Linear Algebra for Mueller Calculus

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Abstract

We give a self-contained exposition of some mathematical aspects of the Mueller-Stokes formalism. In the first part we review some basic notions of linear algebra and establish a proper notation. In the second part we introduce the Mueller-Stokes formalism and derive some useful mathematical relation between physical quantities. Finally a useful decomposition theorem is reviewed.
I. INTRODUCTION AND NOTATION

In these notes we have collected some mathematical results that are not easy to find in the literature. We assume that the reader is already knowledgeable about the Mueller-Stokes formalism. All the results presented here can be found in the following references:

[1] R. W. Schmieder, “Stokes-Algebra Formalism”, J. Opt. Soc. Am. 59, 297-302 (1969).

[2] S. R. Cloude, “Group theory and polarisation algebra”, Optik 75, 26-36 (1986).

[3] K. Kim, L. Mandel, and E. Wolf, “Relationship between Jones and Mueller matrices for random media”, J. Opt. Soc. A 4, 433-437 (1987).

[4] S. R. Cloude, “Conditions for the physical realisability of matrix operators in polarimetry”, in Polarization Considerations for Optical Systems II, R. A. Chipman ed., Proc. Soc. Photo-Opt. Instrum. Eng. 1166, 177-185 (1989).

[5] S. R. Cloude, “Lie Groups in Electromagnetic Wave Propagation and Scattering”, Journal of Electromagnetic Waves and Applications 6, 947-974 (1992).

[6] D. G. M. Anderson and R. Barakat, “Necessary and sufficient conditions for a Mueller matrix to be derivable from a Jones matrix”, J. Opt. Soc. A 11, 2305-2319 (1994).

Different authors use different notations which makes difficult to recognize the same result appearing on different papers. For this reason we have tried to simplify and unify the notation by adopting one which seems (at least to us) to be the closest to the physics (especially the quantum physics) of the problem. For example, we have not adopted the awkward “optical” notation for the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

but we have adopted the standard “quantum” notation

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Of course, as a consequence of this choice, also the Stokes parameters defined in these notes are different from the standard “optical” one. If with \( \mathbf{E} = X \mathbf{x} + Y \mathbf{y} \) we denote the
electric field of an homogeneous plane wave propagating along the axis \( z \), then our Stokes parameters \( \{S_0, S_1, S_2, S_3\} \) are defined as

\[
\begin{align*}
S_0 &= |X|^2 + |Y|^2 = I = S_{0}^{BW} = S_{0}^{H}, \\
S_1 &= XY^* + X^*Y = U = S_{2}^{BW} = S_{2}^{H}, \\
S_2 &= i(XY^* - X^*Y) = -V = -S_{3}^{BW} = -S_{3}^{H}, \\
S_3 &= |X|^2 - |Y|^2 = Q = S_{1}^{BW} = -S_{1}^{H},
\end{align*}
\]

where the last three columns display the traditional \( \{I, Q, U, V\} \), the “Born-Wolf”\(^1\) \( \{S_{0}^{BW}, S_{1}^{BW}, S_{2}^{BW}, S_{3}^{BW}\} \), and the “van de Hulst”\(^2\) \( \{S_{0}^{H}, S_{1}^{H}, S_{2}^{H}, S_{3}^{H}\} \) definitions of the Stokes parameters, respectively. It is curious to notice that this change of notation was already suggested in the sixties [1] but it was unadopted. The last three Stokes parameters form a Cartesian coordinate system on the Poincaré sphere (Fig. 1). This change of notation for the Stokes parameters also causes a change in the definition of the Mueller matrix. For example, if we write the Stokes vectors in ours and in the “van de Hulst” notation as

\[
\vec{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, \quad \vec{S}^H = \begin{pmatrix} S_0^H \\ S_1^H \\ S_2^H \\ S_3^H \end{pmatrix},
\]

then, from Eq. (3) it is easy to see that \( \vec{S} \) and \( \vec{S}^H \) are related by a unitary matrix \( Q \),

\[
\vec{S} = Q\vec{S}^H,
\]

where

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

Now, let us consider a linear optical process described in the two different notations as

\[
\vec{S}^{\text{out}} = M\vec{S}^{\text{in}}, \quad \vec{S}^{H,\text{out}} = M^{H}\vec{S}^{H,\text{in}}.
\]

\(^1\) M. Born, E. Wolf, *Principles of optics*, 7th ed., (Cambridge University Press, Cambridge, 1999).

\(^2\) H. C. van de Hulst, *Light Scattering by Small Particles*, (Dover Publications, Inc., New York, 1981).
Then, it is easy to calculate
\[ \vec{S}_{\text{out}} = Q \vec{S}_{\text{out}}^H = Q M^H \vec{S}_{\text{in}}^H \]
\[ = Q M^H Q^{-1} Q \vec{S}_{\text{in}}^H \]
\[ = [Q M^H Q^{-1}] \vec{S}_{\text{in}} \]
\[ \equiv M \vec{S}_{\text{in}}, \]
from which it follows
\[ M = Q M^H Q^{-1}. \] (9)

This is the sought relation between our definition of Mueller matrix, and the optical one.

These notes aim to be, from mathematical point of view, as self-contained as possible; all formulae are derived, the only omitted derivations are the ones which reduce to an explicit calculation. For example the formula \( \sigma_1 \sigma_2 = i \sigma_3 \) cannot be “demonstrated”, it must be checked by explicit calculation from the definition of the Pauli matrices. However, all the omitted explicit calculations can be easily done in few seconds with a computer program like Mathematica.

As we already said, these notes focus on the mathematical aspects of the Mueller formalism, so no emphasis is given to any physical process. For this reason in the first part of this script we almost exclusively deal with the case of deterministic (or Mueller-Jones) Mueller matrices which requires the knowledge of the same amount of linear algebra results as the more general case. However, all formulae derived here can be straightforwardly extended to the case of non-deterministic Mueller matrices.

A. Notation

A few words about the notation. We use three different kind of indices: Latin, Greek and Calligraphic. Latin indices \( i, j, k, \ldots \) run from 0 to 1 and label the components of \( 2 \times 2 \) matrices and 2-D vectors. Greek indices \( \mu, \nu, \alpha, \ldots \) run from 0 to 3 and label the components of \( 4 \times 4 \) matrices and 4-D vectors. Finally, Calligraphic indices \( A, B, C, \ldots \) run from 0 to 15 and label the components of 16-D vectors. In these notes the Einstein summation convention is used extensively, that is the sum on repeated indices (Latin, Greek and Calligraphic) is understood. For example
\[ a_\mu = \Lambda_{\mu \nu} b_\nu \iff a_\mu = \sum_{\nu=0}^{3} \Lambda_{\mu \nu} b_\nu. \] (10)
We often use the direct product of two matrices $A$ and $B$, indicated with the symbol "⊗":

$$ C = A \otimes B. $$

(11)

For this kind of matrix product, the standard convention for the indices is the following:

$$ c_{ik,jl} = a_{ij} b_{kl}. $$

(12)

It worths to note the order of the indices $j$ and $k$ in both sides of this equation; it will play an important role in these notes.

II. MATRIX BASES

In this section we study two different ways to represent $2 \times 2$ matrices and the relations between different representations.

A. The Standard Basis

Let $A \in \mathbb{C}^{2 \times 2}$ denotes a $2 \times 2$ complex-valued matrix defined in terms of its elements $[A]_{ij} \equiv a_{ij}$, $(i, j = 0, 1)$ as

$$ A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}. $$

(13)

Any $2 \times 2$ matrix can be put in one-to-one correspondence with a complex 4-vector $\vec{a} \in \mathbb{C}^4$ by writing

$$ A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \equiv \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}, $$

(14)

where

$$ \vec{a} = \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} \equiv \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}. $$

(15)

This rule is very simple and can be easily extended to $n \times n$ matrices by defining

$$ a_{ij} \equiv a_{ni+j}, $$

(16)
for \( i, j = 0, \ldots, n - 1 \). This rule is so important that in the remaining part of these notes we shall refer to as the “Rule”. At this point it is important to notice that when we write a vector \( \vec{a} \) as in Eq. (15), we are implicitly assuming that its components \( a_\mu, \mu = 0, \ldots, 3 \) are referred to the so-called standard basis in \( \mathbb{R}^4 \), that is

\[
\vec{a} = a_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]  

(17)

Analogously Eq. (14) naturally suggests the possibility to write

\[
A = a_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\equiv a_\mu \epsilon_\mu, \quad (\mu = 0, 1, 2, 3),
\]

where summation on repeated indices is understood and the basis matrices \( \epsilon_\mu \in \mathbb{R}^{2 \times 2} \) are defined as

\[
\epsilon_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_1 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_2 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \epsilon_3 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(19)

Then the numbers \( \{a_\mu\} \), that we have found by using the Rule Eq.(16), appear to be the components of the matrix \( A \) with respect to the basis \( \{\epsilon_\mu\} \). In order to demonstrate this, it is necessary to define a norm in the vector space \( \mathbb{C}^{2 \times 2} \) of the complex \( 2 \times 2 \) matrices. It is possible to introduce a norm in \( \mathbb{C}^{2 \times 2} \) by defining the scalar product \( \{A, B\} \) between two matrices \( A \) and \( B \) as

\[
\{A, B\} = \text{Tr}\{A^\dagger B\}
\]

\[
= a_{ij}^* b_{ij}, \quad (i, j = 0, 1),
\]

(20)

where summation on repeated indices is again understood and \( A^\dagger \) denotes the Hermitian-conjugate of \( A \); that is \( A^\dagger = (A^T)^* = (A^*)^T \), where \( A^* \) and \( A^T \) are the complex conjugate and the transpose of \( A \) respectively. Moreover, since \( \{A, B\}^* = (a_{ij}^* b_{ij})^* = a_{ij} b_{ij}^* = \text{Tr}\{B^\dagger A\} \), the result \( \{A, B\}^* = \{B, A\} \) follows. By explicit calculation, one can see that the basis vectors \( \{\epsilon_\mu\} \) are orthonormal with respect to that norm:

\[
\{\epsilon_\mu, \epsilon_\nu\} = \text{Tr}\{\epsilon_\mu^T \epsilon_\nu\}
\]

\[
= \delta_{\mu\nu},
\]

(21)
where \( \mu, \nu = 0, \ldots, 3 \) and \( \epsilon^{\mu} = \epsilon_{(\mu)}^T \) follows from Eq. (19). Now, having introduced the norm Eq. (20), it is easy to calculate the components of the matrix \( A \) with respect to the basis \( \{ \epsilon_{(\mu)} \} \) as

\[
\{ \epsilon_{(\mu)} , A \} = \{ \epsilon_{(\mu)} , a_\nu \epsilon_{(\nu)} \}
= a_\nu \{ \epsilon_{(\mu)} , \epsilon_{(\nu)} \}
= a_\mu,
\]

where \( \mu, \nu = 0, 1, 2, 3 \).

Then we have shown that it is possible to associate with any matrix \( A \in \mathbb{C}^{2 \times 2} \) a vector \( \vec{a} \in \mathbb{C}^4 \) and there are two different (but equivalent) ways to calculate \( \vec{a} \): we can either use the Rule given in Eq. (17)

\[
a_{\mu=2i+j} = a_{ij}, \quad (i, j = 0, 1),
\]

or calculate explicitly

\[
a_\mu = \{ \epsilon_{(\mu)} , A \}
= \text{Tr} \{ \epsilon_{(\mu)}^T A \}, \quad (\mu = 0, 1, 2, 3).
\]

Until now, \( A \) was left arbitrary, therefore Eq. (24) holds for any \( 2 \times 2 \) matrix. If \( A \) coincides with one of the basis matrices \( \epsilon_{(\alpha)} \), then Eq. (24) gives the components \( [\epsilon_{(\alpha)}]_\mu \) (\( \mu = 0, \ldots, 3 \)), of the matrix \( \epsilon_{(\alpha)} \) with respect to the basis \( \{ \epsilon_{(\mu)} \} \):

\[
[\epsilon_{(\alpha)}]_\mu = \{ \epsilon_{(\mu)} , \epsilon_{(\alpha)} \}
= \text{Tr} \{ \epsilon_{(\mu)}^T \epsilon_{(\alpha)} \}
= \delta_{\mu\alpha},
\]

where \( \alpha, \mu = 0, 1, 2, 3 \). Therefore we can build the basis 4-vectors \( \vec{e}_{(\alpha)} \in \mathbb{R}^4 \) associated to the basis matrices \( \epsilon_{(\alpha)} \) as

\[
\vec{e}_{(\alpha)} = \begin{pmatrix} [\epsilon_{(\alpha)}]_{00} \\ [\epsilon_{(\alpha)}]_{01} \\ [\epsilon_{(\alpha)}]_{10} \\ [\epsilon_{(\alpha)}]_{11} \end{pmatrix} = \begin{pmatrix} [\epsilon_{(\alpha)}]_0 \\ [\epsilon_{(\alpha)}]_1 \\ [\epsilon_{(\alpha)}]_2 \\ [\epsilon_{(\alpha)}]_3 \end{pmatrix} = \begin{pmatrix} \delta_{0\alpha} \\ \delta_{1\alpha} \\ \delta_{2\alpha} \\ \delta_{3\alpha} \end{pmatrix}, \quad (\alpha = 0, 1, 2, 3).
\]

It is trivial to calculate from Eq. (26) that

\[
\vec{e}_{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

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that is \{\vec{e}(\mu)\} is simply the standard basis in \( \mathbb{R}^4 \). In summary, we have shown that there is a one-to-one correspondence between the standard basis \{\epsilon(\mu) \in \mathbb{R}^{2\times2}\} and the standard basis \{\vec{e}(\mu) \in \mathbb{R}^4\}.

**B. The Pauli Basis**

Another basis commonly used in physics is the so called Pauli basis constituted by the \( 2 \times 2 \) identity matrix and the three Pauli matrices. Here we use a normalized version of the Pauli matrices defined as

\[
\sigma_{(0)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{(1)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{(2)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{(3)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

An explicit calculation shows that they satisfy the following multiplication table:

\[
\begin{array}{c|cccc}
\sqrt{2}\sigma_{(\mu)}\sigma_{(\nu)} & \sigma_{(0)} & \sigma_{(1)} & \sigma_{(2)} & \sigma_{(3)} \\
\hline
\sigma_{(0)} & \sigma_{(0)} & \sigma_{(1)} & \sigma_{(2)} & \sigma_{(3)} \\
\sigma_{(1)} & \sigma_{(1)} & \sigma_{(0)} & i\sigma_{(3)} & -i\sigma_{(2)} \\
\sigma_{(2)} & \sigma_{(2)} & -i\sigma_{(3)} & \sigma_{(0)} & i\sigma_{(1)} \\
\sigma_{(3)} & \sigma_{(3)} & i\sigma_{(2)} & -i\sigma_{(1)} & \sigma_{(0)} \\
\end{array}
\]

Moreover, again an explicit calculation shows that these matrices are orthonormal with respect to the norm defined in Eq. (20):

\[
\{\sigma_{(\mu)},\sigma_{(\nu)}\} = \operatorname{Tr}\{\sigma_{(\mu)}^\dagger\sigma_{(\nu)}\} = \delta_{\mu\nu},
\]

where \( \mu, \nu = 0, 1, 2, 3 \). The Pauli basis is complete; in order to show this we have to calculate the components \( [\sigma_{(\mu)}]_\alpha \) of the matrices \( \sigma_{(\mu)} \) with respect to the basis \( \{\epsilon_{(\mu)}\} \) in the way we learned in the previous subsection (see Eq. (24) with \( A = \sigma_{(\mu)} \)):
where $\mu, \alpha = 0, \ldots, 3$, and in the last line we have defined the $4 \times 4$ transformation matrix $\Lambda$ in terms of its elements $[\Lambda]_{\alpha \mu} \equiv \Lambda_{\alpha \mu} = \text{Tr}\{\epsilon^{T}_{(\alpha)}\sigma_{(\mu)}\}$. An explicit calculation shows that

$$\Lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

(31)

where Eqs. (19, 28) have been used. In the previous subsection we shown how to build the basis vectors $\{\vec{e}_{(\alpha)}\}$ in $\mathbb{R}^4$ associated to the basis matrices $\{\epsilon_{(\alpha)}\}$ in $\mathbb{R}^{2 \times 2}$. Analogously, we can now build the basis vectors $\{\vec{s}_{(\mu)}\}$ in $\mathbb{C}^4$ associated to basis matrices $\{\sigma_{(\mu)}\}$ in $\mathbb{C}^{2 \times 2}$. To this end, for a given $\mu$ we define the four components $[\vec{s}_{(\mu)}]_{\alpha}$, ($\alpha = 0, \ldots, 3$) of the vector $\vec{s}_{(\mu)}$, as $[\vec{s}_{(\mu)}]_{\alpha} \equiv [\sigma_{(\mu)}]_{\alpha}$, that is

$$\vec{s}_{(\mu)} \equiv \begin{pmatrix} [\sigma_{(\mu)}]_{0} \\ [\sigma_{(\mu)}]_{1} \\ [\sigma_{(\mu)}]_{2} \\ [\sigma_{(\mu)}]_{3} \end{pmatrix} = \begin{pmatrix} \Lambda_{0\mu} \\ \Lambda_{1\mu} \\ \Lambda_{2\mu} \\ \Lambda_{3\mu} \end{pmatrix},$$

(32)

where $\mu = 0, \ldots, 3$. From Eq. (32) is clear that the $\mu$-th column of the matrix $\Lambda$ is made of the components $[\vec{s}_{(\mu)}]_{\alpha}$ of the 4-vector $\vec{s}_{(\mu)}$. Alternatively we can find the vectors $\vec{s}_{(\mu)}$ by using the Rule: $[\vec{s}_{(\mu)}]_{\alpha=2i+j} = [\sigma_{(\mu)}]_{ij}$. For example, for $\mu = 2$ we have

$$\sigma_{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} \equiv \begin{pmatrix} [\vec{s}_{(2)}]_{0} & [\vec{s}_{(2)}]_{1} \\ [\vec{s}_{(2)}]_{2} & [\vec{s}_{(2)}]_{3} \end{pmatrix},$$

(33)

from which we deduce that

$$\vec{s}_{(2)} = \begin{pmatrix} [\vec{s}_{(2)}]_{0} \\ [\vec{s}_{(2)}]_{1} \\ [\vec{s}_{(2)}]_{2} \\ [\vec{s}_{(2)}]_{3} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ i \\ 0 \end{pmatrix}.$$  

(34)

From Eq. (31) it is easy to check by explicit calculation that $\Lambda$ is unitary, that is $\Lambda^{\dagger}\Lambda = \Lambda\Lambda^{\dagger} = I_4$, where $I_4$ is the $4 \times 4$ identity matrix. In terms of the components with respect to
the basis \( \{ \epsilon(\mu) \} \) the relation \( I_4 = \Lambda \Lambda^\dagger \) becomes

\[
\delta_{\alpha\beta} = \sum_{\mu=0}^{3} \Lambda_{\alpha\mu} \Lambda_{\mu\beta}^\dagger \\
= \sum_{\mu=0}^{3} \Lambda_{\alpha\mu} \Lambda_{\beta\mu}^* \\
= \sum_{\mu=0}^{3} [\sigma(\mu)]_\alpha [\sigma^*(\mu)]_\beta.
\]  

(35)

The first and the last line of Eq. (35) give us the completeness relation (also called resolution of the identity) we are seeking:

\[
\sum_{\mu=0}^{3} [\sigma(\mu)]_\alpha [\sigma^*(\mu)]_\beta = \delta_{\alpha\beta}.
\]  

(36)

By using Eq. (30) this relation can be written in less involved form as

\[
\sum_{\mu=0}^{3} [\sigma(\mu)]_\alpha [\sigma^*(\mu)]_\beta = \sum_{\mu=0}^{3} \{ \epsilon(\alpha), \sigma(\mu) \} \{ \sigma(\mu), \epsilon(\beta) \} \\
= \{ \epsilon(\alpha), \epsilon(\beta) \} \\
= \delta_{\alpha\beta},
\]  

(37)

which is easier to understand. It is useful for later purposes to write the completeness relation in terms of the “Latin” matrix elements \([\sigma(\mu)]_{ij}, (i, j = 0, 1)\). To this end we first associate the four indices \( i, j, k, l = 0, 1 \) to the two indices \( \alpha, \beta = 0, \ldots, 3 \), by using the Rule:

\[
\alpha = 2i + j, \\
\beta = 2k + l.
\]  

(38)

Then, after noticing that \( \delta_{\alpha\beta} = \delta_{2i+j,2k+l} = \delta_{ik}\delta_{jl} \), we rewrite Eq. (36) as

\[
\sum_{\mu=0}^{3} [\sigma(\mu)]_{ij} [\sigma^*(\mu)]_{kl} = \delta_{ik}\delta_{jl}.
\]  

(39)

