Multipolar universal relations between $f$-mode frequency and tidal deformability of compact stars

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Though individual stellar parameters of compact stars usually demonstrate obvious dependence on the equation of state (EOS), EOS-insensitive universal formulas relating these parameters remarkably exist. In the present paper, we explore the inter-relationship between two such formulas, namely the $f$-I relation connecting the $f$-mode quadrupole oscillation frequency $\omega_f$ and the moment of inertia $I$, and the I-Love-Q relations relating $I$, the quadrupole tidal deformability $\lambda_2$, and the quadrupole moment $Q$, which have been proposed by Lau et al. [Astrophys. J. 714, 1234 (2010)], and Yagi and Yunes [Science, 341, 365 (2013)], respectively. A relativistic universal relation between $\omega_f$ and $\lambda_2$ with the same angular momentum $l = 2, 3, \ldots$, the so called “diagonal $f$-Love relation” that holds for realistic compact stars and stiff polytropic stars, is unveiled here. An in-depth investigation in the Newtonian limit is further carried out to pinpoint its underlying physical mechanism and hence leads to a unified $f$-I-Love relation. We reach the conclusion that these EOS-insensitive formulas stem from a common physical origin — compact stars can be considered as quasi-incompressible when they react to slow time variations introduced by $f$-mode oscillations, tidal forces and rotations.

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I. INTRODUCTION

The structure of neutron stars (NSs), the remnants of supernovas, is dictated by the strong gravity prevailing inside such stars. As a result, the density of the inner core of a typical NS can be several times the normal nuclear density $2.8 \times 10^{14}$ gm cm$^{-3}$, which is not yet achievable (at least in a stable form) in the terrestrial environment. The equation of state (EOS) of nuclear matter in NSs is often masked by various uncertainties in nuclear and particle physics, leading to the associated uncertainties in the physical characteristics of NSs, including their masses, radii, and moments of inertia. Thus, it has become a common practice for nuclear physicists to examine how EOSs of high-density nuclear matter could affect the structure of NSs, e.g., the mass-radius relation, and systematic investigations along such direction have also been carried out (see, e.g., [1–4]). On the other hand, mass efforts have been pooled together to infer the details of nuclear matter from various possibly detectable characteristics (such as mass, radius, moment of inertia and gravitational wave spectrum) of NSs (see, e.g., [2–10]).

However, paralleling these attempts to differentiate the structure of NSs with respect to nuclear EOS and vice versa, several universal EOS-insensitive formulas connecting different physical parameters (e.g., mass, radius and moment of inertia) of NSs or quark stars (QSs) have also been discovered. Such formulas are important and useful as they enable astronomers to infer (or at least to constrain) a physical parameter of NSs (or QSs) from the others that are more amenable to physical measurement even in the absence of exact information of relevant EOSs. For example, Lattimer and Schutz $^\mathfrak{3}$ and Bejger and Haensel $^\mathfrak{11}$ found formulas relating the moment of inertia $I$, the mass $M$, and the radius $R$ for NSs constructed with different realistic EOSs, which could be used to determine the moment of inertia of star A in the double pulsar system J0737-3039 $^\mathfrak{12, 13}$ to about 10% accuracy. Besides, several universal behaviors of the quadrupolar $f$-mode oscillations of NSs have been established and applied to infer the EOS of nuclear matter $\mathfrak{3, 6, 14, 20}$.

It is particularly interesting to note that some of these universal formulas actually relate the dynamical response of a NS (or QS) under external perturbation to its static stellar structure. In particular, Lau et al. $^\mathfrak{20}$ found that the frequency and damping rate of $f$-mode quadrupole oscillation are expressible in terms of $M$ and the effective compactness $\eta \equiv \sqrt{M^3/I}$, hereafter referred to as the “$f$-I relations”. Hence, the values of $M$, $R$, and $I$ of a NS (or QS) can be inferred accurately from the $f$-mode gravitational-wave signals $^\mathfrak{20}$. Since pulsating NSs are expected to be promising sources of gravitational waves, the above-mentioned relations can lead to useful information about the static structure of NSs (or QSs) once the third-generation gravitational wave detectors (e.g., the Einstein Telescope $\mathfrak{21}$) are available in the future.

On the other hand, universal relations expressing the distortion of a NS induced by tidal forces or spin in terms of its static structural parameters have also been

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found recently\textsuperscript{22,25}. The so called “I-Love-Q relations”, discovered by Yagi and Yunes\textsuperscript{22,23}, relate \( I \), the quadrupole tidal Love number \( \lambda_2 \), (or, more precisely, tidal deformability\textsuperscript{26,27}), and the spin-induced quadrupole moment \( Q \), with \( M \) being a proper scaling parameter. The relations are robust and also prevail in several different situations, including binary systems with strong dynamical tidal field\textsuperscript{28}, magnetized NSs with sufficiently high rotation rates\textsuperscript{29}, and rapidly-rotating stars\textsuperscript{30,31}. Besides, there are works extending the I-Love-Q relations to consider the relation between higher-order multipole moments induced by either tidal forces or rotation\textsuperscript{27,32,33}.

Similar to other universal relations, these I-Love-Q relations enable us to infer any two of \( I, \lambda_2 \) and \( Q \) from the detected value of the remaining one even in the absence of prior knowledge of the EOS of NSs. In addition, the I-Love-Q relations could facilitate the analysis of gravitational wave signals emitted during the late stages of NS-NS binary mergers\textsuperscript{22,23,34,35}, and also serve as an indicator to identify the validity of other modified gravity theories\textsuperscript{22,23,36,37}.

Despite the fact that the above mentioned f-I relation and I-Love-Q relations were discovered separately, the moment of inertia \( I \) is involved in both of them. It is physically natural to expect that there should be a link between these two relations. The present paper aims at examining their inter-relationship and establishing universal relations that can directly link the characteristics of multipolar distortions induced respectively by oscillations and tidal forces together. First of all, we investigate the so called “f-Love relation” between the \( l \)-th multipolar \( f \)-mode oscillation frequency \( \omega \), and the \( l' \)-th multipolar tidal deformability \( \lambda_{l'} \), where \( l, l' = 2, 3, 4, \ldots \). We show that there exist approximately EOS-independent and relativistically correct relations between these two if \( l = l' \), which are referred to as the diagonal f-Love relations hereafter. It is obvious that the case with \( l = 2 \) is a direct consequence of the original f-I and I-Love-Q relations\textsuperscript{20,22,23}. For the off-diagonal f-Love relations with \( l \neq l' \), the relation between the two relevant physical quantities become more EOS-sensitive (see Section III and Figs. 1-8). This finding is important and useful. On the one hand, assuming that the mass of a NS is known from astronomical measurements, one can use the diagonal f-Love relation to determine the \( f \)-mode frequency from the measured tidal deformability in the same angular momentum sector \textsuperscript{22,23,34,35} (and vice versa) even in the absence of detailed knowledge of nuclear EOS. On the other, the details of EOS could be inferred from the off-diagonal f-Love relations.

To pursue the physical origin of the EOS-independency of the diagonal f-Love relations, we compare the f-Love relations of realistic stars and polytropic stars with those of incompressible stars (see Section III and Figs. 1-8). We find that the latter can nicely approximate the behavior of realistic stars (including both NSs and QSs) and polytropic stars whose polytropic indices \( N \) are less than 1. As a matter of fact, the polytropic index of most (if not all) popular realistic nuclear EOSs in the high density regime is less than 1. Hence, these prevailing EOSs are stiff enough to guarantee the f-Love relations except for NSs with very low masses. If softer nuclear EOSs were proposed and used to construct NSs, such stars would demonstrate significant deviations from the f-Love relation discovered in the present paper. Conversely, if the f-Love relations were violated, it would hint at the softening of nuclear matter due to physical mechanisms such as kaon condensation or other phase transitions.

