ON FORCING PROJECTIVE GENERIC ABSOLUTENESS FROM STRONG CARDINALS

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Abstract. W.H. Woodin showed that if \( \kappa_1 < \cdots < \kappa_n \) are strong cardinals then two-step \( \Sigma^n_{n+3} \) generic absoluteness holds after collapsing \( 2^{\kappa_n} \) to be countable. We show that this number can be reduced to \( 2^{\kappa_n} \), and to \( \kappa_n^+ \) in the case \( n = 1 \), but cannot be further reduced to \( \kappa_n \).

1. Introduction

The goal of this article is to slightly sharpen a theorem of W.H. Woodin regarding forcing generic absoluteness from strong cardinals. We begin with a brief introduction to generic absoluteness principles and a summary of their relationship to large cardinals. For a positive integer \( n \), one-step \( \Sigma^n_1 \) generic absoluteness says that every \( \Sigma^n_1 \) statement is absolute between \( V \) and all generic extensions, and two-step \( \Sigma^n_1 \) generic absoluteness says that for every generic extension of \( V \), every \( \Sigma^n_1 \) statement is absolute between it and all further generic extensions. Crucially, two-step generic absoluteness allows real parameters from generic extensions. Projective generic absoluteness (either one-step or two-step) says that \( \Sigma^n_1 \) generic absoluteness holds for every positive integer \( n \).

From Shoenfield’s absoluteness theorem (see, for example, Kanamori [5, Theorem 13.15]) it follows that one-step \( \Sigma^n_1 \) generic absoluteness is a theorem of ZFC and therefore two-step \( \Sigma^n_2 \) generic absoluteness is also a theorem of ZFC. Generic absoluteness principles for higher pointclasses in the projective hierarchy are independent of ZFC and are equiconsistent with the existence of large cardinals. For example, one-step \( \Sigma^n_3 \) generic absoluteness is equiconsistent with the existence of a \( \Sigma_2 \)-reflecting cardinal (see Feng, Magidor, and Woodin [2, Corollary 3.1 and Theorem 3.3]).

Two-step \( \Sigma^n_3 \) generic absoluteness has higher consistency strength than one-step \( \Sigma^n_3 \) generic absoluteness: it is equivalent to the statement “every set has a sharp”. Using Jensen’s covering lemma, Woodin [14] showed that if two-step \( \Sigma^n_3 \) generic absoluteness holds then every set has a sharp. The converse implication originates with Martin and Solovay [8, Lemma 5.6]; see Kechris [6, Section 2] or Kanamori [5, Section 15] for a modern treatment of the subject. (A simpler proof of \( \Sigma^n_3 \) generic absoluteness from sharps is given by Caicedo and Schindler [1, Theorem 3]. However, elements of the Martin–Solovay argument will be needed later in this paper.)

At higher levels of complexity, generic absoluteness relates to strong cardinals. By results of Woodin and K. Hauser, one-step and two-step projective generic absoluteness are both equiconsistent with the existence of infinitely many strong cardinals. More specifically, Woodin showed that if \( \lambda \) is a limit of strong cardinals, then two-step projective generic
absoluteness holds after forcing with \( \text{Col}(\omega, \lambda) \) (see Steel [12] Corollary 4.8,) and Hauser [3] Theorem 3.14] showed that if one-step projective generic absoluteness holds, then either there is an inner model with a Woodin cardinal or the core model \( K \) has infinitely many strong cardinals.

For two-step generic absoluteness, the results of Woodin and Hauser give a level-by-level equiconsistency: for every positive integer \( n \), two-step \( \Sigma^1_{n+3} \) generic absoluteness is equiconsistent with the existence of \( n \) strong cardinals. Woodin showed that if there are \( n \) strong cardinals \( \kappa_1 < \cdots < \kappa_n \), then two-step \( \Sigma^1_{n+3} \) generic absoluteness holds after forcing with \( \text{Col}(\omega, 2^{\kappa_2}) \) (see Steel [12] Corollary 4.7,) and Hauser [3] Theorem 3.10] showed that if two-step \( \Sigma^1_{n+3} \) generic absoluteness holds, then either there is an inner model with a Woodin cardinal or the core model \( K \) has at least \( n \) strong cardinals. We will show that the cardinal \( 2^{\kappa_2} \) in Woodin’s theorem can be reduced to \( 2^{\kappa_n} \):

**Theorem 1.1.** Let \( \kappa_1 < \cdots < \kappa_n \) be strong cardinals where \( n \) is a positive integer. Let \( G \subset \text{Col}(\omega, 2^{\kappa_n}) \) be a V-generic filter. Then \( V[G] \) satisfies two-step \( \Sigma^1_{n+3} \) generic absoluteness.

