Applications of BGP-reflection functors:
isomorphisms of cluster algebras *

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Abstract. Given a symmetrizable generalized Cartan matrix $A$, for any index $k$, one can define an automorphism associated with $A$, of the field $\mathbb{Q}(u_1, \ldots, u_n)$ of rational functions of $n$ independent indeterminates $u_1, \ldots, u_n$. It is an isomorphism between two cluster algebras associated to the matrix $A$ (see section 4 for precise meaning). When $A$ is of finite type, these isomorphisms behave nicely, they are compatible with the BGP-reflection functors of cluster categories defined in [Z1, Z2] if we identify the indecomposable objects in the categories with cluster variables of the corresponding cluster algebras, and they are also compatible with the "truncated simple reflections" defined in [FZ2, FZ3]. Using the construction of preprojective or preinjective modules of hereditary algebras by Dlab-Ringel [DR] and the Coxeter automorphisms (i.e., a product of these isomorphisms), we construct infinitely many cluster variables for cluster algebras of infinite type and all cluster variables for finite types.

Key words. Coxeter automorphisms of cluster algebras, BGP-reflection functors, cluster variables.

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1. Introduction

Clusters and cluster algebras are defined and studied by Fomin and Zelevinsky [FZ1-3, BFZ] in order to provide an algebraic framework for total positivity and canonical bases in semisimple algebraic groups. Since they appeared, there have been many interesting connections with other directions [FZ1-3] [BFZ] [CC], amongst them to representation theory of quivers and tilting theory [MRZ] [BMRRT] [BMR] [CC] [CK] [Z1, Z2].

The connections between representation theory of quivers and cluster algebras are firstly discovered by Marsh-Reineke-Zelevinsky through extending the well-known Gabriel’s Theorem; and then by Buan-Marsh-Reineke-Reiten-Todorov who introduced cluster categories, see also [CCS], and related tilting theory with clusters; and by some others. Since the cluster algebras of finite type are classified by Dynkin diagrams [FZ2], there should be some stronger

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links between cluster algebras with representation theory of quivers. In [Z1], see also [Z2], we introduced the BGP-reflection functors in cluster categories by extending the usual BGP-functors of module categories. By applying these functors to cluster algebras of finite type, we gave a one-to-one correspondence from indecomposable objects in cluster categories to the almost positive roots of the corresponding simple Lie algebras, and then to the set of cluster variables of corresponding cluster algebras. This correspondence sends basic tilting objects to clusters.

The aim of the paper is to understand clusters and cluster algebras of any type in terms of representations of quivers based on the works of [BMRRT], [Z1]. From Dlab-Ringel [DR], any preprojective or preinjective indecomposable module can be constructed as an image of projective resp. injective modules under some power of Coxeter functors. Applying to cluster categories, the indecomposable objects coming from preprojective or preinjective modules are also constructed from the objects $P_k[1]$ by acting some powers of Coxeter functors in cluster categories, where $P_k$ is an indecomposable projective module and the Coxeter functors in cluster categories are defined as a composition of BGP-reflection functors introduced in [Z1]. Passing from the set of indecomposable objects in cluster categories to the set of cluster variables, we should get a construction of some cluster variables from $u_i$ by some automorphisms of cluster algebras induced from Coxeter functors in cluster categories. So we should find some automorphisms of cluster algebras corresponding to the Coxeter functors in the corresponding cluster categories.

Let $A = (a_{ij})$ be a symmetrizable generalized Cartan matrix and $\mathcal{F} = Q(u_1, \ldots, u_n)$ the field of rational functions in variables $u_i$ $i = 1, \ldots, n$. For any $k$, we define $T_k$ to be the automorphism of $\mathcal{F}$ by setting $T_k(u_k) = u_k^{-1}(\prod_{a_{kj} < 0} u_j^{-a_{kj}} + 1)$, $T_k(u_j) = u_j$ $\forall j \neq k$. Let $u = (u_1, \ldots, u_n)$ and $B(A)$ a skew-symmetrizable matrix whose Cartan counterpart is $A$ and such that $k$ is a sink in the quiver corresponding to $B(A)$. Then $T_k$ is an isomorphism from the cluster algebra associated to the initial seed $(u, B(A))$ to the cluster algebra associated to the initial seed $(u, s_k B(A))$. When $A$ is of finite type, these automorphisms are compatible with the BGP-reflection functors in cluster categories when we identify the indecomposable objects with cluster variables. By using these isomorphisms, we give a construction of cluster variables from the initial cluster. We extend the construction to cluster algebras of infinite type.

This paper is organized as follows: in Section 2, some basic results on cluster categories which will be needed later on are recalled. In Section 3, we recall the BGP-reflection functors in cluster categories from [Z1, Z2], and extend the result of Dlab-Ringel [DR] to cluster categories, namely, we prove the indecomposable objects coming from preprojective or preinjective modules are some powers of the Coxeter functor on objects $P[1]$ where $P$ is an indecomposable projective module. In Section 4, for any symmetrizable integer matrix and any index $i$, we define an automorphism $T_i$ on $\mathcal{F}$ and the Coxeter automorphism $T$ as a product of these $T_i$, and prove that this Coxeter automorphism is a symmetry of the corresponding cluster algebras. This gives some nice consequences such
as that all cluster variables can be obtained from the initial cluster by some powers of $T$ when $A$ is of finite type. This construction is generalized to the infinite types.

2. Basics on cluster categories.

Let $H$ be a finite-dimensional hereditary algebra over a field $k$, with $n$ pairwise non-isomorphic simple modules. Then there are $n$ pairwise non-isomorphic indecomposable projective $H$-modules $P_1, \ldots, P_n$. We denote by $\mathcal{D} = D^b(H)$ the bounded derived category of $H$ with shift functor $[1]$. For any Krull-Schmidt category $\mathcal{E}$ [Ri], any object can be written as a direct sum of indecomposable objects and such decomposition is unique up to isomorphisms, we will denote by $\text{ind}\mathcal{E}$ the full subcategory of representatives of isomorphism classes of indecomposable objects in $\mathcal{E}$; depending on the context we shall also use the same notation to denote the set of isomorphism classes of indecomposable objects in $\mathcal{E}$.

The cluster category of type $H$ is introduced in [BMRRT], which is defined to be the factor category $\mathcal{D}/F$ of $D^b(H)$, where $F = \tau^{-1}[1]$ and $\tau$ is the Auslander-Reiten translation in $\mathcal{D}$. We simply denote the cluster category of type $H$ by $\mathcal{C}(H)$. This factor category $\mathcal{D}/F$ is a Krull-Schmidt triangulated category [K]. The canonical functor $\pi: \mathcal{D} \to \mathcal{D}/F : X \mapsto \tilde{X}$ is a covering functor of triangulated categories, i.e., it sends triangles to triangles [XZ]. The shift in $\mathcal{D}/F$ is induced by the shift in $\mathcal{D}$, and is also denoted by $[1]$. In both cases we write as usual $\text{Hom}(U, V[1]) = \text{Ext}^1(U, V)$. We then have

$$\text{Ext}^1_{\mathcal{D}/F}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \text{Ext}^1_{\mathcal{D}}(X, F^i Y),$$

where $X, Y$ are objects in $\mathcal{D}$ and $\tilde{X}, \tilde{Y}$ are the corresponding objects in $\mathcal{D}/F$.

**Proposition 2.1.** [BMRRT] Any indecomposable object in $\mathcal{C}(H)$ is of the form $\tilde{M}$ for some indecomposable $H$-module $M$ or $\tilde{P}_i[1]$ for an indecomposable projective module $P_i$, $1 \leq i \leq n$.

