Complex Paths
and
Covariance of Hawking Radiation
in 2D Stringy Black Holes

Elias C. Vagenas

University of Athens
Physics Department
Nuclear and Particle Physics Section
Panepistimioupolis, Ilisia 157 71
Athens, Greece

Abstract
Hawking radiation is computed in different coordinate systems using the method of complex paths. In this procedure the event horizon of the 2D Schwarzschild stringy black hole is treated as a singularity for the semiclassical action functional. After the regularization of the event horizon’s singularity the emission/absorption probabilities and the Hawking temperature in the different coordinate representations are derived. The identical results obtained indicate the covariance of the Hawking radiation.
In 1974, S.W. Hawking observed that black holes radiate as a consequence of quantum effects and their radiation spectrum is thermal [1]. Therefore, the Hawking radiation extracts energy from the black hole interior and the mass of the black hole is diminished. This mass reduction is a physical phenomenon independent of the coordinate representation and thus Hawking radiation should also be covariant.

S.W. Hawking described the radiation process using Bogoliubov coefficients [1]. Their derivation required the knowledge of the wave modes of the quantum field in the Schwarzschild gauge. Anyhow, solving the wave equation in an arbitrary coordinate system in order to get the wave function [2] in terms of simple functions is not always feasible. Therefore, the problem of finding a method for evaluating the spectrum of the Hawking radiation without using the wave modes, turns up.

J.B. Hartle and S.W. Hawking trying to solve this problem introduced a semiclassical analysis [3]. Applying the Feynman path-integral approach they analytically continued the propagator $K(x', x)$ into the complexified Schwarzschild space where the propagator $K(x', x)$ is the amplitude for a particle to propagate from a spacetime point $x$ to a spacetime point $x'$ of the Schwarzschild black hole:

$$K(x', x) \approx \sum_{\text{paths}} \exp \left[ \frac{i}{\hbar} S(x', x) \right]$$

where $S$ is the classical action for a particular path connecting $x$ and $x'$. This analytic continuation led to the evaluation of the probabilities of emission and absorption of a particle with energy $E$:

$$P[\text{emission}] = P[\text{absorption}] e^{-\beta E}.$$  

The different values of these probabilities displayed the thermal spectrum of the Hawking radiation and its temperature. Unfortunately the Kruskal extension used in this method does not fit to all coordinate systems.

K. Srinivasan and T. Padmanabhan introduced a semi-classical treatment for evaluating the Hawking radiation [4] without the usage of wave modes or the complexification of spacetime. Their method stemmed from the non-relativistic semiclassical quantum mechanics [5]. The coordinate singularity at the event horizon of the Schwarzschild spacetime manifests itself as a singularity in the semiclassical propagator. Applying the WKB approximation for the propagator and assuming that the main contribution comes from the first term in the expansion, the method of complex paths appropriately regularizes the singularity in the action. Therefore, taking into account particles tunnelling from the
interior of the event horizon to the exterior of the event horizon, the Hawking radiation is recovered. Recently S. Shankaranarayanan, T. Padmanabhan and K. Srinivasan applied the method of complex paths for the Schwarzschild spacetime [6]. They obtained the correct temperature associated with the Hawking radiation in different coordinate systems.

In this paper, motivated by the work of S. Shankaranarayanan, T. Padmanabhan and K. Srinivasan in the background of a Schwarzschild black hole, we apply the method of complex paths in several different coordinate systems describing the two-dimensional “Schwarzschild” stringy black hole geometry [7].

(i) Schwarzschild gauge

The two-dimensional stringy black hole in the “Schwarzschild” gauge is characterized by the line element:

\[ ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 \]  

where the function \( g(r) \) is given by:

\[ g(r) = 1 - \frac{M}{\lambda} e^{-2\lambda r} \]  

and \( 0 < t < +\infty, r_H < r < +\infty \), with \( r_H = \frac{1}{2\lambda} \ln(\frac{M}{\Lambda}) \) the position of the event horizon of the black hole.

