Generating black holes in the novel 4D Einstein-Gauss-Bonnet gravity

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Recently Glavan and Lin [Phys. Rev. Lett. 124, 081301 (2020)] proposed a Einstein-Gauss-Bonnet gravity (EGB) in which the Gauss-Bonnet coupling has been rescaled as $\alpha/(D-4)$ and the 4D theory is defined as the limit $D \to 4$, namely the novel 4D EGB theory. It preserves the number of degrees of freedom thereby free from the Ostrogradsky instability and admits black hole solutions. We prove a theorem that characterizes a large family of nonstatic or radiating black hole solutions to the novel 4D EGB theory, representing, in general, spherically symmetric Type II fluid. This enables us to identify the recently found Vaidya-black hole solution to the novel theory, as particular cases, but also to find new solutions of the theory, such as the analogous of the Bonnor-Vaidya (de Sitter/anti-de Sitter) solution, a global monopole, and the Husain black holes. A trivial extension of the theorem to generate static spherically symmetric black hole solutions of the theory is also presented.

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I. INTRODUCTION

Amongst, other solutions of general relativity (GR) or any modified theories of gravity, black holes remain one of the most exciting and active areas of study, since they throw challenging questions about the fundamental interaction between gravity and quantum mechanics. The concept of black holes began very shortly after Einstein’s GR came to existence, Schwarzschild [1] found the solution to Einstein’s equations in vacuum. Soon after, the electrovacuum static black hole solution was obtained [2], since then, numerous static black hole solutions sourced by some energy-matter distributions have been reported. Though the black hole’s uniqueness theorems [3] encapsulate that, in the Einstein-Maxwell theory, the unique black hole solutions are stationary and axially symmetric and are completely defined by three parameters, however, in the presence of complicated matter fields distributions, uniqueness theorems or even black hole solutions are difficult to obtain.

Nevertheless, Salgado [4] proved a theorem characterizing a three-parameter family of static and spherically symmetric black hole solutions to Einstein equations by imposing certain conditions on the energy-momentum tensor (EMT). This allows to generate a large family of exact static spherically symmetric black hole solutions, including their generalization to asymptotically de Sitter/Anti-de Sitter (dS/AdS) spacetimes. Salgado [4] work was promptly extended to higher-dimensional spacetime by Gallo [5]. Though the static solutions should represent the eventually steady state of the dynamic evolution of black holes, this is obviously not the most physical scenario and one would like to consider dynamical black hole solutions, i.e., black holes with non-trivial time dependence. However, due to the complexity of the Einstein field equations, such solutions are intractable and very few meaningful dynamical or nonstatic solutions are known. The Vaidya metric [6] is one of the nonstatic solution of Einstein’s equations with spherical symmetry whose metric in the Eddington-Finkelstein coordinates $\{v, r, \theta, \phi\}$ has a form [6],

$$ds^2 = - \left[1 - \frac{2m(v)}{r}\right] dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \epsilon \pm 1$$

for a null fluid (radiation) source (a Type II fluid [7]) described by EMT $T_{ab} = \psi l_a l_b$, $l_a$ being a null vector field, and $m(v)$ is the mass function in advance time $v$. The Vaidya geometry permitting the incorporations of the effects of null fluid or null dust offers a more realistic background than static geometries. The Vaidya solution is commonly used as an exterior solution for gravitational collapse models consisting of heat-conducting matter [8] and useful to get insights in gravitational collapse situations [9], as a testing ground for the Cosmic Censorship Conjecture (CCC) [10], to model the dynamical evolution of a Hawking evaporating black holes [11], and in the stochastic gravity program.

