Coded Consensus Monte Carlo: Robust One-Shot Distributed Bayesian Learning with Stragglers

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Abstract

This letter studies distributed Bayesian learning in a setting encompassing a central server and multiple workers by focusing on the problem of mitigating the impact of stragglers. The standard one-shot, or embarrassingly parallel, Bayesian learning protocol known as consensus Monte Carlo (CMC) is generalized by proposing two straggler-resilient solutions based on grouping and coding. The proposed methods, referred to as Group-based CMC (G-CMC) and Coded CMC (C-CMC), leverage redundant computing at the workers in order to enable the estimation of global posterior samples at the server based on partial outputs from the workers. Simulation results show that C-CMC may outperform G-GCMC for a small number of workers, while G-CMC is generally preferable for a larger number of workers.

Index Terms

Distributed Bayesian learning, stragglers, Consensus Monte Carlo, grouping, coded computing

I. INTRODUCTION

One of the main problems in distributed computing systems [1]–[4] is the presence of stragglers – i.e., working machines whose random computing time is much larger than other machines [5]. The effect of stragglers may be mitigated by leveraging redundant storage and computing at the workers, whereby each worker is allocated, and computes over, multiple data shards. State-of-the-art techniques leverage grouping, whereby groups of workers are assigned the same shards and compute the same output, and/or coding, whereby computed outputs are coded at the workers and jointly decoded at the server [6]–[8].

Existing work on grouping and coded distributed computing for machine learning applications focuses on frequentist learning. In frequentist learning, the goal is to identify a single model parameter vector that approximately minimizes the training loss, e.g., via gradient descent [6]–[8]. Frequentist learning is limited in its ability to quantify uncertainty, incorporate prior knowledge, guide active learning, and enable continual learning. Bayesian learning provides a principled approach to address all these limitations, at the cost of an increase in computational complexity [9]–[13].

Scalable implementations of Bayesian learning are based on either variational inference (VI) – replacing integration with optimization over an approximate posterior distributions – or Monte Carlo (MC) sampling – replacing integration with sampling from the posterior distribution [11]. VI-based protocols for distributed Bayesian learning follow the same general principles of standard distributed frequentist learning (e.g., [14] and references therein). Distributed MC sampling protocols are either one-shot, i.e., embarrassingly parallel [15]–[17]; or else based on iterative gradient-based methods [18], [19].

In this letter, we focus on the standard one-shot protocol known as Consensus Monte Carlo (CMC) [15]. CMC aims at obtaining samples from the global posterior distribution based on local sampling at the workers and aggregation at the server.
Distributed Bayesian learning via Consensus Monte Carlo (CMC).

Fig. 1. Distributed Bayesian learning via Consensus Monte Carlo (CMC).

(see Fig. 1). CMC assumes that all workers respond to the server by delivering their local samples before a global sample can be produced by the server. This letter considers, for the first time, the problem of stragglers for CMC.

Two extensions of CMC are proposed that obtain resilience to stragglers based on grouping and coding. The first protocol, referred to as Group-based CMC (G-CMC), requires the partition of workers into groups, with each group being responsible for the computation of local samples for a given subset of shards. In contrast to grouping methods proposed for frequentist learning and distributed computing [7], [20], a novel feature of G-CMC scheme is that all the computed samples can be eventually utilized, even if produced by straggling workers. The second protocol, Coded-CMC (C-CMC) applies an erasure correcting code to the produced samples, in a manner similar to gradient coding [6]. Unlike gradient coding, C-CMC requires the design of a novel pre-processing step of the local samples in order to enable CMC-based aggregation at the server.

II. System Model

This letter considers the problem of drawing samples from a posterior distribution on a large training data set to implement Bayesian learning via Monte Carlo (MC) sampling. Let \( Z = \{ z_n \}_{n=1}^{N} \) represents the training data and \( \theta \in \mathbb{R}^d \) represents the model parameter vector. The global posterior distribution is given as

\[
\text{(Global Posterior)} \quad p(\theta | Z) \propto p(\theta)p(Z | \theta), \tag{1}
\]

where \( p(\theta) \) is the prior distribution and \( p(Z | \theta) \) is the likelihood. We assume that the data points are conditionally independent and identically distributed (i.i.d.), given the model parameter vector \( \theta \), i.e., \( p(Z | \theta) = \prod_{n=1}^{N} p(z_n | \theta) \). The goal is to draw \( L \) samples \( \{ \theta^l \}_{l=1}^{L} \) from the global posterior \( p(\theta | Z) \).

