Classical signal-flow in cluster-state quantum computation

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Abstract

We study concretely how classical signals should be processed in quantum cluster-state computation. Deforming corresponding quantum teleportation circuit, we find a simple rule of a classical signal-flow to obtain correct quantum computation results.

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Cluster-state quantum computation proposed by Raussendorf and Briegel\cite{1} is a promising scheme for its simplicity. Preparing a cluster-state corresponding to a certain problem, we can perform quantum computation only by successive quantum measurements and feedforward of measurement outcomes. We can obtain a proper computation result only through choosing adequate quantum measurement basis and rectifying a resultant quantum state. These choices of the basis and rectification of the resultant quantum state are done referring to preceding quantum measurement outcomes. Therefore, it is inevitable to treat quantum measurement outcomes properly. As far as the author knows, however, treatment of measurement outcomes has not been studied significantly. The purpose in this paper is to give a concrete classical signal-flow chart that leads to proper computation results in cluster-state quantum computation.

Cluster-state quantum computation is nothing but quantum computation with measurements\cite{2,3}. Actually, we can associate a quantum teleportation circuit with a cluster-state. For a quantum teleportation circuit, dragging back controlled-phase transformations connecting pairs of junction qubits to the starting point, we obtain the corresponding cluster state. We can find a classical signal-flow in cluster-state computation by analyzing the corresponding quantum teleportation circuit.

It is well known that a set of single-qubit unitary transformations and the two-qubits controlled-phase transformation is universal. An arbitrary single-qubit unitary transformation can be carried out by a sequence of three elements in the set \{HZ_α\}, where \( H \) is the Hadamard transformation and \( Z_α = \cos \frac{α}{2} I - i \sin \frac{α}{2} Z \), with \( Z = |0\rangle\langle 0| - |1\rangle\langle 1| \) for quantum computational basis states \{\(|0\rangle, |1\rangle\}\}. This can be seen by noting the identity \( HZ_αHZ_βHZ_γ = HZ_αX_βZ_γ \), where \( X_β = \cos \frac{β}{2} I - i \sin \frac{β}{2} X \), with \( X = |0\rangle\langle 1| + |0\rangle\langle 1| \).

A single-qubit unitary transformation \( HZ_α \) on a quantum state \(|ψ⟩\) can be realized by the single-qubit quantum teleportation circuit\cite{4}. The single-qubit teleportation circuit is nothing but the simplest cluster-state quantum computation. The first qubit of the initial cluster state \(|ψ⟩ − |+_⟩\), where the − sign indicates the controlled-phase transformation, is measured by the basis \{(HZ_α)^†|0⟩, (HZ_α)^†|1⟩\} to transform the second qubit into \( HZ_α|ψ⟩ \).

The single-qubit quantum teleportation circuits can be combined into a multistage type\cite{5} corresponds to a one-dimensional cluster state. Since a product of three successive transformations \( HZ_{α_1}HZ_{α_2}HZ_{α_3} \) causes an arbitrary single-qubit unitary transformation, the number of the sequence can be limited to three.

Starting from an adequate cluster state, we can carry out desired quantum computation by precise quantum measurements and proper processing of measurement outcomes. Each controlled-phase transformation in an associated quantum teleportation circuit corresponds to a controlled-phase transformation between a pair of junction qubits in a cluster state. A multistage single-qubit quantum teleportation circuit replaces a one-dimensional chain in a cluster state. This quantum teleportation circuit shows us how quantum measurement outcomes should be treated in cluster-state computation.

In a cluster state that we are considering, junction qubits appear in pairs connected by controlled-phase transformations. Each junction qubit connects a one-dimensional incoming chain with a one-dimensional outgoing chain. Therefore, a pair of junction qubits forms a fundamental block of an H-branch with two one-dimensional incoming chains and two one-dimensional outgoing chains. Two H-branches are connected by sharing a one-dimensional chain that is outgoing from one H-branch and incoming into the other.
H-branch. Connecting H-branches successively we can form a desired cluster state. Successive measurements of qubits cause a classical signal-flow over the cluster state. Classical signals flow over a one-dimensional chain and branch at junctions. Our purpose is to obtain a classical signal-flow chart for a cluster state using the corresponding quantum teleportation circuit.

First, we study a classical signal-flow for a one-dimensional quantum chain that is simulated by a multistage quantum teleportation circuit. The first qubit on the left is in a state \( |\psi\rangle \) and the other qubits are in the state \( |+_\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) until the controlled-phase transformations are carried out. Except for the last qubit or the last few qubits, we measure each qubit by the basis \( (HZ_0)^\dagger |0\rangle, (HZ_0)^\dagger |1\rangle (i = 1, 2, \cdots) \) from the left.

