Zero Sets of Solutions to Semilinear Elliptic Systems of First Order

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Abstract
Consider a nontrivial solution to a semilinear elliptic system of first order with smooth coefficients defined over an n-dimensional manifold. Assume the operator has the strong unique continuation property. We show that the zero set of the solution is contained in a countable union of smooth \((n - 2)\)-dimensional submanifolds. Hence it is countably \((n - 2)\)-rectifiable and its Hausdorff dimension is at most \(n - 2\). Moreover, it has locally finite \((n - 2)\)-dimensional Hausdorff measure. We show by example that every real number between 0 and \(n - 2\) actually occurs as the Hausdorff dimension (for a suitable choice of operator). We also derive results for scalar elliptic equations of second order.

Mathematics Subject Classification: 35B05

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0 Introduction
Many important geometric objects are defined as zero sets of solutions of certain elliptic differential operators of first order. The most prominent classical example is provided by algebraic geometry. Analytic varieties are zero sets of holomorphic functions which in turn are characterized by the Cauchy-Riemann equations. More recently zero sets of spinors satisfying a Dirac equation have become important. In \(\mathbb{R}^4\) pseudoholomorphic curves in symplectic 4-manifolds have been constructed using zero sets of harmonic spinors. Spinors have become a tool to construct conformal immersions of surfaces in \(\mathbb{R}^3\) with prescribed mean curvature. Here the zeros are the bad points where the construction does not work. All this indicates that a general study of the zero set of solutions of first order elliptic equations should be useful and important.

Studying the structure of the zero set \(\mathcal{N}\) of such a solution splits into two quite different problems. To see this decompose \(\mathcal{N}\) into the set of zeros of finite order, \(\mathcal{N}_{f\text{in}}\), and those of infinite order, \(\mathcal{N}_{\infty}\). There is a vast literature concerned with \(\mathcal{N}_{\infty}\), the upshot being that it is typically empty. An operator with \(\mathcal{N}_{\infty} = \emptyset\) for all its nontrivial solutions is said to have the strong unique continuation property. One knows classical criteria which ensure that an operator has this property, see e.g. \([8]\) Ch. IX. It seems that all operators appearing “naturally” in geometry are of this type \([13]\). For example Dirac operators (in the most general sense) fall into this class.

Elliptic operators of first order do not always have the strong unique continuation property, however. Here is an example. It is the reduction to first order of the second order example given in \([13]\) Ex. 1.11].
Example. Let $M = \mathbb{R}^2$. Let $\Delta_1 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ be the Laplace operator and let $\Delta_2 = -\frac{\partial^2}{\partial x^2} - \frac{1}{2} \frac{\partial^2}{\partial y^2}$, both operators acting on $C^\infty(\mathbb{R}^2, \mathbb{R})$. The fourth order operator $\Delta_1 \Delta_2$ satisfies the assumptions of a theorem by Alinhac [1] which tells us that there exist functions $u, a \in C^\infty(U, \mathbb{R})$, defined on a neighborhood $U \subset \mathbb{R}^2$ of 0, vanishing of infinite order at 0, but not identically zero, and solving

$$\Delta_1 \Delta_2 u = a \cdot u.$$ 

Now let $D_1 = d + \delta$ act on $C^\infty(\mathbb{R}^2, \bigoplus_{j=0}^2 \Lambda^j T^* \mathbb{R}^2)$. This operator is a square root of $\Delta_1$ in the sense that $D_1^2 \big|_{C^\infty(\mathbb{R}^2, \Lambda^0 T^* \mathbb{R}^2)} = \Delta_1$. Similarly, let $D_2$ acting on $C^\infty(\mathbb{R}^2, \bigoplus_{j=0}^2 \Lambda^j T^* \mathbb{R}^2)$ be a square root of $\Delta_2$. Then we have

$$D_1 D_1 D_2 D_2 u = au.$$ 

Therefore $\varphi = (u, D_2 u, D_2^2 u, D_1 D_2^2 u)$ solves the linear elliptic system

$$D \varphi = \begin{pmatrix} D_2 & -1 & 0 & 0 \\ 0 & D_2 & -1 & 0 \\ 0 & 0 & D_1 & -1 \\ -a & 0 & 0 & D_1 \end{pmatrix} \varphi = 0 \quad (1)$$

Since $u$ has a zero of infinite order at 0 the same is true for $\varphi$. Therefore the linear elliptic operator $D$ does not have the strong unique continuation property. Since every “component” in the matrix in (1) acts itself on sections of a 4-dimensional vector bundle, (1) amounts to a system of 16 equations when spelled out.

Such counterexamples always seem somewhat artificial. In interesting cases the strong unique continuation property is usually satisfied. Therefore we focus our attention on the other component of the zero set, $\mathcal{N}_{\text{fin}}$. Unless the operator has analytic coefficients or the underlying manifold is of dimension $n \leq 2$ surprisingly little seems to be known about it. Classical theorems on uniqueness in the Cauchy problem tell us that $\mathcal{N}_{\text{fin}}$ does not contain certain hypersurfaces. But this does not really mean much, $\mathcal{N}_{\text{fin}}$ could still be a very irregular set of any Hausdorff dimension.

It turns out that the zero sets are in general very irregular but also very well-behaved depending on the point of view. We will show by example in Theorem [1] that every closed subset of a submanifold of codimension 2 is the zero set of a solution for some first order elliptic operator. This operator has the strong unique continuation property. Hence the Hausdorff dimension of the zero set can be any real number between 0 and $n - 2$ where $n$ is the dimension of the underlying manifold.

On the other hand, the zero sets have the following regularity properties. Our Main Theorem says that they are contained in a countable union of smooth submanifolds of codimension 2. In particular, they are countably $(n - 2)$-rectifiable and the Hausdorff dimension is at most $n - 2$. Moreover, they have locally finite $(n - 2)$-dimensional Hausdorff measure. If the underlying manifold is a surface, $n = 2$, then $\mathcal{N}_{\text{fin}}$ is discrete. We give an upper bound for the Hausdorff measure in small balls in terms of the vanishing order of the solution. All this shows that one can apply methods from geometric measure theory to such zero sets. We hope that this will be useful in the future.

