We have demonstrated that $\Phi_0$ periodic Aharonov–Bohm oscillations measured in an ensemble of rings may survive after ensemble averaging procedure. The central point is the difference between the preparation stage of the ensemble and the subsequent measurement stage. The robustness of the effect under finite temperature and non–zero charging energy of rings is discussed.

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I. INTRODUCTION

Some of the more interesting results to emerge from the extensive research on quantum mesoscopic systems has been related to the study of multiply connected conductors, threaded by an Aharonov–Bohm magnetic flux $\Phi_0 = \hbar c/e$. Physical observables measured in such systems, are in general, periodically oscillating with the flux. This applies to transport, as well as to thermodynamic quantities (including, e.g., non–dissipative persistent currents that have been shown to exist in such systems). Employing gauge invariance, one may show that these oscillations must be $\Phi_0 = \hbar c/e$ periodic. Under most averaging procedures (most notably impurity averaging), the odd harmonics of the periodic signal are suppressed, and the effective periodicity turns out to be $\Phi_0/2$.

We have recently [3] pointed out that this period halving depend strongly on the averaging procedure employed. One is commonly concerned with a set of macroscopically similar systems (e.g., metallic rings all having the same size, shape and impurity concentration) which are, though, microscopically distinct (e.g., characterized by different microscopic impurity configurations). In order to systematically compare different types of ensembles, we had to distinguish between the preparation stage of a given ensemble and the subsequent measurement stage, where it responds to an external perturbation. The ensemble we termed Grandcanonical–Canonical (GC–C) has turned out to be of particular interest. At the preparation stage each member of the ensemble (“ring”) subject to an external Aharonov–Bohm flux $\Phi$, is brought to equilibrium with an external reservoir (at temperature $T$ and chemical potential $\mu$). That is, as far as preparation is concerned, this ensemble is Grandcanonical. Subsequently the flux is varied from $\Phi$ to $\Phi$ (the latter may represent either a new static value of the flux or a time dependent signal), while the number of electrons on each ring is constrained to remain constant. At the stage of response the ensemble is kept under canonical conditions. We have been able to express the average GC–C persistent current, $I_{\Phi}(\Phi)$ in terms of the average Canonical–Canonical (C–C) current, $I_{cc}(\Phi)$ [4]. The latter refers to a situation where not only measurement, but also preparation is done under canonical con-
ditions. For example, the number of electrons we assign to each ring is a randomly selected number in some interval, uncorrelated with the particular energy spectrum of the ring. The relation we have found is 

\[ I_\Phi(\Phi) = I_{cc}(\Phi) - I_{cc}\left(\frac{\Phi + \Phi}{2}\right) - I_{cc}\left(\frac{\Phi - \Phi}{2}\right). \]  

(1)

Thus, although \( I_{cc}(\Phi) \) was earlier found to be \( \Phi_0/2 \) periodic \([4]\), the invariance under \( \Phi \to \Phi + \Phi_0/2 \) is broken for the GC–C ensemble, as the preparation flux \( \Phi \) needs not satisfy any special symmetry condition. Indeed \( I_\Phi(\Phi) \) is \( \Phi_0 \) periodic rather than \( \Phi_0/2 \) periodic!

It is quite remarkable that all experimental evidence to date supports the common wisdom that ensemble averaged quantities should be \( \Phi_0/2 \) periodic. One may raise the question why our \( \Phi_0 \) periodic prediction has not been observed so far and whether – beyond its academic interest – \( \Phi_0 \) periodic GC-C current may be connected with any experiment. This question is of evident interest as the difference between C-C and GC-C ensembles is qualitative and involves a change in the symmetry of the measured quantity.

Our earlier analysis of the GC-C ensemble \([3]\) which yielded as a result \( \Phi_0 \) periodic averaged signals was confined to rather simplified and idealized systems: we have considered a zero temperature situation in the absence of any Coulomb interaction. It is the purpose of the present note to examine the robustness of this theory (hence of the \( \Phi_0 \) periodic signals) against two factors that play a major role in any experimentally realizable system, namely finite temperature and Coulomb energy. Our two main results are:

1. We derive an expression for the GC-C flux dependent persistent current both when the preparation temperature, \( \mathcal{T} \), and the measurement temperature, \( T \) (at the stage where the system’s response to an external perturbation is measured) are finite \([5]\). We find that the various harmonics of the \( \Phi_0 \) periodic signal are washed out as \( T \) is increased, the first harmonic being the slowest to die out (Fig. (1a)). The harmonics also depend on the preparation temperature, \( \mathcal{T} \). Whereas even harmonics will not die out in the limit of large \( \mathcal{T} \), the odd ones are suppressed as a power low in \( \mathcal{T}/E_c \) if \( \mathcal{T} \ll E_c \) and become exponentially small for \( \mathcal{T} \gg E_c \) (Fig’s. (1b),(1c)). Here \( E_c \) is the Thouless correlation energy.
of the system: \( E_c = D/L_x^2 \), where \( D \) is the system diffusivity and \( L_x \) is the ring perimeter. This behaviour is summarized in Eq. (10) and Fig. (1).

