A Note on Indexing Planar Point Sets for Approximate Bottleneck Distance Queries

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Abstract

The bottleneck distance is a natural measure of the distance between two finite point sets of equal cardinality, defined as the minimum over all bijections between the point sets of the maximum distance between any pair of points put in correspondence by the bijection. In this work, we consider the problem of building a data structure \( D \) that indexes a collection of \( m \) planar point sets (of varying sizes) and supports nearest bottleneck distance queries: given a query point set \( Q \) of size \( n \), we would like to find the point set(s) \( P \in D \) of size \( n \) that are closest in terms of bottleneck distance. Without loss of generality, we assume that all point sets belong to the unit box \([0,1]^2\) in the plane and focus on the \( L_\infty \) norm, although the techniques can also be used for other norms. The main contribution is a trie-based data structure finds a 6-approximate nearest neighbor in \( O(-\lg(d_B(D,Q))n) \) time, where \( d_B(D,Q) \) is the minimum bottleneck distance from \( Q \) to any point set in \( D \).

1 Introduction

The bottleneck distance is a natural measure of the distance between two finite point sets of equal cardinality. The problem of computing the bottleneck distance arises in geometric applications such as comparing persistence diagrams in topological data analysis \[3\]. Bottleneck distance is defined between two point sets \( P \) and \( Q \) as

\[
d_B(P, Q) = \min_{h: P \rightarrow Q} \max_{p \in P} ||h(p) - p||,
\]

where \( h \) is a bijection and \( ||\cdot|| \) is chosen as the \( L_\infty \) norm, as this is common for the persistence diagram comparison application. Given a database \( D \) of point sets, we can also define

\[
d_B(D, Q) = \min_{P \in D, |P| = |Q|} d_B(P, Q).
\]

Finally,

\[
\text{nearest}(D, Q) = \{ P : P \in D, |P| = |Q|, d_B(P, Q) = d_B(D, Q) \}.
\]

The problem considered in this work is to identify approximate nearest neighbor point sets \( P \), whose bottleneck distance from \( Q \) is within a constant factor of
$d_B(D, Q)$. Without loss of generality, we assume that all point sets belong to the unit box $[0, 1]^2$ in the plane. We first describe a simple approach to represent point sets using strings. This suggests using a trie data structure [6, 5] to store strings associated with each point set in the database.

2 Related Work

Bottleneck distance is closely related to the bipartite matching problem, which can be solved by the classic maximum flow technique of Hopcroft and Karp [10]. The current best exact algorithm for bipartite matching of points in the plane is due to Efrat et al. [7] and runs in $O(n^{1.5} \log n)$ time for point sets of size $n$.

Earlier seminal work by Hefferman and Schira [9] considered approximation algorithms for the more general problem in which one of the point sets is mapped by an isometry (translated, rotated, and possibly reflected) prior to being matched. In the case of just computing the bottleneck distance, their methods provide a $O(n^{1.5}(\epsilon/\gamma)^4)$ time algorithm to test if $d_B(P, Q) \leq \epsilon$, where the answer must be correct if $\epsilon \notin [d_B(P, Q) - \gamma, d_B(P, Q) + \gamma]$. A key idea in [9] is to check for bottleneck matchings using a maximum flow computation in graph that arises from “snap-rounding” the point sets to their nearest point in grid. Our approach uses a similar idea in which the maximum flow instance is a planar graph (not true for [9]), so a recent improved algorithm for multi-source, multi-sink maximum flow due to Borradaile [2] that runs in $O(n \log^3 n)$ time can be leveraged.

