Research Article

On the Study of Multiwavelet Deconvolution Density Estimators

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In this paper, multiwavelet deconvolution density estimators are presented by a linear multiwavelet expansion and a nonlinear multiwavelet expansion, respectively. Moreover, the unbiased estimation is shown, and asymptotic normality is discussed for the multiwavelet deconvolution density estimators. Finally, a numerical example is given for our discussion.

1. Introduction and Preliminary

Assume that \((\Omega, F, P)\) is a probability space. \(Y_1, Y_2, \ldots, Y_n\) are independent and identically distributed (i.i.d) random variables. They have the same model \(Y = X + \varepsilon\), where \(X\) is a real random variable and \(\varepsilon\) denotes a random noise (error). Furthermore, \(X\) and \(\varepsilon\) are independent of each other. Let \(f_X\) be the unknown probability density of \(X\) and \(f_\varepsilon\) be the density of \(\varepsilon\). So the probability density \(f_Y\) of \(Y\) reduces to be noise-free. So, approximating the density \(f_X\) by an estimator \(\hat{f}_n(\cdot) = f_n(\cdot; Y_1, Y_2, \ldots, Y_n)\) can be recognized as a deconvolution problem. A wavelet estimator \(f_n\) means that \(\hat{f}_n\) can be expanded by a wavelet basis. Some important work has been done, such as wavelet deconvolution estimators and asymptotic normality (seen in [1-5]). Moreover, multiwavelet deconvolution density estimators are presented by a linear multiwavelet expansion and a nonlinear multiwavelet expansion, respectively.

Firstly, we introduce the concept of multiplicity projections from the space \(L^2(\mathbb{R})\) to \(V_j\) and \(W_j\), respectively. Then,

\[
\sum_{i=1}^{r} c_{ijk} \Phi_{ijk} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} D_{jk} \Phi_{jk} = f
\]
\[ P_{ij}f = \sum_{k \in \mathbb{Z}} C_{jk}^T \Phi_{jk} = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}} c_{ijk} \phi_{ijk}, \]

\[ Q_{ij}f = \sum_{k \in \mathbb{Z}} D_{jk}^T \Phi_{jk} = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}} d_{ijk} \psi_{ijk} = (P_{i+1} - P_i)f. \]  
\[ (2) \]

Thus, \( f = P_{0}f + \sum_{i=1}^{\infty} Q_{ij}f. \)

And we also define the notation \( Q_{0j}f = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}} d_{ijk} \psi_{ijk}. \)

Moreover, the Fourier transform \( f^{FT} \) of \( f \) is defined by

\[ f^{FT}(\omega) := \int_{\mathbb{R}} f(x) e^{-i\omega x} dx. \]  
\[ (3) \]

And the inverse transform of \( f^{FT} \) is denoted by

\[ f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f^{FT}(\omega) e^{i\omega x} d\omega. \]  
\[ (4) \]

So, the Fourier transform \( \Phi^{FT}(\omega) \) of \( \Phi(x) \) is defined as

\[ \Phi^{FT}(\omega) := \int_{\mathbb{R}} \Phi(x) e^{-i\omega x} dx = \left[ \phi^{FT}_{r}(\omega), \phi^{FT}_{r-1}(\omega), \ldots, \phi^{FT}_{0}(\omega) \right]^T. \]  
\[ (5) \]

In this paper, choose a multiscale function \( \Phi \) with multiplicity \( r \) satisfying the following condition:

(C1) \( \Phi = [\phi_1, \phi_2, \ldots, \phi_r]^T \in L^2(\mathbb{R})^r \) and \( |\phi_j^{FT}|_{m+1}(\omega) \leq (1 + |\omega|)^{-(m+1)} \) with \( j = 1, m = 0, 1, 2, i = 1, 2, \ldots, r. \)

Note: \( A \leq B \) denotes two variables \( A, B \) satisfying \( A \leq cB \), for some constant \( c > 0 \); \( A \leq B \) is equivalent to \( B \leq A \), and \( A \sim B \) means both \( A \leq B \) and \( A \geq B \). Obviously, multiwavelets Sa4 (constructed by Shen et al.) [6] and CL (constructed by Chui and Lian) [7, 8] are examples for C1.

According to condition (C1), the corresponding multiresolution \( \Psi = [\psi_1, \psi_2, \ldots, \psi_r]^T \) satisfies \( |\psi_j^{FT}|_{m+1}(\omega) \leq (1 + |\omega|)^{-(m+1)} \) with \( j = 1, m = 0, 1, 2, i = 1, 2, \ldots, r. \).

