THE DECIDABILITY OF THE GENUS OF REGULAR LANGUAGES AND DIRECTED GRAPH EMULATORS

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Abstract. The article continues our study of the genus of a regular language. Here we study the closely related graph-theoretic notion of a directed emulator of a directed graph. Let \( L \) be a regular language. Consider the set \( S \) of all directed emulators of the underlying directed graph of the minimal deterministic automaton for \( L \). We prove that the genus of \( L \) is \( \min_{G \in S} g(G) \). We also consider undirected emulators and prove that if the problem of determining the minimal genus of a directed emulator of a directed graph has a solution then the problem of determining the minimal genus of an undirected emulator of an undirected graph has a solution.

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1. Introduction

In our first article [BD16], we defined the genus of a regular language \( L \): this is the minimal genus of a deterministic automaton recognizing the language \( L \). In particular, a planar language is a language with genus 0. As shown in [BD16], this notion defines a proper hierarchy of (regular) languages. We note that the decidability of the planarity of a regular language is a question asked by R. V. Book and A. K. Chandra [BC76] in 1976.

It should be noted that the genus of a regular language behaves quite differently from its set-theoretical size. First, as we showed in [BD16], a minimal genus deterministic automaton recognizing a given language has not minimal size in general, thus differs from the minimal automaton in the sense of Myhill-Nerode. Secondly, as we showed in [BD19], the size of the minimal automaton recognizing a language \( L \) with minimal genus may be exponentially larger than the minimal automaton itself. This suggests that the computation of the genus of a regular language should have a high complexity.

In [BD16], we showed that a minimal genus deterministic automaton recognizing a given language has not minimal size in general, thus differs from the minimal automaton in the sense of Myhill-Nerode.

In a sequel to the first article [BD19], we showed that the size of the minimal automaton recognizing a language \( L \) with minimal genus may be exponentially larger than the minimal automaton itself. The existence of an upper bound remains open. Anyway, this shows that the witness automaton of minimal genus can be very large with respect to the size of the minimal automaton that serves as reference. That is an insight that computing the genus of a language should have a high complexity. We also proved that under a fairly generic hypothesis\(^1\), the genus of a regular language is computable. In particular, under this hypothesis, the planarity of a regular language is decidable. One should point out that the proof is entirely constructive and yields an implementable algorithm. We conjecture that the genus of a regular language is computable in general.

In this third opus of the series, we study a graph-theoretical approach. The graph-theoretical substance of the relation of the minimal automaton to the genus minimal automaton is the notion of minimal directed emulator (§4). A closely related notion is that of directed cover. Most of this paper is devoted to the systematic study of directed emulators.

Our key result establishes that the original problem of determining the genus of a regular language is equivalent to the problem of determining the minimal genus of a directed cover of its minimal automaton (Theorem 2).

\(^1\)The minimal automaton must not contain small cycles, the size of which depends on the size of the alphabet. Moreover, automata are supposed to be complete.
Three main mathematical structures arise. First, (finite state) automata are used to describe regular languages. Forgetting letters on edges leads to the notion of directed graphs. Forgetting the direction on edges leads to undirected graphs. These two forgetful operations are performed when the genus is computed. So the question arises: what is the right setting to compute the genus since the problem is not about computing the genus of one specific graph but the class of equivalent automata? We show that it is a subcategory of the category of directed graphs, the category of directed emulators and directed emulator morphisms.

The notion of directed emulator is the natural refinement of the notion of graph emulator, introduced by R. M. Fellows in 1985. In order to study the properties of emulators, he also used the notion of graph cover and conjectured that a connected finite graph has a finite planar emulator if and only if it has a finite planar cover. It took more than twenty years before Y. Rieck and Y. Yamashita found a counterexample \cite{RY09} to the conjecture. However, P. Iliniény had found ten years earlier \cite{Hl99} an example of a graph having an emulator of genus strictly less than the minimal genus of any of its covers. We show that the situation for directed graphs is much simpler: a directed graph has a directed emulator of genus $g$ if and only if it has a directed cover of genus $g$.

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2. Outline

Given an automaton $A$, let $L(A)$ denote the language recognized by $A$ and let $G_A$ denote the underlying simple directed graph. The genus $g(L)$ of a regular language $L$ is the minimal genus of a deterministic automaton recognizing the language:

$$g(L) = \min \{g(G_A) \mid L(A) = L, A \text{ deterministic}\}.$$ 

As evidenced in \cite{BD16}, the genus is in general reached at a nonminimal automaton.

A directed emulator morphism between directed graphs is a graph morphism $\pi$ that sends the outgoing edges at each vertex $w$ surjectively onto the outgoing edges at $\pi(w)$. More precisely, a directed graph $G'$ is a directed emulator of a directed graph $G$ if there is a directed graph epimorphism $G' \to G$ such that for any vertex $v$ of $G$, any outgoing edge $e$ starting at $v$ and any vertex $v'$ of $G'$ in the preimage of $v$, there is an outgoing edge $e'$ starting at $v'$. A more restrictive notion is that of directed cover where

\footnote{Actually even four: we will deal with semi-automata, that is automata without the notion of initial and final states. But that does not change the following discussion.}
the outgoing edges at $w$ are sent bijectively onto the outgoing edges at the image of $w$.

Let $L$ be a regular language, let $A_{\text{min}}(L)$ be the Myhill-Nerode minimal automaton recognizing $L$ and let $G(L) = G_{A_{\text{min}}(L)}$ be the underlying directed graph.

**Theorem A.** Let $g \geq 0$. The three following statements are equivalent:

(i) the regular language $L$ has genus $g$

(ii) the directed graph $G(L)$ has a directed emulator of genus $g$

(iii) the directed graph $G(L)$ has a directed cover of genus $g$.

**Corollary 1.** Let $g \geq 0$. The problem of determining whether the genus of a regular language $L$ is lower than $g$ is decidable if the problem of determining whether the genus of some directed emulator of a given directed graph is lower than $g$ is decidable.

There is a related notion of emulator for undirected graphs. An emulator morphism between (undirected) graphs is a graph morphism $\pi$ that sends the set of incident edges of each vertex $w$ surjectively onto the set of incident edges of $\pi(w)$.

The following result shows that a solution to the directed emulation genus problem can be used to solve the emulation genus problem.

**Theorem B.** The problem of determining whether the genus of some emulator of a given graph is lower than $g$ has a solution if the problem of determining whether the genus of some directed emulator of a given directed graph is lower than $g$ has a solution.

Both results are proved in §6.

Throughout the paper, some statements are asserted without proof. We do it only when proofs are immediate consequences of the definitions. Details are provided whenever a more elaborate proof is required or a specific construction.

### 3. Preliminary material

#### 3.1. Graphs.

A **directed graph** (sometimes shortened to *digraph*) $G$ consists of a set $V$ of *vertices* and a set $E$ of *edges* and two maps $s, t : E \rightarrow V$ (resp. “source” and “target”). Since we do not need this extra generality, all along, we suppose that $V$ and $E$ are finite sets even though definitions could be extended to the infinite case.

We think of an edge $e \in E$ as “starting” at $x = s(e)$ and “ending” at $y = t(e)$ which is summed up by the notation $\xrightarrow{e} x \rightarrow y$. We say that $x$ is **adjacent** to $y$. An edge $x \xrightarrow{e} x$ is a **loop** at vertex $x$. Given a graph $G$, we denote by $V_G$ its vertices, by $E_G$ its edges and by $s_G, t_G$ its corresponding

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3This is actually the edge endowed with its source and target. So the notation contains more than the datum of the edge itself.
Given a directed graph $G$, the ordered boundary map is the map $\Delta_G : E_G \rightarrow V_G \times V_G$ on the set of edges defined by $\Delta_G(e) = (s_G(e), t_G(e))$. An edge $x \xrightarrow{e} y \in E_G$ is simple if there is no other edge $e'$ such that $\Delta_G(e) = \Delta_G(e')$. The graph $G$ is simple if any of its edges is simple.

A morphism $G \rightarrow H$ between directed graphs $G$ and $H$ is a pair $(p, q)$ where $p : V_G \rightarrow V_H$ is a map between the set of vertices of $G$ to the set of vertices of $H$ and $q : E_G \rightarrow E_H$ is a map between the set of edges of $G$ to the set of edges of $H$, that satisfies the relations:

\begin{equation}
 p \circ s_G = s_H \circ q \quad \text{and} \quad p \circ t_G = t_H \circ q.
\end{equation}

Setting $p^{\times 2} : V_G \times V_G \rightarrow V_H \times V_H$, $(x, y) \mapsto (p(x), p(y))$, one can state the previous equation in an equivalent way as

\begin{equation}
 p^{\times 2} \circ \Delta_G = \Delta_H \circ q.
\end{equation}

This relation is referred as the adjacency relation.

The identity $(1_{V_G}, 1_{E_G}) : G \rightarrow G$ is a morphism and morphisms compose.

**Lemma 1.** Directed graphs and morphisms between them form a category denoted $\mathcal{D}Gr$. Directed simple graphs and morphisms between them form a full subcategory $\mathcal{D}SGr$ of $\mathcal{D}Gr$.

As defined, the category of graphs is equivalent to the homset category $\text{Hom}_{\text{Fset}}(\cdot, \cdot)$ where $\text{Fset}$ is the category of finite sets and maps between them. We do not use it explicitly but the equivalence occurs between the lines in the sequel.

A graph epimorphism (resp. monomorphism, isomorphism) is a graph morphism $(p, q)$ such that both maps $p$ and $q$ are surjective (resp. injective, bijective).

A subset $W \subseteq V$ of vertices of a graph $G = (V, E, s, t)$ determines the graph

\begin{equation}
 G|_W = (W, E_W, s|_{E_W}, t|_{E_W}) \quad \text{where} \quad E_W = \{ e \in E \mid \Delta(e) \in W \times W \}.
\end{equation}

Similarly, a subset $F \subseteq E$ of edges determines the graph\footnote{The notation is common for restrictions to vertices and to edges. The context will disambiguate it.}

\begin{equation}
 G|_F = (s(F) \cup t(F), F, s|_F, t|_F).
\end{equation}

Alternatively,

A directed subgraph of $G$ is a graph $H$ such that there is a set of vertices $W$ and a set of edges $F$ such that $H = (G|_W)|_F$. We also say that $G$ contains $H$.

Given two sets of vertices $W \subseteq V$, the quotient set $V/W$ is made of elements $[x]$ with $x \in V$ such that $[x] = [y]$ whenever $x, y \in W$. It induces two new maps $[s], [t] : E \rightrightarrows V/W$ defined by $[s](e) = [s(e)]$ and $[t](e) = [t(e)]$. We shall drop subscripts if this does not seem to cause confusion.
respectively. The resulting graph $G/W = (V/W, E, [s], [t])$ is the graph with all the vertices in $W$ identified. The quotient comes with an epimorphism: $([\cdot], 1_{E_G}) : G \to G/W$.

Let us mention that the notation $[x] \in V/W$ does not show the dependency on $V$ and $W$. In case it is necessary, we will write it $[x]_{V/W}$.

When $W$ consists of pairs of adjacent vertices $s(e_i), t(e_i)$ of a subset $F = \{e_i, i \in I\} \subseteq E$ of edges, the resulting graph is the contraction of $G$ along $F$ and is denoted $G/F$. In the case when $W$ is one pair of adjacent vertices of an edge $e$, we denote the contraction of $G$ along the edge $e$ by $G/e$.

Remark 1. It follows from the definition that for any sequence of distinct edges $e_1, \ldots, e_n$, we have $(G/e_1) \cdots /e_n = G/\{e_1, \ldots, e_n\}$.

Remark 2. In the original notion of edge-contraction (see e.g., [GYZ14, Chap.2, p. 156]), an edge between two identified vertices are deleted whereas in our definition, it is transformed into a loop (on the resulting vertex). Nevertheless, the original operation is a composition of the contraction as defined above followed by the removal of these loops. Since both operations are compatible with our problem (see Lemma 16 and Lemma 12), we shall be able to apply Robertson and Seymour’s minors theorem.

Given a morphism $(p, q) : G \to H$, we define its image graph to be $(p, q)(G) = (p(V_G), q(E_G), s, t)$ with $s(q(e)) = p(s_G(e))$ and $t(q(e)) = p(t_G(e))$ for all $e \in E_G$. It is clear that $(p, q)(G)$ is a subgraph of $H$ and that $(p, q) : G \to (p, q)(G)$ is an epimorphism.