Note that Theorem 1.1 implies Woodin’s theorem because \( \text{Col}(\omega, 2^{\kappa_n}) \) regularly embeds into \( \text{Col}(\omega, 2^{\kappa_2}) \).

**Remark 1.2.** Theorem 1.1 is optimal in the sense that two-step (or even one-step) \( \Sigma^1_{n+3} \) generic absoluteness may fail after collapsing every cardinal less than \( 2^{\kappa_n} \) to be countable. Suppose that \( V = K \) and there are \( n \) strong cardinals. Let \( \kappa_1 < \cdots < \kappa_n \) be the first \( n \) strong cardinals. Then GCH holds and in particular \( 2^{\kappa_n} = \kappa_n^+ \), so two-step \( \Sigma^1_{n+3} \) generic absoluteness holds after forcing with \( \text{Col}(\omega, \kappa_n^+) \) by Theorem 1.1, but (as explained below) it fails after forcing with only \( \text{Col}(\omega, \kappa_n) \).

In fact, in this situation even one-step \( \Sigma^1_{n+3} \) generic absoluteness fails after forcing with \( \text{Col}(\omega, \kappa_n) \) because there is a \( \Sigma^1_{n+3} \) statement that is not absolute between generic extensions of \( V \) by \( \text{Col}(\omega, \kappa_n) \) and \( \text{Col}(\omega, \kappa_n^+) \). To see this, note that the number of \( < \kappa_n \)-strong cardinals less than \( \kappa_n \) is \( n - 1 \) because every cardinal that is strong up to a strong cardinal is strong. Therefore by the proof of Hauser [3] Theorem 3.10], if we take a V-generic filter \( G \subset \text{Col}(\omega, \kappa_n) \) and a real \( x \in V[G] \) coding the extender sequence of \( K \) up to \( \kappa_n \), the statement “\( \omega_1 \) is the cardinal successor of \( \kappa_n \) in \( K \)” can be expressed in a \( \Pi^1_{n+3}(x) \) way. This statement is true in \( V[G] \) because \( V = K \), but it becomes false after further forcing to collapse \( \kappa_n^+ \).

One might still ask whether the cardinal \( 2^{\kappa_n} \) in Theorem 1.1 can be replaced by \( \kappa_n^+ \). Although this remains open in the general case, we can prove it in the case \( n = 1 \):

**Theorem 1.3.** Let \( \kappa \) be a strong cardinal and let \( G \subset \text{Col}(\omega, \kappa^+) \) be a V-generic filter. Then \( V[G] \) satisfies two-step \( \Sigma^1_4 \) generic absoluteness.

**Question 1.4.** Let \( \kappa_1 < \cdots < \kappa_n \) be strong cardinals where \( n \geq 2 \). Let \( G \subset \text{Col}(\omega, \kappa_n^+) \) be a V-generic filter. Must \( V[G] \) satisfy two-step \( \Sigma^1_{n+3} \) generic absoluteness?

Theorems 1.1 and 1.3 are proved in Sections 3 and 4 respectively. Some prior knowledge of tree representations of sets of reals would help the reader, although Section 2 contains a brief review of the subject. Section 4 requires some inner model theory, but nothing very technical.
2. Tree representations of sets of reals

In this section we briefly review some definitions regarding tree representations of sets of reals, where by “reals” we mean elements of the Baire space \( \omega^\omega \). For more information on this topic, see Moschovakis [3] or Kechris and Moschovakis [7]. We also define a generalization of semiscales that we call proto-semiscales (see Definition 2.1 below.)

For a class \( X \), a tree on \( X \) is a subset of \( X^{<\omega} \) that is closed under initial segments. When we consider trees on Cartesian products such as \( \omega \times \text{Ord} \), we conflate sequences of pairs with pairs of sequences. Because we require trees to be sets, a tree on \( \omega \times \text{Ord} \) is actually a tree on \( \omega \times \gamma \) for some ordinal \( \gamma \), but for simplicity we avoid naming \( \gamma \) whenever possible.

For a tree \( T \) on \( \omega \times \text{Ord} \) we let \([T]\) denote the set of branches of \( T \), meaning the set of pairs \( (x, f) \in \omega^\omega \times \text{Ord}^\omega \) such that \( (x \upharpoonright n, f \upharpoonright n) \in T \) for all \( n < \omega \). Note that \([T]\) is a closed subset of \( \omega^\omega \times \text{Ord}^\omega \), and conversely every closed subset of \( \omega^\omega \times \text{Ord}^\omega \) is the set of branches of some tree on \( \omega \times \text{Ord} \). For a relation \( R \) we use the notation \( pR \) for the projection of \( R \) along its last coordinate, so \( p[T] \) is the set of all reals \( x \) such that \( (x, f) \in [T] \) for some \( f \in \text{Ord}^\omega \).

