ON UNIQUENESS PROPERTIES OF SOLUTIONS OF THE 
\textit{k}-GENERALIZED KDV EQUATIONS 

L. ESAURIAZA, C. E. KENIG, G. PONCE, AND L. VEGA 

1. Introduction

In this paper we study uniqueness properties of solutions of the \textit{k}-generalized Korteweg-de Vries equations

\begin{equation}
\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad (x, t) \in \mathbb{R}^2, \quad k \in \mathbb{Z}^+.
\end{equation}

Our goal is to obtain sufficient conditions on the behavior of the difference \( u_1 - u_2 \) of two solutions \( u_1, u_2 \) of (1.1) at two different times \( t_0 = 0 \) and \( t_1 = 1 \) which guarantee that \( u_1 \equiv u_2 \).

This kind of uniqueness results has been deduced under the assumption that the solutions coincide in a large sub-domain of \( \mathbb{R} \) at two different times. In [17] B. Zhang proved that if \( u_1(x, t) \) is a solution of the KdV, i.e. \( k = 1 \) in (1.1), such that \( u_1(x, t) = 0 \), \( (x, t) \in (b, \infty) \times \{t_0, t_1\} \) (or \( (-\infty, b) \times \{t_0, t_1\} \)), \( b \in \mathbb{R} \), then \( u_1 \equiv 0 \), (notice that \( u_2 \equiv 0 \) is a solution of (1.1)). His proof was based on the inverse scattering method (IST). In [10] this result was extended to any pair of solutions \( u_1, u_2 \) to the generalized KdV equation, which includes non-integrable models. In particular, if \( u_1, u_2 \) are solutions of (1.1) in an appropriate class with \( u_1(x, t) = u_2(x, t) \), for \( (x, t) \in (b, \infty) \times \{t_0, t_1\} \) (or \( (-\infty, b) \times \{t_0, t_1\} \)), then \( u_1 \equiv u_2 \).

In [13] L. Robbiano proved the following uniqueness result : Let \( u \) be a solution of the equation

\begin{equation}
\partial_t u + \partial_x^3 u + a_2(x, t) \partial_x^2 u + a_1(x, t) \partial_x u + a_0(x, t) u = 0,
\end{equation}

with coefficients \( a_j, j = 0, 1, 2 \) in suitable function spaces. If \( u(x, 0) = 0 \) for \( x \in (b, \infty) \) for some \( b > 0 \) and there exist \( c_1, c_2 > 0 \) such that

\[ |\partial_x^j u(x, t)| \leq c_1 e^{-c_2 x^\alpha}, \quad \forall (x, t) \in (b, \infty) \times [0, 1], \quad j = 0, 1, 2, \]

for some \( \alpha > 9/4 \), then \( u \equiv 0 \). This result applies to the difference \( u = u_1 - u_2 \) of two solutions \( u_1, u_2 \) of (1.1) with the coefficients in (1.2) \( a_0, a_1 \) depending on \( u_1, u_2, \partial_x u_1, k \) and with \( a_2 \equiv 0 \).

In [16], using the IST, S. Tarama showed that if the initial data \( u(x, 0) = u_0(x) \) has an appropriate exponential decay for \( x > 0 \), then the corresponding solution of the KdV becomes analytic in the \( x \)-variable for all \( t > 0 \).
It is interesting to notice that even in the KdV case neither of the results in [13] and [16] described above implies the other one. In [13] the decay assumption is needed in the whole time interval \([0,1]\), and the result in [16] does not apply to the difference of two arbitrary solutions of the KdV.

Our main result concerning the equation (1.1) is the following.

**Theorem 1.1.** Let \(u_1, u_2 \in C([0,1] : H^2(\mathbb{R}) \cap L^2(|x|^2 dx))\) be strong solutions of (1.1) in the domain \((x,t) \in \mathbb{R} \times [0,1]\). If \(k = 1\) in (1.1) also assume that \(u_1, u_2 \in C([0,1] : H^3(\mathbb{R}))\).

If

\[
(1.3) \quad u_1(\cdot,0) - u_2(\cdot,0), \quad u_1(\cdot,1) - u_2(\cdot,1) \in H^1(e^{ax^3/2} dx),
\]

for any \(a > 0\), then \(u_1 \equiv u_2\).

We shall say that \(f \in H^1(e^{ax^3/2} dx)\) if \(f, \partial_x f \in L^2(e^{ax^3/2} dx)\), where \(x_+ = \max\{x; 0\}\), and \(x_- = \max\{-x; 0\}\).

**Remarks**

a) The same result holds if in (1.3) instead of the space \(H^1(e^{ax^3/2} dx)\) one considers \(H^1(e^{-ax^3/2} dx)\).

b) We recall that the solution of the associated linear initial value problem

\[
\partial_t v + \partial_x^3 v = 0, \quad v(x,0) = v_0(x),
\]

is given by the unitary group \(\{U(t) : t \in \mathbb{R}\}\) where

\[
U(t)v_0(x) = \frac{1}{\sqrt{3t}} \text{Ai} \left( \frac{x}{\sqrt{3t}} \right) \ast v_0(x),
\]

and

\[
\text{Ai}(x) = \int_{\mathbb{R}} e^{2\pi ix\xi + 8\xi^3 i/3\pi^3} d\xi
\]

is the Airy function. This satisfies the estimate

\[
|\text{Ai}(x)| \leq c(1 + x_-)^{-1/4} e^{-cx_+^3/2}.
\]

Thus, the exponent 3/2 in (1.3) can be seen as a reflection of the asymptotic behavior of the Airy function. In fact, for the linear equation

\[
\partial_t v + \partial_x^3 v = 0,
\]

it shows that the decay rate in Theorem 1.1 is optimal.

c) In the particular case \(u_2 \equiv 0\) Theorem 1.1 tells us that the only solution of the \(k\)-generalized KdV equation (1.1) which decays, itself and its first derivative, as \(e^{-cx_+^3/2}\).
at two different times is the zero solution. This is in contrast with the solutions of the equation
\[ \partial_t u + \partial_x^2(u^2) + 2u\partial_x u = 0, \]
mapped by Rosenau and Hyman \cite{14} called “compactons”. These are solitary waves of speed \( c \) with compact support
\[ u_c(x,t) = \begin{cases} \frac{4}{3} \cos^2((x - ct)/4), & |x - ct| \leq 2\pi, \\ 0, & |x - ct| > 2\pi. \end{cases} \]  

\[ (1.4) \]

d) In \cite{3} we proved the following result concerning the semi-linear Schrödinger equation
\[ i\partial_t v + \Delta v + F(v, \bar{v}) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \]

\[ (1.5) \]

**Theorem 1.2.** Let \( v_1, v_2 \in C([-1, 1] : H^k(\mathbb{R}^n)), k \in \mathbb{Z}^+, k > n/2 + 1 \) be strong solutions of the equation \[ (1.5) \] in the domain \( (x,t) \in \mathbb{R}^n \times [0,1], \) with \( F : \mathbb{C}^2 \rightarrow \mathbb{C}, \) \( F \in C^k \) and \( F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0. \) If
\[ v_1(\cdot, 0) - v_2(\cdot, 0), \quad v_1(\cdot, 1) - v_2(\cdot, 1) \in H^1(e^{a|x|^2} \, dx), \]
for any \( a > 0, \) then \( v_1 \equiv v_2. \)

The argument of the proof of Theorem 1.2 has two main steps. The first one is based on the exponential decay estimates obtained in \cite{8}. These “energy” estimates are expressed in terms of the \( L^2(e^{\beta|x|^2} \, dx) \)-norm and involve bounds independent of \( \beta. \) In \cite{3} they are used to deduce similar ones with higher order powers in the exponent. The second step is to establish lower bounds for the asymptotic behavior of the \( L^2 \)-norm of the solution and its space gradient in the annular domain \( (x,t) \in \{R - 1 < |x| < R\} \times [0,1]. \) This idea was motivated by the work of Bourgain and Kenig \cite{11} on a class of stationary Schrödinger operators (i.e. \( -\Delta + V(x)\)). Also in this second step we follow some arguments due to V. Izakov \cite{5}.

For the equations \[ (1.1) \] considered here the first step in both \cite{8} and \cite{3}, i.e. weighted energy estimates, is not available. We need to replace it by appropriate versions of Carleman estimates. For example, for \( H_\beta = (\partial_t + \beta x \partial_x e^{-\beta x}) \) one has that
\[ (1.7) \]
\[ \|e^{\beta x} \partial_x e^{-\beta x} v\|_{L^{16/5}_{I_t} L^{16}_{L^1}} \leq c\|H_\beta v\|_{L^{16/15}_{I_t} L^{16/11}_{L^1}}, \]
for functions \( v \in C_0^\infty(\mathbb{R} \times [0,1]), \) see \cite{10}. This kind of estimate resembles those established in \cite{11} and some extensions obtained in \cite{10} related to the “smoothing effect” found in \[ 6, 12 \] (homogeneous version) and in \cite{7} (inhomogeneous version), see also \cite{4}. However, we shall need their extension to functions \( v \in C_0^\infty([0,1] : \mathcal{S}(\mathbb{R})). \) In the case of \[ (1.7) \] we shall prove that there exists \( j \in \mathbb{Z}^+ \) such that
\[ (1.8) \]
\[ \|e^{\beta x} \partial_x e^{-\beta x} v\|_{L^{16}_{I_t} L^{16}_{L^1}} \leq c\beta j(\|J^{1/2} v(\cdot, 0)\|_{L^2} + \|J^{1/2} v(\cdot, 1)\|_{L^2}) + c\|H_\beta v\|_{L^{16/15}_{I_t} L^{16/11}_{L^1}}. \]
It will be crucial in our proof that although in (1.8) the constant in front of the norms involving the function \( v \) evaluated at time \( t = 0 \) and \( t = 1 \) may grow as a power of \( \beta \), the constant in front of the norm of inhomogeneous term, i.e. \( H_\beta v \), is independent of \( \beta > 0 \).

e) Our argument here is direct and does not rely as that in [10] on the unique continuation principle obtained by Saut and Scheurer [15]: if a solution \( v = v(x, t) \) of (1.2) in the domain \( (x, t) \in (a, b) \times (t_1, t_2) \), with the coefficients \( a_j, j = 0, 1, 2 \) in an appropriate class, vanishes on an open set \( \Omega \subseteq (a, b) \times (t_1, t_2) \), then \( v \) vanishes in the horizontal components of \( \Omega \), i.e. the set

\[ \{(x, t) \in (a, b) \times (t_1, t_2) : \exists y \text{ s.t. } (y, t) \in \Omega \}. \]

f) For the existence of solutions and well-posedness results for the IVP associated to the equation (1.1) we refer to [7] and references therein. We recall that the conditions

\[ u_0 \in H^2(\mathbb{R}) \cap L^2(\mathbb{R}) : |x|^2 dx = X_{2,2} \text{ and } u_0 \in X_{2,2} \cap H^3(\mathbb{R}) \]

are locally preserved by the flow of solutions of (1.1), see [6]. For our arguments it suffices to have the decay in only one side of the line, i.e. changing \( |x| \) by \( x_+ \) in the weighted norms. This class is preserved for positive time \( t > 0 \) by the flow of solutions, see [12].

