Local solvability for a quasilinear wave equation with the far field degeneracy: 1D case

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Abstract

We study the Cauchy problem for the quasilinear wave equation \( \partial_t^2 u = u^{2a} \partial_x^2 u + F(u)u_x \) with \( a \geq 0 \) and show a result for the local in time existence under new conditions. In the previous results, it is assumed that \( u(0, x) \geq c_0 > 0 \) for some constant \( c_0 \) to prove the existence and the uniqueness. This assumption ensures that the equation does not degenerate. In this paper, we allow the equation to degenerate at spatial infinity. Namely we consider the local well-posedness under the assumption that \( u(0, x) > 0 \) and \( u(0, x) \to 0 \) as \( |x| \to \infty \). Furthermore, to prove the local well-posedness, we find that the so-called Levi condition appears. Our proof is based on the method of characteristic and the contraction mapping principle via weighted \( L^\infty \) estimates.

1 Introduction

In this paper, we consider the following Cauchy problem of the model quasilinear wave equation in \( \mathbb{R} \):

\[
\begin{aligned}
&\partial_t^2 u = \partial_x (u^{2a} \partial_x u) + F(u)u_x, \quad (t, x) \in (0, T] \times \mathbb{R}, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
&\partial_t u(0, x) = u_1(x), \quad x \in \mathbb{R},
\end{aligned}
\]

(1.1)

where \( F \) is a given function and \( a \geq 0 \). The purpose of this paper is to show the local existence and the uniqueness under new conditions. The existence of solutions to the more general quasi-linear wave equations has been widely known since the 1970s. Kato [12] and Hughes, Kato and Marsden [9] have shown an abstract theorem about the well-posedness of the system of general quasi-linear wave equations in \( L^2 \) Sobolev space. In 1 dimensional case, the well-posedness in \( C^1_b \) class for first order hyperbolic equations has been studied by Dougis [4] and Hartman and Winter [7] (see also Majda [14] and Courant and Lax [3]), where \( C^1_b \) is a set of continuous and bounded functions whose derivatives are also bounded. In order to apply these results to the existence problem of (1.1), the following assumption is required:

\[ u_0(x) \geq c_0 > 0 \]

(1.2)

for a constant \( c_0 \). This condition ensures that the equation in (1.1) is the strictly hyperbolic type near \( t = 0 \). This paper relaxes this condition. We show the the local existence and

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the uniqueness of solutions of (1.1) under the assumption that the equation degenerates at spatial infinity. Namely we weaken (1.2) by $u(0, x) > 0$ and allow that $u(0, x)$ can decay to 0 as $|x| \to \infty$ (more precise assumptions are given later). To the best of my knowledge, the well-posedness has never been studied under these types of assumptions.

1.1 Known results

Let us review some results on the solvability for degenerate wave equations (weakly hyperbolic equations). The existence, nonexistence and regularity of solutions to the following type of linear weakly hyperbolic equations have been studied by many authors (e.g. Oleinik [18], Colombini and Spagnolo [2], Ivrii and Petkov [11] and Taniguchi and Tozaki [22]),

$$\partial_t^2 u - \sum_{i,j=1}^{n} a_{i,j}(t, x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(t, x) u_{x_j} = 0,$$

where $a_{i,j}$ and $b_j$ are smooth functions and $\sum_{i,j=1}^{n} a_{i,j}(t, x) \xi_i \xi_j \geq 0$ is assumed for $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. We note that $\sum_{i,j=1}^{n} a_{i,j}(t, x) \xi_i \xi_j = 0$ corresponds to the degeneracy. In Oleinik [18], (1.3) have been solved under the so-called Levi condition:

$$C_1 \left( \sum_{j=1}^{n} b_j \xi_j \right)^2 \leq C_2 \left( \sum_{i,j=1}^{n} a_{i,j} \xi_i \xi_j + \partial_t a_{i,j} \xi_i \xi_j \right).$$

(1.4)

Even if the Levi condition is assumed, we can only obtain the following energy estimate with the regularity loss for weakly hyperbolic equations:

$$\|u\|_{H^s} + \|u_t\|_{H^{s-1}} \leq C (\|u_0\|_{H^{s+r_1}} + \|u_1\|_{H^{s-1+r_2}}),$$

(1.5)

where $s$ is an arbitrary real number and $r_1$ and $r_2$ are non-negative numbers. It is known that this estimate is optimal in the sense of the regularity by observing some explicit solution to some special linear weakly hyperbolic equations. Ivrii and Petkov in [11] have treated the the following model of the 1D weakly hyperbolic equation:

$$u_{tt} - t^{2l} u_{xx} + t^k u_x = 0.$$

They have shown that the Levi condition ($k \geq l-1$) is necessary for the Cauchy problem of this equations to be $C^\infty$ well-posed. Colombini and Spagnolo in [2] have given an example of a $C^\infty$ function $a(t) \geq 0$ such that

$$u_{tt} - a(t) u_{xx} = 0$$

is not well-posed in $C^\infty$. Roughly speaking, highly oscillatory behaviors of $a(t)$ near the point that $a(t) = 0$ causes the ill-posedness. In [10], Han has derived an energy inequality with a regularity loss for the linear weakly hyperbolic equation:

$$\partial_t^2 u - a(t, x) u_{xx} = 0,$$

where $a(t, x) = t^m + a_1(x)t^{m-1} + a_2(x)t^{m-2} + \cdots + a_{m-1}(x)t + a_m(x)$. 

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Manfrin in [17] have established the local existence and the uniqueness for following 1D degenerate quasilinear wave equations with \( u_0, u_1 \in C_0^\infty(\mathbb{R}^n) \):

\[
u_{tt} = a(u)\Delta u,
\]

where \( a(u) \) is a analytic function satisfying \( a(0) = 0 \). This result can be extended to more general degenerate wave equations (see also Manfrin [15, 16]). In Dreher’s paper [6], he also has shown the local solvability for \( \partial^2_t u = \partial_x(\partial_x(|\partial_x u|^p-2\partial_x)) \) with \( p > 5 \) and under the initial condition that \( u_0, u_1 \in C_0^k(\mathbb{R}^n) \) for a large natural number \( k \). Since their proof is based on the Nash-Moser implicit function theorem and the argument in the Oleinik’s paper [18], the compactness of the support of initial data is essentially used. Hence it does not seem difficult to extend Manfrin’s method to the case that initial data are not compactly supported.

In [8], Hu and Wang have shown the local existence and uniqueness of solutions to the following variational wave equation:

\[
\partial^2_t u = c(u, x)\partial_x(c(u, x)\partial_x u) \tag{1.6}
\]

with initial data and the function \( c(u, x) \) satisfying

\[
c(u(0, x), x) = 0, \\
\partial_t u(0, x) \geq c_0 > 0, \\
c_u(u, x) \geq c_1 > 0
\]

for some constants \( c_0 \) and \( c_1 \). The choice of initial data implies that the equation degenerates at \( t = 0 \) and that \( c(u, x) \) becomes positive uniformly and immediately after \( t = 0 \). The method in [8] is inspired by Zhang and Zheng’s paper [24] which studies the existence of solutions to Euler type equation in gas dynamics. In [24] and [8], they use method of characteristic for a new dependent variable and the fixed point theorem in a special metric space.

