Critical Sets of PL and Discrete Morse Theory: a Correspondence *

Ulderico Fugacci1[0000–0003–3062–997X], Claudia Landi2[0000–0001–8725–4844], and Hanife Varlı3[0000–0002–3122–8444]

1 Polytechnic University of Torino, Torino, Italy
2 University of Modena and Reggio Emilia, Modena, Italy
3 Çankırı Karatekin University, Çankırı, Turkey

Abstract. Piecewise-linear (PL) Morse theory and discrete Morse theory are used in shape analysis tasks to investigate the topological features of discretized spaces. In spite of their common origin in smooth Morse theory, various notions of critical points have been given in the literature for the discrete setting, making a clear understanding of the relationships occurring between them not obvious. This paper aims at providing equivalence results about critical points of the two discretized Morse theories. First of all, we prove the equivalence of the existing notions of PL critical points. Next, under an optimality condition called relative perfectness, we show a dimension agnostic correspondence between the set of PL critical points and that of discrete critical simplices of the combinatorial approach. Finally, we show how a relatively perfect discrete gradient vector field can be algorithmically built up to dimension 3. This way, we guarantee a formal and operative connection between critical sets in the PL and discrete theories.

Keywords: Critical Point · Gradient Vector Field · Relative Perfectness.

1 Introduction

Topological shape analysis is useful to extract information about topological and morphological properties of a shape, naturally finding applications in fields that require shape understanding such as computer graphics, computer vision and visualization. [3] shows that most methods of topological shape analysis are grounded in Morse theory [14], thus investigating critical sets of functions.

The proven effectiveness of Morse theory has led to the development of several discrete counterparts of this theory useful when one works with shapes discretized as cell complexes, in particular simplicial complexes [5]. Among them, two discretized versions of Morse theory have gained a prominent role in the literature, specifically, the piecewise-linear (PL) Morse theory introduced by Banchoff [1] and the discrete Morse theory developed by Forman [9].

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Both the approaches are worth to be addressed as Morse theories on the ground that they satisfy discrete versions of the main theorem valid in the smooth case: topological changes of a shape occur at the critical points of a function defined on it (or, equivalently, at singularities of a gradient field).

In spite of these similarities, several aspects distinguish the two theories. First of all, PL Morse theory is more centered on functions whereas discrete Morse theory is more based on gradient vector fields. Indeed, a PL function is uniquely defined on each point of a polyhedron by linearly interpolation of scalar values given at the vertices. In contrast, the approach by Forman is combinatorial in that it treats simplices as a whole rather than as a set of points, and can produce a collection of simplex pairs to simulate the behaviour of a function gradient independently of whether or not a scalar function is globally defined.

Because of these different standpoints, the notions of critical points the two approaches prompt are quite different. In the PL case, the various definitions given in the literature for critical points of a PL function on a discretized manifold domain always point to vertices and reflect our common intuition of features like a minimum, a maximum, or a saddle. In contrast, in discrete Morse theory, critical points do not consist of just vertices but, more generally, of simplices of any dimension. This way, at a visual level, discrete Morse theory loses its ties with the smooth theory. On the other hand, discrete Morse theory is recently gaining much more visibility than the PL one thanks to its combinatorial nature and to its capability in dealing with arbitrary domains rather than only manifolds.

We think that for these reasons it is not obvious how to relate and interpret the critical sets obtained by the PL and the combinatorial approaches. This represents a real obstruction for experts to exploit the full potentialities of combining the two theories, and for practitioners to knowingly adopt in their application domains one or the other of the two theories. Aim of this paper is to unveil such a relation.

**Contributions.** Firstly, we consider the different definitions of PL critical points given in the literature, starting from the original definition of PL critical point given in [1] that has been later specialized or generalized according to specific tasks and working dimensions. In spite of the intuitive analogy between them, to the best of our knowledge, a formal proof that the proposed definitions are equivalent has never been given, and we make up for this gap in Section 3.

Secondly, in Section 4 we turn our attention to discrete Morse theory and introduce the notion of relative perfectness for a discrete gradient vector field $V$ with respect to a given function $f$. It amounts to require that the number of discrete critical simplices of $V$ coincides with the number of topological changes occurring along a sublevel set filtration by $f$. We show the utility of this notion in Section 4.1 by proving that, for relatively perfect discrete gradient vector fields there is a well-defined correspondence between its discrete critical simplices and the PL critical points of $f$. In particular, such a correspondence is a bijection for PL Morse functions. Moreover, it ensures that each discrete critical simplex
σ and its corresponding PL critical point \( v \) are closely located: each PL critical point \( v \) is necessarily a vertex of the corresponding discrete critical simplex \( \sigma \).

As a third and final contribution, in Section 4.2 we prove that, for manifolds of dimension lower than or equal to 3, it is always possible to build a discrete gradient vector field \( V \) relatively perfect with respect to \( f \). The proof exhibits an algorithmic strategy to build such a gradient and, combined with our second contribution, ensures that a correspondence between the critical sets obtained by the PL and the discrete approach is always established for low dimensions.

Related works. Although the interest in an explicit understanding of the connections between the PL and discrete Morse theories would be most natural, very few papers in the literature address this topic. The work by Lewiner [11] is, at the best of our knowledge, the only work that deals with it. In his work, the author adopts a greedy algorithm for the construction of a discrete gradient vector field [13] to build, after a sequence of barycentric subdivisions, an adjacent discrete critical simplex for each PL critical point. Even if this represents a first encouraging result, the obtained correspondence is affected by some serious constraints. Firstly, it is limited to simplicial complexes up to dimension 2. Secondly, the entire approach is available only for a specific algorithm. Finally, the need of a sequence of barycentric subdivisions to obtain the desired correspondence, with the consequent rapid increase in the number of simplices, is not desirable.

