1 Introduction

Let $X$ be a complex algebraic variety, and $X^\circ \subset X$ be the smooth part of $X$. Consider the scheme $\mathcal{L}(X)$ of formal arcs in $X$. The $\mathbb{C}$-points of $\mathcal{L}(X)$ are just maps $D = \text{Spec } \mathbb{C}[[t]] \to X$ (see, for example, [DL] for a definition of $\mathcal{L}(X)$ as a scheme).

Let $\mathcal{L}^\circ(X)$ be the open subscheme of arcs whose image is not contained in $X \setminus X^\circ$. Fix an arc $\gamma : D \to X$ in $\mathcal{L}^\circ(X)$, and let $\mathcal{L}(X)_{\gamma}$ be the formal neighborhood of $\gamma$ in $\mathcal{L}(X)$. The purpose of this paper is to prove the following.

**Theorem 1.1** There exists a scheme $Y = Y(\gamma)$ of finite type over $\mathbb{C}$, and a point $y \in Y(\mathbb{C})$, such that:

$$\mathcal{L}(X)_{\gamma} \cong Y_y \times D^\infty,$$

the formal neighborhood of $y$ in $Y$, times a product of countably many copies of $D$.

The finite-dimensional piece $Y_y$ may be called the parameter space of a versal deformation of $\gamma$. It gives a model for the singularity of $\mathcal{L}(X)$ at $\gamma$. It is not hard to check that the analytic germ $(Y, y)$ is determined uniquely up to multiplying by a finite-dimensional vector space. We note, as a pitfall, that the embedding $Y_y \to \mathcal{L}^\circ(X)$ given by Theorem 1.1 does not extend to a map of the analytic germ $(Y, y) \to (\mathcal{L}^\circ(X), \gamma)$. Intuitively, this is because the versal deformation captures the breaking-up of singularities, but the formal disc has only one point.

The study of formal arcs in an algebraic variety was originated by J. Nash in [Na]. It has been a focus of much recent activity (see [Ba], [DL]). However, this recent work has mostly concentrated on the spaces of truncated arcs.

We expect the singularities of the spaces $Y(\gamma)$ to arise in many problems involving maps of smooth curves into singular varieties, or other objects which can be locally described as such maps. The original motivating example of this is the work of Feigin, Finkelberg, Kuznetsov, and Mirković (see [K], [FK], [FFKM], and their...
references) on quasimap spaces $Q^D_\alpha$. Conjecturally, the singularities of the $Q^D_\alpha$ are the same as those of $\mathcal{L}^\alpha(X)$, where $X = \overline{G/U}$ is the affine closure of the quotient of an algebraic group by the unipotent radical of a Borel.

Theorem 1.1 was conjectured by V. Drinfeld [Dr]. Drinfeld also proved a weaker version of it, in which the finite-dimensional piece $Y_y$ is not identified as the formal neighborhood of a point of a scheme of finite type.

The paper is organized as follows. In Section 2 we prove a lemma about plane curves depending on parameters on which the proof of Theorem 1.1 is based. Theorem 1.1 is proved in Section 3, and Section 4 contains a few simple examples.

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A notational convention. Throughout this paper, a test-ring $A$ is a finite-dimensional, local commutative $\mathbb{C}$-algebra with 1. If $S$ is a scheme over $\mathbb{C}$, and $p \in S(\mathbb{C})$ is a $\mathbb{C}$-point, we think of the formal neighborhood $S_p$ in terms of its functor of points: $A \mapsto S_p(A)$, from test-rings to sets.

2 Plane Curves Depending on Parameters

In this section, we prove a lemma about plane curves depending on parameters which will be used in the proof of Theorem 1.1. Let $f(x,y)$ be a polynomial vanishing at the origin. Denote the curve $\{f(x,y) = 0\}$ by $C$. Let $U$ be a small neighborhood of 0 in $\mathbb{C}^2$. Assume that $f$ has no critical points in $U \setminus \{0\}$. In addition, assume that $C \cap U = C_1 \cup C_2$, where $C_1$ is a smooth analytic branch of $C$, which is tangent to the $x$-axis at 0, and $C_1 \cap C_2 = \{0\}$. In other words, we have $f|_U = f_1 \cdot f_2$, where $f_1, f_2 : U \to \mathbb{C}$ are analytic functions, the differential $d_0 f_1 = y$, and $f_1(x,y) = f_2(x,y) = 0$ implies $x = y = 0$.

Let $m$ be the multiplicity of 0 as a point of intersection of $C_1$ and $C_2$. In other words, $m$ is the order of vanishing of the restriction $f_2|_{C_1}$ at 0. Pick a large integer $M$, and let $\mathbb{C}[x,y]^{\leq M} \subset \mathbb{C}[x,y]$ be the affine space of all polynomials of degree less than $M$. Denote by $P$ a small analytic neighborhood of $f$ in $\mathbb{C}[x,y]^{\leq M}$. Let $R \subset P$ be the locus of $g \in P$, such that the curve $\{g(x,y) = 0\}$ has an analytic branch near the origin which is $C^\infty$-close to $C_1$. We may choose a neighborhood $D$ of zero in $\mathbb{C}$, such that for any $g \in R$, there is a unique analytic function $h_g : D \to \mathbb{C}$ satisfying $g(t, h_g(t)) = 0$ for all $t \in D$. Let $b_k(g)$ be the $k$-th coefficient of the Taylor series for $h_g$, so that $h_g(t) = \sum_{k=0}^{\infty} b_k(g) \cdot t^k$. Finally, let $R_f \subset P_f$ be the formal neighborhoods of $f$ in $R$ and $P$.