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The EGB gravity [16] is a special case of Lovelock’s theory of gravitation [17], whose Lagrangian contains just the first three terms. The Gauss-Bonnet term yields non-trivial dynamics in $D \geq 5$. It appears naturally in the low energy effective action of heterotic string theory [18]. Boulware and Deser [19] found exact black hole solutions in $D (\geq 5)$-dimensional EGB gravitational theory. Other spherically symmetric black hole solutions in the Gauss-Bonnet gravity have been found and discussed in [20]. In four-dimensional (4D) spacetime, the Gauss-Bonnet term does not contribute to the gravitational dynamics since it becomes a total derivative. However, the EGB gravity theory is reformulated in which the Gauss-Bonnet coupling has been re-scaled as $\alpha/(D - 4)$ [21]. This novel 4D EGB theory is defined as the limit $D \to 4$ at the level of equations of motion, which preserves the number of degrees of freedom and thereby free from the Ostrogradsky instability [21]. Further, this natural extension of Einstein’s gravity bypasses all conditions of Lovelock’s theorem [22] and is also free from the singularity problem.

This stimulates research in 4D EGB gravity and various solutions of theory have been found, namely the static spherically symmetric black holes [21] and their charged extension in AdS space [23], rotating black holes and their shadows [24, 25], Vaidya-like radiating black holes [26], relativistic stars solution [27]. Furthermore, the quasinormal modes, stability and shadows of spherically symmetric black holes [28, 29], the motion of a classical spinning test particle [30] and thermodynamical phase transitions in AdS space [31] have also been investigated. The extension to higher-order Lovelock gravity is presented in Refs. [32, 33].

The main aim of this work is to prove theorems that were presented in [14, 15] so that a large family of exact spherically symmetric Type II fluid solutions are possible, including its generalization to asymptotically dS/AdS within framework of the novel 4D EGB gravity. As a result, we are able to find the analogous several GR solutions in the novel 4D EGB gravity.

The paper is organized as follows: In section II, we review the novel 4D EGB gravity theory and recall the static spherically symmetric black hole solution. In section III, we prove the theorem for generating dynamical black holes in the 4D EGB gravity and discuss some particular cases. In section IV, we investigate the imposition of energy conditions on these metrics. Section V is devoted to the construction of static black holes. We summarize our findings and discuss possible future works in section VI.

**II. THE NOVEL 4D EINSTEIN-GAUSS-BONNET GRAVITY**

The 4D novel EGB gravity is obtained by rescaling the coupling constant, $\alpha/(D - 4)$, of the Gauss-Bonnet term, and then consider the limit $D \to 4$ [21]. The action for EGB gravity theory with re-scaled coupling constant $\alpha/(D-4)$, reads

$$S = \frac{1}{16 \pi} \int d^D x \sqrt{-g} \left[ R - 2\Lambda + \frac{\alpha}{D - 4} (R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) \right] + S_{\text{matt}},$$

where $R$, $R_{ab}$ and $R^{ab}$ are, respectively, the Ricci scalar, Ricci tensor and Riemann curvature tensor, $g$ is the determinant of the metric tensor $g_{ab}$, $\Lambda$ is the cosmological constant, and $S_{\text{matt}}$ is the matter fields action. Varying action Eq. (2) with $g_{ab}$, i.e., $\delta S/g^{ab} = 0$, yield the equations of motion as follow

$$8\pi T_{ab} = G_{ab} = G_{ab}^{(0)} + G_{ab}^{(1)} + G_{ab}^{(2)},$$

where $T_{ab}$ is the EMT associated with the matter-field distribution resulting from the variation $\delta S_{\text{matt}}/\delta g^{ab}$, and

$$G_{ab}^{(0)} = \Lambda g_{ab}$$

$$G_{ab}^{(1)} = R_{ab} - \frac{1}{2} R g_{ab}$$

$$G_{ab}^{(2)} = -\frac{\alpha}{D - 4} \left[ \frac{1}{2} g_{ab} (R_{cde}R^{cde} - 4R_{c}R^{c} + R^2) - 2RR_{ab} + 4R_{ac}R_{b}^{c} + 4R_{ac}R_{b}^{c} - 2R_{acde}R_{b}^{cde} \right],$$

where $G_{ab}^{(1)}$ and $G_{ab}^{(2)}$, respectively, are the Einstein’s tensor and the Lanczos’s tensor [16]. The four-dimensional theory is defined as the limit $D \to 4$, at the level of equations of motion rather than at the level of the action [21].
Taking the $D$ dimensional static and spherically symmetric metric

