ON THE SOLVABILITY OF EULER GRAPHENE BEAM
SUBJECT TO AXIAL COMPRESSION LOAD

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Abstract. In this paper we formulate the equilibrium equation of a beam
made of graphene material subjected to some boundary conditions and
acted upon by axial compression and nonlinear lateral constrains as a
fourth-order nonlinear boundary value problem. We also formulate the
nonlinear eigenvalue for buckling analysis of the beam. We verify the solv-
ability of the buckling problem as an asymptotic expansion in a ratio of
the elastoplastic parameters, that the spectrum is bounded away from zero
and contains a discrete infinite sequence of eigenvalues. We also verify, for
certain ranges of the lateral forces, the solvability of the general equations
using energy methods and a suitable iteration scheme.

1. Introduction

It is well-known from the materials science, physics, and chemistry litera-
ture, that there is intense interest in studying the mechanics of structures made
of graphene. Industrial applications and the potentials for graphene made
structures are abundant. For instance, nanoscale devices that use graphene
as basic components, such as nanoscale resonators, switches, and valves, are
being developed in many industries. Understanding the response of individual
graphene structure elements to applied loads is therefore crucially important
(see [1]-[8] and the references therein for a comprehensive list of applications).
In this paper, we analyze the effects of axial compression and nonlinear lateral
forces upon an idealized graphene beam. We prove the existence of a minimal
"buckling load," which, mathematically speaking, is not obvious due to the
structure of the constitutive law relating the stress and strain upon a beam
made of graphene. Furthermore, we prove the existence and uniqueness of so-
lutions for the equilibrium equation of the elastoplastic beam when the lateral
force satisfies a natural bound in terms of the elastoplastic parameters (and
we prove non-existence, in certain cases, when this bound is not satisfied).

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The Euler buckling load of simply supported straight elastics beam subject to an end axial compressive load can be modeled by the equation:

\[(1.1) \quad EIv'''' + Pv'' = 0, \quad 0 < x < L\]

with boundary conditions

\[(1.2) \quad v(0) = v(L) = v''(0) = v''(L)\]

where \(L\) is the length of the beam, the \(E\) Young’s modulus, and \(I\) the area moment of inertia. Integrating (1.1) twice gives

\[(1.3) \quad EIv'' + Pv = 0\]

when the boundary conditions are taken into account.

Therefore the boundary value problem (1.1)-(1.2) reduces to the well-known eigenvalue problem for the Laplacian in one dimension:

\[(1.4) \quad EIv'' + Pv = 0\]

\[(1.5) \quad v(0) = v(L) = 0\]

As is well known, system (1.4)-(1.5) yields a sequence of eigenvalues and eigenfunctions

\[v_k(x) = \sin(k\pi x), \quad P_k = EI\left(\frac{k\pi}{L}\right)^2, \quad k = 1, 2, 3, \ldots\]

Furthermore, each eigenfunction in (1.4)-(1.5) is simple. The eigenfunction \(v_1\) is called the first buckling mode and \(P_1\) is the well-known Euler critical buckling load, sometimes also called the onset buckling load. It is used widely in engineering practice.

The above Euler critical buckling load is derived based on the Hooke’s law, relating the axial stress \(\sigma_x\) and the axial strain \(\varepsilon_x\): \(\sigma_x = E\varepsilon_x\) and the assumption that during the deformation, the cross-sections of the beam column remains perpendicular to the center line. This classical result is generalized for the Hollomon’s law \(\sigma_x = K|\varepsilon_x|^{n-1}\varepsilon_x\), where equation (1.1) is replaced by:

\[(1.6) \quad (KI_n|v''|^{n-1}v''')'' + Pv'' = 0\]

\[(1.7) \quad v(0) = v(L) = v''(0) = v''(L)\]

The the critical load of (1.6)-(1.7) was found in [10], and is given by:

\[P_{cr} = \frac{2n(\pi_2,1+1/n)^2}{n+1}KI_n\]

where \(I_n = \int_A |y|^{1+n}dydz\) is the generalized area moment of inertia, and \(\pi_{2,1+1/n} = 2\int_0^\pi \cos(\theta)^{\frac{n-1}{n+1}}d\theta\). The first eigenfunction is defined in terms of generalized sine function by using the notation of the two parameter sine function developed in [11].
Graphene material is shown to be modeled by the following quadratic stress-strain constitutive law (see [3] and [8]):

\[ \sigma_x = E\epsilon_x + D|\epsilon_x|\epsilon_x, \]

where \( D \) is related to the Young’s modulus by the relation \( D = -\frac{E^2}{4\sigma_{\text{max}}} \), and \( \sigma_{\text{max}} \) is the material’s ultimate maximal shear stress.