If \( A = p[T] \) then we say \( T \) projects to \( A \), or \( T \) is a tree representation of \( A \).

For a tree \( T \) on \( \omega \times \text{Ord} \) and a real \( x \), we define the section of \( T \) by \( x \) as follows: \( T_x = \{ s \in \text{Ord}^{<\omega} : (x \upharpoonright |s|, s) \in T \} \). Then \( T_x \) is a tree on \( \text{Ord} \) and we have \( x \in p[T] \) if and only if \( T_x \) is illfounded. If \( x \notin p[T] \) (so that \( T_x \) is wellfounded) then we let \( \text{rank}_{T_x} \) denote the rank function on \( T_x \). For convenience we define \( \text{rank}_{T_x}(s) \) to be zero in the case \( s \notin T_x \), so that \( \text{rank}_{T_x} \) is defined on all \( s \in \text{Ord}^{<\omega} \). For various finite sequences \( s \in \text{Ord}^{<\omega} \), the corresponding ordinals \( \text{rank}_{T_x}(s) \) provide various measures of how “close” \( x \) is to \( p[T] \).

If \( T \) and \( \tilde{T} \) are trees on \( \omega \times \text{Ord} \) and \( \mathbb{P} \) is a poset, the pair \((T, \tilde{T})\) is called \( \mathbb{P} \)-absolutely complementing if \( p[\tilde{T}] = \omega^\omega \setminus p[T] \) in every generic extension of \( V \) by \( \mathbb{P} \). Because every generic extension of \( V \) by \( \mathbb{P} \) is contained in a generic extension of \( V \) by \( \text{Col}(\omega, \lambda) \) where \( \lambda = |\mathbb{P}| \), every \( \text{Col}(\omega, \lambda) \)-absolutely complementing pair of trees is \( \mathbb{P} \)-absolutely complementing, so for our purposes it will suffice to consider Levy collapse posets.

For a set of reals \( A \), a norm on \( A \) is a function \( \varphi : A \to \text{Ord} \). For every norm \( \varphi \) on \( A \), we may define a corresponding prewellordering \( \leq_{\varphi} \) of \( A \) by \( x \leq_{\varphi} y \iff \varphi(x) \leq \varphi(y) \). Two norms on \( A \) are equivalent if their corresponding prewellorderings are equal.

A semiscale on \( A \) is a sequence \( \vec{\varphi} = (\varphi_i : i < \omega) \) of norms on \( A \) such that for every sequence of reals \( (x_n : n < \omega) \) in \( A \) and every real \( y \), if \((x_n : n < \omega)\) converges to \( y \) and for each \( i < \omega \) the sequence of ordinals \((\varphi_i(x_n) : n < \omega)\) is eventually constant, then \( y \in A \). Note that replacing each norm by an equivalent norm preserves this defining property of semiscales.

From a semiscale \( \vec{\varphi} = (\varphi_i : i < \omega) \) on \( A \) we may define the tree associated to \( \vec{\varphi} \), which is the tree on \( \omega \times \text{Ord} \) consisting of all finite sequences of pairs

\[
(x(0), \varphi_0(x)), \ldots, (x(n-1), \varphi_{n-1}(x))
\]

where \( x \in A \) and \( n < \omega \). The relevant consequence of this definition is that if \( \tilde{T} \) is the tree associated to a semiscale on \( A \), then \( \tilde{T} \) projects to \( A \). (This consequence is proved for the tree of a scale by Kechris and Moschovakis [7, Section 6.2], and the proof applies equally well to semiscales.)

The following generalization of semiscales will also be useful.

Definition 2.1. A proto-semiscale on a set of reals \( A \) is a set \( \mathcal{C} \) of norms on \( A \) such that for every sequence of reals \((x_n : n < \omega)\) in \( A \) and every real \( y \), if \((x_n : n < \omega)\) converges to
Note that replacing each norm by an equivalent norm preserves the defining property of proto-semiscales. Note also that if the sequence of norms \( (\varphi_i : i < \omega) \) is a semiscale on \( A \) then the set of norms \( \{\varphi_i : i < \omega\} \) is a proto-semiscale on \( A \), and if the set of norms \( C \) is a countable proto-semiscale on \( A \) then every enumeration of \( C \) by \( \omega \) is a semiscale on \( A \). If we have a proto-semiscale \( C \) on \( A \) and we want to define an associated tree representation for \( A \), we will first need to enumerate \( C \) by \( \omega \).

