Other Squashing Deformation and
$\mathcal{N} = 3$ Superconformal Chern-Simons Gauge Theory

Changhyun Ahn

Department of Physics, Kyungpook National University, Taegu 702-701, Korea

ahn@knu.ac.kr

Abstract

We consider one of the well-known solutions in eleven-dimensional supergravity where the seven-dimensional Einstein space is given by a $SO(3)$-bundle over the $\mathbb{CP}^2$. By reexamining the $AdS_4$ supergravity scalar potential, the holographic renormalization group flow from $\mathcal{N} = (0, 1)$ $SU(3) \times SU(2)$-invariant UV fixed point to $\mathcal{N} = (3, 0)$ $SU(3) \times SU(2)$-invariant IR fixed point is reinterpreted. A dual operator in three-dimensional superconformal Chern-Simons matter theories corresponding to this RG flow is described.
1 Introduction

The $\mathcal{N} = 6$ superconformal Chern-Simons matter theory with gauge group $U(N) \times U(N)$ at level $k$ and with two hypermultiplets in the bifundamental representations is found in [1]. This gauge theory is described as the low energy limit of $N$ M2-branes probing $\mathbf{C}^4/\mathbb{Z}_k$ singularity. At large $N$-limit, this theory is dual to the eleven-dimensional M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ where the seven-sphere metric is realized as an $S^1$-fibration over $\mathbb{C}P^3$ [2]. One of the main observations in [1] is to look at the special case of $\mathcal{N} = 3$ superconformal Chern-Simons matter theory with above particular gauge group, matter contents, and particular choice of Chern-Simons levels of two gauge groups. Then naive $SU(2)$ flavor symmetry appearing in the hypermultiplets is enhanced to $SU(2) \times SU(2)$ symmetry that occurs in the whole action of the theory and there exists a $SU(2)_R$ symmetry coming from the original $\mathcal{N} = 3$ superconformal symmetry. It turns out the full theory has $\mathcal{N} = 6$ superconformal symmetry and the full scalar potential is invariant under $SU(4)_R$ coming from this enhanced $\mathcal{N} = 6$ superconformal symmetry.

The simplest spontaneous compactification of the eleven-dimensional supergravity is the Freund-Rubin [3] compactification to a product of $AdS_4$ spacetime and an arbitrary compact seven-dimensional Einstein manifold $X^7$ of positive scalar curvature. The standard Einstein metric of the round seven-sphere $S^7$ yields a vacuum with $SO(8)$ gauge symmetry and $\mathcal{N} = 8$ supersymmetry. There exists a second squashed Einstein metric [4, 5] yielding a vacuum with $SO(5) \times SU(2)$ gauge symmetry and $\mathcal{N} = (1,0)$ supersymmetry [6, 7]. As suggested in [8, 9], in [10], it was shown that the well-known spontaneous (super)symmetry breaking deformation from round $S^7$ to squashed one is mapped to a renormalization group(RG) flow from $\mathcal{N} = (1,0) SO(5) \times SU(2)$-invariant fixed point in the UV to $\mathcal{N} = 8 SO(8)$-invariant fixed point in the IR. In particular, the squashing deformation corresponds to an irrelevant operator at the UV superconformal fixed point and a relevant operator at the IR (super)conformal fixed point respectively. Moreover the RG flow is described geometrically by a static domain wall which interpolates the two asymptotically $AdS_4$ spacetimes with round and squashed $S^7$’s. For the different type of compactifications where the internal space has nonzero four-form field strength, see also [11, 12].

One could ask [13] what happens when we perform $\mathbb{Z}_k$-quotient [1] along the above whole RG flow [10]? Starting from the general, one parameter-family, metric for $\mathbb{C}P^3$ inside of seven-sphere and its seven-dimensional uplift metric on an $S^1$-bundle over this $\mathbb{C}P^3$, the full eleven-dimensional metric with appropriate warp factors was constructed. By analyzing the $AdS_4$ scalar potential, the holographic supersymmetric(or nonsupersymmetric) RG flow from $\mathcal{N} = 1$
(1, 0) $SO(5) \times U(1)$-invariant UV fixed point to $\mathcal{N} = (6, 0) SU(4)_R \times U(1)$-invariant IR fixed point was described in [13]. Each symmetry group is the subset of previous ones respectively. That is, $SO(5) \times U(1)$ is contained in $SO(5) \times SU(2)$ and $SU(4)_R \times U(1)$ is contained in $SO(8)$. The squashing deformation corresponds to the singlet of $20'$ of $SU(4)_R$ and it is given by the quartic term for the matter fields transforming as fundamental representation under the $SU(4)_R$. The dual Chern-Simons matter theory at the $\mathcal{N} = (1, 0) SO(5) \times U(1)$-invariant UV fixed point is constructed in [14].

Now it is natural to ask that are there any other examples where some squashing deformation in $X^7$ might provide a similar RG flow and one can think of some dual operator in three-dimensional boundary conformal field theory? Yes, the manifold $X^7 = N^{0,1,0}_I$ has been studied originally by Castellani and Romans [15] who identified this manifold as a particular coset manifold which is specified by three integers. We consider the particular case where $p = 0, q = 1$ and $r = 0$ in $N^{p,q,r}_I$ manifold. There exists $\mathcal{N} = (3, 0)$ supersymmetry with $SU(3) \times SU(2)$ gauge symmetry or $\mathcal{N} = (1, 0)$ supersymmetry with $SU(3) \times U(1)$ gauge symmetry (See also [16]). Moreover, Page and Pope [17] have completed the coset manifold construction by showing existence of another family of Einstein manifold, $N^{0,1,0}_II$, which can be obtained from geometric squashing of the $N^{0,1,0}_I$ manifold, retains the same gauge group $SU(3) \times SU(2)$ but instead preserves $\mathcal{N} = (0, 1)$ supersymmetry. As in the case of seven-sphere $S^7$, the scalar field corresponding to the squashing deformation acquires a nonzero vacuum expectation value leading to (super)-Higgs mechanism. With left-orientation the squashing interpolates between a $\mathcal{N} = 3$ supersymmetric vacuum and another with $\mathcal{N} = 0$ supersymmetry. With right-orientation it interpolates between a nonsupersymmetric vacuum and a supersymmetry restored one with $\mathcal{N} = 1$ supersymmetry [18].

