The origin and effect of hemispheric helicity imbalance in solar dynamo

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(Received xx; revised xx; accepted xx)

In this paper we study the effects of hemispheric imbalance of magnetic helicity density on breaking the equatorial reflection symmetry of the dynamo generated large-scale magnetic field. Our study employs the axisymmetric dynamo model which takes into account the nonlinear effect of magnetic helicity conservation. We find that the evolution of the net magnetic helicity density, in other words, the magnetic helicity imbalance, on the surface follows the evolution of the parity of the large-scale magnetic field. Random fluctuations of the $\alpha$-effect and the helicity fluxes can inverse the causal relationship, i.e., the magnetic helicity imbalance or the imbalance of magnetic helicity fluxes can drive the magnetic parity breaking. We also found that evolution of the net magnetic helicity of the small-scale fields follows the evolution of the net magnetic helicity of the large-scale fields with some time lag. We interpret this as an effect of the difference of the magnetic helicity fluxes out of the Sun from the large and small scales.

1. Introduction

The reflection asymmetry of the solar magnetic activity about equator is one of the most important property of the solar dynamo. The magnetic fields of the leading and following sunspots’ groups of the solar bipolar regions have predominantly opposite polarities in each hemisphere. This is the so-called Hale polarity rule. The similar asymmetry exist for the polar magnetic fields, which is the most prominent during the sunspot minima. After [Parker (1955)], it is commonly accepted that the reflection properties of the large-scale magnetic field are determined by the dynamo mechanism operating inside the Sun. The essential parts of the large-scale dynamo are govern by the differential rotation and the turbulent convective motions. In the convective zone of a star the global rotation makes turbulent convective motions helical. This results in the reflection asymmetry of the convective vortices about equator and produces the dynamo generation $\alpha$-effect, which transforms the global toroidal magnetic field to the poloidal [Parker (1955), Krause & Rädler (1980)]. The reflection hemispheric asymmetry of the $\alpha$-effect results to the hemispheric asymmetry of the helical properties of solar magnetic field. This phenomena is called the hemispheric helicity rule (hereafter HHR) and it is observed in the number of the magnetic helicity tracers like the current helicity in the solar active regions, hirality of the solar prominences etc. The standard HHR suggests

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the negative sign of the current helicity of solar ARs in the northern hemisphere and
the positive in southern one. For the global magnetic field the opposite HHR is expected
(Blackman & Brandenburg 2003). In ideal situation, there is the hemispheric balance of
distributions of the current and magnetic helicity density.

The origin of the HHR and its impact on the dynamo is extensively discussed in the
literature (see, the recent review Brandenburg (2018)). Recently Singh et al. (2018) found
that in cycle 24 more than 20% of the vector magnetic field synoptic maps show violations
of the expected sign rule. Reversals of the sign rule of the current helicity of solar active
regions during the beginning and the end of cycles 22 and 23 was found by Zhang et al.
(2010). Similar reversals was found at the end of cycle 24 for the magnetic helicity density
by Pipin et al. (2019). The origin of the HHR reversals was addressed in our previous
paper using the mean-field dynamo models (Pipin et al. 2013b). The evolution of the
magnetic helicity is governed by the conservation law. It was found that the reversal of
the sign of the small-scale magnetic helicity follows the dynamo wave propagating inside
the convection zone and the spatial patterns of the magnetic helicity reversals reflect the
processes which contribute to generation and evolution of the large-scale magnetic fields.

In the paper, the HHR will be characterized by the sign distribution and the hemi-
spheric imbalance of the magnetic helicity parameters, such as the current and magnetic
helicity density. For the perfect HHR the imbalance is about zero and the sign rule
is fulfilled. One of the reason of this imbalance could be the hemispheric imbalance of
the magnetic helicity flux from the surface to the outer atmosphere. The existence of
the net helicity flux is still under debate. For example, Georgoulis et al. (2009), found
that the helicity injection through the solar photosphere associated with active region
magnetic fields was well-balanced over the solar cycle 23. On the other hand, Yang
& Zhang (2012) reported significant imbalance between helicity fluxes of northern and
southern hemispheres. Currently, it is unclear to which extent the imbalance of helicity
fluxes impact the dynamo processes inside the convection zone. It is also unclear how
the imbalance of helicity fluxes affect the hemispheric imbalance of the magnetic helicity
density. Another possible reason could be due to redistribution of the magnetic helicity
density over the spatial scale. Both effects (helicity fluxes and helicity cascades) are
governing by the complicated magnetohydrodynamic processes which can easily destroy
the equatorial symmetry from time to time and produce the net magnetic helicity of the
Sun.