Lemma 2.1 (i) $R$ is a smooth analytic subvariety of $P$, of codimension $m$.

(ii) $R$ is a local analytic branch of some algebraic subvariety of $\mathbb{C}[x,y]^{\leq M}$.

(iii) The functions $b_k : R \to \mathbb{C}$ are complex analytic.
(iv) Let $A$ be a test-ring with maximal ideal $m$. Then $R_f(A)$ is the set of all $\bar{f} \subset A[x,y]_{\leq M}$ such that $\bar{f} \equiv f \pmod{m}$, and there exists an $h \in A[[t]]$ satisfying $\bar{h} \equiv h_f \pmod{m}$ and $\bar{f}(x,\bar{h}(x)) \equiv 0$.

Our proof of Lemma 2.1 will use the following basic result of Tougeron (see [To]).

**Lemma 2.2** [AVG, Part I, Ch 6.3] Let $f : (\mathbb{C}^r,0) \to (\mathbb{C},0)$ be a germ of a complex analytic function with an isolated singularity at the origin. Then there exists an $n \in \mathbb{Z}_+$, such that any germ $f' : (\mathbb{C}^r,0) \to (\mathbb{C},0)$, with the same $n$-jet as $f$, is analytically right-equivalent to $f$. \hfill \Box

Both [To] and [AVG] state Lemma 2.2 for real $C^\infty$ functions. However, the proof in [AVG] goes through word-for-word in the complex analytic setting.

**Proof of Lemma 2.1:** The set $P$ gives a right-versal deformation of the singularity of $f$ at the origin (see [AVG] for a discussion of versal deformations of singularities of functions). Therefore, to prove (i), (iii), and (iv), we can replace $f$ by any polynomial $f'$ which satisfies the hypothesis of the lemma, and is analytically right-equivalent to $f$ near zero. Using Lemma 2.2, it is not hard to check that such an $f'$ can be chosen to be divisible by $y$. In other words, in the proof of parts (i), (iii), and (iv), we can assume that $C_1$ is the $x$-axis. We proceed with this assumption.

Part (iii) is a combination of the Cauchy integral formula and the implicit function theorem. To be precise, fix a small $\epsilon > 0$, and let $S_\epsilon$ be the $\epsilon$-circle in the $x$-axis. Then the $\{b_k(g)\}$ can be computed as the positive Fourier coefficients of the restriction $h_g|_{S_\epsilon}$. But this restriction depends analytically on $g$, by the implicit function theorem.

Note that this argument shows that we have a uniform bound on the size of the $b_k = b_k(g)$ in terms of the size of $g - f$. More precisely, let

$$g(x,y) = f(x,y) + \sum_{i,j} c_{i,j} x^i y^j$$

$(0 \leq i,j \leq M)$. Then there exists a $\sigma > 0$ such that:

$$|b_k| < \sigma \cdot \epsilon^{-k} \cdot \max_{i,j} |c_{i,j}|,$$

(1)

for any $g \in R$, and any $k \in \mathbb{Z}_+$.

Turning now to part (i), let $f(x,y) = \sum_{i,j} a_{i,j} x^i y^j \ (a_{i,j} \in \mathbb{C})$. By assumption, $a_{i,0} = 0$ for all $i$, and $a_{i,1} = 0$ for $i < m$. Our proof of part (i) is based on analyzing the identity $g(x,h_g(x)) = 0$. Specifically, setting each coefficient of the power series $g(x,h_g(x)) \in \mathbb{C}[x]$ to zero, we obtain an infinite system of equations in the $\{c_{i,j}\}$ and the $\{b_k\}$. Let us inspect the first few equations of this system. We write $O(b^2)$ for terms that contain a product of at least two of the $b_k$. 

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\[ c_{0,0} + c_{0,1} b_0 + O(b^2) = 0 \]
\[ c_{1,0} + c_{1,1} b_0 + c_{0,1} b_1 + O(b^2) = 0 \]
\[ c_{2,0} + c_{2,1} b_0 + c_{1,1} b_1 + c_{0,1} b_2 + O(b^2) = 0 \]
\[ \ldots \]
\[ c_{m,0} + a_{m,1} b_0 + c_{m,1} b_0 + c_{m-1,1} b_1 + \ldots + c_{0,1} b_m + O(b^2) = 0 \]
\[ c_{m+1,0} + a_{m+1,1} b_0 + c_{m+1,1} b_0 + c_{m,1} b_1 + \ldots + c_{0,1} b_{m+1} + O(b^2) = 0 \]
\[ \ldots \]

(2)