Related to the problems here addressed is also the work by Benedetti [2]. Benedetti proves that, taking a Morse vector to list the number of critical points in each dimension, if a smooth manifold \( M \) admits a Morse vector \( c \), then for any PL triangulation \( \Sigma \) of \( M \) there exists a finite number of barycentric subdivisions of \( \Sigma \) such that the obtained triangulation admits \( c \) as a discrete Morse vector.

From different perspectives, both Benedetti [2] and Lewiner et al. [13] are also interested in perfect functions for which the number of critical points is the minimal one allowed by the homology of the manifold. In applications, functions correspond to measurements and cannot be chosen, so perfection is scarcely interesting. Instead, it may be useful to achieve the minimal number of critical points ensuring the same persistent homology as the given function. In particular, Robins et al. in [15] show that such optimality is achievable for cubical complexes up to dimension 3. In the present paper, we rephrase this kind of optimality in terms of relative homology, hence calling it relative perfecteness, and prove that it is also achievable for simplicial complexes up to dimension 3, improving [10].

2 Basic notions in PL and discrete Morse theory

In this section, we briefly introduce the required background notions on PL and discrete Morse theory with a special emphasis on the definitions of critical points and simplices provided in the literature.

Notations and working hypothesis. From now on, we will adopt the following notations that will also describe the common framework on which the different
versions of the PL Morse theory are settled. Given a simplicial complex $\Sigma$, we denote by $|\Sigma|$ the underlying space of $\Sigma$, also known as the polytope of $\Sigma$ (i.e., the geometric realization of $\Sigma$ as a subspace of the Euclidean space $\mathbb{R}^n$ where it is embedded). Hereafter, depending on the domain on which it is applied, the symbol $H_*$ represents singular or simplicial homology with coefficients in a field. By $\beta_i$, we denote the rank of the $i^{th}$ homology group $H_i$. Analogously, $\tilde{\beta}_i$ denotes the $i^{th}$ group $\tilde{H}_i$ of reduced homology.

We assume that an injective scalar function $f: \Sigma_0 \rightarrow \mathbb{R}$ is give on the set $\Sigma_0$ of vertices of $\Sigma$. It can be extended to two functions: a piecewise-linear function $f_{PL}: |\Sigma| \rightarrow \mathbb{R}$ defined by linear interpolation for all the points of $|\Sigma|$, and a function $f_{max}: \Sigma \rightarrow \mathbb{R}$ defined by mapping each simplex $\sigma \in \Sigma$ to $\max\{f(v) \mid \text{vertex } v \text{ face of } \sigma\}$. Thanks to such a function $f_{max}$, it is possible to filter $\Sigma$ through a collection of sublevel sets where, for $l \in \mathbb{R}$, the $l$-sublevel set of $\Sigma$ w.r.t. $f_{max}$ is the simplicial complex $\Sigma_l := \{\sigma \in \Sigma \mid f_{max}(\sigma) \leq l\}$.

Given a vertex $v$ of $\Sigma$, the star and the link of $v$ represent combinatorial counterparts of an open neighbourhood and of its boundary, respectively. Formally, the star of a vertex $v$ of $\Sigma$, $\text{star}(v)$, is defined as the collection of the cofaces of $v$, while the link of a vertex $v$, $\text{link}(v)$, consists of the collection of the simplices of $\Sigma$ that are faces of an element in $\text{star}(v)$ but not cofaces of $v$.

The function $f$ allows to define the lower star of a vertex $v$ of $\Sigma$, $\text{star}^-(v)$, as the subset of $\text{star}(v)$ on which the function $f_{max}$ takes values not greater than $f(v)$: $\text{star}^-(v) := \text{star}(v) \cap \Sigma f(v)$. Similarly, one can define the lower link of $v$, $\text{link}^-(v)$, as the intersection $\text{link}(v) \cap \Sigma f(v)$. The closure under the face relation of some collection $S$ of simplices (such as the star of a vertex) is denoted by $\overline{S}$ and is the smallest simplicial subcomplex of $\Sigma$ containing $S$.

### 2.1 PL critical points

As previously mentioned, the literature about PL Morse theory proposes several definitions of a critical point. In view of proving their equivalence, we start reviewing these definitions.

Similarly to the smooth theory, PL Morse theory requires working with a manifold, though a combinatorial manifold, i.e., a simplicial complex of some dimension $d \geq 1$ such that the underlying space of the link of each vertex is homeomorphic to the $(d-1)$-sphere $S^{d-1}$. So, in the following, if not differently specified, we will always assume that $\Sigma$ is a combinatorial $d$-manifold.

Our brief survey will consider first two definitions of a PL critical point for the case $d = 2$ (a widely studied case because of its applications in terrain analysis), and then other two definitions for the case of a combinatorial manifolds of arbitrary dimension $d$.

