Random transition-rate matrices for the master equation

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Random-matrix theory is applied to transition-rate matrices in the Pauli master equation. We study the distribution and correlations of eigenvalues, which govern the dynamics of complex stochastic systems. Both the cases of identical and of independent rates of forward and backward transitions are considered. The first case leads to symmetric transition-rate matrices, whereas the second corresponds to general, asymmetric matrices. The resulting matrix ensembles are different from the standard ensembles and show different eigenvalue distributions. For example, the fraction of real eigenvalues scales anomalously with matrix dimension in the asymmetric case.

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I. INTRODUCTION

The Pauli master equation is encountered in many fields of science such as physics, chemistry, and biology. It describes the time evolution of probabilities for a system to be in certain states. Formally identical rate equations describe the dynamics of concentrations or populations of certain entities. The dynamics of probabilities is described by the Pauli master equation

$$\dot{P}_i = \sum_{j \neq i} (R_{ij} P_j - R_{ji} P_i),$$

(1)

where $P_i$ is the probability to find the system in state $i = 1, \ldots, N$ and $R_{ij}$ is the transition rate from state $j$ to state $i$. Evidently, the rates of change of probabilities depend only on the probabilities at time $t$, i.e., Eq. (1) describes a memory-less or Markovian process. Equation (1) ensures that the total probability is conserved,

$$\frac{d}{dt} \sum_i P_i = 0.$$

(2)

Typical applications in physics include lasers [1], disordered conductors [2], microelectronic devices [3], quantum dots [4], and molecular electronics [5]. In these cases one can, in principle, obtain the Pauli master equation by first deriving a quantum master equation for the reduced density matrix of a small system, which is obtained by tracing out the reservoir degrees of freedom from the full density operator [6, 7, 8, 9, 10]. If the off-diagonal components of the reduced density matrix decay rapidly, it is sufficient to keep only the diagonal components representing the probabilities $P_i$ of states $|i\rangle$ of the small system. In certain fields, for example in transport and laser theory, the resulting Eqs. (1) are often called rate equations.

However, if even the small system is complicated, such as a system of interacting enzymes, this route becomes unfeasible. In applications outside of physics, where $i$ could refer to the state of a technical or social process, a quantum-statistical description becomes inappropriate in any case. One would then view Eq. (1) as the fundamental description.

Our goal is to make progress in the understanding of the master equations for complex systems. The number $N$ of possible states will typically be large. It should be noted however that complex behavior can already emerge for moderate $N$. An example is provided by the differential conductance calculated in Ref. [11] for a magnetic molecule with magnetic anisotropy axis not aligned with the applied magnetic field, where $N = 20$, but due to noncommuting terms in the Hamiltonian many rates are nonzero and are distributed over a broad range.

A. Properties of the master equation

We first recount some basic properties. It is clear that one can rewrite Eq. (1) in the form

$$\dot{P}_i = \sum_j A_{ij} P_j$$

(3)

or $\dot{P} = AP$ with the transition-rate matrix, or, for short, rate matrix,

$$A_{ij} \equiv \begin{cases} R_{ij} & \text{for } i \neq j \\ -\sum_{k \neq j} R_{kj} & \text{for } i = j. \end{cases}$$

(4)

It follows that the column sums vanish,

$$\sum_i A_{ij} = 0 \quad \text{for all } j.$$

(5)

Note that $(d/dt) \sum_i P_i = \sum_{i,j} A_{ij} P_j$ vanishes for all $P_j$ if and only if Eq. (5) holds. The constraint (5) is thus dictated by conservation of probability. From Eq. (4) it is also clear that

$$A_{ij} \geq 0 \quad \text{for all } i \neq j$$

(6)
if we interpret the $R_{ij}$ as transition rates. A matrix satisfying the inequalities (6) and $\sum_j A_{ij} \leq 0$ for all $j$ is called a compartmental matrix.

Equation 3 can be solved by the ansatz $P = e^{\lambda t} v$, which leads to the eigenvalue equation $A v = \lambda v$. Since $A$ is generally not symmetric, the eigenvalues $\lambda$ and the components of the right eigenvectors $v$ can be complex. However, since $A$ is real, the equation $A v = \lambda v$ implies $A^* v^* = \lambda^* v^*$. Thus, the eigenvalues are real with real eigenvectors or form complex conjugate pairs with their eigenvectors also being complex conjugates.

Let $v_n$ be the right eigenvector to eigenvalue $\lambda_n$. It is well known that there is always at least one strictly zero eigenvalue, which we call $\lambda_0 = 0$: the constraint (9) implies that $A$ has a left eigenvector $(1, 1, \ldots, 1)$ to the eigenvalue $\lambda_0 = 0$. The corresponding right eigenvector $v_0$ describes the stationary state.

A real eigenvector $v_n$ with real eigenvalue $\lambda_n$ describes a contribution to the probability vector $P(0)$ that decays exponentially with the rate $-\lambda_n$. A complex conjugate pair of eigenvectors $v_n, v_n^*$ with eigenvalues $\lambda_n, \lambda_n^*$ can be combined to form the two independent real solutions $(e^{\lambda_n t} v_n + e^{\lambda_n^* t} v_n^*)/2$ and $(e^{\lambda_n t} v_n - e^{\lambda_n^* t} v_n^*)/2i$. Writing the components of $v_n$ as $v_{nj} = v_{nj}^0 e^{\phi_{nj}}$ with $v_{nj}^0$ real, we obtain the solutions

$$v_{nj}^0 e^{\lambda_n t} \propto \begin{cases} \cos(\Im \lambda_n t + \phi_{nj}) & \text{for } \lambda_n > 0 \\ \sin(\Im \lambda_n t + \phi_{nj}) & \text{for } \lambda_n < 0 \end{cases} \tag{7}$$

The initial values at time $t = 0$ are clearly $\Re v_{nj}$ and $\Im v_{nj}$, respectively. We thus find damped harmonic oscillations with damping rate $-\Re \lambda_n$ and angular frequency $\Im \lambda_n$. We obtain the solution at all times by expanding the initial probability vector $P(t = 0)$ into the basis of real vectors $v_n$ (for real $\lambda_n$) and $P(t = 0)$ into the basis of complex conjugate pairs $\lambda_n, \lambda_n^*$.

An eigenvalue $\lambda_n$ with $\Re \lambda_n > 0$ would be unphysical, since the corresponding contribution to the probabilities would diverge for $t \rightarrow \infty$. However, for any compartmental matrix the spectrum is contained in $\{\lambda | \Re \lambda < 0 \} \cup \{0\}$ [12, 13]. Thus all eigenvalues are either zero or have a strictly negative real part.

The Perron-Frobenius theorem [14, 15] applied to the non-negative matrix $A - a_{\min} I$, where $a_{\min} < 0$ is the minimum of $A_{ij}$ and $I$ is the $N \times N$ unit matrix, shows that the right eigenvector $v_0$ to $\lambda_0$ has only non-negative components. This ensures that the probabilities in the stationary state are non-negative.

### B. Random rate matrices

As noted above, even relatively simple problems lead to master equations with rates $A_{ij}, i \neq j$, distributed over a broad range. In problems with large numbers of states it is often impractical to obtain all independent components $A_{ij}$. This situation is reminiscent of Hamiltonians for complex systems. Difficult problems of this type concern atomic nuclei and quantum dots, where the Hamiltonian is too complicated to write down explicitly, but cannot be simplified by methods restricted to weakly interacting systems. For these systems, random-matrix theory (RMT) [16, 17, 18, 19] has lead to significant progress. The main assumption is that a Hamiltonian of this type is a typical representative of an ensemble of Hamiltonians of appropriate symmetry. While this approach does not allow one to obtain specific eigenvalues, it does provide information about the statistical properties of the spectrum [16, 17, 18, 19].

Our point of departure is to treat the rate matrix $A$ for a complex system as an element of a suitable random-matrix ensemble. In the case of transport through quantum dots, this is complementary to treating the Hamiltonian of the quantum dot as a random matrix, which has been done extensively [17].

