Exact WKB Analysis of Schrödinger Equations with a Stokes Curve of Loop Type

By

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Abstract. Stokes phenomena with respect to a large parameter are investigated for Schrödinger-type ordinary differential equations having a Stokes curve of loop-type. For this purpose, we employ a Bessel-type equation as a canonical form and compute the Voros coefficient of the equation. Combining the formula describing the Stokes automorphism for the Voros coefficient and the formal coordinate transformation connecting the Schrödinger-type equation and the Bessel-type equation, we have some formulas describing the action of alien derivatives and Stokes automorphism for WKB solutions of the Schrödinger-type equation.

Key Words and Phrases. Exact WKB analysis, Stokes automorphism.
2010 Mathematics Subject Classification Numbers. 34E20, 34M25, 34M40.

1. Introduction

The aim of this article is to investigate the Stokes phenomenon relevant to a loop-type degeneration of Stokes curves for the WKB solutions of the Schrödinger-type ordinary differential equations with a large parameter from the viewpoint of the exact WKB analysis. In particular, we will give an explicit formula which describes the action of alien derivatives on (the Borel transform of) the WKB solutions.

In the pioneering work of Voros [27] on the exact WKB analysis, the importance of a degeneration of the Stokes graph, that is, appearance of a Stokes curve connecting turning points, are pointed out. This is because such a degenerate Stokes curve generates a set of singular points of the Borel transform of the WKB solutions, so-called fixed singularities. In terms of the Borel resummation method, such singularities cause a Stokes phenomenon what we are interested in. The first systematic study of such Stokes phenomena is done by Voros [27] for the harmonic oscillator and for the quartic oscillator. Generalizing the result, Delabaere-Dillinger-Pham [6] obtained a formula that describes the Stokes phenomena for the Schrödinger-type equation with polynomial potentials. After these works, several results related to the phenomena have been obtained from various points of view ([2], [3], [11], [12], [17], [21],...
Recently the importance of this type of Stokes phenomenon has been recognized in the theory of cluster algebra ([9], [10]).

In the above articles, the authors analyzed a fundamental type of degeneration; that is, the case where one Stokes curve connects two distinct turning points are discussed. On the other hand, a loop-type degeneration, which is the main subject of this article, is also one of the fundamental types of degeneration of Stokes curves for Schrödinger equations having a regular singular point. This loop-type Stokes curve also generates a set of fixed singularities of the Borel transform of the WKB solution. However, it has not been treated systematically in the previous works. (Some related topics are studied by the third author [23].) This is one of the motivations of this article.

Our analysis consists of two parts. Firstly, in Section 2, we investigate a Bessel-type equation with a large parameter:

$$
\left(- \frac{d^2}{dx^2} + \eta^2 \frac{x + y^2}{4x^2} - \frac{1}{4x^2}\right)\psi = 0.
$$

Here $\nu$ is a non-zero complex parameter and $\eta$ is a large parameter. We compute the Voros coefficient (cf. [3], [6], [11], [12], [17], [25], [27]) for this equation and using the explicit form of it, we obtain a formula for the alien derivative (see [20] for the definition) of the WKB solutions (Theorem 2.5 and 2.10). In Section 3, we study a Schrödinger-type equation

$$
\left(- \frac{d^2}{dx^2} + \eta^2 \tilde{Q}(x, \eta)\right)\tilde{\psi} = 0,
$$

where $\tilde{Q} = \tilde{Q}(x, \eta)$ is a finite sum

$$
\tilde{Q} = \sum_{j=0}^{N} \eta^{-j} \tilde{Q}_j(x)
$$

of rational functions $\tilde{Q}_j$ satisfying the following conditions:

Assumption 1.1. (1) The leading term $\tilde{Q}_0$ of $\tilde{Q}$ has a double pole at the origin and a simple zero $p$ near but distinct from the origin and has no other poles and zeros near the origin.

(2) If $s$ is a pole of $\tilde{Q}_j$ for some $j \geq 1$, then $s$ is a pole of $\tilde{Q}_0$ and satisfying the following:

- If the pole order at $s$ of $\tilde{Q}_0$ is equal to 1, then $\tilde{Q}_j$ for $j \geq 1$ have a pole of order at most 2 at $s$. 

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• If the pole order at \( s \) of \( \tilde{Q}_0 \) is greater than 1, then

\[
\text{(pole order of } \tilde{Q}_0 \text{ at } s \text{)} \geq \text{(pole order of } \tilde{Q}_j \text{ at } s \text{)}
\]

holds for \( j \geq 1 \).

(3) A Stokes curve emanating from \( p \) forms a loop which encircles the origin. Moreover, any other Stokes curve flows into a pole of \( \tilde{Q}_0 \) whose order is greater than or equal to 2.

The third condition can be satisfied only when the residue of \( \sqrt{Q_0} \, dx \) at the origin is a non-zero pure imaginary number (see [22, Section 7]). Under the assumption, we construct a WKB theoretic transformation from (1.2) to (1.1) and, using the result of Section 2, we compute the alien derivatives of the WKB solutions of (1.2) for the fixed singularities (Theorems 3.5 and 3.8). The explicit formula of the alien derivatives enable us to describe the action of the Stokes automorphism on the WKB solutions (Theorem 3.9).

In ending Introduction, the authors express their heartiest thanks to Professor T. Kawai, Professor Y. Takei, Professor T. Koike, Professor S. Kamimoto and Dr. S. Sasaki for the stimulating discussions with them.

2. Stokes graph and analytic properties of WKB solutions of the Bessel-type equation

In this section we study the singularity structure of Borel transform of WKB solutions of the one-dimensional Schrödinger-type equation

\[
(2.1) \quad \left( -\frac{d^2}{dx^2} + \eta^2 Q(x, \eta) \right) \psi(x, \eta) = 0
\]

with the potential

\[
(2.2) \quad Q(x, \nu, \eta) = Q_0(x, \nu) + \eta^{-2} Q_2(x) = \frac{x + \nu^2}{4x^2} - \eta^{-2} \frac{1}{4x^2}.
\]

Here \( \eta \) is a large parameter and \( \nu \) is a non-zero complex parameter. The equation (2.1) is obtained from the Bessel equation

\[
(2.3) \quad \left( \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} - \left( 1 + \frac{x^2}{z^2} \right) \right) \varphi = 0
\]

through the following change of variables

\[
(2.4) \quad z = \eta x^{1/2}, \quad \alpha = \eta \nu
\]

and a certain gauge transformation.
2.1. WKB solutions and Borel resummation method

Our main interest consists in the analysis of the WKB solutions. They are formal solutions of (2.1) of the form

$$\psi_{\pm}(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}(x, \eta)}} \exp\left( \pm \int x S_{\text{odd}}(x, \eta) dx \right),$$

where $S_{\text{odd}}(x, \eta)$ is defined as follows. Let $S_{-1}^{(\pm)}(x, \eta) = \eta S_{-1}^{(\pm)}(x) + S_0^{(\pm)}(x) + \eta^{-1}S_1^{(\pm)}(x) + \cdots$ be the formal solutions the Riccati equation

$$S^2 + \frac{dS}{dx} = \eta^2 Q(x, \eta)$$

with the leading term

$$S_{-1}^{(\pm)}(x; v) = \pm \sqrt{Q_0(x)} = \pm \frac{\sqrt{x + v^2}}{2x}.$$  

Once we fix the sign in (2.7) (or the branch of $\sqrt{x + v^2}$), the higher order terms $\{S_n^{(\pm)}(x)\}_{n \geq 0}$ are determined recursively. Note that, the coefficients are singular at the turning point $x = -v^2$. It is easy to verify that $S_n^{(\pm)}(x)$ is holomorphic at $x = 0$ and $\infty$ if $n \geq 1$ (thanks to the term $Q_2(x)$ in (2.2)).

Then, $S_{\text{odd}}(x, \eta; v)$ and $S_{\text{even}}(x, \eta; v)$ are defined by

$$S_{\text{odd}}(x, \eta; v) = S_{\text{odd}} = \frac{1}{2}(S^{(+)} - S^{(-)}),$$

$$S_{\text{even}}(x, \eta; v) = S_{\text{even}} = \frac{1}{2}(S^{(+)} + S^{(-)}).$$

By a similar argument to [14, Remark 2.2], we have

$$S_{\text{even}} = -\frac{1}{2} \frac{d}{dx} \log S_{\text{odd}}.$$

The WKB solutions (2.5) have the following formal series expansion (with an exponential factor)

$$\psi_{\pm}(x, \eta) = \exp\left( \pm \eta \int x \sqrt{Q_0(x)} dx \right) \sum_{n=0}^{\infty} \eta^{-n-1/2} \psi_{\pm, n}(x)$$

It is known that (2.10) are divergent series in general, and are of Gevrey 1 (or Borel transformable) uniformly on any compact set $K$ which doesn’t contain any turning points and poles of $Q_0(x)$; that is,

$$\sup_{x \in K} |\psi_{\pm, n}(x)| \leq n! C^{n+1}$$
with some constant $C > 0$ (cf. [14, Lemma 2.5]). In exact WKB analysis (cf. [7], [14], [27]), we apply the Borel resummation method (for a formal series in $\eta^{-1}$) to the WKB solutions (2.5) and obtain analytic solutions of (2.1). Here we give a rough explanation of the Borel resummation method (see [5, 20] for more details).

For a fixed $x_0$ in $x$-plane, the WKB solution $\psi_\pm$ is said to be Borel summable near $x_0$ if the following conditions are satisfied (see [14, Definition 1.3]):

- The Borel transform

\[
\mathcal{B}\psi_\pm(x, y) = \psi_{\pm,B}(x, y) = \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}(x)}{n!} (y \pm a(x))^n
\]

of $\psi_\pm$ is holomorphic on a domain $D = \bigcup_{s \geq 0} \{(x, y) \in U \times C \mid |y + a(x) - s| < \epsilon\}$ with a sufficiently small $\epsilon > 0$. Here $U$ is a neighborhood of $x_0$ and $a(x) = \int^x \sqrt{Q_0(x)} \, dx$.

- On the above domain $D$, the function $\psi_{\pm,B}(x, y)$ satisfies $|\psi_{\pm,B}(x, y)| < C_1 e^{C_2|y|}$ with $C_1, C_2 > 0$.

If $\psi_\pm$ is Borel summable, then the following Laplace integral defines an analytic function of both $x$ (on $U$) and $\eta$ ($\gg 1$):

\[
\int_{x=a(x)}^{x_0} e^{-\eta y} \psi_{\pm,B}(x, y) \, dy.
\]

The path of integration in (2.13) is taken along the half line which is parallel to the positive real axis in $y$-plane. The function (2.13) is called the Borel sum of $\psi_\pm$.

