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ANALYTICAL SOLUTIONS IN THE TWO-CAVITY COUPLING PROBLEM

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Abstract

Analytical solutions of precise equations that describe the rf-coupling of two cavities through a co-axial cylindrical hole are given for various limited cases. For their derivation we have used the method of solution of an infinite set of linear algebraic equations, based on its transformation into dual integral equations.

1 Introduction

In the papers [1, 2, 3], we derived precise equations, describing the rf-coupling of two cavities through a centerhole of arbitrary dimensions. On the base of these equations we numerically calculated the relationship of coupling coefficients versus different parameters (frequency, hole radius, etc.). This paper presents analytical solutions of these equations for various limited cases. In particular, it is explicitly shown that in the case of small holes ($a \to 0$) the formulated equations agree with those derived in the papers [4, 5, 6] on the base of quasi-static approach. Besides, expressions are derived for coupling coefficients which are valid up to the second order in the relation of the hole dimension ($a$) with the free-space wavelength ($\lambda$). For derivation of these expressions we have used the method of solving of an infinite set of linear algebraic equations, based on its transformation into dual integral equations.

2 Problem definition. Original equations

Let us consider the coupling of two cavities through a circular hole with the radius $a$ in a separating wall that has the thickness $t$. For simplicity’s sake, we will consider the case of two identical cavities, with $b$ -being the cavity radii and $d$ — their length. In the papers [1, 2, 3] it was demonstrated that if the field is expanded with the short-circuit resonant cavity modes and $E_{010}$-modes are selected as fundamental, the precise set of equations will consist of two equations for the amplitudes of $E_{010}$-modes, where coupling coefficients are defined by the way of solution of an infinite set of linear algebraic equations. Let us generalize the case considered in [1, 2, 3], choosing as fundamental $E_{0qp}$-modes of closed cavities ($q$ is the number of field variations across the radius, $p$ is the number of field variations along the longitudinal coordinates). Using the method, similar to the one in [1, 2, 3], one can show that the set of equations, describing the system under consideration, has the form;

$$
\epsilon_p Z_{q,p} a_{q,p}^{(1)} = -\omega_{q,p}^2 - \omega^2 - \frac{4}{3\pi} \frac{a^3}{b^2 d} \left[ a_{q,p}^{(1)} \Lambda_1 - (-1)^p a_{q,p}^{(2)} \Lambda_2 \right]$

$$
\epsilon_p Z_{q,p} a_{q,p}^{(2)} = -\omega_{q,p}^2 - \omega^2 - \frac{4}{3\pi} \frac{a^3}{b^2 d} \left[ a_{q,p}^{(2)} \Lambda_1 - (-1)^p a_{q,p}^{(1)} \Lambda_2 \right]$

where

$$Z_{q,p} = \omega_{q,p}^2 - \omega^2, \quad \omega_{q,p}^2 = c^2 \left[ \frac{\lambda_q^2}{b^2} + (p\pi/d)^2 \right],$$

$$\epsilon_p = \begin{cases} 
2, & p = 0 \\
1, & p \neq 0
\end{cases}, \quad p = 0, 1, \ldots \infty, \quad J_0(\lambda_s) = 0, \quad s = 1, 2, \ldots \infty,$$
$a^{(i)}_{q,p}$ is the amplitude of $E_{0qp}$-mode in the $i$-th cavity ($i = 1, 2$). The normalized coupling coefficients $\Lambda_i$ are determined by the expression:

$$\Lambda_i = \Lambda_i(\omega) = J_0^2(\theta_q) \sum_{s=1}^{\infty} w_s^{(i)} / \left( \lambda_s^2 - \theta_q^2 \right),$$

where $w_s^{(i)}$ are the solution of the following set of linear equations:

$$w_m^{(1)} + \sum_s G_{m,s} \left( w_s^{(1)} f_m^{(1)} + w_s^{(2)} f_m^{(2)} \right) = 3\pi f_m^{(1)} / \left( \lambda_m^2 - \Omega_s^2 \right),$$

$$w_m^{(2)} + \sum_s G_{m,s} \left( w_s^{(2)} f_m^{(1)} + w_s^{(1)} f_m^{(2)} \right) = 3\pi f_m^{(2)} / \left( \lambda_m^2 - \Omega_s^2 \right),$$

$$f_m^{(j)} = \frac{\mu_m}{sh(q_m)} \left\{ \begin{array}{l}
ch(q_m) - ch(q_m t/l), \ j = 1 \\
ch(2q_m d/l) - 1, \ j = 2,
\end{array} \right.$$  

