Abstract

We comprehensively reveal the learning dynamics of deep neural networks (DNN) with batch normalization (BN) and weight decay (WD), named as Spherical Motion Dynamics (SMD). Our theorem on SMD is based on the scale-invariant property of weights caused by BN, and regularization effect of WD. SMD shows the optimization trajectory of weights is like a spherical motion; and a new indicator, angular update is proposed to measure the update efficiency of DNN with BN and WD. We rigorously prove that the angular update is only determined by pre-defined hyper-parameters (i.e. learning rate, WD parameter and momentum coefficient), and provide their quantitative relationship. Most importantly, the quantitative result of SMD can perfectly match the empirical observation in complex and large scale computer vision tasks like ImageNet and COCO with standard training schemes. SMD can also yield reasonable interpretations on some phenomena about BN from an entirely new perspective, including avoidance of vanishing and exploding gradient, no risk of being trapped into sharp minima, and sudden drop of loss when shrinking learning rate. Further, to present the practical significance of SMD, we discuss the connection between SMD and commonly used learning rate tuning scheme: Linear Scaling Principle.

1 Introduction

Both batch normalization (BN) \cite{12} and weight decay (WD), aka \(l_2\) regularization, are most commonly adopted techniques for training deep neural networks (DNN). The latter is designed to prevent overfitting in classic statistics and machine learning, while BN are especially effective for accelerating training and avoiding gradients vanishing/exploding. Despite of the practical success of BN and WD in deep learning, their underlying mechanisms have not been fully understood. Some works have discussed the function of BN and WD, seperately \cite{12, 24, 17, 1}. However, BN and WD are typically applied \textit{jointly}, and the interaction between them is essential for achieving practical success in various deep learning tasks \cite{26, 10, 4, 14, 28}. Thus, understanding the joint effect of BN and WD could raise both theoretical interests and algorithmic significance for deep learning.

Most existing studies along this line focus on the behavior of the so called “effective learning rate” during training. The concept is invented in \cite{4} due to the scale invariance induced by BN, i.e. with BN,
Figure 1: Illustration of optimization behavior with BN and WD. The gradient of WD (red arrow) tends to reduce the weight’s norm; while the gradient of loss (green arrow) tries to increase weights’ norm. Angular update $\Delta_t$ represents the angle between the updated weight $W_t$ and its former value $W_{t+1}$.

the $l^2$ norm of trainable weight $|w|$ will not affect the performance of DNN at all; and the gradient of weight is always orthogonal to the weight (see Figure 1). Hence, rather than the learning rate $\eta$, the relative value of learning rate $\eta/|w|^2$, i.e. effective learning rate, is perhaps more reasonable to represent the update efficiency. Previous works have shown the property of effective learning rate from different aspects: [26] showed the effective learning rate is inversely proportional to square root of learning rate due to BN and WD; [10] further pointed out that with BN and WD effective learning rate should be inversely proportional to the square of weight’s norm; the authors in [3] showed the effective learning rate should be controlled by learning rate and WD coefficient; [14] proved that training neural networks with BN and WD with constant learning rate is exactly same as with BN using exponentially increasing learning rate schedule. However, these analysis are only able to be verified in toy experiments with specific settings, and their practical significance is not clear. Thus a comprehensive understanding on the learning dynamics of DNN with BN and WD is still required.

Figure 2: Comparison between theoretical and empirical value of angular update in ImageNet and COCO experiments. The solid lines with different colors represent the angular update of weights from different convolution layers (with BN). Note that the training settings rigorously follow [6,7] respectively, except the norm of scale-invariant weight is divided by $\sqrt{10}$ when learning rate is divided by 10 (according to [16]). We emphasize that angular update still can reach theoretical value without any manual adjustment after enough iterations. Further discussion can be seen in Section 4.1.

To address this issue, in this work, we reveal the learning dynamics of DNN with BN and WD, named as Spherical Motion Dynamics (SMD). We find for scale-invariant weight, the optimization trajectory is like a spherical motion with centripetal force (from WD) and centrifugal force (from gradients of loss orthogonal to weight, see Figure 1), the spherical radius ($l_2$ norm of weight) is
continuously changing until equilibrium condition is reached. Therefore, instead of effective learning rate, we discover SMD relies on the behavior of angular update:

**Definition 1. (Angular Update)** Assuming $w_t$ (vectorized) is a scale-invariant weight from a neural network at iteration $t$, then the angular update $\Delta_t$ is defined as

$$\Delta_t = \angle(w_t, w_{t+1}) = \arccos\left(\frac{\langle w_t, w_{t+1} \rangle}{||w_t|| \cdot ||w_{t+1}||}\right),$$

where $\angle(\cdot, \cdot)$ denotes the angle between two vectors, $\langle \cdot, \cdot \rangle$ denotes the inner product, $||\cdot||$ denotes the $l_2$ norm of the vectorized weights.

Compared with learning rate and effective learning rate, angular update summaries the effect of gradient norm, learning rate and weight norm, which make it more accurate to measure the update efficiency of scale-invariant weights. Our main result on angular update is described as follows:

**Theorem 1. (Main, Informal)** Assume a DNN with BN and WD is trained using SGD/SGDM (SGD with momentum, heavy ball method [21]). The WD coefficient is $\lambda > 0$, the momentum coefficient is $\alpha \geq 0$, and the learning rate is $\eta > 0$. Then angular update $\Delta_t$ satisfies:

$$\lim_{t \to \infty} \Delta_t \approx \sqrt{\frac{2\eta\lambda}{1 + \alpha}}$$

Note that theorem [1] is not only a qualitative but quantitative result, it can perfectly match the empirical observations even including large-scale computer vision tasks such as ImageNet classification [22] and COCO detection tasks [16] (see Figure 2). Many previous works [4, 26, 10, 3, 14] can be viewed as deductions of theorem [1]. The theoretical result on SMD can provide us some insights about the function of BN, and has strong practical significance. As an example, we show the connection between SMD and commonly used learning rate tuning scheme Linear Scaling Principle [6].

Specifically, our contribution can be summarized as follows:

- We propose a novel theoretical perspective, Spherical Motion Dynamics (SMD), and angular update to depict the learning dynamics of DNN with BN and WD. We also show it is indispensable to take SMD into account when analyzing the learning dynamics of DNN with BN and WD, since the two components will make the commonly used assumptions indefensible in optimization theory. Comparing with previous most related work [3, 14], we provide more rigorous theoretical justification on SMD, and clarify its practical significance.

- Under the view of SMD, we derive the theoretical value of the angular update, which can perfectly match complex application examples like ImageNet and COCO. Furthermore, we provide new interpretations on some experimental phenomena: why BN and WD can avoid gradient explosion/vanishing, and lead optimization trajectory to flat minima; why manually decaying learning rate is indispensable to reduce the loss, and it can lead to sudden drop of loss in practice;

- To demonstrate the practical significance of SMD, we discuss the effect of Linear Scaling Principle [6] under the view of SMD.

The remaining of the paper is structured as follows. In Section 2, we analyze the scale-invariant property of weight caused by BN; In Section 3, we point out that WD and BN are probably the main reason why many theoretical results about optimization cannot be verified in practice. Then, we reveal and prove the existence of Spherical Motion Dynamics when training DNN with BN and WD. Moreover, we derive the theoretical value of angular update, and the properties of SMD; In Section 4, we show the theoretical results of angular update can perfectly match observations in ImageNet and COCO experiments, and present the performance of Linear Scaling Principle under the view of SMD.

## 2 Scale-invariance with Batch Normalization

Given a neural network and its loss function $\mathcal{L}(X; W)$, $W_l \in W$ is the weight from a linear transform layer $l$ with batch normalization, where

$$Y_l = W_lX_l, \quad Z_l = \frac{Y_l - \mathbb{E}Y_l}{\sqrt{\text{Var}(Y_l)}}.$$ 

(3)
Note with BN, $W_l$ cannot be equal to zero, the domain of $W_l$ is defined on $\mathbb{R}^{m \times n} \backslash \{0\}$. Therefore, we have

**Lemma 1. (Scale-invariant Property)** If transform layer $l$ has BN, then $\forall k > 0$,
\[
L(X; kW_l, W \backslash \{W_l\}) = L(X; W).
\]

Then we say $w_l$ is scale invariant. For ease of demonstration, in the following context we will no longer distinguish $W_l$ and $W$, if the $W$ denotes part of scale-invariant weights, the rest of weights will be omitted in $L(W)$. Besides, we only use the vectorized formulation $w$ to represent weight $W$. Assume $w \in \mathbb{R}^p \backslash \{0\}$, let $\tilde{w}$ denote the projection of $w$ on the unit sphere $S^p$, i.e. $\tilde{w} = w/||w||$.

Lemma[1] shows the scale of weight with BN does not influence the output of DNN at all, and suggests that we should consider unit sphere $S^{p-1}$ rather than $\mathbb{R}^p \backslash \{0\}$ as essential optimizing domain when analyzing the dynamics of training. It also implies the necessity to replace $||w_t - w_{t-1}||$ with angular update $\Delta_t$ to measure the update of DNN within a single iteration. The scale-invariant property can derive important properties of weight $w$ and its gradients w.r.t. $L(w)$, i.e.

