SIGNATURE OF GROTHENDIECK RESIDUE

MOHAMMAD REZA RAHMATI

Abstract. For \( f : \mathbb{R}^{n+1} \to \mathbb{R} \) an algebraically isolated hypersurface singularity germ, it can be assigned a non-degenerate bilinear form \( \langle \cdot, \cdot \rangle_L : A_f \times A_f \to A_f \to \mathbb{R} \) where, \( A_f \) is the Jacobi ring of \( f \), the first map is the usual product in \( A_f \) and the second map is an arbitrary linear map such that it maps the class of Hessian of \( f \) to a positive number. It is a theorem by Grothendieck that this form is non-degenerate, and also another theorem by Eisenbud-Levine that its signature is independent of the choice of the second linear map with the appropriate property. We provide a method to calculate the signature of this form in terms of Hodge numbers of vanishing cohomology associated to fiberation, \( f \). The result also applies to topological indices of singularities of vector fields.

Introduction

Let \( f_0, \ldots, f_n : \mathbb{R}^{n+1} \to \mathbb{R} \) be germs of real analytic functions that form a regular sequence in \( \mathbb{R}\{x_0, \ldots, x_n\} \), and let

\[
A_f := \frac{\mathbb{R}\{x_0, \ldots, x_n\}}{(f_0, \ldots, f_n)}
\]

The class of the Jacobian

\[
J := \det(\partial_j f_i)_{ij} \in A_f
\]

generates the unique minimal ideal of \( A_f \). A symmetric bilinear form

\[
\langle \cdot, \cdot \rangle_L : A_f \times A_f \to A_f \overset{L}{\to} \mathbb{R}, \quad L(J) > 0
\]

can be defined, where the first map is the usual product in \( A_f \) and the second map is an arbitrary linear map. It is a theorem by Grothendieck (Local Duality Theorem, \cite{?}) that this form is non-degenerate, and also another observation by Eisenbud-Levine that its signature is independent of the choice of the second linear map with the appropriate property.

Key words and phrases. Signature, Riemann-Hodge bilinear relations, Lefschetz property, Hodge index theorem, residue pairing.
1. Signature associated to a singular point of a hypersurface

By an algebraically isolated hyper-surface singularity we mean an analytic germ \( f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \) having an isolated singularity at 0 and also the singularity remain isolated for \( f_C \) at the origin. Then \( A_f \) is a complete intersection algebra having Lefschetz property;

**Lemma 1.1. (X. Gomez Mont)\[M\], [GGM]**

There are linear subspaces \( P_j; j = 1, ..., l + 1 \) of \( A \) called primitive subspaces such that

\[
A = \bigoplus_{1 \leq j \leq l+1} M^k_j P_j
\]

This theorem is an application of Jordan-Holder decomposition with the nilpotent operator \( M_f = f : A \to A \)

**Remark 1.2.** By the above theorem \( A_f \) is a graded Artinian algebra having strong Lefschetz property, whose socle has dimension 1 over \( \mathbb{C} \). This shows that \( A_f \) is Gorenstein and therefore has a graded Poincare duality, \([MW]\).

Define a decreasing filtration by ideals in \( A \)

\[
K_m = \text{Ann}(f) \cap (f^{m-1})
\]

Also define a family of bilinear forms

\[
Q_m := \langle a, b \rangle_{f,m} = L\left(\frac{a}{f^{m-1}} \cdot b\right)
\]

where \( L \) is some linear map with \( L(\text{Hess}(f)) > 0 \).

**Theorem 1.3. (X. Gomez Mont)\[M\], [GGM]**

For \( m \geq 1 \), the mapping

\[
M_f^{m-1} : P_m \to K_m / K_{m+1}
\]

is a well-defined isomorphism. The pairings

\[
Q_m : P_m \times P_m \to \mathbb{R}
\]

are non-degenerate symmetric bilinear forms, via the isomorphisms given.

