Local Lipschitz Stability for Inverse Robin Problems in Some Elliptic and Parabolic Systems

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Abstract

In this work, we shall study the nonlinear inverse problems of recovering the Robin coefficients in second order elliptic and parabolic systems with Robin boundary conditions, and establish their local Lipschitz stabilities. We shall first show the local Lipschitz stability for the elliptic inverse Robin problem and give some remarks to demonstrate why we also need to consider the Robin condition but not Neumann or Dirichlet condition on the accessible boundary. Then the new arguments are generalized to help establish a novel local Lipschitz stability for parabolic inverse Robin problems and some counterexamples shall also be given to show why we also consider the Robin condition on the accessible boundary.

Key Words. Inverse Robin problems, local Lipschitz stability, elliptic and parabolic equations

1 Introduction

We are concerned in this work with the determination of a spatially dependent Robin coefficient in both stationary elliptic and time-dependent parabolic systems from measurement data on a partial boundary. This is a highly nonlinear inverse problem and arises in several applications of practical importance. The Robin coefficient may characterize the thermal properties of conductive materials on the interface or certain physical processes near the boundary, e.g., it represents the corrosion damage profile in corrosion detection ([10][13]), and indicates the thermal property in quenching processes ([19]).

For the description of the model problems that are considered in this work, we let $\Omega = B_{r_2}(0) \setminus B_{r_1}(0) \subset R^2$ be an open bounded and connected annular domain, where $0 < r_1 < r_2$ and $B_r(0)$ denotes a circle centered on the origin with radius $r > 0$. The boundary $\partial \Omega$ consists of two disjointed parts $\partial \Omega = \Gamma_i \cup \Gamma_a$, where $\Gamma_i = \partial B_{r_1}(0)$ and $\Gamma_a = \partial B_{r_2}(0)$ are respectively the part of the boundary that is inaccessible and accessible to experimental measurements. Then we shall consider the inverse Robin problems associated with the following elliptic boundary value problem

$$\begin{cases}
-\triangle u = f(x) \quad \text{in} \quad \Omega, \\
\frac{\partial u}{\partial n} + \gamma(x)u = g(x) \quad \text{on} \quad \Gamma_i, \\
\frac{\partial u}{\partial n} + u = h(x) \quad \text{on} \quad \Gamma_a,
\end{cases}$$

(1.1)

and parabolic initial boundary value problem

$$\begin{cases}
\partial_t u - |x|^2 \Delta u = 0 \quad \text{in} \quad \Omega \times [0,T], \\
\frac{\partial u}{\partial n} + \gamma(x)u = g(x,t) \quad \text{on} \quad \Gamma_i \times [0,T], \\
\frac{\partial u}{\partial n} + u = h(x,t) \quad \text{on} \quad \Gamma_a \times [0,T], \\
u(x,0) = 0 \quad \text{in} \quad \Omega.
\end{cases}$$

(1.2)
Functions \( f, \ g \) and \( h \) are the source strength, ambient temperature and heat flux respectively. The coefficients \( \gamma(x) \) in (1.1) and (1.2) represent the Robin coefficients, which will be the focus of our interest and are assumed to stay in the following feasible constraint set:

\[
K := \left\{ \gamma \in L^2(\Gamma_i); \ 0 < \underline{\gamma} \leq \gamma(x) \leq \bar{\gamma} \ a.e. \ on \ \Gamma_i \right\},
\]

where \( \gamma \) and \( \bar{\gamma} \) are two positive constants. For convenience, we often write the solutions of the systems (1.1) and (1.2) as \( u(\gamma) \) to emphasize their dependence on the Robin coefficient \( \gamma \).

We are now ready to formulate the inverse problems of our interest in this work.

**Elliptic Inverse Robin Problem:** recover the Robin coefficient \( \gamma(x) \) in (1.1) on the inaccessible part \( \Gamma_i \) from the measurable data of \( z \) on the accessible part \( \Gamma_a \).

**Parabolic inverse Robin problem:** recover the Robin coefficient \( \gamma(x) \) in (1.2) on the inaccessible part \( \Gamma_i \) from the measurable data of \( z \) on the accessible part \( \Gamma_a \) over the whole time range \([0, T]\).

