On the dependence of the gauge–invariant field–strength correlators in QCD on the shape of the Schwinger string

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Abstract

We study, by numerical simulations on a lattice, the dependence of the gauge–invariant two–point field–strength correlators in QCD on the path used to perform the color parallel transport between the two points.

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1. Introduction

The gauge–invariant two–point correlators of the field strengths in the QCD vacuum are defined as

\[ D_{\mu\rho,\nu\sigma}(x) = g^2 \langle 0 | \text{Tr} \left\{ G_{\mu\rho}(0)S(0, x|\mathcal{C})G_{\nu\sigma}(x)S^\dagger(0, x|\mathcal{C}) \right\} | 0 \rangle, \]  

where \( G_{\mu\rho} = T^a G^a_{\mu\rho} \) and \( T^a \) are the matrices of the algebra of the color group SU(3) in the fundamental representation. The trace in (1.1) is taken on the color indices.

Field–strength correlators play an important role in hadron physics. In the spectrum of heavy Q\(\bar{Q} \) bound states, they govern the effect of the gluon condensate on the level splittings [1, 2, 3]. They are the basic quantities in models of stochastic confinement of color [4, 5, 6] and in the description of high–energy hadron scattering [7, 8, 9, 10, 11, 12, 13]. In some recent works [14, 15], these correlators have been semi–classically evaluated in the single–instanton approximation and in the instanton dilute–gas model, so providing useful information about the role of the semiclassical modes in the QCD vacuum.

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The correlators (1.1) in principle depend on the choice of the path \( \mathcal{C} \). Usually, in developing the “Stochastic Vacuum Model” (SVM), the path \( \mathcal{C} \) appearing in Eqs. (1.1) and (1.2) is assumed to be the straight line connecting the points 0 and \( x \). Lattice data [16, 17, 18, 19], which are the input for these SVM calculations, also refer to the choice of the straight–line parallel transport. With this choice the Schwinger string (1.2) reads:

\[ S(0, x|\mathcal{C}) = \text{P} \exp \left( i g \int_0^1 dz A_\mu(z) \right), \]  

Then, in the Euclidean region, translational, O(4)– and parity invariance require the correlator (1.1) to be of the following form [4, 5, 6]:

\[ D_{\mu\rho,\nu\sigma}(x) = (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\sigma} \delta_{\rho\nu}) \left[ D(x^2) + D_1(x^2) \right] \]

\[ + \left( x_\mu x_\nu \delta_{\rho\sigma} - x_\mu x_\sigma \delta_{\rho\nu} + x_\rho x_\sigma \delta_{\mu\nu} - x_\rho x_\nu \delta_{\mu\sigma} \right) \frac{\partial D_1(x^2)}{\partial x^2}, \]  

(1.4)
where $\mathcal{D}$ and $\mathcal{D}_1$ are invariant functions of $x^2$.

The functions $\mathcal{D}(x^2)$ and $\mathcal{D}_1(x^2)$ have been directly determined by numerical simulations on a lattice in the quenched (i.e., pure–gauge) theory, with gauge group SU(2) \[16\], in the quenched SU(3) theory in the range of physical distances between 0.1 and 1 fm \[17, 18\] and also in full QCD, i.e., including the effects of dynamical fermions \[19\].

No analysis exists of what happens for a different choice of the path $C$ than the straight line, except for qualitative statements that the physics should not strongly depend on the deformations of the path \[4, 5, 6\]. In this paper we compute the gauge–invariant field–strength correlators for deformed paths (see Fig. 1), in the quenched SU(3) lattice gauge theory, for the benefit of the developments of the SVM.

### 2. Computations and results

In principle, every choice of a path $C_{[0,x]}$ connecting the two points 0 and $x$ in the expression \((1.2)\) for the Schwinger string operator will generate a different field–strength correlator, that we have denoted in Eq. \((1.1)\) as $\mathcal{D}_{\mu\rho,\nu\sigma}(x)$.

