PROJECTION OPERATORS ONTO SPACES OF CHEBYSHEV SPLINES

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Abstract. We prove that the Chebyshev spline orthoprojectors are uniformly bounded on $L^\infty$.

1. Introduction

In this paper, we extend Shadrin’s theorem [11] on the boundedness of the polynomial spline orthoprojector on $L^\infty$ by a constant that does not depend on the underlying univariate grid $\Delta$ to the setting of Chebyshev splines. The space of Chebyshev splines $S(\mathcal{U}_k; \Delta)$ of order $k$ consists of functions that, on each grid interval of $\Delta$, are contained in $\mathcal{U}_k = \text{span}\{u_1, \ldots, u_k\}$, the space of linear combinations of generalized monomials $u_1, \ldots, u_k$ that arise by iterated integrals of positive weight functions in the same way as the classical monomials $1, \ldots, x^{k-1}$ arise as iterated integrals of constant functions.

One of the main reasons to consider this extension to spline orthoprojectors onto Chebyshev splines is that in recent years, it turned out that in many cases (see e.g. [11, 9, 7, 6, 8, 5]), sequences of orthogonal projections onto classical spline spaces corresponding to arbitrary grid sequences $\Delta_n$ behave like sequences of conditional expectations (or, more generally, like martingales) and we want to extend martingale type results to an even larger class of orthogonal projections.

In order to explain those martingale type results, we have to introduce a little bit of terminology: Let $k$ be a positive integer, $(\mathcal{F}_n)$ an increasing sequence of $\sigma$-algebras of sets in $[0, 1]$ where each $\mathcal{F}_n$ is generated by a finite partition of $[0, 1]$ into intervals of positive length. Moreover, let $S_n^{(k)} = \{f \in C^{k-2}[0, 1] : f$ is a classical polynomial of order $k$ on each atom of $\mathcal{F}_n\}$ and define $P_n^{(k)}$ as the orthogonal projection operator onto $S_n^{(k)}$ with respect to the $L^2$ inner product on $[0, 1]$ with the Lebesgue measure $|\cdot|$. The space $S_n^{(1)}$ consists of piecewise constant functions and $P_n^{(1)}$ is the conditional expectation operator with respect to the $\sigma$-algebra $\mathcal{F}_n$. Similarly to the definition of martingales, we introduce the following notion: let $(f_n)_{n \geq 0}$ be a sequence of integrable functions, we call this sequence a $k$-martingale spline sequence (adapted to $(\mathcal{F}_n)$), if

$$P_n^{(k)} f_{n+1} = f_n, \quad n \geq 0.$$ 

Classical martingale theorems such as Doob’s inequality, the martingale convergence theorem or Burkholder’s inequality in fact carry over to $k$-martingale spline sequences corresponding to arbitrary filtrations $(\mathcal{F}_n)$ of the above type. Indeed, we have

(i) (Shadrin’s theorem) there exists a constant $C_k$ depending only on $k$ such that

$$\sup_n \|P_n^{(k)} : L^1 \to L^1\| \leq C_k,$$
(ii) there exists a constant $C_k$ depending only on $k$ such that for any $k$-martingale spline sequence $(f_n)$ and any $\lambda > 0$,
\[
|\{\sup_n |f_n| > \lambda\}| \leq C_k \frac{\sup_n \|f_n\|_{L^1}}{\lambda},
\]
(iii) for all $p \in (1, \infty]$ there exists a constant $C_{p,k}$ depending only on $p$ and $k$ such that for all $k$-martingale spline sequences $(f_n)$,
\[
\|\sup_n |f_n|\|_{L^p} \leq C_{p,k} \sup_n \|f_n\|_{L^p},
\]
(iv) if $(f_n)$ is an $L^1$-bounded $k$-martingale spline sequence, then $(f_n)$ converges almost surely to some $L^1$-function.
(v) for all $p \in (1, \infty]$ and all positive integers $k$, scalar-valued $k$-spline-differences converge unconditionally in $L^p$, i.e. for all $f \in L^p$,
\[
\|\sum_n \pm (P_n^{(k)} - P_{n-1}^{(k)})f\|_{L^p} \leq C_{p,k}\|f\|_{L^p},
\]
for some constant $C_{p,k}$ depending only on $p$ and $k$.

Note that the $L^1$-boundedness stated in property (i) and $L^\infty$-boundedness of the operators $P_n^{(k)}$ are equivalent, as $P_n^{(k)}$ being an orthogonal projection operator is also self-adjoint. (i) is proved in [11], vector valued versions of (ii)–(iv) are proved in [9, 6] and (v) is proved in [7]. The basic starting point in proving the results (ii)–(v) independently of the filtration $(F_n)$ is that (i) is true and in this paper we prove the analogue of (i) for Chebyshev spline spaces (see Section 2 for exact definitions and properties of Chebyshev splines):

**Theorem 1.1.** There exists a finite positive constant $C$, depending only on $U_k$, so that for all partitions $\Delta$ of the unit interval $[0, 1]$, the orthogonal projection operator $P_\Delta$ onto the space of Chebyshev splines $S(U_k; \Delta)$ is bounded on $L^\infty$ by the constant $C$, i.e.,
\[
\|P_\Delta f\|_\infty \leq C\|f\|_\infty, \quad f \in L^\infty[0, 1].
\]

The organization of this article is as follows: In Section 2, we recall basic definitions and facts involving Chebyshev spline functions. Next, in Section 3, we prove Theorem 1.1 ‘at the boundary’, meaning that we prove $|P_\Delta f(0)|$ and $|P_\Delta f(1)|$ are bounded by $C\|f\|_\infty$ with a constant $C$ independent of $\Delta$, provided $\Delta$ satisfies some additional conditions. Finally, in Section 4, we prove Theorem 1.1 by reducing the general case to the boundary and showing how the conditions on $\Delta$ from Section 3 can be eliminated.

