Maximum Matching in Two, Three, and a Few More Passes Over Graph Streams

Sagar Kale∗ Sumedh Tirodkar† Sundar Vishwanathan‡

Abstract

We consider the maximum matching problem in the semi-streaming model formalized by Feigenbaum, Kannan, McGregor, Suri, and Zhang [13] that is inspired by giant graphs of today. As our main result, we give a two-pass \((1/2 + 1/16)\)-approximation algorithm for triangle-free graphs and a two-pass \((1/2 + 1/32)\)-approximation algorithm for general graphs; these improve the approximation ratios of \(1/2 + 1/52\) for bipartite graphs and \(1/2 + 1/140\) for general graphs by Konrad, Magniez, and Mathieu [20]. In three passes, we are able to achieve approximation ratios of \(1/2 + 1/10\) for triangle-free graphs and \(1/2 + 1/19.753\) for general graphs. We also give a multi-pass algorithm where we bound the number of passes precisely—we give a \((2/3 - \varepsilon)\)-approximation algorithm that uses \(2/(3\varepsilon)\) passes for triangle-free graphs and \(4/(3\varepsilon)\) passes for general graphs. Our algorithms are simple and combinatorial, use \(O(n \log n)\) space, and (can be implemented to) have \(O(1)\) update time per edge.

For general graphs, our multi-pass algorithm improves the best known deterministic algorithms in terms of the number of passes:

- Ahn and Guha [1] give a \((2/3 - \varepsilon)\)-approximation algorithm that uses \(O(\log(1/\varepsilon) / \varepsilon^2)\) passes, whereas our \((2/3 - \varepsilon)\)-approximation algorithm uses \(4/(3\varepsilon)\) passes;
- they also give a \((1 - \varepsilon)\)-approximation algorithm that uses \(O(\log n \cdot \text{poly}(1/\varepsilon))\) passes, where \(n\) is the number of vertices of the input graph; although our algorithm is \((2/3 - \varepsilon)\)-approximation, our number of passes do not depend on \(n\).

Earlier multi-pass algorithms either have a large constant inside big-O notation for the number of passes [9] or the constant cannot be determined due to the involved analysis [22, 1], so our multi-pass algorithm should use much fewer passes for approximation ratios bounded slightly below 2/3.

∗Department of Computer Science, Dartmouth College, Hanover, NH, USA. Email: sag (at) cs.dartmouth.edu.
†School of Technology and Computer Science, TIFR, Mumbai, India. Email: sumedh.tirodkar (at) tifr.res.in.
‡Department of Computer Science and Engineering, IITB, Mumbai, India. Email: sundar (at) cse.iitb.ac.in.
1 Introduction

Maximum matching is a well-studied problem in a variety of computational models. We consider it in the semi-streaming model formalized by Feigenbaum, Kannan, McGregor, Suri, and Zhang [13] that is inspired by generation of ginormous graphs in recent times. A graph stream is an (adversarial) sequence of the edges of a graph, and a semi-streaming algorithm must access the edges in the given order and use $O(n \text{ polylog } n)$ space only, where $n$ is the number of vertices; note that a matching can have size $\Omega(n)$, so $\Omega(n \log n)$ space is necessary. The number of times an algorithm goes over a stream of edges is called the number of passes. A trivial $(1/2)$-approximation algorithm that can be easily implemented as a one-pass semi-streaming algorithm is to output a maximal matching. Since the formalization of the semi-streaming model more than a decade ago, the problem of finding a better than $(1/2)$-approximation algorithm or proving that one cannot do better has baffled researchers [21]. In a step towards resolving this, Goel, Kapralov, and Khanna [14] proved that for any $\varepsilon > 0$, a one-pass semi-streaming $(2/3 + \varepsilon)$-approximation algorithm does not exist; Kapralov [16], building on those techniques, showed non-existence of one-pass semi-streaming $(1 - 1/e + \varepsilon)$-approximation algorithms for any $\varepsilon > 0$. A natural next question is: Can we do better in, say, two passes or three passes? In answering that, Konrad, Magniez, and Mathieu [20] gave three-pass and two-pass algorithms that output matchings that are better than $(1/2)$-approximate. In this work, we give algorithms that improve their approximation ratios for two-pass and three-pass algorithms. We also give a multi-pass algorithm that does better than the best known multi-pass algorithms for at least initial few passes. We are able to bound the number of passes precisely: we give a $(2/3 - \varepsilon)$-approximation algorithm that uses $2/(3\varepsilon)$ passes for triangle-free graphs and $4/(3\varepsilon)$ passes for general graphs. Earlier works either have a large constant inside the big-O notation for the number of passes [9] or the constant cannot be determined due to the involved analysis [22, 1]. For example, the $(1 - \varepsilon)$-approximation algorithm by Eggert et al. [9] potentially uses $288/\varepsilon^5$ passes, and for the $(1 - \varepsilon)$-approximation algorithms by McGregor [22] and Ahn and Guha [1], the constants inside the big-O bound cannot be determined due to the involved analysis. The $(2/3 - \varepsilon)$-approximation algorithm by Feigenbaum et al. [13] uses $O(\log(1/\varepsilon)/\varepsilon)$ passes, which is $O(\log(1/\varepsilon))$ factor larger than the number of passes we use to get the same approximation ratio. Our algorithms are simple and combinatorial, use $O(n \log n)$ space, and (can be implemented to) have $O(1)$ update time per edge. We also give an explicit and tight analysis of the three-pass algorithm by Konrad et al. [20] that is reminiscent of Feigenbaum et al.’s [13] multi-pass algorithm.

Technical overview: If we can find a matching $M$ such that there are no augmenting paths of length 3 in $M \cup M^*$, where $M^*$ is a maximum matching, then $M$ is $(2/3)$-approximate, i.e., $(1/2 + 1/6)$-approximate. This is because, in each connected component of $M \cup M^*$, the ratio of $M$-edges to $M^*$-edges is at least $2/3$. This is the basis for the $(2/3 - \varepsilon)$-approximation algorithm by Feigenbaum et al. [13] that uses $O(\log(1/\varepsilon)/\varepsilon)$ passes. The same idea is used by Konrad et al. [20] in the analysis of their two-pass algorithms. In the first pass, they find a maximal matching $M_0$ and some subset of support edges, say $S$. If $M_0$ is so bad that $M_0 \cup M^*$ is almost entirely made up of augmenting paths of length 3 (i.e., $|M_0| \approx |M^*|/2$), then by the end of the second pass, they manage to augment (using length-3 augmentations) a constant fraction of $M_0$ using $S$ and a fresh access to the edges, resulting in a better than $(1/2)$-approximation. On the other hand, if $M_0$ is not so bad, then they already have a good matching. One limitation this idea faces is that a fraction of the edges in $S$ may become useless for an augmentation if both its endpoints get matched in $M_0$ by the end of the first pass. Our main result is a two-pass algorithm (described in Section 5) that differs in two ways from the former approach. Firstly, in the first pass, we only find a maximal
advantage over a maximal matching: an algorithm is said to have advantage $\alpha$ if it is a $(1/2 + \alpha)$-approximation algorithm (because a maximal matching is $(1/2)$-approximate).

| Problem                          | Konrad et al. [20]                  | This work              |
|----------------------------------|-------------------------------------|------------------------|
| Bipartite three-pass             | Given, but not analyzed             | 1/10 (in Section 3)    |
| Triangle-free three-pass         | Not considered                      | 1/10 (in Section A)    |
| General three-pass               | Not considered                      | 81/1600 $\approx$ 1/19.753 (in Section B) |
| Bipartite two-pass               | 1/52                                | 1/16 (by triangle-free bound below) |
| Triangle-free two-pass           | Not considered separately           | 1/16 (in Section 5)    |
| General two-pass                 | 1/140                               | 1/32 (in Section 5)    |

Table 2: Multi-pass algorithms—see Section 6.

| Graph               | Results                          | Approx                              | # Passes                        |
|---------------------|----------------------------------|-------------------------------------|---------------------------------|
| Bipartite           | Feigenbaum et al. [13]           | $2/3 - \varepsilon$                 | $O(\log(1/\varepsilon)/\varepsilon)$ |
|                     | Eggert et al. [9]                | $1 - \varepsilon$                   | $288/\varepsilon^5$             |
|                     | Ahn and Guha [1]                 | $1 - \varepsilon$                   | $O(\log \log(1/\varepsilon)/\varepsilon^2)$ |
| Triangle free       | This work                        | $2/3 - \varepsilon$                 | $2/(3\varepsilon)$              |
| General             | McGregor [22] randomized         | $1 - \varepsilon$                   | $O((1/\varepsilon)^{1/\varepsilon})$ |
|                     | Ahn and Guha [1]                 | $2/3 - \varepsilon$                 | $O(\log(1/\varepsilon)/\varepsilon^2)$ |
|                     | Ahn and Guha [1]                 | $1 - \varepsilon$                   | $O(\log n \cdot \text{poly}(1/\varepsilon))$ |
|                     | This work                        | $2/3 - \varepsilon$                 | $4/(3\varepsilon)$              |

matching $M_0$ so that in the second pass, where we maintain a set $S$ of support edges, $S$ would not contain “useless” edges. Secondly, any augmentation in our algorithm happens immediately when an edge arrives if it forms an augmenting path of length 3 with edges in $M_0$ and $S$.

