SOV approach for integrable quantum models associated with general representations on spin-1/2 chains of the 8-vertex reflection algebra

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Abstract
The analysis of the transfer matrices associated with the most general representations of the 8-vertex reflection algebra on spin-1/2 chains is here implemented by introducing a quantum separation of variables (SOV) method, which generalizes to these integrable quantum models the method first introduced by Sklyanin. For representations reproducing in their homogeneous limits the open XYZ spin-1/2 quantum chains with the most general integrable boundary conditions, we explicitly construct representations of the 8-vertex reflection algebras, for which the transfer matrix spectral problem is separated. Then, in these SOV representations we get the complete characterization of the transfer matrix spectrum (eigenvalues and eigenstates) and its non-degeneracy. Moreover, we present the first fundamental step toward the characterization of the dynamics of these models by deriving determinant formulae for the matrix elements of the identity on separated states, which particularly apply to transfer matrix eigenstates. A comparison of our analysis of the 8-vertex reflection algebra with that of (Niccoli G 2012 J. Stat. Mech. P10025, Faldella S et al 2014 J. Stat. Mech. P01011) for the 6-vertex leads to an interesting remark in that there is a profound similarity in both the characterization of the spectral problems and the scalar products, which exists for these two different realizations of the reflection algebra once they are described by the SOV method. As will be shown in a future publication, this remarkable similarity will be the basis of a simultaneous determination of the form factors of...
local operators of integrable quantum models associated with general reflection algebra representations of both 8-vertex and 6-vertex type.

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1. Introduction

In the framework of the quantum inverse scattering method (QISM) [3–15], we analyze the class of one-dimensional (1D) lattice integrable quantum models associated with monodromy matrices, which are the most general solutions of the reflection algebra [16–21] w.r.t. the elliptic 8-vertex R-matrix. It is worth commenting that these models have attracted a large interest that goes beyond the community of quantum integrability. This is in particular true for representations associated with non-diagonal integrable boundary matrices, which have proven to be hard to describe by standard Bethe ansatz analysis [22–29] and which allow us to describe interesting out of equilibrium physical systems. In the 6-vertex case, a very large literature has already been developed to address the different methods of the associated transfer matrix spectral problems3 [30–46]. The homogeneous limit of the 8-vertex reflection algebra representations that we analyze in this paper leads to the description of open XYZ spin-1/2 quantum chains with the most general integrable boundary conditions. For these integrable quantum models, we introduce a quantum version of the separation of variables (SOV) in the spirit of the works [47] pioneered by Sklyanin. In our SOV approach we both obtain the complete characterization of the transfer matrix spectrum (eigenvalues and eigenstates) and derive simple determinant formulae for the scalar products of transfer matrix eigenstates. In particular, starting from the original spin-1/2 representations of the 8-vertex reflection algebra, we explicitly construct a new (SOV) basis of the space of the representations, for which the transfer matrix spectral problem is separated and completely characterized in terms of the set of solutions to a inhomogeneous system of \( N \) quadratic equations in \( N \) unknowns, where \( N \) is the size of the chain. It is also worth remarking that, in our SOV approach, it is simple to prove the complete integrability\(^4\) of the associated quantum models; that is, the fact that the transfer matrix forms a complete set of commuting conserved charges on the space of the representation. One fundamental finding of the SOV analysis developed here is that the pseudo-measure entering in the SOV spectral decomposition of the identity is simply expressed as the inverse of determinants of \( N \times N \) Vandermonde’s matrices. It is, therefore, essential to observe that all of the SOV representations constructed so far in [51–53], which are associated with 6-vertex representations of the Yang–Baxter and reflection algebra, share the same structure of inverse of Vandermonde’s determinant for these pseudo-measures. This observation is even more important once we point out that the SOV spectral decompositions of the identity have a different structure of the pseudo-measures in the case of the transfer matrices associated with both the 8-vertex representation of the Yang–Baxter algebra [57] and the elliptic representations of the 6-vertex dynamical Yang–Baxter algebra [58]. Indeed, in [56] and [59] a different determinant form has been derived for these pseudo-measures. It is, therefore, the combined use of the SOV method and of the reflection algebra that allows an amazing simultaneous description of the spectral and dynamical problems for the 8-vertex and

3 See the papers [1, 2] for a discussion of this point and for more details on the role of the cited references.

4 This definition can be seen as the natural quantum analogous of the classical Liouville complete integrability and was shown in the SOV framework for a series of other integrable quantum models [48–56].
6-vertex transfer matrices, which will be used in future publications to solve simultaneously these dynamical problems.

Let us resume here some results and difficulties appearing in the preexisting literature on the analysis of these 8-vertex integrable quantum models, which will also clarify the reasons of interest and novelty in our SOV analysis. In [57] Baxter has defined the intertwining vectors or gauge transformations in order to be able to use Bethe ansatz techniques to analyze the spectral problem of the transfer matrix associated with 8-vertex Yang–Baxter algebra representations. The use of gauge transformations allows us, in particular, to define pseudo-reference states, thereby opening the possibility to analyze these 8-vertex spectral problems by using the algebraic Bethe ansatz (ABA) [3, 4] as derived in [5]. Baxter’s gauge transformations have also been used in [64] to analyze the spectral problem associated with the 8-vertex reflection algebra in the ABA framework, see also [30] for the 6-vertex case. It is important to remark that in the framework of Bethe ansatz the general problem related to the proof of the completeness of the spectrum description persists and that constrains are required to implement the spectral analysis of these models. In the case of 8-vertex transfer matrices associated with periodic boundary conditions, the following two constrains are introduced: the number of sites of the quantum chains has to be even and the values allowed of the coupling constant $\eta$ are restricted to the elliptic roots of unit. These two constrains do not appear in the description by ABA of the 8-vertex transfer matrices associated with open boundary conditions, which already in this ABA framework reflects a simplification occurring when we consider 8-vertex reflection algebras. However, to make ABA work, it is necessary to introduce constrains between the boundary parameters, as done in [64]; that is, the 8-vertex transfer matrix spectral problems associated with the most general representations of the reflection algebra cannot be analyzed by using ABA. As mentioned above, all of these problems are overcome in our SOV framework and it is, therefore, possible to describe the complete 8-vertex spectrum for all closed [56, 59] and open integrable boundary conditions. Apart from the constrains for the spectral analysis, one central difficulty in the ABA framework is the solution of the dynamical problem. Indeed, in this 8-vertex framework, a scalar product analogue to the 6-vertex Slavnov’s formula [70] is missing for both closed and open boundary conditions. In fact, this is the first fundamental missing step toward the computation of correlation functions according to the Lyon group method developed in [71–75] for the 6-vertex transfer matrix associated with the Yang–Baxter algebra representations and generalized to some classes of 6-vertex reflection algebra in [78–80]. The need to overcome these problems in order to compute matrix elements of local operators on 8-vertex transfer matrix eigenstates is clear, and this leads to the importance of the results derived in the SOV approach, both here for the reflection algebra case and in [56, 59] for the Yang–Baxter algebra case.

2. Reflection algebra

In the framework of the QISM, a class of quantum integrable models characterized by monodromy matrices solutions of the 8-vertex elliptic reflection equations is introduced here.

5 Under periodic boundary conditions, the spectral problems of these transfer matrices have been analyzed. They have also been analyzed by Baxter’s Q-operator techniques, see [57, 60] and also see the series of papers [61–63].

6 The numerical analysis developed in [65] provides some evidence of the completeness of the spectrum description for the periodic 8-vertex transfer matrix.

7 When some special type of double constrains on the boundary parameters are satisfied, some steps in this direction have been done for both 6-vertex and 8-vertex case in [66, 67] and some related analysis appear also in [68, 69].

8 It is worth referring to the following pioneering works [76, 77] developed in a different methodological framework for the computation of correlation functions for semi-infinite open XXZ chains under the same type of boundary conditions.
2.1. Representations of 8-vertex reflection algebra on spin-1/2 chains

Let us start by introducing the following $2 \times 2$ matrix\(^9\) [81]:

\[
K(\lambda; \zeta, \kappa, \tau) = h(\lambda; \zeta) \frac{\tan(\lambda + \zeta)}{\sin(\zeta)} \left( \begin{array}{cc}
    \csc^2 \lambda - \sin^2 \lambda & \frac{k \csc^2 \lambda}{1 - k^2 \csc^2 \lambda} \\
    \frac{k \csc^2 \lambda}{1 - k^2 \csc^2 \lambda} & \csc^2 \lambda - \sin^2 \lambda
\end{array} \right)
\]  

(2.1)

where:

\[
h(\lambda; \zeta) = \theta_4(\lambda + \zeta) \theta_4(\lambda - \zeta) \theta_4(2\lambda), \quad \lambda \equiv 2K_\kappa \lambda, \quad \zeta \equiv 2K_\kappa \eta, \quad \tilde{\zeta} \equiv 2K_\kappa \zeta
\]  

(2.3)

and:

\[
\begin{aligned}
sn \lesssim & = \frac{1}{\sqrt{k} \theta_4(\lambda)} \\
\csc \lesssim & = \sqrt{k} \frac{\theta_2(\lambda)}{\theta_4(\lambda)} \\
dn \lesssim & = \sqrt{k} \frac{\theta_2(\lambda)}{\theta_4(\lambda)}
\end{aligned}
\]  

(2.4)

\[
k \equiv \frac{\theta_2^2(0)}{\theta_4^2(0)}, \quad k' \equiv \frac{\theta_2^2(0)}{\theta_4^2(0)}, \quad k^2 + k'^2 = 1, \quad K_\kappa \equiv \frac{\theta_2^2(0)}{2}.
\]  

(2.5)

Here $\zeta, \kappa$ and $\tau$ are arbitrary complex parameters and $K(\lambda; \zeta, \kappa, \tau)$ is the most general scalar solution\(^10\) the following 8-vertex reflection equation:

\[
R'^B_{12}(\lambda - \mu) K_1(\lambda) R'^B_{12}(\lambda + \mu) K_2(\mu) = K_2(\mu) R'^B_{12}(\lambda + \mu) K_1(\lambda) R'^B_{12}(\lambda - \mu),
\]  

(2.6)

where:

\[
R'^B_{10}(\lambda) = \begin{pmatrix}
    a(\lambda) & 0 & 0 & d(\lambda) \\
    0 & b(\lambda) & c(\lambda) & 0 \\
    0 & c(\lambda) & b(\lambda) & 0 \\
    d(\lambda) & 0 & 0 & a(\lambda)
\end{pmatrix}
\]  

(2.7)

is the elliptic solution of the 8-vertex Yang–Baxter equation:

\[
R'^B_{12}(\lambda_{12}) R'^B_{1a}(\lambda_1) K_{a2}(\lambda_2) = R'^B_{2a}(\lambda_2) K_{a1}(\lambda_1) R'^B_{21}(\lambda_{12}),
\]  

(2.8)

$R_1 \simeq \mathbb{C}^2$ is a 2-dimensional linear space and:

\[
\begin{aligned}
a(\lambda) = & \frac{2\theta_4(\eta) \theta_4(\lambda + \eta)}{\theta_2(0) \theta_4(0) \theta_4(2\lambda)}, \quad b(\lambda) = \frac{2\theta_4(\eta) \theta_4(\lambda + \eta)}{\theta_2(0) \theta_4(0) \theta_4(2\lambda)}, \\
c(\lambda) = & \frac{2\theta_4(\eta) \theta_4(\lambda + \eta)}{\theta_2(0) \theta_4(0) \theta_4(2\lambda)}, \quad d(\lambda) = \frac{2\theta_4(\eta) \theta_4(\lambda + \eta)}{\theta_2(0) \theta_4(0) \theta_4(2\lambda)}
\end{aligned}
\]  

(2.9)

(2.10)

\(^9\) Early studies of the open boundary conditions for the 8-vertex reflection algebra and the associated XYZ open spin chain were presented in [82].

\(^{10}\) The theta functions that are used here are those defined in [83], with the following change of notation in their arguments $(\lambda, \eta)$ instead of $(u, v)$. This analysis, both in the 6-vertex and in the 8-vertex case, was first developed in [84] where only the most general solution for the 6-vertex case was found while the most general solution for the 8-vertex case was found in [81].
As proven in [18], from these monodromy matrices a commuting family of transfer matrices $R_{12}(\lambda)$ also read:

$$a(\lambda) = f(\lambda)\tilde{B}(\lambda), \quad b(\lambda) = f(\lambda)\tilde{B}(\lambda), \quad c(\lambda) = f(\lambda)\tilde{C}(\lambda), \quad d(\lambda) = f(\lambda)\tilde{D}(\lambda),$$

(2.12)

Once we define:

$$f(\lambda) \equiv \frac{2\sqrt{R_0 R_1}}{\theta_2(0)\theta_1(0)},$$

(2.11)

the coefficients of $R_{12}^{(BV)}(\lambda)$ also read:

$$a(\lambda) = f(\lambda)\tilde{B}(\lambda), \quad b(\lambda) = f(\lambda)\tilde{B}(\lambda), \quad c(\lambda) = f(\lambda)\tilde{C}(\lambda), \quad d(\lambda) = f(\lambda)\tilde{D}(\lambda),$$

(2.12)

$$\tilde{a}(\mu) \equiv s\pi(\mu + \eta), \quad \tilde{b}(\mu) \equiv s\pi(\mu), \quad \tilde{c}(\mu) \equiv s\pi(\eta), \quad \tilde{d}(\mu) \equiv \mu s\pi(\mu + \eta) s\pi(\eta);$$

(2.13)

Two classes of solutions to the reflection equation (2.6) are constructed here, following [18] on the $2^N$-dimensional representation space $\mathcal{R}_N \equiv \otimes_{k=1}^{N} R_n$ of the chain. Here, $R_n$ is the 2-dimensional local space associated with the site $n$ of the chain. Let us use introduce the notations:

$$K_{\pm}(\lambda) \equiv K(\lambda \pm \eta/2); \quad \xi_{\pm}, \kappa_{\pm}, \tau_{\pm} \equiv \begin{pmatrix} a_{\pm}(\lambda) & b_{\pm}(\lambda) \\ c_{\pm}(\lambda) & d_{\pm}(\lambda) \end{pmatrix},$$

(2.14)

where $\xi_{\pm}, \kappa_{\pm}, \tau_{\pm}$ are arbitrary complex parameters, the $a_{\pm}(\lambda), b_{\pm}(\lambda), c_{\pm}(\lambda)$ and $d_{\pm}(\lambda)$ are defined by (2.1). The bulk monodromy matrix:

$$M_0(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \in \text{End}(R_0 \otimes \mathcal{R}_N),$$

(2.15)

$$M_0(\lambda) = R_{01}^{(BV)}(\lambda - \xi_N - \eta/2) \cdots R_{02}^{(BV)}(\lambda - \xi_2 - \eta/2) R_{01}^{(BV)}(\lambda - \xi_1 - \eta/2),$$

(2.16)

is the solution of the 8-vertex Yang–Baxter equation:

$$R_{12}^{(BV)}(\lambda - \mu) M_1(\lambda) M_2(\mu) = M_2(\mu) M_1(\lambda) R_{12}^{(BV)}(\lambda - \mu).$$

(2.17)

Then, we define the boundary monodromy matrices $U_{\pm}(\lambda) \in \text{End}(R_0 \otimes \mathcal{R}_N)$, as follows:

$$U_{-}(\lambda) \equiv \begin{pmatrix} A_{-}(\lambda) & B_{-}(\lambda) \\ C_{-}(\lambda) & D_{-}(\lambda) \end{pmatrix} = M_0(\lambda) K_{-}(\lambda) M_0(\lambda),$$

(2.18)

$$U_{+}^{(2)}(\lambda) \equiv \begin{pmatrix} A_{+}(\lambda) & B_{+}(\lambda) \\ C_{+}(\lambda) & D_{+}(\lambda) \end{pmatrix} = [M_0(\lambda)]^{00} [K_{+}(\lambda)]^{00} [\hat{M}_0(\lambda)]^{00},$$

(2.19)

where:

$$\hat{M}(\lambda) = (-1)^N \sigma_0^0[M(-\lambda)]^{00} \sigma_0^0.$$  

(2.20)

$U_{-}(\lambda)$ and $V_{+}(\lambda) \equiv U_{+}^{(2)}(-\lambda)$ are the two solutions of the 8-vertex reflection equation:

$$R_{12}^{(BV)}(\lambda - \mu) U_{-}^{(1)}(\lambda) R_{21}^{(BV)}(\lambda + \mu - \eta) U_{+}^{(2)}(\mu)$$

$$= U_{-}^{(1)}(\mu) R_{12}^{(BV)}(\lambda - \mu - \eta) U_{-}^{(1)}(\lambda) R_{21}^{(BV)}(\lambda - \mu).$$

(2.21)

As proven in [18], from these monodromy matrices a commuting family of transfer matrices $T(\lambda) \in \text{End}(\mathcal{R}_N)$ is defined by:

$$T(\lambda) \equiv \text{tr}_0[K_{+}(\lambda) M(\lambda) K_{-}(\lambda) \hat{M}(\lambda)] = \text{tr}_0[K_{+}(\lambda) U_{-}(\lambda)] = \text{tr}_0[K_{-}(\lambda) U_{+}(\lambda)].$$

(2.22)

We characterize here the eigenvalues and eigenstates of this transfer matrix, as well as the matrix elements of the identity in the transfer matrix eigenstates. Note that, after the homogeneous limit ($\xi_n \rightarrow 0$ for any $n \in \{1, \ldots, N\}$), the analysis here developed applies to
open spin-1/2 XYZ quantum chains under the most general non-diagonal integrable boundary conditions:

\[
H_{\text{XYZ}} = \sum_{i=1}^{N-1} \left( (1 + k \sin^2 \tilde{\eta}) \sigma_i^x \sigma_{i+1}^x + (1 - k \sin^2 \tilde{\eta}) \sigma_i^y \sigma_{i+1}^y + c \sin \tilde{\eta} \sigma_i^z \right) + \frac{\sin \tilde{\eta}}{2\sin \zeta} \left[ \sigma_i^x \cos \zeta \right. \\
\left. + 2k \cos \zeta \cosh \tau_+ \right) \\
+ \frac{\sin \tilde{\eta}}{2\sin \zeta} \left[ \sigma_i^y \cos \zeta \right. \\
\left. + 2k \cosh \tau_+ \sigma_i^z \sinh \tau_- \right].
\]

Indeed, this Hamiltonian is reproduced in this homogeneous limit by the derivative of the transfer matrix (2.22), according to the following explicit formulae:

\[
H_{\text{XYZ}} = T^{-1}(\eta/2) \left( \frac{dT}{d\lambda}(\eta/2) - \frac{d(trK_+)}{d\lambda}(\eta/2) \right) \\
= \sum_{n=1}^{N-1} \frac{1}{2} \frac{dK_-}{d\lambda}(\eta/2) + \frac{tr(K_+^0(\eta/2)H_{N0})}{tr(K_+(\eta/2))}.
\]

where:

\[
H_{n,n+1} = (\sin \tilde{\eta})^{-1} \frac{dR_{n,n+1}}{d\lambda}(0),
\]

which just rewrite the formulae that appeared first in [81] for these most general boundary parameters, once we take into account the different conventions taken in the definitions of the bulk and boundary monodromy matrices.

