COVARIANCE MATRIX ESTIMATION FOR STATIONARY TIME SERIES

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We obtain a sharp convergence rate for banded covariance matrix estimates of stationary processes. A precise order of magnitude is derived for spectral radius of sample covariance matrices. We also consider a thresholded covariance matrix estimator that can better characterize sparsity if the true covariance matrix is sparse. As our main tool, we implement Toeplitz [Math. Ann. 70 (1911) 351–376] idea and relate eigenvalues of covariance matrices to the spectral densities or Fourier transforms of the covariances. We develop a large deviation result for quadratic forms of stationary processes using $m$-dependence approximation, under the framework of causal representation and physical dependence measures.

1. Introduction. One hundred years ago, in 1911, Toeplitz obtained a deep result on eigenvalues of infinite matrices of the form $\Sigma_\infty = (a_{s-t})_{s,t=\infty}^{\infty}$. We say that $\lambda$ is an eigenvalue of $\Sigma_\infty$ if the matrix $\Sigma_\infty - \lambda I_\infty$ does not have a bounded inverse, where $I_\infty$ denotes the infinite-dimensional identity matrix. Toeplitz proved that, interestingly, the set of eigenvalues is the same as the image set \{\(g(\theta), \theta \in [0, 2\pi]\}\}, where
\[
g(\theta) = \sum_{t \in \mathbb{Z}} a_t e^{i\theta} \quad \text{with} \quad i = \sqrt{-1}.
\]

Note that $g(\theta)$ is the Fourier transform of the sequence $(a_t)_{t=\infty}^{\infty}$. For a finite $T \times T$ matrix $\Sigma_T = (a_{s-t})_{1 \leq s,t \leq T}$, its eigenvalues are approximately equally distributed (in the sense of Hermann Weyl) as \{\(g(\omega_s), s = 0, \ldots, T - 1\}\}, where $\omega_s = 2\pi s/T$ are the Fourier frequencies. See the excellent monograph by Grenander and Szegö (1958) for a detailed account.
Covariance matrix is of fundamental importance in many aspects of statistics including multivariate analysis, principal component analysis, linear discriminant analysis and graphical modeling. One can infer dependence structures among variables by estimating the associated covariance matrices. In the context of stationary time series analysis, due to stationarity, the covariance matrix is Toeplitz in that, along the off-diagonals that are parallel to the main diagonal, the values are constant. Let \((X_t)_{t \in \mathbb{Z}}\) be a stationary process with mean \(\mu = \mathbb{E}X_t\), and denote by \(\gamma_k = \mathbb{E}[(X_0 - \mu)(X_k - \mu)], k \in \mathbb{Z}\), its autocovariances. Then

\[
\Sigma_T = (\gamma_{s-t})_{1 \leq s,t \leq T}
\]

is the autocovariance matrix of \((X_1, \ldots, X_T)\). In the rest of the paper for simplicity we also call (2) the covariance matrix of \((X_1, \ldots, X_T)\). In time series analysis it plays a crucial role in prediction [Kolmogoroff (1941), Wiener (1949)], smoothing and best linear unbiased estimation (BLUE). For example, in the Wiener–Kolmogorov prediction theory, one predicts \(X_{T+1}\) based on past observations \(X_T, X_{T-1}, \ldots\) If the covariances \(\gamma_k\) were known, given observations \(X_1, \ldots, X_T\), the coefficients of the best linear unbiased predictor \(\hat{X}_{T+1} = \sum_{s=1}^{T} a_{T,s} X_{T+1-s}\) in terms of the mean square error \(\|X_{T+1} - \hat{X}_{T+1}\|^2\) are the solution of the discrete Wiener–Hopf equation

\[
\Sigma_T a_T = \gamma_T,
\]

where \(a_T = (a_{T,1}, a_{T,2}, \ldots, a_{T,T})^\top\) and \(\gamma_T = (\gamma_1, \gamma_2, \ldots, \gamma_T)^\top\), and we use the superscript \(^\top\) to denote the transpose of a vector or a matrix. If \(\gamma_k\) are not known, we need to estimate them from the sample \(X_1, \ldots, X_T\), and a good estimate of \(\Sigma_T\) is required. As another example, suppose now \(\mu = \mathbb{E}X_t \neq 0\) and we want to estimate it from \(X_1, \ldots, X_T\) by the form \(\hat{\mu} = \sum_{t=1}^{T} c_t X_t\), where \(c_t\) satisfy the constraint \(\sum_{t=1}^{T} c_t = 1\). To obtain the BLUE, one minimizes \(\mathbb{E}(\hat{\mu} - \mu)^2\) subject to \(\sum_{t=1}^{T} c_t = 1\), ensuring unbiasedness. Note that the usual choice \(c_t \equiv 1/T\) may not lead to BLUE. The optimal coefficients are given by \((c_1, \ldots, c_T)^\top = (1^\top \Sigma_T^{-1} 1)^{-1} \Sigma_T^{-1} 1\), where \(1 = (1, \ldots, 1)^\top \in \mathbb{R}^T\); see Adenstedt (1974). Again a good estimate of \(\Sigma_T^{-1}\) is needed.

Given observations \(X_1, X_2, \ldots, X_T\), assuming at the outset that \(\mathbb{E}X_t = 0\), we can naturally estimate \(\Sigma_T\) via plug-in by the sample version

\[
\hat{\Sigma}_T = (\hat{\gamma}_{s-t})_{1 \leq s,t \leq T} \quad \text{where} \quad \hat{\gamma}_k = \frac{1}{T} \sum_{t=|k|+1}^{T} X_{t-|k|} X_t.
\]

To judge the quality of a matrix estimate, we use the operator norm. The term “operator norm” usually indicates a class of matrix norms; in this paper it refers to \(\ell_2/\ell_2\) operator norm or spectral radius defined as \(\lambda(A) := \max_{|x|=1} |Ax|\) for any symmetric real matrix \(A\), where \(x\) is a real vector, and \(|x|\) denotes its Euclidean norm. For the estimate \(\hat{\Sigma}_T\) in (3), un-
fortunately, because too many parameters or autocovariances are estimated and the signal-to-noise ratios are too small at large lags, this estimate is not consistent. Wu and Pourahmadi (2009) showed that $\lambda(\hat{\Sigma}_T - \Sigma_T) \not\to 0$ in probability. In Section 2 we provide a precise order of magnitude of $\lambda(\hat{\Sigma}_T - \Sigma_T)$ and give explicit upper and lower bounds.

The inconsistency of sample covariance matrices has also been observed in the context of high-dimensional multivariate analysis, and this phenomenon is now well understood, thanks to the results from random matrix theory. See, among others, Marčenko and Pastur (1967), Bai and Yin (1993) and Johnstone (2001). Recently, there is a surge of interest on regularized covariance matrix estimation in high-dimensional statistical inference. We only sample a few works which are closely related to our problem. Cai, Zhang and Zhou (2010), Bickel and Levina (2008b) and Furrer and Bengtsson (2007) studied the banding and/or tapering methods, while Bickel and Levina (2008a) and El Karoui (2008) considered the regularization by thresholding. In most of these works, convergence rates of the estimates were established.

However, none of the techniques used in the aforementioned papers is applicable in our setting since their estimates require multiple independent and identically distributed (i.i.d.) copies of random vectors from the underlying multivariate distribution, though the number of copies can be far less than the dimension of the vector. In time series analysis, however, it is typical that only one realization is available. Hence we shall naturally use the sample autocovariances. In a companion paper, Xiao and Wu (2011) established a systematic theory for $L^2$ and $L^\infty$ deviations of sample autocovariances. Based on that, we adopt the regularization idea and study properties of the banded, tapered and thresholded estimates of the covariance matrices. Wu and Pourahmadi (2009) and McMurry and Politis (2010) applied the banding and tapering methods to the same problem, but here we shall obtain a better and optimal convergence rate. We shall point out that the regularization ideas of banding and tapering are not novel in time series analysis and they have been applied in nonparametric spectral density estimation.

In this paper we use the ideas in Toeplitz (1911) and Grenander and Szegö (1958) together with Wu’s (2005) recent theory on stationary processes to present a systematic theory for estimates of covariance matrices of stationary processes. In particular, we shall exploit the connection between covariance matrices and spectral density functions and prove a sharp convergence rate for banded covariance matrix estimates of stationary processes. Using convergence properties of periodograms, we derive a precise order of magnitude for spectral radius of sample covariance matrices. We also consider a thresholded covariance matrix estimator that can better characterize sparsity if the true covariance matrix is sparse. As a main technical tool, we develop a large deviation type result for quadratic forms of stationary processes using $m$-dependence approximation, under the framework of causal representations and physical dependence measures.
The rest of this article is organized as follows. In Section 2 we introduce the framework of causal representation and physical dependence measures that are useful for studying convergence properties of covariance matrix estimates. We provide in Section 2 upper and lower bounds for the operator norm of the sample covariance matrices. The convergence rates of banded/tapered and thresholded sample covariance matrices are established in Sections 3 and 4, respectively. We also conduct a simulation study to compare the finite sample performances of banded and thresholded estimates in Section 5. Some useful moment inequalities are collected in Section 6. A large deviation result about quadratic forms of stationary processes, which is of independent interest, is given in Section 7. Section 8 concludes the paper.

We now introduce some notation. For a random variable $X$ and $p > 0$, we write $X \in L^p$ if
$$
\|X\|_p := \left( \mathbb{E}|X|^p \right)^{1/p} < \infty,
$$
and use $\|X\|_2$ as a shorthand for $\|X\|_2$. To express centering of random variables concisely, we define the operator $E_0$ as
$$
E_0(X) := X - E(X).
$$
Hence $E_0(E_0(X)) = E_0(X)$. For a symmetric real matrix $A$, we use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ for its smallest and largest eigenvalues, respectively, and use $\lambda(A)$ to denote its operator norm or spectral radius. For a real number $x$, $\lfloor x \rfloor := \max\{y \in \mathbb{Z} : y \leq x\}$ denotes its integer part and $\lceil x \rceil := \min\{y \in \mathbb{Z} : y \geq x\}$. For two real numbers $x,y$, set $x \lor y = \max\{x,y\}$ and $x \land y = \min\{x,y\}$. For two sequences of positive numbers $(a_T)$ and $(b_T)$, we write $a_T \asymp b_T$ if there exists some constant $C > 1$ such that $C^{-1} \leq a_T/b_T \leq C$ for all $T$. The letter $C$ denotes a constant, whose values may vary from place to place. We sometimes add symbolic subscripts to emphasize that the value of $C$ depends on the subscripts.