*Banchoff [1]*. Banchoff proposes a definition of PL critical points of an injective function $f$ defined on the vertices of a combinatorial 2-manifold $\Sigma$. Let $\sigma$ be a triangle $[u, v, w]$ in $\text{star}(v)$. $\sigma$ has $v$ middle for $f$ if $f(u) < f(v) < f(w)$. For a vertex $v$ we set $\iota(v, f) := 1 - \frac{1}{2} \cdot \#\{\text{triangles in } \text{star}(v) \text{ with } v \text{ middle for } f\}$.
Hence, vertices of $\Sigma$ are classified as follows:

$$
\iota(v, f) = \begin{cases} 
1 & \iff v \text{ is a point of (local) minimum or maximum}, \\
0 & \iff v \text{ is a regular point}, \\
-1 & \iff v \text{ is a saddle point}, \\
-k < 0 & \iff v \text{ is a multiple saddle point of multiplicity } k.
\end{cases}
$$

*Edelsbrunner et al. [8]*. Edelsbrunner et al. introduce a different definition of a PL critical point of an injective function $f$ defined on the vertices of a combinatorial 2-manifold $\Sigma$. A vertex $v$ of $\Sigma$ is declared critical or not depending on the number of “wedges” in which the lower star of $v$ is subdivided. Formally, a *section* of $\text{star}^{-}(v)$ is an edge or a triangle in $\text{star}^{-}(v)$. Let $S$ be a collection of sections in $\text{star}^{-}(v)$. $S$ is called a “contiguous section” of $\text{star}^{-}(v)$ if $S \setminus \{v\}$ is connected. A *wedge* of $\text{star}^{-}(v)$ is defined as a “contiguous section” of $\text{star}^{-}(v)$ whose boundary in the lower link of $v$ is not a cycle. Letting $W$ the number of wedges of $\text{star}^{-}(v)$, $v$ is classified as follows:

$$
W = \begin{cases} 
0 & \iff v \text{ is a point of (local) minimum or maximum}, \\
1 & \iff v \text{ is a regular point}, \\
2 & \iff v \text{ is a saddle point}, \\
k + 1 > 1 & \iff v \text{ is a multiple saddle point of multiplicity } k.
\end{cases}
$$

For $W = 0$, we can distinguish a minimum or maximum point according to the fact that $\text{star}^{-}(v)$ is $\{v\}$ or it coincides with the entire $\text{star}(v)$.

*Brehm and Kühnel [4]*. A definition of a PL critical point for the case of a combinatorial manifold $\Sigma$ of arbitrary dimension $d$ has been proposed in [4]. Intuitively, the authors define a vertex $v$ of $\Sigma$ as critical by checking if there is a change in homology when one removes $v$ from its sublevel set. Formally, a vertex $v$ of $\Sigma$ is classified as PL critical for $f$ whenever $H_{*}(\mid \Sigma^{l} \mid \setminus \mid \Sigma^{l} \setminus \{v\}) \neq 0$, where $l = f(v)$. Otherwise, $v$ is called regular. Thanks to the following isomorphisms, the criterium can be expressed in various equivalent ways:

$$
H_{*}(\mid \Sigma^{l} \mid \setminus \{v\}) \cong H_{*}(\mid \Sigma^{l} \setminus \text{star}(v)) \\
\cong H_{*}(\Sigma^{l} \setminus \text{star}(v)) \\
\cong H_{*}(\Sigma^{l} \cap \text{star}(v)) \cong H_{*}(\text{star}^{-}(v), \text{link}^{-}(v)).
$$

A PL critical point $v$ of $f$ is said to have index $i$ and multiplicity $k_i$ if $\beta_i(\mid \Sigma^{l} \mid \setminus \{v\}) = k_i$. In general, a PL critical point might be critical with respect to several indices and its total multiplicity is $k := \sum_{i=0}^{d} k_i$. PL critical point of index 0 or $d$ will be called point of (local) minimum or maximum, respectively. Other PL critical points will be addressed as saddle points.
Another definition of a PL critical point of a function $f$ for a combinatorial $d$-manifold $\Sigma$ has been introduced for the case $d = 3$ in [7], and generalized to arbitrary dimension in [6].

Edelsbrunner et al. define a vertex $v$ as PL critical or not depending on the reduced homology of its lower link. More formally, let $\tilde{\beta}_j$ be the rank of the reduced $j^{th}$ homology group of $\text{link}^{-}(v)$. A vertex $v$ of $\Sigma$ is called regular if $\tilde{\beta}_j = 0$ for any $j = -1, 0, 1, \ldots, d$. Otherwise, $v$ is called a PL critical point of index $i$ and multiplicity $k$ of $f$ if

$$
\tilde{\beta}_j = \begin{cases} 
  k & \text{for } j = i - 1, \\
  0 & \text{otherwise}.
\end{cases}
$$

Specifically, a PL critical point of index $i$ is called a (local) minimum if $i = 0$, a (local) maximum if $i = d$, and a $i$-saddle otherwise. A PL critical point with multiplicity $k > 1$ is called a multiple saddle. The function $f$ is called PL Morse if all its PL critical points have multiplicity 1.

Figure 1 illustrates how to classify a vertex of a combinatorial 2-manifold $\Sigma$ according to above definitions. In (a), the star of the vertex $v := f^{-1}(5)$ is given. In (b), we see that $\beta_1(\text{star}^{-}(v), \text{link}^{-}(v)) = 1$, that is, $\beta_1(|\Sigma^5|, |\Sigma^5| \setminus \{v\}) = 1$. So, $v$ is a saddle point according to Brehm and Kühnel in [4]. Since the lower star has two wedges (one consisting of the triangle $[1, 2, 5]$ and its edges $[1, 5]$ and $[2, 5]$, and the other one consisting of the edge $[3, 5]$), $v$ is also a saddle point according to Edelsbrunner et al. in [8]. In (c), we have that $\iota(v, f) = 1 - \frac{1}{2} \cdot 4 = -1$. Thus, $v$ is also a saddle point according to Banchoff in [1].