Since the rate matrix $A$ must satisfy the conditions (3) and (9), we define the exponential general rate-matrix ensemble (EGRE): The EGRE is formed by real $N \times N$ matrices $A$ with independently identically distributed off-diagonal components $A_{ij}$ with the distribution function

$$p(A_{ij}) = \begin{cases} 1/(R!) & \text{for } A_{ij} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and the diagonal components

$$A_{jj} = - \sum_{i \neq j} A_{ij}. \tag{9}$$

The exponential distribution of rates $A_{ij}$ is viewed as the least biased distribution of non-negative numbers. We will also present results that do not depend on the specific distribution function $p$. We will see that the specific distribution becomes irrelevant in the limit of large $N$, at least if all its moments exist. The distribution of components is thus not the most fundamental difference between the EGRE and the well-known ensembles studied in the context of random Hamiltonians. Rather, one such difference lies in the constraint (3) or (9). The other is that the rate matrices are real but not symmetric and thus not hermitian [20].

Ensembles of non-hermitian matrices have been studied in detail, starting with Ginibre’s work on Gaussian ensembles of non-hermitian matrices with real, complex, and quaternion components [21]. We will compare our results to the real Ginibre ensemble.

To be able to analyze the importance of the asymmetry, we also define the exponential symmetric rate-matrix ensemble (ESRE): The ESRE is formed by real symmetric $N \times N$ matrices $A$ with independently identically distributed components $A_{ij}$ above the diagonal ($i < j$) with the distribution function given by Eq. (8) and the diagonal components given by Eq. (9).

Another possible choice is a two-valued distribution of rates, where a transition from state $j$ to state $i$ is either possible or impossible, and all possible transitions have
the same rate. This case with symmetric rates has been studied by various authors [23, 24]. It is essentially equivalent to adjacency matrices of random simple networks.

An ensemble of real symmetric matrices satisfying Eq. (5) but with a Gaussian distribution of $A_{ij}$ has also been studied [24]. This case cannot easily be interpreted in terms of a master equation, since the $A_{ij}$ can be negative. We will compare our results for the eigenvalue spectrum to these works below.

The remainder of this paper is organized as follows: In Sec. II we consider the simpler case of symmetric rate matrices (the ESRE) and obtain results for the eigenvalue density and for the correlations between neighboring eigenvalues. In Sec. III we then study general rate matrices (the EGRE) and obtain results for the eigenvalue density, now in the complex plane, and for the correlations of neighboring eigenvalues. We conclude in Sec. IV. A number of analytical derivations are relegated to appendices.

II. SYMMETRIC RATE-MATRIX ENSEMBLE

We first consider ensembles of symmetric rate matrices $A$. These describe processes where transitions from any state $j$ to state $i$ and from $i$ to $j$ occur with the same rate, $A_{ij} = A_{ji}$.

A. Spectrum

As noted above, the spectrum always contains the eigenvalue $\lambda_0 = 0$. The corresponding eigenvector for symmetric matrices is $(1, 1, \ldots, 1)$ or, normalized to unit probability, $(1/N, 1/N, \ldots, 1/N)$. For symmetric rates, the stationary state is thus characterized by equal distribution over all states $i$. We are interested in the distribution of the other eigenvalues $\lambda_n$, $n = 1, \ldots, N - 1$, which are all real. We have also seen in Sec. I B that $\lambda_n \leq 0$. Since there is no further constraint, the probability of $\lambda_n$ for any $n > 0$ being exactly zero vanishes.

To simplify the calculations, we shift the matrices so that they have zero mean. We discuss this immediately in terms of a master equation, since the average over the matrix ensemble under consideration. Here, $\langle A \rangle$ has the components $\langle A_{ij} \rangle = \langle R \rangle$ for $i \neq j$ and $\langle A_{ii} \rangle = -(N - 1) \langle R \rangle$. Is follows that $\sum_i A_{ij} = 0$ for all $j$. Consequently, $A$ has a left eigenvector $w_0^T \equiv (1, 1, \ldots, 1)$ to the eigenvalue $\lambda_0 = 0$.

Let $v_n$ be the right eigenvectors of $A$ to the eigenvalues $\lambda_n$, $n = 1, \ldots, N - 1$. Since $w_0^T$ is the left eigenvector to the eigenvalue $\lambda_0 = 0$, we have $w_0^T v_n = 0$. Since

$$\langle A \rangle = \langle R \rangle \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} - N \langle R \rangle I,$$  (11)

$v_n$ is a right eigenvector of $\langle A \rangle$ to the eigenvalue $-N \langle R \rangle$. Therefore, $v_n$ is also a right eigenvector of $A$ to the eigenvalue $\lambda_n = \lambda_n + N \langle R \rangle$. The result is that the shifted matrices $\tilde{A}$ also have one eigenvalue $\lambda_0 = 0$ and that the remaining eigenvalues are just the eigenvalues of $A$, shifted by $N \langle R \rangle$.

We now derive the average of eigenvalues $\lambda_n$, here and in the following excluding $\lambda_0 = 0$. We have $\langle \lambda \rangle' = \langle \lambda \rangle' - N \langle R \rangle$, where angular brackets with a prime denote the average over all eigenvalues, excluding the exact zero. Since this leaves $N - 1$ eigenvalues, their average is the trace of the matrix, to which the zero eigenvalue does not contribute, divided by $N - 1$. Consequently,

$$\langle \lambda' \rangle = \frac{1}{N - 1} \text{Tr} \langle \tilde{A} \rangle = \frac{1}{N - 1} \text{Tr} 0 = 0$$  (12)

so that

$$\langle \lambda \rangle' = -N \langle R \rangle.$$  (13)

This result is independent of the specific distribution function of rates, $p$, as long as $\langle R \rangle$ exists.

We next calculate the low-order central moments

$$\mu_m \equiv \langle (\lambda - \langle \lambda \rangle')^m \rangle' = \langle (\lambda + N \langle R \rangle)^m \rangle'$$  (14)

of the eigenvalues $\lambda_n$, $n > 0$. The central moments are identical to the central moments of the shifted values $\lambda_n$. Unless otherwise noted, our results for $\mu_m$ hold for an arbitrary distribution function of rates, $p$, as long as the moments exist. It is instructive to show the calculation of the second moment explicitly. We find

$$\mu_2 = \langle \lambda' \rangle^2 = \frac{1}{N - 1} \text{Tr} \langle \tilde{A}^2 \rangle = \frac{1}{N - 1} \sum_{ij} \langle \tilde{A}_{ij} \tilde{A}_{ji} \rangle$$

$$= \frac{1}{N - 1} \sum_i \left( \sum_{j \neq i} \langle \tilde{A}_{ij} \tilde{A}_{ji} \rangle + \langle \tilde{A}_{ii} \tilde{A}_{ii} \rangle \right).$$  (15)

Using $\tilde{A}_{ij} = \tilde{A}_{ji}$ and $\sum_k \tilde{A}_{ki} = 0$, we obtain

$$\mu_2 = \frac{1}{N - 1} \sum_i \left( \sum_{j \neq i} \langle \tilde{A}_{ij}^2 \rangle + \sum_{k, l \neq i} \langle \tilde{A}_{ki} \tilde{A}_{li} \rangle \right).$$  (16)

With $\langle \tilde{A}_{ij} \rangle = 0$ we finally get

$$\mu_2 = \frac{2}{N - 1} \sum_{i, j \neq i} \langle \delta R^2 \rangle = 2N \langle \delta R^2 \rangle,$$  (17)

where $\langle \delta R^2 \rangle \equiv \langle A_{ij}^2 \rangle - \langle A_{ij} \rangle^2$ for $i \neq j$ is the second central moment of $p(A_{ij})$. For the special case of an
The important consequence is that while the mean of the nonzero eigenvalues of the unshifted matrices $A$ scales with $N$, Eq. (13), the width of their distribution is only $\sqrt{\mu_2} = 2N \langle R^2 \rangle$. Thus for large $N$ the distribution of eigenvalues contains a single eigenvalue $\lambda_0 = 0$ and the remaining $N - 1$ eigenvalues form a narrow distribution around $-N \langle R \rangle$. In physical terms, nearly all deviations from the stationary state decay on the same time scale $1/N\langle R \rangle$.

