COMBINATORIAL FORMULAS FOR PRODUCTS OF THOM CLASSES

VICTOR GUilleMIN* AND CATALIN ZARA **

Abstract. Let $G$ be a torus of dimension $n > 1$ and $M$ a compact Hamiltonian $G$-manifold with $M^G$ finite. A circle, $S^1$, in $G$ is generic if $M^G = M^{S^1}$. For such a circle the moment map associated with its action on $M$ is a perfect Morse function. Let $\{W^+_p \mid p \in M^G\}$ be the Morse-Whitney stratification of $M$ associated with this function, and let $\tau^+_p$ be the equivariant Thom class dual to $W^+_p$. These classes form a basis of $H^*_G(M)$ as a module over $S(g^*)$ and, in particular,

$$\tau^+_p \tau^+_q = \sum c^p_{pq} \tau^+_r$$

with $c^p_{pq} \in S(g^*)$. For manifolds of GKM type we obtain a combinatorial description of these $\tau^+_p$'s and, from this description, a combinatorial formula for $c^p_{pq}$.

1. Products of Thom classes

Let $M^{2d}$ be a compact Hamiltonian $S^1$-manifold with moment map, $\phi : M \to \mathbb{R}$. If $M^{S^1}$ is finite, $\phi$ is a Morse function, and its critical points, $p \in M^{S^1}$, are all of even index. This has important consequences for the topology of $M$: If we put an $S^1$-invariant Riemannian metric, $B$, on $M$ and let

$$v = \nabla_B \phi$$

be the gradient vector field associated with $B$ and $\phi$, then, for every critical point, $p \in M^{S^1}$, the unstable manifold at $p$:

$$W^+_p = \{ q \in M; \lim_{t \to -\infty} (\exp tv)(q) = p \}$$

supports a cohomology class, and these “Thom” classes

(1.1) $$\tau^+_p, \quad p \in M^{S^1}$$

are an additive basis of the cohomology ring, $H^*(M, \mathbb{R})$. The same is true of the stable manifolds

$$W^-_p = \{ q \in M; \lim_{t \to \infty} (\exp tv)(q) = p \}$$

and their dual Thom classes

$$\tau^-_p, \quad p \in M^{S^1}.$$
The main topic of this paper will be the symplectic version of what is sometimes called the multiplicative Morse problem: Given \( p \) and \( q \) in \( M^{S^1} \), \( \tau^+_p \tau^+_q \) can be expanded as a sum
\[
\tau^+_p \tau^+_q = \sum c^r_{pq} \tau^+_r.
\]
What are the \( c^r_{pq} \)'s? Closely related to this is the question of determining the cohomology pairings:
\[
c^pqr = \int \tau^+_p \tau^+_q \tau^-_r. \tag{1.2}
\]
Neither of these questions is easy to answer even when the structure of the cohomology ring itself is well understood. For instance if \( M \) is the coadjoint orbit of a compact Lie group, the computation of the \( c^r_{pq} \)'s is an important open problem in the theory of the Schubert calculus and is the focus of a lot of recent activity. (See, for instance, [BGG], [BH], [Bi], [Ko] and [Kn].)

In this paper we will consider the equivariant version of this problem. We will assume the action of \( S^1 \) on \( M \) can be extended to a Hamiltonian action of a torus, \( G \), of dimension \( n > 1 \), and replace the \( \tau^+ \)'s in (1.2) by their equivariant counterparts. These equivariant Thom classes generate \( H^G(M, \mathbb{R}) \) as a module over the ring, \( H^G(pt) = \mathbb{S}(g^*) \), so one gets as above an identity
\[
\tau^+_p \tau^+_q = \sum c^r_{pq} \tau^+_r,
\]
but now the \( c^r_{pq} \)'s are elements of the polynomial ring, \( \mathbb{S}(g^*) \). When
\[
degree \tau^+_r = degree \tau^+_p + degree \tau^+_q,
\]
they are polynomials of degree zero (i.e. real numbers) and, in fact, coincide with the \( c^r_{pq} \)'s in (1.2). Thus, they are in principle a much larger list of unknown quantities. We will show, however, that they are in some sense, easier to compute due to the fact that, in equivariant cohomology, one has a much richer store of intersection invariants to play around with. More explicitly if \( X \) and \( Y \) are submanifolds of \( M \) and \( \tau_X \) and \( \tau_Y \) their dual Thom classes, the intersection number
\[
\#(X \cap Y) = \int \tau_X \tau_Y \tag{1.3}
\]
is zero except when \( X \) and \( Y \) are of complementary dimension. On the other hand if \( X \) and \( Y \) are \( G \)-invariant and \( \tau_X \) and \( \tau_Y \) their equivariant Thom classes, the expression (1.3) (which is now an element of \( \mathbb{S}(g^*) \)) can be non-zero no matter what the relative dimensions of \( X \) and \( Y \) are. Moreover, by the localization theorem of Atiyah-Bott-Berline-Vergne, (1.3), is a sum of local intersection invariants
\[
\#(X \cap Y)_p \in Q(g^*),
\]
where \( Q(g^*) \) is the quotient field of \( \mathbb{S}(g^*) \) and \( p \) a fixed point, and each of these is itself an intersection invariant.

Of particular interest for us will be certain intersection invariants of this type associated with the moment map, \( \phi \). Suppose that \( p \) and \( q \) are critical points
of \( \phi \) and that there are no critical values of \( \phi \) in the interval \((\phi(p), \phi(q))\). Let \( \phi(p) < c < \phi(q) \) and let
\[
M_c = \phi^{-1}(c)/S^1
\]
be the symplectic reduction of \( M \) at \( c \). By the Marsden-Weinstein theorem, \( M_c \) is a symplectic orbifold, and the action of \( G \) on \( M \) induces an action of the group
\[
G_1 = G/S^1
\]
on \( M_c \). The reduced spaces \((W^+_p)_c\) and \((W^-_q)_c\) are \( G \)-invariant symplectic sub-orbifolds of \( M_c \) and so their equivariant intersection “number”
\[
#((W^+_p)_c \cap (W^-_q)_c)
\]
is well-defined as an element of the subring, \( \mathbb{S}(\mathfrak{g}^*_1) \), of \( \mathbb{S}(\mathfrak{g}^*) \). Moreover, if \( M_{cG_1} \) is finite, then for every \( v \in M_{cG_1} \), the local intersection number
\[
#((W^+_p)_c \cap (W^-_q)_c)v
\]
is well-defined as an element of \( \mathbb{Q}(\mathfrak{g}^*_1) \).

We will now describe the role of these intersection numbers in the computation of the \( c_{pq} \)'s. We will say that \( M \) is a GKM manifold if, for all non-critical values, \( c, M_{cG_1} \) is finite. Thus, being GKM is a necessary and sufficient condition for the invariants \((1.5)\) to be well-defined. We recall some other characterizations of these manifolds.

**Theorem 1.1.** \( M \) is GKM if and only if, for every \( p \in M^G \), the weights, \( \alpha_{i,p} \), of the linear isotropy representation of \( G \) on \( T_p M \) are pair-wise linearly independent; i.e, \( \alpha_{i,p} \) is not a multiple of \( \alpha_{j,p} \) if \( i \neq j \).

**Theorem 1.2.** \( M \) is GKM if and only if, for every codimension one subtorus, \( H \), of \( G \) the connected components of \( M^H \) are either fixed points of \( G \) or imbedded copies of \( S^2 \). Moreover, if a connected component, \( X \), is a copy of \( S^2 \), the action of \( G \) on \( X \) is symplectomorphic to the standard action of \( G/H = S^1 \) on \( S^2 \).

**Theorem 1.3.** \( M \) is GKM if and only if the one skeleton of \( M \)
\[
\{ p \in M, \quad \dim G_p \geq n - 1 \}
\]
is a finite union of embedded \( S^2 \)'s.

**Proof.** Theorem \([1.3]\) is an obvious consequence of Theorem \([1.2]\); and it is easy to see that if the hypotheses of Theorem \([1.2]\) hold, \( M \) is GKM. For the other implications see \([GZ2]\). 

The intersection properties of the embedded \( S^2 \)'s in the set \((1.6)\) can be described by an intersection graph, and a beautiful observation of Goresky-Kottwitz-MacPherson is that one can read off the structure of the equivariant cohomology ring of \( M \) from the “action” of \( G \) on this graph. More explicitly let \( \Gamma \) be the graph whose vertices are the fixed points of \( G \) and whose edges, \( e \), are copies, \( X_e \), of the
The graph structure on this collection of vertices and edges is given by defining the pair of vertices incident to an edge, \( e \), to be the set 
\[
\partial e = \{ p, q \} = X_e^G.
\]
In particular, if \( e \) is an oriented edge, its initial vertex, \( i(e) \), is defined to be the “south pole”, \( p \), of the two-sphere, \( X_e \), and its terminal vertex, \( t(e) \), to be the “north pole”, \( q \) of \( X_e \). The action of \( G \) on the set (1.6) can be described graph theoretically by two pieces of data: a function \( \rho \) which assigns to each oriented edge, \( e \), of \( \Gamma \) a one-dimensional representation, \( \rho_e \), of \( G \), and a function, \( \kappa \), which assigns to each vertex, \( p \), a \( d \)-dimensional representation, \( \kappa_p \). These functions are defined by letting \( \rho_e \) be the representation of \( G \) on \( T_p X_e \), \( p = i(e) \), and letting \( \kappa_p \) be the representation of \( G \) on \( T_p M \). It is easily checked that \( \rho \) and \( \kappa \) satisfy the axioms:

\[
\kappa_p = \bigoplus_{i(e)=p} \rho_e \tag{1.7}
\]
\[
\rho_{\tilde{e}} = \rho_e^* \tag{1.8}
\]
and

\[
\kappa_p|_{G_e} = \kappa_{q|_{G_e}} \tag{1.9}
\]
where \( \tilde{e} \) is the edge obtained from \( e \) by reversing its orientation, \( G_e \) is the kernel of \( \rho_e \) and \( p \) and \( q \) are the vertices \( i(e) \) and \( t(e) \). In particular, by (1.7), \( \kappa_p \) is determined by the \( \rho_e \)’s with \( i(e) = p \). Since \( \rho_e \) is a one-dimensional representation it is determined by its weight, \( \alpha_e \); so the “action” of \( G \) on \( \Gamma \) associated with \( \rho \) and \( \kappa \) consists essentially of a labeling of the oriented edges, \( e \), of \( \Gamma \) by weights, \( \alpha_e \). The axioms (1.7)–(1.9) impose, of course, some condition on this labeling. For instance (1.8) is equivalent to

\[
\alpha_e = -\alpha_{\tilde{e}}.
\]

Now let \( H_G(M) \) be the equivariant cohomology ring of \( M \) and \( H_G(M^G) \) the equivariant cohomology ring of \( M^G \). Since \( M^G \) is a finite disjoint union of fixed points and these fixed points are also the vertices, \( V_\Gamma \), of \( \Gamma \) it follows that

\[
H_G(M^G) = \text{Maps}(V_\Gamma, S(g^*))
\]
Moreover, if \( i : M^G \to M \) is the inclusion, the map \( i^* : H_G(M) \to H_G(M^G) \) is injective, by a well-known result of Kirwan. The theorem of Goresky-Kottwitz-MacPherson which we alluded to above asserts:

**Theorem 1.4.** A map \( h : V_\Gamma \to S(g^*) \) is in the image of \( i^* \) if and only if, for every edge, \( e \), of \( \Gamma \)

\[
h_p|_{g_e} = h_q|_{g_e} \tag{1.10}
\]
p and \( q \) being the vertices of \( e \) and \( g_e \) the annihilator of \( \alpha_e \) in \( g \).

