We deliver a new scheme to compute the quark propagator and the quark-gluon interaction vertex through the coupled Dyson-Schwinger equations (DSEs) of QCD. We take the three-gluon vertex into account in our calculations, and implement the gluon propagator and the running coupling function fitted by the solutions of their respective DSEs. We obtain the momentum and current mass dependence of the quark propagator and the quark-gluon vertex, and the chiral quark condensate which agrees with previous results excellently. We also compute the quark-photon vertex within this scheme and give the anomalous chromo- and electro-magnetic moment of quark. The obtained results also agree with previous ones very well. These applications manifest that the new scheme is realistic and then practical for explaining the QCD-related phenomena.

I. INTRODUCTION

People have made a lot of efforts on studying the non-perturbative phenomena of QCD, such as the dynamical chiral symmetry breaking (DCSB), confinement and the low-lying hadron spectra. As a basic and important ingredient of QCD, the quark propagator is tightly connected with these phenomena. Among all the studies, the non-perturbative continuum QCD approach based on the infinite tower of Dyson-Schwinger equations (DSEs) has made fruitful achievements (see, e.g., Refs. [3–5, 9, 26, 27, 29, 33–35, 42–48]). DSEs can be directly derived from the Lagrangian of QCD, and thus provides a systematic and consistent way from QCD to the phenomena. To solve the DSE for the quark propagator, i.e., the gap equation, people need to know the information of gluon propagator and the quark-gluon vertex, i.e., the gap equation, which agrees with previous results excellently. We also compute the quark-photon vertex within this scheme and give the anomalous chromo- and electro-magnetic moment of quark. The obtained results also agree with previous ones very well. These applications manifest that the new scheme is realistic and then practical for explaining the QCD-related phenomena.

A practical scheme from QCD to phenomena via Dyson-Schwinger equations

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We deliver a new scheme to compute the quark propagator and the quark-gluon interaction vertex through the coupled Dyson-Schwinger equations (DSEs) of QCD. We take the three-gluon vertex into account in our calculations, and implement the gluon propagator and the running coupling function fitted by the solutions of their respective DSEs. We obtain the momentum and current mass dependence of the quark propagator and the quark-gluon vertex, and the chiral quark condensate which agrees with previous results excellently. We also compute the quark-photon vertex within this scheme and give the anomalous chromo- and electro-magnetic moment of quark. The obtained results also agree with previous ones very well. These applications manifest that the new scheme is realistic and then practical for explaining the QCD-related phenomena.
vertex has been verified to be practically equivalent to the ab initio approach \cite{12} and usually referred to as the CLR vertex.

In this paper, we construct a new scheme that combines the DSEs of the quark propagator and the quark-gluon vertex with all the 12 Lorentz structures. Since it has been analyzed that the non-Abelian interaction is crucial in the quark-gluon vertex \cite{53,59–61}, we take then here the contribution of the three-gluon vertex into account in the DSEs of the quark-gluon interaction vertex. After solving the coupled equations, we obtain the momentum and the current quark mass dependence of the quark propagator and the quark-gluon vertex. Our obtained result of the quark condensate is consistent with those from other computations, and the result for the vertex is consistent with the CLR vertex qualitatively. Since this scheme is systematically derived from the QCD Lagrangian, it is easy to be generalized to construct the quark-quark scattering kernel, which is useful for formalizing the Bethe-Salpeter equations beyond RL approximation and constructing a realistic scheme for computing hadron properties.

The remainder of this paper is organized as the follows. In Section \textbf{II} we introduce briefly the coupled DSEs for the quark propagator and the quark-gluon interaction vertex. In Section \textbf{III} we describe the details of solving the coupled DSEs. In Section \textbf{IV} we give the obtained momentum dependence of the quark propagator and the quark-gluon interaction vertex from the coupled DSEs at several values of the current quark mass. In Section \textbf{V} a parameterized expression of the quark-gluon interaction vertex is given. A practical truncation scheme to solve the quark-propagator’s DSE is then built. In Section \textbf{VI} the new truncation method is applied to calculate the anomalous chromo- and electro-magnetic moment of the quark. Finally a summary is given in Section \textbf{VII}.

\section{The Coupled DSEs for the Quark Propagator and the Quark-Gluon Vertex}

\subsection{The coupled DSEs}

The quark’s gap equation in momentum space is usually written as

\begin{equation}
S(p)^{-1} = (i\gamma + m_f) + \Sigma(p),
\end{equation}

with

\begin{equation}
\Sigma(p) = C_F \int \frac{d^4 q}{(2\pi)^4} g_{\gamma \mu} S(q) G_{\mu \nu}(p - q) g_{\Gamma \nu}(p, q) ,
\end{equation}

where $S(p)$ is the quark propagator, $G_{\mu \nu}(p - q)$ is the gluon propagator, $\Gamma_\nu(p, q)$ is the quark-gluon vertex and $g$ is the running coupling of the QCD. The constant $C_F = \frac{N^2 - 1}{2N_c}$ is from the color structure. $m_f$ is the current quark mass.

The quark propagator can be expressed as

\begin{equation}
S(p) = \frac{Z(p^2)}{i\phi + M(p^2)} = \frac{1}{i\phi A(p^2) + B(p^2)} ,
\end{equation}

where $M(p^2)$ is the dynamical mass of the quark.

