Functional delta residuals and applications to functional effect sizes

Fabian Telschow\textsuperscript{1}, Samuel Davenport\textsuperscript{2}, Armin Schwartzman\textsuperscript{1,3}

\textsuperscript{1}Division of Biostatistics, University of California, San Diego
\textsuperscript{2}Department of Statistics, University of Oxford
\textsuperscript{3}Halcioğlu Data Science Institute, University of California, San Diego

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Abstract

Given a functional central limit (fCLT) and a parameter transformation, we use the functional delta method to construct random processes, called functional delta residuals, which asymptotically have the same covariance structure as the transformed limit process. Moreover, we prove a multiplier bootstrap fCLT theorem for these transformed residuals and show how this can be used to construct simultaneous confidence bands for transformed functional parameters. As motivation for this methodology, we provide the formal application of these residuals to a functional version of the effect size parameter Cohen’s $d$, a problem appearing in current brain imaging applications. The performance and necessity of such residuals is illustrated in a simulation experiment for the covering rate of simultaneous confidence bands for the functional Cohen’s $d$ parameter.

Keywords: functional delta method, random field theory, simultaneous confidence bands, effect size, functional data analysis

1 Introduction

The motivation for our work comes from the following problem in spatial functional data analysis. Sommerfeld et al. \cite{Sommerfeld2018}, in the context of climate data, and Bowring et al. \cite{Bowring2019}, in the context of functional magnetic resonance imaging, study confidence statements for estimators of the mean function $\mu(s)$ from a sample $Y_1(s), \ldots, Y_N(s)$ of a signal plus noise model $Y(s) = \mu(s) + \varepsilon(s)$, where $\varepsilon(s)$ is a stochastic error process with variance function $\sigma(s)$, where $s$ is a spatial index. This requires estimation of the quantiles of the maximum of a limiting Gaussian processes. The quantiles are estimated using standardized residuals from the estimated mean function either through a multiplier bootstrap \cite{Chang2009, Chang2017} or the Gaussian kinematic formula \cite{Worsley2004, Adler2009}. These methods successfully approximate the quantiles, since the standardized residuals asymptotically have the same covariance structure as the limiting Gaussian process.

However, this approach no longer works when the object of interest is a nonlinear transformation of the parameters. In order to guarantee comparability between different scanners, Bowring et al. \cite{Bowring2020} extends the work of Bowring et al. \cite{Bowring2019} to the population Cohen’s $d$, i.e., $d(s) = \mu(s)/\sigma(s)$, rather than the mean function $\mu(s)$. This causes a new conceptional problem. While the standard residuals capture the covariance structure for the limiting Gaussian process in estimation of the mean, this no longer holds true for Cohen’s $d$ as we show in Corollary 1. We visualize this effect in Figures 1 and...
In particular, Figure 1 shows samples of Cohen’s $d$ residuals approximating the correct covariance structure.

In this paper, we use the functional delta method to construct random processes, called functional delta residuals, which can be used for obtaining distributional properties of the limiting process whenever the object of inference is a non linear transformation of the functional parameters. The proposed delta residuals are necessary because the nonlinearity not only affects the variance of the limiting transformed process but also its covariance function. As an application, we here use delta residuals and the quantiles of the maximum of the limiting process for construction of simultaneous confidence bands, a problem commonly found in functional data analysis [Degras 2011, Cao et al. 2012, 2014, Chang et al. 2017, Wang et al. 2019], for the Cohen’s $d$ parameter. Its extension to an application to spatial inference using coverage probability excursion sets for the Cohen’s $d$ parameter can be found in Bowring et al. [2020].

Given a functional central limit theorem (fCLT) and a parameter transformation, the construction of the delta residuals is obtained by linearisation in relation to the functional delta method. Our main result, Theorem 1, shows that delta residuals have asymptotically the covariance structure of the limiting process of the transformed parameters. In Section 3 we apply the general theory to the functional Cohen’s $d$ statistic, prove the necessary fCLT in Theorem 2, derive the corresponding delta residuals in Section 3.2 and prove, in Theorem 3, a multiplier functional limit theorem for the delta residuals based on Chang and Ogden [2009]. We use these results to construct simultaneous confidence bands and study the accuracy of their covering rate and the effect of using the wrong residuals in a simulation study in Section 4.

The methods for simultaneous confidence bands for Cohen’s $d$ are implemented in the R-package SCBfda available under [https://github.com/ftelschow/SCBfda](https://github.com/ftelschow/SCBfda) and code reproducing the presented simulation results are available under [https://github.com/ftelschow/DeltaResiduals](https://github.com/ftelschow/DeltaResiduals).

![Figure 1](https://example.com/figure1.png)

**Figure 1:** **Left:** samples of a Gaussian process with square exponential covariance function having a scaled and horizontally shifted Gaussian kernel with standard deviation 0.05 as mean. **Middle:** the standard residuals of this process. **Right:** samples from the delta residuals of Cohen’s $d$ of the same process as given in Corollary 1.

### 2 Functional Delta Residuals

In this section we introduce the construction of functional delta residuals. We develop the idea in the framework of the Banach space $C(S, \mathbb{R}^P)$ of continuous functions with values in $\mathbb{R}^P$ over a compact domain $S \subseteq \mathbb{R}^D$, however the concept can also be generalized to other Banach spaces. The norm on $C(S, \mathbb{R}^P)$ is the maximum norm $\|f\|_\infty = \max_{s \in S} |f(s)|$, where $|\cdot|$ denotes the standard norm on $\mathbb{R}^P$. 
For ease of readability $C(S, \mathbb{R})$ will be denoted by $C(S)$. Since a purely formal treatment hides the basic idea of delta residuals, we motivate them with a special case. Let $\theta \in C(S, \mathbb{R}^P)$ be a functional population parameter and let $\tilde{\theta}^{(1)}, ..., \tilde{\theta}^{(N)} \in C(S, \mathbb{R}^P)$ be estimators of $\theta$. Further, assume that their average $\tilde{\theta}_N = \frac{1}{N} \sum_{n=1}^{N} \tilde{\theta}^{(n)}$ satisfies a fCLT, i.e.,

$$\sqrt{N} \left( \tilde{\theta}_N - \theta \right) = \frac{1}{\sqrt{N}} \left( \sum_{n=1}^{N} \tilde{\theta}^{(n)} - \theta \right) \rightsquigarrow G,$$

where $G$ is a tight zero mean Gaussian process in $C(S, \mathbb{R}^P)$ with covariance function $c$ and $\rightsquigarrow$ denotes weak convergence in $C(S, \mathbb{R}^P)$. We call the processes $R_{N,n} = \tilde{\theta}^{(n)} - \tilde{\theta}_N$ with values in $\mathbb{R}^P$ standard residuals since, by the fCLT, their empirical covariance function converges to the covariance of $G$, i.e.,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} R_{N,n}(s) R_{N,n}^T(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left( \tilde{\theta}^{(n)}(s) - \theta(s) \right) \left( \tilde{\theta}^{(n)}(t) - \theta(t) \right)^T = \text{Cov}[G(s), G(t)] = c(s, t).$$