### 1.2 Assumptions and main theorem

Before stating main theorem of this paper, we introduce assumptions on initial data and the function \( F \). We set \( \gamma = \gamma(a, \alpha) \) as below:

\[
\gamma = \begin{cases} 
0, & a \geq 1, \\
(1 - a)\alpha, & \text{otherwise}.
\end{cases}
\]

For initial data \( u_0 \in C^2(\mathbb{R}) \) and \( u_1 \in C^1_0(\mathbb{R}) \), we assume that

\[
c_1 \langle x \rangle^{-\alpha} \leq u_0(x) \leq c_2, \\
|u_1(x) \pm u_0^0 u_0(x)| \leq c_3 \langle x \rangle^{-\beta}, \\
\left| \frac{d}{dx}(u_1(x) \pm u_0^0 u_0(x)) \right| \leq c_4 \langle x \rangle^{-\gamma}
\]

with conditions on \( \alpha \geq 0 \) and \( \beta \geq 0 \) that

\[
\alpha \leq \beta, \\
\alpha \alpha \leq \beta,
\]

\( 1.10 \)
\( 1.11 \)
simply denoted by $C$ or $L$ of all $X$. Suppose that initial data is $(u_0, u_1)$ with initial position (see Lemma 2.2). We also remark that our approach is applicable to $a \leq b$ and $\alpha \leq \beta$ are satisfied. We also remark that the condition (1.11) is not necessary in the case that $a \leq b$ and $\alpha \leq \beta$ are satisfied, then all assumptions (1.7)-(1.11) on initial data hold. Then there exists a number $T > 0$ depending on the constants in (1.7)-(1.13) such that the Cauchy problem (1.1) has a unique local solution $u \in C^2([0, T] \times \mathbb{R})$ satisfying that for all $(t, x) \in [0, T] \times \mathbb{R}$

$$C_1 \langle x \rangle^{-\alpha} \leq u(t, x) \leq C_2,$$

$$|\partial_1 ((u_t \pm u^a u_x)(t, x))| \leq C_3 \langle x \rangle^{-\beta},$$

$$|\partial_1 ((u_t \pm u^a u_x)(t, x))| + |\partial_2 ((u_t \pm u^a u_x)(t, x))| \leq C_4 \langle x \rangle^{-\gamma},$$

where $C_1, C_2, C_3, C_4$ are positive constants.

Theorem 1.1 asserts the local existence and uniqueness of solutions of (1.1) under the Levi type condition without the regularity loss. Our proof is based on the method of characteristic and the contraction mapping principle via weighted $L^\infty$ estimates. In contrast to previous results on the existence for strictly hyperbolic equations, $1/u$ is not bounded. To avoid this crux, we use the spatial decay of $u_t \pm u^a u_x$. In particular, this property helps to show the boundedness of the derivative of characteristic curves $x(t)$ with initial position (see Lemma 2.2). We also remark that our approach is applicable to various type of 1D quasilinear wave equations (e.g. the equation as $t \leq 1$ under suitable condition on $c(u, x)$).

Remark 1.2. Suppose that initial data is $(u_0(x), u_1(x)) = (\langle x \rangle^{-\alpha_1}, \langle x \rangle^{-\alpha_2})$ with $\alpha_1, \alpha_2 \geq 0$. If $a_2 \geq a_1$ and $a_2a_1 \leq a_2$ are satisfied, then all assumptions (1.7)-(1.11) on initial data are satisfied. We also remark that the condition (1.11) is not necessary in the case that $a \leq 1$, since (1.10) implies (1.11). While if $a \geq 1$, (1.10) is not necessary.

1.3 Notation and plan of the paper

For a domain $\Omega \subset \mathbb{R}^n$, we define $C^m_b(\Omega)$ with $m \in \mathbb{N}$ as follows

$$C^m_b(\Omega) = \{ f \in C^m(\Omega) \mid \sup_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial_\alpha^\nu f(x)| < \infty \}.$$

We write $C_b(\Omega) = C^0_b(\Omega)$ and denote the Lebesgue space for $1 \leq p \leq \infty$ on $\mathbb{R}^n$ by $L^p$. We write $C^m_b(\Omega)$ with the norm $\| \cdot \|_{L^p}$. For a Banach space $X$, $1 \leq p \leq \infty$ and $T > 0$, we denote the set of all $X$-valued $L^p$ functions with $t \in [0, T]$ by $L^p([0, T]; X)$. For convenience, we denote $L^p([0, T]; X)$ by $L^p_T X$. The norm of $L^p_T X$ is denoted by $\| f \|_{L^p_T X}$. Various constants are simply denoted by $C$ or $C_j$ for $j \in \mathbb{N}$. We denote that $\langle x \rangle = (1 + x^2)^{1/2}$. 

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The remainder of the present paper is organized as follows. In Section 2, we review several formulas for the unknown valuable \( R = u_t + u^a u_x \) and \( S = u_t - u^a u_x \), which are called Riemann invariant in the study of the 1D hyperbolic conservation law, and give some estimates for characteristic curves. In Section 3, we show Theorem 1.1 by using the method of characteristic, weighted \( L^\infty \) estimates and the contraction mapping principle. Concluding remarks are given in Section 5.

2 Preliminaries

2.1 Basic formulation for unknown variables \( R \) and \( S \)

We set \( R(t, x) \) and \( S(t, x) \) as follows

\[
\begin{align*}
R &= \partial_t u + u^a \partial_x u, \\
S &= \partial_t u - u^a \partial_x u.
\end{align*}
\]

(2.1)

By (1.1), \( R \) and \( S \) are solutions to the system of the following first order equations:

\[
\begin{align*}
\partial_t R - u^a \partial_x R &= N_1(u, R, S) + L(u, R, S), \\
\partial_t S + u^a \partial_x S &= N_2(u, R, S) + L(u, R, S),
\end{align*}
\]

(2.2)

where we set

\[
L(u, R, S) = \frac{F(u)(R - S)}{2u^a},
\]

\[
N_1(u, R, S) = \frac{a}{2u^a}(R^2 - RS)
\]

and

\[
N_2(u, R, S) = \frac{a}{2u^a}(S^2 - RS).
\]

Let \( x_\pm(t) \) be characteristic curves on the first and third equations of (2.2) respectively. That is, \( x_+(t) \) and \( x_-(t) \) are solutions to the following differential equations respectively:

\[
\frac{d}{dt}x_\pm(t) = \pm u^a(t, x_\pm(t)).
\]

(2.3)

When we emphasize the characteristic curves go through \((s, y)\), we denote \( x_\pm(t) \) by \( x_\pm(t; s, y) \). That is, \( x_\pm(t; s, y) \) satisfies that

\[
x_\pm(t; s, y) = y \pm \int_s^t u^a(\tau, x_\pm(\tau; s, y))d\tau.
\]

(2.4)

On the characteristic curves, \( R \) and \( S \) satisfy that

\[
\begin{align*}
\frac{d}{dt}R(t, x_-(t)) &= N_1(u, R, S)(t, x_-(t)) + L(u, R, S)(t, x_-(t)), \\
\frac{d}{dt}S(t, x_+(t)) &= N_2(u, R, S)(t, x_+(t)) + L(u, R, S)(t, x_+(t)).
\end{align*}
\]

(2.5)
2.2 Some estimates of characteristic curves

We prepare some estimates for characteristic curves for \( u \in C^1([0, T] \times \mathbb{R}) \) satisfying for \( \alpha \geq 0 \)

\[
(x)^{-\alpha} A_0 \leq u(t, x) \leq A_1,
\]

where \( A_0 \) and \( A_1 \) are positive constants. In addition, we assume that

\[
0 \leq u^\alpha |u_x(t, x)| \leq A_2 (x)^{-\alpha},
\]

for a constant \( B_1 \). The boundedness of \( u \) and (2.4) implies the following estimate with \( s, t \in [0, T] \):

\[
x - A_1^\alpha |t - s| \leq x_+(s; t, x) \leq x + A_1^\alpha |t - s|.
\]

Next we show a lemma ensures a uniform Lipschitz continuity of \( x_\pm(t; s, y) \). This lemma helps to show that a sequence of characteristic curves satisfies an assumption of the Arzelà-Ascoli theorem.