### 2. Preliminaries

As a basic reference to Chebyshev splines, we use the book [10], in particular Chapter 9. Suppose that for a non-negative integer $k$, $w = (w_1, \ldots, w_k)$ is a vector consisting of $k$ positive functions (weights) on $I = [a, b]$ with $w_i \in C^{k-i+1}$ for all $i \in \{1, \ldots, k\}$. Then, let $Tf(x) = \int_a^x f(y) dy$ be the operator of integration, and $m_gf(x) = g(x)f(x)$ the multiplication operator by the function $g$ and define the vector $u(w) = (u_1, \ldots, u_k)$ of functions by

\[
u_i = \left(\prod_{t=1}^{i-1} m_{w_t}T\right) w_i = w_1(\cdot) \int_a w_2(s_2) \cdots \int_a^t w_i(s_i) ds_i \cdots ds_2,
\]
for all $i = 1, \ldots, k$. Let $t_1 \leq t_2 \leq \cdots \leq t_k$ be an increasing sequence of real numbers. We set

\[d_i := \max\{0 \leq j \leq k-1 : t_i = \cdots = t_{i-j}\}\]
and define the expression

\[
D\left( t_1, \ldots, t_k \right) \frac{u_1, \ldots, u_k}{D} := \det \left( D^{d_j u_j(t_i)} \right)_{i,j=1}^k,
\]

where \( D \) denotes the ordinary differential operator. The functions \( u(w) \) form an extended complete Chebyshev system (ECT system for short), i.e., for all \( j = 1, \ldots, k \) and all points \( t_1 \leq t_2 \leq \cdots \leq t_j \), we have that

\[
D\left( t_1, \ldots, t_j \right) \frac{u_1, \ldots, u_j}{D} > 0.
\]

An example of an ECT is the set of monomials \( \{1, x, \ldots, x^{k-1}/(k-1)!\} \), which corresponds to the choice \( a = 0 \) and \( w_1 = \cdots = w_k = 1 \). In working with ECT systems, we conveniently define the corresponding differentiation operators

\[
D_i f = D\left( \frac{f}{w_i} \right), \quad i = 1, \ldots, k.
\]

Additionally, put

\[
L_i = D_i D_{i-1} \cdots D_1, \quad i = 1, \ldots, k \text{ and } L_0 f = f.
\]

If \( U_k = \{u_1, \ldots, u_k\} \) is an ECT, then \( U_k := \text{span} U_k \) is the null space of the operator \( L_k \), and, by definition of \( u_i \),

\[
L_j u_i(a) = w_i(a) \delta_{j,i-1}, \quad j = 0, \ldots, i - 1, \quad i = 1, \ldots, k.
\]

We also set

\[
D_{(t_0, \ldots, t_{k-1})} \left( t_1, \ldots, t_k \right) \frac{u_1, \ldots, u_k}{D} := \det \left( L_{d_j u_j(t_i)} \right)_{i,j=1}^k.
\]

In this article, we write \( A(t) \lesssim B(t) \) to mean that there exists a constant \( C \) that depends only on \( \min_{x \in I} w_i(x) \) and on \( \max_{x \in I} |D^j w_i(x)| \) for \( 1 \leq i \leq k \) and \( 0 \leq j \leq k - i + 1 \) so that \( A(t) \leq CB(t) \) for all \( t \), where \( t \) denotes all implicit and explicit dependencies that the objects \( A \) and \( B \) might have. Similarly, we use the notations \( \gtrsim \) and \( \simeq \).

The determinant (2.3) and its derivatives can be compared to the classical Vandermonde determinant \( V(t_1, \ldots, t_k) \) that results from inserting the functions \( u_j = t_j^{-1} \) into (2.1). Indeed, for \( u \in U_k \), we have the following Markov inequality for all \( 1 \leq p, q \leq \infty \):

\[
\|D^j u\|_{L^p(I)} \lesssim |I|^{-j+1/p-1/q} \|u\|_{L^q(I)},
\]

where the implied constant additionally may depend on \( p \) and \( q \). As for algebraic polynomials, any non-zero function \( u \in U_k \) has at most \( k - 1 \) zeros including multiplicities. We can associate a divided difference to the system \( u(w) \)

\[
[t_1, \ldots, t_k]u(w) f = \frac{D\left( t_1, \ldots, t_k \right) f}{u_1, \ldots, u_k} \frac{u_1, \ldots, u_{k-1}}{D}\left( t_1, \ldots, t_k \right)
\]

and observe that \( [t_1, \ldots, t_k]u = 0 \) for all \( u \in \text{span} \{u_1, \ldots, u_{k-1}\} \) and \( [t_1, \ldots, t_k]u_k = 1 \). If \( u(w) = (u_1, \ldots, u_k) \), we define the truncated vector \( \bar{u}(w) = (u_1, \ldots, u_{k-1}) \). Similarly
to the classical divided differences, the following recursion formula for \( k \geq 2, t_1 \neq t_k \) and sufficiently smooth functions \( f \) is true:

\[
[t_1, \ldots, t_k]_{u(w)} f = \frac{[t_2, \ldots, t_k]_{u(w)} f - [t_1, \ldots, t_{k-1}]_{u(w)} f}{[t_2, \ldots, t_k]_{u(w)} u_k - [t_1, \ldots, t_{k-1}]_{u(w)} u_k}.
\]

Given the weights \( w = (w_1, \ldots, w_k) \) and the corresponding ECT-system \( u(w) = (u_1, \ldots, u_k) \), we define the dual canonical ECT-system \( u^*(w) = (u_1^*, \ldots, u_k^*, u_{k+1}^*) \) by

\[
u_i^* = \left( \prod_{\ell=1}^{i-1} T_{m_{w_{k-\ell+1}}} \right) \mathbb{1}_I, \quad i = 1, \ldots, k + 1,
\]

and the associated operators \( D_i^* f = (D f) \/ w_i \) for \( i = 1, \ldots, k \) and \( L_i^* f = D_{k-i+1}^* \cdots D_k^* \) for \( i = 1, \ldots, k \) and \( L_i^* f = f \).

We denote by \( T_y f = \int_y^x f(s) \, ds \) and set

\[
h_j(x, y) = h_j^w (x, y) := \left( \prod_{\ell=1}^{j-1} m_{w_{\ell+1}} T_y \right) w_j(x), \quad g_j(x, y) = g_j^w (x, y) := \mathbb{1}_{x \geq y} (x, y) h_j^w (x, y).
\]

The functions \( g_j \) are the analogues of the truncated power functions \((x - y)^{j-1}\) for polynomials.

