Abstract

We introduce $\Psi$ec, a local spectral exterior calculus that provides a discretization of Cartan’s exterior calculus of differential forms using wavelet functions. Our construction consists of differential form wavelets with flexible directional localization, between fully isotropic and curvelet- and ridgelet-like, that provide tight frames for the spaces of $k$-forms in $\mathbb{R}^2$ and $\mathbb{R}^3$. By construction, these wavelets satisfy the de Rahm co-chain complex, the Hodge decomposition, and that the integral of a $k + 1$-form is a $k$-form. They also enforce Stokes’ theorem for differential forms, and we show that with a finite number of wavelet levels it is most efficiently approximated using anisotropic curvelet- or ridgelet-like forms. Our construction is based on the intrinsic geometric properties of the exterior calculus in the Fourier domain. To reveal these, we extend existing results on the Fourier transform of differential forms to a frequency domain description of the exterior calculus, including, for example, a Parseval theorem for forms and a description of the symbols of all important operators.

Keywords: exterior calculus, wavelets, structure preserving discretization

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1. Introduction

The exterior calculus, first introduced by Cartan \cite{Cartan1945}, provides a formulation of scalar and vector-valued functions that encodes their intrinsic relationships as a differential complex in a coordinate-invariant description. The central objects
of the calculus are differential forms, which formalize the fields, and the exterior derivative, which is a first order differential operator that generalizes gradient, curl, and divergence, and that acts on differential forms. The importance of respecting the exterior calculus’ inherent geometric structure for the numerical solution of partial differential equations, even in $\mathbb{R}^n$ and with a flat metric, has been shown for various applications in the last 30 years. Perhaps the first area where this was understood was electromagnetic theory \cite{2,3} where it was realized that the electric and magnetic fields have different geometric properties and hence also should be discretized differently. Motivated by this, various finite element-based discrete realization of exterior calculus have been developed \cite{4,5,6,7,8}. These have been applied to applications such as fluid mechanics \cite{9,10}, magnetohydrodynamics \cite{11}, complex fluids \cite{11}, and geometry processing, cf. \cite{12}.

In this work, we propose a local space-frequency approach to obtain a form of Cartan’s exterior calculus amenable to numerical calculations. Towards this end, we extend existing results from theoretical physics and develop a description of the calculus in the Fourier domain. This shows that under the Fourier transform it becomes a chain complex with its structure encoded in a simple, geometric way in spherical coordinates. The exterior derivative, for example, only acts on the radial component of a differential forms’s Fourier transform and a form is hence exact, i.e. its exterior derivative vanishes, when tangent to the frequency sphere $S^{n-1}_\xi$. We therefore discretize the exterior calculus in spherical coordinates in the Fourier domain. In particular, we choose aligned differential form basis functions, which are essentially either tangential or radial to the frequency sphere, and localize in space and frequency using scalar wavelet windows that are separable in spherical coordinates. The former enforces by construction the distinction between exact and co-exact forms, i.e. the Hodge-Helmholtz decomposition, and it is in our formulation a key to preserve many other structures. The chosen frequency windows ensure that our differential form wavelets respect the structure of the exterior calculus and that one can seamlessly blend between isotropic and directionally localized, e.g. curvelet- or
ridgelet-like, wavelets. As a functional analytic setting for our construction we use $L_2(\mathbb{R}^n)$ and the homogeneous Sobolev space $\dot{L}_2^1(\mathbb{R}^n)$ that, in our opinion, are together natural for the chain structure of the exterior complex.

The resulting discrete form of the exterior calculus, which we denote by $\Psi_{ec}$, is generated by differential form wavelets $\psi_{\nu,s,a}^{k,n}$ where $n = 2, 3$ is the dimension of the space, $0 \leq k \leq n$ is the degree of the form, $\nu \in \{d, \delta\}$ denotes if the wavelet is exact or co-exact, $s = (j_s, k_s, t_s)$ describes level $j_s$, translation $k_s$ and orientation $t_s$, and $a \in \{1\}$ or $a \in \{1, 2\}$ depending on whether one is tangential or orthogonal to the frequency sphere. These wavelets satisfy (see Sec. 5 for the precise technical statements):

- The $\psi_{\nu,s,a}^{k,n}$ form tight frames for the spaces of exact or co-exact $k$-forms.
- The exterior co-chain complex holds, i.e. $d\psi_{\delta,s,a}^{k,n} = \psi_{d,s,a}^{k+1,n}$ and $d\psi_{d,s,a}^{k,n} = 0$, and the exterior derivative $d$ preserve the localization described by $s$.
- Stokes’ theorem for differential forms,

$$\int_{\partial \mathcal{M}} \alpha = \int_{\mathcal{M}} d\alpha,$$

which subsumes the classical theorems of Green, Gauss and Stokes as special cases, holds with wavelet differential forms as

$$\sum_{s \in S} \alpha_s \langle \langle \psi_{\delta,s}^k, \chi_{\partial \mathcal{M}} \rangle \rangle = \sum_{s \in S} \alpha_s \langle \langle \psi_{d,s}^{k+1}, \chi_{\mathcal{M}} \rangle \rangle$$

where the $\alpha_s$ are the frame coefficients of $\alpha \in \Omega^k(\mathbb{R}^n)$ and $\chi_{\mathcal{M}}$ and $\chi_{\partial \mathcal{M}}$ are the characteristic functions of the tangent space of the manifold $\mathcal{M} \subset \mathbb{R}^n$ and its boundary, respectively. When $\dim(\mathcal{M}) = n$ then Stokes theorem thus becomes the approximation problem for cartoon-like functions, which has been studied extensively in the literature on curvelets, shearlets and related constructions. Moreover, $\star \chi_{\partial \mathcal{M}}$, which arises in the inner product $\langle \langle \psi_{\delta,s}^k, \chi_{\partial \mathcal{M}} \rangle \rangle$, is then exactly the wavefront set.

- The Hodge dual $\star \psi_{\nu,s,a}^{k,n}$ has a simple and practical description.
• The integral of $\psi^{k,n}_{\nu,s,a}$ along an arbitrary direction is again a wavelet differential form $\psi^{k-1,n}_{\nu,s',a}$ and with $s'$ preserving the localization of $s$.

• The Laplace-de Rahm operator $\Delta : \Omega^k \to \Omega^k$ has a closed form representation and Galerkin projection for differential form wavelets.

• The $\psi^{k,n}_{\nu,s,a}$ have closed form expressions in the spatial domain, explaining, e.g., how the angular localization in frequency space carries over to the spatial domain.

Some operators such as the wedge product or the Lie derivative have currently no natural expression in our calculus. We leave a further investigation of these as well as decay estimates to future work.

The remainder of the document is structured as follows. After discussing related work in the next section, we present some background on polar wavelets. Afterwards we develop the Fourier transform of differential forms. We then introduce our wavelet differential forms and develop the exterior calculus, including the exterior derivative, the wedge product, Hodge dual, and Stokes theorem, that is defined on them. Finally, in Sec. 6 we summarize our work and discuss directions for future work.

2. Related Work

Our work builds on ideas from two distinct fields: firstly, wavelet theory and in particular polar wavelets and, secondly, exterior calculus and its discretizations. We will discuss related work in these fields in order.

Polar Wavelets. Polar wavelets, in the sense used in our work, were introduced by Unser and co-workers [13, 14, 15, 16] to provide a comprehensive framework for the different variants of steerable wavelets [17, 18] that have been proposed over the years. Their construction also subsumes second generation curvelets [19, 20]. The idea to work in polar coordinates in the Fourier domain, and that this provides many advantages, goes, however, back much further in
the mathematics literature. For example, Fefferman’s second dyadic decomposition, cf. [21, Ch. IX], already showed its utility. We use the term ‘polar wavelet’, or short ‘polarlet’, since it describes what distinguishes the construction from other multi-dimensional wavelets and what is the key to their utility.

Discretizations of Exterior Calculus. Discretizations of the exterior calculus try to preserve important structures of the continuous theory, e.g. the de Rahm co-chain complex or Stokes’ theorem, to improve the accuracy or robustness of numerical calculations. This goes back to work in electromagnetism, [2, 3], where it was realized that exterior calculus provides a framework to systematically understand and extend earlier results that provided good numerical performance [22]. As in other instances, many precursors can be found in the literature, e.g. [23], but a complete discussion is beyond the scope of our work, see [6] for some remarks on the development of the theory.

Most existing discretizations of the exterior calculus are finite element-based, such as finite element exterior calculus developed by Arnold and collaborators [6] or discrete exterior calculus by Hirani, Desbrun and co-workers [24, 5]. Another approach with a similar spirit are mimetic discretizations [4]. These works all have in common that they construct a discrete analogue that shares structural properties of Cartan’s exterior calculus, e.g. in that it is a co-chain complex or Stokes’ theorem holds. Our construction, in contrast, is best seen as a multi-resolution restriction of the continuous theory that preserve its intrinsic structure. Our wavelet differential forms are hence also forms in the sense of the continuous theory and not “integrated forms” as have to be used in finite element-based discretizations.

Recent work extended finite element-based discretizations of exterior calculus to obtain spectral convergence rates. Rufat, Mason and Desbrun [7] accomplish this by introducing interpolation and histapolation maps to relate discrete exterior calculus and continuous forms. Gross and Atzberger recently presented a similar approach using hyper-interpolation for radial manifolds, i.e. those that can be described as a height field over $S^2$. Another extension of finite
element-based discretizations of exterior calculus is de Goes et al.’s subdivision
discrete exterior calculus [8] which shares the multi-resolution structure with
our approach.

Another recent discretization of the exterior calculus is those by Berry and
Giannakis [25] for manifold learning problems. They employ the eigenfunctions
of the Laplace-de Rahm operator as bases for their forms, an idea which was
already implicitly used in earlier work on structure-preserving discretizations of
fluids [26, 27]. For simple domains, such as the cube or the sphere, for which the
eigenfunctions have closed form expressions, the approach yields basis functions
that are continuous forms, as our differential form wavelets. A method to localize
these results [26, 27] while preserving the advantages of the analytic, closed form
expressions was one of the original motivations for the present work. Compared
to the use the eigenfunctions of the Laplace-de Rahm operator, however, we can
no longer work intrinsically on manifolds.

*Curl- and Divergence Free Wavelets.* Various constructions for curl- and divergence-
free wavelets have been proposed over the years, e.g. [28, 29, 30, 31, 32]. These
are related to our wavelet differential forms because exact differential forms can
be interpreted as such vector fields. A precursors of the presented work is those
in [33] where divergence-free wavelets based on polar wavelets are constructed
and where also the inherent geometric structure in frequency space is exploited,
in this case that a vector field is divergence free when its Fourier transform is
tangential to the frequency sphere.

### 3. Polar Wavelets

Polar wavelets are defined in polar or spherical coordinates in the Fourier do-
main using a compactly supported radial window \( \hat{h}(|\xi|) \), which controls the over-
all frequency localization, and an angular one, \( \hat{\psi}(\tilde{\xi}) \), which controls the direction-
ality, where \( \tilde{\xi} = \xi/|\xi| \). The mother wavelet is thus given by
\[
\hat{\psi}(\xi) = \hat{\psi}(\tilde{\xi}) \hat{h}(|\xi|)
\]
with the whole family of functions being generated by dilation, translation and
rotation.
In two dimensions, the angular window can be described using a Fourier series. A polar wavelet can therefore be given by
\[
\hat{\psi}_s(\xi) \equiv \hat{\psi}_{jkt}(\xi) = \frac{2^j}{2\pi} \left( \sum_n \beta_{j,n}^t e^{in\theta_\xi} \right) \hat{h}(2^{-j}|\xi|) e^{-i(\xi,2^j k)} \tag{3a}
\]
with the $\beta^t_{j,n}$ controlling the angular localization. In the simplest case $\beta_n = \delta_{n0}$ and one has isotropic, bump-like wavelet functions. Conversely, when the support of the $\beta^t_{j,n}$ is all of $\mathbb{Z}$ in $n$, then one can describe compactly supported angular windows. The above formulation also encompasses second generation curvelets [20] and provides more generally a framework to practically realize $\alpha$-molecules-like constructions [34, 35, 36].

A useful property of polar wavelets is that the inverse Fourier transform can be computed in closed form. Using the Fourier transform in polar coordinates, cf. Appendix A.5, one obtains for the spatial representation [37]
\[
\psi_s(x) \equiv \psi_{jkt}(x) = \frac{2^j}{2\pi} \sum_n i^n \beta_{j,n}^t e^{in\theta_x} h_n(|2^j x - k|) \tag{3b}
\]
where $h_n(|x|)$ is the Hankel transform of $\hat{h}(|\xi|)$ of order $n$. For $\hat{h}(|\xi|)$ we will employ the window proposed for the steerable pyramid [38], since $h_n(|x|)$ then has a closed form expression [37]. Note also that the angular localization described by the $\beta^t_{j,n}$ is invariant under the Fourier transform and only modified by the factor of $i^n$ that implements a rotation by $\pi/2$.

When the wavelets in Eq. 3 are suitably augmented using scaling functions $\phi_{j,k}(x)$ to represent a signal’s low frequency part, with $\psi_{-1,k}(x) \equiv \phi_{0,k}(x)$, the polar wavelets in Eq. 3 provide a tight frame for $L_2(\mathbb{R}^2)$ and any function $f(x) \in L_2(\mathbb{R}^2)$ can be represented as [15]
\[
f(x) = \sum_{s \in S} \langle f(y), \psi_s(y) \rangle \psi_s(x) = \sum_{j=-1}^\infty \sum_{k \in \mathbb{Z}^2} \sum_{t=1}^{N_j} \langle f(y), \psi_{jkt}(y) \rangle \psi_{jkt}(x). \tag{4}
\]
Although the above frame is redundant, since it is tight it still affords most of the conveniences of an orthonormal basis.

Analogous to Eq. 3a in three dimensions polar wavelets are defined by
\[
\hat{\psi}_s(\xi) \equiv \hat{\psi}_{j,k,l}(\xi) = \frac{2^{3j/2}}{(2\pi)^{3/2}} \sum_{l,m} \kappa_{j,m}^{l} y_{lm}(\xi) \hat{h}(2^{-j}|\xi|) e^{-i(\xi,2^j k)} \tag{5a}
\]
where $\bar{\xi} = \xi/|\xi|$, the $y_{lm}(\bar{\xi})$ are spherical harmonics, and the coefficients $\kappa_{lm}^{jt}$ control the angular localization. The wavelets in Eq. 5 have again closed form expressions in the spatial domain,

$$\psi_s(x) \equiv \psi_{j,k,t}(x) = \frac{2^{d/2}}{(2\pi)^{3/2}} \sum_{l,m} i^l \kappa_{lm}^{jt} y_{lm}(\bar{x}) \hat{h}_l(|2^j x - k|) \quad (5b)$$

which can be obtained using the Rayleigh formula, cf. Appendix A. The analogue of Eq. 4 holds, i.e. the wavelets in Eq. 5 form a tight frame for $L_2(\mathbb{R}^3)$. We refer to the original works [15, 16] and [37] for a more detailed discussion of polar wavelets.

In the following, we will refer to the wavelets in Eq. 3 and Eq. 5 with windows satisfying the admissibility conditions for a tight frame simply as polar wavelets. Unless mentioned otherwise, the index set $S$ will also run over all scales, translations, and orientations, as in Eq. 4, and this also holds if we just write $s$ without specifying the index set.

4. Differential Forms and their Fourier Transform

In this section we first recall the basic facts about differential forms and the exterior calculus defined on them. Our principle reference for this material will be the book by Marsden, Ratiu, and Abraham [39] and we will also use the notation and conventions from there. Afterwards, starting from existing results in theoretical physics [40, 41, 42], we will study the Fourier transform of the exterior calculus.

4.1. Differential Forms and the Exterior Calculus

A differential $k$-form $\alpha \in \Omega^k(\mathbb{R}^n)$ on $\mathbb{R}^n$, with $0 \leq k \leq n$, is a covariant, anti-symmetric tensor field of rank $k$. When $k = 0$ this corresponds to the usual functions, i.e. $\Omega^0(\mathbb{R}^n) = \mathcal{F}(\mathbb{R}^n)$, and $k = n$ are densities, e.g. $f(x) \, dx$, that yield a Jacobian term under coordinate transformations. All other $k$ correspond to “vector-valued” quantities. For example, a 1-form is a co-vector field, i.e. an object in the cotangent bundle $T^*\mathbb{R}^n$. 

A principal motivation for studying differential forms is that they are closed under the exterior derivative $d$ in that

$$
d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n),
$$

and that $d$ subsumes the ordinary derivative in one dimensions and the first order differential operators of gradient, curl, and divergence in higher dimensions, while ensuring coordinate invariance. The exterior derivative furthermore satisfies $d \circ d = 0$, which subsumes $\nabla \times \nabla = 0$ and related identities in classical vector calculus. The differential forms hence form a co-chain complex,

$$
0 \xrightarrow{d} \Omega^0(\mathbb{R}^n) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(\mathbb{R}^n) \xrightarrow{d} 0
$$

which is known as the de Rahm complex. When $d\alpha = 0$ one say that $\alpha$ is closed and when there exists a $\beta$ such that $\beta = d\alpha$ then is $\alpha$ exact. The Poincaré Lemma states that for every topologically trivial region in $\mathbb{R}^n$ any closed $k$ form is also exact.

The exterior calculus also provides an anti-symmetric product on differential forms known as the wedge product

$$
\wedge : \Omega^k \times \Omega^l \to \Omega^{k+l}
$$

which turns it into a graded algebra.

