On the lambda algebra and Singer’s cohomological transfer

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Abstract: Writing $A$ for the 2-primary Steenrod algebra, which is the algebra of stable natural endomorphisms of the mod 2 cohomology functor on topological spaces. Working at the prime 2, computing the cohomology of $A$ is an important problem of Algebraic topology, because it is the initial page of the Adams spectral sequence converging to stable homotopy groups of the spheres. A relatively efficient tool to describe this cohomology is the Singer algebraic transfer of rank $n$ in [22], which passes from a certain subquotient of a divided power algebra to the cohomology of $A$. Singer predicted that this transfer is a monomorphism, but this remains open for $n \geq 4$. This short note is to verify the conjecture in the ranks 4 and 5 and some generic degrees.

Key words: Primary cohomology operations; Steenrod algebra; Hit problem; Cohomology of Steenrod algebra; Algebraic transfer; Lambda algebra; Adams spectral sequence.

1. Introduction

It is well-known that there is a group homomorphism $Sq^n$ for $n \geq 0$, between mod-2 cohomology groups of a topological space, called Steenrod squares of degrees $n$. They are stable cohomology operations, that is, they commute with suspension maps. All the Steenrod squares form an algebraic structure which is known as the 2-primary Steenrod algebra $A$ subject to the Adem relations. It was applied to the vector fields on spheres and the Hopf invariant one problem, which asks for which $n$ there exist maps of Hopf invariant $\pm 1$. So, the Steenrod algebra is one of the important tools in Algebraic topology. Specifically, its cohomology $\text{Ext}^*(Z/2, Z/2)$ is an algebraic object that serves as the input to the Adams spectral sequence (the ASS) [1] and therefore, computing this cohomology is of fundamental importance to the study of the stable homotopy groups of spheres.