The quark-gluon vertex satisfies its own DSE, which involves 3-point Green’s functions and the full propagators, and the full gluon propagator involves 3-point and 4-point Green’s functions and quark and ghost loops \cite{62}. After neglecting the contribution of ghost and 4-point Green’s functions, one can express the equation for the quark-gluon vertex as (see Fig.1),

\begin{equation}
\Gamma_\nu(p, q) = \gamma_\nu + \Lambda_\nu^A(p, q) + \Lambda_\nu^{NA}(p, q),
\end{equation}

where

\begin{equation}
\Lambda_\nu^A(p, q) = - \frac{g^2}{2N_c} \int \frac{d^4 k}{(2\pi)^4} \Gamma_\beta S(q - p + k) \Gamma_\nu S(k) \Gamma_\alpha G_{\alpha \beta}(p - k),
\end{equation}

\begin{equation}
\Lambda_\nu^{NA}(p, q) = i \frac{N_c g^2}{2} \int \frac{d^4 k}{(2\pi)^4} G_{\alpha \sigma}(p - k) \Gamma_\sigma \gamma_\nu \Gamma_\rho(p - q, k - q) \Gamma_\rho G_{\beta \rho}(k - q) \Gamma_\nu S(k) \Gamma_\alpha.
\end{equation}

The superscript “A”, “NA” denotes the Abelian diagram (the fourth diagram in Fig.1), the non-Abelian one, thus is subleading. Direct numerical calculations \cite{40,63,64} have in fact also shown such a result. Therefore, in this work, except for the bare one, we just consider the contribution from the non-Abelian diagram that includes the three-gluon vertex. After setting this, the coupled DSEs can be expressed diagrammatically in Fig.2.

Now, in this coupled system, the quark-gluon vertex and the quark propagator are all dressed ones with the complete 14 Lorentz structures (2 structures for quark and 12 for the vertex).

\subsection{Renormalization}

For the quark-gluon vertex, which can be expressed as

\begin{equation}
\Gamma_\mu(q, p) = f_1(q, p) \gamma_\mu + \Lambda_\mu(q, p),
\end{equation}

where $\Lambda_\mu$ denotes all the contributions from the other tensors except the $\gamma_\mu$, which are finite, only the first term $f_1(q, p)$ needs to be subtracted for its UV divergence. The renormalization condition for the quark-gluon vertex can then be chosen as

\begin{equation}
Z_{1F} + f_1(p, p)|_{p^2 = M^2} = 1 ,
\end{equation}

where $Z_{1F}$ is the quark propagator. The denominator of $\frac{1}{Z_{1F} + f_1(p, p)|_{p^2 = M^2}}$ is the dynamical mass of the quark.

The quark-gluon vertex satisfies its own DSE, which involves 3-point Green’s functions and the full propagators, and the full gluon propagator involves 3-point and 4-point Green’s functions and quark and ghost loops \cite{62}. After neglecting the contribution of ghost and 4-point Green’s functions, one can express the equation for the quark-gluon vertex as (see Fig.1),

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where $Z_{1F}$ is the quark propagator. The denominator of $\frac{1}{Z_{1F} + f_1(p, p)|_{p^2 = M^2}}$ is the dynamical mass of the quark.
and the renormalized quark-gluon vertex reads,
\[ \Gamma_\mu(q,p) = (f_1(q,p) + Z_1 \Delta \mu(q,p)), \tag{12} \]

For the quark propagator, the renormalized gap equation reads,
\[ S^\nu(p)^{-1} = Z_2 (i\sigma + Z_4 m_f) + Z_1 \Sigma(p), \tag{13} \]

if expressing the self energy \( \Sigma(p) = i\sigma \Sigma_1(p^2) + \Sigma_2(p^2) \) with \( \Sigma_1(p^2) \) and \( \Sigma_2(p^2) \) the scalar functions, the solution of the gap equation can be expressed as,
\[ A(p^2) = Z_2 + Z_1 \Sigma_1(p^2), \quad B(p^2) = Z_2 Z_4 m_f + Z_1 \Sigma_2(p^2), \tag{14} \]

Based on considering the WTI, we set \( Z_1 = Z_2 \). The renormalization condition for the quark propagator is then,
\[ S^\nu(\mu)^{-1} = i\sigma + m_f, \tag{15} \]

and the renormalization factor reads
\[ Z_2 = \frac{1}{\Sigma_1(\mu^2) + 1}, \quad Z_2 Z_4 m_f = m_f - \frac{\Sigma_2(\mu^2)}{\Sigma_1(\mu^2) + 1}. \tag{16} \]

### III. SOLVING THE COUPLED DSES

#### A. Formulation of the quark-gluon vertex

The Lorentz structure of the quark-gluon vertex in Eq. (9), \( \Gamma_\mu(p,q) \), should be expressed as a combination of all the 12 independent Lorenz structures. For convenience, we choose the tensor structures which have been used before \[13, 57, 58\] as:
\[ \Gamma_\mu(q,p) = \Gamma^{L}_\mu(q,p) + \Gamma^{T}_\mu(q,p), \tag{17} \]

with
\[ \Gamma^{L}_\mu(q,p) = \sum_{i=1}^{4} a_i(q,p) L^i_{\mu}(p + q, p - q), \tag{18} \]
\[ \Gamma^{T}_\mu(q,p) = \sum_{i=1}^{8} b_i(q,p) T^i_{\mu}(p + q, p - q). \tag{19} \]

where the matrix-valued tensors \( L^i_{\mu}(q,p) \) and \( T^i_{\mu}(q,p) \) are given, respectively, as Eq. (41) and Eq. (42) in Appendix.