2.2. Properties of reflection algebra generators

The generators of the reflection algebra \(A_-(\lambda), B_-(\lambda), C_-(\lambda)\) and \(D_-(\lambda)\) satisfy some important properties that we prove here. We first define the following functions:

\[
p(\lambda) \equiv \frac{2 \theta_4(2\lambda + \eta|2\omega) \theta_1(2\lambda - \eta|2\omega)}{\theta_2(0|\omega) \theta_4(2\lambda - \eta|2\omega)},
\]

and

\[
\tilde{A}_-(\lambda) \equiv g_-(\lambda) a(\lambda) d(-\lambda), \quad d(\lambda) \equiv a(\lambda - \eta), \quad a(\lambda) \equiv \prod_{n=1}^{N} \theta(\lambda - \xi_n + \eta/2),
\]

where:

\[
g_\pm(\lambda) \equiv h(\lambda; \xi_\pm) \left( \sqrt{\sin(\lambda + \xi_\pm - \tilde{\eta}/2)\sin(-\lambda + \xi_\pm - \tilde{\eta}/2)} \\
+ \kappa_\pm \sin(2\lambda - \tilde{\eta}) \sqrt{(1 - k e^{2\xi_\pm} \sin^2(\lambda - \tilde{\eta}/2))(1 - ke^{-2\xi_\pm} \sin^2(\lambda - \tilde{\eta}/2))} \\
1 - k^2 \sin^2(2\lambda - \tilde{\eta}) \sin^2(\lambda - \tilde{\eta}/2)
\],
\]

then, the following proposition holds.

**Proposition 2.1.** The reflection algebra generators are related by the following parity relation:

\[
A_-(\lambda) = \frac{c(2\lambda)D_-(\lambda) + p(\lambda)D_-(\lambda)}{b(2\lambda)}, \quad D_-(\lambda) = \frac{c(2\lambda)A_-(\lambda) + p(\lambda)A_-(\lambda)}{b(2\lambda)},
\]

(2.30)
where we have defined:

\[ K_-(\lambda) = a(2\lambda)C_-(\lambda) + p(\lambda)C_-(\lambda), \quad C_-(\lambda) = a(2\lambda)B_-(-\lambda) + p(\lambda)B_-(-\lambda), \]  

(2.31)

moreover, the following identities hold:

\[ p(\lambda) = \frac{-c(2\lambda)a_-(-\lambda) + b(2\lambda)d_-(-\lambda)}{a_-(-\lambda)} = \frac{-c(2\lambda)d_-(\lambda) + b(2\lambda)a_-(\lambda)}{d_-(-\lambda)} \]  

(2.32)

\[ = \frac{-a(2\lambda)b_-(-\lambda) + d(2\lambda)c_-(-\lambda)}{b_-(-\lambda)} = \frac{-a(2\lambda)c_-(-\lambda) + d(2\lambda)b_-(-\lambda)}{c_-(-\lambda)}. \]  

(2.33)

Meanwhile, it holds:

\[ U_-^{-1}(\lambda + \eta/2) = \frac{p(\lambda - \eta/2)}{\text{det}_q U_-(\lambda)} U_-(\eta/2 - \lambda), \]  

(2.34)

where in the reflection algebra generated by the elements of \( U_-(\lambda) \) the quantum determinant:

\[ \text{det}_q U_-(\lambda) = \frac{\text{det} U_-(\lambda)}{p(\lambda - \eta/2)} = A_-(\epsilon\lambda + \eta/2)A_-(\eta/2 - \epsilon\lambda) + B_-(\epsilon\lambda + \eta/2)C_-(\eta/2 - \epsilon\lambda) \]  

(2.35)

\[ = D_-(\epsilon\lambda + \eta/2)D_-(\eta/2 - \epsilon\lambda) + C_-(\epsilon\lambda + \eta/2)B_-(\eta/2 - \epsilon\lambda), \]  

(2.36)

where \( \epsilon = \pm 1 \), is central:

\[ [\text{det}_q U_-(\lambda), U_-(\mu)] = 0. \]  

(2.37)

Moreover, it admits the following explicit expression:

\[ \text{det}_q U_-(\lambda) = p(\lambda - \eta/2)\hat{A}_-(\lambda + \eta/2)\hat{A}_-(\lambda - \eta/2). \]  

(2.38)

**Proof.** This proposition is the 8-vertex analogue of proposition 2.1 of [1], as in this last proposition, we can also derive this 8-vertex case following Sklyanin’s article [18]. The following identity holds:

\[ K_-^{-1}(\lambda + \eta/2) = \frac{p(\lambda - \eta/2)}{\text{det}_q K_-(-\lambda)} K_-(\eta/2 - \lambda), \]  

(2.39)

being:

\[ K_-(\eta/2 - \lambda) = \begin{pmatrix} a_-(\eta/2 - \lambda) & b_-(\eta/2 - \lambda) \\ c_-(\eta/2 - \lambda) & d_-(\eta/2 - \lambda) \end{pmatrix} = \begin{pmatrix} d_-(\eta/2 + \lambda) & -b_-(\eta/2 + \lambda) \\ -c_-(\eta/2 + \lambda) & a_-(\eta/2 + \lambda) \end{pmatrix}, \]  

(2.40)

where we have defined:

\[ \text{det}_q K_-(-\lambda) = p(\lambda - \eta/2)(a_-(\lambda + \eta/2)a_-(\eta/2 - \lambda) + b_-(\lambda + \eta/2)c_-(\eta/2 - \lambda)). \]  

(2.41)

Then, the identity (2.34) is obtained by the following chain of identities:

\[ U_-(\eta/2 + \lambda)U_-(\eta/2 - \lambda) \rightarrow \left( \begin{array}{c} \text{det}_q M_0(-\lambda)M_0(\lambda + \eta/2)K_-(\lambda + \eta/2) \\ \times K_-(\eta/2 - \lambda)M_0(\eta/2 - \lambda) \end{array} \right) \]  

\[ = \left( \begin{array}{c} \text{det}_q K_-(-\lambda) \text{det}_q M_0(\lambda + \eta/2)M_0(\eta/2 - \lambda) \end{array} \right) \]  

\[ \rightarrow \left( \begin{array}{c} \text{det}_q K_-(-\lambda) \text{det}_q M_0(\lambda + \eta/2)M_0(\eta/2 - \lambda) \end{array} \right) \]  

\[ \times K_-(\eta/2 - \lambda)M_0(\eta/2 - \lambda) \]  

\[ = \left( \begin{array}{c} \text{det}_q K_-(-\lambda) \text{det}_q M_0(\lambda + \eta/2)M_0(\eta/2 - \lambda) \end{array} \right) \]  

\[ \times \frac{p(\lambda - \eta/2)}{p(\lambda - \eta/2)} I. \]  

(2.42)
where $I$ is the $2 \times 2$ identity matrix in the auxiliary space $R_0$. Here, we have used that:

$$
\tilde{M}(\pm \lambda + \eta/2) = (-1)^N \begin{pmatrix}
D(-\eta/2 \mp \lambda) & -B(-\eta/2 \mp \lambda) \\
-C(-\eta/2 \mp \lambda) & A(-\eta/2 \mp \lambda)
\end{pmatrix}
$$

$$
= (-1)^N \det M_0(\mp \lambda) M^{-1}(\mp \lambda + \eta/2),
$$

where:

$$
det M_0(\lambda) = A(\lambda + \eta/2)D(\lambda - \eta/2) - B(\lambda + \eta/2)C(\lambda - \eta/2)
$$

$$
= a(\lambda + \eta/2) d(\lambda - \eta/2),
$$

is the bulk quantum determinant, which was first proven to be central for the 6-vertex case in [87]. The identity (2.42) implies that up to the numerical factor in the rhs of (2.42) the monodromy matrix $\mathcal{U}_-(\eta/2 - \lambda)$ is the inverse of $\mathcal{U}_-(\eta/2 + \lambda)$. So, the linear combinations of products of generators of the reflection algebra defined at the rhs of (2.35) and (2.36) are indeed central elements of the reflection algebra. Then, the quantum determinant $\det_q \mathcal{U}_-(\lambda)$ is proven to be central; that is, it is a numerical functions of the spectral parameter, which reads:

$$
\det \mathcal{U}_-(\lambda) = \det_q K_-(\lambda) \det_q M_0(\lambda) \det_q M_0(-\lambda),
$$

thanks to the identity (2.42), and its explicit expression (2.38) follows observation that it holds:

$$
\det K_-(\lambda) = p(\lambda - \eta/2)g_-(\lambda + \eta/2)g_-(\lambda + \eta/2).
$$

Sklyanin’s representation (38)[18] for the quantum determinant also clearly works for the 8-vertex case and is defined by

$$
\tilde{\mathcal{U}}_-(\lambda) \equiv -\frac{tr_2 R_{12}(-\eta)(\mathcal{U}_-(\lambda) R_{21}(2\lambda))}{\theta_1(\eta|\lambda|)} = \begin{pmatrix}
\tilde{D}_-(\lambda) & -\tilde{B}_-(\lambda) \\
-\tilde{C}_-(\lambda) & \tilde{A}_-(\lambda)
\end{pmatrix}
$$

the ‘algebraic adjoint’ of the boundary monodromy matrix $\mathcal{U}_-(\lambda), \tilde{\mathcal{U}}_-(\lambda)$ admits the following explicit form in the 8-vertex case:

$$
\tilde{\mathcal{U}}_-(\lambda) = \begin{pmatrix}
\tilde{D}_-(\lambda) b(2\lambda) - A_-(\lambda) c(2\lambda) & \tilde{C}_-(\lambda) d(2\lambda) - B_-(\lambda) a(2\lambda) \\
B_-(\lambda) d(2\lambda) - \tilde{C}_-(\lambda) a(2\lambda) & \tilde{A}_-(\lambda) b(2\lambda) - \tilde{D}_-(\lambda) c(2\lambda)
\end{pmatrix},
$$

and it satisfies the identity (41)[18]:

$$
\tilde{\mathcal{U}}_-(\lambda - \eta/2) \mathcal{U}_-(\lambda + \eta/2) = \det_q \mathcal{U}_-(\lambda),
$$

and so, from the identity (2.34), it follows that:

$$
\tilde{\mathcal{U}}_-(\lambda) = p(\lambda) \mathcal{U}_-(\lambda),
$$

which, by using (2.48), implies the symmetry properties (2.30) and (2.31). Finally, let us remark that the identities in (2.32) and (2.33) can be proven by direct computations. In fact, they just coincide with (2.30) and (2.31) for the scalar case $N = 0$.

Similar statements hold for the reflection algebra generated by $\mathcal{U}_+(\lambda)$ since they are simply consequences of the previous proposition being $\mathcal{U}_+^N(\lambda)$ solution of the same reflection equation of $\mathcal{U}_-(\lambda)$.

Lemma 2.1. The most general boundary transfer matrix $T(\lambda)$ is even in the spectral parameter $\lambda$:

$$
T(-\lambda) = T(\lambda).
$$
Proof. The identity (2.51) can be proven by using the following list of the identities:

\[ T(-\lambda) = \text{tr}_0[K_+(-\lambda)\tilde{M}_-(-\lambda)] = \frac{\text{tr}_0[K_+(-\lambda)\tilde{G}_-(-\lambda)]}{p(\lambda)} \]

\[ p(\lambda) = \begin{pmatrix} d_+(-\lambda) - b(2\lambda) d_+(-\lambda) - a_+(-\lambda)c(2\lambda) \\
-d_+(-\lambda) a_+(-\lambda) b(2\lambda) - d_+(-\lambda) c(2\lambda) \\
-d_+(-\lambda) d_+(-\lambda) a_+(-\lambda) b(2\lambda) - d_+(-\lambda) c(2\lambda) \\
-c_+(-\lambda) d(2\lambda) + b_+(\lambda) b(\lambda) d(2\lambda) \\
-b_+(\lambda) b_+(\lambda) c_+(-\lambda) d(2\lambda) - b_+(\lambda) d(2\lambda) \end{pmatrix} \]

\[ a_+(-\lambda) a_+(-\lambda) + d_+(-\lambda) d_+(-\lambda) + b_+(\lambda) b_+(\lambda) = T(\lambda) \]

Once we observe that the following identities holds:

\[ p(\lambda) = \begin{pmatrix} -c(2\lambda) a_+(-\lambda) - b(2\lambda) d_+(-\lambda) - a_+(-\lambda) c(2\lambda) \\
-c(2\lambda) d_+(-\lambda) - b(2\lambda) a_+(-\lambda) \end{pmatrix} \]

\[ = \begin{pmatrix} -a(2\lambda) b_+(-\lambda) - d(2\lambda) c_+(-\lambda) \\
-a(2\lambda) c_+(-\lambda) - d(2\lambda) b_+(-\lambda) \end{pmatrix} \]

as a direct consequence of the identities (2.32)–(2.33) being:

\[ a_+(-\lambda) a_+(-\lambda) = d_+(-\lambda) d_+(-\lambda) = b_+(\lambda) b_+(\lambda) \]

\[ a_+(-\lambda) a_+(-\lambda) + d_+(-\lambda) d_+(-\lambda) + b_+(\lambda) b_+(\lambda) = T(\lambda) \]

Once we identify \( \zeta_- = \zeta_+, \kappa_- = \kappa_+, \tau_- = \tau_+ \). \[ \square \]

3. Baxter’s gauge transformations and central properties

3.1. Notations

Let us introduce the following \( 2 \times 2 \) matrices:

\[ \tilde{G}(\lambda|\beta) = (X_\beta(\lambda), Y_\beta(\lambda)), \quad \tilde{G}(\lambda|\beta) = (X_{\beta+1}(\lambda), Y_{\beta-1}(\lambda)) \]

\[ \tilde{G}^{-1}(\lambda|\beta) = \left( \tilde{X}_\beta^{-1}(\lambda), \tilde{Y}_\beta^{-1}(\lambda) \right), \quad \tilde{G}^{-1}(\lambda|\beta) = \left( \tilde{X}_{\beta+1}(\lambda), \tilde{Y}_{\beta+1}(\lambda) \right) \]

where:

\[ X_\beta(\lambda) = \begin{pmatrix} \theta_2(\lambda + (\alpha + \beta)\eta|2\omega) \\
\theta_3(\lambda + (\alpha + \beta)\eta|2\omega) \end{pmatrix}, \quad Y_\beta(\lambda) = \begin{pmatrix} \theta_2(\lambda + (\alpha - \beta)\eta|2\omega) \\
\theta_3(\lambda + (\alpha - \beta)\eta|2\omega) \end{pmatrix}, \]

and

\[ \tilde{X}_\beta(\lambda) = \frac{\theta_3(\lambda + (\alpha + \beta)\eta|2\omega) - \theta_2(\lambda + (\alpha + \beta)\eta|2\omega)}{\theta_2(\lambda + (\alpha + \beta)\eta|2\omega)}, \]

\[ \tilde{X}_\beta(\lambda) = \frac{\theta(\lambda + (\alpha + \beta)\eta|2\omega)}{\theta(\lambda + (\alpha + \beta)\eta|2\omega) - \theta_2(\lambda)} \]

\[ \tilde{X}_\beta(\lambda) = \frac{\theta(\lambda + (\alpha - \beta)\eta|2\omega) - \theta_2(\lambda + (\alpha - \beta)\eta|2\omega)}{\theta(\lambda + (\alpha - \beta)\eta|2\omega)}, \]

\[ \tilde{Y}_\beta(\lambda) = \frac{\theta(\lambda + (\alpha + \beta)\eta|2\omega)}{\theta(\lambda + (\alpha + \beta)\eta|2\omega) - \theta_2(\lambda)}, \]

\[ \tilde{Y}_\beta(\lambda) = \frac{\theta(\lambda + (\alpha - \beta)\eta|2\omega)}{\theta(\lambda + (\alpha - \beta)\eta|2\omega) - \theta_2(\lambda)}. \]
Here, $\alpha$ and $\beta$ are arbitrary complex numbers and for simplicity we have introduced the notation $\theta(\lambda) \equiv \theta(\lambda|\omega)$ and we omit the index $\alpha$ because it does not play an explicit role in the following. These covectors/vectors satisfy the following relations:

\[
\begin{align*}
\tilde{Y}_\beta(\lambda)X_\beta(\lambda) &= 1, \\
\tilde{X}_\beta(\lambda)Y_\beta(\lambda) &= 0, \\
\tilde{X}_\beta(\lambda)Y_\beta(\lambda) &= 1,
\end{align*}
\]

and

\[
X_\beta(\lambda)\tilde{Y}_\beta(\lambda) + Y_\beta(\lambda)\tilde{X}_\beta(\lambda) = I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(3.8)

\[
\begin{align*}
\tilde{Y}_{\beta-1}(\lambda)X_{\beta+1}(\lambda) &= 1, \\
\tilde{X}_{\beta-1}(\lambda)Y_{\beta-1}(\lambda) &= 0, \\
\tilde{X}_{\beta+1}(\lambda)Y_{\beta+1}(\lambda) &= 1, \\
&\text{and}
\end{align*}
\]

\[
X_{\beta+1}(\lambda)\tilde{Y}_{\beta-1}(\lambda) + Y_{\beta-1}(\lambda)\tilde{X}_{\beta+1}(\lambda) = I.
\]

(3.9)

3.2. Baxter’s gauge transformation

Baxter’s gauge transformations, which were first introduced in [57], have the following matrix form:

\[
R_{3a}^{(6)}(\alpha_1)S_0(\alpha_2|\alpha, \beta)S_a(\beta_2|\alpha, \beta + \sigma_0^\alpha) = S_a(\alpha_2|\alpha, \beta)S_0(\alpha_1|\alpha, \beta + \sigma_0^\alpha)R_{3a}^{(6)}(\alpha_1|\beta).
\]

(3.10)

where:

\[
S_0(\alpha|\alpha, \beta) \equiv (Y_\beta(\lambda) \quad X_\beta(\lambda)).
\]

(3.11)

In (3.10) $R_{12}^{(6)}(\alpha_1|\beta)$ is the elliptic solution of the following dynamical 6-vertex Yang–Baxter equation [58]:

\[
R_{12}^{(6)}(\alpha_1|\beta + \sigma_0^\alpha)R_{12}^{(6)}(\lambda_1|\beta)R_{12}^{(6)}(\lambda_2|\beta + \sigma_1^\beta) = R_{12}^{(6)}(\lambda_2|\beta)R_{12}^{(6)}(\lambda_1|\beta + \sigma_0^\alpha)R_{12}^{(6)}(\alpha_1|\beta),
\]

(3.12)

and it has the form:

\[
R_{12}^{(6)}(\lambda|\beta) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda|\beta) & c(\lambda|\beta) & 0 \\ 0 & c(\lambda - \beta) & b(\lambda - \beta) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}.
\]

(3.13)

where $a(\lambda)$, $b(\lambda|\beta)$ and $c(\lambda|\beta)$ are defined by:

\[
a(\lambda) = \theta(\lambda + \eta), \quad b(\lambda|\beta) = \frac{\theta(\lambda)\theta((\beta + 1)\eta)}{\theta(\beta\eta)}, \quad c(\lambda|\beta) = \frac{\theta(\eta)\theta(\beta\eta + \lambda)}{\theta(\beta\eta)}.
\]