Before giving the criterion of the Borel summability of the WKB solutions, let us recall the notion of the Stokes graph (see [14, Definition 3.9], [15]).

- A turning point is a zero or simple pole of $Q_0(x) \, dx^2$.
- A Stokes curve is a real one-dimensional integral curve of the direction field

\[
\text{Im}(\sqrt{Q_0(x)} \, dx) = 0
\]

eemanating from a turning point.

- The Stokes graph is defined as a graph on $x$-plane whose vertices are zeros and poles of $Q_0(x) \, dx^2$, and whose edges are Stokes curves.

Here we summarize some facts about the Borel sums. We emphasize that the following facts also hold for more general Schrödinger equations with rational coefficients (at least the equations discussed in Section 3).

**Theorem 2.1** ([16]). (i) The WKB solutions (2.5) are Borel summable near any point on each face of the Stokes graph when the following assumption is satisfied:
The integration path for $S_{\text{odd}}(x, \eta)$ in \((2.5)\) never intersects any Stokes curve connecting turning points.

(ii) Under the assumption (A), the Borel sum \((2.13)\) of the WKB solution $\psi_{\pm}$ defined on each face of Stokes graph gives an analytic solution of the equation \((2.1)\). The original WKB solution $\psi_{\pm}$ is recovered as the asymptotic expansion of the Borel sum \((2.13)\) as $\eta \to +\infty$ (Watson’s lemma).

A proof of Theorem 2.1 will be given in forthcoming paper [16]; see also [26, Section 3.1]. We note that the condition (2) in Introduction is used in the proof of Borel summability in [16].

In the definition of the Borel sum \((2.13)\) the parameter $\eta$ is supposed to be real. However, in the analysis of the Stokes phenomena it is natural to regard $\eta$ as a complex parameter. The argument of $\eta$ is corresponding to the angle of the path of Laplace integral to define the (lateral) Borel sum. Then, the defining direction field of Stokes curves are modified to be

$$\text{Im} \left( e^{-i \arg \eta} \sqrt{Q_0(x)} dx \right) = 0.$$

The configurations of Stokes curves for $\nu = i$ when $\arg \eta$ varies are shown in Figure 2.1. Since $x = -\nu^2$ is a simple (i.e., order 1) turning point, and three Stokes curves emanate from the turning point. In most cases Stokes curves emanating from a turning points flow into poles of $Q_0(x) dx^2$ (i.e., 0 or $\infty$ in the case of (2.1)). However, as we can observe from the figures, a Stokes curve emanates and ends at the same turning point and forms a closed loop around $x = 0$ when $\arg \eta = 0$. An explicit expression of the Stokes curves for the case $\nu = i$ and $\arg \eta = 0$ is given by

$$\text{Im} \left( w + \frac{1}{2} \log \frac{|w - i|}{|w + i|} \right) = 0. \tag{2.14}$$

Here we set $x + \nu^2 = w^2$. In general, it is known that such a loop-type Stokes curve appears around a double pole $b$ of $Q_0(x) dx^2$ when $\text{Res}_{x=b} \sqrt{Q_0(x)} dx \in i \mathbb{R}$ ([22]). Moreover, the topologies of Stokes graphs are different when we vary $\arg \eta$; in particular, the configuration of a Stokes curve emanating from $x = -\nu^2$ abruptly changes at $\arg \eta = 0$.

As is inferred from the above facts on Borel summability, such a loop-type Stokes curve may break the Borel summability of the WKB solutions (cf., Assumption (A) in Theorem 2.1 (i)): In fact, we will see that, although $x$ lies inside of the loop, the WKB solution $\psi_{\pm}$ are not Borel summable since the Borel transform $\psi_{\pm, B}$ of $\psi_{\pm}$ has infinitely many singularities on the path of integral to define the Borel sum \((2.13)\). Our main aim in this article is to analyze the singularity structures of the Borel transform $\psi_{\pm, B}$ when the Stokes graph contains a loop-type Stokes curve. We analyze the particular example \((2.2)\) in
this section, and generalize the result to general Schrödinger equations by WKB theoretic transformation established in [23] (see Section 3).

Remark 2.2. The singularities of Borel transform $\psi_{\pm, B}$ are closely related to the Stokes phenomenon (for a divergent series of $\eta^{-1}$) to be observed on the WKB solutions. Alien derivatives are closely related to connection formula which describes discontinuous change of the Borel sum of WKB solutions caused by the Stokes phenomenon.

2.2. Singularities of the Borel transform of the WKB solutions

In this subsection we consider the following situation:

- $v \in iR_{\neq 0}$ (cf. Condition (3) in Introduction). This implies that a Stokes curve of loop-type appears when $\arg \eta = 0$. 

$\arg \eta = 0.$

$\arg \eta = 0.05.$

$\arg \eta = -0.05.$

$\arg \eta = 0.1.$

$\arg \eta = -0.1.$

Figure 2.1. Stokes graphs of the equation (2.1) for $v = i$ with several $\arg \eta$. A loop-type Stokes curve appears when $\arg \eta = 0$. 

$\arg \eta = 0.$

$\arg \eta = 0.05.$

$\arg \eta = -0.05.$

$\arg \eta = 0.1.$

$\arg \eta = -0.1.$

$\arg \eta = 0.$

$\arg \eta = 0.05.$

$\arg \eta = -0.05.$

$\arg \eta = 0.1.$

$\arg \eta = -0.1.$
The independent variable $x$ is fixed at any point inside of the loop-type Stokes curve, and away from the origin.

The WKB solutions are normalized at the simple turning point $-v^2$; that is,

$$
\psi_{\pm}(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}(x, \eta)}} \exp\left( \pm \int_{-v^2}^{x} S_{\text{odd}}(x, \eta) dx \right).
$$

(2.15)

**Remark 2.3.** Since $S_{2n-1}(x)$ has the square root type singularity at the simple turning point $x = -v^2$ of (2.1), we define the integral in (2.15) as a contour integral:

$$
\int_{-v^2}^{x} S_{\text{odd}}(x, \eta) dx = \frac{1}{2} \int_{C_x} S_{\text{odd}}(x, \eta) dx.
$$

(2.16)

Here $C_x$ is a path on the Riemann surface of $\sqrt{Q_0(x)}$ depicted in Figure 2.2.

In what follows we regard the Borel transform $\psi_{\pm, B}(x, y)$ of the WKB solution (2.15) as a function of $y$ (after fixing $x$ as above), and discuss their singularities on $y$-plane. Note that our result does not depend on the choice of $x$.

Firstly, we have

**Proposition 2.4.** In the situation specified above the Borel transform $\psi_{\pm, B}(x, y)$ of the WKB solution (2.15) can be extended analytically as a multi-valued function (as a function of $y$) on the region

$$
\{ y \in \mathbb{C} | -\epsilon < \text{Im}(y \pm a(x)) < +\epsilon \} \setminus \Omega_\pm,
$$

for a sufficiently small $\epsilon > 0$ and

$$
\Omega_\pm = \{ y = \mp a(x) + 2m\pi iv | m \in \mathbb{Z} \}.
$$

(2.17)

Moreover, $\psi_{\pm, B}(x, y)$ is singular at each point in $\Omega_\pm$. In particular, the WKB solution (2.15) is not Borel summable when $\arg v = \pi/2$.

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**Figure 2.2.** The path $C_x$. The wiggly line designates a branch cut to define the branch of $\sqrt{Q_0(x)}$. The solid part (resp., dotted part) of $C_x$ depicts the part of $C_x$ on the first sheet (resp., second sheet) on the Riemann surface of $\sqrt{Q_0(x)}$. 

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To show Proposition 2.4, we introduce the Voros coefficient of the equation (2.1) defined by

\[ V(n, h) = \int_{-\nu^2}^{0} \left( S_{\text{odd}}(x, \eta) - \eta \sqrt{Q_0(x)} \right) dx. \]  

Here the integral from the simple turning point \( x = -\nu^2 \) is defined by the same way as (2.16). We take the branch of square root \( \sqrt{Q_0(x)} \) so that

\[ \sqrt{Q_0(x)} = \frac{\nu}{2x} (1 + O(|x|)) \]

holds on the first sheet when \( x \) tends to 0. Note also that, since we can show that \( S_{2n-1}(x) \) with \( n \geq 1 \) is holomorphic at \( x = 0 \) by induction on \( n \), it is integrable at the origin.

The following theorem plays a crucially important role in the proof of Proposition 2.4. (and also in studying singularity structure of Borel transform \( \psi_{\pm, B} \) of the WKB solutions of (2.1) which will be discussed in next subsection).

**Theorem 2.5.** The Voros coefficient (2.19) has the following explicit expression:

\[ V(n, \eta) = - \sum_{n=0}^{\infty} \frac{B_{2n}}{2n(2n-1)} (\nu \eta)^{1-2n}. \]

Here \( B_{2n} \) is the \( 2n \)-th Bernoulli number defined by

\[ \frac{w}{e^w - 1} = 1 - \frac{1}{2} w + \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} w^{2n}. \]

Theorem 2.5 is the counterpart of Sato’s conjecture (cf. [25]) for the equation (2.1). Theorem 2.5 is proved in Appendix A.

**Proof of Proposition 2.4.** First, let \( \psi_{\pm}^{(0)}(x, \eta) \) be the WKB solution normalized at the origin:

\[ \psi_{\pm}^{(0)}(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}(x, \eta)}} \exp \left( \pm \eta \int_{-\nu^2}^{x} \sqrt{Q_0(x)} dx \right) \pm \int_{0}^{x} (S_{\text{odd}}(x, \eta) - \eta \sqrt{Q_0(x)}) dx. \]

When \( x \) lies inside of the loop, the integration path from 0 to \( x \) in (2.23) never intersects the loop-type Stokes curve. Therefore, \( \psi_{\pm}^{(0)}(x, \eta) \) is Borel summable since the assumption (A) in Theorem 2.1 (i) is satisfied. Hence, the Borel
transform \( \psi^{(0)}_\pm(b, x, y) \) of \( \psi^{(0)}_\pm(x, \eta) \) has no singularities on the region \( \{ y \in \mathbb{C} \mid -\epsilon < \text{Im}(y + a(x)) < +\epsilon \} \) except for \( y = \mp a(x) \) for a sufficiently small \( \epsilon > 0 \).