Given a directed graph $G$, consider the directed graph defined $R(G) = (V_G, \Delta_G(E_G), \pi_1, \pi_2)$ with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. In particular, the set of edges $E_{R(G)}$ is the subset of ordered pairs of adjacent edges:

$$E_{R(G)} = \{(x, y) \in V_G \times V_G \mid \exists e \in E_G : (s(e), t(e)) = (x, y)\}$$

Lemma 2. $\Delta_{R(G)} = 1_{E_{R(G)}}$.

Proof. $\Delta_{R(G)}(e) = (\pi_1(e), \pi_2(e)) = e$ for all $e \in V_G \times V_G$. □

Corollary 2. The graph $R(G)$ is simple.

Corollary 3. The assignment $G \mapsto R(G)$ extends to a functor $\mathcal{G}Gr \to \mathcal{D}Gr$ that assigns to a morphism $(p, q) : G \to G'$ a morphism $R(p, q) : R(G) \to R(G')$ defined by $R(p, q) = (p \times 2)$. (p, q) : G \to G' a morphism $R(p, q) : \Delta_G \to \Delta_{R(G)}$ defined by $R(p, q) = (p \times 2)$.

Proof. An edge $\Delta_G(e) \in E_{R(G)}$ is mapped to $p \times 2 \circ \Delta_G(e)$. This is a morphism since the adjacency relation is satisfied, for $\Delta_{R(G')} \circ (p \times 2) = p \times 2 = p \times 2 \circ \Delta_{R(G)}$ according to Lemma 2. □

Lemma 3. The pair $\rho_G = (1_{V_G}, \Delta_G) : G \to R(G)$ is an epimorphism. It is an isomorphism when restricted to simple graphs. Moreover, $G \mapsto \rho_G$ defines a natural transformation $1_{\mathcal{G}Gr} \to I \circ R$ where $I$ denotes the inclusion functor $\mathcal{D}Gr \to \mathcal{G}Gr$.
Lemma 4. $R \circ I \circ R = R$.

There is yet another endofunctor we shall need. Let $G = (V, E, s, t)$ be a directed graph. Then $G^\text{op} = (V, E, t, s)$ is another directed graph. A morphism $(f, g) : G \to G'$ of digraphs induces a morphism $(f, g)^\text{op} : G^\text{op} \to G'^\text{op}$ defined by $f^\text{op} = f$ and $g^\text{op} = g$.

Example 1. Let $G = \overset{g}{v} \overset{e}{\longrightarrow} \overset{f}{w}$, then $G^\text{op} = \overset{g}{v} \overset{e}{\longrightarrow} \overset{f}{w}$. In the drawings, $\{v, w\}$ are the vertices, $\{e, f, g\}$ the edges and arrows describe source and targets.

We record a few properties of the graphs that are “invariant” under op.

Lemma 5. The following relations hold:

$$(G^\text{op})^\text{op} = G. \quad (4)$$

$$(G^\text{op}) = R(G)^\text{op}. \quad (5)$$

Proof. The relations follow from the definitions. □

We record the following fact which we shall refine later.

Lemma 6. The functor $R : \mathcal{DGr} \to \mathcal{DGr}$ is essentially surjective and full. The endofunctor $(\cdot)^\text{op}$ is both full and faithful.

Proof. $R$ is essentially surjective since $\rho_G : G \to R(G)$ is an isomorphism for simple graphs. Fullness is a direct consequence of Lemma 4. The second statement follows from the identity $(4)$. □

We shall define yet another operation on graphs. Excising loops in a directed graph consists in removing all loops from the set of edges while keeping other edges and the set of vertices.

Definition 1. Let $G$ be a directed graph. Let $L = \{e \in E_G \mid s_G(e) = t_G(e)\}$ be the subset of loops. The excision of $G$ is the graph $G$ with all loops removed:

$\text{Exc}(G) = G_{|E_G - L}$.

Remark 3. The excision is not yet functorial since we cannot yet define it at the level of graph morphisms. For instance, the graph $\circ \longrightarrow \bigcirc$ maps to $\bigcirc \bigcirc$ and there is no graph morphism if we remove the loop. We shall see later that in the appropriate category, Exc becomes a functor.
3.2. Undirected graphs. An undirected graph \( G = (V, E, \partial) \) consists of a set \( V \) of vertices, a set \( E \) of edges and a map \( \partial : E \to \mathcal{P}_2(V) \) from the edges to the set of unordered pairs of vertices\(^5\). A morphism \( G \to H \) between undirected graphs is a pair of maps \( p : V_G \to V_H \) (between vertices) and \( q : E_G \to E_H \) (between edges) such that:

\[
\partial_H \circ q = p \otimes^2 \circ \partial_G
\]

with \( p \otimes^2 \{x, y\} = \{p(x), p(y)\} \).

Undirected graphs and morphisms between them form a category denoted by \( \text{Gr} \). The canonical projection \( \text{pr}_V : V \times V \to \mathcal{P}_2(V) \) mapping \( (v, w) \mapsto \{v, w\} \) induces a forgetful functor \( U : \mathcal{D}\text{Gr} \to \text{Gr} \). It maps objects \( G \mapsto (V_G, E_G, \text{pr}_{V_G} \circ \Delta_G) \) and it acts as the identity on morphisms.

Lemma 7. \( U(G^{\text{op}}) = U(G) \).

3.3. The genus of a graph. We recall a few definitions. The genus \( g(\Sigma) \) of a closed oriented surface \( \Sigma \) is half the dimension of the first real homology vector space \( H_1(\Sigma; \mathbb{R}) \). Alternatively, it is the maximum number of mutually disjoint simple closed topologically nontrivial curves \( C_1, \ldots, C_g \) such that the complement \( \Sigma - (C_1 \cup \cdots \cup C_g) \) remains connected. This yields a natural notion of genus of a graph.

Definition 2 (Genus of a graph). A graph has genus \( n \) if its geometrical realization is embeddable in a surface of genus \( n \) but cannot be embedded in a surface of strictly smaller genus. We note \( g(G) \) the genus of a graph \( G \).

The definition makes sense for directed and undirected graphs alike.

Lemma 8. For any directed graph, \( g(G) = g(U(G)) \).

Proof. The geometric realization only depends on the underlying undirected graph. \( \square \)

All the proofs dealing with the genus of some graph will rely on one of the following observations.

Lemma 9. If \( G \) is isomorphic to \( H \), then \( g(G) = g(H) \).

Lemma 10. The functor \( (-)^{\text{op}} \) preserves the genus.

Proof. For a directed graph \( G \), \( g(G^{\text{op}}) = g(U(G^{\text{op}})) \) (lemma \( \Box \) = \( g(U(G)) \)) (lemma \( \Box \) = \( g(G) \) (lemma \( \Box \)). \( \square \)

Lemma 11. Removing an edge from a graph does not increase the genus: for any graph \( G \) and \( e \in E_G \), \( g(G - e) \leq g(G) \).

Lemma 12. The excision operation preserves the genus: \( g(\text{Exc}(G)) = g(G) \).

\(^5\)We define \( \mathcal{P}_2(V) = \{\{v, w\} \mid v, w \in V\} \). So, formally, \( X \in \mathcal{P}_2(V) \) is either a singleton or a pair.
Proof. By the previous lemma, \( g(\text{Exc}(G)) \leq g(G) \). On the other hand, (the geometric realization of) a loop at a given vertex embeds as the boundary of a small disc, hence if \( \text{Exc}(G) \) embeds in a surface \( \Sigma \), then \( G \) also embeds there, so \( g(G) \leq g(\text{Exc}(G)) \). \( \square \)

**Lemma 13.** Let \( e \) be an edge of a graph \( G \). Let \( G' \) be the graph defined by \( V_{G'} = V_G \), \( E_{G'} = E_G \cup \{ e' \} \), \( \Delta_G(e') = \Delta_G(e) \). Then \( g(G') = g(G) \).

**Proof.** The graph \( G \) is obtained by removing one edge from the graph \( G' \). By Lemma 11, \( g(G) \leq g(G') \). On the other hand, suppose that \( G \) embeds into a surface \( \Sigma \). Since \( \Delta_G(e') = \Delta_G(e) \), one can embed the extra edge \( e' \) in such a way that the geometrical realization of the union \( \{ e \} \cup \{ e' \} \) is embedded as the boundary of a small disc in \( \Sigma \), hence \( G' \) also embeds into \( \Sigma \), so \( g(G') \leq g(G) \). \( \square \)

**Lemma 14.** The functor \( R \) preserves the genus.

**Proof.** Induction from the previous lemma. \( \square \)

The next observation is that replacing an edge by a loop in a graph does not increase the genus.

**Lemma 15.** Let \( G \) be a directed graph and let \( e \in E_G \) be an edge. Let \( e' \) be a new edge such that \( s(e') = s_G(e) = t(e') \). Let \( G' \) be the graph obtained by removing \( e \) and adding the loop \( e' \). Then \( g(G') \leq g(G) \).

**Proof.** Follows from Lemma 12 and Lemma 11 above. \( \square \)

The last observation is that contracting an edge does not increase the genus.

**Lemma 16.** Contracting an edge from a graph does not increase the genus: for any graph \( G \) and \( e \in E_G \), \( g(G/e) \leq g(G) \).

4. Directed emulators

In this section, we define directed emulators and study their properties. The basic material is introduced in §4.1 Categorical and closure properties are presented in §4.3 and §4.2. Finally we briefly discuss the relation to undirected emulators in §4.4.

4.1. Basic definitions. The following definition is the main object of this section.

**Definition 3.** Let \( \pi = (p, q) : G \to H \) be a directed graph morphism. It is a directed emulator morphism when:
(i) \( p \) is surjective and
(ii) \( \pi \) verifies the edge outgoing lifting property. That is, for any edge \( e \in E_H \) and any vertex \( x' \in V_G \) such that \( p(x') = s_H(e) \), there is an edge \( e' \in E_G \) such that \( q(e') = e \) and \( s_G(e') = x' \).
If $\pi : G \to H$ is a directed emulator morphism (or sometimes shorter: a directed emulator), we say that $G$ is a directed emulator of $H$ and that $H$ is a directed amalgamation of $G$.

*Example 2.* The identity $1_G : G \to G$ is a directed emulator morphism. More generally, an isomorphism $\psi : G \to H$ is a directed emulator morphism.

*Example 3.* For any directed graph $G$, the map $\rho_G : G \to R(G)$ is a directed emulator morphism.

The following observation is a direct consequence of the definition.

**Lemma 17.** A directed emulator morphism is a directed graph epimorphism.

**Proof.** By definition, the vertex map is surjective. Given an edge $e$ in the base graph, since $p$ is surjective, there is a vertex $x \in p^{-1}(s_G(e))$ and then, by the edge outgoing lifting property, there is an edge $e'$ such that $q(e') = e$. \hfill \Box

**Remark 4.** A directed graph epimorphism is not necessarily a directed emulator morphism. For instance, consider the epimorphism below (the vertex function is represented by the dotted arrows, the map on edges is immediate given that the graph is simple), it is not a directed emulator. Indeed, the central node in the base graph has two outgoing edges which is not the case of its antecedent.

*Example 4.* Directed amalgamation can “create” loops as shown by the drawing below.

**Remark 5.** There are in general several directed emulator morphisms between a digraph and a directed amalgamation of it. For instance, the digraph depicted below is a directed emulator of itself in two ways:

The first one is the identity and the other one swaps the two edges. In both cases, the map on the vertices is the identity.

Let us consider a relaxed version of a directed emulator morphism.
Definition 4. Let $G$ and $G'$ be two digraphs. A directed emulator map from $G'$ to $G$ is a pair $(p, q)$ made of a surjective map $p : V_G \to V_G$ and a partial function $q : E_{G'} \to E_G$ such that for any edge $x \xrightarrow{e} y \in E_G$ and any $x' \in p^{-1}(x)$, there is an edge $x' \xrightarrow{e'} y' \in E_{G'}$ such that $p(y') = y$ and $q(e') = e$.

Remark 6. In Definition 3, for the edge outgoing lifting property, we did not ask that $p(y') = y$. But, for a morphism, the equality holds necessarily. Thus a directed emulator morphism is a directed emulator map.

A directed emulator map may not be a digraph morphism as shown by the two examples:

![Diagrams showing two examples of directed emulator maps]

Example 5. Given a directed graph $G$, the pair $(1_{V_G}, 1_{\text{Exc}(G)}) : G \to \text{Exc}(G)$ is a directed emulator map. It is a directed emulator morphism if and only if $G$ has no loops if and only if it is the identity morphism.