(3.14)

Historically, Baxter first used a vectorial representation for these transformations, which explicitly reads:

\[
R_{12}(\alpha_1|\beta_1)X_{1,\beta}(\lambda_1)X_{2,\beta^{-1}}(\lambda_2) = a(\alpha_1|\beta_1)X_{2,\beta}(\lambda_2)X_{1,\beta^{-1}}(\lambda_1),
\]

(3.15)

\[
R_{12}(\alpha_1|\beta_1)X_{1,\beta}(\lambda_1)X_{2,\beta^{-1}}(\lambda_2) = b(\alpha_1|\beta_1)X_{2,\beta}(\lambda_2)X_{1,\beta^{-1}}(\lambda_1)
\]

+ $c(\alpha_1|\beta_1)X_{2,\beta}(\lambda_2)X_{1,\beta^{-1}}(\lambda_1),
\]

(3.16)

\[
R_{12}(\alpha_1|\beta_1)X_{1,\beta}(\lambda_1)X_{2,\beta^{-1}}(\lambda_2) = b(\alpha_1|\beta_1)X_{2,\beta}(\lambda_2)X_{1,\beta^{-1}}(\lambda_1)
\]

+ $c(\alpha_1|\beta_1)X_{2,\beta}(\lambda_2)X_{1,\beta^{-1}}(\lambda_1),
\]

(3.17)

\[
R_{12}(\alpha_1|\beta_1)X_{1,\beta}(\lambda_1)X_{2,\beta^{-1}}(\lambda_2) = a(\alpha_1|\beta_1)X_{2,\beta}(\lambda_2)X_{1,\beta^{-1}}(\lambda_1),
\]

(3.18)

this clarifies the original use of the terminology intertwining vectors for these gauge transformations.
3.3. Gauge transformed boundary operators and their properties

3.3.1. Definitions. Let us define the following bulk gauge transformed monodromy matrices:

\[ M(\lambda|\beta) \equiv \tilde{G}^{-1}_r(\lambda - \eta/2)M(\lambda)\tilde{G}_{\beta+N}(\lambda - \eta/2) = \begin{pmatrix} A(\lambda|\beta) & B(\lambda|\beta) \\ C(\lambda|\beta) & D(\lambda|\beta) \end{pmatrix}, \]  \hspace{1cm} (3.19)

\[ \hat{M}(\lambda|\beta) \equiv \tilde{G}^{-1}_{\beta+N}(\eta/2 - \lambda)\hat{M}(\lambda)\tilde{G}_r(\eta/2 - \lambda) = \begin{pmatrix} \hat{A}(\lambda|\beta) & \hat{B}(\lambda|\beta) \\ \hat{C}(\lambda|\beta) & \hat{D}(\lambda|\beta) \end{pmatrix}, \]  \hspace{1cm} (3.20)

and the following boundary one:

\[ U_- (\lambda|\beta) = \begin{pmatrix} \hat{A}_-(\lambda|\beta + 2) & \hat{B}_-(\lambda|\beta) \\ \hat{C}_-(\lambda|\beta + 2) & \hat{D}_-(\lambda|\beta) \end{pmatrix} \equiv \tilde{G}^{-1}_r(\lambda - \eta/2|\beta)U_- (\lambda)\tilde{G}(\eta/2 - \lambda|\beta). \]  \hspace{1cm} (3.21)

3.3.2. Main symmetries. The rescaled gauge transformed boundary operators:

\[ A_-(\lambda|\beta) \equiv r(\lambda)\hat{A}_-(\lambda|\beta), \quad B_-(\lambda|\beta) \equiv r(\lambda)\hat{B}_-(\lambda|\beta), \]  \hspace{1cm} (3.22)

\[ C_-(\lambda|\beta) \equiv r(\lambda)\hat{C}_-(\lambda|\beta), \quad D_-(\lambda|\beta) \equiv r(\lambda)\hat{D}_-(\lambda|\beta), \]  \hspace{1cm} (3.23)

\[ r(\lambda) \equiv \theta_\alpha(2\lambda - \eta|2\alpha)\theta(\lambda + (\alpha + 1/2)\eta), \]  \hspace{1cm} (3.24)

satisfy the following central properties:

**Proposition 3.1.** \( A_-(\lambda|\beta) \) and \( D_-(\lambda|\beta) \) satisfy the following interrelated parity relations:

\[ A_-(\lambda|\beta) = -\frac{\theta(\eta)\theta((2\lambda - (\beta - 1)\eta)}{\theta(2\lambda)\theta((\beta - 2)\eta)}D_-(-\lambda|\beta), \]  \hspace{1cm} (3.25)

\[ D_-(\lambda|\beta) = \frac{\theta(\eta)\theta((2\lambda + (\beta - 1)\eta)}{\theta(2\lambda)\theta((\beta - 2)\eta)}A_-(-\lambda|\beta), \]  \hspace{1cm} (3.26)

while \( B_-(-\lambda|\beta) \) and \( C_-(-\lambda|\beta) \) satisfy the following independent parity relations:

\[ B_-(-\lambda|\beta) = -\frac{\theta(2\lambda + \eta)}{\theta(2\lambda - \eta)}B_-(\lambda|\beta), \quad C_-(-\lambda|\beta) = -\frac{\theta(2\lambda + \eta)}{\theta(2\lambda - \eta)}C_-(\lambda|\beta). \]  \hspace{1cm} (3.27)

Moreover, it holds:

\[ U_-^{-1}(\lambda + \eta/2|\beta) = \frac{\tilde{U}_-(\lambda + \eta/2|\beta)}{\det_\eta \tilde{U}_-(\lambda)} = \frac{p(\lambda - \eta/2)}{\det_\eta \tilde{U}_-(\lambda)}U_- (\eta/2 - \lambda|\beta), \]  \hspace{1cm} (3.28)

where:

\[ \tilde{U}_-(\lambda|\beta) \equiv \tilde{G}^{-1}_r(-\lambda - \eta/2|\beta)\tilde{U}_-(\lambda)\tilde{G}(\eta/2 + \lambda|\beta) \]  \hspace{1cm} (3.29)

and the quantum determinant admits the representation, for both \( \epsilon = \pm 1 \): \[ \frac{p(\lambda - \eta/2)}{\det_\eta \tilde{U}_-(\lambda)r(\lambda + \eta/2)}r(-\lambda + \eta/2) = A_- (\epsilon\lambda + \eta/2|\beta + 2)A_- (\eta/2 - \epsilon\lambda|\beta + 2) + B_- (\epsilon\lambda + \eta/2|\beta)C_- (\eta/2 - \epsilon\lambda|\beta + 2) \]  \hspace{1cm} (3.30)

\[ = D_- (\epsilon\lambda + \eta/2|\beta)D_- (\eta/2 - \epsilon\lambda|\beta) + C_- (\epsilon\lambda + \eta/2|\beta + 2)B_- (\eta/2 - \epsilon\lambda|\beta). \]  \hspace{1cm} (3.31)
Proof. Let us first prove the equation (3.28), by definition it holds:
\[
\tilde{U}_-(\lambda + \eta/2|\beta) = \tilde{G}_\beta^{-1}(\lambda)\tilde{U}_-(\lambda + \eta/2)\tilde{G}_\beta(\lambda),
\]
and then:
\[
U_-(\lambda + \eta/2|\beta) = \tilde{G}_\beta^{-1}(\lambda)U_-(\lambda + \eta/2)\tilde{G}_\beta(-\lambda),
\]
(3.32)
and similarly:
\[
\tilde{U}_-(\lambda - \eta/2|\beta) = \tilde{G}_\beta^{-1}(\lambda)\tilde{U}_-(\lambda - \eta/2)\tilde{G}_\beta(\lambda)
= \tilde{G}_\beta^{-1}(\lambda)\det U_-(\lambda)\tilde{G}_\beta(\lambda)
= \det U_-(\lambda),
\]
(3.33)
From these identities the expressions for the quantum determinant in terms of gauge transformed operators directly follow. Moreover, defined:
\[
f_\alpha(\lambda) = \frac{\theta((\alpha + 1/2)|\eta + \lambda)}{\theta((\alpha + 1/2)|\eta - \lambda)}
\]
(3.35)
the identities:
\[
(\tilde{U}_-(\lambda|\beta))_{12} = -f_\alpha(\lambda)\theta(2\alpha + \eta)\tilde{B}_-(\lambda|\beta), \quad (\tilde{U}_-(\lambda|\beta))_{21} = -f_\alpha(\lambda)\theta(2\alpha + \eta)\tilde{B}_-(\lambda|\beta),
\]
(3.36)
\[
(\tilde{U}_-(\lambda|\beta))_{22} = f_\alpha(\lambda)\left(\frac{\theta(2\alpha)\theta((\beta - 2)|\eta)}{\theta((\beta - 1)|\eta)}\tilde{A}_-(\lambda|\beta) + \frac{\theta(\eta)\theta(2\alpha - (\beta - 1)|\eta)}{\theta((\beta - 1)|\eta)}\tilde{D}_-(\lambda|\beta)\right),
\]
(3.37)
can be shown by direct computation expanding both the elements of \(\tilde{U}_-(\lambda|\beta)\) and \(U_-(\lambda|\beta)\) in terms of the ungauged elements of \(U_-(\lambda)\). Then, the formulae (3.25) and (3.27) are simply derived by using the above identities and the identity:
\[
\tilde{U}_-(\lambda|\beta) = p(\lambda)\left(\tilde{Y}_{\beta^{-1}}(-\lambda - \eta/2)\right)U_-(\lambda)(X_{\beta^{-1}}(\eta/2 + \lambda) \quad Y_{\beta^{-1}}(\eta/2 + \lambda))
\]
(3.38)
\[
= p(\lambda)U_-(\lambda|\beta).
\]
(3.39)

3.3.3. Commutation relations. All the commutation relations that we need to define the left and right SOV representations of the gauge transformed generators of the reflection algebra are contained in the following lemma.

Lemma 3.1. The following commutation relations are satisfied:
\[
B_-(\lambda_2|\beta)B_-(\lambda_1|\beta - 2) = B_-(\lambda_1|\beta)B_-(\lambda_2|\beta - 2),
\]
(3.40)
and
\[
A_-(\lambda_2|\beta + 2)B_-(\lambda_1|\beta) = \frac{\theta(\lambda_1 - \lambda_2 + \eta)\theta(\lambda_2 + \lambda_1 - \eta)}{\theta(\lambda_1 - \lambda_2)\theta(\lambda_1 + \lambda_2)}B_-(\lambda_1|\beta)A_-(\lambda_2|\beta)
+ \frac{\theta(\lambda_1 + \lambda_2 - \eta)\theta(\lambda_1 - \lambda_2 + (\beta - 1)|\eta)}{\theta(\lambda_1 - \lambda_2)\theta(\lambda_1 + \lambda_2)\theta((\beta - 1)|\eta)}B_-(\lambda_2|\beta)A_-(\lambda_1|\beta)
+ \frac{\theta(\eta)\theta(\lambda_1 + \lambda_2 - \beta)}{\theta(\lambda_1 + \lambda_2)\theta((\beta - 1)|\eta)}B_-(\lambda_2|\beta)D_-(\lambda_1|\beta),
\]
(3.41)
and
\[
B_{-}(\lambda_{1}|\beta)D_{-}(\lambda_{2}|\beta) = \frac{\theta(\lambda_{1} - \lambda_{2} + \eta)\theta(\lambda_{2} + \lambda_{1} - \eta)}{\theta(\lambda_{1} - \lambda_{2})\theta(\lambda_{1} + \lambda_{2})}D_{-}(\lambda_{1}|\beta + 2)B_{-}(\lambda_{2}|\beta)
\]
\[
- \frac{\theta(\lambda_{2} - \lambda_{1} + (1 + \beta)\eta)\theta(\lambda_{2} + \lambda_{1} - \eta)}{\theta(\lambda_{1} - \lambda_{2})\theta(\lambda_{2} + \lambda_{1})\theta((1 + \beta)\eta)}D_{-}(\lambda_{1}|\beta + 2)B_{-}(\lambda_{2}|\beta)
\]
\[
- \frac{\theta(\eta)\theta(\lambda_{2} + \lambda_{1} + \beta\eta)}{\theta(\lambda_{2} + \lambda_{1})\theta((1 + \beta)\eta)}A_{-}(\lambda_{1}|\beta + 2)B_{-}(\lambda_{2}|\beta).
\] (3.42)

\[
A_{-}(\lambda_{1}|\beta + 2)A_{-}(\lambda_{2}|\beta + 2) = \frac{\theta(\eta)\theta(\lambda_{1} + \lambda_{2} - \beta\eta)}{\theta(\lambda_{1} + \lambda_{2})\theta((\beta - 1)\eta)}B_{-}(\lambda_{1}|\beta)C_{-}(\lambda_{2}|\beta + 2)
\]
\[
A_{-}(\lambda_{2}|\beta + 2)A_{-}(\lambda_{1}|\beta + 2) = \frac{\theta(\eta)\theta(\lambda_{1} + \lambda_{2} - \beta\eta)}{\theta(\lambda_{1} + \lambda_{2})\theta((\beta - 1)\eta)}B_{-}(\lambda_{2}|\beta)C_{-}(\lambda_{1}|\beta + 2).
\] (3.43)

**Proof.** The first two commutation relations were first presented in the paper [64], the others can be derived similarly by using Baxter’s gauge transformation properties and the reflection equation.

Note that these commutation relations for the gauge transformed generators of the 8-vertex reflection algebra exactly coincides with those of the gauge transformed 6-vertex ones once we transform the function $\theta()$ in sinh(). This observation and the remark that the first coefficients both in (3.41) and in (3.42) do not depend from the gauge parameters and coincide (under the same elliptic to trigonometric transformation) with those appearing in commutation relations of the original 6-vertex reflection algebra are at the basis of the strong similarity in all the SOV representation of reflection algebra generators. This will clearly appear by comparing the SOV representation of the gauge transformed generators in the 8-vertex reflection algebra here derived with those of the 6-vertex reflection algebra in the gauged [2] and ungauged [1] cases.

3.3.4. $\beta$-parity relations.

**Lemma 3.2.** The gauge transformed generators satisfy the following symmetry:

\[
U_{-}(\lambda|\beta + 2) = \sigma^{T}U_{-}(\lambda|\beta)\sigma^{T}
\] (3.44)

which in terms of matrix elements reads:

\[
B_{-}(\lambda|\beta) = C_{-}(\lambda|\beta + 2), \quad A_{-}(\lambda|\beta) = D_{-}(\lambda|\beta + 2).
\] (3.45)

**Proof.** The proof is a trivial consequence of the following simple identities:

\[
\hat{Y}_{\beta}(\lambda) = \hat{X}_{-\beta}(\lambda), \quad Y_{\beta}(\lambda) = X_{-\beta}(\lambda);
\] (3.46)

for example, we have that:

\[
\hat{B}_{-}(\lambda|\beta) = \hat{Y}_{\beta-1}(\lambda - \eta/2)U_{-}(\lambda)Y_{\beta-1}(\eta/2 - \lambda)
\]
\[
= \hat{X}_{-\beta+2-1}(\lambda - \eta/2)U_{-}(\lambda)X_{-\beta+2-1}(\eta/2 - \lambda)
\]
\[
= C_{-}(\lambda|\beta + 2).
\] (3.47)
3.4. Transfer matrix representations in terms of gauge transformed boundary operators

Let us introduce the vectors:

\[ \hat{Y}_{\beta-1}(\lambda) = \frac{\theta((2 + \beta) \eta) Y_{\beta-1}(\lambda)}{\theta((1 + \beta) \eta) \theta(\lambda + (\alpha + 2) \eta) \theta(2 \lambda|2 \omega)}, \]

\[ \hat{Y}_0(\lambda) = \frac{\hat{Y}_{\beta}(\lambda)}{\theta(2 \lambda|2 \omega) \theta(-\lambda + (\alpha + 1) \eta)}, \]  \hfill (3.48)

\[ \hat{X}_{\beta+3}(\lambda) = \frac{\theta(\beta \eta) X_{\beta+3}(\lambda)}{\theta((1 + \beta) \eta) \theta(\lambda + (\alpha + 2) \eta) \theta(2 \lambda|2 \omega)}, \]

\[ \hat{X}_\beta(\lambda) = \frac{\hat{X}_{\beta}(\lambda)}{\theta(2 \lambda|2 \omega) \theta(-\lambda + (\alpha + 1) \eta)}. \]  \hfill (3.49)

and the following two gauge transformations on the boundary matrix \( K_+ \):

\[ K_+^{(L)}(\lambda|\beta)_{11} = \hat{Y}_{\beta-1}(\eta/2 - \lambda) K_+(\lambda) \hat{X}_{\beta+3}(\lambda - \eta/2), \]

\[ K_+^{(L)}(\lambda|\beta)_{12} = \hat{Y}_{\beta+1}(\eta/2 - \lambda) K_+(\lambda) \hat{Y}_{\beta-1}(\lambda - \eta/2), \]

\[ K_+^{(L)}(\lambda|\beta)_{21} = \hat{X}_{\beta+1}(\eta/2 - \lambda) K_+(\lambda) \hat{X}_{\beta+3}(\lambda - \eta/2), \]

\[ K_+^{(L)}(\lambda|\beta)_{22} = \hat{X}_{\beta+3}(\eta/2 - \lambda) K_+(\lambda) \hat{Y}_{\beta-1}(\lambda - \eta/2), \]  \hfill (3.50)

and

\[ K_+^{(R)}(\lambda|\beta)_{11} = \hat{Y}_{\beta+1}(\eta/2 - \lambda) K_+(\lambda) X_{\beta+1}(\lambda - \eta/2), \]

\[ K_+^{(R)}(\lambda|\beta)_{12} = \hat{Y}_{\beta+1}(\eta/2 - \lambda) K_+(\lambda) Y_{\beta-1}(\lambda - \eta/2), \]

\[ K_+^{(R)}(\lambda|\beta)_{21} = \hat{X}_{\beta+1}(\eta/2 - \lambda) K_+(\lambda) X_{\beta+3}(\lambda - \eta/2), \]

\[ K_+^{(R)}(\lambda|\beta)_{22} = \hat{X}_{\beta+3}(\eta/2 - \lambda) K_+(\lambda) Y_{\beta-1}(\lambda - \eta/2). \]  \hfill (3.51)

then, the following proposition holds.