On the other hand, in [19], the corresponding $\mathcal{N} = 3$ dual gauge theory has gauge group $SU(N) \times SU(N)$ with “three” hypermultiplets transforming as a “triplet” under the $SU(3)$ flavor symmetry which is nothing but one of the global symmetries for $N^{0,1,0}_I$ manifold [4]. In terms of $\mathcal{N} = 2$ superfields, these hypermultiplets can be reorganized as two sets of chiral superfields. For the color representation, one of these transforms as $(N, \overline{N})$ and the other transforms as $(\overline{N}, N)$. Furthermore, these two superfields transform as a doublet of $SU(2)_R$ which is the $SU(2)$ factor in the remaining global symmetry of $N^{0,1,0}_I$ manifold.

As we mentioned before, $SU(2)_R$ corresponds to the $\mathcal{N} = 3$ superconformal symmetry. After integrating out two adjoint fields of the theory, the effective quartic superpotential can be be

---

1We emphasize that this theory is based on Yang-Mills plus Chern-Simons theory with chiral multiplets, contrary to [1] in which the theory is described as Chern-Simons theory with chiral multiplets. For example, the kinetic terms in [19] contain those for the vector multiplet as well as the Chern-Simons term and those for the chiral multiplets. However, in [1], there is no kinetic term for any of the fields in the vector multiplet.
obtained. The coefficient is determined by the $\mathcal{N} = 3$ supersymmetry but breaks $\mathcal{N} = 4$ supersymmetry. For the general discussion on $\mathcal{N} = 3$ superconformal Chern-Simons matter theory, see [20]. The complete $\mathcal{N} = 3$ Kaluza-Klein spectrum is found in [21] and its $OSp(3|4)$ multiplet structure is further explained in [22].

Later, Billo, Fabbri, Fre, Merlatti and Zaffaroni [23] (See also [24]) have constructed an $\mathcal{N} = 3$ long massive spin 3/2 multiplet with conformal dimension 3 from the massless $\mathcal{N} = 3$ graviton multiplet. These two are connected to “shadow” relation: fields of different type, spin and mass are linked by a relation which determines the mass of the one as a function of the other.

In this paper, we will be studying the known example of Kaluza-Klein supergravity vacua and reinterpret it in terms of three-dimensional (super)conformal field theories and associated RG flows. We will be exploring the Freund-Rubin type spontaneous compactification on $AdS_4 \times X^7$. For M2-branes on an eight-dimensional manifold, the near-horizon geometry $X^7$ is expected to change as the M2-branes are placed at or away a conical singularity of the manifold [25, 26]. More specifically, we will consider $X^7$ being 3-Sasaki holonomy manifold, describing near-horizon geometry of M2-branes at relevant conical singularities.

When the work of [18] was completed at that time, it was not possible to analyze the gauge theory description because the Kaluza-Klein spectrum of eleven-dimensional supergravity was not complete. Later, in [23], they have found more mass spectrum in the eleven-dimensional supergravity side that includes the harmonics of the Lichnerowicz scalar with conformal dimension 4. Then one can identify the corresponding fluctuation spectrum for the scalar fields around $\mathcal{N} = 3$ fixed point.

The aim of this paper is to 1) recapitulate the effective scalar potential described in [18] with only breathing mode and squashing mode, and 2) analyze more both the mass spectrum in the eleven-dimensional supergravity and the corresponding Chern-Simons gauge theory operator which gives rise to the squashing deformation, by analyzing the results of [23].

In section 2, we describe the seven-dimensional Einstein space($N_l^{0,1,0}$) and its squashed version($N_l^{0,1,0}$) compactification vacua in eleven-dimensional supergravity. The effective four-dimensional scalar potential looks similar to the one for seven-sphere and the two critical points have nonzero scalar fields. However, the ratio of the squashing parameter at these two critical values(which is equal to 1/5) is the same as the one in seven-sphere case.

In section 3, the squashing deformation of each vacua is described by an irrelevant operator at the $\mathcal{N} = (3, 0)$ conformal fixed point and a relevant operator at the $\mathcal{N} = (0, 1)$ conformal fixed points. The RG flow is described in $AdS_4$ supergravity by a static domain wall interpolating between these two vacua. We identify the corresponding operator in the
boundary conformal field theory in three dimensions by looking at the observations of [23].

## 2 Two seven-dimensional Einstein spaces

A generic eleven-dimensional metric interpolating between two seven-dimensional Einstein spaces with an arbitrary four-dimensional spacetime metric maybe written as

\[
\frac{ds^2}{R^2} = e^{-7u}g_{\alpha\beta}dx^\alpha dx^\beta + e^{2u+3v} \left[ d\mu^2 + \frac{1}{4} \sin^2 \mu (\sigma_1^2 + \sigma_2^2 + \cos^2 \mu \sigma_3^2) \right] + e^{2u-4v} \left[ (\Sigma_1 - \cos \mu \Sigma_1)^2 + (\Sigma_2 - \cos \mu \Sigma_2)^2 + \left( \Sigma_3 - \frac{1}{2} (1 + \cos^2 \mu) \Sigma_3 \right)^2 \right],
\]

where the three real left-invariant one-forms satisfy the SU(2) algebra \( d\sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k \) and those on the manifold SO(3) which is necessary for the regularity of the metric have \( d\Sigma_i = -\frac{1}{2} \epsilon_{ijk} \Sigma_j \wedge \Sigma_k \). The coordinate \( \mu \) and SU(2) one-forms \( \sigma_i \) are the same as CP\(^2\) metric and corresponding isometry is characterized by SU(3). Then the seven-dimensional \( N^0 = 1, 0 \) space is a nontrivial SO(3) bundle over CP\(^2\) and the isometry group for this space is \( SU(3) \times SU(2) \) [17]. The scalar fields \( u(x) \) for the breathing mode and \( v(x) \) for squashing mode depend on the four-dimensional spacetime. Moreover, the squashing is parametrized by \( \lambda^2 \equiv e^{-7v} \).