2. Exact order of operator norms of sample covariance matrices. Suppose $Y$ is a $p \times n$ random matrix consisting of i.i.d. entries with mean 0 and variance 1, which could be viewed as a sample of size $n$ from some $p$-dimensional population; then $YY^\top/n$ is the sample covariance matrix. If $\lim_{n \to \infty} p/n = c > 0$, then $YY^\top/n$ is not a consistent estimate of the population covariance matrix (which is the identity matrix) in term of the operator norm. This is a well-known phenomenon in random matrices literature; see, for example, Marcenko and Pastur (1967), Section 5.2 in Bai and Silverstein (2010), Johnstone (2001) and El Karoui (2005). However, the techniques used in those papers are not applicable here, because we have only one realization and the dependence within the vector can be quite complicated. Thanks to the Toeplitz structure of $\Sigma_T$, our method depends on the connection between its eigenvalues and the spectral density, defined by

$$
f(\theta) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \gamma_k \cos(k\theta).
$$

The following lemma is a special case of Section 5.2 [Grenander and Szegö (1958)].
LEMMA 1. Let $h$ be a continuous symmetric function on $[-\pi, \pi]$. Denote by $\underline{h}$ and $\overline{h}$ its minimum and maximum, respectively. Define $a_h = \int_{-\pi}^{\pi} h(\theta) e^{-ik\theta} d\theta$ and the $T \times T$ matrix $\Gamma_T = (a_{s-t})_{1 \leq s, t \leq T}$; then

$$2\pi \underline{h} \leq \lambda_{\min}(\Gamma_T) \leq \lambda_{\max}(\Gamma_T) \leq 2\pi \overline{h}.$$  

Lemma 1 can be easily proved by noting that

$$\mathbf{x}^T \Gamma_T \mathbf{x} = \int_{-\pi}^{\pi} |\mathbf{x}^T \rho(\theta)|^2 h(\theta) d\theta \quad \text{where} \quad \rho(\theta) = (e^{i\theta}, e^{i2\theta}, \ldots, e^{iT\theta})^T.$$  

The sample covariance matrix (3) is closely related to the periodogram

$$I_T(\theta) = T^{-1} \left| \sum_{t=1}^{T} X_t e^{it\theta} \right|^2 = \sum_{k=1-T}^{T-1} \hat{\gamma}_k e^{ik\theta}.$$  

By Lemma 1, we have $\lambda(\hat{\Sigma}_T) \leq \max_{-\pi \leq \theta \leq \pi} I_T(\theta)$. Asymptotic properties of periodograms have recently been studied by Peligrad and Wu (2010) and Lin and Liu (2009). To introduce the result in the latter paper, we assume that the process $(X_t)$ has the causal representation

$$X_t = g(\varepsilon_t, \varepsilon_{t-1}, \ldots),$$

where $g$ is a measurable function such that $X_t$ is well defined, and $\varepsilon_t, t \in \mathbb{Z}$, are i.i.d. random variables. The framework (6) is very general [see, e.g., Tong (1990)] and easy to work with. Let $\mathcal{F}_t = (\varepsilon_t, \varepsilon_{t-1}, \ldots)$ be the set of innovations up to time $t$; we write $X_t = g(\mathcal{F}_t)$. Let $\varepsilon'_t, t \in \mathbb{Z}$, be an i.i.d. copy of $\varepsilon_t, t \in \mathbb{Z}$. Define $\mathcal{F}'_t = (\varepsilon'_t, \varepsilon_{t-1}', \varepsilon_{t-2}', \ldots)$, obtained by replacing $\varepsilon_0$ in $\mathcal{F}_t$ by $\varepsilon'_0$, and set $X'_t = g(\mathcal{F}'_t)$. Following Wu (2005), for $p > 0$, define

$$\Theta_p(m) = \sum_{t=m}^{\infty} \delta_p(t), \quad m \geq 0, \quad \text{where} \quad \delta_p(t) = \|X_t - X'_t\|_p.$$  

In Wu (2005), the quantity $\delta_p(t)$ is called physical dependence measure. We make the convention that $\delta_p(t) = 0$ for $t < 0$. Throughout the article, we assume the short-range dependence condition $\Theta_p := \Theta_p(0) < \infty$. Under a mild condition on the tail sum $\Theta_p(m)$ (cf. Theorem 2), Lin and Liu (2009) obtained the weak convergence result

$$\max_{1 \leq s \leq q} \left\{ \frac{I_T(2\pi s/T)}{2\pi f(2\pi s/T)} \right\} - \log q \Rightarrow \mathcal{G},$$

where $\Rightarrow$ denotes the convergence in distribution, $\mathcal{G}$ is the Gumbel distribution with the distribution function $e^{-e^{-x}}$, and $q = \lceil T/2 \rceil - 1$. Using this result, we can provide explicit upper and lower bounds on the operator norm of the sample covariance matrix.
Theorem 2. Assume $X_t \in \mathcal{L}^p$ for some $p > 2$ and $\mathbb{E}X_t = 0$. If $\Theta_p(m) = o(1/\log m)$ and $\min_{\theta} f(\theta) > 0$, then
\[
\lim_{T \to \infty} P \left\{ \frac{\pi [\min_{\theta} f(\theta)]^2 \log T}{12 \Theta_2^2} \leq \lambda(\hat{\Sigma}_T) \leq 10 \Theta_2^2 \log T \right\} = 1.
\]

According to Lemma 1, we know $\lambda_{\max}(\Sigma_T) \leq 2\pi \max_{\theta} f(\theta)$. As an immediate consequence of Theorem 2, there exists a constant $C > 1$ such that
\[
\lim_{T \to \infty} P \left[ C^{-1} \log T \leq \lambda(\hat{\Sigma}_T - \Sigma_T) \leq C \log T \right] = 1,
\]
which means the estimate $\hat{\Sigma}_T$ is not consistent, and the order of magnitude of $\lambda(\hat{\Sigma}_T - \Sigma_T)$ is $\log T$. Earlier, Wu and Pourahmadi (2009) also showed that the plug-in estimate $\hat{\Sigma}_T = (\hat{\gamma}_{s-t})_{1 \leq s,t \leq T}$ is not consistent, namely, $\lambda(\hat{\Sigma}_T - \Sigma) \not\to 0$ in probability. Our Proposition 2 substantially refines this result by providing an exact order of magnitude of $\lambda(\hat{\Sigma}_T - \Sigma)$.

An, Chen and Hannan (1983) showed that under suitable conditions, for linear processes with i.i.d. innovations,
\[
\lim_{T \to \infty} \max_{\theta} \left\{ I_T(\theta)/[2\pi f(\theta) \log T] \right\} = 1 \quad \text{almost surely.}
\]
A stronger version was found by Turkman and Walker (1990) for Gaussian processes. Based on (9), we conjecture that
\[
\lim_{T \to \infty} \frac{\lambda(\hat{\Sigma}_T)}{2\pi \max_{\theta} f(\theta) \log T} = 1 \quad \text{almost surely.}
\]
Turkman and Walker (1984) established the following result on the maximum periodogram of a sequence of i.i.d. standard normal random variables:
\[
\max_{\theta} I_T(\theta) - \log T - \frac{\log(\log T)}{2} + \frac{\log(3/\pi)}{2} \Rightarrow \mathcal{G}.
\]
In view of (8) and (10), we conjecture that $\lambda(\hat{\Sigma}_T)$ also converges to the Gumbel distribution after proper centering and rescaling. Note that the Gumbel convergence (10), where the maximum is taken over the entire interval $\theta \in [-\pi, \pi]$, has a different centering term from the one in (8) which is obtained over Fourier frequencies.

If $Y$ is a $p \times n$ random matrix consisting of i.i.d. entries, Geman (1980) and Yin, Bai and Krishnaiah (1988) proved a strong convergence result for the largest eigenvalues of $Y^\top Y$, in the paradigm where $n, p \to \infty$ such that $p/n \to c \in (0, \infty)$. See also Bai and Silverstein (2010) and references therein. If in addition the entries of $Y$ are i.i.d. complex normal or normal random variables, Johansson (2000) and Johnstone (2001) presented an asymptotic distributional theory and showed that, after proper normalization, the limiting distribution of the largest eigenvalue follows the Tracy–Widom law.
[Tracy and Widom (1994)]. Again, their methods depend heavily on the setup that there are i.i.d. copies of a random vector with independent entries, and/or the normality assumption, so they are not applicable here. Bryc, Dembo and Jiang (2006) studied the limiting spectral distribution (LSD) of random Toeplitz matrices whose entries on different sub-diagonals are i.i.d. Solo (2010) considered the LSD of sample covariances matrices generated by a sample which is temporally dependent.

**Proof of Theorem 2.** For notational simplicity we let $f := \min_\theta f(\theta)$ and $\overline{T} := \max_\theta f(\theta)$. It follows immediately from (8) that for any $\delta > 0$

\[
\lim_{T \to \infty} P \left[ \max_{\theta} I_T(\theta) \geq 2(1 - \delta)\pi \overline{T} \log T \right] = 1.
\]

The result in (8) is not sufficient to yield an upper bound of $\max_\theta I_T(\theta)$. For this purpose we need to consider the maxima over a finer grid and then use Lemma 3 to extend to the maxima over the whole real line. Set $j_T = \lfloor T \log T \rfloor$ and $\theta_s := \theta_{T,s} = \pi s/j_T$ for $0 < s < j_T$. Define $m_T = \lfloor T \beta \rfloor$ for some $0 < \beta < 1$.

Now partition the interval $[1, T]$ into blocks $B_1, B_2, \ldots, B_w$ of size $m_T$, where $w_T = \lfloor T/m_T \rfloor$, and the last block may have size $m_T \leq |B|w_T < 2m_T$. Define the block sum $U_{T,k}(\theta) = \sum_{t \in B_k} \tilde{X}_t e^{it\theta}$ for $1 \leq k \leq w_T$. Choose $\beta > 0$ small enough such that for some $0 < \gamma < 1/2$, the inequality

\[
1 - \beta + \beta p - \gamma(p - 1) - 1/2 < 0
\]

holds. We use truncation and define $\overline{U}_{T,k}(\theta) = \mathbb{E}_0[U_{T,k}(\theta)1_{|U_{T,k}(\theta)| \leq T \gamma}]$. Using similar arguments as equation (5.5) [Lin and Liu (2009)] and (13), we have

\[
\max_{0 \leq s \leq j_T} T^{-1/2} \left| \sum_{k=1}^{w_T} [U_{T,k}(\theta_s) - \overline{U}_{T,k}(\theta_s)] \right| = o_P((\log T)^{-1/2}).
\]

Observe that $\overline{U}_{T,k_1}(\theta)$ and $\overline{U}_{T,k_2}(\theta)$ are independent if $|k_1 - k_2| > 1$. Let $\mathfrak{R}(z)$ denote the real part of a complex number $z$. Split the sum $\sum_{k=1}^{w_T} \overline{U}_{T,k}(\theta)$ into four parts

\[
R_{T,1}(\theta) = \sum_{k \text{ odd}} \mathfrak{R}(\overline{U}_{T,k}(\theta)), \quad R_{T,2}(\theta) = \sum_{k \text{ even}} \mathfrak{R}(\overline{U}_{T,k}(\theta))
\]
and $R_{T,3}, R_{T,4}$ similarly for the imaginary part of $\overline{U}_{T,k}$. Since $\mathbb{E}|U_{T,k}(\theta)|^2 \leq |\mathbb{E}|B_{k}|\Theta|^2$, by Bernstein’s inequality [cf. Freedman (1975)],

$$\sup_\theta P\left[ |R_{T,l}(\theta)| \geq \frac{3\Theta_2}{2\sqrt{2}} \sqrt{T \log T} \right] \leq 2 \exp\left\{ -\frac{(9/8) \log T}{1 + 3\Theta_2^{-1} \sqrt{2} \log T \gamma^{-1/2}} \right\}$$

for $1 \leq l \leq 4$. It follows that

$$\lim_{T \to \infty} P\left[ \max_{0 \leq s \leq T} \left| \sum_{k=1}^{w_T} U_{T,k}(\theta_s) \right| \geq 3\Theta_2 \sqrt{T \log T} \right] = 0. \quad (15)$$

