A few remarks on the theory of non-nilpotent graphs

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Abstract

We prove a few results about non-nilpotent graphs of symmetric groups $S_n$ – namely that they satisfy a conjecture of Nongsiang and Saikia (which is likewise proved for alternating groups $A_n$), and that for $n \geq 19$ each vertex has degree at least $\frac{n!}{2}$. We also show that the class of non-nilpotent graphs does not have any “local” properties, i.e. for every simple graph $X$ there is a group $G$, such that its non-nilpotent graph contains $X$ as an induced subgraph.

1 Introduction

Let $G$ be a group. We construct a graph $N_G$ with elements of $G$ as vertices, such that $x, y \in G$ are connected by an edge if and only if the subgroup $\langle x, y \rangle$ is not nilpotent. By removing isolated vertices of $N_G$ we get the non-nilpotent graph of $G$, denoted by $R_G$. Of course in the case when $G$ is weakly nilpotent (i.e. $\langle x, y \rangle$ is nilpotent for any $x, y \in G$), $R_G$ is an empty graph. Thus we will be concerned only about non-weakly nilpotent groups; we also will examine only finite groups (for which notions of nilpotency and weak nilpotency coincide).

Non-nilpotent graphs of groups were first defined by A. Abdollahi and M. Zarrin in the paper [I], where they proved for instance that in the case of finite non-nilpotent groups $R_G$ are not regular graphs, and that they are connected with diameter bounded by 6, stating a conjecture that actually $\text{diam} R_G \leq 2$. This was disproven by A. Davis, J. Kent and E. McGovern – *Undergraduate student; Jagiellonian University, Faculty of Mathematics and Computer Science; Łojasiewicza 6, 30-348 Kraków, Poland.
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in [2] they have shown that \( \text{diam} \mathcal{R}_G \geq 3 \) for \( G \) equal to \( \mathbb{Z}_m \rtimes S_4 \), where \( m \) is odd and the action of \( S_4 \) on \( \mathbb{Z}_m \) is given by \( a^\sigma = a^{|\text{sgn} \sigma|} \). A. Lucchini and D. Nemmi proved in [3] that this bound is indeed optimal, i.e. \( \text{diam} \mathcal{R}_G \leq 3 \) for any finite non-nilpotent group \( G \).

A natural question is: if \( \mathcal{R}_G \simeq \mathcal{R}_H \) for two finite non-nilpotent groups \( G \) and \( H \), does it imply that \( G \simeq H \)? This is not the case – consider the dihedral group of order 18, \( G = D_9 \), and generalized dihedral group of the same order, \( H = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \), with action of the non-trivial element of \( \mathbb{Z}_2 \) on \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) given by \( h \mapsto -h \). In both \( G \) and \( H \) the only maximal nilpotent subgroups are those cyclic of order 2 and those of index two (respectively \( \mathbb{Z}_9 \) and \( \mathbb{Z}_3 \times \mathbb{Z}_3 \)). This means that both \( \mathcal{R}_G \) and \( \mathcal{R}_H \) are complete multipartite graphs with nine parts of size one and one part of size eight.

However, a weaker (and hopefully more true) form of this conjecture was stated in [4] by D. Nongsiang and P. Saikia:

**Conjecture 1** (Nongsiang & Saikia). If \( \mathcal{R}_G \simeq \mathcal{R}_H \) for two finite non-nilpotent groups \( G \) and \( H \), then \( |G| = |H| \).

They managed to prove this conjecture in several cases, including \( G \) being a dihedral group, \( \text{PSL}(2, q) \) where \( q \) is a prime power with \( q^2 \neq 1 \) (mod 16), groups of order \( pq \) and centerless groups of order \( pqr \) where \( p, q, r \) are distinct primes. In this article we append this list by two important families of groups: symmetric groups \( S_n \) and alternating groups \( A_n \) (theorem 3.2).

It is natural to ask how dense graphs \( \mathcal{R}_G \) usually are. For a similar notion of the non-commuting graph of \( G \), it is easy to see that each vertex is connected with at least half of the other vertices. This is not always true in the non-nilpotent case: \((12)(34)\) has degree 8 in \( \mathcal{R}_{S_4} \). However, in the fourth section we prove (Theorem 4.1) that graphs \( \mathcal{R}_{S_n} \) have this property when \( n \geq 19 \). In particular, we can find a Hamiltonian cycle in those graphs by Dirac’s theorem.

Much of the attention given to non-nilpotent graphs of groups was focused on cases where \( \mathcal{R}_G \) has a particularly nice local structure, e.g. if it is a complete multipartite graph. In the last section we prove (theorem 5.1) that we should not hope for these results to be much helpful in the general case: every graph can be found as an induced subgraph of \( \mathcal{R}_G \) for appropriate \( G \).

## 2 Notation and preliminaries

We use the following notations throughout the paper.

- When \( a \) and \( b \) are positive integers, \( a \perp b \) means that they are coprime.
• The identity element of a group will be usually denoted by 1.

• If $S$ is a subset of a group $G$, then $\langle S \rangle$ denotes the subgroup generated by $S$.