To introduce coordinate expressions for differential forms we first need a basis for the tangent space. We will follow the convention from differential geometry and denote it as

$$
\left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \right\}.
$$

The notation implies that one works locally in a suitable chart and this will become important for us for $S^2$ where a single chart longer suffices, and we will use them there in this sense (in contrast to the convention from differential geometry, however, we will work with normalized ). Using the $\partial/\partial x^i$ we can introduce a dual basis formed by $dx^i$ such that the usual biorthogonality
condition holds, i.e.
\[ dx^i \left( \frac{\partial}{\partial x^i} \right) = \delta_{ij}. \] (10)

The coordinate expression for a 1-form \( \alpha \in \Omega^1(\mathbb{R}^n) \) is thus
\[ \alpha(x) = \alpha_1 \, dx^1 + \alpha_2 \, dx^2 + \alpha_3 \, dx^3 \] (11)

The basis functions for higher degree forms are generated using the wedge product in Eq. 8 (and keeping the anti-symmetry in mind). In \( \mathbb{R}^3 \), for example, one has thus the following forms
\[ f = f(x) \in \Omega^0(\mathbb{R}^3) \] (12a)
\[ \alpha = \alpha_1(x) \, dx^1 + \alpha_2(x) \, dx^2 + \alpha_3(x) \, dx^3 \in \Omega^1(\mathbb{R}^3) \] (12b)
\[ \beta = \beta_1(x) \, dx^2 \wedge dx^3 + \beta_2(x) \, dx^1 \wedge dx^3 + \beta_3(x) \, dx^1 \wedge dx^2 \in \Omega^2(\mathbb{R}^3) \] (12c)
\[ \gamma = \gamma(x) \, dx^1 \wedge dx^2 \wedge dx^3 \in \Omega^3(\mathbb{R}^3) \] (12d)

An arbitrary form \( \alpha \in \Omega^k(\mathbb{R}^n) \) can in coordinates be written as
\[ \alpha(x) = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k}(x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \] (13)
where \( i_1 < \cdots < i_k \) ensures anti-symmetry.

Remark 1. In our discussion we assume that our coordinate functions are smooth, or at least sufficiently smooth, and real-valued. Differential forms with coordinate functions that are distributions were introduced by de Rahm [43] and they are known as ‘currents’.

Hilbert Space Structure and Hodge Dual. So far we did not require a metric \( g^{ij} \). When it is available we can use it to identify vectors and co-vectors by “raising” and “lowering” indices. In particular, a vector field \( X \) with components \( X^i \) can be associated with the 1-form \( X^b \) using the flat operator by
\[ (X^b)_j = g_{ij} \, X^i. \] (14)

The inverse of the flat is given by the sharp and it associates a vector with a 1-form,
\[ (\alpha^i)^j = g^{ij} \alpha_i. \] (15)
With the flat and the sharp we can re-write the operators from vector calculus using the exterior derivative,

\[ \nabla f = (df)^\sharp \]  
(16a)

\[ \nabla \cdot u = (d \star u^b)^\sharp \]  
(16b)

\[ \nabla \times u = (du^b)^\sharp \]  
(16c)

Note, however, when one uses one of the classical vector calculus operators one usually has, in fact, already a differential form and there is no need to convert forth and back: one should just use the form (not the least because it ensures coordinate independence of the result).

We can use the metric also to introduce an \( L_2 \) inner product on the space \( \Omega^k(\mathbb{R}^n) \) of differential \( k \)-forms as

\[ \langle \langle \alpha, \beta \rangle \rangle = \int_{\mathbb{R}^n} \alpha \wedge \star \beta = \int_{\mathbb{R}^n} \alpha_{i_1 \cdots i_k} \beta^{i_1 \cdots i_k} \, dx^1 \wedge \cdots \wedge dx^n \]  
(17a)

where

\[ \beta^{i_1 \cdots i_k} = g^{i_1 j_1} \cdots g^{i_k j_k} \beta_{j_1 \cdots j_k} \]  
(17b)

and summation of repeated indices is implied. The space \( L_2(\mathbb{R}^n, \Omega^k) \) is given in the usual sense by all differential forms with finite \( L_2 \) norm. Unless stated otherwise, we will assume that all the forms we work with are in \( L_2(\mathbb{R}^n, \Omega^k) \).

In Eq. (17) we also introduced the Hodge dual

\[ \star : \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{n-k}(\mathbb{R}^n). \]  
(18)

For example, in \( \mathbb{R}^3 \) we have the following correspondences

\[ \star 1 = dx^1 \wedge dx^2 \wedge dx^3 \]  
(19a)

\[ \star dx^1 = dx^2 \wedge dx^3 \]  
(19b)

\[ \star dx^2 = dx^1 \wedge dx^3 \]  
(19c)

\[ \star dx^3 = dx^1 \wedge dx^2 \]  
(19d)
and one has for the composition of the Hodge dual
\[ \star \star = (-1)^{k(n-k)}; \]
see also again Eq. 12a.

A central result that uses the \( L_2 \) structure of differential forms is the Hodge decomposition [39, Thm. 8.5.1] that splits \( L_2(\mathbb{R}^n, \Omega^k) \) into two orthogonal subspaces

\[ \Omega^k = \Omega_d \oplus \Omega_\delta \tag{20} \]

where \( \Omega_d \) is the space of exact forms and \( \Omega_\delta \) the space of co-exact ones, i.e. where \( d\alpha \neq 0 \). The harmonic forms, which usually form the third part of the Hodge decomposition, are not present since \( L_2(\mathbb{R}^n) \) does not contain any nonzero polynomials, cf. [44].

For more details on differential forms and exterior calculus we refer again to Marsden, Ratiu, and Abraham [39].

### 4.2. The Fourier Transform of the Exterior Algebra

Before we get to the formal definition of the Fourier transform of differential forms, let us motivate it. The frequency variable \( \xi \) dual to \( x \in \mathbb{R}^n \) transforms like a co-vector, or 1-form, i.e.

\[ A x \leftrightarrow A^{-T} \xi, \tag{21} \]

for \( A \in \text{GL}(n) \). It should hence be considered as such, i.e. \( \xi \in \mathbb{R}^n_\xi \) where \( \mathbb{R}^n_\xi \) is the (continuous) dual space of \( \mathbb{R}_x^n \). The bases for vector fields and 1-forms for \( \mathbb{R}^n_\xi \) are

\[
\text{span}_{i=1 \ldots n} (d\xi^i) = \mathbb{R}^n_\xi \\
\text{span}_{i=1 \ldots n} \left( \frac{\partial}{\partial \xi^i} \right) = \Omega^1(\mathbb{R}^n_\xi) \tag{22}
\]

Note that since \( \mathbb{R}^n_\xi \) is the dual space the roles of the \( d\xi^i \) and \( \partial/\partial \xi^i \) are interchanged and with respect to \( \xi \) we have an exterior calculus defined over the \( \partial/\partial \xi^i \) with the bases for differential forms of arbitrary degree being generated, as usual, by those for 1-forms. For example, a differential 1-form in frequency space is given by

\[ \hat{\alpha}(\xi) = \hat{\alpha}_1(\xi) \frac{\partial}{\partial \xi^1} + \hat{\alpha}_2(\xi) \frac{\partial}{\partial \xi^2} + \hat{\alpha}_3(\xi) \frac{\partial}{\partial \xi^3}. \tag{23} \]
This also implies that in the Fourier domain integration is over the $\partial/\partial \xi^i$. Furthermore, since, in the sense of distributions, the Fourier transform of a plane is the normal to it, that is, for example,

$$\mathcal{F}\left(\delta_{\mathbb{R}^2}(x)\right) = \delta_{\mathbb{R}^1}(\xi)$$ (24)

we have to have that a 2-form in space $\mathbb{R}^3_x$, which can be integrated over $\mathbb{R}^2_{x_1,2}$, becomes a 1-form in $\mathbb{R}^3_\xi$, where it can be integrated over $\mathbb{R}^1_{\xi_3}$. Analogously, a volume form $\alpha \in \Omega^n(\mathbb{R}^n)$ in space, which can be identified with a function using the Hodge dual, has to become a 0-form in frequency space, since the integral of $\alpha$ over $\mathbb{R}^n$ is given by evaluation of $\hat{\alpha}$ at the origin. With these observations the correspondence between differential forms in space and frequency has to be

$$\Omega^k(\mathbb{R}^n_x) \xrightarrow{\mathcal{F}} \Omega^{n-k}(\mathbb{R}^n_\xi).$$ (25)

**The Fourier Transform of Differential Forms.** With the above motivation in mind we follow Kalkman [41] and define the Fourier transform of differential forms as follows.

**Definition 1.** Let $\alpha(x)$ be a differential form of degree $k$,

$$\alpha(x) = \sum_{i_1 < \cdots < i_k} \alpha_{i_1,\cdots,i_k}(x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(\mathbb{R}^n_x)$$

Then the Fourier transform $\hat{\alpha} = \mathcal{F}(\alpha) \in \hat{\Omega}^{n-k}(\mathbb{R}^n_\xi)$ of $\alpha(x)$ is the $(n-k)$ form on frequency space $\mathbb{R}^n_\xi$ given by

$$\mathcal{F}(\alpha)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_x} \alpha_{i_1,\cdots,i_k}(x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge e^{-i \xi^i x_i} \wedge e^{dx^i \frac{\partial}{\partial \xi^i}}$$ (26a)

where the complex exponential of differential forms is defined as

$$e^{dx^i \frac{\partial}{\partial \xi^i}} = \sum_{a=1}^n \left(\frac{dx^i \frac{\partial}{\partial \xi^i}}{a!}\right)^a$$ (26b)

with summation in $i$ implied. The inverse Fourier transform $\alpha = \mathcal{F}^{-1}(\hat{\alpha}) \in \Omega^k(\mathbb{R}^n_x)$ of $\hat{\alpha}(x) \in \hat{\Omega}^{n-k}(\mathbb{R}^n_\xi)$ is the $k$ form in space given by

$$\mathcal{F}^{-1}(\hat{\alpha})(x) = \frac{-1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_\xi} \hat{\alpha}_{i_1,\cdots,i_k}(\xi) \frac{\partial}{\partial \xi^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial \xi^{n-k}} \wedge e^{i \xi^i x_i} \wedge e^{dx^i \frac{\partial}{\partial \xi^i}}.$$
From a functional analytic perspective the usual requirements for the existence of the Fourier transform apply with respect to the coordinate functions $\alpha_{i_1,\ldots,i_k}(x)$ and $\hat{\alpha}_{i_1,\ldots,i_k}(\xi)$. A direct consequence of our definition is the following.

**Proposition 1.** $F^{-1} \circ F$ is the identity.

*Proof.* See [41, Proposition 5.2].

**Remark 2.** In [41, 42] the exponential $e^{d x_i \frac{\partial}{\partial \xi^j}}$ was defined with a complex unit in the exponent. As mentioned there, it served only “aesthetic” reasons to visually closer match the scalar case. For our purposes, it would lead to an unnecessary cluttering of most equations with powers of the complex unit, which are furthermore difficult to track during calculations, so we use the real definition above.

The powers in Eq. 26b are with respect to the wedge product as multiplication in both $\partial/\partial \xi^j$ and $dx^j$ yielding a double differential form [43, p. 30] on $\Omega(\mathbb{R}^n_\xi) \otimes \Omega(\mathbb{R}^n_x)$. To respect the grading of $\Omega(\mathbb{R}^n_x)$ and $\hat{\Omega}(\mathbb{R}^n_\xi)$ the tensor product has to be such that

$$
(\alpha \otimes \hat{\beta}) \wedge_{x,\xi} (\gamma \otimes \hat{\delta}) = (-1)^{\deg(\gamma) \deg(\hat{\beta})} \left( \alpha \wedge_x \gamma \otimes \hat{\beta} \wedge_{\xi} \hat{\delta} \right)
$$

(27)

where we also wrote $\wedge_{x,\xi}$ to make explicit that the wedge product is in both variables and we will use this notation whenever confusion might arise. For example, in $\mathbb{R}^2$ we have

$$
\left( dx^i \frac{\partial}{\partial \xi^j} \right)^2 = dx^1 \otimes \frac{\partial}{\partial \xi^1} \wedge_{x,\xi} dx^2 \otimes \frac{\partial}{\partial \xi^2} \wedge_{\xi} \frac{\partial}{\partial \xi^2}
$$

(28a)

$$
= (-1)^{1 \cdot 1} dx^1 \wedge_x dx^2 \otimes \frac{\partial}{\partial \xi^1} \wedge_{x,\xi} \frac{\partial}{\partial \xi^2}
$$

(28b)

$$
= - dx^1 \wedge_x dx^2 \otimes \frac{\partial}{\partial \xi^1} \wedge_{x,\xi} \frac{\partial}{\partial \xi^2}
$$

(28c)

with the “square” terms vanishing by the antisymmetry of the wedge product.

We will also typically omit the tensor product symbol as on the left hand side of
the above equation. When the exponential has been expanded, the usual rules for integration of differential forms apply for evaluating Eq. 26. In particular, the integral vanishes unless Eq. 26b together with the \( \text{d}x^1 \wedge \cdots \wedge \text{d}x^k \) of \( \alpha \) yield a volume form. This is also the reason why it suffices to consider the power series in Eq. 26b up to \( n \). Note also that \( e^{\text{d}x^i \frac{\partial}{\partial \xi^i}} \) is used for the Fourier transform and its inverse.

**Example 1.** In \( \mathbb{R}^3 \) the exponential in Eq. 26b is given by

\[
e^{\text{d}x^i \frac{\partial}{\partial \xi^i}} = 1 + \left( \text{d}x^1 \frac{\partial}{\partial \xi^1} + \text{d}x^2 \frac{\partial}{\partial \xi^2} + \text{d}x^3 \frac{\partial}{\partial \xi^3} \right)
\]

\[
+ \left( \text{d}x^2 \frac{\partial}{\partial \xi^2} \wedge \text{d}x^3 \frac{\partial}{\partial \xi^3} - \text{d}x^1 \frac{\partial}{\partial \xi^1} \wedge \text{d}x^3 \frac{\partial}{\partial \xi^3} + \text{d}x^1 \frac{\partial}{\partial \xi^1} \wedge \text{d}x^2 \frac{\partial}{\partial \xi^2} \right)
\]

\[
+ \left( \text{d}x^1 \frac{\partial}{\partial \xi^1} \wedge \text{d}x^2 \frac{\partial}{\partial \xi^2} \wedge \text{d}x^3 \frac{\partial}{\partial \xi^3} \right)
\]

(29)

with the powers being expanded using the multinomial theorem. With Eq. 27 one thus has for the differential form basis functions

\[
\mathcal{F}(1_x) = -\frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3}
\]

(30a)

\[
\mathcal{F}(\text{d}x^i) = \text{sgn} (\sigma) \frac{\partial}{\partial \xi^{i_2}} \wedge \frac{\partial}{\partial \xi^{i_3}}
\]

(30b)

\[
\mathcal{F}(\text{d}x^{i_1} \wedge \text{d}x^{i_2}) = \frac{\partial}{\partial \xi^{i_3}}
\]

(30c)

\[
\mathcal{F}(\text{d}x^1 \wedge \text{d}x^2 \wedge \text{d}x^3) = 1_{\xi}
\]

(30d)

where \( \sigma = (i_1,i_2,i_3) \in S_3 \). As example for the Fourier transform, consider a 1-form \( \alpha = \alpha_1 \text{d}x^1 \). Then we obtain a volume form in the integral in Eq. 26 only from the second line in Eq. 29. In particular,

\[
\mathcal{F}(\alpha)(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \alpha_1(x) \text{d}x^1 \wedge e^{-i \xi^i x_i} \wedge e^{i x_j \frac{\partial}{\partial \xi^j}}
\]

(31a)

\[
= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \alpha_1(x) e^{-i \xi^i x_i} \text{d}x^1 \wedge \text{d}x^2 \frac{\partial}{\partial \xi^2} \wedge \text{d}x^3 \frac{\partial}{\partial \xi^3}
\]

(31b)

\[
= \frac{-1}{(2\pi)^{3/2}} \hat{\alpha}_1(\xi) \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3}
\]

(31c)
where $\hat{\alpha}_1(\xi)$ is the usual Fourier transform of the coordinate function. The result carries over to arbitrary 1-forms by linearity.

**Remark 3.** The prescription in Eq. 26 is not very practical to determine which $\partial/\partial \xi^i$ appear in the Fourier transform of a form basis function. We can borrow a more convenient description of the image of $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ that results under the Fourier transform from the Hodge dual [39, Ch. 7.2]. Consider a permutation

$$
\sigma = \begin{pmatrix}
\sigma_1 & \cdots & \sigma_k & \sigma_{n-k+1} & \cdots & \sigma_n \\
i_1 & \cdots & i_k & \cdots & & 
\end{pmatrix}
$$

whose first $k$ indices match the given differential form; i.e $\sigma \in S_k$ where $S_k$ is the group of permutations of $\{1, \cdots, k\}$. One then has

$$
\mathcal{F}(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \text{sgn}(\sigma) (-1)^{\left\lfloor (n-k)/2 \right\rfloor} \frac{\partial}{\partial \xi^{\sigma_{n-k+1}}} \wedge \cdots \wedge \frac{\partial}{\partial \xi^{\sigma_n}}
$$

with the factor of $(-1)^{\left\lfloor (n-k)/2 \right\rfloor}$ arising from Eq. 27.

