Letter

Quantum simulation of Abelian Wu–Yang monopoles in spin-1/2 systems

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Abstract

With the help of the Berry curvature and the first Chern number ($C_1$), we both analytically and numerically investigate and thus simulate artificial magnetic monopoles formed in parameter space of the Hamiltonian of a driven superconducting qubit. The topological structure of a spin-1/2 system (qubit) can be captured by the distribution of Berry curvature, which describes the geometry of eigenstates of the Hamiltonian. Degenerate points in parameter space act as sources ($C_1 = 1$, represented by quantum ground state manifold) or sinks ($C_1 = -1$, represented by quantum excited state manifold) of the magnetic field. We note that the strength of the magnetic field (described by Berry curvature) has an apparent impact on the quantum states during the process of topological transition. It exhibits an unusual property that the transition of the quantum states is asymmetric when the degenerate point passes from outside to inside and again outside the manifold spanned by system parameters. Our results also pave the way to explore intriguing properties of Abelian Wu–Yang monopoles in other spin-1/2 systems.

Keywords: quantum simulation, fidelity, Berry curvature, Wu–Yang monopole

Supplementary material for this article is available online
(Some figures may appear in colour only in the online journal)

1. Introduction

In nature, magnetic poles always come in twos, a north and a south. Yet their electrostatic cousins, positive and negative charges, exist independently. In 1931, Dirac developed a theory of monopoles consistent with both quantum mechanics and the gauge invariance of the electromagnetic field [1]. The existence of a single Dirac monopole would not only address this seeming imbalance which appears in Maxwell’s equations, but would also explain the quantization of electric charge [1, 2]. To date, magnetic monopole analogues have been created in many different ways, such as superfluid $^3$He [3, 4], exotic spin ice [5–7] and spinor Bose–Einstein condensates [8–15]. Methods in [12–14] could be regarded as excellent examples of quantum simulation of magnetic monopoles. Quantum simulation was originally conceived by Feynman in 1982 [16], which permits the study of quantum systems that are difficult to study in laboratory. For this reason, simulators are especially aimed at providing insight about the behavior of more inaccessible systems appearing in nature. By introducing the point-like topological defects accompanied with a vortex filament into the spin texture of a dilute Bose–Einstein condensate, researchers provided an ideal analogue to Dirac monopole [10].

The topological properties of quantum systems play an extraordinary role in our understanding of the fundamental significance of natural phenomena. For example, the first Chern
number \((C_1)\) [17], which is a kind of robust topological invariants staying the same by small perturbations to the system can be used to help categorize physical phenomena. It is closely related to Berry phase that arises in cyclic adiabatic evolution of a system in addition to the dynamical counterpart [18]. The point-like topological defects as with degeneracy points in Hamiltonian parameter space of a spin-1/2 system could be viewed as the physical counterpart of topological invariant, which can be described by the first Chern number [19]. \(C_1\) can be extracted by integrating Berry curvature over the closed surface. Gritsev et al [20] proposed an effective method to measure the Berry curvature directly via the nonadiabatic response on physical observables to the rate of change of an external parameter. The method provides a powerful and generalized approach to explore topological properties in arbitrary quantum systems where the Hamiltonian can be written in terms of a set of externally controlled parameters. Taking into account this method, some researchers measured the topological transition \(C_1 = 1 \rightarrow 0\) in a single superconducting qubit [21], and others observed the topological transitions in interacting quantum circuits [22]. Experimental schemes have also been proposed to simulate the dynamical quantum Hall effect in a Heisenberg spin chain with interacting superconducting qubits [23], and to realize several-spin one-dimensional Heisenberg chains using magnetic nuclear resonance (NMR) simulators [24].

In this letter, we study the Wu–Yang monopoles [25], which remove out the ‘Dirac string’ by gauge transformation in parameter space of the Hamiltonian of a driven superconducting qubit for both geometry (Berry curvature) and topology (the first Chern number, \(C_1\)). The topological structure of the qubit can be captured by the distribution of Berry curvature, which describes the geometry of eigenstates of the Hamiltonian. We note that degenerate points in parameter space of the Hamiltonian act as the sources (sinks) of \(C_1\) and are analogues to magnetic monopoles \(g_{\text{m}}(g)\) \((C_1 = 1 \leftrightarrow g_0, C_1 = -1 \leftrightarrow g_5)\). We also note that the transition of quantum states is asymmetric during the process when the degeneracy passes from outside to inside and again out of the manifold spanned by system parameters, and the Berry curvature and the fidelity of quantum states have some interesting correlations during the process of topological transition. We give a preliminary explanation to it by introducing the notion of magnetic charges. This general method also can be simulated by other spin-1/2 systems. For example, it can be extended to that in an NMR system and is possible to experimentally investigate more intriguing properties of multi-monopoles, which could be used to construct new kinds of devices based on synthetic magnetic fields.

