QED model of the radiation escape from matter

T Zalialiutdinov¹, D Solovyev¹ and L Labzowsky¹,²

¹ V A Fock Institute of Physics, St Petersburg State University, Petrodvorets, Oulianovskaya 1, 198504 St Petersburg, Russia
² Petersburg Nuclear Physics Institute, 188300 Gatchina, St Petersburg, Russia

E-mail: dimas@landau.phys.spbu.ru

Received 11 April 2012, in final form 25 June 2012
Published 25 July 2012
Online at stacks.iop.org/JPhysB/45/165006

Abstract
A simple model based on quantum electrodynamics (QED) is presented to estimate the contribution of the excited-level few-photon decays to the radiation escape from the matter in the epoch of cosmological hydrogen recombination. It is shown that apart from the widely studied two-photon decays, some specific three-photon decays can contribute with an accuracy of 0.1%, which is required by recent astrophysical observations.

1. Introduction

Theory of the cosmological hydrogen recombination became one of the most intensively discussed fundamental physics problems in the last decade. The interest comes from the accurate measurements of the asymmetry in the temperature and polarization distribution of the cosmic microwave background (CMB) [1, 2]. The launching of the Planck Surveyor, which enables us to perform these measurements with an accuracy of 0.1%, makes the situation even more intriguing.

The modern theory of cosmological hydrogen recombination began with the papers by Zel’dovich et al [3] and Peebles [4]. It was argued that the bound–bound one-photon transitions from the upper levels to the lower ones did not permit the hydrogen atoms to reach their ground states: each photon released in such a transition in one atom was immediately absorbed by another atom. These reabsorption processes did not allow the radiation to ‘escape’ the interaction with the matter. As was first established in [3, 4], the two-photon 2s–1s transition presents the main channel for the radiation ‘escape’ and formation of the CMB. This transition also led to the final hydrogen recombination. Hence, the recent properties of the CMB are essentially defined by the two-photon processes during the cosmological recombination epoch.

In [5], the importance of the two-photon decays from excited states with \( n > 2 \) for the detailed analysis of the properties of CMB was noted. Over the past few years, the theory of cosmological recombination was essentially detalized by many authors. In particular, in [5, 6], it was demonstrated that the two-photon transitions \( ns \rightarrow 1s(n > 2) \) and \( nd \rightarrow 1s \) can also give a sizeable contribution to the radiation ‘escape’. There is a difference between the decay of \( ns(n > 2) \) and \( nd \) states and the decay of the 2s state. This difference is due to the presence of cascade transitions as the dominant decay channels in the cases of \( ns(n > 2) \) and \( nd \) levels. For the 2s level, the cascade transitions are absent. The cascade photons can be effectively reabsorbed and therefore the problem of separation of the ‘pure’ two-photon emission from the cascade photons arises in connection with the ‘escape’ probability. This problem was intensively discussed in the last decade [7–12]. As was proved in [11], the separation of the ‘pure’ two-photon emission for the \( ns(n > 2) \) and \( nd \) levels is an ambiguous procedure. First, this ambiguity was established for the two-photon transitions with cascades in the highly charged ions [13]. To reach an accuracy of 0.1% for the theoretical description of the properties of the CMB, many effects should be taken into account in the astrophysical equations describing the radiation ‘escape’ process: consequences of the universe expansion, thermodynamical properties, induced radiation, processes of the electron scattering, Raman scattering, etc. A detailed analysis of the various distortions of the resonant optical line spectra is also required [14], including the nonresonant (NR) corrections [15].

This very complicated construction requires a careful treatment of the basic principles that this construction stands upon; these principles are given by quantum electrodynamics (QED). In this paper, we analyse these principles and
demonstrate that following them, one can find some additional effects, small but probably sizeable at the level 0.1%. Our treatment will remain in the framework of QED applied to free atoms in the field of photons; the astrophysical aspects will be restricted to the introduction of temperature (i.e. thermodynamical equilibrium). Actually we consider the model universe containing two atoms only. The first atom is in the excited state and the second one in the ground state. The first atom emits the radiation and arrives at the ground state too. If this radiation is not absorbed by the second atom, both atoms appear to be in the ground state: recombination occurs and the radiation has ‘escaped’ the interaction with the matter. This ‘escape’ does not coincide with the definition adopted in astrophysics and has sense only within the framework of our model. We assume however that our model can reproduce correctly the relative role of the higher excited states compared to the 2s state in the process of hydrogen recombination. Our paper is organized as follows. In section 2, we describe the process of the photon scattering (i.e. the process of scattering of the photons emitted by one atom, on another atom) in QED and apply this description to the rescattering of the Lyman-alpha photons. In section 3, we investigate how the two-photon emission from the ns, nd levels is absorbed in the one-photon transitions. This investigation will enable us to compare the probability of the radiation ‘escape’ from the ns(n > 2), nd levels with the ‘escape’ from the 2s level. Section 4 is devoted to the studies of the multi-photon (i.e. three-, four-photon) transitions in the two-photon approximation. We will show that these transitions can give a non-negligible contribution to the radiation ‘escape’. Section 5 contains the discussion of the results and conclusions. In appendix A, we give a rigorous QED derivation of the Lorentz contour for the one-photon transition between the two arbitrary excited levels; such a derivation has so far been absent in the literature. Appendix B is devoted to the derivation of the basic formula employed in section 3.

2. QED theory for the photon rescattering on an atom

2.1. Emission line profile for the transitions between two arbitrary levels

A quantum-mechanical phenomenological description of the line profile (Lorentz profile) has been known since 1930 [16, 17]. A QED derivation of the Lorentz profile for the transition between two excited states was given in [13, 18]. The corresponding expression for the transition \( a \rightarrow a_1 \) looks like [18]:

\[
\begin{align*}
    dW_{a\rightarrow a_1} (\omega) &= \frac{1}{2\pi} \frac{\Gamma_{a\rightarrow a_1} \Gamma_{a_1\rightarrow a} (\Gamma_a + \Gamma_{a_1})}{\Gamma_a \Gamma_{a_1}} \\
    &\times \frac{d\omega}{(\omega - \tilde{\omega}_{a\rightarrow a_1})^2 + \frac{1}{4} (\Gamma_a + \Gamma_{a_1})^2}.
\end{align*}
\]

(1)

In the above equation, it is assumed that the lower level \( a_1 \) decays in turn to the ground state \( a_0 \) via a one-photon decay. Here \( \Gamma_a \) and \( \Gamma_{a_1} \) are the total widths of the levels \( a \) and \( a_1 \), respectively, and \( \Gamma_{a\rightarrow a_1} \) and \( \Gamma_{a_1\rightarrow a} \) are the partial widths corresponding to the transitions \( a \rightarrow a_1 \) and \( a_1 \rightarrow a_0 \). Emission probabilities are connected with the partial widths via the equalities:

\[
\Gamma_{a\rightarrow a_1} = \Gamma_{a_1\rightarrow a}.
\]

(2)

Finally, \( \tilde{\omega}_{a\rightarrow a_1} = E_a + L_a - E_{a_1} - L_{a_1} \), where \( E_a \) and \( E_{a_1} \) are the one-electron energies, and \( L_a \) and \( L_{a_1} \) represent the Lamb shifts of the levels \( a \) and \( a_1 \), respectively. Thus, in principle, the line profile for the transition \( a \rightarrow a_1 \) depends on the further decay channel for the lower state \( a_1 \). Actually, this is the dependence on the branching ratios \( b_{a\rightarrow a_1} = \Gamma_{a\rightarrow a_1}/\Gamma_a \) and \( b_{a_1\rightarrow a} = \Gamma_{a_1\rightarrow a}/\Gamma_{a_1} \).

In the simplest case when both levels \( a \) and \( a_1 \) have only one decay channel \( b_{a\rightarrow a_1} = b_{a_1\rightarrow a} = 1 \), equation (1) simplifies to

\[
    dW_{a\rightarrow a_1} (\omega) = \frac{1}{2\pi} \frac{(\Gamma_a + \Gamma_{a_1}) d\omega}{(\omega - \tilde{\omega}_{a\rightarrow a_1})^2 + \frac{1}{4} (\Gamma_a + \Gamma_{a_1})^2}.
\]

(3)

Here the dependence on the state \( a_0 \) disappeared totally. In appendix A, we also present the derivation of the Lorentz profile for the most general case when the lower level \( a_1 \) decays not directly to the ground state \( a_0 \) but to the intermediate state \( a_2 \), then to the lower intermediate state \( a_3 \) and so on. The total chain of decays (the cascade) is \( a \rightarrow a_1 \rightarrow a_1 \rightarrow a_3 \rightarrow \cdots \rightarrow a_0 \). It is assumed that these decays are of one-photon type.

An important question is as follows: how far from the resonance can the wings of the Lorentz profile be extended? The answer depends on the importance of the so-called nonresonant (NR) corrections which distort, in principle, the line profile. The NR corrections were first introduced in [19] and recently discussed in connection with atomic laboratory experiments in [20–22]. In the astrophysical aspect, the role of the NR corrections was studied in [15]. According to these studies, the Lorentz profile can be extended far from the resonance (actually, to infinity) without any serious errors. We will use this extension throughout this paper.