For small strain, the elastic stress \( E\epsilon_x \) dominates (1.8), while the plastic stress \( D|\epsilon_x|\epsilon_x \) becomes prominent for large strain. Notice that the ratio \( \frac{|D|}{E} = \frac{\sigma_{\text{max}}}{4E} := \alpha \) is the elastoplastic parameter which we will use in our asymptotic analysis of Section 2. When this parameter is small then the material’s ultimate maximal shear stress \( \sigma_{\text{max}} \) is very large, and the elastic behavior dominates. The equilibrium equation for a graphene made Euler-Beam subject to axial compressive load \( P \), lateral force \( f \), and a nonlinear support \( g \) (all per unit length) is given by the fourth order equation:

\[ EIu'''' + DI_2(\|v''\|v''')'' + Pu'' + g(v'') = f(x), 0 < x < L, \]

where \( I = \int_A y^2dydz \), \( I_2 = \int_A y^3dydz \), and the \( z \)-axis being the off-plane direction and \( A \) is the cross sectional area of the bar. We consider (1.9) along with one of the pin-pin (PP), and pin-slide (PS), or the slide-slide (SS) boundary conditions:

(1.10) (PP Conditions) \( v(0) = v(L) = v''(0) = v''(L) \)

(1.11) (PS Conditions) \( v'(0) = v'(L) = v'''(0) = v'''(L) \)

(1.12) (SS Conditions) \( v'(0) = v'(L) = v''(0) = v''(L) \)

Using the non-dimensional variables and parameters:

\[ z = xL^{-1}, u = vL^{-1}, \alpha = \frac{|D|I_2}{EIL}, \lambda = \frac{PL^2}{EI}, \hat{g}(u'') = \frac{g(u'')}{EIL^{-3}}; \hat{f}(z) = \frac{f(z)}{EIL^{-3}}, \]

equation (1.9) can be rewritten as:

\[ u'''' - \alpha(\|u''\|u''')'' + \lambda u'' + \hat{g}(u'') = \hat{f}(z). \]

The boundary conditions (1.10)-(1.12) become:

(1.14) (PP Conditions) \( u(0) = u(1) = u''(0) = u''(1) \)

(1.15) (PS Conditions) \( u'(0) = u'(1) = u'''(0) = u'''(1) \)

(1.16) (SS Conditions) \( u'(0) = u'(1) = u''(0) = u''(1) \)

In the next section we will study a special case of (1.13):

\[ u'''' - \alpha(\|u''\|u''')'' + \lambda u'' = 0 \]
with the boundary condition
\begin{equation}
(1.18) \quad u(0) = u(1) = u''(0) = u''(1) = 0
\end{equation}
Here (1.17)-(1.18) represents the buckling problem for a Euler graphene beam which replaces problems (1.1)-(1.2) and (1.6)-(1.7) for the elastic and Hollomon beams, respectively.

In Section 2, we provide asymptotic expansion of the first eigen-pair of (1.17)-(1.18) in terms of the perturbation parameter \( \alpha \), and prove that, for small enough \( \alpha \), each eigen-pair is simple and continuously dependent upon \( \alpha \), and also establish existence of an infinite sequence of eigen-pairs of (1.17)-(1.18). In section 3, we show that all eigenvalues are positive and derive a lower bound for the smallest eigenvalues. In sections 4, we consider the global existence and uniqueness of solutions for the boundary value problem equations (1.13)-(1.14), for the case of PP boundary condition. Similar techniques are valid for the other boundary conditions. This way, we extend the results established in [9] and [11] for the graphene beam with nonlinear support.

2. Existence of Eigen-pairs and Buckling Analysis of the Graphene Beam

Integrating (1.17) twice, and applying the boundary conditions we obtain the nonlinear eigenvalue problem:
\begin{equation}
(2.1) \quad u'' - \alpha |u''|u'' + \lambda u = 0
\end{equation}
\begin{equation}
(2.2) \quad u(0) = u(1) = 0
\end{equation}
When \( \alpha = 0 \), (2.1)-(2.2) reduces to the eigenvalue problem for the Euler elastic beam:
\begin{equation}
(2.3) \quad u'' + \lambda u = 0
\end{equation}
\begin{equation}
(2.4) \quad u(0) = u(1) = 0
\end{equation}
whose eigenpairs are given by:
\begin{equation}
(2.5) \quad \lambda_k = (k\pi)^2, \quad u_k = \sin(\pi k \alpha), \quad k = 1, 2, 3, \ldots
\end{equation}
In particular this linear problem has a discrete spectrum and each eigen-value is simple.