On the lattice we can define a lattice operator $\mathcal{D}_{\mu\rho,\nu\sigma}^{(C)L}$, which is proportional to $\mathcal{D}_{\mu\rho,\nu\sigma}^{(C)}$ in the continuum limit, when the lattice spacing $a \to 0$. Since the lattice analogue of the field strength is the open plaquette $\Pi_{\mu\rho}(n)$ (the parallel transport along an elementary square of the lattice lying on the $\mu\rho$–plane, starting from the lattice site $n$ and coming back to $n$), $\mathcal{D}_{\mu\rho,\nu\sigma}^{(C)L}$ will be defined as \[17, 18, 19\]

\[
\mathcal{D}_{\mu\rho,\nu\sigma}^{(C)L}(\hat{a}da) = \frac{1}{2} \Re \left\{ \langle \text{Tr}[\Pi_{\mu\rho}(n)S(n, n + \hat{a}a|C)\Omega_{\nu\sigma}(n + \hat{a}a)S^\dagger(n, n + \hat{a}a|C)] \rangle \right\}, \tag{2.1}
\]

where $\Re$ stands for real part and the lattice operator $\Omega_{\nu\sigma}(n)$ is given by \[17, 18\]

\[
\Omega_{\nu\sigma}(n) = \Pi_{\nu\sigma}^\dagger(n) - \Pi_{\nu\sigma}(n) - \frac{1}{3} \text{Tr}[\Pi_{\nu\sigma}^\dagger(n) - \Pi_{\nu\sigma}(n)]. \tag{2.2}
\]

As explained in Refs. \[17, 18\], the inclusion of the operator $\frac{1}{2}\Omega_{\nu\sigma}$ (in place of $\Pi_{\nu\sigma}^\dagger$) on the right–hand side of Eq. \((2.2)\) ensures that the disconnected part and the singlet part of the
correlator are left out, so that we are indeed taking the correlation of two operators with the exchange of the quantum numbers of a color octet.

The lattice site \( n + \hat{d}a \) is the site at a distance \( d \) lattice spacings from \( n \), in the direction of a coordinate axis. The Schwinger string \( S(n, n + \hat{d}a|C) \) is a parallel transport connecting the point \( n \) with the point \( n + \hat{d}a \) along the path \( C \).

We have explored the dependence on the path by measuring the correlators for the various paths shown in Fig. 1. Fig. 1a shows the straight–line path [corresponding to the straight–line Schwinger operator in Eq. (1.3)], which, as explained in the introduction, has already been widely analyzed. Fig. 1b shows our second choice: the transverse size \( \delta \), in units of the lattice spacing \( a \), parametrizes the deviation from the straight–line case. We shall label by \( b_\delta \) the paths in Fig. 1b, for a given value of the transverse size \( \delta \). If an average is made over the orientation of the staple in the plane orthogonal to \( \hat{d} \), the correlator will have the same symmetries (translation, \( \text{O}(4) \) and parity) as the one in Fig. 1a and the parametrization (1.4) will still apply, with, of course, different functions \( D^{(b_\delta)} \) and \( D^{(b_\delta)}_1 \). The same considerations also apply to the collection \( c_\delta \) of paths shown in Fig. 1c, for a given value of the transverse size \( \delta \), if we average over the orientation of the plane containing the path. We have chosen the inversion point \( p \) in Fig. 1c just in the middle of the line \((0, x)\), i.e., \( p = x/2 \), in order to preserve parity invariance. For other choices, i.e., \( p = \alpha x \) with \( \alpha \in [0, 1] \) and \( \alpha \neq 1/2 \), one should average between the path with \( p = \alpha x \) and the path with \( p = (1 - \alpha)x \) in order to preserve parity.