The parameter \( R \) measures the overall radius of curvature. The gauge fields are given by \( A_1 = \cos \mu \sigma_1, A_2 = \cos \mu \sigma_2 \) and \( A_3 = \frac{1}{2} (1 + \cos^2 \mu) \sigma_3 \).

Spontaneous compactification of M-theory to \( AdS_4 \times X^7 \) is obtained from near-horizon geometry of \( N \) coincident M2-branes and the nonvanishing flux of four-form field strength of the Freund-Rubin [3] is given by

\[
\overline{F}_{\alpha\beta\gamma\delta} = Q e^{-7u} \epsilon_{\alpha\beta\gamma\delta} = Q e^{-21u} \epsilon_{\alpha\beta\gamma\delta},
\]

Here the Page charge [31, 32] defined by \( Q \equiv \pi^{-4} \int_{X^7} (\ast \overline{F} + \overline{C} \wedge \overline{F}) \) is related to the total number of M2-branes through \( Q = 96\pi^2 N^6 \ell_p^6 \). Then the eleven-dimensional Einstein equation

\[Recently, the \( \mathcal{N} = 3 \) superconformal Chern-Simons quiver theories are constructed in [27] but these theories do not contain the seven-dimensional Einstein manifold we are considering here. See also other relevant paper [28]. For the earlier studies on \( \mathcal{N} = 3 \) superconformal Chern-Simons theories, there are also some works in [29, 30, 31].

\[They are given by \( \sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi \), \( \sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi \) and \( \sigma_3 = d\psi + \cos \theta d\phi \) and similarly SO(3) one-forms are given by \( \Sigma_1 = \cos \gamma d\alpha + \sin \gamma \sin \alpha d\beta \), \( \Sigma_2 = -\sin \gamma d\alpha + \cos \gamma \sin \alpha d\beta \), and \( \Sigma_3 = d\gamma + \cos \alpha d\beta \) [32, 33]. \]
with (2.3) provides the following Ricci tensor components [34,113]

\[ R^\alpha_\beta = -\frac{4}{3} Q^2 e^{-14\mu} \delta^\alpha_\beta, \quad R^e_\beta = \frac{2}{3} Q^2 e^{-14\mu} \delta^e_\beta, \quad R^a_\beta = R^\gamma_\gamma = 0. \] (2.4)

On the other hand, the Ricci tensor components can be obtained from the following orthonormal basis which can be read off from the eleven-dimensional metric (2.1)

\[ e^1 = e^{-\frac{7}{2}u} \sqrt{g_{11}(x)} dx^1, \quad e^2 = e^{-\frac{7}{2}u} \sqrt{g_{22}(x)} dx^2, \]
\[ e^3 = e^{-\frac{7}{2}u} \sqrt{g_{33}(x)} dx^3, \quad e^4 = e^{-\frac{7}{2}u} \sqrt{g_{44}(x)} dx^4, \]
\[ e^5 = e^{\frac{u}{2} - \frac{5}{2}} v_\mu, \quad e^6 = \frac{1}{2} e^{\frac{u}{2} - \frac{5}{2}} \sin \mu, \]
\[ e^7 = \frac{1}{2} e^{\frac{u}{2} - \frac{5}{2}} \sin \mu, \quad e^8 = \frac{1}{2} e^{\frac{u}{2} - \frac{5}{2}} \cos \mu, \]
\[ e^9 = e^{u - 2v} \left( \Sigma_1 - \cos \mu \Sigma_1 \right), \quad e^{10} = e^{u - 2v} \sin \mu (\Sigma_2 - \cos \mu \Sigma_2), \]
\[ e^{11} = e^{u - 2v} \left[ \Sigma_3 - \frac{1}{2} (1 + \cos^2 \mu) \Sigma_3 \right]. \] (2.5)

The results for Ricci tensor in the basis of (2.5) are summarized by

\[ R^\alpha_\beta = e^{7u} \left( R^\alpha_\beta + \frac{7}{2} \delta^\alpha_\beta u^{\gamma} - \frac{63}{2} u^{\alpha} u_{;\beta} - 21 v^{\alpha} v_{;\beta} \right), \]
\[ R^5 = 6 e^{-2u - 3v} - 6 e^{-2u - 10v} - e^{7u} \left( u^{\alpha} + \frac{3}{2} v^{\alpha} \right) = R^6 = R^7 = R^8, \]
\[ R^9 = \frac{1}{2} e^{-2u + 4v} + 4 e^{-2u - 10v} - e^{7u} \left( u^{\alpha} - 2 v^{\alpha} \right) = R^{10} = R^{11}. \] (2.6)

Note that the only first terms in \( R^5 \) and \( R^9 \) are different from the one for squashed \( S^7 \) space [34,113].

