EXTENDED VISUAL CRYPTOGRAPHY SCHEMES

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ABSTRACT. Visual cryptography schemes have been introduced in 1994 by Naor and Shamir. Their idea was to encode a secret image into $n$ shadow images and to give exactly one such shadow image to each member of a group $P$ of $n$ persons. Whereas most work in recent years has been done concerning the problem of qualified and forbidden subsets of $P$ or the question of contrast optimizing, in this paper we study extended visual cryptography schemes, i.e. shared secret systems where any subset of $P$ shares its own secret.

1. Introduction

A visual cryptography scheme is given by the following set up. Let $P$ be a group of $n$ persons where each participant is given exactly one image (in fact it does not have to be a real image) xeroxed onto a transparency. Stacking all the transparencies together, a secret image is recovered. So in this sense the participants share a secret. This set up can be generalized to the case where some subsets $X \subseteq P$ (which are usually called qualified subsets of $P$) can recover the secret by stacking their transparencies together, whereas other, forbidden subsets cannot. Such structures, called access structures, have been examined very well. In [5] Naor and Shamir analysed so-called $(k, n)$-threshold visual cryptography schemes, i.e. schemes where a subset is qualified if and only if it consists of at least $k$ participants. In [1] and [2] their idea was extended to general access structures.

Most work concerning this subject focuses on two aspects, either the pixel expansion, i.e. the number of subpixels which is needed on the different levels to represent a white or a black pixel, or the contrast, i.e. the difference of subpixels representing a white or a black pixel.

As a further generalization, the existence of a secret image can be concealed by displaying a different image on each slide. Naor and Shamir [5] solved this problem for the $(2, 2)$-threshold scheme. In [3] this problem was considered for a general access structure. In [4] Droste made a further generalization: Stacking the transparencies of each participant together, a secret image is recovered, and there is in fact only this single way to recover it. But moreover, the participants of any arbitrary subset $X$ of $P$ share a secret, too. Hence we have $2^n - 1$ more or less secret images.

We start by briefly recalling the work done by Droste and prove that the scheme proposed in [4] has minimal pixel expansion. Then we prove a trade-off theorem between the contrast of the different images.

Finally we give new constructions for generalized visual cryptography schemes with less then $2^n - 1$ subsets in order to achieve a smaller pixel expansion and a better contrast.

2. Preliminaries

A visual cryptography scheme is based on the fact that each pixel of an image is divided into a certain number $m$ of subpixels. This number $m$ is called the pixel expansion of the image. If the number of black subpixels needed to represent a white pixel in an image is
An extended visual cryptography scheme consists of \( n \) transparencies \( \tau_1, \ldots, \tau_n \) and \( 2^n - 1 \) different images (one for each non-empty subset \( T \subseteq \{1, \ldots, n\} \)). We denote by \( I_T \) the image which is recovered by stacking together exactly the transparencies \( \tau_i \) for \( i \in T \). We generalize this as follows. For any non-empty subset \( S \subseteq \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\} \) an \( S \)-extended visual cryptography scheme consists of \( n \) transparencies \( \tau_1, \ldots, \tau_n \) with the following property: Let \( T \in S \). If we stack together the slides \( \tau_i \) for \( i \in T \), then we recover the image \( I_T \) for which each white pixel is represented by \( l_T \) black subpixels and each black pixel is represented by \( h_T \) black subpixels. Furthermore, for \( T' \) not contained in \( T \), the distribution of subpixels on the transparencies \( \tau_i \) with \( i \in T \) is independent of the image \( I_{T'} \), i.e. the information of the transparencies \( \tau_i \) with \( i \in T \) does not suffice to recover the image \( I_{T'} \).

More formally, we define an \( S \)-extended visual cryptography scheme as follows. (See also [4] for “usual” visual cryptography schemes and [4] for \( S \)-extended visual cryptography schemes.)