$q_m = \mu_m l/a, \ l = 2d + t, \ \mu_m = \sqrt{\lambda_m^2 - \Omega_s^2}, \ \Omega = \omega a/c, \ \Omega_s^2 = \Omega^2 - \pi^2 a^2 p^2 / d^2.$

$$G_{m,s} = B_{m,s} - \frac{1}{2\mu_m} \delta_{m,s} cth \left( \frac{d}{a} \mu_m \right) +$$

$$+ \frac{2\pi a^2 \theta_q^2 J_0^2(\theta_q)}{db \epsilon_p \chi_q (\lambda_m^2 - \theta_q^2)(\lambda_s^2 - \theta_q^2)(\mu_m^2 + \pi^2 a^2 p^2 / d^2)},$$

$$B_{m,s} = \pi \frac{a}{b} \sum_{\ell=1}^{\infty} \frac{\theta_\ell^2 J_0^2(\theta_\ell)}{\ell(\lambda_m^2 - \theta_\ell^2)(\lambda_s^2 - \theta_\ell^2)},$$

$$\theta_\ell = \lambda_\ell a/b, \ \chi_\ell = \pi \lambda_\ell J_1^2(\lambda_\ell)/2, \ \nu_\ell = \sqrt{\theta_\ell^2 - \Omega_s^2},$$

$$R_\ell = \left\{ \begin{array}{l}
\theta_q cth(\nu_q d/a) / \nu_q - 2a \theta_q / \left\{ \epsilon_p d \left( \nu_q^2 + \pi^2 a^2 p^2 / d^2 \right) \right\}, \ \ell = q, \\
\theta_\ell cth(\nu_\ell d/a) / \nu_\ell, \ \ell \neq q.
\end{array} \right.$$  

The coefficients $w_s^{(i)}$ have a simple physical sense. Really, it is easy to show that the tangential electric field component in the left cross-section of the coupling hole $E_{r}^{(-)}(r)$ has the form

$$E_{r}^{(-)}(r) = E_{ind}^{(1)}(r) - E_{ind}^{(2)}(r) = \tilde{E}_{0,q,p}^{(1)}Q^{(1)}(r) - \tilde{E}_{0,q,p}^{(2)}Q^{(2)}(r),$$

where

$$Q^{(i)}(r) = \frac{1}{3\pi} \sum_s \frac{J_1(\lambda_s r/a)}{J_1(\lambda_s)} w_s^{(i)},$$

$\tilde{E}_{0,q,p}$ is the value of the longitudinal (perpendicular to the hole) electric field of $(0, q, p)$-mode in the first cavity on the left coupling hole cross-section at $r = a$, while $\tilde{E}_{0,q,p}^{(2)}$ is the same value for the right-hand cavity on the right coupling hole cross-section at the same radius.

From the expression (3) it follows that the tangential electric field component on the left coupling hole cross-section is equal to the difference of two induced fields, each of

\footnote{The same is true for the right cross section}
which is proportional to the perpendicular electric field components of $E_{0,q,p}$-modes, taken to be fundamental. There, the coefficients $w^{(i)}_s$ are the ones of expansion of the appropriate functions with the complete set of functions $\{J_1(\lambda s/a)\}$.

