Distributed Algorithms that Solve Boolean Equation Systems

Hongsheng Qi, Bo Li, Rui-Juan Jing, Alexandre Proutiere, Guodong Shi

Abstract

In this paper, we propose distributed algorithms that solve a system of Boolean equations over a network, where each node in the network possesses only one Boolean equation from the system. The Boolean equation assigned at any particular node is a private equation known to this node only, and the nodes aim to compute the exact set of solutions to the system without exchanging their local equations. We show that each private Boolean equation can be locally lifted to a linear algebraic equation under a basis of Boolean vectors, leading to a network linear equation that is distributively solvable. A number of exact or approximate solutions to the induced linear equation are then computed at each node from different initial values. The solutions to the original Boolean equations are eventually computed locally via a Boolean vector search algorithm. We prove that given solvable Boolean equations, when the initial values of the nodes for the distributed linear equation solving step are i.i.d selected according to a uniform distribution in a high-dimensional cube, our algorithms return the exact solution set of the Boolean equations at each node with high probability. Furthermore, we present an algorithm for distributed verification of the satisfiability of Boolean equations, and prove its correctness. The proposed algorithms put together become a complete framework for distributively solving Boolean equations: verifying satisfiability and computing the solution set whenever satisfiable.

1 Introduction

Computing the solutions to a system of Boolean equations is a fundamental computation problem. The Boolean satisfiability (SAT) problem for determining whether a Boolean formula is satisfiable or not, was the first computation problem proven to be NP-complete [1]. The solvability of (static) Boolean equations is directly related to problems for gene regulations in biology, and for security of the keystream generation in cryptography [2, 3]. In the meantime, Boolean dynamical systems have found broad applications in modeling epidemic processes of computer networks [4], social opinion dynamics [5], and decision making in economics [6] since logical states are ubiquitous in our world. For such systems, finding their steady
states, if any, again falls to a problem of solving a system of Boolean equations [7, 8]. As a result, static or dynamic Boolean equations attracted much research interest in computing theory and engineering in terms of complexity analysis, efficient algorithm design, and signal processing [3, 9, 10].

We are interested in systems of Boolean equations that are defined over a network, and aim to develop distributed algorithms that can solve such equations, and verify their satisfiability in a distributed manner over the network.

1.1 System of Boolean Equations over a Network

Consider the following system of Boolean equations with respect to decision variable $x = (x_1, \ldots, x_m)$:

$$
\begin{align*}
  f_1(x_1, \ldots, x_m) &= \sigma_1 \\
  \vdots \\
  f_n(x_1, \ldots, x_m) &= \sigma_n
\end{align*}
$$

(1)

where $x_i \in \{0,1\}^m$ for $i = 1, \ldots, m$ and $\sigma_j \in \{0,1\}$, each $f_j$ maps from the $m$-dimensional binary space $\{0,1\}$ to the binary space $\{0,1\}$ for $j = 1, \ldots, n$.

Let there be a network with $n$ nodes indexed in the set $V = \{1, \ldots, n\}$. The communication structure of the network is described by a simple, undirected, and connected graph $G = (V, E)$, where each edge $\{i,j\} \in E$ is an unordered pair of two distinct nodes in the set $V$. The neighbor set of node $i$ over the network is specified as $N_i = \{j : \{i,j\} \in E\}$. The $i$th equation in the Boolean equation system (1), $f_i(x_1, \ldots, x_m) = \sigma_i$ is assumed to be a local and private equation assigned to node $i$ only. We are interested in the following distributed computation problem, where the nodes aim to compute the solutions to the Boolean equation system (1) over the network $G$ distributedly.

**Definition 1.** An algorithm solves the Boolean equation system (1) distributedly if the following conditions are met:

(i) Each node $i \in V$ holds dynamical states $\mathbf{x}_i(t)$ that iterates along discrete time instants $t = 0, 1, 2, \ldots$, which can be shared with their neighboring nodes over the graph $G$;

(ii) Each node carries out local computations on $\mathbf{x}_i(t)$ based on her own and her neighbors’ dynamical states and her private Boolean equation;

(iii) The solution set of the Boolean equation system (1) is obtained at all nodes as the algorithm output.

1.2 Contributions

First of all, a distributed algorithm in the sense of Definition 1 is developed that solves a system of Boolean equations that is satisfiable over a network. This algorithm consists of three ingredients:

(i) Each private Boolean equation is lifted by local computation at each node to a linear algebraic equation under a basis of Boolean vectors;
The network nodes distributedly compute a number of exact solutions to the induced system of linear equations via an exact projection consensus algorithm \cite{11} from randomly selected initial values;

Each node locally computes the solution set of original system of Boolean equations from the linear equation solutions by a Boolean vector search algorithm.

We prove that if the initial values of the nodes for the distributed linear equation solving step are i.i.d generated according to a uniform distribution in a high-dimensional cube, with probability one the proposed algorithm returns the exact solution set of the Boolean equations at each node throughout the network. The complexity of the local Boolean vector search algorithm is also established. Next, we generalize this algorithm based on exact projection consensus, which in theory would require an infinite number of iterations, to an algorithm that relies only on approximate projections with a finite number of iterations. We also prove this extended algorithm can with high probability guarantee the exact solution set of the Boolean equations at each node to be computed at each node. Finally, we propose an algorithm for distributed verification of the satisfiability of the system of Boolean equations, and prove its correctness. Therefore, combining all our algorithms, we have established a framework for distributedly verifying satisfiability of a Boolean equation system, and then computing the exact solution set if indeed it is satisfiable.

1.3 Related Work

Our definition of distributed algorithms for Boolean equation systems is along the same line of research on distributed convex optimization \cite{11}, network linear equation solving \cite{12,13}, distributed signal processing \cite{14}, distributed sensor estimation \cite{15}, and distributed stochastic approximation over networks \cite{16}. This line of studies can be traced back to the seminal work of Tsitsiklis, Bertsekas, and Athans in 1986 \cite{17}. The central idea is, for a global computation task decomposed over a number of nodes in a network, the task can be achieved at each node by the nodes running local computations according to their local tasks which are interconnected by an average consensus process. Since average consensus processes are inherently distributed relying only on local and anonymous communications, and rather robust against factors such as disturbances, communication failures, network structural changes, etc., \cite{18,19,20}, such distributed computations have the advantage of being resilient and scalable. In fact, the construction of our algorithms for solving the Boolean equations distributedly utilizes the framework on distributed linear equation solving \cite{12,13} as a subroutine, and the node communications in our algorithms do not rely on node identities which enables anonymous computing \cite{21}.

We also note that our work is related, but quite different from the work on parallel algorithms for SAT problems \cite{22,23,24}. The focus on parallel SAT solvers is about using a multicore architecture from a set of processing units with a shared memory to solve SAT problems with specific structures \cite{22,23,24}. In comparison with the distributed computation setting, the parallel decomposition of the problem is part of the design of the algorithm, and the shard memory implies an all-to-all communication structure among the processing units. Therefore, the primary motivation of the proposed distributed algorithms for Boolean equations is not acceleration of centralized algorithms, but rather exploring the possibilities of generalizing the distributed computing framework for continuous and convex problems to Boolean equations with a discrete and combinatorial nature.
1.4 Paper Organization

In Section 2, we introduce some preliminary algorithms and results for Boolean equations, average consensus algorithms, and projection consensus algorithms. Section 3 presents the algorithms that are used for distributedly solving a system of Boolean equations, and prove their correctness provided that the equations are satisfiable. Then Section 4 presents a distributed algorithm that can verify satisfiability definitively for a system of Boolean equations over a network. Finally in Section 5 some concluding remarks are given.

2 Preliminaries

We first present some preliminaries on the concepts and tools that are incremental for the construction of our algorithm. First of all, we review methods that can generate a matrix representation for a Boolean mapping when logical variables are represented by certain Boolean vectors (BooleanMatricization), and algorithms that can solve a system of linear algebraic equations distributedly over a network (Distributed-LAE). Although such algorithms are well established in their respective literature, we present some basic form and analysis of the algorithms in order to facilitate a self-contained presentation. Then, we present a novel search algorithm (BooleanVectorSearch) for localizing all Boolean vectors in a given affine subspace, and prove its correctness and computational efficiency.

2.1 Matrix Representation of Boolean Mappings

For any integer \( m \geq 2 \), we introduce two mappings:

(i) \( \lfloor \cdot \rfloor : \{0,1\}^m \to \{1,\ldots,2^m\} \), where \( \lfloor i_1 \cdots i_m \rfloor = \sum_{k=1}^{m} i_k 2^{m-k} + 1 \);

(ii) \( \lceil \cdot \rceil : \{1,\ldots,2^m\} \to \{0,1\}^m \) with \( \lceil i \rceil = [i_1 \ldots i_m] \) satisfying \( i = \sum_{k=1}^{m} i_k 2^{m-k} + 1 \).