In particular, in the case \( u_2 \equiv 0 \) we do not need any decay assumption on \( u_1 \) since this will follow from the hypothesis (1.3).

\[ u_0, u_1, u_2 \in C(\Omega) \text{ and } \partial_x u_1 \text{ are assumed and the value of } k \text{ considered.} \]

In fact, we shall consider (1.2), a more general equation than (1.9).

**Theorem 1.3.** Assume that the coefficients in (1.2) satisfy that

\[ a_0 \in L_{x,t}^{4/3} \cap L_{t}^{16/13} L_{x}^{16/9} \cap L_{x}^{8/7} L_{t}^{8/3}, \]

\[ a_1 \in L_{x}^{16/13} L_{t}^{16/9} \cap L_{x}^{8/7} L_{t}^{8/3} \cap L_{x}^{16/15} L_{t}^{16/3}, \]

\[ a_2 \in L_{x}^{8/7} L_{t}^{8/3} \cap L_{x}^{16/15} L_{t}^{16/3} \cap L_{x}^{1} L_{t}^{\infty}. \]

Also, assume that

\[ a_0, a_1, a_2, \partial_x a_2, \partial_2^2 a_2 \in L^\infty(\mathbb{R}^2), a_2, \partial_t a_2 \in L^\infty(\mathbb{R} : L^1_x(\mathbb{R})). \]

If \( w \in C([0,1] : H^2(\mathbb{R}) \cap L^2(|x|^2 dx)) \) is a strong solution of (1.2) in the domain \( (x, t) \in \mathbb{R} \times [0,1] \) with

\[ w(\cdot, 0), w(\cdot, 1) \in H^1(e^{ax^3/2} dx), \]

for any \( a > 0 \), then \( w \equiv 0 \).
The remark (a) after the statement of Theorem 1.1 also applies here.

As it was pointed out in the remark (b) the decay rate in (1.12) is optimal.

We shall see that under the hypothesis of Theorem 1.1 the coefficients $a_0, a_1$ of the equation in (1.9) belong to the class described in Theorem 1.3 in (1.10) and (1.11). In fact, it will be clear from our proof that the conditions in (1.10) in the $x$-variable are needed only in the positive semi-line, i.e. it suffices to have (1.10) with $a_j \in L^p_t L^q_x([0, \infty) \times [0, 1])$ instead of $a_j \in L^p_t L^q_x(\mathbb{R} \times [0, 1])$.

It is here where the extra hypothesis $u_1, u_2 \in C([0, 1] : H^3)$ in Theorem 1.1 for the power $k = 1$ in (1.12) is needed.

The rest of this paper is organized as follows. In section 2, we deduce upper estimates in the time interval $[0, 1]$ for solutions of the inhomogeneous equation associated to (1.2) from the ones at times $t_0 = 0$ and $t_1 = 1$ and the inhomogeneous term. In section 3, we shall obtain lower bounds for the $L^2$-norm of the solution and its first and second derivatives in the annular domain mentioned above. Finally, in section 4 we combine the results in the previous sections to prove Theorems 1.3 and 1.1.

2. Upper Estimates

We shall use the notations

\[(2.1) \quad H f = (\partial_t + \partial_x^2) f, \quad H_\beta f = (\partial_t + e^{\beta x} \partial_x e^{-\beta x}) f.\]

Our first result in this section is the following lemma.

**Lemma 2.1.** There exists $k \in \mathbb{Z}^+$ such that if $u \in C^\infty([0, 1] : \mathbb{S}(\mathbb{R}))$, then for any $\beta \geq 1$

\[
\|e^{\beta x} u\|_{L^8_t} + \|e^{\beta x} \partial_x u\|_{L^{16}_t L^{16/5}_x} + \|e^{\beta x} \partial^2_x u\|_{L^{\infty}_t L^2_x} 
\]

\[
\leq c \beta^{2k} (\|J(e^{\beta x} u(\cdot, 0))\|_{L^2} + \|J(e^{\beta x} u(\cdot, 1))\|_{L^2})
\]

\[
+ c(\|e^{\beta x} H u\|_{L^{k/7}_t} + \|e^{\beta x} H u\|_{L^{16/15}_t L^{16/11}_x} + \|e^{\beta x} H u\|_{L^{4}_t L^{4}_x}).
\]

where $J g(x) = ((1 + |x|^{1/2}) \hat{g})^\vee$ and the norms in the time variable (i.e. $\| \cdot \|_{L^p_t}$) are restricted to the interval $[0, 1]$.

In order to prove (2.2), we set

\[(2.3) \quad v = e^{\beta x} u,\]

and rewrite (2.2) as

\[
\|v\|_{L^8_t} + \|(e^{\beta x} \partial_x e^{-\beta x}) v\|_{L^{16}_t L^{16/5}_x} + \|(e^{\beta x} \partial^2_x e^{-\beta x}) v\|_{L^{\infty}_t L^2_x} 
\]

\[
\leq c \beta^{2k} (\|Jv(\cdot, 0)\|_{L^2} + \|Jv(\cdot, 1)\|_{L^2})
\]

\[
+ c(\|H_\beta v\|_{L^{k/7}_t} + \|H_\beta v\|_{L^{16/15}_t L^{16/11}_x} + \|H_\beta v\|_{L^{4}_t L^{4}_x}).
\]

To obtain (2.4) we will prove the following string of inequalities

\[
\|v\|_{L^8_t} \leq c(\|v(\cdot, 0)\|_{L^2} + \|v(\cdot, 1)\|_{L^2}) + c\|H_\beta v\|_{L^{k/7}_t},
\]
(2.6) \[ \|e^{\beta x} \partial_x e^{-\beta x} v\|_{L^3_t L^{16/5}_x} \leq c\beta^k(\|J^{1/2} v(\cdot, 0)\|_{L^2} + \|J^{1/2} v(\cdot, 1)\|_{L^2}) + c\|H_\beta v\|_{L^{16/15}_x L^{16/11}_t}, \]

and

(2.7) \[ \|e^{\beta x} \partial_x^2 e^{-\beta x} v\|_{L^\infty_t L^2_x} \leq c\beta^{2k}(\|J v(\cdot, 0)\|_{L^2} + \|J v(\cdot, 1)\|_{L^2}) + c\|H_\beta v\|_{L^1_t L^2_x}. \]

Clearly, \((2.5)-(2.7)\) will imply \((2.2)\).

First we shall prove the following estimate which will be used later

(2.8) \[ \|v\|_{L^\infty_t L^2_x} \leq c(\|v(\cdot, 0)\|_{L^2} + \|v(\cdot, 1)\|_{L^2}) + c\|H_\beta v\|_{L^1_t L^2_x}. \]

**Proof of (2.8).** We have that

(2.9) \[ H_\beta = \partial_t + e^{\beta x} \partial_x^2 e^{-\beta x} = \partial_t + (e^{\beta x} \partial_x e^{-\beta x})^3, \]

with

(2.10) \[ e^{\beta x} \partial_x e^{-\beta x} = \partial_x - \beta, \]

\[ (e^{\beta x} \partial_x e^{-\beta x})^2 = (\partial_x - \beta)^2 = \partial_x^2 - 2\beta \partial_x + \beta^2, \]

\[ (e^{\beta x} \partial_x e^{-\beta x})^3 = (\partial_x - \beta)^3 = \partial_x^3 - 3\beta \partial_x^2 + 3\beta^2 \partial_x - \beta^3 \]

\[ = (\partial_x + 3\beta^2 \partial_x) = (3\beta^2 \partial_x + \beta^3) = \text{skew-symmetric - symmetric part}. \]

The symbol of \(H_\beta\) is

(2.11) \[ i\tau - i\xi^3 + 3i\beta^2 \xi - (\beta^3 - 3\beta \xi^2), \]

whose real part \(\beta^3 - 3\beta \xi^2\) vanishes at

(2.12) \[ \xi_\pm = \pm \beta / \sqrt{3}, \quad (\beta \geq 1). \]

By an approximation argument it suffices to prove \((2.8)\) for \(v \in C^\infty([0, 1] : \mathcal{S}(\mathbb{R}))\) such that \(\hat{v}(\xi, t) = 0\) near \(\xi_\pm\) for all \(t \in [0, 1]\).

Here, we shall denote by \(\hat{f}(\xi, t), \hat{f}(x, \tau), \hat{f}(\xi, \tau)\) the Fourier transform of \(f(\cdot, \cdot)\) with respect to the dual variables \(\xi, \tau, (\xi, \tau)\) respectively, i.e. \(\hat{f}(\cdot, \cdot)\) stands for the Fourier transform of \(f\) with respect to the dual variables where \(\hat{f}\) is evaluated.

