All pairs suffice

Curtis Nelson*, Bryan Shader
Department of Mathematics, University of Wyoming, Laramie, Wyoming, USA

Abstract
A P-set of a symmetric matrix $A$ is a set $\alpha$ of indices such that the nullity of the matrix obtained from $A$ by removing rows and columns indexed by $\alpha$ is $|\alpha|$ more than that of $A$. It is known that each subset of a P-set is a P-set. It is also known that a set of indices such that each singleton subset is a P-set need not be a P-set. This note shows that if all pairs of vertices of a set with at least two elements are P-sets, then the set is a P-set.

Keywords: acyclic matrix, eigenvalues, P-vertex, P-set, Parter-vertex, Parter-Set

2000 MSC: 15A18, 05C50

1. Introduction

Throughout this note, all matrices are real. Let $A = [a_{ij}]$ be an $n \times n$ symmetric matrix. The graph, $G(A)$, of $A$ has vertices $1, 2, \ldots, n$ where $\{i, j\}$ is an edge in the graph if and only if $a_{ij} \neq 0$. Let $\tau$ and $\alpha$ be subsets of $\{1, 2, \ldots, n\}$. Then $A[\tau, \alpha]$ denotes the submatrix of $A$ consisting of the rows indexed by $\tau$ and the columns indexed by $\alpha$. When $\tau = \alpha$, $A[\alpha]$ is used in place of $A[\alpha, \alpha]$. Similarly, $A(\alpha)$ denotes the principal submatrix of $A$ obtained by deleting the rows and columns indexed by $\alpha$. Also, $\nu(A)$ denotes the nullity of $A$ and $\text{RS}(A)$ denotes the row space of $A$.

Vertex $i$ is a downer vertex of $A$ if $\nu(A) - 1 = \nu(A(i))$. Vertex $i$ is a P-vertex of $A$ if $\nu(A) + 1 = \nu(A(i))$. A set of vertices indexed by a set $\alpha$ is

*This research was supported by the University of Wyoming.
*Corresponding author.

Email addresses: curtisgn@gmail.com (Curtis Nelson), bshader@uwyo.edu (Bryan Shader)

Preprint submitted to . December 9, 2014
a *P-set* of $A$ if $\nu(A) + |\alpha| = \nu(A(\alpha))$. In this case, we also say that $\alpha$ is a P-set of $A$. It is known that every non-empty subset of a P-set is a P-set [5, Proposition 5]. It is also known that a set of P-vertices is not necessarily a P-set [2, Example 2.4], [4, Example 4.6]. In [6], we studied the maximal P-sets of matrices whose graphs are trees and also proved that for a symmetric matrix $A$ whose graph is a tree, a set of indices is a P-set of $A$ if and only if every subset of cardinality two (i.e. each pair) is a P-set of $A$. This conclusion came as a corollary of a complicated, technical theorem that identifies the maximal P-sets of $A$. In this note, we show that the previous result holds for all symmetric matrices $A$, regardless of the graph of $A$; i.e. we show that that if $A$ is a symmetric matrix, a set $\alpha$ of at least two indices is a P-set of $A$ if and only if each pair of $\alpha$ is a P-set of $A$.

2. All pairs suffice

We first recall the following known result, Jacobi’s Determinant Identity [1, p. 24].

**Theorem 1.** Let $A$ be an invertible $n \times n$ matrix and let $\alpha$ be a subset of \{1, 2, ..., $n$\}. Then $\det(A[\alpha]) = \det(A^{-1}(\alpha)) \det(A)$.

We next prove a result that shows useful relationships between the linear dependencies among the rows of a matrix and the downer vertices and P-vertices of the matrix.

**Proposition 2.** Let $A$ be a symmetric matrix. Then

(a) $i$ is a downer vertex if and only if row $i$ is a linear combination of other rows of $A$.

(b) If $\alpha$ is a set of P-vertices, then the rows of $A$ indexed by $\alpha$ are linearly independent.

*Proof.* Vertex $i$ is a downer vertex of $A$ if and only if $\text{rank}(A) = \text{rank}(A(i))$. This occurs if and only if row $i$ is a linear combination of other rows of $A$.

For (b), we prove the contrapositive. Assume that the rows of $A$ indexed by $\alpha$ are linearly dependent. Then there is an $i \in \alpha$ such that row $i$ is a linear combination of rows of $A$ indexed by $\alpha \setminus \{i\}$. By (a), $i$ is a downer vertex of $A$ and hence is not a P-vertex.

