ON THE QUARTIC CURVATURE GRAVITY IN THE CONTEXT OF FRW COSMOLOGY

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Abstract

We consider the purely gravitational fourth-order (in the spacetime curvature) quantum corrections to the Einstein-Hilbert gravity action, coming from superstrings in the leading order with respect to the Regge slope parameter, and study their impact on the evolution of the Hubble scale in the context of the FRW cosmology, in four spacetime dimensions. We propose the generalized Friedmann equations, and rule out the most naive (Bel-Robinson tensor squared) gravity. Our new cosmological equations have exact inflationary solutions without a spacetime singularity.

1 Introduction

The homogeneity and isotropy of the Universe, as well as the observed spectrum of density perturbations, are explained by inflationary cosmology [1]. Inflation is usually realised by introducing a scalar field (inflaton) and choosing an appropriate scalar potential. By using Einstein equations it gives rise to a massive violation of the strong energy condition, and the need of an exotic matter with large negative pressure (dark energy). Despite of the simplicity of many inflationary scenarios, the origin of their key ingredients, such as the inflaton and its scalar potential, remain obscure. As is well known, the Standard Model of elementary particles has no inflaton.

Being the strongest candidate for a unified theory of Nature, including a consistent quantum gravity sector, the superstrings/M-theory provide the natural arena for building specific mechanisms of inflation. In recent years, many brane inflation scenarios were proposed (see e.g. ref. [2] for a review), including their embeddings into the (warped) compactified superstring models, in a good package with the phenomenological constraints coming from particle physics (see e.g. ref. [3]). However, it did not contribute to revealing the origin of the key ingredients of inflation. It also greatly increased the number of possibilities up

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to $10^{100}$ (known as the String Landscape), hampering any specific theoretical predictions in the search for signatures of strings and branes in the Universe.

The inflaton driven by a scalar potential, or their engineering by strings and branes, are by no means required. The alternative could be a modification of the gravitational part of Einstein equations by terms of the higher order in the spacetime curvature \[4\]. It needs neither an inflaton nor an exotic matter, while the higher-curvature terms do appear in the effective action of superstrings \[5\].

The perturbative strings are merely defined on-shell (in the form of quantum amplitudes), while they give rise to the infinitely many higher-curvature corrections to the Einstein equations, to all orders in the Regge slope parameter $\alpha'$ and the string coupling $g_s$, whose finite form is unknown and is beyond our control. However, it still makes sense to consider the leading corrections to the Einstein equations, coming from strings and branes. Being valid for limited energy scales, the results to be obtained from them cannot be conclusive, but they may offer both qualitative and technical insights into cosmology, within the well defined and very restrictive framework. We reconsider the fundamentals of that approach towards inflation, based on the Einstein equations modified by the leading superstrings-generated gravitational terms to be treated on equal footing with the Einstein terms, i.e. non-perturbatively.

We consider only geometrical (i.e. pure gravity) terms in the low-energy superstring effective action in four space-time dimensions. We assume that the quantum $g_s$-corrections can be suppressed against the leading $\alpha'$-corrections, whereas all the moduli, including a dilaton and an axion, are somehow stabilized (e.g. by fluxes, after compactification to four dimensions), so that the naive dimensional reduction of the quantum gravity terms is valid.

\section{Superstrings-modified gravity equations}

The only purely gravitational terms, coming from type-II superstrings in four spacetime dimensions, are given by

\[ S_4 = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + \beta J_R \right) \]  

where $\beta$ is a new coupling constant, and

\[ J_R = R^{mijn} R_{pisp} R_{qsp} R_{rsn} + \frac{1}{2} R^{mijn} R_{pplq} R_{mplq} R_{qplq} R_{rsn} + O(R_{mn}) \]  

that can also be rewritten as the Bel-Robinson tensor squared \[6\].

As regards the four-dimensional heterotic strings, the action (2.1) is to be supplemented by the term \[7\]

\[ S_H = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( \frac{1}{2} J_H \right) \]  

where

\[ J_H = R_{ijkl} R^{ijkl} + O(R_{mn}) \]
again modulo Ricci-dependent terms.

The gravitational action is to be added to a matter action, which lead to the modified Einstein equations of motion (in the type II case, for definiteness)

\[ R_{ij} - \frac{1}{2} g_{ij} R + \beta \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ij}} (\sqrt{-g} J_R) = \kappa^2 T_{ij} \]  

(2.5)

where \( T_{ij} \) stands for the energy-momentum tensor of all the matter fields (including dilaton and axion).