Here the convergence is in probability and $v^T$ denotes the transpose of a vector $v \in \mathbb{R}^P$. Almost sure convergence would require additional regularity conditions on the standard residuals $R_{N,1}, ..., R_{N,N}$. We discuss sufficient conditions in the case of Cohen’s $d$ in Section 3.

Let $H \in C^1(\mathbb{R}^P, \mathbb{R}^{P'})$ and suppose we are interested in inferring on $H(\theta) : S \to \mathbb{R}^{P'}$, where $H$ is applied pointwise via $s \mapsto H(\theta(s))$. Let $dH_x \in \mathbb{R}^{P' \times P}$ denote the derivative of $H$ at $x \in \mathbb{R}^P$. Then equations (1) and (2) suggest that the transformed processes

$$\tilde{R}_{N,n}(s) = dH_{\tilde{\theta}_N(s)} R_{N,n}(s),$$

which we call functional delta residuals, can be used to approximate the covariance structure of $dH_\theta G$, which is the limiting process from the delta method, in the sense that

$$\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} \tilde{R}_{N,n}(s) \tilde{R}_{N,n}^T(s') = dH_{\theta(s)} c(s, s') dH_{\theta(s')}^T,$$

with convergence again being in probability.
For illustrative purposes, consider the following more concrete example. Let \( \{X_{N,n} : N \in \mathbb{N}, 1 \leq n \leq N\} \) be a triangular array of random processes in \( C(S, \mathbb{R}^P) \) independent and identically distributed as \( X \) with \( c(s, s') = \text{Cov}[X(s), X(s')] \leq \infty \) and \( \mu = E[X] \). Let \( \hat{\theta}^{(n)} = X_{N,n} \) so that \( \hat{\theta}_N = \hat{X}_N = N^{-1} \sum_{n=1}^{N} X_{N,n} \), and suppose that \( N^{-1/2} (X_N - \mu) \Rightarrow G \) weakly in \( C(S, \mathbb{R}^P) \) for \( N \to \infty \) with \( G \) being a tight, zero mean Gaussian process with covariance \( \hat{c} \). Then \( R_{N,n} = X_{N,n} - \hat{X}_N \) are standard residuals satisfying (2). For \( H : \mathbb{R}^P \to \mathbb{R}^{P'} \) continuously differentiable, the delta residuals are given by \( \hat{R}_{N,n} = dH(\hat{X}_N)(X_{N,n} - \hat{X}_N) \), which can be used to approximate the covariance function \( \hat{c}(s, s') = dH(\hat{\mu})(c(s, s')dH(\hat{\mu}(s')) \). To be even more concrete, let \( P = 2 \). Thus, we can define \( X_{N,n} = (U_{N,n}, V_{N,n}) \). Say we are interested in the asymptotic behavior of the product of the sample means of the two components of the process. Then \( H(x, y) = xy \in \mathbb{R}^1 \) and the delta residuals are given by \( \hat{R}_{N,n} = \hat{U}_N(V_{N,n} - \bar{V}_N) + \hat{V}_N(U_{N,n} - \bar{U}_N) \). These delta residuals can be used to approximate the covariance \( \hat{c}(s, s') = (\mu_V(s), \mu_U(s))c(s, s')(\mu_V(s'), \mu_U(s'))^T \), where \( \mu_U = E[U_{N,n}] \) and \( \mu_V = E[V_{N,n}] \).

The next result is immediate, yet generalizes the previous concept of functional delta residuals to estimators \( \hat{\theta}_N \), which are not averages.

**Theorem 1.** Let \( N \in \mathbb{N} \) and \( \hat{\theta}_N \in C(S, \mathbb{R}^P) \) be an estimator such that as \( N \to \infty \)

\[
\sqrt{N} \left( \hat{\theta}_N - \theta \right) \Rightarrow G, \tag{4}
\]

weakly in \( C(S, \mathbb{R}^P) \), where \( G \) denotes a tight mean zero Gaussian process on \( C(S, \mathbb{R}^P) \) with mean zero and covariance function \( c \). Let \( H : \mathbb{R}^P \to \mathbb{R}^{P'} \) be a continuously differentiable function. Moreover, let \( \{R_{N,n} : N \in \mathbb{N}, 1 \leq n \leq N\} \) be a triangular array of random processes satisfying uniformly in probability that

\[
\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} R_{N,n}(s)R_{N,n}^T(s') = c(s, s'). \tag{5}
\]

Then the functional delta methods yields

\[
\sqrt{N}(H(\hat{\theta}_N) - H(\theta)) \Rightarrow dH_\theta G = \hat{G}, \quad N \to \infty \tag{6}
\]

with \( \hat{G} \) being a zero mean Gaussian process with covariance \( \hat{c}(s, s') = dH_{\theta(s)}c(s, s')dH_{\theta(s')}^T \). Furthermore, the functional delta residuals \( \hat{R}_{N,n}(s) = dH_{\hat{\theta}(s)}R_{N,n}(s), n = 1, ..., N \), satisfy

\[
\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} \hat{R}_{N,n}(s)\hat{R}_{N,n}^T(s') = \hat{c}(s, s')
\]

uniformly in probability.

**Proof.** By a simple Taylor expansion argument \( H \) considered as a function of \( C(S, \mathbb{R}^P) \to C(S, \mathbb{R}^{P'}) \) is Hadamard differentiable tangential to \( C(S, \mathbb{R}^P) \) and therefore [Kosorok, 2008 Theorem 2.8] implies that the functional delta method is applicable.