$H\text{-mod}$ can be embedded into $D^b(H)$ so that the image of a $H$-module is a stalk complex of degree zero. Passing to the cluster category $\mathcal{C}(H)$, $\text{obj}(\text{ind}H)$ can be viewed as a subset of $\text{obj}(\text{ind}\mathcal{C}(H)))$, we fix this inclusion in the rest of paper. Then $\text{ind}\mathcal{C}(H) = \text{ind}H \cup \{P_i[1] \mid 1 \leq i \leq n\}$ (from now on, the tilde notation for objects in $\mathcal{C}(H)$ is dropped). For any hereditary algebra $H$, the indecomposable $H$-modules are either preprojective, or regular, or preinjective; i.e.,

$$\text{ind}H = \mathcal{P} \vee \mathcal{R} \vee \mathcal{I},$$

where $\mathcal{P}$ (or $\mathcal{I}$) denotes the subcategory of indecomposable preprojective modules (resp. preinjective modules), $\mathcal{R}$ denotes the subcategory of indecomposable regular modules. If $H$ is of finite type, then $\mathcal{R}$ disappears, and $\mathcal{P} = \mathcal{I}$. Applying to cluster category, we have the following:
**Proposition 2.2.** \( \text{indC}(H) = \mathcal{P} \lor \{ P_i[1] \mid 1 \leq i \leq n \} \lor \mathcal{I} \lor \mathcal{R} \). If \( H \) is of finite type, then \( \mathcal{R} = \emptyset \), \( \mathcal{P} = \mathcal{I} \), \( \text{indC}(H) = \mathcal{P} \lor \{ P_i[1] \mid 1 \leq i \leq n \} = \mathcal{I} \lor \{ P_i[1] \mid 1 \leq i \leq n \} \).

3. BGP-reflection functors in cluster categories.

Since any hereditary algebra is Morita equivalent to a tensor algebra of some species of a valued quiver, we will use the language of valued quivers and their representations. Firstly we recall some basic notations on representations of valued quivers from [DR].

Let \((\Gamma, d)\) be a valued graph with \( n \) vertices and with an orientation \( \Omega \) (the pair \((\Gamma, d, \Omega)\) or simply the pair \((\Gamma, \Omega)\) is called a valued quiver). For any vertex \( k \in \Gamma \), we can define a new orientation \( s_k \Omega \) of \((\Gamma, d, \Omega)\) by reversing the direction of all edges containing \( k \). A vertex \( k \in \Gamma \) is said to be a sink (or a source) with respect to \( \Omega \) if there are no arrows starting (or ending) at vertex \( k \).

Let \( \mathbf{k} \) be a field and \((\Gamma, \Omega)\) a valued quiver. From now on, we shall always assume that \((\Gamma, \Omega)\) contains no oriented cycles. For any orientation \( \Omega \), there is an ordering \( k_1, \cdots, k_n \) of \( \Gamma \) such that vertex \( k_t \) is a sink with respect to the orientation \( s_{k_{t-1}} \cdots s_{k_1} \Omega \) for all \( 1 \leq t \leq n \). This is also equivalent to that there is an ordering \( k_1', \cdots, k_n' \) of \( \Gamma \) such that the vertex \( k_t' \) is a source with respect to the orientation \( s_{k_{t-1}'} \cdots s_{k_1'} \Omega \) for all \( 1 \leq t \leq n \). Such orderings are called admissible sequences of sinks or admissible sequences of sources and an orientation with an admissible sequence of sinks (equivalently sources) is called an admissible orientation. It is clear that for any admissible sequence \( k_1, \cdots, k_n \) of sinks or sources, \( s_{k_n} \cdots s_{k_1} \Omega = \Omega \).

Let \( \mathbf{M} = (F_i, M_j)_{i,j \in \Gamma} \) be a reduced \( \mathbf{k} \)-species of type \( \Omega \) : that is, for all \( i, j \in \Gamma \), \( i M_j \) is an \( F_i - F_j \)-bimodule, where \( F_i \) and \( F_j \) are finite extensions of \( \mathbf{k} \) and \( \dim(F_i M_j)_{F_j} = d_{ij} \) and \( \dim F_i = e_i \). A \( \mathbf{k} \)-representation \( V = (V_i, \varphi_i) \) of \( \mathbf{M} \) consists of \( F_i \)-vector spaces \( V_i, i \in \Gamma \), and of an \( F_j \)-linear map \( \varphi_i : V_i \otimes_i M_j \rightarrow V_j \) for each arrow \( i \rightarrow j \). Such a representation is called finite dimensional if \( \sum_{i \in \Gamma} \dim_{\mathbf{k}} V_i < \infty \). The category of finite-dimensional representations of \( \mathbf{M} \) over \( \mathbf{k} \) is denoted by \( \text{rep}(\mathbf{M}, \Gamma, \Omega) \). If \( \text{rep}(\mathbf{M}, \Gamma, \Omega) \) contains only finitely many indecomposable representations up to isomorphism, then \( \Gamma \) is called of finite type; otherwise, \( \Gamma \) is called of infinite type. It was proved by Gabriel [ARS][R] that \( \Gamma \) is of finite type if and only if \( \Gamma \) is a disjoint union of Dynkin diagrams.

Now we fix a \( \mathbf{k} \)-species \( \mathbf{M} \) of a given valued quiver \((\Gamma, \Omega)\). Given a sink, or a source \( k \) of the quiver \((\Gamma, \Omega)\), we recall the reflection functor \( S_k^\pm \):

\[
S_k^+ : \text{rep}(\mathbf{M}, \Gamma, \Omega) \rightarrow \text{rep}(\mathbf{M}, \Gamma, s_k \Omega), \quad \text{if } k \text{ is a sink},
\]

or

\[
S_k^- : \text{rep}(\mathbf{M}, \Gamma, \Omega) \rightarrow \text{rep}(\mathbf{M}, \Gamma, s_k \Omega), \quad \text{if } k \text{ is a source}.
\]
We assume $k$ is a sink. For any representation $V = (V_i, \phi_i)$ of $(\mathbf{M}, \Gamma, \Omega)$, its image under $S_k^+ V$ is by definition, $S_k^+ V = (W_i, j\psi_i)$, a representation of $(\mathbf{M}, \Gamma, s_k \Omega)$, where $W_i = V_i$, if $i \neq k$; and $W_k$ is the kernel in the diagram:

$$
(*) \quad 0 \to W_k \xrightarrow{(j \chi_k)_j} \oplus_{j \in \Gamma} V_j \otimes j M_k \xrightarrow{(k \delta_j)_j} V_k
$$

$j \psi_i = j \phi_i$ and $j \psi_k = j \chi_k : W_k \otimes k M_j \to X_j$, where $j \chi_k$ corresponds to $j \chi_k$ under the isomorphism $\text{Hom}_{F_j}(W_k \otimes k M_j, V_j) \approx \text{Hom}_{F_j}(W_k, V_j \otimes j M_i)$.