This is the standard basis that includes the Ball-Chiu ansatz, which satisfies the WTI, and the 8 transversal components hold relation \( k_i \Gamma^{T}_{\mu}(q,p) = 0 \). However, this basis is not completely orthogonal, which brings in the difficulty for numerical calculation. Therefore, we also implement here another set of basis which are given in Eq. (13) and Eq. (14) in the Appendix. Therefore, the quark-gluon interaction vertex can be expressed as
\[ \Gamma_\mu(q,p) = \Gamma^{L}_\mu(q,p) + \Gamma^{T}_\mu(q,p), \tag{20} \]

with
\[ \Gamma^{L}_{\mu}(q,p) = \sum_{i=1}^{4} a_i'(q,p) L^i_{\mu}(p + q, p - q), \tag{21} \]
\[ \Gamma^{T}_{\mu}(q,p) = \sum_{i=1}^{8} b_i'(q,p) T^i_{\mu}(p + q, p - q). \tag{22} \]

The coefficients of the orthogonalized basis can be easily obtained via the following projection:
\[ a'_i(q,p) = \frac{L^i_{\mu}(p + q, p - q) \cdot \Gamma_\mu(q,p)}{L^i_{\mu}(p + q, p - q) \cdot L^i_{\mu}(p + q, p - q)} \tag{23} \]
\[ b'_i(q,p) = \frac{T^i_{\mu}(p + q, p - q) \cdot \Gamma_\mu(q,p)}{T^i_{\mu}(p + q, p - q) \cdot T^i_{\mu}(p + q, p - q)}. \tag{24} \]

After solving the equations, we can convert these coefficients into those with the standard basis for comparison with other results. The relation of the two sets can be given explicitly as
\[ a_1 = a'_1 - \frac{(t \cdot k)^2}{2k^2} a'_2, \quad a_2 = a'_2, \tag{25} \]
\[ a_3 = a'_3, \quad a_4 = a'_4. \tag{26} \]
where $m$ and $t$ with from the ab initio computations \cite{14–21, 65–69} as the gluon vertex. In this work, we use directly a set of results of gluon propagator, the running coupling and the three-momentum, consequently the gluon obtains a mass scale at zero momentum, and goes to the asymptotic form in ultraviolet (UV) region.

$$\begin{align*}
  b_1 &= \frac{2}{k^2}(a'_3 - b'_3), \\
  b_2 &= \frac{1}{k^2}(-a'_3 + 6b'_3), \\
  b_3 &= \frac{1}{k^2} \left[ -a'_1 + \left( \frac{t \cdot k}{k^2} \right)^2 a'_2 + b'_1 - \left( t^2 + \frac{2(t \cdot t)}{k^2} \right) b'_2 + \left( \frac{t \cdot k}{k^2} \right)^2 b'_3 \right], \\
  b_4 &= \frac{4}{k^2} \left[ -\frac{1}{t \cdot k} a'_3 + 2b'_6 + 2b_7 - b'_6 \right], \\
  b_5 &= -2 \left( \frac{t \cdot t}{k^2} \right) b'_6 + 2 \left( t^2 - \left( \frac{t \cdot t}{k^2} \right) \right) b'_7, \\
  b_6 &= t \cdot k \left( \frac{3}{k^2} b'_2 - b'_3 \right), \\
  b_7 &= -\frac{2}{t \cdot k} a'_4 + 4b'_6, \\
  b_k &= b'_4,
\end{align*}$$

with $t = p + q$, $k = p - q$.

### B. Necessary Input

To solve the coupled DSEs, we also need the input of gluon propagator, the running coupling and the three-gluon vertex. In this work, we use directly a set of results from the ab initio computations \cite{14–21, 65–69} as the input, without adding any other parameters.

#### 1. The gluon propagator

Recently, the infrared behavior of the gluon propagator has been investigated a lot, through lattice QCD simulations \cite{63 67} and DSE approach \cite{14–17}. All the computations have reached a common idea that a mass scale is dynamically generated in the infrared region of the gluon propagator. In this article, we choose the numerical result and the fitted formula given in Ref. \cite{14–17}:

$$G_{\mu \nu}(k) = \left( \delta_{\mu \nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right) \Delta(k^2),$$

where

$$\Delta^{-1}(k^2) = m^2(k^2) + k^2 \left[ 1 + \frac{13C_A g^2}{93\pi^2} \ln \left( \frac{k^2 + \mu^2}{k^2} \right) \right],$$

the $m^2(k^2) = \frac{m^4}{k^2 + \mu^2}$ is the dynamical mass of the gluon, $\mu = 4.3$ GeV is the renormalization point which is generally taken in lattice QCD simulations for gluon. The fitted parameters are

$$m = 500 \text{ MeV}, \quad g^2 = 5.68, \quad \rho_1 = 60, \quad \rho_2 = 1.6.$$  (26)

Consequently, the gluon obtains a mass scale at zero momentum, $m(0) = 395$ MeV, and goes to the asymptotic form in ultraviolet (UV) region.

#### 2. The running coupling

It has been shown through the computation of both functional renormalization group approach \cite{68} and DSE approach \cite{18, 69} that the running coupling $\alpha(k^2) = g^2(k^2)/4\pi$ is a constant in the infrared (IF) region, and can be expressed explicitly as \cite{18}:

$$\alpha(k^2) = \frac{1 - d_0 k^2 \ln \frac{k^2}{\Lambda^2} + d_1 k^2 + d_2 k^4}{1 + b_1 k^2 + c_0 d_2 k^4 \ln \frac{k^2}{\Lambda^2}},$$

where

$$d_0 = \frac{14.4}{8\pi} \text{ GeV}^{-2}, \quad d_1 = 3.56 \text{ GeV}^{-2}, \quad d_2 = 2.85 \text{ GeV}^{-4}, \quad b_1 = 7.25 \text{ GeV}^{-2}, \quad c_0 = \frac{11 - \frac{2N_f}{3}}{4\pi}.$$  (28)

After considering the running coupling’s behavior in UV region \cite{69}, one can set the $\Lambda_T$ as $\Lambda_T = 130$ MeV.