To prove the second statement we obtain by linearity of the differential that

\[
N^{-1} \sum_{k=1}^{N} \hat{R}_{N,n}(s)\hat{R}_{N,n}^T(s') = dH_{\hat{\theta}(s)} \left( N^{-1} \sum_{k=1}^{N} R_{N,n}(s)R_{N,n}^T(s') \right) dH_{\theta(s')}^T. \tag{7}
\]

The fCLT (4) yields \( \hat{\theta}_N \to \theta \) uniformly in probability. Hence the claim follows from (7) and \( dH_{\hat{\theta}(s)} \to dH_{\theta(s)} \) uniformly in probability as \( N \to \infty \) by the continuous mapping theorem. \( \square \)
Remark 1. Two observations are noteworthy. First, the factors $\sqrt{N}$ in equation (4) and $N^{-1}$ in equation (5) can be replaced by general factors tending to infinity and zero respectively. We only keep these simple factors for notational simplicity. Secondly, if $dH_{\theta(s)} = 0$ for all $s \in S$, then the delta residuals can be identically equal to zero. Here an assumption of higher differentiability of $H$ can be used to establish a similar result using a second order delta method.

3 Functional delta residuals for Cohen’s $d$

In this section we show how to apply Theorem 1 to the pointwise Cohen’s $d$ statistic for processes with $\mathcal{L}^1$-Hölder continuous paths, see Definition 1 below. This special continuity condition is needed to ensure that the sample mean and the sample variance satisfy a fCLT, which is necessary to obtain the functional delta residuals of Cohen’s $d$. As a second step, we establish a multiplier bootstrap result for the delta residuals. This result implies that the quantiles of the maximum of the limiting process of the functional delta method can be estimated consistently.

The purpose of these considerations is to provide a theoretical basis for the approach taken in Bowring et al. [2020]. Neuroimaging data is typically smoothed with a Gaussian kernel and therefore the assumption on the sample paths is satisfied for the smoothed process provided that the observed data at the voxels has finite 4th moment. To circumvent technicalities from the application in Bowring et al. [2020] we will demonstrate the usefulness of the delta residuals for the task of constructing simultaneous confidence bands for the functional Cohen’s $d$ parameter.

Hereafter, we assume that $X_1, \ldots, X_N \sim X$ is an i.i.d. sample in $C(S)$. The pointwise population Cohen’s $d$ is the function defined by

$$d(s) = \frac{\mathbb{E}[X(s)]}{\sqrt{\text{Var}[X(s)]}} = \frac{\mu(s)}{\sigma(s)} = H(\mu(s), \sigma^2(s))$$

(8)

with $H(x, y) = xy^{-1/2}$. The Cohen’s $d$ parameter is estimated using its corresponding sample counterpart

$$\hat{d}_N(s) = H(\bar{X}_N(s), \hat{\sigma}_N^2(s)) = \frac{N^{-1} \sum_{n=1}^{N} X_n(s)}{\sqrt{N^{-1} \sum_{n=1}^{N} (X_n(s) - N^{-1} \sum_{n=1}^{N} X_n(s))^2}}.$$ 

(9)

The biased variance estimator is used in the denominator, since the delta residuals will be simpler.

Remark 2. Two observations are noteworthy here.

a.) The estimator $\hat{d}_N(s)$ (9) is not unbiased as an estimate of $d(s)$. In the case that $X$ is Gaussian, unbiasedness can be achieved by introducing a bias correcting factor depending on $N$ [Laubscher 1960, p. 1106].

b.) It will be obvious from the proofs that the theorems on delta residuals for Cohen’s $d$ hold true not only for $H(x, y) = x/\sqrt{y}$ but any $H : \mathbb{R}^D \to \mathbb{R}$.

3.1 A Functional Central Limit Theorem

We want to apply Theorem 1 to the function $H(x, y) = xy^{-1/2}$. As such we need to establish a fCLT for the process $(\bar{X}_N, \hat{\sigma}_N^2)$, which takes values in $\mathbb{R}^2$. The following sample path property will be our main assumption on the process $X$ to prove the fCLT.

Definition 1. Let $Z$ be a process in $C(S)$. Given $p \in \mathbb{N}$, we say that $Z$ has $\mathcal{L}^p$-Hölder continuous paths, if

$$|Z(s) - Z(s')| \leq L|s - s'|^\alpha$$

(10)
for a positive random variable $L$ with $\mathbb{E}[L^p] < \infty$ and $0 < \alpha \leq 1$.

**Remark 3.** $L^2$-Hölder continuous paths ensure that $Z$ satisfies a fCLT, i.e., for $Z_1, \ldots, Z_N \sim Z$ iid processes in $C(S)$, the sum $N^{-1/2} \sum Z_n$ converges weakly to a tight mean zero Gaussian process which has the same covariance structure as $Z$, see Jain and Marcus [1973] Theorem 1.

The following Lemma states useful properties of processes with $L^p$-Hölder continuous paths.

**Lemma 1.** Let $Y, Y_1, \ldots, Y_N, \ldots$ and $Z, Z_1, \ldots, Z_N, \ldots$ be iid processes in $C(S)$ having $L^p$-Hölder continuous paths with $p \geq 1$ over a compact set $S$, and assume that there exists $s' \in S$ such that $\mathbb{E}[Y(s')]^p$ is finite. Then $\mathbb{E}[\|Y\|_p^q] < \infty$ for all $q \leq p$ and $\bar{Y}_N \rightarrow \mathbb{E}[Y]$ uniformly almost surely. If $p \geq 2$, then also $N^{-1} \sum_{n=1}^N (Y_n - \bar{Y}_N) (Z_n - \bar{Z}_N) \rightarrow \text{Cov}[Y, Z]$ uniformly almost surely.

**Proof.** First claim: Using the convexity of $| \cdot |^p$ and $\mathbb{E}[Y(s')]^p < \infty$ we have

$$\mathbb{E}[\|Y\|_p^q] \leq 2^{p-1} \left( \mathbb{E}[\|Y - Y(s')\|_p^q] + \mathbb{E}[|Y(s')|^p] \right).$$

where $A$ is the random variable from the $L^p$-Hölder property. This yields $\mathbb{E}[\|Y\|_p^q] < \infty$ for all $q \leq p$.