Our results: In light of the discussion so far, one way to evaluate an algorithm is how much advantage it gains over the $(1/2)$-approximate maximal matching found in the first pass. We summarize our two-pass and three-pass results in Table 1 and multi-pass results in Table 2. We stress that we are able to bound the number of passes precisely, without big-$O$ notation. For general graphs, our multi-pass algorithm improves the best known deterministic algorithms in terms of number of passes—see third multi-row of Table 2.

Note of independent work: We have recently learnt of the almost concurrent work of Esfandiari et al. [11] who claim better approximation ratios for bipartite graphs in two passes and three passes. Our work was done independently and differs in several aspects. We consider triangle-free graphs (superset of bipartite graphs) and general graphs, and we additionally consider multi-pass algorithms. Also, their algorithm has a post-processing step that uses time $O(\sqrt{n} \cdot |E|)$, whereas our
algorithms can be implemented to have $O(1)$ update time per edge. One further detail about this appears in Section D.

1.1 Related Work

Karp, Vazirani, and Vazirani [18] gave the celebrated $(1 - 1/e)$-competitive randomized online algorithm for bipartite graphs in the vertex arrival setting. Goel et al. [14] gave the first one-pass deterministic algorithm with the same approximation ratio, i.e., $1 - 1/e$, in the semi-streaming model in the vertex arrival setting. For the rest of this section, results involving $\epsilon$ hold for any $\epsilon > 0$. As mentioned earlier, Goel, Kapralov, and Khanna [14] proved nonexistence of one-pass $(2/3 + \epsilon)$-approximation semi-streaming algorithms, which was extended to $(1 - 1/e + \epsilon)$-approximation algorithms by Kapralov [16]. On the algorithms side, nothing better than outputting a maximal matching, which is $(1/2)$-approximate, is known. Closing this gap is considered an outstanding open problem in the streaming community [21].

On the multi-pass front, in the semi-streaming model, Feigenbaum et al. [13] gave a $(2/3 - \epsilon)$-approximation algorithm for bipartite graphs that uses $O(\log(1/\epsilon)/\epsilon)$ passes; McGregor [22] improved it to give a $(1 - \epsilon)$-approximation algorithm for general graphs that uses $O((1/e)^{1/\epsilon})$ passes. For bipartite graphs, this was again improved by Eggert et al. [9] who gave a $(1 - \epsilon)$-approximation $O((1/\epsilon)^{5})$-pass algorithm. Ahn and Guha [1] gave a linear-programming based $(1 - \epsilon)$-approximation $O(\log \log(1/\epsilon)/\epsilon^2)$-pass algorithm for bipartite graphs. For general graphs, their $(1 - \epsilon)$-approximation algorithm uses number of passes proportional to $\log n$, so it is worse than that of McGregor [22].

For the problem of one-pass weighted matching, there is a line of work starting with Feigenbaum et al. [13] giving a $6$-approximation semi-streaming algorithm. Subsequent results improved this approximation ratio: see McGregor [22], Zelke [24], Epstein et al. [10], Crouch and Stubbs [8], Grigorescu et al. [15], and most recently in a breakthrough, giving a $(2 + \epsilon)$-approximation semi-streaming algorithm, Paz and Schwartzman [23]. The multi-pass version of the problem was considered first by McGregor [22], then by Ahn and Guha [1]. Chakrabarti and Kale [5] and Chekuri et al. [6] consider a more general version of the matching problem where a submodular function is defined on the edges of the input graph.

The problem of estimating the size of a maximum matching (instead of outputting the actual matching) has also been considered. We mention Kapralov et al. [17], Esfandiari et al. [12], Bury and Schwiegelshohn [4], and Assadi et al. [2].

In the dynamic streams, edges of the input graph can be removed as well. The works of Konrad [19], Assadi et al. [3], and Chitnis et al. [7] consider the maximum matching problem in dynamic streams.

1.2 Organization of the Paper

After setting up notation in Section 2, we give a tight analysis of the three-pass algorithm for bipartite graphs by Konrad et al. [20] in Section 3. In Section 4, we see our simple two-pass algorithm for triangle-free graphs. Then in Section 5, we see our main result—the improved two-pass algorithm, and then we see the multi-pass algorithm in Section 6. The results that are not covered in the main sections are covered in the appendix.
2 Preliminaries

We work on graph streams. The input is a sequence of edges (stream) of a graph \( G = (V,E) \),
where \( V \) is the set of vertices and \( E \) is the set of edges; a bipartite graph is denoted as \( G = (A,B,E) \).
A streaming algorithm may go over the stream a few times (multi-pass) and use space \( O(n \text{ polylog } n) \),
where \( n = |V| \). In this paper, we give algorithms that make two, three, or a few more passes over the input graph stream.
A matching \( M \) is a subset of edges such that each vertex has at most one edge in \( M \) incident to it.
The maximum cardinality matching problem, or maximum matching, for short, is to find a largest matching in the given graph.
Our goal is to design streaming algorithms for maximum matching.

For a subset \( F \) of edges and a subset \( U \) of vertices, we denote by \( U(F) \subseteq U \) the set of vertices in \( U \) that have an edge in \( F \) incident on them. Conversely, we denote by \( F(U) \subseteq F \) the set of edges in \( F \) that have an endpoint in \( U \). For a subset \( F \) of edges and a vertex \( v \in V(F) \), we denote by \( N_F(v) \) the set of \( v \)'s neighbors in the graph \( (V(F),F) \), and we define \( \text{deg}_F(v) := |N_F(v)| \).

In the first pass, our algorithms compute a maximal matching which we denote by \( M_0 \). We use \( M^* \) to indicate a matching of maximum cardinality. Assume that \( M_0 \) and \( M^* \) are given. For \( i \in \{3,5,7,\ldots\} \), a connected component of \( M_0 \cup M^* \) that is a path of length \( i \) is called an \( i \)-augmenting path (nonaugmenting otherwise). We say that an edge in \( M_0 \) is 3-augmentable if it belongs to a 3-augmenting path, otherwise we say that it is non-3-augmentable.

Lemma 2.1 (Lemma 1 in [20]). Let \( \alpha \geq 0 \), \( M_0 \) be a maximal matching in \( G \), and \( M^* \) be a maximum matching in \( G \) such that \( |M_0| \leq (1/2 + \alpha)|M^*| \). Then the number of 3-augmentable edges in \( M_0 \) is at least \( (1/2 - 3\alpha)|M^*| \), and the number of non-3-augmentable edges in \( M_0 \) is at most \( 4\alpha|M^*| \).