**Exterior derivative.** We also require the Fourier transform of the exterior derivative. It is given in the next proposition, which is again based on the discussion by Kalkman [41, Proposition 5.3].

**Proposition 2.** The Fourier transform (or principal symbol) of the exterior derivative $d : \Omega^k(\mathbb{R}^n_x) \mapsto \Omega^{k+1}(\mathbb{R}^n_x)$ is the anti-derivation $\hat{d} : \hat{\Omega}^{n-k}(\mathbb{R}^n_\xi) \mapsto \hat{\Omega}^{n-k-1}(\mathbb{R}^n_\xi)$ given by

$$
\mathcal{F}(d\alpha) = i_\xi \hat{\alpha}
$$

where $i_\xi \hat{\alpha}$ is the interior product and $\xi = \xi_1 \, dx^1 + \xi_2 \, dx^2 + \xi_3 \, dx^3 \in \mathfrak{X}(\mathbb{R}^n_\xi)$. That is, the following commutative diagram holds

$$
\begin{array}{ccc}
\Omega^k(\mathbb{R}^n_x) & \xrightarrow{d} & \Omega^{k+1}(\mathbb{R}^n_x) \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
\hat{\Omega}^{n-k}(\mathbb{R}^n_\xi) & \xrightarrow{\hat{d}=i_\xi} & \hat{\Omega}^{n-k-1}(\mathbb{R}^n_\xi)
\end{array}
$$

and the Fourier transform of the de Rham chain complex $\Omega^k(\mathbb{R}^n_x) \xrightarrow{d} \Omega^{k+1}(\mathbb{R}^n_x)$ is a chain complex $\hat{\Omega}^k(\mathbb{R}^n_\xi) \xrightarrow{\hat{d}=i_\xi} \hat{\Omega}^{k-1}(\mathbb{R}^n_\xi)$.
Proof. The coordinate expression of the exterior derivative is
\[
\begin{align*}
\text{d} \alpha(x) &= \sum_{i=1}^{n} \sum_{i_1 < \cdots < i_k} \frac{\partial \alpha_{i_1, \ldots, i_k}}{\partial x^i} \, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. 
\end{align*}
\] (34a)

with the second sum being over all possible permutations satisfying the anti-symmetry condition \(i_1 < \cdots < i_k\). By linearity it suffices to consider one term in the exterior derivative,
\[
\begin{align*}
\text{d} \alpha_i(x) &= \sum_{i_1 < \cdots < i_k} \frac{\partial \alpha_{i_1, \ldots, i_k}}{\partial x^i} \, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. 
\end{align*}
\] (34b)

Its Fourier transform is
\[
\begin{align*}
\mathcal{F}(\text{d} \alpha_i)(\xi) &= \sum_{i_1 < \cdots < i_k} i \xi^i \hat{\alpha}_{i_1, \ldots, i_k}(\xi) \sgn(\sigma)(-1)^{(n-k)/2} \frac{\partial}{\partial \xi^{\sigma_{k-1}+1}} \wedge \cdots \wedge \frac{\partial}{\partial \xi^{\sigma_{n}}},
\end{align*}
\]
where \(\sigma \in S_k\) is a permutation as discussed in Remark 3. Writing the interior product \(i\xi \hat{\alpha}\) in coordinates yields
\[
\begin{align*}
i \xi \hat{\alpha} &= \sum_{i=1}^{n} i \xi^i d\xi^i \sum_{i_1 < \cdots < i_k} \hat{\alpha}_{i_1, \ldots, i_k}(\xi) \sgn(\sigma)(-1)^{(n-k)/2} \frac{\partial}{\partial \xi^{\sigma_i}} \wedge \cdots \wedge \frac{\partial}{\partial \xi^{\sigma_{k}}},
\end{align*}
\]
It again suffices to consider one component of \(\xi\),
\[
\begin{align*}
i \xi \hat{\alpha} &= \sum_{i_1 < \cdots < i_k} i \xi^i \hat{\alpha}_{i_1, \ldots, i_k}(\xi) \sgn(\bar{\sigma})(-1)^{(n-k)/2} d\xi^i \left( \frac{\partial}{\partial \xi^{\sigma_{k+1}} \wedge \cdots \wedge \frac{\partial}{\partial \xi^{\sigma_{n}}} \right),
\end{align*}
\]
The expression on the right hand side will only be non-zero when \(\partial / \partial \xi^i\) is present in the above permutation. It pairs with \(d\xi^i\) after a permutation \(\bar{\sigma}\). With Remark 3 in mind it can be checked that \(\sigma = \bar{\sigma} \bar{\sigma}'\) which shows Eq. 34a.

That \(\hat{d}\) is an anti-derivation follows immediately from the interior product being one and that the \(\hat{\Omega}^k(\mathbb{R}^n)\) form a chain complex under \(\hat{d}\) follows directly from the definition.

A classical property of the exterior derivative is its nilpotency \(dd = 0\). Hence, in frequency space we have to have \(i\xi i\xi = 0\). That this indeed holds in a well known property of the interior product [39, p. 428]. In fact, the interior product is also referred to as interior derivative [45, Ch. 4.6].
Example 2. Let \( \alpha(x) \in \Omega^1(\mathbb{R}^3) \). Then \( d\alpha(x) \in \Omega^2(\mathbb{R}^3) \) and it corresponds to the curl in classical vector calculus. In frequency space, \( \alpha(x) \) is the 2-form

\[
\hat{\alpha}(\xi) = -\hat{\alpha}_1(\xi) \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} - \hat{\alpha}_2(\xi) \frac{\partial}{\partial \xi^3} \wedge \frac{\partial}{\partial \xi^1} - \hat{\alpha}_3(\xi) \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2},
\]  

(35a)

cf. Example 1. The interior product that defines the exterior derivative in frequency space is

\[
-(i\xi \hat{\alpha})(\xi) = i\xi_1 \hat{\alpha}_2(\xi) d\xi^1 \left( \frac{\partial}{\partial \xi^3} \wedge \frac{\partial}{\partial \xi^1} \right) + i\xi_1 \hat{\alpha}_3(\xi) d\xi^1 \left( \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^1} \right)
+ i\xi_2 \hat{\alpha}_1(\xi) d\xi^2 \left( \frac{\partial}{\partial \xi^3} \wedge \frac{\partial}{\partial \xi^2} \right) + i\xi_2 \hat{\alpha}_3(\xi) d\xi^2 \left( \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \right)
+ i\xi_3 \hat{\alpha}_1(\xi) d\xi^3 \left( \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} \right) + i\xi_3 \hat{\alpha}_2(\xi) d\xi^3 \left( \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^3} \right)
\]

(35b)

where we immediately omitted terms that are necessarily zero by the biorthogonality condition \( d\xi^i(\partial/\partial \xi^j) = \delta_{ij} \). Using again this property and the antisymmetry of the wedge product we obtain

\[
-(i\xi \hat{\alpha})(\xi) = -i\xi_1 \hat{\alpha}_2(\xi) \frac{\partial}{\partial \xi^3} + i\xi_1 \hat{\alpha}_3(\xi) \frac{\partial}{\partial \xi^2} + i\xi_2 \hat{\alpha}_1(\xi) \frac{\partial}{\partial \xi^3} - i\xi_2 \hat{\alpha}_3(\xi) \frac{\partial}{\partial \xi^1} - i\xi_3 \hat{\alpha}_1(\xi) \frac{\partial}{\partial \xi^2} + i\xi_3 \hat{\alpha}_2(\xi) \frac{\partial}{\partial \xi^1}.
\]

(35c)

After collecting terms we thus have

\[
-(i\xi \hat{\alpha})(\xi) = i\hat{\xi} \times \hat{\alpha}(\xi)
\]

(35d)

where \( \hat{\xi} \) and \( \hat{\alpha} \) are the naïve vectors formed by the components in the foregoing equation (this identification can be justified since the metric in \( \mathbb{R}^3 \) is the identity). Up to a sign this corresponds to the classical expression of the curl in the Fourier domain,

\[
\mathcal{F}(\nabla \times \hat{A}) = i\hat{\xi} \times \hat{\alpha}(\xi).
\]

(35e)

Remark 4. Treves [46, Vol. 1, I.7] states that the principal symbol of the exterior derivative is the exterior product, i.e. \( \hat{d}\alpha = i\xi \wedge \hat{\alpha} \). While the naïve
coordinate expression is in $\mathbb{R}^3$ equivalent, not all of the algebraic properties of the exterior product are consistent with the exterior derivative. For example, $d(i\xi \wedge \hat{\alpha})$ satisfies the Leibniz rule while for $di\xi (\hat{\alpha})$ no such expression exists (the interior product does satisfy a Leibniz rule but not involving the exterior derivative).

**Wedge Product.** In the scalar case, multiplication in space becomes convolution in the frequency domain. The next proposition shows that this carries over to the wedge product.

**Proposition 3.** Let $\alpha \in \Omega^k(\mathbb{R}^n)$ and $\beta \in \Omega^l(\mathbb{R}^n)$ and let the convolution of differential forms be $[I]$

$$ (\alpha * \beta)(dy) = \int_{\mathbb{R}^n} \alpha(dx) \wedge \beta(dy - dx) \tag{36} $$

where $\alpha(dx)$ denotes the basis representation of $\alpha$ with respect to the differentials $dx^i$. Then

$$ \mathcal{F}(\alpha \wedge \beta) = \hat{\alpha} \wedge \hat{\beta} = \hat{\alpha} * \hat{\beta} \in \hat{\Omega}^{k+l-n}(\mathbb{R}^n) \tag{37a} $$

and conversely

$$ \mathcal{F}^{-1}(\hat{\alpha} \wedge \hat{\beta}) = \alpha \wedge \beta = \alpha * \beta \in \Omega^{k+l-n}(\mathbb{R}^n), \tag{37b} $$

i.e., the following diagram holds

$$ \begin{array}{ccc}
\hat{\Omega}^{n-k-l} & \wedge = * & \hat{\Omega}^{n-k}, \hat{\Omega}^{n-l} & \wedge = \hat{\wedge} & \hat{\Omega}^{l+k-n} \\
\mathcal{F} & & \mathcal{F} & & \mathcal{F}
\end{array} $$

**Proof.** This can be proved by a direct calculation. See again [I]. \( \square \)

**Example 3.** Let $\alpha = \alpha_3 dx^1 \wedge dx^2 \in \Omega^2(\mathbb{R}^3)$ and $\beta = \beta_1 dx^2 \wedge dx^3 + \beta_2 dx^1 \wedge dx^3 + \beta_3 dx^1 \wedge dx^2 \in \Omega^1(\mathbb{R}^3)$. Then

$$ (\alpha * \beta)(dy) = \int_{\mathbb{R}^n} \alpha(dx) \wedge \beta(dy - dx) \tag{38a} $$
with

\[ \beta(dy - dx) = \beta_1(x - y)(dy^1 - dx^1) \]
\[ + \beta_2(x - y)(dy^2 - dx^2) \]
\[ + \beta_3(x - y)(dy^3 - dx^3) \quad (38b) \]

Using linearity and the anti-symmetry of the wedge product we obtain

\[ (\alpha \ast \beta)(dy) = \int_{\mathbb{R}^n} \alpha_3(x) \, dx^1 \wedge dx^2 \wedge \beta_3(x - y) \, (dy^3 - dx^3) \quad (38c) \]
\[ = \int_{\mathbb{R}^n} \alpha_3(x) \, \beta_3(x - y) \, dx^1 \wedge dx^2 \wedge dx^3 \quad (38d) \]
\[ = \alpha_3 \ast \beta_3 \quad (38e) \]

where the convolution in the last line is those of scalar functions.

**Parseval’s theorem.** Parseval’s theorem is a central result for the scalar Fourier transform. The next proposition establishes the analogue for differential forms.

**Proposition 4** (Parseval’s theorem for differential forms). Let \( \alpha, \beta \in \Omega^k(\mathbb{R}^n) \). Then

\[ \langle\langle \alpha, \beta \rangle\rangle = -\langle\langle \hat{\alpha}, \hat{\beta} \rangle\rangle. \quad (39) \]

**Proof.** As in the classical case, the results can be shown with a direct calculation using that

\[ \langle\langle \alpha, \beta \rangle\rangle = \langle\langle \mathcal{F}^{-1}(\alpha), \mathcal{F}^{-1}(\beta) \rangle\rangle. \quad (40) \]

We will show it for \( \alpha = \alpha_3(x) \, dx^3 \), \( \beta = \beta_3(x) \, dx^3 \). Then

\[ \langle\langle \alpha, \beta \rangle\rangle = \int_{\mathbb{R}^3} \alpha_3(x) \, dx^3 \wedge \beta_3(x) \, dx^1 \wedge dx^2 \quad (41a) \]
\[ = \int_{\mathbb{R}^3} \left( -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\alpha}_3(\xi) \, e^{i \langle \xi, x \rangle} \frac{\partial}{\partial \xi^1} \wedge \xi \frac{\partial}{\partial \xi^2} \wedge \xi \left( \frac{\partial}{\partial \xi^3} \, dx^3 \right) \right) \]
\[ \wedge_x \left( -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\beta}_3(\eta) \, e^{-i \langle \eta, x \rangle} \frac{\partial}{\partial \eta^3} \wedge \eta \left( \frac{\partial}{\partial \eta^1} \, dx^1 \wedge \eta, x \frac{\partial}{\partial \eta^2} \, dx^2 \right) \right) \].

20
Using linearity and separating the terms with $x$ dependence we obtain

$$
\langle\langle \alpha, \beta \rangle\rangle = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \hat{\alpha}_3(\xi) \frac{\partial}{\partial \xi^1} \wedge \xi \frac{\partial}{\partial \xi^2} \right) \left( \hat{\beta}_3^*(\eta) \frac{\partial}{\partial \eta^3} \right)
$$

\hspace{2cm} \wedge \xi, \eta \int_{\mathbb{R}^3} e^{-i(\eta \cdot x)} \frac{\partial}{\partial \eta^3} \mathrm{d}x^3 \wedge \xi e^{i(\xi \cdot x)} \frac{\partial}{\partial \xi^3} \mathrm{d}x^3 \wedge \eta, \xi \frac{\partial}{\partial \eta^2} \mathrm{d}x^2

\hspace{2cm} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \hat{\alpha}_3(\xi) \frac{\partial}{\partial \xi^1} \wedge \xi \frac{\partial}{\partial \xi^2} \right) \left( \hat{\beta}_3^*(\eta) \frac{\partial}{\partial \eta^3} \right)

\hspace{2cm} \wedge \xi, \eta \delta(\xi - \eta) \frac{\partial}{\partial \xi^3} \otimes \frac{\partial}{\partial \eta^3} \wedge \eta \frac{\partial}{\partial \eta^2}.

(41b)

and the last expression is again a differential double form. Combining the differentials in $\xi$ and $\eta$ yields

$$
\langle\langle \alpha, \beta \rangle\rangle = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \hat{\alpha}_3(\xi) \hat{\beta}_3^*(\eta) \delta(\xi - \eta)
$$

\hspace{2cm} \times \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3} \otimes \frac{\partial}{\partial \eta^3} \wedge \frac{\partial}{\partial \eta^3} \wedge \frac{\partial}{\partial \eta^2}.

(41c)

Carrying out the integration over $\eta$ yields

$$
\langle\langle \alpha, \beta \rangle\rangle = - \int_{\mathbb{R}^3} \hat{\alpha}_3(\xi) \hat{\beta}_3^*(\xi) \frac{\partial}{\partial \xi^1} \wedge \frac{\partial}{\partial \xi^2} \wedge \frac{\partial}{\partial \xi^3}
$$

\hspace{2cm} - \int_{\mathbb{R}^3} \hat{\alpha} \wedge \star \hat{\beta}^*

\hspace{2cm} = -\langle\langle \hat{\alpha}, \hat{\beta} \rangle\rangle

(41d)

The other cases follow by analogous calculations. \hfill \Box

Using Parseval’s theorem we can define the Homogeneous Sobolev space $\dot{L}^1_2(\mathbb{R}^n, \Omega^k)$ for differential forms, see Appendix A.3 for the scalar case. The $\dot{L}^1_2(\mathbb{R}^n, \Omega^k)$-inner product is given by

$$
\langle\langle \alpha, \beta \rangle\rangle_{L_2^1} = \langle\langle \mathrm{d}\alpha, \mathrm{d}\beta \rangle\rangle_{L_2} = \frac{-1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} |\xi|^2 \hat{\alpha} \wedge \star \hat{\beta}^*.
$$

(42)
and $\dot{L}^1_2(\mathbb{R}^n, \Omega^k)$ is then defined by

$$\dot{L}^1_2(\mathbb{R}^n, \Omega^k) = \left\{ \alpha \in \Omega^k(\mathbb{R}^n) \mid \|\alpha\|^2_{L^2} = \langle \alpha, \beta \rangle_{\dot{L}^1_2} < \infty \right\}.$$  \hspace{1cm} (43)

As in the scalar case, $\dot{L}^1_2(\mathbb{R}^n, \Omega^k)$ is a Hilbert space.

**Proposition 5.** The Homogeneous Sobolev Space $\dot{L}^1_2(\mathbb{R}^n, \Omega^k)$ is a Hilbert space for $n = 2, 3$.

