Quantum Critical Phenomena in Heat Transport via a Two-State System

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We theoretically study quantum critical behavior in heat transport via a two-state system with sub-ohmic reservoirs. We calculate temperature dependence of thermal conductance near quantum phase transition using the continuous-time quantum Monte Carlo method, and discuss its critical exponents. We also propose a superconducting circuit to realize the sub-ohmic spin-boson model, which can be used for observation of quantum critical phenomena.

I. INTRODUCTION

Quantum critical phenomena (QCP) induced by second-order quantum phase transition (QPT) has been one of central topics in condensed matter physics [1]. Although QPT has been studied in various highly-correlated systems, it is still challenging to realize them in controlled experimental systems. Recently, QCP have been studied for the multi-channel Kondo effect realized in artificial nano-structures [2–7], and quantum critical behavior has been observed via electronic transport properties in good agreement with theories [8–10]. This great success encourages further study of QCP in transport properties using different mesoscopic systems.

Heat transport through nano-structures is another important topics in mesoscopic physics. In particular, heat transport via a two-state system can be realized by phonons (photons) has been studied in several theoretical works [11–19], because it has several similarities to electronic transport through quantum dots. The heat transport via a two-state are described by the spin-boson model, whose properties are characterized by a spectral density function $I(\omega) \propto \omega^s$ [20, 21]. For the sub-ohmic reservoirs ($0 < s < 1$), this model displays QPT at zero temperature when a system-reservoir is tuned at a critical value [21, 29]. In the recent paper by the authors and the other two co-authors [30], temperature dependence of thermal conductances has been studied in detail for all types of the reservoirs (arbitrary $s$) by the continuous-time quantum Monte Carlo (CTQMC) method. For the sub-ohmic reservoirs, however, QCP near the transition point has not been discussed.

It is remarkable that recent rapid development in experimental techniques of nanostructure fabrication and heat measurement has enabled us to access heat current through artificial nano-scale objects experimentally [31, 33]. It has been demonstrated that transmission lines coupled to a superconducting qubit indeed realize the spin-boson model with the ohmic ($s = 1$) reservoir [31, 32, 34–37]. In our knowledge, however, design of the superconducting circuit for realizing of the sub-ohmic spin-boson model has discussed only in Ref. [38], in which the experimental realization of the sub-ohmic reservoirs of $s = 0.5$ was discussed. For study of QCP, it is favorable to consider realization of the sub-ohmic spin-boson model for an arbitrary value of $s$.

In this paper, we investigate QCP in heat transport via a two-state system carried by phonons (photons) for the sub-ohmic reservoirs. We calculate temperature dependence of thermal conductance in the critical regime using the CTQMC method [24, 30, 39], and discuss how its critical exponent is determined. We also consider a superconducting circuit to realize the sub-ohmic spin-boson model with an arbitrary value of $s$.

This paper is organized as follows. We describe the present spin-boson model in Sec. II, and formulate heat current via a two-state system in Sec. III. We show critical temperature dependence of the heat current near quantum phase transition in Sec. IV; this is our main result. We propose a superconducting circuit to realize the spin-boson model with sub-ohmic reservoirs in Sec. V. We summarize our results in Sec. VI. Throughout this paper, we employ the unit of $k_B = \hbar = 1$.

II. MODEL

We consider heat transport between two bosonic reservoirs via a two-state system (see Fig. 1). The model Hamiltonian is given by $H = H_S + \sum_{\nu} H_{B,\nu} + \sum_{\nu} H_{I,\nu}$, where $H_S$, $H_{B,\nu}$, and $H_{I,\nu}$ describe a two-state system, a bosonic reservoir $\nu \ (\equiv L, R)$, and a system-reservoir coupling, respectively. Each term of the Hamiltonian is