Proof. Let the number of 3-augmentable edges in \( M_0 \) be \( k \). For each 3-augmentable edge in \( M_0 \), there are two edges in \( M^* \) incident on it. Also, each non-3-augmentable edge in \( M_0 \) lies in a connected component of \( M_0 \cup M^* \) in which the ratio of the number of \( M^* \)-edges to the number of \( M_0 \)-edges is at most \( 3/2 \). Hence,

\[
|M^*| \leq 2k + \frac{3}{2}(|M_0| - k) \quad \text{since there are } |M_0| - k \text{ non-3-augmentable edges},
\]

\[
\leq 2k + \frac{3}{2}\left(\left(\frac{1}{2} + \alpha\right)|M^*| - k\right) \quad \text{because } |M_0| \leq (1/2 + \alpha)|M^*|,
\]

\[
= \frac{1}{2}k + \left(\frac{3}{4} + \frac{3\alpha}{2}\right)|M^*|,
\]

which, after simplification, gives \( k \geq (1/2 - 3\alpha)|M^*| \). And the number of non-3-augmentable edges in \( M_0 \) is \( |M_0| - k \leq |M_0| - (1/2 - 3\alpha)|M^*| \leq (1/2 + \alpha - 1/2 + 3\alpha)|M^*| = 4\alpha|M^*|. \)

We make the following simple, yet crucial, observation.

Observation 2.1. Let \( M_0 \) be a maximal matching. Then \( V(M_0) \) is a vertex cover, and there is no edge between any two vertices in \( V \setminus V(M_0) \). Therefore, even if the input graph is not a bipartite graph, the set of edges incident on \( V \setminus V(M_0) \), i.e., \( E(V \setminus V(M_0)) \) give rise to a bipartite graph with bipartition \((V \setminus V(M_0), V(M_0))\).

For all the algorithms in this paper, it can be verified that their space complexity is \( O(n \log n) \) and update time per edge is \( O(1) \). We also ignore floors and ceilings for the sake of exposition.
3 Analyzing the Three Pass Algorithm for Bipartite Graphs

We analyze the three-pass algorithm for bipartite graphs given by Konrad et al. [20], i.e., Algorithm 1 by considering the distribution of lengths of augmenting paths. We also give a tight example.

Algorithm 1 Three-pass algorithm for bipartite graphs due to Konrad et al. [20]

1: In the first pass, find a maximal matching $M_0$.
2: In the second pass, find a maximal matching
   
   • $M_A$ in $F_2 := \{ab : a \in A(M_0), b \in B \setminus B(M_0)\}$ (see Figure 1).
3: In the third pass, find a maximal matching
   
   • $M_B$ in $F_3 := \{ab : a \in A \setminus A(M_0) \text{ and } \exists a' \in A(M_A) \text{ such that } a'b \in M_0\}$.
4: Augment $M_0$ using edges in $M_A$ and $M_B$ and return the resulting matching $M$.

Theorem 3.1. Algorithm 1 is a three-pass, semi-streaming, $(1/2 + 1/10)$-approximation algorithm for maximum matching in bipartite graphs.

Proof. Without loss of generality, let $M^*$ be a maximum matching such that all nonaugmenting connected components of $M_0 \cup M^*$ are single edges. For $i = \{3, 5, 7, \ldots\}$, let $k_i$ denote the number of $i$-augmenting paths in $M_0 \cup M^*$, and let $k = |M_0 \cap M^*|$. Then

$$|M_0| = k + \sum_i \frac{i - 1}{2}k_i \quad \text{and} \quad |M^*| = k + \sum_i \frac{i + 1}{2}k_i.$$  \hspace{1cm} (1)

Consider an $i$-augmenting path $b_1a_1b_2a_2b_3 \cdots b_{(i+1)/2}a_{(i+1)/2}$ in $M_0 \cup M^*$, where for each $j$, we have $a_j \in A$ and $b_j \in B$. We call the vertex $a_{(i-1)/2}$ a good vertex, because an edge in $M_A$ incident to $a_{(i-1)/2}$ can potentially be augmented using the edge $b_{(i+1)/2}a_{(i+1)/2}$. To elaborate, consider the set of all edges in $M_A$ incident on good vertices; call it $M'_A$. Consider the set of edges of the type $b_{(i+1)/2}a_{(i+1)/2}$ from each $i$-augmenting path; call it $M_F$. Note that $M_F$ is a matching. Then we can augment $M_0$ using $M'_A$ and $M_F$ by as much as $|M'_A|$.

There is a matching of size $\sum_i k_i$ in $F_2$ formed by edges of the type $b_1a_1$ from each $i$-augmenting path. Since $M_A$ is maximal in $F_2$, we have $|M_A| \geq (\sum_i k_i)/2$. Now, the number of good vertices is $\sum_i k_i$; therefore, the number of bad (i.e., not good) vertices is $|M_0| - \sum_i k_i$. So the number of edges
in $M_A$ incident on good vertices (see Figure 2)

$$|M'_A| \geq \frac{\sum_i k_i}{2} - \left( |M_0| - \sum_i k_i \right) = \frac{3}{2} \sum_i k_i - \frac{|M_0|}{2}. \quad (2)$$

So the output size

$$|M| = |M_0| + |M_B| 
\geq |M_0| + \frac{3}{4} \sum_i k_i - \frac{|M_0|}{2} \quad \text{by (1) and (2),}$$

$$= \frac{|M_0|}{2} + \frac{3}{4} (|M^*| - |M_0|) \quad \text{by (1), } \sum_i k_i = |M^*| - |M_0|,$$

i.e., $|M| \geq 3|M^*|/4 - |M_0|/4$, but we also have $|M| \geq |M_0|$, hence

$$|M| \geq \max \left\{ |M_0|, \frac{3}{4} |M^*| - \frac{1}{4} |M_0| \right\}. \quad \text{□}$$

So the bound is minimized when $|M_0| = 3|M^*|/4 - |M_0|/4 = 3|M^*|/5 = (1/2 + 1/10)|M^*|$.

As we can see in the proof above, the worst case happens when $|M| = |M_0| = 3|M^*|/5$. Setting $k_3 = k_5 \geq 1$, $k = 0$, and $k_i = 0$ for $i > 5$ gives us the tight example shown in Figure 2.

4 A Simple Two Pass Algorithm for Triangle Free Graphs

Before seeing our main result, we see a simple two pass algorithm for triangle-free graphs. The function SEMI() in Algorithm 2 greedily computes a subset of edges such that each vertex in $X$ has degree at most one and each vertex in $Y$ has degree at most $\lambda$; we call such a subset a ($\lambda$, $X$, $Y$)-semi-matching (Konrad et al. [20] call this a $\lambda$-bounded semi-matching). In Algorithm 2, we find a maximal matching $M_0$ in the first pass, and, in the second pass, we find a ($\lambda$, $V(M_0)$, $V \setminus V(M_0)$)-semi-matching $S$. After the second pass, we greedily augment edges in $M_0$ one by one using edges in $S$. 

![Figure 2: Tight example for Algorithm 1: $M_A$ has only one edge that lands on a bad vertex and cannot be augmented in the third pass. So $|M| = |M_0| = 3$ and $|M^*| = 5$.](image)
Algorithm 2 Two-pass algorithm for triangle-free graphs

1: In the first pass: \( M_0 \leftarrow \) maximal matching
2: In the second pass: \( S \leftarrow \text{SEMI}(\lambda, V(M_0), V \setminus V(M_0)) \) (see Figure 3).
3: After the second pass, augment \( M_0 \) greedily using edges in \( S \) to get \( M \); output \( M \).

4: function SEMI(\( \lambda, X, Y \)) ▷ based on Algorithm 7 in Konrad et al. [20]
5: \( S \leftarrow \emptyset \)
6: foreach edge \( xy \) such that \( x \in X, y \in Y \) do
7: if \( \deg_S(x) = 0 \) and \( \deg_S(y) \leq \lambda - 1 \) then
8: \( S \leftarrow S \cup \{xy\} \)

\[ M_0 \quad S \]

Figure 3: Example showing \( M_0 \) and \( S \) at the end of the second pass of Algorithm 2 with \( \lambda = 2 \). When we greedily augment \( M_0 \) after the second pass, we may choose to augment \( u_5v_5 \) and lose two possible augmentations of edges \( u_4v_4 \) and \( u_6v_6 \).

Theorem 4.1. Algorithm 2 is a two-pass, semi-streaming, \((1/2 + 1/20)\)-approximation algorithm for maximum matching in triangle-free graphs.