We now apply the generic uniform convergence result in Davidson [1994, Theorem 21.8]. Since pointwise convergence is obvious by the strong law of large numbers, we only need to establish strong stochastical equicontinuity of the random function $\bar{Y}_N - \mathbb{E}[Y]$. This is established using Davidson [1994, Theorem 21.10 (ii)], since

$$|\bar{Y}_N(s) - \bar{Y}_N(s') - \mathbb{E}[Y(s) - Y(s')]| \leq \left( \sum_{n=1}^N \frac{A_n}{N} + \mathbb{E}[A] \right) |s - s'|^{\alpha} = C_N |s - s'|^{\alpha}$$

for all $s, s' \in S$. Here $A_1, \ldots, A_N \sim A$ iid denote the random variables from the $L^p$-Hölder paths of the $Y_n$’s and $Y$. Hence the random variable $C_N$ converges almost surely to the constant $2\mathbb{E}[A]$ by the strong law of large numbers.

Second claim: With the same strategy and assuming w.l.o.g. $\mathbb{E}[Y] = \mathbb{E}[Z] = 0$, we compute

$$\left| \frac{1}{N} \sum_{n=1}^N Y_n(s)Z_n(s) - Y_n(s')Z_n(s') \right| \leq \left( \frac{1}{N} \sum_{n=1}^N \|Y_n\|_\infty B_n + \|Z_n\|_\infty A_n \right) |s - s'|^{\alpha}$$

$$\leq \left( \sqrt{\sum_{n=1}^N \frac{\|Y_n\|_\infty^2}{N} \sum_{n=1}^N B_n^2 \frac{1}{N}} + \sqrt{\sum_{n=1}^N \frac{\|Z_n\|_\infty^2}{N} \sum_{n=1}^N A_n^2 \frac{1}{N}} \right) |s - s'|^{\alpha},$$

where $B_1, \ldots, B_N \sim B$ iid denote the random variables from the $(L^2, \delta)$-Hölder property of the $Z_n$’s and $Z$. Again by the strong law of large numbers the random Hölder constant converges almost surely and is finite.

Since we are dealing with vector-valued random processes, we need the next lemma in our proof of Theorem 2. It states simple conditions for obtaining weak convergence of a vector-valued process from its components.

**Lemma 2.** Let $X_1, X_2, \ldots, X, Y_1, Y_2, \ldots, Y$ be $C(S)$-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_N \xrightarrow{d} X$ and $Y_N \xrightarrow{d} Y$. If the finite dimensional distributions of $(X_N, Y_N)$ converge to those of $(X, Y)$, we have $(X_N, Y_N) \xrightarrow{d} (X, Y)$ in $C(S) \times C(S)$. 


Proof. Since \( X_N \sim X \) and \( Y_N \sim Y \) in \( C(S) \) and \( C(S) \) is complete and separable, the sequences are tight and so for each \( \epsilon > 0 \), there exist compact \( A, B \in C(S) \) such that for all \( N \), \( \mathbb{P}_{X_N}(A) > 1 - \epsilon \) and \( \mathbb{P}_{Y_N}(B) > 1 - \epsilon \). This implies
\[
\mathbb{P}_{X_N, Y_N}(A \times B) = \mathbb{P}_{X_N, Y_N}(C(S) \times B \cap A \times C(S)) \geq 1 - 2\epsilon.
\]
The latter is true, since in general \( \mathbb{P}(\bar{A} \cap \bar{B}) \geq 1 - \alpha - \beta \), if \( \mathbb{P}(\bar{A}) \geq 1 - \alpha \) and \( \mathbb{P}(\bar{B}) \geq 1 - \beta \), and since
\[
\mathbb{P}_{X_N, Y_N}(C(S) \times B) = \mathbb{P}(X_N \in C(S), Y_N \in B) = \mathbb{P}(Y_N \in B) \geq 1 - \epsilon
\]
and similarly \( \mathbb{P}_{X_N, Y_N}(A \times C(S)) \geq 1 - \epsilon \). This holds for all \( N \) and so the sequence \( (X_N, Y_N) \) is tight. Tightness implies relative compactness by Prohorov’s theorem.

Moreover, the finite dimensional distributions converge and form a separating class in \( C(S) \times C(S) \) (the proof is along the lines of Example 1.3 in [Billingsley, 1999, p. 12]) so in particular the joint distribution converges (arguing as in Example 5.1 in [Billingsley, 1999, p. 57]).

With these preparatory results we are now able to prove the main theorem of this section.

Theorem 2. Let \( S \) be a compact space and \( X_1, \ldots, X_N \sim X \) be an i.i.d. sample in \( C(S) \) satisfying \( \sup_{s \in S} \text{Var}[X^2(s)] < \infty \) and having \( L^1 \)-Hölder continuous paths. Then
\[
\sqrt{N} \left( (\bar{X}_N, \hat{\sigma}_N^2) - (\mu, \sigma^2) \right) \rightsquigarrow G
\]
where \( G \) is a 2D mean zero Gaussian process with covariance matrix function \( \epsilon \) given by
\[
\epsilon(s, s') = \begin{pmatrix}
c_{11}(s, s') & c_{12}(s, s') \\
c_{12}(s, s') & c_{22}(s, s')
\end{pmatrix}
\]
with \( c_{11}(s, s') = \text{Cov}[X(s), X(s')] \), \( c_{12}(s, s') = \text{Cov}[X(s), (X(s') - \mu(s'))^2] \) and \( c_{22}(s, s') = \text{Cov}[(X(s) - \mu(s))^2, (X(s') - \mu(s'))^2] \).

Proof. Since for iid random variables \( Z_1, Z_2, Z_3, \ldots \) with mean \( \mu \) we have that
\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( Z_n - \frac{1}{N} \sum_{n=1}^{N} Z_n \right)^2 = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (Z_n - \mu)^2 + \epsilon_N,
\]
where \( \epsilon_N \rightsquigarrow 0 \) as \( N \to \infty \), we can w.l.o.g replace \( \bar{X}_N \) by \( \mu \) in the definition of \( \hat{\sigma}_N \) from equation (9) and further, for simplicity, assume \( \mu(s) = 0 \) for all \( s \in S \).