The cohomological transfer, defined by Singer [22], could be an useful approach to describe the mysterious structure of the cohomology algebra of $A$. In order to better understand this transfer, we will use the following notations and the relevant concepts. Let denote $V^{\oplus n}$ the $n$-dimensional vector space over the prime field $Z/2$. Then, we write $H^*(V^{\oplus n})$ and $H_*(V^{\oplus n})$ for mod-2 cohomology and homology of $BV^{\oplus n}$ (the classifying space of $V^{\oplus n}$). One should note that $BV^{\oplus n}$ is homotopy equivalent to the cartesian product of $n$ copies of the union of the finite projective spaces. As it is known, $H^*(V^{\oplus n})$ is identified with the polynomial algebra $Z/2[u_1, \ldots, u_n]$ on generators of degree 1, equipped with the canonical unstable algebra structure over the Steenrod algebra (i.e., it is a commutative, associative, graded $Z/2$-algebra equipped with a structure of unstable $A$-module and satisfying two relations, one called the Cartan formula, and the other called the instability relation: $Sq^d(x}(x) = x^2$.) By dualizing, $H_*(V^{\oplus n})$ has a natural basis dual to the monomial basis of $H^*(V^{\oplus n})$. We denote by $a_1, \ldots, a_n$ the basis of $H_1(V^{\oplus n})$ dual to the basis $u_1, \ldots, u_n$ of $H^1(V^{\oplus n}) = \text{Hom}(V^{\oplus n}, Z/2)$, so that $a_i(u_j) = 1$ if $i = j$ and is 0 if $i \not= j$. We write the dual of $u_1^{d_1} \ldots u_n^{d_n}$ as $a_1^{(d_1)} \ldots a_n^{(d_n)}$ where the parenthesized exponents are called divided powers, and be careful that in the corresponding situation over a field of characteristic 0 in place of $Z/2$, $a_i^{(d_i)} = a_i^{d_i/d!}$, which fits with the formula $a(d)a(e) = (d+e)!a(d+e)$. This product gives a commutative graded algebra $H_*(V^{\oplus n})$ over $Z/2$ called a divided power algebra, where $\Gamma(a_1, \ldots, a_n) = H_*(V^{\oplus n})$, and an element $a_1^{(d_1)} \ldots a_n^{(d_n)}$ in $H_*(V^{\oplus n})$ corresponding to a monomial $u_1^{d_1} \ldots u_n^{d_n}$ in $H^*(V^{\oplus n})$ is called $d$-monomial. Now, let $P_2H_*(V^{\oplus n})$ be the subspace of $H_*(V^{\oplus n})$ consisting of all elements that are annihilated by all Steenrod squares of positive degrees. The general linear group $GL_n = GL(V^{\oplus n})$ acts regularly on the classifying space $BV^{\oplus n}$ and therefore on $H^*(V^{\oplus n})$ and $H_*(V^{\oplus n})$. This action commutes with that of the algebra $A$ and so acts
$\mathbb{Z}/2 \otimes_{\Lambda} H^* (V^\otimes n)$ and $P_{\lambda} H_* (V^\otimes n)$. For each $n \geq 0$, Singer constructed in [22] a linear transformation from $P_{\lambda} H_* (V^\otimes n)$ to the $n$-th cohomological group $\text{Ext}^{n, n+*} (\mathbb{Z}/2, \mathbb{Z}/2)$ of $\Lambda$, which commutes with two $S^0$'s on $P_{\lambda} H_* (V^\otimes n)$ and $\text{Ext}^{n, n+*} (\mathbb{Z}/2, \mathbb{Z}/2)$ (see Boardman [2] and Minami [13] for more about this). He shows that this map factors through the quotient of its domain's $GL_n$-coinvariants to give rise the so-called cohomological transfer of rank $n$ $\phi^n_\ast (\mathbb{Z}/2) : \mathbb{Z}/2 \otimes_{GL_n} P_{\lambda} H_* (V^\otimes n) \to \text{Ext}^{n, n+*} (\mathbb{Z}/2, \mathbb{Z}/2)$. The domain of this transfer is dual to the space of $GL_n$-invariants $(\mathbb{Z}/2 \otimes_{\Lambda} H^* (V^\otimes n))^{GL_n}$. It is to be noted that $\phi^n_\ast (\mathbb{Z}/2)$ is induced over the $E_2$-term of the ASS by the geometrical transfer map $\Sigma^\infty (B(V^\otimes n)) \to \Sigma^\infty (S^0)$ in stable homotopy theory (see also Mitchell [14]). The work of Minami [13] indicated that these transfers play a key role in finding permanent cycles in the ASS. In the second cohomology groups of $\Lambda$, following Mahowald [11] and Lin-Mahowald [9], the classes $h_j h_j$ for $j \geq 3$ and $h_j^2$ for $0 \leq j \leq 5$ are known to be the permanent cycles in the ASS. In 2016, Hill, Hopkins, and Ravenel [6] showed that when $j \geq 7$, the class $h_j^2$ is not a permanent cycle in the ASS. It is surprising that so far there is no answer for $j = 6$. The question of whether these $h_j^2$ are the permanent cycles in the ASS or not is called Kervaire invariant problem in literature [4]. This is one of the oldest unresolved issues in Differential and Algebraic topology.

Direct calculating the value of $\phi^n_\ast (\mathbb{Z}/2)$ on any non-zero element is a hard work. It has been demonstrated that $\phi^n_\ast (\mathbb{Z}/2)$ is an isomorphism for $n \leq 2$ by Singer himself [22], and $n = 3$ by Boardman [2]. Most notably, Singer sets up a hypothesis in the same paper [22] that $\phi^n_\ast (\mathbb{Z}/2)$ is a monomorphism, but this is still not confirmed, for all cohomological degrees $n \geq 4$. The cases $n = 4, 5$ are our concern in this paper. Besides Singer's transfer homomorphism, the lambda algebra $\Lambda$ of Bousfield et al. [3] is also a relatively efficient tool to compute the cohomology of the Steenrod algebra. Recall that $\Lambda$ is the quotient of the graded tensor algebra over $\mathbb{Z}/2$ on symbols $\lambda_i$ for $i \geq -1$, modulo the two-sided ideal generated by $\lambda_i \lambda_k - \sum_j \binom{j'-k'-1}{j-k-1} \lambda_{j+k-j} \lambda_j$, for any $s, k \geq -1$ by the right ideal generated by $\lambda_{-1}$. An interesting representation in the algebra $\Lambda$ of the algebraic transfer, established by Chơn and Hà [5], is a $\mathbb{Z}/2$-linear map $\psi_n$ from $P_{\lambda} H_* (V^\otimes n)$ to a subspace of $\Lambda$ spanned by all monomials of length $n$ in all the monomials in $\lambda_i$. The authors showed that the image of an element $\xi \in P_{\lambda} H_* (V^\otimes n)$ under $\psi_n$ is a cycle in $\Lambda$ and $[\psi_n (\xi)] = \phi^n_\ast (\mathbb{Z}/2) ([\xi])$. Note also that this result is a dual version of the one in Hùng [7].