From the definition in Eq. (30), it is obvious that \( \Lambda \) is the matrix that performs the change from the Pauli basis \( \{ \sigma(\mu) \} \) to the standard basis \( \{ \epsilon(\mu) \} \):

\[
\sigma(\mu) = \epsilon(\nu) \Lambda_{\nu\mu}, \\
\epsilon(\nu) = \sigma(\mu) \Lambda_{\mu\nu}^\dagger.
\]  

(40)
where $\mu, \nu = 0, \ldots, 3$ and summation on repeated indices is understood. Previously we learned that to any matrix corresponds a vector, therefore the matrix $\Lambda$ also performs the change from the basis $\{s(\mu)\}$ to the standard basis $\{e(\mu)\}$. In fact, since by definition $[s(\mu)]_\alpha = [\sigma(\mu)]_\alpha = \Lambda_{\alpha\mu}$, then
\[
\tilde{s}(\mu) = e(\alpha)[s(\mu)]_\alpha = \tilde{e}(\alpha)\Lambda_{\alpha\mu},
\]
where $\alpha, \mu = 0, \ldots, 3$. This relation can be written in fully matrix form by noticing that if $U$ denotes the unitary transformation between the two basis: $s(\mu) = U\tilde{e}(\mu)$, then it follows that $U_{\alpha\mu} \equiv (e(\alpha), U\tilde{e}(\mu)) = (e(\alpha), s(\mu)) = \Lambda_{\alpha\mu}$, where the parentheses symbol $(\vec{u}, \vec{v})$ indicates the ordinary Euclidean scalar product in $\mathbb{C}^n$
\[
(\vec{u}, \vec{v}) = \sum_{\alpha=0}^{n-1} u_\alpha^* v_\alpha.
\]
So we have found that $U = \Lambda$ and, therefore,
\[
\tilde{s}(\mu) = \Lambda\tilde{e}(\mu).
\]

**III. THE MUELLER FORMALISM**

Let us consider a doublets of stochastic variables,
\[
E = \begin{pmatrix} E_0 \\ E_1 \end{pmatrix},
\]
which transform under the action of a *deterministic* optical device, as
\[
E \rightarrow E' = TE,
\]
where the $2 \times 2$ complex-valued transformation matrix $T$ is known as the Jones matrix representing the optical device. We do not make any hypothesis on the nature of the matrix $T$, it can be arbitrary. The quantities $E_0, E_1$ in Eq. (44) are complex random variables described by a given ensemble. Starting from $E_0, E_1$ we can build the **covariance matrix** (or polarization matrix) $J \in \mathbb{C}^{2 \times 2}$ whose elements are defined as
\[
J_{ij} = \langle E_i E_j^* \rangle \quad (i, j = 0, 1),
\]
where $\langle \cdot \rangle$ denotes the ensemble average. Note that this average has nothing to do with any random medium, at this stage we are just considering two components of the electromagnetic field as two stochastic variables. By definition, $J$ is Hermitian and nonnegative (or, positive semidefinite), that is $(\mathbf{x}, J\mathbf{x}) \geq 0$, $\forall \mathbf{x} \in \mathbb{C}^2$:

\begin{equation}
(\mathbf{x}, J\mathbf{x}) = x_i^* J_{ij} x_j \\
= x_i^* \langle E_i E_j^* \rangle x_j \\
= \langle x_i^* E_i E_j^* x_j \rangle \\
= \langle (x_i^* E_i)(x_j^* E_j)^* \rangle \\
= \langle |x_i^* E_i|^2 \rangle \\
= \langle |(\mathbf{x}, \mathbf{E})|^2 \rangle \geq 0,
\end{equation}

where $i, j = 0, 1$ and summation on repeated indices is understood. Moreover, in the third line of Eq. (47) we have exploited the fact that, by hypothesis, the vector components $x_i$ are deterministic variables and, therefore, are not affected by the ensemble average.

As any other $2 \times 2$ matrix, $J$ can be written in the basis $\{\sigma(\mu)\}$ as

\begin{equation}
J = S_\mu \sigma(\mu) \quad (\mu = 0, \ldots, 3),
\end{equation}

where the components $S_\mu = \text{Tr}\{\sigma(\mu)J\}$ of the 4-vector $\vec{S}$ are known as the *Stokes parameters* of the field. Explicitly

\begin{equation}
J = \frac{1}{\sqrt{2}} \begin{pmatrix}
S_0 + S_3 & S_1 - iS_2 \\
S_1 + iS_2 & S_0 - S_3
\end{pmatrix}.
\end{equation}

Form the formula above we see that

\begin{equation}
\text{Tr}J = \sqrt{2} S_0,
\end{equation}

while from the definition Eq. (46) we have

\begin{equation}
\text{Tr}J = \langle |E_0|^2 \rangle + \langle |E_1|^2 \rangle \equiv I,
\end{equation}

where with $I$ we denoted the total intensity of the beam. By equating Eq. (50) with Eq. (51) we obtain our definiton of $S_0$:

\begin{equation}
S_0 = \frac{I}{\sqrt{2}}.
\end{equation}
Under the transformation $T$, the polarization matrix $J$ transform as $J \rightarrow J'$ where, by definition,

$$J'_{ij} = \langle E'_i E'_j^* \rangle$$

$$= T_{ik} \langle E_k E_l^* \rangle T_{jl}^*$$

$$= T_{ik} J_{kl} T_{lj}^*,$$

or, in matrix form,

$$J' = T J T^\dagger.$$  \hspace{1cm} (54)

From Eq. (54) is clear that the transformed coherency matrix $J'$ is still Hermitian and non-negative. In correspondence to the transformation $J \rightarrow J'$, the Stokes parameters transform as $S_\mu \rightarrow S'_\mu$ where, by definition,

$$S'_\mu = \text{Tr}\{\sigma(\mu) J'\}$$

$$= \text{Tr}\{\sigma(\mu) T J T^\dagger\}$$

$$= \text{Tr}\{\sigma(\mu) T S_\nu \sigma(\nu) T^\dagger\}$$

$$= \text{Tr}\{\sigma(\mu) T \sigma(\nu) T^\dagger\} S_\nu$$

$$\equiv M_{\mu\nu} S_\nu,$$

where Eq. (48) has been used in the third line and we have defined the 4 $\times$ 4 Mueller matrix $M$ as

$$M_{\mu\nu} = \text{Tr}\left\{\sigma(\mu) T \sigma(\nu) T^\dagger\right\}$$

$$= \left\{\sigma(\mu), T \sigma(\nu) T^\dagger\right\},$$  \hspace{1cm} (56)

where $\mu, \nu = 0, \ldots, 3$. It is easy to see that $M$ has real elements:

$$M^*_{\mu\nu} = \left\{\sigma(\mu), T \sigma(\nu) T^\dagger\right\}^*$$

$$= \left\{T \sigma(\nu) T^\dagger, \sigma(\mu)\right\}$$

$$= \text{Tr}\left\{T \sigma(\nu) T^\dagger \sigma(\mu)\right\}$$

$$= \text{Tr}\left\{\sigma(\mu) T \sigma(\nu) T^\dagger\right\}$$

$$= M_{\mu\nu},$$

where the cyclic property of the trace: $\text{Tr}\{AB\} = \text{Tr}\{BA\}$ has been used. En passant we may note that if we write the Jones matrix $T$ in the Pauli basis as $T = c_\alpha \sigma(\alpha)$, where
\[ c_\alpha = \text{Tr}\{\sigma(\alpha)T\}, \] then Eq. (56) can be written as

\[
M_{\mu\nu} = \text{Tr}\left\{\sigma(\mu)T\sigma(\nu)T^\dagger\right\} \\
= c_\beta c_\alpha^* \text{Tr}\left\{\sigma(\mu)\sigma(\beta)\sigma(\nu)\sigma(\alpha)\right\} \\
\equiv C_{\beta\alpha} \text{Tr}\left\{\sigma(\alpha)\sigma(\mu)\sigma(\beta)\sigma(\nu)\right\} \\
\equiv C_{\beta\alpha}[\Gamma(\mu\nu)]_{\alpha\beta} \\
= \text{Tr}\left\{CT(\mu\nu)\right\},
\]

where the cyclic property of the trace has been used and we have defined the coherency matrix \( C_{\beta\alpha} \equiv c_\beta c_\alpha^* \) and the 16 matrices \( \{\Gamma(\mu\nu)\} : [\Gamma(\mu\nu)]_{\alpha\beta} \equiv \text{Tr}\left\{\sigma(\alpha)\sigma(\mu)\sigma(\beta)\sigma(\nu)\right\} \). In the remaining part of these notes we shall derive again the result in Eq. (58) in two other different ways which are perhaps more complex but also more physically clear.

Note that from Eq. (56) it follows

\[
M_{00} = \text{Tr}\left\{\sigma(0)T\sigma(0)T^\dagger\right\} = \frac{1}{2}\text{Tr}\left\{TT^\dagger\right\},
\]

therefore, when \( T \) is unitary \( \text{Tr}\{TT^\dagger\} = \text{Tr}\{I_2\} = 2 \), which implies

\[
M_{00} = 1.
\]

This is then the “natural” normalization of \( M \).

**A. From the \( M \) matrix to the \( H \) matrix**

The Mueller matrix \( M \) has not, in general, any particular symmetry property. However it is possible to extract from it an Hermitian matrix \( H \) in the way we are going to show. Let us start by writing \( M \) in component form as

\[
M_{\mu\nu} = \text{Tr}\left\{\sigma(\mu)T\sigma(\nu)T^\dagger\right\} \\
= [\sigma(\mu)]_{mn}T_{np}[\sigma(\nu)]_{pq}T_{qm}^\dagger \\
= T_{np}T_{mq}^* [\sigma(\mu)]_{mn} [\sigma(\nu)]_{pq} \\
= (T \otimes T^*)_{nm,pq} [\sigma(\mu)]_{mn} [\sigma(\nu)]_{pq} \\
\equiv F_{nm,pq} [\sigma(\mu)]_{mn} [\sigma(\nu)]_{pq}
\]

where we have defined the matrix \( F \in \mathbb{C}^{4\times4} \) as

\[
F \equiv T \otimes T^*,
\]
which contains all the information about the scattering process. From Eq. (62) it is clear that $F$ is not Hermitian, however, we can extract out of it the Hermitian matrix $H$ by doing a partial exchange of the rows (Per[]) defined in the following way:

$$H = \text{Per}[F] \iff H_{np,mq} = F_{nm,pq}, \quad (63)$$

where the indices $p$ and $m$ have been exchanged. This definition clearly requires the matrices $H$ and $F$ to be written with four indices, as if they were generated by a direct product of two $2 \times 2$ matrices; see, e.g., Eqs. (11-12). However, this is unnecessary; actually after a careful examination of Eq. (63) one can easily convince himself (or herself) that the effect of the “Per[]” operation on an arbitrary $4 \times 4$ matrix can be written explicitly in matrix form as

$$\text{Per} \begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & a_1 & b_1 \\ c_0 & d_0 & c_1 & d_1 \\ a_2 & b_2 & a_3 & b_3 \\ c_2 & d_2 & c_3 & d_3 \end{pmatrix}. \quad (64)$$

This equation can be considered as a definition of the Per[] operation alternative to the one given in Eq. (63). The advantage of Eq. (64) with respect to Eq. (63) is that it does not require the $4 \times 4$ matrix to be written as the direct product of two $2 \times 2$ sub-matrices, but it is applicable to arbitrary matrices.

The matrix $H$ is Hermitian: this can be easily seen by first writing explicitly $F$ in terms of the components $T_{ij}$ of $T$

$$F = \begin{pmatrix} T_{00}T_{00}^* & T_{00}T_{01}^* & T_{01}T_{00}^* & T_{01}T_{01}^* \\ T_{00}T_{10}^* & T_{00}T_{11}^* & T_{01}T_{10}^* & T_{01}T_{11}^* \\ T_{10}T_{00}^* & T_{10}T_{01}^* & T_{11}T_{00}^* & T_{11}T_{01}^* \\ T_{10}T_{10}^* & T_{10}T_{11}^* & T_{11}T_{10}^* & T_{11}T_{11}^* \end{pmatrix}, \quad (65)$$
and then by applying the Per[,] operation to $F$ to obtain $H$:

$$H = \text{Per}[F]$$

$$
= \begin{pmatrix}
T_{00}T_{00} & T_{00}T_{01} & T_{00}T_{10} & T_{00}T_{11} \\
T_{01}T_{00} & T_{01}T_{01} & T_{01}T_{10} & T_{01}T_{11} \\
T_{10}T_{00} & T_{10}T_{01} & T_{10}T_{10} & T_{10}T_{11} \\
T_{11}T_{00} & T_{11}T_{01} & T_{11}T_{10} & T_{11}T_{11}
\end{pmatrix}
$$

(66)

$$
= \begin{pmatrix}
T_{00} \\
T_{01} \\
T_{10} \\
T_{11}
\end{pmatrix}
\begin{pmatrix}
T_{00}^* & T_{01}^* & T_{10}^* & T_{11}^*
\end{pmatrix}
$$

$$
= \vec{h}\vec{h}^\dagger,
$$

where the diad $\vec{h}\vec{h}^\dagger$ is written in terms of the 4-vector $\vec{h}$ defined as

$$\vec{h} = \begin{pmatrix}
T_{00} \\
T_{01} \\
T_{10} \\
T_{11}
\end{pmatrix},
$$

(67)

which is just the 4-vector representing $T$ in the basis $\{\epsilon_{(\mu)}\}$:

$$T = h_{\mu}\epsilon_{(\mu)}.\quad (68)$$

Then, by using Eqs. (66,67) we can write $H$ in component form as

$$H_{\mu\nu} = h_{\mu}h_{\nu}^*,\quad (69)$$

from which its Hermitian character is evident. Finally, by combining Eq. (62) and (68) we get

$$F = h_{\mu}\epsilon_{(\mu)} \otimes h_{\nu}^\ast \epsilon_{(\nu)}$$

$$= h_{\mu}h_{\nu}^*\epsilon_{(\mu)} \otimes \epsilon_{(\nu)}$$

$$\equiv H_{\mu\nu}\epsilon_{(\mu)} \otimes \epsilon_{(\nu)},\quad (70)$$

which shows that $H$ is just the representation of $F$ in the basis $\{\epsilon_{(\mu)} \otimes \epsilon_{(\nu)}\}$. 

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Now we can continue the calculation of $M_{\mu \nu}$ by inserting Eq. (62) in Eq. (61) obtaining

$$M_{\mu \nu} = [\sigma(\mu)]_{nm}(T \otimes T^*)_{nm,pq}[\sigma(\nu)]_{pq}$$

$$= [\sigma^*(\mu)]_{nm}(T \otimes T^*)_{nm,pq}[\sigma(\nu)]_{pq}$$

$$= [\sigma^*(\mu)]_{\alpha}(T \otimes T^*)_{\alpha \beta}[\sigma(\nu)]_{\beta \mu},$$

where in the second line we exploited the fact that the Pauli matrices are Hermitian, so $[\sigma(\mu)]_{nm} = [\sigma^*(\mu)]_{nm}$, and in the last line we used the Rule to define $\alpha = 2n + m$ and $\beta = 2p + q$. But since $[\sigma(\mu)]_{\alpha} = \Lambda_{\alpha \mu}$, then

$$M_{\mu \nu} = \Lambda^*_{\alpha \mu}(T \otimes T^*)_{\alpha \beta}\Lambda_{\beta \mu}$$

$$= \Lambda^\dagger_{\mu \alpha}(T \otimes T^*)_{\alpha \beta}\Lambda_{\beta \mu}$$

$$= [\Lambda^\dagger(T \otimes T^*)\Lambda]_{\mu \nu},$$

or, in matrix form

$$M = \Lambda^\dagger(T \otimes T^*)\Lambda.$$  \hspace{1cm} (73)

This formula is particular relevant because it permits us to define the matrix $F$ even when the Mueller matrix is nondeterministic or, equivalently, when it is not a Mueller-Jones matrix. In fact, by rewriting Eq. (73) as

$$M = \Lambda^\dagger F \Lambda,$$  \hspace{1cm} (74)

it is clear that we can invert it and define, in the general case

$$F \equiv \Lambda M \Lambda^\dagger.$$  \hspace{1cm} (75)

In the same spirit we can define $H$ in the general case by starting from the last line of the Eq. (61) which can be rewritten with the help of the Eq. (63) as

$$M_{\mu \nu} = H_{np,mq}[\sigma(\mu)]_{mn}[\sigma(\nu)]_{pq}$$

$$= H_{np,mq}[\sigma(\mu)]_{mn}[\sigma^*(\nu)]_{qp}$$

$$= H_{np,mq}[\sigma(\mu) \otimes \sigma^*(\nu)]_{mq,np}$$

$$= H_{\alpha \beta}[\sigma(\mu) \otimes \sigma^*(\nu)]_{\beta \alpha}$$

$$= \text{Tr}\left\{H(\sigma(\mu) \otimes \sigma^*(\nu))\right\},$$

where we used the Rule to define $\alpha = 2n + p$ and $\beta = 2m + q$. It is simple to invert this equation by using the completeness relation Eq. (89) that here we rewrite

$$\sum_{\mu=0}^{3}[\sigma(\mu)]_{ij}[\sigma^*(\mu)]_{kl} = \delta_{ik}\delta_{jl}.$$  \hspace{1cm} (77)
Then, by multiplying both sides of Eq. (76) per \[ \sigma^*_{\mu} \] \[ k_i \] \[ \sigma^*_{\nu} \] \[ j_l \] and summing on \( \mu \) and \( \nu \), we obtain

\[
M_{\mu\nu}[\sigma^*_{\mu} k_i][\sigma^*_{\nu} j_l] = H_{np,mq}[\sigma(\mu)]_{mn}[\sigma^*_{\mu} k_i][\sigma^*_{\nu} j_l]
\]

\[
= H_{np,mq} \delta_{mk} \delta_{ni} \delta_{pj} \delta_{ql}
\]

\[
= H_{ij,kl},
\]

which is the desired result. This equation can be put in matrix form by noticing that

\[
[\sigma^*_{\mu} k_i][\sigma^*_{\nu} j_l] = [\sigma_{\mu}]_{ik} [\sigma^*_{\nu}]_{jl}
\]

\[
= (\sigma_{\mu} \otimes \sigma^*_{\nu})_{ij,kl},
\]

which permits us to write

\[
H_{ij,kl} = M_{\mu\nu} (\sigma(\mu) \otimes \sigma^*_{\nu})_{ij,kl}.
\]

This formula is not very appealing because it contains both Latin indices which run from 0 to 1, and Greek indices which run from 0 to 3. This problem can be solved by using again the Rule to define \( \alpha = 2i + j \) and \( \beta = 2k + l \). Finally, we can rewrite Eq. (80) as

\[
H_{\alpha\beta} = M_{\mu\nu} (\sigma(\mu) \otimes \sigma^*_{\nu})_{\alpha\beta},
\]

or, in matrix form

\[
H = \sum_{\mu,\nu} M_{\mu\nu} (\sigma(\mu) \otimes \sigma^*_{\nu}).
\]

We can consider this formula as the definition of \( H \) for arbitrary \( M \).