This leads us to define the equivariant cohomology ring, \( H(\Gamma, \alpha) \) of \( \Gamma \) to be the set of all maps, \( h : V_\Gamma \to S(g^*) \) satisfying (1.10). Each of the Thom classes (1.1) gets mapped by \( i^* \) onto an element of \( H(\Gamma, \alpha) \) and we will continue to use the
notation, $\tau_p^+$, for this “combinatorial” Thom class. The main result of this article is a formula for this Thom class as a kind of path integral over certain paths in $\Gamma$. Before stating this result we’ll describe a few basic properties of these combinatorial Thom classes. Let’s continue to denote by $\phi$ the restriction of the moment map, $\phi$, to $M^G$. Identifying $M^G$ with $V_\Gamma$, one can think of $\phi$ as a real-valued function on $V_\Gamma$. By Theorem 1.2, $\phi$ takes on distinct values on the vertices $i(e)$ and $t(e)$ of an oriented edge, $e$. We will say that this edge is ascending if $\phi(i(e)) < \phi(t(e))$ and descending if the reverse inequality is true. More generally if $\gamma$ is a path in $\Gamma$ we will say that $\gamma$ is ascending if each of its edges is ascending. For every vertex, $p \in V_\Gamma$ define the index, $\sigma_p$, of $p$ to be the number of descending edges, $e$, with $i(e) = p$.

**Theorem 1.5.** The Thom class, $\tau_p^+$, has the following properties:

1. Its support is the set of all vertices of $\Gamma$ which can be joined to $p$ by an ascending path.
2. The value of $\tau_p^+$ at $p$ is
   \[ \nu_p^+ = \prod'_{i(e) = p} \alpha_e, \]
   the product, $\prod'$, being over all descending edges with $i(e) = p$.

In certain instances these properties uniquely characterize $\tau_p^+$.

**Theorem 1.6.** Suppose that the indexing function, $\sigma : V_\Gamma \rightarrow \mathbb{Z}$, $p \rightarrow \sigma_p$, is strictly increasing along ascending paths. Then $\tau_p^+$ is the unique element of $H(\Gamma, \alpha)$ with properties 1 and 2 above.

We will now describe our “path-integral” formula for $\tau_p^+$. As mentioned above this formula will involve the Hamiltonian action of the subgroup, $S^1$, of $G$ on $M$ and the intersection invariants (1.4) and (1.5). Let $\xi \in \mathfrak{g}$ be the infinitesimal generator of this subgroup and let $e$ be an ascending edge of $\Gamma$ with $p = i(e)$. For any point, $c$, on the interval between $\phi(p)$ and $\phi(q)$, the $S^1$-reduced space, $(X_e)_c$ consists of a single point, $v \in M_c$. Let $\iota_e$ be the local intersection number (1.5).

**Theorem 1.7.** For every $q \in V_\Gamma$

\[ \tau_p^+(q) = \sum E(\gamma) \]
the sum being over all ascending paths, $\gamma$, joining $p$ to $q$, and the summands being defined by

\[ E(\gamma) = (-1)^m \nu_q^+ \iota_{e_1} \prod_{k=2}^m \iota_{e_k} \frac{\iota_{e_k}}{\hat{\alpha}_{k-1} - \hat{\alpha}_k}, \]

where $e_1, \ldots, e_m$ are the edges of $\gamma$ and

\[ \hat{\alpha}_k = \frac{\alpha_{e_k}}{\alpha_{e_k}(\xi)}. \]

**Remarks:**
1. The local intersection number, $\iota_e$, is equal to the global intersection number (1.4) provided that there are no ascending paths in $\Gamma$ of length greater than one joining $p = i(e)$ to $q = t(e)$. In particular, if $\gamma$ is a longest path joining $p$ to $q$ all the intersection numbers in (1.12) are global intersection numbers and in particular are elements of $\mathbb{S}(\mathfrak{g}^*)$.

2. In Section 4 we will give a purely combinatorial definition of $\iota_e$.

As a corollary of Theorem 1.5 one gets for (1.2) the formula

$$c_{pqr} = \sum_t \delta_t E(\gamma_1)E(\gamma_2)E(\gamma_3)$$

(1.13)

summed over all configurations of paths, $\gamma_1$, $\gamma_2$ and $\gamma_3$, where $\gamma_1$ is an ascending path from $p$ to $t$, $\gamma_2$ an ascending path from $q$ to $t$, $\gamma_3$ a descending path from $r$ to $t$ and

$$\delta_t = \left( \prod_{i(e) = t} \alpha_e \right)^{-1}.$$
basically just a labeling of the edges of $\Gamma$ by weights. We will denote the function, $e \to \alpha_e$, by $\alpha$ and call it the \textit{axial function} of the action of $G$ on $\Gamma$. The axioms (1.7)–(1.9) can be reformulated as statements about $\alpha$:

**Proposition 2.2.** Axiom (1.8) is satisfied iff

$$\alpha_e = -\alpha_{\bar{e}}$$

and axiom (1.9) is satisfied iff one can order the weights

$$\alpha_{e_k}, \quad i(e_k) = p, \ e_k \neq e$$

and the weights

$$\alpha_{e'_k}, \quad i(e'_k) = q, \ e'_k \neq \bar{e}$$

so that

(2.1)

$$\alpha_{e'_k} = \alpha_{e_k} + c_k \alpha_e.$$  

(We will leave the proof of these assertions as an easy exercise.)

**Definition 2.3.** The action of $G$ on $\Gamma$ is a \textit{GKM action} if, for all vertices, $p$, the weights, $\alpha_{e, i(e)} = p$, are pair-wise linear independent.

From now on we will assume, unless we state otherwise, that the action of $G$ on $\Gamma$ has this property.

For every vertex, $p$, of $\Gamma$ let $E_p$ be the set of oriented edges, $e$, with $i(e) = p$.

**Definition 2.4.** A \textit{connection} on $\Gamma$ is a function which assigns to every oriented edge, $e$, a bijective map $\theta_e : E_{i(e)} \to E_{t(e)}$ satisfying $\theta_{\bar{e}} = \theta_e^{-1}$. This connection is compatible with the action of $G$ if, for every oriented edge, $e$, with $i(e) = p$ and every edge, $e_k \in E_p$, $e_k \neq e$

(2.2)

$$\alpha_{e'_k} = \alpha_{e_k} + c_k \alpha_e,$$

where $e'_k = \theta_e(e_k)$.

Thus the existence of a $G$-compatible connection is a slight sharpening of the identity (2.1). It is easy to see that $G$-compatible connections exist, and we will assume henceforth that $\Gamma$ is equipped with such a connection.

Let $V_\Gamma$ be the set of vertices of $\Gamma$ and $E_\Gamma$ the set of oriented edges. Motivated by the theorem of Goresky–Kottwitz–MacPherson we define the equivariant cohomology ring, $H(\Gamma, \alpha)$, of $\Gamma$ to be the set of maps, $h : V_\Gamma \to S(g^*)$, satisfying the compatibility condition (1.10) for all $e \in E_\Gamma$. This ring has a natural grading

$$H^k(\Gamma, \alpha) = H(\Gamma, \alpha) \cap \text{Maps}(V_\Gamma, S^k(g^*))$$

and contains $S(g^*)$ as a subring: the ring of constant maps of $V_\Gamma$ into $S(g^*)$. The proof of Theorem 1.7 will require a number of results about the structure of $H(\Gamma, \alpha)$

\footnote{This definition is, unfortunately, inconsistent with the topological definition which assigns to $H^k$ the degree $2k$. (It is, however, more natural in this algebraic context.)}
as an $S(\mathfrak{g}^*)$ module. These results were proved in an earlier paper of ours on “equivariant Morse theory on graphs” ([GZ3]), and we will refer to this paper for a detailed treatment of the material in the next few paragraphs. Let

$$\mathcal{P} = \{ \xi \in \mathfrak{g}, \alpha_e(\xi) \neq 0 \text{ for all } e \in E_\Gamma \}. \quad (2.3)$$

Given an element, $\xi \in \mathcal{P}$, we will say that an oriented edge, $e$, is ascending with respect to $\xi$ if $\alpha_e(\xi) > 0$. For every vertex, $p$, let $\sigma_p$, the index of $p$, be the number of ascending edges, $e$, with $t(e) = p$.

**Definition 2.5.** The $k^{th}$ Betti number, $b_k(\Gamma)$, is the number of vertices, $p$, of $\Gamma$ for which $\sigma_p = k$.

**Remark:** The definition of $\sigma_p$ depends upon the choice of $\xi$ but $b_k(\Gamma)$ turns out not to. (See [GZ1, Theorem 2.6]).

A function $\phi : V_\Gamma \to \mathbb{R}$ is a ($\xi$-compatible) Morse function if, for every ascending edge $e$, $\phi(i(e)) < \phi(t(e))$. It is not obvious, and in fact not true, that Morse functions exist. A necessary and sufficient condition for the existence of a Morse function is that, for every ascending path in $\Gamma$ the initial vertex of this path is distinct from its terminal vertex (i.e., there are no ascending “loops”). If a Morse function exists, however, one can easily perturb it so that it is injective as a map of $V_\Gamma$ into $\mathbb{R}$. From now on we will let $\phi$ be a fixed Morse function with this property.

The topological results discussed in Section 1 prompt one to make the following Morse-theoretic conjectures about the equivariant cohomology ring of a graph.