Assume now that \(f \in C^\infty([0, 1] : \mathcal{S}(\mathbb{R}))\) with \(f(x, t) = 0\) for \(t\) near 0 and 1, so we can extend \(f\) as 0 outside the strip \(\mathbb{R} \times [0, 1]\). Also assume that \(\hat{f}(\xi, t) = 0\) for \(\xi\) near \(\xi_\pm\) for all \(t \in \mathbb{R}\). Using our assumptions on \(f\) we define

(2.13) \[ \hat{T}f(\xi, \tau) = \frac{\hat{f}(\xi, \tau)}{i\tau - i\xi^3 + 3i\beta^2 \xi - (\beta^3 - 3\beta \xi^2)}, \]

and claim that the estimate

(2.14) \[ \|Tf\|_{L^\infty_t L^2_x} \leq c\|f\|_{L^1_t L^2_x} \]

implies that in \((2.8)\). To prove it we choose \(\eta_\epsilon \in C^\infty(\mathbb{R}), \epsilon \in (0, 1/4), \)

(2.15) \[ \eta_\epsilon(t) = 1, \quad t \in [2\epsilon, 1 - 2\epsilon], \quad \text{and} \quad \text{supp } \eta_\epsilon \subset [\epsilon, 1 - \epsilon], \]
and define
\[ v_\epsilon(x, t) = \eta_\epsilon(t)v(x, t), \quad f_\epsilon(x, t) = H_\beta(v_\epsilon)(x, t). \]
Then, \( v_\epsilon = Tf_\epsilon \) since both sides have the same Fourier transform and both are in \( L^2_{xt} \) by our assumptions on \( v \), which are inherited by \( v_\epsilon \). Thus, (2.14) gives
\[ \|v_\epsilon\|_{L^\infty_t L^2_x} \leq c\|H_\beta(v_\epsilon)\|_{L^1_t L^2_x} \leq c\|\eta_\epsilon'(t)v\|_{L^1_t L^2_x} + c\|\eta_\epsilon H_\beta(v)\|_{L^1_t L^2_x}. \]

Letting \( \epsilon \downarrow 0 \) in (2.16) the left hand side converges to \( \|v\|_{L^\infty_{t, 1} L^2_x} \), while the right hand side has a limit bounded by \( c(\|v_0\|_{L^2} + \|v_1\|_{L^2}) + c\|H_\beta(v)\|_{L^1_t L^2_x}. \)

Hence, to obtain (2.18) we just need to prove (2.14). In order to prove (2.14) it suffices to show that for \( f(x, t) = f(x) \otimes \delta_{t_0}(t) \), with \( \hat{f}(\xi) = 0 \) near \( \pm \xi \), with \( t_0 \in (0, 1) \) one has that
\[ \|Tf\|_{L^\infty_t L^2_x} \leq c\|f\|_{L^2}, \quad \text{with } c \text{ independent of } t_0. \]

First, we recall the formulas
\[ \left( \frac{1}{\tau + ib} \right)^\gamma(t) = c \begin{cases} \chi_{(-\infty, 0)}(t)e^{ib}, & b > 0, \\
\chi_{(0, \infty)}(t)e^{ib}, & b < 0, \end{cases} \]
and consequently for \( a, b \in \mathbb{R} \)
\[ \left( \frac{e^{ita\tau}}{\tau - a + ib} \right)^\gamma(t) = ce^{ita} \begin{cases} \chi_{(-\infty, 0)}(t - t_0)e^{(t-t_0)b}, & b > 0, \\
\chi_{(0, \infty)}(t - t_0)e^{(t-t_0)b}, & b < 0. \end{cases} \]

Therefore,
\[ \hat{Tf}(\xi, \tau) = \frac{e^{ita\tau}\hat{f}(\xi)}{i((\tau - \xi^3 + 3\beta^2\xi) + i(\beta^3 - 3\beta\xi^2))} = \frac{-i}{\tau - a(\xi) + ib(\xi)}e^{ita\tau}\hat{f}(\xi). \]

Combining (2.19) - (2.20) we see that the operator \( T \) acting on these functions becomes the one variable operator \( R \)
\[ \hat{Rf}(\xi) = (\chi_{\{b(\xi) > 0\}}(\xi)e^{ita(\xi)}e^{(t-t_0)b(\xi)}\chi_{(-\infty, 0)}(t - t_0))\hat{f}(\xi) \]
\[ + (\chi_{\{b(\xi) < 0\}}(\xi)e^{ita(\xi)}e^{(t-t_0)b(\xi)}\chi_{(0, \infty)}(t - t_0))\hat{f}(\xi), \]
for which it needs to be established that
\[ \|Rf\|_{L^2_x} \leq c\|f\|_{L^2_x}, \quad \text{with } c \text{ independent of } \beta, t_0. \]

But this is obvious from the form of the multipliers in (2.21). Therefore (2.8) is proved.

**Proof of (2.18).** Again it suffices to show it for \( v \in C^\infty([0, 1] : \mathcal{S}(\mathbb{R})) \) such that \( \hat{v}(\xi, t) = 0 \) near \( \xi_\pm \). Assume that \( f \in C^\infty([0, 1] : \mathcal{S}(\mathbb{R})) \) with \( f(x, t) = 0 \) for \( t \) near 0 and 1, so we can extend \( f \) as 0 outside the strip \( \mathbb{R} \times [0, 1] \). Also assume that \( \hat{f}(\xi, t) = 0 \) for \( \xi \) near \( \xi_\pm \) for all
Assuming for the moment the inequalities in (2.23) we shall complete the proof of (2.5).

Consider

\[ v_\epsilon(x,t) = \eta_\epsilon(t) v(x,t), \quad H_\beta(v_\epsilon) = \eta_\epsilon'(t) v + \eta_\epsilon H_\beta v = f_1(x,t) + f_2(x,t). \]

Let

\[ v_1(x,t) = T f_1(x,t), \quad v_2(x,t) = T f_2(x,t), \]

where both make sense by our assumption on \( v \). Then,

\[ v_\epsilon(x,t) = v_1(x,t) + v_2(x,t), \]

since both sides are in \( L^2 \) and have the same Fourier transform. Hence, from (2.23) it follows that

\[
\|v_\epsilon\|_{L^8_t} \leq \|v_1\|_{L^2_t} + \|v_2\|_{L^2_t} \\
\leq c\|f_1\|_{L^1_t L^2_x} + c\|f_2\|_{L^{8/7}_t} \leq c\|\eta_\epsilon'(t) v\|_{L^1_t L^2_x} + c\|\eta_\epsilon(t) H_\beta v\|_{L^{8/7}_t},
\]

and letting \( \epsilon \downarrow 0 \) one gets (2.25). So we need to establish (2.23). (2.23)-(a) was proved in [10] (Lemma 2.3). To obtain (2.23)-(b) we again restrict ourselves to consider \( f(x,t) = f(x) \otimes \delta_{t_0}(t) \), and reduce it to show that the operator \( R \) defined in (2.21) satisfies that

\[ \|Rf\|_{L^8_t} \leq c\|f\|_{L^2}, \quad \text{with} \ c \ independent \ of \ \beta, t_0. \]

But, this follows from the proof of Lemma 4.1 in [9].

Proof of (2.7). Again we make our usual assumptions on \( \hat{v}(\xi, \tau) \). For \( f \in S(\mathbb{R}^2) \) with \( \hat{f}(\xi, t) = 0 \) near \( \xi_\pm \) for all \( t \in \mathbb{R} \) we define using (2.13)

\[ \hat{T}_2 \hat{f}(\xi, \tau) = (i\xi - \beta)^2 \hat{T} f(\xi, \tau) = \frac{(i\xi - \beta)^2 \hat{f}(\xi, \tau)}{i\tau - i\xi^3 + 3i\beta^2\xi - (\beta^3 - 3\beta^2\xi^2)}. \]

For the operator

\[ \hat{T}_2 f(x,t) = \chi_{[0,1]}(t)T_2 f(x,t), \]

we claim the following bounds

\[ (a) \quad \|\hat{T}_2 f\|_{L^\infty L^2} \leq c\|f\|_{L^1_t L^2_x}, \]

\[ (b) \quad \|\hat{T}_2 f\|_{L^\infty L^1} \leq c\beta^2 \|f\|_{L^1_t L^2_x}. \]
Assuming (2.29) (a)-(b) we shall prove (2.7). With the notation in (2.24)-(2.26) from (2.31) it follows that
\[
\| \cdt(\beta)^2 v_1 \|_{L^2_t \dot{L}^\infty_x} \leq \| \chi [0,1](t) (\cdt(\beta)^2 v_1) \|_{L^2_t \dot{L}^\infty_x} + \| \chi [0,1](t) (\cdt(\beta)^2 v_2) \|_{L^2_t \dot{L}^\infty_x}
\]
\[
\leq \| \tilde{T}_2 f_1 \|_{L^2_t \dot{L}^\infty_x} + \| \tilde{T}_2 f_2 \|_{L^2_t \dot{L}^\infty_x} \leq c \beta^2 \| Jf_1 \|_{L^1_t L^2_x} + c \| f_2 \|_{L^1_t L^2_x}
\]
so letting \( \epsilon \downarrow 0 \) we obtain (2.7).

The proof of
\[
\| T_2 f \|_{L^2_t \dot{L}^\infty_x} \leq c \| f \|_{L^1_t L^2_x},
\]
and therefore of (2.29)-(a) follows with a minor modification from the argument given in [10]. Notice that the polynomial considered in the numerator of the fraction appearing in (2.21) of [10] is \( \xi (i \xi - \beta) \) with \( \beta = 1 \) while here we are considering \( (i \xi - \beta)^2 \). The proof works in exactly the same way as it can be easily checked. In fact, the use of this polynomial instead of the one in [10] is more convenient for the Littlewood-Paley interpolation argument which appears later on in (2.46)-(2.48) of [10]. Notice that (2.48) in [10] for \( j = 0 \) is not true, but the proof just sketched fixed the error in [10]. Another possible way to bypass this difficulty is to use a Littlewood-Paley decomposition for \( j \in \mathbb{Z} \) instead of \( j = 0, 1, 2, \ldots \).