![Fig. 1.](image-url) (a) The star of $v := f^{-1}(5)$. (b) The lower star of $v$ and its lower link (in red). (c) The triangles in the star of $v$ for which $v$ is middle.

### 2.2 Discrete critical simplices

Discrete Morse theory introduced by Forman in [9] represents the most recently proposed discrete counterpart of the smooth Morse theory. At the price of being a little less intuitive, discrete Morse theory presents some advantages compared to PL Morse theory. First of all, discrete Morse theory can be defined for arbitrary cell complexes non-necessarily discretizing a manifold domain. In spite of this, for the sake of simplicity, we will review discrete Morse theory in the
context of simplicial complexes and we will be forced to work in the common framework of the combinatorial manifolds everytime that a direct comparison between the two theories will be presented. Another great advantage of discrete Morse theory is related to the possibility of describing it in purely combinatorial terms preventing the need of explicitly exhibiting a Morse function defined on the complex. Exploting this fact, in this subsection we introduce by adopting a combinatorial point of view some basic notions of discrete Morse theory.

Given a simplicial complex $\Sigma$, the above mentioned combinatorial approach of describing discrete Morse theory is based on the definition of a collection of simplex pairs simulating the gradient of a function defined on $\Sigma$. Formally, a discrete vector field $V$ on $\Sigma$ is a collection of pairs of simplices $(\sigma, \tau) \in \Sigma \times \Sigma$ such that $\sigma$ is a face of $\tau$ of dimension $\dim(\tau) - 1$. In the following, we will denote that by $\sigma < \tau$ and each simplex of $\Sigma$ is in at most one pair of $V$.

A $V$-path is a closed path if $\sigma_1$ is a face of $\tau_r$ different from $\sigma_r$. A discrete vector field $V$ is called a discrete gradient vector field (of a discrete Morse function) if $V$ is free of closed paths.

Given a simplicial complex $\Sigma$ endowed with a discrete gradient vector field $V$, an $i$-simplex $\sigma \in \Sigma$ is called regular if it belongs to a pair of $V$. Otherwise, $\sigma$ is called discrete critical simplex of index $i$ (equivalently, discrete critical $i$-simplex or $i$-saddle). More specifically, a discrete critical simplex of index 0 is called a minimum, while a discrete critical simplex of index $d = \dim(\Sigma)$ is a maximum.

PL and discrete Morse theories deserve to be addressed as discretized versions of Morse theory since they both adapt the fundamental theorems and properties holding in the smooth case to the combinatorial setting. One among those results, expressed in the following in terms of discrete Morse theory, consists of a collections of inequalities usually called weak Morse inequalities. Properly, given a discrete gradient vector field $V$ on a simplicial complex $\Sigma$, weak Morse inequalities state that $m_i(V) \geq \beta_i(\Sigma)$, for any $i = 0, \ldots, \dim(\Sigma)$, where $m_i(V)$ denotes the number of discrete critical $i$-simplices of $V$. Analogous inequalities hold for a PL Morse function $f$. A discrete gradient vector field $V$ is called perfect if, for any $i = 0, \ldots, \dim(\Sigma)$, the equality $m_i(V) = \beta_i(\Sigma)$ is satisfied.

### 3 Equivalence between the notions of PL critical points

A required first step for establishing a correspondence between critical sets is the proof of the equivalence of the various notions of a PL critical point in the literature. More precisely, we want to show that vertices of a combinatorial $d$-manifold classified in the same way by the different proposed definitions are characterized by equivalent conditions.

**Case $d \geq 1$.** First, we discuss the case in which $\Sigma$ is combinatorial manifold of arbitrary dimension $d$. 

Let us preliminarily notice that in the papers by Banchoff and by Edelsbrunner et al. the only PL critical points which are addressed as multiple are the saddle points. In contrast, in the classification proposed Brehm and Kühnel, the existence of non-saddle PL critical points with total multiplicity greater than 1 is not explicitly excluded. The following results confirm the non-existence of such points and ensure us that all the proposed classifications are complete.

**Lemma 1.** Given a vertex $v$ of $\Sigma$, for every $i$, we have that:

$$H_i(\text{star}^-(v), \text{link}^-(v)) \cong \tilde{H}_{i-1}(\text{link}^-(v)).$$

**Proof.** Thanks to the properties of relative homology, we have the following long exact sequence of the pair $(\text{star}^-(v), \text{link}^-(v))$ for reduced homology

$$\cdots \to \tilde{H}_i(\text{link}^-(v)) \to \tilde{H}_i(\text{star}^-(v)) \to \tilde{H}_i(\text{star}^-(v), \text{link}^-(v)) \to \tilde{H}_{i-1}(\text{link}^-(v)) \to \tilde{H}_{i-1}(\text{star}^-(v)) \to \cdots$$

as well as the isomorphism $H_i(\text{star}^-(v), \text{link}^-(v)) \cong \tilde{H}_i(\text{star}^-(v), \text{link}^-(v))$.

Moreover, since $\text{star}^-(v)$ is a cone, for every $i$, $\tilde{H}_i(\text{star}^-(v)) = 0$. By combining the above facts, we retrieve the thesis.

**Lemma 2.** Let $v$ be a PL critical point of index $i$ with total multiplicity $k := \sum_{i=0}^{d} k_i > 1$ according to [4]. Then, $k_0 = k_d = 0$.