For any integer \( c \), we let \( \delta^i_c \) be the \( i \)-th column of the \( c \times c \) identity matrix \( I_c \), and define

\[ \Delta_c = \{ \delta^i_c : i = 1,\ldots,c \} . \]

We can then establish a one-to-one correspondence between the elements in \( \{0,1\}^m \) and the elements in \( \Delta_{2^m} \). To this end, we define

- The mapping \( \Theta_m(\cdot) \) from \( \{0,1\}^m \) to \( \Delta_{2^m} \):

\[ \Theta_m(x) := \delta^{x_1+1}_{2^1} \otimes \cdots \otimes \delta^{x_m+1}_{2^m} = \delta^{\sum_{i=1}^{m} x_i 2^{m-1}+1}_{2^m} = \delta^{[x]}_{2^m}, \quad (2) \]

for all \( x = [x_1,\ldots,x_m] \in \{0,1\}^m \), where \( \otimes \) represents the Kronecker product.

- The mapping \( \Upsilon_m(\cdot) \) from \( \Delta_{2^m} \) to \( \{0,1\}^m \):

\[ \Upsilon_m(\delta^i_{2^m}) := [i] \quad (3) \]

for all \( i = 1,\ldots,2^m \).
We can easily verify $\Upsilon_m(\Theta_m(x)) = x$ holds for all $x \in \{0,1\}^m$, and therefore, we have obtained a desired bijective mapping between $\{0,1\}^m$ and $\Delta_2^m$. It turns out, with the help of the vectors in $\Delta_2^m$, one can represent any Boolean mapping $g(x_1, \ldots, x_m)$ from $\{0,1\}^m$ to $\{0,1\}$ by a matrix $M_g \in \mathbb{R}^{2 \times 2^m}$.

**Lemma 1.** Let $g(x_1, \ldots, x_m)$ be a Boolean mapping from $\{0,1\}^m$ to $\{0,1\}$. Then there exists $M_g \in \mathbb{R}^{2 \times 2^m}$ as a representation of $g(\cdot)$ in the sense that
\[
M_g(\Theta_m(x)) = \Theta_1(g(x))
\]
for all $x = (x_1, \ldots, x_m) \in \{0,1\}^m$.

Lemma 1 can be viewed as a special case of representing Boolean mappings by matrices in [25] for the analysis of Boolean dynamical networks. However, the existence of the matrix $M_g$ can in fact be directly established from the following commutative diagrams in view of the identity $\Upsilon_m(\Theta_m(x)) = x$:

![Diagram](image)

**Figure 1:** The commutative diagram for mappings over the logical spaces and the vector spaces.

Here in the diagram $M_g$ is a mapping from $\Delta_2^m$ to $\Delta_2$. Note that the elements in $\Delta_2^m$ (resp. $\Delta_2$) form a basis of $\mathbb{R}^{2^m}$ (resp. $\mathbb{R}^2$), $M_g$ can be naturally understood as a linear operator from $\mathbb{R}^{2^m}$ to $\mathbb{R}^2$. As a result, the matrix $M_g$ can be obtained directly as a matrix representation of $M_g$ by
\[
M_g = \begin{bmatrix}
\delta_2^g(0)(1) & \delta_2^g(0)(2) & \ldots & \delta_2^g(0)(2^m) \\
\delta_2^g(1)(1) & \delta_2^g(1)(2) & \ldots & \delta_2^g(1)(2^m)
\end{bmatrix}
\]
where the $i$th column of $M_g$ is the $\Theta_1(g([i]))$, i.e., the coordinate of $M_g(\delta_i^m)$ under the basis $\Delta_1$. From this perspective we also see that the matrix $M_g$ stated in Lemma 1 is in fact unique.

The equation (5) serves also as a brutal-force way of computing the matrix $M_g$ for any $g(x_1, \ldots, x_m)$ from $\{0,1\}^m$ to $\{0,1\}$, where one needs to examine the value of $g$ for all $x \in \{0,1\}^m$. A systematic way of computing the matrix $M_g$ from the original Boolean formula of $g$ is the semi-tensor product approach developed by Cheng and his colleagues [25,26], which puts the study of Boolean dynamical networks into a fully algebraic framework. We refer to [27] for a detailed introduction to semi-tensor product approach and its applications to Boolean networks.

**Remark 1.** Lemma 1 can be easily generalized to Boolean mappings $g(x_1, \ldots, x_m)$ from $\{0,1\}^m$ to $\{0,1\}$, where the resulting $M_g$ becomes a $2^l \times 2^m$ matrix. The existence of such a matrix follows from the same argument, which implies
\[
M_g = \begin{bmatrix}
\delta_2^g(0)(1) & \delta_2^g(0)(2) & \ldots & \delta_2^g(0)(2^m) \\
\delta_2^g(1)(1) & \delta_2^g(1)(2) & \ldots & \delta_2^g(1)(2^m)
\end{bmatrix}
\]
Again, by the semi-tensor product approach, one can alternatively compute this matrix representation for any given Boolean formula [27].

We denote by $\text{BooleanMatricization}$ an algorithm that produces a matrix representation as output from a Boolean mapping as input, e.g., $M_g = \text{BooleanMatricization}(g)$ for any Boolean mapping $g$ from $\{0,1\}^m$ to $\{0,1\}^l$. 


2.2 Projection Consensus for Linear Algebraic Equations

Consider the network $G = (V, E)$. Suppose each node $i \in V$ holds $(H_i, z_i)$ with $H_i \in \mathbb{R}^{k \times d}$ and $z_i \in \mathbb{R}^k$ defining a linear equation

$$H_i y = z_i \quad (6)$$

with respect to an unknown $y \in \mathbb{R}^d$. Then over the network the following system of linear algebraic equations is defined

$$Hy = z, \quad (7)$$

where

$$H = \begin{bmatrix} H_1 \\ \vdots \\ H_n \end{bmatrix} \in \mathbb{R}^{nk \times d}, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^{nk}.$$ 

Define the following affine subspace in $\mathbb{R}^d$ specified by the linear equation (6):

$$E_i := \{ y \in \mathbb{R}^d : H_i y = z_i \}$$

and then let $P_i(\cdot)$ be the projector onto the affine space $E_i$. Let $P_\ast(\cdot)$ be the projector onto the affine subspace

$$E := \{ y \in \mathbb{R}^d : Hy = z \}$$

whenever $E = \bigcap_{i=1}^n E_i$ is nonempty. We can utilize these projectors to obtain a distributed projection consensus solver to the system of linear equations (7).

**Lemma 2.** Suppose the system of linear equations (7) admits at least one exact solutions. Then along the recursion

$$x_i(t+1) = P_i\left(x_i(t) + \epsilon \sum_{j \in N_i} (x_j(t) - x_i(t))\right), \quad i = 1, \ldots, n \quad (8)$$

there holds for $0 < \epsilon < 1/n$ that $\lim_{t \to \infty} x_i(t) = \frac{1}{n} \sum_{k=1}^n P_\ast(x_k(0)) \in E$ for all $i \in V$.

The projection consensus algorithm (8) is in fact a special case of the constrained consensus algorithm presented in [11] for distributively computing a point in the intersection of several convex sets, and the convergence statement in Lemma 2 follows from Lemma 3 of [11]. This algorithm (8) is a form of alternating projection algorithms for convex feasibility problems [28]. The rate of convergence is in fact exponential. The specific representation of the node state limit can be established via Lemma 5 in [13], based on which we can obtain

$$\sum_{i=1}^n P_\ast(x_i(t+1)) = \sum_{i=1}^n P_\ast(x_i(t) + \epsilon \sum_{j \in N_i} (x_j(t) - x_i(t)))$$

$$= \sum_{i=1}^n \sum_{j=1}^n W_{ij} P_\ast(x_j(t))$$

$$= \sum_{i=1}^n P_\ast(x_i(t)) \quad (9)$$

for all $t \geq 0$ along (8). Here $W_{ij} = \epsilon$ for all $\{i, j\} \in E$, $W_{ii} = 1 - |N_i|\epsilon$ for all $i \in V$, and $W_{ij} = 0$ otherwise. Invoking Lemma 3 of [11], we know that all $x_i(t)$ converge to a common limit value in $E$, which has to be $\frac{1}{n} \sum_{k=1}^n P_\ast(x_k(0))$ from (9).
Remark 2. Without the projection in (8), the algorithm
\[ x_i(t + 1) = x_i(t) + \epsilon \sum_{j \in N_i} (x_j(t) - x_i(t)) \] (10)
is a standard average consensus algorithm [29]. If \( 0 < \epsilon < \frac{1}{n} \), then along (10) there holds \( \lim_{t \to \infty} x_i(t) = \sum_{j=1}^{n} x_j(0)/n \) for all \( i = 1, \ldots, n \).

Besides the algorithm (8), there are various other distributed algorithms that produce a solution to (7), e.g., [12, 13]. We denote by DistributedLAE an algorithm that solves a linear algebraic equation (7) with exact solutions distributedly over a graph \( G \), and formally we write
\[ y^*(x(0)) = \text{DistributedLAE}(E_i, x_i(0) : i \in V) = \frac{1}{n} \sum_{k=1}^{n} P_k(x_k(0)) \]

### 2.3 Boolean Vector Search in an Affine Subspace

We term the vectors in \( \Delta_{2^m} \) as Boolean vectors in \( \mathbb{R}^{2^m} \) as noted from their close relationship with the logical states in \( \{0, 1\}^m \). In this subsection, we provide an algorithm to search the Boolean vectors in any affine subspace of \( \mathbb{R}^{2^m} \).