All moments can be obtained by the same method: We first write the average in terms of a trace, split the sum into terms with equal or distinct matrix indices, and use $\sum_k \tilde{A}_{ki} = 0$. With $\tilde{A}_{ij} = A_{ji}$ and $\langle \tilde{A}_{ij} \rangle = 0$ we obtain the moments. Since the enumeration of all possible cases of equal or distinct indices is cumbersome, we have used a symbolic algebra scheme implemented with Mathematica [25]. The results up to $m = 8$ are shown in Table I for a general distribution. The moments are expressed in terms of the central moments $\langle \delta R^n \rangle \equiv \langle (A_{ij} - (A_{ij}))^n \rangle$. Note that in the limit of large $N$, the moments $\mu_m$ for even $m$ only depend on the second moment $\langle \delta R^2 \rangle$. We will return to this point shortly.

Table II shows the central moments $\mu_m$ up to $m = 10$ for the exponential distribution of $A_{ij}$, $i < j$ (ESRE). For the exponential distribution, one has $\langle \delta R^n \rangle = 1/(n! \langle R \rangle^n)$, where $!n \equiv n! \sum_{k=0}^{n+1}(-1)^k/k!$ is the subfactorial. Table II also contains the leading large-$N$ terms for the ESRE. At least up to $m = 10$, the even moments scale as $\mu_m \sim N^{m/2}$ for large $N$, as expected from the scaling of $\mu_2$. However, the odd moments scale only as $\mu_m \sim N^{(m-1)/2}$. If this holds for all $m$, the distribution of $\lambda$ approaches an even function for large $N$. This is indeed the case, as we shall see.

The density of eigenvalues $\tilde{\lambda}_n$ can be obtained from the resolvent $\tilde{G}(z) \equiv (z - \tilde{A})^{-1}$. The density is given by the spectral function

$$
\rho_{\text{all}}(z) = -\frac{1}{\pi N} \text{Im} \text{Tr} \left( \tilde{G}(z + i\eta) \right),
$$

where $\eta \to 0^+$ at the end of the calculation. The density includes the exact zero eigenvalue so that we can write

$$
\rho_{\text{all}}(z) = \frac{1}{N} \delta(z) + \frac{N-1}{N} \rho(z),
$$

where $\rho(z)$ is the normalized density of nonzero eigenvalues. In the limit of large $N$, the eigenvalue density $\rho_{\text{all}}(z) \equiv \rho(z)$ only depends on the second moment $\langle \delta R^2 \rangle$ of the distribution function $p$ of rates, at least as long as all moments of $p$ exist. The proof is sketched in App. A. That the eigenvalue distribution generically becomes independent of $p$ for large $N$ has been conjectured by Mehta (conjecture 1.2.1 in Ref. [19]). However, the second part of this conjecture, stating that the density of eigenvalues is the same as for the Gaussian orthogonal ensemble (GOE), is not true for our ensemble.

Since the density of eigenvalues $\tilde{\lambda}_n$, $n > 0$, of the shifted matrices $A$ only depends on the second moment $\langle \delta R^2 \rangle$ for large $N$, we can obtain the large-$N$ behavior from any distribution with that second moment. We choose the Gaussian distribution

$$
p_G(\tilde{A}_{ij}) = \frac{1}{\sqrt{2\pi \langle \delta R^2 \rangle}} \exp \left( -\frac{A_{ij}^2}{2\langle \delta R^2 \rangle} \right). \tag{20}
$$

For this distribution together with the constraint $\sum_k \tilde{A}_{ij} = 0$, the eigenvalue density is known for large $N$ [24]: the averaged resolvent is the solution of

$$
\langle \tilde{G}(z) \rangle = \frac{1}{\sqrt{N\langle \delta R^2 \rangle}} g \left( \frac{z - N\langle \delta R^2 \rangle \langle \tilde{G}(z) \rangle}{\sqrt{N\langle \delta R^2 \rangle}} \right), \tag{21}
$$

where

$$
g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{e^{-x^2/2}}{z - x}. \tag{22}
$$

This integral can be evaluated,

$$
g(z) = \sqrt{\frac{\pi}{2}} z \sqrt{\frac{1}{z^2 e^{-z^2/2}}} \left( 2 + \text{erfc} \frac{z^2 \sqrt{1/z^2}}{\sqrt{2}} \right). \tag{23}
$$

g(z) has a cut along the whole real axis. The density $\rho(z)$ is thus nonzero for all real $z$. Equations (18) and (21) imply that $\sqrt{N\langle \delta R^2 \rangle} \rho(z)$ is a universal function of $z/\sqrt{N\langle \delta R^2 \rangle}$. The same distribution in the large-$N$ limit was found for adjacency matrices [22, 23]. The corresponding result for the GOE is the well-known semicircle law [16, 19]. It is worth pointing out that the different eigenvalue density results only from the constraint $\sum_k \tilde{A}_{ij} = 0$.

We now study the eigenvalue density for the ESRE for finite $N$. We perform Monte Carlo simulations by generating a number $n_r$ of realizations of matrices from the ESRE for given $N$, shifted according to Eq. (10). The matrices are diagonalized and the eigenvalue with the numerically smallest magnitude, which corresponds to $\lambda_0 = 0$, is dropped. The eigenvalues are rescaled according to $\lambda \to \lambda/\sqrt{N\langle \delta R^2 \rangle}$. Finally, histograms with 500 bins are generated.

Results for $N = 2$, 10, 100, 1000, 10000, and $\infty$ are shown in Fig. 1. For $N \to \infty$, we solve Eq. (21). For $N = 2$, the matrices have a single nonzero eigenvalue $-2A_{12}$ with distribution following from Eq. (8). For each of the other values of $N$, $n_r N = 10^7$ eigenvalues have been generated. Figure 1 shows that the distribution changes smoothly from shifted exponential for $N = 2$ to the known universal function for $N \to \infty$. The inset in Fig. 1 shows the unscaled eigenvalue density of the unshifted ESRE to illustrate that the mean scales with $N$, whereas the width scales with $\sqrt{N}$.

While we have shown that nearly all nonzero eigenvalues lie in a narrow interval around their mean for large $N$, the dynamics after a transient will be dominated by
TABLE I: Central moments \( \mu_m \), \( m = 2, \ldots, 8 \), of the nonzero eigenvalues \( \lambda \) for ensembles of symmetric rate matrices. The results hold independently of the distribution function \( p \) of rates \( A_{ij}, i < j \), as long as the moments exist. Here, \( \langle \delta R^m \rangle \) is the \( m \)-th central moment of \( p \).

| \( m \) | \( \mu_m \) (symmetric matrices, general distribution) |
|------|--------------------------------------------------|
| 2    | \( 2N\langle \delta R^2 \rangle \)               |
| 3    | \( -4N\langle \delta R^3 \rangle \)             |
| 4    | \( N[9(N-2)\langle \delta R^2 \rangle^2 + 8\langle \delta R^4 \rangle] \) |
| 5    | \( -2N[25(N-2)\langle \delta R^2 \rangle^3 + 8\langle \delta R^5 \rangle] \) |
| 6    | \( N[4(14N^2 - 73N + 90)\langle \delta R^2 \rangle^3 + 73(N-2)\langle \delta R^3 \rangle^2 + 132(N-2)\langle \delta R^2 \rangle\langle \delta R^4 \rangle + 32\langle \delta R^6 \rangle] \) |
| 7    | \( -2N[7(41N^2 - 211N + 258)\langle \delta R^2 \rangle^4 + 203(N-2)\langle \delta R^3 \rangle^2 \langle \delta R^4 \rangle + 168(N-2)\langle \delta R^2 \rangle \langle \delta R^5 \rangle + 32\langle \delta R^6 \rangle] \) |
| 8    | \( N[(431N^3 - 4042N^2 + 12021N - 11322)\langle \delta R^2 \rangle^5 + 6(306N^2 - 1561N + 1898)\langle \delta R^3 \rangle^2 \langle \delta R^4 \rangle + 593(N-2)\langle \delta R^4 \rangle^2 
+ 1088(N-2)\langle \delta R^3 \rangle \langle \delta R^5 \rangle + 4(N-2)(507N-1574)\langle \delta R^2 \rangle \langle \delta R^3 \rangle^2 + 832\langle \delta R^2 \rangle \langle \delta R^6 \rangle + 128\langle \delta R^8 \rangle] \) |

The slowest process. The slowest non-stationary process is governed by the eigenvalue \( \lambda_1 < 0 \) which is smallest in magnitude. It is conceivable that matrices from the ESRE typically have an eigenvalue \( \lambda_1 \) close to zero. For example, \( \lambda_1 \) could scale with a lower power of \( N \) compared to the mean \( -N\langle R \rangle \). If the fraction of such anomalously slow rates decreased for large \( N \), they might not be visible in the density plots in Fig. 1.