**Conjecture 1.** $H(\Gamma, \alpha)$ is a free $S(\mathfrak{g}^*)$ module with $b_k(\Gamma)$ generators of degree $k$.

**Conjecture 2.** $H(\Gamma, \alpha)$ is freely generated as an $S(\mathfrak{g}^*)$ module by a family of “Thom classes” $\tau^+_p \in H^k(\Gamma, \alpha)$, $k = \sigma_p$,

$$\text{satisfying} \quad \text{support } \tau^+_p \subseteq F_p \quad (2.4)$$

and

$$\tau^+_p(p) = \prod_{e \in E_\Gamma^-} \alpha_e \quad (:= \nu^+_p), \quad (2.5)$$

$F_p$ being the set of vertices which can be joined to $p$ by an ascending path and $E_\Gamma^-$ being the set of descending edges in $E_\Gamma$.

It is clear that Conjecture 2 implies Conjecture 1, and it is not difficult to prove that Conjecture 1 implies Conjecture 2 (see [GZ2, §2.4.3]). Therefore, since Conjecture 1 doesn’t depend on the choice of an orientation of $\Gamma$ (i.e. the choice of a polarizing vector $\xi \in \mathcal{P}$), the same is true of Conjecture 2. In particular, if we reverse the orientation (replace $\xi$ by $-\xi$), we get from Conjecture 2 the existence of Thom classes, $\tau^-_p$, $p \in V_\Gamma$, associated with the Morse function $-\phi$.

Unfortunately these conjectures are not true in general; however there is a useful necessary and sufficient condition for them to be true involving certain subgraphs of $\Gamma$.
Definition 2.6. A subgraph, $\Gamma_1$, of $\Gamma$ is totally geodesic if, for every pair of edges, $e$ and $e'$, of $\Gamma_1$, with $i(e) = i(e')$, $\theta(e')(e')$ is also an edge of $\Gamma_1$.

Note that if $\Gamma_1$ is a totally geodesic subgraph of $\Gamma$ the restriction to it of $\alpha$ is, by (2.2), an axial function on $\Gamma_1$; so each of these subgraphs is equipped with an action of $G$. An important example of a totally geodesic subgraph is the following. Let $h^*$ be a subspace of $g^*$, and let $\Gamma_{h^*}$ be the subgraph whose edges are the set

$$\{e \in E_G, \alpha_e \in h^*\}.$$  

(It is clear, by (2.1) and (2.2) that this is totally geodesic.) One of the main results of [GZ3] is the following.

Theorem 2.7. Conjecture 2 is true for $\Gamma$ if and only if, for every two-dimensional subspace, $h^*$, of $g^*$, Conjecture 2 is true for $\Gamma_{h^*}$.

Thus, to verify that Conjecture 2 holds for $\Gamma$ it suffices to verify it for these subgraphs (which is usually much easier than verifying it for $\Gamma$ itself).

The proof of Theorem 2.7 involves a graph-theoretic version of symplectic reduction. We will say that $c$ is a critical value of the Morse function $\phi : V_\Gamma \to \mathbb{R}$ if $c = \phi(p)$ for some $p \in V_\Gamma$ and, otherwise, $c$ is a regular value. Let $c$ be a regular value of $\phi$ and let $V_c$ be the set of oriented edges, $e$, of $\Gamma$ with $\phi(i(e)) < c < \phi(t(e))$. We show in [GZ3] that $V_c$ is the set of vertices of a hypergraph, $\Gamma_c$. Thus the elements of $V_c$ are both edges of the graph $\Gamma$ and vertices of this hypergraph. It is useful to distinguish between the two roles they play by saying that “an edge, $e$, intersects $\Gamma_c$ in a vertex, $v_e$.”

Let $g^*_{\xi}$ be the annihilator of $\xi$ in $g^*$. For each oriented edge, $e$, of $\Gamma$ we define a map $\rho_e : g^* \to g^*_{\xi}$ by setting

$$\rho_e \alpha = \alpha - \frac{\alpha(\xi)}{\alpha_e(\xi)} \alpha_e.$$  

This extends to a ring homomorphism

$$\rho_e : \mathcal{S}(g^*) \to \mathcal{S}(g^*_{\xi})$$  

and, from (2.6), we get a ring homomorphism

$$\mathcal{K}_c : H(\Gamma, \alpha) \to \text{Maps}(V_c, \mathcal{S}(g^*_{\xi}))$$  

by setting

$$\mathcal{K}_c(g)(v_e) = \rho_e g_{i(e)} = \rho_e g_{t(e)}.$$  

(The two terms on the right are equal by (1.10).)

We show in [GZ3] that $\mathcal{K}_c$ maps $H(\Gamma, \alpha)$ into the cohomology ring, $H(\Gamma_c, \alpha_c)$, of the hypergraph $\Gamma_c$. We won’t bother to review here the definition of this hypergraph cohomology ring (which is quite tricky) since one of the main theorems of [GZ3] asserts that, if the hypotheses of Theorem 2.7 hold and if $\xi$ satisfies a certain genericity condition (which we will spell out below), the map

$$\mathcal{K}_c : H(\Gamma, \alpha) \to H(\Gamma_c, \alpha_c)$$
is a submersion. Hence, thanks to this theorem, one can define \( H(\Gamma_c, \alpha_c) \) to be the image of \( K_c \).

A key step in the proof of Theorem 2.7 is a theorem which describes how the structure of the ring \( H(\Gamma_c, \alpha_c) \) changes as one passes through a critical value of \( c \). More explicitly suppose \( c \) and \( c' \) are regular values of \( \phi \) and suppose that there exists a unique vertex, \( p \), with \( c < \phi(p) < c' \). In addition suppose that, for \( e_1, e_2, e_3, e_4 \in E_p \)

\[
\frac{1}{\alpha_{e_1}(\xi)} \rho_{e_2} \alpha_{e_1} \neq \frac{1}{\alpha_{e_3}(\xi)} \rho_{e_4} \alpha_{e_3}
\] (2.8)

except when the two sides of (2.8) are forced to be equal (i.e., except when \( e_1 = e_2 \) and \( e_3 = e_4 \) or \( e_1 = e_3 \) and \( e_2 = e_4 \)). The inequality (2.8) is unfortunately not satisfied for all elements, \( \xi \), of the set (2.3), but one can show that those \( \xi \)'s for which it is satisfied form an open dense subset of this set.

Let \( r \) be the index of \( p \) and let \( s = d - r \). Let \( e_i, i = 1, \ldots, r \) be the descending edges in \( E_p \) and \( e_a, a = r + 1, \ldots, d \) be the ascending edges in \( E_p \). Let

\[ \Delta_c = \{ e_i; i = 1, \ldots, r \} \]

and

\[ \Delta_c' = \{ e_a; a = r + 1, \ldots, d \} \].

Then \( \Delta_c \) is a subset of \( V_c \), \( \Delta_c' \) a subset of \( V_c' \) and

\[ V_c - \Delta_c = V_c' - \Delta_c' = V_0 \],

where \( V_0 \) is the intersection of \( V_c \) and \( V_c' \). Let

\[ V^\# = V_0 \cup (\Delta_c \times \Delta_c') \].

Then one has projection maps

\[ \pi_c: V^\# \to V_c \quad \text{and} \quad \pi_{c'}: V^\# \to V_{c'} \]

and, from these projection maps, pull-back maps, \( \pi_c^* \) and \( \pi_{c'}^* \), embedding the rings

(2.9) \[ \text{Maps}(V_c, S(g_\xi^*)) \]

and

(2.10) \[ \text{Maps}(V_{c'}, S(g_\xi^*)) \]

into the ring

(2.11) \[ \text{Maps}(V^\#, S(g_\xi^*)) \].

Moreover the ring

(2.12) \[ \text{Maps}(\Delta_c, S(g_\xi^*)) \]

sits in the ring (2.9) as the set of maps \( h: V_c \to S(g_\xi^*) \) supported on \( \Delta_c \), and the ring

(2.13) \[ \text{Maps}(\Delta_c', S(g_\xi^*)) \]

sits inside the ring (2.10); so all the rings (2.9)–(2.13) can be regarded as subrings of (2.11).
Let $y_1, \ldots, y_{n-1}$ be a basis of $g^*_{\xi}$ and $x$ a fixed element of $g^*_{\xi}$ with $\langle x, \xi \rangle = 1$. Let
\[
\alpha_{e_i} = m_i(x - \beta_a(y)) \quad i = 1, \ldots, r
\]
and
\[
\alpha_{e_a} = m_a(x - \beta_a(y)) \quad a = r + 1, \ldots, d,
\]
with $m_i < 0 < m_a$ and with the $\beta$'s in $g^*_{\xi}$. Consider the maps
\[
\tau_c : \Delta_c \rightarrow g^*_{\xi}, \quad \tau_c(e_i) = \beta_i
\]
\[
\tau_{c'} : \Delta_{c'} \rightarrow g^*_{\xi}, \quad \tau_{c'}(e_a) = \beta_a
\]
and
\[
\tau^# : \Delta_c \times \Delta_{c'} \rightarrow g^*_{\xi}, \quad \tau^#(e_i, e_a) = \beta_i - \beta_a.
\]
The first two of these maps depend on the choice of $x$; however, $\tau^#$ is intrinsically defined since $\tau^#(e_i, e_a)$ is just
\[
\frac{1}{\alpha_{e_i}(\xi)} \rho_a \alpha_{e_a}.
\]
Also, by the genericity condition (2.8) the map, $\tau^#$ sends $\Delta_c \times \Delta_{c'}$ injectively into $g^*_{\xi}$ and, as a consequence, $\tau_c$ and $\tau_{c'}$ map $\Delta_c$ and $\Delta_{c'}$ injectively into $g^*_{\xi}$.

Define the cohomology ring, $H(\Delta_c, \alpha_c)$, to be the set of all maps of $\Delta_c$ into $S(g^*_{\xi})$ of the form
\[
h = \sum_{i=0}^{r-1} g_i \tau_c^i, \quad g_i \in S(g^*_{\xi})
\]
and define $H(\Delta_{c'}, \alpha_{c'})$ to be the set of all maps of $\Delta_{c'}$ into $S(g^*_{\xi})$ of the form
\[
h' = \sum_{i=0}^{s-1} g'_i \tau_{c'}^i, \quad g'_i \in S(g^*_{\xi}).
\]
The theorem we alluded to above asserts

**Theorem 2.8.** For every $f \in H(\Gamma_c, \alpha_c)$ and every $f_i \in H(\Delta_c, \tau_c)$, $i = 1, \ldots, s - 1$, there exists a unique $f' \in H(\Gamma_{c'}, \alpha_{c'})$ and unique $f'_j \in H(\Delta_{c'}, \tau_{c'})$, $j = 1, \ldots, r - 1$ such that
\[
f' + \sum_{j=1}^{r-1} (\tau^#)^j f'_j = f + \sum_{i=1}^{s-1} (\tau^#)^i f_i.
\]

**Remarks:**

1. This theorem gives one a concrete picture of how $H(\Gamma_c, \alpha_c)$ changes as one goes through a critical point of $\phi$. Namely it shows that $H(\Gamma_{c'}, \alpha_{c'})$ can be obtained from $H(\Gamma_c, \alpha_c)$ by a “blow-up” followed by a “blow-down”. (Compare with [GZ2, Theorem 2.3.2].)
2. This theorem can also be used to map cohomology classes in $H(\Gamma_c, \alpha_c)$ into cohomology classes in $H(\Gamma_c', \alpha_c')$. By setting $f_i = 0$, $i = 1, \ldots, s - 1$, in (2.14) one gets from the cohomology class $f \in H(\Gamma_c, \alpha_c)$ a unique cohomology class $f' \in H(\Gamma_c', \alpha_c')$. (This observation will be heavily exploited in the next section.)