Employing the extension of the profiles discussed above, we choose the normalization condition for the Lorentz profile as

\[
    dW (\omega) = L (\omega) d\omega,
\]

(4)

\[
    \int_0^\infty L_{a\rightarrow a_1} (\omega) d\omega = 1,
\]

(5)

in the case of equation (3) and as

\[
    \int_0^\infty L_{a\rightarrow a_1} (\omega) d\omega = b_{a\rightarrow a_1} b_{a_1\rightarrow a_0},
\]

(6)

in the case of equation (1). Equations (5) and (6) represent the absolute probability for the photon to be emitted via the transition \( a \rightarrow a_1 \) with any frequency value. If there are no other decay channels apart from \( a \rightarrow a_1 \), this probability equals 1. If such decay channels exist both for the \( a \) and \( a_1 \) levels, this probability is defined by the product of the branching ratios \( b_{a\rightarrow a_1} b_{a_1\rightarrow a_0} \).
2.2. Re-emission of the photons emitted by one atom by another atom in the same transition

Now we assume that the radiation emitted in the transition \(a \to a_1\) and having the frequency distribution defined by equation (3) is absorbed by another atom via the transition \(a_1 \to a\). The simplified form of the Lorentz profile takes place, in particular, for the most important for the cosmological recombination Lyman-alpha line. The absorption Lorentz profile is defined by equation (3) and the absorption probabilities are connected with partial widths via

\[
W_{a_1a} = \frac{g_{a_1}}{g_a} \Gamma_{aa_1},
\]

where \(g_{a_1}\) and \(g_a\) are the degeneracies for the states \(a_1\) and \(a\), respectively. If the absorbed photons originate from the emission line \(a \to a_1\) of another atom, the probability of the absorption, and hence re-emission of these photons, should be defined as

\[
X_{aa_1}^{(2)} = \int_0^\infty I_{aa_1}(\omega) L_{aa_1}(\omega) \, d\omega,
\]

where \(L_{aa_1}(\omega) = (\Gamma_a + \Gamma_{a_1}) I_{aa_1}(\omega)\) is the dimensionless distribution of the incident photons. This function is normalized according to the condition

\[
\int_0^\infty I_{aa_1}(\omega) \, d\omega = \Gamma_a + \Gamma_{a_1}.
\]

The formal proof of equation (8) on the basis of QED is given in appendix B.

The frequency distribution of the emitted photons was first introduced in the QED S-matrix theory in [19] and later employed in [23, 24] for the studies of the multiple-photon scattering on the hydrogen atom.

In this way, we can also define the probability of the photon emission after the multiple \((n\text{-fold})\) scattering:

\[
X_{aa_1}^{(n)} = \int_0^\infty [I_{aa_1}(\omega)]^{n-1} L_{aa_1}(\omega) \, d\omega,
\]

where

\[
L_{aa_1}(\omega) = (\Gamma_a + \Gamma_{a_1}) I_{aa_1}(\omega) = \frac{1}{\pi} \frac{\Gamma_a + \Gamma_{a_1}}{\omega^2 + (\Gamma_a + \Gamma_{a_1})^2}.
\]

For \(n = 1\), expression (10) reduces to equation (5). The integral can be extended to the interval \(-\infty \leq \omega \leq +\infty\) since the main contribution comes from the pole in expression (1) or (3) \(L_{aa_1}\). Then we can evaluate the integral in the complex plane. Employing the equality

\[
\left(\frac{1}{\omega - \omega_{a_1}}\right) = \left(\frac{1}{\omega - \omega_{a_1}} - \frac{1}{2} (\Gamma_a + \Gamma_{a_1})\right) \frac{1}{1 - \frac{1}{2} (\Gamma_a + \Gamma_{a_1})}
\]

and using Cauchy’s formula, we find

\[
X_{aa_1}^{(n)} = \left[(\Gamma_a + \Gamma_{a_1})\right]^{n-1} \frac{2\pi i}{n-1} \left[\int_0^\infty \frac{d\omega}{\omega - \omega_{a_1}} - \frac{1}{2} (\Gamma_a + \Gamma_{a_1})\right]^n,
\]

and

\[
X_{aa_1}^{(n)} = \left[(\Gamma_a + \Gamma_{a_1})\right]^{n-1} \frac{2\pi i}{n-1} \left[\int_0^\infty \frac{d\omega}{\omega - \omega_{a_1}} - \frac{1}{2} (\Gamma_a + \Gamma_{a_1})\right]^n,
\]

where \([\cdots]^{(n-1)}\) denotes the \((n-1)\)-fold derivative with respect to the variable \(\omega\). The evaluation in equation (12) results in

\[
X_{aa_1}^{(n)} = \frac{(2n - 2)!}{(n-1)! (2\pi)^{n-1}}.
\]

For all \(n > 1\), it is easy to check that \(X_{aa_1}^{(n)} < 1\), so that we can interpret \(X_{aa_1}^{(n)}\) as the absolute probability of the absorption in the process of rescattering and re-emission. Then the quantity

\[
Y_{aa_1}^{(n)} = 1 - X_{aa_1}^{(n)} = 1 - \frac{(2n - 2)!}{(n-1)! (2\pi)^{n-1}}
\]

can be interpreted as the probability of the radiation ‘escape’. In particular,

\[
Y_{aa_1}^{(2)} = 1 - \frac{1}{\pi} = 0.682,
\]

\[
Y_{aa_1}^{(3)} = 1 - \frac{3}{2\pi} = 0.848.
\]

So, in our simple model, the probability of the radiation ‘escape’ directly via the arbitrary one-photon transition is 0.682 already after the first rescattering and becomes close to 1 with the increase in the number of rescatterings. This result does not depend on the particular transition and also corresponds to the Lyman-alpha transition. We should stress that these estimates cannot replace the accurate astrophysical approach to the problem of the photon rescattering on the matter and are presented here only to make the further derivations more obvious.

3. Radiation ‘escape’ in the two-photon transitions

3.1. QED theory of the two-photon transitions

Quantum-mechanical theory for the two-photon transitions was first developed by Göppert-Mayer [25] and the first evaluation of the two-photon \(2s \to 1s + 2\gamma(E1)\) decay rate in hydrogen was performed by Breit and Teller [26]. The accurate nonrelativistic calculation for this transition rate was given in [27]. The fully relativistic calculations also valid for the H-like ions with arbitrary nuclear charge \(1 \leq Z \leq 100\) were performed in [28–30]. The most accurate recent calculation with QED corrections can be found in [31].

The modifications of the theory necessary to describe the two-photon transitions with cascades were discussed in [13] (see also [11, 18]).

The transition rate for the \(2s \to 1s + 2\gamma(E1)\) transition in a H atom looks like (in atomic units)

\[
dW_{2s,1s}^{(2\gamma)}(\omega) = \frac{8\alpha^3}{27\pi} \times \alpha^2 \left|S_{2s,1s}(\omega) + S_{2s,1s}(\omega)\right|^2 d\omega,
\]

where

\[
S_{2s,1s}(\omega) = \sum_{\nu \rho} \langle R_{1s\nu}(r) | R_{2s\rho}(r) | R_{2s\rho}(r) | R_{1s\nu}(r) \rangle \frac{E_{\nu\rho} - E_{2s} + \omega}{E_{\nu\rho} - E_{2s} + \omega},
\]

\[
\langle R_{1s\nu}(r) | R_{2s\rho}(r) | R_{2s\rho}(r) | R_{1s\nu}(r) \rangle = \int_0^\infty r^3 R_{1s\nu}(r) R_{2s\rho}(r) dr.
\]
$\omega_0 = E_{2s} - E_{1s}$, $R_0(r)$ is the radial part of the nonrelativistic hydrogen wavefunction, $E_{ad}$ are the electron energies for the hydrogen atom and $a$ is the fine-structure constant. Due to the absence of the energy levels between 2s and 1s (i.e. cascades), the denominator in equation (18) has no zeros.

The total decay rate for the two-photon transition $2s \to 1s$ can be obtained by the integration of equation (17) over the entire frequency interval,

$$W_{2s,1s} = \frac{1}{2} \int_0^{\infty} dW_{2s,1s} = 8.229 \text{ s}^{-1}. \quad (20)$$

An expression for the transition rate $W_{3s,1s}$ in the presence of cascades was given in [11],

$$W_{3s,1s}^{(2y)} = W_{3s,1s}^{(\text{cascade})} + W_{3s,1s}^{(\text{pure2y})} + W_{3s,1s}^{(\text{interference})}, \quad (21)$$

where

$$W_{3s,1s}^{(\text{cascade})} = \frac{4 \Gamma_{3s} + \Gamma_{2p}}{27\pi} \int_{(II)} \omega^3 (\omega_0 - \omega)^3 \times \left[ \frac{[R_{3s}(r)R_{2p}(r)]^2 |R_{3s}(r')|^2}{E_{2p} - E_{3s} + \omega - \frac{i}{2} (\Gamma_{3s} + \Gamma_{2p})} \right]^2 d\omega + \frac{4 \Gamma_{3s} + \Gamma_{2p}}{27\pi} \int_{(IV)} \omega^3 (\omega_0 - \omega)^3 \times \left[ \frac{[R_{3s}(r)R_{2p}(r)]^2 |R_{3s}(r')|^2}{E_{2p} - E_{1s} + \omega - \frac{i}{2} \Gamma_{2p}} \right]^2 d\omega, \quad (22)$$

$$W_{3s,1s}^{(\text{pure2y})} = \frac{4 \Gamma_{3s} + \Gamma_{2p}}{27\pi} \int_{(I)} \omega^3 (\omega_0 - \omega)^3 \times [S_{2s,3s}(\omega) + S_{1s,3s}(\omega_0 - \omega)]^2 d\omega + \frac{4 \Gamma_{3s} + \Gamma_{2p}}{27\pi} \int_{(IV)} \omega^3 (\omega_0 - \omega)^3 \times [S_{1s,3s}(\omega) + S_{1s,3s}(\omega_0 - \omega)]^2 d\omega, \quad (23)$$

$$dW_{3s,1s}^{(\text{interference})} = \int_{(II)} \frac{4 \omega^3 (\omega_0 - \omega)^3}{27\pi} \times \text{Re} \left[ \frac{[R_{3s}(r)R_{2p}(r)]^2 |R_{3s}(r')|^2}{E_{2p} - E_{3s} + \omega - \frac{i}{2} \Gamma_{2p}} \right] \times \left[ S_{2s,3s}(\omega) + S_{1s,3s}(\omega_0 - \omega) \right] d\omega + \int_{(IV)} \frac{4 \omega^3 (\omega_0 - \omega)^3}{27\pi} \times \text{Re} \left[ \frac{[R_{3s}(r)R_{2p}(r)]^2 |R_{3s}(r')|^2}{E_{2p} - E_{1s} + \omega - \frac{i}{2} \Gamma_{2p}} \right] \times \left[ S_{1s,3s}(\omega) + S_{1s,3s}(\omega_0 - \omega) \right] d\omega. \quad (24)$$

