The integral of complex variables on three defined contours

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Abstract. Complex variables have aroused strong interest in the community of mathematics in the past few decades. Our aim is to obtain the integral results of a particular equation

\[
f(z)=\frac{z^3+(1+2i)z^2-7z+1-2i}{z^4-1},
\]

in three defined contours. To determine the value of integral, the information we need to find is the residue values ABCD on four specific points and find out what residue values each contour includes. After finding the residue values and analysing four contours, we should use Cauchy integral formula to find the integral value. Firstly, by using the partial fraction to separate the equation into four independent parts, we obtain four values on each of the four fractions are the value of integral. Thus we can find exactly the four residue values, which are -1 on (0,0), 3 on (0,i), -2 on (-1,0) and 1 on (-1,i) as shown in the following expression.

\[
f(z)=\frac{z^3+(1+2i)z^2-7z+1-2i}{z^4-1} = \frac{-1}{z-1} + \frac{3}{z-1} + \frac{-2}{z+1} + \frac{1}{z+i},
\]

and then we must analyse each contour and find out the residue values they include. By drawing a diagram of each of the three contours. We find that contour 1 contains A, contour 2 contains A and D, and contour 3 contains ABCD. The final step to obtaining the value of residue is to use Cauchy integral formula

\[
\int y f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res } f(z).
\]

According to Cauchy integral formula, the value of integral of the equation on a specific contour is obtained by multiplying the sum of the residue values included in that contour by 2\pi i, the value of integral of the equation can be found as we now have all the necessary information. For the first contour,

\[
y(t) = 1 + e^{it}, \text{ where } 0 \leq t \leq 2\pi,
\]
the value of integral is calculated by using 2\pi i multiply by -1. We can get the result -2\pi i. For the second contour,

\[
y(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, \text{ where } 0 \leq t \leq 2\pi,
\]
the value of integral is calculated by using 2\pi i multiply by the sum of -1 and 1. As we can see, the sum is 0. We can get the result 0. For the third contour,

\[
y(t) = 2e^{it}, \text{ where } 0 \leq t \leq 2\pi.
\]
the value of integral is calculated by using 2\pi i multiply by the sum of -1,3, -2, and 1. As we can see, the sum of these residue values 1. We can get the result 2\pi i.
f(z)=\frac{z^2+(1+2i)z^2−7z+1−2i}{z^4−1}, \quad (7)

is found.

1. Introduction

In the past decades, the study along curves has attracted a lot of from the community of mathematics. The Kuk, S., Lee, S. studied the endpoint bounds for multilinear integrals Ref. [1]. Next, the Ricci, F. Stein. E.M’s harmonic analysis on fractional integration has been investigated Ref. [2]. Junfeng Li, Peng Liu then study the bilinear integral along cures Ref. [3]. Then, the Chandarana, S transform along curves can be bounded in certain region Ref. [4]. It would appear an additional oscillation along the curve. After Zielinski,M obtain bounds for strongly singular integrals along curves, writers could extend to a higher dimension to explore Ref. [5]. Chen, JC, Fan, DS and M, Zhu have one way of proving something that was bounded during one interval Ref. [6]. Next, The purpose of Cheng,MF, Zang,ZQ is to investigate the bound ability of these integral operators on general modulation spaces Ref. [7]. Therefore, Cheng, MF would concentrate on the study of Fourier integral operators Ref. [8]. Wu, XM, Chen and JC can find the Boundedness fractional integral operators on α-modulation spaces Ref. [9]. Besides this, Stein, EM can use method to calculate its result homogeneous curves Ref. [10]. Laghi, N and Lyall, N could also prove that some terms were bounded on one space, including the in homogeneous spaces and the classical spaces Ref. [11]. In addition, Xiaomei Wu and Xiao Yu can use quite the same way to prove these terms Ref. [12]. At the same time, Grafakos,L can receive the endpoints of multilinear fractional integral Ref. [13]. Gressman, P can obtain the smoothness and inequalities of K-functions Ref. [14]. In the last, Guo, J and Xiao, L can prove that the bilinear Hilbert transforms and maximal functions along certain general plane curves are bounded from certain intervals Ref. [15]. Also, W. Domitrz could obtain that when one developable surface tangent to a surface along a curve on the surface, they call it osculating developable surface along the curve on the surface Ref. [16]. Thus, W. Domitrz and K. Kourliouros study here the relative cohomology and the Gauss-Manin connections associated with an isolated singularity of a function on a manifold with boundary Ref. [17]. Based on the proof of geometric-combinatorial, Natalia Accomazzo, Francesco Di Plinio and Ioannis Parissis could obtain a new way of proving this, and it is singular integrals along lacunary directions Ref. [18]. Then, Alin Bostan, Fernando Chamizo, Mikel P. Sündeqvist furthermore provide a generalization together with an equally elementary proof and discuss some consequences Ref. [19]. Derek Orr then discuss a similar to that of a standard integral Ref. [20]. On the one hand, M. L. Glasser could evaluate the value of certain integrals Ref. [21]. On the other hand, Sergey Goryainov, Elena V. Konstantinova, Honghai Li, Da Zhao could also draw the integral graph by obtaining dual Seidel switching Ref. [22]. Nicola Guglielmi, Maria López-Fernández, Mattia Manucci could determine the integration contour by computing a few pseudo-spectral level sets of the leading operator of the equation Ref. [23]. Gary G. Gundersen, Janne M. Heittokangas, Zhi-Tao Wen found out that The Airy integral is a well-known contour integral solution of Airy’s equation which has several applications and which has been used for mathematical illustrations due to its interesting properties Ref. [24]. Lothar Sebastian Krapp could show that for any o-minimal exponential field (K,exp) satisfying the differential equation, its field is a model of real exponentiation Ref. [25]. Also, Fred Brackx, Hennie De Schepper, Roman Lavicka, Vladimir Soucek as it is the case for the theory of holomorphic functions in the complex plane, the Cauchy Integral Formula has proven to be Clifford’s cornerstone analysis, the monogenic function theory in higher dimensional euclidean space Ref. [26]. Sylwester Zaj continue the research carried out in and the holomorphic hull of L with respect to a domain Ref. [27].