We are interested in finding small solutions \( \{c_{i,j}, b_k\} \) of the system (2). For this, let us first fix a test-ring \( A \) with maximal ideal \( \mathfrak{m} \), and look for solutions \( \{c_{i,j}, b_k\} \) in \( \mathfrak{m} \) (while the \( a_{i,j} \) are still in \( \mathbb{C} \)). Let \( I = \{0, \ldots, M\}^2 \setminus \{0, \ldots, m - 1\} \times \{0\} \). Then it is obvious from inspecting the system (2) that it has a unique solution for any \( \{c_{i,j} \in \mathfrak{m}\}_{(i,j) \in I} \). Indeed, we can first construct the solution modulo \( \mathfrak{m}^2 \) by solving the first equation for \( c_{0,0} \), the second equation for \( c_{1,0} \), \ldots, the \( m \)-th equation for \( c_{m-1,0} \), the \((m+1)\)-st equation for \( b_0 \), the \((m+2)\)-nd equation for \( b_1 \), and so on. Note that at this stage we have \( c_{0,0}, c_{1,0}, \ldots, c_{m-1,0} \in \mathfrak{m}^2 \). Once the solution is constructed modulo \( \mathfrak{m}^2 \), we can go back to the first equation and solve for \( c_{0,0} \) modulo \( \mathfrak{m}^3 \), and so on.

This observation means that there are power series \( s_0, \ldots, s_{m-1} \in \mathbb{C}[\{c_{i,j}\}_{(i,j) \in I}] \), such that for any solution \( \{c_{i,j}, b_k\} \) of (2) in the maximal ideal of any test-ring, we have:

\[ c_{l,0} = s_l(c_{i,j})_{(i,j) \in I}, \quad \text{for } l = 0, \ldots, m - 1. \]

(3)

It is a straightforward exercise in combinatorics to show that the coefficients of the power series \( s_l \) can not grow super-exponentially, and that the \( s_l \) all converge in some neighborhood of zero in \( \mathbb{C}^I \). It follows that the locus \( R \subset P \) is the submanifold defined by the equations (3), where the \( c_{i,j} \) are now small complex numbers. The passage from the formal solution of (2) to the solution in small \( \{c_{i,j}, b_k\} \) presents no difficulties because of the estimate (1). This completes the proof of (i).

For part (iv), note that

\[ P_f(A) = \{ \tilde{f} \subset A[x,y]^{\leq M} \mid \tilde{f} \equiv f \pmod{\mathfrak{m}} \}. \]

Any \( \tilde{f} \in P_f(A) \) can be written as \( \tilde{f}(x,y) = f(x,y) + \sum_{i,j} c_{i,j} x^i y^j \), with \( c_{i,j} \in \mathfrak{m} \)

(0 \leq i, j \leq M). The point \( \tilde{f} \) is in \( R_f(A) \) if and only if the \( \{c_{i,j}\} \) satisfy equations (2). By construction, this means that there exist \( \{b_k \in \mathfrak{m}\}_{k \in \mathbb{Z}^+} \), such that \( \{c_{i,j}, b_k\} \) is a solution of (2). Setting \( \tilde{h} = \sum_{k=0}^{\infty} b_k \cdot t^k \) proves part (iv).
For part (ii), let $Z \subset \mathbb{C}[x,y]^{\leq M}$ be the set of all polynomials $g(x,y)$ such that the curve $\{g(x,y) = 0\}$ has at least $m$ singular points (counted with multiplicities). Then $Z$ is a closed subvariety of $\mathbb{C}[x,y]^{\leq M}$, containing $R$. A generic point $g \in R$ corresponds to a curve with $m$ simple double points $q_1, \ldots, q_m$ near the origin. Let $\tilde{g} \in \mathbb{P}$ be a polynomial very near $g$. For each of the $q_i$, the condition on $\tilde{g}$ that the curve $\{\tilde{g}(x,y) = 0\}$ has a double point near $q_i$ is smooth of codimension 1. Furthermore, if $M$ is sufficiently large, these conditions for different $q_i$ are independent. It follows that the codimension of $Z$ at $g$ is $m$. Therefore, $R$ is a local analytic branch of $Z$. □

Remark 2.3 The proof above shows that the tangent space $T_f R \subset T_f \mathbb{P}$ is the coordinate plane $c_0, \ldots, c_m = 0$.

3 Proof of Theorem 1.1

We break the proof up into six steps.

**Step 1: Approximating by analytic arcs.** Without loss of generality, we may assume that $X$ is a closed subvariety of a finite-dimensional vector space $U$, and that $\gamma(0)$ is the origin. Below, we state three preliminary lemmas whose proofs are routine. For $n \in \mathbb{Z}_+$, let $D_n = \text{Spec} \mathbb{C}[t]/t^{n+1}$, and let $\mathcal{A}(X,\gamma,n)$ be the set of all arcs $\alpha : D \to X$ which agree with $\gamma$ on $D_n$.

**Lemma 3.1** For any $n \in \mathbb{Z}_+$, there exists an analytic $\alpha \in \mathcal{A}(X,\gamma,n)$.

**Lemma 3.2** There is an $n(\gamma) \in \mathbb{Z}_+$, such that the limit $\lim_{t \to 0} T_{\alpha(t)} X$ is the same for all analytic $\alpha \in \mathcal{A}(X,\gamma,n(\gamma))$. We call this limit $V = V(\gamma)$; it is a linear subspace of $U$ of dimension $d = \dim X$.