To check this, we plot the mean \( \langle \lambda \rangle \) as a function of \( N \) in Fig. 2. The average slowest rate \( \langle \lambda \rangle \) is significantly smaller than the average rate \( \langle \lambda \rangle' \) for small \( N \), as one would expect from the width \( \sqrt{\mu_2} \propto \sqrt{\lambda} \). On the other hand, for large \( N \), \( \langle \lambda \rangle \) approaches \( \langle \lambda \rangle' \). Thus we do not find evidence for anomalously slow processes.

Instead, the slowest rate is consistent with the mean and width of the eigenvalue distribution \( \rho(\lambda) \).

B. Eigenvalue correlations

Since the eigenvalue density for the ESRE differs significantly from the GOE, one might ask whether the correlations between eigenvalues are also different. In the GOE, the distribution function of differences of neighboring eigenvalues \( \lambda, \lambda' \) approaches zero as \( \lambda' \rightarrow \lambda \).

Figure 3 shows the distribution function \( \rho_{NN}(\Delta \lambda) \) of separations \( \Delta \lambda \equiv \lambda_{n+1} - \lambda_n \) of neighboring eigenvalues for the ESRE (here, the \( \lambda_n \) are assumed to be ordered by value). The zero eigenvalue \( \lambda_0 = 0 \) is excluded. Since the width of the eigenvalue distribution scales as \( \sqrt{\lambda} \), while the number of eigenvalues for a given realization scales as \( N \), the typical separation should scale as \( 1/\sqrt{\lambda} \). We therefore rescale \( \Delta \lambda \rightarrow \sqrt{\lambda}/\langle \delta R \rangle \Delta \lambda \). Figure 3 shows that the rescaled distribution approaches a limiting form for \( \lambda \rightarrow \infty \). Furthermore, the distribution function \( \rho_{NN}(\Delta \lambda) \) is linear in \( \Delta \lambda \) for small \( \Delta \lambda \) for all \( N \). Thus the distribution of nearest-neighbor separations behaves essentially like for the GOE [17]. The constraint [15], which is responsible for the deviation of the eigenvalue distribution from the GOE result, does not have a comparably strong effect on the eigenvalue correlations. The reason is very likely that the joint probability distribution \( \rho(\lambda_1, \lambda_2, \ldots, \lambda_{N-1}) \) of the eigenvalues [16, while being complicated for the ESRE, does contain the factor \( \Pi_{\lambda_{n+1} > \lambda_n} = \lambda_{1} - \lambda_{n+1} \), which determines the exponent \( \beta = 1 \) in \( \rho_{NN} \sim \Delta \lambda^\beta \).
III. GENERAL RATE-MATRIX ENSEMBLE

We now turn to the ensemble of general, asymmetric rate matrices (EGRE). Compared to the ESRE, it describes the opposite extreme of independent rates $A_{ij}$ and $A_{ji}$ for forward and backward transitions.

A. Spectrum

As noted, there always exists an eigenvalue $\lambda_0 = 0$ with left eigenvector $(1,1,\ldots,1)$. Other than for the symmetric case, the corresponding right eigenvector is different. We are interested in the distribution of the other eigenvalues $\lambda_n$, $n = 1, \ldots, N - 1$, which are now complex with negative real parts. We have already shown in Sec. II that the mean of nonzero eigenvalues equals $-N \langle R \rangle$, see Eq. (25). We shift the matrices according to Eq. (10) so that they have zero mean.

We define the expectation values
\[
\mu_m \equiv \langle \lambda^m \rangle' = \langle (\lambda - \langle \lambda \rangle)^m \rangle' = \langle (\lambda + N \langle R \rangle)^m \rangle'.
\]
in analogy to the ESRE, but they are not the central moments of the distribution of nonzero eigenvalues. Instead, the central moments have to be defined for a two-dimensional distribution in the complex plane,
\[
\mu_{mn} \equiv \langle \langle \text{Re} \lambda + N \langle R \rangle \rangle^m \langle \text{Im} \lambda \rangle^n \rangle'.
\]
Since the eigenvalues are real or form complex conjugate pairs, we have $\mu_{mn} = 0$ for odd $n$. We show in App. B that the shifted eigenvalue distribution only depends on the second moment $\langle \delta R^2 \rangle$ of $p$, like we found for the symmetric case. We here call the $\mu_m$ in Eq. (24) the pseudomoments. They are all real, since the eigenvalues are real or form complex conjugate pairs.

The pseudomoments $\mu_m$ can be obtained in the same way as for symmetric matrices. The results are different, since $\langle A_{ij} A_{ji} \rangle = \langle \delta R^2 \rangle$ for the symmetric case, whereas $\langle A_{ij} A_{ji} \rangle = 0$ for the general case. We present the pseudomoments $\mu_m$ up to $m = 8$ for a general distribution function $p(A_{ij})$ in Table III and up to $m = 10$ for the exponential distribution (EGRE) in Table IV.
the ESRE. In the limit \( N \to \infty \), only the even pseudo-moments survive. Interestingly, at least up to \( m = 10 \), these agree with the central moments of a real Gaussian distribution, \( \mu^G_m = (m-1)!!(N(\delta R^2))^{m/2} \), where \( n!! = n(n-2)(n-4)\ldots \) is the double factorial. We show in App. C that this identity holds for all even \( m \).

The eigenvalue distribution in the complex plane can be obtained from the non-analyticities of the averaged resolvent \( \langle \tilde{G}(z) \rangle = (z - \tilde{A})^{-1} \) \cite{26,27}. However, unlike for symmetric matrices, the non-analyticities are not limited to a branch cut along the real axis. For what follows, it is more convenient to employ the method of hermitization \cite{27}. We define the \( 2N \times 2N \) matrix

\[
\mathcal{H}(z, z^*) \equiv \begin{pmatrix} 0 & A - zI \\ \tilde{A}^T - z^*I & 0 \end{pmatrix},
\]

where \( \tilde{A}^T \) is the transpose of \( \tilde{A} \). \( \mathcal{H}(z, z^*) \) is hermitian for any complex \( z \). With the resolvent of \( \mathcal{H} \),

\[
\mathcal{G}(\eta; z, z^*) \equiv \frac{1}{\eta - \mathcal{H}(z, z^*)},
\]

the density of eigenvalues in the complex plane is \cite{27}

\[
\rho_{all}(x, y) = \frac{1}{\pi N} \frac{\partial}{\partial z^*} \text{Tr}_{2N} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \langle \mathcal{G}(0; z, z^*) \rangle,
\]

where \( z = x + iy \), the derivative with respect to \( z^* \) is to be taken with \( z \) fixed, and \( \text{Tr}_{2N} \) denotes the trace over a \( 2N \times 2N \) matrix. Using this representation, we show that for large \( N \) the eigenvalue density only depends on the second central moment \( \langle \delta R^2 \rangle \) of the distribution of rates \( A_{ij} \). The proof is sketched in App. B. Edelman et al. \cite{28} have conjectured that this is generically the case for asymmetric matrices.

We now present numerical results for \( \rho(x, y) \) for the EGRE, as a function of the matrix dimensions \( N \). As above, \( \rho_{all} \) contains all eigenvalues, whereas \( \rho \) excludes the exact zero. We will compare the results to the Ginibre ensemble of real asymmetric matrices with Gaussian distribution of components (Ginibre orthogonal ensemble, GinOE) \cite{21,29,30,31,32,33}, which is the closest relative of the EGRE that has been studied in detail.

As observed above, the eigenvalues \( \tilde{\lambda} \) of \( \tilde{A} \) can be either real or form complex conjugate pairs. The numerical simulations show that both types of eigenvalues indeed occur. A typical eigenvalue density is shown in Fig. 4 for \( N = 20 \). We assume that the square root of the second pseudomoment, \( \sqrt{\rho_2} = \sqrt{N(\delta R^2)} \), describes the typical width of the distribution and rescale the eigenvalue density accordingly. The real and complex eigenvalues are clearly visible. Here and in the following “complex” should be understood as “not real.” Figure 4 already suggests that the distribution of nonzero eigenvalues of \( \tilde{A} \) becomes a narrow peak around \( -N(\delta R) \) for large \( N \), like for the ESRE. We return to this point below.