Let \( y_1, \ldots, y_k \in \mathbb{R}^{2^m} \). Let \( \text{SubSpace}(y_1, \ldots, y_k) \) be the subspace generated by \( y_1, \ldots, y_k \). By definition, \( \text{SubSpace}(y_1, \ldots, y_k) := \{ \alpha_1 y_1 + \cdots + \alpha_k y_k \mid \alpha_1, \ldots, \alpha_k \in \mathbb{R} \} \). Let \( \text{Aff}(y_1, \ldots, y_k) \) be the generated affine subspace from \( y_1, \ldots, y_k \) in \( \mathbb{R}^{2^m} \), as the minimal affine subspace that contains all \( y_i \). Geometrically, the affine subspace \( \text{Aff}(y_1, \ldots, y_k) \) can be obtained by translating \( \text{SubSpace}(y_2 - y_1, \ldots, y_k - y_1) \) from the origin to the point \( y_1 \). Therefore, a vector \( y \) is contained in \( \text{Aff}(y_1, \ldots, y_k) \) if and only if \( y_1 - y \) is contained in \( \text{SubSpace}(y_2 - y_1, \ldots, y_k - y_1) \). The dimension of the affine subspace \( \text{Aff}(y_1, \ldots, y_k) \) is defined as the dimension of the subspace \( \text{SubSpace}(y_2 - y_1, \ldots, y_k - y_1) \). We are interested in the following Boolean vector search problem.

**Boolean Vector Search.** Suppose \( \text{Aff}(y_1, \ldots, y_k) \cap \Delta_{2^m} \neq \emptyset \). Find all vectors in \( \text{Aff}(y_1, \ldots, y_k) \cap \Delta_{2^m} \).

Without loss of generality, we assume \( \text{SubSpace}(y_2 - y_1, \ldots, y_k - y_1) \) has a basis \( \{v_1, \ldots, v_b\} \) for \( b \geq 1 \), where the matrix \( [v_1, \ldots, v_b] \) has the form
\[
\begin{bmatrix}
V \\
I_b
\end{bmatrix}
\]
To decide whether the vector \( y \) is contained in the affine space \( \text{Aff}(y_1, \ldots, y_k) \) or not is equivalent to determining whether the equation system \( x_1 v_1 + \cdots + x_b v_b = y_1 - y \) has solutions or not. Due to the special structure of the coefficient matrix, the variables \( x_1, \ldots, x_b \) are uniquely associated with the last \( b \) elements of the vector \( y_1 - y \). Based on these observations, we propose the following algorithm.
Algorithm 1 BooleanVectorSearch

Require: A set of generators $y_1, \ldots, y_k$ of an affine subspace;
Ensure: All of the Boolean vectors contained in this affine subspace.

1: Compute a basis $\{v_1, \ldots, v_b\}$ of the linear subspace generated by $\{y_2 - y_1, \ldots, y_k - y_1\}$. Without loss of generality, we assume the last $b$ rows of the matrix $[v_1, \ldots, v_b]$ is the $b \times b$ identity matrix.

2: For $i \in [1, b]$, let $\tilde{v}_i$ and $\tilde{y}_1$ be the column vectors consisting of the first $2^m - b$ rows of $v_i$ and $y_1$, respectively. Let $y_1$ be the column vector consisting of the last $b$ rows of $y_1$.

3: Let $y = \tilde{y}_1 - [\tilde{v}_1, \ldots, \tilde{v}_b]y_1$. Let $S$ be an empty set.

4: If $y = \delta_{2^m-b}^i$ for some $i \in [1, 2^m - b]$, then add $\delta_{2^m}^i$ to $S$.

5: For $i = 2^m - b + 1, \ldots, 2^m$, if $-\tilde{v}_{i-2^m+b} = y$, then add $\delta_{2^m}^i$ to $S$.

6: return $S$.

Lemma 3. Algorithm 1 is correct and terminates.

Proof. The Lines 1 and 2 of Algorithm 1 lead us to solve the following equation system with variables $x_1, \ldots, x_b$:

$$\begin{bmatrix} \tilde{v}_1 & \ldots & \tilde{v}_b \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_b \end{bmatrix} = \begin{bmatrix} \tilde{y}_1 \\ y_1 \end{bmatrix} - \delta_{2^m}^i, \text{ for } i : 1 \leq i \leq 2^m. \tag{11}$$

We discuss Equation 11 in the following two cases:

1. $1 \leq i \leq 2^m - b$
   
   In this case, $(x_1, \ldots, x_b)^t = y_1$. We need to determine whether $\delta_{2^m-b}^i = \tilde{y}_1 - [\tilde{v}_1, \ldots, \tilde{v}_b]y_1$ or not. This leads us to the Line 4.

2. $2^m - b + 1 \leq i \leq 2^m$
   
   In this case, for a fixed $i$, $(x_1, \ldots, x_b)^t = y_1 - \delta_{2^m}^{i-2^m+b}$. We need to determine whether $-\tilde{v}_{i-2^m+b} = \tilde{y}_1 - [\tilde{v}_1, \ldots, \tilde{v}_b]y_1$ or not. This leads us to the Line 5.

The correctness of the algorithm follows immediately. The algorithm terminates obviously.

Lemma 4. Algorithm 1 requires at most $O(2^m kb)$ field operations, where $b$ is the dimension of the affine subspace.

Proof. The column reduced echelon form of $[y_1, \ldots, y_k]$ can be obtained by Gauss elimination. The classical Gauss elimination requires $O(2^m kb)$ field operations, where $b$ is the rank of $[y_1, \ldots, y_k]$. Once we have the column reduced echelon form, the only remaining computations lie in Line 3 which requires at most $O(2^m b)$ field operations. The lemma follows immediately.

We write BooleanVectorSearch as our algorithm that outputs all points in the intersection of $\text{Aff}(y_1, \ldots, y_k)$ and $\Delta_{2^m}$ from $y_1, \ldots, y_k \in \mathbb{R}^{2^m}$ as the input, i.e.,

$$\text{BooleanVectorSearch}(y_1, \ldots, y_k) = \text{Aff}(y_1, \ldots, y_k) \cap \Delta_{2^m}.$$
3 Distributed Boolean Equation Solvers

In this section, we present our algorithms that solve the system of Boolean equations distributedly based on exact or approximate projection consensus algorithms, given satisfiability of the Boolean equations.

3.1 The Algorithm with Exact Projection Consensus

We first present a distributed algorithm for solving the system of Boolean equations (1) utilizing the three algorithms BooleanMatricization, DistributedLAE, BooleanVectorSearch as subroutines. Let uniform([0, 1]2m) be the uniform distribution over the 2m-dimensional cube [0, 1]2m. Let k∗ be a positive integer.

Algorithm 2 DistributedBooleanEquationSolver

Require: Over the network G, node i holds Boolean equation \( f_i(x) = \sigma_i \); nodes communicate with only neighbors on G about their dynamical states.

Ensure: Each node i computes \{x \in \{0, 1\}^m : f_j(x) = \sigma_j, j = 1, \ldots, n\}.

1: Each node i locally computes \( M_{f_i} = \text{BooleanMatricization}(f_i) \in \mathbb{R}^{2 \times 2^m} \) for all \( i \in V \);

2: Each node i assigns linear equation \( \mathcal{E}_i^b : H_i y = z_i \) and then the projection onto its solution space \( \mathcal{E}_i^b \) by \( H_i \leftarrow M_{f_i} \) and \( z_i \leftarrow \Theta_1(\sigma_i) \);

3: For \( s = 1, \ldots, k^* \), each node i randomly and independently selects \( x_i(0) = \beta_{i,s} \sim \text{uniform}([0, 1]^{2m}) \), and runs DistributedLAE to produce \( y_s = \sum_{i=1}^{n} P_*(\beta_{i,s})/n \) at each node \( i \in V \);

4: Each node i locally computes \( S = \text{BooleanVectorSearch}(y_1, \ldots, y_{k^*}) \);

5: \text{return} \( S = \Upsilon_m(S) \) for all nodes.

The intuition behind Algorithm 2 is as follows. First of all, the algorithm BooleanMatricization manages to convert each Boolean equation locally into the following network linear algebraic equation:

\[
\mathcal{E}_i^b : M_{f_i} y = \Theta_1(\sigma_i), \quad i = 1, \ldots, n
\]  

which admits a non-empty affine solution subspace \( \mathcal{E}_i^b = \bigcap_{i=1}^n \mathcal{E}_i^b \) when the system of Boolean equations (1) is solvable. Then, for any given initial values \( (\beta_{1,s}, \ldots, \beta_{n,s}) \), the algorithm DistributedLAE produces an exact algebraic solution \( y_s = \sum_{i=1}^{n} P_*(\beta_{i,s})/n \) to (12) in a distributed manner, where with slight abuse of notation \( P_* \) is the projection onto \( \mathcal{E}_i^b \). However, a single algebraic solution cannot be used to infer any solution to the original Boolean equations in the logical space. To overcome this, we randomly select initial values of the nodes for DistributedLAE to produce \( k^* \) algebraic solutions, in the hope that these random algebraic solutions are useful enough to reconstruct the entire solution affine space of the linear equations. Finally, by BooleanVectorSearch we search all Boolean vectors in that affine space, from which we eventually uncover the solution set of the Boolean equations by the mapping \( \Upsilon_m(\cdot) \).