**Definition 2.1.** Let \( S \subseteq \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\} \).

An \( S \)-extended visual cryptography scheme is described by multi-sets \( C^S \) of \( n \times m \) Boolean matrices for \( \mathcal{T} \subseteq S \). (For given \( \mathcal{T} \) each Boolean matrix in \( C^S \) describes the colors of the subpixels on each transparency, where the corresponding pixel in image \( I_T \) is black if and only if \( T \in \mathcal{T} \). For encoding, each matrix in \( C^S \) is chosen with the same probability.)

The multi-sets \( C^S \) must satisfy the following conditions:

1. Let \( B \in C^S \). For \( \{i_1, \ldots, i_q\} \in S \) the Hamming weight of the OR of the rows \( i_1, \ldots, i_q \) of \( B \) is \( h_{\{i_1, \ldots, i_q\}} \) if \( \{i_1, \ldots, i_q\} \in \mathcal{T} \) and \( l_{\{i_1, \ldots, i_q\}} \) otherwise, i.e. \[
\begin{align*}
\omega_{\text{Ham}}((b_{i_1,1}, \ldots, b_{i_1,m}) \text{ OR } \ldots \text{ OR } (b_{i_q,1}, \ldots, b_{i_q,m})) &= \\
&= \begin{cases}
h_{\{i_1, \ldots, i_q\}} & \text{if } \{i_1, \ldots, i_q\} \in \mathcal{T} \\
l_{\{i_1, \ldots, i_q\}} & \text{if } \{i_1, \ldots, i_q\} \notin \mathcal{T}
\end{cases}. \\
\end{align*}
\]
(This means stacking the transparencies \( \tau_{i_1}, \ldots, \tau_{i_q} \) together we recover the image \( I_{\{i_1, \ldots, i_q\}} \).

2. For \( \{i_1, \ldots, i_q\} \subseteq \{1, \ldots, n\} \) and \( \mathcal{T}, \mathcal{T}' \subseteq S \) with \( \mathcal{T} \cap \mathcal{P}(\{1, \ldots, n\}) = \mathcal{T}' \cap \mathcal{P}(\{1, \ldots, n\}) \) we obtain the same multi-sets if we restrict the matrices in \( C^S \) and \( C^{S'} \) respectively to the rows \( i_1, \ldots, i_q \).

(This condition guarantees the security of the different images.)

If \( S = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\} \) we simply call this an extended visual cryptography scheme.

In [4] Droste gives the following construction for \( S \)-extended visual cryptography schemes using \((k,k)\)-threshold schemes.

**Construction 2.2.** For each \( T \in S \) we take \( 2^{|T|-1} \) subpixels and use them to construct a \((|T|,|T|)\)-threshold visual cryptography scheme. If \( i \notin T \) the corresponding subpixels on \( \tau_i \) will be black. The \( S \)-extended visual cryptography scheme is achieved by putting all these schemes together. Since we shall not need the details of this construction in the sequel, we omit a formal definition and refer to [4].

The scheme obtained by this construction has pixel expansion

\[
m = \sum_{T \in S} 2^{|T|-1}
\]

and the contrast of all encoded images is \( \frac{1}{m} \). We shall prove in the following sections that this construction is optimal if \( S = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\} \) but it is not optimal for general \( S \).
It is sufficient to consider the case of one single pixel. For a given non-empty subset \( T \subseteq \{1, \ldots, n\} \) let \( x_T \) be the number of subpixels which are black exactly on the transparencies \( i \) for \( i \in T \) and let us denote by \( x \) the vector of all the \( x_T \). For a given non-empty subset \( T \subseteq \{1, \ldots, n\} \) let \( r_T \) be the number of black subpixels needed for the image \( I_S \). Formally, we set \( r_\emptyset = 0 \). We write \( r \) for the vector of all the \( r_T \) with \( \emptyset \neq S \subseteq \{1, \ldots, n\} \).