In the paper we model effects of magnetic helicity imbalance using the mean-field
magneto-hydrodynamic framework. In this case it is important to distinguish magnetic
helicity of the small-scale and the large-scale (global) field of the Sun. We represent the
magnetic field \( \mathbf{B} \) and its vector potential \( \mathbf{A} \) (\( \mathbf{B} = \nabla \times \mathbf{A} \)) to the sum of the mean and fluctuating parts: \( \mathbf{B} = \bar{\mathbf{B}} + \mathbf{b}, \mathbf{A} = \bar{\mathbf{A}} + \mathbf{a} \), where the overbar denotes the mean quantities. Next, the magnetic helicity is defined as integral over the closed domain \( \mathcal{H} = \int \mathbf{A} \cdot \mathbf{B} dV \), and the \( \mathbf{A} \cdot \mathbf{B} \) is the magnetic helicity density. Assuming the validity of the Reynolds
rule for averaging of the products and sum of the turbulent quantity we can distinguish
between the contributions of the large-scale and the small-scale magnetic field to the
magnetic helicity density:

\[
\chi^{(tot)} = \bar{\mathbf{A}} \cdot \bar{\mathbf{B}} = \bar{\mathbf{A}} \cdot \bar{\mathbf{B}} + \mathbf{a} \cdot \mathbf{b}.
\]  

(1.1)

Hereafter, we denote the small-scale and large-scale parts of the magnetic helicity density
as follows, \( \chi = \mathbf{a} \cdot \mathbf{b}, \chi^{(m)} = \bar{\mathbf{A}} \cdot \bar{\mathbf{B}}. \)

Following to Hubbard & Brandenburg (2012), Pipin et al. (2013a), we employ the
conservation law for \( \chi^{(tot)} : \)
\[
\frac{d}{dt} \int \chi^{(\text{tot})} dV = -2\eta \int \{ \mathbf{B} \cdot \mathbf{j} + \mathbf{b} \cdot \mathbf{j} \} dV - \int \nabla \cdot \mathbf{F}^{\chi} dV
\]  
(1.2)

where the \( \mathbf{F}^{\chi} \) denotes the helicity flux. In above cited papers it was shown, that with this formulation of the magnetic helicity conservation the dynamo evolution avoids the catastrophic quenching regimes. The differential equation that corresponds to Eq. (1.2) is:

\[
\frac{\partial \chi^{(\text{tot})}}{\partial t} = -\frac{\chi}{R_m \tau_c} - 2\eta \mathbf{B} \cdot \mathbf{j} - \nabla \cdot \mathbf{F}^{\chi} - (\mathbf{U} \cdot \nabla) \chi^{(\text{tot})}
\]  
(1.3)

In deriving the the Eq. (1.2) we assumed \( 2\eta \mathbf{b} \cdot \mathbf{j} = \frac{\chi}{R_m \tau_c} \) (see Kleeorin & Rogachevskii (1999)), where the magnetic Reynolds number \( R_m = 10^{3-6} \) and \( \eta \) is the microscopic diffusivity. Note that conservation law given by the Eq. (1.2) take into account the possibility of the magnetic helicity fluxes out of the dynamo domain. In the stationary state we have locally:

\[
\chi \approx -\mathbf{A} \cdot \mathbf{B} = -\chi^{(m)}
\]  
(1.4)

This balance can be changed in any direction by the helicity fluxes either on the small or the large scales.

We assume that the magnetic helicity density balance is following to the Eqs. (1.3, 1.4). Clearly, there are important unknown details in the the Eq. (1.3), in particular, those are related to the helicity density fluxes. Further it will be shown that breaking of the equatorial symmetry of the global magnetic field can result in the hemispheric imbalance of the magnetic helicity density, as well. We study the mutual effect this imbalance and the magnetic parity breaking using mean-field dynamo models.

In Section 2 we describe some specific details of our dynamo model. Section 3 is devoted to description of the main results and to discussion of those results in the light of the available observational proxies. Section 4 summarizes our findings.

