Discrete Convexity and Polynomial Solvability in Minimum 0-Extension Problems

Hiroshi HIRAI

The University of Tokyo
hirai@mist.i.u-tokyo.ac.jp

RIMS, Kyoto, October 17, 2012
Minimum 0-extension problems
Known results on P/NP-hard classification
Main result
Proof idea from Discrete Convex Analysis & Valued-CSP
Some technical detail

Notation

- \((\text{semi})\text{metric}\) \(d\) on \(X \iff d : X \times X \to \mathbb{R}_+\),

\[
\begin{align*}
d(x, x) &= 0, & d(x, y) &= d(y, x), & d(x, z) + d(z, y) &\geq d(x, y). \\
\end{align*}
\]
- \((X, d)\): (semi)metric space
Minimum 0-Extension Problem (Karzanov 1998)

\((S, \mu)\): finite metric space

**Def:** extension of \((S, \mu)\)
\[
\iff \text{metric space } (X, d) \text{ s.t. } X \supseteq S \text{ and } d|_S = \mu
\]

**Def:** 0-extension of \((S, \mu)\)
\[
\iff \text{extension } (X, d) \text{ s.t. } \exists \rho : X \to S: \rho|_S = \text{id} \text{ and } d = \mu \circ \rho
\]
Minimum 0-Extension Problem (Karzanov 1998)

$(S, \mu)$: finite metric space

**Def:** extension of $(S, \mu)$

$\Leftrightarrow$ metric space $(X, d)$ s.t. $X \supseteq S$ and $d|_S = \mu$

**Def:** 0-extension of $(S, \mu)$

$\Leftrightarrow$ extension $(X, d)$ s.t. $\exists \rho: X \rightarrow S$: $\rho|_S = id$ and $d = \mu \circ \rho$

**Minimum 0-Extension Problem on $(S, \mu)$**

Given a finite set $X \supseteq S$ and $c: \binom{X}{2} \rightarrow \mathbb{Q}_+$,

minimize $\sum_{xy} c(xy)d(x, y)$

subject to $(X, d)$: 0-extension of $(S, \mu)$
$0$-Ext$[\mu]$ 

Minimize $\sum_{xy} c(xy) \mu(\rho(x), \rho(y))$

subject to $\rho : X \to S, \rho|_S = id.$
0-Ext[μ] Minimize \[ \sum_{xy} c(xy) \mu(\rho(x), \rho(y)) \]
subject to \[ \rho : X \to S, \ \rho|_S = id. \]

→ MRF optimization form:

Minimize \[ \sum_{i} \sum_{q \in S} c_{iq} \mu(q, \rho_i) + \sum_{i < j} c_{ij} \mu(\rho_i, \rho_j) \]
subject to \[ (\rho_1, \rho_2, \ldots, \rho_n) \in S \times S \times \cdots \times S \]

Applications: computer vision, clustering, learning theory,...
(S, μ) := (V_Γ, d_Γ) for undirected graph Γ

0-Ext[Γ]  Minimize \[ \sum_{xy} c(xy)d_Γ(\rho(x), \rho(y)) \]
subject to \( ρ : X \to S, \, ρ|_S = id. \)
\((S, \mu) := (V_\Gamma, d_\Gamma)\) for undirected graph \(\Gamma\)

\[
\begin{align*}
0\text{-Ext}[\Gamma] & \quad \text{Minimize} \quad \sum_{xy} c(xy) d_\Gamma(\rho(x), \rho(y)) \\
\text{subject to} \quad & \quad \rho : X \to S, \ \rho|_S = id.
\end{align*}
\]
\((S, \mu) := (V_\Gamma, d_\Gamma)\) for undirected graph \(\Gamma\)

\[
\text{0-Ext}[\Gamma] \quad \text{Minimize} \quad \sum_{xy} c(xy) d_\Gamma(\rho(x), \rho(y))
\]

subject to \(\rho : X \to S, \quad \rho|_S = \text{id}.