• $G'$ is the commutator subgroup of $G$.

• $Z^*(G)$ is the hypercenter (limit of upper central series) of a group $G$.

• The semi-direct product of groups $G$ and $H$ is denoted by $G \rtimes \nu H$ or $G \rtimes H$, where $\nu : H \to \text{Aut} G$ is a homomorphism. (In the second case notation itself does not specify the homomorphism, but we will indicate its choice every time we use it.)

• Choose a finite group $G$, an element $g \in G$ and $p$ a prime. Suppose that $g$ has order $n$. Let $n_p$ be equal to $n$ divided by the greatest power of $p$ that divides $n$ – so that $n_p$ is not divisible by $p$ and $\frac{n}{n_p}$ is a power of $p$. We will denote $g^{n_p}$ by $g_p$. Of course, its order is $\frac{n}{n_p}$, which is a power of $p$. We point out that only finitely many choices of $p$ give $g_p \neq 1$.

• If $\sigma$ is a permutation in $S_n$, then its support is the set of elements $x \in \{1, \ldots, n\}$ such that $\sigma(x) \neq x$.

We will use the following result extensively:

**Lemma 2.1** (Centralizer structure in symmetric groups). If $\sigma \in S_n$ has $a_k$ cycles of length $k$, for $k = 1, 2, \ldots, n$, then the centralizer of $\sigma$ has

$$\prod_{k=1}^{n} k^{a_k} a_k!$$

elements, and is a direct product of groups isomorphic to $C_k \wr S_{a_k}$, where $S_{a_k}$ permutes cycles of size $k$ in $\sigma$ (here $C_k$ denotes the cyclic group of order $k$).

**Proof.** Classical. The first part can be found for instance in [5, p. 78, Ex. 3]; for the second part, we just observe that the direct product is contained in the centralizer and has the same order. \hfill \square

We assume the following characterizations of finite nilpotent groups to be widely known:

**Lemma 2.2.** Let $G$ be a finite group. The following are equivalent:
1. $G$ is nilpotent.

2. Every Sylow subgroup of $G$ is normal.

3. $G$ is a direct product of its Sylow subgroups.

4. If $x, y$ are elements of $G$ with coprime order, then $xy = yx$.

Proof. (1) $\iff$ (2) $\iff$ (3) can be found in [6, p. 24, Thm 1.26], while (3) $\Rightarrow$ (4) $\Rightarrow$ (2) is straightforward.

Corollary 2.1. Choose $g, h \in G$. Then subgroup $\langle g, h \rangle$ is nilpotent if and only if the following conditions hold:

1. For $p \neq q$ primes, elements $g_p$ and $h_q$ commute.

2. For any prime $p$, $\langle g_p, h_p \rangle$ is a $p$-group.

Let $G$ be a finite group. As in the introduction, we define $N_G$ to be a graph with vertices being elements of $G$, such that $x$ and $y$ are connected if and only if $\langle x, y \rangle$ is not nilpotent. The non-nilpotent graph of $G$, denoted by $\mathcal{R}_G$, is formed by removing isolated vertices from $N_G$.

The crucial role of the hypercenter $Z^*(G)$ in investigating $\mathcal{R}_G$ was pointed out by Abdollahi and Zarrin in [1]. They prove in Proposition 2.1 and Lemma 2.2 of their paper the following facts:

**Lemma 2.3.** Isolated vertices of $N_G$ correspond to the elements of $Z^*(G)$. Moreover, $x$ and $y$ are connected in $\mathcal{R}_G$ if and only if their projections on $G/Z^*(G)$ are connected in $\mathcal{R}_{G/Z^*(G)}$.

Quotienting can be rephrased using graph theoretical terms as blow-ups. For a graph $X$ and positive integer $c$, denote by $c \circ X$ the graph composed in the following way: for each vertex $a$ of $X$, we put $c$ vertices $a_1, a_2, \ldots, a_c$ in $c \cdot X$, in such a way that $a_i$ and $a_j$ are never connected, and $a_i$ is connected with $b_j$ if $a$ is connected with $b$ in $X$. This construction is called the blow-up of $X$ of order $c$. The graph $c \circ X$ is the only graph such that there is a $c$-to-1 map from $c \circ X$ to $X$ that preserves adjacency relations.

**Corollary 2.2.** Let $G$ be a finite group. Then $\mathcal{R}_G \simeq |Z^*(G)| \circ \mathcal{R}_{G/Z^*(G)}$. 

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Suppose $G$ and $H$ are two finite non-nilpotent groups with $\mathcal{R}_G \simeq \mathcal{R}_H$. Corollary 2.2 allows us to draw many conclusions from that fact. In particular, by comparing the number of vertices, $|G| - |Z^*(G)| = |H| - |Z^*(H)|$. Therefore instead of proving $|G| = |H|$, we can focus on proving $|Z^*(G)| = |Z^*(H)|$.

**Lemma 3.1.** Let $G$ be a finite non-nilpotent group. Then $|Z^*(G)|$ divides the number of vertices of $\mathcal{R}_G$ and the degree of each vertex.