2.1. Chebyshevian spline functions. Let \( \Delta = \{ a = t_0 = \cdots = t_{k-1} < t_k < \cdots < t_n < t_{n+1} = \cdots = t_{n+k} = b \} \) and define the Chebyshevian spline space \( S(\mathcal{U}_k; \Delta) \) as

\[
S(\mathcal{U}_k; \Delta) = \{ s \in C^{k-2}(I) : \text{there exist functions} \}
\]

\[
s_k-1, \ldots, s_n \in \mathcal{U}_k \text{ so that } s|_{(t_j, t_{j+1})} = s_j \text{ for all } j = k - 1, \ldots, n,\}
\]

where \( \mathcal{U}_k = \text{span} \, \mathcal{U}_k = \text{span} \{ u_1, \ldots, u_k \} \). Denote by \(|\Delta| = \max_{i=0}^n (t_{i+1} - t_i)\) the maximal grid size of \( \Delta \). We define the Chebyshevian B-spline function \( M_i^w \) for weights \( w = (w_1, \ldots, w_k) \) and for \( x \in [a, b] \) by

\[
M_i(x) = M_i^w(x) = (-1)^i [t_i, \ldots, t_{i+k}]_{u^*(w)} g_k(x, y), \quad i = 0, \ldots, n,
\]

where the divided difference is taken with respect to the variable \( y \). We define \( M_i(b) \) by extending \( M_i \) continuously. The system of functions \( \{ M_i \} \) forms an algebraic basis of the space \( S(\mathcal{U}_k; \Delta) \) and we have

\[
M_i > 0 \text{ on } (t_i, t_{i+k}), \quad M_i = 0 \text{ on } (t_i, t_{i+k})^c.
\]

We also have the Peano representation

\[
[t_i, \ldots, t_{i+k}]_{u^*(w)} f = \int_{t_i}^{t_{i+k}} M_i(x) L_k^w f(x) \, dx
\]

for sufficiently smooth functions \( f \). Setting \( f = u_{k+1}^* \), we obtain that \( M_i \) is \( L^1 \)-normalized:

\[
\int_{t_i}^{t_{i+k}} M_i(x) \, dx = 1.
\]

Similar to classical polynomial splines, a nonzero function in \( S(\mathcal{U}_k; \Delta) \) has at most \( \dim S(\mathcal{U}_k; \Delta) - 1 = n \) zeros.

Let \( K : \Omega \times \Sigma \to \mathbb{R} \) be defined on two totally ordered sets \( \Omega, \Sigma \). Then \( K \) is called totally positive if for any integer \( r \) and any choice \( \omega_1 \leq \cdots \leq \omega_r \) of elements in \( \Omega \) and \( \sigma_1 \leq \cdots \leq \sigma_r \) of elements in \( \Sigma \), the determinant of the matrix \( (K(\omega_i, \sigma_j))_{i,j=1}^r \) is non-negative. We note that the kernel \( K(i, t) = M_i(t) \) is totally positive (see e.g. [10 Theorem 9.34] or [3 pp. 527]). Using the basic composition formula for determinants (see e.g. [3 p. 17]), we also get that the kernel \( K(i, j) = \langle M_i, M_j \rangle \) is totally positive.
Additionally, we are able to perform Hermite interpolation; the simplest form is the following: assume that \( y_0 < \cdots < y_n \) are points that satisfy \( y_i \in (t_i, t_{i+k}) \). Then, for all vectors \((v_j)_{j=0}^n\), there exists a unique function \( s \in S(U_k; \Delta) \) so that
\[
s(y_i) = v_i, \quad i = 0, \ldots, n.
\]

2.2. Renormalized B-spline functions. In the following, let \( \phi_{i,r} \) be the function contained in \( \text{span}\{u_1^*, \ldots, u_r^*\} \) with zeros at the points \( t_i, \ldots, t_{i+r-2} \) (including multiplicities) and having leading coefficient 1, i.e., \( \phi_{i,r} \) is of the form
\[
\phi_{i,r} = u_r^* + \sum_{j=1}^{r-1} c_j u_j^*
\]
for some numbers \((c_j)_{j=1}^{r-1}\). Observe that, if \( s \) is not contained in \( \{t_i, \ldots, t_{i+r-2}\} \),
\[
(2.7)
\]
\[
\phi_{i,r}(s) = \frac{D(t_1, \ldots, t_{i+r-2}, s)}{D(t_1, \ldots, t_{i+r-2})} \frac{D(t_0^*, \ldots, t_{r-1}^*)}{D(t_0^*, \ldots, t_{r-1})} \frac{D(t_1^*, \ldots, t_{i+r-2}^*)}{D(t_1^*, \ldots, t_{i+r-2})} \frac{D(t_1^*, \ldots, t_{i+r-2})}{D(t_1^*, \ldots, t_{i+r-2})},
\]
and the difference \( \phi_{i,k+1} - \phi_{i+1,k+1} \) is contained in \( \text{span}\{u_1^*, \ldots, u_k^*\} \) and has zeros at \( t_{i+1}, \ldots, t_{i+k-1} \) including multiplicities. Therefore, there exists a number \( \alpha_i \) so that
\[
(2.9)
\phi_{i,k+1} - \phi_{i+1,k+1} = \alpha_i \phi_{i+1,k}.
\]

We now verify that \( \alpha_i \) is positive. If \( \mu \) denotes the multiplicity of \( t_{i+k} \) in the set \( \{t_{i+1}, \ldots, t_{i+k-1}\} \), then
\[
D^j \phi_{i+1,k+1}(t_{i+k}) = 0, \quad 0 \leq j \leq \mu,
\]
\[
D^j \phi_{i,k+1}(t_{i+k}) = D^j \phi_{i+1,k}(t_{i+k}) = 0, \quad 0 \leq j \leq \mu - 1.
\]

We additionally have that
\[
D^\mu \phi_{i,k+1}(t_{i+k}) \neq 0, \quad \text{and} \quad D^\mu \phi_{i+1,k}(t_{i+k}) \neq 0,
\]
for otherwise this, together with the other conditions imposed on \( \phi_{i,k+1} \) and \( \phi_{i+1,k} \) would imply \( \phi_{i,k+1} \equiv 0 \) or \( \phi_{i+1,k} \equiv 0 \) which is impossible. Now we also know for \( s > t_{i+k} \) that \( \phi_{i,k+1}(s) > 0 \) and \( \phi_{i+1,k}(s) > 0 \) by formula (2.8) and the fact that \( \{u_1^*, \ldots, u_k^*\} \) is an ECT system. Thus, by Taylor expansion,
\[
D^\mu \phi_{i,k+1}(t_{i+k}) > 0 \quad \text{and} \quad D^\mu \phi_{i+1,k}(t_{i+k}) > 0,
\]
or equivalently
\[
L_\mu^* \phi_{i,k+1}(t_{i+k}) > 0 \quad \text{and} \quad L_\mu^* \phi_{i+1,k}(t_{i+k}) > 0,
\]
which yields \( \alpha_i > 0 \) after inserting into formula (2.9):
\[
(2.10)
\]
\[
L_\mu^* \phi_{i+1,k}(t_{i+k}) = \alpha_i L_\mu^* \phi_{i+1,k}(t_{i+k}).
\]

Equation (2.8) and the comparison to Vandermonde determinants (2.4) imply that
\[
(2.11)
\alpha_i \asymp t_{i+k} - t_i.
\]