(2) The charging energy, \( E_Q \), of the rings plays a crucial role at the stage of preparation. We find that the larger \( E_Q \) the further we deviate from GC preparation conditions. The amplitude of the odd harmonics of the periodic Aharonov–Bohm signal are consequently suppressed by a factor

\[
\frac{\Delta}{\Delta + 2E_Q},
\]

where \( \Delta \) is a mean level spacing of the system. This factor is practically temperature independent.

We thus conclude that under certain experimental requirements and within a certain temperature range (see the discussion below) \( \Phi_0 \) periodic averaged Aharonov–Bohm oscillations are observable.

II. TEMPERATURE DEPENDENCE OF THE GC–C CURRENT

Let us first neglect the charging energy. Extending the analysis of Ref. [3] to finite temperatures, we consider an ensemble of macroscopically equivalent rings, each having a fixed, flux independent number of electrons. The current measured at temperature \( T \) and flux \( \Phi \) is (cf. Eq. (4) of Ref. [3])

\[
I_T(\Phi) = \langle I_{GC}^{T}(\Phi) \rangle + \langle \frac{\partial I_{GC}^{T}(\Phi)}{\partial \mu} \delta \mu_T, \bar{T}(\Phi, \bar{\Phi}) \rangle,
\]

where angular brackets \( \langle \ldots \rangle \) stand for the ensemble averaging. The first term on the r.h.s. is the vanishingly small averaged grand canonical current to be neglected hereafter. Let the (sample specific) particle number be \( N_T(\Phi) \) (where \( T \) and \( \Phi \) are the preparation temperature and flux respectively). The variation of the chemical potential under canonical measurement conditions is given by

\[
\sum_n f_T(\epsilon_n(\Phi) - \mu - \delta \mu_T, \bar{T}(\Phi, \bar{\Phi})) = N_T(\bar{\Phi}),
\]
where \( \delta \mu_{T,T}(\Phi, \Phi) \equiv 0 \) (by our choice of the preparation procedure), \( T \) and \( \Phi \) being the measurement temperature and flux respectively. Here \( f_T \) is the standard Fermi–Dirac distribution function at temperature \( T \). Defining the grand canonical particle number as 
\[
N_T(\Phi) = \sum_n f_T(\epsilon_n(\Phi) - \mu),
\]
we find after expanding Eq. (3) to first power in \( \delta \mu \)
\[
\delta \mu_{T,T}(\Phi, \Phi) = \Delta \left[ N_T(\Phi) - N_T(\Phi) \right].
\]
(5)

Using a simple thermodynamic relation 
\[
\partial I^{GC}_T(\Phi)/\partial \mu = \partial N_T(\Phi)/\partial \Phi
\]
and substituting it, together with Eq. (5), into Eq. (3), we obtain for the GC-C current.
\[
I_{\Phi}(\Phi) = -\frac{1}{2} \Delta \frac{\partial}{\partial \Phi} \left[ \langle N^2_T(\Phi) \rangle - 2\langle N_T(\Phi)N_T(\Phi) \rangle \right].
\]
(6)

We next evaluate the correlator
\[
K_{T,T}(\Phi, \Phi, \gamma) = \langle N_T(\Phi)N_T(\Phi) \rangle,
\]
(7)
where the dependence on the inelastic broadening \( \gamma \) has been introduced explicitly. Within perturbation theory we find
\[
K_{T,T}(\Phi, \Phi, \gamma) = \Re \int d\epsilon K_{0,0}(\Phi, \Phi, \gamma - i\epsilon) \xi_{T,T}(\epsilon),
\]
(8)
where the thermal function is \[3\]
\[
\xi_{T,T}(\epsilon) = -\frac{\partial^2}{\partial \epsilon^2} \int dx f_T(x + \frac{\epsilon}{2}) f_T(x - \frac{\epsilon}{2})
\]
(9)
and \( K_{0,0}(\Phi, \Phi, \gamma) \) is a well-known zero temperature correlator, consisting to leading order of a sum of Diffuson and Cooperon contributions \[4\]. In terms of the zero temperature C-C persistent current \( I_{cc}(\Phi, \gamma) \), the GC-C current is given by
\[
I_{\Phi}(\Phi) = \Re \int d\epsilon I_{cc}(\Phi, \gamma - i\epsilon) \xi_{T,T}(\epsilon) - \Re \int d\epsilon \left[ I_{cc} \left( \frac{\Phi + \Phi}{2}, \gamma - i\epsilon \right) + I_{cc} \left( \frac{\Phi - \Phi}{2}, \gamma - i\epsilon \right) \right] \xi_{T,T}(\epsilon).
\]
(10)