In this paper, we study integrals in particular contours using the partial fraction. The main purpose of using partial fraction is to divide the target function into four small parts, which allow us to find the exact values of integral.

That is,
\[ f(z) = \frac{z^3 + z^2(1+2i) - 7z + 1 - 2i}{z^4 - 1}. \]  

By using partial fraction to this equation, we can get the following form:

\[ f(z) = \frac{A}{z-1} + \frac{B}{z-i} + \frac{C}{z+1} + \frac{D}{z+i}. \]

Besides using The Partial Fraction, we still have to depend on one formula to this project:

\[ \int y f(z) dz = 2\pi i \sum_{k=1}^{n} \text{Res} f(z). \]

Based on this formula, we could resolve those three equations one by one:

\[ y(t) = 1 + e^{i t}, \text{ where } 0 \leq t \leq 2\pi. \]
\[ y(t) = \frac{1-i}{2} + \sqrt{2} e^{i t}, \text{ where } 0 \leq t \leq 2\pi. \]
\[ y(t) = 2e^{i t}, \text{ where } 0 \leq t \leq 2\pi. \]

After analyzing those three equations, we could draw them on graph that would be easy and clear to see.

Also, we have to be careful about the center of each function of them A, B, C and D. However, some are not.

Thus, after using partial fraction, integral formula and obtaining information on the graph, we could solve this project.

2. Mechanism

In this section, we introduce the mechanism of this paper. It consists of two parts, namely the partial fraction and Cauchy’s Residue theorem.

2.1 Partial fraction

In mathematics, there are many complex rational expressions. If we try to solve the problems in a complex form, it will take a lot of time to find the solution. In this case, we need to calculate the integral of the function \( f(z) = \frac{z^3 + z^2(1+2i) - 7z + 1 - 2i}{z^4 - 1} \) in different contours. We need to find the singular points of this functional. Then we use the residue theorem. A singular point is when a given function of a complex variable has no derivative but of which every neighborhood contains points at which the function has derivatives. About this function, singular points are zero when the denominator is 0, which means the function is not analytic. At this moment, partial fraction decomposition is needed, which is used to decompose the rational expressions into simpler partial fractions. This process is beneficial to calculate each value of residues.