This leads to the linear equation system given by

\[
M x = r
\]

where \( M = (m_{S,T})_{\emptyset \neq S,T \subseteq \{1, \ldots, n\}} \) is defined by \( m_{S,T} = 1 \) if \( S \cap T \neq \emptyset \) and \( m_{S,T} = 0 \) otherwise.

**Lemma 3.1.** Equation (1) has a unique integral solution.

**Proof.** We prove this by induction on the number \( n \) of transparencies. Writing \( M_1 = (1) \), we obtain the following recursion formula which follows directly from the definition:

\[
M_{n+1} = \begin{pmatrix}
M_n & 0 & M_n \\
0 & 1 & 1 \\
M_n & 1 & 1
\end{pmatrix}
\]

Here the index denotes the number of transparencies.

Let \( e_n \) denote the \((2^n - 1)\)-dimensional vector \((1, \ldots, 1)^t\). With \( M_1^{-1} = (1) \) we obtain the following recursion formula for \( M_n^{-1} \):

\[
M_n^{-1} = \begin{pmatrix}
0 & -M_n^{-1}e_n & M_n^{-1} \\
-e_n^tM_n^{-1} & 0 & e_n^tM_n^{-1} \\
M_n^{-1}e_n & -M_n^{-1}
\end{pmatrix}
\]

We notice that the components of \( M_n^{-1}e_n \) are only \(-1, 0 \) and \( 1 \) and that \( e_n^tM_n^{-1}e_n = 1 \). Then the formula can be proved by induction.

Thus \( M_n^{-1} \) contains only the entries \(-1, 0 \) and \( 1 \) and therefore equation (1) has an integral solution. \( \square \)

**Lemma 3.2.** The solution of (1) is non-negative if and only if for each \( S \subseteq \{1, \ldots, n\} \) the condition

\[
\sum_{S \subseteq T \subseteq \{1, \ldots, n\}} (-1)^{|S|+|T|} r_T \leq 0
\]

is satisfied.

**Proof.** We claim that \( x = (x_S) \) with

\[
x_S = \sum_{\{1, \ldots, n\} \setminus S \subseteq T \subseteq \{1, \ldots, n\}} (-1)^{|T|+|S|+n+1} r_T
\]

solves equation (1) and due to Lemma 3.1 this solution is unique.
To prove this we substitute $x$ in equation (1). For $\emptyset \neq U \subseteq \{1, \ldots, n\}$ the line of the system of linear equations corresponding to $U$ yields

$$\sum_{\emptyset \neq S \subseteq \{1, \ldots, n\}} m_{U,S} x_S = \sum_{S \subseteq \{1, \ldots, n\}} m_{U,S} \sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|T| + |S| + n + 1} r_T$$

$$= \sum_{T \subseteq \{1, \ldots, n\}} \sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|T|} r_T m_{U,S}$$

$$= \sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} (-1)^{|T|} r_T m_{U,S}$$

If $T \not\subseteq U$ we choose $t \in T \setminus U$ and obtain

$$\sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|S| + n + 1} m_{U,S} = \sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|S| + n + 1} m_{U,S} + (-1)^{|S| + n + 1} m_{U,S \cup \{t\}} = 0$$

since $m_{U,S} = m_{U,S \cup \{t\}}$.

If $T \subseteq U$ and $T \neq \emptyset$ we find

$$\sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|S| + n + 1} m_{U,S} = \sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|S| + n + 1} = 0$$

since $m_{U,S} = 1$.

But for $\emptyset \neq T = U$ we find

$$\sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|S| + n + 1} m_{U,S} = \sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|S| + n + 1} - (-1)^{|1, \ldots, n\} \setminus U + n + 1$$

$$= (-1)^{|1, \ldots, n\} \setminus U + n = (-1)^{|U|} r_U$$

since $m_{U,S} = 1$ for $S \neq \{1, \ldots, n\} \setminus U$.