Thanks for the unique continuation theorem [14], the uniqueness of the elliptic and parabolic inverse Robin problems has been well established [2] [15] [16]. The stability of the inverse Robin problems has also been studied for several years, but most of the studies are global logarithmic type stability estimate, see [1] [3] [6]. There exists only a few results on the Lipschitz stability of inverse Robin problems associated with the following elliptic equation:

\[
\begin{cases}
-\Delta u &= 0 \quad \text{in} \quad D, \\
\frac{\partial u}{\partial n} + \gamma(x)u &= 0 \quad \text{on} \quad \Gamma_1, \\
\frac{\partial u}{\partial n} &= g(x) \quad \text{on} \quad \Gamma_2,
\end{cases} \tag{1.3}
\]

where \( D \) is an open bounded and connected domain in \( \mathbb{R}^2 \) and \( \Gamma_1 \) and \( \Gamma_2 \) are the inaccessible and accessible part of the boundary \( \partial D \) respectively. In [2], Chaabane and Jaoua proved that the global Lipschitz stability estimate. Simply speaking, for \( i = 1, 2 \), let \( u(\gamma_i) \) denote the solution of the boundary value problem (1.3) corresponding to \( \gamma = \gamma_i \). If \( \gamma_1 \leq \gamma_2 \), then we have an estimate of the form: \( ||\gamma_1 - \gamma_2||_1 \leq C ||(u(\gamma_1) - u(\gamma_2))||_{\gamma_i} ||2, \) where \( || \cdot ||_1 \) and \( || \cdot ||_2 \) are some appropriate norms, which are different from the standard Sobolev \( H^1 \) and \( H^2 \) norms. In [3], Choulli established a local Lipschitz stability estimate for an arbitrary smooth domain without the monotony condition \( \gamma_1 \leq \gamma_2 \). The essential technique in the proof of the local Lipschitz stability is the construction of a mapping, which is proved to be a \( C^1 \)-diffeomorphism in a neighborhood of a fixed element \( \gamma_0 \in K \). Very recently, Hu and Yamamoto also established in [12] a global Hölder stability estimate for an elliptic inverse Robin problem from a single Cauchy data on an accessible boundary. The main arguments rely on the Schwarz reflection principle with the Robin boundary condition and a novel interior estimate derived from the elliptic Carleman estimate.

However, to our best knowledge, there are still no results available for the local Lipschitz stability for parabolic inverse Robin problems. This will be one of the main novelties and contributions of the present work. Another main novelty is that we construct many counterexamples creatively to show why we also consider the Robin condition but not Neumann or Dirichlet condition on the accessible boundary for both the elliptic Robin inverse problem and parabolic Robin inverse problem. We shall first prove the local Lipschitz stability of the proposed elliptic inverse Robin problem and the new arguments are then generalized to help us establish a novel local Lipschitz stability for the inverse Robin problem associated with the parabolic system (1.2), with some tricky and delicate detailed modifications due to the complication of time dependence. It is worth of mention that it is the first time in literature to establish the local Lipschitz stabilities for the inverse Robin problems associated with time-dependent parabolic equations.

The rest of the paper is organized as follows. In Section 2, the local Lipschitz stability estimate for the elliptic inverse Robin problem will be verified and some remarks to show why we also consider the Robin condition but not Neumann or Dirichlet condition on the accessible boundary are given. In Section 3, we shall establish a newly local Lipschitz stability estimate for the parabolic inverse Robin problem and also give some remarks to show why we also consider the Robin condition on the accessible boundary.

Throughout this work, \( C \) is often used for a generic constant. We shall write the norms of the spaces \( H^s(\Omega), L^2(\Omega), H^s(\Gamma) \) and \( L^2(\Gamma) \) (for some \( \Gamma \subset \partial \Omega \)) respectively as \( || \cdot ||_{s,\Omega}, || \cdot ||_{\Omega}, || \cdot ||_{s,\Gamma} \) and \( || \cdot ||_{\Gamma} \).

## 2 Local Lipschitz stability for elliptic inverse Robin problem

In this section, we shall establish the local Lipschitz stability for the proposed elliptic inverse Robin problem. We first give a preliminary lemma for recalling the classical well-posedness of the forward
solution \(u\) to system (1.1).

**Lemma 2.1.** (see [9] [11]) Let \(\Omega\) be an open bounded and connected domain with \(C^\infty\) boundary \(\partial\Omega\), \(\gamma(x) \in K, f(x) \in L^2(\Omega), g(x) \in L^2(\Gamma_1)\) and \(h(x) \in L^2(\Gamma_a)\), then there exists a unique solution \(u \in H^2(\Omega)\) to system (1.1) and it satisfies

\[
\|u\|_{2, \Omega} \leq C(f_{\Omega} + \|g\|_{\Gamma_1} + \|h\|_{\Gamma_a}).
\] (2.1)

Then we study the differentiability of the solution \(u(\gamma)\) to system (1.1) and give its Fréchet derivative.