In fact, \( \phi_j(x) = (1/2\pi) \int_{\mathbb{R}} \phi_j^{FT}(\omega) e^{i\omega x} d\omega. \) By using integration by parts,

\[ |\phi_j(x)| \leq (1 + |x|^2)^{-1}. \]  
\[ (6) \]

Then,

\[ \sup_{x \in \mathbb{R}} \sum_k |\phi_j(x - k)| \leq \sup_{x \in \mathbb{R}} \sum_k (1 + |x - k|^2)^{-1} \leq 1 \]
\[ (7) \]

According to multiplicity multiresolution analysis (MMRA),

\[ \sum_k |P_{ij}| = \sum_{j} |\phi_j, \phi_{i,j,k}| = \sum_{j} \left| \sqrt{2} \int_{\mathbb{R}} \phi_j(x) \phi_j(2x - k) dx \right| \leq 1, \]
\[ (8) \]

where \( \Phi(x) = \sqrt{2 \sum_k P_k \Phi(2x - k)}, P_k = (p_{i,j,k})_{i,k \in \mathbb{Z}}, \Psi(x) = \sqrt{2 \sum_k Q_k \Phi(2x - k)}, Q_k = (q_{i,j,k})_{i,k \in \mathbb{Z}}, \phi_j(x) = \sqrt{2 \sum_k \sum_{j=1}^{r} P_{i,j,k} \phi_j(2x - k)}, \) and \( \psi_j(x) = \sqrt{2 \sum_k \sum_{j=1}^{r} Q_{i,j,k} \psi_j(2x - k)}. \)

Moreover, \( P_{ij}(\omega) = (1/\sqrt{2}) \sum_k p_{i,j,k} e^{-ik\omega} \) are bounded. So, \( Q_{ij}(\omega) = (1/\sqrt{2}) \sum_k q_{i,j,k} e^{-ik\omega} \) are bounded, where \( Q_{ij}(\omega) \) is constructed by \( P_{ij}(\omega) \) (seen in [7–9]). Thus,

\[ |\psi_j^{FT}(\omega)| = \sum_{i=1}^{r} |Q_{ij}(\omega)| \phi_j^{FT}(\omega) \leq \sum_{i=1}^{r} |Q_{ij}(\omega)| \phi_j^{FT}(\omega) \leq (1 + |\omega|) \leq (1 + |\omega|)^{-(1/2)}. \]  
\[ (9) \]

In addition, the density function \( f_\varepsilon \) of the random noise \( \varepsilon \) satisfies the following conditions [2]:

(C2) \( \int_{\mathbb{R}} f_\varepsilon^{FT}(\omega) \geq (1 + |\omega|)^{-(\beta/2)} \)

(C3) \( \int_{\mathbb{R}} (f_\varepsilon^{FT}(\omega))(1 + |\omega|)^{-(\beta + m/2)} \), \( m = 0, 1, 2 \)

Under these two conditions, the random noise \( \varepsilon \) is said to be ill-posed.

2. Multiwavelet Deconvolution
Density Estimators

In this section, we discuss the multiwavelet deconvolution density estimators. And some lemmas are deduced for the discussion of asymptotic normality in Section 3.

Similar to the discussion in [2, 3, 5], if \( 1 > \beta + 1 \), the estimators can be defined as

\[ \hat{\phi}_{ij} := \frac{2}{\sqrt{n}} \sum_{p=1}^{n} K_{ij} \phi_1(2^p Y p - k), \]
\[ (10) \]

\[ K_{ij} \phi_1(y) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega y} \phi_j^{FT}(\omega) d\omega. \]

\[ (11) \]

According to equation (10), the linear multiwavelet estimator can be defined by

\[ \hat{f}_\varepsilon(x) := \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}} \hat{c}_{i,j,k} \phi_{ijk}. \]  
\[ (12) \]

By deducing simply, we have
\[ E_{ij,k} = E \left( \frac{2^{ij}}{n} \sum_{p=1}^{n} K_{ij,p} \Phi(2^{i}Y_p - k) \right) = \frac{2^{ij}}{n} \sum_{p=1}^{n} EK_{ij,p}(2^{i}Y_p - k) \]

\[
\begin{align*}
  &= \frac{2^{ij}}{n} \sum_{p=1}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\omega(2^{i}Y_p-k)} \frac{\Phi^{FT}_i(\omega)}{f^{FT}_g(-2^{i}\omega)} d\omega f_y(Y) dy \\
  &= \frac{2^{ij}}{n} \sum_{p=1}^{n} \int_{\mathbb{R}} e^{-i\omega(2^{i}Y_p-k)} \Phi^{FT}_i(\omega) f^{FT}_g(-2^{i}\omega) d\omega \\
  &= \frac{1}{n} \sum_{p=1}^{n} \int_{\mathbb{R}} f_X(x)2^{ij}K_{ij,p}(2^{i}x-k) d\omega \\
  &= c_{ij,k}.
\end{align*}
\]

So \( E\tilde{f}_n(x) = E \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} c_{i,k} = \sum_{k \in \mathbb{Z}} c_{i,k} = P_i f_X \). And we have the following conclusion.