Choose a direct sum decomposition $U = V \oplus W$. Let $p : U \to V$ be the projection along $W$, and $p_* : \mathcal{L}(X) \to \mathcal{L}(V)$ be the map induced by $p$. We will use the abbreviation $p_* \gamma = p\gamma$. Denote by $\mathcal{A}(V,p\gamma,n)$ the set of all arcs $\beta : D \to V$ which agree with $p\gamma$ on $D_n$.

**Lemma 3.3** The number $n(\gamma)$ in Lemma 3.2 can be chosen so that, for any choice of $W$, the map $p_*$ induces a bijection $\mathcal{A}(X,\gamma,n(\gamma)) \cong \mathcal{A}(V,p\gamma,n(\gamma))$ which takes analytic arcs to analytic arcs.

**Proof:** Fix a Hermitian metric on $U$. Let $B(x,r)$ ($x \in X$, $r > 0$) be the $r$-ball around $x$. It suffices to choose $n(\gamma)$ so that for some analytic $\alpha \in \mathcal{A}(X,\gamma,n(\gamma))$, and every small $t \neq 0$, the restriction of $p$ to $X \cap B(\alpha(t), |t|^n(\gamma)-1)$ is injective. □
Step 2: Truncating the arc. Since $V$ is a vector space, an arc in $V$ has well-defined higher derivatives. Therefore, for any $N \in \mathbb{Z}_+$, we can write

$$\mathcal{L}(V) = \mathcal{L}(V)^{\leq N} \times \mathcal{L}(V)^{> N},$$

where $\mathcal{L}(V)^{\leq N} = \{ \beta : D \to V \mid \beta^{(n)}(0) = 0, \text{ for } n > N \}$, and $\mathcal{L}(V)^{> N} = \{ \beta : D \to V \mid \beta^{(n)}(0) = 0, \text{ for } n \leq N \}$.

Lemma 3.4 For any sufficiently large $N$, there exists an analytic arc $\alpha : D \to X$ such that:

(i) $\alpha$ agrees with $\gamma$ on $D_N$;
(ii) $p_* \alpha \in \mathcal{L}(V)^{\leq N}$; and
(iii) $\mathcal{L}(X)_\alpha$ is isomorphic to $\mathcal{L}(X)_\gamma$.

Proof: The idea is to use the fact that $\mathcal{L}(X)$ has a very rich set of symmetries. Specifically, any map $t \mapsto \xi_t$ from the formal disc to the vector fields on $X$ produces a vector field $\hat{\xi}$ on $\mathcal{L}(X)$. Such a vector field, in turn, produces a time-1 flow map $\Xi_1 : \mathcal{L}(X)^{\hat{\xi}} \to \mathcal{L}(X)$. This $\Xi_1$ should be understood as follows. Let $X^{\hat{\xi}}(\mathbb{C})$ be the set of $\mathbb{C}$-points of $X$ where the time-1 flow of $\xi_0$ is defined (it is an open subset of $X(\mathbb{C})$ in the analytic topology). Let now $A$ be a test-ring. Denote by $\mathcal{L}(X)^{\hat{\xi}}(A) \subset \mathcal{L}(X)(A)$ the set of all maps $\text{Spec } A \times D \to X$ which send the unique $\mathbb{C}$-point of the domain to $X^{\hat{\xi}}(\mathbb{C})$. Then $\Xi_1$ is a natural transformation between the functor $A \mapsto \mathcal{L}(X)^{\hat{\xi}}(A)$ and the functor $A \mapsto \mathcal{L}(X)(A)$.

Pick a set of coordinates $(u_1, \ldots, u_d)$ on $V$, and consider the basic vector fields $(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_d})$. They lift to meromorphic vector fields $(p^* \frac{\partial}{\partial u_1}, \ldots, p^* \frac{\partial}{\partial u_d})$ on $X^\circ$. We may choose regular functions $(g_1, \ldots, g_d)$ on $X$ to ‘cancel out the poles’ of $(p^* \frac{\partial}{\partial u_1}, \ldots, p^* \frac{\partial}{\partial u_d})$, so that each $\eta_i = g_i \cdot p^* \frac{\partial}{\partial u_i}$ is a regular vector field on $X$. Note that the image of $\gamma$ is not contained in the closure of the critical set of the restriction of $p$ to $X^\circ$. This means that the $g_i$ can be chosen not to vanish identically along $\gamma$. Let $n_i$ be the order of vanishing of $g_i$ along $\gamma$, that is, the smallest $n$ such that the $n$-th coefficient $(g_i \circ \gamma)_n \neq 0$. Set $N_0 = \max(n_1, \ldots, n_d, n(\gamma))$, where $n(\gamma)$ is as in Step 1. The claims of the lemma will hold for any $N \geq N_0$. Fix such a number $N$. We construct the arc $\alpha$ in a series of $d$ steps. As a first approximation we take $\alpha_0 = \gamma$. The first step is accomplished by the following claim.