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{A schematic of the model considered herein, composed of a two-state system coupled to two bosonic reservoirs ($L$ and $R$), with temperatures $T_L$ and $T_R$, respectively. If $T_L < T_R$, a heat current flows from the reservoir $L$ to the reservoir $R$ via the two-state system.}
\end{figure}
given as

\[ H_S = -\frac{\Delta}{2} \sigma_z - \varepsilon \sigma_z, \]  
\[ H_{B,\nu} = \sum_k \omega_{vk} b_k^\dagger b_k, \]  
\[ H_{L,\nu} = -\frac{\alpha}{2} \sum_k \lambda_{vk} (b_k^\dagger b_{\nu k} + b_{\nu k} b_k), \]

respectively, where \( \sigma_{\alpha} \) (\( \alpha = x, y, z \)) is the Pauli matrix, and \( b_{\nu k} (b_k^\dagger) \) is an annihilation (a creation) operator of bosonic excitation with the wavenumber \( k \) in the reservoir \( \nu \). The Hamiltonian of the two-state system, \( H_S \), is obtained by truncating a double-well potential system with the lowest two eigenstates, where \( \Delta \) and \( \varepsilon \) are a tunneling amplitude and a detuning energy, respectively. The energy dispersion of the reservoirs and the system-reservoir coupling strength are denoted with \( \omega_{vk} \) and \( \lambda_{vk} \), respectively. Throughout this paper, we consider heat transport for the symmetric case \( (\varepsilon = 0) \). The detuning energy \( \varepsilon \) is used only for detailed discussion on critical exponents in Appendix A.

The property of the reservoirs is determined by the spectral density function

\[ I_\nu(\omega) = \sum_k \lambda_{\nu k}^2 \delta(\omega - \omega_{vk}). \]

For simplicity, the spectral density function is taken in the form

\[ I_\nu(\omega) = \alpha_\nu \tilde{I}(\omega), \]
\[ \tilde{I}(\omega) = 2\omega_c^{-1} \omega^s e^{-\omega/\omega_c}, \]

where \( \alpha_\nu \) is a dimensionless system-reservoir coupling strength, and \( \omega_c \) is a cutoff frequency which takes as much larger than other characteristic energies. Herein, we focus on the sub-ohmic case \( (0 < s < 1) \), for which second-order quantum phase transition occurs.

### III. FORMULATION

The heat current operator from the reservoir \( \nu \) into the two-state system is defined as

\[ J_\nu \equiv -\frac{dH_{B,\nu}}{dt} = i[H_{B,\nu}, H] = -i\frac{\alpha}{2} \lambda_{\nu k} \omega_k (b_{\nu k} + b_k^\dagger). \]

With the standard procedure of the Keldysh formalism, the Meir-Wingreen-Landauer-type exact formula [13] for the heat current is derived as follows [13 14 15]:

\[ \langle J_L \rangle = \frac{\alpha \gamma_\alpha}{8} \int_0^\infty d\omega \omega \text{Im} \{\chi(\omega)\} \tilde{I}(\omega) \{n_L(\omega) - n_R(\omega)\}, \]

where \( \alpha = \alpha_L + \alpha_R, \gamma_\alpha = 4\alpha_L \alpha_R / \alpha^2 \) is an asymmetric factor, \( n_\nu(\omega) \) is the Bose-Einstein distribution in the reservoir \( \nu \), and \( \chi(\omega) \) is the dynamic susceptibility of the two-state system defined by

\[ \chi(\omega) = -i \int_0^\infty dt e^{i\omega t} \langle [\sigma_z(t), \sigma_z(0)] \rangle. \]

The thermal conductance is obtained from Eq. (8) as

\[ \kappa = \lim_{\Delta T \to 0} \frac{\langle J_L \rangle}{\Delta T} = \frac{\alpha \gamma_\alpha}{8} \int_0^\infty d\omega \text{Im} \{\chi(\omega)\} \tilde{I}(\omega) \left[ \frac{\beta \omega/2}{\sinh(\beta \omega/2)} \right]^2, \]

where \( \Delta T = T_L - T_R \), and \( \beta = 1/T \) (= \( 1/T_L = 1/T_R \)). To evaluate the thermal conductance, one needs to calculate the dynamic susceptibility \( \chi(\omega) \) in thermal equilibrium.