**Theorem 1.** Assume that \( \tilde{c}_{i,k} \) is defined in equation (10). Then, \( \tilde{c}_{i,k} \) is an unbiased estimation of \( \epsilon_{i,k} = \int f_{g_X}(x)2^{i}\Phi(2^{i}x-k) d\omega \), i.e., \( E\tilde{c}_{i,k} = \epsilon_{i,k} \) and \( E\tilde{f}_n(x) = P_i f_X \).

The detailed proof of Theorem 1 is similar to the proof of Lemma 2.2 in [1].

According to the definition of \( \tilde{c}_{i,k} \) in equation (10), the estimator \( \tilde{f}_n(x) \) can be rewritten as

\[ \tilde{f}_n(x) = \frac{2^{ij}}{n} \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} (K_{ij,p}(2^{i}Y_p-k)) \theta_i(2^{i}x-k) \]

To simplify the above expansion, the function

\[ K^*_i(x,y) := \sum_{k \in \mathbb{Z}} (K_{ij,p}(x-k)) \theta(y-k) \]

is introduced. It is similar to the discussion in [2, 3, 5]. Then,

\[ \tilde{f}_n(x) = \frac{2^{ij}}{n} \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} K^*_i(2^{i}x,2^{i}x). \]

We introduce \( \bar{K}^*_i(Y,x) := K^*_i(Y,x) -EK^*_i(Y,x) \).

Then,

\[ \tilde{f}_n(x) - \bar{f}_n(x) = \frac{2^{ij}}{n} \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} \bar{K}^*_i(2^{i}Y_p,2^{i}x). \]

Next, the properties of the above functions are discussed in the following lemmas. Some conclusions are similar to the discussion in [2, 3, 5].

**Lemma 1.** For \( l > \beta + 1, \beta > 1 \), the conditions (C1–C3) hold. Then, for every \( i = 1, 2, \ldots, r \), \( K_{ij,p} \) satisfies:

\[
|K_{ij,p}| \leq 2^{\beta l} (1 + |y|^2)^{-l}. \tag{18}
\]

Proof. Assume that \( \zeta_i(\omega) = (\Phi^{FT}_i(\omega)/f^{FT}_g(-2^{i}\omega)) \), for every \( i = 1, 2, \ldots, r \). Then,

\[ K_{ij,p}(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega Y} \zeta_i(\omega) d\omega. \tag{19} \]

Since

\[
\left| f^{FT}_g(-2^{i}\omega) \right| \geq (1 + |\omega|^2)^{\beta/2} \implies \left| f^{FT}_g(-2^{i}\omega) \right| \geq 2^{-\beta/2} (1 + |\omega|^2)^{\beta/2}
\]

and \( \left| \Phi^{FT}_i(\omega) \right| \leq (1 + |\omega|^2)^{-\beta/2} \),

\[
\zeta_i(\omega) \leq 2^{\beta l} (1 + |\omega|^2)^{-l - \beta/2} \to 0, \quad \omega \to \infty. \tag{21}
\]

We compute the derivative

\[
\zeta''_i(\omega) = (\Phi^{FT}_i)'(\omega) \left[ f^{FT}_g(-2^{i}\omega) \right]^{-l} + 2^{i+1} \Phi^{FT}_i(\omega) \left( f^{FT}_g \right)' \left[ (-2^{i}\omega) \right]^{-l - \beta/2}.
\]

According to the conditions (C1–C3) and \( l > \beta + 1, \beta > 1 \),

\[
\zeta''_i(\omega) \leq (1 + |\omega|^2)^{-l/2} \left[ (1 + |2^{i}\omega|^2)^{\beta/2} + 2^{l+1} \left( 1 + |2^{i}\omega|^2 \right)^{-(\beta+1/2)} \right] \to 0, \quad \omega \to \infty.
\]