Claim: There is a unique $f \in \mathbb{C}[[t^{N+1-n_1}]]$ with the following property. Define a family $\xi_t$ of vector fields on $X$ by $\xi_t = f(t) \cdot \eta_1$. Let $\hat{\xi}$ be the corresponding vector field on $\mathcal{L}(X)$, and $\Xi_1$ be the time-1 flow of $\hat{\xi}$. Let $\alpha_1 = \Xi_1(\alpha_0)$. Then $\alpha_1$ agrees with $\alpha_0$ on $D_N$, and $u_1 \circ p_* \alpha_1$ is a polynomial of degree $\leq N$.

To prove the claim, we construct the power series

$$f = \sum_{i=N+1-n_1}^{\infty} f_i \cdot t^i$$

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inductively, coefficient by coefficient. Let \((g_1 \circ \gamma)_{n_1}\) be the first non-zero coefficient of \(g_1 \circ \gamma\), and \((u_1 \circ p_\ast \gamma)_{N+1}\) be the \((N + 1)\)-st coefficient of \(u_1 \circ p_\ast \gamma\). We set

\[
f_{N+1-n_1} = - \frac{(u_1 \circ p_\ast \gamma)_{N+1}}{(g_1 \circ \gamma)_{n_1}}.
\]

This ensures that \((u_1 \circ p_\ast \alpha_1)_{N+1} = 0\). Once \(f_{N+1-n_1}\) is fixed, the next coefficient, \(f_{N+2-n_1}\), is determined uniquely by the requirement \((u_1 \circ p_\ast \alpha_1)_{N+2} = 0\), and so on. Note that the time-1 flow map \(\Xi_1\) restricts to an isomorphism of formal neighborhoods: \(\mathcal{L}(X)_{\alpha_0} \cong \mathcal{L}(X)_{\alpha_1}\).

The next \(d - 1\) steps of the construction are completely analogous. Each time we obtain an arc \(\alpha_k\) \((k = 2, \ldots, n)\) which agrees with \(\alpha_{k-1}\) on \(D_N\), and satisfies:

\[u_i \circ p_\ast \alpha_k = u_i \circ p_\ast \alpha_{k-1}, \text{ for } i \neq k, \text{ and } u_k \circ p_\ast \alpha_k \text{ is a polynomial of degree } \leq N.\]

In the end, we obtain the required arc \(\alpha = \alpha_d\). Claims (i)-(iii) of the lemma follow from the construction, and the complex analyticity of \(\alpha\) follows from Lemma 3.3. □

**Step 3: General position assumptions.** Using Lemma 3.4, we can assume from now on that the arc \(\gamma\) is analytic, and that the image of \(\gamma\) is a local branch of some algebraic curve. For the arguments in Steps 5 and 6, we will also need to assume that the complement \(W\) is in general position with respect to \(\gamma\). The purpose of Step 3 is to specify this general position assumption.

Given any \(x_1 \neq x_2\) in \(X\), denote by \(L(x_1, x_2) \subset U\) the straight line through \(x_1\) and \(x_2\). For \(x \in X\), let

\[K(x) = \bigcup_{x' \in X \setminus \{x\}} L(x, x') \subset U,
\]

and let \(\bar{K}(x)\) be the closure of \(K(x)\) in \(U\). Given an analytic arc \(\alpha \in \mathcal{L}(X)\), let \(K(\alpha) = \lim_{t \to 0} \bar{K}(\alpha(t)) \subset U\). Note that \(K(\alpha)\) automatically contains \(K(\alpha(0))\).

**Lemma 3.5** The set \(K(\gamma)\) is a closed algebraic cone, with \(\dim K(\alpha) \leq d + 1\). □

Given an analytic arc \(\alpha \in \mathcal{L}(X)\), we denote by \(D_\alpha \subset \mathbb{C}\) a small disc around the origin, such that \(\alpha\) can be viewed as a map \(D_\alpha \to X\). We also let \(D_\alpha^0 = D_\alpha \setminus \{0\}\).

**Lemma 3.6** For a suitably generic choice of the complement \(W\) to \(V\) in \(U\), there is a positive integer \(N = N(W)\), such that for any analytic \(\alpha \in \mathcal{A}(X, \gamma, N)\), there is a neighborhood \(U\) of 0 in \(U\), such that:

(i) \(\dim(W \cap K(\alpha)) \leq 1\).

(ii) \(W \cap X \cap U = \{0\}\).

(iii) The intersection \(p^{-1}(p \circ \alpha(D_\alpha^0)) \cap X \cap U\) in contained in \(X^\circ\).

(iv) The intersection in part (iii) is transverse. That is, \(p^{-1}(p \circ \alpha(D_\alpha^0)) \cap U\) is transverse to \(X^\circ \cap U\) as (non locally closed) submanifolds of \(U\).
Proof: This is a standard general position argument. The complement \( W \) is chosen as follows. Let \( C(X) \subset U \) be the tangent cone of \( X \) at zero; it is a conical subvariety of \( U \) of dimension \( d \). Write \( X^\text{sing} = X \setminus X^0 \). For \( x \in X \), let
\[
S(x) = \bigcup_{x' \in X^\text{sing} \setminus \{x\}} L(x, x') \subset U,
\]
and let \( \bar{S}(x) \) be the closure of \( S(x) \) in \( U \). Let \( S(\gamma) = \lim_{t \to 0} \bar{S}(\alpha(t)) \). By analogy with Lemma 3.5, \( S(\gamma) \) is a closed algebraic cone, with \( \dim S(\gamma) \leq d \).