Considering now a massless scalar field \( \Phi(t, r) \) satisfying the Klein-Gordon equation \( \Box \Phi = 0 \) we have in the Schwarzschild gauge:

\[ -\frac{1}{g(r)} \frac{\partial^2}{\partial t^2} \Phi(t, r) + \frac{\partial}{\partial r} \left[ g(r) \frac{\partial}{\partial r} \Phi(t, r) \right] = 0. \]  

In order to obtain the solution of equation (5) we make the ansatz:

\[ \Phi(t, r) = \exp \left[ \frac{i}{\hbar} S(t, r) \right] \]  

where \( S \) is a function which following the WKB approximation will be expanded in powers of \( \hbar \) as follows:

\[ S(t, r) = S_0(t, r) + \hbar S_1(t, r) + \hbar^2 S_2(t, r) + \ldots \]  

Substituting (6) in the Klein-Gordon equation (5) we get:

\[ 0 = -\frac{1}{g(r)} \frac{i}{\hbar} S e^{\frac{i}{\hbar} S} \left( \frac{\partial^2 S}{\partial t^2} \right) - \frac{1}{g(r)} \left( \frac{i}{\hbar} \right)^2 S^2 e^{\frac{i}{\hbar} S} \left( \frac{\partial S}{\partial t} \right)^2 - \frac{1}{g(r)} \left( \frac{i}{\hbar} \right) e^{\frac{i}{\hbar} S} \left( \frac{\partial S}{\partial t} \right)^2 \]
\[ + \left( \frac{\partial g(r)}{\partial r} \right) \left( \frac{i}{\hbar} \right) S e^{\pm S} \left( \frac{\partial S}{\partial r} \right) + g(r) \left( \frac{i}{\hbar} \right) e^{\pm S} \left( \frac{\partial S}{\partial r} \right)^2 \]

\[ + g(r) \left( \frac{i}{\hbar} \right)^2 S^2 e^{\pm S} \left( \frac{\partial S}{\partial r} \right)^2 + g(r) \left( \frac{i}{\hbar} \right) S e^{\pm S} \left( \frac{\partial^2 S}{\partial r^2} \right). \hspace{1cm} (8) \]

If we now use the expansion of \( S \) given in (7) and we drop all terms of order \( \hbar \) and greater, we get:

\[ - \frac{1}{g(r)} \left( \frac{\partial S_0}{\partial t} \right)^2 + g(r) \left( \frac{\partial S_0}{\partial r} \right)^2 = 0. \hspace{1cm} (9) \]

This is the well known Hamilton-Jacobi equation describing the motion of a massless particle in the two-dimensional “Schwarzschild” stringy black hole geometry \( \mathbb{S} \).

The general solution of this equation reads:

\[ S_0 = E \left( -t \pm \int dr \frac{dr}{g(r)} \right). \hspace{1cm} (10) \]

Applying the saddle point method, the semiclassical propagator \( K(x', x) \) in equation (1) for the particle propagating from spacetime point \( x = (t_1, r_1) \) to spacetime point \( x' = (t_2, r_2) \) is given as:

\[ K(t_2, r_2; t_1, r_1) = N \exp \left[ \frac{i}{\hbar} S_0(t_2, r_2; t_1, r_1) \right] \hspace{1cm} (11) \]

where \( S_0 \) is the action functional satisfying the Hamilton-Jacobi equation and \( N \) is the appropriate normalization constant. Therefore if we solve equation (9) we will be able to evaluate the amplitudes and the probabilities of emission/absorption through the event horizon of the two-dimensional stringy black hole.

If we solve (9), we get:

\[ S_0(t_2, r_2; t_1, r_1) = -E(t_2 - t_1) \pm E \int_{r_1}^{r_2} \frac{dr}{g(r)} \]

\[ = -E(t_2 - t_1) \pm E \int_{r_1}^{r_2} \frac{dr}{1 - \frac{M}{\lambda} e^{-2\lambda r}}. \hspace{1cm} (12) \]

The sign ambiguity in equation (12) corresponds to the two different directions of motion of the massless particle with respect to the event horizon of the two-dimensional black hole.

For the outcoming massless particle, i.e. for \( r_1 < r_H \) the leading term \( S_0 \) of the action must fulfill the equation:

\[ \frac{\partial S_0}{\partial r} > 0 \hspace{1cm} (13) \]
thus we get:

$$S_0(t_2, r_2; t_1, r_1) = -E(t_2 - t_1) - E \int_{r_1}^{r_2} \frac{dr}{g(r)}.$$  \hspace{1cm} (14)$$

Trying to evaluate the definite integral we use the theorem of residues and choosing the contour to lie in the upper complex plane, the leading term $S_0$ for the outcoming massless particle is given as:

$$S_0[\text{emission}] = (\text{real part}) + i \frac{\pi}{2\lambda} E. \hspace{1cm} (15)$$

For the ingoing particle, i.e. for $r_2 > r_H$ the leading term $S_0$ of the action must fulfill the equation:

$$\frac{\partial S_0}{\partial r} < 0 \hspace{1cm} (16)$$

thus we get:

$$S_0(t_2, r_2; t_1, r_1) = -E(t_2 - t_1) + E \int_{r_1}^{r_2} \frac{dr}{g(r)}.$$  \hspace{1cm} (17)$$

