On the Cauchy problem for the magnetic Zakharov system

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Abstract In this paper, we study the well-posedness results for the magnetic type Zakharov system. Such system describes the pondermotive force and magnetic field generation effects resulting from the nonlinear interaction between plasma-wave and particles. By using energy methods together with commutator estimate, we first derive a priori estimates for a regularized system. Then by approximation arguments, we obtain local existence results as well as global existence for small initial data.

Keywords Magnetic Zakharov system · Commutator estimate · Local/global well-posedness

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1 Introduction and the main results

The present paper mainly focuses on the well-posedness results for the Cauchy problem of the following magnetic Zakharov system

\[
\begin{aligned}
&iE_t + \nabla(\nabla \cdot E) - \alpha \nabla \times (\nabla \times E) - nE + iE \times B = 0, \\
n_{tt} - \Delta n = \Delta |E|^2, \\
B_{tt} + \Delta^2 B - \Delta B = -i \Delta^2 (E \times \overline{E})
\end{aligned}
\]  

with initial data

\[
E(0, x) = E_0, \ (n(0, x), n_t(0, x)) = (n_0, n_1), \ (B(0, x), B_t(0, x)) = (B_0, B_1),
\]

where \(\alpha \geq 1\) is a constant, \(x \in \mathbb{R}^d, d = 2, 3\). Here, the function \(E : \mathbb{R}^{1+d} \rightarrow \mathbb{C}^3\) is the slowly varying amplitude of the high-frequency electric field, and the function \(n : \mathbb{R}^{1+d} \rightarrow \mathbb{R}\) denotes the fluctuation of the ion-density from its equilibrium, and \(B : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^3\) is the magnetic field. \(\overline{E}\) denotes the conjugate complex of \(E\), and the notation \(\times\) in the equation means the cross product for \(\mathbb{R}^3\) or \(\mathbb{C}^3\)-valued vectors. If the space dimension \(d = 2\), \(E\) and \(B\) are always taken as the form \(E(t, x) = (E_1(t, x), E_2(t, x), 0)\), \(B(t, x) = (0, 0, B_3(t, x))\), \(x \in \mathbb{R}^2\).

Omitting the magnetic field \(B\) and taking \(\alpha = 1\), then system (1.1) reduces to the classical Zakharov system [19]

\[
\begin{aligned}
iE_t + \Delta E = nE, \\
n_{tt} - \Delta n = \Delta |E|^2.
\end{aligned}
\]  

This system has been studied by many mathematicians in the past decades. For (1.3), local existence and uniqueness of smooth solution \((E, n, n_t) \in L^\infty(0, T ; H^m \oplus H^{m-1} \oplus H^{m-2})\) with \(m \geq 3\) integer were first obtained by Sulem and Sulem [17], moreover, global existence of smooth solution for \(d = 1\) was also obtained in [17]. We also refer to [8] for the results of classical solution in one spatial dimension. In [1], H. Added and S. Added proved that the smooth solution can be extended globally in time when \(\|E_0\|_{L^2}\) is small in the case \(d = 2\). Local well-posedness in \(H^2 \oplus H^1 \oplus L^2\) was shown in [16]. In [2], by using Fourier restriction spaces, J. Bourgain and J. Colliander obtained local well-posedness results in the energy space \(H^1 \oplus L^2 \oplus \dot{H}^{-1}\) and showed the solution is global under small assumption on \(E_0\) for \(d = 2, 3\). For more well-posedness results for (1.3) in lower regular spaces, we refer to [3,4,6,9] and the references therein.

However, the system (1.3) ignores the effect of the magnetic field which is generated in the laser plasma. In fact, it is meaningful to consider the self-generated magnetic field in the Zakharov system from physical viewpoint, e.g. we can study whether the magnetic field can promote the formation of soliton in three dimensions or whether it can affect the collapse process of wave packet in plasma. The magnetic field \(B\) has different expressions in different plasmas. In a cold plasma, the magnetic field satisfies

\[
\Delta B - i\eta \nabla \times (\nabla \times (E \times \overline{E})) + \beta B = 0, \quad \beta \leq 0, \quad \eta > 0
\]  

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while in a hot plasma, $B$ obeys

$$\Delta B - i\eta \nabla \times (\nabla \times (E \times \vec{E})) - \gamma \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \frac{B(t, y)}{|x-y|^2} dy = 0, \quad \eta, \gamma > 0. \tag{1.5}$$

One can see [13] for the derivation of the above magnetic equations. In [15], Laurey studied the existence and uniqueness of the solution for the Zakharov system with the magnetic field given by (1.4) or (1.5), see also [14]. Starting from Vlasov–Maxwell equations, He [10] first obtained the magnetic Zakharov type system (1.1). Such system describes the pondermotive force and magnetic field generation effects resulting from the nonlinear interaction between plasma-wave and particles. In [11], the author discussed the collapse dynamics of wave packet for the Zakharov system with magnetic filed effect.

In the present work, we are interested in proving local existence in time for strong solutions as well as global existence result for weak solutions for (1.1)–(1.2). To obtain such well-posedness results, we use energy methods together with commutator estimate to derive a priori estimates for an approximate problem of the original system. After obtaining the uniform bounds for the approximate solutions, we prove strong convergence of these solutions, then we can obtain the desired results. Since system (1.1) is the classical Zakharov system (1.3) coupled to the magnetic field $B$ by an additional term $i E \times B$ in the first equation of (1.1), the main difficulty during the proof is to get the estimate for $B$. However, we find that the magnetic field $B$ behaves like $E \times \vec{E}$ in some sense (see (2.2) in the next section). And in this way, we can extend the known results for (1.3) to the magnetic Zakharov system (1.1). Now we state our main results.

**Theorem 1.1** Assume that $s > \frac{d}{2}$, and let $D_R(0)$ be the set of $(E_0, n_0, n_1, B_0, B_1) \in H^{s+1} \oplus H^s \oplus (H^{s-1} \cap \dot{H}^{-1}) \oplus (H^s \cap \dot{H}^{-1}) \oplus (H^{s-2} \cap \dot{H}^{-2})$ such that

$$\|E_0\|_{H^{s+1}_t} + \|n_0\|_{H^s} + \|n_1\|_{H^{s-1} \cap \dot{H}^{-1}} + \|B_0\|_{H^s \cap \dot{H}^{-1}} + \|B_1\|_{H^{s-2} \cap \dot{H}^{-2}} \leq R.$$ 

Then for all $R > 0$, there exists $T_{\text{max}} = T_{\text{max}}(R) > 0$ such that for all $(E_0, n_0, n_1, B_0, B_1) \in D_R(0)$ the magnetic Zakharov system (1.1)–(1.2) has a unique solution $(E, n, B)$ with

$$(E, n, B) \in L^\infty(0, T_{\text{max}}; H^{s+1} \oplus H^s \oplus (H^s \cap \dot{H}^{-1})),$$

$$(E_t, n_1, B_t) \in L^\infty(0, T_{\text{max}}; H^{s-1} \oplus (H^{s-1} \cap \dot{H}^{-1}) \oplus (H^{s-2} \cap \dot{H}^{-2})). \tag{1.6}$$

Theorem 1.1 needs the additional condition $n_1 \in \dot{H}^{-1}$, $B_0 \in \dot{H}^{-1}$ and $B_1 \in \dot{H}^{-2}$. The reason is that the energy $\Psi$ defined in (2.2) below includes the $\dot{H}^{-2}$ norm of $B_t$ and the $\dot{H}^{-1}$ norm of $n_1$ and $B$. However, since the Schwarz class $\mathcal{S}(\mathbb{R}^3) \not\subset \dot{H}^{-1}(\mathbb{R}^3)$ and $\mathcal{S}(\mathbb{R}^3) \not\subset \dot{H}^{-2}(\mathbb{R}^3)$, such additional assumption on the initial data seems unnatural. In fact, inspired by [7], this condition can be removed by splitting the initial data into low-frequency part and high-frequency part. Therefore, we can also obtain the following local well-posedness results for (1.1)–(1.2).
Theorem 1.2 Assume $s > \frac{d}{2}$, and let $\tilde{D}_R(0)$ be the set of $(E_0, n_0, n_1, B_0, B_1) \in H^{s+1} \oplus H^{s} \oplus H^{s-1} \oplus H^{s} \oplus H^{s-2}$ such that

$$\|E_0\|_{H^{s+1}} + \|n_0\|_{H^{s}} + \|n_1\|_{H^{s-1}} + \|B_0\|_{H^{s}} + \|B_1\|_{H^{s-2}} \leq R.$$ 

Then for all $R > 0$, there exists $T_{\text{max}} = T_{\text{max}}(R) > 0$ such that for all $(E_0, n_0, n_1, B_0, B_1) \in \tilde{D}_R(0)$ the magnetic Zakharov system (1.1)–(1.2) has a unique solution $(E, n, B)$ satisfying

$$(E, n, B) \in L^\infty(0, T_{\text{max}}; H^{s+1} \oplus H^{s} \oplus H^{s}),$$

$$(E_1, n_1, B_1) \in L^\infty(0, T_{\text{max}}; H^{s-1} \oplus H^{s-1} \oplus H^{s-2}).$$

Throughout the paper, the square root of the Laplacian $(-\Delta)^{\frac{1}{2}}$ will be denoted by $\Lambda$ and obviously $\mathcal{F}(\Lambda f) = |\xi|^{\frac{1}{2}} \hat{f}$. We denote the inner product of $f$ and $g$ by $(f, g) \equiv \int_{\mathbb{R}^d} f(x) \cdot \bar{g}(x) dx$. We define, for $s \in \mathbb{R}$ and $1 < p < \infty$, the inhomogeneous Sobolev space $H^{s,p}(\mathbb{R}^d)$ or simply $H^{s,p}$ of tempered distributions $f$ such that

$$\|f\|_{H^{s,p}} = \|(I - \Delta)^{\frac{s}{2}} f\|_{L^p} < \infty,$$

where $(I - \Delta)^{\frac{s}{2}} f$ is defined by $(I - \Delta)^{\frac{s}{2}} f = \mathcal{F}^{-1}\left((1 + |\xi|^2)^{\frac{s}{2}} \hat{f}\right)$. If $p = 2$, we write $H^s$ instead of $H^{s,2}$ for short, and by Plancherel’s theorem $\|f\|_{H^s} = \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}\|_{L^2}$. For $s \in \mathbb{R}$, one can define the homogeneous Sobolev space $\dot{H}^{s,p}$ or $\dot{H}^s (= \dot{H}^{s,2})$ with

$$\|f\|_{\dot{H}^{s,p}} = \|\Lambda^s f\|_{L^p} < \infty.$$

This paper is organized as follows. In the next section, we derive some conserved quantities for (1.1), and give global existence result of weak solutions. In Sect. 3, we introduce a regularized system for our magnetic Zakharov system which admits a unique smooth solution globally. We derive a priori estimates for this regularized system and obtain strong convergence property of the approximate solution in Sect. 4. Section 5 is concerned with the proof of the main theorem.