Similarly,

\[
\zeta''_i(\omega) = (\Phi^{FT}_i)'(\omega) \left[ f^{FT}_g(-2^{i}\omega) \right]^{-l} + 2^{i+1} \Phi^{FT}_i(\omega) \left( f^{FT}_g \right)' \left[ (-2^{i}\omega) \right]^{-l - \beta/2} + 2^{i+1} \Phi^{FT}_i(\omega) \left[ (f^{FT}_g)' \left[ (-2^{i}\omega) \right]^{-l} \right]^{-1}.
\]

Then,

\[
\zeta''_i(\omega) \leq (1 + |\omega|^2)^{-l/2} \left[ (1 + |2^{i}\omega|^2)^{\beta/2} + 2^{l+1} \left( 1 + |2^{i}\omega|^2 \right)^{-(\beta+1/2)} \right] \cdot \left( 1 + |2^{i}\omega|^2 \right)^{-l/2} + 2^{i+1} \Phi^{FT}_i(\omega) \left[ (f^{FT}_g)' \left[ (-2^{i}\omega) \right]^{-l} \right]^{-1}.
\]

So, the derivative functions \( \zeta'_i(\omega) \) and \( \zeta''_i(\omega) \) are bounded. By integration by parts,
For $l > eta + 1, \beta > 1$, under the conditions (C1–C3), define the function $F(|x|): = (1 + |x|^{-1})$. Then, for every $i = 1, 2, \ldots, r$, $K_{ij}\phi_i$ and $K_i^*$ satisfy the following:

1. $|K_i^*(x, y)| \leq 2^\beta F(|x - y|)$
2. $\sum_{k \in \mathbb{Z}} (K_{ij}\phi_i)(x - k)K_{ij}\phi_i(y - k) \leq 2^\beta F(|x - y|)$
3. $\sum_{i=1}^r EK_i^*(2|x, 2y|) = 2^{-\beta} P_j f x (x) (f \in L^2 (R))$, where $P_j f x$ is defined in equation (2).

Proof. According to the conclusion of Lemma 1,  

$$|K_{ij}\phi_i(x - k)| \leq 2^\beta (1 + |x - k|^2)^{-1},$$

for every $i = 1, 2, \ldots, r$. Then,

$$\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |K_{ij}\phi_i(x - k)| \leq 2^\beta,$$

where $Z = \{k \in \mathbb{Z}, |x - k| \geq (|x - y|/2) \cup \{k \in \mathbb{Z}, |y - k| \geq (|x - y|/2)\}$ for fixed $x, y \in \mathbb{R}$.

According to the definition of $K_i^*(x, y)$,  

$$|K_i^*(x, y)| \leq \sum_{|x - k| \leq (|x - y|/2)} |K_{ij}\phi_i(x - k)| |\phi_i(y - k)|$$

+ $\sum_{|y - k| \leq (|x - y|/2)} |K_{ij}\phi_i(x - k)| |\phi_i(y - k)|$.

On the other hand, $\phi_i(x) = (1/2\pi) \int_{\mathbb{R}} F^T(\omega)e^{i\omega x} d\omega$. According to the condition C1 and integration by parts, we have

$$|\phi_i(x)| \leq (1 + |x|^2)^{-1} = F(|x|),$$

that is,

$$|\phi_i(y - k)| \leq F(|y - k|) \leq F(|x - y|).$$

So,

$$\sum_{|y - k| \leq (|x - y|/2)} |K_{ij}\phi_i(x - k)| |\phi_i(y - k)| \leq 2^\beta F(|x - y|).$$

Since

$$|K_{ij}\phi_i(x - k)| \leq 2^\beta (1 + |x - k|^2)^{-1} = 2^\beta F(|x - k|)$$

$$\leq 2^\beta \frac{|x - y|}{2},$$

$$\sum_{|x - k| \leq (|x - y|/2)} |K_{ij}\phi_i(x - k)| |\phi_i(y - k)|$$

$$\leq \sum_{k \in \mathbb{Z}} 2^\beta F\left(\frac{|x - y|}{2}\right) |\phi_i(y - k)|.$$
Lemma 3 Assume that $H_n(\cdot,\cdot)$, $(n = 1, 2, \ldots)$ are symmetric functions, $X_1, X_2, \ldots, X_n$ are i.i.d random variables and $G_n(x, y)$: $= E[H_n(x, X_1)H_n(y, X_1)]$. The conclusion (11) holds.

\[ |K_{r_i} \phi_i \cdot (y - k)| \leq 2^{\beta \phi} (1 + |y - k|)^{-1} = 2^{\beta \phi} F(|y - k|), \]
\[ \sum_{k \in Z} |K_{r_i} \phi_i \cdot (x - k)|K_{r_i} \phi_i \cdot (y - k)| \]
\[ \leq \sum_{k \in Z} |K_{r_i} \phi_i \cdot (x - k)|2^{\beta \phi} F(|y - k|) \leq 2^{\beta \phi} F(|x - y|). \]