**Lemma 2.2.** The solution \(u(\gamma)\) of system (1.1) is continuously Fréchet differentiable and its derivative \(u'(\gamma)\) with direction \(d \in L^\infty(\Gamma_i)\) solves the following system:

\[
\begin{align*}
-\Delta u'(\gamma)d &= 0 \text{ in } \Omega, \\
\frac{\partial (u'(\gamma)d)}{\partial n} + \gamma (u'(\gamma)d) &= -d u(\gamma) \text{ on } \Gamma_i, \\
u'(\gamma)d &= 0 \text{ on } \Gamma_a.
\end{align*}
\] (2.2)

**Proof.** For any \(\gamma \in K\) and \(d \in L^\infty(\Gamma_i)\) such that \(\gamma + d \in K\), let \(v \equiv u(\gamma + d) - u(\gamma) - u'(\gamma)d\), then we have

\[
\begin{align*}
-\Delta v &= 0 \text{ in } \Omega, \\
\frac{\partial v}{\partial n} + \gamma v &= -d(u(\gamma + d) - u(\gamma)) \text{ on } \Gamma_i, \\
v &= 0 \text{ on } \Gamma_a.
\end{align*}
\] (2.3)

From estimate (2.1) and the Sobolev embedding theorem, we obtain

\[
\|v\|_{1, \Omega} \leq C\|d(u(\gamma + d) - u(\gamma))\|_{\Gamma_i} \leq C\|d\|_{L^\infty(\Gamma_i)}\|u(\gamma + d) - u(\gamma)\|_{L^2, \Gamma_i}.
\]

As \(\psi \equiv u(\gamma + d) - u(\gamma)\) satisfies the following elliptic equation

\[
\begin{align*}
-\Delta \psi &= 0 \text{ in } \Omega, \\
\frac{\partial \psi}{\partial n} + \gamma \psi &= -d u(\gamma + d) \text{ on } \Gamma_i, \\
\psi &= 0 \text{ on } \Gamma_a.
\end{align*}
\] (2.4)

Similarly, we can show from estimate (2.1) and the Sobolev embedding theorem that

\[
\|\psi\|_{1, \Omega} \leq C\|d(u(\gamma + d))\|_{\Gamma_i} \leq C\|d\|_{L^\infty(\Gamma_i)}\|u(\gamma + d)\|_{L^2, \Gamma_i} \leq C\|d\|_{L^\infty(\Gamma_i)}\|u(\gamma + d)\|_{1, \Omega} \leq C\|d\|_{L^\infty(\Gamma_i)}.
\]

Thus it follows directly that

\[
\frac{\|u(\gamma + d) - u(\gamma) - u'(\gamma)d\|_{1, \Omega}}{\|d\|_{L^\infty(\Gamma_i)}} \to 0 \text{ as } \|d\|_{L^\infty(\Gamma_i)} \to 0,
\]

which means that \(u(\gamma)\) is Fréchet differentiable and \(u'(\gamma)d\) is its derivative.

Next, we verify the continuity of \(u'(\gamma)d\). Let \(\phi \in L^\infty(\Gamma_i)\), then \(y \equiv u'(\gamma + \phi)d - u'(\gamma)d\) satisfies

\[
\begin{align*}
-\Delta y &= 0 \text{ in } \Omega, \\
\frac{\partial y}{\partial n} + \gamma y &= -d(u(\gamma + \phi) - u(\gamma)) - \phi u(\gamma + \phi) \text{ on } \Gamma_i, \\
y &= 0 \text{ on } \Gamma_a.
\end{align*}
\] (2.5)

Once again estimate (2.1) implies that

\[
\|y\|_{1, \Omega} \leq C\|d(u(\gamma + \phi) - u(\gamma))\|_{\Gamma_i} + \|\phi u(\gamma + \phi)\|_{\Gamma_i} \leq C\|d\|_{L^\infty(\Gamma_i)}\|u(\gamma + \phi) - u(\gamma)\|_{1, \Omega} + C\|\phi\|_{L^\infty(\Gamma_i)}\|u(\gamma + \phi)\|_{\Gamma_i} \leq C\|d\|_{L^\infty(\Gamma_i)}\|\phi\|_{L^\infty(\Gamma_i)} + C\|\phi\|_{L^\infty(\Gamma_i)},
\]

which tends to 0 when \(\|\phi\|_{L^\infty(\Gamma_i)}\) tends to 0.