**Lemma 2.** If $w$ is scale-invariant with respect to $L(w)$, then for all $k > 0$, we have:
\[
\langle w, \nabla L \bigg|_{w=w_t} \rangle = 0, \quad \frac{\partial L}{\partial w} \bigg|_{w=kw_t} = \frac{1}{k} \frac{\partial L}{\partial w} \bigg|_{w=w_t}.
\]

Proof can be seen in Appendix [A.1] These properties are also discussed in [10, 26, 14]. For easy demonstration, we call $\frac{\partial L}{\partial w}$ as unit gradient of $w_t$. According to Lemma [2], unit gradient satisfies
\[
\frac{\partial L}{\partial w} \bigg|_{w \in S^p} = \frac{1}{||w_t||} \frac{\partial L}{\partial w} \bigg|_{w=w_t}.
\]

We can easily observe an interesting phenomenon when using GD/SGD to update DNN without WD but BN and fixed learning rate $\eta$: on the one hand, $||w_t||$ monotonically increases as the number of iterations increases because $||w_{t+1}||^2 = ||w_t||^2 + ||\nabla L||^2 \leq ||w_t||^2 + \eta^2$. On the other hand, according to Lemma [2] if $\tilde{w}_t$ (direction of $w_t$) is fixed, $||\nabla L/\partial w||$ is inversely proportional to the weight norm $||w_t||$. Increasing $||w_t||$ will also reduce $||\nabla L/\partial w||$. Hence if we find an $\epsilon$-stationary point $w^*$ via optimization, i.e. $||\nabla L/\partial w|| < \epsilon$, it is hard to clarify whether we achieve a true stationary point (i.e. is small) in $S^{p-1}$ or just because $||w^*||$ is too large. Similar phenomenon is also discussed in [11, 10]. It can explain why in practice we still need WD to control the norm of weights.

### 3 Spherical Motion Dynamics

#### 3.1 Singularity of Loss Landscape

We have demonstrated the necessity of WD for DNN with BN. Now we show that with BN and WD, two assumptions, which are commonly used in optimization theory, do not hold anymore. Assume the objective loss function is,
\[
F(w) = L(w) + \frac{\lambda}{2} ||w||^2,
\]
where $w$ is scale-invariant with respect to $L(w)$, then we have the following lemmas.

**Lemma 3.** $\forall w \in \mathbb{R}^p \backslash \{0\}, \forall k \in (0, 1), F(kw) < F(w)$.

**Lemma 4.** $\frac{\partial F}{\partial w}$ is not Lipschitz continuous in any neighborhood of 0.

Proof of lemma [3] and [4] can be seen in Appendix [A.2, A.3]. They imply that the existing optimization analysis is not suitable for the function $F(w)$: Lemma [3] shows optimization trajectory should get close to 0 but the optimal point doesn’t exists; Lemma [4] shows $F(w)$ does not satisfy the Lipschitz continuous condition in the neighborhood of 0, which is commonly used in most of optimization theories [2] [25] [19]. These absurd results are caused by the existence of the singularity 0. We believe the singularity 0 is one of main reasons why it is extremely hard to verify existing theoretical results of DNN on DNN with BN and WD in practice.
3.2 Spherical Motion Dynamics and Equilibrium Condition

Since it is not proper to study the learning dynamics of $F(w)$ on $\mathbb{R}^p \setminus \{0\}$, directly exploring the optimization behavior of $\mathcal{L}(\dot{w})$ on spherical domain $\mathbb{S}^{p-1} = \{ \dot{w} \in \mathbb{R}^p ||\dot{w}|| = 1 \}$ seems to be a reasonable substitute, for $\mathcal{L}(\dot{w})$ is well defined on $\mathbb{S}^{p-1}$ (continuously smooth on a compact set). However, we emphasize that the optimization behavior of commonly used SGD/SGDM on $\mathcal{L}(w)$ with BN and WD is not similar to the optimization behavior of manifold-based SGD/SGDM on $\mathcal{L}(\dot{w})$. Take SGD as an example:

$$w_{t+1} = w_t - \eta (\frac{\partial L}{\partial w} |_{w=w_t}) + \lambda w_t = (1 - \eta \lambda)w_t - \eta (\frac{\partial L}{\partial w} |_{w=w_t}).$$

If $\eta \ll 1$, according to Lemma 2, the angular update $\Delta_t$ is:

$$\Delta_t = \tan(\Delta_t) \approx \tan(\Delta_t) = \frac{\eta}{(1 - \eta \lambda) ||w_t||^2} \cdot ||\frac{\partial L}{\partial w} |_{w=w_t}||$$

The effective learning rate $\tilde{\eta} = \frac{\eta}{(1 - \eta \lambda) ||w_t||^2}$ is influenced by $||w_t||$, which is same as the claim of previous work. Now we explore the norm of updated weight $w_{t+1}$:

$$||w_{t+1}||^2 = (1 - \eta \lambda)^2 ||w_t||^2 + \eta^2 ||\frac{\partial L}{\partial w} |_{w=w_t}||^2 = (1 - \eta \lambda)^2 ||w_t||^2 + \frac{\eta^2}{||w_t||^2} \cdot ||\frac{\partial L}{\partial w} |_{w=w_t}||^2.$$

Rewrite (10) as

$$\frac{||w_{t+1}||^2}{||w_t||^2} = 1 + \left[ \frac{\eta^2}{||w_t||^4} \cdot ||\frac{\partial L}{\partial w} |_{w=w_t}||^2 - (2\eta \lambda - \eta^2 \lambda^2) \right]$$

Notice $\partial L/\partial w |_{w=w_t}$ is the unit gradient of $w_t$, so its norm $||\partial L/\partial w |_{w=w_t}||$ is independent of $||w_t||$. The update rule (8) is like a spherical motion, see Fig. 1 for illustration. The gradient of loss $-\eta (\partial L/\partial w |_{w=w_t})$ provides the centrifugal force to increase $||w_{t+1}||$, while the gradient of weight decay $-\eta \lambda w_t$ provides the centripetal force to reduce $||w_{t+1}||$. The centrifugal force is inversely proportional to $||w||^3$, the centripetal force is proportional to $||w||$. Intuitively, due to the positive (negative) correlation between centripetal (centrifugal) force and weight norm, the equilibrium condition of centripetal and centrifugal force must be reached, i.e.

$$\frac{\eta^2}{||w_t||^4} \cdot ||\frac{\partial L}{\partial w} |_{w=w_t}||^2 = (2\eta \lambda - \eta^2 \lambda^2).$$

When the equilibrium condition holds, combining with Eq. (9), and $\eta \lambda \ll 1$, we have

$$\Delta_t = \frac{\eta}{(1 - \eta \lambda) ||w_t||^2} \cdot ||\frac{\partial L}{\partial w} |_{w=w_t}|| = \frac{\sqrt{2\eta \lambda - \eta^2 \lambda^2}}{1 - \eta \lambda} \approx \sqrt{2\eta \lambda}.$$

Eq. (13) shows that when equilibrium condition is reached, the $||w_t||$ keeps unchanged if unit gradient is steady (however, it does not mean $w_t$ converges), and the angular update $\Delta_t$ is only determined by the predefined learning rate $\eta$ and WD coefficient $\lambda$. Such phenomenon is caused by joint effect of WD and gradient (orthogonal to weight), which automatically adjusts the weight norm until equilibrium condition is reached during optimization. This is why we call the learning dynamics of DNN with BN and WD as Spherical Motion Dynamics.

Eq. (12), (13) are derived from SGD case, similar result can be derived on SGDM case, we can provide rigorous proof under mild conditions, shown in the following theorem.

**Theorem 2. (Equilibrium Condition in Spherical Motion Dynamics)** Assume the loss function is $\mathcal{L}(X; w)$ with scale-invariant weight $w$, $g_t = \frac{\partial L}{\partial w} |_{X_t, w_t}$. Considering the heavy ball method [27],

$$w_t = w_{t-1} - \eta_1 v_{t-1}, \quad v_t = \alpha v_{t-1} + g_{t-1} + \lambda w_{t-1}$$

where $\alpha \in [0, 1)$, $\eta_1, \lambda \in (0, 1)$. Assume 1) $\exists \delta, L \in \mathbb{R}^+$, unit gradient $\tilde{g}_t = g_t \cdot ||w_t||$ satisfies $\lim_{t \to \infty} ||\tilde{g}_t||/L - 1 < \delta$, $\delta \ll 1$, 2) $\lambda \eta \ll 1$. Then For SGD ($\alpha = 0$), we have

$$\lim_{t \to \infty} \frac{||w_t||^2}{L \sqrt{\eta}/(2\lambda)} - 1 < \delta, \quad \lim_{t \to \infty} \frac{||\Delta_t||}{\sqrt{2\lambda \eta}} - 1 \leq \delta.$$
For SGDM ($\alpha \neq 0$), assume 3) $\lambda \eta < (1 - \sqrt{\alpha})^2$, 4) $\exists \varepsilon, |\langle g_t, v_{t-1} \rangle| \leq \varepsilon ||g_t||^2$, $\varepsilon \ll 1$, and assumptions 1), 2) also hold, let $K_\alpha = \frac{7(1+\alpha)}{(1-\alpha)}$, we have
\[
\lim_{t \to \infty} \left| \frac{||w_t||^2}{L \sqrt{\frac{\eta}{\lambda(1-\alpha)}(2 - \lambda \eta \frac{1}{1+\alpha})}} - 1 \right| \leq K_\alpha (\delta + \varepsilon),
\lim_{t \to \infty} \frac{\Delta_t}{\sqrt{2 \lambda \eta / (1 + \alpha)}} - 1 \leq K_\alpha (\delta + \varepsilon) \quad (16)
\]

Proof. See Appendix A.4.