Next, to prove the conclusion (11),
\[ \int_R K_i^* (2^j y - k) f_Y(y)dy = \int_R \sum_{k \in Z} (K_i \phi_i)(2^j y - k) f_Y(y)dy. \]

Note that $|\phi_i(x)| \leq F(|x|) \leq 1$, then
\[ \int_R \sum_{k \in Z} (K_i \phi_i)(2^j y - k) f_Y(y)dy \leq 2^{2 \beta \phi} \int_R f_Y(y)dy = 2^{2 \beta \phi}. \]

So
\[ \int_R \sum_{k \in Z} (K_i \phi_i)(2^j y - k) f_Y(y)dy \approx \sum_{k \in Z} (K_i \phi_i)(2^j y - k) f_Y(y)dy \cdot \phi_i(2^j x - k). \]

By the definition of $K_i \phi_i$ in equation (10) and Fubini theorem,
\[ \int_R (K_i \phi_i)(2^j y - k) f_Y(y)dy = \frac{1}{2\pi} \int_R e^{-i\omega(2^j y - k)} \phi_i^{FT}(\omega) f_Y^{FT}(-\omega) d\omega \int_R f_Y^{FT}(y)dy \]
\[ = \frac{1}{2\pi} \int_R e^{-ij\omega} f_Y^{FT}(y)dy \cdot \frac{\phi_i^{FT}(\omega)}{f_Y^{FT}(\omega)} d\omega \]
\[ = \frac{1}{2\pi} \int_R e^{-ij\omega} f_Y^{FT}(\omega) \phi_i^{FT}(\omega) d\omega \cdot \frac{1}{2\pi} \int_R f_Y^{FT}(y)dy \]
\[ = \int_R f_X(x) \phi_i(2^j x - k) dx. \]

The final equality is due to Plancherel formula:
\[ \sum_{i=1}^2 \int_R (K_i \phi_i)(2^j y - k) f_Y(y)dy = \sum_{k \in Z} \int_R f_X(x)2^{j/2} \phi_i(2^j x - k) d\omega \]
\[ \cdot 2^{j/2} \phi_i(2^j x - k) \]
\[ = \sum_{i=1}^2 \sum_{k \in Z} <f_X, \phi_{i,k}> \phi_{i,k}(x) \]
\[ = P_j f_X(x). \]

The conclusion (11) holds.

Lemma 3 (see 2, Theorem 3.1). Assume that $H_n(\cdot,\cdot)$, $(n = 1, 2, \ldots)$ are symmetric functions, $X_1, X_2, \ldots, X_n$ are i.i.d random variables and $G_n(x, y)$: $= E[H_n(x, X_1)H_n(y, X_1)]$. Then
\[ \sum_{i=1}^2 \int_R \int_R f_X(x)2^{j/2} \phi_i(2^j x - k) d\omega \cdot 2^{j/2} \phi_i(2^j x - k) \]
\[ = \sum_{i=1}^2 \sum_{k \in Z} <f_X, \phi_{i,k}> \phi_{i,k}(x) \]
\[ = P_j f_X(x). \]

Theorem 2. Under the condition (C1–C3), with $\beta > 1$ and $l > \beta + 1$, if $f_X \in L^p (R)$, $(p > 2)$, then the linear estimator $\hat{f}_n$ satisfies
\[ n^{2^{-j(2^{j+1})}} \left[ \left\| \tilde{f}_n - f \right\|_2^2 - E \left\| \tilde{f}_n - f \right\|_2^2 \right] \xrightarrow{d} N \left( 0, \sum_{1 \leq i < j \leq n} s_{i,j,n}^2 \right), \] (51)

with \( j \to \infty \) and \( n^{-1/2j} \to 0 \), where \( s_{i,j,n}^2 = (1/2)(n-1)EH_{i,j,n}^2(Y_1, Y_2) \) and \( H_{i,j,n}^2(Y_{p_1}, Y_{p_2}) \) are defined by equations (55) and (62).

