QUANTUM INVARIANT FOR TORUS LINK AND MODULAR FORMS

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ABSTRACT. We consider an asymptotic expansion of Kashaev’s invariant or of the colored Jones function for the torus link $T(2, 2m)$. We shall give $q$-series identity related to these invariants, and show that the invariant is regarded as a limit of $q$ being $N$-th root of unity of the Eichler integral of a modular form of weight $3/2$ which is related to the $\tilde{su}(2)_{m-2}$ character.

1. INTRODUCTION

Recent studies reveal an intimate connection between the quantum knot invariant and “nearly modular forms” especially with half integral weight. In Ref. 9 Lawrence and Zagier studied an asymptotic expansion of the Witten–Reshetikhin–Turaev invariant of the Poincaré homology sphere, and they showed that the invariant can be regarded as the Eichler integral of a modular form of weight $3/2$. In Ref. 19, Zagier further studied a “strange identity” related to the half-derivatives of the Dedekind $\eta$-function, and clarified a role of the Eichler integral with half-integral weight. From the viewpoint of the quantum invariant, Zagier’s $q$-series was originally connected with a generating function of an upper bound of the number of linearly independent Vassiliev invariants [17], and later it was found that Zagier’s $q$-series with $q$ being the $N$-th root of unity coincides with Kashaev’s invariant [5, 6], which was shown [14] to coincide with a specific value of the colored Jones function, for the trefoil knot. This correspondence was further investigated for the torus knot, and it was shown [3] that Kashaev’s invariant for the torus knot $T(2, 2m + 1)$ also has a nearly modular property; it can be regarded as a limit $q$ being the root of unity of the Eichler integral of the Andrews–Gordon $q$-series, which is theta series with weight $1/2$ spanning $m$-dimensional space. As the torus knot is not hyperbolic, studies of the torus knot may not be attractive for the “Volume Conjecture” [5, 14] which states that an asymptotic limit of Kashaev’s invariant coincides with the hyperbolic volume of the knot complement, but they are rather absorbing from the point of view of the number theory, $q$-series and modular forms.

Motivated by our previous result on the torus knot $T(2, 2m + 1)$, we study Kashaev’s invariant for the torus link $T(2, 2m)$ (see Fig. 1) in this article. We shall show that the invariant is now re-
garded as the half-integration or the Eichler integral of a modular form of weight $3/2$. Remarkable is that this modular form is related to the $\tilde{su}(2)_{m-2}$ character. It is noted that recent studies [20, 21] reveal a relation with Ramanujan’s mock theta functions. We also propose a $q$-series identity, which is new as far as we know, and study an asymptotic expansion thereof.

This paper is organized as follows. In section 2 we construct the colored Jones polynomial for the torus link $T(2, 2m)$. Using the Jones–Wenzl idempotent, we give an explicit formula of the invariant. It is known [14] that Kashaev’s invariant coincides with a specific value of the colored Jones polynomial. This correspondence enables us to give an integral form of Kashaev’s invariant for the torus link $T(2, 2m)$ in section 3. We further give an asymptotic expansion of the invariant, and see that the invariant for $T(2, 2m)$ also has a nearly modular property. We give an explicit form of Kashaev’s invariant for this torus link using the enhanced Yang–Baxter operator. Combining these results we obtain an asymptotic expansion of a certain $\omega$-series. In section 4 we introduce the $q$-series related to Kashaev’s invariant for torus link, and prove a new $q$-series identity. We study the modular property of these $q$-series, and discuss how Kashaev’s invariant for $T(2, 2m)$ may be regarded as the Eichler integral of a modular form with weight $3/2$ which is the affine $\tilde{su}(2)_{m-2}$ character, in section 5. In the last section, we collect some examples.

2. COLORED JONES POLYNOMIAL FOR TORUS LINK $(2, 2m)$

The $N$-colored Jones polynomial for torus knot $T(m, p)$ was studied in Refs. 12, 15. Following these methods, we compute the colored Jones polynomial for torus link $T(2, 2m)$ in this section.
We use the Jones–Wenzl idempotent, and use following formulae (see, e.g., Ref. 10);

\[ (2.1) \]

where each label denotes a color, and we mean that

\[ (a, b, c) \text{ is admissible} \iff a + b + c \text{ is even}, \quad \begin{cases} a \leq b + c, \\ b \leq c + a, \\ c \leq a + b. \end{cases} \]

We have a \( \theta \)-net

\[ (2.3) \]

which is given as

\[ \theta(a, b, c) = \frac{\Delta_c}{\theta(a, b, c)} \]

with

\[ a = y + z, \quad b = z + x, \quad c = x + y. \]

**Proposition 1.** The \( N \)-colored Jones polynomial \( J_N(h; K) \) for the torus link \( K = T(2, 2m) \) is given by

\[ (2.4) \]

where a parameter \( q \) is set to be

\[ q = A^4 = e^h, \]
and \( O \) denotes unknot whose invariant is given by

\[
J_N(h; O) = \frac{\text{sh}(N h/2)}{\text{sh}(h/2)}.
\]

**Proof.** We first apply eq. (2.2) in the torus link \( T(2, 2m) \) (see Fig. 2), and untangle crossings recursively using eq. (2.1). We see that \( \theta(a, b, c) \) vanishes at the end. We have

\[
J_N(h; K) = \sum_{c:(N - 1, N - 1, c) \text{ is admissible}} \Delta_c \left( (-1)^{N-1-\frac{c}{2}} A^{-2(N-1)+c-(N-1)^2+\frac{1}{2}c^2} \right)^{2m}
\]

\[
= \frac{A^{-2m(N^2-1)}}{A^2 - A^{-2}} \sum_{j=0}^{N-1} A^{4mj(j+1)} \left( A^{2(2j+1)} - A^{-2(2j+1)} \right).
\]

This proves eq. (2.4). \( \square \)

**Figure 2.** We apply eq. (2.2) to the torus link \( T(2, 4) \). We set \( n = N - 1 \).

### 3. Kashaev Invariant and Asymptotic Expansion

It is known [14] that Kashaev’s invariant is given from the colored Jones polynomial at a specific value \( h \to 2 \pi i / N \). By use of a result of Prop. 1 we obtain an integral form of the invariant as follows.

**Proposition 2.** The Kashaev invariant for torus link \( T(2, 2m) \) is given by an integral form as

\[
\langle T(2, 2m) \rangle_N = e^{\frac{\pi}{2} N \left( m - \frac{1}{2} \right)} 8 \sqrt{2} \left( m N \right)^{\frac{3}{2}} e^{-\pi i / 4}
\]

\[
\times \int_C dw w^2 e^{8m\pi N w^2 + 4m\pi N w} \frac{2 \text{sh}(4 m \pi N w) \text{sh}(4 \pi w)}{\text{sh}(4 m \pi w)}, \quad (3.1)
\]

where \( C \) denotes a path passing through the origin in the steepest descent direction.
Proof. A proof is essentially same with those given in Refs. 2, 7, 8, 16 for studies in asymptotic behavior of the quantum invariants.

To rewrite eq. (2.4) into an integral form, we use an integral formula

\[ e^\frac{h}{2}w^2 = \frac{1}{\sqrt{\pi h}} \int_C dz \ e^{-\frac{z^2}{2h}} + 2wz, \]

where an integration path \( C \) is an infinite line passing through the origin in the steepest descent direction. Then we get

\[
2 \sinh \left( \frac{N h}{2} \right) \frac{J_N(h; \mathcal{K})}{J_N(h; \mathcal{O})} = e^{-\frac{1}{2}mN^2} \sum_{\varepsilon = \pm 1} \sum_{j=0}^{N-1} \varepsilon e^{m(j + \frac{m+1}{2m})^2} \\
= \frac{1}{\sqrt{\pi m} h} e^{-\frac{1}{2}mN^2} \sum_{\varepsilon = \pm 1} \sum_{j=0}^{N-1} \varepsilon \int_C dz \ e^{-\frac{z^2}{2m} + 2(j + \frac{m+1}{2m})z} \\
= \frac{1}{\sqrt{\pi m} h} e^{-\frac{1}{2}mN^2} \int_C dz \ e^{-\frac{z^2}{2m} + N \pi z} 2 \sinh(N z) \frac{\sin(z/m)}{\sin(z)}.
\]