The WKB solutions (2.15) and (2.23) are related as
\[
\psi_\pm(x, \eta) = e^{\pm V} \psi^{(0)}_\pm(x, \eta),
\]
where \( V = V(v, \eta) \) is the Voros coefficient (2.19). Then, taking the Borel transform, we have
\[
\psi_{\pm, B}(x, y) = ((e^{\pm V})_B \ast \psi^{(0)}_{\pm, B})(x, y) = \int_{\mp a(x)}^{y} (e^{\pm V})_B(y - t) \psi^{(0)}_{\pm, B}(x, t) dt,
\]
where \((e^{\pm V})_B\) is the Borel transform of the formal series \( e^{\pm V} \) and \( \ast \) is the convolution product for the variable \( y \) (see Remark 2.6). As is mentioned above, \( \psi^{(0)}_{\pm, B} \) is holomorphic on the path of integrals defining the Borel sum. Thus, the singularity of \( \psi_{\pm, B}(x, y) \) comes from those of \((e^{\pm V})_B\).

Thanks to Theorem 2.5, we can compute the Borel transform \( V_B \) of \( V \) explicitly as follows:
\[
V_B(v, y) = -\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} y^{1-2n} \frac{y^{2n-2}}{(2n-2)!} = -\left( \frac{y/v}{e^{y/v} - 1} - 1 + \frac{y}{2v} \right) \frac{v}{y^2}
\]
Here we used the definition (2.22) of the Bernoulli numbers. Since \( V_B \) has simple poles at \( y = 2m\pi iv \) for each \( m \in \mathbb{Z}_{\neq 0} \) and holomorphic except for these points, so is \((e^{\pm V})_B\) (see [20, Example 27.4]). Then, the relation (2.25) proves Proposition 2.4.

**Remark 2.6.** Let \( f(y) \) (resp., \( g(y) \)) be a germ of holomorphic function (or a function with a mild singularity) defined near \( y = a \) (resp., \( y = b \)); that is, \( f(y) = \sum_{n \geq 0} f_n(y - a)^n \) (resp., \( g(y) = \sum_{n \geq 0} g_n(y - b)^n \)). Then, the convolution product \( \ast \) for \( f \) and \( g \) is defined by
\[
(f \ast g)(y) = \int_{a}^{y-b} f(t) g(y - t) dt = \int_{b}^{y-a} g(t) f(y - t) dt = (g \ast f)(y)
\]
when the domains of convergence of \( f \) and \( g \) are large enough to define the above integral.

Proposition 2.4 tells that the relative location of singular points of \( \psi_{\pm, B} \) from the reference point \( y = \mp a(x) \) are independent of \( x \) (but depend on \( v \)). Such singularities are called fixed singularity (cf., [7]). In the next subsection, to analyze the singularity structure of the Borel transform \( \psi_{\pm, B} \), we compute the alien derivatives (at these fixed singularities) of the Borel transform of the WKB solutions (2.15) normalized at the simple turning point \(-v^2\).
Remark 2.7. If we take the point \( x \) outside of the loop-type Stokes curve, then the WKB solutions (2.15) are Borel summable: This is because the path \( C_x \) can be deformed to a path which never intersects with the loop-type Stokes curve (see Figure 2.3). Hence, in that case the alien derivative becomes trivial. However, as we will see later, the WKB solutions (2.15) are not Borel summable when \( x \) lies inside of the loop-type Stokes curve.

2.3. Alien derivatives of the Borel transform of the WKB solutions

We impose the same assumptions given in the beginning of the previous subsection. We also assume that \( \text{Im} \nu > 0 \) to specify the situation.

Here we briefly review the definition of alien derivatives introduced by Écalle [8] (see also [20], [24], [25]). We only consider \( \psi_+ \) and its Borel transform since \( \psi_- \) can be treated in a similar way. Let \( \Omega = \{ y = -a(x) - 2m\pi i \nu \mid m \in \mathbb{Z}_{\neq 0} \} \) be the set of the fixed singularities of \( \psi_+ \), and \( \omega_m = -a(x) - 2m\pi i \nu \) (for some \( m \in \mathbb{Z}_{\geq 1} \)) be one of the fixed singularities of \( \psi_+ \) lying on the half line \( \{ y = -a(x) + \rho \mid \rho \geq 0 \} \). Then, the alien derivative \( D_{y = \omega_m} \psi_+ \) of \( \psi_+ \) (in the convolutive model) at the point \( y = \omega_m \) is defined by

\[
(2.28) \quad (D_{y = \omega_m} \psi_+)(y) = \sum_{(e_1, \ldots, e_{m-1}) \in \{ \pm \}^{m-1}} \frac{p_+!p_-!}{m!} \text{sing}_{y = \omega_m} (\text{cont}_{(e_1, \ldots, e_{m-1})} \psi_+)(y),
\]

where

- \( \text{cont}_{(e_1, \ldots, e_{m-1})} \psi_+ \) is a germ of analytic function defined near \( y = \omega_m - \delta \) for a sufficiently small \( \delta > 0 \) given as the analytic continuation of the germ \( \psi_{+ \nu}(x, y) \) defined near \( y = -a(x) \) along the path \( \gamma_{(e_1, \ldots, e_{m-1})} \) defined as follows: For a given \( (e_1, \ldots, e_{m-1}) \in \{ \pm \}^{m-1} \), the path \( \gamma_{(e_1, \ldots, e_{m-1})} \) starts form \( -a(x) \) to \( \omega_m - \delta \) getting around the intermediate singularities \( y = -a(x) - 2k\pi i \nu \) (for \( k = 1, \ldots, m - 1 \)) with small upper (resp., lower) half circle when \( e_k = + \) (resp., \( e_k = - \)), as indicated in Figure 2.4. It follows from Proposition 2.4 that \( \text{cont}_{(e_1, \ldots, e_{m-1})} \psi_+ \) is well-defined for any \( (e_1, \ldots, e_{m-1}) \). Moreover, \( p_+ \) stands for the number of signs \( \pm \) in the given \( (e_1, \ldots, e_{m-1}) \).
For the germ $f = \text{cont}_{e_1, \ldots, e_m} \Psi_+, B$, its singularity $\text{sing}_y = \omega_m(f)$ at $y = \omega_m$ is defined by

$$\text{sing}_y = \omega_m(f)(y) = (\text{cont}_{e_1, \ldots, e_m} \phi - \text{cont}_{e_1, \ldots, e_m} \phi)(y - 2m\pi i v).$$

Here $\text{cont}_{e_1, \ldots, e_m} \phi$ (resp., $\text{cont}_{e_1, \ldots, e_m} \phi$) is the branch defined by an analytic continuation of $\phi$ along the path $\gamma_{e_1, \ldots, e_m}$ (resp., $\gamma_{e_1, \ldots, e_m}$) from $\omega_m - \delta$ to $\omega_m + \delta$ avoiding $\omega_m$ from the above (resp., below).

Note that, due to the shift $y \rightarrow y - 2m\pi i v$, $\text{sing}_y = \omega_m(f)$ (and hence $\Delta_{y = \omega_m} \psi_{e_1, \ldots, e_m}$) is a germ of holomorphic function defined near $y = -a(x) + \delta$ for a sufficiently small $\delta > 0$. We also introduce the dotted alien derivatives ([20, § 29.2])

$$\dot{A}_{y = \omega_m} = \tau_m A_{y = \omega_m},$$

where $\tau_m$ is a shift operator $f(y) \rightarrow f(y + 2m\pi i v)$. Hence $\dot{A}_{y = \omega_m} \psi_{e_1, \ldots, e_m}(x, y)$ is a germ of holomorphic function defined near $y = \omega_m + \delta$. We note that, in the articles [2], [17], [25] the notation $\Delta_{y = \omega_m}$ is used for our dotted alien derivative $\dot{A}_{y = \omega_m}$.

Alien derivatives are originally introduced as the logarithm of the Stokes automorphism (cf., [8], [20]; see also (2.37) below). One of important property is that alien derivatives satisfy the Leibnitz rule:

$$\Delta_{y = \omega_m}(f \ast g) = (\Delta_{y = \omega_m} f) \ast g + f \ast (\Delta_{y = \omega_m} g).$$

See [20] for more properties of alien derivatives.

**Remark 2.8.** We follow the formulation given by [25] in the above definition of alien derivatives. In [20], the alien derivatives are defined in a different manner for resurgent functions with simple singularities. These two definitions are essentially the same, but slightly different due to the factor $\eta^{-1/2}$ in the WKB solution (2.10). We summarize the relation between them in Appendix B.

**Remark 2.9.** Alien derivatives can also be defined in the formal model through the Borel transformation operator $B$. That is, the alien derivatives in the formal model are operator acting on the space of (resurgent) formal power series whose action is defined so that

$$A_{y = \omega_m}^{\text{Conv}} \circ B = B \circ A_{y = \omega_m}^{\text{formal}}.$$
holds (cf., [20, §27]). Here $A_{y=\omega_m}^{\text{conv}}$ (resp., $A_{y=\omega_m}^{\text{formal}}$) is the alien derivative in the convolutive model (resp., formal model). We identify alien derivatives in the convolutive model with those in the formal model, and use the same symbol $A_{y=\omega_m}$ for simplicity.

Our main result in this section is the following:

**Theorem 2.10.** Under the situation as above, for the Borel transform $\psi_{\pm, B}(x, y)$ of the WKB solution $\psi_{\pm}(x, y)$ with the normalization (2.15), the alien derivative at the fixed singularity $\omega_m = \mp a(x) - 2m\pi i$ $(m \in \mathbb{Z}_{\geq 1})$ is given by

\[
\left(A_{y=\omega_m}\psi_{\pm, B}\right)(x, y) = \pm \frac{1}{m} \psi_{\pm, B}(x, y).
\]

(2.33)

\[
\left(A_{y=\omega_m}\psi_{\pm, B}\right)(x, y) = \pm \frac{1}{m} \psi_{\pm, B}(x, y + 2m\pi i).
\]

(2.34)

**Proof.** It follows from the explicit expression (2.26) that $V_B$ has simple pole at $y = 2m\pi i$ for each $m \in \mathbb{Z}_{\neq 0}$ and satisfies

\[
\text{Res}_{y=-2m\pi i} V_B(y)dy = \frac{1}{2m\pi i}.
\]

(2.35)

Then, a similar argument presented in [25, Theorem 2.2] or [17, Theorem 4.1] (by using the alien calculus; e.g., [20, Theorem 30.9, Example 30.1]) gives

\[
A_{y=-2m\pi i}^{B}(e^V_B)(y) = \frac{1}{m} (e^V_B)(y).
\]

(2.36)

We also note that the Borel summability of $\psi_{\pm}^{(0)}$ implies $A_{y=\omega_m}\psi_{\pm, B}^{(0)} = 0$. Then, the relation (2.25) and the property (2.31) show the desired equalities (2.33) and (2.34).