Note that the coefficients $w^{(i)}_m$ can be re-defined which will cause changes in Eqs.(4,5), in the form of relationship (9), and, consequently, in the above nature of proportionality. For example, in (9) one can obtain proportionality $E^{(i)}_r$ to the longitudinal electric field component of $(0,q,p)$-mode at $r = 0$. While defining $w^{(i)}_m$, we proceeded from the following condition: for the two different cavities, in the case $t = 0$ (infinitely thin wall) and taking $E_{0,q,0}$-modes to be fundamental: $Q^{(1)}(r) = Q^{(2)}(r) = Q(r)$, then we have in this case

$$E^{(-)}_r(r) = E^{(+)}_r(r) = Q(r)\left(\hat{E}^{(1)}_{0,q,0} - \hat{E}^{(2)}_{0,q,0}\right).$$

This condition determines the tangential electric field component on the hole while having $E_{0,q,p}$-modes as fundamental. With such a normalization of $w^{(i)}_m$, their determining set of infinite linear equations (4,5) acquires the most symmetrical form.

Thus, the two-cavity coupling problem, rigorously formulated on the base of the electric field expansion with the short-circuit resonant cavity mode, is reduced to the induced field definition on the right and left cylindrical hole cross-section.

3 Infinitely thin wall case

An important role in the problem of cavity coupling plays the case of infinitely thin wall, dividing the cavities ($t = 0$). In this case, from Eqs.(4,5) it follows that $w^{(1)}_m = w^{(2)}_m = w_m$. In this case the set of equations for $w_m$ will take on the form:

$$\sum_{s=1}^{\infty} w_s B_{m,s} = 3\pi / \left\{2 \left(\lambda^2_m - \Omega^2_s\right)\right\}. \quad (10)$$

For the case $t = 0$ $\Lambda_1 = \Lambda_2 = \Lambda$, where

$$\Lambda = J^2_0(\theta_q) \sum_s w_s \left(\lambda^2_s - \theta^2_q\right). \quad (11)$$

3.1 Small coupling hole case ($a \to 0$)

If in Eqs.(10,11) the hole radius tends to zero\(^4\), then Eq.(10) will become:

$$\sum_{s=1}^{\infty} w_s \int_0^{\infty} \frac{\theta^2 J^2_0(\theta) d\theta}{(\lambda^2_s - \theta^2) (\lambda^2_m - \theta^2)} = \frac{3\pi}{2\lambda^2_m}. \quad (12)$$

In order to get the solution for Eq.(12) we will introduce an integer odd function $f_1(z)$ the values of which in the points $z = \lambda_s$ are equal

$$f_1(\lambda_s) = w_s J_1(\lambda_s). \quad (13)$$

\(^3\)We have neglected terms of order $a^3$ in the expression (6) for $G_{m,s}$

\(^4\)In this case, as follows from Eqs.(4,5), the coupling coefficients will be proportional to $a^3$
Let us assume that at $|z| \to \infty$ $f_1(z)$ grows not faster than $\exp(z)$, then, in accordance with Cauchy theorem, the function $(f(z)/J_0(z))$ can be expanded in a partial fraction series

$$
f_1(z)/J_0(z) = 2z \sum_{n=1}^{\infty} w_n/(\lambda_n^2 - z^2).$$

(14)

Using (14), and, also, multiplying Eq.(12) by $J_1(\lambda_m x)/J_1(\lambda_m)$, where $0 < x < 1$, and doing summation over sub-index $m$, we will get

$$
\int_0^\infty f_1(z)J_1(xz)dz = 3\pi x/2, \ 0 < x < 1.
$$

(15)

By multiplying (14) by $zJ_1(xz)$ and integrating over $z$ from 0 to $\infty$, we will obtain (see the Appendix) at $x > 1$:

$$
L_1(x) = \int_0^\infty z f_1(z) J_1(xz) dz = 0, \ x > 1.
$$

(16)

In this way, the set of linear algebraic equations (12) with a complicated coefficients matrix that cannot be expressed via elementary functions and can be calculated only numerically, has been reduced to two integral equations (15),(16). Having determined the kind of function $f_1(z)$, there is no need in calculating the sum (11), since

$$
\Lambda = \sum_s w_s/\lambda_s^2 = \lim_{z \to 0} f_1(z)/(2zJ_0(z)).
$$

(17)

The method of solving the dual integral equations of the type (15,16) on the base of the Mellin transformation, as well as the property of Cauchy-type integrals, can be found in [7]. The brief summery of their solutions is given in [8]. We shall dwell briefly on a simpler method of resolving the system, because it will be used in Sec.3 for the analysis of the infinitely thick wall case.