3. Proof of Theorem 3.3

The main ingredient in the proof of Theorem 3.3 will be the following lemma. Woodin proved a version of this lemma with \( 2^{2^{\omega n}} \) in place of \( 2^{\omega n} \): see Steel [12, Theorem 4.5]. (That version also required the codomain \( M \) of the elementary embedding \( j : V \to M \) to have a bit more agreement with \( V \), but this difference is not relevant to our desired application where \( \kappa \) is a strong cardinal.)

**Lemma 3.1.** Let \( \kappa \) and \( \lambda \) be cardinals with \( 2^n < \lambda \). Let \( j : V \to M \) be an elementary embedding such that \( \text{crit}(j) = \kappa \) and \( j(\kappa) > \lambda \) and \( M \) is a transitive class with \( \mathcal{P}(\lambda) \subseteq M \). Let \( T \) be a tree on \( \omega \times \text{Ord} \) and let \( G \subseteq \text{Col}(\omega, 2^n) \) be a \( V \)-generic filter. Then in \( V[G] \) there is a tree \( \hat{T} \) on \( \omega \times \text{Ord} \) such that the pair \( (j(T), \hat{T}) \) is \( \text{Col}(\omega, \lambda) \)-absolutely complementing.

**Remark 3.2.** Woodin used a system of measures to build an absolute complement \( \hat{T} \) for \( j(T) \) via a Martin–Solovay construction, and the cardinal \( 2^{2^n} \) appeared as an upper bound on the number of measures on \( \kappa \). In our proof, we will instead build a semiscale from norms corresponding to rank functions, and the cardinal \( 2^n \) will appear as an upper bound on the number of norms required. Wilson [13, Lemma 3.1] used a similar argument. (A version of Lemma 3.1 was initially included in that paper, but later removed because it threatened to take the paper on an overly long tangent.)

Once Lemma 3.1 is proved, Theorem 3.3 will follow routinely as in Steel [12, Section 4]. The argument is identical except for the substitution of \( 2^{\omega n} \) for \( 2^{2^n} \), so we only give a brief summary here. We begin with the Martin–Solovay tree representations of \( \Pi^1_2 \) sets, which exist because every set has a sharp, and may be taken to project to the intended sets in arbitrarily large generic extensions. Then we build tree representations of \( \Sigma^1_3, \Pi^1_3, \Sigma^1_4, \Pi^1_4, \ldots, \Sigma^1_{n+2}, \Pi^1_{n+2} \) sets, which may also be taken to project to the intended sets in arbitrarily large generic extensions. In the end, the existence of such tree representations of \( \Pi^1_{n+2} \) sets will imply two-step \( \Sigma^1_{n+3} \) generic absoluteness by a standard argument involving the absoluteness of wellfoundedness.

To go from \( \Pi^1_{i+1} \) to \( \Sigma^1_{i+2} \) is trivial. To go from \( \Sigma^1_{i+2} \) to \( \Pi^1_{i+2} \) we collapse \( 2^{\kappa_i} \), where \( \kappa_i \) is the \( i \)th strong cardinal, and apply Lemma 3.1. Because we may take \( \lambda \) to be arbitrarily large in Lemma 3.1, this step preserves the property that our trees project to the intended sets in arbitrarily large generic extensions. Note that the difference between the two trees \( T \) and \( j(T) \) in Lemma 3.1 is not a problem here. In the case \( i = 1 \) for example, let \( G \subseteq \text{Col}(\omega, 2^{\kappa_1}) \) be a \( V \)-generic filter, let \( H \subseteq \text{Col}(\omega, \lambda) \) be a \( V[G] \)-generic filter, let \( A \) be a \( \Sigma^1_3 \) property, and let \( T \) be a tree in \( V \) such that every generic extension of \( V \) by a poset of cardinality at most \( \lambda \) satisfies \( p[T] = \{ x \in \omega^\omega : A(x) \} \). Then the model \( V[G][H] \) satisfies \( p[T] = \{ x \in \omega^\omega : A(x) \} \)
and by the elementarity of \( j \) the model \( M[G][H] \) satisfies \( p[j(T)] = \{ x \in \omega^\omega : A(x) \} \). These two models have the same reals because \( \mathcal{P}(\lambda) \subset M \), so \( A \) defines the same \( \Sigma^1_3 \) set in both models.

