Abstract

A simple method for maintaining balance in ternary search tries is presented. The new kind of self-balancing ternary search trie, called an $r$-trie, generalizes the balancing strategy of Aragon and Seidel’s randomized binary search trees (treaps). This means that the shape of an $r$-trie for a string set $S$ is a random variable with the same probability distribution as a ternary search trie built by inserting the strings of $S$ in random order. It also means that searching, inserting, or deleting a string of length $k$ in an $r$-trie for $n$ strings takes at most $O(k + \log n)$ time with high probability, no matter from which sequence of insertions and deletions of strings the $r$-trie results.

1 Introduction

In computer science, tries are an important data structure for storing character strings. If viewed as an abstract structure, the trie for a set $S$ of strings can be defined to be the smallest ordered tree such that each node except the root is labeled with a character and each string in $S$ is spelled out by the characters on a path from the root to a node. Each node in the trie corresponds to a prefix of the strings in $S$ (namely the prefix spelled out by the path from the root to the node) and vice versa. A node that corresponds to a string in $S$ is called a terminal node. Any data associated with a string in $S$ is stored at the terminal node of the string.

Because of their special structure, tries can be used for a wide variety of search problems, from simple membership queries to more complex problems such as finding all strings that approximately match a given string. However, since a trie node may have as many children as there are characters in the underlying alphabet, the time- and space-efficient implementation of a trie can be a challenge. A solution to this problem is to organize the children of a trie node as a binary search tree. The resulting data structure, known as a ternary search trie or ternary search tree, needs only three pointers per node: one to the left child and one to the right child as in a binary search tree, and one to the middle child which is the root of the next binary search tree. As shown in the figure below, a ternary search trie can be viewed as a cross between a trie and a binary search tree.

![Figure 1](image-url)
A search for a string in a ternary search trie compares the current character in the search string with the character at the current node: if the character in the search string is less, the search goes to the left child; if the character in the search string is greater, the search goes to the right child; and if the two characters are equal, the search goes to the middle child and proceeds to the next character in the search string. So, if a ternary search trie for \( n \) strings is balanced such that each step to a left or right child cuts the number of terminal nodes in the subtree at least in half, then searching for a given string of length \( k \) takes at most \( O(k + \log n) \) time.

The algorithm for inserting a string into a ternary search trie works analogously to the algorithm for inserting an element into a binary search tree: it searches for the string in the tree, attaching missing nodes at the end of the search path to the tree. Like a binary search tree, a ternary search trie can thus easily become unbalanced. If the strings are inserted in lexicographic order, each of the binary search trees within a ternary search trie degenerates into a linked list, significantly increasing the cost of searches. Fortunately, also like a binary search tree, a ternary search trie can be rebalanced via rotations. A rotation reverses the parent–child relationship between a node and its left or right child without violating the order property of the tree.

The literature contains various approaches to balancing a ternary search trie that attempt to keep the cost of searches, insertions, and deletions within a small constant factor of the optimum. Many if not all of these approaches generalize balancing concepts used in binary search trees. Mehlhorn [4], for example, described a weight-balanced ternary search trie. Vaishnavi [8] presented a height-balanced ternary search tree. And Sleator and Tarjan [7] proposed a self-adjusting ternary search trie in which, whenever we search for a string, we splay at each node that corresponds to a prefix of the string, rotating the node to the root of the binary search tree it belongs to.

The existing methods for balancing a ternary search trie may be difficult to implement in practice. Weight-balanced ternary search tries and height-balanced ternary search tries have relatively complicated insertion and deletion algorithms. And with self-adjusting ternary search tries, all search algorithms must include a splay operation. For this reason, this paper presents a new kind of self-balancing ternary search trie, the \( r \)-trie, which generalizes the balancing strategy of Aragon and Seidel’s randomized binary search trees (treaps) [6].

Section 2 defines \( r \)-tries, describes their specific insertion and deletion algorithms, and gives a pseudocode implementation of the algorithms. Section 3 analyzes the relationship between ternary search tries and binary search trees and derives a high-probability bound on both the shape of \( r \)-tries and the cost of their basic operations.
2 \hspace{1em} r\text{-Tries}

Let $r$ be a positive integer, and let $S$ be a set of strings. An $r$-trie for $S$ is a ternary search trie for $S$ with the following three additional properties:

1. Each string in $S$ has a priority – a random, uniformly distributed integer between 1 and $r$, with a higher number meaning a higher priority.
2. Each node has the priority of the highest-priority string that starts with the prefix corresponding to the node.
3. No node has a lower priority than its left or right child; that is, each of the binary search trees within the ternary search trie is a heap with respect to the node priorities.

If not too many strings have equal priority, the shape of an $r$-trie is uniquely determined by the strings and their priorities, and is that of a ternary search trie built by inserting the strings in order of decreasing priority. For large enough $r$, the shape of an $r$-trie is thus a random variable with the same probability distribution as a ternary search trie built by inserting the strings in random order.

An $r$-trie node has fields `char` (character), `prio` (priority of the node), `strPrio` (priority of the string corresponding to the node), and `left`, `mid`, `right` (pointers to children). A node may also have a pointer to its parent, depending on implementation. For a nonterminal node, `strPrio` is 0. Nonexistent nodes are represented by a sentinel node with priority 0, the so-called `nil` node.