Proof. As in the proof of Theorem 3.1, let \( M^* \) be a maximum matching such that all nonaugmenting connected components of \( M_0 \cup M^* \) are single edges. For \( i = \{3, 5, 7, \ldots\} \), let \( k_i \) denote the number of \( i \)-augmenting paths in \( M_0 \cup M^* \), and let \( k \) denote the number of edges in \( M^* \cap M_0 \).

Now, we define good vertices. Consider an \( i \)-augmenting path \( x_1y_1x_2y_2x_3 \cdots x_{(i+1)/2}y_{(i+1)/2} \) in \( M_0 \cup M^* \). We call the vertices \( y_1 \in V(M_0) \) and \( x_{(i+1)/2} \in V(M_0) \) good vertices, because the edges \( x_1y_1 \in M^* \) and \( x_{(i+1)/2}y_{(i+1)/2} \in M^* \) can potentially be added to \( S \) by our algorithm. Denote by \( V_G \) the set of good vertices and by \( V_B := V(M_0) \setminus V_G \) the set of bad vertices. Then \( |V_G| = 2 \sum_i k_i \).

Note that \( V_G \cap V_B = \emptyset \) and \( V_G \cup V_B = V(M_0) \) by definition.

Let \( V_{NC} := V_G \setminus V(S) \) be the set of good vertices not covered by \( S \). An edge \( uv \in M^* \) with \( u \in V \setminus V(M_0) \) and \( v \in V_{NC} \) was not added to \( S \), because \( \deg_S(u) = \lambda \). Hence

\[ \lambda |V_{NC}| \leq |V(M_0)| - |V_{NC}| \quad \text{i.e.,} \quad |V_{NC}| \leq \frac{2}{\lambda + 1} |M_0|, \quad (3) \]
because at most \( |V(M_0)| - |V_{NC}| \) vertices in \( V(M_0) \) are covered by \( S \). Now,

\[
|V(M_0) \setminus V(S)| = |V_G \setminus V(S)| + |V_B \setminus V(S)| \quad \therefore \quad V_G \cap V_B = \emptyset \text{ and } V_G \cup V_B = V(M_0),
\]

\[
\leq |V_{NC}| + |V_B| \quad \therefore \quad V_{NC} = V_G \setminus V(S) \text{ and } |V_B \setminus V(S)| \leq |V_B|,
\]

\[
\leq \frac{2}{\lambda + 1}|M_0| + |V(M_0)| - |V_G| \quad \text{by (3) and the fact } |V_B| = |V(M_0)| - |V_G|,
\]

\[
= \frac{2}{\lambda + 1}|M_0| + |V(M_0)| - 2\sum_i k_i \quad \text{because } |V_G| = 2\sum_i k_i.
\]

Using \( |V(M_0)| = |V(M_0) \setminus V(S)| + |V(M_0) \cap V(S)| \) and the above, we get

\[
|V(M_0) \cap V(S)| \geq |V(M_0)| - \left(\frac{2}{\lambda + 1}|M_0| + |V(M_0)| - 2\sum_i k_i\right)
\]

\[
= 2\left(\sum_i k_i - \frac{1}{\lambda + 1}|M_0|\right). \tag{4}
\]

We observe that at most \( |M_0| \) vertices in \( V(M_0) \) (one endpoint of each edge) can be covered by \( S \) without having both endpoints of an edge in \( M_0 \) covered. Hence, at least \( |V(M_0) \cap V(S)| - |M_0| \) edges in \( M_0 \) have both their endpoints covered by \( S \), which, by (4), is at least

\[
2\left(\sum_i k_i - \frac{1}{\lambda + 1}|M_0|\right) - |M_0| = 2\sum_i k_i - \frac{\lambda + 3}{\lambda + 1}|M_0|. \tag{5}
\]

After the second pass, when we greedily augment an edge from the above edges, i.e., edges whose both endpoints are covered by \( S \), we may potentially lose \( 2(\lambda - 1) \) other augmentations (see Figure 3). To see this, consider \( uv \in M_0 \) such that \( u, v \in V(S) \) and \( au \in S \) and \( vb \in S \). The graph is triangle free, so we know that \( a \neq b \), and we can augment \( M_0 \) using the 3-augmenting path \( auvb \); but we may lose at most \( \lambda - 1 \) edges incident to \( a \) in \( S \) and at most \( \lambda - 1 \) edges incident to \( b \) in \( S \). Therefore the number of augmentations \( c \) we get after the second pass is at least \( 1/(2\lambda - 1) \) times the right hand side of (5), i.e.,

\[
c \geq \frac{2}{2\lambda - 1}\sum_i k_i - \frac{\lambda + 3}{(2\lambda - 1)(\lambda + 1)}|M_0|.
\]

So the output size \( |M| = |M_0| + c \), and using the above bound on \( c \) and simplifying we get:

\[
|M| \geq \frac{2}{2\lambda - 1}\sum_i k_i + \frac{2(\lambda^2 - 2)}{(2\lambda - 1)(\lambda + 1)}|M_0|;
\]

substituting \( \sum_i k_i = |M^*| - |M_0| \), by (1), in the above,

\[
|M| \geq \frac{2}{2\lambda - 1}|M^*| + \frac{2(\lambda^2 - \lambda - 3)}{(2\lambda - 1)(\lambda + 1)}|M_0|.
\]

Using \( \lambda = 3 \) and the fact that \( M_0 \) is 2-approximate, we get

\[
|M| \geq \frac{2}{5}|M^*| + \frac{3}{10}|M_0| \geq \frac{2}{5}|M^*| + \frac{3}{20}|M^*| = \frac{11}{20}|M^*| = \left(\frac{1}{2} + \frac{1}{20}\right)|M^*|.
\]

\[\square\]
5 Improved Two Pass Algorithm

We present our main result that is a two pass algorithm in this section. In the first pass, we find a maximal matching $M_0$. In the second pass, we maintain a set $S$ of support edges $xy$, such that $x \in V \setminus V(M_0)$, $y \in V(M_0)$, and $\text{deg}_S(y) \leq \lambda_M$ and $\text{deg}_S(x) \leq \lambda_U$, where $\lambda_M \geq 1$ and $\lambda_U \geq 1$ are parameters denoting maximum degree allowed in $S$ for matched and unmatched vertices (with respect to $M_0$), respectively. Whenever a new edge forms a 3-augmenting path with an edge in $M_0$ and an edge in $S$, we augment. Algorithm 3 gives a formal description.

Algorithm 3 Improved two-pass algorithm: input graph $G$

1: In the first pass, find a maximal matching $M_0$.
2: if $G$ is triangle-free then
3: Return IMPROVE-MATCHING($M_0, 2, 1$)
4: else
5: Return IMPROVE-MATCHING($M_0, 4, 2$)
6: function IMPROVE-MATCHING($M_0, \lambda_U, \lambda_M$)
7: $M \leftarrow M_0$, $S \leftarrow \emptyset$, $I \leftarrow \emptyset$ and $I_B \leftarrow \emptyset$
8: foreach edge $xy$ in the stream do
9: if $x$ or $y \in I \cup I_B$ then
10: Continue, i.e., ignore $xy$.
11: else if $x \in V(M_0)$ and $y \in V(M_0)$ then
12: Continue, i.e., ignore $xy$.
13: else if there exist $v$ and $b$ such that $yv \in M_0$ and $vb \in S$ then
14: $M \leftarrow M \setminus \{yv\} \cup \{xy, vb\}$ \hspace{1cm} $\triangleright$ a 3-augmentation
Let $I_x \leftarrow \{ux, vx : xu_x \in S$ and $u_xv_x \in M_0\}$. Let $I_b \leftarrow \{ub, vb : ubvb \in M_0$ and $vbvb \in S\}$. Then $I \leftarrow I \cup \{x, y, v, b\}$ and $I_B \leftarrow I_B \cup I_x \cup I_b$.
15: else
16: Without loss of generality, assume that $x \in V \setminus V(M_0)$ and $y \in V(M_0)$.
17: if $\text{deg}_S(x) < \lambda_U$ and $\text{deg}_S(y) < \lambda_M$ then \hspace{1cm} $\triangleright$ See Figure 4.
18: $S \leftarrow S \cup \{xy\}$ \hspace{1cm} $\triangleright$ Note: Once an edge is added to $S$, it is never removed from it.
19: Return $M$.