Combining (12), (14) and (15), we have

$$\lim_{T \to \infty} P\left[ \max_{0 \leq s \leq T} I_T(\theta_s) \leq 9.5\Theta_2^2 \log T \right] = 1,$n

which together with Lemma 3 implies that

$$\lim_{T \to \infty} P\left[ \max_\theta I_T(\theta) \leq 10\Theta_2^2 \log T \right] = 1. \quad (17)$$

The upper bound in Theorem 2 is an immediate consequence in view of Lemma 1. For the lower bound, we use the inequality

$$\lambda(\Sigma_T) \geq \max_\theta \{ T^{-1} \rho(\theta)^* \Sigma_T \rho(\theta) \},$$

where $\rho(\theta)$ is defined in (5), and $\rho(\theta)^*$ is its Hermitian transpose. Note that

$$\rho(\theta)^* \Sigma_T \rho(\theta) = \sum_{s,t=1}^{T} \hat{\gamma}_{s-t} e^{is\theta} e^{-it\theta}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{s,t=1}^{T} I_T(\omega) e^{-i(s-t)\omega} e^{i(s-t)\theta} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} I_T(\omega) \left| \sum_{t=1}^{T} e^{it(\omega-\theta)} \right|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi-\theta}^{\pi-\theta} I_T(\omega + \theta) \left| \sum_{t=1}^{T} e^{it\omega} \right|^2 d\omega.$$ 

By Bernstein’s inequality on the derivative of trigonometric polynomials [cf. Zygmund (2002), Theorem 3.13, Chapter X], we have

$$\max_\theta |I_T'(\theta)| \leq T \cdot \max_\theta I_T(\theta).$$

Let $\theta_0 = \arg \max_\theta I_T(\theta)$. Set $c = (1 - \delta)\pi f / (10\Theta_2^2)$. By Lemma 1 and (40), we know $2\pi f \leq \Theta_2^2$, and hence $c \leq 1/20$. If $I_T(\theta_0) \geq 2(1 - \delta)\pi f \log T$ and $\max_\theta I_T(\theta) \leq 10\Theta_2^2 \log T$, then for $|w| \leq c/T$, we have

$$I_T(\theta_0 + \omega) \geq [2(1 - \delta)\pi f - 10c\Theta_2] \log T = (1 - \delta)\pi f \log T.$$
Since $|\sum_{j=1}^{T} e^{i\omega_j}|^2 \geq 10T^2/11$ when $|w| \leq c/T$, it follows that
\[
\rho(\theta_0)\Sigma_T \rho(\theta_0) \geq \frac{1}{2\pi} \cdot (1 - \delta) \pi f \log T \cdot \frac{10T^2}{11} \cdot \frac{2c}{T}
\]
which implies that $\lambda(\hat{\Sigma}_T) \geq \pi(1 - \delta)^2 f^2 T \log T/\Theta_2^2$. The proof is completed by selecting $\delta$ small enough. □

**Remark 1.** In the proof, as well as many other places in this article, we often need to partition an integer interval $[s, t] \subset \mathbb{N}$ into consecutive blocks $B_1, \ldots, B_b$ with the same size $m$. Since $s - t + 1$ may not be a multiple of $m$, we make the convention that the last block $B_b$ has the size $m \leq |B_b| < 2m$, and all the other ones have the same size $m$.

### 3. Banded covariance matrix estimates

In view of Lemma 1, the inconsistency of $\hat{\Sigma}_T$ is due to the fact that the periodogram $I_T(\theta)$ is not a consistent estimate of the spectral density $f(\theta)$. To estimate the spectral density consistently, it is very common to use smoothing. In particular, consider the lag window estimate
\[
\hat{f}_{T,B_T}(\theta) = \frac{1}{2\pi} \sum_{k=-B_T}^{B_T} K(k/B_T) \hat{\gamma}_k \cos(k\theta),
\]
where $B_T$ is the bandwidth satisfying natural conditions $B_T \to \infty$ and $B_T/T \to 0$, and $K(\cdot)$ is a symmetric kernel function satisfying
\[
K(0) = 1, \quad |K(x)| \leq 1 \quad \text{and} \quad K(x) = 0 \quad \text{for} \quad |x| > 1.
\]
Correspondingly, we define the tapered covariance matrix estimate
\[
\hat{\Sigma}_{T,B_T} = [K((s - t)/B_T) \hat{\gamma}_{s-t}]_{1 \leq s,t \leq T} = \hat{\Sigma}_T \star W_T,
\]
where $\star$ is the Hadamard (or Schur) product, which is formed by element-wise multiplication of matrices. The term “tapered” is consistent with the terminology of the high-dimensional covariance regularization literature. However, the reader should not confuse this with the notion “data taper” that is commonly used in time series analysis. Our tapered estimate parallels a lag-window estimate of the spectral density with a tapered window. As a special case, if $K(x) = 1_{\{|x| \leq 1\}}$ is the rectangular kernel, then $\hat{\Sigma}_{T,B_T} = (\hat{\gamma}_{s-t} 1_{|s-t| \leq B_T})$ is the banded sample covariance matrix. However, this estimate may not be nonnegative definite. To obtain a nonnegative definite estimate, by the Schur product theorem in matrix theory [Horn and
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Johnson \((1990)\), since \(\hat{\Sigma}_T\) is nonnegative definite, their Schur product \(\hat{\Sigma}_{T,T}\) is also nonnegative definite if \(W_T = [K((s - t)/B_T)]_{1 \leq s, t \leq T}\) is nonnegative definite. The Bartlett or triangular window \(K_B(u) = \max(0, 1 - |u|)\) leads to a positive definite weight matrix \(W_T\) in view of

\[
K_B(u) = \int \omega(x) \omega(x + u) \, dx,
\]  

where \(\omega(x) = 1_{\{|x| \leq 1/2\}}\) is the rectangular window. Any kernel function having form (19) must be positive definite.

Here we shall show that \(\hat{\Sigma}_{T,B}\) is a consistent estimate of \(\Sigma_T\) and establish a convergence rate of \(\lambda(\hat{\Sigma}_{T,B} - \Sigma)\). We first consider the bias. By the Geršgorin theorem \([\text{cf. Horn and Johnson } (1990), \text{Theorem 6.1.1}]\), we have

\[
\lambda(E\hat{\Sigma}_{T,B} - \Sigma) \leq b_T,
\]  

where

\[
b_T = 2 \sum_{k=1}^{B_T} \left| 1 - K\left(\frac{k}{B_T}\right) \right| \gamma_k + \frac{2}{T} \sum_{k=1}^{B_T} k|\gamma_k| + 2 \sum_{k=B_T+1}^{T-1} |\gamma_k|.
\]  

The first term on the right-hand side in (20) is due to the choice of the kernel function, whose order of magnitude is determined by the smoothness of \(K(\cdot)\) at zero. In particular, this term vanishes if \(K(\cdot)\) is the rectangular kernel. If 

\[
1 - K(u) = O(u^2) \text{ at } u = 0 \quad \text{and} \quad \gamma_k = O(k^{-\beta}), \quad \beta > 1,
\]

then \(b_T = O(B_T^{1-\beta})\) if \(1 < \beta < 2\), \(b_T = O(B_T^{1-\beta} + T^{-1})\) if \(2 < \beta < 3\) and \(b_T = O(B_T^{-2} + T^{-1})\) if \(\beta > 3\). Generally, if \(\sum_{k=1}^{\infty} |\gamma_k| < \infty\), then \(b_T \to 0\) as \(B_T \to \infty\) and \(B_T < T\).

It is more challenging to deal with \(\lambda(E\hat{\Sigma}_{T,B} - \Sigma)\). If \(X_t \in L^p\) for some \(2 < p \leq 4\) and \(E X_t = 0\), Wu and Pourahmadi \((2009)\) obtained

\[
\lambda(E\hat{\Sigma}_{T,B} - \Sigma) = O_P\left(\frac{B_T \Theta_p^2}{T^{1-2/p}}\right).
\]  

The key step of their method is to use the inequality

\[
\lambda(E\hat{\Sigma}_{T,B} - \Sigma) \leq 2 \sum_{k=0}^{B_T} |K(k/B_T)||\hat{\gamma}_k - E\hat{\gamma}_k|,
\]

which is also obtained by the Geršgorin theorem. It turns out that the above rate can be improved by exploiting the Toeplitz structure of the autocovariance matrix. By Lemma 1,

\[
\lambda(E\hat{\Sigma}_{T,B} - \Sigma) \leq 2\pi \max_{\theta} |\hat{f}_{T,B_T}(\theta) - E\hat{f}_{T,B_T}(\theta)|.
\]  

Since \(\hat{f}_{T,B_T}(\theta)\) is a trigonometric polynomial of order \(B_T\), we can bound its maximum by the maximum over a fine grid. The following lemma is adapted from Zygmund \((2002)\), Theorem 7.28, Chapter X.
Lemma 3. Let \( S(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \) be a trigonometric polynomial of order \( n \). For any \( x^* \in \mathbb{R} \), \( \delta > 0 \) and \( l \geq 2(1 + \delta)n \), let \( x_j = x^* + 2\pi j/l \) for \( 0 \leq j \leq l \); then
\[
\max_x |S(x)| \leq (1 + \delta^{-1}) \max_{0 \leq j \leq l} |S(x_j)|.
\]

For \( \delta > 0 \), let \( \theta_j = \pi j / [(1 + \delta)B_T] \); then by Lemma 3,
\[
\max_{\theta_j} |\hat{f}_{T,B_T}(\theta) - \mathbb{E}\hat{f}_{T,B_T}(\theta)| \leq (1 + \delta^{-1}) \max_{\theta_j} |\hat{f}_{T,B_T}(\theta_j) - \mathbb{E}\hat{f}_{T,B_T}(\theta_j)|.
\]

Theorem 4. Assume \( X_t \in \mathcal{L}^p \) with some \( p > 4 \), \( \mathbb{E}X_t = 0 \), and \( \Theta_p(m) = O(m^{-\alpha}) \). Choose the banding parameter \( B_T \) to satisfy \( B_T \rightarrow \infty \), and \( B_T = O(T^{\gamma}) \), for some
\[
0 < \gamma < 1, \quad \gamma < \alpha p/2 \quad \text{and} \quad (1 - 2\alpha)\gamma < (p - 4)/p.
\]