Bottleneck distance arises naturally in the comparison of persistence diagrams in topological data analysis [3]. Fasy et al. [8] consider the related problem of building a database of persistence diagrams that permits approximate querying. Their approach is also based on representing point sets by snap-rounding each point to neighboring grid points at each level in a multilevel grid data structure. All combinations of snap-roundings are considered and the resulting grid point configurations are stored in a database. Nearest point set queries are done in $O((n \log m + n^2) \log \tau)$ time, where $\tau$ is number of grid levels used and $m$ is the number of point sets stored. The resulting matches are shown to provide a 6-approximation to the nearest point set in the database. Rather than using a hashing scheme, it is possible to use a trie data structure (described in §3) to achieve $O(-\log(d_B(D, Q))n)$ time queries.

Approximation results are known for general bipartite matching in metric spaces; in [1] the authors show that for any $\delta > 0$, there is an algorithm that computes a $O(\frac{1}{\delta})$-approximate matching, where $\alpha = \log_2 2 \approx 0.631$ in $O(n^2 \log n \log^2 \frac{1}{\delta})$ time. A variation on minimum-distance bottleneck matching, with the additional constraint that the matched edges cannot cross, was recently shown to be NP-hard to approximate within a factor of less than 1.277 [4].
3 Preliminaries

Without loss of generality, all point sets are contained within the unit box $B = [0, 1]^2$ in the plane. Following the general approach of [8], we recursively divide $B$ into finer grids. The corner $(0, 0)$ is designated as the origin. The four corners of $B$ are the grid points at level 1. The grid at level $d$ is subdivided by 2 to form the grid at level $d + 1$. Thus, level 2 contains 9 grid points and in general level $d$ contains $(2^{d-1} + 1)^2$ grid points. The grid length at level $d$ is $\delta_d = 2^{1-d}$.

Let $p$ be a point in or on $B$, for $d \geq 1$, we define $n_d(p)$ as the nearest level $d$ grid point to $p$, breaking ties by going in the S and/or W direction with respect to $p$. Observe that $n_d(p)$ is unique and that if $p$ is already a level $d$ grid point, then $n_d(p) = p$. We also define $n_0(p)$ as the origin.

Suppose $P$ is a point set to be stored in $D$. For each $p \in P$, let

$$n_d^4(p) = \{g : g \text{ is a level } d \text{ grid point, } \|g - p\|_\infty < \delta_d\}.$$ 

Note $|n_d^4(p)| \leq 4$. We consider all ways of snapping each $p$ to some grid point $\text{snap}(p) \in n_d^4(p)$.

**Definition 1.** A query point set $Q$ is said to hit a point set $P$ at level $d$, if there is a snapping of $P$ such that $|q : q \in Q, n_d(q) = g| = |p : p \in P, \text{snap}(p) = g|$ for all level $d$ grid points $g$.

Versions of the following two lemmas appear in [8], and a similar analysis is also found in [9].

**Lemma 1.** If $Q$ hits $P$ at depth $d$, then $d_B(P, Q) \leq \frac{1}{2}\delta_d$.

**Proof.** Since $Q$ hits $P$, there is a snapping of $P$ that produces the grid configuration $n_d(Q) = \{n_d(q) : q \in Q\}$ (repeats allowed); define a bijection $h : P \rightarrow Q$ by mapping each $p$ that snapped to some grid point $g$ to a unique $q$ such that $g = n_d(q)$. Then $\|h(p) - p\|_\infty = \|q - p\|_\infty \leq \|q - n_d(q)\|_\infty + \|n_d(q) - p\|_\infty \leq \frac{\delta_d}{2} + \delta_d$. Thus $d_B(P, Q) \leq \frac{1}{2}\delta_d$. $\Box$

**Lemma 2.** If $Q$ does not hit $P$ at level $d$, then $d_B(P, Q) \geq \frac{\delta_d}{2}$.

**Proof.** Let $h : P \rightarrow Q$ be a bijection that realizes $d_B(P, Q)$. We prove the contrapositive: Suppose $d_B(P, Q) < \frac{\delta_d}{4}$. Let $p \in P$ and $q = h(p)$. Then $\|p - n_d(q)\|_\infty \leq \|p - q\|_\infty + \|q - n_d(q)\|_\infty \leq d_B(P, Q) + \frac{\delta_d}{2} < \delta_d$. It follows that $n_d(q) \in n_d^4(p)$, and so snapping each $p$ to $n_d(q)$ provides a hit to $Q$ at level $d$. $\Box$

