New type of monogenic polynomials and associated spheroidal wavelets

Sabrine Arfaoui

Department of Informatics, Higher Institute of Applied Sciences and Technology of Mateur, Street of Tabarka, 7030 Mateur, Tunisia.
and
Research Unit of Algebra, Number Theory and Nonlinear Analysis UR11ES50, Faculty of Sciences, Monastir 5000, Tunisia.

Anouar Ben Mabrouk

Higher Institute of Applied Mathematics and Informatics, University of Kairouan, Street of Assad Ibn Alfourat, Kairouan 3100, Tunisia.
and
Research Unit of Algebra, Number Theory and Nonlinear Analysis UR11ES50, Faculty of Sciences, Monastir 5000, Tunisia.

Abstract

In the present work, new classes of wavelet functions are presented in the framework of Clifford analysis. Firstly, some classes of new monogenic polynomials are provided based on 2-parameters weight functions. Such classes englobe the well known Jacobi, Gegenbauer ones. The discovered polynomial sets are next applied to introduce new wavelet functions. Reconstruction formula as well as Fourier-Plancherel rules have been proved.

Key words: Continuous Wavelet Transform, Clifford analysis, Clifford Fourier transform, Fourier-Plancherel, Monogenic functions.

PACS: 42B10, 44A15, 30G35.

Email addresses: arfaoui.sabrine@issatm.rnu.tn (Sabrine Arfaoui), anouar.benmabrouk@fsm.rnu.tn (Anouar Ben Mabrouk).
1 Introduction

Fourier analysis has been for many decades the essential mathematical tool in harmonic analysis and related applications’ fields such as physics, engineering, signal/image processing, etc. Next, new extending mathematical tool has been introduced to generalize Fourier one and to overcome in some ways the disadvantages of Fourier analysis. It consists of wavelet analysis.

Compared to Fourier theory, wavelets are mathematical functions permitting themselves to cut up data into different components relatively to the frequency spectrum and next focus on these components somehow independently, extract their characteristics and lift to the original data. One main advantage for wavelets is the fact that they are able more than Fourier modes in analyzing discontinuities and/or singularities efficiently and non-stationarity.

Wavelets were developed independently in the fields of mathematics, physics, electrical engineering, and seismic geology. Next, interchanges between these fields have yielded more understanding of their theory and applications.

Nowadays, wavelets are reputable and successful tools in quasi all domains. The particularity in a wavelet basis is the fact that all the elements of a basis are deduced from one source function known as the wavelet mother. Next, such a mother gives raise to all the elements necessary to analyze objects by simple actions of translation, dilatation and rotation. The last parameter is firstly introduced in [3] (see also [4]) to obtain some directional selectivity of the wavelet transform in higher dimensions and to analyze/characterize spherical data. Indeed, construction of wavelets related to manifolds such as or essentially spheres is based on the geometric structure of the surface where the data lies. This gives raise to the so-called isotropic and anisotropic wavelets.

The present work lies in the whole scope of the study of spherical data. We propose to develop methods based on harmonic structures to define the so-called ultraspheroidal wavelets. One important and actual motivation is issued from 3D-images processing which is noadays a revolutionary task in informatics.

Mathematically, spheroidal functions such as Gegenbauer polynomials which are the starting point in the present extension are solutions with separated variables of the wave equation

$$\nabla^2 w + k^2 w = 0$$

in an elliptic cylinder coordinates system, prolate and/or oblate spheroids. In such systems (mainly radial and angular variables), the wave equation above
may be transformed to a second order ODE of the form

\[(1 - t^2)w'' + 2\alpha tw' + (\beta - \gamma^2 t^2)w = 0.\]

(See [1], [25], [28], [32], [37], [38]). This last equation leads to special functions such as Bessel, Airy, ... and special polynomials such as Gegenbauer, Legendre, Chebyshev, .... and constitutes a first idea behind the link between these functions and a first motivation of the present work where construction of some new spheroidal mother wavelets are done. Besides, spheroidal functions have been in the basis of modeling physical phenomena where the wave behaviour is pointed out such as radars, antennas, 3D-images, ... Recall also that Gegenbauer polynomials themselves are called ultraspherical polynomials. See [3], [5], [7], [17], [18], [24], [25], [27], [29], [35].

The main idea consists in adopting Clifford analysis to introduce or more precisely to extend some existing works on Clifford wavelets for more general cases. Clifford analysis, in its most basic form, is a refinement of harmonic analysis in higher dimensional Euclidean space. By introducing the so-called Dirac operator, researchers introduced the notion of monogenic functions extending holomorphic ones. In this context, different concepts of real and complex analysis have been extended to the Clifford case such as Fourier transform (extended to Clifford Fourier transform, Derivation of functions, ....). For example, Clifford Fourier transform is related or expressed in terms of an exponential operator. For the even dimensional case, it yields a kernel based on Bessel functions. Compared to the classical Fourier transform, the new kernel satisfies here also a system of differential equations.

In the present work, one aim is to provide a rigourous development of wavelets adapted to the Clifford calculus. The frame is somehow natural as wavelets are characterized by scale invariance of approximation spaces. Clifford algebra is one mathematical object that owns this characteristic. Recall that multiplication of real numbers scales their magnitudes according to their position in or out from the origin. However, multiplication of the imaginary part of a complex number performs a rotation, it is a multiplication that goes round and round instead of in and out. So, a multiplication of spherical elements by each other results in an element of the sphere. Again, repeated multiplication of the imaginary part results in orthogonal components. Thus, we need a coordinates system that results always in the object, a concept that we will see again and again in the Algebra. In other words, Clifford algebra generalizes to higher dimensions by the same exact principles applied at lower dimensions, by providing an algebraic entity for scalars, vectors, bivectors, trivectors, and there is no limit to the number of dimensions it can be extended to. More details on Clifford algebra, origins, history, developments may be found in [2], [15], [16], [18], [24], [33].

Let \( \Omega \) be an open subset of \( \mathbb{R}^m \) or \( \mathbb{R}^{m+1} \) and \( f : \Omega \to A \), where \( A \) is the real
Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. $f$ may be written in the form

$$f = \sum_A f_A e_A$$

(1)

where the functions $f_A$ are $\mathbb{R}$-valued or $\mathbb{C}$-valued and $(e_A)_A$ is a suitable basis of $\mathbb{A}$.

In the literature, there are several techniques available to generate monogenic functions in $\mathbb{R}^{m+1}$ such as the Cauchy-Kowalevski extension (CK-extension) which consists in finding a monogenic extension $g^*$ of an analytic function $g$ defined on a given subset in $\mathbb{R}^{m+1}$ of positive codimension. For analytic functions $g$ on the plane $\{(x_0, x) \in \mathbb{R}^{m+1}, \quad x_0 = 0\}$ the problem may be stated as follows: Find $g^* \in \mathbb{A}$ such that

$$\partial_{x_0} g^* = -\partial_x g^* \quad \text{in} \quad \mathbb{R}^{m+1} \quad \text{and} \quad g^*(0, x) = g(x).$$

(2)

A formal solution is

$$g^*(x_0, x) = \exp(-x_0 \partial_x)g(x) = \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!} \partial_x^k g(x).$$

(3)