Trying to evaluate the definite integral we use the theorem of residues and choosing the contour to lie in the upper complex plane, the leading term $S_0$ for the ingoing massless particle is given as:

$$S_0[\text{absorption}] = (\text{real part}) - i \frac{\pi}{2\lambda} E. \hspace{1cm} (18)$$

It is well known that the probabilities are given modulus square the amplitudes and for the case of emission and absorption are given by:

$$P[\text{emission}] \approx e^{-\frac{2}{\hbar} ImS_0[\text{emission}]} \hspace{1cm} (19)$$

$$P[\text{absorption}] \approx e^{-\frac{2}{\hbar} ImS_0[\text{absorption}]} \hspace{1cm} (20)$$

Therefore the ratio between these probabilities is:

$$P[\text{emission}] = P[\text{absorption}] e^{-\frac{2\pi}{\hbar\lambda} E}. \hspace{1cm} (21)$$

Comparing equation (21) with equation (2) and setting $\hbar = 1$ we obtain the expression for the temperature:

$$T_H = \beta^{-1} = \frac{\lambda}{2\pi} \hspace{1cm} (22)$$
which the correct formula of the Hawking temperature for the two-dimensional stringy black hole \[8\].

(ii) Unitary gauge

The line element of the two-dimensional stringy black hole in the unitary gauge is:

\[
ds^2 = -\tanh^2(\lambda y)dt^2 + dy^2
\]

(23)

where the “unitary” variable \(y\) is given in terms of \(r\):

\[
y = \frac{1}{\lambda} \ln \left[ e^{\lambda(r-r_H)} + \sqrt{e^{2\lambda(r-r_H)} - 1} \right]
\]

(24)

and \(0 < y < +\infty\).

The Klein-Gordon equation for a massless scalar field \(\Phi(t, y)\) in the unitary gauge reads:

\[
-\frac{1}{\tanh^2(\lambda y)} \frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{\tanh(\lambda y)} \frac{\partial}{\partial y} \left[ \tanh(\lambda y) \frac{\partial \Phi}{\partial y} \right] = 0.
\]

(25)

Substituting the ansatz (6) for the massless scalar field \(\Phi(t, y)\), using the expansion (7) for the action \(S\) and neglecting all terms of order \(\hbar\) and greater, the Klein-Gordon equation (25) will be:

\[
\frac{1}{\tanh^2(\lambda y)} \left( \frac{\partial S_0}{\partial t} \right)^2 + \left( \frac{\partial S_0}{\partial y} \right)^2 = 0
\]

(26)

This is the Hamilton-Jacobi equation describing (in the unitary gauge) the motion of a massless particle in the two-dimensional “Schwarzschild” stringy black hole. Solving equation (26) we get:

\[
S_0(t_2, y_2; t_1, y_1) = -E(t_2 - t_1) \pm E \int_{y_1}^{y_2} \frac{dy}{\tanh(\lambda y)}.
\]

(27)

The sign ambiguity as before refers to the two different directions of motion of the massless particle with respect to the event horizon of the black hole.

For the outcoming massless particle, i.e. for \(r_1 < r_H\) the leading term \(S_0\) of the action satisfying equation (13) and using the complex path method is given by:

\[
S_0[\text{emission}] = (\text{real part}) + i\frac{\pi}{\lambda}E
\]

(28)

while for the ingoing massless particle, i.e. for \(r_2 > r_H\) the leading term \(S_0\) of the action is given as:

\[
S_0[\text{absorption}] = (\text{real part}) - i\frac{\pi}{\lambda}E.
\]

(29)
Comparing equations (28) and (29) with (15) and (18) respectively it is easily seen that there is an additional contribution in the former case. The reason for this fact is the double mapping of a part of the spacetime.

The R-region [9], i.e. $r > r_H$, of the two-dimensional stringy black hole spacetime in unitary gauge is uniquely mapped to the corresponding R-region of the specific spacetime in Schwarzschild gauge. The T-region [9], i.e. $r < r_H$, of the two-dimensional stringy black hole spacetime in the unitary gauge is doubly mapped to the T-region in the Schwarzschild gauge of the specific spacetime [6]. Hence, it is possible to find one point that is common to paths contributing to absorption/emission. In the theory of complex paths all paths are considered.