Thus equation (4) yields

$$\sum_{\emptyset \neq S \subseteq \{1, \ldots, n\}} m_{U,S} x_S = \sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} (-1)^{|T|} r_T \sum_{\{1, \ldots, n\} \setminus T \subseteq \{1, \ldots, n\}} (-1)^{|S| + n + 1} m_{U,S}$$

$$= (-1)^{|U|} r_U (-1)^{|1, \ldots, n\} \setminus U + n$$

$$= (-1)^{2n} r_U = r_U$$.

This proves that $x$ is a solution of equation (1) and the lemma follows. \qed

Now we can solve (2) to derive bounds for the pixel expansion and the contrast.

**Theorem 3.3.** An extended visual cryptography scheme with $n$ transparencies needs at least $\frac{1}{8}(3^n - 1)$ subpixels. Hence Construction 2.2 is optimal with respect to the pixel expansion.

**Proof.** First we note that the inequality (2) is as strong as possible if $r_T = h_T$ for $|S| + |T|$ even and $r_T = l_T$ for $|S| + |T|$ odd.
Thus an extended visual cryptography scheme exists if and only if
\[
\sum_{S \subseteq T \subseteq \{1, \ldots, n\}} h_T \leq \sum_{S \subseteq T \subseteq \{1, \ldots, n\}} l_T
\]
holds for each \( S \subseteq \{1, \ldots, n\} \).

For each \( \emptyset \neq T \subseteq \{1, \ldots, n\} \) let \( \delta_T = h_T - l_T \). Our goal is to prove that
\[
m \geq h_{\{1, \ldots, n\}} \geq \sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} \delta_T 2^{|T| - 1}.
\]

As the first step in this proof we show that, for given values \( \delta_T \) (for \( \emptyset \neq T \subseteq \{1, \ldots, n\} \)), the number \( h_{\{1, \ldots, n\}} \) is minimal if for all \( S \subseteq \{1, \ldots, n\} \) inequality (5) is satisfied with equality.

To this end, suppose
\[
\sum_{S \subseteq T \subseteq \{1, \ldots, n\}} h_T < \sum_{S \subseteq T \subseteq \{1, \ldots, n\}} l_T
\]
for some \( S \subseteq \{1, \ldots, n\} \). But the contrast levels
\[
\tilde{h}_T = \begin{cases} h_T & \text{for } T \subseteq S \\ h_T - 1 & \text{otherwise} \end{cases}
\]
and
\[
\tilde{l}_T = \begin{cases} l_T & \text{for } T \subseteq S \\ l_T - 1 & \text{otherwise} \end{cases}
\]
satisfy (5), since
\[
|\{T \mid S \subseteq T \subseteq \{1, \ldots, n\}; |T| \equiv |S| \mod 2; T \not\subseteq S\}| = \\
|\{T \mid S \subseteq T \subseteq \{1, \ldots, n\}; |T| \not\equiv |S| \mod 2; T \not\subseteq S\}|.
\]

Thus we may assume that inequality (5) is satisfied with equality for each \( S \subseteq \{1, \ldots, n\} \).

Next we claim that
\[
h_T = \sum_{\emptyset \neq T' \subseteq \{1, \ldots, n\}} \delta_{T'} 2^{|T'|-1} - \sum_{T \subseteq T' \subseteq \{1, \ldots, n\}} \delta_{T'} 2^{|T'|-1-|T|}
\]
for \( \emptyset \neq S \subseteq \{1, \ldots, n\} \) satisfy (5) with equality.

