De Sitter Cosmic Strings and Supersymmetry

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January 19, 2013

Abstract

We study massive spinor fields in the geometry of a straight cosmic string in a de Sitter background. We find a hidden \( N = 2 \) supersymmetry in the fermionic solutions of the equations of motion. We connect the zero mode solutions to the heat-kernel regularized Witten index of the supersymmetric algebra.

Introduction

Cosmic strings are topological defects that could have been created during cosmological phase transitions in the early universe \[1\]. Beyond all question, these defects cannot be responsible for the primordial density perturbations in the Cosmic Microwave Background radiation. However the interest in cosmic strings has been renewed since they can be related to a number of physical phenomena \[2,3,5\].

Cosmic strings cause gravitational phenomena due to the extremely large mass per unit length these have. Therefore, even if the string is a straight line, it affects drastically spacetime around it. For a straight string, spacetime is flat except from a small deficit angle where curvature has a conical singularity. This angle can cause many astrophysical effects, such as doubling images of distant objects (for example quasars), or even cause gravitational lensing. Furthermore, wiggly cosmic strings could explain structure formation. In addition, emission of gravitational waves can be explained with the help of cosmic strings.

Astrophysical phenomena could find their origin in cosmic string theories. Examples of such astrophysical phenomena are high energy cosmic rays in our galaxy and primordial galactic magnetic fields. The later are related to superconducting cosmic strings \[6\] from which ultra-high currents (that consist of charged localized matter) are emitted.

In our study we shall use straight cosmic strings. Although real strings are not straight, they can be thought to be chains of small straight segments. Thus, calculations for straight strings can be very useful. The background we shall use is that of de Sitter spacetime. This spacetime is a maximal symmetric solution to Einstein’s equations, with \( R^1 \times S^3 \) topology.

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Due to the high symmetry that this gravitational background has, many problems are exactly solvable. Therefore studies of such spacetimes can shed light to more difficult problems. We focus our study on massive fermions around cosmic strings in de Sitter spacetime. We find a hidden $N = 2$ supersymmetric quantum mechanics algebra underlying the system and we relate the number of zero mode solutions of the equation of motion to the Witten index of the supersymmetric algebra.

The connection of a supersymmetric algebra with a Dirac fermionic system is not accidental. It is known that Dirac operators $H$ can be split into even and odd parts, that is, $H = H_+ + H_-$ (with $H_+$ denoting the even part and $H_-$ denoting the odd part), and this fact is actually closely related with the general notion of supersymmetry [7]. Particularly, when $H_+$ anti-commutes with $H_-$, then $H$ is called a supersymmetric Dirac operator. Supersymmetric quantum mechanics in Dirac and gauge theories related has been studied in recent works. For the connection of extra dimensional gauge models and $N = 2$ supersymmetric quantum mechanics algebra ($N = 2$ susy QM thereafter) see [8]. Also localized fermions around superconducting cosmic strings are connected with $N = 2$ susy QM algebra, see [9].

This article is organized as follows: First we briefly review the $N = 2$ supersymmetric quantum mechanics algebra, next we derive the equations of motion of the fermions around the cosmic strings and relate the fermionic system with the $N = 2$ susy QM algebra. Finally we present the conclusions along with a discussion.

**Supersymmetric Quantum Mechanics**

We briefly review the $N = 2$ supersymmetric quantum mechanics [10][11] algebra, relevant to our analysis.

Consider a quantum system, described by the self-adjoint Hamiltonian operator $H$ and characterized by the set of self-adjoint operators $\{Q_1, \ldots, Q_N\}$. The quantum system is supersymmetric, if,

$$\{Q_i, Q_j\} = H \delta_{ij} \quad (1)$$

with $i = 1, 2, \ldots, N$. The $Q_i$ are the supercharges and the Hamiltonian “$H$” is called supersymmetric (from now on “susy”) Hamiltonian. The algebra (1) constitutes the $N$-extended supersymmetry. Owing to the anti-commutativity one has,

$$H = 2Q_1^2 = 2Q_2^2 = \cdots = 2Q_N^2 = \frac{2}{N} \sum_{i=1}^{N} Q_i^2. \quad (2)$$