Since $f_1(z)$ is the odd function it can be represented in the form

$$
f_1(z) = \int_0^\infty \sin(zt)\eta(t)dt.
$$

(18)

Substituting this expression in Eq.(16) we obtain such integral equation for $\eta(t)$:

$$
\int_x^\infty \frac{\eta(t)dt}{\sqrt{t^2 - x^2}} = 0, \ x > 1.
$$

The solution of this equation is $\eta(t) = 0$ for $t > 1$. Consequently, any function of the type

$$
f_1(z) = \int_0^1 \sin(zt)\eta(t)dt
$$

(19)

satisfies Eq.(16). Substituting (19) into (13), we obtain the first kind Volterra equation Abelian type

$$
\int_0^x \frac{t \eta(t)dt}{\sqrt{x^2 - t^2}} = \frac{3\pi}{2} x^2, \ 0 < x < 1,
$$

(20)
the solution for which can be found in the analytical form. Omitting the intermediate formulae, we shall give the final expression for the function \( f_1(z) \)

\[
f_1(z) = \frac{6}{z^2} \{ \sin(z) - z \cos(z) \} \approx 2z \left(1 - \frac{z^2}{10}\right) \quad (21)
\]

The normalized coupling coefficients, as follows from (17), is equal to \( \Lambda = 1 \). Since \( w_s = f(\lambda_s)/J_1(\lambda_s) \), then, from (9), we will obtain

\[
E_r(−) (r) = E_0^{(1)} (r) - E_0^{(2)} (r) \quad (22)
\]

Thus, on the base of a rigorous electrodynamic description of the two cavity coupling system we are the first to prove, by the way of the limit transition \( a \to 0 \), the correctness of the equations formulated in the papers [4, 5, 6] on the basis of the quasi-static approximation, and to obtain the expression for the tangential electric field on the hole.

### 3.2 The case of small, though finite, values of coupling hole radius

The above method presents the opportunity to obtain analytical expressions for the normalized coupling coefficients with an accuracy on the order of \( (a/\lambda)^2 \). If \( a/\lambda \) is small, though finite, then, the coefficients \( w_s \) in (10) will be dependent on the hole radius value \( a \): \( w_s = w_s(a) \). Let’s introduce the function of two variables:

\[
\psi(a, z) = 2z J_0(z) \sum_{n=1}^{\infty} \frac{w_n(a)}{\lambda_n^2 - z^2} \quad (23)
\]

We will assume that relative to the variable \( z \) the function \( \psi(a, z) \) will obey the conditions formulated in Subsec.2.1. Using the technique, similar to that described in Subsec.2.1, the set (10) can be reduced to:

\[
\sum_{\ell=1}^{\infty} \frac{\theta \ell J_1(x\theta)}{\chi \ell} \psi(a, \theta \ell) = 0, \quad 1 < x < b/a, \quad (24)
\]

\[
\frac{a}{b} \sum_{\ell=1}^{\infty} \frac{J_1(x\theta)}{\chi \ell} \psi(a, \theta \ell) R_{\ell} = \frac{3\pi J_1(x\Omega_*)}{\Omega_* J_0(\Omega_*)}, \quad 0 < x < 1. \quad (25)
\]

Letting \( a \to 0 \) in Eqs.(24), (25), we derive a set of equations (15,16), and, consequently, \( \psi(0, z) = f_1(z) \), where \( f_1(z) \) is determined by Eq.(21). Let’s represent \( \psi(a, z) \) in the form

\[
\psi(a, z) = \psi(0, z) + a^2 \varphi(a, z), \quad (26)
\]

where \( \varphi(a, z) \) is a function which has the same conditions imposed upon that \( \psi(a, z) \) does.

