Subcritical laminar-turbulence transition with wide domains in simple two-dimensional Navier-Stokes flow without walls

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(Dated: May 14, 2018)

We have confirmed numerically that a subcritical laminar-turbulence transition that belongs to directed percolation (DP) universality class occurs in a purely two-dimensional (2D) simple Navier-Stokes (NS) flow without any walls. The flow is called (extended) Kolmogorov flow which is governed by 2D NS equation in a doubly periodic box with a linear drag and a finite flow rate in the direction in which Galilean invariance is broken. We examine the mechanism of DP class transition focusing on the role of the additional control parameters: the drag coefficient and the flow rate. The drag kills coherent active structures of the system size. The flow rate interferes the growth of weak disturbances. These two effects control two essential and intrinsic elements of an absorbing state phase transition, i.e., the existence of an absorbing state and the locality of active dynamical structures. We also discuss what physically corresponds to the additional parameters in general flow.

Turbulence, macroscopically complex behavior, is seen in a wide range of phenomena in nature not only in a scale of our daily life such as a streamflow of a river and experiments in laboratory [1, 2] but also from quantum turbulence of superfluid [3, 4] to cosmic scales [5]. It can be regarded as a class of solutions of hydrodynamic equations whose energy is distributed into a wide range of scales. However, it is not completely uncovered for a long time how to sustain turbulence even for a flow in a circular pipe driven by pressure gradient [1, 6, 7].

At the middle of 20 century, the scenarios of supercritical transition have been submitted from the point of bifurcation of dynamical systems [8–10]. In a phase space a fixed point corresponding to a laminar flow becomes linearly unstable at a finite critical non-dimensional velocity, called Reynolds number in general, and a cascade of bifurcations leads chaotic behavior corresponding to turbulence. This type of scenario explains a part of turbulence transition well such as thermal convection and wake behind cylinder [11].

However, linear stability theory is unable to explain subcritical transition to turbulence. It is well known that representative canonical wall-bounded flows such as pipe flow and Couette flow have sustained turbulent solutions starting from a finite amplitude of disturbance even when the laminar flow is linearly stable. Such subcritical transition is difficult to be elucidated since dominant modes are unclear because of nonlinearity.

At the end of 20th century, a conjecture was submitted to connect (subcritical) laminar-turbulence transition with nonequilibrium phase transition common in statistical physics [12]. This idea is that turbulence transition would belong to Directed percolation (DP) universality class if there exist absorbing laminar state and turbulent regions which expand and decay stochastically and locally. Universality class tells us approximate predictions of spatio-temporal structure of disordered regions at least [13].

Recent developments of understandings of spatially-localized turbulent states (SLT) enable us to identify candidates for physical correspondence to a unit of turbulence such as puff in pipe flow [14]. Under detailed laboratory and numerically experimental investigations of subcritical turbulence transition, critical exponents for turbulence fraction and spatio-temporal structure are satisfactorily consistent with those of DP universality class for sustained wall turbulence [15], channel flow turbulence [16] and other flow [17].

DP universality class is supposed to appear under simple conjectures like locality and stochastic behavior of dynamics [18]. So, DP behavior arises for a wide range of models of epidemics and synchronization, experiments for turbulent liquid crystals [19, 20]. However, from the point of governing equations of fluid dynamics, which are nonlinear partial differential equations, the dynamics of fluid should strongly depend on way to drive, boundary conditions and spatial dimension. Thus, how to emerge and sustain SLT and locality of dynamics are not obvious and remain open problem [21]. Generally, flows with material wall boundaries are difficult to be studied since the walls induce both explicit and implicit inhomogeneities in the flows. This difficulty prevent us from relating large scale behavior of turbulent system such as subcritical regime of turbulence to relatively well-known dynamics of small systems [21–31]. Thus, a tractable Navier-Stokes model to investigate subcritical transition is promised.

One of the most simple two-dimensional (2D) flows sustaining turbulence called Kolmogorov flow was submitted to investigate routes to turbulence [32–37]. Kolmogorov flow is used to obtain many characteristics of solutions of Navier-Stokes equation and as a test field of novel idea of understanding complex solutions [38–41]. Recently, solutions in which spatially-localized chaotic regions and steady regions coexist were found even in 2D Kolmogorov flow [42–44].