Substituting the last two relations in (2.6) into (2.4) implies the field equations for the breathing mode \( u(x) \) and the squashing mode \( v(x) \) as follows:

\[ u^{\alpha}_{;\alpha} = \frac{3}{14} e^{-9u + 4v} + \frac{24}{7} e^{-9u - 3v} - \frac{12}{7} e^{-9u - 10v} - \frac{2}{3} Q^2 e^{-21u}, \]
\[ v^{\alpha}_{;\alpha} = -e^{-9u + 4v} + 12 e^{-9u - 3v} - 20 e^{-9u - 10v}. \] (2.7)

Note that the only first two terms in \( u^{\alpha}_{;\alpha} \) and \( v^{\alpha}_{;\alpha} \) are different from the one for squashed \( S^7 \) space [34,113]. The vanishing of the right hand side of second equation of (2.7) implies that either \( v = v_1 = \frac{1}{2} \ln 2 \) or \( v = v_2 = \frac{1}{2} \ln 10 \). Furthermore, substituting the field equation

\[ 4 \text{ When the scalar fields } u(x) \text{ and } v(x) \text{ are constant, then } R^5 = 6 e^{-7v} = 6 - 6 \lambda^2 = R^6 = R^7 = R^8 \text{ and } R^9 = 4 e^{-7v} + \frac{1}{2} e^{7v} = 4 \lambda^2 + \frac{1}{2} \lambda^2 = R^{10} = R^{11} \text{ where (2.22) is used. Then } \lambda^2 = \frac{1}{2} \text{ corresponds to the Einstein metric in } [15] \text{ and } \lambda^2 = \frac{1}{10} \text{ corresponds to the squashed Einstein metric } [17]. \]
for $u(x)$ in (2.7) into the first equation of (2.6) together with (2.4) implies the following four-dimensional Ricci tensor

$$R^\alpha_\beta = \frac{63}{2} u^\alpha u_\beta + 21 v^\alpha v_\beta + \delta^\alpha_\beta e^{-9u} \left( -\frac{3}{4} e^{4v} - 12 e^{-3v} + 6 e^{-10v} + Q^2 e^{-12u} \right). \quad (2.8)$$

Now the field equations (2.7) and (2.8) are equivalent to the Euler-Lagrange equations for the following effective Lagrangian

$$L = \sqrt{-g} \left[ R - \frac{63}{2} (\partial u)^2 - 21 (\partial v)^2 - V(u, v) \right] \quad (2.9)$$

with the scalar potential

$$V(u, v) = e^{-9u} \left( -\frac{3}{2} e^{4v} - 24 e^{-3v} + 12 e^{-10v} + 2Q^2 e^{-12u} \right). \quad (2.10)$$

Note that the first two terms in (2.10) are different from the one for squashed $S^7$ space [34, 10, 13].

One analyzes two vacua of this scalar potential as follows:

$$N_0^{0,1,0} : u = u_1 = \frac{1}{12} \ln \left( \frac{2}{20} Q^2 \right), \quad v = v_1 = \frac{1}{7} \ln 2, \quad \lambda^2 = \frac{1}{2},$$

$$\Lambda_1 = -9 \left| \frac{2Q^3}{3} \right|^{-\frac{1}{2}},$$

and

$$N_{II}^{0,1,0} : u = u_2 = \frac{1}{12} \ln \left( \frac{10}{34} Q^2 \right), \quad v = v_2 = \frac{1}{7} \ln 10, \quad \lambda^2 = \frac{1}{10},$$

$$\Lambda_2 = -3^6 \cdot 5^{-2} |10Q^3|^{-\frac{1}{2}}.$$

The two supergravity solutions are classically stable under the changes of the size and squashing parameter of seven-dimensional space [36]. The $N_0^{0,1,0}$ is a saddle point, corresponds to a minimum along the $v$-direction and is invariant under the $SU(3) \times SU(2)$ isometry group while $N_{II}^{0,1,0}$ is a maximum and is invariant under the same $SU(3) \times SU(2)$ isometry group. The left-handed seven-dimensional space $N_{LL}^{0,1,0}$ gives rise to a theory with $\mathcal{N} = 3$ supersymmetry while the right-handed seven-dimensional space $N_{L,R}^{0,1,0}$ gives rise to a theory with no supersymmetry ($\mathcal{N} = 0$) [17].

Moreover, the left-handed squashed seven-dimensional space $N_{II,L}^{0,1,0}$ gives rise to a theory with no supersymmetry while the right-handed squashed seven-dimensional space $N_{II,R}^{0,1,0}$ gives

---

[5] This form for the scalar potential was observed also previously in [35].
rise to a theory with $\mathcal{N} = 1$ supersymmetry. That is, with the choice of left-handed orientation of $N_{1}^{0,1,0}$, one regards the $\lambda^{2} = \frac{1}{2}$ metric as giving the unbroken vacuum state with $\mathcal{N} = 3$ supersymmetry which can be broken spontaneously to $\lambda^{2} = \frac{1}{10}$ metric yielding $\mathcal{N} = 0$ supersymmetry. On the other hand, with the choice of opposite orientation of $N_{R}^{0,1,0}$ the $\lambda^{2} = \frac{1}{10}$ metric provides a vacuum state with $\mathcal{N} = 1$ supersymmetry which can be broken to the $\lambda^{2} = \frac{1}{2}$ metric with $\mathcal{N} = 0$ supersymmetry.

Therefore, for the squashing with left-handed orientation, the RG flow interpolates between the boundary conformal field theories with $\mathcal{N} = 3$ and $\mathcal{N} = 0$ supersymmetry while for the squashing with right-handed orientation, the RG flow interpolates between conformal field theories with $\mathcal{N} = 0$ and $\mathcal{N} = 1$.[4]

3 Three-dimensional (super)conformal field theories

Using the previous results on the Kaluza-Klein spectrum under squashing deformations, an operator giving rise to a RG flow associated with the supersymmetry breaking will be identified and it turns out that the operator is relevant at the $N_{H}^{0,1,0}$ fixed point and irrelevant at the $N_{1}^{0,1,0}$ fixed point with same $SU(3) \times SU(2)$ symmetry groups.

