In the antecedent paper to this it was established that there is an algebraic number \( \xi \approx 2.30522 \) such that while there are uncountably many growth rates of permutation classes arbitrarily close to \( \xi \), there are only countably many less than \( \xi \). Here we provide a complete characterization of the growth rates less than \( \xi \). In particular, this classification establishes that \( \xi \) is the least accumulation point from above of growth rates and that all growth rates less than or equal to \( \xi \) are achieved by finitely based classes. A significant part of this classification is achieved via a reconstruction result for sum indecomposable permutations. We conclude by refuting a suggestion of Klazar, showing that \( \xi \) is an accumulation point from above of growth rates of finitely based permutation classes.

1. Introduction

We are concerned here with the problem of determining the complete list of all growth rates of permutation classes. To be concrete, a permutation class is a downset of permutations under the containment order, in which \( \sigma \) is contained in \( \pi \) if \( \pi \) has a (not necessarily consecutive) subsequence that is order isomorphic to \( \sigma \) (i.e., has the same relative comparisons). If \( \sigma \) is contained in \( \pi \) then we write \( \sigma \leq \pi \); otherwise we say that \( \pi \) avoids \( \sigma \).

Given a permutation class \( \mathcal{C} \) we denote by \( \mathcal{C}_n \) the subset of \( \mathcal{C} \) consisting of its members of length \( n \) (we think of permutations in one-line notation, so length means the number of symbols). The Marcus–Tardos Theorem [12] (formerly the Stanley–Wilf Conjecture) shows that for every proper permutation class \( \mathcal{C} \) (meaning, every class except that containing all permutations), the cardinalities \( |\mathcal{C}_n| \) grow at most exponentially. Thus the upper and lower growth rates of the permutation class \( \mathcal{C} \),

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defined respectively by

\[ \text{gr}(C) = \limsup_{n \to \infty} \sqrt[n]{|C_n|} \quad \text{and} \quad \text{gr}(C) = \liminf_{n \to \infty} \sqrt[n]{|C_n|} \]

are finite for every proper permutation class \( C \). When these two quantities are equal (which is conjectured to hold for all classes and is known to hold for all classes in this work) we denote their common value by \( \text{gr}(C) \) and call it the growth rate of \( C \).

Work on determining the set of growth rates of permutation classes has identified several notable phase transitions where both the set of growth rates and the corresponding permutation classes undergo dramatic changes:

- Kaiser and Klazar [10] showed that 2 is the least accumulation point of growth rates and determined all growth rates below 2. They also showed that every class of growth rate less than the golden ratio \( \phi \) has eventually polynomial enumeration, and thus growth rate 0 or 1.
- Vatter [21] established that there are uncountably many permutation classes of growth rate \( \kappa \approx 2.20557 \) (a specific algebraic integer), but only countably many of growth rate less than \( \kappa \). In the same paper, all growth rates under \( \kappa \) are characterized, showing that \( \kappa \) is the least second-order accumulation point of growth rates, meaning that there is a sequence of accumulation points of growth rates which themselves accumulate to \( \kappa \).
- Albert, Ruškuc, and Vatter [3] established that every class of growth rate less than \( \kappa \) has a rational generating function, while there are (by an elementary counting argument) classes of growth rate \( \kappa \) with non-rational (and even non-D-finite) generating functions.
- Bevan [6], refining work of Albert and Linton [1] and Vatter [20], established that the set of growth rates contains every real number above \( \lambda_B \approx 2.35698 \) and that the set contains an infinite sequence of intervals whose infimum is \( \theta_B \approx 2.355256 \) (both are specific algebraic integers).

Most recently, in the antecedent paper to this, Vatter [19] established that while there are uncountably many growth rates of permutation classes in every neighborhood of the algebraic integer

\[ \xi = \text{the unique positive root of} \quad x^5 - 2x^4 - x^2 - x - 1 \approx 2.30522, \]

there are only countably many growth rates under \( \xi \). In this work we completely determine the set of growth rates under \( \xi \). The known set of growth rates of permutation classes, including those characterized here, is shown in Figure 1.

While our work builds on [19], we use only one result from it. To state this result we need a few definitions. First, the (direct) sum of the permutations \( \pi \) of length \( k \) and \( \sigma \) of length \( \ell \) is defined by

\[ (\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{for } i \in [1, k], \\ \sigma(i - k) + k & \text{for } i \in [k + 1, k + \ell]. \end{cases} \]

The sum of \( \pi \) and \( \sigma \) is shown pictorially on the left of Figure 2. The analogous operation depicted on the right of Figure 2 is called the skew sum and denoted \( \sigma \ominus \pi \). The permutation \( \pi \) is said to be sum indecomposable if it cannot be written as the sum of two nonempty permutations. Otherwise, \( \pi \) is called sum decomposable and in this case we can write \( \pi \) uniquely as \( \alpha_1 \oplus \cdots \oplus \alpha_k \) where the \( \alpha_i \) are sum indecomposable; in this case the \( \alpha_i \) are called the sum components of \( \pi \).
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Figure 1: The set of all growth rates of permutation classes between the golden ratio $\varphi$ and 2.5, as presently known, including the results of this paper.

\[
\begin{array}{l}
\varphi & 2 & \approx 2.06 & \kappa & \xi & \lambda_B & \lambda_A \\
\hline
\text{Kaiser and Klazar [10]} & \text{Vatter [21]} & \text{found here} & \text{unknown} & \approx 0.05 & \text{Bevan [6]} & \text{V [20]}
\end{array}
\]

Figure 2: The sum and skew sum operations.

We say that the class $C$ is sum closed if $\pi \oplus \sigma \in C$ for all $\pi, \sigma \in C$. Given a permutation class $C$, its sum closure, denoted by $\oplus C$, is the smallest sum closed permutation class containing $C$. Equivalently,

\[
\oplus C = \{ \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k : \pi_i \in C \text{ for all } i \}.
\]

It follows from the supermultiplicative version of Fekete’s Lemma that every sum closed permutation class has a proper growth rate (a fact first observed by Arratia [4]). The only result we use from [19] is the following.

**Theorem 1.1** (Vatter [19, Theorem 9.7]). There are only countably many growth rates of permutation classes below $\xi$ but uncountably many growth rates in every open neighborhood of it. Moreover, every growth rate of a permutation class less than $\xi$ is achieved by a sum closed permutation class.

Every permutation class can be specified by a set of permutations that its members avoid, i.e., as

\[
\text{Av}(B) = \{ \pi : \pi \text{ avoids all } \beta \in B \}.
\]

Indeed, we can always take $B$ to be an antichain (a set of pairwise incomparable elements), and in this case $B$ is called the basis of $C$. Note that a class is sum closed if and only if all of its basis elements are sum indecomposable. One can also define a permutation class as the downward closure of a set $X$ of permutations, i.e., as

\[
\text{Sub}(X) = \{ \pi : \pi \leq \tau \text{ for some } \tau \in X \}.
\]

We also make use of the proper downward closure of a set $X$ of permutations,

\[
\text{Sub}^< (X) = \{ \pi : \pi < \tau \text{ for some } \tau \in X \}.
\]

We define the generating function for a class $C$ of permutations as

\[
\sum_{\pi \in C} x^{||\pi||} = \sum_{n \geq 0} |C_n| x^n,
\]
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where $|\pi|$ denotes the length of $\pi$. It is easy to compute generating functions for sum closed classes, assuming we know enough about the sum indecomposable members, as we record below.

**Proposition 1.2.** The generating function for a sum closed permutation class is $1/(1 - g)$ where $g$ is the generating function for nonempty sum indecomposable permutations in the class.

Growth rates can be determined from generating functions via the following result.

**Exponential Growth Formula** (see Flajolet and Sedgewick [7, Section IV.3.2]). The upper growth rate of a permutation class is equal to the reciprocal of the least positive singularity of its generating function.

We say that the sequence $(s_n)$ can be realized if there is a permutation class with precisely $s_n$ sum indecomposable permutations for every $n$. In light of Theorem 1.1, our task in this paper is to determine the realizable sequences corresponding to sum closed permutation classes of growth rate less than $\xi$.

In the next section we review the notions of monotone intervals and quotients and state basic consequences of the connection between sum indecomposable permutations and connected graphs. In Section 3 we describe several classes with growth rates near $\kappa$ and $\xi$ that reappear later in our arguments. In Section 4 we establish that, with two notable exceptions, sum indecomposable permutations are uniquely determined by their sum indecomposable subpermutations. In Section 5 we employ this result to establish certain necessary conditions on realizable sequences. These conditions are further refined in Sections 6 and 7. In Section 8 we show that all sequences not eliminated by these considerations are realizable, and we present the complete list of growth rates under $\xi$ in Section 9. In Section 10 we finish showing that all growth rates less than or equal to $\xi$ are achieved by finitely based classes and exhibit a counterexample to a suggestion of Klazar [11]. We conclude in Section 11 with a discussion of the obstacles that would have to be overcome to extend our characterization.

2. **Monotone Intervals and Inversion Graphs**

An *interval* in the permutation $\pi$ is a set of contiguous indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous, and an interval is *nontrivial* if it contains at least 2 and fewer than all the entries of $\pi$. The *substitution decomposition* describes how a permutation is built up from a *simple permutation* (one with no nontrivial intervals) via repeated inflations by intervals. (Given a permutation $\sigma$ of length $m$ and nonempty permutations $\alpha_1, \ldots, \alpha_m$, the inflation of $\sigma$ by $\alpha_1, \ldots, \alpha_m$ — denoted $\sigma[\alpha_1, \ldots, \alpha_m]$ — is the permutation obtained by replacing each entry $\sigma(i)$ by an interval that is order isomorphic to $\alpha_i$.)
For the results we establish, a different and in some sense weaker decomposition is required. A monotone interval in a permutation is an interval whose entries are monotone (increasing or decreasing). Given a permutation $\pi$, we define the monotone quotient of $\pi$ to be the shortest permutation $\sigma$ such that $\pi$ is an inflation of $\sigma$ by monotone permutations. Alternatively, the monotone quotient of $\pi$ can be found by contracting all maximal length monotone intervals to single entries. For example, the monotone quotient of $345216789$ is $213$, because $345216789 = 213[123, 21, 1234]$. This construction is unique because if two monotone intervals intersect then their union must itself be monotone. Note that the monotone quotient of a sum indecomposable permutation is itself sum indecomposable and that monotone quotients may have nontrivial monotone intervals themselves (unlike the case with the usual substitution decomposition).

Given a permutation $\pi$ and an entry $x$ of $\pi$, we denote by $\pi - x$ the permutation that is order isomorphic to $\pi$ with $x$ removed, and we call $\pi - x$ a child of $\pi$. For example, the set of children of $2314$ is

$$\{2314 - 2, 2314 - 3, 2314 - 4\} = \{213, 123, 231\}.$$  

Our interest in monotone intervals and quotients comes from the following fact, which is easily established using induction on the number of entries between $x$ and $y$.