### B. The Coherency matrix \( C \)

The relation between \( H \) and \( M \) is linear but quite involved, as can be seen by writing explicitly \( H \) in terms of the components \( M_{\mu\nu} \) of \( M \):

\[
H_{00} = \frac{1}{2} (M_{00} + M_{03} + M_{30} + M_{33}),
\]

\[
H_{01} = \frac{1}{2} (M_{01} + M_{31} + iM_{02} + iM_{32}),
\]

\[
H_{02} = \frac{1}{2} (M_{10} + M_{13} - iM_{20} - iM_{23}),
\]

\[
H_{03} = \frac{1}{2} (M_{11} + M_{22} + iM_{12} - iM_{21}),
\]

\[
(83)
\]
\[
H_{10} = \frac{1}{2} (M_{01} + M_{31} - iM_{02} - iM_{32}) , \\
H_{11} = \frac{1}{2} (M_{00} - M_{03} + M_{30} - M_{33}) , \\
H_{12} = \frac{1}{2} (M_{11} - M_{22} - iM_{12} - iM_{21}) , \\
H_{13} = \frac{1}{2} (M_{10} - M_{13} - iM_{02} + iM_{32}) , \\
H_{20} = \frac{1}{2} (M_{10} + M_{13} + iM_{20} + iM_{32}) , \\
H_{21} = \frac{1}{2} (M_{11} - M_{22} + iM_{12} + iM_{21}) , \\
H_{22} = \frac{1}{2} (M_{00} + M_{03} - M_{30} - M_{33}) , \\
H_{23} = \frac{1}{2} (M_{01} - M_{31} + iM_{02} - iM_{32}) , \\
H_{30} = \frac{1}{2} (M_{11} + M_{22} - iM_{12} + iM_{21}) , \\
H_{31} = \frac{1}{2} (M_{10} - M_{13} + iM_{20} - iM_{33}) , \\
H_{32} = \frac{1}{2} (M_{01} - M_{31} - iM_{02} + iM_{32}) , \\
H_{33} = \frac{1}{2} (M_{00} - M_{03} - M_{30} + M_{33}) .
\]

(84)

From this formula we see that

\[ \text{Tr}\{H\} = 2M_{00}. \]  

(87)

If we choose the “natural” normalization \( M_{00} = 1 \), it follows \( \text{Tr}\{H\} = 2 \). The matrix \( H \) is not the only Hermitian matrix we can extract from \( M \), actually there are infinitely many Hermitian matrices generated by \( M \) which differ from \( H \) by a unitary transformation. A particularly relevant Hermitian matrix is the Coherency matrix \( C \) defined as the representation of \( F \) in the basis \( \{\sigma(\mu) \otimes \sigma(\mu)^*\} \). In order to find this representation, let us first write the transformation matrix \( T \) in both the bases \( \{\sigma(\mu)\} \) and \( \{\epsilon(\mu)\} \) as

\[
T = c_\mu \sigma(\mu) = h_\mu \epsilon(\mu), \quad (\mu = 0, 1, 2, 3),
\]

(88)

and then let us calculate

\[
F = T \otimes T^* \\
= c_\mu \sigma(\mu) \otimes c_\nu^* \sigma(\nu)^* \\
= c_\mu c_\nu^* \sigma(\mu) \otimes \sigma(\nu)^* \\
= C_{\mu\nu} \sigma(\mu) \otimes \sigma(\nu)^*,
\]

(89)

where we have defined the coherency matrix elements as

\[
C_{\mu\nu} \equiv c_\mu c_\nu^*, \quad (\mu, \nu = 0, 1, 2, 3).
\]

(90)
By comparing Eq. (70) with Eq. (89) it appears evident that $C$ and $H$ are different representations of the same matrix $F$, with respect to different bases. Therefore they must be related by a unitary transformation: we want to find it. To this end, let us recall that if $A$ is an arbitrary $2 \times 2$ matrix in $\mathbb{C}^2$ which can be represented in the two different bases $\{\epsilon_{(\mu)}\}$ and $\{\sigma_{(\mu)}\}$ as

$$A = a_\mu \epsilon_{(\mu)} = b_\mu \sigma_{(\mu)}, \quad (\mu = 0, 1, 2, 3),$$

then the expansion coefficients $a_\mu$ and $b_\mu$ are related by the change of basis matrix $\Lambda$ as

$$a_\mu = \{\epsilon_{(\mu)}, A\}$$
$$= \{\epsilon_{(\mu)}, b_\nu \sigma_{(\nu)}\}$$
$$= \{\epsilon_{(\mu)}, \sigma_{(\nu)}\} b_\nu$$
$$= \Lambda_{\mu\nu} b_\nu,$$

or, in more compact form,

$$\vec{a} = \Lambda \vec{b}.$$  \hspace{1cm} (93)

In our specific case we find, by using Eq. (40),

$$F = C_{\mu\nu} \sigma_{(\mu)} \otimes \sigma_{(\nu)}$$
$$= C_{\mu\nu} \epsilon_{(\alpha)} \Lambda_{\alpha \mu} \otimes \epsilon_{(\beta)} \Lambda_{\beta \nu}^*$$
$$= \Lambda_{\alpha \mu} C_{\mu\nu} \Lambda_{\nu \beta}^* \epsilon_{(\alpha)} \otimes \epsilon_{(\beta)}$$
$$= [\Lambda C \Lambda^\dagger]_{\alpha \beta} \epsilon_{(\alpha)} \otimes \epsilon_{(\beta)}.$$  \hspace{1cm} (94)

The comparison of Eq. (70) with Eq. (94) reveals that

$$H = \Lambda C \Lambda^\dagger.$$  \hspace{1cm} (95)

Finally, we can combine the results in Eq. (82) and (95) to obtain the sought relation between $H$, $C$ and $M$:

$$C = \Lambda^\dagger H \Lambda$$
$$= \sum_{\mu, \nu}^{0,3} M_{\mu\nu} \Lambda^\dagger (\sigma_{(\mu)} \otimes \sigma_{(\nu)}^*) \Lambda$$
$$= \sum_{\mu, \nu}^{0,3} M_{\mu\nu} \Gamma_{(\mu\nu)},$$

where we have defined the 16 Hermitian matrices $\{\Gamma_{(\mu\nu)}\}$ as

$$\Gamma_{(\mu\nu)} \equiv \Lambda^\dagger (\sigma_{(\mu)} \otimes \sigma_{(\nu)}^*) \Lambda.$$  \hspace{1cm} (97)
Note that since the Pauli basis is complete in $\mathbb{C}^{2 \times 2}$, the direct products $\{\sigma_{(\mu)} \otimes \sigma^*_{(\nu)}\}$ form a complete basis in $\mathbb{C}^{4 \times 4}$. Moreover, since $\Lambda$ is unitary, the matrices $\sigma_{(\mu)} \otimes \sigma^*_{(\nu)}$ and $\Gamma_{(\mu\nu)}$ are equivalent, therefore the 16 matrices $\{\Gamma_{(\mu\nu)}\}$ form a complete basis in $\mathbb{C}^{4 \times 4}$. From the matrix rule

\[
(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger,
\]

and the definition Eq. (97), it follows that the matrices $\{\Gamma_{(\mu\nu)}\}$ are Hermitian. Moreover, with the help of the general matrix rules

\[
\text{Tr}\{A\}\text{Tr}\{B\} = \text{Tr}\{A \otimes B\},
\]

and

\[
(A \otimes B)(C \otimes D) = AC \otimes BD,
\]

we can show that the matrices $\Gamma_{(\mu\nu)}$ are also orthonormal:

\[
\text{Tr}\{\Gamma_{(\mu\nu)}\Gamma_{(\alpha\beta)}\} = \text{Tr}\{\Lambda^\dagger(\sigma_{(\mu)} \otimes \sigma^*_{(\nu)})\Lambda\Lambda^\dagger(\sigma_{(\alpha)} \otimes \sigma^*_{(\beta)})\Lambda\}
\]

\[
= \text{Tr}\{\Lambda^\dagger(\sigma_{(\mu)} \otimes \sigma^*_{(\nu)})\sigma^*_{(\nu)}(\sigma_{(\alpha)} \otimes \sigma^*_{(\beta)})\Lambda\}
\]

\[
= \text{Tr}\{\Lambda\Lambda^\dagger(\sigma_{(\mu)} \otimes \sigma^*_{(\nu)}(\sigma_{(\alpha)} \otimes \sigma^*_{(\beta)})\}
\]

\[
= \text{Tr}\{\sigma_{(\mu)}(\sigma_{(\alpha)} \otimes \sigma^*_{(\nu)}(\sigma^*_{(\beta)})\}
\]

\[
= \text{Tr}\{\sigma_{(\mu)}(\sigma_{(\alpha)} \otimes \sigma^*_{(\nu)}(\sigma^*_{(\beta)})\}
\]

\[
= \delta_{\mu\alpha}\delta_{\nu\beta},
\]

where $\Lambda\Lambda^\dagger = I_4$ and the cyclic property of the trace have been used.

Now we use Eq. (101) to invert Eq. (96) and express $M$ as function of $C$. By multiplying both members of Eq. (96) by $\Gamma_{(\alpha\beta)}$ and by tacking the trace, we obtain

\[
\text{Tr}\{\Gamma_{(\alpha\beta)}C\} = \sum_{0,3} M_{\mu\nu}\text{Tr}\{\Gamma_{(\alpha\beta)}\Gamma_{(\mu\nu)}\}
\]

\[
= \sum_{0,3} M_{\mu\nu}\delta_{\mu\alpha}\delta_{\nu\beta}
\]

\[
= M_{\alpha\beta},
\]

that is

\[
M_{\mu\nu} = \text{Tr}\{\Gamma_{(\mu\nu)}C\}.
\]

The Eqs. (96) and (103) can be put in a more compact form by using the Rule for $n = 4$:

\[
(\mu\nu) \rightarrow 4\mu + \nu \equiv A \in \{0,\ldots,15\}.
\]
Then we can rewrite
\[ C = \sum_{A=0}^{15} \Gamma_{(A)} m_A, \]
where
\[ m_A = \text{Tr}\{\Gamma_{(A)} C\}, \]
and where
\[ \vec{m} = \begin{pmatrix} M_0 \\ \vdots \\ M_{15} \end{pmatrix}, \]
is the 16-vector associated to the matrix \( M \) in the basis \( \{\Gamma_{(A)}\} \).

We conclude this section by writing explicitly the relation between the matrix elements of \( C \) and \( M \). An explicit calculation shows that if \( C \) is written as
\[ C = \begin{pmatrix} a_0 + a & c - id & h + ig & i - ij \\ c + id & b_0 + b & e + if & k - il \\ h - ig & e - if & b_0 - b & m + in \\ i + ij & k + il & m - in & a_0 - a \end{pmatrix}, \]
then \( M \) has the following form:
\[ M = \begin{pmatrix} a_0 + b_0 & c + n & h + l & i + f \\ c - n & a + b & e + j & k + g \\ h - l & e - j & a - b & m + d \\ i - f & k - g & m - d & a_0 - b_0 \end{pmatrix}. \]

C. Alternative version

In the literature can be found another method, more geometrical, to find the matrices \( \Gamma_{(\mu\nu)} \) and the result shown in Eq. 105. In this subsection we expose that method.

Let \( X, Y \) two matrices in \( \mathbb{C}^{2\times2} \) and let us consider their product \( Z = XY \). With \( \vec{x}, \vec{y} \) and \( \vec{z} \) we denote the 4-vectors associated to \( X, Y \) and \( Z \) respectively, with respect to the Pauli basis:
\[ X = x_\mu \sigma_{(\mu)} \Rightarrow x_\mu = \text{Tr}\{\sigma_{(\mu)} X\}; \]
\[ Y = y_\mu \sigma_{(\mu)} \Rightarrow y_\mu = \text{Tr}\{\sigma_{(\mu)} Y\}; \]
\[ Z = z_\mu \sigma_{(\mu)} \Rightarrow z_\mu = \text{Tr}\{\sigma_{(\mu)} Z\}; \]

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where $\mu = 0, \ldots, 3$ and summation on repeated indices is understood. We want to find a formula which expresses $\vec{z}$ as a function of $\vec{x}$ and $\vec{y}$. To this end let us write

$$z_\mu = \text{Tr}\{\sigma(\mu)Z\} = \text{Tr}\{\sigma(\mu)XY\} = x_\alpha y_\beta \text{Tr}\{\sigma(\mu)\sigma(\alpha)\sigma(\beta)\} \equiv x_\alpha y_\beta [\Upsilon(\mu)]_{\alpha\beta},$$

(111)

where we have defined the four matrices $\Upsilon(\mu) \in \mathbb{C}^{4 \times 4}$ as

$$[\Upsilon(\mu)]_{\alpha\beta} \equiv \text{Tr}\{\sigma(\mu)\sigma(\alpha)\sigma(\beta)\}. \quad (112)$$

Then we can rewrite Eq. (111) in a compact form as

$$z_\mu = (\vec{x}^*, \Upsilon(\mu)\vec{y}). \quad (113)$$

Let us notice that because of the cyclic property of the trace

$$\text{Tr}\{\sigma(\mu)\sigma(\alpha)\sigma(\beta)\} = \text{Tr}\{\sigma(\beta)\sigma(\mu)\sigma(\alpha)\} = \text{Tr}\{\sigma(\alpha)\sigma(\beta)\sigma(\mu)\},$$

(114)

we can write

$$[\Upsilon(\mu)]_{\alpha\beta} = [\Upsilon(\beta)]_{\mu\alpha} = [\Upsilon(\alpha)]_{\beta\mu}. \quad (115)$$

We shall exploit this property in a moment. Now, let us consider the special case in which $Y = \sigma(\nu) \Rightarrow y_\beta = \delta_\beta\nu$. Then, from Eq. (113) follows that

$$z_\mu = x_\alpha [\Upsilon(\mu)]_{\alpha\nu} = [\Upsilon(\nu)]_{\mu\alpha} x_\alpha = [\Upsilon(\nu)\vec{x}]_\mu, \quad (116)$$

where Eq. (115) has been used. Another special case is the transposed one, that is when $X = \sigma(\gamma) \Rightarrow x_\alpha = \delta_\alpha\gamma$ and

$$z_\mu = [\Upsilon(\mu)]_{\gamma\beta} y_\beta = [\Upsilon(\gamma)]_{\beta\mu} y_\beta = [\Upsilon^T(\gamma)\vec{y}]_\mu. \quad (117)$$

The previous results can be summarized as follows:

$$Z = X\sigma(\nu) \Rightarrow \vec{z} = \Upsilon(\nu)\vec{x}$$

$$Z = \sigma(\mu)Y \Rightarrow \vec{z} = \Upsilon^T(\mu)\vec{y}. \quad (118)$$
Now we are equipped to consider the last, most complicated case \( Z = \sigma(\mu)T\sigma(\nu) \), where 

\[
T = c_\alpha \sigma(\alpha) \quad \text{is a given} \quad 2 \times 2 \quad \text{matrix associated to the vector} \quad \vec{c},
\]

where \( \vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \) \hspace{1cm} (119)

Then, by putting \( Y = T\sigma(\nu) \Rightarrow \vec{y} = \Upsilon(\nu)\vec{c} \), it is easy to see that

\[
\vec{z} = \Upsilon^T(\mu)\vec{y} = \Upsilon^T(\mu) \Upsilon(\nu)\vec{c},
\]

where Eqs. (118) have been used. To summarize, we can write

\[
\sigma(\mu)T\sigma(\nu) = \Upsilon^T(\mu) \Upsilon(\nu)\vec{c} \equiv \Gamma(\mu\nu)\vec{c},
\]

and the symbol “\( \doteq \)” stands for “is represented by”.

Now we want to use these equations to calculate the matrix elements \( M_{\mu\nu} \) of the Mueller matrix, by using Eq. (61) that here we rewrite:

\[
M_{\mu\nu} = \text{Tr} \{ T^\dagger \sigma(\mu)T\sigma(\nu) \}. \hspace{1cm} (123)
\]

Before doing that, notice that if \( T = c_\alpha \sigma(\alpha) \doteq \vec{c} \) then \( T^\dagger = c^*_\alpha \sigma(\alpha) \doteq \vec{c}^* \); and notice that if \( A \doteq \vec{a} \) and \( B \doteq \vec{b} \), then

\[
\{A, B\} = \text{Tr} \{ A^\dagger B \} = a^*_\mu b_\nu \text{Tr} \{ \sigma(\mu)\sigma(\nu) \} = a^*_\mu b_\mu = (\vec{a}, \vec{b}).\hspace{1cm} (124)
\]

Finally, from Eqs. (121-124) it follows straightforwardly that

\[
M_{\mu\nu} = \text{Tr} \{ T^\dagger \sigma(\mu)T\sigma(\nu) \}
= (\vec{c}, \Gamma(\mu\nu)\vec{c})
= c_\beta c^*_\alpha [\Gamma(\mu\nu)]_{\alpha\beta}
\equiv C_{\beta\alpha} [\Gamma(\mu\nu)]_{\alpha\beta}
= \text{Tr} \{ C \Gamma(\mu\nu) \},
\]

\( \hspace{1cm} (125) \)
which coincides with the result found in Eq. (103). To complete the calculation we have to demonstrate that the \( \{\Gamma_{\mu\nu}\} \) matrices found in Eq. (122) coincide with the ones found in Eq. (97). To this end we calculate the matrix elements in both cases and then compare them. Let us start from Eq. (122) to write

\[
[\Gamma_{\mu\nu}]_{\alpha\beta} = \sum_{\gamma=0}^{3} [\Upsilon_{T}(\mu)]_{\alpha\gamma} [\Upsilon(\nu)]_{\gamma\beta}
\]

\[
= \sum_{\gamma=0}^{3} \text{Tr}\{\sigma(\mu)\sigma(\gamma)\sigma(\alpha)\} \text{Tr}\{\sigma(\nu)\sigma(\gamma)\sigma(\beta)\}
\]

\[
= \sum_{\gamma=0}^{3} [\sigma(\mu)]_{ij} [\sigma(\gamma)]_{j\gamma} [\sigma(\nu)]_{\gamma\gamma} \text{Tr}\{\sigma(\gamma)\}_{mn} [\sigma(\gamma)]_{nl}
\]

\[
= \left( \sum_{\gamma=0}^{3} [\sigma(\gamma)]_{j\gamma} [\sigma(\gamma)]_{mn} \right) [\sigma(\mu)]_{ij} [\sigma(\alpha)]_{k\gamma} [\sigma(\nu)]_{l\gamma} [\sigma(\beta)]_{nl}.
\]

From the completeness Eq. (39) we know that

\[
\sum_{\gamma=0}^{3} [\sigma(\gamma)]_{j\gamma} [\sigma(\gamma)]_{mn} = \sum_{\gamma=0}^{3} [\sigma(\gamma)]_{j\gamma} [\sigma(\gamma)]_{nm}
\]

\[
= \delta_{jn} \delta_{km},
\]

so that Eq. (122) becomes

\[
[\Gamma_{\mu\nu}]_{\alpha\beta} = \delta_{jn} \delta_{km} [\sigma(\mu)]_{ij} [\sigma(\alpha)]_{k\gamma} [\sigma(\nu)]_{l\gamma} [\sigma(\beta)]_{nl}
\]

\[
= [\sigma(\mu)]_{ij} [\sigma(\alpha)]_{k\gamma} [\sigma(\beta)]_{l\gamma} [\sigma(\nu)]_{j\gamma}
\]

\[
= \text{Tr}\{\sigma(\alpha)\sigma(\mu)\sigma(\beta)\sigma(\nu)\}.
\]

The equality

\[
[\Upsilon_{T}(\mu)] [\Upsilon(\nu)]_{\alpha\beta} = \text{Tr}\{\sigma(\alpha)\sigma(\mu)\sigma(\beta)\sigma(\nu)\};
\]

(129)

can be also easily checked by explicit calculation. Now we repeat the calculation of \([\Gamma_{\mu\nu}]_{\alpha\beta}\) starting from Eq. (97):

\[
[\Gamma_{\mu\nu}]_{\alpha\beta} = [\Lambda^\dagger(\sigma(\mu) \otimes \sigma^*(\nu))\Lambda]_{\alpha\beta}
\]

\[
= [\Lambda^\dagger(\sigma(\mu) \otimes \sigma^*(\nu))\Lambda]_{\gamma\epsilon} [\Lambda]_{\epsilon\beta}
\]

\[
= [\sigma^*(\alpha)]_{\gamma\epsilon} [\Lambda^\dagger(\sigma(\mu) \otimes \sigma^*(\nu))\Lambda]_{\epsilon\beta}
\]

\[
= [\sigma^*(\alpha)]_{\gamma\epsilon} [\sigma(\mu) \otimes \sigma^*(\nu)]_{\gamma\epsilon} [\sigma(\beta)]_{\epsilon\beta},
\]

(130)

where we have used Eq. (30) in the last line. Now we use the Rule to pass from the dummy
4-D Greek indices to the dummy 2-D Latin indices and write

$$[\Gamma_{(\mu\nu)}]_{\alpha\beta} = [\sigma^*_{(\alpha)}]_{ik} [\sigma_{(\mu)} \otimes \sigma^*_{(\nu)}]_{jl} [\sigma_{(\beta)}]_{jl}$$
$$= [\sigma^*_{(\alpha)}]_{ik} [\sigma_{(\mu)}]_{ij} [\sigma^*_{(\nu)}]_{jl} [\sigma_{(\beta)}]_{jl}$$
$$= [\sigma_{(\alpha)}]_{ki} [\sigma_{(\mu)}]_{ij} [\sigma_{(\beta)}]_{jl} [\sigma_{(\nu)}]_{lk}$$
$$= \text{Tr}\{\sigma_{(\alpha)} \sigma_{(\mu)} \sigma_{(\beta)} \sigma_{(\nu)}\}. \quad (131)$$

This complete our demonstration.