Now we take a limit

\[ h \to \frac{2 \pi i}{N}, \]

to compute Kashaev’s invariant. As LHS vanishes in this limit, we take a derivative of both sides to have

\[
\langle T(2, 2m) \rangle_N = \frac{1}{4\sqrt{2} \pi^3} \left( \frac{N}{m} \right)^{3/2} e^{\frac{\pi i}{2m}(m - \frac{1}{m})} \int_C dz \ z^2 e^{N \left( \frac{m^2}{2m} + z \right)} \frac{2 \sinh(N z) \sin(z/m)}{\sin(z)}.
\]

Rescaling an integral variable, we obtain eq. (3.1).

\[ \Box \]

Theorem 3. Kashaev’s invariant \( \langle \mathcal{K} \rangle_N \) for the torus link \( \mathcal{K} = T(2, 2m) \) has an asymptotic expansion in \( N \to \infty \) as

\[
\langle T(2, 2m) \rangle_N \sim e^{\frac{\pi i}{2m}(m - \frac{1}{m})} N^{3/2} \sqrt{\frac{2}{m}} \sum_{k=1}^{m-1} (-1)^{k} (k - m) \sin \left( \frac{k}{m} \pi \right) e^{-\frac{k^2}{2m} \pi i N} \\
+ e^{-\frac{(m-1)^2}{2m} \pi i} N \sum_{k=0}^{\infty} \frac{E_k^{(m;0)}}{k!} \left( \frac{\pi i}{2m N} \right)^k, \quad (3.2)
\]

where \( E_k^{(m;0)} \) is defined from a generating function (see also eq. (3.16)) as

\[
\frac{m \sinh(z)}{\sinh(mz)} = \sum_{k=0}^{\infty} \frac{E_k^{(m;0)}}{(2k)!} z^{2k} \\
= 1 + \frac{1 - m^2}{6} z^2 + \frac{(3 - 7m^2)(1 - m^2)}{360} z^4 + \ldots.
\]

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Proof. When we decompose $\text{sh}(4m \pi N w)$ in the integrand (3.1) into $(e^{4m \pi N w} - e^{-4m \pi N w})/2$, the integral reduces to $I_1 - I_2$ up to constant where $I_1$ and $I_2$ are

$$I_1 = \int_C dw \, w^2 e^{8m \pi N (i w^2 + w)} \frac{\text{sh}(4 \pi w)}{\text{sh}(4 m \pi w)},$$

$$I_2 = \int_C dw \, w^2 e^{8m \pi N i w^2} \frac{\text{sh}(4 \pi w)}{\text{sh}(4 m \pi w)}.$$

In $I_1$ we deform an integration path $C \to C + \frac{i}{2}$. In this deformation we have contributions from residues at $w = \frac{k}{4m} i$ for $k = 1, 2, \ldots, 2m - 1$. Then we get

$$I_1 = 2 \pi i \cdot \text{(Residues)} + \int_{C + \frac{i}{2}} dw \, w^2 e^{8m \pi N (i w^2 + w)} \frac{\text{sh}(4 \pi w)}{\text{sh}(4 m \pi w)}$$

$$= \frac{1}{32 m^3} \sum_{k=1}^{2m-1} (-1)^k k^2 \sin \left( \frac{k}{m} \pi \right) e^{-\frac{k^2}{2m^2} \pi i N} + \int_C dz \left( z + \frac{i}{2} \right)^2 e^{8m \pi i N z^2} \frac{\text{sh}(4 \pi z)}{\text{sh}(4 m \pi z)}$$

$$= \frac{1}{8 m^2} \sum_{k=1}^{m-1} (-1)^k (k - m) \sin \left( \frac{k}{m} \pi \right) e^{-\frac{k^2}{2m^2} \pi i N} + I_2 - \frac{1}{4} \int_C dz e^{8m \pi i N z^2} \frac{\text{sh}(4 \pi z)}{\text{sh}(4 m \pi z)}.$$

In the last equality, we have used a symmetry $z \leftrightarrow -z$ of the integrand. Substituting a series expansion (3.3) into above expression, we obtain an assertion of theorem. \qed

Asymptotic expansion for the torus knot $T(2, 2m + 1)$ was studied in Ref. 3. In view of these results, a tail of asymptotic expansion of Kashaev’s invariant is given in an infinite series of $N^{-1}$ with coefficients, whose generating function seems to be related to

$$\frac{1 - s}{\Delta_K(s)}.$$

Here $\Delta_K(s)$ is the Alexander polynomial for knot $K$, and in a case of the torus link $K = T(2, 2m)$ we have

$$\Delta_K(s) = \frac{1 - s^{2m}}{1 + s}.$$

We have shown that the volume conjecture [5, 14] is fulfilled for the torus link $T(2, 2m)$,

$$\lim_{N \to \infty} \frac{2 \pi}{N} \log |\langle T(2, 2m) \rangle_N| = 0,$$

because the torus link $T(2, 2m)$ is not hyperbolic. Rather we have interests in an asymptotic expansion of the $q$-series. To this aim, we compute the quantum invariant by another method;
explicit form of Kashaev’s invariants can be directly computed from a set of the enhanced Yang–Baxter operators [5] (see also Refs. 13, 14, 18);

\[
R^{ij}_{k\ell} = \frac{N \omega^{1-(k-j+1)(\ell-i)}}{(\omega)_{[\ell-k-1]}(\omega)_{[j-\ell]}(\omega)_{[i-j]}(\omega)_{[k-i]}} \cdot \theta \left[ \begin{array}{c} i \\ j \\ k \\ \ell \end{array} \right], \tag{3.4a}
\]

\[
(R^{-1})^{ij}_{k\ell} = \frac{N \omega^{-1+(\ell-i-1)(k-j)}}{(\omega)_{[\ell-k-1]}(\omega)_{[j-\ell]}(\omega)_{[i-j]}(\omega)_{[k-i]}} \cdot \theta \left[ \begin{array}{c} i \\ j \\ k \\ \ell \end{array} \right], \tag{3.4b}
\]

\[
\mu^{k}_{\ell} = -\delta_{k,\ell+1}\omega^{1/2}, \tag{3.4c}
\]

where we have defined the \(N\)-th root of unity as

\[
\omega = \exp\left(\frac{2\pi i}{N}\right), \tag{3.5}
\]

and * means a complex conjugation. We have also used \([x] \in \{0, 1, \ldots, N - 1\}\) modulo \(N\), and

\[
\theta \left[ \begin{array}{c} i \\ j \\ k \\ \ell \end{array} \right] = 1, \text{ if and only if } \begin{cases} i \leq k < \ell \leq j, \\ j \leq i < k < \ell, \\ \ell \leq j \leq i \text{ (with } \ell < k), \\ k < \ell \leq j \leq i. \end{cases} \tag{3.6}
\]

Remark that used is the standard notation of the \(q\)-product and the \(q\)-binomial coefficient;

\[
(\omega)_{n} = \prod_{k=1}^{n} (1 - \omega^{k}), \tag{3.6}
\]

\[
[m \atop n] = \begin{cases} \frac{(\omega)_{m}}{(\omega)_{n}(\omega)_{m-n}}, & \text{if } m \geq n \geq 0, \\ 0, & \text{others.} \end{cases} \tag{3.7}
\]

These operators are assigned to a projection of knot as follows:

\[
R^{ij}_{k\ell} = \begin{array}{c} i \\ k \\ \ell \end{array}, \quad (R^{-1})^{ij}_{k\ell} = \begin{array}{c} i \\ k \\ \ell \end{array}, \quad \mu^{k}_{\ell} = \begin{array}{c} k \\ \ell \end{array}, \quad (\mu^{-1})^{k}_{\ell} = \begin{array}{c} k \\ \ell \end{array}.
\]
Proposition 4. Kashaev’s invariant \( \langle \mathcal{K} \rangle_N \) for the torus link \( \mathcal{K} = T(2, 2m) \) is explicitly given by

\[
\langle T(2, 2m) \rangle_N = N \sum_{N-1 \geq c_{m-1} \geq \cdots \geq c_2 \geq c_1 \geq 0} (-1)^{c_{m-1}} \omega^{\frac{1}{2}c_{m-1}(c_{m-1}+1)} \left( \prod_{i=1}^{m-2} \omega^{c_i(c_i+1)} \left[ \begin{array}{c} c_{i+1} \\ c_i \end{array} \right] \right). 
\]

(3.8)

Proof. We get this result from a direct computation using \( R \)-matrix (3.4) for \((1,1)\)-tangle of \((2,2m)\)-torus link. See Refs. 13, 18. 

We note that Kashaev’s invariant for the Hopf link \( T(2,2) \) is given by

\[
\langle T(2,2) \rangle_N = N.
\]

Combining Prop. 4 with Thm. 3, we obtain an asymptotic expansion of \( \omega \)-series.