From (24), (25) it follows that \( \varphi(0, z) \) satisfies the following equations

\[
\int_{0}^{\infty} \theta J_1(x\theta) \varphi(0, \theta) d\theta = 0, \quad x > 1, \quad (27)
\]

\[
\int_{0}^{\infty} J_1(x\theta) \varphi(0, \theta) d\theta = F(x), \quad 0 < x < 1, \quad (28)
\]
where
\[ F(x) = \lim_{a \to 0} \frac{1}{a^2} \left[ \frac{3\pi J_1(x\Omega_\ast)}{\Omega_\ast J_0(\Omega_\ast)} - \pi a \sum_{\ell=1}^\infty \frac{J_1(x\theta_\ell)}{\chi_\ell} \psi(0, \theta_\ell) R_\ell \right]. \]

The coefficients \( R_\ell \) can be represented as:
\[ R_\ell = 1 + \frac{\Omega_\ast^2}{2\theta_\ell^2} + \hat{R}_\ell. \]

It can be shown that the following estimations are true
\[ \pi a b \sum_{\ell=1}^\infty \frac{J_1(x\theta_\ell)}{\chi_\ell} \psi(0, \theta_\ell) = 3\pi x + O(a^3), \]
\[ \pi a b \sum_{\ell=1}^\infty \frac{J_1(x\theta_\ell)}{\chi_\ell} \psi(0, \theta_\ell) \hat{R}_\ell = O(a^3). \]

Then
\[ F(x) = \frac{3\pi}{8} x \left[ \frac{\Omega_\ast^2 - \Omega^2}{a^2} - \frac{2\Omega_\ast^2 - \Omega^2}{4a^2} x^2 \right]. \]

The solution of Eqs.(27,28) has the form
\[ \varphi(0, z) = \frac{\Omega_\ast^2 - \Omega^2}{4a^2} f_1(z) - \frac{2\Omega_\ast^2 - \Omega^2}{2a^2} f_2(z), \]
where \( f_1(z) \) is determined by the formula (27), while \( f_2(z) \) is
\[ f_2(z) = \frac{(3z^2 - 6)\sin(z)}{z^4} - \frac{z(z^2 - 6)\cos(z)}{z^4} \approx \frac{z}{5}. \quad (29) \]

The function \( \psi(a, z) \approx \psi(0, z) + a^2 \varphi(0, z) \), accurate to the order \((a/\lambda)^2\), has the form
\[ \psi(a, z) \approx \left( 1 + \frac{\Omega_\ast^2 - \Omega^2}{4} \right) f_1(z) - \frac{2\Omega_\ast^2 - \Omega^2}{2} f_2(z). \quad (30) \]

The normalized coupling coefficients \( \Lambda \) is determined by the relationship
\[ \Lambda = J_0 \left( \frac{a}{b} \lambda_q \right) \psi \left( a, \frac{a}{b} \lambda_q \right) / \left( 2 \frac{a}{b} \lambda_q \right) \approx \]
\[ \approx 1 - \frac{1}{5} \left( \frac{a}{b} \lambda_q \right)^2 - \frac{3}{20} \left( \frac{a}{c} \omega_{q,p} \right)^2 - \frac{1}{20} \left( \frac{a}{c} \omega \right)^2. \quad (31) \]

For the case \( \omega \approx \omega_{q,p} \) the expression (31) agrees with that for the generalized polarizability, obtained in [9] at \( b \to \infty \) via the variation technique. Note that the expression (31) is true for the frequency \( \omega \) that is not close to the resonant frequencies of the non-fundamental modes of closed cavities: \( \omega \neq \omega_{m,n} \), if \( (m, n) \neq (q, p) \).