Therefore taking into account all these and dividing $S_0$’s by two, we obtain the ratio between the corresponding probabilities of emission and absorption:

$$P[\text{emission}] = P[\text{absorption}] e^{-\frac{2\pi}{\hbar} E}$$  \hspace{1cm} (30)

which gives again the correct formula for the Hawking temperature of the two-dimensional stringy black hole.

(iii) Asymmetric gauge

The line element of the two-dimensional stringy black hole in the asymmetric gauge is written as:

$$ds^2 = -\frac{X}{X + 1} dt^2 + \frac{dX^2}{4\lambda^2 X(X + 1)}$$  \hspace{1cm} (31)

where the “asymmetric” variable is given by:

$$X = e^{2\lambda(r-r_H)} - 1$$  \hspace{1cm} (32)

and $0 < X < +\infty$.

The Klein-Gordon equation for a massless scalar field $\Phi(t, X)$ in this gauge is:

$$- \left( \frac{X + 1}{X} \right) \frac{\partial^2}{\partial t^2} \Phi(t, X) + 2\lambda (X + 1) \frac{\partial}{\partial X} \left[ 2\lambda X \frac{\partial}{\partial X} \Phi(t, X) \right] = 0.$$  \hspace{1cm} (33)

Substituting the ansatz (6) for the massless scalar field $\Phi(t, X)$, using the expansion (7) for the action $S$ and neglecting all terms of order $\hbar$ and greater the Klein-Gordon equation (33) becomes:

$$- \left( \frac{\partial S_0}{\partial t} \right)^2 + 4\lambda X^2 \left( \frac{\partial S_0}{\partial X} \right)^2 = 0$$  \hspace{1cm} (34)
This is the Hamilton-Jacobi equation describing (in the asymmetric gauge) the motion of a massless particle in the two-dimensional “Schwarzschild” stringy black hole. Solving equation (34) we get:

\[ S_0(t_2, X_2; t_1, X_1) = -E(t_2 - t_1) \pm E \int_{X_1}^{X_2} \frac{dX}{2\lambda X}. \]  

(35)

The sign ambiguity as before refers to the two different directions of motion of the massless particle with respect to the event horizon of the black hole.

For the outgoing massless particle, i.e. for \( r_1 < r_H \) the leading term \( S_0 \) of the action satisfying equation (13) and using the complex path method is given as:

\[ S_0[\text{emission}] = (\text{real part}) + i \frac{\pi}{2\lambda} E \]  

(36)

while for the ingoing massless particle, i.e. for \( r_2 > r_H \) the leading term \( S_0 \) of the action is given as:

\[ S_0[\text{absorption}] = (\text{real part}) - i \frac{\pi}{2\lambda} E. \]  

(37)

The ratio between the corresponding probabilities of emission and absorption will be as before:

\[ P[\text{emission}] = P[\text{absorption}] e^{-\frac{2\pi}{\hbar \lambda} E} \]  

(38)

giving again the correct formula for the Hawking temperature of the two-dimensional stringy black hole.

(iv) Painlevé gauge

The line element of the two-dimensional stringy black hole in the Painlevé gauge is written as:

\[ ds^2 = -\left(1 - \frac{M}{\lambda} e^{-2\lambda r}\right) d\tau_P^2 + 2\sqrt{\frac{M}{\lambda} e^{-2\lambda r}} d\tau_P dr + dr^2 \]  

(39)

where the Painlevé coordinate \( \tau_P \) satisfies the equation:

\[ d\tau_P = dt - \sqrt{\frac{M}{\lambda} e^{-2\lambda r}} dr \]  

(40)

and \( 0 < t < +\infty, r_H < r < +\infty \).

Considering now a massless scalar field \( \Phi(\tau_P, r) \) satisfying the Klein-Gordon equation we have in the Painlevé gauge:

\[ \frac{\partial^2 \Phi}{\partial \tau_P^2} - \sqrt{\frac{M}{\lambda} e^{-2\lambda r}} \frac{\partial^2 \Phi}{\partial \tau_P \partial r} - \frac{\partial}{\partial r} \left[ \left(1 - \frac{M}{\lambda} e^{-2\lambda r}\right) \frac{\partial \Phi}{\partial r} \right] - \frac{\partial}{\partial r} \left[ \sqrt{\frac{M}{\lambda} e^{-2\lambda r}} \frac{\partial \Phi}{\partial \tau_P} \right] = 0. \]  

(41)
Repeating the actions we have done in the previous gauges we obtain the Hamilton-Jacobi equation:

\[
\left( \frac{\partial S_0}{\partial \tau_P} \right)^2 - 2 \sqrt{\frac{M}{\lambda} e^{-2\lambda r}} \left( \frac{\partial S_0}{\partial r} \right) \left( \frac{\partial S_0}{\partial \tau_P} \right) - \left( 1 - \frac{M}{\lambda} e^{-2\lambda r} \right) \left( \frac{\partial S_0}{\partial r} \right)^2 = 0. \tag{42}
\]

