Rescaling of applied oscillating voltages in small Josephson junctions

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Abstract
The standard theory of dynamical Coulomb blockade [P(E) theory] in ultra-small tunnel junctions has been formulated on the basis of phase-phase correlations by several authors. It was recently extended by several experimental and theoretical works to account for novel features such as electromagnetic environment-based renormalization effects. Despite this progress, aspects of the theory remain elusive especially in the case of linear arrays. Here, we apply path integral formalism to re-derive the Cooper-pair current and the BCS quasi-particle current in single small Josephson junctions and extend it to include long Josephson junction arrays as effective single junctions. We consider renormalization effects of applied oscillating voltages due to the impedance environment of a single junction as well as its implication to the array. As is the case in the single junction, we find that the amplitude of applied oscillating electromagnetic fields is renormalized by the same complex-valued weight $\Xi(\omega) = |\Xi(\omega)| \exp{\im \eta(\omega)}$ that rescales the environmental impedance in the $P(E)$ function. This weight acts as a linear response function for applied oscillating electromagnetic fields driving the quantum circuit, leading to a mass gap in the thermal spectrum of the electromagnetic field. The mass gap can be modeled as a pair of exotic ‘particle’ excitation with quantum statistics determined by the argument $\eta(\omega)$. In the case of the array, this pair corresponds to a bosonic charge soliton/anti-soliton pair injected into the array by the electromagnetic field. Possible application of these results is in dynamical Coulomb blockade experiments where long arrays are used as electromagnetic power detectors.

1. Introduction
Since the pioneering theoretical work by Likharev et al. [1, 2] small Josephson junctions have been thought of as a dual system to large Josephson junctions—the roles of current and voltage are interchanged. In the case of large Josephson junctions, their effective interaction with oscillating electromagnetic fields has been intensively studied, demonstrating their unique suitability for microwave-based applications such as the metrological standard for the Volt (in terms of voltage Shapiro steps) and other microwave-based devices [3]. The dual system, on the other hand, holds enormous promise for complementary applications such as a metrological standard for the Ampère in terms of the current Shapiro steps [4].

However, observation of dual phenomena in small junctions faces daunting experimental and theoretical challenges due, in part, to the lack of an approach that consistently covers both regimes [5]. In particular, small tunnel junctions are prone to quantum and thermal fluctuations—their characteristics cannot be analyzed separate from their dissipative environment [6, 7]. As a consequence of Heisenberg uncertainty principle, their current-voltage $I-V$ characteristics is highly sensitive to energy changes in the environment [8]. Heuristically, tunneling of a single charge $e$ across a tunnel junction of capacitance $C$ and conductance $1/R$ is restricted unless the maximum energy it can absorb from zero-point fluctuations in the vacuum, $\hbar/RC$, is sufficient to offset its own charging energy $e^2/2C = E_c$ in the absence of other energy sources, where $\hbar$ is Planck’s constant. Thus, the heuristic condition for Coulomb blockade is $\hbar/RC < E_c$, which corresponds to $R_Q < R$, where $R_Q = \hbar/e^2$ is the quantum resistance.
In the case of Josephson junctions, quantum tunneling is due to paired electrons (Cooper pairs). Thus, two distinct ratios parameterize the Josephson junction at zero temperature: (1) The phase/charge regime characterized by the ratio $E_j/E_c$, where $E_j$ is the Josephson coupling energy and $E_c$ the charging energy, which are simply the coefficients of the potential energy and kinetic energy respectively in the Hamiltonian of the Josephson junction; and (2) the superconducting/Coulomb blockade regime characterized by the aforementioned ratio $R_j/R$ where $R$ is the real part of the environmental impedance, $Z(\omega)$. In this picture, the small junction is defined in the regime $E_j/E_c < 1$ and simultaneously $R_j/R < \kappa^2$, where $\kappa = 2$ is the number of electrons in a single Cooper pair.

Consequently, lifting of Coulomb blockade occurs when other sources of energy are present. For instance, energy is easily supplied by thermal fluctuations $\beta^{-1} = \hbar k_B T > 0$, strong coupling between pairs across the junction $E_j \gg E_c$ or external constant voltages $V_{\text{dc}} > V_0 \sim E_c$ above the Coulomb blockade threshold voltage, thus resulting in current-voltage characteristics highly dependent on these environmental parameters. Formally, this implies that the action for large junctions $\mathcal{I}(\phi) \approx \ln \int \mathcal{D}[\phi] \exp i \sum \mathcal{S}_n(\phi, \phi_0)$ is effective, meaning it emerges from tracing out environmental degrees of freedom $\mathcal{S}_n(\phi, \phi_0)$ that act as energy sources for the tunnel junction. This leads to a theory of dynamical Coulomb blockade [P(E) theory] in single small Josephson junctions formulated on the basis of $\mathcal{I}(\phi(t)\phi(0)) = Z^{-1} \int \mathcal{D}[\phi] \mathcal{I}(\phi(t)\phi(0)) \exp i \mathcal{I}(\phi) \exp i \mathcal{S}_n(\phi, \phi_0)$ where tunneling across the barrier is influenced by a high impedance environment treated within the Caldeira-Leggett model [13].

$P(E)$ theory has successfully been tested to a great degree of accuracy in a myriad of experiments [14–18]. This has lead to its widespread application in describing progressively complex tunneling processes such as dynamical Coulomb blockade in small Josephson junctions and quantum dots [19, 20]. Moreover, owing to significant improvement in microwave precision measurement technology such as near-quantum-limited amplification [21, 22] and progress in theory, recently published works suggest novel features in the framework ranging from time reversal symmetry violation [23] and Tomonaga-Luttinger Liquid (TLL) physics [24], to renormalization of electromagnetic quantities appearing in the $P(E)$ function [25–28]. Despite this progress, aspects of the theory remain elusive especially in the case of linear arrays.

Here, we apply path integral formalism to derive the Cooper-pair current and the BCS quasi-particle current in single small Josephson junctions. We consider renormalization effects of applied oscillating voltages due to wavefunction renormalization/Lehmann weights [29] that rescale the environmental impedance of the single junction as well as the array. The array is treated as infinitely long [30] and transformed into an effective circuit. As is the case for the single junction, we show that the Lehmann weight, $\Xi(\omega) = \exp(-\Lambda^{-3})\exp[-\beta M(\omega)]\exp i\beta\xi_m$ also acts as a linear response function for oscillating electromagnetic fields, and can be interpreted as the probability amplitude of exciting a ‘particle’ of mass $M$ from the junction ground state by the radio-frequency (RF) field [31]. The quantum statistics of this ‘particle’ are determined by the argument $\beta\xi_m$ where $\xi_m$ is identified as the Matsubara frequency [32]. In the case of the infinite array, this ‘particle’ corresponds to a bosonic charge soliton injected into the array. Possible application of these results is in accurately determining the absorbed RF power in dynamical Coulomb blockade experiments especially where long arrays are used as on-site electromagnetic power detectors [33, 34].

The paper is organized as follows:

Section 2 deals with the rescaling of oscillating voltages applied on single junctions. In the sections, 2.1 explains the basis for this rescaling. 2.3 introduces the finite temperature propagator and the environmental impedance as Green’s function, 2.4 interprets the impedance Lehmann weight as a complex-valued probability amplitude for applied oscillating voltages (RF field) exciting a ‘particle’ with quantum statistics given by the argument of the factor, or equivalently as a linear response function and the RF field as the external force leading the full expression for the current-voltage characteristics that includes the RF field and renormalization effects.

Finally, section 3 considers the Lehmann weight in infinitely long arrays. The difference from the single junction is an additional Lehmann weight $\exp(-\Lambda^{-3})$ representing a finite range of the electromagnetic field due to the presence of a charge soliton of length $\Lambda$ injected into the infinitely long array.

Note that, units where Planck’s constant, Swihart velocity [35] and Boltzman constant are set to unity ($\hbar = c = k_B = 1$) and Einstein summation convention are used through out unless otherwise stated with $\delta_{\mu\nu} = \delta_{\mu\nu}$ the Kronecker delta symbol.

2. Rescaling of oscillating voltages applied on single junctions

2.1. Introduction

Within the Caldeira-Leggett model [13], the environment of a dissipative voltage-biased single junction shown in figure 1 is modeled by the action $S_2 = \int dt \mathcal{L}_{s2}$, where the Lagrangian is given by,
Figure 1. Mesoscopic tunnel junction, J with capacitance C driven by a voltage source $V_x$ via an environmental impedance $Z(\omega)$ composed of infinite number of parallel $L_n C_n$ circuits. The circuit stores a flux $\Phi = \sum_i \phi_i = \phi_1 + \phi_2 + \phi_3$ related to a topological potential $\lambda(t)$ by $\int_{-\infty}^\infty d\lambda(s) = \Phi(t)$.

The effective action,

$$S'_k = -i \ln \int \prod_{n=1}^k D\phi_n \exp i S_k(\phi_\alpha, \phi')$$

$$= \int dt \left\{ \frac{C}{2\epsilon^2} \left( \frac{\partial \phi'_\alpha}{\partial t} \right)^2 - \frac{1}{4\epsilon^2} \int ds \phi'(s) \left[ \frac{\partial Z^{-1}(t-s)}{\partial t} \right] \phi'(t) \right\}$$

$$= \frac{2\pi}{2\epsilon^2} \int d\omega \phi'(\omega) i\omega Z_{eff}^{-1}(\omega) \phi'(\omega)$$

requires the impedance Green’s function in the $P(E)$ function to be modified by a wavefunction renormalization (Lehmann) weight, $[29]$, $P_k(E) = \frac{1}{2\pi} \int dt \exp (\kappa^2 J(t) + iEt)$

$$J(t) = \frac{2\epsilon^2}{2\pi} \int \frac{d\omega}{\omega} \text{Re} \{ Z_{eff}(\omega) \} \frac{\exp(-i\omega t) - 1}{1 - \exp(-\beta \omega)}$$

$$Z_{eff}(\omega) = [Z^{-1}(\omega) + i\omega C]^{-1} = \Xi(\omega)Z(\omega)$$

where $\beta$ is the inverse temperature, $\kappa e = 1e$, $2e$ is the quasi-particle, Cooper-pair charge, $E$ is the energy exchanged between the junction, $\Xi(\omega)$ is a Lehmann weight and $L_n C_n$ circuits acting as the environment, $\omega$ is the Fourier transform frequency that also plays the role of the thermal photon frequency at finite temperature.
It is known—at least since the work of Callen and Welton [7]—that the (causal) response function
\[ \Xi(\omega) \equiv \int_{-\infty}^{\infty} dt \, \theta(t) \chi(t) \exp(i\omega t) dt \] for a system driven by oscillating electromagnetic fields appears as the coefficient\(^2\) of the black body spectrum. Consequently, this requires that the response \( V_{RF}^{\mu}(t) \) as seen by the junction J in figure 1 be a weighted function of \( \chi(t) \) and the applied oscillating voltage \( V_{RF}(t) \): \( V_{RF}^{\mu}(t) = \int_{-\infty}^{t} ds \, \chi(t - s) \, V_{RF}(s) \). Therefore, to accurately describe the I–V characteristics of J driven by an applied oscillating voltage \( V_{RF}(t) \), it is not enough to simply rescale the impedance \( Z(\omega) \) in the \( P(E) \) function: the amplitude and phase of the applied oscillating voltage \( V_{RF}(t) \) has to be renormalized accordingly. In subsequent sections, we first consider tracing our steps from standard quantum electrodynamics (QED) and ease our way into circuit–QED and hence \( P(E) \) theory. We then proceed to introduce the finite temperature propagator for the junction and consider how the Lehmann weight for the impedance in \( P(E) \) theory, and its implications for single junctions and long arrays driven by \( V_{RF}(t) \). We find that, a finite time varying flux \( \Phi(t) \) stored by the circuit consistently implements the aforementioned wavefunction renormalization by guaranteeing the circuit responds linearly to \( V_{RF}(t) \).

2.2. Connection of \( P(E) \) theory to quantum electrodynamics (QED).

Here, we shall connect the Caldeira-Leggett model to QED. We shall find out that circuit-QED is merely the 1 dimensional space time version of QED. This also allows us to link the propagator introduced in equation (B42) with the photon propagator in QED.

The Fourier transform of the summed terms in Caldeira-Leggett action given in equation (1a) is \( S_0 + S_{\text{int}} \) where,
\[
S_0 = \frac{2\pi}{2e^2} \sum_n \int d\omega C_n \phi_n(\omega)[\omega^2 - \omega_n^2] \phi_n(-\omega) + \frac{2\pi}{2e^2} \sum_n \frac{1}{L_n} \int d\omega \phi_n(\omega) \phi'(-\omega) + O(\phi'^2),
\]
\( O(\phi'^2) \) is a term with \( \phi'^2 \) that we initially neglect, \( \phi_n \) are the Caldeira-Leggett phases of \( L_n C_n \) circuits in figure 1, \( \omega_n = 1/L_n C_n \) and the interaction term \( S_{\text{int}} \) is given by,
\[
S_{\text{int}} = \frac{2\pi}{2e^2} \int d\omega \omega^2 \phi'(-\omega) \phi'(-\omega) + \frac{2\pi}{e} \int d\omega I_E(-\omega) \phi'(-\omega),
\]
where \( I_E \) is the fluctuation current which we shall later set, \( I_E = 0 \).

Integrating out the fluctuating degrees of freedom, \( \phi_{in} \)
\[
\prod_n \int D\phi_n \exp(iS_0) = \exp(iS_0'),
\]
we arrive at,
\[
S_0' = \frac{(2\pi)^2}{4\pi} \int d\omega \omega^2 \phi'(-\omega) G^{-1}(\omega) \phi'(-\omega),
\]
\[
G^{-1}(\omega) = \frac{1}{e^2} \sum_n \frac{1}{L_n \omega^2 - \omega_n^2} \equiv i\omega e^{-2}Z^{-1}(\omega),
\]
where we have converted the Green’s function \( G(\omega) \), from the \( \phi_n \) degrees of freedom, to the environmental impedance \( Z(\omega) \).

Proceeding to combine the two actions yields,
\[
S'_0 + S_{\text{int}} = \frac{2\pi}{2e^2} \int d\omega \omega^2 [\epsilon^2 G^{-1}(\omega) + \omega^2 C] \phi'(-\omega) + \frac{2\pi}{e} \int d\omega I_E(-\omega) \phi'(-\omega).
\]
By defining the electromagnetic vector potential \( A_\mu = (V, \vec{A}) \), the electric field \( \vec{E} = \partial A / \partial t - \vec{\nabla} V \), the magnetic field \( \vec{B} = \vec{\nabla} \times \vec{A} \) and the fluctuation current density \( j_\mu^E \), it can be seen that the interaction term given by \( S_{\text{int}} \) above is actually Maxwell’s action in disguise,
\[
S_{\text{int}} \propto \int dt \, dAdl \left[ \frac{\epsilon_\mu \epsilon_\nu}{2} (\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}) + \epsilon j_\mu^E A_\mu \right] = \frac{1}{2} \int d^4k \epsilon_\mu \epsilon_\nu A_\mu(k) G_{\nu 1}^{-1} A_\nu(-k) + \epsilon \int d^4k \epsilon_\mu \epsilon_\nu j_\mu^E(-k) A_\nu(k),
\]
with the conditions \( \vec{B} = 0, \partial \phi' / \partial t = -\epsilon \vec{A} \cdot \vec{E}, \phi' = \epsilon \int dt \vec{n} \cdot \vec{A}, C = \epsilon A / l \) and \( \int dA \vec{n} \cdot \vec{l}_E = l_k \) where \( \vec{n} = (1, 0, 0) \) is the normal vector to the junction barrier, \( l \equiv d_{\text{eff}} \) the effective barrier thickness and \( A \) the junction area. The last term is the Fourier transform of the action where \( k \equiv k^{\mu} = (\omega, \vec{k}) \) is the photon

\(^1\) Appendix A.
\(^2\) This co-efficient can be computed using the driven system’s equation of motion.
energy–momentum satisfying \( k^2 \equiv k^\mu k_\mu = \varepsilon^2 \) and the Green’s function in 1 + 3 space-time takes the form [36],

\[
G_{\mu\nu} = \lim_{\varepsilon \to 0} -\frac{\eta_{\mu\nu} + k_\mu k_\nu / \varepsilon^2}{k^2 - \varepsilon^2}.
\] (8)

Integrating out \( \phi^I(\omega) \) degrees of freedom in \( S_0 + S_{\text{int}} \) given by equation (6) as in equation (5a) leads to a circuit–QED term,

\[
\frac{\pi}{\varepsilon^2} \int d\omega \eta_\omega(\omega) G_{\text{eff}}(\omega) I^\dagger_{\text{f}}(-\omega)
\] (9)

where \( G_{\text{eff}}^{-1} = G_{\omega}^{-1} + e^{-2}\omega^2C \) is reminiscent of the Coulomb interaction term in QED,

\[
\alpha \int \frac{d^4k}{(2\pi)^3} \langle \eta^\dagger_{\xi}(k) G_{\mu\nu}(k) I^\dagger_{\text{f}}(-k) \rangle
\] (10)

where \( \alpha = e^2 / 4\pi\varepsilon_0\varepsilon_r \) is the fine structure constant. The QED term is obtained in a similar fashion by integrating out \( A_\mu \) instead of \( \phi^I \). Nonetheless, both expressions are essentially describing the same process. The difference is the dimensionality of the theory: QED is in 1 + 2 space–time dimensions whereas circuit–QED is solely in the time dimension (circuit–QED). Thus, we have showed that \( 1/\omega^2C \) and hence \( G(\omega) \) both have an interpretation as photon propagator in circuit–QED.