The Singer transfer we are discussing is closely related to the hit problem in literature [16] of determination of a minimal generating set for the unstable $\Lambda$-module $H^* (V^\otimes n)$. The reader can find an excellent list of publications about this problem in the works by Kameko [8], Mothebo-Uys [15], the present writer [17, 18, 21], Singer [23], Sum [24, 26], Walker-Wood [28], Wood [29] and others. Hit problems are motivated by several problems in Topology and Algebra. It was completely studied for the cases $n \leq 4$ by Peterson [16], Kameko’s thesis [8] and Sum [24]. Nevertheless, the general answer seems to be out of reach with the present techniques. Therefore, it is renowned as a difficult problem, even with the help of a computer. In fact, when $\mathbb{Z}/2$ is a trivial $\Lambda$-module, solving the hit problem is equivalent to determining the "cohits" $\mathbb{Z}/2 \otimes_\Lambda H^* (V^\otimes n)$ as a graded vector space, or more generally as a graded module over the group algebra $\mathbb{Z}/2[GL_n]$. Frank Peterson [16] conjectured that $\mathbb{Z}/2 \otimes_\Lambda H^* (V^\otimes n)$ is a trivial for any $n$ where $\alpha (n + *) \leq n$, where $\alpha (k)$ is the number of $1$’s in the dyadic expansion of a positive integer $k$. His motivation for this was to prove that if $M$ is a smooth manifold of dimension $r$ such that all products of length $n$ of Stiefel-Whiney classes of its normal bundle vanish, then either $\alpha (r) \leq n$ or $M$ is cobordant to zero. The conjecture was established by Wood [29]. Therefrom, to study $\mathbb{Z}/2 \otimes_\Lambda H^* (V^\otimes n)$ in each $n$ and degree $*, \geq 0$, it suffices by Peterson’s conjecture and iteration of the Kameko map [8] to consider degrees $*$ in the following "generic" form:

\begin{enumerate}
  \item $k(t - 1) + \ell, 2t$, for $k, \ell \geq 0$, $\mu (\ell) < k \leq n$, where $\mu (\ell)$ is a smallest number $r \in \mathbb{N}$ such that $\alpha (\ell + r) \leq r$. By Minami [12], hit problems are also considered as an useful tool for studying permanent cycles in the ASS.
\end{enumerate}

In the present work, we explicitly determine the structure of the coinvariant $\mathbb{Z}/2 \otimes_{GL_n} P_{\lambda} H_* (V^\otimes n)$ and the behavior of the cohomological transfer of ranks 4 and 5 in some generic degrees of the form (1) by using techniques of the hit problem and the representation in the algebra $\Lambda$ of these transfers.

2. Main results To begin with, we remark that by Sum [24], it is enough to depict the behavior

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of the fourth transfer in the following degrees $d$:

(i) $d = 2t + r - 1$, for $1 \leq r \leq 3$,

(ii) $d = 2t + s + 1 + 2t + 1 - 3$,

(iii) $d = 2t + s + 2 + 2t - 2$,

(iv) $d = 2t + s + u + 2t + s + 2t - 3$,

whenever $r, t, s, u$ are positive integers. It is not difficult to check that the above degrees can be rewritten as (1). The cases of (i) are known by Sum [25]. The results for (ii) and (iii) have been partially probed in [19, 20]. This note is to investigate the case (iv).

Now, let us state the main results of this text. We first study the fourth cohomological transfer in degrees $d_{t,s,u} := 2t + s + u + 2t + s + 2t - 3$. To make this, we give an explicit description of a minimal generating set for the domain of $\phi_4^*(\mathbb{Z}/2)$ in degree $d_{t,s,u}$ and obtain the following, which is proved in many steps by using computational techniques of the hit problem of four variables and some previous results by the present author [19, 20] and Sum [25].