**Proof.** With the definition of the Hodge dual in terms of the metric tensor [39, Ch. 7.2], Eq. 42 can be written as

$$-\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^n^\prime} |\xi|^2 \hat{\alpha} \wedge \hat{\beta}^* = -\frac{1}{(2\pi)^{3/2}} \sum_{i=1}^{n} \int_{\mathbb{R}^n^\prime} |\xi|^2 \hat{\alpha}_i(\xi) \hat{\beta}^i(\xi) d\xi \hspace{1cm} (44)$$

where we also used classical notation for the integral on the right hand side. But thus the scalar result [48, Ch. II.6] applies to each coordinate function. Note that in general one has to consider co-sets of polynomials in the definition of homogeneous Sobolev space for these to be Hilbert. This is not necessary in our cases since these are excluded from the outset. \hspace{1cm} \Box

*Hodge dual.* The following result characterizes the Fourier transform of the Hodge dual.

**Proposition 6.** Let $\alpha \in \Omega^k(\mathbb{R}^n)$. Then

$$\mathcal{F}(\hat{\star} \alpha) = -\hat{\star} \hat{\alpha}$$  \hspace{1cm} (45)

where the Hodge dual on the right hand side is the natural one on $\hat{\Omega}(\mathbb{R}^n_\xi)$.

**Proof.** Since the Hodge dual leaves the coordinate functions unchanged in $\mathbb{R}^n$ it suffices by linearity to consider the form basis functions. The result then follows from Remark 3. \hspace{1cm} \Box

*Stokes’ theorem.* An important result of exterior calculus is Stokes’ theorem, which has the classical theorems of Green, Gauss, and Stokes in vector calculus as special cases. Let $\mathcal{M}$ be a $k$-dimensional, smooth submanifold of $\mathbb{R}^n$ with
smooth boundary $\partial \mathcal{M}$ and $\alpha \in \Omega^k(\mathbb{R}^n)$. Then Stokes’ theorem is given by [39, Theorem 8.2.8]

$$\int_{\partial \mathcal{M}} i^* \alpha = \int_{\mathcal{M}} d\alpha$$

(46)

where $i : \partial \mathcal{M} \to \mathcal{M}$ is the inclusion map. Using the change of variable theorem [39, Theorem 8.1.7] we obtain for the left hand side

$$\int_{\partial \mathcal{M}} i^* \alpha = \int_{i(\partial \mathcal{M})} \alpha.$$  

(47)

Since $\partial \mathcal{M}$ is smooth we can always choose local coordinates such that $\alpha$ can be written as a canonical $(k-1)$-form. It is then clear that this form can be completed to a volume form with the characteristic form $\chi_{\partial \mathcal{M}} \in \Omega^{n-k}(\mathbb{R}^n)$ of $\partial \mathcal{M}$ such that one has

$$\int_{i(\partial \mathcal{M})} \alpha = \int_{\mathbb{R}^n} \alpha \wedge \chi_{\partial \mathcal{M}}$$

(48)

and we return to the question of how $\chi_{\partial \mathcal{M}}$ is defined momentarily. Using an analogous argument for the right hand side of Eq. (46) we obtain the following form for Stokes’ theorem

$$\int_{\mathbb{R}^n} \alpha \wedge \chi_{\partial \mathcal{M}} = \int_{\mathbb{R}^n} d\alpha \wedge \chi_{\mathcal{M}}.$$  

(49)

When $k + 1 = n$ then $\chi_{\mathcal{M}}$ is a 0-form, i.e. the usual characteristic function of $\mathcal{M}$, and it is well known from the classical divergence theorem that $\chi_{\partial \mathcal{M}}$ is given by the weak gradient of $\chi_{\mathcal{M}}$, i.e.

$$\chi_{\partial \mathcal{M}} = d\chi_{\mathcal{M}} = n_i \delta_{\partial \mathcal{M}} \, dx^i$$

(50)

where the $n_i$ are the components of the normal of $\partial \mathcal{M}$ and $\delta_{\partial \mathcal{M}}$ is its Dirac distribution. In other words, when $k + 1 = n$ then $\chi_{\partial \mathcal{M}}$ is nothing but the wavefront set $WF(\chi_{\mathcal{M}})$ of $\chi_{\mathcal{M}}$ [49, Ch. VI]. We will return to this connection in Sec. 5. In the general case, that is when $k + 1$ is not necessarily $n$, we can define $\chi_{\partial \mathcal{M}}$ as the contraction

$$\chi_{\mathcal{M}} = dx^1 \wedge \cdots \wedge dx^n \left( \frac{\partial}{\partial u^1}, \cdots, \frac{\partial}{\partial u^{k+1}}, \cdots \right)$$

(51)
where the $\partial / \partial u^i$ span the tangent space $T \mathcal{M}$ and $dx^1 \wedge \cdots \wedge dx^n$ is the canonical volume form on $\mathbb{R}^n$. Note that the definition also applies to $\chi_{\partial \mathcal{M}}$ since it is a $k$-dimensional manifold (with then a contraction with $k$ tangent vectors).

We can write Stokes’ theorem in Eq. 49 also using the inner product for differential forms. It then takes the form

$$\langle \langle \alpha, \chi_{\partial \mathcal{M}} \rangle \rangle = \langle \langle d\alpha, \chi_{T \mathcal{M}} \rangle \rangle$$

(52)

and it follows from Eq. 51 that $\chi_{T \partial \mathcal{M}} = \star \chi_{\partial \mathcal{M}}$ is a local representation for the tangent space of $\partial \mathcal{M}$ and similarly $\chi_{T \mathcal{M}} = \star \chi_{\mathcal{M}}$. For example, in $\mathbb{R}^3$ let the local tangent space be aligned with the $x_1$-$x_2$ plane then $\chi_{\partial \mathcal{M}} = \delta(x_3) \, dx_3$ and $\chi_{T \partial \mathcal{M}} = \delta(x_3) \, dx^1 \wedge dx^2$. Comparing to the classical form of Stokes’ theorem in Eq. 46, the inner product one in Eq. 52 provides a very similar geometric description since in both cases the left hand side is the pairing of $\alpha$ with the tangent space $T \partial \mathcal{M}$ and the right hand side those of $d\alpha$ with $T \mathcal{M}$.

Example 4. Let $\mathcal{M} = B^2$, $\partial \mathcal{M} = S^1$ and $\alpha = \alpha_\theta(\theta, r) \, d\theta \in \Omega^1(\mathbb{R}^2)$. Thus $d\alpha = -(\partial \alpha / \partial r) \, d\theta \wedge dr$. The completion of $\alpha$ to a volume form is then given by the 1-form $\chi_{\partial \mathcal{M}} = \chi_{S^1} = \delta_{S^1}(r) \, dr$, which, as required by Eq. 50, is in the direction of the normal of $B^2$. The completion of $d\alpha$ is the 0-form $\chi_{\mathcal{M}} = \chi_{B^2}(r)$. Stokes’ theorem in the form in Eq. 49 then becomes

$$\int_{\mathbb{R}^2} \alpha_\theta(\theta, r) \, d\theta \wedge \delta_{S^1}(r) \, dr = \int_{\mathbb{R}^2} -\frac{\partial \alpha(\theta, r)}{\partial r} \, d\theta \wedge dr \wedge \chi_{B^2}(r)$$

(53a)

and re-arranging terms yields

$$\int_{\mathbb{R}^2} \alpha_\theta(\theta, r) \, d\theta \wedge dr = \int_{\mathbb{R}^2} -\frac{\partial \alpha(\theta, r)}{\partial r} \, \chi_{B^2}(r) \, d\theta \wedge dr$$

(53b)

from which the equality also follows by integration by parts. In the current example one set of global coordinates suffices. In general one has to work with local charts.

Compared to the classical form of Stokes’ theorem in Eq. 46 one advantage of those in Eq. 49 is that it can be combined with Parseval’s theorem in Eq. 4 to transfer the result to the frequency domain. This yields

$$\int_{\mathbb{R}^n} \hat{\alpha} \wedge \hat{\chi}_{\partial \mathcal{M}} = \int_{\mathbb{R}^n} \hat{d\alpha} \wedge \hat{\chi}_{\mathcal{M}} = \int_{\mathbb{R}^n} \hat{d\alpha} \wedge \hat{\chi}_{\mathcal{M}}.$$  

(54)
Example 5. Continuing the previous example, the Fourier transforms of the
differential forms involved are

\[ \hat{\alpha} = -i \hat{\alpha}(\theta, r) \frac{\partial}{\partial r} \]  
\[ (55a) \]

\[ \hat{\chi}_{\alpha_M} = -i \hat{\delta}_{\alpha}(\xi) \frac{\partial}{\partial \theta} = -i J_1(r) \frac{\partial}{\partial \theta} \]  
\[ (55b) \]

\[ \hat{\alpha} = \frac{\partial \alpha}{\partial r}(\theta, r) = \hat{\alpha}(\theta, r) r \]  
\[ (55c) \]

\[ \hat{\chi}_{\nu M} = \hat{\delta}_{\nu}(\xi) \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial r} = \frac{J_1(r)}{r} \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial r} \]  
\[ (55d) \]

where \( J_1(\cdot) \) is the (cylindrical) Bessel function of order 1 and for \( \hat{\alpha} \) we used Proposition 3 and that \( \xi = r \xi \, dr \xi \) in polar coordinates. Inserting the Fourier
transforms into Eq. 54 we obtain

\[ \int_{\mathbb{R}^2} \hat{\alpha}(\theta, r) \frac{\partial}{\partial r} \wedge J_1(r) \frac{\partial}{\partial \theta} = \int_{\mathbb{R}^2} \hat{\alpha}(\theta, r) r \frac{\partial}{\partial \theta} \wedge \frac{J_1(r)}{r} \frac{\partial}{\partial r} \]  
\[ (56a) \]

and rearranging terms yields

\[ \int_{\mathbb{R}^2} \hat{\alpha}(\theta, r) J_1(r) \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial r} = \int_{\mathbb{R}^2} \hat{\alpha}(\theta, r) r \frac{J_1(r)}{r} \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial r}. \]  
\[ (56b) \]

By cancelling the \( r \) factor on the right hand side the equality holds trivially.

The above example provides some insight into the mechanics behind Stokes’
theorem in the Fourier domain: The growth in \( \hat{\alpha} \) compared to \( \hat{\alpha} \) introduced
by the exterior derivative is compensated by the difference in the decays of \( \hat{\chi}_{\alpha_M} \)
and \( \hat{\chi}_{M} \).

5. Local Spectral Exterior Calculus

The results of the previous section show that the exterior derivative acts
in the Fourier domain only on the radial component of a differential form and
that the co-chain structure of the de Rahm complex becomes there hence a
pointwise condition on the “direction” of the form basis functions: a form is
exact when it has no radial component. Motivated by these insights we define
our wavelet differential forms in the frequency domain using differential form basis functions in spherical coordinates, i.e. $\partial/\partial \hat{\theta}$, $\partial/\partial \hat{\phi}$, $\partial/\partial \hat{r}$, and using polar wavelets, i.e. with window functions separable in polar coordinates, since these respect the structure of the exterior calculus in the Fourier domain, allow for flexible angular localization, and ensure that the wavelet differential forms have closed form expressions in the spatial domain. In the following we will formally define wavelet differential forms and study their properties. Stokes’ theorem and other results related to differential forms are considered subsequently.

5.1. Wavelet Differential Forms

We define our wavelet differential forms as follows.

**Definition 2.** Let $\hat{\psi}_s(\xi)$ be a polar wavelet in the sense of Sec. 3 and $\{\partial/\partial \hat{\theta}, \partial/\partial \hat{\phi}, \partial/\partial \hat{r}\}$ be an orthonormal frame for $T^*\mathbb{R}^n_\xi$ in spherical coordinates, i.e. $\{\partial/\partial \hat{\theta}, \partial/\partial \hat{\phi}\}$ spans $T^*S^{n-1}_\xi$. Furthermore, let $\nu \in \{d, \delta\}$ and $A_{\nu,k} \in \{1, 2\}$ for $\nu = \delta, k = 1$ and $\nu = d, k = 2$ and otherwise $A_{\nu,k} = \{1\}$, and $a$ is implicitly 1 when not explicitly written.

Then the **wavelet differential k-forms** $\psi_{\nu,k}^{d}(\xi)$ in $\mathbb{R}^2$ are given by

\[
\Omega^0(\mathbb{R}^2) : \quad \hat{\psi}_{d,s}^{0,2} = \frac{1}{|\xi|} \hat{\psi}_s(\xi) \frac{\partial}{\partial \hat{\theta}} \wedge \frac{\partial}{\partial \hat{r}} \quad \in \hat{L}_2
\]

\[
\Omega^1(\mathbb{R}^2) \oplus \Omega^1(\mathbb{R}^2) : \quad \hat{\psi}_{d,s}^{1,2} = i \hat{\psi}_s(\xi) \frac{\partial}{\partial \hat{\theta}} \otimes \hat{\psi}_{d,s}^{1,2} = i \frac{1}{|\xi|} \hat{\psi}_s(\xi) \frac{\partial}{\partial \hat{r}} \quad \in L_2 \oplus \hat{L}_2
\]

\[
\Omega^2(\mathbb{R}^2) : \quad \hat{\psi}_{d,s}^{2,2} = -\hat{\psi}_s(\xi) \quad \in L_2
\]

and in $\mathbb{R}^3$ they are
Remark 5. As discussed in Sec. 4.2, the Fourier transform of a \( k \)-form is an \( n - k \) form in the frequency domain. We use the degree of the form in space in our nomenclature for the form frame functions, i.e. \( \hat{\psi}_{k,n}^{\nu,s} \in \Omega^{n-k}(\mathbb{R}^n, \Omega^\nu) \).

It follows immediately from the definition that wavelet differential \( k \)-forms are differential forms in sense of the continuous theory. The principal utility of wavelet differential forms stems from the following result.

**Proposition 7.** Wavelet differential \( k \)-forms \( \psi_{k,n}^{\nu,s,a} \) provide a tight frame for the space \( L^2(\mathbb{R}^n, \Omega^\nu) \) when \( \nu = d \) and \( \dot{L}^1(\mathbb{R}^n, \Omega^\nu) \) when \( \nu = \delta \) and an \( \alpha \in \Omega^\nu(\mathbb{R}^n) \) in the respective space has the representation

\[
\alpha(x) = \sum_{a \in A_{\nu,k}} \sum_{s \in S} \langle \alpha, \psi_{\nu,s,a}^{k,n} \rangle \psi_{\nu,s,a}^{k,n}(x) \tag{57}
\]

where \( \psi_{\nu,s,a}^{k,n} = \mathcal{F}^{-1}(\hat{\psi}_{\nu,s,a}^{k,n}) \).

The proof is relegated to Sec. 4.2. A corollary of the result is that our wavelet differential forms inherently respect the Hodge-Helmholtz decomposition. The following proposition establishes that wavelet differential forms satisfy important properties of the exterior calculus.
Proposition 8. Let $\alpha \in \Omega^k(\mathbb{R}^n)$ and $E(n)$ be the Euclidean group of rigid body transformations on $\mathbb{R}^n$. Then wavelet differential $k$-forms satisfy:

i) Closure under the exterior derivative:

$$d\psi_{k,s,a}^k(x) = \psi_{k+1,s,a}^k(x)$$

$$d\psi_{d,s,a}^k(x) = 0$$

and furthermore

$$\|\psi_{d,s,a}^k\|_{L^2} = \|d\psi_{d,s,a}^k\|_{L^2} = \|\psi_{d,s,a}^{k+1}\|_{L^2}.$$  (58c)

ii) Hodge dual:

$$\mathcal{F}(\star \psi_{\nu,s,a}^k) = -\star \hat{\psi}_{\nu,s,a}^k(\xi)$$

and

$$\star \hat{\psi}_{d,s,a}^k(\xi) = \frac{1}{|\xi|} \hat{\psi}_{d,s,a}^{n-k}(\xi)$$

iii) Interior product with radial vector $r$:

$$i_r \psi_{\nu,s,a}^k = d\hat{\psi}_{\nu,s}^k$$

iv) Covariance under rigid body transformations:

$$\alpha = \sum_{j,k,t} \alpha_s \psi_{s,a}^k \Leftrightarrow A\alpha = \sum_{j,k,t} \alpha_s' \psi_{s,a}^k,$$  (59e)

where $A \in E(n)$, for some coefficients $\alpha'_s$; i.e. the translation and/or rotation of a $J$-bandlimited representation in wavelet differential forms can be represented in the same frame with the same bandlimit.

v) Wavelet differential forms are real-valued in space. In $\mathbb{R}^2$ they are given by

$$\psi_{\delta,s}^{0,2}(x) = \frac{2j}{2\pi} \sum_m i^m \beta_m e^{im\theta} h_m(2^j x - k_s)$$  (60a)
\[
\psi_{d,s}^{1,2}(x) = \frac{2^j}{4\pi} \sum_{\sigma \in \{-1,1\}} \sum_{m} i^{m_{\sigma}} \beta_{m} e^{im_{\sigma}\theta} h_{m_{\sigma}}(|2^j x - k_s|) (-\sigma \, dx^1 + i \, dx^2)
\]

(60b)

\[
\psi_{S,s}^{2,2}(x) = \frac{2^j}{4\pi} \sum_{\sigma \in \{-1,1\}} \sum_{m} -i^{m_{\sigma}} \beta_{m} e^{im_{\sigma}\theta} h_{m_{\sigma}}(|2^j x - k_s|) (i \, dx^1 + \sigma \, dx^2)
\]

(60c)

\[
\psi_{d,s}^{2,2}(x) = \frac{2^j}{2\pi} \sum_{m} i^{m} \beta_{m} e^{im\theta} h_{m}(|2^j x - k_s|) \, dx^1 \wedge dx^2
\]