We numerically calculate the dynamic susceptibility \( \chi(\omega) \) by the CTQMC method (for details of the CTQMC method, refer Refs. [24 30]). Using the CTQMC method, we calculate the spin-spin correlation function \( C(\tau) = \langle \sigma_z(\tau) \sigma_z(0) \rangle_{\text{eq}} \), where \( \sigma_z(\tau) \) is the imaginary time path \( (0 < \tau < \beta) \), and \( \langle \cdots \rangle_{\text{eq}} \) indicates thermal average. The dynamic susceptibility is obtained as

\[ \tilde{C}(i\omega_n) = \int_0^\beta d\tau C(\tau), \]
\[ \chi(\omega) = \tilde{C}(i\omega_n \to \omega + i\delta). \]

The analytic continuation is performed numerically by the Padé approximation [46 47].

### IV. RESULT

For the sub-ohmic case \( (0 < s < 1) \), there occurs quantum phase transition at zero temperature when the reservoir-system coupling reaches a critical value \( \alpha_c \), where \( \alpha_c \) is a function of \( s \) and \( \Delta/\omega_c \) [21 23 25]. For \( \alpha < \alpha_c \), the ground state is described by a coherent superposition of two wave functions localized at each well \( (\sigma_z = \pm 1) \), and is called a “delocalized state”. For \( \alpha > \alpha_c \), the ground-state becomes two-fold degenerate because the coherent superposition is completely broken due to disappearance of quantum tunneling between the two wells. This state is called a “localized state”. The phase diagram of the spin-boson model determined by the CTQMC method is shown in Fig. 2 for \( \Delta/\omega_c = 0.1 \) (for details of determination of the critical value \( \alpha_c \), refer Refs. [24 30 48]). The transition separating the two phases is of second-order for the sub-ohmic case (the empty squares) or of Kosterlitz-Thouless-type [49 50] for the ohmic case (the filled circle). This phase diagram is consistent with the previous numerical studies [24 25].

In Fig. 3, we show the temperature dependence of the thermal conductance for \( s = 0.5 \) and \( \Delta/\omega_c = 0.1 \), where the critical system-reservoir coupling is \( \alpha_c = 0.1074 \). Fig. 3 (a) and (b) show the delocalized-phase side \( (\alpha \leq \alpha_c) \) and the localized-phase side \( (\alpha \geq \alpha_c) \), respectively.
FIG. 2. The phase diagram of the sub-ohmic spin-boson model for $\Delta/\omega_c = 0.1$. The solid line indicates the second-order transition line separating the delocalized and localized phases. The empty squares indicate the critical system-reservoir coupling numerically determined for the sub-ohmic case ($0 < s < 1$), whereas the filled circle represents the known transition point $\alpha_c = 1$ for the ohmic case ($s = 1$).

In general, at the critical point, the thermal conductance shows the power-law behavior:

$$\kappa \propto T^c, \quad (\alpha = \alpha_c),$$

where $c$ is a critical exponent dependent on $s$. As shown in Fig. 3, the exponent $c$ is 1 for $s = 0.5$. As the system-reservoir coupling is reduced below the critical value ($\alpha < \alpha_c$), the temperature dependence of the thermal conductance deviates from one at the critical point. For a sufficiently small system-reservoir coupling (e.g. $\alpha = 0.07$ in Fig. 3(a)), the thermal conductance becomes proportional to $T^{2s+1}$ at low temperature as expected for heat transport due to co-tunneling (see Appendix B). The temperature dependence of the thermal conductance also deviates as the system-reservoir coupling is increased above the critical value ($\alpha > \alpha_c$). Its temperature dependence cannot be fit by a simple formula such as the non-interacting blip approximation, which is expected to hold in the localized phase [30], up to $\alpha = 0.13$.