As is well-known, to teleport a quantum state successfully, we need to operate the Pauli matrices \( X \) and \( Z \) according to measurement outcomes. In a multistage quantum teleportation circuit we need to teleport intermediate quantum states that are corrected by the Pauli matrices. Using the identity \( (HZ_0 \otimes I)C_{\text{phase}}(Z \otimes I) = (X \otimes I)(HZ_0 \otimes I)C_{\text{phase}}, \) that comes from \( HZ = XH, \) the correction \( Z \) in the first qubit can be changed into a correction \( X \) after the controlled-phase transformation and the \( HZ_0 \) transformation. In the same way using the identity \( (HZ_0 \otimes I)C_{\text{phase}}(X \otimes I) = (Z \otimes Z)(HZ_0 \otimes I)C_{\text{phase}}, \) the correction \( X \) can be postponed. Moreover, we can see that the operation \( Z \otimes Z \) can be removed by adding a \( Z \) after the controlled-\( X \) operation in the second qubit. Thus the quantum teleportation circuit in Fig.1(a) can be transformed into the circuit in Fig.1(b).

If the measurement outcome of the \( (n-1) \)-th qubit is 1, the \( n \)-th qubit is measured after the \( HZ_0 \) transformation instead of the \( HZ_\alpha \) transformation. If the measurement outcome of the \( (n-2) \)-th qubit is 1, the additional transformation \( X \) is required before the meter.

The situation is simple when the present qubit is not teleported. If the \( n \)-th qubit is not measured, the \( (n-1) \)-th qubit measurement outcome 1 brings the \( X \) correction to the \( n \)-th qubit, and the \( (n-2) \)-th qubit measurement outcome 1 brings the \( Z \) correction to the \( n \)-th qubit. Even if the \( (n-1) \)-th qubit is not measured the \( n \)-th qubit is corrected by \( Z \) from the the \( (n-2) \)-th qubit measurement outcome 1.

In anyway, it is important to note that the measurement outcome of a qubit affects only the following two qubits. This rule holds also for two-dimensional cluster states.

Second, we consider two-dimensional cluster states. The H-branch in Fig.2(a) is a fundamental constituent of two-dimensional cluster states. We study how the measurement outcomes affect the qubits around the junctions. The only distinct of the two-dimensional case from the one-dimensional case originates from the controlled-phase transformation connecting a pair of junction qubits. In cluster-state quantum computation these controlled-phase transformations have been done at the outset. The corresponding quantum teleportation circuit with measurements having been done up to the left side of the pair of junction qubits is depicted in Fig.2(b). In Fig.2(b), two single-qubit quantum-teleportation circuits are connected by the central controlled-phase transformation. The controlled-phase transformation connecting the two junction qubits can be dragged back to the left as the controlled-phase transformation in the quantum teleportation circuit. The only extra effect is to add the extra \( X \) transformation on the \( n \)-th\((m \)-th\) qubit due to a measurement outcome 1 of the \( (m-1) \)-th\((n-1) \)-th qubit. If both of the measurement outcomes of the \( (m-1) \)-th and the \( (n-1) \)-th qubits are 1, an extra harmless
overall minus factor appears due to an exchange of $Z$ and $X$. In Fig.4(c) the two extra $X$ corrections through the junctions are depicted before the meters. Note that the $(n+1)$-th qubit, which is next to the junction qubit, is not affected by the $m$-th qubit. If the $n$-th qubit is not measured this qubit is corrected by $X$ from the $(n-1)$-th qubit measurement outcome 1 and by $Z$ from the $(n-2)$-th qubit measurement outcome 1 and by $Z$ from the $(m-1)$-th qubit measurement outcome 1.

Our result of classical signal-flow is summarized in Table I. Table I shows how the $n$-th qubit should be corrected according to the measurement outcomes of prior qubits. This table partially applies also for the case the $n$-th qubit not being a junction qubit.

**TABLE I.** Corrections to the $n$-th qubit of the H-branch in Fig.2(a). The first column indicates control qubits. The middle row has meaning only when the $(n-1)$-th qubit is measured. The final row has meaning only when the $n$-th qubit is a junction qubit.

| $n$  | Not Measured | Measured          |
|------|--------------|-------------------|
| $n-2$| $Z$          | $X$ before the meter |
| $n-1$| $X$          | $Z_\alpha \rightarrow Z_{-\alpha}$ |
| $m-1$| $Z$          | $X$ before the meter |

**References**

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FIGURE CAPTIONS
FIG 1. (a) The single-qubit quantum teleportation circuit with the ZX correction in the incoming state. The vertical line means the controlled-pase transformation $C_{\text{phase}}$. (b) A transformed circuit of (a). The arrows mean classical signal-flow.

FIG 2. (a) An H-branch. (b),(c) Quantum teleportation circuits that simulate the H-branch.