However, a question still remains: is there any significance of this trivial Fourier space transformation given by \( G^{-1}(\omega) \rightarrow G_{\text{eff}}^{-1}(\omega) ? \) We notice that we can define a factor \( \Xi(\omega) = G(\omega) / G_{\text{eff}} \) and claim that this factor renormalizes the propagator \( G(\omega) \) (of the \( \phi^I \) degrees of freedom) to \( G_{\text{eff}}(\omega) \) due to the presence of the Maxwell term, \( S_{\text{int}} \). Since this renormalization takes a photon propagator into a different photon propagator, the Feynman rules to calculate it resemble photon self–energy interactions.

In particular, in self–energy interactions, the photon polarizes the QED vacuum by creating electron–positron pairs which subsequently annihilate. Such pairs can be created an infinite number of times, thus the contribution to the amplitude of all the processes takes the form:

\[
G_{\text{eff}} \equiv G + G U G + G U G U G \cdots = G / (1 - U G) = 1 / (G^{-1} - U)
\]

where \( G \) is the photon propagator and \( U \) the vacuum polarization energy (interaction term) [36]. Bearing this in mind, we formulate the following Feynman rules for the propagator:

1. The photon propagator \( G(\omega) = e^2 Z(\omega) / i\omega \) is represented by: \( \bigcirc \)
2. The vacuum polarization energy term, \( U(\omega) = e^{-2}\omega^2C \) is represented by: \( \bigcirc \)
3. Therefore, the leading order interaction term, \( G(\omega) U(\omega) G(\omega) \) is drawn as, \( \bigcirc \)

where time increases from left to right. Note that diagram 1 reads as follows: A photon of energy \( \omega \) is created, propagates with a probability amplitude \( G(\omega) \) and annihilates at a later time. Thus the amplitude must be assigned a photon creation operator and an annihilation operator, \( a^\dagger \) and \( a \) respectively and, in mathematical form, it should be written as \( a^\dagger G(\omega) a \).

Likewise, diagram 3 represents an interaction whereby a photon of frequency \( \omega \) is produced by acting on the vacuum state with \( a^\dagger \), it propagates with an amplitude given by \( G(\omega) U(\omega) G(\omega) \) then it annihilates by acting on the vacuum with \( a(\omega) \). We emphasize that reversing \( \omega \) reverses the aforementioned processes. This implies that the operators themselves should also be defined accordingly as,

\[
a(-i\omega) = a^\dagger,
\]

\[
a(i\omega) = a,
\]

\[
[a(\omega), a(\omega')] = [\theta(\omega') - \theta(\omega)]\delta_{\omega, -\omega'},
\] (11c)

where \( \theta(\omega) \) is the Heaviside function. This clearly displays the roles of the positive and negative frequencies. Note that negative frequencies are allowed since we are interested in energy differences due to single photon emission and absorption processes. For instance, processes where a photon is created before annihilation are related to processes where a photon is annihilated before creation by reversing the sign of the frequency \( \omega \) and re-ordering the \( a^\dagger, a \) operators appropriately.

2.3. Connection to \( P(E) \) theory: The finite Temperature Green’s Function and Propagator

Observe that a straightforward regularization procedure verifies \( Z_{\text{eff}}(\omega) \) plays the role of effective Green’s function of the \( P(E) \) function,

\( \text{3 This procedure is invalid when arguments for time–reversal asymmetry e.g. discussed in [33] apply.} \)
\[
Z_{\text{eff}}(\omega) = Z(\omega) + Z(\omega)[-i\omega CZ(\omega)] + Z(\omega)[-i\omega CZ(\omega)]^2 + \cdots + Z(\omega)[-i\omega CZ(\omega)]^n + \cdots + Z(\omega)[-i\omega CZ(\omega)]^p+^{1\rightarrow\infty}
= Z(\omega) \sum_{n=0}^{+\infty} [-i\omega CZ(\omega)]^n = \frac{Z(\omega)}{1 + i\omega CZ(\omega)} = \Xi(\omega)Z(\omega),
\]
(12)

analogous to the renormalization of the propagator in QED which often leads to a (Lehmann) factor \([29, 37]\) analogous to \(\Xi(\omega)\).

To elucidate this, consider the finite temperature propagator for \(S_0'\) given by \(\langle \cdots \rangle\),
\[
D_{\text{eff}}(\omega) = \frac{-\kappa^2}{2\pi i} \left[ G(\omega) + G(\omega)U(\omega)G(\omega) + \cdots \right] \langle a(\omega)a(-\omega) \rangle
+ \frac{-\kappa^2}{2\pi i} \left[ G(-\omega) + G(-\omega)U(-\omega)G(-\omega) + \cdots \right] \langle a(-\omega)a(\omega) \rangle
= \frac{-\kappa^2}{2\pi i} \left[ G(\omega)(1 - U(\omega)G(\omega))^{-1} \langle a(\omega)a(-\omega) \rangle \right]
+ \frac{-\kappa^2}{2\pi i} \left[ G(-\omega)(1 - U(-\omega)G(-\omega))^{-1} \langle a(\omega)a(-\omega) \rangle \right]
= \frac{-\kappa^2}{2\pi i} \left[ G_{\text{eff}}(\omega) \langle a(\omega)a(-\omega) \rangle + G_{\text{eff}}(-\omega) \langle a(-\omega)a(\omega) \rangle \right].
\]
(13a)

The term proportional to \([U(\pm\omega)G(\pm\omega)]^n\) is the finite temperature propagator for the photon interacting \(n\) times with the junction impedance and the perturbation series
\[
\sum_{n=0}^{+\infty} \left[ U(\pm\omega)G(\pm\omega) \right] = [1 + y(\pm\omega)Z(\pm\omega)]^{-1} = 1 - y(\pm\omega)Z_{\text{eff}}(\pm\omega) \equiv \Xi(\pm\omega),
\]
(14b)
is computed by analytic continuation of the series \(1 + x + x^2 \cdots + x^n \rightarrow [1 - x]^{-1}\), where \(x\) is given by the diagram, \(\cdots\). Thus, the effective Green’s function \(G_{\text{eff}}(\omega)\) becomes,
\[
\begin{align*}
\cdots &= \cdots \times \frac{\cdots}{1 - \cdots} = \frac{1}{\left[\cdots\right]^{-1} - \cdots}.
\end{align*}
\]

Comparing equation (13) to (14a), we find that \(G(\omega)\) is rescaled to \(G_{\text{eff}}(\omega) = [G^{-1}(\omega) - U(\omega)]^{-1}\). Consequently, the effective action is given by
\[
S'_1|_{\beta=0} = S'_0 + S_{\text{ent}}|_{\beta=0} = \pi \int_{-\infty}^{+\infty} d\omega \phi'(\omega)G_{\text{eff}}^{-1}(\omega)\phi'(-\omega),
\]
(15)
which is equivalent to equation (B37a) with \(I_p = 0\).

Indeed we recover \(\mathcal{J}(t) = e^t[D_{\text{eff}}(t) - D_{\text{eff}}^0(0)]\), where \(D_{\text{eff}}^0(t) = \int d\omega D_{\text{eff}}^{0\omega}(\omega)\exp(-i\omega t)\) is the Fourier transform of \(D_{\text{eff}}^{0\omega}(\omega)\), by substituting \(\langle a(\omega)a(-\omega) \rangle = n(\omega) = \left[\exp(-\beta\omega) - 1\right]^{-1}\) and \(\langle\cdots\rangle\) for negative frequencies.

\[\text{has been used to regularize the divergent sum when calculating averages, }\langle\cdots\rangle\text{, for negative frequencies.}\]

\[\text{4 Also confer equation (B37a) and (B42b).}\]
In section 2.4, we have established that the fraction of photons absorbed by a tunnel junction is remains to be investigated. Consequently, we can take \( G_{\text{eff}}(\omega) \) to be real and \( y(\omega) = i\omega C \), corresponding to equation (B14), yields

\[ \arctan \omega R C = \eta(\omega) = \beta e_m. \] (18)

We introduce quantum statistics of the ‘particle’ by identifying \( e_m = (2m + 1)\pi \beta^{-1} \) or \( e_m = 2\pi m \beta^{-1} \) as the fermionic or bosonic Matsubara frequency [32] respectively where \( m \in \mathbb{Z} \) is an integer. This requires the oscillation period \( 2\pi / \omega \) of the electromagnetic field to greatly exceed the relaxation time \( RC \) of the circuit, \( 2\pi / \omega \gg 2\pi RC \). Thus, when this condition is not satisfied, it leaves the possibility for ‘anyons’ [38] with exotic statistics. Consequently, we can take \( \Xi(\omega) \) as the amplitude [39] that a photon of frequency \( \omega \) is absorbed by the junction creating a ‘particle’ of mass \( M \) and statistics according to the Matsubara frequency \( e_m \). This realization, together with the fact that the field theory introduced in appendix B.2.2 lives in 1 + 2 dimensions, suggests that anyonic excitations cannot be ignored [38].

Finally, notice that the exponent can be re-written conveniently as,

\[ G_{\text{eff}}^{-1}(M, i e_m) + \frac{1}{2} \coth(\beta \omega / 2) G_{\text{eff}}^{-1}(M, i e_m) + \sum_{m=-\infty}^{\infty} \frac{1}{2\pi mi - \beta \omega} \] (19a)

where \( G_M, i e_m = \beta (M - i e_m) \) is the Green’s function of the ‘particle’. Thus, its excitation statistics can be computed as,

\[ \sum_{m=-\infty}^{\infty} G_M, i e_m = \frac{\beta^{-1}}{M - i e_m} \] (19b)

where equation (19b) is the inverse thermal Green’s function of the ‘particle’ with mass \( M \). Whether these ‘particles’ in the single junction are anything more than a tool to implement the renormalization scheme above remains to be investigated.

### 2.4.1. Applied alternating voltages

In section 2.4, we have established that the fraction of photons absorbed by a tunnel junction is \( |\Xi(\omega)|^2 = \left[ 1 + y(\omega) Z(\omega) \right]^{-2} \). A straightforward way to experimentally measure \( |\Xi(\omega)|^2 \) is applying an external oscillating electric field in the form of ac voltage \( V_{\text{rf}}(t) \) supplying power \( P = \int_{-T/2}^{T/2} dt V_{\text{rf}}^2(t) \) where \( T \) is the oscillation period. Whether the ac power is efficiently transferred to the junction from this ac source ought to depend on \( \Xi(\omega) \). We proceed to formally express the explicit form of the effective alternating voltage at the junction.

This can be done by substituting \( \Delta \phi_0(t) = \Delta \phi_0 (t) - 0^+ = \int_{dy} V_x(t) d\tau \) in equation (B10) where \( V_x = V + V_{\text{rf}}(t) \) and \( V_{\text{rf}}(t) \) is an alternating voltage corresponding to the effect of the RF field given by

\[ V_{\text{rf}}(t) = \int_{-\infty}^{\infty} V_{\text{rf}}(\omega) \exp(-i\omega t) d\omega = V_{\text{ac}} \cos \Omega t \] (20a)

5 Photon amplitude here refers to a wavefunction renormalization (Lehmann) weight, where the photon wavefunction is taken to be the \( x \) component of the Riemann—Silberstein vector.
\[ V_{\text{RF}}(\omega) = \frac{V_c}{2} \{ \delta(\Omega - \omega) + \delta(\Omega + \omega) \} \]  

(20b)

where \( V_{\text{RF}}(\omega), V_c \) and \( \Omega \) is the spectrum, the amplitude and frequency of the RF field respectively and the significance of the + sign is that \( 0^+ \) is experimentally, not computationally equal to 0; it is the limit \( 0^+ \equiv t \rightarrow 0 \).

Proceeding, the applied power \( P \) of the RF field is the mean-square value of \( V_{\text{RF}}(t) \) given by,

\[ P \propto \frac{1}{2\pi/\Omega} \int_{-\pi/\Omega}^{\pi/\Omega} [V_{\text{RF}}(t)]^2 dt = \frac{V_c^2}{2}, \]

(21)

where the proportionality factor is the admittance of the junction.

### 2.4.2. Lehmann weight and linear response

However, we are interested in the power absorbed by the junction \( P_I \) instead since it modifies the \( I - V \) characteristics. In section 2.4, we argued that this power \( P_I \) is proportional to \( |\Xi(\Omega)|^2 \) in the presence of bosonic, anyonic or fermionic excitations.

Within the context of linear response theory [40], this means that the applied voltage \( V_{\text{RF}}(t) \) acts as an external force, and the effective voltage as a linear response \( V'_{\text{RF}}(t) \) of the circuit,

\[ V'_{\text{RF}}(t) = \int_{-\infty}^{t} \chi(t - s) V_{\text{RF}}(s) ds, \]

(22a)

\[ V'_{\text{RF}}(\omega) = \Xi(\omega) V_{\text{RF}}(\omega) \]

(22b)

\[ |\Xi(\omega)| = \int_{0}^{+\infty} \chi(t) \exp(i\omega t) dt, \]

(22c)

where \( \chi(t - s) \) is the response function, making \( \Xi(\omega) \) the susceptibility. For the special case discussed in equation (18), we have \( \chi(t) = \langle 1/RC \rangle \exp(-t/RC) \). Thus, the RF spectrum above gets modified to

\[ V'_{\text{RF}}(\omega) = \frac{V_c}{2} \Xi(\omega) \{ \delta(\Omega - \omega) + \delta(\Omega + \omega) \}, \]

(23a)

\[ V'_{\text{RF}}(t) = \int_{-\infty}^{+\infty} V'_{\text{RF}}(\omega) \exp(-i\omega t) d\omega', =|\Xi(\Omega)| V_c \cos(\Omega t + \eta), \]

(23b)

and \( P_I \) is given by

\[ P_I \propto \frac{1}{2\pi/\Omega} \int_{0}^{2\pi/\Omega} [V'_{\text{RF}}(t)]^2 dt = \frac{V_c^2}{2} |\Xi(\Omega)|^2 \propto P|\Xi(\Omega)|^2, \]

(24)

where we have used equation (20), (21) and (23), as expected.

### 2.4.3. Unitarity and the topological potential

This section displays the unitary nature of the renormalization effect. In particular, we show that the renormalization effect can be split into two: The fraction of ac voltage drop at the junction and the fraction of ac drop at the environment. This entails taking the quantum states of the environment and the junction as two orthonormal states \( |\psi_1 \rangle = |1\rangle \) and \( |\psi_2 \rangle = |0\rangle \) respectively (figure 2) undergoing a time dependent unitary transformation.