2 Conserved quantities and weak solutions

As we know, conserved laws play an important role in the analytic theory(e.g. well-posedness theory and asymptotic behavior) for nonlinear PDEs of physical origin. For the magnetic Zakharov system (1.1), we have the following conserved results.
**Proposition 2.1** For sufficiently regular solutions of the system (1.1), there hold two conserved quantities:

\[
\Phi(t) := \| E(t) \|_{L^2}^2 = \Phi(0), \tag{2.1}
\]

\[
\Psi(t) := \| \nabla \cdot (E(t)) \|_{L^2}^2 + \alpha \| \nabla \times (E(t)) \|_{L^2}^2 + \frac{1}{2} \| n(t) \|_{L^2}^2 + \frac{1}{2} \| \Lambda^{-1} n_t(t) \|_{L^2}^2
\]
\[
+ \frac{1}{2} \| B(t) \|_{L^2}^2 + \frac{1}{2} \| \Lambda^{-1} B(t) \|_{L^2}^2 + \frac{1}{2} \| \Lambda^{-2} B_t(t) \|_{L^2}^2
\]
\[
+ \int \frac{n(t)}{\| E(t) \|_{L^2}}^2 dx + i \int (E(t) \times \overline{E(t)}) \cdot B(t) dx
\]
\[
= \Psi(0). \tag{2.2}
\]

**Proof** Note that \( E \times \overline{E} \) is purely imaginary, so the value of the term \( i (E \times B) \cdot \overline{E} = \overline{i (E \times E)} \cdot B \) is real. Multiplying the first equation of (1.1) by \( \overline{E} \) and integrating the imaginary part over \( \mathbb{R}^d \), we then obtain

\[
\frac{1}{2} \frac{d}{dt} \| E(t) \|_{L^2}^2 = 0
\]

which yields (2.1) immediately.

To prove (2.2), we multiply the first equation of (1.1) by \( -E_t \) and integrate the real part, then we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla \cdot (E(t)) \|_{L^2}^2 + \alpha \| \nabla \times (E(t)) \|_{L^2}^2 \right) + \frac{1}{2} \int n |E_t|^2 dx - \operatorname{Re} i \int (E \times B) \cdot \overline{E_t} dx = 0. \tag{2.3}
\]

On the other hand, we take inner product of the third equation of (1.1) with \( \Lambda^{-4} B_t \) and obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \Lambda^{-2} B_t \|_{L^2}^2 + \| B \|_{L^2}^2 + \| \Lambda^{-1} B \|_{L^2}^2 \right) + i \int (E \times \overline{E}) \cdot B_t dx = 0. \tag{2.4}
\]

Similarly, taking inner product of the second equation of (1.1) with \( \Lambda^{-2} n_t \), then one gets

\[
\frac{1}{2} \frac{d}{dt} \left( \| \Lambda^{-1} n_t \|_{L^2}^2 + \| n \|_{L^2}^2 \right) + \int n_t |E|^2 dx = 0. \tag{2.5}
\]

Since \( (E \times \overline{E})_t = 2i \operatorname{Im}(E \times \overline{E}_t) \), then the third term in (2.3) can be written as

\[
-\operatorname{Re} i \int (E \times B) \cdot \overline{E_t} dx = -\operatorname{Im} \int (E \times \overline{E}_t) \cdot B dx = \frac{i}{2} \int (E \times \overline{E})_t \cdot B dx. \tag{2.6}
\]

Combining the equalities (2.3)–(2.6), we finally get \( \frac{d}{dt} \Psi(t) = 0 \) which implies that \( \Psi(t) = \Psi(0) \). \( \square \)
The conserved quantities (2.1)–(2.2) are the main tool in establishing the global existence of weak solutions for (1.1)–(1.2). Before doing so, we need the following lemma.

**Lemma 2.1** Assume \( f(t) \) is a continuous function defined on \( \mathbb{R}^+ \), moreover, \( f(t) \) is nonnegative and satisfies

\[
f(t) \leq a + bf^\kappa(t), \quad a, b > 0, \quad \kappa > 1.
\]

If \( a\kappa^{-1}b < \frac{(\kappa-1)\kappa^{-1}}{\kappa^\kappa} \) and \( f(0) \leq a \), then \( f(t) \) is bounded for all \( t \geq 0 \).

The proof of Lemma 2.1 is easy, so we omit it. Combining Proposition 2.1 and Lemma 2.1, we can obtain the following estimate for \((E, n, B)\) in the energy space \( H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1}) \). Before stating Lemma 2.2, we recall the following Sobolev best constant inequality (see [18])

\[
\|f\|_{L^4(\mathbb{R}^d)}^4 \leq K^4(d)\|f\|_{L^2(\mathbb{R}^d)}^4\|\nabla f\|_{L^2(\mathbb{R}^d)}^d,
\]

and \( K^4(d) = \frac{2}{\|Q\|_{L^2}^2} \) with \( Q \) the ground state solution of

\[
\frac{d}{2}\Delta Q - \left(2 - \frac{d}{2}\right)Q + Q^3 = 0.
\]

**Lemma 2.2** Let \((E, n, B)\) be a sufficiently regular solution to the magnetic Zakharov system (1.1)–(1.2) with \( \|E_0\|_{L^2} \) small \( (d = 2) \) or \( \|E_0\|_{H^1} \) small \( (d = 3) \), more precisely,

\[
2\|E_0\|_{L^2}^2 < \|Q\|_{L^2}^2, \quad \text{if } d = 2,
\]

\[
\|E_0\|_{L^2}^2 < \frac{1}{27K^8(3)|\Psi(0)|}, \quad \|\nabla E_0\|_{L^2}^2 \leq |\Psi(0)|, \quad \text{if } d = 3,
\]

where \( Q = Q(x) \) is the ground state solution of \( \Delta Q - Q + Q^3 = 0, \quad x \in \mathbb{R}^2 \). Then there holds that

\[
\|E(t)\|_{H^1}^2 + \|n(t)\|_{L^2}^2 + \|n(t)\|_{\dot{H}^{-1}}^2 + \|B(t)\|_{L^2 \cap \dot{H}^{-1}}^2 + \|B(t)\|_{\dot{H}^{-2}}^2 \leq C, \quad (2.8)
\]

where the constant \( C \) depends on \( \|E_0\|_{H^1}, \|n_0\|_{L^2}, \|n_1\|_{\dot{H}^{-1}}, \|B_0\|_{L^2 \cap \dot{H}^{-1}} \) and \( \|B_1\|_{\dot{H}^{-2}} \).
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Proof By Cauchy–Schwarz inequality, we have

\[
\left| \int_{\mathbb{R}^d} n |E|^2 \, dx \right| \leq \epsilon \|n\|^2_{L^2} + \frac{1}{4\epsilon} \|E\|^4_{L^4} \leq \epsilon \|n\|^2_{L^2} + \frac{1}{4\epsilon} K^4(d) \|E\|^4_{L^2} \|\nabla E\|^d_{L^2},
\]  

(2.9)

\[
i \int_{\mathbb{R}^d} (E \times \overline{E}) \cdot B \, dx \leq \epsilon \|B\|^2_{L^2} + \frac{1}{4\epsilon} K^4(d) \|E\|^4_{L^2} \|\nabla E\|^d_{L^2},
\]  

(2.10)

where we have used the Sobolev best constant inequality in (2.9) and (2.10). Note that

\[
\|\nabla \cdot E\|^2_{L^2} + \alpha \|\nabla \times E\|^2_{L^2} \geq \|\nabla \cdot E\|^2_{L^2} + \|\nabla \times E\|^2_{L^2} = \|\nabla E\|^2_{L^2}.
\]

Now we deduce from (2.9), (2.10) and Proposition 2.1 that if \(d = 2\)

\[
\|\nabla E\|^2_{L^2} + \left(\frac{1}{2} - \epsilon\right) \|n\|^2_{L^2} + \left(\frac{1}{2} - \epsilon\right) \|B\|^2_{L^2} + \frac{1}{2} \|B\|^2_{\dot{H}^{-1}} + \frac{1}{2} \|n\|^2_{\dot{H}^{-1}} + \frac{1}{2} \|B\|^2_{\dot{H}^{-2}}
\]

\[
\leq |\Psi(0)| + \frac{1}{2\epsilon} K^4(2) \|E_0\|^2_{L^2} \|\nabla E\|^2_{L^2}
\]

\[
= |\Psi(0)| + \frac{1}{\epsilon} \|E_0\|^2_{L^2} \|\nabla E\|^2_{L^2}, \quad 0 < \epsilon < \frac{1}{2},
\]  

(2.11)

If \(2 \|E_0\|^2_{L^2} < \|Q\|^2_{L^2}\), then we can choose \(\epsilon\) very close to \(\frac{1}{2}\) such that \(\epsilon \|E_0\|^2_{L^2} < \frac{1}{2} \|Q\|^2_{L^2}\), thus (2.8) follows from the above estimate in the case \(d = 2\).

If \(d = 3\), we choose \(\epsilon = \frac{1}{4}\) and obtain

\[
\|\nabla E\|^2_{L^2} + \frac{1}{4} \|n\|^2_{L^2} + \frac{1}{4} \|B\|^2_{L^2} + \frac{1}{2} \|B\|^2_{\dot{H}^{-1}} + \frac{1}{2} \|n\|^2_{\dot{H}^{-1}} + \frac{1}{2} \|B\|^2_{\dot{H}^{-2}}
\]

\[
\leq |\Psi(0)| + 2 K^4(3) \|E_0\|^3_{L^2} \|\nabla E\|^3_{L^2},
\]  

(2.12)

If we take \(f(t) = \|\nabla E(t)\|^2_{L^2}, a = |\Psi(0)|, b = 2 K^4(3) \|E_0\|^2_{L^2}\) and \(\kappa = \frac{3}{2}\), then Lemma 2.1 gives \(\|\nabla E(t)\|^2_{L^2} \leq C\) for all \(t \geq 0\). Inserting this estimate into (2.12), we see that (2.8) also holds in the case \(d = 3\).