To prove this we have to show that
\[
\sum_{S \subseteq T \subseteq \{1, \ldots, n\}} \left[ \sum_{\emptyset \neq T' \subseteq \{1, \ldots, n\}} \delta_{T'} 2^{|T'|-1} - \sum_{T \subseteq T' \subseteq \{1, \ldots, n\}} \delta_{T'} 2^{|T'|-1-|T|} \right] = \\
\sum_{S \subseteq T \subseteq \{1, \ldots, n\}} \left( \sum_{\emptyset \neq T' \subseteq \{1, \ldots, n\}} \delta_{T'} 2^{|T'|-1} - \sum_{T \subseteq T' \subseteq \{1, \ldots, n\}} \delta_{T'} 2^{|T'|-1-|T|} \right) - \delta_T
\]
or equivalently

\[
\sum_{\emptyset \neq T' \subseteq \{1, \ldots, n\}} \delta_{T'} \left[ \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} \frac{2^{|T'|-1} - \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} 2^{|T'|-1} - |T|}{|S| \equiv |T| \mod 2} \right] = \\
\sum_{\emptyset \neq T' \subseteq \{1, \ldots, n\}} \delta_{T'} \left[ \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} \frac{2^{|T'|-1} - \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} 2^{|T'|-1} - |T|}{|S| \equiv |T| \mod 2} \right] + \delta_{T'} \left( \frac{(-1)^{|T'|+|S| - 1}}{2} \right).
\]

(Note that the last summand is equal to \(-\delta_{T'}\) for \(|T'| \neq |S| \mod 2\) and equal to 0 otherwise.)

Comparing coefficients for each \(\delta_{T'}\) we obtain

\[
2^n - |S| - 1 \cdot 2^{|T'|-1} - \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} \frac{2^{|T'|-1} - |T|}{|S| \equiv |T| \mod 2} = \\
2^n - |S| - 1 \cdot 2^{|T'|-1} - \left( \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} \frac{2^{|T'|-1} - |T|}{|S| \equiv |T| \mod 2} \right) + \frac{(-1)^{|T'|+|S| - 1}}{2},
\]

but this is true since

\[
(-1)^{|T'|-|S|} = (1 - 2)^{|T'|-|S|} = \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} \frac{2^{|T'|-|T|} - \sum_{S \subseteq T' \subseteq \{1, \ldots, n\}} 2^{|T'|-|T|}}{|T'| \equiv |T| \mod 2}.
\]

Suppose that \(\bar{h}_T\) (for \(\emptyset \neq T \subseteq \{1, \ldots, n\}\)) satisfy \(\delta\) with equality, too. For \(S \neq \emptyset\) inequality \(\delta\) gives

\[
\bar{h}_S = h_S + \bar{h}_{\{1, \ldots, n\}} - \bar{h}_{\{1, \ldots, n\}}.
\]

But for \(S = \emptyset\) inequality \(\delta\) yields \(\bar{h}_{\{1, \ldots, n\}} = h_{\{1, \ldots, n\}}\) and therefore \(\bar{h}_T = h_T\) for all \(\emptyset \neq T \subseteq \{1, \ldots, n\}\).

This proves that \((8)\) is the only solution of \((\delta)\) that satisfies all inequalities with equality. Thus we find

\[
m \geq h_{\{1, \ldots, n\}} \geq \sum_{\emptyset \neq T' \subseteq \{1, \ldots, n\}} \delta_{T'} 2^{|T'|-1} \geq \sum_{\emptyset \neq T' \subseteq \{1, \ldots, n\}} 2^{|T'|-1} = \frac{1}{2}(3^n - 1).
\]

Next we prove a trade-off between the contrast of the different images.

**Theorem 3.4.** For \(\emptyset \neq T \subseteq \{1, \ldots, n\}\) let \(\alpha_T = \frac{\bar{h}_T - h_T}{m}\) be the contrast of the image \(I_T\). The contrast levels of the images satisfy

\[
\sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} 2^{|T'|-1} \alpha_T \leq 1.
\]