The following gives useful characterizations of a P-set.
Theorem 3. Let
\[ A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix} \]
be a symmetric \( n \times n \) matrix where \( B \) is \( k \times k \). Then the following are equivalent:
(a) \( \{1, 2, \ldots, k\} \) is a P-set
(b) \( \text{rank}(A) = \text{rank}(D) + 2k \)
(c) \( \{x : x^TC \in \text{RS}(D)\} = \{0\} \)

Proof. Note that \( \{1, 2, \ldots, k\} \) is a P-set of \( A \)
\[ \begin{align*}
&\iff \nu(D) = \nu(A) + k \\
&\iff n - \nu(D) = n - \nu(A) - k \\
&\iff \text{rank}(D) + k = \text{rank}(A) - k \\
&\iff \text{rank}(A) = \text{rank}(D) + 2k
\end{align*} \]
Thus (a) and (b) are equivalent.

Now note
\[ \text{rank}(A) \leq k + \text{rank} \left( \begin{bmatrix} C \\ D \end{bmatrix} \right) \leq k + \text{rank}(D) = 2k + \text{rank}(D) \]
with equality if and only if (b) holds. Hence (b) and (c) are equivalent. \( \square \)

We use Theorem 3 to give sufficient conditions for a P-set of a principal submatrix to be a P-set of the entire matrix.

Corollary 4. Let \( A \) be an \( n \times n \) symmetric matrix, \( \alpha \) be a set of P-vertices of \( A \), and \( \beta \) be a subset of \( \{1, 2, \ldots, n\} \) such that \( \alpha \subseteq \beta \) and the rows of \( A \) indexed by \( \beta \) are a basis of \( \text{RS}(A) \). If \( \alpha \) is a P-set of \( A[\beta] \), then \( \alpha \) is a P-set of \( A \).

Proof. Assume \( \alpha \) is a P-set of \( A[\beta] \). Without loss of generality,
\[ A = \begin{bmatrix} B & C \\ C^T & D \\ E^T & F^T & G \end{bmatrix}, \quad A[\alpha] = B \text{ and } A[\beta] = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}. \]
Since \( \alpha \) is a P-set of \( A[\beta] \), by Theorem 3 \( \{x : x^TC \in \text{RS}(D)\} = \{0\} \). Hence \( \{x : x^T[C \quad E] \in \text{RS}(D \quad F)\} = \{0\} \).
We now show that
\[ \text{RS}([D \ F]) = \text{RS}\left(\begin{bmatrix} D & F \\ F^T & G \end{bmatrix}\right). \]

Since the rows of \( A \) indexed by \( \beta \) are a basis of \( \text{RS}(A) \),
\[ \text{RS}([E^T \ F^T \ G]) \subseteq \text{RS}\left(\begin{bmatrix} B & C & E \\ C^T & D & F \end{bmatrix}\right). \]

Since \( \alpha \) is a set of P-vertices of \( A \), by Proposition \[2\] no row of \( [B \ C \ E] \) is a linear combination of other rows of \( A \). It follows that \( \text{RS}([E^T \ F^T \ G]) \subseteq \text{RS}([C^T \ D \ F]) \). In particular, each row of \( [F^T \ G] \) is in \( \text{RS}([D \ F]) \) and hence
\[ \text{RS}([D \ F]) = \text{RS}\left(\begin{bmatrix} D & F \\ F^T & G \end{bmatrix}\right). \]

Therefore
\[ \left\{ x : x^T [C \ E] \in \text{RS}\left(\begin{bmatrix} D & F \\ F^T & G \end{bmatrix}\right) \right\} = \{0\} \]
and thus, by Theorem \[3\] \( \alpha \) is a P-set of \( A \).

We now show that all pairs suffice in the case the matrix is nonsingular.

**Theorem 5.** Let \( M \) be a nonsingular, symmetric \( n \times n \) matrix and \( \alpha \) be a set of cardinality at least two such that each pair of \( \alpha \) is a P-set of \( M \). Then \( \alpha \) is a P-set of \( M \).

**Proof.** Let \( N = M^{-1} \). Since each non-empty subset of a P-set is a P-set, the hypotheses imply that \( \det M(i) = 0 \) for each \( i \in \alpha \). By Theorem \[4\] \( \det N[i] = 0 \) for each \( i \in \alpha \). Thus \( n_{ij} = 0 \) for each \( i \in \alpha \). Also, for each subset \( \gamma \) of \( \alpha \) of cardinality 2, we know that \( \det M(\gamma) = 0 \). Thus \( \det N[\gamma] = 0 \). Since the diagonal entries of the \( 2 \times 2 \), symmetric matrix \( N[\gamma] \) are 0, \( N[\gamma] = 0 \) for each such \( \gamma \). Hence \( N[\alpha] = 0 \).