We remark that in the case \(d = 2\), the solution of the usual Zakharov system (1.3) satisfies

\[
\|E(t)\|_{H^1} + \|n(t)\|_{L^2} + \|n_t(t)\|_{\dot{H}^{-1}} \leq C (\|E_0\|_{H^1}, \|n_0\|_{L^2}, \|n_1\|_{\dot{H}^{-1}})
\]

provided that \(\|E_0\|^2_{L^2} < \|Q\|^2_{L^2}\) (see [1, 7]). This condition is slightly different from (2.7). This is because the equation for \(E\) in (1.1) contains the magnetic term \(i E \times B\) which is absent in the usual Zakharov system. Moreover, the magnetic field \(B\) in some
way behaves like $E \times \overline{E}$, which plays a similar role as the term $|E|^2$, so here we need
the condition $2\|E_0\|_{L^2} < \|Q\|^2_{L^2}$ to ensure the uniform boundedness of the solution.
In fact, as pointed out in [11], the generation of the magnetic field $B$ will accelerate
the collapse process of wave packet in plasma.

Applying Lemma 2.2, it is easy to establish global existence of weak solutions for
the magnetic Zakharov system (1.1)–(1.2).

**Theorem 2.1** If $E_0 \in H^1$, $(n_0, n_1) \in L^2 \oplus \dot{H}^{-1}$, $(B_0, B_1) \in (L^2 \cap \dot{H}^{-1}) \oplus \dot{H}^{-2}$, and
the initial data satisfying (2.7), then there exists a weak solution $(E, n, B)$ for
(1.1)–(1.2) in the distributional sense such that

$$E \in L^\infty(\mathbb{R}^+; H^1), \quad (n, n_1) \in L^\infty(\mathbb{R}^+; L^2 \oplus \dot{H}^{-1}), \quad (B, B_1) \in L^\infty(\mathbb{R}^+; (L^2 \cap \dot{H}^{-1}) \oplus \dot{H}^{-2}).$$

Using the estimate (2.8), Theorem 2.1 can be proved by applying Galerkin method
and compactness argument, since this procedure is standard, the proof of Theorem 2.1
is omitted here.

### 3 Regularization for the original system

In this section, we introduce a regularized system for our original system (1.1). Now
we consider the following system $(0 < \epsilon < 1)$

$$i E^\epsilon_t + i \epsilon^2 \Delta^2 E^\epsilon_t + \nabla(\nabla \cdot E^\epsilon) - \alpha \nabla \times (\nabla \times E^\epsilon) - n^\epsilon E^\epsilon + i E^\epsilon \times B^\epsilon = 0, \quad \text{(3.1)}$$

$$n^\epsilon_{tt} - \Delta n^\epsilon = \Delta |E^\epsilon|^2, \quad \text{(3.2)}$$

$$B^\epsilon_{tt} + \Delta^2 B^\epsilon - \Delta B^\epsilon = -i \Delta^2 (E^\epsilon \times \overline{E^\epsilon}), \quad \text{(3.3)}$$

with smooth initial data

$$E^\epsilon(0) = E^\epsilon_0, \quad n^\epsilon(0) = n^\epsilon_0, \quad n^\epsilon_t(0) = n^\epsilon_1, \quad B^\epsilon(0) = B^\epsilon_0, \quad B^\epsilon_t(0) = B^\epsilon_1. \quad \text{(3.4)}$$

With the same argument as Proposition 2.1, we can obtain the following results for
the regularized system (3.1)–(3.3).

**Proposition 3.1** Assume $(E^\epsilon, n^\epsilon, B^\epsilon)$ is a regular solution for the system (3.1)–(3.3),
then we have

$$\Phi^\epsilon(t) := \|E^\epsilon(t)\|^2_{L^2} + \epsilon^2 \|\Delta E^\epsilon(t)\|^2_{L^2} = \Phi^\epsilon(0), \quad \text{(3.5)}$$

$$\Psi^\epsilon(t) := \|
abla \cdot E^\epsilon(t)\|^2_{L^2} + \alpha \|
abla \times E^\epsilon(t)\|^2_{L^2} + \frac{1}{2} \|n^\epsilon(t)\|^2_{L^2} + \frac{1}{2} \|\Lambda^{-1} n^\epsilon_t(t)\|^2_{L^2} + \frac{1}{2} \|\Lambda^{-1} B^\epsilon(t)\|^2_{L^2} + \frac{1}{2} \|\Lambda^{-2} B^\epsilon_t(t)\|^2_{L^2} + \int_{\mathbb{R}^d} n^\epsilon(t)|E^\epsilon(t)|^2 dx + i \int_{\mathbb{R}^d} (E^\epsilon(t) \times \overline{E^\epsilon(t)}) \cdot B^\epsilon(t) dx \quad \text{(3.6)}$$

$$= \Psi^\epsilon(0).$$
Set $\mathcal{L} = (I + e^2 \Delta^2)^{-1}$, and let $\mathcal{A}$ be the linear operator defined by $\mathcal{A}E = -\nabla(\nabla \cdot E) + \alpha \nabla \times (\nabla \times E)$. Since the operator $\mathcal{L}\mathcal{A}$ is self-adjoint, then the linear equation

$$iE_t = \mathcal{L}\mathcal{A}E, \quad E(0) = E_0$$

generates a unitary group $U(t)$ in $H^r(\mathbb{R}^d)$. Therefore, we can transform the regularized system (3.1)–(3.4) into the following integral equation

$$E^\epsilon(t) = U(t)E_0^\epsilon + \int_0^t U(t-\tau)f(E^\epsilon(\tau))d\tau,$$

where $f(E^\epsilon(t)) = -i\mathcal{L}(n^\epsilon E) - \mathcal{L}(E^\epsilon \times B^\epsilon)$, and $n^\epsilon = n(E^\epsilon)$, $B^\epsilon = B(E^\epsilon)$ are the solution of equations (3.2), (3.3), respectively.

The main result in this section is the global existence of smooth solution for (3.1)–(3.4).

**Theorem 3.1** Suppose that $E_0^\epsilon \in H^{r+1}$, $(n_0^\epsilon, n_1^\epsilon) \in H^r \oplus (H^{r-1} \cap \dot{H}^{-1})$, $(B_0^\epsilon, B_1^\epsilon) \in (H^{r-1} \cap \dot{H}^{-1}) \oplus (H^{r-3} \cap \dot{H}^{-2})$, and $r$ is a large integer. Then there exists a unique smooth solution $(E^\epsilon, n^\epsilon, B^\epsilon)$ for the regularized system (3.1)–(3.4) such that

$$(E^\epsilon, n^\epsilon, B^\epsilon) \in C(\mathbb{R}^+; H^{r+1} \oplus H^r \oplus H^{r-1}).$$

**Proof** By contraction argument, we first show that equation (3.7) has a unique solution locally, then we extend this solution globally in time based on some uniform estimates.

Denote $M = \|E_0^\epsilon\|_{H^{r+1}} + \|n_0^\epsilon\|_{H^r} + \|n_1^\epsilon\|_{H^{r-1}} + \|B_0^\epsilon\|_{H^{r-1}} + \|B_1^\epsilon\|_{H^{r-3}}$. Let $0 < T \leq 1$ to be determined later, and set $X = C([0, T]; H^{r+1})$, $X_M = \{E \in X; \|E\|_X \leq 2M\}$. Now we define the map $T$ acting on $X$ by

$$T(E^\epsilon) = U(t)E_0^\epsilon + \int_0^t U(t-\tau)f(E^\epsilon(\tau))d\tau, \forall E^\epsilon \in X. \quad (3.7)$$

Our aim is to show that $T$ has a unique fixed point on $X_M$ if $T$ is small enough. Given $E^\epsilon \in X$, we obtain from (3.2) and (3.3) that

$$\|n^\epsilon(t)\|_{H^r} \leq (1 + T)(\|n_0^\epsilon\|_{H^r} + \|n_1^\epsilon\|_{H^{r-1}}) + T\|E^\epsilon\|^2_X, \quad (3.8)$$
$$\|B^\epsilon(t)\|_{H^{r-1}} \leq (1 + T)(\|B_0^\epsilon\|_{H^{r-1}} + \|B_1^\epsilon\|_{H^{r-3}}) + T\|E^\epsilon\|^2_X, \quad (3.9)$$

for all $t \in [0, T]$. Note also that $\mathcal{L}$ is a bounded linear operator from $H^{k-4}$ to $H^k$, namely, there exists $K > 0$ not depending on $k$ such that

$$\|\mathcal{L}f\|_{H^k} \leq K\|f\|_{H^{k-4}}, \forall f \in H^{k-4}. \quad (3.10)$$

Using the estimates (3.8)–(3.10), it is not hard to show that $T$ is a contraction map on $X_M$ provided that $T = T(M)$ is small enough. So by fixed point theorem, we
know that the equation (3.7) has a unique solution $E^e \in C([0, T]; H^{r+1})$, and by (3.2)–(3.3), we also know $\eta^e \in C([0, T]; H^r)$, $B^e \in C([0, T]; H^{r-1})$.

Assume now $T_{\text{max}}$ is the maximal existence time of the solution $(E^e, \eta^e, B^e)$ for the regularized system (3.1)–(3.4), hence in order to complete the proof of Theorem 3.1, we have to show that $T_{\text{max}} = \infty$. To this goal, it suffices to prove the term $\|E^e\|_{H^{r+1}}$ can be controlled on the time interval $[0, T_{\text{max}})$.

It follows from (3.5) that

$$\|E^e\|_{H^2} \leq C (\Rightarrow \|E^e\|_{L^\infty} \leq C). \quad (3.11)$$

By (3.11), we deduce from (3.8) and (3.9)

$$\|\eta^e\|_{H^1} \leq C, \; \|B^e\|_{L^2} \leq C. \quad (3.12)$$

Besides, one also deduces from (3.6) that

$$\|\eta^e_t\|_{H^{-1}} + \|B^e_t\|_{H^{-2}} \leq C. \quad (3.13)$$

We emphasize that the constant $C$ in the above estimates depends on the parameter $\epsilon$.