**Proposition 2.1.** We have $\pi - x = \pi - y$ for entries $x$ and $y$ of a permutation $\pi$ if and only if $x$ and $y$ lie in the same monotone interval of $\pi$.

As a consequence of Proposition 2.1, if $\pi$ has monotone quotient $\mu$ then by deleting entries of $\pi$ corresponding to distinct entries of $\mu$ we obtain distinct children.

We also make frequent use of the connection between permutations and graphs and a few basic graph-theoretic results. Given a permutation $\pi$ on $[n]$, its inversion graph $G_\pi$ is the graph on the vertices $[n]$ in which $\pi(i) \sim \pi(j)$ if and only if $\pi(i)$ and $\pi(j)$ form an inversion in $\pi$, i.e., if $i < j$ and $\pi(i) > \pi(j)$. Note that the graph $G_\pi$ is connected if and only if $\pi$ is sum indecomposable (because two entries that form an inversion must lie in the same component). As each entry of $\pi$ corresponds to a vertex of $G_\pi$, we commit a slight abuse of language by referring (for example) to the degree of an entry of $\pi$ when we mean the degree of the corresponding vertex of $G_\pi$. In a similar way, we talk about the leaves of $\pi$ when we mean the entries of $\pi$ that correspond to leaves of $G_\pi$.

The containment order on permutations corresponds to the induced subgraph order on inversion graphs in the sense that if $\sigma \leq \pi$ then $G_\sigma$ contains an induced subgraph isomorphic to $G_\pi$. Note that this is true even though the mapping $\pi \to G_\pi$ is not injective; in particular, $G_\pi$ and $G_{\pi^{-1}}$ are isomorphic for all permutations $\pi$. We extend our definition of child to graphs by saying that $H$ is a child of $G$ if $H = G - v$ for some vertex $v$.

If the inversion graph $G_\pi$ is a path, we call $\pi$ an increasing oscillation. This term dates back to Murphy’s thesis [13], though note that under our definition, the permutations $1, 21, 231$, and $312$ are increasing oscillations while in other works they are not. By direct construction, the increasing oscillations can be seen to be precisely the sum indecomposable permutations that are order isomorphic to subsequences of the increasing oscillating sequence,

$$2, 4, 1, 6, 3, 8, 5, \ldots, 2k, 2k - 3, \ldots.$$  

We denote by $O_I$ the downward closure of the set of increasing oscillations. There are two increasing oscillations of each length $n \geq 3$, and they are inverses of each other. We arbitrarily choose those beginning with $2$ to call the primary type. The two primary type increasing oscillations are

$$2416385 \cdots n(n - 3)(n - 1)$$
for even \( n \geq 4 \), and
\[
2416385 \cdots (n - 4)n(n - 2).
\]
for odd \( n \geq 5 \).

We conclude this section by stating two graph-theoretic results we appeal to later.

**Proposition 2.2.** Every connected graph with at least two vertices contains at least two vertices that are not cut vertices, and the only connected graphs with precisely two non-cut vertices are paths. Correspondingly, every sum indecomposable permutation has at least two entries whose removal leaves the permutation sum indecomposable (though the resulting children may be identical).

In the following result, we call the complete bipartite graph \( K_{1,3} \) a **claw**, and thus the inversion graphs of the permutations 2341 and 4123 are claws. Also, the only permutations whose inversion graphs are cycles are 321 and 4123.

**Proposition 2.3.** Every connected graph that is neither a path, a cycle, nor a claw has a connected child that is not a path. Correspondingly, every sum indecomposable permutation that is not an increasing oscillation, 321, 2341, 3412, or 4123 contains a sum indecomposable child that is not an increasing oscillation.

### 3. The Neighborhoods of \( \kappa \) and \( \xi \)

Here we briefly survey certain classes that lie at or near the growth rates \( \kappa \) and \( \xi \). First, because \( \mathcal{O}_I \) is sum closed and contains precisely two sum indecomposable permutations of each length \( n \geq 3 \), we see from Proposition 1.2 that the generating function of \( \mathcal{O}_I \) is
\[
1 \over 1 - \left( x + x^2 + {2x^3 \over 1-x} \right) = 1 - x \over 1 - 2x - x^3.
\]
We can use the Exponential Growth Formula to compute the growth rate of \( \mathcal{O}_I \). In order to express this growth rate as the root of a polynomial (as opposed to the reciprocal of the root of a polynomial), we take the factor responsible for the least positive singularity, \( 1 - 2x - x^3 \), replace \( x \) by \( x^{-1} \) to obtain \( 1 - 2x^{-1} - x^{-3} \), and then multiply by \( x^3 \) to get \( x^3 - 2x^2 - 1 \). The growth rate of \( \mathcal{O}_I \) is the greatest positive root of this polynomial, namely \( \kappa \).

We can use increasing oscillations to build a variety of infinite antichains whose closures also have growth rate \( \kappa \). Two such antichains, one contained in the other, are central to this work; these are
denoted $U^o$ and $U$. To construct $U^o$ we take a primary type increasing oscillation (i.e., beginning with 2) of odd length at least $2k - 1 \geq 5$ and inflate the two leaves by copies of 12 (these pairs of entries are called anchors). This produces a permutation $\mu_{2k+1}$ of length $2k + 1 \geq 7$ (whether $U^o$ contains 23451 differs from paper to paper; we exclude it throughout this work). An example is shown in Figure 4.

The inversion graph of $\mu_{2k+1} \in U^o$ is therefore a split-end path, meaning that it is formed from a path on $2k - 3$ vertices by adding four leaves, two adjacent to each endpoint of the original path. The set of split-end paths clearly forms an infinite antichain under the induced subgraph order. Because $G_\sigma$ is an induced subgraph of $G_\pi$ whenever $\sigma \preceq \pi$, it follows that the set $U^o$ also forms an infinite antichain of permutations.

We further denote by $U$ the set of all permutations whose inversion graphs are isomorphic to split-end paths on 6 or more vertices. It follows that $U$ contains $U^o$ together with three other types of members: inflated increasing oscillations of primary type and even length, $U^e$, and the inverses of both $U^o$ and $U^e$. Members of these three additional sets are shown in Figure 5. That figure also includes (on the right) a member of a different antichain based on increasing oscillations. Indeed, there are a great many ways to form infinite antichains from increasing oscillations; the reason that $U^o$ is central to this work while the others are not is essentially because the growth rate of $\bigoplus \text{Sub}(U^o)$ is small.

Consider a particular member, say $\mu_{2k+7}$, of $U^o$ for $k \geq 0$. The downward closure of this permutation contains the sum indecomposable permutations 1, 21, 231, 312, 2341, 2413, and 3142. It also contains 4 sum indecomposable permutations of length $n$ for each $n$ satisfying $4 \leq n \leq 2k + 4$, of the forms shown in Figure 6. As we increase $n$ beyond $2k + 4$, we begin to lose some of these sum indecomposable subpermutations. For $n = 2k + 5$, $\text{Sub}_n(\mu_{2k+7})$ no longer contains the non-primary body and for $n = 2k + 6$, $\text{Sub}_n(\mu_{2k+7})$ no longer contains the primary body. A similar analysis applies to members of $U^e$, though they contain 4 sum indecomposable subpermutations of length 4 (the additional permutation, 4123, is contained in the tails of members of $U^e$). We collect these
completions below.

**Proposition 3.1.** The sequence of sum indecomposable permutations contained in $\mu_{2k+7}$ for $k \geq 0$ is $1, 1, 2, 3, 4^{2k}, 3, 2, 1$. The sequence of sum indecomposable permutations contained in $\mu_{2k+6}$ for $k \geq 0$ is $1, 1, 2, 4^{2k}, 3, 2, 1$.

Moreover, it is easy to establish that
$$\text{gr}(\text{Sub}(U)) = \text{gr}(\text{Sub}(U^\omega)) = \text{gr}(O_1) = \kappa;$$

one way is the following. The permutation $\pi$ is said to be a $p$-point extension of $\pi$ if $\pi$ can be obtained from $\pi$ by inserting $p$ or fewer entries. The class of all $p$-point extensions of members of the class $C$ is further denoted $C^+p$. Clearly $|C^+_n| \leq (n+1)^2|C_n|$, from which it follows that $\text{gr}(C^+p) = \text{gr}(C)$ and $\text{gr}(C^+p) = \text{gr}(C)$ for all classes $C$ and natural numbers $p$. The growth rate claim above holds because $\text{Sub}(U^\omega) \subseteq \text{Sub}(U) \subseteq O_{1+2}^+$.

Because $U^\omega$ is an antichain, no member of $U^\omega$ is contained in any other permutation of Sub$(U^\omega)$. Thus every class of the form Sub$(U^\omega) \setminus T$ for $T \subseteq U^\omega$ has growth rate $\kappa$, establishing the following result which was first noted in the conclusion of Kaiser and Klazar [10].

**Proposition 3.2.** There are uncountably many permutation classes of growth rate $\kappa$.

The results of [21] establish no classes of growth rate less than $\kappa$ may contain infinite antichains (i.e., they are well quasi-ordered), and thus Proposition 3.2 is best possible.

The antichain $U^\omega$ allows us to construct uncountably many permutation classes, but they all have the same growth rate, $\kappa$. In order to construct uncountably many classes with different growth rates, we examine classes lying between $\bigoplus \text{Sub}^<(U^\omega)$ and $\bigoplus \text{Sub}(U^\omega)$. It follows from our justification of Proposition 3.1 that the sequence of sum indecomposable permutations in Sub$^<(U^\omega)$ is $1, 1, 2, 3, 4^2$. (Note that this is not the sequence of sum indecomposable permutations in Sub$^<(U)$ as that class contains two different types of tails of each length as well as an extra sum indecomposable permutation of length 4.) Consequently, the generating function of $\bigoplus \text{Sub}^<(U^\omega)$ is
$$\frac{1}{1 - (x + x^2 + 2x^3 + 3x^4 + 4x^5)} = \frac{1 - x}{1 - 2x - x^3 - x^4 - x^5}.$$

This shows that the growth rate of $\bigoplus \text{Sub}^<(U^\omega)$ is the greatest positive root of $x^5 - 2x^4 - x^2 - x - 1$, i.e., $\xi$.