\qed

### 2.4. Action of Stokes automorphism

As a corollary of Theorem 2.10, we can describe the action of *Stokes automorphism* on the WKB solutions. The Stokes automorphism $\mathcal{E}$ for the WKB solutions is defined as

\[
\mathcal{E} = \exp \left( \sum_{m=1}^{\infty} A_{y=\mp a(x)-2m\pi i} \right).
\]

(2.37)

Here

$$A_{y=\mp a(x)-2m\pi i} = e^{2\pi iny} A_{y=\mp a(x)-2m\pi i}.$$
is the dotted alien derivative at \( y = \omega_m \) in the formal model (see Remark 2.9). The factor \( e^{2\pi i n} \) is understood as an external symbol satisfying

\[
\hat{A}_{y = \omega_m(x) - 2\pi i n} (e^{2\pi i n} \psi) = e^{2(n+m)\pi i n} A_{y = \omega_m(x) - 2\pi i n} \psi.
\]

When we discuss the Stokes phenomenon, it is natural to extend the alien derivatives to an operator acting on a certain graded algebra consists of formal series with exponential factors as above ([20, §29.4]). Actually, we have

\textbf{Theorem 2.11.} Under the situation as above, the action of the Stokes automorphism \( \Xi \) on the WKB solutions with the normalization (2.15) is given by

\[
(2.38) \quad \Xi \psi_{\pm} = (1 - e^{2\pi i n})^{-1} \psi_{\pm}.
\]

Theorem 2.11 is immediately follows from (2.33) by a similar discussion given in [25, Theorem 2.1] or [17, Theorem 4.2]. Similarly, we also have

\[
(2.39) \quad \Xi \exp(V) = (1 - e^{2\pi i n})^{-1} \exp(V).
\]

(Cf., [20, Example 30.1].)

\textbf{Remark 2.12.} In [7] the Stokes automorphism is defined in a different manner: They are defined as the discrepancy of the Borel resummation as

\[
(2.40) \quad \mathcal{S}_{\pm} = \mathcal{S} \circ \Xi.
\]

Here \( \mathcal{S}_{\pm} \) is the lateral Borel resummation operator defined by rotating the path of Laplace integral in (2.13) with angle \( \pm \epsilon \) for a sufficiently small \( \epsilon > 0 \). However, in our situation the lateral Borel sum of the WKB solution for a sufficiently small angle \( \pm \epsilon \) is not well-defined. This is because there are infinitely many \( \epsilon > 0 \) which is small enough such that the Stokes curve forming a logarithmic spiral hits \( x \) when \( \arg \eta = \pm \epsilon \). Thus we adopt (2.37) for the definition of the Stokes automorphism since (2.37) can be defined in our situation.

3. WKB theoretic transformation of Schrödinger equations with a loop-type Stokes curve to a canonical form

In this section we will formulate the WKB theoretic transformation which reduces a general Schrödinger equation to the Bessel-type equation (2.1) on a loop-type Stokes curves. Since we simultaneously deal with two different Schrödinger equations, we put \( \sim \) over variables or functions relevant to the general equation in order to avoid confusion.
We consider the following Schrödinger-type equation with a large parameter $\eta$:

$$
\left( -\frac{d^2}{d\tilde{x}^2} + \eta^2 \tilde{Q}(\tilde{x}, \eta) \right) \tilde{\psi}(\tilde{x}, \eta) = 0.
$$

(3.1)

Here $\tilde{Q} = \tilde{Q}(\tilde{x}, \eta)$ is a finite sum

$$
\tilde{Q} = \sum_{j=0}^{N} \eta^{-j} \tilde{Q}_j(\tilde{x})
$$

of rational functions $\tilde{Q}_j$ satisfying Assumption 1.1 given in Introduction. We denote by $l$ the loop-type Stokes curve around the origin appearing in the Stokes graph of (3.1).

The goal of this section is to analyze the fixed singularities of the WKB solutions of (3.1) normalized at the turning point $p_0$

$$
\tilde{\psi}_\pm(\tilde{x}, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp \left( \pm \int_{p_0}^{\tilde{x}} \tilde{S}_{\text{odd}} \, d\tilde{x} \right)
$$

when $\tilde{x}$ lies inside of $l$. Here $\tilde{S}_{\text{odd}}$ is defined in a similar manner as (2.8) (cf. [2]). For the purpose, we will construct a WKB theoretic transformation from (3.1) to our canonical form

$$
\left( -\frac{d^2}{d\tilde{x}^2} + \eta^2 x + \frac{v(\eta)^2}{4\tilde{x}^2} - \frac{1}{4\tilde{x}^2} \right) \varphi(x, \eta) = 0
$$

(3.3)

in a neighborhood of the closure of the Stokes region bounded by $l$. Here, unlike the equation (2.1) discussed in Section 2, we consider the situation where

$$
v(\eta) = v_0 + v_1 \eta^{-1} + v_2 \eta^{-2} + \cdots
$$

might be an infinite series in $\eta^{-1}$ while $v$ in (2.1) is a genuine constant. To avoid confusion, we call (3.3) the $\infty$-Bessel equation.

### 3.1. Construction of the transformation series

Since two of the Stokes curves emanating from $p_0$ form the loop, we can apply Proposition 2.2 in [23] to construct a formal transformation from (3.1) to (3.3) in an annular neighborhood of the loop $l$. Since the canonical form employed in [23] has a slightly different form in appearance, we restate the construction here. The construction is based on an argument given in [1].
Proposition 3.1. There exists a small open annular neighborhood $V$ of the loop $\ell$ and two formal power series

\begin{equation}
\sum_{j \geq 0} x_j(\tilde{x}) \eta^{-j}, \quad \sum_{j \geq 0} v_j \eta^{-j}
\end{equation}

which enjoy the following properties:

1. $x(\tilde{x}, \eta)$ is Gevrey 1 in the sense of (2.11), and each $x_j(\tilde{x})$ ($j \geq 1$) is holomorphic on $V$.

2. The function $x_0(\tilde{x})$ is holomorphic on $V$ and satisfies

\begin{equation}
x'_0(\tilde{x}) \neq 0 \quad \text{on } V,
\end{equation}

and

\begin{equation}
x_0(p_0) = -\nu_0^2,
\end{equation}

where $\nu_0$ is the leading term of $v(\eta)$ which is a non-zero pure imaginary number defined by

\begin{equation}
\nu_0 = 2 \text{Res}_{\tilde{x} = 0} \sqrt{\tilde{Q}_0(\tilde{x})} d\tilde{x}
\end{equation}

with a suitable choice of the branch of square root (cf., (3.10) below).

3. The following relation between formal series of $\eta^{-1}$ holds when $\tilde{x} \in V$:

\begin{equation}
\tilde{Q}(\tilde{x}, \eta) = \left(\frac{\partial x}{\partial \tilde{x}}(\tilde{x}, \eta)\right)^2 \left(\frac{x(\tilde{x}, \eta) + v(\eta)}{4x(\tilde{x}, \eta)^2} - \eta^{-2} \frac{1}{4x(\tilde{x}, \eta)^2}\right)
\end{equation}

\begin{equation}
- \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \eta); \tilde{x}\}.
\end{equation}

Here $\{x; \tilde{x}\}$ designates the Schwarzian derivative defined as follows:

\begin{equation}
\{x; \tilde{x}\} = \frac{x'''}{x'} - \frac{3}{2} \left(\frac{x''}{x'}\right)^2.
\end{equation}

Here and hereafter we denote $d^2 x_j / d\tilde{x}^2$ by $x_j^{(2)}$ ($x = 0, 1, 2, \ldots$). We also denote, as usual, $x_j^{(1)}, x_j^{(2)}, \ldots$ by $x_j, x_j', x_j'', \ldots$, respectively.

We review the definitions of holomorphic functions $x_n(\tilde{x})$ ($n \geq 0$) in an annular domain $V$ and a formal series $v(\eta)$ of $\eta^{-1}$ with constant coefficients. It is noteworthy that, in the argument, we determined each $v_n$ in order that $x_n(\tilde{x})$ to be holomorphic in $V$. The function $x_0(\tilde{x})$ is determined as a biholomorphic function on $V$ by solving the following differential equation for unknown function $z = z(\tilde{x})$:

\begin{equation}
\tilde{Q}_0(\tilde{x}) = \frac{z + \nu_0^2}{4z^2} (z')^2,
\end{equation}
where \( v_0 \) is defined by (3.7) under suitable choice of the branch of \( \sqrt{Q_0} \). Note that, taking square root and integration of the both members, we have

\[
\int_{p_0}^{\tilde{x}} \sqrt{Q_0(\tilde{x})} d\tilde{x} = \int_{-v_0^2}^{v_0^2} \frac{\sqrt{z + v_0^2}}{2z} dz.
\]  

(3.10)

Here the branch of the square root in the right-hand side is taken so that \( \sqrt{z + v_0^2} \sim v_0 \) holds near the origin.

If \( x_j(\tilde{x}) \ (j < n) \) are obtained, \( x_n(\tilde{x}) \) is defined as a holomorphic solution in \( V \) of a differential equation of the form

\[
(x_0 + v_0^2) x_n' - \frac{2v_0^2 + x_0}{2x_0} x_0' x_n = -x_0^2 v_n v_0 + \frac{2x_0^2}{x_0} \tilde{Q}_n(\tilde{x}) + R_n(\tilde{x}).
\]  

(3.11)

Here \( R_n(\tilde{x}) \) is a holomorphic function in \( V \) depending on \( x_j(\tilde{x}) \ (j < n) \). See [23] for the concrete form of \( R_n \). (Some factors concerning \( v_j \) should be trivially modified.) Using a new coordinate \( t = x_0(\tilde{x}) \) and replacing \( 2x_0^2/x_0^2 \tilde{Q}_n(\tilde{x}) + R_n(\tilde{x})/x_0' \) with \( \tilde{R}_n(t) \) in (3.11), we obtain

\[
(t + v_0^2) \frac{dx_n}{dt} - \frac{2v_0^2 + t}{2t} x_n = -v_n v_0 + \tilde{R}_n(t).
\]  

(3.12)

Then the function \( x_n \) and the constant \( v_n \) are obtained in the following forms:

\[
x_n = \frac{t}{(t + v_0^2)^{1/2}} \int_{-v_0^2}^{t} \frac{-v_n v_0 + \tilde{R}_n(s)}{s(s + v_0^2)^{1/2}} ds,
\]  

(3.13)

\[
v_n = -\lambda^{-1} \tilde{\lambda}_n,
\]  

(3.14)

where we set

\[
\lambda = \int_{\gamma_0} \frac{-v_0}{s(s + v_0^2)^{1/2}} ds = -2\pi i,
\]  

(3.15)

\[
\tilde{\lambda}_n = \int_{\gamma_0} \frac{R_n(s)}{s(s + v_0^2)^{1/2}} ds,
\]  

(3.16)

and \( \gamma_0 \) is a positively oriented small circle around the origin. See [23] for the estimation of \( x_n \) and \( v_n \).