**Theorem 2.1.** The domain of $\phi_4^*(\mathbb{Z}/2)$ in degree $d_{t,s,u}$ is determined by

$$Z/2 \otimes_{GL_4} P_h H_{d_{t,s,u}}(V^{\otimes 4}) = 0 \quad \text{if} \quad s = 1, u = 1 \text{ and } t \geq 1,$$

$$0 \quad \text{if} \quad s = 2, u = 1 \text{ and } t \geq 2,$$

$$0 \quad \text{if} \quad s \geq 3, u = 1 \text{ and } t \geq 1,$$

$$0 \quad \text{if} \quad s = 1, u = 2 \text{ and } t \geq 1,$$

$$\langle \{\zeta_{1,2}\} \rangle \quad \text{if} \quad s = 1, u = 2 \text{ and } t \geq 2,$$

$$\langle \{\zeta_{1,2}\} \rangle \quad \text{if} \quad s = 1, u \geq 3 \text{ and } t \geq 1,$$

$$\langle \{\zeta_{2,1}\} \rangle \quad \text{if} \quad s = 2, u \geq 1 \text{ and } t \geq 1,$$

$$\langle \{\zeta_{1,2}\} \rangle \quad \text{if} \quad s \geq 3, u \geq 2 \text{ and } t \geq 1,$$

$$\langle \{\zeta_{1,2}\} \rangle \quad \text{if} \quad s \geq 2, u \geq 2 \text{ and } t \geq 2,$$

where

$$\zeta_{1,2} = a_{1}^{(0)}(2^{t+2} - 1) a_{2}^{(2^{t+2} - 1)} a_{3}^{(2^{t+2} - 1)} a_{4}^{(2^{t+2} - 1)} + a_{1}^{(0)}(2^{t+2} - 1) a_{3}^{(2^{t+2} - 1)} a_{4}^{(2^{t+2} - 1)} + a_{1}^{(0)}(6.2^{t+2} - 1) a_{2}^{(3.2^{t+2} - 1)} a_{3}^{(2^{t+2} - 1)} a_{4}^{(2^{t+2} - 1)} + a_{1}^{(0)}(7.2^{t+2} - 1) a_{3}^{(2^{t+2} - 1)} a_{4}^{(2^{t+2} - 1)} a_{4}^{(2^{t+2} - 1)},$$

$$\zeta_{1,2,u} = a_{1}^{(3)}(2^{t+3} - 1) a_{2}^{(3)} a_{3}^{(3)} a_{4}^{(2)} + a_{1}^{(2)}(2^{t+3} - 1) a_{2}^{(3)} a_{3}^{(3)} a_{4} + a_{1}^{(2)}(2^{t+3} - 1) a_{2}^{(3)} a_{3}^{(3)} a_{4} + a_{1}^{(2)}(2^{t+3} - 1) a_{2}^{(3)} a_{3}^{(3)} a_{4},$$

$$\zeta_{t,s,u} = a_{1}^{(0)}(2^{t+1} - 1) a_{2}^{(3)} a_{3}^{(3)} a_{4}^{(2^{t+1} - 1)} a_{5}^{(2^{t+1} - 1)} + a_{1}^{(0)}(2^{t+1} - 1) a_{2}^{(3)} a_{3}^{(3)} a_{4}^{(2^{t+1} - 1)} a_{5}^{(2^{t+1} - 1)}.$$
Now according to [10], it is easily seen that 
\[ \text{Ext}^5_{\mathbb{Z}/2} (\mathbb{Z}/2, \mathbb{Z}/2) = \langle h_0 h_{t+1} h_{t+2} h_{t+3} h_{t+5} \rangle = 0, \]
for arbitrary \( t \geq 0 \), and so, by Theorem 2.3, we immediately obtain

**Corollary 2.4.** The Singer transfer is a trivial isomorphism in bidegree \((5, 5 + d_t)\) for every non-negative integer \( t \).

Thus, Corollaries 2.2 and 2.4 favor the Singer conjecture in bidegrees \((4, 4 + d_t, s, u)\) and \((5, 5 + d_t)\) for any \( t, s, u \).

Detailed proofs of all the results of this note will be published elsewhere.

3. **Conclusion** Although our work does not apparently lead to either a proof or a refutation of the Singer conjecture in general, we feel that it represents an interesting note about an application of hit problem and the lambda algebra for studying this conjecture. Perhaps a continuation of our methods may prove fruitful. It is our belief that further research can spring from these ideas.

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