For \( d \in \mathbb{N} \) and any \( s_1, \ldots, s_d \in S \), applying the multivariate CLT to the sequence of vectors
\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( X_n(s_1), X_n^2(s_1) - \sigma^2(s_1), \ldots, X_n(s_d), X_n^2(s_d) - \sigma^2(s_d) \right)^T
\]
yields convergence to the finite dimensional distributions of \( G \) from the statement of the theorem. Hence the finite dimensional distributions of \( (\bar{X}_N, \hat{\sigma}_N^2) \) converge to those of \( G \). Since the process \( X \) is \( L^1 \)-Hölder continuous, we have \( L^2 \)-Hölder bounds on \( X \) and \( X^2 - \sigma^2 \) as shown in the proof of Theorem 5 in [Telschow and Schwartzman, 2019]. Thus, by Jain and Marcus [1975] both satisfy the CLT in \( C(S) \). In particular, by Lemma 2 we obtain the iCLT for \( (X_N, \hat{\sigma}_N^2) \).}

The functional delta method yields the following corollary.
yielding the simplified version of the limiting covariance structure. Moreover, if $X$ is a Gaussian process, then $\tilde{c}$ simplifies to

$$
\tilde{c}(s, s') = \left( \sigma(s)^{-1}, -\mu(s)\sigma(s)^{-3/2} \right) \sigma(s, s') \left( \sigma(s')^{-1}, -\mu(s')\sigma(s')^{-3/2} \right)^T
$$



(12)

Hence we can identify

$$
\hat{\theta}(n) = \left( X_n(s), (X_n(s) - \bar{X}_N(s))^2 \right)
$$

with standard residuals

$$
R_{N,n} = \hat{\theta}(n) - \bar{\theta}_N = \left( X_n - \bar{X}_N, (X_n - \bar{X}_N)^2 - \sigma_N^2 \right).
$$

(14)

3.2 Functional Delta Residuals

In the previous section we established that the estimator $\hat{d}_N$ of $d$ given in (9) satisfies a fCLT. Therefore, we now derive the corresponding functional delta residuals. Theorem 2 states that

$$
\sqrt{N} \left( N^{-1} \sum_{n=1}^{N} \left( X_n(s), (X_n(s) - \bar{X}_N(s))^2 \right) - (\mu, \sigma^2) \right)
$$

converges to a tight mean zero Gaussian process. This is an average estimator as discussed in Section 2. Hence we can identify

$$
\hat{\theta}(n) = \left( X_n(s), (X_n(s) - \bar{X}_N(s))^2 \right)
$$

with standard residuals

$$
R_{N,n} = \hat{\theta}(n) - \bar{\theta}_N = \left( X_n - \bar{X}_N, (X_n - \bar{X}_N)^2 - \sigma_N^2 \right).
$$

(14)
The functional delta residuals for $H(x,y) = x/\sqrt{y}$ therefore are

$$\tilde{R}_{N,n} = \left(\hat{\sigma}_N^{-1} - \bar{X}_N \hat{\sigma}_N^{-3}/2\right)^T R_{N,n} = \frac{X_n - \bar{X}_N}{\hat{\sigma}_N} - \frac{\hat{d}}{2} \left(\frac{(X_n - \bar{X}_N)^2}{\hat{\sigma}_N^2} - 1\right). \tag{15}$$

We call these residuals Cohen’s $d$ residuals. It is easy to show that $\sum_{n=1}^N \tilde{R}_{N,n} = 0$. To prove that these residuals satisfy Theorem 1, the following result is necessary.

**Lemma 3.** Suppose $X$ has $\mathcal{L}^4$-Hölder continuous paths and $\mathbb{E}[|X^4(s)|] < \infty$ for some $s \in S$, then the standard residuals \[r\] are componentwise $\mathcal{L}^2$-Hölder continuous.

**Proof.** For the component $(X_n - \bar{X}_N)$ it is clear that the process has $\mathcal{L}^4$-Hölder continuous paths. Therefore we only prove the claim for $(X_n - \bar{X}_N)^2$. Note that for all $s, t \in S$

$$\begin{align*}
(X_n(s) - \bar{X}_N(s))^2 - (X_n(t) - \bar{X}_N(t))^2 &= X_n^2(s) - X_n^2(t) + 2(X_n(s)\bar{X}_N(s) - X_n(t)\bar{X}_N(t)) + \bar{X}_N^2(s) - \bar{X}_N^2(t).
\end{align*}$$

Here each term can be bounded in a similar manner. As such we only provide the bound for the middle term, which is

$$|X_n(s)\bar{X}_N(s) - X_n(t)\bar{X}_N(t)| \leq |X_n(s)\bar{X}_N(s) - X_n(s)\bar{X}_N(t)| + |X_n(s)\bar{X}_N(t) - X_n(t)\bar{X}_N(t)|$$

$$\leq \|X_n\|_\infty |\bar{X}_N(s) - \bar{X}_N(t)| + \|\bar{X}_N\|_\infty |X_n(s) - X_n(t)|$$

$$\leq \left(\frac{\|X_n\|_\infty}{N} \sum_{n=1}^N L_n + \frac{L_\alpha}{N} \sum_{n=1}^N \|X_n\|_\infty\right) |s - t|^\alpha = A |s - t|^\alpha.$$ 

Applying the inequality $ab \leq a^2 + b^2$, using that $\mathbb{E}\left[\|X_n\|_\infty^4\right] < \infty$ by Lemma 1 and $\mathbb{E}[L_n^4] < \infty$ shows that $\mathbb{E}[A^2] < \infty$. \hfill \Box

Lemma 3 together with Lemma 1 implies almost sure uniform convergence of

$$N^{-1} \sum_{n=1}^N R_{N,n}(s)R_{N,n}^T(s') \to \epsilon(s, s'), \tag{16}$$

where $\epsilon$ is given in Theorem 2. Hence Theorem 1 holds true for the Cohen’s $d$ residuals.