2. Basic equations

2.1. Dynamo model

The dynamo model follows outline and it employs the same dynamo parameters, which are given in our previous papers, Pipin et al. (2013a, b). We study the mean-field induction equation in the turbulent perfectly conducting medium:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{E} + \mathbf{U} \times \mathbf{B}),
\]  
(2.1)

where \( \mathbf{E} = \mathbf{u} \times \mathbf{b} \) is the mean electromotive force, with \( \mathbf{u}, \mathbf{b} \) being fluctuating velocity and magnetic field, respectively, \( \mathbf{U} \) is the mean velocity field, which is represented by the differential rotation. The meridional circulation is neglected. A large-scale axisymmetric magnetic field is represented by decomposition on the sum of the toroidal and poloidal parts:

\[
\mathbf{B} = \mathbf{e}_\phi B + \nabla \times \frac{A \mathbf{e}_\phi}{r \sin \theta},
\]
where $\theta$ is the polar angle. The mean electromotive force $E$ is expressed as follows:

$$E_i = \left( \alpha_{ij} + \gamma_{ij}^{(A)} \right) \overline{B}_j + \eta_{ijk} \nabla_j \overline{B}_k. \tag{2.2}$$

The tensor $\alpha_{ij}$ represents the $\alpha$-effect, $\gamma_{ij}^{(A)}$ is the turbulent pumping, and $\eta_{ijk}$ is the diffusivity tensor. The $\alpha$ effect includes hydrodynamic and magnetic helicity contributions,

$$\alpha_{ij} = C_\alpha \left( 1 + \xi_{N,S}^{(\alpha)}(t) \right) \sin^2 \theta \alpha_{ij}^{(H)} + \alpha_{ij}^{(M)}. \tag{2.3}$$

The details in expressions for the kinetic part of the $\alpha$ effect $\alpha_{ij}^{(H)}$, as well as $\gamma_{ij}^{(A)}$, $\alpha_{ij}^{(M)}$ and $\eta_{ijk}$ can be found in Pipin et al. (2012, 2013a, b). Note, that in addition to the turbulent generation of the poloidal magnetic field by the $\alpha$-effect, the model employs the generation effect induced by the large-scale current and the global rotation. It is usually called as the $\Omega \times J$- or the $\delta$ dynamo effect (Radler 1969). Following to results of Pipin & Seehafer (2009), addition of the $\delta$ dynamo effect increases the efficiency of the turbulent generation of the poloidal magnetic field. Including this effect in the model allows to bring the simulated dynamo period and the time-latitude diagrams of the near-surface toroidal magnetic field in a better agreement with observations. We use the same, slightly overcritical dynamo parameter $C_\alpha$ as in above cited papers. Similar to (Pipin & Seehafer 2009; Pipin et al. 2013b), the dynamo number of the $\delta$ effect is chosen as follows $C_\delta = C_\alpha/3$.

The bottom of the integration domain is $r_b = 0.715R_\odot$ and the top of the integration domain is $r_e = 0.99R_\odot$. We matched the dynamo solution to the potential field outside, and assume the perfect conductivity at the bottom boundary. For the magnetic helicity density we employ $\chi^{(m)} = 0$ at the bottom and $\nabla_r \chi^{(tot)} = 0$ at the top. The rest details about our models can be found in Pipin et al. (2013b).

The random terms $\xi_{N,S}(t)$ takes into account the possible fluctuations of the kinetic $\alpha$-effect in the northern and southern hemispheres. The specific details about it are given in the next subsection.