In particular, defining matrices \( V_2 \) and \( U_{\text{RF}} \) and two quantum states \( |\psi_1 \rangle \) and \( |\psi_2 \rangle \) of the junction and the environment respectively,

\[ V_2 = \frac{1}{2} \{ V_0 \sigma_0 + V_u U_{\text{RF}} \}, \]

(25a)

\[ U_{\text{RF}}(t) = \begin{pmatrix} \Xi(\Omega) e^{it\eta} & -\sqrt{1 - |\Xi(\Omega)|^2} e^{it\eta} \\ \sqrt{1 - |\Xi(-\Omega)|^2} e^{-it\eta} & \Xi(-\Omega) e^{-it\eta} \end{pmatrix}, \]

(25b)

such that,

\[ \begin{pmatrix} |\psi'_1 \rangle \\ |\psi'_2 \rangle \end{pmatrix} = U_{\text{RF}}(t) \begin{pmatrix} |\psi_1 \rangle \\ |\psi_2 \rangle \end{pmatrix}. \]

(25c)

\[ V' = \text{tr} \{ V_2 \} = V + \int_{-\infty}^{+\infty} V'_{\text{RF}}(\omega) \exp(-i\omega t) d\omega, \]

(25d)

\[ \det (U_{\text{RF}}) = 1, \]

(25e)

\[ U_{\text{RF}}^\dagger U_{\text{RF}} = U_{\text{RF}}U_{\text{RF}}^\dagger = 1, \]

(25f)

where \( \sigma_0 \) is the \( 2 \times 2 \) identity matrix, we find that these states \( |\psi_1 \rangle = |1\rangle \), \( |\psi'_1 \rangle = |1'\rangle \) and \( |\psi_2 \rangle = |0\rangle \), \( |\psi'_2 \rangle = |0'\rangle \) are normalized \( \langle 0 | 0 \rangle = \langle 1 | 1 \rangle = 1 \) and orthogonal to each other, \( \langle 0 | 1 \rangle = \langle 0' | 1' \rangle = 0 \) while
orthogonality is preserved under the unitary transformation $U_R(t = 0)$ leading to a renormalized external voltage $V_k = V + V_{ac} \cos(\Omega t) \rightarrow V_k' = V_k + A(t)$. This requires the topological flux $\Delta \Phi(t) \neq 0$ in equation (B40) not vanish in the presence of oscillating electromagnetic fields. Solving for the topological potential $A(t)$, we find

$$A(t) = V_k' - V_k = V_{ac} \{v(\Omega) \sin \Omega t - u(\Omega) \cos \Omega t\}$$ \hspace{1cm} (26a)

$$\Gamma(\Omega) = u(\Omega) + iv(\Omega)$$ \hspace{1cm} (26b)

$$\Xi(\Omega) = 1 - \Gamma(\Omega) = 1 - u(\Omega) - iv(\Omega)$$ \hspace{1cm} (26c)

with equation (26c) relating the impedance Lehmann weight $\Xi(\Omega)$ to the topological potential amplitude factors $u(\Omega), v(\Omega)$.

Thus, the purpose of the topological potential is to implement the renormalization scheme above. By equation (26), we find that the topological flux $\Phi(t) = \int_{-\infty}^{t} A(\tau) d\tau$ is ill-defined for $\tau = -\infty$ since $\sin(\omega \infty)$ and $\cos(\omega \infty)$ both oscillate rapidly without converging. Nonetheless, this poses no problem since it is the flux difference $\Delta \Phi(t) = \int_{t_0}^{t} A(\tau) d\tau$ that appears in the correlation function in equation (B40) rendering the $I-V$ characteristics in equation (B44) perfectly well-defined.

Moreover, by equation (14b), we find that

$$\Xi(\Omega) = \frac{y^{-1}(\Omega)}{y^{-1}(\Omega) + z(\Omega)} = \frac{1}{1 + z(\Omega)y(\Omega)}$$ \hspace{1cm} (27)

$$\Gamma(\Omega) = \frac{z(\Omega)}{y^{-1}(\Omega) + z(\Omega)} = y(\Omega)Z_{\text{eff}}(\Omega)$$ \hspace{1cm} (28)

are ratios of impedances. Thus, in the simple model in equation (B14), power renormalization is negligible ($\Xi(\Omega) \approx 1$) only for extremely low frequencies satisfying $1/RC \gg \Omega$. However, for samples exhibiting Coulomb blockade that satisfy the Lorentzian-delta function approximation $\text{Re} \{Z_{\text{eff}}\} = R/(1 + \Omega^2 C^2 R^2) \sim \pi C^{-1} \Omega$, the conductance $R^{-1}$ is extremely small ($1/RC \ll \Omega$) and thus we should expect power renormalization for virtually all applied frequencies.

Figure 2. Diagrammatic representation of the unitary transformation implemented by the matrix $U_R(t = 0)$ in equation (25). (a) The equivalent circuit of the Josephson junction labeled by the coupling energy $E_J$, the capacitance $C$ and inductance $L = \sum_1 L_n$ where the admittance $y(\omega) = \omega C + \sum_1 1/\omega L_n$. The junction is coupled serially to the environmental impedance $Z(\omega)$ and symmetrically biased by an external voltage $V_0$ where the quantum states of the environment and the junction can be represented by $|0\rangle$ and $|1\rangle$ respectively; (b) The equivalent circuit of the Josephson junction and its environment. The bias voltage, the quantum states and the environmental impedance are all renormalized by the unitary transformation given by $\{\hat{U}\}$ in equation (23); (c) A Bloch sphere representing the action of the unitary transformation given by $\hat{U}(t = 0)$ in equation (23), where $\eta$ is the argument of the renormalization factor $\Xi = |\Xi| \exp(i\eta)$ and $\theta = 2\arccos|\Xi|$.
2.5. Current–Voltage Characteristics with finite RF Field

Now that we have the form of the voltage \( V'_c = V + V_{ac} + A = V + |\Xi|V_{ac}\cos(\Omega t + \eta) \), where \( |\Xi|V_{ac} = V_{ac}^{\text{eff}} \), it should be substituted into equation (B44) to determine the Cooper-pair and quasi-particle tunneling current in the presence of microwaves. Thus, substituting \( V'_c \) into equation (B44) and using the identities, \( \sin(x \sin y) = \sum_{n=-\infty}^{\infty} j_0(x) \sin ny \) and \( \cos(x \sin y) = \sum_{n=-\infty}^{\infty} j_0(x) \cos ny \) where \( j_0(x) = (-1)^{-n} j_{-n}(x) = \frac{1}{\sqrt{2\pi}} \int ds \exp(i sx - ns) \) is the Bessel function of the first kind, \( x, y \) are arbitrary functions and \( n \in \mathbb{Z} \) is an integer, the \( I-V \) characteristics of the irradiated junction can be expressed in terms of the \( I-V \) characteristics \( I_1 \) and \( I_2 \) of the unirradiated junction,

\[
I(V) = \sum_{n=-\infty}^{\infty} j_0(x) \left( \frac{eV_{ac}}{\Omega} \right) I_1 \left( V - \frac{n\Omega}{e} \right) + \sum_{n=-\infty}^{\infty} j_0(x) \left( \frac{2eV_{ac}}{\Omega} \right) I_2 \left( V - \frac{n\Omega}{2e} \right),
\]

(29)

Here, \( I_{1,2} \) are the quasi-particle, Cooper pair RF-free \( I-V \) characteristics given in equation (B44), \( j_0(x) \) are Bessel functions of the first kind where the order \( n \) is the number of actual photons absorbed by the junction, \( V_{ac} \) is the amplitude of the alternating voltage, \( \Omega \) is the energy quantum of individual photons, \( V \) and \( I \) respectively are the applied dc voltage and tunneling current of the junction. The total current is shifted by the number of photons reflecting energy conservation and is proportional to the square of the Bessel function reflecting the modification of density of states. This equation neglects higher harmonics derived in [25] which were shown to be largely suppressed.

3. Renormalization of applied oscillating voltages in linear arrays of Josephson junctions

3.1. Renormalization factor and soliton length

We consider the action for \( N_0 \) number of Josephson junction elements where the action resembles that for a single junction given in equation (1b) where all the admittance elements in the expression are replaced by \((N_0 - 1) \times (N_0 - 1)\) matrices,

\[
S^A = \sum_{j, k=1}^{N_0-1} \int dt \left\{ \frac{1}{2e^2} G_{jk} \frac{\partial \phi_j^*(t) \partial \phi_k(t)}{\partial t} - \frac{1}{4\pi e^2} \int ds \phi_j^*(s) \left( \frac{\partial (Z^{-1})_{jk}(t-s)}{\partial t} \right) \phi_k(t) \right\}
- E_1 \sum_{i=1}^{N_0-1} \int dt \cos(\phi_i^*(t) - \phi_{i+1}^*(t)),
\]

(30a)

where \( G_{jk} = (C_0 + 2C) \delta_{jk} - C_\delta_{j+1,k} - C_\delta_{j-1,k} \) is the capacitance matrix of the array with Josephson junctions of equal capacitance \( C \), \( N_0 - 1 \) is the number of islands, \( C_0 \) is the stray capacitance of each island, \( Q_j^t \) and \( Q_j^\ell \) are \( e^2/2C \) the phase and charge of each island respectively, \( E_1 \ll e^2/2C \) the Josephson coupling energy of each island and \( (Z^{-1})_{jk} \) is the unspecified environment admittance matrix of the array. The Fourier transform of this action with \( E_1 \to 0 \) as a perturbation parameter takes the general form,

\[
S^A = \frac{2\pi}{2e^2} \sum_{j, k=1}^{N_0-1} \int dw \phi_j^*(\omega) \phi_k^*(\omega) \Omega \omega (Z_{eff}^{-1})_{jk} \phi_j^*(\omega),
\]

(30b)

where \( (Z_{eff}^{-1})_{jk} \) is the effective admittance matrix. It is now straightforward to determine the phase-phase correlation function \( \langle \phi_j^*(t) \phi_k^*(t) \rangle \) by recalling that its Fourier transform is equal to the imaginary part of the Green’s function of the action read-off from equation (30b) to yield \( G_{jk}^{\text{eff}}(\omega) = 2e^2\omega^{-1}(Z_{eff}^{-1})_{jk}(\omega) \) where \( (Z_{eff}^{-1})_{jk}(\omega) \) is the inverse matrix of \( (Z_{eff}^{-1})_{jk}(\omega) \). This procedure yields,

\[
\int dt \langle \phi_j^*(t) \phi_k^*(0) \rangle \exp(i\Omega t) = G^{\text{eff}}_{jk}(\omega) - G^{2\text{eff}}(\omega) = 2e^2\omega^{-1}(\Omega Z_{eff}^{-1})_{jk}(\omega) \Omega = n^f.1.
\]

This suggests that the Lehmann/wavefunction renormalization weight is a matrix of the form, \( \Omega Z_{eff}^{-1} \). The amplitude of the applied ac voltage will be modified by its determinant \( \Xi = \det(|\Xi|\omega) \).

For Gaussian correlated phases, \( \langle \phi_j^*(t) \phi_k^*(s) \rangle = 0 \) with \( k \neq j \), the impedance matrix \( (Z_{eff}^{-1})_{jk}(\omega) \) has to be diagonalized, with \( (Z_{eff}^{-1})_{jj}(\omega) = 0 \) for \( j = k \). This is akin to setting all the phase-phase interaction terms to zero. However, this is not the case since the islands will effectively interact when a charge soliton propagates along the array constituting a current. The injection of a soliton/anti-soliton pair into the array depends on the electrostatic potential at the junction at the edge and the one at the center of the array labeled 1 and 2. We shall approximate the array as infinite with \( N_0 \gg 1 \) junctions, where each junction has a capacitance \( C \) and

---

6 The amplitude of the higher (current) harmonics has been shown to vanish in [25], to correspond to \( |\Xi|n^f |\Xi| = |Z_{eff}(n\Omega)|/|Z_{eff}(n\Omega)| \) where \( n^f \) is the \( n^f \)-th harmonic (assumed to be positive). Using the impedances \( Z(\Omega) = R \) and \( Z_{eff}(\Omega) = 1/(R + i\Omega C + 1/i\Omega L) \), the renormalization factor \( |\Xi|n^f |\Xi| \) is a rapidly decreasing function of \( n \).

7 This action has been considered before within the context of one dimensional XY model of topological phase transitions, e.g. in [9] with \( (Z^{-1})_{jk} = 0 \). This reference can be consulted for introduction on how to approach dynamics and phase transitions in such a linear array.
environmental impedance $Z(\omega) = R$, leading to the effective circuit depicted in figure 4. Thus, the capacitance of the rest of the array is computed by recognizing that for an infinite array, neglecting the capacitance of the first junction $C$ and the self-capacitance of the first island $C_0$ does not alter the capacitance $C_i$ of the rest of the array, $C_i^{-1} = C_0^{-1} + (C_0 + C_r)^{-1}$. Solving for $C_i$, we find

$$
\frac{1}{(C_0 + C_r + C_i)C_i} = \frac{1}{(C_0 + C_r)C_r} \rightarrow C_i^2 + C_0C_r - C_0C = 0 \rightarrow C_i = \frac{1}{2}(C_0 + \sqrt{C_0^2 + 4CC_0}).
$$

The total capacitance of the infinite array $C_A$ (excluding the first junction) is given by,

$$
C_A = C_0 + C_r = \frac{1}{2}(C_0 + \sqrt{C_0^2 + 4CC_0}).
$$

We thus set the capacitance of half the array as $C_3 = C_A/2$.

Taking the junction phases as $\phi_1$ and $\phi_2$, we can write the exact action of the effective circuit as,

$$
S_{\text{eff}} = \frac{C}{2e^2} \int dt \left\{ \left( \frac{\partial \phi_1}{\partial t} \right)^2 + \left( \frac{\partial \phi_2}{\partial t} \right)^2 \right\} + \int dt \frac{C_3}{2e^2} \left( \frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t} \right)^2
$$

$$
- \frac{1}{4\pi e^2} \int dt ds \phi_1(t) \frac{\partial Z^{-1}(t-s)}{\partial t} \phi_1(s) - \frac{1}{4\pi e^2} \int dt ds \phi_2(t) \frac{\partial Z^{-1}(t-s)}{\partial t} \phi_2(s). 
$$

(31)

Note that, when figure 4 instead of figure 3 is used to evaluate the effective impedance of the array, any possible corrections to equation (31) ought to be negligible. Such corrective terms in equation (31) ought to be small but finite if and only if one perturbatively moves from the finite array towards the infinite array, neglecting the $N_0 > 1$ regime of the infinite array. Thus, equation (31) leads to the following $2 \times 2$ capacitance and admittance matrices respectively,

$$
i\omega C_{jk} = i\omega \begin{pmatrix} C + C_3 & -C_3 \\ -C_3 & C + C_3 \end{pmatrix}, \quad (Z^{-1})_{jk} = \frac{1}{R} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

where $j, k = 1, 2$. The effective admittance matrix is thus given by their sum, $(Z_{\text{eff}})^{jk} = (Z^{-1})^{jk} + i\omega C_{jk}$.

Thus, the array via the interaction term with $C_3 = C_A/2$ modifies the phase-phase correlations by destroying their Gaussian nature as the impedance matrix is shifted as $(i\omega C + 1/R) \delta_{ji} \rightarrow (Z_{\text{eff}})^{jk}$. In turn, this corresponds to a Lehmann weight, $\Xi_{\lambda} = \sum_{-\infty}^{\infty} (i\omega C + 1/R) \delta_{ji} (Z_{\text{eff}})^{jk} = (i\omega C + 1/R) (Z_{\text{eff}})^{jk}$. This will affect the amplitude of the applied oscillating voltage by a Lehmann weight $\det(\Xi_{\lambda})$. When the angular frequency $\omega$ is much larger than the inverse of the time constant $RC$, we find,

$$
\Xi_{\lambda} = \lim_{\omega \rightarrow \infty} \det(\Xi_{\lambda}) = \frac{C}{C + C_A} = \exp(-\Lambda^{-1}),
$$

(32)

where $\Lambda$ is the soliton length of the array. When an alternating voltage is applied across the array, the amplitude of the oscillating voltage will be renormalized by $\Xi_{\lambda} \sim \exp(-\Lambda^{-1})$. This represents a damping of the applied power of applied oscillating voltage.