Corollary 5.

\[
\sum_{N-1 \geq c_{m-1} \geq \cdots \geq c_2 \geq c_1 \geq 0} (-1)^{c_{m-1}} \omega^{\frac{1}{2}c_{m-1}(c_{m-1}+1)} \left( \prod_{i=1}^{m-2} \omega^{c_i(c_i+1)} \left[ \begin{array}{c} c_{i+1} \\ c_i \end{array} \right] \right)
\]

\[
\simeq \sqrt{N} e^{\frac{2 \pi i (m-1)^2}{2mN}} \sqrt{\frac{2}{m}} \sum_{k=1}^{m-1} (-1)^k (k - m) \sin \left( \frac{k \pi}{m} \right) e^{-\frac{2}{m} \pi i N} e^{-\frac{(m-1)^2}{2m} \pi i} \sum_{k=0}^{\infty} \frac{E_k^{(m;0)}}{k!} \left( \frac{\pi i}{2mN} \right)^k.
\]

(3.9)

In the rest of this paper, we shall reveal a meaning of this asymptotic expansion from a point of view of the modular form. As a generalization of \( \omega \)-series defined by Kashaev’s invariant we introduce

\[
Y_m^{(a)}(\omega) = \sum_{c_1, \ldots, c_{m-1} = 0}^{N-1} (-1)^{c_{m-1}} \omega^{\frac{1}{2}c_{m-1}(c_{m-1}+1)} \times \omega^{c_1^2 + \cdots + c_{m-2}^2 + c_2 + \cdots + c_{m-2}} \prod_{i=1}^{m-2} \left[ \begin{array}{c} c_{i+1} + \delta_i,a \\ c_i \end{array} \right],
\]

(3.10)

for \( m \geq 2 \) and \( a = 0, 1, \ldots, m-2 \). See that Kashaev’s invariant for \( T(2, 2m) \) corresponds to a case of \( a = 0 \),

\[
\langle T(2, 2m) \rangle_N = N \cdot Y_m^{(0)}(\omega).
\]

(3.11)

It is unclear whether the \( \omega \)-series \( Y_m^{(a)}(\omega) \) for \( a \neq 0 \) represent the quantum invariant for any three manifolds.
**Conjecture 1.** Let $Y_m^{(a)}(\omega)$ be defined by eq. (3.10). An asymptotic expansion of this $\omega$-series in $N \to \infty$ is given by

$$
e^{\frac{2\pi i}{mN} (m-1-a)^2} Y_m^{(a)}(\omega)
\approx \sqrt{N} e^{\frac{2\pi i}{mN}} \cdot \sqrt{2} \sum_{k=1}^{m-1} (-1)^k (k - m) \sin \left( \frac{k}{m} (a + 1) \pi \right) e^{-\frac{4\pi^2 k^2}{m^2 N}}
\quad + \sum_{k=0}^{\infty} E_k^{(m;a)} \left( \frac{\pi i}{2 m N} \right)^k,
(3.12)$$

where a generalized Euler number $E_k^{(m;a)}$ is given from a generating function as

$$m \frac{\sin((a + 1)z)}{\sin(mz)} = \sum_{k=0}^{\infty} \frac{E_k^{(m;a)}}{(2k)!} z^{2k},
(3.13)$$

$$= (a + 1) + \frac{1}{6} (1 + a) ((1 + a)^2 - m^2) z^2
\quad + \frac{1}{360} (1 + a) ((1 + a)^2 - m^2) (3 (1 + a)^2 - 7 m^2) z^4
\quad + \frac{1}{15120} (1 + a) ((1 + a)^2 - m^2) (3 (1 + a)^4 - 18 (1 + a)^2 m^2 + 31 m^4) z^6 + \cdots.$$  

The case $a = 0$ of this conjecture is proved in Corollary 5.

Note that we have

$$\frac{\sin((a + 1)z)}{\sin(mz)} = \sum_{n=0}^{\infty} \chi_{2m}^{(a)}(n) e^{-nz},
(3.14)$$

where the odd periodic function $\chi_{2m}^{(a)}(n)$ is written as

$$\begin{array}{c|ccc}
 n \mod 2m & m - 1 - a & m + 1 + a & \text{others} \\
\hline
\chi_{2m}^{(a)}(n) & 1 & -1 & 0
\end{array}
(3.15)$$

Applying the Mellin transformation to eqs. (3.13) and (3.14), we have an expression of the generalized Euler number in terms of the $L$-function associated to $\chi_{2m}^{(a)}(n)$;

$$E_k^{(m;a)} = m \cdot L(-2k; \chi_{2m}^{(a)})
= -m \left( \frac{2m}{2k+1} \right)^{2k} \left( B_{2k+1} \left( \frac{m - 1 - a}{2m} \right) - B_{2k+1} \left( \frac{m + 1 + a}{2m} \right) \right),
(3.16)$$

where $B_n(x)$ is the $n$-th Bernoulli polynomial. It should be remarked that the colored Jones polynomial (2.4) for the torus link $K = T(2, 2m)$ is rewritten using the periodic function as

$$2 \sin \left( \frac{N h}{2} \right) \frac{J_N(h; K)}{J_N(h; O)} = -e^{-\frac{1}{4} m (N^2 - 1) h - \frac{m^2 + 1}{4m} h} \sum_{k=0}^{2mN} \chi_{2m}^{(0)}(k) e^{\frac{k^2}{2m} h}.
(3.17)$$
Based on this expression, we find that Kashaev’s invariant is given by

$$\langle T(2, 2m) \rangle_N = -\frac{1}{4mN} e^{-\frac{(m-1)^2}{2mN} \pi i} \sum_{k=0}^{2mN} k^2 \chi_{2m}^{(0)}(k) e^{\frac{k^2}{2mN} \pi i}. \quad (3.18)$$

Later we shall clarify a relationship between above conjecture and the modular form.

4. q-Series Identity

In this section we study a q-series identity, which is closely related with $Y_m^{(a)}(\omega)$ defined in eq. (3.10). We use standard notation as in eqs. (3.5) – (3.7), but in this section we replace $\omega$, the $N$-th primitive root of unity, with generic $q$.

We define the q-series

$$K_m^{(a)}(x) = \sum_{c_1, \ldots, c_{m-1}=0}^{\infty} (-1)^{c_{m-1}} q^{\frac{1}{2}(c_{m-1}+1)} x^{c_1+\cdots+c_{m-1}} \times q^{c_1^2+\cdots+c_{m-2}^2+c_{a+1}+\cdots+c_{m-2}} \left( \prod_{i=1}^{m-2} \left[ c_{i+1} + \delta_{i,a} \right] / c_i \right), \quad (4.1)$$

for $m \geq 2$ and $a = 0, 1, 2, \ldots, m-2$. We simply replace $\sum_{c_{m-1}=0}^{N-1}$ in eq. (3.10) with an infinite sum $\sum_{c_{m-1}=0}^{\infty}$, though we have introduced an additional variable $x$.

**Theorem 6.** Let the q-series $K_m^{(a)}(x)$ be defined in eq. (4.1). We have

$$K_m^{(a)}(x) = \sum_{n=0}^{\infty} \chi_{2m}^{(a)}(n) q^{\frac{n^2-(m-1-a)^2}{4m} x^{\frac{n-(m-1-a)}{2}}}, \quad (4.2)$$

where $\chi_{2m}^{(a)}(n)$ is a periodic function in eq. (3.15).

**Proof.** We prove this statement by showing that both sides satisfy the same q-difference equation.

It is easy to see from a periodicity of the function $\chi_{2m}^{(a)}(n)$ that RHS of eq. (4.2) solves a difference equation (see, e.g., Refs. 1, 3, 19),

$$K_m^{(a)}(x) = 1 - q^{a+1} x^{a+1} + \sum_{n=2m}^{\infty} \chi_{2m}^{(a)}(n) q^{\frac{n^2-(m-1-a)^2}{4m} x^{\frac{n-(m-1-a)}{2}}}$$

$$= 1 - q^{a+1} x^{a+1} + x^m q^{2m-1-a} K_m^{(a)}(q^2 x). \quad (4.3)$$
We shall show that LHS of eq. (4.2) also fulfills a same difference equation. To this aim, we introduce

\[
K^{(a)}_m(x_1, \ldots, x_{m-1}) = \sum_{c_1, \ldots, c_{m-1}=0}^{\infty} (-1)^{c_{m-1}} q^{\frac{1}{2}c_{m-1}(c_{m-1}+1)} x_{m-1}^{c_{m-1}} \\
\times \left( \prod_{i=1}^{m-2} q^{c_i^2} x_i^{c_i} \left[ \begin{array}{c} c_{i+1} \\ c_i \end{array} \right] \right) \cdot q^{c_a^2} x_a^{c_a} \left[ \begin{array}{c} c_{a+1} + 1 \\ c_a \end{array} \right] \\
\times \left( \prod_{i=a+1}^{m-2} q^{c_i^2+c_i} x_i^{c_i} \left[ \begin{array}{c} c_{i+1} \\ c_i \end{array} \right] \right). \tag{4.4}
\]

See that by definition we have

\[
K^{(a)}_m(x) = K^{(a)}_m(x, \ldots, x). \tag{4.5}
\]

We use same symbol \(K^{(a)}_m\), but we believe there is no confusion.