Given infinite sequences or finite sequences padded with zeros, $(r_n)$ and $(t_n)$, we say that $(t_n)$ dominates $(r_n)$, and write $(r_n) \preceq (t_n)$, if $r_n \leq t_n$ for all $n$. Because $U^\omega$ contains one sum indecomposable permutation of each odd length at least 7, the sequence of sum indecomposable permutations in Sub$(U^\omega)$ is $1, 1, 2, 3, 4, 5, 4, 5, 4, \ldots$. Therefore we can construct sum closed subclasses of $\bigoplus \text{Sub}(U^\omega)$ with $s_n$ sum indecomposable permutations of length $n$ for every sequence $(s_n)$ satisfying
$$(1, 1, 2, 3, 4^2) \leq (s_n) \leq (1, 1, 2, 3, 4, 5, 4, 5, 4, \ldots).$$

Our next result shows that each of these classes has a different growth rate.

**Proposition 3.3.** For every different sequence $(s_n)$ satisfying
$$(1, 1, 2, 3, 4^2) \leq (s_n) \leq (1, 1, 2, 3, 4, 5, 4, 5, 4, \ldots),$$
the growth rate of $1/(1 - \sum s_n x^n)$ is different. Therefore every open neighborhood of $\xi$ contains uncountably many such growth rates.
Proof. Let \((s_n)\) and \((t_n)\) be two such sequences and suppose that the least index for which they disagree is \(2k + 1\). Without loss of generality we may assume that \(s_{2k + 1} = 4\) while \(t_{2k + 1} = 5\). Define
\[
g(x) = s_1 x + s_2 x^2 + \cdots + s_{2k} x^{2k} + 4x^{2k+1} + 4x^{2k+2} + \cdots \\
= t_1 x + t_2 x^2 + \cdots + t_{2k} x^{2k} + 4x^{2k+1} + 4x^{2k+2} + \cdots .
\]
The growth rate of the series expansion of \(1/(1 - \sum s_n x^n)\) is therefore at most the growth rate of the series expansion of
\[
\frac{1}{1 - (g(x) + \frac{x^{2k+3}}{1-x})}
\]
while the growth rate of the series expansion of \(1/(1 - \sum t_n x^n)\) is at least the growth rate of the series expansion of
\[
\frac{1}{1 - (g(x) + x^{2k+1})}.
\]
We claim that the first quantity is smaller than the second. Because both growth rates are known to be at least 2, it suffices to consider \(0 < x < 1/2\). For these values of \(x\), we have
\[
\frac{x^{2k+3}}{1-x} < \frac{4}{3} x^{2k+3} < \frac{1}{3} x^{2k+1} < x^{2k+1}.
\]
Therefore the smallest real singularity of \(1/(1 - \sum s_n x^n)\) is greater than the smallest real singularity of \(1/(1 - \sum t_n x^n)\), implying the inequality of growth rates claimed above.

This gives us the following fact, first established in [20].

**Proposition 3.4.** Every open neighborhood of \(\xi \approx 2.30522\) contains uncountably many growth rates of permutation classes.

4. Reconstruction

As we have seen, the study of growth rates under \(\xi\) is intimately related to the properties of sum indecomposable permutations. Here we establish a general result about sum indecomposable permutations themselves which we apply in the next section to restrict sequences of sum indecomposable permutations in permutation classes.

The permutation \(\pi\) is said to be *set reconstructible* if it is the unique permutation with its set of children. Smith [18] was the first to establish that all permutations of length at least 5 are set reconstructible (this is the permutation analogue of the graph-theoretical Set Reconstruction Conjecture first stated by Harary [8]), and shortly thereafter Raykova [16] gave a simpler proof. Among several examples, the two increasing oscillations of length 4, 2413 and 3142, show that this result is best possible.

**Theorem 4.1** (Smith [18]). Every permutation of length at least 5 is uniquely determined by its set of children.

We establish an analogous result for sum indecomposable permutations here. To this end, given a sum indecomposable permutation \(\pi\) we let \(K(\pi)\) denote its set of *sum indecomposable* children,
\[
K(\pi) = \{\pi - x : x \text{ is an entry of } \pi \text{ and } \pi - x \text{ is sum indecomposable}\}.
\]
We also define a closely related set,

\[ K^{(m)} = \{ \text{sum indecomposable } \pi : \pi \text{ has at most } m \text{ sum indecomposable children} \} \]

\[ = \{ \text{sum indecomposable } \pi : |K(\pi)| \leq m \} . \]

The main result of this section, below, establishes that \( K(\pi) \) uniquely determines \( \pi \) except in one notable case.

**Theorem 4.2.** Every sum indecomposable permutation of length at least 5 that is not an increasing oscillation is uniquely determined by its set of sum indecomposable children \( K(\pi) \).

We prove Theorem 4.2 via a sequence of propositions. The last of these, Proposition 4.9, represents the majority of the cases and its proof is an adaptation of Raykova’s proof of Theorem 4.1. Before that, we establish the result for the members of \( K^{(1)} \) and \( K^{(2)} \).

**Proposition 4.3.** For \( n \geq 3 \), the permutations of length \( n \) in \( K^{(1)} \) are \( n \cdot \cdots \cdot 21, 1 \ominus (12 \cdots (n - 1)), \) and \( (12 \cdots (n - 1)) \ominus 1 \).

*Proof.* Let \( \pi \in K^{(1)} \) have monotone quotient \( \mu \). Our first goal is to show that \( \mu \) must be 1 or 21. Indeed, if \( |\mu| \geq 3 \), then Proposition 2.2 shows that the inversion graph \( G_{\mu} \) has at least two non-cut vertices. Label the corresponding entries of \( \mu \) as \( \mu(i) \) and \( \mu(j) \). By deleting entries of \( \pi \) corresponding to \( \mu(i) \) and \( \mu(j) \) we obtain two distinct sum indecomposable children of \( \pi \), a contradiction.

If \( \mu = 1 \), \( \pi \) must be decreasing because it is sum indecomposable. In the case where \( \mu = 21 \) we see that at most one of the entries of \( \mu \) may be inflated as otherwise \( \pi \) would have more than one sum indecomposable child, completing the proof.

**Proposition 4.4.** The set \( K^{(2)} \setminus K^{(1)} \) consists of the sum indecomposable permutations formed by inflating the leaves of an increasing oscillation by monotone intervals.

*Proof.* Take \( \pi \in K^{(2)} \) and let \( \mu \) denote the monotone quotient of \( \pi \). The inversion graph \( G_{\mu} \) must have at most two non-cut vertices, for if \( G_{\mu} \) had three non-cut vertices then deleting corresponding entries in \( \pi \) would give three distinct sum indecomposable children.

Proposition 2.2 states that a connected graph has at most two non-cut vertices if and only if it is a path. Thus \( \mu \) must be an increasing oscillation. If a non-leaf of \( \mu \) is inflated to form \( \pi \) then \( \pi \) has at least three distinct sum indecomposable children: two from deleting or shrinking a leaf (depending on whether the leaf was inflated), and one from shrinking the inflated non-leaf. Therefore only leaves of \( \mu \) may be inflated, and they may only be inflated by monotone intervals because \( \mu \) is the monotone quotient of \( \pi \). To complete the proof, note that if \( \pi \) is formed by inflating the leaves of an increasing oscillation then \( \pi \) has at most two sum indecomposable children, each formed by deleting an entry corresponding to a leaf of \( \mu \).

Our next result allows us to reconstruct most members of \( K^{(2)} \) from their sets of sum indecomposable children.
Proposition 4.5. Let $\pi \in K^{(2)}$ have length at least 5. From $K(\pi)$ one can determine whether $\pi$ is an increasing oscillation. Moreover, if $\pi$ is not an increasing oscillation, it is uniquely determined by $K(\pi)$.

Proof. Obviously, we can determine from $K(\pi)$ whether $\pi \in K^{(1)}$ or $\pi \in K^{(2)} \backslash K^{(1)}$. If $\pi \in K^{(1)}$ then it is one of the three permutations specified by Proposition 4.3 and determining which is trivial. Suppose that $\pi \in K^{(2)} \backslash K^{(1)}$. If both members of $K(\pi)$ are increasing oscillations then $\pi$ must be an increasing oscillation by Proposition 2.3, but we cannot determine which of the two increasing oscillations it is. Otherwise at least one permutation in $K(\pi)$ is not an increasing oscillation, and thus by Proposition 4.4, $\pi$ is the result of inflating the leaves of an increasing oscillation by monotone intervals. By examining the corresponding leaves in both children we can determine the monotone quotient of $\pi$, and then the two children together determine $\pi$ uniquely.

Our next goal is to recognize from $K(\pi)$ for permutations $\pi$ of length $n$ whether $\pi - n$ is sum decomposable, and then to reconstruct those permutations from their sets of sum indecomposable children. In the proofs of these results we make frequent use of the following observation.

Proposition 4.6. Suppose that the permutation $\pi$ of length $n$ is sum indecomposable.

(a) If $\pi$ contains at least one entry to the left of $n$ then there is an entry $x$ to the left of $n$ such that $\pi - x$ is sum indecomposable.

(b) If $\pi$ contains at least two entries to the right of $n$ then there is an entry $x$ to the right of $n$ such that $\pi - x$ is sum indecomposable.

Proof. Both parts of the proposition follow from the same analysis. Write $\pi$ as the concatenation $\lambda n \rho$. While $\lambda$ is a word over distinct integers instead of a permutation, we can extend the notion of sum components to $\lambda$ in a natural way by defining a sum component of $\lambda$ to be the entries corresponding to a sum component in the permutation that is order isomorphic to $\lambda$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the sum components of $\lambda$.

In both cases, $\rho$ must be nonempty (in case (a) this follows from sum indecomposability and in case (b) it follows by our hypotheses). Let $b$ denote the bottommost entry of $\rho$. Thus we have the picture shown in Figure 7, though note that $\lambda$ may be empty in case (b).

For case (a), because $\pi$ is sum indecomposable, at least one entry of $\lambda_1$ must lie above $b$. If $|\lambda_1| = 1$ then every entry of $\lambda$ lies above $b$, and thus we can take $x$ to be the single entry of $\lambda_1$. Otherwise, because every nontrivial connected graph has at least two non-cut vertices (Proposition 2.2), there are at least two entries of $\lambda_1$ whose removal leaves $\lambda_1$ sum indecomposable; let $x$ denote the bottommost of these entries. Removing $x$ leaves at least one entry of $\lambda_1$ above $b$, so $\pi - x$ is sum indecomposable.
Case (b) is trivial if $\lambda$ is empty because then we may remove any entry of $\rho$. Otherwise let $x$ denote any entry of $\rho$ except $b$. In $\pi - x$ there is still an entry of $\lambda_1$ above $b$, so $\pi - x$ is sum indecomposable.