Setting up a charging scheme to lower bound the number of augmentations

We first lay the groundwork and give a charging scheme.

Observation 5.1. For general graphs (that are possibly not triangle-free), we need to set $\lambda_M \geq 2$.

To see why, suppose $\lambda_M = 1$. Let $uv$ be a 3-augmentable edge in $M_0$. Then, for the edge $uv$, we might end up storing the edges $ub$ and $vb$ in $S$, and the edge $uv$ would not get augmented. If $\lambda_M \geq 2$, and we store at least $\lambda_M$ edges incident to $u$, then an edge incident to $v$ will not form a triangle with at least one of those and $uv$ would get augmented. So, for general graphs, we need to set $\lambda_M \geq 2$.

Let $|M_0| = (1/2 + a)|M^*|$. For a 3-augmentable edge $uv \in M_0$, let $auvb$ be the 3-augmenting path such that $au, vb \in M^*$. Without loss of generality, assume that $au$ arrived before $vb$. Then we make the following observation.
Observation 5.2. When au arrived, it may not be added to S for one of the following reasons:

- The vertex a was already matched.
- There were $\lambda_M$ edges incident to u in S.
- There were $\lambda_U$ edges incident to a in S.

We call some edges in $M_0$ good, some partially good, and some bad. An edge is good if it got augmented. An edge $uv \in M_0$ is bad if it is 3-augmentable, not good, and vertex a or b had $\lambda_U$ edges incident to them in S when edge au or vb arrived. An edge $uv \in M_0$ is partially good if it is 3-augmentable, but neither good nor bad ("partially" good because, as we will see later, we can hold some good edge $u'v' \in M_0$ responsible for $uv$ not getting augmented). Note that all 3-augmentable edges get some label according to our labeling. We require the following Lemma to describe the charging scheme.

Lemma 5.1. Suppose au was not added to S because there were already $\lambda_M$ edges incident to u in S. If, later, uv did not get augmented when vb arrived, then

- b was already matched via augmenting path $a'u''v''b$, or
- there exists $a' \in S$ and $u' \in M_0$ such that $a'$ was matched via augmenting path $a'u'v'\ b'$.

Proof. When au arrived, $|N_S(u)| \geq \lambda_M$. If b was unmatched when vb arrived, then some $a' \in N_S(u) \setminus \{b\}$ must have been matched, otherwise we would have augmented $uv$. Now for triangle-free graphs $b \notin N_S(u)$, so $|N_S(u) \setminus \{b\}| = |N_S(u)| \geq 1$, and for general graphs, by Observation 5.1, $\lambda_M \geq 2$, so $|N_S(u) \setminus \{b\}| \geq \lambda_M - 1 \geq 1$. 

Figure 4: Example showing $M_0$ and some of the edges in $M^*$ and S during the second pass of Algorithm 3 for triangle-free graphs with $\lambda_U = 2$ and $\lambda_M = 1$. At most one of $u_i$ and $v_i$ can have positive degree in S, because we would rather augment $u_iv_i$ instead of adding the latter edge to S. By our convention, $a_4u_4$ arrived before $v_4b_4$, and $a_6u_6$ arrived before $v_6b_6$. Since $a_4u_4$ was not added to S, we have $\deg_S(a_4) = \lambda_U$ (S edges incident to $a_4$ are not shown).
and consider a was not augmented and when a''u'v' arrived, deg_S(a'') = 2, so a''u'v' is a bad edge. For uv, we did not take au in S, because deg_S(u) = 1, so uv is a partially good edge, and we can charge uv to u'v' using Lemma 5.1.

**Charging Scheme.** As alluded to earlier, we charge a partially good edge to some good edge. Recall that for a 3-augmentable edge uv ∈ M_0, we denote by au, vb ∈ M* the edges that form the 3-augmenting path with uv such that au arrived before vb. We use Observation 5.2 and consider the following cases. See Figure 5.

- Suppose au was not added to S because a was already matched. Then, let u'v' ∈ M_0 was augmented using au'v'b'. If deg_S(a') ≤ λ_U − 1, then we charge uv to u'v'. Otherwise, uv is bad.
- Suppose au was not added to S because deg_S(u) = λ_M. Then we use Lemma 5.1. We either charge uv to u'v', or if deg_S(b) ≤ λ_U − 1, then we charge uv to u''v''. Otherwise, uv is bad.
- Suppose au was not added to S because deg_S(a) = λ_U, then uv is bad.
- Otherwise, au was added to S, but uv did not get augmented when vb arrived. Then:
  - Either there exists a' ∈ N_S(u) that was matched via augmenting path a'u'v'b' (note that a' may be same as a), then we charge uv to u'v';
  - or b was already matched via augmenting path a''u''v''b, and vb was ignored; in this case, if deg_S(b) ≤ λ_U − 1, then we charge uv to u''v'', otherwise, uv is bad.

We now bound the number of bad edges in M_0 from above.

**Lemma 5.2.** The number of bad edges is at most λ_M|M_0|/λ_U.

*Proof.* We claim that for any uv ∈ M_0, deg_S(u) + deg_S(v) ≤ λ_M, hence |S| ≤ λ_M|M_0|. A short argument is that the (λ_M + 1)th edge would cause an augmentation and will not be added to S. Let us assume the claim. By the definition of a bad edge, at most λ_U edges in S are “responsible” for one bad edge in M_0. Also, an edge au' in S can be responsible for at most one bad edge that can only be uv if au ∉ S (considering the 3-augmenting path auv'b). Hence, the total number of bad edges is at most |S|/λ_U ≤ λ_M|M_0|/λ_U. Now we prove the claim.

We first prove the claim for triangle-free graphs by contradiction. Let deg_S(u) + deg_S(v) > λ_M, and let vy ∈ S be the (λ_M + 1)th edge incident to one of u and v that was added to S. Since λ_M ≥ 1 and deg_S(v) ≤ λ_M, we have deg_S(u) ≥ 1, i.e. N_S(u) ≠ ∅. Now when vy arrived:
• the vertex \( y \) was unmatched, otherwise \( vy \) would not be added to \( S \);
• no vertex \( x \in N_S(u) \) was matched, otherwise \( u, v \in I_B \) and \( vy \) would not be added to \( S \).

The above implies that when \( vy \) arrived, due to some \( x \in N_S(u) \) the if condition on Line 14 became true, and we augmented \( uv \) via \( xuvy \) instead of adding \( vy \) to \( S \). This is a contradiction.

For general graphs, we argue by contradiction slightly informally for the sake of brevity. By Observation 5.1, for general graphs, \( \lambda_M \geq 2 \). Let \( \deg_S(v) + \deg_S(u) > \lambda_M \geq 2 \). Let \( vy \) be the second edge incident to one of \( u \) and \( v \) that was added to \( S \); the first edge can be \( xu \) or \( vy' \).

Suppose \( xu \) was the first edge. If \( x \neq y \), then we would have augmented \( uv \) via \( xuvy \) instead of adding \( vy \) to \( S \)—a contradiction. If \( x = y \), then after \( vy \) was processed, \( N_S(u) = N_S(v) = \{ y \} \), and a third edge incident to one of \( u \) and \( v \) would not be added to \( S \), because it would have formed a 3-augmenting path with either \( yu \) or \( vy \), resulting in a contradiction that \( \deg_S(v) + \deg_S(u) = 2 \).

Otherwise, suppose \( vy' \) was the first edge; then \( N_S(v) = \{ y, y' \} \) after \( vy \) was processed. Since eventually \( \deg_S(u) + \deg_S(v) > \lambda_M + 1 \geq 3 \) and \( \deg_S(v), \deg_S(v) \leq \lambda_M \), we would eventually have \( \deg_S(v) \geq 1 \), so let \( xu \in S \). When \( xu \) arrived, it would have formed an 3-augmenting path with either \( vy \) or \( vy' \) (here, taking care of the fact that one of \( y \) and \( y' \) can be same as \( x \)), resulting in a contradiction that \( xu \) was not added to \( S \).