### 3.3 A Multiplier Bootstrap Functional Limit Theorem

Our main application of delta residuals is to approximate statistics that depend on the limiting process $\tilde{G}$ in (6) or quantiles thereof such as quantiles of the maximum of the process. The latter are used in Bowring et al. [2020] to construct coverage probability excursion sets for Cohen’s $d$. In order to justify their construction we establish weak conditional convergence for the multiplier process based on the delta residuals. For $N \geq 1$, the multiplier bootstrap process is defined by

$$\tilde{G}_{N}^r = \frac{1}{\sqrt{N}} \sum_{n=1}^N r_{N,n}\tilde{R}_{N,n}, \tag{17}$$

where $\{r_{N,1}, ..., r_{N,N} : N \in \mathbb{N}\}$ is an iid triangular array of multipliers satisfying $\mathbb{E}[r_{N,1}] = 0$ and $\mathbb{E}[r_{N,1}^2] = 1$. Moreover, the multipliers are assumed to be independent of the $X_n$’s and thereby independent of the delta residuals $\tilde{R}_{N,n}$ defined in equation (15).
The following theorem is based on the proofs from [Chang and Ogden, 2009] and implies that the multiplier bootstrap process \( \hat{G}_N^r \) conditioned on the delta residuals asymptotically has similar sample path properties as the limiting process \( \hat{G} \) from the delta method. As such it can, for example, be used to estimate quantiles of the maximum, see Remark 3.

**Theorem 3.** Under the assumptions of Theorem 2 the following statements hold

\[
(i) \quad \hat{G}_N^r \rightsquigarrow \hat{G} \quad \quad (ii) \quad \sup_{h \in BL_1(S)} \left| \mathbb{E}_r h(\hat{G}_N^r) - \mathbb{E} h(\hat{G}) \right| \rightarrow 0 .
\]

Here convergence in (ii) is in outer probability, \( \mathbb{E}_r \) is the expectation over \( r_N,1, ..., r_N, N \) conditional on \( \bar{R}_{N,1}, ..., \bar{R}_{N,N} \) and \( BL_1(S) \) is the set of all \( h : C(S) \rightarrow \mathbb{R} \) such that \( \sup_{f \in C(S)} |h(f)| \leq 1 \) and \( |h(f) - h(f')| \leq \|f - f'\|_\infty \) for all \( s, s' \in S \).

**Theorem 3** (i) It suffices to prove the result for the multiplier bootstrap process defined by the standard residuals \( R_{N,1}, ..., R_{N,N} \) with weak convergence towards \( G \). This can be seen as follows: \( dH_{\hat{\theta}_N} \) converges to \( dH_{\theta} \) uniformly almost surely by Theorem 2 and the continuous mapping theorem. Here \( \hat{\theta}_N = (\hat{X}_N, \hat{\sigma}_N^2) \) and \( \theta = (\mu, \sigma^2) \). Thus, applying Theorem 18.10(v) from [Van der Vaart, 2000], we obtain the weak convergence

\[
\left( dH_{\hat{\theta}_N}, \frac{1}{\sqrt{N}} \sum_{n=1}^{N} r_{N,n} R_{N,n} \right) \rightsquigarrow \left( dH_{\theta}, G \right) ,
\]

where \( G \) is given in Theorem 2. The continuous mapping Theorem [Van der Vaart, 2000, Theorem 18.11], then yields

\[
dH_{\hat{\theta}_N} \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} r_{N,n} R_{N,n} \right) \rightsquigarrow dH_{\theta}G = \hat{G} .
\]

Let us define the unobservable iid samples

\[
R_{N,n} = (X_n - \mu, (X_n - \mu)^2 - \sigma^2) ,
\]

where \( \mu = \mathbb{E}[X] \) and \( \sigma = \text{Var}[X] \). By definition these samples satisfy \( \mathbb{E}[R_{N,n}] = 0 \). Since \( X \) has \( L^4 \)-Hölder continuous paths and \( \mathbb{E}[\|X^4\|_\infty] < \infty \), both components of \( R_{N,n} \) divided by \( \sqrt{N} \) satisfy (A), (B), (C) and (D) from [Chang and Ogden, 2009] meaning their Theorem 1 and 2 are applicable. In particular, applying Lemma 2 this means that \( \sum R_{N,n}/\sqrt{N} = \sum r_{N,n} R_{N,n}/\sqrt{N} \) is a random process converging uniformly to zero as \( N \) tends to infinity. Thus, \( \sum r_{N,n} R_{N,n}/\sqrt{N} \) converge weakly to \( G \) in the space of bounded functions over \( S \). Since \( G \) and all \( R_{N,n} \) are assumed to be continuous processes, the convergence is also in \( C(S) \) by [Van Der Vaart and Wellner, 1996, Theorem 1.3.10]. This finishes the proof of part (i).

(ii) Let \( \tilde{G}_N^r = \sum r_{N,n} \tilde{R}_{N,n}/\sqrt{N} \) and \( \hat{G}_N^r = \sum r_{N,n} \hat{R}_{N,n}/\sqrt{N} \). Then

\[
\sup_{h \in BL_1(S)} \left| \mathbb{E}_r h(\tilde{G}_N^r) - \mathbb{E} h(\tilde{G}) \right| \leq \sup_{h \in BL_1(S)} \left| \mathbb{E}_r h(\hat{G}_N^r) - h(dH_{\hat{\theta}_N} \hat{G}_N^r) \right| + \sup_{h \in BL_1(S)} \left| \mathbb{E}_r h(dH_{\hat{\theta}_N} \hat{G}_N^r) - \mathbb{E} h(\hat{G}) \right| \\
\leq \|dH_{\hat{\theta}_N}\|_\infty \mathbb{E}_r \|\hat{G}_N^r - \tilde{G}_N^r\|_\infty + \sup_{h \in BL_1(S)} \left| \mathbb{E}_r h(dH_{\hat{\theta}_N} \hat{G}_N^r) - \mathbb{E} h(\hat{G}) \right| .
\]
The weak convergence of $dH_{\theta_N}G_N^r$ to $\tilde{G}$ proved in part (i) implies that the second summand converges to zero as $N$ tends to infinity. Moreover, $\|dH_{\theta_N}\|_\infty$ converges almost surely to $\|dH_{\theta}\|_\infty < \infty$ by the continuous mapping theorem. Thus, it remains to prove that $\mathbb{E}_r \| G^r_N - G^r_{N_N} \|_\infty$ converges to zero. We treat the summands derived from the two components of the vector valued processes separately. The first component yields

$$\mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N r_{N,n} (X_{n} - \mu) - \sum_{n=1}^N r_{N,n} (X_{n} - \bar{X}_N) \right\|_\infty \right]$$

$$= \mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N r_{N,n} (\mu - \bar{X}_N) \right\|_\infty \right]$$

$$= \mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N r_{N,n} \right\|_\infty \| \mu - \bar{X}_N \|_\infty \right].$$

This converges to zero, since the factor $\frac{1}{\sqrt{N}} \sum_{n=1}^N r_{N,n} \sim N(0, 1)$ and the second converges to zero by Lemma[1].