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*Note: The text contains mathematical expressions and equations that are part of the solution to a problem in a specific field, possibly involving partial differential equations and functional analysis.*
Now let $γ^* \in K$ be the true Robin coefficient for the proposed elliptic Robin inverse problem. Then from Lemma 2.2 we see that $u(γ^*)$ is continuously Fréchet differentiable and its derivative $w(d) = u'(γ^*)d$ satisfies the following elliptic system

$$
\begin{cases}
-\triangle w(d) = 0 & \text{in } \Omega, \\
\frac{\partial w(d)}{\partial n} + γ^* w(d) = -du(γ^*) & \text{on } Γ_i, \\
\frac{\partial w(d)}{\partial n} + w(d) = 0 & \text{on } Γ_a.
\end{cases}
$$

(2.6)

With the help of $ω(d)$, we define a bounded and linear operator from $L^2(Γ_i)$ to $L^2(Γ_a)$ as follows:

$$
N(d) = \frac{\partial w(d)}{\partial n} \text{ on } Γ_a, \quad \forall d \in L^2(Γ_i).
$$

For establishing the local Lipschitz stability estimate, we need to make some assumptions.

**Assumption 2.1.** $u(γ^*) \neq 0$ almost everywhere on $Γ_i$.

This assumption is very natural, as one can see from the second equation of system (1.1) that if $u(γ^*) = 0$ a.e. on $Γ_i$ then the true Robin coefficient $γ^*$ is not identifiable.

**Assumption 2.2.** When we consider the polar coordinates $(r, θ)$, functions $f(x), g(x), h(x)$ and true Robin coefficient $γ^*$ are only dependent on $r$ but independent of $θ$.

**Lemma 2.3.** Under Assumptions 2.1-2.2, the operator $N$ is bijective and $\|N^{-1}\| \leq C$.

**Proof.** We first show $N$ is injective, i.e., if $N(d) = 0$ then $d = 0$. Indeed, if

$$
N(d) = \frac{\partial w(d)}{\partial n} = 0 \text{ on } Γ_a,
$$

then the third equation of (2.6) implies that $w(d) = 0$ on $Γ_a$. Therefore, we know that $w(d)$ is also the solution of the following system

$$
\begin{cases}
-\triangle w = 0 & \text{in } \Omega, \\
\frac{\partial w}{\partial n} = 0 & \text{on } Γ_a, \\
\frac{\partial w}{\partial n} = 0 & \text{on } Γ_a,
\end{cases}
$$

(2.7)

which implies that $w(d) = 0$ in $Ω$ by the unique continuation principle [14] and thus $du(γ^*) = 0$ on $Γ_i$. Then we easily get $d = 0$ on $Γ_i$ by Assumption 2.1.

Now we prove $N$ is surjective, which means that for any element $φ = \frac{\partial w}{\partial n} \in L^2(Γ_a)$, we want to seek $d ∈ L^2(Γ_i)$ such that $N(d) = φ$. To do so, we use the separation of variables in polar coordinates to solve (2.6) and thus the first equation transforms to the following equation with variables $r$ and $θ$

$$
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial θ^2} = 0.
$$

(2.8)

The Assumption 2.2 implies that the solutions $u$ and $w$ of system (1.1) and (2.6) respectively are all just dependent on $r$ but independent on $θ$ and thus we have

$$
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} = 0.
$$

(2.9)

It is easy to get the general solutions of (2.9) that

$$
w = c_1 + c_2 \ln r,
$$

(2.10)

where $c_1$ and $c_2$ are two real number. Hence we get on $Γ_a$, i.e., $r = r_2$ that

$$
φ = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial r} = c_2 \frac{1}{r_2},
$$

$$
w = c_1 + c_2 \ln r_2,
$$

and thus

$$
0 = \frac{\partial w}{\partial n} + w = c_2 \frac{1}{r_2} + (c_1 + c_2 \ln r_2) = c_1 + c_2 \left(\frac{1}{r_2} + \ln r_2\right).
$$
For simplicity, we let $c_2 = -1$, then $c_1 = \frac{1}{r_2} + \ln r_2$ and obtain

$$w = \frac{1}{r_2} + \ln r_2 - \ln r \mbox{ in } \Omega.$$  

Hence, we have on $\Gamma_1$, i.e., $r = r_1$ that

$$-du(\gamma^*)(r_1) = \frac{\partial w}{\partial n} + \gamma^*(r_1)w = -\frac{\partial w}{\partial r} + \gamma^*(r_1)w$$

$$= \frac{1}{r_1} + \gamma^*(r_1)\left(\frac{1}{r_2} + \ln r_2 - \ln r_1\right),$$

which with Assumption 2.1 implies that

$$d = -\frac{1}{r_1 u(\gamma^*)(r_1)} \frac{\gamma^*(r_1)}{u(\gamma^*)(r_1)} \left(\frac{1}{r_2} + \ln r_2 - \ln r_1\right).$$

As $N$ is linear, bounded and bijective, then by the Open Mapping Theorem [7], we know that $N^{-1}$ exists and $\|N^{-1}\|$ is bounded, i.e., there exists a positive constant $C$ such that

$$\|N^{-1}\| \leq C.$$ 

Finally, for convenience, we shall write for any positive constant $b$ that

$$N(\gamma^*, b) = \{\gamma \in K; \|\gamma - \gamma^*\|_{\Gamma_1} \leq b\}.$$ 

We are now ready to establish the local Lipschitz stability for elliptic inverse Robin problem.