Furthermore, regarding the physical origin of the multipolar f-I, I-Love and f-Love formulas in the Newtonian limit, we consider a model star characterized by a density profile \( \rho(r) = \rho_0(1-\delta x^2) \) with \( x \equiv r/R \) and \( \rho_0 \) being the central density (see Section IV A). Here \( 1 \geq \delta \geq 0 \) is a parameter controlling the stiffness of the star. With such a stellar model, we can approximately reproduce the behavior of QSs and NSs whose polytropic index is less than 1, while rendering calculations of physical quantities more manageable and simplified (see the discussion in Section IV A). First of all, we analytically derive the f-I formula relating \( \omega_l \) and \( I_n \), where \( I_n = \int_0^R \rho(r)^{\gamma+1} r \, dr \) is the \( n \)-th mass moment (see Section IV B), and the I-Love formula connecting \( I_n \) and \( \lambda_n \) (see Section IV C). In general, these two relations explicitly depend on the value of \( \delta \), i.e., the underlying EOS of the star, and lack of universality. However, if \( n = 2l - 2 \), to first-order in \( \delta \) both of them are independent of \( \delta \). Consequently, combining the f-I and I-Love formulas, we can show that the diagonal f-Love formula is also to first-order independent of \( \delta \), corroborating the EOS-independence of the diagonal f-Love formula as discovered here (see Section IV D).

The analytic studies developed in the present paper clearly explains why these universal formulas are so insensitive to changes in EOSs. We arrive at the conclusion that the crux of the observed universality is: (i) these universal relations actually derive from the behavior of incompressible stars; (ii) they are stationary with respect to changes in \( N \) around the incompressible limit (i.e. \( N = 0 \)) and hence are EOS-insensitive; and (iii) the prevailing EOSs for nuclear matter are stiff enough to be considered as almost incompressible (termed as quasi-incompressible here) in certain slow physical processes (e.g., \( f \)-mode oscillations, tidal and rotational deformation).

A remark about NSs with low-compactness is in order. Near the low mass limit, the star is primarily made of soft nuclear matter with adiabatic index \( \Gamma = 1 + N^{-1} < 1.4 \) and therefore marked deviation from the incompressible limit is expected. However, the behavior of low-density nuclear matter is quite well studied and there is not much dispute. Accordingly, NSs still demonstrate universality, which is of course different from the incompressible counterpart, in that limit.

Physically speaking, if an external perturbation is applied to a compact star, the response of the star to such perturbation is given by the Green’s function, which con-
sists of the contributions arising from different oscillation modes \[38, 39\]. Tidal deformation of NSs is merely the zero-frequency component of the Green’s function. We show in the present paper that for quasi-incompressible stars in the Newtonian limit, the tidal field couples only to the \( f \)-mode oscillation and establish a robust \( f \)-I-Love relation. Such a relation readily shows that the \( f \)-I and I-Love relations imply each other, thus unifying these two independently discovered universal relations.

To put our work into proper perspective, we note that several recent studies have been performed to extend the I-Love-Q universality to multipolar sectors. For example, Yagi \[27\] found that there exists certain degree of correlation between two tidal deformabilities with different angular momentum indices albeit with more obvious EOS-dependence. On the other hand, in an attempt to generalize the no-hair theorem for blackholes to NSs (or QSs), the so called “three-hair theorem” has been proposed \[32, 33, 40\], which states that higher multipole moments can be expressed in terms of just the mass monopole, spin current dipole, and mass quadrupole moments through EOS-independent relations. However, the accuracy of the three-hair theorem was also found to deteriorate with the order of multipole. Using the terminology coined here, we note that these two relations are actually off-diagonal ones. In the present paper, we link \( \omega_1, \lambda_1 \) and \( I_{2l-2} \) together in the Newtonian limit. Apart from some trivial numerical factors due to angular integrals, the mass moment \( I_n \) is equal to the \( 2^n \)-pole moment under the isodensity approximation \[32, 33, 10, 11\], and \( I_2 \) is indeed the moment of inertia. Thus, the discovery reported here generalize the well known \( f \)-I and I-Love relations to higher angular momentum sectors. Most interestingly, after eliminating the mass moment from the \( f \)-I and I-Love relations, a fully relativistic multipolar diagonal \( f \)-Love relation, which is accurate to within 3\% for \( l \leq 5 \) as long as the effective polytropic index of nuclear matter is less than unity (see Section III), is established. Besides, the present work also relates the multipole moments considered in the three-hair theorem \[32, 33, 10\] to respective \( f \)-mode oscillation frequencies. Since the multipole moments of compact stars could be inferred by measuring the atomic spectra observed from such stars with future x-ray telescopes such as LOFT and NICER \[12, 13\], the relevant \( f \)-mode oscillation frequencies could likewise be derived from the \( f \)-I universal formula.

The plan of the paper is as follows. In Section II, we briefly review the previously discovered universal behaviors of the quadrupole \( f \)-I and I-Love-Q relations for NSs. We present the multipolar \( f \)-Love universal relations in Section III and study how the stiffness of nuclear matter could affect the accuracy of the I-Love relation. Newtonian analytic analysis will be carried out in Section IV to establish the multipolar \( f \)-I, I-Love and \( f \)-Love relations. A unified Green’s function approach to multipolar tidal deformation in the Newtonian regime and hence a novel multipolar \( f \)-I-Love relation are given in Section V. The conclusions of the present paper are presented in Section VI. Unless otherwise stated explicitly, we use geometric units where \( G = c = 1 \).

## II. QUADRUPOLE F-I AND I-LOVE-Q RELATIONS

When compact stars are perturbed away from their equilibrium state, their subsequent oscillations can be analyzed in terms of quasi-normal modes (QNMs) \[14, 48\]. Each QNM is characterized by a complex eigenfrequency \( \omega = \omega_r + i\omega_i \), with \( \omega_i \) measuring its decay rate due to emission of gravitational waves. For typical NSs, the oscillation frequency of the fundamental (\( f \)) mode usually lies in the kilohertz range, which is lower than other QNMs such as the pressure (\( p \)) modes and the spacetime (\( w \)) modes. Hence, as far as gravitational waves emitted from oscillating NSs are concerned, \( f \)-mode oscillations are probably most likely to be detectable with advanced gravitational wave observatories such as the Einstein telescope \[21\]. Owing to this, numerous approximate formulas attempting to relate \( \omega_r \) and \( \omega_i \) of \( f \)-mode oscillation to the physical parameters of NSs have been proposed \[3, 6, 14, 16\]. In most of these cases, \( M \) and \( R \) were used as the independent parameters to express the \( f \)-mode frequency. It is not until Lau et al. \[20\] introduced \( M \) and \( I \) as the two parameters to study quadrupolar \( f \)-mode oscillations and found two nearly EOS-independent relations

\[
\bar{\omega}_2 \equiv \frac{M \omega_r}{I} = -0.0047 + 0.133 \eta + 0.575 \eta^2, \quad (2.1)
\]

\[
I^2 \omega_i/M^5 = 0.00694 - 0.0256 \eta^2, \quad (2.2)
\]

where the effective compactness \( \eta \equiv \sqrt{M^3/I} \) replaces the role of the traditionally defined compactness \( C \equiv M/R \). (In (2.1) we have corrected a typographical error in \[20\].) As shown in Table 1 of Lau et al. \[20\], equations (2.1) and (2.2) are less sensitive to the EOS in comparison with the other universal relations using \( R \) as a parameter. The typical deviation of the frequency and the damping rate of realistic NSs built with a particular EOS from the above two formulas is less about a few percent. In the present discussion we focus our attention on (2.1) and refer it as the \( f \)-I relation.