Let $(p, q) : G' \to G$ be a directed emulator map. We say that an edge $x' \xrightarrow{e'} y' \in E_{G'}$ is an emulating edge of an edge $x \xrightarrow{e} y \in E_G$ if $q(e') = e$, $p(x') = x$ and $p(y') = y$. We write $e' \downarrow e$ such a configuration. More globally, we say that such an edge $e'$ is an emulating edge.

Lemma 18. Let $(p, q) : G' \to G$ be a directed emulator map. Then there is a directed emulator morphism $(p, q') : G'' \to G$ with $G''$ a subgraph of $G'$. In particular, a directed emulator map $G' \to G$ is a directed emulator morphism if and only if the set of emulating edges in $G'$ is the whole set of edges in $G'$.

Proof. We define $G''$ as follows. Let

$$E'' = \{ e' \in E_{G'} \mid \text{there is } e \in E_G : e' \downarrow e \}$$

be the subset of emulating edges. Then, let $G'' = G'_{|E''}$ and $q' = q_{|E''}$. We claim that $(p, q') : G'' \to G$ is a directed emulator. First, it is a morphism. Indeed, elements in $E''$ verify the adjacency relation. Second, given $x \xrightarrow{e} y \in G$ and $x' \in p^{-1}(x)$, there is $x' \xrightarrow{e'} y' \in G'$ such that $q(e') = e$, $p(x') = x$ and $p(y') = y$. But, then, $e' \downarrow e$ and thus, $e' \in E''$ and $q'(e') = q(e') = e$.

In the case $E'' = E_{G'}$, $q' = q$ and the conclusion follows. \qed

Definition 5. Let $G$ be a directed graph and $x \in V_G$ a vertex. The centripetal star of $x$ is the set $\text{out}E_G(x)$ of outgoing edges from $x$, that is $\text{out}E_G(x) = \{ e \in E_G \mid x \xrightarrow{e} y \text{ for some } y \in V_G \}$. 
A directed graph morphism \( (p, q) : G \to H \) induces, for each vertex \( x \in V_G \), a map
\[
q_x : \text{out}_{E_G}(x) \to \text{out}_{E_H}(p(x)), \; e \mapsto q(e).
\]

**Definition 6.** If at each \( x \in V_G \), \( q_x \) is injective (resp. surjective), then we say that \( (p, q) : G \to H \) is a directed immersion (resp. a directed submersion).

A directed emulator morphism \( (p, q) : G' \to G \) lifts any outgoing edge \( e \in E_G \) to an outgoing edge \( e' \) from any specified vertex in the preimage \( p^{-1}(s_G(e)) \in V_{G'} \). This implies the following observation.

**Lemma 19.** A directed graph morphism is a directed emulator morphism if and only if it is a directed submersion.

**Definition 7.** Given a graph morphism \( \psi : G' \to G \). If the map \( \text{out}_{E_G'}(x') \to \text{out}_{E_G}(x) \) is actually bijective for any vertex \( x' \in G' \), then we say that the morphism \( \psi \) is a directed covering morphism and that the digraph \( G' \) is a directed cover of \( G \).

**Lemma 20.** A directed covering morphism is an epimorphism that is both a submersion and an immersion.

### 4.2. Closure properties.

**Proposition 1.** The composition of two directed emulators (resp. directed covering) morphisms is a directed emulator (resp. directed covering) morphism. Directed emulator maps also compose.

**Proposition 2.** If \( G \to H \) is a directed emulator and \( H' \) is a subgraph of \( H \), there is a subgraph \( G' \) of \( G \) that is an emulator of \( H' \).

**Proof.** If \( V_H = V_{H'} \), it is a direct consequence of Lemma 18. If \( H' \) is the empty graph, take \( G' = H' \). Otherwise, suppose \( (p, q) : G \to H \) is the directed emulator. Set \( p'(x) = p(x) \) if \( p(x) \in V_{H'} \) and \( p'(x) = x_0 \) for an arbitrary vertex \( x_0 \in H' \). And define \( q'(e) = q(e) \) if \( e \in E_{H'} \) and let it undefined otherwise. Then, we are back to Lemma 18 with the pair \( (p', q') \). \( \square \)

**Proposition 3.** The forgetful functor \( R \) preserves directed emulators. The statement holds for directed emulators maps.

**Proof.** Let \( (p, q) : G' \to G \) be a directed emulator. Let us verify that \( R(p, q) = (p, p^{\times 2}) : R(G') \to R(G) \) is a directed emulator. Let \( \Delta_G(e) \) be an edge in \( R(G) \) and let \( x' \in R(G') \) such that \( p(x') = s_{R(G)}(\Delta_G(e)) = s_G(e) \). Since \( (p, q) \) is an emulator, there is \( e' \in G' \) such that \( q(e') = e \) and \( s_{G'}(e') = x' \). By definition, \( s_{R(G')}(\Delta_{G'}(e')) = s_{G'}(e') = x' \) and \( p^{\times 2}(\Delta_{G'}(e')) = \Delta_{G}(q(e')) = \Delta_G(e) \). Thus \( \Delta_{G'}(e') \) is the expected emulating edge.

The proof applies to directed emulator maps since we use adjacency only on emulating edges (that verify the adjacency relation). \( \square \)
Remark 7. The forgetful functor $R$ does not preserve directed covers. For instance, the directed graph $G = \circlearrowleft$ covers the directed graph $H = \circlearrowleft$. However, $R(G) = G$ only emulates (and does not cover) the simple directed graph $R(H) = \circlearrowleft$.

The next result explains how the forgetful functor $R$ can be “reversed” with respect to directed emulators. For a pullback interpretation, see below §4.3 Example 6.

Lemma 21. Suppose $\psi = (p,q) : G' \to G$ is a directed emulator morphism between simple directed graphs. Assume that $H$ is a directed graph such that $R(H) = G$. Then there exists a directed emulator morphism $\varphi : H' \to H$ such that the following properties are verified:

(i) $R(H') = R(G')$;
(ii) The diagram

$$
\begin{array}{ccc}
H' & \xrightarrow{\rho_H} & R(H') \\
\varphi \downarrow & & \downarrow \psi \\
H & \xrightarrow{\rho_H} & R(H)
\end{array}
$$

is commutative;
(iii) $g(H') = g(G')$.

Proof. We define $H' = (V_{G'}, E, s, t)$ as follows. Let

$$E = \{(e', \tilde{e}) \in E_{G'} \times E_H \mid \Delta_G(q(e')) = \Delta_H(\tilde{e})\}.$$

We set $s(e', \tilde{e}) = s_{G'}(e')$ and $t(e', \tilde{e}) = t_{G'}(e')$. In other words, $\Delta_{H'}(e', \tilde{e}) = \Delta_{G'}(e')$. We define $\varphi = (p, q')$ with $q'(e', \tilde{e}) = \tilde{e}$.

It is a morphism. Indeed, take $(e', \tilde{e}) \in H'$, we have $\Delta_H(\tilde{e}) = \Delta_{H'}(e', \tilde{e}) = \Delta_G(q(e')) = p^x(\Delta_{G'}(e')) = p^x(\Delta_{H'}(e', \tilde{e}))$.

Let us verify it is a directed emulator. Given $x \xrightarrow{\tilde{e}} y \in H$, and $x' \in H'$ such that $p(x') = x$. Then, by the definition of $\rho$, we have $x \xrightarrow{\rho_H(\tilde{e})=\Delta_H(\tilde{e})} y \in G$ for some $y \in G$. Since $\tilde{e}$ is a directed emulator and $x' \in V_{H'} = V_{G'}$ verifies $p(x') = x$, there is an edge $x' \xrightarrow{e'} y' \in G$ such that $q(e') = \Delta_H(\tilde{e})$ and $s_{G'}(e') = x'$. We have $\Delta_G(q(e')) = \Delta_G(\Delta_H(\tilde{e})) = \Delta_H(\tilde{e})$. The last equation is due to the fact that $G = R(H)$ (see Lemma 3). Thus, $(e', \tilde{e})$ is an edge within $H'$. By definition, $s(e', \tilde{e}) = s_{G'}(e') = x'$. The edge has expected source. Furthermore, $q'(e', \tilde{e}) = \tilde{e}$ is as expected.

Now, the left part of the diagram commutes as a corollary of Lemma 3. Second, since for all $(e', \tilde{e}) \in E_{H'}$ we have $\Delta_{H'}(e', \tilde{e}) = \Delta_{G'}(e')$, then $\Delta_{H'}(E_{H'}) = \Delta_{G'}(E_{G'})$. As a consequence, $R(H') = (V_{G'}, \Delta_{H'}(E_{H'}), \pi_1, \pi_2) = (V_{G'}, \Delta_{G'}(E_{G'}), \pi_1, \pi_2) = R(G')$. This proves (i).

So, it remains to prove the commutation of the right hand side of the diagram to complete the proof of (ii). For vertices, the diagram reads:
$p \circ 1_{V_G} = 1_{V_G} \circ p$ which holds. For edges, we have $\Delta_G \circ q = p \times 2 \circ \Delta_{G'}$ since $(p, q)$ is a morphism. But since $\Delta_G = 1_{E_G}$, this leads to $q = p \times 2 \circ \Delta_{G'}$. Since $G'$ is a simple graph, $\Delta_{G'}$ is an isomorphism. Thus, $q \circ \Delta^{-1}_{G'} = p \times 2$. This is commutation for edges.

The statement (iii) about the genus follows from genus invariance of $R$ (see Lemma 14) and Lemma 9. □

The next result states that it is possible to extend a directed cover over an excised graph and preserve the genus of the original cover.

**Lemma 22** (Extension of a directed cover over an excised graph). Suppose that $\psi : G' \to G$ is a directed cover morphism between directed graphs. Furthermore, assume that $G = \text{Exc}(H)$. Then there exists a directed graph $H'$ and a directed cover morphism $\varphi : H' \to H$ such that

1. $\text{Exc}(H') = G'$;
2. The directed cover morphism $\varphi$ extends the directed cover morphism $\psi$, that is $\varphi|_{G'} = \psi$;
3. $g(H') = g(G')$;
4. The directed cover $H'$ has minimal genus among all directed covers of $H$ extending the directed cover of $G$.

The statement holds replacing covers by emulators.

**Proof.** The construction is explicit. Let $\psi = (p, q) : G' \to G$ and let $E^0$ be the set of loops of $H$. For each loop $e \in E^0$, create a loop $e_{x'}$ in $G'$ at each preimage $x' \in p^{-1}(s(e))$ and set $s(e_{x'}) = t(e_{x'}) = x'$. Set

$$H' = \left( V_{G'}, E_{G'}, \bigcup_{e \in E^0} \bigcup_{v' \in p^{-1}(s(e))} e_{v'}, s_{G'}, t_{G'} \cup s, t \cup t \right).$$

Extend the directed cover morphism $\psi$ to $H'$ by sending each edge $e_{x'}$ to $e$. Rename the extended direct cover morphism $\varphi$. By construction, $\text{Exc}(H') = G'$ and $\varphi|_{G'} = \psi$. This proves (1) and (2). Genus invariance (Property (3)) follows from Genus invariance under excision (Property (1) and Lemma 12).

Now consider any directed cover morphism $H'' \to H$ extending $\psi : G' \to G$. As a graph, $H''$ differs from $H'$ at most with respect to the preimage of $E^0$. Consider $e \in E^0$ and let $v' \in V'$ be a vertex in the preimage of $s(e)$. Since $e$ is a loop, the preimage of $e$ starting at $v'$ is an edge $e'$ such that $t(e')$ lies in the same fibre as $v' = s(e')$. So $H''$ coincides with $H'$ on the edge $e'$ if and only if $e'$ is a loop. Then, according to Lemma 15, $g(H') \leq g(H'')$, as claimed.

The construction is valid for directed emulators. □
Next we consider what happens when one removes an edge in the target directed graph. Given a graph $G$ and an edge $e$, recall that we denote by $G - e$ the subgraph induced by the removal of the edge $e$.

**Lemma 23** (Removal of an edge in the target). Let $(p, q) : G' \to G$ be a directed emulator morphism and let $e \in E_G$ be an edge in $G$. Then there is a directed emulator $(p, q') : G'' \to G - e$ such that $G''$ is a subgraph of $G'$.

