Quantum State Complexity Measure

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Abstract

The complexity measures role has become much clearer in recent years as they help to better understand complex systems dynamical behavior. Even though the large number of measures proposed to tackle this issue for classical systems, for quantum systems only Kolmogorov’s algorithm complexity extensions have been proposed. Hence, the present approach makes use of a new and mathematically well-established complexity measure for classical systems and extends it to assess quantum states complexity as well. Then the proposed extension is applied to a mixed state constructed with a W-state together with controlled white noise, showing a convex behavior of quantum state complexity. Thus, this reinforces the differences from previous known quantum complexities.

Keywords: complexity measure, quantum state, information geometry

2000 MSC: 81P45, 94A17,
1. Introduction

In the last few decades, the scientific community interest in quantifying systems complexity has grown. And a significant number of those attempts make use of information-theoretic tools to address this issue (Prokopenko et al., 2007; Haken, 2006).

Their application scope is just larger than their number, ranging from biological, economic and social systems (Prokopenko et al., 2007; Gros, 2008; Piqueira et al., 2009; Piqueira, 2009) to quantum physics (Angulo et al., 2008; Chatzisavvas et al., 2005; Lopez-Ruiz et al., 2009; Nagy et al., 2009), justifying the analysis power of complexity quantification approaches to better understand complex systems, sometimes bridging together very distinct systems.

Besides, in the last years, some new complexity measures of classical systems were developed by the use of the modern extension of Shannon’s information theory, the quantum information theory (Vedral, 2006), taking advantage of the latter unique properties such as superposition of states to build a deeper characterization of those systems and to assess systems in which these features take a central role, e.g. quantum systems.

Although there are some proposed quantum informational complexity measures (Rogers et al., 2008; Mora and Briegel, 2005; Mora et al., 2007), they do not have the desired generality of the modern complexity measures proposed for classical systems because all of them are a quantum extension of Kolmogorov’s complexity (Kolmogorov, 1965). For that reason, they all will present the latter feature: monotonically increasing function of disorder (Shiner et al., 1999).

That is an undesirable feature for a modern complexity measure to have since its objective is to measure the degree of organization between the periodic and the random, and not how disordered a system is. To this goal, there are a number of entropies to be used (Tsallis, 2002).

This shows that to build a modern quantum complexity measure is still a new path to follow, by the use of concepts from quantum information theory as well as from modern complexity measures.

Thus, in this work the combination of a promising new complexity measure to quantum information-theoretic procedures is proposed to tackle the quantum state complexity issue.

The rest of this paper is organized as follows; in the next section there are a brief introduction to the chosen complexity measure, Interaction Structures
Kahle et al. (2009), and the definition of an iterative successive measurement process necessary for interaction structures application to quantum states. In Section 3 the proposed full method is applied to a mixed tripartite state of qubits, the W-state, and the results are discussed. Lastly, Section 4 brings a summary of this paper with conclusions and future works addressed.

2. Theory

2.1. Interaction Structures

Proposed by Kahle et al. (2009), this is an approach essentially based on quantifying the degree of interaction among various parts of one system, in a fashion that as many parts contribute to the system dynamics, higher orders of interaction rise and more complex this system is. Albeit brief, it is necessary to introduce some concepts of information geometry for a proper definition of this approach, among them, exponential families and the Kullback-Leibler divergence. For a complete definition, see Kahle et al. (2009).

Let a configuration be formed by elements, and an index \( v \) in it in the set \( V := \{1, \ldots, N\} \). For each index \( v \), there is a finite configuration space \( \mathcal{X}_v \). Hence, by constructing the product space such that \( \mathcal{X}_V := \times_{v \in V} \mathcal{X}_v \), it is possible to obtain the space of all possible configurations \( \mathcal{X}_V \).

It is thus possible to consider real functions in \( \mathcal{X}_V \) as elements in a vector space \( \mathbb{R}^{|\mathcal{X}_V|} := \{f : \mathcal{X}_V \to \mathbb{R}\} \); hence, probability distribution functions in \( \mathcal{X}_V \) can also be elements in \( \mathbb{R}^{|\mathcal{X}_V|} \). Thus, let

\[
\mathcal{P}(\mathcal{X}_V) := \left\{ P \in \mathbb{R}^{|\mathcal{X}_V|} : P(x) \geq 0, \sum_{x \in \mathcal{X}_V} P(x) = 1 \right\},
\]

(1)

be probability distributions in \( \mathcal{X}_V \).