The configuration of this paper proceeds as follows. In section 2, we introduce the quantum geometric metric tensor and show its relation to the Berry curvature. In section 2.1, we describe how the first Chern numbers are obtained from Berry curvatures. In section 2.2, we outline an effective method to measure the Berry curvature directly via the nonadiabatic response on physical observables to the rate of change of an external parameter. In section 2.1, we describe how the first Chern numbers are obtained from Berry curvatures. As a useful example, we introduce a physical model for the simulation of the Abelian Wu–Yang monopoles by a driven superconducting qubit in section 3. In section 4 we explain how Wu–Yang monopoles differ from the Dirac monopoles through two kinds of quantum state manifolds. Finally, in section 5, we discuss some interesting correlations between the Berry curvature and the quantum states during the process of topological transition, we then describe the experimental feasibility of this theoretical method.

2. Geometry and topology in the specific state manifold

Consider a family of parameter-dependent Hamiltonian \(\tilde{\Lambda}\) for a quantum system and require \(\tilde{\Lambda}\) to depend smoothly on a set of parameters \(\tilde{\Lambda} = (\lambda_1, \lambda_2, \cdots) \in \mathcal{M}\) \((\mathcal{M}\) denotes the Hamiltonian parameters base manifold\) and act over the Hilbert space. The outline font \(\bar{a}\) and \(\textbf{2}\) indicate different indices. The distance between the two neighbouring specific (say, ground) state wave functions \(|\psi(\tilde{\Lambda})\rangle\) and \(|\psi(\tilde{\Lambda} + d\tilde{\Lambda})\rangle\) over \(\mathcal{M}\) is [26–28]

\[
d s^2 = 1 - |\langle \psi(\tilde{\Lambda}) | \psi(\tilde{\Lambda} + d\tilde{\Lambda}) \rangle|^2 = \sum_{\mu \nu} g_{\mu \nu} d\lambda^\mu d\lambda^\nu, \tag{2.1}\]

where the quantum (Fubini–Study) metric tensor \(g_{\mu \nu}\) associated with the ground state manifold is the symmetric real part of the quantum geometric tensor \(Q_{\mu \nu}\):

\[
Q_{\mu \nu} = \langle \partial_{\lambda} \psi(\tilde{\Lambda}) | \partial_{\lambda} \psi(\tilde{\Lambda}) \rangle - \langle \partial_{\lambda} \psi(\tilde{\Lambda}) | \psi(\tilde{\Lambda}) | \partial_{\lambda} \psi(\tilde{\Lambda}) \rangle, \tag{2.2}\]

\[
g_{\mu \nu} = \text{Re}(Q_{\mu \nu}) = (Q_{\mu \nu} + Q_{\nu \mu})/2, \tag{2.3}\]

with \(\partial_{\lambda}(\mu) \equiv \partial(\partial_{\lambda} \lambda^\mu)\). The Hermitian metric tensor \(Q_{\mu \nu}\) remains unchanged under arbitrary \(\lambda\)-dependent \(U(1)\) local gauge transformation of \(|\psi(\tilde{\Lambda})\rangle\). In another pioneering work [18], Berry introduced the concept of the geometric phase and the related geometric curvature (also called Berry phase and Berry connection). The Abelian Berry curvature \(F_{\mu \nu}\) is given by the antisymmetric imaginary part of \(Q_{\mu \nu}\):

\[
F_{\mu \nu} = -2i \text{Im}(Q_{\mu \nu}) = i(Q_{\mu \nu} - Q_{\nu \mu}) = \partial_{\lambda} A_{\mu} - \partial_{\lambda} A_{\nu}, \tag{2.4}\]

where \(A_{\mu(\nu)} = i(\langle \psi(\tilde{\Lambda}) | \partial_{\lambda} \psi(\tilde{\Lambda}) \rangle |\psi(\tilde{\Lambda})\rangle)\) is just the Berry connection.