Minimum cut
\((S, \mu) := (V_\Gamma, d_\Gamma)\) for undirected graph \(\Gamma\)

\[\text{0-Ext}[\Gamma] \quad \text{Minimize} \quad \sum_{xy} c(xy) d_\Gamma(\rho(x), \rho(y))\]

subject to \(\rho : X \rightarrow S, \rho|_S = id\).
$(S, \mu) := (V_\Gamma, d_\Gamma)$ for undirected graph $\Gamma$

**0-Ext$[\Gamma]$**

Minimize \[
\sum_{xy} c(xy) d_\Gamma(\rho(x), \rho(y))
\]

subject to $\rho : X \to S$, $\rho|_S = \text{id}$.
\[(S, \mu) := (V_\Gamma, d_\Gamma)\] for undirected graph \(\Gamma\)

\[
\text{0-Ext}[\Gamma] \quad \text{Minimize} \quad \sum_{xy} c(xy)d_\Gamma(\rho(x), \rho(y))
\]

subject to \(\rho : X \to S, \: \rho|_S = id.\)
\( \Gamma = K_2 \Rightarrow \text{minimum cut} \Rightarrow P \)

\( \Gamma = K_n \Rightarrow \text{multi-terminal cut} \Rightarrow \text{NP-hard} \quad (n \geq 3) \)

**Question**

*What is \( \Gamma \) for which 0-Ext[\( \Gamma \)] is in \( P \) ?*
A classical result in location theory

Theorem (Picard-Ratliff 1978)

If $\Gamma$ is a tree, then $0\text{-Ext}[\Gamma]$ is in $P$. 

Obs. $2P$ and $2P'$ are expressed as Cartesian product $\Gamma \times \Gamma'$. 

$\frac{7}{27}$
A classical result in location theory

**Theorem (Picard-Ratliff 1978)**

If $\Gamma$ is a tree, then $\textbf{0-Ext} [\Gamma]$ is in $\mathbb{P}$.

**Obs.** $\Gamma \in \mathbb{P}$ and $\Gamma' \in \mathbb{P} \Rightarrow \Gamma \times \Gamma' \in \mathbb{P}$

($\times : = \text{Cartesian product}$)
Median graph

- **Median** of $x_1, x_2, x_3$ $\Leftrightarrow$ $v \in V_{\Gamma}$ satisfying

$$d_{\Gamma}(x_i, x_j) = d_{\Gamma}(x_i, v) + d_{\Gamma}(v, x_j) \quad (1 \leq i < j \leq 3)$$

- **Median graph** $\Leftrightarrow \forall$ triple has a *unique* median.
Median graph

- **Median** of $x_1, x_2, x_3 \iff v \in V_\Gamma$ satisfying

  $$d_\Gamma(x_i, x_j) = d_\Gamma(x_i, v) + d_\Gamma(v, x_j) \quad (1 \leq i < j \leq 3)$$

- **Median graph** $\iff \forall$ triple has a *unique* median.

  c.f. Median graph $\simeq$ graph of CAT(0) cube complex (Chepoi 2000)
Median graph

- **Median** of $x_1, x_2, x_3 \iff v \in V_\Gamma$ satisfying

$$d_\Gamma(x_i, x_j) = d_\Gamma(x_i, v) + d_\Gamma(v, x_j) \quad (1 \leq i < j \leq 3)$$

- **Median graph** $\iff \forall$ triple has a *unique* median.

C.f. Median graph $\simeq$ graph of CAT(0) cube complex (Chepoi 2000)
Median of $x_1, x_2, x_3 \iff v \in V_\Gamma$ satisfying

$$d_\Gamma(x_i, x_j) = d_\Gamma(x_i, v) + d_\Gamma(v, x_j) \quad (1 \leq i < j \leq 3)$$

Median graph $\iff \forall$ triple has a unique median.

Theorem (Chepoi 1996)

If $\Gamma$ is a median graph, then 0-Ext[$\Gamma$] is in P.
Metric relaxation (Karzanov 1998)

\[ \text{0-Ext}[\Gamma] : \quad \text{Min.} \quad \sum_{xy} c(xy) d(x, y) \]
\[ \text{s.t.} \quad (X, d): \text{0-extension of } (V_\Gamma, d_\Gamma) \]
Metric relaxation (Karzanov 1998)

\[ \text{Ext}[\Gamma] : \quad \text{Min.} \quad \sum_{xy} c(xy)d(x, y) \]

\[ \text{s.t.} \quad (X, d): \text{extension of} \quad (V_\Gamma, d_\Gamma) \]
Metric relaxation (Karzanov 1998)

\[
\text{Ext}[\Gamma]: \quad \text{Min. } \sum_{xy} c(xy)d(x, y)
\]
\[
\text{s.t. } d(x, x) = 0 \quad (x \in X)
\]
\[
d(x, y) = d(y, x) \geq 0 \quad (x, y \in X)
\]
\[
d(x, y) + d(y, z) \geq d(x, z) \quad (x, y, z \in X)
\]
\[
d(s, t) = d_\Gamma(s, t) \quad (s, t \in V_\Gamma \subseteq X)
\]

