Optimal quantum state discrimination with confidentiality

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We investigate quantum state discrimination with confidentiality. $N$ observers share a given quantum state belonging to a finite set of known states. The observers want to determine the state as accurately as possible and send a discrimination result to a receiver. However, the observers are not allowed to get any information about which state was given. $N$ – 1 or fewer observers might try to steal the information, but if $N$ observers coexist, the honest ones will keep the dishonest ones from doing anything wrong. Assume that the state set has a certain symmetry, or more precisely, is Abelian geometrically uniform; this letter describes the case of three linearly independent cyclic pure states as a special case. We propose a protocol that realizes any optimal inconclusive measurement, which is a generalized version of a minimum-error measurement and an optimal unambiguous measurement, for such a state set and ensures that any combined state of $N$ – 1 or fewer observers has absolutely no information about the given state. Our protocol provides a method of performing a quantum measurement securely, which could be useful in quantum information applications.

Suppose that a sender wants to send a classical message to a receiver in the harsh environment, such as deep space. They cannot communicate directly; thus, the sender sends the message to a third party, called an observer, and the observer sends it to the receiver. Consider that the observer receives a quantum state $\rho_m$ belonging to a set of known quantum states, $\rho_0, \rho_1, \ldots, \rho_{M-1}$, which are mutually non-orthogonal. The observer performs a quantum measurement on $\rho_m$ and sends its result to the receiver using classical communication. However, the message is highly private and/or sensitive (e.g., a classified message), and so the observer is not allowed to get any information about $m$. What can the observer do to send as precise information about $m$ as possible to the receiver while ensuring that the observer obtains no information?

Consider that there are two observers, Alice and Bob, instead of one and they receive the state $\rho_m$. We propose a protocol where they tell a discrimination result to the receiver, Charlie, while ensuring information security. We assume that Alice or Bob might try to steal the information about $m$ by illegal means, but if they coexist, the honest one will keep the dishonest one from doing anything wrong. Despite this assumption, our protocol will demonstrate that absolutely no information is leaked to Alice or Bob. We explain our protocol using Fig. 1. Alice and Bob first transform a given state $\rho_m$ into $\rho'_m$ as a preprocessing step, where $\rho'_m$ is a (generally entangled) state of their composite system. In this step, they cannot perform a wrong or evil action since they coexist. They next independently measure their individual systems. In this step, a dishonest observer may try to extract information about $m$. They tell their outcomes to Charlie via classical communication, and Charlie determines $m$ from their outcomes. We refer to such a measurement as a bipartite secure measurement if neither Alice nor Bob obtains any information about $m$ even if they act dishonestly in their individual measurements.

As is well known, non-orthogonal states cannot be perfectly distinguished; thus, we want to find a measurement that performs best in a certain strategy. In one strategy, a measurement that maximizes the average correct probability [1], denoted by a minimum-error measurement, has been investigated [2, 3]. In another strategy, a measurement that achieves unambiguous, i.e., error-free, discrimination with the minimum average failure probability [4], denoted by an optimal unambiguous measurement, has also been studied [5]. Recently, as a more general measurement, a measurement that maximizes the average correct probability with a fixed average failure probability, which we refer to as an optimal inconclusive measurement (OIM), has been investigated [6]. Minimum-error measurements and optimal unambiguous measurements can be interpreted as special cases of OIMs.

Remarkably, we show that any OIM, with any average

![FIG. 1. Data flow for a bipartite secure measurement.](image-url)
failure probability, for the state set \{ρ_m\} can be realized with a bipartite secure measurement if \{ρ_m\} has certain symmetry properties, or more precisely, if it is a (not necessarily pure) Abelian geometrically uniform (AGU) state set [7]. Such a state set is a broad class of quantum state sets, including optical phase shift keyed coherent state sets, pulse position modulated state sets, and linear codes with binary letter-states [8]. We also investigate the multipartite case and derive that a multipartite secure measurement can realize any OIM if \{ρ_m\} is an AGU state set. For simplicity, throughout this Letter we consider only three linearly independent cyclic pure states, but our technique can be extended to AGU states (see Supplemental Material [9]).