An important ingredient in the proof of Theorem 2.8 is the following.

**Theorem 2.9.** If $f$ is in $H(\Gamma_c, \alpha_c)$, then its restriction to $\Delta_c$ is in $H(\Delta_c, \tau_c)$.

(It is in the proof of this result that the hypotheses of Theorem 2.7 are needed.)

3. **Combinatorial formulas for Thom classes**

We will describe in this section how to compute the combinatorial Thom class, $\tau_{p_0}^+$, at an arbitrary point $p$ on the flow-up $F_{p_0}$. We recall that $\tau_{p_0}^+$ is canonically defined only if the index function $\sigma : V_\Gamma \to \mathbb{Z}$, is strictly increasing along ascending paths in $\Gamma$. Assuming that $\sigma$ has this property, we will show below that there is a simple inductive method for computing $\tau_{p_0}^+$ on a critical level, $c$, of $\phi$ if one knows the values of $\tau_{p_0}^+$ on lower critical levels. Then, later in this section, we will show that this method works even when the hypothesis about $\sigma$ is dropped. Let $\phi(p_0) = c_0$ and $\sigma(p_0) = m$. The first step in this induction is to set $\tau_{p_0}^+(q) = 0$ for all vertices, $q$, with $\phi(q) < c_0$ and set $\tau_{p_0}^+(p_0)$ equal to $\nu_{p_0}^+$, as in (2.3). Now let $c > c_0$ and suppose, by induction, that $\tau_{p_0}^+$ is defined for all $q$ with $\phi(q) < c$ and is zero unless $q$ is in $F_{p_0}$. Let $p$ be a vertex with $\phi(p) = c$. Let $\sigma_p = r$ and let $e_k$, $k = 1, \ldots, r$, be the descending edges in $\Gamma$ with $p = i(e_k)$. Then the vertices, $q_k = t(e_k)$, are points where $\tau_{p_0}^+$ is already defined. We will prove below.

**Lemma 3.1.** There exists a unique polynomial, $\psi \in S(g^*)$ such that

\[
\psi \equiv \tau_{p_0}(q_k) \mod \alpha_{e_k}, \quad k = 1, \ldots, r.
\]

**Remark:** The “uniqueness” part of this lemma is where the hypothesis on $\sigma$ is used. If $f \in S^m(g^*)$ and

\[
f = 0 \mod \alpha_{e_k}, \quad k = 1, \ldots, r,
\]

then

\[
f = h\alpha_{e_1} \ldots \alpha_{e_r}, \quad h \in S^{m-r}(g^*)
\]

Hence, if $m < r$, $f$ is identically zero.

Using this lemma, set $\tau_{p_0}^+(p) = \psi$, and continue with the induction until the set of vertices of $\Gamma$ is exhausted. It is clear from (3.1) that this construction gives us a map: $\tau_{p_0}^+ : V_\Gamma \to S^m(g^*)$ satisfying (1.10) and that this map is supported on $F_p$.

By giving a constructive proof of the “existence” part of Lemma 3.1, the induction argument we just sketched can be converted into a formula for $\tau_{p_0}^+$, and this will be the main goal of this sections. Note that the solution of (3.1) is basically an interpolation problem: finding a polynomial with prescribed values at $\alpha_{e_1}, \ldots, \alpha_{e_r}$. 
To solve this problem constructively, we review a few elementary facts about “interpolation”. The basic problem in interpolation theory is to find a polynomial

\[ p(x) = \sum_{i=1}^{n} g_i x^{i-1} \]  

which takes prescribed values

\[ p(x_i) = f_i \]  

at \( n \) distinct points, \( x_i \), on the complex line. The solution of this problem is more or less trivial. The polynomial

\[ p(x) = \sum_{j} \prod_{k \neq j} \frac{x - x_k}{x_j - x_k} f_j \]  

satisfies (3.3) and is the only polynomial of degree less than \( n \) which does satisfy (3.3). Moreover, from (3.4) one gets an explicit formula for the \( g_i \)’s in (3.2). Let

\[ \prod_{\ell \neq j} (x - x_\ell) = \sum_{i=1}^{n-1} (-1)^{n-1-i} \sigma_{n-1-i}^j x^i, \]

where \( \sigma_j^r \) is the \( r \)-th elementary symmetric function in the variables, \( x_\ell, \ell \neq j \). Then by (3.4)

\[ g_i = (-1)^{n-i} \sum_{j=1}^{n} \frac{\sigma_{n-i}^j}{\prod_{\ell \neq j} (x_j - x_\ell)} f_j. \]

One consequence of (3.5) is an inversion formula for the Vandermonde matrix, \( A \), with entries

\[ a_{ij} = x_i^{j-1}, \ 1 \leq j, i \leq n. \]

If \( B = A^{-1} \) then by (3.5) and (3.3):

\[ b_{ij} = (-1)^{n-i} \frac{\sigma_{n-i}^j}{\prod_{\ell \neq j} (x_j - x_\ell)} f_j. \]

In particular

\[ b_{nj} = \prod_{\ell \neq j} \frac{1}{x_j - x_\ell} \]

and

\[ b_{1j} = \prod_{\ell \neq j} \frac{-x_\ell}{x_j - x_\ell}. \]

It is sometimes convenient to write the inversion formula (3.6) in terms of the elementary symmetric functions \( \sigma_r = \sigma_r(x_1, \ldots, x_n) \) rather than in terms of the \( \sigma_j^r \)’s.
To do so, we note that

\[ \sigma_j^k = \sum_{r=0}^{k} (-1)^r \sigma_{k-r} x_j^r. \]  

(3.9)

(To derive (3.9) observe that \( \prod_{\ell \neq j} (x - x_\ell) = \prod_{\ell} (x - x_\ell) \frac{1}{x - x_j} = \prod_{\ell} (x - x_\ell) \frac{1}{x} \sum_{i=0}^{\infty} \left( \frac{x_j}{x} \right)^i \) and compare coefficients of \( x^{n-k-1} \) on both sides.) Substituting (3.9) into (3.6) one gets an alternative inversion formula for the Vandermonde matrix

\[ b_{ij} = \sum_{r=0}^{n-i} (-1)^{n-i-r} \sigma_{n-i-r} x_j^r \prod_{\ell \neq j} (x_j - x_\ell). \]  

(3.10)

Finally we note a couple of trivial consequences of (3.7) and (3.8). From (3.7) and the identity

\[ \sum_j b_{nj} a_{jk} = \delta_k^n, \]

we conclude that the sum

\[ \sum_j \frac{x_j^{k-1}}{\prod_{\ell \neq j} (x_j - x_\ell)} \]

is zero if \( k \) is less than \( n \) and 1 if \( k = n \); and from (3.8) and the identity

\[ \sum_j b_{1j} a_{jk} = 1, \]

we conclude that

\[ \sum_j \prod_{\ell \neq j} \frac{-x_\ell}{x_j - x_\ell} = 1. \]  

(3.11)

In the applications which we will make of these identities below the \( x_i \)'s will be indeterminants and the \( f_i \)'s polynomials in these indeterminants, and we will want to know when the \( g_i \)'s are also polynomials in these indeterminants. To answer this question we will show that these identities have a simple “topological” interpretation: Suppose one is given a graph, \( \Gamma \), and an action of \( G \) on \( \Gamma \) defined by an axial function, \( \alpha : E_\Gamma \to g^* \). One of the main results of an earlier paper of ours is that there is a canonical integration operation

\[ \int_{\Gamma} : H(\Gamma, \alpha) \to S(g^*) \]

defined by

\[ \int_{\Gamma} f = \sum_{p \in V_\Gamma} f_p \delta_p \]  

(3.12)
where
\[ \delta_p = \left( \prod_{i(e)=p} \alpha_e \right)^{-1}. \]

(See [GZ1, § 2.4]. This formula is the formal analogue of the standard localization theorem [AB - BV] in equivariant DeRham theory.)

In particular let \( \Delta \) be the complete graph on \( n \) vertices. Denote these vertices by \( 1, \ldots, n \), and let \( x_1, \ldots, x_n \) be a basis of \( \mathfrak{g}^* \). It is easy to check that the map
\[ \alpha : E_\Delta \to \mathfrak{g}^*, \]
which assigns the weight \( x_i - x_j \) to the edge joining \( i \) to \( j \), is an axial function, and that the map
\[ \tau : V_\Delta \to \mathfrak{g}^*, \quad \tau(i) = x_i \]
is an element of \( H^1(\Delta, \alpha) \). We claim that \( 1, \tau, \ldots, \tau^{n-1} \) generate \( H(\Delta, \alpha) \) as a module over \( \mathbb{S}(\mathfrak{g}^*) \). To see this let \( \nu_i \) be the cohomology class
\[ \nu_i = \sum_{r=0}^{n-i} (-1)^{n-i-r} \sigma_{n-i-r} \tau^r, \quad i = 1, \ldots, n. \]

Then (3.10) simply asserts that
\[ \int_\Delta \nu_i \tau^{j-1} = \delta_i^j. \]

In particular if \( f \) is any cohomology class, then one can express \( f \) as a sum
\[ f = \sum_{i=1}^n g_i \tau^{i-1} \]
where
\[ g_i = \int_\Delta \nu_i f \in \mathbb{S}(\mathfrak{g}^*). \]

This proves the assertion:

**Proposition 3.2.** If \( f_1, \ldots, f_n \) are polynomials in \( x_1, \ldots, x_n \) and the function
\[ p(x) = \sum_{i=1}^n g_i x^{i-1} \]
solves the interpolation problem
\[ p(x_i) = f_i \]
then the \( g_i \)'s are polynomials in \( x_1, \ldots, x_n \) if and only if \( x_i - x_j \) divides \( f_i - f_j \).