Remark 1. In practical implementation, learning rate $\eta$ and WD coefficient $\lambda$ are typically very small, so the assumptions 2), 3) usually hold; Assumption 1) and 4) are technical assumptions, but they all relies on empirical observations, we conduct extensive experiments to show the reasonableness of assumption 1) and 4), the results can be seen in Appendix B.1.

Comparison with existing works. [3] also analyzes the equilibrium condition (Eq. (13)), but their discussion is limited on SGD without momentum, and lack of rigorous proof such equilibrium will be certainly achieved with constant learning rate; [14] derives similar results as Eq. (16) in momentum case, but their proof is based on very strong technical assumptions that the convergence of weight norm $||w_t||$ has been known. While in our theorem, convergence of weight norm is a crucial result, and the assumptions are much weaker (see further discussions in Appendix B.1). Besides, our analysis is of both theoretical and practical significance, providing reasonable explanations on the deep learning behaviors (Sec. 3.3) and algorithmic design (Sec. 4).

3.3 Properties of Spherical Motion Dynamics

Theorem 2 presents the theoretical value of the angular update, but its significance are far beyond that. Corollaries derived from Theorem 2 can provide new theoretical explanations on some phenomena about BN that have not been well understood before.

Corollary 2.1. The scale-invariant weight cannot suffer from vanishing or exploding gradients.

It has been a long time since researchers realize BN can prevent the gradients of weight from vanishing or exploding (relative scale of gradient norm is too large or small) [24, 12], but the mechanism is ambiguous before. Now under the view of SMD, the reason is clear. As long as the scale-invariant weight and its gradient are not exactly equal to zero, SMD will adaptively adjust the weight norm to reach the equilibrium condition (Eq. (16)), no matter what the scale of unit gradient is. Consequently, the relative scale of gradient and weight norm at initialization will not affect the angular update in equilibrium condition at all.

Corollary 2.2. With BN and WD, SGD/SGDM cannot get trapped in a sharp local minimum, whose diameter is smaller than $\sqrt{\frac{2\eta \lambda}{1+\alpha}}$.

It is well-known that gradient-based method may get trapped in a sharp local minimum or saddle point due to tiny gradient norm [13], there have been a large number of works [5, 29, 27] discussing that SGD/SGDM can escape from sharp local minima or saddle point with high probability. However, no previous works have ever explicitly demonstrated that with BN and WD, gradient-based approaches can certainly escape from sharp local minima (in this paper, the local minima is defined on the unit sphere domain $S^{p-1}$). According to Theorem 2 when equilibrium condition holds, the SGD/SGDM in SMD is similar to SGD/SGDM on unit sphere $S^{p-1}$ with fixed update (same as angular update), therefore the optimization trajectory can only oscillate in flat local minima whose diameter is larger than $\sqrt{\frac{2\eta \lambda}{1+\alpha}}$.

Corollary 2.3. With BN and WD, optimization will not converge unless manually decreasing the learning rate $\eta$.

With constant learning rate, SMD can guarantee that SGD/SGDM will certainly move to flat minima. However, such phenomenon can yield side effect: it also prevents the optimization of DNN from entirely convergence. Even SGD/SGDM finds a flat minimum, the angular update can be too large to allow optimization trajectory to reach the bottom of minimum. Therefore, in practice sophisticated schemes are required to manually shrink learning rate during training. SMD can also explain why
with multi-step learning rate decay schedule [6], the value of loss function will immediately drop as soon as shrinking learning rate: by Eq. (16), smaller learning rate means yields smaller angular update, which allows optimization trajectory to slip to the bottom of minima.

4 Experiments

4.1 Empirical Study on ImageNet and COCO

In this section, we will show that our results can be perfectly verified in ImageNet [22] Classification and COCO [16] detection and segmentation tasks, with the most commonly used models and experimental settings in computer vision community as a strong baseline.

In ImageNet classification task, we adopt Resnet50 [9] as baseline for it is a widely recognized network structure. The training settings rigorously follow [6]: learning rate is initialized as 0.1, and divided by 10 at 30, 60, 80-th epoch; the WD coefficient is $10^{-4}$; the momentum coefficient is 0.9. In COCO experiment, we conduct experiments on Mask-RCNN [8] benchmark using a Feature Pyramid Network (FPN) [15], ResNet50 backbone and SyncBN [20] following the 4x setting in [7]: total number of iteration is 360,000, learning rate is initialized as 0.02, and divided by 10 at iteration 300, 000, 340, 000; WD coefficient is $10^{-4}$. We also evaluate the performance of other structures, further experiment results can be seen in Appendix B.2.

![Figure 3](a) Resnet50 on ImageNet (b) Mask-RCNN on COCO

Figure 3: (a) Angular update of Resnet50 on ImageNet, training settings follow [6]; (b) Angular update of Mask-RCNN on COCO, training settings follow [7]. In both figures, the solid line with various colors represent weights of different convolutional layers as long as they are scale-invariant (followed by BN). Black dashed line represent the theoretical value of angular update within each learning rate stage. The theoretical value is computed by $\sqrt{2\lambda\eta/(1 + \alpha)}$.

![Figure 4](a) ResNet50 on ImageNet (b) Mask-RCNN on COCO

Figure 4: The change of norm of weight from "layer1.0.conv2" in ResNet50 Backbone. Blue line represents the change in standard training setting following [6]; Orange line represents the change in same training setting as [6], except the norm of weight is divided by $\sqrt{10}$ when learning rate is divided by 10 to skip the process back to new equilibrium condition.
Figure 3 presents the angular update of weights in all convolution layers with BN in different tasks. In ImageNet experiments, during the first ($\eta = 0.1$) and second ($\eta = 0.01$) learning rate stage, we can observe that the angular update of the all weights eventually comes to be steady, and perfectly match the theoretical values though the weights are of different size or layer position. Similar phenomenon can be observed in the first learning rate stage ($\eta = 0.02$) in COCO experiment. These results can strongly support our main theoretical results.

However, there seems to be some mismatch between the theoretical prediction and the practical performance in Figure 3, the angular update in the last two learning rate stage is smaller than theoretical value. The mismatch can be well interpreted by theoretical deduction. Here we only discuss the case without momentum ($\alpha = 0$): according to Eq (17) and (18): when the equilibrium condition is reached, we have $$||\partial L/\partial w||/\sqrt{2\lambda} = ||w_t||/\sqrt{\eta}$$. This implies that if learning rate is divided by $k$ ($k > 1$) (assume the current learning rate is $\eta$), the balance between centripetal and centrifugal force of SMD will break up, hence the weight norm has to be $\sqrt{k}$ times smaller to reach the new equilibrium condition. But new equilibrium condition cannot be achieved immediately (see blue lines in Figure 3), we can roughly estimate the convergence rate from previous equilibrium condition to the new one based on the fact that $||w_{t+m}|| > [1 - \lambda \eta^m]||w_t||$, at least $m = \lceil[\log(k)]/(4\lambda \eta)\rceil$ iterations are required to reach the new equilibrium condition (Note it is only a lower bound of convergence rate, proof can be seen in Appendix A.5). In the third learning rate stage ($\lambda = 10^{-4}$, $\eta = 10^{-3}$) in Figure 3(a) at least 5,756,463 iterations are required to reach the new equilibrium condition, which is far more than actual iterations($10^5$) during the third learning rate stage. Similar reasons hold for the fourth learning rate stage in Figure 3(a) and the second, third learning rate stages in Figure 3(b).

In fact, we can let SMD skip the process to reach the new equilibrium condition after learning rate decay via a simple modification: divide the norm of weight by $\sqrt{k}$ as long as dividing learning rate by $k$. In our experiments, when the learning rate is shrunk to 10 times smaller, norm of all scale-invariant weights will be divided by $\sqrt{10}$ to make sure that equilibrium condition still holds (see orange lines in Figure 3). The angular update in rescaling cases is shown in Fig. 2(a) 2(b) The experiment results prove our theorem on SMD can perfectly match empirical observation.

### 4.2 Rethinking Linear Scaling Principle in Spherical Motion Dynamics

In this section, we will discuss the effect of Linear Scaling Principle (LSP) under the view of SMD. Linear Scaling Principle is proposed by [6] to tune the learning rate $\eta$ with batch size $B$ by $\eta \propto B$. The intuition of LSP is if weights do not change too much within $k$ iterations, then $k$ iterations of SGD with learning rate $\eta$ and minibatch size $B$ (Eq. (17)) can be approximated by a single iteration of SGD with learning rate $k\eta$ and minibatch size $kB$ (Eq. (18)).

$$w_{t+k} = w_t - \eta \sum_{j<k} \frac{1}{B} \sum_{x \in B_j} \frac{\partial L}{\partial w_t} \Big|_{w_{t+j},x} + \lambda w_{t+j}, \quad \text{(17)}$$

$$w_{t+1} = w_t - k\eta \sum_{j<k} \frac{1}{kB} \sum_{x \in B_j} \frac{\partial L}{\partial w_t} \Big|_{w_{t},x} + \lambda w_{t}. \quad \text{(18)}$$

[6] shows that combining with gradual warmup, LSP can enlarge the batch size up to 8192 ($256 \times 32$) without severe degradation on ImageNet experiments.