Starting from the real space $\mathbb{R}^m$, $(m > 1)$ (or the complex space $\mathbb{C}^m$) endowed with an orthonormal basis $(e_1, \ldots, e_m)$, the Clifford algebra $\mathbb{R}_m$ (or its complexification $\mathbb{C}_m$) starts by introducing a suitable interior product. Let

$$e_j^2 = -1, \quad j = 1, \ldots, m,$$

$$e_j e_k + e_k e_j = 0, \quad j \neq k, \quad j, k = 1, \ldots, m.$$

Two anti-involutions on the Clifford algebra are important. The conjugation is defined as the anti-involution for which

$$\overline{e_j} = -e_j, \quad j = 1, \ldots, m$$

with the additional rule in the complex case,

$$\overline{i} = -i.$$

The inversion is defined as the anti-involution for which

$$e_j^+ = e_j, \quad j = 1, \ldots, m.$$

A basis for the Clifford algebra $(e_A : A \subset \{1, \ldots, m\})$ where $e_0 = 1$ is the identity element. As these rules are defined, the Euclidian space $\mathbb{R}^m$ is then embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying the vector $x =
The product of two vectors is given by

\[ \mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y} \]

where

\[ \mathbf{x} \cdot \mathbf{y} = - \langle \mathbf{x}, \mathbf{y} \rangle = - \sum_{j=1}^{m} x_j y_j \]

and

\[ \mathbf{x} \wedge \mathbf{y} = \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k (x_j y_k - x_k y_j) \]

is the wedge product. In particular,

\[ \mathbf{x}^2 = - \langle \mathbf{x}, \mathbf{x} \rangle = -|\mathbf{x}|^2. \]

An \( \mathbb{R}_m \) or \( \mathbb{C}_m \)-valued function \( F(x_1, \ldots, x_m) \), respectively \( F(x_0, x_1, \ldots, x_m) \) is called right monogenic in an open region of \( \mathbb{R}^m \), respectively, or \( \mathbb{R}^{m+1} \), if in that region

\[ F \partial_\mathbf{x} = 0, \quad \text{respectively} \quad F(\partial_{x_0} + \partial_\mathbf{x}) = 0. \]

Here \( \partial_\mathbf{x} \) is the Dirac operator in \( \mathbb{R}^m \):

\[ \partial_\mathbf{x} = \sum_{j=1}^{m} e_j \partial x_j, \]

which splits the Laplacian in \( \mathbb{R}^m \)

\[ \Delta_m = -\partial_\mathbf{x}^2, \]

whereas \( \partial_{x_0} + \partial_\mathbf{x} \) is the Cauchy-Riemann operator in \( \mathbb{R}^{m+1} \) for which

\[ \Delta_{m+1} = (\partial_{x_0} + \partial_\mathbf{x})(\partial_{x_0} + \overline{\partial_\mathbf{x}}) \]

Introducing spherical co-ordinates in \( \mathbb{R}^m \) by

\[ \mathbf{x} = r \mathbf{\omega}, \quad r = |\mathbf{x}| \in [0, +\infty[, \quad \mathbf{\omega} \in S^{m-1}, \]

the Dirac operator takes the form

\[ \partial_\mathbf{x} = \mathbf{\omega} \left( \partial_r + \frac{1}{r} \Gamma_{\mathbf{\omega}} \right) \]

where

\[ \Gamma_{\mathbf{\omega}} = - \sum_{i<j} e_i e_j (x_i \partial x_j - x_j \partial x_i) \]
is the so-called spherical Dirac operator which depends only on the angular coordinates.
Throughout this article the Clifford-Fourier transform of $f$ is given by

$$
\mathcal{F}(f(x))(y) = \int_{\mathbb{R}^m} e^{-i\langle \bar{x}, y \rangle} f(x) dV(x),
$$

where $dV(x)$ is the Lebesgue's measure on $\mathbb{R}^m$.

### 2 A 2-parameters Clifford-Gegenbauer-Jacobi polynomials and associated wavelets

In this section we propose to introduce a new family of orthogonal polynomials in the Clifford context that generalizes the well-known Jacobi polynomials as well as Clifford-Jacobi polynomials. In the sequel the new polynomials will be denoted by $S_{\ell,m}^{\mu,\alpha}(x)$. Here, the indexation on $l, m$ is related to the classic indexes relative to the degree and the kind of the polynomial, and $\mu, \alpha$ are related to the new Clifford algebra weight

$$
\omega_{\mu,\alpha}(x) = |x|^{2\mu}(1 + |x|^2)^\alpha.
$$

The polynomials $S_{\ell,m}^{\mu,\alpha}(x)$ are generated as usually by the CK-extension of the monogenicity property of the function $F^*$ which can be written as

$$
F^*(t, x) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} S_{\ell,m}^{\mu,\alpha}(x) \omega_{\mu-\ell,\alpha-\ell}(x).
$$

The Dirac operator acting on the CK-extension yields the time derivative of $F^*$ as

$$
\frac{\partial F^*(t, x)}{\partial t} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} S_{\ell+1,m}^{\mu,\alpha}(x) \omega_{\mu-\ell-1,\alpha-\ell}(x),
$$

and the Clifford algebra variable derivative as

$$
\frac{\partial F^*(t, x)}{\partial x} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \left( \partial_x S_{\ell,m}^{\mu,\alpha}(x) \omega_{\mu-\ell,\alpha-\ell}(x) + S_{\ell,m}^{\mu,\alpha}(x) \partial_x \omega_{\mu-\ell,\alpha-\ell}(x) \right).
$$

Immediate computations yield that

$$
\partial_x \omega_{\mu-\ell,\alpha-\ell}(x) = 2x \left[ (\mu - \ell) \omega_{\mu-\ell-1,\alpha-\ell}(x) + (\alpha - \ell) \omega_{\mu-\ell,\alpha-\ell-1}(x) \right].
$$
Therefore,

\[ \frac{\partial F^*(t, \mathbf{x})}{\partial \mathbf{x}} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \left[ \frac{\partial_x S^\mu_\ell,\alpha_\ell (x)}{\partial x} \omega_{\mu-\ell,\alpha-\ell}(x) \right. \]

\[ + S^\mu_\ell,\alpha_\ell (x) 2 x \left[ (\mu - \ell) \omega_{\mu-\ell-1,\alpha-\ell}(x) \right. \]

\[ \left. + (\alpha - \ell) \omega_{\mu-\ell,\alpha-1}(x) \right] \right]. \]  

From the monogenicity relation applied to \( F \) we derive the recurrence relation

\[ S^\mu_{\ell+1,\alpha}(x) \omega_{\mu-\ell-1,\alpha-\ell-1}(x) + \omega_{\mu-\ell,\alpha-\ell}(x) \frac{\partial_x S^\mu_\ell,\alpha_\ell (x)}{\partial x} \]

\[ + 2 x \left[ (\mu - \ell) \omega_{\mu-\ell-1,\alpha-\ell}(x) + (\alpha - \ell) \omega_{\mu-\ell,\alpha-1}(x) \right] S^\mu_{\ell,\alpha} (x) = 0. \]

We thus obtain

\[ S^\mu_{\ell+1,\alpha}(x) = -2x[(\mu - \ell)(1 + |x|^2) + (\alpha - \ell)|x|^2] S^\mu_{\ell,\alpha} (x) \]

\[ - |x|^2(1 + |x|^2) \frac{\partial_x S^\mu_\ell,\alpha_\ell (x)}{\partial x}. \]  

(5)

Starting from \( S^\alpha_{0,m}(x) = 1 \), a straightforward calculation yields for example that

\[ S^\alpha_{1,m}(x) = -2x[\mu(1 + |x|^2) + \alpha|x|^2] = -2[\mu x - (\mu + \alpha) x^3]. \]