Here $S_{2s,3s}(\omega)$ is expression (18) with the term in $n'p = 2p$ excluded from the summation, $\omega_0 = E_{3s} - E_{1s}$. The intervals of integration over $\omega$ (I)-(V) are defined as

(I) $0 \leq \omega \leq \omega_0 - l(\Gamma_{2p} + \Gamma_{3s})$, \quad (25)

(II) $\omega_0 - l(\Gamma_{2p} + \Gamma_{3s}) \leq \omega \leq \omega_0 - l(\Gamma_{2p} + \Gamma_{3s})$, \quad (26)

(III) $\omega_0 + l(\Gamma_{2p} + \Gamma_{3s}) \leq \omega \leq \omega_0 + l(\Gamma_{2p} + \Gamma_{3s})$, \quad (27)

(IV) $\omega_0 + l(\Gamma_{2p} + \Gamma_{3s}) \leq \omega \leq \omega_0 + l(\Gamma_{2p} + \Gamma_{3s})$, \quad (28)

(V) $\omega_0 + l(\Gamma_{2p} + \Gamma_{3s}) \leq \omega \leq \omega_0$, \quad (29)

where $\omega_0 = E_{3s} - E_{2p}$ and $\omega_{2p} = E_{2p} - E_{1s}$ are the frequencies for the two links of the cascade, $l$ is an integer chosen to separate the cascade contribution from the ‘pure’ two-photon contribution. As was shown in [11], this separation is not unique, i.e. the contributions $W_{3s,1s}^{(\text{cascade})}$, $W_{3s,1s}^{(\text{pure2y})}$ and $W_{3s,1s}^{(\text{interference})}$ change essentially depending on the choice of $l$, but the total sum $W_{3s,1s}^{(2y)}$ remains invariant:

$$W_{3s,1s}^{(2y)} = \frac{1}{2} \int_0^{\infty} dW_{3s,1s} d\omega = 6.317 \times 10^6. \quad (30)$$

Note that the factor $\frac{\Gamma_{3s} + \Gamma_{2p}}{\omega_{2p}}$ in the first line in equation (22) was omitted in [11], so the numerical value in (30) was also different. We should stress that in our derivations, the total width of the level $\Gamma_{3s}$ does not coincide with the value given by equation (30), but coincides with the transition rate $\Gamma_{3s,2p}$ as in the atomic spectroscopy. The value $\Gamma_{3s,2p}$, in principle, defines the total width in the laboratory experiments when the one-photon transition rate for the photons with frequency $\omega_{3s,2p} = E_{3s} - E_{2p}$ is measured. The one-photon decay $3s \to 2p$ appears to be faster than the decay $3s \to 2p \to 1s$ due to the destructive interference of the cascade decay with the ‘pure’ two-photon decay in equation (30). However, this difference can be traced only in the fifth digit. The same picture holds for the decays of the other ns(n > 2), nd levels. Similar expressions can be written for the transition 3d − 1s with the cascade 3d−2p−1s, for the transition 4s−1s with two cascades 4s−3p−1s and 4s−2p−1s, etc.

3.2. Radiation ‘escape’ via two-photon decays

We define the ‘escape’ probability for the incoming two-photon radiation via the Lyman-alpha channel similarly to equation (8):}

$$X_{2s,1s}^{(2y)} = \frac{1}{2} \int_0^{\infty} I_{2s,1s}(\omega) L_{2p,1s}(\omega) d\omega, \quad (31)$$

where $I_{2s,1s}(\omega) = dW_{2s,1s}(\omega)$, $\omega_0 = E_{2s} - E_{1s}$. The result of the integration

$$X_{2s,1s}^{(2y)} = 6.50 \times 10^{-22} \quad (32)$$

shows that the two-photon 2s−1s radiation emitted by one atom cannot be absorbed by another atom. This means that the radiation ‘escape’ via the two-photon 2s−1s transition is absolutely full:

$$X_{2s,1s}^{(2y)} = 1 - X_{2s,1s}^{(2y)} = 1. \quad (33)$$

The superscript (2) here, as in section 2, means that we consider only one scattering (re-emission) of the photons. This is enough to understand the relative importance of different decay channels for the radiation ‘escape’.
Table 1. Contributions of the multiphoton cascade processes, having one two-photon link, to the radiation ‘escape’. Here $k$ is the number of photons; $nl$—initial state of an atom; $W^{(2k)p}_{nl,1s}$—total transition rate for the cascade transition; $W^{(2k)p}_{nl,1s}$—transition rate via ‘pure two-photon links’, $\Gamma_u$—total width of the upper level in the two-photon link; $\nu_{k,l}^{(2k)p}$—re-emission probability for the photons from the two-photon link of the k-photon cascade; $\nu_{k,l}^{(2k)p}$—‘escape’ probability for the photons from the two-photon link of the k-photon cascade.

| $k$ | $nl$ | $W^{(2k)p}_{nl,1s}$, s$^{-1}$ | $W^{(2k)p}_{nl,1s}$, s$^{-1}$ | $\Gamma_u$, s$^{-1}$ | $\nu_{k,l}^{(2k)p}$ | $\nu_{k,l}^{(2k)p}$ |
|-----|------|----------------------------|----------------------------|-----------------|----------------|----------------|
| 1   | 2    | 8.229 35                   | 8.229 35                   | 0.063 17 × 10$^8$ | 0.046 97      | 1.0000        |
| 2   | 3s   | 0.063 17 × 10$^8$          |                           | 0.064 86 × 10$^8$ | 0.046 52      | 0.953 49      |
| 3   | 3d   | 0.041 71 × 10$^8$          |                           | 0.044 16 × 10$^8$ | 0.031 01      | 0.985 69      |
| 4   | 4d   | 0.260 13 × 10$^8$          |                           | 0.276 77 × 10$^8$ | 0.021 18      | 0.978 82      |
| 5   | 3p   | 1.019 09                   |                           | 1.898 03 × 10$^8$ | 2.334 32 × 10$^{-22}$ | 1.0000       |
| 7   | 4p   | 0.003 929 × 10$^8$         |                           | 0.813 11 × 10$^8$ | 0.011 00      | 0.989 95      |
| 8   | 4f   | 0.784 812 × 10$^8$         |                           | 0.137 95 × 10$^8$ | 0.084 35      | 0.953 48      |
| 9   | 4s   | 0.615 71                   |                           | 0.044 16 × 10$^8$ | 2.479 54 × 10$^{-22}$ | 1.0000       |
| 10  | 4d   | 0.411 32                   |                           | 0.276 78 × 10$^8$ | 1.757 24 × 10$^{-22}$ | 1.0000       |

Now we can repeat the same for the transition $3s \rightarrow 1s + 2\gamma (E1)$. In this case, we evaluate the probability of re-emission of the $3s$–$1s$ two-photon radiation via all possible one-photon cascades within the frequency range $[0, \omega_0]$, i.e. $3s$–$2p$, $3d$–$2s$, $3p$–$2s$ and $2p$–$1s$,

$$X_{3s,1s}^{(2\gamma)p} = \frac{1}{2} \int_0^{\omega_0} I_{3s,1s}(\omega) \times [I_{3s,2p}(\omega) + I_{2p,1s}(\omega) + I_{3d,2p}(\omega) + I_{3p,2s}(\omega)] \, d\omega,$$

(34)

where $\omega_0 = E_{3s} - E_{1s}$. The numerical result is

$$X_{3s,1s}^{(2\gamma)p} = 0.004 97.$$

(35)

The value $X_{3s,1s}^{(2\gamma)p}$ is much larger than $X_{2s,1s}^{(2\gamma)p}$ but still essentially smaller than 1. This means that the ‘escape’ probability is very high:

$$Y_{3s,1s}^{(2\gamma)p} = 0.995 04.$$

(36)

The same picture holds for the two-photon decays of the other ns ($n > 2$), nd levels: for the transition 3 d $\rightarrow$ 1 s + 2 $\gamma$ (E1) which occurs as one cascade with two links; for the transition 4 s $\rightarrow$ 1 s + 2 $\gamma$ (E1) which includes two cascades each with two links 4 s $\rightarrow$ 3 p $\rightarrow$ 1 s and 4 s $\rightarrow$ 2 p $\rightarrow$ 1 s and for the transition 4 d $\rightarrow$ 1 s + 2 $\gamma$ (E1) which includes two cascades each with two links 4 d $\rightarrow$ 3 p $\rightarrow$ 1 s and 4 d $\rightarrow$ 2 p $\rightarrow$ 1 s. The corresponding total decay rates $W_{3d,1s}^{(2\gamma)p}$, $W_{4s,1s}^{(2\gamma)p}$, $W_{4d,1s}^{(2\gamma)p}$ and the total widths of the levels $\Gamma_{3d,1s}$, $\Gamma_{4s,1s}$, $\Gamma_{4d,1s}$ as well as the probabilities of the re-emission $X_{nl,1s}^{(2\gamma)p}$ and the ‘escape’ probabilities $Y_{nl,1s}^{(2\gamma)p}$ are given in Table 1.

Thus, all the levels ns, nd with $n = 3, 4$ seem to be nearly as effective for the radiation ‘escape’ via the two-photon transitions to the 1 s state as the 2 s level. The smallest ‘escape’ (difference about 3% with the 2 s level) occurs for the 3 d level. This corresponds to the maximum ‘death probability’ for the Lyman-alpha photons due to the transitions to the 3 d level, as found in [14]. However, the role of all these levels in the cosmological radiation ‘escape’ is strongly suppressed by the thermodynamical factor (see section 5).