If we solve (42), we obtain:

\[
S_0(\tau_{P2}, r_2; \tau_{P1}, r_1) = -E(\tau_{P2} - \tau_{P1}) \pm E \int_{r_1}^{r_2} \left( \frac{1 \pm \sqrt{\frac{M}{\lambda} e^{-2\lambda r}}}{1 - \frac{M}{\lambda} e^{-2\lambda r}} \right) dr. \tag{43}
\]

The sign ambiguity is again attributed to the two different directions of motion of the massless particle with respect to the event horizon of the two-dimensional black hole.

For the outcoming massless particle, i.e. for \( r_1 < r_H \) the leading term \( S_0 \) of the action must fulfill the equation (13). In order to evaluate the definite integral we use as before the theorem of residues and choosing the contour to lie in the upper complex plane, the leading term \( S_0 \) for the outcoming massless particle is given as:

\[
S_0[\text{emission}] = (\text{real part}) + i\frac{\pi}{\lambda} E. \tag{44}
\]

For the ingoing particle, i.e. for \( r_2 > r_H \) the leading term \( S_0 \) of the action must fulfill the equation (16). In order again to evaluate the definite integral we use the theorem of residues and choosing the contour to lie in the upper complex plane, the leading term \( S_0 \) for the ingoing massless particle is given as:

\[
S_0[\text{absorption}] = (\text{real part}) - i\frac{\pi}{\lambda} E. \tag{45}
\]

Note that the \( \frac{1}{2} \) factor that seems to be missing in equations (44) and (45) is accounted for by the fact that there is a double counting in the complex paths \([6, 9]\).

Therefore the ratio between the probabilities of emission and absorption is the same as in all other cases:

\[
P[\text{emission}] = P[\text{absorption}] e^{-\frac{2\pi}{\hbar \lambda} E}. \tag{46}
\]

Comparing equation (46) with equation (2) and setting \( \hbar = 1 \) we obtain the right expression for the Hawking temperature:

\[
T_H = \beta^{-1} = \frac{\lambda}{2\pi} \tag{47}
\]

for the two-dimensional stringy black hole in the Painlevé gauge.
For the case of the corresponding charged two-dimensional stringy black hole \[11\] similar results hold: The corresponding ratio between the probabilities of emission and absorption through the outer horizon of the charged black hole is given as:

\[ P[\text{emission}] = P[\text{absorption}]e^{-\frac{2\pi}{\lambda\mu}E} \]  \hspace{1cm} (48)

where \(E\) is the energy of a massless particle propagating in the charged two-dimensional black hole spacetime and \(\mu\) is a function of the charge \(Q\) and the mass \(M\) of the charged black hole. Comparing equation (48) with (2) we obtain the corresponding Hawking temperature:

\[ T_H = \frac{\lambda}{2\pi} \mu \]  \hspace{1cm} (49)

which is the standard formula for the temperature of a charged two-dimensional stringy black hole. These results were also obtained by T. Christodoulakis et al using the method of Bogoliubov coefficients \[12\].

It is obvious that the method introduced by the K. Srinivasan and T. Padmanabhan is applicable to several static and non-static coordinate systems of the two-dimensional stringy black hole spacetimes. As was mentioned in the introduction the Hawking radiation is a physical effect related to the decrease of the black hole mass. Therefore it is expected that the energy (mass) radiated outwards the event horizon, i.e. Hawking radiation should be covariant. Using here the method of complex paths the thermal spectrum and the temperature of the Hawking radiation are recovered in different coordinate systems indicating the covariance of the Hawking radiation of the two-dimensional stringy black hole. We have deliberately omitted the conformal gauge in which the event horizon is not at a finite distance and therefore the method of complex paths cannot be applied, since there can be no crossing of the horizon by particles. This shows one more that the Hawking radiation is interlinked with the presence of black hole horizons.

A semiclassical approach for computing the Hawking radiation in the Painlevé gauge was recently introduced by Parikh and Wilczek \[13\]. This method considers the Hawking radiation as a pair creation outside the event horizon with the negative energy particle tunnelling into the black hole. We have performed the method of Parikh and Wilczek for the charged two-dimensional stringy black hole background \[14\]. In the present paper we have also taken into consideration the contribution to the amplitudes of the pair creation from within the event horizon.
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