We adopt a data center computing platform that consists of a server and \( K \) workers, as shown in Fig. 1. The training data \( Z \) is partitioned into \( K \) disjoint shards \( Z = \{ Z_s \}_{s=1}^{K} \), each of size \( N/K \) where \( K \) assumed to be an integer divisor of \( N \), and allocated to the workers by following a data allocation scheme. We allow for a redundant shard allocation, so that each shard is allocated to \( r \) workers, where \( r \in [K] = \{1, \ldots, K\} \) is referred to as the redundancy parameter. Each worker \( k \) has a set \( S_k \subseteq [K] \) of \( |S_k| = r \) shards, which are denoted as \( Z_{S_k} = \{ Z_s \}_{s \in S_k} \). Following CMC, we assume that the sampling from
each subposterior, 

\[
\text{(Subposterior)} \quad \tilde{p}(\theta|Z_s) \propto p(\theta)^{1/K} p(Z_s|\theta), \tag{2}
\]
is tractable, where \(p(Z_s|\theta) = \prod_{z \in Z_s} p(z|\theta)\) represents the local likelihood function for the \(s\)-th shard. In \(\text{(2)}\), the prior is underweighted as \(p(\theta)^{1/K}\) in order to preserve the total prior, so that the global posterior \((\text{1})\) can be expressed as the product of subposteriors \(p(\theta|Z) \propto \prod_{s=1}^{K} \tilde{p}(\theta|Z_s)\).

Each worker \(k\) computes in parallel \(r\) samples \(\theta^l_{k,s} \sim \tilde{p}(\theta|Z_s)\) for all the \(r\) allocated shards \(Z_s, s \in S_k\), where index \(l \in \{1, 2, \ldots\}\) runs over the generated samples. We refer to the collection of such samples as \(\Theta^l_k = \{\theta^l_{k,s}\}_{s \in S_k}\). The produced samples may be processed at the worker, and the outcome of this calculation is sent to the server. The server uses this information to produce global samples \(\theta^l\), that are approximately distributed according to the global posterior distribution \((\text{1})\).

We assume that the wall-clock time \(\Delta T^l_k\) required to compute any \(l\)-th batch \(\Theta^l_k\) of \(r\) local samples from the subposteriors \((\text{2})\) of the shards allocated to each worker \(k\) is random with mean \(\eta r\), for some \(\eta > 0\). The computing times \(\{\Delta T^l_k\}_{k \in [K]}\) are i.i.d. across the workers and across index \(l\). An example distribution of the computing time is Pareto with scale-shape parameters \((\eta r (\beta - 1)/\beta, \beta)\), which gives mean \(\eta r\), with \(\eta > 0, \beta > 1\) as constants.

For any continuous time \(t\), define as \(L(t)\) the number of global samples produced at the server based on the information received so far from the workers. Following the prior works \([15, 21]\), we evaluate the error of a CMC algorithm by fixing a test function \(f(\cdot)\), and comparing the empirical average obtained with the produced global samples, \(\{\theta^l\}_{l=1}^{L(t)}\), available at time \(t\) with the corresponding ensemble average \(\mathbb{E}_{p(\theta|Z)}[f(\theta)]\) with respect to the true posterior distribution \(p(\theta|Z)\) in \((\text{1})\). This can be written as 

\[
\text{err}(t) = \frac{\frac{1}{T(t)} \sum_{t'=1}^{T(t)} f(\theta^l) - \mathbb{E}_{p(\theta|Z)}[f(\theta)]}{\mathbb{E}_{p(\theta|Z)}[f(\theta)]}, \tag{3}
\]

III. CMC WITH STRAGGLERS

In this section, we describe the standard CMC protocol in the context of the system under study with random computing times. The purpose of this novel formulation is to study the effect of stragglers in the CMC protocol \([15]\). Following \([15]\), we focus here on the standard case with no computing redundancy, i.e., \(r = 1\), and we let each data shard, \(Z_s\), be allocated only to worker \(k = s\). Throughout this section, we accordingly simplify the notation by writing \(\theta^l_k\) for the sample \(\theta^l_{k,k}\) generated at worker \(k\) for the \(k\)-th shard \(Z_k\). Note also that we have \(\Theta^l_k = \{\theta^l_{k,k}\} = \{\theta^l_k\}\).