Next, for \( \ell = 1 \) we obtain

\[ S^\alpha_{2,m}(x) = (4\mu(\mu - 1) + 2m\mu)x^2 - [4(\mu - 1)(\mu + \alpha) + 4\mu(\mu + \alpha - 2) \]

\[ + 2(\mu + \alpha)(m + 2) + 2m\mu]x^4 \]

\[ + [4(\mu + \alpha)(\mu + \alpha - 2) + 2(\mu + \alpha)(m + 2)]x^6. \]
For $\ell = 2$ we get

$$S_{3,m}^{\alpha,\mu}(x)$$

$$= -2[\mu(2\mu m + 4\mu(\mu - 1)) - 2\mu m - 8\mu(\mu - 1)]x^3$$

$$+ [28\mu m + 54\mu(\mu - 1) + 48\mu + 12\alpha m + (\mu + \alpha - 4)[4\mu m$$

$$+ 8\mu(\mu - 1)]x^5 + [2(\mu - 2)[6\mu m + 12\mu(\mu - 1) + 16\mu \alpha$$

$$+ 4\alpha m + 4\alpha(\alpha - 1)] + 2(\alpha - 2)[4\mu m + 8\mu(\mu - 1) + 8\mu \alpha + 2\alpha m]$$

$$+ 28\mu m + 56\mu(\mu - 1) + 80\mu + 20\alpha m + 24\alpha(\alpha - 1)]x^7$$

$$+ [2(\mu + \alpha - 4)(2\mu m + 4\mu(\mu - 1) + 8\mu \alpha + 2\alpha m + 4\alpha(\alpha - 1))]$$

$$+ 12\mu m + 24\mu(\mu - 1)48\mu + 12\alpha m + 24\alpha(\alpha - 1)]x^9.$$  

**Remark 1** $S_{\ell,m}^{\alpha,\mu}(x)$ is a polynomial of degree $3\ell$ in $x$.

As for the classical cases, here also we may derive an analogue Rodrigues formulation for the polynomials $S_{\ell,m}^{\mu,\alpha}(x)$.

**Proposition 2** The $(\mu, \alpha)$ Clifford-Jacobi polynomials $S_{\ell,m}^{\mu,\alpha}(x)$ may be expressed as

$$S_{\ell,m}^{\mu,\alpha}(x) = (-1)^{\ell}(|x|^2)^{\ell-\mu}(1 - x^2)^{\ell-\alpha} \partial_x^\ell \omega_{\mu,\alpha}(x).$$

**Proof.** We proceed by recurrence on $\ell$. For $\ell = 0$, we have

$$|x|^2(1 - x^2)^{\alpha} = (-1)^0 \omega_{\mu-0,\alpha-0} \times 1$$

$$= (-1)^0 \omega_{\mu-0,\alpha-0} \times S_{0,m}^{\alpha,\mu}(x).$$

Thus,

$$S_{0,m}^{\alpha,\mu}(x) = (-1)^0 \omega_{0-\mu,0-\alpha}(x) \partial_x \omega_{\mu,\alpha}(x).$$
For $\ell = 1$, we have
\[
\partial_x (|x|^{2\mu} (1 + |x|^2)^\alpha) = 2\mu x \omega_{\mu-1, \alpha}(x) + 2\alpha x \omega_{\mu, \alpha-1}(x)
\]
\[
= (-1)\omega_{\mu-1, \alpha-1}(x)[-2\mu x(1 - x^2) + 2\alpha x^3]
\]
\[
= (-1)\omega_{\mu-1, \alpha-1}(x)[-2\mu x + 2(\mu + \alpha)x^3]
\]
\[
= (-1)\omega_{\mu-1, \alpha-1}(x) S^{\mu, \alpha}_{1, m}(x).
\]
Thus,
\[
S^{\alpha, \mu}_{1, m}(x) = (-1)\omega_{1-\mu, 1-\alpha}(x) \partial_x \omega_{\mu, \alpha}(x).
\]
For the convenience, we push the calculus to the order $\ell = 2$.
\[
\partial_x \omega_{\mu, \alpha}(x)
\]
\[
= 2\partial_x (\mu x \omega_{\mu-1, \alpha}(x) + \alpha x \omega_{\mu, \alpha-1}(x))
\]
\[
= 2\mu \partial_x (x \omega_{\mu-1, \alpha}(x)) + 2\alpha \partial_x (x \omega_{\mu, \alpha-1}(x))
\]
\[
= -2m\mu \omega_{\mu-1, \alpha}(x) + 4\mu(\mu - 1)x^2 \omega_{\mu-2, \alpha}(x) + 4\alpha x^2 \omega_{\mu-1, \alpha-1}(x)
\]
\[
-2m\alpha \omega_{\mu, \alpha-1}(x) + 4\mu x^2 \omega_{\mu-1, \alpha-1}(x) + 4\alpha(\alpha - 1)x^2 \omega_{\mu, \alpha-2}(x)
\]
\[
= (-1)^2 \omega_{\mu-2, \alpha-2}(x)[-2m \mu \omega_{1, 2}(x) + 4\mu(\mu - 1)x^2 \omega_{0, 2}
\]
\[
+4\mu x^2 \omega_{1, 1}(x) - 2m \alpha \omega_{2, 1}(x) + 4\mu x^2 \omega_{1, 1}(x)
\]
\[
+4\alpha(\alpha - 1)x^2 \omega_{2, 0}(x)]
\]
\[
= (-1)^2 \omega_{\mu-2, \alpha-2}(x)([2\mu m + 4\mu(\mu - 1)]x^2 - [4\mu m + 8\mu(\mu - 1)
\]
\[
+8\mu + 2\alpha m ]x^4 + [2\mu m + 4\mu(\mu - 1) + 8\mu + 2\alpha m + 4\alpha(\alpha - 1)]x^6)
\]
\[
= (-1)^2 \omega_{\mu-2, \alpha-2}(x) S^{\alpha, \mu}_{2, m}(x).
\]
Thus,
\[
S^{\alpha, \mu}_{2, m}(x) = (-1)^2 \omega_{2-\mu, 2-\alpha}(x) \partial_x^2 \omega_{\mu, \alpha}(x).
\]
Now, assume that
\[
S^{\mu, \alpha}_{\ell, m}(x) = (-1)^\ell \omega_{\ell-\mu, \ell-\alpha}(x) \partial_x^\ell \omega_{\mu, \alpha}(x).
\]
and denote
\[ \Upsilon = -2x[\mu - \ell)(1 - x^2) + (\alpha - \ell)|x|^2](-1)^\ell \omega_{\ell - \mu, \ell - \alpha}(x) \partial_x^\ell \omega_{\mu, \alpha}(x) \]
and
\[ \Psi = (-1)^\ell \omega_{1,1}(x)[2(\ell - \mu)\omega_{\ell - 1, \ell - \alpha}(x) + 2m(\ell - \alpha)\omega_{\ell - \mu, \ell - \alpha - 1}(x)]. \]

From equations (5) and (6) we get
\[ S_{\ell+1,m}^{\mu,\alpha}(x) = \int \omega_{\ell+1,\ell+1}(x) \partial_x^{\ell+1} \omega_{\mu, \alpha}(x) \]
\[ = \Upsilon - \Psi \partial_x^{\ell} \omega_{\mu, \alpha}(x) - (-1)^\ell \omega_{\ell+1, \ell+1}(x) \partial_x^{\ell+1} \omega_{\mu, \alpha}(x). \]

On the other hand, we have
\[ \Psi = (-1)^\ell \omega_{1,1}(x)[2(\ell - \mu)\omega_{\ell - 1, \ell - \alpha}(x) + 2(\ell - \alpha)|x|\omega_{\ell - \mu, \ell - \alpha - 1}(x)] \]
\[ = (-2x)(-1)^\ell[(\mu - \ell)(1 - x^2) - (\alpha - \ell)|x|^2] \omega_{\ell - \mu, \ell - \alpha}(x) \]
\[ = \Upsilon \partial_x^{\ell} \omega_{\mu, \alpha}(x). \]