**Proof.** Let us prove that by reductio ad absurdum. Let us suppose that $k_0 \neq 0$. This is true if and only if (by Lemma 1) $\tilde{\beta}_{i-1}(\text{link}^-(v)) \neq 0$ if and only if $\text{link}^-(v) = \emptyset$ if and only if $\text{star}^-(v) = \{v\}$. Thus,

$$\beta_i(|\Sigma^d|, |\Sigma^d| \setminus \{v\}) = \beta_i(\text{star}^-(v), \text{link}^-(v)) = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to a contradiction since by hypothesis $k > 1$. Now, let us suppose that $k_d \neq 0$. This is true if and only if (by Lemma 1) $\tilde{\beta}_{d-1}(\text{link}^-(v)) \neq 0$ if and only if (by the fact that $\Sigma$ is a combinatorial $d$-manifold and so the underlying space of the link of each vertex is homeomorphic to the $(d-1)$-sphere $S^{d-1}$) $\text{link}^-(v) = \text{link}(v)$ if and only if $\text{star}^-(v) = \text{star}(v)$. Thus,

$$\beta_i(|\Sigma^d|, |\Sigma^d| \setminus \{v\}) = \beta_i(\text{star}^-(v), \text{link}^-(v)) = \begin{cases} 1 & \text{for } i = d, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to a contradiction since by hypothesis $k > 1$.

By combining Lemma 1 and Lemma 2, we obtain the equivalence between the notions of PL critical points proposed in [4] and in [6].

**Proposition 1.** A vertex $v$ is a PL critical point of index $i$ and multiplicity $k$ according to [4] if and only if $v$ is a PL critical point of the same index and the same multiplicity according to [6].
Case $d = 2$. We now prove the equivalence between the definitions of PL critical points for a combinatorial 2-manifold.

**Proposition 2.** For $d = 2$, the definitions of PL critical points proposed in [1], in [8], in [4], and in [6] are equivalent.

**Proof.** Let us start by comparing the two point classifications proposed for combinatorial 2-manifolds in [1] and [8]. Their equivalence is easily achieved by noticing that by definition, for a vertex $v$, the number $W$ of wedges of star$^-$($v$) coincides with half the number of triangles in star($v$) with $v$ middle for $f$. In order to conclude the proof, we show the equivalence between the definitions proposed in [8] and in [4]. Let us consider the various possible cases. Let $v$ be a point of local minimum for [8]. By definition, we have that $W = 0$ and star$^-$($v$) = {v}. This is equivalent to the fact that link$^-$($v$) is empty. Since the only simplicial complex having the same reduced homology of the empty complex is the empty complex itself, thanks to Lemma 1 the previous condition is satisfied if and only if $v$ is a PL critical point of index 0 and multiplicity 1 for [4], i.e., $v$ is a point of local minimum for [4]. A vertex $v$ is a point of local maximum for [8] if and only if $W = 0$ and star$^-$($v$) = star($v$), i.e., if and only if link$^-$($v$) = link($v$). Since $\Sigma$ is a combinatorial $d$-manifold, this is true if and only if link$^-$($v$) has the same reduced homology of the sphere $S^1$. Thanks to Lemma 1 this is satisfied if and only if $v$ is a PL critical point of index 2 and multiplicity 1 for [4], i.e., $v$ is a point of local maximum for [4]. Given a vertex $v$ which is not a point of local minimum or maximum for [8], we have that $W > 0$. Let us noticing that in such a case, link$^-$($v$) is simplicial complex consisting of exactly $W$ connected components and free of higher dimensional homological cycles. I.e., for $W > 0,$

$$\tilde{\beta}_i(\text{link}^- (v)) = \begin{cases} W - 1 & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So, the number of wedges of star$^-$($v$) is $W = 1$ if and only if $v$ is a regular point for [4]; while $W > 1$ if and only if $v$ is a PL critical point of index 1 and multiplicity $W - 1$ for [4].

The just proven results ensure us an equivalence between all the definitions of PL critical point proposed in the literature. So, in the following, it will not be ambiguous to address a vertex as a PL critical point without specifying which definition we are adopting.

### 4 Relating PL and discrete critical sets

Aim of this section is to investigate under which conditions it is possible to establish a correspondence between the set of the PL critical points of a function $f$ on a domain $\Sigma$ and the set of the discrete critical simplices of a discrete gradient vector field on $\Sigma$. For the rest of the section, unless differently specified, we assume that $\Sigma$ is a combinatorial $d$-manifold, $f : \Sigma_0 \to \mathbb{R}$ is an injective function defined on the vertices of $\Sigma$, and $V$ is a discrete gradient vector field on $\Sigma$. 
4.1 An explicit correspondence

Given a value \( l \) in the image of \( f \), let us denote as \( l' \) the greatest value in the image of \( f \) among the ones lower than \( l \). The number \( \beta_i(\Sigma^l, \Sigma^{l'}) \) denotes the number of variations in the \( i^{th} \) homology space occurred at value \( l \).

Equivalently, a simple calculation shows that, denoting by \( \phi_i^l \) the homology map in dimension \( i \) induced by the inclusion of \( \Sigma^{l'} \) into \( \Sigma^l \), \( \beta_i(\Sigma^l, \Sigma^{l'}) = \dim(\ker \phi_{i-1}^l) + \dim(\coker \phi_i^l) \).