Each worker \(k\) communicates a generated sample \(\theta^l_k\) to the server as soon as it is produced, where index \(l \in \{1, 2, \cdots\}\) runs over the samples. Given the model described in Section II we denote as \(L_k(t)\) the number of samples \(\{\theta^l_k\}_{l=1}^{L_k(t)}\) received by the server up to time \(t\) from worker \(k\), which is given by

\[
L_k(t) = \max \left\{ l : \sum_{l'=1}^{l} \Delta T^{l'}_k \leq t \right\}. \tag{4}
\]

As soon as all the \(K\) local samples \(\{\theta^l_k\}_{k=1}^K\) are received, a global sample \(\theta^l\) can be computed at the server by aggregating the corresponding local samples \(\{\theta^l_k\}_{k=1}^K\). Therefore, at time \(t\), the number of global samples aggregated at the server is
$L(t) = \min_{k \in [K]} L_k(t)$. An illustration of the computing times of each worker and the server are shown in Appendix A with $K = 3$ workers.

CMC makes the working assumption that the local samples $\theta_k^l$ are Gaussian $\mathcal{N}(\mu_k, C_k)$ with mean $\mu_k$ and covariance $C_k$. Under this condition, which is practically and approximately valid only as $N/K \to \infty$, the optimal aggregation function is

$$\theta^t = \sum_{k=1}^{K} W_k \theta_k^l,$$

with weight matrices

$$W_k = \left( \sum_{k=1}^{K} C_k^{-1} \right)^{-1} C_k^{-1}. \quad (6)$$

The weight matrices in (6) are not directly computable, since the parameters \{\mu_k, C_k\}_{k=1}^{K} are unknown. However, at time $t$, the server can estimate the mean $\mu_k(t)$ and covariance $C_k(t)$ of the subposterior $p(\theta|Z_k)$ by using the $L_k(t)$ samples $\{\theta_k^l\}_{l=1}^{L_k(t)}$ received from the worker $k$ up to time $t$ as

$$\hat{\mu}_k(t) = \frac{1}{L_k(t)} \sum_{t=1}^{L_k(t)} \theta_k^l, \quad (7a)$$

$$\hat{C}_k(t) = \sigma^2 I + \frac{1}{L_k(t)} \sum_{t=1}^{L_k(t)} (\theta_k^l - \hat{\mu}_k(t)) (\theta_k^l - \hat{\mu}_k(t))^T, \quad (7b)$$

respectively, where $\sigma^2$ is a regularization parameter. Using these estimates CMC approximates the weight matrices in (6) and obtain an estimate of the global sample $\theta^l$, for each $l \in [L(t)]$, at time $t$, using (5), as

$$\theta^l(t) = \sum_{k=1}^{K} \left( \sum_{k=1}^{K} (\hat{C}_k(t))^{-1} \right)^{-1} (\hat{C}_k(t))^{-1} \theta_k^l. \quad (8)$$

At any time $T^l$ at which the server has received a new set of local samples $\{\theta_k^l\}_{k \in [K]}$, the server computes all global samples $\{\theta^l(T^l)\}_{t=1}^{T^l}$ using (8), with the updated estimates of covariance matrices, $\{\hat{C}_k(T^l)\}_{k \in [K]}$, calculated using (7b). CMC is summarized in Algorithm 1 and Appendix B.

IV. GROUP-BASED CMC (G-CMC)

In this section, we propose a protocol named $G$-$CMC$ that aims at leveraging the redundancy in data allocation to mitigate the effect of stragglers on the performance of CMC. The approach clusters all the workers into groups, similar to [7], [20], [22], [23], and allocates the same set of $r$ shards to all the workers in a group. In this way, in order to generate the $l$-th global sample $\theta^l$, the server must only wait to receive one batch of $r$ local samples from the fastest worker in each group.

To elaborate, G-CMC partitions the set of $K$ shards, $\{Z_s\}_{s \in [K]}$, into $G$ disjoint groups $\{Z_g\}_{g \in [G]}$ each having $r$ shards, and the set of workers, $[K]$, into $G$ disjoint groups $\{K_g\}_{g \in [G]}$ each having $r$ workers. Accordingly, we have $K = Gr$. For any $g \in [G]$, the group of shards $Z_g$ is allocated exclusively to all the workers in group $K_g$, i.e., $Z_{Sk} = Z_g$ for all $k \in K_g$. Therefore, each data shard is available at $r$ workers, and each worker has access to exactly $r$ shards.