Given a set $X$ of permutations (always a set of sum indecomposable children in what follows) we define its set of max locations by

$$\text{ml}(X) = \{i : \text{some } \pi \in X \text{ has its maximum entry at index } i\}.$$ 

Suppose now that $\pi$ is sum indecomposable but $\pi - n$ is not and let $\pi(i) = n$. It follows that $\pi$ has the form shown in Figure 8, where $\alpha_1$ and $\alpha_2$ are nonempty, $\alpha_1$ is sum indecomposable, and there is an entry in $\alpha_1$ lying to the right of $n$. For all such permutations we have

$$\text{ml}(K(\pi)) \subseteq \{i - 1, i\}.$$ 

Note that this property is recognizable from $K(\pi)$. We now prove that we can also recognize from $K(\pi)$ whether $\pi - n$ is sum decomposable.

**Proposition 4.7.** Let $\pi$ be a sum indecomposable permutation of length at least 5 that is not an increasing oscillation. It can be determined from $K(\pi)$ whether $\pi - n$ is sum decomposable.

**Proof.** We make use of the following properties of $K(\pi)$ when $\pi$ is sum indecomposable but $\pi - n$ is sum decomposable, all of which follow readily from the form of such permutations depicted in Figure 8:

- $\text{ml}(K(\pi)) \subseteq \{i - 1, i\}$ for some $i$;
- $K(\pi)$ contains at most one permutation in which the greatest and second greatest entries form a monotone interval;
- $K(\pi)$ contains at most one permutation that has its greatest entry in its second-to-last index; and
- if $\text{ml}(K(\pi)) = \{1, 2\}$ then $\pi(2) = n$ and thus $K(\pi)$ would contain at most one child beginning with its maximum entry.

Now take $\pi$ to be a sum indecomposable permutation of length $n \geq 5$ for which $\pi - n$ is sum indecomposable. We seek to show that $K(\pi)$ violates one of the conditions above. Thus we may begin by assuming that $\text{ml}(K(\pi)) \not\subseteq \{i - 1, i\}$ for some $i$. We are done if $\pi \in K^{(2)}$ by Proposition 4.5, so we may assume that $\pi$ has at least three sum indecomposable children. Throughout the proof we let $k$ denote the position of $n$ in $\pi$ and $j$ the position of $n - 1$. We distinguish three cases, depending on whether $|k - j|$ is 1, 2, or at least 3.
We first consider the cases where $|k - j| = 1$, i.e., where $n$ and $n - 1$ form a monotone interval, cases (1) and (2) of Figure 9. Because $\pi$ has at least three sum indecomposable children, it has at least two sum indecomposable children whose greatest and second greatest entries form a monotone interval. However, we have already remarked that this could not occur if $\pi - n$ were sum decomposable.

Next we handle the most difficult of the cases, where $|k - j| = 2$, cases (3) and (4) of Figure 9. In case (3), the sum indecomposability of $\pi - n$ implies that $j \in \text{ml}(K(\pi))$. Furthermore, if the rightmost box has at least two entries, then one may be deleted to give a sum decomposable child of $\pi$ that has its maximum entry at index $k$ (by Proposition 4.6 (b)). Since the rightmost box cannot be empty ($\pi$ is sum indecomposable), it follows that this case is complete unless the rightmost box contains exactly one entry. In that case, there are at least two sum indecomposable children that have their maximum entries in the second-to-last index, which could not occur if $\pi - n$ were sum decomposable.

In case (4), because we are assuming that $\pi - n$ is sum indecomposable, we have $j - 1 = k + 1 \in \text{ml}(K(\pi))$. If there are any entries to left of $n$ in $\pi$ then Proposition 4.6 (a) shows that one of those entries can be deleted to obtain a sum indecomposable child of $\pi$, implying that $k - 1 \in \text{ml}(K(\pi))$, and thus we are done with that case. Otherwise $\pi$ begins with its maximum, and it follows that $\text{ml}(K(\pi)) = \{1, 2\}$. Because $\pi$ has at least three sum indecomposable children, there are at least two permutations in $K(\pi)$ that begin with their maximum entries, which could not occur if $\pi - n$ were sum decomposable.

This leaves us with the cases where $|k - j| \geq 3$, cases (5) and (6) of Figure 9. In this case, the sum indecomposability of $\pi - n$ shows that either $j$ or $j - 1$ lies in $\text{ml}(K(\pi))$. Because $\pi$ has another sum indecomposable child, we see that either $k$ or $k - 1$ lies in $\text{ml}(K(\pi))$. Therefore $\text{ml}(K(\pi))$ is not of the form $\{i - 1, i\}$, completing the proof. 

Next we reconstruct the permutations recognized by the previous result.

**Proposition 4.8.** Let $\pi$ be a sum indecomposable permutation of length at least 5 that is not an increasing oscillation. If $\pi - n$ is sum decomposable then $K(\pi)$ uniquely determines $\pi$.

**Proof.** By our previous proposition, we can recognize from $K(\pi)$ whether $\pi - n$ is sum decomposable. Assuming that it is sum decomposable, it has the form shown in Figure 8, where $\alpha_1$ is nonempty and sum indecomposable and $\alpha_2$ is also nonempty.

As $\pi - x$ is sum indecomposable for all entries $x$ in $\alpha_2$, in this case, we can determine the subpermutation contained in $\alpha_1$ by looking for the longest initial sum indecomposable subpermutation (ignoring the maximum entry) in any member of $K(\pi)$. From this we also identify the location of the entry $n$ in $\pi$. We can then determine the contiguous set of indices and values that make up the entries in $\alpha_2$. As we know there is some entry $x$ in $\alpha_1$ such that $\pi - x$ is sum indecomposable (Proposition 4.6
Lastly we show that \( \pi \) is a permutation in \( K(\pi) \) that contains \( \sigma_2 \) intact. After identifying this child, we have all the information necessary to reconstruct \( \pi \) from \( K(\pi) \).

Because the inverse of a sum indecomposable permutation is also sum indecomposable, Proposition 4.7 shows that we can also recognize from \( K(\pi) \) whether \( \pi - \pi(n) \) is sum indecomposable, and Proposition 4.8 shows that we can reconstruct \( \pi \) from \( K(\pi) \) whenever \( \pi - \pi(n) \) is sum decomposable. It remains only to reconstruct those permutations \( \pi \) for which both \( \pi - n \) and \( \pi - \pi(n) \) are sum indecomposable, which we do in the final result of this section to complete the proof of Theorem 4.2. The proof we give is an adaptation of that given in the general reconstruction context by Raykova [16].

**Proposition 4.9.** If \( \pi \) is a sum indecomposable permutation of length at least 5 and \( \pi - n \) and \( \pi - \pi(n) \) are both sum indecomposable then \( K(\pi) \) uniquely determines \( \pi \).

**Proof.** We prove the claim by induction on \( n \), the base case of \( n = 5 \) being readily verified by computer.

Let \( \pi \) be as in the statement of the proposition and suppose that \( \tau \) is a sum indecomposable permutation such that \( K(\tau) = K(\pi) \). We will show that this implies \( \tau = \pi \). Proposition 4.7 shows that \( \tau - n \) must be sum indecomposable. Additionally, by applying Proposition 4.7 to the set of inverses of \( K(\tau) = K(\pi) \) we see that \( \tau - \tau(n) \) must also be sum indecomposable. If both \( \pi - n = \tau - n \) and \( \pi - \pi(n) = \tau - \tau(n) \) then it follows that either \( \pi = \tau \) or \( \pi \) or \( \tau \) ends in \( n \). As \( \pi \) and \( \tau \) are assumed to be sum indecomposable, neither can end in \( n \). Therefore in this case, \( \pi = \tau \).

Thus we can assume that either \( \pi - n \neq \tau - n \) or \( \pi - \pi(n) \neq \tau - \tau(n) \). By symmetry, we may assume the former, and therefore by induction, we have that \( K(\pi - n) \neq K(\tau - n) \). Therefore there is some permutation in one set but not the other; suppose without loss of generality that \( \sigma \in K(\pi - n) \setminus K(\tau - n) \), so there is an entry \( x \) of \( \pi \) such that \( \pi - n - x = \sigma \). We claim that \( \pi - x \in K(\pi) \setminus K(\tau) \), a contradiction which will complete the proof.

To prove this claim we must first show that \( \pi - x \in K(\pi) \). Suppose otherwise, so \( \pi \), \( \pi - n \), and \( \pi - n - x \) are all sum indecomposable, but \( \pi - x \) is sum decomposable. The only way this can happen is if \( n \) is the second to last entry of \( \pi \), which violates the assumption that \( \pi - \pi(n) \) is sum indecomposable. Hence, \( \pi - x \in K(\pi) \).

Lastly we show that \( \pi - x \notin K(\tau) \). Suppose to the contrary that \( \pi - x \in K(\tau) \). Then there exists an entry \( y \) in \( \tau \) such that \( \pi - x = \tau - y \). If \( y = n \), then \( \pi - n - x = \tau - n - (n - 1) \in K(\tau - n) \), a contradiction. If \( y \neq n \), then \( \pi - n - x = \tau - n - y \in K(\tau - n) \), again a contradiction. Therefore, since \( \pi - x \in K(\pi) \setminus K(\tau) \), we have derived a contradiction. It must be the case that \( \pi = \tau \).

## 5. Hereditary Facts About Sum Indecomposable Permutations

The purpose of this section is to establish results of the form

\[
\text{If the class } \mathcal{C} \text{ contains } m \text{ sum indecomposable permutations of length } n \text{ then } \mathcal{C} \text{ also contains } m \text{ sum indecomposable permutations of length } n - 1.
\]

One such result is obvious—because every connected graph contains a connected induced subgraph with one fewer vertex, if a permutation class contains a sum indecomposable permutation of length \( n \geq 2 \) then it also contains a sum indecomposable permutation of length \( n - 1 \).
In general though, we must be careful with our choices of \( m \) and \( n \). For a trivial example, there are 3 sum indecomposable permutations of length 3, but only 1 of length 2. Thus our results must impose restrictions on \( n \). More significantly, no such result holds for \( m = 5 \), with any restrictions on \( n \), as witnessed by the set of permutations depicted in Figure 10. Note that the permutations from this example are contained in \( \text{Sub}(U^o) \).

The main results of this section are collected below.

- **Proposition 5.1.** If the permutation class \( C \) contains 2 sum indecomposable permutations of length \( n \geq 4 \) then it also contains at least 2 sum indecomposable permutations of length \( n - 1 \).

- **Proposition 5.3.** If the permutation class \( C \) contains 3 sum indecomposable permutations of length \( n \geq 5 \) then it also contains at least 3 sum indecomposable permutations of length \( n - 1 \).

- **Proposition 5.5.** If the permutation class \( C \) contains 4 sum indecomposable permutations of length \( n \geq 6 \) then it also contains at least 4 sum indecomposable permutations of length \( n - 1 \).