**Proof.** The map $(p, q) : G' \to G - e$ is a directed emulator map. The lemma is a direct consequence of Lemma 18. □

**Remark 8.** A directed emulator $\psi = (p, q) : G' \to G$ and an edge $e \in E_G$ induces a graph epimorphism $\psi_e = (p_e, q_e) : G'/q^{-1}(e) \to G/e$ that is not necessarily a directed emulator. (example)

Finally, covers and emulators are strongly related.

**Lemma 24** (Extraction of a directed cover). Let $G'$ be a directed emulator of a directed graph $G$. Then there is a directed subgraph $G''$ of $G$ with the following properties:

1. $G''$ is a directed cover of $G$;
2. $V_G = V_G''$.

**Proof.** Let $x$ be a vertex of $G$. Let $x' \in p^{-1}(x)$. By Lemma 19, the emulator morphism induces a surjection $\text{OutE}(x') \to \text{OutE}(x)$. Remove if necessary some outgoing edges from $x'$ (while keeping the same vertices) so that a bijection $\text{OutE}(x') \to \text{OutE}(x)$ is obtained. Proceed thus on each vertex in the fibre $p^{-1}(x)$ for each vertex $x \in G$. The restriction of the directed emulator morphism is then a directed cover morphism. □

**Corollary 4.** Let $g \geq 0$. A directed graph has a directed emulator of genus $g$ if and only if it has a directed cover of genus $g$.

**Proof.** Lemma 24 shows that if a genus $g$ directed graph $G'$ emulates $G$, then it contains a directed subgraph $G''$ that covers $G$. Therefore $g(G'') \leq g(G') = g$. Therefore the class of directed emulators of $G$ contains a directed cover of minimal genus. □

4.3. The category of emulators. The composition of emulators is ensured by Proposition 1. The identities are directed emulators, see Example 2. Thus, directed graphs and directed emulator morphisms between them form a subcategory $\text{Em}_{\mathcal{D}Gr}$ of $\mathcal{D}Gr$. Simple directed graphs and directed simple emulator morphisms between them form a subcategory of $\mathcal{DSGr}$. Lemma 25.

**The category $\text{Em}_{\mathcal{D}Gr}$ is a full subcategory of $\text{Em}_{\mathcal{D}Gr}$.**

In the same way, directed graphs and directed emulator maps between them form a category $\text{Em}^0_{\mathcal{D}Gr}$ and $\text{Em}^0_{\mathcal{DSGr}}$ is a full subcategory of $\text{Em}^0_{\mathcal{D}Gr}$. Furthermore, since a directed emulator morphism is a directed emulator map, the category $\text{Em}_{\mathcal{D}Gr}$ is a subcategory of $\text{Em}^0_{\mathcal{D}Gr}$. 
Proposition 4. The categories $\text{Em}_{D\text{Gr}}$ and $\text{Em}_{D\text{Gr}}$ have pullbacks.

Proof. We prove the statement for $\text{Em}_{D\text{Gr}}$. The proof is similar for $\text{Em}_{S\text{Gr}}$. Let $(p, q) : G \to K$ and $(p', q') : H \to K$ two directed emulator morphisms. Since the categories $S\text{Gr}$ and $D\text{Gr}$ have pullbacks and directed emulator morphisms are graph morphisms, there is a pullback for $(p, q)$ and $(p', q')$, namely a directed graph $L = G \times_K H$ defined by

$$V_L = \{(u, v) \in V_G \times V_H \mid p(u) = p'(v)\},$$

$$E_L = \{(e, f) \in E_G \times E_H \mid q(e) = q'(f)\},$$

$$\Delta_L(e, f) = ((s_G(e), s_H(f)), (t_G(e), t_H(f)))$$

so that the diagram

$$
\begin{array}{ccc}
G \times_K H & \xrightarrow{\pi_2} & H \\
\downarrow{\pi_1} & & \downarrow{(p, q)} \\
G & \xrightarrow{(p', q')} & K
\end{array}
$$

is commutative. The maps $\pi_i|_L$, $i = 1, 2$, are the restriction of the projection morphisms $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ respectively. There only remains to see is that $\pi_i|_L$, $i = 1, 2$, are directed emulator morphisms. Let $u \in V_G$ and let $u \xrightarrow{e} u'$ be an edge in $G$. Let $(u, v) \in V_L$ a vertex in the preimage. Note by construction of $L$, such a vertex verifies $p'(v) = p(u)$. We have to show that there is some outgoing edge in $E_L$ from $(u, v)$ that emulates $e$. Applying first $(p, q)$ to $u \xrightarrow{e} u'$ yields an edge $p(u) \xrightarrow{q(e)} p(u')$ in $K$. Since $(p', q')$ is a directed emulator morphism, there is an edge $v \xrightarrow{f} v'$ in $H$ that emulates $q(e)$, that is, $(p'(v) \xrightarrow{q'(f)} p'(v')) = (p(u) \xrightarrow{q(e)} p(u'))$

for any $v \in p'^{-1}(p(u))$. But this implies that $(u, v) \xrightarrow{(e, f)} u', v'$ is an edge in $L$ that emulates $u \xrightarrow{e} u'$.

Example 6. The directed emulator $H'$ built in Lemma 21 is isomorphic to $H \times_{R(H')} R(H')$. It follows that $H'$ is the unique directed emulator, up to isomorphism, that satisfies the conditions of Lemma 21 with minimal number of edges.

The removal of loops (Exc) does not determine a functor in the category of directed graphs (Remark 3). The next observation is that Exc becomes one in the category $\text{Em}_{D\text{Gr}}$ of directed emulators maps.

Proposition 5. The excision $\text{Exc}$ is a endofunctor of $\text{Em}_{D\text{Gr}}^0$ that restricts to an endofunctor of $\text{Em}_{S\text{Gr}}^0$. In particular, Exc sends $\text{Em}_{D\text{Gr}}$ (resp. $\text{Em}_{S\text{Gr}}$) onto $\text{Em}_{D\text{Gr}}^0$ (resp. $\text{Em}_{S\text{Gr}}^0$). Furthermore,

1. $\text{Exc} \circ \text{Exc} = \text{Exc}$;
2. $\text{Exc} \circ R = R \circ \text{Exc}$.
Proof. Exc acts on a graph $G$ by removal of all loops; for any directed graphs $G, G'$, Exc acts on the set $\text{Hom}_{\text{Em}^0_{\mathcal{DG}}} (G', G)$ of directed pre-emulator maps as the identity. The results follow. □

Remark 9. The category $\text{Em}^0_{\mathcal{DG}}$ does not have pullbacks.

There is a dual notion to the definition of a directed emulator. The original definition distinguishes the outgoing edges. One could distinguish the incoming edges instead. Hence, parallel to Lemma 19, we consider the set $\text{InE}(y) = \{ e \in E \mid t(e) = y \}$ of incoming edges at $e$ and require that the map $\text{InE}(y') \to \text{InE}(y)$ induced by the epimorphism be surjective. This defines a new notion of directed emulator, which we call incoming directed emulator, while the original notion is renamed an outgoing directed emulator. If this does not seem to cause confusion, we shall keep the terminology of directed emulator for an outgoing directed emulator. In contexts where we need to be more precise, we shall use the full terminology. A bidirected emulator is a directed graph morphism that is both an incoming directed emulator and an outgoing directed emulator.

Lemma 26. A directed graph $G'$ is an incoming directed emulator of $G$ if and only if $G'^{\text{op}}$ is a outgoing directed emulator of $G^{\text{op}}$.

Proof. Follows from the definitions. □

4.4. The case of undirected graphs. In this section, we discuss the relationship between directed and undirected graph emulators. First, we recall the original definition of an undirected graph emulator introduced by M. R. Fellows in his PhD thesis in 1985 (see [FL88]).

Fellows’ definition. Let $G$ be an undirected graph. We say that an undirected graph $G'$ is an emulator of $G$ if there is a graph epimorphism $(p, q) : G' \to G$ such that for any edge $e \in E_G$ with $\partial(e) = \{x, y\}$ and any $x' \in V_{G'}$ such that $p(x') = x$, there is an edge $e' \in E_{G'}$ such that $q(e') = e$.

Definition 8. Let $G$ be an undirected graph and let $v$ be a vertex in $G$. The incidence star of $v$ is the set $\text{Edg}_G(v) = \{ e \in E_G \mid v \in \partial e \}$ of all edges incident to $v$.

Given a vertex $v' \in G'$, an undirected graph morphism $(p, q) : G' \to G$ induces by definition a map $q_{v'} : \text{Edg}_{G'}(v') \to \text{Edg}_G(p(v'))$, $e \mapsto q(e)$.

We can now repeat verbatim the definition in the directed case (Definition 3).

Definition 9. If at each $v' \in V_{G'}$, $q_{v'}$ is injective (resp. surjective), then we say that $(p, q) : G' \to G$ is an immersion (resp. a submersion).
Let us observe the following fact.

**Lemma 27.** An undirected graph epimorphism is an emulator morphism if and only if it is a submersion.

**Definition 10.** If the map \( q_v : \text{Edg}_{G'}(v') \to \text{Edg}_G(v) \) is bijective for any vertex \( v \in V_G \) and any vertex \( v' \in p^{-1}(v) \), then we say that the pair \((p, q)\) is a covering morphism and that the graph \( G' \) is a covering of \( G \).

Note that the functor \( U \) does not send directed emulators to emulators (and neither directed covers to covers). For instance, on the left, we have a directed emulator. But, on the right, node \( u_1 \) has only one adjacent edge where \( w_1 \) has two.

**Definition 11.** The bidirection of an undirected graph \( G \) is the directed graph \( \vec{G} = (V, E, s, t) \) defined by

\[
V = V_G, \\
E = \{(e, x, y) \mid e \in E_G \text{ and } \partial_G(e) = \{x, y\}\}, \\
s(e, x, y) = x \text{ and } t(e, x, y) = y.
\]

Notice that nonloop edges in \( G \) are duplicated in \( \vec{G} \) whereas the set \( \{\partial e = \{x\} \mid e \in E_G\} \) of loops in \( G \) are in one-one correspondence with the set \( \{(e, x, x) \mid e \in E_G, \partial_G(e) = \{x\}\} \) of loops in \( \vec{G} \).

Let \( \phi = (p, q) : G \to H \) be a graph morphism. We define \( \vec{\phi} = (p, q') \) with \( q'((e, x, y)) = (q(e), p(x), p(y)) \). It is clear that \( \Delta_H(q'(e, x, y)) = \Delta_H(q(e), p(x), p(y)) = (p(x), p(y)) = p^x(\Delta_G(e)) \).

**Lemma 28.** The assignment \( \vec{-} \) yields a functor \( \text{Gr} \to \mathcal{D}\text{Gr} \) that is right adjoint to the functor \( U \).

**Proof.** Composition is left to the reader. For the statement about adjoints, let \( G \) be a directed graph and let \( H \) be an undirected graph. We define a map \( \Phi_{G,H} : \text{Hom}_{\text{Gr}}(G, \vec{H}) \to \text{Hom}_{\text{Gr}}(U(G), H) \) as follows. Let \((p, q) : G \to \vec{H}\) be a directed graph morphism. Let \( \pi_3^1 : E_H \to E_H, \pi_3^1(e, x, y) = e \). Then set

\[
\Phi_{G,H}(p, q) = (p, \pi_3^1 \circ q) : U(G) \to H.
\]

Let us verify that \((p, \pi_3^1 \circ q)\) is a morphism. Indeed, take \( e \in E_{U(G)} = E_G \) such that \( x \xrightarrow{e} y \in G \). Let \( q(e) = (e', x', y') \in \vec{H} \). Then, \( \partial_H \circ \pi_3^1(q(e)) = \{x', y'\} \). Since \((p, q)\) is a morphism, \((x', y') = (p(x), p(y))\) so that \( \partial_H \circ \pi_3^1(q(e)) = \)}
\[ p^{\times 2}(\partial_U(G)(e)) \text{ as expected. Conversely, given a morphism } (p', q') : U(G) \to H, \text{ define a map } \Psi_{G,H}(p', q') : G \to \widehat{H} \text{ by } \]

\[ \Psi_{G,H}(p', q')(x) = (q'(x), p'(s_{G}(x)), p'(t_{G}(x))). \]

Let us verify that the map \( q' \) is well-defined, that is, an edge \( e \in E_G \) is sent to an edge \( q'(e) \in E_{\widehat{H}} \). Indeed, since \( (p', q') \) is a morphism, \( \partial_H(q'(e)) = p' \circ \partial_U(G)(e) = \{p'(s_{G}(e)), p'(t_{G}(e))\} \) so \( (q'(e), p'(s_{G}(e)), p'(t_{G}(e))) \in E_{\widehat{H}} \).