Let us discuss the critical exponent $c$ defined in Eq. (13) for general values of $s$. The spin susceptibility is expressed by

$$\chi_0 = \beta \langle \bar{m}^2 \rangle_{eq},$$

$$\bar{m} = \frac{1}{\beta} \int_0^\beta d\tau \sigma_z(\tau).$$

Combining Eq. (14) with Eq. (15), the spin susceptibility is expressed as $\chi_0 = \int_0^\beta d\tau C(\tau)$ with the spin-spin correlation function $C(\tau) = \langle \sigma_z(\tau)\sigma_z(0) \rangle_{eq}$. At the critical point, the spin-spin correlation function shows the power-law decay as

$$C(\tau) = C(\beta - \tau) \sim \tau^{-\eta}, \quad (\omega_c^{-1} \ll \tau \ll \beta/2),$$

where $\eta$ is the critical exponent related to the spin dynamics. Then, the temperature dependence of the static susceptibility at the critical point is obtained as

$$\chi_0 \sim \beta^{1-\eta}.$$  

By using Eqs. (11) and (12), the critical behavior of the imaginary part of the dynamic susceptibility is obtained as

$$\text{Im}[\chi(\omega)] \sim \omega^{\eta-1}.$$  

Substituting this into Eq. (10), the thermal conductance at the critical point behaves as $\kappa \sim T^c$, where the exponent is given by

$$c = s + \eta.$$
was evaluated by the $\varepsilon$-expansion [51] (see Appendix A). In summary, the exponent of the thermal conductance is given as

$$c = \begin{cases} s + 1/2, & (s \leq 1/2), \\ 1 - \varepsilon/2 - \varepsilon^2 A(s)/3s + O(\varepsilon^3), & (s > 1/2), \end{cases} \quad (20)$$

where $\varepsilon = 2s - 1$ and $A(s) = s[\psi(1) - 2\psi(s/2) + \psi(s)]$.

Finally, we point out that the critical behavior can be observed for other physical quantities [24, 38, 51]. We summarize the critical exponents for measurable quantities in Appendix A.

V. EXPERIMENTAL REALIZATION

In this section, we discuss a superconducting circuit to realize the spin-boson Hamiltonian with the sub-ohmic reservoirs. The previous theoretical study [52] have shown that a spatially-uniform transmission line can realize the sub-ohmic reservoir with $s = 0.5$. For a controlled experiment of the QPT, however, it is favorable to realize the sub-ohmic reservoir with an arbitrary value of $s$. In this section, we propose a superconducting circuit to realize the sub-ohmic reservoirs for arbitrary $s$ by introducing spacial dependence to circuit elements.

We consider a flux qubit coupled to two transmission lines (or two junction arrays) shown in Fig. 4 (a). The flux qubit is composed of three small Josephson junctions [52]. By tuning the external magnetic field, the flux qubit is described by a double-well potential system, and its effective Hamiltonian is given by Eq. (1) (for detailed derivation, see Appendix C). Then, the flux qubit coupled to the transmission lines can be described by the spin-boson model effectively. Using the linear response theory [38] [53] [54], the spectral density function is expressed with the joint impedance of the two transmission lines ($Z(\omega)^{-1} = \sum_{\nu} Z_{\nu}(\omega)^{-1}$) as

$$I(\omega) = \sum_{\nu} I_{\nu}(\omega) = \frac{4\omega_0^2 \langle \varphi^2 \rangle^2}{\pi} I_0(\omega), \quad (21)$$

$$I_0(\omega) = \omega \text{Re}[Z(\omega)^{-1}], \quad (22)$$

where $\phi_0 = h/2e$, and $\pm \langle \varphi_- \rangle$ is an expectation value of the phase at the flux qubit. Detailed discussion is given in Appendix C.

To realize the sub-ohmic reservoir with an arbitrary exponent $s$, we propose a superconducting circuit given in Fig. 4 (b). The circuit is composed by a resistances $R_j$, an inductances $L_j$, and a capacitances $C_j$ ($j = 1, 2, \cdots, N$). For simplicity, we assume that the two transmission lines are constructed by the same circuit. The joint impedance of the two transmission lines is then calculated as $Z(\omega)^{-1} = 2Z_N(\omega)^{-1}$, where $Z_j(\omega)$ ($j = 1, 2, \cdots, N$) is given by a recurrence relation

$$Z_j(\omega) = R_j + i\omega L_j + \frac{1}{Z_{j-1}(\omega)^{-1} + i\omega C_j}, \quad (23)$$

with $Z_0(\omega)^{-1} = 0$.

