Scalar-Tensor Gravity in Two 3-brane System

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Abstract

We derive the low-energy effective action of four-dimensional gravity in the Randall-Sundrum scenario in which two 3-branes of opposite tension reside in a five-dimensional spacetime. The dimensional reduction with the Ansatz for the radion field by Charmousis et al., which solves five-dimensional linearized field equations, results in a class of scalar-tensor gravity theories. In the limit of vanishing radion fluctuations, the effective action reduces to the Brans-Dicke gravity in accord with the results of Garriga and Tanaka: Brans-Dicke gravity with the corresponding Brans-Dicke parameter \(0 < \omega < \infty\) (for positive tension brane) and \(-3/2 < \omega < 0\) (for negative tension brane). In general the gravity induced a brane belongs to a class of scalar-tensor gravity with the Brans-Dicke parameter which is a function of the interval and the radion. In particular, gravity on a positive tension brane contains an attractor mechanism toward the Einstein gravity.

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I. INTRODUCTION

Randall and Sundrum recently proposed a mechanism to solve the hierarchy problem \cite{1} using two 3-branes of opposite tension residing embedded in a five dimensional spacetime. In this scenario we are assumed to live in the negative tension brane. They also found that even the graviton is trapped on the positive tension brane in the same set up \cite{2} (see also \cite{3}).

It is natural to ask how gravity look like on the brane. In a recent paper, Garriga and Tanaka have analyzed the metric perturbation equation on the 3-brane in the background spacetime of Randall and Sundrum and have shown that gravity on the brane is described by Brans-Dicke gravity \cite{4} (for a single 3-brane, see \cite{5}). Charmousis et al. recently proposed an Ansatz for the radion which solves five-dimensional linearized equations of motion \cite{6}.

The purpose of this note is to provide alternative considerations on the brane gravity based on the derivation of the low-energy effective action on each 3-brane from the five dimensional action \textit{à la} Kaluza-Klein. We take into account the degrees of freedom of the modulus field as well as four-dimensional graviton fluctuations following the metric Ansatz by Charmousis et al. We will see that gravity on the brane is actually a more general scalar-tensor gravity theory, where the Brans-Dicke parameter is a function of the Brans-Dicke scalar field, rather than the Brans-Dicke gravity, where the Brans-Dicke parameter is a constant. The modulus field (the radion) plays the role of the Brans-Dicke scalar field. We believe that our approach is technically simpler. The calculations required in our approach are reduced considerably, and it is easy to generalize to include matter fields. We show that the cosmological attractor mechanism toward the Einstein gravity is realized in gravity on a positive tension brane. We also make a brief comment on the cosmological constant problem in the brane-world view of the universe.

II. REDUCTION IN TWO THREE-BRANE SYSTEM

A. Background

We briefly review the background metric found by Randall and Sundrum \cite{1} to introduce our notations. The set up is that the five-dimensional Einstein gravity with a cosmological constant with two 3-branes located in the orbifold $S^1/Z_2$ at the fifth dimensional coordinate $z = 0$(positive tension brane), $z = r_c$(negative tension brane). The action is

$$S = 2 \int d^4x \int_0^{r_c} d\phi \sqrt{-g_5} \left[ \frac{M^3}{2} R_5 - 2\Lambda \right] - \sigma(+) \int d^4x \sqrt{-g(+) - \sigma(-) \int d^4x \sqrt{-g(-)}, \quad (1)$$

where $M$ is the five-dimensional Planck mass defined in term of the five-dimensional gravitational constant $G_5$ as $M^{-3} = 8\pi G_5$. $\sigma(\pm)$ and $g(\pm)$ are the brane tension and the induced metric on the brane, respectively. Randall and Sundrum have shown that there exists a solution that respects four-dimensional Poincaré invariance \cite{1}:

$$ds^2 = e^{-2k|z|} \eta_{\mu\nu} dx^\mu dx^\nu + dz^2, \quad (2)$$

only if $\Lambda, \sigma(+) \text{ and } \sigma(-)$ are related as

$$\Lambda = -3M^3k^2, \sigma(+) = -\sigma(-) = 3M^3k. \quad (3)$$
B. Naive Ansatz

We shall derive the four-dimensional low-energy effective action on a 3-brane. In order to include massless gravitational degrees of freedom (the zero modes about the background spacetime Eq. (2)), we replace the Minkowski metric $\eta_{\mu\nu}$ with a general metric $\bar{g}_{\mu\nu}(x)$ and $z$-direction length $r_c$ with a modulus field $T(x)$:

$$ds^2 = e^{-2kT(x)}|z|\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu + T(x)^2dz^2.$$  (4)

Although this metric ansatz does not correctly describe the linearized dynamics of massless fields, we present the analysis here because the results nevertheless seem simple and attractive. The induced metric on the positive (or negative) tension brane is thus

$$g^{(+)}_{\mu\nu} = \bar{g}_{\mu\nu}$$

or

$$g^{(-)}_{\mu\nu} = e^{-2kr_cT}\bar{g}_{\mu\nu},$$

respectively. Since we impose $\mathbb{Z}_2$ symmetry ($z \leftrightarrow -z$), massless vector fluctuations associated with the off-diagonal part of the metric are absent.