**Proof.** If \( x_i - x_j \) divides \( f_i - f_j \) the map
\[ f : V_\Delta \to \mathbb{S}(\mathfrak{g}^*), \quad i \to f_i \]
is in \( H(\Delta, \alpha) \). \( \square \)
Let’s come back now to Theorem 2.8 and the application of it which we discussed at the end of Section 2. As in Theorem 2.8 let \( c \) and \( c' \) be regular values of \( \phi \), and suppose that there is just one vertex, \( p \), of \( \Gamma \) with \( c < \phi(p) < c' \). By setting \( f_1 = f_2 = \cdots = f_{s-1} = 0 \) in (2.14) one gets a map

\[
T_{c,c'} : H(\Gamma_c, \alpha_c) \to H(\Gamma_{c'}, \alpha_{c'})
\]

(3.13)

sending \( f_0 \) to \( f'_0 \), and by the results above one can give a fairly concrete description of this map. Let’s order the edges \( e_1, \ldots, e_d \in E_p \) so that \( e_1, \ldots, e_r \) are descending and \( e_{r+1}, \ldots, e_d \) are ascending, and let \( \Delta_c \) and \( \Delta'_{c'} \) be the vertices of \( \Gamma_c \) and \( \Gamma_{c'} \) corresponding to the \( e_j \)'s, \( 1 \leq j \leq r \), and the \( e_a \)'s, \( r+1 \leq a \leq d \). Then

\[
V_c = V_0 \cup \Delta_c
\]

and

\[
V_{c'} = V_0 \cup \Delta'_{c'},
\]

\( V_0 \) being the vertices which are common to \( \Gamma_c \) and \( \Gamma_{c'} \). To simplify notation we will identify \( \Delta_c \) with the set \( \{1, \ldots, r\} \) and \( \Delta'_{c'} \) with the set \( \{r+1, \ldots, d\} \). Let \( f_0 \) be in \( H(\Gamma_c, \alpha_c) \) and let \( f'_0 = T_{c,c'}(f_0) \). Then, by (2.14) and (3.8)

\[
f'_0(a) = \sum_{j=1}^{r} \prod_{k \neq j} \frac{\beta_a - \beta_k}{\beta_j - \beta_k} f_0(j),
\]

(3.14)

for \( a \) in \( \Delta_{c'} \) and \( j \) and \( k \) in \( \Delta_c \); and

\[
f'_0 = f_0 \quad \text{on } V_0.
\]

The identity (3.14) has the following simple interpretation. Let

\[
p(x) = \sum_{j=1}^{r} \prod_{k \neq j} \frac{x - \beta_k}{\beta_j - \beta_k} f_0(j).
\]

(3.16)

Then, by (3.4), \( p(x) \) solves the interpolation problem

\[
p(\beta_j) = f_0(j).
\]

(3.17)

On the other hand, by Theorem 2.9

\[
f_0|_{\Delta_c} \in H(\Delta_c, \tau_c);
\]

so, by Proposition 3.2, \( p(x) \) is a polynomial in \( x, \beta_1(y), \ldots, \beta_r(y) \) and hence also a polynomial in \( (x, y_1, \ldots, y_{n-1}) \), i.e., an element of the ring, \( S(\mathfrak{g}^*) \). In fact, if \( f_0 \in H^m(\Gamma_c, \alpha_c) \) and \( r > m \), \( p(x) \) is the unique element of \( S^m(\mathfrak{g}^*) \) satisfying (3.17). Now by (3.14), \( p(\beta_a) = f'_0(a) \), so (3.14) simply says that

\[
f'_0|_{\Delta_{c'}} = p(\tau_{c'}).
\]

Thus to summarize, we have proved:

**Theorem 3.3.** The map

\[
T_{c,c'} : H^m(\Gamma_c, \alpha_c) \to H^m(\Gamma_{c'}, \alpha_{c'})
\]
is the identity map on \( V_0 \) and on \( \Delta_c \) is the “flip–flop”

(3.18) \( f_0 = p(\tau_c) \to p(x) \to f'_0 = p(\tau_{c'}). \)

Since \( V_c \) and \( V_{c'} \) are finite sets, the map \( T_{c,c'} \) is defined by a matrix with entries

\[
T_{c,c'}(v,v'), \quad (v,v') \in V_c \times V_{c'}.
\]

An important property of this matrix is the Markov property:

(3.19) \[ \sum_{v \in V_c} T_{c,c'}(v,v') = 1. \]

**Proof.** It suffices to check this for \( a \in \Delta_{c'} \), i.e. it suffices to check that

\[
\sum_{j=1}^{r} T_{c,c'}(j,a) = 1.
\]

However, by (3.14), this sum is equal to

\[
\sum_{j=1}^{r} \prod_{k \neq j} \frac{\beta_a - \beta_k}{\beta_j - \beta_k}
\]

which is equal to 1 by (3.11), with \( x_\ell = \beta_\ell - \beta_a. \)

We will next give a more intrinsic description of \( T_{c,c'} \) and of the polynomial \( p \) in (3.16). We recall that

\[
\alpha_{e_i} = m_i(x - \beta_i(y)), \quad i = 1, \ldots, r
\]

and

\[
\alpha_{e_a} = m_a(x - \beta_a(y)), \quad a = r + 1, \ldots, d.
\]

Hence

\[
T_{c,c'}(j,a) = \prod_{k \neq j} \frac{\beta_a - \beta_k}{\beta_j - \beta_k} = \prod_{k \neq j} \frac{\alpha_{e_k} - (m_k/m_a)\alpha_{e_a}}{\alpha_{e_k} - (m_k/m_j)\alpha_{e_j}}
\]

and therefore

\[
T_{c,c'}(j,a) = \frac{\rho_{e_a}(\prod_{k \neq j} \alpha_{e_k})}{\rho_{e_j}(\prod_{k \neq j} \alpha_{e_k})} f_0(j),
\]

where \( \rho_e \) is the map (2.6). Similarly the polynomial \( p \) is just

(3.20) \[ \sum_{j} \frac{\prod_{k \neq j} \alpha_{e_k}}{\rho_{e_j}(\prod_{k \neq j} \alpha_{e_k})} f_0(j). \]

By iterating (3.13) we will extend the definition of \( T_{c,c'} \) to arbitrary regular values of \( \phi \) with \( c < c' \). Let

\[
c_i, i = 0, \ldots, \ell
\]

be regular values of \( \phi \) with \( c_0 = c \) and \( c_\ell = c' \) such that there exists a unique vertex, \( p_i \), with \( c_{i-1} < \phi(p_i) < c_i. \) Let \( T_i = T_{c_{i-1}, c_i} \) and let

(3.21) \[ T : H(\Gamma_c, \alpha_c) \to H(\Gamma_{c'}, \alpha_{c'}) \]
be the map
\[(3.22) \quad T = T_\ell \circ \cdots \circ T_1.\]

We will list a few properties of this map:

1). This map is defined by a matrix with entries
\[T(v, v') \in V_c \times V_{c'}\]
and since all the factors on the right hand side of (3.22) have the Markov property (3.19), this matrix also has this property.

2). The matrix version of (3.22) asserts that
\[T(v, w) = \sum T_\ell(v_{\ell-1}, w) \cdots T_2(v_1, v_2)T_1(v, v_1)\]
summed over all sequences \(v_1, \ldots, v_{\ell-1}\) with \(v_k \in V_c\). By (3.14) and (3.15), a large number of the matrix entries in this formula are either 1 or 0: If \(e_k\) is an ascending edge which intersects \(\Gamma_c\) in \(v_k\) and \(\Gamma_{c-1}\) in \(v_{k-1}\), then
\[T(v', v_k) = \begin{cases} 0 & \text{if } v' \neq v_{k-1}, \\ 1 & \text{if } v' = v_{k-1}. \end{cases}\]

This fact can be exploited to write the sum above more succinctly. For every pair of edges, \(e\) and \(e'\), with \(t(e) = i(e') = p\), let \(e_1, \ldots, e_r\) be the descending edges in \(E_p\), ordered so that \(e_r = \bar{e}\) and let
\[Q(e, e') = \frac{\rho_{e'}(\prod_{i=1}^{r-1} \alpha_{e_i})}{\rho_e(\prod_{i=1}^{r-1} \alpha_{e_i})}.\]

Then \(T(v, w)\) can be written as a weighted sum:
\[T(v, w) = \sum \gamma Q(\gamma)\]
over all ascending paths, \(\gamma\), in \(\Gamma\) whose initial edge intersects \(\Gamma_c\) in \(v\) and whose terminal edge intersects \(\Gamma_{c'}\) in \(w\), the weighting of the path, \(\gamma\), being given by
\[(3.23) \quad Q(\gamma) = \prod_{i=1}^{m} Q(e_{i-1}, e_i),\]
where the \(e_i\)'s are the edges of \(\gamma\), ordered so that for \(i > 1\), \(t(e_{i-1}) = i(e_i)\).

3). The map (3.22) can also be viewed as a series of “flip–flops”. Let \(f_0\) be an element of \(H(\Gamma_c, \alpha_c)\) and let \(f_i = T_i \circ \cdots \circ T_1 f\). Then \(T_i\) maps \(f_{i-1}\) to \(f_i\) by a map of the form (3.18). Let's denote the polynomial, \(p\), in (3.18) by \(\psi_{p_i}\). We claim:

**Proposition 3.4.** If \(p_i\) is joined to \(p_j\) by an ascending edge, \(e,\)
\[\psi_{p_i} \equiv \psi_{p_j} \mod \alpha_e.\]
Proof. This is equivalent to asserting that
(3.24) \[ \rho e \psi_{p_i} = \rho e \psi_{p_j}; \]
however, (3.24) is, by definition, the common value of \( f_k(v_k), i \leq k < j, \) at the
vertices, \( v_k, \) at which \( e \) intersects \( \Gamma_{c_k}. \)

4). In particular let \( p_0 \) be an arbitrary vertex of \( \Gamma; \) and choose \( c \) and \( c' \)
such that there are no critical values of \( \phi \) on the interval, \((\phi(p_0), c)\) and such that
\( c' > \max \phi(p), p \in V_{\Gamma}. \) Order the edges \( e_1, \ldots, e_d \) in \( E_{p_0} \) so that \( e_1, \ldots, e_r \) are
descending and \( e_{r+1}, \ldots, e_d \) are ascending. For \( r+1 \leq a \leq d \) let \( v_a \) be the vertex
at which \( e_a \) intersects \( \Gamma_{c_a} \) and let
(3.25) \[ f_0 : V_{\Gamma_{c}} \rightarrow S^r(g^*) \]
be the map defined by
(3.25) \[ f_0(v) = \begin{cases} 0, & \text{if } v \notin \{v_{r+1}, \ldots, v_d\} \\ \rho e_a (\prod_{i=1}^{r} \alpha_{e_i}), & \text{if } v = v_a. \end{cases} \]

Proposition 3.5. \( f_0 \) is an element of \( H(\Gamma_{c}, \alpha_{c}). \)

Proof. By (2.7), \( f_0 = K_{c} \tau_{p_0}^+. \) (This proof assumes that there exists a Thom class, \( \tau_{p_0}^+, \)
having the properties listed in Theorem 1.5. Alternatively, Proposition 3.5 can
be proved directly using a more sophisticated definition of \( H(\Gamma_{c}, \alpha_{c}) \) than that which
we gave in Section 1. For more details see [GZ3, § 4].)