If $\alpha = (\alpha_i) : V \to V'$ is a morphism in $\text{rep}(\mathbf{M}, \Gamma, \Omega)$, then $S_k^+ \alpha = (\beta_i)$, where $\beta_i = \alpha_i$ for $i \neq k$ and $\beta_k : W_k \to W_k'$ is the restriction of $\oplus_{j \in \Gamma} (\alpha_j \otimes 1)$ given in the following commutative diagram:

$$
\begin{array}{ccc}
0 & \to & W_k \xrightarrow{(j \chi_k)_j} \oplus_{j \in \Gamma} V_j \otimes j M_k \xrightarrow{(k \delta_j)_j} V_k \\
& & \downarrow \beta_k \\
0 & \to & W_k' \xrightarrow{(j \chi_k')_j} \oplus_{j \in \Gamma} V_j' \otimes j M_k \xrightarrow{(k \delta_j')_j} V_k'
\end{array}
$$

If $k$ is a source, the definition of $S_k^- V$ is dual to that of $S_k^+ V$, we omit it and refer to [DR].

For simplicity, we denote by $\mathcal{H}$ the category $\text{rep}(\mathbf{M}, \Gamma, \Omega)$ and by $\mathcal{H}'$ the category $\text{rep}(\mathbf{M}, \Gamma, s_k \Omega)$, where $k$ is a sink (or source) of $(\Gamma, \Omega)$. The cluster categories $D^b(\mathcal{H})/F$, $D^b(\mathcal{H}')/F$ are denoted by $\mathcal{C}(\Omega)$ and $\mathcal{C}(s_k \Omega)$ respectively.

Let $P_i$, $I_i$ (or $P_i'$, $I_i'$) be the projective, injective indecomposable representation in $\mathcal{H}$ (resp. $\mathcal{H}'$) corresponding to the vertex $i \in \Gamma$, and $E_i$ (resp. $E_i'$) the corresponding simple representation in $\mathcal{H}$ (resp. $\mathcal{H}'$). We denote by $H$ (resp. $H'$) the tensor algebra of $(\mathbf{M}, \Gamma, \Omega)$ (resp. $(\mathbf{M}, \Gamma, s_k \Omega)$). Note that $H - \text{mod}$ is Morita equivalent to $\mathcal{H}$, and if $k$ is a sink (or source), then $P_k = E_k$ (resp. $I_k = E_k$) is simple projective (resp. injective) $H$-module.

Let $T = \oplus_{i \in \Gamma - \{k\}} P_i \oplus \tau^{-1} P_k$. Suppose $k$ is a sink, then $T$ is a tilting $H$-module which is called BGP- or APR-tilting module and $S_k^+ = \text{Hom}(T, -)$ as functors. The following theorem was proved in [Z1] (in a more general case).

**Theorem 3.1.** Let $k$ be a sink (or a source) of a valued quiver $(\Gamma, \Omega)$. Then the BGP-reflection functor $S_k^+$ (resp. $S_k^-$) induces a triangle equivalence $R(S_k^+)$ (resp., $R(S_k^-)$) from $\mathcal{C}(\Omega)$ to $\mathcal{C}(s_k \Omega)$. Moreover we have that

$$R(S_k^+(X)) = \begin{cases} 
P_k'[1], & X \cong E_k \\
E_k', & X \cong P_k[1] \\
P_j'[1], & X \cong P_j[1], j \neq k \\
S_k^+(X), & X \in \text{ind}H - \{E_k\} 
\end{cases}$$
Moreover, we have that
\[ \text{ind}(\Omega) = \{ C^m P_i[1] \mid m > 0, i \in \Gamma \} \]
\[ \text{I}(\Omega) = \{ C^m P_i[1] \mid m > 0, i \in \Gamma \}. \]

**Theorem 3.3.** If \( \Gamma \) is of finite type, then \( \text{ind}C(\Omega) = \mathcal{P} \lor \{ P_i[1] \mid i \in \Gamma \} = \mathcal{I} \lor \{ P_i[1] \mid i \in \Gamma \}; \) if \( \Gamma \) is of infinite type, then \( \text{ind}C(\Omega) = \mathcal{P} \lor \{ P_i[1] \mid i \in \Gamma \} \lor \mathcal{I} \lor \mathcal{R}; \)

Moreover, we have that
\[ \mathcal{P} = \{ C^{-m} P_i[1] \mid m > 0, i \in \Gamma \}, \quad \mathcal{I} = \{ C^m P_i[1] \mid m > 0, i \in \Gamma \}. \]

**Proof.** The first part follows from Proposition 2.2. We prove the second part. For any indecomposable projective representation \( P \in \mathcal{H} \), we have that \( \tau^{-1} P[1] \simeq P \in \mathcal{C}(\Omega) \). It follows that \( C^{-1} P[1] \simeq P \in \mathcal{C}(\Omega) \). From [DR], we know that for any preinjective indecomposable module \( M \), there are an indecomposable projective module \( P \) and an integer \( m \geq 0 \), such that \( M \simeq C^{-m} P \). Therefore we have that \( M = C^{-m+1} P[1] \). This finishes the proof of the description of \( \mathcal{P} \). Dually, for any preinjective \( H \)-module \( N \), there are an indecomposable injective module \( I \) and an integer \( m \geq 0 \) such that \( N \simeq C^m I \). We also have that \( I_i = \tau P_i[1] \in \mathcal{C}(\Omega) \) for \( i \in \Gamma \) (since in derived category \( D^b(\mathcal{H}) \), \( \tau P_i[1] = I_i \)). Therefore we have that \( N \simeq C^{-m} I \). This finishes the proof for \( \mathcal{I} \). The proof is finished.

For this reason, we denote the union \( \mathcal{P} \lor \{ P_i[1] \} \lor \mathcal{I} \lor \mathcal{R} \) by \( \mathcal{P} \mathcal{I}(\Omega) \). Note that
\[ \mathcal{P} \mathcal{I}(\Omega) = \{ C^m P_k[1] \mid m \in \mathbb{Z}, 1 \leq k \leq n \}. \]

If \( \Gamma \) is of finite type, then \( \text{ind}C(\Omega) = \mathcal{P} \mathcal{I}(\Omega) \), otherwise \( \text{ind}C(\Omega) = \mathcal{P} \mathcal{I}(\Omega) \lor \mathcal{R}. \)

**Corollary 3.4.** If \( \Gamma \) is of finite type, then for any orientation \( \Omega \) on \( \Gamma \),
\[ \text{ind}C(\Omega) = \{ C^m P_k[1] \mid m \in \mathbb{Z}, 1 \leq k \leq n \} = \{ C^m P_k[1] \mid m \geq 0, 1 \leq k \leq n \} = \{ C^{-m} P_k[1] \mid m \geq 0, 1 \leq k \leq n \}. \]

**Example 3.5.** Let \( \Gamma \) be \( B_2 : 2 \to 1 \). We give it an orientation \( \Omega : 2 \to 1 \). The AR-quiver of the cluster category \( \mathcal{C}(\Omega) \) is the following:
with the valuation \((1, 2)\) on all arrows like \(\nearrow\), and the valuation \((2, 1)\) on all arrows like \(\searrow\).

The Coxeter functor \(C = R_2^+ R_1^+\), and we have \(P_1 = C^{-1} P_1[1]\), \(P_2 = C^{-1} P_2[1]\), \(I_1 = C^{-2} P_1[1]\), \(I_2 = C^{-2} P_2[1]\); also \(P_1 = C^2 P_1[1]\), \(P_2 = C^2 P_2[1]\), \(I_1 = CP_1[1]\), \(I_2 = CP_2[1]\).

