Hyperfactored of The arrangements $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$

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Abstract. The purpose of this paper is to study the hyperfactored of the complex reflection arrangements $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$. Depending on the lattices of the arrangements $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$, the basis of these arrangements has been found and then partitioned. Also, showed that $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$, are not hyperfactored and are not inductively factored.

Keywords: Complex reflection arrangement, nice partition, Factored arrangement, Inductively Factored.

1. Introduction

In [1] Al-Aleyawee found the lattices of $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$. In this paper the basis of $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$ has been found by using programs. Proved that the arrangements $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$ are not factored depending on lattice, and proved that $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$ are not inductively factored depending on triple arrangement. The exponent vector and partition of $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$ have been computed.

Throughout this paper, $V$ is a finite dimensional complex vector space over field $C$. [5] A hyperplane HP in $V$ is an affine subspace of dimension $(n - 1)$. A hyperplane arrangement $\mathcal{A} = (\mathcal{A}, V)$ is a finite set of hyperplanes in $V$. The product $Q(\mathcal{A}) = \prod_{HP \in \mathcal{A}} \alpha_{HP}$ (where $\alpha_{HP}$ is a linear form and $HP = \text{Ker} (\alpha_{HP})$) is called a defining polynomial of $\mathcal{A}$. We agree that $Q(\emptyset_n) = 1$ is the defining polynomial of $\emptyset_n$, where $\emptyset_n$ is empty 1-arrangement. A reflection on $V$ is a linear transformation on $V$ of finite order with exactly $\ell$-1 eigenvalues equal to 1. A reflection group $G$ on $V$ is a finite group generated by reflection on $V$. The lattice of $\mathcal{A}$ denoted by $L_\mathcal{A}$ is the finite set of hyperplanes in $V$. The order being reverse inclusion; that is, $x \leq y \iff y \subseteq x$, for each , $y \in L_\mathcal{A}$. A subarrangement of $\mathcal{A}$ is $\mathcal{A}_X = \{HP \in \mathcal{A} | X \subseteq HP\}$. The restriction arrangement $\mathcal{A}^X = \{X \cap HP: HP \in \mathcal{A} - \mathcal{A}_X \text{ and } X \cap HP \neq \emptyset \}$ is the arrangement within the vector space $X$. ($\mathcal{A},\mathcal{A}',\mathcal{A}'')$ is a triple of arrangements that is, $HP \in \mathcal{A}, \mathcal{A}' = \mathcal{A} - \{HP_0\}$ and $\mathcal{A}'' = \mathcal{A}'HP_0$ (where $HP_0$ distinguished hyperplane). The rank function is a function $rk: L_\mathcal{A} \rightarrow \mathbb{Z}_+$ defined by $rk(X)$.
= \text{cod}(X), \forall X \in L_{\mathcal{A}}. The symmetric algebra $S = S(V^*)$ (where $V^*$ the dual vector space of $V$), which is isomorphic to the polynomial algebra $K[x_1, x_2, ..., x_n]$. For more details on hyperplane arrangement see [2].

2. Factored and inductively factored of $\mathcal{A}(G_{24})$

Definition 2.1: [2] [4]

Let $\pi = (\pi_1, ..., \pi_s)$ be a partition of arrangement. Then $\pi$ is called independent, for any choice $HP_i \in \pi_i$, $1 \leq i \leq s$, $\text{rk}(HP_1 \cap ... \cap HP_s) = s$.

Definition 2.2: [2] [8] [9]

Let $\pi = (\pi_1, ..., \pi_s)$ be a partition of $\mathcal{A}$ and let $x \in L_{\mathcal{A}}$. The induced partition $\pi_x$ of $\mathcal{A}_x$ is given by the non-empty block of the form $\pi_i \cap \mathcal{A}_x$.

Definition 2.3: [2] [4] [7]

The partition $\pi$ of $\mathcal{A}$ is a nice arrangement if $\pi$ is independent and for each $X \in L_{\mathcal{A}} \setminus \{V\}$, $\pi_x$ admits a block which is a singleton.

Definition 2.4: [2] [6]

Let $\{e_1, e_2, ..., e_n\} \subset V$ be the dual basis of $\{x_1, x_2, ..., x_n\}$. Then define $D_i = D_{e_i}$, $1 \leq i \leq n$, to be the derivation $\frac{\partial}{\partial x_i}, D_i(f) = \frac{\partial f}{\partial x_i}, f \in S$. Notice that $\{D_1, D_2, ..., D_n\}$ is a basis for $Der_K(S)$ over $S$.

Thus, any derivation $\theta$ of $S$ over $K$ is $\theta = f_1D_1 + ... + f_nD_n$, where $f_1, ..., f_n \in S$. Therefore, $Der_K(S)$ is free $S$-module of $\text{rk} n$.

