Noncommutative Quantum Cosmology

H. García-Compeán\textsuperscript{a}, O. Obregón\textsuperscript{b} and C. Ramírez\textsuperscript{c}

\textsuperscript{a} Departamento de Física,  
Centro de Investigación y de Estudios Avanzados del IPN  
P.O. Box 14-740, 07000, México D.F., México  
E-mail address: compean@fis.cinvestav.mx  
\textsuperscript{b} Instituto de Física de la Universidad de Guanajuato  
P.O. Box E-143, 37150, León Gto., México  
E-mail address: octavio@ifug3.ugto.mx  
\textsuperscript{c} Facultad de Ciencias Físico Matemáticas,  
Universidad Autónoma de Puebla  
P.O. Box 1364, 72000, Puebla, México  
E-mail address: cramirez@fcfm.buap.mx  
(Dated: November 1, 2018)

We propose a model for noncommutative quantum cosmology by means of a deformation of minisuperspace. For the Kantowski-Sachs metric we are able to find the exact wave function. We construct wave packets and show that new quantum states that “compete” to be the most probable state appear, in clear contrast with the commutative case. A tunneling process could be possible among these states.

PACS numbers: 04.60.Kz, 11.10.Lm, 11.25.Sq, 98.80.Hw

The old proposal of noncommutativity in space-time \[ [1] \] has been recently subject of renewed interest (see reviews \[ [2, 4] \]). This is a consequence of the developments in M(atrix) theory \[ [5] \] and string theory, from which noncommutativity arises in the low energy effective field theory on a D-brane in a constant background B-field \[ [6, 7, 8, 9] \].

One of the most exciting recent applications of the idea of a minimal size to field theory, is that concerning the description of Yang-Mills instantons in four dimensional noncommutative spacetimes. It has been shown that in these spaces instantons acquire an effective size in terms of the noncommutativity parameter \( \theta \). As a consequence, the moduli space of noncommutative instantons no longer has the singularities corresponding to small instantons \[ [10] \]. This effect has a nice stringy interpretation \[ [11] \].

Noncommutative gravity has been considered in \[ [12, 13] \]. In particular in reference \[ [13] \] a deformed Einstein gravity is constructed by using the Seiberg-Witten map \[ [9] \], gauging the noncommutative ISO(3,1) group. Some aspects of noncommutative 2 + 1-dimensional Chern-Simons gravity have also been studied \[ [14, 15] \].

One of the puzzles in quantum gravity is the measurement of length, which seems to be limited to distances greater than the Planck length \( L_P \), because to locate a particle we would need an energy greater than the Planck mass \( M_P \). The corresponding gravitational field will have an horizon given by the Schwarzschild radius \( R = \frac{2GM_P}{c^2} = 2L_P \) and, whatever happens inside, this radius is shielded and therefore a minimal size should exist for quanta of space and time configuration.

Consequently, at very early times of the universe, before the Planck time, nontrivial effects of noncommutativity can be expected.

In the study of homogeneous universes, the metric depends only on the time parameter. Thus, the space dependence can be integrated out in the action and a model with a finite dimensional configuration space arises, called also minisuperspace, whose variables are the three-metric components. These theories have been considered by themselves, and their quantization is performed following the rules of quantum mechanics.

The minisuperspace construction is a procedure to define quantum cosmology models in the search to describe the quantum behavior of the very early stages of the universe \[ [16, 17] \]. By defining these models one necessarily freezes out degrees of freedom, so that these are only simple and probably approximate models of full quantum gravity at Planckian times. Formally, a kind of Klein Gordon equation arises in these models, which describes the quantum behavior of the universe. Actually, the validity of this approach remains as an open question to date. For example, within the context of Bianchi IX cosmology, it has been shown that imposing additional symmetry on the model alters its physical predictions \[ [18] \]. On the other hand, it has been argued that one can find conditions that must be satisfied to justify the minisuperspace approximation \[ [19] \]. In string theory formalism, general relativity, and consequently the Wheeler-DeWitt equation, corresponds to the s-wave approximation \[ [20] \]. Nevertheless, by considering a more general analysis, \[ [21] \], it seems that we can expect that the fundamental behavior of the wave function will be preserved.