By applying the sequence of flip–flops, \( T_i, \) to the \( f_0 \) above, we get a polynomial,
\( \psi_{p_i} \in S^r(g^*) \) for each vertex, \( p_i, \) of \( \Gamma \) with \( \phi(p_i) > c. \) On the other hand, we can
define \( \tau_p \) for \( \phi(p) < c \) to be equal to (2.3) at \( p_0 \) and equal to zero otherwise. By
(3.24) \( \tau_p \) satisfies the cocycle condition (1.10) at all vertices except \( p_0, \) and by (3.25)
it satisfies this condition at \( p_0 \) as well. Thus, if the index function, \( \sigma : V_{\Gamma} \rightarrow \mathbb{Z}, \)
is strictly increasing along ascending paths, this settles the existence part of Lemma 3.1
and justifies the induction method for constructing \( \tau_{p_0}^+ \) which we outlined at the
beginning of this section. On the other hand, if \( \sigma \) fails to satisfy this hypothesis, the
assignment, \( p \rightarrow \tau_p, \) still defines an element of \( H(\Gamma, \alpha) \) with the properties listed in
Theorem 1.5; however, it won’t be the only element with these properties and may
not even be the optimal element with these properties.

5). From (3.23) one gets the following “path integral” formula for \( \tau_{p}^+. \) If \( e \) is an
ascending edge of \( \Gamma, \) let \( p = t(e) \) and let \( e_1, \ldots, e_r \) be the descending edges in \( E_p, \)
ordered so that \( e_r = \bar{e}. \) Let
(3.26) \[ Q(e) = \frac{\prod_{i=1}^{r-1} \alpha_{e_i}}{\rho e (\prod_{i=1}^{r-1} \alpha_{e_i})}. \]
Then by (3.20), (3.23) and (3.25)
(3.27) \[ \tau_{p_0}^+(p) = \sum E(\gamma) \]
summed over all ascending paths in $\Gamma$ joining $p_0$ to $p$, $E(\gamma)$ being defined by
\[(3.28) \quad E(\gamma) = Q(e_m)Q(\gamma)\rho_{e_1}(\nu_{p_0}^+) ,\]
where $e_1$ is the initial edge of $\gamma$ and $e_m$ is the terminal edge of $\gamma$.

4. Combinatorial intersection numbers

We will show below how to recast the formula (3.28) into the form (1.12) and will also show that, if the hypothesis of Theorem 1.6 is satisfied, one can deduce from (1.12) the formula that we described in Section 1 for the products of Thom classes. First, however, we will examine this hypothesis in more detail: Suppose the graph $\Gamma$ is connected and admits a family of Thom classes, $\tau_p^+, p \in V_\Gamma$, which generates $H(\Gamma, \alpha)$ as a free module over the ring $\mathbb{S}(g^*)$, and have the properties (2.4) and (2.5).

By (1.10)
\[\dim H^0(\Gamma, \alpha) = 1 ;\]
hence there is a unique vertex, $p_0$, with $\sigma_{p_0} = 0$. Let $p$ be an arbitrary vertex of $\Gamma$ and let $\gamma$ be an ascending path with terminal endpoint $p$. If $\gamma$ is of maximal length, its initial vertex has to be $p_0$, since every other vertex has a descending edge. Let $\phi(p)$ be the length of this longest path. If $p$ can be joined to $q$ by an ascending edge, $\phi(p)$ is strictly less than $\phi(q)$, so the map
\[\phi : V_\Gamma \to \mathbb{Z} \quad , \quad p \to \phi(p) ,\]
is a Morse function.

**Theorem 4.1.** The index function, $\sigma$, is strictly increasing along ascending paths if and only if $\phi = \sigma$ (i.e. if and only if the Morse function, $\phi$, is self-indexing.) Moreover, if $\phi$ has this property then, for every pair of vertices, $p \in V_\Gamma$ and $q \in F_p$, the length of the longest ascending path from $p$ to $q$ is $\sigma_q - \sigma_p$.

It suffices to prove the last assertion, and it suffices by induction to prove this assertion for paths of length one. This we will do by proving a slightly stronger assertion.

**Theorem 4.2.** Let $e$ be an ascending edge joining $p$ to $q$. If $e$ is the only ascending path from $p$ to $q$ then
\[\sigma_q \leq \sigma_p + 1 .\]

**Proof.** Let $\Gamma_e$ be the totally geodesic subgraph of $\Gamma$ consisting of the single edge, $e$, and vertices $p$ and $q$. The Thom class, $\tau_e$, of $\Gamma_e$ is defined by
\[\tau_e(p) = \prod_{\substack{i(e') = p \quad \text{if} e' \neq e}} \alpha_{e'} , \quad \tau_e(q) = \prod_{\substack{i(e'') = q \quad \text{if} e'' \neq \bar{e}}} \alpha_{e''} \]
and
\[\tau_e(r) = 0 \quad \text{if} \quad r \neq p, q .\]
It is easily checked that $\tau_e \in H^{d-1}(\Gamma, \alpha)$.

**Lemma 4.3.** A cohomology class, $\tau \in H(\Gamma, \alpha)$ is supported on $\{p, q\}$ iff $\tau = h\tau_e$, $h \in H(\Gamma_e, \alpha)$.
Suppose now that \( e \) satisfies the hypotheses of Theorem 4.2. Then \( \tau_p^+ \tau_q^- \) is supported on \( \{ p, q \} \); so, by the lemma,

\[
\tau_p^+ \tau_q^- = h \tau_e , \quad h \in H(\Gamma_e, \alpha) .
\]

In particular

\[
\sigma_p + d - \sigma_q = \deg \tau_p^+ + \deg \tau_q^- \geq \deg \tau_e = d - 1 ,
\]

so \( \sigma_q \leq \sigma_p + 1 \). \( \square \)

Coming back to the formula (3.27) lets first consider the simplest summands in this formula, those associated with paths, \( \gamma \), of length one. For each \( q \in \Gamma \) denote by \( E_q^- \) and \( E_q^+ \) the descending and ascending edges in \( E_q \) and let \( \nu_q \) be defined as in (2.5). Let \( \gamma \) be an ascending path of length one consisting of a single edge, \( e \), with \( i(e) = p \) and \( t(e) = q \). Then by (3.26) and (3.28)

\[
E(\gamma) = \frac{\nu_q \cdot \prod' \rho_e(\alpha e_i)}{-\alpha_e \prod'' \rho_e(\alpha e'_j)} ,
\]

where \( \prod' \) in the numerator is a product over the edges \( e_i \in E_p^- \) and \( \prod'' \) in the denominator is the product over the edges \( e'_j \in E_q^- - \{ qp \} \). Let

\[
\theta_e : E_p \to E_q
\]

be the connection along this edge and let \( \theta_e = \theta_e^{-1} : E_q \to E_p \). We define

\[
E_{p,q} = \{ e' \in E_p^- ; \theta_e(e') \notin E_q^- \}
\]

and

\[
E_{q,p} = \{ e'' \in E_q^- ; \theta_e(e'') \notin E_p^- \} - \{ e \} .
\]

Note that \( \theta_e \) restricts to a bijection

\[
\theta_e : E_p - E_{p,q} \to E_q - E_{p,q} .
\]

If \( e' \in E_p \), then (2.3) implies

\[
\rho_e(\alpha e') = \rho_e(\alpha \theta_e(e')) .
\]

Therefore if \( e_i \in E_p - E_{p,q} \), then the terms corresponding to \( e_i \) and \( \theta_e(e_i) \) in (4.2) cancel each other and we obtain

\[
\frac{E(\gamma)}{\nu_p} = \frac{\nu_q \cdot \rho_e(Z_{p,q})}{-\alpha_e \nu_p \rho_e(Z_{q,p})} = \frac{-\nu_q}{\alpha_{pq} \nu_p} \Theta_{pq} ,
\]

where

\[
Z_{p,q} = \prod_{e' \in E_{p,q}} \alpha e' , \quad Z_{q,p} = \prod_{e'' \in E_{q,p}} \alpha e'' ,
\]

and

\[
\Theta_{pq} = \frac{\prod' \rho_e(\alpha e_i)}{\prod'' \rho_e(\alpha e'_j)} = \frac{\rho_e(Z_{p,q})}{\rho_e(Z_{q,p})} .
\]
If $\gamma$ is the only ascending path from $p$ to $q$, then $\Theta_{p,q}$ has an interpretation as an "intersection number": By (4.1) the quotient,

$$\frac{\tau_p^+ \tau_q^-}{\tau_e}$$

is an element of $H(\Gamma_e, \alpha_e)$. Let $e$ be a point on the interval $(\phi(p), \phi(q))$ and let $v_e$ be the vertex of $\Gamma_e$ corresponding to $e$. If we apply the Kirwan map

$$K_e : H(\Gamma, \alpha) \rightarrow H(\Gamma_e, \alpha_e)$$

to this quotient and evaluate at $v_e$ we get an element of $S(g^*_\xi)$. We claim

$$(4.7) \quad \frac{\Theta_{p,q}}{\alpha_e(\xi)} = K_e\left(\frac{\tau_p^+ \tau_q^-}{\tau_e}\right)(v_e).$$

**Proof.** A direct computation shows that

$$K_e(\tau_p^+)(v_e) = \prod' \rho_e(\alpha_{e'}) \quad \text{and} \quad K_e(\tau_q^-)(v_e) = \frac{K_e(\tau_e)(v_e)}{\prod'' \rho_e(\alpha_{e''})},$$

hence, (4.7) follows from (4.6). □

We will now show that the right hand side of (4.7) can be interpreted as a “pairing” of the cohomology classes $K_e(\tau_p^+)$ and $K_e(\tau_q^-)$. We pointed out in Section 3 that the localization formula in equivariant DeRham theory enables one to define an integration operation on $H(\Gamma, \alpha)$. The analogue of this result for $\Gamma_e$ asserts that there is an integration operation

$$\int_{\Gamma_e} : H(\Gamma_e, \alpha_e) \rightarrow S(g^*_\xi)$$

mapping $f \in H(\Gamma_e, \alpha_e)$ to the sum

$$\sum_{v \in V_e} f(v)\delta_v,$$

where

$$\delta_v = (K_e(\tau_e)(v))^{-1},$$

$e$ being the edge of $\Gamma$ which intersects $\Gamma_e$ of the vertex $v = v_e$.