Our main tool is the Malgrange Preparation Theorem which states that the zero set of a smooth real-valued function vanishing of finite order can locally be described as the zero set of another function which is polynomial in one of its variables. From this one gets countable $(n - 1)$-rectifiability fairly easily. The difficulty in the proof is to show that the zero sets of the various components of our solution intersect in such a way that they decrease the dimension once more. This requires an algebraic study of certain “resultants”. This approach has first been used by the author in [1]
to investigate zero sets of Dirac operators. In this paper we extend those results in two respects. Firstly, we enlarge the class of operators from (linear) Dirac operators to semilinear elliptic operators of first order. This seems to be the largest natural class of operators for which our results can be expected to hold. Secondly, we get more precise information about the zero sets such as bounds on their Hausdorff measure.

By a suitable reduction of order we obtain corollaries for scalar elliptic equations of second order as well. Decompose the zero set $N$ of a solution, often also called its nodal set, into a smooth hypersurface and into the critical zero set $N_{\text{crit}}$ where the function and its gradient vanish simultaneously. We give simple new proofs of the following facts: $N$ is countably $(n-1)$-rectifiable with locally finite $(n-1)$-dimensional Hausdorff measure and $N_{\text{crit}}$ is countably $(n-2)$-rectifiable with locally finite $(n-2)$-dimensional Hausdorff measure. Again, we get upper bounds for the Hausdorff measure in small balls. These facts have been obtained by various authors over the time, some of them only very recently. It is amusing that they follow easily from our results on first order systems by a simple reduction of order.

The paper is organized as follows. In the first section we give the necessary definitions and a few examples. We state the Main Theorem and deduce some immediate corollaries. The second section contains the proof of the Main Theorem. In the third section we construct examples with very irregular zero sets. The last section contains the discussion of scalar equations of second order.

All differential operators in this paper are assumed to have $C^\infty$-coefficients.

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1 Semilinear Elliptic Differential Operators of First Order

Let $M$ be an $n$-dimensional connected differential manifold, let $E, F \to M$ be real vector bundles over $M$. The case of complex vector bundles is included in our discussion because we can simply forget the complex structures and regard the bundles as real vector bundles. Let $D : C^\infty(M, E) \to C^\infty(M, F)$ be a linear differential operator of first order. Here $C^\infty(M, E)$ and $C^\infty(M, F)$ denote the spaces of smooth sections in the bundles $E$ and $F$ resp. If we introduce local coordinates $x_1, \ldots, x_n$ on $M$ and trivialize the bundles, then $D$ takes the form

$$D = \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x_j} + B(x)$$

where $A_j$ and $B$ are smooth matrix-valued functions.

The principal symbol $\sigma_D$ of $D$ associates to each covector $\xi \in T^*_x M$ with base point $x$ a homomorphism $\sigma_D(\xi) : E_x \to F_x$ which is characterized as follows:

Choose a smooth function $f$ defined in a neighborhood of $x$ such that $f(x) = 0$ and $df(x) = \xi$. Take an arbitrary $\varphi \in E_x$ and extend it smoothly to a section $\Phi$ of $E$ in a neighborhood of $x$. Then

$$\sigma_D(\xi) \varphi = D(f \Phi)(x).$$

1Actually, this is not quite true. Once our Main Theorem is shown to hold for semilinear equations it is clear it also holds for the larger class of quasilinear equations. The obvious details are left to the reader.
It is easy to see that \( \sigma_D(\xi) \) does not depend on the choices of \( f \) and \( \Phi \). In coordinates, for \( \xi = \sum_{j=1}^{n} \xi_j dx_j \), this homomorphism is given by the matrix

\[
\sigma_D(\xi) = \sum_{j=1}^{n} A_j(x) \xi_j.
\]

Very frequently in the literature there is an additional factor of \( i \) in the definition of the principal symbol. But since we are dealing with real bundles and operators introducing a factor of \( i \) would seem somewhat artificial. The only disadvantage of our convention is a minus sign in the formula for the symbol of the formally adjoint operator, \( \sigma_D(\xi) = -\sigma_D(\xi)^* \). But this will be of no relevance to our discussion.

Note that the map \( T^*_x M \to \text{Hom}(E_x, F_x), \xi \mapsto \sigma_D(\xi) \), is linear. This is a special property of first order operators.

For a section \( \varphi \in C^\infty(M, E) \) and a function \( f \in C^\infty(M, \mathbb{R}) \) we have the formula

\[
D(f \varphi) = f D(\varphi) + \sigma_D(df) \varphi.
\]

The operator \( D \) is called \textit{elliptic} if the symbol \( \sigma_D(\xi) \) is an isomorphism for all \( \xi \neq 0 \). In particular, \( E \) and \( F \) must then have the same rank.

Now let \( V : E \to F \) be a smooth fiber-preserving map, i.e. \( V(E_x) \subset F_x \) for all \( x \in M \). We say that \( V \) respects the zero section if \( V(0) = 0 \) for \( 0 \in E_x \) and all \( x \). Of course, this is automatic if \( V \) is fiberwise linear.

An operator of the form

\[
L = D + V : C^\infty(M, E) \to C^\infty(M, F)
\]

where \( D \) is a linear first order differential operator is called a \textit{semilinear} differential operator of first order. We define \( \sigma_L := \sigma_D \) as the principal symbol of \( L \). We say \( L \) is \textit{elliptic} if \( D \) is and we say \( L \) respects the zero section if \( V \) does.

Note that the decomposition \( L = D + V \) of a semilinear operator is unique up to linear zero-order terms. Hence all concepts are well-defined.

For a section \( \varphi \in C^\infty(M, E) \) we define its \textit{zero set}

\[
\mathcal{N}(\varphi) = \{ x \in M \mid \varphi(x) = 0 \}.
\]

We say that \( \varphi \) vanishes at \( x \) of order \( k \) if when expressed in a local trivialization all derivatives up to order \( k - 1 \) of all components \( \varphi_i, \varphi = (\varphi_1, \ldots, \varphi_N) \), vanish at \( x \),

\[
\frac{\partial^m \varphi_j}{\partial x_{j_1} \cdots \partial x_{j_m}}(x) = 0,
\]

for \( 1 \leq i \leq N, 1 \leq j_\nu \leq n, \) and \( 0 \leq m \leq k - 1 \). This condition is independent of the choice of coordinates and local trivialization.

We split the zero set into two parts,

\[
\mathcal{N}(\varphi) = \mathcal{N}_{fin}(\varphi) \cup \mathcal{N}_\infty(\varphi),
\]

where \( \mathcal{N}_{fin}(\varphi) \) is the set of zeros of \( \varphi \) of finite order, i.e. the set of those zeros for which there exists \( k \in \mathbb{N} \) such that \( \varphi \) vanishes of order \( k \) but not of order \( k + 1 \). Accordingly, \( \mathcal{N}_\infty(\varphi) \) is set of zeros of infinite order.

