The Q-generating Function for Graphs with Application

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Abstract

For a simple connected graph $G$, the Q-generating function of the numbers $N_k$ of semi-edge walks of length $k$ in $G$ is defined by $W_Q(t) = \sum_{k=0}^{\infty} N_k t^k$. This paper reveals that the $Q$-generating function $W_Q(t)$ may be expressed in terms of the $Q$-polynomials of the graph $G$ and its complement $\overline{G}$. Using this result, we study some $Q$-spectral properties of graphs and compute the $Q$-polynomials for some graphs obtained from various graph operations, such as the complement graph of a regular graph, the join of two graphs and the (edge)corona of two graphs. As another application of the $Q$-generating function $W_Q(t)$, we also give a combinatorial interpretation of the $Q$-coronal of $G$, which is defined to be the sum of the entries of the matrix $(\lambda I_n - Q(G))^{-1}$. This result may be used to obtain the many alternative calculations of the $Q$-polynomials of the (edge)corona of two graphs. Further, we also compute the $Q$-generating functions of the join of two graphs and the complete multipartite graphs.

Keywords  Signless Laplacian matrix · $Q$-polynomial · $Q$-spectrum · $Q$-generating function · $Q$-coronal · Semi-edge walk

Mathematics Subject Classification 05C50 · 05C90

1 Introduction

Throughout this paper, we consider only simple connected graphs. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Two vertices $v_i$ and $v_j$ of $G$ are

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called adjacent, denoted by \( v_i \sim v_j \), if they are connected by an edge. The adjacency matrix \( A(G) \) of \( G \) is a square matrix of order \( n \), whose entry \( a_{i,j} \) is defined as follows: 
\[ a_{i,j} = 1 \text{ if } v_i \sim v_j, \text{ and } a_{i,j} = 0 \text{ otherwise.} \]
Let \( D(G) \) be the diagonal degree matrix of \( G \). The matrix \( Q(G) = D(G) + A(G) \) is called the signless Laplacian matrix of \( G \). The \( Q \)-spectrum of \( G \) is defined to

\[ S(G) = (q_1(G), q_2(G), \ldots, q_n(G)), \]

where \( q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \) are the eigenvalues of \( Q(G) \). They also are the roots of the \( Q \)-polynomial \( f_Q(\lambda) = \det(\lambda I_n - Q(G)) \) of \( G \). If it is clear from the context, then we use \( Q \) and \( q_i \) instead of \( Q(G) \) and \( q_i(G) \), respectively. Denote \( Q \)-polynomial of the complement graph \( \overline{G} \) of \( G \) by \( f_{\overline{G}}(\lambda) = \det(\lambda I_n - Q(\overline{G})) \). For more information on the \( Q \)-spectrum and \( Q \)-polynomial, we refer the reader to [1,5–10,16] and the references therein.

Let \( G \) be a simple connected graph and \( A(G) \) be its adjacency matrix. A walk (of length \( k \)) in \( G \) is an alternating sequence \( v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} \) of vertices \( v_1, v_2, \ldots, v_{k+1} \) and edges \( e_1, e_2, \ldots, e_k \) such that for any \( i = 1, 2, \ldots, k \) the vertices \( v_i \) and \( v_{i+1} \) are distinct end-vertices of the edge \( e_i \). It is well known [10] that the \((i, j)\)-entry of the matrix \( A(G)^k \) equals the number of walks of length \( k \) starting at vertex \( v_i \) and terminating at vertex \( v_j \). Let \( M_k \) denote the total number of all walks of length \( k \) in \( G \). \( H_G(t) = \sum_{k=0}^{\infty} M_k t^k \) is called the generating function of the numbers \( M_k \) of all walks of length \( k \) in \( G \). In [10], the generating function \( H_G(t) \) is expressed in terms of the characteristic polynomials of the graph \( G \) and its complement \( \overline{G} \), and many spectral properties are obtained. For example, the characteristic polynomials of some graphs are computed by employing the generating function \( H_G(t) \) in [10].