Recently, attempts to connect M(atrix)-string theory to cosmology on the brane \[ [22] \] have been done. The fact that in the former theory noncommutativity has been shown to be present, motivated us to consider it also in models of the universe. In this paper we make a pro-
positional deformation of space-time, we consider a “deformation of minisuperspace”.

We will assume that the minisuperspace variables do not commute, as it has been proposed for the space-time coordinates. Our proposal is inspired in various related results in the literature, some of them already referred here, as is the case of the Seiberg-Witten map, where by demanding gauge invariance, a redefinition of the gauge fields as an expansion on the noncommutativity parameters is obtained. As a consequence of space-time noncommutativity, the fields do not commute among them in a specific manner dictated by the Moyal product. In our ansatz, we propose a simple and direct noncommutativity among certain components of the gravitational field.

It is well known that already in the usual spacetime, noncommutativity is usually defined in the “preferred” frame of cartesian coordinates, where the noncommutative parameters are taken to be constant. For any other coordinate systems, the corresponding noncommutativity will be, in general, in terms of parameters related in a complicated manner to those of cartesian coordinates. For the even-dimensional case, the spacetimes can be interpreted as symplectic manifolds with the noncommutativity parameters playing the role of a symplectic form. In this case, Durboux’s theorem ensures that always, locally, there exists a coordinate system in which the components of the symplectic form are constant. In string theory this is assumed when a ‘constant’ B-field is considered.

On the other hand, the Seiberg-Witten map for gravity has been proposed in [24], where noncommutative tetrads and connections are computed. In the case of quantum cosmology, the minisuperspace variables play the role of the “coordinates” of the configuration space. Thus, it seems reasonable to propose a kind of noncommutativity among these specific gravitational variables, as it is the case in standard spacetime, when cartesian coordinates are selected. From the canonical quantization of a cosmological model a quantum mechanical version, of the general Wheeler-DeWitt equation for general relativity arises. In this way, quantum cosmology is usually understood as a quantum mechanical model of the universe. We will then further assume, that it can be proceeded as in standard noncommutative quantum mechanics.

The noncommutativity we propose, can be reformulated in terms of a Moyal deformation of the Wheeler-DeWitt equation, similar to the case of the noncommutative Schrödinger equation [25]. Actually, cosmology depends only on time and a Moyal product of functions of only one variable is trivially realized. Other authors have worked out noncommutativity in the early universe, however without considering the gravitational field. In Ref. [26] the noncommutativity of space-time is interpreted as a magnetic field on a horizon scale. In [25, 26], it is argued that noncommutativity affecting matter or gauge fields could have played an important role to produce inflation.

As an example of our proposal, we will consider the cosmological model of the Kantowski-Sachs metric. In the parametrization due to Misner, this metric looks like [27]:

$$ds^2 = -N^2dt^2 + e^{2\sqrt{3}\beta}dr^2 + e^{-2\sqrt{3}\beta}e^{-2\sqrt{3}\Omega}(d\theta^2 + \sin^2\theta d\varphi^2).$$

The corresponding Wheeler-DeWitt equation, in a particular factor ordering, can be written as

$$\exp(\sqrt{3}\beta + 2\sqrt{3}\Omega)[-P_\Omega^2 + P_\beta^2 - 48\exp(-2\sqrt{3}\Omega)]\psi(\Omega, \beta) = 0,$$

where $P_\Omega = -i\frac{\partial}{\partial \Omega}$ and $P_\beta = -i\frac{\partial}{\partial \beta}$. Thus, in this parametrization the Wheeler-DeWitt equation has a simple form, which can be formally identified with usual quantum mechanics in cartesian coordinates.

The solutions to this Wheeler-DeWitt equation are given by [27]

$$\psi_\nu^\pm(\beta, \Omega) = e^{\pm i\nu \sqrt{3}\beta}K_{i\nu}(4e^{-\sqrt{3}\Omega}),$$

where $K_{i\nu}$ is the modified Bessel function. Packet waves of these solutions have been constructed as superpositions of these solutions. Summing over $e^{i\nu \sqrt{3}\beta}$ and $e^{-i\nu \sqrt{3}\beta}$ to make real trigonometric functions, the “Gaussian” state

$$\Psi(\beta, \Omega) = 2iN\int_0^\infty \nu \left[\psi_\nu^+(\beta, \Omega) - \psi_\nu^-(\beta, \Omega)\right] d\nu$$

has been obtained [27, 28]. A possible connection with quantum black holes [29, 30] and quantum wormholes [31, 32] has been suggested.