Proof. Since \( E\tilde{f}_n(x) = P_j f, \tilde{f}_n(x) - E\tilde{f}_n(x) \in V_j \) and

\[ E\tilde{f}_n - f = -\sum_{j \neq j}^\infty Q_f f \cdot x. \] (52)

So \( \tilde{f}_n(x) - E\tilde{f}_n(x) \) and \( E\tilde{f}_n - f \) are orthogonal in \( L^2(R) \). We have

\[ \left\| \tilde{f}_n - f \right\|_2^2 = \left\| \tilde{f}_n - E\tilde{f}_n + E\tilde{f}_n - f \right\|_2^2 = \left\| \tilde{f}_n - E\tilde{f}_n \right\|_2^2 + \left\| E\tilde{f}_n - f \right\|_2^2. \] (53)

Assume that

\[ J_n := \left\| \tilde{f}_n - f \right\|_2^2 - E \left\| \tilde{f}_n - f \right\|_2^2 = \left\| \tilde{f}_n - E\tilde{f}_n \right\|_2^2 - E \left\| \tilde{f}_n - E\tilde{f}_n \right\|_2^2, \]

\[ \left\| \tilde{f}_n(x) - E\tilde{f}_n(x) \right\|^2 = \frac{2^{2j}}{n^2} \left( \sum_{i=1}^r \sum_{p=1}^n \bar{R}^*_i \left( 2^i Y_p, 2^i x \right) \right)^2 \]

\[ = \frac{2^{2j}}{n^2} \sum_{i=1}^r \sum_{p=1}^n \left( \bar{R}^*_i \left( 2^i Y_p, 2^i x \right) \right)^2 \]

\[ + \frac{2^{2j}}{n^2} \sum_{1 \leq i < j \leq r} \sum_{1 \leq p_1 < p_2 \leq n} \left[ \bar{R}^*_i \left( 2^i Y_{p_1}, 2^i x \right) \bar{R}^*_i \left( 2^i Y_{p_2}, 2^i x \right) \right. \]

\[ + \left. \bar{R}^*_i \left( 2^i Y_{p_1}, 2^i x \right) \bar{R}^*_i \left( 2^i Y_{p_2}, 2^i x \right) \right] \]

\[ + \frac{2^{2j+1}}{n^2} \sum_{1 \leq i < j \leq r} \sum_{1 \leq p_1 < p_2 \leq n} \bar{R}^*_i \left( 2^i Y_{p_1}, 2^i x \right) \bar{R}^*_i \left( 2^i Y_{p_2}, 2^i x \right) \]

\[ + \frac{2^{2j+1}}{n^2} \sum_{1 \leq i < j \leq r} \sum_{1 \leq p_1 < p_2 \leq n} \bar{R}^*_i \left( 2^i Y_{p_1}, 2^i x \right) \bar{R}^*_i \left( 2^i Y_{p_2}, 2^i x \right). \] (54)

Define

\[ H_{i,j,n}^* \left( Y_{p_1}, Y_{p_2} \right) := \int_\mathbb{R} \bar{R}^*_i \left( 2^i Y_{p_1}, 2^i x \right) \bar{R}^*_i \left( 2^i Y_{p_2}, 2^i x \right) dx. \] (55)

According to the independence of \( \left\{ Y_p \right\}_{p=1}^n \) and \( E\tilde{K}_i^* \left( Y, x \right) = 0 \), we have

\[ J_n = \frac{2^{2j}}{n^2} \sum_{1 \leq i < j \leq r} \sum_{1 \leq p_1 < p_2 \leq n} H_{i,j,n}^* \left( Y_{p_1}, Y_{p_2} \right) \]

\[ + \frac{2^{2j-1}}{n^2} \sum_{i=1}^r \sum_{p=1}^n \left[ H_{i,i,n}^* \left( Y_p, Y_p \right) - EH_{i,i,n}^* \left( Y_p, Y_p \right) \right] \]

\[ + \frac{2^{2j}}{n^2} \sum_{1 \leq i < j \leq r} \sum_{1 \leq p_1 < p_2 \leq n} H_{i,j,n}^* \left( Y_{p_1}, Y_{p_1} \right) \]

\[ + \frac{2^{2j}}{n^2} \sum_{1 \leq i < j \leq r} \sum_{1 \leq p_1 < p_2 \leq n} \left[ H_{i,i,n}^* \left( Y_{p_1}, Y_{p_2} \right) - EH_{i,i,n}^* \left( Y_{p_1}, Y_{p_2} \right) \right] \]

\[ := J_n^{(1)} + J_n^{(2)} + J_n^{(3)} + J_n^{(4)}. \] (56)
According to Lemma 2 and $K^*_i(Y, x)$, we have

$$
|K^*_i(2^j Y_p, 2^j x) - E K^*_i(Y, x)| \leq 2^{2j/2} \left(2^j |Y_p - x|\right)^F
$$