Now consider a subset \( \tau \) of \( \{1, 2, \ldots, n\} \setminus \alpha \) of cardinality at most \( |\alpha| - 1 \). It is easy to verify that the columns of \( N[\tau, \alpha] \) are linearly dependent, and hence the columns of \( N[\tau \cup \alpha] \) indexed by \( \tau \) are linearly dependent. We conclude that \( \det N[\tau \cup \alpha] = 0 \). Thus, by Theorem \[4\] every principal minor of \( M(\alpha) \) of order at least \( n - 2|\alpha| + 1 \) is 0. Since \( M(\alpha) \) is a symmetric matrix of order \( n - |\alpha| \), the nullity of \( M(\alpha) \) is greater than \( |\alpha| - 1 \). We conclude that \( \alpha \) is a P-set of \( M \).
Lastly, we prove the main result of the note.

**Theorem 6.** Let $A$ be an $n \times n$ symmetric matrix and $\alpha$ a set of indices of cardinality at least two. Then $\alpha$ is a P-set of $A$ if and only if every pair of $\alpha$ is a P-set of $A$.

**Proof.** First assume $\alpha$ is a P-set of $A$. Since every subset of a P-set is a P-set, each pair is a P-set of $A$.

Conversely, assume each pair of $\alpha$ is a P-set of $A$. By Proposition 2 the rows of $A$ indexed by $\alpha$ are linearly independent. Thus there exists a set $\beta$ such that $\alpha \subseteq \beta$, $|\beta| = n - \nu(A)$, and $A[\beta]$ is nonsingular. It follows that the rows of $A$ indexed by $\beta$ are a basis of the row space of $A$.

Consider a subset $\gamma$ of $\alpha$ of cardinality 2. By assumption, $A(\gamma)$ has nullity $\nu(A) + 2$. Thus $A[\beta \setminus \gamma]$, which is obtained from $A(\gamma)$ by deleting $\nu(A)$ rows and columns, has nullity at least $\nu(A) + 2 - \nu(A) = 2$. Hence for each pair $\gamma$ of $\alpha$, $\gamma$ is a P-set of $A[\beta]$. By Theorem 5 $\alpha$ is a P-set of the nonsingular $A[\beta]$. By Corollary 4 $\alpha$ is a P-set of $A$.

Thus if $A$ is a symmetric matrix and $\alpha$ is a set of indices of cardinality at least two, then the condition that $\nu(A) + 2 = \nu(A\{i, j\})$ for each subset $\{i, j\}$ of $\alpha$ implies $\alpha$ is a P-set of $A$. We note that if $\alpha$ is a set of P-vertices of $A$, then in order to show $\alpha$ is a P-set of $A$, it suffices to show the weaker condition that $\nu(A) < \nu(A\{i, j\})$ for every subset $\{i, j\}$ of $\alpha$. This is seen as follows. In Proposition 4.7, it is shown that if $i$ and $j$ are P-vertices of $A$, then $\nu(A) - \nu(A\{i, j\}) \in \{-2, 0\}$. Combining this with Theorem 6 shows that if $\alpha$ is a set of P-vertices of $A$, then $\alpha$ is a P-set of $A$ if and only if $\nu(A) < \nu(A\{i, j\})$ for every subset $\{i, j\}$ of $\alpha$.

**References**

[1] R. Horn and C.R. Johnson. *Matrix Analysis*. 2nd ed. Cambridge University Press, New York, 2013.

[2] C.R. Johnson, A. Leal Duarte, C.M. Saiago, B.D. Sutton, and A.J. Witt. On the relative position of multiple eigenvalues in the spectrum of an Hermitian matrix with a given graph. *Linear Algebra and its Applications*, 363 (2003) 147-159.

[3] C.R. Johnson and B.D. Sutton. Hermitian matrices, eigenvalue multiplicities, and eigenvector components. *SIAM Journal on Matrix Analysis and Applications*, 26 (2004) 390-399.
[4] I.J. Kim and B.L. Shader. On Fiedler- and Parter-vertices of acyclic matrices. *Linear Algebra and its Applications*, 428 (2008) 2601-2613.

[5] I.J. Kim and B.L. Shader. Non-singular acyclic matrices. *Linear and Multilinear Algebra*, 57:4 (2009) 399-407.

[6] C.G. Nelson and B.L. Shader. Maximal P-sets of matrices whose graph is a tree. *Linear Algebra and its Applications*, under review.