Then for \( B_T \) defined in (20), and \( c_\gamma = (p + 4)e^{p/4}\Theta_4^2 \),
\[
\lim_{T \to \infty} \mathbb{P} \left[ \lambda(\hat{\Sigma}_{T,B_T} - \Sigma_T) \leq 12c_\gamma \sqrt{B_T \log B_T / T} + b_T \right] = 1.
\]

In particular, if \( K(x) = 1_{\{|x| \leq 1\}} \) and \( B_T \propto (T/\log T)^{1/(2\alpha + 1)} \), then
\[
\lambda(\hat{\Sigma}_{T,B_T} - \Sigma_T) = O_p \left( \left( \frac{\log T}{T} \right)^{\alpha/(2\alpha + 1)} \right).
\]

Proof. In view of (20), to prove (25) we only need to show that
\[
\lim_{T \to \infty} \mathbb{P} \left[ \lambda(\hat{\Sigma}_{T,B_T} - \mathbb{E}\hat{\Sigma}_{T,B_T}) \leq 12c_\gamma \sqrt{B_T \log B_T / T} \right] = 1.
\]

By (22) and (23) where we take \( \delta = 1 \), the problem is reduced to
\[
\lim_{T \to \infty} \mathbb{P} \left[ (2\pi) \cdot \max_j |\hat{f}_{T,B_T}(\theta_j) - \mathbb{E}\hat{f}_{T,B_T}(\theta_j)| \leq 6c_\gamma \sqrt{B_T \log B_T / T} \right] = 1.
\]

By Theorem 10 (where we take \( M = 2 \)), for any \( \gamma < \beta < 1 \), there exists a constant \( C_{p,\beta} \) such that
\[
\max_j P \left[ (2\pi) \cdot |\hat{f}_{T,B_T}(\theta_j) - \mathbb{E}\hat{f}_{T,B_T}(\theta_j)| \geq 6c_\gamma \sqrt{B_T \log B_T / T} \right] \leq C_{p,\beta}(TB_T)^{-p/4}(\log T)[(TB_T)^{p/4}T^{-\alpha\beta p/2} + TB_T^{p/2 - 1 - \alpha\beta p/2} + T] + C_{p,\beta}B_T^{-2}.
\]
If (24) holds, there exist a $0 < \beta < 1$ such that $\gamma - \alpha/2 < 0$ and $(p/4 - \alpha/2)\gamma - (p/4 - 1) < 0$. It follows that by (29),

$$P \left[ \max_j |\hat{f}_{T,B_T}(\theta_j) - E\hat{f}_{T,B_T}(\theta_j)| \geq 6c_p \sqrt{B_T \log B_T} \right]$$

$$\leq C_{p,\beta}(\log T)^{T\gamma - \alpha/2 + T^{1-p/4} + T^{p/4 - \alpha/2}\gamma - (p/4 - 1)} + C_{p,\beta}B_T^{-1}$$

$$= o(1).$$

Therefore, (28) holds and the proof of (25) is complete. The last statement (26) is an immediate consequence. Details are omitted. □

**Remark 2.** In practice, $EX_1$ is usually unknown, and we estimate it by $X_T = T^{-1} \sum_{t=1}^T X_t$. Let $\hat{\gamma}_k = T^{-1} \sum_{k=1}^T (X_t - X_T) (X_{t-k} - X_T)$, and $\Sigma_{T,B_T}$ be defined as $\Sigma_{T,B_T}$ by replacing $\hat{\gamma}_k$ therein by $\hat{\gamma}_c$. Since $X_T - EX_1 = O_P(T^{-1/2})$, it is easily seen that $\lambda(\hat{\Sigma}_{T,B_T} - \hat{\Sigma}_{T,B_T}) = O_P(B_T/T)$. Therefore, the results of Theorem 4 hold for $\hat{\Sigma}_{T,B_T}$ as well.

**Remark 3.** In the proof of Theorem 4, we have shown that, as an intermediate step from (28) to (27),

$$\lim_{T \to \infty} P \left[ \max_{0 \leq \theta \leq 2\pi} |\hat{f}_{T,B_T}(\theta) - E\hat{f}_{T,B_T}(\theta)| \leq 6\pi^{-1}c_p \sqrt{T^{-1}B_T \log B_T} \right] = 1.\tag{30}$$

The above uniform convergence result is very useful in spectral analysis of time series. Shao and Wu (2007) obtained the weaker version

$$\max_{0 \leq \theta \leq 2\pi} |\hat{f}_{T,B_T}(\theta) - E\hat{f}_{T,B_T}(\theta)| = O_P\left(\sqrt{T^{-1}B_T \log B_T}\right)$$

under a stronger assumption that $\Theta_p(m) = O(\rho^m)$ for some $0 < \rho < 1$.

**Remark 4.** For linear processes, Woodroofe and Van Ness (1967) derived the asymptotic distribution of the maximum deviations of spectral density estimates. Liu and Wu (2010) generalized their result to nonlinear processes and showed that the limiting distribution of

$$\max_{0 \leq j \leq B_T} \sqrt{T \frac{B_T}{B}} \frac{|\hat{f}_{T,B_T}(\pi j/B) - E\hat{f}_{T,B_T}(\pi j/B)|}{f(\pi j/B)}$$

is Gumbel after suitable centering and rescaling, under stronger conditions than (24). With their result, and using similar arguments as Theorem 2, we can show that for some constant $C_p$,

$$\lim_{T \to \infty} P \left[ C_p^{-1} \sqrt{\frac{B_T \log B_T}{T}} \leq \lambda(\hat{\Sigma}_{T,B_T} - E\hat{\Sigma}_{T,B_T}) \leq C_p \sqrt{\frac{B_T \log B_T}{T}} \right] = 1,$$

which means that the convergence rate we have obtained in (27) is optimal.
Remark 5. The convergence rate $\sqrt{T^{-1}B_T \log B_T + b_T}$ in Theorem 4 is optimal. Consider a process $(X_t)$ which satisfies $\gamma_0 = 3$ and when $k > 0$,
$$
\gamma_k = \begin{cases} 
A^{-\alpha j}, & \text{if } k = A^j \text{ for some } j \in \mathbb{N}, \\
0, & \text{otherwise},
\end{cases}
$$
where $\alpha > 0$ and $A > 0$ is an even integer such that $A^{-\alpha} \leq 1/5$. Consider the banded estimate $\hat{\Sigma}_{T,B_T}$ with the rectangular kernel. As shown in the supplementary article [Xiao and Wu (2012)], there exists a constant $C > 0$ such that
$$
\lim_{T \to \infty} P \left[ \lambda(\hat{\Sigma}_{T,B_T} - \Sigma_T) \geq C \sqrt{B_T \log B_T + b_T/5} \right] = 1,
$$
suggesting that the convergence rate given in (25) of Theorem 4 is optimal. This optimality property can have many applications. For example, it can allow one to derive convergence rates for estimates of $a_T$ in the Wiener–Hopf equation, and the optimal weights $c_T = (c_1, \ldots, c_T)^\top$ in the best linear unbiased estimation problem mentioned in the Introduction.

Remark 6. We now compare (21) and our result. For $p = 4$, (21) gives the order $\lambda(\hat{\Sigma}_{T,B_T} - \Sigma_T) = O_P(B_T/\sqrt{T})$. Our result (27) is sharper by moving the bandwidth $B_T$ inside the square root. We pay the costs of a logarithmic factor, a higher order moment condition ($p > 4$), as well as conditions on the decay rate of tail sum of physical dependence measures (24). Note that when $\alpha \geq 1/2$, the last two conditions of (24) hold automatically, so we merely need $0 < \gamma < 1$, allowing a very wide range of $B_T$. In comparison, for (21) to be useful, one requires $B_T = o(T^{1-2/p})$.

Remark 7. The convergence rate (21) of Wu and Pourahmadi (2009) parallels the result of Bickel and Levina (2008b) in the context of high-dimensional multivariate analysis, which was improved in Cai, Zhang and Zhou (2010) by constructing a class of tapered estimates. Our result parallels the optimal minimax rate derived in Cai, Zhang and Zhou (2010), though the settings are different.

Remark 8. Theorem 4 uses the operator norm. For the Frobenius norm see Xiao and Wu (2011) where a central limit theory for $\sum_{k=1}^{B_T} \hat{\gamma}_k^2$ and $\sum_{k=1}^{B_T} (\hat{\gamma}_k - \mathbb{E}\hat{\gamma}_k)^2$ is established.

4. Thresholded covariance matrix estimators. In the context of time series, the observations have an intrinsic temporal order and we expect that observations are weakly dependent if they are far apart, so banding seems to be natural. However, if there are many zeros or very weak correlations within the band, the banding method does not automatically generate a sparse estimate.
The rationale behind the banding operation is sparsity, namely autoco-
variances with large lags are small, so it is reasonable to estimate them
as zero. Applying the same idea to the sample covariance matrix, we can
obtain an estimate of $\Sigma_T$ by replacing small entries in $\hat{\Sigma}_T$ with zero. This
regularization approach, termed *hard thresholding*, was originally developed
in nonparametric function estimation. It has recently been applied by Bickel
and Levina (2008a) and El Karoui (2008) as a method of covariance regular-
arization in the context of high-dimensional multivariate analysis. Since they
do not assume any order of the observations, their sparsity assump-
tions are permutation-invariant. Unlike their setup, we still use $\Theta_p(m)$ [cf. (7)] and

$$
\Psi_p(m) = \left( \sum_{t=m}^{\infty} \delta_p(t)t^{p'} \right)^{1/p'}, \quad \Delta_p(m) = \sum_{t=0}^{\infty} \min\{C_p\Psi_p(m), \delta_p(t)\}
$$

as our weak dependence conditions, where $p' = \min(2, p)$ and $C_p$ is given
in (38). This is natural for time series analysis.

Let $A_T = 2c_p\sqrt{\log T/T}$, where $c_p$ is the constant given in Lemma
6. The thresholded sample autocovariance matrix is defined as

$$
\hat{\Gamma}_{T,A_T} = (\hat{\gamma}_{s-t} - 1_{|\hat{\gamma}_{s-t}| \geq A_T}) \text{ for } 1 \leq s,t \leq T
$$

with the convention that the diagonal elements are never thresholded. We
emphasize that the thresholded estimate may not be positive definite. The
following result says that the thresholded estimate is also consistent in terms
of the operator norm, and provides a convergence rate which parallels the
banding approach in Section 3. In the proof we compare the thresholded
estimate $\hat{\Gamma}_{T,A_T}$ with the banded one $\Sigma_{T,B_T}$ for some suitably chosen $B_T$.
This is merely a way to simplify the arguments. The same results can be
proved without referring to the banded estimates.