Two comments are necessary concerning the accuracy of the results given above. First, we have fully neglected the two-photon transitions other than E1E1. For the neutral hydrogen atom, it is well justified (see, for example, [32]). Second, we have neglected the difference between $\Gamma_{nl,1s}$ and $\Gamma_{u}$ in equation (1) for the one-photon transitions 3 s–2 p, 3 d–2 p, 4 s–3 p, 4 s–2 p, 4 d–3 p and 4 d–2 p. For example, in equation (34), the Lorentz profile $L_{3s,2p}$ should be defined as

$$L_{3s,2p} = \frac{1}{2\pi} \frac{\Gamma_{3s}^{(1\gamma)} + \Gamma_{2p}^{(1\gamma)}}{\Gamma_{3s}^{(1\gamma)} + \Gamma_{2p}^{(1\gamma)}}.$$

(37)

As we have discussed above, in our derivations we have to put $\Gamma_{3s} = \Gamma_{3s}^{(1\gamma)}$. In the latter equality, the transitions $3 s \rightarrow 1 s + \gamma (M1)$ and $3 s \rightarrow 2 s + \gamma (M1)$ are neglected, which gives only an extremely small contribution to the total width $\Gamma_{3s}$ [33].

4. Radiation ‘escape’ in the multiphoton transitions

4.1. Contribution of the three-photon transitions

In [34, 35], it was suggested that the multiphoton transitions which contain cascades with the two-photon links can also contribute to the radiation ‘escape’ in the process of cosmological recombination. This approach was called ‘two-photon approximation’ since the contribution of the ‘pure’ multiphoton transitions with more than two photons was neglected. One of the examples described in [34] was the two-photon approximation for the three-photon 3 p $\rightarrow$ 1 s transition. The 3 p level decay can occur either as a one-photon transition $3 p \rightarrow 1 s + \gamma (E1)$ or as a three-photon transition $3 p \rightarrow 1 s + 3 \gamma (E1)$. These channels do not interfere due to the different number of photons in the final state. The one-photon decay rate is

$$W_{3p,1s}^{(1\gamma)} = 195.61 \nu^2 (\alpha Z)^4 \text{ r.u.} = 1.673 42 \times 10^8 \text{ s}^{-1},$$

(38)

where $\nu$ is the electron mass, $\alpha$ is the fine-structure constant and Z is the charge of the nucleus ($Z = 1$ for the hydrogen).

The three-photon decay rate $3 p \rightarrow 1 s + 3 \gamma (E1)$ consists of the ‘pure’ three-photon contribution, two cascade contributions $3 p \rightarrow 2 s + \gamma (E1) \rightarrow 1 s + 2 \gamma (E1)$, $3 p \rightarrow 2 p + 2 \gamma (E1) \rightarrow 1 s + \gamma (E1)$ and the interference terms. The ‘pure’ three-photon
The three-photon contribution was evaluated in [36] for
the 2p → 1s + 3γ(E1) transition which does not contain any
cascade contributions:

\[
W^{(3\gamma)}_{2p,1s} = 0.4946ma^3(\alpha Z)^8 \text{ r.u.}
\]  

(39)

In principle, for the 3p → 1s + 3γ(E1) transition rate, the
contributions of the 'pure' three-photon decay channel and the
cascade contributions are again inseparable, similar to the case
of the two-photon 3s → 1s + 2γ(E1), 3d → 1s + 2γ(E1)
transitions as discussed in section 3. However, unlike the two-
photon decays in section 3, where at the level of accuracy of the
two-photon approximation we were interested in all of the
contributions, in the case of the three-photon transitions
the same is the concern with the equation for the other
two-photon decay rates. This mistake was noted in [37]. The
order of the magnitude of the three-photon cascade transition rate is
defined by the fact that the cascade transition rate is
determined by transition rate of the slowest cascade link, i.e.
in our case by the two-photon transitions.

The total decay rate of the 3p level \( \Gamma_{3p} \) is defined as (see
the discussion concerning the width \( \Gamma_{3s} \) in section 2)

\[
\Gamma_{3p} = \Gamma^{(1\gamma)}_{3p} + \Gamma^{(2\gamma)}_{3p,2s}.
\]  

(41)

The two-photon transition rate \( W^{(2\gamma)}_{3p,2s} \) should be evaluated
similarly to the \( W^{(2\gamma)}_{3p,1s} \) transition rate since it is a 'pure' two-
photon transition rate. Hence,

\[
W^{(2\gamma)}_{3p,2s} = \frac{1}{2} \int_{0}^{\infty} \text{d}W^{(2\gamma)}_{3p,2s}(\omega),
\]  

(42)

where \( \omega_0 = E_{3p} - E_{2p} \). The two-photon frequency distribution
\( \text{d}W^{(2\gamma)}_{3p,2p} \equiv I_{3p,2p} \) looks like (in au)

\[
\text{d}W^{(2\gamma)}_{3p,2p}(\omega) = \frac{8\alpha^3(\omega_0 - \omega)^3}{9\pi} \times 2 \sum_{m_q m_{m'}} S^{q q'}_{3p,2p}(\omega) \left[ 3\alpha^2 \right]^{m_q m_{m'}} d\omega,
\]  

(43)

where

\[
S^{q q'}_{3p,2p}(\omega) = \left( 5 C_{1q 0 0}^0 C_{q 0 0}^0 \right) \sum_{n' s'} \frac{\langle R_{3p}|\bar{r}|R_{n's'}\rangle}{E_{n's'} - E_{3p} + \omega} + \frac{1}{2} \sum_{m_q m_{m'}} C_{1q 0 0} m_q m_{m'} \left( 2 C_{q 0 0}^0 \left| \tilde{r}_{m_q m_{m'}} \right| \right) \times \sum_{n'd} \frac{\langle R_{3p}|\bar{r}|R_{n'd}\rangle}{E_{n'd} - E_{3p} + \omega}.
\]  

(44)

The integrals \( I_0, I_2 \) in equation (46) are defined as

\[
I_0(\nu) = \int_{0}^{\infty} \text{d}r_1 \text{d}r_2 \text{d}^3 r R_{2s}(r_1) g_0(\nu) \int_{0}^{\infty} \text{d}\omega
\]  

(47)

\[
I_2(\nu) = \int_{0}^{\infty} \text{d}r_1 \text{d}r_2 \text{d}^3 r R_{2s}(r_1) g_2(\nu) \int_{0}^{\infty} \text{d}\omega
\]  

(48)

In equation (48), \( L_{\nu+1}^{2\nu+1} \) are the Laguerre polynomials The
radiation 'escape' via the three-photon transition 3p → 1s +
3γ(E1) should be defined as

\[
\nu(2\gamma)^{1\gamma}_{3p,1s} = 1 - \nu(2\gamma)^{1\gamma}_{3p,2s}.
\]  

(49)

\[
\nu(2\gamma)^{2\gamma}_{3p,2s} = \frac{1}{2} \int_{0}^{\infty} I_{3p,2s}(\omega) \times \left[ L_{3p,2s}(\omega) + L_{3s,2p}(\omega) + L_{3d,2p}(\omega) \right] d\omega + \frac{W^{(1\gamma)}_{3p,2s}}{\Gamma_{3p}} X_{3s,2s}.
\]  

(50)

In equation (50), the decay rate \( W^{(1\gamma)}_{3p,2s} \equiv \Gamma^{(1\gamma)}_{3p,2s} = 2.24603 \times 10^7 \text{ s}^{-1} \), the decay rate \( \Gamma^{(1\gamma)}_{3p,1s} = 1.6734 \times 10^8 \text{ s}^{-1} \) and according to equation (41), \( \Gamma_{3p} = 1.8983 \times 10^8 \text{ s}^{-1} \). Note that the difference between \( W^{(3\gamma)}_{3p,1s} \) and \( \Gamma_{3p} \) becomes much more significant than the difference between \( W^{(2\gamma)}_{3d,1s} \), \( W^{(2\gamma)}_{4s,1s} \), \( W^{(2\gamma)}_{4d,1s} \) and \( \Gamma_{3d}, \Gamma_{4s}, \Gamma_{4d} \), respectively. The direct one-
photon decays of the levels 3d, 4s and 4d are very much less
than the corresponding cascades (see table 1). This
happens because these cascades contain a 'pure two-photon'
link, whose transition rate is as slow as the 2s→1s transition
rate. Another situation occurs for the three-photon decays
Similarly, the two-photon links of the cascade 4p → 2p + 2γ(E1) → 1s + 3γ(E1), 4f → 2p + 2γ(E1) → 1s + 3γ(E1) are similar to the 3s → 1s + 2γ(E1) transition and are comparable by the magnitude. With it again the total decay rates \( W(4p, 2p) \) differ from the total widths \( \Gamma = \Gamma_{4p, 1s} + \Gamma_{4p, 3s} + \Gamma_{4p, 3d} \) and \( \Gamma_{4f} = \Gamma_{4f, 1s} \), respectively. The two other decay channels of 4p, 4f levels, i.e. 4p → 3p + 2γ(E1) → 1s + 3γ(E1) and 4f → 3p + 2γ(E1) → 1s + 3γ(E1), contain ‘pure’ two-photon cascade links and, therefore, have very low transition rates compared to \( \Gamma_{4p}, \Gamma_{4f} \). The results for \( X(4s, 1s) \) and \( Y(4s, 1s) \) are also given in table 1.