**Proof.** An easy consequence of corollary 2.2.

Suppose that the group $G$ is centerless (hence $Z^*(G)$ is trivial) and that $\mathcal{R}_G \simeq \mathcal{R}_H$ for some finite non-nilpotent group $H$. Then if the greatest common divisor of degrees and the number of vertices in $\mathcal{R}_G$ is equal to 1, we may immediately conclude by lemma 3.1 that $|Z^*(H)| = 1$, and thus $|G| = |H|$.

It turns out this is precisely the case when $G = S_n$ or $G = A_n$. To prove the Conjecture 1 in this case, we simply need to look at degrees of certain permutations in the graph $\mathcal{R}_G$.

**Lemma 3.2.** Let $\sigma \in G \in \{S_n, A_n\}$. Suppose that $\sigma$ cycle lengths are $p_1, p_2, \ldots, p_k$, where $p_i$ are distinct and either prime or equal to 1 (hence $\sum p_i = n$). Then for any $\tau \in S_n$ the subgroup $\langle \sigma, \tau \rangle$ is nilpotent if and only if $\sigma \tau = \tau \sigma$, and there are $\prod p_i$ such permutations $\tau$.

**Proof.** Implication $(\Leftarrow)$ is obvious, so assume $\langle \sigma, \tau \rangle$ is nilpotent. Choose a prime $q$ and consider $\tau_q$. By Corollary 2.1 for all $i$ such that $p_i \neq 1, q$ the element $\sigma_{p_i}$ (which is the corresponding $p_i$-cycle raised to a certain power) commutes with $\tau_q$. By Lemma 2.1 $\tau_q$ and $\sigma_{p_i}$ have disjoint supports.

If $q$ is distinct from all $p_i$-s, this immediately forces $\tau_q$ to be trivial. If, however, we have $q = p_i$ for some $i$, then the support of $\tau_q$ is contained in the union of $q$-cycle of $\sigma$ and its possible fixed point. By Corollary 2.1 we know that $\langle \tau_q, \sigma_q \rangle$ is a $q$-subgroup of $S_n$ with support not bigger than $q + 1$. But $q^2 \not| (q + 1)!$, so this subgroup is cyclic. Thus $\tau_q$ is generated by $\sigma_q$.

As it is true for all $q$, and they are pairwise coprime, we get $\tau \in \langle \sigma \rangle$ and the result follows.

Lemma 3.2 provides us with permutations for which we can easily enumerate degrees in $\mathcal{R}_G$. A similar family is given by permutations with two cycles of length two and other cycles with lengths being distinct primes, as shown in the following lemma.
Lemma 3.3. Let $\sigma \in G \in \{S_n, A_n\}$. Suppose that $\sigma$ cycle lengths are $2, 2, p_1, \ldots, p_k$, where $p_i$ are distinct odd primes. Then there are $16 \prod p_i$ such permutations $\tau \in S_n$ that $\langle \sigma, \tau \rangle$ is nilpotent, and $4 \prod p_i$ of them lie in $A_n$.

Proof. Without loss of generality, $(12)$ and $(34)$ are the transpositions in $\sigma$. Similarly to the proof of Lemma 3.2, if $\tau$ is taken in such a way that $\langle \sigma, \tau \rangle$ is nilpotent, then $\tau_q = 1$ for $q \notin \{2, p_1, \ldots, p_k\}$, and its support is contained in the support of $\sigma_q$ if $q$ is among these primes, with $\langle \tau_q, \sigma_q \rangle$ being a $q$-group. For $q = p_i$ we get that $\tau_q$ is a power of the $q$-cycle in $\sigma$. For $q = 2$ the element $\tau_2$ acts only on $\{1, 2, 3, 4\}$ in such a way that $\langle (12)(34), \tau_2 \rangle$ is a 2-group. We can check that all elements of $S_4$ of order being a power of two can be taken here (due to the three copies of $D_4$ sitting inside $S_4$), and there are 16 of them (but only four inside $A_4$). In total, we get the desired claim.

Now we just need to state a final, number-theoretical lemma.

Lemma 3.4. Every integer $m \geq 28$ can be expressed as a sum of distinct prime numbers greater than 3.

Proof. We can check that all integers in the interval $[28, 89]$ can be expressed as a sum of distinct prime numbers from the interval $[5, 31]$.

We can proceed by induction. For $n \leq 89$, by above, a stronger claim holds, hence we can assume $n \geq 90$. By Bertrand’s postulate we may find a prime $p$ in the interval $\left(\frac{n-27}{2}, n-27\right)$. Then we may write $n-p$ as a sum of distinct primes $\geq 5$ by induction assumption. It suffices to check that these primes are different from $p$.

As $p \geq \frac{n-27}{2} > 31$, we need to worry only about primes generated in the induction steps. But they are decreasing: indeed, in the next step we will choose a prime $q$ in the interval $\left(\frac{n-p-27}{2}, n-p-27\right)$, and $q < n-p-27 \leq \frac{n-27}{2} \leq p$.