**FIG. 4:** (Color online) Scaled distribution function of nonzero eigenvalues \( \tilde{\lambda} \) of shifted general rate matrices \( \tilde{A} \) of dimension \( N = 20 \). More specifically, a two-dimensional histogram with \( 500 \times 500 \) bins was populated for \( n_r \) matrices randomly chosen from the EGRE, with \( n_r N = 4 \times 10^7 \).

The question arises of what fraction \( f_R \) of the nonzero eigenvalues are real. For the GinOE, this fraction is known analytically \cite{28}. (The probability of finding exactly \( N_R \) real eigenvalues for \( N \times N \) matrices from the GinOE is also known \cite{32}.) Edelman et al. \cite{28} derive various equivalent expressions for the expected number of real eigenvalues, \( \langle N_R \rangle \), from which we obtain

\[
f_R^{\text{GinOE}} = \langle N_R \rangle / N.
\]

We here quote an expression in terms of the hypergeometric function \( _2F_1 \) \cite{29}:

\[
f_R^{\text{GinOE}} = \frac{1}{2N} + \frac{\sqrt{2}}{\pi} \frac{\Gamma(N + 1/2)}{\Gamma(N + 1)} _2F_1 \left( 1, -1/2, N; \frac{1}{2} \right).
\]

For large \( N \), this becomes \cite{29}

\[
f_R^{\text{GinOE}} \approx \frac{\sqrt{2}}{\pi N}.
\]

For the GinOE, the fraction of real eigenvalues thus asymptotically decays with a simple exponent of \(-1/2\).

Figure 5 shows the fraction \( f_R \) as a function of \( N \) for the EGRE and for comparison the exact result for the GinOE. For \( N = 2 \), \( f_R \) must be unity, since the single nonzero eigenvalue cannot be a complex conjugate pair. The results clearly differ from the GinOE and decay more slowly for large \( N \). A fit of a power law \( f_R \sim f_0 N^{-\alpha} \) to the data points for \( N = 2000 \) and 5000 is also included in Fig. 5. We obtain \( f_0 \approx 1.37 \) and \( \alpha \approx 0.460 \). The large-\( N \) behavior is inconsistent with the exponent \( 1/2 \) found for the GinOE. This is remarkable, since all other scaling relations we have so far found, as well as the ones for the GinOE, only contain integer powers of \( \sqrt{N} \). Physically, this means that the fraction of eigenvectors describing purely exponentially decaying deviations from the sta-
TABLE III: Pseudomoments $\mu_m$, $m = 2, \ldots, 8$, of the nonzero eigenvalues $\lambda$ for ensembles of general rate matrices. The results hold independently of the distribution function $p$ of rates $A_{ij}$, $i \neq j$, as long as the moments exist.

| $m$ | $\mu_m$ (general matrices, general distribution) |
|-----|-----------------------------------------------|
| 2   | $N\langle\delta R^2\rangle$                   |
| 3   | $-N\langle\delta R^3\rangle$                  |
| 4   | $N[3(N-1)\langle\delta R^2\rangle^2 + \langle\delta R^4\rangle]$ |
| 5   | $-N[10(N-1)\langle\delta R^2\rangle + \langle\delta R^4\rangle]$ |
| 6   | $N[(15N^2 - 49N + 38)\langle\delta R^2\rangle^3 + 10(N-1)\langle\delta R^4\rangle^2 + 15(N-1)\langle\delta R^2\rangle\langle\delta R^4\rangle + \langle\delta R^6\rangle]$ |
| 7   | $-N[21(5N^2 - 17N + 14)\langle\delta R^2\rangle^3 + 35(N-1)\langle\delta R^4\rangle^2 + 21(N-1)\langle\delta R^2\rangle\langle\delta R^4\rangle + \langle\delta R^6\rangle]$ |
| 8   | $N[35N^3 - 240N^2 + 551N - 422\langle\delta R^2\rangle^4 + 36(5N^2 - 121N + 102)\langle\delta R^4\rangle^2\langle\delta R^2\rangle + 35(N-1)\langle\delta R^4\rangle^2 + 56(N-1)\langle\delta R^6\rangle + 56(5N^2 - 18N + 16)\langle\delta R^2\rangle\langle\delta R^4\rangle^2 + 28(N-1)\langle\delta R^2\rangle\langle\delta R^6\rangle + \langle\delta R^8\rangle]$ |

TABLE IV: Second column: pseudomoments $\mu_m$, $m = 2, \ldots, 10$, of the nonzero eigenvalues $\lambda$ for ensembles of general rate matrices, assuming an exponential distribution of rates (EGRE). Third column: leading term of $\mu_m$ for large $N$.

| $m$ | $\mu_m$ (EGRE) | $\mu_m$ (EGRE, $N \gg 1$) |
|-----|----------------|----------------------------|
| 2   | $N\langle R \rangle^2$ | $N\langle R \rangle^2$ |
| 3   | $-2N\langle R \rangle^3$ | $-2N\langle R \rangle^3$ |
| 4   | $3N\langle N + 2 \rangle\langle R \rangle^4$ | $3N^2\langle R \rangle^4$ |
| 5   | $-4N(5N + 6)\langle R \rangle^5$ | $-20N^2\langle R \rangle^5$ |
| 6   | $N(15N^2 + 126N + 128)\langle R \rangle^6$ | $15N^3\langle R \rangle^6$ |
| 7   | $-6N(35N^2 + 140N + 148)\langle R \rangle^7$ | $-210N^4\langle R \rangle^7$ |
| 8   | $N(105N^3 + 2290N^2 + 6270N + 7476)\langle R \rangle^8$ | $105N^4\langle R \rangle^8$ |
| 9   | $-8N(315N^3 + 2953N^2 + 6741N + 9018)\langle R \rangle^9$ | $-2520N^5\langle R \rangle^9$ |
| 10  | $N(945N^4 + 42494N^3 + 249174N^2 + 532840N + 774744)\langle R \rangle^{10}$ | $945N^5\langle R \rangle^{10}$ |

To pinpoint the origin of the anomalous scaling, we have also evaluated $f_R$ for ensembles of matrices of dimension $N = 5, 50, 500$ satisfying the constraint $\langle \lambda \rangle = 0$, but with Gaussian distribution of rates $A_{ij}$, $i \neq j$. This is the asymmetric analogue of the symmetric ensemble studied by Stärig et al. [24]. The results are shown as crosses in Fig. 5. They clearly approach the EGRE results for large $N$, not the GinOE. It is thus the constraint $\langle \lambda \rangle = 0$ that leads to the anomalous scaling.

In the following, we will consider the real and complex eigenvalues separately. Figure 6 shows the density $\rho_R$ of shifted real nonzero eigenvalues $\tilde{\lambda}$, normalized to unity and rescaled with the square root of the pseudomoment $\sqrt{\mu_2} = \sqrt{N\langle R \rangle^2}$, for $N = 2, 10, 100, 1000, 5000$. For $N = 2$, the single nonzero eigenvalue is $\lambda = -A_{12} - A_{21}$. In the EGRE, its distribution function is $\rho_R(\lambda) = (2\langle R \rangle - \lambda\langle R \rangle^2)\exp(\lambda\langle R \rangle - 2)\lambda ^{\leq 2\langle R \rangle}$ and zero otherwise. For the other values of $N$, Fig. 6 shows numerical noise. The noise increases for large $N$, not only because $n_rN$ was smaller for $N = 5000$ but also because $\rho_R$ decreases with increasing $N$. It is obvious however that the distribution for large $N$ is quite different from the eigenvalue density for the ESRE, Fig. 1.

The distribution clearly becomes more symmetric for $N \to \infty$, as it must, since the large-$N$ result only depends on the width of the distribution of rates $A_{ij}$. There is an indication that the distribution develops non-analyticities with sudden changes of slope in the limit $N \to \infty$. This is not unexpected, since the scaled distribution of real eigenvalues of the GinOE is uniform on the interval $[-1, 1]$ and zero otherwise [24, 33] and thus also shows non-analyticities. Compared to the ESRE (Fig. 1), the convergence to the large-$N$ limit is slower for the EGRE (Fig. 6). In fact, from Fig. 6 we cannot exclude the possibility that the width scales with an anomalous power of $N$, different from $1/2$.