For a simple connected graph \( G \), let \( Q(G) \) be its signless Laplacian matrix. Similarly, a semi-edge walk (of length \( k \)) [6] in an (undirected) graph \( G \) is an alternating sequence \( v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1} \) of vertices \( v_1, v_2, \ldots, v_{k+1} \) and edges \( e_1, e_2, \ldots, e_k \) such that for any \( i = 1, 2, \ldots, k \) the vertices \( v_i \) and \( v_{i+1} \) are end-vertices (not necessarily distinct) of the edge \( e_i \). Let \( Q(G)^k = (q_{ij}^{(k)}) \) and \( N_k(i, j) \) denote the number of semi-edge walks of length \( k \) starting at vertex \( v_i \) and terminating at vertex \( v_j \) in \( G \). It is proved [6] that the \((i, j)\)-entry of the matrix \( Q(G)^k \) equals the number of semi-edge walks of length \( k \) starting at vertex \( v_i \) and terminating at vertex \( v_j \), that is, \( q_{ij}^{(k)} = N_k(i, j) \).

The \( Q \)-generating function of the numbers \( N_k \) of semi-edge walks of length \( k \) in \( G \) is defined to \( W_Q(t) = \sum_{k=0}^{\infty} N_k t^k \), where \( N_k \) denotes the total number of semi-edge walks of length \( k \) in \( G \). The following problem seems interesting:

\text{Study the } Q \text{-generating function } W_Q(t) \text{ for the numbers } N_k \text{ of semi-edge walks of length } k \text{ in } G \text{ and compute the } Q \text{-polynomials of some graphs by employing the } Q \text{-generating function } W_Q(t). \]

This paper reveals that the \( Q \)-generating function \( W_Q(t) \) may be expressed in terms of the \( Q \)-polynomials of the graph \( G \) and its complement \( \overline{G} \). Using this result, we obtain some \( Q \)-spectral properties of graphs and compute the \( Q \)-polynomials for some graphs obtained by the use of some operation on graphs, such as the complement of a graph, the join of two graphs and the (edge)corona of two graph.
As another application of the $Q$-generating function $W_Q(t)$, we also give a combinatorial interpretation of the $Q$-coronal of a graph $G$, which is defined to be the sum of the entries of the matrix $(\lambda J_n - Q(G))^{-1}$. This result may be used to obtain the many alternative calculations of the $Q$-polynomials of the (edge)corona of two graphs. Further, we also compute the $Q$-generating functions of the join of two graphs and complete multipartite graphs.

2 The $Q$-generating Functions and $Q$-polynomials of Graphs

For a simple connected graph $G$, Proposition 2.1 reveals that the $Q$-generating function $W_Q(t)$ may be expressed in terms of the $Q$-polynomials of the graph $G$ and its complement $\overline{G}$.

**Proposition 2.1** Let $G$ be a simple connected graph on $n$ vertices. Then, for any $-\frac{1}{q_1} < t < \frac{1}{q_1}$, we have

$$W_Q(t) = \frac{1}{t} \left( (-1)^n f_{\overline{G}} (n - 2 - \frac{t}{1}) f_Q \left( \frac{1}{t} \right) - 1 \right).$$

**Proof** The proof is totally similar to Theorem 1.11 in [10]. Let $B$ be a nonsingular $n$-by-$n$ square matrix and $J$ be a square matrix each of whose entries is 1. Then, for arbitrary number $x$,

$$\det(B + xJ) = \det B + x \sum(\text{adj} B),$$

where $\sum(K)$ denotes the sum of all entries of a matrix $K$ and $\text{adj} K$ denotes its adjoint matrix.