For two distributions \( P, Q \in \mathcal{P}(\mathcal{X}_V) \) a semi-metric in this space is defined by

\[
D(P||Q) := \begin{cases} 
\sum_{x \in \mathcal{X}_V} P(x) \log_2 \frac{P(x)}{Q(x)}, & \text{supp}(P) \subseteq \text{supp}(Q) \\
\infty, & \text{else}
\end{cases}
\]

(2)

known as Kullback-Leibler divergence or relative entropy. It is not a full metric, but \( D(P||Q) = 0 \iff P \equiv Q \).
In addition, in this space it is possible to define families of probability measures that compose sub-manifolds in $\mathcal{P}(\mathcal{X}_V)$. The exponential families are one of them, they are constructed through the operation

$$\exp : \mathbb{R}^{\mathcal{X}_V} \to \mathcal{P}(\mathcal{X}_V),\ f \mapsto \frac{e^{f}}{\sum_{x \in \mathcal{X}_V} e^{f}},$$

which maps the space $\mathbb{R}^{\mathcal{X}_V}$ into the set of probability distributions by component wise exponentiation and normalization. Let $\mathcal{I}$ be a linear sub-space in $\mathbb{R}^{\mathcal{X}_V}$, the exponential family $\mathcal{E}_{\mathcal{I}}$ is defined as the image of $\mathcal{I}$ through the exponential mapping.

The degree of interaction between parts of one system can be quantified by the definition of interaction spaces. For any $A \subseteq V$, it is possible to split the space into two; thus, any $x \in \mathcal{X}_V$ is written as $x = (x_A, x_{V/A})$, composed by parts in $A$ and outside it. In this manner, space $\mathcal{I}_A$ is defined as the sub-space of functions only of configurations inside $A$

$$\mathcal{I}_A := \left\{ f \in \mathbb{R}^{\mathcal{X}_V} : f(x_A, x_{V/A}) = f((x_A, x'_{V/A}) \right. \forall x_A \in \mathcal{X}_A, \forall x_{V/A}, x'_{V/A} \in \mathcal{X}_{V/A} \}.$$  

(4)

Then, by defining $A$ in a proper way it is possible to construct spaces in which the probability distributions are functions only of any $k$ elements

$$\mathcal{I}_k := \text{span}_{A \subseteq V, |A| = k} \mathcal{I}_A.$$  

(5)

There is a exponential family $\mathcal{E}_k = \exp(\mathcal{I}_k)$ associated with each interaction space $\mathcal{I}_k$, $k = 1, \cdots, N$. It is clear that they form a structure such that $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_N$, therefore the exponential families do, too, $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_N$.

Exponential families have very desirable statistical properties, mainly related to inference methods. In this approach, given an inferred probability distribution function $\hat{P}$ from an observation procedure, it is desired to know how far this function is from each exponential family $\mathcal{E}_k$ to then quantify the information being shared by $k$ elements.

Therefore, this distance is computed by minimizing the Kullback-Leibler divergence $D(\hat{P}||Q)$ between function $\hat{P}$ and another function $Q$ in a given exponential family $\mathcal{E}$

$$D(P||\mathcal{E}) := \inf_{Q \in \mathcal{E}} D(P||Q).$$  

(6)
It can be proved (Kahle et al., 2009) that this minimization procedure is equivalent to the maximum-likelihood estimation of $P$ in set $E$.

Hence, this minimization procedure is equivalent to $P$ projection in the set $E$, meaning that $Q$ is the best approximation of $P$ in the set $\mathcal{E}$. Therefore, there are a number of methods to compute this minimization. The one used in this work is the same as that in the original work by Kahle et al. (2009), iterative proportional fitting algorithm.

Then, the interaction structures complexity is defined as a vector-valued quantity $I(P) := (I^{(1)}(P), \ldots, I^{(k)}(P))$, with

$$I^{(k)}(P) := D(P||E_{k-1}) - D(P||E_k),$$

for $k = 1, \ldots, N$; due to the divergence continuity for all $P \in \overline{P(\mathcal{X})}$.

In this way, the interaction structures complexity $I_k$ is established as a quantification of information being shared by $k$ dynamical structures in the system.