2.1. The Chern–Gauss–Bonnet theorem

Let \(\mathcal{M}^m\) be a compact oriented Riemann manifold of even dimension \((m = 2n)\) and define on \(\mathcal{M}^m\) a global m form, the Chern–Gauss–Bonnet (C–G–B) formula says that

\[
\int_{\mathcal{M}^m} \epsilon(\Omega) = \chi(\mathcal{M}), \tag{2.5}\]

where \(\epsilon(\Omega)\) is the Euler class, \(\chi(\mathcal{M}) \equiv 2(1 - g)\) is the integer Euler characteristic describing the topology of the smooth manifold \(\mathcal{M}\) and \(g\) is the genus that also can be considered as the number of holes of the manifold. As shown in figure 1, two simplest closed manifolds are taken for example. In the lower dimensional version, the C–G–B theorem reduces to the Gauss–Bonnet (G–B) theorem. The Fubini–Study tensor \(g_{\mu \nu}\)
defines a Riemannian manifold related to the ground state. In particular, the structure of the Riemannian manifold provides a different topological integer, given by using the G–B theorem to the metric tensor in quantum version [29]:

\[
\frac{1}{2\pi} \left( \int_M K \, dS + \int_{\partial M} \kappa_\parallel \, d\ell \right) = \chi(M),
\]

(2.6)

where \( K \) (Gauss curvature), \( dS \) (area element), \( \kappa_\parallel \) (geodesic curvature), and \( d\ell \) (line element) are geometric invariants, meaning that they remain unchanged under any change of variables. The left side of equation (2.6) are the bulk \((M)\) and boundary \((\partial M)\) contributions to \( \chi(M) \) of the Riemannian manifold. If the manifold \( M \) is compact and without boundary (closed), then the boundary Euler integrals vanish, as we prove in detail in appendix I of the supplementary data. In this letter we will focus only on the two-dimensional \( (m = 2 \) in equation (2.5)) version and the dimensionality here is that of parameter space (i.e. \( S^2 \)) which is composed by the polar angle \( \theta \) and the azimuthal angle \( \phi \) of a magnetic field applied to a spin-1/2 system. Then we get the global G–B theorem on the sphere

\[
\frac{1}{2\pi} \oint_{S^2} K \, dS = \chi(S^2).
\]

(2.7)

To catch the significance of the first Chern number \( C_1 \), we need to adiabatically change these parameters around a loop that bounds a sphere \( S^2 \) to acquire a Berry phase, which can be written as

\[
\varphi_{\text{Berry}} = \int \int_{S^2} F_{\mu\nu} dS_{\mu\nu} = \int \int_{S^2} \tilde{F} \cdot d\tilde{S},
\]

(2.8)

where \( dS_{\mu\nu} \) is a directed surface element, \( \tilde{S} \) is a vector normal to the sphere \( S^2 \) and \( \tilde{F} \) is a vector known as the Berry curvature analogous to the magnetic field in electromagnetism, which is given by the off-diagonal components of the electromagnetic tensor \( F_{\mu\nu} \) see in equation (2.4). For example, the Berry curvatures \( F_{\mu\nu}^{(0)} \) and \( F_{\mu\nu}^{(2)} \) only have off-diagonal components.

As we all know by now, Berry phase depends on the local gauge choice \( |\psi_i\rangle \rightarrow e^{i\varphi(\theta, \phi)}|\psi_i\rangle \), where \( |\psi_i\rangle \) is a certain eigenstate in this letter (subscript \( i = 0, 1 \), showing that the Berry curvature is gauge invariant. Therefore, we obtain the integral

\[
C_1 = \frac{1}{2\pi} \oint_{S^2} F_{\mu\nu} dS_{\mu\nu} = \frac{1}{2\pi} \oint_{S^2} \tilde{F} \cdot d\tilde{S},
\]

(2.9)

which is a kind of robust topological invariant known as the first Chern number, and it could be viewed as counting the number of times an eigenstate circles around a sphere in the Hilbert space [21].