In this letter, we report observations of subcritical laminar-turbulence transition belonging to DP universality class in pure 2D flow as shown in FIG. 1. We also dis
cuss the origin of linear stability of the laminar solution and the locality of dynamics in this Navier-Stokes model.

![Visualization of the flow for (Re,Reγ, U_y, n, α) = (242, 30, 0.5, 4, 1/256). (a) Typical time evolution of turbulent region(black). White region denotes laminar region. (b) Snapshot of a part of the whole domain, x ∈ [16π, 20π], using vorticity ω. The whole domain is x ∈ [0, 512π].](image)

FIG. 1. Visualization of the flow for (Re,Reγ, U_y, n, α) = (242, 30, 0.5, 4, 1/256). (a) Typical time evolution of turbulent region(black). White region denotes laminar region. (b) Snapshot of a part of the whole domain, x ∈ [16π, 20π], using vorticity ω. The whole domain is x ∈ [0, 512π].

We focus on two-dimensional (2D) Kolmogorov flow with the linear drag force in a doubly periodic box (x, y) ∈ [0, 2π/α] × [0, 2π]. The velocity field \( \mathbf{u} = (u_x, u_y) \) governed by 2D Navier-Stokes equation with a steady sinusoidal force in non-dimensionalized form as follows:

\[
\begin{align*}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla P - \gamma (\mathbf{u} - U_y \hat{y}) \\
&\quad + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} + \sin(ny)\hat{x}, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

where \( \alpha, \text{Re}, \gamma, n, \hat{x} \) and \( \hat{y} \) denote the aspect ratio of the rectangle domain, Reynolds number, the coefficient of drag force, the wave number of the external force, the unit vectors in \( x \) and \( y \), respectively. The pressure function \( P \) is doubly periodic. The velocity averaged over the box and the flow rate in \( y \)-direction are denoted by \( \mathbf{U} \) and \( U_y \) which are defined as follows:

\[
\mathbf{U} = \frac{\alpha}{4\pi^2} \int dxdy \mathbf{u} = (0, U_y).
\]

Note that since the \( y \) dependence of the external force breaks Galilean invariance in \( y \), the flow rate \( U_y \) is a control parameter of the system. To nondimensionalize the governing equation we use \( L_y/2\pi \) for the characteristic length scale where \( L_y \) is the length of the box in \( y \)-direction. For the work rate of the force to be \( O(1) \), the characteristic time is set to \( \sqrt{L_y/2\pi \chi} \) where \( \chi \) is the amplitude of the sinusoidal force. The linear drag force is typically originated from weak but inevitable effects of material walls. For the scales longer than the non-dimensional length \( L_{\text{drag}} \sim (\gamma \text{Re})^{-1/2} \), the drag term dominates the viscous term in Eq. (1). We expect that the drag force kills large scale structures of the system size formed by the energy transport toward large scales as occurring in 2D inverse-cascade turbulence[15-17].

Our DNS solves the governing equation for the vorticity \( \omega = \partial_y u_x - \partial_x u_y \) with the pseudo-spectral method for spatial discretization using the two-thirds rule for dealiasing and the 2nd order Runge-Kutta (Heun) method for time evolution. The time and spatial resolutions used for DNSs are \( 2 \times 10^{-3} \) and 64 points per \( 2\pi \), respectively.

When a subcritical laminar-turbulence transition is realized in 2DKF, the laminar state should be linearly stable at least. First, we check the stability of the following laminar state which is the stationary solution with global \( x \)-translational symmetry and denoted by \( u_{\text{lam}}(U_y) \):

\[
u_{\text{lam}}(U_y) = \frac{\text{Re}}{D} \sin(ny + \theta)\hat{x} + U_y\hat{y}.
\]

Here, \( \tan \theta = \text{Re}U_y n/(n^2 + \gamma) \) and \( D^2 = (n^2 + \gamma\text{Re})^2 + (n\text{Re}U_y)^2 \). This one-parameter form of Eq. (4) can be applied approximately for a part of the whole system and this locally-embedded laminar profile can take a certain value of the local flow rate, i.e., \( \beta \) which is not necessarily equal to the system flow rate \( U_y \). Hereafter, we call \( u_{\text{lam}}(U_y) \) as the global laminar (GL) state and \( u_{\text{lam}}(\beta) \) as a local laminar state, respectively.

For \( \gamma = 0 \), we can estimate the critical Reynolds number for the linearly unstable GL state \( \text{Re}_l \) under the assumption that the GL state is unstable for a long-wave disturbance in \( x \)-direction.