Let us conclude this subsection by calculating explicitly the 4 matrices \{\(\Upsilon_{(\mu)}\)\}. First of all we notice that in the case in which one of the indices is zero, then we have

$$[\Upsilon_{(0)}]_{\alpha\beta} = \frac{1}{\sqrt{2}} \text{Tr}\{\sigma_{(\alpha)} \sigma_{(\beta)}\} = \frac{1}{\sqrt{2}} \delta_{\alpha\beta},$$
$$[\Upsilon_{(\mu)}]_{0\beta} = \frac{1}{\sqrt{2}} \text{Tr}\{\sigma_{(\mu)} \sigma_{(\beta)}\} = \frac{1}{\sqrt{2}} \delta_{\mu\beta}, \quad (132)$$
$$[\Upsilon_{(\mu)}]_{\alpha0} = \frac{1}{\sqrt{2}} \text{Tr}\{\sigma_{(\mu)} \sigma_{(\alpha)}\} = \frac{1}{\sqrt{2}} \delta_{\mu\alpha}.$$  

In the case in which all indices are different from zero, we use the following well known property of the Pauli matrices

$$\sigma_{(i)} \sigma_{(j)} = \frac{1}{\sqrt{2}} \left( \delta_{ij} \sigma_{(0)} + i \varepsilon_{ijl} \sigma_{(l)} \right), \quad (133)$$

where \(i, j, l = 1, 2, 3\) and \(\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{231} = \varepsilon_{213} = 1\) is the completely antisymmetric Levi-Civita pseudo-tensor (all the unwritten components are zero); to show that

$$[\Upsilon_{(i)}]_{jk} = \text{Tr}\{\sigma_{(i)} \sigma_{(j)} \sigma_{(k)}\}$$
$$= \frac{1}{\sqrt{2}} \delta_{ij} \text{Tr}\{\sigma_{(0)} \sigma_{(k)}\} + \frac{i}{\sqrt{2}} \varepsilon_{ijl} \text{Tr}\{\sigma_{(l)} \sigma_{(k)}\} \quad (134)$$
$$= \frac{i}{\sqrt{2}} \varepsilon_{ijk}.$$
Finally, by collecting all these results, we can write explicitly:

\[
\Upsilon_{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Upsilon_{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \\
\Upsilon_{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \Upsilon_{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]  

(135)

IV. THE DECOMPOSITION THEOREM

In this section we show that a given Mueller matrix \( M \) can be written as a linear combination with positive coefficients of at most 4 Mueller-Jones matrices. Here we do not adopt the Einstein summation convention, therefore repeated indices must not be summed. All sums will be written explicitly as in the right side of Eq. (10).

In the Eq. (90) we have defined the Hermitian matrix \( C \) and we have shown its relation with \( H \) and \( M \). Since \( C \) is Hermitian it can be diagonalized. Let \( \vec{u}_{(\alpha)} \), \( (\alpha = 0, \ldots, 3) \) the four eigenvectors of \( C \) associated with the four real eigenvalues \( \lambda_\alpha \)

\[
C \vec{u}_{(\alpha)} = \lambda_\alpha \vec{u}_{(\alpha)}, \quad (\alpha = 0, \ldots, 3),
\]

(136)

where there is not sum on repeated indices. The eigenvectors of an Hermitian matrix can always be chosen orthonormal, so we assume

\[
(\vec{u}_{(\alpha)}, \vec{u}_{(\beta)}) = \delta_{\alpha\beta}, \quad (\alpha, \beta = 0, \ldots, 3).
\]

(137)

By tacking the scalar product of both sides of Eq. (136) with \( \vec{u}_{(\beta)} \), we obtain

\[
(\vec{u}_{(\beta)}, C \vec{u}_{(\alpha)}) = \lambda_\alpha (\vec{u}_{(\beta)}, \vec{u}_{(\alpha)}) = \lambda_\alpha \delta_{\beta\alpha}, \quad (\alpha, \beta = 0, \ldots, 3),
\]

(138)
If we write explicitly the left side of this equation we get

\[(\vec{u}(\beta), C\vec{u}(\alpha)) = \sum_{\mu, \nu}^{0,3} [\vec{u}(\beta)]_{\mu} C_{\mu \nu} \vec{u}(\alpha)_{\nu} \]

\[\equiv \sum_{\mu, \nu}^{0,3} U^\ast_\mu C_{\mu \nu} U_\nu \alpha \]

\[\equiv \sum_{\mu, \nu}^{0,3} U^\dagger_\beta \mu C_{\mu \nu} U_\nu \alpha \]

\[\equiv [U^\dagger C U]_{\beta \alpha}, \] (139)

where we have defined the matrix \(U\) as

\[U : \quad U_{\beta \alpha} \equiv [\vec{u}(\alpha)]_{\beta}, \quad (\alpha, \beta = 0, \ldots, 3). \] (140)

The matrix \(U\) is unitary by definition:

\[[U^\dagger U]_{\alpha \beta} = \sum_{\mu=0}^{3} U^\ast_\mu \alpha U_{\mu \beta} \]

\[\equiv \sum_{\mu=0}^{3} [\vec{u}^\ast(\alpha)]_{\mu} [\vec{u}(\beta)]_{\mu} \]

\[\equiv (\vec{u}(\alpha), \vec{u}(\beta)) \]

\[\equiv \delta_{\alpha \beta}. \] (141)

By comparing Eq. (138) with Eq. (139) we immediately obtain

\[[U^\dagger C U]_{\beta \alpha} = \lambda_\alpha \delta_{\beta \alpha}, \quad (\alpha, \beta = 0, \ldots, 3), \] (142)

or, in matrix form

\[U^\dagger C U = D, \] (143)

where \(D = \text{diag}\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}\) or, explicitly

\[D = \begin{pmatrix}
\lambda_0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}. \] (144)

Since \(C\) is positive semidefinite, all its eigenvalues are nonnegative: \(\lambda_\mu \geq 0, (\mu = 0, \ldots, 3)\). Moreover, since from Eqs. (87,95) follow that

\[\text{Tr}\{C\} = \text{Tr}\{A^\dagger H A\} = \text{Tr}\{H\} = 2, \] (145)
then $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 2$. Now we want to write $C$ in terms of its eigenvalues; to this end we have to invert Eq. (143) obtaining

$$C = UDU^\dagger,$$  \hspace{0.5cm} (146)

or, in components form

$$C_{\alpha\beta} = [UDU^\dagger]_{\alpha\beta} = \sum_{\mu,\nu} U_{\alpha\mu} [D]_{\mu\nu} U^*_{\beta\nu} = \sum_{\mu,\nu} [\bar{u}_{(\mu)}]_{\alpha} \lambda_{\mu} \delta_{\mu\nu} [\bar{u}^*_{(\nu)}]_{\beta} = \sum_{\mu=0}^3 \lambda_{\mu} [\bar{u}_{(\mu)}]_{\alpha} [\bar{u}^*_{(\mu)}]_{\beta}. \hspace{0.5cm} (147)$$

If we indicate with $\Omega_{(\mu)} \equiv \bar{u}_{(\mu)} \bar{u}^\dagger_{(\mu)}$ the $4 \times 4$ Hermitian diad whose elements are

$$[\Omega_{(\mu)}]_{\alpha\beta} \equiv [\bar{u}_{(\mu)}]_{\alpha} [\bar{u}^*_{(\mu)}]_{\beta} = U_{\alpha\mu} U^\dagger_{\mu\beta}, \hspace{0.5cm} (148)$$

we can rewrite Eq. (147) in matrix form as

$$C = \sum_{\mu=0}^3 \lambda_{\mu} \Omega_{(\mu)}. \hspace{0.5cm} (149)$$

It is easy to see that the matrices $\{\Omega_{(\mu)}\}$ are orthogonal:

$$\{\Omega_{(\alpha)}, \Omega_{(\beta)}\} = \text{Tr}\{\Omega^\dagger_{(\alpha)} \Omega_{(\beta)}\} = \sum_{\mu,\nu} [\Omega_{(\alpha)}]_{\nu\mu} [\Omega_{(\beta)}]_{\mu\nu} = \sum_{\mu,\nu} U_{\nu\alpha} U^\dagger_{\alpha\mu} U_{\mu\beta} U^\dagger_{\beta\nu} = \sum_{\nu=0}^3 U^\dagger_{\beta\nu} U_{\nu\alpha} \sum_{\mu=0}^3 U^\dagger_{\alpha\mu} U_{\mu\beta} = \delta_{\beta\alpha} \delta_{\alpha\beta} = \delta_{\alpha\beta} \hspace{0.5cm} (150)$$

where in the second line we have exploited the fact that the $\{\Omega_{(\alpha)}\}$ are Hermitian. We are
now close to our goal; let us notice that from Eqs. (74-89-97) follow that

\[ M = \Lambda F \Lambda^\dagger = \sum_{\mu,\nu} C_{\mu\nu} \Lambda \left( \sigma_{(\mu)} \otimes \sigma_{(\nu)}^* \right) \Lambda = \sum_{\mu,\nu} C_{\mu\nu} \Gamma_{(\mu\nu)}. \]  

(151)

Now we insert Eq. (149) in Eq. (151) to obtain

\[ M = \sum_{\alpha=0}^3 \lambda_{\alpha} \sum_{\mu,\nu} [\Omega_{(\alpha)}]_{\mu\nu} \Gamma_{(\mu\nu)} \equiv \sum_{\alpha=0}^3 \lambda_{\alpha} \Phi_{(\alpha)}, \]  

(152)

where we have defined the four Mueller-Jones matrices \( \Phi_{(\alpha)} \), \( (\alpha = 0, \ldots, 3) \) as

\[ \Phi_{(\alpha)} \equiv \sum_{\mu,\nu} [\Omega_{(\alpha)}]_{\mu\nu} \Gamma_{(\mu\nu)}. \]  

(153)

These matrices are real, in fact

\[ \Phi_{(\alpha)}^* = \sum_{\mu,\nu} [\Omega_{(\alpha)}^*]_{\mu\nu} \Gamma_{(\mu\nu)}^* = \sum_{\mu,\nu} [\Omega_{(\alpha)}]_{\nu\mu} \Gamma_{(\nu\mu)} = \Phi_{(\alpha)}, \]  

(154)

since both \( \Omega_{(\alpha)} \) and \( \Gamma_{(\nu\mu)} \) are Hermitian matrices. Actually we have still to demonstrate that the \( \{ \Phi_{(\alpha)} \} \) are Mueller-Jones matrices. To do that we need two simple partial results. The first comes from Eq. (131) which shows that

\[ [\Gamma_{(\mu\nu)}]_{00} = \text{Tr} \{ \sigma_{(0)}\sigma_{(\mu)}\sigma_{(0)}\sigma_{(\nu)} \} = \text{Tr} \{ \sigma_{(\mu)}\sigma_{(\nu)} \} / 2 = \delta_{\mu\nu} / 2. \]  

(155)
The second result we need is the orthonormality of the \( \{ \Phi(\alpha) \} \):

\[
\{ \Phi(\alpha), \Phi(\beta) \} = \text{Tr}\{ \Phi^\dagger(\alpha) \Phi(\beta) \} \\
= \sum_{\mu,\nu} \sum_{\gamma,\tau} \Omega^*(\alpha)_{\mu\nu} \Omega(\beta)_{\gamma\tau} \text{Tr}\{ \Gamma(\mu\nu) \Gamma(\gamma\tau) \} \\
= \sum_{\mu,\nu} \sum_{\gamma,\tau} \Omega^*(\alpha)_{\mu\nu} \Omega(\beta)_{\gamma\tau} \delta_{\mu\gamma} \delta_{\nu\tau} \\
= \sum_{\mu,\nu} [\Omega(\alpha)]_{\mu\nu} [\Omega(\beta)]_{\mu\nu} \\
= \{ \Omega(\alpha), \Omega(\beta) \} \\
= \delta_{\alpha\beta}.
\]

It is now straightforward to calculate from Eqs. (153,155)

\[
[\Phi(\alpha)]_{00} = \sum_{\mu,\nu} \Omega(\alpha)_{\mu\nu} \Gamma(\mu\nu)_{00} \\
= \sum_{\mu=0} \Omega(\alpha)_{\mu\mu}/2 \\
= \sum_{\mu=0} U_{\mu\alpha} U_{\alpha\mu}^\dagger/2 \\
= 1/2,
\]

while from Eqs. (156) we get \( \text{Tr}\{ \Phi^T(\alpha) \Phi(\alpha) \} = 1 \). A necessary and sufficient condition for a Mueller matrix \( M \) to be a Mueller-Jones matrix is \( \text{Tr}\{ M^T M \} = (2M_{00})^2 \). In our case we have

\[
\frac{\text{Tr}\{ \Phi^T(\alpha) \Phi(\alpha) \}}{(2[\Phi(\alpha)]_{00})^2} = 1, \quad (\alpha = 0, \ldots, 3),
\]

therefore the \( \{ \Phi(\alpha) \} \) are genuine Mueller-Jones matrices. This step complete the demonstration of the decomposition theorem. In the next subsection we shall derive this result once more by explicit construction of the matrices \( \{ \Phi(\alpha) \} \).

A. A step backward: from \( M \) to \( T \)

Now that we learned how to decompose \( M \), we want to make a step backward in order to see if it is possible to find such a kind of decomposition for the \( 2 \times 2 \) matrix \( J' \) introduced in Eq. (54). To this end we seek a different form for the matrices \( \{ \Phi(\alpha) \} \). We start by rewriting
Eq. (153) with the help Eq. (97) of as

\[
\Phi(\alpha) = \sum_{\mu, \nu}^{0, 3} \left[ \Omega(\alpha) \right]_{\mu \nu} \Gamma(\mu \nu)
\]

\[
= \sum_{\mu, \nu}^{0, 3} \left[ \bar{u}(\alpha) \right]_{\mu} \left[ \bar{u}^{*}(\alpha) \right]_{\nu} A^\dagger \left( \sigma(\mu) \otimes \sigma^{*}(\nu) \right) A
\]

\[
= A^\dagger \left( \sum_{\mu=0}^{3} \left[ \bar{u}(\alpha) \right]_{\mu} \sigma(\mu) \otimes \sum_{\nu=0}^{3} \left[ \bar{u}^{*}(\alpha) \right]_{\nu} \sigma^{*}(\nu) \right) A
\]

\[
= A^\dagger \left( T(\alpha) \otimes T^*(\alpha) \right) A,
\]

(159)

where we have defined the four 2 × 2 Jones matrices \( \{T(\alpha)\} \) as

\[
T(\alpha) \equiv \sum_{\mu=0}^{3} \left[ \bar{u}(\alpha) \right]_{\mu} \sigma(\mu).
\]

(160)

The result in Eq. (159) shows once again that the \( \Phi(\alpha) \) are genuine Mueller-Jones matrices.

At this point we can rewrite Eq. (152) as

\[
M = \sum_{\alpha=0}^{3} \lambda_{\alpha} A^\dagger \left( T(\alpha) \otimes T^*(\alpha) \right) A,
\]

(161)

and compare it with Eq. (73). Then it appears that in the general case comprising also nondeterministic Mueller matrices the single Jones matrix \( T \) must be substituted by the set of the four Jones matrices \( \{T(\alpha)\} \) following the recipe given above. In the same way, if we \textit{assume a priori} that in the general case Eq. (52) must be substituted by

\[
J' = \sum_{\alpha=0}^{3} \lambda_{\alpha} T(\alpha) J T^\dagger(\alpha),
\]

(162)

and rewrite Eqs. (117, 118, 119), we obtain again Eq. (161). Then the decomposition of \( J' \) we were looking for has been found. Note that since \( \lambda_{\alpha} \geq 0 \), we can always rewrite Eq. (162) as

\[
J' = \sum_{\alpha=0}^{3} \lambda_{\alpha} A(\alpha) J A^\dagger(\alpha)
\]

\[
\equiv \sum_{\alpha=0}^{3} A(\alpha) J A^\dagger(\alpha)
\]

where we have defined \( A(\alpha) = \sqrt{\lambda_{\alpha}} T(\alpha) \). In quantum optics and quantum information Eq. (168) is known as \textit{“Kraus decomposition”}. 

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At this point there are two things to be noted. The first is about the normalization of the Jones matrices $T^{\alpha}$. In fact, it is easy to see

$$
\text{Tr}\{T^\dagger(\alpha)T(\alpha)\} = \text{Tr}\{\sum_{\mu=0}^{3}[\vec{u}(\alpha)]^*_\mu\sigma(\mu)\sum_{\nu=0}^{3}[\vec{u}(\alpha)]_\nu\sigma(\nu)\}
$$

$$
= \sum_{\mu,\nu=0}^{3}U^*_{\mu\alpha}U_{\nu\alpha}\text{Tr}\{\sigma(\mu)\sigma(\nu)\}
$$

$$
= \sum_{\mu=0}^{3}U^\dagger_{\mu\alpha}U_{\mu\alpha}
$$

$$
= [U^\dagger U]_{\alpha\alpha} = 1.
$$

(164)

This result may seem surprising because if the $T^{\alpha}$ were unitary, then the result would have been $\text{Tr}\{T^\dagger(\alpha)T(\alpha)\} = 2$. However, surprising or not, this result is correct and consistent with the normalization we adopted. The second thing is about trace-preserving processes. A Kraus decomposition maintains the trace of the coherency matrix $J$, if and only if

$$
\sum_{\alpha=0}^{3}A^\dagger(\alpha)A(\alpha) = I_2,
$$

(165)

where $I_2$ is the $2 \times 2$ identity matrix. Let us see whether this is true or not in our case:

$$
\sum_{\alpha=0}^{3}A^\dagger(\alpha)A(\alpha) = \sum_{\alpha=0}^{3}\lambda_\alpha T^\dagger(\alpha)T(\alpha)
$$

$$
= \sum_{\alpha=0}^{3}\lambda_\alpha \sum_{\mu,\nu}^{0,3}[\vec{u}(\alpha)]^*_\mu\sigma(\mu)[\vec{u}(\alpha)]_\nu\sigma(\nu)
$$

$$
= \sum_{\mu,\nu}^{0,3}\sigma(\mu)\sigma(\nu)\sum_{\alpha=0}^{3}\lambda_\alpha[\vec{u}(\alpha)]^*_\nu[\vec{u}(\alpha)]_\mu
$$

$$
= \sum_{\mu,\nu}^{0,3}\sigma(\nu)C_{\nu\mu}\sigma(\mu),
$$

(166)

where Eq. (147) has been used. From the definition Eq. (90), we can write $C_{\nu\mu} = \langle c_\nu^* c_\mu \rangle$, where the brackets indicate the average with respect to an ensemble that represent a generic
medium. Then, Eq. (166) can be rewritten as

\[
\sum_{\alpha=0}^{3} A_{(\alpha)} A_{(\alpha)}^\dagger = \sum_{\mu, \nu}^{0,3} \sigma_{(\nu)} C_{\nu \mu} \sigma_{(\mu)}
\]

\[
= \sum_{\mu, \nu}^{0,3} \sigma_{(\nu)} \langle c_{\nu} c_{\mu}^* \rangle \sigma_{(\mu)}
\]

\[
= \left\langle \sum_{\mu, \nu}^{0,3} \sigma_{(\nu)} c_{\nu} c_{\mu}^* \sigma_{(\mu)} \right\rangle
\]

\[
= \left\langle \sum_{\mu=0}^{3} c_{\mu} \sum_{\nu=0}^{3} c_{\nu}^* \sigma_{(\mu)} \sigma_{(\nu)} \right\rangle
\]

\[
= \langle TT^\dagger \rangle,
\]

where Eq. (88) has been used. The it is clear that for a non-depolarizing medium \( TT^\dagger = I_2 \), which implies

\[
\sum_{\alpha=0}^{3} A_{(\alpha)} A_{(\alpha)}^\dagger = I_2.
\]  

(168)

In summary, for any Mueller \( M \) we can calculate the associate Hermitian matrix \( C \). Then, by diagonalizing \( C \) we find its eigenvectors \( \{ \vec{u}_{(\alpha)} \} \) whose components constitutes the Jones matrices \( \{ T_{(\alpha)} \} \). Finally we can find the transformation rule for the covariance matrix \( J \) as in Eq. (162).

A small comment is in order. Until now we have used the matrix \( C \) instead of \( H \) because it is expressed in terms of measurable quantities. However, from computational point of view the use of the matrix \( H \) reveals to be more advantageous. This can be seen in the following manner: let us multiply both sides of Eq. (136) by \( \Lambda \) and exploit the fact that \( \Lambda \) is unitary:

\[
(\Lambda C \Lambda^\dagger) \Lambda \vec{u}_{(\alpha)} = \lambda_\alpha \Lambda \vec{u}_{(\alpha)} \Leftrightarrow H \vec{v}_{(\alpha)} = \lambda_\alpha \vec{v}_{(\alpha)}, \quad (\alpha = 0, \ldots, 3),
\]

(169)

where Eq. (95) has been used and we have written the eigenvectors \( \vec{v}_{(\alpha)} \) of \( H \) as

\[
\vec{v}_{(\alpha)} = \Lambda \vec{v}_{(\alpha)}, \quad (\alpha = 0, \ldots, 3),
\]

(170)
At this point we can jump directly to Eq. (160) to write $T(\alpha)$ in terms of $\vec{v}(\alpha)$ as

\[
T(\alpha) = \sum_{\mu=0}^{3} [\bar{u}(\alpha)]_{\mu} \sigma(\mu)
\]
\[
= \sum_{\mu=0}^{3} [\Lambda^{\dagger} \vec{v}(\alpha)]_{\mu} \sigma(\mu)
\]
\[
= \sum_{\mu,\nu} \sigma(\mu) \Lambda^{\dagger}_{\mu\nu} [\vec{v}(\alpha)]_{\nu}
\]
\[
= \sum_{\nu=0}^{3} \epsilon(\nu) [\vec{v}(\alpha)]_{\nu},
\]

where Eq. (160) has been used. It is clear then, that the representation of $T(\alpha)$ in the basis $\{\epsilon(\mu)\}$ is very simple, being

\[
T(\alpha) = \begin{pmatrix} [\vec{v}(\alpha)]_{0} & [\vec{v}(\alpha)]_{1} \\ [\vec{v}(\alpha)]_{2} & [\vec{v}(\alpha)]_{3} \end{pmatrix}, \quad (\alpha = 0, \ldots, 3),
\]

which is very advantageous from computational point of view.