Suppose $d^*$ is the maximum depth at which there is a hit $P \in D$ for a query point set $Q$. Lemma 1 implies $d_B(P, Q) \leq \frac{\delta_d}{4}$. On the other hand, since no hits were found at depth $d^* + 1$, by Lemma 2 $d_B(D, Q) \geq \frac{\delta_{d^*+1}}{2} = \frac{\delta_d}{2}$. Thus $d_B(P, Q) \leq 6d_B(D, Q)$, and so for a query point set $Q$, the point set returned $P$ is guaranteed to be a 6-approximation to the nearest point set in $D$. 

3
4 A Trie-based Data Structure

We propose an indexing approach based on representing configurations of grid points as strings. We first define a string representation for a single grid point at level $d$ as a length $d$ string and then interleave $n$ such strings to represent a set of $n$ grid points in the level $d$ grid. The interleaving is done so that the string first describes the level 1 grid points, then level 2, etc.

Let $g$ be a grid point at level $d \geq 1$. We define $N_d(g)$ as the grid point neighbor at level $d$ directly north of $g$, provided this point belongs to the grid. Define similarly for all eight principal compass wind directions and let $I_d(g) = g$ ($I$ for identity). We introduce a string encoding of any grid point $g$ at some level $d \geq 1$. The string, $s_d(g)$ is constructed in left-to-right order, in $O(1)$ time per symbol by “walking” in the grid toward $g$, starting at the origin, following the grid points $n_0(g), n_1(g), \ldots, n_d(g) = g$. Observe that for $1 \leq i \leq d$,

$$n_i(g) = \text{dir}_i(g)(n_{i-1}(g)),$$

where $\text{dir}_i(g) \in \{I, N, S, E, W, NE, SE, NW, SW\}$. Thus, we can compactly describe a level $d$ grid point $g$ by a unique string $s_d(g)$ of length $d$ over the nine symbols $\{I, N, S, E, W, NE, SE, NW, SW\}$, where the $i$th symbol indicates $\text{dir}_i(g)$.

We now consider how to use the above string encoding to represent grid point configurations. Let $G$ be a set of $n$ grid points (repeats allowed) at level $d > 0$ and let $S_d(G) = \{s_d(g) | g \in G\}$ be the set of length $d$ strings that encode each grid point in $G$. Consider $S_d(G)$ sorted into lexicographic order, i.e. $S_d(G) = \{s_d(g_1) \leq s_d(g_2) \leq \ldots \leq s_d(g_n)\}$. $S_d(G)$ can be encoded as a single interleaved string of length $nd$, defined as:

$$S_{d,G} = s_d(g_1) \ldots s_d(g_n)_{1} s_d(g_1)_{2} \ldots s_d(g_n)_{2} \ldots s_d(g_1)_{n} \ldots s_d(g_n)_{n}.$$ 

Notice that the first $n$ characters in $S_{d,G}$ describe the level 1 nearest neighbor grid points for $G$, the next $n$ characters describe the level 2 nearest neighbor grid points for $G$ and so on. Any distinguishable level $d$ grid point configuration $G$ is encoded uniquely by $S_{d,G}$. The time required to generate $S_{d,G}$ is $O(dn)$ (e.g. by using radix sort).

Lemma 3. Let $p(G) = \{n_d-g : g \in G\}$. Then $S_{d,G} = [S_{d-1,p(G)}]s_{d}(g_1)_{n} \ldots s_{d}(g_n)_{n}$.

Proof. This can be seen by noting that each string in $S_d(G)$ is formed from a string in $S_{d-1}(G)$ with a single symbol appended to the end, so the lexicographic sortings of $S_d(G)$ and $S_{d-1}(p(G))$ agree up to position $d-1$. \hfill $\square$

A natural approach to storing a collection of point sets, each represented as a string, is to use a trie-based data structure [5].