1. Effective Four-dimensional Action

We can perform the $z$ integral to obtain the four-dimensional action on the positive tension brane described by $g^{(+)}_{\mu\nu}$:

$$S^{(+)} = 2\int d^4x \int_0^{r_c} dz \sqrt{-g^{(+)}_5} \left[ \frac{M^3}{2}R_5 - 2\Lambda \right] - \sigma^{(+)} \int d^4x \sqrt{-g^{(+)}_5} - \sigma^{(-)} \int d^4x \sqrt{-g^{(-)}_5}$$

$$= \int d^4x \sqrt{-g^{(+)}_5} \left[ \frac{M^3}{2}e^{2kTz} \left( R^{(+)} + 6e^{kTz}\Box^{(+)}e^{-kTz} \right) - 2\Lambda \right]$$

$$- \sigma^{(+)} \int d^4x \sqrt{-g^{(+)}_5} - \sigma^{(-)} \int d^4x \sqrt{-g^{(+)}}e^{-4kr_cT}$$

$$= \int d^4x \sqrt{-g^{(+)}_5} \left[ \frac{M^3}{2k}(1 - e^{-2kr_cT})R^{(+)} - 3kr_c^2M^3e^{-2kr_cT} \left( \nabla^{(+)}T \right)^2 \right]$$

$$- \frac{(1 - e^{-4kr_cT})}{k^\Lambda} - \sigma^{(+)} - \sigma^{(-)}e^{-4kr_cT}$$

$$= \int d^4x \sqrt{-g^{(+)}_5} \left[ \frac{M^3}{2k}(1 - e^{-2kr_cT})R^{(+)} - 3kr_c^2M^3e^{-2kr_cT} \left( \nabla^{(+)}T \right)^2 \right],$$  (5)

where we have used the relation Eq.(3) in the last equation.

The case of the negative tension brane is obtained by the conformal transformation such that $g^{(-)}_{\mu\nu} = e^{-2kr_cT}\bar{g}_{\mu\nu}$, and the result is

$$S^{(-)} = \int d^4x \sqrt{-g^{(-)}_5} \left[ \frac{M^3}{2k}(e^{2kr_cT} - 1)R^{(-)} + 3kr_c^2M^3e^{2kr_cT} \left( \nabla^{(-)}T \right)^2 \right].$$  (6)

The above result is also easily obtained by the change of signature such that $k \rightarrow -k$, the meaning of which may be intuitively clear: exchange of the locations of the branes.
2. Brans-Dicke parameter

The action of the scalar-tensor gravity theory is given by

\[ S = \int d^4x \sqrt{-g} \frac{1}{16\pi} \left( \Phi_{BD} R - \omega(\Phi_{BD}) \frac{\Phi_{BD}}{\Phi_{BD}} (\nabla \Phi_{BD})^2 \right). \]  (7)

The correspondence of the gravity theory on the brane to the Brans-Dicke gravity is then immediate. The Brans-Dicke scalar field (or the inverse of the effective gravitational constant) is

\[ \frac{1}{G_{\text{eff}}^{(\pm)}} = \Phi_{BD}^{(\pm)} = \frac{16\pi M^3}{k} e^{\mp kr_cT} \sinh(kr_cT) = \frac{2}{kG_5} e^{\mp kr_cT} \sinh(kr_cT), \]  (8)

while the corresponding Brans-Dicke function is

\[ \omega(\pm)(T) = \pm 3 e^{\pm kr_cT} \sinh(kr_cT). \]  (9)

Remarkably, despite the defect of the metric Ansatz, both expressions are in perfect agreement with those of Garriga and Tanaka if the length parameter between the branes is replaced with the modulus field \( r_c T(x) \): the Brans-Dicke gravity with the corresponding Brans-Dicke parameter \( 0 < \omega < \infty \) (for positive tension brane) and \( -3/2 < \omega < 0 \) (for negative tension brane).