The insertion of a string $s$ into an $r$-trie can be described as follows:

1. Insert $s$ with priority 0 using the usual insertion algorithm for ternary search tries.
2. Set the priority of $s$ to a random integer between 1 and $r$.
3. On the path from the terminal node of $s$ to the root, for each node $x$ that corresponds to a prefix of $s$:
   a) Set the priority of $x$ to $\max(x.strPrio, x.mid.prio)$.
   b) While $x$ is the left or right child of its parent and has a higher priority than its parent:
      Rotate $x$ with its parent.
The deletion of a string \( s \) from an \( r \)-trie can be achieved by "inverting" the insertion operation for each node that corresponds to a prefix of \( s \):

1. Search for \( s \) and set the priority of \( s \) to 0.
2. On the path from the terminal node of \( s \) to the root, for each node \( x \) that corresponds to a prefix of \( s \):
   a) Set the priority of \( x \) to \( \max(x.\text{strPrio}, x.\text{mid}.\text{prio}) \).
   b) While \( x \) has a lower priority than its left or right child:
      Rotate \( x \) with the higher-priority child.
   c) If \( x \) has priority 0, then unlink \( x \) from the tree (that is, replace \( x \) by \( \text{nil} \)).

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{trie_deletion}
\caption{Deletion of string "AL" from an \( r \)-trie.}
\end{figure}
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A recursive pseudocode implementation of the insertion and deletion algorithms for \( r \)-tries is shown below. To insert/delete string \( s \) into/from an \( r \)-trie with root \( x \), we call \( \text{insert}(s, 1, x) \) or \( \text{delete}(s, 1, x) \). The expression \( s[i] \) denotes the \( i \)-th character of \( s \).

\begin{verbatim}
\textbf{insert}(s, i, x):
  IF x = nil
    x ← newNode()
    x.char ← s[i]
    x.prio ← 0, \ x.\text{strPrio} ← 0
    x.left ← nil, \ x.mid ← nil, \ x.right ← nil
  IF s[i] < x.char
    x.left ← \text{insert}(s, i, x.left)
  IF x.left.prio > x.prio
    x ← rotateWithLeft(x)
  ELSE IF s[i] > x.char
    x.right ← \text{insert}(s, i, x.right)
  IF x.right.prio > x.prio
    x ← rotateWithRight(x)
  ELSE
    IF i < s.length
      x.mid ← \text{insert}(s, i + 1, x.mid)
    ELSE IF x.strPrio = 0
      x.strPrio ← \text{randomInteger}(1, r)
      x.prio ← \max(x.strPrio, x.mid.prio)
  RETURN x

\textbf{delete}(s, i, x):
  IF x ≠ nil
    IF s[i] < x.char
      x.left ← \text{delete}(s, i, x.left)
    ELSE IF s[i] > x.char
      x.right ← \text{delete}(s, i, x.right)
    ELSE
      IF i < s.length
        x.mid ← \text{delete}(s, i + 1, x.mid)
      ELSE
        x.strPrio ← 0
        x.prio ← \max(x.strPrio, x.mid.prio)
      x ← heapifyOrDelete(x)
    RETURN x

\textbf{heapifyOrDelete}(x):
  IF x.prio < x.left.prio OR x.prio < x.right.prio
    IF x.left.prio > x.right.prio
      x ← rotateWithLeft(x)
    ELSE
      x.right ← \text{heapifyOrDelete}(x.right)
  ELSE
    x ← rotateWithRight(x)
    x.left ← \text{heapifyOrDelete}(x.left)
  ELSE IF x.prio = 0
    x ← \text{nil}
  RETURN x
\end{verbatim}
Most of the above implementation is adapted from Bentley and Sedgewick’s ternary search trie implementation [1] and Aragon and Seidel’s treap implementation [6].

Function \texttt{insert}(s, i, x) inserts string \(s\) from its \(i\)-th character on into the subtree rooted at node \(x\), returning the root of the updated subtree. The function compares the \(i\)-th character of \(s\) with the character at \(x\) and then recursively calls itself for the left, right, or middle subtree of \(x\). If the function runs off the end of the tree, it creates a new node, initializes the node, and then falls through to the ELSE branch of the standard case. If in the ELSE branch \(s\) has no more characters and \(s\) is a new string in the tree, then the \texttt{strPrio} field of the terminal node of \(s\) is set to a random integer between 1 and \(r\). Finally, when the recursive calls return and the nodes are revisited in reverse order, the code for the node priority updates and the rotations is executed.

Function \texttt{delete}(s, i, x) works analogously. The subroutine \texttt{heapifyOrDelete}(x) recursively rotates node \(x\) to its proper position in the heap, and then replaces \(x\) by \texttt{nil} if \(x\) has priority 0.

3 \hspace{1cm} Analysis

Clément et al. [3] and Broutin and Devroye [2] thoroughly analyzed ternary search tries that are built by inserting random strings over a given alphabet. In many situations, their theoretical models should predict quite well the expected behavior of \(r\)-tries on real-world data. However, for the purposes of this paper, we will use a more general approach to analyzing \(r\)-tries that makes no assumptions about the distribution of the strings.

