(d, i, j) \) and \( C_F(d, i, j) \) a homogeneous \((q + 1)d\)-form \( v(d, i, j) B v(d, i, j)^{(q)} \) and a tetranomial curve having a parametrization given by \( F v(d, i, j) \), respectively. It is obvious that \( C_F(d, i, j) \) is contained in a smooth \( k \)-Hermitian surface \( X_A \) if and only if \( f_{F_B}(d, i, j) \) is identically 0 for all \( s, t \). For a positive integer \( n \) and the 4-by-4 anti-diagonal matrix \( J \) whose entries are entirely 1, we identify 4-tuples \((d, i, j, n, B)\) and \((d, d - j, d - i, nB)\) with \((d, i, j, B)\) because both \( C(d, i, j, n) \) and \( C_J(d, d - j, d - i) \) become \( C(d, i, j) \) by the changes of coordinates \((s, t) \mapsto (s^{1/n}, t^{1/n})\) and \((s, t) \mapsto (t, s)\), respectively. Then we will prove the following lemma:

**Lemma 2.1.** If \( f_B(d, i, j) \) is identically 0 for all \( s, t \) then \((d, i, j, B)\) is equal to one of the following three cases:

I. \( d = q + 1, i = 1, j = q \).

\[
B = \begin{pmatrix}
0 & a_{11} & a_{12} & a_{13} \\
0 & a_{21} & a_{22} & a_{23} \\
-a_{11} & -a_{12} & -a_{13} & 0 \\
-a_{21} & -a_{22} & -a_{23} & 0
\end{pmatrix},
\]
where the first two rows are $k$-linearly independent, $(a_{11}, a_{21}), (a_{13}, a_{23}) \neq (0, 0)$, and if $q \geq 3$ then $a_{12} = a_{22} = 0$.

II. $d = (q + 1)q$, $i = q + 1$, $j = q^2 + 1$ for $q$ even.

$$B = \begin{pmatrix} 0 & b_1 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \\ -b_1 & -b_2 & 0 & 0 \end{pmatrix},$$

where $b_1, b_3 \neq 0$, and if $q \geq 3$ then $b_2 = 0$.

III. $d = (q + 1)q/2$, $i = (q + 1)/2$, $j = (q^2 + 1)/2$ for $q$ odd.

The matrix $B$ is the same with the case II.

**Proof.** The exponents of $t$ appearing in $f_B(d, i, j)$ are formed in the following matrix:

$$E = \begin{pmatrix} 0 & qi & qj & qd \\ i & (q+1)i & i + qj & i + qd \\ j & qi + j & (q+1)j & j + qd \\ d & qi + d & qj + d & (q+1)d \end{pmatrix}.$$

Here denoting the entries of $B$ by $B_{lm}$ and those of $E$ by $E_{lm}$, $B_{lm}$ indicates the coefficient of the monomial containing $t^{E_{lm}}$. Then by assumption, there are some linear expressions $F_{lm'l'm'} := E_{lm} - E_{lm'}$ with respect to $d, i, j$ such that $F_{lm'l'm'} = 0$, and then $B_{lm} + B_{lm'} = 0$. Hence if we choose three $F_{lm'l'm'}$’s to be linearly independent over rational numbers then $(d, i, j) = (0, 0, 0)$, but this is a contradiction. On the other hand, if one chooses two linearly independent $F_{lm'l'm'}$’s then $d, j$ can be represented by only $i, q$. In fact we can choose two appropriate $F_{lm'l'm'}$’s as follows: First of all, note that one can exclude $E_{11}, E_{21}, E_{33}, E_{43}$ since these are not equal to any other entries of $E$, and furthermore $E_{31}$ and $E_{24}$ can be equal to only $E_{12}$ or $E_{22}$ and $E_{33}$ or $E_{43}$, respectively. When one puts $F_{3112} = 0$ and $F_{2443} = 0$ as $E$ to be rank 4, one has $d = (q + 1)i$, $j = qi$, $F_{4122} = 0$, $F_{1433} = 0$, and furthermore if $q = 2$ then $F_{3213} = 0$, $F_{4223} = 0$. Then $B$ and $f_B(d, i, j)$ become a form of the case I under the identification $f_B((q + 1)i, i, qi) = f_B(q + 1, q)$. Similarly, putting $F_{4112} = 0$ and $F_{3324} = 0$, one has $d = qj$, $(q + 1)j = (q^2 + 1)i$, and furthermore if $q = 2$ then $F_{4214} = 0$, and thus $B$ is of the case II. Noting that $\gcd(q^2 + 1, q + 1) = 1$ or 2 by whether $q$ is even or odd, respectively, one has $d = (q + 1)qn$, $i = (q + 1)n$, $j = (q^2 + 1)n$ for the former and $d = (q + 1)qn/2$, $i = (q + 1)n/2$, $j = (q^2 + 1)n/2$ for the latter, where $n$ is a positive integer. By the identification, each of them becomes a form of the case II or III. One can easily verify that all possible choices of $(d, i, j, B)$ up to the identification become the case I, II or III, and thus the lemma has been proven.