To prove the assertion of the theorem, we use formulae for the \(q\)-binomial coefficients;

\[
\left[ \begin{array}{c} n+1 \\ c \end{array} \right] = q^n \left[ \begin{array}{c} n \\ c \end{array} \right] + \left[ \begin{array}{c} n \\ c-1 \end{array} \right] \tag{4.6a}
\]

\[
= \left[ \begin{array}{c} n \\ c \end{array} \right] + q^{n+1-c} \left[ \begin{array}{c} n \\ c-1 \end{array} \right]. \tag{4.6b}
\]

Applying eq. (4.6a) to \(\left[ \begin{array}{c} c_{a+1} + 1 \\ c_a \end{array} \right]\) in eq. (4.4), we get

\[
K^{(a)}_m(x_1, \ldots, x_{m-1}) = K^{(0)}_m(q^{-1} x_1, \ldots, q^{-1} x_{a-1}, x_a, \ldots, x_{m-1}) \\
+ q x_a \cdot K^{(a-1)}_m(x_1, \ldots, x_{a-1}, q x_a, x_{a+1}, \ldots, x_{m-1}). \tag{4.7}
\]

On the other hand, using eq. (4.6b), we have

\[
K^{(a)}_m(x_1, \ldots, x_{m-1}) = K^{(0)}_m(q^{-1} x_1, \ldots, q^{-1} x_a, x_{a+1}, \ldots, x_{m-1}) \\
+ q x_a \cdot K^{(a-1)}_m(x_1, \ldots, x_a, q x_{a+1}, x_{a+2}, \ldots, x_{m-1}). \tag{4.8}
\]

Another difference equation is given as follows;

\[
K^{(0)}_m(x_1, \ldots, x_{m-1}) \\
= 1 + \sum_{c_{m-1}=1}^{\infty} \sum_{c_1, \ldots, c_{m-2}=0}^{\infty} (-1)^{c_{m-1}} q^{\frac{1}{2}c_{m-1}(c_{m-1}+1)} x_{m-1}^{c_{m-1}} \left( \prod_{i=1}^{m-2} q^{c_i(c_i+1)} x_i^{c_i} \left[ \begin{array}{c} c_{i+1} \\ c_i \end{array} \right] \right) \\
= 1 - q x_{m-1} \cdot K^{(m-2)}_m(q x_1, \ldots, q x_{m-1}). \tag{4.9}
\]
We can prove eq. (4.3) by use of eqs. (4.7)—(4.9) as follows. Recursive use of eq. (4.7) gives

\[
K_m^{(a)}(x_1, \ldots, x_a, q x_{a+1}, \ldots, q x_{m-1})
= K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{a-1}, x_a, q x_{a+1}, \ldots, q x_{m-1})
+ q x_a \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{a-2}, x_{a-1}, q x_a, \ldots, q x_{m-1})
+ q^2 x_{a-1} x_a \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{a-3}, x_{a-2}, q x_{a-1}, \ldots, q x_{m-1})
+ \cdots + q^a x_1 \cdots x_a \cdot K_m^{(0)}(q x_1, \ldots, q x_{m-1}). \quad (4.10)
\]

Substituting above equation for \( a = m - 2 \) into eq. (4.9), we get

\[
K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{m-2}, x_{m-1})
+ q x_{m-1} \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{m-3}, x_{m-2}, q x_{m-1})
+ q^2 x_{m-2} x_{m-1} \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{m-4}, x_{m-3}, q x_{m-2}, q x_{m-1})
+ \cdots + q^{m-2} x_2 \cdots x_{m-1} \cdot K_m^{(0)}(x_1, q x_2, \ldots, q x_{m-1})
= 1 - q^{m-1} x_1 \cdots x_{m-1} \cdot K_m^{(0)}(q x_1, \ldots, q x_{m-1}). \quad (4.11)
\]

In the same way, iterated use of eq. (4.8) gives

\[
K_m^{(a)}(x_1, \ldots, x_{a+1}, q x_{a+2}, \ldots, q x_{m-1})
= K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_a, x_{a+1}, q x_{a+2}, \ldots, q x_{m-1})
+ q x_a \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{a-1}, x_a, q x_{a+1}, \ldots, q x_{m-1})
+ q^2 x_{a-1} x_a \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{a-2}, x_{a-1}, q x_a, \ldots, q x_{m-1})
+ \cdots + q^a x_1 \cdots x_a \cdot K_m^{(0)}(x_1, q x_2, \ldots, q x_{m-1}). \quad (4.12)
\]

Substituting above equation for \( a = m - 2 \) into eq. (4.9), we find

\[
K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{m-2}, x_{m-1})
+ q x_{m-2} \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{m-3}, x_{m-2}, q x_{m-1})
+ q^2 x_{m-3} x_{m-2} \cdot K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{m-4}, x_{m-3}, q x_{m-2}, q x_{m-1})
+ \cdots + q^{m-2} x_1 \cdots x_{m-2} \cdot K_m^{(0)}(x_1, q x_2, \ldots, q x_{m-1})
= \frac{1}{x_{m-1}} (1 - K_m^{(0)}(q^{-1} x_1, \ldots, q^{-1} x_{m-1})). \quad (4.13)
\]

Combining two equations (4.11) and (4.13) with \( x \equiv x_1 = \cdots = x_{m-1} \), we obtain

\[
K_m^{(0)}(x) = 1 - q x + q^{2m-1} x^m \cdot K_m^{(0)}(q^2 x).
\]

This proves eq. (4.2) for \( a = 0 \).
Other cases can be shown by using eqs. (4.11) and (4.13) with eqs. (4.10) and (4.12).

Corollary 7. We define the q-series $\tilde{\Phi}_m^{(a)}(\tau)$ by $K_m^{(a)}(x = 1)$ up to constant, i.e.,

$$
\tilde{\Phi}_m^{(a)}(\tau) = m \frac{(m-1-a)^2}{4m} \sum_{c_1,\ldots,c_{m-1}=0}^{\infty} (-1)^{c_{m-1}} q^{\frac{1}{4}c_{m-1}(c_{m-1}+1)} \times q^{c_1^2+c_2^2+c_m + \cdots + c_{m-2}} \left( \prod_{i=1}^{m-2} \left[ c_i + \delta_{i,a} \right] \right),
$$

(4.14)

where $m \geq 2$ and $a = 0, 1, \ldots, m - 2$, and we mean

$$
q = e^{2\pi i \tau}.
$$

Then we have

$$
\tilde{\Phi}_m^{(a)}(\tau) = m \sum_{n=0}^{\infty} \lambda_{2m(a)}(n) q^{\frac{n}{4m} \tau^2}.
$$

(4.15)

Factor $m$ in above definition is merely for our later convention.

Corollary 8. Let the q-series $\Phi_m^{(a)}(\tau)$ be defined by

$$
\Phi_m^{(a)}(\tau) = \sum_{n \in \mathbb{Z}} n \lambda_{2m(a)}(n) q^{\frac{n}{4m} \tau^2},
$$

(4.16)

for $m \geq 2$ and $a = 0, 1, \ldots, m - 2$. Then we have

$$
\Phi_m^{(a)}(\tau) = 4 q^{(m-1-a)^2/4m} \sum_{c_1,\ldots,c_{m-1}=0}^{\infty} \left( c_1 + c_2 + \cdots + c_{m-1} + \frac{m - 1 - a}{2} \right) (-1)^{c_{m-1}} q^{\frac{1}{4}c_{m-1}(c_{m-1}+1)} \times q^{c_1^2+c_2^2+c_m + \cdots + c_{m-2}} \left( \prod_{i=1}^{m-2} \left[ c_i + \delta_{i,a} \right] \right).
$$

(4.17)

Proof. We differentiate eq. (4.2) with respect to $x$ and substitute $x \to 1$. 