Consider the nonlinear graphene operator defined by:
\[
N_G(\alpha, u) = u'' - \alpha |u''|u'' - \lambda u, \quad u \in H^2(0, 1) \cap H^1_0(0, 1)
\]
Ideally, we would like to prove that \( N_G \) has a discrete spectrum. The next proposition is a first step in this direction. We show that for each eigenvalue \( \lambda_k \) of the linear operator (the laplacian) there exists a continuously differentiable
curve of eigenvalues to $N_G(\alpha, \cdot)$, for small $\alpha$. The proof of these facts is based on the implicit function theorem as demonstrated below.

**Proposition 2.1.** For each eigenpair $(u_1, \lambda_1)$ of (2.2), there exists $\alpha_0$ small so that there exists a unique smooth curve $(u(\alpha), \lambda(\alpha))$ of eigenpairs of $N_G(\alpha, u)$ defined for $\alpha \leq \alpha_0$ such that $\lambda(0) = \lambda_1$ and $u(0) = u_1$.

**Proof** Define $F : \mathbb{R} \times H^2 \cap H^1_0 \times \mathbb{R} \to L^2 \times \mathbb{R}$ in the following way: $F(\alpha, u, \lambda) = (u'' - \alpha |u''|)u'' + \lambda u, < u', u' > - < u_1', u_1' >)$

$F$ is obviously continuously differentiable and $F(0, u_1, \lambda_1) = (0, 0)$. We seek to prove that $F_{u,\lambda}(0,u_1,\lambda_1) = \begin{bmatrix} cc(c)'' + \lambda_1(\cdot) & u_1 \\ < u_1', (\cdot)' > & 0 \end{bmatrix}$ is invertible.

Now, if $F_{u,\lambda}(0,u_1,\lambda_1) \begin{bmatrix} ccu \\ \lambda \end{bmatrix} = 0$ then $u$ and $\lambda$ have to satisfy the following system:

$$u'' + \lambda_1 u + \lambda u_1 = 0$$

and

$$< u_1', u' > = 0.$$ 

Now, multiply the first equation by $u_1$, integrate from 0 to 1, and integrate by parts in the first term. Using the fact that $u_1'' + \lambda_1 u_1 = 0$, we get: $\lambda \int_0^1 u_1^2 dx = 0$ so that $\lambda = 0$. Then the first equation becomes

$$u'' + \lambda_1 u = 0.$$ 

But since $< u_1', u' > = 0$ and since $u_1$ is simple, $u \equiv 0$. The theorem then follows from the implicit function theorem in Banach spaces.

We now seek to find an asymptotic expansion of the solution of (2.1) in powers of $\alpha$. The zeroth order boundary value problem is (2.3)-(2.4) whose solution is given by (2.5). The first order equation then reads:

$$u_2'' + \lambda_1 u_2 = u_1'' - \lambda_2 u_1$$

$$u_2(0) = u_2(1) = 0$$

whose solvability condition gives:

$$\lambda_2 = -\frac{\int_0^1 |u_0'|^2 dz}{\int_0^1 |u_0|^2 dz}$$

In this way we obtain an asymptotic expansion:

$$u(z) = u_1(z) + \alpha u_2(z) + O(\alpha^2)$$

$$\lambda = \lambda_1 + \alpha \lambda_2 + O(\alpha^2)$$

valid for small enough $\alpha$, where $u_2$ is the unique solution of the first order problem above. We denote be $X$ the Banach space $H^2(0,1)$ and define $K : X \to X$
by $K(u)(v) = \int_0^1 u'v' \, dx$. Consider the existence of solutions of (2.1),(2.2) with the following additional constraint:

$$
\gamma = \langle u, w \rangle_X, \forall w \in K^{-1}(u)
$$

We now prove the following theorem:

**Theorem 2.2.** For any given $\gamma > 0$, there exists a real number $\lambda \in \mathbb{R}$, and a weak solution $w \in X$ of (2.1)-(2.2) satisfying (2.6). Furthermore, there exist infinitely many distinct eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ for which $\lambda_i \to \infty$, as $i \to \infty$.