The main idea underlying G-CMC is to treat each group as a “super-worker”, and apply the CMC protocol presented in Section III across the $G$ “super-workers”. The $l$-th batch of $r$ local samples received from the $g$-th group $K_g$ is $\Theta^{l}_{K_g} = \ldots$
To this end, the server estimates the mean $\hat{L}$ from all the $\{\Theta^l_{K_g}\}_{l \in [L_{K_g}(t)]}$ obtained during the $t$th time instant. The $L_{K_g}(t)$ batches of samples $\{\Theta^l_{K_g}\}_{l \in [L_{K_g}(t)]}$ are received at the server from the group $K_g$ at time $t$. Here, $t$ is the $l$th order statistic, i.e., the $l$th smallest value of the variables $\{\{\sum_{i=1}^{l} \Delta T^g_k\}_{i \in [L_k(t)]}\}_{k \in K_g}$.

As soon as all the $l$th batches of samples, $\{\Theta^l_{K_g}\}_{g \in [G]}$, are received at the server, we assume that workers share common randomness, i.e., common random seeds, so that two workers $l, s : s \in {\mathcal{S}}$, are equal to $\hat{\mu}_s(t)$, and covariance $C_s(t)$ of the subposterior $\tilde{p}(\theta | Z_s)$, for $Z_s \in Z_g$, by using the $L_{K_g}(t)$ samples $\{\theta^l_{K_g,s}\}_{l \in [L_{K_g}(t)]}$ received from the group $K_g$ up to time $t$ as

$$\hat{\mu}_s(t) = \frac{1}{L_{K_g}(t)} \sum_{l=1}^{L_{K_g}(t)} \theta^l_{K_g,s}, \quad \text{and}$$

$$\hat{C}_s(t) = \sigma^2 I + \frac{1}{L_{K_g}(t)} \sum_{l=1}^{L_{K_g}(t)} (\theta^l_{K_g,s} - \hat{\mu}_s(t))(\theta^l_{K_g,s} - \hat{\mu}_s(t))^T,$$

respectively, where $\sigma^2$ is a regularization parameter. Using these estimates, the weight matrices in (6) are approximated to obtain the global sample $\theta^l$, for each $l \in [L(t)]$, at time $t$, using (8), as

$$\theta^l(t) = \left( \sum_{s=1}^{K_g} (\hat{C}_s(t))^{-1} \right)^{-1} (\hat{C}_s(t))^{-1} \theta^l_{K_g,s}. \quad (10)$$

At any time $T^l$ at which the server has received a new set of batches of samples $\{\Theta^l_{K_g}\}_{g \in [G]}$, the server computes all global samples $\{\theta^l(T^4)\}_{l=1}^{l}$ using (10) with the updated estimates of covariance matrices, $\{\hat{C}_s(T^4)\}_{s \in [K]}$, calculated using (9b).

G-CMC is summarized in Algorithm 2.

V. CODED CMC (C-CMC)

In this section, we introduce C-CMC. To start, we fix a $K \times K$ encoding matrix $B$ and a $F \times K$ decoding matrix $A$ that define a gradient coding scheme robust to $r - 1$ stragglers, with $F = \binom{K}{K-r+1}$. Accordingly, matrices $A$ and $B$ satisfy the equality $AB = 1$, where $1$ is the $F \times K$ all-1 matrix. The shards are allocated to the workers according to the non-zero entries of the encoding matrix $B$, i.e., a shard $Z_s$ is allocated to the worker $k$ if the $(k, s)$-th entry of $B$ is not equal to zero.

The row weight and column weight of $B$ are all equal to $r$, accounting for the facts that each shard is available to $r$ workers and that each worker has access to exactly $r$ shards.

We assume that workers share common randomness, i.e., common random seeds, so that two workers $k$ and $k'$ assigned the same shard $Z_s$ produce the same $l$-th sample $\theta^l_{k,s} = \theta^l_{k',s}$. Accordingly, we henceforth write $\theta^l_{k,s} = \theta^l_{k',s} = \theta^l_s$. Note that common randomness is a requirement for C-CMC and not for G-CMC, which assumes the independence of the samples $\{\theta^l_{k,s}\}_{k : s \in {\mathcal{S}}}$ produced by all workers that are assigned the same shard $Z_s$ (see Section IV).