The vorticity equation for \( \gamma = 0 \) is rewritten in terms of the doubly periodic stream function \( \Psi \) as follows:

\[
\begin{align*}
\partial_t \nabla^2 \Psi - \{\Psi, \nabla^2 \Psi\}_{xy} &= U_y \partial_y \nabla^2 \Psi - \frac{1}{\text{Re}} (\nabla^2 \Psi)^2 = n \cos(ny),
\end{align*}
\]

where \( \{f, g\}_{xy} = \partial_x f \partial_y g - \partial_y f \partial_x g \).

We search a stationary solution that is the GL state as follows:

\[
\Psi(x, y) = \Psi_{\text{lam}}(y) + \sum_{i=0} \epsilon^i \Psi_i(\epsilon x, y),
\]

where \( u_{\text{lam}}(U_y) = (\partial_y \Psi_{\text{lam}}(y), U_y) \) and \( \epsilon \) is the small parameter.

Since \( \partial_y \Psi_0 = 0 \) in the limit of \( \epsilon \to 0 \), we set \( \Psi_0 = A(\epsilon x) \). Executing the perturbation expansion with Eq. (5) yields

\[
\mathcal{L}_y \Psi_i = F_i(\Psi_{\text{lam}}, \Psi_0, \ldots, \Psi_{i-1}),
\]

\[
\mathcal{L}_y = \frac{1}{\text{Re}} \partial_y^4 - U_y \partial_y^3,
\]

for the i-th order of \( \epsilon \). We assume that there exists a non-trivial solution of \( \mathcal{L}_y \Psi_0 \) = 0, then \( \Psi_0 \) should be a function independent of \( y \). A necessary condition that each order of Eq. (7) has a solution is that the inhomogeneous term of Eq. (7) is orthogonal to \( \Psi_0 \), as follows:

\[
\int dy F_i(\Psi_{\text{lam}}, \Psi_0, \ldots, \Psi_{i-1}) = 0
\]

at any \( X = \epsilon x \). This condition (9) is rewritten as

\[
\int dy (\{\Psi, \partial_y^2 \Psi\}_{xy} + \epsilon \frac{1}{\text{Re}} \partial_y^4 \Psi) = 0
\]
for each order of $\epsilon$. At the zeroth order Eq.(10) is automatically satisfied and

$$\Psi_{\text{lam}} = -\frac{Re}{D_n} \cos(ny + \delta). \quad (11)$$

At the first order

$$\partial_X^2 \int dy (- (\partial_y \Psi_L) \Psi_1 + \frac{1}{Re} \partial_X A) = 0. \quad (12)$$

We assume that $\Psi_1 = (Re/D)^2 \partial_X A \sin(ny + 2\delta)$, then we obtain the following condition for Reynolds number $Re$

$$(1 - 2n^2 U_y^4) Re^4 - 4U_y^2 n^4 Re^2 - 2n^6 = 0. \quad (13)$$

Therefore, the condition for $\Psi_1$ to exist is that $Re$ of Eq.(13) is positive real. When $|U_y| < U_y^c(n) = 1/(\sqrt{2} \sqrt{n})$, there exists the following unique real solution of Eq.(13):

$$Re = Re_1 = \sqrt{\frac{2U_y^2 n^4 + n^3 \sqrt{2}}{1 - 2n^2 U_y^2}}. \quad (14)$$

This solution is a likely candidate for the neutral stability curve of the GL state: $Re_c = Re_1(n, U_y)$. In fact, in the case $U_y = 0$, we obtain the same result as the previous works. Equation (14) shows the flow rate stabilizes the GL state which becomes linearly stable at any $Re$ when $|U_y| > U_y^c(n) = (2n^2)^{(-1/4)}$. We expect this lower bound, i.e., $U_y^c(n)$ at $\gamma = 0$ also comes into effect at $\gamma > 0$ for most cases.

The stabilization of the GL state for large $U_y$ can be understood as follows. The amplitude of the laminar flow has an upper bound proportional to $U_y^{-1}$. Thus, for large $U_y$ the amplitude of the GL state is too small to be unstable. In the following numerical experiments at the subcritical regime, we choose $U_y > U_y^c$ in order for laminar states to be stable.