Now, from Theorem 4.1 in [6], one gets $N_k = \sum(Q^k)$. Note that, for $-\frac{1}{q_1} < t < \frac{1}{q_1}$,

$$\sum_{k=0}^{\infty} Q^k t^k = (I - tQ)^{-1} = \frac{\text{adj}(I - tQ)}{\det(I - tQ)}.$$

Thus, we obtain

$$W_Q(t) = \sum_{k=0}^{\infty} N_k t^k = \sum_{k=0}^{\infty} \sum(Q^k) t^k = \frac{\sum(\text{adj}(I - tQ))}{\det(I - tQ)}.$$

With $B = I - tQ$, $x = t$, formula (1) yields

$$\sum(\text{adj}(I - tQ)) = \frac{1}{t} \left( \det(I - tQ + tJ) - \det(I - tQ) \right)$$

$$= \frac{1}{t} \left( \det(((1 - (n - 2)t)I + tQ) - \det(I - tQ) \right),$$
where $Q = (n - 2)I + J - Q$ is the signless Laplacian matrix of the complement $\overline{G}$ of $G$. Hence,

$$W_Q(t) = \frac{1}{t} \left( \frac{\det((1 - (n - 2)t)I + t\overline{Q})}{\det(I - tQ)} - 1 \right)$$

$$= \frac{1}{t} \left( (-1)^n \frac{\det \left( (n - 2 - \frac{1}{t})I - \overline{Q} \right)}{\det \left( \frac{1}{t}I - Q \right)} - 1 \right)$$

$$= \frac{1}{t} \left( (-1)^n \frac{f_Q(n - 2 - \frac{1}{t})}{f_Q \left( \frac{1}{t} \right)} - 1 \right).$$

This completes the proof of Proposition 2.1. \hfill \Box

**Theorem 2.2** Let $G$ be an $r$-regular graph on $n$ vertices and $\overline{G}$ be its complement graph. Then

$$f_Q(\lambda) = (-1)^n \left( 1 + \frac{n}{n - 2 - 2r - \lambda} \right) f_Q(n - 2 - \lambda).$$

Moreover, if the signless Laplacian spectrum of $G$ is $2r, q_2, \ldots, q_n$, then the signless Laplacian spectrum of $\overline{G}$ is $2(n - r - 1), n - 2 - q_2, \ldots, n - 2 - q_n$.

**Proof** Since $G$ is an $r$-regular graph on $n$ vertices, then $\sum (A(G)^i) = nr^i$ for any integer $i$ (see [10]). Thus, the total number $N_k$ of semi-edge walks of length $k$ in $G$ equals

$$N_k = \sum (rI_n + A(G))^k = \sum_{i=0}^{k} \binom{k}{i} r^i (A(G))^{k-i} = \sum_{i=0}^{k} \binom{k}{i} nr^k = n(2r)^k.$$

Hence, for $|t| < \frac{1}{2r}$,

$$W_Q(t) = \sum_{k=0}^{\infty} N_k t^k = \sum_{k=0}^{\infty} n(2r)^k t^k = \frac{n}{1 - 2rt}.$$  

From Proposition 2.1, one has

$$\frac{1}{t} \left( (-1)^n \frac{f_Q(n - 2 - \frac{1}{t})}{f_Q \left( \frac{1}{t} \right)} - 1 \right) = \frac{n}{1 - 2rt}.$$  

(2)

With $\lambda = (n - 2) - \frac{1}{t}$, that is, $\frac{1}{t} = (n - 2) - \lambda$ in (2), we obtain the required result. Moreover, if the signless Laplacian spectrum of $G$ contains $2r, q_2, \ldots, q_n$, then it is easy to see that the signless Laplacian spectrum of $\overline{G}$ contains $2(n - r - 1), n - 2 - q_2, \ldots, n - 2 - q_n$. \hfill \Box
Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The join $G_1 \vee G_2$ of $G_1$ and $G_2$ is the direct sum of $G_1$ and $G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$.