Although our scheme and secret sharing might seem somewhat similar, they are quite different. In classical secret sharing [10], a classical, i.e., perfectly distinguishable, secret is split among several parties. A method for sharing an arbitrary unknown quantum state has also been proposed [11], which provides a quantum version of secret sharing. In such schemes, the parties share a classical or quantum state, which can be perfectly reconstructed when a sufficient number of parties cooperate. In contrast, in a bipartite secure measurement, the observers share a classical message encoded in quantum states that are not perfectly distinguishable, i.e., a given quantum state cannot be perfectly reconstructed from the measurement outcome. Moreover, in our scenario, the observers cannot communicate with each other after preprocessing. Our scheme provides a method for optimally discriminating between quantum states with confidentiality using the basic idea of secret sharing, though our technique is drastically different from that of secret sharing.

Let us begin by considering a minimum-error measurement, and later extend it to an OIM.

**Theorem 1** A minimum-error measurement for three linearly independent cyclic pure states can be realized with a bipartite secure measurement.

**Proof** Assume that \(Ψ = \{|ψ_m⟩ : m ∈ I_3\}\) is a three linearly independent pure state set, where \(I_t = \{0, 1, . . . , t − 1\}\). Let \(H\) and \(P\), respectively, be the space spanned by \(Ψ\) and the projection operator onto \(H\). Also, assume that \(Ψ\) is cyclic; i.e., there exists a unitary operator \(V\) with \(V^M = P\) such that \(|ψ_0⟩ = V^m |ψ_0⟩\), and \(\{|ψ_m⟩\}\) have equal probabilities. Let \(Π^{(c)} = \{Π_m^{(c)} : m ∈ I_3\}\) be a positive operator-valued measure (POVM) representing a minimum-error measurement on \(H\). \(Π^{(c)}\) is always projective, and rank \(Π_m^{(c)} = 1\) holds [12], which means that \(Π_m^{(c)}\) is expressed by \(Π_m^{(c)} = |π_m^{(c)}⟩⟨π_m^{(c)}|\) with an orthonormal basis (ONB) \(\{|π_m^{(c)}⟩\}\) in \(H\). Moreover, \(Π^{(c)}\) is cyclic, i.e., \(|π_m^{(c)}⟩ = V^m |π_0^{(c)}⟩\) holds [3].

We can see that a necessary and sufficient condition for a bipartite secure measurement is that the candidate states after preprocessing, \(\{ρ'_m\}\), satisfy

\[Tr_A ρ'_j = Tr_A ρ'_k, \quad Tr_B ρ'_j = Tr_B ρ'_k, \quad (1)\]

for any \(j, k ∈ I_3\), where \(Tr_A\) and \(Tr_B\), respectively, represent the partial traces over Alice’s and Bob’s systems, which implies that whether a given procedure is a bipartite secure measurement is determined only by the preprocessing. Indeed, suppose by contradiction that \(Tr_A ρ'_j ≠ Tr_A ρ'_k\) holds for certain \(j, k ∈ I_3\); then, there exists Bob’s measurement that gives some information to distinguish \(ρ'_j\) and \(ρ'_k\). It follows that, in order to obtain a bipartite secure measurement that can realize the minimum-error discrimination, we must consider a measurement such that neither Alice nor Bob knows any information about the outcome obtained by Charlie. This means that Alice’s measurement outcome must be independent of Charlie’s outcome, and so must that of Bob. To realize this, let us consider preprocessing that transforms \(|π_m^{(c)}⟩\), corresponding to Charlie’s outcome, into \(|η_m⟩\) such that \(Tr_A |η_m⟩⟨η_m|\) and \(Tr_B |η_m⟩⟨η_m|\) are independent of \(m\). To be concrete, let