2.2 Cauchy’s Residue theorem

The significance of the residue theorem lies in the transformation of the closed curve integral of the complex variable function into the residue of the integrand at the isolated singularity. Since the general integrand has only a few isolated singularities in the corresponding region, it is relatively easy to find the residue of these isolated singularities, so the residue theorem is a very effective method to calculate the closed curve integral of a complex function.

\[ \int_C f(z) dz = 2\pi i \sum_{k=1}^{n} \text{Res} f(z). \]

Here is the proof. The following integral

\[ \int_{-\infty}^{\infty} e^{itx} \frac{dx}{z^4 + 1}, \]

occurs when you compute the proper function of the Cauchy distribution, and it’s impossible to compute it with elementary calculus. We express this integral as the limit of a path integral from \(-A\) to \(A\) along a real line, and then from \(A\) to \(-A\) in a counterclockwise direction along a semicircle centered at zero. Let \(a\) be greater than 1, so that the imaginary unit I surrounds the curve. The path integral is

\[ \int_C f(z) dz = \int_C e^{itx} \frac{dz}{z^4 + 1}. \]
Since $e^{itz}$ is an integral function (without any singularities), this function has a singularity only if the denominator $z^2 + 1$ is zero. Since $z^2 + 1 = (z + i)(z - i)$, this function has a singularity at $z = i$ or $z = -i$. Only one of these two points is in the region surrounded by the path. Since $f(z)$ is

$$\frac{e^{itz}}{z^2 + 1} = \frac{e^{itz}}{2i} \left( \frac{1}{z-i} - \frac{1}{z+i} \right) = \frac{e^{itz}}{2i} \frac{1}{z-i} - \frac{e^{itz}}{2i(z+i)},$$

(17)

The residue of $f(z)$ at $z = i$ is:

$$\text{Res}_{z=i} f(z) = \frac{e^{-t}}{2i}.$$  

(18)

By the residue theorem, we have

$$\oint_C f(z) \, dz = 2\pi i \cdot \text{Res}_{z=i} f(z) = 2\pi i \frac{e^{-t}}{2i} = \pi e^{-t}.$$  

(19)

Path C can be divided into a "straight" section and an arc of a curve such that

$$\int_{\text{straight}} + \int_{\text{arc}} \pi e^{-t}.$$  

(20)

Therefore,

$$\int_{\text{straight}} + \int_{\text{arc}} = \pi e^{-t}.$$  

(21)

If $t>0$, then the integral along the semicircle path goes to zero as the radius of the semicircle goes to infinity

$$\int_{1}^{\infty} e^{itz} \, dz = 0 \text{ as } a \to \infty.$$  

(22)

Similarly, if the curve is bypassing $-i$ instead of $i$, then it can be proved that if $t<0$, then

$$\int_{-\infty}^{\infty} e^{itz} \, dz = \pi e^{t}.$$  

(23)

So we have

$$\int_{-\infty}^{\infty} e^{itz} \, dz = \pi e^{-|t|}.$$  

(24)

Finally, we need to know the Euler form of the complex number and draw the contour of the specific function.

3. Result

We use a partial fraction to do some work on the original function,

$$f(z) = \frac{z^3 + (1+2i)z^2 - 7z + 1 - 2i}{z^4 - 1}.$$  

(25)

Because the denominator is $z^4 - 1$, we can use the reverse of square variation formula to obtain the denominator is $(z+1)(z-1)(z+i)(z-i)$. Here the singular points are 1, -1, i, -i. Then we assume

$$f(z) = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z+i} + \frac{D}{z-i}.$$  

(26)

Here A, B, C, D are the value of residue when the points are at 1, i, -1, -i. Secondly, we need to express the whole numerator to find the exact value of A, B, C, D. For the numerator, we assume

$$z^3 + (1+2i)z^2 - 7z + 1 - 2i = A(z-i)(z+1)(z+i) + B(z-1)(z+i)(z-i) + C(z-1)(z+i)(z+i) + D(z-1)(z-i)(z+1).$$  

(27)

To calculate each value of A, B, C, D individually, we can make $z$ equal to different values, to make the coefficients of three of A, B, C, D zero. If $z=1$ then Eq. (3) becomes,

$$1^3 + (1+2i)1^2 - 7*1 + 1 - 2i = A(1-i)(1+1)(1+i),$$

(28)

$$-4i = 4A,$$  

(29)

$$A = -1.$$  

(30)

If $z=i$ then Eq. (3) becomes,
\[i^3 + (1+2i) \cdot i^2 - 7 \cdot i + 1 - 2i = B(i-1)(i+1)(i+i),\]  
\[-12i = 4iB, \]  
\[B = 3.\] (33)

If \(z = -1\) then Eq. (3) becomes,
\[-1)^3+(1+2i))(-1)^2(-1)+1-2i= C(-1-1)(-1-i)(-1+i),\]  
\[8 = 4C, \]  
\[C = -2.\] (36)

If \(z = -i\) then Eq. (3) becomes,
\[-i)^3+(1+2i))(-i)2-7(-i)+1-2i= D(-i-1)(-i-i)(-i+1),\]  
\[4i = 4iD, \]  
\[D = 1.\] (39)