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2d\Omega_{D-2}^2,$$  \hspace{1cm} (7)

with

$$d\Omega_{D-2}^2 = d\theta_i^2 + \sum_{i=2}^{D-2} \prod_{j=1}^{i-1} \sin^2 \theta_j \, d\theta_i^2,$$  \hspace{1cm} (8)
as ansatz, and solving the field equation (3) in the limit $D \to 4$, we get the static spherically symmetric black hole solution in the novel 4D EGB gravity with $[21, 23]$

$$F(r) = 1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{8M\alpha}{r^3} + \frac{4\Lambda\alpha}{3}} \right).$$ \hspace{1cm} (9)

Here, $M$ is the black hole mass and the two branches of solutions corresponds for the “±” sign. At large distances, Eq. (9) reduces to

$$F_-(r) = 1 - \frac{2M}{r\sqrt{1 + 4\alpha}} + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{4\Lambda\alpha}{3}} \right) + \mathcal{O}\left( \frac{1}{r^2} \right),$$

$$F_+(r) = 1 + \frac{2M}{r\sqrt{1 + 4\alpha}} + \frac{r^2}{2\alpha} \left( 1 + \sqrt{1 + \frac{4\Lambda\alpha}{3}} \right) + \mathcal{O}\left( \frac{1}{r^2} \right),$$ \hspace{1cm} (10)
which shows that only the “−” branch solution, $F_-(r)$, that asymptotically goes over to the Schwarzschild black hole with the correct mass sign, i.e., has the correct limit for $\alpha \to 0$. We consider that the Gauss-Bonnet coupling parameters $\alpha$ is positive and $\Lambda$ is a constant quantity. Though the semi-classical gravity with a conformal anomaly $[34]$ and the theory of gravity with quantum corrections $[35]$ also admit similar black hole solutions as found in Eq. (9), the EGB gravity can be considered as a classical modified gravity theory on equal footing with GR. Regularized Lovelock gravity with an arbitrary curvature order, when truncated at quadratic order, also admits the similar black hole solution $[33]$.

Next, we consider a theorem, so that a large family of exact spherically symmetric dynamical black hole solutions, for the novel 4D EGB gravity, are possible. The generated solutions represent generalization of Vaidya-like solutions to this theory.

### III. RADIATING BLACK HOLES SOLUTIONS

**Theorem I:** Let $(\mathcal{M}, g_{ab})$ be a $D$-dimensional space-time such that: i) it satisfies in the limit $D \to 4$ the Einstein-Gauss-Bonnet gravity equations obtained by re-scaled coupling constant $\alpha/(D-4)$, ii) it is spherically symmetric, iii) in the Eddington-Bondi coordinates, where the metric reads $ds^2 = -A^2(v, r)F(v, r)dv^2 + 2\epsilon A(v, r)dvdr + r^2d\Omega_{D-2}^2$, the EMT $T^a_b$ satisfies the conditions $T^v_v = 0$, and $T^\gamma_\gamma = \gamma T^r_r$, $(\gamma = \text{const} \in \mathbb{R})$, iv) if $\alpha \to 0$, the solution converges to the 4D GR limit. Then the metric of the space-time is given by

$$ds^2 = -F(v, r)dv^2 + 2\epsilon dvdr + r^2d\Omega_2^2, \hspace{1cm} (\epsilon = \pm 1),$$ \hspace{1cm} (11)