\[|η_m⟩ = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} |a_k⟩|b_{m ∘ k}\rangle, \quad (2)\]

where \(∘\) denotes the subtraction modulo 3, and \(\{|a_m⟩\}\) and \(\{|b_m⟩\}\) are ONBs in Alice’s and Bob’s spaces, respectively. Such preprocessing can be realized with the completely positive trace-preserving (CPTP) map \(L^{(c)}(X) = A^{(c)} X A^{(c)\dagger}\), where \(A^{(c)} = \sum_k |η_k⟩⟨π_k^{(c)}|\). This preprocessing turns the given state \(|ψ_m⟩\) into \(|ψ'_m⟩ = A^{(c)} |ψ_m⟩\). Now, we show that \(ρ'_m = |ψ'_m⟩⟨ψ'_m|\) satisfies Eq. (1). Let \(χ_k = |π_k^{(c)}⟩⟨ψ_0|\); then, since \(\{|π_m^{(c)}⟩\}\) and \(\{|ψ_m⟩\}\) are cyclic, \(|π_k^{(c)}⟩ |ψ_m⟩ = χ_{k ∘ m}\) holds, which gives \(|ψ'_m⟩ = \sum_k χ_{k ∘ m} |η_k⟩\). Thus, from Eq. (2), we have

\[Tr_A ρ'_m = \sum_{j,k,l=0}^{2} \frac{X_{jl}χ_{kj}^\ast}{3} |b_{j ∘ l}⟩⟨b_{k ∘ l}| \quad \Rightarrow \quad \sum_{j',k',l'=0}^{2} \frac{X_{j'k'}χ_{j'k'}^\ast}{3} |b_{j' ∘ l'}⟩⟨b_{k' ∘ l'}|, \quad (3)\]

where \(j' = j \in m, k' = k \in m, l' = l \in m\). This equation means that \(Tr_A ρ'_m\) is independent of \(m\). In the same way, we can easily derive that \(Tr_B ρ'_m\) is also independent of \(m\). Therefore, Eq. (1) holds.

The last thing we have to show is that the minimum-error discrimination can be realized with only the local operations to the state \(|ψ'_m⟩\). Now, we consider the following procedure: Alice and Bob independently perform measurements for \(|ψ'_m⟩\) in the ONBs \(\{|a_m⟩\}\) and \(\{|b_m⟩\}\), and then send their outcomes \(j\) and \(k\), corresponding to \(|a_j⟩\) and \(|b_k⟩\), to Charlie, respectively. Charlie records his result as \(j ⊕ k\), where \(⊕\) is the addition modulo 3.
It follows that this procedure can be represented by the POVM $\Phi^{(e)} = \{\Phi^{(e)}_m\}$ with
\[
\Phi^{(e)}_m = \sum_{k=0}^2 |a_k\rangle \langle a_k| \otimes |b_{m\otimes k}\rangle \langle b_{m\otimes k}|.
\]
We obtain
\[
\langle \psi'_m | \Phi^{(e)}_n | \psi'_m \rangle = \sum_{k=0}^2 \left| \langle a_k| b_{n\otimes k}\rangle \right|^2 \chi_{n\otimes m} |\eta_m\rangle^2 = |\chi_{n\otimes m}|^2 = \langle \psi_m | \Pi^{(e)}_n | \psi_m \rangle,
\]
which indicates that this procedure can realize the minimum-error discrimination.

We extend this argument to an OIM in the following theorem.

**Theorem 2** An OIM with any average failure probability for three linearly independent cyclic pure states can be realized with a bipartite secure measurement.