**Lemma 2.1.** Let \( u \in C^1([0, T] \times \mathbb{R}) \). Suppose that (2.6) and (2.7) hold. Then the characteristic curves fulfill that for \( x_1, x_2 \in \mathbb{R} \) and \( t_1, t_2, t_3, t_4 \in [0, T] \)

\[
|x_\pm(t_3; t_1, x_1) - x_\pm(t_4; t_2, x_2)| \leq 3(1 + A_1^\alpha)(|x_1 - x_2| + |t_1 - t_2| + |t_3 - t_4|),
\]

if \( T > 0 \) is sufficiently small.

**Proof.** First we show the case that \( t_3 = t_4 = t \in [0, T] \) and \( t \geq t_1, t_2 \). From (2.4), we can easily compute that

\[
|x_\pm(t; t_1, x_1) - x_\pm(t; t_2, x_2)| \leq |x_1 - x_2|
\]

\[
+ \left| \int_{t_1}^{t} u^\alpha(\tau, x_\pm(\tau; t_1, x_1))d\tau - \int_{t_2}^{t} u^\alpha(\tau, x_\pm(\tau; t_2, x_2))d\tau \right|
\]

\[
\leq |x_1 - x_2| + \left| \int_{t_2}^{t} u^\alpha(\tau, x_\pm(\tau; t_2, x_2))d\tau \right|
\]

\[
+ \int_{t_2}^{t} |u^\alpha(\tau, x_\pm(\tau; t_1, x_1)) - u^\alpha(\tau, x_\pm(\tau; t_2, x_2))|d\tau.
\]

From (2.6) and (2.7), we have that \( u^\alpha u_x \) is bounded, from which we have for the third term of the right hand side in (2.10) that

\[
|u^\alpha(\tau, x_\pm(\tau; t_1, x_1)) - u^\alpha(\tau, x_\pm(\tau; t_2, x_2))| \leq \int_{x_\pm(\tau; t_2, x_2)}^{x_\pm(\tau; t_1, x_1)} u^\alpha u_x(\tau, y)dy
\]

\[
\leq C |x_\pm(\tau; t_1, x_1) - x_\pm(\tau; t_2, x_2)|.
\]

Hence we have

\[
|x_\pm(t; t_1, x_1) - x_\pm(t; t_2, x_2)| \leq |x_1 - x_2| + A_1^\alpha |t_1 - t_2|
\]

\[
+ C \int_{0}^{t} |x_\pm(\tau; t_1, x_1) - x_\pm(\tau; t_2, x_2)|d\tau.
\]

\[
\leq (1 + A_1^\alpha)(|x_1 - x_2| + |t_1 - t_2|)
\]

\[
+ C \int_{0}^{t} |x_\pm(\tau; t_1, x_1) - x_\pm(\tau; t_2, x_2)|d\tau.
\]
Thus we have (2.9) from the Gronwall inequality

$$|x_+(t; t_1, x_1) - x_+(t; t_2, x_2)| \leq (1 + A_1^2)(|x_1 - x_2| + |t_1 - t_2|) e^{Ct}. $$

Hence if $T$ is small, then we have that with $t_1 \geq t_2$

$$|x_+(t; t_1, x_1) - x_+(t; t_2, x_2)| \leq 2(1 + A_1^2)(|x_1 - x_2| + |t_1 - t_2|). \quad (2.11)$$

In the same way as above, we can show (2.11) with the case that $t < t_1$ or $t < t_2$. We omit the proof of this case. Next we show (2.13). The left hand side of (2.13) is written by

$$|x_+(t_3; t_1, x_1) - x_+(t_4; t_2, x_2)| \leq |x_+(t_3; t_1, x_1) - x_+(t_3; t_2, x_2)|$$

$$+ |x_+(t_3; t_1, x_1) - x_+(t_4; t_1, x_2)|$$

From (2.11), the first term of the right hand side is estimated by $2(1 + A_1^2)(|x_1 - x_2| + |t_1 - t_2|)$. From (2.4) and (2.6), the second term is estimated by $A_1^2|t_3 - t_4|$. Therefore, we have the desired inequality. \hfill \Box

Following lemma is used to show the boundedness of the derivatives of $R$ and $S$.

**Lemma 2.2.** Let $u \in C^1([0, T] \times \mathbb{R})$. Suppose that (2.6) and (2.7) hold. Then the characteristic curves $x_\pm(t; s, x)$ are differentiable with $x$ and $\partial_x x_\pm(t; s, x)$ satisfies that with $(t, x) \in [0, T] \times \mathbb{R}$ and $s \in [0, T]$ for small $T > 0$

$$\left\{ \begin{array}{l}
\frac{d}{ds} \partial_x x_\pm(s; t, x) = \pm a u^{a-1} u_x(t, x_\pm(s; t, x)) \partial_x x_\pm(s; t, x), \\
\partial_x x_\pm(t; t, x) = 1
\end{array} \right. \quad (2.12)$$

and

$$|\partial_x x_\pm(s; t, x)| \leq e^{C|t-s|}, \quad (2.13)$$

where the positive constant $C$ is depending on $A_0, A_1$ and $A_2$.

**Proof.** The differentiability of $x_\pm(s; t, x)$ and (2.12) are well-known as a basic fact(e.g. textbook of Sideris [19]). We estimate $\partial_x x_\pm(s; t, x)$. We only show (2.13) with the case that $t \geq s$. From (2.6) and the boundedness of $(x)^\alpha u^{a} u_x$, we obtain that

$$|\partial_x x_\pm(s; t, x)| \leq 1 + a \int_s^t |u^{a-1} u_x| |\partial_x x_\pm(\tau; t, x)| d\tau$$

$$\leq 1 + C \int_s^t |\partial_x x_\pm(\tau; t, x)| d\tau.$$ 

Hence, from the Gronwall inequality, we obtain (2.13) for small $T$. \hfill \Box

**3 Proof of the main theorem**

As in Introduction, we set

$$\gamma = \begin{cases} 
0, & a \geq 1, \\
(1 - a)\alpha, & \text{otherwise}.
\end{cases}$$

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We treat functions satisfying the following conditions for \( \alpha, \beta \geq 0 \) with \( \alpha \leq \beta \) such that

\[
A_0 \langle x \rangle^{-\alpha} \leq f(t, x) \leq A_1
\]

and

\[
f^n(t, x)|f_x(t, x)| \leq A_2 \langle x \rangle^{-\beta}, \quad \text{(3.2)}
\]

\[
|f_t(t, x)| \leq A_3 \langle x \rangle^{-\beta} \quad \text{(3.3)}
\]
or

\[
|f(t, x)| \leq A_3 \langle x \rangle^{-\beta}, \quad \text{(3.4)}
\]

and

\[
|f_x(t, x)| \leq A_4 \langle x \rangle^{-\gamma}, \quad \text{(3.5)}
\]

\[
|f_t(t, x)| \leq A_5 \langle x \rangle^{-\gamma}, \quad \text{(3.6)}
\]

where \( A_j \) are positive constants with \( j = 1, \ldots, 5 \). We define sets of \( C^1 \) functions \( X_{\alpha}, Y_{\beta,1}, Y_{\beta,2} \) as follows:

\[
X_{\alpha} = \{ f \in C^1 \cap C_b \mid f(0, x) = u_0(x) \text{ and } (3.1), (3.2) \text{ and } (3.6) \text{ hold.}, \}
\]

\[
Y_{\beta,1} = \{ f \in C^1_b \mid f(0, x) = R_0(x) \text{ and } (3.1), (3.5) \text{ and } (3.6) \text{ hold.}, \}
\]

\[
Y_{\beta,2} = \{ f \in C^1_b \mid f(0, x) = S_0(x) \text{ and } (3.4), (3.5) \text{ and } (3.6) \text{ hold.}, \}
\]

where given functions \((u_0, R_0, S_0)\) belongs to \( C^1 \cap C_b \times C^1_b \times C^1_b \). For given functions \((v, R, S)\) \( X_{\alpha} \times Y_{\beta,1} \times Y_{\beta,2} \), we consider the first order linear hyperbolic equation:

\[
\begin{cases}
R_t - v^a R_x = N_1(v, R, S) + L(v, R, S), \\
S_t + v^a S_x = N_2(v, R, S) + L(v, R, S)
\end{cases}
\]

with initial condition \((R(0, x), S(0, x)) = (R_0, S_0) \in C^1_b \times C^1_b \). We set

\[
u = u_0(x) + \int_0^t \frac{R + S}{2} (s, x) ds.
\]