Thus, we get the claim and complete the proof.

As a consequence, we get the following.

**Observation 5.3.** In any call to \textsc{Improve-Matching()}, we need to set \( \lambda_U > \lambda_M \), i.e., \( \lambda_U \geq 2 \).

To see why, suppose \( \lambda_U \leq \lambda_M \). Then by Lemma 5.2, potentially all 3-augmentable edges in \( M_0 \) could become bad edges.

Recall that a 3-augmentable edge is good, partially good, or bad; so by Lemmas 2.1 and 5.2,

\[
\# \text{ good or partially good edges} \geq \left( \frac{1}{2} - 3\alpha \right) |M^*| - \frac{\lambda_M |M_0|}{\lambda_U} \\
= \left( \frac{1}{2} - 3\alpha \right) |M^*| - \frac{\lambda_M}{\lambda_U} \left( \frac{1}{2} + \alpha \right) |M^*| \\
= \left( \frac{\lambda_U - \lambda_M}{2\lambda_U} - \left( \frac{3\lambda_U + \lambda_M}{\lambda_U} \right) \alpha \right) |M^*| .
\]

In the following Lemma, we bound the number of partially good edges in \( M_0 \) that are charged to one good edge.

**Lemma 5.3.** At most \( 2\lambda_U - 1 \) partially good edges in \( M_0 \) are charged to one good edge in \( M_0 \).

**Proof.** Suppose \( uv \in M_0 \) was augmented by edges \( xu \) and \( vy \) such that \( xu \) arrived before \( vy \), then \( xu \in S \). Now \( |N_S(x)|, |N_S(y)| \leq \lambda_U \). Since \( xu \in S \), we have \( |N_S(x) \setminus \{u\}| \leq \lambda_U - 1 \). Let \( B := (N_S(x) \setminus \{u\}) \cup N_S(y) \), then \( |B| \leq 2\lambda_U - 1 \). Now, the set of partially good edges that are charged to \( uv \) is a subset of \( M_0(B) \). Observing that \( |M_0(B)| \leq |B| \leq 2\lambda_U - 1 \) finishes the proof.

The following lemma characterizes the improvement given by \textsc{Improve-Matching()}.

**Lemma 5.4.** Let \( |M_0| = (1/2 + \alpha) |M^*| \) and \( M = \textsc{Improve-Matching}(M_0, \lambda_U, \lambda_M) \), then

\[
|M| \geq \left( \frac{1}{2} + \frac{\lambda_U - \lambda_M}{4\lambda_U} + \left( 1 - \frac{3\lambda_U + \lambda_M}{2\lambda_U} \right) \alpha \right) |M^*| \geq \left( \frac{1}{2} + \frac{\lambda_U - \lambda_M}{4\lambda_U} \right) |M^*| .
\]
Proof. By (6) and Lemma 5.3, the total number of augmentations during one call to \textsc{Improve-Matching()} is at least
\[
\frac{1}{2\lambda_U} \left( \frac{\lambda_U - \lambda_M}{2\lambda_U} - \left( \frac{3\lambda_U + \lambda_M}{\lambda_U} \right) \alpha \right) |M^*| = \left( \frac{\lambda_U - \lambda_M}{4\lambda_U^2} - \left( \frac{3\lambda_U + \lambda_M}{2\lambda_U^2} \right) \alpha \right) |M^*|.
\]
Hence, we get the following bound on the size of the output matching \( M \):
\[
|M| \geq |M_0| + \left( \frac{\lambda_U - \lambda_M}{4\lambda_U^2} - \frac{3\lambda_U + \lambda_M}{2\lambda_U^2} \alpha \right) |M^*| = \left( \frac{1}{2} + \frac{\lambda_U - \lambda_M}{4\lambda_U^2} + \left( 1 - \frac{3\lambda_U + \lambda_M}{2\lambda_U^2} \right) \alpha \right) |M^*| \text{ because } |M_0| = (1/2 + \alpha)|M^*|,
\]
\[
\geq \left( \frac{1}{2} + \frac{\lambda_U - \lambda_M}{4\lambda_U^2} \right) |M^*| \text{ since } \lambda_U \geq 2 \text{ (see Observation 5.3).}
\]

Now we state and prove our main result.

\textbf{Theorem 5.5.} Algorithm 3 uses two passes and has an approximation ratio of \( 1/2 + 1/16 \) for triangle-free graphs and an approximation ratio of \( 1/2 + 1/32 \) for general graphs for maximum matching.

\textit{Proof.} By Lemma 5.4, after the second pass, the output size \( |M| \geq (1/2 + (\lambda_U - \lambda_M)/(4\lambda_U^2))|M^*| \); we use \( \lambda_U = 2 \) and \( \lambda_M = 1 \) for triangle-free graphs and \( \lambda_U = 4 \) and \( \lambda_M = 2 \) (see Observation 5.1) for general graphs to get the claimed approximation ratios. \( \square \)

6 Multi Pass Algorithm

We run the function \textsc{Improve-Matching()} in Algorithm 3 with increasing values of \( \lambda_U \), and the approximation ratio converges to \( 1/2 + 1/6 \).

\textbf{Algorithm 4} Multi-pass algorithm: input graph \( G \)

\begin{algorithmic}
1: In the first pass, find a maximal matching \( M_1 \).
2: \( M \leftarrow M_1 \)
3: \textbf{if} \( G \) is triangle-free \textbf{then}
4: \hspace{1em} \textbf{for} \( i = 2 \) to \( \lceil 2/(3\varepsilon) \rceil \) \textbf{do}
5: \hspace{2em} \( M \leftarrow \text{Improve-Matching}(M, i, 1) \)
6: \textbf{else}
7: \hspace{1em} \textbf{for} \( i = 2 \) to \( \lceil 4/(3\varepsilon) \rceil \) \textbf{do}
8: \hspace{2em} \( M \leftarrow \text{Improve-Matching}(M, i + 1, 2) \)
9: \textbf{return} \( M \).
\end{algorithmic}

\textbf{Theorem 6.1.} For any \( \varepsilon > 0 \), Algorithm 4 is a semi-streaming \( (1/2 + 1/6 - \varepsilon) \)-approximation algorithm for maximum matching that uses \( 2/(3\varepsilon) \) passes for triangle-free graphs and \( 4/(3\varepsilon) \) passes for general graphs.

\textit{Proof.} We prove the theorem for triangle-free case; the general case is similar. Let \( M_i \) be the matching computed by Algorithm 4 after \( i \)th pass, and let \( p := \lceil 2/(3\varepsilon) \rceil \), so \( \varepsilon \leq 2/(3p) \). Since \( M_1 \) is maximal, it is \( (1/2) \)-approximate. Let \( \alpha_1 := 0 \), and for \( i \in \{2, 3, \ldots, p\} \), let
\[
\alpha_i := \frac{i - 1}{4i^2} + \left( 1 - \frac{3i + 1}{2i^2} \right) \alpha_{i-1}\]
(see Lemma 5.4 with \( \lambda_U = i \) and \( \lambda_M = 1 \)). Then, by Lemma 5.4 and the logic of Algorithm 4, for \( i \in [p] \), the matching \( M_i \) is \( (1/2 + \alpha_i) \)-approximate (by a trivial induction). Now we bound \( \alpha_p \) by induction. We claim that for \( i \in [p] \),

\[
\alpha_i \geq \frac{1}{6} - \frac{2}{3i},
\]

which we prove by induction on \( i \).

Base case: For \( i = 1 \), we have \( 1/6 - \alpha_1 = 1/6 - 0 = 1/6 \leq 2/(3 \cdot 1) \).