For a star with a given mass \( M \), the effective radius \( R_e \equiv \sqrt{T/M} \) measures its average size weighted by its mass distribution and is therefore more relevant to the dynamics of the star than the geometric radius \( R \). Hence, the introduction of the effective compactness \( \eta = M/R_e \) is expected to be a crucial reason to lead to the better performance of (2.1) and (2.2).

On the other hand, owing to the frequency sensitivity limit of currently available gravitational wave observatories, only the low-frequency part of gravitational waves with typical frequency less than about 100 Hz emitted in the early stage of binary inspirals of compact stellar objects could be detected in the near future. However, as suggested by Flanagan and Hinderer \[54\], such signals are
still likely to be affected by the internal structure of the binary systems. For the same reason, it was proposed that EOS of nuclear matter could also be constrained from the gravitational wave signals emitted in the last few orbits of binary inspirals. In particular, the influence of stellar structure on the phase of gravitational wave signal in the early stage of inspirals is uniquely determined by the quadrupole tidal deformability \( \lambda_2 \) defined by:

\[
Q_{ij} = -\lambda_2 \mathcal{E}_{ij}
\]

where \( Q_{ij} \) is the traceless quadrupole moment tensor of the star and \( \mathcal{E}_{ij} \) is the tidal field tensor inducing the deformation. Besides, the term tidal Love number \( k_2 \equiv 3\lambda_2/(2R^5) \) is also often referred to in the literature. Numerous investigations have been performed to study how the tidal deformability (or Love number) depends on \( R, M, \mathcal{C} \) and the EOS. In general, obvious EOS-dependence is observed if the tidal deformability (or Love number) is considered as a function of any one of \( R, M, \) and \( \mathcal{C} \).

Nonetheless, a set of almost universal relations, the I-Love-Q relations, were discovered recently by Yagi and Yunes (2022). In such relations, three dimensionless scaled physical quantities, namely, the scaled moment of inertia \( I \equiv I/M^3 \), the scaled quadrupole tidal deformability \( \lambda_2 \equiv \lambda_2/M^2 \), the scaled rotational quadrupole moment \( Q \equiv -MQ/J^2 \), where \( Q \) and \( J \) are respectively the spin-induced quadrupole moment and the angular moment of the star (see, e.g., 22, 24, 26, 34, 33, 50-52 for methods to evaluate these quantities in general relativity), are related through nearly three nearly EOS-independent empirical formulas, which can be cast into the following form:

\[
\ln y_i = a_i + b_i \ln x_i + c_i \left( \ln x_i \right)^2 + d_i \left( \ln x_i \right)^3 + e_i \left( \ln x_i \right)^4,
\]

where \( a_i, b_i, c_i, d_i, \) and \( e_i \) are some fitting coefficients (see 22, 23). Thus, given any one of these three scaled quantities, the remaining two can be obtained. In addition, the I-Love-Q relations also imply that the degeneracy between the quadrupole moments and spins of NSs in binary inspiral waveforms can be removed. Hence, with the advent of second-generation gravitational wave detectors, the averaged (dimensionless) spin of binary systems could be measured with accuracy up to 10% (22, 23). Motivated by the possible applications of the I-Love-Q relations in astrophysics, general relativity and fundamental physics, there has recently been a surge of interest in this field.

On the other hand, it is remarkable that \( I \) considered in the I-Love-Q relations is actually equal to the inverse-square of the effective compactness (i.e. \( \eta^{-2} \)) used in the f-I relations (2.1) and (2.2). This prompts us to study the possibility whether universal relations directly linking \( \lambda_2, Q \) and \( \omega_2 \) exist. An additional question also arises naturally. Could such relations exist in higher-order multipoles? The focus of the present paper is to perform an in-depth investigation on these issues. Specifically, we study the so-called f-Love relation between the f-mode frequency and the electric tidal deformability in the context of multipolar distortion in the following discussion.

### III. F-LOVE RELATIONS

In the present paper, we adopted the convention for terminologies introduced in 26, 27 to discuss multipolar tidal deformation. We consider the \( l \)-th order electric tidal deformability (see 26, 27, 54 for its definition), \( \lambda_l \), with \( l \) being the angular momentum index of the relevant tidal field in consideration, as well as the dimensionless \( l \)-th order electric tidal deformability

\[
\bar{\lambda}_l \equiv \frac{\lambda_l}{M^{2l+1}},
\]

which is also related to \( l \)-th order electric tidal Love number \( k_l \) through the relation 26:

\[
k_l \equiv \frac{\left(2l+1\right)!}{2} \mathcal{C}^{2l+1} \bar{\lambda}_l.
\]

As highlighted above, we investigate the relationship between the \( l \)-th multipole scaled f-mode frequency \( \bar{\omega}_l \equiv M \omega_l \) and the \( l \)-th multipole dimensionless deformability \( \bar{\lambda}_l \). In general, the values of \( l \) and \( l' \) can be distinct. In Figs. 1-4, \( \bar{\omega}_l \) is plotted against \( \ln \bar{\lambda}_l \), where \( l = 2, 3, 4, 5, \) and \( l' = 2 \) (Fig. 1), \( l' = 3 \) (Fig. 2), \( l' = 4 \) (Fig. 3) and \( l' = 5 \) (Fig. 4), for five kinds of realistic NSs (respectively constructed with AU 53, UU 53, WS 53, 54, BBB2 57 and FPS 54, 58 EOS), one QS described by the MIT bag model (see, e.g., 54, 61), and incompressible stars. At first sight, all such \( \omega_l \) versus \( \ln \bar{\lambda}_l \) graphs demonstrate certain degree of EOS-independence. In particular, all data points of realistic NSs and QSs nearly coalesce onto the solid line representing the data of incompressible stars. To further examine the degree of accuracy of the universal relations manifested in these curves, we use the case of incompressible stars as a reference and show in Fig. 5-8 that the fraction of deviation \( E \equiv (\bar{\omega}_l - \bar{\omega}_l^{(0)})/\bar{\omega}_l^{(0)} \), where \( \bar{\omega}_l^{(0)} \) is the scaled frequency of incompressible stars, in each of these plots. It is clearly shown that for a fixed \( l \), \( |E| \) is the smallest for the diagonal case where \( l = l' \) and, in addition, also least sensitive to EOS (including both NSs and QSs). In fact, \( |E| \) is less than 0.01 in all diagonal cases considered here. Generally speaking, for the off-diagonal cases where \( l \neq l' \), the fractional deviation \( E \) of QSs deviates obviously from those of NSs.