Knowing \( \psi(a, z) \), and, consequently, \( w_s(a) = \psi(a, \lambda_s) / J_1(\lambda_s) \), the form of the tangential electric field around the hole can be reconstructed
\[ E_r^{(-)}(r) = \frac{E_{0,p,q}(r = 0) - E_{0,p,q}(r = 0)}{\pi} \times \]
\[
\times \left\{ \left[ 1 - \frac{1}{4} \left( \frac{a}{b} \mu \right)^2 + \frac{\Omega^2 - 2\Omega^2}{12} \right] \frac{r}{\sqrt{a^2 - r^2}} + \frac{2\Omega^2 - \Omega^2}{6} \frac{r}{a} \sqrt{1 - \left( \frac{r}{a} \right)^2} \right\}. \tag{32}
\]

Since in our approach the case \( \Omega \to 0 \) corresponds to the quasi-static method of field calculation, then, the formulae \( \text{(31,32)} \) at \( \Omega = 0 \) present the solution for the appropriate quasi-static problem up to the second-order approximation in \( (a/b) \) in the case when the “far” field (see, \( \text{[5,6]} \)) is not homogeneous.

4  Infinitely thick wall case

The above analytical results pertain to the case of infinitely thin wall, when the singularity at the rim of the hole has the form \( (r - a)^{-1/2} \). It is of interest to consider the possibility to apply the above method to a non-zero thickness wall, when the singularity at the rim of the hole has the form \( (r - a)^{-1/3} \) \( \text{[10]} \). The simplest problem in this class, although representing an important application, is research into the coupling of a cylindrical cavity with a co-axial cylindrical waveguide of the radius \( a < b \). Such a system can be studied upon the base the above equations in the limit case \( t \to \infty \):

\[
\epsilon_p Z_{q,p}a_{q,p} = -\omega^2 \frac{q}{3\pi J_1^2(\lambda_q)} \frac{a^3}{b^2} \Lambda a_{q,p}, \tag{33}
\]

where the normalized coupling coefficients at \( a \to 0 \) is equal to

\[
\Lambda = \sum_{s=1}^{\infty} \frac{w_s}{\lambda_s^2}, \tag{34}
\]

while \( w_s \) are the equation solutions

\[
\frac{w_m}{2\lambda_m} + \sum_{s=1}^{\infty} w_s \int_{0}^{\infty} \frac{\theta^2 J_0^2(\theta)d\theta}{(\lambda_s^2 - \theta^2)(\lambda_s^2 - \theta^2)} = \frac{3\pi}{\lambda_m^2}. \tag{35}
\]

The set \( \text{(35)} \) is different from the above-studied \( \text{(12)} \) in additional addends in the diagonal matrix elements. Introducing a function of the type \( \text{(14)} \), we obtain the following set of equations:

\[
\int_{0}^{\infty} z f(z) J_1(xz)dz = 0, \ x > 1. \tag{36}
\]

\[
\sum_{m=0}^{\infty} \frac{f(\lambda_m) J_1(x\lambda_m)}{\lambda_m J_1^2(\lambda_m)} + \frac{1}{2} \int_{0}^{\infty} f(z) J_1(xz)dz = \frac{3\pi x}{2}, \ 0 < x < 1. \tag{37}
\]

As indicated above, from \( \text{(36)} \) it follows that \( f(z) \) should be sought in the form of \( \text{(19)} \), then, from \( \text{(37)} \) we get the fact that \( \eta(t) \) has to be the solution of the Fredholm equation of the second kind

\[
\eta(u) + \frac{4}{\pi} \int_{0}^{1} \eta(t) \sum_{m=1}^{\infty} \frac{\sin(\lambda_m t) \sin(\lambda_m u)}{\lambda_m J_1^2(\lambda_m)} dt = 12 u. \tag{38}
\]
Since the kernel of this integral equation is degenerate, then its solution have the appearance

$$\eta(u) = 12u - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(\lambda_m u)}{\lambda_m J_1^2(\lambda_m)} c_m,$$