- $SU(3) \times SU(2)$-invariant conformal fixed point

Let us consider the harmonic fluctuations of spacetime metric and $u(x)$ and $v(x)$ scalar fields around $AdS_{4} \times N_{1}^{0,1,0}$. Following [34, 10, 13], it is more convenient to rewrite (2.9) in terms of the unscaled M-theory metric $g_{\alpha\beta} = e^{-7u}g_{\alpha\beta}$ in (2.1):

$$\mathcal{L} = \sqrt{-\bar{g}}e^{7u} \left[ R - 2\Lambda_{1} - 105(\partial u)^{2} - 21(\partial v)^{2} - 2\nabla_{1}(u, v) \right],$$  \hspace{1cm} (3.1)

where the scalar potential is written as

$$\nabla_{1}(u, v) = -\Lambda_{1} \left[ 1 - \frac{1}{4}e^{-2(u-u_{1})}(e^{4(v-v_{1})} + 8e^{-3(v-v_{1})} - 2e^{-10(v-v_{1})}) + \frac{3}{4}e^{-14(u-u_{1})} \right]$$

It is known that the $N_{1}^{0,1,0}$ can be written as an $U(1)$-fibration over a Kahler-Einstein six-manifold that is an $S^{2}$-bundle over $\mathbb{CP}^{2}$ [37]. For the general squashed case $N_{1}^{0,1,0}$ including $N_{H}^{0,1,0}$, one can also get $U(1)$-fibration over a six-manifold by reorganizing the last line of (2.1):

$$ds^{2} = 2\lambda^{2}\left[ -\frac{1}{2}d\psi - \frac{1}{2}\cos \mu \sin \theta (\cos \phi \sigma_{1} - \sin \phi \sigma_{2}) + \frac{1}{4}\cos \theta (1 + \cos^{2} \mu)\sigma_{3} - \frac{1}{2}\cos \theta d\phi \right]^{2} + \frac{\lambda^{2}}{2}\left( |d\theta - \cos \mu (\sin \phi \sigma_{1} + \sin \phi \sigma_{2})|^{2} + \sin^{2} \theta \left( d\phi - \cos \mu \cot \theta (\cos \phi \sigma_{1} - \sin \phi \sigma_{2}) - \frac{1}{2}(1 + \cos^{2} \mu)\sigma_{3} \right) \right)^{2} + \frac{1}{2}\left( |d\mu + \frac{1}{4}\sin^{2} \mu (\sigma_{1}^{2} + \sigma_{2}^{2} + \cos^{2} \mu \sigma_{3}^{2}) | \right),$$

where we choose the same parametrization given in [37]. The Euler angles $\theta, \phi$ and $\psi$ correspond to the one-forms $\Sigma_{i}$. Of course, for the case of $\lambda^{2} = \frac{1}{2}$, this metric leads to the one in [37].
in which the un-rescaled cosmological constant $\Lambda_1 = e^{r_{u_1}} \Lambda_1 = \frac{1}{2} e^{r_{u_1}} V(u_1, v_1)$ is given by

$$\overline{\Lambda}_1 \equiv -3m_1^2 \ell_p^2 = -3 \left( \frac{|Q|}{6} \right)^{\frac{4}{3}} \ell_p^2$$

where $m_1 = \frac{1}{\overline{r}_{IR}}$. (3.2)

Here $\overline{r}_{IR}$ is related to $N$ and Planck scale $\ell_p$ as $\overline{r}_{IR} = \ell_p \frac{1}{2} (32 \pi^2 N)^{1/6}$.

By rescaling the scalar fields from the kinetic terms in the Lagrangian (3.1) as

$$\sqrt{210} u \equiv u, \sqrt{42} v \equiv v,$$

one obtains the fluctuation spectrum for $\tau$-field around the $N_1^{0,1,0}$ which takes a positive value:

$$M_{\tau\tau}(N_1^{0,1,0}) = \left( \frac{\partial^2}{\partial \tau^2} 2V_1 \right)_{\tau = \nu, \tau = \nu} = -\frac{4}{3} \overline{\Lambda}_1 \ell_p^2 = 4m_1^2$$

where the relation (3.2) is used.

Recall that in the compactification of $AdS_4 \times S^7$, the $\tau$-field represents the squashing of $S^7$ and hence ought to correspond to $300$ (that is the Young tableaux $[\begin{array}{c} 3 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]$ of $SO(8)$ or the $SO(8)$ Dynkin label is given by $(0,2,0,0)$: the lowest mode of the transverse, traceless symmetric tensor representation. The branching rule of the representation $300$ in terms of $SO(7)$ Dynkin labels is given by $[38, \ 39] \ 300([\begin{array}{c} 3 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]) \rightarrow [27([\begin{array}{c} 2 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]), 105([\begin{array}{c} 3 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]), 168([\begin{array}{c} 1 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}])]$ and the Lichnerowicz operator $\Delta_L$ in $SO(7)$ representation becomes $[27([\begin{array}{c} 2 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]), [32]$ which has a Dynkin label $(2,0,0)$ and its branching rule in terms of maximal subgroup $SU(4)$ Dynkin labels is given by $[27([\begin{array}{c} 2 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]) \rightarrow [1([\begin{array}{c} 1 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]), 6([\begin{array}{c} 0 \end{array} \begin{array}{c} 1 \end{array} \begin{array}{c} 0 \end{array}]), 20'([\begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}])$. Note that the $20'$ in $SU(4)$ is represented by a traceless symmetric matrix and the squashing should correspond to nonzero expectation value for the $20'$ of $SU(4)$. Then the branching rule of this $20'$ in terms of $SU(3)$ which is a global symmetry for $N_1^{0,1,0}$ is given by

$$20'([\begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]) \rightarrow 6([\begin{array}{c} 0 \end{array} \begin{array}{c} 1 \end{array} \begin{array}{c} 0 \end{array}]) \oplus 6([\begin{array}{c} 1 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]) \oplus 8([\begin{array}{c} 2 \end{array} \begin{array}{c} 0 \end{array} \begin{array}{c} 0 \end{array}]).$$

Moreover, the $6$ in $SU(3)$ is represented by a traceless symmetric matrix and the squashing should eventually correspond to nonzero expectation value for the $6$ of $SU(3)$.