Theorem 3.1. Let $n \geq 32$ be an integer, and $H$ be a finite group. Then $\mathfrak{R}_{S_n} \simeq \mathfrak{R}_H$ implies that $|H| = |S_n|$. Similarly, $\mathfrak{R}_{A_n} \simeq \mathfrak{R}_H$ implies that $|H| = |A_n|$.

Proof. We just need to prove that $Z^*(H)$ is trivial.

By lemma 3.4 we can write $n-4 = \sum_{i=1}^k p_i$, where $p_i$ are distinct primes greater or equal to 5. Let $\sigma$ be any permutation with cycle lengths $1, 3, p_1, p_2, \ldots, p_k$. Similarly let $\tau$ be any permutation with cycle lengths $2, 2, p_1, p_2, \ldots, p_k$. 

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Consider in the first place the case of $S_n$. By Lemma 3.2, $\deg \sigma = n! - 3 \prod p_i$. By Lemma 3.3, $\deg \tau = n! - 16 \prod p_i$. Hence, by Lemma 3.1

$$|Z^*(H)| \mid \deg \sigma - \deg \tau = 13 \prod p_i.$$ 

But by the same lemma $|Z^*(H)| \mid n! - 1$, so $|Z^*(H)| \perp n!$. However, every prime divisor of $13 \prod p_i$ is also a divisor of $n!$ (as $n \geq 13$), thus $|Z^*(H)| = 1$.

For $A_n$ using the same results we get $|Z^*(H)| \mid \frac{n!}{2} - 1$, $\deg \sigma = \frac{n!}{2} - 3 \prod p_i$ (note that all permutations mentioned in the Lemma 3.2 are even) and $\deg \tau = \frac{n!}{2} - 4 \prod p_i$, hence $|Z^*(H)| \mid \prod p_i$, but again $|Z^*(H)| \perp n!$, thus we again get trivial $Z^*(H)$. □

**Theorem 3.2.** • Let $n \geq 3$ be an integer, and $H$ be a finite group. Then $\mathcal{R}_{S_n} \simeq \mathcal{R}_H$ implies $|H| = |S_n|$.

• Let $n \geq 4$ be an integer, and $H$ be a finite group. Then $\mathcal{R}_{A_n} \simeq \mathcal{R}_H$ implies $|H| = |A_n|$.

**Proof.** By Theorem 3.1 we just need to check cases $n \leq 31$. The only thing we were using in its proof were $n \geq 13$ and the ability to express $n - 4$ as a sum of distinct primes greater than 3. This can be done if $n \geq 15$ and $n \neq 18, 19, 25, 31$.

Consider the part about $S_n$. Let $z := |Z^*(H)|$. In general Lemma 3.2 means that for any partition of $n = \sum p_i$ as a sum of distinct primes or ones, we have $z \mid n! - \prod p_i$, while Lemma 3.1 adds divisibility $z \mid n! - 1$. If $n$ has a form $p$ or $p + 1$, where $p$ is an odd prime, then those imply $z \mid n! - p$ and $z \mid n! - 1$, therefore $z \mid p - 1 \mid n!$, thus $z = 1$. This leaves us with cases $n = 9, 10, 25$.

For $n = 25$ we can use partitions $25 = 17 + 7 + 1 = 17 + 5 + 3$, which imply $z \mid 25! - 17 \cdot 7 \cdot z \mid 25! - 17 \cdot 5 \cdot z \mid 17 \cdot 8$, which is a contradiction with $z \mid 25! - 1$. For $n = 10$ the partition $10 = 3 + 7$ implies $z \mid 10! - 21$, and since $z \mid 10! - 1$ we get $z \mid 20$, a contradiction. For $n = 9$ we take $9 = 5 + 3 + 1$, thus $z \mid 9! - 15$ and $z \mid 9! - 1$, thus $z \mid 14$, a contradiction.

Note that the above paragraphs work with $A_n$ as well, because all primes we have used in our partitions were odd – and in this case Lemma 3.2 works just as well for $A_n$. The only other difference is that we have $\frac{1}{2} n!$ instead of $n!$ in this case, but the reasoning carries through (since $n \geq 4$ and $\frac{1}{2} n!$ is even). □
4 Large degrees

Theorem 4.1. Let \( n \geq 19 \) be an integer. Then for every vertex \( v \) in \( R_{S_n} \) we have \( \deg v \geq \frac{1}{2}n! \).

This whole section is aimed to prove Theorem 4.1. Some restriction on \( n \) being large enough is indeed necessary: we already observed in the previous section that \( (12)(34) \) has degree eight in \( R_{S_4} \). It is compelling to try shrinking the bound of 19 using i.e. computer calculations, but it was not attempted by the author.

A direct consequence of Theorem 4.1 is the existence of Hamiltonian cycles in \( R_{S_n} \) for \( n \geq 19 \). Indeed, Dirac’s theorem assures us that if a graph has \( m \) vertices, each with degree at least \( \frac{1}{2}m \), then the graph has a Hamiltonian cycle. Using Bondy-Chvatal theorem [7], which is a generalization of Dirac’s theorem, it is possible to prove that \( R_{S_n} \) has a Hamiltonian cycle for every \( n \geq 3 \). However, the proof found by the author, while being elementary, is also long and dependent on calculations. Consequently, we decided to omit the proof from this paper; more details can be found in the first version of the article’s preprint published on arXiv. The existence of Hamiltonian cycles in non-nilpotent graphs is in line with similar results about non-generating graphs. F. Erdem proved in [8] that the non-generating graph of \( S_n \) has a Hamiltonian cycle for any \( n \geq 104 \), while earlier Breuer et al. [9] obtained the same result for any large enough simple group.