Now multiplying (3.1) by $\Lambda^{2r-2} E^e$, and integrating the imaginary part, then we obtain from (3.12)–(3.13) that

$$\frac{1}{2} \frac{d}{dt} (\|\Lambda^{r-1} E^e\|_{L^2}^2 + \epsilon^2 \|\Lambda^{r+1} E^e\|_{L^2}^2) \leq \text{Im} \int_{\mathbb{R}^d} \Lambda^{r-1} (\eta^e E^e) \Lambda^{r-1} \overline{E^e} dx - \text{Im} \int_{\mathbb{R}^d} \Lambda^{r-1} (E^e \times B^e) \Lambda^{r-1} \overline{E^e} dx \leq C (\|\eta^e\|_{H^{r-1}} \|E^e\|_{L^\infty} + \|\eta^e\|_{L^4} \|E^e\|_{H^{r-1,4}}) \|\Lambda^{r-1} \overline{E^e}\|_{L^2} + C (\|B^e\|_{H^{r-1}} \|E^e\|_{L^\infty} + \|B^e\|_{L^2} \|E^e\|_{H^{r-1,\infty}}) \|\Lambda^{r-1} \overline{E^e}\|_{L^2} \leq C (\|\eta^e\|_{H^r}^2 + \|B^e\|_{H^{r-1}}^2 + \|E^e\|_{H^{r+1}}^2). \quad (3.14)$$

We then multiply (3.3) by $\Lambda^{2r-6} B^e_t$ and obtain

$$\frac{1}{2} \frac{d}{dt} (\|\Lambda^{r-3} B^e_t\|_{L^2}^2 + \|\Lambda^{r-1} B^e\|_{L^2}^2 + \|\Lambda^{r-2} B^e\|_{L^2}^2) \leq \int_{\mathbb{R}^d} \Lambda^{r+1} (E^e \times \overline{E^e}) \Lambda^{r-3} B^e_t d x \leq C (\|E^e\|_{H^{r+1}} \|\overline{E^e}\|_{L^\infty} + \|E^e\|_{L^\infty} \|\overline{E^e}\|_{H^{r+1}}) \|\Lambda^{r-3} B^e_t\|_{L^2} \leq C (\|B^e_t\|_{H^{r-3}}^2 + \|E^e\|_{H^{r+1}}^2). \quad (3.15)$$
Similarly, by taking inner product of (3.2) with $\Lambda^{2r-2}n^\varepsilon_t$, then we can get

$$
\frac{1}{2} \frac{d}{dt} (\|\Lambda^{r-1}n^\varepsilon_t\|_{L^2}^2 + \|\Lambda^r n^\varepsilon\|_{L^2}^2) = - \int_{\mathbb{R}^d} \Lambda^{r+1}(E^\varepsilon \cdot \overline{E^\varepsilon})\Lambda^{r-1}n^\varepsilon_t \, dx \\
\leq C(\|n^\varepsilon_t\|_{H^{r-1}}^2 + \|E^\varepsilon\|_{H^{r+1}}^2).
$$

(3.16)

Now summing the estimates (3.11)–(3.16), and integrating the result, we can obtain

$$
\|E^\varepsilon\|_{H^{r+1}}^2 + \|n^\varepsilon_t\|_{H^{r-1}}^2 + \|n^\varepsilon\|_{H^r}^2 + \|B^\varepsilon_t\|_{H^{r-3}}^2 + \|B^\varepsilon\|_{H^{r-1}}^2 \leq C + C \int_0^t (\|E^\varepsilon\|_{H^{r+1}}^2 + \|n^\varepsilon_t\|_{H^{r-1}}^2 + \|n^\varepsilon\|_{H^r}^2 + \|B^\varepsilon_t\|_{H^{r-3}}^2 + \|B^\varepsilon\|_{H^{r-1}}^2) \, d\tau,
$$

hence by Gronwall’s inequality, we get

$$
\|E^\varepsilon\|_{H^{r+1}}^2 + \|n^\varepsilon_t\|_{H^{r-1}}^2 + \|n^\varepsilon\|_{H^r}^2 + \|B^\varepsilon_t\|_{H^{r-3}}^2 + \|B^\varepsilon\|_{H^{r-1}}^2 \leq C, \\
\forall \, t \in [0, T_{\text{max}}).
$$

(3.17)

The estimate (3.17) implies that the solution $(E^\varepsilon, n^\varepsilon, B^\varepsilon)$ can be extended to the interval $[0, T_{\text{max}} + \delta]$ for some $\delta > 0$, which contradicts the definition of $T_{\text{max}}$, hence $T_{\text{max}} = \infty$. Therefore, the solution for the regularized system (3.1)–(3.4) exists globally in time. This ends the proof of Theorem 3.1.

$\square$

### 4 A priori estimates

We will approximate the solution of the magnetic Zakharov system (1.1)–(1.2) by smooth solutions for the regularized system introduced in Sect. 3. Hence, in order to get strong or weak limit of these smooth solutions in the $L^\infty_t H^r_x$ topology, one must show that the approximate solutions are uniformly bounded in this energy norm. Therefore, we are devoted to establishing a priori estimates for (3.1)–(3.4) in this section.

**Proposition 4.1** Let $s > \frac{d}{2}$, $E_0 \in H^{s+1}$, $(n_0, n_1) \in H^s \oplus (H^{s-1} \cap \dot{H}^{-1})$, $(B_0, B_1) \in (H^s \cap \dot{H}^{-1}) \oplus (H^{s-2} \cap \dot{H}^{-2})$. Assume the sequence $\{(E_0^\varepsilon, n_0^\varepsilon, n_1^\varepsilon, B_0^\varepsilon, B_1^\varepsilon)\}$ satisfying $E_0^\varepsilon \in H^{r+1}$, $(n_0^\varepsilon, n_1^\varepsilon) \in H^r \oplus (H^{r-1} \cap \dot{H}^{-1})$, $(B_0^\varepsilon, B_1^\varepsilon) \in (H^{r-1} \cap \dot{H}^{-1}) \oplus (H^{r-3} \cap \dot{H}^{-2})$ with $r$ large enough, and

$$
E_0^\varepsilon \to E_0 \text{ in } H^{s+1}, \quad (n_0^\varepsilon, n_1^\varepsilon) \to (n_0, n_1) \text{ in } H^s \oplus (H^{s-1} \cap \dot{H}^{-1}), \\
(B_0^\varepsilon, B_1^\varepsilon) \to (B_0, B_1) \text{ in } (H^s \cap \dot{H}^{-1}) \oplus (H^{s-2} \cap \dot{H}^{-2}).
$$

(4.1)
as $\epsilon \to 0$. If $(E^\epsilon, n^\epsilon, B^\epsilon)$ is the smooth solution for the regularized system (3.1)–(3.3) with initial data $(E_0^\epsilon, n_0^\epsilon, n_1^\epsilon, B_0^\epsilon, B_1^\epsilon)$, then there exist $T > 0$ and $C > 0$ such that

\[
\|E^\epsilon\|_{C([0,T]; H^{s+1})} + \|n^\epsilon\|_{C([0,T]; H^s)} + \|B^\epsilon\|_{C([0,T]; H^s \cap \dot{H}^{-1})} \leq C, \\
\|E_t^\epsilon\|_{C([0,T]; H^{s-1})} + \|n_t^\epsilon\|_{C([0,T]; H^{s-1} \cap \dot{H}^{-1})} + \|B_t^\epsilon\|_{C([0,T]; H^{s-2} \cap \dot{H}^{-2})} \leq C, 
\]

where $T$ and $C$ are dependent of the norm of $(E_0, n_0, n_1, B_0, B_1)$, but independent of $\epsilon$.

**Proof** Note that the constant $C$ given in (3.17) depends on $\epsilon$, hence, one can not use the same argument that leads to (3.17) to obtain the desired estimate (4.2). In order to derive a bound independent of $\epsilon$, we first rewrite (3.1) in the following form

\[
i E_t^\epsilon + \mathcal{L} \nabla (\nabla \cdot E^\epsilon) - \alpha \mathcal{L} \nabla \times (\nabla \times E^\epsilon) - \mathcal{L} (n^\epsilon E^\epsilon) + i \mathcal{L} (E^\epsilon \times B^\epsilon) = 0, 
\]

where $\mathcal{L} = (I + \epsilon^2 \Delta^2)^{-1}$. It is easy to see that $\mathcal{L}$ satisfies the following properties:

\[
(1) \|\mathcal{L} f\|_{H^k} \leq \|f\|_{H^k}, \quad \forall k \in \mathbb{R}, \\
(2) (\mathcal{L} f, f) = \int_{\mathbb{R}^d} (\mathcal{L} f) \cdot \bar{f} \, dx \geq 0, \\
(3) (\mathcal{L} f, g) = (f, \mathcal{L} g), \\
(4) \int_{\mathbb{R}^d} (\mathcal{L} \Lambda^s f) g \, dx = \int_{\mathbb{R}^d} (\Lambda^s \mathcal{L} f) g \, dx, \quad \forall s \in \mathbb{R}.
\]

Property (4) in (4.4) says that $\mathcal{L}$ commutes with the Fourier multiplier $\Lambda^s$. Indeed, this property still holds with $\Lambda^s$ replaced by other Fourier multipliers such as $\nabla$. Since (4.1) holds, we have

\[
\|E_0^\epsilon\|_{H^{s+1}} + \|n_0^\epsilon\|_{H^s} + \|n_1^\epsilon\|_{H^{s-1} \cap \dot{H}^{-1}} + \|B_0^\epsilon\|_{H^s \cap \dot{H}^{-1}} + \|B_1^\epsilon\|_{H^{s-2} \cap \dot{H}^{-2}} \leq c_0,
\]

where the constant $c_0$ depends only on $\|E_0\|_{H^{s+1}}$, $\|n_0\|_{H^s}$, $\|n_1\|_{H^{s-1} \cap \dot{H}^{-1}}$, $\|B_0\|_{H^s \cap \dot{H}^{-1}}$ and $\|B_1\|_{H^{s-2} \cap \dot{H}^{-2}}$. Now we divide the proof into the following three steps.