We quote the following important proposition (cf. e.g. [1] and [7] Proposition 2.5.) for a proof):
Proposition 2.2. For each $A \in \text{GL}_n(k)$, there is an element $B \in \text{GL}_n(k)$ such that $A = B B'(q)$. If $A = A(q)$ then $B$ is defined over $\mathbb{F}_q$.

By Lemma 2.1 and Proposition 2.2, if $T_d \neq \emptyset$, all curves in $T_d$ are projectively isomorphic over $k$ to $C(q+1,1,q), C(q+1,q+1,q^2+1)$ or $C(q+1)/2,q+1/(q^2+1)/2$.

Remark 2.3. Every curve $C \subseteq T_{q+1}$ is smooth. In fact, $C$ is projectively isomorphic over $k$ to the curve $C(q+1,1,q)$ whose defining equations are presented as

$$x_1^q = x_0 q^{-1} x_2, \quad x_2^q = x_1 x_3 q^{-1}, \quad x_1 x_2 = x_0 x_3$$

and whose Jacobian matrix always has rank 2. Hence $C(q+1,1,q)$ is smooth, and thus so is $C$. On the other hand, all curves in $T_{q(q+1)}$, $T_{q(q+1)/2}$ are singular, because every $C' \subseteq T_{q(q+1)}$ is projectively isomorphic over $k$ to the curve $C((q+1)q+1,q^2+1)$ whose defining equations are presented as

$$x_1^q = x_0 q^{-1} x_3, \quad x_2^{q+1} = x_1 x_3 q, \quad (x_1 x_2)^{q+1} = x_0 q q + 1) = x_0 q q + 1) q + 1) q + 1) q + 1)$$

and whose Jacobian matrix has rank 1 at $\frac{q}{2}(0,0,0,1)$. Hence $C((q+1)q+1,q^2+1)$ has a singular point, and thus so does $C'$. The case of $T_{q(q+1)/2}$ is the same too.

3. The action of $\text{Aut}(X_A)$ on $T_d$

We denote the sets of matrices $B$ of I, II, III of Lemma 2.1 by $B_1, B_2, B_3$, respectively. Further we denote by $Q'_i$ the set of $\vdash-(q+2)$-by-$\vdash-(q+2)$ matrices whose entries are 0 except for the $(i,j)$-entries with $i,j \in \{1,2,q+1,q+2\}$ which form a submatrix belonging to $B_1$, and for $q$ even, denote by $Q'_2$ the set of $\vdash-(q+2)$-by-$\vdash-(q+2)$ matrices whose entries are 0 except for the $(i,j)$-entries with $i,j \in \{1,q+2,q^2+2,(q+1)q+1\}$ which form a submatrix belonging to $B_2$, and for $q$ odd, denote by $Q'_3$ the set of $\vdash-(q+2)$-by-$\vdash-(q+2)$ matrices whose entries are 0 except for the $(i,j)$-entries with $i,j \in \{1,(q+1)/2+1, (q^2+1)/2+1, (q+1)q/2+1\}$ which form a submatrix belonging to $B_3$.

For each $M \in Q'_i$, we denote by $M^*$ the 4-by-4 submatrix of $M$ belonging to $B_i$ for $i = 1,2,3$. We also use the same symbol for the inverse correspondence, namely $M^{**} = M$. Then $M \in Q'_i$ if and only if $M^* \in B_i$.

For a positive integer $d$, we denote by $\text{Im}(\varphi_d)$ the image of the group homomorphism $\varphi_d : \text{GL}_2(k) \ni g \mapsto \varphi_d(g) \in \text{GL}_{d+1}(k)$ defined by

$$\left( u^d, u^{d-1} v, \ldots, v^d \right) = \varphi_d(g) \left( s^d, s^{d-1} t, \ldots, t^d \right) \text{ for } \left( u, v \right) = g \left( s, t \right).$$

We denote the set of $m$-by-$n$ matrices with entries in $k$ by $M_{m,n}$, particularly by $M_m$ if $m = n$. Putting $d_1 = q+1, d_2 = (q+1)q, d_3 = (q+1)q/2$, we write $\varphi_1, \varphi_2, \varphi_3$ instead of $\varphi_{d_1}, \varphi_{d_2}, \varphi_{d_3}$, respectively.