**Theorem 5.** Assume $X_t \in \mathcal{L}^p$ with some $p > 4$, $E X_t = 0$, and $\Theta_p(m) = O(m^{-\alpha})$, $\Delta_p(m) = O(m^{-\alpha'})$ for some $\alpha \geq \alpha' > 0$. If

$$
\alpha > 1/2 \quad \text{or} \quad \alpha' p > 2,
$$

then

$$
\lambda(\hat{\Gamma}_{T,A_T} - \Sigma_T) = O_P \left[ \left( \frac{\log T}{T} \right)^{\alpha/(2(1+\alpha))} \right].
$$

**Remark 9.** Condition (33) is only required for Lemma 6. As commented
by Xiao and Wu (2011), it can be reduced to $\alpha p > 2$ for linear processes.
See Remark 2 of their paper for more details.

The key step for proving Theorem 5 is to establish a convergence rate for
the maximum deviation of sample autocovariances. The following lemma
is adapted from Theorem 3 of Xiao and Wu (2011), where the asymptotic
distribution of the maximum deviation was also studied.
Lemma 6. Assume the conditions of Theorem 5. Then
\[
\lim_{T \to \infty} P \left( \max_{1 \leq k < T} |\hat{\gamma}_k - E\hat{\gamma}_k| \leq c'_p \sqrt{\frac{\log T}{T}} \right) = 1,
\]
where \(c'_p = 6(p + 4)e^{p/4}\|X_0\|_4\Theta_4\).

Proof of Theorem 5. Let \(B_T = \lfloor (T/\log T)^{1/(2(1+\alpha))} \rfloor\), and \(\hat{\Sigma}_{T,B_T}\) be the banded sample covariance matrix with the rectangular kernel. Recall that \(b_T = (2/T) \sum_{k=1}^{B_T} k|\gamma_k| + 2 \sum_{k=B_T+1}^{T-1} \gamma_k\) from (20). By Lemma 6, we have
\[
\lambda(\hat{\Sigma}_{T,B_T} - \Sigma_T) = O_P \left( B_T \sqrt{\frac{\log T}{T}} + b_T \right).
\]
Write the thresholded estimate \(\hat{\Gamma}_{T,A_T} = \hat{\Gamma}_{T,A_T,1} + \hat{\Gamma}_{T,A_T,2}\), where
\[
\hat{\Gamma}_{T,A_T,1} = (\hat{\gamma}_{s-t} 1_{|\hat{\gamma}_{s-t}| \geq A_T, |s-t| \leq B_T})_{1 \leq s,t \leq T}
\]
and
\[
\hat{\Gamma}_{T,A_T,2} = (\hat{\gamma}_{s-t} 1_{|\hat{\gamma}_{s-t}| > B_T})_{1 \leq s,t \leq T}.
\]
By Geršgorin’s theorem, it is easily seen that
\[
\lambda(\hat{\Gamma}_{T,A_T,1} - \hat{\Sigma}_{T,B_T}) \leq A_T B_T = O \left( B_T \sqrt{\frac{\log T}{T}} \right).
\]
On the other hand,
\[
\lambda(\hat{\Gamma}_{T,A_T,2}) \leq 2 \left( \sum_{k=B_T+1}^{T} |\hat{\gamma}_k - E\hat{\gamma}_k| 1_{|\gamma_k| < A_T/2, |\hat{\gamma}_k| \geq A_T} \right.
\]
\[
+ \sum_{k=B_T+1}^{T} |\hat{\gamma}_k - E\hat{\gamma}_k| 1_{|\gamma_k| \geq A_T/2, |\hat{\gamma}_k| \geq A_T} + \sum_{k=B_T+1}^{T} |E\hat{\gamma}_k| \bigg)
\]
\[
= : 2(I_T + II_T + III_T).
\]
The term \(III_T\) is dominated by \(b_T\). By Lemma 6, we know
\[
\lim_{T \to \infty} P(I_T > 0) \leq \lim_{T \to \infty} P \left( \max_{1 \leq k \leq T-1} |\hat{\gamma}_k - E\hat{\gamma}_k| \geq A_T/2 \right) = 0.
\]
For the remaining term \(II_T\), note that the number of \(\gamma_k\) such that \(k > B_T\) and \(|\gamma_k| \geq A_T/2\) is bounded by \(2 \sum_{k=B_T+1}^{T} |\gamma_k|/A_T\); therefore by Lemma 6
\[
II_T \leq C(B_T^\alpha/A_T) \cdot \max_{1 \leq k \leq T-1} |\hat{\gamma}_k - E\hat{\gamma}_k| = O_P(B_T^{-\alpha}).
\]
Putting (34), (35), (36) and (37) together, the proof is complete. \(\square\)
Remark 10. If the mean $\mathbb{E}X_1$ is unknown, we need to replace $\hat{\gamma}_k$ by $\hat{\gamma}_c^k$ (Remark 2). Since Lemma 6 still holds when $\hat{\gamma}_k$ are replaced by $\hat{\gamma}_c^k$ [Xiao and Wu (2011)], Theorem 5 remains true for $\hat{\gamma}_c^k$.

5. A simulation study. The thresholded estimate is desirable in that it can lead to a better estimate when there are a lot of zeros or very weak autocovariances. Unfortunately, due to technical difficulties, the theoretical result (cf. Theorem 5) does not reflect this advantage. We show by simulations that thresholding does have a better finite sample performance over banding when the true autocovariance matrix is sparse.

Consider two linear processes $X_t = \sum_{s=0}^{\infty} a_s \varepsilon_{t-s}$ and $Y_t = \sum_{s=0}^{\infty} b_s \varepsilon_{t-s}$, where $a_0 = b_0 = 1$, and when $s > 0$ 

$$a_s = cs^{-(1+\alpha)}, \quad b_s = c(s/2)^{-(1+\alpha)}1_s \text{ is even}$$

for some $c > 0$ and $\alpha > 0$; and $\varepsilon_s$'s are taken as i.i.d. $\mathcal{N}(0,1)$. Let $\gamma^X_k, \Sigma^X_T$, and $\gamma^Y_k, \Sigma^Y_T$ denote the autocovariances and autocovariance matrices of the two processes, respectively. It is easily seen that $\gamma^Y_k = 0$ if $k$ is odd. In fact, $(Y_t)$ can be obtained by interlacing two i.i.d. copies of $(X_t)$. For a given set of centered observations $X_1, X_2, \ldots, X_T$, assuming that its true autocovariance matrix is known, for a fair comparison we choose the optimal bandwidth $B_T$ and threshold $A_T$ as

$$\hat{A}_T^X = \arg\min_{l \in \{|\hat{\gamma}_1^X|, |\hat{\gamma}_2^X|, \ldots, |\hat{\gamma}_{T-1}^X|\}} \lambda(\hat{\Gamma}_T^X - \Sigma_T^X), \quad \hat{B}_T^X = \arg\min_{1 \leq k \leq T} \lambda(\hat{\Sigma}_T^X - \Sigma_T^X).$$

The two parameters for the $(Y_t)$ process are chosen in the same way. In all the simulations we set $c = 0.5$. For different combinations of the sample size $T$ and the parameter $\alpha$ which controls the decay rate of autocovariances, we report the average distances in term of the operator norm of the two estimates $\hat{\Sigma}_T^X, \hat{\Sigma}_T^Y$ and $\hat{\Gamma}_T^X, \hat{\Gamma}_T^Y$ from $\Sigma_T^X, \Sigma_T^Y$, as well as the standard errors based on 1000 repetitions. We also give the average bandwidth of $\hat{\Sigma}_T^X, \hat{\Sigma}_T^Y$. Instead of reporting the average threshold for $\hat{\Gamma}_T^X, \hat{\Gamma}_T^Y$, we provide the average number of nonzero autocovariances appearing in the estimates, which is comparable to the average bandwidth of $\hat{\Sigma}_T^X, \hat{\Sigma}_T^Y$.

From Table 1, we see that for the process $(X_t)$, the banding method outperforms the thresholding one, but the latter does give sparser estimates. For the process $(Y_t)$, according to Table 2, we find that thresholding performs better than banding when the sample size is not very large ($T = 100, 200$), and yields sparser estimates as well. The advantage of thresholding in error disappears when the sample size is 500. Intuitively speaking, banding is a way to threshold according to the truth (autocovariances with large lags are small), while thresholding is a way to threshold according to the data. Therefore, if the autocovariances are nonincreasing as for the process $(X_t)$, or if the sample size is large enough, banding is preferable. If the autocovariances do not vary regularly as for the process $(Y_t)$ and the sample size is
Table 1
Errors under operator norm for \( (X_t) \)

| \( T = 100 \) | \( T = 200 \) | \( T = 500 \) |
|---------------|---------------|---------------|
| Error \( \times 10^2 \) | Error \( \times 10^2 \) | Error \( \times 10^2 \) |
| BW \( \times 10^2 \) | BW \( \times 10^2 \) | BW \( \times 10^2 \) |
| 0.2 | 2.94 (1.17) 9.55 (6.60) | 3.01 (1.22) 13.4 (7.67) | 2.96 (1.23) 23.4 (13.1) |
| 3.66 (1.07) 5.40 (4.87) | 3.88 (1.14) 7.39 (5.81) | 4.08 (1.17) 12.5 (10.1) |
| 6.98 (2.63) | 8.12 (2.85) | 10.57 (3.93) |
| 0.5 | 1.52 (0.68) 6.31 (4.58) | 1.38 (0.60) 8.46 (5.57) | 1.15 (0.50) 11.9 (7.67) |
| 1.90 (0.64) 3.49 (2.56) | 1.89 (0.59) 4.15 (3.07) | 1.74 (0.54) 5.15 (3.27) |
| 5.55 (2.37) | 6.73 (2.91) | 8.88 (3.28) |
| 1 | 0.82 (0.39) 4.04 (2.33) | 0.69 (0.32) 4.62 (2.47) | 0.52 (0.24) 5.68 (3.06) |
| 1.03 (0.38) 2.24 (0.87) | 0.95 (0.32) 2.29 (0.74) | 0.81 (0.29) 2.58 (0.83) |
| 4.80 (2.14) | 6.05 (2.25) | 7.81 (2.64) |

“Error” refers to the average distance between the estimates and the true autocovariance matrix under the operator norm, and “BW” refers to the average bandwidth of the banded estimates, and the average number of nonzero sub-diagonals (including the diagonal) for the thresholded ones. The numbers 0.2, 0.5 and 1 in the first column are values of \( \alpha \).

For each combination of \( T \) and \( \alpha \), three lines are reported, corresponding to banded estimates, thresholded ones and sample autocovariance matrices, respectively. Numbers in parentheses are standard errors.

Apparently our simulation is a very limited one, because we assume that the true autocovariance matrices are known. Practitioners would need a method to choose the bandwidth and/or threshold from the data. Although theoretical results suggest convergence rates of banding and thresholding moderate, thresholding is more adaptive. As a combination, in practice we can use a thresholding-after-banding estimate which enjoys both advantages.