To investigate how the fractional deviation \( E \) in \( \bar{\omega}_l \) depends on the stiffness of a star, we show in Fig. 5(E) \( E \) versus \( \ln \bar{\lambda}_l \) for polytropic stars with different values of relativistic adiabatic index \( \Gamma \equiv (\rho p)/(dp/d\rho) = 1+1/N \), where \( \rho, p, \) and \( N \) are energy density, pressure and the polytropic index, respectively. Save for some “atypical cases” which will be further discussed later, the diagonal f-Love relation again yields the smallest deviation and...
demonstrate the least dependence on the adiabatic index $\Gamma$. We also note that $|E|$ increases with the polytropic index $N$ and markedly grows if $\Gamma < 2$, especially for dense stars close to the maximum mass limit (i.e., stars with small scaled tidal deformability). As a rule of thumb, general relativistcal effects tend to soften EOS. Hence, the polytropic index effectively increases with compactness and this explains the increase in $E$ towards the maximum mass limit. These observations clearly indicate that the stiffness of nuclear EOS is a crucial factor affecting the f-Love relation discovered here.

Moreover, some interesting behaviors of $E$ can be observed from the graphs shown in Figs. 5-8. For all diagonal cases, $E$ is negative (except near the maximum mass limit in Fig. 5) and its magnitude decreases with the stiffness (or $\Gamma$) of the EOS. For non-diagonal cases showing $E$ in the plots of $\bar{\omega}$ versus $\ln \bar{\lambda}$, if $l > l'$, the above-mentioned behavior of $E$ remains unchanged. In fact, $|E|$ also increases with $l$ for a fixed $l'$. However, the situation for cases with $l < l'$ becomes more complicated. For the diagrams of $E$ with $(l, l') = (2, 3), (2, 4), (2, 5), (3, 5)$, $E$ becomes positive although its magnitude still decreases with increasing stiffness (or increasing $\Gamma$). On the other hand, for the “atypical cases” with $(l, l') = (3, 4), (4, 5)$ there are some crossings between lines representing EOSs of different stiffness and the sign of $E$ is not uniquely defined. Thus, the magnitude of $E$ is no longer a good indicator for the stiffness of EOS.

Based on these observations, we figure out a phenomenological method, whose analytical basis will be provided in Section IV to explain the above-mentioned behavior of $E$. Schematically we expand $E$ in power series of $N$,

$$E(\ln \bar{\lambda}; l, l'; N) = E_1(l, l') N + E_2(l, l') N^2,$$  \hspace{1cm} (3.3)

where the coefficients $E_1(l, l')$ and $E_2(l, l')$ in general also depend on $\bar{\lambda}$, and only terms up to $N^2$ are kept in the expansion. As $E$ is insensitive to changes in $N$ for the diagonal case, it is reasonable to expect that $E_1$ is proportional to $l - l'$, or any of its positive powers. Without loss of generality, we rewrite (3.3) as

$$E(\ln \bar{\lambda}; l, l'; N) = -\alpha(l - l') N - \beta N^2,$$  \hspace{1cm} (3.4)

and $\alpha$ and $\beta$ (both are dependent on $l, l'$) are assumed to be positive constants in order to explain the numerical data shown in the figures. Therefore, for the diagonal cases with $l = l'$, $E = -\beta N^2 < 0$ and $|E|$ increases with $N$. This successfully captures the trend observed in Figs. 5-8. In fact, we have verified this point (within the accuracy of available numerical data) in Figs. 5-8 in the Newtonian limit (see also Table III).

On the other hand, for off-diagonal cases with $l > l'$, it follows from from (3.4) that $E$ is still negative and $|E|$ also increases with $N$. Complication arises for off-diagonal cases with $l < l'$. Several situations could be possible. Two of them are: (i) If either $N$ is small or $|l - l'|$ is large, then the first-order term dominates the second-order term. Consequently, $E$ is positive and increases with $N$ (see the diagrams of $E$ with $(l, l') = (2, 3), (2, 4), (2, 5), (3, 5)$). (ii) If the first-order and the second-order terms are of similar magnitudes, they could partly cancel each other out. Hence, $E$ is not of a definite sign and its magnitude may be rather small due to the cancellation of these two terms. Such is the explanation for the “atypical cases” mentioned above (cases with $(l, l') = (3, 4), (4, 5)$). Hence, in these atypical cases the magnitude of the deviation could not used as a direct measure of the stiffness of nuclear matter.

However, (3.4) has to be properly modified in order to account for relativistic effects. As can be observed from the diagonal cases in Figs. 1-4 for a fixed EOS $|E|$ usually increases towards the maximum mass limit, which could be understood as the consequence of the general relativistical softening effect mentioned above. Therefore, the polytropic index $N$ used in (3.4) should be replaced by its modified value, which is expected to be larger than the original value. Such replacement can qualitatively explain the increase in $|E|$ observed for dense stars. Empirically, we find that the modified polytropic index $N$ increases by an amount of the order of the compactness of the star, which is in agreement with physical intuition. On the other hand, a subtlety can also be observed from these figures especially for stiff stars in the small $\lambda$ regime (i.e., dense stars). It seems that an extra anomalous (positive) contribution emerges in RHS of (3.3). For example, in the diagonal case of Fig. 5 both the curves with $\Gamma = 2, 2.2$ crosses zero in the small $\lambda$ regime. We expect the magnitude of this extra term, which is more important for stiff stars, grows with compactness and leads to this anomalous behavior. However, the effect of this anomalous term is quite small and we cannot extract its quantitative dependence from our data.

To sum up the above observations, we find that both realistic NSs and QSs obey the f-Love universal relation as shown in Figs. 1-4. Such universal behavior, which is shown to be insensitive to changes in polytropic index $N$ as long as $N$ is less than unity (see Figs. 5-8), is attributable to the fact that realistic NSs far from the theoretical low mass limit are made of stiff nuclear material with $N < 1$. On the other hand, for QSs obeying the simple MIT bag model (see, e.g., 1-3, 6-8), it is straightforward to show that $N = \rho/(4B) - 1$, where $B$ is the bag constant. Unless for QSs with central density much higher than the bag constant, $N$ is also less than unity. In fact, $N$ is almost zero for low mass QSs. Hence, QSs also reveal similar f-Love universal behavior.

### IV. NEWTONIAN ANALYSIS

#### A. Generalized Tolman model

In the previous section, we have shown that the diagonal f-Love relation is particularly insensitive to variations
in EOS and the behaviors of realistic stars and polytropic stars tends to that of incompressible stars as long as these stars are sufficiently stiff. On the other hand, the off-diagonal counterpart displays more sensitive dependence on stiffness. To provide these observations a proper theoretical basis, at least in the Newtonian limit, we consider a model compact star, referred to as the generalized Tolman model (GTM) in the present paper, whose density distribution \( \rho(r) \) depends on the radial coordinate \( r \) as follows:

\[
\rho(r) = \rho_0 f(x, \delta), \tag{4.1}
\]

with \( \rho_0 \) being the central density, \( x \equiv r/R \), \( 0 \leq \delta \leq 1 \) and

\[
f(x, \delta) \equiv 1 - \delta x^2. \tag{4.2}
\]

Here \( \delta \) is a parameter determining the stiffness of the EOS of the star, which can be quantitatively measured by a position-dependent effective polytropic index \( N_e(\delta; x) \), where

\[
[N_e(\delta; x)]^{-1} = \frac{\rho \frac{dp}{d\rho}}{\rho \frac{dp}{d\rho}} - 1
= \frac{(1 - \delta x^2)(5 - 3\delta x^2)}{\delta[5(1 - x^2) - 4\delta(1 - x^2) + \delta^2(1 - x^6)]} - 1. \tag{4.3}
\]