If we define the renormalized B-spline function
\[
(2.12)
N_i(x) = \alpha_i M_i(x),
\]
then the collection of those functions forms a partition of unity in the sense
\[
(2.13)
\sum_i N_i(x) = u_1(x).
\]
2.3. Dual functionals to $N_i$. Let $J_i = [t_i, t_{i+1}]$ be a largest grid interval contained in the support $[t_i, t_{i+k}]$ of $N_i$. Let $g$ be a smooth function on the real line with $g = 0$ on $(-\infty, 0]$ and $g = 1$ on $[1, \infty)$. We define the functionals

$$\lambda_i f = \frac{1}{\alpha_i} \int_{t_i}^{t_{i+k}} f(x)L_k^\psi(x) \, dx,$$

where $\psi_i(x) = \alpha_i \phi_{i+1,k}(x)G_i(x)$ and $G_i(x) := g(\overline{\sup_{x \in J_i} f})$. Those functionals are dual to the B-spline functions $N_i$ in the sense that $\lambda_i N_j = \delta_{i,j}$ and, for $p \in [1, \infty]$, they satisfy the inequality

$$|\lambda_i f| \lesssim C|J_i|^{-1/p}\|f\|_{L^p(J_i)},$$

provided that $|\Delta| \leq 1$ and where the implied constants additionally depend on $p$. An important consequence is the following stability of the functions $(N_i)$ under the condition that $|\Delta| \leq 1$: for all $p \in [1, \infty]$,

$$\|\sum_i c_i N_i\|_p \simeq \left( \sum_i |c_i|^p \alpha_i \right)^{1/p},$$

where, again, the implied constants may additionally depend on the parameter $p$. The proof of this result proceeds in the same way as for polynomial splines (see e.g. [2, p. 145]).

3. Boundary points

The main result of this section is the following theorem:

**Theorem 3.1.** There exist constants $\varepsilon, K_1 > 0$, depending only on $U_k$, so that for all partitions $\Delta$ of the interval $[a, b]$ satisfying $|\Delta| \leq \varepsilon$, the orthogonal projection operator $P_\Delta$ onto the space of Chebyshev splines $S(U_k; \Delta)$ satisfies

$$|P_\Delta f(a)| + |P_\Delta f(b)| \leq K_1 \|f\|_{\infty}, \quad f \in L^\infty[a, b].$$

Similarly as it is done in [11] and [3] for polynomial splines, we construct a function $\phi \in S(U_k; \Delta)$ with the properties that the sign of $\langle \phi, M_j \rangle$ is alternating in $j$ and that, for all $j$, $|\langle \phi, M_j \rangle| \geq 1$. The basic idea of the construction of such a function $\phi$ is the same as in [11], but we have to deal with the fact that the weights $(w_i)$ now are arbitrary functions. In Section 3.1, we give a few estimates and formulas on the B-spline functions $(M_i)$ that are needed subsequently. In Section 3.2, the function $\phi$ is defined and it is shown that it has the desired properties and in Section 3.3, we show how those properties imply Theorem 3.1.

3.1. Estimates for the derivatives of B-spline functions. The following formula for the derivative of the B-spline functions $M_i$ in terms of B-splines of lower order is a consequence of the recursion formula for divided differences.

**Lemma 3.2.** Let $w = (w_1, \ldots, w_k)$, $\bar{w} = (w_2, \ldots, w_k)$ and $u^*(w) = (u_1^*, \ldots, u_{k+1}^*)$ be the dual system to $u(w)$. Then, for any $x \notin \{t_i, \ldots, t_{i+k}\}$,

$$D_1 M_i^w(x) = \left( \frac{M_i^w(x)}{w_1(x)} \right)' = \frac{M_i^w(x) - M_{i+1}^w(x)}{h_i^w},$$

where

$$h_i^w = [t_{i+1}, \ldots, t_{i+k}][u^*(\bar{w})]u_{k+1}^* - [t_i, \ldots, t_{i+k-1}][u^*(\bar{w})]u_k^*.$$

Additionally,

$$0 < h_i^w \lesssim t_{i+k} - t_i.$$
Proof. We use the definitions of $M_i^w$ and of $g_k(x, y)$ to get
\[
M_i^w(x) = (-1)^k [t_i, \ldots, t_{i+k}]_{u^*(w)} g_k^w(x, y)
\]
\[
= (-1)^k [t_i, \ldots, t_{i+k}]_{u^*(w)} \left( \mathbb{1}_{x \geq y}(x, y) \left( \prod_{\ell=1}^{k-1} m_{w_\ell} T_{y_\ell} w_k(x) \right) \right),
\]
where the divided difference is applied to the $y$-variable. If we divide by $w_1$ and differentiate, we obtain
\[
D_1 M_i^w(x) = (-1)^k [t_i, \ldots, t_{i+k}]_{u^*(w)} \left( \mathbb{1}_{x \geq y}(x, y) \left( \prod_{\ell=2}^{k-1} m_{w_\ell} T_{y_\ell} w_k(x) \right) \right)
\]
\[
= (-1)^k [t_i, \ldots, t_{i+k}]_{u^*(w)} g_{k-1}^w(x, y).
\]
Now, we can use the recursion formula for divided differences to get for $\bar{u}^*(w) = (u_1^*, \ldots, u_k^*)$
\[
D_1 M_i^w(x) = (-1)^k \frac{[t_{i+1}, \ldots, t_{i+k}]_{\bar{u}^*(w)} - [t_i, \ldots, t_{i+k-1}]_{\bar{u}^*(w)} u_{k+1}^*}{t_{i+1}, \ldots, t_{i+k}} u_k^* \bar{g}_{k-1}^w(x, y).
\]
Since $u^*(\bar{w}) = \bar{u}^*(w)$ we obtain the desired formula for $D_1 M_i^w$ by using the definition of $M_i^w$.

Next, we show $h_i^w > 0$ and first consider the case $t_{i+k} > t_{i+k-1}$. Here, for $x \in (t_{i+k-1}, t_{i+k})$ we have by the above formula
\[
\frac{M_i(x)}{w_1(x)} = \frac{1}{h_i^w} \int_{t_i}^{x} (M_i^w(s) - M_{i+1}^w(s)) \, ds = \frac{1}{h_i^w} \left( 1 - \int_{t_i}^{x} M_{i+1}^w(s) \, ds \right).
\]
Since the left hand side is positive and the term in the brackets on the right hand side is positive, we conclude that $h_i^w$ is positive as well.