Table 2
Error under operator norm for \( (Y_t) \)

| \( T = 100 \) | \( T = 200 \) | \( T = 500 \) |
|---------------|---------------|---------------|
| Error \( \times 10^2 \) | Error \( \times 10^2 \) | Error \( \times 10^2 \) |
| BW \( \times 10^2 \) | BW \( \times 10^2 \) | BW \( \times 10^2 \) |
| 0.2 | 3.33 (0.86) 9.87 (6.89) | 3.54 (0.95) 13.7 (7.67) | 3.61 (1.07) 24.7 (13.1) |
| 3.15 (0.93) 3.95 (2.50) | 3.43 (1.00) 5.69 (4.72) | 3.75 (1.08) 9.23 (8.04) |
| 7.21 (4.28) | 8.69 (4.79) | 11.1 (5.31) |
| 0.5 | 1.98 (0.61) 7.26 (5.32) | 1.88 (0.59) 9.95 (6.44) | 1.63 (0.53) 16.3 (10.1) |
| 1.81 (0.60) 2.93 (2.41) | 1.81 (0.59) 3.44 (2.22) | 1.71 (0.54) 4.64 (2.97) |
| 5.88 (3.27) | 7.25 (3.59) | 9.25 (3.72) |
| 1 | 1.19 (0.41) 5.31 (3.33) | 1.01 (0.35) 6.20 (3.58) | 0.79 (0.28) 8.28 (4.95) |
| 1.02 (0.39) 2.16 (0.65) | 0.92 (0.32) 2.21 (0.57) | 0.80 (0.28) 2.52 (0.77) |
| 5.09 (2.77) | 6.39 (2.79) | 8.18 (2.91) |
parameters which lead to optimal convergence rates of the estimates, they
do not offer much help for finite samples. The issue was addressed by Wu
and Pourahmadi (2009) incorporating the idea of risk minimization from
Bickel and Levina (2008b) and the technique of subsampling from Politis,
Romano and Wolf (1999), and by McMurry and Politis (2010) using the rule
introduced in Politis (2003) for selecting the bandwidth in spectral density
estimation. An alternative method is to use the block length selection pro-
dure in Bühmann and Künsch (1999) which is designed for spectral density
estimation. We shall study other data-driven methods in the future.

6. Moment inequalities. This section presents some moment inequalities
that will be useful in the subsequent proofs. In Lemma 7, the case \( 1 < p \leq 2 \)
follows from Burkholder (1988) and the other case \( p > 2 \) is due to Rio (2009).
Lemma 8 is adopted from Proposition 1 of Xiao and Wu (2011).

**Lemma 7** [Burkholder (1988), Rio (2009)]. Let \( p > 1 \) and \( p' = \min\{p, 2\} \); 
let \( D_t, 1 \leq t \leq T \), be martingale differences, and \( D_t \in \mathcal{L}^{p'} \) for every \( t \). Write
\( M_T = \sum_{t=1}^{T} D_t \). Then

\[
\|M_T\|_{p'} \leq C_p' \sum_{t=1}^{T} \|D_t\|_{p'} \quad \text{where} \quad C_p' = \begin{cases} \frac{(p-1)^{-1}}{\sqrt{p-1}}, & \text{if } 1 < p \leq 2, \\
\frac{p}{p-1}, & \text{if } p > 2. \end{cases}
\]

It is convenient to use \( m \)-dependence approximation for processes with
the form (6). For \( t \in \mathbb{Z} \), define \( \mathcal{F}_t = \langle \varepsilon_t, \varepsilon_{t+1}, \ldots \rangle \) be the \( \sigma \)-field generated
by the innovations \( \varepsilon_t, \varepsilon_{t+1}, \ldots \), and the projection operator \( \mathcal{H}_t(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_t) \)
and \( \mathcal{P}_t(\cdot) = \mathcal{H}_t(\cdot) - \mathcal{H}_{t+1}(\cdot) \). Observe that \( (\mathcal{P}_t(\cdot))_{t \in \mathbb{Z}} \) is a martingale differ-
ence sequence with respect to the filtration \( (\mathcal{F}_t) \). For \( m \geq 0 \), define
\( \tilde{X}_t = \mathcal{H}_{t-m} X_t \); then \( \|X_t - \tilde{X}_t\|_p \leq C_p \Psi_p(m + 1) \) [see Proposition 1 of Xiao
and Wu (2011) for a proof], and \( (\tilde{X}_t)_{t \in \mathbb{Z}} \) is an \( (m+1) \)-dependent sequence.

**Lemma 8.** Assume \( \mathbb{E}X_t = 0 \) and \( p > 1 \). For \( m \geq 0 \), define \( \tilde{X}_t = \mathcal{H}_{t-m} X_t = \mathbb{E}(X_t | \mathcal{F}_{t-m}) \). Let \( \delta_p(\cdot) \) be the physical dependence measure for the sequence \( (\tilde{X}_t) \).
Then

\[
\|P_0 X_t\|_p \leq \delta_p(t) \quad \text{and} \quad \tilde{\delta}_p(t) \leq \delta_p(t),
\]

\[
|\gamma_k| \leq \zeta_2(k) \quad \text{where} \quad \zeta_p(k) := \sum_{j=0}^{\infty} \delta_p(j)\delta_p(j+k),
\]

\[
\left\| \sum_{s,t=1}^{T} c_{s,t} (X_s X_t - \gamma_{s-t}) \right\|_{p/2} \leq 4C_{p/2}C_p \Theta_p^2 B_T \sqrt{T} \quad \text{when } p \geq 4,
\]

\[
\left\| \sum_{t=1}^{T} c_t (X_t - \tilde{X}_t) \right\|_p \leq C_p A_T \Theta_p(m + 1) \quad \text{when } p \geq 2,
\]
where
\[ A_T = \left( \sum_{t=1}^{T} |c_t|^2 \right)^{1/2} \quad \text{and} \quad B_T^2 = \max \left\{ \max_{1 \leq t \leq T} \sum_{s=1}^{T} c_{s,t}^2, \max_{1 \leq s \leq T} \sum_{t=1}^{T} c_{s,t}^2 \right\}. \]

7. Large deviations for quadratic forms. In this section we prove a result on probabilities of large deviations of quadratic forms of stationary processes, which take the form
\[ Q_T = \sum_{1 \leq s \leq t \leq T} a_{s,t}X_sX_t. \]
The coefficients \( a_{s,t} = a_{T,s,t} \) may depend on \( T \), but we suppress \( T \) from subscripts for notational simplicity. Throughout this section we assume that \( \sup_{s,t} |a_{s,t}| \leq 1, \) and \( a_{s,t} = 0 \) when \( |s - t| > B_T \), where \( B_T \) satisfies \( B_T \to \infty \), and \( B_T = O(T^\gamma) \) for some \( 0 < \gamma < 1 \).

Large deviations for quadratic forms of stationary processes have been extensively studied in the literature. Bryc and Dembo (1997) and Bercu, Gamboa and Rouault (1997) obtained the large deviation principle [Dembo and Zeitouni (1998)] for Gaussian processes. Gamboa, Rouault and Zani (1999) considered the functional large deviation principle. Bercu, Gamboa and Lavielle (2000) obtained a more accurate expansion of the tail probabilities. Zani (2002) extended the results of Bercu, Gamboa and Rouault (1997) to locally stationary Gaussian processes. In fact, our result is more relevant to the so-called moderate deviations according to the terminology of Dembo and Zeitouni (1998). Bryc and Dembo (1997) and Kakizawa (2007) obtained moderate deviation principles for quadratic forms of Gaussian processes. Djellout, Guillin and Wu (2006) studied moderate deviations of periodograms of linear processes. Bentkus and Rudzkis (1976) considered the Cramér-type moderate deviation for spectral density estimates of Gaussian processes; see also Saulis and Statulevičius (1991). Liu and Shao (2010) derived the Cramér-type moderate deviation for maxima of periodograms under the assumption that the process consists of i.i.d. random variables.

For our purpose, on one hand, we do not need a result that is as precise as the moderate deviation principle or the Cramér-type moderate deviation. On the other hand, we need an upper bound for the tail probability under less restrictive conditions. Specifically, we would like to relax the Gaussian, linear or i.i.d. assumptions which were made in the precedent works. Rudzkis (1978) provided a result in this fashion under the assumption of boundedness of the cumulant spectra up to a finite order. While this type of assumption holds under certain mixing conditions, the latter themselves are not easy to verify in general and many well-known examples are not strong mixing [Andrews (1984)]. We mean to impose alternative conditions through physical dependence measures, which are easy to use in many applications [Wu (2005)]. Furthermore, our result can be sharper; see Remark 11.
Our main tool is the \( m \)-dependence approximation. In the next lemma we use dependence measures to bound the \( L^p \) norm of the distance between \( Q_T \) and the \( m \)-dependent version \( \tilde{Q}_T \). The proof and a few remarks on the optimality of the result are given in the supplementary article [Xiao and Wu (2012)].

**Lemma 9.** Assume \( X_t \in L^p \) with \( p \geq 4 \), \( \mathbb{E}X_t = 0 \) and \( \Theta_p < \infty \). Let \( \tilde{X}_t = \mathcal{H}_{t-m_T}X_t \) and \( \tilde{Q}_T = \sum_{1 \leq s \leq t \leq T} a_{s,t} \tilde{X}_s \tilde{X}_t \); then
\[
\| \mathbb{E}_0 Q_T - \mathbb{E}_0 \tilde{Q}_T \|_{p/2} 
\leq 4\Theta_p (m_T)^2 + 11(p-2)\Theta_p \sqrt{T} B_T \Theta_p (m_T)
\]
\[
+ (p-2)\sqrt{T} B_T [3\Theta_p ([m_T/2]) \Delta_p (m_T) + 3\Theta_p (m_T) \Delta_p ([m_T/2])].
\]

The following theorem is the main result of this section.

**Theorem 10.** Assume \( X_t \in L^p \), \( p > 4 \), \( \mathbb{E}X_t = 0 \), and \( \Theta_p (m) = O(m^{-\alpha}) \). Set \( c_p = (p+4)e^{p/4} \Theta_p^2 \). For any \( M > 1 \), let \( x_T = 2c_p \sqrt{T} M B_T \log B_T \). Assume that \( B_T \to \infty \) and \( B_T = O(T^\gamma) \) for some \( 0 < \gamma < 1 \). Then for any \( \gamma < \beta < 1 \), there exists a constant \( C_{p,M,\beta} > 0 \) such that
\[
P\left( \| \mathbb{E}_0 Q_T - \mathbb{E}_0 \tilde{Q}_T \|_{p/2} \geq x_T \right) 
\leq C_{p,M,\beta} x_T^{-p/2} (\log T) \left( (TB_T)^{p/4} T^{-\alpha \beta p/2} + T B_T^{p/2-1-\alpha \beta p/2} + T \right)
\]
\[
+ C_{p,M,\beta} B_T^{-M}.
\]

**Remark 11.** Rudzkis (1978) proved that if \( p = 4k \) for some \( k \in \N \), then
\[
P(\| \mathbb{E}_0 Q_T \|_{p/2} \geq x_T) \leq C x_T^{-p/2} (TB_T)^{p/4},
\]
which can be obtained by using Markov inequality and (41) under our framework. The upper bound given in Theorem 10 has a smaller order of magnitude. We note that Rudzkis (1978) also proved a stronger exponential inequality under strong moment conditions. They required the existence of every moment and the absolute summability of cumulants of every order.