LSP has been proven extremely effective in a wide range of applications. However, from the perspective of SMD, the angular update mostly relies on the pre-defined hyper-parameters, and it is hardly affected by batch size. To clarify the connection between LSP and SMD, we explore the learning dynamics of DNN with different batch size by conducting extensive experiments with ResNet50 on ImageNet, the training settings rigorously follow [6]: momentum coefficient is $\alpha = 10^{-4}$; WD coefficient is $\lambda = 10^{-4}$; Batch size is denoted by $B$; learning rate is initialized as $\frac{B}{256} \cdot 0.1$; Total training epoch is 90 epoch, and learning rate is divided by 10 at 30, 60, 80 epoch respectively.

The results of experiments(Figure 5)[6] suggests that the assumption of LSP does not always hold in practice because of three reasons: first, the approximate equivalence between a single iteration in large batch setting, and multiple iterations in small batch setting can only hold in pure SGD formulation, but momentum method is far more commonly used; Second, according Theorem 2 the enlargement ratio of angular update is only determined by the increase factor of learning rate. Figure
shows in practice, the accumulated angular update $\angle (\mathbf{w}_t, \mathbf{w}_{t+k})$ in small batch batch setting is much larger than angular update $\angle (\mathbf{w}_t, \mathbf{w}_{t+1})$ of a single iteration in larger batch setting when using Linear Scaling Principle; Third, even in pure SGD cases, the enlargement of angular update still relies on the increase of learning rate, and has no obvious connection to the enlargement of gradient’s norm when equilibrium condition is reached (see Figure 6).

In conclusion, though LSP usually works well in practical applications, SMD suggests we can find more sophisticated and reasonable schemes to tune the learning rate when batch size increases.

5 Conclusion

In this paper, we comprehensively reveal the learning dynamics of DNN with BN and WD, Spherical Motion Dynamics (SMD). Our theorem on SMD is only based on mild conditions with practical significance, and it can perfectly match the empirical observations in various large and complex computer vision tasks like ImageNet and COCO. We believe our theoretical results on SMD can bridge the gap between current theoretical progress and practical usage on deep learning techniques when involving BN and WD. From the view of SMD, we can rethink other important normalization and regularization schemes, which will be left as future work.
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A Proof of Theorems

Remark 2. In the following context, we will use the following conclusion multiple times: \( \forall \delta, \varepsilon \in \mathbb{R}, \) if \( |\delta| \ll 1, |\varepsilon| \ll 1, \) then we have:

\[
(1 + \delta)^2 \approx 1 + 2\delta, \quad \sqrt{1 + \delta} \approx 1 + \frac{\delta}{2}, \quad \frac{1}{1 + \delta} \approx 1 - \delta, \quad (1 + \delta)(1 + \varepsilon) \approx 1 + \delta + \varepsilon. \tag{19}
\]

A.1 Proof of Lemma 2

Proof. Given \( w_0 \in \mathbb{R}^p \setminus \{ 0 \}, \) since \( \forall k > 0, \mathcal{L}(w_0) = \mathcal{L}(kw_0), \) then we have

\[
\frac{\partial \mathcal{L}(w)}{\partial w} \bigg|_{w=w_0} = \frac{\partial \mathcal{L}(kw)}{\partial w} \bigg|_{w=w_0} = \frac{\partial \mathcal{L}(w)}{\partial w} \bigg|_{w=kw_0}, \quad k
\]

\[
\frac{\partial \mathcal{L}(kw)}{\partial k} \bigg|_{w=w_0} = \langle \frac{\partial \mathcal{L}(w)}{\partial w} \bigg|_{w=kw_0}, w_0 \rangle = k \cdot \left( \frac{\partial \mathcal{L}(w)}{\partial w} \bigg|_{w=w_0}, w_0 \right) = 0 \tag{21}
\]

\[
\square
\]

A.2 Proof of Lemma 3

Proof. Since \( \mathcal{L}(w) \) is scale-invariant, then \( \forall k \in (0, 1), \) we have

\[
F(kw) = \mathcal{L}(kw) + \frac{\lambda}{2} k^2 ||w||^2
= \mathcal{L}(w) + \frac{\lambda}{2} k^2 ||w||^2
< \mathcal{L}(w) + \frac{\lambda}{2} ||w||^2
= F(w) \tag{22}
\]

\[
\square
\]

A.3 Proof of Lemma 4

Proof. Assume \( \exists w_1, w_2 \) satisfying \( \partial \mathcal{L}/\partial w \big|_{w=w_1} \neq \partial \mathcal{L}/\partial w \big|_{w=w_2}, \) then \( \forall L > 0, \) set \( k < \min \left( \frac{||\partial \mathcal{L}/\partial w \big|_{w=w_1} - \partial \mathcal{L}/\partial w \big|_{w=w_2}||}{(L+\lambda)||w_1-w_2||}, 1 \right), \) we have

\[
\geq \frac{1}{k} \left( \frac{\partial \mathcal{L}(w)}{\partial w} \bigg|_{w=kw_1} - \frac{\partial \mathcal{L}(w)}{\partial w} \bigg|_{w=kw_2} \right)
> (L + \lambda - \lambda k) ||w_1 - w_2||
\geq Lk ||w_1 - w_2||
\]

Since \( k \) is arbitrarily small, \( \partial F(x)/\partial w \) is not Lipschitz continuous in any neighborhood of 0. \( \square \)

A.4 Proof of Theorem 2

Lemma 5. If the sequence \( \{x_t\}_{t=1}^\infty \) satisfies

\[
x_t = \alpha x_{t-1} + \frac{L_t}{x_{t-1}} \tag{24}
\]

. If \( x_1 > 0, \lim_{t \to \infty} L_t < l_2, \lim_{t \to \infty} L_t > l_1, 0 < l_1 < l_2, \alpha > \frac{1}{2} \).
Then, we have
\[
\lim_{t \to \infty} x_t \geq \sqrt{\frac{l_1}{1 - \alpha}} \tag{25}
\]
\[
\lim_{t \to \infty} x_t \leq \sqrt{\frac{l_2}{1 - \alpha}} \tag{26}
\]

**Proof.** Consider $t$ is sufficient large, then $l_1 \leq L_t \leq l_2$. If $x_t \geq \sqrt{\frac{l_1}{1 - \alpha}}$, since $\sqrt{l_1/(1 - \alpha)} \geq \sqrt{l_1/\alpha}$ then we have
\[
x_{t+1} \geq \alpha x_t + \frac{l_1}{x_t} \geq \alpha \sqrt{\frac{l_1}{1 - \alpha}} + \frac{l_1}{\sqrt{l_1/(1 - \alpha)}} = \sqrt{\frac{l_1}{1 - \alpha}}, \tag{27}
\]
which means $\forall k > t, x_k \geq \sqrt{\frac{l_1}{1 - \alpha}}$. If $x_t < \sqrt{\frac{l_1}{1 - \alpha}}$, then
\[
x_{t+1} - x_t = \frac{L_t}{x_t} - (1 - \alpha)x_t > \sqrt{\frac{l_1}{\sqrt{l_1/(1 - \alpha)}}} - (1 - \alpha)\sqrt{\frac{l_1}{1 - \alpha}} = 0, \tag{28}
\]
Therefore, if $\forall t, x_t < \sqrt{\frac{l_1}{1 - \alpha}}$, $\{x_t\}$ is monotonically increasing, that means $x^* = \lim_{t \to \infty} x_t$ exists, and we have
\[
x^* \geq \alpha x^* + \frac{l_1}{x^*} \geq \sqrt{\frac{l_1}{1 - \alpha}}. \tag{29}
\]
Thus, $\lim_{t \to \infty} x_t \geq \sqrt{\frac{l_1}{1 - \alpha}}$. Similarly, we can prove $\lim_{t \to \infty} x_t \leq \sqrt{\frac{l_2}{1 - \alpha}}$. \qed