C. CGR Ansatz

Although the previous analysis based on the naive Ansatz seems simple and attractive, we seek after another metric Ansatz which does solve the linearized equations of motion. As an example, we adopt the metric Ansatz proposed by Charmousis et al. (CGR) (we shall consider the region \( 0 < z < r_c \); the other region by the orbifold symmetry):

\[ ds^2 = e^{-2kh(x,z)} \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu + h^2_c dz^2, \]  (10)

\[ h(x, z) = z + f(x) e^{2kz}. \]  (11)

The induced metric on each brane is thus respectively

\[ g_{(+)}_{\mu\nu} = e^{-2kf} \bar{g}_{\mu\nu}, \]  (12)

\[ g_{(-)}_{\mu\nu} = \alpha^{-1} e^{-2akf} \bar{g}_{\mu\nu}. \]  (13)

Here we have introduced the notation \( \alpha \equiv e^{2kr_c} \) for convenience.

1. Effective Four-dimensional Action

We then perform the \( z \) integral to obtain the four-dimensional action on the positive tension brane in terms of \( g_{(+)}_{\mu\nu} \):
\[ S_{(+)} = \int d^4x \sqrt{-g_{(+)}} \left[ \frac{M^3}{2k} \left( 1 - \frac{1}{\alpha} e^{-2(\alpha-1)kf} \right) R_{(+)} - 3kM^3 \left[ (\alpha + \frac{1}{\alpha}) e^{-2(\alpha-1)kf} - 2 \right] \left( \nabla_{(+)} f \right)^2 \right], \]

where we have again used the relation Eq.(3).

The case of the negative tension brane is obtained by the conformal transformation such that \( g_{(-)\mu\nu} = \alpha^{-1} e^{-2\alpha kf} \tilde{g}_{\mu\nu} = \alpha^{-1} e^{-2(\alpha-1)kf} \bar{g}_{(+)} \bar{\mu}\bar{\nu} \), and the result is

\[ S_{(-)} = \int d^4x \sqrt{-g_{(-)}} \left[ \frac{M^3}{2k} \left( \alpha e^{2(\alpha-1)kf} - 1 \right) R_{(-)} + 3\alpha^2 kM^3 \left[ (\alpha + \frac{1}{\alpha}) e^{2(\alpha-1)kf} - 2 \right] \left( \nabla_{(-)} f \right)^2 \right]. \]

### 2. Brans-Dicke parameter

The Brans-Dicke scalar field (or the inverse of the effective gravitational constant) is

\[ \frac{1}{G_{eff}(\pm)} = \Phi_{BD}(\pm) = \frac{2}{kG_5} e^{\mp k[(\alpha-1)f + r_c]} \sinh k[(\alpha - 1)f + r_c] \]

while the corresponding Brans-Dicke function is

\[ \omega_{(\pm)}(f) = \pm 3e^{\pm k[(\alpha-1)f + r_c]} \sinh k[(\alpha - 1)f + r_c] \left( 1 \mp \frac{e^{\pm(\alpha-1)kf} \sinh[(\alpha - 1)kf]}{\sinh^2(kr_c)} \right). \]

Both expressions are in agreement with those of Garriga and Tanaka [4] in the limit where the radion fluctuations are vanishing, \( f \to 0 \).

Now let us consider what kind of gravity is induced on the brane if the radion fluctuations are present. As long as \( e^{2\alpha kf} \ll \alpha \equiv e^{2kr_c} \), the metric ansatz Eq.(10) and Eq.(11) correctly describes the linearized dynamics of massless fields. For the positive tension brane, gravity on the brane is a class of scalar-tensor gravity theories with \( \omega > 0 \) (for \( e^{2\alpha kf} \ll \alpha \) we have \( \omega = 3\alpha e^{2\alpha kf}/2 \)) and can satisfy the constraint by the solar system experiment (\( |\omega| > 3000 \)) if \( kr_c > 4 \). In the limit \( kr_c \to \infty \), the Einstein gravity is recovered with the gravitational constant \( G_5 = kG_5 \).

On the other hand, gravity on the negative tension brane is a class of scalar tensor theories with \( \omega = -3/2 + 9\alpha^{-1} e^{2\alpha kf}/2 < 0 \) (in the lowest order of \( \mathcal{O}(\alpha^{-1}e^{2\alpha kf}) \)) which does not satisfy the constraint by the solar system experiment. This does not immediately mean that the scenario of [1] is not valid because we do not include massive degrees of freedom that may give rise to a stabilizing potential for the modulus (see [10,11]).