(60d)

with \(m_{\sigma} = m + \sigma\). With \(\partial/\partial \hat{\theta}, \partial/\partial \hat{\phi}, \partial/\partial \hat{r}\) being now the vectors induced by standard spherical coordinates, the \(\psi_{v,s,\alpha}^{k,3}\), for angular localization windows \(\hat{v}_s(\xi)\) defined away from the poles, is given by

\[
\psi_{d,s,2}^{0,3}(x) = \frac{2^{3j/2}}{(2\pi)^{3/2}} \sum_{l,m} i^{l} \kappa_{l,m}^{s} y_{l,m}(\bar{x}) h_{l}(|2^j x - k_s|)
\]

(61a)

\[
\psi_{d,s,1}^{2,3}(x) = \sum_{i_1=1}^{3} \sum_{l,m} \kappa_{l,m}^{s} \theta_{l,m}^{i_1} \times \sum_{l_2,m_2} i^{l_2} y_{l_2,m_2}(\bar{x}) G_{l_2||l_1,m_1}^{l_1,m_2} h_{l_2}(|2^j x - k_s|) dx^{i_2} \wedge dx^{i_3}
\]

(61b)

\[
\psi_{d,s,2}^{2,3}(x) = \frac{2^{3j/2+1}}{\sqrt{3}(2\pi)^{3/2}} \sum_{l,m} i^{l} \kappa_{l,m}^{s} \sum_{l_2=(-1)}^{l+1} i^{l_2} h_{l_2}(|2^j x - k_s|)
\]

\[
\times \sum_{\sigma \in \pm 1} G_{l_2,m+1\sigma}^{l,m+1\sigma} y_{l_2,m+1\sigma}(\bar{x}) \left( -dx^2 \wedge dx^3 + i \sigma \, dx^1 \wedge dx^3 \right)
\]

(61c)

\[
\psi_{d,s,2}^{2,3}(x) = \frac{2^{3j/2+1}}{\sqrt{3}(2\pi)^{3/2}} \sum_{l,m} i^{l} \kappa_{l,m}^{s} \sum_{l_2=(-1)}^{l+1} i^{l_2} h_{l_2}(|2^j x - k_s|)
\]

\[
\times \sum_{\sigma \in \pm 1} G_{l_2,m+1\sigma}^{l,m+1\sigma} y_{l_2,m+1\sigma}(\bar{x}) \left( i \sigma \, dx^2 \wedge dx^3 + i dx^1 \wedge dx^3 \right)
\]

\[
+ \frac{2^{3j/2+2}i}{\sqrt{3}(2\pi)^{3/2}} \sum_{l,m} i^{l} \kappa_{l,m}^{s} \sum_{l_2=(-1)}^{l+1} i^{l_2} h_{l_2}(|2^j x - k_s|) G_{l_2,m}^{l,m} y_{l_2,m}(\bar{x}) \, dx^1 \wedge dx^2
\]

(61d)

\[
\psi_{d,s}^{3,3}(x) = \frac{2^{3j/2}}{(2\pi)^{3/2}} \sum_{l,m} i^{l} \kappa_{l,m}^{s} y_{l,m}(\bar{x}) h_{l}(|2^j x - k_s|) \, dx^1 \wedge dx^2 \wedge dx^3
\]

(61e)

with \((i_1, i_2, i_3) \in S_3\) and \(\theta_{l,m}^{i_1}\) being the spherical harmonics coefficients for
the $i^{th}$ component of $\partial / \partial \hat{\theta}$. The spatial radial windows $h_m(|x|)$ and $h_l(|x|)$ have closed form expressions when $\hat{h}(|\xi|)$ is the window from the steerable pyramid [53] and the expressions for the $\psi_k^{3,3}$ that are not listed can be obtained using the Hodge dual.

The proof of Proposition 8 is again relegated to Sec. 4.2.

Remark 6. The form basis vector $\partial / \partial \hat{\theta}$ and $\partial / \partial \hat{\phi}$ have singularities at the poles and for $\partial / \partial \hat{\theta}$ this made the indirect computation of the inverse Fourier transform necessary. In [33] it has been proposed to use the frame we used in this calculation to span $T S^2$, see Appendix A.6. While possible, in the context of exterior calculus some extra work is required, however, because it is usually assumed that the tangent space is spanned by a basis (albeit an arbitrary biorthogonal one). We leave it to future work to either re-write the necessary parts of exterior calculus using a frame or find suitable coordinate charts for $S^2$ that work well with polar wavelets and avoid the singularities. With compactly supported windows one can, in fact, choose $\{\partial / \partial \hat{\theta}, \partial / \partial \hat{\phi}\}$ such that the singularities are outside the support.

5.2. Exterior Calculus using Wavelet Differential Forms

We continue with a discussion of the wavelet differential form interpretation of important results in exterior calculus.

5.2.1. Stokes theorem

As discussed in Sec. 4.2 Stokes’ theorem can be written as

$$\int_{\mathbb{R}^n} \alpha \wedge \chi_{\partial M} = \int_{\mathbb{R}^n} d\alpha \wedge \chi_M. \quad (62)$$

The representation of $\alpha \in \Omega^k(\mathbb{R}^n)$ and $d\alpha \in \Omega^{k+1}(\mathbb{R}^n)$ in wavelet differential forms is given by

$$\alpha = \sum_s \alpha_s \psi_{\delta,s}^k \quad (63a)$$

$$d\alpha = \sum_s \alpha_s \psi_{d,s}^{k+1} \quad (63b)$$
Figure 1: Exact different form wavelets $\psi_{d_i,s}^{2,3}$ in the frequency domain (top) and the spatial one (bottom). The left function is isotropic around the $x_3$ axis and the other ones are directional modeling a high frequency feature across the $x_1$-$x_2$ plane.

where we immediately used that $d^2 = 0$ and $\partial \partial M = \emptyset$ so that it suffices to consider the co-exact part of $\alpha$. Thus we have

\[
\sum_s \alpha_s \int_{\mathbb{R}^n} \psi_{d_i,s}^k \land \chi_{\partial M} = \sum_s \alpha_s \int_{\mathbb{R}^n} \psi_{d_i,s}^{k+1} \land \chi_M
\]  
(64a)

\[
\sum_s \alpha_s \chi_{s}^{\partial M} = \sum_s \alpha_s \chi_{s}^{M}.
\]  
(64b)

Stokes’ theorem hence becomes the scalar product between the coefficients $\alpha_s$ of the form $\alpha$ (or its exterior derivative $d\alpha$) and those of the characteristic functions $\tilde{\chi}_{\partial M}$ and $\chi_M$,

\[
\tilde{\chi}_{s}^{\partial M} = \int_{\mathbb{R}^n} \psi_{\delta_i,s}^k \land \chi_{\partial M}
\]  
(65a)

\[
\chi_{s}^{M} = \int_{\mathbb{R}^n} \psi_{d_i,s}^{k+1} \land \chi_M
\]  
(65b)

we use the tilde for the $\tilde{\chi}_{\partial M}$ since they differ from the frame coefficients by the use of the $L_2$ inner product in Eq. 65. It follows immediately from the equality
in Eq. 64 that the coefficients $\tilde{\chi}^M_s$ and $\chi^M_s$ are in fact equal, i.e.

$$\tilde{\chi}^M_s = \chi^M_s.$$ (66)

When $\psi_{k+1}^{d,s}$ is a volume form, i.e. $k+1 = n$, then the integral on the right hand side describes the frame coefficient of the scalar characteristic function $\chi^M$ with the scalar polar wavelet $\psi_{k+1}^{d,s}$, i.e. we can write

$$\chi^M_s = \int_{\mathbb{R}^n} \psi_{k+1}^{d,s} \chi^M dx.$$ (67)

The behavior of these coefficients $\chi^M_s$ has been studied extensively in the literature on ridgelets (e.g. [50, 51]), curvelets (e.g. [52, 53, 19, 20, 54]), shearlets (e.g. [55, 56, 57]), contourlets (e.g. [58, 59]) and related constructions such as $\alpha$-molecules (e.g. [34, 60]). From these results it is known that, for sufficiently fine levels $j$, the coefficients $\chi^M_s$ are non-negligible only when $\psi_{k+1}^{d,s}$ is in the neighborhood of the boundary $\partial M$ and, in the anisotropic case, when it is oriented along it. By Eq. 64b, the coefficients $\tilde{\chi}^M_s = \chi^M_s$ thus “select” the coefficients $\alpha_s$ of $\alpha$ (and hence also $d\alpha$) in the vicinity of $\partial M$, in that $\alpha_s$ will provide a significant contribution to the integral only when $\chi^M_s$ is significant. This selection implements the pullback and the integral over the submanifold $\partial M$ in the original form of Stokes’ theorem in Eq. 64, which becomes, perhaps, most explicit with the inner product form of Stokes’ theorem in Eq. 72

$$\langle \langle \alpha, \chi^M_{\partial M} \rangle \rangle = \langle \langle d\alpha, \chi^M_{M} \rangle \rangle$$ (68a)

$$\sum_{s \in S} \alpha_s \langle \langle \psi_{d,s}^{k}, \chi^M_{\partial M} \rangle \rangle = \sum_{s \in S} \alpha_s \langle \langle \psi_{d,s}^{k+1}, \chi^M_{M} \rangle \rangle$$ (68b)

Eq. 65 also shows that anisotropic wavelets are more efficient for realizing the boundary integral numerically than isotropic ones since the $\chi^M_s$ are then sparser and the sums in Eq. 64b contain fewer non-negligible terms. For $\partial M$ being $C^2$, for example, curvelets, shearlets, and contourlets yield quasi-optimally sparse representations of $\chi^M$. For curvelets and shearlets it is, furthermore, known that these resolve the wavefront set $[19, 61]$. But $\chi_{\partial M}$ is just the wavefront set $WF(\chi^M)$ of $\chi^M$, cf. Eq. 50 and Eq. 65a thus a projection of $WF(\chi^M)$.
into the frame \( \{ \psi^k_{\delta,s} \} \). This is consistent with our results since from Eq. \[64\] one would, indeed, expect that Eq. \[65b\] in determined by \( \text{WF}(\chi_{\mathcal{M}}) \), and Stokes’ theorem provides hence, in our opinion, an interesting perspective on the results in \[19\] \[61\].

Eq. \[64\] is valid for any \( k \). To our knowledge, however, there are only few results on the approximation power of “vector-valued” wavelets or curvelets. We believe that the above discussion for \( k + 1 = n \) should also apply to arbitrary \( k \), in that one has once again non-negligible coefficients only in the vicinity of \( \partial \mathcal{M} \) and with an orientation aligned with it, but it is beyond the scope of the present paper.

**Example 6.** We consider Kelvin’s circulation theorem in fluid mechanics, e.g. \[62\] Ch. 1.2] for which the velocity and vorticity forms are related by Stokes’ theorem. Using differential forms it is given by,

\[
\int_{\partial \Sigma} u^b = \int_{\Sigma} \omega \tag{69a}
\]

where \( u^b \) is the 1-form field associated with the velocity vector field \( \vec{u} \) and \( \omega = du^b \) is the vorticity. Representing the velocity 1-form using the 1-form frame and assuming the flow domain is a subset of \( \mathbb{R}^2 \) we have

\[
u^b = \sum_s u^b_s \psi^1,2_{\delta,s} \tag{69b}
\]

and hence the representation for the vorticity is given by

\[
\omega = du^b = \sum_s u^b_s \psi^{2,2}_{d,s}. \tag{69c}
\]

Thus

\[
\sum_s u^b_s \int_{\mathbb{R}^2} \psi^{1,2}_{\delta,s} \wedge \chi_{\partial \Sigma} = \sum_s u^b_s \int_{\mathbb{R}^2} \psi^{2,2}_{d,s} \wedge \chi_{\Sigma}. \tag{69d}
\]

As in Eq. \[50\] the characteristic function associated with \( \partial \Sigma \) is the weak gradient

\[
\chi_{\partial \Sigma} = d\chi_{\Sigma} = n_i \delta_{\partial \chi} dx^i \tag{69e}
\]

where the \( n_i \) are the components of the normal of \( \partial \Sigma \) and \( \delta_{\partial \chi} \) its Dirac-distribution. In components we hence have for the integrand of the left hand
Figure 2: Left: Conceptual depiction of partial integration of wavelet differential forms for a volume form in two dimensions. Middle and right: Directional form wavelet $\psi_{\delta,s}^{2,2}(x)$ in two dimensions (middle) and its “sliced” counter-part $\psi_{\delta,0,0}^{1,1}(x_1)$, which can be considered as a wavelet differential form in one dimension, obtained by projecting along the $x_2$-axis (right).

\[
\psi_{\delta,s}^{1,2} \wedge \chi_{\partial \Sigma} = \left( \psi_{\delta,s,1}^{1,2} \, dx^1 + \psi_{\delta,s,2}^{1,2} \, dx^2 \right) \wedge \left( n_1 \delta_{\partial \Sigma} \, dx^1 + n_2 \delta_{\partial \Sigma} \, dx^2 \right) \quad \text{(69f)}
\]

\[
= \left( \psi_{\delta,s,1}^{1,2} \, n_2 - \psi_{\delta,s,2}^{1,2} \, n_1 \right) \delta_{\partial \Sigma} \, dx^1 \wedge dx^2. \quad \text{(69g)}
\]

Since for a vector $\vec{a} = (a_1, a_2)$ the one with components $(a_2, -a_1)$ is orthogonal to it, the above wedge product vanishes when $\psi_{\delta,s}^{1,2}$ is in the normal direction and it is maximized when it is parallel (and no metric is required). Hence, as expected, the integral over $\mathbb{R}^2$ implements the line integral along $\partial \Sigma$.

The integral on the right hand side of Eq. (69d) is the representation problem for the cartoon-like function $\chi_{\Sigma}$, which, as discussed above, has been studied extensively in the literature. That a co-exact 1-form wavelet attains the same approximation behavior for the boundary $\partial \Sigma$ is consistent with the results in [33] where it was shown that divergence free wavelets, which correspond to our exact 1-forms, have the same approximation rates as scalar curvelets.

5.2.2. Partial Integration of Differential Forms

A natural property of differential forms is that the integral of a $k$-form over one direction yields a $(k-1)$-form. Our wavelet differential forms verify this property in that for integration along direction $\mathbb{R}_\nu$ one has

\[
\psi_{\nu,s}^k \xrightarrow{\int_{\mathbb{R}_\nu}} \psi_{\nu,s}^{k-1}
\]
and the integral of a wavelet $k$-form is a wavelet $(k-1)$-form whose structure is determined $ψ^k$ and that has a closed form expression. In the Fourier domain, the integral is equivalent to “slicing” the form along the plane with normal $ν$, which in the case of volume forms was already used in [63] for a local Fourier slice theorem based on polar wavelets. The result carries over to arbitrary polar differential form wavelets since slicing in the Fourier domain preserves spherical coordinates, i.e. a planar slice of a vector field separable in spherical coordinates in $\mathbb{R}^3$ is a vector field separable in polar coordinates in $\mathbb{R}^2$.

We will demonstrate this for $ψ^{2,3}_{δ,s}$. Its Fourier representation is

$$\hat{ψ}^{2,3}_{δ,s}(ξ) = \frac{1}{|ξ|} \hat{ψ}_{s}(ξ) \hat{h}(|ξ|) \frac{∂}{∂r} \hat{ψ}_{s}(ξ) \hat{h}(|ξ|) \frac{∂}{∂r}. \quad (71a)$$

Without loss of generality, let $ν$ be along the $x_3$ axis. The integration is then the restriction to the $ξ_1-ξ_2$ plane in frequency space and in spherical coordinates this corresponds to $θ = π/2$. Thus,

$$\hat{ψ}^{2,3}_{δ,s}(ξ) \bigg|_{ξ_{1,2}} = \frac{1}{|ξ_{1,2}|} \sum_{lm} κ_{lm} y_{lm}(ξ) \hat{h}(|ξ_{1,2}|) \frac{∂}{∂r_{ξ_{1,2}}}. \quad (71b)$$

where $|ξ_{1,2}|$ and $∂/∂r_{ξ_{1,2}}$ are the restriction of $|ξ|$ and $∂/∂r_ξ$ to the $ξ_1-ξ_2$ plane, which, importantly, are the natural radial variables there. Expanding the spherical harmonics $y_{lm}(ξ)$ using Eq. A.6 we obtain

$$\hat{ψ}^{2,3}_{δ,s}(ξ) \bigg|_{ξ_{1,2}} = \frac{1}{|ξ_{1,2}|} \sum_{lm} κ_{lm} (C_{lm} P^m_l(0)) e^{imφ} \hat{h}(|ξ_{1,2}|) \frac{∂}{∂r_{ξ_{1,2}}}. \quad (71c)$$

$$= \frac{1}{|ξ_{1,2}|} \sum_{m} \left( \sum_{l} κ_{lm} C_{lm} P^m_l(0) \right) e^{imφ} \hat{h}(|ξ_{1,2}|) \frac{∂}{∂r_{ξ_{1,2}}} \quad (71d)$$

$$= \frac{1}{|ξ_{1,2}|} \sum_{m} \tilde{β}_m e^{imφ} \hat{h}(|ξ_{1,2}|) \frac{∂}{∂r_{ξ_{1,2}}}. \quad (71e)$$

Comparing to the definition of $ψ^{1,2}_{δ,s}$ in Def. 2 and using those of polar wavelets, cf. Eq. 3 we see that this a co-exact 1-form in $\mathbb{R}^2$. The angular localization coefficient $β_m$ differ from the $β_m$ we proposed in Sec. 3. From the restriction it is, nonetheless, clear that $ψ^{2,3}_{δ,s}(ξ) \bigg|_{ξ_{1,2}}$ will be non-negligible only when the
angular window $\hat{\psi}_s(\xi)$ is approximately centered on the $\xi_1$-$\xi_2$ plane and that the projected window characterized by the $\bar{\beta}_m$ will be localized there around its original location. The localization is hence, in an appropriate sense, preserved under integration. A precise, quantitative analysis of this property is beyond the scope of the present work. By the closure of the spherical harmonics bands under rotation, cf. Sec. Appendix A.4, the above discussion carries over to arbitrary $\nu$.