### 2.2. Random sources of the helicity density imbalance

In our model we explore a few possible sources of the helicity density imbalance. The non-symmetric fluctuations of the kinetic $\alpha$-effect, (see the Eq. (2.3)) can be one of the natural origin of it. For the deterministic problems like the dynamo equations system, the Eqs. (1.2, 2.1) which are solved by the standard numerical integration schemes, the spatial and temporal fluctuations of the model parameters are the sources of the potential numerical pitfalls because the meaning of the derivative is rather different for deterministic and the random functions. Practically, without going deep in to details, we are safe if the typical spatial and temporal scales of fluctuations are much larger when the size of the spatial mesh and the size of the time-step. The numerical scheme employs the spatial mesh with 70 nodes in the radius. We use the pseudospectral approach for the differentiation operators along latitude and the 64 nodes in latitude are located in the collocation points of the Legendre polynomial. To simulate the randomness of the $\alpha$-effect distribution over hemispheres we generate the spatially random sequences, $\xi_{N,S}^{(\alpha)}$ where N and S denote sequences of fluctuations of the $\alpha$-effect at the North and South respectively. The ensemble of the $\xi_{N,S}^{(\alpha)}$ follows the Gaussian probability distribution with zero mean and the the standard deviation $\sigma = 0.2$. The renewal time for the sequences $\xi_{N,S}^{(\alpha)}$ is also taken in form of the random sequence. From that we pick up the values larger than 1 Yr intervals, which is safe for the numerical scheme with the time step about 1 day.
Figure 1. Probability distributions: panel (a), the distribution of the renewal time; panel (b) distribution of the fluctuations of the $\alpha$-effect averaged over latitudes.

The probability distribution of the renewal time is shown in the Figure. The resulted distribution of the fluctuations of the averaged $\alpha$-effect is shown on the same Figure.

The nonlinear feedback of the large-scale magnetic field to the $\alpha$-effect is described by a dynamical quenching due to the constraint of magnetic helicity conservation given by the Eq.(1.3). It is likely that there are a number of mechanisms which can cause the hemispheric imbalance of the helicity density in the axisymmetric mean-field dynamo. In our paper we restrict consideration to the simple idea that the net helicity density at the surface could be caused by the asymmetric about equator flux of the helicity density from the dynamo domain to the corona. Following to suggestions by Guerrero et al. (2010) we model this by subtracting the fraction of the helicity density from the local helicity density in the upper parts of the convection zone. Thus, the modified equation for the helicity density evolution is

$$\frac{\partial \chi^{(\text{tot})}}{\partial t} = -\frac{\chi}{R_m \tau_c} - 2 \eta B \cdot \mathbf{J} - \nabla \cdot \mathbf{F}^\chi$$

$$- \frac{\tau\xi (r)}{T_D} \left( \xi^{(N,S)} (t) \bar{\chi} + \xi^{(m)}_{(N,S)} (t) \bar{\chi} \right) ,$$

where, $\mathbf{F}^\chi = -\eta \chi \nabla \bar{\chi}$, with $\eta = 0.1 \eta^{(p)}$. Similarly to Pipin et al. (2013b), we employ $R_m = 10^6$. The last term in the Eq.(2.4) takes into account the helicity density flux out of the solar convection zone. It is assumed that this flux is working mainly in the upper part of the solar convection zone. To take this into account we introduce the function:

$$\tau\xi (r) = \frac{1}{2} \left[ 1 + \text{erf} \left( 50 (r_0 - r) \right) \right] ,$$

where $r_0 = 0.9 R_\odot$ and the dimension factor $T_D = \frac{R_\odot^2}{\eta^{(0)}}$.

The sequence of the renewal times of the helicity density outflows will be determined in the same way as for the $\alpha$-effect except for the low limit which is put about ten times smaller and it is equal to one month. The helicity density flux out hemisphere is computed as integral over the shell which includes subsurface region between $r_0$ and $R_\odot$. The $\xi^{(N,S)}$ and $\xi^{(m)}_{(N,S)}$ are defined in the same way as $\xi^{(\alpha)}_{(N,S)}$ and we use the ensembles of the spatial fluctuations with the Gaussian probability function distribution, the mean value of $\xi^{(N,S)} = 1$ and the standard deviation of $\sigma \left( \xi^{(N,S)} \right) = 1$. These fluctuations are driven with the random renewal time interval, which has the same probability distribution as the $\alpha$ effect fluctuations.
3. Observational proxies of the magnetic helicity imbalance

Following to Introduction the hemispheric helicity rule is characterized by the surface integral of the magnetic helicity proxies. Figure 2(a) shows the integral of the current helicity density of the solar active regions obtained from the reduced data set of Huairou
Table 1. Parameters of the model runs. In the models D2, D3 and D4 we neglect the hemispheric asymmetry of the \( \alpha \)-effect fluctuations. In this case the fluctuating part of the \( \alpha \)-effect is equal to \( \xi_{N,S}(t) \) where we use average over latitudes, see the Eq. (2.3).