We proceed to the proof of Theorem 4.1. Firstly, for any finite group \( G \), \( x, y \in G \) and \( n \in \mathbb{Z}_+ \) we have \( \langle x^n, y \rangle \subset \langle x, y \rangle \), and thus \( \deg_{N_G} x \geq \deg_{N_G} x^n \).

Hence

**Observation.** It is enough to check Theorem 4.1 for permutations of prime order.

Any permutation \( \sigma \) of order \( p \) is a product of disjoint \( p \)-cycles, and for any \( \tau \in S_n \) we can analyse cycle lengths of \( \tau \) to get some results about nilpotence.

**Definition.** For \( \sigma \in S_n \) and \( i \in \{1, 2, \ldots, n\} \) denote by \( c(\sigma, i) \) the length of the cycle in \( \sigma \) in which \( i \) is contained (in other words, \( c(\sigma, i) \) is the smallest \( k \in \mathbb{Z}_+ \) such that \( \sigma^k(i) = i \)).

**Lemma 4.1.** Let \( \sigma, \tau \in S_n \) be such that \( \langle \sigma, \tau \rangle \) is nilpotent, and \( \sigma \) has order \( p \) being prime. Take any \( i, j \in \{1, 2, \ldots, n\} \) that belong to the same cycle in \( \sigma \) (with \( i \neq j \)). Then quotient \( \frac{c(\tau i)}{c(\tau j)} \) is a rational power of \( p \) (so is of form \( p^t \), where \( t \) is an integer, not necessarily nonnegative).
Proof. Consider an element \( \rho \) equal to \( \tau^{p^k} \), where \( p^k \) is the largest power of \( p \) dividing \( \text{ord} \tau \). Then \( c(\tau, i) / c(\rho, i) \) and \( c(\tau, j) / c(\rho, j) \) are rational powers of \( p \), and the claim will follow if we prove that \( c(\rho, i) = c(\rho, j) \). We know that \( p \nmid \text{ord} \rho \).

Choose \( m \) so that \( \sigma^m(i) = j \). Then \( \sigma^m \) has order \( p \), so it commutes with \( \rho \) (as both are elements of nilpotent subgroup \( \langle \sigma, \tau \rangle \)). Then \( c(\rho, i) = c(\sigma^{-m}\rho\sigma^m, i) \), because \( \rho = \sigma^{-m}\rho\sigma^m \). But conjugation does not change the length of cycles, while it changes their placement, hence \( c(\sigma^{-m}\rho\sigma^m, i) = c(\rho, j) \). Therefore \( c(\rho, i) = c(\rho, j) \) and our claim is proven. \( \square \)

Our strategy now will be to prove that for a fixed \( \sigma \) of order \( p \), \( \tau \) chosen uniformly at random from \( S_n \) violates Lemma 4.1 with probability at least \( \frac{1}{2} \). Therefore we need a result about the distribution of \( (c(\tau, i), c(\tau, j)) \) for a random \( \tau \) and fixed \( i, j \). For one coordinate the result is well-known: \( c(\tau, i) \) is uniformly distributed in \( \{1, 2, \ldots, n\} \). Indeed, in exactly \( \frac{(n-1)!}{(k-1)!} \) ways we can choose a \( k \)-cycle containing \( i \), so \( (n-1)! = \frac{(n-1)!}{(k-1)!} \) permutations have a \( k \)-cycle containing \( i \), and the answer does not depend on \( k \).

For two coordinates, the answer still has the same elegant symmetry. We can assume \((i, j) = (1, 2)\) without loss of generality.

Lemma 4.2. Let \( \tau \) be a permutation chosen uniformly at random from \( S_n \). Then \( 1 \) and \( 2 \) are in the same cycle with probability \( \frac{1}{2} \), and if they are not, then \( (c(\sigma, 1), c(\sigma, 2)) \) has a uniform distribution on the set \( \{a, b \in \mathbb{Z}_+ : a + b \leq n\} \).

Proof. We start with enumerating the cases where \( 1 \) and \( 2 \) are in the same cycle. If this cycle has length \( k \geq 2 \), then we can choose the other elements of the cycle in \( \binom{n-2}{k-2} \) ways, the \( k \)-cycle on those \( k \) elements in \( (k-1)! \) ways, and the rest of the permutation in \( (n-k)! \) ways. In total, this gives

\[
\frac{(n-2)!}{(k-2)!(n-k)!} \cdot (k-1)! \cdot (n-k)! = \frac{(n-2)!(n-k)!}{(k-1)!} \cdot \frac{1}{2} n!
\]

possibilities. Summing for \( k \) from 2 to \( n \) we get

\[
(n-2)! \sum_{k=2}^{n} \frac{(n-2)!}{(k-1)!} = \frac{n(n-1)}{2} = \frac{1}{2} n!
\]

options, as claimed.