**A.4.1 SGD Cases**

*Proof of Theorem 2 ($\alpha = 0$).* Denote $\tilde{g}_t$ as the unit gradient of $w_t$, then $\tilde{g}_t = ||w_t|| \cdot g_t$, the update rule is
\[
w_t = w_{t-1} - \eta\left(\frac{\tilde{g}_{t-1}}{||w_{t-1}||} + \lambda w_{t-1}\right), \tag{30}
\]
since $\langle w_{t-1}, g_{t-1} \rangle = 0$, then we have:
\[
||w_t||^2 = (1 - \eta \lambda)^2 ||w_{t-1}||^2 + \frac{||\tilde{g}_{t-1}||^2 \eta^2}{||w_{t-1}||^2} \tag{31}
\]
Note when $t$ is sufficiently large, $||\tilde{g}_{t-1}|| \in (L - \delta L, L + \delta L)$, according to lemma we have
\[
\lim_{t \to \infty} ||w_t||^2 \geq (1 - \delta) L \sqrt{\frac{\eta}{2 \lambda}} \tag{32}
\]
\[
\lim_{t \to \infty} ||w_t||^2 \leq (1 + \delta) L \sqrt{\frac{\eta}{2 \lambda}} \tag{33}
\]
Thus we have
\[
\lim_{t \to \infty} \left| \sqrt{\frac{4}{L^2 \eta/(2 \lambda)}} - 1 \right| < \frac{\delta}{2} \tag{34}
\]
when $t$ is sufficiently large, we have
\[
\frac{||w_t||}{||w_{t-1}||} = \sqrt{(1 - \eta \lambda)^2 + \frac{||\tilde{g}_{t-1}||^2 \eta^2}{||w_{t-1}||^4}} \leq \sqrt{1 + \frac{4 \delta \eta \lambda}{1 - \delta}} \approx 1 + 2 \delta \eta \lambda, \tag{35}
\]
\[
\frac{||w_t||}{||w_{t-1}||} = \sqrt{(1 - \eta \lambda)^2 + \frac{||\tilde{g}_{t-1}||^2 \eta^2}{||w_{t-1}||^4}} \geq \sqrt{1 - \frac{4 \delta \eta \lambda}{1 + \delta}} \approx 1 - 2 \delta \eta \lambda. \tag{36}
\]

(37)
Due the fact that $\langle w_{t-1}, g_{t-1} \rangle = 0$, we have
\[
\langle w_t, w_{t-1} \rangle = \langle w_t, w_{t-1} - \eta(\frac{\dot{g}_{t-1}}{\|w_{t-1}\|} + \lambda w_{t-1}) \rangle = (1 - \eta \lambda)\|w_{t-1}\|^2,
\] (38)
then angular update $\Delta_{t-1}$ satisfies
\[
\cos(\Delta_{t-1}) = \frac{\langle w_t, w_{t-1} \rangle}{\|w_t\| \cdot \|w_{t-1}\|} = (1 - \lambda \eta) \frac{\|w_{t-1}\|}{\|w_t\|}.
\] (39)
Since $\delta << 1$, $\lambda \eta << 1$, we can estimate the bound of $\cos(\Delta_t)$ by
\[
1 - (1 + 2\delta)\lambda \eta < \cos(\Delta_t) < 1 - (1 - 2\delta)\lambda \eta,
\] (40)
which means
\[
(1 - \delta)\sqrt{\frac{\lambda \eta}{2}} \leq \sin(\Delta_t/2) \leq \sqrt{\frac{(1 + 2\delta)\lambda \eta}{2}} \approx (1 + \delta)\sqrt{\frac{\lambda \eta}{2}}.
\] (41)
Since $\Delta_t$ is close to 0, $\sin(\Delta_t/2) \approx \Delta_t/2$, thus
\[
\lim_{t \to \infty} \frac{\Delta_t}{\sqrt{2\lambda \eta}} - 1 \leq \delta
\] (42)
\[\square\]

A.4.2 SGDM Cases

Lemma 6. Assume $0 < \alpha < 1, 0 < \beta < (1 - \sqrt{\alpha})^2$, $\beta \ll 1$, $\exists \delta, L > 0, \delta \ll 1$, $\forall t, \lim_{t \to \infty} |L_t / L - 1| < \delta$, given the following iterative sequence:

\[
a_{t+1} = (1 + \alpha - \beta)^2a_t - 2\alpha(1 + \alpha - \beta)b_t + \alpha^2c_t + \frac{L_t}{a_t},
\]
\[
b_{t+1} = (1 + \alpha - \beta)a_t - \alpha b_t,
\]
\[
c_{t+1} = a_t,
\]
where $a_1 = b_1 = c_1 > 0$, then the sequence $\{(a_t, b_t, c_t)\}_{t=1}^{\infty}$ satisfies:
\[
\lim_{t \to \infty} |a_t - 1| \leq \delta \frac{1 + \alpha}{1 - \alpha}
\] (44)

Proof of Lemma A2. First of all, let $X_t$, $A$, $e$ denote:
\[
X_t = \begin{pmatrix} a_t \\ b_t \\ c_t \end{pmatrix},
\]
\[
A = \begin{pmatrix} (1 + \alpha - \beta)^2 & -2\alpha(1 + \alpha - \beta) & \alpha^2 \\ 1 + \alpha - \beta & -\alpha & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
\[
e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\] (47)
respectively. The orginal iterative map [43] can be formulated as
\[
X_{t+1} = AX_t + \frac{L_t}{e^T X_t} e.
\] (49)

When $\beta < (1 - \sqrt{\alpha})^2$, the eigen value of $A$ are all real number:
\[
\lambda_1 = \frac{(1 + \alpha - \beta)^2 + (1 + \alpha - \beta)\sqrt{(1 + \alpha - \beta)^2 - 4\alpha}}{2} - \alpha,
\] (50)
\[
\lambda_2 = \alpha,
\] (51)
\[
\lambda_3 = \frac{(1 + \alpha - \beta)^2 - (1 + \alpha - \beta)\sqrt{(1 + \alpha - \beta)^2 - 4\alpha}}{2} - \alpha,
\] (52)
and it’s easy to prove

\[ 0 < \lambda_3 < \lambda_2 = \alpha < \lambda_1 < 1. \]  \tag{53}

Therefore, \( A \) can be formulated as

\[ S^{-1} AS = \Lambda, \]  \tag{54}

where \( \Lambda \) is a diagonal matrix whose diagonal elements are the eigen value of \( A \); the column vector of \( S \) is the eigen vectors of \( A \), note the formuation of \( S, \Lambda \) are not unique. Specifically, we set \( \Lambda, S \) as

\[ \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \]  \tag{55}

\[ S = \begin{pmatrix} \frac{1}{\lambda_1} & \frac{1}{\alpha + \beta} & \frac{1}{\alpha + \beta} \\ \frac{1}{\alpha + \lambda_2} & \frac{1}{\alpha + \lambda_3} & \frac{1}{\alpha + \lambda_3} \\ \frac{1}{\alpha + \lambda_1} & \frac{1}{\alpha + \lambda_2} & \frac{1}{\alpha + \lambda_3} \end{pmatrix}. \]  \tag{56}

Moreover, the inverse of \( S \) exists, and can be explicitly expressed as

\[ S^{-1} = \begin{pmatrix} \frac{(\alpha + \lambda_1)(\lambda_2 - \lambda_3)}{(\lambda_1 - \alpha)(\lambda_2 - \lambda_3)} & -\frac{2\lambda_1(\alpha + \lambda_1)(\alpha + \lambda_2)}{2\alpha(\alpha + \lambda_1)(\alpha + \lambda_2)} & \frac{\lambda_1\lambda_2(\alpha + \lambda_1)}{2\alpha\lambda_1\lambda_2} \\ -\frac{(\lambda_1 - \alpha)(\alpha - \lambda_3)}{(\alpha + \lambda_2)(\alpha + \lambda_3)} & \frac{2\lambda_2(\alpha + \lambda_2)(\alpha + \lambda_3)}{2\alpha(\alpha + \lambda_1)(\alpha + \lambda_2)} & \frac{-\lambda_1\lambda_2(\alpha + \lambda_1)}{2\alpha\lambda_1\lambda_2} \\ -\frac{(\alpha + \lambda_1)(\alpha - \lambda_3)}{(\lambda_1 - \alpha)(\alpha + \lambda_3)} & \frac{2\lambda_3(\alpha + \lambda_3)(\alpha + \lambda_1)}{2\alpha(\alpha + \lambda_1)(\alpha + \lambda_2)} & \frac{-\lambda_1\lambda_2(\alpha + \lambda_1)}{2\alpha\lambda_1\lambda_2} \end{pmatrix}, \]  \tag{57}

let \( Y_t = S^{-1}X_t \), combining with \( (49) \), we have

\[ Y_{t+1} = \Lambda Y_t + \frac{L_t}{(S^T e^T Y_t)^2}S^{-1}e. \]  \tag{58}

Combining with \( (50) \) and \( (57) \), and set \( Y_t = (\tilde{a}_t, \tilde{b}_t, \tilde{c}_t)^T \), we can rewritten \( (43) \) as

\[ \begin{align*}
\tilde{a}_{t+1} &= \lambda_1 \tilde{a}_t + \frac{L_t}{\tilde{a}_t + \tilde{b}_t + \tilde{c}_t}, \\
\tilde{b}_{t+1} &= \alpha \tilde{b}_t - \frac{L_t}{\tilde{a}_t + \tilde{b}_t + \tilde{c}_t}, \\
\tilde{c}_{t+1} &= \lambda_3 \tilde{c}_t + \frac{L_t}{\tilde{a}_t + \tilde{b}_t + \tilde{c}_t}.
\end{align*} \]  \tag{59, 60, 61}