A differential operator \( L \) respecting the zero section is said to have the \textit{strong unique continuation property} if for all local solutions \( \varphi \) of \( L \varphi = 0 \) either \( \mathcal{N}_\infty(\varphi) = \emptyset \) or \( \varphi \equiv 0 \).

If \( L \) is a semilinear elliptic differential operator of first order on a connected 1-dimensional manifold, then the uniqueness theorem for ordinary differential equations tells us that \( \mathcal{N}(\varphi) = \emptyset \) unless \( \varphi \equiv 0 \). In particular, \( L \) has the strong unique
continuation property. In dimension \( n \geq 2 \), the strong unique continuation property sometimes fails even for linear elliptic operators of first order as we have seen in the introduction. There is a vast literature on conditions on differential operators which imply the strong unique continuation property. Fortunately, operators arising from geometric problems always seem to satisfy it \[15\].

**Example.** Let the underlying manifold \( M \) carry a Riemannian metric \( g \). If the principal symbol \( \sigma_D \) of a linear differential operator \( D : C^\infty(M, E) \to C^\infty(M, E) \) of first order satisfies the Clifford relations

\[
\sigma_D(\xi) \circ \sigma_D(\eta) + \sigma_D(\eta) \circ \sigma_D(\xi) + 2g(\xi, \eta) \cdot \text{Id}_E = 0
\]

for all \( \xi, \eta \in T^*_x M, x \in M \), then \( D \) is called a Dirac operator. If, in addition, \( V \) respects the zero section, then we call \( L = D + V \) a semilinear Dirac operator.

Such a Dirac operator is certainly elliptic because the Clifford relations imply \( \sigma_D(\xi)^2 = -|\xi|^2 \cdot \text{Id}_E \).

Moreover, solutions \( \varphi \) of a semilinear Dirac equation \( L\varphi = 0 \) locally satisfy a differential inequality

\[
|D^2\varphi| = |D(V(\varphi))| \leq C \cdot (|\varphi| + |\nabla \varphi|).
\]

Since \( D^2 \) is an elliptic differential operator of second order with scalar symbol Aronszajn’s theorem \[3\] applies and tells us that \( L \) has the strong unique continuation property.

**Example.** Let \( M \) be an oriented surface equipped with a spin structure. Let \( D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M) \) be the Dirac operator acting on spinors and let \( H : M \to \mathbb{R} \) be a smooth function. Define \( V : \Sigma M \to \Sigma M \) by

\[
V(\varphi) = -H|\varphi|^2\varphi.
\]

Then \( L = D + V \) is a semilinear Dirac operator (which respects the zero section).

This operator has attracted much attention in recent years because the solutions of \( L\varphi = 0 \) give rise to conformal immersions of the universal cover of \( M - \mathcal{N}(\varphi) \) into \( \mathbb{R}^3 \) with mean curvature \( H \). This can be regarded as a generalization of the classical Weierstrass representation of minimal surfaces, see e.g. \[6, 11, 16, 18\].

The operator \( L \) has the strong unique continuation property. Corollary \[4\] to our Main Theorem then says that \( \mathcal{N}(\varphi) \) is discrete. Thus \( \varphi \) defines a multivalued immersion of \( M \) into \( \mathbb{R}^3 \) branched along the discrete set \( \mathcal{N}(\varphi) \).

In this paper we study the set \( \mathcal{N}_{fin}(\varphi) \) for solutions of arbitrary semilinear elliptic operators of first order which respect the zero section. For operators having the strong unique continuation property such as semilinear Dirac operators this means that we will be able to to control all of \( \mathcal{N}(\varphi) \).

Recall that a subset of \( \mathbb{R}^n \) is called countably \( k \)-rectifiable if it can be written as a countable union of images under Lipschitz maps of bounded closed subsets of \( \mathbb{R}^k \), c.f. \[8\]. A subset of a manifold is countably \( k \)-rectifiable if it is so in coordinate charts.

In fact, we will prove something stronger rather than countable rectifiability. Therefore we make the following

**Definition.** A subset of a differential manifold is called countably \( k-C^\infty \)-rectifiable if it is contained in a countable union of smooth \( k \)-dimensional submanifolds.

Of course, a set which is countably \( k-C^\infty \)-rectifiable is also countably \( k \)-rectifiable.
Also recall the definition of Hausdorff measure density [8, 2.10.19]. Let $\alpha(m)$ denote the $m$-dimensional volume of the unit ball in $\mathbb{R}^m$. Denote the $m$-dimensional Hausdorff measure by $\mathcal{H}^m$. If $N$ is a subset of a Riemannian manifold and $p \in N$, then the limit
\[
\Theta^*(N, p) = \limsup_{r \to 0} \frac{\mathcal{H}^m(N \cap B(p, r))}{\alpha(m) r^m} \in [0, \infty]
\]
is called $m$-dimensional upper Hausdorff density of $N$ at $p$.

If for example $N$ is an $m$-dimensional submanifold, then $\Theta^*(N, p) = 1$.

**Main Theorem.** Let $M$ be a connected $n$-dimensional differential manifold. Let $L$ be a semilinear elliptic differential operator defined over $M$ which respects the zero section. Let $\varphi \not\equiv 0$ satisfy $L\varphi = 0$.

Then $N_{\text{fin}}(\varphi)$ is a countably $(n-2)$-$C^\infty$-rectifiable set. At each point $p \in N_{\text{fin}}(\varphi)$ the $(n-2)$-dimensional upper Hausdorff density has a bound
\[
\Theta^{n-2}(N(\varphi), p) \leq C(n) k^3
\]
where $C(n)$ is a constant depending only on the dimension $n$ and $k$ is the order of vanishing of $\varphi$ at $p$.

In particular, we have for the Hausdorff dimension
\[
\dim(N_{\text{fin}}(\varphi)) \leq n - 2
\]
and if $n = 2$, then $N_{\text{fin}}(\varphi)$ is a discrete set.

The proof will deliver an explicit value for $C(n)$. For example, we can take $C(n) = \frac{2^{n-3} n (n-1)}{\alpha(n-2)}$. This constant is probably not optimal.