According to the nice observations of [23], the harmonics of Lichnerowicz scalars with conformal dimension $\Delta = 4$ has been obtained. The harmonics are eigenfunctions of the Lichnerowicz operator with eigenvalue $M_{200} = 96m^2$ in their notation. Recall that the representation $(2,0,0)$ refers to $SO(7)$ representation $27$. With the help of [40], the value of $M_{200}$ can be obtained from $M_{\frac{1}{2} \frac{1}{2} \frac{1}{2}}$ which is an eigenvalue for Rarita Schwinger operator and it turns out $-16$. The representation $(\frac{3}{2} \frac{1}{2} \frac{1}{2})$ refers to $SO(7)$ representation $48$. The explicit
form is given by $M_{200} = (M^{311\over 3\over 3\over 3} + 4)(M^{311\over 3\over 3\over 3} + 8)m^2$. By plugging $M^{311\over 3\over 3\over 3} = -16$, one gets $M_{200} = 96m^2$. Note that $m^2$ is mass-squared parameter of a given $AdS_4$ spacetime and the mass of a scalar field $\phi$ in [40] is defined as $(\Delta_{AdS} + M_{200} - 32m^2)\phi = 0$. Then $96m^2$ goes to $64m^2$ by subtracting $32m^2$ and by dividing out 16 further in order to compare with the usual normalization in AdS/CFT correspondence, one arrives that the correct result for the mass-squared is $4m_2^2$ which is the same normalization used in [18]. For example, see also the recent paper [41] for the normalization between the conformal dimension and the mass-squared term. Therefore, the mass-squared for the representation $\overline{6}$ of $SU(3)$ is given by

$$M_{\overline{6}}^2 = 4m_1^2,$$

which is exactly equal to (3.3). Note that the assignment of $SU(2)_R$ isospin $J$ is related to the $SU(3)$ assignment and the R-symmetry group is the maximal $SU(2)$ subgroup of $SU(3)$. Under this embedding, the $SU(3)$ representations decompose into as follows:

$$1, \overline{1} \to J = 0, \quad 3, \overline{3} \to J = 1, \quad 6, \overline{6} \to J = 0 \oplus J = 2.$$

Or we have $\overline{6}(\overline{\Box}) = 1(\overline{\Box}) \oplus 5(\Box)$ under the breaking of $SU(3)$ into the $SU(2)_R$.

One concludes that, in three-dimensional conformal field theory with $N = 3$ supersymmetry, the $SU(3) \times SU(2)$ symmetric left-handed squashing should be an irrelevant perturbation of conformal dimension $\Delta = 4$. Note that this gives a non-supersymmetric theory for the right-handed orientation for seven-dimensional Einstein manifold $N^0_{1,R}$.

- $SU(3) \times SU(2)-$invariant conformal fixed point

Due to the skew-whipping, the theory will be either left-squashed $N_{0,LL}$ with $N = 0$ supersymmetry or right-squashed $N_{0,1,R}$ with $N = 1$ supersymmetry. The isometry of the squashed seven-dimensional Einstein manifold is given by $SU(3) \times SU(2)$. In terms of the unrescaled M-theory metric, the Lagrangian (2.9) can be rewritten as

$$\mathcal{L} = \sqrt{-g} e^{7u} \left[ \bar{R} - 2\bar{\Lambda}_2 - 105(\partial u)^2 - 21(\partial v)^2 - 2V_2(u,v) \right],$$

where the scalar potential is given by

$$V_2(u,v) = -\bar{\Lambda}_2 \left[ 1 - \frac{1}{36} e^{-2(u-u_2)} \left( 25e^{4(v-v_2)} + 40e^{-3(v-v_2)} - 2e^{-10(v-v_2)} \right) + \frac{3}{4} e^{-14(u-u_2)} \right],$$

and the un-rescaled cosmological constant $\bar{\Lambda}_2 = e^{7u_2} \Lambda_2 = \frac{1}{2} e^{7u_2} V(u_2,v_2)$ is given by

$$\bar{\Lambda}_2 \equiv -3m_2^2 \frac{1}{\ell_p^2} = -3 \left[ 3^7 \cdot 5^{-\frac{2}{3}} \left( \frac{|Q|}{6} \right)^{-\frac{2}{3}} \right] \frac{1}{\ell_p^2}, \quad \text{where} \quad m_2 = \frac{1}{\tau_{UV}}.$$
The mass spectrum of the $\overline{\nu}(x)$ field is calculated similarly

$$M^2_{\overline{\nu}}(N^0_{1,0}) \equiv \left( \frac{\partial^2}{\partial v^2} 2V_2 \right)_{\overline{\nu}=\nu=\overline{\nu}} = \frac{20}{27} \Lambda_2^2 = \frac{20}{9} m_2^2. \quad (3.4)$$

From the mass formula for the $SU(3) \times SU(2)$ representation and the eigenvalues of the Lichnerowicz operator, one should obtain the mass-squared for the singlet as follows: $M^2_{(1,1)} = -\frac{20}{9} m_2^2$, and this coincides with (3.4). The perturbation that corresponds to squashing around $N^0_{1,0}$ has a scaling dimension either $\Delta = 4/3$ or $5/3$ and hence corresponds to a relevant operator.