**Proof** Let $I_3^2$ be the set formed by adding element `?' to the set $I_3$ and $\Psi$ be a set of three linearly independent cyclic pure states. Also, let $\Pi = \{\Pi_m : m \in I_3^2\}$ be an OIM on $H$ for $\Psi$. The operator $\Pi_m$ ($m \in I_3$) corresponding to identification of the state $|\psi_m\rangle$, is rank one and thus can be expressed in the form $\Pi_m = |\pi_m\rangle \langle \pi_m|$ [13]. In contrast, $\Pi_?$ corresponding to failure, is generally not rank one. Assume without loss of generality that $\Pi$ is cyclic, i.e., $|\pi_m\rangle = V^m |\pi_0\rangle$ holds [13].

In the proof of Theorem 1, to realize a minimum-error measurement with a bipartite secure measurement, we exploited the fact that the POVM $\Pi^{(e)}$ is projective and cyclic. We want to apply a similar approach to an OIM; however, $\Pi$ is generally non-projective. We consider, instead of $\Pi$, an OIM that is projective and cyclic. Let $\Omega = \{\Omega_m : m \in I_3^2\}$ be an projective measurement expressed as
\[
\Omega_m = |\omega_m(0)\rangle \langle \omega_m(0)|, \quad m \in I_3,
\]
\[
\Omega_? = \sum_{m=0}^2 |\omega_m(1)\rangle \langle \omega_m(1)|,
\]
where $\{|\omega_m(s)\rangle : m \in I_3, s \in I_2\}$ is an ONB in $H_{ex}$ ($H_{ex}$ is a six-dimensional Hilbert space including $H$). Assume that $\Omega$ is an OIM for $\Psi$, which satisfies $P\Omega_m \Pi = \Pi_m$ for any $m \in I_3$, and that for each $s \in I_2$, $\{P|\omega_m(s)\rangle : m \in I_3\}$ is cyclic, i.e., we have
\[
P|\omega_m(s)\rangle = V^m P|\omega_0(s)\rangle.
\]
As will be described later, these assumptions hold if we properly choose an ONB $\{|\omega_m(s)\rangle\}$. Now, under these assumptions, we show that a bipartite secure measurement can realize an OIM for $\Psi$.

First, we show preprocessing in which a bipartite secure measurement is possible, i.e., Eq. (1) holds. Consider that Alice and Bob perform the preprocessing represented by the CPTP map $\mathcal{E}(X) = AXA^\dagger$ with $A = \sum_{s,m} |\eta_m(s)\rangle \langle \omega_m(s)|$, where
\[
|\eta_m(s)\rangle = \frac{1}{\sqrt{6}} \sum_{q=0}^1 \sum_{k=0}^2 |a_k^{(q)}\rangle |b_{m\otimes k}^{(q,s)}\rangle, \quad s \in I_2, m \in I_3.
\]

○ is the addition modulo 2, and $\{|a_m(s)\rangle\}$ and $\{|b_m(s)\rangle\}$ are ONBs in Alice’s and Bob’s systems. Since $\{P|\omega_m(s)\rangle\}$ and $\{|\psi_m\rangle\}$ are cyclic, $\langle \psi_m | \Pi_m | \psi_m \rangle = (\omega_m | \psi_m \rangle, | \psi_m \rangle)$ holds. Thus, we can verify that the state after preprocessing $|\psi'_m\rangle = A|\psi_m\rangle$ satisfies Eq. (1) in the same way as in Eq. (3).