V. MUELLER MATRIX IN THE STANDARD BASIS

In this Section we introduce a new Mueller matrix $\mathcal{M}$ defined with respect to the standard basis. Let $J$ and $J'$ be the covariance matrices that describe the input and output light beams entering and leaving a given optical system, respectively. We assume the system to be a linear, passive optical element described by the linear map $\mathcal{M}$:

\[
\mathcal{M} : J \rightarrow J' = \mathcal{M}[J].
\]

The above linear relation can be explicitly written in terms of cartesian components as

\[
J'_{ij} = \mathcal{M}_{ij,kl} J_{kl}, \quad (i, j, k, l \in \{0, 1\}),
\]

or, by using the Rule

\[
J'_{\mu} = \mathcal{M}_{\mu\nu} J_{\nu}, \quad (\mu, \nu \in \{0, 1, 2, 3\}),
\]

where $\mu = 2i + j$, $\nu = 2k + l$, and $J_{\alpha} = \{\epsilon(\alpha), J\} = \text{Tr} \{\epsilon^{T}(\alpha) J\}$ are the components of the covariance matrix $J$ with respect to the standard basis $\{\epsilon(\alpha)\}$, $(\alpha = 0, \ldots, 3)$. Equation (175) is analogous to Eq. (55), the difference being that the former is written with respect
to the standard basis, while the latter with respect to the Pauli basis. Then, it is clear that $\mathcal{M}_{\mu\nu}$ is just the Mueller matrix written in the standard basis. This statement can be easily proved by calculating

$$J_\mu = \{\epsilon(\mu), J\}$$

$$= \text{Tr}\{\epsilon(\mu) J\}$$

$$= \Lambda_{\mu\nu} \text{Tr}\{\sigma(\nu) J\}$$

$$\equiv \Lambda_{\mu\nu} S_\nu,$$

where Eq. (40) was used in the third line (in fact, we have just rewritten Eq. (92)). Now, if we insert Eq. (176) for both $J_\nu$ and $J'_\mu$ into Eq. (175) we obtain

$$\Lambda_{\mu\alpha} S'_\alpha = \mathcal{M}_{\mu\nu} \Lambda_{\nu\beta} S_\beta,$$

which reads, in vectorial form

$$\Lambda \tilde{S}' = \mathcal{M} \Lambda \tilde{S} \Rightarrow \tilde{S}' = \Lambda^\dagger \mathcal{M} \Lambda \tilde{S}.$$

Since we know that $\tilde{S}' = M \tilde{S}$, then from Eq. (178) it straightforwardly follows the desired relation between $\mathcal{M}$ and $M$:

$$M = \Lambda^\dagger \mathcal{M} \Lambda.$$  

Finally, from Eqs. (63, 74) it follows that

$$\mathcal{M} = F, \quad H = \text{Per}[\mathcal{M}].$$

It is possible to write $\mathcal{M}$ directly in terms of the matrix elements of $H$. To this end, let us indicate with $\{E(\mu\nu)\}$ the standard basis in $\mathbb{R}^{4 \times 4}$ defined as

$$[E(\mu\nu)]_{\alpha\beta} = \delta_{\mu\alpha} \delta_{\nu\beta}.$$  

An explicit calculation shows that

$$\text{Per}[E(\mu\nu)] = \epsilon(\mu) \otimes \epsilon(\nu).$$

However, this equality can also be easily proved in the following way: Let us write

$$[E(\mu\nu)]_{\alpha\beta} = \delta_{\mu\alpha} \delta_{\nu\beta}$$

$$= [\epsilon(\mu)]_\alpha [\epsilon(\nu)]_\beta$$

(183)
where Eq. (25) has been used. Now we can use the Rule to write $\alpha = 2i + j$ and $\beta = 2k + l$ and rewrite Eq. (183) as

$$
\begin{align*}
[E_{(\mu\nu)}]_{\alpha\beta} &= [E_{(\mu\nu)}]_{ij,kl} \\
&= [\epsilon(\mu)]_{ij}[\epsilon(\nu)]_{kl} \\
&= [\epsilon(\mu) \otimes \epsilon(\nu)]_{ik,jl} \\
&= [\text{Per}[\epsilon(\mu) \otimes \epsilon(\nu)]]_{ij,kl} \\
&= [\text{Per}[\epsilon(\mu) \otimes \epsilon(\nu)]]_{\alpha\beta}
\end{align*}
$$

(184)

where Eq. (63) has been used. By comparing the first and the last row of Eq. (184) we obtain

$$
E_{(\mu\nu)} = \text{Per}[\epsilon(\mu) \otimes \epsilon(\nu)].
$$

(185)

Since for an arbitrary matrix $A \in \mathbb{C}^{4 \times 4}$ the following relations hold

$$
\text{Per}[\text{Per}[A]] = A, \quad A = \sum_{\alpha,\beta}^{0,3} A_{\alpha\beta} E_{(\alpha\beta)},
$$

(186)

then Eq. (182) follows and, moreover, we can write

$$
\mathcal{M} = \text{Per}[H] \\
= \sum_{\alpha,\beta}^{0,3} H_{\alpha\beta} \text{Per}[E_{(\alpha\beta)}] \\
= \sum_{\alpha,\beta}^{0,3} H_{\alpha\beta} (\epsilon(\alpha) \otimes \epsilon(\beta)),
$$

(187)

which is just the sought relation.

A. $\mathcal{M}$ as a positive map

In this subsection, we assume that the linear map $\mathcal{M}$ is a completely positive (CP) map. In this case we can write the transformation law of $J$ as a Kraus decomposition:

$$
J' = \sum_{\alpha=0}^{3} A_{(\alpha)} J A_{(\alpha)}^\dagger.
$$

(188)

In the standard basis

$$
A_{(\alpha)} = \sum_{\beta=0}^{3} \epsilon(\beta) A_{\beta\alpha},
$$

(189)
where

\[ A_{\beta\alpha} \equiv [A_{(\alpha)}]_\beta = \text{Tr}\{\epsilon^\dagger_{(\beta)} A_{(\alpha)}\}. \tag{190} \]

If we substitute Eq. (189) into Eq. (188) we obtain

\[
J' = \sum_{\alpha=0}^{3} A_{(\alpha)} J A^\dagger_{(\alpha)}
\]
\[
= \sum_{\alpha=0}^{3} \left( \sum_{\beta=0}^{3} \epsilon_{(\beta)} A_{\beta\alpha} \right) J \left( \sum_{\gamma=0}^{3} \epsilon^\dagger_{(\gamma)} A^\ast_{\gamma\alpha} \right)
\]
\[
= \sum_{\alpha,\beta,\gamma} A_{\beta\alpha} A^\dagger_{\beta\gamma} \left( \epsilon_{(\beta)} J \epsilon^\dagger_{(\gamma)} \right)
\]
\[
= \sum_{\beta,\gamma} \chi_{\beta\gamma} \left( \epsilon_{(\beta)} J \epsilon^\dagger_{(\gamma)} \right), \tag{191}
\]

where we have defined the Hermitian, positive semidefinite $4 \times 4$ matrix $\chi$ as:

\[ \chi \equiv AA^\dagger. \tag{192} \]

Now, in order to compare Eq. (191) with Eq. (174) we have to write the latter in terms of cartesian components as

\[
J'_{ij} = \sum_{\beta,\gamma} \chi_{\beta\gamma} \left( \epsilon_{(\beta)} J \epsilon^\dagger_{(\gamma)} \right)_{ij}
\]
\[
= \sum_{\beta,\gamma} \chi_{\beta\gamma} [\epsilon_{(\beta)}]_{ik} J_{kl} [\epsilon^\dagger_{(\gamma)}]_{lj}
\]
\[
= \left\{ \sum_{\beta,\gamma} \chi_{\beta\gamma} [\epsilon_{(\beta)}]_{ik} [\epsilon^\dagger_{(\gamma)}]_{jl} \right\} J_{kl}
\]
\[
\equiv M_{ij,kl} J_{kl}, \tag{193}
\]

where

\[
M_{ij,kl} = \sum_{\beta,\gamma} \chi_{\beta\gamma} [\epsilon_{(\beta)}]_{ik} [\epsilon^\dagger_{(\gamma)}]_{jl}
\]
\[
= \sum_{\beta,\gamma} \chi_{\beta\gamma} [\epsilon_{(\beta)} \otimes \epsilon_{(\gamma)}]_{ij,kl}, \tag{194}
\]

or, in matrix form

\[ M = \sum_{\beta,\gamma} \chi_{\beta\gamma} \left( \epsilon_{(\beta)} \otimes \epsilon_{(\gamma)} \right). \tag{195} \]

This Equation should be compared with Eq. (187) to write the identity

\[ H = \chi. \tag{196} \]
Therefore, we conclude that when $\mathcal{M}$ is a completely positive map, its associated $H$ matrix is positive semidefinite.

At this point it may be instructive to write explicitly the relation between $\mathcal{M}$ and $\chi$ (or $H$) in terms of their elements. Since

$$[\epsilon_{(\mu)}]_{ij} = \delta_{\mu, 2i+j},$$

then

$$\mathcal{M}_{ij,kl} = \sum_{\beta,\gamma} \chi_{\beta\gamma} [\epsilon_{(\beta)}]_{ik} [\epsilon_{(\gamma)}]_{jl}$$

$$= \sum_{\beta,\gamma} \chi_{\beta\gamma} \delta_{\beta, 2i+k} \delta_{\gamma, 2j+l}$$

$$= \chi_{2i+k, 2j+l};$$

or, in matrix form

$$\chi = H = \begin{pmatrix} \mathcal{M}_{00,00} & \mathcal{M}_{00,01} & \mathcal{M}_{01,00} & \mathcal{M}_{01,01} \\ \mathcal{M}_{00,10} & \mathcal{M}_{00,11} & \mathcal{M}_{01,10} & \mathcal{M}_{01,11} \\ \mathcal{M}_{10,00} & \mathcal{M}_{10,01} & \mathcal{M}_{11,00} & \mathcal{M}_{11,01} \\ \mathcal{M}_{10,10} & \mathcal{M}_{10,11} & \mathcal{M}_{11,10} & \mathcal{M}_{11,11} \end{pmatrix}.$$  

(199)

As expected, we found again the relation $H = \text{Per}[\mathcal{M}]$, as it is clear from a visual inspection of Eq. (199).

VI. CLASSICAL MUELLER MATRICES AND QUANTUM ENTANGLED STATES OR: QUANTUM MEASUREMENT OF A CLASSICAL MUELLER MATRIX

In this section we deal with the problem of determining the $4 \times 4$ density matrix representing a two-photon state, when the photon pair is scattered by a “medium” classically describable by a Mueller matrix. Here, with the word “medium” we denote any linear optical device, either deterministic or random, which scatters the photons. We consider two possible configurations: In the first one, a single scatterer interacts with only one of the two photons. In the second configuration there are two spatially separated media, each of them interacting with a single photon belonging to the photon pair. The relevant literature for the problem under consideration is listed below:
As it is in the style of these notes, we shall follow a didactic approach, so all the main formulas will be explicitly calculated step by step.

A. Rewriting the decomposition theorem

Let us begin by rewriting Eq. (162) as:

\[ J \rightarrow J' = \sum_{\alpha=0}^{3} p_{\alpha} S_{(\alpha)} J S_{(\alpha)}^{\dagger}, \tag{200} \]

where we have defined

\[ p_{\alpha} = \frac{\lambda_{\alpha}}{2M_{00}}, \quad S_{(\alpha)} = \sqrt{2M_{00}} T_{(\alpha)}, \tag{201} \]

in such a way that

\[ \sum_{\alpha=0}^{3} p_{\alpha} = 1, \quad \text{Tr}\{S_{(\alpha)}^{\dagger} S_{(\alpha)}\} = 2M_{00}, \quad (\alpha = 0, 1, 2, 3), \tag{202} \]

where Eq. (164) has been used. Now, we exploit the isomorphism between the classical covariance matrix \( J \) and the quantum density matrix \( \rho \) and make the ansatz that a single photon initially prepared in the quantum state \( \rho \), after the interaction with a medium classically described by Eq. (200) can be described by the density matrix \( \rho' \) defined as:

\[ \rho \rightarrow \rho' = \sum_{\alpha=0}^{3} p_{\alpha} S_{(\alpha)} \rho S_{(\alpha)}^{\dagger}. \tag{203} \]

B. Single- and two-photon quantum states

Let us denote with

\[ \{|i\} = \{|0\}, \{1\} \}, \quad (i = 0, 1), \tag{204} \]
the basis kets representing two orthogonal linear polarization states of a photon. These states are often indicated as horizontal $|H\rangle$ and vertical $|V\rangle$, respectively. Here we follow the convention

$$|0\rangle = |H\rangle, \quad |1\rangle = |V\rangle.$$  \hspace{1cm} (205)

By definition these states form an orthonormal and complete basis:

$$\langle i|j \rangle = \delta_{ij}, \quad (i, j \in \{0, 1\}), \quad \sum_{i=0}^{1} |i\rangle\langle i| = \hat{1}.$$  \hspace{1cm} (206)

As usual, we put them in correspondence with the standard basis in $\mathbb{R}^2$ $\{|\alpha\rangle\}$:

$$|0\rangle \triangleq f(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle \triangleq f(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (207)

In a similar manner, the dual basis $\{\langle i|\}$, $(i = 0, 1)$ is associated with $\{f^\dagger(\alpha)\}$:

$$\langle 0| \triangleq f^\dagger(0) = (1\ 0), \quad \langle 1| \triangleq f^\dagger(1) = (0\ 1).$$  \hspace{1cm} (208)

The two-photon polarization standard basis can be built by tacking the direct product between single photon states, as follows:

$$|\alpha = 2i + j\rangle = |i\rangle \otimes |j\rangle \equiv |ij\rangle, \quad (i, j \in \{0, 1\}, \alpha \in \{0, \ldots, 3\}),$$  \hspace{1cm} (209)

where the Rule has been used to write $\alpha = 2i + j$. It is straightforward to show that

$$\langle \alpha|\beta \rangle = \langle \langle i| \otimes \langle j| \rangle(|k\rangle \otimes |l\rangle) \rangle \hspace{1cm} = \langle i|k\rangle \langle j|l\rangle \hspace{1cm} = \delta_{ik}\delta_{jl} \hspace{1cm} = \delta_{2i+j,2k+l} \hspace{1cm} = \delta_{\alpha\beta}.$$  \hspace{1cm} (210)

In the literature it is often used the so-called Bell basis $\{|b(\alpha)\rangle\}$ defined as

$$|b(\alpha)\rangle = \hat{B}|\alpha\rangle, \quad (\alpha = 0, \ldots, 3),$$  \hspace{1cm} (211)

where the unitary operator $\hat{B}$ is represented with respect to the standard basis $\{|\alpha\rangle\}$ by the unitary matrix $B$

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (212)
In explicit form we have

\[
|\psi^+\rangle = |b(0)\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle),
|\psi^-\rangle = |b(1)\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle),
|\phi^+\rangle = |b(2)\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle),
|\phi^-\rangle = |b(3)\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle),
\] (213)

where the first column displays the most common notation for the Bell states.

Four single-photon operators \(\{\hat{\epsilon}(\alpha), (\alpha = 0, \ldots, 3)\}\) may be formed by tacking the direct product between a single-photon bra and a single-photon ket as follows:

\[
\hat{\epsilon}(\alpha) \equiv |i\rangle\langle j|, \quad (\alpha = 2i + j; \ i, j \in \{0, 1\}).
\] (214)

These operators can be straightforwardly put in a one-to-one correspondence with the elements of the standard basis \(\{\epsilon(\alpha)\}\) in \(\mathbb{R}^{2\times 2}\):

\[
\hat{\epsilon}(\alpha) \cong \epsilon(\alpha) = \tilde{f}(i) \otimes \tilde{f}^\dagger(j), \quad (\alpha = 2i + j; \ i, j \in \{0, 1\}),
\] (215)

where

\[
[\epsilon(\alpha)]_{kl} = [\tilde{f}(i) \otimes \tilde{f}^\dagger(j)]_{kl} = [\tilde{f}(i)]_k[\tilde{f}^\dagger(j)]_l = \delta_{ik}\delta_{jl} = \delta_{2i+j,2k+l} = \delta_{\alpha\beta},
\] (216)

where \(\beta = 2k + l\), in agreement with Eq. (25).
C. Two-photon density matrix and scattering processes

An arbitrary two-photon state can be described by a density operator $\hat{\rho}$ as

$$
\hat{\rho} = \sum_{\alpha,\beta}^{0,3} D_{\alpha\beta} |\alpha\rangle \langle \beta| \\
= \sum_{i,j,k,l}^{0,1} D_{ijkl} |i\rangle \langle k| \otimes |j\rangle \langle l| \\
= \sum_{i,j,k,l}^{0,1} \tilde{D}_{ikjl} |i\rangle \langle k| \otimes |j\rangle \langle l| \\
= \sum_{\mu,\nu}^{0,3} \tilde{D}_{\mu\nu} \hat{\epsilon}(\mu) \otimes \hat{\epsilon}(\nu),
$$

(217)

where $\mu = 2i + k$, $\nu = 2j + l$ and

$$
\tilde{D}_{ikjl} = D_{ijkl} \iff \tilde{D} = \text{Per}[D].
$$

(218)

At this point we can work directly with the matrix representation of the operators and deal with the density matrix $\rho$ corresponding to the operator $\hat{\rho}$:

$$
\hat{\rho} \equiv \rho = \sum_{\mu,\nu}^{0,3} \tilde{D}_{\mu\nu} \epsilon(\mu) \otimes \epsilon(\nu),
$$

(219)

where Eq. (216) has been used. Before going ahead, we need to derive two intermediate results. The first one is a simple calculation: Because of the completeness of the Pauli basis we can always write:

$$
\sigma_{(\alpha)} \sigma_{(\mu)} \sigma_{(\beta)} = \sum_{\nu=0}^{3} K_{\alpha\mu\beta\nu} \sigma_{(\nu)},
$$

(220)

where, by definition

$$
K_{\alpha\mu\beta\nu} = \text{Tr}\{\sigma_{(\alpha)} \sigma_{(\mu)} \sigma_{(\beta)} \sigma_{(\nu)}\} = [\Gamma_{(\mu\nu)}]_{\alpha\beta},
$$

(221)

where Eq. (131) has been used. Moreover, we note that from the definition (221) it immediately follows that

$$
[\Gamma_{(\mu\nu)}]_{\alpha\beta} = \text{Tr}\{\sigma_{(\alpha)} \sigma_{(\mu)} \sigma_{(\beta)} \sigma_{(\nu)}\}
= \text{Tr}\{\sigma_{(\beta)} \sigma_{(\nu)} \sigma_{(\alpha)} \sigma_{(\mu)}\}
= [\Gamma_{(\nu\mu)}]_{\beta\alpha}.
$$

(222)
The second result we need is also a simple calculation: First, from Eq. (201) we write

\[ S(\alpha) = \sum_{\beta=0}^{3} S_{\alpha\beta} \sigma(\beta), \]  

(223)

where, by definition,

\[ S_{\alpha\beta} = \text{Tr}\{\sigma(\beta)S(\alpha)\} \]

\[ = \sqrt{2M_{00}} \sum_{\mu=0}^{3} [\bar{u}(\alpha)]_{\mu} \text{Tr}\{\sigma(\beta)\sigma(\mu)\} \]

\[ = \sqrt{2M_{00}} [\bar{u}(\alpha)]_{\beta} \]

\[ = \sqrt{2M_{00}} U_{\beta\alpha}, \]

(224)

and Eqs. (201, 160, 140) have been used. Then, by using Equations (40, 221, 224) we can calculate the following quantity that will be used later:

\[ \sum_{\gamma=0}^{3} p_{\gamma} S(\gamma) \epsilon(\eta) S(\gamma)^\dagger = \sum_{\gamma,\alpha,\beta}^{0,3} p_{\gamma} S_{\gamma\alpha \gamma\beta}^* \sigma(\alpha) \epsilon(\eta) \sigma(\beta) \]

\[ = \sum_{\gamma,\alpha,\beta,\mu}^{0,3} \Lambda_{\mu}^{\dagger} p_{\gamma} S_{\gamma\alpha \gamma\beta}^* \sigma(\alpha) \sigma(\mu) \sigma(\beta) \]

\[ = \sum_{\gamma,\alpha,\beta,\mu,\nu}^{0,3} \Lambda_{\mu}^{\dagger} p_{\gamma} S_{\gamma\alpha \gamma\beta}^* \Gamma(\mu\nu) \sigma(\nu) \]

\[ = \sum_{\alpha,\beta,\mu,\nu,\tau} \Lambda_{\mu}^{\dagger} \left( \sum_{\gamma=0}^{3} p_{\gamma} S_{\gamma\alpha \gamma\beta}^* \Gamma(\mu\nu) \right) \Lambda_{\tau\nu} \epsilon(\tau). \]

(225)

Moreover, from Eqs. (142, 143, 146) it follows that

\[ \sum_{\gamma=0}^{3} p_{\gamma} S_{\gamma\alpha \gamma\beta}^* = \sum_{\gamma=0}^{3} \frac{\lambda_{\gamma}}{2M_{00}} S_{\gamma\beta}^* \]

\[ = \sum_{\gamma=0}^{3} \sqrt{2M_{00}} U_{\alpha\gamma} \frac{\lambda_{\gamma}}{2M_{00}} \sqrt{2M_{00}} U_{\beta\gamma}^* \]

\[ = \sum_{\gamma=0}^{3} U_{\alpha\gamma} \lambda_{\gamma} U_{\gamma\beta}^\dagger \]

\[ = \sum_{\gamma=0}^{3} U_{\alpha\gamma} \Lambda_{\gamma\beta}^\dagger \]

\[ = \sum_{\gamma=0}^{3} U_{\alpha\gamma} \delta_{\gamma\beta} U_{\gamma\beta}^\dagger \]

\[ = [UDU^\dagger]_{\alpha\beta} \]

\[ = C_{\alpha\beta}, \]
therefore we can rewrite Eq. (225) as

\[
3 \sum_{\gamma=0} p_{\gamma} S_{(\gamma)} \epsilon(\eta) S_{(\gamma)}^\dagger = 3 \sum_{\alpha,\beta,\mu,\nu,\tau} \Lambda^{\dagger}_{\mu \eta} C_{\alpha \beta} \left[ \Gamma_{(\mu \nu)} \right]_{\alpha \beta} \Lambda_{\tau \nu} \epsilon(\tau)
\]

\[
= 3 \sum_{\alpha,\beta,\mu,\nu,\tau} \Lambda^{\dagger}_{\mu \eta} C_{\alpha \beta} \left[ \Gamma_{(\nu \mu)} \right]_{\beta \alpha} \Lambda_{\tau \nu} \epsilon(\tau)
\]

\[
= 3 \sum_{\mu,\nu,\tau} \Lambda_{\tau \nu} M_{\nu \mu} \Lambda^{\dagger}_{\mu \eta} \epsilon(\tau)
\]

\[
= 3 \sum_{\tau=0}^{3} \left[ \Lambda M \Lambda^{\dagger} \right]_{\tau \eta} \epsilon(\tau)
\]

\[
= 3 \sum_{\tau=0}^{3} M_{\tau \eta} \epsilon(\tau),
\]

(227)

where Eqs. (103, 179, 226) have been used.