#### 3. The three-gluon vertex

There have been a lot of works to investigate the behavior of the three-gluon vertex through different methods \cite{10–21}, and shown that it involves a “zero-crossing” in the IF region. Therefore, when solving the DSEs of the quark propagator and the quark-gluon interaction vertex, we choose the following form as the input of the three-gluon vertex

$$\Gamma_{\nu \sigma \tau}^{3g}(p - k; k - q) = G(l^2) \left( (q + k - 2p)_\nu \delta_{\sigma \tau} \right) + (p + q - 2k)_\nu \delta_{\sigma \tau} + (k - p - 2q)_\nu \delta_{\sigma \tau},$$

where

$$G(l^2) = \begin{cases} 0.865 \log_{10}(l^2/2 + 1) \text{ for } l^2 < 55.47 \text{ GeV}^2, \\
1 \text{ for } l^2 > 55.47 \text{ GeV}^2, 
\end{cases}$$

in which $l^2 = (p - q)^2 + (k - p)^2 + (k - q)^2$. This expression guarantees that $G(l^2) = 0$ when $l^2 = 0$. As $l^2$ increases, $G(l^2)$ becomes larger and goes to 1 when $l^2 > 3 \times 4.3^2 \text{ GeV}^2$.

### C. Approximation on the dependence of the angle

Previous works have shown that the coefficients of the Lorentz structures of the quark-gluon vertex do not have considerable dependence on the angle between the two momentums \cite{54 57}. We then intend to employ the assumption:

$$\begin{align*}
  a_i(q^2, p^2) &= a_i(q^2, p^2, \theta), \\
  b_j(q^2, p^2) &= b_j(q^2, p^2, \theta),
\end{align*}$$

(30)
with \( \bar{\theta} \) being a fixed value for the angle.

Nevertheless, it has also been shown numerically that there exists few difference among the quark-gluon vertex’s longitudinal behaviors in different angles [22]. To be more careful, we check it by analyzing the solutions of the coupled DSEs. The obtained results of the dynamical masses in case of \( \bar{\theta} = 0.25\pi, 0.49\pi \) and \( 0.8\pi \) are shown in Fig. [3]. It can be found easily that, for light quark, there exists observable difference between the dynamical masses from the coupled DSEs with different angles, and the difference diminishes gradually as the current quark mass increases. Then, considering the monotonous behavior of the dynamical mass \( M(p^2) \) with respect to the angle, one can still take the assumption in Eq.(30) with choosing the central value at \( \bar{\theta} = 0.49\pi \) even though the effect of the angle in case of light quark can not be completely ignored.

IV. SOLUTIONS OF THE COUPLED DSES

In this part, we discuss the numerical results of the quark propagator and the quark-gluon interaction vertex. Firstly, we search the quark propagator’s multiple solutions to build the complete picture of the DCSB, and analyze the current quark mass dependence to show the DCSB effect in case of that the explicit chiral symmetry breaking (ECSB) has already there. Secondly, we calculate the quark-gluon vertex and analyze the results in some different cases.

A. Quark Propagator

It is generally believed that there are multiple solutions for the quark propagator [9, 10, 26, 29, 70–73], at least, the Nambu solutions and the Wigner solutions. After managing the initial conditions carefully, we obtain the multiple solutions of the quark propagator in chiral limit and also in case of a small current quark mass. The obtained results are shown in Fig. [4].

![Fig. 3. Calculated dynamical masses \( M(p^2) \) in case of \( \theta = 0.25\pi, 0.49\pi \) and \( 0.8\pi \). Upper panel: the result for \( m_f = 0.0 \) GeV. Lower panel: the result for \( m_f = 1.27 \) GeV.](image1)

![Fig. 4. Calculated multiple solutions of the quark propagator in case of chiral limit \( m_f = 0.0 \) MeV (upper panel), and those of a small current mass \( m_f = 3.4 \) MeV (lower panel).](image2)

Fig. [3] shows that, in chiral limit, the dynamical mass is generated and demonstrated within Nambu solution and its negative counterpart. Meanwhile, there is also a meta-stable Wigner solution with vanishing mass function, which is the chiral symmetry preserving state. It indicates apparently that there exists DCSB. We then compute the quark condensate from the Nambu+ solution in the chiral limit, and obtain \(-\langle \bar{q}q \rangle^{1/3}(m_f = 0) = 290 \) MeV. This is definitely consistent with the other calculations (e.g., given in Ref.[53]). It has also been shown that, if the current quark mass is not large...
enough, the two kind solutions for the quark propagator still exist [3, 10, 29]. The difference from that in the chiral limit is mainly that the Winger solution in IF region becomes non-zero, which shows the ECSB effect brought by the finite current quark mass. And the DCSB still exists and exhibited also by the Nambu solution. We have, in turn, $-\langle \bar{q}q \rangle^{1/3} (m_f = 3.4 \text{ MeV}) = 290 \text{ MeV}$. This is evidently consistent with that constrained by the GOR relation [74]. Noticing that here we take directly the gluon propagator and the coupling strength from the ab initio computation and do not include any other parameters, the consistency indicates that our presently obtained quark-gluon interaction vertex with only the three-gluon interaction being involved would be sufficient to describe the phenomena of QCD.

To show the properties of the quark propagator more thoroughly, we also computed the solutions for different current quark masses up to that of bottom quark. The obtained results of the scalar functions $A(p^2)$ and $B(p^2)$ in the quark propagators at several values of the current quark mass are displayed in Fig. 5. It is apparent that, in the quark propagators at several values of the current quark mass, the consistency indicates that our presently obtained quark-gluon interaction vertex with only the three-gluon interaction being involved would be sufficient to describe the phenomena of QCD.

![Graph](Image)

FIG. 5. Calculated scalar functions $A(p^2)$ and $B(p^2)$ in the quark propagators at several values of the current quark mass.

$\frac{M(0)}{m_q}$. This indicates that the effect of the DCSB becomes smaller as the current mass increases, and the ECSB effect related to the current quark mass becomes dominant as the current mass gets as large as that of bottom quark and further.