**Proof.** We define $F : X \to X$ by defining $F(u) = u - \alpha |u|u$, and $\phi(u) = \frac{1}{2}u^2 - \frac{2}{3}|u|^3$. The proof follows by observing that the operators $K$, $F$, and $\phi$ satisfy the properties stated in the following theorem of (Amann, 1972) and that the eigenvalue problem (2.1) and (2.2) can be written in the operator form:

$$
\lambda w = KF(w), w \in X
$$

$$
\gamma = \langle u, w \rangle_X, \forall w \in K^{-1}(u)
$$

3. **Positivity of Eigenvalues and Lower Bound on the First Buckling Load**

In the following two theorems we present a bound on the maximum norm of an eigenfunction of the eigenvalue problem (1.17)-(1.18) and prove that all the eigenvalues are positive. In particular, we study the following eigenvalue/eigenfunction problem (by calling $v = u''$ in (1.17)):

$$
(v - \alpha v|v|)'' + \lambda v = 0
$$

$$
v(0) = v(1) = 0
$$

It is not obvious that all eigenvalues of (3.1)-(3.2) are positive. Indeed, one may imagine that the eigenvalues should change sign. On the other hand,
physically, it makes sense that there be a minimal positive eigenvalue—a so-called buckling load. In this section we show that all eigenvalues of (3.1)-(3.2) are positive and bounded below by a constant (depending on \( \alpha \)) and we give a-priori estimates on the eigenfunctions in the \( L^\infty \) norm.

One technicality which gives us a little bit of trouble is that an eigenfunction \( v \) is not necessarily smooth in \((0,1)\). In fact, if it has interior zeros it cannot be smoother than of class \( W^{2,\infty} \) near those zeros (due to the presence of the \(|v|\) in our equation). Nonetheless, away from the zeros of a continuous eigenfunction, and away from points where \(|v| = \frac{1}{2\alpha}\), the eigenfunction must be smooth. This can be proved using the same techniques as are used in the regularity part of the proof of theorem 4.1 of section 4.

**Proposition 3.1.** For every given \( \alpha \), let \( \lambda \) be an eigenvalue of (3.1)-(3.2). Then \( \lambda > 0 \).

**Proof.** It is obvious that \( \lambda \neq 0 \). Now let \((\lambda, v)\) be an eigenpair of (2.7) and (2.8). Then

\[
(v - \alpha(|v|))'' = -\lambda v, \quad 0 < z < 1
\]

and \( v(0) = v(1) = 0 \). Multiply both sides of the differential equation by \((v - \alpha(|v|))'\) and integrate, we get:

\[
\frac{1}{2}[(v - \alpha(|v|))^2] + \lambda[\frac{1}{2}v^2 - \frac{2\alpha}{3}|v|^3] = A,
\]

where \( A \) is an integration constant.

Suppose that \( \exists c \in (0,1) \), such that \(|v(c)| = \frac{1}{2\alpha}\). Then, upon evaluating at \( x = 0 \) and \( x = c \), we get \( \frac{1}{2}[v'(0)]^2 = A = \frac{1}{24\alpha^2}\lambda \), which gives \( \lambda \geq 0 \). However, since \( \lambda = 0 \) gives \( v = 0 \), we conclude that, in this case, we must have \( \lambda > 0 \).

If there is no number \( c \) satisfying \(|v(c)| = \frac{1}{2\alpha}, c \in [0,1]\), then, it must be that \(|v| < \frac{1}{2\alpha}\). We then multiply both sides of (3.1) by \( v \) and integrate over \([0,1]\). Upon integrating by parts we see

\[
\int_0^1 v^2(1 - 2\alpha|v|)dx = \lambda \int_0^1 v^2dx
\]

which leads again to the conclusion that \( \lambda > 0 \). This completes the proof of the proposition. \( \square \)

**Theorem 3.2.** Let \( v \) be a continuous eigenfunction of (3.1)-(3.2) corresponding to an eigenvalue \( \lambda \geq 0 \). Then \(|v| \leq \frac{1}{2\alpha} \) on \([0,1]\).

**Proof.** Suppose that the conclusion of the theorem is false. Then without loss of generality we may assume that \( v \) has a maximum at \( x = c \) and that \( v(c) > \frac{1}{2\alpha} \). Note that \( v \) must be smooth in a neighborhood of \( c \) by elliptic regularity (one may mimic the regularity proof in the next section). Now, because \( v \) has a local maximum at \( c \), \( v''(c) \leq 0 \) and \( v'(c) = 0 \). Expanding
Theorem 3.3. There exists an absolute constant $c$ so that if $\lambda$ is an eigenvalue of (3.1)-(3.2) then $\lambda \geq c$. $c$ can be taken to be $\frac{1}{4}$.