Now \( p' \circ \Delta_G(e) = (p'(s_{G}(e)), p'(t_{G}(e))) = \Delta_{\widehat{H}}(q'(e), p'(s_{G}(e)), p'(t_{G}(e))) = \Delta_{\widehat{H}}(q'(e)) \) so \( (p', q') \) is a morphism. Finally, the maps \( \Phi_{G,H} \) and \( \Psi_{G,H} \) are inverse of each other, hence \( \Phi_{G,H} : \text{Hom}_{\mathcal{G}_r}(G, \widehat{H}) \to \text{Hom}_{\mathcal{G}_d}(U(G), H) \) is bijective. Naturality of the isomorphism \( \Phi \) should be clear. \qed

**Lemma 29.** The morphism \( \phi = (p, q) : G \to H \) is an emulator if and only if \( \Phi : G \to \widehat{H} \) is a directed emulator.

**Remark 10.** Note that \( \Phi : G \to \widehat{H} \) is a directed emulator if and only if it is a bidirected emulator. (Proof: use the natural isomorphism \( (\widehat{G})^{\text{op}} \simeq \widehat{G} \) for any \( G \) and Lemma 26)

**Proof.** Suppose \( \phi \) is an emulator. Since \( p \) is surjective, we only have to verify the outgoing edge lifting property. Suppose now \( (e, x, y) \in E_{\widehat{H}} \) and \( p(x') = x \) with \( x' \in G \). Since \( \phi \) is an emulator, there is an edge \( e' \in E_G \) such that \( q(e') = e \) and \( \partial_G(e') = \{x', y'\} \) for some \( y' \in V_G \). We have \( q'(e', x', y') = (q(e'), p(x'), p(y')) = (e, x, p(y')) \). Do we have \( p(y') = y \)? Since \( \Delta_{\widehat{H}}(q(e)) = p^{\times 2}(e) = \{p(x'), p(y')\} = \{x, y\} \), either \( x = p(x') = p(y') = y \) or \( p(x') \neq p(y') \) and since \( p(x') = x \), we have \( p(y') = y \).

Suppose now that \( \phi = (p, q) \) is a directed emulator. Again, since \( p \) is surjective, we only have to verify the edge lifting property. Let \( e \in E_H \) with \( \partial_H(e) = \{x, y\} \) and \( x' \in V_G \) with \( p(x') = x \). Then, \( (e, x, y) \in E_{\widehat{H}} \) and \( x' \in V_G \) leads to an edge \( (e', x', y') \in E_{\widehat{G}} \) with \( q'(e', x', y') = (e, x, y) \). By definition of \( q' \), that means that \( q(e') = e \). \qed

**Lemma 30.** Suppose that \( \phi : G \to \widehat{H} \) is a directed emulator, then, there is an emulator \( \phi' : G' \to H \) with \( g(G') = g(G) \).

**Proof.** Since \( U \) is left adjoint to \( \widehat{-} \), Lemma 28 provides an undirected graph morphism \( \phi' : U(G) \to H \). It is clear that \( g(U(G')) = g(G) \). It remains to check that if \( \phi' \) is an emulator morphism. For this, let \( \phi = (p, q) \) so that \( \phi' = (p, \pi_3 \circ q) \) with \( \pi_3^1(e, x, y) = e \). Let \( e \in E_H \) such that \( \partial_H(e) = \{x, y\} \) and \( x' \in V_{U(G)} = V_G \) such that \( p(x') = x \). Then, \( x \overset{(e, x, y)}{\longrightarrow} y \in \widehat{H} \) and \( p(x') = x \) implies the existence of \( e' \in E_G \) such that \( q(e') = (e, x, y) \) with \( s_{G}(e') = x' \). We have \( \pi_3 \circ q(e') = e \) and \( p(e') = x' \) as expected. \qed

**Remark 11.** The converse of Lemma 30 does not hold: the directed graph morphism \( G \to \widehat{H} \) induced by an emulator morphism \( U(G) \to H \) is not a
directed emulator morphism in general. A counterexample is provided by $U(G) = H = \circ - \circ - \circ$.

A direction $\vec{G}$ of an undirected graph $G$ is a subgraph of $\vec{G}$.

**Lemma 31** (Lifting lemma). Let $\phi : G' \to G$ be an undirected graph emulator morphism. Given a direction $\vec{G}$ of $G$, there is a directed emulator $\phi' : \vec{G}' \to \vec{G}$ where $\vec{G}'$ is a direction of $G'$.

**Proof.** By Lemma 29, $\vec{\phi} : \vec{G}' \to \vec{G}$ is a directed emulator morphism. Thus, $\vec{\phi} : \vec{G}' \to \vec{G}$ is a directed emulator map. Then, apply Lemma 18. □

The undirected version of Lemma 24 fails to hold. The following picture depicts a 4-vertex undirected emulator of a simple undirected 3-vertex graph that does not contain a 4-vertex cover as a subgraph.

![Undirected Graph](image)

It is left to the reader to verify on this example that any direction of the bottom graph yields a directed emulator on the top by lifting directions (Lemma 31) and that from it a directed cover with all the original vertices can be extracted (Lemma 24).

In other words, each time we have an undirected emulator, we have a directed emulator for each direction of the base. Roughly speaking, it is much easier to find directed emulators compared to emulators (or bidirected emulators).

For the remainder of the paragraph, let us define, for a directed (resp. undirected) graph $G$,

$$g_{\text{cover}}(G) = \min \{ g(G') \mid G' \text{ covers } G \}, \quad g_{\text{em}}(G) = \min \{ g(G') \mid G' \text{ emulates } G \}.$$

By Corollary 4 above, $g_{\text{cover}}(G) = g_{\text{em}}(G)$ for any directed graph $G$. One can ask whether this equality also holds for undirected graphs. In 1999, P. Hliněný [HL99] found an emulator such that

$$g_{\text{em}}(G) \leq 3 < g_{\text{cover}}(G).$$

In particular, there exist undirected emulators of $G$ that do not contain covers of $G$.

M. Fellows proposed the following conjecture for undirected graphs in 1985.

**Conjecture 1.** A connected graph has a finite planar emulator if and only if it has a finite planar cover.
The conjecture was proved false in 2009 by Y. Rieck and Y. Yamashita who found a counterexample [RY09]. Therefore, there are even graphs \( G \) such that \( g_{\text{sem}}(G) = 0 \) and \( g_{\text{cover}}(G) \geq 1 \).

5. Transition systems and regular languages

Transition systems and automata provide a rich source of examples of directed emulators. In this section we study the relationship between automata, regular languages and emulators. For basic definitions about automata, we refer to [Elk74] and [Sak03]. Given a map \( A \to B \) between sets, we shall frequently use and denote the free extension \( A^* \to B^* \) by the same letter without further notice.

A *semi-automaton* \( A \) is a 3-tuple \( A = (G, A, \ell) \) made of

- a directed graph \( G \) whose vertices and edges are respectively called *states* and *transitions*,
- a fixed set \( A \), the *alphabet*, a surjective map \( \ell : E \to A \) (labelling of the edges in \( A \)).

Again, given a semi-automaton \( A \), we write \( G_A \) its underlying graph, an operation that will turn out to be functorial. Finally, let \( V_A \) be the set of states of \( A \), \( E_A \) its transitions, \( A_A \) its alphabet and \( \ell_A \) its labeling function.

**Remark** 12. Some authors do not assume the labelling \( \ell \) to be onto. In this case, we say the alphabet is *underused*. In this paper, we assume that all semi-automata have no underused alphabet. The hypothesis is required for Lemma 34.

**Example 7.** The semi-automaton \( A = (G, A, \ell) \) is illustrated:

\[
\begin{array}{c}
\circlearrowleft \quad e : a \\
\longleftarrow \quad e' \quad \quad \rightarrow \quad e'' : b
\end{array}
\]

with \( G = e \circlearrowleft \circ e' \), \( A = \{a, b\} \) and \( \ell \) such that \( \ell(e) = \ell(e') = a \) and \( \ell(e'') = b \).

Let \( A \) and \( B \) be two semi-automata. A *morphism* \( (f, g, \alpha) : A \to B \) is a directed graph morphism \( (f, g) : G_A \to G_B \) together with \( \alpha : A_A \to A_B \) between alphabets such that

\[
\alpha \circ \ell_A = \ell_B \circ g.
\]

The relation (7) ensures that the map \( g \) on the set of edges and the map \( \alpha : A_A \to A_B \) on alphabets are compatible with both labels on the source and the target. If \( \alpha \) is the identity, we say that the morphism is *strict*.

The identity \((1_{V_A}, 1_{E_A}, 1_A) : A \to A\) is a (strict) morphism. And it is easily seen that the composition of two morphisms (resp. strict morphisms) is a morphism (resp. a strict morphism). We denote by \( \text{Semi} \) the category of semi-automata and their morphisms and by \( \text{Semi}^0 \) the category of semi-automata with their strict morphisms.

**Lemma 32.** The category \( \text{Semi}^0 \) is a faithful subcategory of \( \text{Semi} \)
We defined semi-automata in a slightly unorthodox way because of the tropism towards graphs in this paper. Indeed, the following example

![Diagram of a semi-automaton]

is a semi-automaton for which there are two transitions labelled $a$ between $q$ and $q'$. This is not compatible with the standard definition of the transitions as a function $\delta : V \times A \to 2^V$ with $V$ the set of states and $A$ the alphabet. But actually, when we will restrict to deterministic automata, such a case will vanish and our definition will actually turn out to be equivalent to the standard one.

**Example 8.** Let $A$ be a semi-automaton. Let $W$ be a subset of $V_A$. Form

$$E_W = \{ v \xrightarrow{e} w \in E_A \mid v, w \in W^2 \}, \quad \ell_W = \ell|_{E_W}, \quad A_W = \ell(E_W).$$

Then $A_W = (W, E_W, A_W, \ell_W, I \cap W, F \cap W)$ is a semi-automaton. It is easily seen that the inclusion maps $W \subseteq V_A, E_W \subseteq E_A, A_W \subseteq A_A$ induce an injective morphism $A_W \to A$. There is a similar observation starting from a subset $E'$ of transitions. Form

$$V_{E'} = \{ v \mid \exists v \xrightarrow{e} w \in E' \} \cup \{ v \mid \exists w \xrightarrow{e} v \in E' \}, \quad A_{E'} = \ell(E').$$

Then $A_{E'} = (V_{E'}, E_A', \ell|_{E'}, I \cap V_{E'}, F \cap V_{E'})$ is a semi-automaton and the inclusions induce an injective morphism $A_{E'} \to A$. A semi-automaton produced in this fashion will be called a **sub-semi-automaton** of $A$.

Finally, given a semi-automaton morphism $(f, g, \alpha) : A \to B$, we define the semi-automaton $(f, g, \alpha)(A) = ((f, g)(G_A), \alpha(A), \ell)$ with $\ell(g(e)) = \alpha(\ell_A(e))$ for all $e \in E_A$ (see[4]). To be correct, we have to verify that $\ell$ is surjective. By definition, $\ell_A(E_A) = A_A$ so that $\ell(g(E_A)) = \alpha(A_A)$ and thus $\ell$ is surjective. It is clear that $(f, g, \alpha) : A \to (f, g, \alpha)(A)$ is a semi-automaton morphisms and that $(f, g, \alpha)(A)$ is sub-semi-automaton of $B$.

**Definition 12.** A morphism that is the identity on the vertices and the edges is called a **relabelling** morphism.

**Lemma 33.** Given a function $\alpha : A \to B$ and a semi-automaton $A = (G, A, \ell)$, the triple $A_{\alpha} = (G, \alpha(A), \alpha \circ \ell)$ is a semi-automaton and furthermore, $(1_{V_A}, 1_{E_A}, \alpha) : A \to A_{\alpha}$ is a relabelling morphism.