For $i = 1,2,3$, the sets $Q_i$ are defined by

$$Q_i := \left\{ M \in M_{d_i+1} \mid \varphi_i(g) M \varphi_i(g) \cap (q) \in Q'_i \text{ for some } g \in \text{GL}_2(k) \right\}.$$
By definition \(Q_i' \subset Q_i\), and \(\text{Im}(\varphi_i)\) acts on \(Q_i\) as \(M \mapsto \varphi_i(g)M\varphi_i(g)^q\). We denote by \(\sim\) the equivalence relation induced by this action, and denote by \(M^{\varphi_i}\) the equivalence class of \(M \in Q_i\).

Further for \(i = 1, 2, 3\) we define the sets \(S_i\) by

\[
S_i := \{F \in \mathcal{M}_{4d_i+1} \mid \phi F AF(q) \in Q_i\}
\]

By definition, \(\text{Im}(\varphi_i)\) acts on \(S_i\) as \(F \mapsto F\varphi_i(g)\), and then notice that the set \(T_{d_i}\) is in one-to-one correspondence with \(k^\times \setminus S_i/\text{Im}(\varphi_i), \) where \(k^\times = k \setminus \{0\}\) (cf. [4, Section 3]).

We consider the action of the group \(\text{Aut}(X_A)\) on \(T_{d_i}\), where \(\text{Aut}(X_A)\) is represented by

\[
\{P \in \text{GL}_4(k) \mid \phi P A P(q) = \lambda A, \ \lambda \in k^\times / k^\times I\}
\]

Then \(\text{Aut}(X_A)\) acts on \(T_{d_i}\) by acting on \(k^\times \setminus S_i/\text{Im}(\varphi_i)\) by the multiplication from the left.

**Lemma 3.1.** For \(i = 1, 2, 3\), there is the following bijection:

\[
\text{Aut}(X_A)k^\times \setminus S_i/\text{Im}(\varphi_i) \longrightarrow k^\times \setminus Q_i/\sim
\]

\[
\text{Aut}(X_A)k^\times F \text{Im}(\varphi_i) \longrightarrow k^\times (\phi F AF(q))^{\varphi_i}.
\]

**Proof.** It is obvious by definition that the map is well-defined.

**Surjectivity:** For every \(k^\times M^{\varphi_i} \in k^\times \setminus Q_i/\sim\), we can choose \(M \in Q_i\) such that \(M \in Q_i'\), if necessary, by acting an appropriate \(\varphi_i(g)\) on \(M\). Then by definition \(M^* \in \mathcal{B}_i \subset \text{GL}_4(k)\), and also \(A \in \text{GL}_4(k)\). Hence by Proposition [2] there are \(G, H \in \text{GL}_2(k)\) such that \(M^* = \phi G(q)\) and \(A = \phi H(q)\), and putting \(F^* = H^{-1}G\), one has \(\phi F^* AF^*(q) = M^*\). Then we may take \(F \in S_i\) such that \(\phi F AF(q) = M\) by placing 4 columns of \(F^*\) into the appropriate columns of \(F\).

**Injectivity:** For \(F_1, F_2 \in S_i\) such that \(k^\times (\phi F_1 AF_1(q))^{\varphi_i} = k^\times (\phi F_2 AF_2(q))^{\varphi_i}\) with \(\phi F_1 AF_1(q), \phi F_2 AF_2(q) \in Q_i'\), we have

\[
\lambda^\phi \varphi_i(g)^{\phi F_1 AF_1(q)} = \varphi_i(g)^{\phi F_2 AF_2(q)}
\]

for \(\lambda \in k^\times, g \in \text{GL}_2(k)\), where \(F_1^*, F_2^*, \varphi_i(g)^* \in \text{GL}_4(k)\) such that

\[
\phi F_1 AF_1(q) = \left(\phi F_2 AF_2(q)\right)^*
\]

and \(\phi F_2 AF_2(q) = (\phi F_2 AF_2(q))^*\). Then \(k^\times F_1^* \varphi_i(g) F_2^* = k^\times P F_2^*\) for some \(k^\times P \in \text{Aut}(X_A)\). Therefore

\[
\text{Aut}(X_A)k^\times F_1 \text{Im}(\varphi_i) = \text{Aut}(X_A)k^\times F_2 \text{Im}(\varphi_i).
\]