Proof: 
Set $f = -\lambda v$. Then $v$ solves

$$(v - \alpha v|v|)^{''} = f, u(0) = u(1) = 0$$

Integrating by parts twice, we see that

$$v - \alpha v|v| = F,$$

where $F$ solves $F^{''} = f, F(0) = F(1) = 0$.

We want to see now that if $F$ is small then $u$ is small. Indeed, if $|F|_{L^\infty} \leq \frac{1}{8\alpha}$, then $|v|_{L^\infty} \leq \frac{1}{4\alpha}$ (just by solving the equation $v - \alpha v|v| = F$).

Since we know a-priori that $|v| \leq \frac{1}{2\alpha}$, if we take $\lambda \leq \frac{1}{4}$, then

$$|F|_{L^\infty} \leq |f|_{L^2} \leq |f|_{L^\infty} \leq \frac{\lambda}{2\alpha} \leq \frac{1}{8\alpha}.$$ 

Therefore, we have shown that if $\lambda \leq \frac{1}{4}$, then $|v| \leq \frac{1}{4\alpha}$.

Now assume $\lambda \leq \frac{1}{4}$ and multiply (3.1) by $v$ and integrate from 0 to 1. Then, integration by parts tells us:

$$\int_{0}^{1} |v'|^2(1 - 2\alpha|v|) = \lambda \int_{0}^{1} |v|^2.$$ 

But now that $|v| \leq \frac{1}{4\alpha}$,

$$\int_{0}^{1} |v'|^2 = 4\lambda \int_{0}^{1} |v|^2.$$ 

So, if $4\lambda < \pi^2$, we have that $v \equiv 0$. That is, if $\lambda < \frac{\pi^2}{4}$ then $v \equiv 0$. But remember that we had to assume that $\lambda \leq \frac{1}{4}$ to get this. Therefore, the theorem is proved with $c = \frac{1}{4}$. □

This Theorem gives us a lower bound on the first buckling load for the graphene beam as

$$P_{cr} \geq \frac{I^2 \sigma_{max}}{I_2 L}$$
4. Existence and Uniqueness of the Beam with Nonlinear Support

In this section we want to prove existence and uniqueness of solutions for the elastic beam equations with compression below the first buckling load, with a nonlinear foundational support, and subject to a mild external force. In the next section we will show that the conditions we assume to prove existence and uniqueness are more or less optimal. Consider the following non-linear elliptic boundary value problem:

\[
(4.1) \quad ((1 - 2\alpha |v|)v')' + \lambda v + g(v) = f, \quad \text{in } (0, 1)
\]

\[
(4.2) \quad v(0) = v(1) = 0
\]

with \( \alpha \geq 0, \lambda < \frac{\pi^2}{2}, \) and \( f \) is a bounded function. Furthermore, \( g \) is a differentiable function which is homogeneous of degree 2 or more and satisfies the following inequalities:

\[
tg(t) \leq 0, \quad g'(t) \leq 0 \quad \text{for all } t.
\]

The main result of this section is that if \( f \) is small enough in \( L^2(0, 1) \), then \( (4.1)-(4.2) \) has a unique \( H^2 \) solution. Moreover, we show by example, that our result is in some sense optimal: if \( f \) is positive and large enough then no solution exists. We prove the uniqueness before we prove existence.

We prove that if we have two solutions of \( (4.1)-(4.2) \) which are both small enough then the two solutions must coincide. Define the following classes of functions:

\[
B_\delta \equiv \{ k \in W^{1,\infty}(0, 1) : |k|_{W^{1,\infty}} \leq \delta \}
\]

**Proposition 4.1.** If \( \delta < \frac{(1+\frac{1}{\alpha})}{2\alpha} \), then \( (4.1)-(4.2) \) has at most one weak solution in \( B_\delta \).

**Proof:** Suppose that \( v_1, v_2 \in B_\delta \) solve \( (1.1)-(1.2) \).

Then, \( v = v_1 - v_2 \) satisfies the following equation:

\[
v'' + \lambda v + g(v_1) - g(v_2) = 2\alpha[(|v_1| - |v_2|)|v_1'| - (|v_1|v_1')].
\]

Now multiply by \( u \) and integrate by parts. Since \( v = 0 \) at 0 and 1, all the boundary terms vanish and we get:

\[
(4.3) \quad \int_0^1 |v'|^2 + |\lambda| \int_0^1 |v|^2 - \int_0^1 (g(v_1) - g(v_2))(v_1 - v_2) \\
\leq 2\alpha \int_0^1 |v||v_1'||v'| + |v_1||v'|^2
\]

where we used

\[
||v_1| - |v_2|| \leq |v_1 - v_2|
\]
Now, because $g' \leq 0$, we have that $(g(v_1) - g(v_2))(v_1 - v_2) \leq 0$, so we can drop the last term on the left hand side of (4.3).