### 2.2. Measuring the Berry curvature

In analogy to electrodynamics, the local gauge-dependent Berry connection \( A_\mu \) can never be physically observed, while Berry curvature \( F_{\mu\nu} \) is gauge-invariant and may be related to a physical observable that manifests the local geometric property of the eigenstates in the parameter space. The first Chern number reveals the global topological property of such a Hamiltonian manifold. In fact, \( C_1 \) exactly counts the number of degenerate points enclosed by parameter space \( S^2 \), see appendix II of the supplementary data, where we endow it with physical meaning by using the conception of the magnetic monopole. We substitute \( A_\mu \) into \( F_{\mu\nu}(\tilde{F}) \) and rewrite the Berry curvature as

\[
F_{\mu\nu} = i \sum_{n=0} \frac{\langle \psi_0 | \partial_\mu \hat{H} | \psi_n \rangle \langle \psi_n | \partial_\nu \hat{H} | \psi_0 \rangle}{(E_n - E_0)^2} - (\nu \leftrightarrow \mu),
\]

(2.10)

where \( E_n \) and \( |\psi_n\rangle \) are the \( n \)th eigenvalue and its corresponding eigenstate of the Hamiltonian \( \hat{H} \), respectively. Equation (2.10) indicates that degeneracies are some singularities that will contribute nonzero terms to \( C_1 \) in equation (2.9).

In order to extract the Chern number of closed manifolds in the parameter space of the two-level system Hamiltonian, we analytically describe a simple topological structure of a superconducting qubit driven by a microwave field. In [20], it states that Berry curvature can be extracted from the linear response of the qubit to nonadiabatic manipulations of its Hamiltonian \( \hat{H}(\mu = \theta, \nu = \phi) \), which leads to a general force

\[
M_\theta \equiv -\partial_\nu |\psi(t)\rangle \langle \partial_\mu |\psi(t)\rangle, \quad \text{given by} \quad [20, 21, 30]
\]

\[
M_\phi = \text{const} + \nu \hbar F_{\psi\phi} + O(\nu^2),
\]

(2.11)

where \( \nu \) is the rate of change of the parameter \( \theta \) (quench velocity) and \( F_{\psi\phi} \) is a component of the Berry curvature tensor. To neglect the nonlinear term, the system parameters should be ramped slowly enough or quasi-adiabatically.

### 3. From Dirac monopole to Wu–Yang monopole

In order to discuss in more detail the Dirac monopole, we first consider a monopole with the magnetic field sitting at the origin

\[
\nabla \cdot \hat{B} = 4\pi \delta(\vec{r}).
\]

(3.1)

It follows from \( \nabla^2 (1/r) = -4\pi \delta(\vec{r}) \) and \( \nabla (1/r) = -\vec{r}/r^3 \) that the solution of this equation is...
If we transform it to the Coulomb gauge, the magnetic field lines emerge radially outwards in the same way as electric field lines emerge from an electric point charge, so that the endpoint acts as a magnetic monopole. (b) Wu–Yang monopole. By selecting different coordinate systems to eliminate the singularity of Dirac string.

For example, let us introduce the singular vector potential with its positive pole which has strength $g$ at the origin [31]. Nitely long and thin solenoid placed along the negative $z$ axis is possible, it may reasonably consider the field due to an infinitely thin vector potential $A = g\vec{r}/r^3$, where $g = \pm 1/2$. The magnetic flux $\Phi$ is obtained by integrating over a sphere $S^2$ of radius $r$ so that

$$\Phi = \oint_{S^2} \vec{B} \cdot d\vec{S} = 4\pi g. \tag{3.3}$$

But if $\vec{B} = \nabla \times \vec{A}$, this integral would have to vanish. Thus magnetic vector potential $\vec{A}$ cannot exist everywhere on $S^2$, even though $\nabla \cdot \vec{B}$ is only non-zero at the origin, and the best we can do is to find an $\vec{A}$ defined everywhere except on a line joining the origin to infinity, such that $\vec{B} = \nabla \times \vec{A}$. To see this is possible, it may reasonably consider the field due to an infinitely long and thin solenoid placed along the negative $z$ axis with its positive pole which has strength $g$ at the origin [31]. For example, let us introduce the singular vector potential

$$A_r = A_\theta = 0, \quad A_\phi = \frac{g(1 - \cos \theta)}{r \sin \theta}, \tag{3.4}$$

and verify that

$$\nabla \times \vec{A} = g\vec{r}/r^3 + \vec{B}_s, \tag{3.5}$$

where $\vec{B}_s$ is the singular vector field along $z$-axis, with the expression

$$\vec{B}_s = \begin{cases} 4\pi g\delta(x)\delta(y)\delta(z), & z < 0, \theta = \pi \\ 0, & z > 0, \theta = 0 \end{cases} \tag{3.6}$$