As is proved in [23], \( x_n(\tilde{x}) \) are holomorphic in \( V \). Now we shall show that \( x_n(\tilde{x}) \) can be analytically continued to a neighborhood of the origin containing \( V \). Set \( \tilde{V} = V \cup D \), where \( D \) is one of connected components of \( C \setminus \phi \) which contains the origin.
Proposition 3.2. (1) The functions \( x_n \) (\( n = 0, 1, 2, \ldots \)) are holomorphic in \( \tilde{V} \). Moreover \( x_0(0) = 0 \) and \( x_0'(\tilde{x}) \neq 0 \) \( (\tilde{x} \in \tilde{V}) \) hold.

(2) The series \( x(\tilde{x}, \eta) = \sum_{n=0}^{\infty} x_n(\tilde{x}) \eta^{-n} \) is Gevrey 1 with respect to \( \eta^{-1} \) on \( \tilde{V} \).

(3) The series \( v(\eta) = \sum_{n=0}^{\infty} v_n \eta^{-n} \) is convergent.

Proof. By the definition, \( x_0 \) is a biholomorphic function defined in \( V \). It maps the loop-type Stokes curve of (3.1) to that of the canonical form. If we vary the argument of \( \eta \) so that one of the Stokes curves emanating from \( p_0 \) goes to the origin, that Stokes curve is mapped to a Stokes curve of (3.3) which is also going to the origin (cf., Figure 2.1). Since \( x_0 \) does not depend on \( \eta \), this means that \( x_0 \) can be analytically continued to \( \tilde{V}_0 \) as a single-valued function and \( \lim_{\tilde{x} \to 0} x_0(\tilde{x}) = 0 \). This also implies \( x_0 \) is bounded inside the loop \( \ell \). Thus the origin is a removable singularity of \( x_0 \). Hence the function \( x_0 \) has an expansion the form

\[
x_0(\tilde{x}) = a(\tilde{x}) + O(\tilde{x}), \quad (\tilde{x} \to 0, a \neq 0).
\]

Here \( m \) is a non-negative integer. By the definition of \( v_0 \), we have

\[
\sqrt{Q_0(\tilde{x})} = \frac{1}{\tilde{x}} \left( \frac{v_0}{2} + O(\tilde{x}) \right).
\]

Putting these expansions into the equation

\[
(3.17) \quad \sqrt{Q_0(\tilde{x})} = \frac{\sqrt{x_0(\tilde{x}) + v_0^2}}{2x_0(\tilde{x})} x_0'(\tilde{x})
\]

and comparing the coefficients of the leading terms of the both members, we have \( m = 1 \). This implies \( x_0' \neq 0 \) at the origin. Furthermore, \( x_0' \neq 0 \) also holds on \( \tilde{V} \) thank to (3.17).

Next we prove that the functions \( x_n \) (\( n \geq 1 \)) are holomorphic in \( \tilde{V} \) by induction. By the construction of \( x_n \), they are holomorphic in \( \tilde{V}_0 \). We suppose that \( x_j \) (\( 0 \leq j \leq n - 1 \)) are holomorphic at the origin. Then \( \tilde{R}_n \) which appeared in (3.12) is holomorphic in \( \tilde{V} \) and it can be written as

\[
\tilde{R}_n(t) = \tilde{R}_n(0) + t h_n(t),
\]

where \( h_n(t) \) is a holomorphic function at the origin. Then we compute the integrand of (3.13). Since \( R_n \) is holomorphic at the origin, (3.16) yields

\[
\tilde{\lambda}_n = \frac{2\pi i}{v_0} \tilde{R}_n(0).
\]
By the definition, we have \(v_n v_0 = \tilde{R}_n(0)\) and hence we obtain
\[-v_n v_0 + \tilde{R}_n(s) = -\tilde{R}_n(0) + \tilde{R}_n(0) + s h_n(s) = s h_n(s)\]
Therefore, the integrand of (3.13) is integrable at the origin and hence \(x_n\) is holomorphic there.

In [23], it is proved that for any compact set \(K\) in \(V\), there is a positive constant \(C\) such that
\[
\sup_{x \in K} |x_n(\tilde{x})| \leq n! C^{n+1}
\]
holds for \(n = 0, 1, 2, \ldots\). The maximum principle shows that the same inequality holds for any compact set \(K\) in \(\tilde{V}\).

By our assumption, we may write \(\tilde{Q}_j(\tilde{x}) = \tilde{q}_j/4\tilde{x}^2 + \cdots\) for some constants \(\tilde{q}_j\) when \(\tilde{x}\) tends to 0 \((j = 0, 1, \ldots, N)\). We set \(c(\eta) = v(\eta)^2 - \eta^{-2}\).

**Lemma 3.3.** The formal series \(c(\eta)\) becomes a polynomial in \(\eta^{-1}\):
\[c(\eta) = c_0 + c_1 \eta^{-1} + \cdots + c_n \eta^{-n}\]
and \(c_j = \tilde{q}_j\) holds.

**Proof of Lemma 3.3.** Let \(T^{(\pm)}\) be formal solutions of the Riccati equation
\[
T^2 + \frac{dT}{dx} = \eta^2 \frac{x + c(\eta)}{4x^2},
\]
and set \(T_{\text{odd}} = (T^{(+)} - T^{(-)})/2\) (cf., (2.8)). Then we obtain the following relation (cf., [23]):
\[
\tilde{S}_{\text{odd}}(\tilde{x}, \eta) = \left(\frac{\partial x(\tilde{x}, \eta)}{\partial \tilde{x}}\right) T_{\text{odd}}(x(\tilde{x}, \eta), \eta).
\]
Here the branch of the leading term of \(T_{\text{odd}}\) is chosen consistently. Taking the contour integral around the origin of the both members, we obtain
\[
\pi i \eta \sqrt{\sum_{j=0}^{N} \eta^{-j} \tilde{q}_j + \eta^{-2}} = \pi i \eta \sqrt{c + \eta^{-2}}.
\]
(Cf., [14, Proposition 3.6].) Thus we have \(c(\eta) = \sum_{j=0}^{N} \eta^{-j} \tilde{q}_j\), and hence \(c_j = \tilde{q}_j\).

Therefore we have
\[v^2(\eta) - \eta^{-2} = c(\eta)\]
and we conclude the infinite series \(v(\eta)\) becomes a convergent series. This completes the proof of Proposition 3.2.
We also note that the formal power series $v(\eta)$ has the following expression in terms of $\tilde{S}_{\text{odd}}$:

$$v(\eta) = 2\eta^{-1} \text{Res}_{\tilde{x}} \tilde{S}_{\text{odd}} \, d\tilde{x}. \tag{3.21}$$

**Remark 3.4.** It follows from the regularity of the Borel summation method, $v(\eta)$ is Borel summable and its Borel sum coincides with $v(\eta)$.

Let

$$\varphi_{\pm}(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \pm \int_{-v_0^2}^{x} S_{\text{odd}} \, dx \right) \tag{3.22}$$

be the WKB solution of (3.3) with $v(\eta)$ being constructed in Proposition 3.1, which is normalized at the (simple) turning point $x = -v_0^2$. The infinite series $x(\tilde{x}, \eta)$ constructed above defines a transformation of WKB solutions $\tilde{\psi}_{\pm}$ of (3.1) normalized as (3.2) to WKB solutions $\psi_{\pm}$ of (3.3) normalized as (3.22) (cf. [1], [2], [14]).

**Theorem 3.5.** Under the notation as above, the relation

$$\tilde{\psi}_{\pm}(\tilde{x}, \eta) = \left( \frac{\vartheta x(\tilde{x}, \eta)}{\vartheta \tilde{x}} \right)^{-1/2} \varphi_{\pm}(x(\tilde{x}, \eta), \eta) \tag{3.23}$$

holds on the domain $\tilde{V} \setminus \{0\}$.

Let us also mention about the Borel summability of the formal series $x(\tilde{x}, \eta)$ proved by S. Sasaki recently:

**Theorem 3.6 ([18]).** Under Assumption 1.1, the series $x(\tilde{x}, \eta)$ is Borel summable near any point $\tilde{x}$ in the region $V$. In particular, the Borel transform $x_B(\tilde{x}, y)$ is holomorphic on a domain

$$\bigcup_{s \geq 0} \{(\tilde{x}, y) \in V \times C \mid |y - s| < \varepsilon \}$$

with a sufficiently small $\varepsilon > 0$.

Using the fact that the coefficients $x_n(\tilde{x})$ of $x(\tilde{x}, \eta)$ are holomorphic on $\tilde{V}$ (see Proposition 3.2 (1)), we can show that $x(\tilde{x}, \eta)$ is Borel summable near any point in $\tilde{V}$ by the Hartogs-type theorem for the Borel summability due to Kamimoto and Koike (see [13, Theorem 2.9]). Therefore, $x_B(\tilde{x}, y)$ is in fact holomorphic on the domain $\bigcup_{s \geq 0} \{(\tilde{x}, y) \in \tilde{V} \times C \mid |y - s| < \varepsilon \}$ with a sufficiently small $\varepsilon > 0$. We will employ this result in next subsection to compute the alien derivatives of the Borel transform $\tilde{\psi}_{\pm, B}(\tilde{x}, y)$ of the WKB solution of the general equation (3.1).
3.2. Alien derivatives of WKB solutions

In this section we will describe the alien derivatives at fixed singularities of the WKB solution $\hat{\psi}(\tilde{x}, \eta)$ of the general equation (3.1).

In what follows we consider a similar situation to that in Section 2.2:
- The independent variable $\tilde{x}$ is fixed at any point inside of the loop-type Stokes curve $\gamma$, and away from the origin.
- The pure imaginary number $v_0$ given in (3.7) satisfies $\text{Im} \ v_0 > 0$.

Then, as is explained in the end of Section 2.2, the WKB solution (3.2) is not Borel summable due to the existence of the loop-type Stokes curve $\gamma$. In what follows, as similar to Sections 2.2 and 2.3, we regard $\hat{\psi}(\tilde{x}, h)$ as a function of $y$ (after fixing $\tilde{x}$ as above), and analyze the fixed singularity on the $y$-plane with the aid of the WKB theoretic transformation established in the previous subsection.

Firstly, we recall the relationships between two of
(a) the WKB solutions $\hat{\psi}_\pm(\tilde{x}, \eta)$ of the general equation (3.1) (normalized at $\tilde{x} = p_0$ as in (3.2)),
(b) the WKB solutions $\varphi_\pm(x, \eta)$ of $\infty$-Bessel equation (3.3) with $v(\eta)$ constructed in Proposition 3.1 being substituted (normalized at $x = -v_0^2$ as in (3.22)), and
(c) the WKB solutions $\psi_\pm(x; \eta; v)$ of the Bessel-type equation (2.1) with $v$ being a complex parameter (normalized at $x = x_0(\tilde{x})$ as in (2.15)).