5. Modular Property

We shall reveal the modular property of the q-series $\Phi_m^{(a)}(\tau)$ and $\tilde{\Phi}_m^{(a)}(\tau)$ defined by eq. (4.16) and eq. (4.15) respectively. The theta series $\Phi_m^{(a)}(\tau)$ have weight $3/2$, and span $(m-1)$-dimensional space; it is straightforward to get

$$
\Phi_{m-1-a}^{(a)}(\tau + 1) = e^{a^2/4m \pi i} \Phi_{m}^{(m-1-a)}(\tau),
$$

(5.1)
and from the standard method using the Poisson summation formula we have
\[ \Phi_m(\tau) = \left( \frac{i}{\tau} \right)^{3/2} M_m \cdot \Phi_m\left( -\frac{1}{\tau} \right), \quad (5.2) \]
where
\[ \Phi_m(\tau) = \begin{pmatrix} \Phi_{m-2}(\tau) \\ \vdots \\ \Phi_{m}^{(1)}(\tau) \\ \Phi_{m}^{(0)}(\tau) \end{pmatrix}, \]
and \( M_m \) is an \((m-1) \times (m-1)\) matrix,
\[ (M_m)_{1 \leq a, b \leq m-1} = \sqrt{\frac{2}{m}} \sin \left( \frac{a b}{m} \pi \right). \quad (5.3) \]
Remarkable is that the theta series defined by
\[ \text{ch}_m^\lambda(\tau) = \Phi_{m-\lambda}(\tau) + \frac{\Phi_{m+\lambda}(\tau)}{2 (\eta(\tau))^2}, \quad (5.4) \]
where \( \eta(\tau) \) is the Dedekind \( \eta \)-function (6.3), is the affine \( \widehat{su}(2)_m \) character (see Examples in Section 6) [4, 11].

As studied in Ref. 9, we have interests in the Eichler integral of the modular form \( \Phi_m(\tau) \). Generally when the \( q \)-series
\[ F(\tau) = \sum_{n=1}^{\infty} a_n q^n, \]
is a modular form with weight \( k \in \mathbb{Z}_{\geq 2} \), the Eichler integral defined as \( k-1 \) integrations of \( F(\tau) \) with respect to \( \tau \), or explicitly defined by
\[ \tilde{F}(\tau) = \sum_{n=1}^{\infty} \frac{a_n}{\eta^{k-1}} q^n, \]
satisfies
\[ (c \tau + d)^{k-2} \cdot \tilde{F}(\gamma(\tau)) - \tilde{F}(\tau) = G_\gamma(\tau), \quad (5.5) \]
where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}) \), and \( G_\gamma(\tau) \) is the period polynomial
\[ G_\gamma(z) = \frac{(2 \pi i)^{k-1}}{(k-2)!} \int_{\gamma^{-1}(i \infty)}^{i \infty} F(\tau)(z - \tau)^{k-2} \, d\tau. \]
In our case, the modular form \( \Phi_m(a)(\tau) \) in eq. (4.16) has a half-integral weight, and above story does not work any more. But the Eichler integral as an infinite \( q \)-series can be defined in a naive sense, and we may find that
\[ \tilde{\Phi}_m^{(a)}(\tau) = m \sum_{n=0}^{\infty} \chi_{2m}(n) q^{\frac{1}{2m}n^2}, \]
which is nothing but a definition (4.15). We note that a prefactor in above definition is for our convention. We can regard $\widetilde{\Phi}_m(a)\tau$ as the Eichler integral of the modular form $\Phi_m(a)\tau$ with weight $3/2$.

To study a nearly modular property (see eq. (5.5)) of this Eichler integral of the half-integral weight modular form, we first recall a following result.

**Proposition 9.** Let $C_f(n)$ be a periodic function with mean value 0 and modulus $f$. Then we have an asymptotic expansion as $t \to 0$

$$
\sum_{n=1}^{\infty} C_f(n) e^{-n^2 t} \simeq \sum_{k=0}^{\infty} L(-2k, C_f)(-t)^k \div k!,
$$

where $L(k, C_f)$ is the $L$-function associated with $C_f(n)$, and is given by

$$
L(-k, C_f) = -f^k \div k+1 \sum_{n=1}^{f} C_f(n) B_{k+1} \left( \frac{n}{f} \right).
$$

**Proof.** It is a standard result using the Mellin transformation. See, e.g., Ref. 9. \hfill $\square$

Using this property, we obtain the Eichler integral $\widetilde{\Phi}_m(a)\tau$ near at a root of unity as follows.

**Proposition 10.** The Eichler integral (4.15) for $\tau = \frac{M}{N} \in \mathbb{Q}$ ($N > 0$, and $M, N$ are coprime integers) reduces to

$$
\widetilde{\Phi}_m^a \left( \frac{M}{N} \right) = m \sum_{n=0}^{mN} \chi_2^{(a)}(n) \left( 1 - \frac{n}{mN} \right) e^{n^2 \pi M N i}.
$$

**Proof.** We have from eq. (4.15)

$$
\widetilde{\Phi}_m^a \left( \frac{M}{N} + i \frac{t}{2 \pi} \right) = m \sum_{n=0}^{\infty} C_{2mN}(n) e^{-\frac{n^2}{4m}t},
$$

where

$$
C_{2mN}(n) = \chi_2^{(a)}(n) e^{\frac{Mn^2}{2mN} \pi i}.
$$

We see that $C_{2mN}(n + 2mN) = C_{2mN}(n)$ and $C_{2mN}(2mN - n) = -C_{2mN}(n)$, and we can apply Prop. 9 to get an asymptotic expansion in $t \to 0$ as

$$
\widetilde{\Phi}_m^a \left( \frac{M}{N} + i \frac{t}{2 \pi} \right) \simeq \sum_{k=0}^{\infty} L(-2k, C_{2mN})(-t)^k \div 4m^k.
$$

Then we obtain a limiting value

$$
\widetilde{\Phi}_m^a \left( \frac{M}{N} \right) = m \cdot L(0, C_{2mN}).
$$
Recalling an explicit form of the Bernoulli polynomial $B_1(x) = x - \frac{1}{2}$ and a property $C_{2mN}(2mN - n) = -C_{2mN}(n)$, we obtain eq. (5.6).

**Conjecture 2.** Let $\tilde{\Phi}_m^{(a)}(\tau)$ be the Eichler integral defined by eq. (4.15) or (4.14). When $q$ is the $N$-th root of unity, $\tilde{\Phi}_m^{(a)}(1/N)$, which was computed as eq. (5.6), coincides with an expression (3.10) up to constant, i.e.,

$$\tilde{\Phi}_m^{(a)}\left(\frac{1}{N}\right) = e^{\frac{(m-1-a)^2}{2mN} \pi i} \cdot Y_m^{(a)}(\omega).$$  \hspace{1cm} (5.7)

**Proof for a case of $a = 0$.** As a case of $a = 0$ is related to Kashaev’s invariant for the torus link as in eq. (3.11), this case can be directly proved by using eq. (3.18) as follows;

$$e^{\frac{(m-1)^2}{2mN} \pi i} Y_m^{(0)}(\omega) = \frac{1}{N} e^{\frac{(m-1)^2}{2mN} \pi i} \langle T(2, 2m) \rangle N$$

$$= -\frac{1}{4mN^2} \sum_{k=0}^{2mN} k^2 \chi_{2m}(k) e^{\frac{k^2}{2mN} \pi i}$$

$$= \sum_{k=0}^{2mN} \chi_{2m}(k) \left( \frac{m}{2} - \frac{k}{2N} \right) e^{\frac{k^2}{2mN} \pi i}$$

$$= \sum_{k=0}^{mN} \chi_{2m}(k) \left( m - \frac{k}{N} \right) e^{\frac{k^2}{2mN} \pi i} = \tilde{\Phi}_m^{(0)}\left(\frac{1}{N}\right).$$

In the third equality, we have summed an expression with $k \rightarrow 2mN - k$. As a result of eq. (5.6) the statement of Conjecture is true for $a = 0$. \hfill \Box

We now discuss how Conjectures 1 follows from Conjecture 2. We first recall from eq. (5.6) that for $N \in \mathbb{Z}$

$$\tilde{\Phi}_m^{(a)}(N) = (1 + a) e^{\frac{(m-1-a)^2}{2m} \pi i N}. \hspace{1cm} (5.8)$$