There is about a hundred of the Ricci-dependent terms in the most general off-shell gravitational effective action quartic in the curvature. It also means about 100 of the new coefficients, which makes fixing the off-shell action to be extremely difficult. The quartic curvature terms are thus different from the quadratic curvature terms, present in the on-shell heterotic string effective action (2.3), whose off-shell extension is very simple (see below).

The absence of the higher order time derivatives is usually desirable to prevent possible unphysical solutions to the equations of motion, as well as preserve the perturbative unitarity, but it is by no means necessary. As is well known, the standard Friedmann equation of General Relativity is an evolution equation, i.e. it contains only the first-order time derivatives of the scale factor [1, 8]. It happens due to the cancellation of terms with the second-order time derivatives in the mixed 00-component of Einstein tensor — see e.g. Appendix of ref. [9] for details. It can also be seen as the consequence of the fact that the second-order dynamical (Raychaudhuri) equation for the scale factor in General Relativity can be integrated once, by the use of the continuity equation, thus leading to the evolution (Friedmann) equation. As regards the quadratic curvature terms present in the heterotic case, their unique off-shell extension is given by the Gauss-Bonnet-type combination [10]

\[ J_H \rightarrow G = R_{ijkl} R^{ijkl} - 4 R_{ij} R^{ij} + R^2 \]  

(2.6)

In the expansion around Minkowski space, \( g_{ij}(x) = \eta_{ij} + h_{ij}(x) \), the fourth-order derivatives (at the leading order in \( O(h^2) \)) coming from the first term in eq. (2.6) cancel against those in the second and third terms [11]. As a result, the off-shell extension (2.6) appears to be ghost-free in any dimensions. As regards four space-time dimensions, the terms (2.6) can be rewritten as the four-dimensional Euler density. Therefore, being a total derivative, eq. (2.6) does not contribute to the four-dimensional effective action.

The matter equations of motion in General Relativity imply the covariant conservation law of the matter energy-momentum tensor,

\[ (T^{ij})_{;j} = 0 \]  

(2.7)

By the well known identity \( (R^{ij} - \frac{1}{2} g^{ij} R)_{;j} = 0 \), eqs. (2.5) and (2.7) yield

\[ \left[ \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ij}} (\sqrt{-g} J) \right]_{;j} = 0 \]  

(2.8)
For instance, when $J = G$ as in eq. (2.6), eq. (2.8) reads

\[-\frac{1}{2}(R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2)_{;m} + 2(R_{mijkl}R^{ijkl})_{;n} \]
\[-4(R_{mjkl}R^{mjkl})_{;n} - 4(R_{i}R_{ij})_{;n} + 2(RR_{mn})_{;n} = 0 \quad (2.9)\]

By the use of Bianchi identities for the curvature tensor, we found that the left-hand-side of eq. (2.9) identically vanishes in the case of Gauss-Bonnet gravity. Equation (2.8) should be identically satisfied by any off-shell gravitational correction $J$.

Given the quartic curvature terms (2.2), the modified Einstein equations of motion (2.5) are given by

\[\kappa^2 T_{ij} = R_{ij} - \frac{1}{2}g_{ij} R + \beta \left[ -\frac{1}{2}g_{ij} J_R - R_{mhk(i} R_{j)rt}^m (R^{kqrt} R^{r}_{hq} + R^{ksqt} R^{hr}_{qs}) \right. \]
\[\left. - R_{qg(i} R_{j)rnt} (R^{hsqt} R^{krm} k - R^{thsq} R^{rmk}_h) + (R_{i}R_{j})_{;k} R^{k}_{;h} \right] \]
\[+ (R_{isqt} R^{tkm} R_{j}^{sq})_{;kr} - (R^{hrs} (i R_j)_{mn} R^{m}_{nk} + R^{sht} (i R_j)_{mn} R^{k}_{mn} R^{h}_{nk})_{;k} \] \quad (2.10)

3 Off-shell quartic terms in FRW cosmology

The main Cosmological Principle of a spatially homogeneous and isotropic (1 + 3)-dimensional universe (at large scales) gives rise to the standard Friedmann-Robertson-Walker (FRW) metric

\[\text{ds}^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \quad (3.1)\]

where the function $a(t)$ is known as the scale factor in ‘cosmic’ coordinates $(t, r, \theta, \phi)$; we use $c = 1$ and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, while $k$ is the FRW topology index taking values (−1, 0, +1). Accordingly, the FRW metric (3.1) admits a 6-dimensional isometry group $G$ that is either $SO(1, 3)$, $E(3)$ or $SO(4)$, acting on the orbits $G/SO(3)$, with the spatial 3-dimensional sections $H^3$, $E^3$ or $S^3$, respectively. By the coordinate change, $dt = a(t)d\eta$, the FRW metric (3.1) can be rewritten to the form