**Step 1. Low order norm estimates.**

By the conserved quantities (3.5) and (3.6), there holds that

\[
\|E^\epsilon\|_{H^1}^2 + \|n^\epsilon\|_{L^2}^2 + \|n_t^\epsilon\|_{\dot{H}^{-1}}^2 + \|B^\epsilon\|_{L^2}^2 + \|B_t^\epsilon\|_{\dot{H}^{-1}}^2 + \|B_t^\epsilon\|_{\dot{H}^{-2}}^2 \leq C(c_0) + \int_{\mathbb{R}^d} n^\epsilon |E^\epsilon|^2 \, dx + \int_{\mathbb{R}^d} (E^\epsilon \times \overline{E^\epsilon}) \cdot B^\epsilon \, dx. 
\]

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Then by Cauchy–Schwarz inequality, we have

\[
\left| \int_{\mathbb{R}^d} (E^e \times \overline{E^e}) \cdot B^e \, dx \right| \leq \frac{1}{2} \|B^e\|_{L^2}^2 + \frac{1}{2} \|E^e\|_{L^4}^4, \tag{4.6}
\]

\[
\left| \int_{\mathbb{R}^d} n^e |E^e|^2 \, dx \right| \leq \frac{1}{2} \|n^e\|_{L^2}^2 + \frac{1}{2} \|E^e\|_{L^4}^4.
\]

Since \(|E^e|_t^2 = 2 \text{Im}(i E^e_t \cdot \overline{E^e})\), then (4.3) yields

\[
|E^e|_t^2 = 2 \text{Im} \left[ \left( -\mathcal{L} \nabla (\nabla \cdot E^e) + \alpha \mathcal{L} \nabla \times (\nabla \times E^e) + \mathcal{L}(n^e E^e) \right) \cdot \overline{E^e} \right], \tag{4.7}
\]

hence we can obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^d} |E^e|^4 \, dx = 2 \int_{\mathbb{R}^d} |E^e|^2 |E^e|_t^2 \, dx \leq C (\|E^e\|_{H^{s+1}}^4 + \|E^e\|_{H^{s+1}}^4 (\|n^e\|_{H^{s}} + \|B^e\|_{H^{s}}))^3 d\tau.
\]

which implies that

\[
\|E^e\|_{L^4}^4 \leq C + C \int_{0}^{t} (\|E^e\|_{H^{s+1}}^2 + \|n^e\|_{H^{s}}^2 + \|B^e\|_{H^{s}}^2 + 1)^3 d\tau. \tag{4.8}
\]

Inserting the above estimate into (4.5)–(4.6), then we have

Step 2. High order norm estimates.
Applying the operator \(\Lambda^s\) to equation (4.3), then one has

\[
i \Lambda^s E^e_t + \Lambda^s \mathcal{L} \nabla (\nabla \cdot E^e) - \alpha \Lambda^s \mathcal{L} \nabla \times (\nabla \times E^e) - \Lambda^s \mathcal{L}(n^e E^e) \\
+ i \Lambda^s \mathcal{L}(E^e \times B^e) = 0.
\]
Taking inner product of this equation with $-\Lambda^s \nabla (\nabla \cdot E^\epsilon) + \alpha \Lambda^s \nabla \times (\nabla \times E^\epsilon)$, and using the properties for $\mathcal{L}$ given by (4.4), one can obtain

$$\frac{d}{dt} (\| \Lambda^s (\nabla \cdot E^\epsilon) \|^2_{L^2} + \alpha \| \Lambda^s (\nabla \times E^\epsilon) \|^2_{L^2})$$

$$= 2 \text{Im} \int_{\mathbb{R}^d} \mathcal{L} \Lambda^s (n^\epsilon E^\epsilon) \cdot (-\Lambda^s \nabla (\nabla \cdot E^\epsilon) + \alpha \Lambda^s \nabla \times (\nabla \times E^\epsilon)) dx$$

$$+ 2 \text{Im} i \int_{\mathbb{R}^d} \mathcal{L} \Lambda^s (E^\epsilon \times B^\epsilon) \cdot (\Lambda^s \nabla (\nabla \cdot E^\epsilon) - \alpha \Lambda^s \nabla \times (\nabla \times E^\epsilon)) dx$$

$$=: I_1 + I_2. \quad (4.9)$$

Multiplying equation (3.3) by $\Lambda^{2s-4} B^\epsilon_t$, and using the fact

$$(E^\epsilon \times E^\epsilon)_t = 2i \text{Im} (E^\epsilon_t \times E^\epsilon)$$

$$= 2i \text{Re} \left[ (\mathcal{L} \nabla (\nabla \cdot E^\epsilon) - \alpha \mathcal{L} \nabla \times (\nabla \times E^\epsilon)) \times E^\epsilon \right], \quad (4.10)$$

then we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \| \Lambda^{s-2} B^\epsilon_t \|^2_{L^2} + \| \Lambda^s B^\epsilon \|^2_{L^2} + \| \Lambda^{s-1} B^\epsilon \|^2_{L^2} + 2i \int_{\mathbb{R}^d} \Lambda^s (E^\epsilon \times E^\epsilon) \Lambda^s B^\epsilon dx \right)$$

$$= i \int_{\mathbb{R}^d} \Lambda^s B^\epsilon \Lambda^s (E^\epsilon \times E^\epsilon)_t dx$$

$$= -2 \text{Re} \int_{\mathbb{R}^d} \Lambda^s B^\epsilon \Lambda^s \left[ (\mathcal{L} \nabla (\nabla \cdot E^\epsilon) - \alpha \mathcal{L} \nabla \times (\nabla \times E^\epsilon)) \times E^\epsilon \right] dx$$

$$- 2 \text{Re} \int_{\mathbb{R}^d} \Lambda^s B^\epsilon \Lambda^s \left[ ( -\mathcal{L}(n^\epsilon E^\epsilon) + i \mathcal{L}(E^\epsilon \times B^\epsilon)) \times E^\epsilon \right] dx$$

$$=: I_3 + I_4. \quad (4.11)$$

It is easy to see that

$$|I_4| \leq C \| B^\epsilon \|_{H^s} (\| n^\epsilon \|_{H^s} \| E^\epsilon \|^2_{H^{s+1}} + \| B^\epsilon \|_{H^s} \| E^\epsilon \|^2_{H^{s+1}})$$

$$\leq C (1 + \| n^\epsilon \|^2_{H^s} + \| B^\epsilon \|^2_{H^s} + \| E^\epsilon \|^2_{H^{s+1}}). \quad (4.12)$$
Now we estimate $I_2 + I_3$. Rewrite $I_2 + I_3$ in the form

$$I_2 + I_3 = -2\text{Re} \int_{\mathbb{R}^d} \Lambda^{s+1} (E^\epsilon \times B^\epsilon) \cdot \left( \Lambda^{s-1} \mathcal{L} \nabla (\nabla \cdot E^\epsilon) - \Lambda^{s-1} \alpha \mathcal{L} \nabla \right) dx$$

$$+ 2\text{Re} \int_{\mathbb{R}^d} \Lambda^s B^\epsilon \Lambda^s \left[ \overline{E^\epsilon} \times \left( \mathcal{L} \nabla (\nabla \cdot E^\epsilon) - \alpha \mathcal{L} \nabla \times (\nabla \times E^\epsilon) \right) \right] dx,$$

if we take $f = E^\epsilon$, $g = \Lambda^{-2} \mathcal{L} \nabla (\nabla \cdot E^\epsilon) - \Lambda^{-2} \alpha \mathcal{L} \nabla \times (\nabla \times E^\epsilon)$, $\bar{h} = B^\epsilon$, then by (4.22) in Lemma 4.1 below, we have

$$I_2 + I_3 = -2\text{Re} \int_{\mathbb{R}^d} \Lambda^{s+1} (f \times \bar{h}) \Lambda^{s+1} \bar{g} dx + 2\text{Re} \int_{\mathbb{R}^d} \Lambda^s (\bar{f} \times \Lambda^2 g) \Lambda^s \bar{h} dx$$

$$\leq C \| f \|_{H^{s+1}} \| g \|_{H^{s+1}} \| \bar{h} \|_{H^s}$$

$$\leq C (\| E^\epsilon \|_{H^{s+1}}^2 + \| B^\epsilon \|_{H^s}^2 + 1)^2. \quad (4.13)$$

Next, we multiply equation (3.2) by $\Lambda^{2s-2} n^\epsilon_t$ and get

$$\frac{1}{2} \frac{d}{dt} \left( \| \Lambda^{s-1} n^\epsilon_t \|_{L^2}^2 + \| \Lambda^s n^\epsilon \|_{L^2}^2 + 2 \int \Lambda^s |E^\epsilon|^2 \Lambda^s n^\epsilon dx \right) = \int \Lambda^s n^\epsilon \Lambda^s |E^\epsilon|^2 dx$$

$$= 2\text{Im} \int_{\mathbb{R}^d} \Lambda^s n^\epsilon \Lambda^s \left[ - \mathcal{L} \nabla (\nabla \cdot E^\epsilon) + \alpha \mathcal{L} \nabla \times (\nabla \times E^\epsilon) \right] \cdot \overline{E^\epsilon} dx$$

$$+ 2\text{Im} \int_{\mathbb{R}^d} \Lambda^s n^\epsilon \Lambda^s \left[ (\mathcal{L}(n^\epsilon E^\epsilon) - i \mathcal{L}(E^\epsilon \times B^\epsilon)) \cdot \overline{E^\epsilon} \right] dx$$

$$=: I_5 + I_6. \quad (4.14)$$

Similar to the argument for (4.12), we obtain

$$|I_6| \leq C (1 + \| n^\epsilon \|_{H^s}^2 + \| B^\epsilon \|_{H^s}^2 + \| E^\epsilon \|_{H^{s+1}}^2)^2. \quad (4.15)$$