**Definition 1.** A discrete gradient vector field \( V \) on \( \Sigma \) is called relatively perfect (RP) w.r.t. \( f : \Sigma_0 \to \mathbb{R} \) if \( m_i^l(V) = \beta_i(\Sigma^l, \Sigma^{l'}) \), for every \( i \in \mathbb{N} \) and every value \( l \in \text{Im } f \), where \( m_i^l(V) \) denotes the number of discrete critical \( i \)-simplices for \( V \) in \( \Sigma^l \setminus \Sigma^{l'} \).

Whenever there is no ambiguity about the considered discrete gradient vector field, we will write \( m_i^l \) in place of \( m_i^l(V) \).

The above definition and the equivalences shown in the previous subsection (Proposition 1 and Proposition 2) enable us to find a correspondence between PL critical points and discrete critical simplices.

**Theorem 1.** Let \( V \) be a relatively perfect discrete gradient vector field on \( \Sigma \) w.r.t. \( f \). Then, a vertex \( v \in \Sigma_0 \) is a PL critical point of index \( i \) and multiplicity \( k \) of \( f \) if and only if there are exactly \( k \) discrete critical \( i \)-simplices \( \sigma \) of \( V \) such that \( \sigma \in \text{star}(v) \) and \( f_{\text{max}}(\sigma) = f(v) \).

**Proof.** By definition, a vertex \( v \) is PL critical of index \( i \) and multiplicity \( k \) if and only if, setting \( l = f(v) \),

\[
\dim(H_j(|\Sigma^l|, |\Sigma^l| \setminus \{v\})) = \begin{cases} k & \text{for } j = i, \\ 0 & \text{otherwise.} \end{cases}
\]

Note that \( \Sigma^{l'} = \Sigma^l \setminus \text{star}(v) \). Indeed, \( \sigma \in \text{star}(v) \) implies \( f_{\text{max}}(\sigma) \geq f(v) = l \).

So, on one hand, \( \Sigma^{l'} \leq \Sigma^l \setminus \text{star}(v) \) because \( l' < l \). On the other hand, for each \( \sigma \in \Sigma^l \setminus \text{star}(v) \), \( f_{\text{max}}(\sigma) < l \) by the injectivity of \( f \), yielding \( f_{\text{max}}(\sigma) \leq l' \) by definition of \( l' \). Thus, \( \dim(H_j(|\Sigma^l|, |\Sigma^l| \setminus \{v\})) = \beta_j(\Sigma^l, \Sigma^{l'}) = m_j^l \), for every \( j \in \mathbb{Z} \), where the first equality follows from \( 1 \) and the second equality from relatively perfectness of \( V \) w.r.t. \( f \). Thus, the claim follows by recalling that \( m_j^l \) is the number of discrete critical \( j \)-simplices \( \sigma \) of \( V \) such that \( \sigma \in \Sigma^l \setminus \Sigma^{l'} \), \( \Sigma^l \setminus \Sigma^{l'} = \text{star}(v) \), and \( l = f(v) \).

As a consequence of the above results, we can give the following corollary.

**Corollary 1.** Let \( V \) be a discrete gradient vector field on \( \Sigma \) relatively perfect w.r.t. \( f \). Then, there is a 1-to-\( k \) correspondence between PL critical points of index \( i \) and multiplicity \( k \) of \( f \) and discrete critical \( i \)-simplices \( \sigma \) of \( V \) such that \( f_{\text{max}}(\sigma) = f(v) \). In particular, if \( f \) is PL Morse, then the correspondence is bijective.
Remark 1. Worth to be noticed that the above corollary ensures also that a PL critical point \( v \) of \( f \) and the discrete critical \( i \)-simplices in correspondence with \( v \) are closely located. More properly, given any discrete critical \( i \)-simplex \( \sigma \) in correspondence with \( v \), we have that \( \sigma \) belongs to \( \text{star}(v) \) or, equivalently, that \( v \) is a vertex of \( \sigma \).

4.2 Construction of RP discrete gradient vector fields

In this subsections, we prove that, for combinatorial manifolds of dimension \( d = 2, 3 \), the existence of an RP discrete gradient vector field (and, consequently, a correspondence between PL and discrete critical sets) is always ensured.

Lemma 3. Let \( \Sigma \) be a simplicial complex of dimension 2 such that \( |\Sigma| \subseteq S^2 \). Then, there exists a triangle in \( \Sigma \) admitting a free face.

Proof. Suppose that none of the triangles in \( \Sigma \) admits a free face. Then, every face of each triangle in \( \Sigma \) belongs to exactly two triangles. Thus, the collection of all triangles forms at least one 2-cycle in \( \Sigma \) which is not a boundary and \( \dim(H_2(\Sigma)) \geq 1 \). Let \( \Sigma' \) be a simplicial complex such that \( |\Sigma'| = S^2 \setminus |\Sigma| \).

Since \( S^2 = |\Sigma'| \cup |\Sigma| \) and \( |\Sigma| \subseteq S^2 \) is of dimension 2, then \( |\Sigma'| \subseteq S^2 \) and it has dimension 2. Similar to the case of \( \Sigma \), \( \dim(H_2(\Sigma')) \geq 1 \). Thus, we get \( \dim(H_2(|\Sigma'|) \oplus H_2(|\Sigma|)) \geq 2 \). Since \( \dim(|\Sigma'| \cap |\Sigma|) \leq 1 \) and \( H_2((|\Sigma'| \cap |\Sigma|)) = 0 \), thanks to the following Mayer-Vietoris sequence for homology

\[
0 \to H_2(|\Sigma'| \cap |\Sigma|) \to H_2(|\Sigma'|) \oplus H_2(|\Sigma|) \xrightarrow{\phi} H_2(S^2) \to H_1(|\Sigma'| \cap |\Sigma|) \to \cdots
\]

we get that the map \( \phi : H_2(|\Sigma'|) \oplus H_2(|\Sigma|) \to H_2(S^2) \) is injective. But this is not possible because \( \dim(H_2(|\Sigma'|) \oplus H_2(|\Sigma|)) \geq 2 \) and \( \dim(H_2(S^2)) = 1 \).

