Multiplicative signal induces double and quasi-periodic vibrational resonance in the overdamped oscillator

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Abstract. The vibrational resonance in the overdamped oscillator with the triple-well potential driven by a multiplicative low-frequency force and an additive high-frequency force is investigated by numerical simulation. The double VR and quasi-periodic oscillator are observed. There exists a resonant range, which is increased with the amplitude and frequency of the high-frequency force, but the response amplitude decreases as the high-frequency force becomes strong.

1. Introduction
The vibrational resonance (VR) is that a resonant behavior appears in some nonlinear systems driven by a low-frequency force and a high-frequency force, and the optimal amplitude of high-frequency driving enhances the response of an excitable system to a low-frequency subthreshold signal. VR originally is found by Landa and McClintock [1], which is motivated by stochastic resonance (SR) [2], and closely resembles SR in that a noise term is replaced by a high-frequency harmonic signal. The high-frequency signal input is deterministic and more controllable than the noise, therefore, more and more attentions are payed to the VR in many fields. Such as, laser physics [3], acoustics [4], neuroscience [5], physics of the ionosphere [6], or the biological field [7, 8, 9]. The experimental evidence of the VR in the overdamped Duffing oscillator [10] and an optical system [11] have also been achieved. The theoretic method given by Blechman [12] has been widely applied to many nonlinear systems. For example, a bistable system Ref [1, 13], double-well Duffing oscillator [14], signal transmission by VR in one-way coupled bistable systems [15], a damped quintic oscillator with monostable potentials [16, 17], and so on. However, for the complex nonlinear dynamical system, the numerical simulation method is general used. For example, the delay time induces quasi-periodic VR [18, 19]. Two coupled overdamped an anharmonic oscillator driven by an amplitude modulated force [20]. Moreover, VR in a noise induced structure has been studied in Ref [21, 22]. Especially, VR is also been observed in other system, e.g. a two-fluid plasma model [23], an inhomogeneous medium [24], and a discrete neuronal model [25, 26].

In the VR phenomenon in the nonlinear dynamical systems as mentioned above, the low-frequency signal is general taken into an additive fashion (state independent) in most cases for the studies. However, in some situations, the signal should be taken into a multiplicative fashion (state dependent)–such as in the membrane-protein system investigated by Ref. [27], where the incoming signal is coupled to the transmitting process in a multiplicative way. Since the low-frequency signal in a nonlinear system has an external origin in many cases, considering a multiplicative signal is more realistic. The effects of the multiplicative signal on the SR have been widely investigated in some stochastic systems [28-32]. Nicolis et al introduced a multiplicative aperiodic signal into a symmetric
bistable potential with the presence of additive noise, and found aperiodic SR [28]. VR closely resembles SR in that a high frequency harmonic signal plays the role of noise. Motivating by this idea, in this letter, the effect of the multiplicative low-frequency signal on the VR in the overdamped oscillator with triple-well potential is investigated by numerical simulation.

This paper is organized as follows. The model and numerical simulation method are introduced in the Section 2. In the Section 3, the VR in the overdamped oscillator with the triple-well potential is discussed. A brief conclusions are given in final section.

2. Model and method

In the present work, we only consider the overdamped oscillator with the triple-well potential, which is a basic model, and analyze occurrence of VR. The equation of motion of the overdamped oscillator with the triple-well potential and two periodic forces is given by

\[ \gamma \ddot{x} = -U'(x) + x(t)A\cos(\omega t) + B\cos(\Omega t) \]  

(1)

Where the term \( B\cos(\Omega t) \) is a high-frequency force with amplitude \( B \), and \( x(t)A\cos(\omega t) \) is a low-frequency signal of amplitude \( A \) and frequency \( \omega \). Assuming that \( \Omega >> \omega \) and \( A << B \). \( \gamma \) is the dissipation constant. \( U(x) \) is given by [33]

\[ U(x) = x^2(bx^2 - c)^2 \]  

(2)

which is a symmetric triple-well potential, \( b \) and \( c \) are the parameters of the potential, which are chosen \( b > 0 \) and \( c > 0 \), or \( b < 0 \) and \( c < 0 \). In order to calculated simply in this paper, we set the parameters \( b = 0.1 \) and \( c = 1 \). The symmetric triple-well potential is shown by figure 1 for the fixed parameter value \( b \) and \( c \).