A supersymmetric quantum system is said to have unbroken supersymmetry, if its ground state vanishes, that is $E_0 = 0$. When $E_0 > 0$, susy is said to be broken. In order susy is unbroken, the ground states that belong in the Hilbert space of all the eigenstates, must be annihilated by the supercharges,

$$Q_i |\psi_0\rangle = 0. \quad (3)$$
$N = 2$ supersymmetric quantum mechanics algebra

The $N = 2$ algebra consists of two supercharges $Q_1$ and $Q_2$ and a Hamiltonian $H$, which obey,

$$\{Q_1, Q_2\} = 0, \quad H = 2Q_1^2 = 2Q_2^2 = Q_1^2 + Q_2^2$$

We introduce the operator,

$$Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$$

and the adjoint,

$$Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2)$$

The above two operators satisfy,

$$Q^2 = Q^{\dagger 2} = 0$$

and are related to the Hamiltonian as,

$$\{Q, Q^\dagger\} = H$$

The Witten parity, $W$, for a $N = 2$ algebra is defined as,

$$[W, H] = 0$$

and

$$\{W, Q\} = \{W, Q^\dagger\} = 0$$

Also $W$ satisfies,

$$W^2 = 1$$

By using $W$, the Hilbert space $\mathcal{H}$ of the quantum system is spanned to positive and negative Witten parity subspaces which are defined as,

$$\mathcal{H}^\pm = P^\pm \mathcal{H} = \{|\psi\rangle : W|\psi\rangle = \pm|\psi\rangle\}$$

Therefore, the Hilbert space $\mathcal{H}$ is decomposed into the eigenspaces of $W$, hence $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Each operator acting on the vectors of $\mathcal{H}$ can be represented by $2N \times 2N$ matrices. We use the representation:

$$W = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

with $I$ the $N \times N$ identity matrix. Bring to mind that $Q^2 = 0$ and $\{Q, W\} = 0$, hence the supercharges are of the form,

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

and

$$Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix}$$
which imply,

\[ Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \]  

(16)

and also,

\[ Q_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix} \]  

(17)

The \( N \times N \) matrices \( A \) and \( A^\dagger \), are generalized annihilation and creation operators. The action of \( A \) is defined as \( A : H^- \to H^+ \) and that of \( A^\dagger \) as, \( A^\dagger : H^+ \to H^- \). In the representation \((13), (14), (15)\) the Hamiltonian \( H \), can be cast in a diagonal form \([1]\),

\[ H = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \]  

(18)

Therefore the total supersymmetric Hamiltonian \( H \), consists of two superpartner Hamiltonians,

\[ H_+ = A A^\dagger, \quad H_- = A^\dagger A \]  

(19)

We define the operator \( P^\pm \). The eigenstates of \( P^\pm \), denoted as \( |\psi^\pm\rangle \) are called positive and negative parity eigenstates which satisfy,

\[ P^\pm |\psi^\pm\rangle = \pm |\psi^\pm\rangle \]  

(20)

Using the representation \((13)\), the parity eigenstates are represented in the form,

\[ |\psi^+\rangle = \begin{pmatrix} |\phi^+\rangle \\ 0 \end{pmatrix} \]  

(21)

and also,

\[ |\psi^-\rangle = \begin{pmatrix} 0 \\ |\phi^\pm\rangle \end{pmatrix} \]  

(22)

with \( |\phi^\pm\rangle \in H^\pm \).

In order to have unbroken supersymmetry, there must be at least one state in the Hilbert space (we denote it as \( |\psi_0\rangle \) ) with vanishing energy eigenvalue, that is \( H|\psi_0\rangle = 0 \). This implies that \( Q|\psi_0\rangle = 0 \) and \( Q^\dagger |\psi_0\rangle = 0 \). For a ground state with negative parity,

\[ |\psi^-_0\rangle = \begin{pmatrix} 0 \\ |\phi^\pm_0\rangle \end{pmatrix} \]  

(23)

this would imply that \( A|\phi^-_0\rangle = 0 \), while for a positive parity ground state,

\[ |\psi^+_0\rangle = \begin{pmatrix} |\phi^+_0\rangle \\ 0 \end{pmatrix} \]  

(24)