**Theorem 2.3** Let $G_1$ and $G_2$ be two simple connected graphs with $n_1$ and $n_2$ vertices, respectively. Then

$$f_Q(G_1 \vee G_2) = (-1)^{n_2} f_Q(G_1)(\lambda - n_2) f_Q(G_2)(n_1 + n_2 - \lambda - 2)$$

$$+ (-1)^{n_1} f_Q(G_2)(\lambda - n_1) f_Q(G_1)(n_1 + n_2 - \lambda - 2)$$

$$- (-1)^{n_1+n_2} f_Q(G_1)(n_1 + n_2 - \lambda - 2) f_Q(G_2)(n_1 + n_2 - \lambda - 2),$$

where $Q_1$ and $Q_2$ are the signless Laplacian matrices of $G_1$ and $G_2$, respectively.

**Proof** Clearly, $W_Q(G_1 \oplus G_2)(t) = W_Q(G_1)(t) + W_Q(G_2)(t)$, where $G_1 \oplus G_2$ denotes the direct sum of $G_1$ and $G_2$. Proposition 2.1 implies that, for $|t| < \min\{\frac{1}{Q_1}, \frac{1}{Q_2}\}$,

$$\frac{1}{t} \left( (-1)^{n_1+n_2} f_Q(G_1 \oplus G_2) \left( \frac{n_1 + n_2 - 2 - \frac{1}{t}}{\lambda} \right) - 1 \right)$$

$$= \sum_{i=1}^{2} \frac{1}{t} \left( (-1)^{n_i} f_Q(G_i) \left( \frac{n_i - 2 - \frac{1}{t}}{\lambda} \right) - 1 \right).$$

(3)

Note that $G_1 \oplus G_2 = G_1 \vee G_2$. Setting $n_1 + n_2 - 2 - \frac{1}{t} = \lambda$ and substituting $G_1$, $G_2$ for $G_1$, $G_2$ in (3), we obtain the required result. \qed

**Corollary 2.4** [11] For $i = 1, 2$, let $G_i$ be a regular graph of degree $r_i$ with $n_i$ vertices. Then

$$f_Q(G_i \vee G_2) = \left( 1 - \frac{n_1 n_2}{\lambda - n_1 - 2 r_2}(\lambda - n_2 - 2 r_1) \right) f_Q(G_1)(\lambda - n_2) f_Q(G_2)(\lambda - n_1).$$

**Proof** This is an immediate consequence of Theorems 2.2 and 2.3, omitted. \qed

Let $Q(G) = (q_{ij})_{n \times n}$ be the signless Laplacian of a simple graph $G$. Assume that $x_1, x_2, \ldots, x_n$ are mutually orthogonal normalized eigenvectors of $Q(G)$ associated with eigenvalues $q_1, q_2, \ldots, q_n$, respectively. Also let $\Lambda = \text{diag}(q_1, q_2, \ldots, q_n)$ and $P = (x_1, x_2, \ldots, x_n) = (x_{ij})_{n \times n}$. Then $Q(G) = \Lambda P P^T$, which implies that the number $N_k$ of all semi-edge walks of length $k$ in $G$ equals

$$N_k = \sum_{i,j} N_k(i, j) = \sum_{i,j} q_{ij}^{(k)} = \sum_{i=1}^{n} \left( \sum_{i=1}^{n} x_{ii} \right)^2 q_i^k.$$

Thus, we arrive at:
Theorem 2.5 The total number $N_k$ of semi-edge walks of length $k$ in $G$ equals

$$N_k = \sum_{l=1}^{n} \gamma_l q_l^k \quad (k = 0, 1, 2, \ldots),$$

where $\gamma_l = \left(\sum_{i=1}^{n} x_{ii}\right)^2$.

It is clear to see that $N_k = n(2r)^k$ whenever $G$ is an $r$-regular graph with $n$ vertices. In this case, the signless Laplacian spectral radius $q_1$ of $G$ is equal to

$$q_1 = \sqrt[k]{\frac{N_k}{n}} = 2r.$$ 

In general case, we have Theorem 2.6, which is analogous to an existing result related to the adjacency spectrum (see Theorem 1.12 in [10]).