For the correctness of the Algorithm 2 we present the following theorem.

Theorem 1. Suppose the system of Boolean equations (1) admits at least one exact solutions. Let \( k^* = 2^m + 1 \). Then with probability one, the set \( S \) that the algorithm DistributedBooleanEquationSolver returns is exactly the set of solutions to (1).

Example 1. We present an example illustrating the computation process of Algorithm 2. Let “\( \land \)”, “\( \lor \)”, and “\( \neg \)” be the logical “AND”, “OR”, “NOT” operations. Let “\( \leftrightarrow \)” be the logical equivalence operation.
with \(x \leftrightarrow y = (\neg x \lor y) \land (\neg y \lor x)\). Consider the following Boolean equations

\[
\begin{align*}
    f_1 &= x_1 \lor x_2 \lor \neg x_3 = 1, \\
    f_2 &= x_1 \land (x_1 \leftrightarrow x_2) = 0, \\
    f_3 &= x_2 \land x_3 = 0.
\end{align*}
\]

Let there be a network with three nodes in \(V = \{1, 2, 3\}\) with node \(i\) holding the \(i\)th equation. A detailed breakdown of Algorithm 2 is as follows.

**S1.** Each node \(i\) computes the \(M_{f_i}\) based on her local Boolean equation \(f_i\) as

\[
M_{f_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},
\]

\[
M_{f_2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},
\]

\[
M_{f_3} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.
\]

**S2.** Each node \(i\) locally assigns \(\delta_i^h\): \(H_i y = z_i\) by \(H_i \leftarrow M_{f_i}\) and \(z_i \leftarrow \Theta_1(\sigma_i)\), where

\[
z_1 = \Theta_1(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
z_2 = \Theta_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
z_3 = \Theta_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

**S3.** Then for \(s = 1, \ldots, 2^3 + 1\), each node \(i\) randomly and independently selects \(x_i(0) = \beta_{i,s}\) from \(\text{Uniform}([0, 1]^8)\), and runs DistributedLAE to produce \(y_s\). A sample obtained by such randomization for the \(y_s\) is

\[
\begin{bmatrix} y_1 & \ldots & y_9 \end{bmatrix} = \begin{bmatrix} 0.3837 & 0.0299 & -0.0509 & 0.4616 & 0.2897 & 0.1139 & 0.1043 & 0.3578 & 0.0277 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.1019 & 0.4604 & 0.2581 & 0.1935 & 0.3299 & 0.2565 & 0.4819 & 0.3105 & 0.1353 \\ 0.0640 & 0.0604 & 0.1110 & 0.1110 & 0.0974 & 0.0806 & 0.1506 & 0.0511 & 0.0300 \\ 0.0944 & 0.1918 & 0.3854 & 0.2492 & -0.1419 & 0.1938 & 0.0559 & 0.2378 & 0.3244 \\ 0.3561 & 0.3116 & 0.2964 & -0.0201 & 0.4249 & 0.3551 & 0.2073 & 0.0429 & 0.4827 \\ 0.0640 & 0.0604 & 0.1110 & 0.1157 & 0.0974 & 0.0806 & 0.1506 & 0.0511 & 0.0300 \\ -0.0640 & -0.0604 & -0.1110 & -0.1157 & -0.0974 & -0.0806 & -0.1506 & -0.0511 & -0.0300 \end{bmatrix}.
\]

**S4.** Each node then locally runs the algorithm **BooleanVectorSearch**, which gives \(S = \{\delta_1^1, \delta_2^3, \delta_3^5, \delta_8^6\}\).

**S5.** Each node finally computes and returns \(S = \Upsilon_m(S) = \{[000], [010], [100], [101]\}\).

We can easily verify that \(S = \{[000], [010], [100], [101]\}\) is indeed the solution set of the equation (13), which provides a validation to the correctness of the Algorithm 2.
3.2 Extended Algorithm with Approximate Projection Consensus

In Algorithm 2, the step for distributed linear equation solving assumes accurate computation of the projections, i.e., for $x_i(0) = \beta_{i,s} \sim \text{uniform}(0, 1)^{2m}$, the algorithm DistributedLAE produces $y_s = \sum_{i=1}^{n} P_*(\beta_{i,s})/n$ at each node $i \in V$. However, based on Lemma 2, the convergence of DistributedLAE is only asymptotic, and therefore on the face value, the accurate projection $y_s = \sum_{i=1}^{n} P_*(\beta_{i,s})/n$ can only be obtained by an infinite number of steps. Suppose the DistributedLAE only runs at each node for $T$ steps. Then for $x_i(0) = \beta_{i,s} \sim \text{uniform}(0, 1)^{2m}$, the output at each node $i$ becomes

$$\hat{y}_{i,s} = \sum_{k=1}^{n} P_*(\beta_{k,s})/n + r_{i,s}(T) = y_s + r_{i,s}(T)$$

where $r_{i,s}(T)$ is a computation residual incurred at node $i$. Now it is of interest to develop an extended algorithm for solving the Boolean equation with only a finite number of iterations for the projection consensus step.

Remark 3. It is also possible to construct the exact projection $y_s = \sum_{i=1}^{n} P_*(\beta_{i,s})/n$ from a series of node states along the DistributedLAE algorithm, e.g., [30], utilizing the idea of finite-time consensus [31, 32]. This method would however rely on additional observability conditions for the network structure $G$, and the structure $G$ should also be known to all nodes.

Based on Lemma 2 and noting the exponential rate of convergence for algorithm (8), there holds

$$\|r_{i,s}(T)\| \leq C_0 e^{-\gamma_0 T}$$

for some constants $C_0 > 0$ and $\gamma_0 > 0$, where $\| \cdot \|$ represents the $\ell_2$ norm. Now, the new challenge lies in how we should construct the affine space $\mathcal{E}_b$ of the solutions to the linear equation (12) from the $\hat{y}_{i,s}, s = 1, \ldots, 2^m + 1$. There are two important properties that $\mathcal{E}_b$ should satisfy:

(i) The distance between $\hat{y}_{i,s}$ and $\mathcal{E}_b$ is upper bounded by $C_0 e^{-\gamma_0 T}$ which decays exponentially as $T$ increases;

(ii) $\mathcal{E}_b \cap \Delta_{2^m} \neq \emptyset$.

If only these two properties are utilized, we end up with a trivial solution of $\mathcal{E}_b$ from the $\hat{y}_{i,s}, s = 1, \ldots, 2^m + 1$ as the $\mathbb{R}^{2^m}$. Therefore, in order to compute the actual solution space $\mathcal{E}_b$, we propose to find an affine subspace with the minimal rank among all the affine spaces satisfying the two properties. Let $\mathfrak{A}$ denote the set of all affine spaces of $\mathbb{R}^{2^m}$. Fixing $\epsilon > 0$, we define the following optimization problem:

$$\min_{\mathcal{A} \in \mathfrak{A}} \dim(\mathcal{A}) \quad \text{s.t.} \quad \sum_{s=1}^{2^m+1} \text{dist}(\hat{y}_{i,s}, \mathcal{A}) \leq \epsilon.$$

The optimization problem (16) is obviously feasible for all $\epsilon > 0$. We present the following lemma on whether solving (16) can potentially give us the true $\mathcal{E}_b$. 

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Lemma 5. Suppose the algorithm DistributedLAE runs at each node for $T$ steps, and produces $\hat{y}_{i,s}$ at each node $i$ for $x_i(0) = \beta_{i,s} \sim \text{uniform}([0,1]^{2^m})$, $s = 1, \ldots, 2^m + 1$. Let $\epsilon := \epsilon_T = C_\epsilon e^{-\gamma_T} (2^m + 1)$ for some $C_\epsilon \geq C_0$ and some $0 < \gamma_\epsilon \leq \gamma_0$. Then

$$\lim_{T \to \infty} \mathbb{P}(\epsilon^b \text{ is a solution to } (16)) = 1$$

under the probability measure introduced by the random initial values.

Based on Lemma 5 with $\epsilon := \epsilon_T = C_\epsilon e^{-\gamma_T} (2^m + 1)$, the optimization problem (16) achieves its optimum at a family of affine spaces $\mathfrak{A}_{i, \epsilon}^*$. Lemma 5 also suggests that with high probability, $\epsilon^b \in \mathfrak{A}_{i, \epsilon}^*$ for small $\epsilon$ (i.e., large $T$). It remains unclear how we can localize the exact $\epsilon^b$ from $\mathfrak{A}_{i, \epsilon}^*$, for which we propose to solve the following optimization problem:

$$\max_{\mathcal{A} \in \mathfrak{A}_{i, \epsilon}^*} \text{Card}(\mathcal{A} \cap \Delta_{2^m}). \quad (17)$$

Here Card($\cdot$) represents the cardinality of a finite set.

Lemma 6. Suppose the algorithm DistributedLAE runs at each node for $T$ steps, and produces $\hat{y}_{i,s}$ at each node $i$ for $x_i(0) = \beta_{i,s} \sim \text{uniform}([0,1]^{2^m})$, $s = 1, \ldots, 2^m + 1$. Let $\epsilon := \epsilon_T = C_\epsilon e^{-\gamma_T} (2^m + 1)$ for some $C_\epsilon \geq C_0$ and some $0 < \gamma_\epsilon \leq \gamma_0$. There holds

$$\lim_{T \to \infty} \mathbb{P}(\epsilon^b \text{ is the unique solution to } (17)) = 1$$

under the probability measure introduced by the random initial values.