Here, to show the dependency on $v$ more manifestly, we use the notation which is different from that used in Section 2. Note that, the above condition for $\tilde{x}$ implies that $x = x_0(\tilde{x})$ lies inside of the loop-type Stokes curve of (2.1). Hence, we can use the results for $\hat{\psi}_\pm(x, \eta; v)$ established in Sections 2.2 and 2.3.

The relation between (a) and (b) has already been described in Theorem 3.5. On the other hand, since the equations (3.3) and (2.1) are related by the substitution $v \mapsto v(\eta)$, the WKB solutions in (b) and (c) satisfies

$$
\varphi_\pm(x, \eta) = \psi_\pm(x; \eta; v(\eta)).
$$

As is shown in [1, 2], the relation (3.23) (or (3.24)) through the WKB theoretic transformation is converted into a relation between the Borel transformed WKB solutions via a microdifferential operator ([19]), where the large parameter $\eta$ plays the role of the symbol of $\partial/\partial y$ in the Borel transform. We have shown in Proposition 3.2 that the formal power series $x(\tilde{x}, \eta)$ and $v(\eta)$ satisfy the similar estimate to [2, Theorem B.4] (our $v(\eta)$ plays the similar role of $E(\eta)$ in [2]; our $v(\eta)$ satisfies stronger estimate since $E(\eta)$ in [2] may be divergent). Therefore, the arguments given in [2] enable us to compute the
alien derivatives for the WKB solution $\psi_{\pm}(\hat{x}, \eta)$ of the general equation (3.1) as follows.

By a similar argument to Section 4 in [2], the Borel transforms $\varphi_{\pm,B}$ of the WKB solutions $\varphi_{\pm}$ of (3.3) and the Borel transforms $\psi_{\pm,B}$ of the WKB solutions $\psi_{\pm}$ are related by a microdifferential operator $\mathcal{T}$ defined by

$$ (3.25) \quad \mathcal{T} = \exp(\tilde{\psi}(\eta)\theta), $$

as

$$ (3.26) \quad \varphi_{\pm,B}(x, y) = \mathcal{T}\psi_{\pm,B}(x, y; v_0). $$

Here $\tilde{\psi}$ denotes $\sum_{j \geq 1} v_j \eta^{-j} = \psi(\eta) - v_0$, $\theta$ being the symbol of $\partial/\partial v_0$ and we used the notation $*: \ast$ of the normal ordered product (cf. [2], [4]). Although $v_0$ is a constant, we regard it as a variable lying on a sufficiently small neighborhood of the original $v_0$ when we consider the action of the microdifferential operator. The action of the microdifferential operator $\mathcal{T}$ is represented by an integral transformation

$$ (3.27) \quad \mathcal{T}\psi_{\pm,B}(x, y; v_0) = \psi_{\pm,B}(x, y; v_0) $$

$$ + \int_{y}^{y_0} K_{\mathcal{T}} \left(y - y', \frac{\partial}{\partial v_0}\right) \psi_{\pm,B}(x, y'; v_0) dy', $$

where

$$ (3.28) \quad a(x) = \int_{-v_0^2}^{x} \sqrt{Q_0(x, v_0)} dx \quad \left(= \int_{-v_0^2}^{x} \frac{\sqrt{x + v_0^2}}{2x} dx\right) $$

and $K_{\mathcal{T}}(y, \partial/\partial v_0)$ is an differential operator of infinite order for the variable $v_0$ defined as follows: If we write

$$ (3.29) \quad \exp(\tilde{\psi}(\eta)\theta) = 1 + \sum_{l=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(\frac{\tilde{\psi}(\eta)}{n}\right)^n\right) \theta^l, $$

then

$$ (3.30) \quad K_{\mathcal{T}}(y, \frac{\partial}{\partial v_0}) = \sum_{l=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(\frac{\tilde{\psi}(\eta)}{n}\right)^{n-1} \frac{\tilde{\psi}(\eta)}{n-1}!\right) \left(\frac{\partial}{\partial v_0}\right)^l. $$

The microdifferential operator $\mathcal{T}$ is a-priori defined locally so that the equality (3.27) holds only on $\{\ y \in \mathbb{C} \ | \ 0 < |y \pm a(x)| < \epsilon \}$ for sufficiently small $\epsilon > 0$. However, the convergence of $\tilde{\psi}(\eta)$ (see Proposition 3.3) implies that the convolution with the kernel function $K_{\mathcal{T}}$ does not generate new singularities
on the \(y\)-plane. Therefore the relation (3.27) holds on a (punctured) strip domain

\[
\bigcup_{s \geq 0} \{ y \in \mathbb{C} \mid |y \pm a(x) - s| < \epsilon \} \setminus \Omega_{\pm},
\]

where \(\Omega_{\pm} = \{ y = -a(x) + 2m\pi i \nu_0 \mid m \in \mathbb{Z} \} \) is the set of the fixed singularities of \(\psi_{\pm, B}\). This implies that \(\varphi_{\pm, B}(x, y)\) also has fixed singularities at the points in \(\Omega_{\pm}\). Moreover we have

**Proposition 3.7.** Let \(\varphi_{\pm}(x, \eta)\) be the WKB solution of \(\infty\)-Bessel equation (3.3) normalized as in (3.22). Then, the alien derivative at the fixed singularity \(y = -a(x) - 2m\pi i \nu_0\) \((m \in \mathbb{Z}_{\geq 1})\) is given by

\[
\begin{align*}
(3.31) & \quad A_{y = -a(x) - 2m\pi i \nu_0} \varphi_{\pm, B}(x, y) = \pm \frac{1}{m} : \exp(2m\pi i \nu(\eta) \eta) : \varphi_{\pm, B}(x, y). \\
(3.32) & \quad \dot{A}_{y = -a(x) - 2m\pi i \nu_0} \varphi_{\pm, B}(x, y) = \pm \frac{1}{m} : \exp(2m\pi i \nu(\eta) \eta) : \varphi_{\pm, B}(x, y, y + 2m\pi i \nu_0). 
\end{align*}
\]

The equality is derived from Theorem 2.10 and (3.27) (cf., [2, Theorem 4.1]).

Next, using Theorem 3.5 and Proposition 3.7, we will compute the alien derivative of the Borel transformed WKB solution \(\tilde{\psi}_{\pm, B}\). Set \(r = r(\tilde{x}, \eta) := x(\tilde{x}, \eta) - x_0(\tilde{x})\). As was employed in [2], [4], we take \(x_0 = x_0(\tilde{x})\) as a new independent variable instead of \(\tilde{x}\) and denote it simply by \(x\). Let \(\tilde{x} = g(x)\) denote the inverse function of \(x_0(\tilde{x})\). In view of (3.10), the phase functions of WKB solutions are related as

\[
(3.33) \quad a(x) = \tilde{a}(g(x)) \quad \text{(i.e., } a(x_0(\tilde{x})) = \tilde{a}(\tilde{x})\text{)},
\]

where \(a(x)\) is defined in (3.28) and

\[
(3.34) \quad \tilde{a}(\tilde{x}) = \int_{p_0}^{\tilde{x}} \sqrt{Q_0(\tilde{x})} d\tilde{x}.
\]

Taking Borel transform of both sides of (3.23), we may write the relation of the Borel transforms of them as follows:

\[
(3.35) \quad \tilde{\psi}_{\pm, B}(g(x), y) = \mathcal{X} \varphi_{\pm, B}(x, y).
\]

Here \(\mathcal{X}\) denotes a microdifferential operator

\[
g'(x)^{1/2} \left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} \exp(r(x, \eta) \xi) : g'(x)^{1/2} \left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} \exp(r(x, \eta) \xi):
\]
and \( \xi \) denotes the symbol of \( \partial / \partial x \) (cf. [2], [4]). As well as (3.27), we can rewrite (3.35) in the form

\[
\tilde{\psi}_{\pm, B}(g(x), y) = g'(x)^{1/2} \varphi_{\mp, B}(x, y) \\
+ \int_{\tilde{\tau}_a(x)}^y K_x \left( x, y - y', \frac{\partial}{\partial x} \right) \varphi_{\mp, B}(x, y') dy'
\]

by using a differential operator \( K_x(x, y, \partial / \partial x) \) of infinite order (cf., [2]). Because of the Borel summability of \( x(x, y) \) (see Theorem 3.6), we can verify that there exists a small constant \( \epsilon > 0 \) such that (3.36) holds on

\[
\bigcup_{s \geq 0} \left\{ y \in \mathbb{C} \mid |y + a(x) - s| < \epsilon \right\} \setminus \Omega_\pm,
\]

and furthermore, the convolution with \( K_x \) preserves the singularity structure of the Borel transformed WKB solution \( \varphi_{\mp, B} \) on the \( y \)-plane. Therefore, the singularities of \( \psi_{\pm, B}(g(x), y) \) are confined to

\[
y = \mp a(x) + 2m\pi i v_0 \quad (m \in \mathbb{Z}).
\]

Combining the above arguments and going back to the original coordinate \( \tilde{x} \), we have

**Theorem 3.8.** For \( m \in \mathbb{Z}_{\geq 1} \) and the Borel transform \( \tilde{\psi}_{\pm, B}(\tilde{x}, y) \) of the WKB solution \( \tilde{\psi}_{\pm}(\tilde{x}, \eta) \) of (3.1) normalized as (3.2), the following relation holds.

\[
\Delta_{y = \mp a(\tilde{x}) - 2m\pi iv_0} \tilde{\psi}_{\mp, B}(\tilde{x}, y) = \pm \frac{1}{m} : \exp(2m\pi iv(\eta)) : \tilde{\psi}_{\mp, B}(\tilde{x}, y),
\]

\[
\Delta_{y = \mp a(\tilde{x}) - 2m\pi iv_0} \tilde{\psi}_{\mp, B}(\tilde{x}, y) = \pm \frac{1}{m} : \exp(2m\pi iv(\eta)) : \tilde{\psi}_{\mp, B}(\tilde{x}, y + 2m\pi iv_0).
\]

This theorem can be obtained by Proposition 3.7 and using a similar computation which is used to obtain Theorem 5.1 in [2] and we do not repeat it.

**3.3. Stokes automorphism for \( \tilde{\psi}_{\pm} \)**

We can compute the action of Stokes automorphism of \( \tilde{\psi}_{\pm} \) by straightforward calculation. We use the same notation \( \Xi \) to designate the Stokes automorphism introduced in Section 2.4.