Next, we check the effect of the drag forcing to coherent structures of the flow numerically. We focus on the distribution of velocity in $y$-direction, since $U_y$ is a conserved and effective indicator of redistribution of momentum. When $\gamma Re$ is small, the whole state is composed of kink-antikink arrays connecting between adjacent local laminar states with $\beta \neq U_y$ as shown in FIG 2. This state is reminiscent of Cahn-Hilliard like flow emerged at the weak nonlinear regime for $U_y = 0$ and $\gamma = 0$ where the GL state is linearly unstable for large $Re$. When $\gamma Re$ is large, the most part of the whole domain are filled with the GL state and strong vorticity regions emerge spatially-intermittently as shown in FIGS 1 and 2. These kinds of structures are strongly connected with the locality of the dynamics of SLT.

To simplify the situation, we assume the whole domain is separated in $x$-direction into two types of regions in terms of $V(x) = \int_0^{2\pi} dy u_y(x, y)/2\pi$. First type is the region which is a laminar state with $V \sim V_0$ and occupies most of the whole domain. The other type is SLT whose characteristic width is $L_{\text{SLT}}$. There are $N$ turbulent states and $V$ averaged over the width of the i-th state is denoted by $V_i$. Then, integrating $V(x)$ over the whole domain, we obtain the relationship among $V_0$ and $V_i$:

$$\sum_{i=1}^{N} V_i - V_0 \sim (U_y - V_0)/\rho \quad (15)$$

FIG. 2. Average flow over $y$ direction, $V(x) = \int_0^{2\pi} dy u_y(x, y)/2\pi$ for $Re = 240$. (a) $\gamma Re = 0$, (b) $\gamma Re = 30$. Straight line ($V = 0.5$) denotes the GL state.

Therefore, the condition $V_0 \sim U_y$ should be satisfied for a subcritical transition occurred in 2DKF to belong to DP universality class. This condition holds for large $\gamma Re$ in our system.

We numerically conduct quenching experiments and confirm the occurrence of the DP class transition under the conditions that $n = 4$, $U_y = 0.5 > U_y^c(4) \sim 0.42$ and $\gamma Re = 30$ in large domains of $1/\alpha = 64$ and 256. Initial velocity fields are obtained from sustaining turbulence consisting of many SLTs at $Re = 1000$, then $Re$ is reduced to the target values of $Re \in [230, 250]$. Typically, SLTs make their own replicas and decay to local laminar states as randomly and independently.

As the numerical definition of the local turbulence state by binalization, we adopt the following local distance in 1D form to the local laminar state denoted by $L_{\text{loc}}(x)$:

$$L_{\text{loc}}(x) = \frac{1}{2\pi \delta x} \int_0^{2\pi} dy \int_{x-\delta x}^{x+\delta x} d\omega (\omega(x') - \omega_{\text{lam}})^2, \quad (16)$$

where $\omega_{\text{lam}}$ denotes the vorticity of GL state. Then, turbulent states are defined by satisfying the condition that
\( \{ x | L_{oc}(x) > \text{const} \} \). The threshold constant is not sensitive to the results so that we choose \( \delta x = \pi / 16 \) which is comparable to the characteristic width of the typical SLT, \( L_{SLT} \). FIG 4 shows typical visualization of turbulent region in this definition.

To confirm some aspects of the DP critical phenomenon observed in our 2D system, we estimate the turbulence fraction and the distributions of laminar gaps both in space and time. Their dependence on Re is also examined. Based on the estimated critical Reynolds number \( Re_c = 241 \), we confirm that the three independent critical indices \( (\beta, \mu_\perp, \mu_\parallel) \) are consistent with the predictions by the \((1+1)D\) DP universality class as shown in FIGs 3 and 4. Note that these indices are estimated by the following scaling relations with \( \epsilon = Re/Re_c - 1 \):

\[
\rho^*(\epsilon) \sim \epsilon^\beta, \quad (17)
\]

\[
N(S_{lam}) \sim S_{lam}^{-\mu_\perp}, \quad (18)
\]

\[
N(T_{lam}) \sim T_{lam}^{-\mu_\parallel}, \quad (19)
\]

where \( \rho^* \) is \( \rho(t) \) for large \( t \), and \( N(S_{lam}) \) and \( N(T_{lam}) \) are the histograms of the laminar gaps between turbulent region in spatial direction \( S_{lam} \) and temporal direction \( T_{lam} \), respectively. For \((1+1)D\) DP class, the values of the indexes we used are \( \beta = 0.276, \mu_\perp = 1.097, \mu_\parallel = 1.734, \mu_\parallel = 1.748 \) and \( \mu_\parallel = 1.841 \) which were estimated in ref. [48].