For a valued graph \(\Gamma\), we denote by \(\Phi\) the set of roots of the corresponding Kac-Moody Lie algebra. Let \(\Phi_{\geq -1}\) denote the set of almost positive roots, i.e. the positive roots together with the negatives of the simple roots. Let \(s_i\) be the Coxeter generator of the Weyl group of \(\Phi\) corresponding to \(i \in \Gamma\). We recall from \([FZ3]\) that the "truncated reflections" \(\sigma_i\) of \(\Phi_{\geq -1}\) are defined as follows:

\[
\sigma_i(\alpha) = \begin{cases} 
\alpha & \text{if } \alpha = -\alpha_j, j \neq i \\
 s_i(\alpha) & \text{otherwise.}
\end{cases}
\]

On the one hand, when \(\Gamma\) is of finite type, there is a bilinear form \((- \| -\) on \(\Phi_{\geq -1}\) which is called the "compatibility degree" of \(\Phi_{\geq -1}\) (for details, we refer to \([FZ3, FZ2]\)). \(\alpha, \beta \in \Phi_{\geq -1}\) are called compatible if \((\alpha \| \beta) = 0\). Any maximal mutually compatible subset is called a cluster of \(\Phi_{\geq -1}\). It was proved in \([FZ3]\) that any cluster in \(\Phi_{\geq -1}\) contains \(n\) elements, where \(n\) is the number of simple roots of \(\Phi_{\geq -1}\).

On the other hand, in the cluster category \(C(H)\), there is a tilting machinery. An object \(T\) is called tilting if \(\text{Ext}^{\leq 1}_{C(H)}(T, T) = 0\) and it has a maximal number of non-isomorphic indecomposable direct summands. A multiplicity-free tilting object is called a basic tilting object. In the following we assume tilting objects are always basic. Any tilting object contains \(n\) indecomposable direct summands.

We have seen that \(\text{ind}H \subset \text{ind}C(H)\), and \(\Phi \subset \Phi_{\geq -1}\). The well-known Gabriel’s Theorem gives a one-to-one correspondence from \(\text{ind}H\) of hereditary algebra of finite type to the root system \(\Phi^+\) of the corresponding simple Lie algebra by taking dimension vectors of modules. This correspondence was generalized to cluster categories of finite type, which induces a bijection between the set of tilting objects to the set of clusters in \(\Phi_{\geq -1}\), in the simple-laced case in \([BMRRZ]\), and to all Dynkin cases in \([Z1]\) (see Proposition 3.7. latter for precise meaning). In fact this map can be defined for any cluster category (finite and infinite types) as follows: for any \(X \in \text{ind}(\text{mod}H \vee H[1])\),

\[
\gamma_\Omega(X) = \begin{cases} 
\dim X & \text{if } X \in \text{ind}H; \\
-\dim E_i & \text{if } X = P_i[1],
\end{cases}
\]
where $\dim X$ denotes the dimension vector of the representation $X$. In general, this map $\gamma_\Omega : \text{ind} C(\Omega) \to \Phi_{\geq -1}$ is not injective, but it is surjective in all cases, and is a bijection in the finite type case.

Let $\Phi'_{\geq -1}$ be the subset of $\Phi_{\geq -1}$ consisting of $\gamma_\Omega(X)$ for all $X \in PI(\Omega)$. Then the restriction of $\gamma_\Omega$ to $PI(\Omega)$ is a bijection from $PI(\Omega)$ to $\Phi'_{\geq -1}$ by [DR], [Kac], this map is also denoted by $\gamma_\Omega$.

When $\Gamma$ is of finite type, one can choose a skew-symmetrizable integer matrix $B$ from $\Gamma$ such that the cluster variables of type $\Gamma$ are in one-to-one correspondence with the elements of $\Phi_{\geq -1}$ (compare [FZ2]). For any orientation $\Omega$, $\text{ind} C(\Omega)$ is in one-to-one correspondence with $\Phi_{\geq -1}$ [BMRRT] [Z1]. We will relate these two results and generalize partially these one-to-one correspondences to infinite type in the next section.

By using Theorem 3.1, one gets the following commutative diagram which explains that $R^k_{\pm}$ is the realization of the "truncated reflection" $\sigma_k$ (for proof, we refer to [Z1, Z2]).

**Proposition 3.6.** Let $k$ be a sink (or a source) of a valued quiver $(\Gamma, \Omega)$. Then we have the commutative diagram:

$$
\begin{array}{cccc}
\text{ind} C(\Omega) & \xrightarrow{R^k_{\pm}} & \text{ind} C(s_k \Omega) \\
\gamma_\Omega \downarrow & & \downarrow \gamma_{s_k \Omega} \\
\Phi_{\geq -1} & \xrightarrow{\sigma_k} & \Phi_{\geq -1}
\end{array}
$$

The following result is proved for simply-laced Dynkin diagram in [BMRRT], and is generalized to all Dynkin diagram in [Z1].

**Proposition 3.7** [BMRRT] [Z1]. Let $(\Gamma, \Omega)$ be any valued Dynkin quiver. Then the one-to-one correspondence $\gamma_\Omega$ sends tilting objects of $C(\Omega)$ to clusters in $\Phi_{\geq -1}$.
4. Coxeter automorphisms of cluster algebras.

We recall some basic notation on cluster algebras which can be found in the series of papers by Fomin and Zelevinsky [FZ1, FZ2, FZ3, BFZ]. The cluster algebras we deal with in this paper are defined on a trivial semigroup of coefficients, since it is enough for the connection with representation theory of quivers [BMRRT]. These cluster algebras are called reduced cluster algebras in [CC]. We will call these algebras just cluster algebras.

The definition is as follows: Let \( F = \mathbb{Q}(u_1, u_2, \cdots, u_n) \) be the field of rational functions in indeterminates \( u_1, u_2, \cdots, u_n \). Set \( \underline{u} = (u_1, u_2, \cdots, u_n) \). Let \( B = (b_{ij}) \) be an \( n \times n \) skew-symmetrizable integer matrix. A pair \( (\underline{x}, B) \), where \( \underline{x} = (x_1, x_2, \cdots, x_n) \) is a transcendence base of \( F \) and where \( B \) is an \( n \times n \) skew-symmetrizable integer matrix, is called a seed. Fix a seed \( (\underline{x}, B) \) and an element \( z \) in the base \( \underline{x} \). Let \( z' \) in \( F \) be such that

\[
zz' = \prod_{b_{xz}>0} x^{b_{xz}} + \prod_{b_{xz}<0} x^{-b_{xz}}.
\]

Now, set \( x' := \underline{x} - \{z\} \cup \{z'\} \) and \( B' = (b'_{xy}) \) such that

\[
b'_{xy} = \begin{cases} 
-b_{xy} & \text{if } x = z \text{ or } y = z, \\
b_{xy} + 1/2(|b_{xz}|b_{zy} + b_{xz}|b_{zy}|) & \text{otherwise}.
\end{cases}
\]

The pair \( (\underline{x}', B') \) is called the mutation of the seed \( (\underline{x}, B) \) in direction \( z \), it is also a seed. The "mutation equivalence \( \approx \)" is an equivalence relation on the set of all seeds generated by \( (\underline{x}, B) \approx (\underline{x}', B') \) if \( (\underline{x}', B') \) is a mutation of \( (\underline{x}, B) \).

The cluster algebra \( A(B) \) associated to the skew-symmetrizable matrix \( B \) is by definition the subalgebra of \( F \) generated by all \( \underline{x} \) such that \( (\underline{x}, B) \approx (\underline{u}, B) \). Such \( \underline{x} = (x_1, x_2, \cdots, x_n) \) is called a cluster of the cluster algebra \( A(B) \) or simply of \( B \), and any \( x_i \) is called a cluster variable. If the set \( \chi \) of all cluster variables is finite, then the cluster algebra \( A(B) \) is said to be of finite type.