![Figure 1. The symmetric triple-well potential \( U(x) \) for \( b = 0.1, c = 1 \).](image)

To evaluate the amplitude of the input frequency in the output signal, we calculate the response amplitude \( Q \) (e.g. the Fourier coefficient for the input frequency). We use the \( Q \) parameter instead of the power spectrum because we are interested in the signal transmission in the system at the lower frequency \( \omega \). For this task the response amplitude \( Q \) is a much more compact tool than the power spectrum.

The response amplitude \( Q \) is given by [1]

\[ Q = \frac{\sqrt{B_i^2 + B_s^2}}{A} \]  

(3)

and the phase shift is as follow,

\[ \Psi = -\arctan\left(\frac{B_i}{B_s}\right) \]  

(4)
with
\[ B_x = \frac{2}{nT} \int_0^{\alpha T} x(t) \sin(\omega t) dt \]
and
\[ B_c = \frac{2}{nT} \int_0^{\alpha T} x(t) \cos(\omega t) dt \]
where \( T = \frac{2\pi}{\omega} \) and \( n \) is a positive integer.

Equation (1) can be discretized through Euler scheme for iterations
\[
x(n+1) = x(n) + \left[ x(n)^2 (b x(n)^2 - c)^2 + x(n) A \cos(\omega n \Delta t) + B \cos(\Omega n \Delta t) \right] \Delta t,
\]

\[ n = 1, 2, ..., N, N = \frac{t}{\Delta t}. \]

We calculate the response amplitude \( Q \) and the phase shift \( \Psi \) numerically. In the following simulation process, the parameters are chosen as \( b = 0.1, c = 1, \gamma = 1 \). Without special notes, the initial condition \( x(1) \) is chosen randomly. The fixed step sizes \( \Delta t = 0.01 \) and the total time is \( t = 2 \times 10^6 \).

3. Results and discussions
Applying the numerical simulation method given in section 2, the results are shown in figures 2-5.

Figure 2. The numerical simulations of the response amplitude \( Q \) of this system is plotted as functions of \( B \). The parameters are chosen as \( A = 0.01, \gamma = 1, \omega = 0.1, \Omega = 5, 10 \). The other parameters are the same as those in figure 1.

Figure 2 shows the response amplitude \( Q \) as a function of the amplitude \( B \) of the high-frequency signal with different frequency of the high-frequency signal \( \Omega = 5, 10 \). A double VR is observed in figure 2a. For \( \Omega = 5 \), \( Q \) has two maxima in the range \( B \in [7, 15] \), and the first maximum is at the position \( B = 7 \) and the second one is at \( B = 15 \) (see figure 2a). \( Q \) increases very slow in the range \( B \in [0, 7] \) and decreases in the range \( B \in [15, 40] \) (see figure 2c). However, the values of \( Q \) in the
range $B \in [7,15]$ are greater than ones in the other ranges. For $\Omega = 10$, the behavior of $Q$ with $B$ increasing is similar to the case $\Omega = 5$, but there is a distinct difference between them. e.g., the range at which the VR occurs becomes large. i.e., the range is $B \in [7,15]$ for $\Omega = 5$, and $B \in [14.4, 29.4]$ for $\Omega = 10$. Moreover, for $\Omega = 10$, $Q$ increases monotonously in $B \in [0, 14.4]$ and $B \in [29.4, 40]$ (see figure 2b). The other hand, the maxima of the response amplitude $Q$ become smaller as the $\Omega$ increases. That is to say, $\Omega$ can restrain VR, but the range at which the VR occurs is broaden.

![Figure 3](image1.png) ![Figure 3](image2.png)

**Figure 3.** The response amplitude $Q$ of this system is plotted as functions of $\Omega$. The parameters are $A = 0.1, \gamma = 1, \omega = 0.01, B = 2, 4, 8, 15$. The other parameters are the same as those in figure 1. figure 3(b) is an additional remarks to figure 3(a).