Let \( c = \dim U - \dim V \). Define \( P^0 \) to be the set of all affine \( c \)-planes in \( U \) which have a point of non-transverse intersection with \( X^0 \). Let \( P \) be the closure of \( P^0 \) in the Grassmannian of all affine \( c \)-planes. For any \( x \in X \), let \( P(x) \) be the set of all \( P \in P \) passing through \( x \). By standard general position, \( P(x) \) is a proper, closed subvariety of the Grassmannian of all \( c \)-planes through \( x \). Set \( P(\gamma) = \lim_{t \to 0} P(\gamma(t)) \); it is a proper, closed subvariety of the Grassmannian of linear \( c \)-planes in \( U \). To satisfy conditions (i) - (iv) of the lemma, it is enough to choose \( W \) so that:

1. \( \dim(W \cap K(\gamma)) \leq 1 \);
2. \( W \cap C(X) = \{0\} \);
3. \( W \cap S(\gamma) = \{0\} \);
4. \( W \notin P(\gamma) \).

Manifestly, each of these four conditions is Zariski open and dense.

Using Lemmas 3.6 and 3.4, we assume from now on that the arc \( \gamma \) and the complement \( W \) satisfy conditions (i) - (iv) of Lemma 3.6, and that \( p_\gamma \in L(V)^{\leq N} \) for some large \( N \in \mathbb{Z}_+ \).

Step 4: A product decomposition.

Lemma 3.7 The map \( p_* : \mathcal{L}(X) \to \mathcal{L}(V) \) induces a closed immersion
\[
\mathcal{L}(X)_{\gamma} \hookrightarrow \mathcal{L}(V)_{p_\gamma}.
\]
The differential \( d_* p_* : T_{\gamma} \mathcal{L}(X) \to T_{p_\gamma} \mathcal{L}(V) \) is an isomorphism.

Proof: This is an exercise in the inverse function theorem. \( \square \)

Decomposition \((i)\) induces a decomposition of formal neighborhoods
\[
\mathcal{L}(V)_{p_\gamma} = \mathcal{L}(V)^{\leq N}_{p_\gamma} \times \mathcal{L}(V)^{> N}_{p_\gamma}.
\]
Let \( p_* \mathcal{L}(X)_{\gamma} \subset \mathcal{L}(V)^{> N}_{p_\gamma} \) be the image of the closed immersion in Lemma 3.7. Set \( F = p_* \mathcal{L}(X)_{\gamma} \cap \mathcal{L}(V)^{\leq N}_{p_\gamma} \).

Lemma 3.8 We have \( p_* \mathcal{L}(X)_{\gamma} \supset \mathcal{L}(V)^{> N}_{p_\gamma} \), and
\[
p_* \mathcal{L}(X)_{\gamma} \cong F \times \mathcal{L}(V)^{> N}_{p_\gamma}.
\]
**Proof:** Containment \( p_* \mathcal{L}(X)_\gamma \subseteq \mathcal{L}(V)_{p\gamma}^{>N} \) follows from Lemma 3.3. The proof of the product decomposition is similar to the proof of Lemma 3.4. We sketch it below, continuing with the notation of the proof of Lemma 3.4.

Define an index set \( J \subset \{1, \ldots, d\} \times \mathbb{Z}_+ \) by \( J = \{(i, k) \mid k > N - n_i\} \). Write \( C^J \) for the set of all maps \( a : J \to \mathbb{C} \). For any \( a \in C^J \), let

\[
\xi^a_t = \sum_{(i,k) \in J} a(i, k) t^k \cdot \eta_i ;
\]

this gives a family of vector fields on \( X \) parametrized by \( D \). Let \( \hat{\xi}^a \) be the corresponding vector field on \( \mathcal{L}(X) \), and \( \Xi^a_t : \mathcal{L}(X)^{\xi^a} \to \mathcal{L}(X) \) be the time-1 flow of \( \xi^a \). Define a map \( \Psi : F \times C^J \to \mathcal{L}(V) \) by \( \Psi(\alpha, a) = p_* \Xi^a_t(\alpha) \), where \( \alpha \in F(A) \) for some test-ring \( A \). It is not hard to check that \( \Psi \) induces an isomorphism \( \Psi_{(\gamma,0)} : F \times D^J \cong p_* \mathcal{L}(X)_\gamma \), and that the image \( \Psi_{(\gamma,0)}(\{\gamma\} \times D^J) = \mathcal{L}(V)_{p\gamma}^{>N} \). \( \square \)

**Step 5: Generic projections.** In order to prove Theorem 1.1, we now need to identify the finite-dimensional piece \( F \subseteq \mathcal{L}(V)_{p\gamma}^{\leq N} \). Our strategy is to present \( F \) as an intersection of finitely many closed subschemes of \( \mathcal{L}(V)_{p\gamma}^{\leq N} \); then to analyze each of them using the results of Section 2.