The second component can be bounded as follows:

$$\mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N r_{N,n} \left( (X_n - \mu)^2 - \sigma^2 \right) - \sum_{n=1}^N r_{N,n} \left( (X_n - \bar{X}_N)^2 - \tilde{\sigma}_N^2 \right) \right\|_\infty \right]$$

$$= \mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N r_{N,n} \left( (X_n - \mu)^2 - (X_n - \bar{X}_N)^2 \right) - \sum_{n=1}^N r_{N,n} \left( \sigma^2 - \tilde{\sigma}_N^2 \right) \right\|_\infty \right]$$

$$\leq \mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N r_{N,n} \left( \mu^2 - \bar{X}_N^2 + 2X_n(\bar{X}_N - \mu) \right) \right\|_\infty \right] + \mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \sum_{n=1}^N r_{N,n} \right] \| \mu^2 - \tilde{\sigma}_N^2 \|_\infty$$

As argued before, the second summand in the last row converges to zero. To establish the same result for the first summand note that

$$\mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N r_{N,n} \left( \mu^2 - \bar{X}_N^2 + 2X_n(\bar{X}_N - \mu) \right) \right\|_\infty \right]$$

$$\leq \mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \sum_{n=1}^N r_{N,n} \right] \| \mu^2 - \tilde{\sigma}_N^2 \|_\infty + \mathbb{E}_r \left[ \frac{1}{\sqrt{N}} \sum_{n=1}^N r_{N,n} X_n \right] \| \bar{X}_N - \mu \|_\infty$$

which again converges to zero by Lemma[1]. This finishes the proof of (ii).

The usefulness of the above theorem is mainly due to the following corollary.

**Corollary 2.** Given any continuous function $F : C(S) \rightarrow \mathbb{R}$, for every point $a \in \mathbb{R}$ at which $\mathbb{P}(F(\tilde{G}) \leq a)$ is continuous, $\mathbb{P}(F(G_N^r) \leq a) \rightarrow \mathbb{P}(F(\tilde{G}) \leq a)$.

**Proof.** Suppose the claim is false, then there exists $a \in \mathbb{R}, \epsilon > 0$ and a subsequence $(N_j)_{j \in \mathbb{N}}$ such that for all $j$, $|\mathbb{P}(F(G_N^r) \leq a) - \mathbb{P}(F(\tilde{G}) \leq a)| > \epsilon$. Now, applying Theorem 2(ii) and Lemma 1.9.2(ii) from Van Der Vaart and Wellner [1996], it follows that there exists a subsequence $(N_{jk})_{k \in \mathbb{N}}$ such that $\sup_{h \in BL_1(S)} |\mathbb{E}_r h(G_N^r) - \mathbb{E}_r h(\tilde{G})|$ converges outer almost surely to 0. In particular by Theorem 1.12.2 in Van Der Vaart and Wellner [1996] and the continuous mapping theorem, $F(G_N^r)$ converges weakly to $F(\tilde{G})$. This gives a contradiction.

\[\square\]
Remark 4. The above corollary applies to $F$ being the maximum norm $\| \cdot \|_{\infty}$. This means that the multiplier bootstrap consistently estimates the quantiles of the maximum, which we will use for the construction of simultaneous confidence bands in the next section.

3.4 Simultaneous confidence bands

An application of delta residuals is the construction of simultaneous confidence bands for the population Cohen’s $d$ or $H(\theta)$ in general. Corollary 1 suggests that the asymptotic variance of Cohen’s $d$ can be estimated from a sample $X_1, \ldots, X_N$ using

$$\hat{\sigma}_N(s) = \sqrt{1 + \bar{X}_N(s) \hat{\sigma}_N(s)/2(2\hat{\sigma}_N(s)^2)}.$$ 

This converges uniformly almost surely to the asymptotic variance of Cohen’s $d$ by Lemma 1 and the continuous mapping theorem. Thus, we consider the collection of intervals $SCB(s, N, q_\alpha)$ with endpoints

$$\hat{d}_N(s) \pm q_\alpha \sqrt{\frac{\hat{\sigma}_N(s)}{N}},$$

where $q_\alpha$ satisfies $P(\max_{s \in S} |\hat{G}(s)| > q_\alpha) = \alpha$ and with $\hat{G}$ being the mean zero Gaussian process such that

$$\text{Cov}[\hat{G}(s), \hat{G}(s')] = \frac{\hat{\sigma}(s, s')}{\sqrt{\hat{\sigma}(s)'\hat{\sigma}(s')}}.$$ (19)

From similar arguments as in Chang et al. [2017], it is possible to derive the following result.

Theorem 4. Under the assumptions of Theorem 2 and the notation above we have that

$$\lim_{N \to \infty} P\left( \hat{d}_N(s) \in SCB(s, N, q_\alpha) \text{ for all } s \in S \right) = 1 - \alpha,$$

meaning that these intervals form asymptotic $(1 - \alpha)$-simultaneous confidence bands for $H(\theta)$.

Estimation of the quantile $q_\alpha$ can be approached by estimating the quantile of the maximum of the multiplier bootstrap process based on the delta residuals from Section 3.3. This is an adaptation of Chang et al. [2017] and Corollary 2 implies that this estimate is consistent for $q_\alpha$.