For our noncommutative proposal of quantum cosmology, we will follow the procedure outlined above. We will assume that the “cartesian coordinates” $\Omega$ and $\beta$ of the Kantowski-Sachs minisuperspace obey a kind of commutation relation, like the ones in noncommutative quantum mechanics [23]

$$[\Omega, \beta] = i\theta.$$

As stated above, this is a particular ansatz in these configuration coordinates. The relation with other minisuperspace coordinates would follow in a similar way as in standard spacetime.
As usual, this deformation can be reformulated in terms of a noncommutativity of minisuperspace functions, with the Moyal product,

\[ f(\Omega, \beta) \ast g(\Omega, \beta) = f(\Omega, \beta)e^{i\frac{\bar{\theta}}{2} (\bar{\partial}_\alpha \bar{\partial}_\beta - \bar{\partial}_\beta \bar{\partial}_\alpha)} g(\Omega, \beta). \]

(6)

Now, our noncommutative Wheeler-DeWitt equation will be

\[
\exp(\sqrt{3}\beta + 2\sqrt{3}\Omega) \ast \left[-P_\Omega^2 + P_\beta^2 - 48\exp(-2\sqrt{3}\Omega)\right] * \psi(\Omega, \beta) = 0. 
\]

(7)

Then, as is well known in noncommutative quantum mechanics, the original phase-space, as well as its symplectic structure, is modified. It is possible to reformulate it in terms of commutative variables and the ordinary product of functions, if new variables are introduced, \( \Omega \rightarrow \Omega - \frac{3}{2} \theta P_\beta \) and \( \beta \rightarrow \beta - \frac{1}{2} \theta P_\Omega \), the momenta remain the same. As a consequence, the original equation changes, with a potential modified due to these new coordinates.

\[ V(\Omega, \beta) \ast \psi(\Omega, \beta) = V(\Omega - \frac{1}{2} \theta P_\beta, \beta - \frac{1}{2} \theta P_\Omega) \psi(\Omega, \beta). \]

(8)

Thus, we get

\[
[-\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \beta^2} + 48\exp(-2\sqrt{3}\Omega + \sqrt{3}\theta P_\beta)] \psi(\Omega, \beta) = 0. 
\]

(9)

Assuming a separation of variables with the ansatz

\[ \psi(\Omega, \beta) = \exp(\sqrt{3} \nu \beta) \chi(\Omega), \]

(10)

we observe that the operator \( P_\beta \) in the exponential in (9) will shift the wave function by a factor

\[ \psi(\Omega, \beta - i\sqrt{3} \theta) = \exp(-3i \nu \theta) \psi(\Omega, \beta), \]

(11)

thus \( \chi(\Omega) \) must satisfy the equation

\[
-\frac{d^2}{d\Omega^2} + 48\exp(-3i \nu \theta) \exp(-2\sqrt{3}\Omega + 3\nu^2) \chi(\Omega) = 0. 
\]

(12)

This equation can be solved, in the same manner as in the commutative case, by a modified Bessel function \( K_{i\nu} \). Therefore the wave function will be given by

\[
\psi_{\nu}^\pm(\Omega, \beta) = e^{\pm i\sqrt{3} \nu \theta} K_{i\nu} \left\{ 4\exp \left[-\sqrt{3} \left( \Omega \mp \frac{\sqrt{3}}{2} \nu \theta \right) \right] \right\}. 
\]

(13)

These are the solutions to the Wheeler-DeWitt equation (9). Note that noncommutativity induces a difference of the arguments of the Bessel functions in these solutions. Moreover, from its form, we can expect that the noncommutativity effects are enhanced for \( \psi^+ \). Thus, for the particular model we have chosen, the solution (13) allows an exact analysis, without the need of a \( \theta \) expansion.

In order to see the consequences of noncommutativity, let us consider a wave packet weighted by a Gaussian,

\[
\Psi(\Omega, \beta) = N \int_{-\infty}^{\infty} e^{-a(\nu - \nu_0)^2} \psi_{\nu}^+ (\Omega, \beta) \, d\nu. \]

(14)

This integral is performed numerically. We are interested to see which is the influence of the \( \theta \) parameter, but we are as well interested in consequences for the values of the \( \Omega \) and \( \beta \) variables, which in particular can provide information about the anisotropy.