### 3.2. Soliton field theory origin of the Lehmann weight in an infinite array

Consider the charge soliton lagrangian of the array,

$$
\mathcal{L}_{\text{sol}} = \frac{1}{2\pi} \int dx \left\{ \frac{1}{2} \left( \frac{\partial \chi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \chi}{\partial x} \right)^2 - \frac{2}{\Lambda^2} \sin^2(\chi/2) \right\},
$$

(33a)

where the co-ordinates $x \equiv x/a, t \equiv vt/a$ are dimensionless, and $a = 1$ is the length of the islands (lattice constant), $v_0 = 1$ is the velocity of electromagnetic radiation along $x$. The Euler–Lagrange equations of equation (33a) yield the solution,
\[ \chi = 4 \arctan \{ \Lambda^{-1} \gamma (x \pm \nu t) \} + 2\pi n, \quad (33b) \]

with \( \gamma = \sqrt{1 - v^2} \) the Lorentz factor. Plugging in equation (33b) into equation (33a), we can eliminate \( \partial / \partial t \) in favor of \( \partial / \partial x \),

\[ \left( \frac{\partial \chi}{\partial t} \right)^2 = v^2 \left( \frac{\partial \chi}{\partial x} \right)^2 \quad (34a) \]

\[ \mathcal{L}_{\text{sol}} = -\frac{1}{\pi} \int dx \left( \frac{\partial \chi}{\partial x} \right)^2 + \Lambda^2 \sin^2 (\chi / 2) \right) \quad (34b) \]

Observing that since the integrand is quadratic, we can apply the Bogomol’nyi inequality \[ (36) \]

\( (A^2 + B^2 \geq 2|AB|) \) to evaluate the mass, \( M \) given by,

\[ \mathcal{L}_{\text{sol}} \geq M = \frac{1}{\pi} \int dx \left( \frac{\partial \chi}{\partial x} \right) \sin (\chi / 2) \left| \frac{\partial \cos (\chi / 2)}{\partial x} \right| = \frac{1}{\gamma \Lambda \pi} \left| \cos (\pi) - \cos (0) \right| = \frac{2}{\gamma \Lambda} \equiv E_0 \quad (36c) \]

with \( E_0 = 1/\pi v \). Treating the solitons as charged dust of mass density, \( M / V \) where \( u^\mu = dx^\mu / d\tau = (\gamma, \pm v \gamma, 0, 0) \) is the four-velocity, \( V = \Lambda l \) is the volume and \( l \) the length of the array, we can introduce the energy-momentum tensor,

\[ T^{\mu \nu} = \frac{2M}{V} u^\mu u^\nu + \varepsilon_0 \varepsilon_i \left\{ F_{\mu \nu} F_i^{\mu \nu} - \frac{1}{4} \delta^{\mu \nu} \delta_{\alpha \beta} \right\} \quad (35) \]

The total energy is given by,

\[ E = \int_V d^3x \left\langle T^{00} \right\rangle = \frac{1}{V} \int_V d^3x \left( 2Mu^0 u^0 \right) + \frac{1}{V} \sum_{s=\pm 1} \int_V d^3x \omega \left\langle a_s (\omega) a_s (-\omega) \right\rangle \omega + \frac{\text{sgn}(\omega)}{2} \quad (36a) \]

with \( s = \pm 1 \) the photon polarization states. Note that,

\[ \langle Mu^0 u^0 \rangle = \frac{E_0}{\Lambda} \langle \gamma \rangle \simeq \frac{E_0}{\Lambda} + \frac{E_0}{\Lambda} \frac{v^2}{2} = \frac{1}{\Lambda} \left( E_0 + \frac{1}{2\beta} \right) \quad (36b) \]

where,

\[ \langle \cdots \rangle \equiv \left( \int_{-\infty}^{+\infty} dv \exp -\beta E_0 v^2 / 2 \right)^{-1} \int_{-\infty}^{+\infty} dv (\cdots \exp -\beta E_0 v^2 / 2), \]

is the Boltzmann average for a gas of anti-)solitons in \( 1 + 1 \) dimensions. Comparing equation (36) to equation (17) (neglecting the vacuum energy \( E_0 / \Lambda \) and considering only one photon polarization mode), we conclude that the array is an effective single junction with \( \Xi_\Lambda = \exp (-\Lambda^{-1} \exp (2\pi\alpha)) \).

### 4. Application: optimization of linear arrays for classical RF field power detection

We treat the array of Josephson junctions as an effective single junction based on the arguments presented in appendix C. This entails using the \( l-V \) characteristics given in equation (C1) under irradiation of the RF field, together with the renormalized external voltage \( V'_\chi (t) = V + V'_{\text{ac}} \cos (\Omega t + \eta) \) in equation (C1), which simply results in equation (29),

\[ I_\chi (V) = \sum_{k=1}^{2} \sum_{n=-\infty}^{\infty} \int_n^\infty \left( \kappa e V'_{\text{ac}} / \Omega \right) I_k \left( V - \frac{n \Omega}{\kappa e} \right) \quad (37) \]

where \( V'_{\text{ac}} = |\Xi(\Omega)| V_{\text{ac}}, \quad I_k (V) \) are the \( l-V \) characteristics of the array with \( V_{\text{ac}} = 0 \) and the \( P_s (E) = \frac{1}{2\pi \gamma} \int dt \Re(T(t) \exp (iEt)) \) functions appearing in \( I_\chi \), \( I_1 \) in equation (B44) are rescaled as \( P_s (t) \rightarrow [P_s (t)]^{1/\Gamma} \). This is the celebrated Tien-Gordon equation \[ (41) \] describing photon-assisted tunneling of Cooper-pairs or quasi-particles of charge \( e \) across a barrier. However, in the case of the array, current is predominantly generated by charge solitons. Thus, equation (37) ought to correspond to charge solitons/anti-solitons injected into the array by the influence of the RF field.

The universality of equation (29) and hence equation (37) is apparent when we observe that the necessary and sufficient conditions for reproducing it are:
1. The unirradiated characteristics of the sample take the form $I_s(V) = \frac{i e}{\hbar} \int_{-\infty}^{\infty} dt \alpha_n(t) P_n(t) \sin \left[ \kappa e V_{cb} \cos (\Omega t + \eta) \right]$, where the structure coefficients $\alpha_n(t)$ depend on the density of states of the tunneling particles;

2. The effect of the RF field be to shift $V$ to $V_s'(t) = V + V_{sc}^{\text{eff}} \cos (\Omega t + \eta)$.

Thus, the aforementioned rescaling of $P_n(t)$ in the case of the array merely corresponds to a modification of the structure coefficients $\alpha_n(t)$ in condition 1 by a multiplicative factor $[P_n(t)]^{[\text{const}]}$. Moreover, the renormalization effect highlighted in this paper concerns only condition 2. As a consequence, it is sufficient to work in the classical limit $\Omega \ll \kappa e V_{cb}$, corresponding to multi-photon absorption by the sample (single junction or array). Setting $\kappa e V_{cb} \sin \theta = n \Omega$, the sum over photon number $n$ can be approximated by an integral formula corresponding to the classical detection of radiation [42],

$$I_s(V) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I_0(V - \lambda \sin \theta) d\theta,$$

where $I_0(V) = \sum_i I_i(V)$ is the sum of the quasi-particle and Cooper-pair currents. For the simulation, we used the measured characteristics, $I_s(V)$ of a linear array of 10 small Josephson junctions exhibiting distinct Coulomb blockade characteristics, thus by-passing simulating condition 1 which merely corresponds to standard $PE$ theory [12, 43].

The target parameters of the linear array per junction displayed in table 1 were determined by design during electron-beam lithography and oxidation during shadow evaporation. The tunnel resistance $R_T$ and $E_V$ are calculated respectively from the offset voltage and the differential conductance $dI_0(V)/dV$ [44, 45]. The differential conductance, alongside the measured $\Delta-H$ dependence determine the superconducting gap $\Delta$. The Josephson coupling energy, $E_J$ is then determined by the Ambegaokar-Baratoff relation [46] using the measured superconducting gap. Note that $E_J < E_C$ and $R_T > R_0/4 \approx 6.45 \text{ k\Omega}$ satisfy the small junction and Coulomb blockade conditions respectively.

The $I_s(V)$ characteristics of the linear array (given by the black bold curve in figure 5) was measured via the well-known $r$-bias method [47]. It entails incorporating a fixed resistance given by $r = 1 \text{ M\Omega}$ or $5 \text{ M\Omega}$ serially connected to the array and biasing both (the resistor and the array) with a voltage, and measuring the current and voltage values employing differential amplifiers with high input impedance. Noise reduction was achieved by applying half the dc voltage in each terminal with opposite polarity ($-V/2$ and $V/2$) relative to the ground.

The simulated curves given in figure 5 were numerically produced using equation (38), by applying Simpson’s rule to approximate the integral,

$$I_s \approx \frac{\Delta \theta}{3\pi} \left[ I_0(\theta_0) + I_0(\theta_{2\ell}) + 2 \sum_{m=1}^{\ell} I_0(\theta_{2m}) + 4 \sum_{m=1}^{\ell} I_0(\theta_{2m+1}) \right],$$

where $\theta_m = -\pi/2 + m \pi/2\ell$ defined between the entire integration interval $\pi = 2\ell \Delta \theta$ for $2\ell + 1$ values bounding $2\ell$ equally spaced intervals of width $\Delta \theta$. The sum was then carried out using spline interpolation [48]. The results of the simulation have been plotted in figure 5 for the amplitude range $0 \mu \text{V} \leq V_{cb} \leq 260 \mu \text{V}$, where the interval between adjacent curves is $\Delta V_{cb} = 10 \mu \text{V}$ and $|\Xi| = 1$.

Figure 5 displays the lifting of Coulomb blockade characteristics by the power of the RF field. Such characteristics have been experimentally observed [33, 45, 48] for both quasi-particle and Cooper-pair tunneling. This phenomenon is dual to microwave-enhanced phase diffusion [33, 49, 50]. The simulated results for $|\Xi| < 1$ exhibit exactly the same Coulomb blockade lifting behaviour.

The lifting of Coulomb blockade characteristics with applied RF power can be exploited to design a microwave power detector by defining the Coulomb blockade threshold voltage $V_{cb}$ at a given threshold current $I_{th}$ and tracking its value in the presence of RF power. To illustrate this, we have plotted the $V_{cb}-V_{sc}$ dependence at $I_{th} = 3 \text{ pA}$ in figure 6 for simulated curves using equation (38) with selected values of the renormalization factor, $|\Xi| \leq 1$. The sensitivity of such as detector is given by $|dV_{cb}/dV_{sc}| \propto |\Xi|$. However, observe that detection range for $|\Xi! = 1$ given by $0 \text{ mV} \leq V_{sc} \leq 0.26 \text{ mV}$ is significantly smaller than that for $|\Xi| < 1$. For instance, the viable microwave power detection range for $|\Xi| = 0.1$ is $0 \text{ mV} \leq V_{sc} \leq 1.2 \text{ mV}$. This implies that the optimum microwave detector, depending on use, should lie between $0.1 < |\Xi| < 1$. Since the renormalization factor for a linear array is predicted to be $|\Xi| \sim \exp(-\Lambda^{-1})$, this corresponds to an optimization condition for the soliton length of a suitable linear array for microwave detection. Such a linear array with soliton length $\Lambda \approx 9$, corresponding to $|\Xi| \sim \exp(-\Lambda^{-1}) \approx 0.89$, has been experimentally shown to have a sensitivity slightly greater than $10^6 \text{ V/}\text{W}$ [48].

5. Discussion

5.1. Implication
In the case of the single Josephson junction, the renormalization of the amplitude of applied oscillating electromagnetic fields is implemented by linear response. This entails the excitation of ‘particles’ appearing as a
mass gap $M - i\varepsilon_m = -\beta^{-1}\ln X(\omega)$ in the thermal radiation spectrum, where $\varepsilon_m$ is the Matsubara frequency \[32\], $X(\omega) = [1 + \gamma(\omega)Z(\omega)]^{-1}$ is the linear response function, $Z(\omega)$ is the environmental impedance of the junction and $\gamma(\omega) \simeq \omega C$ is the impedance of the junction. Likewise, when an infinitely long array \[30, 45\] is modeled as half the infinite array interacting with two junctions, one at one edge of the the array and the other at the center, as illustrated in figure 4, we find an additional Lehmann weight \[\Xi_A = \exp(-\Lambda^{-1})\]. This requires that applied oscillating electric fields are damped by the same factor over a finite range of electric fields along the array. This is dual to the Meissner effect where the Cooper-pair order parameter leads to a finite range of the magnetic field.

A Josephson junction circuit that exhibits a large Coulomb blockade voltage is ideal for the observation of the renormalization effect. In particular, for the single junction, power renormalization is negligible only for extremely low microwave frequencies satisfying $1/RC \gg \Omega$. However, for samples exhibiting Coulomb blockade that also satisfy the Lorentzian-delta function approximation $\text{Re} \{Z_{\text{eff}}\} = R/(1 + \Omega^2C^2R^2) \sim \pi C^{-1}\Lambda(\Omega)$, the conductance $1/R$ is extremely small ($1/RC \ll \Omega$) and thus we should expect power renormalization for virtually all applied frequencies. In the case of long arrays ($N_0 \gg \Lambda$) with $\Xi \simeq 1$, RF amplitude renormalization should be readily observed due to the additional factor $[\Xi(\Omega)] \sim \exp(-\Lambda^{-1})$. \[48\] Finally, a list of the electromagnetic quantities and their rescaled formulae is displayed in table 2.

### Table 1. Average parameters per junction the array of 10
Aluminium (Al)/Aluminium Oxide (Al$_2$O$_3$)/Aluminium (Al)
Josephson junctions whose measured characteristics, $I_0(V)$ have been used in the simulation with equation (38). The parameters consecutively are, the capacitance $C$, tunnel resistance $R_T$, Josephson coupling energy $E_J$, charging energy $E_c$ and $E_J/E_c$ ratio.

| C [fF] | $R_T$ [k Ω] | $E_J$ [μeV] | $E_c$ [μeV] | $E_J/E_c$ |
|--------|-------------|-------------|-------------|-----------|
| 0.8    | 10          | 62.5        | 96          | 0.65      |

\[5.2. Backaction considerations\]

The form of the admittance $\gamma(\omega)$ given in equation (B41c) neglects the back-action of the Josephson junctions on the environment (with the bath and the junction becoming entangled) which has been reported to dramatically change the predictions of the $P(E)$ theory \[28, 51, 52\]. This back-action manifests through the non-linear inductive response of the junction where the Josephson coupling energy is renormalized, $E_J^{\text{eff}} = E_J(\cos(\Delta2\phi)) = E_J\exp(-\Delta2\phi^2)$. In our work, we have made an implicit assumption that, whenever the Josephson coupling energy $E_J$ is considerably small compared to all relevant energy scales such as the charging energy $E_c$, and the current-voltage characteristics of the junction do not exhibit a superconducting branch, this back-reaction can be taken to be small. Nonetheless,
considering this back-reaction in our theoretical framework especially in the case of a single Josephson junction is certainly warranted since the back-reaction has been suggested to dramatically alter the superconductor-insulator transition conditions for the Josephson junction\(^{52}\).

### Table 2. Electromagnetic quantities and their Rescaled Expressions.

| Quantity                        | Expression                                      | Rescaled Expression          |
|---------------------------------|--------------------------------------------------|-----------------------------|
| Environmental impedance (Single Junction) | \(Z(\omega)\)                                    | \(\Xi(\omega) = [1 + i\omega C Z(\omega)]^{-1}\) |
| Lehmann weight (Single Junction)  | 1                                                | \(\Xi(\omega) = [1 + i\omega C Z(\omega)]^{-1}\) |
| Microwave amplitude (single junction) | \(V_{ac}\)                                     | \(V_{eff} = |\Xi(\Omega)|V_{ac}\) |
| Environmental Impedance (long array: figure 4) | \((i\omega C + 1/R)\delta_{jk}\) \(Z_{eff}\) \(Z_{eff}\) | \(V_{eff} = |\Xi(\Omega)|V_{ac}\) |
| Lehmann Weight (long array)      | \(\det(\delta_{jk})\) = 1                      | \(\Xi_{jk} = \det(i\omega C + 1/R)(Z_{eff})_{jk} \sim \exp(-\Lambda^{-1})\) |
| Microwave amplitude (long array) | \(V_{ac}\)                                      | \(V_{eff} \sim \exp(-\Lambda^{-1})V_{ac}\) |

Figure 5. Simulated \(I-V\) curves using equation (38) and the measured characteristics \(I(V)\) (black bold curve) of a linear array of 10 Josephson junctions with average parameters per junction given in table 1, for amplitude range \(0 \mu V \leq V_{ac} \leq 260 \mu V\), where the interval between adjacent curves is \(\Delta V_{ac} = 10 \mu V\) and the renormalization factor is set to unity \(\Xi = 1\). These characteristics simulate the lifting of Coulomb blockade by RF power.