Note \( \tilde{a}_t = \tilde{a}_t + \tilde{b}_t + \tilde{c}_t \). Now we can prove the following equations \( (62), (63), (64), (65), (66) \) by Mathematical Induction

\[ \tilde{b}_t < 0, \]  \tag{62}

\[ \tilde{c}_t > 0, \]  \tag{63}

\[ (\alpha - \lambda_1) \tilde{b}_t > (\lambda_1 - \lambda_3) \tilde{c}_t, \]  \tag{64}

\[ \tilde{a}_t + \tilde{b}_t + \tilde{c}_t > 0, \]  \tag{65}

\[ \tilde{a}_{t+1} + \tilde{b}_{t+1} + \tilde{c}_{t+1} > \lambda_1 (\tilde{a}_t + \tilde{b}_t + \tilde{c}_t) + \frac{L_t}{\tilde{a}_t + \tilde{b}_t + \tilde{c}_t}, \]  \tag{66}

Since the start point \( X_1 = (a_1, a_1, a_1)^T \) \((k_0 > 0)\), the start point \( Y_1 = S^{-1}X_1 \). Combining with \( (57) \), we have

\[ \begin{align*}
\tilde{b}_1 &= -\frac{2\alpha^2 \beta}{(\lambda_1 - \alpha)(\alpha - \lambda_3)}a_1, \\
\tilde{c}_1 &= \frac{\lambda_3(\lambda_3 + \alpha)(1 - \alpha + \beta)}{(\lambda_1 - \alpha)(\lambda_1 - \lambda_3)(1 + \alpha - \beta)} \left( \frac{1 - \alpha - \beta}{1 - \alpha + \beta} - \lambda_1 \right) a_1,
\end{align*} \]  \tag{67, 68}

by which it’s easy to prove \((\alpha - \lambda_1) \tilde{b}_1 > (\lambda_1 - \lambda_3) \tilde{c}_1\). Besides \( \tilde{a}_1 + \tilde{b}_1 + \tilde{c}_1 = e^T SY_1 = e^T X_1 = a_1 = k_0 > 0 \).
Suppose for $t = T$, equations (62), (63), (64), (65) hold, combining with (60), (61), we can derive $\tilde{b}_{T+1} < 0, \tilde{a}_{T+1} > 0$, so (62), (65) hold for $t = T + 1$; Besides, we have

$$\begin{align*}
(a - \lambda_1)\tilde{b}_{T+1} &= a(a - \lambda_1)\tilde{b}_T + \frac{L_t}{\tilde{a}_T + \tilde{b}_T + \tilde{c}_T} \cdot \frac{2\alpha^2}{(\alpha - \lambda_3)} \\
&> \lambda_3(a - \lambda_3)\tilde{c}_T + \frac{L_t}{\tilde{a}_T + \tilde{b}_T + \tilde{c}_T} \cdot \frac{(\alpha + \lambda_3)\lambda_3}{(\alpha - \lambda_3)} \\
&= (\alpha - \lambda_3)\tilde{c}_{T+1},
\end{align*}$$

(69)

thus (64) holds for $T = 1$. Sum (59), (60), (61), due to (64) we have

$$\begin{align*}
\tilde{a}_{T+1} + \tilde{b}_{T+1} + \tilde{c}_{T+1} &= \lambda_1\tilde{a}_T + \alpha\tilde{b}_T + \lambda_3\tilde{c}_T + \frac{L_t}{\tilde{a}_T + \tilde{b}_T + \tilde{c}_T} \\
&> \lambda_1(\tilde{a}_T + \tilde{b}_T + \tilde{c}_T) + \frac{L_t}{\tilde{a}_T + \tilde{b}_T + \tilde{c}_T},
\end{align*}$$

(70)

(66) holds for $t = T + 1$, combining with the fact that $\tilde{a}_T + \tilde{b}_T + \tilde{c}_T > 0$, we have $\tilde{a}_{T+1} + \tilde{b}_{T+1} + \tilde{c}_{T+1} > 0$.

According to Lemma, we can estimate the lower bound of $\tilde{a}_t + \tilde{b}_t + \tilde{c}_t$ when $t$ is sufficiently large by

$$\lim_{t \to \infty} \tilde{a}_t + \tilde{b}_t + \tilde{c}_t \geq \sqrt{\frac{(1 - \delta)L}{1 - \lambda_1}}.$$  

(71)

Now we can analyze the distance($l^\infty$ norm) between $Y_t = (\tilde{a}_t, \tilde{b}_t, \tilde{c}_t)^T$ and the fixed point $Y^* = (\tilde{a}^*, \tilde{b}^*, \tilde{c}^*)^T$ which satisfies

$$Y^* = AY^* + \frac{L}{(S^T e)^T} Y^* S^{-1} e,$$

(72)

and $\tilde{a}^* + \tilde{b}^* + \tilde{c}^* = a^* > 0$, then we have:

$$Y_{t+1} - Y^* = \begin{pmatrix}
\tilde{a}_{t+1} - \tilde{a}^* \\
\tilde{b}_{t+1} - \tilde{b}^* \\
\tilde{c}_{t+1} - \tilde{c}^*
\end{pmatrix}$$

$$= \begin{pmatrix}
(\lambda_1 - \frac{Lk_1}{\alpha^2})(\tilde{a}_t - \tilde{a}^*) \\
-\frac{Lk_2}{\alpha^2}(\tilde{a}_t - \tilde{a}^*) \\
-\frac{Lk_3}{\alpha^2}(\tilde{a}_t - \tilde{a}^*)
\end{pmatrix}$$

$$+ \begin{pmatrix}
\frac{Lk_1}{\alpha^2}(\tilde{b}_t - \tilde{b}^*) \\
-\frac{Lk_2}{\alpha^2}(\tilde{b}_t - \tilde{b}^*) \\
-\frac{Lk_3}{\alpha^2}(\tilde{b}_t - \tilde{b}^*)
\end{pmatrix}$$

$$+ \begin{pmatrix}
(\frac{L_1 - L}{\alpha})k_1 \\
(\frac{L_1 - L}{\alpha})k_2 \\
(\frac{L_1 - L}{\alpha})k_3
\end{pmatrix},$$

(73)

where $(k_1, k_2, k_3)^T = S^{-1} e$. In the following context, will omit the $O(\beta^2)$ part since $\beta \ll 1$. Then $\lambda_1, \lambda_3, k_1, k_2, k_3$ can be approximated as

$$\lambda_1 = 1 - \frac{2}{1 - \alpha} \beta + O(\beta^2)$$

(74)

$$\lambda_3 = \alpha^2 + \frac{2\alpha^2}{(1 - \alpha)} \beta + O(\beta^2)$$

(75)

$$k_1 = \frac{1}{(1 - \alpha)^2} + O(\beta)$$

(76)

$$k_2 = -\frac{2\alpha}{(1 - \alpha)^2} + O(\beta)$$

(77)