So, we have obtained the value of \(A, B, C, D.\)

Next, we need to find the contours. The residue values ABCD and the contours for a defined function are obtained. \(A = -1, B = 3, C = -2, D = 1.\)

The first contour is
\[y(t) = 1 + e^{it}, \text{ where } 0 \leq t \leq 2\pi,\] (40)

The coefficient of \(e^{it}\) is 1, so the radius of it is 1. Because \(0 \leq t \leq 2\pi,\) the outline of \(e^{it}\) is a circle with a center at origin and a radius of 1. But when we look back at the contour in (3.15), there’s an extra 1 added. So we need to shift our circle, the circle of \(e^{it}\), one to the right in the horizontal direction. Then we obtain the contour of \(y(t) = 1 + e^{it}, \text{ where } 0 \leq t \leq 2\pi.\) The graph below shows the contour of \(y(t) = 1 + e^{it}, \text{ where } 0 \leq t \leq 2\pi.\)

![Graph of contour in (3.15)](image)

Figure 1. The graph of contour in (3.15).

It contains point \((1,0),\) so we should include residue value \(A.\) The result can be calculated by Cauchy integral formula,
\[
\int y f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} f(z) = -1 \cdot 2\pi i = -2\pi i.
\]
while the second contour is

\[ y(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, \text{ where } 0 \leq t \leq 2\pi, \]  

(43)

The coefficient of \( e^{it} \) is \( \sqrt{2} \), so the radius of it is \( \sqrt{2} \). Because \( 0 \leq t \leq 2\pi \), the outline of \( \sqrt{2}e^{it} \) is a circle with a center at origin and a radius of \( \sqrt{2} \). But when we look back at the contour in (44), there’s an extra \( \frac{1-i}{2} \) added. \( \frac{1-i}{2} \) can be regard as \( \frac{1}{2} - \frac{i}{2} \). So we need to shift the circle of \( \sqrt{2}e^{it}, \frac{1}{2} \) to the right in the horizontal direction, \( \frac{i}{2} \) downwards in the vertical direction. Then we obtain the contour of \( y(t) = \frac{1-i}{2} + \sqrt{2}e^{it} \), where \( 0 \leq t \leq 2\pi \). The graph below shows the contour of \( y(t) = \frac{1-i}{2} + \sqrt{2}e^{it} \), where \( 0 \leq t \leq 2\pi \).

It contains points (1,0) and (-1,0), so we should include residue values A and D. The result can be calculated by the Cauchy integral formula,

\[ \int yf(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res } f(z) \]

(44)

\[ = (-1+1) \ast 2\pi i = 0, \]

(45)

Therefore, the integral value is 0.

Next, the third contour is

\[ y(t) = 2e^{it}, \text{ where } 0 \leq t \leq 2\pi. \]  

(46)

The coefficient of \( e^{it} \) is 2, so the radius of it is 2. Because \( 0 \leq t \leq 2\pi \), the outline of \( 2e^{it} \) is a circle with a center at origin and a radius of 2. The graph below shows the contour of \( y(t) = 2e^{it} \), where \( 0 \leq t \leq 2\pi \).
Figure 3. The graph of contour in (3.21).

It contains points (1,0), (i,0), (-1,0) and (-i,0) so we should include residue values ABCD. The result can be calculated by Cauchy integral formula,

\[ \int y f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} \, f(z) \]

\[ = (-1+3-2+1) \times 2\pi i = 2\pi i, \]

Therefore, the integral value is \( 2\pi i \). Hence we find all of the values of integral.

4. Conclusion

The integral results of a particular equation

\[ f(z) = \frac{z^3 + (1+2i)z^2 - 7z + 1 - 2i}{z^4-1} \]

over three defined contours,

\[ y(t) = 1 + e^{it}, \text{ where } 0 \leq t \leq 2\pi, \]

\[ y(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, \text{ where } 0 \leq t \leq 2\pi, \]

and

\[ y(t) = 2e^{it}, \text{ where } 0 \leq t \leq 2\pi. \]

are obtained through Cauchy integral formula

\[ \int e f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} \, f(z). \]

For the first contour,

\[ y(t) = 1 + e^{it}, \text{ where } 0 \leq t \leq 2\pi, \]

the value of integral is calculated by using \( 2\pi i \) multiply by the residue value on A which is -1. We can get the result \(-2\pi i\).

For the second contour,

\[ y(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, \text{ where } 0 \leq t \leq 2\pi, \]
the value of integral is calculated by using $2\pi i$ multiply by the sum of residue values on A and D which is -1 and 1. As we can see the sum of these residue values on A and D is 0. We can get the result 0.