At this point we have collected all the results necessary to calculate explicitly the transformation law of the density matrix:

\[
\rho' = 3 \sum_{\gamma=0}^{3} p_{\gamma} \left( S_{(\gamma)} \otimes I_2 \right) \rho \left( S_{(\gamma)}^\dagger \otimes I_2 \right)
\]

\[
= 3 \sum_{\gamma,\eta,\zeta} D_{\eta \zeta} p_{\gamma} \left( S_{(\gamma)} \otimes I_2 \right) \left( \epsilon(\eta) \otimes \epsilon(\zeta) \right) \left( S_{(\gamma)}^\dagger \otimes I_2 \right)
\]

\[
= 3 \sum_{\eta,\zeta} D_{\eta \zeta} \left( 3 \sum_{\gamma=0}^{3} p_{\gamma} S_{(\gamma)} \epsilon(\eta) S_{(\gamma)}^\dagger \right) \otimes \epsilon(\zeta)
\]

\[
= 3 \sum_{\eta,\zeta,\tau} M_{\tau \eta} D_{\eta \zeta} \epsilon(\tau) \otimes \epsilon(\zeta)
\]

\[
= \sum_{\eta,\zeta,\tau} [M \tilde{D}]_{\tau \zeta} \epsilon(\tau) \otimes \epsilon(\zeta)
\]

\[
= \sum_{\zeta,\tau} [M \tilde{D}]_{\tau \zeta} \text{Per}[E_{(\tau \zeta)}]
\]

\[
= \text{Per} \left[ 3 \sum_{\zeta,\tau} [M \tilde{D}]_{\tau \zeta} E_{(\tau \zeta)} \right]
\]

\[
= \text{Per}[M \tilde{D}],
\]

(228)

where Eq. (185) has been used. Let us note that, by definition, from Eq. (219) it trivially
follows that

\[ \rho' = \sum_{\alpha,\beta} \tilde{D}_{\alpha\beta}' \epsilon_{(\alpha)} \otimes \epsilon_{(\beta)} \]

\[ = \sum_{\alpha,\beta} \tilde{D}_{\alpha\beta}' \text{Per}[E_{(\alpha\beta)}] \]

\[ = \text{Per}[\tilde{D}']. \]  

Finally, by equating Eq. (228) with Eq. (229), we obtain

\[ \tilde{D}' = M \tilde{D} \]

\[ = \Lambda M \Lambda^\dagger \tilde{D}, \]  

which, when \( \text{det}\{\tilde{D}\} \neq 0 \), can be inverted to give:

\[ M = \Lambda^\dagger \tilde{D}'(\tilde{D})^{-1}\Lambda. \]  

This result shows that the knowledge given by a single input quantum state (represented in this case by \( D \)) is sufficient to uniquely determine the classical Mueller matrix representing the scatterer.

Equation (230) relates the Cartesian coordinates in the standard basis of the input and output density matrices \( \rho \) and \( \rho' \) respectively. However, in the classical Mueller-Stokes formalism the observables are referred to the Pauli basis rather than to the standard one. To illustrate this point let us consider the density matrices \( \rho^A \) and \( \rho^B \) of two independent photons

\[ \rho^F = \sum_{\alpha=0}^{3} S_{\alpha}^F \sigma_{(\alpha)}, \quad (F = A, B), \]  

and let build the corresponding two-photon density matrix \( \rho^{AB} \) in the usual way:

\[ \rho^{AB} = \rho^A \otimes \rho^B \]

\[ = \sum_{\alpha,\beta} S_{\alpha}^A S_{\beta}^B \sigma_{(\alpha)} \otimes \sigma_{(\beta)} \]

\[ \equiv \sum_{\alpha,\beta} D_{\alpha\beta}^{AB} \sigma_{(\alpha)} \otimes \sigma_{(\beta)}, \]  

where we have defined the 16 two-photon Stokes parameters as:

\[ D_{\alpha\beta}^{AB} \equiv S_{\alpha}^A S_{\beta}^B. \]
From now on we suppress the superscript $AB$ and we seek the relation between the two $4 \times 4$ matrices $\tilde{D}$ and $D$ defined by the following relations:

$$\rho = \sum_{\alpha,\beta}^{0,1} D_{\alpha\beta} \sigma_{(\alpha)} \otimes \sigma_{(\beta)}$$

$$= \sum_{\alpha,\beta}^{0,1} \tilde{D}_{\alpha\beta} \epsilon_{(\alpha)} \otimes \epsilon_{(\beta)} \cdot \tag{235}$$

By using Eq. (40) it trivially follows

$$\rho = \sum_{\alpha,\beta}^{0,1} D_{\alpha\beta} \sigma_{(\alpha)} \otimes \sigma_{(\beta)}$$

$$= \sum_{\alpha,\beta,\mu,\nu}^{0,1} \Lambda_{\mu\alpha} D_{\alpha\beta} \Lambda_{\nu\beta} \epsilon_{(\mu)} \otimes \epsilon_{(\nu)}$$

$$= \sum_{\mu,\nu}^{0,1} [\Lambda D \Lambda^T]_{\mu\nu} \epsilon_{(\mu)} \otimes \epsilon_{(\nu)}$$

$$= \sum_{\mu,\nu}^{0,1} \tilde{D}_{\mu\nu} \epsilon_{(\mu)} \otimes \epsilon_{(\nu)} \cdot \tag{236}$$

So, we found

$$\tilde{D} = \Lambda D \Lambda^T \iff D = \Lambda^\dagger \tilde{D} \Lambda^* \cdot \tag{237}$$

where we have used the fact that $\Lambda^T \Lambda^* = I_4$. Finally, by multiplying both sides of Eq. (230) from left by $\Lambda^\dagger$ and from right by $\Lambda^*$ we obtain

$$D' = MD \cdot \tag{238}$$

This relation is the “quantum-equivalent” to the classical one relating input and output Stokes vectors. Then, in the Pauli basis the expression for the Mueller matrix becomes very simple:

$$M = D'D^{-1} \cdot \tag{239}$$

We can use alternatively Eq. (231) or Eq. (239) to determine what classical Mueller matrix is necessary to achieve a certain quantum state. For example, suppose that we seek a scatterer that produces a Maximally Entangled Mixed State (MEMS) when interacting with an individual photon belonging to an entangled pair prepared in the “singlet state”, namely $|b(3)\rangle$ as given in the last row of Eq. (213). The output MEMS is characterized by
the density matrix $D'$ in the standard basis

$$D' = \begin{pmatrix} g(\gamma) & 0 & 0 & \gamma/2 \\ 0 & 1 - 2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & g(\gamma) \end{pmatrix},$$

(240)

where

$$g(\gamma) = \begin{cases} \gamma/2, & \gamma \geq 2/3, \\ 1/3, & \gamma < 2/3, \end{cases}$$

(241)

while the input singlet state is described by

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{242}$$

If we substitute Eq. (240) and Eq. (242) into Eq. (231) we obtain straightforwardly

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 - 2g(\gamma) \\ 0 & -\gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 1 - 2g(\gamma) & 0 & 0 & 1 - 4g(\gamma) \end{pmatrix}. \tag{243}$$

As a last example, we consider the case of an output Werner state represented by

$$D' = \begin{pmatrix} (1 - p)/4 & 0 & 0 & 0 \\ 0 & (1 + p)/4 & -p/2 & 0 \\ 0 & -p/2 & (1 + p)/4 & 0 \\ 0 & 0 & 0 & (1 - p)/4 \end{pmatrix}, \tag{244}$$

and again a singlet input state. In this case it is easy to see that the required Mueller matrix can be written as

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{245}$$

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3 W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, *Maximizing the entanglement of two mixed qubits*, Phys. Rev. A 64 R030302 (2001).
D. Multi-mode states

Until now we dealt with two- and four-dimensional Hilbert spaces, since we considered only polarization degrees of freedom of photons. However, photons also possess other degrees of freedom that, although apparently irrelevant, may play an important role. In this subsection we consider photons as physical systems with many degrees of freedom, including the polarization ones that will be regarded as the relevant ones.

Let us consider a finite-dimensional “bare bones” version of the electromagnetic field. It consists of $2N$ independent one-dimensional harmonic oscillators each of them characterized by two quantum numbers: the “mode” number $n \in \{0, 1, \ldots, N-1\}$ and the “polarization” number $\alpha \in \{0, 1\}$. For a given $n$ the two oscillators labelled by the pairs $\{n, \alpha = 0\}$ and $\{n, \alpha = 1\}$ “oscillate” along two mutually orthogonal directions fixed by the two (possibly complex) unit vectors $\epsilon_{n0}$ and $\epsilon_{n1}$, respectively:

$$\langle \epsilon_{n\alpha}, \epsilon_{n\beta} \rangle = \delta_{\alpha\beta}, \quad (\alpha, \beta \in \{0, 1\}).$$

(246)

A third unit vector $\epsilon_{n3}$ orthogonal to the other two remains automatically fixed by the relation

$$\epsilon_{n2} = \epsilon_{n0} \times \epsilon_{n1}.$$  

(247)

It is important to note that in the theory there is not a third harmonic oscillator labelled by $\{n, \alpha = 2\}$ that oscillates along $\epsilon_{n2}$. However, from a geometrical point of view the introduction of $\epsilon_{n2}$ is necessary to write the resolution of the identity in a 3-dimensional space as

$$\sum_{i=0}^{2} \epsilon_{ni} \epsilon_{ni}^\dagger = I_3,$$

(248)

where $I_3$ is the $3 \times 3$ identity matrix. A set of $N$ projection matrices $\{P_n\}$ (and the complementary ones $\{Q_n\}$) projecting onto the physical directions of oscillation of the system, can be easily build as

$$P_n = \sum_{\alpha=0}^{1} \epsilon_{n\alpha} \epsilon_{n\alpha}^\dagger$$

(249)

and $P_n + Q_n = I_3$. Each harmonic oscillator is characterized by its annihilation and creation operators $\hat{a}_{n\alpha}$ and $\hat{a}_{n\alpha}^\dagger$ respectively, that satisfy the canonical commutation rules:

$$[\hat{a}_{n\alpha}, \hat{a}_{m\beta}^\dagger] = \delta_{nm} \delta_{\alpha\beta}.$$  

(250)
The Hamiltonian of the system is just the sum of the Hamiltonians of the $2N$ harmonic oscillators:

$$
\hat{H} = \frac{1}{2} \sum_{n=0}^{N-1} \sum_{\alpha=0}^{1} \omega_n (\hat{a}_{n\alpha}^\dagger \hat{a}_{n\alpha} + \hat{a}_{n\alpha} \hat{a}_{n\alpha}^\dagger),
$$

(251)

where $\hbar = 1$ and $\omega_n \geq 0$. The single-particle states $\{ |n\alpha\rangle \}$ are built from the vacuum state $|0\rangle$ in the usual way

$$
|n\alpha\rangle = \hat{a}_{n\alpha}^\dagger |0\rangle.
$$

(252)

Finally, the resolution of the identity can be written as

$$
\mathbb{I} = \mathbb{I}_0 + \mathbb{I}_1 + \ldots = |0\rangle \langle 0| + \sum_{n=0}^{N-1} \sum_{\alpha=0}^{1} |n\alpha\rangle \langle n\alpha| + \sum \{\text{multiparticle states}\}.
$$

(253)

Now that our system is well defined, we try to build a Positive Operator Valued Measure (POVM) in order to determine the relevant density matrix pertaining to the relevant polarization degrees of freedom. Let $\{ f_i \}$ denotes an orthonormal and complete basis in $\mathbb{C}^3$:

$$
(f_i, f_j) = \delta_{ij}, \quad \sum_{i=0}^{2} f_i f_i^\dagger = I_3, \quad (i, j \in \{0, 1, 2\}).
$$

(254)

By using Eq. (248) for each mode $n$ we can write

$$
f_i = I_3 \cdot f_i = \sum_{j=0}^{2} \epsilon_{nj}^\dagger \epsilon_{nj} \cdot f_i = \sum_{j=0}^{2} \epsilon_{nj} (\epsilon_{nj}, f_i) \equiv \sum_{j=0}^{2} \epsilon_{nj} F_{nji},
$$

(255)

where $F_{nji} \equiv (\epsilon_{nj}, f_i)$. Then, we define the physical vectors $f_{ni}$ associated to the mode $n$ as

$$
f_{ni} = \mathcal{P}_n f_i = \sum_{\alpha=0}^{1} \epsilon_{n\alpha} \epsilon_{n\alpha}^\dagger \cdot f_i = \sum_{\alpha=0}^{1} \epsilon_{n\alpha} (\epsilon_{n\alpha}, f_i) \equiv \sum_{\alpha=0}^{1} \epsilon_{n\alpha} F_{nai},
$$

(256)
These vectors are not of unit length nor mutually orthogonal:

\[(f_{ni}, f_{nj}) = (\mathcal{P}_n f_i, \mathcal{P}_n f_j) = (f_i, \mathcal{P}_n f_j) = \mathcal{P}_{nij},\]

where, by definition, \(\mathcal{P}_{nii} \geq 0\).

Now we are ready to write the single-mode operator \(\hat{F}_{ni}\) acting on the physical states of the system as

\[
\hat{F}_{ni} = \begin{cases} 
\sum_{\alpha=0}^{1} \frac{f_{ni}}{\sqrt{(f_{ni}, f_{ni})}} (f_{ni}, \epsilon_{n\alpha}) \hat{a}_{n\alpha}, & (f_{ni}, f_{ni}) \neq 0, \\
0, & (f_{ni}, f_{ni}) = 0.
\end{cases}
\] (258)

Then, we can use this operator to build the multi-mode Hermitian positive semidefinite “intensity” operator \(\hat{F}_i\) as

\[
\hat{F}_i = \sum_{n=0}^{N-1} \hat{F}_{ni} \cdot \hat{F}_{ni}
= \sum_{n=0}^{N-1} \sum_{\alpha,\beta=0,1} \left[ \frac{f_{ni}^\dagger}{\sqrt{(f_{ni}, f_{ni})}} (f_{ni}, \epsilon_{n\alpha})^* \hat{a}_{n\alpha}^\dagger \right] \cdot \left[ \frac{f_{ni}}{\sqrt{(f_{ni}, f_{ni})}} (f_{ni}, \epsilon_{n\beta}) \hat{a}_{n\beta} \right]
= \sum_{n=0}^{N-1} \sum_{\alpha,\beta=0,1} (f_{ni}, \epsilon_{n\alpha})^* (f_{ni}, \epsilon_{n\beta}) \hat{a}_{n\alpha}^\dagger \hat{a}_{n\beta}
= \sum_{n=0}^{N-1} \sum_{\alpha,\beta=0,1} (\epsilon_{n\alpha}, f_{ni}) (f_{ni}, \epsilon_{n\beta}) \hat{a}_{n\alpha}^\dagger \hat{a}_{n\beta}
= \sum_{n=0}^{N-1} \sum_{\alpha,\beta=0,1} (\epsilon_{n\alpha}, f_{ni}) (f_{ni}, \epsilon_{n\beta}) \hat{a}_{n\alpha}^\dagger \hat{a}_{n\beta},
\] (259)

where the last step trivially follows from the fact that \(\mathcal{P}_n \epsilon_{n\alpha} = \epsilon_{n\alpha}\) and, therefore,

\[
(\epsilon_{n\alpha}, f_{ni}) = (\mathcal{P}_n \epsilon_{n\alpha}, f_{ni}) = (\epsilon_{n\alpha}, \mathcal{P}_n f_{ni}) = (\epsilon_{n\alpha}, f_{ni}).
\] (260)

At this point it is easy to see that the three operators \(\{\hat{F}_0, \hat{F}_1, \hat{F}_2\}\) form a POVM in the
one-particle space:

\[
\hat{F} = \sum_{i=0}^{2} \hat{F}_i
\]

\[
= \sum_{n=0}^{N-1} \sum_{\alpha, \beta} \sum_{i=0}^{2} (\epsilon_{\alpha \beta}, f_{\alpha \beta}) (f_{\alpha \beta}, \epsilon_{\alpha \beta}) \hat{a}_{\alpha \beta} \hat{a}_{\alpha \beta}
\]

\[
= \sum_{n=0}^{N-1} \sum_{n=0}^{1} \sum_{\alpha, \beta} (\epsilon_{\alpha \beta}, \epsilon_{\alpha \beta}) \hat{a}_{\alpha \beta} \hat{a}_{\alpha \beta}
\]

\[
= \sum_{n=0}^{N-1} \sum_{\alpha=0}^{1} \sum_{\alpha=0}^{1} (\epsilon_{\alpha \beta}, \epsilon_{\alpha \beta}) \hat{a}_{\alpha \beta} \hat{a}_{\alpha \beta}
\]

\[
= \hat{N},
\]

where \(\hat{N}\) is the particle-number operator and Eqs. (246) and (254) have been used.