We next prove (2.29)-(b). Let \( \theta_r \in C_0^\infty(\mathbb{R}) \) with \( \theta_r(x) = 1 \), for \( |x| \leq 2r \) and \( \text{supp} \theta_r \subset \{|x| \leq 3r\} \), and consider
\[
\tilde{T}_2 f(\xi, \tau) = \frac{\theta_{\beta}(\xi)(i \xi - \beta)^2 \hat{f}(\xi, \tau)}{i \tau - i \xi^3 + 3i \beta^2 \xi - (\beta^3 - 3 \beta \xi^2)} + \frac{(1 - \theta_{\beta}(\xi)(i \xi - \beta)^2 \hat{f}(\xi, \tau)}{i \tau - i \xi^3 + 3i \beta^2 \xi - (\beta^3 - 3 \beta \xi^2)}
\]
\[
= \tilde{T}_{2,1} f(\xi, \tau) + \tilde{T}_{2,2} f(\xi, \tau).
\]
Now, using Sobolev lemma one gets that
\[
\| \tilde{T}_2 f \|_{L^2_t \dot{L}^\infty_x} \leq c \| \tilde{J} \tilde{T}_2 f \|_{L^2_t L^2_x} = c \| \tilde{J} \tilde{T}_{2,1} f \|_{L^2_t L^2_x} \leq c \| \tilde{J} T_{2,1} f \|_{L^2_t L^2_x},
\]
where \( \tilde{T}_{2,1} = \chi [0,1](t) T_{2,1} \). Now let
\[
\tilde{g}_1(\xi, \tau) = \theta_{\beta}(\xi)(1 + |\xi|^2)^{1/2} (i \xi - \beta)^2 \hat{f}(\xi, \tau).
\]
Then
\[
\tilde{J} T_{2,1} f(x, t) = T g_1(x, t),
\]
so by (2.14) and (2.30) it follows that
\[
\| \tilde{T}_2 f \|_{L^2_t \dot{L}^\infty_x} \leq c \| g_1 \|_{L^1_t L^2_x} \leq c \beta^2 \| Jf \|_{L^1_t L^2_x}.
\]
To complete (2.29)-(b) it suffices to prove that
\[
\| T_{2,2} f \|_{L^2_t \dot{L}^\infty_x} \leq c \| Jf \|_{L^1_t L^2_x}.
\]
We again reduce ourselves to consider functions of the form \( f(x,t) = f(x) \otimes \delta_{t_0}(t) \), so we just need to bound the operator
\[
\widehat{R_{2,2}f}(\xi,t) = (1 - \theta_\beta(\xi))(i\xi - \beta)^2 \chi_{b(\xi)>0}(\xi)e^{it\alpha(\xi)}e^{(t-t_0)b(\xi)}\chi_{(-\infty,0)}(t - t_0)\hat{f}(\xi)
\]
as
\[
\|R_{2,2}f\|_{L^2_x L^2_t} \leq c\|Jf\|_{L^2_x}, \text{ with } c \text{ independent of } \beta, t_0.
\]
We write
\[
R_{2,2}f(x,t) = e^{ix\xi}(1 - \theta_\beta(\xi))(i\xi - \beta)^2 \chi_{b(\xi)>0}(\xi)e^{it\alpha(\xi)}e^{(t-t_0)b(\xi)}\chi_{(-\infty,0)}(t - t_0)\hat{f}(\xi)d\xi,
\]
and recall that \( a(\xi) = (\xi^3 - 3\beta^2\xi) \). Now we change variables
\[
\lambda = \xi^3 - 3\beta^2\xi, \quad d\lambda = (3\xi^2 - 3\beta^2)d\xi = 3(\xi^2 - \beta^2)d\xi.
\]
From the definition of \( \theta_\beta(\cdot) \) the domain of integration in (2.33) is equal to \( \{ |\xi| \geq 2\beta \} \), where \( |\xi^2 - \beta^2| \simeq |\xi|^2 \), and the transformation is one to one. Thus, we have \( \xi = \xi(\lambda) \) and \( R_{2,2}f(x,t) = \int e^{it\lambda} \hat{\gamma}_2(\lambda)\Psi(\lambda,t)d\lambda, \) where
\[
\hat{\gamma}_2(\lambda) = \frac{e^{ix\xi}(1 - \theta_\beta(\xi))(i\xi - \beta)^2 \hat{f}(\xi)}{3(\xi^2 - \beta^2)}\chi_{\{b(\xi)>0\}}(\xi)e^{it\alpha(\xi)}e^{(t-t_0)b(\xi)}\chi_{(-\infty,0)}(t - t_0),
\]
with
\[
\Psi(\lambda,t) = \chi_{\{b(\xi)>0\}}(\xi)e^{(t-t_0)b(\xi)}\chi_{(-\infty,0)}(t - t_0).
\]
We observe that
\[
|\Psi(\lambda,t)| \leq c, \quad \forall (\lambda,t) \in \mathbb{R}^2 \quad \text{and} \quad \int |\partial_t\Psi(\lambda,t)|dt \leq c \quad \forall \lambda \in \mathbb{R}.
\]
Therefore, using the result in [22] (page 26) and taking adjoint one gets that
\[
\| \int e^{it\lambda}\hat{\gamma}_2(\lambda)\Psi(\lambda,t)d\lambda \|_{L^2_x} \leq \| \hat{\gamma} \|_{L^2}
\]
\[
\leq c \left( \int \frac{|e^{ix\xi}(1 - \theta_\beta(\xi))(i\xi - \beta)^2 \hat{f}(\xi)|^2}{|3(\xi^2 - \beta^2)||3(\xi^2 - \beta^2)|}d\lambda \right)^{1/2}
\]
\[
\leq c \left( \int \frac{|(1 - \theta_\beta(\xi))|\xi^2 + \beta^2| \hat{f}(\xi)|^2}{|3\xi^2 - \beta^2|}d\xi \right)^{1/2}
\]
\[
\leq c\|Jf\|_{L^2},
\]
which finishes the proof of \((2.29)-(b)\)

Proof of (2.6). We make the usual assumptions on \(v\) and \(\dot{v}\). For \(f \in S(\mathbb{R}^2)\) with \(\hat{f}(\xi, t) = 0\) near \(\xi \pm \) for all \(t \in \mathbb{R}\) we define using (2.13)

\[
(2.34) \quad \hat{T}_1 f(\xi, \tau) = (i\xi - \beta) \hat{T} f(\xi, \tau) = \frac{(i\xi - \beta) \hat{f}(\xi, \tau)}{i\tau - i\xi^3 + 3i\beta^2 \xi - (\beta^3 - 3\beta^2 \xi^2)}.
\]

For the operator

\[
(2.35) \quad \hat{T}_1 f(x, t) = \chi_{[0,1]}(t) T_1 f(x, t),
\]

we claim the following bounds

\[
(2.36) \quad \begin{align*}
(a) \quad & \|\hat{T}_1 f\|_{L_t^{16} L_x^{16/5}} \leq c\|f\|_{L_t^{16/15} L_x^{16/11}}, \\
(b) \quad & \|\hat{T}_1 f\|_{L_t^{16} L_x^{16/5}} \leq c\beta \|J^{1/2} f\|_{L_t^{4} L_x^{2}}.
\end{align*}
\]

As above it is easy to see that (2.6) follows from (2.36). Next, we recall that in \([10]\) (see also the second paragraph after (2.29)) it was proved that

\[
\|T_1 f\|_{L_t^{16} L_x^{16/5}} \leq c\|f\|_{L_t^{16/15} L_x^{16/11}},
\]

which implies (2.36)-(a). To obtain (2.36)-(b) we write

\[
\hat{T}_1 f(\xi, \tau) = \frac{\theta_\beta(\xi)(i\xi - \beta) \hat{f}(\xi, \tau)}{i\tau - i\xi^3 + 3i\beta^2 \xi - (\beta^3 - 3\beta^2 \xi^2)} + \frac{(1 - \theta_\beta(\xi)(i\xi - \beta) \hat{f}(\xi, \tau)}{i\tau - i\xi^3 + 3i\beta^2 \xi - (\beta^3 - 3\beta^2 \xi^2)}
\]

\[
= \hat{T}_{1,1} f(\xi, \tau) + \hat{T}_{1,2} f(\xi, \tau),
\]

and consider first \(\hat{T}_{1,1} = \chi_{[0,1]}(t) T_{1,1}\). From (2.31)

\[
\|\hat{T}_{1,1} f\|_{L_t^{4} L_x^{2}} \leq c\beta^2 \|J f\|_{L_t^{4} L_x^{2}},
\]

and from (2.23)-(b) it follows that

\[
\|\hat{T}_{0,1} f\|_{L_t^{6} L_x^{2}} \leq c\|J f\|_{L_t^{4} L_x^{2}}.
\]

Hence using the interpolation argument based on the Littlewood-Paley decomposition as in (2.46)-(2.48) of \([10]\) one gets

\[
\|\hat{T}_{1,1} f\|_{L_t^{16} L_x^{16/5}} \leq c\beta \|J^{1/2} f\|_{L_t^{4} L_x^{2}}.
\]

Finally, we interpolate between

\[
\|\hat{T}_{0,2} f\|_{L_t^{6} L_x^{2}} \leq c\|J f\|_{L_t^{4} L_x^{4}},
\]

which follows from (2.23)-(a), with (2.32) to get that

\[
\|\hat{T}_{1,2} f\|_{L_t^{16} L_x^{16/5}} \leq c\|J^{1/2} f\|_{L_t^{4} L_x^{2}}.
\]

which yields (2.36)-(b).

This finished the proof of Lemma 2.1.
Our next goal is to extend the estimates (2.22) in Lemma 2.1 to solutions of the linear equation with variable coefficients

\[ \partial_t u + \partial_x^3 u + a_2(x,t) \partial_x^2 u + a_1(x,t) \partial_x u + a_0(x,t)u = 0. \]

We introduce the notation

\[ H_a = \partial_t + \partial_x^3 + a_2(x,t) \partial_x^2 + a_1(x,t) \partial_x + a_0(x,t), \]

and try to find conditions which guarantee that multiplication by \( a_0(x,t) \) maps

\[ L^8_{xt} \to L^{8/7}_{xt}, \quad L^8_{xt} \to L^{16/15}_{x} L^{16/11}_t, \quad L^8_{xt} \to L^1_x L^2_t, \]

multiplication by \( a_1(x,t) \) maps

\[ L^{16}_{x} L^{16/5}_t \to L^{8/7}_{xt}, \quad L^{16}_{x} L^{16/5}_t \to L^{16/15}_{x} L^{16/11}_t, \quad L^{16}_{x} L^{16/5}_t \to L^1_x L^2_t, \]

and multiplication by \( a_2(x,t) \) maps

\[ L^\infty_x L^2_t \to L^{8/7}_{xt}, \quad L^{16}_{x} L^{16/5}_t \to L^{16/15}_{x} L^{16/11}_t, \quad L^{16}_{x} L^{16/5}_t \to L^1_x L^2_t. \]

So it suffices to have that

\[ a_0 \in L^{4/3}_{xt} \cap L^{16/9}_{x} L^{8/7}_t L^{8/3}_t, \]
\[ a_1 \in L^{16/13}_{x} L^{16/5}_t \cap L^{8/7}_x L^{3/3}_t \cap L^{16/15}_x L^{16/3}_t, \]
\[ a_2 \in L^{8/7}_x L^{3/3}_t \cap L^{16/15}_x L^{16/3}_t \cap L^1_x L^\infty_t. \]

Thus, if \( a_0, a_1, a_2 \) are in these spaces with small norm, then the inequality (2.22) will hold with \( H_a \) in (2.38) instead of \( H \), and one has the following result.