This means that the map is injective. \(\square\)

**Lemma 3.2.**

\[
|k^\times \setminus Q_i/\sim| = \begin{cases} \infty & \text{for } q = 2, i = 1, 2, \\ 1 & \text{for } q \geq 3, i = 1, 2, 3, \end{cases}
\]
and furthermore when \( q \geq 3 \) one can take a representative \( M_i \) of \( k^x \setminus Q_1 / \sim \) such that

\[
M_1^* = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad M_2^* = M_3^* = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

Proof. For every \( k^x M^\varphi \in k^x \setminus Q_1 / \sim \), acting an appropriate \( \varphi_i(g) \) on \( M \) if necessary, one may assume that \( M \in Q'_i \), and then \( M^* \in B_i \). We refer to the entries of \( M^* \) as \( a_{ij}(M^*) \) or \( b_i(M^*) \) by following the description of the matrices \( B \) of Lemma 2.1.

**The case \( q \geq 3 \):** In this case, \( (a_{12}(M^*), a_{22}(M^*)) = (0, 0) \) and \( b_2(M^*) = 0 \). When \( i = 1 \) has been already proven in [4]. When \( i = 2, 3 \), there is a matrix

\[
g = \begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix} \in \text{GL}_2(k)
\]

such that \( t^{\varphi_i(g)} M \varphi_i(g)^{(q)} = M_i \). In fact, this implies that

\[
b_1(M^*) \lambda^{d_i q} \mu^{d_i} = 1 \quad \text{and} \quad b_3(M^*) \lambda^{(d_i - j)(q + 1)} \mu^{j(q + 1)} = 1,
\]

where \( j = q^2 + 1 \) or \((q^2 + 1)/2\) by whether \( i = 2 \) or \( 3 \), respectively, and solutions \( \lambda, \mu \) of above equations can be found in \( k \) since \( k \) is algebraically closed. This completes the proof for \( i = 2, 3 \).

**The case \( q = 2 \):** When \( i = 1 \), we consider the following matrices:

\[
M^*(x) = \begin{pmatrix}
0 & x & 1 & 0 \\
0 & 1 & 0 & 1 \\
x & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix} \in B_1 \quad \text{for} \quad x \in k.
\]

Then one has by a direct calculation that if \( k^x M(x)^{\varphi_1} = k^x M(x')^{\varphi_1} \) then \( x = x' \). Therefore \( M(x) \) and \( M(x') \) with \( x \neq x' \) belong to distinct classes of \( k^x \setminus Q_1 / \sim \). Since there is an infinite number of choices of \( x, x' \), we conclude that \( |k^x \setminus Q_1 / \sim | = \infty \). When \( i = 2 \), for all \( M \in Q'_2 \) we may assume that \( b_1(M^*) = b_3(M^*) = 1 \) by the same argument as \( q \geq 3 \). Then a direct computation shows that if \( k^x M^{\varphi_2} = k^x N^{\varphi_2} \) for \( M, N \in Q'_2 \) then \( b_2(M^*) = b_2(N^*) \xi \), where \( \xi \) is a cube root of unity. Hence \( M \) and \( N \) in \( Q'_2 \) such that \( b_2(M^*) \neq b_2(N^*) \xi \) belong to distinct classes of \( k^x \setminus Q_2 / \sim \), and thus \( |k^x \setminus Q_2 / \sim | = \infty \). \( \square \)

**Remark 3.3.** As we have seen in the above proof, when \( q = 2 \), for \( i = 1, 2 \) the set \( k^x \setminus Q_1 / \sim \), and thus the set \( T_{d_i} \) modulo \( \text{Aut}(X_4) \), is one-dimensional over \( k \). That is, \( k^x \setminus Q_2 / \sim \) is in one-to-one correspondence with \( k/\langle \omega \rangle \), where \( \langle \omega \rangle \) denotes the cyclic group generated by a primitive cube root \( \omega \) of unity. Moreover denoting a matrix \( B \) of the case I of Lemma 2.1 by

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{pmatrix},
\]

then
By Lemmas 3.1 and 3.2 one sees immediately that $\mod 4$.