\[\text{ds}^2 = a^2(\eta) \left[ d\eta^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2 \right] \quad (3.2)\]

which is manifestly (4-dim) conformally flat in the case of $k = 0$. Therefore, the 4-dim Weyl tensor of the FRW metric obviously vanishes in the ‘flat’ case of $k = 0$. In fact, the FRW Weyl tensor vanishes in the other two cases, $k = -1$ and $k = +1$, too [9], thus

\[C_{ijkl}^{\text{FRW}} = 0 \quad (3.3)\]
The inflation in early universe is defined as an epoch during which the scale factor is accelerating [1],

\[ \dddot{a}(t) > 0, \quad \text{or equivalently} \quad \frac{d}{dt} \left( \frac{H^{-1}}{a} \right) < 0 \quad (3.4) \]

where the dots denote time derivatives, and \( H = \frac{\dot{a}}{a} \) is Hubble ‘constant’. The amount of inflation is given by a number of e-foldings [1],

\[ N = \ln \frac{a(t_{\text{end}})}{a(t_{\text{start}})} = \int_{t_{\text{start}}}^{t_{\text{end}}} H \, dt \quad (3.5) \]

which should be around 70 [1].

On the experimental side, it is known that the vacuum energy density \( \rho_{\text{inf}} \) during inflation is bounded from above by a (non)observation of tensor fluctuations of the Cosmic Microwave Background (CMB) radiation [14],

\[ \rho_{\text{inf}} \leq (10^{-3} M_{\text{Pl}})^4 \quad (3.6) \]

It severely constrains but does not exclude the possibility of the geometrical inflation originating from the purely gravitational sector of string theory, because the factor of \( 10^{-3} \) above may be just due to some numerical coefficients.

Due to a single arbitrary function \( a(t) \) in the FRW Ansatz (3.1), it is enough to take only one gravitational equation of motion in eq. (2.5) without matter, namely, its mixed 00-component. As is well known [1], the spatial (3-dimensional) curvature can be ignored in a very early universe, so we choose the manifestly conformally-flat FRW metric (3.1) with \( k = 0 \) in our Ansatz. It leads to a purely gravitational equation of motion having the form

\[ 3H^2 \equiv 3 \left( \frac{\dot{a}}{a} \right)^2 = \beta P_8 \left( \frac{\dddot{a}}{a}, \frac{\ddot{a}}{a}, \frac{\dot{a}}{a}, \frac{a}{a} \right), \quad (3.7) \]

where \( P_8 \) is a polynomial with respect to its arguments,

\[ P_8 = \sum_{n_1 + 2n_2 + 3n_3 + 4n_4 = 8, \quad n_1, n_2, n_3, n_4 \geq 0} c_{n_1 n_2 n_3 n_4} \left( \frac{\dot{a}}{a} \right)^{n_1} \left( \frac{\ddot{a}}{a} \right)^{n_2} \left( \frac{\dddot{a}}{a} \right)^{n_3} \left( \frac{a}{a} \right)^{n_4} \quad (3.8) \]

Here the sum goes over the integer partitions \( (n_1, 2n_2, 3n_3, 4n_4) \) of 8, the dots stand for the derivatives with respect to time \( t \), and \( c_{n_1 n_2 n_3 n_4} \) are some real coefficients. The highest derivative can enter only linearly, \( n_4 = 0, 1 \).

The FRW Ansatz with \( k = 0 \) yields the relevant curvatures as follows:

\[ R_{0i0j}^0 = -\delta_{ij} \dddot{a} a, \quad R^i_{jkl} = (\delta_k^i \delta_{j}^l - \delta_k^l \delta_{j}^i) (\dddot{a})^2, \quad R^i_j = \delta^i_j \left[ \frac{\dddot{a}}{a} - 2 \left( \frac{\dot{a}}{a} \right)^2 \right] \quad (3.9) \]
where $i, j = 1, 2, 3$. For example, in the case of the $(BR)^2$ gravity (2.10), after a straightforward (though quite tedious) calculation of the mixed 00-equation without matter and with the curvatures (3.9), we find

$$3H^2 + \beta \left[ 9 \left( \frac{\dddot{a}}{a} \right)^4 - 72H^2 \left( \frac{\ddot{a}}{a} \right)^3 + 96H^4 \left( \frac{\ddot{a}}{a} \right)^2 - 36H \left( \frac{\ddot{a}}{a} \right) \left( \frac{\dddot{a}}{a} \right) \right]$$

$$+ 75H^8 - 72H^3 \left( \frac{\ddot{a}}{a} \right) \left( \frac{\dddot{a}}{a} \right) + 24H^6 \left( \frac{\dddot{a}}{a} \right) - 24H^5 \left( \frac{\dddot{a}}{a} \right) \right] = 0 \quad (3.10)$$