(39)

where

$$c_m = \int_0^1 \eta(t) \sin(\lambda_m t) dt$$

the constant coefficients which are the solution of the infinite linear set of equations to be easily obtained by way of the appropriate integration (39). Since $c_m = f(\lambda_m) = w_mJ_1(\lambda_n)$ (see (19)), this system can be represented as:

$$w_m + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{w_n}{\lambda_n J_1(\lambda_n) J_1(\lambda_m)} \left[ \frac{\sin(\lambda_n - \lambda_m)}{\lambda_n - \lambda_m} - \frac{\sin(\lambda_n + \lambda_m)}{\lambda_n + \lambda_m} \right] =$$

$$= \frac{12 \{\sin(\lambda_m) - \lambda_m \cos(\lambda_m)\}}{\lambda_m^2 J_1(\lambda_m)},$$

(40)

Thus, for fields with the singularity at the rim of the hole of the type $(r - a)^{-1/3}$ we have obtained an analogous system (40) instead of the initial linear set of equations (15). Comparing (40) and (15), we can see that in the system (15) the matrix coefficients are expressed through the improper integrals, whereas in the modified set (40) they are determined by well studied functions. This considerably facilitates both analytical studies and numerical simulation.

We have carried out numerical calculations of the normalized coupling coefficient $\Lambda$ determined by the formula (34) both on the base of (33) and (40). For a 200*200 matrix, the calculations based on (33) gives $\Lambda = 0.85835$, based on (40) — $\Lambda = 0.85854$, that is the results agree up to $2 \times 10^{-4}$. This result confirms the correctness of the above analytical method. Note that the obtained $\Lambda$-value corresponds to the results of a purely static analysis [11] carried out on the base of the variation technique for the infinite plane with a cylindrical hole.

5 Conclusion

On the base of our method of reduction of the infinite linear algebraic equation set to dual integral equations, we obtained, in different limited cases, the rigorous analytical solutions regarding the two-cavity coupling problem. Alongside with general theory significance, the obtained solutions are of applied interest, since they can be used for a better convergence of the original equation solution (13), which are true for arbitrary dimensions of the coupling hole.

6 Acknowledgment

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7 Appendix

If $f_1(z)$ can be representable as series (13), then

$$L_1(x) = \int_0^\infty zJ_1(xz)f_1(z)dz = 2 \sum_{n=0}^\infty w_n G_n(x). \tag{A.1}$$

where

$$G_n(x) = \int_0^\infty \frac{z^2 J_1(xz)J_0(z)dz}{\lambda_n^2 - z^2} =$$

$$= - \int_0^\infty J_1(xz)J_0(z)dz + \lambda_n^2 \int_0^\infty \frac{J_1(xz)J_0(z)dz}{\lambda_n^2 - z^2} = -G^{(1)}(x) + \lambda_n^2 G^{(2)}(x). \tag{A.2}$$

It is easy to demonstrate (see [8]), that the integral $G^{(1)}$ at $x > 1$ is equal $G^{(1)} = 1/x$. Let’s consider the integral $G^{(2)}(x)$ at $x > 1$:

$$G^{(2)}_n(x) = \int_0^\infty \frac{J_1(xz)J_0(z)dz}{\lambda_n^2 - z^2}. \tag{A.3}$$

Having made the substitution $y = xz$, we obtain

$$G^{(2)}_n(x) = \alpha \int_0^\infty \frac{J_1(y)J_0(\alpha y)dy}{\lambda_n^2 - \alpha^2 y^2}, \tag{A.4}$$

where $\alpha = 1/x$, $0 < \alpha < 1$. Using the expansion

$$J_1(y) = 2yJ_0(y) \sum_{m=1}^\infty \frac{1}{\lambda_m^2 - y^2},$$

(A.4) will take the form

$$G^{(2)}_n(x) = 2\alpha \sum_{m=1}^\infty \int_0^\infty \frac{yJ_0(y)J_0(\alpha y)dy}{(\lambda_n^2 - \alpha^2 y^2)(\lambda_m^2 - y^2)} =$$

$$= 2\alpha \sum_{m=1}^\infty \frac{1}{\alpha^2 \lambda_m^2 - \lambda_n^2} \left[ \alpha^2 \int_0^\infty \frac{yJ_0(y)J_0(\alpha y)dy}{(\lambda_n^2 - \alpha^2 y^2)} - \int_0^\infty \frac{yJ_0(y)J_0(\alpha y)dy}{(\lambda_m^2 - y^2)} \right] =$$

$$= 2\alpha \sum_{m=1}^\infty \frac{1}{\alpha^2 \lambda_m^2 - \lambda_n^2} \left[ \alpha^2 P^{(1)}_n - P^{(2)}_n \right]. \tag{A.5}$$