We find that \( (3.7) \) with \( C^1 \) initial data has unique and time-global solutions such that \( R, S \in C^1([0, T] \times \mathbb{R}) \cap C_b([0, T] \times \mathbb{R}) \) with arbitrary fixed \( T > 0 \) from the method of characteristic. From \( (3.8) \), it holds that \( u \in C^1 \cap C_b \). Namely we can define the map

\[
\Phi : X_{\alpha} \times Y_{\beta,1} \times Y_{\beta,2} \to C^1 \times C^1 \times C^1
\]

such that \( \Phi(v, R, S) = (u, R, S) \). We take four positive numbers \( A_0, A_1, A_3, A_4 \) satisfying that

\[
2A_0 \langle x \rangle^{-\alpha} \leq u_0(x) \leq \frac{A_1}{2}, \quad \text{(3.9)}
\]

\[
\| \langle x \rangle^\beta R_0 \|_{L^\infty} + \| \langle x \rangle^\beta S_0 \|_{L^\infty} \leq \frac{A_3}{4}, \quad \text{(3.10)}
\]

\[
\| \langle x \rangle^\gamma R'_0 \|_{L^\infty} + \| \langle x \rangle^\gamma S'_0 \|_{L^\infty} \leq \frac{A_4}{8}. \quad \text{(3.11)}
\]
The constants $A_2$ and $A_3$ in (3.10) will be taken later. Moreover, we assume that

\[ \| \langle x \rangle^\beta u_0^j u_0^l \|_{L^\infty} \leq B_1 \]  

(3.12)

for a positive constant $B_1$. In the following, we show that $(u, R, S) \in X_\alpha \times Y_{\beta,1} \times Y_{\beta,2}$ and $\Phi$ is a contraction mapping in the topology of $L^\infty$ for sufficient small $T$. $X_\alpha$ and $Y_{\beta,j}$ with $j = 1, 2$ are not closed set of $L^\infty$ space. Nevertheless it is possible to show that the fixed point belongs to $X_\alpha \times Y_{\beta,1} \times Y_{\beta,2}$. Furthermore we will show that the regularity is improved as $u \in C^2$. First we show the following proposition.

**Proposition 3.1.** Let $(u_0, R_0, S_0) \in C^1 \times C^1 \times C^1$ satisfying (3.9)–(3.11) and (3.12). Suppose that $v, \bar{R}, \bar{S} \in X_\alpha \times Y_{\beta,1} \times Y_{\beta,2}$. Then $\Phi(v, \bar{R}, \bar{S}) = (u, R, S) \in X_\alpha \times Y_{\beta,1} \times Y_{\beta,2}$ for sufficiently small $T > 0$.

**Proof.** From the method of characteristic, we can see that the solution of (3.7) can be written by

\[
\begin{align*}
R(t, x) &= R(0, x_-(0)) + \int_0^t N_1(v, \bar{R}, \bar{S})(s, x_-(s)) + L(v, \bar{R}, \bar{S})(s, x_-(s))ds, \\
S(t, x) &= S(0, x_+(0)) + \int_0^t N_2(v, \bar{R}, \bar{S})(s, x_+(s)) + L(v, \bar{R}, \bar{S})(s, x_+(s))ds,
\end{align*}
\]

(3.13)

where the characteristic curves for the linear equation (3.7) are defined as follows:

\[
\frac{dx_\pm(t)}{dt} = \pm v^a(t, x_\pm(t))
\]

with initial data $x_\pm(t) = x$. From this expression, we have that $(u, R, S) \in C^1 \times C^1 \times C^1$. Now we estimate $\| \langle x \rangle^\beta R \|_{L^\infty_L}$. From (2.8), if $T$ is small, we have

\[
\frac{\langle x \rangle^\beta}{2} \leq \langle x_-(s; t, x) \rangle^\beta \leq 2 \langle x \rangle^\beta.
\]

Using these inequalities, from (3.13), we have that if $T$ is small

\[
\| \langle x \rangle^\beta R(t, x) \| \leq 2\| \langle x \rangle^\beta R(0, \cdot) \|_{L^\infty_L}
\]

\[
+ \int_0^t (\langle x_-(s) \rangle^\beta \frac{R^2 - \bar{R}S}{v} + \langle x_-(s) \rangle^\beta F(v) \frac{R - \bar{R}}{v^a} |R - \bar{R}|)ds
\]

\[
\leq \frac{A_3}{2} + \int_0^t (\langle x_-(s) \rangle^\alpha + \beta \frac{R^2 - \bar{R}S}{v} + C \langle x_-(s) \rangle^\beta |R - \bar{R}|)ds
\]

\[
\leq \frac{A_3}{2} + CT \left( \| \langle x \rangle^\beta R \|_{L^\infty_L} \| \langle x \rangle^\beta S \|_{L^\infty_L} + \| \langle x \rangle^\beta \bar{S} \|_{L^\infty_L} \right)
\]

\[
+ CT(\| \langle x \rangle^\beta R \|_{L^\infty_L} + \| \langle x \rangle^\beta \bar{S} \|_{L^\infty_L})
\]

\[
\leq \frac{A_3}{2} + CT,
\]

where $C$ is a positive constant depending on $A_0, A_1, A_3$ and the assumptions that $\beta \geq \alpha$ and (1.12) are used. Hence we obtain that for sufficiently small $T$

\[
\| \langle x \rangle^\beta R \|_{L^\infty_L} \leq A_3.
\]

(3.14)
Similarly for $S$, we have that
\[
\| (x)^\beta S \|_{\mathcal{L}_T^\infty \mathcal{L}^\infty} \leq A_3.
\] (3.15)

Next we estimate $\| (x)^{\gamma} R_x \|_{\mathcal{L}_T^\infty \mathcal{L}^\infty}$ and $\| (x)^{\gamma} S_x \|_{\mathcal{L}_T^\infty \mathcal{L}^\infty}$. Differentiating the both side of the equations (3.7) with $x$, we can obtain integral equations for $R_x$ and $S_x$ as follows:
\[
V(t, x) = V_0(x_- (0; t, x)) \partial_x x_- (0; t, x)
+ \int_0^t \partial_x x_- (s; t, x) \left( N_1 u v_x + N_1 R \tilde{V} + N_1 S \tilde{W} \right) (t, x_- (s; t, x)) ds
+ \int_0^t \partial_x x_- (s; t, x) \left( L u v_x + L R \tilde{V} + L S \tilde{W} \right) (t, x_- (s; t, x)) ds
\] (3.16)

and
\[
W(t, x) = W_0(x_- (0; t, x)) \partial_x x_+ (0; t, x)
+ \int_0^t \partial_x x_+ (s; t, x) \left( N_2 u v_x + N_2 R W + N_2 S V \right) (t, x_+ (s; t, x)) ds
+ \int_0^t \partial_x x_+ (s; t, x) \left( L u v_x + L R \tilde{V} + L S \tilde{W} \right) (t, x_+ (s; t, x)) ds,
\] (3.17)

where we denote $\tilde{V} = \tilde{R}_x$ and $\tilde{W} = \tilde{S}_x$ and $(V_0, W_0) = (R'_0 (\cdot), S'_0 (\cdot))$ and $N_{j u}, N_{j S}, N_{j R}$ $(j = 1, 2)$ are partial derivatives of $N_j = N_j (u, R, S)$ with $u, S, R$ respectively (the same manners are also used for $L$). From Lemma 2.2, we obtain that $| \partial_x x_+ (0; t, x) |$ is bounded by 2, if $T$ is small (note that smallness of $T$ depends on $A_0, A_1$ and $A_2$). Hence we have that
\[
|W(t, x)| \leq \frac{A_4 (x)^{-\gamma}}{2} + 2 \int_0^t |N_1 u v_x + N_1 R \tilde{V} + N_1 S \tilde{W}| (t, x_- (s; t, x)) ds
+ 2 \int_0^t |L u v_x + L R \tilde{V} + L S \tilde{W}| (t, x_- (s; t, x)) ds.
\] (3.18)