**Proposition 3.2.** In terms of the gauge transformed reflection algebra generators, the boundary transfer matrix \( T(\lambda) \) admits the decompositions:

\[
T(\lambda) = K_+^{(L)}(\lambda|\beta)_{11} A_-(\lambda|\beta + 2) + K_+^{(L)}(\lambda|\beta)_{21} B_-(\lambda|\beta) \\
+ K_+^{(L)}(\lambda|\beta)_{12} C_-(\lambda|\beta + 4) + K_+^{(L)}(\lambda|\beta)_{22} D_-(\lambda|\beta + 2),
\]  \hfill (3.52)

and

\[
T(\lambda) = K_+^{(R)}(\lambda|\beta)_{11} A_-(\lambda|\beta + 2) + K_+^{(R)}(\lambda|\beta)_{21} B_-(\lambda|\beta + 2) \\
+ K_+^{(R)}(\lambda|\beta)_{12} C_-(\lambda|\beta + 2) + K_+^{(R)}(\lambda|\beta)_{22} D_-(\lambda|\beta + 2). \]  \hfill (3.53)

**Proof** To prove the two decompositions of the transfer matrix we first remark that the following identities hold:

\[
\begin{pmatrix}
\hat{Y}_{\beta-1}(\lambda) \\
\hat{X}_{\beta+3}(\lambda)
\end{pmatrix}
\begin{pmatrix}
\hat{X}_{\beta+3}(\lambda) \\
\hat{Y}_{\beta-1}(\lambda)
\end{pmatrix} = \frac{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\theta(\lambda + (\alpha + 1) \eta) \theta(2 \lambda|2 \omega)}, \]

\[
\begin{pmatrix}
\hat{X}_{\beta+3}(\lambda) \\
\hat{Y}_{\beta-1}(\lambda)
\end{pmatrix} \hat{Y}_{\beta-1}(\lambda) + \hat{Y}_{\beta-1}(\lambda) \hat{X}_{\beta+3}(\lambda) = \frac{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\theta(\lambda + (\alpha + 1) \eta) \theta(2 \lambda|2 \omega)}. \]  \hfill (3.54)
The formulae (3.9) and (3.54) imply the following chain of identities:
\[
\mathcal{A}_- (\lambda | \beta + 2) + K_+^{(L)} (\lambda | \beta) K_{+}^{(L)} (\lambda | \beta)_{21} + B_-(\lambda | \beta) K_+^{(L)} (\lambda | \beta)_{21} + D_-(\lambda | \beta + 2) K_+^{(L)} (\lambda | \beta)_{22}
\]
\[+ C_- (\lambda | \beta + 4) K_+^{(L)} (\lambda | \beta)_{12} = \tilde{Y}_{\beta-1} (\lambda - \eta/2) U_- (\lambda) K_+ (\lambda) \tilde{X}_{\beta+3} (\lambda - \eta/2) + \tilde{X}_{\beta+3} (\lambda - \eta/2) U_- (\lambda) K_+ (\lambda) \tilde{Y}_{\beta-1} (\lambda - \eta/2)
\]
\[= \text{tr}_0 \left\{ \left( \tilde{Y}_{\beta-1} (\lambda - \eta/2) U_- (\lambda) K_+ (\lambda) \tilde{X}_{\beta+3} (\lambda - \eta/2) \right) \right\}
\]
\[= \text{tr}_0 \left\{ \left( \tilde{Y}_{\beta-1} (\lambda - \eta/2) U_- (\lambda) K_+ (\lambda) \tilde{X}_{\beta+3} (\lambda - \eta/2) \right) \right\}
\]
\[= \text{tr}_0 [U_- (\lambda) K_+ (\lambda)] = \mathcal{T} (\lambda).
\]

(3.56)

Similarly, the formulae (3.9) and (3.8) imply the following chain of identities:
\[
K_+^{(R)} (\lambda | \beta)_{11} \mathcal{A}_- (\lambda | \beta + 2) + K_+^{(R)} (\lambda | \beta)_{12} C_- (\lambda | \beta + 2) + K_+^{(R)} (\lambda | \beta)_{22} D_- (\lambda | \beta + 2)
\]
\[+ K_+^{(R)} (\lambda | \beta)_{21} B_- (\lambda | \beta + 2)
\]
\[= \tilde{Y}_{\beta+1} (\eta/2 - \lambda) K_+ (\lambda) U_- (\lambda) X_{\beta+1} (\eta/2 - \lambda)
\]
\[+ \tilde{X}_{\beta+1} (\eta/2 - \lambda) U_- (\lambda) K_+ (\lambda) Y_{\beta+1} (\eta/2 - \lambda)
\]
\[= \text{tr}_0 \left\{ \left( \tilde{Y}_{\beta+1} (\eta/2 - \lambda) U_- (\lambda) X_{\beta+1} (\eta/2 - \lambda) \right) \right\}
\]
\[= \text{tr}_0 \left\{ \left( \tilde{Y}_{\beta+1} (\eta/2 - \lambda) U_- (\lambda) X_{\beta+1} (\eta/2 - \lambda) \right) \right\}
\]
\[= \text{tr}_0 [K_+ (\lambda) U_- (\lambda)] = \mathcal{T} (\lambda).
\]

(3.57)

**Proposition 3.3.** The following two, explicitly even in \( \lambda \), representations of the transfer matrix hold:
\[
\mathcal{T} (\lambda) = a_+ (\lambda) A_- (\lambda | \beta + 2) + a_+ (-\lambda) A_- (-\lambda | \beta + 2)
\]
\[+ K_+^{(L)} (\lambda | \beta)_{12} C_- (\lambda | \beta + 4) + K_+^{(L)} (\lambda | \beta)_{21} B_- (\lambda | \beta),
\]

(3.58)
\[
\mathcal{T} (\lambda) = d_+ (\lambda) D_- (\lambda | \beta + 2) + d_+ (-\lambda) D_- (-\lambda | \beta + 2)
\]
\[+ K_+^{(R)} (\lambda | \beta)_{12} C_- (\lambda | \beta + 2) + K_+^{(R)} (\lambda | \beta)_{21} B_- (\lambda | \beta + 2),
\]

(3.59)

where we have defined:
\[
a_+ (\lambda) = \frac{\theta (2 \lambda + \eta) \theta ((\beta + 1) \eta)}{\theta (2 \lambda) \theta ((\beta + 2) \eta)} K_+^{(L)} (\lambda | \beta)_{22},
\]
\[d_+ (\lambda) = \frac{\theta (2 \lambda + \eta) \theta ((\beta + 1) \eta)}{\theta (2 \lambda) \theta (\beta \eta)} K_+^{(R)} (\lambda | \beta)_{11}.
\]

(3.60)

**Proof.** The decompositions of the transfer matrix given in the previous proposition can be rewritten in the following way:
\[
\mathcal{T} (\lambda) = \left( K_+^{(L)} (\lambda | \beta)_{11} + \frac{\theta (\eta) \theta (2 \lambda + (\beta + 1) \eta)}{\theta (2 \lambda) \theta ((\beta + 2) \eta)} K_+^{(L)} (\lambda | \beta)_{22} \right) A_- (\lambda | \beta + 2)
\]
\[+ A_- (-\lambda | \beta + 2) \frac{\theta (2 \lambda - \eta) \theta ((\beta + 1) \eta)}{\theta (2 \lambda) \theta ((\beta + 2) \eta)} K_+^{(L)} (\lambda | \beta)_{22}
\]
\[+ K_+^{(L)} (\lambda | \beta)_{21} B_- (\lambda | \beta) + K_+^{(L)} (\lambda | \beta)_{12} C_- (\lambda | \beta + 4),
\]

(3.61)
that one can verify by direct computations, imply the announced results.

\[ T(\lambda) = \begin{pmatrix} K_+^{(R)}(\lambda|\beta)_{12} - \frac{\theta(\eta)\theta(2\lambda - (\beta + 1)\eta)}{\theta(2\lambda)\theta(\beta\eta)} K_+^{(R)}(\lambda|\beta)_{11} \end{pmatrix} \]

\[ + D_-(\lambda|\beta + 2) \begin{pmatrix} \frac{\theta(2\lambda - \eta)((\beta + 1)\eta)}{\theta(2\lambda)\theta(\beta\eta)} K_+^{(R)}(\lambda|\beta)_{11} \\ + K_+^{(R)}(\lambda|\beta)_{12} \end{pmatrix} \]

\[ + K_+^{(R)}(\lambda|\beta)_{12} C_-(\lambda|\beta + 2), \]  

once we use the properties (3.25)–(3.26). Then, the identities:

\[ K_+^{(L)}(\lambda|\beta)_{11} + \frac{\theta(\eta)\theta(2\lambda + (\beta + 1)\eta)}{\theta(2\lambda)\theta((\beta + 1)\eta)} K_+^{(L)}(\lambda|\beta)_{22} = \frac{\theta(2\lambda + \eta)\theta((\beta + 1)\eta)}{\theta(2\lambda)\theta(\beta\eta)} K_+^{(L)}(\lambda|\beta)_{22}, \]

\[ K_+^{(R)}(\lambda|\beta)_{12} - \frac{\theta(\eta)\theta(2\lambda - (\beta + 1)\eta)}{\theta(2\lambda)\theta(\beta\eta)} K_+^{(R)}(\lambda|\beta)_{11} = \frac{\theta(2\lambda + \eta)\theta((\beta + 1)\eta)}{\theta(2\lambda)\theta(\beta\eta)} K_+^{(R)}(\lambda|\beta)_{11}, \]

that one can verify by direct computations, imply the announced results.

The functions \( a_+(\lambda) \) and \( d_+(\lambda) \) will be crucial in the SOV description of the transfer matrix spectrum, as will be the following properties:

**Lemma 3.3.** Using the freedom in the choice of the gauge parameters to fix:

\[ K_+^{(L)}(\lambda|\beta)_{12} = 0, \]  

keeping completely arbitrary the six boundary parameters, the following quantum determinant conditions are satisfied:

\[ \det_q K_+^{(L)}(\lambda) p(\lambda - \eta/2) = a_+(\lambda + \eta/2)a_+(-\lambda + \eta/2) = d_+(\lambda + \eta/2)d_+(-\lambda + \eta/2). \]

Proof. Let us first prove that we can indeed use the freedom in the choice of the gauge parameters to satisfy the condition (3.65) while keeping completely arbitrary boundary parameters. The following explicit formula:

\[ K_+^{(L)}(\lambda|\beta)_{12} = \frac{\theta_1(2\lambda + \eta|2\omega)\theta_2(\xi_+|2\omega)}{\theta_2(0|2\omega)\theta((\beta + 1)\eta)((\alpha + 3/2)\eta - (\beta + 1)\eta)((\alpha + 3/2)\eta + \lambda)\theta_1(\xi_+|2\omega)} \]

\[ \times \left\{ \theta_5^2(0|2\omega)\theta_2(\xi_+ + (\alpha - \beta)\eta|2\omega)\theta_5(\xi_+ - (\alpha - \beta)\eta|2\omega) \right\}, \]

is proven by using standard formulae in theta function literature. In particular, by the definition (3.50), we have:

\[ K_+^{(L)}(\lambda|\beta)_{12} = F(\lambda)(-\theta_3((\alpha - \beta - 1/2)\eta - \lambda|2\omega)\theta_2((\alpha - \beta + 1/2)\eta + \lambda|2\omega)a_+(-\lambda) \]

\[ + \theta_2((\alpha - \beta - 1/2)\eta - \lambda|2\omega)\theta_2((\alpha - \beta + 1/2)\eta + \lambda|2\omega)d_+(\lambda) \]

\[ + \theta_2((\alpha - \beta - 1/2)\eta - \lambda|2\omega)\theta_2((\alpha - \beta + 1/2)\eta + \lambda|2\omega)c_+(\lambda) \]

\[ - \theta_1((\alpha - \beta - 1/2)\eta - \lambda|2\omega)\theta_1((\alpha - \beta + 1/2)\eta + \lambda|2\omega)b_+(\lambda)) \]  

(3.70)
where:
\[ F(\lambda) = (\theta((\beta + 1)\eta)\theta((\alpha + 3/2)\eta - \lambda)\theta((\alpha + 3/2)\eta + \lambda))^{-1}. \]  (3.71)

Now by using the formula\(^{12}\):
\[
\theta_4(0|t)\theta_1(u + v|t)\theta_1(v + w|t) = \theta_1(u|t)\theta_1(v|t)\theta_2(w|t)\theta_4(u + v + w|t)
+ \theta_1(u|t)\theta_2(v|t)\theta_1(w|t)\theta_1(u + v + w|t),
\]  (3.72)

we can transform the first two terms in the rhs of (3.70) in the term in the second line of (3.69) times the prefactor in the first line of (3.69) once we fix:
\[ u \equiv \lambda - (\alpha - \beta - 1/2)\eta, \quad w \equiv \lambda + (\alpha - \beta + 1/2)\eta, \quad v \equiv -\lambda - \eta/2 + \zeta, \]  (3.73)

and we use the identities (8.183.1)–(8.183.4) of [83]. While we get the term containing \(e^{-r}\) in (3.69) once we use the formula (3.72) taking:
\[ u \equiv \lambda - (\alpha - \beta - 1/2)\eta, \quad v \equiv \lambda + (\alpha - \beta + 1/2)\eta, \quad w \equiv -\lambda - \eta/2. \]  (3.74)

Finally, using the above identifications (3.74) and the following formula:
\[
\theta_4(0|t)\theta_4(v|t)\theta_4(w|t)\theta_4(u + v + w|t) = \theta_1(u|t)\theta_1(v|t)\theta_1(w|t)\theta_1(u + v + w|t)
+ \theta_1(u|t)\theta_2(v|t)\theta_1(w|t)\theta_1(u + v + w|t),
\]  (3.75)

we get the term containing \(e^{r}\) in (3.69).

The main point to stress here is that \(K^{(\lambda)}_{\beta}(\lambda | \beta)_{12}\) has functional dependence in the spectral parameter \(\lambda\) factorized w.r.t. the other parameters and it is an elliptic polynomial in the boundary parameters and the gauged parameters. Then, for any fixed value of the \(\xi, \kappa\), and \(\tau\) one can always fix \(\beta\) such that the condition (3.65) is satisfied for any value of \(\lambda\).

Let us now prove only the identity (3.66) since the other identity follows similarly. From the very definitions of these functions, it holds that:
\[
a_-(\lambda + \eta/2)a_+(\eta/2 - \lambda)
= \frac{\tilde{X}_{\beta+1}(\eta + \lambda)K_\lambda(-\lambda - \eta/2)\tilde{Y}_{\beta+1}(\eta - \lambda)K_\lambda(\lambda - \eta/2)Y_{\beta+1}(\lambda + \eta)}{r(\lambda + \eta/2)r(-\lambda + \eta/2)\theta(\eta - 2\lambda)\theta(2\lambda + \eta)(p(-\lambda - \eta/2)p(\lambda - \eta/2))^{-1}},
\]  (3.76)

\[
\tilde{X}_{\beta+3}(\eta + \lambda)K_\lambda(-\lambda - \eta/2)K_\lambda(\lambda - \eta/2)Y_{\beta+1}(\lambda + \eta)
= \frac{r(\lambda + \eta/2)r(-\lambda + \eta/2)\theta(\eta - 2\lambda)\theta(2\lambda + \eta)(p(-\lambda - \eta/2)p(\lambda - \eta/2))^{-1}}{\det_\lambda K_\lambda(\lambda)p(\lambda - \eta/2)}.
\]  (3.77)

The second line is obtained by using the identity (3.9) once we add to the first line the following term:
\[
\tilde{X}_{\beta+3}(\eta + \lambda)K_\lambda(-\lambda - \eta/2)\tilde{Y}_{\beta+1}(\lambda + \eta)
= \frac{r(\lambda + \eta/2)r(-\lambda + \eta/2)\theta(\eta - 2\lambda)\theta(2\lambda + \eta)(p(-\lambda - \eta/2)p(\lambda - \eta/2))^{-1}}{\det_\lambda K_\lambda(\lambda)p(\lambda - \eta/2)}.
\]  (3.78)

for the condition (3.65). Then, the third line follows since by direct computation one can prove:
\[
\frac{\det_\lambda K_\lambda(\lambda)}{p(-\lambda - \eta/2)} = K_\lambda(\lambda - \eta/2)K_\lambda(-\lambda - \eta/2),
\]  (3.82)

and the last identity is once again due to (3.9).

---

\(^{12}\) See, for example, equation (L.4–L.5) of [5] and references [85] and [86].
4. SOV representations

Let us introduce the following gauge transformed matrices, starting from the $K_-(\lambda)$ boundary matrix:

\begin{align}
K_-(\lambda|\beta)_{11} &= \tilde{Y}_{\beta+N-1}(\lambda - \eta/2)K_-(\lambda)X_{\beta+N-1}(\eta/2 - \lambda), \\
K_-(\lambda|\beta)_{12} &= \tilde{Y}_{\beta+N-1}(\lambda - \eta/2)K_-(\lambda)Y_{\beta+N-1}(\eta/2 - \lambda), \\
K_-(\lambda|\beta)_{21} &= \tilde{X}_{\beta+N-1}(\lambda - \eta/2)K_-(\lambda)X_{\beta+N-1}(\eta/2 - \lambda), \\
K_-(\lambda|\beta)_{22} &= \tilde{X}_{\beta+N-1}(\lambda - \eta/2)K_-(\lambda)Y_{\beta+N-1}(\eta/2 - \lambda),
\end{align}

and

\begin{align}
\tilde{K}_-(\lambda|\beta)_{11} &= \tilde{Y}_{\beta+N-3}(\lambda - \eta/2)K_-(\lambda)X_{\beta+N-1}(\eta/2 - \lambda), \\
\tilde{K}_-(\lambda|\beta)_{12} &= \tilde{Y}_{\beta+N-3}(\lambda - \eta/2)K_-(\lambda)Y_{\beta+N-1}(\eta/2 - \lambda), \\
\tilde{K}_-(\lambda|\beta)_{21} &= \tilde{X}_{\beta+N-1}(\lambda - \eta/2)K_-(\lambda)X_{\beta+N-1}(\eta/2 - \lambda), \\
\tilde{K}_-(\lambda|\beta)_{22} &= \tilde{X}_{\beta+N-1}(\lambda - \eta/2)K_-(\lambda)Y_{\beta+N-1}(\eta/2 - \lambda),
\end{align}

then the following theorem holds.

**Theorem 4.1.** Let the following conditions be satisfied:

$$\xi_a \neq \pm \xi_b + r\eta \mod (\pi, \pi \omega) \forall a \neq b \in \{1, \ldots, N\} \text{ and } r \in \{-1, 0, 1\}$$

then:

(1) for all the gauge parameters $\alpha, \beta \in \mathbb{C}$ such that:

$$K_-(\lambda|\beta)_{12} \neq 0,$$

(2) for all the gauge parameters $\alpha, \beta \in \mathbb{C}$ such that:

$$\tilde{K}_-(\lambda|\beta)_{21} \neq 0,$$

(3) for all the gauge parameters $\alpha, \beta \in \mathbb{C}$ such that:

$$K_-(\lambda|\beta + 2)_{12} \neq 0,$$

(4) for all the gauge parameters $\alpha, \beta \in \mathbb{C}$ such that:

$$\tilde{K}_-(\lambda|\beta + 2)_{21} \neq 0,$$

(5) $B_-(\lambda|\beta + 2)$ is right pseudo-diagonalizable and with simple pseudo-spectrum.

(6) $\tilde{C}_-(\lambda|\beta + 2)$ is right pseudo-diagonalizable and with simple pseudo-spectrum.

(1) $B_-(\lambda|\beta + 2)$ is left pseudo-diagonalizable and with simple pseudo-spectrum.

(2) $\tilde{C}_-(\lambda|\beta + 2)$ is left pseudo-diagonalizable and with simple pseudo-spectrum.