Definition 2.5: [2]

$0 \neq \theta \in Der_K(S)$ is homogeneous of polynomial degree $p$ if $\theta = \sum_{j=1}^{n} f_jD_j$ and $f_j \in S_p$ for $1 \leq j \leq n$, and defined by $p \deg \theta = p$ and $t \deg \theta = p \deg \theta - 1$.

Definition 2.6: [2] [10]

Let $\mathcal{A}$ be an arrangement with defining polynomial $Q(\mathcal{A}) = \prod_{\mathcal{H} \in \mathcal{A}} \alpha_\mathcal{H}$, a submodule $D_S(\mathcal{A})$ of $Der_K(S)$ is $D_S(\mathcal{A}) = \{\theta \in Der_K(S) | \theta(Q) \in QS\}$. $D_S(\mathcal{A})$ is called the module of $\mathcal{A}$-derivations.

Definition 2.7: [2]

The class $\text{IFAC}$ of inductively factored is the smallest class of pairs $(\mathcal{A}, \pi)$ of $\mathcal{A}$ together with a partition $\pi$ subject to

1. $(\emptyset_n, (\emptyset)) \in \text{IFAC}, \forall n \geq 0$, (where $\emptyset_n$ is empty $n$-arrangement)

2. If there exists a partition $\pi$ of $\mathcal{A}$ and $HP_0$ the restriction map $\sigma = \sigma_\pi$.

$HP_0: \mathcal{A} \setminus \pi_1 \rightarrow A''$ is injective and for the induced partition $\pi'$ of $\mathcal{A}'$ and $\pi''$ of $\mathcal{A}''$ both $(\mathcal{A}', \pi')$ and $(\mathcal{A}'', \pi'') \in \text{IFAC}$, then $(\mathcal{A}, \pi)$.

Definition 2.8: [3]
A real arrangement \( \mathcal{A} \) of hyperplane is said to be factored if there exists a partition \( \pi = (\pi_1, ..., \pi_n) \) of \( \mathcal{A} \) into \( n \) disjoint subsets such that Orlik-Solomon algebra of \( \mathcal{A} \) factors according to this partition.

Theorem 2.1: [3]

If \( \mathcal{A} \) is a nice partition, then an arrangement \( \mathcal{A} \) is factored arrangement.

3. The Complex Reflection Arrangement (G

The complex Reflection Group \( G_{24} \) [1]

Kleins simple group of order "168" is a subgroup GL(3,C). It give rise to a complex reflection group \( G \subset U(C^3) \cong U(3,C) \subset GL(3,C) \), of order "336" it is generated by reflections

\[
S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } S_3 = 1/2 \begin{bmatrix} 1 & -1 & -\beta \\ -1 & 1 & -\beta \\ -\beta & -\beta & 0 \end{bmatrix}
\]

Where \( -\beta \) is a root of the equation \( t^2 - t + 2 = 0 \) i.e.

\[
\beta = 1/2(1-i\sqrt{7}), \quad \beta' = 1/2 (1+i\sqrt{7}) \text{ but the group } G_{24} \text{ is not the complexification of a real group.}
\]

The corresponding reflection arrangement has "21" hyperplanes and defined by:

\[
\mathcal{A} (G_{24}) \text{ is } Q(\mathcal{A}(G_{24})) = x_1 x_2 x_3 \prod_{i,j,k=1,2,3} (x_i \mp x_j)(\beta x_i \mp x_j \mp x_k).
\]

The hyperplanes of \( \mathcal{A} (G_{24}) \) where \( HP_i = \text{Ker} \alpha_{H_i}, 1 \leq i \leq 21 \) are:

| Table 1. The hyperplanes of \( \mathcal{A} (G_{24}) \) |
|-----------------------------------------------|
| HP₁ : \( x_2 = 0 \)                         | HP₉ : \( x_1 - x_3 = 0 \)                        | HP₁₅ : \( \beta x_2 - x_1 - x_3 = 0 \) |
| HP₂ : \( x_2 = 0 \)                         | HP₁₀ : \( x_2 = x_3 = 0 \)                       | HP₁₆ : \( \beta x_2 - x_1 + x_3 = 0 \) |
| HP₃ : \( x_3 = 0 \)                         | HP₁₁ : \( \beta x_1 + x_2 + x_3 = 0 \)          | HP₁₇ : \( \beta x_2 + x_1 - x_3 = 0 \) |
| HP₄ : \( x_1 + x_2 = 0 \)                  | HP₁₂ : \( \beta x_1 - x_2 + x_3 = 0 \)          | HP₁₈ : \( \beta x_3 + x_1 + x_3 = 0 \) |
| HP₅ : \( x_1 + x_3 = 0 \)                  | HP₁₃ : \( \beta x_1 + x_2 - x_3 = 0 \)          | HP₁₉ : \( \beta x_3 - x_1 + x_2 = 0 \) |
| HP₆ : \( x_2 + x_3 = 0 \)                  | HP₁₄ : \( \beta x_2 - x_1 + x_3 = 0 \)          | HP₂₀ : \( \beta x_3 - x_1 + x_2 = 0 \) |
| HP₇ : \( x_1 - x_2 = 0 \)                  | HP₁₅ : \( \beta x_2 + x_1 + x_3 = 0 \)          | HP₂₁ : \( \beta x_3 + x_1 - x_2 = 0 \) |