**Theorem 2.1.** Under Assumptions 2.4, 2.5 there exists a positive constant $b$ such that the following stability estimate holds:

$$\|u(\gamma_1) - u(\gamma_2)\|_{r_a} \geq C\|\gamma_1 - \gamma_2\|_{r_1}, \quad \forall \gamma_1, \gamma_2 \in N(\gamma^*, b).$$  

(2.11)

**Proof.** We first introduce an important mapping

$$\theta: \gamma \in L^2(\Gamma_1) \to \frac{\partial u(\gamma)}{\partial n} \in L^2(\Gamma_a),$$

which is continuously Fréchet-differentiable from Lemma 2.2 and it is obviously to see

$$\theta'(\gamma^*)d = \frac{\partial u'(\gamma^*)d}{\partial n} = N(d).$$

Then it follows from lemma 2.3 that $\theta'(\gamma^*)^{-1} = N^{-1}$ and

$$\|\theta'(\gamma^*)^{-1}\| = \|N^{-1}\| \leq C.$$ 

By the inverse function theorem [8] we find that $\theta(\gamma^*)$ is $C^1$-diffeomorphism on a neighborhood of $\gamma^*$, consequently $\theta(\gamma^*)^{-1}$ is locally Lipschitz continuous and $\|\theta(\gamma^*)^{-1}\| = \|\theta'(\gamma^*)^{-1}\| \leq C$. Thus, there exists a neighborhood $N(\gamma^*, b)$ of $\gamma^*$ such that the Lipschitz constant is less equal to $2\|\theta(\gamma^*)^{-1}\|$, i.e., for any $\gamma_1, \gamma_2 \in N(\gamma^*, b)$, it holds that

$$\|\gamma_1 - \gamma_2\|_{r_1} \leq 2\|\theta(\gamma^*)^{-1}\|\|\frac{\partial u(\gamma_1)}{\partial n} - \frac{\partial u(\gamma_2)}{\partial n}\|_{r_a}$$

$$\leq 2C\|u(\gamma_2) - u(\gamma_1)\|_{r_a},$$  

(2.12)

where we used the fact that $\frac{\partial u(\gamma)}{\partial n} = -u(\gamma)$ on $\Gamma_a$ in the second inequality.  

\[\]
3 Local Lipschitz stability for parabolic inverse Robin problem

In this section, we shall establish the local Lipschitz stability for the proposed parabolic inverse Robin problem and give some counterexamples to show why we also consider the Robin condition on accessible part. We first give a preliminary lemma for recalling the classical well-posedness of the forward solution $u$ to system (1.2).

**Lemma 3.1.** (see [9][17]) Let $\Omega$ be an open bounded and connected domain with $C^\infty$ boundary $\partial \Omega$, $\gamma(x) \in K$, $g(x,t) \in L^2(0,T;L^2(\Gamma_i))$ and $h(x,t) \in L^2(0,T;L^2(\Gamma_a))$, then there exists a unique solution $u \in L^2(0,T;H^2(\Omega))$ to system (1.2) and it satisfies

$$
\|u\|_{L^2(0,T;H^2(\Omega))} \leq C(\|g\|_{L^2(0,T;L^2(\Gamma_i))} + \|h\|_{L^2(0,T;L^2(\Gamma_a)))}).
$$

(3.1)

Now we study the differentiability of the solution $u(\gamma)$ to system (1.2) and give its Fréchet derivative.

**Lemma 3.2.** The solution $u(\gamma)$ of system (1.2) is continuously Fréchet differentiable and its derivative $u'(\gamma)d$ with direction $d \in L^\infty(\Gamma_i)$ solves the following system:

$$
\begin{aligned}
\partial_t(u'(\gamma)d) - |x|^2 u'(\gamma)d &= 0 &\text{in } \Omega \times [0,T], \\
\frac{\partial(u'(\gamma)d)}{\partial n} + \gamma u'(\gamma)d &= -d u(\gamma) &\text{on } \Gamma_i \times [0,T], \\
\frac{\partial(u'(\gamma)d)}{\partial n} + u'(\gamma)d &= 0 &\text{on } \Gamma_a \times [0,T], \\
u'(\gamma)d(x,0) &= 0 &\text{in } \Omega,
\end{aligned}
$$

(3.2)