From (3.9)-(3.11), $|N_1 R \tilde{V}|$ and $|N_1 S \tilde{W}|$ are trivially estimated as
\[
|N_1 R \tilde{V}| + |N_1 S \tilde{W}| \leq \frac{C}{\nu} (|\tilde{R}| + |\tilde{S}|) (|\tilde{V}| + |\tilde{W}|)
\leq C \langle x \rangle^{\alpha - \beta - \gamma}
\leq C \langle x \rangle^{-\gamma}.
\]

From (3.9)-(3.11) and (1.10) and (1.11), we have that for $|N_1 u v_x|
\[
|N_1 u v_x| \leq \frac{C_1 u^a v_x}{u^2 + a} (|\tilde{R}|^2 + |\tilde{S}|^2)
\leq \frac{C}{u^{2 + a}} (|\tilde{R}|^3 + |\tilde{S}|^3)
\leq C \langle x \rangle^{(2 + a) \alpha - 3 \beta}
\leq C \langle x \rangle^{-\gamma}.
\]
From (1.10), (1.11), (1.12) and (1.13), we estimate $L_{uvx}$ as
\[
|L_{uvx}| \leq C \left( \frac{|F(v)v_x|}{v^a} + \frac{|F'(v)v_x|}{v^a} \right)
\]
\[
\leq C(|R| + |S|)
\]
\[
\leq C \langle x \rangle^{a-\beta}
\]
\[
\leq C \langle x \rangle^{-\gamma}.
\]
(1.12) and (3.11) directly imply that
\[
|L_{RV}| + |L_{SW}| \leq C \langle x \rangle^{-\gamma}.
\]
Applying these estimates to (3.18), we obtain that with small $T$
\[
\| \langle x \rangle^\gamma W \|_{L^\infty_T L^\infty_x} \leq \frac{A_1}{2}.
\]
Similarly
\[
\| \langle x \rangle^\gamma V \|_{L^\infty_T L^\infty_x} \leq \frac{A_1}{2}.
\]
Estimates of $R_t$ and $S_t$ are obtained from (3.7). In fact, we have that
\[
|R_t(t, x)| + |S_t(t, x)| \leq |v^a R_x| + |v^a S_x| + |N_1(v\tilde{R}, \tilde{S})|
\]
\[
+ |N_2(v\tilde{R}, \tilde{S})| + 2|L(v\tilde{R}, \tilde{S})|
\]
\[
\leq C \langle x \rangle^{-\gamma} + C \langle x \rangle^{-\beta}
\]
\[
\leq C_A \langle x \rangle^{-\gamma},
\]
(3.21)
where $C_A$ is a positive constant depending on $A_1$, $A_3$ and $A_4$ (independent of $A_5$). Here we choose $A_5$ as $A_5 = C_A$. From the above estimates, we have that $(R, S) \in Y_{\beta,1} \times Y_{\beta,2}$. Next we show that $u \in X_\alpha$. From (3.8), (3.14) and (3.15), it follows for sufficiently small $T$
\[
\langle x \rangle^\alpha u(t) \geq \langle x \rangle^\alpha u_0(x) - \int_0^t \langle x \rangle^\alpha \left( \frac{|R| + |S|}{2} \right) ds
\]
\[
\geq 2A_0 - \int_0^t \langle x \rangle^{\beta} \left( \frac{|R| + |S|}{2} \right) ds
\]
\[
\geq 2A_0 - TA_3
\]
\[
\geq A_0.
\]
(3.22)
Similarly we can easily check that $\|u\|_{L^\infty_T L^\infty_x} \leq A_1$, if $T$ is small. Next we show that
\[
\| \langle x \rangle^\beta u^a u_x \|_{L^\infty_T L^\infty_x} \leq A_2,
\]
(3.23)
(3.8) directly implies that
\[
u^a u_x = \left( u_0 + \int_0^t \frac{R + S}{2} ds \right)^a u_0
\]
\[
+ \left( u_0 + \int_0^t \frac{R + S}{2} ds \right)^a \int_0^t \frac{R_x + S_x}{2} ds.
\]
(3.24)
From (3.9) and the boundedness of \( x^{\beta} u_0' |v_0| \) the first term of (3.24) is estimated as

\[
\left( u_0 + \int_0^t \frac{R + S}{2} ds \right) \beta |u_0'| \leq 2^\alpha (u_0^a + CT^a \langle x \rangle ^{-a\beta}) |v_0'|
\]

\[
\leq (2^\alpha + CT^a) u_0^a |v_0'|
\]

\[
\leq C_{1,A} \langle x \rangle ^{-\beta},
\]

where we note that the positive constant \( C_{1,A} \) does not depend on \( A_2 \). Deducing \( R_x + S_x \) via (3.7), we also obtain

\[
\left( u_0 + \int_0^t \frac{R + S}{2} ds \right) \beta |u_0'| \leq 2^\alpha (u_0^a + T^a \langle x \rangle ^{-a\beta}) \left| \int_0^t \frac{1}{2v^a} (R_t - S_t + N_2 - N_1) ds \right|
\]

Since the spatial decay of \( R_t \) and \( S_t \) is not enough to show (3.23), we need to change this term. From the integration by parts and the property of \( X_\alpha \) that \( v_0 = u_0 \), we obtain that

\[
\left| \int_0^t \frac{R_t - S_t}{2v^a} ds \right| = \left| \int_0^t \frac{R_t - S_t}{2v^a} ds \right| \leq C_2 + A_0 \langle x \rangle ^{-\beta},
\]

Hence, with the help of (3.1), it holds that

\[
\frac{1}{2} \leq 1 - A_3 A_0 T \leq \frac{v}{u_0} \leq 1 + A_3 A_0 T \leq 2
\]

for small \( T \). The third term of the right hand side in (3.25) is estimated as

\[
\left| \int_0^t \frac{a(R - S)}{2v^a + 1} ds \right| \leq C \int_0^t \langle x \rangle ^{-\beta} (R - S) |v_t| ds
\]

\[
\leq C \langle x \rangle ^{-\beta} \int_0^t v^{-a} ds.
\]

Applying (3.26) and (3.27) to (3.25), we have

\[
2^\alpha (u_0^a + T^a \langle x \rangle ^{-a\beta}) \left| \int_0^t \frac{1}{2v^a} (R_t - S_t) ds \right| \leq C_{2,A} \langle x \rangle ^{-\beta}
\]

and similarly

\[
2^\alpha (u_0^a + T^a \langle x \rangle ^{-a\beta}) \left| \int_0^t \frac{1}{2v^a} (N_2 - N_1) ds \right| \leq C_{3,A} \langle x \rangle ^{-\beta},
\]
where positive constants $C_{2,A}$ and $C_{3,A}$ are independent of $A_2$.

$$\| (x)^{−\beta} u^a u_x \|_{\mathcal{L}^\infty} \leq C_{1,A} + C_{2,A} + C_{3,A}.$$ 