By the Poincaré inequality, we have $|v|_{L^2} \leq \frac{1}{\pi} |v'|_{L^2}$. This and using the fact that $|v_1|, |v_1'| \leq \delta$, we see that

$$|v'|_{L^2} \leq 2\alpha \delta (1 + \frac{1}{\pi}) |v'|_{L^2}$$

Therefore, if $\delta < (1 + \frac{1}{\pi})^{-1} \frac{1}{2\alpha}$, then $v' \equiv 0$ and, using the boundary condition, the uniqueness theorem is proven. \qed

The proof of existence will rely upon energy estimates and a suitable iteration scheme.

We will begin by proving the existence of a small solution in $H^1_0$ under a suitable condition on $f$.

First of all, in (4.1)-(4.2), we write $v = \frac{1}{2\alpha} w$ and $F = 2\alpha f$. Then we get that $v$ is a solution of (4.1)-(4.2) if and only if $w$ is a solution of:

(4.4) $$((1 - |w|)w')' + \lambda w + 2\alpha g\left(\frac{1}{2\alpha} w\right) = F, \text{ in } (0,1)$$

(4.5) $$w(0) = w(1) = 0.$$

Recall that $\lambda < \frac{\pi^2}{2}$ and $G(\cdot) := 2\alpha g(\frac{1}{2\alpha} \cdot)$ satisfies the same conditions as $g$.

The main idea we want to use is that if $F$ is smooth and small enough, then, using the maximum principle, $w$ must also be small. Once $w$ is small, the equation becomes uniformly elliptic and we will then be able to deduce the existence and uniqueness of a small solution. We now prove existence of an $H^1$ weak solution.

**Proposition 4.2.** Let $h$ be a bounded, measurable function with $|h| \leq \frac{1}{2}$. Then if $w$ solves the following semi-linear boundary-value problem

(4.6) $$((1 - |h|)w')' + \lambda w + G(w) = F, \text{ in } (0,1)$$

(4.7) $$w(0) = w(1) = 0,$$

with $tG(t) \leq 0$, for all $t$. Assume further that $\lambda < \frac{\pi^2}{2}$. Then

$$|w'|_{L^2} \leq \frac{1}{\pi(\frac{1}{2} - \frac{1}{\pi^2})} |F|_{L^2}$$

**Proof:**

Multiply (4.6) by $w$ and integrate from 0 to 1. Upon integrating by parts we see

(4.8) $$\int_0^1 (1 - |h|)|w'|^2 dz - \lambda \int_0^1 w^2 dz - \int_0^1 G(w)wdz = -\int_0^1 Fwdz$$
Using the condition on $h$ and that $tG(t) \leq 0$, we see that
\[
\frac{1}{2} \int_0^1 |w'|^2 dx \leq | \int_0^1 Fw dx | + \lambda \int_0^1 |w|^2 dx.
\]
Now, using the best constant in the Poincaré inequality on $[0, 1]$, we know that
\[
\int_0^1 |w|^2 dz \leq \frac{1}{\pi^2} \int_0^1 |w'|^2 dz
\]
This implies that
\[
\frac{1}{2} \int_0^1 |w'|^2 dz \leq | \int_0^1 Fw dz | + \frac{\lambda}{\pi^2} \int_0^1 |w'|^2 dz
\]
Since, by assumption, $\lambda < \frac{\pi^2}{2}$,
\[
\left(\frac{1}{2} - \frac{\lambda}{\pi^2}\right) \int_0^1 |w'|^2 dz \leq | \int_0^1 Fudz |
\]
Using the Cauchy-Schwarz inequality and the Poincaré inequality once more we see:
\[
|w'|_{L^2} \leq \frac{1}{\pi\left(\frac{1}{2} - \frac{\lambda}{\pi^2}\right)} |F|_{L^2}
\]
We now need the fact that $H^1_0$ is imbedded in $L^\infty$ in dimension one:
\[
|f|_{L^\infty} \leq |f'|_{L^2}
\]
for all $f \in H^1_0(0, 1)$. This is just a consequence of the Cauchy-Schwarz inequality. Therefore,
\[
|w|_{L^\infty} \leq \frac{1}{\pi\left(\frac{1}{2} - \frac{\lambda}{\pi^2}\right)} |F|_{L^2}
\]
One may also try to prove this proposition using the maximum principle by seeing that the condition on $h$ implies that the ellipticity constant of our equation is $1 - |h| \geq \frac{1}{2}$. However we wanted a simple way to get exact constants in our bounds. Now assume $|F|_{L^2} \leq \frac{\pi\left(\frac{1}{2} - \frac{\lambda}{\pi^2}\right)}{2}$ and define the following sequence of functions $w_n$:
\[
w_0 = 0, ((1 - |w_{n-1}|)w_n')' + w_n + G(u_n) = F, \text{ in } (0, 1)
\]
\[
w_n(0) = w_n(1) = 0
\]
Using the theory of semi-linear elliptic equations in one dimension, we see that the sequence $w_n$ can be defined for all $n$ (see, for example, [9]). Moreover, by Proposition 4.2, $|w_n| \leq \frac{1}{\pi\left(\frac{1}{2} - \frac{\lambda}{\pi^2}\right)} |F|_{L^2}$ and $|w_n|_{H^1} \leq \frac{2}{\pi\left(\frac{1}{2} - \frac{\lambda}{\pi^2}\right)} |F|_{L^2}$ for all $n$.