Lemma 4. Let \( \Sigma \) be a simplicial complex such that \( |\Sigma| \subset S^2 \). Then, \( \Sigma \) admits a perfect discrete gradient vector field.

Proof. We can assume, without loss of generality, that \( \Sigma \) is connected. If if it is not the case, the lemma can be proven by considering each component separately. If \( |\Sigma| = S^2 \), then \( \Sigma \) admits a perfect discrete gradient vector field by [12]. If \( |\Sigma| \subset S^2 \), then \( \Sigma \) admits a triangle \( \tau_1 \) with a free face \( \sigma_1 \) by Lemma 3.

Now, we let \( \Sigma^1 = \Sigma \setminus \{\sigma_1, \tau_1\} \). If \( \dim(\Sigma^1) \leq 1 \), then \( \Sigma^1 \) admits a perfect discrete gradient vector field \( V^1 \) by [12]. Since \( \Sigma = \Sigma^1 \cup \{\sigma_1, \tau_1\} \) and removal of the pair \( \{\sigma_1, \tau_1\} \) is a collapse, then \( \Sigma \) and \( \Sigma^1 \) have isomorphic homology groups. Thus, \( V = V^1 \cup \{\{\sigma_1, \tau_1\}\} \) is a perfect discrete gradient vector field on \( \Sigma \) with \( m_i(V) = m_i(V^1) \) for \( i = 0, 1 \). If \( \dim(\Sigma^1) = 2 \), then we remove pairs of simplices \( \{\sigma_i, \tau_i\} \) successively from \( \Sigma^1 \) (which is possible by Lemma 3) up to getting a subcomplex \( \Sigma^n \) of \( \Sigma^1 \) such that \( \dim(\Sigma^n) \leq 1 \) and \( \Sigma^1 = \Sigma^n \cup \{\sigma_2, \tau_2, \ldots, \sigma_n, \tau_n\} \) where \( \{\sigma_i, \tau_i\} \) is the edge-triangle pair removed at the \( i \)-th step of the successive operation. Since each removal of a pair of simplices \( \{\sigma_i, \tau_i\} \) is a collapse, \( \Sigma^1 \) and \( \Sigma^n \) have isomorphic homology groups. Let \( V^n \) be a perfect discrete gradient vector field on \( \Sigma^n \). Then, \( V^1 = V^n \cup \{\{\sigma_2, \tau_2\}, \ldots, \{\sigma_n, \tau_n\}\} \).
is a perfect discrete gradient vector field on $\Sigma^1$ with $m_i(V^1) = m_i(V^n)$ for $i = 0, 1$. Since $\Sigma = \Sigma^1 \cup \{\sigma_1, \tau_1\}$, $\Sigma$ and $\Sigma^1$ have isomorphic homology groups. So, $V = V^1 \cup \{\{\sigma_1, \tau_1\}\}$ is a perfect discrete gradient vector field on $\Sigma$ with $m_i(V) = m_i(V^1)$ for $i = 0, 1$.

**Proposition 3.** Let $\Sigma$ be a combinatorial $d$-manifold for $d = 2, 3$ and let $f : \Sigma_0 \to \mathbb{R}$ be an injective function. Then, there exists a discrete gradient vector field $V$ on $\Sigma$ that is RP with respect to $f$.

We give a proof of Proposition 3 for $d = 3$. The proof for $d = 2$ is similar to the case for $d = 3$.

**Proof.** Since $f$ is injective, then $\cap_{v \in \Sigma_0} \text{star}^-(v) = \emptyset$. Hence, $\Sigma$ can be constructed as a disjoint union of lower stars, that is, $\Sigma = \bigsqcup_{v \in \Sigma_0} \text{star}^-(v)$. Since $\Sigma$ is combinatorial 3-manifold, $|\text{link}^-(v)| \subset S^2$. If $\text{link}^-(v) = \emptyset$, then $\text{star}^-(v) = \{v\}$ and any discrete gradient vector field $W$ on $\text{star}^-(v)$ admits exactly one critical simplex, which is the 0-simplex $v$. So $W$ is a perfect discrete gradient vector field on $\text{star}^-(v)$. Assume that $\text{link}^-(v) \neq \emptyset$. By Lemma 4, $\text{link}^-(v)$ admits a perfect discrete gradient vector field $W$. By Lemma 1, $H_{i+1}(\text{star}^-(v), \text{link}^-(v)) \cong H_i(\text{link}^-(v))$ for $i \geq 0$. Moreover, because $H_0(\text{star}^-(v)) = 0$, the long exact sequence used in Lemma 1 implies that

$$H_0(\text{star}^-(v), \text{link}^-(v)) \cong H_0(\text{link}^-(v)) = 0.$$ 

So, $\beta_{i+1}(\text{star}^-(v), \text{link}^-(v)) = \beta_i(\text{link}^-(v))$ for $i > 0$, $\beta_1(\text{star}^-(v), \text{link}^-(v)) = \beta_0(\text{link}^-(v)) - 1$ and $\beta_0(\text{star}^-(v), \text{link}^-(v)) = 0$. Since $W$ is a perfect discrete gradient vector field, then $m_i(W) = \beta_i(\text{link}^-(v))$ for $i = 0, 1, 2$.