where

$$F_\pm(v, r) = \begin{cases} 
1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{4\Lambda\alpha}{3} + \frac{8M(v)a}{r^3} - \frac{32\pi\alpha C(v)}{(1+2\gamma)r^{2(1-\gamma)}}} \right) & \text{if } \gamma \neq -\frac{1}{2}, \\
1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{4\Lambda\alpha}{3} + \frac{8M(v)a}{r^3} - \frac{32\pi\alpha C(v)\ln r}{r^3}} \right) & \text{if } \gamma = -\frac{1}{2},
\end{cases}$$ \hspace{1cm} (12)
with the diagonal components of EMT $T^a_b$ given by

$$T^a_b(\text{Diag}) = \frac{C(v)}{r^{2(1-\gamma)}} \text{diag}[1, 1, \gamma, \gamma],$$ \hspace{1cm} (13)
and only non-vanishing off-diagonal element as 

\[ T^r_v = \begin{cases} \frac{1}{4\pi r^2} \frac{dM(v)}{dv} - \frac{2}{2\gamma + 1} \frac{dC(v)}{dv} & \text{if } \gamma \neq -\frac{1}{2}, \\ \frac{1}{4\pi r^2} \frac{dM(v)}{dv} - \frac{\ln(r)}{r^2} \frac{dC(v)}{dv} & \text{if } \gamma = -\frac{1}{2}, \end{cases} \] (14)

where \( M(v) \) and \( C(v) \) are two arbitrary functions depending on the distribution of the underlying matter.

**Proof:** By the hypothesis iii) of the Theorem I, we start with the metric for higher dimensional spherically symmetric spacetime in Eddington coordinates

\[ ds^2 = -A^2(r,v)F(v,r)dv^2 + 2\epsilon A(r,v)dvdr + r^2d\Omega^2_{D-2}, \] (15)

where \( \epsilon = -1, +1 \), respectively, correspond to the outgoing and ingoing null fluid. Due to the hypothesis ii), metric (15) must satisfy the EGB field equations (3). Considering the special case \( T^r_v = 0 \) (hypothesis), Eq. (3) yields

\[ G^v_r = (D - 2) \left[ r^2 + 2(D - 3)\alpha(1 - F) \right] \frac{1}{r^3\epsilon A^2} \left( \frac{\partial A}{\partial r} \right), \] (16)

which implies that in the limit \( D \to 4 \), Eq. (16) solves to \( A(v,r) = g(v) \). However, re-defining the null coordinate as \( \Pi = \int g(v)dv \), we can always set, without the loss of generality, \( A(v,r) = 1 \).

Now, from the \((r, \ r)\) and \((v, \ v)\) components of the field equations (3), we obtain that \( G^v_r = G^r_v \), which further ensure that

\[ T^r_v = T^v_r. \]

Thus the EMT can be written as :

\[ T^b_a = \begin{pmatrix} T^v_v & T^r_r & 0 & 0 \\ T^v_v & T^r_r & 0 & 0 \\ 0 & 0 & T^{\theta_1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \]

which in general belongs to a Type II fluid with \( T^{\theta_1}_{\theta_1} = T^{\theta_2}_{\theta_2} = \ldots = T^{\theta_{D-2}}_{\theta_{D-2}} \). It may be worthwhile to recalled that the EMT of a Type II fluid has a double null eigenvector, whereas that for a Type I fluid has only one time-like eigenvector [7]. If we impose the conservation laws, \( \nabla_a T^a_b = 0 \), and using again the hypothesis iii), \( T^{\theta_1}_{\theta_1} = \gamma T^r_r \), we have that

\[ \frac{\partial}{\partial v} T^v_v + \frac{\partial}{\partial r} T^r_r + \frac{1}{2\epsilon} (T^r_r - T^v_v) \frac{\partial}{\partial r} F + \frac{(D - 2)}{r} T^r_r = 0, \] (17)

\[ \frac{\partial}{\partial r} T^r_r + \frac{(D - 2)(1 - \gamma)}{r} T^r_r = 0. \] (18)