Thus the sequence $w_n$ is uniformly bounded in $H^1 \cap L^\infty$. Thus we may extract a subsequence of $w_n$ which converges weakly in $H^1$, strongly in $L^p$, for
some $p > 2$ and pointwise to a function $w \in H^1 \cap L^\infty$. Moreover, $|w|_{H^1} \leq \frac{1}{\pi(\frac{1}{2} - \frac{1}{p})} |F|_{L^2}$ and $|w|_{L^\infty} \leq \frac{1}{\pi(\frac{1}{2} - \frac{1}{p})} |F|_{L^\infty}$. Therefore, $w$ is a bounded weak solution of (4.6)-(4.7).

We now want to show that $w$, in fact, belongs to $H^2$ with a certain smallness estimate. We aim to show that $w \in H^2(0, 1)$ with an appropriate bound. We will first show that $w - \frac{|w|w}{2} \in H^2(0, 1)$. Notice that $(w - \frac{|w|w}{2})' = ((1 - |w|)w)'$. Therefore, we can write or equation as

$$\left(w - \frac{|w|w}{2}\right)'' = H$$

where $H$ is an $L^2$ function ($H = F - \lambda w - G(w)$). Using standard elliptic theory, $w - \frac{|w|w}{2} \in H^2(0, 1)$.

Call $v = w - \frac{|w|w}{2}$. Define the function $\Phi$ with $\Phi(x) = x - \frac{1}{2}x|x|$. Now, $\Phi$ is not invertible on the whole real line. However, it is invertible on $|x| \leq \frac{1}{2}$. From Theorem 3.2, we have $|w| \leq \frac{1}{2}$. Then we see, upon integration by parts in the second and third term s,

$$\int_0^1 |(1 - |w|)w'|^2dz - \lambda \int_0^1 |w'|^2(1 - |w|)dz + \int_0^1 G'(w)w'|^2(1 - |w|)dz$$

$$= \int_0^1 F((1 - |w|)w')dz$$

Therefore,

$$\int_0^1 |(w - \frac{|w|w}{2})''|^2dz - \lambda \int_0^1 |w'|^2(1 - |w|)dz - \int_0^1 G'(w)|w'|^2(1 - |w|)dz$$

$$= \int_0^1 F(w - \frac{|w|w}{2})''dz$$

Using the Cauchy-Schwarz inequality,

$$\int_0^1 |(w - \frac{|w|w}{2})''|^2dz - \lambda \int_0^1 |w'|^2dz \leq \frac{1}{2} \int_0^1 F^2 + |(w - \frac{|w|w}{2})''|^2dz$$

$$\int_0^1 |(w - \frac{|w|w}{2})''|^2dz \leq \int_0^1 F^2dz + 2\lambda \int_0^1 |w'|^2dz$$
Now recall that \( \lambda \leq \frac{\pi}{2} \) and \( |w''|_{L^2} \leq (\frac{1}{\pi(\frac{1}{2}-\frac{\lambda}{\pi^2})})^2 |F|_{L^2}^2 \). Therefore,

\[
\int_0^1 |(w - \frac{|w|}{2})''|^2 dz \leq (1 + (\frac{1}{(\frac{1}{2} - \frac{\lambda}{\pi^2})})^2) \int_0^1 F^2 dz
\]

So,

\[
\int_0^1 |v''| dz \leq (1 + (\frac{1}{(\frac{1}{2} - \frac{\lambda}{\pi^2})})^2) \int_0^1 F^2 dz
\]

and \( w = \Psi(v) \). By simple calculations, \( |w''|_{L^2} \leq 2(|v''|_{L^2} + |v''|_{L^2}) \).