The first term on the r.h.s of Eq. (10) is the (measurement temperature dependent) C-C current, which contains only even harmonics. Most interesting, it does not depend on
the preparation temperature. The second term on the r.h.s of Eq. (10) gives rise to both odd and even harmonics. These are suppressed as the preparation temperature is increased (corresponding to the broadening of $\xi$). As a result the odd harmonics are suppressed with the preparation temperature, unlike the even ones [8]. The temperature dependence of the four low harmonics is shown in Fig. (1).

### III. EFFECT OF CHARGING ENERGY

Incorporating Coulomb interactions is crucial for a correct description of the thermodynamics of small rings. We now account for the characteristic charging energy $E_Q = \frac{e^2}{2\varepsilon}$, $C$ being the capacitance (see below).

The expression for the sample specific particle number $N_T(\Phi)$ (Eq. (4)) is now replaced by

$$N_T(\Phi) = \frac{\sum_N N \exp \left\{ -\frac{1}{T} \left[ F(N, T, \Phi) - \mu N + E_Q(N - N_0)^2 \right] \right\}}{\sum_N \exp \left\{ -\frac{1}{T} \left[ F(N, T, \Phi) - \mu N + E_Q(N - N_0)^2 \right] \right\}}.$$  \hspace{1cm} (11)

Here $F(N, T, \Phi)$ is the free energy of a ring with $N$ particles (in the absence of any charging energy); $N_0$ represents the charge (in electron units) of the positive background. At high enough temperature one may evaluate the sum, Eq. (11), employing a saddle point approximation. We obtain

$$\frac{\partial F(N, T, \Phi)}{\partial N} \bigg|_{N = N_T(\Phi)} = \mu - \mu(N, \Phi) + 2E_Q(N - N_0) = 0.$$  \hspace{1cm} (12)

Here $\frac{\partial F}{\partial N} \bigg|_{\bar{N}} \equiv \mu(\bar{N}, \Phi)$, is to be determined from

$$\sum_n f_T(\epsilon_n(\Phi) - \mu(\bar{N}, \Phi)) = \bar{N}.$$  \hspace{1cm} (13)

($\epsilon_n$ are the single particle energies in the absence of charging energy). The particle number $\bar{N}$ is to be determined self-consistently from Eq. (12) and Eq. (13). Expanding Eq. (13) to the leading order in $\mu - \bar{\mu}$ and $\bar{N} - N_T(\Phi)$ (cf. Eq. (4)) and substituting it in Eq. (12) we find
\[
\overline{N_T(\Phi)} = \frac{1}{\Delta + 2E_Q} [\Delta N_T(\Phi) + 2E_QN_0].
\]  

(14)

Repeating the procedure outlined above (for the \(E_Q = 0\) scenario), namely substituting Eq. (14) in Eq. (3) and that, in turn, in Eq. (3) we obtain for the current

\[
I(\Phi) = -\frac{1}{2} \frac{\Delta}{\partial \Phi} \left[ \langle N_T^2(\Phi) \rangle - \frac{2\Delta}{\Delta + 2E_Q} \langle N_T(\Phi)N_T(\Phi) \rangle \right],
\]

which, for \(E_Q = 0\), reduces to Eq. (6). Eq. (15) is our main result. Two remarks are now due:

(i) The derivation of this equation, following a saddle point approximation (Eq. (12)), assumes that many terms in the grand canonical sums (Eq. (11)) are almost equally important. A necessary condition for that is that \(E_Q \ll T\) [9]. In the low preparation temperature limit, \(E_Q \gg T\), there is generically only one term \((N)\) in the grand canonical sum which is of importance. It is this \(N\) rather than the solution of the saddle point equation, \(N\) (cf. Eq. (14)), which represents the actual number of electrons in the system. Since the typical preparation flux \(\Phi\) dependent meander of a single electron level is \(\sim \Delta (\ll E_Q)\), we expect that for \(E_Q \gg T\) the charging energy indeed quenches the \(\Phi\) dependence of the initial particle number, leading to \textit{canonical} preparation condition. Only a fraction \(\sim \frac{\Delta}{2E_Q}\) of the ensemble members should still have \(\Phi\) sensitive \(N_T(\Phi)\), qualitatively satisfying the \(\frac{\Delta}{\Delta + 2E_Q}\) factor in Eq. (13) even at low \(T\).