For \( l \in W^* \), define \( \pi_l : U \to V \oplus \mathbb{C} \) by \((v, w) \mapsto (v, l(w))\). Let \( W^{*,0} \subseteq W^* \) be the set of all \( l \), such that \( \gamma(t) \) lands in the domain of injectivity of the restriction \( \pi_l|_X \) (that is, \( X \cap \pi_l^{-1}(\pi_l \circ \gamma(t)) = \{\gamma(t)\} \), for small \( t \).

**Lemma 3.9** The subset \( W^{*,0} \subseteq W^* \) is non-empty and Zariski open.

**Proof:** This follow from condition (i) of Lemma 3.6. \( \square \)

Pick a basis \( L \) of \( W^* \), such that \( L \subseteq W^{*,0} \). For each \( l \in L \), let \( X_l = \pi_l(X) \subseteq V \oplus \mathbb{C} \), and \( \gamma_l = \pi_l \circ \gamma : D \to X_l \). We can make all the constructions of Step 4 for each of the arcs \( \gamma_l \). In particular, we obtain a finite piece \( F_l = p_* \mathcal{L}(X_l)_{\gamma_l} \cap \mathcal{L}(V)_{p\gamma}^{\leq N} \), for each \( l \).

**Lemma 3.10** We have \( F = \bigcap_{l \in L} F_l \), as subschemes of \( \mathcal{L}(V)_{p\gamma}^{\leq N} \).

**Proof:** Let \( A \) be a test-ring with maximal ideal \( m \), and \( \alpha \in \mathcal{L}(V)_{p\gamma}^{\leq N}(A) \) be a map \( \text{Spec} A \times D \to V \). Let \( D^o = \text{Spec} \mathcal{C}((t)) \subseteq D \) be the punctured formal disc, \( \gamma^o \) be the restriction of \( \gamma \) to \( D^o \), and \( \alpha^o \) be the restriction of \( \alpha \) to \( \text{Spec} A \times D^o \). Then there is a unique map \( \tilde{\alpha}^o : \text{Spec} A \times D^o \to X \) such that \( \tilde{\alpha}^o \) restricted to \( D^o (= \text{Spec} A/m \times D^o) \) equals \( \gamma^o \), and \( p \circ \tilde{\alpha}^o = \alpha^o \).

By definition, \( \alpha \in F(A) \) if and only if \( \tilde{\alpha}^o \) extends to a map of \( \text{Spec} A \times D \). This will happen if and only if \( l \circ \tilde{\alpha}^o \in A[[t]] \), for all \( l \in L \). But for each individual \( l \), this condition is equivalent to saying that \( \alpha \in F_l(A) \). \( \square \)
Step 6: Reduction to plane curves. We now proceed to identify each of the $F_i$ ($i \in L$). Consider coordinates $(u_1, \ldots, u_d, z)$ on $V \oplus \mathbb{C}$, such that $\mathbb{C}$ is the $z$-axis, and $V$ is the plane $\{z = 0\}$. Let $f_l(u_1, \ldots, u_d, z)$ be the defining equation of the hypersurface $X_l \subset V \oplus \mathbb{C}$. Define an evaluation map $\psi : \mathcal{L}(V)^{\leq N} \to \mathbb{C}[x, y]$ by

$$\psi : \beta \mapsto \phi_l((u_1 \circ \beta)(x), \ldots, (u_d \circ \beta)(x), y),$$

and let $f_l = \psi_l(p\gamma)$.

Lemma 3.11 The polynomial $f_l \in \mathbb{C}[x, y]$ has no critical points in some punctured neighborhood of the origin.

Proof: This follows from conditions (ii) - (iv) of Lemma 3.6.

The rest of the proof is based on applying Lemma 2.1 to $f = f_l$. Let $C \subset \mathbb{C}^2$ be the curve $\{f(x, y) = 0\}$, and $(C_1, 0) \subset (C, 0)$ be the germ of the graph $\{y = z \circ \gamma_l(x)\}$. Fix an integer $M \gg N \cdot \deg(\phi_l)$, and let $P = P_l$ be a small analytic neighborhood of $f$ in $\mathbb{C}[x, y]^{\leq M}$. Based on this data, we can define the locus $R = R_l \subset P_l$ as in Section 2. Consider the formal neighborhoods $(R_l)_f \subset (P_l)_f$, and the map $(\psi_l)_\gamma : \mathcal{L}(V)^{\leq N} \to (P_l)_f$ induced by $\psi_l$.

Lemma 3.12 We have: $F_l = (\psi_l)_\gamma^{-1}(R_l)_f$.

Proof: Begin with the inclusion $F_l \subset (\psi_l)_f^{-1}(R_l)_f$. Let $A$ be a test-ring, and $\alpha \in F_l(A)$. This means that $\alpha \in \mathcal{L}(V)^{\leq N}(A)$ admits a lift $\tilde{\alpha} : \text{Spec } A \times D \to X_l$ such that $\tilde{\alpha}|_D = \gamma_l$, and $p \circ \tilde{\alpha} = \alpha$. Consider the image

$$\tilde{f} = (\psi_l)_\gamma(\alpha) \in (P_l)_f(A) \subset A[x, y]^{\leq M}.$$

Let $\tilde{h} \in A[[t]]$ be the composition $z \circ \tilde{\alpha} : \text{Spec } A \times D \to \mathbb{C}$. It is a tautology that $\tilde{f}$ and $\tilde{h}$ satisfy the conditions of Lemma 2.1 (iv). Inclusion $\subset$ follows. The opposite inclusion in analogous.