Moreover, gravity on the negative tension brane exhibits some peculiar behavior in the limit \( kr_c \to \infty \): \( G_{eff} \to 0 \) and \( \omega \to -3/2 \). The vanishing of the gravitational constant would imply the absence of gravity force. Furthermore, \( \omega = -3/2 \) means that the scalar field degrees of freedom is completely absorbed by the conformal transformation; the action is reduced to that of vacuum Einstein gravity (no scalar field) by the conformal transformation
such that $g_{\mu\nu}^E = [\alpha e^{2(\alpha-1)kf} - 1]g_{(-)\mu\nu}$. In this sense, the scalar field degrees of freedom is also frozen.

3. Attractor Mechanism

The Einstein gravity is generically a cosmological attractor in scalar-tensor theories (attractor mechanism \cite{12}). Therefore, as far as gravity is concerned, the dilaton stabilization is not always necessary. It is interesting to examine whether gravity on the positive tension brane contains a similar attractor mechanism although it requires slight abuse the action Eq.(14) beyond the weak field approximation. To do so, we assume matter on the positive tension brane and shall work in the Einstein conformal frame defined by

$g_{\mu\nu}^E = [1 - \alpha^{-1}e^{-2(\alpha-1)kf}]g_{(+)\mu\nu} \equiv e^{-2a(\phi)}g_{(+)\mu\nu}, \quad 2(da/\kappa d\phi)^2 = (2\omega(f) + 3)^{-1}$

so that the action Eq.(14) reduces to

$$S_{(+)} = \int d^4x \sqrt{-g_E} \left[ \frac{1}{2\kappa^2} R_E - \frac{1}{2} (\nabla_E \phi)^2 \right],$$

where $\kappa^2 = 8\pi G_4 \equiv 8\pi kG_5$. An important point of the attractor mechanism by Damour and Nordvedt \cite{12} is that the function $a(\phi)$ determines the cosmological dynamics of the dilaton $\phi$ and the minimum of it is the Einstein gravity ($a(\phi) = 0 \rightarrow \omega = \infty$). Therefore it is sufficient to see the shape of $a(\phi)$. For $e^{2\phi k} \ll \alpha$, we find that $2a(\phi) = -\ln(1 - 2\kappa^2 \phi^2/3)$ with $2\kappa^2 \phi^2 = 3\alpha^{-1}e^{-2\alpha k f}$; that is, $a(\phi)$ indeed has a minimum at $\phi = 0$, which corresponds to $\omega \rightarrow \infty$.

4. Adjusting Mechanism?

Finally let us consider the situation where the bulk cosmological constant $\Lambda$ is slightly perturbed from the assumed value Eq.(3), $\Lambda \rightarrow \Lambda + \delta \Lambda$, while the brane tensions are fixed. Such a mismatch would induce an effective potential for the modulus $f(x)$ and give rise to the dynamics of $f(x)$. However, because the modulus $f(x)$ couples non-minimally to the curvature, the effective potential for $f(x)$ is given by performing the conformal transformation to the canonical Einstein frame $g_{\mu\nu}^E = [1 - \alpha^{-1}e^{-2(\alpha-1)kf}]g_{(+)\mu\nu}$ (for simplicity we consider the positive tension brane), and it is given by

$$V_{\text{eff}}(f) = \frac{\delta \Lambda}{k} \coth k[(\alpha - 1)f + r_c].$$

\footnote{We note that in the limit $kr_c \rightarrow \infty$ the Einstein conformal frame metric $g_{\mu\nu}^E$ coincides with the metric on the positive tension brane $g_{(+)\mu\nu}$.}

\footnote{The attractor mechanism is also realized in the action Eq.(3).}

\footnote{The potential of this type has been discussed in the context of inflation \cite{16}.}
Since the minimum of the potential is not zero, there is no dynamical mechanism to enforce \( V_{\text{eff}} = 0 \). Hence, contrary to the argument in [13], brane-world picture of the universe seems to be compatible with a nonzero four-dimensional cosmological constant (similar observations are made in [14,15]): the four-dimensional background metric can be Minkowski spacetime, de Sitter spacetime, or anti-de Sitter spacetime, depending both on the bulk cosmological constant and on the brane tensions. The cosmological constant problem (in the present context the fine-tuning between the bulk five-dimensional cosmological constant and the brane tension) remains the problem.

III. SUMMARY

We have given simple considerations on the brane gravity based on the dimensional reduction. We have shown that gravity on the brane belongs to a class of scalar-tensor gravity theory where the radion field plays the role of the Brans-Dicke scalar field. We have pointed out the possibility of the cosmological attractor mechanism on the positive tension brane and have also made a brief comment on the cosmological constant problem in the brane world scenario.

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4For stability of the classical Minkowski background, see [17].
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