**Proof.** Let $E = E_{A_{\alpha}}$. We have $\alpha \circ \ell(E) = \alpha(A)$ since $\ell$ is surjective. So, $A_{\alpha}$ is a semi-automaton. Second, we have $(\alpha \circ \ell) \circ 1_{E_{A_{\alpha}}} = \alpha \circ \ell$, it is a morphism. By definition, it is a relabelling morphism. \[\square\]

**Proposition 6.** For any morphism $\phi$, there is a strict morphism $\pi$ and a relabelling morphism $\lambda$ such that $\phi = \lambda \circ \pi$ and a strict morphism $\psi$ and a relabelling $\mu$ such that $\phi = \psi \circ \mu$.\[\text{Note that } \ell = \ell_{\alpha(A_A)} \text{ due to Equation }[4].\]
Proof. Given \((f, g, \alpha) : A \to B\), one of the decomposition is seen within the commutative diagram:

\[
\begin{array}{c}
A \xrightarrow{(1_V, 1_E, \alpha)} A_{\alpha} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(\ell, \xi) & (f, g, \alpha) & (f, g, \alpha) & \xi \\
\end{array}
\]

where \(V = V_A\), \(E = E_A\) and \(B\) is the alphabet \(A_B\). The other decomposition is illustrated as follows:

\[
\begin{array}{c}
A \xrightarrow{(f, g, \ell \setminus g)} A_{\alpha} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(\ell, \xi) & (f, g, \alpha) & (f, g, \alpha) & \xi \\
\end{array}
\]

where \(\ell \setminus g(e) = \ell(e)\) for all \(e \in E_A\). The triple \((1_V, 1_E, \alpha)\) is a proper morphism. Indeed, for all \(g(e) \in E_{(f, g, \ell \setminus g)(A)}\), we have \(\alpha(\ell \setminus g(e)) = \alpha(\ell_A(e)) = \ell_B(g(e)) = \ell_B(1_{E_B}(g(e)))\). Thus, Equation 7 holds. □

By definition, a semi-automaton \(A\) is a directed graph with extra structure. Forgetting this extra structure yields the underlying directed graph \(G_A\) of the automaton.

Lemma 34. The assignment \(A \mapsto G_A\) yields a forgetful faithful functor \(G : \text{Semi} \to \text{DGr}\).

Proof. Let \(A\) and \(B\) be two semi-automata. Consider the induced map \(\text{Hom}_{\text{Semi}}(A, B) \to \text{Hom}_{\text{DGr}}(G_A, G_B)\). Suppose two morphisms \((f_i, g_i, \alpha_i) : A \to B\), \(i = 0, 1\), yield the same graph morphism \((f, g) : G_A \to G_B\): they coincide at the level of the underlying graph hence the maps on the set \(V\) of vertices in \(A\) and the set \(E\) of edges in \(A\) coincide: \(f_0 = f_1 = f\) and \(g_0 = g_1 = g\). Let \(\alpha_i : A_i \to A_B\), \(i = 0, 1\), be the respective maps between the alphabets. Both maps satisfy the relation \(\alpha_i \circ \ell_A = \ell_B \circ g\), \(i = 0, 1\). Therefore \(\alpha_0|_{\ell_A(E)} = \alpha_1|_{\ell_A(E)}\). Since \(A_0\) and \(A_1\) have no underused alphabet, \(\alpha_0 = \alpha_1\). □

Remark 13. The functor \(G(-)\) is not full. As an example, the induced map

\[
\text{Hom}_{\text{Semi}} \left( \begin{array}{c}
e \cdot a \cdot e \cdot g \cdot \delta \\
e' \cdot a' \cdot e' \cdot g' \cdot \delta' 
\end{array} \right) \to \text{Hom}_{\text{DGr}} \left( \begin{array}{c}
e \cdot a \cdot e \cdot g \cdot \delta \\
e' \cdot a' \cdot e' \cdot g' \cdot \delta' 
\end{array} \right)
\]

is not onto. For instance, the identity on the directed graph is not the image of a semi-automaton morphism.

Remark 14. The functor \(G(-)\) preserves surjectivity (resp. injectivity).

Definition 13. A directed graph \(G\) yields a semi-automaton \(A_G = (G, A, \ell)\) by setting

\[A = E_G, \quad \ell = 1_{E_G}\]

This transition system is called the tautological semi-automaton.
The morphism \((f, g) : G \rightarrow H\) between two directed graphs is sent to \(A_{(f,g)} = (f,g,g) : A_G \rightarrow A_H\). Indeed, we have \(g \circ 1_{E_G} = 1_{E_H} \circ g\), thus is a morphism. So, the tautological assignment forms a functor

\[ A_{(\cdot)} : \mathcal{D}G \rightarrow \text{Semi}. \]

**Proposition 7.** The functor \(G \rightarrow A_G\) is full and faithful.

**Proof.** Consider two morphisms \(\phi = (f, g) : G \rightarrow H\) and \(\psi = (f', g') : G \rightarrow H\) such that \(A_\phi = (f, g, g) = (f', g', g') = A_\psi\). Then, \(f = f'\) and \(g = g'\).

Suppose now \((f, g, \alpha) : A_G \rightarrow A_H\). It satisfies \(\alpha \circ \text{id}_E = \text{id}_{E'} \circ g\), that is, \(\alpha = g\) so that \((f, g, \alpha) = A_{(f,g)}\). \(\Box\)

**Lemma 35.** The following properties hold:

(i) The identity \(1_{\mathcal{D}G} : G \rightarrow G\) is a natural transformation \(1_{\mathcal{D}G} \rightarrow G_{A_{(\cdot)}}\).

(ii) There is a natural transformation \(\epsilon : A_{G_{(\cdot)}} \rightarrow 1_{\text{Semi}}\).

**Proof.** (i) follows from the definition. For (ii), we define \(\epsilon_A = (1_{V_A}, 1_{E_A}, \ell_A)\). It is a (relabelling) morphism \(A_{G_A} \rightarrow A\). Indeed, due to (i), the two automata share the same graph: \(G_{A_{G_A}} = G_A\). Thus \((1_{V_A}, 1_{E_A})\) is a graph morphism.

Second, we have \(\ell_{A_{G_A}} = 1_{E_{G_A}} = 1_{E_A} : E_A \rightarrow E_A\), again due to (i). Thus, \(\ell_A \circ \ell_{A_{G_A}} = \ell_A \circ 1_{E_A}\) that leads to Equation 9.

Let us verify naturality. Given \((f, g, \alpha) : A \rightarrow B\), by definition, \(A_{G_{(\cdot)}}(f,g,\alpha) = (f, g, g)\). We have to verify that the diagram commute:

\[ \begin{array}{ccc}
A_{G_A} & \xrightarrow{(1_{V_A}, 1_{E_A}, \ell_A)} & A \\
(f,g,g) \downarrow & & \downarrow (f,g,\alpha) \\
A_{G_B} & \xrightarrow{(1_{V_B}, 1_{E_B}, \ell_B)} & B \\
\end{array} \]

For vertices and edges, this is trivial. The last equation is read \(\alpha \circ \ell_A = \ell_B \circ g\) which is Equation 9. \(\Box\)

**Corollary 5.** The tautological assignment \(G \mapsto A_G\) is left-adjoint to the forgetful functor \(G_{(\cdot)} : \text{Semi} \rightarrow \mathcal{D}Gr\).

A \((\text{length } n)\) computation in \(A\) starting at state \(q\), ending at state \(q'\), is a path \(\pi = e_0 e_1 \ldots e_n\) of transitions in \(A\) where \(s(e_0) = q\), \(t(e_i) = s(e_{i+1})\), \(i = 0, \ldots, n-1\) and \(t(e_n) = q'\). The label of the computation \(\pi = e_0 e_1 \ldots e_n\) is \(\ell(\pi) = t(e_0)\ell(e_1) \cdots \ell(e_n) \in A^*\).

**Lemma 36.** The image of a computation \(\pi\) in \(A\) by a semi-automaton morphism \((f, g, \alpha) : A \rightarrow B\) is a computation \(\pi' = g(\pi)\) in \(B\). Furthermore, the label of the image of the computation is the image of the label of the computation: \(\ell_B(\pi') = \ell_B(g(\pi)) = \alpha(\ell_A(\pi))\).

**Proof.** The function \(g\) preserving adjacency, \(g(\pi)\) is a path within \(G_B\). The Equality of images is a direct consequence of Equation 9. \(\Box\)
Definition 14. A semi-automaton morphism is \textit{conformal} if every computation in the target is the image of a computation in the source. A directed graph morphism \( \psi : G \to H \) is \textit{conformal} if there exists a conformal semi-automaton morphism \( \varphi : A \to B \) such that \( \psi = G \varphi \).

Remark 15. Applying the definition on a path of length 1, we can already state that a conformal semi-automaton morphism (or a conformal directed graph morphism) is an epimorphism.

Lemma 37. The three following facts hold:

(i) The composition of two conformal morphisms is conformal.

(ii) Relabelling morphisms are conformal.

(iii) Given a semi-automaton morphism \( \phi : A \to B \) and a relabelling \( \lambda : B \to C \) such that \( \lambda \circ \phi \) is conformal, then \( \phi \) is conformal.

Proof. (i) is immediate. Concerning (ii), since a relabelling acts as the identity on the underlying graph, any path in the image is a path in the antecedent.

For (iii), suppose that \( \pi \) is a path within \( B \). Then \( \lambda(\pi) = \pi \) is a path in \( C \) since \( \lambda \) acts as the identity on the underlying graph. Thus, there is a path \( \pi' \) in \( A \) such that \( \lambda \circ \phi(\pi') = \phi(\pi') = \pi \) and the conclusion follows.

Lemma 38. A directed graph morphism \( \psi : H \to K \) is conformal if and only if the tautological morphism \( A\psi : A_H \to A_K \) is.

Proof. By definition, if \( A\psi : A_H \to A_K \) is a conformal semi-automaton morphism, \( \psi \) is conformal.

Suppose now that \( \psi \) is conformal. There is a conformal semi-automaton morphism \( \varphi : A \to B \) such that \( G\varphi = \psi \). On a diagram:

Given (i-ii) of the preceding lemma, we can state that \( \phi \circ \epsilon_A \) is conformal. Thus, \( \epsilon_B \circ A\psi = \phi \circ \epsilon_A \) is conformal. Item (iii) leads to the conclusion.

Lemma 39. Suppose that we have a morphism \( \phi : A \to B \) such that \( G\phi : G_A \to G_B \) is conformal. Then, \( \phi \) is conformal.

Proof. Given the preceding Lemma, \( A\psi \) is conformal. We have \( \phi \circ \epsilon_A = \epsilon_B \circ A\psi \). By Lemma 37 (i-ii), \( \epsilon_B \circ A\phi \) is conformal, and by (iii), \( \phi \) is. □
Remark 16. It should be noted that an epimorphism need not be conformal. The epimorphism map induced by merging the vertices \( v_0 \) and \( v_2 \) in the directed graph \( \begin{array}{c} v_0 \mathrel{\rightarrow} \mathrel{\rightarrow} v_1 \end{array} \begin{array}{c} a \mathrel{\rightarrow} b \mathrel{\rightarrow} v_2 \end{array} \) is not conformal. Indeed, in the resulting automaton \( \begin{array}{c} v \mathrel{\rightarrow} \mathrel{\rightarrow} v_1 \end{array} \begin{array}{c} a \mathrel{\rightarrow} b \mathrel{\rightarrow} \mathrel{\rightarrow} \end{array} \), the computation \( ba \) is not the image of a computation in the first automaton.

Lemma 40. A directed emulator morphism is a conformal morphism.

Proof. Given an emulator \( (p,q) : G' \to G \). Let \( A_{(p,q)} = (p,q,q) : A_{G'} \to A_G \). Suppose \( \pi = e_1 \ldots e_n \) is a path in \( A_G \). Since the vertex function \( p \) is surjective, there is an antecedent \( x' \) in \( G' \) (that is also a state in \( A_{G'} \)) such that \( p(x') = s_G(e_1) \). So, we can apply the outgoing edge property to \( G' \) that leads to a first edge \( d_1 \) with \( q(d_1) = e_1 \) and \( s_G'(d_1) = x' \). We can apply successively the outgoing edge property to the newly extracted vertex \( t(d_1) \) up to the end. □

Remark 17. A conformal morphism may not induce a directed emulator morphism. For instance, let \( G' \) be the disjoint union of two copies of \( \begin{array}{c} v \mathrel{\rightarrow} \mathrel{\rightarrow} \end{array} \). There is a conformal epimorphism that sends (any automaton with underlying graph) \( G' \) to (the automaton having underlying graph) the directed graph \( \begin{array}{c} v \mathrel{\rightarrow} \mathrel{\rightarrow} \end{array} \).