In both cases, we define \( \mathcal{D}_\parallel^{(C)}(x^2) \) and \( \mathcal{D}_\perp^{(C)}(x^2) \) as follows. We go to a reference frame in which \( x^\mu \) is parallel to one of the coordinate axes, say \( \mu = 0 \). Then

\[
\mathcal{D}_\parallel^{(C)} \equiv \frac{1}{3} \sum_{i=1}^{3} \mathcal{D}_{i0,0i}^{(C)}(x) = \mathcal{D}^{(C)} + \mathcal{D}_1^{(C)} + x^2 \frac{\partial \mathcal{D}_1^{(C)}}{\partial x^2},
\]

\[
\mathcal{D}_\perp^{(C)} \equiv \frac{1}{3} \sum_{i<j=1}^{3} \mathcal{D}_{ij,ij}^{(C)}(x) = \mathcal{D}^{(C)} + \mathcal{D}_1^{(C)}, \quad (2.3)
\]

where \( \mathcal{D}^{(C)} = \mathcal{D}^{(C)}(x^2) \) and \( \mathcal{D}_1^{(C)} = \mathcal{D}_1^{(C)}(x^2) \) are the two invariant functions entering in the parametrization (1.4) for the given collection of paths \( C \).

In the naïve continuum limit \( (a \to 0) \)

\[
\mathcal{D}_{\mu\nu,\nu\sigma}^{(C)L}(\hat{d}a) \sim a^4 \mathcal{D}_{\mu\nu,\nu\sigma}^{(C)}(\hat{d}a) + \mathcal{O}(a^6). \quad (2.4)
\]
Making use of the definition (2.3) we can also write, in the same limit,

\[ D_{||}^{(C)} (d^2 a^2) \sim a^4 D_{||}^{(C)} (d^2 a^2) + O(a^6), \]
\[ D_{\perp}^{(C)} (d^2 a^2) \sim a^4 D_{\perp}^{(C)} (d^2 a^2) + O(a^6). \]  

(2.5)

In order to remove the lattice artefacts, i.e., the terms \( O(a^6) \) in Eqs. (2.4) and (2.5), we shall make use of the cooling technique, described in previous papers (see Refs. [20, 21] and [17, 18, 19]). Cooling is a local procedure, which affects correlations at distances that grow as the square root of the number of cooling steps, as in a diffusion process. If the distance at which we observe the correlation is sufficiently large, we then expect that lattice artefacts are frozen by cooling long before the correlation is affected: this will produce a plateau in the dependence on the cooling step. Our data are the values of the correlators at the plateau; the error is the typical statistical error at the plateau, plus a systematic error which is estimated as the difference between neighbouring points at the plateau.

We have measured the correlations on a \( 16^4 \) lattice at distances ranging from 3 to 8 lattice spacings and at \( \beta = 6 \). The value of the lattice spacing \( a \) in physical units can be extracted from the value of the string tension \([22, 23]\) and it is \( a \simeq 0.1 \) fm at our value of \( \beta \). At this value of \( \beta \) scaling has been already tested for the case of the straight–line paths in Refs. [17, 18]. The data of Refs. [17, 18] came from several different values of \( \beta \) (including also \( \beta = 6 \)) and two lattice sizes, \( 16^4 \) and \( 32^4 \). They showed no visible dependence neither on the ultraviolet, nor on the infrared scale. We therefore do not expect significant scaling violations.

The results are shown in Figs. 2, 3, 4 and 5, versus the distance \( d \) in units of the lattice spacing \( a \). The lines are drawn as an eye–guide. Figs. 2 and 3 refer to the paths in Fig. 1b, for various values of the transverse size \( \delta \) of the staple. Figs. 4 and 5 give the analogous results for the paths shown in Fig. 1c.

The very result of this paper is the unexpectedly strong dependence of the correlators on the shape of the path. The correlator with the straight–line path (1.3) (see Fig. 1a), has the largest signal (for every distance \( d \)), compared with the two other choices for the path. Every deformation from the straight–line path seems to produce a sharp decrease of the value of the correlator. What seems to be path–independent is the slope of the

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† In Figs. 2, 3, 4 and 5 we have plotted only the points corresponding to a clear plateau in the cooling process. For some points at large distances, our cooling proved to be not long enough to reach the plateau.
curves, which look parallel to one another in the linear–log plots of Figs. 2, 3, 4 and 5. This means that the correlation length $\lambda_A$ defined in Refs. [17, 18, 19] is approximately path–independent, at least for the classes of paths that we have considered.