We first consider the database scheme proposed in [8], in which many snap-roundings of each point set are stored and use the aforementioned string representation and trie data structure. To represent a point set $P$, snap-roundings to grid point configurations (up to some maximum grid level $d_{\text{max}}$) are stored
Proof. Let \( G = \{g_1, \ldots, g_n\} \) be a snap-rounding of \( P \) at level \( d > 1 \). Each \( g_i = \text{snap}(p) \in n_i^1(p) \) for some \( p \in P \). Let \( g'_i = n_{d-1}(g_i) \). Clearly, \( g'_i \in n_{d-1}^1(p) \).

It follows that the grid configuration \( G' = n_{d-1}(G) \) will be snapped to by \( P \) in the level \( d - 1 \) grid. Furthermore, \( S_{d,G} = [S_{d-1,G'}s_d(g_1) \ldots s_d(g_n)], \) by Lemma 4.

Each trie node will also store a pointer to the list of point sets (initialized to \text{null}). As snapped grid point configurations for \( P \) are added to \( T \), \( P \) is appended to this list at each trie node that “finishes” describing a grid point configuration for some level, e.g. if \(|P| = k\), then a trie node at depth \( dk \) describes a level \( d \) grid point configuration for \( P \).

The time required to add a new point set \( P \) of size \( n \) to \( T \) is \( O(4^{d\max n}d_{\max}^{n}) \), since at most \( O(4^{d\max n}) \) snapped grid configuration strings are stored and each is generated in \( O(d_{\max} n) \) time. The additional space requirement for \( T \) is also \( O(4^{d\max n}d_{\max}^{n}) \).

4.1 Handling Queries

Let \( Q \) be a query point set of size \( n \); our objective is to find those \( P \in \mathbb{D} \) that approximate nearest(\( \mathbb{D} \), \( Q \)), where the database \( \mathbb{D} \) is represented using a trie \( T \), as described above. A query string \( S_Q \) is constructed in left-to-right order in blocks of size \( n \) as follows: For each point \( q \in Q \), we consider the sequence of grid points \( n_0(q), n_1(q), \ldots, n_{d_{\max}}(q) \); the sequence gets monotonically closer to \( q \). As before, we can represent this sequence as a string \( s_{d_{\max}}(q) \), whose \( i \)th symbol is \( \text{dir}_i(q) \) and \( S_{d_{\max}}(Q) \) is the collection of these strings for all \( q \in Q \). In order to produce the query string \( S_Q \), \( S_{d_{\max}}(Q) \) must be sorted lexicographically, however this can be done \text{ lazily} using radix sort. First, \( s_{d_{\max}}(q)_1 \) is found for all \( q \in Q \) and the strings are sorted on index 1. The resulting sorted column provides the first \( n \) symbols in \( S_Q \). Next, the trie \( T \) is searched on this block. If there is a hit, then the search continues to the next index position, the string symbols are computed at that position (in \( O(n) \) time) and the radix sort is continued at the next index. This produces the next size \( n \) block of \( S_Q \) and \( T \) is probed from where the previous hit was found. If \( d^* < d_{\max} \) is the maximum hit depth, then \( d^* = -\lg(d_B(\mathbb{D}, Q)) \) and the query runs in \( O(-\lg(d_B(\mathbb{D}, Q))n) \) time.
5 Discussion

An approach to indexing planar point sets that supports approximate nearest bottleneck distance queries using a trie-based data structure to compactly represent point configurations in a multi-level grid is described. The obvious drawback is the exponential space complexity: up to $4^{d_{\max}} n$ strings are stored for each point set of size $n$. A natural question is whether a more space-efficient database scheme is possible. It would also be interesting to consider if an indexing approach and querying procedure can be found that permits one of the point sets to be transformed by an isometry, such as done in [9].

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