Hence, we get
\[ S_{\ell+1,m}^{\mu,\alpha}(x) = (-1)^\ell \omega_{\ell+1, \ell+1}(x) \partial_x^{\ell+1} \omega_{\mu, \alpha}(x). \]

**Proposition 3** Let the integral
\[ P_{\ell, t, p}^{\mu, \alpha} = \int_{\mathbb{R}^m} \omega_{\ell+1, \ell+1}(x) \partial_x^{\ell+1} \omega_{\mu, \alpha}(x) dV(x). \]
Then, the following orthogonality relation holds.
\[ I_{\ell, t, t}^{\mu, \alpha} = 0 \] (7)
whenever \( 4t < 1 - m - 2(\mu + \alpha) \).
Proof. Denote

\[ I_{\ell,t} = \int_{\mathbb{R}^m} x^{\ell} \partial_{x}^t (\omega_{\mu+t,\alpha+t}(x)) dV(x). \]

Using Stokes’s theorem, we obtain

\[
\begin{aligned}
&\int_{\mathbb{R}^m} x^{\ell} \partial_{t,m}^t \omega_{\mu+t,\alpha+t}(x) \omega_{\mu,a}(x) dV(x) \\
= &\int_{\mathbb{R}^m} x^{\ell} (-1)^t \omega_{\mu-t,\alpha-t}(x) \partial_{x}^t (\omega_{\mu+t,\alpha+t}(x)) \omega_{\mu,a}(x) dV(x) \\
= &(-1)^t \int_{\mathbb{R}^m} x^{\ell} \partial_{x}^t (\omega_{\mu+t,\alpha+t}(x)) dV(x) \\
= &(-1)^t \int_{\mathbb{R}^m} x^{\ell} \partial_{x}^{t-1} (\omega_{\mu+t,\alpha+t}(x)) dV(x) \\
= &(-1)^t \left[ \int_{\partial\mathbb{R}^m} x^{\ell} \partial_{x}^{t-1} (\omega_{\mu+t,\alpha+t}(x)) \partial \Gamma(x) \\
&\quad - \int_{\mathbb{R}^m} \partial_{x}(x^{\ell}) \partial_{x}^{t-1} (\omega_{\mu+t,\alpha+t}(x)) dV(x) \right].
\end{aligned}
\]

Denote

\[ I = \int_{\partial\mathbb{R}^m} x^{\ell} \partial_{x}^{t-1} (\omega_{\mu+t,\alpha+t}(x)) \partial \Gamma(x) \]

and

\[ II = \int_{\mathbb{R}^m} \partial_{x}(x^{\ell}) \partial_{x}^{t-1} (\omega_{\mu+t,\alpha+t}(x)) dV(x). \]

The integral \( I \) vanishes due to the assumption \( 4t < 1 - m - 2(\mu + \alpha) \). We now apply the following technical result.

Lemma 4 For all \( n \in \mathbb{N} \), we have

\[ \partial_{x}(x^n) = \gamma_{n,m} x^{n-1}, \quad (8) \]

where

\[
\gamma_{n,m} = \begin{cases} 
-n & \text{if } n \text{ is even}, \\
-(m+n-1) & \text{if } n \text{ is odd}.
\end{cases}
\]

Due to Lemma 4, we get

\[
II = \gamma_{l,m} \int_{\mathbb{R}^m} x^{\ell-1} \partial_{x}^{t-1} (\omega_{\mu+t,\alpha+t}(x)) dV(x) \\
= \gamma_{l,m} I_{l-1,t-1}.
\]

11
Hence we obtain

\[
\int_{\mathbb{R}^m} S_{t,m}^\mu S_{t,m}^{\mu+\ell,\alpha+\ell}(x) \omega_{\mu,\alpha}(x) dV(x) = (\mu + \alpha) \int_{\mathbb{R}^m} S_{t,m}^{\mu+\ell,\alpha+\ell}(x) \omega_{\mu,\alpha}(x) dV(x) = (-1)^{\ell+1} \gamma_{\ell, \ell - 1, t} \gamma_{\ell - 1, \ell - 2, t - 2} \gamma_{\ell - 2, \ell - 2, t - 2} \gamma_{\ell - 2, \ell - 2, t - 2} = C(m, \ell, t) I_0 = 0,
\]

where

\[
C(m, \ell, t) = (-1)^{m+1} \prod_{k=0}^{m} \gamma_{k, m}.
\]  

We now introduce the generalized \((\mu, \alpha)\)-Clifford-Gegenbauer-Jacobi wavelets associated to the polynomials introduced previously. Note that proposition 3 implies that for 0 < \( t < \frac{1 - m - 2(\mu + \alpha)}{4} \),

\[
\int_{\mathbb{R}^m} S_{t,m}^\mu S_{t,m}^{\mu+\ell,\alpha+\ell}(x) \omega_{\mu,\alpha}(x) dV(x) = 0.
\]

**Definition 5** The generalized \((\mu, \alpha)\) Clifford-Gegenbauer-Jacobi analyzing wavelet is defined by

\[
\psi_{t,m}^{\mu,\alpha}(x) = S_{t,m}^\mu S_{t,m}^{\mu+\ell,\alpha+\ell}(x) \omega_{\mu,\alpha}(x) = (-1)^{\ell} \partial_{x}^\ell \omega_{\mu+\ell,\alpha+\ell}(x).
\]

The wavelet \(\psi_{t,m}^{\mu,\alpha}\) has vanishing moments as is shown in the next proposition.

**Proposition 6** The following assertions are true.

(1) Whenever 0 < \( k < m - \ell - 2(\mu + \alpha) \) and \( k < \ell \) we have

\[
\int_{\mathbb{R}^m} x^k \psi_{t,m}^{\mu,\alpha}(x) dV(x) = 0.
\]

(2) The Clifford-Fourier transform of \(\psi_{t,m}^{\mu,\alpha}\) takes the form

\[
\hat{\psi}_{t,m}^{\mu,\alpha}(\mu) = (-i)^{\ell} \xi^{\ell} (2\pi)^{\frac{\mu}{2}} \rho^{1 + \frac{\mu}{2} + \ell} \int_{0}^{\infty} \bar{\omega}_{\alpha,\mu}(r) J_{\frac{\mu}{2} - 1}(r \rho) dr
\]

where

\[
\bar{\omega}_{\alpha,\mu}(r) = r^{2(\mu + \ell) + \frac{\mu}{2}} (1 + r^2)^{\alpha + \ell}.
\]
Proof. The first assertion is a natural consequence of proposition 3. We prove the second. We have
\[ \hat{\psi}_{\ell,m}^{\mu,\alpha}(u) = (-1)^\ell (iu)^\ell \omega_{\mu+\ell,\alpha+\ell}(u). \]
This Fourier transform can be simplified by using the spherical co-ordinates. By definition, we have
\[ \hat{\omega}_{\mu+\ell,\alpha+\ell}(u) = \int_{\mathbb{R}^m} |x|^{2(\mu+\ell)}(1 + |x|^2)^{\alpha+\ell} e^{-i<x,u>} dV(x). \]
(12)
Introducing spherical co-ordinates
\[ x = r\omega, \ u = \rho\xi, \ r = |x|, \ \rho = |u|, \ \omega \in S^{m-1}, \ \xi \in S^{m-1} \]
(where \( S^{m-1} \) is the unit sphere of \( \mathbb{R}^m \)) expression (12) becomes
\[ \hat{\omega}_{\mu+\ell,\alpha+\ell}(u) = \int_0^\infty r^{2(\mu+\ell)+m-1}(1 + r^2)^{\alpha+\ell} dr \int_{S^{m-1}} e^{-i<r\omega,\rho\xi>} d\sigma(\omega) \]
where \( d\sigma(\omega) \) stands for the Lebesgue measure on \( S^{m-1} \).
We now use the following technical result which is known in the theory of Fourier analysis of radial functions and the theory of Bessel functions.