A semi-automaton \( A \) is complete (resp. deterministic) if given any state \( q \in V_A \), the map \( \ell_q : \text{Out}(q) \to A \) is surjective (resp. injective). The following observation is a direct consequence of the definitions.

Lemma 41. For any directed graph \( G \), the tautological semi-automaton \( A_G \) is deterministic.

Lemma 42. If \( A \) is complete and deterministic, then, for all \( q \in V_A \), \( \text{out}(q) \simeq A_A \).

Proof. By definition, \( \ell_q \) is both surjective and injective. □

Corollary 6. Suppose that \( \phi : A \to B \) is strict morphism, \( A \) is both complete and deterministic and \( G_{\phi} \) is a directed emulator, then, it is a cover.

Proof. For all \( q \in V_A \), we have \( \ell_B \circ g_q = \ell_q \) that is an isomorphism. Thus \( g_q \) is an isomorphism. □

Lemma 43. Suppose that \( \phi : A \to B \) is a semi-automaton epimorphism. Assume the following conditions:

1. The source semi-automaton \( A \) is complete;
2. The target semi-automaton \( B \) is deterministic.

We rule out multiple transitions with the same source, target and label for a deterministic semi-automaton. (Not only this is consistent with the traditional definition, but this is required for the next theorem to hold.)
Then the morphism $\phi$ is a directed emulator, thus conformal.

Proof. Suppose that $\phi = (f, g, \alpha)$ with $(f, g)$ an epimorphism. Then, $f$ is surjective. Let us check the outgoing edge lifting property. Let $q_0 \xrightarrow{e} q_0' \in E_B$ be a transition in $B$ and $p_0 \in f^{-1}(q_0)$. We search an antecedent of $e$ starting at $p_0$. Since $(f, g)$ is an epimorphism, there exists $q_1 \in f^{-1}(q)$ and $q_1 \xrightarrow{\tilde{e}} q_1' \in E_A$ such that $g(\tilde{e}) = e$, $f(q_1) = q_0$ and $f(q_1') = q_0'$. Furthermore, $\alpha(\ell_A(\tilde{e})) = \ell_B(e)$. If $p_0 = q_1$, we are done. Otherwise, since $A$ is complete, there is some edge $p_0 \xrightarrow{e'} p_1 \in E_A$ starting from $p_0$ with the same label $\ell_A(\tilde{e})$. Consider its image $g(e')$. We have $s_B(g(e')) = q_0$ and $\ell_B(g(e')) = \alpha(\ell_A(\tilde{e})) = \alpha(\ell_A(e)) = \ell_B(e)$. Therefore the edge $g(e')$ has same source and same label as the edge $g(\tilde{e})$. Since $B$ is deterministic, this implies that $g(e') = g(\tilde{e}) = e$ and we are done. 

5.1. Finite state automata. An automaton $A$ is a semi-automaton equipped with a state $q_0 \in V_G$ that is called the initial state and a set $F \subseteq V_A$ of final states. Furthermore, we suppose that any state $q \in V_G$ is accessible from $q_0$, that is there is a computation starting at $q_0$ ending at $q$.

Example 9. The initial state is represented with an input arrow while final states have an output arrow:

\[ \begin{array}{c}
\bullet & \xrightarrow{\bullet} & A
\end{array} \]

Let $A$ and $B$ be two automata. A morphism between automata $A \to B$ is a semi-automaton morphism $(f, g, \alpha)$ verifying

1. $f^{-1}(\{q_{0B}\}) = \{q_{0A}\}$ and
2. $F_A = f^{-1}(F_B)$.

In other words, $f(q) = q_{0B} \iff q = q_{0A}$ and $q \in F_A \iff f(q) \in F_B$.

The identity is an automaton morphism. Automata morphism compose componentwise. We denote $\text{Auto}$ the category of automata and their morphisms.

Lemma 44. The assignment that forgets the sets of initial and final states induces a forgetful faithful functor $\text{Auto} \to \text{Semi}$.

Remark 18. The forgetful functor is not full. For instance, the induced map

\[ H_1 = \text{Hom}_{\text{Auto}} \left( \begin{array}{c}
\bullet & \xrightarrow{\bullet} & A
\end{array} \right) \to \text{Hom}_{\text{Semi}} \left( \begin{array}{c}
\bullet & \xrightarrow{\bullet} & A
\end{array} \right) = H_2 \]

is not onto. Indeed, $H_1$ contains only the identity while $H_2$ contains also the semi-automaton morphism induced by exchanging the two vertices (and the two edges).

We denote the category of automata (resp. semi-automata) and their strict morphisms by $\text{Auto}^0$ (resp. $\text{Semi}^0$).
Lemma 45. The category $\text{Auto}^0$ (resp. $\text{Semi}^0$) is a faithful subcategory of $\text{Auto}$ (resp. $\text{Semi}$) such that the diagram

\[
\text{Auto}^0 \longrightarrow \text{Semi}^0
\]

\[
\downarrow \quad \downarrow
\]

\[
\text{Auto} \longrightarrow \text{Semi}
\]

is commutative.

A computation in an automaton is successful if it starts at some initial state and ends at some final state.

Definition 15. Given an automaton $A$ the subset $L(A) \subseteq A^*$ of the words $w$ for which there is a a successful path $\pi$ such that $\ell(\pi) = w$ is called the language represented by the automaton.

From Kleene’s Theorem, we know that the languages defined by (such) automata are the regular languages. They are well known to be closed by several operations such as union, concatenation, intersection, complementation, transposition.

Lemma 46. The image of a successful computation by a morphism $(f, g, \alpha) : A \to B$ is a successful computation.

Proof. Let $e_0$ be the first transition of a path $\pi = e_0 \cdots e_n$ in $A$. We have $s_A(e_0) = q_0A$. First, we compute $s_B(g(e_0)) = f(s_A(e_0)) = f(q_0A) = q_0B$. In an analogous way, from $t_A(e_n) \in F_A$, we get $t_B(g(e_n)) = f(t_A(e_n)) \in F_B$ for the final transition. Thus, the path $g(\pi)$ is successful. \qed

We record a consequence:

Lemma 47. If there is an automaton morphism $(f, g, \alpha) : A \to B$ then $\alpha(L(A)) \subseteq L(B)$. In particular, if there is a strict morphism $A \to B$ then $L(A) \subseteq L(B)$.

Proposition 8. Suppose that $(f, g, \alpha) : A \to B$ is a conformal morphism between two automata $A$ and $B$, then $\alpha(L(A)) = L(B)$. In particular, if two automata are related by a strict conformal morphism, they compute the same language.

Proof. Given Lemma 47, we have $\alpha(L(A)) \subseteq L(B)$. Now, suppose $w \in L(B)$, there is a sequence $\pi = e_0e_1 \cdots e_n$ with $s_B(e_0) = q_0B$ and $t_B(e_n) = F_B$ with $w = \ell_B(\pi)$. By definition of a conformal morphism, there is a corresponding computation $\psi = d_0d_1 \cdots d_n$ in $A$ such that $g(\psi) = \pi$. Since $f(s_A(d_0)) = s_B(g(d_0)) = s_B(e_0) = q_0B$, $s_A(d_0) = q_0A$. In an analogous way, we have $f(t_A(d_n)) = t_B(g(d_n)) = t_B(e_n) \in F_B$, thus $t_A(d_n) \in f^{-1}(F_B) = F_A$. To conclude, the computation $\psi$ is successful. By definition, $w = \ell_B \circ g(\psi) = \alpha \circ \ell_A(\psi)$. Thus, $w \in \alpha(L(A))$. \qed
Corollary 7. Let $\phi : A \to B$ be a strict morphism between two automata. If the induced graph morphism $G_{\phi} : G_A \to G_B$ is a directed emulator morphism, then $L(A) = L(B)$.

Proof. By Lemma 40, $G_{\phi}$ is conformal. By Lemma 39, $\phi$ itself is conformal. $\square$

Remark 19. In Sakarovitch’s book, a strict automaton morphism that induces a directed emulator morphism is called a totally surjective morphism [Sak03, Chap. II, Def. 3.2].

Theorem 1. Suppose that $\phi : A \to B$ is an automaton morphism. Assume the following conditions:

(1) The source automaton $A$ is complete;
(2) The target automaton $B$ is deterministic.

Then the three assertions are equivalent:

(i) the morphism $G_{\phi}$ is an epimorphism,
(ii) the morphism $\phi$ is conformal,
(iii) $G_{\phi}$ is a directed emulator morphism.

Proof. Lemma 43 shows that $(i) \Rightarrow (iii)$. By Lemma 40, $(iii) \Rightarrow (ii)$ and by Remark 15, $(ii) \Rightarrow (i)$. $\square$

Corollary 8. If $\phi : A \to B$ is a strict epimorphism between complete and deterministic automata, then the induced map $G_{\phi} : G_A \to G_B$ is a directed covering map.

Proof. First, by the preceding theorem, it is a directed emulator morphism. Then, by Corollary 6, it is a covering map. $\square$

From the whole theory of regular languages, we will keep only one fact. We refer to Sakarovitch (see for instance, [Sak03, Chap. I, §3.3]). Let $A$ be a deterministic automaton. There is a minimal complete deterministic automaton, denoted $A_{\text{min}}$, such that $L(A_{\text{min}}) = L(A)$ together with the canonical projection $A \to A_{\text{min}}$ that sends equivalent states to their equivalence class; furthermore, $A_{\text{min}}$ is unique up to automaton (strict) isomorphism. Note that the canonical projection $A \to A_{\text{min}}$ is a strict morphism in our terminology.

Corollary 9. Let $A$ be a deterministic automaton and let $\pi : A \to A_{\text{min}}$ be the canonical epimorphism to the minimal automaton. There exists an automaton $\hat{A}$ such that the following properties are satisfied:

(1) $\hat{A}$ is complete and deterministic;
(2) $\hat{A}$ contains $A$ as a subautomaton;
(3) $L(\hat{A}) = L(A)$;
(4) The underlying graph morphism $G_{\hat{A}} \to G_{A_{\text{min}}}$ is a covering morphism.

Proof. Each time there is a state $q$ in $A$ and a letter $a \in A$ without edges $q \xrightarrow{e} q' \in E_A$ with $\ell_A(e) = a$, take $p = \pi(q)$. Since $A_{\text{min}}$ is complete, there is
an outgoing edge $p \xrightarrow{e'} p'$ with $\ell_{a\min}(e') = a$. Take some node $q'$ in $Q_A$ in the fibre over $p'$ (it exists since the morphism is onto). We add to $A$ a new edge $e''$ between $p$ and $p'$ with label $a$ and we set $\pi(e'') = e'$. After modification, the automaton $A$ still verify the hypotheses of the Corollary. We continue the process until $A$ is complete, call the result $\hat{A}$. Since $\hat{A}$ is complete and deterministic, Corollary 8 leads to (4).

Theorem 1 and Corollary 9 yield our main source of directed emulators in this paper. We now seek a reconstruction of an automaton morphism from a directed emulator map. The following lemma is the first step.

**Lemma 48.** Consider a directed graph $G_A$ induced by a finite automaton $A$. Suppose that there is a directed emulator morphism $\varphi : G' \to G_A$. Then there exists a finite automaton $A'$ and a strict automaton morphism $\pi : A' \to A$ such that

1. $G_\pi = \varphi$;
2. $L(A') = L(A)$.

Here, we do not ask the automaton $A$ to be deterministic nor complete.

**Proof.** We begin with the commutative diagram:

$$
\begin{array}{ccc}
A_{G'} & \xrightarrow{A_{G'}} & A_G \\
\downarrow{G'} & & \downarrow{G} \\
G' & \xrightarrow{\varphi} & G_A \\
\end{array}
$$

By Lemma 40, $A_{G'}$ is conformal and by Lemma 37, $\epsilon_A \circ A_{G'}$ is conformal. So, by Proposition 6, the morphism $\epsilon_A \circ A_{G'} = \pi \circ \lambda$ for a strict morphism $\pi$ and $\lambda$ a relabelling. Then, set $A' = \lambda(A_{G'})$. We have $G' = G_{A_{G'}} = G'$ and $G_\pi = \varphi$. However, as defined, $A'$ is a semi-automaton. We define $\ell_{\varphi} = \pi^{-1}(\ell_{\phi})$ and $F' = \pi^{-1}(F_A)$ so that $\pi$ is actually an automaton morphism. By Lemma 40, $\pi$ is conformal and thus by Proposition 8, $L(A') = L(A)$. \qed

**Lemma 49.** Consider a directed graph $G = G_A$ induced by some finite deterministic automaton $A$. Suppose that there is a directed emulator morphism $G' \to G$. Then there exists a finite deterministic automaton $A''$ and a strict automaton epimorphism $\pi : A'' \to A$ such that

1. $G_\pi : G_{A''} \to G_A$ is a directed covering morphism;
2. $G_{A''}$ is a subgraph of $G'$;
3. $g(A'') \leq g(G')$.