### 5. The role of the excited levels in the radiation ‘escape’ in the epoch of the cosmological recombination

In this section, we estimate the relative importance of the multiphoton (two-, three-, four-photon) decays of the ns, np(2 ≤ n ≤ 4), nd, n(n ≤ 4) levels in the radiation ‘escape’ in the epoch of cosmological recombination. We assume, as usually, that the thermodynamical equilibrium existed and the electron temperature \( T_e \) was approximately equal to the photon temperature \( T_\gamma \) [40]. For defining \( T_\gamma \), we employ the formula [40]

\[
T_\gamma = T_0(1 + z),
\]

where \( T_0 = 2.725K \) is the recent CMB temperature and \( z \) is the redshift, which for the estimates can be taken as \( z \approx 1000 \) for the cosmological recombination epoch.

Our aim is to compare the role of the excited states in the radiation ‘escape’ with the well-known role of the 2s level [3, 4]. For this comparison, any characteristics of the radiation ‘escape’ can be employed, which we will not specify here. Denoting these characteristics as \( R_{2s} \), we can suggest the following formula for the estimate of the relative role of the excited states:

\[
R = R_{2s}\left[1 + \left(\frac{\nu}{\nu_{2s, 1s}} + \frac{\nu}{\nu_{2s, 1s}} + \frac{\nu}{\nu_{2s, 1s}}\right)e^{-\nu \alpha_{2s} - \nu \alpha_{2s}} + \left(\frac{\nu}{\nu_{4s, 1s}} + \frac{\nu}{\nu_{4s, 1s}} + \frac{\nu}{\nu_{4s, 1s}}\right)e^{-\nu \alpha_{4s} - \nu \alpha_{4s}}\right].
\]

In equation (58), the degeneracy numbers for the nl states are \( g_n = 2, g_p = 6, g_d = 10 \) and \( g_f = 14 \). The ‘escape’ probabilities \( Y_{nl, 1s} \) are compiled in table 1. For the rough estimate, we can set all \( Y_{nl, 1s} \) values equal to 1 (see table 1).
For $T_e = 2725$ K, the first exponential factor on the right-hand side of equation (58) equals

$$e^{-\frac{E_e - E_a}{kT_e}} \approx 0.000 321 05$$

(59)

and the second exponential factor equals

$$e^{-\frac{E_a - E_b}{kT_e}} \approx 0.000 019 225.$$  

(60)

Hence,

$$R = R_2 [1 + 0.001 8501 + 0.000 2941] = R_2 [1 + 0.002 1442].$$

(61)

The numbers in equation (61) mean that the radiation ‘escape’ from all the excited levels with $n = 3$ can contribute at the level 0.18% and the radiation ‘escape’ from all the levels with $n = 4$ can contribute about 0.03%. Both these numbers may be essential on the recent level of accuracy of the astrophysical observations. It is also evident that the levels with $n > 4$ can hardly give a sizeable contribution. From this formula, we excluded the three-photon decay of the 3p level and the four-photon decays of the $\Gamma_{4s}$, $\Gamma_{4d}$ levels which were considered in sections 4.1 and 4.2. The branching ratios for these decays, according to table 1, are too small. To contribute essentially to the ‘escape’ probability, the cascade transition has to contain a two-photon link which is not a ‘pure’ two-photon. The only three-photon decays which satisfy this condition are the decays of the $\Gamma_{4p}$, $\Gamma_{4f}$ levels. In other words, essential contribution comes from the three-photon cascade processes. The four-photon decays, compatible with this condition, can arise only for the $n\ell$ levels with $n \geq 5$, for example, $5s \rightarrow 1s + 4\gamma(E1)$.

6. Conclusion

In this paper, we have presented a simple model, based on QED, to estimate the relative role of the excited-level few-photon decays in the radiation ‘escape’ in the cosmological recombination epoch. These estimates cannot replace the accurate solution of the astrophysical balance equations but can give a hint regarding which processes (decays) should be included in these equations. In particular, it appears that the three-photon cascade decays of the $n\ell(n \leq 4)$ levels can give a contribution comparable with the widely discussed contribution of the two-photon decays. Our studies are based on the ‘two-photon approximation’ when we take into account the cascades which, apart from the one-photon links, also have one two-photon link. This approximation seems to be well justified due to the relative smallness of the ‘pure’ three-photon, four-photon, etc processes and due to the smallness of the processes with several two-photon cascade links.

In our investigations, we also used an idea of the photon re-emission probability first introduced in QED by Low [19]. According to this idea, if an atom emits a photon whose frequency is distributed as a result of the preceding absorption, the total probability of the photon emission can be smaller than 1. The deviation of this probability from the unity reflects the fact that the incoming photon was not necessarily absorbed. We associate this deviation with the radiation ‘escape’. We also assume that the use of one re-emission ($Y_{\text{re-emission}}^{(2\gamma)}$) value is enough to characterize the importance of a certain decay channel for the radiation ‘escape’.

The total contribution of the excited $N_s(n > 2)$, $n\ell(l = 1, 2, 3, 4, n \leq 4)$ levels to the radiation ‘escape’ compared to the contribution of the $2s \rightarrow 1s + 2\gamma(E1)$ process according to our estimates reaches the value 0.21% which is not negligible in view of the growing accuracy of the recent astrophysical observations. The smallness of this contribution is due exclusively to the relatively low equilibrium temperature during the epoch of cosmological recombination.

Acknowledgments

The authors are indebted to V K Dubrovich for helpful consultations. This work was supported by RFBR grant no 11-02-00168-a and by Goskontrakt I1334. TZ acknowledges the support by the Non-profit Foundation ‘Dynasty’ (Moscow).

Appendix A. Resonant scattering of photons on an atomic electron and the line profile for the emission process

We consider first an $n$-photon elastic scattering process, depicted in the Feynman graph in figure A1. The corresponding S-matrix element can be written as

$$\tilde{S}^{(2n)} = (-ie)^{2n} \int d^4x_1 d^4x_2 \cdots d^4x_{2n-1} d^4x_{2n} \bar{\psi}_{a_0}(x_1) \gamma_{\mu_1} \psi_{a_1}(x_1) \times A^{(k_{1\mu_1})}_{a_1}(x_1) S(x_1,x_2) \gamma_{\mu_2} A^{(k_{2\mu_2})}_{a_2}(x_2) S(x_2,x_3) \cdots \gamma_{\mu_{2n}} A^{(k_{2n\mu_{2n}})}_{a_{2n}}(x_{2n}) \psi_{a_0}(x_{2n}) \psi_{a_0}(x_1),$$

(A.1)

where $\tilde{S}^{(2n)}$ is the S-matrix of second order, $e$ is the charge of electron, $\psi_{a}(x) = \psi_{a}^{\dagger} (r) e^{-i\omega t}$ is the solution of the Dirac equation for the atomic electron and $E_a$ is the Dirac energy. $\psi_{a}^{\dagger} (r) = \psi_{a}^{\dagger} (r) \gamma_0$ is the Dirac-conjugated wavefunction with $\psi_{a}^{\dagger}$ being its Hermitian conjugate and $\gamma_0 \equiv (\gamma_1, \gamma_2)$ are the Dirac matrices. The wavefunction of the photon $A_{\mu}(x)$ looks like

$$A_{\mu}(x) = \sqrt{\frac{2\pi}{\omega}} e^{i\omega(t-x)} A^{(k_{\mu})}(x),$$

(A.2)

where $A^{(k_{\mu})}$ is the photon polarization four-vector, $k = (k, \omega)$ is the photon momentum four-vector ($k$ is the wave vector, $\omega = |\vec{k}|$ is the photon frequency) and $x \equiv (\vec{r}, t)$ is the coordinate four-vector ($\vec{r}$ and $t$ are the space and time coordinates).

Function (A.2) corresponds to the absorbed photon and the function $A^{\dagger}_{\mu}(x)$ corresponds to the emitted one. $S(x_1,x_2)$ is the Feynman propagator for the atomic electron. In the Furry picture, the eigenmode decomposition for this propagator reads

$$S(x_1,x_2) = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{i\omega(1-\frac{1}{2})} \sum_x \frac{\psi_x(\vec{r}_2)|\psi_x(\vec{r}_2)}{\omega - E_x (1 - i\delta)},$$

(A.3)
where summation in (A.3) extends over the entire Dirac spectrum of the electron states \( s \) in the field of nucleus. Integration over frequency and time variables in (A.1) leads to

\[
\hat{\mathcal{S}}^{(2n)} = (-2\pi i)\delta \left( \sum_{i=1}^{n} \omega_i - \sum_{i=n+1}^{2n} \omega_i \right) U^{\text{inc.}(2n)}. \tag{A.5}
\]

The energy conservation in this process is implemented by the condition

\[
\sum_{i=1}^{n} \omega_i = \sum_{i=n+1}^{2n} \omega_i \tag{A.6}
\]

and the resonance frequencies are given by

\[
\omega_{2n} = \omega_1 = E_{s0} - E_{a0},
\]

\[
\omega_{2n-1} = \omega_2 = E_{s1} - E_{a1},
\]

\[\vdots\]

\[
\omega_{n+1} = \omega_n = E_{s_n} - E_{a_{n-1}} \quad (E_{a0} \equiv E_a).
\]

Accordingly, we will obtain for the scattering amplitude the expression

\[
\begin{align*}
U^{\text{sc.}(2n)} &= \sum_{s_1, s_2, \ldots, s_n} \frac{(U^{a}_{s_1})_{s_0n}(U^{a}_{s_2})_{s_1n} \cdots (U^{a}_{s_n})_{s_{n-1}n}}{(E_{a0} - E_{a1} + \omega_1)(E_{a1} - E_{a2} + \omega_2) \cdots (E_{a_{n-1}} + \omega_{n-1} + \omega_n)} \times \\
&\times \frac{(U^{a}_{s_n})_{s_{n-1}n}(U^{a}_{s_{n-1}})_{s_{n-2}n} \cdots (U^{a}_{s_2})_{s_1n}(U^{a}_{s_1})_{s_0n}}{(E_{a_{n-1}} + \sum_{i=1}^{n-1} \omega_i - \omega_{n-1})(E_{a_{n-1}} + \sum_{i=1}^{n-2} \omega_i - \omega_{n-2}) \cdots (E_{a0} - E_{a1} + \omega_1)}.
\end{align*}
\tag{A.8}
\]

where we abbreviate the transition matrix element as

\[
(U_{ab})_{ab} \equiv (\mathcal{A}^{(\alpha, \beta)})_{ab}. \tag{A.9}
\]