Taking $A_5 = C_{1,A} + C_{2,A} + C_{3,A}$, we obtain $3.23$. $3.38$ and $3.4$ directly yield that

$$\| (x)^{−\beta} u \|_{\mathcal{L}^\infty} \leq \frac{\| (x)^{−\beta} (R + S) \|_{\mathcal{L}^\infty}}{2} \leq A_3.$$ 

Therefore we have that $(u, R, S) \in X_\beta \times Y_{\beta,1} \times Y_{\beta,2}$. In the end of the proof, we show that $(u, R, S)$ is Lipschitz continuous. From $3.10$, $3.20$, and $3.21$, we can obviously check that $R$ and $S$ satisfies the following uniform Lipschitz estimate:

$$|R(t, x) − R(s, y)| + |S(t, x) − S(s, y)| \leq 2(A_4 + A_5)(|t − s| + |x − y|).$$  

(3.28)

Next we check that $u$ is Lipschitz continuous. From $3.23$ and the condition that $\beta \geq a\alpha$, we have that

$$|u(t, x) − u(t, y)| \leq \int_y^x |u_x(t, z)|dz \leq C\int_y^x (z)^{a\alpha} u^a |u_x|(t, z) dz \leq C|x − y|.$$ 

Combining this estimate with the boundedness of $R$ and $S$, we have from $3.5$ that $u$ is Lipschitz continuous such that for any $t_1, t_2 \in [0, T], x_1, x_2 \in \mathbb{R}$ with

$$|u(t_1, x_1) − u(t_2, x_2)| \leq C(|t_1 − t_2| + |x_1 − x_2|),$$  

(3.29)

where $C$ is a positive constant depending on $A_0, A_3, A_4$ and $A_5$. These additional properties are used to show that the fixed point of $\Phi$ satisfies integral equations.

**Proposition 3.2.** Under the same assumptions on $3.1$, $\Phi$ is a contraction mapping in the topology of $\mathcal{L}^\infty$-norm with small $T > 0$. Namely, If $T > 0$ is small, then there exists a constant $c \in (0, 1)$ such that $\Phi$ satisfies that

$$\|u_1 − u_2\|_{\mathcal{L}^\infty} + \|R_1 − R_2\|_{\mathcal{L}^\infty} + \|S_1 − S_2\|_{\mathcal{L}^\infty} \leq c \left(\|v_1 − v_2\|_{\mathcal{L}^\infty} + \|\tilde{R}_1 − \tilde{R}_2\|_{\mathcal{L}^\infty} + \|\tilde{S}_1 − \tilde{S}_2\|_{\mathcal{L}^\infty}\right),$$

where $(u_j, R_j, S_j) = \Phi(v_j, \tilde{R}_j, \tilde{S}_j)$ with $j = 1, 2$.

**Proof.** Put $\tilde{u} = u_1 − u_2, \tilde{R} = R_1 − R_2, \tilde{S} = S_1 − S_2$. From $3.7$, we have

$$\tilde{R}_t − v_1^a \tilde{R}_x = N_1(v_1, \tilde{R}_1, \tilde{S}_1) − N_1(v_2, \tilde{R}_2, \tilde{S}_2) + L(v_1, \tilde{R}_1, \tilde{S}_1) − L(v_2, \tilde{R}_2, \tilde{S}_2) + (v_1^a − v_2^a)R_{2x}.$$
From the method of characteristic, we have that

$$\tilde{R}(t, x) = \int_0^t \left( N_1(v_1, \tilde{R}_1, \tilde{S}_1) - N_1(v_2, \tilde{R}_2, \tilde{S}_2) \right) ds$$

$$+ \int_0^t (L(v_1, \tilde{R}_1, \tilde{S}_1) - L(v_2, \tilde{R}_2, \tilde{S}_2)) ds$$

$$+ \int_0^t (v_1^q - v_2^q) R_{2x} ds.$$  \hspace{1cm} (3.30)

The second term of the right hand side in (3.30) can be written as

$$\int_0^t (L(v_1, \tilde{R}_1, \tilde{S}_1) - L(v_2, \tilde{R}_2, \tilde{S}_2)) ds = \int_0^t (G(v_1) - G(v_2))(\tilde{R}_1 - \tilde{S}_1) ds$$

$$+ \int_0^t G(v_2)(\tilde{R}_1 - \tilde{S}_1 - \tilde{R}_2 + \tilde{S}_2) ds,$$  \hspace{1cm} (3.31)

where we set $G(\theta) = F(\theta) / 2\theta^a$. Using (1.12), (1.13) and (3.1), we obtain that

$$|G(v_1) - G(v_2)| \leq \int_0^1 |G'(\theta v_1 + (1 - \theta)v_2)|d\theta|v_1 - v_2|$$

$$\leq \int_0^1 \frac{C d\theta}{\theta v_1 + (1 - \theta)v_2} |v_1 - v_2|$$

$$\leq C \langle x \rangle^a |v_1 - v_2|,$$

which implies that the first term of the right hand side in (3.31) is estimated as

$$\int_0^t |G(v_1) - G(v_2)||\tilde{R}_1 - \tilde{S}_1| ds \leq CTA_3\|v_1 - v_2\|_{L^p L^\infty}.$$  

From (1.12), the second term is estimated by

$$\int_0^t |G(v_2)||\tilde{R}_1 - \tilde{S}_1 - \tilde{R}_2 + \tilde{S}_2| ds \leq CT(\|\tilde{R}_1 - \tilde{R}_2\|_{L^p L^\infty} + \|\tilde{S}_1 - \tilde{S}_2\|_{L^p L^\infty}).$$

Setting $N_1(v, \tilde{R}, \tilde{S}) = \frac{\alpha}{2\pi} Q(\tilde{R}, \tilde{S})$ and $Q(\tilde{R}, \tilde{S}) = (\tilde{R}^2 - \tilde{R}\tilde{S})$, we change the first term of the right hand side in (3.31) to

$$\int_0^t (N_1(v_1, \tilde{R}_1, \tilde{S}_1) - N_1(v_2, \tilde{R}_2, \tilde{S}_2)) ds = \int_0^t a \left( \frac{v_2 - v_1}{2v_1v_2} \right) Q(\tilde{R}_1, \tilde{S}_1) ds$$

$$+ \int_0^t \frac{a}{2v_2} (Q(\tilde{R}_1, \tilde{S}_1) - Q(\tilde{R}_2, \tilde{S}_2)) ds.$$  \hspace{1cm} (3.32)

The first term of the right hand side in (3.32) is estimated as

$$\left| \int_0^t a \left( \frac{v_2 - v_1}{2v_1v_2} \right) Q(\tilde{R}_1, \tilde{S}_1) ds \right| \leq \int_0^t a \frac{|v_1 - v_2|}{2 \langle x \rangle^{2\alpha} |v_1v_2|} \langle x \rangle^{2\beta} |Q(\tilde{R}_1, \tilde{S}_1)| ds$$

$$\leq \int_0^t a \frac{2|v_1 - v_2|}{A_0^2} \langle x \rangle^{2\beta} |Q(\tilde{R}_1, \tilde{S}_1)| ds \leq CT\|v_1 - v_2\|_{L^p L^\infty}.$$
The second term can be estimated as
\[
\left| \int_0^t \left( \frac{a}{2|v_0^2|} \right) (Q(R_1, S_1) - Q(R_2, S_2)) \right| ds \leq \int_0^t \frac{a}{\alpha} \left| (x)^\alpha \right| |Q(R_1, S_1) - Q(R_2, S_2)| ds
\]
\[
\leq \int_0^t \frac{a}{\alpha} \left| (x)^\alpha \right| |Q(R_1, S_1) - Q(R_2, S_2)| ds
\]
\[
\leq CT \left( \|R_1 - R_2\|_{L^\infty} + \|S_1 - S_2\|_{L^\infty} \right).
\]