The GTM \( f = 2 \) reduces to nearly incompressible and Tolman VII model stars in the limits \( \delta \ll 1 \) and \( \delta \approx 1 \), respectively \[2, 49, 64, 66\]. For \( \delta \ll 1 \), it can be shown that \( N_e \approx \delta \) for \( x < \delta \). In fact, near the center of the star, \( f = 2 \) closely resembles the density distribution of a polytropic star with polytropic index \( N = \delta \), whose density is, to leading order in \( N \), given by \[64, 65\]

\[
\rho(r) = \rho_0[1 + N \ln(1 - x^2)]. \tag{4.4}
\]

Therefore, this model can nicely approximate nearly incompressible stars in the small-\( \delta \)-limit. On the other hand, if \( \delta \) is equal to unity, the model obviously reduces to the standard Tolman VII model, which has been shown to be a good approximation of realistic NSs \[2, 49, 64, 66\]. This is the reason why the model \( f = 2 \) is coined here as the generalized Tolman model. Besides, unless \( \delta = 1 \), there is a density discontinuity developed at the stellar surface, which prevails in bare QSs (see, e.g., \[1, 63\] and references therein). Hence, the GTM is also capable of reproducing such typical behavior of QSs.

Furthermore, we have also verified that the effective polytropic index \( N_e \) in \[43\] is always less than or equal to unity for all physical choices of \( \delta \) and \( x \). For example, for the case with \( \delta = 1 \) (i.e., the the standard Tolman VII model), \( N_e = 1 \) at the stellar surface and decreases monotonically to 2/3 at the center \( x = 0 \) \[67\]. For \( \delta < 1 \), the corresponding value of \( N_e \) is also less than unity everywhere inside the star. Hence, we expect that GTM is a valid model to emulate sufficiently stiff NSs and QSs whose effective polytropic index is less than unity. In the following discussion, we shall make use of the simplicity and manageability of GTM to study the underlying physical mechanism of the universal \( f \)-Love relation discovered here. First, two universal formulas respectively connecting the \( f \)-mode frequency and the tidal deformability to mass moments of suitable orders are derived. Each of these analytic formula constitutes the generalization of \( f \)-I and I-Love relations to multipolar cases. The mass moment is then eliminated from these two formulas and hence the \( f \)-Love relation in the Newtonian limit is obtained.

### B. \( f \)-I relation

It can be shown from the variational method proposed by Chandrasekhar \[38\] that the frequency of the \( l \)-th multipole \( f \)-mode oscillation, \( \omega_0 \equiv \omega_l \) (the subscript “0” stands for \( f \)-mode and is suppressed unless otherwise stated), of compact stars is approximately given by \[68\]

\[
\omega_l^2 = \frac{2l(l-1)(2l-1)}{2l+1} \frac{\int_0^R \rho r^{2l-2} dr}{\int_0^R \rho r^{2l} dr}. \tag{4.5}
\]

Here \( p \) is the pressure at radius \( r \), which can be obtained from the hydrodynamic equilibrium condition (see, e.g., \[69\]). In \[43\] the quasi-incompressible fluid approximation that the Lagrangian displacement \( \xi_0 \propto \nabla (r^l Y_{lm}) \), where \( Y_{lm} \) is the spherical harmonic function, has been made. Such approximation stems from the observation that in \( f \)-mode oscillations of stiff stars the Lagrangian change in density is negligible except perhaps near the stellar surface. As shown in Table I, the accuracy of \( \omega_l^2 \) is excellent as long as the star concerned is stiff. For example, for polytropic stars with polytropic index \( N \) less than unity and a realistic star (with FPS EOS) the percentage error in \( \omega_l^2 \) \((l = 2 \) in the table) is less than 1.6\%. Meanwhile, the accuracy of the approximation deteriorates with increasing value of the polytropic index. As the polytropic index for typical NSs whose mass is greater than \( 1M_\odot \) is less than 1, the quasi-incompressible approximation is justified.

For GTM, whose the density is given by \[4.1\], \( \omega_l^2 \) acquires the following explicit form:

\[
\omega_l^2 = \frac{8\pi \rho_0 G l(l-1) \int_0^1 x^{2l} f(x, \delta) dx}{3} \frac{\int_0^R \rho r^{2l} dr}{\int_0^R \rho r^{2l+2} dr}; \tag{4.6}
\]

where \( f(y) = f(x = 1, y) \). As expected, \( \omega_l^2 \) is proportional to \( \rho_0 \). However, in general, the proportional constant is dependent on the value of \( \delta \), i.e., the EOS.

The central density \( \rho_0 \) in \[4.6\] and the stellar radius \( R \) can be expressed in terms of \( M \) and the \( n \)-th mass moment:

\[
I_n = \int_0^R \rho(r)r^{2+n} dr = \rho_0 R^{3+n} \frac{\int f(n + 3 \delta)}{n + 3}, \tag{4.7}
\]
It is interesting to note that to first-order in \( \delta \), \( (\bar{\omega}_2)^2 = A_l^{(0)} (M^3/I_3)^{3/2} \propto n^3 \), which can be considered as the Newtonian extension of the quadrupole f-I relation \( (2.11) \). In Fig. 9 (top panel), we show \( \Delta_\alpha \equiv A_l(\delta)/A_l^{(0)} - 1 \) as a function of \( \delta \). It is obviously that \( \Delta_\alpha \) is limited to a few percent for all physical values of \( \delta \). Thus, the validity of the f-I relation is established.

### C. I-Love relation

Similarly, we can evaluate the tidal deformability of GTM as follows. In the Newtonian limit, the metric coefficient \( H_l \) satisfies \( \bar{\omega}_l = \omega_l \equiv (r H_l/dr)_{r=R^+} = \frac{3}{2} \frac{dH_l}{H_l} \frac{dF}{F}. \) (4.13)

where

\[
\bar{\omega}_l = \frac{1}{(2l-1)!} \frac{l - y_l}{y_l}, \quad \text{and} \quad y_l = \frac{r \frac{dH_l}{H_l}}{r}. \quad \text{For the GTM considered above, Eq. (4.13) can be exactly solved,} \]

\[
H_l(r) = r^l F_l(a; b; d; \zeta), \quad (4.16)
\]

where \( 2F_1(a; b; d; \zeta) \) is the standard hypergeometric function, with

\[
a = \frac{1 + 2l - \Xi}{4}, \quad (4.17)
\]

\[
b = \frac{1 + 2l + \Xi}{4}, \quad (4.18)
\]

\[
\Xi = \sqrt{4l^2 + 4l + 41}, \quad (4.19)
\]

\[
d = \frac{3}{2} + l, \quad (4.20)
\]

\[
\zeta = \frac{3\delta^2}{5R^2}, \quad (4.21)
\]

and, as usual, regularity of \( H_l \) at the origin is assumed. Hence, we can analytically find \( y_l \),

\[
y_l = l - \frac{6\delta^2 F_1(a_1, b_1; d_1; \frac{3\delta}{5})}{(2l + 3) F_1(a, b; d; \frac{4\delta}{5})} - \frac{15(1 - \delta)}{5 - 3\delta}. \quad (4.22)
\]

### Table 1: \( R^l \omega^2_l/(GM) \), where \( \omega_2 \) is the quadrupole f-mode oscillation frequency, is shown for various stars, including polytropic stars with different polytropic indices \( N \), and a FPS NS (with central density \( 10^{14} \text{g cm}^{-3} \)). The three columns indicate the EOS of the star, the exact values of \( R^l \omega^2_l/M \) and the corresponding approximate values obtained from (4.5), respectively. Besides, the values in the brackets are the percentage errors of the approximation.