For the term $I_5 + I_6$, we take $f = E^\epsilon$, $g = -\Lambda^{-2} \mathcal{L} \nabla (\nabla \cdot E^\epsilon) + \Lambda^{-2} \alpha \mathcal{L} \nabla \times (\nabla \times E^\epsilon)$, $h = n^\epsilon$, then we use Lemma 4.1 below and obtain

$$I_1 + I_5 = 2\text{Im} \int_{\mathbb{R}^d} \Lambda^{s+1} (f h) \Lambda^{s+1} \bar{g} dx + 2\text{Im} \int_{\mathbb{R}^d} \Lambda^s (\bar{f} \Lambda^2 g) \Lambda^s \bar{h} dx$$

$$\leq C \| f \|_{H^{s+1}} \| g \|_{H^{s+1}} \| h \|_{H^s}$$

$$\leq C (\| E^\epsilon \|_{H^{s+1}}^2 + \| n^\epsilon \|_{H^s}^2 + 1)^2. \quad (4.16)$$
Putting (4.9) and (4.11)–(4.16) together, we arrive at
\[
\| \nabla \cdot E^\epsilon \|^2_{H^s} + \alpha \| \nabla \times E^\epsilon \|^2_{\dot{H}^s} + \| n^\epsilon \|^2_{H^s} + \| n_i^\epsilon \|^2_{\dot{H}^{s-1}} + \| B_i^\epsilon \|^2_{\dot{H}^{s-2}} + \| B^\epsilon \|^2_{\dot{H}^s \cap \dot{H}^{s-1}} \\
\leq C + C \int_0^t (1 + \| n^\epsilon \|^2_{H^s} + \| B^\epsilon \|^2_{\dot{H}^s} + \| E^\epsilon \|^2_{\dot{H}^{s+1}}) \, d\tau
\]
\[
+ 2 \left| \int_{\mathbb{R}^d} \Lambda^s (E^\epsilon \times \overline{E^\epsilon}) \Lambda^s B^\epsilon \, dx \right| + 2 \left| \int_{\mathbb{R}^d} \Lambda^s (E^\epsilon \times \overline{E^\epsilon}) \Lambda^s B^\epsilon \, dx \right| .
\]  
(4.17)

Now we estimate the last two terms in (4.17). By Cauchy–Schwarz inequality, we have
\[
\left| \int_{\mathbb{R}^d} \Lambda^s (E^\epsilon \times \overline{E^\epsilon}) \Lambda^s B^\epsilon \, dx \right| \leq \frac{1}{2} \| \Lambda^s B^\epsilon \|^2_{L^2} + \frac{1}{2} \| \Lambda^s (E^\epsilon \times \overline{E^\epsilon}) \|^2_{L^2} .
\]  
(4.18)

Then by (4.10) we have
\[
\frac{d}{dt} \int_{\mathbb{R}^d} (\Lambda^s (E^\epsilon \times \overline{E^\epsilon}))^2 \, dx = 2 \int_{\mathbb{R}^d} \Lambda^s (E^\epsilon \times \overline{E^\epsilon}) \Lambda^s (E^\epsilon \times \overline{E^\epsilon}) \, dx \\
\leq C (\| E^\epsilon \|^4_{H^{s+1}} + \| E^\epsilon \|^4_{H^{s+1}} (\| n^\epsilon \|_{H^s} + \| B^\epsilon \|_{H^s})).
\]

So the left-hand side of (4.18) can be bounded by
\[
\frac{1}{2} \| \Lambda^s B^\epsilon \|^2_{L^2} + C + C \int_0^t (\| E^\epsilon \|^2_{H^{s+1}} + \| n^\epsilon \|^2_{H^s} + \| B^\epsilon \|^2_{H^s} + 1)^3 \, d\tau.
\]

In a similar way, we can treat the term \( \left| \int_{\mathbb{R}^d} \Lambda^s |E^\epsilon|^2 \Lambda^s n^\epsilon \, dx \right| \). Now combining the above estimates, we have
\[
\text{LHS of (4.17)} \leq C + C \int_0^t (\| E^\epsilon \|^2_{H^{s+1}} + \| n^\epsilon \|^2_{H^s} + \| B^\epsilon \|^2_{H^s} + 1)^3 \, d\tau .
\]  
(4.19)

**Step 3. Conclusion**

We deduce from (4.8) and (4.19) that
\[
\| E^\epsilon \|^2_{H^{s+1}} + \| n^\epsilon \|^2_{H^s} + \| B^\epsilon \|^2_{H^s \cap H^{-1}} + \| n_i^\epsilon \|^2_{H^{s-1} \cap H^{-1}} + \| B_i^\epsilon \|^2_{H^{s-2} \cap H^{-2}} \\
\leq C + C \int_0^t (\| E^\epsilon \|^2_{H^{s+1}} + \| n^\epsilon \|^2_{H^s} + \| B^\epsilon \|^2_{H^s} + 1)^3 \, d\tau ,
\]
where \( C \) depends on \( c_0 \), hence we know there exist \( T > 0 \) and \( C > 0 \) both independent of \( \epsilon \) such that for all \( t \in [0, T] \)
\[
\| E^\epsilon \|_{\tilde{H}^{s+1}}^2 + \| n^\epsilon \|_{\tilde{H}^s}^2 + \| B^\epsilon \|_{\tilde{H}^{s-1} \cap \tilde{H}^{-1}}^2 + \| n_t^\epsilon \|_{\tilde{H}^{s-1} \cap \tilde{H}^{-1}}^2 + \| B_t^\epsilon \|_{\tilde{H}^{s-2} \cap \tilde{H}^{-2}}^2 \leq C, \quad (4.20)
\]
from which Proposition 4.1 follows. \( \square \)

Now we prove the following lemma, which plays an important role in our proof of Proposition 4.1.

**Lemma 4.1** Assume that \( f, g \in H^{s+1}(\mathbb{R}^d) \) are \( \mathbb{C}^3 \) valued functions, and \( \tilde{h} \in H^s(\mathbb{R}^d) \) is a \( \mathbb{R}^3 \) valued function, and \( h \in H^s(\mathbb{R}^d) \) is a real valued function, \( s > \frac{d}{2} \). Then the following two estimates hold:
\[
\left| \text{Im} \int_{\mathbb{R}^d} \Lambda^{s+1}(f h) \cdot \Lambda^{s+1} \bar{g} dx + \text{Im} \int_{\mathbb{R}^d} \Lambda^s(\bar{f} \cdot \Lambda^2 g) \Lambda^s h dx \right| \leq C \| f \|_{H^{s+1}} \| g \|_{H^{s+1}} \| h \|_{H^s}, \tag{4.21}
\]
\[
\left| \text{Re} \int_{\mathbb{R}^d} \Lambda^{s+1}(f \times \tilde{h}) \cdot \Lambda^{s+1} \bar{g} dx - \text{Re} \int_{\mathbb{R}^d} \Lambda^s(\bar{f} \times \Lambda^2 g) \cdot \Lambda^s \tilde{h} dx \right| \leq C \| f \|_{H^{s+1}} \| g \|_{H^{s+1}} \| \tilde{h} \|_{H^s}. \tag{4.22}
\]

To prove Lemma 4.1, we need the following lemma, the proof of which can be found in [5,12].

**Lemma 4.2** Assume that \( s > 0 \) and \( p \in (1, +\infty) \). If \( f, g \in \mathcal{S}(\mathbb{R}^d) \), the Schwarz class, then
\[
\| \Lambda^s(f g) - f(\Lambda^s g) \|_{L^p} \leq C \| \nabla f \|_{L^{p_1}} \| g \|_{\tilde{H}^{s-1}, p_2} + \| f \|_{\tilde{H}^{s}, p_3} \| g \|_{L^{p_4}} \tag{4.23}
\]
with \( p_2, p_3 \in (1, +\infty) \) such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

**Proof of Lemma 4.1** We first show (4.21). Denote the LHS of (4.21) by \( |J| = |J_1 + J_2| \).

The term \( J_1 \) can be written as
\[
J_1 = \text{Im} \int_{\mathbb{R}^d} [\Lambda^{s+1}(f h) - f \Lambda^{s+1} h] \cdot \Lambda^{s+1} \bar{g} dx + \text{Im} \int_{\mathbb{R}^d} f \Lambda^{s+1} h \cdot \Lambda^{s+1} \bar{g} dx
\]
\[
= \text{Im} \int_{\mathbb{R}^d} [\Lambda^{s+1}(f h) - f \Lambda^{s+1} h] \cdot \Lambda^{s+1} \bar{g} dx
\]
\[ + \operatorname{Im} \int_{\mathbb{R}^d} \Lambda^s h[\Lambda(f \cdot \Lambda^{s+1}g) - f \cdot \Lambda^{s+2}g] \, dx + \operatorname{Im} \int_{\mathbb{R}^d} f \Lambda^s h \cdot \Lambda^{s+2}g \, dx \]
\[ =: J_{11} + J_{12} + J_{13}. \quad (4.24) \]

From the commutator estimate (4.23), we have
\[ |J_{11}| \leq C(\|\nabla f\|_{L^\infty} \|h\|_{H^s} + \|f\|_{H^{s+1}} \|h\|_{L^\infty}) \|g\|_{H^{s+1}} \leq C \|f\|_{H^{s+1}} \|g\|_{H^{s+1}} \|h\|_{H^s}, \quad (4.25) \]

since \( s > \frac{d}{2} \). The term \( J_{12} \) can be estimated by commutator estimate (4.23) again
\[ |J_{12}| \leq C \|h\|_{H^s} (\|\nabla f\|_{L^\infty} \|g\|_{H^{s+1}} + \|f\|_{H^{1,p}} \|g\|_{H^{s+1,q}}), \]
where \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \). Since \( s > \frac{d}{2} \), we have \( H^{s+1} \hookrightarrow H^{1,p} \) for all \( p \in [2, \infty) \), then
\[ |J_{12}| \leq C \|h\|_{H^s} (\|f\|_{H^{s+1}} \|g\|_{H^{s+1}} + \|f\|_{H^{s+1}} \|g\|_{H^{s+1,q}}), \]

letting \( q \to 2^+ \) in the above inequality, then we get
\[ |J_{12}| \leq C \|f\|_{H^{s+1}} \|g\|_{H^{s+1}} \|h\|_{H^s}. \quad (4.26) \]

Now we estimate \( J_2 \). It is obvious that
\[ J_2 = \operatorname{Im} \int_{\mathbb{R}^d} [\Lambda^s (\bar{f} \cdot \Lambda^2 g) - \bar{f} \cdot \Lambda^{s+2} g] \Lambda^s h \, dx + \operatorname{Im} \int_{\mathbb{R}^d} \bar{f} \cdot \Lambda^{s+2} g \Lambda^s h \, dx \]
\[ =: J_{21} + J_{22}. \quad (4.27) \]