As mentioned, a minimal size should exist for quanta of space and time configuration, and we are considering the very early time of the universe, where the influence of noncommutativity could have played a role in its quantum behavior. The figures 1, 2 and 3, computed for the values \( a = 1.5 \) and \( b = 1.3 \), show the dramatic changes that the universe could have had if noncommutativity was present.

Figure 1 corresponds to \( \theta = 0 \), the standard commutative case, and is presented for \( \Omega \) in the range \([0,10]\) and \( \beta \) in the range \([-10,10]\), it shows only one preferred state of the universe around \( \beta = 0 \). For \( \theta = 4 \), things have drastically changed, more peaks appear that seem to compete to be the most probable state of the universe. The peaks are no more around only of \( \beta = 0 \), but they appear, symmetrically, around other values of \( \beta \). Moreover, the original peak has changed its \( \Omega \) value. Furthermore, figure 3 shows more clearly the existence of the possibility of tunneling among these states.

Noncommutativity in minisuperspace creates then, new possible states of the universe. So the universe we live today could have evolved not only from the state for the commutative case (figure 1), but from any of the other states of the noncommutative model, like those of figure 2. It could have jumped, by means of a tunneling process, as figure 3 shows more clearly, from one universe (one state) to other universe (other state).

Although quantum cosmology, as discussed above, is only a limited model in an attempt to describe some of the features of the quantum theory of the universe, the consideration of noncommutativity seems to be one way to take into account the presence of constant Neveu-Schwarz background \( B \)-fields in M(atrix) and string theory, at early times in the universe.

As already mentioned, there are in the literature proposals for a noncommutative theory of gravity and string theory, at early times in the universe.
map was used to construct a deformed Einstein gravity. By means of this result, one could try to find the corresponding noncommutative Wheeler-DeWitt equation for specific cosmological models. In that case the $\theta$ terms corresponding to spacetime noncommutativity could be considered in order to search for another way to define a noncommutative cosmological model, even though cosmology depends only on time. The computation of this Wheeler-DeWitt equation is a very complicated task, because at each $\theta$ order higher derivative terms will appear. If such a noncommutative Hamiltonian model could be defined, its quantum cosmological solutions and the corresponding states could be compared with those obtained by means of our noncommutative minisuperspace proposal. In this context the $\theta$ parameter we have proposed (5), could be a kind of effective noncommutative parameter. Such an approach is currently under exploration.

Further work in more realistic cosmological models, including matter, will be needed to search for constraints on the range of values of the $\theta$ parameter in the early stages of evolution of the universe. In our proposal this is in principle possible, because it means to enlarge the minisuperspace configuration space and then consider an appropriate commutativity among its coordinates. Following these lines, we will consider also the supersymmetric extension [32] to these models.

In previous works, the Wheeler-DeWitt equation of the Kantowski-Sachs model has been related to quantum black holes [28, 29] and wormholes [30, 31]. It could be interesting to search for an extension of our results to possible noncommutative versions of these quantum gravitational systems.

Our simple proposal provides a picture of the dramatic influence that noncommutativity could have played at early stages of the universe. We have been able to obtain and work with an exact quantum solution for the Kantowski-Sachs metric, without the need to expand on the $\theta$ parameter. The corresponding wave packet exhibits new stable states of the universe with similar probabilities, showing peaks around $\beta$ values different from zero. A tunneling process could happen between these states. By these means there are different possible universes (states) from which our present universe could have evolved and also could have tunneled in the past, from one universe (state) to other one. Further work is needed to analyze other physical consequences of this and other more realistic quantum cosmological models, taking into account the influence of matter. It will be also necessary to extend these ideas to other, more general gravitational models. In particular, it would be very interesting to reinterpret the results of reference [33], concerning the influence of a primordial magnetic field in classical cosmology. The corresponding quantum model should be constructed and compared with our proposal in terms of the noncommutativity of the minisuperspace. Results in these directions and those mentioned above
will be presented elsewhere.

Acknowledgments
This work was supported in part by CONACyT Mexico Grant Nos. 28454E and 33951E.