B. Quark-Gluon Interaction Vertex

1. 3D distribution of the coefficients in chiral limit

The obtained momentum dependence of the coefficients of the longitudinal Lorentz structures, $a_1$, $a_2$ and $a_3$ are illustrated in Fig. 6 and Fig. 7. It is apparent that, at arbitrary momentums $q^2$ and $p^2$, the $a_1$ takes always non-negative value, but the $a_2$ and $a_3$ get definitely non-positive values. All the amplitudes (absolute values) of the coefficients decrease monotonously as the momentums increase, and tend to 0 in the UV region.

![Graph](Image)

FIG. 6. Calculated momentum dependence of the coefficient $a_1$ (with the scale parameter $v = 1.0 \text{ GeV}$).

The obtained momentum dependence of the coefficient $a_4$ is shown in Fig. 8. It is evident that, even though the $a_4$ oscillates as the momentums vary, the absolute value of the $a_4$ is quite small (lower than 0.05). This fact accords with the constraint both from the WTI (as shown in Eq. (11)) and the STI in Eq. (2). The momentum dependence of the transverse structure of the quark-gluon interaction vertex has not yet been discussed in detail in the past decades, even though some works have shown the importance of the transversal part [11, 12, 57, 58]. In the following, we analyze the momentum dependence of the coefficients of the $8$ transverse Lorentz structures.

The calculated results of the momentum dependence of the coefficients $b_3$, $b_5$, $b_7$ and $b_8$ are displayed in Fig. 9. From the Fig. 9, one can notice easily that, at any momentum $q^2$ and $p^2$, the $b_3$ takes positive definite value, but the $b_5$, $b_7$ and $b_8$ take negative definite values. Their variation behaviors with respect to the momentums are quite similar: there is a flat-form in the IF region, and the amplitudes (the absolute values) reduce to approach...
The scale parameter is also taken as $v = 1.0 \text{ GeV}$. However, the value of the amplitude is quantitatively different, specifically, $b_5(0,0) = 0.48$, $b_7(0,0) = -1.22$, $b_7(0,0) = -0.4$, $b_8(0,0) = -2.63$. This result matches qualitatively with the CLR vertex [57]:

\begin{align}
    b_5(q^2, p^2) &= \frac{\eta \Delta_B(q^2, p^2)}{2a_3(q^2, p^2)}, \\
    b_8(q^2, p^2) &= \frac{\eta \Delta_B(q^2, p^2)}{2a_3(q^2, p^2)},
\end{align}

where $\eta = 0.65$ and $M(x, y) = [x + y + M(x^2 + y^2)]^2 + 2$. The scale parameter is also set at $v = 1.0 \text{ GeV}$.

$M(q^2)^2/2[M(x) + M(y)]$, and leads then to the similar behavior of the quark’s anomalous magnetic moment as shown in next section.

The obtained results of the momentum dependence of the coefficients $b_1$, $b_2$, $b_3$, and $b_6$ are illustrated in Fig. 10. One can observe from Fig. 10 that, similar to the behaviors of $b_1$, $b_2$, $b_3$, and $b_6$, at arbitrary momentum $q^2$ and $p^2$, the $(p^2 - q^2)b_4$ holds positive definite value, but the $(p^2 - q^2)b_4$ and $(p^2 - q^2)b_4$ have negative definite values. They also have their own similar behaviors with respect to the momentums: there is a flat-form in the
IF region for each of the \((p^2 - q^2)b_2, (p^2 - q^2)b_4\) and \((p^2 - q^2)b_8\), and their amplitudes (the absolute values) fall quickly to 0 as the momentums get larger. The infrared finite behaviours also reveal that these three coefficients tend to \(1/(p^2 - q^2)\) when \(p^2 - q^2\) goes to zero. As for the coefficient \(b_6\), its behavior is qualitatively the same as that of the coefficient \(a_4\), except for that its magnitude is larger than that of \(a_4\).

2. Current quark mass dependence of the vertex

In this part, we discuss the current quark mass dependence of the 12 coefficients with respect to the momentum. For easy access, we here just show the single momentum dependence \(a_i(0, p^2)\) \((i = 1, 2, 3, 4)\) and \(b_j(0, p^2)\) \((j = 1, 2, \cdots, 8)\) in case of several values of the current quark mass. While the 3-dimensional variation behavior of the quark-gluon vertex with non-vanishing current quark mass \(m_f\) delivers certainly the similar behavior as that in case of the chiral limit.

Calculated momentum dependence of the longitudinal structures’ coefficients is illustrated in Fig. 11. It is worth noticing that all the amplitudes (the absolute values) of the four coefficients are monotonously suppressed as the current quark mass increases. As for the momentum dependence, the \(a_i(0, p^2)\) \((i = 1, 2, 3)\) vary with respect to \(p^2\) monotonously, which is different from that given in BC vertex (in which \(a^2_{BC} = A(p^2)\), while the \(A(p^2)\) from the BC vertex is not monotonous), but consistent with the results from the lattice QCD simulations \[22\], qualitatively. The reason might be that the WTI is not a correct constraint to the vertex any more, after taking the non-Abelian nature of the interaction into consideration. However, it is exciting to see that the curves are in accordance with the STI results \[22\] qualitatively. Another remarkable feature is that, in all the cases of the current quark masses, the contribution of the \(L^4\) to the full vertex is quite small.