The singularity along the $z$-axis is called the Dirac string and reflects the poor choice of the coordinate system, as is shown in figure 2(a). This magnetic field differs from $\vec{B}$ only by the singular magnetic flux along the solenoid but it is clearly source-free; while at the origin, $\vec{B}$ vanishes. Thus it may be represented by a vector potential, $\vec{A}$ (say), everywhere and we may write

$$\vec{B} = \nabla \times \vec{A} - \vec{B}_s. \tag{3.7}$$

Now, let us describe how these monopoles differ from the standard Dirac monopoles. Under the condition of quantum excited state manifold, and from equations (3) and (4) in appendix I of the supplementary data, we obtain the magnetic field of the south monopole

$$F_{\theta \phi}^{(S)} = -2\text{Im}[Q_{ho}^2] = \frac{1}{2} \begin{pmatrix} 0 & -\sin \theta \\ \sin \theta & 0 \end{pmatrix}. \tag{3.8}$$

The corresponding Berry curvature is $F_{\theta \phi}^{(S)} = -1/2 \sin \theta d\theta \wedge d\phi$, which is a symplectic form on $S^2$. If we transform it to the Coulomb-like magnetic field

$$F_{\theta \phi}^{(S)}(r, \theta, \phi) = g_\pi r^3 = -\vec{r}/2r^3, \tag{3.9}$$

it turns out to be the magnetic field originating from a monopole located at the origin with magnetic charge $g_S = 1/2$ [19]. Similarly, if we take another eigenstate which corresponds to the quantum ground state manifold

$$|\psi_0(\theta, \phi)\rangle = -\sin(\theta/2)|0\rangle + e^{i\phi} \cos(\theta/2)|1\rangle, \tag{3.10}$$

where we set $\sin(\theta/2) = -\frac{\sqrt{1 + (E_0 - \frac{\Delta}{2})^2}}{1 + (E_0 - \frac{\Delta}{2})^2}$, and $\cos(\theta/2) = -(E_0 - \frac{\Delta}{2})\sqrt{1 + (E_0 - \frac{\Delta}{2})^2}$, then we have the magnetic field of the north monopole

$$F_{\theta \phi}^{(N)} = \frac{1}{2} \begin{pmatrix} 0 & \sin \theta \\ -\sin \theta & 0 \end{pmatrix}. \tag{3.11}$$

The corresponding Berry curvature is $F_{\theta \phi}^{(N)} = 1/2 \sin \theta d\theta \wedge d\phi$, and the magnetic field

$$F_{\theta \phi}^{(N)}(r, \theta, \phi) = g_\pi r^3 = \vec{r}/2r^3, \tag{3.12}$$

with the magnetic charge $g_N = 1/2$.

Wu and Yang [25] noticed that it may employ more than one vector potential to describe monopoles. For example, we may avoid singularities if we adopt $\vec{A}_N$ in the northern hemisphere and $\vec{A}_S$ in the southern hemisphere of the sphere $S^2$ surrounding the monopole, as depicted in figure 2(b). It shows that the vector potential $\vec{A}_N$ in the region of $S^2_N$ can be expressed as

$$(A_r)_{\theta} = (A_\theta)_{\phi} = 0, \quad (A_\phi)_{\theta} = \frac{g(1 - \cos \theta)}{r \sin \theta}, \tag{3.13}$$

and the vector potential $\vec{A}_S$ in the region of $S^2_S$ can be expressed as

$$(A_r)_{\theta} = (A_\theta)_{\phi} = 0, \quad (A_\phi)_{\theta} = -\frac{g(1 + \cos \theta)}{r \sin \theta}. \tag{3.14}$$

Obviously, the two vector potentials yield the magnetic field $\vec{B} = g\vec{r}/r^3$, which is non-singular everywhere on the sphere [32].

Of special note is the fact that the magnetic monopoles we simulate here are the Abelian Wu–Yang monopoles, which
by selecting different coordinate systems to eliminate the singularity of Dirac string. The two coordinate systems are characterized by the choice of two different Berry curvatures, see more in appendix II of the supplementary data.

4. Physical model for implementation

As we have mentioned above, the degenerate points emerging from the Berry curvature \( F_{\mu\nu} \) act as the sources (the north magnetic charge \( g_N \)) and sinks (the south magnetic charge \( g_S \)) of \( C_4(\pm1) \) and are analogous to Wu–Yang monopoles in parameter space. We reconsider the proposal that use a superconducting transmon qubit described in [21]. As seen in figure 3, where an anharmonicity of 280 MHz makes the qubit an effective two-level system in the parameter scope. In the rotating frame of a microwave drive with frequency \( \omega_{m0} \), the Hamiltonian for the qubit can be written as (\( \hbar \equiv 1 \)) [33, 34]

\[
\hat{H} = 1/2[\Delta \hat{\sigma}_z + \Omega \hat{\sigma}_x \cos \phi + \Omega \hat{\sigma}_y \sin \phi],
\]

where \( \Delta = \omega_m - \omega_{m0}, \hat{\sigma}_i(i = x, y, z) \) is the Pauli spin matrix, \( \phi \) and \( \Omega \) are the phase of the drive tone and the amplitude of the drive tone as the Rabi frequency, respectively.