It therefore remains to prove Lemma 3.3. In fact, we will prove a sharper version that will be needed for the proof of Theorem 1.3 in Section 4.

**Lemma 3.3.** Let \( \kappa, \eta, \) and \( \lambda \) be cardinals with \( \kappa \leq \eta < \lambda \). Let \( j : V \to M \) be an elementary embedding such that \( \text{crit}(j) = \kappa \) and \( j(\kappa) > \lambda \) and \( M \) is a transitive class with \( \mathcal{P}(\lambda) \subset M \). Let \( T \) be a tree on \( \omega \times \text{Ord} \) with \( |V_{\kappa+1} \cap L(j(T), V_\kappa)| \leq \eta \) and let \( G \subset \text{Col}(\omega, \eta) \) be a \( V \)-generic filter. Then in \( V[G] \) there is a tree \( \bar{T} \) on \( \omega \times \text{Ord} \) such that the pair \( (j(T), \bar{T}) \) is \( \text{Col}(\omega, \lambda) \)-absolutely complementing.

Because \( \kappa \) is inaccessible we have \( |V_{\kappa+1} \cap L(j(T), V_\kappa)| \leq 2^\kappa \) for every tree \( T \), so Lemma 3.3 with \( \eta = 2^\kappa \) implies Lemma 3.1. To complete this section it remains to prove Lemma 3.3.

**Remark 3.4.** The proof will resemble a well-known construction which, given a countable tree representation \( T \) of a \( \Sigma^1_1 \) set, builds a semiscale \( \tilde{\varphi} \) on \( T \) (and thereby an associated tree representation \( \tilde{T} \) of) the complementary \( \Pi^1_1 \) set. The main difference here is that in Lemma 3.3 we may not assume \( T \) is countable; indeed, \( T \) will have cardinality at least \( \kappa \) in the desired applications. Therefore what we first get is only a proto-semiscale. To get a semiscale, we will need to collapse the proto-semiscale to be countable. If the tree itself is collapsed, then it’s not clear this proto-semiscale will be useful; however, by using a given degree of strongness of \( \kappa \), we may “inflate” \( T \) to \( j(T) \) and show that it suffices to collapse something less than the cardinality of \( j(T) \). As noted previously, the difference between \( T \) and \( j(T) \) will not matter for applications within the projective hierarchy.

**Proof of Lemma 3.3.** Note that for every generic extension \( V[g] \) of \( V \) by a poset in \( V_\kappa \), the map \( j \) extends to an elementary embedding \( V[g] \to M[g] \), giving \( p[T]^{V[g]} = p[j(T)]^{M[g]} \). Of course, \( p[j(T)]^{M[g]} = p[j(T)]^{V[g]} \). Let \( H \subset \text{Col}(\omega, \lambda) \) be a \( V[G] \)-generic filter. By the elementarity of \( j \), the fact that \( p[T]^{V[g]} = p[j(T)]^{V[g]} \) for every generic extension \( V[g] \) of \( V \) by a poset in \( V_\kappa \), and the fact that \( j(\kappa) > \lambda \), we have \( p[j(T)]^{M[G][H]} = p[j(j(T))]^{M[G][H]} \). The models \( M[G][H] \) and \( V[G][H] \) have the same reals because \( \mathcal{P}(\lambda) \subset M \), so

\[
p[j(T)]^{V[G][H]} = p[j(T)]^{M[G][H]} = p[j(j(T))]^{M[G][H]} = p[j(j(T))]^{V[G][H]}.
\]

Let \( A \) denote the complement of this common set of reals:

\[
A = (\omega^\omega \setminus p[j(T)])^{V[G][H]} = (\omega^\omega \setminus p[j(T)])^{M[G][H]} = \ldots.
\]

Take an ordinal \( \gamma \) such that \( T \) is a tree on \( \omega \times \gamma \). Then for each \( t \in j(\gamma)^{<\omega} \), we may define a norm \( \varphi_t \) on \( A \) by

\[
\varphi_t(x) = \text{rank}_{j(j(T))_x}(j(t)).
\]

Recall that this rank is defined as zero if \( j(t) \notin j(j(T))_x \).

**Claim 3.5.** *The set of norms \( \{ \varphi_t : t \in j(\gamma)^{<\omega} \} \) is a proto-semiscale on \( A \).*

**Proof.** Let \( (x_n : n < \omega) \) be a sequence of reals in \( A \) and let \( y \) be a real in \( V[G][H] \). Assume that \( (x_n : n < \omega) \) converges to \( y \) and for every \( t \in j(\gamma)^{<\omega} \) the sequence of ordinals \( \{ \varphi_t(x_n) : n < \omega \} \) is eventually constant; let \( \lambda_t \) denote its eventual value. We want to show \( y \in A \).