Let \( \underline{x} \) be a seed. The Laurent phenomenon, see [FZ1], asserts that any cluster variables are Laurent polynomials with integer coefficients in variables \( x_1, \cdots, x_n \). It implies that \( A(B) \subset \mathbb{Z}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \).

Fix any integer square matrix \( B = (b_{ij}) \). Its Cartan counterpart is by definition, a generalized Cartan matrix \( A = A(B) = (a_{ij}) \) of the same size defined by

\[
a_{ij} = \begin{cases} 
2 & \text{if } i = j, \\
-|b_{ij}| & \text{if } i \neq j.
\end{cases}
\]

**Theorem 4.1.** [FZ2] A cluster algebra \( A \) is of finite type if and only if there is a seed \( (\underline{x}, B) \) of \( A \) such that the Cartan counterpart of the matrix \( B \) is a Cartan matrix of finite type.

For any \( \alpha \in \Phi \), we write \( \alpha \) as a sum of simple roots \( \alpha = \sum_{i \in I} a_i \alpha_i \), then we use \( u^\alpha \) to denote \( \prod_{i \in I} u_i^{a_i} \).
Theorem 4.2. [FZ2] Fix a Dynkin diagram $\Gamma$ and a distinguished seed $(u, B)$. Then there exists a bijection

$$P : \Phi_{\geq -1} \rightarrow \chi_{\Gamma} : \alpha \mapsto u[\alpha] = \frac{P_\alpha(u)}{u^\alpha},$$

where $P_\alpha(u)$ is a polynomial with nonzero constant term. Under this correspondence, $-\alpha_i$ corresponds to $u_i$, and clusters in $\Phi_{\geq -1}$ correspond to clusters of the corresponding cluster algebra $A$.

Corollary 4.3. Let $\Gamma$ be a Dynkin graph with the orientation $\Omega_0$ (i.e., such that any vertex is a sink or source). Then the composition (denoted by $\phi_{\Omega_0}$) of $\gamma_{\Omega_0}$ and $P$ gives a one-to-one correspondence between $\text{ind}\mathcal{C}(\Omega_0)$ and $\chi_{\Gamma}$. Under this correspondence, $P_i[1]$ corresponds to $u_i$, and tilting objects correspond to clusters.

Proof. This is a consequence of Proposition 3.7. and Theorem 4.2.

Note that when $\Gamma$ is a simply-laced Dynkin diagram, Corollary 4.3 is also proved in [CC] (compare Theorem 3.4. there) in which the correspondence is given in explicit expressions of indecomposable objects by Laurent polynomials of $u_1, \cdots, u_n$. In the following, for any orientation, we will give an explicit one-to-one correspondence from cluster categories to the set of cluster variables in a different spirit, which works in simply-laced case and non-simply-laced case.

Firstly we need to define some isomorphisms between cluster algebras.

Given a generalized Cartan matrix $A$ of size $n \times n$, its Coxeter graph $\Delta$ is by definition, a valued graph consisting of $n$ vertices, named $1, 2, \cdots, n$, and edges $i - j$ with a valuation $(a_{ij}, a_{ji})$ if $a_{ij} \neq 0$. If $A$ is a Cartan matrix of finite type, then its Coxeter graph is a tree.

Let $B = (b_{ij})$ be a skew-symmetrizable integer matrix and $A = A(B)$, its Cartan counterpart. Then we say that $B$ and $A$ form a matched pair $(B, A)$. Note that $A$ is a generalized Cartan matrix and there are different skew-symmetrizable matrices $B$ and $B'$ with the same Cartan counterpart $A$.

For any matched pair $(B, A)$ of matrices with $B$ a skew-symmetrizable integer matrix, one can give an orientation $\Omega$ on its Coxeter graph $\Delta$ as follows: if $b_{ij} > 0$, then there is an arrow $i \rightarrow j$. With this orientation, $(\Delta, \Omega)$ becomes a valued quiver. This quiver is called the quiver of $(B, A)$. Note that if we can choose $B = B(A)$ with the property "$b_{ij}b_{ik} \geq 0$, for all $i, j, k$," then the quiver of $(B, A)$ is such that any vertex is a sink or a source. If a Coxeter graph is a tree, then we can choose such an orientation, this orientation was considered in [FZ2]. More generally, the orientation on a Coxeter graph corresponds to the skew-symmetrizable matrix $B = (b_{ij})$ such that $A = A(B)$ is as indicated in the next lemma.

Lemma 4.4. Fix a Coxeter diagram $\Delta$, equivalently, a generalized Cartan matrix $A$. Then the orientations of $\Delta$ are in bijection with the matched pairs $(B, A)$. Moreover, the orientation contains an orientated cycle if and only if there
is a sequence of indices $i_1, i_2, \cdots i_t$ such that $b_{i_1, i_2}, b_{i_2, i_3}, \cdots, b_{i_t-1, i_t}, b_{i_t, i_1}$ are positive integers.

**Proof.** For any matched pair $(B, A)$, we can define an orientation of $\Delta$ as above. Conversely, for any orientation of $\Delta$, the matrix $B = (b_{ij})$ can be defined uniquely in the way:

$$b_{ij} = \begin{cases} 0 & \text{if } i = j \\ |a_{ij}| & \text{if } i \rightarrow j \\ a_{ij} & \text{if } j \rightarrow i. \end{cases}$$

The final statement follows from the correspondence between orientations and $B$. The proof is finished.

**Definition 4.5.** Let $A$ be a generalized Cartan matrix and $B$ one of the skew-symmetrizable matrix with $A = A(B)$. For any index $i$, we define an automorphism $T_i$ of $F = \mathbb{Q}(u_1, \cdots, u_n)$ by defining the images of the indeterminates $u_1, \cdots, u_n$ as follows:

$$T_i(u_j) = \begin{cases} u_j & \text{if } j \neq i, \\ \frac{\prod_{a_{ik} < 0} u_k^{-a_{ik} + 1}}{u_i} & \text{if } j = i. \end{cases}$$

It is easy to check that all $T_i$ are involutions of $F$, i.e., $T_i^2 = \text{id}_F$.

From the definition, all $T_i$ are independent of $B$, and only depend on the matrix $A$.

In the rest of paper, we study the properties of automorphisms $T_i$ with respect to clusters and cluster algebras. Since any orientation of $\Gamma$ is admissible, for such an orientation $\Omega$ on $\Gamma$, by Lemma 4.4, we have a pair $(B, A)$ with $A$ a generalized Cartan matrix, the matrix $B$ corresponding to $\Omega$ is sometimes denoted by $B_\Omega$. The cluster algebra $A$ associated with the seed $(u, B_\Omega)$ is called the cluster algebra associated with the orientation $\Omega$, and is denoted by $A_\Omega$; the set of cluster variables of $A_\Omega$ is denoted by $\chi_\Omega$. By [FZ2], $A_\Omega$ is isomorphic to $A_{\Omega_0}$.

**Remark 4.6.** $\chi_\Omega$ is different from $\chi_{\Omega_0}$ in general, see Example 4.9. below.

In the following, we prove that for any orientation of a Dynkin graph $\Gamma$ there is a bijection $\phi_\Omega$ from $\text{ind}C(\Omega)$ to $\chi_\Omega$ such that $P_i[1]$ corresponds to $u_i$, which generalizes Corollary 4.3.

**Theorem 4.7.** Let $k$ be a sink (or source) in $\Omega$ on a Dynkin diagram $\Gamma$. Then (1). there is a bijection $\phi_\Omega$ from $\text{ind}C(\Omega)$ to $\chi_\Omega$ such that $P_i[1]$ corresponds to $u_i$, for any $i \in \Gamma$; inducing a one-to-one correspondence between basic tilting objects and clusters.