Next, we show that an OIM can be realized with the following procedure: Alice and Bob independently perform the measurements for the state $|\psi_m\rangle$ in the ONBs $\{\eta_m(s)\rangle\}$ and $\{\eta_m(s)\rangle\}$, and send their outcomes (denoted by $|a_j^{(q)}\rangle$ and $|b_k^{(s)}\rangle$) to Charlie, respectively. Charlie records his result as $j \oplus k$ if $q = s$ and “failure” otherwise. This procedure can be represented by the POVM $\Phi = \{\Phi_m : m \in I_3\}$ with
\[
\Phi_m = \sum_{q=0}^1 \sum_{k=0}^2 |a_k^{(q)}\rangle \langle a_k^{(q)}| \otimes |b_{m\otimes k}^{(q,s)}\rangle \langle b_{m\otimes k}^{(q,s)}|, \quad m \in I_3,
\]
\[
\Phi_? = \sum_{q=0}^1 \sum_{k=0}^2 |a_k^{(q)}\rangle \langle a_k^{(q)}| \otimes |b_{m\otimes k}^{(q,s)}\rangle \langle b_{m\otimes k}^{(q,s)}|.
\]
In a similar way to Eq. (4), we can easily verify that $\langle \psi'_m | \Phi_k | \psi'_m \rangle = \langle \psi_m | \Omega_k | \psi_m \rangle$. Therefore, this procedure realizes an OIM.

Finally, we have to show that an ONB $\{|\omega_m(s)\rangle\}$ exists such that $\Omega$ is an OIM for $\Psi$ and $\{P|\omega_m(s)\rangle\}$ is cyclic. Let $\{\phi_m : m \in I_2\}$ be an ONB in $H$ such that the Schatten decomposition of $\Pi$ is represented by $\Pi = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|$. We choose an ONB $\{|\phi_m(s)\rangle\}$ in $H_{ex}$ such that
\[
P|\phi_m^{(0)}\rangle = \sqrt{1 - \lambda_m} |\phi_m\rangle,
\]
\[
P|\phi_m^{(1)}\rangle = \sqrt{\lambda_m} |\phi_m\rangle.
\]
This implies that the one-dimensional subspace span($|\phi_m\rangle$) of $H$ is associated with the two-dimensional subspace span($|\phi_m^{(0)}\rangle, |\phi_m^{(1)}\rangle$) of $H_{ex}$. Let $F_s = \sum_k |\phi_s^{(s)}\rangle \langle \phi_s^{(s)}|$, which is an isometric mapping from $H$ to span($|\phi_0^{(0)}\rangle, |\phi_1^{(0)}\rangle, |\phi_2^{(0)}\rangle$), and $\{\nu_m\}$ be an ONB in $H$ satisfying $\Lambda |\nu_m\rangle = |\pi_m\rangle$, where $\Lambda = (P - \Pi_?)^{1/2}$ (such an ONB always exists [14]). We choose $|\omega_m^{(0)}\rangle$ as $|\omega_m^{(0)}\rangle = F_0 |\nu_m\rangle$ and $|\omega_m^{(1)}\rangle = F_1 |\nu_m\rangle$, where $|\pi_m\rangle$ is a detection vector of the minimum-error measurement $\Pi^{(e)}$. We show that $\{|\omega_m^{(s)}\rangle\}$ is an ONB in $H_{ex}$ that we sought. From Eq. (5) and the definition of $F_s$, we can easily verify that $P F_0 = \Lambda$ and $P F_1 = \Pi_?^{1/2}$ hold. The former equation yields
\[
P|\omega_m^{(0)}\rangle = P F_0 |\nu_m\rangle = \Lambda |\nu_m\rangle = |\pi_m\rangle,
\]
which indicates $P \Omega_m \Pi = \Pi_m$. Also, $P \Omega_? = P (P_{ex} - \sum_{m=0}^2 \Omega_m) \Pi = \Pi_?$ holds ($P_{ex}$ is the projection operator.
Let $A$. Chefles and S. M. Barnett, J. Mod. Opt. M. Ban, K. Kurokawa, R. Momose, and O. Hirota, Int. I. D. Ivanovic, Phys. Lett. A. G. Blakley, independently perform measurements in the ONBs $\{|\psi_m\rangle\}$.