Define

\[
\sigma_i = \text{sign } Q_i
\]

The signature of Grothendieck pairing is \( \sigma = \sum \sigma_i \). It is known that the signature of local residue is equal to the index of the gradient vector field of \( f \), namely \( \nabla(f) = (\partial_0(f), ..., \partial_n(f)) \), \([V]\).
2. Hodge theory and Residue pairing

For a holomorphic germ \( f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) with an isolated critical point; the local residue \( \text{Res}_{f,0}(\omega, \eta) \) where, \( \omega, \eta \) are differential forms; defines a symmetric bilinear pairing (Grothendieck Pairing= residue form) which is non-degenerate (Proved by Grothendieck). In fact after division by \( df \) each of the forms \( \omega \) and \( \eta \) define a middle dimensional cohomology class of every local level hypersurface of the function \( f \). In this way, the forms \( \omega \) and \( \eta \) define two sections of the vanishing cohomology bundle. The asymptotic of the polarization form on vanishing cohomology gives a meromorphic function on a neighborhood of the critical value \( 0 \in \mathbb{C} \). The residue of this function at \( 0 \in \mathbb{C} \) is equal to \( \text{Res}_{0,f}(\omega, \eta) \).

Assume \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) is a germ of isolated singularity. As previously mentioned suppose,

\[
H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda}(I^{p,q})^\lambda
\]

be the Deligne-Hodge \( C^\infty \)-splitting, and generalized eigen-spaces. Consider the isomorphism obtained by composing the two maps,

\[
\Phi : H^n(X_\infty, \mathbb{C}) \xrightarrow{\hat{\Phi}} \bigoplus_{-1 \leq \beta < n} \text{Gr}^\beta V^H \to \bigoplus_{-1 \leq \beta < n} \text{Gr}^\beta V^H / \text{Gr}^\beta \partial^{-1} H^\prime = \Omega_f
\]

\[
\hat{\Phi}|_{I^{p,q}} := \partial_t^{-n} \circ \psi_\alpha(I^{p,q})^\lambda
\]

**Theorem 2.1.** Let \( f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \), be a holomorphic germ with isolated singularity at \( 0 \). Then, the isomorphism \( \Phi \) makes the following diagram commutative up to a complex constant;

\[
\begin{array}{ccc}
\widehat{\text{Res}}_{f,0} : \Omega(f) \times \Omega(f) & \longrightarrow & \mathbb{C} \\
\downarrow(\Phi^{-1}, \Phi^{-1}) & & \parallel \\
S : H^n(X_\infty) \times H^n(X_\infty) & \longrightarrow & \mathbb{C}
\end{array}
\]

where,

\[
\widehat{\text{Res}}_{f,0} = \text{res}_{f,0}(\bullet, \hat{C}\bullet)
\]

and \( \hat{C} \) is defined relative to the Deligne-Hodge decomposition of \( \Omega_f \), via the isomorphism \( \Phi \).

\[
\Omega_f = \bigoplus_{p,q} J^{p,q} \quad \hat{C}|_{J^{p,q}} = (-1)^p
\]

In other words;

\[
S(\frac{\omega}{df}, \frac{\eta}{df}) = \text{Const} \times \text{res}_{f,0}(\omega, \hat{C}\eta),
\]
Theorem 2.2. Assume \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for asymptotic fiber \( \Omega_f \), via the aforementioned isomorphism \( \Phi \). Moreover, there exists a set of forms \( \{ \text{Res}_k \} \) giving a graded polarization for \( \Omega_f \).

Remark 2.3. Let \( G \) be the Gauss-Manin system associated to a polarized variation of Hodge structure \( (\mathcal{L}_Q, \nabla, F, S) \) of weight \( n \), with \( S : \mathcal{L}_Q \otimes \mathcal{L}_Q \to \mathbb{Q}(-n) \) the polarization. Then we have the isomorphism

\[
\bigoplus_{k \in \mathbb{Z}} \text{Gr}_k^k \mathcal{G} \to \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\text{Gr}_k^{n-k} \mathcal{G}, \mathcal{O}_X)
\]

given by (up to a sign factor) \( \lambda \to S(\lambda, -) \), for \( \lambda \in \text{Gr}_k^k \mathcal{G} \).