| EOS   | Exact Approximation (4.5) |
|-------|---------------------------|
| \( N = 3.0 \) | \( 5.880 \) 10.60 (80.3%) |
| \( N = 2.5 \) | \( 4.311 \) 5.722 (32.7%) |
| \( N = 2.0 \) | \( 3.026 \) 3.431 (13.4%) |
| \( N = 1.7 \) | \( 2.441 \) 2.635 (7.9%) |
| \( N = 1.5 \) | \( 1.320 \) 1.331 (0.8%) |
| \( N = 1.0 \) | \( 1.397 \) 1.402 (0.4%) |
| FPS   | \( 1.160 \) 1.164 (0.3%) |
|       | \( 1.160 \) 1.164 (0.3%) |
|       | \( 1.397 \) 1.402 (0.4%) |

where for simplicity the solid angle \( 4\pi \) is omitted in the definition of \( I_n \), as follows:

\[
\rho_0 = \frac{(n + 3)^{3/n} I_n^{1+3/n}}{3l^{1+3/n}} \left[ \hat{\omega}_l \right]^{1+3/n}, \quad (4.8)
\]

and, most interestingly, the coefficient of \( \delta \) in the above expansion vanishes if \( n = 2l - 2 \). Besides, it can also be shown that \( \delta = 1 \) is always a stationary point of \( \omega_2^l \) irrespective of the values of \( l \) and \( n \). Therefore we expect that the dependence of \( \omega_2^l \) on \( \delta \) is relatively weak if \( n = 2l - 2 \). Hence, we arrive at the universal “diagonal” multipolar f-I relation

\[
\bar{\omega}_l^2 = A_l(\delta) \left( \frac{M^{2l-1}}{I_{2l-2}} \right)^{3/2}, \quad (4.11)
\]

where the coefficient \( A_l(\delta) \) is to first-order independent of \( \delta \), and, in particular,

\[
A_l(\delta = 0) \equiv A_l^{(0)} = \frac{2l(l - 1)}{(2l + 1)^{3/2}} \left( \frac{3}{4\pi} \right)^{3/2}. \quad (4.12)
\]
where \( a_1 = a + 1, b_1 = b + 1, \) and \( d_1 = d + 1 \). Expressing \( R \) in terms of \( I_n \), we find that to first-order in \( \delta \),

\[
(2l - 1)!!M^{2l+1}\tilde{\lambda}_l \left[ \frac{(n+3)I_n}{3I_0} \right]^{\frac{2l+1}{2l}} = \left[ \frac{3}{2l - 2} + \frac{3(2l + 1)(2l - 2 - n)\delta}{5(l - 1)(2l + 3)(n + 5)} \right].
\]

(4.23)

If \( n = 2l - 2 \), to first-order in \( \delta \), \( \tilde{\lambda}_l \) is simply proportional to \( (I_n/M)^{(2l+1)/n} \) with a \( \delta \)-independent proportional constant. Such a case also holds if \( 1 - \delta \ll 1 \) for all values of \( n \). The situation is similar to the analysis developed previously for the f-I relation. As a consequence, the universal “diagonal” multipolar I-Love formula can be expressed as:

\[
\tilde{\lambda}_l = B_l \left( \frac{I_{2l-2}}{M^{2l-1}} \right)^{\frac{2l+1}{2l}},
\]

(4.24)

where \( B_l(\delta) \) depends only weakly on \( \delta \), with \( (dB_l/d\delta)_{\delta=0} = 0 \) and

\[
B_l(\delta = 0) = B_l^{(0)} = \frac{3}{2(l - 1)(2l - 1)!!} \left[ \frac{4\pi(2l + 1)}{3} \right]^{\frac{2l+1}{2l}}.
\]

(4.25)

For \( l = 2 \), Eq. (4.24) leads to \( \tilde{\lambda}_2 = B_2^{(0)}(I_2/M^3)^{5/2} \) to first-order in \( \delta \), which is in agreement with the result obtained in \([22]\). Figure 9 (middle panel) plots \( \Delta_d \) against \((\bar{\omega}_l^2 - \bar{\omega}_l^2)/(2D_l/M^3)\) to first-order in \( \delta \), which guarantees the accuracy of the multipolar I-Love relation. However, it is obvious that EOS-dependence of such diagonal relations grows gradually with increasing value of \( l \).

**D. f-Love relation**

Eliminating the mass moment \( I_{2l-2} \) from (4.11) and (4.24), we arrive at the Newtonian form of the diagonal multipolar f-Love universal relation:

\[
\bar{\omega}_l^2 \tilde{\lambda}_l^{\frac{2l+1}{2l}} = D_l(\delta),
\]

(5.1)

**V. F-I-LOVE RELATION**

Instead of eliminating the mass moment \( I_{2l-2} \), we can remove mass \( M \) from (4.11) and (4.24) to obtain an equation linking \( \lambda_l, \bar{\omega}_l^2 \) and \( I_{2l-2} \) together,

\[
\lambda_l \bar{\omega}_l^2 = \frac{4\pi l}{(2l - 1)!^l} I_{2l-2}.
\]

(5.1)

This equation, coined as the f-I-Love relation in the present paper, indeed connects three physical quantities, namely the f-mode frequency \( \bar{\omega}_l \), tidal deformation \( \lambda_l \),
and the mass moment \( I_{2l-2} \) together. It is noteworthy that each of them carries proper dimensions and the mass \( M \) completely disappears in (5.1). Comparing the f-I-Love relation (5.1) with the f-Love relation (4.26), the mass moment \( I_{2l-2} \) in the former actually plays the role of \( M \) in the latter.

We have derived the f-I-Love relation (5.1) from the GTM assumed above. However, to show the robustness of (5.1), in the following we apply the linear response theory (Green’s function method) to quasi-incompressible stars to develop an independent proof for it.

In Newtonian theory, the steady state response of a star to an external time-dependent potential \( U(\mathbf{r}) \exp(-i\omega t) \) of frequency \( \omega \) can be obtained from the Green’s function method to be detailed as follows. First of all, the normal modes of a star are defined by the solutions to the eigenvalue equation:

\[
(\mathcal{L} - \rho \omega_n^2) \xi_{ln} = 0, \tag{5.2}
\]

where \( \rho \) is the density distribution of the equilibrium state, \( \omega_n \) and \( \xi_{ln} \) are the eigenfrequency and the Lagrangian displacement of the \( n \)-th \( (n = 0, 1, 2, \ldots) \) oscillation mode carrying angular momentum \( l = 0, 1, 2, 3, \ldots \).

The \( l = 0 \) case corresponds to radial oscillation, \( l = 1 \) case usually represents translational motion, and for \( l \geq 2 \) cases the star undergoes non-radial oscillations. Unless otherwise stated explicitly, the \( z \)-component of angular momentum, \( m \), is suppressed in our equations. We adopt the convention where \( \omega_0 < \omega_1 < \omega_2, \ldots \). For barotropic stars, which is always assumed in our analysis, the 0-th mode is the \( f \)-mode. Besides, \( -\mathcal{L} \xi \) in general represents the net internal restoring force (including pressure force and the gravitational force due to the star itself) acting on a fluid element, which can be obtained from the Lagrangian displacement \( \xi \). The explicit form of \( \mathcal{L} \) can be found in [38]. Most importantly, \( \mathcal{L} \) is a hermitian operator guaranteeing the orthogonality relation of \( \xi_{ln} \) [38]:

\[
\langle \xi_{lj} | \xi_{ln} \rangle = \int_V \rho \xi_{lj}^* \cdot \xi_{ln} d^3r = \delta_{lj} \delta_{jn}, \tag{5.3}
\]

where \( V \) here denotes the volume of the star and the displacement fields are properly normalized to accord with (5.3).