Figure 6. Simulated response of the Coulomb blockade threshold voltage \(V_{cb}\) at a threshold current \(I_{th} = 3\) pA to RF power \(V_{ac}\) for selected values of the renormalization factor in the range \(0.1 < |\Xi| < 1\). These characteristics can be exploited to design an RF power detector with a sensitivity \(|dV_{cb}/dV_{ac}|\) proportional to \(|\Xi|^{48}\).
Within our path integral approach, considering such effects entails performing the path integral for the rescaled $P(E)$ function given by $P_{\kappa=2}^{\text{eff}}(E) = \int dt \, P_{\kappa=2}^{\text{eff}}(t) \exp(-iEt)$ where,

$$P_{\kappa=2}^{\text{eff}}(t) = \int D\phi(t) \cos 2\Delta \phi(t) \exp(iS_{\text{CL}}) \times \exp\left(-i \int dt \, E_\text{c} \cos 2\phi(t)\right),$$

which is challenging to carry out successfully to all orders of perturbation. Typically, the exponent is linearized as $\cos(2\phi(t)) \simeq 1 - (2\phi(t))^2/2 + (2\phi(t))^4/4 + \cdots$ which becomes the $\phi^4$ theory [33]. In turn, at order $\phi^2$, the renormalized $E_\text{c}$ is expected to enter the usual Caldeira-Leggett expression as an inductance, $L_{\text{eff}} = 1/(2\epsilon)^2 E_{\text{c}}^{\text{eff}}$, $P_{\kappa=2}(E) = \int dt P_{\kappa=2}(t) \exp(-iEt)$ where, $P_{\kappa=2}(t) = \int D\phi(t) \cos \Delta \phi(t) \exp(iS_{\text{CL}}^{\text{eff}})$, yields a $P_{\kappa=2}(E)$ function with a linear inductive part in the impedance as given by $\gamma(\omega)$ in equation (B41e). Further discussion on the experimental and theoretical results of $E_\text{c}$ based renormalization, the back-action and other possible effects in Shapiro step-based experiments when $E_\text{c}/E_\text{c} > 1$ is beyond the scope of this work.

5.3. Summary
We have employed path integral formalism to derive the Cooper-pair current and the BCS quasi-particle current in small Josephson junctions and introduced a model which transforms the in finite array, this particle corresponds to a bosonic charge soliton injected into the array. This analysis does not take into account random offset charges which are known to act as static or dynamical background charges in the islands of the array, resulting in shifting of the threshold voltage $V_\text{th}$ and noise generation affecting the soliton flow along the array [54, 55].

5.4. Application
Since the quasi-particle current naturally reduces to the normal current and the supercurrent vanishes when the superconducting gap goes to $\Delta = 0$, the final expression of the tunnel current equation (29) is essentially the time averaged current result previously proposed in [25]. In the classical limit when the RF frequency $\Omega$ is small compared to the amplitude of the alternating voltage $(\kappa e V_{\text{ac}} \sin \theta = n\Omega)$, the sum over photon number can be approximated by an integral formula that corresponds to a classical detection of the RF field,[42]

$$I(V) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I_0(V - |\Xi(\Omega)| V_{\text{ac}} \sin \theta ) \, d\theta,$$

where $I_0(V)$ is given by equation (B44). This result offers a way to measure the magnitude of the Lehmann weight $|\Xi(\Omega)|$, where $|\Xi(\Omega)|$ is proportional to the sensitivity of the detector to RF power [48]. Conversely, this implies that our results are indispensable in dynamical Coulomb blockade experiments where long arrays are used as detectors of oscillating electromagnetic fields [33, 34].

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Appendix A. Causal linear response
Appendix A is meant for the skimming reader, who wants a quick reference to linear response (and its relevance to microwave power renormalization). Thus, we do not strive to introduce the entire subject of linear response and its subtleties. (For a comprehensive introduction to the subject, see [40])

8 Appendix 3.2.
Within Linear Response Theory, the response $\tilde{R}(t)$ of a system is related to the driving force, $\tilde{F}(t)$ by the central causal relation
\[ \tilde{R}(t) = \int_{-\infty}^{t} \chi(t-s)\tilde{F}(s)ds, \tag{A1} \]
where $\chi(\tau)$ is the response function. The system variable, $\tilde{R}(t)$ obeys some equation of motion,
\[ f(\partial/\partial t)\tilde{R}(t) = \tilde{F}(t), \tag{A2} \]
with $f(\partial/\partial t)$ a function of $\partial/\partial t$. Introducing the Green’s function of the system, $G_R(t-s)$, satisfying,
\[ f(\partial/\partial t)G_R(t-s) = \delta(t-s), \tag{A3} \]
we see that,
\[ f(\partial/\partial t)\tilde{R}(t) = \int_{-\infty}^{t} f(\partial/\partial t)G_R(t-s)\tilde{F}(s)ds = \int_{-\infty}^{t} \delta(t-s)\tilde{F}(s)ds = \int_{0}^{+\infty} \delta(\tau)\tilde{F}(t-\tau)d\tau \]
\[ = \int_{-\infty}^{+\infty} \delta(\tau)\theta(\tau+0^+)\tilde{F}(t-\tau)d\tau = \theta(0^+)\tilde{F}(t) = \tilde{F}(t) \tag{A4} \]
with $\theta(\tau)$ the Heaviside function. Thus, we can equate the Green’s function to the response function: $\chi(t-s) = G_R(t-s)$.

Substituting the Fourier transforms of $\tilde{R}(t) = \int d\omega \tilde{R}(\omega)\exp(-i\omega t)$ and $\tilde{F}(t) = \int d\omega \tilde{F}(\omega)\exp(-i\omega t)$ into equation (A1),
\[ \int d\omega \tilde{R}(\omega)\exp(-i\omega t) = \int_{-\infty}^{t} \int \tilde{F}(\omega)\exp(-i\omega s)\chi(t-s)dsd\omega \]
\[ \int \left\{ \tilde{F}(\omega) \int_{0}^{+\infty} \chi(\tau)\exp(i\omega\tau)d\tau \right\} \exp(-i\omega t)d\omega = \int \{ \tilde{F}(\omega)\Xi(\omega) \} \exp(-i\omega t)d\omega, \tag{A5} \]
where $\tau = t-s$ and,
\[ \tilde{R}(\omega) = \Xi(\omega)\tilde{F}(\omega), \tag{A6a} \]
\[ \Xi(\omega) = \int_{0}^{+\infty} \chi(\tau)\exp(i\omega\tau)d\tau = \int_{-\infty}^{+\infty} \theta(\tau)\chi(\tau)\exp(i\omega\tau)d\tau, \tag{A6b} \]
\[ 2\pi\theta(\tau)\chi(\tau) = \Xi(\tau). \tag{A6c} \]
Finally, that $\Xi(\omega) = \pm \exp(-\beta M)\exp -i\eta(\omega)$ acts as the response function to the applied oscillating electromagnetic field is to be understood as the result of the arguments in section 2.4, and not necessarily the converse. This leaves the possibility that linear response is violated in complicated circuits, where novel physics may lurk.

**Appendix B. The Electromagnetic Environment in Large and Small Josephson Junctions**

Despite the existence of excellent reviews on the subject and techniques [11, 12, 56], the authors found much of the techniques and prior concepts useful in following the arguments in this thesis scattered in various literature [13, 36, 40, 57–59]. In particular, the techniques used in the subsequent chapters include path integral formalism [60] and Green’s functions [36, 59] to calculate phase-phase correlation functions and four-vector notation [61] where Maxwell’s equations appear for compactness. Thus, we include this section as a preamble for completeness and/or compactness. Hopefully it offers a more nuanced understanding of the Caldeira-Leggett model and $P(E)$ theory in the context of Green’s functions and generally a path integral framework.

**B.1. Organization**

Section B.2 considers how the effects of the electromagnetic environment arises via a normal current in large junctions. In the sections, B.2.1 introduces a 2-spinor and Pauli matrices that act on the spinor to derive the well-known Josephson equations. We proceed in B.2.2 to introduce the effect of the environment as a normal current proportional to the electric field in the tunneling direction and a fluctuating noise current whose degrees of freedom we introduce in B.3 as a Caldeira-Leggett heat bath.

Section B.4 tackles the environment in small junctions. In the sections, B.4.1 introduces the total Hamiltonian of the Josephson junction including the environment and derives an expression for the tunneling current, B.4.2 expands this expression into a perturbation series and explicitly calculates the Cooper-pair kernel using the Pauli matrices introduced in B.2.1 while B.4.3 uses path integral formalism to calculate phase-phase correlation functions in the $P(E)$ function. The expression for the Cooper-pair and quasi-particle tunneling current at finite temperature is derived in B.4.4.
Note that, units where Planck’s constant, Swihart velocity [35] and Boltzman constant are set to unity ($\hbar = \varepsilon = k_B = 1$) and Einstein summation convention are used through out unless otherwise stated with \(\delta_{\mu\nu} = \delta^\mu_\nu\) the Kronecker delta symbol.

### B.2. Josephson effect and the electromagnetic environment

#### B.2.1. The Josephson effect (large junction)

The physics of Josephson junctions (schematic shown in figure B1) is described by the well known Josephson equations [62],

\[
I_s = 2eE_J \sin 2\phi_x(t) \tag{B1a}
\]

\[
\frac{\partial \phi_x(t)}{\partial t} = eV_x \tag{B1b}
\]

Here, \(2\phi_x(t)\) denotes the phase difference across the junction, where the subscript \(x\) distinguishes it from quantum phases of other circuit elements defined later in the manuscript, and \(E_J\) is the Josephson coupling energy [63]. The simplest derivation of equation (B1) follows from the real and imaginary parts of these two coupled Schrodinger equations

\[
\frac{i}{\hbar} \frac{\partial \psi_1}{\partial t} = \mu_1 \psi_1 + m_0 \psi_2, \tag{B2a}
\]

\[
\frac{i}{\hbar} \frac{\partial \psi_2}{\partial t} = \mu_2 \psi_2 + m_0 \psi_1. \tag{B2b}
\]

Here, \(\mu_1\) and \(\mu_2\) and the chemical potentials of the left (1) and right (2) junction respectively, \(\psi_1\) and \(\psi_2\) are the Cooper-pair wavefunctions of the left and right superconductors respectively and \(m_0\) is a coupling energy term characterizing magnitude of overlap for the two wavefunctions across the insulator. When a potential difference (voltage) \(V_x\) is applied across the junction, the two chemical potentials shift relative to each other in order to accommodate this change. This means that we can set \(\mu - \mu_2 = 2eV_x\), where \(2e\) is the Cooper pair charge. Based on this, it is instructive to define an average chemical potential, \(\bar{\mu} \equiv (\mu_1 + \mu_2)/2\), and solve for \(\mu_1\) and \(\mu_2\) in terms of \(\bar{\mu}\). This yields, \(\mu_1 = \bar{\mu} + eV_x\) and \(\mu_2 = \bar{\mu} - eV_x\). Plugging this back to equation (B2) yields,

\[
\frac{i}{\hbar} \frac{\partial \psi_1}{\partial t} = (\bar{\mu} + eV_x)\psi_1 + m_0 \psi_2, \tag{B3a}
\]

\[
\frac{i}{\hbar} \frac{\partial \psi_2}{\partial t} = (\bar{\mu} - eV_x)\psi_2 + m_0 \psi_1. \tag{B3b}
\]

From this, it is clear \(\bar{\mu}\) is simply the common chemical potential relative to which the voltage drop is measured. This observation implies we can set it to zero without loss of generality, \(\bar{\mu} = 0\).

The Cooper pair wavefunctions are defined as \(\psi_1 = \sqrt{n_1} \exp(i2\varphi_1)\) and \(\psi_2 = \sqrt{n_2} \exp(i2\varphi_2)\) where \(n_1\) and \(n_2\) are the number of Cooper-pairs in the left (1) and right (2) superconductor respectively, and \(2\varphi_1\) and \(2\varphi_2\) is their respective macroscopic quantum phases. Plugging these definitions into equation (B3), we find,

\[
\frac{\partial m_1}{\partial t} = 2m_0 \sqrt{n_1n_2} \sin(2[\varphi_2 - \varphi_1]) \tag{B4a}
\]

\[
\frac{\partial m_2}{\partial t} = 2m_0 \sqrt{n_1n_2} \sin(2[\varphi_1 - \varphi_2]) \tag{B4b}
\]

and,

\[
2\frac{\partial \varphi_1}{\partial t} = -m_0 \sqrt{\frac{n_2}{n_1}} \cos(2[\varphi_2 - \varphi_1]) - eV_x, \tag{B5a}
\]

Figure B1. Schematic of a Josephson junction (S: superconductor, I: insulator, S: superconductor) depicting a Cooper pair from the left/right electrode tunneling through the insulator to the right/left electrode.
Taking the approximation that any tunneling currents that arise have the effect of varying $n_1$ and $n_2$ by only a small amount \( \Delta n \), we see that the supercurrent across the barrier is given by \( \Delta n \psi \) and the voltage drop by \( \Delta V = eV \), which yield equation (B1) when \( E = nm \) and \( \phi_2 - \phi_1 = \phi_\lambda \).

There is another advantage of setting \( \mu = 0 \). In particular, equation (B3) becomes a spinor equation,

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = H_{cp} \psi 
\]

(B6a)

\[
H_{cp} = eV \sigma_3 + m_0 \sigma_1, 
\]

(B6b)

\[
\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]

(B6c)

where \( \sigma_1 \) and \( \sigma_3 \) are the Pauli matrices,

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(B7a)

Since \( E \propto m_0 \), \( \sigma_1 \) plays the role of the coupling term that results to Cooper-pair tunneling. Using the last Pauli matrix,

\[
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, 
\]

(B7b)

we can define two operators \( \sigma_x \equiv (\sigma_1 + i\sigma_2)/2 \) and \( \sigma_- \equiv (\sigma_1 - i\sigma_2)/2 \). These operators form tunneling matrix elements with the spinor and a transpose conjugate spinor defined as,

\[
\psi^\dagger \equiv \psi^* \text{T} = (\psi_1^*, \psi_2^*). 
\]

(B8)

For instance, tunneling from left to right requires replacing \( \psi_1 \) with \( \psi_2 \) and annihilating \( \psi_1 \), which corresponds to \( \sigma_x \psi \). The inverse process process corresponds to \( \sigma_\psi \psi \). These matrices will be useful when calculating Cooper-pair tunneling rates for small junctions (See equation (B34)).

B.2.2. Sources of the electromagnetic field as the Josephson junction environment. Equation (B1) only considers the superconducting current and thus neglects the environment that lead to effects such as Coulomb blockade. The environment consists of all sources of the electromagnetic field (including the field itself) which couple to the Cooper-pair wavefunction via the phase difference thus determining the \( I - V \) characteristics satisfying equation (B1). Specifically, the environment arises from processes such as the alternating currents and voltages, thermal fluctuations in the form of Johnson-Nyquist noise and coupled high impedance circuit environments \([6, 49]\).