**Proof of Theorem 10.** Without loss of generality, assume \( B_T \leq T^\gamma \). For \( \gamma < \beta < 1 \), let \( m_T = \lfloor T^{\gamma/2} \rfloor \), \( \tilde{X}_t = \mathcal{H}_{t-m_T}X_t \) and
\[
\tilde{Q}_T = \sum_{1 \leq s \leq t \leq T} a_{s,t} \tilde{X}_s \tilde{X}_t.
\]
By Lemma 9 and (41), we have
\[
P(\| \mathbb{E}_0 (Q_T - \tilde{Q}_T) \|_{p/2} \geq c_p M^{1/2} \sqrt{T} B_T (\log B_T))
\]
\[
\leq C_{p,M} x_T^{-p/2} (TB_T)^{p/4} T^{-\alpha \beta p/2}.
\]
Split $[1, T]$ into blocks $B_{1}, \ldots, B_{b_{T}}$ of size $2m_{T}$, and define

$$Q_{T,k} = \sum_{t \in B_{k}} \sum_{1 \leq s \leq t} a_{s,t} \tilde{X}_{s} \tilde{X}_{t}.$$ 

By Corollary 1.7 of Nagaev (1979) and (41), we know for any $M > 1$, there exists a constant $C_{p,M,\beta}$ such that

$$P[|\mathbb{E}_0 Q_T| \geq x_T / C_{p,M,\beta}]$$

$$\leq \sum_{k=1}^{b_T} P \left( |\mathbb{E}_0 Q_{T,k}| \geq x_T / C_{p,M,\beta} \right) + \left[ C_{p,M,\beta} T_{m_{T}}^{-1} (m_{T} B_{T})^{p/4} \right] C_{p,M,\beta}$$

$$+ C_{\beta} \exp \left( \frac{c_{p}^{2} (\log B_{T})}{(p+4)^{2} e^{p/2} \Theta_{4}^{2}} \right)$$

$$\leq \sum_{k=1}^{b_T} P(|\mathbb{E}_0 Q_{T,k}| \geq x_T / C_{p,M,\beta}) + C_{p,M,\beta}(B_{T}^{-M} + T^{-M}).$$

By Lemma 11, we have

$$P[ |\mathbb{E}_0 Q_{T,k}| \geq x_T / C_{p,M,\beta}]$$

$$\leq C_{p,s,\beta} x_T^{-p/2} (\log T)$$

$$\times [(T \sqrt{\beta} B_{T})^{p/4} T^{-\alpha \beta p/2} + T \sqrt{\beta} B_{T}^{p/2 - 1 - \alpha \beta p/2} + T \sqrt{\beta}].$$

Combining (43), (44) and (45), the proof is complete. □

**Lemma 11.** Assume $X_t \in \mathcal{L}_{p}$ with $p > 4$, $\mathbb{E} X_t = 0$, and $\Theta_{p}(m) = O(m^{-\alpha})$. If $x_T > 0$ satisfies $T^{\delta} \sqrt{T B_{T}} = o(x_T)$ for some $\delta > 0$, then for any $0 < \beta < 1$, there exists a constant $C_{p,s,\beta}$ such that

$$P(|\mathbb{E}_0 Q_T| \geq x_T) \leq C_{p,s,\beta} x_T^{-p/2} (\log T)$$

$$\times [(T \sqrt{\beta} B_{T})^{p/4} T^{-\alpha \beta p/2} + T B_{T}^{p/2 - 1 - \alpha \beta p/2} + T \sqrt{\beta}].$$

**Proof.** For $j \geq 1$, define $m_{T,j} = \lfloor T^{\beta j} \rfloor$, $X_{t,j} = H_{t-m_{T,j}, X_t}$ and

$$Q_{T,j} = \sum_{1 \leq s \leq t \leq T} a_{s,t} X_{s,j} X_{t,j}.$$ 

Let $j_T = \lfloor -\log(\log(T) / (\log(\beta)) \rfloor$. Note that $m_{T,j_T} \leq e$. By Lemma 9 and (41),

$$P[|\mathbb{E}_0 (Q_T - Q_{T,1})| \geq x_T / j_T] \leq C_{p,\beta} (\log T)^{1/2} x_T^{-p/2} (T B_{T})^{p/4 - \alpha \beta p/2}.$$ 

Let $j_T'$ be the smallest $j$ such that $m_{T,j} < B_{T}/4$. For $1 \leq j < j_T'$, split $[1, T]$ into blocks $B_{1}^{(j)}, \ldots, B_{b_{T}}^{(j)}$ of size $B_{T} + m_{T,j}$. Define

$$R_{T,j,b} = \sum_{t \in B_{b}^{(j)}} \sum_{1 \leq s \leq t} a_{s,t} X_{s,j} X_{t,j} \quad \text{and} \quad R'_{T,j,b} = \sum_{t \in B_{b}^{(j)}} \sum_{1 \leq s \leq t} a_{s,t} X_{s,j+1} X_{t,j+1}.$$
By Corollary 1.6 of Nagaev (1979) and (41), we have for any $C > 2$

\begin{equation}
P \left( \left| \mathbb{E}_0(Q_{T,j} - Q_{T,j+1}) \right| > \frac{x_T}{2JT} \right) \leq \sum_{b=1}^{b_{T,j}} P \left( \left| \mathbb{E}_0(R_{T,j,b} - R'_{T,j,b}) \right| \geq \frac{x_T}{C_{JT}} \right)
\end{equation}

\begin{equation}
+ 2 \left[ \frac{64C e^2 \Theta_4^2 T B_{T,j}^2}{x_T^2} \right]^{C/4}.
\end{equation}

It is clear that for any $M > 1$, there exists a constant $C_{M,\delta,\beta}$ such that the term in (48) is less than $C_{M,\delta,\beta} x_{T}^{-M}$. For (47), by Lemma 9 and (41)

\begin{equation}
\sum_{b=1}^{b_{T,j}} P \left( \left| \mathbb{E}_0(R_{T,j,b} - R'_{T,j,b}) \right| \geq \frac{x_T}{C_{JT}} \right)
\end{equation}

\leq C_{p,\delta} T^{-1} \cdot (\log T)^{1/2} \cdot x_T^{-p/2} \cdot (m_{T,j} B_T)^{p/4} \cdot m_{T,j+1}^{-\alpha p/2}

\leq C_{p,\delta} x_T^{-p/2} \cdot (\log T)^{1/2} B_T^{p/4} \cdot (m_{T,j})^{p/4-1-\alpha \beta p/2}.

Depending on whether the exponent $p/4-1-\alpha \beta p/2$ is positive or not, the term $(m_{T,j})^{p/4-1-\alpha \beta p/2}$ is maximized when $j = 1$ or $j = j'_T - 1$, respectively, and we have

\begin{equation}
\sum_{b=1}^{b_{T,j}} P \left( \left| \mathbb{E}_0(R_{T,j,b} - R'_{T,j,b}) \right| \geq \frac{x_T}{C_{JT}} \right)
\end{equation}

\leq C_{p,\beta} x_T^{-p/2} \cdot (\log T)^{1/2} \cdot [(TB_T)^{p/4} T^{1/2} ]^{p/2-1-\alpha \beta p/2}.

Combining (46), (47), (48) and (49), we have shown that

\begin{equation}
P(\left| \mathbb{E}_0 Q_{T,j'_{T}} \right| \geq x_T)
\end{equation}

\leq P(\left| \mathbb{E}_0 Q_{T,j'_{T}} \right| \geq x_T/2) + C_{p,M,\delta,\beta} x_{T}^{-M}

+ C_{p,M,\delta,\beta} x_{T}^{-p/2} \cdot (\log T)^{1/2} \cdot [(TB_T)^{p/4} T^{1/2} ]^{p/2-1-\alpha \beta p/2}.

To deal with the probability concerning $Q_{T,j'_{T}}$ in (50), we split $[1, T]$ into blocks $B_1, \ldots, B_{b_{T}}$ with size $2B_T$, and define the block sums

\begin{equation}
R_{T,j'_{T},b} = \sum_{t \in B_b} \sum_{1 \leq s \leq t} a_{s,t} X_{t,j'_T} X_{t,j'_T}.
\end{equation}

Similarly as (47) and (48), there exists a constant $C_{p,M,\delta,\beta} > 2$ such that

\begin{equation}
P(\left| \mathbb{E}_0 Q_{T,j'_{T}} \right| \geq x_T/2) \leq \sum_{b=1}^{b_{T'}} P \left( \left| \mathbb{E}_0 R_{T,j'_{T},b} \right| \geq \frac{x_T}{C_{p,M,\delta,\beta}} \right) + C_{p,M,\delta,\beta} x_{T}^{-M}.
By Lemma 12, we have
\[ P(\|E_0R_{T,j_T}\| \geq C_{p,M,\delta,\beta}^{-1}x_T) \leq C_{p,M,\delta,\beta}x_T^{-p/2}(\log T)(B_T^{p/2-\alpha\beta p/2} + B_T); \]
and it follows that for some constant \( C_{p,\delta,\beta} > 0 \),
\[ P(\|E_0Q_T,j_T\| \geq x_T/2) \leq C_{p,\delta,\beta}x_T^{-p/2}(\log T)(B_T^{p/2-1-\alpha\beta p/2} + 1). \]
The proof is completed by combining (50) and (51). 

In the next lemma we consider \( Q_T \) when the restriction \( a_{s,t} = 0 \) for \( |s-t| > B_T \) is removed. To avoid confusion, we use a new symbol. Let
\[ R(T,m) = \sum_{1 \leq s \leq t \leq T} c_{s,t}(H_{s-m}X_s)(H_{t-m}X_t). \]
For \( x_T > 0 \), define
\[ U(T,m,x_T) = \sup_{\{c_{s,t}\}} P(\|E_0R(T,m)\| \geq x_T), \]
where the supremum is taken over all arrays \( \{c_{s,t}\} \) such that \( |c_{s,t}| \leq 1 \). We use \( R_T \) and \( U(T,x_T) \) as shorthands for \( R(T,\infty) \) and \( U(T,\infty,x_T) \), respectively.