\(^1\)The diagonal form of a Hamiltonian is most welcome, since the spectral analysis of the Hamiltonian can be reduced to the analysis of simpler operators. However if an off diagonal form is preferred, one can perform a Foldy-Wouthuysen transformation \([7]\).
it would imply that $A|\phi_0^+\rangle = 0$. A ground state can either have positive or negative Witten parity. Nevertheless, when the ground state is degenerate, both cases can occur. When $E \neq 0$, the number of positive parity eigenstates is equal to the negative parity eigenstates. Yet, this does not hold for the zero modes. Zero modes are fully described by the Witten index. Let $n_\pm$ be the number of zero modes of $H_\pm$ in the subspace $\mathcal{H}_\pm$. For a finite number of zero modes (which implies the operator $A$ is Fredholm\footnote{If an operator $A$ is Fredholm, this implies that it has discrete spectrum. In addition the Fredholm property is ensured if $\dim \ker A < \infty$. Equivalently if an operator is trace-class, then it is by definition Fredholm. For a extensive analysis on these issues, see \cite{7}.}), $n_+$ and $n_-$, the quantity,

$$\Delta = n_- - n_+$$ (25)

is called the Witten index. When the Witten index is non-zero integer, supersymmetry is unbroken and if it is zero, it is not clear whether supersymmetry is broken. If $n_+ = n_- = 0$ supersymmetry is obviously broken, but if $n_+ = n_- \neq 0$ supersymmetry is not broken.

The Fredholm index of the operator $A$ is closely related to the Witten index. The former is defined as,

$$\text{ind} A = \dim \ker A - \dim \ker A^\dagger = \dim \ker A^\dagger A - \dim \ker A A^\dagger$$ (26)

Indeed we have,

$$\Delta = \text{ind} A = \dim \ker H_- - \dim \ker H_+$$ (27)

If the operator $A$ is not Fredholm, then the Witten index is not defined as in (25) and (27). However there exists a heat-kernel regularized index, both for the operator $A$ (which we denote $\text{ind}_t A$) and for the Witten index, $\Delta_t$. The regularized index for the operator $A$ is defined as:

$$\text{ind}_t A = \text{tr} e^{-tA^\dagger A} - \text{tr} e^{-tAA^\dagger}$$ (28)

with $t > 0$ and the trace is taken over the eigenfunctions corresponding to the zero modes of $A$. The regularized Witten index is equal to:

$$\Delta_t = \lim_{t \to \infty} \text{ind}_t A$$ (29)

In the following we shall extensively use the definitions we gave and the notation we used in this section.

### Fermions Around de Sitter Cosmic Strings and $N = 2$ SUSY QM

Consider and infinitely long straight cosmic string. Due to the cylindrical symmetry, the line element in cylindrical coordinates $(r, \phi, z)$ is (we use the notation of \cite{5}):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - e^{2t/a}(dr^2 + r^2 d\phi^2 + dz^2),$$ (30)
with $r \geq 0$, $-\infty < z < \infty$. The points $\phi$ and $\phi_0$ on the hypersurface $z = \text{const}$ and $r = \text{const}$ are considered to be identical. The parameter $a$ is equal to $a = \sqrt{3/\Lambda}$, with $\Lambda$ the cosmological constant.

The Dirac equation in the curved spacetime reads \[ i\gamma^\mu \nabla_\mu \psi - m\psi = 0, \quad \nabla_\mu = \partial_\mu + \Gamma_\mu \] where $\gamma^\mu$ and $\Gamma^\mu$, stand for the curved spacetime gamma matrices and spin connection respectively. Using the vierbeins $e^\mu_{(a)}$, we can connect the curved spacetime gamma matrices, to the flat spacetime ones:

\[ \gamma^\mu = e^\mu_{(a)} \gamma^{(a)}, \quad \Gamma^\mu = \frac{1}{4} \gamma^{(a)} \gamma^{(b)} e^\nu_{(a)} e_\nu^{(b)} \eta_{\mu\nu} \]

with $g^{\mu\nu} = e^\mu_{(a)} e_\nu^{(b)} \eta^{ab}$ and $\mu = 0, 1, 2, 3, 4$. The flat space spacetime Dirac matrices are,

\[ \gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix} \quad \alpha = 1, 2, 3 \]

with $\sigma_i$ the Pauli matrices. Using the vierbeins,

\[ e^\mu_{(a)} = e^{-t/a} \begin{pmatrix} 1 & 0 \\ 0 & e^{-iq\phi} \end{pmatrix} \begin{pmatrix} 0 & -i \sigma^x \\ i \sigma^x & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \sigma^y \\ i \sigma^y & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \sigma^z \\ i \sigma^z & 0 \end{pmatrix} \]

with $q = 2\pi/\phi_0$, the curved spacetime gamma matrices can be written as, $\gamma^0 = \gamma^{(0)}$ and

\[ \gamma^i = e^{t/a} \begin{pmatrix} 0 & -\beta^i \\ -\beta^i & 0 \end{pmatrix} \]