We gave a nonzero expectation value to a supergravity scalar in the $\overline{6}$ of $SU(3)$. Using the AdS/CFT correspondence, one identifies this perturbation with a composite operator of $N=3$ superconformal Chern-Simons matter theory with a mass term for the symmetric and traceless product between two $\overline{3}$'s: $\lambda^{AB} \int d^3x O_{AB}$ where $\lambda^{AB}$ is in the $6$ of $SU(3)$. Note that the tensor product of these leads to $\overline{3}(\overline{3}) \times \overline{3}(\overline{3}) = 3([\overline{3}] \oplus \overline{6}(\overline{3}))$. Then one can construct a $\overline{3}$ (that is the Young tableau $[\overline{3}]$) representation by using the Clebsch-Gordan coefficient $\Gamma_{AIJ}(A=1,2,3)$ which transforms two $\overline{3}$'s into $\overline{3}$ of $SU(3)$ ($3([\overline{3}] \times 3([\overline{3}]) = 3([\overline{3}] \oplus 6([\overline{3}])$ together with matter field $C^I$: $\Gamma_{AIJ} C^I C^J$ where $C^I(I=1,2,3)$ are three complex scalars ($3$ under the $SU(3)$) transforming as $(N,\overline{N})$ with gauge group $SU(N) \times SU(N)$ in $\mathcal{N} = 3$ superconformal Chern-Simons gauge theory $[19]$. The perturbation is given by

$$O_{AB} \sim \text{Tr} \Gamma_{AIJ} C^I C^J \Gamma_{BKL} C^K C^L.$$ 

The singlet of this operator $O_{AB}$ which is $\overline{6}$ of $SU(3)$ corresponds to the supergravity field $v(x)$ (or $\overline{\nu}(x)$) and the conformal dimensions are given by $\Delta_{UV} = \frac{4}{3}$ (or $\frac{5}{3}$) and $\Delta_{IR} = 4$ respectively as we computed before. The other five states with $J = 2$ among $\overline{6}$ of $SU(3)$ are non-diagonal and correspond to deformations of the seven-dimensional metric $[23]$. Since the Lichnerowicz operator provides nine-dimensional space, the remaining three states correspond to the eigenvalue $M^{200} = 0$. These are organized in a triplet ($J = 1$) of $SU(2)_R$ and the massless scalars belong to the additional massless vector multiplet. So far we considered only FR compactification where there are no internal components for the four-form field strength. In $[23]$, they further described the deformation from turning on an internal three-form.

## 4 Conclusions and outlook

We have constructed the full eleven-dimensional metric given by (2.1) and obtained the scalar potential in (2.10) by using the Freund-Rubin ansatz (2.3). The holographic supersymmetric (or nonsupersymmetric) RG flow from $\mathcal{N} = (0,1)$ $SU(3) \times SU(2)$-invariant UV fixed point
to $\mathcal{N} = (3, 0)$ $SU(3) \times SU(2)$-invariant IR fixed point was described. The corresponding operator in three-dimensional Chern-Simons matter theories is identified.

For the seven-dimensional Einstein metric, we considered $SO(3)$-bundle over the base $\mathbb{C}P^2$ in (2.1). Also the base $\mathbb{C}P^2$ can be replaced by either $S^4$ or $\mathbb{C}P^1 \times \mathbb{C}P^1$. It would be interesting to find out whether these metrics provide the new nontrivial eleven-dimensional solutions. Are there any new general $AdS_4$ vacua corresponding to any supersymmetric Chern-Simons matter theories? Are there any new critical points in the context of $SO(3)$ gauged supergravity(which might be related to $\mathcal{N} = 3$ Chern-Simons matter theory) or $SO(4)$ gauged supergravity?

Acknowledgments

This work was supported by grant No. R01-2006-000-10965-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

References

[1] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” arXiv:0806.1218 [hep-th].

[2] B. E. W. Nilsson and C. N. Pope, “Hopf Fibration Of Eleven-Dimensional Supergravity,” Class. Quant. Grav. 1, 499 (1984).

[3] P. G. O. Freund and M. A. Rubin, “Dynamics Of Dimensional Reduction,” Phys. Lett. B 97, 233 (1980).

[4] G. Jensen, “Einstein metrics on principal fire bundles,” J. Diff. Geom. 8 (1973) 599.

[5] J. P. Bourguignon and H. Karcher, Ann. Sci. Normale Sup. 11 (1978) 71.

[6] M. A. Awada, M. J. Duff and C. N. Pope, “N = 8 supergravity breaks down to N = 1,” Phys. Rev. Lett. 50, 294 (1983).

[7] M. J. Duff, B. E. W. Nilsson and C. N. Pope, “Spontaneous Supersymmetry Breaking By The Squashed Seven Sphere,” Phys. Rev. Lett. 50, 2043 (1983); 51, 846 (1983) (errata).

[8] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, “Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics,” JHEP 9812, 022 (1998).
[9] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, Adv. Theor. Math. Phys. 3, 363 (1999).

[10] C. Ahn and S. J. Rey, “Three-dimensional CFTs and RG flow from squashing M2-brane horizon,” Nucl. Phys. B 565, 210 (2000).

[11] C. Ahn, “Holographic Supergravity Dual to Three Dimensional N=2 Gauge Theory,” JHEP 0808, 083 (2008).

[12] C. Ahn, “Towards Holographic Gravity Dual of N=1 Superconformal Chern-Simons Gauge Theory,” JHEP 0807, 101 (2008).