Worker $k$ estimates the mean $\mu_k(s)$ and covariance $C_k(s)$ of the subposterior $\tilde{p}(\theta | Z_s)$, with $s \in {\mathcal{S}}$, by using $L_k(t)$
batches of samples computed by it up to time \( t \) as

\[
\hat{\mu}_{k,s}(t) = \frac{1}{L_k(t)} \sum_{t'=1}^{L_k(t)} \theta_{s,t}', \quad \text{and}
\]

\[
\hat{C}_{k,s}(t) = \sigma^2 I + \frac{1}{L_k(t)} \sum_{t'=1}^{L_k(t)} (\theta_{s,t}' - \hat{\mu}_{k,s}(t))(\theta_{s,t}' - \hat{\mu}_{k,s}(t))^T
\]

respectively, where \( \sigma^2 \) is a regularization parameter. This is in contrast to CMC and G-CMC, where the mean vector and covariance matrix of the subposterior are estimated at the server.

At time \( t = T_k^l = \sum_{t=1}^{l} \Delta T_k^l \), worker \( k \) computes the \( l \)-th batch of samples \( \Theta^l_k = \{ \theta_{s,t}' \}_{s \in [S_k]} \). Then, it updates the covariance matrices \( \hat{C}_{k,s}(t) \) using [11] for all \( s \in S_k \). Given the assumption of common random seeds described above, the estimates \( \hat{C}_{k,s}(t) \) and \( \hat{C}_{k',s}(t') \) evaluated at any two workers \( k \) and \( k' \), with \( s \in S_k \), \( s \in S_{k'} \) and \( L_k(t) = L_{k'}(t') \), are equal.

Let \( b_k \) be the \( 1 \times K \) row vector of the encoding matrix \( B \) corresponding to the worker \( k \). Let \( \Theta^l_k = \{ \theta_{s,t}' \}_{s \in [r], \ t} \), with \( s_i \in S_k \) for each \( i \in [r] \), be the \( l \)-th batch of \( r \) samples computed at the worker \( k \). Each sample \( \theta_{s,t}' \) is pre-processed as \( (\hat{C}_{k,s}(T_k^l))^{-1} \theta_{s,t}' \) and the resulting processed samples are encoded using the encoding vector \( b_k \) as

\[
\tilde{\theta}_k^l = [(\hat{C}_{k,s_1}(T_k^l))^{-1} \theta_{s_1,t}' \cdots \cdots (\hat{C}_{k,s_r}(T_k^l))^{-1} \theta_{s_r,t}']^T (b_k)^T,
\]

with \( \tilde{b}_k \) being the \( 1 \times r \) row vector given as \([b_k(s_1) \ b_k(s_2) \cdots b_k(s_r)]\), where \( b_k(s_i) \) is the \( s_i \)-th element of \( b_k \).

The server waits until it receives transmissions corresponding to \( l \)-th samples from at least \( K - r + 1 \) workers, and proceeds to decoding to finally compute the \( l \)-th global sample \( \theta^l \). This occurs at time \( T^l \) equal to the \( (K - r + 1) \)-th order statistic, i.e., the \( (K - r + 1) \)-th smallest value, of the variables \( \{ T^l_k \}_{k \in [K]} \). Let the set of \( K - r + 1 \) non-stragglers for the \( l \)-th sample be indexed by an integer \( j \in [1 : K - r + 1] \) and \( a_j \) be the corresponding \( 1 \times K \) row vector of the decoding matrix \( A \). Let \( K_j^l = \{ k_1^l, k_2^l, \cdots, k_{K-r+1}^l \} \subseteq [K] \) be the corresponding subset of \( K - r + 1 \) non-stragglers. The server decodes the sum of the processed \( l \)-th samples, by using the transmissions from the workers in \( K_j^l \), as

\[
[\tilde{\theta}_{k_1^l} \tilde{\theta}_{k_2^l} \cdots \tilde{\theta}_{k_{K-r+1}^l}]^T (a_j)^T = \sum_{s=1}^{K} (\hat{C}_s^l)^{-1} \theta_{s,t}' = \phi^l.
\]

with \( \tilde{a}_j \) being the \( 1 \times (K - r + 1) \) row vector given by \([a_j(k_1^l) a_j(k_2^l) \cdots a_j(k_{K-r+1}^l)]\), where \( a_j(k_i^l) \) is the \( k_i^l \)-th element of the \( 1 \times K \) row vector \( a_j \) for any \( i \in [K - r + 1] \). The matrix \( \hat{C}_s^l \) is the empirical covariance matrix of the subposterior \( \hat{p}(\theta|Z_s) \) computed using the first \( l \) samples, computed at any worker \( k \) in \( K_j^l \), and is equal to \( \hat{C}_{k_i^l,s}(T_{k_i^l}^l) \) for any \( i \in [K - r + 1] \).