Now suppose \( 1 \) is in a cycle of length \( k \), \( 2 \) is in a cycle of length \( l \), and those are distinct cycles (though not necessarily \( k \neq l \)). As in the previous case, we choose elements of the \( k \)-cycle in \( \binom{n-2}{k-1} \) ways, the cycle itself in
(k − 1)! ways; the elements of the l-cycle in $\binom{n-k-1}{l-1}$ ways, times (l − 1)! ways of choosing the cycle, and finally (n − k − l)! for the rest of the permutation. This gives in total
\[
\binom{n - 2}{k - 1} \cdot (k - 1)! \cdot \binom{n - k - 1}{l - 1} \cdot (l - 1)! \cdot (n - k - l)! = (n - 2)!
\]
options, regardless of the values of k and l.

Lemma 4.1 has an additional assumption on i and j lying in the same cycle of $\sigma$. If the same is true for $\tau$ as well, then the lemma becomes useless. Therefore we need some control over that event.

**Lemma 4.3.** Let $k \leq n$ be positive integers. Each element $\sigma$ of $S_n$ induces a partition on $\{1, 2, \ldots, k\}$ – elements in the same cycle of $\sigma$ are put to the same set of the partition. Then, when $\sigma$ is a random permutation in $S_n$, the distribution of partitions will be the same regardless of $n$ (in particular, the same as in the case of $n = k$).

**Proof.** We will proceed by induction on $n$. For $n = k$ there is nothing to prove. Suppose $n > k$ and consider the following function $f : S_n \to S_{n-1}$:
\[
f(\sigma)(j) = \begin{cases} 
\sigma(j), & \sigma(j) \neq n, \\
\sigma(n), & \text{else.} 
\end{cases}
\]
In other words, if we imagined $\sigma$ as a union of disjoint cycles, $f(\sigma)$ is created by deleting $n$ from the permutation, and if this creates a hole in some cycle, we “sew it up”, putting $\sigma(n)$ as a value of the permutation in the point $\sigma^{-1}(n)$.

It is clear that $f(\sigma)$ yields the same partition on $\{1, 2, \ldots, k\}$ as $\sigma$ does, thus we only need to prove that $f(\sigma)$ is uniformly distributed whenever $\sigma$ is uniformly distributed.

But $f$ is n-to-1 as for any $\tilde{\sigma} \in S_{n-1}$ and $m \in \{1, 2, \ldots, n\}$ there is exactly one permutation $\sigma \in S_n$ such that $\tilde{\sigma} = f(\sigma)$ and $n = \sigma(m)$. For $m = n$ we get it by adding $n$ as a fixed point, and for $m \neq n$ we need to have $\sigma(n) = \tilde{\sigma}(m)$ and $\sigma$ has to agree with $\tilde{\sigma}$ outside of $\{m, n\}$.

Now we have all the machinery needed to prove Theorem 4.1.

**Proof of Theorem 4.1.** We already know it is enough to check that $\deg_{R_{S_n}} \sigma \geq \frac{1}{2} n!$, where $\ord \sigma = p$ is prime. Without loss of generality, 1 and 2 are in the same cycle of $\sigma$. We will have several cases.
1. \( p \) is odd;
2. \( p = 2 \), \( \sigma \) is not a transposition;
3. \( \sigma \) is a transposition.

In each case, we choose \( \tau \) uniformly at random from \( S_n \), and we claim \( \langle \sigma, \tau \rangle \) is nilpotent with probability at most \( \frac{1}{2} \).

**Case 3.** Suppose \( \langle \sigma, \tau \rangle \) is nilpotent. Let \( k \) be an integer such that the order of \( \tau^k \) is odd. Then \( \sigma \) has coprime order with \( \tau^k \), hence they commute and by Lemma 2.1 we have \( c(\tau^k, 1) = 1 \). Therefore \( c(\tau, 1) \) is a power of two. But \( c(\tau, 1) \) is uniformly distributed on \( \{1, \ldots, n\} \), hence \( c(\tau, 1) \) is a power of two with probability at most \( \left\lfloor \frac{\log_2 n + 1}{n} \right\rfloor + 1 \), which is less than \( \frac{1}{2} \) when \( n \geq 19 \) (actually, six is enough).

**Cases 1. and 2.** For now, assume that 1 and 2 lie in different cycles of \( \tau \). Then, by Lemma 4.1, \( \langle \sigma, \tau \rangle \) being nilpotent implies that \( c(\tau, 1) c(\tau, 2) \) is \( p^k \) for an integer \( k \). Let \( (a, b) := (c(\tau, 1), c(\tau, 2)) \). By Lemma 4.2 pairs \( (a, b) \) are distributed uniformly on the set \( S := \{(a, b) \in \mathbb{Z}_+: a + b \leq n\} \). Set \( S \) has \( \frac{n(n-1)}{2} \) elements; how many of them have \( a/b = p^k \)? If \( k = 0 \), we have \( a = b \leq n/2 \), so we have \( \lfloor n/2 \rfloor \) possibilities. If \( k \neq 0 \), then we can assume by symmetry \( k > 0 \). Then \( a = b p^k \), so \( n \geq a + b = b(p^k + 1) \), which gives us \( \left\lfloor \frac{n}{p^k + 1} \right\rfloor \) pairs \( (a, b) \).