In the multiparty scenario, there are Bob cooperate to gain the information. In the multiparty measurement is not sufficiently secure; for example, if Alice and states. As a preprocessing step, Alice, Bob, and Charlie transform a given state $|\psi_m\rangle$ into $|\psi'_m\rangle$ by the preprocessing $L$. They next perform measurements in the ONBs $\{|\omega_m^{(s)}\rangle\}$ and $\{|\varpi_m^{(s)}\rangle\}$, and send their outcomes to Charlie. The average failure probability can be controlled by properly choosing the ONB $\{|\omega_m^{(s)}\rangle\}$. In this discussion, we consider the preprocessing $L$ that transforms $|\psi_m\rangle$ into a generally entangled state. We can also show that a bipartite secure measurement that realizes an OIM exists such that $\rho_m'$ is always separable [9].

We consider extending this scenario to the multipartite case. The more observers there are, the more secure the scheme would become. Thus, it is desirable to increase the number of observers if a bipartite secure measurement is not sufficiently secure; for example, if Alice and Bob cooperate to gain the information. In the multipartite scenario, there are $N \geq 3$ observers and one receiver. Let us consider the following protocol: $N$ observers first share a given state by preprocessing. After that, they independently perform measurements and send their results to the receiver. We refer to the measurement as an $N$-partite secure measurement if any combined state of $N - 1$ or fewer observers has absolutely no information about which state was given. Here, we describe the case of $N = 3$ for three linearly independent cyclic pure states. As a preprocessing step, Alice, Bob, and Charlie transform $\rho_m$ into $\rho_m' = L(\rho_m)$ with the CPTP map $L_N(X) = A_N X A_N^\dagger$, where $A_N = \sum_{s,m} |\varpi_m^{(s)}\rangle \langle \omega_m^{(s)}|$ and

$$|\varpi_m^{(s)}\rangle = \frac{1}{\sqrt{6}} \sum_{q_1,q_2=0}^2 \sum_{k_1,k_2=0}^2 |a_{q_1}^{(k_1)}\rangle |b_{q_2}^{(k_2)}\rangle |c_{m\oplus k_1\oplus k_2}^{(s)}\rangle.$$  

$\{|c_m^{(s)}\rangle\}$ is an ONB in Charlie’s system. They independently perform measurements in the ONBs $\{|a_{m}^{(s)}\rangle\}$, $\{|b_{m}^{(s)}\rangle\}$, and $\{|c_{m}^{(s)}\rangle\}$, and send their outcomes (denoted by $|a_{q_1}^{(q_2)}\rangle$, $|b_{k_1}^{(k_2)}\rangle$, and $|c_{m\oplus k_1\oplus k_2}^{(r)}\rangle$) to Dave. Dave records his result as $j\oplus r$ if $q \oplus s \oplus r = 0$ and “failure” otherwise. 

In a similar way to the bipartite case, we can see that using this procedure, any OIM can be realized with a tripartite secure measurement. We can show that if possible candidate states are AGU, then any OIM can be realized with an $N$-partite secure measurement for any $N \geq 3$ (proof in Supplemental Material [9]).

In summary, we have provided a quantum measurement scheme, called an $N$-partite secure measurement, that provides information security. In our bipartite protocol, Alice and Bob first share a quantum state obtained with preprocessing. They next independently perform the measurements, in which neither Alice nor Bob gets any information about which state was given even if one does anything wrong, and send their results to Charlie. We stated that an OIM for AGU states can be realized with an $N$-partite secure measurement.

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[14] Let $|\varpi_m\rangle = A^+ |\varpi_m\rangle$, where $A^+$ is the Moore-Penrose inverse of $A$. Since $\sum_m |\varpi_m\rangle \langle \varpi_m| = A^+ A^2 A^+ = P_A$ (PA is the projection operator onto the support of $A$), $\{|\varpi_m\rangle \langle \varpi_m| : m \in \mathcal{I}_3\}$ is a POVM. From Naimark’s theorem, an ONB $\{|m\rangle : m \in \mathcal{I}_3\}$ exists such that $P_A |m\rangle = |m\rangle$.

This ONB satisfies $A |\varpi_m\rangle = A |\varpi_m\rangle = |\varpi_m\rangle$.