[13] C. Ahn, “Squashing Gravity Dual of N=6 Superconformal Chern-Simons Gauge Theory,” arXiv:0809.3684 [hep-th].

[14] H. Ooguri and C. S. Park, “Superconformal Chern-Simons Theories and the Squashed Seven Sphere,” arXiv:0808.0500 [hep-th].

[15] L. Castellani and L. J. Romans, “N=3 And N=1 Supersymmetry In A New Class Of Solutions For D = 11 Supergravity,” Nucl. Phys. B 238, 683 (1984).

[16] L. Castellani, “The Mass Spectrum In The SU(3) X U(1) Compactifications Of D = 11 Supergravity,” Nucl. Phys. B 254, 266 (1985).

[17] D. N. Page and C. N. Pope, “New Squashed Solutions Of D = 11 Supergravity,” Phys. Lett. B 147, 55 (1984).

[18] C. Ahn and S. J. Rey, “More CFTs and RG flows from deforming M2/M5-brane horizon,” Nucl. Phys. B 572, 188 (2000).

[19] M. Billo, D. Fabbri, P. Fre, P. Merlatti and A. Zaffaroni, “Rings of short N = 3 superfields in three dimensions and M-theory on AdS(4) x N(0,1,0),” Class. Quant. Grav. 18, 1269 (2001).

[20] D. Gaiotto and X. Yin, “Notes on superconformal Chern-Simons-matter theories,” JHEP 0708, 056 (2007).

[21] P. Termonia, “The complete N = 3 Kaluza-Klein spectrum of 11D supergravity on AdS(4) x N(010),” Nucl. Phys. B 577, 341 (2000).
[22] P. Fre’, L. Gualtieri and P. Termonia, “The structure of $N = 3$ multiplets in $\text{AdS}(4)$ and the complete $\text{Osp}(3—4) \times \text{SU}(3)$ spectrum of $\text{M}$-theory on $\text{AdS}(4) \times N(0,1,0)$,” Phys. Lett. B 471, 27 (1999).

[23] M. Billo, D. Fabbri, P. Fre, P. Merlatti and A. Zaffaroni, “Shadow multiplets in $\text{AdS}(4)/\text{CFT}(3)$ and the super-Higgs mechanism,” Nucl. Phys. B 591, 139 (2000).

[24] D. Fabbri and P. Fre, “Shadow multiplets and superHiggs mechanism,” Fortsch. Phys. 49, 475 (2001).

[25] B. S. Acharya, J. M. Figueroa-O’Farrill, C. M. Hull and B. J. Spence, “Branes at conical singularities and holography,” Adv. Theor. Math. Phys. 2, 1249 (1999).

[26] D. R. Morrison and M. R. Plesser, “Non-spherical horizons. I,” Adv. Theor. Math. Phys. 3, 1 (1999).

[27] D. L. Jafferis and A. Tomasiello, “A simple class of $N=3$ gauge/gravity duals,” arXiv:0808.0864 [hep-th].

[28] K. M. Lee and H. U. Yee, “New $\text{AdS}(4) \times X(7)$ geometries with $N = 6$ in $\text{M}$ theory,” JHEP 0703, 012 (2007).

[29] B. M. Zupnik and D. V. Khetselius, “Three-dimensional extended supersymmetry in harmonic superspace,” Sov. J. Nucl. Phys. 47 (1988) 730 [Yad. Fiz. 47 (1988) 1147].

[30] H. C. Kao, “Selfdual Yang-Mills Chern-Simons Higgs systems with an $N=3$ extended supersymmetry,” Phys. Rev. D 50, 2881 (1994).

[31] H. C. Kao, K. M. Lee and T. Lee, “The Chern-Simons coefficient in supersymmetric Yang-Mills Chern-Simons theories,” Phys. Lett. B 373, 94 (1996).

[32] M. J. Duff, B. E. W. Nilsson and C. N. Pope, “Kaluza-Klein Supergravity,” Phys. Rept. 130, 1 (1986).

[33] C. Ahn, “More Penrose limit of $\text{AdS}(4) \times N(0,1,0)$ and $N = 3$ gauge theory,” Mod. Phys. Lett. A 17, 1847 (2002).

[34] D. N. Page, “Classical Stability Of Round And Squashed Seven Spheres In Eleven-Dimensional Supergravity,” Phys. Rev. D 28, 2976 (1983).

[35] V. L. Campos, G. Ferretti, H. Larsson, D. Martelli and B. E. W. Nilsson, “A study of holographic renormalization group flows in $d = 6$ and $d = 3$,” JHEP 0006, 023 (2000).
[36] O. Yasuda, “Classical Stability Of M (Pqr), Q (Pqr) And N (Pqr) In D = 11 Supergravity,” Phys. Rev. Lett. 53, 1207 (1984).

[37] J. P. Gauntlett, S. Lee, T. Mateos and D. Waldram, “Marginal deformations of field theories with AdS(4) duals,” JHEP 0508, 030 (2005).

[38] R. Slansky, “Group Theory For Unified Model Building,” Phys. Rept. 79, 1 (1981).

[39] J. Patera and D. Sankoff, Tables of Branching Rules for Representations of Simple Lie Algebras (L’Université de Montréal, Montréal, 1973); W. MacKay and J. Patera, Tables of Dimensions, Indices and Branching Rules for representations of Simple Algebras (Dekker, New York, 1981).

[40] R. D’Auria and P. Fre, “Universal Bose-Fermi Mass Relations In Kaluza-Klein Supergravity And Harmonic Analysis On Coset Manifolds With Killing Spinors,” Annals Phys. 162, 372 (1985).

[41] I. Klebanov, T. Klose and A. Murugan, “AdS$_4$/CFT$_3$ – Squashed, Stretched and Warped,” arXiv:0809.3773 [hep-th].