It is remarkable that the 4th order time derivatives (present in various terms of eq. (2.10)) cancel, whereas the square of the 3rd order time derivative of the scale factor, $\left( \frac{\dddot{a}}{a} \right)^2$, does not appear at all in this equation. 4

Our generalized Friedmann equation (3.7) applies to any combination of the quartic curvature terms in the action, including the Ricci-dependent terms. The coefficients $c_{n_1n_2n_3n_4}$ in eq. (3.8) can be thought of as linear combinations of the coefficients in the most general quartic curvature action. The polynomial (6.3) has just about 10 coefficients to be determined.

Moreover, eqs. (3.7) and (3.8) have the structure that allows the existence of generic exact inflationary solutions without a spacetime singularity. Indeed, when using the most naive (de Sitter) Ansatz for the scale factor,

$$a(t) = a_0 e^{At}\quad (3.11)$$

with some real positive constants $a_0$ and $A$, and substituting eq. (3.11) into eq. (3.7) we get $3A^2 = (\#)\beta A^8$, whose coefficient $(\#)$ is just a sum of all $c$-coefficients in eq. (3.8). Assuming the $(\#)$ to be positive, we find a solution

$$A = \left( \frac{3}{\#\beta} \right)^{1/6}\quad (3.12)$$

This solution in non-perturbative in $\beta$, i.e. it is impossible to get it when considering the quartic curvature terms as a perturbation. Of course, the assumption that we are dealing with the leading correction, implies $At \ll 1$. It leads to the natural hierarchy

$$\kappa M_{KK} \ll 1 \quad \text{or} \quad l_{Pl} \ll l_{KK} \quad (3.13)$$

where we have introduced the four-dimensional Planck scale $l_{Pl} = \kappa$ and the compactification scale $l_{KK} = M_{KK}^{-1}$.

The exact solution (3.11) is non-singular, while it describes an inflationary isotropic and homogeneous early universe 5.

In the case of the quartic terms given by the Bel-Robinson tensor squared, i.e. when ignoring all the Ricci tensor dependent terms in eq. (2.2), we find that the coefficient $(\#)$ is negative, thus ruling out that option because it does not allow the inflationary solution (3.12).

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4 Taking Weyl tensors instead of Riemann curvatures leads to all vanishing coefficients.

5 The exact de Sitter solutions in the special case (2.2) were also found in ref. [15].
4 Conclusion

The higher curvature terms in the gravitational action defy the famous Hawking-Penrose theorem [16] about the existence of a spacetime singularity in any exact solution to the Einstein equations. As we demonstrated here, the initial cosmological singularity can be easily avoided by considering the superstrings-motivated higher curvature terms on equal footing (i.e. non-perturbatively) with the Einstein-Hilbert term.

As regards inflation, though we showed the natural existence of inflationary (de Sitter) exact solutions without a spacetime singularity under rather generic conditions on the coefficients in the higher-derivative terms, it is by no means sufficient, because our geometrical inflation is very short (not enough e-foldings), and has no end. In fact, we assumed the dominance of the higher curvature gravitational terms over all matter contributions in a very early universe at the Planck scale, and ignored all Kaluza-Klein modes beyond four spacetime dimensions. Given the expansion of a four-dimensional universe under the geometrical inflation, the spacetime curvatures decrease, so that the matter terms could no longer be ignored. The latter may effectively replace the geometrical inflation by another matter-dominated mechanism, allowing the inflation to continue substantially below the Planck scale. Needless to say, more research is needed in order to submit a specific mechanism for that.

The higher curvature terms are also relevant for the alternative (to inflation) Brandenberger-Vafa scenario of string gas cosmology [17] — see e.g. ref. [18] for a recent (perturbative) investigation of the higher curvature corrections there.

The higher time derivatives in the cosmological equations are unavoidable with the higher curvature terms in the action, they should not be ignored (as e.g. in ref. [19]), while they do not necessarily constitute a trouble.

The higher curvature terms may offer the alternative to the use of dynamical moduli, warped compactifications, or engineering desirable brane configurations [2, 3, 12], since all those popular mechanisms need many ad hoc assumptions about a non-perturbative off-shell effective action of M-theory, while no explanation is provided why certain brane configurations are to be considered and how they arise.

Gravity with the quartic curvature terms is a good playground for going beyond the Einstein equations. Our analysis may be part of a more general approach based on superstrings, including moduli and extra dimensions.

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