Moreover,

$$
\text{var}(H^*_i(Y_p, Y_p)) \leq E \left(H^*_i(Y_p, Y_p)^2\right) \leq 2^{j(4^j - 2)}
$$

$$
\text{var}(H^*_i(Y_p, Y_p)) \leq E \left(H^*_i(Y_p, Y_p)^2\right) \leq 2^{j(4^j - 2)}.
$$

(57)

By Markov’s inequality, $\forall \varepsilon > 0$,

$$
P\left\{ \left| f_n^{(2)} \right| \geq n^{-1}2^{j(2^j+1/2)\varepsilon} \right\}
$$

$$
= P\left\{ \left| \sum_{i=1}^r \sum_{j=1}^n [H^*_i(Y_p, Y_p) - EH^*_i(Y_p, Y_p)] \right| \geq n2^{j(2^j-(3/2))\varepsilon} \right\}
$$

$$
\leq P\left\{ \left| \sum_{i=1}^r \sum_{j=1}^n [H^*_i(Y_p, Y_p) - EH^*_i(Y_p, Y_p)] \right| \geq n2^{j(2^j-(3/2))\varepsilon} \right\}
$$

$$
\leq \frac{\text{var}(\sum_{i=1}^r \sum_{j=1}^n H^*_i(Y_p, Y_p))}{n^2 2^{j(4^j-3) \varepsilon^2}}.
$$

(59)

$$
P\left\{ \left| f_n^{(4)} \right| \geq n^{-1}2^{j(2^j+1/2)\varepsilon} \right\}
$$

$$
= P\left\{ \left| \sum_{i=1}^r \sum_{j=1}^n [H^*_i(Y_p, Y_p) - EH^*_i(Y_p, Y_p)] \right| \geq n2^{j(2^j-(3/2))\varepsilon} \right\}
$$

$$
\leq \frac{\text{var}(\sum_{i=1}^r \sum_{j=1}^n H^*_i(Y_p, Y_p))}{n^2 2^{j(4^j-3) \varepsilon^2}}.
$$

According to the independence of $\{Y_p\}_{p=1}^n$,

$$
P\left\{ \left| f_n^{(2)} \right| \geq n^{-1}2^{j(2^j+1/2)\varepsilon} \right\} \leq \frac{\text{var}(\sum_{p=1}^n \sum_{i=1}^r H^*_i(Y_p, Y_p))}{n^2 2^{j(4^j-3) \varepsilon^2}} \leq \frac{n2^{j(4^j-3)} \varepsilon^2}{n^2 2^{j(4^j-3) \varepsilon^2}} = n^{-1}2^j \varepsilon^{-2},
$$

(60)

$$
P\left\{ \left| f_n^{(4)} \right| \geq n^{-1}2^{j(2^j+1/2)\varepsilon} \right\} \leq \frac{\text{var}(\sum_{i=1}^r \sum_{j=1}^n H^*_i(Y_p, Y_p))}{n^2 2^{j(4^j-3) \varepsilon^2}} \leq \frac{n2^{j(4^j-2)} \varepsilon^2}{n^2 2^{j(4^j-3) \varepsilon^2}} \leq \frac{2^j \varepsilon^{-2}}{n^2 \varepsilon^2}.
If \( n \to \infty \), then for arbitrary given \( \varepsilon > 0 \),
\[ n^{-2} \| \alpha_j - \alpha \|_2^2 \to 0. \] Moreover, \( n^{-2} \frac{(4j+1)}{2} \alpha_j < \varepsilon \) and
\( n^{-2} \frac{(4j+1)}{2} \alpha_j < \varepsilon \), a.s. So \( J_n \) can be denoted by

\[ \sum_{1 \leq i < j} H_{i, i, n} (yp, yp) + o(1) \]

It is easy to check that \( H_{i, i, n} (yp, yp) \) are symmetric functions.
It is similar to the work of Theorem A in [2] that
\( EH_k^{i, i, n} (yp, yp) \) and \( \mathbb{E}G_k^{i, i, n} (yp, yp) \) satisfy the condition of
Lemma 3. According to Lemma 2 and Lemma 3,
\[ \sum_{1 \leq i < j} H_{i, i, n} (yp, yp) \to N(0, 1) \]

where \( \gamma_{i, i, n}^2 = (1/2)n(n - 1)EH_k^{i, i, n} (Y_1, Y_2) \).
Thus,

\[ n^{-2} \frac{(4j+1)}{2} \alpha_j \to 0 \]

with \( \lambda_j \sim (j/n)^{2j}, j_0, i_0 \to \infty \), and \( n^{-2} \frac{(4j+4)}{2} \alpha_j \to 0 \),
where \( \gamma_{i, i, n}^2 = (1/2)n(n - 1)EH_k^{i, i, n} (Y_1, Y_2) \) and
\( H_{i, i, n} (yp, yp) \) are defined by equations (55) and (62).