Proof. For any $\gamma \in K$ and $d \in L^\infty(\Gamma_i)$ such that $\gamma + d \in K$, let $v \equiv u(\gamma + d) - u(\gamma) - u'(\gamma)d$, then we have

$$
\begin{aligned}
\partial_t v - |x|^2 v &= 0 &\text{in } \Omega \times [0,T], \\
\frac{\partial v}{\partial n} + \gamma v &= -d(u(\gamma + d) - u(\gamma)) &\text{on } \Gamma_i \times [0,T], \\
\frac{\partial v}{\partial n} + v &= 0 &\text{on } \Gamma_a \times [0,T], \\
v(x,0) &= 0 &\text{in } \Omega,
\end{aligned}
$$

(3.3)

From estimate (3.1) and the Sobolev embedding theorem, we obtain

$$
\begin{aligned}
\|v\|_{L^2(0,T;H^1(\Omega))} &\leq C\|d(u(\gamma + d) - u(\gamma))\|_{L^2(0,T;L^2(\Gamma_i))} \\
&\leq C\|d\|_{L^\infty(\Gamma_i)}\|(u(\gamma + d) - u(\gamma))\|_{L^2(0,T;H^1(\Gamma_i))} \\
&\leq C\|d\|_{L^\infty(\Gamma_i)}\|(u(\gamma + d) - u(\gamma))\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}
$$

As $\psi \equiv u(\gamma + d) - u(\gamma)$ satisfies the following parabolic equation

$$
\begin{aligned}
\partial_t \psi - |x|^2 \Delta \psi &= 0 &\text{in } \Omega \times [0,T], \\
\frac{\partial \psi}{\partial n} + \gamma \psi &= -d u(\gamma + d) &\text{on } \Gamma_i \times [0,T], \\
\frac{\partial \psi}{\partial n} + \psi &= 0 &\text{on } \Gamma_a \times [0,T], \\
\psi(x,0) &= 0 &\text{in } \Omega,
\end{aligned}
$$

(3.4)

Similarly, we can show from estimate (3.1) and the Sobolev embedding theorem that

$$
\begin{aligned}
\|\psi\|_{L^2(0,T;H^1(\Omega))} &\leq C\|d u(\gamma + d)\|_{L^2(0,T;L^2(\Gamma_i))} \leq C\|d\|_{L^\infty(\Gamma_i)}\|u(\gamma + d)\|_{L^2(0,T;H^2(\Gamma_i))} \\
&\leq C\|d\|_{L^\infty(\Gamma_i)}\|u(\gamma + d)\|_{L^2(0,T;H^1(\Omega))} \leq C\|d\|_{L^\infty(\Gamma_i)}.
\end{aligned}
$$

Thus it follows directly that

$$
\frac{\|u(\gamma + d) - u(\gamma) - u'(\gamma)d\|_{L^2(0,T;H^1(\Omega))}}{\|d\|_{L^\infty(\Gamma_i)}} \rightarrow 0 \text{ as } \|d\|_{L^\infty(\Gamma_i)} \rightarrow 0,
$$

which means that $u(\gamma)$ is Fréchet differentiable and $u'(\gamma)d$ is its derivative.

Next, we verify the continuity of $u'(\gamma)d$. Let $\phi \in L^\infty(\Gamma_i)$, then $y \equiv u'(\gamma + \phi)d - u'(\gamma)d$ satisfies

$$
\begin{aligned}
\partial_t y - |x|^2 \Delta y &= 0 &\text{in } \Omega \times [0,T], \\
\frac{\partial y}{\partial n} + \gamma y &= -d(u(\gamma + \phi) - u(\gamma)) - d\phi u(\gamma + \phi) &\text{on } \Gamma_i \times [0,T], \\
\frac{\partial y}{\partial n} + y &= 0 &\text{on } \Gamma_a \times [0,T], \\
y(x,0) &= 0 &\text{in } \Omega,
\end{aligned}
$$

(3.5)
Once again estimate (3.1) implies that
\[ \|y\|_{\Gamma} \leq C \left( \|d(u(\gamma + \phi) - u(\gamma))\|_{L^2(0,T;L^2(\Gamma)))} + \|\phi u(\gamma + \phi)\|_{L^2(0,T;L^2(\Gamma)))} \right), \]
which tends to 0 when \( \|\phi\|_{L^\infty(\Gamma)} \) tends to 0.