Suppose toward a contradiction that \( y \in p[j(T)] \) as witnessed by \( f \in j(\gamma)^\omega \). That is, for all \( i < \omega \) we have \( f \upharpoonright i \in j(T)_y \). Then we have \( j(f \upharpoonright i) \in j(j(T))_y \), and because
Let a version of the Martin–Solovay tree representations of $\Sigma^1_3$ Steel [11, Section 7D]. In particular we will need the following definition and lemma regarding $\lambda_{j[i]}(\bar{x})$. The sequence of nodes $j(V)$ projects to the set $\bar{x}$ and child node respectively in the well-founded tree $j(j(T))_{x_n}$, so the eventual norm values satisfy the inequality $\lambda_{j[i]} > \lambda_{j[i+1]}$. This inequality holds for all $i < \omega$, so we get an infinite decreasing sequence of ordinals, which is a contradiction. \hfill \Box

We can strengthen Claim 3.5 as follows:

**Claim 3.6.** There is a set $\sigma \subset j(\gamma)^{<\omega}$ in $V$ such that $|\sigma|^V \leq \eta$ and the set of norms $\{\phi_t : t \in \sigma\}$ is a proto-semiscale on $A$.

**Proof.** For every $t \in j(\gamma)^{<\omega}$ we may define, in any model containing the tree $j(T)$, a norm $\psi_t$ on the set of reals $\omega_\omega \setminus p[j(T)]$ in that model by $\psi_t(x) = \text{rank}_{j(T)}(t)$. Note that $\psi_t$ is defined like $\phi_t$ but with one fewer level of applications of $j$. Define an equivalence relation $\sim$ on $j(\gamma)^{<\omega}$ by $t \sim t'$ if and only if in every generic extension of $V$ by a poset in $V_\kappa$ the norms $\psi_t$ and $\psi_{t'}$ are equivalent (meaning their corresponding prewellorderings $\leq_{\psi_t}$ and $\leq_{\psi_{t'}}$ are equal.) Note that $t \sim t'$ if and only if $S_t = S_{t'}$, where for every $t \in j(\gamma)^{<\omega}$ we define the set

$$S_t = \{(\mathbb{P}, \bar{p}, \bar{x}, \bar{y}) \in V_\kappa : p \models_{\mathbb{P}} \bar{x} \leq_{\psi_t} \bar{y}\}.$$  

Because $S_t \in V_{\kappa+1} \cap L(j(T), V_\kappa)$ for all $t \in j(\gamma)^{<\omega}$, the number of equivalence classes of $\sim$ is at most $\eta$, so we may take a set $\sigma \subset j(\gamma)^{<\omega}$ in $V$ such that $|\sigma|^V \leq \eta$ and for every $t \in j(\gamma)^{<\omega}$ there is some $t' \in \sigma$ with $t \sim t'$. Note that if $t \sim t'$, then from the elementarity of $j$ and the fact that $j(\kappa) > \lambda$ it follows that the two norms $\phi_t$ and $\phi_{t'}$ on the set of reals $(\omega_\omega \setminus p[j(j(T))])^{M[G][H]}$ (which is equal to $A$) are equivalent. Therefore every norm in the set $\{\phi_t : t \in j(\gamma)^{<\omega}\}$ is equivalent to a norm in the set $\{\phi_t : t \in \sigma\}$. Because replacing every norm by an equivalent norm preserves the defining property of proto-semiscales, the claim now follows from Claim 3.5. \hfill \Box

Fixing an enumeration $(t_i : i < \omega)$ of $\sigma$ in $V[G]$, the corresponding sequence of norms $\bar{\phi} = (\phi_t : i < \omega)$ is therefore a semiscale on $A$. Let $\bar{T}$ be its associated tree. In $V[G][H]$ the tree $\bar{T}$ is definable from the semiscale $\bar{\phi}$, which in turn is definable from the tree $j(j(T)) \in V$ and the sequence of nodes $(j(t_i) : i < \omega) \in V[G]$. Because $\bar{T}$ is a subset of $V[G]$ (in fact of $V$) and the poset $\text{Col}(\omega, \lambda)$ is almost homogeneous, we therefore have $\bar{T} \in V[G]$. Because $\bar{T}$ projects to the set $A = \omega_\omega \setminus p[j(T)]$ in $V[G][H]$, it is a $\text{Col}(\omega, \lambda)$-absolute complement for $j(T)$ in $V[G]$.