In order to compute the \( l \)-th global sample \( \theta^l \) in [8] the server has to pre-multiply the decoded sample \( \phi^l \) in [13] with the matrix \( (\sum_{s=1}^{K} (\hat{C}_s^l)^{-1})^{-1} \). We propose that the server estimates the matrix \( \sum_{s=1}^{K} (\hat{C}_s^l)^{-1} \) as the empirical covariance of the \( l \) decoded samples \( \phi^l \) for \( l' \in [l] \), i.e., as the matrix

\[
\hat{D}^l = \frac{1}{l} \sum_{l'=1}^{l} (\phi^l - \bar{\phi})(\phi^l - \bar{\phi})^T \approx \sum_{s=1}^{K} C_s^{-1}
\]

with \( \bar{\phi} = \frac{1}{l} \sum_{l'=1}^{l} \phi^l \). The approximate equality in [14] can be seen to be exact when \( l \to \infty \). The rationale for the proposed estimate [14] is provided in Appendix C. Accordingly the final global sample is \( \theta^l = (\sigma^2 I + \hat{D}^l)^{-1}\phi^l \), where \( \sigma^2 \) is a
VI. EXPERIMENTS

In this section, we evaluate the performance of the considered CMC schemes in the presence of stragglers. We present two experiments on distributed computing systems with $K = 5$ and $K = 20$ workers. In both experiments, we assume that the subposterior $\tilde{p}(\theta|Z_s)$ for each shard $Z_s$ where $s \in [K]$ to be a $5$-dimensional multivariate Gaussian distribution $\mathcal{N}(0, C_s)$ with a symmetric Toeplitz covariance matrix $C_s$ using first column $[1 \ \rho_s \ \rho^2_s \ \rho^3_s \ \rho^4_s]^T$ with $\rho_s = (s - 1)/K$ [21]. As in [16], [21], we calculate the error in (3) for multiple test functions, with each test function $f_{i,j}(\theta) = \theta[i] \theta[j]$ being an element in the outer product matrix $\theta\theta^T$ for $i,j \in [d]$, and average them to obtain the final error. We consider Pareto distribution with $\eta = 0.1$ and shape parameter $\beta = 1.2$, which corresponds to scale parameter $r/60$ and mean $0.1r$ (see Section II), for the computing time $\Delta T^i_k$ at the workers (see, e.g., [7]).

In Fig. 2 and Fig. 3, we plot the test error averaged over 50 random realizations of computing times, as a function of time, at $K = 5$ and $K = 20$ workers respectively, following a Pareto distribution with $\eta = 0.1$ and shape parameter $\beta = 1.2$, which corresponds to scale parameter $r/60$, for the Pareto distribution with mean $0.1r$. The curve represents the average error, and shaded region represents the error bars corresponding to $0.15$ times the standards deviation of the error across the realizations.

We plot the curves for CMC, C-CMC and G-CMC. For $K = 5$, as $K/r$ is not an integer, G-CMC is applied to three groups with one of the groups having a single worker. Note that C-CMC can be directly applied as there is no such restriction on $r$ in C-CMC. From Fig. 2 and Fig. 3 we can conclude that both G-CMC and C-CMC can effectively mitigate stragglers, with G-CMC being more efficient for larger values of $K$.

![Fig. 2. Average test error versus time with $K = 5$ workers.](image)
VII. CONCLUSIONS

In this letter, we have considered, for the first time, the problem of stragglers in Consensus Monte Carlo (CMC) for distributed Bayesian learning. We studied the effect of stragglers in the standard CMC protocol, and proposed two schemes, Group-based CMC (G-CMC) and Coded CMC (C-CMC) that effectively leverage the redundancy in data allocation.