Solving Eq. (18) for \( T^r_r \), we obtain:

\[ T^r_r = \frac{C(v)}{r^{(D - 2)(1 - \gamma)}}. \] (19)

where \( C(v) \) is an arbitrary function. Using these results, the diagonal elements of \( T^a_b \) can be written as follow

\[ T^a_{b(\text{Diag})} = \frac{C(v)}{r^{(D - 2)(1 - \gamma)}} \text{diag}[1, 1, \gamma, \ldots, \gamma]. \]

Now, in the limit \( D \to 4 \), the EGB equations \( G^v_r = 8\pi T^v_r \), reduces to

\[ \Lambda + \left[ r^2 + 2\alpha(1 - F) \right] \frac{1}{r^3} \frac{\partial F}{\partial r} - \left[ r^2 - \alpha(1 - F) \right] \frac{1 - F}{r^4} = \frac{8\pi C(v)}{r^{2(1 - \gamma)}}. \] (20)

After solving this differential equation and making some algebraic simplifications, we get

\[ F_{\pm}(r,v) = \begin{cases} 1 + \frac{r^2}{2\alpha} \left\{ 1 \pm \sqrt{1 + \frac{4\Delta v}{3} + \frac{8M(v)\alpha}{r^2} - \frac{32\pi\alpha C(v)}{(1 + 2\gamma)r^{2(1 - \gamma)}}} \right\} & \text{if } \gamma \neq -\frac{1}{2}, \\ 1 + \frac{r^2}{2\alpha} \left\{ 1 \pm \sqrt{1 + \frac{4\Delta v}{3} + \frac{8M(v)\alpha}{r^2} - \frac{32\pi\alpha C(v)\ln r}{r^2}} \right\} & \text{if } \gamma = -\frac{1}{2}. \end{cases} \] (21)
Some important comments are in order. The result of the Theorem I represents a general class of non-static, spherically symmetric solutions to the novel 4D EGB theory describing radiating black-holes with the EMT satisfying the conditions in accordance with the hypothesis (iii). In general, the family of the solutions outlined by the Theorem I generates solutions of the novel 4D EGB theory, for instance, Bonnor-Vaidya-like [26], dS/AdS [36], global monopole-like [37], Husain and Dadhich-Ghosh Vaidya solution on brane-like [38]. Clearly, by proper choice of the functions $M(v)$ and $C(v)$, and $\gamma$-index, one can generate solutions of the theory and some of them are presented in the Table I. They include most of the known Vaidya-based spherically symmetric solutions of the novel 4D EGB theory. These solutions could be very useful to study the collapse of different matter fields, or the formation of naked...
singularities. Furthermore, these solutions can be used to get some insights into the semi-classical approaches for black holes evaporation.

To further illustrate the theorem, we generate the two known solution of the novel 4D EGB theory.

a. **Glavan and Lin solution [21]:** As an immediate consequence of the theorem, we generate the static spherically symmetric black hole solution of the novel 4D EGB theory [21]. For this we have to choose, \( M(v) \equiv M = \text{constant}, \) \( C(v) = 0, \) and \( \Lambda = 0, \) solution (21), becomes static and is given by

\[
F_\pm(r) = 1 + \frac{r^2}{2a} \left( 1 \pm \sqrt{1 + \frac{8Ma}{r^3}} \right).
\]

(23)

The metric (15) with \( F(r) \) in (23) represents the static spherically symmetric black hole solution obtained by Glavan and Lin [21], but in the Einstein-Finkelstein coordinates.

b. **Fernandes solution [23]:** The charged counterpart of the spherically symmetric black hole of the novel 4D EGB theory [21] were also found by Fernandes [23]. To generate this, we should choose \( M(v) \equiv M = \text{constant}, \) \( C(v) = -Q^2/8\pi, \) \( \Lambda = -3/l^2 \) and \( \gamma = -1. \)

c. **GR limit:** At this point, it is important to note that we have obtained two branches of solution (21), namely, \( F_+ \) and \( F_- , \) which correspond to \pm signs in front of the square root term. However, the positive branch, \( F_+, \) does not converge to GR, henceforth, we will be considering only negative branch. In this case, the limit \( \alpha \to 0, \) reduces \( F(v, r) \) to