Lemma 7 [39]
\[ \int_{S^{m-1}} e^{-i<r\omega,\rho\xi>} d\sigma(\omega) = \frac{(2\pi)^\frac{m}{2} J_{\frac{m}{2}-1}(r\rho)}{(r\rho)^\frac{m}{2}-1} \]
where \( J_{\frac{m}{2}-1} \) is the Bessel function of the first kind of order \( \frac{m}{2} - 1 \) and \( d\sigma \) is the Lebesgue measure on the sphere \( S^{m-1} \).

Now, according to lemma 7, we obtain
\[ \hat{\omega}_{\mu+\ell,\alpha+\ell}(u) = (2\pi)^\frac{m}{2} \rho^{1-\frac{m}{2}} \int_0^\infty r^{2(\mu+\ell)+\frac{m}{2}}(1 + r^2)^{\alpha+\ell} J_{\frac{m}{2}-1}(r\rho) dr \]
\[ = (2\pi)^\frac{m}{2} \rho^{1-\frac{m}{2}} \int_0^\infty \tilde{\omega}_{\alpha,\mu}(r) J_{\frac{m}{2}-1}(r\rho) dr. \]
Consequently, we obtain the following expression for the Fourier transform of the \((\mu, \alpha)\)-clifford-jacobi wavelets
\[ \hat{\psi}_{\ell,m}^{\mu,\alpha}(u) = (-i)^\ell \xi^\ell (2\pi)^\frac{m}{2} \rho^{1-\frac{m}{2}+\ell} \int_0^\infty \tilde{\omega}_{\alpha,\mu}(r) J_{\frac{m}{2}-1}(r\rho) dr. \]

Proof of Lemma 7. Denote firstly \( I \) the right hand side integral. Recall next that for \( \lambda \in \mathbb{R} \), the Bessel function \( J_\lambda \) may be written in the integral form
\[ J_\lambda(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(\lambda - xsint)} dt. \]
Next, as the measure $d\sigma$ is invariant under rotations, we may assume without loss of the generality that $\xi = (1, 0, \ldots, 0)$. As a result, we obtain

$$I(\xi) = \int_{S^{m-1}} e^{-ir\rho\cos\theta} d\sigma(\omega),$$

where $\theta$ is the angle $(\omega, e_1)$, with $e_1 = (1, 0, \ldots, 0)$. In spherical coordinates, this means that

$$I(\xi) = \omega_{m-1} \int_0^\pi e^{-ir\rho\cos\theta} \sin^{m-2}\theta \, d\theta.$$

Denote next $t = \cos \theta$. We get

$$I(\xi) = \omega_{m-1} \int_{-1}^1 e^{-ir\rho t} (1 - t^2)^{(m-3)/2} \, dt.$$

Observing next that the area $\omega_{m-1}$ of the unit sphere $S^{m-1}$ is

$$\omega_{m-1} = \frac{\pi^{m-1/2}}{\Gamma(m/2)}$$

and in the other hand,

$$J_\lambda(x) = \frac{|\frac{x}{2}|^\lambda}{\pi^{1/2} \Gamma(\lambda + 1/2)} \int_{-1}^1 e^{-ixt} (1 - t^2)^{-1/2} \, dt,$$

we obtain the desired result.

A first question in wavelet theory is the admissibility of the wavelet mother. This is checked in the following lemma.

**Lemma 8** The wavelet mother $\psi^\mu,\alpha_{\ell,m}$ satisfies the dmissibility assumption

$$A^\mu,\alpha_{\ell,m} = \frac{1}{\omega_{m-1}} \int_{\mathbb{R}^m} \left| \overline{\psi^\mu,\alpha_{\ell,m}(x)} \right|^2 \frac{dV(x)}{|x|^m} < +\infty. \quad (13)$$

Indeed,

Now, we introduce the Generalized $(\mu, \alpha)$-Clifford-Gegenbauer-Jacobi Continuous Wavelet Transform. For $a > 0$ and $\underline{b} \in \mathbb{R}_m$, the $(a, \underline{b})$-copy of the wavelet mother $\psi^\mu,\alpha_{\ell,m}$ is defined by

$$a^{\underline{b}} \psi^\mu,\alpha_{\ell,m}(x) = a^{-\frac{m}{2}} \psi^\mu,\alpha_{\ell,m}(\frac{x - \underline{b}}{a}). \quad (14)$$

**Definition 9** The generalized $(\mu, \alpha)$-Clifford-Gegenbauer-Jacobi CWT applies to functions $f \in L_2$ by means of the wavelet coefficient

$$C_{a,\underline{b}}(f) = \langle a^{\underline{b}} \psi^\mu,\alpha_{\ell,m}, f \rangle = \int_{\mathbb{R}^m} f(x) a^{\underline{b}} \psi^\mu,\alpha_{\ell,m}(x) dV(x).$$
Introducing the inner product
\[ < C_{a,b}(f), C_{a,b}(g) > = \frac{1}{A_{\ell,m}} \int_{\mathbb{R}^m} \int_{0}^{+\infty} C_{a,b}(f) C_{a,b}(g) \frac{da}{a^{m+1}} dV(b) \]
we obtain the following result.

**Theorem 10** Any function \( f \) in \( L_2(\mathbb{R}_m) \) may be reconstructed in the \( L_2 \)-sense as
\[ f(x) = \frac{1}{A_{\ell,m}} \int_{a>0} \int_{b \in \mathbb{R}^m} C_{a,b}(f) \psi \left( \frac{x - b}{a} \right) \frac{dV(b)}{a^{m+1}}. \] (15)

The proof reposes on the following result.

**Lemma 11** It holds that
\[ \int_{a>0} \int_{b \in \mathbb{R}^m} C_{a,b}(f) C_{a,b}(g) \frac{da}{a^{m+1}} dV(b) = A_{\ell,m} \int_{\mathbb{R}^m} f(x) g(x) dV(x). \]

**Proof.** Using the Clifford Fourier transform we observe that
\[ C_{a,b}(f)(b) = a^{\frac{m}{2}} \hat{f}(\omega) \hat{\psi}(a\omega)(b), \]
where, \( \tilde{h}(u) = h(-u), \forall h \). Thus,
\[ C_{a,b}(f) C_{a,b}(g) = (\hat{f}(\omega) a^{\frac{m}{2}} \hat{\psi}(a\omega)) (-b) (\hat{g}(\omega) a^{\frac{m}{2}} \hat{\psi}(a\omega)) (-b). \]
Consequently,
\[ < C_{a,b}(f), C_{a,b}(g) > = \int_{a>0} \int_{\mathbb{R}^m} (\hat{f}(\omega) a^{\frac{m}{2}} \hat{\psi}(a\omega)) (\hat{g}(\omega) a^{\frac{m}{2}} \hat{\psi}(a\omega)) \frac{dV(b)}{a^{m+1}} \]
\[ = \int_{a>0} \int_{\mathbb{R}^m} \frac{f(b) \hat{g}(\omega) a^m |\hat{\psi}(ab)|^2}{a^{m+1}} da V(b) \]
\[ = A_{\ell,m} \int_{\mathbb{R}^m} f(b) \hat{g}(b) dV(b) \]
\[ = A_{\ell,m} \langle \hat{f}, \hat{g} \rangle \]
\[ = < f, g > . \]