APPENDIX A

AN ILLUSTRATION OF THE COMPUTING TIMES

Fig. 4 describes the computing times of local samples at 3 workers and generation of global samples at the server. For instance, the first local samples $\theta_1^1, \theta_2^1, \theta_3^1$ are generated at time instants $\Delta T_1^1, \Delta T_2^1, \Delta T_3^1$ at the workers 1, 2, 3 respectively. The first global sample $\theta^1$ is generated at the server at the time instant $T^1 = \max_k \{\Delta T_k^1\}$.
**Algorithm 1** CMC with straggling workers

1: **Input:** Number of workers $K$, data shards $\{Z_k\}_{k=1}^K$
2: **Data allocation:**
3: for each $k \in [K]$ do
4: Allocate shard $Z_k$ to worker $k$
5: end for
6: At worker $k$: set $l = 1$
7: repeat
8: compute $l$-th sample $\theta^l_k \sim \tilde{p}(\theta | Z_k)$
9: when completed, i.e., at time $\sum_{l'=1}^l \Delta T^l_{k'}$, send sample $\theta^l_k$ to the server
10: end repeat
11: At the server: set $l = 1$
12: repeat
13: when receiving all $l$-th local samples $\{\theta^l_k\}_{k \in [K]}$, i.e., at time $T^l = \max_{k \in [K]} (\sum_{l'=1}^l \Delta T^l_{k'})$, compute the covariance matrices $\{\hat{C}_k(T^l)\}_{k \in [K]}$ using (7),
14: for each $l' \in [l]$ compute $\theta^{l'}(T^l)$ using (8) with covariance matrices $\{\hat{C}_s(T^l)\}_{s \in [K]}$
15: end repeat

**Algorithm 2** Group-based CMC (G-CMC)

1: **Input:** Partition the set of workers $[K]$ into $G$ groups $\{K_g\}_{g \in [G]}$ each having $r$ workers, Partition the set of shards $\{Z_s\}_{s=1}^S$ into $G$ groups $\{Z_g\}_{g \in [G]}$ each having $r$ shards, where $r = \frac{K}{G}$ is the redundancy parameter
2: **Data allocation:**
3: for each $g \in [G]$ do
4: allocate all $r$ shards in the group $Z_g$ to all $r$ workers in group $K_g$, i.e., $Z_{S_k} = Z_g$ for all $k \in K_g$
5: end for
6: At the group of workers $K_g$: set $l = 1$
7: repeat
8: $l$-th batch of $r$ local samples $\Theta^l_{K_g} = \{\theta^l_{k,s} \sim \tilde{p}(\theta | Z_s)\}_{Z_s \in Z_g}$ is computed at any of the worker in the group $K_g$
9: when completed, i.e., at time $T^l_g$, send $\Theta^l_{K_g}$ to the server, where $T^l_g$ denote the $l$-th order statistic of the variables $\{\sum_{l'=1}^l \Delta T^l_{k'}\}_{t \in [L_s(t)]}$ for all $k \in K_g$
10: end repeat
11: At the server: set $l = 1$
12: repeat
13: when receiving all the $l$-th batches of samples $\{\Theta^l_{K_g}\}_{g \in [G]}$, at time $T^l = \max_{g \in [G]} T^l_g$, compute the covariance matrices $\{\hat{C}_s(T^l)\}_{s \in [K]}$ using (9),
14: for each $l' \in [l]$ compute $\theta^{l'}(T^l)$ using (8) with covariance matrices $\{\hat{C}_s(T^l)\}_{s \in [K]}$
15: end repeat
Algorithm 3 Coded CMC (C-CMC)

1: **Input:** Number of workers $K$, Maximum number of stragglers $r − 1$, Data shards $\{Z_s\}_{s=1}^K$
2: Consider two matrices $A, B$ such that it forms a GC scheme robust to $r − 1$ stragglers \[6\]
3: **Data allocation:**
4: for each $k, s \in [K]$ do
5: if $B(k,s) \neq 0$ then
6: Allocate shard $Z_s$ to worker $k$
7: end if
8: end for
9: **At worker $k$:** set $l = 1$
10: **repeat**
11: computes the $l$-th batch of samples $\Theta^l_s = \{\Theta^l_s\}_{s \in [S_a]}$
12: when completed, i.e., at time $T^l_k = \sum_{l'=1}^l \Delta T^l_k$, worker $k$ estimates the covariance matrices $\{\hat{C}^l_s\}_{s \in S_a}$ of the subposters $\{\hat{p}(\theta|Z_s)\}_{s \in S_a}$ using the $l$ batches of samples $\{\Theta^l_s\}_{v \in [l]}$ computed at the worker $k$.
13: worker $k$ transmits $\hat{\theta}^l_k$ using (12)
14: end repeat
15: **At the server:** set $l = 1$
16: **repeat**
17: using the transmissions from $K-r+1$ non-stragglers, compute $\sum_{s=1}^K (\hat{C}^l_s)^{-1}\theta^l_s$ using (13) at time $T^l$ being equal to $(K-r+1)$-th order statistic of the variables $\{T^l_k\}_{k \in [K]}$
18: Estimate $\sum_{s=1}^K (\hat{C}^l_s)^{-1}$ using (14) and compute global sample $\hat{\theta}^l$ using (8).
19: end repeat