Theorem 2.6

$$q_1 = \lim_{k \to \infty} k^{\frac{1}{k}} \sqrt[k]{\frac{N_k}{n}} = \lim_{k \to \infty} \sqrt[k]{N_k}.$$ 

Proof Firstly, it is easy to see that

$$q_1 \sqrt[k]{\frac{1}{n}} \leq \sqrt[k]{\frac{N_k}{n}} = \sqrt[k]{\frac{\sum_{l=1}^{n} \gamma_l q_l^k}{n}} \leq q_1 \sqrt[k]{\frac{\sum_{l=1}^{n} \gamma_l}{n}}.$$ 

The squeeze theorem implies that

$$q_1 = \lim_{k \to \infty} k^{\frac{1}{k}} \sqrt[k]{\frac{N_k}{n}}.$$ 

Note that $\lim_{k \to \infty} k^{\frac{1}{k}} = 1$, the required result follows. \qed

The following statement and its proof is analogous to an existing result related to the adjacency spectrum (see Theorem 2.5 in [10]).

Theorem 2.7 If the $Q$-spectrum of a graph $G$ contains a signless Laplacian eigenvalue $q_0$ with multiplicity $s \geq 2$, then the $Q$-spectrum of its complementary graph $\overline{G}$ contains a signless Laplacian eigenvalue $n - 2 - q_0$ with multiplicity $\tau$, where $s - 1 \leq \tau \leq s + 1$.

Proof By Theorem 2.5, the $Q$-generating function of the numbers $N_k$ of semi-edge walks of length $k$ in $G$ is
\[ W_Q(t) = \sum_{k=0}^{\infty} N_k t^k = \sum_{k=0}^{\infty} \left( \sum_{l=1}^{n} q_l^k \right) t^k = \sum_{l=1}^{n} \frac{\gamma_l}{1 - tq_l} \quad \text{for} \quad -\frac{1}{q_1} < t < \frac{1}{q_1}. \]

Set
\[ \Phi(u) = (-1)^n \frac{f_Q(n - 2 - u)}{f_Q(u)}. \]

From Proposition 2.1, one has
\[ \Phi(u) = 1 + \frac{1}{u} W_Q \left( \frac{1}{u} \right) = 1 + \sum_{l=1}^{n} \frac{\gamma_l}{u - q_l} = \frac{\varphi_1(u)}{\varphi_2(u)}, \]

where \( \varphi_1(u) \) and \( \varphi_2(u) \) are polynomials in \( u \) and the roots of \( \varphi_2(u) \) are all simple. Note that \( q_0 \) is a signless Laplacian eigenvalue of \( G \) with multiplicity \( s \geq 2 \). Then \( f_Q(u) = (u - q_0)^s g(u) \), where \( g(q_0) \neq 0 \). Therefore,
\[ \Phi(u) = (-1)^n \frac{f_Q(n - 2 - u)}{(u - q_0)^s g(u)} = \frac{\varphi_1(u)}{\varphi_2(u)}, \]

which implies that \( f_Q(n - 2 - u) (u - q_0)^s \), \( \tau \geq s - 1 \) as the roots of \( \varphi_2(u) \) are all simple. Thus, \( f_Q(u) \) must have a factor \( (u - (n - 2 - q_0))^\tau \). Hence, the \( Q \)-spectrum of the complementary graph \( \overline{G} \) contains a signless Laplacian eigenvalue \( n - 2 - q_0 \) with multiplicity \( \tau \geq s - 1 \).

Next, we shall prove \( \tau \leq s + 1 \). Assume that the \( Q \)-spectrum of the complementary graph \( \overline{G} \) contains a signless Laplacian eigenvalue \( n - 2 - q_0 \) with multiplicity \( \tau > s + 1 \).

According to the above statement, \( \overline{G} = G \) contains a signless Laplacian eigenvalue \( n - 2 - (n - 2 - q_0) = q_0 \) with multiplicity \( r \geq \tau - 1 > s + 1 - 1 = s \), a contradiction. Hence, \( \tau \leq s + 1 \).