Further let \(\alpha'_T \geq 0\) (for \(\emptyset \neq T \subseteq \{1, \ldots, n\}\) satisfy \(\delta\)). Then for every \(\varepsilon > 0\) there exists a generalized visual cryptography scheme with contrast levels \(\alpha_T\) (for \(\emptyset \neq T \subseteq \{1, \ldots, n\}\)) where \(\alpha_T - \alpha'_T < \varepsilon\) for nonempty subsets \(T\) of \(\{1, \ldots, n\}\).
Proof. Let $\delta_T = h_T - l_T$. By (7) we conclude
\[
m \geq h_{\{1, \ldots, n\}} \geq \sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} \delta_T 2^{|T|-1}
\]
and therefore
\[
\sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} 2^{|T|-1} \alpha_T = \frac{1}{m} \sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} \delta_T 2^{|T|-1} \leq 1.
\]
Now assume (8) holds for $\alpha_T$. Then we choose $\delta_T \in \mathbb{N}$ and $M \in \mathbb{N}$ with
\[
0 \leq \alpha_T' - \frac{\delta_T}{M} \leq \varepsilon.
\]
By (7) we know that there exists an extended visual cryptography scheme with contrast levels $h_T - l_T = \delta_T$ and minimal pixel expansion
\[
m = \sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} 2^{|T|-1} \delta_T.
\]
Since $\frac{\delta_T}{M} \leq \alpha_T$ and $\alpha_T$ satisfy (8) we find $m < M$.

If we add useless subpixels (e.g. subpixels that are always black) to the extended visual cryptography scheme constructed above, we obtain a scheme with contrast $\alpha_T = \frac{\delta_T}{M}$. This proves the theorem. \(\square\)

4. Pixel Expansion and Contrast for the $\mathcal{G}$-Extended Scheme

In the previous section we proved that Construction 2.2 is optimal if $\mathcal{G} = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\}$.

If we set $\delta_T = 1$ for $T \in \mathcal{G}$ and $\delta_T = 0$ for $T \notin \mathcal{G}$ in equation (7) we obtain the same contrast values as in Construction 2.2. In this sense Construction 2.2 can be viewed as a $(\mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\})$-extended visual cryptography scheme with degenerated contrast values.

Theorem 4.1. Let $\mathcal{G} = \mathcal{P}(\{1, \ldots, n\}) \setminus \emptyset, \{1, \ldots, n\}$ then Construction 2.2 is optimal if and only if $n$ is odd.

Proof. Let $\delta_T = 1$ for $\emptyset \neq T \subseteq \{1, \ldots, n\}$ and $\delta_{\{1, \ldots, n\}} = 0$. Then $h_T$ and $l_T = h_T - \delta_T$ as in (7) satisfy (8) and these are the solutions given by 2.2.

Let us assume $n$ is even. We show that in this case we can find a better construction in the sense that fewer subpixels are required. Setting $\overline{h}_T = h_T - 1$ and $\overline{l}_T = l_T - 1$ for $\emptyset \neq T \subseteq \{1, \ldots, n\}$ and $\overline{h}_\emptyset = \overline{l}_\emptyset = 0$ we observe that inequality (8) still holds for all $S \neq \emptyset$. Thus the solution $x = (x_T)_{\emptyset \neq T \subseteq \{1, \ldots, n\}}$ of equation (1) satisfies $x_T \geq 0$ for $T \neq \{1, \ldots, n\}$.

The value of $x_{\{1, \ldots, n\}}$ will be non-negative unless $r_T = \overline{h}_T$ for $|T|$ even and $r_T = \overline{l}_T$ for $|T|$ odd. In this special case equation (3) gives the solution $x_{\{1, \ldots, n\}} = -1$. To obtain a solution with positive $x_{\{1, \ldots, n\}}$ we adjust the value of $r_{\{1, \ldots, n\}}$ from $\overline{h}_{\{1, \ldots, n\}}$ to $\overline{h}_{\{1, \ldots, n\}} - 1$. (This is possible, since $\{1, \ldots, n\} \notin \mathcal{G}$ which means that the number of black subpixels in the stack of all transparencies does not matter.) Now (3) reveals the solution
\[
x_{\{1, \ldots, n\}} = \sum_{T \subseteq \{1, \ldots, n\}} (-1)^{|T|+1} r_T
\]
\[
= \sum_{T \subseteq \{1, \ldots, n\}, |T| \text{ odd}} \overline{l}_T - \left( \sum_{T \subseteq \{1, \ldots, n\}, |T| \text{ even}} \overline{h}_T \right) + 1 = -1 + 1 = 0.
\]
For $S \neq \{1, \ldots, n\}$ and $|S|$ even we obtain