Rem: Ext[\Gamma] is polynomial size LP \rightarrow P
Metric relaxation (Karzanov 1998)

\[
\textbf{Ext}[\Gamma] : \quad \text{Min.} \quad \sum_{xy} c(xy) d(x, y) \\
\text{s.t.} \quad \begin{align*}
    d(x, x) &= 0 \quad (x \in X) \\
    d(x, y) &= d(y, x) \geq 0 \quad (x, y \in X) \\
    d(x, y) + d(y, z) &\geq d(x, z) \quad (x, y, z \in X) \\
    d(s, t) &= d_{\Gamma}(s, t) \quad (s, t \in V_{\Gamma} \subseteq X)
\end{align*}
\]

Rem: Ext[\Gamma] is polynomial size LP \rightarrow P

Q. What is \(\Gamma\) for which \(\textbf{Ext}[\Gamma]\) is exact (for every \(X, c\))?

c.f. Multicommodity flows (Karzanov 1998, H. 2009 ~)}
Frame = graph for which $\text{Ext}[\Gamma]$ is exact

\[ \Gamma: \text{frame} \iff \]
- bipartite
- no isometric cycle of length $> 4$
- orientable $\iff \exists$ orientation: $\forall$ 4-cycle

Rem: frame is obtained by gluing $K_{2,m}$ and $K_2$ (in a certain way)
Frame = graph for which $\text{Ext}[\Gamma]$ is exact

\[ \Gamma: \text{frame} \iff \]

- bipartite
- no isometric cycle of length $> 4$
- orientable $\iff \exists$ orientation: $\forall$ 4-cycle

Rem: frame is obtained by gluing $K_{2,m}$ and $K_2$ (in a certain way)
Frame = graph for which $\text{Ext}[\Gamma]$ is exact

$\Gamma$: frame \iff
- bipartite
- no isometric cycle of length $> 4$
- orientable \iff \exists orientation: $\forall$ 4-cycle

\[
\begin{array}{c}
\text{c.f. frame} \simeq \text{CAT(0)-complex of folders} \text{ (Chepoi 2000)}
\end{array}
\]
Frame = graph for which \( \text{Ext}[\Gamma] \) is exact

\[ \Gamma: \text{ frame} \iff \]
- bipartite
- no isometric cycle of length \( > 4 \)
- \( \text{orientable} \iff \exists \text{ orientation: } \forall \text{ 4-cycle} \)

Theorem (Karzanov 1998)

\( \Gamma \) is a frame if and only if \( \text{Ext}[\Gamma] = 0-\text{Ext}[\Gamma] \).

Corollary (Karzanov 1998)

If \( \Gamma \) is a frame, then \( 0-\text{Ext}[\Gamma] \) is in \( \mathbb{P} \).
Rem. \{frames\} is not closed under Cartesian product
Rem. frame \not\in\{\text{median graphs}\}, \not\subseteq\{\text{frames}\}

\[ \begin{array}{c}
\text{modular,} \\
\text{triple has a median.}
\end{array} \]

\[ \begin{array}{c}
\text{orientable,} \\
\text{orientation:} \\
4\text{-cycle}
\end{array} \]

Rem. frame = orientable hereditary modular graph (c.f. Bandelt 85)
Ex. Hasse diagram of modular lattice
Rem. \{frames\} is not closed under Cartesian product
Rem. frame \neq median graph
\[\in \{\text{median graphs}\}, \not\in \{\text{frames}\}\]

\[\not\in \{\text{median graphs}\}, \in \{\text{frames}\}\]

\(\Gamma: \text{modular} \iff \forall \text{ triple has a median.}\)
\(\Gamma: \text{orientable} \iff \exists \text{ orientation: } \forall \text{ 4-cycle}\)

Rem. frame = orientable *hereditary* modular graph (c.f. Bandelt 85)
Ex. Hasse diagram of modular lattice
Theorem (Karzanov 1998)

If $\Gamma$ is not modular or not orientable, then $0\text{-Ext}[\Gamma]$ is NP-hard.
Main result

Theorem (H. 2012, SODA 2013)
If $\Gamma$ is orientable modular, then $0$-Ext[$\Gamma$] is in $P$. 