Turning to complex eigenvalues, we note that for large $N$ nearly all eigenvalues belong to this class, since the fraction $f_R$ of real eigenvalues approaches zero. We plot their distribution function $p_C$ in the complex plane for $N = 100$ and 2000 in Fig. 7. The scaled distribution for $N = 5000$ is virtually indistinguishable from the one for $N = 2000$. From Figs. 4 and 7 we see that the distribution becomes more symmetric with respect to inversion of the real part as $N$ increases.

The widths of the distribution in the real direction, $\sqrt{\mu_2}$, and in the imaginary direction, $\sqrt{\mu_4}$, see Eq. (25), both scale with $\sqrt{N\langle R \rangle^2}$. This means that the typical decay rate is $\langle \lambda_f \rangle = N\langle R \rangle$, whereas the typical oscillation frequency is of the order of $\sqrt{N\langle R \rangle}$. For large $N$ it will thus be difficult to observe the oscillations.
FIG. 5: (Color online) Fraction $f_R$ of nonzero eigenvalues that are real, as a function of $N$ for the EGRE. The solid circles denote numerical values obtained for $n_r$ realizations with $n_rN = 4 \times 10^7$ for $N \leq 2000$, $n_rN = 10^7$ for $N = 5000$, and $n_rN = 4 \times 10^5$ for $N = 10000$. The solid square represents the exact result $f_R = 1$ for $N = 2$. The dashed line denotes a power law $f_R \propto N^{-\alpha}$ fitted to the two points for $N = 2000$ and $N = 5000$. The solid line is the exact result for the GinOE, Eq. (29). The crosses denote numerical results for ensemble of rate matrices with Gaussian instead of exponential distribution of rates $A_{ij}$, $i \neq j$.

FIG. 6: (Color online) Scaled density of nonzero real eigenvalues of shifted general rate matrices $\tilde{A}$ of dimension (a) $N = 100$ and (b) $N = 2000$. Specifically, two-dimensional histograms with $500 \times 500$ bins were populated for $n_r$ matrices randomly chosen from the EGRE, where $n_rN = 4 \times 10^7$. Note the different scales of the axes.

FIG. 7: (Color online) Scaled distribution function of complex eigenvalues $\tilde{\lambda}$ of shifted general rate matrices $\tilde{A}$ of dimension (a) $N = 100$ and (b) $N = 2000$. We observe that for the EGRE the complex eigenvalues are repelled by the real axis with the same exponent of unity. We note that the distribution of the real part of complex eigenvalues is distinct from both the distribution of real eigenvalues, Fig. 6, and the distribution of eigenvalues for the ESRE, Fig. 1.

It is instructive to compare the distribution to the one for the GinOE. For the GinOE, the distribution function $\rho_C$ of complex eigenvalues for finite $N$ has been obtained by Edelman \cite{31} in terms of a finite sum of $N-1$ terms, which can be rewritten as a simple integral \cite{33}. The distribution function $\rho_C$ is found to contain a factor $|\text{Im} \tilde{\lambda}|$, showing that the density goes to zero linearly for $\tilde{\lambda}$ approaching the real axis. Complex eigenvalues are thus repelled by the real axis with a characteristic exponents of unity. Figures 4 and 7 clearly show that complex eigenvalues are also repelled by the real axis for the EGRE. In Fig. 8 we plot the density of complex eigenvalues, projected onto the real and imaginary axes, for $N = 100$ and $N = 2000$. We observe that for the EGRE the complex eigenvalues are repelled by the real axis with the same exponent of unity. We note that the distribution of the real part of complex eigenvalues is distinct from both the distribution of real eigenvalues, Fig. 6, and the distribution of eigenvalues for the ESRE, Fig. 1.

For the GinOE, the scaled distribution approaches a uniform distribution on the unit disk in the complex plane for $N \to \infty$. This was conjectured by Girko \cite{28} for an arbitrary distribution of components with zero mean and proven by Bai \cite{30}. The EGRE result is clearly much more complicated. The histograms for various values of
Fig. 7 onto the real and imaginary axes. The inset shows the $N = 2000$. The curves are projections of the data shown in values of shifted general rate matrices with $N$ lines) and the imaginary part (dashed lines) of complex eigenvalues, while the $N$ eigenvalues, while the typical imaginary part of $\lambda_1$, i.e., the oscillation frequency, decreases for large $N$, mainly because the probability of $\lambda_1$ being real increases. While the fraction of real eigenvalues approaches zero for large $N$, the eigenvalue with the largest real part becomes more likely to be real.

To end this section, we again consider the slowest process. The dynamics at late times is typically governed by the eigenvalue $\lambda_1$ with the largest (smallest in magnitude) real part. In Fig. 9 we show the mean of the real part $\mathbb{E} \lambda_1$ and of the magnitude of the imaginary part, $\mathbb{E} |\lambda_1|$ for random matrices from the ESRE, as functions of $N$. The behavior of the real part, i.e., the rate, is very similar to the ESRE. Again, the slowest rate is consistent with the mean and width of the eigenvalue distribution $\rho(\lambda)$. The typical imaginary part of $\lambda_1$, i.e., the oscillation frequency, decreases for large $N$, mainly because the probability of $\lambda_1$ being real increases. While the fraction of real eigenvalues approaches zero for large $N$, the eigenvalue with the largest real part becomes more likely to be real.

B. Eigenvalue correlations

The eigenvalue density for the ESRE is quite different from the GinOE. Like for the ESRE, we again ask whether the eigenvalue correlations are also different. We consider the real and complex eigenvalues separately. The main effect of correlations between real and complex eigenvalues is seen in Fig. 8. The complex eigenvalues are repelled by the real axis with a characteristic exponent of unity.
Figure 10 shows the distribution function $\rho_{NN}^R(\Delta \lambda)$ of separations of neighboring real eigenvalues. Note that the distribution is not rescaled with a power of $N$. The typical separation of real eigenvalues depends only weakly on $N$ for large $N$ for the EGRE, whereas it scales with $N^{-1/2}$ for the ESRE. This can be understood as follows: The expected number of real eigenvalues of a randomly chosen matrix is $N f_R \sim N^{1-\alpha}$, while the width of their distribution scales with $N^{1/2}(R)$. Consequently, the typical nearest-neighbor separation should scale with $N^{\alpha-1/2}(R)$. Since $\alpha$ is close to $1/2$, we obtain a weak dependence on $N$. The dependence on separation $\Delta \lambda$ is again linear for small $\Delta \lambda$, though. Thus real eigenvalues repel each other with a characteristic exponent of unity, like for the GinOE [33].

![Figure 10](image1.png)

**FIG. 10:** (Color online) Distribution of nearest-neighbor separations $\Delta \lambda$ of nonzero real eigenvalues for the EGRE for $N = 10, 100, 1000, 5000$ for the same data sets as in Fig. 8. The axes are not rescaled with a power of $N$.

In Figs. 11(a) and (b), we plot the distribution function $\rho_{NN}(\Delta \lambda)$ of complex differences of neighboring eigenvalues with positive imaginary part for $N = 20$ and $N = 2000$. More specifically, for each eigenvalue $\lambda$ with positive imaginary part, we determine the eigenvalue $\lambda'$ with positive imaginary part that minimizes $|\lambda' - \lambda|$. We then collect the complex differences $\Delta \lambda \equiv \lambda' - \lambda$ of all such pairs in a two-dimensional histogram. The eigenvalues with negative imaginary part just form a mirror image. Correlations between eigenvalues with positive and negative imaginary parts are dominated by their repulsion by the real axis and a $\delta$-function from complex conjugate pairs and are not considered further.