| Model | \( \xi_{N,S}^{(\alpha)} \) | \( \xi_{N,S}^{(\chi)} \) | \( \xi_{N,S}^{(m)} \) |
|-------|-----------------|-----------------|-----------------|
| D1    | yes             | no              | no              |
| D2    | \( \xi_{N,S}^{(\alpha)} \) | no              | yes             |
| D3    | \( \xi_{N,S}^{(\alpha)} \) | yes             | no              |
| D4    | \( \xi_{N,S}^{(\alpha)} \) | yes             | yes             |

Solar Observing Station given in Zhang et al. (2010),

\[
\delta h_C = \int_0^1 h_C d\mu \tag{3.1}
\]

where \( \mu = \cos \theta \) and \( \theta \) is the polar angle, and the same for the magnetic helicity density of the global magnetic field which was reconstructed by Pipin and Pevtsov (2014) using the SOHO/MDI data set,

\[
\delta \chi^{(m)} = \int_0^1 \chi^{(m)} d\mu \tag{3.2}
\]

We observe the solar cycle variations of the HHR parameters in both cases. The low cadence of data set of \( \delta h_C \) and the limited time interval in both data sets result into uncertainty in our conclusions about the long-term behaviour of these parameters. To get a rough idea we apply the empirical mode decomposition (EMD). Because of the mentioned issue of our data sets, our analysis is rather rough and it can be subjected to systematic aliasing errors. We show results on the same Figure 2(a). The information about the contribution of the empirical modes to the energy of the signal is illustrated by Figure 2(b). In the signal of the \( \delta h_C \), the "small-scale" modes of short periods 1-3 year are the strongest. Their effect on the whole \( \delta h_C \) is rather strong over the sunspot minima. The first three modes of \( \delta h_C \) show the variation with the solar cycle period. The large-scale magnetic helicity density imbalance has a strong signal with period of about 9 years and the first three modes quite accurately reproduce the total signal. The sum of the first three modes of current helicity density imbalance \( \sum_0^2 \delta h_{C_i} \) has a similar period. It is seen that the \( \sum_0^2 \delta h_{C_i} \) goes ahead of the \( \sum_0^2 \delta \chi^{(m)} \) with the phase shift about \( \pi \). This rough analysis show a possibility of the quasi-regular variations of \( \delta h_C \) and \( \delta \chi^{(m)} \) in the dynamo cycle. We shall see if this can be reproduced in our dynamo models.

4. Results
The simulations parameters in the model are shown in Table 1. In all the runs we consider a slightly overcritical dynamo regimes using the same dynamo parameters set as in the our previous papers (Pipin et al. 2013a,b). Similar to those papers our models are weakly nonlinear with \( \beta_{\text{max}} = |B| / \sqrt{4\pi \rho u'^2} < 0.3 \).  

For the purpose of analysis it will be useful to introduce the parity index

\[
E_B^S = \frac{1}{4} \int_{-1}^1 \left[ B(\mu, t) + B(-\mu, t) \right]^2 d\mu,
\]
Figure 3. a) The time-latitude diagram of the toroidal magnetic field in the upper part of the solar convection zone and the radial magnetic field at the surface (contours); b) the time-latitude evolution of the small-scale magnetic helicity density and the toroidal magnetic field (contours) for the “active” Sun

\[ E_N^B = \frac{1}{4} \int_{-1}^{1} [\mathcal{B}(\mu,t) - \mathcal{B}(-\mu,t)]^2 d\mu, \]

where \( \mathcal{B}(\mu,t) = \langle B(r,\mu,t) \rangle_{r=(0.9:1)}R \) is the mean strength of the toroidal large-scale magnetic field in the upper part of the solar convection zone. Then the parity index, or the reflection symmetry index of the toroidal magnetic field is

\[ P = \frac{E_S^B - E_N^B}{E_S^B + E_N^B}. \]  