Of these results, the first was proved by Kaiser and Klazar [10], though we present a shorter proof using reconstruction. The second of these results was claimed in [21, Proposition A.16], but the argument presented there is faulty and our proof corrects this error. Proposition 5.5 is original, and best possible in light of the example depicted in Figure 10.

**Proposition 5.1.** If the permutation class \( C \) contains 2 sum indecomposable permutations of length \( n \geq 4 \) then it also contains at least 2 sum indecomposable permutations of length \( n - 1 \).

**Proof.** Suppose that \( C \) contains two sum indecomposable permutations of length \( n \geq 4 \), say \( \pi_1 \) and \( \pi_2 \). As the result is easily checked for \( n = 4 \), we may assume that \( n \geq 5 \). If either \( \pi_1 \) or \( \pi_2 \) has two sum indecomposable children then we are immediately done. Thus we may assume that they each have precisely one sum indecomposable child. Theorem 4.2 (which guarantees reconstruction from sum indecomposable children) implies that these children must be different, completing the proof. \( \square \)
As a consequence of our characterization of $K^{(1)}$ in Proposition 4.3, we obtain the proposition below which we use in what follows.

**Proposition 5.2.** Every sum indecomposable permutation of length at least 5 that does not lie in $K^{(1)}$ has at least one sum indecomposable child that does not lie in $K^{(1)}$.

**Proof.** Suppose that $\pi \notin K^{(1)}$ has length at least 5 and let $\sigma_1$ and $\sigma_2$ denote two sum indecomposable children of $\pi$. If $\sigma_1, \sigma_2 \in K^{(1)}$ then by Proposition 4.3 we must, up to relabeling, have $\sigma_1 = 1 \odot (12 \cdots (n-2))$ and $\sigma_2 = (12 \cdots (n-2)) \ominus 1$, because $\pi$ cannot contain one of these permutations together with $(n-1) \cdots 21$. This however implies that $\pi = 1 \odot (12 \cdots (n-2)) \ominus 1$, which has $1 \odot (12 \cdots (n-3)) \ominus 1$ as a sum indecomposable child, and as $n \geq 5$, this permutation does not lie in $K^{(1)}$—in fact, it lies in $K^{(3)} \setminus K^{(2)}$. \hfill \Box

We now prove our second result.

**Proposition 5.3.** If the permutation class $C$ contains 3 sum indecomposable permutations of length $n \geq 5$ then it also contains at least 3 sum indecomposable permutations of length $n-1$.

**Proof.** Suppose that $C$ contains three sum indecomposable permutations of length $n \geq 5$, say $\pi_1, \pi_2,$ and $\pi_3$, labeled so that $|K(\pi_1)| \geq |K(\pi_2)| \geq |K(\pi_3)|$. We may assume that $\pi_1, \pi_2, \pi_3 \in K^{(2)}$ because otherwise we are done. If $\pi_1$ (and consequently also $\pi_2$ and $\pi_3$) has only a single sum indecomposable child, then reconstruction (Theorem 4.2) shows that these three children must be distinct, and we are done.

Suppose now that $\pi_1$ has two sum indecomposable children. If $\pi_2$ also has two sum indecomposable children then we are done by reconstruction unless both $\pi_1$ and $\pi_2$ are increasing oscillations. In this case, $\pi_3$ cannot be an increasing oscillation (there are only two of each length), and so by reconstruction it must have a sum indecomposable child that is not a child of $\pi_1$ or $\pi_2$.

If $\pi_1$ has two sum indecomposable children and $\pi_2$ and $\pi_3$ have one sum indecomposable child each, we see from Proposition 5.2 that $\pi_1$ has a sum indecomposable child that does not lie in $K^{(1)}$ and therefore differs from the sum indecomposable children of $\pi_2$ and $\pi_3$, which must also differ from each other by reconstruction. \hfill \Box

From our characterizations of $K^{(1)}$ and $K^{(2)}$ in Propositions 4.3 and 4.4, we obtain the following analogue of Proposition 5.2.

**Proposition 5.4.** Every sum indecomposable permutation of length at least 7 that does not lie in $K^{(2)}$ has at least one sum indecomposable child that does not lie in $K^{(2)}$.\footnote{The restriction on length is needed because the permutation 324651 $\in K^{(3)} \setminus K^{(2)}$ has three sum indecomposable children, 23541, 32451, and 32541, all of which belong to $K^{(2)}$.}

**Proof.** Suppose that $\pi \notin K^{(2)}$ has length at least 7 and let $\mu$ denote its monotone quotient. If $\mu$ is an increasing oscillation then $\pi$ must contain a non-trivial monotone interval corresponding to a non-leaf of $\mu$. Since $|\pi| \geq 7$, either $|\mu| \geq 4$ or some entry of $\mu$ is inflated by an interval of length at least 3. In either case, an entry of $\pi$ can be deleted to yield a child of $\pi$ not in $K^{(2)}$.

In the case where $\pi$ is not a path, we consider two subcases. If $|\mu| \geq 5$, then by Proposition 2.3, $\mu$ contains a sum indecomposable child that is not an increasing oscillation, from which it follows...
immediately that $\pi$ contains a sum indecomposable child that does not lie in $K^{(2)}$. Otherwise $|\mu| \leq 4$. If $\pi$ has four non-trivial monotone intervals, we see from the characterization of $K^{(2)}$ that $\pi$ has no sum indecomposable children in $K^{(2)}$ and thus satisfies the conclusion of the proposition. Otherwise, because $|\pi| \geq 7$ and $|\mu| \leq 4$, $\pi$ contains a monotone interval of length at least 3. Deleting an entry from this interval yields a sum indecomposable child of $\pi$ that does not lie in $K^{(2)}$. \[ \square \]

We now establish the final result of the section.

**Proposition 5.5.** If the permutation class $C$ contains 4 sum indecomposable permutations of length $n \geq 6$ then it also contains at least 4 sum indecomposable permutations of length $n - 1$.

**Proof.** The case $n = 6$ can be checked by computer. Suppose that $C$ contains sum indecomposable permutations $\pi_1, \pi_2, \pi_3, \pi_4$ of length $n \geq 7$, labeled so that $|K(\pi_1)| \geq |K(\pi_2)| \geq |K(\pi_3)| \geq |K(\pi_4)|$. We may assume that $\pi_1 \in K^{(3)}$, as otherwise we are done. We use the observation that the sets $K^{(1)}$ and $K^{(2)}$ are closed downward$^2$ in the sense that if $\pi$ and $\sigma$ are sum indecomposable and $\sigma \leq \pi$, then $\pi \in K^{(1)}$ implies $\sigma \in K^{(1)}$, and $\pi \in K^{(2)}$ implies $\sigma \in K^{(2)}$. This is easily seen by examining the forms of such permutations established by Propositions 4.3 and 4.4.

The proof is via case analysis. First we treat the case $|K(\pi_1)| = 3$. If $|K(\pi_2)| = 3$ as well, then reconstruction (Theorem 4.2) guarantees that $\pi_2$ has a child different from those of $\pi_1$, and the proof is complete. Otherwise let $K(\pi_1) = \{\sigma_1, \sigma_2, \sigma_3\}$. By Proposition 5.4 we may assume that $\sigma_1 \notin K^{(2)}$. Proposition 5.3 then shows that $\pi_2, \pi_3, \pi_4$ together have at least three sum indecomposable children, none equal to $\sigma_1$ because $\sigma_1 \notin K^{(2)}$, completing this case.

We proceed to the case where $|K(\pi_1)| = 2$, which is more delicate than the previous case. Suppose first that $|K(\pi_2)| = 2$ and that $\pi_1$ and $\pi_2$ are not both increasing oscillations. Then together $\pi_1$ and $\pi_2$ have at least three sum indecomposable children, say $\sigma_1, \sigma_2, \sigma_3$, by reconstruction.

Suppose that $|K(\pi_3)| = 2$. If $K(\pi_3) \subseteq \{\sigma_1, \sigma_2, \sigma_3\}$ then by Proposition 5.2 and reconstruction, at least two of these children do not lie in $K^{(1)}$; suppose that $\sigma_1, \sigma_2 \notin K^{(1)}$. If $|K(\pi_4)| = 2$ then we are done by reconstruction as four permutations from $K^{(2)}$ cannot together have only 3 sum indecomposable children. Thus we may assume that $\pi_4 \in K^{(1)}$ and so we are done unless $K(\pi_4) = \{\sigma_3\}$. However, simple case analysis based on the three possibilities for $\sigma_3$ shows that this is not possible.

If instead $\pi_3, \pi_4 \in K^{(1)}$ then (up to labeling) the only way the $\pi_i$ could share 3 sum indecomposable children is as in the diagram below.

```
\[
\begin{array}{c}
\pi_3 \\
\sigma_1
\end{array}
\begin{array}{ccc}
\pi_1 \\
\sigma_2
\end{array}
\begin{array}{c}
\pi_2 \\
\sigma_3
\end{array}
\begin{array}{c}
\pi_4
\end{array}
\]
```

In this case we may assume by symmetry that $\sigma_1 = 1 \ominus (12 \cdots (n - 2))$. Moreover, $\sigma_3$ is obtained from $\sigma_1$ by inserting two entries and deleting two entries. Thus $\sigma_3$ cannot be decreasing, so the only possibility is that $\sigma_3 = (12 \cdots (n - 2)) \ominus 1$. In this case it is straight-forward to see that no choice of $\pi_1$ and $\pi_2$ makes this diagram possible.

If $\pi_1$ and $\pi_2$ are increasing oscillations, we may assume that $|K(\pi_3)| = |K(\pi_4)| = 1$ as otherwise we are in a case that we have already treated. By reconstruction, the sum indecomposable children of

$^2$It can be shown that all sets $K^{(m)}$ are closed downward in this sense, but we only use this fact for $K^{(1)}$ and $K^{(2)}$. 

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\( \pi_3 \) and \( \pi_4 \) must be different, and neither of the two can be increasing oscillations because they lie in \( K^{(1)} \), so \( \pi_1, \pi_2, \pi_3, \) and \( \pi_4 \) together have 4 sum indecomposable children, as desired.

There are two final cases. If \( |K(\pi_1)| = 2 \) and \( |K(\pi_2)| = |K(\pi_3)| = |K(\pi_4)| = 1 \) then \( \pi_2, \pi_3, \) and \( \pi_4 \) together have 3 sum indecomposable children by reconstruction, all of which belong to \( K^{(1)} \), while \( \pi_1 \) has a sum indecomposable child that does not lie in \( K^{(1)} \), giving the 4 sum indecomposable children we need. Finally, we cannot have \( |K(\pi_i)| = 1 \) for all \( i \) because there are precisely 3 such permutations of each length.

\[ \square \]

6. Narrowing the Search I — Basic Calculations

Recall that we say the sequence \( (s_n) \) can be realized if there is a permutation class with \( s_n \) sum indecomposable permutations of every length \( n \geq 1 \). In this section we begin the investigation of sequences of sum indecomposable permutations that lead to classes with growth rates less than \( \xi \) and of those, which can be realized. We separate this endeavor into roughly three tasks. In this section, we establish basic results on which sequences of sum indecomposable permutations lead to growth rates greater than \( \xi \). In the next section, we show that certain sequences that are not eliminated in the first step cannot be realized. Finally, in Section 8, we realize the remaining sequences.