**Proof.** For $n = 1$, $2$, $3$ of Theorem 1.1 has been completed.

4. THE STABILIZER OF A CURVE

It remains to determine the number of curves in $T_{d_i}$ of the case $q \geq 3$ and $i = 2, 3$. In this case one has that

$$|T_{d_i}| = |Aut(X_A)|/|Stab(C_i)|,$$

where $Stab(C_i) = k^\times \backslash \Gamma(C_i)$ is the subgroup of $Aut(X_A)$ stabilizing the curve $C_i$ corresponding to $k^\times F_i1m(\varphi_i)$ such that $tF_iAF_i^{(q)} = M_i$ for $M_i$ of Lemma 3.2. We have already known that $Aut(X_A) \simeq PGU_4(F_q^2)$ cf. [4], and thus

$$|Aut(X_A)| = q^6(q^4 - 1)(q^3 + 1)(q^2 - 1).$$

**Lemma 4.1.** The group $Stab(C_2)$ is a cyclic group of order $q^3 + 1$, and $Stab(C_3)$ is a cyclic group of order $(q^3 + 1)/2$ or $(q^3 + 1)/4$ by whether $q$ modulo 4 is 1 or 3, respectively.

**Proof.** For $i = 2, 3$ and $\gamma \in \Gamma(C_i)$, there are $\lambda, \lambda' \in k^\times$ and $g \in GL_2(k)$ such that $t\gamma A \gamma^{(q)} = \lambda A$ and $\gamma = \lambda' F_i^{(q)} \varphi_i(g)^* F_i^{*^{-1}}$ with $tF_i^{* -1}AF_i^{*(q)} = M_i^*$. Hence one has

$$t\varphi_i(g)^* M_i^* \varphi_i(g)^{*(q)} = \lambda M_i^*$$

for some $\lambda \in k^\times$.

We will find a matrix $g$ satisfying this equation. For $i = 2$, the matrix $\varphi_i(g)^*$ is of the following form:

$$\begin{pmatrix}
\alpha & 0 & 0 & bq^2+q \\
\beta & 0 & 0 & b^q, \\
\gamma & 0 & 0 & dq^2+q \\
0 & 0 & 0 & dq^2+q
\end{pmatrix}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

where

$$\alpha = aq^2 - q - 2 \det(g)^{q+1}, \quad \beta = a^{q-2}q^{-q}d^{q} \det(g),$$

$$\gamma = 0, \quad \delta = a^{q-2}d^{q} \det(g).$$

In the equation (1), multiplying $g$ by an appropriate element of $k^\times$ we may assume that $\lambda = 1$. Then by a straightforward computation, it turns out that $g = (a, b; c, d)$ satisfies (1) if and only if $b = c = 0$, $a = \xi^{nq^2}$ and $d = \xi^n$, where $n$ is an integer such that $1 \leq n \leq (q^3 + 1)(q + 1)$ and $\xi$ is a primitive $(q^3 + 1)(q + 1)$-th root of unity. Further one has that $\xi^{nq^2} = \xi^n$ if and only if $n$ is a multiple of $q^3 + 1$. Hence there is an isomorphism

$$\mathbb{Z}/(q^3 + 1)\mathbb{Z} \cong \mathbb{F}_q \longrightarrow k^\times \gamma F_{2}^{*} \varphi_{2}((\xi^{nq^2}, 0; 0, \xi^n))^{*} F_{2}^{*^{-1}} \in Stab(C_2).$$
Similarly, for $i = 3$ one has that $g = (a, b; c, d)$ satisfies (1) if and only if $b = c = 0$, $a = \xi^{nq^2}$ and $d = \xi^n$, where $n$ is an integer such that $1 \leq n \leq (q^3 + 1)(q + 1)/2$ and $\xi$ is a primitive $(q^3 + 1)(q + 1)/2$-th root of unity. Further one has that $\xi^{nq^2} = \xi^n$ if and only if $n$ is a multiple of $(q^3 + 1)/2$ or $(q^3 + 1)/4$ by whether $q$ modulo 4 is 1 or 3, respectively. Hence there is an isomorphism from $\mathbb{Z}/2^{-1}(q^3 + 1)\mathbb{Z}$ or $\mathbb{Z}/4^{-1}(q^3 + 1)\mathbb{Z}$ to Stab($C_3$) as described above.

By Lemma 4.1 it follows immediately that $|T_{d_2}| = q^6(q^4 - 1)(q^2 - 1)$ and $|T_{d_3}| = 2q^6(q^4 - 1)(q^2 - 1)$ or $4q^6(q^4 - 1)(q^2 - 1)$. Thus the proof of Theorem 1.1 has been completed.

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