The calculated momentum dependence of the transverse structures’ coefficients in case of several values of the current quark mass are shown in Fig. 12 and Fig. 13. The figures clearly manifest that the \(b_1, b_2\) and \(b_4\) fall quite rapidly to 0 with the increasing of both the momentum and the current mass. The \(b_3, b_5\) and \(b_6\) especially the \(b_5\) and \(b_6\) for the \(u, d\) and \(s\)-quarks, hold considerable values in the IF region. While the value of \(b_8\) keeps small, and the maximum of the peak appearing at mediate-momentum region is only 0.65. It indicates that the magnitudes of the coefficients of the \(T^3, T^2\),...
can be written explicitly as: the fitted expressions \( i \) with

\[ T_4 \quad \text{and} \quad T_6 \] terms are small and can usually be neglected. However, the coefficients of the \( T_4 \) and \( T_6 \) terms which are included in the CLR vertex \([57]\), are large enough in the infrared area, especially for the \( u, d, \) and \( s \)-quark. This provides another positive evidence for the CLR model of the quark-gluon interaction vertex. Meanwhile, the values of the coefficients of the \( T_4 \) and \( T_6 \) terms cannot be ignored for light quark, either.

It is worth noticing that in Figs. 12 and 13 generally, with the increasing of the current quark mass, the contributions of the transverse parts to the full vertex get weakened drastically. Eventually, the transverse parts can be ignored for very heavy quark (e.g., \( b \)-quark) system \([48]\).

V. PARAMETERIZING THE QUARK-GLUON INTERACTION VERTEX

To make use of the presently determined quark-gluon interaction vertex elsewhere, it is necessary to parameterize the above results as analytical functions. We then model the vertex with functions including some parameters. After some tedious calculations, the coefficients of the 12 Lorentz structures can be fitted as:

\[
\begin{align*}
a_i(p, q) &= c_i(p^2, q^2)e^{-(p^2+q^2)/d_i(m)}, \\
b_j(p, q) &= b_j(p^2, q^2)e^{-(p^2+q^2)/d_j(m)},
\end{align*}
\]

with \( i = 1, 2, 3, 4 \), and \( j = 1, 2, \cdots, 8 \).

The current quark mass effect can also be modeled in analytic expression of the vertex. The fitted expressions can be written explicitly as:

\[
\begin{align*}
a_1(p^2, q^2) &= 1.0 + \left(0.058 + \frac{0.136}{m_f + 0.15}\right) \\
&\quad \times e^{-(p^2+q^2)/(379 - \frac{244}{m_f + 0.49})}, \\
a_2(p^2, q^2) &= -\left(0.027 + \frac{0.06}{m_f + 0.08}\right) \\
&\quad \times e^{-(p^2+q^2)/(5.84 - \frac{1.58}{m_f + 0.34})}, \\
a_3(p^2, q^2) &= \left(0.013 - \frac{0.095}{m_f + 0.133}\right) \\
&\quad \times e^{-(p^2+q^2)/(55.5 - \frac{264}{m_f + 0.83})}, \\
a_4(p^2, q^2) &= \left(-0.0056 + \frac{0.0034}{m_f + 0.026}\right) \\
&\quad (p^2 - q^2)e^{-(p^2+q^2)}, \\
b_1(p^2, q^2) &= -0.01 \frac{e^{-(p^2+q^2)}}{p^2 - q^2}, \\
b_2(p^2, q^2) &= 0.0005 + \frac{0.0058}{m_f + 0.34} e^{-(p^2+q^2)}, \\
b_3(p^2, q^2) &= \left(-0.029 + \frac{0.011}{m_f + 0.021}\right) \\
&\quad \times e^{-(p^2+q^2)/(1.33 + \frac{0.067}{m_f + 0.25})}, \\
b_4(p^2, q^2) &= 0.1 \frac{e^{-(p^2+q^2)}}{p^2 - q^2}, \\
b_5(p^2, q^2) &= \left(0.062 - \frac{0.52}{m_f + 0.44}\right) \\
&\quad \times e^{-(p^2+q^2)/(0.67 - \frac{3.48}{m_f + 0.77})}, \\
b_6(p^2, q^2) &= \left(0.073 - \frac{0.127}{m_f + 0.077}\right) \\
&\quad (p^2 - q^2)e^{-(p^2+q^2)}, \\
b_7(p^2, q^2) &= \left(0.036 - \frac{0.067}{m_f + 0.14}\right) \\
&\quad \times e^{-(p^2+q^2)/(1.62 - \frac{0.53}{m_f + 0.37})}, \\
b_8(p^2, q^2) &= \left(0.075 - \frac{0.43}{m_f + 0.16}\right) \\
&\quad \times e^{-(p^2+q^2)/(3.72 - \frac{0.23}{m_f + 0.75})}.
\end{align*}
\]

With the expression of the gluon and the running coupling described in Section III, a new truncation scheme of solving the DSE of the quark propagator can be built.

To check the efficiency of the presently built truncation scheme of the DSEs, we calculated the dynamical mass of the quark in beyond the chiral limit with the full parameterized expression of the vertex listed above, the parameterized vertex with simplification so as to include the Lorentz structures \( L^1, L^2, L^3, T^5 \) and \( T^8 \), and the further simplified model including only the \( L^1, L^5 \) and \( T^8 \) terms. The obtained results and the comparison with that coming from the full vertex directly from the coupled DSEs are displayed in Fig. 14. The figure manifests clearly that the dynamical mass obtained via the parameterized expression of the full vertex is very close to that directly from the vertex given by the coupled DSEs. We also find that the contributions from the Lorentz structure \( L^1, L^2, L^3, T^5 \) and \( T^8 \) (just the same as in the CLR vertex \([57]\)) are dominant for obtaining the same DCSB effect in quark propagator.