Since the fraction of complex eigenvalues approaches unity for $N \to \infty$, the number of complex eigenvalues of a chosen matrix scales with $N$. The widths of the distribution in both the real and the imaginary direction scale with $\sqrt{N}$, see Fig. 7. The typical nearest-neighbor distance should thus approach a constant for large $N$. This is indeed seen in Fig. 11. We observe that the distribution of differences becomes rotationally symmetric for large $N$. This is perhaps surprising since the distribution of the eigenvalues themselves is far from symmetric, see Fig. 7. Also, small differences are suppressed, i.e., the eigenvalues repel each other. To find the characteristic exponent, we plot the distribution of the magnitudes $|\Delta \lambda| = |\lambda' - \lambda|$ of differences of neighboring eigenvalues in Fig. 12. We observe that the distribution behaves like $|\Delta \lambda|^\alpha$ for small $|\Delta \lambda|$. Together with the rotational symmetry this implies that the two-dimensional distribution in the complex plane, Fig. 11(b), approaches zero like $|\Delta \lambda|^2$. The exponent of two is the same as for the GinOE [33]. We conclude that the constraint [5] and the exponential distribution of rates in the EGRE do not change the repulsion of neighboring eigenvalues compared to the GinOE, while the eigenvalue density is very different. The origin of this is likely the same as to the ESRE: The correlations are governed by “local” properties of the joint distribution function of eigenvalues, which are not strongly affected by the constraint.

![Figure 11](image2.png)

**FIG. 11:** (Color online) Distribution function of complex differences $\Delta \lambda$ of neighboring eigenvalues with positive imaginary part for the EGRE for (a) $N = 20$ and (b) $N = 2000$. 

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* [33]: Reference to the source material.
For symmetric rate matrices, the density of eigenvalues has been studied numerically as a function of $N$ and found to approach the same limiting form for $N \to \infty$ as obtained earlier for Gaussian and two-valued distributions \cite{22,23,24}, but very different from the semi-circle law for the GOE \cite{16,19}. This difference is due to the constraint \cite{4}. On the other hand, the correlations between eigenvalues are dominated by a repulsion with a characteristic exponent of unity, as for the GOE.

For general rate matrices, we have numerically studied the eigenvalue density in the complex plane as a function of $N$. For large $N$, it approaches a non-trivial distribution different from the disk found for the GinOE \cite{28,30}. Interestingly, the fraction of nonzero eigenvalues that are real decays as $N^{-\alpha}$ with an anomalous exponent $\alpha \approx 0.460$, unlike for the GinOE, where $\alpha = 1/2$. Thus the fraction of eigenvectors describing purely exponentially decaying deviations from the stationary state scales with a nontrivial power of the number of possible states. Both the non-trivial distribution and the anomalous scaling for large $N$ are due to the constraint \cite{5}. The density of real eigenvalues is also different from the GinOE. We have obtained simple analytical results for the expectation values $\langle (\lambda - \langle \lambda \rangle)^m \rangle = (m-1)! (N\langle R^2 \rangle)^{m/2}$ of all even powers of shifted nonzero eigenvalues in the limit of large $N$. Interestingly, they agree with the central moments of a real Gaussian distribution. The central moments of the eigenvalue density in the complex plane are shown to satisfy exact sum rules involving these expectation values.

Correlations between eigenvalues are found to agree with the GinOE: Real eigenvalues repel each other with an exponent of unity, complex eigenvalues are repelled by the real axis with an exponent of unity and by each other with an exponent of two.

In view of the power of RMT for Hamiltonians, we hope that this approach will also benefit our understanding of complex stochastic processes. Comparisons with real processes are now called for.

**APPENDIX A: LARGE-$N$ LIMIT FOR SYMMETRIC RATE MATRICES**

In the limit of large $N$, the density of eigenvalues $\hat{\lambda}$ of $\hat{A}$ only depends on the second moment $\langle R^2 \rangle$ of the distribution of components $\hat{A}_{ij}$, $i \neq j$, for any distribution function of $\hat{A}_{ij}$, as long as all its central moments exist. In this appendix, we sketch the proof of this statement.

The eigenvalue density is given by Eq. (18). In the expansion of the geometric series for the resolvent \cite{26},

$$
\langle \hat{G}(z) \rangle = \sum_{n=0}^{\infty} \frac{\text{Tr}(\hat{A}^n)}{z^{n+1}},
$$

(A1)

the $n = 0$ term is independent of the distribution of $\hat{A}_{ij}$, while the $n = 1$ term vanishes. Since $\sum_i (\hat{A}^n)_{ij} = 0$ for
\[ n \geq 1 \text{ we can write} \]
\[ \langle \tilde{G}(z) \rangle = \frac{1}{z} - \sum_{n=2}^{\infty} \frac{1}{z^{n+1}} \sum_{i,j,i \neq j} \langle \tilde{A}^n \rangle_{ij} \]
\[ = \frac{1}{z} - \sum_{n=2}^{\infty} \frac{1}{z^{n+1}} \sum_{i,j,i \neq j} \sum_{k_1,k_2,...} \langle \tilde{A}_{ik_1} \tilde{A}_{k_1 k_2} \cdots \tilde{A}_{k_{n-1} j} \rangle. \]
\[ (A2) \]

We now introduce a diagrammatic representation for the expectation values \( \langle \tilde{A}^n \rangle_{ij}, i \neq j \):
\[ \equiv \sum_{i,j,i \neq j} \langle \tilde{A} \rangle_{ij} = 0, \quad (A3) \]
\[ \equiv \sum_{i,j,i \neq j} \langle \tilde{A}^2 \rangle_{ij} = \sum_{i,j,i \neq j} \sum_{k} \langle \tilde{A}_{ik} \tilde{A}_{kj} \rangle, \quad (A4) \]
\[ \equiv \sum_{i,j,i \neq j} \langle \tilde{A}^4 \rangle_{ij} \text{ etc.} \quad (A5) \]

Here, an arrow represents a factor of \( \tilde{A} \), a vertex (filled circle or cross) represents a matrix index, and all indices are summed over \( 1, \ldots, N \), subject to the constraint that indices corresponding to filled circles are distinct. Vertices drawn as crosses do not imply any constraint.

In Eq. (A2), we now decompose the sums over indices into terms with equal and distinct indices. For equal indices we attach the arrows to the same filled-circle vertex, whereas distinct indices are denoted by distinct filled-circle vertices. For example,
\[ \sum_{i,j,i \neq j} \langle \tilde{A}^2 \rangle_{ij} = \equiv \sum_{i,j,i \neq j} \langle \tilde{A} \rangle_{ij} \]
\[ = \sum_{i,j,i \neq j} \langle \tilde{A}_{ij} \rangle + \sum_{i,j,i \neq j} \langle \tilde{A}_{ij} \rangle \quad (A6) \]
The constraint \( \tilde{A}_{jj} = - \sum_{i \neq j} \tilde{A}_{ij} \) assumes the form
\[ \equiv = - \quad (A7) \]
where the open circle denotes an index that is different from the one connected to it but not otherwise constrained. Applying this rule to all terms, we obtain open-circle vertices, which we dispose of by again distinguishing between equal and distinct indices. For example,
\[ \sum_{i,j,i \neq j} \langle \tilde{A}^2 \rangle_{ij} = \equiv \sum_{i,j,i \neq j} \langle \tilde{A} \rangle_{ij} \]
\[ = \sum_{i,j,i \neq j} \langle \tilde{A}_{ij} \rangle + \sum_{i,j,i \neq j} \langle \tilde{A}_{ij} \rangle \quad (A8) \]

We have achieved that factors of \( \tilde{A} \) with two equal indices are no longer present and that all indices to be summed over are distinct.

Since different off-diagonal components \( \tilde{A}_{ij} \) are independent, except for \( \tilde{A}_{ii} = \tilde{A}_{ij} \), the expectation value of each term decays into a product of expectation values of powers of components, \( \langle \delta R^m \rangle = \langle (\tilde{A}_{ij})^m \rangle \). The corresponding diagrams are of the forms
\[ \equiv = 0, \quad (A9) \]
\[ \equiv = \equiv = \langle \delta R^2 \rangle, \quad (A10) \]
\[ \equiv = \equiv = \langle \delta R^3 \rangle \text{ etc.} \quad (A11) \]

Finally, any term containing \( m \) vertices obtains a factor \( N(N-1)(N-2) \cdots (N-m+1) \) from the sum over distinct indices. In the limit of large \( N \) this becomes \( N^m \).