\hfill \Box

4. PROOF OF THEOREM 1.3

In this section we will need to use parts of the proof of $\Sigma^1_3$ correctness of the core model in Steel [11, Section 7D]. In particular we will need the following definition and lemma regarding a version of the Martin–Solovay tree representations of $\Sigma^1_3$ sets.

**Definition 4.1.** Let $\kappa$ be an inaccessible cardinal such that every set in $V_\kappa$ has a sharp.

1. For every ordinal $\alpha \geq 1$, let $u_\alpha$ be the $\alpha^{th}$ uniform indiscernible for the models $L[x]$ where $x \in V_\kappa$. 

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In order to show that condition (∗) holds already in \( V \), we will show that condition (∗) holds in \( V \) and therefore two-step \( \Sigma^1_3 \) generic absoluteness holds already in \( V \). Let \( B \) be a \( \Pi^1_3 \) property, let \( \lambda \) be an infinite cardinal, and let \( H \subset \text{Col}(\omega,\lambda) \) be a \( V \)-generic filter. Work in \( V[H] \). By \( \Delta^1_2 \) determinacy and the second periodicity theorem of Moschovakis \([9, \text{Corollary 6C.4}]\) applied to the pointclass \( \Sigma^1_3 \), the pointclass \( \Pi^1_3 \) has the scale property. In particular the \( \Pi^1_3 \) set \( \{ x \in \omega^\omega : B(x) \} \) has a definable semiscale and is therefore the projection of a definable tree \( \tilde{T} \) on \( \omega \times \text{Ord} \). Because the poset \( \text{Col}(\omega,\lambda) \) is almost homogeneous we have \( \tilde{T} \in V \), witnessing condition (∗) for \( B \) and \( \lambda \) in \( V \).

The other case is where \( \Delta^1_2 \) determinacy fails in some generic extension of \( V \). In this case we will need to collapse \( \kappa^+ \) and apply Lemmas \([1, \text{Lemma 7.10}]\) to the Martin–Solovay trees. Because \( V_\kappa \prec_{\Sigma^3_2} V \) it follows in this case that \( \Delta^1_2 \) determinacy fails in some generic extension of \( V \) by a poset in \( V_\kappa \). Because we are ultimately interested in \( V^{\text{Col}(\omega,\kappa^+)} \), by passing to such a small generic extension we may assume without loss of generality that \( \Delta^1_2 \) determinacy fails already in \( V \).

Fix a real \( z_0 \in V \) such that \( \Delta^1_2(z_0) \) determinacy fails in \( V \). Because every set has a sharp, we have two-step \( \Sigma^1_3 \) generic absoluteness and in particular \( \Sigma^1_3(z_0) \) generic absoluteness. Therefore if we take two \( \Sigma^1_3(z_0) \) properties that are complementary in \( V \) and define a failure of \( \Delta^1_2(z_0) \) determinacy in \( V \), they remain complementary in every generic extension of \( V \) and define a failure of \( \Delta^1_2(z_0) \) determinacy there also. From the failure of \( \Delta^1_2(z_0) \) determinacy in every generic extension of \( V \), it follows by a theorem of Woodin (see Neeman \([10, \text{Corollary 4.13}]\)).

\(^{1}\text{Steel defines the Martin–Solovay tree representations of } \Pi^1_2 \text{ sets instead of } \Sigma^1_3 \text{ sets, but we may use the fact that tree representations of } \Sigma^1_{n+1} \text{ sets are definable from tree representations of } \Pi^1_n \text{ sets in a local and uniform way.} \)
2.3], which relativizes to an arbitrary real) that no generic extension of \( V \) has a proper class inner model containing \( z_0 \) with a Woodin cardinal.

Let \( B \) be a \( \Pi^1_3 \) property, say \( B(x) \leftrightarrow \neg A(x) \) where \( A \) is \( \Sigma^1_3 \), and let \( T_A = T_A^V \) be the Martin–Solovay tree representation of \( A \).

**Claim 4.3.** For every generic extension \( V[g] \) of \( V \) by a poset in \( V_\kappa \) and every real \( z \in V[g] \) such that \( z_0 \in L[z] \), we have \( V[g] \models |\omega^\omega \cap L[T_A, z]| \leq \omega_1 \).

**Proof.** Take an inner model \( N \) of ZFC such that \( V_\kappa \subset N, \kappa \) is measurable in \( N \), and \( N \) has a measurable cardinal \( \Omega > \kappa \). (Because \( \kappa \) is strong, we may take an elementary embedding \( j : V \to N \) with critical point \( \kappa \) where \( V_{\kappa+2} \subset N \), which implies the desired properties for \( N \).) Let \( V[g] \) be a generic extension of \( V \) by a poset in \( V_\kappa \). Note that \( \kappa \) and \( \Omega \) remain measurable cardinals in \( N[g] \).