Next we estimate the third term of the right hand side in \((3.32)\). When \(a \geq 1\), from the mean-value theorem for \(|v_1^2 - v_2^2|\) and the boundedness of \(R_x\), we obtain that
\[
|(v_1^2 - v_2^2)R_{2x}| \leq C|v_1 - v_2|_{L^\infty}.
\]

While, \(a < 1\), by using the boundedness of \(|x|^\gamma R_x\), we have that
\[
|(v_1^2 - v_2^2)R_{2x}| \leq a \int_0^1 (\theta v_1 + (1 - \theta)v_2)\alpha - 1|v_1 - v_2||R_{2x}|d\theta
\]
\[
\leq C |x| \gamma |v_1 - v_2||R_{2x}|
\]
\[
\leq C\|v_1 - v_2\|_{L^\infty}.
\]

Therefore, we obtain that for sufficiently small \(T\)
\[
\|\tilde{R}\|_{L^\infty} \leq \frac{1}{6} \left( \|v_1 - v_2\|_{L^\infty} + \|\tilde{R}_1 - \tilde{R}_2\|_{L^\infty} + \|\tilde{S}_1 - \tilde{S}_2\|_{L^\infty} \right).
\]

In the same way as in the estimate of \(\tilde{R}\), we have that
\[
\|\tilde{S}\|_{L^\infty} \leq \frac{1}{6} \left( \|v_1 - v_2\|_{L^\infty} + \|\tilde{R}_1 - \tilde{R}_2\|_{L^\infty} + \|\tilde{S}_1 - \tilde{S}_2\|_{L^\infty} \right).
\]

From \((3.33)\), the above two estimates on \(\tilde{R}\) and \(\tilde{S}\) imply that for sufficiently small \(T\)
\[
\|\tilde{u}\|_{L^\infty} \leq \frac{T}{2} \left( \|\tilde{R}\|_{L^\infty} + \|\tilde{S}\|_{L^\infty} \right)
\]
\[
\leq \frac{1}{6} \left( \|v_1 - v_2\|_{L^\infty} + \|\tilde{R}_1 - \tilde{R}_2\|_{L^\infty} + \|\tilde{S}_1 - \tilde{S}_2\|_{L^\infty} \right).
\]

Therefore, we find that \(\Phi\) is a contraction mapping for sufficiently small \(T > 0\).

Next we construct a unique solution \((u, R, S)\) of the nonlinear problem and the characteristic curves \(x_{\pm}(\cdot; t, x)\).

**Proposition 3.3.** Under the same assumptions as in Proposition 3.2, if \(T\) is small, then there uniquely exist \((u, R, S) \in X_\alpha \times Y_{\beta,1} \times Y_{\beta,2}\) and \(x_{\pm}(s) = x_{\pm}(s; t, x)\) satisfying that
\[
\begin{align*}
R(t, x) &= R(0, x_{-}(0)) + \int_0^t N_1(u, R, S)(s, x_{-}(s)) + L(u, R, S)(s, x_{-}(s))ds, \\
S(t, x) &= S(0, x_{+}(0)) + \int_0^t N_2(v, R, S)(s, x_{+}(s)) + L(u, R, S)(s, x_{+}(s))ds,
\end{align*}
\]
\(\tag{3.33}\)

\[
u(t, x) = u_0(x) + \int_0^t \frac{R + S}{2} ds \tag{3.34}\]

and
\[
x_{\pm}(s; t, x) = x \pm \int_t^s u^{\alpha}(\tau, x_{\pm}(\tau; t, x)) d\tau. \tag{3.35}\]
In the same way as in the proof of (3.19) and (3.20), we achieve the boundedness of 
\( u_{n+1}, R_{n+1}, S_{n+1} \) = \( \Phi(u_n, R_n, S_n) \)
with initial term \( (u_0, S_0, R_0) \). By Proposition 3.2, \( (u_n, R_n, S_n) \) converges the fixed point 
\( (u, R, S) \) in the topology of \( L^\infty \). While we can define a sequence of the characteristic 
curves \( \{x_{\pm,n}(\cdot; t, x)\}_{n \in \mathbb{N}} \). We note that the characteristic curves can be defined uniquely 
on \([0, T] \times [0, T] \times [-K, K] \) with arbitrarily fixed \( t, x \) by the Lipschitz continuity, 
the boundedness of \( u_n^n \). For arbitrarily fixed \( K \geq 1 \), we see that \( \{x_{\pm,n}(\cdot; t, x)\}_{n \in \mathbb{N}} \) is the uniform equicontinuous and uniform bounded from Lemma 2.1 and (2.8). Thus the Arzelá-Ascoli theorem implies 
that there exists a subsequence of \( \{x_{\pm,n}(\cdot; t, x)\}_{n \in \mathbb{N}} \) (we use the same suffix as in the original sequence) such that \( x_{\pm,n}(\cdot) \) converges \( x_{\pm}(\cdot) \) uniformly on \([0, T] \times [0, T] \times [-K, K] \) as \( n \to \infty \). Note that this choice of the subsequence is depending on \( K \). However, from Cantor’s diagonal argument, we can reselect a subsequence independently of \( K \) such that the convergence holds on \([0, T] \times [0, T] \times [-K', K'] \) with any \( K' \geq 1 \). From (3.25) and (3.29), we see that as \( n \to \infty \)
\( (u_n(t, x_{\pm,n}(t)), R_n(t, x_{\pm,n}(t)), S_n(t, x_{\pm,n}(t))) \to (u(t, x_{\pm}(t)), R(t, x_{\pm}(t)), S(t, x_{\pm}(t))) \).

Hence (3.33) and (3.37) are satisfied. Now we check that \( (u, R, S) \in \mathcal{X}_\alpha \times Y_{\beta,1} \times Y_{\beta,2} \). It is obvious that the properties (3.1) and (3.4) are satisfied. From the Lipschitz continuity, 
\( u, R, S \) are differentiable almost everywhere. In the same way as in the proof of Proposition 3.1 we can obtain the boundedness of \( \langle x \rangle^\beta u^\alpha, u_t \) and \( \langle x \rangle^\beta u_t \), since the constant \( A_2 \) is taken independently of \( A_1 \) and \( A_3 \). Thus we have \( u \in \mathcal{X}_\alpha \). To show the boundedness of \( \langle x \rangle^\gamma R_x \) and \( \langle x \rangle^\gamma S_x \), differentiating the both side of (3.33) with \( x \), we obtain that

\[
V(t, x) = V_0(x_-(0; t, x)) \partial x_-(-0; t, x) \\
+ \int_0^t \partial x_-(s; t, x) (N_1 u_x + N_1 R V + N_1 S W) (t, x_-(s; t, x)) ds \\
+ \int_0^t \partial x_-(s; t, x) (L_n u_x + L_R V + L_S W) (t, x_-(s; t, x)) ds
\]

and

\[
W(t, x) = W_0(x_-(0; t, x)) \partial x_+(0; t, x) \\
+ \int_0^t \partial x_+(s; t, x) (N_2 u_x + N_2 R V + N_2 S V) (t, x_+(s; t, x)) ds \\
+ \int_0^t \partial x_+(s; t, x) (L_n u_x + L_R V + L_S W) (t, x_+(s; t, x)) ds.
\]

In the same way as in the proof of (3.19) and (3.20), we achieve the boundedness of \( \langle x \rangle^\gamma W \)
and \( \langle x \rangle^\gamma Y \). The estimates of \( \langle x \rangle^\gamma R_x \) and \( \langle x \rangle^\gamma S_x \) can be shown by similarity to in (3.21).

Thus we have \( (R, S) \in Y_{\beta,1} \times Y_{\beta,2} \). The uniqueness can be shown in the same way as in the proof of Proposition 3.2.