In the resonance case, we have to retain the terms \( s_1 = a_1, s_2 = a_2, \ldots, s_n = a_n, s_{n-1} = a_2, \) \( s_{2n-2} = a_2, \) \( s_{2n-1} = a_1 \) in equation (A.8), which yields

\[
\begin{align*}
\sum_{s_1, s_2, \ldots, s_n} \frac{(U^{a}_{s_1})_{s_0n}(U^{a}_{s_2})_{s_1n} \cdots (U^{a}_{s_n})_{s_{n-1}n}}{(E_{a0} - E_{a1} + \omega_1)(E_{a1} - E_{a2} + \omega_2) \cdots (E_{a_{n-1}} + \omega_{n-1} + \omega_n)} \times \\
&\times \frac{(U^{a}_{s_n})_{s_{n-1}n}(U^{a}_{s_{n-1}})_{s_{n-2}n} \cdots (U^{a}_{s_2})_{s_1n}(U^{a}_{s_1})_{s_0n}}{(E_{a_{n-1}} + \sum_{i=1}^{n-1} \omega_i - \omega_{n-1})(E_{a_{n-1}} + \sum_{i=1}^{n-2} \omega_i - \omega_{n-2}) \cdots (E_{a0} - E_{a1} + \omega_1)}.
\end{align*}
\tag{A.10}
\]

In order to describe the line profile for the multiphoton emission, we have to consider first the amplitude for the multiphoton emission in the resonance approximation, which can be defined as

\[
U^{\text{em.}} = \sum_{s_1, s_2, \ldots, s_n} \frac{(U^{a}_{s_1})_{s_0n}(U^{a}_{s_2})_{s_1n} \cdots (U^{a}_{s_n})_{s_{n-1}n}}{(E_{a0} - E_{a1} + \omega_1)(E_{a1} - E_{a2} + \omega_2) \cdots (E_{a_{n-1}} + \omega_{n-1} + \omega_n)}.
\tag{A.11}
\]

The resonance approximation for this \( n \)-photon emission process assumes actually the existence of the cascade transition \( a \rightarrow a_{n-1} \rightarrow a_{n-2} \rightarrow \cdots \rightarrow a_1 \rightarrow a_0 \). The problem of cascades will be investigated below. An expression, similar to equation (A.11), can be defined for the corresponding absorption amplitude,

\[
\begin{align*}
U^{\text{ab}} &= \sum_{s_1, s_2, \ldots, s_n} \frac{(U^{a}_{s_1})_{s_0n}(U^{a}_{s_2})_{s_1n} \cdots (U^{a}_{s_n})_{s_{n-1}n}}{(E_{a0} - E_{a1} + \omega_1)(E_{a1} - E_{a2} + \omega_2) \cdots (E_{a_{n-1}} + \omega_{n-1} + \omega_n)} \times \\
&\times \frac{(U^{a}_{s_n})_{s_{n-1}n}(U^{a}_{s_{n-1}})_{s_{n-2}n} \cdots (U^{a}_{s_2})_{s_1n}(U^{a}_{s_1})_{s_0n}}{(E_{a_{n-1}} + \sum_{i=1}^{n-1} \omega_i - \omega_{n-1})(E_{a_{n-1}} + \sum_{i=1}^{n-2} \omega_i - \omega_{n-2}) \cdots (E_{a0} - E_{a1} + \omega_1)}.
\end{align*}
\tag{A.12}
\]

The factors in the denominators of equation (A.11) generate simple poles at the resonance frequencies. These singularities are removed (from the real axis) by inserting radiative corrections into the central and all upper electron propagators, i.e. by introducing ‘radiative dressed’ propagators. The insertion of the lowest-order radiative corrections (the vacuum polarization, represented by the Uehling potential and the electron self-energy) in the central propagator (see figure A2) yields

\[
\begin{align*}
U^{\text{sc.}(2n+2)} &= \sum_{s_1, s_2, \ldots, s_n} \frac{(U^{a}_{s_1})_{s_0n}(U^{a}_{s_2})_{s_1n} \cdots (U^{a}_{s_n})_{s_{n-1}n}}{(E_{a0} - E_{a1} + \omega_1)(E_{a1} - E_{a2} + \omega_2) \cdots (E_{a_{n-1}} + \omega_{n-1} + \omega_n)} \times \\
&\times \frac{(U^{a}_{s_n})_{s_{n-1}n}(U^{a}_{s_{n-1}})_{s_{n-2}n} \cdots (U^{a}_{s_2})_{s_1n}(U^{a}_{s_1})_{s_0n}}{(E_{a_{n-1}} + \sum_{i=1}^{n-1} \omega_i - \omega_{n-1})(E_{a_{n-1}} + \sum_{i=1}^{n-2} \omega_i - \omega_{n-2}) \cdots (E_{a0} - E_{a1} + \omega_1)}.
\end{align*}
\tag{A.13}
\]
where $V^U$ is the Uehling potential and $(\hat{\Sigma}(\xi))_{\nu,\beta}$ represents the matrix element of the energy-dependent self-energy operator

$$
(\hat{\Sigma}(\xi))_{\nu,\beta} = e^2 \sum_{n} \frac{i}{2\pi} \int d\Omega \frac{(H(\Omega)))_{\nu,\alpha}}{\xi - \Omega - E_n(1 - i0)}.
$$

(A.14)

Here the shorthand notation

$$(H(\Omega))_{\nu,\beta} = \sum_{\mu,\gamma} \int d^3r_1 d^3r_2 \psi_{\mu,\gamma}(r_1) \psi_{\nu,\beta}(r_2) r_1^\mu r_2^\gamma I_{\mu,\beta} ;$$

(A.15)

is used with

$$I_{\mu,\beta} = g_{\mu,\beta} \frac{1}{\Gamma_1} e^{i\Omega_1/2},$$

(A.16)

$r_{12} = |r_1 - r_2|$ and the metric tensor $g_{\mu,\beta}$.

In the resonance case, equation (A.13) reduces to

$$U^{res(2n+2)} = \frac{(U^*_{\nu,\alpha})_{\nu,\beta}(U^*_{\gamma,\delta})_{\gamma,\delta} \cdots (U^*_{\nu,\alpha})_{\nu,\beta}}{(E_{\nu} - E_{\mu} + \omega_1)(E_{\nu} - E_{\mu} + \omega_1 + \omega_2) \cdots (E_{\nu} - E_{\mu} + \sum_{i=1}^{n} \omega_i)} \left[ \frac{\hat{\Sigma}(\sum_{i=1}^{n} \omega_i) + V^U}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n} \omega_i} \right] \times \left[ \frac{1}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n-1} \omega_i} \right] \cdots \left[ \frac{1}{E_{\nu} - E_{\mu} + \omega_n} \right]$$

(A.17)

Resummation of an infinite sequence of radiative insertions to all orders of the perturbation theory (geometric progression) leads to the following expression for the emission amplitude:

$$U^{em} = \frac{(U^*_{\nu,\alpha})_{\nu,\beta}(U^*_{\gamma,\delta})_{\gamma,\delta} \cdots (U^*_{\nu,\alpha})_{\nu,\beta}}{(E_{\nu} - E_{\mu} + \omega_1)(E_{\nu} - E_{\mu} + \omega_1 + \omega_2) \cdots (E_{\nu} - E_{\mu} + \sum_{i=1}^{n} \omega_i)} \left[ \frac{\hat{\Sigma}(\sum_{i=1}^{n-1} \omega_i) + V^U}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n-1} \omega_i} \right] \times \left[ \frac{1}{E_{\nu} - E_{\mu} + \omega_n} \right] \cdots \left[ \frac{1}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n-1} \omega_i} \right]$$

(A.18)

with the (in general complex-valued) energy corrections

$$V_a \left( \sum_{i=1}^{n} \omega_i \right) = E_{\alpha} + \left( \hat{\Sigma}(\sum_{i=1}^{n} \omega_i) \right)_{\alpha\beta} + V^U_{\alpha\beta}.$$

(A.19)

Expanding the expression for the matrix element of the operator $\hat{\Sigma}$ into Taylor series around the resonance energy $E_{\alpha} + \sum_{i=1}^{n} \omega_i = E_{\alpha}$ and retaining only the leading term in correction (A.19) yield

$$V^{em} = \frac{(U^*_{\nu,\alpha})_{\nu,\beta}(U^*_{\gamma,\delta})_{\gamma,\delta} \cdots (U^*_{\nu,\alpha})_{\nu,\beta}}{(E_{\nu} - E_{\mu} + \omega_1)(E_{\nu} - E_{\mu} + \omega_1 + \omega_2) \cdots (E_{\nu} - E_{\mu} + \sum_{i=1}^{n} \omega_i - V_{\alpha\beta})},$$

(A.20)

where

$$V_{\alpha} = E_{\alpha} + \left( \hat{\Sigma}(E_{\alpha}) \right)_{\alpha\beta} + V^U_{\alpha\beta}.$$

(A.21)