This completes the proof of theorem.
where $I_n$ and $Q$ are the identity matrix of order $n$ and the signless Laplacian matrix of $G$, respectively. Using this concept, we computed the $Q$-polynomials of the corona $G_1 \circ G_2$ and edge corona $G_1 \diamond G_2$ (for more details about this aspect, see [2–4,13–15,17,18]) as follows.

**Theorem 2.8** [3] Let $G_1$ and $G_2$ be two graphs on $n_1$ and $n_2$ vertices, respectively. Also let $\Gamma_Q(\lambda)$ be the $Q_2$-coronal of $G_2$ and $G = G_1 \circ G_2$. Then the $Q$-polynomial of $G$ is

$$f_Q(\lambda) = (f_Q(\lambda - 1))^{n_1} f_{Q_1}(\lambda - n_2 - \Gamma_Q(\lambda - 1)).$$

**Theorem 2.9** [3] Let $G_1$ be an $r$-regular graph with $n_1$ vertices, $m_1$ edges and $G_2$ be any graph with $n_2$ vertices, $m_2$ edges. Also let $\Gamma_Q(\lambda)$ be the $Q_2$-coronal of $G_2$ and $G = G_1 \diamond G_2$. If $\lambda$ is not a pole of $\Gamma_Q(\lambda - 2)$, then the $Q$-polynomial of $G$ is

$$f_Q(\lambda) = (f_Q(\lambda - 2))^{m_1} f_{Q_1} \left( \frac{\lambda - r_1 n_2}{1 + \Gamma_Q(\lambda - 2)} \right) (1 + \Gamma_Q(\lambda - 2))^{n_1}.$$  

It is well known that it is difficult for us to compute the inverse of matrices, especially high-order matrices, which results in a difficulty when we need to compute the $Q$-coronal $\Gamma_Q(\lambda)$ in Theorems 2.8 and 2.9. In [3], we computed the $Q$-coronal of some special graphs and gave the $Q$-polynomials of their (edge)coronae.

Next, we shall give a combinatorial interpretation of the $Q$-coronal of a graph $G$ of order $n$, which is used to obtain the many alternative calculations of the $Q$-polynomials of the corona $G_1 \circ G_2$ and edge corona $G_1 \diamond G_2$ for any graphs $G_1$ and $G_2$.

**Proposition 2.10** Let $G$ be a simple connected graph of order $n$. Then its $Q$-coronal equals

$$\Gamma_Q(\lambda) = -1 + (-1)^n \frac{f_Q(n - 2 - \lambda)}{f_Q(\lambda)}.$$

**Proof** Let $Q$ be the signless Laplacian matrix of $G$ and $1_n$ denote the column $n$-vector, whose each element equals 1. By a simple calculation,

$$\Gamma_Q(\lambda) = 1_n^T (\lambda I_n - Q)^{-1} 1_n$$

$$= \lambda^{-1} 1_n^T (I_n - \lambda^{-1} Q)^{-1} 1_n$$

$$= \frac{1}{\lambda} 1_n^T \left( \sum_{k=0}^{\infty} Q^k \left( \frac{1}{\lambda} \right)^k \right) 1_n$$

$$= \frac{1}{\lambda} \sum_{k=0}^{\infty} \left( 1_n^T Q^k 1_n \right) \left( \frac{1}{\lambda} \right)^k.$$

(4)
Since the sum $\sum_{k=0}^{\infty} N_k \left(\frac{1}{\lambda}\right)^k$ of all elements of $Q^k$ is the total number $N_k$ of all semi-edge walks of length $k$ in $G$, then equality (4) becomes

$$\Gamma_Q(\lambda) = \frac{1}{\lambda} \sum_{k=0}^{\infty} N_k \left(\frac{1}{\lambda}\right)^k = \frac{1}{\lambda} W_Q \left(\frac{1}{\lambda}\right).$$

From Proposition 2.1, the required result follows. \hfill \Box

Now applying Proposition 2.10, Theorems 2.8 and 2.9 may be rewritten as Theorems 2.11 and 2.12, respectively.