(2). $T_k$ sends cluster variables and clusters in $\chi_\Omega$ to those in $\chi_{s_k}$.
\( (3). \) \( T_k \) is induced from the reflection functor \( R_k^+ \) indicated in the following commutative diagram:

\[
\begin{array}{ccc}
\text{indC}(\Omega) & \xrightarrow{R_k^+} & \text{indC}(s_k \Omega) \\
\phi_\Omega & \downarrow & \downarrow \phi_{s_k \Omega} \\
\chi_\Omega & \xrightarrow{T_k} & \chi_{s_k \Omega}
\end{array}
\]

\( (4). \) \( T_k \) induces an isomorphism from the cluster algebra \( A_\Omega \) to the cluster algebra \( A_{s_k \Omega} \) (\( T_k \) induces a so-called strongly isomorphism from \( A_\Omega \) to \( A_{s_k \Omega} \)).

**Proof.** By definition, \( \chi_\Omega \) and \( \chi_{s_k \Omega} \) are the sets of cluster variables of the initial seeds \( (u, B_\Omega) \) and \( (u, B_{s_k \Omega}) \) respectively. Let \( u'_k = (u_1, \cdots, u_{k-1}, u_k', u_{k+1}, \cdots, u_n) \) be the cluster of \( (u, B_{s_k \Omega}) \) obtained by mutation once in direction \( k \). Then \( u'_k u_k = \prod_{a_{ik} < 0} u_i a_{ik} + 1 \) since \( k \) is a source of \( s_k \Omega \), hence \( u'_k = \prod_{a_{ik} < 0} u_i u_k \cdot u_k, \) and the matrix \( B_{s_k \Omega} \) after this mutation is \( B_\Omega \). Therefore we have that \( T_k(u) = u'_k \) and the automorphism \( T_k \) of \( \mathcal{F} \) sends the seed \( (u, B_\Omega) \) to the seed \( (u'_k, B_\Omega) \). Dually \( T_k \) sends the seed \( (u'_k, B_\Omega) \) to the seed \( (u, B_\Omega) \). Therefore, \( T_k \) sends \( \chi_\Omega \) to \( \chi_{(u'_k, B_\Omega)} \), the latter is the set of cluster variables associated to the initial seed \( (u'_k, B_\Omega) \). Since the seed \( (u'_k, B_\Omega) \) is obtained by mutation from \( (u, B_{s_k \Omega}) \), i.e., \( (u'_k, B_\Omega) \) \( \xrightarrow{\text{mutation}} (u, B_{s_k \Omega}) \), \( \chi_{(u'_k, B_\Omega)} = \chi_{s_k \Omega} \). Hence \( T_k \) induces a bijection from \( \chi_\Omega \) to \( \chi_{s_k \Omega} \). For any seed \( (x, B) \) which is mutation equivalent to the initial seed \( (u, B_\Omega) \), denote by \( T_k(x) \) the vector \( (T_k(x_1), \cdots, T_k(x_n)) \), then \( (T_k(x), B) \) is a seed which is mutation equivalent to the seed \( (u'_k, B_\Omega) \), and then it is mutation equivalent to the initial seed \( (u, B_{s_k \Omega}) \). This implies that \( T_k(x) \) is a cluster of \( (u, B_{s_k \Omega}) \). Then \( T_k \) sends clusters in \( \chi_\Omega \) to clusters in \( \chi_{s_k \Omega} \). This proves part (2). Since \( \Omega \) and \( \Omega_0 \) are orientations on \( \Gamma \), there is a sequence of vertices \( i_1, \cdots, i_t \) such that \( s_{i_1} \cdots s_{i_t} \Omega_0 = \Omega \), where \( i_j \) is a sink of \( s_{i_{j-1}} \cdots s_{i_j} \Omega_0 \) for any \( j \). It follows from part 2 proved above that \( T_{i_1} \cdots T_{i_t} \chi_\Omega = \chi_\Omega \). Combining with Corollary 4.3, we have the bijection \( \phi_\Omega \) (obtained by taking \( \phi_\Omega = T_{i_1} \cdots T_{i_t} \phi_{\Omega_0}(R_{i_t}^+ \cdots R_{i_1}^+)^{-1} \)) from \( \text{indC}(\Omega) \) to \( \chi_\Omega \) which satisfies the commutative diagram:

\[
\begin{array}{ccc}
\text{indC}(\Omega_0) & \xrightarrow{R_{i_t}^+ \cdots R_{i_1}^+} & \text{indC}(\Omega) \\
\phi_{\Omega_0} & \downarrow & \downarrow \phi_\Omega \\
\chi_{\Omega_0} & \xrightarrow{T_{i_t} \cdots T_{i_1}} & \chi_\Omega
\end{array}
\]

Since all isomorphisms \( T_{i_t} \) send clusters to clusters, all \( R_{i_t}^+ \) are triangle equivalences sending basic tilting objects to basic tilting objects, and \( \phi_{\Omega_0} \) send basic tilting objects to clusters (by Corollary 4.3.), we have that \( \phi_\Omega \) sends basic tilting objects to clusters. This proves part (1).

Let \( i_1, \cdots, i_t \) be the vertices of \( \Gamma \) such that \( s_{i_t} \cdots s_{i_1} \Omega_0 = \Omega \) with \( i_j \) a sink of \( s_{i_{j-1}} \cdots s_{i_j} \Omega_0 \) for any \( j \). By part (1), we have that \( \phi_\Omega = T_{i_t} \cdots T_{i_1} \phi_{\Omega_0}(R_{i_t}^+ \cdots R_{i_1}^+)^{-1} \).
and $\phi_{s_k\Omega} = T_kT_i \cdots T_i \phi_{\Omega_0} (R^+_i R^+_i \cdots R^+_i)^{-1}$. Then $T_k\phi_{\Omega} = T_kT_i \cdots T_i \phi_{\Omega_0} (R^+_i \cdots R^+_i)^{-1} = T_kT_i \cdots T_i \phi_{\Omega_0} (R^+_i R^+_i) R^+_k = \phi_{s_k\Omega} R^+_k$. This proves (3).

Since cluster algebras $A_\Omega$ and $A_{s_k\Omega}$ are generated as subalgebras of $F$ by $\chi_\Omega$ and $\chi_{s_k\Omega}$ respectively, $T_k$ induces an isomorphism from $A_\Omega$ to $A_{s_k\Omega}$. This proves the final part. The whole proof is completed.

**Remark 4.8.** By using the commutative diagram in Theorem 4.7., one can get that the bijection $\phi_\Omega$ is of the form: $C^k(P_i[1]) \mapsto T^k(u_i)$ for any $k \in \mathbb{Z}$, $i = 1, \cdots, n$.

**Remark 4.9.** In general, for any diagram $\Gamma$ (finite type or infinite type), $T_k$ sends cluster variables and clusters in $\chi_\Omega$ to those in $\chi_{s_k\Omega}$ and $T_k$ induces an isomorphism from $A_\Omega$ to $A_{s_k\Omega}$. The proof for this general result is same as that for part (2) (4) of Theorem 4.7.