In Figs. 2, 3, 4 and 5 the best fit to the data for straight–line Schwinger string of Ref. [18], Eqs. (2.10) and (2.11), is displayed for comparison (see also Ref. [24], where a critical comparison among different best fits is performed).

In order to quantitatively test the independence of the correlation length $\lambda_A$ on the path $C$, we have tried a best fit to the data for $D_\perp = D + D_1$, at intermediate distances $5 \leq d \leq 8$, i.e., about 0.5 fm $\leq x \leq 0.8$ fm, with the same function used in Ref. [18], Eq. (2.10):

$$D_\perp(x^2) = A_\perp e^{-|x|/\lambda_A} + \frac{a_\perp}{|x|^4} e^{-|x|/\lambda_a}.$$  \hspace{1cm} (2.6)

Keeping $\lambda_A$ fixed to the value of Ref. [18], Eq. (2.11), i.e., $\lambda_A = 1/182 \Lambda_L$, where $\Lambda_L \simeq 4.9$ MeV is the lattice scale [22, 23], a good fit results with a reasonable $\chi^2/N_{d.o.f.}$. The path–dependence of the correlator reflects in a rather strong path–dependence of the coefficient $A_\perp$ of the exponential term. In the range of distances chosen, the second term of Eq. (2.6) is compatible with zero, within the errors. The coefficient $A_\perp$ changes up to a factor of four, going from $\delta = 0$ to $\delta = 6$ for the collection of paths $b_\delta$ (Fig. 1b); and it changes up to a factor of two, going from $\delta = 0$ to $\delta = 6$ for the collection of paths $c_\delta$ (Fig. 1c).

In Ref. [18] Eq. (2.6) was considered as a split–point regulator of the gluon condensate, which was extracted from the coefficient $A_\perp$ of the exponential term (see also Ref. [24]). It is not clear to us how this determination depends on the shape of the path.

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FIGURE CAPTIONS

Fig. 1. The different paths for the Schwinger string $S$ that we have considered for our analysis.

Fig. 2. The function $a^4D^{(b_3)}_\parallel$ versus the distance $d$ in units of the lattice spacing $a$. Circles correspond to the straight-line path of Fig. 1a; triangles to Fig. 1b with $\delta = 1$; squares to Fig. 1b with $\delta = 4$; diamonds to Fig. 1b with $\delta = 6$. Lines are drawn as eye-guides. The thick continuum line has been obtained using the parameters of the best fit obtained in Ref. [18], Eqs. (2.10) and (2.11).

Fig. 3. The function $a^4D^{(b_3)}_\perp$ versus the distance $d$ in units of the lattice spacing $a$. The symbols are the same as in Fig. 2.

Fig. 4. The function $a^4D^{(c_3)}_\parallel$ versus the distance $d$ in units of the lattice spacing $a$. The symbols are the same as in Fig. 2, except that they now refer to the path of Fig. 1c.

Fig. 5. The function $a^4D^{(c_3)}_\perp$ versus the distance $d$ in units of the lattice spacing $a$. The symbols are the same as in Fig. 2, except that they now refer to the path of Fig. 1c.
Figure 1

a)

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

b)

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

c)

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]

\[ \begin{array}{c}
\delta \\
0 \\
\delta
\end{array} \]
\[ a^4 D_{\text{para}}^{(b_\delta)} \]

lattice $16^4$ \quad $\beta = 6$
$a^4 D_{\text{perp}}^{(b_\delta)}$

lattice $16^4$, $\beta = 6$
$a^4 D_{\text{para}}^{(c_\delta)}$

lattice $16^4$  \( \beta = 6 \)
$a^4 D_{\text{perp}}^{(c_\delta)}$

lattice $16^4$ \quad $\beta = 6$