**Theorem 2.11** Let $G_1$ and $G_2$ be two graphs on $n_1$ and $n_2$ vertices, respectively. Also let $G = G_1 \circ G_2$. Then the $Q$-polynomial of $G$ is

$$f_Q(\lambda) = (f_{Q_2}(\lambda - 1))^n_1 f_{Q_1}(\lambda - n_2 + 1 - (-1)^{n_2} \frac{f_{Q_2}(n_2 - \lambda - 1)}{f_{Q_2}(\lambda - 1)}).$$

**Theorem 2.12** Let $G_1$ be an $r_1$-regular graph with $n_1$ vertices, $m_1$ edges and $G_2$ be any graph with $n_2$ vertices, $m_2$ edges. Also let $G = G_1 \circ G_2$. Then the $Q$-polynomial of $G$ is

$$f_Q(\lambda) = (f_{Q_2}(\lambda - 2))^n_1 f_{Q_1}(\lambda - n_2 + 1 - (-1)^{n_2} \frac{f_{Q_2}(n_2 - \lambda - 1)}{f_{Q_2}(\lambda - 1)}).$$

Proposition 2.13 exhibits the $Q$-coronal of the join $G_1 \vee G_2$ of two regular graphs $G_1$ and $G_2$.

**Proposition 2.13** Let $G_1$ be an $r_1$-regular graph on $n_1$ vertices and $G_2$ be an $r_2$-regular graph on $n_2$ vertices. Also let $G = G_1 \vee G_2$. Then

$$\Gamma_Q(\lambda) = \frac{(\lambda - n_2 - 2r_1)n_2 + (\lambda - n_1 - 2r_2)n_1 + 2n_1n_2}{(\lambda - n_2 - 2r_1)(\lambda - n_1 - 2r_2) - n_1n_2}.$$

**Proof** This follows directly from Theorem 2.2, Corollary 2.4 and Proposition 2.10. Namely, from Theorem 2.2, we have

$$f_{Q_i}(\lambda) = (-1)^{n_i} \left(1 + \frac{n_i}{n_i - 2 - 2r_i - \lambda}\right) f_{Q_i}(n_i - 2 - \lambda), \quad (i = 1, 2).$$

It follows from $f_{Q}(\lambda) = f_{Q_1}(\lambda) f_{Q_2}(\lambda)$ that

$$f_{Q}(n - 2 - \lambda) = (-1)^{n_1+n_2} \left(1 + \frac{n_1}{\lambda - n_2 - 2r_1}\right) \left(1 + \frac{n_2}{\lambda - n_1 - 2r_2}\right) f_{Q_1}(\lambda - n_2) f_{Q_2}(\lambda - n_1).$$
By Corollary 2.4, one gets

$$f_Q(\lambda) = \left(1 - \frac{n_1n_2}{(\lambda - n_1 - 2r_2)(\lambda - n_2 - 2r_1)}\right) f_Q_1(\lambda - n_2) f_Q_2(\lambda - n_1).$$

Now the result follows easily from Proposition 2.10. \hfill \Box

Next, we shall derive the $Q$-generating function for some graphs obtained by the use of some operation on graphs, such as the complement of a graph, the direct sum and the join of two graphs.

**Theorem 2.14** For the $Q$-generating function $W_Q(t)$ for the numbers $N_k$ of semi-edge walks of length $k$ in a graph $G$, we have

$$W_Q(t) = -W_Q\left(\frac{t}{(n-2)t-1}\right), \quad (5)$$

$$W_{Q_1 \oplus Q_2}(t) = W_{Q_1}(t) + W_{Q_2}(t), \quad (6)$$

$$W_{Q_1 \vee Q_2}(t) = \frac{M}{1 - tM}, \quad (7)$$

where

$$M = \sum_{i=1}^{2} \frac{W_{Q_i}\left(\frac{t}{(n_i - n)t+1}\right)}{(n_i - n)t + 1 + tW_{Q_i}\left(\frac{t}{(n_i - n)t+1}\right)}.$$ 