In particular, consider the product in $H(\Gamma_e, \alpha_e)$ of $K_e(\tau_p^+)$ and $K_e(\tau_q^-)$. If $e$ is the only ascending path in $\Gamma$ joining $p$ to $q$ this product is zero except at the point $v_e$; so by (4.7)

$$(4.8) \quad \frac{\Theta_{p,q}}{\alpha_e(\xi)} = \int_{\Gamma_e} K_e(\tau_p^+)K_e(\tau_q^-),$$

which is the formal analogue of the intersection number (1.4).

**Remarks:**

1. By Theorem 1.4, $\sigma_q \leq \sigma_p + 1$. One can see by inspection that the right hand side of (4.8), which is by definition an element of $S(g^*_\xi)$, is of degree $\sigma_p + 1 - \sigma_q$.

In particular, if $\sigma$ is a self-indexing Morse function, the right hand side of (4.8) is just a constant.
2. If the edge, \( e \), is not the only path joining \( p \) to \( q \), the identity (4.7) is still true; however the right hand side of (4.7) is in \( Q(q^*_e) \) and has to be interpreted as the formal analogue of the local intersection number (4.5).

We now return to the general case.

Let \( p \stackrel{\gamma'}{\rightarrow} q \) be an ascending path from \( p \) to \( q \), let \( q \stackrel{\gamma''}{\rightarrow} r \) be an ascending path from \( q \) to \( r \), and let \( \gamma : p \stackrel{\gamma'}{\rightarrow} q \stackrel{\gamma''}{\rightarrow} r \) be the ascending path from \( p \) to \( r \) obtained by joining \( \gamma' \) and \( \gamma'' \). A direct computation shows that

\[
\frac{E(\gamma)}{\nu_p} = \frac{E(\gamma')}{\nu_p} \cdot \frac{E(\gamma'')}{\nu_q} \cdot \frac{\alpha_{e_i}}{\rho_{e_a}(\alpha_{e_i})},
\]

where \( e_i \) is the last edge of \( \gamma' \) and \( e_a \) is the first edge of \( \gamma'' \), both pointing upward.

Let \( \gamma : p = p_0 \rightarrow p_1 \rightarrow \ldots \rightarrow p_{m-1} \rightarrow p_m = q \) be an ascending path. We will express the contribution \( E(\gamma) \) by breaking up the path \( \gamma \) into its constituent edges. Then

\[
\frac{E(\gamma)}{\nu_p} = \frac{E(pp_1)}{\nu_p} \cdot \ldots \cdot \frac{E(p_{m-1}q)}{\nu_{p_{m-1}}} \cdot \prod_{k=1}^{m-1} \frac{\alpha_{p_{k-1}p_k}}{\rho_{p_{k+1}p_k}(\alpha_{p_{k-1}p_k})}
\]

Therefore the contribution \( E(\gamma) \) of the path \( \gamma \) is

\[
(4.10) \quad E(\gamma) = \nu_q \cdot \left( \prod_{k=1}^{m} \Theta_{p_{k-1}p_k} \right) \cdot \frac{(-1)^m}{\alpha_{p_{m-1}q} \prod_{k=1}^{m-1} \rho_{p_{k+1}p_k}(\alpha_{p_{k-1}p_k})}.
\]

In view of (4.8) we can also write this in the form (1.12), \( e_i \) being the edge of \( \Gamma \) joining \( p_{i-1} \) to \( p_i \) and \( \iota_e \) being the local intersection number (4.7).

If we reverse the orientation of \( \Gamma \) replacing \( \xi \) with \( -\xi \) and the Morse function \( \phi \) by \( -\phi \), we get a formula similar to (1.11) for \( \tau_p^- \)

\[
(4.11) \quad \tau_p^-(q) = \sum E(\gamma),
\]

the sum being over descending paths from \( p \) to \( q \).

Moreover, the \( E(\gamma) \)'s in (4.11) are easy to compute in terms of the \( E(\gamma) \)'s in (1.12). To see this lets consider as above the simplest example of an ascending path in \( \Gamma \), an ascending edge, \( e \), joining \( p \) to \( q \). By (4.3) and (4.4)

\[
\theta_e E_{p,q} = \{ e'' \in E_q^+ \mid \theta_e e'' \in E_p^+ \}
\]

and

\[
\theta_e E_{q,p} = \{ e' \in E_p^+ \mid \theta_e e' \in E_q^- \} - \{ e \}
\]

so, by (4.5) and (4.6)

\[
(4.12) \quad \Theta_{q,p} = \Theta_{p,q}.
\]
Now let $\gamma$ be an ascending path of length $m$ from $p$ to $q$ and let $\bar{\gamma}$ be the same path traced in the reverse direction. Then by (4.10) and (4.12)

$$E(\bar{\gamma}) = (-1)^m \frac{\hat{\alpha}_m}{\hat{\alpha}_1} \cdot \frac{\nu^-}{\nu_q} \cdot E(\gamma)$$

where

$$\nu^- = \prod_{e' \in E_p^+} \alpha_{e'} \ , \ \hat{\alpha}_m = \frac{\alpha_{e_m}(\xi)}{\alpha_{e_m}(\xi)} \ , \ \hat{\alpha}_1 = \frac{\alpha_{e_1}(\xi)}{\alpha_{e_1}(\xi)} \ .$$

We are now finally in position to compute the cohomology pairing (1.2). By (1.5), (3.12) and (4.11) the integral

$$c_{pqr} = \int_{\Gamma} \tau_p^{+} \tau_q^{-} \tau_r^{-}$$

is equal to the sum

$$\sum \delta_t E(\gamma_1) E(\gamma_2) E(\gamma_3)$$

summed over all triples $\gamma_1, \gamma_2, \gamma_3$ consisting of an ascending path, $\gamma_1$, from $p$ to $t$, an ascending path, $\gamma_2$, from $q$ to $t$, and a descending path, $\gamma_3$, from $r$ to $t$. (See Figure 4.1.)

![Figure 4.1. Configuration of paths](image)

Thus, in particular, if there exist no such configurations, $c_{pqr} = 0$. Now suppose that the hypothesis of Theorem 1.6 is satisfied, i.e. $\sigma$ is a self-indexing Morse function. Then we claim that

(4.13)

$$\int \tau_p^{+} \tau_q^{-} = \delta_{pq} \ .$$

In fact if $q \notin F_p$ the supports of $\tau_p^{+}$ and $\tau_q^{-}$ are non-overlapping so (4.13) is automatically zero; and if $q = p$, then the support of $\tau_p^{+} \tau_q^{-}$ consists of the single point $p$ and it is easy to verify that (4.13) is equal to one. Thus (4.13) is trivially true except when $q \in F_p$ and $q \neq p$. In this case however, $\sigma_q > \sigma_p$ so

$$k = \text{degree} \ \tau_p^{+} \tau_q^{-} = \text{degree} \ \tau_p^{+} + \text{degree} \ \tau_q^{-} = d - \sigma_q + \sigma_p < d$$
so the integral (1.13) is zero just by degree considerations. Thus if we substitute the sum
\[ \sum c_{pq}^s \tau_s^+ \]
for \( \tau_p^+\tau_q^+ \) in (1.12) we obtain for \( c_{pq}^s \) the formula (1.13).

5. Examples

Each of the summands in (1.12) is a rational function: an element of the quotient field, \( Q(g^*) \); however, the sum itself is a polynomial, so the singularities in the individual summands are mysteriously cancelling each other out. We will discuss below a few simple examples in which one can see how some of these cancellations are happening.

5.1. Cancellations occurring in the individual terms. Suppose \( \gamma \) is a longest ascending path from \( p \) to \( q \). Let \( e_1, \ldots, e_m \) be the edges of \( \gamma \) ordered so that \( t(e_{k-1}) = p_k = i(e_k) \). Then \( e_k \) is the only path joining \( p_k \) to \( p_{k+1} \); hence the intersection numbers, \( \iota_{e_k} \), are all global intersection numbers of the form (4.8) and are in \( S(g^*) \). Hence the factor
\[ \prod \iota_{e_k} \]
in the formula (1.12) is in \( S(g^*) \). If, in addition, the Morse function \( \phi \) is self-indexing, this factor is a polynomial of degree zero, i.e. is just a constant.

5.2. Nearby paths. Suppose \( \Gamma \) contains a totally geodesic subgraph of the form shown in Figure 5.1.

![Figure 5.1. Nearby paths](image)

Let \( \gamma \) be the path consisting of the single edge, \( e \), joining \( p \) to \( q \) and let \( \gamma_1 \) be the path \( p \to r \to q \). Assume \( \gamma_1 \) is a longest path joining \( p \) to \( q \) and that \( \sigma_q = \sigma_p + 2 \). We claim that
\[ E(\gamma) + E(\gamma_1) = \frac{\nu_q}{\alpha_e \alpha_{e''}} \].
Proof. We first note that
\begin{equation}
\alpha_e = \alpha_e' + \alpha_e'' .
\end{equation}
(This is a consequence of the compatibility conditions
\[-\alpha_e'' = \alpha_e' + c_1 \alpha_e \quad \text{and} \quad \alpha_e'' = \alpha_e + c_2 \alpha_e'\]
from which one concludes that \(c_1 = c_2 = -1\).)