Calculation of the integrals under consideration is based on such expansion (see [8]),

$$\frac{\pi}{4} \frac{J_0(\alpha \beta z)}{J_0(z)} \left[ J_0(z)Y_0(\beta z) - J_0(\beta z)Y_0(z) \right] =$$

$$= \sum_{s=1}^\infty \frac{J_0(\alpha \beta \lambda_s) J_0(\beta \lambda_s)}{J_1^2(\lambda_s)(z^2 - \lambda_s^2)}, \tag{A.6}$$

where $0 < \alpha < 1$, $0 < \beta < 1$. 


Let’s consider (A.6) at $z = \lambda_m / \beta$

$$
\sum_{s=1}^{\infty} \frac{J_0(\alpha \beta \lambda_s) J_0(\beta \lambda_s) \beta^2}{J_1^2(\lambda_s)(\lambda_m^2 - \beta^2 \lambda_s^2)} = \frac{1}{2} \frac{J_0(\alpha \lambda_m)}{J_1(\lambda_m) \lambda_m}
$$

or

$$
\pi \beta \sum_{s=1}^{\infty} \frac{J_0(\alpha \beta \lambda_s) J_0(\beta \lambda_s) \beta \lambda_s}{\pi \lambda_s J_1^2(\lambda_s)/2} \left(\lambda_m^2 - \beta^2 \lambda_s^2\right) = \frac{J_0(\alpha \lambda_m)}{J_1(\lambda_m) \lambda_m}
$$

At $\beta \to 0$, we have

$$
P^{(2)}_m = \int_0^{\infty} \frac{y J_0(y) J_0(\alpha y) dy}{(\lambda_m^2 - y^2)} = \frac{J_0(\alpha \lambda_m)}{J_1(\lambda_m) \lambda_m}. \tag{A.7}
$$

The value $P^{(1)}_n = 0$ can be obtained from (A.7) at $\lambda_m \to \lambda_n / \alpha$. Then

$$
G^{(2)}_n(x) = 2\alpha \sum_{m=1}^{\infty} \frac{J_0(\alpha \lambda_m)}{(\lambda_n^2 - \alpha^2 \lambda_m^2) \lambda_m J_1(\lambda_m)}. \tag{A.7}
$$

From the expansion

$$
\frac{J_0(\alpha z)}{J_0(z)} = 1 - 2z^2 \sum_{m=1}^{\infty} \frac{J_0(\alpha \lambda_m)}{\lambda_m J_0(\lambda_m)} \frac{1}{z^2 - \lambda_m^2}.
$$

at $z = \lambda_n / \alpha$ we have

$$
\sum_{m=1}^{\infty} \frac{J_0(\alpha \lambda_m)}{\lambda_m J_0(\lambda_m)} \frac{1}{\lambda_n^2 - \alpha^2 \lambda_m^2} = \frac{1}{2 \lambda_n^2}.
$$

Consequently,

$$
G^{(2)}_n(x) = \frac{\alpha}{\lambda_n^2} = \frac{1}{x \lambda_n^2}.
$$

and

$$
G_n(x) = \int_0^{\infty} \frac{z^2 J_1(x z) J_0(z) dz}{\lambda_n^2 - z^2} = -G^{(1)}(x) + \lambda_n^2 G^{(2)}(x) = -\frac{1}{x} + \frac{1}{x} = 0.
$$

Finally,

$$
L_1(x) = \int_0^{\infty} z J_1(x z) f_1(z) dz = 2 \sum_{n=0}^{\infty} w_n G_n(x) = 0, \ x > 1, \tag{A.8}
$$

which is what needed to be proven.
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