$$k_3 = \frac{\alpha^2}{(1 - \alpha)^2} + O(\beta)$$

(78)
It's easy to obtain the value of fixed point \( a^* = \tilde{a}^* + \tilde{b}^* + \tilde{c}^* = \sqrt{\frac{L}{\beta(1-\alpha)/(2(\tau+\frac{1}{\tau+\beta})}} \). With the fact that 
\[
a_t \geq \sqrt{\frac{1-\delta L}{1-\lambda_1}}
\]
when \( t \) is sufficient large, we have
\[
0 < \frac{L}{a^* a_t} \leq \sqrt{(1-\lambda_1)\beta(1-\alpha)/(1-\delta)} = 2(1+\delta/2)\beta + O(\delta^2). \tag{79}
\]
Moreover,
\[
\begin{align*}
|L_t - L| &= \frac{\delta L}{a_t} \leq \delta \sqrt{\frac{L(1-\lambda_1)}{1-\delta}} \leq \delta \sqrt{\frac{2\beta L}{1-\delta} + O(\delta^2) \sqrt{3}} \tag{80}
\end{align*}
\]
Since \( \beta \ll 1 \), we can assume \( \lambda_1 \frac{L}{a_t a^2}, \lambda_3 > \frac{L}{a_t a^2} \). Then according to (73), we have
\[
|\tilde{a}_{t+1} - \tilde{a}^*| \leq (1 - \frac{2\beta}{1-\alpha} - \frac{Lk_1}{a_t a^2})|\tilde{a}_t - \tilde{a}^*| + \frac{Lk_1}{a_t a^2} |\tilde{b}_t - \tilde{b}^*| + \frac{Lk_1}{a_t a^2} |\tilde{c}_t - \tilde{c}^*| + k_1 \delta \sqrt{\frac{2\beta L}{1-\alpha}}, \tag{81}
\]
\[
|\tilde{b}_{t+1} - \tilde{b}^*| \leq \frac{-Lk_2}{a_t a^2} |\tilde{a}_t - \tilde{a}^*| + (\alpha - \frac{Lk_2}{a_t a^2}) |\tilde{b}_t - \tilde{b}^*| + \frac{Lk_2}{a_t a^2} |\tilde{c}_t - \tilde{c}^*| - k_2 \delta \sqrt{\frac{2\beta L}{1-\alpha}}, \tag{82}
\]
\[
|\tilde{c}_{t+1} - \tilde{c}^*| \leq \frac{Lk_3}{a_t a^2} |\tilde{a}_t - \tilde{a}^*| + \frac{Lk_3}{a_t a^2} |\tilde{b}_t - \tilde{b}^*| + (\alpha^2 + \frac{2\alpha^2}{(1-\alpha)^2} \beta|\tilde{c}_t - \tilde{c}^*| + k_3 \delta \sqrt{\frac{2\beta L}{1-\alpha}}, \tag{83}
\]
Note we omit all \( o(\beta^{3/2}) \) part. For fixed \( \alpha \in (0,1) \), there must \( \exists M \), which satisfies \( \alpha^M + \alpha^{2M} < 1 \). Denote max\( \{|\tilde{a}_t - \tilde{a}^*|, |\tilde{b}_t - \tilde{b}^*|, |\tilde{c}_t - \tilde{c}^*|\} \) as \( ||Y_t - T^*||_{\infty} \), then we have
\[
|\tilde{a}_{t+1} - \tilde{a}^*| \leq (1 - \frac{2\beta}{1-\alpha} - \frac{Lk_1}{a_t a^2})|\tilde{a}_t - \tilde{a}^*| + \frac{Lk_1}{a_t a^2} |\tilde{b}_t - \tilde{b}^*| + \frac{Lk_1}{a_t a^2} |\tilde{c}_t - \tilde{c}^*| + k_1 \delta \sqrt{\frac{2\beta L}{1-\alpha}}
\]
\[
\leq (1 - \frac{2\beta}{1-\alpha} - \frac{Lk_1}{a_t a^2})(||Y_t - Y^*||_{\infty} + \frac{Lk_1}{a_t a^2} (\alpha^M + \alpha^{2M})||Y_t - Y^*||_{\infty} + k_1 \delta \sqrt{\frac{2\beta L}{1-\alpha}}
\]
\[
\leq (1 - \frac{2\beta}{1-\alpha} \cdot \max\{||Y_t - Y^*||_{\infty}, ||Y_t - M - Y^*||_{\infty} \} + k_1 \delta \sqrt{\frac{2\beta L}{1-\alpha}} \tag{84}
\]
According to (82), (83), we have
\[
|\tilde{b}_{t+1} - \tilde{b}^*| \leq (\alpha^2 + \frac{4\alpha}{(1-\alpha)^2} \beta)|||Y_t - Y^*||_{\infty} - k_2 \delta \sqrt{\frac{2\beta L}{1-\alpha}} \tag{85}
\]
\[
|\tilde{c}_{t+1} - \tilde{c}^*| \leq (\alpha^2 + \frac{3\alpha^2 - 2\alpha^3}{(1-\alpha)^2} \beta)|||Y_t - Y^*||_{\infty} + k_3 \delta \sqrt{\frac{2\beta L}{1-\alpha}} \tag{86}
\]
set \( \kappa = \max\{1 - \frac{2\beta}{1-\alpha}, \alpha + \frac{4\alpha}{(1-\alpha)^2} \beta, \alpha^2 + \frac{3\alpha^2 - 2\alpha^3}{(1-\alpha)^2} \beta \}, \tau = \frac{1+\alpha}{1-\alpha^2} > \max\{k_1, -k_2, k_3\} \), then by (84), (85), (86), we have
\[
||Y_{t+1} - Y^*||_{\infty} \leq \kappa \cdot \max\{||Y_t - Y^*||_{\infty}, ||Y_t - M - Y^*||_{\infty} \} + \tau \delta \sqrt{\frac{2\beta L}{1-\alpha}} \tag{87}
\]
where \( \kappa \in (0,1) \) and \( \kappa, M \) are only determined by \( \alpha, \beta \). Moreover, by (87), \( \exists \Pi > 0 \), which satisfies:
\[
||Y_t - Y^*||_{\infty} \leq \kappa^{(t/(M+1))} \Pi + \frac{\tau \delta}{1-\kappa} \sqrt{\frac{2\beta L}{1-\alpha}} \tag{88}
\]
Combining with the fact that \( X_t = SY_t \), then we have:
\[
|a_t - a^*| < |\tilde{a}_t - \tilde{a}^*| + |\tilde{b}_t - \tilde{b}^*| + |\tilde{c}_t - \tilde{c}^*| \leq 3\kappa^{(t/(M+1))} \Pi + \frac{3\tau \delta}{1-\kappa} \sqrt{\frac{2\beta L}{1-\alpha}} \tag{89}
\]
When \( \beta \ll 1, \kappa = 1 - \frac{2\beta}{1-\alpha} \), then we have
\[
\lim_{t \to \infty} |a_t - a^*| \leq 3\tau \delta \sqrt{\frac{(1-\alpha)L}{2\beta}} \tag{90}
\]
Note $a^* = \sqrt{\frac{L \eta}{\beta(1-\alpha)(2-\alpha)}}$, combing with (90), we have
\[ \lim_{t \to \infty} \left| \frac{a_t}{a^*} - 1 \right| \leq 3\tau\delta(1-\alpha) = 3\delta^{\frac{1+\alpha}{1-\alpha}} \] (91)

\[ \text{Proof of Theorem 2 ($\alpha > 0$). The update rule is} \]
\[ w_t = w_{t-1} - \eta v_{t-1} \]
\[ = w_{t-1} - \eta(\alpha v_{t-2} + \frac{\hat{g}_{t-1}}{||w_{t-1}||} + \lambda w_{t-1}) \]
\[ = w_{t-1} - \eta(\alpha \frac{w_{t-2} - w_{t-1}}{\eta} + \frac{\hat{g}_{t-1}}{||w_{t-1}||} + \lambda w_{t-1}) \]
\[ = (1 - \eta\lambda + \alpha)w_{t-1} - \alpha w_{t-2} - \frac{\hat{g}_{t-1}}{||w_{t-1}||}. \] (92)

Then we have
\[ ||w_t||^2 = (1 - \eta\lambda + \alpha)^2 ||w_{t-1}||^2 - 2\alpha(1 + \eta\lambda)(w_{t-1}, w_{t-2}) + \alpha^2 ||w_{t-2}||^2 \]
\[ + \frac{||\hat{g}_{t-1}||^2 \eta^2}{||w_{t-1}||^2} + \langle \alpha w_{t-2}, \frac{\hat{g}_{t-1}}{||w_{t-1}||} \rangle \]
\[ = (1 - \eta\lambda + \alpha)^2 ||w_{t-1}||^2 - 2\alpha(1 + \eta\lambda)(w_{t-1}, w_{t-2}) + \alpha^2 ||w_{t-2}||^2 \]
\[ + \frac{||\hat{g}_{t-1}||^2 \eta^2}{||w_{t-1}||^2} + \langle \alpha(w_{t-1} + \eta v_{t-2}), \frac{\hat{g}_{t-1}}{||w_{t-1}||} \rangle \]
\[ = (1 - \eta\lambda + \alpha)^2 ||w_{t-1}||^2 - 2\alpha(1 + \eta\lambda)(w_{t-1}, w_{t-2}) + \alpha^2 ||w_{t-2}||^2 \]
\[ + \frac{||\hat{g}_{t-1}||^2 \eta^2}{||w_{t-1}||^2} + (1 + \alpha \frac{\langle v_{t-2}, \hat{g}_{t-1} \rangle}{||\hat{g}_{t-1}||^2}). \] (93)

Note we have $\langle v_{t-2}, \hat{g}_{t-1} \rangle < \varepsilon ||g_{t-1}||^2$, therefore $(1 + \alpha \frac{\langle v_{t-2}, \hat{g}_{t-1} \rangle}{||\hat{g}_{t-1}||^2}) \in (1 - \alpha\varepsilon, 1 + \alpha\varepsilon)$, $||\hat{g}_t||^2/L^2$. $(1 + \alpha \frac{\langle v_{t-2}, \hat{g}_{t-1} \rangle}{||\hat{g}_{t-1}||^2}) \in (1 - 2\delta - \alpha\varepsilon, 1 + 2\delta + \alpha\varepsilon)$.