Therefore, \( |w|_{H^2} \leq 2\sqrt{(1 + (\frac{1}{(\frac{1}{2} - \frac{\lambda}{\pi^2})})^2)|F|_{L^2}^2 + (1 + (\frac{1}{(\frac{1}{2} - \frac{\lambda}{\pi^2})})^2)|F|_{L^2}^2} \)

Thus we have proven the following theorem.

**Theorem 4.3.** Let \( \alpha > 0 \) be given. Suppose that \( g \) is a continuous function on the real line which is homogeneous of degree 2 or more. Suppose further that \( tg(t) \leq 0 \) and \( g'(t) \leq 0 \) for all \( t \). Suppose \( \lambda \leq \frac{\pi}{2} \). Then there exists \( c_1 > 0 \) small (explicitly given below) so that if \( f \) is a measurable \( L^2 \) function on \([0, 1]\) with \( |f|_{L^2} \leq \frac{c}{\alpha} \), then the following non-linear boundary-value problem has a unique solution belonging to \( H^2 \).

\begin{align}
(4.9) & \quad ((1 - 2\alpha|v|)v')' + \lambda v + g(v) = f \\
(4.10) & \quad v(0) = v(1) = 0.
\end{align}

Moreover, there exists a constant \( c_2 \) so that \( |v|_{H^2} \leq \frac{c_1}{c_2} |f|_{L^2} \). Here, \( c_1 = \frac{\pi(\frac{1}{2} - \frac{\lambda}{\pi^2})}{2} \), and \( c_2 = 4\sqrt{(1 + (\frac{1}{(\frac{1}{2} - \frac{\lambda}{\pi^2})})^2)} \).

5. **Nonexistence for Large External Force**

**Proposition 5.1.** Consider the system (4.9)-(4.10). Take \( \lambda = 0 \) and \( g = 0 \). Then there exists a universal constant \( c_3 > 0 \) so that if we take \( f = \frac{c}{\alpha} \), for \( c > c_3 \), then there exists no solution to (4.9) – (4.10).

**Proof**

(4.9)-(4.10) reduces to

\[
(1 - \alpha|v|)v'' = \frac{c}{\alpha},
\]

\[
v(0) = v(1) = 0.
\]

Integrating twice and using the boundary condition yields

\[
v - \alpha|v|v = \frac{c}{\alpha} z(z - 1)
\]
Factoring we get:
\[ v(1 - \alpha |v|) = \frac{c}{\alpha} z(z - 1). \]

Since the right hand side is never zero in \((0,1)\), the left hand side can never be zero either. Therefore, \(v\) is either positive or negative in \((0,1)\). Moreover, since \(v(0) = 0, (1 - 2\alpha |v|) > 0\) for \(z\) close to 0. The right hand side is negative in \((0,1)\). Therefore \(v\) is negative for \(z > 0\) small. Therefore \(v\) is negative in the entire interval \((0,1)\). Therefore,
\[ v + \alpha v^2 = \frac{c}{\alpha} z(z - 1) \]

Take \(z = \frac{1}{2}\). Then,
\[ v\left(\frac{1}{2}\right) + \alpha v^2\left(\frac{1}{2}\right) = -\frac{1}{4} \cdot \frac{c}{\alpha} \]

So, if \(c > 1\), we see that the discriminant of this equation is negative so that no solutions exist. \(\square\).

Note that in the case that \(\lambda = 0, c_1 = \frac{\pi}{8} \alpha < 2\) so that if the external force is less than \(\frac{\pi}{8} \alpha\) in \(L^2\), then we have existence and uniqueness of an \(H^2\) solution. Moreover, we have an example of an external force larger than \(\frac{1}{4} \alpha\) in \(L^2\) so that there exists no \(H^1\) solution to (4.9)-(4.10). Thus our result in theorem 3.3 is, essentially, optimal both in the mathematical and physical sense. Physically, this says that for a small enough lateral force we have a smooth deformation, but for a large lateral force—'small' and 'large' being determined by the basic physical constants in the system such as the maximal strain \(\sigma_{\text{max}}\), there is no smooth deformation.

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