In the next sections we will show the theorem and clarify the terminology by an explicit construction in the cases (1) and (2). Note that the construction in the cases (1) and (2) can be induced from the cases (1) and (2) thanks to the $\beta$-symmetries (3.44). Let us here anticipate that the conditions (4.9) assure that the separate quantum variables have spectrum (set of corresponding eigenvalues) separated; that is, the set of the eigenvalues corresponding to any two different separate quantum variables are disjoint sets, which is an essential requirement to develop the SOV approach.
4.1. Gauge transformed reflection algebra in $B_\mathcal{B}(\beta)$-SOV representations

4.1.1. Simultaneous $B(\lambda|\beta)$ and $B(\lambda|\beta)$ bulk left reference state. Let us define the following state:

$$\langle \beta | \equiv N_\beta \otimes_{n=1}^{N} \tilde{Y}^{(\eta)}_{p+N-n}(\xi_n), \quad N_\beta = 2^{N} \prod_{n=1}^{N} \theta(N-n+\beta)\eta$$ (4.14)

where $\tilde{Y}^{(\eta)}_{p+N-n}(\xi_n)$ is the covector $\tilde{Y}^{(\eta)}_{p+N-n}(\xi_n)$ in the local $\mathcal{L}$ quantum covector space and $N_\beta$ is a normalization factor.

**Proposition 4.1.** The state $\langle \beta |$ is a simultaneous $B(\lambda|\beta)$ and $B(\lambda|\beta)$ eigenstate associated with the eigenvalue zero, for which the following identities hold:

$$\langle \beta | B(\lambda|\beta) = \langle \beta | B(\lambda|\beta) = \{ \theta \}$$ (4.15)

$$\langle \beta | A(\lambda|\beta) = \theta(\xi - \eta/2) \bigg( 1 - \frac{\theta(\xi - \eta/2)}{\theta(\xi - \eta/2)} \bigg)$$ (4.16)

$$\langle \beta | D(\lambda|\beta) = \prod_{n=1}^{N} \theta(\lambda - \xi_n - \eta/2)(\beta + 1)$$ (4.17)

$$\langle \beta | \tilde{A}(\lambda|\beta) = \theta(\lambda|\beta) \prod_{n=1}^{N} \theta(\lambda + \xi_n + \eta/2)(\beta + 1)$$ (4.18)

$$\langle \beta | \tilde{D}(\lambda|\beta) = \prod_{n=1}^{N} \theta(\lambda + \xi_n - \eta/2)(\beta - 1).$$ (4.19)

**Proof.** The proposition is a consequence of the following identities for local operators:

$$\tilde{Y}^{(\eta)}_{s}(\xi_n) \tilde{G}^{-1}_{s-1}(\xi_n - \eta/2) \tilde{R}_{0}(\lambda - \xi_n - \eta/2) \tilde{G}_{s+1}(\lambda - \eta/2)$$ (4.20)

$$= \left( \begin{array}{c}
\frac{\theta((x+1+\beta)\xi - \eta)}{\theta((x+1+\beta)\xi - \eta)} \tilde{Y}^{(\eta)}_{s-1}(\xi_n) \\
\frac{\theta((x+1+\beta)\xi - \eta)}{\theta((x+1+\beta)\xi - \eta)} \tilde{Y}^{(\eta)}_{s-1}(\xi_n)
\end{array} \right)$$ (4.21)

where we have used:

$$\tilde{Y}^{(\eta)}_{s}(\xi_n) \tilde{Y}^{(\eta)}_{s-1}(\lambda - \eta/2) \tilde{R}_{0}(\lambda - \xi_n - \eta/2) \tilde{X}^{(0)}_{s+1}(\lambda - \eta/2)$$ (4.22)

$$= \frac{\theta((x+1+\beta)\xi - \eta)}{\theta((x+1+\beta)\xi - \eta)} \tilde{Y}^{(\eta)}_{s}(\xi_n)$$

and similarly:

$$-\tilde{Y}^{(\eta)}_{s}(\xi_n) \tilde{G}^{-1}_{s+1}(\eta/2 - \lambda) \tilde{R}_{0}(\lambda - \xi_n - \eta/2) \tilde{G}_{s}(\eta/2 - \lambda)$$ (4.24)

$$= \tilde{Y}^{(\eta)}_{s}(\xi_n) \tilde{G}^{-1}_{s+1}(\eta/2 - \lambda) \tilde{R}_{0}(\lambda + \xi_n - \eta/2) \tilde{G}_{s}(\eta/2 - \lambda)$$ (4.25)

$$= \left( \begin{array}{c}
\frac{\theta((x+1+\beta)\xi - \eta)}{\theta((x+1+\beta)\xi - \eta)} \tilde{Y}^{(\eta)}_{s+1}(\xi_n) \\
\frac{\theta((x+1+\beta)\xi - \eta)}{\theta((x+1+\beta)\xi - \eta)} \tilde{Y}^{(\eta)}_{s+1}(\xi_n)
\end{array} \right).$$ (4.26)
Let us assume that (4.9) and (4.10) are boundary operator $\mathcal{B}$. 

4.1.3. Gauge transformed reflection algebra in left $\mathcal{B}(|\beta\rangle)$-SOV representations. 

4.1.2. Simultaneous $\mathcal{J}$. 

Left Theorem 4.2. 

$$C(\lambda|\beta\rangle|\beta + 1\rangle = 0.$$ 

(4.28) 

$$A(\lambda|\beta\rangle|\beta + 1\rangle = \prod_{n=1}^{N} \theta(\lambda - \xi_n + \eta/2)|\beta + 2\rangle.$$ 

(4.29) 

$$D(\lambda|\beta\rangle|\beta + 1\rangle = \frac{\theta(\eta(N + \beta))}{\theta(\eta\beta)} \prod_{n=1}^{N} \theta(\lambda - \xi_n - \eta/2)|\beta\rangle.$$ 

(4.30) 

$$\tilde{A}(\lambda|\beta\rangle|\beta + 1\rangle = \prod_{n=1}^{N} \theta(\lambda + \xi_n + \eta/2)|\beta\rangle.$$ 

(4.31) 

$$\tilde{D}(\lambda|\beta\rangle|\beta + 1\rangle = \frac{\theta(\eta\beta)}{\theta(\eta(N + \beta))} \prod_{n=1}^{N} \theta(\lambda + \xi_n - \eta/2)|\beta + 2\rangle.$$ 

(4.32) 

4.1.3. Gauge transformed reflection algebra in left $\mathcal{B}(|\beta\rangle)$-SOV representations. 

The left $\mathcal{B}(|\beta\rangle)$-pseudo-eigenbasis is here constructed and the representation of the gauge transformed boundary operator $\mathcal{A}_-(\lambda|\beta\rangle)$ in this basis is determined. In the following we will need the notations: 

$$\xi_{-1} = \eta/2, \quad \xi_{-2} = (\eta - \pi)/2, \quad \xi_{-3} = (\eta - \pi \omega)/2, \quad \xi_{-4} = (\eta - \pi - \pi \omega)/2.$$ 

(4.33) 

$$\xi_{a} = \xi_{a} + \pi \omega \text{ for } a \in \{1, 2, 3, 4\} \text{ and also}$$ 

$$\xi_n^{(h_n)} \equiv \phi_n \left[ \xi_n + \left( h_n - \frac{1}{2} \right) \eta \right] \forall n \in \{1, \ldots, 2N\}, h_n \in \{0, 1\} \text{ with}$$ 

$$h_{N+a} \equiv h_a \forall a \in \{1, \ldots, N\},$$ 

(4.34) 

(4.35) 

Moreover, we define the states: 

$$\langle \beta, h_1, \ldots, h_N \rangle = \prod_{n=1}^{N} \left( \frac{A_-(\eta/2 - \xi_n)|\beta + 2\rangle}{A_-(\eta/2 - \xi_n)} \right)^{h_n}$$ 

(4.36) 

where we have defined $A_-(\lambda) \equiv r(\lambda) \tilde{A}_-(\lambda)$ and (at this stage) $N_{\beta+2}$ is just an arbitrary normalization function of $\beta$ and $|\beta\rangle$ is the reference state defined in (4.14). It is important to point out that the states $|\beta, h\rangle$ are well defined states; that is, their definition does not depend on the order of the operator $A_-(\xi^{(0)}b + 2)$ since one can directly verify from the commutation relations (3.43). 

Theorem 4.2. Left $\mathcal{B}_-(|\beta\rangle)$-SOV-representations. 

Let us assume that (4.9) and (4.10) are satisfied; then, the states (4.36) define a basis formed out of pseudo-eigenstates of $\mathcal{B}_-(\lambda|\beta\rangle)$: 

$$\langle \beta, h|\mathcal{B}_-(\lambda|\beta\rangle) = \delta_{\beta,h}(\lambda)|\beta - 2, h\rangle,$$ 

(4.37)
Moreover, \(B_-(\lambda|\beta)\) is an order 4N + 8 elliptic polynomial of periods \(\pi\) and \(2\pi \omega\):
\[B_-(\lambda + \pi|\beta) = B_-(\lambda|\beta), B_-(\lambda + 2\pi \omega|\beta) = (e^{-2(\lambda - \eta/2)} / q^2)^{4N+8} B_- (\lambda|\beta),\]
where \(q \equiv e^{\alpha \omega}\). \(A_-(\lambda|\beta)\) is an order 4N + 8 elliptic polynomial of periods \(\pi\) and \(2\pi \omega\):
\[A_-(\lambda + 2\pi \omega|\beta) = (-e^{-2\omega} / q^2)^{4N+8} e^{2\omega} \alpha A_-(\lambda|\beta),\]
\(A_-(\lambda + \pi|\beta) = A_-(\lambda|\beta)\), where \(\alpha A_-(\beta) \equiv 2(N + \beta)\eta\).

Moreover, we have defined the operator \(A_-^{(0)}(\lambda|\beta + 2)\) by the following action on the generic state \(|\beta, h\rangle\):
\[
|\beta, h|A_-^{(0)}(\lambda|\beta + 2) \equiv \sum_{a=1}^{8} \frac{\theta_1(2(N + \beta + 2) - \lambda - \sum_{b=1, b \neq h}^{8} \xi_{a}|2\omega)}{\theta_1(2(N + \beta + 2) - \sum_{b=1}^{8} \xi_{a}|2\omega)} a_h(\lambda)a_h(-\lambda)
\times \sum_{b=1, b \neq h}^{8} \frac{\theta_1(\lambda - \xi_{a}|2\omega)}{\theta_1(\xi_{a} - \xi_{a}|2\omega)} |\beta, h|A_-(\xi_{a} - \lambda|\beta + 2),
\]
then the operator:
\[
\tilde{A}_-(\lambda|\beta + 2) \equiv A_-(\lambda|\beta + 2) - A_-^{(0)}(\lambda|\beta + 2),
\]
has the following action on the generic state \(|\beta, h\rangle\):
\[
|\beta, h|\tilde{A}_-(\lambda|\beta + 2) \equiv \sum_{a=1}^{2N} \frac{\theta_1(2\lambda - \eta|2\omega)\theta_1(2\lambda - \eta|2\omega)\theta_1(2(N + \beta + 2) + \xi_{a}^{(h)} - \lambda - \sum_{b=1}^{8} \xi_{a}^{(b)}|2\omega)}{\theta_1(2\xi_{a}^{(h)} + \eta|2\omega)\theta_1(2\xi_{a}^{(h)} - \eta|2\omega)\theta_1(2(N + \beta + 2) - \sum_{b=1}^{8} \xi_{a}^{(b)}|2\omega)} \times \frac{\theta_1(\lambda + \xi_{a}^{(h)}|2\omega)\theta_1(\xi_{a}^{(h)}|2\omega)}{\theta_1(2\xi_{a}^{(h)}|2\omega)\theta_2(2\xi_{a}^{(h)}|2\omega)} \prod_{\ell \neq a \text{ and } N} \frac{\theta_2(\xi_{a}^{(h)}|2\omega)}{\theta_2(2\xi_{a}^{(h)}|2\omega)} A_-^{(h)}(\xi_{a}^{(h)})|\beta, h|T^{\nu} a
\]
and:
\[
|\beta, h_1, \ldots, h_a, \ldots, h_N|T_a^{\pm} = |\beta, h_1, \ldots, h_a \pm 1, \ldots, h_N|.
\]

**Proof**
The following boundary-bulk decomposition:
\[
B_-(\lambda|\beta) \equiv K_- (\lambda|\beta) \otimes B(\lambda|\beta) \tilde{D}(\lambda|\beta - 1)
\]
\[
+ K_- (\lambda|\beta)_{12} A(\lambda|\beta) \tilde{B}(\lambda|\beta - 1) + K_- (\lambda|\beta)_{12} B(\lambda|\beta) \tilde{B}(\lambda|\beta - 1)
\]
\[
+ K_- (\lambda|\beta)_{12} A(\lambda|\beta) \tilde{D}(\lambda|\beta - 1),
\]

of the gauge transformed reflection algebra generator \(B_-(\lambda|\beta)\) in terms of the gauge transformed bulk generators and the formulae (4.15)–(4.19) implies that \(|\beta\rangle\) is a \(B_-(\lambda|\beta)\)-pseudo-eigenstate:
\[
|\beta|B_-(\lambda|\beta) \equiv b_{\beta, \lambda}(\lambda|\beta - 2),
\]

(4.48)
with non-zero pseudo-eigenvalue:
\[
 b_{\beta,\theta}(\lambda) = (-1)^N \theta_2(\lambda - \eta + 2\omega) \theta(\lambda + (\alpha + 1/2)\eta) a_\beta(\lambda) a_\theta(-\lambda). 
\] (4.49)

To prove the validity of (4.37) we can now use, step by step, the procedure described in [54], starting from the gauge transformed reflection algebra commutation relations. The conditions (4.9) imply that the operator zeros of \( \mathcal{B}_-(\lambda|\beta) \) (the separate quantum variables) have a disjoint spectrum. Indeed, under these conditions we can prove that the set of non-zero pseudo-eigenvalues of \( \mathcal{B}_-(\lambda|\beta) \), as defined by the \( \eta_{\beta,h}(\lambda) \) when \( h = (h_1, \ldots, h_N) \), takes values in \([0,1]^{\otimes N} \), which defines a set of \( 2^N \) different elliptic polynomials from which we can prove that the set of states \( \langle \beta \| h \rangle \) forms a set of \( 2^N \) independent states; that is, a \( \mathcal{B}_-(\lambda|\beta) \)-pseudo-eigenbasis of the left representation space. Let us stress the fact that this last property is essential to apply the SOV method since we need to use this SOV basis to represent all of the transfer matrix eigenstates. Moreover, the definition of the states \( \langle \beta \| h \rangle \) and the commutation relation (3.41) allow us to define the action of \( A_-((\xi_1 h)_{(b)}|\beta + 2) \) for \( b \in \{1, \ldots, 2N\} \) once we use the quantum determinant relations and the conditions:

\[
\langle \beta | A_-((\xi - \eta/2)|\beta + 2) \rangle = 0, \quad \langle \beta | A_-((\eta/2 - \xi)|\beta + 2) \rangle \neq 0 
\] (4.50)

which trivially follow from the boundary-bulk decomposition:

\[
A_-((\lambda|\beta + 2) = \tilde{\theta}_2(\lambda - \eta + 2\omega) \tilde{\theta}(\lambda + (\alpha + 1/2)\eta) \lambda A_1((\lambda|\beta) + 1)
\]

\[
+ \tilde{K}_-(\lambda|\beta)_{22} B(\lambda|\beta) C(\lambda|\beta + 1) + \tilde{K}_-(\lambda|\beta)_{21} B(\lambda|\beta) \lambda (\lambda|\beta + 1) + \tilde{K}_-(\lambda|\beta)_{12} A(\lambda|\beta) \lambda (\lambda|\beta + 1), 
\] (4.51)

where we have defined

\[
\tilde{K}_-(\lambda|\beta)_{11} \equiv \tilde{Y}_{\beta + N - 1}(\lambda - \eta/2) K_-(\lambda) X_{\beta + N - 1}(\eta/2 - \lambda), 
\]

\[
\tilde{K}_-(\lambda|\beta)_{12} \equiv \tilde{Y}_{\beta + N - 1}(\lambda - \eta/2) K_-(\lambda) Y_{\beta + N - 1}(\eta/2 - \lambda), 
\]

\[
\tilde{K}_-(\lambda|\beta)_{21} \equiv \tilde{X}_{\beta + N - 1}(\lambda - \eta/2) K_-(\lambda) X_{\beta + N - 1}(\eta/2 - \lambda), 
\]

\[
\tilde{K}_-(\lambda|\beta)_{22} \equiv \tilde{X}_{\beta + N - 1}(\lambda - \eta/2) K_-(\lambda) Y_{\beta + N - 1}(\eta/2 - \lambda). 
\] (4.52)

The fact that the operator \( \mathcal{B}_-(\lambda|\beta) \) is an order \( 4N + 8 \) elliptic polynomials of periods \( \pi \) and \( 2\pi \omega \) that satisfies (4.40) can be simply derived from the functional form of its pseudo-eigenvalues once we recall the identities [13]:

\[
\theta_\alpha(x + \pi|2\omega) = (-1)^{\delta_{\alpha,1} + \delta_{\alpha,2}} \theta_\alpha(x|2\omega), 
\]

\[
\theta_\alpha(x + 2\pi \omega|2\omega) = (-1)^{\delta_{\alpha,1} + \delta_{\alpha,2}} e^{-2i(x + \pi \omega)} \theta_\alpha(x|2\omega), 
\] (4.53)

from which it also follows that:

\[
\theta(x + \pi) = -\theta(x), \quad \theta(x + 2\pi \omega) = e^{-4i(x + \pi \omega)} \theta(x). 
\] (4.54)