The condition that $L$ respect the zero section is obviously necessary for the theorem to hold as one can see already in the 1-dimensional case. There are many solutions of ordinary differential equations of first order not respecting the zero section which do have zeros. Of course, if the ordinary differential equation respects the zero section, i.e. it is of the form $\frac{du}{dt} + F(t, u(t)) = 0$ with $F(t, 0) = 0$, then since $u_0 \equiv 0$ is a solution, a nontrivial solution $u$ cannot have any zeros by the uniqueness theorem.

In [3] it was shown that the bound $n - 2$ in the Main Theorem is sharp already in the class of linear Dirac operators. In Section 3 we will see by example that $N_{\text{fin}}(\varphi)$ can be very irregular and can have any real number $d \in [0, n - 2]$ as its Hausdorff dimension.

Combining the Main Theorem with the strong unique continuation property yields the following corollary which applies in particular to semilinear Dirac operators.

**Corollary 1.** Let $M$ be a connected $n$-dimensional differential manifold. Let $L$ be a semilinear elliptic differential operator defined over $M$ which respects the zero section and which has the strong unique continuation property. Let $\varphi \not\equiv 0$ satisfy $L\varphi = 0$.

Then $N(\varphi)$ is countably $(n-2)$-$C^\infty$-rectifiable and has locally finite $(n-2)$-dimensional Hausdorff measure. In particular, we have for the Hausdorff dimension
\[
\dim(N(\varphi)) \leq n - 2
\]
and if $n = 2$, then $N(\varphi)$ is a discrete set.

In the linear case the difference of two solutions is again a solution, so that the Main Theorem can be regarded as a strong version of the uniqueness part in the Cauchy problem.
Corollary 2. Let $M$ be a connected $n$-dimensional differential manifold. Let $A \subset M$ be a closed subset of Hausdorff dimension $\dim A > n - 2$. Let $D$ be a linear elliptic differential operator of first order defined over $M$ which has the strong unique continuation property. Let $\varphi_1$ and $\varphi_2$ satisfy $L\varphi_i = 0$.

If $\varphi_1|_A = \varphi_2|_A$, then $\varphi_1 = \varphi_2$ on all of $M$.

Here $A$ replaces the hypersurface in the classical Cauchy problem.

Applications of the Main Theorem to scalar elliptic equations of second order will be given in Section 4. We conclude this section with a corollary for a very special but important elliptic second order system.

Corollary 3. Let $M$ be a complete connected $n$-dimensional Riemannian manifold. Let $\Delta = d\delta + \delta d$ be the Laplace-Beltrami operator acting on $p$-forms. Let $\omega \neq 0$ be a square-integrable harmonic $p$-form, $\Delta \omega = 0$.

Then $N(\omega)$ is countably $(n - 2)$-$C^\infty$-rectifiable and has locally finite $(n - 2)$-dimensional Hausdorff measure. In particular, the Hausdorff dimension of $N(\omega)$ is at most $n - 2$ and if $n = 2$, then $N(\omega)$ is discrete.

Note that Corollary 3 fails if one drops the assumption of square-integrability. For example, $\omega = x_1 dx_1 \wedge \ldots \wedge dx_n$ is a harmonic $p$-form on $\mathbb{R}^n$ but its zero set has codimension 1. The corollary also fails if one replaces harmonic forms by eigenforms for positive eigenvalues.

Proof of Corollary 3. By [4, Thm. 26] harmonic $L^2$-forms are closed and coclosed, $(d + \delta)\omega = 0$.

The operator $d + \delta$ is a Dirac operator. Thus it has the strong unique continuation property, $N(\omega) = \emptyset$, and the Main Theorem gives the result for $\tilde{N}(\omega) = N_{fin}(\omega)$.

\section{The Proof}

In this section we prove the Main Theorem. We first consider a very special class of differential operators. Let $X$, $E$, and $F$ be real vector spaces with $\dim(X) = n$ and $\dim(E) = \dim(F) = N$. Let $\sigma : X^* \to \text{Hom}(E, F)$ be a linear map. This induces a differential operator of first order $\sigma(\partial) : C^\infty(X, E) \to C^\infty(X, F)$ as follows: Choose a basis $e_1, \ldots, e_n$ of $X$, let $e_1^*, \ldots, e_n^*$ be the dual basis and put

$$\sigma(\partial) = \sum_{j=1}^n \sigma(e_j^*) \partial e_j.$$ 

This definition is easily seen to be independent of the choice of basis $e_1, \ldots, e_n$. Such an operator is called an operator with constant coefficients. Of course, the principal symbol of $\sigma(\partial)$ is precisely given by $\sigma$. We now look at the zero set of polynomial solutions of elliptic operators with constant coefficients. For elements $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ we use the notation $(x_2, \ldots, x_n) = x'$ and denote the projection $x \mapsto x'$ by $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$.

Lemma 1. Let $\sigma(\partial) : C^\infty(\mathbb{R}^n, \mathbb{R}^N) \to C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ be an elliptic first order differential operator with constant coefficients. Let $\varphi = (\varphi_1, \ldots, \varphi_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ be a solution of $\sigma(\partial)\varphi = 0$ such that all components $\varphi_\nu$ are homogeneous polynomials of the same degree $k$ of the form

$$\varphi_\nu(x_1, x') = \alpha_\nu \left( x_1^k + \sum_{j=0}^{k-1} a_{\nu,j}(x') x_1^j \right),$$
where \( \alpha_1, \ldots, \alpha_N \in \mathbb{R} \) and \( u_{\nu,j} \) are homogeneous polynomials of degree \( k-j \). Then:

If \( \pi(\mathcal{N}(\varphi)) = \mathbb{R}^{n-1} \), then \( \varphi \equiv 0 \).

**Proof.** We write \( \varphi \) as a polynomial in \( x_1 \) with vector-valued coefficients

\[
\varphi(x) = \sum_{j=0}^{k} Y_j(x') \cdot x_1^j
\]

where \( Y_k(x') = (\alpha_1, \ldots, \alpha_N) \) and \( Y_j(x') = (\alpha_1 u_{1,j}(x'), \ldots, \alpha_N u_{N,j}(x')) \), \( j = 0, \ldots, k-1 \).