Let $n_i$ denote the number of $i$-simplices in $\text{link}^-(v)$ for $i = 0, 1, 2$, and $n'_i$ denote the number of $i$-simplices in $\text{star}^-(v)$ for $i = 0, 1, 2, 3$. Since there is a 1-to-1 correspondence between the $i$-simplices in $\text{link}^-(v)$ and $(i + 1)$-simplices in $\text{star}^-(v)$, then $n'_0 = 1, n'_1 = n_0, n'_2 = n_1$ and $n'_3 = n_2$.

Now, we construct a vector field $W'$ on $\text{star}^-(v)$ as follows.

- If $(\alpha, \beta) \in W$, then we set $(v * \alpha, v * \beta) \in W'$, where $*$ denotes the usual join of simplices.
- If $\gamma$ is a discrete critical $i$-simplex of $W$ with $i > 0$, then we set $v * \gamma$ as critical for $W'$.
- If $\gamma_1, \gamma_2, \ldots, \gamma_{m_0(W)}$ are the discrete critical 0-simplices of $W$, then we set $(v, v * \gamma_1) \in W'$ and $v * \gamma_2, \ldots, v * \gamma_{m_0(W)}$ as critical for $W'$.

Since $W$ is a discrete gradient vector field, it does not admit any closed path. Hence, $W'$ does not admit any closed path thanks to its construction. The numbers of discrete critical simplices of $W'$ are as follows:

- $m_0(W') = 0 = \beta_0(\text{star}^-(v), \text{link}^-(v)))$,
- $m_1(W') = m_0(W) - 1 = \beta_0(\text{link}^-(v)) - 1 = \beta_1(\text{star}^-(v), \text{link}^-(v)))$,
- $m_i(W') = m_{i-1}(W) = \beta_{i-1}(\text{link}^-(v)) = \beta_i(\text{star}^-(v), \text{link}^-(v)))$ for $i = 2, 3$. 

Hence, $W'$ is a discrete gradient vector field on $\text{star}^{-}(v)$. Let $V$ be the collection of all discrete gradient vector fields $W'$ on $\text{star}^{-}(v)$ for each $v$. Since $\Sigma = \bigsqcup_{v \in \Sigma_0} \text{star}^{-}(v)$, $V$ is a discrete gradient vector field on $\Sigma$ whose restriction to each $\text{star}^{-}(v)$ is $W'$. Let $f(v) = l$. Since $\Sigma' = \Sigma' \setminus \text{star}^{-}(v)$, in accordance with Equation (1), we get $m_i(V) = m_i(W') = \beta_i((\text{star}^{-}(v), \text{link}^{-}(v))) = \beta_i(|\Sigma'|, |\Sigma'| \setminus \{v\})$. Thus, $V$ a discrete gradient vector field on $\Sigma$ that is RP with respect to $f$.

By combining Proposition 3 with Corollary 1, we get the following result.

**Corollary 2.** Let $\Sigma$ be a combinatorial $d$-manifold for $d = 2, 3$ and let $f : \Sigma_0 \to \mathbb{R}$ be an injective function. There exists a discrete gradient vector field $V$ on $\Sigma$ (RP w.r.t. $f$) such that there is a 1-to-$k$ correspondence (a 1-to-1 correspondence if $f$ is PL Morse) between PL critical points of index $i$ and multiplicity $k$ of $f$ and discrete critical $i$-simplices $\sigma$ of $V$ such that $f_{\text{max}}(\sigma) = f(v)$.

The correspondence given in Corollary 1 is shown in Figure 2. In (a), we depict a portion of an RP discrete vector field given on a combinatorial 2-manifold $\Sigma$. The vertex $v := f^{-1}(5)$ is a PL critical saddle and it is on the discrete critical edge $[3, 5]$ as it is stated in Corollary 1. The nature of the vertices 3 and 7 depends on the rest of the vector field. To get this RP discrete gradient vector field, we can use the algorithm given in Proposition 3, as shown in (b): we extend the perfect discrete gradient vector field (in yellow) in the lower link of the vertex $v$ to the lower star of $v$ (in red). The discrete gradient vector field given in (c) is not RP because $v$ is a discrete critical 0-simplex of $\Sigma^5 \setminus \Sigma^3$, that is $m_0^5 = 1$, but $\beta_0(\Sigma^5, \Sigma^3) = 0$.

**Fig. 2.** (a) An RP discrete gradient vector field. (b) An RP discrete gradient vector field on the lower star of $v := f^{-1}(5)$. (c) A non-RP discrete vector field.

5 Conclusions and future developments

In this paper, we have studied the link between PL critical points and discrete critical simplices. Our investigation has identified a condition (which can be always satisfied in the case of domains of dimension 2 and 3) under which a correspondence between the two critical sets exists. Moreover, if the function is PL Morse, the retrieved correspondence is a bijection.
In the near future, we plan to leverage the results obtained in this paper in order to understand the relationships among the various notions of Morse complexes introduced in the literature (e.g., ascending/descending Morse complexes and Morse-Smale complexes). Based on this study, we would like to offer a tool to translate all the known results about discrete Morse theory to PL Morse theory and viceversa.

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