5.2.3. Laplace–de Rham operator

The Laplace operator acting on differential forms is known as Laplace–de Rham operator $\Delta : \Omega^k(\mathbb{R}^n) \mapsto \Omega^k(\mathbb{R}^n)$. It is defined as [39, Def. 8.5.1]

$$\Delta = d\delta + \delta d = (-1)^{n(k-1)+1}(d\star d + \star d\star d)$$

(72)

and, as the usual Laplace operator, it plays a fundamental role in many applications, e.g [39, Ch. 9]. Since our wavelet differential forms are closed under the exterior derivative and the Hodge dual has a well defined form, also the Laplace–de Rham operator can be computed in closed form. For example, for a co-exact 2-form in $\mathbb{R}^3$ we have

$$\Delta\psi^{2,3}_{\delta,s} = (-1)^{n(k-1)+1}(d\star d\star \psi^{2,3}_{\delta,s} + \star d\star \psi^{2,3}_{\delta,s})$$

(73a)

We will compute the two terms independently using the results of Proposition 8. For the first one we have

$$d\star d\star \psi^{2,3}_{\delta,s} = d\star d\psi^{1,3}_{d,s} = 0$$

(73b)

since $\psi^{1,3}_{d,s}$ is exact. In fact, $d\star d\star \psi^{k,3}_{d,s} = 0$ for any $k$. The second term is

$$\star d\star \psi^{2,3}_{\delta,s} = \star d\star \psi^{3,3}_{d,s}$$

(73c)

$$= \star d\mathcal{F}^{-1}(-|\xi|\hat{\psi}_{\delta,s}^{0,3})$$

(73d)

$$= \star \mathcal{F}^{-1}(-|\xi|\hat{\psi}_{d,s}^{1,3})$$

(73e)

$$= \mathcal{F}^{-1}(|\xi|^2\hat{\psi}_{\delta,s}^{2,3}).$$

(73f)
Figure 3: Decay of the Galerkin projection of the Laplace–de Rahm operator, Eq. (76) as a function of the separation $|k_s - k_t|$ for $\mathbb{R}^2$. The decay for isotropic wavelets corresponds to $m = 0$, for anisotropic ones one has a linear combination of the different $m$. Note that the decay does not depend on the wavelet type since our wavelet differential forms use the same frequency window independent of the type of the form.

Thus

$$\Delta \psi^{2,3}_{\delta,s} = F^{-1}(\xi^2 \hat{\psi}^{2,3}_{\delta,s}) \quad (74)$$

and the Laplace–de Rham operator leads to a form of the same type but with a $|\xi|^2$ weight. This is what one would expect from the scalar Laplace operator but it is non-trivial given Eq. (72). One can check that this holds for all $\psi^{k,n}_{\nu,s}(x)$ and we hence have the following the proposition.

**Proposition 9.** The Laplace–de Rahm operator $\Delta = d\delta + \delta d$ of a polar wavelet differential form $\psi^{k,n}_{\nu,s}$ is

$$\Delta \psi^{k,n}_{\nu,s} = (-1)^{n(k-1)+1} F^{-1}(|\xi|^2 \hat{\psi}^{k,n}_{\nu,s}) \quad (75)$$

**Remark 7.** The symbol $\hat{\Delta}$ of the Laplace–de Rahm operator $\Delta = d\delta + \delta d$ is hence $\hat{\Delta} = |\xi|^2$ and this holds not only for wavelet differential forms frame functions but for differential forms in general.

Important for numerical calculations is the Galerkin projection,

$$D_{sr} = \langle \Delta \psi^{n,k}_{\nu,s}, \psi^{n,k}_{\nu,r} \rangle , \quad (76)$$
of the Laplace–de Rahm operator. With the definition of our wavelet differential forms in the Fourier domain and $\hat{\Delta} = |\xi|^2$ it has a closed form solution, see the supplementary material for the expression. Furthermore, by the compact support of our wavelets in the frequency domain the matrix elements $D_{sr}$ are nonzero only when $\min(-1,j_r-1) \leq j_s \leq j_r + 1$ and when $t_s \approx t_r$ (assuming compactly supported windows; otherwise nonzero has to be replaced by non-negligible). The coefficients $D_{sr}$ as a function of the separation $|k_s - k_r|$ are shown in Fig. 3 where it can be seen that these decay rather fast as $|k_s - k_r|$ grows. The wavelet differential form representation of the Laplace–de Rahm operator is hence well localized in space and frequency by our construction.

Example 7. One of the first applications where the importance of differential forms for numerical modeling was recognized was electromagnetic theory [2, 3].

The resonant cavity problem, which we consider in this example, serves thereby as a reference problem [64]. It seeks the electromagnetic field in a simple domain consisting of a perfect conductor containing no enclosed charges. Maxwell’s equations then reduce to an eigenvalue problem for which naïve vector calculus discretizations fail to provide the correct answer [2, 3, 64].

Let $\mathcal{M} \subset \mathbb{R}^2$ be the square domain with side length $\pi$. The resonant cavity problem then seeks the electric 1-form field $E \in \Omega^1(\mathcal{M})$ such that

$$\Delta E = \delta dE = \lambda E \quad \text{in } \mathcal{M} \quad (77)$$

$$i^* E = 0 \quad \text{on } \partial \mathcal{M} \quad (78)$$

where the pullback $i^* E$ describes that $E$ is tangential to the boundary and $E$ is co-exact since the electric field is divergence free in a region free of charges.

Writing $E$ using the co-exact 1-form basis this becomes

$$\left\langle \psi^{1,2}_{\delta,r}, \sum_s E_s \Delta \psi^{1,2}_{\delta,s} \right\rangle = \lambda \left\langle \psi^{1,2}_{\delta,r}, E \right\rangle \quad (79a)$$

$$\sum_s E_s \left\langle \psi^{1,2}_{\delta,r}, \Delta \psi^{1,2}_{\delta,s} \right\rangle = \lambda E_r \quad (79b)$$

$$\sum_s E_s D_{sr} = \lambda E_r \quad (79c)$$
where the $D_{sr}$ are again the matrix elements of the Galerkin projection of the Laplace–de Rahm operator. As before, these can be determined in closed form in the Fourier domain. By linearity of the pullback, the boundary conditions become in the frame representation

$$\sum_s E_s i^* \psi^{1,2}_{\delta,s} = 0 \quad \text{on } \partial M$$

which can be enforced by choosing frame functions such that

$$i^* \psi^{1,2}_{\delta,s} = 0$$

for all $\psi_{\delta,s}$ that are used to represent the solution $E$, see also the discussion on Stokes’ theorem in Sec. 4.2 where the same pullback arose. Fig. 4 shows that suitably constructed anisotropic 1-form frame functions satisfy this to good approximation and these can hence be used to model the boundary conditions by construction.

It has to be noted that, compared to the discretizations of Eq. 7 in the literature [2, 64], no weak formulation is required in our case since our wavelet differential forms have sufficient regularity.

5.3. Other Operators on Wavelet Differential Forms

In the following we will briefly discuss other important operations on differential forms where we do not yet have an elegant realization in our calculus.

Figure 4: Anisotropic co-exact 1-form frame functions $\psi^{1,2}_{\delta,s}$ for different levels (visualized as divergence free vector fields). See [33] for more details on the windows used for directional localization.
5.3.1. Wedge product

Our wavelet differential forms $\psi_{\nu,s,a}^{k,n}$ are continuous forms and hence the wedge product $\psi_{\nu,s,a}^{k,n} \wedge \psi_{\nu,s,a}^{l,n}$ is a $k + l$ form. For numerical calculations one would hope that the nonzero coefficients of the product are sparse, at known location, and can be computed efficiently. As shown in Proposition 3, the wedge product becomes a convolution in frequency space. Hence, considerable sparsity is lost by the wedge product and precise conclusions are not easily established.

In practice we currently use the closed form formulas for the spatial representations of the differential form wavelets in Eq. 60 and Eq. 61. With these, the multiplication of the coordinate functions that arises as part of the wedge product can be computed in closed form. Although rather complicated, it can be combined with numerical quadrature to implement the wedge product. While practical, because of the complexity of the expressions the solution is rather unelegant and we currently have no insight into the sparsity of the coefficients of the product or their decay properties.

5.3.2. Lie derivative

The Lie derivative $\mathcal{L}_X \alpha$ describes the infinitesimal advection of the differential form $\alpha$ along the vector field $X$. It hence plays a fundamental role in the description of physical systems using exterior calculus. Using Cartan’s formula, the Lie derivative can be written as

$$\mathcal{L}_X \alpha = d i_X \alpha + i_X d \alpha.$$  \hspace{1cm} (80)

While the exterior derivative is well defined for wavelet differential forms the interior product with an arbitrary vector field is not. Similar to the wedge product, the multiplication of the coordinate functions has a closed form solution in the spatial domain, and this can be used to implement it in practice.

Example 8. The Euler equation in vorticity form is given by

$$\frac{\partial \omega}{\partial t} - \mathcal{L}_u \omega = 0$$  \hspace{1cm} (81a)
where \( \vec{u} \in X_{\text{div}}(\mathbb{R}^n) \) is the divergence free velocity vector field and \( \omega = \text{d}u^b \) is the vorticity, cf. Example \([6]\). In \( \mathbb{R}^2 \), vorticity is a volume form, i.e. \( \omega \in \Omega^2(\mathbb{R}^2) \), and using Cartan’s formula we hence have

\[
\mathcal{L}_{\vec{u}} \omega = \text{d} i_{\vec{u}} \omega + i_{\vec{u}} \text{d} \omega = \text{d} i_{\vec{u}} \omega.
\] (81b)

In coordinates this equals

\[
\mathcal{L}_{\vec{u}} \omega = \text{d}(i_{\vec{u}} \omega \, \text{d}x^1 \wedge \text{d}x^2 - i_{\vec{u}} \omega \, \text{d}x^2 \wedge \text{d}x^1)
\] (81c)

\[
= \text{d}(u_1 \omega \, \text{d}x^1 - u_2 \omega \, \text{d}x^2)
\] (81d)

\[
= \left( \frac{\partial u_1}{\partial x^2} \omega + u_1 \frac{\partial \omega}{\partial x^2} \right) \, \text{d}x^2 \wedge \text{d}x^1 - \left( \frac{\partial u_2}{\partial x^1} \omega + u_2 \frac{\partial \omega}{\partial x^1} \right) \, \text{d}x^1 \wedge \text{d}x^2
\] (81e)

\[
= - \left( \frac{\partial u_1}{\partial x^2} \omega + u_1 \frac{\partial \omega}{\partial x^2} + \frac{\partial u_2}{\partial x^1} \omega + u_2 \frac{\partial \omega}{\partial x^1} \right) \, \text{d}x^1 \wedge \text{d}x^2.
\] (81f)

Writing \( \vec{u} \) with the divergence free wavelets from \([33]\), which we denote by \( \psi_s \), and \( \omega \) with the 2-form frame we obtain

\[
\frac{\partial \omega_q}{\partial t} = \sum_s \sum_r u_s \omega_r \left( \mathcal{L}_{\psi_s} \psi_{d,r}^{2,2}, \psi_{d,q}^{2,2} \right).
\] (82a)

The advection coefficients

\[
C_{sr}^{q} = \left< \mathcal{L}_{\psi_s}, \psi_{d,r}^{2,2}, \psi_{d,q}^{2,2} \right>
\] (82b)

can then be computed with a closed form expression for \( \mathcal{L}_{\psi_s}, \psi_{d,r}^{2,2} \) using Eq.81f, see the supplementary material, and implementing the reprojection onto \( \psi_{d,q}^{2,2} \) using a fast transform type computation \([33]\).

5.3.3. Pullback

The pullback \( \eta^* \alpha \in \Omega^k(\mathcal{N}) \) of a differential form \( \alpha \in \Omega^k(\mathcal{M}) \) by a map \( \eta : \mathcal{N} \to \mathcal{M} \) between two manifolds \( \mathcal{N}, \mathcal{M} \) is another important operation in the exterior calculus, in particular since it commutes with the exterior derivative, i.e. \( \text{d}\eta^* \alpha = \eta^* \text{d} \alpha \). For special maps, such as rotations or shears, closed form solutions for the pullback of a wavelet differential form can be derived in
the Fourier domain. For general diffeomorphisms a result by Candès and De- manet [65, Thm. 5.3] shows that curvelet-like frames are essentially preserved. It would be interesting to extend this result into a numerically practical form. A special case that is of particular relevance are volume preserving diffeomorphisms. Then \( \eta \) can be associated with a unitary operator \([66]\), which might enable an easier analysis. Furthermore, volume preserving diffeomorphisms also have important applications, e.g. in fluid dynamics.

5.4. Proofs

Proof of Proposition [7] The case of 0-forms and volume forms are equivalent to the existing results in the literature \([14, 15, 16]\), we provide an alternative, more direct proof in Appendix B. As we show below, the case of tight frames for exact forms can be handled analogous to \([33]\) by reducing it to the scalar one. The co-exact case then follows and the proposition follows by the Hodge-Helmholtz decomposition.

**Exact forms.** Let \( \alpha \in L_2(\mathbb{R}^3, \Omega^2) \) be an exact 2-form with Fourier transform

\[
\hat{\alpha} = \hat{\alpha}_\theta \frac{\partial}{\partial \theta} + \hat{\alpha}_\phi \frac{\partial}{\partial \phi}
\]  

(83a)

We want to show that

\[
\alpha = \sum_{s \in S} \left\langle \alpha, \psi^{2,3}_{d,s,1} \right\rangle \psi^{2,3}_{d,s,1} + \sum_{s \in S} \left\langle \alpha, \psi^{2,3}_{d,s,2} \right\rangle \psi^{2,3}_{d,s,2}.
\]  

(83b)

Taking the Fourier transform and using Parseval’s theorem we obtain

\[
\hat{\alpha} = -\sum_{s \in S} \left\langle \hat{\alpha}, \psi^{2,3}_{d,s,1} \right\rangle \psi^{2,3}_{d,s,1} - \sum_{s \in S} \left\langle \hat{\alpha}, \psi^{2,3}_{d,s,2} \right\rangle \psi^{2,3}_{d,s,2}.
\]  

(83c)

Expanding the first term yields

\[
\sum_{s \in S} \left\langle \alpha, \psi^{2,3}_{d,s,1} \right\rangle \psi^{2,3}_{d,s,1} = \sum_{s \in S} \left\langle \alpha, \psi^{2,3}_{d,s,1} \right\rangle \left( \hat{\alpha}_\theta \frac{\partial}{\partial \theta} + \hat{\alpha}_\phi \frac{\partial}{\partial \phi} - i\hat{\psi}_s(\xi) \frac{\partial}{\partial \theta} \right) \psi^{2,3}_{d,s,1}
\]  

(83d)

and writing out the inner product with the Hodge dual we obtain

\[
\sum_{s \in S} \left\langle \alpha, \psi^{2,3}_{d,s,1} \right\rangle \psi^{2,3}_{d,s,1} = \sum_{s \in S} \int_{\mathbb{R}^3} \left( \hat{\alpha}_\theta \frac{\partial}{\partial \theta} + \hat{\alpha}_\phi \frac{\partial}{\partial \phi} - i\hat{\psi}_s(\xi) \frac{\partial}{\partial \theta} \right) \psi^{2,3}_{d,s,1}
\]
The integral is nonzero only for the first term of $\alpha$, since only then one obtains a volume form. Hence
\[
\sum_{s \in S} \left\langle \hat{\alpha}, \psi_{d,s,1}^{2,3} \right\rangle \psi_{d,s,1}^{2,3} = \sum_{s \in S} \int_{\mathbb{R}^3} \left( -i \hat{\alpha}_\phi(\xi) \hat{\psi}_s(\xi) \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \psi} \wedge \frac{\partial}{\partial r} \right) \psi_{d,s,1}^{2,3}. \quad (83c)
\]
Also expanding $\psi_{d,s,1}^{2,3}$ we can write
\[
\sum_{s \in S} \left\langle \hat{\alpha}, \psi_{d,s,1}^{2,3} \right\rangle \psi_{d,s,1}^{2,3} = - \sum_{s \in S} \left\langle \hat{\alpha}_\phi(\xi), \hat{\psi}_s(\xi) \right\rangle \hat{\psi}_s(\xi) \frac{\partial}{\partial \theta} \quad (83f)
\]
where the inner product is now those of scalar functions. The term with the brace is equivalent to the case of 0-forms, i.e. the known scalar result apply.