E. Reconstruction of the density matrix

Let \(\mathcal{R} = \{x, y, z\}\) be an orthonormal Cartesian coordinate system in \(\mathbb{R}^3\) and let \(\mathcal{U}, \mathcal{V}\) and \(\mathcal{W}\) three mutually unbiased bases for \(\mathbb{C}^3\) defined as

\[
\mathcal{U} = \{u_0, u_1, u_2\} = \{x, y, z\},
\]

\[
\mathcal{V} = \{v_0, v_1, v_2\} = \left\{\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}, z\right\},
\]

\[
\mathcal{W} = \{w_0, w_1, w_2\} = \left\{\frac{x+iy}{\sqrt{2}}, \frac{x-iy}{\sqrt{2}}, z\right\}.
\]

(262)

From a physical point of view, these three bases correspond to the three pairs of mutually orthogonal polarization directions (\(\mathcal{U}, \mathcal{V}, \mathcal{W}\): linear horizontal-vertical, linear 45°-135°, and circular right-left, respectively), selected by a polarizer whose planar surface is orthogonal to \(z\). We want to calculate the Stokes parameter of a beam of light (either classical or quantum). To this end, let us imagine to repeat the construction of the POVM outlined in the previous section for each of the basis set \(\mathcal{U}, \mathcal{V}\) and \(\mathcal{W}\), thus obtaining three different POVMs denoted with \(\hat{U}_i, \hat{V}_i\) and \(\hat{W}_i\), respectively. For example, if in Eq. (259) we substitute \(f_{ni}\) with \(u_{ni}\), we obtain

\[
\hat{U}_i = \sum_{n=0}^{N-1} \sum_{\alpha, \beta} (\epsilon_{\alpha \beta}, u_{ni}) (u_{ni}, \epsilon_{\alpha \beta}) \hat{a}_{\alpha \beta} \hat{a}_{\alpha \beta}, \quad (i = 0, 1, 2).
\]

(263)
In exactly the same manner we may obtain $\hat{V}_i$ and $\hat{W}_i$. As a subsequent step we introduce, in analogy with classical optics, four Hermitian “Stokes” operators defined as follows:

$$
\hat{S}_0 = \frac{1}{\sqrt{2}} \left( \hat{U}_0 + \hat{U}_1 \right),
\hat{S}_1 = \frac{1}{\sqrt{2}} \left( \hat{V}_0 - \hat{V}_1 \right),
\hat{S}_2 = \frac{1}{\sqrt{2}} \left( \hat{W}_0 - \hat{W}_1 \right),
\hat{S}_3 = \frac{1}{\sqrt{2}} \left( \hat{U}_0 - \hat{U}_1 \right).
$$

(264)

For sake of clarity, we introduce the six operators $\{\hat{E}_x\}$, $(X = 0, \ldots, 5)$ defined as

$$
\begin{bmatrix}
\hat{E}_0 \\
\hat{E}_1 \\
\hat{E}_2 \\
\hat{E}_3 \\
\hat{E}_4 \\
\hat{E}_5
\end{bmatrix} = \begin{bmatrix}
\hat{U}_0 \\
\hat{U}_1 \\
\hat{V}_0 \\
\hat{V}_1 \\
\hat{W}_0 \\
\hat{W}_1
\end{bmatrix},
$$

(265)

in such a way that we can rewrite Eq. (264) in a compact form as

$$
\hat{S}_A = \sum_{X=0}^{5} P_{AX} \hat{E}_X, \quad (A \in \{0, \ldots, 3\}),
$$

(266)

where we have defined the $4 \times 6$ matrix $P$ as

$$
P = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

(267)

and $PP^\dagger = I_4$. It is instructive to write explicitly the operators $\{\hat{S}_A\}$:

$$
\hat{S}_A = \sum_{X=0}^{5} P_{AX} \hat{E}_X
$$

(268)

$$
= \sum_{X=0}^{5} P_{AX} \sum_{n=0}^{N-1} \sum_{\alpha,\beta} (\mathbf{e}_{n\alpha}, \mathbf{e}_{n\beta}) (\mathbf{e}_{nx}, \mathbf{e}_{n\beta}) \hat{a}_{n\alpha}^\dagger \hat{a}_{n\beta},
$$

where $\mathbf{e} = \xi(X) \in \{u, v, w\}$ and $x = x(X) \in \{0, 1\}$. Then, we can rewrite Eq. (268) as

$$
\hat{S}_A = \sum_{n=0}^{N-1} \sum_{\alpha,\beta} (\mathbf{e}_{n\alpha}, \mathbf{e}_{n\beta}) \left[ \sum_{X=0}^{5} P_{AX} \mathbf{e}_{nx} \mathbf{e}_{n\beta}^\dagger \right] \cdot \mathbf{e}_{n\beta} \hat{a}_{n\alpha}^\dagger \hat{a}_{n\beta},
$$

(269)
where an explicit calculation shows that

\[
\sum_{X=0}^{5} P_{AX} \xi_{nx} \xi_{nx}^\dagger = \begin{pmatrix}
[\sigma(A)]_{00} & [\sigma(A)]_{01} & 0 \\
[\sigma(A)]_{10} & [\sigma(A)]_{11} & 0 \\
0 & 0 & 0
\end{pmatrix}
\equiv \Omega(A),
\]  

(270)

where \(\{\sigma(A)\}, (A \in \{0, \ldots, 3\})\) are the \(2 \times 2\) Pauli matrices, and Eqs. (262) and (267) have been used. Finally we can write

\[
\hat{S}(A) = \sum_{n=0}^{N-1} \sum_{\alpha, \beta} (\epsilon_{n\alpha}, \Omega(A)\epsilon_{n\beta}) \hat{a}_{n\alpha}^\dagger \hat{a}_{n\beta}
\]

(271)

where with \(\epsilon_{n\beta} \in \mathbb{C}^2\) we have denoted the restriction of \(\epsilon_{n\beta}\) to a two-dimensional subspace:

\[
\epsilon_{n\beta} \equiv \begin{pmatrix}
[\epsilon_{n\beta}]_0 \\
[\epsilon_{n\beta}]_1
\end{pmatrix}.
\]

(272)

Of course, the two-dimensional vectors \(\{\epsilon_{n\alpha}\}\) are not unit length nor mutually orthogonal. Now we can use this result to calculate

\[
\langle n\nu | \hat{S}(A) | m\mu \rangle = \sum_{p=0}^{N-1} \sum_{\alpha, \beta} (\epsilon_{p\alpha}, \sigma(A)\epsilon_{p\beta}) \langle n\nu | \hat{a}_{p\alpha}^\dagger \hat{a}_{p\beta} | m\mu \rangle
\]

(273)

\[
= \sum_{p=0}^{N-1} \sum_{\alpha, \beta} (\epsilon_{p\alpha}, \sigma(A)\epsilon_{p\beta}) \langle 0 | \hat{a}_{n\alpha}^\dagger \hat{a}_{p\beta}^\dagger \hat{a}_{m\mu} | 0 \rangle
\]

\[
= \delta_{nm}(\epsilon_{n\nu}, \sigma(A)\epsilon_{n\mu}).
\]

At this point we have all the ingredients necessary to calculate the expectation value \(\langle \hat{S}(A) \rangle\) with respect to the generic state described by \(\hat{\rho}\):

\[
\hat{\rho} = \sum_{m,n} \sum_{\mu,\nu} \rho_{m\mu,n\nu} |m\mu\rangle \langle n\nu|.
\]

(274)
Then
\[
\langle \hat{S}(A) \rangle = \text{Tr} \left\{ \hat{\rho} \hat{S}(A) \right\}
\]
\[
= \sum_{m,n} \sum_{\mu,\nu} \rho_{m\mu,n\nu} \text{Tr} \left\{ |m\mu\rangle \langle n\nu| \hat{S}(A) \right\}
\]
\[
= \sum_{m,n} \sum_{\mu,\nu} \rho_{m\mu,n\nu} \langle n\nu| \hat{S}(A) |m\mu\rangle
\]
\[
= \sum_{n=0}^{N-1} \sum_{\mu,\nu} \rho_{n\mu,n\nu} (\epsilon_{n\nu}, \sigma(A) \epsilon_{n\mu})
\]
\[
\equiv \sum_{n=0}^{N-1} \text{Tr} \left\{ D_n \sigma_n(A) \right\},
\]
where we have defined the 2 \times 2 single-mode matrices \(D_n\) and \(\sigma_n(A)\) as:
\[
[D_n]_{\alpha\beta} = \rho_{n\alpha,n\beta},
\]
\[
[\sigma_n(A)]_{\alpha\beta} = (\epsilon_{n\alpha}, \sigma(A) \epsilon_{n\beta}),
\]
and \(\alpha, \beta \in \{0, 1\}\).

In a \textit{paraxial regime} of propagation there is a “dominant” mode of the field, say \(n = n_0\), and one can assume
\[
(\epsilon_{n_0\alpha}, \sigma(A) \epsilon_{n_0\beta}) \cong (\epsilon_{n_0\alpha}, \sigma(A) \epsilon_{n_0\beta}), \quad \forall n \in \{0, \ldots, N-1\}.
\]

Since we always have the freedom to choose our reference frame, in this case it is convenient to choose the two polarization vectors \(\{\epsilon_{n_0\alpha}\}\) associated to the mode \(n_0\) in such a way that:
\[
\epsilon_{n_00} = x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_{n_01} = y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

From the definition \([283]\) it trivially follows
\[
\epsilon_{n_00} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \epsilon_{n_01} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
which implies
\[
(\epsilon_{n_0\alpha}, \sigma(A) \epsilon_{n_0\beta}) = [\sigma(A)]_{\alpha\beta}.
\]
Then, from Eqs. (275, 277, 280) it follows that

\[ \langle \hat{S}_{(A)} \rangle = \sum_{n=0}^{N-1} \text{Tr}\{D_n \sigma_{n(A)} \} \]

\[ \cong \sum_{n=0}^{N-1} \text{Tr}\{D_n \sigma_{(A)} \} \]

\[ \equiv \text{Tr}\{D \sigma_{(A)} \}, \quad (281) \]

where we have defined the $2 \times 2$ matrix $D$ as

\[ D \equiv \sum_{n=0}^{N-1} D_n, \quad \text{or,} \quad [D]_{\alpha\beta} = \sum_{n=0}^{N-1} \rho_{n\alpha,n\beta}, \quad (282) \]

which coincides with the naive definition of reduced density matrix.

F. The relevant density matrix

At this point we are ready to calculate the relevant density operator $\hat{\rho}^R$ for the polarization degrees of freedom. However, before doing so, let us apply the formulas written above to the simple case in which a single mode of the field, say again $n_0$, is excited. In this case, by definition

\[ \rho_{m\mu,n\nu} = \delta_{mn_0} \delta_{n0} \rho_{n_0,m\mu,n_0,n\nu} \equiv \delta_{mn_0} \delta_{n0} \rho_{m\mu}^{n_0}, \quad (283) \]

and if we substitute Eq. (283) into Eq. (275) we obtain

\[ \hat{\rho} = \sum_{\mu,\nu}^{0,1} \rho_{0,\mu}^{0,0} |n_0\mu\rangle \langle n_0\nu|, \quad (284) \]

From Eq. (275) we can easily calculate

\[ \langle \hat{S}_{(A)} \rangle = \text{Tr}\{\hat{\rho} \hat{S}_{(A)} \} \]

\[ = \sum_{\mu,\nu}^{0,1} \rho_{\mu\nu}^{0,0} \text{Tr}\{|n_0\mu\rangle \langle n_0\nu| \hat{S}_{(A)} \} \]

\[ = \sum_{\mu,\nu}^{0,1} \rho_{\mu\nu}^{0,0} \langle n_0\nu| \hat{S}_{(A)} |n_0\mu\rangle \]

\[ = \sum_{\mu,\nu}^{0,1} \rho_{\mu\nu}^{0,0} (\epsilon_{n_0\nu}, \sigma_{(A)} \epsilon_{n_0\mu}). \quad (285) \]

By substituting Eq. (280) into Eq. (285) we immediately obtain

\[ \langle \hat{S}_{(A)} \rangle = \sum_{\mu,\nu}^{0,1} \rho_{\mu\nu}^{0,0} [\sigma_{(A)}]_{\nu\mu} \quad (286) \]

\[ = \text{Tr}\{\rho_{0}^{0} \sigma_{(A)} \}. \]
since, by definition, $\text{Tr}\{\rho^0\} = \text{Tr}\{\hat{\rho}\} = 1$, then $\langle \hat{S}_{(0)} \rangle = 1/\sqrt{2}$ and, after a simple calculation, we obtain explicitly

$$
\hat{\rho} = \rho^0 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\langle \hat{S}_{(0)} \rangle + \langle \hat{S}_{(3)} \rangle & \langle \hat{S}_{(1)} \rangle - i\langle \hat{S}_{(2)} \rangle \\
\langle \hat{S}_{(1)} \rangle + i\langle \hat{S}_{(2)} \rangle & \langle \hat{S}_{(0)} \rangle - \langle \hat{S}_{(3)} \rangle
\end{pmatrix}.
$$

(287)

for later purposes it is useful to define the four scaled real parameters $\{s_A\}$ as

$$
s_A = \frac{1}{\sqrt{2}} \frac{\langle \hat{S}_{(A)} \rangle}{\langle \hat{S}_{(0)} \rangle}, \quad (A \in \{0, \ldots, 3\}),
$$

(288)

and rewrite Eq. (287) as

$$
\rho^0 = \frac{1}{\sqrt{2}} \begin{pmatrix}
s_0 + s_3 & s_1 - is_2 \\
s_1 + is_2 & s_0 - s_3
\end{pmatrix}.
$$

(289)

Now we go back to the general multi-mode case. Let us suppose that we have measured or calculated the four values $\{\langle \hat{S}_{(A)} \rangle\}$. Two cases are possible: Either

$$
\langle \hat{S}_{(0)} \rangle^2 < \sum_{B=1}^{3} \langle \hat{S}_{(B)} \rangle^2, \quad \text{or} \quad \langle \hat{S}_{(0)} \rangle^2 \geq \sum_{B=1}^{3} \langle \hat{S}_{(B)} \rangle^2.
$$

(290)

If the first case occurs, then it is not possible to calculate a relevant density matrix for the system (this may happen because of unwanted experimental errors). Vice versa, if the second case occur, then $\hat{\rho}^R$ can be obtained straightforwardly. This result can be achieved in three steps: First, by using Eq. (288) we calculated the four scaled parameters $\{s_A\}$; second, we introduce the four relevant observables $\{\hat{\sigma}_{(A)}^R\}$ such that

$$
\text{Tr}\{\hat{\rho}^R \hat{\sigma}_{(A)}^R\} = s_A,
$$

(291)

where, inspired by Eq. (271), we make the ansatz:

$$
\hat{\sigma}_{(A)}^R = \sigma_{(A)}, \quad (A \in \{0, \ldots, 3\}).
$$

(292)

Finally, we calculate the less biased $\hat{\rho}^R$ by using the maximum entropy criterion which leads to

$$
\hat{\rho}^R = \rho^R = \exp \left[ -\Psi - \sum_{B=1}^{3} \gamma_B \sigma_{(B)} \right],
$$

(293)

---

4. R. Balian, *Incomplete descriptions and relevant entropies*, Am. J. Phys. 67 (12), 1078 (1999).
where the normalization term $\Psi$ ensure the condition $\text{Tr}\hat{\rho}^R = 1$, and each of the three lagrange multipliers $\gamma_B$, ($B = 1, 2, 3$) is associated with each constraint Eq. (291). After a straightforward calculation it follows that

$$\hat{\rho}^R = D^R = \sum_{A=0}^{3} s_A \sigma(A).$$

This is our result, now we seek a relation between input and output relevant density operator in a multi-mode scattering process.

G. Input and output relations

Let us consider now a generic linear scattering process that transform the input single-photon density operator $\hat{\rho}^{\text{in}}$ into the output single-photon density operator $\hat{\rho}^{\text{out}}$:

$$\hat{\rho}^{\text{out}} = L[\hat{\rho}^{\text{in}}] = \sum_i \hat{A}_i \hat{\rho}^{\text{in}} \hat{A}_i^\dagger, \quad (295)$$

where

$$\sum_i \hat{A}_i \hat{A}_i^\dagger = \hat{I}. \quad (296)$$

The relevant quantities to calculate are the transition amplitudes

$$\langle m\mu | \hat{A}_i | n\nu \rangle \equiv A_{i,\mu\nu}(m,n), \quad (297)$$

where with $A_i(m,n)$ we have denoted the $2 \times 2$ matrix whose elements are $A_{i,\mu\nu}(m,n)$. Then, we can rewrite Eq. (295) as

$$\hat{\rho}_{\text{out}} = \sum_{m,n} \sum_{\mu,\nu} \sum_i A_{i,\alpha\mu}(a,m) \rho_{m\mu,n\nu}^{\text{in}} A_{i,\nu\beta}^\dagger(n,b). \quad (298)$$

From the algebra of the Pauli matrices it is easy to see that for any given pair of modes $\{m,n\}$ one can always write

$$\rho_{m\mu,n\nu} = \sum_{A=0}^{3} S_A(m,n)[\sigma(A)]_{\mu\nu}, \quad (299)$$

where we have defined

$$S_A(m,n) = \sum_{\mu,\nu} \rho_{m\mu,n\nu}[\sigma(A)]_{\nu\mu}, \quad (A \in \{0, \ldots, 3\}). \quad (300)$$

58
If we use Eq. (299) in both sides of Eq. (298) we obtain

\[ S_{A}^{\text{out}}(a, b) = \sum_{\alpha, \beta} \rho_{a\alpha, b\beta}^{\text{out}}[\sigma_{(A)}]_{\beta\alpha} \]

\[ = \sum_{\alpha, \beta} \sum_{m, n} \sum_{\mu, \nu} \sum_{i} A_{i, \alpha\mu}(a, m) \rho_{m\mu, n\nu}^{\text{in}} A_{i, \nu\beta}^{\dagger}(n, b) [\sigma_{(A)}]_{\beta\alpha} \]

\[ = \sum_{\alpha, \beta} \sum_{m, n} \sum_{\mu, \nu} \sum_{i} A_{i, \alpha\mu}(a, m) \sum_{B=0}^{3} \sum_{\mu, \nu} \sum_{i} \sigma_{B}^{\text{in}}(m, n) [\sigma_{(B)}]_{\mu\nu} A_{i, \nu\beta}^{\dagger}(n, b) [\sigma_{(A)}]_{\beta\alpha} \]

\[ = \sum_{B=0}^{3} \sum_{m, n} \sum_{\alpha, \beta} \sum_{\mu, \nu} \sum_{i} A_{i, \alpha\mu}(a, m) [\sigma_{(B)}]_{\mu\nu} A_{i, \nu\beta}^{\dagger}(n, b) [\sigma_{(A)}]_{\beta\alpha} \]

\[ = \sum_{B=0}^{3} \sum_{m, n} \sum_{\alpha, \beta} \sum_{\mu, \nu} \sum_{i} \sigma_{B}^{\text{in}}(m, n) \sum_{i} \text{Tr} \left\{ \sigma_{(A)} A_{i}(a, m) \sigma_{(B)} A_{i}^{\dagger}(n, b) \right\} \]

\[ = \sum_{B=0}^{3} \sum_{m, n} M_{AB}(a, b, m, n) \sigma_{B}^{\text{in}}(m, n), \tag{301} \]

where, by analogy with the definition of a classical Mueller matrix, we have defined

\[ M_{AB}(a, b, m, n) \equiv \sum_{i} \text{Tr} \left\{ \sigma_{(A)} A_{i}(a, m) \sigma_{(B)} A_{i}^{\dagger}(n, b) \right\}. \tag{302} \]

From Eqs. (276) and (299) it follows that

\[ \langle \hat{S}_{(A)} \rangle = \sum_{B=0}^{N-1} \sum_{n=0}^{0,1} \sum_{\mu, \nu} \rho_{n\mu, n\nu}(\epsilon_{n\nu}, \sigma_{(A)} \epsilon_{n\mu}) \]

\[ = \sum_{B=0}^{3} \sum_{n=0}^{N-1} \sum_{\mu, \nu} \sigma_{B}(n, n) [\sigma_{(B)}]_{\mu\nu} [\sigma_{n(A)}]_{\nu\mu} \]

\[ = \sum_{B=0}^{3} \sum_{n=0}^{N-1} \sigma_{B}(n, n) \text{Tr} \left\{ \sigma_{(B)} \sigma_{n(A)} \right\} \]

\[ = \sum_{B=0}^{3} \sum_{n=0}^{N-1} \Delta_{AB}(n) \sigma_{B}(n, n), \tag{303} \]

where

\[ \Delta_{AB}(n) \equiv \text{Tr} \left\{ \sigma_{n(A)} \sigma_{(B)} \right\}. \tag{304} \]
Note that in the paraxial approximation (see Eq. (277)) \( \Delta_{AB}(n) \cong \delta_{AB} \). By using the results in Eqs. (301) and (303) we can write

\[
\langle \hat{S}_{(A)} \rangle^{\text{out}} = \sum_{B=0}^{3} \sum_{n=0}^{N-1} \sum_{n=0}^{N-1} \Delta_{AB}(n) S_{B}^{\text{out}}(n, n)
\]

\[
= \sum_{B=0}^{3} \sum_{n=0}^{N-1} \sum_{c=0}^{N-1} \sum_{p,q} M_{BC}(n, n, p, q) S_{C}^{\text{in}}(p, q)
\]

\[
= \sum_{B,C} \sum_{p,q,n} \Delta_{AB}(n) M_{BC}(n, n, p, q) S_{C}^{\text{in}}(p, q).
\]

In a realistic experimental configuration, the input beam of light has a well defined polarization irrespective of the spatial and temporal coherency properties of the beam itself. This means that it is possible to write

\[
\rho_{\mu\nu} = \rho_{\mu\nu} \cong R_{\mu\nu},
\]

namely

\[
\rho \cong R \otimes r,
\]

where \( R \) and \( r \) are a \( N \times N \) and a \( 2 \times 2 \) matrices, respectively, and \( \text{Tr}\{R\} = 1 = \text{Tr}\{r\} \).