For inductive step, we want to show that

\[
\frac{1}{6} - \alpha_i = \frac{1}{6} - \frac{i - 1}{4i^2} - \left( 1 - \frac{3i + 1}{2i^2} \right) \alpha_{i-1} \leq \frac{2}{3i},
\]

which is implied by the following (using inductive hypothesis)

\[
\frac{1}{6} - \frac{i - 1}{4i^2} + \left( 1 - \frac{3i + 1}{2i^2} \right) \left( \frac{2}{3(i-1)} - \frac{1}{6} \right) \leq \frac{2}{3i},
\]

which is implied by

\[
\frac{1}{6} - \frac{i - 1}{4i^2} + \left( \frac{2i^2 - 3i - 1}{2i^2} \right) \left( \frac{4 - i + 1}{6(i-1)} \right) \leq \frac{2}{3i},
\]

multiplying both sides by \( 12i^2(i-1) \), we then need to show that,

\[
2i^2(i-1) - 3(i-1)^2 + (2i^2 - 3i - 1)(-i + 5) \leq 8i(i-1)
\]

which is implied by

\[
2i^3 - 2i^2 - 3(i^2 - 2i + 1) + (-2i^3 + 10i^2 + 3i^2 - 15i + i - 5) \leq 8i^2 - 8i
\]

which is implied by

\[
2i^3 - 5i^2 + 6i - 3 + (-2i^3 + 13i^2 - 14i - 5) \leq 8i^2 - 8i
\]

which is true, so we get the claim. Therefore \( \alpha_p \geq 1/6 - 2/(3p) \geq 1/6 - \varepsilon \), and by our earlier observation, \( M_p \) is \( (1/2 + \alpha_p) \)-approximate, and this finishes the proof for triangle-free case. The proof for general case is very similar. We define \( p := \lceil 4/(3\varepsilon) \rceil \) and \( \alpha_1 := 0 \), and for \( i \in \{2,3,\ldots,p\} \), we define

\[
\alpha_i := \frac{i - 1}{4(i+1)^2} + \left( 1 - \frac{3(i+1) + 2}{2(i+1)^2} \right) \alpha_{i-1},
\]

i.e., we use \( \lambda_U = i + 1 \) and \( \lambda_M = 2 \). The corresponding claim then is that for \( i \in [p] \),

\[
\alpha_i \geq \frac{1}{6} - \frac{4}{3i},
\]

which can be verified by induction on \( i \).

\( \Box \)

**Acknowledgements**

The first author would like to thank his advisor Amit Chakrabarti and Andrew McGregor for helpful discussions. The first and the second author would like to thank Ashish Chiplunkar for helpful discussions.
References

[1] Kook Jin Ahn and Sudipto Guha. Linear programming in the semi-streaming model with application to the maximum matching problem. *Inf. Comput.*, 222:59–79, January 2013.

[2] Sepehr Assadi, Sanjeev Khanna, and Yang Li. On estimating maximum matching size in graph streams. In *Proc. 28th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1723–1742, 2017.

[3] Sepehr Assadi, Sanjeev Khanna, Yang Li, and Grigory Yaroslavtsev. Maximum matchings in dynamic graph streams and the simultaneous communication model. In *Proc. 27th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1345–1364, 2016.

[4] Marc Bury and Chris Schwiegelshohn. Sublinear estimation of weighted matchings in dynamic data streams. In *Proc. 23rd Annual European Symposium on Algorithms*, pages 263–274, 2015.

[5] Amit Chakrabarti and Sagar Kale. Submodular maximization meets streaming: matchings, matroids, and more. *Mathematical Programming*, 154(1):225–247, 2015.

[6] Chandra Chekuri, Shalmoli Gupta, and Kent Quanrud. Streaming algorithms for submodular function maximization. In *Proc. 42nd International Colloquium on Automata, Languages and Programming*, pages 318–330, 2015.

[7] Rajesh Chitnis, Graham Cormode, Hossein Esfandiari, MohammadTaghi Hajiaghayi, Andrew McGregor, Morteza Monemizadeh, and Sofya Vorotnikova. Kernelization via sampling with applications to finding matchings and related problems in dynamic graph streams. In *Proc. 27th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1326–1344, 2016.

[8] Michael Crouch and Daniel M. Stubbs. Improved streaming algorithms for weighted matching, via unweighted matching. In *Proc. 17th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems*, volume 28, pages 96–104, 2014.

[9] Sebastian Eggert, Lasse Kliemann, Peter Munstermann, and Anand Srivastav. Bipartite matching in the semi-streaming model. *Algorithmica*, 63(1):490–508, 2012.

[10] Leah Epstein, Asaf Levin, Julian Mestre, and Danny Segev. Improved approximation guarantees for weighted matching in the semi-streaming model. *SIAM Journal on Discrete Mathematics*, 25(3):1251–1265, 2011.

[11] H. Esfandiari, M. Hajiaghayi, and M. Monemizadeh. Finding large matchings in semi-streaming. In *2016 IEEE 16th International Conference on Data Mining Workshops (ICDMW)*, pages 608–614, Dec 2016.

[12] Hossein Esfandiari, Mohammad T. Hajiaghayi, Vahid Liaghat, Morteza Monemizadeh, and Krzysztof Onak. Streaming algorithms for estimating the matching size in planar graphs and beyond. In *Proc. 26th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1217–1233, 2015.

[13] Joan Feigenbaum, Sampath Kannan, Andrew McGregor, Siddharth Suri, and Jian Zhang. On graph problems in a semi-streaming model. *Theor. Comput. Sci.*, 348(2):207–216, December 2005.
A Three Pass Algorithm for Triangle Free Graphs

For completeness, we present our three-pass algorithm for triangle-free graphs.

**Theorem A.1.** Algorithm 5 is a three-pass, semi-streaming, \((1/2 + 1/10)\)-approximation algorithm for maximum matching in triangle-free graphs, and the analysis is tight.

**Proof.** Let \(|M_0| = (1/2 + \alpha)|M^*|\). The number of edges in \(M^*\) incident on \(V(M^*) \setminus V(M_0)\) is

\[
|V(M^*) \setminus V(M_0)| \geq |V(M^*)| - |V(M_0)| = 2|M^*| - 2|M_0| = (1 - 2\alpha)|M^*|; \tag{7}
\]

and these edges also belong to \(F_1\). Since \(M_1\) is a maximal matching in \(F_1\),

\[
|M_1| \geq (1 - 2\alpha)|M^*| / 2 = (1/2 - \alpha)|M^*| \tag{8}
\]
Algorithm 5 Three-pass algorithm for triangle-free graphs

1: In the first pass, find a maximal matching $M_0$.
2: In the second pass, find a maximal matching $M_1$ in $F_1 := \{ uv : u \in V \setminus V(M_0), v \in V(M_0) \}$.
3: After the second pass:
   - $M'_1 \leftarrow$ arbitrary largest subset of $M_1$ such that there is no 3-augmenting path in $M'_1 \cup M_0$ with respect to $M_0$
   - $V_2 \leftarrow \{ x \in V(M_0) : \exists v, w \text{ such that } vw \in M'_1 \text{ and } wx \in M_0 \}$
   - For $x \in V_2$, denote by $P(x)$ the vertex $v$ such that there exists $w$ with $vw \in M'_1$ and $wx \in M_0$. See $x$ and $P(x)$ in Figure 6.
4: In the third pass: $F_2 := \{ xy : x \in V_2, y \in V \setminus V(M_0) \}$
5: $M_2 \leftarrow \emptyset$
6: for edge $xy \in F_2$ do
   - if $x$, and $y$ are unmarked then
     - $M_2 \leftarrow M_2 \cup \{ xy \}$; since the graph is triangle free, $y \neq P(x)$, and we can augment $M_0$ using $xy$.
     - Mark $P(x)$, $x$, $y$, and $P^{-1}(y)$ (if exists).
   - Let $M$ be largest of $M_3$ and $M'_3$ which are computed below.
     - Augment $M_0$ using edges in $M_1$ to get $M_3$.
     - Augment $M_0$ using edges in $M'_1$ and $M_2$ to get $M'_3$.
11: Output $M$.

Let $c$ be the number of 3-augmenting paths in $M_1 \cup M_0$, so $|M'_1| = |M_1| - c$ by the definition of $M'_1$. By Lemma 2.1, there are at most $4\alpha|M^*|$ non-3-augmentable edges in $M_0$. So at least $|M_1| - c - 4\alpha|M^*|$ edges of $M'_1$ are incident on 3-augmentable edges of $M_0$. Therefore there is a matching of size at least $|M_1| - c - 4\alpha|M^*|$ in $F_2$; consider one such matching $M_F$. We claim that $|M_2| \geq |M_F|/4$. See Figure 6. Let $xy \in M_2$; we note that $xy$ disallows at most four edges in $M_F$ from being added to $M_2$ due to the (at most) four marks that it adds, because a marked vertex can disallow at most one edge in $M_F$ (due to it being a matching), which shows the claim. Hence:

Figure 6: An edge $xy \in M_2$ disallows at most four edges in $M_F$ from being added to $M_2$. 