We define an elliptic differential operator with constant coefficients by

\[
D : C^\infty(\mathbb{R}^{n-1}, \mathbb{R}^N) \to C^\infty(\mathbb{R}^{n-1}, \mathbb{R}^N),
\]

\[
D = -\sigma(e_1^*)^{-1} \sum_{j=2}^{n} \sigma(e_j^*) \frac{\partial}{\partial x_j}.
\]

We compute

\[
0 = \sigma(e_1^*)^{-1} \sigma(\partial) \varphi = \left( \frac{\partial}{\partial x_1} - D \right) \left( \sum_{j=0}^{k} Y_j(x') \cdot x_1^j \right)
\]

\[
= \sum_{j=0}^{k-1} ((j+1)Y_{j+1}(x') - DY_j(x')) x_1^j
\]

We conclude \( Y_{j+1} = \frac{1}{j+1}DY_j \) and hence \( Y_j = \frac{1}{j!}D^jY_0 \). Thus we can rewrite (3) as

\[
\varphi(x) = \sum_{j=0}^{k} \frac{1}{j!}D^jY_0(x') \cdot x_1^j
\]

(4)

The assumption \( \pi(\mathcal{N}(\varphi)) = \mathbb{R}^{n-1} \) means that for any \( x' \in \mathbb{R}^{n-1} \) we can find an \( x_1(x') \in \mathbb{R} \) such that

\[
\sum_{j=0}^{k} \frac{x_1(x')^j}{j!}D^jY_0(x') = 0
\]

(5)

Roots of polynomials do not depend smoothly on the coefficients of the polynomial everywhere but smoothness fails only when multiple zeros branch to distinct zeros. Hence we can assume that \( x_1(x') \) depends smoothly on \( x' \) on a nonempty open subset of \( \mathbb{R}^{n-1} \). On this subset we apply \( D \) to equation (5) and obtain

\[
0 = D \sum_{j=0}^{k} \frac{x_1(x')^j}{j!}D^jY_0(x')
\]

\[
= \sum_{j=0}^{k-1} \frac{x_1(x')^j}{j!}D^{j+1}Y_0(x') - \sum_{j=1}^{k} \sigma(e_1^*)^{-1} \sigma \left( D \left( \frac{x_1(x')^j}{j!} \right) \right) D^jY_0(x')
\]

\[
= (1 - \sigma(e_1^*)^{-1} \sigma(dx_1(x'))) \sum_{j=0}^{k-1} \frac{x_1(x')^j}{j!}D^{j+1}Y_0(x')
\]

(6)

Now observe

\[
\det[1 - \sigma(e_1^*)^{-1} \sigma(dx_1(x'))] = \det[\sigma(e_1^*)]^{-1} \det[\sigma(e_1^*) - \sigma(dx_1(x'))]
\]

\[
= \det[\sigma(e_1^*)]^{-1} \det[\sigma(e_1^* - dx_1(x'))]
\]

\[
\neq 0
\]
since $e_1^+ - dx_1(x') \neq 0$ and $\sigma(\partial)$ is elliptic. Hence (5) yields

$$\sum_{j=0}^{k-1} \frac{x_1(x')^j}{j!} D^{j+1}Y_0(x') = 0$$

(7)

Equation (7) is the same as (5) with $k$ replaced by $k - 1$ and $Y_0$ replaced by $DY_0$. Repeating this inductively we eventually get

$$0 = D^kY_0(x') = k!Y_k(x') = k!(\alpha_1, \ldots, \alpha_N).$$

Hence $\varphi \equiv 0$. □

To study local properties of differential operators it can be useful to approximate the operator in a neighborhood of a given point by an operator with constant coefficients. This freezing of coefficients is done as follows.

Let $L : C^\infty(M, E) \to C^\infty(M, F)$ be a semilinear differential operator which respects the zero section. Let $p \in M$ be a point. Then the principal symbol $\sigma_{L,p} : T^*_pM \to \text{Hom}(E_p, F_p)$ gives rise to a linear differential operator with constant coefficients

$$\hat{L}_p = \sigma_{L,p}(\partial) : C^\infty(T_pM, E_p) \to C^\infty(T_pM, F_p)$$

defined on the tangent space of $M$ at $p$.

To a section $\varphi \in C^\infty(M, E)$ and a point $p \in M$ we associate a homogeneous polynomial $\hat{\varphi}_p \in C^\infty(T_pM, E_p)$ as follows. Choose local coordinates near $p$ and look at the Taylor series expansion of $\varphi$ at $p$. Then $\hat{\varphi}_p$ is the homogeneous part of lowest degree in the expansion which does not identically vanish,

$$\varphi(p + \xi) = \hat{\varphi}_p(\xi) + \text{terms of higher order in } \xi$$

One checks that this gives rise to a well-defined homogeneous polynomial on $T_pM$ with coefficients in $E_p$. E.g., if $\varphi(p) \neq 0$, then $\hat{\varphi}_p$ is constant, $\hat{\varphi}_p(\xi) = \varphi(p)$. If $\varphi$ vanishes of first order at $p$, then $\hat{\varphi}_p(\xi) = \nabla_\xi \varphi$ for any connection $\nabla$ on $E$. If $\varphi$ vanishes of infinite order at $p$, then we set $\hat{\varphi}_p \equiv 0$.

The next lemma tells us how to pass from an arbitrary solution of a semilinear first order equation to a polynomial solution of an operator with constant coefficients.

**Lemma 2.** Let $M$ be an $n$-dimensional differential manifold. Let $E$ and $F$ be real vector bundles over $M$. Let $p \in M$ be a point. Let $L : C^\infty(M, E) \to C^\infty(M, F)$ be a semilinear elliptic differential operator which respects the zero section. Then:

If $\varphi \in C^\infty(M, E)$ satisfies

$$L\varphi = 0,$$

then $\hat{\varphi}_p \in C^\infty(T_pM, E_p)$ satisfies

$$\hat{L}_p \hat{\varphi}_p = 0.$$

**Proof.** Expand $\varphi$ and the coefficients of $L$ in their Taylor series at $p$, use $V(\varphi) = O(|\varphi|)$ and take the lowest order term of $L\varphi$. □

**Lemma 3.** Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0. Let $f : U \to \mathbb{R}$ be a smooth function vanishing of finite order $k$ at 0, but not of order $k + 1$. Write $\mathcal{N}(f) = f^{-1}(0)$ for the zero set.

Then for sufficiently small $r > 0$ the set $\mathcal{N}(f) \cap B(0, r)$ is countably $(n - 1)$-$C^\infty$-rectifiable and

$$\Theta^{n-1}(\mathcal{N}(f), 0) \leq C(n)k.$$
Here $C(n)$ is a constant depending only on $n$. For example, $C(n) = \frac{2^{n-1}}{\alpha(n-1)}$ works where $\alpha(m)$ is the volume of the unit ball in $\mathbb{R}^m$. See [3] for the definition of $\Theta^{*n-1}$.