Let \( z \) be a real in \( V[g] \) such that \( z_0 \in L[z] \) and let \( K^c(z) \) be the background certified core model built over \( z \) in \( N[g] \) up to the measurable cardinal \( \Omega \). The model \( K^c(z) \) has no Woodin cardinal: otherwise we could iterate a measure on \( \Omega \) through the ordinals to obtain a proper class inner model containing \( z \) (and therefore also containing \( z_0 \)) with a Woodin cardinal, contradicting our assumption. Therefore the core model \( K(z) \) over \( z \) in \( N[g] \) up to \( \Omega \) exists. By the measurability of \( \kappa \) in \( N[g] \) and the proof of Steel [11, Theorem 7.9] we have \( u_2^{K^c(z)} = u_2^{N[g]} \). Moreover, it is easy to see that \( u_2^{N[g]} = u_2^V = u_2^V \), so by Lemma 4.2 we have

\[
T_A^{K(z)} = T_A^{N[g]} = T_A^V = T_A.
\]

Therefore \( T_A \in K(z) \), so \( L[T_A, z] \subset K(z) \subset V[g] \) and the claim follows by CH in \( K(z) \). \( \square \)

Let \( \lambda > \kappa^+ \) be a cardinal. Because \( \kappa \) is strong, we may take an elementary embedding \( j : V \to M \) such that \( \text{crit}(j) = \kappa \) and \( j(\kappa) > \lambda \) and \( M \) is a transitive class with \( \mathcal{P}(\lambda) \subset M \).

**Claim 4.4.** \( |V_{\kappa+1} \cap L(j(T_A), V_\kappa)| \leq \kappa^+ \).

**Proof.** Take a generic extension \( V[g] \) of \( V \) by \( \text{Col}(\omega, \kappa) \) and take a real \( z \in V[g] \) coding \( V_\kappa \). Because \( z \) is in a generic extension of \( M \) by a poset of cardinality less than \( j(\kappa) \) and \( z_0 \in L[z] \), by Claim 4.3 and the elementarity of \( j \) we have \( M[g] \models |\omega^\omega \cap L[j(T_A), z]| \leq \omega_1 \). Note that \( \omega_1^M[g] = \kappa^+ \). Because \( L(j(T_A), V_\kappa) \subset L[j(T_A), z] \) and \( V_\kappa \) is countable in \( L[j(T_A), z] \), every subset of \( V_\kappa \) in \( L(j(T_A), V_\kappa) \) produces a different real in \( L[j(T_A), z] \) and the claim follows. \( \square \)

Now by Lemma 3.3 with \( \eta = \kappa^+ \), if we let \( G \subset \text{Col}(\omega, \kappa^+) \) be a \( V \)-generic filter then in \( V[G] \) there is a tree \( \tilde{T} \) on \( \omega \times \text{Ord} \) such that the pair \( (j(T_A), \tilde{T}) \) is \( \text{Col}(\omega, \lambda) \)-absolutely complementing. Let \( H \subset \text{Col}(\omega, \lambda) \) be a \( V[G] \)-generic filter. Note that every generic extension of \( V \) by a poset in \( V_\kappa \) satisfies \( p[T_A] = \{ x \in \omega^\omega : A(x) \} \) (this follows from Lemma 4.2 because small forcing preserves \( u_2 \)). Therefore by the elementarity of \( j \) and the fact that \( j(\kappa) > \lambda \) it follows that \( M[G][H] \) satisfies \( p[j(T_A)] = \{ x \in \omega^\omega : A(x) \} \).

The models \( M[G][H] \) and \( V[G][H] \) have the same reals because \( \mathcal{P}(\lambda) \subset M \), so \( V[G][H] \) also satisfies \( p[j(T_A)] = \{ x \in \omega^\omega : A(x) \} \). Because the pair \( (j(T_A), \tilde{T}) \) projects to complements in \( V[G][H] \), it follows that \( V[G][H] \) satisfies \( p[\tilde{T}] = \{ x \in \omega^\omega : B(x) \} \). Therefore the tree \( \tilde{T} \) witnesses condition (*) for \( B \) and \( \lambda \) in \( V[G] \). The proof of Theorem 4.3 is complete.

5. **Acknowledgments**

The author thanks Paul Larson and Menachem Magidor for their helpful comments.
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