Figure 7: Tight example for Algorithm 5: $M_1$ has only two edges that land on bad vertices and cannot be augmented in the third pass. So $|M| = |M_0| = 3$ and $|M^*| = 5$.

$$|M_2| \geq \frac{|M_F|}{4}$$
$$\geq \frac{|M_1| - c - 4\alpha |M^*|}{4}$$
$$\geq \frac{1}{4} \left( \left( \frac{1}{2} - \alpha \right) |M^*| - c - 4\alpha |M^*| \right)$$
$$= \frac{1}{4} \left( \left( \frac{1}{2} - 5\alpha \right) |M^*| - c \right).$$

Now, each edge in $M_2$ gives one augmentation after the second pass. To see this, we observe that for any $x \in V_2$, at any point in the algorithm, $x$ and $P(x)$ are either both marked or both unmarked. So when an edge $xy \in M_2$ arrives, $x$ and $y$ are unmarked, and $P(x)$ and $P^{-1}(y)$ (if it exists) are also unmarked, otherwise one of $x$ and $y$ would have been marked and $xy$ would not have been added to $M_2$. Since both $P(x)$ and $P^{-1}(y)$ were unmarked, we can use the augmenting path $\{ M'_1(\{P(x)\}), M_0(\{x\}), xy \}$. Hence we get at least

$$\max \left\{ c, \frac{1}{4} \left( \left( \frac{1}{2} - 5\alpha \right) |M^*| - c \right) \right\}$$

augmentations after the third pass. This is minimized by setting

$$c = \frac{1}{4} \left( \left( \frac{1}{2} - 5\alpha \right) |M^*| - c \right)$$
$$= \frac{1}{5} \left( \left( \frac{1}{2} - 5\alpha \right) |M^*| \right)$$
$$= \left( \frac{1}{10} - \alpha \right) |M^*|. $$

So we get the following bound:

$$|M| \geq |M_0| + \left( \frac{1}{10} - \alpha \right) |M^*| \geq \left( \frac{1}{2} + \alpha \right) |M^*| + \left( \frac{1}{10} - \alpha \right) |M^*| = \left( \frac{1}{2} + \frac{1}{10} \right) |M^*|. \quad \square$$

The tight example is shown in Figure 7.
B Three Pass Algorithm for General Graphs

We find a maximal matching $M_1$ in the first pass. Then we use IMPROVE-MATCHING() function from Algorithm 3, i.e.,

- in the second pass, $M_2 \leftarrow$ IMPROVE-MATCHING($M_1, 4, 2$), and
- in the third pass, $M_3 \leftarrow$ IMPROVE-MATCHING($M_2, 5, 2$).

We observe that $M_1$ is $(1/2)$-approximate. Then by double application of Lemma 5.4, we get that $M_3$ is $(1/2 + 81/1600)$-approximate.

C Three Pass Algorithm for Bipartite Graphs: Suboptimal Analysis

We now give an analysis of Algorithm C Three Pass Algorithm for Bipartite Graphs: Suboptimal Analysis.

We find a maximal matching $M$ of size at least $\alpha$ and since $M$ is maximal, we then get the following:

$$|M| \geq |M_0| + \frac{1}{2}(|M_0| - 4\alpha |M^*|)$$

$$\geq |M_0| + \frac{1}{2}\left(\frac{1}{2} \cdot \frac{1}{2} - \frac{9}{4}\alpha\right) |M^*|$$

$$= (\frac{1}{2} + \alpha) |M^*| + \left(\frac{1}{8} - \frac{9}{4}\alpha\right) |M^*|$$

because $|M_0| = (1/2 + \alpha) |M^*|$, we get the following bound:

$$|M| \geq |M_0| + \frac{1}{2}(|M_0| - 4\alpha |M^*|)$$

$$\geq |M_0| + \frac{1}{2}\left(\frac{1}{2} \cdot \frac{1}{2} - \frac{9}{4}\alpha\right) |M^*|$$

$$= (\frac{1}{2} + \alpha) |M^*| + \left(\frac{1}{8} - \frac{9}{4}\alpha\right) |M^*|$$

$$= \left(\frac{1}{2} + \frac{1}{8} - \frac{5}{4}\alpha\right) |M^*|.$$  

We also have $|M| \geq |M_0| = (1/2 + \alpha) |M^*|$. As $\alpha$ increases, the former bound deteriorates and the latter improves, so the worst case $\alpha$ is when these two bounds are equal, which happens at $\alpha = 1/18$, and the approximation ratio we get is $1/2 + 1/18$. 

\[ \square \]
C.1 Improved Analysis Without Considering Longer Augmenting Paths

We can analyze Algorithm 1 better if we bound $|M_A|$ more carefully. The claim is that at least $(1/2 - 7\alpha)|M^*|/2$ edges of $M_A$ are incident on 3-augmentable edges of $M_0$. Let $A_G \subseteq A(M_0)$ be the set of vertices in $A$ that are endpoints of 3-augmentable edges of $M_0$; also, let $A_N = A(M_0) \setminus A_G$. So there is a matching of size at least $|A_G|$ in $F_2$ that covers $A_G$. Any maximal matching in $F_2$ has at least $(|A_G| - |A_N|)/2$ edges that are incident on $A_G$. To see the claim, we use the facts $|A_G| \geq (1/2 - 3\alpha)|M^*|$ and $|A_N| \leq 4\alpha|M^*|$. So there is a matching of size at least $(1/2 - 7\alpha)|M^*|/2$ in $F_3$. We output a maximal matching in $F_3$; hence we get at least $(1/2 - 7\alpha)|M^*|/4$ augmentations after the third pass. So we get the following bound:

$$|M| \geq |M_0| + \frac{1}{4} \left( \frac{1}{2} - 7\alpha \right) |M^*|$$

$$= \left( \frac{1}{2} + \alpha \right) |M^*| + \frac{1}{4} \left( \frac{1}{2} - 7\alpha \right) |M^*|$$

$$= \left( \frac{1}{2} + \frac{1}{8} - \frac{3}{4}\alpha \right) |M^*|.$$

where the second inequality is by (9). We also have $|M| \geq |M_0| = (1/2 + \alpha)|M^*|$, so the worst case $\alpha$ is when these two bounds are equal, which happens at $\alpha = 1/14$ and the approximation ratio we get is $1/2 + 1/14$, and we get the following theorem.

**Theorem C.2.** Algorithm 1 is a three-pass, semi-streaming, $(1/2 + 1/14)$-approximation algorithm for maximum matching in bipartite graphs.

D A Note on the Analysis by Esfandiari et al. [11]

We demonstrate with an example that the analysis of the algorithm by Esfandiari et al. [11] given for bipartite graphs cannot be extended for triangle-free graphs to get the same approximation ratio. See Figure 8. Lemma 6 in their paper, as they correctly claim, holds only for bipartite graphs and not for triangle-free graphs. Our algorithm in Section 4 is essentially the same algorithm except for the post-processing step; we augment the maximal matching computed in the first pass greedily, whereas they use an offline maximum matching algorithm. We have highlighted some other comparison points in Section 1.
Figure 8: Example demonstrating that Lemma 6 in Esfandiari et al. [11] does not hold when the input graph is not bipartite but is triangle-free. We use $k = 3$. For an $M$ edge $u_i v_i$, there are two $M^*$ edges incident on it, which are $a_i u_i$ and $v_i b_i$, and some of the $M^*$ edges are not shown, but all golden edges are shown, which we call support edges or denote by $S$ in our terminology. It can be seen from this example that their algorithm is not a $(1/2 + 1/12)$-approximation algorithm for triangle free graphs, because out of the seven 3-augmentable edges in $M$, only one will get augmented, thereby giving a worse approximation ratio.