**Proof of Theorem 10.** It follows immediately from lemma 11 and Riesz rule.
In this section, we develop a second class of orthogonal polynomials and associated wavelets already generalising the well known classes of Gegenbauer and Jacobi and also based on a 2-parameters weight function on the Clifford algebra. Polynomials elements of the new class will be denoted by $K^{\alpha, -\beta}_{\ell,m}(x)$. These are generated by the CK-extension $F^*$ of the 2-parameters weight function

$$\omega_{\alpha, -\beta}(x) = (1 + |x|^2)^{\alpha} e^{-\beta|x|^2}, \quad \beta > 0.$$ 

The CK-extension then takes the form

$$F^*(t, x) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} K^{\alpha, -\beta}_{\ell,m}(x) \omega_{\alpha-\ell, -\beta}(x).$$

The Dirac operator acting on the CK-extension can be written as

$$\frac{\partial F^*(t, x)}{\partial t} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} K^{\alpha, -\beta}_{\ell+1,m}(x) \omega_{\alpha-\ell-1, -\beta}(x)$$

and

$$\frac{\partial F^*(t, x)}{\partial x} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \left[ \omega_{\alpha-\ell, -\beta}(x) \partial_x K^{\alpha, -\beta}_{\ell,m}(x) + [2(\alpha - \ell) x \omega_{\alpha-\ell-1, -\beta}(x) - 2\beta x \omega_{\alpha-\ell, -\beta}(x)] K^{\alpha, -\beta}_{\ell,m}(x) \right].$$

From the monogenicity relation, we get

$$K^{\alpha, -\beta}_{\ell+1,m}(x) \omega_{\alpha-\ell-1, -\beta}(x) + \omega_{\alpha-\ell, -\beta}(x) \partial_x K^{\alpha, -\beta}_{\ell,m}(x)$$

$$+ [2(\alpha - \ell) x \omega_{\alpha-\ell-1, -\beta}(x) - 2\beta x \omega_{\alpha-\ell, -\beta}(x)] K^{\alpha, -\beta}_{\ell,m}(x)$$

$$= 0.$$ 

Finally, the following recurrence relation is obtained.

$$K^{\alpha, -\beta}_{\ell+1,m}(x) = -[2(\alpha - \ell) x - 2\beta x \omega_{1,0}(x)] K^{\alpha, -\beta}_{\ell,m}(x)$$

$$- \omega_{1,0}(x) \partial_x K^{\alpha, -\beta}_{\ell,m}(x).$$

(16)

As $K^{1, -\beta}_{0,m}(x) = 1$, a straightforward calculation yields that

$$K^{1, -\beta}_{1,m}(x) = -2(\alpha - \beta) x - 2\beta x^3.$$
Next, for $\ell = 1$, we obtain

$$K_{2,m}^{\alpha,-\beta}(x) = \partial_x[2\alpha \omega_{\alpha,1,-\beta}(x) - 2\beta \omega_{\alpha,-\beta}(x)]$$

$$= [4(\alpha - 1 - \beta)(\alpha - \beta) - 2\beta(m + 2) + 2m(\alpha - \beta)]x^2 + [4(\alpha - 1 - \beta)\beta + 4(\alpha - \beta)\beta + 2\beta(m + 2)]x^4 + 4\beta^2x^6 - 2m(\alpha - \beta) = [2m + 4\alpha(\alpha - 1) - 8\alpha\beta - 4m\beta + 4\beta^2]x^2 + (8\alpha\beta + 2m\beta - 8\beta^2)x^4 + 4\beta^2x^6 - 2m(\alpha - \beta).$$

For $\ell = 2$, we get

$$K_{3,m}^{\alpha,-\beta}(x)$$

$$= [4m(\alpha^2 + \beta^2) + 4(2\alpha^2 + \beta^2) + 4m(\beta - \alpha) - 8\alpha - 8\alpha\beta(1 + m)]x$$

$$- 8\alpha\beta(1 + m)x + [8\alpha^3 - \beta^3] + 8\alpha\beta(1 + m) + 4\alpha m(1 - \alpha) - 24\beta^2(1 + \alpha) - 8\alpha(1 - 2\alpha) + 24\alpha^2\beta - 12m\beta^2]x^3$$

$$+ [-24\alpha^2\beta - 8\alpha\beta + 48\alpha^2\beta + 12m\beta^2 - 24\beta^3 + 8\alpha\beta + 24\beta^2]x^5$$

$$+ (-24\alpha^2\beta - 8\beta^2 - 8\beta^3 - 4m\beta^2)x^7 + 8\beta^3x^9.$$

Remark that $K_{\ell,m}^{\alpha,-\beta}(x)$ is a polynomial of degree $3\ell$ in $x$.

**Proposition 12** The Generalized Clifford-Gauss-Gagenbauer-Jacobi polynomials $K_{\ell,m}^{\alpha,-\beta}(x)$ may be expressed by

$$K_{\ell,m}^{\alpha,-\beta}(x) = (-1)^\ell \omega_{\ell,\alpha,\beta}(x) \partial_x^\ell(\omega_{\alpha,-\beta}(x)).$$

**Proof.** For $\ell = 0$, we have

$$K_{0,m}^{\alpha,-\beta}(x) = 1$$

and on the right hand side we have

$$\omega_{\alpha,\beta}(x) \omega_{\alpha,-\beta}(x) = 1.$$
For $\ell = 1$, we have

$$\partial_1^{\ell}(\omega_{\alpha,-\beta}(x)) = 2\alpha x \omega_{\alpha-1,-\beta}(x) - 2\beta x \omega_{\alpha,-\beta}(x)$$

$$= (-1) \omega_{\alpha-1,-\beta}(x)(-2(\alpha - \beta)x - 2\beta x^3)$$

$$= (-1) \omega_{\alpha-1,-\beta}(x) K_{1,m}^{\alpha,-\beta}(x).$$

Therefore,

$$K_{1,m}^{\alpha,-\beta}(x) = (-1) \omega_{1-\alpha,-\beta}(x) \partial_1^{\ell}(\omega_{\alpha,-\beta}(x)).$$

Now, as previously, we explain the case $\ell = 2$. We have

$$\partial_2^{\ell}(\omega_{\alpha,-\beta}(x))$$

$$= \partial_2^{\ell}[2\alpha x \omega_{\alpha-1,-\beta}(x) - 2\beta x \omega_{\alpha,-\beta}(x)]$$

$$= -2m\alpha \omega_{\alpha-1,-\beta}(x) + 4\alpha(\alpha - 1)x^2 \omega_{\alpha-2,-\beta}(x)$$

$$- 4m^2\alpha\beta x^2 \omega_{\alpha-1,-\beta}(x) + 2m^2\beta \omega_{\alpha,-\beta}(x)$$

$$- 4\alpha\beta x^2 \omega_{\alpha-1,-\beta}(x) + 4\beta^2 \omega_{\alpha,-\beta}(x)$$

$$= (-1)^2 \omega_{\alpha-2,-\beta}(x)[2\alpha m + 4\alpha(\alpha - 1) - 8\alpha\beta - 4m\beta + 4\beta^2]x^2$$

$$+ (8\alpha\beta + 2m\beta - 8\beta^2)x^4 + 4\beta^2 x^6 - 2\alpha m + 2m\beta]$$

$$= (-1)^2 \omega_{\alpha-2,-\beta}(x) K_{2,m}^{\alpha,-\beta}(x).$$

Consequently,

$$K_{2,m}^{\alpha,-\beta}(x) = (-1)^2 \omega_{2-\alpha,-\beta}(x) \partial_2^{\ell}(\omega_{\alpha,-\beta}(x)).$$