Now let \( \gamma^* \in K \) be the true Robin coefficient for the proposed parabolic Robin inverse problem, then from Lemma 3.2, we see that \( u(\gamma^*) \) is continuously Fréchet differentiable and its derivative \( w(p) = u'(\gamma^*)p \) satisfies the following parabolic system
\[
\begin{aligned}
\partial_t w(p) - |x|^2 \Delta w(p) &= 0 & \text{in } \Omega \times [0,T], \\
\frac{\partial w(p)}{\partial n} + \gamma^* w(p) &= -pu(\gamma^*) & \text{on } \Gamma_i \times [0,T], \\
\frac{\partial w(p)}{\partial n} + w(p) &= 0 & \text{on } \Gamma_a \times [0,T], \\
w(p)(x,0) &= 0 & \text{in } \Omega.
\end{aligned}
\] (3.6)

Next we define a bounded and linear operator from \( L^2(\Gamma_i) \) to \( L^2(0,T;L^2(\Gamma_a)) \) as follows:
\[ N(p) = \frac{\partial w(p)}{\partial n} \text{ on } \Gamma_a \times [0,T], \quad \forall p \in L^2(\Gamma_i). \] (3.7)

For establishing the local Lipschitz stability estimate, we need to make some assumptions.

**Assumption 3.1.** \( u(\gamma^*) \neq 0 \) almost everywhere on \( \Gamma_i \times [0,T] \).

This assumption is very natural, as one can see from the second equation of system (3.2) that if \( u(\gamma^*) = 0 \) a.e. on \( \Gamma_i \times [0,T] \) then the true Robin coefficient \( \gamma^* \) is not identifiable.

**Assumption 3.2.** The solution \( u(\gamma^*) \) to system (3.2) is completely separated into its spatial and temporal components and has the formulation \( u(\gamma^*) = te^{t-in}V(x) \). When we consider the polar coordinates \((r,\theta), V(x)\) and true Robin coefficient \( \gamma^* \) are only dependent on \( r \) but independent of \( \theta \).

**Lemma 3.3.** Under Assumptions 3.1-3.2, \( N \) is bijective and \( \|N^{-1}\| \) is bounded.

**Proof.** We first show \( N \) is injective, i.e., if \( N(p) = 0 \) then \( p = 0 \). Indeed, if
\[ N(p) = \frac{\partial w(p)}{\partial n} = 0 \text{ on } \Gamma_a \times [0,T], \]
then the third equation of (3.7) implies that \( w(p) = 0 \) on \( \Gamma_a \times [0,T] \). Therefore, we know that \( w(p) \) is also the solution of the following system
\[
\begin{aligned}
\partial_t w(p) - |x|^2 \Delta w(p) &= 0 & \text{in } \Omega \times [0,T], \\
\frac{\partial w(p)}{\partial n} &= 0 & \text{on } \Gamma_a \times [0,T], \\
w(p)(x,0) &= 0 & \text{in } \Omega.
\end{aligned}
\] (3.8)

which implies that \( w(p) = 0 \) in \( \Omega \times [0,T] \) by the unique continuation principle [14] and thus \( pu(\gamma^*) = 0 \) on \( \Gamma_i \times [0,T] \). Then we easily get \( p = 0 \) on \( \Gamma_i \) by Assumption 3.1.

Now we prove \( N \) is surjective, which means that for any element \( \varphi = \frac{\partial w}{\partial n} \in L^2(0,T;L^2(\Gamma_a)) \), we want to seek \( p \in L^2(\Gamma_i) \) such that \( N(p) = \varphi \). To do so, we first see from Assumption 3.2 that \( w(p) \) should also have the expression \( w(p) = te^{t-in}V(x) \) and when we consider the polar coordinates \((r,\theta), V(x)\) is only dependent on \( r \).

Next, we use the separation of variables in polar coordinates to solve (3.7) and thus get
\[ 0 = \partial_t w(p) - |x|^2 \Delta w(p) = te^{t-in} \left( V - r^2 \frac{\partial^2 V}{\partial r^2} - r \frac{\partial V}{\partial r} - \frac{\partial^2 V}{\partial \theta^2} \right), \]
\[ = te^{t-in} \left( V - r^2 \frac{\partial^2 V}{\partial r^2} - \frac{\partial V}{\partial r} \right). \]

This gives that
\[ r^2 \frac{\partial^2 V}{\partial r^2} + r \frac{\partial V}{\partial r} - V = 0, \]
which is an Euler ordinary differential equation and its general solutions, for any constants \(c_1, c_2\),
\[
V = c_1 r + c_2 r^{-1}.
\]

Hence we obtain on \(\Gamma_a \times [0, T]\), i.e., \(r = r_2\) that
\[
\varphi = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial r} = t e^{t - \ln t}(c_1 - c_2 \frac{1}{r_2}),
\]
\[
w = t e^{t - \ln t}(c_1 r_2 + c_2 \frac{1}{r_2}),
\]
and thus
\[
0 = \frac{\partial w}{\partial n} + w = t e^{t - \ln t}\{c_1(1 + r_2) - c_2\left(\frac{1}{r_2^2} - \frac{1}{r_2}\right)\}.
\]