**Lemma 12.** Assume \( X_t \in L^p \) with \( p > 4 \), \( EX_t = 0 \), and \( \Theta_p(m) = O(m^{-\alpha}) \). If \( x_T > 0 \) satisfies \( T^{1+\delta} = o(x_T) \) for some \( \delta > 0 \), then for any \( 0 < \beta < 1 \), there exists a constant \( C_{p,\delta,\beta} \) such that
\[ P(\|E_0R_T\| \geq x_T) \leq C_{p,\delta,\beta}x_T^{-p/2}(\log T)(T^{p/2-\alpha\beta p/2} + T). \]

**Proof.** Let \( m_T = [T^{3/2}] \) and \( \tilde{R}_T := R(T,m_T) \). By Lemma 9 and (41),
\[ P(\|E_0(R_T - \tilde{R}_T)\| \geq x_T/2) \leq C_{p,\beta}x_T^{-p/2}T^{p/2-\alpha\beta p/2}. \]
We claim that there exists a constant \( C_{p,\delta,\beta} \) such that
\[ U(T,m_T,x_T/2) \leq C_{p,\delta,\beta}x_T^{-p/2}(T\log T)(m_T^{p/2-1-\alpha\beta p/2} + 1). \]
Therefore, the proof is complete by using
\[ P(\|E_0R_T\| \geq x_T) \leq P(\|E_0(R_T - \tilde{R}_T)\| \geq x_T/2) + U(T,m_T,x_T/2). \]
We need to prove the claim (52). Let \( z_T \) satisfy \( T^{1+\delta} = o(z_T) \). Let \( \beta_T = [\log(\log T)/(\log \beta)] \), and note that \( T^{\beta_T} \leq \epsilon \). Set \( y_T = z_T/(2\beta_T) \). We consider \( U(T,m,z_T) \) for an arbitrary \( 1 < m < T/4 \). Set \( X_{t,1} := H_{t-m}X_t \) and \( X_{t,2} := H_{t-[m^\beta]}X_t \). Define
\[ Y_{t,1} = \sum_{s=1}^{t-3m-1} c_{s,t}X_{s,1} \quad \text{and} \quad Z_{t,1} = \sum_{s=1}^{t} c_{s,t}X_{s,1}. \]
and $Y_{t,2}, Z_{t,2}$ similarly by replacing $X_{s,1}$ with $X_{s,2}$. Observe that $X_{t,k}$ and $Y_{t,l}$ are independent for $k, l = 1, 2$. We first consider $\sum_{t=1}^{T}(X_{t,1}Z_{t,1} - X_{t,2}Z_{t,2})$. Split $[1,T]$ into blocks $B_{1}, \ldots, B_{b_T}$ with size $4m$, and define $W_{T,b} = \sum_{t \in B_{b}}(X_{t,1}Z_{t,1} - X_{t,2}Z_{t,2})$. Let $y_T$ satisfy $y_T < z_T/2$ and $T^{1+\delta/2} = o(y_T)$.

Since $W_{T,b}$ and $W_{T,b'}$ are independent if $|b - b'| > 1$, by Corollary 1.6 of Nagaev (1979), (41) and Lemma 9, similarly as (47) and (48), we know for any $M > 1$, there exists a constant $C_{p,M,\delta,\beta}$ such that

$$P\left[ \left| \mathbb{E}_{0}\left( \sum_{t=1}^{T} X_{t,1}Z_{t,1} - X_{t,2}Z_{t,2} \right) \right| \geq y_T \right]$$

$$\leq C_{p,M,\delta,\beta} y_T^{-M} + \sum_{b=1}^{b_T} P\left( \left| \mathbb{E}_{0}W_{T,b} \right| \geq y_T/C_{M,\delta} \right)$$

$$\leq C_{p,M,\delta,\beta} y_T^{-M} + C_{p,M,\delta,\beta} y_T^{-p/2} T^{p/2 - 1 - \alpha \beta p/2}. \tag{53}$$

Now we deal with the term $\sum_{t=1}^{T}(X_{t,1}Y_{t,1} - X_{t,2}Y_{t,2})$. Split $[1,T]$ into blocks $B_{1}^{*}, \ldots, B_{b_T}^{*}$ with size $m$. Define $R_{T,b} = \sum_{t \in B_{e}}(X_{t,1}Y_{t,1} - X_{t,2}Y_{t,2})$. Let $\xi_b$ be the $\sigma$-fields generated by $\{\varepsilon_{b_{-1}}, \varepsilon_{b_{-2}}, \ldots\}$, where $b_b = \max\{B_{e}^{*}\}$. Observe that $(R_{T,b})_b$ is odd is a martingale sequence with respect to $(\xi_b)_b$ is odd, and so are $(R_{T,b})_b$ is even and $(\xi_b)_b$ is even. By Lemma 1 of Haeusler (1984) we know for any $M > 1$, there exists a constant $C_{M,\delta,\beta}$ such that

$$P\left[ \left| \sum_{t=1}^{T}(X_{t,1}Y_{t,1} - X_{t,2}Y_{t,2}) \right| \geq y_T \right]$$

$$\leq C_{M,\delta} y_T^{-M} + 4P\left( \sum_{b=1}^{b_T} \mathbb{E}(R_{T,b}^2|\xi_{b-2}) > \frac{y_T^2}{(\log y_T)^{3/2}} \right)$$

$$+ \sum_{b=1}^{b_T} P\left( |R_{T,b}| \geq \frac{y_T}{\log y_T} \right)$$

$$=: I_T + II_T + III_T. \tag{54}$$

Since $(X_{t,1}, X_{t,2})$ and $(Y_{t,1}, Y_{t,2})$ are independent, $R_{T,b}$ has finite $p$th moment. Using similar arguments as Lemma 9, we have

$$\|R_{T,b}\|_p \leq C_p(mT)^{p/2} m^{-\alpha \beta p};$$

and it follows that

$$III_T \leq C_p y_T^{-p}(\log y_T)^{p/2} T^{p/2 - 1 - \alpha \beta p}. \tag{55}$$
For the second term, let $r_{s-t,k} = \mathbb{E}(X_{s,k}X_{t,k})$ for $k = 1, 2$; we have

$$
\sum_{b=1}^{b_{s}} \mathbb{E}(R_{T,b}^{2}(\xi_{b-2}) \leq 2 \sum_{b=1}^{b_{s}} \left[ \sum_{s,t \in B_{b}} (r_{s-t,1}Y_{s,1}Y_{t,1} + r_{s-t,2}Y_{s,2}Y_{t,2}) \right]
$$

(56)

$$
= \sum_{1\leq s \leq t \leq T} a_{s,t,1}X_{s,1}X_{t,1} + \sum_{1\leq s \leq t \leq T} a_{s,t,2}X_{s,2}X_{t,2}.
$$

By (39) and (40), we know $\sum_{t \in \mathbb{Z}} \lvert r_{t,k} \rvert < \infty$ for $k = 1, 2$, and hence $|a_{s,t,k}| \leq CT$. It follows that the expectations of the two terms in (56) are all less than $CT^2$, and

$$
II_T \leq C_{\beta}U\left[ T, m, \frac{y_T^{2}}{(T \log y_T)^{2}} \right] + C_{\beta}U\left[ T, m, \frac{y_T^{2}}{(T \log y_T)^{2}} \right]
$$

(57)

$$
+ C_{p,M,\delta,\beta}[y_T^{-m} + y_T^{-\frac{p}{2}}T^{m/p/2-1-\alpha}\beta/p/2
$$

$$
+ y_T^{-p}(\log y_T)^{p/2+1}m/p/2-1-\alpha\beta]
$$

Combining (53), (54), (55) and (57), we have shown that $U(T, m, z_T)$ is bounded from above by

$$
U(T, m^\beta, z_T - 2y_T)
$$

$$
+ C_{\beta}U\left[ T, m^\beta, \frac{y_T^{2}}{(T \log y_T)^{2}} \right] + C_{\beta}U\left[ T, m, \frac{y_T^{2}}{(T \log y_T)^{2}} \right]
$$

(58)

$$
+ C_{p,M,\delta,\beta}[y_T^{-m} + y_T^{-\frac{p}{2}}T^{m/p/2-1-\alpha}\beta/p/2
$$

$$
+ y_T^{-p}(\log y_T)^{p/2+1}m/p/2-1-\alpha\beta]
$$

Since $\sup \{c_{s,t} \lVert E_0 R_T \rVert_{p/2} \leq C_pT$ by (41), by applying (58) recursively to deal with the last term on the first line of (58) for $q$ times such that $(y_T/T)^{-2q} = O[y_T^{-(M+q)}]$, we have

$$
U(T, m, z_T) \leq C_{p,M,\delta,\beta}[U(T, m^\beta, z_T - 2y_T) + y_T^{-\frac{p}{2}}T^{m/p/2-1-\alpha}\beta/p/2
$$

(59)

$$
+ y_T^{-p}(\log y_T)^{p/2+1}m/p/2-1-\alpha\beta + y_T^{-M}]
$$

Using the preceding arguments similarly, we can show that when $1 \leq m \leq 3$

$$
U[T, m, z_T/(2JT)] \leq C_{p,M,\delta}[z_T^{-\frac{p}{2}}(\log T)T + z_T^{-p}(\log z_T)^{p+1}T^{p/2+1} + z_T^{-M}]
$$

The details of the derivation are omitted. Applying (59) recursively for at most $j_T - 1$ times, we have the first bound for $U(T, m, z_T)$,

$$
U(T, m, z_T)
$$

$$
\leq C_{p,M,\delta,\beta}^\prime U[T, 3, z_T/(2JT)] + z_T^{-\frac{p}{2}}(\log z_T)T^{m/p/2-1-\alpha}\beta/p/2 + 1
$$

(60)

$$
+ z_T^{-p}(\log z_T)^{p+1}T^{p/2+1}(m/p/2-1-\alpha\beta + 1) + z_T^{-M}
$$

$$
\leq C_{p,\delta,\beta}^\prime (\log z_T)^{p+1}(z_T^{-p/2}T + z_T^{-pT^{p/2+1}})(m/p/2-1-\alpha\beta/p/2 + 1).
$$
Now plugging (60) back into (58) for the last two terms on the first line and using the condition $T^{1+\delta/2} = o(y_T)$, we have

\begin{equation}
U(T, m, z_T) \leq U(T, \lfloor m^{\beta} \rfloor, z_T - 2y_T) + C_{p,\delta,\beta}[y_T^{-p/2}T(m^{p/2-1-\alpha\beta p/2} + 1)].
\end{equation}

Again by applying (61) for at most $j_T - 1$ times, we obtain the second bound for $U(T, m, z_T)$:

$U(T, m, z_T) \leq C_{p,\delta,\beta}z_T^{-p/2}(T \log T)(m^{p/2-1-\alpha\beta p/2} + 1)$.

The proof of the claim (52) is complete. □

8. Conclusion. In this paper we use Toeplitz’s connection of eigenvalues of matrices and Fourier transforms of their entries, and obtain optimal bounds for tapered covariance matrix estimates by applying asymptotic results of spectral density estimates. Many problems are still unsolved; for example, can we improve the convergence rate of the thresholded estimate in Theorem 5? What is the asymptotic distribution of the maximum eigenvalues of the estimated covariance matrices? We hope that the approach and results developed in this paper can be useful for other high-dimensional covariance matrix estimation problems in time series. Such problems are relatively less studied compared to the well-known theory of random matrices which requires i.i.d. entries or multiple i.i.d. copies.

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SUPPLEMENTARY MATERIAL

Additional technical proofs (DOI: 10.1214/11-AOS967SUPP; .pdf). We give the proofs of Remark 5 and Lemma 9, as well as a few remarks on Lemma 9.

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