The fact that the operator \( A_-((\lambda|\beta) \) is an order \( 4N + 8 \) elliptic polynomial of periods \( \pi \) and \( 2\pi \omega \), which satisfies (4.41)–(4.42), can be simply derived from (4.40) by using the commutation relations (3.41). Indeed, by shifting the variable \( \lambda_2 \) in \( \lambda_2 + 2\pi \omega \) and using the transformation properties (4.40) and (4.54), we get:

\[
f_{A_-((\beta + 1)|\lambda_2)} A_-((\lambda_2|\beta + 2) \mathcal{B}_-(\lambda_1|\beta) 
\]

\[
= \frac{\theta(\lambda_1 - \lambda_2 + \eta) \theta(\lambda_2 + \lambda_1 - \eta)}{\theta(\lambda_1 - \lambda_2) \theta(\lambda_1 + \lambda_2)} e^{8i\theta} f_{A_-((\beta)|\lambda_2)}(\mathcal{B}_-(\lambda_1|\beta) A_-((\lambda_2|\beta) 
\]

\[
+ \frac{\theta(\lambda_1 + \lambda_2 - \eta) \theta(\lambda_1 - \lambda_2 + (\beta + 1)\eta)}{\theta(\lambda_2 - \lambda_1) \theta(\lambda_1 + \lambda_2) \theta((\beta + 1)\eta)} e^{-4i\theta} f_{\mathcal{B}_-(\beta)|\lambda_2)}(\mathcal{B}_-(\lambda_2|\beta) A_-((\lambda_1|\beta) 
\] (4.55)

\[\text{[13]} \text{See the equations 8.182–1, 8.182–3 and 8.183–5, 8.183–6 at page 878 of [83].} \]
Theorem 4.3.\) where \(f_{A_{-}(|\beta\rangle)}(\lambda)\) is defined by:
\[
A_{-}(|\lambda + 2\pi \omega\rangle|\beta\rangle) = f_{A_{-}(|\beta\rangle)}(\lambda) A_{-}(|\lambda\rangle|\beta\rangle),
\] (4.57)
which implies:
\[
f_{A_{-}(|\beta\rangle)}(\lambda) \equiv (e^{-2i\varphi/q^2})^{4N+8}\text{e}^{2i\alpha A_{-}(|\beta\rangle)} \quad \text{where} \quad \alpha_{A_{-}(|\beta\rangle)} \equiv 2(N + \beta)\eta.\] (4.58)
Moreover, by the definition (4.43), it is simple to argue that the operator \(A_{-}^{(0)}(|\lambda\rangle|\beta\rangle)\) is also an order 4N + 8 elliptic polynomial of periods \(\pi\) and \(2\pi \omega\), which satisfies (4.41) and (4.42), and then the same is true for \(\tilde{A}_{-}(|\lambda\rangle|\beta\rangle)\). These properties together with the identities:
\[
\tilde{A}_{-}(|\zeta - a\rangle|\beta\rangle) \equiv 0 \quad \text{for any} \quad a \in \{1, \ldots, 8\},
\] (4.59)
imply the interpolation formula (4.45) by using the following interpolation formula:
\[
P(\lambda) = \sum_{a=1}^{M} \frac{\theta(\alpha_p + x_a - \lambda - \sum_{n=1}^{M} x_n)}{\theta(\alpha_p - \sum_{n=1}^{M} x_n)} \prod_{b \neq a} \theta(x_a - x_b) P(x_a),
\] (4.60)
which holds true for any order \(M\) elliptic polynomial, such that:
\[
P(\lambda + \pi) = (-1)^M P(\lambda), \quad P(\lambda + 2\pi \omega) = (e^{-2i\varphi/q^2})^M \text{e}^{2i\varphi P(\lambda)}.\] (4.61)


4.1.4. Gauge transformed reflection algebra in right \(B_{-}(|\beta\rangle\rangle\text{-SOV representations.}\) The right \(B_{-}(|\beta\rangle\rangle\text{-pseudo-eigenbasis is here constructed and the representation of the gauge transformed boundary operator} \(D_{-}(|\lambda\rangle|\beta\rangle)\) in this basis is determined. Let us use the following notation:
\[
|\beta\rangle \equiv | - \beta + 2\rangle,
\] (4.62)
where \(|\beta\rangle\rangle\) is the right reference state defined in (4.27). Further, let us introduce the states:
\[
|\beta, h_1, \ldots, h_N\rangle \equiv \frac{1}{N!} \prod_{n=1}^{N} \frac{\left(D_{-}(\xi_n + \eta/2)|\beta\rangle\right)^{(1-h_n)}}{\left(k_{n}^{(1)} A_{-}(\eta/2 - \xi_n)|\beta\rangle\right)},
\] (4.63)
where:
\[
k_{n}^{(1)} = \frac{\theta(2\xi_n + \eta)\theta(\beta\eta)\theta_{1}(2(N + 2 - \beta) - \sum_{b=1}^{8} \xi_{b} + 2\xi_n|2\omega)\theta_{1}(\eta|2\omega)\theta^{2N}(\zeta^{(1)}|2\omega)}{\theta(\eta)\theta(2\xi_n + \beta + \eta)\theta_{1}(2(N + 2 - \beta) - \sum_{b=1}^{8} \xi_{b} + 2\xi_n|2\omega)\theta_{1}(2\xi_n|2\omega)\theta^{2N}(\zeta^{(0)}|2\omega)},
\] (4.64)
h\(_n\) \in \{0, 1\}, \(n \in \{1, \ldots, N\}\). It is important to point out that the states \(|\beta, h\rangle\rangle\) are well defined states because their definition is independent of the order of operator \(D_{-}(-\zeta^{(0)}|\beta\rangle)\), as one can verify directly by using the commutation relations (3.43) and the \(\beta\text{-parity relation (3.45).}\)

**Theorem 4.3.** **Right** \(B_{-}(|\beta\rangle\rangle\text{-SOV-representations If (4.9) and}^{14}\)
\[
K_{-}(|\lambda - \beta + 2\rangle_{21} \neq 0,
\] (4.65)
are satisfied, then the states \(|\beta, h\rangle\rangle\) defines a basis formed out of \(B_{-}(|\lambda\rangle|\beta\rangle\text{-pseudo-eigenstates:}\)
\[
B_{-}(|\lambda\rangle|\beta\rangle) h \equiv |\beta + 2, h\rangle \tilde{B}_{\beta, h}(\lambda),
\] (4.66)

\(^{14}\)Note that, this is the condition (4.11) in \(\beta'\) for \(\beta' = \beta - 2\).
where:
\[
\tilde{b}_{\beta,h}(\lambda) \equiv (-1)^N \tilde{K}_-(\lambda) - N + 2}_{21} \\
\times \frac{\theta(2(\lambda - N \lambda 2 \omega) \theta(\lambda + (\alpha + 1/2) \eta) \theta(\eta(\beta - N))}{\theta(\eta(\beta)) (N/N + 2)} \left( \prod_{n=1}^{N} \theta_n^p(\beta)/k_n^{(\beta + 2)} \right) a_h(\lambda) a_h(-\lambda). \tag{4.67}
\]

Moreover, \( \mathcal{D}_-(\lambda|\beta) \) is an order \( 4N + 8 \) elliptic polynomial of periods \( \pi \) and \( 2\pi \omega \):
\[
\mathcal{D}_-(\lambda + \pi \omega |\beta) = (-e^{-2i \omega} / q^2)^{4N + 8} e^{2i \omega \pi / 4} \mathcal{D}_-(\lambda|\beta), \tag{4.68}
\]
\[
\mathcal{D}_-(\lambda + \pi \beta |\beta) = \mathcal{D}_-(\lambda|\beta), \quad \text{where} \quad \alpha_{\mathcal{D}_-(\beta)} \equiv 2(N + 2 - \beta) \eta. \tag{4.69}
\]

We have defined the operator \( \mathcal{D}^{(0)}_-(\lambda|\beta) \) by the following action on the generic state \( |\beta, h\rangle \):
\[
\mathcal{D}^{(0)}_-(\lambda|\beta)|\beta, h\rangle \equiv \sum_{a=1}^{2N} \theta_1(2(\lambda + 2 - \beta) - \lambda - \sum_{b=1, b \neq a}^{2N} \theta_1(2(\lambda + 2 - \beta) = \sum_{b=1, b \neq a}^{2N} \theta_1(\lambda - \xi - 2 \omega) a_h(\lambda) a_h(-\lambda) \tag{4.70}
\]
\[
\times \prod_{b=1, b \neq a}^{2N} \theta_1(\lambda - \xi - 2 \omega) D_-(\xi - \beta)|\beta, h\rangle, \tag{4.71}
\]
then, the operator:
\[
\tilde{\mathcal{D}}_-(\lambda|\beta) \equiv \mathcal{D}_-(\lambda|\beta) - \mathcal{D}_-(\lambda|\beta), \tag{4.72}
\]
has the following action on the generic state \( |\beta, h\rangle \):
\[
\tilde{\mathcal{D}}_-(\lambda|\beta)|\beta, h\rangle \equiv \sum_{a=1}^{2N} T_{\alpha}^{\beta} |\beta, h\rangle D_-(\xi |\beta)|\beta, h\rangle \tag{4.73}
\]
\[
\times \prod_{b=1, b \neq a}^{2N} \frac{\theta_1(2(\lambda + 2 - \beta) - \lambda - \sum_{b=1, b \neq a}^{2N} \theta_1(\lambda - \xi - 2 \omega) a_h(\lambda) a_h(-\lambda) \tag{4.74}
\]

\[
\times \prod_{b=1, b \neq a}^{2N} \theta_1(\lambda - \xi - 2 \omega) D_-(\xi - \beta)|\beta, h\rangle, \tag{4.71}
\]

where:
\[
\mathcal{D}_-(\xi |\beta) = (\kappa |\beta) A_\alpha (\xi |\beta) - 2 \varphi_\xi |\beta\rangle, \tag{4.74}
\]
\[
\prod_{a=1}^{N} |\beta, h_1, \ldots, h_a, \ldots, h_N\rangle = |\beta, h_1, \ldots, h_a \pm 1, \ldots, h_N\rangle. \tag{4.74}
\]

**Proof.** The proof follows as in the previous theorem. Let us first prove that \( |\beta\rangle \) is a right \( \mathcal{B}_-(\lambda|\beta) \) pseudo-eigenstate. From the proposition 4.2 and the following boundary-bulk decomposition:
\[
C_-(\lambda|\beta) \tag{4.76}
\]

\[
\frac{1}{\theta_4(2\lambda - N \lambda 2 \omega) \theta(\lambda + (\alpha + 1/2) \eta)} \equiv \tilde{K}_-(\lambda|\beta) C(\lambda|\beta - 2) \tilde{A}(\lambda|\beta - 1) + \tilde{K}_-(\lambda|\beta) D(\lambda|\beta - 2) \tilde{C}(\lambda|\beta - 1)
\]
\[
+ \tilde{K}_-(\lambda|\beta) D(\lambda|\beta - 2) \tilde{A}(\lambda|\beta - 1), \tag{4.75}
\]

it follows that the state \( |\beta\rangle \) is a right \( \mathcal{C}_-(\lambda|\beta) \)-pseudo-eigenstate; that is, it holds:
\[
C_-(\lambda|\beta)|\beta\rangle = \mathcal{C}_-(\lambda|\beta)|\beta\rangle, \tag{4.76}
\]

\[
\mathcal{C}_-(\lambda|\beta)|\beta\rangle = |\beta - 2\rangle C_\beta(\lambda) \tag{4.76}
\]
where:
\[ c_\beta(\lambda) = (-1)^N \tilde{B}_-^N(\lambda|\beta) a_1(2\lambda - \eta)(2\omega)\theta(\lambda + (\alpha + 1/2)\eta) \]
\[ \times \frac{\theta(\eta(N + \beta - 2))}{\theta(\eta(N - 2))} a_1(\lambda) a_1(-\lambda). \]  
(4.77)

Then, from the identity (3.45), it follows that the formula (4.76) is equivalent to the following:
\[ \mathcal{B}_-(\lambda|\beta) \equiv (\beta + 2) c_{-\beta+2}(\lambda). \]  
(4.78)

Then, by using the identities (4.78) and the commutation relations (3.42) and the formulae:
\[ D_- (\xi_n - \eta/2|\beta) \equiv 0, \quad D_-(\xi_n + \eta/2|\beta) \not\equiv 0, \]  
(4.79)

the states (4.63) are proven to be non-zero \( \mathcal{B}_-(\lambda|\beta) \)-pseudo-eigenstates with pseudo-eigenvalues \( \tilde{b}_\beta(\lambda) \), which then forms a basis of \( \mathcal{R}_N \). The fact that the operator \( D_- (\lambda|\beta) \) is an order \( 4N \) elliptic polynomial of periods \( \pi \) and \( 2\pi \omega \) that satisfies (4.41)–(4.42) can be simply derived from (4.40) by using the commutation relations (3.42). Indeed, by shifting the variable \( \lambda_2 \) in \( \lambda_2 + 2\pi \omega \) and using the transformation properties (4.40) and (4.54), we get:
\[ f_{D_-(\beta)}(\lambda_2) \mathcal{B}_-(\lambda|\beta) D_- (\lambda_2|\beta) = \theta(\lambda_1 - \lambda_2 + \eta) \theta(\lambda_2 + \lambda_1 - \eta) \frac{e^{i2\eta f_{D_-(\beta)}(\lambda_2)}}{\theta(\lambda_2 + \lambda_1)} f_{D_-(\beta+2)}(\lambda_2) \]
\[ \times D_- (\lambda_2|\beta + 2) \mathcal{B}_-(\lambda|\beta) = \theta(\lambda_1 - \lambda_2 + \eta) \theta(\lambda_2 + \lambda_1 - \eta) \frac{e^{i2\eta f_{D_-(\beta+2)}(\lambda_2)}}{\theta(\lambda_2 + \lambda_1)} f_{D_-(\beta)}(\lambda_2) \]
\[ \times e^{-4i\eta f_{D_-(\beta)}(\lambda_2)} f_{D_-(\beta)}(\lambda_2) \mathcal{B}_-(\lambda|\beta + 2) \mathcal{B}_-(\lambda_2|\beta) \]  
(4.80)

where we have defined:
\[ D_- (\lambda + 2\pi \omega|\beta) = f_{D_-(\beta)}(\lambda) D_- (\lambda|\beta), \]  
(4.81)

which implies:
\[ f_{D_-(\beta)}(\lambda) = (-e^{-2i\kappa f_{D_-}})^{4N+8} e^{2i\kappa f_{D_-}(\beta)} \quad \text{where} \quad \alpha_{D_-}(\beta) = 2(N + 2 - \beta)\eta. \]  
(4.82)

Moreover, by the definition (4.71) it is simple to argue that the operator \( D_-^{(0)} (\lambda|\beta) \) is also an order \( 4N + 8 \) elliptic polynomial of periods \( \pi \) and \( 2\pi \omega \), which satisfies (4.68) and (4.69); and then, the same is true for \( D_- (\lambda|\beta) \). This property, together with the identities:
\[ D_- (\xi_n|\beta) \equiv 0 \text{ for any } a \in \{1, \ldots, 8\}, \]  
(4.83)
implies the interpolation formula (4.73).

### 4.2. SOV-decomposition of the identity

We can derive some important information by analyzing the change of basis from the spin basis:
\[ (|h\rangle \equiv \otimes_{n=1}^{N} (2h_n - 1, n) \text{ and } |h\rangle \equiv \otimes_{n=1}^{N} (2h_n - 1, n), \]  
(4.84)

to the SOV-basis. This change of basis can be characterized in terms of the \( 2^N \times 2^N \) matrices \( U^{(L,\beta)} \) and \( U^{(R,\beta)} \):
\[ \langle \beta, h| = (h| U^{(L,\beta)} = \sum_{i=1}^{2^N} U_{i,\beta}^{(L,\beta)} (\theta_1^{-1}(i)) \text{ and } \]
\[ |\beta, h\rangle = U^{(R,\beta)} |h\rangle = \sum_{i=1}^{2^N} U_{i,\beta}^{(R,\beta)} (\theta_1^{-1}(i)), \]  
(4.85)
where:
\[ \gamma : h \in \{0,1\}^N \rightarrow \gamma(h) \equiv 1 + \sum_{a=1}^{N} 2^{(a-1)}h_a \in \{1,\ldots,2^N\}, \]

is an isomorphism between the sets \( \{0,1\}^N \) and \( \{1,\ldots,2^N\} \). The pseudo-diagonalizability of \( B_\lambda(\lambda|\beta) \) implies that the matrices \( U_{J(L,c)} \) and \( U_{J(R,c)} \) are invertible matrices satisfying the following identities:
\[ U_{J(L,c)}^{(L,\beta)} B_\lambda(\lambda|\beta) U_{J(L,c)}^{(L,\beta-2)} = U_{J(R,c)}^{(R,\beta)} B_\lambda(\lambda|\beta) U_{J(R,c)}^{(R,\beta+2)} = \Delta_{B_\lambda(\lambda|\beta)}. \]

Here, \( \Delta_{B_\lambda(\lambda|\beta)} \) is the \( 2^N \times 2^N \) diagonal matrix whose elements, for the simplicity of the \( B_\lambda \)-pseudo-spectrum, read:
\[ (\Delta_{B_\lambda(\lambda|\beta)})_{i,j} = \delta_{ij} B_{\lambda,x_i} (\lambda|\beta), \quad \forall i,j \in \{1,\ldots,2^N\}. \]

Moreover, we can prove.

**Proposition 4.3.** Let us define the following \( 2^N \times 2^N \) matrix:
\[ M \equiv U_{J(L,c)}^{(L,\beta-2)} U_{J(R,c)}^{(R,\beta)}. \]

Then, it is diagonal and it explicitly reads:
\[ M_{\gamma(h_\lambda)(k_\lambda)}(\beta, h|\beta, k) = \delta_{\gamma(h_\lambda)(k_\lambda)} \prod_{1 \leq b, a \leq N} \frac{1}{\eta_a(h_b)} \]

once the function \( \gamma_\beta \) entering in the pseudo-eigenstates normalization is defined by:
\[ \gamma_\beta = \left[ \prod_{1 \leq b, a \leq N} \left( \eta_a^{(1)} - \eta_a^{(0)} \right) / \left( \eta_a^{(0)} \right) \right]^{1/2}, \]
and
\[ \eta_a^{(h_a)} = \frac{\theta_2^2((\xi_a + (h_a - \frac{1}{2})\eta_2 \omega)}{\theta_2^2((\xi_a + (h_a - \frac{1}{2})\eta_2 \omega)}. \]

**Proof.** The occurrence of \( \delta_{\gamma(h_\lambda)(k_\lambda)} \) in (4.90) follows the following identities of matrix elements:
\[ \tilde{\nu}_{\beta,h}(\lambda|\beta)(\beta, h|\beta + 2, k) = (\beta, h|\beta)(\beta, k) = \nu_{\beta,h}(\lambda|\beta)(\beta - 2, h|\beta, k). \]

Indeed, the condition \( h \neq k \) implies \( \exists n \in \{1,\ldots,N\} \) such that \( h_n \neq k_n \) and then it implies:
\[ \tilde{\nu}_{\beta,h}(\xi_n^{(k_n)}(\beta)) = 0, \quad \nu_{\beta,h}(\xi_n^{(k_n)}(\beta)) \neq 0, \]
and so:
\[ (\beta - 2, h|\beta, k) \propto \delta_{\gamma(h_\lambda)(k_\lambda)}. \]