**Example 4.10.** Let $\Gamma$ be $A_3 : 3 \rightarrow 2 \rightarrow 1$ with Cartan matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Let $\Omega$ be an orientation of $\Gamma : 3 \rightarrow 2 \rightarrow 1$. Then the corresponding skew-symmetrizable matrix $B$ is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. The AR-quiver of $\mathcal{C}(\Omega)$ has the following shape:

![AR-quiver of C(\Omega)](image)

The correspondence $\phi_\Omega$ from ind$\mathcal{C}(\Omega)$ to the set $\chi_\Omega$ is indicated as follows:

- $P[1] \mapsto u_i$, for $i = 1, 2, 3$;
- $P_2 \mapsto \frac{u_1 + u_2 + u_3}{u_2}$;
- $P_3 \mapsto \frac{u_1 + u_2 + u_3}{u_3}$;
- $E_2 \mapsto \frac{u_2 + u_3}{u_2}$;
- $I_2 \mapsto \frac{u_1 + u_2 + u_3}{u_2 u_3}$;
- $I_3 \mapsto \frac{1 + u_1}{u_3}$.

The cluster algebra $A_\Omega$ is the $\mathbb{Q}$–subalgebra of $F$ generated by all cluster variables above.

If we reflect the orientation $\Omega$ at vertex 1, we get the quiver $(\Gamma, s_1\Omega) : 3 \rightarrow 2 \leftarrow 1$. It corresponds to the skew-symmetrizable matrix $B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

The AR-quiver of $\mathcal{C}(s_1\Omega)$ has the following shape:
In this case, the correspondence $\phi_{s_{1}\Omega}$ from $\text{indC}(s_{1}\Omega)$ to the set $\chi_{s_{1}\Omega}$ is as follows:

\[
P_{1}[1] \mapsto u_{i}, \text{ for } i = 1, 2, 3; \quad P_{1} \mapsto \frac{1+u_{1}+u_{2}+u_{3}}{u_{2}}, \quad P_{2} \mapsto \frac{1+u_{1}+u_{3}}{u_{2}}; \\
P_{3} \mapsto \frac{1+u_{1}+u_{2}+u_{3}}{u_{2}u_{3}}; \quad E_{1} \mapsto \frac{1+u_{1}+u_{2}}{u_{2}}, \quad I_{2} \mapsto \frac{1+u_{2}+u_{3}}{u_{1}u_{2}u_{3}}, \quad E_{3} \mapsto \frac{1+u_{1}+u_{3}}{u_{2}}.
\]

The corresponding cluster algebra $A_{s_{1}\Omega}$ is the $\mathbb{Q}$-subalgebra of $\mathcal{F}$ generated by all cluster variables above. $\chi_{\Omega} \neq \chi_{s_{1}\Omega}$. It is easy to see that $\phi_{s_{1}\Omega}(R_{i}^{+}X) = T_{i}(\phi_{\Omega}(X))$ for all $X \in \text{indC}(\Omega)$. Therefore the isomorphism $T_{1}$ induces an isomorphism from the cluster algebra $A_{\Omega}$ to the cluster algebras $A_{s_{1}\Omega}$.

Let $\Omega$ be an orientation of $\Gamma$ and $C = R_{k_{0}}^{+} \cdots R_{k_{2}}^{+} R_{k_{1}}^{+}$ the corresponding Coxeter functor on $C(\Omega)$. We define an automorphism $T_{\Omega}$ of $\mathcal{F}$ as $T_{\Omega} = T_{k_{n}} \cdots T_{k_{2}} T_{k_{1}}$. By Theorem 4.7 (4), $T_{\Omega}$ induces an automorphism of cluster algebra $A_{\Omega}$.

**Definition 4.10.** $T_{\Omega}$ and its inverse $T_{\Omega}^{-1}$ are called the Coxeter automorphisms of the cluster algebra $A_{\Omega}$. $T_{\Omega}$ is simply denoted by $T$.

We recall that $\Omega_{0}$ and $\Omega_{0}'$ denote the two orientations of $\Gamma$ such that in any of these two orientations, any vertex is a sink or source. For such an orientation $\Omega$, we denote by $\Gamma_{+}$ the set of sinks in $\Omega$, by $\Gamma_{-}$ the set of sources in $\Omega$. Then $\Gamma = \Gamma_{+} \cup \Gamma_{-}$. Dually we have $\Gamma = \Gamma_{+}' \cup \Gamma_{-}'$. Now we set $T_{2} = \prod_{i \in \Gamma_{+}} T_{i}$ for $\varepsilon \in \{+, -\}$. Note that $T = T_{+} T_{-}$.

**Corollary 4.11.** $\chi_{\Omega_{0}} = \chi_{\Omega_{0}'}$; $T_{\pm}$ are automorphisms of $A_{\Omega_{0}}$ and induce a bijection from $\chi_{\Omega_{0}}$ to itself, which sends clusters to clusters.

**Proof.** We prove firstly that $\chi_{\Omega_{0}} = \chi_{\Omega_{0}'}$. Let $B_{\Omega_{0}}$ and $B_{\Omega_{0}'}$ be the skew-symmetric integer matrices corresponding to the quivers $(\Gamma, \Omega_{0})$ and $(\Gamma, \Omega_{0}')$ respectively. Then $B_{\Omega_{0}'} = -B_{\Omega_{0}}$. From the initial seed $(u, B_{\Omega_{0}})$ and the initial seed $(u', B_{\Omega_{0}'}')$ respectively, the new seeds $(u', B_{\Omega_{0}})$ and $(u', B_{\Omega_{0}'}')$ obtained by one step mutation in any direction $k$ contain the same cluster variables $u_{1}, \ldots, u_{k-1}, u_{k}', \ldots, u_{n}$, and their matrices $B_{\Omega_{0}}'$ and $B_{\Omega_{0}'}'$ also satisfy the relation $B_{\Omega_{0}'}' = -B_{\Omega_{0}}'$. By induction, we have that $\chi_{\Omega_{0}} = \chi_{\Omega_{0}'}$. The second and the third statements follow easily from the first one and statements (2) and (4) in Theorem 4.7. The proof is finished.

**Remark. 4.12.** If the number of vertices of $\Gamma$ is 2, the automorphisms $T_{\pm}$ of $A_{\Omega_{0}}$ are defined in [SZ], these automorphisms are used to study the positivity and canonical bases in rank 2 cluster algebras there.
In the following, we will generalize Theorem 4.7. to arbitrary valued graph \( \Gamma \). Let \( \Omega \) be an orientation of \( \Gamma \), \( C = R_1^{k_1} \cdots R_n^{k_n} R_k^{k_k} \) the corresponding Coxeter functor on \( C(\Omega) \) and \( T = T_1 k_1 \cdots T_n k_n \) the corresponding Coxeter automorphism of \( A_\Omega \).

We define:
\[
\chi'_\Omega = \{ T^m(u_k) \mid m \in \mathbb{Z}, 1 \leq k \leq n \}.
\]

When \( \Gamma \) is of finite type, then \( \chi'_\Omega = \chi_\Omega \) is the set of cluster variables of \( (\Gamma, \Omega) \) by Corollary 3.4. and Theorem 4.7. For infinite type, we prove that elements in \( \chi'_\Omega \) are cluster variables of the initial seed \( (u, B) \) and prove the map \( \phi_\Omega : C^k(P_1) \rightarrow T^k(u_1), \) for any \( k \in \mathbb{Z}, \) sends tilting objects in \( \mathcal{P}_\mathcal{I}(\Omega) \) to clusters in \( \chi'_\Omega \) (compare Remark 4.8.).