(ii) The magnitude of the odd harmonics in \(\Phi\), calculated from the second term on r.h.s. of Eq. (15), is by a factor \(\Delta/(\Delta + 2E_Q)\) smaller compared with \(E_Q = 0\) case. The temperature dependence of the harmonics remains practically the same as we discussed in the previous sections.

A. What is the capacitance

In order to maximize the odd harmonic effect discussed above, one should try to minimize the magnitude of charging energy \(E_Q\). We recall that the typical experiment will consist of coupling the system to a “particle reservoir” (a large conductor), and then cutting it off from
the reservoir. For a two dimensional electron gas this may be achieved by varying a gate voltage, as is shown schematically in Fig. (2). The similar scheme has already been realized experimentally (with a single ring only) cf. Ref. [10]. The ring – reservoir capacitance is quite large (owing to the very small distance between the ring and the electron gas at the moment when they are just being decoupled from each other). We use the parameters given in Ref. [10] to estimate the reduction factor Eq. (3). The circumference of the ring $L = 10 \mu m$, its width $W = 0.16 \mu m$, the distance between the ring and the conductive substrate $d = 72 nm$ and the Fermi wavelength $\lambda_F = 42 nm$. With these parameters we estimate the level spacing to be $\Delta = 4.5 \times 10^{-6} eV$ and the charging energy as $E_Q = 2.5 \times 10^{-6} eV$, implying a reduction factor of the order of 0.5.

IV. CONCLUSION

We have thus demonstrated here that although the observation of $\Phi_0$ periodic ensemble averaged signal requires more care in devising the experimental setup than the observation of a $\Phi_0/2$ periodic signal, it is nonetheless possible. The two main requirements are: (i) the preparation temperature $T$, as well as the measurement temperature $T$ should be smaller or of the order of Thouless energy $E_c$; (ii) the charging energy of the system $E_Q$ should not exceed much the mean level spacing $\Delta$. Observation of $\Phi_0$ periodic oscillations is facilitated provided the ensemble “remembers” the preparation conditions (flux $\Phi$), which breaks the symmetry with respect to $\Phi \rightarrow \Phi + \frac{\Phi_0}{2}$. We suggest that differences between GC–C and C–C conditions (grand canonical versus canonical preparation) should be observable for other physical quantities defined for finite size systems as well.

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[5] In Ref. [3] we have considered a \( T = T = 0 \) scenario.

[6] The thermal function assumes particularly simple form for \( T = T \) and for \( T \) and/or \( T = 0 \),

\[
\xi_{T,T}(\epsilon) = \frac{1}{2} \partial_{\epsilon}^2 \left( \epsilon \coth \frac{\epsilon}{2T} \right); \quad \xi_{T,0}(\epsilon) = \xi_{0,T}(\epsilon) = \frac{1}{4T \cosh^2 \frac{\epsilon}{2T}}.
\]

[7] In Ref. [3] \( K_{0,0}(\Phi, \overline{\Phi}, \gamma) \) was denoted by \( \tilde{K}(\Phi, \overline{\Phi}) = K(\Phi - \overline{\Phi}) + K(\Phi + \overline{\Phi}) \) and given by the diagram Fig. (1a). It is straightforward to show that for a quasi one–dimensional ring (“short cylinder”)

\[
K(\Phi, \gamma) = \frac{2}{\pi^2} \sum_{p=1}^{\infty} \frac{1}{e^{-p\sqrt{\gamma/E_c}} \cos 4\pi p \frac{\Phi}{\Phi_0}}.
\]
[8] Higher preparation temperature implies that the number of electrons assigned to each ring is less correlated with its specific spectrum; the preparation is then closer to being canonical.

[9] We note that in that case the solution of the saddle point equation, Eq. (12), $\overline{N}$ may be non-integer. This is of no real consequence since one may always choose for the particle number the integer closest to $\overline{N}$.

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FIGURES

FIG. 1. The first four harmonics of the GC–C current as a functions of the measurement temperature $T$ ($T = 0$) (a); and the preparation temperature $\mathcal{T}$ ($T = 0$) (b), ($T = .5E_c$) (c). Index of the harmonics is indicated. The preparation flux $\mathcal{F}$ is taken to be zero in all the cases.

FIG. 2. Schematic plot of the experimental setup. The rings are etched in a 2D electron gas, (1); metallic contact (particle reservoir), (2) and the gate, (3).