**Proof** From Proposition 2.1, one has

$$W_Q(t) = \frac{1}{t} \left((-1)^n \frac{f_Q(n - 2 - \frac{1}{t})}{f_Q(\frac{1}{t})} - 1\right)$$

and

$$W_Q\left(\frac{1}{n - 2 - \frac{1}{t}}\right) = \left(n - 2 - \frac{1}{t}\right) \left((-1)^n \frac{f_Q(\frac{1}{t})}{f_Q(n - 2 - \frac{1}{t})} - 1\right),$$

which implies that the required result (5). Formula (6) is obvious. Next, we shall prove (7). According to (5) and (6), one gets

$$W_{Q_1 \vee Q_2}(t) = \frac{-W_{Q_1 \oplus Q_2}\left(\frac{t}{(n-2)t-1}\right)}{(n-2)t - 1 + tW_{Q_1 \oplus Q_2}\left(\frac{t}{(n-2)t-1}\right)}.$$
\[
W_{Q_i} \left( \frac{t}{(n-2)t-1} \right) = -\frac{W_{Q_i} \left( \frac{t}{n_i-n+2} \right) - 1}{(n_i-n)t + 1 + tW_{Q_i} \left( \frac{t}{n_i-n+2} \right)}. \tag{9}
\]

Substituting (9) back into (8), we obtain the required result (7). \[ \square \]

**Remark 2.15** In view of formulas (5) and (6), the formula in (7) may be generalized to the case \(k > 2\), that is,

\[
W_{Q_1 \lor Q_2 \lor ... \lor Q_k} (t) = \frac{M}{1 - tM}, \tag{10}
\]

where

\[
M = \sum_{i=1}^{k} \frac{W_{Q_i} \left( \frac{t}{n_i-n+2} \right) - 1}{(n_i-n)t + 1 + tW_{Q_i} \left( \frac{t}{n_i-n+2} \right)}. \]

**Example 2.16** Consider the complete multipartite graph \(G = K_{n_1,n_2,...,n_k}\), which can be represented as the join of graphs \(G_1, G_2, \ldots, G_k\), all of which contain only isolated vertices. For an \(r\)-regular graph \(G\), its \(Q\)-generating function \(W_Q (t) = \frac{n-2}{1-2rt}\) for the numbers \(N_k\) of semi-edge walks of length \(k\) in \(G\) (see the proof of Theorem 2.2). Hence, the \(Q_i\)-generating function for the numbers \(N_k\) of semi-edge walks of length \(k\) in \(G_i\) equals \(n_i\) for \(i = 1, 2, \ldots, k\). Now applying (10), we obtain

\[
W_Q (t) = \left( \sum_{i=1}^{k} \frac{n_i}{(n_i-n)t + 1 + tn_i} \right)^{-1} - t \right)^{-1}. \]

According to the proof of Proposition 2.10, the \(Q\)-coronal of the complete multipartite graph \(G = K_{n_1,n_2,...,n_k}\) equals

\[
\Gamma_Q (\lambda) = \left( \sum_{i=1}^{k} \frac{n_i}{\lambda - n + 2n_i} \right)^{-1} - 1 \right)^{-1}. \]

Finally, Liu and Lu [14] introduced the definitions of the subdivision-vertex neighbourhood corona and subdivision-edge neighbourhood corona for two graphs \(G_1\) and \(G_2\), and their \(Q\)-polynomials are determined by using the \(Q\)-coronal of \(G_2\). Clearly,
Applying the combinatorial interpretation of the $Q$-coronal of graphs (see Proposition 2.10), we may obtain many alternative calculations of the $Q$-polynomials of the subdivision-vertex neighbourhood corona and subdivision-edge neighbourhood corona of $G_1$ and $G_2$. These contents are omitted.

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