It remains to consider $t_{i+k} = t_{i+k-1}$, in which case we define the function
\[
g(t) = D \begin{pmatrix} t_{i+1} & \ldots & t_{i+k-1} & t \\ u_1^* & \ldots & u_{k-1}^* & u_k^* \end{pmatrix} \cdot D \begin{pmatrix} t_i & \ldots & t_{i+k-1} \\ u_1 & \ldots & u_k \end{pmatrix}
\]
\[
- D \begin{pmatrix} t_i & \ldots & t_{i+k-2} & t_{i+k-1} \\ u_1 & \ldots & u_{k+1}^* & u_k^* \end{pmatrix} \cdot D \begin{pmatrix} t_{i+1} & \ldots & t_{i+k-1} & t \\ u_1 & \ldots & u_k^* \end{pmatrix},
\]
for $t \notin \{t_i, \ldots, t_{i+k}\}$ and extend it smoothly to $\{t_i, \ldots, t_{i+k}\}$. Then, all the zeros of $g$ (including multiplicities) are the points $\{t_i, \ldots, t_{i+k-1}\}$ and in particular $D^j g(t_{i+k}) = 0$ for $j = 0, \ldots, \mu - 1$ where $\mu$ denotes the multiplicity of $t_{i+k}$ in the set $\{t_i, \ldots, t_{i+k-1}\}$.

By the definition of $h_i^w$ and (2.2), the sign of $h_i^w$ is the same as the sign of $D^\mu g(t_{i+k})$. Since $g(t_{i+k} + \varepsilon) > 0$ by the case $t_{i+k} > t_{i+k-1}$, we also deduce $D^\mu g(t_{i+k}) > 0$.

The upper estimate for $h_i^w$ now follows from
\[
h_i^w = \int_{t_i}^{t_{i+k}} w_1(x) \left( \int_{t_i}^{x} M_i^w(s) \, ds - \int_{t_i}^{x} M_{i+1}^w(s) \, ds \right) \, dx,
\]
where we have used the normalization $\int M_i^w = 1$. 

Let us denote by $M_j^{(\ell)}$ the $j$th B-spline corresponding to the weights $(w_{k-\ell+1}, \ldots, w_k)$ and denote by $h_j^{(\ell)}$ the corresponding factor in formula (3.1), i.e. formula (3.1) reads
\[
D_1 M_j^{(\ell)} = \frac{M_{j+1}^{(\ell-1)} - M_j^{(\ell-1)}}{h_j^{(\ell)}}.
\]
Lemma 3.3. We have \( \text{sgn } D_{k-1} \cdots D_1 M_j^{(k)}(t) = (-1)^i \) for \( t \in (t_{j+i}, t_{j+i+1}) \) with \( i = 0, \ldots, k-1 \), and for any \( \ell = 0, \ldots, k-1 \) and \( t \in [t_j, t_{j+k}] \setminus \{t_j, \ldots, t_{j+k}\} \),

\[
|D_{k-1} \cdots D_1 M_j^{(k)}(t)| \gtrsim (t_{j+k} - t_j)^{-k} \cdot \max \left( 1, (t_{j+k} - t_j)^{\ell+1} |D_\ell \cdots D_1 M_j^{(k)}(t)| \right).
\]

Proof. Let us first observe that by (3.3), for \( M \) in terms of the functions \((M_m^{(k-\ell)})\) as

\[
D_\ell \cdots D_1 M_j^{(k)}(t) = \sum_m \beta_m^{(\ell)} M_m^{(k-\ell)}(t),
\]

where the coefficients \( \beta_m^{(\ell)} \) are given by the following recursion formulas:

\[
\beta_j^{(0)} = 1, \quad \beta_m^{(\ell)} = 0 \quad \text{for } m \notin \{j, \ldots, j + \ell\}
\]

(3.4)

\[
\beta_m^{(\ell)} = \frac{\beta_m^{(\ell-1)}}{h_m^{(k-\ell+1)}} - \frac{\beta_{m-1}^{(\ell-1)}}{h_{m-1}^{(k-\ell+1)}}, \quad m = j, \ldots, j + \ell.
\]

Using (3.2), this implies in particular that \( \text{sgn } \beta_m^{(\ell)} = (-1)^{m-j} \) and

\[
|\beta_m^{(k-1)}| \gtrsim (t_{j+k} - t_j)^{-(k-1)+1} \sum_{r=m-(k-\ell)+1}^m |\beta_r^{(\ell)}| \quad \text{for } m = j, \ldots, j + \ell.
\]

(3.5)

Taking a point \( t \in (t_m, t_{m+1}) \) for \( m = j, \ldots, j + k-1 \) and choosing \( \ell = 0 \) in formula (3.5), yields

\[
|D_{k-1} \cdots D_1 M_j^{(k)}(t)| = \left| \sum_r \beta_r^{(k-1)} \cdot M_r^{(1)}(t) \right| = |\beta_m^{(k-1)}| \cdot M_m^{(1)}(t)
\]

\[
\gtrsim (t_{j+k} - t_j)^{-k+1} \sum_{r=m-k+1}^m |\beta_r^{(0)}|(t_{m+1} - t_m)^{-1}
\]

\[
\gtrsim (t_{j+k} - t_j)^{-k},
\]

which shows the first part of the desired inequality.

On the other hand, we can compute for \( t \in (t_m, t_{m+1}) \), by the support property \( \text{supp } M_r^{(k-\ell)} = [t_r, t_{r+k-\ell}] \) of the B-spline functions,

\[
|D_\ell \cdots D_1 M_j^{(k)}(t)| = \left| \sum_{r=m-(k-\ell)+1}^m \beta_r^{(\ell)} M_r^{(k-\ell)}(t) \right|.
\]

As a consequence of (2.11), (2.12), (2.13), we get that \( |M_r^{(k-\ell)}(t)| \lesssim (t_{r+k-\ell} - t_r)^{-1} \), which, together with (3.5), implies

\[
|D_\ell \cdots D_1 M_j^{(k)}(t)| \lesssim (t_{m+1} - t_m)^{-1} \sum_{r=m-(k-\ell)+1}^m |\beta_r^{(\ell)}|
\]

\[
\lesssim (t_{m+1} - t_m)^{-1} (t_{j+k} - t_j)^{k-\ell-1} |\beta_m^{(k-1)}|
\]

\[
\lesssim (t_{j+k} - t_j)^{k-\ell-1} |D_{k-1} \cdots D_1 M_j^{(k)}(t)|,
\]

which also shows the second part of the desired inequality. \( \square \)
3.2. Definition and Properties of the Chebyshev spline \( \phi \). Let \( \Delta = \{ a = t_{-k+1} = \cdots = t_{k-1} < t_k < \cdots < t_n = t_{n+1} = \cdots = t_{n+2k-1} = b \} \) and \( \sigma \) be the Chebyshev spline on the grid \( \Delta \) so that in the interior of each grid interval,

\[
\sigma \in \ker D_k D_{k-1} \cdots D_2 D_2 \cdots D_k D_{k+1},
\]

(where \( D_j = D \circ m_{1/w_j} \) and \( D_{k+1} = D \)) with the properties

1. \( D_3 \cdots D_k D_{k+1} \sigma(a) = 1 \),
2. \( D_i \cdots D_k D_{k+1} \sigma(a) = D_i \cdots D_k D_{k+1} \sigma(b) = 0 \) for any \( i = 4, \ldots, k + 1 \).
3. \( \sigma(t_j) = 0 \) for any \( j = k - 1, \ldots, n + 1 \).