Following Ref. 9 (see also Ref. 21) we define the period function

$$r_m^{(a)}(z; \alpha) = \sqrt{\frac{m}{8i}} \int_{\alpha}^{\infty} \frac{\tilde{\Phi}_m^{(a)}(\tau)}{\sqrt{\tau - z}} d\tau, \hspace{1cm} (5.9)$$

where $\alpha \in \mathbb{Q}$. It is defined for $z$ in the lower half plane, $z \in \mathbb{H}^-$, but it is analytically continued to $\mathbb{R} = \partial \mathbb{H}^-$. To see a modular property of $\tilde{\Phi}_m^{(a)}(\alpha + iy)$ in $y \searrow 0$, we further define

$$\tilde{\Phi}_m^{(a)}(z) = \sqrt{\frac{m}{8i}} \int_{z^*}^{\infty} \frac{\tilde{\Phi}_m^{(a)}(\tau)}{\sqrt{\tau - z}} d\tau, \hspace{1cm} (5.10)$$
where \( z \in \mathbb{H}^- \). We can find that this function is nearly modular of weight \( 1/2 \) by
\[
\sum_{b=1}^{m-1} (M_m)_{a,b} \hat{\Phi}^{(m-1-b)}_m \left( -\frac{1}{z} \right) = \sum_{b=1}^{m-1} (M_m)_{a,b} \sqrt{\frac{m}{8i}} \int_0^s \frac{\Phi^{(m-1-b)}_m \left( -\frac{1}{s} \right) ds}{\sqrt{-s-1+z^{-1}} / s^2}
\]
\[
= \sqrt{\frac{1}{z}} \left( \hat{\Phi}^{(m-1-a)}_m \left( z \right) - r^{(m-1-a)}_m \left( z; 0 \right) \right). \tag{5.11}
\]

On the other hand, we have for \( z = \alpha + iy \)
\[
\hat{\Phi}^{(a)}_m (z) = \sqrt{\frac{m}{8i}} \sum_{n \in \mathbb{Z}} n \chi_2(m)(n) \int_{z^*}^{\infty} \frac{e^{\frac{n^2\pi i r}{m}}}{\sqrt{\tau - z}} d\tau
\]
\[
= m \sum_{n=1}^{\infty} \chi_4(m)(n) e^{\frac{n^2\pi i}{2m}} \text{erfc} \left( n \sqrt{\frac{-\pi y}{m}} \right),
\]
where \( \text{erfc}(x) \) is the complementary error function
\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt.
\]
As \( \text{erfc}(0) = 1 \) and we know eq. (5.6) from a definition of \( \tilde{\Phi}^{(a)}_m (\alpha) \), we get
\[
\hat{\Phi}^{(a)}_m (\alpha) = \tilde{\Phi}^{(a)}_m (\alpha), \tag{5.12}
\]
for \( \alpha \in \mathbb{Q} \). We stress that LHS is a limit from the lower half plane \( \mathbb{H}^- \) while RHS is analytically continued from the upper half plane \( \mathbb{H} \).

Taking a limit \( z \to 1/N \) for \( N \in \mathbb{Z} \) in eq. (5.11), we obtain
\[
\hat{\Phi}^{(m-1-a)}_m \left( \frac{1}{N} \right) = \sqrt{-iN} \sum_{b=1}^{m-1} (M_m)_{a,b} \hat{\Phi}^{(m-1-b)}_m \left( -N \right) + r^{(m-1-a)}_m \left( \frac{1}{N}; 0 \right). \tag{5.13}
\]
When we recall eq. (5.8) and use an asymptotic expansion of \( r^{(a)}_m \left( \frac{1}{N}; 0 \right) \) in \( N \to \infty \), we obtain a following proposition.

**Proposition 11.** Let the Eichler integral \( \tilde{\Phi}^{(a)}_m (\tau) \) be defined by eq. (4.15). Asymptotic expansion in \( N \to \infty \) is then given by
\[
\tilde{\Phi}^{(a)}_m \left( \frac{1}{N} \right) \simeq \sqrt{-iN} \sum_{b=1}^{m-1} \sqrt{\frac{2}{m}} (m - b) \sin \left( \frac{(m - 1 - a) b}{m} \pi \right) e^{-\frac{2^2\pi}{2m}\pi i N}
\]
\[
+ \sum_{k=0}^{\infty} \frac{E^{(m; a)}_k}{k!} \left( \frac{\pi i}{2mN} \right)^k. \tag{5.14}
\]
Based on this proposition, we get a conjecture (3.12) assuming a conjecture (5.7).
6. EXAMPLES

6.1 (2, 4)-Torus Link: $m = 2$

- $q$-series:

The modular form with weight $3/2$ is

\[
\Phi_2^{(0)}(\tau) = \sum_{n \in \mathbb{Z}} n \chi_4^{(0)}(n) q^{\frac{1}{2} n^2} = \sum_{n \in \mathbb{Z}} (-1)^n (2n + 1) q^{\frac{1}{2}(n^2 + n + \frac{1}{4})}
\]

\[
= 2 q^{\frac{1}{8}} \left(1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + \cdots\right),
\]

where $\chi_4^{(0)}(n)$ is the primitive character modulo 4,

| $n \mod 4$ | 1   | 3   | others |
|------------|-----|-----|--------|
| $\chi_4^{(0)}(n)$ | 1   | -1  | 0      |

It is known by Jacobi that we can write

\[
\Phi_2^{(0)}(\tau) = 2 \left(\eta(\tau)\right)^3,
\]

where $\eta(\tau)$ is the Dedekind $\eta$-function,

\[
\eta(\tau) = q^{1/24} \cdot (q)_{\infty}.
\]

From a modular property of the $\eta$-function we see that

\[
\Phi_2^{(0)}(\tau + 1) = e^{\frac{4\pi i}{N}} \Phi_2^{(0)}(\tau),
\]

\[
\Phi_2^{(0)}(-1/\tau) = \left(\frac{\tau}{1}\right)^{3/2} \Phi_2^{(0)}(\tau).
\]

The Eichler integral is given by

\[
\tilde{\Phi}_2^{(0)}(\tau) = 2 \sum_{n=0}^{\infty} \chi_4^{(0)}(n) q^{\frac{1}{8} n^2} = 2 q^{\frac{1}{8}} \sum_{k=0}^{\infty} (-1)^k q^{\frac{3}{8}k(k + 1)}
\]

\[
= 2 q^{\frac{1}{8}} \left(1 - q + q^3 - q^6 + q^{10} - q^{15} + \cdots\right).
\]

- root of unity:

As a limit of $q$ being the $N$-th root of unity, the Eichler integral (6.6) coincides with Kashaev’s invariant for torus link,

\[
\tilde{\Phi}_2^{(0)} \left(\frac{1}{N}\right) = e^{\frac{2\pi i}{N}} \sum_{c=0}^{N-1} (-1)^c \omega^{\frac{1}{2}c(c+1)} = \frac{1}{N} e^{\frac{2\pi i}{N}} \langle T(2, 4) \rangle_N.
\]

We see that the Eichler integral (6.6) with $q$ being the $N$-th root of unity takes a same form with the original Eichler integral up to constant, only an infinite sum reduces to a finite


sum. Using $\Phi_2^{(0)}(N) = e^{\pi i N/4}$, we have the nearly modular property (see eq. (3.9)),

$$\Phi_2^{(0)}\left(\frac{1}{N}\right) \simeq \sqrt{-i N} \Phi_2^{(0)}(-N) + \sum_{n=0}^{\infty} \frac{E_n^{(2,0)}}{n!} \left(\frac{\pi i}{4 N}\right)^n,$$

(6.8)

where $E_n^{(2,0)}$ is the Euler number defined by

$$E_n^{(2,0)} = \frac{2^{4n+1}}{2n+1} \left( B_{2n+1} \left(\frac{1}{4}\right) - B_{2n+1} \left(\frac{3}{4}\right) \right),$$

some of which are given as

$$\frac{1}{\text{ch}(x)} = \sum_{n=0}^{\infty} \frac{E_n^{(2,0)}}{(2 n)!} x^{2n} = 1 - \frac{1}{2} x^2 + \frac{5}{24} x^4 - \frac{61}{720} x^6 + \cdots.$$



6.2 \text{(2,6)-Torus Link: } m = 3

• q-series:

A set of the theta series is given by

$$\Phi_3^{(0)}(\tau) = \sum_{n \in \mathbb{Z}} n \chi_6^{(0)}(n) q^{\frac{1}{12} n^2}$$

$$= 4 q^{\frac{1}{12}} \left( 1 - 2 q + 4 q^5 - 5 q^8 + 7 q^{16} - 8 q^{21} + \cdots \right),$$

(6.9a)

$$\Phi_3^{(1)}(\tau) = \sum_{n \in \mathbb{Z}} n \chi_6^{(1)}(n) q^{\frac{1}{12} n^2}$$

$$= 2 q^{\frac{1}{12}} \left( 1 - 5 q^2 + 7 q^4 - 11 q^{10} + 13 q^{14} - 17 q^{24} + \cdots \right),$$

(6.9b)

where

$$\begin{array}{c|ccc} n \mod 6 & 2 & 4 & \text{others} \\ \hline \chi_6^{(0)}(n) & 1 & -1 & 0 \\ \hline \end{array} \quad \begin{array}{c|ccc} n \mod 6 & 1 & 5 & \text{others} \\ \hline \chi_6^{(1)}(n) & 1 & -1 & 0 \\ \hline \end{array}$$

We note [11] that these series can be written in terms of the Dedekind $\eta$-function as

$$\Phi_3^{(0)}(\tau) = 4 e^{\frac{i}{12} \pi i} \frac{(\eta(2 \tau))^5}{(\eta(\tau + \frac{1}{2}))^2} = 4 \left( \frac{\eta(\tau) \eta(4 \tau)}{\eta(2 \tau)} \right)^2,$$

(6.10a)

$$\Phi_3^{(1)}(\tau) = 2 \left( \frac{\eta(2 \tau)}{\eta(4 \tau)} \right)^2.$$

(6.10b)
The modular property is written as
\[
\begin{pmatrix} \Phi_3^{(1)}(\tau + 1) \\ \Phi_3^{(0)}(\tau + 1) \end{pmatrix} = \begin{pmatrix} e^{\frac{i\pi}{6}} & 0 \\ 0 & e^{\frac{i\pi}{3}} \end{pmatrix} \begin{pmatrix} \Phi_3^{(1)}(\tau) \\ \Phi_3^{(0)}(\tau) \end{pmatrix}, \tag{6.11}
\]

\[
\begin{pmatrix} \Phi_3^{(1)}(-1/\tau) \\ \Phi_3^{(0)}(-1/\tau) \end{pmatrix} = \left( \frac{\tau}{i} \right)^{3/2} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \Phi_3^{(1)}(\tau) \\ \Phi_3^{(0)}(\tau) \end{pmatrix}. \tag{6.12}
\]

The Eichler integrals are then defined by
\[
\tilde{\Phi}_3^{(0)}(\tau) = 3 q^{\frac{1}{12}} \sum_{a=0}^{\infty} (-1)^a q^{\frac{a}{2}(a+1)} \sum_{b=0}^{a} q^{b(b+1)} \left[ \frac{a}{b} \right]
\]
\[= 3 \sum_{n=0}^{\infty} \chi_6^{(0)}(n) q^{\frac{1}{12}n^2} \tag{6.13a}\]
\[= 3 q^\frac{1}{12} \left( 1 - q + q^5 - q^8 + q^{16} - q^{21} + \cdots \right),
\]

\[
\tilde{\Phi}_3^{(1)}(\tau) = 3 q^{\frac{1}{12}} \sum_{a=0}^{\infty} (-1)^a q^{\frac{a}{2}(a+1)} \sum_{b=0}^{a+1} q^{b^2} \left[ \frac{a+1}{b} \right]
\]
\[= 3 \sum_{n=0}^{\infty} \chi_6^{(1)}(n) q^{\frac{1}{12}n^2} \tag{6.13b}\]
\[= 3 q^\frac{1}{12} \left( 1 - q^2 + q^4 - q^{10} + q^{14} - q^{24} + \cdots \right).
\]

We note that Zagier’s identity [19] leads us to find
\[
\tilde{\Phi}_3^{(1)}(\tau) = 3 q^{\frac{1}{12}} \sum_{n=0}^{\infty} (-1)^n (-1; q^2)_{n+1}.
\]

• root of unity:

From Prop. 10, the Eichler integrals (6.13) reduce in a case of \( q \) being \( N \)-th root of unity to
\[
\tilde{\Phi}_3^{(a)} \left( \frac{1}{N} \right) = 3 \sum_{n=0}^{3N} \chi_6^{(a)}(n) \left( 1 - \frac{n}{3N} \right) e^{\frac{a^2}{6N} \pi i}. \tag{6.14}
\]

This coincides with
\[
\tilde{\Phi}_3^{(0)} \left( \frac{1}{N} \right) = e^{\frac{2\pi i}{N}} \sum_{a,b=0}^{N-1} (-1)^a \omega^{\frac{a}{2}(a+1)+b(b+1)} \left[ \frac{a}{b} \right] = \frac{1}{N} e^{\frac{2\pi i}{N}} \langle T(2, 6) \rangle_N, \tag{6.15a}\]
\[
\tilde{\Phi}_3^{(1)} \left( \frac{1}{N} \right) = e^{\frac{2\pi i}{6N}} \sum_{a,b=0}^{N-1} (-1)^a \omega^{\frac{a}{2}(a+1)+b^2} \left[ \frac{a+1}{b} \right]. \tag{6.15b}\]
where the second identity remains to be proved (the first identity was established from an asymptotic expansion due to a discussion in the previous section). We have checked numerically a validity of this identity.

As we have

$$(\tilde{\Phi}_3^{(1)}(N)) = (2e^{\frac{\pi i N}{6}})$$

modular property and eqs. (6.15) supports that

$$(\tilde{\Phi}_3^{(1)}(\frac{1}{N})) \simeq \sqrt{-1 N} \cdot \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \cdot (\tilde{\Phi}_3^{(1)}(-N)) + \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{E_n^{(3;1)}}{E_n^{(3;0)}} \right) \left( \frac{\pi i}{6 N} \right)^n , \quad (6.16)$$

where the generalized Euler number is defined by

$$E_n^{(3;\alpha)} = -\frac{3 \cdot 6^{2n}}{2n + 1} \left( B_{2n+1} \left( \frac{2 - \alpha}{6} \right) - B_{2n+1} \left( \frac{4 + \alpha}{6} \right) \right) ,$$

or some of them are as follows;

$$\frac{3}{\text{sh}(3 \, x)} \left( \frac{\text{sh}(2 \, x)}{\text{sh}(x)} \right) = \sum_{k=0}^{\infty} \left( \frac{E_k^{(3;1)}}{E_k^{(3;0)}} \right) \frac{x^{2k}}{(2k)!} = \left( \begin{array}{c} 2 \\ 1 \end{array} \right) - \left( \begin{array}{c} 10 \\ 3 \end{array} \right) \frac{x^2}{2} + \left( \begin{array}{c} 34 \\ 32 \end{array} \right) \frac{x^4}{24} - \left( \begin{array}{c} 910 \\ 896 \end{array} \right) \frac{x^6}{720} + \cdots .$$

6.3 (2, 8)-Torus Link: $m = 4$

- $q$-series:
A set of theta series is given by

\[ \Phi_4^{(0)}(\tau) = \sum_{n \in \mathbb{Z}} n \chi_8^{(0)}(n) q^{\frac{1}{16} n^2} \]  
\[ = \left( \frac{(\eta(\tau))^3}{\eta(\frac{\tau}{2})} \right)^3 - \left( \frac{\eta(\frac{\tau}{2})}{\eta(2 \tau)} \right)^3 \]
\[ = 2 q^{\frac{9}{16}} (3 - 5q + 11q^7 - 13q^{10} + 19q^{22} - 21q^{27} + \cdots), \]

\[ \Phi_4^{(1)}(\tau) = \sum_{n \in \mathbb{Z}} n \chi_8^{(1)}(n) q^{\frac{1}{16} n^2} \]  
\[ = 4 \left( \eta(2 \tau) \right)^3 \]
\[ = 4 q^{\frac{1}{4}} (1 - 3q^2 + 12q^4 - 7q^{12} + 9q^{20} - 11q^{30} + \cdots), \]

\[ \Phi_4^{(2)}(\tau) = \sum_{n \in \mathbb{Z}} n \chi_8^{(2)}(n) q^{\frac{1}{16} n^2} \]  
\[ = \left( \frac{(\eta(\tau))^3}{\eta(\frac{\tau}{2}) \eta(2 \tau)} \right)^3 + \left( \frac{\eta(\tau)}{\eta(\frac{\tau}{2})} \right)^3 \]
\[ = 2 q^{\frac{1}{16}} (1 - 7q^3 + 9q^5 - 15q^{14} + 17q^{18} - 23q^{33} + \cdots), \]