In total (remembering about cases for negative \( k \)) we get

\[
h(n, p) := \left\lfloor \frac{n}{2} \right\rfloor + 2 \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k + 1} \right\rfloor
\]

pairs \( (a, b) \in S \) such that \( a/b \) is a power of \( p \). We denote this expression by \( h(n, p) \). In this part of the proof, we can deduce that there are at most \( h(n, p) \cdot (n - 2)! \) permutations \( \tau \) such that \( \langle \sigma, \tau \rangle \) is nilpotent and 1, 2 are in distinct cycles of \( \tau \).

**Case 1.** In the case when \( p \) is odd, without loss of generality we can assume that 1, 2 and 3 are in the same cycle of \( \sigma \). Suppose again that \( \tau \) is a general permutation such that \( \langle \sigma, \tau \rangle \) is nilpotent.

- If 1 and 2 are in distinct cycles of \( \tau \), we have at most \( h(n, p)(n - 2)! \) possibilities for \( \tau \).
- The same happens if 2 and 3 are in distinct cycles of \( \tau \).
- 1, 2 and 3 are in the same cycle for \( \frac{1}{3} n! \) different choices of \( \tau \), according to Lemma 1.3.
In total, $\sigma$ is not connected to at most $n! + 2h(n, p)(n - 2)!$ other vertices of $N_{S_n}$. We want this to be smaller or equal to $n! / 2$, which is equivalent to $h(n, p) \leq n(n - 1)/12$. But

$$h(n, p) \leq \frac{n}{2} + 2 \sum_{k=1}^{\infty} \frac{n}{3^k} = \frac{3}{2}n,$$

and this is at most $n(n - 1)/12$ as long as $n \geq 19$.

**Case 2.**
We proceed analogously but with different calculations. Without loss of generality, (12) and (34) are two transpositions in $\sigma$. Let $\tau$ be any permutation with $\langle \sigma, \tau \rangle$ nilpotent.

- If 1 and 2 are in distinct cycles of $\tau$, we have at most $h(n, 2)(n - 2)!$ possibilities for $\tau$.
- The same number appears if 3 and 4 are in distinct cycles of $\tau$.
- Assume both (1, 2) and (3, 4) are in the same cycles of $\tau$. By Lemma 4.3, there are exactly $\frac{7}{24}n!$ such $\tau$ (because their fraction is the same as in $S_4$, where we have six 4-cycles plus (12)(34)).

This in total gives at most $2h(n, 2)(n - 2)! + \frac{7}{24}n!$ vertices of $N_{S_n}$ that are not connected to $\sigma$. We want to bound this by $n!/2$ from above, which is equivalent to $h(n, 2) \leq 5n(n - 1)/48$.

However,

$$h(n, 2)/n \leq \frac{1}{2} + \frac{2}{3} + \frac{2}{5} + 2 \sum_{k=3}^{\infty} \frac{1}{2^k} = \frac{31}{15},$$

which is enough for $n \geq 21$. For $n = 19$ and 20, we can evaluate $h(n, 2)$ directly and check that the claim still holds.

\[\square\]

5 **Induced subgraphs**

It is natural to ask if all graphs $\mathcal{R}_G$ share some particular structure. For instance, $\mathcal{R}_G$ is a complete multipartite graph for a wide class of cases, including dihedral and dicyclic groups, as well as the groups of order $pq$ with prime $p, q$. The smallest non-example is $\mathcal{R}_{S_4}$.

While being a complete multipartite graph may seem like a global condition, it is actually a local one – among any three vertices we can find either
no or at least two edges. Therefore we can wonder if the class of all graphs $\mathcal{R}_G$ can have a characterization by forbidden induced subgraphs, or if we could find any forbidden induced subgraphs at all. This, as the following theorem shows, is not the case.

**Theorem 5.1.** For any simple graph $X$ there is a group $G$ such that $\mathcal{R}_G$ has $X$ as an induced subgraph.

**Proof.** Let us label vertices of $X$ by numbers from $\{1, 2, \ldots, k\}$. Our construction will go as follows: we will create sequences of groups $\{G_t\} \simeq G_0 < G_1 < \ldots < G_k$ and $N_1, N_2, \ldots, N_k$ so that $G_t = G_{t-1} \rtimes N_t$, and choose $x_i \in N_i \setminus \{1\}$. Our groups $N_t$ will have a form $\mathbb{Z}_{p^t}$, where $p_1, p_2, \ldots, p_k$ are distinct primes to be chosen later. In other words, our main idea is to extend $G$ by an elementary abelian group in each step.

We would want to do it inductively so that the following conditions hold:

1. vertices numbered $i, j$ are not connected by an edge in $X$ \iff $[x_i, x_j] = 1$,
2. $G_t'$ does not contain any $x_i$ with $1 \leq i \leq t$.