Denote

$$\Lambda(x) = -[2(\alpha - \ell)x - 2\beta x \omega_{1,0}(x)](-1)^{\ell}\omega_{\ell,-\alpha,-\beta}(x) \partial_2^{\ell}\omega_{\alpha,-\beta}(x),$$

and

$$\Theta(x) = (-1)^{\ell}\omega_{1,0}(x)[2(\ell - \alpha)x \omega_{\ell-1,-\beta}(x) + 2\beta x \omega_{\ell,-\alpha,-\beta}(x)] \partial_2^{\ell}(\omega_{\alpha,-\beta}(x)).$$
From equations (16) and (17) we observe that

\[
K_{\ell+1,m}^{\alpha,-\beta}(x) = -(2(\alpha - \ell)x - 2\beta x_0) \omega_{\alpha,-\beta}(x)
\]

\[
\omega_{1,0}(x) \partial_\ell \left( -(1)^{\ell} \omega_{\ell-\alpha,-\beta}(x) \partial_\ell \omega_{\alpha,-\beta}(x) \right)
\]

\[
= \Lambda(x) - \Theta(x) + (1)^{\ell+1} \omega_{\ell-\alpha+1,-\beta}(x) \partial_{\ell+1} \omega_{\alpha,-\beta}(x).
\]

In the other hand, we have

\[
\Theta(x) = -(1)^{\ell} \left[ 2(\ell - \alpha)x \omega_{\ell-\alpha,-\beta}(x) + 2\beta x_0 \omega_{\ell-\alpha,-\beta}(x) \omega_{1,0}(x) \right] \partial_\ell \omega_{\alpha,-\beta}(x)
\]

\[
= -(1)^{\ell} \omega_{\ell-\alpha,-\beta}(x) \left[ -(2(\alpha - \ell)x - 2\beta x_0) \omega_{1,0}(x) \right] \partial_\ell \omega_{\alpha,-\beta}(x)
\]

\[
= \Lambda(x).
\]

As a result,

\[
K_{\ell+1,m}^{\alpha,-\beta}(x) = -(1)^{\ell+1} \omega_{\ell-\alpha+1,-\beta}(x) \partial_{\ell+1} \omega_{\alpha,-\beta}(x).
\]

**Proposition 13** Let

\[
I_{\ell,t,p}^{\alpha,-\beta} = \int_{\mathbb{R}^m} x^\ell K_{t,m}^{\alpha+p,\beta}(x) \omega_{\alpha,-\beta}(x) dV(x).
\]

Whenever \(2t < 1 - m - 2\alpha\) we have the orthogonality relation

\[
I_{\ell,t,t}^{\alpha,-\beta} = 0.
\]

(18)

**Proof.** Denote

\[
I_{t,t} = \int_{\mathbb{R}^m} x^\ell \partial_\ell \omega_{\alpha+t,\beta}(x) dV(x).
\]

Using Stokes’s theorem, we obtain

\[
\int_{\mathbb{R}^m} x^\ell K_{t,m}^{\alpha+t,\beta}(x) \omega_{\alpha,-\beta}(x) dV(x)
\]

\[
= -(1)^{\ell} \int_{\mathbb{R}^m} x^\ell \omega_{\ell-\alpha-t,\beta}(x) \partial_\ell \omega_{\alpha+t,\beta}(x) \omega_{\alpha,-\beta}(x) dV(x)
\]

\[
= -(1)^{\ell} \left[ \int_{\partial \mathbb{R}^m} x^\ell \partial_\ell \omega_{\alpha+t,\beta}(x) dV(x) 
\right.
\]

\[
- \int_{\mathbb{R}^m} x^\ell \partial_\ell \omega_{\alpha+t,\beta}(x) dV(x)
\]
Denote here also

\[ I = \int_{\partial \mathbb{R}^m} x^\ell \partial_{x^\ell}^{-1}(\omega_{\alpha, t, \beta}(x)) \, dV(x) \]

and

\[ II = \int_{\mathbb{R}^m} \partial_x(x^\ell) \partial_{x^\ell}^{-1}(\omega_{\alpha, t, \beta}(x)) \, dV(x). \]

The integral \( I \) vanishes due to the assumption \( 2t < 1 - m - 2\alpha \) and for the next, we have

\[ II = \gamma_{\ell, m} \int_{\mathbb{R}^m} x^\ell \partial_{x^\ell}^{-1}(\omega_{\alpha, t, \beta}(x)) \, dV(x) \]

\[ = \gamma_{\ell, m} I_{\ell-1, t-1}. \]

Hence, we obtain

\[ \int_{\mathbb{R}^m} x^\ell K_{\ell, m}^{\alpha, t, \beta}(x) \omega_{\alpha, \beta}(x) \, dV(x) = (-1)^{t+1} \gamma_{\ell, m} I_{\ell-1, t-1} \]

\[ = (-1)^{t+1} \gamma_{\ell, m} (-1)^t \gamma_{t-1, m} I_{t-2, t-2} \]

\[ = (-1)^{2t+1} \gamma_{\ell, m} \gamma_{t-1, m} I_{t-2, t-2} \]

\[ : \]

\[ = C(m, \ell, t) I_0 \]

\[ = 0 \]

where \( C(m, \ell, t) \) is defined by (9).

**Definition 14** The generalized Clifford-Gauss-Gegenbauer-Jacobi Wavelet mother is defined by

\[ \psi_{\ell, m}^{\alpha, \beta}(x) = K_{\ell, m}^{\alpha, t, \beta}(x) \omega_{\alpha, \beta}(x) = (-1)^{t} \partial_{x^\ell}^{-1} \omega_{\alpha, \beta}(x). \]

As for the previous class, we may prove that the wavelet \( \psi_{\ell, m}^{\alpha, \beta}(x) \) possesses further vanishing moments as is shown in the next proposition.

**Proposition 15** The following assertions are true.

1. Whenever \( 0 < k < -m - \ell - 2\alpha \) and \( k < \ell \), we have

\[ \int_{\mathbb{R}^m} x^k \psi_{\ell, m}^{\alpha, \beta}(x) \, dV(x) = 0. \] (19)

2. The Clifford-Fourier transform of the generalized Clifford-Gauss-Gegenbauer-Jacobi wavelet is

\[ \psi_{\ell, m}^{\mu, \alpha}(u) = (-i)^{\ell} \xi^\ell (2\pi)^{\frac{m}{2}} \rho^{1+\frac{m}{2}+\ell} \int_0^\infty \tilde{\omega}_{\alpha, \beta}(r) J_{\frac{m}{2}-1}(r \rho) \, dr \] (20)
where
\[ \tilde{\omega}^{l,m}_{\alpha,\beta}(r) = (1 + r^2)^\alpha + \ell r \frac{m}{2} e^{-\beta r^2}. \]

**Definition 16** Let \( a > 0 \) and \( b \in \mathbb{R}^m \). The copy of the generalized Clifford-Gauss-Gegenbauer-Jacobi wavelet mother at the scale \( a \) and the position \( b \) is defined by
\[ b_a \psi^{\alpha,-\beta}_{\ell,m}(x) = a^{-\frac{m}{2}} \psi^{\alpha,-\beta}_{\ell,m}(\frac{x - b}{a}). \]

The generalized Clifford-Gauss-Gegenbauer-Jacobi wavelet transform of a function \( f \in L_2 \) is defined by
\[ C_{a,b}(f) = \left< b_a \psi^{\alpha,-\beta}_{\ell,m}, f \right>. \]

The following result holds.

**Theorem 17** Let \( \psi^{\mu,\alpha}_{\ell,m} \) be the wavelet defined in Definition 14. The following assertions hold.