The proof is similar to Theorem B in [2].

4. Numerical Example

In this section, an example is given for discussing the results of
multiwavelet deconvolution density estimators.

Choose the model \( Y = X + \varepsilon \). Construct the data \( X \) by the
function “randn” and error data \( \varepsilon \) by the function “rand”
in Matlab. That is, \( X \sim N(0, 1) \) is a standard normal random variable
and \( \varepsilon \sim U(0, 1) \) is a uniform random variable. So \( f_Y \)
is the convolution of \( f_X \) and \( f_\varepsilon \), where
\[ f_X (x) = \left( \frac{1}{\sqrt{2\pi}} \right) e^{-x^2/2} \]

By the formula of the convolution, we have density function \( f_Y \) of random variable \( Y \) as follows:

\[ f_Y (y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \]

In Figure 1, random data \( Y \) is shown at the left side and
its sampling number is 2048. At the right side of Figure 1, the
blue dotted curve denotes the empirical density of data \( Y \)
and the density of data \( Y \) is shown by the red solid curve.
According to Theorem 1 and \( E\tilde{f}_n (x) = P_j f_X \), we choose
the multiwavelet Sa4 to estimate the expectation of the linear
multiwavelet deconvolution density estimators. The sampling
data are decomposed into 4 levels by multiwavelet transform.

The density \( f_X \) of \( X \) is shown in the second row and
second column of Figure 2. In the first row and first column of
Figure 2, the linear multiwavelet deconvolution density estimator
\( \hat{f}_n \) of \( X \) is given by the black solid line and the expectation
\( E\tilde{f}_n \) of linear multiwavelet deconvolution density estimator \( \tilde{f} \) defined by equation (12) is shown by
the red solid line. In the first row and second column of Figure 2,
nonlinear multiwavelet deconvolution density estimator \( \hat{f}_{ln}^{\text{non}} \) is given by the black solid line and the expectation \( E\hat{f}_{ln}^{\text{non}} \) of nonlinear multiwavelet density estimator \( \widehat{f}_{ln}^{\text{non}} \) is shown by the red solid line, where \( \hat{f}_{ln} \) can be denoted by

\[
\hat{f}_{ln}^{\text{non}}(x) := \sum_{i=1}^{r} \sum_{k \in Z} \bar{c}_{ij,k} \phi_{ij,k} + \sum_{i=1}^{r} \sum_{k \in Z} \bar{d}_{ij,k} \psi_{ij,k}. \tag{70}
\]
In the second row and first column of Figure 2, nonlinear multiwavelet deconvolution density estimator \( \hat{f}_{non}^{2n} \) is given by the black solid line and the expectation \( E \hat{f}_{non}^{2n} \) of nonlinear multiwavelet density estimator \( \hat{f}_{non}^{2n} \) is shown by the red solid line, where \( \hat{f}_{non}^{2n} \) can be denoted by

\[
\hat{f}_{non}^{2n}(x) := \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}} c_{ij, k} \phi_{ij, k} + \sum_{j_{i}} \sum_{k \in \mathbb{Z}} \sum_{i=1}^{r} d_{ij, k} \psi_{ij, k}.
\]

(71)

Moreover, asymptotic normality is identified by the Jarque–Bera test. The results of the J-B test are given for \( \hat{f}_{n} - f_X \), \( \hat{f}_{non}^{1n} - f_X \), and \( \hat{f}_{non}^{2n} - f_X \) in Table 1.

| Estimators   | P value | Results of normality |
|--------------|---------|----------------------|
| \( \hat{f}_{n} - f_X \) | 0.5000  | 0                    |
| \( \hat{f}_{non}^{1n} - f_X \) | 0.4642  | 0                    |
| \( \hat{f}_{non}^{2n} - f_X \) | 0.5000  | 0                    |

Note: this table shows the results of the J-B test for multiwavelet estimators \( \hat{f}_{n} - f_X \), \( \hat{f}_{non}^{1n} - f_X \), and \( \hat{f}_{non}^{2n} - f_X \). If the result of Jarque–Bera test is zero, it indicates that it obeys normal distribution at significant level 0.05. If P value of the Jarque–Bera test is closer to zero, it indicates that the original assumption of normal distribution can be rejected.

In Table 1, all results of normality are zero, and the original assumption of normal distribution can be accepted by the P value of the Jarque–Bera test. These imply the conclusions of Theorem 2 and Theorem 3.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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