In the presence of an external time-dependent gravitational potential \( U(\mathbf{r}) \exp(-i\omega t) \), the steady state linear response of the Lagrangian displacement \( \xi \) is given by solution of the inhomogeneous equation

\[
(\mathcal{L} - \rho \omega_n^2) \xi = -\rho \nabla U. \tag{5.4}
\]

The RHS of the above equation is actually the external gravitational force exerted on the star. Expanding \( \xi \) in terms of the complete set of \( \xi_{ln} \) and using (5.2), (5.3), we find that

\[
\xi = \sum_{l,n} \frac{\langle \xi_{ln} | \nabla U \rangle \xi_{ln}}{\omega_n^2 - \omega_n^2} = \sum_{l,n} B_{ln} \xi_{ln}. \tag{5.5}
\]

In particular, if the external potential is a time-independent tidal potential in the \( l \)-th \( (l \geq 2) \) multipolar sector, namely \( U_\delta = r^l Y_{lm}(\theta, \phi) \), where \( Y_{lm}(\theta, \phi) \) is the standard spherical harmonics of angles \( \theta \) and \( \phi \), a corresponding multipole moment defined by

\[
Q_l = \frac{4\pi}{2l+1} \int_V r^l \delta \rho Y_{lm}^* (\theta, \phi) d^3r \tag{5.6}
\]

with \( \delta \rho \) being the Eulerian change in density, is induced. The induced multipole moment in turn sets an additional potential \( U_\delta = Q_l Y_{lm}(\theta, \phi) / r^{l+1} \) outside the star.

In the linear regime response \( \delta \rho(r) = -\nabla \cdot (\rho \delta \xi) \). The expansion in (5.5) then readily leads to

\[
Q_l = \frac{4\pi}{2l+1} \sum_{l',n} B_{ln} \langle \nabla (r^l Y_{lm}) | \xi_{ln} \rangle = \frac{-4\pi}{2l+1} \sum_{l',n} \| \nabla (r^l Y_{lm}) \| \xi_{ln} \|^2 \tag{5.7}
\]

So far the calculation has been exact up to the leading order of the perturbing potential. For \( f \)-mode oscillations of quasi-incompressible stars, the Lagrangian change in the density vanishes and hence \( \nabla \cdot \xi_0 = 0 \). Taking into consideration the fact that \( \xi \) is derivable from a scalar potential (see, e.g., [60]), we arrive at the approximate formula

\[
\xi_0 = \nabla (r^l Y_{lm}) / \| \nabla (r^l Y_{lm}) \| (r^l Y_{lm})^{1/2}. \]

We note that such approximation has been put forward by Chandrasekhar [38] as the trial input of a variational principle, which is used in Section III to evaluate \( f \)-mode oscillation frequencies. By orthogonality of the normal modes, only the \( f \)-mode with the same angular momentum index \( l \) has to be included in the sum in (5.7), and after
integrating by parts, we show that
\[ Q_l \omega_{l0}^2 = -4\pi l I_{2l-2}. \] (5.8)
By virtue of (3.2), (5.8) and the definition of tidal Love number \( k_l \),
\[ k_l = -\frac{Q_l}{2 R^{2l+1}} = -\left( \frac{U_p}{2\rho_c} \right)_{r=R}, \] (5.9)
the general f-I-Love relation involving three dimensionless quantities \( M \omega_l \equiv \bar{\omega}_l, \) \( I_{2l-2}/M^{2l-1} \) and \( \lambda_l \)
\[ \lambda_l \bar{\omega}_l^2 = \frac{4\pi l}{(2l-1)!} \left( \frac{I_{2l-2}}{M^{2l-1}} \right) \] (5.10)
is established, which holds for sufficiently stiff stars and \( l = 2, 3, 4, \ldots \).

It is remarkable that (5.1) and (5.10) are in fact equivalent to each other. While the former is a consequence of f-I and I-Love relations, the latter is derived directly from the Green’s function method. Given the f-I-Love relation (5.10) obtained from the linear response theory, the two seemingly independent f-I and I-Love relations are in fact the consequence of each other for quasi-incompressible stars characterized by stiff EOSs. Hence, the inter-relationship between the f-I and I-Love relations is clearly shown from the f-I-Love relation.

VI. CONCLUSION AND DISCUSSION

Although the f-I relation and the I-Love-Q relations have recently been discovered in the quadrupolar sector and various potential applications of them have been proposed \([20, 22, 23]\), the reasons for the validity of these two relations and their inter-relationship are not yet fully understood. Yagi and Yunes \([22, 23]\) suggested two possible reasons for the I-Love-Q relations: (i) the relations are most sensitive to the stellar matter in an outer layer between 70% and 90% of the radius of a NS and the EOS there is quite unified; and (ii) the internal stellar structure of NSs is gradually effaced as the black-hole limit is approached and hence NSs reveal similar behavior. On the other hand, more recently Yagi et al. \([70]\) found that the I-Love-Q relations are actually dominated by the outer-core lying in a region bounded between 50% and 90% of the stellar radius. However, given this finding, the I-Love-Q relations can no longer be attributed to the similarity of EOSs in the low-density regime. In addition, both f-I relation and the I-Love-Q relations are valid for QSs \([20, 27]\), whose EOS and stellar structure are completely different from those of NSs especially in the outer layer. Formally speaking, the outer layer of bare QSs can be considered as incompressible, while that of NSs is rather soft with an adiabatic index of about 1.4.

In the present paper we perform an in depth examination on the relationship between the f-I relation and the I-Love-Q relations, in turn propose a robust multipolar f-Love relation, and study the physical origin of these universal relations for compact stars in multipolar distortions. The multipolar f-Love relation discovered here is generally valid in any angular momentum sector with \( I \geq 2 \) (albeit EOS-dependence increases with \( I \)) and for realistic compact stars (including both NSs and QSs) constructed with different prevailing nuclear EOSs. We pinpoint that such universal behavior of realistic stars indeed follows closely that of incompressible stars, which are chosen as the standard to benchmark other stars. As shown in Figs. \([58]\) as long as the polytropic index \( N \) of a star is not greater 1, the fractional deviation in \( f \)-mode frequency, as compared with the incompressible counterpart, is less than 2% and is inert to changes in \( N \). Therefore, we claim here that the stiffness of nuclear matter at large densities is the crux of these multipolar universal relations.

Through the GTM, which is able to mimic realistic stars with \( N \leq 1 \), we carry out detailed Newtonian analysis to show that both \( \omega_l \) and \( \lambda_l \) are related to \( I_{2l-2} \), the \((2l-2)\)-th mass moment, in a way insensitive to changes in the parameter \( \delta \). Hence, after eliminating \( I_{2l-2} \) from the f-I and I-Love relations, the Newtonian form of the diagonal multipolar f-Love relation \((4.26)\) is readily established, which provides a strong support to the relativistic f-Love relation discovered here. On the other hand, we also use the linear response theory to derive a f-I-Love relation for Newtonian stars with high stiffness (say, \( N \leq 1 \)). Any two of the f-I, I-Love and f-I-Love relations will imply the validity of the other one. More interestingly, in the f-Love relation \((4.26)\), the \( f \)-mode frequency and the tidal deformability are related with the mass as a parameter in the formula. However, in the f-I-Love relation \((5.11)\), the \( f \)-mode frequency, the tidal deformability and the \((2l-2)\)-th mass moment are directly linked together in the absence of the knowledge of the mass.