**Lemma 2.2.** Assume that the coefficients in (2.37) \( a_0, a_1, a_2 \) satisfy (2.42) with small enough norms. There exists \( k \in \mathbb{Z}^+ \) such that if \( u \in C^\infty([0, 1] : \mathcal{S}(\mathbb{R})) \), then for any \( \beta \geq 1 \)

\[
\|e^{\beta x} u\|_{L^8_{xt}} + \|e^{\beta x} \partial_x u\|_{L^{16}_{x} L^{16/5}_t} + \|e^{\beta x} \partial_x^2 u\|_{L^\infty_x L^2_t} \\
\leq c\beta^{2k} (\|J(e^{\beta x} u(\cdot, 0))\|_{L^2} + \|J(e^{\beta x} u(\cdot, 1))\|_{L^2}) \\
+ c(\|e^{\beta x} H_a u\|_{L^{8/7}_{xt}} + \|e^{\beta x} H_a u\|_{L^{16/15}_{x} L^{16/11}_t} + \|e^{\beta x} H_a u\|_{L^1_x L^2_t}).
\]

**Proof.** First we introduce the notation

\[ \|\|h\|\|_1 \equiv \|e^{\beta x} h\|_{L^2_{xt}} + \|e^{\beta x} \partial_x h\|_{L^{16}_{x} L^{16/5}_t} + \|e^{\beta x} \partial_x^2 h\|_{L^\infty_x L^2_t}, \]
\[ \|\|h\|\|_2 \equiv \|h\|_{L^{8/7}_{xt}} + \|h\|_{L^{16/15}_{x} L^{16/11}_t} + \|h\|_{L^1_x L^2_t}, \]
\[ \|\|h\|\|_3 \equiv \|e^{\beta x} h\|_{L^{8/7}_{xt}} + \|e^{\beta x} \partial_x h\|_{L^{16}_{x} L^{16/5}_t} + \|e^{\beta x} \partial_x^2 h\|_{L^\infty_x L^2_t}. \]
From Lemma 2.1 and our assumptions it follows that
\[
\|u\|_1 \leq c\beta^2 \left( \|J(e^{\beta x}u(\cdot, 0))\|_{L^2} + \|J(e^{\beta x}u(\cdot, 1))\|_{L^2} \right) \\
+ \|e^{\beta x} H u\|_2 \\
\leq c\beta^2 \left( \|J(e^{\beta x}u(\cdot, 0))\|_{L^2} + \|J(e^{\beta x}u(\cdot, 1))\|_{L^2} \right) \\
+ \|e^{\beta x} H_0 u\|_2 + \|c\beta^2 (a_2(x, t)\partial_x^2 u + a_1(x, t)\partial_x + a_0(x, t))u\|_2 \\
\leq c\beta^2 \left( \|J(e^{\beta x}u(\cdot, 0))\|_{L^2} + \|J(e^{\beta x}u(\cdot, 1))\|_{L^2} \right) \\
+ \|e^{\beta x} H_0 u\|_2 + \frac{1}{2} \|u\|_1.
\]
(2.45)

Hence,
\[
\|e^{\beta x} u\|_1 \leq c\beta^2 \left( \|J(e^{\beta x}u(0))\|_{L^2} + \|J(e^{\beta x}u(1))\|_{L^2} \right) + c\|e^{\beta x} H_0 u\|_2,
\]
(2.46)
whic yields the desired result. \(\square\)

We now start with \(u\) solving
\[
\partial_t u + \partial_x^3 u + a_2(x, t)\partial_x^2 u + a_1(x, t)\partial_x u + a_0(x, t)u = 0, \quad (x, t) \in \mathbb{R} \times [0, 1],
\]
with \(u_0 = u(\cdot, 0), u_1 = u(\cdot, 1) \in H^1(e^{ax^2})\) for some \(a > 0, \alpha > 1\), and \(a_0, a_1, a_2\) just in the spaces in (2.42).

Choose \(R\) so large that in the \(x\)-interval \((R, \infty)\) the coefficients \(a_0, a_1, a_2\) in the corresponding spaces (2.42) have small norms. Let \(\mu \in C^\infty(\mathbb{R})\) with \(\mu(x) = 0\) if \(x < 1\), and \(\mu(x) = 1\) if \(x > 2\). For \(\mu_R(x) = \mu(x/R)\) we have that
\[
u_R(x, t) = \mu_R(x)u(x, t),
\]
satisfies the equation
\[
\partial_t \nu_R + \partial_x^3 \nu_R + a_2(x, t)\partial_x^2 \nu_R + a_1(x, t)\partial_x \nu_R + a_0(x, t) \nu_R = e_R(x, t),
\]
where
\[
e_R(x, t) = \mu_R^{(3)} \frac{1}{R^3} u + 3\mu_R^{(2)} \frac{1}{R^2} \partial_x u + 3\mu_R^{(1)} \frac{1}{R} \partial_x^2 u \\
+ a_2(x, t)(2\mu_R^{(1)} \frac{1}{R} \partial_x u + \mu_R^{(2)} \frac{1}{R} u) + a_1(x, t)\partial_x \mu_R^{(1)} \frac{1}{R} u.
\]
Notice that \(\text{supp } e_R \subset \{x : R < x < 2R\}\). We will take
\[
\beta = \frac{a}{2} R^{(\alpha - 1)}.
\]

Now we apply our inequality (2.43) to \(u_R\), with
\[
H\tilde{\mu}_R = \partial_t + \partial_x^3 + a_2(x, t)\tilde{\mu}_R(x)\partial_x^2 + a_1(x, t)\tilde{\mu}_R(x)\partial_x + a_0(x, t)\tilde{\mu}_R(x),
\]
where \(\tilde{\mu}_R(x)\mu_R(x) = \mu_R(x)\), and so that \(a_j(x, t)\tilde{\mu}_R(x)\) with \(j = 0, 1, 2\) have small norm in the corresponding spaces in (2.42) for \(R > R_0\). From Lemma 2.2 it follows that for \(R\) large
\[
\|u_R\|_1 \leq c\beta^2 \left( \|J(e^{\beta x}u_R(\cdot, 0))\|_{L^2} + \|J(e^{\beta x}u_R(\cdot, 1))\|_{L^2} \right) + \|e^{\beta x} e_R\|_2.
\]
(2.48)
To bound the first term in the right hand side of (2.49) we use that
\[ (1 + \beta^{2k}) \| J(e^{\beta x} u_R(0)) \|_{L^2} \leq c(1 + \beta^{2k+1})(\| e^{\beta x} u_R(0) \|_{L^2} + \| e^{\beta x} \partial_x u_R(0) \|_{L^2}) \]
(2.49)
\[ \leq c(1 + \beta^{2k+1})(\| e^{\beta x} u(0) \|_{L^2(x>R)} + \| e^{\beta x} \partial_x u(0) \|_{L^2(x>R)}) \]
\[ \leq c(1 + \beta^{2k+1})(\| e^{a R^{\alpha-1} x/2} u(0) \|_{L^2(x>R)} + \| e^{a R^{\alpha-1} x/2} \partial_x u(0) \|_{L^2(x>R)}) \]

Since \( k \in \mathbb{Z}^+ \) is fixed and \( \beta = a R^{(\alpha-1)/2} \) for \( R \) sufficiently large, depending on \( \alpha \) and \( a \), one has
\[ a^{2k+1} R^{(2k+1)(\alpha-1)/2} \leq c_{a,\alpha} e^{a x^\alpha}, \quad \text{for} \ R > R > 0. \]
(2.50)
Then the right hand side of (2.49) is bounded by \( c_{a,\alpha} \).

A similar argument shows that
\[ (1 + \beta^{2k}) \| J(e^{\beta x} u_R(1)) \|_{L^2} \leq c_{a,\alpha}. \]

Thus, it remains to bound \( \| e^{\beta x} e_R \|_{L^2} \). Since \( \text{supp } e_R \subset \{ x : R < x < 2R \} \) combining Hölder inequality and Minkowski’s integral inequality it follows that
\[ \| e^{\beta x} e_R \|_{L^2} \leq c e^{a R^{\alpha-1} R} \| (|u| + |\partial_x u| + |\partial^2_x u|) \chi_{\{ x : R < x < 2R \}} \|_{L^\infty L^2} \leq c e^{a R^{\alpha-1} R}. \]

Inserting these estimates in (2.48) we obtain that
\[ \| e^{\beta x} u \|_{L^8_{x>4R} L^8_t} + \| e^{\beta x} \partial_x u \|_{L^8_{x>4R} L^{16/5}_t} + \| e^{\beta x} \partial^2_x u \|_{L^8_{x>4R} L^2_t} \]
\[ \leq c_{a,\alpha} + c e^{R^{\alpha-1} R} = c_{a,\alpha} + c e^{a R^\alpha}. \]

If \( x > 4R \), then \( e^{a R^{\alpha-1} x/2} \leq e^{a R^\alpha} \), so we get
\[ e^{a R^\alpha} \left( \| u \|_{L^8_{x>4R} L^8_t} + \| \partial_x u \|_{L^{16/5}_{x>4R} L^2_t} + \| \partial^2_x u \|_{L^2_{x>4R} L^2_t} \right) \]
\[ \leq c_{a,\alpha}. \]
(2.51)

Therefore, using Hölder inequality in (2.51) it follows that for \( R \) sufficiently large
\[ \| u \|_{L^2(\{ 4R < x < 4R+1 \} \times (0,1))} + \| \partial_x u \|_{L^2(\{ 4R < x < 4R+1 \} \times (0,1))} \]
\[ + \| \partial^2_x u \|_{L^2(\{ 4R < x < 4R+1 \} \times (0,1))} \leq c_{a,\alpha} e^{-a R^\alpha}. \]

Now changing \( 4R \) by \( R' \) we get that for any \( R' > 0 \) sufficiently large
\[ \| u \|_{L^2(\{ R' < x < R'+1 \} \times (0,1))} + \| \partial_x u \|_{L^2(\{ R' < x < R'+1 \} \times (0,1))} + \| \partial^2_x u \|_{L^2(\{ R' < x < R'+1 \} \times (0,1))} \]
\[ \leq c_{a,\alpha} e^{-a (R')^\alpha / R^\alpha}. \]

So we have proved the following upper estimates for solutions of (2.37).