By Hermite interpolation \( \text{(2.7)} \) such a function \( \sigma \) exists and is uniquely determined. Indeed, denote by \( M_j^{2k-1} \) the B-splines with respect to the partition \( \Delta \) corresponding to the differential operator in \( (3.1) \). Then, conditions (2) and \( \sigma(a) = \sigma(b) = 0 \) for \( \sigma = \sum_{j=-k+1}^n a_j M_j^{2k-1} \) amount to solving two triangular systems resulting in \( a_{-k+1} = \cdots = a_{-2} = 0 \) and \( a_n = \cdots = a_{n-k+3} = 0 \), since \( D_i \cdots D_k M_j^{2k-1}(a) \) is only non-zero for \( j = -k + 1, \ldots, -i + 2 \) and \( D_i \cdots D_k M_j^{2k-1}(b) \) is only non-zero for \( j = n + i - k - 1, \ldots, n \). Next, we choose \( a = 0 \) so that \( D_3 \cdots D_k D_{k+1} \sigma(a) = 1 \). Then, we can use standard Hermite interpolation \( \text{(2.7)} \) to obtain uniquely determined coefficients \( (a_j)_{j=0}^{n+k+2} \) satisfying \( \sigma(t_k) = \cdots = \sigma(t_n) = 0 \).

Next, we define the operators

\[
S_i := \begin{cases} 
D_{k+2-i} \cdots D_{k+1}, & \text{if } i \leq k \\
D_{i-k+1} \cdots D_2 D_2 \cdots D_{k+1}, & \text{if } k + 1 \leq i \leq 2k + 1
\end{cases}
\]

and the function \( \phi \) by using \( \sigma \) and the operators \( S_i \) as follows:

\[
\phi := \frac{w_1}{w_2} S_{k-1} \sigma.
\]

Note that \( D_k \cdots D_1 \phi = D_k \cdots D_2 D_2 S_{k-1} \sigma = 0 \) on each interval \( (t_i, t_{i+1}) \).

We now observe that \( \sigma \) has \( n + k - 1 \) zeros and this is the highest number of zeros a nonzero spline of this order can have (including multiplicities), so we conclude that \( \text{sgn} \sigma = (-1)^{j-k+1} \) on \( (t_j, t_{j+1}) \) for all indices \( j \), where we also used the fact that \( \sigma \) is positive on \( (t_{k-1}, t_k) \) by conditions (1), (2) and (3).

**Lemma 3.4.** If \( |\Delta| \leq 1 \), then

\[
\int_{t_j}^{t_{j+1}} |\sigma(t)| \, dt \gtrsim (t_{j+1} - t_j)^k, \quad j = k - 1, \ldots, n.
\]

**Proof.** We introduce the function

\[
H = \left( \frac{S_{k-1} \sigma}{w_2} \right)^2 + 2 \sum_{q=1}^{k-1} (-1)^q \frac{S_{k-1-q} \sigma}{w_{q+2}} \cdot \frac{S_{k-1+q} \sigma}{w_{q+1}}.
\]

Since all the functions \( \sigma, S_1 \sigma, \ldots, S_{2k-2} \sigma \) are continuous at the grid points \( t_j \) and \( \sigma(t_j) = 0 \), the function \( H \) is continuous at the grid points. Moreover, \( H \) is differentiable in the interior \( (t_j, t_{j+1}) \) of the grid intervals. This yields

\[
H' = 2 \frac{S_{k-1} \sigma}{w_2} S_k \sigma + 2 \sum_{q=1}^{k-1} (-1)^q \left( S_{k-q} \sigma \cdot \frac{S_{k+q-1} \sigma}{w_{q+1}} + \frac{S_{k-q-1} \sigma}{w_{q+2}} \cdot S_{k+q} \sigma \right)
\]

\[
= 2 \sum_{q=2}^{k-1} (-1)^q S_{k-q} \sigma \cdot \frac{S_{k+q-1} \sigma}{w_{q+1}} + 2 \sum_{q=1}^{k-2} (-1)^q S_{k-q-1} \sigma \cdot \frac{S_{k+q} \sigma}{w_{q+2}}.
\]
\[ + (-1)^{k-1} \frac{\sigma}{w_{k+1}} \cdot S_{2k-1} \sigma = (-1)^{k-1} \frac{\sigma}{w_{k+1}} \cdot S_{2k-1} \sigma = 0, \]

where the last equality holds due to the condition \( \sigma \in \ker D_k \cdots D_2 D_2 \cdots D_{k+1} \). So we deduce that \( H \) is constant on all of the interval \([a, b]\) and \( H(a) = 1/w_2^2(a) \).

By Markov’s inequality \((2.5)\), we have that for all \( r = 0, \ldots, 2k-2 \) and \( I_j = (t_j, t_{j+1}) \),

\[ \| S_t \sigma \|_{L^\infty(I_j)} \lesssim |I_j|^{-r-1} \int_{I_j} |\sigma(t)| \, dt. \]

Thus, for all \( t \in (t_j, t_{j+1}) \) and all \( j \),

\[ \frac{1}{w_2(a)^2} = H(t) \lesssim |I_j|^{-2k} \left( \int_{I_j} |\sigma(t)| \, dt \right)^2, \]

which is the conclusion. \( \Box \)

Now we are ready to prove the two desired properties of the Chebyshev spline \( \phi \).