The first condition is natural and implies that $x_i$-s induce $X$ in $\mathcal{R}_G$ as, since their orders are distinct prime numbers, we have $\langle x_i, x_j \rangle$ nilpotent if and only if $[x_i, x_j] = 1$. The second one is a technical assumption important for the construction, but not for the nilpotence itself.

We go by induction on $k$. For $k = 1$ choose $N_1 = G_1 = \mathbb{Z}_2$, and $x_1$ as the non-trivial element.

Now suppose the sequences up to index $t$ are constructed, and we want to perform the inductive step. At first, we consider condition 2. We claim $G_{t+1}' \cap G_t = G_t'$ regardless of how we choose $N_{t+1}$ and extension $G_t \rtimes N_{t+1}$. Direction $\supset$ is easy; to show direction $\subset$ we write epimorphism $\phi : G_{t+1} \to G_t$ (as $G_t \simeq G_{t+1}/N_{t+1}$) and observe that $\phi([x, y]) = [\phi(x), \phi(y)] \in G'_t$. Therefore if $x_r \notin G'_r$, then $x_r \notin G'_s$ for any $s \geq r$. Thus while performing the step $t \to t + 1$ we need to worry only about the case $x_{t+1} \in G'_{t+1}$, which we will sort out at the end.

We claim there is a normal subgroup of $G_t$ (we denote it by $H$) which contains exactly those $x_i$-s, such that vertex $i$ and vertex $t+1$ are not joined by an edge in $X$. Consider the quotient morphism $\mu : G_t \to G_t/G'_t$. We know, by the induction assumption and condition 2, that all $x_i$-s are mapped to non-trivial elements. Group $G_t/G'_t$ is abelian and orders of $x_i$-s (thus also of their images) are distinct primes, therefore we can find a $K \leq G_t/G'_t$ containing images of those $x_i$ we want – for instance, any abelian group is a
direct product of its Sylow subgroups, and we take subgroups corresponding
to the respective primes – and choose simply $H = \mu^{-1}(K)$. As $K$ is normal
in $G_t / G'_t$ (all subgroups of an abelian group are normal), $H$ is normal in $G_t$
by the third isomorphism theorem.

Let $d = [G_t : H]$. We can set $\alpha_{t+1} = d$. By Cayley theorem we have a
monomorphism $G_t / H \to S_d$; moreover, we also have a natural embedding
$S_d \to \text{Aut}(\mathbb{Z}^d_{pt+1})$ by permuting the coordinates. Composing these maps
together we get

$$\xi : G_t \to G_t / H \to S_d \to \text{Aut}(\mathbb{Z}^d_{pt+1})$$

Since only the first arrow is not a monomorphism, $\ker \xi = H$. Now let us
define $G'_{t+1} = \mathbb{Z}^d_{pt+1} \rtimes \xi G_t$. Take $x_{t+1}$ to have coordinates that are pairwise
different and have a sum different from zero (it can be easily done when
$pt+1 > d + 1$, as either $(1, 2, \ldots, d-1, d)$ or $(1, 2, \ldots, d-1, d+1)$ will work).

Then:

- If vertices $i$ and $t+1$ are not connected by an edge in $X$, then $\xi(x_i)$ is
  trivial as $x_i \in H = \ker \xi$, so $[x_i, x_{t+1}] = 1$.

- If they are connected by an edge, then $\xi(x_i)$ is a non-trivial permuta-
tion of coordinates on $\mathbb{Z}^d_{pt+1}$, so it acts non-trivially on $x_{t+1}$ (as it has
  pairwise different coordinates). Therefore $[x_i, x_{t+1}]$ is non-trivial.

The only thing left is to ensure that $x_{t+1} \notin G'_{t+1}$. $G'_{t+1}$ is the intersection
of all normal subgroups $K$ of $G_{t+1}$ such that $G_{t+1} / K$ is abelian. Now look
at the subset $S$ of $G_{t+1} = N_{t+1} \rtimes G_t$ composed of elements of the form
$n * g$ where the sum of coordinates of $n$ is equal to 0. As, firstly, such $n$-s
form a subgroup in $N_{t+1}$ (which is abelian), and secondly, $G_t$ acts without
changing the sum of coordinates, hence fixes this set, our set is indeed a
normal subgroup. Since $|G_{t+1} / S| = pt+1$, this quotient has to be abelian,
thus $G'_{t+1} \subset S \not\ni x_{t+1}$, qed.

This way we have shown that construction of $G_k$ is possible. Now the
only thing left is to embed $G_k$ in $G := S_{|G_k|}$ (by Cayley theorem this is
possible), so that all $x_i$-s are outside of $Z^*(G)$ (which is now trivial), so their
respective vertices belong not only to $N_G$, but also to $\mathfrak{R}_G$ (in case of $k = 1$
we need to additionally embed $S_2$ in $S_3$). This ends the proof. \[\square\]

**Declaration of competing interest**

The authors declare that they have no known competing financial interests
or personal relationships that could have appeared to influence the work
reported in this paper.
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