For simplicity, we let \(c_2 = 1\), then \(c_1 = \frac{1 - r_2}{r_2^2(1 + r_2)}\) and obtain
\[
w = t e^{t - \ln t}\left(\frac{1 - r_2}{r_2^2(1 + r_2)} r + 1\right) \text{ in } \Omega.
\]

Therefore, with Assumption 3.2 we have on \(\Gamma_i \times [0, T]\), i.e., \(r = r_1\) that
\[
-d t e^{t - \ln t} v(r_1) = \frac{\partial w}{\partial n} + \gamma^*(r_1) w = -\frac{\partial w}{\partial r} + \gamma^*(r_1) w
\]
\[
= t e^{t - \ln t}\left\{-\frac{1 - r_2}{r_2^2(1 + r_2)} + \frac{1}{r_1^2} + \gamma^*(r_1)\left(\frac{1 - r_2}{r_2^2(1 + r_2)} r_1 + 1\right)\right\},
\]

which with Assumption 3.1 implies that
\[
d = -\frac{1}{v(r_1)}\left\{\frac{r_2 - 1}{r_2^2(1 + r_2)} + \frac{1}{r_1^2} + \gamma^*(r_1)\left(\frac{1 - r_2}{r_2^2(1 + r_2)} r_1 + 1\right)\right\}.
\]

As \(N\) is linear, bounded and bijective, then by the Open Mapping Theorem \(\text{[7]}\), we know that \(N^{-1}\) exists and \(\|N^{-1}\|\) is bounded, i.e., there exists a positive constant \(C\) such that
\[
\|N^{-1}\| \leq C.
\]

We are now ready to establish the local Lipschitz stability for parabolic inverse Robin problems.

**Theorem 3.1.** Under Assumption 3.1-3.2 we have
\[
\|\gamma_1 - \gamma_2\|_{\Gamma_i} \leq C\|u(\gamma_1) - u(\gamma_2)\|_{L^2(0, T; L^2(\Gamma_a))}.
\]

**Proof.** We first introduce an important mapping
\[
\theta : \gamma \in L^2(\Gamma_i) \rightarrow \frac{\partial u(\gamma)}{\partial n} \in L^2(0, T; L^2(\Gamma_a)),
\]
which is continuously Fréchet-differentiable from Lemma 3.3 and get
\[
\theta'(\gamma^*) p = \frac{\partial u'(\gamma^*) p}{\partial n} = N(p).
\]
Then it follows from lemma 3.3 that \(\theta'(\gamma^*)^{-1} = N^{-1}\) and
\[
\|\theta'(\gamma^*)^{-1}\| = \|N^{-1}\| \leq C.
\]

By the inverse function theorem \(\text{[8]}\) we find that \(\theta'(\gamma^*)\) is \(C^1\)-diffeomorphism on a neighborhood of \(\gamma^*\), consequently \(\theta'(\gamma^*)^{-1}\) is locally Lipschitz continuous and \(\|\theta'(\gamma^*)^{-1}\| = \|\theta'(\gamma^*)^{-1}\| \leq C\). Thus, there exists a neighborhood \(N(\gamma^*, \eta)\) of \(\gamma^*\) such that the Lipschitz constant is less equal to \(2\|\theta'(\gamma^*)^{-1}\|\), i.e., for any \(\gamma_1, \gamma_2 \in N(\gamma^*, \eta)\), it holds that
\[
\|\gamma_1 - \gamma_2\|_{\Gamma_i} \leq 2\|\theta'(\gamma^*)^{-1}\|\|\frac{\partial u(\gamma_1)}{\partial n} - \frac{\partial u(\gamma_2)}{\partial n}\|_{L^2(0, T; L^2(\Gamma_a))}
\]
\[
\leq 2C\|u(\gamma_2) - u(\gamma_1)\|_{L^2(0, T; L^2(\Gamma_a))}.
\]

\[
\|\gamma_1 - \gamma_2\|_{\Gamma_i} \leq 2C\|u(\gamma_2) - u(\gamma_1)\|_{L^2(0, T; L^2(\Gamma_a))}.
\]
Remark 3.1. Assumption 3.2 can be improved to more general situation: the solution $u(\gamma^*)$ to system (1.2) is completely separated into its spatial and temporal components and has the formulation $u(\gamma^*) = F(t)v(x)$ with $F(0) = 0$, $F(t) \neq 0$ and $F'(t) = F(t)$. When we consider the polar coordinates $(r, \theta)$, $v(x)$ and true Robin coefficient $\gamma^*$ are only dependent on $r$ but independent of $\theta$.

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