The diagonal elements \( M_{\gamma(h_\lambda)(k_\lambda)} \) are obtained by computing
\[ \theta_2^{(\beta)} \equiv (\beta - 2, h_1, \ldots, h_\lambda = 1, \ldots, h_N|\tilde{D}_-(\xi_a + \eta/2|\beta)) \beta, h_1, \ldots, h_\lambda = 0, \ldots, h_N) \]
for any \( a \in \{1,\ldots,N\} \). Being:
\[ (\beta - 2, h_1, \ldots, h_\lambda = 1, \ldots, h_N|\tilde{D}_-(\xi_a + \eta/2|\beta)) \beta, h_1, \ldots, h_\lambda = 0, \ldots, h_N) = (\beta - 2, h_1, \ldots, h_\lambda) = 1, \ldots, h_N|\tilde{D}_-(\xi_a + \eta/2|\beta)). \]
Then, using the decomposition (3.26) and the fact that:
\[
\langle \beta - 2, h_1, \ldots, h_a = 1, \ldots, h_N|A_-(\xi_a + \eta/2)|\beta \rangle = 0
\]
(4.97)
it holds:
\[
\langle \beta - 2, h_1, \ldots, h_a = 1, \ldots, h_N|\tilde{D}_-(\xi_a + \eta/2)|\beta \rangle
\]
(4.98)
\[
= \frac{\theta(\eta)}{\theta(2\xi_a + \eta)} \langle \beta - 2, h_1, \ldots, h_a = 1, \ldots, h_N|A_-(\xi_a + \eta/2)|\beta \rangle
\]
(4.99)
\[
= \frac{\theta(\eta)}{\theta(2\xi_a + \eta)} \langle \beta - 2, h_1, \ldots, h_a = 0, \ldots, h_N|A_-(\eta/2 + \xi_a)|\beta \rangle
\]
(4.100)
and then we get:
\[
\theta_{A}^{(\beta)} = \frac{\theta(\eta)}{\theta(2\xi_a + \eta)} \frac{\theta(2\xi_a + \eta)}{\theta(\beta)} \langle \beta - 2, h_1, \ldots, h_a = 0, \ldots, h_N|\tilde{D}_-(\xi_a + \eta/2)|\beta \rangle
\]
(4.101)
On the other hand, the right action of the operator \(\tilde{D}_-(\xi_a + \eta/2)|\beta \rangle\) and the condition (4.95) implies:
\[
\theta_{A}^{(\beta)} = (k_d^{(\beta)} | A_-(\eta/2 + \xi_a)
\]
\[
\times \frac{\theta_1(2(N + 2 - \beta) - \sum_{b=1}^{N} \xi_b - 2\xi_a | 2\omega) }{\theta_1(2(N + 2 - \beta) - \sum_{b=1}^{N} \xi_b - 2\xi_a | 2\omega)} \frac{\theta_1(2\xi_a | 2\omega) }{\theta_1(2\xi_a | 2\omega)} \frac{\theta_2^{2N}(\beta | 2\omega) }{\theta_2^{2N}(\beta | 2\omega)} 
\]
\[
\times \prod_{b=1}^{N} \frac{\theta_2^{2N}(\beta | 2\omega) }{\theta_2^{2N}(\beta | 2\omega)} \frac{\theta_2^{2N}(\beta | 2\omega) }{\theta_2^{2N}(\beta | 2\omega)} 
\]
\[
\langle \beta - 2, h_1, \ldots, h_a = 1, \ldots, h_N|\beta \rangle
\]
(4.102)
so that it holds that:
\[
\frac{\langle \beta - 2, h_1, \ldots, h_a = 0, \ldots, h_N|\beta \rangle}{\langle \beta - 2, h_1, \ldots, h_a = 1, \ldots, h_N|\beta \rangle} = \prod_{b=1}^{N} \frac{\theta_2^{2N}(\beta | 2\omega) }{\theta_2^{2N}(\beta | 2\omega)} 
\]
(4.103)
from which one can prove:
\[
\frac{\langle \beta - 2, h_1, \ldots, h_n|\beta \rangle}{\langle \beta - 2, 1, \ldots, 1|\beta | 1, \ldots, 1 \rangle} = \prod_{b=1}^{n} \frac{\theta_2^{2N}(\beta | 2\omega) }{\theta_2^{2N}(\beta | 2\omega)} 
\]
(4.104)
This last identity implies (4.90) being
\[
\frac{\langle \beta - 2, 1, \ldots, 1|\beta | 1, \ldots, 1 \rangle}{\prod_{b=1}^{n} \frac{\theta_2^{2N}(\beta | 2\omega) }{\theta_2^{2N}(\beta | 2\omega)}} 
\]
(4.105)
by our definition of the normalization \(n_\beta\).

The previous results allow us to write the following spectral decomposition of the identity \(I\):
\[
\mathbb{I} = \sum_{i=1}^{2N} \mu|\beta, \chi^{-1}(i)| \langle \beta - 2, \chi^{-1}(i) | 
\]
(4.106)
where $\mu \equiv \left( (\beta - 2, \kappa^{-1} (i) \mid \beta, \kappa^{-1} (i)) \right)^{-1}$ is the analogous (pseudo-measure) of the so-called Sklyanin’s measure in the 8-vertex reflection algebra representations, which reads explicitly:

$$I \equiv \sum_{k_1, \ldots, k_N=0}^{1} \prod_{1 \leq b < c \leq N} (\eta_{a}^{(k_b)} - \eta_{a}^{(k_c)}) \langle \beta, h_1, \ldots, h_N \rangle \langle \beta - 2, h_1, \ldots, h_N \rangle. \quad (4.107)$$

5. Separate variable characterization of transfer matrix spectrum

In this section, we show how the SOV approach allows us to write eigenvalues and eigenstates for the transfer matrix associated with the most general representation of the 8-vertex reflection algebra once the gauge transformations are used. The SOV characterization here presented is the natural generalization to the 8-vertex reflection algebra case of those first derived for the 6-vertex case in [1].

**Theorem 5.1.** Keeping completely arbitrary the six boundary parameters, and using the freedom in the choice of the gauge parameters to impose (3.65), then:

- $(I_a)$ is the left representation, for which the one parameter family $B_-(\lambda | \beta)$ is pseudo-diagonal and which defines a left SOV representation for the spectral problem of the transfer matrix $T(\lambda)$.

- $(II_a)$ is the right representation, for which the one parameter family $B_-(\lambda | \beta + 2)$ is pseudo-diagonal and which defines a right SOV representation for the spectral problem of the transfer matrix $T(\lambda)$.

By keeping the six boundary parameters completely arbitrary and using the freedom in the choice of the gauge parameters to impose:

$$K^{(I_a)}_+(\lambda | \beta)_{21} = 0, \quad (5.1)$$

then:

- $(I_c)$ is the left representation, for which the one parameter family $C_-(\lambda | \beta + 4)$ is pseudo-diagonal and which defines a left SOV representation for the spectral problem of the transfer matrix $T(\lambda)$.

- $(II_c)$ is the right representation, for which the one parameter family $C_-(\lambda | \beta + 2)$ is pseudo-diagonal and which defines a right SOV representation for the spectral problem of the transfer matrix $T(\lambda)$.

Here, we will present these SOV constructions in this way, proving the theorem only in the cases $(I_a)$ and $(II_a)$, while for cases $(I_c)$ and $(II_c)$ these can be inferred mainly by using the $\beta$-symmetries defined in lemma 3.2.

**Lemma 5.1.** Let us denote with $\Sigma_T$ the set of the eigenvalue functions of the transfer matrix $T(\lambda)$. Then, any $t(\lambda) \in \Sigma_T$ is even in $\lambda$ and it satisfies the following quasi-periodicity properties in $\lambda$ w.r.t. the periods $\pi$ and $\pi \omega$:

$$t(\lambda + \pi) = t(\lambda), \quad t(\lambda + \pi \omega) = (e^{-2i \lambda}/q)^{2N+2} t(\lambda). \quad (5.2)$$

Moreover, the following identities hold:

$$t(\pm \zeta_{-1}) = \frac{2q_2(q|\omega)q_{2}(\zeta_{-} | 2\omega)q_{2}(\zeta_{+} | 2\omega)}{q_2(0|\omega)q_{1}^{-1}(2\eta | 2\omega)q_{1}^{-1}(0| 2\omega)} \det M(0), \quad (5.3)$$

$$t(\pm \zeta_{-2}) = \frac{2q_2(q|\omega)q_2(\zeta_{-} | 2\omega)q_2(\zeta_{+} | 2\omega)}{q_2(0|\omega)q_{1}(\zeta_{-} | 2\omega)q_{1}(\zeta_{+} | 2\omega)q_{2}^{-1}(2\eta | 2\omega)q_{4}^{-1}(0| 2\omega)} \det M(\pi / 2), \quad (5.4)$$
while the following identities:

\[
\lim_{\lambda \to \pm \zeta, -} \theta_4(2\lambda + \eta|2\omega)\theta_4(2\lambda - \eta|2\omega)t(\lambda) = 4\kappa_\pm \sinh \tau \pm \sinh \tau \pm e^{-2i\sum_{\epsilon=1}^{N} \epsilon \phi^{(0)}} \det M(-\pi \omega/2)
\]
\[
\times \frac{\theta_1(\pi \omega|2\omega)\theta_1(2\eta - \pi \omega|2\omega)\theta_2^2(\pi(\omega + 1)/2|2\omega)\theta_2^3(\zeta|2\omega)\theta_2^3(\zeta + 2\omega)\theta_2^3(0|2\omega)}{\theta_1(\zeta - 2\omega)\theta_1(\zeta + 2\omega)\left[\theta_2^2(\eta - \pi \omega/2|2\omega) + \theta_2^2(\eta - \pi \omega/2|2\omega)\right]^{-1}},
\]
\[
\lim_{\lambda \to \pm \zeta, -} \theta_4(2\lambda + \eta|2\omega)\theta_4(2\lambda - \eta|2\omega)t(\lambda) = 4\kappa_\pm \cos \tau \pm \cosh \tau \pm e^{-2i\sum_{\epsilon=1}^{N} \epsilon \phi^{(0)}} \times \det M(-\pi (\omega + 1)/2)
\]
\[
\times \frac{\theta_1(\pi \omega|2\omega)\theta_1(2\eta - \pi \omega|2\omega)\theta_2^2(\pi(\omega + 1)/2|2\omega)\theta_2^3(\zeta|2\omega)\theta_2^3(\zeta + 2\omega)\theta_2^3(0|2\omega)}{\theta_1(\zeta - 2\omega)\theta_1(\zeta + 2\omega)\left[\theta_2^2(\eta - \pi (\omega + 1)/2|2\omega) - \theta_2^2(\eta - \pi (\omega + 1)/2|2\omega)\right]^{-1}},
\]
\[
(5.5)
\]

**Proof.** The transfer matrix \( T(\lambda) \) is an even function of \( \lambda \) so the same is true for the \( t(\lambda) \in \Sigma_T \). Moreover, it is simple to verify the following identities:

\[
U_{\pm}(\eta/2) = \theta_4^2(\zeta|2\omega)\det M(0)\theta_1, \quad U_{\pm}(\eta/2 + \pi/2) = \frac{\theta_1(\zeta|2\omega)\theta_2(\zeta|2\omega)}{\theta_1(\zeta|2\omega)\theta_1^3(\zeta|2\omega)} \det M(\pi/2)\phi_i,
\]
\[
(5.7)
\]

from which the following identities are derived:

\[
T(\pm \zeta^{(0)}_{-1}) = \frac{2\theta_2(\eta|\omega)\theta_2^2(\zeta|2\omega)\theta_2^3(\zeta|2\omega)}{\theta_2(0|\omega)\theta_2^3(2\eta|2\omega)\theta_2^3(0|2\omega)} \det M(0),
\]
\[
(5.8)
\]

\[
T(\pm \zeta^{(0)}_{-2}) = \frac{2\theta_2(\eta|\omega)\prod_{\epsilon=1}^{N} \theta_2(\xi|2\omega)\theta_1(\xi|2\omega)\theta_2(\xi|2\omega)}{\theta_2(0|\omega)\theta_1(\zeta - 2\omega)\theta_1(\zeta + 2\omega)\theta_1^3(2\eta|2\omega)\theta_1^3(0|2\omega) \det M(\pi/2),
\]
\[
(5.9)
\]

in this way proving (5.3) and (5.4). The boundary matrix \( K_\pm(\lambda; \zeta, \kappa, \tau) \) contains the function \( \theta_2(2\lambda + \epsilon \eta|2\omega) \), with \( \epsilon = + \) or \(-\), at the denominator of the off-diagonal elements. So, it is simple to argue that, for general values of the boundary parameters, the transfer matrix \( T(\lambda) \) may have poles in the zeros of the functions \( \theta_4(2\lambda - \eta|2\omega)\theta_4(2\lambda + \eta|2\omega) \). The residues associated with these poles follow from the identities:

\[
\lim_{\lambda \to \pm \zeta, -} \theta_4(2\lambda - \eta|2\omega)U_{\mp}(\lambda) = -\kappa_{-} \theta_4(\pi \omega + (a - 3)\pi|2\omega)\theta_2^3(\pi(\omega + a - 3)/2|2\omega) \frac{\theta_2^3(\zeta|2\omega)}{\theta_2^3(\zeta - 2\omega)\theta_2^3(0|2\omega)} \frac{e^{-2i\sum_{\epsilon=1}^{N} \epsilon \phi^{(0)}}}{\theta_2(0|\omega)}\left[\theta_2^2(\eta - \pi (\omega + a - 3)/2|2\omega)\right]^{-1}
\]
\[
\times \left(e^\tau + (2a - 7)e^{-\tau}\right) \det M(-\pi (\omega + a - 3)/2) \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \left[\begin{array}{c} 2a - 7 \\ 0 \end{array}\right],
\]
\[
(5.10)
\]

for \( a = 3 \) and \( 4 \), which are derived by using the following identities:

\[
\lim_{\lambda \to \pm \zeta, -} \theta_4(2\lambda - \eta|2\omega)K_\pm(\lambda) = -\kappa_{-} \frac{\theta_4(\pi \omega + (a - 3)\pi|2\omega)\theta_2^3(\pi(\omega + a - 3)/2|2\omega)}{\theta_2(0|\omega)\theta_2^3(\zeta - 2\omega)\theta_2^3(0|2\omega)}
\]
\[
\times \left(e^\tau + (2a - 7)e^{-\tau}\right) \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \left[\begin{array}{c} 2a - 7 \\ 0 \end{array}\right],
\]
\[
(5.12)
\]

and

\[
M(\zeta_{-a}) = (-1)^{N}e^{-2i\sum_{\epsilon=1}^{N} \epsilon \phi^{(0)}} \left[\begin{array}{cc} 0 & 2a - 7 \\ 1 & 0 \end{array}\right] M(\eta - \zeta_{-a}) \left[\begin{array}{cc} 0 & 1 \\ 2a - 7 & 0 \end{array}\right],
\]
\[
(5.14)
\]
for $a = 3$ and $4$ where this last identity follows from:

\[ a \left( \frac{-\pi}{2} \left( \omega + \frac{1 - e}{2} \right) - \xi_a \right) = -e^{-2i\pi a b} \left( \frac{\pi}{2} \omega + \frac{1 - e}{2} \right) - \xi_a \].

(5.15)

\[ c \left( \frac{-\pi}{2} \left( \omega + \frac{1 - e}{2} \right) - \xi_a \right) = e^{-2i\pi a b} d \left( \frac{\pi}{2} \omega + \frac{1 - e}{2} \right) - \xi_a \].

(5.16)

\[ \square \]

Let us associate to any $t(\lambda) \in \Sigma_T$ the following even functions in $\lambda$:

\[ \hat{t}(\lambda) \equiv \theta_4(2\lambda + \eta|2\omega)\theta_4(2\lambda - \eta|2\omega)t(\lambda), \]

(5.17)

then for the previous lemma $\hat{t}(\lambda)$ is an elliptic polynomial in $\lambda$ of order $2N + 6$, which satisfies the following quasi-periodicity properties in $\lambda$ w.r.t. the periods $\pi$ and $\pi\omega$:

\[ \hat{t}(\lambda + \pi) = \hat{t}(\lambda), \quad \hat{t}(\lambda + \pi \omega) = (e^{-2i\lambda}/q)^{2N+6}\hat{t}(\lambda). \]

(5.18)

Moreover, $\hat{t}(\lambda)$ has values in the points $\pm \xi_{-a}$ for $a = 1, 2, 3$ and $4$, which are independent from the particular choice of $t(\lambda) \in \Sigma_T$ and are completely fixed by the previous lemma. Then, by defining:

\[ j(\lambda) \equiv \frac{4}{\lambda} \sum_{a=1}^{4} l_a(\lambda)\hat{t}(\xi_{-a}), \]

(5.19)

where:

\[ l_a(\lambda) \equiv \prod_{\eta, \eta'} \frac{\theta(\lambda - \xi_{-b})\theta(\lambda + \xi_{-b})}{\theta(\xi_{-a} - \xi_{-b})\theta(\xi_{-a} + \xi_{-b})} \prod_{\lambda = 1}^{N} \frac{\theta(\lambda - \xi_{b}^{(0)})\theta(\lambda + \xi_{b}^{(0)})}{\theta(\xi_{a}^{(0)} - \xi_{b}^{(0)})\theta(\xi_{a}^{(0)} + \xi_{b}^{(0)})} \forall a \in \{-4, \ldots, N\}, \]

(5.20)

one can observe that the elliptic polynomial $j(\lambda)$ is independent from the particular choice of $t(\lambda) \in \Sigma_T$. We can now prove the following complete characterization of the transfer matrix spectrum.

**Theorem 5.2.** $T(\lambda)$ has simple spectrum if (4.9) is satisfied and $\Sigma_T$ admits the following characterization:

\[ \Sigma_T \equiv \left\{ t(\lambda) : t(\lambda) = \frac{j(\lambda) + \sum_{a=1}^{N} l_a(\lambda)x_a}{\theta_4(2\lambda + \eta|2\omega)\theta_4(2\lambda - \eta|2\omega)}, \quad \forall [x_1, \ldots, x_N] \in \Sigma_T \right\}, \]

(5.21)

where $\Sigma_T$ is the set of solutions to the following inhomogeneous system of $N$ quadratic equations:

\[ x_n \sum_{a=1}^{N} l_a(\xi_{n}^{(1)})x_a + x_n j(\xi_{n}^{(1)}) = q_n, \quad q_n \equiv \tilde{\lambda}(\xi_{n}^{(1)})\tilde{\lambda}(-\xi_{n}^{(0)}), \quad \forall n \in \{1, \ldots, N\}, \]

(5.22)

in the $N$ unknowns $[x_1, \ldots, x_N]$, where $\tilde{\lambda}(\lambda)$ is defined by:

\[ \tilde{\lambda}(\lambda) \equiv \theta_4(2\lambda + \eta|2\omega)\theta_4(2\lambda - \eta|2\omega)\tilde{\lambda}(\lambda), \quad \tilde{\lambda}(\lambda) \equiv a_+(\lambda)A_-(\lambda), \]

(5.23)

where $\lambda(\lambda)$ satisfies the quantum determinant condition:

\[ \det_{\eta}[K_1(\lambda)]

\[ \frac{\det_{\eta}[K_1(\lambda)]}{\det_{\eta}[K_2(\lambda)]} = \lambda(\eta/2 - \lambda)\lambda(\eta/2). \]

(5.24)

(R) If (4.11) is verified, the vector:

\[ |t\rangle = \prod_{h_1, \ldots, h_N} \prod_{a=1}^{N} Q(a(\xi_{a}^{(h_a)}) \prod_{1 \leq |h| \leq \delta} (\eta_{a}^{(h_a)} - \eta_{b}^{(h_a)})\beta + 2, h_1, \ldots, h_N), \]

(5.25)
with coefficients:
\[ Q_t(\xi_a^{(1)})/Q_t(\xi_a^{(0)}) = t(\xi_a^{(0)})/\lambda(-\xi_a^{(0)}), \]
(5.26)
is the right \( T \)-eigenstate corresponding to \( t(\lambda) \in \Sigma_T \) which is uniquely defined up to an overall normalization.