The proof will show a somewhat stronger statement then just countably $(n-1)$-$C^\infty$-rectifiability of $\mathcal{N}(f) \cap B(0, r)$. Namely, for a generic choice of cartesian coordinate system on $\mathbb{R}^n$ the following is true:

Pick any index $j \in \{1, \ldots, n\}$. Then $B(0, r)$ can be written as a countable union of subsets in which the $x_j$-component of the points in $\mathcal{N}(f)$ is a smooth function defined on bounded subsets of the hyperplane perpendicular to the $x_j$-axis. In slightly other words, near 0, $\mathcal{N}(f)$ is contained in a countable union of smooth graphs over the hyperplane $e_j$.

Case $n=2$

Fig. 1

Proof. Since $f$ vanishes of finite order $k$ at 0, we can choose an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ such that $D^k f(0)(e_i, \ldots, e_i) \neq 0$ for all $i = 1, \ldots, n$. We use cartesian coordinates on $\mathbb{R}^n$ with respect to which $e_1, \ldots, e_n$ are the standard basis.

By Malgrange’s Preparation Theorem [17, Ch. V] we can write $f$ in a neighborhood of 0 as

$$f(x) = v(x) \left( x_1^k + \sum_{j=0}^{k-1} u_j(x') x_j^j \right)$$

where $v, u_0, \ldots, u_{k-1}$ are smooth functions defined in a neighborhood of 0, $v$ nowhere vanishing and $u_j$ vanishing at 0 of order $k - j$ at least. So $\mathcal{N}(f)$ is given near 0 as the zero locus of a polynomial in the $x_1$-variable of degree $k$. Hence if we define the $m$-dimensional cube $W^m(0, r) = [-r, r] \times \cdots \times [-r, r]$, then under the projection onto the $x'$-hyperplane each $x' \in W^{n-1}(0, r)$ has at most $k$ preimages contained in $\mathcal{N}(f) \cap W^n(0, r)$.

Now recall the following known facts about how roots of polynomials depend on the coefficients of the polynomial [21, L. 6]: $\mathbb{R}^k$ can be covered by countably many bounded closed sets $\mathbb{R}^k = \bigcup A_\mu$ such that number of real roots of the polynomial $F_u(t) = t^k + \sum_{j=0}^{k-1} u_j t^j$ is the same number $k_\mu \in \{0, \ldots, k\}$ for all $u = (u_0, \ldots, u_{k-1}) \in A_\mu$. Moreover, there exist smooth functions $\xi_{\mu, i} : \mathbb{R}^k \to \mathbb{R}$, $i = 1, \ldots, k_\mu$, such that $\xi_{\mu, 1}(u) < \cdots < \xi_{\mu, k_\mu}(u)$ are precisely the roots of $F_u$ whenever $u \in A_\mu$. 

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This together with (8) shows that \( \mathcal{N}(f) \cap W^n(0, r) \) is contained in the union of the graphs of the smooth functions \( x_1 = \xi_{\nu,i}(u(x')) \), \( x' \in W^{n-1}(0, r) \). Hence \( \mathcal{N}(f) \) is countably \( (n-1)\text{-}C^\infty \)-rectifiable near 0.

To estimate the Hausdorff density we note that by the choice of the coordinate system we can apply Malgrange’s preparation theorem to any of the variables \( x_j \), not just \( x_1 \). Therefore \( \mathcal{N}(f) \cap W^n(0, r) \) contains for each \( x' \in W^{n-1}(0, r) \) at most \( k \) preimages under each projection to the hyperplane \( \mathbb{R}^{n-1} = e_1^* \subset \mathbb{R}^n \). Now the second estimate in [3, 3.2.27] yields for small \( r > 0 \)

\[
\mathcal{H}^{n-1}(\mathcal{N}(f) \cap W^n(0, r)) \leq nk \cdot \text{vol}(W^{n-1}(0, r)) = nk(2r)^{n-1}.
\]

Thus

\[
\Theta^{n-1}(\mathcal{N}(f), 0) = \limsup_{r \to 0} \frac{\mathcal{H}^{n-1}(N \cap \mathbb{B}(p, r))}{\alpha(n-1)r^{n-1}}
\]

\[
\leq \limsup_{r \to 0} \frac{nk(2r)^{n-1}}{\alpha(n-1)r^{n-1}}
\]

\[
= \frac{n2^{n-1}}{\alpha(n-1)} k
\]

\[\square\]

With these preparations we can now proceed to the proof of the Main Theorem.

**Proof of Main Theorem.** Let \( L : C^\infty(M, E) \to C^\infty(M, F) \) be a semilinear elliptic operator respecting the zero section and let \( \varphi \in C^\infty(M, E) \) solve \( L\varphi = 0 \). Let \( p \in N_{fin}(\varphi) \). We will show that \( \mathcal{N}(\varphi) \) is countably \( (n-2)\text{-}C^\infty \)-rectifiable near \( p \).

Since \( \varphi \) vanishes at \( p \) of some order \( k \) but not of order \( k+1 \) we can choose the coordinates near \( p = 0 \) in such a way that the homogeneous part of order \( k \) has the form

\[\hat{\varphi}_p(x) = a \cdot x_1^k + \text{ terms of lower degree in } x_1\]

with \( a \in \mathbb{E}_p, a \neq 0 \). We trivialize \( E \) near \( p \) in such a way that \( a = (1, \ldots, 1) \). Now Malgrange’s Preparation Theorem says that there are smooth real-valued functions \( v_\nu \) and \( u_{\nu, j} \) defined in a neighborhood of \( p = 0 \) such that

\[\varphi_\nu(x) = v_\nu(x) \left( x_1^k + \sum_{j=0}^{k-1} u_{\nu, j}(x') x_1^j \right)\]

and \( v_\nu(0) = 1 \). The functions \( u_{\nu, j} \) vanish of order \( k - j \) at \( x' = 0 \). We see that in a neighborhood of 0 the components \( \varphi_\nu \) have the same zero set as the \( x_1 \)-polynomials \( x_1^k + \sum_{j=0}^{k-1} u_{\nu, j}(x') x_1^j \). Lemma 3 already tells us that \( \mathcal{N}(\varphi_\nu) \) is countably \( (n-1)\text{-}C^\infty \)-rectifiable near 0.