Now let us turn to the insertion in the first upper electron propagator. After performing time and frequency integrations, the corresponding $S$-matrix element reads

$$U^{S(2n+2)} = \sum_{\nu,\beta,\mu,\gamma} \left( \frac{1}{E_{\nu} - E_{\mu} + \omega_1} \right) \left( \frac{1}{E_{\nu} - E_{\mu} + \omega_1 + \omega_2} \right) \cdots \left( \frac{1}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n} \omega_i} \right) \times \left[ \frac{\hat{\Sigma}(\sum_{i=1}^{n-1} \omega_i) + V^U}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n-1} \omega_i} \right] \times \left[ \frac{1}{E_{\nu} - E_{\mu} + \omega_n} \right] \cdots \left[ \frac{1}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n-1} \omega_i} \right]$$

$$\times \left[ \frac{1}{E_{\nu} - E_{\mu} + \omega_n} \right] \cdots \left[ \frac{1}{E_{\nu} - E_{\mu} + \sum_{i=1}^{n-1} \omega_i} \right]$$

(A.21)
Inserting radiative corrections into the remaining electron propagators and repeating the procedure described above we finally arrive at the following expression for the emission amplitude:

\[ \Gamma_{Ea} = \sum_{i} \frac{1}{2\pi} \text{Im} \frac{1}{E - E_i - \Sigma_{\text{em}}(E) - \Sigma_{\text{em}}(E) + i\epsilon} \]

Expanding again the operator \( \Sigma \) in equation (A.23) into a Taylor series close to the point of the resonance, and replacing \( \Sigma \) by \( \Sigma_{\text{em}} \) and also using equation (A.28) we obtain together with

\[ \Gamma_{Ea} = \sum_{i} \frac{1}{2\pi} \text{Im} \frac{1}{E - E_i - \Sigma_{\text{em}}(E) - \Sigma_{\text{em}}(E) + i\epsilon} \]

where the energies \( E_i \) are the partial widths, associated with the transition \( a_{1} \rightarrow a \). This is the line profile for the cascade transitions when all photons are assumed to be present. From equation (A.31) we can also obtain an expression for the single-photon transition \( a_{1} \rightarrow a \), in the case when the photon directions and summing over polarizations, we obtain an expression for the \( \Gamma_{Ea} \) as the probability one has to take the square modulus of the amplitude \( V_{\alpha_{1}} = E + L_{\alpha_{1}} + \frac{1}{2}\Gamma_{\alpha_{1}} \).

The resonant case is characterized by the conditions \( a_{1} = a_{1} \), \( a_{2} = a_{2} \), \( a_{3} = a_{3} \), \( a_{4} = a_{4} \).

As the next step towards the evaluation of transition probabilities, we integrate over all polarization \( \epsilon \), which imply the corresponding scattering amplitudes, one has to take the square modulus of the amplitude \( V_{\alpha_{1}} = E + L_{\alpha_{1}} + \frac{1}{2}\Gamma_{\alpha_{1}} \).

We can assume that all necessary resonant insertions into the electron line in the resonance approximation and performing re-summation of the resulting geometrical progression, finally, we find

\[ \Gamma_{Ea} = \sum_{i} \frac{1}{2\pi} \text{Im} \frac{1}{E - E_i - \Sigma_{\text{em}}(E) - \Sigma_{\text{em}}(E) + i\epsilon} \]

where the energies \( E \) and \( V_{\alpha_{1}} \) are improved by the radiative corrections.
is depicted in figure A1 for equation (A.7): approximation is defined by the condition following from $n \rightarrow a$ Appendix B. Re-emission of the photon, emitted by one atom, by another atom: QED derivation

We start with the derivation of the Lorentz profile for the emission line from the QED description of the photon scattering on the atomic electron [13, 19].

The Feynman graph for the one-photon scattering process is depicted in figure A1 for $n = 1$. The resonance approximation is defined by the condition following from equation (A.7):

$$\omega_1 = \omega_2 = E_a - E_{a_0}. \quad (B.1)$$

A scattering amplitude $U^{sc,(2)}$ corresponding to equation (A.8) with $n = 1$ looks like

$$U^{sc,(2)} = \sum_{\alpha} \frac{(U_{a\alpha})_{\alpha}^* (U_{\alpha a})_{\alpha}}{E_{a_0} - E_{a_1} + \omega}. \quad (B.2)$$

In the resonance approximation,

$$U^{sc,(2)} = \frac{(U_{a\alpha})_{\alpha}^* (U_{\alpha a})_{\alpha}}{E_{a_0} - E_{a_1} + \omega}. \quad (B.3)$$

and the emission amplitude is defined as

$$U^{em,(2)} = \frac{(U_{\alpha a})_{\alpha}}{E_{a_0} - E_{a_1} + \omega}. \quad (B.4)$$

Thus, the emission amplitude can be obtained from the scattering amplitude in the resonance approximation by omitting the absorption matrix element $(U)_{a\alpha}$. A transition probability for the one-photon emission process $a \rightarrow a_0 + \gamma$ follows from (A.33) and looks like

$$dW_{a_0} = \frac{1}{2\pi} \frac{\Gamma_a}{(\omega - \omega_{a0})^2 + \frac{\Gamma_a^2}{4}}. \quad (B.5)$$

It is assumed that there are no other decay channels for the transition $a \rightarrow a_0$ apart from the one-photon decay $a \rightarrow a_0 + \gamma$. Actually the unique example of such a situation is the Lyman-alpha transition $2p \rightarrow 1s + \gamma$. The normalization condition (5) is valid for the Lorentz profile (B.5).

Now we will describe the situation when the Lyman-alpha photon, absorbed and emitted by one atom, is then reabsorbed and re-emitted by another atom. The corresponding Feynman graph is depicted in figure B1. The resonance condition (B1) is now modified as

$$\omega' = \omega = E_{a'} - E_{a_0} = E_a - E_{a_0}, \quad (B.6)$$

where the quantities with or without 'prime' index correspond to the electrons in two different atoms. The $S$-matrix element, corresponding to the graph in figure B1 looks like

$$S^{(4)} = (-ie)^4 \int d^4x_1 \cdots d^4x_2 \frac{\bar{\psi}_{a_0}(x_1)}{\Gamma_1} \hat{A}_{a_0}(x_1) S(x_1,x_2) \bar{\psi}_{a_0}(x_2) \frac{\hat{A}_{a_0}^*(x_2)}{\Gamma_1} \frac{\bar{\psi}_{a_0}(x_3)}{\Gamma_2} \hat{A}_{a_0}(x_3) S(x_3,x_4) \bar{\psi}_{a_0}(x_4) \frac{\hat{A}_{a_0}^*(x_4)}{\Gamma_2}. \quad (B.7)$$

In this matrix element, the variables $x_1$ and $x_2$ correspond to one atom and the variables $x_3$ and $x_4$ correspond to another one. $D'_{\mu\nu}$ denotes the transverse photon propagator in the Coulomb gauge. We employ this propagator since we want to describe the emission of the real (transverse) photon by one atom and the absorption of this photon by another atom. This propagator can be presented in the form [41]

$$D'_{\mu\nu}(x_1,x_2) = \frac{1}{2\pi^4} \int_{-\infty}^{\infty} d\Omega'_{\mu\nu} \langle \Omega', r_{12} \rangle e^{-i\Omega'_{\mu\nu} r_{12}}, \quad (B.8)$$

$$H_{\mu\nu}(\Omega, r_{12}) = \left( \frac{\delta_{\mu\nu} \omega_{\Omega_{12}}}{r_{12}} - \frac{\partial}{\partial \Omega_{12}} \right) \left( \frac{\partial}{\partial \Omega_{12}} - 1 \right) \times (1 - \delta_{\mu\nu}) (1 - \delta_{\mu(0)}). \quad (B.9)$$

Performing the standard integration over the time and frequency variables and using relation (A.5), we arrive at the following expression for the scattering amplitude:

$$U^{sc} = e^{i\delta} \sum_{\alpha} \frac{(U_{a\alpha})_{\alpha}^* (U_{\alpha a})_{\alpha}}{E_{a_0} + \omega - E_{a_1} + \omega_a}. \quad (B.10)$$

In the resonance approximation,

$$U^{sc} = e^{i\delta} \left( \frac{\bar{\psi}_{a_0}(x_1)}{\Gamma_1} \hat{A}_{a_0}(x_1) S(x_1,x_2) \bar{\psi}_{a_0}(x_2) \frac{\hat{A}_{a_0}^*(x_2)}{\Gamma_1} \frac{\bar{\psi}_{a_0}(x_3)}{\Gamma_2} \hat{A}_{a_0}(x_3) S(x_3,x_4) \bar{\psi}_{a_0}(x_4) \frac{\hat{A}_{a_0}^*(x_4)}{\Gamma_2}. \quad (B.11)$$
According to equations (B.3)–(B.5), we obtain the emission amplitude in the resonance approximation by omitting the absorption matrix element in equation (B.11),

\[
U^{\text{em}} = e^{\frac{d}{2}} \frac{e^{i(E_{a} - E_{0})} - (\nabla_{2} \alpha_{1}^{*})(\nabla_{3} \alpha_{1}^{*})}{(E_{f} - E_{0} + \omega)(E_{f} - E_{0} + \omega)} \left( \epsilon_{2} \alpha_{2}^{\text{dir}}(\tilde{r}) \right)_{\alpha_{1} \alpha_{1}}.
\]

(B.12)

Singlarities in the denominators in equation (B.11) should be avoided by the summation of the electron self-energy and vacuum polarization insertions in the electron lines in figure B1. Then, taking the square modulus of equation (B.12), summation over the polarizations, integration over the angles for the emitted photon and finally, after the integration over the frequency \( \omega' \), we obtain the following result (setting \( \alpha'_{0} = \alpha_{0} = 1 \), \( \alpha' = a = 2p \):

\[
dW_{\text{2p-1s}}^{(1)}(\omega) = \frac{1}{2\pi} \left[ \int \frac{R_{2p}}{R_{1s}} d\omega \alpha_{0}(\omega) \right] [I_{1s,2p}^{2p,1s}(\omega)^{2}],
\]

(B.13)

where

\[
I_{1s,2p}^{2p,1s} = \int_{\text{Vcell}} d\tilde{r} \frac{1}{2\pi} \left[ \int \frac{R_{2p}}{R_{1s}} d\omega \alpha_{0}(\omega) \right] [I_{1s,2p}^{2p,1s}(\omega)^{2}]
\]

(B.14)

In equation (B.14), the one-electron Dirac wavefunctions \( \psi_{2p} \) and \( \psi_{1s} \) for the electrons in the two different atoms are present. Thus, integral (B.14) depends on the distance between two atoms, i.e. on the density of the atomic gas. It is convenient to fix the origin of the coordinate system at the nucleus of an atom which absorbs the photon.