The results of the previous section impose restrictions on how a sequence \( (s_n) \) of sum indecomposable permutations can taper. In particular, we know that (for large \( n \)) once the sequence drops below 4, it stays below 4, once it drops below 3, it stays below 3, once it drops below 2, it stays below 2, and once it hits 0, it stays 0. More precisely:

- for \( n \geq 1 \), if \( s_n = 0 \) then \( s_{n+1} = 0 \) (trivially),
- for \( n \geq 3 \), if \( s_n \leq 1 \) then \( s_{n+1} \leq 1 \) (by Proposition 5.1),
- for \( n \geq 4 \), if \( s_n \leq 2 \) then \( s_{n+1} \leq 2 \) (by Proposition 5.3), and
- for \( n \geq 5 \), if \( s_n \leq 3 \) then \( s_{n+1} \leq 3 \) (by Proposition 5.5).

We say that the sequence \( (s_n) \) is legal if it satisfies these five rules as well as \( s_1 \leq 1 \), \( s_2 \leq 1 \), and \( s_3 \leq 3 \) (these are the numbers of sum indecomposable permutations of those lengths). Only legal sequences of sum indecomposable permutations can be realized, though as we will see, not all legal sequences can be realized.

Recall from Section 3 that we say that the sequence \( (t_n) \) dominates \( (r_n) \), written \( (r_n) \preceq (t_n) \), if \( r_n \leq t_n \) for all \( n \). If the sequence \( (r_n) \) of sum indecomposable permutations leads to a growth rate greater than \( \xi \), every sequence dominating \( (r_n) \) also leads to a growth rate greater than \( \xi \).

Table 1 gives several \( \preceq \)-minimal legal sequences of sum indecomposable permutations that lead to growth rates equal to or greater than \( \xi \) and therefore do not need to be considered here. This table leaves open the sequence 1, 1, 2, 6 (and two sequences which dominate it, 1, 1, 2, 6, 1 and 1, 1, 2, 7) but it is easy to see that these sequences cannot be realized: in order to have only 2 sum indecomposable permutations of length 3, a class must avoid 231, 312, or 321, but each of the classes \( \text{Av}(231), \text{Av}(312), \) and \( \text{Av}(321) \) contains only 5 sum indecomposable permutations of length 4.

Table 2 presents several more families of legal sequences that lead to growth rates equal to or greater than \( \xi \). It differs from Table 1 in that the sequences it presents can be arbitrarily long. Moreover,
Consider for example the final sequence in this chart, 1

except for the first two sequences in this table, the growth rates of the corresponding classes (if the sequences can be realized, a question we do not need to consider) converge from above to $\xi$. To demonstrate this we need a basic analytic fact.

Consider for example the final sequence in this chart, 1, 1, 2, 3, 4, 8. If this sequence could be realized, the generating function for the sum closed class realizing it would be

$$
\frac{1}{1 - \left(x + x^2 + 2x^3 + 3x^4 + 4x^5 \frac{1-x^2}{1-x} + 8x^{i+5}\right)} = \frac{1-x}{1-2x - x^3 - x^4 - 4x^{i+5} + 8x^{i+6}}.
$$

Proceeding as usual, the growth rate of this generating function is the greatest real root of

$$x^{i+1} \left(x^5 - 2x^4 - x^2 - x - 1\right) - 4x + 8.$$

In fact, all of the polynomials in Table 2 except the first two are of the form $x^i f(x) + g(x)$ where the greatest real root of $f(x)$ is $\xi$ and $g(x)$ is a polynomial with negative leading term. In this case we see that $g(x)$ is negative for all $x > 2$ and thus the growth rates corresponding to sequences in this family converge to $\xi$ from above by the following.

**Proposition 6.1.** Suppose that $f(x)$ and $g(x)$ are polynomials, that the largest real root of $f$ is $r > 1$, and that $f'(x)$ is positive for $x > r$. Define $h_i(x) = x^i f(x) + g(x)$. If there exists a $\delta > 0$ such that $g$ is negative on the interval $[r, r + \delta]$ then the largest roots of the functions $h_i$ converge from above to $r$.

### Table 1: Short legal sequences leading to growth rates of at least $\xi$.

| sequence | growth rate is the greatest real root of | bound |
|----------|-----------------------------------------|-------|
| 1, 1, 2, 4, 3, 3, 2, 1 | $x^5 - 2x^4 - x^2 - x - 1$ | $\xi \approx 2.30522$ |
| 1, 1, 2, 4, 3, 3, 3 | $x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 3$ | $\xi > 2.30688$ |
| 1, 1, 2, 4, 4, 1, 1, 1, 1, 1 | $x^{11} - x^{10} - x^9 - 2x^8 - 4x^7 - 4x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ | $\xi > 2.30525$ |
| 1, 1, 2, 4, 4, 2 | $x^6 - x^5 - x^4 - 2x^3 - 4x^2 - 4x - 2$ | $\xi > 2.30692$ |
| 1, 1, 2, 4, 4, 5 | $x^5 - x^4 - x^3 - 2x^2 - 4x - 5$ | $\xi > 2.30902$ |
| 1, 1, 2, 5, 2, 1, 1 | $x^6 - 2x^5 + x^4 - 3x^3 - 2x^2 - 1$ | $\xi > 2.30790$ |
| 1, 1, 2, 5, 2, 2 | $x^6 - x^5 - x^4 - 2x^3 - 5x^2 - 2x - 2$ | $\xi > 2.31179$ |
| 1, 1, 2, 5, 3 | $x^5 - x^4 - x^3 - 2x^2 - 5x - 3$ | $\xi > 2.31392$ |
| 1, 1, 3, 3, 4, 1, 1, 1, 1, 1 | $x^{10} - x^9 - x^8 - 3x^7 - 3x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ | $\xi > 2.30528$ |
| 1, 1, 3, 3, 2 | $x^5 - x^4 - x^3 - 3x^2 - 3x - 2$ | $\xi > 2.30939$ |
| 1, 1, 3, 4 | $x^3 - 2x^2 + x - 4$ | $\xi > 2.31459$ |

### Table 2: Long legal sequences leading to growth rates of at least $\xi$.

| sequence | growth rate is the greatest real root of |
|----------|-----------------------------------------|
| 1, 1, 2, 3, 4 | $x^5 - 2x^4 - x^2 - x - 1$ |
| 1, 1, 2, 3, 4, 5, 3, 2, 1 | $x^5 - 2x^4 - x^2 - x - 1$ |
| 1, 1, 2, 3, 4, 5, 3, 3, 3 | $x^{i+3} \left(x^5 - 2x^4 - x^2 - x - 1\right) - x^4 + 2x^3 + 3$ |
| 1, 1, 2, 3, 4, 5, 4, 1, 1, 1, 1, 1 | $x^{i+8} \left(x^5 - 2x^4 - x^2 - x - 1\right) - x^4 + x^3 + 3x^2 + 1$ |
| 1, 1, 2, 3, 4, 5, 4, 2 | $x^{i+3} \left(x^5 - 2x^4 - x^2 - x - 1\right) - x^4 + x^3 + 2x + 2$ |
| 1, 1, 2, 3, 4, 5, 5 | $x^{i+2} \left(x^5 - 2x^4 - x^2 - x - 1\right) - x^4 + 5$ |
| 1, 1, 2, 3, 4, 6, 2, 1, 1 | $x^{i+4} \left(x^5 - 2x^4 - x^2 - x - 1\right) - x^4 + 4x^3 + x^2 + 1$ |
| 1, 1, 2, 3, 4, 6, 2, 2 | $x^{i+3} \left(x^5 - 2x^4 - x^2 - x - 1\right) - x^4 + 4x^2 + 2$ |
| 1, 1, 2, 3, 4, 6, 3 | $x^{i+2} \left(x^5 - 2x^4 - x^2 - x - 1\right) - x^2 + 3x + 3$ |
| 1, 1, 2, 3, 4, 7, 1 | $x^{i+2} \left(x^5 - 2x^4 - x^2 - x - 1\right) - 3x^2 + 6x + 1$ |
| 1, 1, 2, 3, 4, 8 | $x^{i+1} \left(x^5 - 2x^4 - x^2 - x - 1\right) - 4x + 8$ |
Proof. Because \( h_i(r) < 0 \) and \( h_i(x) \to \infty \) as \( x \to \infty \), we may conclude by the Intermediate Value Theorem that the largest real root of \( h_i(x) \) is strictly greater than \( r \) for all \( i \).

To prove convergence, let \( \epsilon > 0 \) be given, and assume without loss of generality that \( \epsilon < \delta \). We seek an integer \( n_0 \) such that for all \( n \geq n_0 \), the largest real root of \( h_n(x) \) lies in the interval \((r, r + \epsilon)\). As \( h_i(r) < 0 \) for all \( i \), it suffices to find \( n_0 \) such that \( h_n(r + \epsilon) > 0 \) for all \( n \geq n_0 \), or equivalently,

\[
(r + \epsilon)^n f(r + \epsilon) + g(r + \epsilon) > 0.
\]

Moreover, to ensure that the root found in the interval \((r, r + \epsilon)\) is the largest real root, we also choose \( n_0 \) large enough so that \( h_n'(x) > 0 \) for \( x > r \) for all \( n \geq n_0 \), or equivalently,

\[
x^{n-1} (xf'(x) + nf(x)) + g(x) > 0.
\]

As \( f(x) > 0 \) and \( f'(x) > 0 \) for \( x > r \), such an \( n_0 \) can be chosen to satisfy both demands.

\[\square\]

7. Narrowing the Search II — The Insertion Encoding

There are several families of legal sequences that would lead to growth rates less than \( \xi \) yet are not realizable. We eliminate these sequences here by way of a computer search.\(^3\) The results of this section are collected below.

**Proposition 7.1.** No realizable sequence beginning with 1, 1, 2, 3 may contain an entry greater than 5 before it contains an entry equal to 5.