NP-hard
Proof idea: *Discrete Convex Analysis & Valued-CSP*

- **Discrete Convex Analysis** (Murota 1996 ~)
  
  → *Our approach suggests “Discrete Convex Analysis on $\Gamma$”*
Proof idea: *Discrete Convex Analysis & Valued-CSP*

- **Discrete Convex Analysis** (Murota 1996 ~)
  → *Our approach suggests “Discrete Convex Analysis on $\Gamma$”*

- **Valued-CSP**
  ≃ Minimization of a sum of functions with *bounded arity*

\[
\begin{align*}
\text{Min.} & \quad \sum_{1 \leq i_1 < i_2 < \cdots < i_K \leq N} f^{i_1,i_2,\ldots,i_K}(\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_K}) \\
\text{s.t.} & \quad (\rho_1, \rho_2, \ldots, \rho_N) \in D \times D \times \cdots \times D = \bar{D}
\end{align*}
\]
Proof idea: *Discrete Convex Analysis & Valued-CSP*

- **Discrete Convex Analysis** (Murota 1996 ~)
  → *Our approach suggests “Discrete Convex Analysis on $\Gamma$”*
- **Valued-CSP**
  $\simeq$ Minimization of a sum of functions with *bounded arity*

\[
\begin{align*}
\text{Min.} & \quad \sum_{1 \leq i_1 < i_2 < \cdots < i_K \leq N} f^{i_1, i_2, \ldots, i_K}(\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_K}) \\
\text{s.t.} & \quad (\rho_1, \rho_2, \ldots, \rho_N) \in D \times D \times \cdots \times D = \overline{D}
\end{align*}
\]

$0\text{-Ext}[\Gamma] \simeq$

\[
\begin{align*}
\text{Min.} & \quad \sum_{i} \sum_{q \in V_{\Gamma}} c_{iq} d_{\Gamma}(q, x_i) + \sum_{i < j} c_{ij} d_{\Gamma}(x_i, x_j) \\
\text{s.t.} & \quad (x_1, x_2, \ldots, x_n) \in V_{\Gamma} \times V_{\Gamma} \times \cdots \times V_{\Gamma}.
\end{align*}
\]
Proof idea: *Discrete Convex Analysis & Valued-CSP*

- **Discrete Convex Analysis** (Murota 1996 ~)
  → *Our approach suggests “Discrete Convex Analysis on $\Gamma$”*

- **Valued-CSP**
  ≃ Minimization of a sum of functions with *bounded arity*

\[
\begin{align*}
\text{Min.} & \quad \sum_{1 \leq i_1 < i_2 < \cdots < i_K \leq N} f_{i_1,i_2,\ldots,i_K}(\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_K}) \\
\text{s.t.} & \quad (\rho_1, \rho_2, \ldots, \rho_N) \in D \times D \times \cdots \times D = \bar{D}
\end{align*}
\]

≃ Minimization of a sum of functions with *bounded arity*

\[
\begin{align*}
\text{Min.} & \quad \sum_{i_1,i_2,\ldots,i_K} \sum_{p \in D^K} \lambda_{i_1,i_2,\ldots,i_K}^{p_1,p_2,\ldots,p_K} f_{i_1,i_2,\ldots,i_K}(p_1, p_2, \ldots, p_K) \\
\text{s.t.} & \quad \ldots \lambda_{i_1,i_2,\ldots,i_K}^{p_1,p_2,\ldots,p_K} \in \{0, 1\}
\end{align*}
\]

→ Poly($N^K|D|^K$)-size IP formulation
Basic LP-relaxation ($\rightarrow \text{Poly}(N^K|D|^K)$-size LP)

Min. $\sum_{i_1, i_2, \ldots, i_K} \sum_{p_1, p_2, \ldots, p_K} \lambda_{i_1, i_2, \ldots, i_K}^{p_1, p_2, \ldots, p_K} f_{i_1, i_2, \ldots, i_K}(p_1, p_2, \ldots, p_K)$

s.t. $\lambda_{p_1, p_2, \ldots, p_K}^{i_1, i_2, \ldots, i_K} \in [0, 1]$
Basic LP-relaxation ($\rightarrow \text{Poly}(N^K|D|^K)$-size LP)