**Proof.** By the previous lemma, there is a finite automaton $B$ together with a strict epimorphism $B \to A$ inducing the directed emulator morphism $G' \to G$. By Lemma 24, one can extract from the directed emulator $G'$ a directed cover $G'' \subseteq G'$ over $G$. This determines a subautomaton $A''$ of $B$ with underlying directed graph $G_{A''} = G''$ that is a subgraph of $G'$. Therefore $g(A'') = g(G'') \leq g(G')$. It remains to verify that $A''$ is deterministic. Since
the strict epimorphism $A'' 	o A$ induces a covering morphism $G'' \to G$, for any state $q' \in Q''$ and its image $q \in Q_A$, the induced map $\text{OutE}(q') \to \text{OutE}(q)$ is a bijection between sets of labelled outgoing transitions. Since $A$ is deterministic, $\ell_A|_{\text{OutE}(q)}$ is injective and so must be $\ell_{A''}|_{\text{OutE}(q')}$. Therefore $A''$ is deterministic. □

We can get rid of the multiple edges in the previous lemma, i.e., a simple graph theoretical version of the previous lemma holds.

**Lemma 50.** Consider a directed graph $G = G_A$ induced by some finite deterministic automaton $A$. Suppose that there is a directed emulator morphism $H \to R(G)$. Then there exists a finite deterministic automaton $A'$ and a strict automaton epimorphism $A' \to A$ such that

1. The induced directed emulator morphism $G' \to G_A$ is a directed covering morphism;
2. $g(A') \leq g(H)$.

**Proof.** Let $\phi : H \to R(G)$ denote a directed emulator morphism. Since $R$ preserves directed emulators, $R(\phi) : R(H) \to R(R(G)) = R(G)$ is a directed emulator. We apply Lemma 21 on $G$ and $R(\phi)$ leading to the diagram:

\[
\begin{array}{ccc}
H' & \xrightarrow{\rho_H} & R(H') \\
\downarrow{\psi} & & \downarrow{R(\psi)} \\
G & \xrightarrow{\rho_G} & R(G)
\end{array}
\quad
\begin{array}{ccc}
R(G) & \xrightarrow{1} & R(G) \\
\downarrow{R(\phi)} & & \downarrow{\phi} \\
H & \xrightarrow{1} & H
\end{array}
\]

The lemma provides a graph $H'$ with a directed emulator $\psi : H' \to G$. We conclude with Lemma 49 and Lemma 24. □

Finally we get rid of loops in the following sense.

**Lemma 51.** Consider a directed graph $G = G_A$ induced by some finite deterministic automaton $A$. Suppose that there is a simple directed cover morphism $H \to \text{Exc}(R(G))$. Then there exists a finite deterministic automaton $A'$ and a strict automaton epimorphism $A' \to A$ such that

1. The induced directed emulator morphism $G' \to G_A$ is a directed covering morphism;
2. $g(A') \leq g(H)$.

**Proof.** Apply Lemma 22 to obtain a directed cover $H'$ over $R(G)$ such that $g(H') = g(H)$. Applying Lemma 50 yields the result. □

6. The genus of a regular language

In this section, we state and prove the two main results of the paper (Theorem 2 and Corollary 14).

First, we recall the definition that we introduced in [BD16].

**Definition 16.** The genus $g(L)$ of a regular language $L$ over alphabet $A$ is the minimum of all genera of finite deterministic automata computing $L$. 

Remark 20. The word “deterministic” is essential in the definition of the genus, for it is known that any regular language has a genus 0 (planar) non-deterministic automaton that computes it, a nice result due to R. V. Book and A. K. Chandra [BC76].

Remark 21. It can be proved ([BD16]) that there always exists a complete deterministic automaton $A$ such that $g(L) = g(A)$. Basically, given a deterministic (but not complete) automaton, for any missing transition, one may add a transition to a fresh trash state whose outgoing transitions are loops. That leaves the genus unchanged.

Definition 17. The directed graph $G(L)$ associated to a regular language $L$ is the directed graph underlying the minimal automaton $A_{\text{min}}(L)$ canonically associated to $L$: $G(L) = G_{A_{\text{min}}(L)}$.

Remark 22. For any regular language $L$, $g(L) \leq g(G(L))$. The inequality is an equality if and only if the minimal automaton for $L$ has minimal genus among all finite deterministic automata computing $L$.

Theorem 2. Let $L$ be a regular language. Let $n \in \mathbb{N}$. The following assertions are equivalent:

1. $g(L) \leq n$;
2. The directed graph $G(L)$ has a directed cover $G$ of genus $g(G) \leq n$;
3. The directed graph $G(L)$ has a directed emulator $G$ with $g(G) \leq n$;
4. The directed simple graph $R(G(L))$ has a directed simple cover $G$ such that $g(G) \leq n$;
5. The directed simple graph $\text{Exc}(R(G(L)))$ without loop has a directed cover $G$ such that $g(G) \leq n$.

Proof. Suppose that $L$ has genus $g(L) \leq n$. There is some finite deterministic complete automaton $A$ computing $L$ such that $g(A) \leq n$. This automaton comes naturally with an automaton epimorphism $A \to A_{\text{min}}(L)$. Applying the functor $G(-)$ yields (Theorem 1) a directed covering epimorphism $G_A \to G(L)$. Hence (1) $\implies$ (2). The implication (2) $\implies$ (3) is obvious. Assume (3). The functor $R$ preserves directed emulation (Lemma 3) and genus (Lemma 14). Thus applying $R$ shows that $R(G(L))$ has a directed (simple) emulator $G$ such that $g(G) \leq n$. Using Lemma 31 extract from $G$ a directed (simple) cover $G'$ over the same directed graph $R(G(L))$. Then $g(G') \leq g(G) \leq n$. This proves (4). Assuming (4), we have a directed cover morphism $\psi : G \to R(G(L))$ with $g(G) \leq n$. Excizing $R(G(L))$ yields a simple graph $\text{Exc}(R(G(L)))$ without loop.Restricting $\psi$ to the preimage of $\text{Exc}(R(G(L)))$ yields a directed cover $\psi^{-1}(\text{Exc}(R(G(L)))) \to \text{Exc}(R(G(L)))$. Let $G' = \psi^{-1}(\text{Exc}(R(G(L)))$. Since $G'$ is a subgraph of $G$, $g(G') \leq g(G) \leq n$. This proves (5). Finally, assume that $G$ is a directed cover over the directed simple graph $\text{Exc}(R(G(L)))$. Lemma 31 provides a deterministic automaton $A$ such that $L(A) = L$ and $g(A) \leq g(G) \leq n$. Hence $g(L) \leq n$. 

Corollary 10. If $L, L'$ are two regular languages such that $\text{Exc}(R(G(L))) = \text{Exc}(R(G(L')))$, then $g(L) = g(L')$. 

Corollary 11. Let \( L, L' \) be regular languages such that \( \text{Exc}(RG(L')) \) is a subgraph of \( \text{Exc}(RG(L)) \). Then \( g(L') \leq g(L) \).

Proof. A directed emulator \( H \) of \( \text{Exc}(RG(L)) \) of minimal genus contains a directed emulator \( H' \) of \( \text{Exc}(RG(L')) \) as a subgraph. Hence \( g(L) \geq g(H') \geq g(L') \). \( \square \)

Corollary 11 can be used to bound genera of languages as well.

Corollary 12. Any regular language \( L \) of size \( |L|_{\text{set}} \leq 6 \) is planar.

Proof. Let \( Z_6 \) be the language that consists of the words in \( \{-2, -1, 0, 1, 2, 3\}^* \) such that the sum of their letters is 0 modulo 6. The underlying graph of its minimal automaton is the complete simple graph \( X_6 \) of size 6:

![Graph](image)

This graph has a planar directed emulator [BD19, §2.1]. Let \( L \) be any regular language whose minimal automaton \( A \) has size at most 6. The simple graph \( R(G_A) = R(G(L)) \) is then a subgraph of \( X_6 = R(G(Z_6)) \), thus (Corollary 11) has a planar directed emulator. Hence by Theorem 2, \( L \) is planar. \( \square \)

Corollary 13. Let \( L_1 \) and \( L_2 \) be two regular languages on disjoint alphabets. Then \( g(L_1 \cup L_2) \geq \max(g(L_1), g(L_2)) \).

Proof. The minimal automaton \( A_{\text{min}}(L_1 \cup L_2) \) for \( L_1 \cup L_2 \) contains both the minimal automaton \( A_{\text{min}}(L_1) \) and the minimal automaton \( A_{\text{min}}(L_2) \) as subgraphs. \( \square \)

The Language Genus Problem is the following problem:

**Inputs:** a regular language \( L \) and \( n \in \mathbb{N} \),

**Output.** YES if \( g(L) \leq n \), otherwise NO.

The regular language can be described alternatively via a regular expression or by some finite state (possibly non-deterministic) automaton. In any case, we can compute the minimal automaton out of that data.

The Directed Emulation Genus Problem is:

**Inputs:** a strongly connected directed graph \( G \) and \( n \in \mathbb{N} \),
Output: YES if there is a directed emulator $G'$ of $G$ such that $g(G') \leq n$, otherwise NO.

Theorem 2 allows to reduce the problem of the determining the genus of a language to a graph-theoretic problem in terms of directed emulators.

**Corollary 14.** The Language Genus Problem has a solution if and only if the Directed Emulation Genus Problem has a solution.

**Proof.** Theorem 2 shows a solution for the Directed Emulation Genus Problem implies a solution for the Language Genus Problem. We need to show the converse. Suppose that $G = (V, E)$ is a strongly connected directed graph. We turn $G$ into a minimal deterministic automaton as follows. First, consider the associated tautological semi-automaton $A_G$ (see Definition 13).

Next, we define an arbitrary vertex $v_0 \in V$ to be an initial state and all vertices $v \in V$ to be final states. Let $A$ be the resulting automaton. By Lemma 41, $A$ is deterministic. Strong connectedness ensures that every state is accessible in $A$. Since every state is final, every state is trivially co-accessible. Furthermore, since every outgoing edge has a unique label, there cannot be two distinct and equivalent states, so $A$ is minimal. We have $G = G(L(A))$. By Theorem 2, $L(A)$ has genus $g$ if and only if $G$ has a directed emulator of genus $g$. Since we assumed to know a solution to the Language Genus problem, this completes the proof. □

We now consider the undirected version. The Emulation Genus Problem is:

**Inputs:** a connected undirected graph $G$ and $n \in \mathbb{N}$,

**Output:** YES if there is an emulator $G'$ of $G$ such that $g(G') \leq n$, otherwise NO.

**Corollary 15.** The Emulation Genus Problem has a solution if the Directed Emulation Genus Problem has a solution.

**Proof.** Let $G$ be a graph. It suffices to prove that there is emulator $G'$ of $G$ such that $g(G') \leq n$ if and only if there is a directed emulator $H$ of $G$ such that $g(H) \leq n$.

Suppose that $\phi : G' \rightarrow G$ is an emulator such that $g(G') \leq n$. By Lemma 29, $\phi : G' \rightarrow G$ is a directed emulator. Furthermore, it is clear that $g(G') = g(G')$.

Conversely, suppose that $\phi : G' \rightarrow G$ is a directed emulator with $g(G') \leq n$. According to Lemma 30, there is an emulator $G'' \rightarrow G$ with $g(G'') = g(G') \leq n$.

□

7. Conclusion

We have shown that the Language Genus Problem is decidable if and only if the Directed Emulation Genus Problem is decidable. However, we do not have yet a complete proof of decidability. A general approach consists in properly defining directed minors and proving a “directed graph minor”
theorem analogous to the celebrated graph minor theorem of Robertson and Seymour [RS04, §10.5]. This is the approach aimed at in [Kup]. Even in the case this approach would be successful, one would need to find the minors of nonplanar emulable directed graphs. In some way, we would face the kind of issues that are discussed by M. Chimani, M. Derka, P. Hliněný and M. Klusáček in their article [CDHK13]. Concerning emulators, the relationship between undirected and directed graphs is another promising direction.

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