Let $C_\epsilon \geq C_0, \gamma_\epsilon \leq \gamma_0$ be fixed. Let $T$ be given. We present the following algorithm which relies only on a finite number of steps for the DistributedLAE subroutine.

**Algorithm 3 ExtendedDistributedBooleanEquationSolver**

**Require:** Over the network $G$, node $i$ holds Boolean equation $f_i(x) = \sigma_i$; nodes communicate with only neighbors on $G$ about their dynamical states.

**Ensure:** Each node $i$ computes $\{x \in \{0,1\}^m : f_j(x) = \sigma_j, \ j = 1, \ldots, n\}$ with high probability.

1. Each node $i$ locally computes $M_i = \text{BooleanMatricization}(f_i) \in \mathbb{R}^2 \times \mathbb{R}^{2^m}$ for all $i \in V$;
2. Each node $i$ assigns $\theta^b_i : \mathbf{H}_{i} \mathbf{y} = \mathbf{z}_i$ by $\mathbf{H}_i \leftarrow M_{f_i}$ and $\mathbf{z}_i \leftarrow \Theta_1(\sigma_i)$;
3. For $s = 1, \ldots, 2^m + 1$, each node $i$ randomly and independently selects $x_i(0) = \beta_{i,s} \sim \text{uniform}([0,1]^{2^m})$, and runs DistributedLAE for $T$ steps to produce $\hat{y}_{i,s} = \sum_{k=1}^n P_s(\beta_{k,s})/n + r_{i,s}(T)$ at each node $i \in V$;
4. Each node $i$ assigns $\epsilon = C_\epsilon e^{-\gamma_T} (2^m + 1)$ to produce $\mathfrak{A}_{i, \epsilon}^*$ by solving (16) locally;
5. Each node $i$ solves (17) to generate $S_i = \arg \max_{\mathcal{A} \in \mathfrak{A}_{i, \epsilon}^*} \text{Card}(\mathcal{A} \cap \Delta_{2^m})$ locally;
6. return $S_i = \Upsilon_{m}(S_i)$ at node $i$ for all $i \in V$.

**Theorem 2.** Suppose the system of Boolean equations (11) admits at least one exact solutions. The Algorithm 3 returns the set of solutions to (11) at each node $i$ with high probability for large $T$. To be precise, there holds for all $i = 1, \ldots, n$ that

$$\lim_{T \to \infty} \mathbb{P}(S_i \text{ is the set of solutions to } (11)) = 1.$$
Remark 4. Algorithm 3 relies on knowledge at each node on an upper bound of $C_0$, and a lower bound of $\gamma_0$. Since the number of Boolean equations in the form of (1) is essentially finite, such knowledge can be obtained from the network structure $G$. Note that Algorithm 2 on the other hand does not rely on any information of $G$.

It is straightforward to see that Theorem 2 is a direct result of Lemma 5 and Lemma 6. The details of the proofs of Lemma 5 and Lemma 6 have been put in the appendices.

3.3 Prior Knowledge of Solution Set

From Theorem 1, to find all solutions to the Boolean equations (1), the projection consensus step for solving the induced linear algebraic equation needs to run $2^m + 1$ rounds with randomly selected initial values. It turns out, if we know certain structure of the Boolean equations (1), we can reduce the number of rounds of solving linear equations.

Let $\chi_0$ be the cardinality of the image of the Boolean mapping $[f_1 \ldots f_n]^T$, i.e.,

$$\chi_0 = \text{Card}\{(f_1(x), \ldots, f_n(x)) : x \in \{0, 1\}^m\}.$$  \hspace{1cm} (18)

We present the following result.

**Theorem 3.** Suppose the system of Boolean equations (1) admits at least one exact solutions. Let $k^* = 2^m - \chi_0 + 1$. Then with probability one, the set $S$ that the algorithm DistributedBooleanEquationSolver returns is the solution set to (1).

The detailed proof of Theorem 3 is in the appendix. As an illustration of Theorem 3 we present the following example.

**Example 2.** Let $\rightarrow$ denote the logical implication operation with $x \rightarrow y = \neg x \lor y$. Consider the following Boolean equations

$$\begin{align*}
  f_1(x_1, x_2, x_3) &= (x_1 \lor x_2) \land \neg x_3 = 1, \\
  f_2(x_1, x_2, x_3) &= (x_1 \rightarrow x_2) \lor x_3 = 0, \\
  f_3(x_1, x_2, x_3) &= x_1 \land x_3 = 0
\end{align*}$$

which has a unique solution $[100] \in \{0, 1\}^3$. We can verify that $\chi_0 = 4$ for the given $f_i, i = 1, \ldots, 3$. Along the Algorithm 2 each node $i$ computes the $M_{f_i}$ from $f_i$ as

$$M_{f_1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$M_{f_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$M_{f_3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 \ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.\hspace{1cm} (20)$$

\footnote{There can be an infinite number of formulas representing a single Boolean mapping $g$ from $\{0, 1\}^m$ to $\{0, 1\}$. However, the matrix representation $M_g$ in the sense of Lemma 1 is unique which does not depend on a particular formula of $g$.}
Then node $i$ locally assigns $\ell_i^b : H_i y = z_i$ by $H_i \leftarrow M_{f_i}$ and $z_i \leftarrow \Theta(\sigma)_i$, where

$$z_1 = \Theta(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad z_2 = \Theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad z_3 = \Theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. $$

Next, for $s = 1, \ldots, 2^3 - \chi_0 + 1 = 5$, each node $i$ randomly and independently selects $x_i(0) = \beta_{i,s}$ from $\text{Uniform}(0, 1]^8$, and runs $\text{DistributedLAE}$ to produce $y_s$ as

$$[y_1 \ldots y_5] = \begin{bmatrix} -0.1558 & 0.0871 & -0.1208 & -0.0962 & -0.1209 \\ 0.1417 & -0.1609 & 0.1522 & 0.1201 & -0.0082 \\ -0.0003 & 0.0835 & 0.0835 & -0.1127 & 0.1244 \\ 0.0141 & 0.0738 & -0.0314 & -0.0239 & 0.1292 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ -0.0813 & 0.0717 & -0.1067 & 0.1856 & -0.0769 \\ 0.0003 & -0.0835 & -0.0835 & 0.1127 & -0.1244 \\ 0.0813 & -0.0717 & 0.1067 & -0.1856 & 0.0769 \end{bmatrix}. $$

Each node then locally runs the algorithm $\text{BooleanVectorSearch}$, which gives $S = \{\delta_5^3\}$. Eventually nodes return $S = \Upsilon_m(S) = \{[100]\}$. We can randomly select other $x_i(0) = \beta_{i,s}$ for $s = 1, \ldots, 5$, and $S = \{[100]\}$ is always correctly computed. Therefore, we have provided a verification of Theorem 3.

### 4 Distributed Satisfiability Verification

The correctness of the Algorithm 2 and Algorithm 3 relies on the crucial fact that the system of Boolean equations (1) admits at least one exact solutions. This, however, has no guarantee in the first place. Even with all the $f_i$, verification of this satisfiability is a classical SAT problem. For the network that runs Algorithm 2 and Algorithm 3 we also need to develop a method that can verify the satisfiability of (1) in a distributed manner.

#### 4.1 A Distributed SAT Algorithm

We present the following algorithm for distributedly verifying the satisfiability of (1).
Algorithm 4 DistributedBooleanSatisfiability

Require: Over the network G, node \( i \) holds Boolean equation \( f_i(x) = \sigma_i \); nodes communicate with only neighbors on G about their dynamical states.

Ensure: Each node \( i \) verifies definitively satisfiability of \( \{ x \in \{0,1\}^m : f_j(x) = \sigma_j, j = 1, \ldots, n \} \) for all \( i \).

1: Each node \( i \) locally computes \( M_{f_i} = \text{BooleanMatricization}(f_i) \in \mathbb{R}^{2 \times 2^m} \) for all \( i \in V \);
2: Each node \( i \) assigns \( E_{b_i} : H_i, y = z_i \) by \( H_i \leftarrow M_{f_i} \) and \( z_i \leftarrow \Theta_1(\sigma_i) \);
3: Each node \( i \) randomly and independently selects \( x_i(0) \sim \text{uniform}([0,1]^{2^m}) \), and runs the recursion (8) to produce an output \( \tilde{y}_i \);
4: The network runs the average consensus algorithm (10) with \( x_i(0) = \tilde{y}_i \) to produce an output \( \tilde{y}_\text{ave} = \sum_{j=1}^n \tilde{y}_j / n \) at each node \( i \);
5: return unsatisfiable at node \( i \) if \( \tilde{y}_\text{ave} = \tilde{y}_i \); and go to Step 6 otherwise;
6: The network runs \( \text{DistributedLAE} \) to produce \( y_s = \sum_{i=1}^n P_*(\beta_{i,s}) / n \) at each node \( i \in V \) with \( x_i(0) = \beta_{i,s} \sim \text{uniform}([0,1]^{2^m}) \), \( s = 1, \ldots, 2^m + 1 \), and then each node \( i \) locally computes \( S = \Upsilon_m(\text{BooleanVectorSearch}(y_1, \ldots, y_{2^m+1})) \);
7: return unsatisfiable if \( S = \emptyset \), and satisfiable otherwise at each node \( i \).