If we denote the homogeneous part of degree \( k \) of \( u_{\nu, j} \) by \( \hat{u}_{\nu, j} \), then we have

\[\hat{\varphi}_{\nu,p}(x) = x_1^k + \sum_{j=0}^{k-1} \hat{u}_{\nu, j}(x') x_1^j.\]

By Lemma 2 \( \hat{\varphi}_p \) solves

\[L_p\hat{\varphi}_p = 0.\]

Since, by assumption, \( \hat{\varphi}_p \neq 0 \) Lemma 1 says \( \pi(\mathcal{N}(\hat{\varphi}_p)) \neq \mathbb{R}^{n-1} \). Recall that \( \pi : \mathbb{R}^n = T_p M \to \mathbb{R}^{n-1}, \pi(x) = x', \) is the projection on the orthogonal complement of the \( x_1 \)-axis. Hence there exists an \( x'_0 \) such that the \( x_1 \)-polynomials
\[ x_1^k + \sum_{j=0}^{k-1} \hat{u}_{\nu,j}(x')x_1^j, \nu = 1, \ldots, N, \] have no common root. In other words, we can find linear combinations

\[
\begin{align*}
\hat{F}(x) &= \sum_{\nu=1}^{N} A_{\nu} \left( x_1^k + \sum_{j=0}^{k-1} \hat{u}_{\nu,j}(x')x_1^j \right), \\
\hat{G}(x) &= \sum_{\nu=1}^{N} B_{\nu} \left( x_1^k + \sum_{j=0}^{k-1} \hat{u}_{\nu,j}(x')x_1^j \right)
\end{align*}
\]

which do not have a common root \( x_1 \) for \( x' = x_0' \).

Recall that two polynomials have a common root if and only if their resultant vanishes. Let us denote the resultant of the two \( x_1 \)-polynomials \( \hat{F} \) and \( \hat{G} \) by \( R_{\hat{F},\hat{G}}(x') \). It is a homogeneous polynomial in \( x' \) with \( R_{\hat{F},\hat{G}}(x_0') \neq 0 \). Moreover, \( R_{\hat{F},\hat{G}}(x') \) is the lowest order term in the Taylor expansion of the resultant \( R_{F,G}(x') \) of the \( x_1 \)-polynomials

\[
\begin{align*}
F(x) &= \sum_{\nu=1}^{N} A_{\nu} \left( x_1^k + \sum_{j=0}^{k-1} u_{\nu,j}(x')x_1^j \right), \\
G(x) &= \sum_{\nu=1}^{N} B_{\nu} \left( x_1^k + \sum_{j=0}^{k-1} u_{\nu,j}(x')x_1^j \right)
\end{align*}
\]

If \( x = (x_1, x') \in \mathcal{N}(\varphi) \) near \( p \), then \( F(x) = G(x) = 0 \) and hence \( R_{F,G}(x') = 0 \). In other words, near \( p, \pi(\mathcal{N}(\varphi)) \subset \mathcal{N}(R_{F,G}) \). For each \( x' \in \mathcal{N}(R_{F,G}) \) there are at most \( k \) roots \( x_1 \) such that \( \varphi(x_1, x') = 0 \). From the proof of Lemma 3 we know that \( \mathcal{N}(\varphi) \) is contained in a countable union of graphs of smooth functions defined on the \( x' \)-hyperplane. Hence \( \mathcal{N}(\varphi) \) is countably \((n-2)\)-\( C^\infty \)-rectifiable if \( \mathcal{N}(R_{F,G}) \) is.

From \( (\tilde{R}_{F,G})_0 = \tilde{R}_{F,G} \neq 0 \) we see that \( R_{F,G} \) vanishes at \( x' = 0 \) of finite order. Applying Lemma 3 once more we conclude that \( \mathcal{N}(R_{F,G}) \) is countably \((n-2)\)-\( C^\infty \)-rectifiable.

![Fig. 2](image-url)
The resultant $R_{\hat{\mathcal{F}}, \hat{\mathcal{C}}}$ is a homogeneous polynomial of degree $k^2$ in $x_2, \ldots, x_n$. By counting preimages and applying \[3.2.27\] as in the proof of Lemma 3 we obtain the following bound on the $(n-2)$-dimensional Hausdorff measure:

$$
\mathcal{H}^{n-2}(\mathcal{N}(\varphi) \cap W^n(0, r)) \leq \frac{1}{2} n(n-1) \cdot k \cdot k^2 \cdot \text{vol}(W^{n-2}(0, r))
$$

$$
= \frac{1}{2} n(n-1)k^3(2r)^{n-2}.
$$

Hence

$$
\Theta^{n-2}(\mathcal{N}(\varphi), p) \leq \frac{2n^3n(n-1)k^3}{\alpha(n-2)}.
$$

\[\square\]

3 An Example with a Wild Zero Set

In this section we want to give evidence that our Main Theorem is more or less all that one can say about local properties of zero sets of solutions to first order elliptic equations. The following theorem will show that the zero set can be extremely irregular and that its Hausdorff dimension can be any real number $d \in [0, n-2]$.

**Theorem 1.** Let $n \geq 3$. Let $A \subset \mathbb{R}^{n-2}$ be any closed subset. Then there exists a linear elliptic differential operator $D$ defined over $\mathbb{R}^n$ and a solution $\varphi$ of $D\varphi = 0$ such that

$$
\mathcal{N}(\varphi) = \{(0,0)\} \times A.
$$

Moreover, $D$ has the strong unique continuation property.

**Proof.** Let $d$ be exterior differentiation of differential forms on $\mathbb{R}^n$ and let $\delta$ be its adjoint operator (with respect to the Euclidean metric). Then

$$
D_1 = d + \delta : C^\infty(\mathbb{R}^n, \bigoplus_{j=0}^n \Lambda^j T^* \mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \bigoplus_{j=0}^n \Lambda^j T^* \mathbb{R}^n)
$$

is a Dirac operator. Let us denote the coordinates of $\mathbb{R}^n$ by $(x, y, z)$ with $x, y \in \mathbb{R}$ and $z \in \mathbb{R}^{n-2}$. Now $\varphi_1 = xy + ydx$ is a solution of $D_1 \varphi_1 = 0$ with $\mathcal{N}(\varphi_1) = \{(0,0)\} \times \mathbb{R}^{n-2}$.