In the discussions so far, we do not assume any relations between \( u_0 \), \( R_0, S_0 \). To show \( u \) in (3.34) is a solution of (1.1), we assume that

\[
\begin{aligned}
u_0 &= \frac{R_0 - S_0}{2 u_0^2}, \\
u_1 &= \frac{R_0 + S_0}{2}.
\end{aligned}
\]
Moreover, we improve the regularity of the solution if \((u_0, u_1) \in C^2 \times C^1\). The following proposition completes the proof of Theorem 1.1.

**Proposition 3.4.** Addition to the assumption of Proposition 3.3, we assume for \((u_0, u_1) \in C^2 \times C^1\) that (3.38) is satisfied. Then the function \(u\) defined in (3.34) is \(C^2\) on \([0, T] \times \mathbb{R}\) and is the classical solution of (1.1).

**Proof.** From the Lipschitz continuity of \(R, S\), these are differentiable almost everywhere and satisfy that
\[
\begin{aligned}
R_t - u^a R_x &= N_1(u, R, S) + L(u, R, S), \\
S_t + u^a S_x &= N_2(u, R, S) + L(u, R, S).
\end{aligned}
\]
(3.39)

Since \(u\) is also differentiable almost everywhere, differentiating (3.34), we have that
\[
\partial_x u = u'_0(x) + \int_0^t \frac{R_x + S_x}{2} ds.
\]
(3.40)

From the first and third equation of (3.39), we have that
\[
\int_0^t R_x + S_x ds = \int_0^t \frac{1}{u^a} (N_2(u, R, S) - N_1(u, R, S) + R_t - S_t) ds
\]
\[
= \int_0^t \frac{1}{u^a} \left( \frac{a}{2u} (S^2 - R^2) + R_t - S_t \right) ds.
\]
(3.41)

From the integration by parts, \(u_t = \frac{R + S}{2}\) and (3.38), it follows that
\[
\int_0^t \frac{1}{u^a} (R_t - S_t) ds = -2u'_0(x) + \frac{R - S}{u^a} + \int_0^t \frac{au}{u^a + 1} (R - S) ds
\]
\[
= -2u'_0(x) + \frac{R - S}{u^a}
\]
\[
+ \int_0^t \frac{a}{2u^a + 1} (R^2 - S^2) ds.
\]
(3.42)

From (3.40), (3.41) and (3.42), we have that
\[
\partial_x u = \frac{R - S}{2u^a}.
\]
(3.43)

Combining (3.39), (3.41) and (3.43), we have that the function \(u\) satisfies (1.1). Lastly, applying Theorem 4 in Douglis [5], we obtain the continuity of the \(W = R_x\) and \(V = S_x\). From the equations of \(R, S\), we see that \(R_t\) and \(S_t\) are also continuous. Hence we have the continuity of \(u_{xx}, u_{tx}, u_{tt}\). Therefore we have that \(u \in C^2([0, T] \times \mathbb{R})\).

\[\square\]

4 Concluding remarks

4.1 Physical background

We set a function \(G\) as a primitive function of \(F\) such that \(G(0) = 0\). Integrating with \(x\) over \([-\infty, x]\), we formally obtain that
\[
\int_{-\infty}^x u_{tt} dx = \partial_x \left( \frac{u^{a+1}}{a + 1} \right) + G(u).
\]

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Setting $v = \int_{-\infty}^{x} u_t \, dx$ and $\sigma(u) = v^{a+1}/a + 1$, we have the following 1st order hyperbolic equation:

\[
\begin{cases}
u_t - v_x = 0, \\
\nu_t - \partial_x(\sigma(u)) = G(u).
\end{cases}
\] (4.1)

This equation governs the motion for one dimensional elastic waves with the case that the density of material is equal to 1. Unknown functions $u$ and $v$ describe the differentiations of the displacement $X$ with $x$ and $t$ respectively. Namely $u = X_t(t, x)$ and $v = X_{tx}(t, x)$. The first equation means the relation $u_t = X_{xt}(t, x) = v_x$. The second equation is Newton’s second since $v_t$ is the acceleration. From the definition of $u$, $u$ is the strain (more precisely, $(1, 1)$ component of the strain matrix) and $\sigma(u)$ is so-called stress-strain relation. $G$ is an external force term depending only on the strain. The detailed derivation with $G \equiv 0$ is given in Cristescu’s book [4].

### 4.2 On the generalization of the main theorem

The our existence theorem is also applicable to $\partial_t^2 u = (c(u)^2u_x)_x + F(u)u_x$ under the following assumptions on $c(\cdot) \in C([0, \infty)) \cap C^2((0, \infty))$ and $F \in C([0, \infty)) \cap C^1((0, \infty))$

\[
C_{1,K}\theta^a \leq c(\theta) \leq C_{2,K},
\] (4.2)

\[
|c'(\theta)| \leq C_{3,K}\theta^{a-1},
\] (4.3)

\[
|c''(\theta)| \leq C_{4,K}\theta^{a-2},
\] (4.4)

and

\[
|F(\theta)| \leq C_{5,K}\theta^a,
\] (4.5)

\[
|F'(\theta)| \leq C_{6,K}\theta^{a-1},
\] (4.6)

where $a \geq 0$, $\theta \in [0, K]$ for $K > 0$ and $C_{j,K}$ are positive constants depending on $K$ for $j = 1, \ldots, 6$. For this equation, the unknown valuable $R$ and $S$ are defined by

\[
R = u_t + c(u)u_x,
\]

\[
S = u_t - c(u)u_x
\]

and $R$ and $S$ satisfy that

\[
\begin{cases}
\partial_t R - u^a \partial_x R = \frac{d}{dx}(RS - S^2) + F(u)\frac{R - S}{2c}, \\
\partial_t S + u^a \partial_x S = \frac{d}{dx}(RS - R^2) + F(u)\frac{R - S}{2c}.
\end{cases}
\] (4.7)

Since we have that $|\frac{\nu'(u)}{c(u)}| \leq C \langle x \rangle^a$ from the assumption on $c$ and initial data, we can obtain weighted $L^\infty$ estimated for $R$ and $S$. The assumption (4.4) is used in the proof of the construction of the contraction mapping.

### 4.3 Finite time blow-up or degeneracy

We define $T^*$ as the maximal existence time of the solution constructed by Theorem 1.1. When $T^* < \infty$, we have the following criterion of the break-down:

\[
\limsup_{t \to T^*} \| \langle x \rangle^\beta u_t \|_{L^\infty} + \| \langle x \rangle^\beta u_x \|_{L^\infty} = \infty
\] (4.8)
or

\[ \lim \inf_{t \to T^*} \inf_{x \in \mathbb{R}} u(t, x) = 0. \quad (4.9) \]

We call (4.8) and (4.9) the blow-up and the degeneracy respectively. In the case that \( F \equiv 0 \), we can obtain the non-trivial solutions blow up in finite time, if \( R(0, x) \) and \( S(0, x) \) are non-negative. In fact, we can show that the non-negativity of \( R(0, x) \) and \( S(0, x) \) preserves as time goes by, from which we have \( u_t(t, x) \geq 0 \). Thus we find that (4.9) does not occur in finite time. Therefore, using the method of Lax [13] or [23] (see also Chen [1]), we have the conclusion. While, in the case that \( F \equiv 0 \), we can apply main theorems to the equation in (1.1) and find that (4.9) can occur in finite time for non-trivial solution, if \( R(0, x) \) and \( S(0, x) \) are non-positive. Sufficient conditions for the occurrence of (4.9) have been studied in the author’s papers [20, 21].

4.4 Multi-dimensional case

The multi-dimensional version of the equation in (1.1) is

\[ \partial_t^2 u = u^{2a} \Delta u + F(u) \cdot \nabla u = 0. \]

The method of characteristic (and Riemann invariant) does not work, even with radial initial data. In the forthcoming paper, we deal this problem via a local-energy argument.

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