A different approach assumes that the residuals have $C^3$ sample paths and utilizes the Gaussian kinematic formula. Here the quantile $q_\alpha$ is approximated by exploiting the fact that for large $u$,

$$P\left( \max_{s \in S} \hat{G}(s) > u \right) \approx \mathcal{L}_0 \Phi^+(u) + \sum_{d=1}^D \mathcal{L}_d \rho_d(u),$$

as shown in Taylor et al. [2005]. The functions $\rho_d(u) = (2\pi)^{-(d+1)/2} H_{d-1}(u) e^{-u^2/2}$, $d = 1, \ldots, D$, are the so-called Euler characteristic densities, where $H_d$ is the $d$-th probabilistic Hermite polynomial and $\Phi^+(u) = P(N(0, 1) > u)$. The coefficients $\mathcal{L}_0, \ldots, \mathcal{L}_D$ are referred to as the Lipschitz Killing curvatures of $S$, which are intrinsic volumes of $S$ considered as a Riemannian manifold endowed with a Riemannian metric induced by $\hat{G}$ [Adler and Taylor, 2009, Chapter 12]. In particular, $\mathcal{L}_0 = \chi(S)$ is the Euler characteristic of the set $S$, which is usually known.

Given consistent estimators $\hat{\mathcal{L}}_1, \ldots, \hat{\mathcal{L}}_D$ of the Lipschitz Killing curvatures an estimate $\hat{q}_\alpha$ of $q_\alpha$ can be found by finding the largest $u$ solving

$$\mathcal{L}_0 \Phi^+(u) + \sum_{d=1}^D \hat{\mathcal{L}}_d \rho_d(u) = \alpha.$$
Currently there are only a few works dealing with estimation of Lipschitz Killing curvatures for nonstationary processes and arbitrary dimensional domains \( S \), see Taylor and Worsley [2007], Telschow et al. [2020]. The estimators from the last two sources require residuals having asymptotically the covariance structure of the limiting process \( \hat{G} \). Such residuals are the Cohen’s \( d \) residuals, if we normalize them to have empirical variance 1.

4 Simulations for Simultaneous Confidence bands of Cohen’s \( d \)

In this section we study the covering rate of simultaneous confidence bands for Cohen’s \( d \). We use 5 000 Monte Carlo simulations to assess the coverage and evaluate the processes on a grid of \([0, 1]\) composed of 175 equally spaced points. The bands are based on the methodology from Section 3.4.

The quantile \( q_\alpha \) is estimated using either the standardized residuals or the Cohen’s \( d \) residuals through the multiplier bootstrap given in Chang et al. [2017] with Rademacher multipliers or the Gaussian kinematic formula.

4.1 Delta Residuals vs. Standard Residuals

Using two different stochastic processes defined on \( S = [0, 1] \), we compare results for the covering rate of simultaneous confidence bands constructed using the techniques described in Section 3.4. The processes are given by

\[
X(s) = 3 + \frac{3\phi_{0.25,0.05}(s)}{2\phi_{0.05}(0)} \cdot \frac{a^T K^G(s)}{\|K^G(s)\|},
\]

\[
Y(s) = 3 + (s + 0.3) \cdot \frac{b^T K^B(s)}{\|K^B(s)\|},
\]

where \( \phi_{\mu,\sigma^2} \) is the density of a \( N(\mu, \sigma^2) \) random variable and \( a \sim N(0, I_{21 \times 21}), b \sim N(0, I_{7 \times 7}) \).

Moreover, \( K^G(s) \in \mathbb{R}^{21} \) is a vector with \( i \)-th entry given by \( \exp \left( - (s - x_i)^2 / (2h_i^2) \right) \) with \( x_i = i/21, \) \( h_1 = 0.04 \) for \( i < 10, h_{11} = 0.2 \) and \( h_i = 0.08 \) for \( i > 10 \). \( K^B(s) \in \mathbb{R}^7 \) is a vector with entries \( \binom{6}{i} s^i (1 - s)^{6-i} \) the \((i,6)\)-th Bernstein polynomial for \( i = 0, ..., 6 \). Sample paths from these two processes are shown on the left hand side of Figure 3. Its right hand side shows that the Gaussian kinematic formula method using the delta residuals (GKF (delta)) gives very accurate covering rates. The multiplier bootstrap overestimates the covering rate for small sample sizes, yet converges for \( N \approx 100 \) to the targeted nominal level. The methods using the standard residuals GKF (std) and rMult (std), have the wrong nominal asymptotical coverage. This is not surprising, since by Corollary the standard residuals do not have the same covariance structure as the limiting process from the fCLT for Cohen’s \( d \).
Figure 3: **Left:** sample paths from processes (20) (top) and (21) (bottom). **Right:** dependence of covering rate on the sample size and method for quantile estimation for processes (20) (top) and (21) (bottom).

### 4.2 Dependence on Cohen’s d values

In order to study the dependence of the covering of the simultaneous confidence bands on Cohen’s d values, we use the following stochastic process

\[
X(s) = \eta \cdot \left(1.5e^{\frac{(s-0.5)^2}{0.045}} - 0.5\right) + Z_e(s) \quad \text{with} \quad \eta \in \{0.1, 0.3, 0.7, 1, 2, 3\},
\]

where \(Z_e\) is a Gaussian mean zero process with covariance function \(C(h) = \exp\left(-h^2/10\right)\). In this parametrization \(\eta\) corresponds to the maximum of Cohen’s d over all \(s\). Since the GKF with the delta residuals was the best method in the previous simulations, we only provide results for this method, compare Figure 4. Our simulations show that achieving the nominal coverage rate seems to be independent of the maximum of Cohen’s d.
Figure 4: **Left:** sample paths from the processes \( 22 \). **Right:** dependence of covering rate on the sample size and maximal signal-to-noise ratio.

## 5 Discussion

Despite their mathematical simplicity, functional delta residuals are a powerful and necessary tool for inference for functional data. They are useful for spatial coverage probability excursion sets, as demonstrated in [Bowring et al., 2020], and for the construction of simultaneous confidence bands for Cohen’s \( d \), as done in this article. Future research can extend the application of these residuals to other parameter estimators satisfying a fCLT derived from the functional delta method. Potential extensions are coverage probability excursion sets or simultaneous confidence bands for \( R^2 \) processes from linear models such as those usually fitted in functional magnetic resonance imaging analysis. Another potential application in this context are new statistical methods such as LISA [Lohmann et al., 2018]. The latter introduces spatial smoothing of the z-score field of brain activation together with an false discovery rate controlled inference. In this context, any inference based on random field theory, e.g., cluster inference or coverage probability excursion sets, that is used to detect activation of their smoothed z-score process, will require residuals having the correct asymptotic correlation structure in order to perform valid inference. These residuals can be derived through our Theorem [1].

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