**Theorem 2.1.** Assume that the coefficients in (2.17) \( a_0, a_1, a_2 \) satisfy (2.42). If \( u = u(x,t) \) is a solution of (2.17) with \( u \in C([0,1] : H^2(\mathbb{R})) \) satisfying that
\[ u(\cdot,0), \ u(\cdot,1) \in H^1(e^{a x^\alpha}) \]
for some $\alpha > 1$ and $a > 0$, then there exist $c_0$ and $R_0 > 0$ sufficiently large such that for $R \geq R_0$

$$
\|u\|_{L^2(\{R < x < R+1\} \times (0,1))} + \|\partial_x u\|_{L^2(\{R < x < R+1\} \times (0,1))}
$$

$$
\partial_t \alpha \geq \alpha + A \|\partial_x \alpha\|_{L^2(\{R < x < R+1\} \times (0,1))} \leq c_0 e^{-\alpha R^3 / 4a}.
$$

3. Lower Bounds

This section is concerned with lower bound estimates for the $L^2$-norm of a solution $u$ of the equation (1.2) and its first order space derivative $\partial_x u$ in the box $\{R - 1 < x < R\} \times [0,1]$.

**Lemma 3.1.** Assume that $\varphi : [0,1] \rightarrow \mathbb{R}$ is a smooth function. Then, there exist $c > 0$ and $M_1 = M_1(\|\varphi'\|_{\infty}; \|\varphi''\|_{\infty}) > 0$ such that the inequality

$$
\frac{\alpha^{5/2}}{R^3} \|e^{\alpha(\frac{\varphi}{R} + \varphi(t))^2} (\frac{\varphi}{R} + \varphi(t))^2 g\|_{L^2(\{x < R\} \times (0,1))} + \frac{\alpha^3}{R^2} \|e^{\alpha(\frac{\varphi}{R} + \varphi(t))^2} (\frac{\varphi}{R} + \varphi(t))^2 \partial_x g\|_{L^2(\{x < R\} \times (0,1))}
$$

$$
\leq c \|e^{\alpha(\frac{\varphi}{R} + \varphi(t))^2} (\partial_t + \partial_x^2) g\|_{L^2(\{x < R\} \times (0,1))}
$$

(3.1)

holds, for $R \geq 1$, $\alpha$ such that $\alpha^2 \geq M_1 R^3$, and $g \in C^\infty_0(\mathbb{R}^2)$ supported in

$$
\{ (x, t) \in \mathbb{R}^2 : |\frac{x}{R} + \varphi(t)| \geq 1 \}.
$$

**Proof.** We define $f = e^{\alpha \theta(x,t)} g$, for a general smooth function $\theta(x,t)$, and consider the expression

$$
e^{\alpha \theta(x,t)} (\partial_t + \partial_x^2)(e^{-\alpha \theta(x,t)} f(x,t)) = S_\alpha f + A_\alpha f,
$$

(3.2)

where

$$
S_\alpha = -3\alpha \partial_x (\partial_x \theta(x,t) \partial_x \cdot) + (-\alpha^3 (\partial_x \theta(x,t))^3 - \alpha \partial_x^3 \theta(x,t) - \alpha \partial_t \theta(x,t))
$$

$$
A_\alpha = \partial_t + \partial_x^2 + 3\alpha^2 (\partial_x \theta(x,t))^2 \partial_x + 3\alpha^2 \partial_x \theta(x,t) \partial_x^2 \theta(x,t).
$$

Thus,

$$
A_\alpha^* = -A_\alpha, \quad S_\alpha^* = S_\alpha,
$$

(3.3)

and one has

$$
\|e^{\alpha \theta(x,t)} (\partial_t + \partial_x^2) g\|_{L^2(\{x < R\} \times (0,1))}^2 = \|(A + S) f\|_{L^2(\{x < R\} \times (0,1))}^2
$$

$$
= \langle (A + S) f, (A + S) f \rangle
$$

$$
= \|Af\|^2 + \|Sf\|^2 + \langle Af, Sf \rangle + \langle Sf, Af \rangle
$$

$$
\geq \langle (SA - AS) f, f \rangle.
$$

(3.4)
A computation shows that
\[
(SA - AS)f = [S; A]f = 9\alpha \partial^2_x (\partial^2_x \theta \partial_x^2 f) \\
+ \partial_x((6\alpha \partial^2_x \theta + 6\alpha \partial^2_x \theta - 18\alpha^3 (\partial_x \theta)^2 \partial_x^2 \theta) \partial_x f) \\
+ (-3\alpha^3 (\partial^2_x \theta)^3 - 18\alpha^3 \partial_x \theta \partial^2_x \theta \partial^2_x \theta - 3\alpha^3 (\partial_x \theta)^2 \partial_x^3 \theta + 2\alpha \partial_x \theta + 6\alpha^3 (\partial_x \theta)^2 \partial_x^2 \theta + 9\alpha^5 (\partial_x \theta)^4 \partial_x^2 \theta) f.
\]
(3.5)

Now taking \(\theta(x, t) = (x/R + \varphi(t))^2\) it follows from (3.5) and integrations by parts that
\[
\langle (SA - AS)f, f \rangle = \frac{18\alpha}{R^2} \| \partial_x^2 f \|_{L^2(dxdt)}^2 \\
- \frac{12\alpha}{R} \int \int \varphi'(t)(\partial_x f)^2 dxdt + \frac{144\alpha^3}{R^4} \int \int (\frac{\varphi}{R} + \varphi(t))^2 (\partial_x f)^2 dxdt \\
- \frac{24\alpha^3}{R^6} \int \int f^2 dxdt + 2\alpha \int \int (\varphi'(t)f)^2 dxdt + 2\alpha \int \int (\frac{\varphi}{R} + \varphi(t))\varphi''(t)f^2 dxdt \\
+ \frac{48\alpha^3}{R^3} \int \int \varphi'(t)(\frac{\varphi}{R} + \varphi(t))^2 f^2 dxdt + \frac{288\alpha^5}{R^6} \int \int (\frac{\varphi}{R} + \varphi(t))^4 f^2 dxdt
\]
(3.6)

We first observe that
\[
I_5 + I_7 + I_8 = 2 \int \int (\alpha^{1/2} \varphi'(t)f + \frac{12\alpha^{5/2}}{R^3} (\frac{\varphi}{R} + \varphi(t))^2 f)^2 dxdt.
\]

Therefore, since \(|\frac{\varphi}{R} + \varphi(t)| > 1\) on the support of \(f\), for
\[
\alpha^2 \geq \| \varphi' \|_{\infty} R^3,
\]
(3.7)

it follows that
\[
I_5 + I_7 + I_8 \geq \frac{242\alpha^5}{R^6} \int \int (\frac{\varphi}{R} + \varphi(t))^4 f^2 dxdt.
\]

Similarly, since \(|\frac{\varphi}{R} + \varphi(t)| > 1\) on the support of \(f\) for
\[
\alpha^2 \geq (\| \varphi'' \|_{\infty}^{1/2} + 1) R^3,
\]
(3.8)

it follows that
\[
2\alpha^5 \int \int (\frac{\varphi}{R} + \varphi(t))^4 f^2 dxdt \geq |I_6|,
\]

and that
\[
\frac{24\alpha^5}{R^6} \int \int (\frac{\varphi}{R} + \varphi(t))^4 f^2 dxdt \geq |I_4|.
\]

Also from (3.7) one has that
\[
I_2 + I_3 \geq \frac{132\alpha^3}{R^4} \int \int (\frac{\varphi}{R} + \varphi(t))^2 (\partial_x f)^2 dxdt.
\]
Hence, gathering the above information we conclude that for
\begin{equation}
\alpha^2 \geq (\|\varphi\|_{\infty} + \|\varphi''\|_{\infty}^{1/2} + 1)R^3,
\end{equation}

one has that
\begin{equation}
\|e^{\alpha(\frac{x}{R} + \varphi(t))^2}(\partial_t + \partial_x^3)g\|^2_{L^2(dxdt)} = \|(A + S)f\|^2_{L^2(dxdt)}
\end{equation}
\begin{equation}
\geq \langle (SA - AS)f, f \rangle \geq \frac{18\alpha}{R^2} \int \int (\partial_x^2 f)^2 dxdt
\end{equation}
\begin{equation}
+ \frac{132\alpha^3}{R^4} \int \int (\frac{x}{R} + \varphi(t))^2(\partial_x f)^2 dxdt + \frac{216\alpha^5}{R^6} \int \int (\frac{x}{R} + \varphi(t))^4 f^2 dxdt.
\end{equation}

Next, we rewrite (3.10) in terms of \( g = e^{-\alpha(\frac{x}{R} + \varphi(t))^2} f \). In fact, it follows from (3.9) and (3.10) that there exits a universal constant \( c_0 > 0 \) such that
\begin{equation}
\|e^{\alpha(\frac{x}{R} + \varphi(t))^2}(\partial_t + \partial_x^3)g\|_{L^2(dxdt)} \geq \frac{c_0\alpha^{1/2}}{R} \left( \int \int e^{2\alpha(\frac{x}{R} + \varphi(t))^2}(\partial_x^2 g)^2 dxdt \right)^{1/2}
\end{equation}
\begin{equation}
\quad + \frac{c_0\alpha^{3/2}}{R^2} \left( \int \int (\frac{x}{R} + \varphi(t))^2 e^{2\alpha(\frac{x}{R} + \varphi(t))^2}(\partial_x g)^2 dxdt \right)^{1/2}
\end{equation}
\begin{equation}
\quad + \frac{c_0\alpha^{5/2}}{R^3} \left( \int \int (\frac{x}{R} + \varphi(t))^4 e^{2\alpha(\frac{x}{R} + \varphi(t))^2} g^2 dxdt \right)^{1/2},
\end{equation}
which completes the proof of Lemma 3.1.