**Theorem 3.9.** The action of \( \Xi \) on \( \tilde{\psi}_{\pm} \) is given by

\[
\Xi \tilde{\psi}_{\pm} = \left( 1 - \exp(2\pi iv(\eta)) \right)^{\mp 1} \tilde{\psi}_{\pm}.
\]

**Proof.** We compute the action of the Stokes automorphism in the formal model. Let \( \hat{A} \) be the sum of the dotted alien derivatives at fixed singularities
in the formal model:

\[ \Delta = \sum_{m=1}^{\infty} \exp \left( \int_{y=\tilde{y}(x)-2m\pi i} \cdot \right) \]

Then we obtain

| Equation Number | Description |
|-----------------|-------------|
| (3.42)          | \( \dot{\psi}_\pm = \pm \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i m \eta} \beta^{-1} \left( \exp(2\pi i \nu(\eta) \eta) \cdot \psi_{\pm, B}(\tilde{x}, y) \right) \) |
|                 | = \pm \sum_{m=1}^{\infty} \frac{1}{m} \exp(2\pi i \nu(\eta) \eta) \psi_{\pm} |
|                 | = \mp \log(1 - \exp(2\pi i \nu(\eta) \eta)) \psi_{\pm} |

Using the equality \( \Xi = \exp \dot{\Delta} \), we conclude that (3.41) holds.

**Remark 3.10.** Thanks to the equality (3.21), the formula (3.41) can also be written as

| Equation Number | Description |
|-----------------|-------------|
| (3.43)          | \( \Xi \dot{\psi}_\pm = \left( 1 - \exp \left( \int_{\gamma} \mathcal{S}_{\text{odd}}(\tilde{x}', \eta) d\tilde{x}' \right) \right)^{\mp 1} \dot{\psi}_\pm \) |

Here \( \gamma_\ell \) is a closed cycle around the loop \( \ell \) depicted in Figure 3.1. The cycle \( \gamma_\ell \) is called the saddle class for the loop-type Stokes curve in [9, Section 3.2].

**A. Proof of Theorem 2.5**

Here we give a proof of Theorem 2.5 on the explicit expression of the Voros coefficient of the equation (2.1). We emphasize that \( \nu \) appearing in Appendix A is a complex parameter (not a formal series such as (3.4)).

The following proposition is a key in the proof of Theorem 2.5:
Proposition A.1. $V(v, \eta)$ is a formal solution of the following difference equation:

$$V(v + \eta, \eta) - V(v, \eta) = -1 + \left(v\eta + \frac{1}{2}\right) \log \left(1 + \frac{1}{v\eta}\right).$$  \hspace{1cm} (A.1)

Theorem 2.5 immediately follows from Proposition A.1 by the same argument presented in [3]. Namely, the formal power series appearing in the right-hand side of (2.21) is the unique formal power series solution of the difference equation (A.1) without constant terms. Hence, our task is to show Proposition A.1.

To prove Proposition A.1, we use the idea of [25]; that is, we use a shift operator which yields $v \mapsto v + \eta^{-1}$ to compute the Voros coefficient. Hence, in this subsection the parameter $v$ plays an important role. To show the dependency on $v$ more manifestly, we use the notation $S^{(\pm)}(x; \eta; v)$ for solutions (2.6) of the Riccati equation, $\psi_\pm(x; \eta; v)$ for WKB solutions, etc. in what follows.

First, we introduce the following cut-off integrals:

$$I^{(\pm)}(x, \eta; v) = \int_{C_x} S^{(\pm)}(x, \eta; v) dx,$$  \hspace{1cm} (A.2)

$$I_{-1}(x; v) = \int_{C_x} S_{-1}(x; v) dx,$$  \hspace{1cm} (A.3)

where $C_x$ is a contour shown in Figure 2.2. Then, we have

$$V(v, \eta) = \frac{1}{2} \lim_{x \to 0} \left( \frac{I^{(+)}(x, \eta; v) - I^{(-)}(x, \eta; v)}{2} - \eta I_{-1}(x; v) \right).$$  \hspace{1cm} (A.4)

We want to determine the asymptotic behaviors of $I^{(\pm)}(x, \eta; v + \eta^{-1}) - I^{(\pm)}(x, \eta; v)$ etc. when $x \to 0$. For the purpose, we use the following shift operator:

**Lemma A.2.** Let $L(v)$ and $\mathcal{A}(v)$ be differential operators given by

$$L(v) = -\frac{d^2}{dx^2} + \eta^2 \left(\frac{x + v^2}{4x^2} - \eta^{-2} \frac{1}{4x^2}\right),$$  \hspace{1cm} (A.5)

$$\mathcal{A}(v) = 2\eta^{-1} x^{1/2} \frac{d}{dx} - (v + \eta^{-1}) x^{-1/2}.$$  \hspace{1cm} (A.6)

Then, for any solution $\psi(x, v, \eta)$ of the equation (2.1) (i.e., $L(v)\psi(x, v, \eta) = 0$), the equality $L(v + \eta^{-1})\mathcal{A}(v)\psi(x, v, \eta) = 0$ holds.

We can show Lemma A.2 by direct computation. Using the shift operator $\mathcal{A}(v)$, we have
Lemma A.3. As \( x \) tends to 0, we have

\[
I^{(\pm)}(x, \eta; v + \eta^{-1}) - I^{(\pm)}(x, \eta; v) = \pm \log \left( -\frac{x}{4v(v + \eta^{-1})} \right) + O(|x|)
\]

and

\[
\eta I_{-1}(x; v + \eta^{-1}) - \eta I_{-1}(x; v) = 2 - 2v \log \left( \frac{v + \eta^{-1}}{v} \right) - \log \left( \frac{4(v + \eta^{-1})^2}{x} \right) + O(|x|).
\]

Proof. First we prove (A.7) by using the shift operator. Lemma A.2 implies that there exists formal power series \( C_\pm(\eta) \) with \( x \)-independent coefficients for which

\[
(2\eta^{-1}x^{1/2} \frac{d}{dx} - (v + \eta^{-1})x^{-1/2}) \psi_\pm(x, \eta; v) = C_\pm(\eta)\psi_\pm(x, \eta; v + \eta^{-1})
\]

holds. Since the left-hand side of (A.9) is

\[
(2\eta^{-1}x^{1/2}S^{(\pm)}(x, \eta; v) - (v + \eta^{-1})x^{-1/2})\psi_\pm(x, \eta; v),
\]

the logarithmic derivatives of both sides of (A.9) give

\[
S^{(\pm)}(x, \eta; v + \eta^{-1}) = S^{(\pm)}(x, \eta; v) + \frac{d}{dx} \log(2\eta^{-1}x^{1/2}S^{(\pm)}(x, \eta; v) - (v + \eta^{-1})x^{-1/2}).
\]

Therefore we obtain

\[
I^{(\pm)}(x, \eta; v + \eta^{-1}) - I^{(\pm)}(x, \eta; v) = \pm \log \left( \frac{2\eta^{-1}x^{1/2}S^{(+)}(x, \eta; v) - (v + \eta^{-1})x^{-1/2}}{2\eta^{-1}x^{1/2}S^{(-)}(x, \eta; v) - (v + \eta^{-1})x^{-1/2}} \right).
\]

To determine the asymptotic behavior of the right-hand side of (A.10), we use the following explicit formulas as \( x \to 0 \) which are easily obtained from the Riccati equation (2.6):

\[
S_{-1}(x; v) = \frac{\sqrt{x + v^2}}{2x} = \frac{v}{2x} + \frac{1}{4v} - \frac{x}{16v^3} + O(|x|^2),
\]

\[
S_0(x; v) = \frac{1}{4} \left( \frac{2}{x} - \frac{1}{x + v^2} \right) = \frac{1}{2x} - \frac{1}{4v^2} + \frac{x}{4v^4} + O(|x|^2),
\]

\[
S_1(x; v) = \frac{1}{16(x + v^2)\sqrt{x + v^2}} \left( 4 - \frac{5x}{x + v^2} \right) = \frac{1}{4v^3} - \frac{11x}{16v^5} + O(|x|^2),
\]

\[
S_n(x; v) = a_n(v) + b_n(v)x + O(|x|^2) \quad (n \geq 2),
\]
with some \(a_n(v)\) and \(b_n(v)\) which are independent of \(x\) \((n \geq 2)\). These asymptotic formulas show

\[
S^{(\pm)}(x, \eta; v) = \left( \pm \frac{\eta v}{2} + \frac{1}{2} \right) \frac{1}{x} + \eta A^{(\pm)}(v, \eta) + \eta B^{(\pm)}(v, \eta) x + O(|x|^2),
\]

where \(A^{(\pm)}(v, \eta)\) and \(B^{(\pm)}(v, \eta)\) are formal power series in \(\eta^{-1}\) whose coefficients are independent of \(x\). To obtain expressions of \(A^{(\pm)}(v, \eta)\) and \(B^{(\pm)}(v, \eta)\), let us substitute the expansion (A.11) into the equation (2.6) which is satisfied by \(S^{(\pm)}(x, \eta; v)\) and compare the coefficients of \(x^0\) and \(x^1\). Then we have

\[
\begin{align*}
A^{(\pm)}(v, \eta) &= \frac{1}{4(\pm v + \eta^{-1})}, \\
B^{(\pm)}(v, \eta) &= - \frac{1}{16(\pm v + 2\eta^{-1})(\pm v + \eta^{-1})^2}.
\end{align*}
\]

The formula (A.7) readily follows from (A.10), (A.11) and (A.12).

Next we show the equality (A.8). We can compute the integral \(I_{-1}(x; v)\) explicitly:

\[
I_{-1}(x; v) = \int_{C_v} S_{-1}(x; v) dx = 2\sqrt{x + v^2} - v \log \left( \frac{v + \sqrt{x^2 + v^2}}{v - \sqrt{x^2 + v^2}} \right).
\]

Therefore we have

\[
I_{-1}(x; v) = 2v - v \log \left( - \frac{4v^2}{x} \right) + O(|x|)
\]

which implies (A.8).

We now prove Proposition A.1. Lemma A.3 implies

\[
\left( \frac{I^{(+)}(x, \eta; v + \eta^{-1}) - I^{(-)}(x, \eta; v + \eta^{-1})}{2} - \frac{I^{(+)}(x, \eta; v) - I^{(-)}(x, \eta; v)}{2} \right)
- \eta (I_{-1}(x; v + \eta^{-1}) - I_{-1}(x; v))
= -2 + (2v\eta + 1) \log \left( 1 + \frac{1}{v\eta} \right) + O(|x|).
\]

This relation and (A.4) gives (A.1). This completes the proof of Proposition A.1 together with that of Theorem 2.5.