**Proposition 7.2.** No realizable sequence beginning with 1, 1, 2, 3, 4, 4 may have an even-indexed entry equal to 5.

**Proposition 7.3.** No realizable sequence beginning with 1, 1, 2, 3, 4, 4 and containing a 5 may end with 1, 1.

We establish Propositions 7.1–7.3 using an implementation of the insertion encoding, which is a length-preserving bijection between permutation classes and formal languages introduced by Albert, Linton, and Ruškuc [2]. Importantly, there is a characterization of the classes whose insertion encodings are regular languages. To state this characterization we need a definition: a vertical alternation is a permutation in which every entry of odd index lies above every entry of even index, or the complement of such a permutation. For example, the permutations shown in Figure 11 are all vertical alternations. These are precisely the obstructions to having a regular insertion encoding:

\(^3\)The Python code to perform the computer search can be found at the first author’s website here: http://jaypantone.com/publications/010-xi-grs/.
Theorem 7.4 (Albert, Linton, and Ruškuc [2]). The insertion encoding of a permutation class $C$ forms a regular language if and only if $C$ does not contain arbitrarily long vertical alternations.

It follows readily from the Erdős–Szekeres Theorem that a permutation class contains arbitrarily long vertical alternations if and only if it contains arbitrarily long vertical alternations of one of the four types shown on the right of Figure 11. We call these vertical wedge alternations and vertical parallel alternations.

In order to prove Propositions 7.1–7.3, we need to consider classes whose sequences of sum indecomposable permutations begin with $1, 1, 2, 3$. It is easy to see that every permutation class that contains arbitrarily long vertical wedge or parallel alternation contains at least 4 sum indecomposable permutations of length 4, and thus the classes we are interested in have regular insertion encodings.

Moreover, an algorithm is described in Vatter [22]—using the Myhill–Nerode Theorem [14, 15]—which can compute the generating functions of classes with regular insertion encodings and the generating functions enumerating sum indecomposable permutations in such classes. An implementation of this algorithm is available in a Python package developed by Homberger and Pantone [9]. We use this algorithm to inspect all potential counterexamples to Propositions 7.1–7.3.

We begin by examining the classes to which Proposition 7.1 applies. In order for the sequence of sum indecomposable permutations to begin with $1, 1, 2, 3$, the class must avoid one of the 3 sum indecomposable permutations of length 3. In each of the three resulting classes—Av(231), Av(312), and Av(321)—there are 5 sum indecomposable permutations of length 4. In order for the sequence to begin with $1, 1, 2, 3$ we must choose 2 of these 5 to add to the basis.

At this point we have 30 permutation classes whose sequences of sum indecomposable permutations begin with $1, 1, 2, 3$. For each such class $C = Av(B)$ we create a list of the sum indecomposable permutations in the class up to length 10. If the sequence contains no entry greater than 5, we use the insertion encoding to verify that in fact the sum indecomposable sequence of the class has no entries greater than 5. If the sequence contains an entry of value greater than 5 that follows an entry of value 5, letting $i$ denote the length corresponding to occurrence of value 5, we repeat the test on each class whose basis consists of $B$ together with one of the sum indecomposable permutations of length $i$ in $C$. Obviously, if we encounter a sequence that contains an entry of value greater than 5 that does not follow a 5 we have found a counterexample.

For example, one of the 30 initial classes is $C = Av(231, 4312, 4321)$, and the enumeration of the sum indecomposable members of this class is given by the Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. The 5 sum indecomposable permutations of length 5 are $51234, 51243, 51324, 52134, \text{and } 52143$. To attempt to find a counterexample to Proposition 7.1 we add each of these to the basis one at a time, bringing the number of sum indecomposable permutations of length 5 down to 4, leading us to 5 new classes to check.

Continuing this example, when we add 51234 to the basis, the result is a class with a finite sequence of sum indecomposable permutations, $1, 1, 2, 3, 4, 3, 1$. Adding either 51243, 51324, or 52134 to the basis instead yields a class with sum indecomposable sequence $1, 1, 2, 3, 4^i$. More interestingly, the class $Av(231, 4312, 4321, 52143)$ has sum indecomposable sequence $1, 1, 2, 3, 4, 5, 6, 7, \ldots$, and thus its subclasses require further examination. Repeating the process, we form each class whose basis consists of $\{231, 4312, 4321, 52143\}$ together with one of the 5 sum indecomposable permutations of length 6. As none of the sum indecomposable sequences of these 5 subclasses contains an entry 5 or greater, none of them are counterexamples to the statement. Therefore, all subclasses of
Av(231, 4312, 4321) satisfy Proposition 7.1.

This process terminates after the insertion encoding has been applied to 178 permutation classes. We then use the insertion encoding to compute the generating function for the sum indecomposable members of each of these classes. In every case the sequence is either finite or ends with an infinite repeating sequence of the same number. For example, the generating function we compute for sum indecomposable members of Av(231, 4312, 4321, 52143, 613245) is

\[ x + x^2 + 2x^3 + 3x^4 + 4\frac{x^5}{1-x}. \]

Upon verifying that none of the 178 generating functions obtained in this manner contains a term greater than 5, the proof of Proposition 7.1 is complete.

The proofs of Propositions 7.2 and 7.3 are related to each other and similar to that of Proposition 7.1. We start with all classes whose bases consist of permutations of length 6 or less and whose sequences of sum indecomposable members begins 1, 1, 2, 3, 4, 4. Of these 173 classes we find via the insertion encoding that only two contain 5 sum indecomposable permutations of any length, Av(321, 3412, 4123, 23451, 314625) and Av(321, 2341, 3412, 51234, 251364).

Note that these classes are inverses of each other. Via the insertion encoding we can compute that the sequence of sum indecomposable permutations in each of these two classes is

\[ 1, 1, 2, 3, 4, 4, 5, 4, 5, 4, \ldots \]

As neither class contains an even-indexed entry equal to 5, we have proved Proposition 7.2.

It follows by direct observation that \( \oplus \text{Sub}(U) \) lies in the first of these classes, while its inverse class lies in the second. As we observed in Section 3, the sequence of sum indecomposable permutations in \( \oplus \text{Sub}(U) \) is also 1, 1, 2, 3, 4, 4, 5, 4, 5, 4, \ldots, so we immediately obtain the following result which we use in the next section.

**Proposition 7.5.** The basis of the class \( \oplus \text{Sub}(U) \) consists of 321, 3412, 4123, 23451, and 314625.

From inspection, we see that every sum indecomposable permutation of length at least 4 in this class (or its inverse) has at least 2 sum indecomposable children. Thus no subclass of \( \oplus \text{Sub}(U) \) may have its sequence of sum indecomposable members end with 1, 1, proving Proposition 7.3.

### 8. Realizing the Remaining Sequences

We now show that all remaining legal sequences can be realized, which we do with two constructions. The first of these consists of four chains of sum indecomposable permutations together with 5 sporadic sum indecomposable permutations. Specifically, the four chains are of the following forms:

- Type 1: 12 \( \cdots \) \((n - 1) \oplus 1\) for \( n \geq 1 \),
- Type 2: \((21 \oplus 12 \cdots (n - 3)) \oplus 1\) for \( n \geq 3 \),
- Type 3: \((1 \oplus 21 \oplus 12 \cdots (n - 4)) \oplus 1\) for \( n \geq 4 \), and
Figure 12: A set of sum indecomposable permutations that can be used to realize any legal sequence dominated by $1, 1, 3, 5, 5, 5, 4^\infty$.

- Type 4: $(12 \oplus 21 \oplus 12 \cdots (n - 5)) \ominus 1$ for $n \geq 5$.

The sporadic elements are $312, 3421, 4321, (21 \oplus 21) \ominus 1 = 32541$, and $(123 \oplus 21) \ominus 1 = 234651$.

The Hasse diagram displaying the containment relations between these permutations is shown in Figure 12. Importantly, all containment relations in this diagram run up and to the right. In particular, elements of type $j$ only contain elements of type $i$ for $i \leq j$. Therefore any legal sequence $(s_n)$ dominated by $1, 1, 3, 5, 5, 5, 4^\infty$ can be realized by taking the sum indecomposable permutations of length $n$ to be the leftmost $s_n$ permutations on level $n$ of Figure 12.

Moreover, we can show that these sequences are realized by finitely based classes as follows. It can be computed that the sum indecomposable permutations in this construction are all contained in the sum closed class whose finite basis consists of $2413, 3142, 3412, 4123, 4132, 4213, 4231, 4312, 24531, 25431, 34251, 34521, 35421, 43251, 43521, 45321, 54321, 54321, and $2346571$. Using the insertion encoding, it can then be verified that the sum indecomposable permutations in this class have the same enumeration as those of our construction, and thus this is the basis for the sum closure of all permutations used in the construction. The four types of sum indecomposable permutations in this class form chains, and thus we need only add to this basis the shortest excluded permutation of each type. This yields the following result.

**Proposition 8.1.** All legal sequences of sum indecomposable permutations that are dominated by $1, 1, 3, 5, 5, 5, 4^\infty$ are realizable. Moreover, all of these sequences are realized by finitely based classes.

While the construction used to prove Proposition 8.1 might seem arbitrary, it is far from it. Up to symmetry there is only one class whose sequence of sum indecomposable permutations begins $1, 1, 2, 3, 5$. That class is $Av(312, 4321, 3412)$, and its sum indecomposable permutations are all of the form $(\oplus \{1, 21\}) \ominus 1$. Because we need to realize sequences that begin $1, 1, 2, 3, 5$, our choice of the four chains is essentially forced.
Our second construction uses subclasses of $\bigoplus \text{Sub}(U^o)$.

**Proposition 8.2.** All legal sequences of sum indecomposable permutations that are dominated by $1, 1, 2, 3, 4^{2i}, 5, 4^\infty$ and do not end in $1, 1$ are realizable. Moreover, all of these sequences that lead to a growth rate of less than $\xi$ are realized by finitely based classes.

**Proof.** Let $(s_n)$ be a sequence of this form. If $(s_n)$ does not contain a 5 then it is realizable by Proposition 8.1. Thus (by the results of Section 5) we may assume that $(s_n)$ begins $1, 1, 2, 3, 4^{2i}, 5$, and ends with zero or more 4s, followed by zero or more 3s, followed by zero or more 2s, and then possibly a single 1 (though any one of these portions of $(s_n)$ other than the 1s might be infinite).

We realize such a sequence by taking the sum indecomposable members $\text{Sub}_p U^o$ of lengths up to $2i+5$ (giving a sequence beginning with $1, 1, 2, 3, 4^{2i+1}$) together with $\mu_{2i+5}$ itself, which gives us 5 sum indecomposable permutations of length $2i + 5$.

Later 4s in $(s_n)$ can be realized with members of $\text{Sub}_p U^o$, later 3s with the two increasing oscillations of each length together with heads of members of $U^o$ (increasing oscillations of primary type with their first entries inflated), and later 2s simply with the two increasing oscillations of each length. As we have assumed that $(s_n)$ does not end in $1, 1$, it can end in at most one 1, and for this sum indecomposable permutation we can take either increasing oscillation.