Min. $\sum_{i_1, i_2, \ldots, i_K} \sum_{p \in D^K} \lambda^{i_1, i_2, \ldots, i_K}_{p_1, p_2, \ldots, p_K} f^{i_1, i_2, \ldots, i_K}(p_1, p_2, \ldots, p_K)$

s.t. $\sum_{i_1, i_2, \ldots, i_K} \lambda^{i_1, i_2, \ldots, i_K}_{p_1, p_2, \ldots, p_K} \in [0, 1]$

Theorem (Thapper-Živný, FOCS 2012)

BLP is exact if

$\exists$ convex combination $\sum_g \omega(g) g$ of operations $g : D \times D \rightarrow D$ s.t.

1. $\frac{1}{2}(f(p) + f(q)) \geq \sum_g \omega(g) f(g(p, q))$
2. $(f \in \{ f^{i_1, i_2, \ldots, i_K} \}, (p, q) \in \bar{D} \times \bar{D})$,
3. $\{ g \mid \omega(g) > 0 \} \ni$ semilattice operation.
Basic LP-relaxation ($\rightarrow \text{Poly}(N^K|D|^K)$-size LP)

Min. $\sum_{i_1, i_2, \ldots, i_K} \sum_{p \in D^K} \lambda_{p_1, p_2, \ldots, p_K}^{i_1, i_2, \ldots, i_K} f_{i_1, i_2, \ldots, i_K}(p_1, p_2, \ldots, p_K)$

s.t. $\lambda_{p_1, p_2, \ldots, p_K}^{i_1, i_2, \ldots, i_K} \in [0, 1]$

---

**Theorem (Thapper-Živný, FOCS 2012)**

BLP is exact if

$\exists$ convex combination $\sum_g \omega(g) g$ of operations $g: D \times D \to D$ s.t.

- $\frac{1}{2}(f(p) + f(q)) \geq \sum_g \omega(g) f(g(p, q))$
- $(f \in \{f_{i_1, i_2, \ldots, i_K}\}, (p, q) \in \overline{D} \times \overline{D})$, $\{g \mid \omega(g) > 0\} \ni$ semilattice operation.

... but I don’t know whether this is directly applicable to 0-Ext.
0-Ext$[\Gamma] \simeq$

\[
\text{Min. } \sum_i \sum_{q \in V_\Gamma} c_{iq} d_\Gamma(q, x_i) + \sum c_{ij} d_\Gamma(x_i, x_j) \text{ s.t. } (x_i) \in (V_\Gamma)^n
\]

Obs. If $\Gamma = \text{path}$, then $\Gamma \times \Gamma \times \cdots \times \Gamma \simeq [1, m]^n \subseteq \mathbb{Z}^n$,

\[
\text{Min. } \sum_i \sum_{q \in [1, m]} c_{iq} |q - x_i| + \sum_{i<j} c_{ij} |x_i - x_j| \text{ s.t. } x \in [1, m]^n \subseteq \mathbb{Z}^n.
\]

→ This is $L$-convex function minimization in DCA
Idea from Discrete Convex Analysis

0-Ext$[\Gamma]$ $\simeq$

$$\min \sum_{i} \sum_{q \in V_\Gamma} c_{iq} d_\Gamma(q, x_i) + \sum c_{ij} d_\Gamma(x_i, x_j) \text{ s.t. } (x_i) \in (V_\Gamma)^n$$

**Obs.** If $\Gamma = \text{path}$, then $\Gamma \times \Gamma \times \cdots \times \Gamma \simeq [1, m]^n \subseteq \mathbb{Z}^n$,

$$\min \sum_{i} \sum_{q \in [1, m]} c_{iq} |q - x_i| + \sum_{i < j} c_{ij} |x_i - x_j| \text{ s.t. } x \in [1, m]^n \subseteq \mathbb{Z}^n.$$  

$\rightarrow$ *This is L-convex function minimization in DCA*

**Fact.** L-convex functions have many nice properties:

- *Local* optimality $\Rightarrow$ *Global* optimality
- Checking Local optimality $\simeq$ Submodular Function Minimization
- Descent algorithm by successive SFM.
• We define **submodular functions on modular semilattices** & **L-convex functions on orientable modular graphs**:

  - **Local** optimality ⇒ **Global** optimality
  - Checking local optimality ≃ SFM on modular semilattice (? ∈ P or ∉ P ?)
  - Descent algorithm by successive SFM.
  - **0-Ext[Γ]** is **L-convex function minimization** on \( Γ \times \cdots \times Γ \)

C.f. Modular semilattice (Bandelt-Van De Vel-Verheul 1993)
We define submodular functions on modular semilattices & L-convex functions on orientable modular graphs:

- **Local optimality** ⇒ **Global optimality**
- Checking local optimality ≃ SFM on modular semilattice (? ∈ P or ̸∈ P ?)
- Descent algorithm by successive SFM.
- **0-Ext[Γ]** is **L-convex function minimization** on $Γ \times \cdots \times Γ$

C.f. Modular semilattice (Bandelt-Van De Vel-Verheul 1993) 

*Thapper-Živný theorem is applicable to our submodular functions!*

→ A sum of submodular functions with bounded arity can be minimized in polynomial time.
We define **submodular functions on modular semilattices** & **L-convex functions on orientable modular graphs**:

- *Local* optimality ⇒ *Global* optimality
- Checking local optimality ≃ SFM on modular semilattice (? ∈ P or ∉ P ?)
- Descent algorithm by successive SFM.
- **0-Ext[Γ]** is **L-convex function minimization** on \( Γ × ⋯ × Γ \)

c.f. Modular semilattice (Bandelt-Van De Vel-Verheul 1993)

*Thapper-Živný theorem is applicable to our submodular functions!*

→ A sum of submodular functions with bounded arity can be minimized in polynomial time.

→ Descent algorithm (with scaling) finds a global optimum of **0-Ext[Γ]** in polynomial time.
How to define \( L \)-convex function \( g \) on \( \Gamma, o \)

- **Modular complex** \( \Delta(\Gamma, o) \) of \( (\Gamma, o) \)

- **Lovász extension** \( \bar{g} : \Delta(\Gamma, o) \rightarrow \mathbb{R}_+ \) of \( g : V_{\Gamma} \rightarrow \mathbb{R} \).

- **Neighborhood semilattice** \( \mathcal{L}^*_p \) (\( \leftarrow \) modular semilattice)

**Def:** \( g \) is \( L \)-convex \( \Leftrightarrow \) \( \bar{g} \) is submodular on \( \mathcal{L}^*_p \) (\( \forall p \in V_{\Gamma} \)).
Some technical detail

1. Definition of submodular functions on modular semilattices
2. How to apply Thapper-Živný theorem

Min. \[ \sum_{i_1, i_2, \ldots, i_K} f^{i_1, i_2, \ldots, i_K}(\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_K}) \]

s.t. \((\rho_1, \rho_2, \ldots, \rho_N) \in \mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L} = \bar{\mathcal{L}}\)

Theorem (Thapper-Živný, FOCS 2012)

BLP is exact if

\exists \text{ convex combination } \sum_g \omega(g)g \text{ of operations } g : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \text{ s.t. }

1. \[ \frac{1}{2}(f(p) + f(q)) \geq \sum_g \omega(g)f(g(p, q)) \]

\[ (f \in \{ f^{i_1, i_2, \ldots, i_K} \}, (p, q) \in \bar{\mathcal{L}} \times \bar{\mathcal{L}}), \]

2. \( \{ g \mid \omega(g) > 0 \} \ni \text{ semilattice operation.} \)
Meet-semilattice $L$ is **modular** $\iff$

- every lower ideal is a modular lattice
- $u \lor v, v \lor w, w \lor u \in L \implies u \lor v \lor w \in L$
Meet-semilattice $\mathcal{L}$ is **modular** $\iff$

- every lower ideal is a modular lattice
- $u \lor v, v \lor w, w \lor u \in \mathcal{L} \Rightarrow u \lor v \lor w \in \mathcal{L}$

$(p, q) \in \mathcal{L} \times \mathcal{L}$ is said to be
- **bounded** if $p \lor q \in \mathcal{L}$
- **antipodal** if $\forall$ bounded $(a, b) \in [p \land q, p] \times [p \land q, q]$, $r[a, p]r[b, q] \geq r[a, a \lor b]r[b, a \lor b]$.