**Theorem 3.5.** There exist two numbers \( \varepsilon, c > 0 \) depending only on the weights \( w_i \), so that if \( |\Delta| \leq \varepsilon \), we have

1. \( \text{sign} \langle \phi, M_j \rangle \) is alternating,
2. \( |\langle \phi, M_j \rangle| \geq c. \)

**Proof.** We note that by the definition of \( \phi \) and \( \sigma \), we obtain by iterated partial integration that

\[ \int_{t_j}^{t_{j+k}} \phi(t)M_j(t) \, dt = (-1)^{k-1} \int_{t_j}^{t_{j+k}} \sigma(t)u(t) \, dt, \]

with \( u = D_k \cdots D_2 w_i M_j \). Note that the function \( u \) can be written in the form

\[ u = \sum_{\ell=0}^{k-1} \tilde{w}_\ell \cdot D_\ell \cdots D_1 M_j, \]

where \( \tilde{w}_\ell \) are bounded in terms of the weight functions and \( \tilde{w}_{k-1} = \frac{w_{k-1}}{w_k} \cdots \frac{w_2}{w_3} = \frac{w_k^2}{w_k} \) additionally is bounded away from zero. Using Lemma \ref{lem:3.3} we obtain that there exists \( \varepsilon > 0 \) depending only on the weight functions \( w_1, \ldots, w_k \) so that, if \( |\Delta| \leq \varepsilon \) then

\(\quad \text{(3.7)} \quad \text{sgn} \ u = \text{sgn} \ D_{k-1} \cdots D_1 M_j \quad \text{and} \quad |u| \gtrsim |D_{k-1} \cdots D_1 M_j|. \)

We have \( \text{sgn} \ \sigma(t) = (-1)^{j+i-k+1} \) for \( t \in (t_{j+i}, t_{j+i+1}) \) and also, by using Lemma \ref{lem:3.3} we obtain \( \text{sgn} \ D_{k-1} \cdots D_1 M_j(t) = (-1)^i \) for \( t \in (t_{j+i}, t_{j+i+1}) \). Hence,

\[ (-1)^j \langle \phi, M_j \rangle = \sum_{i=0}^{k-1} \int_{t_{j+i}}^{t_{j+i+1}} (-1)^{j+i-k+1} \sigma(t) \cdot (-1)^i u(t) \, dt \]

\[ = \sum_{i=0}^{k-1} \int_{t_{j+i}}^{t_{j+i+1}} |\sigma(t) \cdot u(t)| \, dt = \int_{t_j}^{t_{j+k}} |\sigma(t) \cdot u(t)| \, dt. \]

If \( J_j = (t_i, t_{i+1}) \) is a largest subinterval of \([t_j, t_{j+k}]\), we obtain by \(\text{(3.7)}\), Lemma \ref{lem:3.4} and Lemma \ref{lem:3.3},

\[ (-1)^j \langle \phi, M_j \rangle \gtrsim \int_{t_j}^{t_{j+1}} |\sigma(t)| \cdot |D_{k-1} \cdots D_1 M_j| \, dt \gtrsim 1, \]

which shows both (1) and (2). \( \Box \)
3.3. **Proof of Theorem 3.1.** Now, let $K(\tau, t)$ be the Dirichlet kernel associated to the orthogonal projection operator $P_\Delta$ onto $S(U_k; \Delta)$, i.e., $P_\Delta$ is given by the formula

$$P_\Delta f(\tau) = \int_a^b K(\tau, t) f(t) \, dt.$$  

Then, by duality,

$$\sup_{\|f\|_\infty \leq 1} |P_\Delta f(\tau)| = \int_a^b |K(\tau, t)| \, dt. \tag{3.8}$$

We will need the following result concerning the sign of the B-spline coefficients of $K(a, \cdot)$.

**Lemma 3.6.** The coefficients $(c_i)$ of $K(a, t) = \sum_{i=0}^{n} c_i M_i(t)$ satisfy

$$(-1)^i c_i \geq 0, \quad i = 0, \ldots, n.$$  

**Proof.** Observe that $c_i$ is given by

$$c_i = \sum_{j=0}^{n} \beta_{ij} M_j(a) = \beta_{i0}$$

where $(\beta_{ij})$ is the inverse to the Gram matrix $B = (\langle M_i, M_j \rangle)$. Since $B$ is totally positive (see Section 2.1) and the inverse of a totally positive matrix is checkerboard (i.e. alternates in sign), the lemma is proved.

Now let us turn to the proof of Theorem 3.1. By definition of $\phi$, we have $\phi \in S(U_k; \Delta)$ and hence

$$w_1(a) = \phi(a) = P_\Delta \phi(a) = \int_a^b K(a, t) \phi(t) \, dt = \sum_i c_i \langle \phi, M_i \rangle.$$  

If $|\Delta|$ is sufficiently small, then, collecting Theorem 3.5 Lemma 3.6 we can conclude

$$w_1(a) = \sum_i |c_i| \cdot |\langle \phi, M_i \rangle| \geq c \cdot \sum_i |c_i|$$

$$\geq c \cdot \| \sum_i c_i M_i \|_1 = \int_a^b |K(a, t)| \, dt = \sup_{\|f\|_\infty \leq 1} \|P_\Delta f(a)\|.$$  

The estimate for $P_\Delta f(b)$ is proved similarly by exchanging condition (1) on page 9 for $\sigma$ by $D_3 \cdots D_k D_{k+1} \sigma(b) = 1$.

4. **From boundary points to the general case**

In this section, we are going to prove Theorem 4.1. As an intermediate step, we show the following theorem, which still has a restriction on the size of the partition $\Delta$ in its assumptions.

**Theorem 4.1.** There exist constants $\varepsilon, K_2 > 0$, only depending on $U_k$, so that if the partition $\Delta$ satisfies $|\Delta| \leq \varepsilon$, then the orthogonal projection operator $P_\Delta$ onto $S(U_k; \Delta)$ satisfies

$$\|P_\Delta f\|_\infty \leq K_2 \|f\|_\infty, \quad f \in L^\infty.$$  

In the case of polynomial splines (i.e. the weights $w_1, \ldots, w_k$ each set to be the constant function 1), it was shown in [3] that Theorem 4.1 (in the form for polynomial splines without the condition on $|\Delta|$) implies this result. By analyzing the proof in [3], [3],
we see that this transition can be obtained in a more general setting which we now describe:

For a partition $\Delta = \{a = t_{k-1} < t_k < \cdots < t_n < t_{n+1} = b\}$ and for $x \in (a, b)$, we let $m(x)$ be the unique index so that $x \in (t_{m(x)}, t_{m(x)+1})$ and define the subpartition $\Delta_{x,t} = \{t_{k-1} < t_k < \cdots < t_{m(x)} < x\}$ to the left of $x$ and the subpartition $\Delta_{x,r} = \{x \leq t_{m(x)+1} < \cdots < t_{n+1}\}$ to the right of $x$. To each partition $\Delta$, we associate a vector space $V(\Delta)$ of real valued functions defined on $[a, b]$ and with it a basis $(Q^\Delta_i^t)^n_{i=0}$ of $V(\Delta)$ consisting of functions with local support $\text{supp} \ Q^\Delta_i^t$ consisting of functions with local support $\text{supp} \ Q^\Delta_i^t$.