Turbulence fraction \( \rho^* \) shown in FIG 3 is consistent with a continuous transition of \((1+1)D\) DP especially for \( \epsilon > 0 \). Relatively larger error bars are originated from the divergence of long time correlation at the critical point. For a smaller domain \( \alpha = 1/64, \rho(t) \) quickly decays to zero for \( \epsilon < -0.01 \). This is because large fluctuations cause accidentally transitions to the absorbing state. To estimate a finite size effect, we conducted a finite size scaling estimation by considering spatial correlation. As shown in FIG 3, function \( L^{\beta/\nu_\parallel} \rho^*(\epsilon/L^{1/\nu_\parallel}) \) does not depend on the system size \( L \) and scales as \( \epsilon^\beta \). Then, we can expect that as \( L \rightarrow \infty \), \( \rho^*(\epsilon) \) approaches the scaling since the \( \epsilon^\beta \) part in \( \epsilon > 0 \) remains invariant but \( \rho \) in \( \epsilon < 0 \) goes to zero.

Each of the two critical exponents \( \mu_\perp, \mu_\parallel \) of the GL state gaps is connected with the fractal box-counting dimension in each direction. Both of the histograms in FIG 4 show consistency with the DP prediction. The slope for the spatial gaps in FIG 4 is slightly less steeper as discussed in ref. [17].

As shown in FIG 5, by the scaling with the DP critical exponents, the decays of the turbulence fraction for quenching processes for a wide range of Re values collapse approximately to a single universal function, \( F(t) = \rho(t-\epsilon/\nu_\parallel) t^{\beta/\nu_\parallel} \). All these results support that the subcritical transition observed in 2DKF belongs to \((1+1)D\) DP universality class.

In summary, laminar-turbulence transition belonging to DP universality class, which is commonly seen in wall-bounded flows with large domains, also occurs in a 2D flow with periodic boundary conditions. This result explicitly shows the concrete form(s) of self sustained process(SSP) for near-wall dynamics is not necessarily essential for subcritical transition observed in wall-bounded flows. Moreover, we can also conclude the DP class transition realized for subcritical turbulence transition is not universal. This is supported by counterexamples observed for Kolmogorov flow with \( \gamma = 0 \).

The essential characteristics of DP, the existence of the absorbing state and the locality, are related in our system to the flow rate \( U_y \) and the drag forcing \( -\gamma u \). This parameter dependency is difficult to be identified for wall-bounded flows since they typically have only one independent parameter Re.
We now consider what plays the roles of $U_y$ and $\gamma$ in wall-bounded flows or more general flows. As discussed above, the drag forcing can be interpreted as a frictional force by the walls. From the governing equation (1), $U_y$ can be interpreted as an advection speed of a weak disturbance in the frame of zero flow rate: the forcing term is $\sin(n(y - U_y t))$ in this frame. This advection effect disturbs an efficient supply of kinetic energy for disturbances to grow. This interpretation is consistent with the fact that the tails of SLT do not develop for large $U_y$ flows \[43, 44\]. These kinds of effects can be seen in wall-bounded flow but not be controlled. An advantage of our system is that the important effects can be easily separated in terms of a multidimensional parameter space. Thus, we expect that even more general flows share the characteristics of our simple system and that more detailed researches help us to grasp fundamental aspects of sustenance of turbulence.

To confirm the above conjecture, we need to estimate or describe theoretically and quantitatively the interaction or correlation among localised turbulent structures. These estimation and description are important not only for the consideration of minimum elements of DP transition but also for analyses of collective behaviors of localised turbulent structures which appear as large scale structures such as turbulence stripes in the wall-bounded turbulence\[2 \ 49 \ 52\].

We have dealt with laminar-turbulence transition observed in 2DKF which is governed by the 2D Navier-Stokes equation. This governing equation must be the simplest in the physically-supported spatio-temporal evolution equations which can describe the transition of DP universality class. In this sense, we expect that we can study the relationship between governing law and universality class of nonequilibrium phase transition from more general contexts.

This work was supported by JSPS KAKENHI Grant Number JP18J13334, Grant-in-Aid for JSPS Research Fellow; the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

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