The Figures 3 and 4 illustrate the time-latitude diagram of the toroidal magnetic field in the upper part of the solar convection zone, the radial magnetic field at the surface, the time-latitude evolution of the small-scale magnetic helicity density for the periods of the active and quiet magnetic activity in the model D4. Evolution patterns of the magnetic field and HHR for the large- and small-scale magnetic helicity density are qualitatively similar during these periods. The results for the “active” periods show the same pattern as in Pipin et al. (2013b). During the “quiet period we see penetrations of the dynamo wave patterns from one hemisphere to another. The same effect is seen in magnetic helicity evolution. Figure 5 shows the simulated value of the sunspot number

\[ W = \bar{B}_{\text{max}}(t) \exp \left( -\frac{\bar{B}_{\text{max}}(t)}{B_0} \right), \]

where \( B_0 = 600 \text{G} \) and \( \bar{B}_{\text{max}}(t) = \max_{\mu=0:\pi} \mathcal{B}(\mu,t) \), the parity index, \( P \) and the net magnetic helicity density at the surface for the small-scales and the large-scale magnetic fields, \( \delta\chi \) and \( \delta\chi^{(m)} \), respectively, for the model D4. Results for the other models look qualitatively similar.

It is seen that during the epochs of the centennial magnetic minima the distributions of the magnetic field and the magnetic helicity density is not symmetric about equator and the parity index is greater than -1 during the most part of the cycle, oscillating around \([-0.8:-0.6]\). The increase of the parity index of magnetic fields results to increase of the oscillation magnitude of the imbalances \( \delta\chi \) and \( \delta\chi^{(m)} \). The effect is further illustrated by Figure 6 (top), where we show results for \( \langle \delta\chi(t)\delta\chi^{(m)}(t+\tau) \rangle \). We see that in the model D1 they vary in anti-phase both on the time scale of the magnetic cycle and on the centennial time scales. This is because of the perfect magnetic helicity density conservation which is resulted from the Eq. (2.4) in the absence of the magnetic helicity density fluxes.
Figure 4. The same as Figure 3 for the “quiet” Sun.

Figure 5. a) The simulated value of the sunspot number; (b) the parity index; c) the net magnetic helicity density at the surface
Figure 6. Top, the cross-correlations \( \langle \delta \mathcal{X}(t) \delta \mathcal{X}^{(m)}(t + \tau) \rangle \) for the dynamo models and the correlations of the first three empirical modes of \( h_C \) and \( \mathcal{X}^{(m)} \) from observations, the curve M2, and the M3 is the same for the forth empirical modes of \( h_C \) and \( \mathcal{X}^{(m)} \) which are shown in Figure 2a; Middle, the phase diagram for phases of the analytical signals of the \( \delta \mathcal{X}(t) \) and \( \delta \mathcal{X}^{(m)}(t) \) in the models D1 and D4; Bottom, the parity index in the model D4 vs the difference of the phases of the analytical signals of the \( \delta \mathcal{X}(t) \) and \( \delta \mathcal{X}^{(m)}(t) \).
We see that in all the models there is an anti-correlation between the $\delta \chi(t)$ and $\delta \chi(m)(t)$. The effect of the anti-phase synchronization is largest in case D1. The same effect exists in the data as well, in particular, when we restrict ourselves with the first three empirical modes of $h_C$ and $\chi(m)$ (the curve M2). This conclusion is not robust because the correlation changes sign to positive after adding the forth empirical mode (see the curve M3). Also the amount of the observational data is not enough for the robust conclusion. The anti-phase synchronization in the models is further illustrated by the phases of the analytical signals of $\delta \chi(t)$ and $\delta \chi(m)(t)$, which are computed using the Hilbert transform, and denoted as $\Phi(\chi)$ and $\Phi(\chi(m))$, respectively. In the model D1 the synchronization persists on the longer time intervals than in the model D4. This is reflected in the clustering of the points in the phase diagram to the two bands. The effect is less for the model D4. The relation of the synchronization between the $\delta \chi(t)$ and $\delta \chi(m)(t)$ with the magnetic parity is further illustrated in Figure 6(bottom). We see that the dispersion of the difference $\Phi(\chi) - \Phi(\chi(m))$ is large (and possibly random nature) when the parity index varies around 1. The dispersion decreases when the parity index grows.

5. Discussion and Summary

Results of our models predict the anti-correlation between variations of magnetic helicity imbalance on the small- and large-scales. The similar effect is demonstrated by the observational data (see, Figure 2). However, observations are rather noisy and cover a small period of time which is not enough to robustly determine the given effect. The results of the models show a connection between the hemispheric asymmetry of the magnetic activity, which is indicated by the parity parameter $P$ (see, Eq(4.1)), and the helicity imbalance parameters. The parity parameter of the dynamo generated magnetic field is related with mixing of the fundamental dynamo modes corresponding to the symmetric and antisymmetric about the solar equator magnetic field [Sokoloff & Nesme-Ribes (1994)].