By using Program (1), it is found that

\[
D_1(f) = \frac{\partial f}{\partial x}, \quad D_2(f) = \frac{\partial f}{\partial y}, \quad D_3(f) = \frac{\partial f}{\partial z}
\]
of $\mathcal{A}$ ($G_{24}$) and found degree of $\mathcal{A}(G_{24})$ is $\{2,4,12\}$. Thus, the exponent vector of $\mathcal{A}$ ($G_{24}$) is $\{3,5,13\}$ and the partition of this arrangement is $\pi = \{\Pi_1, \Pi_2, \Pi_3\}$ where

$$
\pi_i = \{\text{HP}_1, \text{HP}_2, \text{HP}_3\}, \\
\pi_2 = \{\text{HP}_4, \text{HP}_5, \text{HP}_6, \text{HP}_7, \text{HP}_8\}, \\
\pi_3 = \{\text{HP}_9, \text{HP}_{10}, \text{HP}_{11}, \text{HP}_{12}, \text{HP}_{13}, \text{HP}_{14}, \text{HP}_{15}, \text{HP}_{16}, \text{HP}_{17}, \text{HP}_{18}, \text{HP}_{19}, \text{HP}_{20}, \text{HP}_{21}\}.
$$

The $\mathcal{A}_{x_i}$ for each $x_i \in \text{rk} 2$ has been found.

Theorem

i. The induced partition $\pi_x$ of $\mathcal{A}$ ($G_{24}$) has no singleton.

ii. $\mathcal{A}$ ($G_{24}$) is not factored arrangement.

Proof:

i. By the intersection of the partitions $\pi_i, i = 1,2,3$, with arrangement of $\text{rk} 2$ in Table (2) the result is deduced.

ii. This part is direct result from Part i.

**Table 2.** $\mathcal{A}_{x_i}$ for each $x_i \in \text{rk} 2$

| $\mathcal{A}_{x_1}$ = \{HP$_1$, HP$_2$, HP$_3$, HP$_7$\} | $\mathcal{A}_{x_2}$ = \{HP$_1$, HP$_3$, HP$_5$, HP$_B$\} | $\mathcal{A}_{x_3}$ = \{HP$_1$, HP$_6$, HP$_{10}$, HP$_{13}$\} | $\mathcal{A}_{x_4}$ = \{HP$_1$, HP$_9$, HP$_{11}$, HP$_{12}$\} | $\mathcal{A}_{x_5}$ = \{HP$_2$, HP$_3$, HP$_6$, HP$_9$\} | $\mathcal{A}_{x_6}$ = \{HP$_2$, HP$_5$, HP$_{14}$, HP$_{15}$\} | $\mathcal{A}_{x_7}$ = \{HP$_2$, HP$_6$, HP$_{16}$, HP$_{17}$\} | $\mathcal{A}_{x_8}$ = \{HP$_3$, HP$_4$, HP$_{18}$, HP$_{19}$\} | $\mathcal{A}_{x_9}$ = \{HP$_3$, HP$_7$, HP$_{20}$, HP$_{21}$\} | $\mathcal{A}_{x_{10}}$ = \{HP$_4$, HP$_{10}$, HP$_{17}$, HP$_{20}$\} | $\mathcal{A}_{x_{11}}$ = \{HP$_4$, HP$_{12}$, HP$_{14}$, HP$_{21}$\} | $\mathcal{A}_{x_{12}}$ = \{HP$_5$, HP$_{10}$, HP$_{16}$, HP$_{21}$\} | $\mathcal{A}_{x_{26}}$ = \{HP$_2$, HP$_{10}$, HP$_{11}$\} | $\mathcal{A}_{x_{27}}$ = \{HP$_2$, HP$_{12}$, HP$_{13}$\} | $\mathcal{A}_{x_{28}}$ = \{HP$_2$, HP$_{18}$, HP$_{21}$\} | $\mathcal{A}_{x_{29}}$ = \{HP$_3$, HP$_{10}$, HP$_{12}$\} | $\mathcal{A}_{x_{30}}$ = \{HP$_3$, HP$_{11}$, HP$_{13}$\} | $\mathcal{A}_{x_{31}}$ = \{HP$_3$, HP$_{14}$, HP$_{17}$\} | $\mathcal{A}_{x_{32}}$ = \{HP$_3$, HP$_{15}$, HP$_{16}$\} | $\mathcal{A}_{x_{33}}$ = \{HP$_4$, HP$_{5}$, HP$_9$\} | $\mathcal{A}_{x_{34}}$ = \{HP$_4$, HP$_{6}$, HP$_9$\} | $\mathcal{A}_{x_{35}}$ = \{HP$_4$, HP$_{11}$, HP$_{15}$\} | $\mathcal{A}_{x_{36}}$ = \{HP$_4$, HP$_{13}$, HP$_{16}$\} | $\mathcal{A}_{x_{37}}$ = \{HP$_5$, HP$_{6}$, HP$_7$\} |
Every factored arrangement is a nice partition.