Now, we assume that circuit elements have spatial dependence as

$$R_j = R_0(1 - j/N)^n, \quad (24)$$

$$L_j = L_0, \quad (25)$$

$$C_j = C_0(1 - j/N)^m, \quad (26)$$

where $n$ and $m$ are non-negative real numbers. We show the spectral density function $I_0(\omega)$ of this circuit in Fig. 5 for $(n, m) = (2, 2)$ and $(6, 6)$. The parameters are set as $R_0 = 1 \text{k}\Omega$, $L_0 = 13 \text{nH}$, $C_0 = 1 \text{pF}$, and $N = 10^4$, referring experimental studies of Josephson junction arrays [54]. In Fig. 5, we added 1% relative randomness to each circuit element to indicate tolerance to the circuit parameter fluctuation.
We find that the spectral density function is approximately proportional to \( \omega^s \) in a certain range of the frequency with the exponent \( 0 < s < 1 \). This indicates that the present circuit can realize the sub-ohmic reservoir with an arbitrary value of \( s \). Actually, analytic calculation concludes

\[
I(\omega) \propto \omega^{2/(m+2)}, \quad (\omega^s \ll \omega \ll \omega_c).
\]  

(27)

Detailed calculation is given in Appendix D. This result is in good agreement with Fig. 5; \( \omega^s \ll \omega \ll \omega_c \).\footnote{The lower frequency limit for the sub-ohmic spectral density function, \( \omega^s \), is calculated as \[
\omega^s = \left( \frac{m}{L_0} \right)^{2n} \frac{B_0^{m+2}}{C_0^{m+n+2}} \right)^{1/(m+2n+2)}. \]

(28)

So, the exponent \( n \) for the resistance controls the lower limit of the sub-ohmic spectral density function. In contrast, the higher frequency limit \( \omega_c \) is a complex function of circuit parameters.

Finally, we summarize the condition for realization of the quantum phase transition. First, the tunneling amplitude \( \Delta \) has to be in the range of \( \omega^s \ll \Delta \ll \omega_c \). Second, the dimensionless system-reservoir coupling \( \alpha \) has to be tuned around the predicted critical point \( \alpha_c \). For a typical value of the tunneling amplitude \( \Delta = 25 \Gamma_0 \) for the flux qubit \cite{32}, we find that both of the conditions are satisfied for the parameters used in Fig. 5 for \( s = 0.5 \) (\( m = 2 \)).

VI. SUMMARY

We studied quantum critical phenomena in heat transport by the spin-boson model with the sub-ohmic reservoirs. By the continuous-time quantum Monte Carlo simulation, we showed that a thermal conductance at the critical point has a characteristic power-law temperature dependence, whose critical exponent depends on the exponent of the spectral density function. We also clarified how the critical exponent of the thermal conductance is related to other critical exponents discussed in previous theoretical studies. Finally, we proposed a superconducting circuit realizing the sub-ohmic reservoirs for an arbitrary value of the exponent \( s \).

We expect that our study provides a new platform for experiments accessing quantum phase transition directly in transport property of mesoscopic devices. Although we have used the flux qubit to realize the spin-boson model, other types of qubits such as a charge qubit and a transmon qubit can be considered similarly. We will give detailed description for the other types of qubits elsewhere.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Exponent & Definition & Condition \\
\hline
\( \gamma \) & \( \chi_0 \propto (\sigma - \alpha)^{-\gamma} \) & \( \alpha < \alpha_c, \ T = 0 \) \\
\hline
\( \beta' \) & \( m_z \propto (\sigma - \alpha)^{\beta'} \) & \( \alpha > \alpha_c, \ T = 0 \) \\
\hline
\( \eta \) & \( m_z \propto T^{\eta/2} \) & \( \alpha = \alpha_c, \ T = 0 \) \\
\hline
\( x \) & \( \chi_0 \propto T^x \) & \( \alpha = \alpha_c, \ T > 0 \) \\
\hline
\end{tabular}
\caption{Summary of the critical exponents.}
\end{table}

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Appendix A: Critical Exponents