The net magnetic helicity density of the large-scale magnetic field can be related with the parity parameter $P$ as well. To see it, lets decompose $r$ and $\phi$ components of the magnetic field and its vector potential on series of the Legendre polynomial $P_n$ and $P^1_n$ (also see Pipin & Pevtsov (2014)):

$$\bar{A}_\phi(t, \theta) = \sum_{n=1}^{N} a^{(n)}_\phi(t) P^1_n(\cos \theta),$$  \hspace{1cm} (5.1)

$$\bar{B}_r(t, \theta) = \sum_{n=1}^{N} b^{(n)}_r(t) P_n(\cos \theta),$$  \hspace{1cm} (5.2)

$$\bar{B}_\phi(t, \theta) = \sum_{n=1}^{N} b^{(n)}_\phi(t) P^1_n(\cos \theta),$$  \hspace{1cm} (5.3)

$$\bar{A}_r(t, \theta) = \sum_{n=1}^{N} a^{(n)}_r(t) P_n(\cos \theta).$$  \hspace{1cm} (5.4)

Using the standard relations between $P_n$ and $P^1_n$ we can find expressions for coefficients
of the vector-potential:

\[ a_{\phi}^{(n)}(t) = -\frac{Rb_{r}^{(n)}(t)}{n(n+1)}, \quad (5.5) \]

\[ a_{r}^{(n)}(t) = -Rb_{\phi}^{(n)}(t), \quad (5.6) \]

Then restricting ourselves only with two first terms of expansions we have

\[ \bar{A}_{\phi}(t,\theta) = -\frac{R_{\odot}b_{r}^{(1)}(t)}{2}P_{1}^{1} - \frac{R_{\odot}b_{r}^{(2)}(t)}{6}P_{2}^{1} + \ldots \]

\[ \bar{B}_{\phi}(t,\theta) = b_{\phi}^{(1)}(t)P_{1}^{1} + b_{\phi}^{(2)}(t)P_{2}^{1} + \ldots \]

Note that, \[ \int_{-1}^{1} A_{\phi}B_{\phi}d\mu = \int_{-1}^{1} A_{r}B_{r}d\mu \] because of the symmetry properties (Blackman & Brandenburg 2003). Therefore, the net magnetic helicity density will be

\[ \delta\chi^{(m)} = 2\int_{-1}^{1} \bar{A}_{\phi}\bar{B}_{\phi}d\mu \approx \frac{4R_{\odot}}{3}b_{r}^{(1)}(t)b_{\phi}^{(1)}(t) - \frac{8R_{\odot}}{15}b_{r}^{(2)}(t)b_{\phi}^{(2)}(t) + \ldots \quad (5.7) \]

In this Equation, \( b_{r}^{(1)} \) is the dipole mode of the radial magnetic field and \( b_{\phi}^{(1)} \) is the quadrupole mode of the toroidal magnetic field. Therefore the magnetic parity \( P \) is readily related with magnetic helicity imbalance. In the recent paper of Pipin & Kosovichev (2018), the parameter \( P \) was calculated from the data set including the last 4 solar cycles. It was found that \( P \approx 0 \) (strong asymmetry of the magnetic activity) near the maxima of the sunspot activity. Taking into account the data presented in Figure 2 we conclude that the models prediction about the connection of the parity and helicity imbalance parameter roughly agrees with observations.