By changing these parameters (\( \Delta \) and \( \Omega \)), we can create arbitrary single-qubit Hamiltonians that can be represented in a set of parameters as an ellipsoidal manifold. The eigenstates of this Hamiltonian are

\[
|\psi_0\rangle = \frac{\Omega/2 |0\rangle}{\sqrt{\frac{\Omega^2}{4} + (E_0 - \frac{\Delta}{2})^2}} - e^{i\phi} \frac{\Omega/2 |1\rangle}{\sqrt{\frac{\Omega^2}{4} + (E_0 - \frac{\Delta}{2})^2}},
\]

\[
|\psi_1\rangle = \frac{\Omega/2 |0\rangle}{\sqrt{\frac{\Omega^2}{4} + (E_1 - \frac{\Delta}{2})^2}} + e^{i\phi} \frac{\Omega/2 |1\rangle}{\sqrt{\frac{\Omega^2}{4} + (E_1 - \frac{\Delta}{2})^2}},
\]

where \( |0\rangle = |e\rangle = (1, 0)^T \) is the excited state and \( |1\rangle = |g\rangle = (0, 1)^T \) is the ground state. The corresponding eigenvalues of the eigenstates \( |\psi_{01}\rangle \) are \( E_{01} = \pm \frac{1}{2} \sqrt{\Omega^2 + \Delta^2} \).

We notice that for \( E_1 = E_0 \), equation (2.10) clearly shows that degeneracies are some singular points that will contribute nonzero terms to \( C_4 \) in equation (2.9). In particular, with the choice

\[
\Delta = \Delta_1 \cos \theta + \Delta_2, \quad \Omega = \Omega_0 \sin \theta,
\]

the Hamiltonian can be presented in parameter space as an ellipsoidal manifold with cylindrical symmetry about the \( z \)-axis [21]. Here, we set ellipsoids of size \( \Delta_q = 2\pi \times 30 \text{ MHz} \), and \( \Omega_0 = 2\pi \times 10 \text{ MHz} \). The topological properties are independent of deformations of the manifold that includes the degenerate point and the choice of these particular parameters does not really matter.

Figure 4(a) depicts an implementable pulse sequence used to measure the Berry curvature. We respectively initialize the qubit in its bare ground state \( |g\rangle \) and bare excited state \( |e\rangle \) at \( \theta(t = 0) = 0 \) (this method works for arbitrary eigenstates of the initial Hamiltonian, so the particular state targeted is irrelevant), fix \( \phi(t) = 0 \), and linearly ramp the angle \( \theta(t) = \pi t/t_{\text{ramp}} \) in time, stopping the ramp at various times \( t_{\text{ramp}} \) to execute qubit tomography. From equation (2.10), the Berry curvature reads

\[
F_{\phi \theta} = \frac{\langle \partial_{\theta} \hat{H} \rangle}{\nu_{\theta}} = \frac{\Omega_0 \sin \theta}{2 \nu_{\phi}} \langle \hat{\sigma}_y \rangle,
\]

where \( \nu_{\theta} = \nu(t) = \pi t/t_{\text{ramp}} \). Figure 4(b) shows the results of different Berry curvatures with \( \Omega_1 \) and \( \Omega_2 \), respectively, for a protocol with \( t_{\text{ramp}} = 1 \mu s \) and \( \Delta_2 = 0 \). We extract the Berry curvatures \( F_{\phi \theta} \) from the measured values of \( \langle g|\hat{\sigma}_y|g\rangle \) and...
The Berry curvature is positive when the curving of the surface is elliptic. The sharper the elliptic curving, the greater the Berry curvature. And if the surface starts to show hyperbolic such as a saddle, then the Berry curvature becomes negative, and the sharper the hyperbolic curving of the surface, the smaller the Berry curvature, just the same as the Gauss curvature in figure 1.