We present the following theorem.

**Theorem 4.** With probability one, the Algorithm 4 correctly returns the satisfiability of (1) at all nodes.

There are a few points behind the Algorithm 4 that are worth emphasizing. First of all, satisfiability of the Boolean equations (1) is not equivalent to the satisfiability of the induced algebraic equation \( E_{b} \) in (12). In fact, \( E_{b} \) may be satisfiable in \( \mathbb{R}^{2^m} \), but not in \( \Delta_{2^m} \) which corresponds to the solutions to the Boolean equations (1). Secondly, Algorithm 2 cannot be directly applied when the Boolean equations (1) is unsatisfiable since the Step 3 of Algorithm 2 depends crucially on the fact that \( E_{b} \) is solvable. Therefore, in Algorithm 4 we embed a component where nodes can first distributively verify the satisfiability of the induced algebraic equation \( E_{b} \) as a preliminary evaluation of satisfiability for (1), and then the subroutines of Algorithm 2 can be utilized to produce a further and final decision on the satisfiability for (1). In order to establish the desired theorem, we would require the following technical lemma.

**Lemma 7.** Suppose the system of linear equations (7) is not satisfiable. Then along the recursion (8), there hold

(i) Each \( x_i(t) \) converges to a finite value \( y_i^* \) as \( t \to \infty \).

(ii) There exist at least two nodes \( j, k \in V \) such that \( y_j^* \neq y_k^* \).

The proof of Lemma 7 is put in the appendix, followed by the proof of Theorem 4.

### 4.2 Numerical Examples

**Example 3.** Consider the following Boolean equations

\[
\begin{align*}
f_1 &= x_1 \land x_2 \land x_3 = 1, \\
f_2 &= \neg x_1 \lor (x_2 \leftrightarrow x_3) = 1, \\
f_3 &= x_1 \land (x_2 \lor x_3) = 0.
\end{align*}
\] (21)

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which is unsatisfiable. The $f_i, i = 1, \ldots, 3$ lead to $M_{f_i}$ as:

\[
M_{f_1} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
M_{f_2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix},
\]

\[
M_{f_3} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix};
\]

and the $z_i$ give

\[
z_1 = \Theta_1(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
z_2 = \Theta_1(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
z_3 = \Theta_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Then node $i$ locally assigns $\delta_i^b: H_i y = z_i$ by $H_i \leftarrow M_{f_i}$ and $z_i \leftarrow \Theta_1(\sigma_i)$.

![Figure 2: The three-node path graph.](image)

Let the network $G$ be a three-node path graph as shown in Figure 2. Each node $i$ randomly and independently selects $x_i(0) \sim \text{uniform}(0, 1)^{2m}$, and runs the algorithm (8) with $\epsilon = 0.2$, where each $x_i(t)$ is in $\mathbb{R}^8$. Denote $x_i^t(t) := (x_i(t)[1], x_i(t)[2])^\top$, where $x_i(t)[1]$ and $x_i(t)[2]$ are, respectively, the first and second entries of $x_i(t)$. We run the algorithm (8) for 50 steps, and plot the trajectories of $x_i^t(t)$, $t = 0, 1, \ldots, 50$ for $i = 1, 2, 3$, respectively, in Figure 3.

![Figure 3: The trajectories of $x_i^t(t)$, $t = 0, 1, \ldots, 50$ for $i = 1, 2, 3$ with a randomly selected initial condition.](image)

From the trajectories in Figure 3, it can be seen that $x_i^t(t)$ (and thus $x_i(t)$) converge to different limits. As a result, Algorithm 4 returns unsatisfiable in Step 5.
Example 4. Let us consider the following Boolean equations

\[
\begin{aligned}
  f_1(x_1, x_2, x_3) &= (x_1 \lor x_2) \land \neg x_3 = 0, \\
  f_2(x_1, x_2, x_3) &= (x_1 \rightarrow x_2) \lor x_3 = 0, \\
  f_3(x_1, x_2, x_3) &= x_1 \land x_3 = 1
\end{aligned}
\] (22)

which is unsatisfiable. The \(f_i\)'s are the same as those used in Example 2, and therefore lead to the same \(M_{f_i}\)'s in (20). Then node \(i\) locally assigns \(\mathcal{E}_i^b : H_i y = z_i \) by \(H_i \leftarrow M_{f_i}\) and \(z_i \leftarrow \Theta_1(\sigma_i)\), where

\[
\begin{align*}
  z_1 &= \Theta_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
  z_2 &= \Theta_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
  z_3 &= \Theta_1(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{align*}
\]

Although (22) is unsatisfiable, we can verify that the linear equation \(\mathcal{E}_i^b\) from the three \(\mathcal{E}_i^b\)'s is actually satisfiable since

\[
\text{rank} \begin{bmatrix} M_{f_1} \\ M_{f_2} \\ M_{f_3} \end{bmatrix} = \text{rank} \begin{bmatrix} M_{f_1} & z_1 \\ M_{f_2} & z_2 \\ M_{f_3} & z_3 \end{bmatrix}.
\]

Thus, Algorithm 4 proceeds to Step 6 – 7 after Step 5, and eventually returns unsatisfiable in Step 7.

5 Conclusions

We have proposed distributed algorithms that solve a system of Boolean equations over a network, where each node in the network possesses only one Boolean equation from the system. The Boolean equation for a particular node is a private equation known to this only, and the network nodes aim to compute the exact set of solutions to the system of Boolean equations without exchanging their local equations. Based on an algebraic representation of Boolean mappings and existing algorithms for network linear equations, we show that distributed algorithms can be constructed that return the exact solution set of the Boolean equations at each node with high probability. We also presented an algorithm for distributed verification of the satisfiability of Boolean equations. Therefore, the proposed algorithms put together may serve as a comprehensive framework for distributedly treating Boolean equations: verifying satisfiability and then finding the exact solution set whenever possible.

Appendices

A. Proof of Theorem 1

First of all, let \(x_*\) be an exact solution to the system of Boolean equations (1). Then by Lemma 1 there holds

\[
M_{f_i}(\Theta_m(x_*)) = \Theta_1(\sigma_i), \quad i = 1, \ldots, n.
\]

As a result, the system of linear equations \(\mathcal{E}_i^b : H_i y = z_i\) with \(H_i = M_{f_i}\) and \(z_i = \Theta_1(\sigma_i)\) contains at least one exact solution \(\Theta_m(x_*)\). This leads to \(\mathcal{E}^b = \bigcap_{i=1}^n \mathcal{E}_i^b \neq \emptyset\). Therefore, Lemma 2 is applicable to the DistributedLAE algorithm, and each node \(i\) indeed obtains the \(y_s = \sum_{i=1}^n P_s(\beta_{i,s})/n \in \mathcal{E}\) for \(s = 1, \ldots, 2^m + 1\).
Next, we prove that with probability one there holds \( \text{Aff}(\mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1}) = \mathcal{E}^b \). Since \( \mathcal{E}^b \) is an affine subspace from its definition, and \( \mathbf{y}_s \in \mathcal{E}^b \) for all \( s = 1, \ldots, 2^m+1 \), we conclude that \( \text{Aff}(\mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1}) \subseteq \mathcal{E} \) is a sure event. We continue to show that \( \dim(\text{Aff}(\mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1})) = \dim(\mathcal{E}^b) \) with probability one, where the randomness arises from the random initial values \( \mathbf{x}_i(0) \) for Step 3 of the algorithm DistributedBooleanEquationSolver.

Note that, there holds from the basic properties of projections onto affine subspaces that

\[
\mathbf{y}_s = \sum_{i=1}^{n} P_s(\beta_{i,s})/n = P_s\left(\frac{\sum_{i=1}^{n} \beta_{i,s}}{n}\right) = \sum_{i=1}^{n} P_s \beta_{i,s}/n + \mathbf{a}_s.
\]

where \( P_s \in \mathbb{R}^{2m \times 2m} \) is a projection matrix, and \( \mathbf{a}_s \) is a vector in \( \mathbb{R}^{2m} \). In fact, letting \( \mathbf{e} \in \mathcal{E} \), \( P_s \) is the projector onto the linear subspace \( \mathcal{E}^b_s := \{ \mathbf{y} - \mathbf{e} : \mathbf{y} \in \mathcal{E} \} \). We thus have

\[
(\mathbf{y}_2 - \mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1} - \mathbf{y}_1) = P_s(\mathbf{l}_2 - \mathbf{l}_1, \ldots, \mathbf{l}_{2^m+1} - \mathbf{l}_1)
\]

where \( \mathbf{l}_j = \sum_{i=1}^{n} \beta_{i,j}/n \) for \( j = 2, \ldots, 2^m+1 \). Now, since the \( \beta_{i,s} \) \( \sim \) uniform\(([0,1]^{2m})\) are selected randomly and independently, it is trivial to see that \( \text{rank}(\mathbf{l}_2 - \mathbf{l}_1, \ldots, \mathbf{l}_{2^m+1} - \mathbf{l}_1) = 2^m \) holds with probability one. This implies

\[
\text{rank}(\mathbf{y}_2 - \mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1} - \mathbf{y}_1) = \text{rank}(\mathcal{E}^b)
\]

with probability one. Therefore, \( \dim(\text{Aff}(\mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1})) = \dim(\mathcal{E}^b) \) with probability one, leading to \( \text{Aff}(\mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1}) = \mathcal{E}^b \) in view of the fact \( \text{Aff}(\mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1}) \subseteq \mathcal{E} \).