Next, we shall extend the result of Lemma 3.1 to operators of the form
\begin{equation}
L = \partial_t + \partial_x^3 + a_0(x, t) + a_1(x, t)\partial_x,
\end{equation}
with
\[ a_0, a_1 \in L^\infty(\mathbb{R}^2). \]

**Lemma 3.2.** Assume that \( \varphi : [0, 1] \rightarrow \mathbb{R} \) is a smooth function. Then, there exist \( c > 0, R_0 = R_0(\|\varphi\|_{\infty}; \|\varphi''\|_{\infty}; \|a_0\|_{\infty}; \|a_1\|_{\infty}) > 1 \) and \( M_1 = M_1(\|\varphi\|_{\infty}; \|\varphi''\|_{\infty}) > 0 \) such that the inequality
\begin{equation}
\frac{\alpha^{5/2}}{R^3} \left\| e^{\alpha(\frac{x}{R} + \varphi(t))^2}(\frac{x}{R} + \varphi(t))^2 g \right\|_{L^2(dxdt)} + \frac{\alpha^{3/2}}{R^2} \left\| e^{\alpha(\frac{x}{R} + \varphi(t))^2} \partial_x g \right\|_{L^2(dxdt)}
\leq c \left\| e^{\alpha(\frac{x}{R} + \varphi(t))^2}(\partial_t + \partial_x^3 + a_0(x, t)\partial_x + a_1(x, t)\partial_x)g \right\|_{L^2(dxdt)}
\end{equation}
holds, for \( R \geq R_0, \alpha \) such that \( \alpha^2 \geq M_1 R^3 \) and \( g \in C_0^\infty(\mathbb{R}^2) \) supported in
\[ \{(x, t) \in \mathbb{R}^2 : |\frac{x}{R} + \varphi(t)| \leq 1\}. \]
Proof. From (3.1), Lemma 3.1 it follows that

\[
\frac{\alpha^{5/2}}{R^3} \left| e^{\alpha(t^2)} \sum_{i=1}^{2} \frac{\partial^2}{\partial r^2} u \right|_{L^2(\mathbb{R}^3)} + \frac{\alpha^{3/2}}{R^2} \left| e^{\alpha(t^2)} \sum_{i=1}^{2} \frac{\partial^2}{\partial r^2} u \right|_{L^2(\mathbb{R}^3)}
\]

Then there exist constants $R_0, c_0, c_1 > 0$ depending on

(3.14)

Since our hypothesis guarantee that $\alpha^{5/2}/R^3$ and $\alpha^{3/2}/R^2$ growth as a positive (fractional) power of $R$ for $R$ sufficiently large the last two terms in the right hand side of (3.14) can be hidden in the left hand side to obtain the desired result.

Theorem 3.1. Let $u \in C([0, 1] : H^2(\mathbb{R}))$ be a solution of

(3.15)

with $a_0, a_1, a_2, \partial_x a_2, \partial^2_x a_2 \in L^\infty(\mathbb{R}^2)$ and $a_2, \partial_t a_2 \in L^\infty(\mathbb{R}^2 : L^1_x(\mathbb{R}))$. Assume that

\[
\int_\mathbb{R} \int_0^1 (u^2 + (\partial_x u)^2 + (\partial^2_x u)^2)(x, t)dxdt \leq A^2,
\]

and

\[
\int_{1/2 - 1/8}^{1/2 + 1/8} \int_0^1 u^2(x, t)dxdt \geq 1.
\]

Then there exist constants $R_0, c_0, c_1 > 0$ depending on

(3.16)

such that for $R \geq R_0$

(3.17)

\[
\delta(R) = \delta_u(R) = \left( \int_{0}^{1} \int_{R-1 < x < R} (u^2 + (\partial_x u)^2 + (\partial^2_x u)^2)dxdt \right)^{1/2} \geq c_0 e^{-c_1 R^{3/2}}.
\]

Proof. First, we use a gauge transformation (i.e. a change of the dependent variable) to reduce the equation in (3.15) to an “equivalent” one which does not involve second order derivative. Define

(3.18)

\[
v(x, t) = u(x, t)e^{t^2 \int_0^t a_2(s, t)ds}.
\]
Thus multiplying the equation in (3.15) by $e^{\frac{1}{3} \int_0^x a_2(s,t)ds}$ and using that

\[
e^{\frac{1}{3} \int_0^x a_2(s,t)ds} \partial_t u = \partial_t v - \frac{1}{3} \left( \int_0^x \partial_t a_2(s,t)ds \right) v,
\]

\[
e^{\frac{1}{3} \int_0^x a_2(s,t)ds} \partial_x u = \partial_x v - \frac{1}{3} a_2 v,
\]

\[
e^{\frac{1}{3} \int_0^x a_2(s,t)ds} \partial_x^2 u = \partial_x^2 v + a_2 \partial_x v - \frac{1}{6} \partial_x a_2 + \left( \frac{1}{3} a_2 \right)^2 v,
\]

\[
e^{\frac{1}{3} \int_0^x a_2(s,t)ds} \partial_x^3 u = \partial_x^3 v - a_2 \partial_x^2 v + \left( \left( \frac{1}{3} a_2 \right)^2 - \frac{1}{3} \partial_x a_2 \right) \partial_x v + \left( - \left( \frac{1}{3} a_2 \right)^3 + \frac{1}{3} a_2 \partial_x a_2 - \frac{1}{3} \partial_x^3 a_2 \right) v.
\]

the equation for $v = v(x,t)$ can be written as
\[
\partial_t v + \partial_x^2 v + \tilde{a}_1(x,t) \partial_x v + \tilde{a}_0(x,t) v = 0,
\]

where from our hypothesis on $a_0, a_1, a_2$ it follows that $\tilde{a}_1, \tilde{a}_0 \in L^\infty(\mathbb{R}^2)$.

Next, we shall follow the arguments in [3].

For $R > 2$ let $\theta_R \in C^\infty(\mathbb{R})$ with $\theta_R(x) = 1$ if $x < R - 1$, $\theta_R(x) = 0$ if $x > R$.

Let $\mu \in C^\infty(\mathbb{R})$ with $\mu(x) = 0$ if $x < 1$ and $\mu(x) = 1$ if $x > 2$, and $\varphi \in C^\infty_0(\mathbb{R})$, $\varphi: \mathbb{R} \to [0, 3]$ with

\[
(3.19) \quad \varphi(t) = \begin{cases} 
0, & t \in [0, 1/4] \cap [3/4, 1], \\
3, & t \in [1/2 - 1/8, 1/2 + 1/8].
\end{cases}
\]

We define
\[
(3.20) \quad g(x,t) = \theta_R(x) \mu(\frac{x}{R} + \varphi(t)) v(x,t), \quad (x,t) \in \mathbb{R} \times [0, 1]
\]

and observe that

\[
\begin{align*}
&\text{if } x > R, \text{ then } g(x,t) = 0, \\
&\text{if } x < R \text{ and } t \in [0, 1/4] \cap [3/4, 1], \text{ then } g(x,t) = 0, \\
&\text{if } \frac{x}{R} + \varphi(t) < 1, \text{ then } g(x,t) = 0, \text{ so that } g \text{ has support on } \mathbb{R} \times (0,1) \text{ and can be assumed to satisfy the hypothesis of Lemma 3.1.}
\end{align*}
\]

Also, for $(x,t) \in (0, R - 1) \times [1/2 - 1/8, 1/2 + 1/8]$, $g(x,t) = v(x,t)$ and $|\frac{x}{R} + \varphi(t)| \geq 2$.

From (3.20) one has that

\[
(\partial_t + \partial_x^2 + \tilde{a}_1 \partial_x + \tilde{a}_0) g
\]

\[
= \mu(\frac{x}{R} + \varphi(t)) \left[ 3\theta_R^{(1)}(\partial_x^2 v + 3\theta_R^{(2)} \partial_x v + \theta_R^{(3)} v + \tilde{a}_1 \theta_R^{(1)} v) \\
+ \theta_R(x) \left( \mu^{(1)}(\cdot) \left( \varphi^{(1)} + \frac{\tilde{a}_1}{R} \right) v + 3\mu^{(2)}(\cdot) \frac{1}{R^{(2)}} \partial_x^2 v + 3\mu^{(3)}(\cdot) \frac{1}{R^{(3)}} \partial_x v + \mu^{(4)}(\cdot) \frac{1}{R^{(4)}} v \right) \\
+ 3\mu^{(1)}(\cdot) \frac{1}{R} \theta_R^{(2)} v + 3\mu^{(2)}(\cdot) \frac{1}{R} \theta_R^{(1)} v + 6\mu^{(1)}(\cdot) \frac{1}{R} \theta_R^{(1)} \partial_x v, \right]
\]

where the first term in the right hand side of (3.21) is supported in $[R - 1, R] \times [0, 1]$, where $|\frac{x}{R} + \varphi(t)| \leq 4$, and the remaining terms in the right hand side of (3.21) are supported in $(x,t) : 1 \leq |\frac{x}{R} + \varphi(t)| \leq 2$.
Using the notation
\[(3.22) \quad \delta_v(R) = \left( \int_0^1 \int_{R-1 < x < R} (v^2 + (\partial_x v)^2 + (\partial_x^2 v)^2)(x, t) \, dx \, dt \right)^{1/2}, \]
from (3.13) and (3.21) it follows that
\[c\alpha \frac{5}{2} R^3 \leq c \frac{5}{2} \alpha \frac{R^3}{2} e^{4\alpha} \leq c_1 e^{16\alpha} \delta_v(R) + c_2 e^{4\alpha} A, \]
therefore
\[c\alpha \frac{5}{2} R^3 \leq c_1 e^{12\alpha} \frac{5}{2} R^3 \delta_v(R) + c_2 A. \]

Taking \(\alpha = M_1 R^{3/2}\) with \(M_1\) as in Lemma 3.2 we get
\[(3.23) \quad cM_1^{5/2} R^{3/4} \leq c_1 e^{12M_1 R^{3/2}} \delta_v(R) + c_2 A. \]

For \(R\) sufficiently large the last term in the right hand side of (3.23) can be absorbed into the left hand side to get that
\[\delta_v(R) \geq \frac{c}{2} M_1^{5/2} R^{3/4} e^{-12M_1 R^{3/2}}. \]
Finally, from (3.18), (3.20), (3.22) and our hypothesis one has that \( \delta_u \sim \delta_v \), i.e. there exists \( c > 1 \) such that
\[c^{-1} \delta_v(R) \leq \delta_u(R) \leq c \delta_v(R), \quad \forall R \geq R_0, \]
which yields the desired result. \(\square\)

4. PROOF OF THEOREMS 1.3 AND 1.1

Proof of Theorem 1.3.

If \( u \not\equiv 0 \), we can assume, after a possible translation, dilation, and multiplication by a constant, that \( u = u(x, t) \) satisfies the hypothesis of Theorem 3.1. Hence, we have that
\[(4.1) \quad \delta_u(R) = \left( \int_0^1 \int_{R-1 < x < R} (u^2 + (\partial_x u)^2 + (\partial_x^2 u)^2)(x, t) \, dx \, dt \right)^{1/2} \geq c_0 e^{-c_1 R^{3/2}}, \]
for all \( R \) sufficiently large where the constants \( c_0, c_1 \) depend on the quantities in (3.16).