Using (4.23), we obtain
\[ |J_{21}| \leq C(\|\nabla f\|_{L^\infty} \|g\|_{H^{s+1}} + \|f\|_{H^{s,p}} \|g\|_{H^{2,q}}) \|h\|_{H^s}, \]

where we select \( p, q \) satisfying
\[ \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \frac{d}{2} \leq 1 + \frac{d}{p}, \quad 1 + \frac{d}{2} - \frac{d}{q} \leq s, \]

hence we have \( H^{s+1} \hookrightarrow H^{s,p} \) and \( H^{s+1} \hookrightarrow H^{2,q} \), and we get
\[ |J_{21}| \leq C \|f\|_{H^{s+1}} \|g\|_{H^{s+1}} \|h\|_{H^s}. \quad (4.28) \]
Since $J_{13} + J_{22} = 0$, now (4.21) follows from (4.24)–(4.28). If we replace $f$ by $i \cdot f$, then from (4.21) we can get
\[
\text{Re} \int_{\mathbb{R}^d} \Lambda^{s+1}(f \cdot h) \cdot \Lambda^{s+1} \tilde{g} dx - \text{Re} \int_{\mathbb{R}^d} \Lambda^s(f \cdot \Lambda^2 g) \Lambda^s h dx
\]
\[
\leq C \|f\|_{H^{s+1}} \|g\|_{H^{s+1}} \|h\|_{H^s}.
\]
Moreover, if we expand the term $\Lambda^{s+1}(f \times \tilde{h}) \cdot \Lambda^{s+1} \tilde{g}$ and $\Lambda^s(f \times \Lambda^2 g) \cdot \Lambda^s \tilde{h}$ by the definition of dot product and cross product, then one can easily see that (4.22) reduces to the above estimate. We thus finish the proof of Lemma 4.1. □

Under the estimates given in Proposition 4.1, we can prove that the solutions $(E^\epsilon, n^\epsilon, B^\epsilon)$ to the regularized system (3.1)–(3.4) form a Cauchy sequence in the space $C([0, T]; H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1}))$.

**Proposition 4.2** With the same assumptions as Proposition 4.1, then the family $(E^\epsilon, n^\epsilon, B^\epsilon)$ forms a Cauchy sequence in $C([0, T]; H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1}))$, namely,

\[
\sup_{t \in [0, T]} (\|E^\epsilon - E^{\epsilon'}\|_{H^1} + \|n^\epsilon - n^{\epsilon'}\|_{L^2} + \|B^\epsilon - B^{\epsilon'}\|_{L^2 \cap \dot{H}^{-1}}) \to 0, \quad \epsilon, \epsilon' \to 0,
\]

\[
\sup_{t \in [0, T]} (\|E^\epsilon_t - E^{\epsilon'}_t\|_{H^{-1}} + \|n^\epsilon_t - n^{\epsilon'}_t\|_{\dot{H}^{-1}} + \|B^\epsilon_t - B^{\epsilon'}_t\|_{\dot{H}^{-2}}) \to 0, \quad \epsilon, \epsilon' \to 0.
\]

We give a sketch of the proof of Proposition 4.2. For brevity, we set $E = E^\epsilon - E^{\epsilon'}, n = n^\epsilon - n^{\epsilon'}, B = B^\epsilon - B^{\epsilon'}$. Since $(E^\epsilon, n^\epsilon, B^\epsilon)$ and $(E^{\epsilon'}, n^{\epsilon'}, B^{\epsilon'})$ both satisfy the regularized system (3.1)–(3.3), then $(E, n, B)$ satisfies

\[
iE_i + \mathcal{L}(\nabla \cdot E) - \alpha \mathcal{L}(\nabla \times (\nabla \times E)) - \mathcal{L}(nE^\epsilon + n^{\epsilon'}E) + i\mathcal{L}(E \times B^\epsilon + E^{\epsilon'} \times B) = 0,
\]

\[
n_{tt} - \Delta n = \Delta(|E^\epsilon|^2 - |E^{\epsilon'}|^2), \quad B_{tt} + \Delta^2 B - \Delta B = -i\Delta^2(E^\epsilon \times E^{\epsilon'} - E^{\epsilon'} \times E^{\epsilon'})
\]

with initial data

\[
E(0) = E^\epsilon_0 - E^{\epsilon'}_0, \quad n(0) = n^\epsilon_0 - n^{\epsilon'}_0, \quad n_t(0) = n^\epsilon_1 - n^{\epsilon'}_1,
\]

\[
B(0) = B^\epsilon_0 - B^{\epsilon'}_0, \quad B_t(0) = B^\epsilon_1 - B^{\epsilon'}_1.
\]
For the above equations, applying similar energy estimate as Proposition 4.1 and using the estimate (4.2), then it is not hard to derive the following estimate

\[
\|E\|_{H^1}^2 + \|n\|_{L^2}^2 + \|B\|_{L^2 \cap H^{-1}}^2 + \|n_t\|_{H^{-1}}^2 + \|B_t\|_{\dot{H}^{-2}}^2 \\
\leq C(\|E(0)\|_{H^1}^2 + \|n(0)\|_{L^2}^2 + \|B(0)\|_{L^2 \cap H^{-1}}^2 + \|n_t(0)\|_{H^{-1}}^2 + \|B_t(0)\|_{\dot{H}^{-2}}^2) \\
+ C \int_0^t (\|E\|_{H^1}^2 + \|n\|_{L^2}^2 + \|B\|_{L^2}^2) d\tau.
\]

Hence, by Gronwall’s inequality, we see that (4.29) follows from the above estimate.

5 Proof of the main theorem

Proof of Theorem 1.1 For given \( E_0 \in H^{s+1}, \ (n_0, n_1) \in H^s \oplus (H^{s-1} \cap \dot{H}^{-1}), \)
\((B_0, B_1) \in (H^s \cap \dot{H}^{-1}) \oplus (H^{s-2} \cap \dot{H}^{-2}), \) we choose \((E^\epsilon_0, n^\epsilon_0, n^\epsilon_1, B^\epsilon_0, B^\epsilon_1)\) sufficiently regular such that (4.1) holds. Then by the strong convergence result (4.29) and the interpolation in Sobolev spaces, we know that there exists \((E, n, B)\) satisfying

\[
\sup_{t \in [0, T]} (\|E^\epsilon - E\|_{H^{\tilde{s}+1}} + \|n^\epsilon - n\|_{H^{\tilde{s}}} + \|B^\epsilon - B\|_{H^{\tilde{s} \cap \dot{H}^{-1}}}) \to 0,
\]

(5.1)

for all \(\tilde{s} < s\) as \(\epsilon \to 0\). Moreover, for all \(\tilde{s} < s\), we have

\[
n^\epsilon E^\epsilon \to nE, \quad E^\epsilon \times B^\epsilon \to E \times B, \quad \text{in } C([0, T]; H^{\tilde{s}}),
\]

\[
|E^\epsilon|^2 \to |E|^2, \quad E^\epsilon \times E^\epsilon \to E \times E, \quad \text{in } C([0, T]; H^{\tilde{s}+1}).
\]

Now letting \(\epsilon \to 0\) in (3.1)–(3.4), and using the above strong convergence properties, we finally see that \((E, n, B)\) is a solution of the original magnetic Zakharov system (1.1)–(1.2). From Proposition 4.1, we know that the existence time \(T\) depends on the norm of the initial data. In fact, if \(T_{\text{max}}\) is the maximal lifespan of the solution, then either \(T_{\text{max}} = \infty\) or \(T_{\text{max}} < \infty\) and

\[
\|E(t)\|_{H^{s+1}} + \|n(t)\|_{H^{s}} + \|n_t(t)\|_{H^{s-1} \cap \dot{H}^{-1}} + \|B(t)\|_{H^{s} \cap \dot{H}^{-1}} \\
+ \|B_t(t)\|_{H^{s-2} \cap \dot{H}^{-2}} \to \infty
\]

as \(t \to T_{\text{max}}\). Hence, the local existence part of Theorem 1.1 is proved. The proof of uniqueness is essentially the same as the proof of (4.29), thus we omit the details. This ends the proof of Theorem 1.1.

Note that Theorem 1.1 needs the additional assumption \(n_1, B_0 \in \dot{H}^{-1}, \ B_1 \in \dot{H}^{-2}\). As stated in Sect. 1, this assumption is rather strong. In fact, this condition can be removed by splitting the initial data into low-frequency part and high-frequency part.
Denote \( \varphi(\xi) \in C^\infty_c(\mathbb{R}^d) \) such that \( 0 \leq \varphi(\xi) = \varphi(|\xi|) \leq 1, \) \( \varphi \equiv 1 \) if \( |\xi| \leq 1 \) and \( \varphi \equiv 0 \) if \( |\xi| \geq 2. \) For any given \( f \in H^r(\mathbb{R}^d), \ r \in \mathbb{R}, \) we decompose \( f = f_L + f_H, \) where
\[
\hat{f}_L = \varphi(\xi) \hat{f}, \quad \hat{f}_H = (1 - \varphi(\xi)) \hat{f}.
\]

So one can easily see that \( f_L \in H^k \) for all \( k \in \mathbb{R} \) and \( f_H \in \dot{H}^l \cap H^l \) for all \( l \leq r. \) In this way, we can decompose \( n_1 \in H^s \) as \( n_1 = n_{1L} + n_{1H} \) with \( n_{1L} \in H^k \) for all \( k \in \mathbb{R}, \) \( n_{1H} \in \dot{H}^l \cap H^l \) for all \( l \leq s, \) and in particular \( n_{1H} \in H^s \cap \dot{H}^{-1}. \) Moreover we have
\[
\|n_{1L}\|_{H^s} \leq C(k, s)\|n_1\|_{H^s}, \quad \|n_{1H}\|_{H^s \cap \dot{H}^{-1}} \leq \|n_1\|_{H^s}. \tag{5.2}
\]

Similarly, for given \( B_0 \in H^s, \ B_1 \in H^{s-2}, \) we have \( B_0 = B_{0L} + B_{0H}, \ B_1 = B_{1L} + B_{1H}, \) where \( B_{0L} \) and \( B_{1L} \) are good functions, \( B_{0H} \in H^s \cap \dot{H}^{-1} \) and \( B_{1H} \in H^{s-2} \cap H^{-2}. \)