The matrices $\beta^i$ are equal to,

\[ \beta^1 = \begin{pmatrix} 0 & e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \quad \beta^2 = -\frac{i}{r} \begin{pmatrix} 0 & e^{-iq\phi} \\ -e^{iq\phi} & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Finally the spin connections are,

\[ \Gamma_0 = 0, \quad \Gamma_i = -\frac{1}{2a} \gamma^0 \gamma_i + \frac{1}{2} \gamma^{(1)} \gamma^{(2)} q^2_i, \quad i = 1, 2, 3 \]

Decomposing the spinor $\psi$ in the following form,

\[ \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \]

the fermionic equations of motion around a cosmic string, in de Sitter background, can be cast as:

\[ D_+ \phi + \left( \beta^i \partial_i + \frac{1-q}{2r} \beta^1 \right) \chi = 0 \]
\[ D_- \chi + \left( \beta^i \partial_i + \frac{1-q}{2r} \beta^1 \right) \phi = 0 \]
In the above equations, \( D_{\pm} \), stand for:

\[
D_{\pm} = \partial_r - \frac{1}{\tau} \left( \frac{3}{2} \pm i \alpha \right)
\]  

(40)

with \( \tau = -ae^{t/a}, -\infty < \tau < 0 \). Note that \( D_+ = D^*_-, \) which means the complex conjugate of \( D_+ \) is \( D_- \). Also the operator \( B = \beta^i \partial_i + \frac{1-\eta}{2\tau} \beta^1 \) is self-adjoint. Using the operators \( D_{\pm} \) and also \( B \), we can form an \( N = 2 \) susy QM algebra. Indeed, we define,

\[
D = \left( \begin{array}{cc}
D_+ & \beta^1 \partial^1 + \frac{1-\eta}{2\tau} \beta^1 \\
\beta^1 \partial^1 + \frac{1-\eta}{2\tau} \beta^1 & D_-
\end{array} \right)
\]  

(41)

acting on,

\[
|\phi^-\rangle = \left( \begin{array}{c}
\phi \\
\chi
\end{array} \right).
\]  

(42)

Upon taking the adjoint we obtain,

\[
D^\dagger = \left( \begin{array}{cc}
D_- & \beta^1 \partial^1 + \frac{1-\eta}{2\tau} \beta^1 \\
\beta^1 \partial^1 + \frac{1-\eta}{2\tau} \beta^1 & D_+
\end{array} \right)
\]  

(43)

acting on,

\[
|\phi^+\rangle = \left( \begin{array}{c}
\chi \\
\phi
\end{array} \right).
\]  

(44)

The equation \( D|\phi^-\rangle = 0 \) (we used the notation of (23) for reasons that will become clear shortly) yields the solutions of the equation of motion (39). Hence, it is easy to see that the zero modes of the operator \( D \) correspond to the solutions of the equation of motion (39). Notice that, the zero modes of the operator \( D^\dagger \) are \( \chi \) and \( \phi \), as can be easily checked (actually the equation \( D^\dagger|\phi^+\rangle = 0 \) yields the equations of motion (39)). Using the operators \( D \) and \( D^\dagger \), we can define the supercharges \( Q \) and \( Q^\dagger \),

\[
Q = \left( \begin{array}{cc}
0 & D \\
0 & 0
\end{array} \right), \quad Q^\dagger = \left( \begin{array}{cc}
0 & 0 \\
D^\dagger & 0
\end{array} \right)
\]  

(45)

Also the Hamiltonian of the system can be written in terms of \( D \) and \( D^\dagger \), in the following diagonal form,

\[
H = \left( \begin{array}{cc}
DD^\dagger & 0 \\
0 & D^\dagger D
\end{array} \right)
\]  

(46)