\[
F_-(v, r) = \begin{cases} 1 - \frac{2M(v)}{r} - \frac{\Lambda r^2}{3} + \frac{8\pi C(v) r^{2\gamma}}{(1+2\gamma)} & \text{if } \gamma \neq -\frac{1}{2}, \\ 1 - \frac{2M(v)}{r} - \frac{\Lambda r^2}{3} + \frac{8\pi C(v) \ln r}{r} & \text{if } \gamma = -\frac{1}{2}. \end{cases}
\]

(24)

These are solutions for the 4D GR version of the Theorem, i.e., solution of \( G_{ab}^{(0)} + G_{ab}^{(1)} = 8\pi T_{ab}, \) instead of the EGB gravity, which are the same as those found in Ref. [14].

**IV. ENERGY CONDITIONS**

The family of solutions discussed here, in general, belongs to Type II fluid defined in Ref. [7]. To discuss the energy conditions, let us introduce two independent future null vectors, \( l_a \) and \( n_a, \) where \( l^a \) is tangent to the null surface constructed by \( v, \) and \( n^a \) is an another independent null vector such that

\[
l_a = -\delta_a^v, \quad n_a = -\frac{1}{2} F(v, r) \delta_a^v + \delta_a^r,
\]

\[
l_a l^a = n_a n^a = 0, \quad l_a n^a = -1, \quad l^a n_a = 1,
\]

(25)

(26)

and the EMT, with the help of these null vectors, reads [7, 36]

\[
T_{ab} = \epsilon \mu(v, r) l_a l_b - P_r(v, r) (l_a n_b + l_b n_a) + P_\theta(v, r) (g_{ab} + l_a n_b + l_b n_a),
\]

(27)

where

\[
\mu = T_{rr}^v, \quad P_r = T_r^r \equiv C(v) r^{2(\gamma-1)}, \quad P_\theta = \gamma P_r,
\]

(28)

(29)

(30)

where \( \mu \) corresponds for the radiating energy along the null direction \( l^a; \) \( P_r \) and \( P_\theta, \) respectively, are the radial and transverse pressures components generated by the charges of the fluids. All these physical quantities are measured in the reference frame of an observer moving along a time-like direction \( u^a \) given by

\[
u^a = \frac{1}{\sqrt{2}} (l^a + n^a).
\]
The energy density $\rho$ measured by this observer is defined by the projection of $T_{ab}$ along the $u^a$, as follow

$$\rho = - T_{ab} u^a u^b = - P_t.$$  

**i.** The *weak energy condition (WEC):* For any timelike vector $w^a$, the EMT $T_{ab} w^a w^b \geq 0$ [7, 36]. Equation (27) can be recast as

$$\mu \geq 0, \quad \rho \geq 0 \quad \text{and} \quad P_\theta \geq 0.$$  

The WEC and strong energy condition (SEC) are identical for the Type II fluid [7, 14, 36].

**ii.** The *dominant energy condition (DEC):* For any timelike vector $w^a$, $T_{ab} w^a w^b \geq 0$ and also $T_{ab} w^b$ is a non-spacelike vector, i.e.,

$$\mu \geq 0 \quad \text{and} \quad \rho \geq P_\theta \geq 0.$$  

For the radiating fluid, the WEC condition (31) is satisfied if $C(v) \leq 0$ and $\gamma \leq 0$. However, $\mu > 0$, leads to

$$\frac{1}{4\pi r^2} \frac{dM(v)}{dv} - \frac{r^{2\gamma-1}}{2\gamma + 1} \frac{dC(v)}{dv} > 0 \quad \text{if} \quad \gamma \neq -\frac{1}{2},$$  

and

$$\frac{1}{4\pi r^2} \frac{dM(v)}{dv} - \frac{\ln(r)}{r^2} \frac{dC(v)}{dv} > 0 \quad \text{if} \quad \gamma = -\frac{1}{2},$$