In this Appendix, we briefly discuss critical exponents of several observables at the quantum phase transition for the sub-ohmic reservoirs following Refs. \cite{24-26}. Fig. 6 shows schematics of \( \langle \sigma_z \rangle \) as a function of the detuning energy \( \varepsilon \) near the critical point \( \varepsilon = 0 \). In the delocalized phase (blue line; \( \varepsilon < \varepsilon_c \)), \( \langle \sigma_z \rangle \) is a continuous function of \( \varepsilon \), and a susceptibility \( \chi_0 \) can be defined by a slope at \( \varepsilon = 0 \). At a critical point (green line; \( \varepsilon = \varepsilon_c \)), \( \langle \sigma_z \rangle \) is continuous, but the susceptibility diverges at \( \varepsilon = 0 \). For the localized phase (red line; \( \varepsilon > \varepsilon_c \)), \( \langle \sigma_z \rangle \) has discontinuity at \( \varepsilon = 0 \).

\[
\chi_0 = \lim_{\varepsilon \to 0} \langle \sigma_z \rangle_{\text{eq}}. \tag{A1}
\]

The spin susceptibility \( \chi_0 \) diverges as \( \alpha \) approaches \( \alpha_c \) from below. In the localized phase (\( \alpha > \alpha_c \)), \( \langle \sigma_z \rangle \) jumps from \(-m_z\) to \( m_z \) at \( \varepsilon = 0 \), where \( m_z = \langle \sigma_z \rangle |_{\varepsilon \to 0} \) is the spontaneous magnetization.

In Table A, we summarize the critical exponents. All the exponents can be determined experimentally by measurement of \( \langle \sigma_z \rangle \). By using two exponents related to
the fix point, \( y^*_b \) and \( y^*_r \), these critical exponents are expressed as \( 24 \)

\[
\beta' = (1 - y^*_b) / y^*_r, \quad \gamma = (2y^*_b - 1) / y^*_r, \quad \eta = 1 - x = 2 - 2y^*_b.
\]

(A2) \( \gamma = 1, \eta = 1/2, x = 1/2 \). (A6)

For \( s > 0.5 \), \( y^*_s \) and \( y^*_h \) is a nontrivial function of \( s \). By the \( \varepsilon \)-expansion, the exponents are calculated as \( 21 \)

\[
y^*_s = s + \varepsilon / 6 - 4^{2}A(s) / 9s + O(\varepsilon^3),
\]

(A7)

\[
y^*_h = (1 + s) / 2 + \varepsilon / 4 - \varepsilon^2 A(s) / 6s + O(\varepsilon^3),
\]

(A8)

where \( \varepsilon = 2s - 1 \), \( A(s) = s[\psi(1) - 2\psi(s/2) + \psi(s)] \) and \( \psi(x) \) is the Digamma function. The results of the critical exponents are conformal numerically in the previous studies \( 24, 26, 51, 56 \).

Appendix B: Asymptotically exact formula for co-tunneling

When the ground-state is a delocalized state (\( \alpha < \alpha_c \)), heat transport is induced by the virtual excitation of the two-state system for \( T \ll \Delta_{\text{eff}} \), where \( \Delta_{\text{eff}} \) is a renormalized tunneling amplitude. This process is called co-tunneling. By utilizing the generalized Shiba relation \( 57 \), the asymptotically-exact formula for the thermal conductance in the co-tunneling regime (\( T \ll \Delta_{\text{eff}} \)) is derived as \( 30 \)

\[
\kappa_{co} = \frac{\pi A_0^2}{8} \int_0^\infty d\omega \, I_L(\omega) I_R(\omega) \left[ \frac{\beta \omega / 2}{\sinh(\beta \omega / 2)} \right]^2.
\]

(B1)

where \( \chi_0 \) is the susceptibility defined by Eq. \( A1 \). This formula leads to the thermal conductance proportional to \( T^{2s+1} \).

Appendix C: Circuit Model

In this appendix, we consider a flux qubit coupled to transmission lines (see Fig. 3), and derive the effective spin-boson model \( 37 \). As a simple example, we consider a uniform transmission line with constant capacitances and inductances (\( C_i = C \), \( L_i = L \)), neglecting resistances (\( R_i = 0 \)). Finally, we derive a general linear-response relation between the spectral density function and the circuit impedance.
where $\langle \uparrow \Phi_- | \uparrow \rangle \equiv \phi_0 \langle \varphi_+ \rangle$, $\langle \downarrow \Phi_- | \downarrow \rangle \equiv -\phi_0 \langle \varphi_- \rangle$, and $\langle \uparrow | \Phi_- | \uparrow \rangle = (\downarrow | \Phi_- | \downarrow) = 0$.