VI. APPLICATION EXAMPLES OF THE NEW TRUNCATION METHOD

A. Quark-photon vertex

The quark-photon vertex can be obtained by solving the Bethe-Salpeter Equation (BSE) with the vector channel interaction. Once the quark-gluon vertex and the quark-quark scattering kernel are available, the BSEs can
be solved. As described in Refs. [76, 77], the BS kernel can be obtained through the variation of the self energy:

\[ K(p, q; 0) = -\frac{\delta \Sigma(p)}{\delta S(q)}. \]  

(35)

Since we now have both the DSEs for the quark propagator and the quark-gluon vertex, this variation can be done directly. However, for simplicity, we will here make use of directly the parameterized vertex above, and thus take into account only the derivative with respect to the quark propagator and neglect the derivatives against the vertex, the gluon propagator and the running coupling. The BSE of the quark-photon system can then be simplified as shown in Fig. 15. In general, such a quark-photon interaction vertex can still be written as

\[ \Gamma_\mu^\gamma(q, p) = \Gamma_\mu^\gamma(L)(q, p) + \Gamma_\mu^\gamma(T)(q, p). \]

(36)

We compute the solution of the gap equation with the parameterized form of the presently obtained quark-gluon interaction vertex, the gluon propagator and the running coupling described in Section III being substituted into, and the same ansatz for the BSE as implemented in the gap equation. With the constraint from the WTI being taken into account, the longitudinal part of the quark-photon vertex takes the similar form as the Ball-Chiu ansatz, i.e.,

\[ \Gamma_\mu^\gamma(L)(q, p) = \sum_{i=1}^{4} c_i(q^2, p^2) L_i(p + q, p - q) \]

\[ = \Sigma_A(q^2, p^2) L_1(p + q, p - q) + \delta_A(q^2, p^2) L_2(p + q, p - q) \]

\[ + \delta_B(q^2, p^2) L_3(p + q, p - q), \]

(37)

It means that \( c_1 = \Sigma_A \), \( c_2 = \delta_A \), \( c_3 = \delta_B \), and \( c_4 = 0 \). For the transverse part, it can be generally written as

\[ \Gamma_\mu^\gamma(T)(q, p) = \sum_{i=1}^{8} d_i(q^2, p^2) T_i(p + q, p - q). \]

(38)

The obtained results of the coefficients \( c_i(q^2, p^2) \) (\( i = 1, 2, 3 \)) and \( d_i(q^2, p^2) \) (\( i = 1, 2, \cdots, 8 \)) at the symmetric momentum \( (p^2 = q^2) \) are shown in Fig. 16 and Fig. 17.

FIG. 15. The BSE of the quark-photon vertex, the bolded vertex and all the propagators are the fully dressed ones.

FIG. 16. Calculated momentum dependence of the longitudinal structure’s coefficients of the quark-photon vertex. The ones marked with \( \Sigma_A \), \( \delta_A \) and \( \delta_B \) are the results from the presently obtained quark-gluon interaction vertex, while the \( \Sigma_A' \), \( \delta_A' \) and \( \delta_B' \) are those from the RL approximation.
\( \delta_A \) is close to 0 at low momentum, while the \( \delta_A \) given with the present truncation scheme holds negative values with quite large amplitude in the IF region and increases monotonously to 0 as the momentum increases. Such a behaviour of \( \delta_A \) is actually consistent with the results given in lattice QCD simulations qualitatively.

Numerical data indicate that the momentum dependence of the coefficients of the Lorentz Structure \( T^1, T^2 \) and \( T^4 \) are similar to that of the corresponding term of the quark-gluon interaction vertex, i.e., they involve \( 1/(p^2 - q^2) \) divergence as \( p^2 - q^2 \) approaches to 0. Then in Fig. 17 we show the variation behaviors of the coefficients \( d_1, d_2 \) and \( d_4 \) with a factor \( (p^2 - q^2) \) being multiplied. After comparing the amplitudes and variation behaviors of the coefficients \( d_i \) \( (i = 1, 2, \cdots, 8) \) displayed in Fig. 17 one can notice that the contribution of the Lorentz Structure \( T^1, T^3, T^5 \) and \( T^8 \) are dominant to the quark-photon vertex which is also similar to the quark-gluon interaction vertex. These quite large contributions of the transverse structure of the quark-photon vertex will lead to observable signals.

\[ \kappa(\zeta) = \frac{2\varepsilon F_2 + 2\varepsilon^2 F_4 + A_k(\zeta)}{a_1 + F_1 - A_k(\zeta)} , \]  

where \( A_k(\zeta) = 2\varepsilon^2 a_2 - 2\varepsilon a_3 - \varepsilon F_5 - \varepsilon^2 F_7 \) with the \( a_i \) and \( F_i \) being evaluated at \( q^2 = p^2 = M(p^2) = \zeta^2 \). In which the functions \( a_1, a_2 \) and \( a_3 \) are just correspondingly the coefficients of the \( L^1, L^2 \) and \( L^3 \) defined in Eq. 37, namely the longitudinal structure of the vector vertex. The \( F_i \) \( (i = 1, 2, \cdots, 8) \) are the coefficients of the tensors redefined as

\[ \Gamma_{\mu T}(q, p) = \gamma_\mu^T F_1 + \sigma_{\mu \rho} k_\rho F_2 + T_{\mu \rho} \sigma_{\rho \nu} l_\nu k \cdot F_3 \]

\[ + \left[ i \gamma^\mu \cdot k + i \gamma^\rho \sigma_{\rho \nu} l_\nu \right] F_4 - i \gamma^\mu F_5 \]

\[ + \left[ i \gamma^\mu \cdot k + k \cdot F_6 - i \gamma^\mu \cdot F_7 \right] \]

\[ + i \gamma^\mu \sigma_{\nu \rho} l_\nu k_\rho F_8 \]

where \( \sigma_{\mu \nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\mu \gamma_\nu) \); \( l \) and \( k \) are respectively the \( 2l = p + q \) and \( k = q - p \). The conversion from the coefficients \( a_i \) \( (i = 1, 2, 3, 4) \) and \( b_i \) \( (j = 1, 2, \cdots, 8) \) of the standard tensors given in Section III to the coefficients \( F_i \) here is the same as that described in Section III. The obtained quark’s anomalous chromo- and electro-magnetic moment from the the presently proposed truncation scheme and the comparison with that given by the CLR vertex is illustrated in Fig. 18. It can be seen, typically, that the larger anomalous chromo-magnetic moment leads to the larger anomalous electromagnetic moment. This is because both types of the anomalous magnetic moments are related to the DCSB effect embodied in the vertex. Looking through the figure more cautiously, one can observe that the presently