(L) If (4.10) is verified, then the covector
\[ \langle t | = \sum_{h_1, \ldots, h_N = 0} \prod_{a=1}^{\lambda} \bar{Q}_t(\xi_a^{(h_a)}) \prod_{1 \leq b, c \leq \lambda} (\eta_b^{(h_b)} - \eta_c^{(h_c)}) \langle \beta, h_1, \ldots, h_N |, \]
(5.27)
with coefficients:
\[ \tilde{Q}_t(\xi_a^{(1)})/\tilde{Q}_t(\xi_a^{(0)}) = t(\xi_a^{(0)})/(d(\xi_a^{(1)})D_+ \xi_a^{(1)}) \]
(5.28)
is the left \( T \)-eigenstate corresponding to \( t(\lambda) \in \Sigma_T \) which is uniquely defined up to an overall normalization.

**Proof.** The separate variables characterization of the spectral problem for \( T(\lambda) \) is reduced to the discrete system of \( 2^N \) Baxter-like equations:
\[ t(\xi_n^{(h_n)}) \Psi_t(h) = \lambda(\xi_n^{(h_n)}) \Psi_t(T^+_n(h)) + \lambda(-\xi_n^{(h_n)}) \Psi_t(T^-_n(h)), \]
(5.29)
for any \( n \in \{1, \ldots, N\} \) and \( h \in [0, 1]^N \). Here, the \( \psi \) (wave-functions) \( \Psi_t(h) \) are the coefficient of the \( T \)-eigenstate \( |t| \) corresponding to the \( t(\lambda) \in \Sigma_T \) in the right \( \beta^- \)-SOV representation, and the following notations are introduced:
\[ T^+_n(h) = (h_1, \ldots, h_n \pm 1, \ldots, h_N). \]
(5.30)
This system of separate equations is derived from the identities:
\[ \lambda(\xi_n^{(0)}) = \lambda(-\xi_n^{(0)}) = 0, \]
(5.31)
once we compute the matrix elements:
\[ \langle \beta, h_1, \ldots, h_n, \ldots, h_N | T (\pm \xi_n^{(h_n)}) | t \rangle. \]
(5.32)
Indeed, (3.52) implies:
\[
\begin{align*}
t(\pm \xi_n^{(0)}) & \Psi_t(h_1, \ldots, h_n = 0, \ldots, h_N) \\
& = \langle \beta, h_1, \ldots, h_n = 0, \ldots, h_N | T (\pm \xi_n^{(0)}) | t \rangle \\
& = \lambda(\pm \xi_n^{(0)}) \Psi_t(h_1, \ldots, h_n = 0, \ldots, h_N) \\
& = \lambda(\pm \xi_n^{(0)}) \Psi_t(h_1, \ldots, h_n = 1, \ldots, h_N) \\
& = \lambda(\pm \xi_n^{(0)}) \Psi_t(h_1, \ldots, h_n = 1, \ldots, h_N) \\
& = + \lambda(\pm \xi_n^{(0)}) \Psi_t(h_1, \ldots, h_n = 1, \ldots, h_N),
\end{align*}
\]
and
\[
\begin{align*}
t(\pm \xi_n^{(1)}) & \Psi_t(h_1, \ldots, h_n = 1, \ldots, h_N) \\
& = \langle \beta, h_1, \ldots, h_n = 1, \ldots, h_N | T (\pm \xi_n^{(1)}) | t \rangle \\
& = \lambda(\pm \xi_n^{(1)}) \Psi_t(h_1, \ldots, h_n = 1, \ldots, h_N) \\
& = \lambda(\pm \xi_n^{(1)}) \Psi_t(h_1, \ldots, h_n = 0, \ldots, h_N) \\
& = + \lambda(\pm \xi_n^{(1)}) \Psi_t(h_1, \ldots, h_n = 0, \ldots, h_N). (5.34)
\end{align*}
\]
The system (5.29) is clearly equivalent to the system of homogeneous equations:
\[
\begin{pmatrix}
\begin{pmatrix} t(\pm \xi_n^{(0)}) & -\lambda(\pm \xi_n^{(0)}) \\
-\lambda(\pm \xi_n^{(1)}) & t(\pm \xi_n^{(1)}) \end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\Psi_t(h_1, \ldots, h_n = 0, \ldots, h_N) \\
\Psi_t(h_1, \ldots, h_n = 1, \ldots, h_N)
\end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}. (5.35)
\]
for any \( n \in \{1, \ldots, N\} \) with \( h_{g\neq a} \in \{0,1\} \). Then, the determinants of the \( 2 \times 2 \) matrices in (5.35) must be zero for any \( n \in \{1, \ldots, N\} \) if \( t(\lambda) \in \Sigma_T \); that is, it holds:

\[
t(\pm \epsilon_0) t(\pm \epsilon_1) = \lambda (\epsilon_0) \lambda (-\epsilon_0), \quad \forall a \in \{1, \ldots, N\}.
\]

Being

\[
\lambda (-\epsilon_0) \neq 0 \quad \text{and} \quad \lambda (\epsilon_1) \neq 0,
\]

then the matrices in (5.35) have all rank 1 and, up to an overall normalization, the solution is unique:

\[
\frac{\Psi_t(h_1, \ldots, h_N)}{\Psi_t(h_1, \ldots, h_N)} = \frac{t(\epsilon_0)}{\lambda (-\epsilon_0)}, \quad \forall n \in \{1, \ldots, N\}
\]

for any \( n \in \{1, \ldots, N\} \) with \( h_{g\neq a} \in \{0,1\} \). So, for any fixed \( t(\lambda) \in \Sigma_T \) the associated eigenspace is 1D (\( T(\lambda) \) has simple spectrum) and \( |t| \) defined by (5.25)–(5.26) is the only corresponding eigenstate up to normalization. It is simple now to prove that the set \( \Sigma_T \) is included in the set of functions characterized by (5.21) and (5.22). Indeed, for any \( t(\lambda) \in \Sigma_T \) the associated elliptic polynomial defined in (5.17) admits the following interpolation formula:

\[
\hat{t}(\lambda) = j(\lambda) + \sum_{a=1}^{N} l_a(\lambda) \hat{t}(\epsilon_0)
\]

since the functions \( j(\lambda) \) and \( l_a(\lambda) \), as well as \( \hat{t}(\lambda) \), are even elliptic polynomials in \( \lambda \) of order \( 2N + 6 \), which satisfies the same quasi-periodicity properties (5.18), and the interpolation formula is given on the \( 2(N + 4) \) points:

\[
\pm \epsilon_0, \ldots, \pm \epsilon_1, \pm \epsilon_0, \ldots, \pm \epsilon_0.
\]

Then, using (5.39), the system of equation (5.36) is equivalent to (5.22).

Let us now prove the reverse inclusion of the set of functions; that is, let us prove that if \( t(\lambda) \) is in the set of functions characterized by (5.21) and (5.22) then it is an element of \( \Sigma_T \). Indeed, taking the state \( |t| \) defined by (5.25)–(5.26) the following identities are satisfied:

\[
\langle \beta, h_1, \ldots, h_N | T(\pm \epsilon_0) | t \rangle = t(\pm \epsilon_0) \langle \beta, h_1, \ldots, h_N | t \rangle \quad \forall n \in \{1, \ldots, N\},
\]

and

\[
\lim_{\lambda \to \pm \epsilon_0} \theta_4(2\lambda + \eta) \theta_4(2\lambda - \eta) = \hat{t}(\pm \epsilon_0) \langle \beta, h_1, \ldots, h_N | t \rangle
\]

for any \( a = 1, 2, 3, 4 \), which implies:

\[
\langle \beta, h_1, \ldots, h_N | T(\lambda) | t \rangle = t(\lambda) \langle \beta, h_1, \ldots, h_N | t \rangle \quad \forall \lambda \in \mathbb{C},
\]

for any \( B(\lambda) \)-pseudo-eigenstate \( \langle \beta, h_1, \ldots, h_N | t \rangle \); that is, \( t(\lambda) \in \Sigma_T \) and \( |t| \) is the corresponding \( T \)-eigenstate. Finally, let us point out that the quantum determinant condition (5.24) follows from the definition (4.46) and the quantum determinant conditions (2.38) and (3.66), where this last identity holds when (3.65) is satisfied, as proven in lemma 3.3. Concerning the left \( T \)-eigenstates, the proof is done as above. Here, one has to compute the matrix elements:

\[
\langle t | T(\epsilon_0) | \beta + 2, h_1, \ldots, h_N \rangle,
\]

which by using the right \( B(\lambda) \)-representation reads:

\[
t(\epsilon_0) \tilde{\Psi}_t(h) = d(\epsilon_0) \tilde{\Psi}_t(T_\beta^+(h)) + d(-\epsilon_0) \tilde{\Psi}_t(T_\beta^-(h)), \quad \forall n \in \{1, \ldots, N\}
\]
where:
\[
\hat{\Psi}_t(h) \equiv \langle \{\beta, h_1, \ldots, h_N\}, d(\pm \varepsilon^{(h)}_a) \equiv d_t(\pm \varepsilon^{(h)}_a) D_\pm(\pm \varepsilon^{(h)}_a). \quad (5.45)
\]

Under the most general boundary conditions the above inhomogeneous system of quadratic equations provides the characterization of the spectrum and replaces the Bethe ansatz formulation, which applies only when the parameters satisfy the linear relation derived in [64]. It is, however, interesting to get a reformulation of this characterization by functional equations and the construction of a Baxter Q-operator can be important in this direction. In our next paper, we will provide this construction based only on the SOV characterization following the approach defined first in [49] and generalized in [50] for cyclic 6-vertex representations. In the roots of unit case, and for the most general boundary conditions, this construction will be proven to lead to a Baxter Q-operator that is an elliptic polynomial in spectral parameter \(\lambda\), and so to a proof of completeness of the spectrum (eigenvalues and eigenstates) characterization in terms of a system of Bethe ansatz equations. Finally, we want to report that after the completion of this manuscript, we have remarked the interesting paper [88] that follows a series of recent papers [89] on integrable quantum models associated with the spin-1/2 representations of both Yang–Baxter and reflection algebras. For these integrable quantum models, T-Q functional equations have been introduced for the characterization of the transfer matrix eigenvalues by an ansatz, using as a starting point the identities relating the products of the transfer matrix eigenvalues and the quantum determinant in special points related to the inhomogeneities of the models. These identities can be proven directly at the operator level; for example, by using the annihilation identities of the generators of both the Yang–Baxter and reflection algebras for both the 6-vertex and 8-vertex cases. This approach was described, for example, in [56] in the case of the periodic transfer matrices associated with spin-1/2 representation of the 8-vertex Yang–Baxter algebra and in the case of the antiperiodic transfer matrix associated with the spin-1/2 representation of the dynamical 6-vertex Yang–Baxter algebra. In [88] these identities are derived using the reduction in zero to the permutation operator of both the 8-vertex and 6-vertex R-matrix. The link with the SOV approach is very simple to explain in all the integrable quantum models analyzed so far, and is associated with representations defined on spin-1/2 quantum chains [1, 2, 54, 56, 59] and the compatibility conditions of the transfer matrix separate equations; that is, the system of Baxter like equations of type (5.29), are just the mentioned identities involving product of transfer matrices and quantum determinant (5.36). In the SOV framework these equations are proven to reconstruct the full spectrum (eigenvalues and eigenstates) of the transfer matrix when one analyses the full class of solutions to (5.36) in a known and model dependent class of functions. The clear interest in the paper [88] is that it proposes an ansatz\(^{15}\) to associate to the equation of type (5.36) the functional T-Q equations in terms of elliptic polynomials, allowing a more traditional analysis of the eigenvalue problem by the analysis of a system of Bethe equations.

6. Scalar Products

The above analysis in SOV allows us to get the following scalar product formulae for separate states. One interesting point is that they are mainly automatically derived and universal in this SOV framework.

\(^{15}\) An analysis of the open problem of completeness of such a type of ansatz has been addressed recently in [90] for the case of the inhomogeneous XXX spin chains.
Theorem 6.1. Let $|u\rangle$ and $|v\rangle$ be arbitrary states with the following separate forms:

\[
|u\rangle = \prod_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} u_a(\xi_a^{(h_a)}) \prod_{1\leq b<a\leq N} (\eta_a^{(h_a)} - \eta_b^{(h_b)})(\beta, h_1, \ldots, h_N),
\]

\[
|v\rangle = \prod_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} v_a(\zeta_a^{(h_a)}) \prod_{1\leq b<a\leq N} (\eta_a^{(h_a)} - \eta_b^{(h_b)})\beta + 2, h_1, \ldots, h_N),
\]

in the $B$-pseudo-eigenbasis; then, the action of $|u\rangle$ on $|v\rangle$ reads:

\[
\langle u|v \rangle = \det_N ||\mathcal{M}_{a,b}^{(a,v)}|| \text{ with } \mathcal{M}_{a,b}^{(a,v)} \equiv \sum_{b=0}^{1} u_a(\xi_a^{(h)}) v_a(\zeta_a^{(h)}) (\eta_a^{(h)})(\beta-1). \tag{6.3}
\]

The above formula particularly holds in cases where the left and right states are transfer matrix eigenstates.

Proof. Formula (4.90) and the definitions of the states $|u\rangle$ and $|v\rangle$ imply:

\[
\langle u|v \rangle = \sum_{h_1, \ldots, h_N=0}^{1} V(\eta_1^{(h_1)}, \ldots, \eta_N^{(h_N)}) \prod_{a=1}^{N} u_a(\xi_a^{(h_a)}) v_a(\zeta_a^{(h_a)}), \tag{6.4}
\]

where $V(\eta_1^{(h_1)}, \ldots, \eta_N^{(h_N)}) \equiv \prod_{1\leq b<a\leq N}(\eta_a^{(h_a)} - \eta_b^{(h_b)})$ is the determinant of the following $N \times N$ Vandermonde matrix:

\[
V_{ij}^{(h_1, \ldots, h_N)} = (\eta_i^{(h_i)})^{j-1} \forall i, j \in \{1, \ldots, N\}. \tag{6.5}
\]

Now, by using the multilinearity of the determinants, we can rewrite the rhs of (6.4) in the form (6.3). In more detail, implementing the sum on $h_1$ we can write:

\[
\langle u|v \rangle = \sum_{h_1, \ldots, h_N=0}^{1} \tilde{V}_1(\eta_1^{(h_1)}, \eta_2^{(h_2)}, \ldots, \eta_N^{(h_N)}) \prod_{a=2}^{N} u_a(\xi_a^{(h_a)}) v_a(\zeta_a^{(h_a)}), \tag{6.6}
\]

where $\tilde{V}(\eta_1, \eta_2^{(h_2)}, \ldots, \eta_N^{(h_N)})$ is the determinant of the following $N \times N$ matrix:

\[
\tilde{V}_{ij}^{(h_1, h_2, \ldots, h_N)} \equiv \begin{cases} \sum_{b=0}^{1} u_a(\xi_a^{(h)}) v_a(\zeta_a^{(h)}) (\eta_a^{(h)})(\beta-1) & \forall j \in \{1, \ldots, N\} \\ (\eta_a^{(h)})^{j-1} & \forall i \in \{2, \ldots, N\}, j \in \{1, \ldots, N\} \end{cases}, \tag{6.7}
\]

and implementing the sum on the remaining $h_i$ we get our result (6.3).

The explicit and manageable determinant formulae that are derived here are central toward the achievement of exact computation on the dynamics associated with this quantum system. Indeed, it is worth recalling that left and right transfer matrix eigenstates are of separate form (6.1) and (6.2), respectively. Then, these determinant formulae apply in particular to transfer matrix eigenstates, in this way providing the first fundamental step toward the exact computation of form factors of local operators. A direct comparison with the results obtained in [1, 2] for the scalar products associated with the transfer matrix eigenstates of the 6-vertex reflection algebra allows us to clarify the above statement of universality and opens the possibility of mainly using the same procedure introduced in [1] to generate some first form factors of local operators.

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7. Conclusion and outlook

In this paper we have considered representation of the 8-vertex reflection algebra and we have studied the quantum models associated with the most general integrable boundary conditions on the spin-1/2 quantum chains and have developed for them the SOV method, obtaining the following results.

- The complete integrability of these quantum models and the complete characterization of their spectrum (transfer matrix eigenvalues and eigenstates) in terms of the set of solutions to an inhomogeneous system of \( N \) quadratic equations in \( N \) unknowns, where \( N \) is the number of sites of the chain.

  It is important to remark here that, for the most general boundary conditions and values of the coupling constant \( \eta \), the previous characterization is not yet proven to be equivalent to a characterization in terms of Bethe equations and this equivalence can surely only be proven by imposing some constrains on the boundary parameters or on the coupling constant. In particular, in a future paper we will show, as in the case \( \eta \) an elliptic root of unit, that we can derive for the most general integrable boundary conditions a Baxter Q-operator and rewrite the SOV spectrum characterization in terms of solutions to a system of Bethe equations.

- The action of left separate states on right separate states are written in terms of one determinant formulae of \( N \times N \) matrices. These matrices have elements given by sums over the spectrum of quantum separate variables of products of the corresponding left/right separate coefficients.

These results define the required setup to compute matrix elements of local operators on transfer matrix eigenstates. The remarked similarities in the SOV representations of the gauge transformed reflection algebras and the form of the pseudo-measure entering in the SOV spectral decomposition of the identity for both the 8-vertex and 6-vertex case imply the possibility of solving in parallel these two \textit{a priori} very different dynamical problems. In particular, in a future publication we will address the analysis of the following steps.

- Reconstruction of local operators in terms of Sklyanin’s quantum separate variables.

- Representation of form factors of local operators on transfer matrix eigenstates in determinant form.

Let us comment that (I) is a fundamental step in the solution of the dynamical problem since it allows us to identify the local operators writing them in terms of the global generators of the SOV representation. In fact, this identification has represented a longstanding problem in the S-matrix formulation\(^{16}\) of the dynamics of infinite volume quantum field theories and the lattice approach seems to give an advantage to make it solvable. Moreover, once it is solved it allows us to compute algebraically the actions of local operators on transfer matrix eigenstates and write them as linear combinations of separate states, from which the form factors can be computed by using our results on the action of left separate states on right separate states. Let us also point out that the reconstructions derived in the 6-vertex reflection algebra case also apply to the 8-vertex reflection algebra and that, being both the gauge transformed 8-vertex and 6-vertex reflection algebra generators written as linear combinations of the ungauged ones, the solution of the reconstruction problem for the most general integrable boundary conditions is simply derived once it is solved for the ungauged 6-vertex one following the

\(^{16}\) A large literature has been dedicated to this problem and several results are known \([91–96]\), which confirm the characterization \([97, 98]\) of these models as (superrenormalizable) massive perturbations of conformal field theories by relevant local fields from which a classification of their local field content (solutions to the form factor equations \([99, 100]\)) can be developed by using the corresponding ultraviolet conformal field theories.
approach described in [1]. This last observation implies that we are already able to describe the matrix elements of a class of quasi-local operators for the most general reflection algebra representations of both 8-vertex and 6-vertex type; indeed, in order to do so we just need to elaborate the results of this paper with those of [2] and the matrix elements in the ungauged SOV framework derived in [1].

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