$u = a \lor b \iff (r(a), r(b)) \in \mathbb{R}^2$
$f : \mathcal{L} \rightarrow \mathbb{R}$ is **submodular** $\iff$

- **submodularity inequality:**
  \[ f(p) + f(q) \geq f(p \wedge q) + f(p \vee q) \quad (\forall (p, q):\text{bounded}) \]

- **\wedge\text{-convexity inequality:}**
  \[ r[p \wedge q, q]f(p) + r[p \wedge q, p]f(q) \geq (r[p \wedge q, p] + r[p \wedge q, q])f(p \wedge q) \]
  \[ (\forall (p, q):\text{antipodal}) \]

Ex. submodular set functions, bisubmodular functions, multimatroid rank functions (Bouchet 1997), $k$-submodular functions (Huber-Kolmogorov 2012)

Rem. closed under restriction/extension/sum

- $f$: submo on $\mathcal{L} \times \mathcal{L}' \Rightarrow f(\cdot, q)$: submo on $\mathcal{L}$
- $f$: submo on $\mathcal{L} \Rightarrow f'$: submo on $\mathcal{L} \times \mathcal{L}'$ where $f'(p, q) := f(p)$. 
Toward Thapper-Živný criterion

From Def. of submodular functions:

\[
\begin{align*}
    f(p) + f(q) & \geq f(p \land q) + f(p \lor q) \\
    f(p) + f(q) & \geq f(p \land q) + \frac{r[p \land q, p]f(p) + r[p \land q, q]f(q)}{r[p \land q, p] + r[p \land q, q]} \\
    & \quad \text{for } ((p, q) : \text{bounded}) \\
    f(p) + f(q) & \geq f(p \land q) + \frac{r[p \land q, p]f(p) + r[p \land q, q]f(q)}{r[p \land q, p] + r[p \land q, q]} \\
    & \quad \text{for } ((p, q) : \text{antipodal})
\end{align*}
\]
Toward Thapper-Živný criterion

From Def. of submodular functions:

\[
\begin{align*}
    f(p) + f(q) & \geq f(p \wedge q) + f(p \vee q) \\
    f(p) + f(q) & \geq f(p \wedge q) + \frac{r[p \wedge q, p]f(p) + r[p \wedge q, q]f(q)}{r[p \wedge q, p] + r[p \wedge q, q]} \\
\end{align*}
\]

((p, q) : bounded)

((p, q) : antipodal)

Step 1: unified inequalities:

\[
f(p) + f(q) \geq f(p \wedge q) + \sum_{u \in E^{p,q}} \nu_u f(u) \\
\]  

((p, q) \in \bar{\mathcal{L}} \times \bar{\mathcal{L}})
Toward Thapper-Živný criterion

From Def. of submodular functions:

\[
\begin{align*}
    f(p) + f(q) & \geq f(p \land q) + f(p \lor q) & ((p, q) : \text{bounded}) \\
    f(p) + f(q) & \geq f(p \land q) + \frac{r[p \land q, p]f(p) + r[p \land q, q]f(q)}{r[p \land q, p] + r[p \land q, q]} & ((p, q) : \text{antipodal})
\end{align*}
\]

Step 1: unified inequalities:

\[
f(p) + f(q) \geq f(p \land q) + \sum_{u \in E_{p,q}} \nu_{u}^{p,q} f(u) \quad ((p, q) \in \tilde{L} \times \tilde{L})
\]

Step 2: fractional polymorphism \( \frac{1}{2} \land + \frac{1}{2} \sum_{g} \nu(g)g \):

\[
f(p) + f(q) \geq f(p \land q) + \sum_{g: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}} \nu(g) f(g(p, q)) \quad ((p, q) \in \tilde{L} \times \tilde{L})
\]
$l(p, q) := \{u \mid u = a \lor b, \exists (a, b) \in [p \land q, p] \times [p \land q, q]\}$

$l(p, q) \ni u = a \lor b \mapsto (r(a), r(b)) \in \mathbb{R}^2$

$(p, q)$-envelope $\mathcal{E}^{p,q} :=$

$\{ a \lor b \mid (r(a), r(b)) \text{ is a maximal extreme point of } \text{Conv}(l(p, q)) \}$
\[ I(p, q) := \{ u \mid u = a \lor b, \exists (a, b) \in [p \land q, p] \times [p \land q, q] \} \]
\[ I(p, q) \ni u = a \lor b \mapsto (r(a), r(b)) \in \mathbb{R}^2 \]