$$
x_S = \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}} (-1)^{|T|+|S|+n+1} r_T
$$

$$
= \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}, |T| \text{ odd}} \tilde{T}_T - \left( \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}, |T| \text{ even}} \tilde{T}_T \right) + 1 = 0 + 1 = 1.
$$

For $|S|$ odd we obtain

$$
x_S = \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}} (-1)^{|T|+|S|+n+1} r_T
$$

$$
= \left( \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}, |T| \text{ even}} \tilde{T}_T \right) - 1 - \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}, |T| \text{ odd}} \tilde{T}_T
$$

$$
> \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}, |T| \text{ even}} \tilde{T}_T - \left( \sum_{\{1, \ldots, n\}\setminus S \subseteq T \subseteq \{1, \ldots, n\}, |T| \text{ odd}} \tilde{T}_T \right) - 1 = -1.
$$

Thus all possible values of $r$ lead to non-negative solutions for $x$, hence an $\mathcal{S}$-extended visual cryptography scheme with $m = H_{\{1, \ldots, n\}} < h_{\{1, \ldots, n\}}$ exists, i.e. the solution given by 2.2 is not optimal.

Now we assume $n$ is odd. Let $\tilde{h}_T$ and $\tilde{l}_T$ ($\emptyset \neq T \subsetneq \{1, \ldots, n\}$) be a solution of 5 different from $h_T$ and $l_T$. Let $H_{\{1, \ldots, n\}}$ be the maximal number of black subpixels occurring when all the transparencies are stacked together.

Since $\tilde{h}_T$ and $\tilde{l}_T$ solve inequality 5, we can apply the same arguments leading from 5 to 7 and find

$$
\bar{h}_{\{1, \ldots, n\}} - \tilde{h}_T \geq h_{\{1, \ldots, n\}} - h_T
$$

for all $\emptyset \neq T \subseteq \{1, \ldots, n\}$. But for $S = \emptyset$ in 5, we obtain

$$
\sum_{T \subseteq \{1, \ldots, n\}, |T| \text{ even}} \tilde{T}_T \leq \sum_{T \subseteq \{1, \ldots, n\}, |T| \text{ odd}} \tilde{T}_T
$$

and hence, together with 6,

$$
\tilde{l}_{\{1, \ldots, n\}} \geq l_{\{1, \ldots, n\}} = h_{\{1, \ldots, n\}}.
$$

But by definition $\bar{h}_{\{1, \ldots, n\}} \geq \tilde{l}_{\{1, \ldots, n\}}$ and therefore this solution needs at least $h_{\{1, \ldots, n\}}$ subpixels, i.e. the solution given by 2.2 is optimal. \hfill \square

We notice that in the first part of the proof of 4.1 we did not need the assumption $\delta_T = 1$ for $\emptyset \neq T \subsetneq \{1, \ldots, n\}$. In fact, it is sufficient to assume $h_T \neq 0$ where $h_T$ is defined by 7. A short calculation proves that this is the case if $\mathcal{S} \not\subseteq \mathcal{P}(S)$ for a proper subset $S$ of $\{1, \ldots, n\}$. Thus the first part of 4.1 proves:

**Corollary 4.2.** For $\mathcal{S} \subseteq \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset, \{1, \ldots, n\}\}$, $\mathcal{S} \not\subseteq S$ for any proper subset $S$ of $\{1, \ldots, n\}$ and $n$ even, Construction 2.2 is not optimal.
Even more general we can prove:

**Theorem 4.3.** Let $\mathcal{S} \subseteq \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\}$. Let us assume that there exists a nonempty subset $T \in \mathcal{P}(\{1, \ldots, n\}) \setminus \mathcal{S}$ with $|T|$ even and that $\mathcal{S} \cap \mathcal{P}(T) \not\subseteq \mathcal{P}(T')$ for each proper subset $T'$ of $T$. Then Construction 2.2 is not optimal.