Finally, based on Lemma 3 BooleanVectorSearch\((\mathbf{y}_1, \ldots, \mathbf{y}_{2^m+1})\) with probability one returns the set \( S \) as \( \mathcal{E}^b \cap \Delta_{2^m} \). This fact can be established from the following two aspects.

(i) Let \( x \in \Upsilon_m(\mathcal{E}^b \cap \Delta_{2^m}) \). Then there holds \( M_{f_i}(\Theta_m(x)) = \Theta_1(\sigma_i) \) for all \( i = 1, \ldots, n \), or equivalently, \( f_i(x) = \sigma_i \) for all \( i = 1, \ldots, n \) based on Lemma 1.

(ii) Let \( x \in \{0,1\}^m \) satisfying \( f_i(x) = \sigma_i \) for all \( i = 1, \ldots, n \). Then \( \Theta_m(x) \in \Delta_{2^m} \) from the definiton of \( \Theta_m(\cdot) \), and there must also hold \( \Theta_m(x) \in \mathcal{E}^b \) again from Lemma 1. As a result, we also have \( x \in \Upsilon_m(\mathcal{E}^b \cap \Delta_{2^m}) \).

We have now concluded that \( \hat{S} = \Theta_m(S) \) is the solution set of the system of Boolean equations (1) with probability one. The proof is complete.

B. Proof of Lemma 5

In view of (14) and (15), there always holds

\[
\sum_{s=1}^{2^m+1} \text{dist}(\mathbf{y}_{i,s}, \mathcal{E}^b) \leq (2^m + 1)C_\epsilon e^{-\gamma \epsilon T} = \epsilon.
\]

Therefore, the true solution space \( \mathcal{E}^b = \bigcap_{s=1}^{n} \mathcal{E}^b_s \) will always be a feasible point of (16). Let us assume for the sake of building up a contradiction argument that for any \( \epsilon > 0 \), there exists \( \mathcal{A}_* \in \mathcal{A} \) as a solution to (16), and \( \text{rank}(\mathcal{A}_*) < \text{rank}(\mathcal{E}^b) \) with at least a probability \( p > 0 \) (which does not depend on \( T \) or \( \epsilon \)).

\footnote{Technically, this \( \mathcal{A}_* \) is a random set and depends on the particular \( \epsilon \).}
We have proved in the proof of Theorem 1 that there holds
\[
\text{rank}(y_2 - y_1, \ldots, y_{2m+1} - y_1) = \text{rank}(E_b) \tag{25}
\]
with probability one. Therefore, if indeed $A_\star \in \mathfrak{A}$ is a solution to (16), there must hold
\[
2m+1 \sum_{s=1}^{2m+1} \text{dist}(y_s, A_\star) \leq 2m+1 \sum_{s=1}^{2m+1} \left( \text{dist} \left( \hat{y}_{i,s}, A_\star \right) + \|y_s - \hat{y}_{i,s}\| \right)
\leq \sum_{s=1}^{2m+1} \left( \text{dist} \left( \hat{y}_{i,s}, A_\star \right) + \|r_{i,s}(T)\| \right)
\leq 2\epsilon \tag{26}
\]
where the first inequality is directly from the triangle inequality, the second inequality is from (14), and the last inequality follows from (15) and (16). Noting the fact that $\beta_{i,s} \sim \text{uniform}([0,1])$ are selected randomly and independently, and the identity that for $l_j = \sum_{i=1}^n \beta_{i,j}/n$

\[
y_2 - y_1, \ldots, y_{2m+1} - y_1 = P_* (l_2 - l_1, \ldots, l_{2m+1} - l_1),
\]
the probability that (26) and $\text{rank}(A_\star) < \text{rank}(E_b)$ simultaneously holds goes to zero as $\epsilon$ tends to zero. We therefore have established a contradiction, and proved that $\lim_{T \to \infty} P(E_b \text{ is a solution to (16)}) = 1$.

C. Proof of Lemma 6

Suppose that $A_\star \in \mathfrak{A}_{t,\epsilon}$ is a solution to (16). Then there hold
\[
\text{(i) rank}(A_\star) = \text{rank}(E_b);
\]
\[
\text{(ii) } \sum_{s=1}^{2m+1} \text{dist}(y_s, A_\star) \leq 2\epsilon.
\]
We prove the desired lemma by establishing $(A_\star \cap \Delta_{2m}) \subseteq (E_b \cap \Delta_{2m})$ with probability one when $\epsilon$ is sufficiently small.

Note that the $y_s$ are distributed over the convex polyhedron $P_*([0,1]^{2m}) \subseteq E$ due to the fact that $y_s = \sum_{i=1}^n P_* \beta_{i,s}/n + a_*$. The condition (i) and (ii) imply that for sufficiently small $\epsilon$, there holds

\[
\text{Aff}(P_{A_*}(y_1), \ldots, P_{A_*}(y_{2m+1})) = A_\star
\]
with probability one since $\text{Aff}(y_1, \ldots, y_{2m+1}) = E_b$ with probability one from the proof of Theorem 1. Let $\delta \in A_\star \cap \Delta_{2m}$. Then there are $\lambda_k, k = 1, \ldots, 2m+1$ which are upper bounded by some absolute constant $B > 0$, so that

\[
\delta = \sum_{k=1}^{2m+1} \lambda_k P_{A_*}(y_k). \tag{27}
\]
This implies

\[
\text{dist}(\delta, \mathcal{E}^b) \leq B \sum_{k=1}^{2^m+1} \text{dist}(\mathcal{P}_{A_\star}(y_k), \mathcal{E}^b) \\
\leq B \sum_{k=1}^{2^m+1} \text{dist}(\mathcal{P}_{A_\star}(y_k), y_k) \\
= B \sum_{k=1}^{2^m+1} \text{dist}(y_k, A_\star) \\
\leq 2(2^m + 1)B\epsilon.
\]  

(28)

Now, (28) suggests that

\[
\|M_{f_i}(\delta) - \Theta_1(\sigma_i)\| \leq 2(2^m + 1)B\epsilon, \quad i = 1, \ldots, n.
\]

The only possibility for this hold with small enough \(\epsilon\) is

\[
f_i(\Upsilon_m(\delta)) = \sigma_i, \quad i = 1, \ldots, n
\]

because \(M_{f_i}(\delta) = \Theta_1 f_i(\Upsilon_m(\delta))\) by Lemma \(\text{[1]}\). As a result, \(\delta \in \mathcal{E} \cap \Delta_{2m}\), which implies \((A_\star \cap \Delta_{2m}) \subseteq (\mathcal{E}^b \cap \Delta_{2m})\) with probability one. Therefore, \(\mathcal{E}^b\) must be the unique solution to (17) if indeed \(\mathcal{E}^b \in \mathcal{A}_{i,\epsilon}\).

By Lemma \(\text{[5]}\) there holds

\[
\lim_{T \to \infty} \mathbb{P}(\mathcal{E}^b \in \mathcal{A}_{i,\epsilon}^*) = 1
\]

and this concludes the proof of the desired lemma.

**D. Proof of Theorem 3**

If \(\chi_0\) is the cardinality of the image of the Boolean mapping \([f_1 \ldots f_n]^{\top}\), then the matrix \(M_f\) has at most \(\chi_0\) distinct columns. Since each column of \(M_f\) is a vector in \(\mathbb{R}^{2n}\) with the form of

\[
\begin{bmatrix}
v_1 \\ \vdots \\ v_n
\end{bmatrix}
\]

where \(v_k \in \{(1 0)^{\top}, (0 1)^{\top}\}\), we conclude that

\[
\text{rank}(M_f) \leq \chi_0.
\]

This implies \(\dim(\mathcal{E}^b) \geq d - \chi_0\). Repeating the proof of Theorem \(\text{[1]}\) we know that

\[
\text{rank}(y_2 - y_1, \ldots, y_{k_\star} - y_1) = \dim(\mathcal{E}^b)
\]

(29)

with probability one with \(k_\star = d - \chi_0 + 1\). The desired theorem thus follows from the same argument as the proof of Theorem \(\text{[1]}\).
E. Proof of Lemma 7

Denote \( x(t) = (x_1^T(t) \ldots x_n^T(t))^\top \). Let \( A^\dagger \) be the M-P pseudoinverse of a matrix \( A \). Let \( \rho(A) \) and \( \sigma(A) \) represent the spectral radius and spectrum of a matrix \( A \), respectively. Then from the basic representations of the affine projections \( P_i \), the algorithm (8) can be written in a compact form as

\[
x(t + 1) = PWx(t) + b
\]  

(30)

where \( P = \text{diag}(P_1 \ldots P_n) \) is a block diagonal matrix with \( P_i = I_d - H_i^\dagger H_i \) being a projection matrix, \( W = W \otimes I_d \) with \( W \in \mathbb{R}^{n \times n} \) defined in (9), and

\[
b = \begin{bmatrix}
H_1^\dagger z_1 \\
H_2^\dagger z_2 \\
\vdots \\
H_n^\dagger z_n
\end{bmatrix}.
\]

Note that, the matrix \( P_i \) is the projector onto the linear subspace \( \mathcal{L}_i := \{ y : H_i y = 0 \} \). For any vector \( u = (u_1^\top \ldots u_n^\top)^\top \) with \( w_i \in \mathbb{R}^d \), there holds

\[
\| PWu \|_2^2 = \sum_{i=1}^n \| P_i \| \sum_{j=1}^n W_{ij} u_j \|_2^2 \leq \sum_{i=1}^n \sum_{j=1}^n W_{ij} \| u_j \|_2^2 = \sum_{j=1}^n \| u_j \|_2^2 = \| u \|_2^2.
\]  

(31)

This implies \( \| PW \|_2 \leq 1 \), and consequently, we have \( \rho(PW) \leq 1 \). Moreover, all eigenvalues of \( PW \) on the unit circle of the complex plain must have equal algebraic and geometric multiplicities. We proceed to establish the following claims on the matrix \( PW \).