At low latitudes the predicted patterns of the small- and large-scale magnetic helicity in our dynamo model are in qualitative agreement with results of observations of the current helicity of solar active regions (see, Zhang et al. 2010 and results of Pipin et al. 2019). The latter show results of the magnetic helicity density evolution of the small- and large-scale magnetic field in solar cycle 24. The model can not correctly reproduce the polar parts of the helicity density butterfly diagrams shown in that paper because we neglect the effect of the meridional circulation. A different kind of the time latitude diagrams for the density of the magnetic helicity flux is found in the recent paper of Hawkes & Yeates (2019) (cf., ourFig3a and Fig3d in their paper). The difference is likely due to different nature of the helicity density flux in their model. In our model the helicity density results from the dynamo process due to magnetic helicity conservation. In the model of Hawkes & Yeates (2019) (also, see, Hawkes & Berger 2018) it mainly comes from the effect of differential rotation acting on the magnetic flux emerging at the surface. The interesting feature of the helicity flux found in that paper is the presence of both signs of magnetic helicity simultaneously as the dynamo cycle progress from high to low latitudes. The equatorial part of diagrams satisfy the standard HHR. It is interesting that a rather similar pattern can be found in the nonaxisymmetric dynamo model of Pipin & Kosovichev (2018). Pipin et al. (2019) used this model as a benchmark before processing the magnetic vector synoptic maps of SDO/HMI. The effect of the magnetic helicity due to differential rotation on the long-term magnetic activity evolution and the equatorial symmetry properties of the dynamo generated magnetic field has to be studied further using the nonaxisymmetric dynamo models.

Our results show that helicity imbalance can be connected with reverse of the hemispheric helicity rule. Miesch et al. (2016) investigated the role of magnetic helicity in cyclic magnetic activity in a global 3D magnetohydrodynamic (MHD) simulation of a
convective dynamo model. They also show that this hemispheric rule reverses during the declining phase of each cycle. In their simulations, the net magnetic helicity in one solar cycle is also not zero, and even behave differently from one cycle to another; it alternates sign with times even shorter than the 11 year solar cycle. This implies that the conservation of magnetic helicity could manifest in at a longer time, not just in one solar cycle. In our model we also find that the evolution of the net magnetic helicity density on the surface follows the evolution of the parity of the large-scale magnetic field. The causal relationship can be reversed, i.e., the net magnetic helicity density or the net magnetic helicity fluxes can drive the magnetic parity breaking.

According to dynamo theory (Blackman & Brandenburg 2003), the magnetic helicity of large-scale field, is, in general but not completely, determined by the sign of α-effect and the opposite helicity sign is expected for the small-scale magnetic field. A complicated connection between small- and large-scale properties of the magnetic helicity fluxes in solar cycles 23-24 was discussed earlier by Yang & Zhang (2012) and Zhang & Yang (2013). Results of Brandenburg et al. (2017), Singh et al. (2018) and Pipin et al. (2019) show that the bi-helical property can be violated and it was violated in solar cycle 24. As a result, the sign of the surface magnetic helicity density of the large- and small-scale field can be the same. The origin of this phenomenon is unclear. In general, we can assume its relationship with fluctuations of magnetic helicity fluxes. Our results about anti-correlation between variations of magnetic helicity imbalance on the small- and large-scales support this idea. With some reservation, it can be suggested that there is a relationship between violation of the bi-helical property on the surface and the equatorial parity breaking of the magnetic activity evolution. In this study we show the theoretical possibility of such a relation. However, the strength of our prediction is rather limited because the amplitude of the helicity flux fluctuations remains unconstrained in the model. This opens an interesting theoretical and observational prospects for the future studies.

Acknowledgements This study is supported by the RFBR of Russia and NNSF of China bilateral grant 19-52-53045, also grants 11427901, 10921303, 11673033, U1731113,11611530679, and 11573037 of the National Natural Science Foundation of China; by grant No. XDB09040200, XDA15010700 of the Strategic Priority Research Program of Chinese Academy of Sciences; by the Max-Planck Society Inter-Institutional Research Initiative Turbulent transport and ion heating, reconnection, and electron acceleration in solar and fusion plasma of Project No. MIF-IF-A-AERO8047; by Max-Planck-Princeton Center for Plasma Physics PS AERO 8003; by ISSI International Team on Magnetic Helicity estimations in models and observations of the solar magnetic field. The authors would also like to thank the Supercomputer Center of the Chinese Academy of Sciences (SCCAS) and the Max-Planck Computing and Data Facility (MPCDF) for the allocation of computing time. The dynamo model was tested during Solar Helicities in Theory and Observations: Implications for Space Weather and Dynamo Theory Program at Nordic Institute for Theoretical Physics (NORDITA). DDS thanks support of the RFBR grant 18-02-00085. VVP conducted this work as a part of research project II.16.3.1 of ISTP SB RAS.

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