**APPENDIX C**

**DERIVATION OF THE APPROXIMATION IN (14)**

In this Appendix, we detail the approximation in (14). To this end, we write the empirical covariance of the $l$ decoded samples $\phi^l_{l'}$, for $l' \in [l]$, as

\[
\hat{D}^l = \frac{1}{l} \sum_{l'=1}^l (\phi^l_{l'} - \bar{\phi})(\phi^l_{l'} - \bar{\phi})^T = \frac{1}{l} \sum_{l'=1}^l (\phi^l_{l'} (\phi^l_{l'}^T) - \bar{\phi}(\bar{\phi})^T \\
= \frac{1}{l} \sum_{l'=1}^l \left[ \left( \sum_{s=1}^K (\hat{C}^l_s)^{-1}\theta^l_s \right) \left( \sum_{m=1}^K (\theta^l_m)^T C_m^{-1} \right) \right] - \frac{1}{l} \sum_{l'=1}^l \sum_{s=m=1}^K (\hat{C}^l_s)^{-1}\theta^l_s \left( \frac{1}{l} \sum_{l'=1}^l \sum_{m=1}^K (\theta^l_m)^T C_m^{-1} \right) \\
\approx \frac{1}{l} \sum_{l'=1}^l \left[ \left( \sum_{s=1}^K C_s^{-1}\theta^l_s \right) \left( \sum_{m=1}^K (\theta^l_m)^T C_m^{-1} \right) \right] - \frac{1}{l} \sum_{l'=1}^l \sum_{s=m=1}^K C_s^{-1}\theta^l_s \left( \frac{1}{l} \sum_{l'=1}^l \sum_{m=1}^K (\theta^l_m)^T C_m^{-1} \right) \\
\]

(15)

where, the approximation in (a) is justified by the fact that, as $l \to \infty$, the covariance matrix $\hat{C}^l_s$ converges to the ground-truth covariance matrix $C_s$. Letting $\mu^s_s = \frac{1}{l} \sum_{l'=1}^l \theta^l_{s,l'}$ be the mean of first $l$ samples of $\bar{p}(\theta|Z_s)$, we can write $\hat{D}^l$ as

\[
\hat{D}^l \approx \frac{1}{l} \sum_{l'=1}^l \left[ \left( \sum_{s=1}^K C_s^{-1}\theta^l_s \right) \left( \sum_{m=1}^K (\theta^l_m)^T C_m^{-1} \right) \right] - \frac{1}{l} \sum_{l'=1}^l \left[ \left( \sum_{s=1}^K C_s^{-1}\mu^s_s \right) \left( \sum_{m=1}^K (\mu^s_m)^T C_m^{-1} \right) \right] \\
= \frac{1}{l} \sum_{l'=1}^K \sum_{s=1}^K \left[ C_s^{-1}\theta^l_s (\theta^l_{s,l'})^T C_s^{-1} - C_s^{-1}\mu^s_s (\mu^s_{s,l'})^T C_s^{-1} \right] + \frac{1}{l} \sum_{l'=1}^K \sum_{s,m=1}^K \left[ C_s^{-1}\theta^l_s (\theta^l_{s,l'})^T C_s^{-1} - C_s^{-1}\mu^s_s (\mu^s_{s,l'})^T C_s^{-1} \right] \\
\approx \frac{\sum_{s=1}^K \sum_{s,m=1}^K \left[ C_s^{-1}\theta^l_s (\theta^l_{s,l'})^T - \mu^s_s (\mu^s_{s,l'})^T \right] + \sum_{s,m=1}^K \sum_{s \neq m} \left[ C_s^{-1}\theta^l_s (\theta^l_{s,l'})^T - \mu^s_s (\mu^s_{s,l'})^T \right] C_s^{-1} \]
\begin{align*}
K \sum_{s=1}^{K} C^{-1}_s C^{-1}_s + K \sum_{s, m=1 \atop s \neq m}^{K} C^{-1}_s (0) C^{-1}_m = K \sum_{s=1}^{K} C^{-1}_s,
\end{align*}

where the approximation (b) is exact as \( l \to \infty \).

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