Pick a smooth function $F : \mathbb{R}^{n-2} \to \mathbb{R}$ with $F^{-1}(0) = A$. The map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, $\Psi(x, y, z) = (x, y - F(z), z)$, is a diffeomorphism. This diffeomorphism maps the linear subspace $\{(0,0)\} \times \mathbb{R}^{n-2}$ to the submanifold $E = \{ (\xi, \eta, \zeta) \mid \xi = 0, \eta = -F(\zeta) \}$. By pulling back the Euclidean metric via $\Psi$ we obtain a new Riemannian metric on $\mathbb{R}^n$. Let $D_2$ be the corresponding Dirac operator with respect to this new metric, i.e. $D_2 = d + \delta_\Psi$ where $\delta_\Psi$ is the adjoint operator of $d$ with respect to the new metric. There exists an analogous solution $\varphi_2$ of $D_2 \varphi_2 = 0$ with $\mathcal{N}(\varphi_2) = E$.

Hence the operator

$$
D = \begin{pmatrix}
D_1 & 0 \\
0 & D_2
\end{pmatrix}
$$

acting on $C^\infty(\mathbb{R}^n, \bigoplus_{j=0}^n \Lambda^j T^* \mathbb{R}^n \oplus \bigoplus_{j=0}^n \Lambda^j T^* \mathbb{R}^n)$ has the solution $\varphi = (\varphi_1, \varphi_2)$ with zero set

$$
\mathcal{N}(\varphi) = \mathcal{N}(\varphi_1) \cap \mathcal{N}(\varphi_2) = \{(0,0)\} \times A.
$$

Since both $D_1$ and $D_2$ are Dirac operators and hence have the strong unique continuation property so has $D$. \[\square\]
4 Scalar Elliptic Equations of Second Order

In this section we study implications of our Main Theorem for scalar elliptic equations of second order. For a smooth real-valued function $u : M \to \mathbb{R}$ the critical zero set is defined by

$$N_{\text{crit}}(u) = \{ x \in M \mid u(x) = 0, du(x) = 0 \}$$

Obviously, $N(u) - N_{\text{crit}}(u)$ is a smooth hypersurface.

By a scalar semilinear elliptic differential operator of second order we mean a map

$$L : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$

which is locally of the form

$$Lu(x) = -\sum_{ij} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + F(x, u(x), du(x))$$

where $(a_{ij})$ is a symmetric and positive definite matrix and $F$ is smooth and respects the zero section, $F(x, 0, 0) = 0$.

**Theorem 2.** Let $M$ be a connected $n$-dimensional differential manifold. Let $L : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ be a scalar semilinear elliptic differential operator of second order. Let $u \in C^\infty(M, \mathbb{R})$, $u \not\equiv 0$, be a solution of $Lu = 0$.

Then $N(u)$ is countably $(n-1)$-C$^\infty$-rectifiable with locally finite $(n-1)$-dimensional Hausdorff measure. Moreover, $N_{\text{crit}}(u)$ is a countably $(n-2)$-C$^\infty$-rectifiable set with locally finite $(n-2)$-dimensional Hausdorff measure. In particular, we have for the Hausdorff dimension

$$\dim(N_{\text{crit}}(u)) \leq n-2$$

and if $n = 2$, then $N_{\text{crit}}(u)$ is a discrete set.

This theorem has been shown by Hardt and Simon using different methods in [12, Thm. 1.10] except for the statement that $N_{\text{crit}}$ has locally finite $(n-2)$-dimensional Hausdorff measure. This latter statement has been shown for $n = 3$ by M. and T. Hoffmann-Ostenhof and Nadirashvili [13]. The case $n \geq 4$ has been settled very recently by Hardt [11].

**Proof.** The principal symbol of $L$ defines a Riemannian metric on $M$ with respect to which $L$ can be written as

$$Lu = \Delta u + \tilde{F}(x, u, du)$$

where $\Delta = \delta d$ is the Laplace-Beltrami operator and $\tilde{F}$ is smooth and respects the zero section.

Let $E = \bigoplus_{j=1}^n \Lambda^j T^*M$ be the exterior form bundle. We define the smooth map $V : E \to E$ by

$$V \left( \sum_{j=0}^n \omega_j \right) = \tilde{F}(x, \omega_0, \omega_1) - \omega_1$$

where $\omega_j \in \Lambda^j T^*_x M$. Note that $V$ respects the zero section.

Now if $u$ solves $Lu = 0$, then $\omega = u + du \in C^\infty(M, E)$ solves $(d + \delta + V)(\omega) = 0$. The operator $D = d + \delta + V$ is a semilinear Dirac operator acting on $C^\infty(M, E)$. Thus Corollary 1 yields the statements on $N_{\text{crit}}(u) = N(\omega)$.

Since Dirac operators have the strong unique continuation property Lemma 8 shows that $N(u) = N_{\text{fin}}(u)$ has locally finite $(n-1)$-dimensional Hausdorff measure. □
Theorem 3. Let $M$ be a closed $n$-dimensional Riemannian manifold.

Then there exists a constant $C = C(M)$ such that for any eigenfunction $u \in C^\infty(M, \mathbb{R})$ of the Laplace operator, $\Delta u = \lambda u$, and any $p \in M$ we have

$$
\Theta^{n-1}(\mathcal{N}(u), p) \leq C \sqrt{\lambda},
\Theta^{n-2}(\mathcal{N}_{\text{crit}}(u), p) \leq C \lambda^{3/2}.
$$

Proof. A theorem of Donnelly and Fefferman [5, Thm. 1.1] tells us that there is a constant $C' = C'(M)$ such that $u$ can vanish only up to order $C' \sqrt{\lambda}$. Now the Main Theorem and Lemma 3 give the bounds on the Hausdorff density. □

This theorem gives us upper bounds on the Hausdorff measures of $\mathcal{N}(u)$ and $\mathcal{N}_{\text{crit}}(u)$ in small balls. Unfortunately, we have no explicit control on how small the radius of these balls must be such that the estimates hold. Therefore we cannot deduce bounds on $\mathcal{H}^{n-1}(\mathcal{N}(u))$ and $\mathcal{H}^{n-2}(\mathcal{N}_{\text{crit}}(u))$ in terms of a suitable power of $\lambda$. It is an old conjecture of Yau [20, Probl. 38] that

$$
\mathcal{H}^{n-1}(\mathcal{N}(u)) \leq C(M) \sqrt{\lambda}.
$$

In the case of analytic Riemannian metrics this has been shown by Donnelly and Fefferman [5, Thm. 1.2] whereas in the smooth case it is still open. See e.g. [12, Thm. 5.3] and [6, Cor. 1.3] for partial results. See also [10] for a bound on the Hausdorff measure of $\mathcal{N}_{\text{crit}}(u)$.

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