Claim A. \( |\lambda_i(PW)| < 1 \) if for all \( \lambda_i(PW) \neq 1 \in \sigma(PW) \).

Claim B. If \( 1 \in \sigma(PW) \), then the eigenspace corresponding to the eigenvalue 1 is a subspace of \( \mathcal{M} : \{ 1_m \otimes y : y \in \mathbb{R}^d \} \).

In fact, let us consider the linear dynamical system

\[
x(t + 1) = PWx(t),
\]  

(32)

which defines a special projection consensus algorithm in the following form:

\[
\tilde{x}_i(t + 1) = \tilde{P}_i \left( \sum_{j=1}^n W_{ij} \tilde{x}_j(t) \right), \quad i = 1, \ldots, n.
\]  

(33)

Here each \( \tilde{P}_i \) is the projection onto the linear subspace \( \mathcal{L}_i \). As the \( \mathcal{L}_i \)'s are linear subspaces, there always holds \( \bigcap_{i=1}^n \mathcal{L}_i \neq \emptyset \). Therefore, we can directly invoke Lemma 3 of [11] to conclude that along (32), all \( \tilde{x}_i(t) \) converges to a common static value for all initial conditions at time grows to infinity. Since (32) is a linear time-invariant system, the above two claims must hold.

Proof of (i). We divide the proof into two cases.
(a). Suppose \( \rho(\mathbf{P} \mathbf{W}) < 1 \). Then with \( \mathbf{y}^*_t = (\mathbf{I}_{nd} - \mathbf{P} \mathbf{W})^{-1} \mathbf{b} \), (30) becomes
\[
x(t + 1) - \mathbf{y}^*_t = \mathbf{P} \mathbf{W} (\mathbf{x}(t) - \mathbf{y}^*_t).
\]
Obviously there holds \( \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{y}^*_t \).

(b). Suppose \( \rho(\mathbf{P} \mathbf{W}) = 1 \). From Claim A, we know that the only eigenvalue of \( \mathbf{P} \mathbf{W} \) with magnitude one is 1, which has equal algebraic and geometric multiplicity. Thus, we can find a real orthogonal matrix \( \mathbf{T} \) such that
\[
\mathbf{T}^{-1} \mathbf{P} \mathbf{W} \mathbf{T} = \begin{bmatrix}
\mathbf{I}_c & 0 \\
0 & \mathbf{P}_W
\end{bmatrix}
\]
where \( c \) is the multiplicity of the eigenvalue one, and all eigenvalues of \( \mathbf{P}_W \) are strictly within the unit circle. Letting \( \mathbf{y}(t) = \mathbf{T} \mathbf{x}(t) \), (30) is written as
\[
\mathbf{y}(t + 1) = \mathbf{T}^{-1} \mathbf{P} \mathbf{W} \mathbf{y}(t) + \mathbf{T}^{-1} \mathbf{b}.
\]
The first \( c \) columns of the matrix \( \mathbf{T} \), \( \mathbf{T}_1, \ldots, \mathbf{T}_c \), are eigenvectors of the matrix \( \mathbf{P} \mathbf{W} \) corresponding to eigenvalue one. We write
\[
\mathbf{T} = (\mathbf{T}_1 \ldots \mathbf{T}_c \mathbf{T}_s).
\]
Now, from Claim B, each \( \mathbf{T}_k \) can be written into \( 1_n \otimes \mathbf{h}_k \) with \( \mathbf{h}_k \in \mathbb{R}^d \). This implies
\[
\mathbf{P}_i \mathbf{h}_k = (\mathbf{I}_d - \mathbf{H}_i^\dagger \mathbf{H}_i) \mathbf{h}_k = \mathbf{h}_k
\]
for all \( i = 1, \ldots, n \) and all \( k = 1, \ldots, c \). As a result, \( \mathbf{H}_i^\dagger \mathbf{H}_i \mathbf{h}_k = 0 \) for all \( i = 1, \ldots, n \) and all \( k = 1, \ldots, c \), which further implies
\[
\mathbf{T}_k^\top \mathbf{b} = \mathbf{T}_k^\top \begin{bmatrix}
\mathbf{H}_1^\dagger \mathbf{z}_1 \\
\mathbf{H}_2^\dagger \mathbf{z}_2 \\
\vdots \\
\mathbf{H}_n^\dagger \mathbf{z}_n
\end{bmatrix} = \sum_{i=1}^{n} \mathbf{h}_k^\top \mathbf{H}_i^\dagger \mathbf{z}_i = 0, \quad k = 1, \ldots, c
\]
since \( \mathbf{H}_i^\dagger \mathbf{H}_i \mathbf{h}_k = 0 \) implies \( \mathbf{h}_k^\top \mathbf{H}_i \) utilizing the basic properties of M-P pseudoinverse. The system (36) can therefore be further written as
\[
\mathbf{y}_a(t + 1) = \mathbf{y}_a(t) \\
\mathbf{y}_b(t + 1) = \mathbf{P}_W \mathbf{y}_b(t) + \mathbf{b}
\]
where \( \mathbf{y}_a(t) \) consists of the first \( c \) entries of \( \mathbf{y}(t) \), and \( \mathbf{y}_b(t) \) has the remaining entries of \( \mathbf{y}(t) \). From (38), the case has been reduced to Case (a), and each \( \mathbf{x}_i(t) \) must converge to a static value because \( \mathbf{y}(t) \) does.

This concludes the proof of statement (i).

Proof of (ii). Suppose at the limits of the \( \mathbf{x}_i(t) \) there holds \( \mathbf{y}^*_1 = \cdots = \mathbf{y}^*_n = \mathbf{u}^* \). This means \( \mathbf{u}^* \in \mathcal{E}_i \) for all \( i \), and thus \( \bigcap_{i=1}^n \mathcal{E}_i \neq \emptyset \). Consequently, the system of linear equations (7) admits at least one exact solution, which is a contradiction with our standing assumption of the lemma. There must exist at least two nodes \( j, k \in \mathcal{V} \) such that \( \mathbf{y}^*_j \neq \mathbf{y}^*_k \), and this concludes the proof.
F. Proof of Theorem 4

Suppose the system of Boolean equation (1) is satisfiable. Then the linear equation $M_i y_i = \Theta_1(\sigma_i), \quad i = 1, \ldots, n$ admits at least one exact solution. Therefore, applying Lemma 2 we conclude that in Step 4 of the algorithm 4, there holds $y_i = \sum_{k=1}^n p_k(x_k(0))/n$ for all $i = 1, \ldots, n$. This obviously leads to $y_{ave} = y_i$ for all $i$. As a result, Algorithm 4 proceeds to Step 6 and Step 7. Based on Theorem 1 with probability one $S$ will be returned as the exact solution set of the Boolean equations (1), which certainly satisfies $S \neq \emptyset$. Therefore, Algorithm 4 correctly returns *satisfiable*.

Now suppose on the other hand the system of Boolean equation (1) is not satisfiable. There will be two cases.

(a) Let the linear equation $M_i y_i = \Theta_1(\sigma_i), \quad i = 1, \ldots, n$ admit no exact solutions. Based on Lemma 7 we know that the Step 4 of the algorithm 4 does produce a finite value $\tilde{y}_i$ at each node $i$ as the limit of the algorithm (8), but there exist at least two nodes $j$ and $k$ such that $\tilde{y}_j \neq \tilde{y}_k$. As a result, except for a set of initial values with measure zero, there holds $y_{ave} = \tilde{y}_i$ for any node $i$. Consequently, with probability one, Algorithm 4 correctly returns *unsatisfiable* at Step 5.

(b) Let the linear equation $M_i y_i = \Theta_1(\sigma_i), \quad i = 1, \ldots, n$ admit at least one exact solutions. Again, from Theorem 1 with probability one $S$ will be returned as the exact solution set of the Boolean equations (1), in which case there must hold $S = \emptyset$. Therefore, Algorithm 4 correctly returns *unsatisfiable* at Step 7.

We have now proved the desired theorem.

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