### B. Alien derivatives for simple resurgent functions

Here we give a rough review of the definition of alien derivatives for the simple resurgent function following [20]. We also relate them to the alien derivatives given in Section 2.3 (following the formulation of [25]).
Let \( \psi^\pm(x, \eta) \) be the WKB solution of the Bessel-type equation (2.1) which is normalized at turning point as (2.15). Now we introduce a slightly different WKB solution:

\[
\varphi^\pm(x, \eta) = \eta^{1/2} \psi^\pm(x, \eta) = \exp(\pm a(x) \eta) \sum_{n=0}^\infty \eta^{-n} \psi^\pm_{\pm n}(x).
\]

(Note that \( \varphi^\pm \) is different from (3.22) which appeared in Section 3.) Since the power series part \( \sum_{n=0}^\infty \eta^{-n} \psi^\pm_{\pm n}(x) \) in (B.1) contains the constant term \( \psi^\pm_{\pm 0}(x) \), its Borel transform should contain the \( \delta \)-function (i.e., the unit for the convolution product *):

\[
\varphi^\pm_B(x, y) = \delta(y \pm a(x)) + \sum_{n=1}^\infty \frac{\psi^\pm_{\pm n}(x)}{(n-1)!} (y \pm a(x))^{n-1}
\]

Then, their Borel transforms satisfy

\[
\psi^\pm_B = (\eta^{-1/2})_B \ast \varphi^\pm_B,
\]

where \( (\eta^{-1/2})_B(y) = y^{1/2} / \Gamma(1/2) \) is the Borel transform of \( \eta^{-1/2} \). Due to the square root singularity of \( (\eta^{-1/2})_B(y) \), the singularity structure of \( \psi^\pm_B \) and that of \( \varphi^\pm_B \) are different. The goal of Appendix B is to show the alien derivatives for \( \varphi^\pm_B \) is also described by the same formula (2.33) for \( \psi^\pm_B \), if we adopt the definition of alien derivatives given in [20] instead of that given in Section 2.3.

In what follows, we impose the same assumption given in Section 2.2. Also, we treat only \( \varphi^+ B \) and \( \psi^+ B \) (\( \varphi^- B \) and \( \psi^- B \) can be treated in the same manner). Then, we can show that the Borel transform \( \varphi^+ B \) of (B.1) is holomorphic on the following domain (cf., Proposition 2.4)

\[
\{ y \in C \mid -\epsilon < \text{Im}(y + a(x)) < +\epsilon \} \setminus \Omega^*
\]

Here \( \epsilon > 0 \) is a sufficiently small number and

\[
\Omega^* = \{ y = -a(x) + 2m\pi iv \mid m \in Z \neq 0 \}.
\]

We also know that \( \varphi^+ B \) has singularities at points in \( \Omega^*_\pm \). Moreover, it turns out that these are simple singularities in the sense of [20, Section 26]; that is, for any point \( \omega \in \Omega^*_\pm \) and any path \( \gamma \) from \( -a(x) \) to \( y \) which is close to \( \omega \), there exists \( c \in C \) and a germ of holomorphic function \( f(y) \) at \( y = 0 \) such that

\[
\text{cont}_y \varphi^+ B(y) = \frac{c}{2\pi i(y - \omega)} + f(y - \omega) \frac{\log(y - \omega)}{2\pi i} + (\text{holomorphic at } y = \omega)
\]
holds when $y$ tends to $\omega$. Here the left-hand side of (B.6) represents the analytic continuation of the germ

$$\varphi_{+,B} - \delta(y + a(x)) = \sum_{n=1}^{\infty} \frac{\psi_{\pm,n}(x)}{(n-1)!} (y \pm a(x))^{n-1}$$

along the path $\gamma$. The simple singularity property (B.6) follows from [20, Corollary 30.7] and the following facts:

- The WKB solution $\varphi^{(0)} = \eta^{1/2}\psi^{(0)}$ with $\psi^{(0)}$ being given in (2.23) is Borel summable (cf., Proof of Proposition 2.4). This WKB solution satisfies $\varphi_{+,B} = (e^V)_B * \varphi^{(0)}$.
- Since $V_B$ given in (2.26) has only simple singularities, so does $(e^V)_B$ ([(20, Theorem 30.9)]).

Again we assume that $\text{Im} \nu > 0$ to fix the situation. For the function $\phi(y) = \text{cont}_y \varphi_{+,B}(y)$ given in (B.6) which has a simple singularity at $\omega_m = -a(x) - 2m\pi i \nu \in \Omega^*$ with $m \in \mathbb{Z}_{\geq 1}$, its singularity $\text{sing}_{y=\omega_m}(\phi)$ (instead of (2.29)) is defined as follows (cf., [20, Section 25]):

$$\text{sing}_{y=\omega_m}(\phi)(y) = c\delta(y + a(x)) + f(y + a(x)),$$

where $c \in \mathbb{C}$ and $f(y)$ is defined through the behavior (B.6) of $\phi(y)$ near $y = \omega_m$. We note that, although the function $\phi$ is defined near $y = \omega_m$, $\text{sing}_{y=\omega_m}(\phi)(y) - c\delta(y + a(x))$ is the germ of holomorphic function defined on a disc $\{y \in \mathbb{C} \mid |y + a(x)| < \epsilon\}$ for a sufficiently small $\epsilon > 0$.

Our main claim in this appendix is the following:

**Proposition B.1.** Let $\omega_m = -a(x) - 2m\pi i \nu \ (m \in \mathbb{Z}_{\geq 1})$ be one of the fixed singularities of the WKB solutions. Then, for any $\epsilon_1, \ldots, \epsilon_{m-1} \in \{\pm\}$, the equality

$$\text{sing}_{y=\omega_m}(\text{cont}_{(\epsilon_1, \ldots, \epsilon_{m-1})} \psi_{+,B}) = (\eta^{-1/2})_B * \text{sing}_{y=\omega_m}(\text{cont}_{(\epsilon_1, \ldots, \epsilon_{m-1})} \varphi_{+,B})$$

holds. Here $\text{sing}_{y=\omega_m}$ in the left (resp., right)-hand side is defined as in (2.29) (resp., (B.7)).

**Proof.** Suppose that $y$ lies near the point $\omega_m + \epsilon$ for a sufficiently small $\epsilon > 0$ (and hence $y + 2m\pi i \nu$ lies near $-a(x) + \epsilon$). Using the relation (B.3), we have

$$\text{sing}_{y=\omega_m}(\text{cont}_{(\epsilon_1, \ldots, \epsilon_{m-1})} \psi_{+,B})(y + 2m\pi i \nu)$$

$$= \int_{y^{(m)}_{(\epsilon_1, \ldots, \epsilon_{m-1})}} \varphi_{+,B}(t) \frac{(y - t)^{-1/2}}{\Gamma(1/2)} \ dt$$
Here \( g_m \) and \( g(\epsilon_1, \ldots, \epsilon_m) \) are the paths given in Section 2.3, and \( g_m \) \( g(\epsilon_1, \ldots, \epsilon_m) \) denotes the composition of these paths. \( C \) is also a path starting from \( y \) and going back to the initial point \( y \) after encircling \( \omega_m \) in the positive direction (see Figure B.1). Then, we can decompose the integral (B.9) into the following three parts:

\[
\int_C \text{cont}_{(\epsilon_1, \ldots, \epsilon_m)} \varphi_{+, B}(t) \frac{(y - t)^{-1/2}}{\Gamma(1/2)} dt = I_1(y) + I_2(y) + I_3(y),
\]

where

\[
I_1 = \int_C c \frac{2\pi i(t - \omega_m)}{\Gamma(1/2)} dt,
\]

\[
I_2 = \int_C f(t - \omega_m) \frac{\log(t - \omega_m)}{2\pi i} \frac{(y - t)^{-1/2}}{\Gamma(1/2)} dt,
\]

\[
I_3 = \int_C h(t) \frac{(y - t)^{-1/2}}{\Gamma(1/2)} dt.
\]

Here \( c \) and \( f(y) \) are determined by

\[
\text{sing}_{y = \omega_m} (\text{cont}_{(\epsilon_1, \ldots, \epsilon_m)} \varphi_{+, B})(y) = c \delta(y + a(x)) + f(y + a(x)),
\]

and \( h(y) \) is a holomorphic function at \( y = \omega_m \) (cf., (B.6)).

Let us compute these integrals. Thanks to the residue theorem, we have

\[
I_1(y) = \text{Res}_{t = \omega_m} \left( \frac{c}{t - \omega} \frac{(y - t)^{-1/2}}{\Gamma(1/2)} dt \right) = c \frac{(y - \omega)^{-1/2}}{\Gamma(1/2)}, \quad I_3(y) = 0.
\]

Since the integrand in \( I_2(y) \) has logarithmic singularity at \( t = \omega \), we have
\[ I_2(y) = \int_{\omega_m}^y f(t - \omega_m) \frac{\log(t - \omega_m) \ (y - t)^{-1/2}}{2\pi i} \frac{t - \omega_m}{\Gamma(1/2)} \, dt \]
\[ + \int_{\omega_m}^y f(t - \omega_m) \frac{\log(t - \omega_m) + 2\pi i \ (y - t)^{-1/2}}{2\pi i} \frac{t - \omega_m}{\Gamma(1/2)} \, dt \]
\[ = \int_{\omega_m}^y f(t - \omega_m) \frac{(y - t)^{-1/2}}{\Gamma(1/2)} \, dt. \]

Thus, for \( y \) which is close to \(-a(x) + \epsilon\), we obtain
\[ \text{sing}_{y=\omega_m} (\text{cont}_{(e_1, \ldots, e_m)} \psi_{+, B})(y) \]
\[ = (I_1 + I_2 + I_3)(y - 2\pi i \epsilon) \]
\[ = \epsilon \left( \frac{y + a(x)}{\Gamma(1/2)} \right)^{-1/2} + \int_{-a(x)}^y f(t + a(x)) \frac{(y - t)^{-1/2}}{\Gamma(1/2)} \, dt \]
\[ = ((y^{-1/2})_B * \text{sing}_{y=\omega_m} (\text{cont}_{(e_1, \ldots, e_m)} \varphi_{+, B}))(y). \]

For a resurgent function having simple singularities, its alien derivatives are defined by the formula (2.28) after replacing \( \text{sing}_{y=\omega_m} \) to the one defined above (B.7). Then, Proposition B.1 implies that we obtain the same formula as that shown in Theorem 2.10:
\[ A_{y=a(x)-2\pi i \epsilon} \varphi_{\pm, B}(x, y) = \frac{1}{m} \varphi_{\pm, B}(x, y). \]

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(Recevita la 30-an de majo, 2017)
(Revizita la 19-an de junio, 2017)