Let \(\gamma'\) be the path joining \(p\) to \(r\) and \(\gamma''\) the path joining \(r\) to \(q\). By (4.2)
\[E(\gamma) = \frac{\nu_q}{\alpha_e} \cdot \frac{1}{\rho_e(\alpha_e')} , \quad E(\gamma') = -\frac{\nu_r}{\alpha_e'} , \quad E(\gamma'') = -\frac{\nu_q}{\alpha_e''} \]
and by (4.3)
\[E(\gamma_1) = \frac{E(\gamma')}{\nu_r} \cdot \frac{E(\gamma'')}{\rho_e''(\alpha_e')} \cdot \alpha_e' .\]

Hence
\[E(\gamma_1) = \frac{1}{\nu_r} \left( -\frac{\nu_r}{\alpha_e'} \right) \left( -\frac{\nu_q}{\alpha_e''} \right) \frac{\alpha_e'}{\rho_e''(\alpha_e')} = \frac{\nu_q}{\alpha_e'' \rho_e''(\alpha_e')} ;\]

However, by (5.1), \(\rho_e''(\alpha_e') = \rho_e''(\alpha_e - \alpha_e'') = \rho_e''(\alpha_e)\); hence we can rewrite this as
\[E(\gamma_1) = \frac{\nu_q}{\alpha_e'' \rho_e''(\alpha_e)} ;\]

so \(E(\gamma) + E(\gamma_1)\) is equal to the expression :
\[\nu_q \left( -\frac{1}{\alpha_e \rho_e(\alpha_e')} + \frac{1}{\alpha_e'' \rho_e''(\alpha_e)} \right) .\]

However,
\[\rho_e(\alpha_e''') = -\frac{\alpha_e'''(\xi)}{\alpha_e(\xi)} \rho_e''(\alpha_e) ,\]

so the term in parentheses can be rewritten
\[
\frac{1}{\alpha_e \rho_e(\alpha_e''')} - \frac{\alpha_e''(\xi)}{\alpha_e(\xi)} \frac{1}{\alpha_e'' \rho_e''(\alpha_e'')} = \frac{1}{\rho_e''(\alpha_e)} \left( \frac{1}{\alpha_e} - \frac{\alpha_e''(\xi)}{\alpha_e(\xi)} \right) \frac{1}{\alpha_e''} =
\]
\[
= \frac{1}{\alpha_e''} \cdot \frac{\alpha_e''(\xi)}{\alpha_e(\xi)} \frac{1}{\alpha_e''} = \frac{1}{\alpha_e''} . \quad \square
\]

5.3. The flag variety \(G = SL(n, \mathbb{C})/B\). Graph theoretically, the flag variety \(SL(n, \mathbb{C})/B\) is the permutedehedron: a Cayley graph associated with the Weyl group of \(SL(n, \mathbb{C})\), the symmetric group \(S_n\). Each vertex of this graph corresponds to a permutation \(\pi \in S_n\), and two permutations \(\pi\) and \(\pi'\), are adjacent in \(\Gamma\) if and only if there exists a transposition \(\tau_{ij}\), \(1 \leq i < j \leq n\) with \(\pi' = \pi \tau_{ij}\). Moreover, if \(e\) is the edge joining \(\pi\) to \(\pi \tau_{ij}\), the weight labeling \(e\) is
\[\alpha_e = \begin{cases} 
\epsilon_j - \epsilon_i , & \text{if } \pi(j) > \pi(i) \\
\epsilon_i - \epsilon_j , & \text{if } \pi(j) < \pi(i),
\end{cases}\]
where \(\epsilon_1, \ldots, \epsilon_n\) is the standard basis vectors of the lattice \(\mathbb{Z}^n\). The connection \(\theta_e\) along this edge is given by
\[\theta_{\pi, \pi \tau}(\pi, \pi' \tau) = (\pi \tau, \pi' \tau) .\]
If $\xi = (\xi_1, ..., \xi_n) \in \mathcal{P}$, with $\xi_1 < ..., \xi_n$, then the function
\[ \phi : V_\Gamma \to \mathbb{Z}, \quad \phi(\pi) = \text{length}(\pi) \]
is a self-indexing $\xi$-compatible Morse function on $\Gamma$.

The permutahedron is a bi-partite graph, with the two sets of vertices corresponding to even, respective odd permutations. In the special case $n = 3$, this graph is a complete bi-partite graph, and the corresponding labeling is shown in Figure 5.2.

![Figure 5.2. The flag variety](image)

Here $\alpha_1 = \epsilon_2 - \epsilon_1$ and $\alpha_2 = \epsilon_3 - \epsilon_2$, and we have used the notation (231) for the cycle $1 \to 2 \to 3 \to 1$.

The quantities $\Theta_{pq}$ given by (4.6) are all equal to 1, with the exception of $\Theta_{1,(13)}$, which is
\[ \Theta_{1,(13)} = \frac{1}{\rho_{\alpha_1 + \alpha_2}(\alpha_1 \alpha_2)} = -\frac{(\alpha_1(\xi) + \alpha_2(\xi))^2}{(\alpha_2(\xi)\alpha_1 - \alpha_1(\xi)\alpha_2)^2} \]

There are two ascending paths from (12) to (13), namely
\[ \gamma_1 : (12) \to (231) \to (13) \quad \text{and} \quad \gamma_2 : (12) \to (312) \to (13) \]
and their contributions to $\tau_{(12)(13)}$ are
\[ E(\gamma_1) = -\alpha_1 \alpha_2(\alpha_1 + \alpha_2) \cdot \frac{1}{\alpha_1} \cdot \frac{1}{\rho_{\alpha_1}(\alpha_1 + \alpha_2)} = \frac{\alpha_1(\xi)\alpha_2(\alpha_1 + \alpha_2)}{\alpha_2(\xi)\alpha_1 - \alpha_1(\xi)\alpha_2} \]
and
\[ E(\gamma_2) = -\alpha_1 \alpha_2(\alpha_1 + \alpha_2) \cdot \frac{1}{\alpha_2} \cdot \frac{1}{\rho_{\alpha_2}(\alpha_1 + \alpha_2)} = -\frac{\alpha_2(\xi)\alpha_1(\alpha_1 + \alpha_2)}{\alpha_2(\xi)\alpha_1 - \alpha_1(\xi)\alpha_2}, \]
so

\[ \tau_{(12)}(13) = E(\gamma_1) + E(\gamma_2) = -\alpha_1 - \alpha_2. \]

The other classes can be computed similarly and are given by

\[
\begin{array}{cccccccc}
\tau_1 & \tau_{(12)} & \tau_{(23)} & \tau_{(231)} & \tau_{(312)} & \tau_{(13)} \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
(12) & 1 & -\alpha_1 & 0 & 0 & 0 & 0 \\
(23) & 1 & 0 & -\alpha_2 & 0 & 0 & 0 \\
(231) & 1 & -\alpha_1 - \alpha_2 & -\alpha_2 & \alpha_2(\alpha_1 + \alpha_2) & 0 & 0 \\
(312) & 1 & -\alpha_1 & -\alpha_1 - \alpha_2 & 0 & \alpha_1(\alpha_1 + \alpha_2) & 0 \\
(13) & 1 & -\alpha_1 - \alpha_2 & -\alpha_1 - \alpha_2 & \alpha_2(\alpha_1 + \alpha_2) & \alpha_1(\alpha_1 + \alpha_2) & -\alpha_1\alpha_2(\alpha_1 + \alpha_2)
\end{array}
\]

5.4. The zero-dimensional Thom class. Suppose the graph \( \Gamma \) is connected and hence has a unique vertex, \( p_0 \), of index zero. Then the Thom class, \( \tau_{p_0} \), is the unique generator of \( H^0(\Gamma, \alpha) \), with \( \tau_{p_0}(p_0) = 1 \). Thus

\[ \tau_{p_0}(p) = 1 \]

for all vertices \( p \). We will show how to deduce (5.2) from (1.11). Choose the constants, \( c \) and \( c' \), in (3.21), so that \( p_0 \) is the only vertex with \( \phi(p_0) < c \) and such that \( \phi(p) \) is the smallest critical value of \( \phi \) greater than \( c' \). By the Markov property of the map (3.22)

\[ 1 = \sum_{w \in V_c} Q(v, w) \]

for every vertex \( v \in V_c \). In particular let

\[ E^-_p = \{e_i \quad i = 1, \ldots, r\} \]

and let \( v_i \in V_c \) be the vertex at which \( e_i \) intersects \( \Gamma_c \). Then by (1.28)

\[ \tau_{p_0}(p) = \sum_{i=1}^{r} Q(e_i) \sum_{w} Q(v_i, w) = \sum_{w} Q(e_i) \]

and by (3.26)

\[ \tau_{p_0}(p) = \sum_{i=1}^{r} \prod_{j \neq i} \frac{\alpha_{e_j}}{\alpha_{e_j} - \left( \frac{\alpha_{e_j}(\xi)\alpha_{e_i}(\xi)}{\alpha_{e_i}(\xi)\alpha_{e_i}(\xi)} \right)\alpha_{e_j}}. \]

Letting

\[ x_i = -\frac{1}{\alpha_{e_i}(\xi)} \alpha_{e_i} \]

this becomes

\[ \sum_{i=1}^{r} \prod_{j \neq i} -\frac{x_j}{x_i - x_j}, \]

which is equal to 1 by (3.11). Thus \( \tau_{p_0}(p) = 1 \).
5.5. **The \((n-1)\)-dimensional projective space.** Graph theoretically this is just the complete graph, \(\Delta\), on \(n\) vertices. Let us denote these vertices by \(p_1, \ldots, p_n\) and as in Section 3 assign to the edge, \(e\), joining \(p_i\) to \(p_j\), the weight
\[
\alpha_e = x_i - x_j .
\]
(As we pointed out in Section 3 this defines an axial function on \(\Delta\).) Let \(\xi\) be an \(n\)-tuple of real numbers with \(\xi_1 > \xi_2 > \ldots > \xi_n\) and orient the edges of \(\Delta\) by decreeing that an edge, \(e\), is ascending if \(\alpha_e(\xi) > 0\). With this orientation, the function mapping \(p_i\) to \(i\) is a \(\xi\)-compatible Morse function. Lets compute the Thom class, \(\tau_{p_i}\). If \(i = 1\), we get from the computation above
\[
\tau_{p_1}(p) = 1
\]
for all vertices \(p\). If \(i > 1\), we can regard the vertices \(p_i, p_{i+1}, \ldots, p_n\) as the vertices of a complete graph, \(\Delta'\), having the same axial function as above. Consider the sum
\[
\sum_{\gamma} E(\gamma)
\]
over all ascending paths joining \(p = p_i\) to \(q = p_j\), where \(j > i\). The individual summands can be written in the form
\[
\frac{\nu_q}{\nu'_q} E'(\gamma) ,
\]
where by (1.12)
\[
E'(\gamma) = (-1)^m \nu_q \frac{\ell_{e_1}}{\alpha_m} \prod_{k=2}^{m} \frac{l_{e_k}}{\alpha_{k-1} - \alpha_k}
\]
and \(\nu'_q\) is the product
\[
\prod' \alpha_{e'}
\]
over all descending edge, \(e' \in E_q^-\), which join \(q\) to vertices in \(\Delta'\). Then (5.3) becomes
\[
\frac{\nu_q}{\nu'_q} \left( \sum E'(\gamma) \right) .
\]
However, by (1.11), the expression in parentheses computes the zeroth Thom class of the subgraph \(\Delta'\), at \(q\), and hence is equal to one. Thus
\[
\tau_{p_i}(q) = \frac{\nu_q}{\nu'_q} = \prod' \alpha_{e''} ,
\]
where \(\prod'\) is the product over all the edges \(e''\) of \(\Delta\), which join \(q\) to the vertices \(p_k\), \(k = 1, \ldots, i-1\).

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Department of Mathematics, MIT, Cambridge, MA 02139

E-mail address: vwg@math.mit.edu

Department of Mathematics, MIT, Cambridge, MA 02139

E-mail address: czara@math.mit.edu