The response amplitude $Q$ as a function of the frequency $\Omega$ of the high-frequency signal with different amplitude of the high-frequency signal $B = 2, 4, 8, 15$ is shown in figure 3. From the figure 3a, it can be observed that the appearance of a quasi-periodic oscillation is only for the small amplitude of the high-frequency signal ($B = 2$). For the large amplitude of the high-frequency signal ($B = 4,8,15$), the double VR phenomenon is found. Moreover, with $B$ increasing, the height of the two peaks decrease gradually, but the range of $\Omega$ at which the VR occurs enlarged, which is similar to the case in figure 2. It implicates that the VR is more easy to occur for the strong high-frequency signal. It is worth noting that the quasi-periodic oscillation is still existed for the large amplitude of the high-frequency signal (see figure 3b). The response amplitude $Q$ into the range at which the VR occurs is far greater than one outside range, and leads to the quasi-periodic oscillation can’t be shown in figure 3a for the large amplitude of the high-frequency signal. The quasi-periodic oscillation can be explained as follow. Since $\Omega \gg \omega$ and $A << B$, the response amplitude $Q$ of the system are mainly excited by the high-frequency signal $B \cos(\Omega t)$. In this case, $Q$ is a function of $B$, $\sin(\Omega t)$ and $\cos(\Omega t)$. Therefore, $Q$ is a quasi-periodic function of $\Omega$. It is emphasized that when the low-frequency signal is chosen an additive signal, there is no the quasi-periodic oscillation in the overdamped oscillator with the triple-well potential [34].

In order to verify the double VR and the quasi-periodic oscillation, the phase shift $\Psi$ corresponding to $Q$ as the function of the $B$ and $\Omega$ are plotted in figure 4 and figure 5. It is evident that there is a resonant range in figure 4 and figure 5, and with the $B$ and $\Omega$ increasing, the range become large. Moreover, a quasi-periodic oscillator is also presented in the $\Psi - \Omega$ plot.

The mechanism of double VR is can be understood as follow. The overdamped oscillator with the triple-well potential in our investigation have three minima and two maximum as shown in figure 1. It
means there are three potential wells for the system. When the high-frequency signal are taken account into this model, The depth and location of the minima of the potential wells have a distinct effect on VR. That is to say, with the amplitude $B$ and frequency $\Omega$ varying, the depth and location of the minima of the potential wells are changed. If we apply a weak aperiodic forcing to the particle, which is moving in the potential wells, the potential wells are tilted asymmetrically up and down, raising and lowering the depth of the potential wells. Although the aperiodic forcing is too weak to let the particle transit periodically from one potential well into the other ones, the transition of the high-frequency signal induced hopping between the potential wells can become synchronized with the weak aperiodic forcing. This yields the time-scale matching condition for VR. The time-scale matching condition can be fulfilled and the VR occurs.

**Figure 4.** The numerical simulations of the phase shift $\Psi$ of this system is plotted as functions of $B$ with a multiplicative low-frequency signal and a high-frequency signal. The parameters are $A = 0.01$, $\gamma = 1$, $\omega = 0.1$, $\Omega = 5, 10$. The other parameters are the same as those in figure 1.

**Figure 5.** The numerical simulations of the phase shift $\Psi$ of this system is plotted as functions of $\Omega$ with a multiplicative low-frequency signal and a high-frequency signal. The parameters are $A = 0.01$, $\gamma = 1$, $\omega = 0.1$, $B = 5, 8$. The other parameters are the same as those in figure 1.

**4. Conclusions**

To conclude, the occurrence of VR in the overdamped oscillator with the triple-well potential driven by a multiplicative low-frequency force and an additive high-frequency force is investigated by
numerical simulation. The double VR is found, and the range at which the VR occur is given for the fixed parameters. With $B$ and $\Omega$ increasing, the range becomes large and the height of the peak decreases. It is worthy of note that a quasi-periodic oscillator is observed in the $Q - \Omega$ plot, and the amplitude $Q$ attenuates with $\Omega$ increasing. On the contrary the amplitude $Q$ enhances with $B$ increasing. The phase shift $\Psi$ corresponding to $Q$ is shown. There exists a resonant range, and with the $B$ and $\Omega$ increasing, the range become large. Moreover, a quasi-periodic oscillator is also presented in the $\Psi - \Omega$ plot.

Since the high-frequency signal input is deterministic and is more controllable, we except that the scheme could be of great significance for potential applications due to its simplicity and high efficiency. Moreover, due to the aperiodic signal is general for the real system in a variety of science and engineering fields, we hope our study will be applicable in solving some problems and arousing more techniques to deal with the aperiodic signal issues in different disciplines.

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