Similarly, the second term in Eq. (83c) can be written as
\[
\sum_{s \in S} \left\langle \hat{\alpha}, \psi_{d,s,1}^{2,3} \right\rangle \psi_{d,s,1}^{2,3} = - \sum_{s \in S} \left\langle \hat{\alpha}_\phi(\xi), \hat{\psi}_s(\xi) \right\rangle \hat{\psi}_s(\xi) \frac{\partial}{\partial \phi} \quad (83g)
\]
\[
= - \sum_{s \in S} \left\langle \hat{\alpha}_\phi(\xi), \hat{\psi}_s(\xi) \right\rangle \hat{\psi}_s(\xi) \frac{\partial}{\partial \phi} \quad (83h)
\]
The signs in Eq. (83f) and Eq. (83g) cancel with those in Eq. (83c) and Eq. (83b) thus follows from the scalar theory for polar wavelets. For different $k$ and $\mathbb{R}^2$ the results can be established using analogous calculations. This shows Proposition 7 for the case of exact forms.

**Co-exact forms.** Let $\Psi_{\nu}^{k,n}$ be the frame operator associated with the $\psi_{\nu,s}^{k,n}$ for fixed $\nu$, $k$, $n$. For the case of co-exact forms we will use the Gramian $G = \Psi \Psi^*$ whose entries are $G_{s,r} = \langle \psi_s, \psi_r \rangle$. Using Parseval’s theorem it is easy to see that
\[
G_{\delta,sr}^{k,n} = \left\langle \psi_{\delta,s}^{k,n}, \psi_{\delta,r}^{k,n} \right\rangle_{L_2^2} = \left\langle \psi_{d,s}^{k+1,n}, \psi_{d,r}^{k+1,n} \right\rangle_{L_2} = G_{d,sr}^{k+1,n}, \quad (84a)
\]
i.e. the Gramians for co-exact $k$-form frame functions and exact $(k+1)$-form ones are identical. The lower and upper frame bounds, $A$ and $B$, respectively, of a frame can be characterized using the Gramian as
\[
A = \inf \left\{ \langle a, Ga \rangle_{L_2}, a \in \text{ran}(\Psi), |a| = 1 \right\} \quad (84b)
\]
\[ B = \sup \{ \langle a, Ga \rangle_{L^2} \mid a \in \text{ran}(\Psi), \ |a| = 1 \} \]  

By our decay conditions and the Poincaré lemma, \( \text{ran}(\Psi_{\delta}^{k,n}) = \text{ran}(\Psi_{\delta}^{k,n}) \), and thus the tightness of the frame for exact \((k+1)\) forms carries over to co-exact \(k\)-forms. \[ \square \]

**Proof of Proposition** 4. We have:

i) *Closure under exterior derivative:* The property holds because the \( \hat{d} = i \xi \) only acts on the \( \partial / \partial \hat{r} \) component. For instance,

\[ \hat{d} \hat{\psi}^{0,3}_{\delta,s} = i \xi \hat{\psi}^{s}_{\delta,s} = i \xi \hat{\psi}^{s}_{\delta,s} = i \xi |\xi| \partial / \partial \theta \wedge \partial / \partial \phi \wedge \partial / \partial r \]  

where we used that in spherical coordinate \( \xi = |\xi| d\hat{r} \) and that the basis vector satisfy the Kronecker property \( \partial / \partial \hat{r} (d\hat{r}) = 1 \). The other cases are analogous. That \( d \) is a unitary operator on co-exact forms follows immediately from our choice of function spaces.

ii) The first part is just Proposition 4. The second part holds since we are in Euclidean space where the metric is the identity. The Hodge dual is thus also the identity w.r.t the coordinate functions and Eq. 59 follows then immediately from the usual rules for the Hodge dual.

iii) This follows directly from our choice of function spaces.

iv) This follows from the covariance of polar wavelets under rigid body transformations [67].

v) That the wavelet differential forms are real-values follows from the usual symmetry properties as well as the explicit calculations. The inverse Fourier transforms can be computed in closed form using the Jacobi-Anger and Rayleigh formulas that describe the complex exponential in polar and spherical coordinates, respectively, see Appendix A.5 and by expanding the form basis function in spherical coordinates.
For example in \( \mathbb{R}^2 \),

\[
\hat{\psi}^{1,2}_{d,s}(\xi) = -i \sum_m \beta_m e^{im\theta} \hat{h}(|\xi|) \frac{\partial}{\partial \theta}
\]

\[
= -i \sum_m \beta_m e^{im\theta} \hat{h}(|\xi|) \left( \sin \theta \frac{\partial}{\partial \xi^1} - \cos \theta \frac{\partial}{\partial \xi^2} \right)
\]

(86a)

By expanding the trigonometric functions as Fourier series, the components of \( \hat{\psi}^{1,2}_{d,s}(\xi) \), which we simply denote as \( \hat{\psi}_1(\xi) \) and \( \hat{\psi}_2(\xi) \) here, become

\[
\hat{\psi}_1(\xi) = \frac{1}{2} (e^{-i\theta} - e^{i\theta}) \left( \sum_n \beta_n e^{in\theta} \right) \hat{h}(|\xi|) \frac{\partial}{\partial \xi^1}
\]

(86c)

\[
= \frac{1}{2} \left( \sum_n \beta_n e^{i(n-1)\theta} \hat{h}(|\xi|) - \sum_n \beta_n e^{i(n+1)\theta} \hat{h}(|\xi|) \right) \frac{\partial}{\partial \xi^1}
\]

\[
\hat{\psi}_2(\xi) = \frac{i}{2} (e^{-i\theta} + e^{i\theta}) \left( \sum_n \beta_n e^{in\theta} \right) \hat{h}(|\xi|) \frac{\partial}{\partial \xi^2}
\]

(86d)

To obtain the inverse Fourier transform of the 1-forms above we require those of each of the sums in the above formulas. These be computed using the Jacobi-Anger formula. For example,

\[
\mathcal{F}^{-1} \left( \sum_n \beta_n e^{i(n-1)\theta} \hat{h}(|\xi|) \right)
\]

\[
= \int_{\mathbb{R}_+} \int_{\theta = 0}^{2\pi} \sum_n \beta_n e^{i(n-1)\theta \xi} \hat{h}(|\xi|) \sum_{m \in \mathbb{Z}} i^m e^{im(\theta_x - \theta_\xi)} J_m(|\xi| |x|) |\xi d\xi| d\theta
\]

\[
= \sum_n \sum_{m \in \mathbb{Z}} i^m \beta_n e^{i\theta_x} \int_{\theta = 0}^{2\pi} e^{i(n-1)\theta_\xi} e^{-im\theta_\xi} d\theta \int_{\mathbb{R}_+} \hat{h}(|\xi|) J_m(|\xi| |x|) |\xi d\xi|
\]

\[
= \sum_n i^{n-1} \beta_n e^{i(n-1)\theta_x} h_m(|x|).
\]

(88)

Carrying this out for all terms and combining it with the Fourier transform of the form basis functions we obtain

\[
\psi_1(x) = \frac{1}{2} \sum_{\sigma \in \{-1, 1\}} \sum_m \sigma i^{m+\sigma} \beta_m e^{i(m+\sigma)\theta} h_{m+\sigma}(|x|) dx^1
\]

(89a)
\[ \psi_2(x) = \frac{i}{2} \sum_{\sigma \in \{-1, 1\}} \sum_m \iota^{m+\sigma} \beta_m e^{i(\iota m + \sigma) \theta} h_{m+\sigma}(|x|) \, dx^2 \]  \hspace{1cm} (89b)

Rewriting this in vector form yields the result in Proposition \[ \text{8} \]

An analogous but somewhat more tedious calculation applies in \( \mathbb{R}^3 \) when the Jacobi-Anger formula is replaced by the Rayleigh formula and one uses a spherical harmonics expansion instead of a Fourier series one. Since, to our knowledge, it did not appear before in the literature we will present it here in detail for \( \psi_{2,3}^{2,3} \). In coordinates it is given by

\[ \hat{\psi}_{d,s,2}^2(x) = -i \left( \sum_{l=0}^{L} \sum_{m=-l}^{l} \kappa^s_{lm} y_{lm}(\bar{\xi}) \right) \hat{h} \left( 2^{-j_s} |\xi| \right) \frac{\partial}{\partial \phi} \tag{90a} \]

and without loss of generality we will assume in the following that \( j_s = 0 \).

To obtain the inverse Fourier transform, the spherical form basis function is again written in Cartesian form, i.e.

\[ \frac{\partial}{\partial \phi} = (- \sin \theta \sin \phi) \frac{\partial}{\partial \xi^1} + (\sin \theta \cos \phi) \frac{\partial}{\partial \xi^2}, \tag{90b} \]

and it will be convenient to write the components in their spherical harmonics expansions,

\[ -\sin \theta \sin \varphi = -i \sqrt{\frac{2\pi}{3}} \left( y_{1,-1}(\omega) - i y_{1,1}(\omega) \right) \tag{90c} \]
\[ \sin \theta \cos \varphi = \sqrt{\frac{2\pi}{3}} \left( y_{1,-1}(\omega) - y_{1,1}(\omega) \right). \tag{90d} \]

The first component of the inverse Fourier transform, which we denote simply by \( \psi_{s,1} \), is thus given by

\[ \psi_{s,1}(x) = -\frac{1}{2\pi \sqrt{3}} \sum_{l,m} \kappa^s_{lm} \int_{\mathbb{R}^3_{\xi}} y_{lm}(\bar{\xi}) \left( y_{1,-1}(\bar{\xi}) + y_{1,1}(\bar{\xi}) \right) \hat{h}(|\xi|) e^{i(\xi, x)} \, d\xi. \]

Changing to spherical coordinates and using the Rayleigh formula then yields

\[ \psi_{s,1}(x) = -\frac{2}{\sqrt{3}} \sum_{l,m} \kappa^s_{lm} \sum_{l_2,m_2} \iota^{l_2} y_{l_2 m_2}(\bar{x}) \]
\[ \times \sum_{\sigma \in \{-1, 1\}} \int_{S^2_{\bar{\xi}}} y_{lm}(\bar{\xi}) y_{l_1 \sigma}(\bar{\xi}) y_{l_2 m_2}(\bar{\xi}) \, d\bar{\xi} \int_{\mathbb{R}^3_{\xi}} \hat{h}(|\xi|) j_{l_1}(|\xi|) |\xi|^2 |d\xi|. \]
The angular integral is the projection of the product of two spherical harmonics into a third one, which is given by the spherical harmonics product coefficient $G_{l\sigma,m}^{l_\sigma,m_\sigma}$, cf. Appendix A.4. From the symmetry properties of the coefficients it follows that $G_{l\sigma,m}^{l_\sigma,m_\sigma}$ is non-zero only when $l-1 \leq l_\sigma \leq l+1$ and $m + 1 = m_\sigma$. Hence, the product is not diagonal, which would correspond to $\delta_{l,1} \delta_{m,1}$ and entirely collapse the sum over $l_\sigma, m_\sigma$, but it “diffuses” only to directly adjacent bands. For the inverse Fourier transform of the first component we thus have

$$\psi_{s,1}(x) = -\frac{2}{\sqrt{3}} \sum_{l,m} \sum_{\sigma \in \pm 1} G_{l\sigma}^{l_\sigma,m_\sigma} y_{l_\sigma,m_\sigma}(\bar{x}) h_{l_\sigma}(|x|).$$ (90f)

An analogous calculation can be carried out for the second component. Combining this with the Fourier transform of the form basis functions we have

$$\psi_{d,3}^{s,2}(x) = \frac{2}{\sqrt{3}} \sum_{l,m} \sum_{\sigma \in \pm 1} G_{l\sigma}^{l_\sigma,m_\sigma} y_{l_\sigma,m_\sigma}(\bar{x})(- dx^2 \wedge dx^3 + i \sigma dx^1 \wedge dx^3).$$ (90g)

6. Conclusion

We introduced wavelet differential $k$-forms $\psi_{p,s,a}^{k,n}$ that provide tight frames for the spaces $\dot{L}^1_2(\mathbb{R}^n, \Omega^k_4)$ and $L_2(\mathbb{R}^n, \Omega^k_4)$ and that satisfy important properties of Cartan’s exterior calculus, such as closure under the exterior derivative. Wavelet differential forms hence establish a “discrete” exterior calculus, which we call $\Psi_{ec}$, that is amenable for numerical calculations. We showed that Stokes’ theorem holds in this calculus and that with a finite number of levels anisotropic curvelet- or ridgelet-like frame functions provide more efficient approximations. Operators that are currently not well described in our calculus are the wedge product and the Lie derivative.
The present work provides, in our opinion, only a first step towards a more complete theory. For example, except when \( n = 0 \) or \( k = n \) and the existing results for cartoon-like functions apply, we did not establish approximation rates for differential forms in our representations. Similarly, for the Laplace–de Rahm operator we are missing decay estimates for its Galerkin projection \( D_{sr} \), which are, for example, necessary for error bounds for Example 7. We also did not yet present numerical experiments. An implementation of the differential form wavelets is available, cf. Fig. 2 and Fig. 4, and we will present its details as well as numerical results for example applications in a forthcoming publication. Since our wavelets have non-compact support in space, the efficiency of numerical calculations depends critically on the decay of the radial window. It should hence be optimized, cf. [68], or an ansatz should be developed that yields compactly supported wavelets.

Some technical extensions of our work would also be interesting. We defined our wavelet differential forms only for \( n = 2, 3 \) but the construction carries over to \( n > 3 \). One then would require more generic proofs then the ones currently used by us that typically proceed explicitly with the individual cases. However, with the tools introduced in Sec. 4 this is possible, although also, perhaps, less insightful. It would also be interesting to extend the functional analytic setting of our work. Troyanov [44] considers, for example, \( L^p(\mathbb{R}^n, \Omega) \) and for applications spaces with variable regularity, for example in the sense of Hölder, would be relevant.

Various extensions of our treatment of Stokes’ theorem are possible as well. We only considered smooth manifolds but using the tools of geometric measure theory [69] it should be possible to establish a similar result also for non-smooth ones. Given that we are limited to an extrinsic treatment of manifolds, it would also be interesting to explore if our construction can be combined with those by Berry and Giannakis [25] who use the eigenfunctions of the Laplace–Beltrami operator to define a spectral exterior calculus on arbitrary manifolds. A special case of particular relevance is the sphere \( S^2 \) where one still has an extensive analytic theory available. Preliminary investigations indicate that there one could
use needlets [70], which, interestingly, provide the angular localization windows \( \hat{\psi}_\mu(\vec{\xi}) \) used for wavelet differential forms in \( \mathbb{R}^3 \), instead of polar wavelets as a scalar frame. A local spectral exterior calculus on \( S^2 \) would have applications, for example, to climate simulations.

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URL http://link.springer.com/10.1007/978-3-642-62010-2
Appendix A. Conventions

Appendix A.1. Notation

We denote the norm in $\mathbb{R}^n$ by $| \cdot |$ and those in a function space by $\| \cdot \|$. Unit vectors are written as $\bar{x}$, i.e. for $x \in \mathbb{R}^n$ we have $\bar{x} = x / |x|$. We will not always distinguish between a unit vector and its corresponding coordinates in polar or spherical coordinates and depending on the context $\bar{x}$ might thus be a geometric vector or its spherical coordinates $(\theta, \phi)$.

We use standard spherical coordinates with latitude $\theta \in [0, \pi]$ and longitude $\phi \in 2\pi$. The standard area form in polar coordinates is $r \, d\theta \, dr$ and in spherical coordinates $r^2 \sin \theta \, d\theta \, d\phi \, dr$. For example, the integral of $f(x)$ takes in spherical
coordinates thus the form

\[
\int_{\mathbb{R}^3} f(x) \, dx = \int_{\mathbb{R}^+_{|x|}} \int_{S^2_\theta} f(r, \theta, \phi) \, r^2 \sin \theta \, d\theta \, d\phi \, dr
\]

(A.1a)

\[
= \int_{\mathbb{R}^+_{|x|}} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r, \theta, \phi) \, r^2 \sin \theta \, d\theta \, d\phi \, dr
\]

(A.1b)

where \(\mathbb{R}^+_{|x|}\) means that the radial variable \(|x|\) is integrated over the positive real line. We will use similar notation whenever confusion might arise otherwise.