![Fig. 17. Calculated momentum dependence of the transverse structures’s coefficients of the quark-photon vertex.](image)

![Fig. 18. Calculated momentum dependence of the anomalous chromo- and electro-magnetic moment of a dressed-quark.](image)
obtained vertex leads to larger anomalous magnetic moments than the CLR vertex, while the RL approximation gives only nearly zero anomalous magnetic moment. These results reveal that our presently obtained quark-gluon interaction vertex carries more DCSB effect than the CLR model and the RL approximation, even though the DCSB effect demonstrated by the quark propagator, i.e., quark condensates, are similar.

VII. SUMMARY

We construct a scheme that couples the DSEs of the quark propagator and the quark-gluon interaction vertex in this paper. In this scheme, the physical coupling strength and gluon propagator from ab initio computation can be implemented directly, which then builds a connection from the QCD Lagrangian to the phenomenology numerically. We study the quark-gluon interaction vertex which includes all the 12 Lorentz structures, and give the variation behaviors of the full vertex with respect to the momentums (including the angle between them) and the current quark mass. Our direct computation gives consistent result with that from phenomenological ansatz and the lattice QCD simulations. We also give a parameterized expression for the obtained quark-gluon vertex to the convenience of using it elsewhere. We show furthermore the practicality of this new scheme by combining it with the BSEs and producing reasonable quark-photon interaction vertex and anomalous chromo- and electro-magnetic moments of quark. This may shed light on building the realistic scheme beyond RL approximation to study hadron properties. The work is under progress.

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APPENDIX

We list here the standard tensor structures of the full quark-gluon interaction vertex. Those of the longitudinal part are given in Eqs. (42) and the transverse ones are listed in Eqs. (43).

\[
\begin{align*}
T^1_\mu(t,k) &= \frac{i}{2}(k^2 t_\mu - t \cdot k k_\mu), \\
T^2_\mu(t,k) &= \frac{1}{2}(k^2 t_\mu - t \cdot k k_\mu), \\
T^3_\mu(t,k) &= k^2 \gamma_\mu - k_\mu \gamma_5, \\
T^4_\mu(t,k) &= \frac{i}{8}(k^2 t_\mu - t \cdot k k_\mu)(f \cdot k - f k), \\
T^5_\mu(t,k) &= \frac{i}{2}(\gamma_\mu - \gamma_5 f k), \\
T^6_\mu(t,k) &= t \cdot k \gamma_\mu - f k_\mu, \\
T^7_\mu(t,k) &= \frac{i}{2}(\gamma_\mu - \gamma_5 f k) + \frac{i}{4} t_\mu(f \cdot k - f k), \\
T^8_\mu(t,k) &= \frac{1}{2}(k t_\mu - f k_\mu) + \frac{\gamma_\mu}{4}(f \cdot k - f k), \\
\end{align*}
\]

where \(I_D\) is the \(4 \times 4\) identity matrix in Dirac space.

The tensor set whose constituents are orthogonal with each other are listed in Eqs. (43) and (44), where the longitudinal part reads

\[
\begin{align*}
L^1_\mu(t,k) &= \frac{k k_\mu}{k^2}, \\
L^2_\mu(t,k) &= \frac{t \cdot k f - (t \cdot k)^2 \gamma_5}{2 k^2} k_\mu, \\
L^3_\mu(t,k) &= -i \frac{t \cdot k k_\mu}{k^2}, \\
L^4_\mu(t,k) &= \frac{(f \cdot k - f k) t_\mu}{2 k^2}. \\
\end{align*}
\]

The transverse part is

\[
\begin{align*}
T^1_\mu(t,k) &= \gamma_\mu \frac{k_\mu}{k^2}, \\
T^2_\mu(t,k) &= 3\left(\left(t_\mu - \frac{t \cdot k k_\mu}{k^2}\right)(f - \frac{k t \cdot k}{k^2}) - \left(\gamma_\mu - \frac{t \cdot k t_\mu}{k^2}\right)(t - \frac{t \cdot k}{k^2})\right) + \frac{5}{2}, \\
T^3_\mu(t,k) &= t \cdot (t_\mu + \frac{t \cdot k k_\mu}{k^2}) k, \\
T^4_\mu(t,k) &= \frac{1}{2}(k t_\mu - f k_\mu) + \frac{\gamma_\mu}{4}(f k - f k), \\
T^5_\mu(t,k) &= -i(t_\mu - \frac{t \cdot k}{k^2}), \\
T^6_\mu(t,k) &= -i t \cdot k \left(\gamma_\mu - \frac{k k_\mu}{k^2}\right)(f - \frac{k t \cdot k}{k^2}) - (f - \gamma_\mu)(\gamma_\mu - \frac{k k_\mu}{k^2}), \\
T^7_\mu(t,k) &= -i t^2 \left(\frac{t \cdot k}{k^2}\right)^2(\gamma_\mu \cdot k - \gamma_5 f k) + 2i(t_\mu - \frac{t \cdot k k_\mu}{k^2})(f k - t \cdot k), \\
T^8_\mu(t,k) &= -i(t_\mu - \frac{t \cdot k}{k^2})(f k - t \cdot k).
\end{align*}
\]

Such an expression of the transverse part of the orthogonalized tensors are from the orthogonalized vector basis.
of the vector meson [44, 46]. After changing the form of the $L_{ij}^i$, the 12 independent orthogonalized tensors can be built.