Now we apply Theorem 2.1 with \( \alpha = 3/2 \) and \( a >> 4^{3/2} c_1 \) with \( c_1 \) as in (1.1) to conclude that
\[(4.2) \quad \delta_u(R) \leq c e^{-c R^{3/2}/4^{3/2}}, \]
for all \( R \) sufficiently large. Combining (4.1) and (4.2) and letting \( R \uparrow \infty \) we get a contradiction. Therefore \( u \equiv 0 \).

Proof of Theorem 1.1.
It will be shown that Theorem 1.3 applies to the equation of the type (1.2) satisfied by the difference $u_1 - u_2$ of the solutions. Thus, one just needs to prove that the coefficients $a_0, a_1$ satisfy the assumptions (1.10) and (1.11). We recall that in this case $a_2 \equiv 0$.

Since for any $k \in \mathbb{Z}^+$, $a_0, a_1$ are polynomials of order $k$ in $u_1, u_2, \partial_x u_1$, with $u_1, u_2 \in C([0, 1] : H^2(\mathbb{R}))$, and $a_2 \equiv 0$ it is clear that the hypothesis (1.11) holds. So it remains to check the conditions (1.10), i.e.

\begin{equation}
(4.3)
\begin{align*}
a_0 & \in L^{4/3}_{xt} \cap L^{16/13}_x L^{16/9}_t \cap L^{8/7}_x L^{8/3}_t, \\
a_1 & \in L^{16/13}_x L^{16/9}_t \cap L^{8/7}_x L^{8/3}_t \cap L^{16/15}_x L^{16/3}_t.
\end{align*}
\end{equation}

First, we consider the KdV equation, i.e. $k = 1$ in (1.1), for which we have

$$a_0(x, t) = \partial_x u_1(x, t) \quad \text{and} \quad a_1(x, t) = u_2(x, t).$$

Using the hypothesis $u_1, u_2 \in C([0, 1] : H^2 \cap L^2(|x|^2 dx)$ it follows by interpolation (or integration by parts) that

\begin{equation}
(4.4)
\begin{align*}
a_0 & \in L^\infty([0, 1] : H^2) \quad \text{and} \quad |x|^{2/3} a_0, |x|^{1/3} \partial_x a_0 \in L^\infty([0, 1] : L^2_x), \\
\end{align*}
\end{equation}

and by Sobolev lemma that

\begin{equation}
(4.5)
|x|^{1/3} a_0 \in L^\infty([0, 1] : L^\infty_x).
\end{equation}

Thus, (4.4) and Hölder inequality yields

$$\|a_0(t)\|_{L^{4/3}_{xt}} \leq c \sup_{t \in [0,1]} \|(1 + |x|)^{1/2} a_0(\cdot, t)\|_{L^2}, \quad t \in \mathbb{R},$$

which proves that $a_0 \in L^{4/3}_{xt}$. Next, the string of inequalities,

\begin{equation}
(4.6)
\begin{align*}
&\|a_0\|_{L^{16/13}_x L^{16/9}_t} \\
&= \left( \int \frac{1}{(1 + |x|)^{8/13}} (1 + |x|)^{8/13} \left( \int |a_0(x, t)|^{16/9} \, dt \right)^{9/13} \, dx \right)^{13/16} \\
&\leq c \left( \int \|(1 + |x|)^{1/2} a_0(\cdot, t)\|_{L^2_x}^{16/9} \, dt \right)^{9/16} \\
&\leq \sup_{t \in [0,1]} \|(a_0(\cdot, t))\|_{L^2} + |||x|^{1/2} a_0(\cdot, t)||_{L^2_x}),
\end{align*}
\end{equation}

and (4.4) show that $a_0 \in L^{16/13}_x L^{16/9}_t$. 

Hence, that \(u\) since \(u\) (4.8) \(|\nabla u| \leq 1\) which together with (4.4) and (4.5) imply that \(\epsilon > 0\) for any \(a\) which in turn is \(a\) (4.10) \(L^2\). Therefore, inserting (4.8), (4.9) in (4.10) one obtains the desired result.

Now we consider \(a_1(x, t) = u_2 \in C([0, 1] : H^2 \cap L^2(\mathbb{R}^2)dx).\) Thus, it follows that

\[
|x|a_1, \ |x|^{2/3} \partial_x a_1, \ |x|^{1/3} \partial_x^2 a_1, \partial_x^3 a_1 \in L^\infty([0, 1] : L^2_x),
\]

and by Sobolev lemma that

\[
|x|^{2/3} a_1 \in L^\infty([0, 1] : L^\infty_x).
\]

The same arguments used in (4.8) and (4.9) show that \(a_1 \in L_x^{16/13} L_t^{16/9} \cap L_x^{8/7} L_t^{8/3}\). So it only remains to prove that \(a_1 \in L_x^{16/15} L_t^{16/13}\). A familiar process leads to

\[
\|a_0\|_{L_x^{16/5} L_t^{16/3}} \leq \left( \int \int (1 + |x|)\partial_x (a_1(x, t))^{16/3} dxdt \right)^{3/16} \leq c \left( \int \int (1 + |x|^2) a_1(x, t)^{2} dxdt \right)^{3/16} \leq c \|1 + |x|\|_{L_x^{16/15}} \|a_1\|_{L_t^{16/13}}^{5/8} \leq c \|1 + |x|a_1\|_{L_x^{16/15}}^{3/8} \|x^{1/3} a_1\|_{L_t^{16/13}}^{5/8}.
\]

Therefore, inserting (4.8), (4.9) in (4.10) one obtains the desired result.

We have completed the proof of Theorem 1.1. in the case of the KdV equation.

Next, we turn to the proof of Theorem 1.1. for the equations in (1.1) with \(k \geq 2\). Using that \(u_1, u_2 \in L^\infty(\mathbb{R} \times [0, 1])\) it suffices to consider the case \(k = 2\) where

\[
a_0(x, t) = (u_1 + u_2) \partial_x u_1, \quad a_1(x, t) = u_2^2.
\]

Since \(u_1, u_2 \in C([0, 1] : H^2 \cap L^2(\mathbb{R}^2)dx)\) by interpolation and Sobolev lemma it follows that

\[
|x|u_j, \ |x|^{1/2} \partial_x u_j \in L^\infty([0, 1] : L^2_x), \quad |x|^{1/2} u_j \in L^\infty([0, 1] : L^\infty_x), \quad j = 1, 2.
\]

Hence,

\[
|x|^{3/2} a_0 \in L^\infty([0, 1] : L^2_x), \quad |x|^{1/2} a_0 \in L^\infty([0, 1] : L^\infty_x),
\]
and
\[ |x|a_1 \in L^\infty([0,1]:L^2_x), \quad |x|^{2/3}a_1 \in L^\infty([0,1]:L^\infty_x), \]
which were the conditions used to obtain the result in the case \( k = 1 \).
This completes the proof of Theorem 1.1.

**Acknowledgments**

L. E. and L. V. were supported by a MEC grant and by the European Commission via the network Harmonic Analysis and Related Problems. C. E. K. and G. P. were supported by NSF grants.

**References**

[1] Bourgain, J., and Kenig, C. E., *On localization in the continuous Anderson-Bernoulli model in higher dimensions*, Invent. Math. **161** (2005), 389-342

[2] Coifman, R. R., and Meyer, Y. *Au delà des opérateurs pseudodifférentiels*, Asterisque 57, Société Mathématique de France (1973)

[3] Escauriaza, L., Kenig, C. E., Ponce, G., and Vega, L., *On unique continuation of solutions of Schrödinger equations*, to appear in Comm. PDE

[4] Ionescu, I. D., and Kenig, C. E., *L^p Carleman inequalities and uniqueness of solutions of nonlinear Schrödinger equations*, Acta Math. **193** (2004), 193-239

[5] Isakov, V., *Carleman type estimates in anisotropic case and applications* J. Diff. Eqs. **105** (1993), 217–238

[6] Kato, T., *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Advances in Mathematics Supplementary Studies, Studies in Applied Math. **8** (1983), 93-128

[7] Kenig, C. E., Ponce, G., and Vega, L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527-620

[8] Kenig, C. E., Ponce, G., and Vega, L., *On the support of solutions of nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **60** (2002), 1247-1262

[9] Kenig, C. E., Ponce, G., and Vega, L., *On the suport of solutions to the generalized KdV equation*, Annales de l'I.H.P. Analyse Non Linéaire **19** (2002), 191-208

[10] Kenig, C. E., Ponce, G., and Vega, L., *On unique continuation of solutions to the generalized KdV equation*, Math. Res. Letters **10** (2003), 833-846

[11] Kenig, C. E., Ruiz, A., and Sogge, C., *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J. **55** (1987), 329–347

[12] Kruzhkov, S. N., and Faminskii, A. V., *Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation*, Math. U.S.S.R. Sbornik, **48** (1984), 93–138

[13] Robiano, L., *Unicité forte à l’infini pour KdV*, Control Opt. and Cal. Var. **8** (2002), 933-939

[14] Rosenau, P., and Hyman, J. M., *Compactons: Solitons with finite wavelength*, Physical Rev. Lett. **70** (1993), 564-567

[15] Saut, J.-C., and Schuur, B. *Unique continuation for some evolution equations* J. Diff. Eqs. **66** (1987), 118-139
[16] Tarama, S., Analytic solutions of the Korteweg-de Vries equation, J. Math. Kyoto Univ. 44 (2004), 1-32
[17] Zhang, B.-Y., Unique continuation for the Korteweg-de Vries equation, SIAM J. Math. Anal. 32 (1992), 55-71

Luis Escauriaza
Departamento de Matematicas
Universidad del Pais Vasco
Apartado 644
48080 Bilbao
Spain
E-mail: mtpeszul@lg.ehu.es

Carlos E. Kenig
Department of Mathematics
University of Chicago
Chicago, Il. 60637
USA
E-mail: cek@math.uchicago.edu

Gustavo Ponce
Department of Mathematics
University of California
Santa Barbara, CA 93106
USA
E-mail: ponce@math.ucsb.edu

Luis Vega
Departamento de Matematicas
Universidad del Pais Vasco
Apartado 644
48080 Bilbao
Spain
E-mail: mtpvegol@lg.ehu.es