Corollary 2.4. Assume \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) defines an isolated singularity germ. Then polarization form of MHS of vanishing cohomology and the modified residue pairing on the limit fiber \( \Omega_f \) are given by the same matrix in corresponding basis’s.

Theorem 2.5. The signature \( \sigma \) of Grothendieck pairing on the real vector-space \( A_f \) associated to algebraically isolated singularity germ; is equal to the signature of the polarization form associated to the vanishing cohomology of the complex fibration; \( f : \mathbb{C}^{n+1} \to \mathbb{C} \), the complexification of \( f \).

Proof: Trivial.

The above mentioned signature may be calculated via Hodge index theorem, \[\text{JS}\], as

\[
\sigma = \sum_{p+q=n+2} (-1)^q h_1^{pq} + 2 \sum_{p+q \geq n+3} (-1)^q h_1^{pq} + \sum (-1)^q h_{\neq 1}^{pq}
\]

in case \( n \) is even.

Corollary 2.6. The index of the gradient vector field associated to a real analytic germ \( f : \mathbb{R}^n \to \mathbb{R} \) with an algebraically isolated singularity at \( 0 \in \mathbb{R}^n \) is equal to the Signature calculated from the PMHS associated to the Milnor fibration \( f_C : \mathbb{C}^n \to \mathbb{C} \).

The significance of this theorem is that, it allows to calculate the topological invariants of \( f \) in terms of Hodge numbers. It allows various applications to calculations related to Euler characteristic formulas computing different indices associated to isolated singularities of holomorphic germs, analytic vector fields and complete intersections.
Remark 2.7. If \( X = \sum_{i=0}^{n} X^i \frac{\partial}{\partial x_i} \) is a real analytic vector field, with an algebraically isolated zero at 0, then the Poincare-Hopf index of of \( X \) at 0 is the signature of the bilinear form:

\[
\langle \cdot, \cdot \rangle_L : A_f \times A_f \to A_f \xrightarrow{L} \mathbb{R}, \quad A_f := \mathbb{R}\{x_0, \ldots, x_n\}^{(X^0, \ldots, X^n)}
\]

Where \( L \) is a linear map such that \( L(J) > 0 \). If \( X \) is tangent to the fiber \( V_0 := f^{-1}(0) \), then \( df(X) = h.f \), with \( h \) a real analytic function namely co-factor. If 0 is a smooth point of \( V_0 \), then the P-H index of \( X \) is the signature of the bilinear form:

\[
\langle \cdot, \cdot \rangle_L : \frac{A}{\text{ann}(h)} \times \frac{A}{\text{ann}(h)} \to A \xrightarrow{L} \mathbb{R}, \quad A := \mathbb{R}\{x_0, \ldots, x_n\}^{(X^0, \ldots, X^n)}
\]

When 0 is an isolated singularity in \( V_0 \) the signature may be calculated in different cases of \( n \) even, or odd, [GGM].

References

[G] P. Griffiths, J. Harris. Principles of algebraic geometry. Wiley Classics Library. John Wiley-Sons Inc., New York, 1994. Reprint of the 1978 original.

[GGM] L. Giraldo, X. Gomez Mont, P. Mardesic, Flags in zero dimensional complete intersection algebras and indices of real vector fields, arxiv:math/0612275v2, Jan 2008

[M] P. Mardesic; Index of singularities of real vector fields on singular hypersurfaces,

[MW] T. Maeno, J. Watanabe, Lefschetz elements of Artinian Gorenstein algebras and Hessians of Homogeneous polynomials, arXiv:0903.3581v3, Dec 2009

[R] M. Rahmati, Asymptotic polarization, Opposite filtration and Primitive forms, arxiv.org submit/0927050 , 5 March 2014

[JS] J. Steenbrink; Mixed Hodge structure on the vanishing cohomology, 1976

[V] On the local residue and the intersection form on the vanishing cohomology, A. Varchenko, 1986

Centro de Investigacion en Matematicas, CIMAT

E-mail address: mrahmati@cimat.mx