Using equation (B1), one can define a conserved energy by treating the junction as a capacitance

\[
E = \frac{Q_x^2}{2C} - E_I \cos(2\phi_\lambda) 
\]

(B9a)

\[
Q_x = -CV \]

(B9b)

\[
\frac{\partial Q_x}{\partial t} = I_x 
\]

(B9c)

where \( C \) is the capacitance of the junction. Modifying the last equation in (B9) to

\[
\frac{\partial Q_x}{\partial t} = \sum_a I_a, 
\]

(B10)

one can then include all the environmental sources of energy in the form of currents. In fact, to arrive at equation (B10), the phase-difference needs to couple to the electromagnetic field (Maxwell equations) in a straight-forward manner

\[
\frac{\partial \phi_\lambda}{\partial x^\mu} = e \epsilon_\mu \eta N^\nu F_{\mu \nu} 
\]

(B11a)

\[
N^\nu N_\nu = -\sum_y N_i N_j \delta_{ij} = -1 
\]

(B11b)
Here, $F_{\mu\nu} = \partial A_\mu / \partial x_\nu - \partial A_\nu / \partial x_\mu = -F_{\mu\nu}$ is the electromagnetic tensor, $d_{\text{eff}}$ is the thickness of the barrier and $N^\mu = (0, 0, 0, N)$ points in the direction $N^i$ normal to the tunnel barrier. We have used Einstein notation where only the Greek indices are summed over and the Minkowski space-time signature is diag($\eta_{\mu\nu} =$ $\{\pm 1, -1, -1, -1\}$).

Taking the total derivative $\eta_{\mu\nu} \partial / \partial x^\nu = \partial / \partial x_\nu$ of equation (B11a) (with $\eta_{\mu\nu} \eta_{\nu\nu} = \delta_{\mu}^\nu$) and using $\partial N^\nu / \partial x^\nu = 0$, we arrive at

$$\eta_{\mu\nu} \partial^2 \phi_\mu / \partial x^\mu \partial x^\nu = \partial^2 \phi_\nu / \partial x^\nu \partial x^\mu = \frac{ed_{\text{eff}}}{\varepsilon_0 \varepsilon_\infty} \sum_a N_a I_a^\nu = -wj$$

which is equation (B10) in disguise. Note that $F_{0\alpha} = \vec{E}$ and $\sum_{\alpha=1}^3 \varepsilon_{\alpha\beta} F_{0\beta} = \vec{B}$ where $\vec{E}$ and $\vec{B}$ are the $x$, $y$, $z$ components of the electric and magnetic fields respectively and $\varepsilon_{\alpha\beta}$ is the Levi–Civita symbol. Equation (B12) is the sourced Klein–Gordon equation with $j = \sum_a N_a I_a^\nu$ the source and $w = ed_{\text{eff}} / \varepsilon_0 \varepsilon_\infty$ the coupling constant.

The vector $N_a$ and the anti-symmetry of the electromagnetic tensor $F_{\mu\nu}$ guarantees that, unlike Maxwell equations, the coupled Klein–Gordon equation lives in 2 + 1 dimensions instead of 3 + 1. For instance, when the tunnel barrier is aligned to the $y$-$z$ direction, $N^\nu = (0, 1, 0, 0)$ and equation (B11a) and by extension equation (B12) become independent of $x$.

$$\frac{\partial^2 \phi_x}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 \phi_x}{\partial x^\mu \partial x^\nu} = \frac{4\beta^{-1}}{R} R(t - t')$$

Furthermore, taking the limit for small junctions which corresponds to taking the area of the barrier $A$ to be small such that the phase neither varies with $y$ nor $z$, we arrive at equation (B10)

$$\frac{\partial^2 \phi_x}{\partial t^2} = -4E_i E_j \sin(2\phi_x) - \frac{1}{RC} \partial \phi_x / \partial t = 2E_i I_0 / \varepsilon$$

$$\sum_a I_a^\nu = I_0^\nu + I_N^\nu + I_b^\nu$$

$$\langle I_b(t) I_b(t') \rangle = \frac{4\beta^{-1}}{R} R(t - t')$$

with $I_b^\nu = I_0 \sin 2\phi_0$ the supercurrent, $I_N^\nu = \sigma_{\alpha\nu} F_{0\nu}$ the normal current and $\sigma_{\alpha\nu}$ the effective conductivity of the barrier along the $\alpha$ direction. Here, we have used the cross-sectional area of the junction, $A$ to define $I_0^\nu A = I_0^\nu$, the junction capacitance $C = \varepsilon_0 \varepsilon_\infty A / d_{\text{eff}}$, the charging energy $E_i = \varepsilon^2 / 2C$ and the junction conductance $1/R = \sigma_{\alpha\nu} A / d_{\text{eff}}$. Finally, $\beta^{-1} = k_B T$ is the inverse temperature and we have assumed the fluctuation current $I_b^\nu A = I_b^\nu$ is Gaussian-correlated over a bath (B stands for bath or Boltzmann distribution) with the thermal correlation function given by equation (B14c).

### B.3. Generalized impedance environment and the thermal bath

It is straight forward to generalize the conductance $1/R$ in equation (B14) using a spectral function $K(x) = [Z^{-1} (\omega) + Z^{-1} (-\omega)] / (2\pi)$ describing the macroscopic physics of the microscopic degrees of freedom of the system undergoing Brownian motion due to a heat bath comprising $kBn$ harmonic oscillators [57],

$$H_B = \sum_{n=1}^k \left\{ \frac{Q_n^2}{2C_n} + \frac{(\phi_n - \phi_s)^2}{2e^2L_n} \right\}$$

$$K(t) = \sum_{n=1}^k L_n^{-1} \cos(\omega_n t) = \int_{-\infty}^{\infty} dw K(\omega) \exp(-i\omega t)$$

where $H_B$ is the Hamiltonian of the heat bath consisting of $L_n C_n$ circuits in parallel where $\omega_n = 1/L_n C_n Q_n \phi_n$ are the charges stored by and the phases of the elements and $K(t)$ is referred to as the Kernel representing the dissipative nature of the circuit. The generalized Lagrangian for the system is given by

$$\mathcal{L} = \frac{C}{2e^2} \left( \frac{\partial \phi_s}{\partial t} \right)^2 - \frac{1}{2e^2} \int_{-\infty}^{\infty} \phi_s(t) \frac{dK(s - t)}{ds} \phi_s(s) ds - \frac{1}{e} \int_{-\infty}^{\infty} I_b(t - s) \phi_s(s) ds + E_i \cos(2\phi_0),$$

where the fluctuation current is given by,

$$I_b(t) = \sum_{n=1}^k \omega_n Q_n \sin(\omega_n t) + e^{-1} L_n^{-1} \left( \phi_n - \phi_s \right) \cos(\omega_n t) \langle I_b(t) I_b(t') \rangle = \sum_{n=1}^k 2L_n^{-1} \langle H_B(\omega_m) \rangle \cos(\omega_n (t - t')).$$

The average is over the thermal bath degrees of freedom. For the Ohmic conductance above, we have $Z^{-1} (\omega) = 1/R$ and $\langle H_B(\omega_m) \rangle = \beta^{-1}$ where the continuous, large $k$ limit
\[
\lim_{k \to \infty} \sum_{n=1}^{k} L_n^{-1} \times \rightarrow \int_{-\infty}^{\infty} d\omega K(\omega) \times (B17a)
\]

is taken in accordance with equation (B15b) thus recovering equation (B14c). The fluctuation current density certainly satisfies the Green-Kubo relation \[40, 58],
\[
\sigma_{xx} = \frac{\beta}{4} \int d^4x N'\nu'(q) F_p(t) F_p(t') = \frac{\beta}{4} \mathcal{A}^{-1} \text{d}t \text{d}t' \int dt \langle F_p(t) F_p(t') \rangle = \frac{\text{d}t \text{d}t'}{RA} \quad (B17b)
\]

where we have used equation (B14c) in the last line. Note that to obtain the correct equation of motion, integration by parts of the second term in equation (B16) should be performed after applying the Euler–Lagrangian equations, then the boundary term is dropped
\[
\frac{1}{e^2} \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} \{ K(s-t) \phi_x(s) \} ds = 0 \quad (B18)
\]

**B.4. Coulomb blockade and the electromagnetic environment**

**B.4.1. Hamiltonian.** Consider a mesoscopic tunnel junction with capacitance \(C\) driven by a voltage source \(V_s\) via an environmental impedance \(Z(\omega)\). Each circuit element is characterized by a phase \(\phi_\alpha\) related to the voltage drop \(V_s\) of the element in the circuit by \(\phi_\alpha(t) = \int_{-\infty}^{t} e^{V_s} d\tau\), where the subscript \(\alpha = J, x\) or \(z\) corresponds to the junction, voltage source and environment impedance and \(\kappa = 2e\), \(e\) corresponds to Cooper pair, quasiparticle charge respectively. That the effect of the environmental impedance \(Z(\omega)\) can be represented by a single quantum phase \(\phi\) defined by the voltage drop over \(Z(\omega)\) is not at all obvious. At this stage, we treat it as an ansatz. It will not appear in the equations until we impose the topological constraint \(\sum_\alpha \phi_\alpha = e\Phi\) on the circuit.

The voltages \(V_s\) and \(V_fL\) decrease as one moves clockwise along the circuit, whereas the value increases for the voltage source \(V_s\) in the same direction. The corresponding charge on the junction is defined as \(Q_j = CV_f\) where \(C\) is the capacitance of the junction. The circuit can store a topological flux \(e\Phi = \phi_1 + \phi_x + \phi_z = \sum_\alpha \phi_\alpha\) related to a topological potential \(\int_a^b A(\tau) d\tau = \Phi(t)\), which leads to the renormalization effect in section 2.4.3.

The total Hamiltonian, \(\mathcal{H}\) of the circuit (figure 1) is given by the expression
\[
\mathcal{H} = \sum_{\kappa=1}^{2} \mathcal{H}_\kappa + \mathcal{H}_1 + \mathcal{H}_z \quad (B19)
\]

Here, \(\sum_{\kappa=1}^{2} \mathcal{H}_\kappa = \mathcal{H}_1 + \mathcal{H}_z\) where the Cooper-pair Hamiltonian \(\mathcal{H}_z = \mu \psi^\dagger \psi = \psi^\dagger i[\partial / \partial t + iH_{\text{qp}}] \psi\) depends on the chemical potential \(\mu\) and the 2-spinor \(\psi\) and the quasi-particle Hamiltonian \(\mathcal{H}_1\) is given by,
\[
\mathcal{H}_1 = \mathcal{H}_L + \mathcal{H}_R = \sum_{\nu, \sigma} \epsilon_{\nu, \sigma} \gamma^\dagger_{\nu, \sigma} \gamma_{\nu, \sigma} + \sum_{\nu, \sigma} \epsilon_{\nu, \sigma} \gamma^\dagger_{\nu, \sigma} \gamma_{\nu, \sigma} \quad (B20)
\]

where \(\gamma_{\nu, \sigma}\) or \(\gamma^\dagger_{\nu, \sigma}\) are the annihilation and creation operators respectively of a quasi-particle state with energy \(\epsilon_{\nu, \sigma}\) or \(\epsilon_{\nu, \sigma}\), momentum \(\nu\) or \(\sigma\) and spin \(\sigma\) in the left or right electrode,

\[
\mathcal{H}_1 = \sum_{\kappa=1}^{2} \Theta_\kappa \exp(-i\kappa \phi_\kappa) + h.c. \quad (B21)
\]

is the tunneling Hamiltonian where
\[
\Theta_1 = \sum_{\nu, q, \sigma} M_{pq} \gamma^\dagger_{\nu, \sigma} \gamma_{pq} \quad (B22a)
\]
\[
\Theta_2 = \frac{E_i}{2} (\phi_1 - i\sigma_2) = \frac{E_i}{2} \sigma_z \quad (B22b)
\]

\(\sigma_{1,2}\) are the \(x, y\) Pauli matrices acting on the 2-spinor given in equation (B7), \(M_{pq}\) is a dimensionful spin-conserving complex-valued quasi-particle tunneling matrix, \(p \neq q\) enforces the condition \([\mathcal{H}_L, \mathcal{H}_R] = 0\) and \(E_i\) is the Josephson coupling energy [63],

\[
\mathcal{H}_z = \frac{(Q_1 + CV_x + CA)^2}{2C} + \sum_{n=1}^{\infty} \left\{ \frac{Q_1^2}{2Cn} + e^{-2}(\phi_x - \phi_1 + \phi_x + e\Phi)^2 \right\} \quad (B23)
\]

is the Hamiltonian describing the environmental impedance \(Z(\omega)\) and junction capacitance \(C\), where \(Z(\omega)\) is characterized by an infinite number of parallel \(L_n C_n\) circuits coupled serially to the tunnel junction. One can define \(Q = Q_1 - CV_x\) and \(\phi = \phi_1 - \phi_2\) as the fluctuation variables of the junction charge \(Q_j = CV_f\) and junction phase \(\phi_1(t) = e\int_{-\infty}^{t} dt' V_f(t')\) around the mean value determined by the voltage source \(V_s\), where \(V_f(t')\) is the voltage drop across the junction. (See [12] on page 27. Note that \(\phi_1\) and \(\phi_2\) are related by a suitable unitary transformation \(U\) of the Hamiltonian,
\[ \mathcal{H}' = iU \frac{\partial}{\partial t} U^\dagger + U H U^\dagger, \]  
(B24)

where \( \mathcal{H}' = H_0' + H_1 + H_\nu, \) \( H_\nu = \sum_{\nu=1}^{2} \epsilon_{\nu} \phi_{\nu} \exp(-i\nu \phi_{\nu}) + \text{h.c.}, \) \( H_0' = \sum_{\nu=\rho=+,-} \epsilon_{\nu} \phi_{\nu} + \sum_{\nu=\rho=+,-} \phi_{\nu} \phi_{\rho}, \) and \( \epsilon_{\rho} = \epsilon_{\rho} + eV_x, \) \( Q = Q_1 - CV_x, \) \( Q_0, \) are the conjugate variables to \( \phi = \phi_1 - \phi_2, \) \( \phi_3, \) satisfying the charge-phase commutation relation,

\[ [\phi_\mu, Q_m] = i\delta_{\mu m} \epsilon, \]
(B25)

\[ [\phi, Q] = ie \]
(B26)

where \( \delta_{\mu \nu} \) is the Kroneker delta. Operators, \( O(t) \) in the Heisenberg picture are related to the ones in the Schrödinger picture, \( O(0) \) by \( O(t) = U_0(t)O(0)U_\dagger(0) \) with the unitary evolution operator \( U_0(t) \) given by \( U_0(t) = \exp \{- \sum_{\nu=\rho=+,-} \mathcal{H}_\nu t \} \) in the absence of tunneling.

In what follows, we assume the Cooper-pair ground state energy \( \mu = 0, \) as we did in equation (B3). The tunneling current \( I(V) \) at the junction is given by

\[ I(V, s) = \text{tr} \langle T U^{\dagger} I_i(0) U \rangle \]
(B27)

Here, \( U = U_0 + U_{\text{int}} \) where

\[ U_{\text{int}} = \exp \left( -i \int_{s-s^-}^{s+s^-} dt H_1(t) \right) = \exp \left( -i \int_{s^-}^{0} dt H_1(t) \right) \exp \left( -i \int_{0}^{s} dt H_1(t) \right) \]
\[ = \exp \left( i \int_{0}^{s^-} dt H_1(t) \right) \exp \left( -i \int_{0}^{s} dt H_1(t) \right) = U_{\text{int}}^0(-s)U_{\text{int}}^0(+s) \]
(B28)

\( T \) is the time ordering operator with the property given by \( T \mathcal{E}_\nu(t) \mathcal{E}_\nu^\dagger(0) = \mathcal{E}_\nu(t) \mathcal{E}_\nu^\dagger(0), \) \( T \mathcal{E}_\nu^\dagger(t) \mathcal{E}_\nu(0) = \mathcal{E}_\nu^\dagger(t) \mathcal{E}_\nu(0). \) Here, \( s \) is the elapsed time after switching on the interaction term \( U_{\text{int}}(s) \) and takes the range \( 0 \leq s \leq +\infty. \) Note that \( U_{\text{int}}^0(s) \) takes care of c.c. term in equation (B30), updating the integral range as discussed: \( \int_{s^-}^{0} dt \rightarrow \int_{s^-}^{0} dt + \int_{0}^{s} dt = \int_{s^-}^{s} dt. \) We shall be interested in the current \( I(V, s \rightarrow +\infty) = I(V) \) at equilibrium (equation (B44)).