**Theorem 4.13.** Let \( \Gamma \) be any valued graph, \( \Omega \) an orientation of \( \Gamma \). Then any element in \( \chi'_\Omega \) is a cluster variable of the initial seed \( (u, B) \), where \( B \) is the skew-symmetric matrix corresponding to \( (\Gamma, \Omega) \). Furthermore, the assignment \( \phi_\Omega : C^m(P_1[1]) \rightarrow T^m(u_k), \forall m \in \mathbb{Z}, k \in \Gamma \) is a surjection from \( \mathcal{P}_\mathcal{I}(\Omega) \) to \( \chi'_\Omega \) such that under this correspondence, \( P_k[1] \) corresponds to \( u_k \).

**Proof.** Firstly, we show the map \( \phi_\Omega : \mathcal{P}_\mathcal{I}(\Omega) \rightarrow \chi'_\Omega \) is well-defined.

Since any indecomposable object in \( C(\Omega) \) is of the form \( P_k[1] \) or \( X \) for some indecomposable \( H \)-module \( X \) [BMRRT], it follows from [G] [BrB] that \( C^m(X) \cong \tau^m X \) for any \( X \) in \( \text{indC}(\Omega) \).

If \( \Gamma \) is of finite type, then \( \chi'_\Omega = \chi_\Omega \), \( \mathcal{P}_\mathcal{I}(\Omega) = \text{indC}(\Omega) \), and the map \( \phi_\Omega \) is well defined, bijective and sends tilting objects to clusters by Theorem 4.7.

If \( \Gamma \) is of infinite type, then for any pair of indecomposable objects \( X, Y \) in \( \mathcal{P}_\mathcal{I}(\Omega) \), \( C^s X \cong C^t Y \) if and only if \( \tau^s X \cong \tau^t Y \) if and only if \( s = t \) and \( X \cong Y \). Therefore the map \( \phi_\Omega \) is well-defined and is surjective, and under this map, \( P_k[1] \) is sent to \( u_k \) for all \( k \).

In the following, we suppose that \( \Gamma \) is of infinite type. We will prove that \( (T^m(u_1), \ldots, T^m(u_n)) \) is a cluster. For \( m = 0 \), we know \( (u_1, \ldots, u_n) \) is the initial cluster, this corresponds to the slice \( (P_1[1], \ldots, P_n[1]) \) in the AR-quiver of the cluster category \( C(\Omega) \). Since \( C^m \) is a triangle auto-equivalence of the cluster category \( C(\Omega) \), it sends cluster-tilting set \( (P_1[1], \ldots, P_n[1]) \) to cluster-tilting set \( (C^m P_1[1], \ldots, C^m P_n[1]) \). By the proof of Theorem 4.7.(2) and Remark 4.9., we have that \( T^m \) sends the initial cluster \( (u, B_\Omega) \) to the cluster \( (T^m(u), B_\Omega) \), where \( T^m(u) = (T^m(u_1), \ldots, T^m(u_n)) \) is the image of \( (C^m P_1[1], \ldots, C^m P_n[1]) \) under \( \phi_\Omega \). This proves that all elements of the form \( T^m(u_i) \) are cluster variables. The proof is finished.

Note that \( \phi_\Omega \) should be injective, but up to now, we could not find a proof for this (compare [Z3]).

We remind that \( \Phi'_{\geq -1} \) denotes the subset of \( \Phi_{\geq -1} \) consisting of \( -\alpha_i, \) \( i \in \Gamma \) and \( \text{dim} X \) for any \( X \in \mathcal{P} \cup \mathcal{I} \), where \( \mathcal{P} \) and \( \mathcal{I} \) are those in Theorem 3.3.. By Proposition 2.1., \( \gamma_\Omega \) gives a bijection from \( \mathcal{P}_\mathcal{I}(\Omega) \) to \( \Phi'_{\geq -1} \). Set \( \sigma = \sigma_1 \cdots \sigma_\ell \). It is a bijection of \( \Phi_{\geq -1} \). Let \( \Omega \) be a map from \( \Phi'_{\geq -1} \) to \( \chi'_\Omega \) defined as \( \phi_\Omega \gamma_\Omega^{-1} \).
**Proposition 4.14.** Let $\Gamma$ be any valued graph, $\Omega$ an orientation of $\Gamma$. Then $P_\Omega : \Phi_{\geq -1} \to \chi'_\Omega$ is surjective, and the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{P}I(\Omega) & \xrightarrow{C} & \mathcal{P}I(\Omega) \\
\gamma_\Omega & \downarrow & \gamma_\Omega \\
\Phi'_\Omega & \xrightarrow{\sigma} & \Phi'_\Omega \\
\downarrow P_\Omega & & \downarrow P_\Omega \\
\chi'_\Omega & \xrightarrow{T} & \chi'_\Omega
\end{array}
\]

**Proof.** Since $\phi_\Omega$ is surjective and $\gamma_\Omega$ is bijective, $P_\Omega = \phi_\Omega \gamma_\Omega^{-1}$ is surjective. The upper square in the diagram is commutative due to Proposition 3.6. We verify the commutativity of the lower square. We verify that $T \phi_\Omega = \phi_\Omega C$.

For any $C^m(P_k[1]) \in \mathcal{P}I(\Omega)$, $T \phi_\Omega(C^m(P_k[1])) = T(T^m(u_k)) = T^{m+1}(u_k) = \phi_\Omega(C^{m+1}(P_k[1])) = \phi_\Omega(C^m(P_k[1]))$. Then $TP_\Omega = T \phi_\Omega \gamma_\Omega^{-1} = \phi_\Omega C \gamma_\Omega^{-1} = \phi_\Omega \gamma_\Omega^{-1} \sigma = P_\Omega \sigma$. The proof is finished.

Let $< C >$ and $< T >$ be the groups generated by $C$ and $T$ respectively. Let $h$ be the Coxeter number of a finite root system $\Phi$ [FZ3].

**Theorem 4.15.** Let $\Gamma$ be a valued graph of finite type and $\Omega$ an orientation of $\Delta$. Then the order of $C$ is equal to $\frac{h+2}{2}$ if the longest element of Weyl group is $-1$, and is equal to $h + 2$ otherwise.

**Proof.** For any two orientations $\Omega, \Omega'$, the corresponding Coxeter functors $C_\Omega$ and $C_{\Omega'}$ are conjugated with each other by Proposition 4.14. So we choose $\Omega$ is the orientation such that any vertex is a sink or source. It follows from Theorem 2.6 in [FZ3] that the order of $\sigma$ is equal to $\frac{h+2}{2}$ if the longest element of Weyl group is $-1$, and is equal to $h + 2$ otherwise. Applying Proposition 4.14 to $(\Gamma, \Omega)$, we have the order of $C$ is the same as that of $\sigma$. The proof is finished.

**Corollary 4.16.** Let $\Gamma$ be a valued graph of finite type and $\Omega$ an orientation of $\Gamma$. Then the order of $T$ is equal to $\frac{h+2}{2}$ if the longest element of Weyl group is $-1$, and is equal to $h + 2$ otherwise.

**Proof.** If $\Gamma$ is of finite type, the maps $\phi_\Omega$ and $P_\Omega$ in Proposition 4.14. are bijections. Then the order of $T$ is the same as $C$. This finishes the proof.

**Corollary 4.17.** Let $\Gamma$ be a valued graph of finite type and $\Omega$ an orientation of $\Gamma$. Then $\chi_\Omega = \{T^m(u_k) \mid k \in \Gamma, \ 0 \leq m \leq \frac{h+2}{2}\}$ if the longest element of Weyl group is $-1$, and $\chi_\Omega = \{T^m(u_k) \mid k \in \Gamma, \ 0 \leq m \leq h + 2\}$ otherwise.

**Proof.** This is a consequence of Theorem 4.13. and the Corollary 4.16.

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