1. \( \mathcal{A}_\psi = \frac{1}{\tilde{\omega}_m} \int_{\mathbb{R}^m} \left| \tilde{\psi}^{\alpha,-\beta}_{\ell,m}(\varphi) \right|^2 dV(\varphi) < +\infty. \)

2. Any \( L_2 \) function \( f \) may be reconstructed as
\[ f(x) = \frac{1}{\mathcal{A}_\psi} \int_{a>0} \int_{b \in \mathbb{R}^m} C_{a,b}(f) \psi^{\mu,\alpha}_{\ell,m}(\frac{x - b}{a}) \frac{da dV(b)}{a^{m+1}}. \]

**Proof.**

4 Conclusion

In this paper new classes of monogenic orthogonal polynomials have been introduced relatively to different weights in the context of Clifford analysis. The new classes generalize the well known Jacobi and Gegenbauer polynomials. Such polynomial are proved to be good candidates to construct new wavelets in Clifford analysis. Fourier-Plancherel type results are generalized for the new classes of wavelets.

**References**

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards, Applied Mathematics Series, 55, USA, 1964.
[2] R. Abreu-Blaya, J. Bory-Reyes and P. Bosch, Extension Theorem for Complex Clifford Algebras-Valued Functions on Fractal Domains, Boundary Value Problems, 2010, Article ID 513186, 9 pages.

[3] J.-P. Antoine, R. Murenzi and P. Vandergheynst, Two-dimensional directional wavelets in image processing, Int. J. of Imaging Systems and Technology, 7(3) (1996), 152-165.

[4] J.-P. Antoine and P. Vandergheynst, Wavelets on the n-sphere and related manifolds, J. Math. Phys. 39 (1998), 3987-4008.

[5] S. Arfaoui and A. Ben Mabrouk, Some old orthogonal polynomials revisited and associated wavelets Part I, Submitted to Advances in Applied Clifford Algebra. In revised version, 2017.

[6] S. Arfaoui, I. Rezgui and A. Ben Mabrouk, Harmonic wavelet analysis on the sphere, spheroidal wavelets, Degryuter, 2016, ISBN 978-11-048188-4.

[7] D. Barlet and J.-L. Clerc, Le comportement à l’infini des fonctions de Bessel généralisées. Advances in Mathematics 61 (1986), 165–183.

[8] G. M. Bisci, V. D. Radulescu and R. Servadei, Variational Methods for Nonlocal Fractional Problems. Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2016.

[9] F. Brackx, N. De Schepper and F. Sommen, The Clifford-Gegenbauer polynomials and the associated continuous wavelet transform. Integral Transforms and Special Functions, 15(5) (2004), 387-404.

[10] F. Brackx, N. De Schepper and F. Sommen, The Two-Dimensional Clifford-Fourier Transform. J Math Imaging, 26 (2006), 5-18.

[11] F. Brackx, N. De Schepper and F. Sommen, The Fourier Transform in Clifford analysis. Advances in Imaging and Electron Physics, 156 (2009), 55-201.

[12] F. Brackx, N. De Schepper and F. Sommen, Clifford-Jacobi Polynomials and the associated continuous wavelet transform in eucllidean space, (2006), 185-198.

[13] F. Brackx, N. De Schepper and F. Sommen, The Clifford-Laguerre continuous wavelet transform, (2003), 201-215.

[14] M. J. Craddock and J. A. Hogan, The fractional Clifford-Fourier kernel. The Erwin Schrödinger Internationa Institute for Mathematical Physics ESI, Vienna, Preprint ESI 2411, 2013.

[15] H. De Bie, Clifford algebras, Fourier transforms and quantum mechanics, arXiv: 1209.6434v1, September 2012, 39 pages.

[16] H. De Bie and Y. Xu, On the Clifford-Fourier transform. ArXiv: 1003.0689, December 2010, 30 pages.

[17] N. De Schepper, The generalized Clifford-Gegenbauer polynomials revisited. Adv. appl. Clifford alg. 19 (2009), 253-268.
[18] R. Delanghe, Clifford Analysis: History and Perspective, Computational Methods and Function Theory, 1(1) (2001), 107-153.

[19] A. Friedman, A new proof and generalizations of the Cauchy-Kowalewski theorem. ...

[20] E. Hitzer and S. J. Sangwine, Quaternion and Clifford Fourier Transforms and Wavelets. Eckhard Hitzer Stephen J. Sangwine Editors, Trends in Mathematics, Birkhäuser, Springer Basel, 2013.

[21] A. A. Kilbas, H. A. Srivastava and J. I. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies 204, Editor: Jan van Mill, Faculteit der Exacte Wetenschappen, Amsterdam, The Netherlands, 2006.

[22] D. Kumar, Prolate Spheroidal Wavelet Coefficients, Frames and Double Infinite Matrices. European J. of Pure and Applied Mathematics, 3(4) (2010), 717-724.

[23] D. Kumar, Convergence of Prolate Spheroidal Wavelets in a Generalized Sobolev Space and Frames. European J. of Mathematical Sciences, 2(1) (2013), 102-114

[24] S. Lehar, Clifford Algebra: A visual introduction. A topnotch WordPress.com site, March 18, 2014.

[25] L.-W. Li, X.-K. Kang, M.-S. Leong, Spheroidal Wave Functions in Electromagnetic Theory. Wiley-Interscience Publication, 2002.

[26] H. R. Malonek and M. I. Falcao, On special functions in the context of Clifford analysis. AIP Conference Proceedings, 2010, 1281, 1492–1495, DOI: 10.1063/1.3498054

[27] V. Michel, Lectures on constructive approximation, Birkhäuser, 2013.

[28] J. Morais, K.I. Kou and W. Sprößig, Generalized holomorphic Szegö kernel in 3D spheroids. Computers and Mathematics with Applications, 65(4) (2013), 576-588.

[29] M.-M. Moussa, Calcul efficace et direct des représentations de maillages 3D utilisant les harmoniques sphériques, Thèse de Doctorat de l’université Claude Bernard, Lyon 1, France, 2007.

[30] F. W. J. Olver, Bessel Functions of Integer Order, ....

[31] M. D. Ortigueira, J. A. Tenreiro Machado, What is a fractional derivative, Journal of Computational Physics 293 (2015), 4-13.

[32] A. Osipov, V. Rokhlin and H. Xiao, Prolate Spheroidal Wave Functions of Order Zero. Mathematical Tools for Bandlimited Approximation, Applied Mathematical Sciences Volume 187, Springer 2013.

[33] D. P. Pena, Cauchy-Kowalevski extensions, Fueters theorems and boundary values of special systems in Clifford analysis, A PhD thesis in Mathematics, Ghent University, 2008.
[34] J. Ryan and W. Sprößig, Clifford Algebras and their Applications in Mathematical Physics, Volume 2: Clifford Analysis, Springer Science and Business Media, 2000.

[35] J. Saillard and G. Bunel, Apport des fonctions sphéroïdales pour l’estimation des paramètres d’une cible radar. 12ème Colloque Gretsi-Juan-Les-Pins, 12-19 Juin 1989, 4 pages.

[36] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives Theory and applications, Gordon and Breach Science Publisher, Amsterdam, The Netherlands, 1993.

[37] J. A. Stratton, Spheroidal functions. Physics, 21 (1935), 51-56.

[38] J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little and F. J. Corbato, Spheroidal wave functions, New York, Wiley, 1956.

[39] E. M. Stein and G. Weiss, Introduction to Fourier Analysis On Euclidiens Spaces. Princeton, New Jersey Princeton University Press, 1971.

[40] J. Winkler, A uniqueness theorem for monogenic functions, Annales Academiæ Scientiarum Fennicæ, Series A. I. Mathematica, 18 (1993), 105-116.

[41] J. Zhao and L. Peng, Clifford Algebra-valued Admissible Wavelets Associated to More than 2-dimensional Euclidean Group with Dilations. in Operator Theory: Advances and Applications, Vol. 167, 183190, 2006 Birkhäuser Verlag, Basel, Switzerland.