The tunneling current operator is

\[ I_i(0) = -i[Q_1(0), H_1(0)], \]
(B29)

and the average \( \langle \ldots \rangle \) is over the quasi-particle equilibrium states, whose density matrix is given by \( \rho_1 = \rho_1 \rho_2 = \mathcal{Z}_1^{-1} \exp(-\beta H_1) \) with \( \mathcal{Z}_1 = \mathcal{Z}_1 \times \mathcal{Z}_2 = \Pi_n [1 + \exp(-\beta \epsilon_{\nu})] \times \Pi_{n} [1 + \exp(-\beta \epsilon_{\rho})], \) and the environment \( \rho_{\text{env}} = \mathcal{Z}_2^{-1} \exp(-\beta H_2) \) where \( \beta = 1/k_B T \) is the inverse temperature while the trace (tr) is over the Pauli matrices.

**B.4.2. Perturbation expansion.** We can then expand equation (B27) as a perturbation series in the tunneling Hamiltonian \( H_1(t), \)

\[ I = \left\langle I_i(0) - i \int_{s}^{+\infty} [I_i(0), H_1(t)] dt + O(H_1^2) \right\rangle. \]
(B30)

Using \( T \mathcal{E}_\nu^\dagger(t) \mathcal{E}_\nu(0) = \mathcal{E}_\nu^\dagger(t) \mathcal{E}_\nu(0), \) \( \mathcal{E}_\nu^\dagger(t) \mathcal{E}_\nu(0) = \mathcal{E}_\nu(t) \mathcal{E}_\nu(0) = \alpha_\nu(t) \alpha_\nu^\dagger(0), \) we find that

\[ I \simeq \int_{s}^{+\infty} dt [H_1(t), [Q_1(0), H_1(t)]] \]
\[ = i\epsilon \sum_{\nu=\rho} \int_{s}^{+\infty} dt (\alpha_\nu(t) \langle \sin[\kappa \Delta \phi_1(t)] \rangle_{\nu}), \]
(B31)

where \( \Delta \phi_1(t) = \phi_1(t) - \phi_1(0). \) Thus, the ‘particle’ degrees of freedom \( \alpha_\nu(t) \) and the environment are decoupled and the trace over the environment \( \langle \ldots \rangle_{\nu} \) has been re-written as \( \langle \ldots \rangle_{\nu} \) in terms of the junction phase \( \phi_1 \) degree of freedom. (section B.4.3)

For the quasi-particle current, the kernel \( \alpha_3(t) \) scales with the dimensionless tunneling conductance \( e^2 R_T^{-1} \) but its functional form depends on the gap, reflecting the corresponding structures in the quasi-particle \( I - V \) characteristics. It can be computed by taking the continuous limit, \( 2\pi e^2 \mathcal{R}_1(0) \mathcal{R}_0(0) M_{33} M_{3q} \rightarrow 1, \)
\[ \alpha_2(t) = \langle \Theta_1(t) \Theta_2(0) \rangle = \sum_{p=q,d} \sum_{p'=q,d} M_{pq} M_{p'd}' [R, \{ L, s \} \gamma_{q}(t) \gamma_{p}(0) \rho_{q} L, s \{ R, \} \gamma_{p'}(t) \gamma_{q'}(0) \rho_{q'} R \} ] \]

\[ = 2 \sum_{p=q,d} \sum_{p'=q,d} M_{pq} M_{p'd}' [R \{ \gamma_{q}(t) \} \{ L, s \} \gamma_{p}(0) \rho_{q} L \{ R, \} \gamma_{p'}(t) \gamma_{q'}(0) \rho_{q'} R \} ] \]

\[ = 2 \sum_{p=q,d} \sum_{p'=q,d} f(\epsilon_p) \exp(-i\epsilon_p t) \sum_{p'=q,d} M_{pq} M_{p'd}' [R \{ \gamma_{q}(t) \} \{ L, s \} \gamma_{p}(0) \rho_{q} L \{ R, \} \gamma_{p'}(t) \gamma_{q'}(0) \rho_{q'} R \} ] \]

\[ = 2 \sum_{p=q,d} \sum_{p'=q,d} \{ f(\epsilon_p)[1 - f(\epsilon_p)] \exp[i(\epsilon_q - \epsilon_p)t] \} \sum_{p'=q,d} M_{pq} M_{p'd}' \delta_{q'q} \delta_{p'p} \]

\[ = 2 \sum_{p=q,d} \sum_{p'=q,d} f(\epsilon_p)[1 - f(\epsilon_p)] \sum_{p'=q,d} M_{pq} M_{p'd}' \exp[i(\epsilon_q - \epsilon_p)t] \]

\[ = -\frac{1}{\pi e^2 R^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\epsilon_{q} d\epsilon_{p} \frac{N_{q}(\Delta)}{N_{q}(0)} \frac{N_{p}(\Delta)}{N_{p}(0)} f(\epsilon_p)[1 - f(\epsilon_p)] \exp[i(\epsilon_q - \epsilon_p)t]. \] (B32)

Here, \( e^2 R \) is the dimensionless tunnel resistance, \( f(E) = [1 + \exp(\beta E)]^{-1} \) is the Fermi–Dirac function and \( N_{q}(\Delta) \), \( N_{p}(\Delta) \) is the left, right BCS density of states [64] which reduce to the electron density of states \( N_{q}(0) \), \( N_{p}(0) \) when the superconducting gap \( \Delta = 0 \) vanishes,

\[ \frac{dE_p}{d\epsilon_p} = \frac{N_{q}(\Delta)}{N_{q}(0)} \frac{N_{p}(\Delta)}{N_{p}(0)} \] (B33a)

\[ E_p = \sqrt{\epsilon_p^2 - \Delta^2}, \quad E_q = \sqrt{\epsilon_q^2 - \Delta^2} \] (B33b)

where \( E_p = p^2/2m, \quad E_q = q^2/2m \) is the kinetic energy of the electrons above the Fermi sea.

Likewise, calculating \( \alpha_2(t) \), we find,

\[ \alpha_2 = \langle \Theta_1(t) \Theta_2(0) \rangle = \langle \Theta_2(0) \Theta_2(0) \rangle = \left( \frac{E}{2} \right)^2 \text{tr}[(\sigma_1 + i\sigma_2)(\sigma_1 - i\sigma_2)] \]

\[ = \frac{E^2}{4} \text{tr}[2\sigma_0 + i[\sigma_2, \sigma_1]] = \frac{E^2}{2} \text{tr}[\sigma_0 + \sigma_3] = E_0^2. \] (B34)

We discover that, unlike \( \alpha_1(t) \), \( \alpha_2(t) = \alpha_2(0) = E_0^2 \) is time independent and only depends on the strength of Cooper pair tunneling, \( E_0 \).

**B.4.3. Path integrals and phase correlations.** To calculate the remaining average over \( \phi \) in equation (B31), we work in Minkowski time at zero temperature (thus by-passing a rigorous but otherwise tedious Wick rotation to Euclidean time) since the finite temperature propagator is trivially related to the zero temperature result (equation (B42b) for the trivial relation and appendix D for the formalism).

In this formalism, given an observable \( O(\phi) \), its average at zero temperature is given by the functional/path integral,

\[ \lim_{\beta \to \infty} \langle O(\phi) \rangle_\beta = \mathcal{Z}^{-1} \prod_{n=1}^{k} \int D\phi_n D\phi_0 O(\phi) \exp iS_x(\phi_0, \phi_n), \] (B35)

where \( \mathcal{Z} = \prod_{n=1}^{k} \int D\phi_0 D\phi_0 \exp iS_x(\phi_0, \phi_0) \) is the partition function normalizing equation (B35) and the Lagrangian in the action for the environment \( S_x(\phi_0, \phi_n) \) is given by the (inverse) Legendre transform of the environment Hamiltonian in equation (B23)

\[ S_x = \int L_x dt = \int \left\{ (Q_t + Q_k - CA) \frac{\partial H_k}{\partial Q_t} - H_x \right\} dt, \] (B36a)

\[ C \frac{\partial \phi_0(t)}{\partial t} = eQ_k, \quad C \frac{\partial \phi(t)}{\partial t} = eQ_t, \quad C \frac{\partial \Phi(t)}{\partial t} = A(t). \] (B36b)

The effective action \( S_\epsilon' (\phi) \) resulting from performing first the functional integral product over \( \phi_n \) is given by

\[ S_\epsilon' (\phi - e\Phi) = S_\epsilon' (\phi - e\Phi) + S_\epsilon'^{m} (\phi - e\Phi) = \frac{C}{2e^2} \int_{-\infty}^{+\infty} \left( \frac{\partial (\phi(t) - e\Phi(t))}{\partial t} \right)^2 dt \]

\[ - \frac{1}{2e^2} \int_{-\infty}^{+\infty} \sum_{n} L_n dt \]

\[ - \frac{1}{4\pi e^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\phi(t) - e\Phi(t)] \frac{\partial Z^{-1}(s - t)}{\partial t} \{ \phi(t) - e\Phi(t) \} dt \]

\[ + \frac{1}{e} \int_{-\infty}^{+\infty} I_\epsilon (t) (\phi(t) - e\Phi(t)) dt \] (B37a)

with a fluctuation current \( I_\epsilon (t) = 0 \) and \( \phi_1 + \phi = \phi_2 \). Here, \( Z^{-1}(\omega) \) is the Fourier transform of a generalized admittance function \( Z^{-1}(\omega) \) given by

23
\[ Z^{-1}(\omega) = \sum_{n=1}^{k} \frac{\omega_n}{i\omega L_n} \left\{ \frac{\omega_n}{(\omega + i\varepsilon)^2 - \omega_n^2} \right\} = \sum_{n=1}^{k} \frac{\omega_n}{i\omega L_n} \left\{ \frac{1}{\omega - \omega_n + i\varepsilon} - \frac{1}{\omega + \omega_n + i\varepsilon} \right\} \]  

(B37b)

\[ \frac{1}{\omega + \omega_n \pm i\varepsilon} = \mp i\pi \delta(\omega + \omega_n) + \text{p.p.} \left( \frac{1}{\omega + \omega_n} \right). \]  

(B37c)

where equation (B37c) is the Sokhotski-Plemelj formula and p.p. stands for Cauchy principal part.

Equation (B37b) is related to the spectral function \( K(\omega) = [Z^{-1}(\omega) + Z^{-1}(-\omega)]/(2\pi) \) given in section B.3 where \( \varepsilon \) is the infinitesimal satisfying \( \omega \varepsilon = \varepsilon \) and the nilpotent condition \( \varepsilon^2 = 0 \). Note, the spectral function is the sum of negative and positive frequency impedance accounting for emission and absorption processes respectively by the circuit. Thus, equation (B16) differs slightly from equation (B37a) where the real-valued spectral function \( K(t) \) in the classical Lagrangian gets replaced with the complex valued admittance \( Z^{-1}(t)/(2\pi) \) in the quantum case.

Introducing the Dirac delta function \( \delta(x) \) for functional integrals with the property

\[ \int Dx \; f(x) \delta(x - y) = f(y) \]  

(B38)

for any functional \( f(x) \), we may proceed to insert \( \int D\phi_x \delta(\phi_t + \phi_x + \phi_z - e\Phi) = 1 \) into equation (B35) thus introducing the constraint \( \sum_a \phi_a = \phi_t + \phi_x + \phi_z = e\Phi \) guaranteed by the circuit in figure 1. Consequently, the average in equation (B31) is now taken over both \( \phi_t \) and \( \phi_z \):

\[ \lim_{\beta \to +\infty} \langle \sin [\kappa \Delta \phi_t(t)] \rangle_{\phi_t \phi_z} = Z^{-1} \int D\phi_t \int D\phi_x \phi_z \left( \sum_a \phi_a - e\Phi \right) \sin [\kappa \Delta \phi_t(t)] \exp iS_x^\prime(\phi - e\Phi). \]  

(B39)

We find,

\[ -\langle \sin [\kappa \Delta \phi_t(t)] \rangle_{\phi_t \phi_z} = \langle \sin [\kappa \Delta \phi_x(t) + \kappa \int_0^t A(\tau)d\tau + \kappa \Delta \phi_z(t)] \rangle_{\phi_t \phi_z} \]

\[ = \langle \sin [\kappa \Delta \phi_x(t)] \rangle_{\phi_t \phi_z} \cos \left[ \kappa \Delta \phi_x(t) + \kappa \int_0^t A(\tau)d\tau \right] \]

\[ + \langle \cos [\kappa \Delta \phi_z(t)] \rangle_{\phi_t \phi_z} \sin \left[ \kappa \Delta \phi_z(t) + \kappa \int_0^t A(\tau)d\tau \right]. \]

(B40)

with \( \Delta \Phi(t) = e \int_0^t A(\tau)d\tau \). We have assumed Fubini’s theorem for interchange of integration order applies and thus performed first the integral over \( \phi_x \). Using the fact that \( S_x^\prime \) is quadratic, the resulting functional integral over \( \phi_z \) in equation (B40) is Gaussian resulting in \( \langle \sin [\kappa \Delta \phi_z(t)] \rangle_{\phi_t \phi_z} = 0 \) term vanishing. Likewise, \( \langle \cos [\kappa \Delta \phi_z(t)] \rangle_{\phi_t \phi_z} \) satisfies Wick’s theorem [12]

\[ \langle \cos [\kappa \Delta \phi_x(t)] \rangle_{\phi_t \phi_z} = \exp \left( \kappa^2 \int_{\phi(0)}^{\phi(t)} d\phi_x(\tau) \right), \]

(B41a)

\[ \int D\phi_x \exp iS_x^\prime(\phi_x L_x) = \exp iS_x^\prime(L_x), \]  

(B41b)

\[ S_x^\prime(\phi_x L_x) = \frac{2\pi}{2\varepsilon \omega} \int_{-\infty}^{+\infty} L_x(\omega) G_{\text{eff}}(\omega) \phi_x(\omega) d\omega \]  

(B41c)

\[ G_{\text{eff}}(\omega) = -e^2 i\omega^{-2} Z_{\text{eff}}(\omega), \]  

(B41d)

\[ Z_{\text{eff}}(\omega) = \frac{1}{Z^{-1}(\omega) + y(\omega)} \]  

(B41e)

where \( y(\omega) = i\omega - i\omega^{-1} \sum_{n=1}^{\infty} \frac{L_n}{\omega} \).

We introduce the zero temperature propagator \( D_{\tau \infty}(t) \) given by

\[ D_{\tau \infty}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \exp -i\omega \{ Z_{\text{eff}}(\omega) + n.f. \}, \]  

(B42a)

where \( n.f. \) stands for negative frequency. The finite temperature propagator is related to \( D_{\tau \infty}(t) \) by a sum over the photon number states

\[ D_{\tau \infty}(t) \to D_{\tau}(t) = \sum_{n=0}^{+\infty} D_{\tau \infty}(t - in) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \exp -i\omega \{ Z_{\text{eff}}(\omega) + n.f. \}. \]  

(B42b)

Thus, computing the phase–phase correlation function, we find

\[ \langle \phi_z(\tau) \phi_z(t) \rangle_{\phi_t \phi_z} = \frac{Z^{-1} b^2 \exp iS_x^\prime(\phi_x L_x)}{e^{-b^2 L_x(\phi_x L_x)}} \bigg|_{L_x = 0, z = 1} = e^2 D_{\tau \infty}(s - t) \to e^2 D_{\tau}(s - t), \]  

(B43)

which satisfies the well-know fluctuation-dissipation theorem [7].
B.4.4. Cooper Pair and BCS quasi-particle tunneling current. Finally, plugging in results (B40) and (B42b) in equation (B31), and using \(\Delta \phi_k(t) = \int_0^t V(\tau) d\tau = V t\) where \(V_c = V\) is a constant external voltage and \(\Delta \Phi(t) = 0\), the total \(I - V\) characteristics is given by

\[
I_0(V) = I_i(V) + I_c(V) = e^{-R_i} e^{-i \pi/2} \int_{-\infty}^{+\infty} \frac{d\epsilon_p d\epsilon_q}{N^2(\epsilon_p)} f(\epsilon_p) (1 - f(\epsilon_q)) \left[ P_1(\epsilon_q - \epsilon_p - eV) - P_1(\epsilon_q - \epsilon_p + eV) \right] + e\pi e_i^2 \left[ P_2(2eV) - P_2(-2eV) \right]
\]

where we have introduced the so-called \(P(E)\) function \[12\]

\[
P_p(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \exp \kappa \beta \mathcal{J}(t) \exp iEt,
\]

\[
e^{-\kappa \beta \mathcal{J}(t) = D_J(t) - D_J(0)}
\]

with \(E\) some arbitrary energy. It gives the probability that the junction will absorb energy \(E\) from the environment. Note that equation (B44) reduces to the normal junction \(I - V\) characteristics

\[
I(V)|_{\Delta = 0} = e^{-R_i} e^{-i \pi/2} \int_{-\infty}^{+\infty} d\epsilon_p d\epsilon_q f(\epsilon_p) (1 - f(\epsilon_q)) \left[ P_1(\epsilon_q - \epsilon_p + eV) - P_1(\epsilon_q - \epsilon_p - eV) \right]
\]

when the superconducting gap vanishes \(\Delta = 0\), since \(E_j(\Delta = 0) = 0, \mathcal{N}(\Delta = 0)/\mathcal{N}(0) = 1\) and \(E_p = \epsilon_p, \epsilon_q = \epsilon_q\).