which are modular coinvariant;

\[ \begin{pmatrix} \Phi_4^{(2)}(\tau + 1) \\ \Phi_4^{(1)}(\tau + 1) \\ \Phi_4^{(0)}(\tau + 1) \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{4} \pi} & 0 & 0 \\ 0 & e^{\frac{3}{4} \pi i} & 0 \\ 0 & 0 & e^{\frac{5}{4} \pi i} \end{pmatrix} \begin{pmatrix} \Phi_4^{(2)}(\tau) \\ \Phi_4^{(1)}(\tau) \\ \Phi_4^{(0)}(\tau) \end{pmatrix}, \quad (6.18) \]

\[ \begin{pmatrix} \Phi_4^{(2)}(-1/\tau) \\ \Phi_4^{(1)}(-1/\tau) \\ \Phi_4^{(0)}(-1/\tau) \end{pmatrix} = \left( \frac{\tau}{i} \right)^{3/2} \cdot \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \Phi_4^{(2)}(\tau) \\ \Phi_4^{(1)}(\tau) \\ \Phi_4^{(0)}(\tau) \end{pmatrix}. \quad (6.19) \]
We then have the Eichler integral as

\[ \tilde{\Phi}^{(0)}_4(\tau) = 4 q^{\frac{a}{16}} \sum_{a=0}^{\infty} (-1)^a q^{\frac{1}{2}(a+1)} \sum_{b=0}^{a} q^{b(b+1)} \left[ \begin{array}{c} a \\ b \end{array} \right] \sum_{c=0}^{b} q^{c(c+1)} \left[ \begin{array}{c} b \\ c \end{array} \right] \]

\[ = 4 \sum_{n=0}^{\infty} \chi_n^{(0)}(n) q^{\frac{1}{16} n^2} \]

\[ = 4 q^{\frac{9}{16}} \left( 1 - q + q^7 - q^{10} + q^{22} - q^{27} + \cdots \right), \quad (6.20a) \]

\[ \tilde{\Phi}^{(1)}_4(\tau) = 4 q^{\frac{1}{2}} \sum_{a=0}^{\infty} (-1)^a q^{\frac{1}{2}(a+1)} \sum_{b=0}^{a} q^{b(b+1)} \left[ \begin{array}{c} a \\ b \end{array} \right] \sum_{c=0}^{b} q^{c^2} \left[ \begin{array}{c} b+1 \\ c \end{array} \right] \]

\[ = 4 \sum_{n=0}^{\infty} \chi_n^{(1)}(n) q^{\frac{1}{16} n^2} \]

\[ = 4 q^{\frac{1}{4}} \left( 1 - q^2 + q^6 - q^{12} + q^{20} - q^{30} + \cdots \right), \quad (6.20b) \]

\[ \tilde{\Phi}^{(2)}_4(\tau) = 4 q^{\frac{1}{4}} \sum_{a=0}^{\infty} (-1)^a q^{\frac{1}{2}(a+1)} \sum_{b=0}^{a+1} q^{b^2} \left[ \begin{array}{c} a+1 \\ b \end{array} \right] \sum_{c=0}^{b} q^{c^2} \left[ \begin{array}{c} b \\ c \end{array} \right] \]

\[ = 4 \sum_{n=0}^{\infty} \chi_n^{(2)}(n) q^{\frac{1}{16} n^2} \]

\[ = 4 q^{\frac{1}{4}} \left( 1 - q^3 + q^5 - q^{14} + q^{18} - q^{33} + \cdots \right). \quad (6.20c) \]

One sees that

\[ \tilde{\Phi}^{(1)}_4(\tau) = 2 \tilde{\Phi}^{(0)}_2(2\tau). \]

- root of unity:

Limiting value of the Eichler integrals when \( q \) goes to the \( N \)-th primitive root of unity is given by

\[ \tilde{\Phi}^{(c)}_4 \left( \frac{1}{N} \right) = 4 \sum_{n=0}^{4N} \chi_n^{(c)}(n) \left( 1 - \frac{n}{4N} \right) e^{\frac{\pi i}{8N} \langle T(2, 8) \rangle_N}, \quad (6.21) \]

which is rewritten as

\[ \tilde{\Phi}^{(0)}_4 \left( \frac{1}{N} \right) = e^{\frac{\pi i}{8N}} \sum_{a,b,c=0}^{N-1} (-1)^a \omega^a \left[ a \omega^{b(b+1)+c(c+1)} \right] \left[ \begin{array}{c} a \\ b \end{array} \right] \left[ \begin{array}{c} b \\ c \end{array} \right] = \frac{1}{N} e^{\frac{\pi i}{8N}} \langle T(2, 8) \rangle_N, \quad (6.22a) \]

\[ \tilde{\Phi}^{(1)}_4 \left( \frac{1}{N} \right) = e^{\frac{\pi i}{8N}} \sum_{a,b,c=0}^{N-1} (-1)^a \omega^a \left[ a \omega^{b(b+1)+c^2} \right] \left[ \begin{array}{c} a+1 \\ b \end{array} \right] \left[ \begin{array}{c} b+1 \\ c \end{array} \right], \quad (6.22b) \]

\[ \tilde{\Phi}^{(2)}_4 \left( \frac{1}{N} \right) = e^{\frac{\pi i}{8N}} \sum_{a,b,c=0}^{N-1} (-1)^a \omega^a \left[ a \omega^{b^2+c^2} \right] \left[ \begin{array}{c} a+1 \\ b \end{array} \right] \left[ \begin{array}{c} b \\ c \end{array} \right]. \quad (6.22c) \]
Though we have checked these three equalities numerically, we proved only eq. (6.22a) in this article.

The nearly modular properties are written as

\[
\begin{pmatrix}
\tilde{\Phi}_4^{(2)}\left(\frac{1}{N}\right) \\
\tilde{\Phi}_4^{(1)}\left(\frac{1}{N}\right) \\
\tilde{\Phi}_4^{(0)}\left(\frac{1}{N}\right)
\end{pmatrix}
\approx \sqrt{-1}N \cdot \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \cdot
\begin{pmatrix}
\tilde{\Phi}_4^{(2)}\left(-N\right) \\
\tilde{\Phi}_4^{(1)}\left(-N\right) \\
\tilde{\Phi}_4^{(0)}\left(-N\right)
\end{pmatrix}
\]

\[+
\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} E_{n}^{(4;2)} \\ E_{n}^{(4;1)} \\ E_{n}^{(4;0)} \end{pmatrix} \left(\frac{\pi i}{8N}\right)^{n}, \quad (6.23)
\]

where we have

\[
\begin{pmatrix}
\tilde{\Phi}_4^{(2)}\left(N\right) \\
\tilde{\Phi}_4^{(1)}\left(N\right) \\
\tilde{\Phi}_4^{(0)}\left(N\right)
\end{pmatrix}
= \begin{pmatrix} 3 e^{\frac{3}{8}\pi i N} \\ 2 e^{\frac{1}{8}\pi i N} \\ e^{\frac{1}{2}\pi i N} \end{pmatrix},
\]

and the generalized Euler number is defined by

\[
E_{n}^{(4;a)} = \frac{2^{6n+2}}{2n+1} \left( B_{2n+1} \left( \frac{3 - a}{8} \right) - B_{2n+1} \left( \frac{5 + a}{8} \right) \right).
\]

Some of them are explicitly given as follows;

\[
\frac{4}{\text{sh}(4x)} \begin{pmatrix}
\text{sh}(3x) \\
\text{sh}(2x) \\
\text{sh}(x)
\end{pmatrix}
= \sum_{k=0}^{\infty} \begin{pmatrix} E_{k}^{(4;2)} \\ E_{k}^{(4;1)} \\ E_{k}^{(4;0)} \end{pmatrix} \frac{x^{2k}}{(2k)!} 
\]

\[
= \begin{pmatrix}
3 \\ 2 \\ 1
\end{pmatrix} - \begin{pmatrix} 7 \\ 8 \\ 5
\end{pmatrix} \frac{x^2}{2} + \begin{pmatrix} 119 \\ 160 \\ 109
\end{pmatrix} \frac{x^4}{24} - \begin{pmatrix} 5587 \\ 7808 \\ 5465
\end{pmatrix} \frac{x^6}{720} + \cdots.
\]

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