To induce a topological transition in the qubit, the detuning offset $\Delta_2$ is first changed. At the same time, the ground and excited states evolution are quantitatively modified. But for $|\Delta_2| < |\Delta_1|$, the corresponding Berry curvature as we see in figure 5(a) shows the Berry curvature acts like the magnetic field produced by a north magnetic charge $g_N$ (sources), while figure 5(b) shows that it acts like the magnetic field produced by a south magnetic charge $g_S$ (sinks). The scale of the Berry curvature corresponds to the strength of the magnetic field and it falls with the square of distance between the manifold and the magnetic poles. However, for $|\Delta_2| > |\Delta_1|$, it gives the zero Berry curvature, meaning that the system undergoes a topological transition at $|\Delta_2| = |\Delta_1|$. Such a transition only occurs when the Berry curvature becomes ill defined at the point $\Delta = \Omega = 0$ in equation (2.10).

By integrating equation (4.5), we obtain the first Chern number

$$C_1 = \frac{1}{2\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi F_{\theta\phi} = \int_0^\pi F_{\theta\phi} d\theta.$$  

(4.6)

The measured Chern number $C_1$ is plotted in figure 6(b), showing a relatively sharp transition at the expected value.
Figure 7. The fidelity of the target states versus \(\theta/\pi\) and \(\Delta g/\Delta_{1}\). The initial state \(|e\rangle\) (will evolve within \(|\psi_0\rangle\)) is set in (a) and (c). The initial state \(|g\rangle\) (will evolve within \(|\psi_1\rangle\)) is set in (b) and (d). In (a) and (b), the density matrix of the target state is \(|\rho(t_f)\rangle = |g\rangle\langle g|\) while in (c) and (d) the density matrix is \(|\rho(t_f)\rangle = |e\rangle\langle e|\). The parameters chosen here are \(\Delta_{1} = 2\pi \times 30\) MHz, \(\Delta_{2} = 2\pi \times 10\) MHz, and \(\Delta_{2}\) ramps from \(-2\Delta_{1}\) to \(2\Delta_{1}\). The Berry curvature only has relatively strong influence around \(|\Delta g/\Delta_{1}| = 1\) which is shown circled in (a) and (c).

\(|\Delta g/\Delta_{1}| = 1\). We find that the topological transition in the elliptical manifold (the green line) is sharper (faster) than that in the sphere manifold (the red line) shown in figure 6(b), and it shows that the topological invariant \(C_1\) is strongly robust against variations in Hamiltonian parameters, such as in Rabi frequency \(\Omega_n\) and in detuning \(\Delta_i\). The topological transition corresponds to degeneracies moving from outside to inside and again outside the elliptical manifold. In other words, the Chern number is nonzero as long as there exists Berry curvature. From this point of view, we can set up the corresponding relation between topological invariants and magnetic monopoles [10, 21, 22]. Then we can draw such a conclusion, as shown in figure 6, with a formula [19]

\[ C_1 = \text{magnetic number} = \pm 1, \tag{4.7} \]

where ‘1’ is the number of the degeneracy points in parameter space of the Hamiltonian, and the sign ‘±’ corresponds to the polarity of the magnetic charge in parameter space (\(C_1 = +1 \leftrightarrow g_1, C_1 = -1 \leftrightarrow g_2\)).

5. Results and discussion

In equation (4.4), \(\theta = 0\) and \(\pi\) corresponds to \(\Delta = \Delta_1 + \Delta_2\) and \(\Delta = -\Delta_1 + \Delta_2\), respectively. For the case with \(\Delta = 0\) and \(\Omega = 0\), i.e. the microwave drive induces the resonant transition between the two states \(|0\rangle\) and \(|1\rangle\) of the qubit, the two eigenstates in equation (4.2) and equation (4.3) become a degenerate state \(|\psi_0\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)\).

Based on this point, we track and investigate the change of the quantum states accompanied with the change of the Berry curvatures. In figure 7, the fidelity of the target state \(|g\rangle\) and \(|e\rangle\) is plotted versus \(\theta/\pi\) and \(\Delta g/\Delta_{1}\) at \(\theta = \pi\). (a) The fidelity: \(|\psi_0\rangle^2\) or \(|\psi_1\rangle^2\). (b) The fidelity: \(|\psi_0\rangle^2\) or \(|\psi_1\rangle^2\). (c) The fidelity of the degenerate state: \(|\psi_0\rangle^2\). (d) The fidelity of the degenerate state: \(|\psi_1\rangle^2\).

is plotted versus \(\theta/\pi\) and \(\Delta g/\Delta_{1}\), where the fidelity is defined as \(f = (\langle\psi_j|\hat{\rho}(t_f)|\psi_j\rangle)^2\) (\(j = 0, 1\)). We note that the quantum state flips at \(\Delta g/\Delta_{1} = -1\), when the monopole in parameter space passes from outside to inside the spherical manifold, except the area where the Berry curvatures (the magnetic fields) exist. However, the quantum state does not flip at \(\Delta g/\Delta_{1} \geq 1\), because the Berry curvatures no longer exist in the manifold and the Gauss theorem of magnetic field turns into the Stokes theorem, see in (a) and (c) of figure 6.