Then we can present \( r_{23} \) in the form

\[
r_{23} = |\tilde{r} - \tilde{r}'| = |\tilde{r} - \tilde{r} + \tilde{R}|,
\]

(B.15)

where \( \tilde{r}' \) and \( \tilde{r} \) are the distances between the electrons and the nuclei in the emitting and absorbing atoms, respectively, and \( \tilde{R} \) is the distance between the two nuclei. Assuming that \( \tilde{r}' \), \( r \ll \tilde{R} \), we replace the distance \( r_{23} \) by \( \tilde{R} \).

We employ the equalities

\[
(\nabla_{2} \alpha_{1}^{*})(\nabla_{3} \alpha_{1}^{*}) f(r_{23})|_{\tilde{R}, \tilde{R}} = -[\frac{\partial \tilde{R}}{\partial \tilde{R}} f(r_{23})]|_{\tilde{R}, \tilde{R}} \alpha_{2}^{\text{dir}} = (E_{d0}E_{d0} - E_{a0}E_{a0} + E_{a0}E_{d0} + E_{d0}E_{a0})(f(r_{23}) \alpha_{0} \alpha_{0}),
\]

(B.16)

where \( \tilde{R} \) is the one-electron Dirac Hamiltonian for the bound electron with the arbitrary potential \( V(\tilde{r}) \),

\[
\tilde{h}(\tilde{r}) = -i\hbar \nabla + \beta m - eV(\tilde{r}),
\]

(B.17)

where \( \alpha \) and \( \beta \) are the Dirac matrices, and \( m \) and \( e \) are the mass and the charge of the electron, respectively. The wavefunctions in the matrix elements are assumed to be the eigenfunctions of Hamiltonian (B.17) with the eigenvalues \( E_{d0}, E_{a0}, E_{d0} \), and \( E_{a0} \). In the case when \( E_{d0} = E_{a0}, E_{d0} = E_{a0} \), equation (B.16) reduces to

\[
((\nabla_{2} \alpha_{1}^{*})(\nabla_{3} \alpha_{1}^{*}) f(r_{23}) \alpha_{2}^{\text{dir}} = (E_{a0} - E_{d0})^{2} f(r_{23}) \alpha_{0} \alpha_{0},
\]

(B.18)

Now, employing equation (B.18) with the evident approximation

\[
\frac{1}{r_{23}} = \frac{1}{\tilde{R}}
\]

(B.19)

and taking into account the orthogonality of the wavefunctions \( \psi_{a0}, \psi_{a0} \), we find that the second term in the brackets in equation (B.14) turns to zero. Then

\[
I_{1s,2p}^{2p,1s}(\tilde{R}) = \frac{4}{\tilde{R}}(\alpha_{2}^{\text{dir}})^{2}.
\]

(B.20)

In equation (B.21), one of the matrix elements \( \alpha_{2}^{\text{dir}} \) originates from the emitting atom and another matrix element originates from the absorbing atom. In the nonrelativistic limit,

\[
|\alpha_{2}^{\text{dir}}|^{2} \approx (E_{2p} - E_{1s})^{3}|(\tilde{r})_{2p,1s}|^{2}.
\]

(B.22)

Hence,

\[
I_{1s,2p}^{2p,1s}(\tilde{R}) = \frac{3}{4}(E_{2p} - E_{1s})\tilde{R}.
\]

(B.23)

We can average the result over the positions of the emitting atoms, surrounding the absorbing atom, assuming the distribution of these atoms spherically symmetrical and introducing the density of the emitting atoms \( \rho(\tilde{R}) \). Then

\[
dW_{\text{2p-1s}}^{(1)}(\omega) = \int_{\text{Vcell}} d\tilde{r} \frac{1}{2\pi} \left[ \int \frac{R_{2p}}{R_{1s}} d\omega \alpha_{0}(\omega) \right] [I_{1s,2p}^{2p,1s}(\omega)^{2}]
\]

(B.25)

where

\[
I_{2p,1s}(\omega) = \int_{0}^{\infty} \frac{\rho(\tilde{R})\tilde{R}^{2} d\tilde{R}}{4\pi} \frac{\Gamma_{2p}^{2}}{(E_{2p} - E_{1s})^{2} + \Gamma_{2p}^{2}}
\]

(B.26)

is the dimensionless function which represents the frequency distribution for the re-emitted photon in equation (8). In a simple model, employed in section 2.2 the normalization of the function \( \rho(\tilde{R}) \) was chosen as

\[
\frac{\rho(\tilde{R})^{2} d\tilde{R}}{4\pi} \frac{\Gamma_{2p}^{2}}{(E_{2p} - E_{1s})^{2} + \Gamma_{2p}^{2}} = \frac{9\pi}{4}(E_{2p} - E_{1s})^{2} \int_{0}^{\infty} \rho(\tilde{R}) d\tilde{R} = 1.
\]

(B.27)

References

[1] Hinshaw G et al 2007 Astrophys. J. Suppl. 170 288
[2] Page L et al 2007 Astrophys. J. Suppl. 170 335
[3] Zel’dovich Y B, Kurt V G and Syunyaev R A 1968 Zh. Eksp. Teor. Fiz. 55 278
Zel’dovich Y B, Kurt V G and Syunyaev R A 1969 Sov. Phys.—JETP 28 146 (Engl. transl.)
[4] Peebles P J E 1968 Astrophys. J. 153 1
[5] Dubrovich V K and Grachev S I 2006 Astron. Lett. 31 359
[6] Wong W Y and Scott D 2007 Mon. Not. R. Astron. Soc. 375 1441
[7] Chluba J and Sunyaev R A 2008 Astron. Astrophys. 480 629
[8] Hirata C M 2008 Phys. Rev. D 78 023001
[9] Jentschura U D 2007 J. Phys. A: Math. Theor. 40 F223
[10] Jentschura U D and Surzhykov A 2008 Phys. Rev. A 77 042507
[11] Labzowsky L, Solovyev D and Plunien G 2009 Phys. Rev. A 80 062514
[12] Amaro P, Santos J P, Parente F, Surzhykov A and Indelicato P 2009 Phys. Rev. A 79 062504
[13] Labzowsky L N and Shonin A V 2004 Phys. Rev. A 69 012503
[14] Chluba J and Sunyaev R A 2010 Astron. Astrophys. A 53 512
[15] Karshenboim S G and Ivanov V G 2009 Astron. Lett. 34 289
[16] Weisskopf V and Wigner E 1930 Z. Phys. 63 54
[17] Heitler W 1954 The Quantum Theory of Radiation (Oxford: Oxford University Press)
[18] Andreev O Y, Labzowsky L N, Plunien G and Solovyev D A 2008 Phys. Rep. 455 135
[19] Low F 1952 Phys. Rev. 88 53
[20] Labzowsky L, Solovyev D, Plunien G and Soff G 2001 Phys. Rev. Lett. 87 143003
[21] Jentschura U D and Mohr P J 2002 Can. J. Phys. 80 633
[22] Labzowsky L, Schedrin G, Solovyev D, Chernovskaya E, Plunien G and Karshenboim S 2009 Phys. Rev. A 79 052506
[23] Labzowsky L and Solovyev D 2004 J. Phys. B: At. Mol. Opt. Phys. 37 3271
[24] Labzowsky L, Solovyev D, Plunien G, Andreev O and Schedrin G 2007 J. Phys. B: At. Mol. Opt. Phys. 40 525
[25] Goppert-Mayer M 1931 Ann. Phys., Lpz. 9 273
[26] Breit G and Teller E 1940 Astrophys. J. 91 215
[27] Klarsfeld S 1969 Phys. Lett. A 30 382
[28] Goldman S P and Drake G W F 1981 Phys. Rev. A 24 183
[29] Parpia F A and Johnson W R 1982 Phys. Rev. A 26 1142
[30] Santos J P, Parente F and Indelicato P 1998 Eur. Phys. J. D 3 43
[31] Jentschura U D 2004 Phys. Rev. A 69 052118
[32] Labzowsky L N, Solovyev D A, Plunien G and Soff G 2006 Eur. Phys. J. D 37 335
[33] Sucher J 1978 Rep. Prog. Phys. 41 1781
[34] Solovyev D and Labzowsky L 2010 Phys. Rev. A 81 062509
[35] Solovyev D and Labzowsky L 2011 Can. J. Phys. 89 123
[36] Solovyev D, Labzowsky L, Plunien G and Sharipov V 2010 J. Phys. B: At. Mol. Opt. Phys. 43 074005
[37] Karshenboim S G, Ivanov V G and Chluba J 2011 arXiv:1104.486v1 [physics.atom-ph]
[38] Rapoport L P, Zon B A and Manakov N L 1978 Teorija mnogofotonnych prozessov v atomach Theory of the Multiphoton Processes in Atoms (Moscow: Atomizdat) (in Russian)
[39] Solovyev D, Labzowsky L, Volotka A and Plunien G 2011 Eur. Phys. J. D 61 297
[40] Seager S, Sasselov D and Scott D 2000 Astrophys. J. Suppl. 128 407
[41] Labzowsky L, Klimchitskaya G and Dmitriev Y 1993 Relativistic Effects in the Spectra of Atomic Systems (Bristol: Institute of Physics Publishing)