Note that this factorization has been made upon the matrix \( \rho \) representing the operator \( \hat{\rho} \) and not on the operator itself, where it would have been meaningless. With this assumption we can write

\[
S_{C}^{\text{in}}(p, q) = \sum_{\alpha, \beta}^{0,1} \rho_{\alpha\beta}^{\text{in}} [\sigma_{(C)}]_{\beta\alpha}
\]

\[
\cong R_{pq}^{\text{in}} \sum_{\alpha, \beta}^{0,1} r_{\alpha\beta}^{\text{in}} [\sigma_{(C)}]_{\beta\alpha}
\]

\[
\equiv R_{pq}^{\text{in}} S_{C}^{\text{in}},
\]

where \( S_{C}^{\text{in}} \equiv \text{Tr}\{r^{\text{in}} \sigma_{(C)}\} \), and we use this result in Eq. (305) to obtain

\[
\langle \hat{S}_{(A)} \rangle^{\text{out}} = \sum_{B,C} \sum_{n,p,q} \sum_{0, N-1} \sum_{3} \Delta_{AB}(n) M_{BC}(n, n, p, q) S_{C}^{\text{in}}(p, q)
\]

\[
= \sum_{B,C} \sum_{n,p,q} \sum_{0, N-1} \sum_{3} \Delta_{AB}(n) M_{BC}(n, n, p, q) R_{pq}^{\text{in}} S_{C}^{\text{in}}
\]

\[
= \sum_{C=0}^{3} \left[ \sum_{n,p,q} \sum_{B=0}^{3} \Delta_{AB}(n) M_{BC}(n, n, p, q) R_{pq}^{\text{in}} \right] S_{C}^{\text{in}}
\]

\[
\equiv \sum_{C=0}^{3} M_{AC} S_{C}^{\text{in}},
\]

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where we have defined the effective $4 \times 4$ Mueller matrix $M$ as

$$M_{AC} \equiv \sum_{p,q,n} \sum_{B=0}^{0,N-1} \Delta_{AB}(n) M_{BC}(n,n,p,q) R_{pq}^{in},$$

From Eqs. (303,306,308), it follows that for a paraxial input beam

$$\langle \hat{S}(A) \rangle^{in} = \sum_{B=0}^{N-1} \sum_{n=0}^{N-1} \Delta_{AB}(n) S^{in}_B(n,n)$$

$$= \sum_{B=0}^{N-1} \sum_{n=0}^{N-1} \delta_{AB} R^{in}_{nn} S^{in}_B$$

$$= S^{in}_A \sum_{n=0}^{N-1} R^{in}_{nn}$$

$$= S^{in}_A \text{Tr}\{R^{in}\}$$

Finally, by comparing Eq. (309) with Eq. (311) we found the sought relation between $\langle \hat{S}(A) \rangle^{out}$ and $\langle \hat{S}(A) \rangle^{in}$:

$$\langle \hat{S}(A) \rangle^{out} = \sum_{B=0}^{N-1} M_{AB} \langle \hat{S}_B \rangle^{in},$$

where

$$M_{AB} \equiv \sum_{p,q,n} \sum_{C=0}^{0,N-1} \Delta_{AC}(n) M_{CB}(n,n,p,q) R_{pq}^{in}$$

$$= \sum_{p,q,n} \sum_{C=0}^{0,N-1} \Delta_{AC}(n) \sum_i \text{Tr}\{\sigma(C) A_i(n,p)\sigma(B) A_i^\dagger(q,n)\} R_{pq}^{in}$$

$$\simeq \sum_{p,q,n} \sum_i \text{Tr}\{\sigma(A) A_i(n,p)\sigma(B) A_i^\dagger(q,n)\} R_{pq}^{in},$$

where the last, approximate equality is valid only in the limit of paraxial detection.

**H. Two-photon scattering**

Let us consider now the case of two photons, say $A$ and $B$, that are scattered by two independent, spatially separated media. We denote with $|a\alpha\rangle$ and $|b\beta\rangle$ the single-photon basis states for photons $A$ and $B$ respectively, where $a, b \in \{0, \ldots, N-1\}$ and $\alpha, \beta \in \{0, 1\}$. A two-photon basis state will be indifferently written as

$$|a\alpha\rangle \otimes |b\beta\rangle = |a\alpha\rangle |b\beta\rangle = |a\alpha, b\beta\rangle = |AB\rangle,$$
where $A$ and $B$ are cumulative indices for the pairs of indices $(a\alpha)$ and $(b\beta)$, respectively. Let $\hat{\rho}^\text{in}$ denotes the density operator describing the input two-photon state:

$$\hat{\rho}^\text{in} = \sum_{a,b} \sum_{\alpha,\beta} \sum_{a',b'} \rho_{a\alpha,b\beta,a'\alpha',b'\beta'} |a\alpha, b\beta\rangle \langle a'\alpha', b'\beta'|$$

(315)

$$= \sum_{A,B} \sum_{A',B'} \rho_{AB,A'B'} |AB\rangle \langle A'B'|,$$

where

$$\rho_{a\alpha,b\beta,a'\alpha',b'\beta'} = \langle a\alpha, b\beta | \hat{\rho}^\text{in} | a'\alpha', b'\beta' \rangle = \rho_{\alpha\beta,\alpha'\beta'} (ab, a'b').$$

(316)

A linear scattering process due to two independent, spatially separated media, transforms the input two-photon density operator $\hat{\rho}^\text{in}$ into the output two-photon density operator $\hat{\rho}^\text{out}$:

$$\hat{\rho}^\text{out} = \mathcal{L}[\hat{\rho}^\text{in}] = \sum_{i,j} \left( \hat{A}_i \otimes \hat{B}_j \right) \hat{\rho}^\text{in} \left( \hat{A}_i^\dagger \otimes \hat{B}_j^\dagger \right),$$

(317)

where

$$\sum_i \hat{A}_i \hat{A}_i^\dagger = \hat{I} = \sum_j \hat{B}_j \hat{B}_j^\dagger.$$  

(318)

The relevant quantities to calculate are the transition amplitudes

$$\langle a\alpha|\hat{A}_i|a'\alpha'\rangle = A_{i,\alpha\alpha'} (a, a'),$$

$$\langle b\beta|\hat{B}_j|b'\beta'\rangle = B_{j,\beta\beta'} (b, b'),$$

(319)

where with $A_i(a, a')$ and $B_j(b, b')$ we have denoted the $2 \times 2$ matrices whose elements are $A_{i,\alpha\alpha'} (a, a')$ and $B_{j,\beta\beta'} (b, b')$, respectively. Then, we can rewrite Eq. (317) as

$$\rho_{a\alpha,b\beta,a'\alpha',b'\beta'}^\text{out} = \sum_{i,j} \sum_{\alpha',\alpha''} \sum_{\beta',\beta''} A_{i,\alpha\alpha''} (a, a'') B_{j,\beta\beta''} (b, b'') \rho_{a\alpha'',b\beta'',a''\alpha',b''\beta'}^\text{in}$$

$$\times A_{i',\alpha''\alpha'} (a', a'') B_{j',\beta''\beta'} (b'', b') \rho_{\alpha''\alpha',\beta''\beta'} (ab, a'b').$$

(320)

From the algebra of the Pauli matrices it is easy to see that if we define the $4 \times 4$ matrices $\Sigma_{(AB)}$ as

$$\Sigma_{(AB)} \equiv \sigma_{(A)} \otimes \sigma_{(B)},$$

(321)
they form a complete (by definition) and orthonormal set of basis matrices in $\mathbb{C}^{4\times4}$:

$$\text{Tr} \left\{ \Sigma_{(AB)} \Sigma_{(A'B')} \right\} = \text{Tr} \left\{ \left( \sigma_{(A)} \otimes \sigma_{(B)} \right) \left( \sigma_{(A')} \otimes \sigma_{(B')} \right) \right\}$$

$$= \text{Tr} \left\{ \sigma_{(A)} \sigma_{(A')} \right\} \text{Tr} \left\{ \sigma_{(B)} \sigma_{(B')} \right\}$$

$$= \delta_{AA'} \delta_{BB'}.$$  \hfill (322)

Then, it is clear that it is always possible to write

$$\rho_{a\alpha,b\beta; a'\alpha', b'\beta'} = 0,$$  \hfill (323)

where we have defined

$$S_{AB}(ab, a'b') = \sum_{\alpha, \beta, \alpha', \beta'} \rho_{a\alpha,b\beta; a'\alpha', b'\beta'} \Sigma_{(AB)}[a\alpha, b\beta],$$  \hfill (324)

If we use Eq. (323) in both sides of Eq. (320), we obtain, after a lengthy but straightforward calculation,

$$S_{AB}^{\text{out}}(ab, a'b') = \sum_{0,3} \sum_{A'B'} \sum_{a''b'', a'''b'''} [\mathcal{M}(ab, a'b'; a''b'', a'''b''')],_{AB, A'B'} S_{AB}^{\text{in}}(a''b'', a'''b'''),$$  \hfill (325)

where the $16 \times 16$ matrix $\mathcal{M}(ab, a'b'; a''b'', a'''b''')$ is defined as:

$$\mathcal{M}(ab, a'b'; a''b'', a'''b''') \equiv M^{(A)}(aa', a''a'') \otimes M^{(B)}(bb', b''b'''),$$  \hfill (326)

and where, as in Eq. (302), we have defined the $4 \times 4$ matrices $M^{(A)}(aa', a''a'')$ and $M^{(B)}(bb', b''b''')$ as

$$M^{(A)}(aa', a''a'') = \sum_{i} \text{Tr} \left\{ \sigma_{(A)} A_{i}^{\dagger}(a, a'') \sigma_{(A')} A_{i}^{\dagger}(a'', a') \right\},$$

$$M^{(B)}(bb', b''b''') = \sum_{j} \text{Tr} \left\{ \sigma_{(B)} B_{j}^{\dagger}(b, b'') \sigma_{(B')} B_{j}^{\dagger}(b'', b') \right\}.$$  \hfill (327)

Now we want to relate the quantities displayed in Eq. (325) with quantities that are actually measured which, therefore, corresponds to mean values of Hermitian operators. To this end,
we calculate step by step the mean value of the two-photon Stokes operator $\hat{S}_{(A)} \otimes \hat{S}_{(B)}$:

$$\langle \hat{S}_{(A)} \otimes \hat{S}_{(B)} \rangle = \text{Tr}\{\hat{\rho} \left( \hat{S}_{(A)} \otimes \hat{S}_{(B)} \right) \}$$

$$= \text{Tr}\left\{ \sum_{A,B} \sum_{A',B'} \rho_{AB,A'B'} |AB\rangle \langle A'B'| \left( \hat{S}_{(A)} \otimes \hat{S}_{(B)} \right) \right\}$$

$$= \sum_{A,B} \sum_{A',B'} \rho_{AB,A'B'} \text{Tr}\left\{ |AB\rangle \langle A'B'| \left( \hat{S}_{(A)} \otimes \hat{S}_{(B)} \right) \right\}$$

$$= \sum_{A,B} \sum_{A',B'} \rho_{AB,A'B'} \langle A'| \hat{S}_{(A)} \rangle \langle B'| \hat{S}_{(B)} \rangle$$

$$= \sum_{A,B} \sum_{A',B',a',b',\alpha',\beta',\alpha,\beta} \rho_{a\alpha,b\beta; a'\alpha', b'\beta'} \langle a'\alpha' | \hat{S}_{(A)} | a\alpha \rangle \langle b'\beta' | \hat{S}_{(B)} | b\beta \rangle$$

$$\equiv \sum_{a,b} \sum_{\alpha,\beta} \sum_{a',b',\alpha',\beta'} \rho_{a\alpha,b\beta; a'\alpha', b'\beta'} \delta_{a' a} \left( \varepsilon_{a'\alpha'} \sigma_{(A)} \varepsilon_{a\alpha} \right) \delta_{b' b} \left( \varepsilon_{b'\beta'} \sigma_{(B)} \varepsilon_{b\beta} \right)$$

$$\equiv \sum_{a,b} \sum_{\alpha,\beta} \sum_{a',b',\alpha',\beta'} \rho_{a\alpha,b\beta; a'\alpha', b'\beta'} \left[ \sigma_{a(A)} \right]_{\alpha' \alpha} \left[ \sigma_{b(B)} \right]_{\beta' \beta}$$

$$= \sum_{a,b} \sum_{\alpha,\beta} \sum_{a',b',\alpha',\beta'} \rho_{a\alpha,b\beta; a'\alpha', b'\beta'} \left( ab, ab \right) \left[ \sigma_{a(A)} \otimes \sigma_{b(B)} \right]_{\alpha' \beta', \alpha \beta}$$

$$= \sum_{a,b} \sum_{\alpha,\beta} \sum_{a',b',\alpha',\beta'} \rho_{a\alpha, b\beta; a'\alpha', b'\beta'} \left( ab, ab \right) \left( \sigma_{a(A)} \otimes \sigma_{b(B)} \right)$$

$$\equiv \sum_{a,b} \text{Tr}\{\hat{\rho} (ab, ab) (\sigma_{a(A)} \otimes \sigma_{b(B)}) \}$$.

From Eq. (304) it is easy to see that

$$\sigma_{f(F)} = \sum_{F' = 0}^{3} \Delta_{F,F'}(f) \sigma_{(F')}$$

$$\Delta_{F,F'}(f) = \text{Tr}\{\sigma_{f(F)} \sigma_{(F')}\}$$,
where \( f \in \{a, b\} \), and \( F \in \{A, B\} \). By using this result and Eq. (324) we can rewrite Eq. (328) as

\[
\langle \mathcal{S}_A \otimes \mathcal{S}_B \rangle = \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} \Delta_{AA'}(a) \Delta_{BB'}(b) \text{Tr}\{\rho(ab, ab) \left( \sigma_{a(A)} \otimes \sigma_{b(B)} \right) \}
\]

\[
= \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} \Delta_{AA'}(a) \Delta_{BB'}(b) \text{Tr}\{\rho(ab, ab) \left( \sigma(A') \otimes \sigma(B') \right) \}
\]

\[
= \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} \Delta_{AA'}(a) \Delta_{BB'}(b) \text{Tr}\{\rho(ab, ab) \Sigma_{(A'B')} \}
\]

\[
= \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} \Delta_{AA'}(a) \Delta_{BB'}(b) \mathcal{S}_{A'B'}(ab, ab)
\]

\[
\approx \sum_{a,b}^{0,N-1} \mathcal{S}_{AB}(ab, ab),
\]

where the last, approximate equality holds in the paraxial limit where \( \Delta(a) \cong I_4 \cong \Delta(b) \).

By using Eqs. (325) and (330) it is easy to see that

\[
\langle \mathcal{S}_{(A)} \otimes \mathcal{S}_{(B)} \rangle_{\text{out}} = \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} \Delta(\Delta \otimes \Delta(b))_{AB,A'B'} \mathcal{S}_{A'B'}(ab, ab)
\]

\[
= \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} \left\{ \left[ \Delta(\Delta \otimes \Delta(b))_{AB,A'B'} \times \left[ M(ab, ab; a'b', a''b'')_{A'B',A''B''} \mathcal{S}_{A''B''}^{\text{in}}(a'b', a''b'') \right] \right]_{AB,A'B'} \right\}
\]

\[
\times \left[ M^{(A)}(aa, a'a'' \otimes M^{(B)}(bb, b'b'')_{A'B',A''B''} \mathcal{S}_{A''B''}^{\text{in}}(a'b', a''b'') \right]_{AB,A'B'} \right\}
\]

\[
= \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} \left\{ \left[ \Delta(ab, ab; a'b', a''b'')_{A'B',A''B''} \mathcal{S}_{A''B''}^{\text{in}}(a'b', a''b'') \right] \right\}
\]

\[
\times \left[ M^{(A)}(aa, a'a'' \otimes M^{(B)}(bb, b'b'')_{A'B',A''B''} \mathcal{S}_{A''B''}^{\text{in}}(a'b', a''b'') \right]_{AB,A'B'} \right\}
\]

\[
\equiv \sum_{a,b}^{0,N-1} \sum_{A',B'}^{0,3} M_{AB,A'B'}(a'b', a''b') \mathcal{S}_{A''B''}^{\text{in}}(a'b', a''b'),
\]

(331)
where we have defined

\[ M_{AB,A'B'}(a'b',a''b'') \equiv \left[ \sum_{a=0}^{N-1} \Delta(a) M^{(A)}(aa,a'a'') \otimes \sum_{b=0}^{N-1} \Delta(b) M^{(B)}(bb,b'b'') \right]_{AB,A'B'} , \tag{332} \]

or, in compact matrix form:

\[ M(a'b',a''b'') \equiv \sum_{a=0}^{N-1} \Delta(a) M^{(A)}(aa,a'a'') \otimes \sum_{b=0}^{N-1} \Delta(b) M^{(B)}(bb,b'b''). \tag{333} \]

Then, we can rewrite Eq. (331) as

\[ \langle \hat{S}_{(A)} \otimes \hat{S}_{(B)} \rangle^{\text{out}} = \sum_{a',b'} \sum_{a''=0}^{N-1} M_{AB,A'B'}(a'b',a''b'') S_{A'B'}^{\text{in}}(a'b',a''b''). \tag{334} \]

When the input state is not hyperentangled, one can write

\[ \rho_{\alpha',\beta';\alpha'\beta'}^{\text{in}} = R_{\alpha',\beta'}^{\text{in}} \rho_{\alpha',\beta'}^{\text{in}} , \tag{335} \]

where \( R^{\text{in}} \) and \( r^{\text{in}} \) are a \( N^2 \times N^2 \) and a \( 4 \times 4 \) matrices, respectively, and \( \text{Tr}\{R^{\text{in}}\} = 1 = \text{Tr}\{r^{\text{in}}\} \).

In this case, a straightforward calculation shows that

\[ \langle \hat{S}_{(A)} \otimes \hat{S}_{(B)} \rangle^{\text{in}} = \sum_{A',B'} \Delta_{AB,A'B'} S_{A'B'}^{\text{in}} , \tag{336} \]

and

\[ S_{A'B'}^{\text{in}}(a'b',a''b'') = \sum_{\alpha',\beta'} R_{\alpha',\beta'}^{\text{in}} \rho_{\alpha',\beta'}^{\text{in}} \Delta_{A'B'}(\alpha',\beta') \tag{337} \]

where we have defined

\[ S_{A'B'}^{\text{in}} = \text{Tr}\{r^{\text{in}} \Sigma(A'B')\} , \tag{338} \]

and

\[ \Delta_{AB,A'B'} \equiv \sum_{a,b} R_{ab,ab}^{\text{in}} [\Delta(a) \otimes \Delta(b)]_{AB,A'B'} \tag{339} \]

\[ \cong \delta_{AA'} \delta_{BB'}. \]
where the last, approximate equality holds in the paraxial limit only. In this limit, we have

$$\langle \hat{S}(A) \otimes \hat{S}(B) \rangle^{\text{in}} = S^{\text{in}}_{AB},$$  \hspace{1cm} (340)

and, by combining this result with Eq. (337), we obtain

$$S^{\text{in}}_{A'B'}(a'b',a''b'') = R^{\text{in}}_{a'b',a''b'}\langle \hat{S}(A') \otimes \hat{S}(B') \rangle^{\text{in}}.$$  \hspace{1cm} (341)

Finally, we substitute Eq. (341) into Eq. (334) to obtain

$$\langle \hat{S}(A) \otimes \hat{S}(B) \rangle^{\text{out}} = \sum_{a',b'}^{0,N-1} \sum_{a'',b''}^{0,3} M_{AB,A'B'}(a'b',a''b'') R^{\text{in}}_{a'b',a''b'}\langle \hat{S}(A') \otimes \hat{S}(B') \rangle^{\text{in}}$$

$$\equiv \sum_{A',B'} M_{AB,A'B'}\langle \hat{S}(A') \otimes \hat{S}(B') \rangle^{\text{in}},$$  \hspace{1cm} (342)

where we have defined the field-dependent two-photon 16 \times 16 Mueller matrix $M$ as:

$$M_{AB,A'B'} \equiv \sum_{a',b'}^{0,N-1} \sum_{a'',b''}^{0,3} M_{AB,A'B'}(a'b',a''b'') R^{\text{in}}_{a'b',a''b''}$$  \hspace{1cm} (343)

Equation (342) is our final result: It represents the linear relation between input and output measured quantities. This equation is the two-photon quantum analogue of the classical Mueller-Stokes relation. This similarity can be made manifest if we define the 16 two-photon Stokes parameters as

$$\langle \hat{S}(A) \otimes \hat{S}(B) \rangle \equiv S_{\Phi},$$  \hspace{1cm} (344)

where we introduced the cumulative index $\Phi = (AB) \in \{0, \ldots, 15\}$. Then, Eq. (342) can be rewritten as

$$S_{\Phi}^{\text{out}} = \sum_{\Phi=0}^{15} M_{\Phi\Phi'} S_{\Phi'}^{\text{in}},$$  \hspace{1cm} (345)

which is formally equivalent to the classical one.
FIG. 1: The Poincaré sphere. To each point on the sphere it is possible to associate a definite pure polarization state of the light. Moreover, internal points are associate with mixed (or partially polarized) states.