For the third contour,

$$y(t) = 2e^{it}, \text{ where } 0 \leq t \leq 2\pi.$$  \hspace{1cm} (56)

the value of integral is calculated by using $2\pi i$ multiply by the sum of residue values on A, B, C and D which is -1, 3, -2 and 1. As we can see the sum of these residue values on A, B, C and D is 1. We can get the result $2\pi i$. The methods in our paper can be applied to integrals over more complex functions.

References

[1] Kuk, S., Lee, S, endpoint bounds for multilinear fractional integrals. Math. Res. Lett. 1145–1154 (2012).

[2] Ricci, F. Stein, E.M., harmonic analysis on nilpotent groups and singular integrals.III, fractional integration along manifolds. J. Funct. Anal. 86(2), 360–389 (1989).

[3] Junfeng Li, Peng Liu, Bilinear Fractional Integral Along Homogeneous Curves, J Fourier Anal Appl 23, 1465–1479 (2017).

[4] Chandarana. S, Lp-bounds for hyper singular integral operators along curves, Pac. J. Math. 175(2), 389-416 (1996).

[5] Ziepinski,M, Highly oscillatory singular integrals along curves, PhD thesis, University of Wisconsin, Madison (1995).

[6] Chen. JC, Fan. DS, Wang. M, Zhu. XR, Lp bounds for oscillatory hyper-Hilbert transform along curves, Proc. Am. Math.Soc. 136(9), 3145-3153 (2008).

[7] Cheng. MF, Zhang. ZQ, Hypersingular integrals along homogeneous curves on modulation spaces, Acta. Math. Sin. Chin. Ser. 53(3), 531-540 (2010).

[8] Cheng. MF, Hypersingular integral operators on modulation spaces for 0<p<1, J.Inequal.Appl. 2012, 165 (2012).

[9] Wu. XM, Chen. JC, Boundedness of fractional integral operators on $\alpha$-modulation spaces, Appl. Math. J. Chin. Univ. 29(3), 339-351 (2014).

[10] Stein. EM, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton (1993).

[11] Laghi. N, Lyall. N: Strongly singular integrals along curves, Pac. J. Math. 233, 403-415(2007).

[12] Xiaomei Wu and Xiao Yu, Strongly singular integrals along curves on $\alpha$-modulation spaces, Journal of Inequalities and Applications 2017:185 (2017).

[13] Grafakos. L, On multilinear fractional integrals. Stud. Math. 102 (1), 49–56 (1992).

[14] Gressman. P, Convolution and fractional integration with measures on homogeneous curves on Rn. Math. Res. Lett. 11(5–6), 869–881 (2004).

[15] Guo. J, Xiao. L, Bilinear Hilbert transforms associated with plane curves. J. Geom. Anal. 26(2), 967–995 (2016).

[16] W. Domitrz, Flat approximations of surfaces along curves. Dema-201-0018(2015).

[17] W. Domitrz and K. Kouriouros, Gauss-Manin connections for boundary Singularities and isochore deformations (2015).

[18] Natalia Accomazzo, Francesco Di Plinio and Ioannis Parissis, Singular integrals along lacunary directions in R n.March 2021 DOI: 10.1016 (2021).

[19] Alin Bostan, Fernando Chamizo, Mikael P. Sundqvist, On an integral identity.25 Feb (2020).

[20] Derek Orr, Redefining the integral.on 4 May (2018).

[21] M. L. Glasser, Evaluation of an Integral. 28 Feb (2018).

[22] Sergey Goryainov, Elena V. Konstantinova, Honghai Li, Da Zhao, Integral graphs obtained by dual Seidel switching.14 Jun (2019).

[23] Nicola Guglielmi, Maria López-Fernánde, Mattia Manucci, Pseudospectral roaming contour integral methods for convection-diffusion equations, 2020.
[24] Gary G. Gundersen, Janne M. Heittokangas, Zhi-Tao Wen, Contour integral solutions of linear differential equations which include a generalization of the Airy integral, 2019.
[25] Lothar Sebastian Krapp, Value Groups and Residue Fields of Models of Real Exponentiation, 2018.
[26] Fred Brackx, Hennie De Schepper, Roman Lavicka, Vladimír Soucek, The Cauchy Integral Formula in Hermitian, Quaternionic and osp(4|2) Clifford Analysis, 2019.
[27] Sylwester Zaj, The product of domains in several complex variables, 2021.