It is obvious that the above matrices obey, \( \{Q, Q^\dagger\} = H, \quad Q^2 = 0, \quad Q^\dagger 2 = 0, \quad \{Q, W\} = 0, \quad W^2 = I \) and \( [W, H] = 0 \). Thus we can see that an \( N = 2 \) susy QM algebra underlies the fermionic system. This property is it self particularly useful, especially for spectral problems of Dirac fields around defects (straight on deformed). Let us now see what are the implications of supersymmetry in the fermionic system and it’s zero mode solutions. We are particularly interested in the zero mode solutions but, we shall also discuss at the end of this section the implications of supersymmetry on the eigenfunctions of the Hamiltonian with \( E \neq 0 \).
The operator \( D \) is not Fredholm because it has not discrete spectrum. This can be easily seen from equation (39). Indeed it can be written as:

\[
\left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial \phi^2 + \partial_r^2 + \frac{q-1}{r} \beta_1 \beta_2 \partial \phi - \frac{(q-1)^2}{4r^2} - D_+ D_- \right) \phi = 0
\]  

(47)

The solution reads,

\[
\psi_\sigma(x) \sim \left( \begin{array}{c}
C_1 H^{(1)}_{1/2-\text{ima}} (\gamma \eta) J_\beta_1 (\lambda r) \\
C_2 H^{(1)}_{1/2-\text{ima}} (\gamma \eta) J_\beta_1 (\lambda r) e^{iq\phi} \\
C_3 H^{(1)}_{1/2-\text{ima}} (\gamma \eta) J_\beta_1 (\lambda r) e^{iq\phi} \\
C_4 H^{(1)}_{-1/2-\text{ima}} (\gamma \eta) J_\beta_1 (\lambda r) e^{iq\phi}
\end{array} \right)
\]  

(48)

with “\( \sigma \)” characterizing the quantum numbers of the system, two of which continuously vary from zero to infinity (\( \lambda \) and \( k \)). For a complete analysis on the solutions see [5]. Accordingly, the definitions (28) and (29) for the regularized indices hold.

Since the two equations \( D|\phi^-\rangle = 0 \) and \( D^\dagger|\phi^+\rangle = 0 \) have the same solutions, namely \( \chi \) and \( \phi \), this implies that \( \ker D = \ker D^\dagger \), which in turn implies \( \ker DD^\dagger = \ker DD^\dagger^\dagger \). Hence the operators \( e^{-tD^\dagger D} \) and \( e^{-tDD^\dagger} \) have the same trace, that is \( \tr e^{-tD^\dagger D} = \tr e^{-tDD^\dagger} \). This means that the regularized index of the operator \( D \) is zero and hence the regularized Witten index is zero. Therefore, supersymmetry is unbroken. In order to further clarify this point, recall that supersymmetry is unbroken in two cases: when the Witten index is non-zero integer and if it is zero, but the number of zero modes satisfy \( n_+ = n_- \neq 0 \). If on the contrary \( n_+ = n_- = 0 \) supersymmetry is obviously broken. In the non-Fredholm operator case, the numbers \( n_+ \) and \( n_- \), that is, the zero modes of the operator \( D \) and \( D^\dagger \) respectively, are replaced by \( \ker D \) and \( \ker D^\dagger \). Since \( \ker D = \ker D^\dagger \neq 0 \), we conclude that just as in the Fredholm operators case, supersymmetry is unbroken. Clearly this corresponds to the case we described below equation (48).

The relation \( D|\phi^-\rangle = 0 \) (we use the notation of relations (22) and (23)) implies \( Q|\psi^-\rangle = 0 \). This means that the ground state \( \Psi^- \) is actually a negative parity eigenstate. In such a way, the negative Witten parity zero mode eigenstate \( |\psi_0^-\rangle \), of the Hamiltonian \( H^- \), is:

\[
|\psi_0^-\rangle = \left( \begin{array}{c}
0 \\
0 \\
\phi \\
\chi
\end{array} \right)
\]  

(49)

In the same vain, the positive parity zero mode eigenstate is,

\[
|\psi_0^+\rangle = \left( \begin{array}{c}
\chi \\
\phi \\
0 \\
0
\end{array} \right)
\]  

(50)