On the other hand, combining with (92), we have
\[ \langle w_t, w_{t-1} \rangle = (1 - \alpha - \lambda\eta)||w_{t-1}||^2 - \alpha (w_{t-1}, w_{t-2}), \] (94)

According to Lemma 6, we have
\[ \lim_{t \to \infty} \left| \frac{||w_t||^2}{a^*} - 1 \right| \leq 3(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha}(\delta + \varepsilon) \leq \frac{7(1 + \alpha)}{1 - \alpha}(\delta + \varepsilon) \] (95)

where
\[ a^* = L \sqrt{\frac{\eta}{\lambda(1-\alpha)(2-\alpha)}}. \] (96)

Denote $||\hat{g}_t||^2/L^2 \cdot (1 + \alpha \frac{\langle v_{t-2}, \hat{g}_{t-1} \rangle}{||\hat{g}_{t-1}||^2})$ as $k_t$, then when $t$ is sufficiently large, $k_t \in (1 - 2\delta - \alpha\varepsilon, 1 + 2\delta + \alpha\varepsilon)$, then combining (93) and (94) by (93), $\frac{||w_{t-1}||^2}{94}$, we have
\[ ||w_t||^2 ||w_{t-1}||^2 - ||w_t, w_{t-1}||^2 = \alpha^2 (||w_t||^2 ||w_{t-1}||^2 - ||w_t, w_{t-1}||^2) + L^2 \eta^2 k_t. \] (97)

which implies
\[ \lim_{t \to \infty} \frac{||w_t||^2 ||w_{t-1}||^2 - ||w_t, w_{t-1}||^2}{L^2 \eta^2} \leq \frac{L^2 \eta^2}{1 - \alpha^2}(1 + 2\delta + \alpha\varepsilon), \] (98)
\[ \lim_{t \to \infty} \frac{||w_t||^2 ||w_{t-1}||^2 - ||w_t, w_{t-1}||^2}{L^2 \eta^2} \geq \frac{L^2 \eta^2}{1 - \alpha^2}(1 - 2\delta - \alpha\varepsilon). \] (99)
According to (95) we have
\[
L^2\eta^2 \leq \frac{L^2\eta^2}{(1 - \alpha^2)a^2} \left[ 1 + 6(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right]
\]
\[
\approx \frac{2\lambda\eta}{1 + \alpha} \left[ 1 + 6(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right].
\]

Similarly, we have
\[
L^2\eta^2 \geq \frac{L^2\eta^2}{(1 - \alpha^2)a^2} \left[ 1 - 6(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right]
\]
\[
\approx \frac{2\lambda\eta}{1 + \alpha} \left[ 1 - 6(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right].
\]

Therefore, we can get
\[
\lim_{t \to \infty} \left( 1 - \frac{|\langle \mathbf{w}_t, \mathbf{w}_{t-1} \rangle|}{||\mathbf{w}_t||^2||\mathbf{w}_{t-1}||^2} \right) \leq \frac{2\lambda\eta}{1 + \alpha} \left[ 1 + 7(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right].
\]
\[
\lim_{t \to \infty} \left( 1 - \frac{|\langle \mathbf{w}_t, \mathbf{w}_{t-1} \rangle|}{||\mathbf{w}_t||^2||\mathbf{w}_{t-1}||^2} \right) \geq \frac{2\lambda\eta}{1 + \alpha} \left[ 1 - 7(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right].
\]

Note \(\frac{|\langle \mathbf{w}_t, \mathbf{w}_{t-1} \rangle|}{||\mathbf{w}_t||^2||\mathbf{w}_{t-1}||^2} = \cos^2(\Delta_t)\), therefore we have
\[
\lim_{t \to \infty} \sin^2(\Delta_t) \leq \frac{2\lambda\eta}{1 + \alpha} \left[ 1 + 7(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right],
\]
\[
\lim_{t \to \infty} \sin^2(\Delta_t) \geq \frac{2\lambda\eta}{1 + \alpha} \left[ 1 - 7(2\delta + \alpha\varepsilon) \frac{1 + \alpha}{1 - \alpha} \right].
\]

since \(\sin(\Delta_t)\) is close to 0, \(\sin(\Delta_t) \approx \Delta_t\), combining with (104), (105), we have
\[
\lim_{t \to \infty} \left[ \frac{\Delta_t}{\sqrt{2\lambda\eta}/(1 + \alpha)} - 1 \right] \leq (2\delta + \alpha\varepsilon) \frac{7(1 + \alpha)}{2(1 - \alpha)} \leq \frac{7(1 + \alpha)}{2(1 - \alpha)} (\delta + \varepsilon)
\]

\[\Box\]

### A.5 Convergence Rate to Equilibrium Condition after learning rate decay

**Corollary 2.4** (SGD case(\(\alpha = 0\))). Assume the equilibrium condition (12) holds with learning rate \(\eta\) and WD coefficient \(\lambda\). If the learning rate is shrunk to \(k\) times smaller, then at least \(m = \left\lceil \log(k) / (4\lambda\eta) \right\rceil\) iterations are required to reach the new equilibrium condition.

**Proof.** According to (31), we have
\[
||\mathbf{w}_{t+1}||^2 > (1 - \lambda\eta)^2||\mathbf{w}_t||^2,
\]
which means
\[
||\mathbf{w}_{t+T}||^2 > (1 - 2\lambda\eta)^T||\mathbf{w}_t||^2.
\]

On the other hand, by (33) we know that when \(\eta\) is divided by \(k\), \(||\mathbf{w}_t||^2\) should be divided by \(\sqrt{k}\) to reach the new equilibrium condition, therefore we have
\[
\frac{||\mathbf{w}_{t+T}||^2}{||\mathbf{w}_t||^2} = \sqrt{k} > (1 - 2\lambda\eta)^T.
\]

Since \(\lambda\eta \ll 1\), \(\log(1 - 2\lambda\eta) \approx -2\lambda\eta\), thus
\[
T > \frac{\log(k)}{4\lambda\eta}
\]

\[\Box\]
Corollary 2.5 (SGDM case($\alpha > 0$)). Assume the equilibrium condition (42) holds with learning rate $\eta$, WD coefficient $\lambda$, and momentum coefficient $\alpha$. If the learning rate is shrunk to $k$ times smaller, then at least $m = \lceil(1 - \alpha) \log(k)/(4\lambda\eta)\rceil$ iterations are required to reach the new equilibrium condition.

Proof. According to (70) and (74), we have
\[
||w_{t+1}||^2 > (1 - \frac{2\lambda\eta}{1 - \alpha})||w_t||^2,
\]
which means
\[
||w_{t+T}||^2 > (1 - \frac{2\lambda\eta}{1 - \alpha})^T||w_t||^2.
\]
On the other hand, by (34) we know that when $\eta$ is divided by $k$, $||w_t||^2$ should be divided by $\sqrt{k}$ to reach the new equilibrium condition, therefore we have
\[
\frac{||w_{t+T}||^2}{||w_t||^2} = \sqrt{k} > (1 - \frac{2\lambda\eta}{1 - \alpha})^T.
\]
Since $\lambda\eta \ll 1$, $\log(1 - \frac{2\lambda\eta}{1 - \alpha}) \approx -\frac{2\lambda\eta}{1 - \alpha}$, thus
\[
T > \frac{(1 - \alpha) \log(k)}{4\lambda\eta}
\]
\[
\square
\]

B Experiments

B.1 Empirical study on Assumptions of Theorem 2

In this section, we shows the reasonableness of assumption 1), 4) by conducting empirical study on ImageNet datasets with ResNet50, the training settings rigorously follows [6]: learning rate is initialized as $\eta = 0.1$, and divided by 10 at 30, 60, 80 epoch; the WD coefficient is $\lambda = 10^{-4}$; the momentum coefficient is 0.9.

Assumption 1. $\exists \delta, L \in \mathbb{R}^+$, unit gradient $\tilde{g}_t = g_t \cdot ||w_t||$ satisfies $\lim_{t \to \infty} ||\tilde{g}_t||/L - 1 < \delta, \delta \ll 1$.

![Figure 7: Norm of gradient and unit gradient of weight from layer1.0.conv2 in ResNet50 [9]. Training settings rigorously follow [6].](image)

Assumptions 1) requires the norm of gradients to become steady as training continues. As an example, figure 7 presents the change of gradient’s norm and unit gradient’ norm of weight from layer1.0.conv2 in ResNet50. It can be seen from Figure 7 that though unit gradient’s norm varies greatly across the whole training process, but it gradually become steady within each learning rate stage. Under the view of traditional optimization theory, regardless of effect of scale-invariant property, the norm of
unit gradient should get close to zero as the number of iterations grows. It’s seems very wired that unit gradient’s norm converge to a constant number larger than zero. But under the view of Spherical Motion Dynamic (SMD), optimization trajectory will oscillate along the edge of a local optimum on the unit sphere, leading to a approximately fixed angular update, therefore the norm of gradient will not get close to zero.

**Assumption 2.** \( \exists \varepsilon, |\langle g_t, v_{t-1} \rangle| \leq \varepsilon \|g_t\|^2. \)

Figure 8: The related indicators are computed based on weight from layer1.0.conv2 in ResNet50 [9]. Training settings rigorously follow [6]. Assumptions 2 relies on the fact that momentum at iteration \( t - 1 \) and gradient at iteration \( t \) is almost orthogonal to each other (see Figure 8). Such phenomenon is probably caused by two reasons: first, the dimensionality of \( v_{t-1} \) and \( g_t \) is extremely large in practice; second, \( g_t \) is computed based on the mini-batch of samples at iteration \( t \), hence the randomness of mini-batch is too large, and independent of momentum at previous iterations (before \( t \)). Therefore the cosine similarity \( \frac{\langle v_{t-1}, g_t \rangle}{\|v_{t-1}\| \cdot \|g_t\|} \) is close to 0.

### B.2 Spherical Motion Dynamics with Different Network Structures

Figure 9: The angular update \( \Delta_t \) of MobileNet-V2 [23] and ShuffleNet-V2+ [18]. The solid lines with different colors represent all scale-invariant weights from the model; The dash black line represents the theoretical value of angular udpate, which is computed by \( \sqrt{\frac{2\lambda}{1+\alpha}} \). Learning rate \( \eta \) is initialized as 0.5, and divided by 10 at epoch 30, 60, 80 respectively; WD coefficient \( \lambda \) is \( 4 \times 10^{-5} \); Momentum parameter \( \alpha \) is set as 0.9.