Now we set
\[
\tilde{n} = n - tn_{1L}, \quad \tilde{B} = B - B_{0L} - tB_{1L}, \tag{5.3}
\]
and consider the system
\[
\begin{align*}
    iE_t + \nabla(\nabla \cdot E) - \alpha \nabla \times (\nabla \times E) - (\tilde{n} + tn_{1L})E + iE \times (\tilde{B} + B_{0L}) + tB_{1L} &= 0, \\
    \tilde{n}_{tt} - \Delta \tilde{n} - \Delta |E|^2 - t\Delta n_{1L}, \\
    \tilde{B}_{tt} + \Delta^2 \tilde{B} - \Delta \tilde{B} &= -i\Delta^2(E \times \tilde{E}) - \Delta^2(B_{0L} + tB_{1L}) + \Delta(B_{0L} + tB_{1L})
\end{align*} \tag{5.4}
\]

with initial data
\[
E(0, x) = E_0, \ (\tilde{n}(0, x), \tilde{n}_t(0, x)) = (n_0, n_{1H}), \ (\tilde{B}(0, x), \tilde{B}_t(0, x)) = (B_{0H}, B_{1H}). \tag{5.5}
\]

Note that the initial data (5.5) satisfies the condition \( \tilde{n}_t(0), \ B(0) \in \dot{H}^{-1} \) and \( \tilde{B}_t(0) \in H^{-2}. \) We also remark that if \( (E, n, B) \) solves (1.1)–(1.2), then \( (E, \tilde{n}, \tilde{B}) \) defined by (5.3) solves (5.4)–(5.5), and vice versa.

For the regular solution of (5.4), a similar argument as Proposition 2.1 gives that
\[
\|E(t)\|_{L^2} = \|E_0\|_{L^2} \quad \text{and}
\]
\[
\frac{d}{dt} \left( \|\nabla \cdot E(t)\|_{L^2}^2 + \alpha \|\nabla \times E(t)\|_{L^2}^2 + \frac{1}{2} \||\tilde{n}(t)||_{L^2}^2 + \frac{1}{2} \|\Lambda^{-1} \tilde{n}_t(t)\|_{L^2}^2 \\
    + \frac{1}{2} \|\Lambda^{-2} \tilde{B}_t(t)\|_{L^2}^2 + \frac{1}{2} \|\tilde{B}(t)\|_{L^2}^2 + \frac{1}{2} \|\Lambda^{-1} \tilde{B}(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} \tilde{n}(t) |E(t)|^2 dx \\
    + \int_{\mathbb{R}^d} tn_{1L} |E(t)|^2 dx + \int_{\mathbb{R}^d} \left( E(t) \times \overline{E(t)} \right) \cdot \tilde{B}(t) dx \right)
\]
+ i \int_{\mathbb{R}^d} (E(t) \times \overline{E(t)}) \cdot (B_{0L} + t B_{1L}) dx \\
= \int_{\mathbb{R}^d} n_{1L} |E(t)|^2 dx + i \int_{\mathbb{R}^d} B_{1L} \cdot (E(t) \times \overline{E(t)}) dx - t \int_{\mathbb{R}^d} \Lambda n_{1L} \Lambda^{-1} \tilde{n}_t dx \\
- \int_{\mathbb{R}^d} \Lambda^2 (B_{0L} + t B_{1L}) \Lambda^{-2} \tilde{B}_t dx - \int_{\mathbb{R}^d} (B_{0L} + t B_{1L}) \Lambda^{-2} \tilde{B}_t dx. \quad (5.6)

Using (5.2) and the fact \( \| E \|_{L^2} = \| E_0 \|_{L^2} \), we have

\[
\text{RHS of } (5.6) \leq C + C(1 + t)(\| \Lambda^{-1} \tilde{n}_t \|_{L^2}^2 + \| \Lambda^{-2} \tilde{B}_t \|_{L^2}^2)
\]

and

\[
\int_{\mathbb{R}^d} tn_{1L} |E(t)|^2 dx \leq t \| n_{1L} \|_{L^\infty} \| E_0 \|_{L^2}^2 \leq C t,
\]

\[
i \int_{\mathbb{R}^d} (E(t) \times \overline{E(t)}) \cdot (B_{0L} + t B_{1L}) dx \leq C(1 + t).
\]

Integrating (5.6), and applying the same method given in Lemma 2.2 and Gronwall’s inequality, we can bound the quantity \( \| E \|_{H^{1}}^2 + \| \tilde{n} \|_{L^2}^2 + \| \tilde{n}_t \|_{H^{-1}}^2 + \| \tilde{B} \|_{L^2 \cap H^{-1}}^2 + \| \tilde{B}_t \|_{H^{-2}}^2 \) by the norm of initial data (5.5). If we return to our original system, then we can obtain

\[
\| E(t) \|_{H^1}^2 + \| n(t) \|_{L^2}^2 + \| n_t(t) \|_{H^{-1}}^2 + \| B(t) \|_{L^2}^2 + \| B_t(t) \|_{H^{-2}}^2 \leq C
\]

with \( C \) depending on \( t, \| E_0 \|_{H^1}, \| n_0 \|_{L^2}, \| n_1 \|_{H^{-1}}, \| B_0 \|_{L^2} \) and \( \| B_1 \|_{H^{-2}}, \) provided that \( 2 \| E_0 \|_{L^2}^2 \leq \| Q \|_{L^2}^2 \) in the case \( d = 2 \) with \( Q \) the ground state solution of \( \Delta Q - Q + Q^3 = 0 \) or \( \| E_0 \|_{H^1} \) small in the case \( d = 3 \). Hence, the above estimate implies global existence of weak solutions for the magnetic Zakharov system.

**Theorem 5.1** If \( E_0 \in H^1, (n_0, n_1) \in L^2 \oplus H^{-1}, (B_0, B_1) \in L^2 \oplus H^{-2}, \) and the initial data satisfying \( 2 \| E_0 \|_{L^2}^2 \leq \| Q \|_{L^2}^2 \) in the case \( d = 2 \) and \( \| E_0 \|_{H^1} \) small in the case \( d = 3 \), then there exists a weak solution \( (E, n, B) \) for (1.1)–(1.2) in the sense of distributions such that

\[
E \in L^\infty_{loc}(\mathbb{R}^+; H^1), \quad (n, n_t) \in L^\infty_{loc}(\mathbb{R}^+; L^2 \oplus H^{-1}), \quad (B, B_t) \in L^\infty_{loc}(\mathbb{R}^+; L^2 \oplus H^{-2}).
\]

Due to (5.2), we see that the low-frequency part of \( n_1, B_0 \) and \( B_1 \) appearing in the equation (5.4) can be well controlled. Therefore, one can follow the same procedure as in Sects. 3, 4, 5 and then get the existence and uniqueness of solution for the system.
(5.4)–(5.5), which in turn leads to Theorem 1.2. Since this process is much the same as the proof of Theorem 1.1, the details are omitted. Hence, in this way, Theorem 1.2 is proved.

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References

1. Added, H., Added, S.: Existence globle de solutions fortes pour les équations de la turbulence de Langmuir en dimension 2. C. R. Acad. Sci. Paris 299, 551–554 (1984)
2. Bourgain, J., Colliander, J.: On wellposedness of the Zakharov system. Int. Math. Res. Notices 11, 515–546 (1996)
3. Bejenaru, I., Herr, S., Holmer, J., Tataru, D.: On the 2d Zakharov system with $L^2$ Schrödinger data. Nonlinearity 22, 1063–1089 (2009)
4. Colliander, J., Holmer, J., Tzirakis, N.: Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems. Trans. AMS 360, 4619–4638 (2008)
5. Coifman, R., Meyer, Y.: Nonlinear harmonic analysis operator theory and P.D.E. In: Beijing Lectures in Harmonic Analysis. Princeton University Press (1986)
6. Fang, D., Pecher, H., Zhong, S.: Low regularity global well-posedness for the two-dimensional Zakharov system. Analysis 29, 265–282 (2009)
7. Glangetas, L., Merle, F.: Concentration properties of blow-up solutions and instability results for Zakharov equation in dimension two. Part II. Commun. Math. Phys. 160, 349–389 (1994)
8. Guo, B., Shen, L.: The existence and uniqueness of the classical solution on the periodic initial value problem for Zakharov equation (in Chinese). Acta Math. Appl. Sin. 5, 310–324 (1982)
9. Ginibre, J., Tsutsumi, Y., Velo, G.: On the Cauchy problem for the Zakharov system. J. Funct. Anal. 151, 384–436 (1997)
10. He, X.: The pondermotive force and magnetic field generation effects resulting from the non-linear interaction between plasma-wave and particles (in Chinese). Acta Phys. Sin. 32, 325–337 (1983)
11. He, X.: The modulation instability and the collapse process of wave packet in plasma (in Chinese). Acta Phys. Sin. 32, 627–639 (1983)
12. Kato, T.: Liapunov functions and monotonicity in the Euler and Navier-Stokes equations. In: Lecture Notes in Mathematics, vol. 1450. Springer, Berlin (1990)
13. Kono, M., Skoric, M.M., Ter Haar, D.: Spontaneous excitation of magnetic fields and collapse dynamics in a Langmuir plasma. J. Plasma Phys. 26, 123–146 (1981)
14. Kenig, C., Wang, W.: Existence of local smooth solution for a generalized Zakharov system. J. Fourier Anal. Appl. 4, 469–490 (1998)
15. Laurey, C.: The Cauchy problem for a generalized Zakharov system. Differ. Integ. Equ. 8, 105–130 (1995)
16. Ozawa, T., Tsutsumi, Y.: Existence and smooth effect of solutions for the Zakharov equations. Pub. RIMS. Kyoto Univ. 28, 329–361 (1992)
17. Sulem, C., Sulem, P.L.: Quelques résultats de régularité pour les équation de la turbulence de Langmuir. C. R. Acad. Sci. Paris 289, 173–176 (1979)
18. Weinstein, M.I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Commun. Math. Phys. 87, 567–576 (1983)
19. Zakharov, V.E.: Collapse of Langmuir waves. Sov. Phys. JETP 35, 908–914 (1972)