[1] H. Snyder, Phys. Rev. 71, 38 (1947).
[2] A. Connes, Noncommutative Geometry (Academic Press 1994); Noncommutative Geometry: Year 2000, math.qa/0011193; A. Connes, J. Math. Phys. 41, 3832 (2000).
[3] J.C. Varilly, physics/9709045.
[4] M.R. Douglas and N.A. Nekrasov, “Noncommutative Field Theory”, Rev. Mod. Phys. 73 977 (2002), hep-th/0106048.
[5] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, Phys. Rev. D 55, 5112 (1997).
[6] A. Connes, M.R. Douglas and A. Schwarz, JHEP 02, 003 (1998).
[7] C.S. Chu, P.M. Ho, Nucl. Phys. B 550, 151 (1999).
[8] V. Schomerus, JHEP 06, 030 (1999).
[9] N. Seiberg and E. Witten, JHEP 09, 032 (1999).
[10] N. Nekrasov and A. Schwarz, Commun. Math. Phys. 198, 689 (1998).
[11] J.W. Moffat, Phys. Lett. B 491, 345 (2000).
[12] J.W. Moffat, Phys. Lett. B 493, 142 (2000).
[13] A.H. Chamessianne, Phys. Lett. B 504, 33 (2001).
[14] M. Bañados, O. Chandia, N. Grandi, F.A. Schaposnik and G.A. Silva, Phys. Rev. D 64, 084012 (2001).
[15] N. Grandi and G.A. Silva, Phys. Lett. B 507, 345 (2001).
[16] M. Ryan, Hamiltonian Cosmology (Springer Berlin, 1972).
[17] J. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983).
[18] K.V. Kuchar and M.P. Ryan, Phys. Rev D 40, 3982 (1989).
[19] S. Sinha and B.L. Hu, Phys. Rev. D 44, 1028 (1991); B.L. Hu and S. Sinha Directions in General Relativity, ed B.L. Hu, M.P. Ryan and C.V. Vishveshwara (Cambridge: Cambridge University Press, 1993).
[20] L. Suskind and J. Uglum, Talk presented at the PAS-COS meeting in Syracuse, New York, May 1994, hep-th/9410074.
[21] J.J. Halliwell, Proc. 13th. Int. Conf. on General Relativity, ed R.J. Gleisser, C.N. Kozameh and O.M. Moreschi (Bristol: IOP Publishing, 1993).
[22] C. Grojean, F. Quevedo, G. Tasinato and L. Zavala, JHEP 0108, 005 (2001).
[23] See for example, J. Gamboa, M. Loewe and J.C. Rojas, Phys. Rev. D 64, 067901 (2001); M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001).
[24] A. Mazumdar and M.M. Sheikh-Jabbari, Phys. Rev. Lett. 87, 011301 (2001).
[25] Chong-Sun Chu, B.R. Greene and G. Shiu, Mod. Phys. Lett. A 16, 2231 (2001).
[26] F. Lizzi, G. Mangano, G. Miele and G. Sparano, Int. J. Mod. Phys. A 11, 2907 (1996).
[27] C. Misner, Minisuperspace in Magic without magic: John Archibald Wheeler (Freeman, 1972).
[28] O. Obregón and M.P. Ryan, Mod. Phys. Lett A 13, 3251 (1998).
[29] M. Cavaglia, V. De Alfaro and A.T. Filippov, Int. J. Mod. Phys. D 4, 661 (1995).
[30] M. Cavaglia, Mod. Phys. Lett. A 9, 1897 (1994).
[31] L.M. Campbell and L. Garay, Phys. Lett. B 254, 49 (1991).
[32] A. Macías, O. Obregón and M.P. Ryan, Class. Quantum Grav. 4, 1477 (1987); O. Obregón and C. Ramírez, Phys. Rev. D 57, 1015 (1998); P.D. D’Eath, S.W. Hawking and O. Obregón, Phys. Lett. B 300, 44 (1993); R. Graham, Phys. Rev. Lett. 67, 1381 (1991); P.D. D’Eath, Supersymmetric Quantum Cosmology (Cambridge University Press, Cambridge, England, 1996); P.V. Moniz, Int. J. Mod. Phys. A 11, 4321 (1996).
[33] C.G. Tsagas and Roy Maartens, Class. Quant. Grav. 17, 2215 (2000); D. R. Matravers and C.G. Tsagas, Phys. Rev. D 62, 103519 (2000); C.G. Tsagas, Phys. Rev. Lett. 86, 5421 (2001).