For simplicity, we consider the continuous limit $\Delta x \to 0$ keeping the length of the transmission line, $L_t = N \Delta x$, constant, where $\Delta x$ is the size of each elementary island. Then, the system-reservoir coupling can be rewritten by \[ \mathcal{I}_0 \equiv \mathcal{I}(x = 0) = \sum_{k} -\frac{i \lambda_k}{2 \phi_0 \langle \varphi_- \rangle}(b_k + b_k^\dagger). \] (C13)

From Eqs. (4) and (C11)-(C13), the spectral density function can be rewritten as

\[ I(\omega) = \frac{4 \phi_0^2 \langle \varphi_- \rangle^2}{\pi} \text{Im}[G^{R}_{\mathcal{I}_0}(\omega)], \]

where $G^{R}_{\mathcal{I}_0}(\omega)$ is the Fourier transformation of the current-current correlation function defined by $G^{R}_{\mathcal{I}_0}(t) = -i \theta(t) \langle \mathcal{I}_0(t), \mathcal{I}_0(0) \rangle$. Using the linear response theory \[ G^{R}_{\mathcal{I}_0}(\omega) \] can be related with the total impedance of the transmission lines as

\[ \frac{1}{Z(\omega)} = \frac{i}{\omega} G^{R}_{\mathcal{I}_0}(\omega). \]

Substituting Eq. (C15) into Eq. (C14), we can derive Eqs. (21) and (22) in the main text. Although we have derived it for a special case, i.e., the case of uniform transmission lines without damping, Eqs. (21) and (22) hold for arbitrary circuits of the transmission lines.

Appendix D: Analytic expression of the spectral function

We analyze the frequency dependence of the spectral density function for the circuit model discussed in Sec. V. Assuming $|\omega c_j Z_{j-1}(\omega)| \ll 1$, the recurrence relation \[ (D1) \]

In the continuous limit $N \to \infty$, this recurrence relation reduces to the differential equation

\[ \frac{dZ(\omega, x)}{dx} = r(x) + i\omega l(x) - i\omega c(x)Z(\omega, x), \]

where $r(x)$, $l(x)$, and $c(x)$ $(0 \leq x = j/N \leq 1)$ are the resistance, the inductance, and the capacitance per unit length, respectively. From Eq. (24), they are given as

\[ r(x) = r_0(1-x)^n, \]

\[ l(x) = l_0, \]

\[ c(x) = c_0(1-x)^m, \]

where $r_0 = R_0/L_t$, $l_0 = L_0/L_t$, and $c_0 = C_0/L_t$ ($L_t$ is the length of the transmission line). We note that $Z(\omega) = Z(\omega, x \to 1)$. For

\[ 1 - x^* \equiv \left( \frac{n}{2\omega \sqrt{l_0 c_0}} \right)^{2/(m+2)} \ll 1 - x \ll \left( \frac{\omega l_0}{r_0} \right)^{1/n} \]

we have

\[ Z(\omega) \approx Z_B(\omega, x \to 1), \]

where $Z_B(\omega, x) = \frac{r_0}{l_0(1-x)^m} + \sqrt{\frac{l_0}{c_0}(1-x)^{-m/2}}$. (D9)

From Eq. (21), we obtain the spectral density function

\[ I(\omega) \propto \omega \text{Re}[Z(\omega)^{-1}] \propto \omega^{2/(m+2)}. \]

This frequency dependence appears for $\omega^* \ll \omega \ll \omega_c$, where the lower bound $\omega^*$ is obtained from the condition [D6] as

\[ \omega^* = \left( \frac{m}{2} \right)^{2n} \frac{r_0^{m+2}}{c_0^{m+n+2}} \right]^{1/(m+2n+2)}. \]

This corresponds to Eq. (28) in the main text.