Therefore, owing to supersymmetry, the \( \chi \) and \( \phi \) components of the Dirac spinor constitute the positive and negative parity solutions of the \( N = 2 \) supersymmetric system.
Let us see the implications of supersymmetry on the eigenfunctions of the Hamiltonian with \( E \neq 0 \). The Hamiltonians \( H_+ \) and \( H_- \), are known to be isospectral for eigenvalues different from zero \([7, 10]\), that is,
\[
\text{spec}(H_+) \setminus \{0\} = \text{spec}(H_-) \setminus \{0\}
\]
In addition, the following relations hold,
\[
Q|\psi_0^-\rangle = \sqrt{E}|\psi_0^+\rangle \quad \text{and} \quad Q^\dagger|\psi_0^+\rangle = \sqrt{E}|\psi_0^-\rangle,
\]
with \( E \) the common eigenvalues of the Hamiltonians \( H_+ \) and \( H_- \). In turn these imply,
\[
D|\phi^-\rangle = \sqrt{E}|\phi^+\rangle \quad \text{and} \quad D^\dagger|\phi^+\rangle = \sqrt{E}|\phi^-\rangle.
\]
Apart from these, there are many interesting mathematical properties that the supersymmetric system possesses, but these are beyond the scope of this article\(^3\).

We have to note that supersymmetry of any kind around topological defects and in various gravitational backgrounds, has been studied in many works, for example \([15–17]\). In reference \([15]\), there was found that there is a close connection between supersymmetric quantum mechanics and the mechanics of a spinning Dirac particle moving in Schwarzschild spacetime. In addition in reference \([16]\) it was found that a fermion in a monopole field possesses a rich supersymmetry structure. The connection of supersymmetry with the Taub-NUT spacetime was studied in \([17]\).

Studies involving de Sitter space are of particular importance. Indeed there has been a great deal of work on de Sitter space solutions in supergravity and string theory (see \([18]\) and references therein). The connection of de Sitter space and supercritical string theory was presented in \([19]\). In addition, de Sitter space is known to have finite entropy. De Sitter entropy can be understood as the number of degrees of freedom in a quantum mechanical dual \([20]\). Moreover, de Sitter space is closely connected with \( N = 2 \) supersymmetry \([21]\) and also these two are connected with hybrid inflation solutions. Furthermore, \( N = 4 \) supersymmetric quantum mechanics is very closely related to the description of particle dynamics in de Sitter space.

Obviously spacetime supersymmetry and supersymmetric quantum mechanics are not the same, nevertheless the connection is profound, since extended (with \( N = 4, 6... \)) supersymmetric quantum mechanics models describe the dimensional reduction to one (temporal) dimension of \( N = 2 \) and \( N = 1 \) Super-Yang Mills models \([22]\). The interconnection of supersymmetric theories is an interesting mathematical property that is closely connected with the construction of superconformal quantum mechanical models \([22]\). In addition, a number of physical applications, renders studies of supersymmetric quantum mechanics models to be of valuable importance (due to the simplicity these have), such as low energy dynamics of black hole models, see \([22]\). Localized fermion solutions around vortices and other defects frequently appear in superconductor studies. Thus in view of a micro-description of superconductors, through

\(^3\)For example the Krein’s spectral function \( \xi(\lambda) \), which is equal to zero because the Witten index is zero.
holographic superconductors \[23\], the existence of a supersymmetric quantum mechanic
algebra could be of particular importance.

From the above, it is clear that supersymmetric quantum mechanics is a very useful tool
to make complex field theories more easy to handle. In addition, studying fermionic
solutions around defects in de Sitter space is very interesting, since these problems may
be reductions of more evolved problems.

But there is another valuable property of the supersymmetric quantum mechanics su-
percharges that we did not mentioned. The set of the \(N=2\) supersymmetric quantum
mechanics supercharges are invariant under an R-symmetry. Furthermore, the Hamilto-
nian is invariant under this symmetry \[10\]. Particularly, the real superalgebra \[1\] and \[4\]
is invariant under the transformation,

\[
\begin{pmatrix}
Q_1' \\
Q_2'
\end{pmatrix} =
\begin{pmatrix}
cos a & \sin a \\
-sin a & \cos a
\end{pmatrix} 
\begin{pmatrix}
Q_1 \\
Q_2
\end{pmatrix}
\]

Correspondingly the complex supercharges \(Q\) and \(Q^\dagger\) are transformed under a global \(U(1)\)
transformation:

\[
Q' = e^{-ia} Q, \quad Q'^\dagger = e^{ia} Q^\dagger
\]