To complete the proof of Theorem 1.1, we need a basic result of M. Artin.

Lemma 3.13 [Ar, Corollary 2.1] Let $E \cong \mathbb{C}^n$ be an affine space, $Z \subset E$ be a closed algebraic subvariety, and $e$ be a point in $Z$. Suppose the analytic germ $(Z, e)$ is reducible: $(Z, e) = (Z_1, e) \cup (Z_2, e)$. Then there exists an etale neighborhood $\pi : \tilde{E} \to E \subset E$ of $e$ in $E$, and a point $\tilde{e} \in \pi^{-1}(e)$, such that the preimage $\tilde{Z} = \pi^{-1}(Z)$ is reducible as an algebraic variety: $\tilde{Z} = \tilde{Z}_1 \cup \tilde{Z}_2$, and $\pi$ induces an isomorphism of analytic germs $(\tilde{Z}_1, \tilde{e}) \cong (Z_1, e)$.
Let now $E = \prod_{i \in L} \mathbb{C}[x, y]_{\leq M}$, where $M \gg N \cdot \deg(\phi_i)$ for all $l$. We have a map $\psi : \mathcal{L}(V)_{\leq N} \to E$ given, in components, by the $\{\psi_l\}_{l \in L}$. Let $e = \psi(\gamma) = \{f_l\}_{l \in L}$. For each $l \in L$, let $Z_l \subset \mathbb{C}[x, y]_{\leq M}$ be a closed subvariety having $R_l$ as an analytic irreducible component near $f_l$ (cf. Lemma 3.13). Let $Z = \prod_{i \in L} Z_i \subset E$, and $Z_1 = \prod_{i \in L} R_i$. Then $Z_1$ is an analytic component of $Z$ near $e$. Applying Lemma 3.13, we obtain an etale neighborhood $\pi : E^{\circ} \to E^{\circ} \subset E$ of $e$, a point $\tilde{e} \in \pi^{-1}(e)$, and an irreducible component $\tilde{Z}_1$ of $\tilde{Z} = \pi^{-1}(Z)$, such that $(\tilde{Z}_1, \tilde{e}) \cong (Z_1, e)$. The scheme $Y$ of Theorem 1.1 is defined as the fiber product $Y = \mathcal{L}(V)_{\leq N} \times_E \tilde{Z}_1$, and the point $y$ is taken to be $(\gamma, \tilde{e})$. The theorem follows from Lemmas 3.8, 3.10, and 3.12.

4 Examples

The proof of Theorem 1.1 described above is far from being constructive. Here are a few simple examples where we can explicitly present the versal deformation of an arc.

Example 1. Let $X = \{(x, y, z) \in \mathbb{C}^3 \ | \ xy = z^2\}$, and $\gamma \in \mathcal{L}^c(X)$ be the arc $\gamma(t) = (t, 0, 0)$. Then the scheme $Y = Y(\gamma)$ of Theorem 1.1 is just a double point: $Y = \text{Spec } \mathbb{C}[a]/(a^2)$. The point $y \in Y(\mathbb{C})$ is the only $\mathbb{C}$-point of $Y$, and a versal deformation $\alpha : Y \times D \to X$ of $\gamma$ is given by $\alpha(a, t) = (t, 0, a)$. Note that this means that “topologically” $\gamma$ is a smooth point of $\mathcal{L}^c(X)$.

Example 2. Let now $X = \{(x, y, z) \in \mathbb{C}^{2+r} \ | \ xy = z^2\}$ be the quadric cone in $\mathbb{C}^{2+r}$, for any $r \geq 1$. Here $z \in \mathbb{C}^r$, and $z^2 \in \mathbb{C}$ is the standard dot-product $z \cdot z$. Let $\gamma \in \mathcal{L}^c(X)$ be the arc $\gamma(t) = (t, 0, 0)$. Then $Y = Y(\gamma)$ is the quadric cone in $\mathbb{C}^r$: $Y = \{w \in \mathbb{C}^r \ | \ w^2 = 0\}$ (in its reduced structure, if $r > 1$). The point $y \in Y(\mathbb{C})$ is zero, and a versal deformation $\alpha : Y \times D \to X$ of $\gamma$ is given by $\alpha(w, t) = (t, 0, w)$. Note that this means that $\gamma$ “lies on a stratum of codimension $r - 1$ in $\mathcal{L}^c(X)$.”

Example 3. Let $X$ be as in Example 2, and $\gamma \in \mathcal{L}^c(X)$ be the arc $\gamma(t) = (t^2, 0, 0)$. Then $Y = \{(a, b, v, w) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^r \ | \ aw^2 = v^2, bw^2 = 2vw\}$ (in its reduced structure). The point $y \in Y(\mathbb{C})$ is zero, and a versal deformation $\alpha : Y \times D \to X$ of $\gamma$ is given by $\alpha(a, b, v, w, t) = (a + bt + t^2, w^2, v + tw)$. This means that $\gamma$ “lies on a stratum of codimension $2r$ in $\mathcal{L}^c(X)$.”

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