It remains to show that the sequences of this form that lead to growth rates under $\xi$ are realized by finitely based classes. First recall that $\xi$ is the growth rate of the sum closed class whose sequence of sum indecomposable permutations is $1, 1, 2, 3, 4^\infty$. Therefore every sequence of the form $1, 1, 2, 3, 4^{2i}, 5, 4^\infty$ leads to a growth rate greater than $\xi$. The sequences leading to growth rates less than $\xi$ therefore either do not contain a 5 (and thus are included in Proposition 8.1) or do not end in $4^\infty$.

By Proposition 7.5 we have

$$\bigoplus \text{Sub}(U^o) = \text{Av}(321, 3412, 4123, 23451, 314625).$$

Therefore the basis of $\bigoplus \text{Sub}^<(U^o)$ consists of the five basis elements of $\bigoplus \text{Sub}(U^o)$ together with $U^o$ itself. We are only interested in sequences that include a term less than 4 after their 5, so our construction above instructs us to omit all sufficiently long tails (increasing oscillations with their greatest entries inflated). This can be imposed by a single additional restriction. Moreover, once this tail is omitted, we do not need the members of $U^o$ that are longer than this tail in our basis. From that point, depending on how the sequence tapers, we need add at most three additional basis elements—a head, and one or two of the increasing oscillations—establishing that these classes all have finite bases.

Again, the construction used to prove Proposition 8.2 is far from arbitrary. The proofs of Propositions 7.2 and 7.3 show that our only choice is whether to use the antichain $U^o$ or its inverse, which permutation to use in addition to increasing oscillations to realize any 3s in the sequence (i.e., to choose between heads and tails), and then which of the two increasing oscillation to use to realize the final 1 of a sequence, if it ends with a 1.

### 9. The Growth Rates Under $\xi$

Taking into account the $\leq$-minimal sequences that lead to growth rates equal to or greater than $\xi$ from Tables 1 and 2, we can begin to describe the complete list of growth rates of permutation
classes smaller than \( \xi \). First, Table 3 lists all realizable sequences that are dominated by one of the short sequences from Table 1; by Proposition 8.1 all of the candidate sequences are realizable except 1, 1, 2, 6 (as established in Section 6).

Not all sequences dominated by one of the long sequences in Table 2 are realizable. All that begin with 1, 1, 2, 3, 4, 5 and end with 1, 1 (or 1, 1 or 1, 2) can be realized for \( i = 1 \) by Proposition 8.1, but cannot be realized for \( i \geq 2 \) by Proposition 7.3. In addition, sequences beginning 1, 1, 2, 3, 4, 5 for \( i \geq 2 \) are realizable only if \( i \) is even by Proposition 7.2.

All other sequences that are dominated by one of the long sequences in Table 2 can be realized by Proposition 8.2. This produces the list of realizable legal sequences in Table 4, and completes the listing of all growth rates of permutation classes at most \( \xi \).

| sequence | restriction | growth rate is the greatest real root of | growth rate |
|----------|-------------|----------------------------------------|-------------|
| 1, 1, 3, 3, 1 | \( i \leq 5 \) | \( x^i(x^5 - 2x^4 - 2x^2 + 2) + 1 \) | \( \leq 2.30503 \) |
| 1, 1, 3, 3, 2 | \( x^4 - 2x^3 - 2x + 1 \) | | \( \approx 2.29663 \) |
| 1, 1, 3, 3, 5 | \( x^3(x^4 - 2x^3 - 2x + 1) + 1 \) | | \( \approx 2.29663 \) |
| 1, 1, 3, 3, 1 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 4 | \( x^5 - 2x^4 - x^3 - 2x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 5 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 6 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 7 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 8 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 9 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 10 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 11 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 12 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 13 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 14 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 15 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 16 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 17 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 18 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 19 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 20 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 21 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 22 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 23 | \( x^6 - 2x^5 - x^4 - 2x^3 + x^2 + 1 \) | | \( \approx 2.30326 \) |
| 1, 1, 3, 3, 24 | \( x^7 - x^6 - x^5 - 2x^4 - 4x^3 - 3x^2 - 3x - 2 \) | | \( \approx 2.30326 \) |

*Table 3: Legal realizable sequences dominated by a sequence in Table 1 leading to growth rates under \( \xi \).*
Therefore computation shows that this class has growth rate $\xi$.
and Linton [1]—disproved a conjecture of Balough, Bollobás, and Morris [5] who had conjectured (in the more general context of ordered graphs) that the set of growth rates could not have such accumulation points. Klazar [11], who denotes the set of growth rates of permutation classes by $E^*$, suggested a possible way to revive their conjecture, writing “it seems that the refuted conjectures should have been phrased for finitely based downsets.” In particular, his Problem 2.7 reads

Let $E^*$ be the countable subset of $E$ consisting of the growth [rates] of finitely based [permutation classes]. Show that every $\alpha$ in $E^*$ is an algebraic number and that for every $\alpha$ in $E^*$ there is a $\delta > 0$ such that $(\alpha, \alpha + \delta) \cap E^* = \emptyset$.

While the first part of this problem regarding the algebraicity of growth rates remains open, we conclude by providing a counterexample to the second part.

Recall that Proposition 7.5 shows that

$\bigoplus \text{Sub}(U^o) = \text{Av}(321, 3412, 4123, 23451, 314625)$.

Therefore each of the classes $\bigoplus (\text{Sub}(U^o) \setminus \{\mu_7, \ldots, \mu_{2i+7}\})$ has a finite basis,

$\{321, 3412, 4123, 23451, 314625, \mu_7, \ldots, \mu_{2i+7}\}$.

Moreover, these classes have generating functions of the form

$$\frac{1}{1 - \left(x + x^2 + 2x^3 + 3x^4 + 4x^5 + 7 + \frac{x^2+1}{1-x^2}\right)} = \frac{1 - x^2}{1 - x^2 - x^3 - 2x^4 - 2x^5 - x^6 - x^{2i+7}},$$

and consequently their growth rates are the greatest real roots of the polynomials

$$x^{2i+7} - x^{2i+6} - 2x^{2i+5} - x^{2i+4} - 2x^{2i+3} - 2x^{2i+2} - x^{2i+1} - 1 = (x^5 - 2x^4 - x^2 - x - 1)(x + 1)x^{2i+1} - 1.$$

Thus these growth rates are solutions to

$$x^5 - 2x^4 - x^2 - x - 1 = \frac{1}{(x + 1)x^{2i+1}},$$

and it follows that these growth rates accumulate at $\xi$ from above. Combining this with our observation that there is a finitely based class with growth rate $\xi$, we obtain the following.

**Proposition 10.2.** There is a sequence of finitely based permutation classes whose growth rates accumulate at $\xi$ from above. Moreover, $\xi$ is itself the growth rate of a finitely based permutation class.

11. Concluding Remarks

In [19] it was proved that $\xi \approx 2.30522$ represents the phase transition from countably to uncountably many growth rates of permutation classes and that every growth rate below $\xi$ is achieved by a sum closed class. Here we have used this result to determine the complete list of growth rates below $\xi$. 

establishing that \( \xi \) is the least accumulation point of growth rates from above, and showing that each of these growth rates is achieved by a finitely based class, while there are growth rates arbitrarily close to \( \xi \) that cannot be achieved by finitely based classes.

Given that Bevan [6] has shown that every real number at least \( \lambda_B \approx 2.35698 \) is the growth rate of a permutation class, it is natural to try to close this gap of approximately 0.05176. The first challenge would be to extend the results of [19] to this range, i.e., to show that every growth rate between \( \xi \) and \( \lambda_B \) is achieved by a sum closed class. More generally, [19] presents a conjecture that every growth rate of a permutation class is achieved by a sum closed class; this is known to be true for growth rates up to \( \xi \) by [19] and for growth rates between \( \lambda_B \) and approximately 3.79 by [6, 20].

Supposing that this conjecture were established, extending the results of this paper up to \( \lambda_B \) still appears to be a difficult task. For a start, determining the realizable sequences of sum indecomposable permutations would require much more nuance than was required in Sections 5–8. While we only had to use one infinite antichain (\( U^o \), in Proposition 8.2) to realize the sequences below \( \xi \), one would need to consider several infinite antichains (some of which are introduced in the conclusion of [19]) to extend this work to \( \lambda_B \).

Indeed, even specifying the realizable sequences in this range is more challenging. Recall that when we considered sum closed classes lying between \( \bigoplus \text{Sub}^\infty(U^o) \) and \( \bigoplus \text{Sub}(U^o) \) in Section 3, we were able to say that these classes had sequences \( (s_n) \) of sum indecomposable members for all sequences \( (p_n,q_n) \) satisfying

\[
(1, 1, 2, 3, 4^2) \leq (s_n) \leq (1, 1, 2, 3, 4, 5, 4, 5, 4, \ldots).
\]

Now consider the antichain formed by inflating both leaves of odd-length increasing oscillations of primary type by 123. A typical member, of length 13, in this antichain is shown on the left of Figure 13. This antichain (or a symmetry of it) would have to be considered because the growth rate of the sum closure of its proper downward closure is only approximately 2.34.

However, the antichain member on the left of Figure 13 has two sum indecomposable children of length 12 and one sum indecomposable child of length 11 that are not contained in any other member of the antichain. Therefore if we choose to include the antichain member, we must also include these three additional permutations. Of course, we may also choose to take only the grandchild, one child and the grandchild, or both children and the grandchild. Thus we cannot characterize the realizable sequences arising from this antichain in terms of domination of sequences.

Instead, it seems likely one would have to express the realizable sequences as Bevan [6] did, using the “\( \beta \) bases” of Rényi [17]. To briefly introduce this perspective, note that if the sequence of sum
indecomposable permutations in a sum closed class is \((s_n)\) then its growth rate \(\gamma\) is the unique real number \(\gamma\) such that
\[
\sum_{n \geq 1} s_n \gamma^{-n} = 1,
\]
or in other words, 1 can be represented in base \(\gamma\) by the sequence \((s_n)\) of digits. In the case of sum closed classes lying between \(\bigoplus \text{Sub}^*(U^o)\) and \(\bigoplus \text{Sub}(U^o)\), we are allowed digits \(s_{2n} = 4\) for \(n \geq 3\) and \(s_{2n+1} \in \{4, 5\}\) for \(n \geq 3\). In this viewpoint, we can handle the multitude of choices we have in choosing antichain members, children, and grandchildren in antichains such as that shown in Figure 13 by allowing for “generalized digits”, such as 7.21 (choosing the antichain member, its two children, and its grandchild), 6.21 (choosing only the two children and the grandchild), 6.11 (choosing one child and the grandchild), 6.01 (choosing only the grandchild), and 6 (choosing none).

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