Appendix C. The array as an effective single junction

It is prudent to highlight the ingredients that went into deriving the \(I - V\) characteristics of the single small Josephson junction given in equation (B44):

1) The Caldeira-Leggett action \(S_F(\phi, t)\) that is varied with respect to \(\phi = \phi_1 + \phi_2\) to obtain the equation of motion for the single large Josephson junction; 2) the correlation \(\langle \sin [\kappa \Delta \phi_1(t)]\rangle_{\phi_1}\) calculated with respect to \(\phi_1\); 3) The condition \(\sum \phi_j = e\Phi\) enforced by the circuit.

In the case of a linear array of \(N_0\) small Josephson junctions, it is clear that constraint 3) has to include all the phases \(\phi_{j = 1} \cdots \phi_{j = N_0}\) of the junctions along the array, and the quantum average in 2) taken over each phase where the action in 1) is the sum of the action of individual junctions in the array. To simplify the calculation, one assumes all the junctions have the same structure coefficients \(\alpha_{k}(t)\) and calculates the quantum average \(\langle \sin [\kappa \Delta \phi_{\phi}(t)]\rangle_{\phi_1 \cdots \phi_{N_0}}\) (step 2). Since the same current passes through all the junctions in the array, the calculation is carried out at any one of them [e.g. the \(j = 1\) junction] while assuming that the circuit forms a loop that imposes condition 3) as before. Hence, treating the other junctions as effective single junctions with an effective action of the form \(S'(\phi_j)\), we have,

\[
I_A(V) = i e \sum_{\phi_{j = 1}}^{N_0} \int_{-\infty}^{+\infty} dt \alpha_j(t) \langle \sin [\kappa \Delta \phi_{\phi}(t)]\rangle_{\phi_1 \cdots \phi_{N_0}}
\]

\[
= -i e \sum_{\phi_{j = 1}}^{N_0} \int_{-\infty}^{+\infty} dt \alpha_j(t) \sin \left[ \kappa e \int_0^t A(\tau) d\tau + \kappa \Delta \phi_{\phi}(t) \right] \int \prod_{\phi = 1}^{N_0} D\phi \exp i S'(\phi_j) [\cos \kappa \Delta \phi_{\phi}(t)],
\]

where,

\[
= \prod_{\phi = 1}^{N_0} \left[ \cos \kappa e \int_0^t A(\tau) d\tau + \kappa \Delta \phi_{\phi}(t) \right]
\]

\[
\exp \left[ -\frac{\kappa}{2} \sum_{\phi = 1}^{N_0} \Delta \phi_{\phi}(t) \right] \sin \left[ \kappa e \int_0^t A(\tau) d\tau + \kappa \Delta \phi_{\phi}(t) \right]
\]

\[
= \exp \left[ -\frac{\kappa}{2} \sum_{\phi = 1}^{N_0} \Delta \phi_{\phi}(t) \right] \sin \left[ \kappa e \int_0^t A(\tau) d\tau + \kappa \Delta \phi_{\phi}(t) \right]
\]

\[
= \exp \left[ \kappa e \sum_{\phi = 1}^{N_0} \Delta \phi_{\phi}(t) \right] \sin \left[ \kappa e \int_0^t A(\tau) d\tau + \kappa \Delta \phi_{\phi}(t) \right] = \prod_{\phi = 1}^{N_0} P'$(t) \sin \left[ \kappa e \int_0^t A(\tau) d\tau + \kappa \Delta \phi_{\phi}(t) \right].
\]
Evidently, the current depends on the product
\[ P^A_\gamma(t) = \prod_x P^\gamma_x(t) \]
as expected, since it is comprised of individual tunneling events at each junction. Since, \( 1/2\pi \int dt P^A_\gamma(t) \exp iEt \) is the probability that the array will absorb energy \( E \) from the environment, we discover that the tunnel current obeys the product rule of probabilities. For identical junctions of capacitance \( C \) and impedance \( Z(\omega) \),
\[ \langle \phi_z(t) \phi_0(0) \rangle \equiv \langle \phi_{z-2}(t) \phi_{z-2}(0) \rangle = \cdots = \langle \phi_{z-N}(t) \phi_{z-N}(0) \rangle, \]
we have \( \prod_{x=1}^{N} P^\gamma_x(t) = [B_\gamma(t)]^N \) where
\[ P_\gamma(t)^N = \exp (N_0\kappa^2 [\langle \phi_z(t) - \phi_0(0) \rangle \phi_0(0)]_{\phi_0}) = \exp (N_0\kappa^2 J_\gamma(t)) \]  
(C2)
where \( Z^A_\gamma(\omega) \) in \( J_\gamma(t) = \langle [\phi_z(t) - \phi_0(0)] \phi_0(0) \rangle_{\phi_0} \) will differ from \( Z^\text{eff}(\omega) \) in equation (B45a) due to possible interaction terms. Neglecting these interactions by setting \( J_\gamma(t) \simeq J(t) \), equation (C2) implies that at zero temperature, Cooper pair Coulomb blockade threshold voltage \( V^A_{cb} = N_0 V_{cb} \) for the array is a factor \( N_0 \) larger than for the single junction.

However, the rest of the array \( \langle \phi_{z-2} \cdots \phi_{z-N} \rangle \) acts as the environment for the single junction \( \langle \phi_{z-1} \rangle \) thus introducing interaction terms. In particular, the single junction interacts with the rest of the array electromagnetically. Since, in the presence of Cooper pair solitons \([45, 65]\) and the Meissner effect, the electromagnetic field has a finite range within an infinitely long array \( (N_0 \gg 1) \), the array has a cut-off number of junctions beyond which no electromagnetic interactions occur. Consequently, the effective number of junctions \( N_e \ll N_0 - 1 \) acting as the environment will be determined by the range of the electromagnetic field. \( N_e(\Lambda) \) is independent of the magnetic field, \( H \) when the superconducting islands are shorter than the penetration depth of the magnetic field, \( H \). It can be evaluated by equating the Coulomb blockade voltage \( V^A_{cb} \) (estimated by replacing \( N_0 \) with \( N_e(\Lambda) \) and setting \( \text{Re} \{ Z^A_{cb}(\omega) \} \simeq \text{Re} \{ Z^\text{eff}(\omega) \} \) in equation (C2) to the standard expression for the soliton threshold voltage \([30]\) of the array, \( eV^A_{cb} = eV_{cb} \simeq 2E_c[\exp(\Lambda^{-1}) - 1]^{-1} \), leading to
\[ N_0 \rightarrow N_e(\Lambda) = \frac{1}{\exp(\Lambda^{-1}) - 1}, \]  
(C3a)
which approaches the soliton length \( N_e(\Lambda) \rightarrow \Lambda \) when \( \Lambda \gg 1 \). However, the infinite array effectively has
\[ N_e(\Lambda) + 1 = \frac{1}{\exp(-\Lambda^{-1})} \]  
(C3b)
junctions. This means that \( N_0 \) in equation (C2) is instead rescaled to \( N_e(\Lambda) \). Since \( V^A_{cb}(\Lambda) \) should be invariant under the transformation \( N_e(\Lambda) \rightarrow N_e(\Lambda) + 1 \), we find
\[ \lim_{r \rightarrow +\infty} \text{Re} \{ Z^A_{cb}(\omega) \} \simeq \lim_{r \rightarrow +\infty} \text{Re} \{ Z^\text{eff}(\omega) \} \rightarrow \lim_{r \rightarrow +\infty} \text{Re} \{ Z^A_{cb}(\omega) \} \simeq \exp(-\Lambda^{-1}) \lim_{r \rightarrow +\infty} \text{Re} \{ Z^\text{eff}(\omega) \}, \]  
(C4a)
we discover that switching on electromagnetic interactions leads to a rescaled impedance and a rescaled response function given by \( \Xi(\omega) \rightarrow \Xi_e(\omega) \Xi(\omega) = \exp(-\Lambda^{-1}) \Xi(\omega) \).

Finally, \( N_e(\omega) \) (equation (C3b)) is given by (confer: equation (19b)),
\[ N_e(\Lambda) = \sum_{m=-\infty}^{+\infty} \frac{1}{\Lambda^{-1} - 2\pi mi} = \frac{1}{2} \coth(\frac{1}{2\Lambda}) - \frac{1}{2}. \]  
(C5)
This result is not surprising, since we have determined the \( I-V \) characteristics of the array by treating it as a single junction with the rest of the array acting as its environment. This means that the \( N_0 - 1 \) junctions themselves act as bosonic excitations whose (average) number \( \langle N_0 \rangle - 1 = N_e \) determines the electromagnetic cut-off, which is also the effective number of junctions that can be approximated as the environment of the effective single junction.

Appendix D. Path integral formalism with gaussian functional integral

For completeness, this section summarizes how to compute correlation functions with Gaussian functional integrals such as the ones used in section B.4.3 in the derivation of the propagator \( D_{\gamma,\omega}(t) \) in equation (B42).

Our approach differs from typical procedures with imaginary time \([59]\). We work with real time instead since the finite temperature propagator is \textit{trivially} related to the zero temperature propagator (equation (B42b)).

Consider a quadratic action \( S(X, \dot{Y}) \) with \( X \) as the coordinate variable, \( Y \) as a fluctuation force, \( a \) as a mass term and \( g \) a coupling constant. The computation procedure is then as follows:
1. Take the Fourier transform of the action by substituting the Fourier or inverse Fourier transforms

\[ X(t) = \int d\omega X(\omega) e^{-i\omega t}, \quad X(\omega) = \frac{1}{2\pi} \int dt X(t) e^{i\omega t}, \]
\[ Y(t) = \int d\omega Y(\omega) e^{-i\omega t}, \quad Y(\omega) = \frac{1}{2\pi} \int dt Y(t) e^{i\omega t} \quad (D1) \]

in the action,

\[ S(X, Y) = \int dt \left[ \frac{\alpha}{2} \left( \frac{\partial X(t)}{\partial t} \right)^2 - \frac{\alpha}{2} \omega^2 X^2(t) + gX(t)Y(t) \right] \]
\[ = 2\pi \int d\omega \left[ \frac{1}{2} \omega X G_X^{-1} Y(-\omega) + gX(\omega) Y(-\omega) \right], \quad (D2) \]

where \( a^{-1}G_X^{-1}(\omega) = (\omega + i\epsilon)^2 - \omega_0^2 \) and \( G_X(t) = \int d\omega G_X(\omega) e^{-i\omega t} \);

2. Perform the functional integral \( \int DX \exp iS(X, Y) \propto \exp iS'(Y) \) emulating a typical Gaussian integral

\[ \int dx \exp \left[ \frac{a^2}{2} + gxy \right] \propto \exp \left[ \frac{i(g)^2y^2}{2a} \right] = \exp \left[ \frac{-i(g)^2y^2}{2a} \right] \rightarrow S'(Y) \]
\[ = 2\pi \int d\omega \left( \frac{(g)^2}{2} Y(\omega) G_X(\omega) Y(-\omega) \right) = \frac{(g)^2}{2 \times 2\pi} \int dt ds Y(s) G_X(s-t) Y(t); \quad (D3) \]

3. Compute the correlation functions with the quadratic part of the action as follows,

\[ \langle X(t_1) \cdots X(t_n) \rangle = \mathbb{Z}^{-1} \int DX[X(t_1) \cdots X(t_n)] \exp iS(X, Y = 0) \]
\[ = \left[ \frac{1}{(ig)^n} \delta \frac{\delta}{\delta Y(t_1)} \cdots \delta \frac{\delta}{\delta Y(t_n)} \mathbb{Z}^{-1} \int DX \exp iS(X, Y = 0) \right]_{Y = 0, Z = 1} \]
\[ = \left[ \frac{1}{(ig)^n} \delta \frac{\delta}{\delta Y(t_1)} \cdots \delta \frac{\delta}{\delta Y(t_n)} \exp iS'(Y) \right]_{Y = 0} = \left[ \frac{1}{(ig)^n} \delta \frac{\delta}{\delta Y(t_1)} \cdots \delta \frac{\delta}{\delta Y(t_n)} \sum_{m=0}^{m=\infty} (iS'(Y))^m \right]_{Y = 0} \cdot \quad (D4) \]

We require the variation \( \delta / \delta Y(t) \) to satisfy the anti-commutation rule,

\[ \frac{\delta}{\delta Y(t)} + \frac{\delta}{\delta Y(s)} = 0, \quad (D5a) \]

where \( \delta Y(s) / \delta Y(t) = \delta(t - s) \) is the Dirac delta function. Note that the anti-commutation rule accounts for time ordering. Since \( S'(Y) \) is quadratic in \( Y \), the integral vanishes for odd number of variables \( n = 2N - 1 \) where \( N \) is a positive integer. For even number of variables \( n = 2N \) we have the continuation,

\[ \langle X(t_1) \cdots X(t_n) \rangle = \left[ \frac{i^n}{(ig)^nN!} \delta \frac{\delta}{\delta Y(t_1)} \cdots \delta \frac{\delta}{\delta Y(t_n)} \mathcal{S}^N(Y) \right]_{Y = 0, n = 2N} = \frac{(ig)^n}{(ig)^n N! (2 \times 2\pi)^N} \delta \frac{\delta}{\delta Y(t_1)} \cdots \delta \frac{\delta}{\delta Y(t_n)} \prod_{m=1}^{m=N} \int ds m_{m-1} ds m_{m-1} G_X(s_{m-1} - s_{2m}) Y(s_{2m}); \quad (D6) \]

4. For illustration, we compute the case \( N = 1 \),

\[ \langle X(t_1) X(t_2) \rangle_{t_1 = t_2} = \frac{i}{2 \times 2\pi} \delta \frac{\delta}{\delta Y(t_1)} \int ds_1 ds_2 Y(s_1) G_X(s_1 - s_2) Y(s_2) \]
\[ = \frac{i}{2 \times 2\pi} \int ds_1 ds_2 \left[ \frac{\delta Y(s_1)}{\delta Y(t_1)} G_X(s_1 - s_2) \frac{\delta Y(s_2)}{\delta Y(t_1)} \right] = \frac{i}{2 \times 2\pi} \int ds_1 ds_2 \left[ \frac{\delta Y(s_1)}{\delta Y(t_1)} G_X(s_1 - s_2) \frac{\delta Y(s_2)}{\delta Y(t_1)} \right] \]
\[ = \frac{i}{2 \times 2\pi} \left[ G_X(t_2 - t_1) - G_X(t_1 - t_2) \right] = \frac{i}{2 \times 2\pi} \int d\omega \left[ G_X(\omega) - G_X(-\omega) \right] \exp(-i\omega t) \quad (D7) \]

and
\begin{align}
\langle X(0)X(0) \rangle & \equiv \langle X(t_1)X(t_2) \rangle\big|_{t_1=t_2} = \frac{i}{2 \times 2 \pi} \int d\omega \left[ G_X(\omega) - G_X(-\omega) \right] = 0,
\end{align}

where $t = t_1 - t_2$ and we have used the anti-commutation rule given in 3. Note that, after Fourier transforming the action given in equation (B37a) and the delta functional integral (B38) performed in equation (B40), we simply have $X(t) \rightarrow \phi(t) / \sqrt{2}$ and $G_X(\omega) \rightarrow G_{\text{eff}}(\omega) = -e^{2i\omega^{-1}Z_{\text{eff}}(\omega)}$ to yield equation (B43).

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