\((p, q)\)-envelope \( E^{p, q} := \{ a \lor b \mid (r(a), r(b)) \text{ is a maximal extreme point of } Conv(I(p, q)) \}\)

\[
\begin{align*}
\sum_{u \in E^{p,q}} \nu([u])f(u) & \geq f(p) + f(q) - f(p \land q) \\
\nu : \{ \text{convex cones in } \mathbb{R}^2_+ \} & \rightarrow \mathbb{R}_+ \text{ satisfying} \\
\nu(\mathbb{R}_+) & = 1, \quad \nu(C) + \nu(C') = \nu(C \cap C') + \nu(C \cup C') \quad (C \cap C' \neq \emptyset)
\end{align*}
\]
\[ l(p, q) := \{u \mid u = a \vee b, \exists (a, b) \in [p \wedge q, p] \times [p \wedge q, q]\} \]
\[ l(p, q) \ni u = a \vee b \mapsto (r(a), r(b)) \in \mathbb{R}^2 \]

\((p, q)\)-envelope \(E^{p,q} := \{a \vee b \mid (r(a), r(b)) \text{ is a maximal extreme point of } \text{Conv}(l(p, q))\}\)

\[ f(p) + f(q) \geq f(p \wedge q) + \sum_{u \in E^{p,q}} \nu([u])f(u) \quad ((p, q) \in \mathcal{L} \times \mathcal{L}) \]

\[ \nu := \frac{\sin \alpha}{\cos \alpha + \sin \alpha} - \frac{\sin \beta}{\cos \beta + \sin \beta} \]
cone-decomposition of $\mathbb{R}^2_+ \rightarrow$ fractional polymorphism

\[
\begin{align*}
&u_3 = q \\
&u_0 = p \land q \\
&\text{Conv}(I(p, q)) \\
&u_1
\end{align*}
\]
cone-decomposition of $\mathbb{R}^2_+ \rightarrow$ fractional polymorphism

$$C(\mathcal{L}) := \begin{cases} \bigtriangleup \quad (p, q) \in \mathcal{L} \times \mathcal{L} \\ \end{cases}$$

$$C(\mathcal{L}) \ni C \mapsto g_C : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$$

$$g_C(p, q) := u \in \mathcal{E}^{p, q}, \text{ where } C \subseteq [u].$$
cone-decomposition of $R^2_+ \rightarrow$ fractional polymorphism

\[ C(\mathcal{L}) := (p, q) \in \mathcal{L} \times \mathcal{L} \]

$C(\mathcal{L}) \ni C \mapsto g_C : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$

$g_C(p, q) := u \in \mathcal{E}^{p, q}$, where $C \subseteq [u]$.

\[ f(p) + f(q) \geq f(p \wedge q) + \sum_{C \in C(\mathcal{L})} \nu(C)f(g_C(p, q)) \quad ((p, q) \in \widetilde{\mathcal{L}} \times \widetilde{\mathcal{L}}) \]

\[ \rightarrow \frac{1}{2} \wedge + \frac{1}{2} \sum_{C \in C(\mathcal{L})} \nu(C)g_C \text{ is a fractional polymorphism } \ni \wedge. \]
Question

What is $\Gamma$ for which $0\text{-Ext}[\Gamma]$ is in $P$?

Answer

$0\text{-Ext}[\Gamma] \in \begin{cases} P & \text{if } \Gamma \text{ is orientable modular} \\ \text{NP-hard} & \text{otherwise} \end{cases}$
Many interesting aspects & future work:

- Combinatorial min-max theorems in multicommodity flows
- Tight spans of metric spaces
- Modular lattices, and modular semilattices
- Connection to CAT(0)-complexes
- Dichotomy theorems in Valued-CSP

Thank you for your attention!
Many interesting aspects & future work:

- Combinatorial min-max theorems in multicommodity flows
- Tight spans of metric spaces
- Modular lattices, and modular semilattices
- Connection to CAT(0)-complexes
- Dichotomy theorems in Valued-CSP
- Toward a new Discrete Convex Analysis including them!
Many interesting aspects & future work:

- Combinatorial min-max theorems in multicommodity flows
- Tight spans of metric spaces
- Modular lattices, and modular semilattices
- Connection to CAT(0)-complexes
- Dichotomy theorems in Valued-CSP

*Toward a new Discrete Convex Analysis including them!*

*Thank you for your attention!*