which is satisfied if $dM(v)/dv > 0$, and either $dC(v)/dv > 0$ with $\gamma < -1/2$, or $dC(v)/dv < 0$ with $\gamma > -1/2$. On the other hand, the Eq. (34) is satisfied if

$$\frac{dM(v)}{dv} > 4\pi \ln(r) \frac{dC}{dv}.$$

Finally, the DEC conditions (32), $\rho \geq P_\theta \geq 0$, are satisfied only if $C(v) \leq 0$ and $-1 \leq \gamma \leq 0$.

V. **STATIC BLACK HOLES SOLUTIONS**

The *Theorem I* generates a general class of non-static, spherically symmetric solutions to the novel 4D EGB gravity representing radiating black holes with the EMT, which satisfies the conditions in accordance with hypothesis (iii). One can also generate the static solutions in the Eddington-Finkelstein coordinates by setting $M(v) = M$, $C(v) = C$, with $M$ and $C$ as constants, in which case matter is Type I. Then metric (15) can be transformed in the usual spherically symmetric form by the transformation

$$ds^2 = -F(r) \, dt^2 + \frac{dr^2}{F(r)} + r^2 (d\Omega_{D-2})^2,$$

by the coordinate transformation

$$dv = A(r)^{-1} \left( dt + \epsilon \frac{dr}{F(r)} \right).$$

In case of spherical symmetry, even when $F(r)$ is replaced by $F(t, r)$, one can cast the metric in the form (15) [39]. Thus, one would like to have the above theorem to generate static spherically symmetric solutions which we state without proof.

**Theorem II:** Let $(\mathcal{M}, g_{ab})$ be a $D$-dimensional space-time such that: i) it satisfies, $D \rightarrow 4$, the Einstein-Gauss-Bonnet gravity equations obtained by re-scaled coupling constant $\alpha/(D-4)$, ii) it is spherically symmetric, iii) in the *spherical polar* coordinates, where the metric reads $ds^2 = -F(r) dt^2 + N(r) dr^2 + r^2 d\Omega_{D-2}$, the EMT $T_{ab}$ satisfies the conditions $T_{ab} = 0$, and $T_{\theta \theta} = \gamma T_{\gamma} \gamma$, ($\gamma = \text{const} \in \mathbb{R}$), iv) if $\alpha \rightarrow 0$, the solution converges to the 4D GR limit. Then the metric of the space-time is given by

$$ds^2 = -F(r) dt^2 + \frac{1}{F(r)} dr^2 + r^2 d\Omega_2^2,$$

where
\[ F_\pm(r) = \begin{cases} \frac{1 + \alpha^2}{2a} & \left(1 \pm \sqrt{1 + \frac{4\Lambda}{3} + \frac{8M}{r^\alpha} - \frac{32\pi\alpha C}{(1 + 2\gamma) r^{2(1+\gamma)}}}\right) \text{ if } \gamma \neq -\frac{1}{2}, \\ \frac{1 + \alpha^2}{2a} & \left(1 \pm \sqrt{1 + \frac{4\Lambda}{3} + \frac{8M}{r^\alpha} - \frac{32\pi\alpha C \ln r}{r^\alpha}}\right) \text{ if } \gamma = -\frac{1}{2}, \end{cases} \] (38)

with the components of EMT \( T^a_b \) given by

\[ T^a_b = \frac{C}{r^{2(1-\gamma)}} \text{diag}[1, 1, \gamma, \gamma], \] (39)

where \( M \) and \( C \) are two arbitrary constant depending on the distribution of the underlying matter.

The **Theorem II** generates a general class of static, spherically symmetric black hole solutions to the novel 4D EGB theory with the EMT, which satisfies the conditions in accordance with the hypothesis (iii). The family of solutions outlined here contains the novel 4D EGB version, for instance, of Glavan-Lin [21] static spherically symmetric black hole when \( C = 0, \Lambda = 0 \) and charged counterpart of spherically symmetric AdS black hole due to Fernandes [23] by choosing \( C(v) = -Q^2/8\pi, \Lambda = -3/l^2 \) and \( \gamma = -1 \).