**Proof.** We apply [4] to $\mathcal{S} \cap \mathcal{P}(T)$. For a proper subset $T'$ of $T$ we define $\tilde{t}_{T'}$ and $\tilde{t}_T$ as in the proof of [1]. Formally we define $\tilde{t}_T = h_T - 2$ and $\tilde{t}_T = h_T - 1$ where $h_T$ is defined by (7). Let $\delta_T = -1$ and for $T \neq S \subseteq \{1, \ldots, n\}$ we define $\delta_S = 0$ for $S \not\in \mathcal{S}$ and $\delta_S = 1$ otherwise.

Corresponding to (7) we define

$$\tilde{h}_S = \begin{cases} 
\tilde{t}_T + \sum_{\emptyset \neq S' \subseteq \{1, \ldots, n\} \mid -|S'|} \delta_S' 2^{|S'| - 1} - \sum_{S \subseteq S' \subseteq \{1, \ldots, n\} \mid -|S'|} \delta_S' 2^{|S'| - 1} - |S| & \text{for } S \not\subseteq T \\
\tilde{t}_T + \sum_{\emptyset \neq S' \subseteq \{1, \ldots, n\} \mid -|S'|} \delta_S' 2^{|S'| - 1} - \sum_{S \subseteq S' \subseteq \{1, \ldots, n\} \mid -|S'|} \delta_S' 2^{|S'| - 1} - |S| & \text{for } S \subseteq T 
\end{cases}$$

for $\emptyset \neq S \subseteq \{1, \ldots, n\}$ and $\hat{t}_S = \tilde{h}_S - \delta_S$.

An easy but tedious calculation corresponding to the one that was used in the proof of Theorem 3.3 shows that $\hat{h}_S, \hat{t}_S$ satisfy (5). Hence an $\mathcal{S}$-extended visual cryptography system with contrast levels $\hat{h}_S, \hat{t}_S$ exists. This scheme has a smaller pixel-expansion than the scheme given by Construction 2.2. \qed

5. Conclusions and further remarks

Equation (6) gives us a simple method to construct an $\mathcal{S}$-extended visual cryptography scheme with given contrast values $l_T$ and $h_T$. Furthermore, for fixed $n$ and $\mathcal{S}$, equation (6) leads to a linear programming problem which describes all possible $\mathcal{S}$-extended visual cryptography schemes. For small values of $n$ this problem can easily be solved.

In this article we have given a full solution for the special cases $\mathcal{S} = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\}$ and $\mathcal{S} = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset, \{1, \ldots, n\}\}$. We close by presenting the following open problems:

1. We conjecture:
   Let $\mathcal{S} \subseteq \mathcal{P}(\{1, \ldots, n\}) \setminus \{\emptyset\}$. Then Construction 2.2 is optimal for an $\mathcal{S}$-extended visual cryptography scheme if and only if for all $\emptyset \neq T \not\in \mathcal{S}$ we have either $|T|$ odd or $\mathcal{S} \cup \mathcal{P}(T) \not\subseteq \mathcal{P}(T')$ for some proper subset $T'$ of $T$.

2. An even harder problem is a full characterization of $\mathcal{S}$-extended visual cryptography schemes with minimal pixel expansion for arbitrary subsets $\mathcal{S}$ of $\mathcal{P}(\{1, \ldots, n\})$, i.e. to find a formula for the minimal pixel expansion depending on $\mathcal{S}$.

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