We note that, the strength of the magnetic field (Berry curvature) has an apparent impact on the quantum state \(|\psi_0\rangle\) in (b) and (d) of figure 7, while it only has relatively strong influence on the state \(|\psi_i\rangle\) around \(|\Delta g/\Delta_{1}| = 1\) in (a) and (c) of figure 7 (the dashed circle).

In order to illustrate the change of the quantum states in the process of topological transition in more detail, we choose a special position at \(\theta = \pi\), and thus get \(\Delta_{1} = \Delta_{2}\). For such a case, the initial state evolves to the degenerate state \(|\psi_0\rangle\). Figure 8(a) depicts the status of quantum states in (a) and (d) of figures 7, and figure 8(b) depicts the status of quantum states in (b) and (c) of figure 7. From figures 8(c) and (d), we note that the fidelity is fluctuating around \(|\Delta g/\Delta_{1}| = 1\). We attribute this interesting phenomenon to the influence of the magnetic fields resulting from the magnetic charges. When the charges pass from inside to outside the Hamiltonian manifold, the quantum states influenced by the Berry curvatures will cause ripples in the Hilbert space, a detailed discussion will be presented in future works. While for the position at \(\Delta g/\Delta_{1} = -1\), there is no such apparent fluctuating because the quantum states still have not been affected by the magnetic field. More vividly speaking, the quantum states have not yet been ‘magnetized’ by the magnetic monopoles. Actually, according to these phenomena, we find a new way to control the evolution of system quantum states by manipulating (moving) the monopoles (degenerate points) in the manifolds.
Hereinbefore upwards, our main consideration about how to simulate Abelian Wu–Yang monopoles in parameter space just relies on a driven superconducting qubit. However, this general method also could be simulated by other spin-1/2 systems, such as a NMR system in a synthetic magnetic field. A simple experimental scheme is shown in appendix III of the supplementary data.

6. Conclusion

We have simulated the Abelian Wu–Yang monopoles in parameter space of the Hamiltonian of a superconducting qubit controlled by a microwave drive for both geometry (Berry curvature) and topology (Chern number). The topological structure of the qubit can be captured by the distribution of Berry curvature, which describes the geometry of the eigenstates of the Hamiltonian. We note that during the process of topological transition, the Berry curvature and the fidelity of quantum states have some interesting correlations due to the influence of the magnetic fields resulting from the magnetic charges. We also note that the quantum state flips at the position where the topological transition occurs, when the monopole in parameter space passes from outside to inside and again outside the spherical manifold, except the area where the Berry curvatures (the magnetic fields) exist. This phenomenon might provide a promising perspective to flexibly manipulate the qubit states by designing the specific synthetic magnetic fields.

Degenerate points in parameter space of the Hamiltonian act as the sources (sinks) of $C_1$ and are analogues to magnetic monopoles. We also note that the transition of quantum states is asymmetric during the process when the monopole passes from outside to inside and again outside the Hamiltonian manifold. For example, when the monopole passes from inside to outside the Hamiltonian manifold, the quantum states influenced by the Berry curvatures cause ripples in the Hilbert space. However, when the monopole passes from outside to inside the Hamiltonian manifold, there is no such apparent fluctuating. We give a preliminary explanation to this interesting phenomenon by introducing the notion of magnetization distribution (Berry curvature) and topology (Chern number). The topological transition, the Berry curvature and the contribution of Berry curvature, which describes the geometry of the eigenstates of the Hamiltonian. We note that during the process of topological transition, the Berry curvature and the fidelity of quantum states have some interesting correlations due to the influence of the magnetic fields resulting from the magnetic charges. We also note that the quantum state flips at the position where the topological transition occurs, when the monopole in parameter space passes from outside to inside and again outside the spherical manifold, except the area where the Berry curvatures (the magnetic fields) exist. This phenomenon might provide a promising perspective to flexibly manipulate the qubit states by designing the specific synthetic magnetic fields.

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