Obviously, by proper choice of the constant \( M \) and \( C \), and \( \gamma \)-index, one can generate as many solutions as required. The above **Theorem II** can generate several spherically symmetric solutions to the novel 4D EGB theory with the EMT satisfying conditions mentioned in the theorem.

**VI. DISCUSSION AND CONCLUSIONS**

EGB gravity is a natural extension of GR to higher dimensions that has several additional nice properties than Einstein’s GR [40] and is the first nontrivial term of low energy limit of string theory. But EGB gravity is topological in 4D and does not make a contribution to the gravitational dynamics. However, as recently demonstrated that rescaling the Gauss-Bonnet coupling parameter as \( \alpha \to \alpha/(D-4) \), and taking the limit \( D \to 4 \) at the level of the equation of motion, the Gauss-Bonnet term in the Einstein-Hilbert action makes a non-trivial contribution to the gravitational dynamics in 4D. The static spherically symmetric black hole solution of this novel 4D EGB gravity in contrast to the Schwarzschild black hole solution of GR is free from the singularity pathology. Whilst for finding the exact solutions of Einstein equations in the 4D space-time several powerful mathematical tools were developed, it would be interesting how to develop some of these methods to get exact solutions of the complicated higher curvature gravities say to the EGB gravity.

In addition, knowledge of exact solutions helps us to understand their properties in a better way, and they greatly enhanced our understanding of gravitational physics and also any new meaningful exact solutions, in any theory, are always desirable and valuable. With this motivation, we have proved a theorem, which, with certain restrictions on the EMT characterizes a large family of radiating black hole solutions to this novel 4D EGB gravity, representing, in general, spherically symmetric Type II fluid. The solutions depend on one parameter \( \gamma \), and two arbitrary functions \( M(v) \) and \( C(v) \) (modulo energy conditions). It is easy to generate various solutions by suitable choice of these functions and the parameter \( \gamma \). In particular, we have demonstrated that some known solutions of the theory are generated as the particular case from the **Theorem I**, and also some other solutions that are listed in the table I, which means that there exists realistic matter that follows the restrictions of the theorem. Whilst, we have generated a set of solutions of the novel 4D EGB gravity, it is always desirable to see if there exist physically valid new solutions to extend this list.

The family of solutions generated by the **Theorem I**, in general, belongs to Type II fluid. However, if \( M(v) = C(v) = \text{constant} \), the matter field degenerates to Type I fluid with no off-diagonal component of the EMT and one can generate static black hole solutions in the Eddington-Finkelstein coordinates, of the novel 4D EGB gravity, with appropriate choices of \( M, C \) and \( \gamma \). A trivial extension of the **Theorem I** is also stated as **Theorem II**, without proof, being similar to that of **Theorem I**, which allows one to generate a three-parameter family of static, spherically symmetric solutions of the novel 4D EGB gravity in the Schwarzschild coordinates.

The 4D-EBG solution (23) in [21] actually was also found earlier in the gravity with a conformal anomaly [34], in EGB gravity with quantum corrections [35] and recently in the third order Lovelock gravity [33]. Hence, the theorems presented here, with appropriate modifications, may also be relevant in these theories and one can generate family of both static and dynamical spherical symmetric solutions of these theories.

There are many interesting avenues that are amenable for future work from the solutions generated, it will be intriguing to analyse the causal structure and thermodynamics. Also, it should be interesting to apply these metrics to study effect of the higher order curvature in semi-classical analysis of the black hole evaporation in 4D. One should
also see possibility of generalization of these results more general Lovelock gravity theories. Further, the solutions presented provide an excellent setting to get insights into more general gravitational collapse situations and in better understanding of cosmic censorship conjecture [41].

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