We will work

**Appendix A.2. The Fourier transform**

The unitary Fourier transform of a function \(f \in L_2(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)\) is defined as

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i(x,\xi)} \, dx
\]

(A.2a)

with inverse transform

\[
\mathcal{F}^{-1}(f)(\xi) = f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i(\xi,x)} \, d\xi.
\]

(A.2b)

**Appendix A.3. The Homogeneous Sobolev Space \(\dot{L}^1_2\)**

The homogeneous Sobolev space \(\dot{L}^1_2(\mathbb{R}^n)\) is defined as

\[
\dot{L}^1_2 = \left\{ f \in L_1(\mathbb{R}^n) \mid ||f||_{\dot{L}^1_2} < \infty \right\}
\]

(A.3a)

with the norm being those induced by the inner product [71, Ch. 1.2.1]

\[
\langle f, g \rangle_{\dot{L}^1_2} = \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\xi) |\xi|^{2s} \, d\xi
\]

(A.3b)

and \(||f||_0\) denoting the co-sets of functions modulo constant polynomials [48, Ch. II.6]. As is customary, the co-sets are usually omitted in the notation. The so defined space \(\dot{L}^1_2(\mathbb{R}^n)\) is a Hilbert space [48, Ch. II.6].
Appendix A.4. Spherical harmonics

The analogue of the Fourier transform in Eq. A.2 on the sphere is the spherical harmonics expansion. For any $f \in L^2(S^2)$ it is given by

$$f(\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle f(\eta), y_{lm}(\eta) \rangle y_{lm}(\omega)$$  \hspace{1cm} (A.4a)

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} y_{lm}(\omega)$$  \hspace{1cm} (A.4b)

where $\langle \cdot, \cdot \rangle$ denotes the standard $L^2$ inner product on $S^2$ given by

$$\langle f(\omega), g(\omega) \rangle = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(\theta, \phi) g(\theta, \phi) \sin \theta d\theta d\phi.$$  \hspace{1cm} (A.5)

We use standard (geographic) spherical coordinates with $\theta \in [0, \pi]$ being the polar angle and $\phi \in [0, 2\pi]$ the azimuthal one. The spherical harmonics basis functions in Eq. A.4 are given by

$$y_{lm}(\omega) = y_{lm}(\theta, \phi) = C_{lm} P^l_m(\cos \theta) e^{im\phi}$$  \hspace{1cm} (A.6)

where the $P^l_m(\cdot)$ are associated Legendre polynomials and $C_{lm}$ is a normalization constant so that the $y_{lm}(\omega)$ are orthonormal over the sphere. The associated Legendre polynomials are defined as\footnote{The Condon-Shortley phase factor of $(-1)^m$ is included to be consistent with Mathematica.}

$$P^l_m(\cos \theta) = (-1)^m \sum_{p=0}^{r} c_{lmp} \sin^m(\cos \theta) t^{l-m-2p}$$  \hspace{1cm} (A.7a)

where $r = \lfloor (l - m)/2 \rfloor$ and

$$c_{lmp} = (-1)^p \frac{2^{-l} (2l - 2p)!}{p! (l - p)! (l - m - 2p)!}.$$  \hspace{1cm} (A.7b)

The associated Legendre polynomials are not $L^2$-normalized by satisfy

$$\int_{-1}^{1} P^l_{m_1}(x) P^l_{m_2}(x) dx = \frac{2}{2l + 1} \frac{(l + m)!}{(l - m)!} \delta_{l_1 l_2}. \hspace{1cm} (A.7c)$$
**Spherical Harmonics Addition Theorem.** The spherical harmonics addition theorem is given by

\[
P_l(\vec{x}_1 \cdot \vec{x}_1) = \frac{4\pi}{2l + 1} \sum_{m=-l}^{l} y_{lm}(\omega_1) y^*_{lm}(\omega_2) \tag{A.8}
\]

where \(\omega_1\) are the spherical coordinates for the unit vector \(\vec{x}_1 \in \mathbb{R}^3\) and \(\omega_2\) are those for \(\vec{x}_2 \in \mathbb{R}^3\). It follows immediately from the above formula that \(P_l(\vec{x}_1 \cdot \vec{x}_1)\) is the reproducing kernel for the space spanned by all spherical harmonics of fixed \(l\).

**Rotation of spherical harmonics representations and Wigner-D matrices.** The spaces \(\mathcal{H}_l\) spanned by all spherical harmonics with fixed degree \(l\) are closed under rotation, i.e. \(f \in \mathcal{H}_l \Rightarrow Rf \in \mathcal{H}_l\) for \(R \in SO(3)\) and the action is given by \((Rf)(\omega) = f(R^T \omega)\). The rotation is represented by Wigner-D matrices \(W_{lm}(R)\) that act as

\[
f_{lm}(R) = \sum f_{lm'} \tag{A.9}
\]

where the \(f_{lm}(R)\) are the spherical harmonics coefficients of the rotated signal.\(^4\)

When the rotation is specified in Euler angles \((\alpha, \beta, \gamma)\), the Wigner-D function may be written as

\[
D^{m'}_{lm}(\alpha, \beta, \gamma) = \sum_{m} d^{m'}_{lm}(\beta) e^{-im'\gamma} \tag{A.10}
\]

where \(d^{m',s}_{m,-s}(\beta)\) is the Wigner small-d matrices that are given by the Wigner formula.\(^3\)

A special case of the Wigner-D matrices also follows from the spherical harmonics addition theorem. Let

\[
f(\theta, \phi) = \sum_l \kappa_l y_{l0}(\theta) \tag{A.11a}
\]

\(^4\)Since the evaluation of the elements of the Wigner-D matrices is computationally expensive and numerically not stable, for a numerical implementation of the rotation in the spherical harmonics domain are schemes are typically better suited, see e.g.\(^3\).
so that it has longitudinal symmetry. By definition of the spherical harmonics we have

\[ f(\theta, \phi) = \sum_l \kappa_l P_l(\cos \theta) \]  
(A.11b)

\[ = \sum_l \kappa_l P_l(x_3 \cdot \omega) \]  
(A.11c)

where in the second line \( x_3 \) denotes the third axis and \( \omega \) the unit vector with spherical coordinates \((\theta, \phi)\) as a finite series in trigonometric factors. The rotation of \( f \) to be centered at a direction \( \lambda \) is hence given by

\[ f_\lambda(\theta, \phi) = \sum_l \kappa_l P_l(\lambda \cdot \omega). \]  
(A.11d)

The Legendre polynomial \( P_l(\lambda \cdot \omega) \) can be expanded using the spherical harmonics addition theorem in Eq. A.8,

\[ f_\lambda(\theta, \phi) = \sum_l \kappa_l \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} y^*_l m(\lambda) y_{lm}(\omega) \]  
(A.11e)

\[ = \sum_l \sum_{m=-l}^{l} \left( \kappa_l \sqrt{\frac{4\pi}{2l+1}} y^*_l m(\lambda) \right) y_{lm}(\omega) \]  
(A.11f)

The spherical harmonics coefficients \( f_{lm}(\lambda) \) of the rotated signal \( f_\lambda(\theta, \phi) \) are thus

\[ f_{lm}(\lambda) = \sqrt{\frac{4\pi}{2l+1}} y^*_l m(\lambda) \kappa_l \]  
(A.11g)

and this immediately implies

\[ D^\theta_{lm}(\lambda) = \sqrt{\frac{4\pi}{2l+1}} y^*_l m(\lambda). \]  
(A.11h)

Comparing to Eq. A.10 and recognizing that the Euler angles for the rotation from the North pole are \((\alpha, \beta, \gamma) = (0, \theta, \phi)\) we see that the Wigner small-D matrices in fact coincide with the Legendre polynomials in this case.

**Clebsch-Gordon coefficients.** In contrast to the Fourier series, where the product of two basis functions \( e^{im_1 \theta} \) and \( e^{im_2 \theta} \) is immediately given by one other Fourier
series function $e^{i(m_1 + m_2)\theta}$, for spherical harmonics the product is not diagonal and characterized by Clebsch-Gordan coefficients $C_{l_1, m_1; l_2, m_2}^{l, m}$. In particular, the product of two functions on the sphere is in the spherical harmonics domain given by

$$ (f \cdot g)(\omega) = \sum_{l m} \left( \sum_{l_1 m_1 l_2 m_2} f_{l_1 m_1} g_{l_2 m_2} G_{l_1, m_1; l_2, m_2}^{l, m} \right) y_{l m}(\omega) $$

(A.12)

where the spherical harmonics product coefficients $G_{l_1, m_1; l_2, m_2}^{l, m}$ are given by

$$ G_{l_1, m_1; l_2, m_2}^{l, m} = \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)}} C_{l, 0; l_1, 0}^{0, 0} C_{l_1, m_1; l_2, m_2}^{l, m} $$

(A.13)

and the $C_{l_1, m_1; l_2, m_2}^{l, m}$ are Clebsch-Gordan coefficients. These are the projection of the product of $y_{l_1, m_1}(\omega)$ and $y_{l_2, m_2}(\omega)$ onto the spherical harmonics $y_{l m}(\omega)$,

$$ \int_{S^2} y_{l_1, m_1}(\omega) y_{l_2, m_2}(\omega) y_{l m}^*(\omega) d\omega = \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)}} C_{l_1, 0; l_2, 0}^{0, 0} C_{l_1, m_1; l_2, m_2}^{l, m}. $$

The Clebsch-Gordon coefficients are sparse and non-zero only when

$$ m = m_1 + m_2, $$

(A.14a)

that is the $m$ parameter is superfluous but conventionally used, and

$$ l_1 + l_2 - l \geq 0 $$

(A.14b)

$$ l_1 - l_2 + l \geq 0 $$

(A.14c)

$$ -l_1 + l_2 + l \geq 0. $$

(A.14d)

Appendix A.5. Fourier Transform in Polar and Spherical Coordinates

Jacobi-Anger formula. In the plane, the Fourier transform can also be written in polar coordinates by using the Jacobi-Anger formula [74],

$$ e^{i(\xi x)} = \sum_{m \in \mathbb{Z}} i^m e^{im(\phi_x - \phi_e)} J_m(|\xi| \mid x), $$

(A.15)
that relates the complex exponential in Euclidean and polar coordinates. The ordering of the \( \phi_x \) and \( \phi_\xi \) on the right hand side is arbitrary and when the left hand side is conjugated \( i^m \) becomes \( i^{-m} \) \(^4\). In Eq. A.15, \( J_m(z) \) is the Bessel function of the first kind and \((\phi_x, |x|)\) and \((\phi_\xi, |\xi|)\) are polar coordinates for the spatial and frequency domains, respectively.

The Jacobi-Anger formula allows one to compute the Fourier transform in polar coordinates. Let \( f(\phi, |x|) \equiv f(\bar{x}|x|) = f(x) \) with \( \bar{x} = (\cos \phi, \sin \phi) \), for fixed radius \(|x|\). The function \( f(x) \) can then be written as the Fourier series expansion

\[
f(x) = f(\phi_x, |x|) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} f_n(|x|) e^{in\phi_x}. \tag{A.16a}
\]

Inserting this together with the Jacobi-Anger formula into Eq. A.2a and performing a change of variables to polar coordinates we obtain

\[
\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^+} \int_{\phi_x=0}^{2\pi} \left( \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} f_n(|x|) e^{2\pi in\phi_x} \right) \times \left( \sum_{m \in \mathbb{Z}} i^m e^{-i m(\phi_x - \phi_\xi)} J_m(|\xi| |x|) \right) |x| \, d\phi_x \, d|x|. \tag{A.16b}
\]

The integral over \( \phi_x \) is trivial since it only involves the complex exponentials \( e^{im\phi_x} \) and \( e^{-im\phi_x} \), giving \( 2\pi \delta_{nm} \) and also collapsing the product of sums into a single sum. The Fourier transform \( \hat{f}(\xi) = \hat{f}(\phi_\xi, |\xi|) \) is thus

\[
\hat{f}(\xi) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} i^m e^{-im\phi_\xi} \int_{\mathbb{R}^+} f_m(|x|) J_m(|\xi| |x|) |x| \, d|x| \tag{A.16c}
\]

\[
= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} i^m \hat{f}_m(|\xi|) e^{im\phi_\xi}. \tag{A.16d}
\]

with an analogous expression for the inverse transform. Note that the Fourier transform preserves the polar structure: \( f(\phi_x, |x|) \) described in polar coordinates is mapped to \( \hat{f}(\phi_\xi, |\xi|) \) in polar coordinates in the frequency domain.

\(^4\)This follows since the sum over \( m \) runs over all integers, cancelling the imaginary part.
Rayleigh formula. The analogue of the Jacobi-Anger formula in three dimensions is the Rayleigh formula,

\[ e^{i\langle \xi, x \rangle} = \sum_{l=0}^{\infty} i^l (2l + 1) P_l(|\xi| |x|) j_l(|\xi| |x|) \]  

(A.17a)

\[ = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l y_{lm}(\xi) y_{lm}(\bar{x}) j_l(|\xi| |x|) \]  

(A.17b)

where \( j_l(\cdot) \) is the spherical Bessel function and the second line follows by the spherical harmonics addition theorem in Eq. A.8; conjugation leads to a factor of \( i^{-l} \) instead of \( i^l \). Using a calculation analogous to those in Eq. A.16, the Rayleigh formula enables the calculation of the Fourier transform in spherical coordinates.

Appendix A.6. Hedgehog frame

The canonical coordinate vectors \( \partial/\partial \theta \) and \( \partial/\partial \phi \) for the tangent space \( TS^2 \) induced by the standard spherical coordinates,

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  \sin \theta \cos \phi \\
  \sin \theta \sin \phi \\
  \cos \theta
\end{pmatrix},
\]  

(A.18)

are singular at the poles. This reflects the well known fact that \( S^2 \) cannot be covered by a single chart. An alternative is provided by the Hedgehog frame. Let

\[
\tau_1 = \begin{pmatrix}
  0 \\
  \sin \theta \sin \phi \\
  \sin \theta \cos \phi
\end{pmatrix} \quad \tau_2 = \begin{pmatrix}
  -\sin \theta \sin \phi \\
  0 \\
  \sin \theta \cos \phi
\end{pmatrix} \quad \tau_3 = \begin{pmatrix}
  -\sin \theta \sin \phi \\
  \sin \theta \cos \phi \\
  0
\end{pmatrix}
\]  

(A.19)

i.e., \( \tau_a \) is the canonical (non-unit norm) tangent vector \( \partial/\partial \phi \) with the \( a \)th axis as up-axis. Then

\[ H(\theta, \phi) = \{\tau_1, \tau_2, \tau_3\} \]  

(A.20)

is a tight frame for \( TS^2 \), which we call the Hedgehog frame. Note that \( H \) also provides a frame for the cotangent space. Geometrically, \( H \) can also...
be interpreted as generated by three charts for $S^2$ defined by standard spherical coordinates with respect to the three Cartesian coordinate axes with $\sin \theta$, defined locally, as weight function for each of it.

Appendix B. Admissibility of Polar Wavelets for $L_2(\mathbb{R}^3)$

**Proposition 10.** Let $u_{j,t}$ be the $(L_j + 1)^2$-dimensional vector formed by the $\kappa_{lm}^{j,t}$ for fixed $j, t$. Then the wavelets in Eq. 5 form a Parseval tight frame when

$$\sum_{j \in \mathbb{Z}} |\hat{h}(2^{-j|\xi|})|^2 = 1, \quad \forall \xi \in \mathbb{R}_\xi^2$$

(B.1a)

and

$$\delta_{l,0} \delta_{m,0} = M_j \sum_{t=0}^{M_j} u_{j,t} G_{lm} u_{j,t}$$

(B.1b)

where $G_{lm}$ is the matrix formed by the spherical harmonics product coefficients in Eq. A.14 for fixed $(l,m)$.

**Proof.** Our proof will be for the scale-discrete, continuous frame generated by the wavelets. Since our window functions are bandlimited, the result carries over to the discrete case using standard arguments, see for example [20, Sec. 4] or [14, Sec. IV].

We want to show

$$u = \sum_j (u \ast \psi_j) \ast \psi_j$$

(B.2)

for $u \in L_2(\mathbb{R}^3)$. Taking the Fourier transform of both sides we have

$$\hat{u} = \sum_j (\hat{u} \hat{\psi}_j) \hat{\psi}_j$$

(B.3)

Using linearity we hence have

$$\hat{u} = \hat{u} \sum_j \hat{\psi}_j^* \hat{\psi}_j$$

(B.4)
and it suffices to show that the scalar window functions satisfy the Caldéron admissibility condition

$$\sum_j \sum_t |\hat{\psi}_j|^2 = 1, \quad \forall \xi \in \mathbb{R}^3. \quad (B.5)$$

With the definition of the window functions and after re-arranging terms one obtains

$$\sum_j M_j \sum_t |\hat{\psi}_j|^2 = \sum_j \sum_{l_1,m_1} \sum_{l_2,m_2} \kappa_{l_1,m_1}^{j,t} y_{l_1,m_1} (\bar{\xi}) \kappa_{l_2,m_2}^{j,t} y_{l_2,m_2}^* (\bar{\xi}) |\hat{h}(2^{-j}|\xi|)|^2$$

Assuming the Caldéron condition in Eq. (B.1a) is satisfied, the lemma holds when the product of the angular part evaluates to the identity for every band $j$ and every direction $\bar{\xi}$. This means that for every $j$ the projection of the angular part in the above equation onto spherical harmonics has to satisfy

$$\delta_{l,0} \delta_{m,0} = M_j \sum_t \sum_{l_1,m_1} \kappa_{l_1,m_1}^{j,t} \kappa_{l_2,m_2}^{j,t*} \left\langle y_{l_1,m_1} (\bar{\xi}) \right| \left. y_{l_2,m_2}^* (\bar{\xi}) \right\rangle, y_{l_0,m_0} (\bar{\xi}) \right\rangle. \quad (B.6a)$$

Rearranging terms we obtain

$$\delta_{l,0} \delta_{m,0} = \sum_{l_1,m_1} \sum_{l_2,m_2} \kappa_{l_1,m_1}^{j,t} \kappa_{l_2,m_2}^{j,t*} \left\langle y_{l_1,m_1} (\bar{\xi}) \right| \left. y_{l_2,m_2}^* (\bar{\xi}) \right\rangle, y_{l_0,m_0} (\bar{\xi}) \right\rangle. \quad (B.6b)$$

The product of two spherical harmonics projected into another spherical harmonic is given by the product coefficents in Eq. A.14. We can hence write

$$\delta_{l,0} \delta_{m,0} = \sum_{l_1,m_1} \sum_{l_2,m_2} \kappa_{l_1,m_1}^{j,t} \kappa_{l_2,m_2}^{j,t*} \left\langle y_{l_1,m_1} (\bar{\xi}) \right| \left. y_{l_2,m_2}^* (\bar{\xi}) \right\rangle \kappa_{l_3,m_3}^{j,t}, G_{l_1,m_1,l_2,m_2}^{l_3,m_3}. \quad (B.6c)$$

Collecting the $\kappa_{l_1,m_1}^{j,t}$ into vectors we obtain the condition in the lemma. The remaining sum over $j$ yields the identity by construction of the $\hat{h}$ windows. \qed