Next, we eliminate the restriction $|\Delta| \leq \varepsilon$ in Theorem 3.1. For this purpose, we will need the following geometric decay inequality of the inverse of the Gram matrix of B-spline functions:

Corollary 4.3. Suppose that $\Delta$ is such that $\|P_\Delta : L^\infty \to L^\infty\| \leq K$ for some constant $K$. Then, there exist two constants $C$ and $q < 1$ depending only on $K$ so that
where \((a_{ij})\) denotes the inverse of the matrix \((\langle N_i, N_j \rangle)\).

This inequality is a consequence of Theorem 4.1 in the same way as its polynomial spline counterpart in [1] (see also [9]) is a consequence of Shadrin’s theorem. Using this geometric decay inequality, we can prove the main theorem:

**Proof of Theorem 1.1.** We assume that \(\tilde{\Delta}\) is a partition and \(\Delta\) is a partition that we get from adding one point to \(\tilde{\Delta}\) in the middle of the largest grid interval \(I\) in \(\tilde{\Delta}\) with length > \(\varepsilon\) and we assume that \(\|P_\Delta : L^\infty \to L^\infty\| \leq K\). Then, by definition of the orthogonal projections \(P = P_\Delta\) and \(\tilde{P} = P_{\tilde{\Delta}}\), we have, if \((N_i)\) denotes the Chebyshevian B-spline basis of \(S(U_k; \Delta)\),

\[
\langle (\tilde{P} - P) f, N_i \rangle = 0
\]

if \(|I \cap \text{supp } N_i| = 0\) since in this case \(N_i\) is both in the range of \(P\) and \(\tilde{P}\) and thus \(\langle \tilde{P} f - f, N_i \rangle = \langle P f - f, N_i \rangle = 0\). Thus, we can expand

\[
(\tilde{P} - P) f = \sum_{j: \text{supp } N_j \cap I > 0} c_j N_j^*
\]

with \(c_j = \langle (\tilde{P} - P) f, N_j \rangle\) and \((N_j^*)\) being the dual basis to \((N_i)\) which is given by \(N_j^* = \sum_i a_{ij} N_i\). Here, as in Corollary 4.3, \((a_{ij})\) denotes the inverse of the matrix \((\langle N_i, N_j \rangle)\). Since \(\tilde{P}\) and \(P\) are orthogonal projections, their \(L^2\)-norm equals 1 and we can estimate, for \(j\) with \(|\text{supp } N_j \cap I| > 0\),

\[
|c_j| \lesssim \|f\|_2 \|N_j\|_2 \lesssim \|f\|_\infty \|\text{supp } N_j\|_1 \lesssim \|f\|_\infty |I|^{1/2}.
\]

On the other hand, by the above result on the entries \(a_{ij}\) of the inverse matrix to \((\langle N_i, N_j \rangle)\) and \((2.11)\),

\[
|N_j^*(x)| = |\sum_i a_{ij} N_i(x)| \lesssim \frac{q^{j-\ell}}{\alpha_j} \lesssim \frac{q^{j-\ell}}{|I|}
\]

for \(j\) with \(|\text{supp } N_j \cap I| > 0\), where the index \(\ell\) is chosen such that \(x \in [t_\ell, t_{\ell+1}]\). Therefore, we get

\[
\|((\tilde{P} - P) f\|_\infty \lesssim |I|^{-1/2} \|f\|_\infty \lesssim \varepsilon^{-1/2} \|f\|_\infty
\]

and we conclude

\[
\|\tilde{P} f\|_\infty \lesssim \|P f\|_\infty + \|P - \tilde{P}\|_\infty \lesssim (K + C \varepsilon^{-1/2}) \|f\|_\infty,
\]

where \(C\) is some constant that depends only on \(U_k\).

In order to go from an arbitrary partition \(\tilde{\Delta}\) to a partition \(\Delta\) with \(\|P_\Delta : L^\infty \to L^\infty\| \leq K_2\), we can apply the above construction iteratively, until we arrive at a partition \(\Delta\) with \(|\Delta| \leq \varepsilon\), where \(\varepsilon > 0\) is from Theorem 4.1. Then, it is guaranteed by Theorem 4.1 that \(\|P_\Delta : L^\infty \to L^\infty\| \leq K_2\). The number of iteration steps depends only on \(\varepsilon\) and thus, only on \(U_k\). Therefore,

\[
\|\tilde{P} f\|_\infty \lesssim \|f\|_\infty,
\]

which finishes the proof of the main theorem. \(\square\)

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REFERENCES

[1] Z. Ciesielski. Orthogonal projections onto spline spaces with arbitrary knots. In Function spaces (Poznań, 1998), volume 213 of Lecture Notes in Pure and Appl. Math., pages 133–140. Dekker, New York, 2000.

[2] R. A. DeVore and G. G. Lorentz. Constructive approximation, volume 303 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1993.

[3] M. v. Golitschek. On the $L_\infty$-norm of the orthogonal projector onto splines. A short proof of A. Shadrin’s theorem. J. Approx. Theory, 181:30–42, 2014.

[4] S. Karlin. Total positivity. Vol. I. Stanford University Press, Stanford, Calif, 1968.

[5] K. Keryan and M. Passenbrunner. Unconditionality of periodic orthonormal spline systems in $L^p$. preprint arXiv:1708.09294, to appear in Studia Math., 2017.

[6] P. F. X. Müller and M. Passenbrunner. Almost everywhere convergence of spline sequences. preprint arXiv:1711.01859, 2017.

[7] M. Passenbrunner. Unconditionality of orthogonal spline systems in $L^p$. Studia Math., 222(1):51–86, 2014.

[8] M. Passenbrunner. Orthogonal projectors onto spaces of periodic splines. Journal of Complexity, 42:85–93, 2017.

[9] M. Passenbrunner and A. Shadrin. On almost everywhere convergence of orthogonal spline projections with arbitrary knots. J. Approx. Theory, 180:77–89, 2014.

[10] L. L. Schumaker. Spline functions: basic theory. John Wiley & Sons Inc., New York, 1981. Pure and Applied Mathematics, A Wiley-Interscience Publication.

[11] A. Shadrin. The $L_\infty$-norm of the $L_2$-spline projector is bounded independently of the knot sequence: a proof of de Boor’s conjecture. Acta Math., 187(1):59–137, 2001.

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