We begin with a lemma on the ancestor relation in binary search trees. It is adapted from an analogous lemma for treaps, given by Aragon and Seidel [6].

\textbf{Lemma 1.} In a binary search tree for a set of elements, the node containing element \(x\) is an ancestor of the node containing element \(y\) if and only if \(x\) was inserted into the tree before \(y\) and before all elements \(z\) with \(\min(x, y) < z < \max(x, y)\).

\textit{Proof.} The element at the root of the tree was inserted before all other elements. So, the lemma is obviously true if \(x\) or \(y\) is at the root, or if \(x\) is in the left subtree of the root and \(y\) is in the right subtree of the root or vice versa. And since the left and right subtrees of the root are binary search trees themselves, we get by recursion that the lemma is also true if \(x\) and \(y\) are both in the left subtree or both in the right subtree. ■

Now we can analyze the relationship between a ternary search trie and the corresponding binary search tree of strings.

\textbf{Lemma 2.} Let \(S\) be a set of strings, let \(\sigma\) be a permutation of \(S\), and let \(T_\sigma\) and \(B_\sigma\) be the ternary search trie and binary search tree resulting from inserting the strings of \(S\) in the order defined by \(\sigma\). Then for each string \(s\) in \(S\), the depth of the terminal node of \(s\) in \(T_\sigma\) is less than \(k + d\), where \(k\) is the length of \(s\) and \(d\) is the depth of the node containing \(s\) in \(B_\sigma\).

\textit{Proof.} Let \(P\) be the path from the root of \(T_\sigma\) to the terminal node of string \(s\), and let \(x_1, \ldots, x_m\) be the nodes on \(P\) whose left or right child belongs to \(P\) too (we assume there is at least one such node). Further, let \(i = 1, \ldots, m\) and let \(t_i\) be the string in \(S\) during whose insertion \(x_i\) was added to the tree. Then the strings \(t_1, \ldots, t_m\) and \(s\) are all distinct. By applying Lemma 1 to the binary search tree containing \(x_i\) we get that \(t_i\) was inserted into the ternary search trie before \(s\) and before all strings in \(S\) that are lexicographically between \(s\) and \(t_i\). Since \(B_\sigma\) results from the same insertion order, we obtain by applying Lemma 1 to \(B_\sigma\) that in \(B_\sigma\) the node containing \(t_i\) is an ancestor of the node containing \(s\). Consequently, the node containing \(s\) has at least \(m\) ancestors in \(B_\sigma\). ■
Reed [5] has shown that a binary search tree built by inserting \( n \) elements in random order has a height of \( O(\log n) \) with high probability. Combining this with Lemma 2, we get the following:

**Theorem 1.** For large enough \( r \), with high probability, the depth of each terminal node in an \( r \)-trie for \( n \) strings is at most \( O(k + \log n) \), where \( k \) is the length of the string corresponding to the node.

How about the cost of the basic operations on \( r \)-tries? A successful search for a string takes time proportional to the depth of the terminal node of the string. An unsuccessful search for a string terminates somewhere on the path from the root to the terminal node of the string’s lexicographic predecessor or successor. The insertion of a string begins with an unsuccessful search for the string and requires at most as many rotations as there are edges to left and right child nodes on the initial path from the root to the terminal node of the string. The deletion operation basically reverses an insertion operation and thus requires the same number of rotations (ignoring the case of ties between nodes with equal priority). So, putting all this together we obtain the following theorem:

**Theorem 2.** For large enough \( r \), with high probability, the time cost of searching, inserting, or deleting a string of length \( k \) in an \( r \)-trie for \( n \) strings is at most \( O(k + \log n) \).

**References**

[1] Bentley, J., Sedgewick, R. (1997), *Fast Algorithms for Sorting and Searching Strings*, Proceedings of the 8th ACM-SIAM Symposium on Discrete Algorithms, 360–369.

[2] Broutin, N., Devroye, L. (2007), *The Height of List-tries and TST*, Discrete Mathematics and Theoretical Computer Science Proceedings AH, 253–262.

[3] Clément, J., Flajolet, P., Vallée, B. (1998), *The Analysis of Hybrid Trie Structures*, Proceedings of the 9th ACM-SIAM Symposium on Discrete Algorithms, 531–539.

[4] Mehlhorn, K. (1979), *Dynamic Binary Search*, SIAM Journal on Computing 8 (2), 175–198.

[5] Reed, B. (2003), *The Height of a Random Binary Search Tree*, Journal of the ACM 50 (3), 306–332.

[6] Seidel, R., Aragon, C. (1996), *Randomized Search Trees*, Algorithmica 16 (4/5), 464–497.

[7] Sleator, D. D., Tarjan, R. E. (1985), *Self-Adjusting Binary Search Trees*, Journal of the ACM 32 (3), 652–686.

[8] Vaishnavi, V. K. (1984), *Multidimensional Height-Balanced Trees*, IEEE Transactions on Computers C-33 (4), 334–343.