In general, for NSs far from their upper and lower mass limits, they are stiff enough to guarantee the universal formulas studied here. However, deviations from these universal formulas for stars with masses close to either of these two mass limits could arise for the following reasons. Near the maximum mass limit, the strong gravitational attraction effectively softens the nuclear matter and hence the star can no longer be approximated by an quasi-incompressible star. From Figs. \([58]\) we can see that the magnitude of \( E \) gradually grows larger and displays stronger dependence on the polytropic index as the star concerned approaches the maximum mass limit. In fact, such softening mechanism seems to reduce the adiabatic index by an amount of the order of the compactness of the star and hence, following directly from \([34]\), leads to larger deviation form the incompressible star. Notwithstanding this, the universal formulas still hold around the maximum mass limit because the softening effect will at the same time destabilize the star. As a result, the star becomes unstable before it could further deviate from the universal formulas. On the other hand, near the low mass limit, the star is primarily made of soft...
nuclear matter with adiabatic index $\Gamma \approx 1.4$ and therefore deviations from the behavior of incompressible stars are expected and have been observed (see, e.g., Fig. 9 of [27]). However, there is not much uncertainty in the EOS of low-density NS nuclear matter, which is well studied. Such deviations are unimportant and merely lead to modifications of the universal formulas instead of breaking them. On the other hand, NSs and QSs behave differently in the low-mass limit and hence do not follow the same universal formulas. Researchers could make use of the difference in the universal trends obeyed respectively by NSs and QSs in the low-mass limit to distinguish these two kinds of compact stars.

Finally, we note that two kinds of relations have been studied here, namely the diagonal and the off-diagonal ones. In the former case, the scaled $f$-mode frequency $\tilde{\omega}$ and the scaled tidal deformability $\tilde{\lambda}$ with the same angular momentum index $l$ are linked together in an almost EOS-independent way. In the latter case, $\tilde{\omega}$ is related to $\tilde{\lambda}$ with $l \neq l'$. However, as shown in Figs. 1-8, such off-diagonal relations usually display stronger EOS-dependence. On the other hand, by combining these diagonal and off-diagonal relations, other off-diagonal relations, such as $\tilde{\omega}_l$ against $\tilde{\omega}_{l'}$, or $\tilde{\lambda}_l$ against $\tilde{\lambda}_{l'}$, with $l \neq l'$ can be readily established. In fact, the multipole Love relations discovered by Yagi [27], which connect tidal deformabilities of different angular momenta together, is an example of such off-diagonal relations. We end our paper by showing the other off-diagonal relation connecting $\tilde{\omega}_l$ of two different $l$ in Fig. 11 where $\tilde{\omega}_l/\tilde{\omega}_2$ ($l = 3, 4, 5$) is plotted against $\tilde{\omega}_2$ for incompressible stars ($N = 0$), and polytropic stars with $N = 0.67, 1.0$. As expected, EOS-dependence of these off-diagonal relations become more obvious, especially for cases with larger difference in the two angular momentum indices.

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FIG. 1: In the upper panel of each figure the scaled $f$-mode frequency $\bar{\omega}_l$ for cases with $l = 2$ (left-top), 3 (left-bottom), 4 (right-top), and 5 (right-bottom) is plotted against the logarithm of the scaled tidal deformability $\bar{\lambda}_2$, while in the lower panel the corresponding fractional deviation $E$ from incompressible stars in each case is shown. The data are obtained from five realistic NSs (including AU, UU, WS, BBB2 and FPS EOSs), QSs, and incompressible stars (indicated by the solid line).
FIG. 2: In the upper panel of each figure the scaled $f$-mode frequency $\bar{\omega}$ for cases with $l = 2$ (left-top), 3 (left-bottom), 4 (right-top), and 5 (right-bottom) is plotted against the logarithm of the scaled tidal deformability $\bar{\lambda}$, while in the lower panel the corresponding fractional deviation $E$ from incompressible stars in each case is shown. The symbols and EOSs used in the figure have been defined in Fig. 1.
FIG. 3: In the upper panel of each figure the scaled $f$-mode frequency $\bar{\omega}_l$ for cases with $l = 2$ (left-top), 3 (left-bottom), 4 (right-top), and 5 (right-bottom) is plotted against the logarithm of the scaled tidal deformability $\bar{\lambda}_4$, while in the lower panel the corresponding fractional deviation $E$ from incompressible stars in each case is shown. The symbols and EOSs used in the figure have been defined in Fig. 1.
FIG. 4: In the upper panel of each figure the scaled $f$-mode frequency $\bar{\omega}_l$ for cases with $l = 2$ (left-top), 3 (left-bottom), 4 (right-top), and 5 (right-bottom) is plotted against the logarithm of the scaled tidal deformability $\bar{\lambda}_5$, while in the lower panel the corresponding fractional deviation $E$ from incompressible stars in each case is shown. The symbols and EOSs used in the figure have been defined in Fig. 1.

FIG. 5: The fractional deviation $E$ from incompressible stars in $\bar{\omega}_l$ ($l = 2, 3, 4, 5$) is plotted against $\ln \bar{\lambda}_2$ for polytropic stars with adiabatic index $\Gamma = 1.8, 2, 2.2, 2.5$. 
FIG. 6: The fractional deviation $E$ from incompressible stars in $\bar{\omega}_l$ ($l = 2, 3, 4, 5$) is plotted against $\ln \bar{\lambda}_3$ for polytropic stars with adiabatic index $\Gamma = 1.8, 2, 2.2, 2.5$.

FIG. 7: The fractional deviation $E$ from incompressible stars in $\bar{\omega}_l$ ($l = 2, 3, 4, 5$) is plotted against $\ln \bar{\lambda}_4$ for polytropic stars with adiabatic index $\Gamma = 1.8, 2, 2.2, 2.5$. 
FIG. 8: The fractional deviation $E$ from incompressible stars in $\tilde{\omega}$ ($l = 2, 3, 4, 5$) is plotted against $\ln \tilde{\lambda}_5$ for polytropic stars with adiabatic index $\Gamma = 1.8, 2, 2.2, 2.5$.

FIG. 9: Plots of $\Delta_a$ (top), $\Delta_b$ (middle) and $\Delta_d$ (bottom) versus $\delta$ for GTM with $l = 2, 3, 4$ and $5$. In all cases, $\Delta$'s are measured in percent.
FIG. 10: Plot of $2 \ln \bar{\omega}_l$ against $[3/(2l+1)] \ln \bar{\lambda}_l$ for incompressible stars with $l = 2, 3, 4$ and 5.

FIG. 11: Plot of $\bar{\omega}_l/\bar{\omega}_2$ against $\bar{\omega}_2$ with $l = 3$ (black lines), $l = 4$ (red lines), and $l = 5$ (green lines) for incompressible stars with $N = 0$ (solid lines), and polytropic stars with $N = 0.67$ (dashed lines) and $N = 1.0$ (dot-dashed lines).