We conclude that at any order \( n \geq 2 \) in Eq. (A2), the largest terms for large \( N \) are the non-vanishing ones with the maximum number of vertices. Note that the diagrams generated by this procedure are always connected. Diagrams containing single arrows connecting two vertices vanish because of Eq. (A9). For even \( n \), the maximum number of vertices is \( n/2 + 1 \), which is obtained if all connections are double arrows. In this case the contribution is proportional to \( N^{n/2+1} \langle \delta R^2 \rangle^{n/2} \). The next smaller terms have two triple arrows and contribute \( \propto N^{n/2} \langle \delta R^2 \rangle^{n/2-3} \delta R^2 \). For odd \( n \), the largest terms have one triple arrow and all other connections are double arrows. Their contribution is proportional to \( N^{n/2+1/2} \langle \delta R^2 \rangle^{n/2-3/2} \delta R^3 \).

Since Eq. (13) contains an explicit factor of \( 1/N \), the leading contributions to the density scale as \( N^{n/2} \) (\( N^{n/2-1/2} \) for even (odd) \( n \)). If we rescale the density so that the width approaches a constant, the odd terms in the expansion (A2) vanish like \( N^{-1/2} \), showing that the rescaled density approaches an even function. Furthermore, the leading even terms only depend on the second moment \( \langle \delta R^2 \rangle \), which is what we set out to prove.

Rewriting Eq. (A1) in terms of the moments \( \mu_n \),
\[ \langle \tilde{G}(z) \rangle = \frac{1}{z} + (N-1) \sum_{n=2}^{\infty} \frac{\mu_n}{z^{n+1}}, \quad (A12) \]
we see that the terms of order \( n \) contribute exclusively to the moment \( \mu_n \). The result proved here is consistent with the calculated moments in Table II.

### APPENDIX B: LARGE-\( N \) LIMIT FOR GENERAL RATE MATRICES

For ensembles of general, asymmetric rate matrices, it is also true that the density of eigenvalues only depends on the second moment \( \langle \delta R^2 \rangle \) for large \( N \). We here sketch the proof of this assertion.
The distribution of eigenvalues in the complex plane is given by Eqs. (26)–(28). We define
\[ g(\eta; z, z^*) \equiv \text{Tr}_{2N} \left( \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \langle G(0; z, z^*) \rangle \right) \] (B1)
so that \( \rho(x, y) = (1/\pi N) \partial g(0; z, z^*)/\partial z^* \) and expand the resolvent,
\[ g(\eta; z, z^*) = \sum_{\eta \text{ odd}} \frac{1}{\eta^{n+1}} \text{Tr} \left( [(\tilde{A}^T - z^* I)(\tilde{A} - z I)]^n \right) \times (\tilde{A}^T - z^* I). \] (B2)

Expanding the products, we obtain a linear combination of expressions of the form \( \text{Tr} \langle \cdots \tilde{A}^T \cdots \tilde{A} \cdots \rangle \) containing any number of factors \( \tilde{A}^T \) and \( \tilde{A} \) in any order. Now the arguments of App. A go through with few changes. We can group the terms according to the total order \( m \) of \( \tilde{A} \) and \( \tilde{A}^T \). The term of order zero is independent of the distribution of \( A_j \). The terms of first order are \( \text{Tr} \langle \tilde{A} \rangle = \text{Tr} \langle \tilde{A}^T \rangle = 0 \). In all other terms we can use cyclic permutation under the trace and the identity \( \text{Tr} B^T = \text{Tr} B \) to make sure that a factor \( \tilde{A} \) and not \( \tilde{A}^T \) is appearing first under the trace. We can then use Eq. (B1) to write \( \text{Tr} \langle \tilde{A} \cdots \rangle = - \sum_{i,j, i \neq j} \langle \tilde{A} \cdots \rangle_{ij} \).

Now we can apply the diagrammatics of App. A where \( (\tilde{A}^T)_{ij} = A_{ji} \) is drawn as an arrow pointing in the opposite direction. In the evaluation of expectation values corresponding to Eqs. (A10), (A11) we have to take into account that \( A_{ij} \) and \( \tilde{A}_{ji} \) are now independent so that we instead have
\[ = 0, \] (B3)
\[ = \langle \delta R^2 \rangle, \] (B4)
\[ = 0, \] (B5)
\[ = \langle \delta R^2 \rangle, \] (B6)
\[ ... = 0 \text{ etc.} \] (B7)

We note that all terms of the same order \( m \) in Eq. (B2) have the same sign and thus cannot cancel. We thus find that to any order \( m \) the leading terms in Eq. (B2) for large \( N \) have the same form as for symmetric matrices. In particular, for even \( m \) the leading term in the density \( \rho(x, y) \) scales with \( N^{m/2} \langle \delta R^2 \rangle^{m/2} \) and the odd terms scale with a lower power of \( N \). Finally, it is conceivable that taking the derivative of \( g(0; z, z^*) \) with respect to \( z^* \) in order to obtain the density could remove the leading-\( N \) term. This is not the case, since for any even order \( m \geq 2 \) there is at least a contribution from \( m = n - 1 \) in Eq. (B2), which is linear in \( z^* \).

**APPENDIX C: PSEUDOMOMENTS FOR THE EGRE**

In this appendix, we use the diagrammatics of App. A to calculate the pseudomoments
\[ \mu_m = \langle \tilde{A}^m \rangle' = \frac{1}{N-1} \text{Tr} \langle \tilde{A}^m \rangle, \] (C1)
\[ m \geq 2, \] to leading order for large \( N \) for the EGRE. Appendix B shows that for large \( N \) only the even pseudomoments are relevant. We write
\[ \mu_m = -\frac{1}{N-1} \sum_{i,j, i \neq j} \langle \tilde{A}^m \rangle_{ij} \]
\[ = -\frac{1}{N-1} \sum_{i,j, i \neq j} \sum \langle \tilde{A}_{ik_1} \tilde{A}_{k_1 k_2} \cdots \tilde{A}_{k_{m-1} j} \rangle. \] (C2)

It was shown in App. B that for large \( N \) the distribution of \( A_j \) only enters through its second moment \( \langle \delta R^2 \rangle \). We decompose all terms into a sum of contributions with equal or distinct indices, see Eq. (A10). For each term, some or none of the indices in \( \{i, k_1, k_2, \ldots, k_{m-1}, j\} \) are equal. Contributions for which two equal indices are separated by other, distinct indices in this string, correspond to diagrams of the type
\[ \text{(C3)} \]
and are of lower order in \( N \). All remaining diagrams are of the form of chains leading from \( j \) to \( i \) with any number of single-vertex loops \( (\tilde{A}_{kk}) \) decorating the vertices. We call these single-vertex loops “leaves.”

Next, we prove
\[ \text{(C4)} \]
for \( N \to \infty \), where the left-most vertex in the first term carries \( l \geq 1 \) leaves, while the second vertex in the second term carries \( l - 1 \geq 0 \) leaves. The shaded circle is an arbitrary diagram part. The proof proceeds as follows: Applying the rule (A7), we obtain
\[ \text{(C5)} \]
with the upper (lower) signs for even (odd) \( l \). In the leading large-\( N \) term, all connections must be of the form of two arrows pointing in the same direction, as in Eq. (B4). This is only possible if we pair up the open-circle vertices among themselves, not with any vertices in the right-hand part of the diagrams. This requires \( l \) to be even. Furthermore, for the first diagram there are \((l - 1)!\) ways to partition \( l \) leaves into pairs. For the second diagram
there are \(l-1\) ways to pair one of the leaves with the leftmost vertex and \((l-3)!!\) ways to partition the remaining \(l-2\) leaves into pairs. With these factors we obtain

\[
(l-1)!! - (l-1)(l-3)!! = 0. \tag{C6}
\]

All diagrams of leading order in \(N\) are of the form of one of the two diagrams in Eq. \((\text{C4})\). Thus all diagrams cancel, except if only one of the two forms exists. This is only the case for

\[
\begin{array}{c}
\text{...}\\
\text{...}
\end{array}
\] \(\cong -(m-1)!! \) \[\text{...}\\\text{...}\]

\[
\begin{array}{c}
\text{...}\\
\text{...}
\end{array}
\] \(\cong -(m-1)!! N^{m/2+1} (\delta R^2)^{m/2}, \tag{C7}\)

since its partner would contain only a single vertex, which is excluded by \(i \neq j\).

With the prefactor from Eq. \((\text{C2})\), we obtain

\[
\mu_m \cong (m-1)!! N^{m/2} (\delta R^2)^{m/2}. \tag{C8}\]