2.2. Quantum state sampling

The use of interaction structures complexity with quantum states is not straightforward. It is necessary to define a measurement process similar to Quantum State Tomography in experimental quantum information science to use it in a proper way. This procedure is required to characterize the underlying quantum state probability distributions and will be called Quantum State Sampling (QSS).

QSS is described in Figure 1. The analyzed quantum state is produced by a perfect Source. The source generates identical quantum states and for each one of them a sequential eigenstate projective measurement procedure occurs in its sub-systems, which constitute the original quantum state. After the last sub-system measurement takes place, the original state is completely destroyed and another one is generated by the source to undergo the same routine. This is done iteratively until the probability distribution of measurement outcomes is securely estimated. Finally, this estimated probability distribution is the input of the interaction structures complexity measure.

This approach to adapt interaction structures complexity measure to quantum systems can be applied to any $n$-partite quantum state of qudits by a proper configuration in the interaction structures method and maintaining this quantum state sampling process as it is.
3. Results and Discussion

A mixed tripartite state constructed with a maximally entangled W-state with white noise is used to test the approach as to quantum state complexity. This is state is defined as

$$\rho_{W3} = \alpha |W\rangle \langle W| + \frac{1 - \alpha}{8} I^{\otimes 3},$$

with $0 \leq \alpha \leq 1$.

The use of a tripartite state is important for this quantification for every qubit to be an interaction structure and then interactions up to $k = 3$ to be assessed. One can thus observe the behavior of $I_3$ which is the first complexity index in the interaction structures measurement because it describes the first high order interactions $k > 2$ (Kahle et al., 2009).

The QSS procedure is modified accordingly to Figure 2 to accommodate the $\alpha$-controlled state $\rho_{W3}$.

Figures 3 and 4 show the interaction structures complexity measure obtained via QSS for the state $\rho_{W3}$. The first complexity index $I_3$ is convex between the maximally entangled and the maximally mixed state. Thus, at some point between state $\rho_{W3}$ leaves the maximum mixedness, becomes entangled and then maximally entangled; there is a peak in high order interactions between the structures, the qubits.
This shows the difference between already proposed approaches (Rogers et al., 2008; Mora and Briegel, 2005; Mora et al., 2007) for quantum state complexity. In the present approach, the complexity is not a monotonically increasing function of disorder, instead it is a convex function capable of indicating the structure underlying the system dynamics between order and disorder.

The other two quantities $I_{1,2}$ responsible for low order interactions are monotonically increasing, Figure 4. This was expected due to the similarity between the analysis done by Kahle et al. (2009) for synchronizing dynamical systems networks and a maximally entangled quantum state. In these two extremes, the pair interactions are decisive for a complete description of the system dynamics. Indicating that maximally entangled qubits behave as being synchronized, or at least their measurement outcomes do.

This shows that the entanglement phenomenon has close connection with quantum complexity, as well as the quantity of mixedness and entanglement itself in entangled mixed states complexity. All of them playing non-trivial roles still to be explored.

4. Conclusions

The proposed method for quantum state complexity measure is capable of tackling any $n$–partite state composed by qudits. However, the iterative pro-
Figure 3: Interaction of order $n = 3$ in state $\phi_{W3}$. 
Figure 4: Interactions of order $n = 1, 2$ in state $g_{W3}$. 
portional fitting algorithm is unfeasible for $|X_V| \gg 10^6$ (Kahle et al., 2009).

Hence, new methods for Kullback-Leiber’s divergence minimization stands as a future development to ensure assessing higher dimensional quantum states.

Issues such as mixedness and entanglement roles in quantum complexity still need to be better clarified through their quantification along with quantum state complexity. However, the observed convexity in $q_{W3}$ state complexity suggests a duality mixedness/entanglement responsible for it. In addition, more exploration is necessary in a variety of quantum states to assess these issues properly.

Finally, the proposed method to quantify quantum state complexity stands out of previous attempts due to its convexity behavior of order/disorder. Thus, the present approach does not measure how disordered a quantum state is, but indicates how much structures interaction is responsible for observed system dynamics.
ACKNOWLEDGEMENTS

This work is sponsored by the Brazilian National Council for Scientific and Technological Development (Conselho Nacional de Desenvolvimento Científico e Tecnológico) - CNPq.

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