This R-symmetry is a symmetry of the Hilbert states corresponding to the spaces \(H^+\) and
\(H^-\). Furthermore the two spaces can have different transformation parameters. To make
this clear, let \(\psi^+\) and \(\psi^-\) denote the Hilbert states corresponding to the spaces \(H^+\) and
\(H^-\). Then the \(U(1)\) transformation of the states is,

\[
\psi'^+ = e^{-i\beta^+} \psi^+, \quad \psi'^- = e^{-i\beta^-} \psi^-
\]

It is clear that the parameters \(\beta^+\) and \(\beta^-\) are different global parameters. Consistency
with relation \[55\] requires that \(a = \beta^+ - \beta^-\). We must note that a global \(U(1)\) symmetry
is expected around a cosmic string \[24\]. Furthermore the breaking of a local \(U(1)\) leaves
a global \(U(1)\) in cosmic string models \[1\]. Thus this global \(U(1)\) symmetry is really
interesting. Nevertheless we must further study whether there is a connection between
the susy R-symmetry and the remnant global \(U(1)\) symmetry of phenomenological models
around strings. It would then be interesting to study more realistic models, but this is
beyond the scopes of this article.

The cylindrical line element \[30\] has played a crucial role in our analysis. An important
question that is raised is whether the outcomes of the de-Sitter cosmic string framework
hold for more general spacetimes. The answer in this question is not so trivial and we
should be very cautious in generalizing our results. We have applied our results to cosmic
strings in flat spacetime and also in anti-de Sitter spacetimes and no supersymmetry
algebra underlies the fermionic system in these two cases. Therefore such an analysis
must be carried away with great detail and caution and certainly would be invaluable. Also
possible generalizations to include bosonic or vector configurations could be interesting.
But this analysis is beyond the purposes of this article.

Before closing this section we must mention some supersymmetric quantum mechanics
developments, relevant to our work. The supersymmetry studied in this article is similar
to what takes place in some periodic systems where \( n_+ = n_- = 1 \neq 1 \), that is, susy is exact though the Witten index is equal to zero \([25,27]\).

The peculiarity of the periodic systems \([25,27]\) is that besides an \( N = 2 \) supersymmetry, these systems possess much more rich structure (related to their special nature being finite gap systems) which is related to the existence of a hidden supersymmetry. Since in our case, our system is characterized by the same property \( n_+ = n_- \neq 0 \), maybe it also possess a hidden, bosonized supersymmetry. Hidden, bosonized supersymmetry is related to the existence of a grading operator, which has a nonlocal nature (reflection operator) \([28,30]\).

We hope we pursuit further these issues in a future publication.

**Conclusions**

In this paper we found a hidden \( N = 2 \) susy QM algebra, underlying the fermionic solutions of the Dirac equation around a straight cosmic string in de Sitter spacetime. The negative and positive Witten parity states of the susy algebra can be written in terms of the fermionic solutions of the equations of motion. The Witten index is zero and additionally the number of zero modes for the two supercharges are equal. We therefore concluded that the system has unbroken supersymmetry. It was tempting to check whether this susy algebra underlies the fermion system around a cosmic string but with a flat background. It turns out that there is no supersymmetry in the flat case. It stands to reason to argue that, the presence of spacetime curvature is responsible for the susy structure in the de Sitter fermion system. However this must not be the case, since in the superconducting string case, the background spacetime is also flat, still, an \( N = 2 \) susy QM algebra is present \([9]\). We hope to comment on this issues in the future.

Before closing we must note that there is a mathematical property stemming from \( N = 2 \) susy QM, that can be very useful perhaps when studying perturbations of the metric around the string, or in the case the string is not straight (also perhaps in the case of an electromagnetic field around the string. However we must further study the last case to be sure. We hope to do so in a future work). If these effects can be described by a matrix \( C \), which anti-commutes with the Witten operator and is also symmetric\(^4\), then,

\[
\text{ind}_t(D + C) = \text{ind}_tD \tag{57}
\]

This means that there is a correspondence between the solutions of the equation \((D + C)\psi = 0\) and these of the equation \(D\psi = 0\). This is very useful, regarding fermionic spectral problems around one dimensional defects.

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\(^4\) Also the operator \(Ce^{-D^*D} \) must be trace-class
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