Let $A_{x_{10}} = \{HP_5, HP_{13}, HP_{17}, HP_{18}\}$ and $A_{x_{13}} = \{HP_5, HP_{12}, HP_{19}\}$.

Let $A_{x_{14}} = \{HP_6, HP_{13}, HP_{14}, HP_{20}\}$ and $A_{x_{16}} = \{HP_5, HP_{13}, HP_{20}\}$.

Let $A_{x_{14}} = \{HP_6, HP_{12}, HP_{16}, HP_{18}\}$ and $A_{x_{18}} = \{HP_6, HP_{15}, HP_{21}\}$.

Let $A_{x_{16}} = \{HP_7, HP_{11}, HP_{14}, HP_{19}\}$ and $A_{x_{19}} = \{HP_6, HP_{17}, HP_{19}\}$.

Let $A_{x_{17}} = \{HP_7, HP_{13}, HP_{15}, HP_{18}\}$ and $A_{x_{20}} = \{HP_7, HP_{10}, HP_{18}\}$.

Let $A_{x_{18}} = \{HP_8, HP_{12}, HP_{15}, HP_{20}\}$ and $A_{x_{19}} = \{HP_7, HP_{10}, HP_{18}\}$.

Let $A_{x_{20}} = \{HP_9, HP_{10}, HP_{15}, HP_{19}\}$ and $A_{x_{21}} = \{HP_9, HP_{10}, HP_{18}\}$.

Let $A_{x_{21}} = \{HP_9, HP_{13}, HP_{17}, HP_{21}\}$ and $A_{x_{22}} = \{HP_9, HP_{10}, HP_{18}\}$.

Let $A_{x_{22}} = \{HP_1, HP_{14}, HP_{16}\}$ and $A_{x_{23}} = \{HP_9, HP_{14}, HP_{18}\}$.

Let $A_{x_{23}} = \{HP_1, HP_{15}, HP_{17}\}$ and $A_{x_{24}} = \{HP_9, HP_{16}, HP_{20}\}$.

Let $A_{x_{24}} = \{HP_1, HP_{16}, HP_{20}\}$ and $A_{x_{25}} = \{HP_2, HP_{10}, HP_{10}\}$.

Let $A_{x_{25}} = \{HP_1, HP_{10}, HP_{21}\}$.

\[\delta: \mathcal{A}\setminus\pi_1' \to \mathcal{A}\] is injective.

Let $HP_4$ be distinguished hyperplane then by Definition (2.7) $\delta$ is not injective since $\exists \alpha,\beta \in \mathcal{A}\setminus\pi_1'$ such that $\delta(\alpha) = \delta(\beta)$ and $\alpha \neq \beta$. Thus, $\mathcal{A}(G_{24})$ is not inductively factored.

Theorem 2.1

Every factored arrangement is a nice partition.
Proof:
Suppose that $\mathcal{A}$ is factored arrangement. Then $\exists \pi = (\pi_1, ..., \pi_n)$ of $\mathcal{A}$ such that $\pi = \bigoplus \pi_i$, $i = 1, ..., n$. Thus, $\pi$ is independent. Without loss of generality let $\pi_1 = \{ HP_i \}$, $i = 1, ..., n$. Then $\pi_x = \pi_1 \cap \mathcal{A}_{x_k}$ is singleton $\forall \mathcal{A}_{x_k} \in L_\mathcal{A}$, where $x_k$ of rank two. Therefore, By Definition (2.1.3) $\mathcal{A}$ is nice arrangement.

4. The Complex Reflection Arrangement ($G_{27}$).

The complex Reflection Group $G_{27}$ [1]

$G_{27}$ is a subgroup of $GL(3, \mathbb{C})$. It give rise to a complex reflection group $G \subset U(\mathbb{C}^3) \cong U(3, \mathbb{C}) \subset GL(3, \mathbb{C})$, of order 2160. It is generated by the reflections:

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & -w^2t & -wt \\ -w^2t & -t^{-1} & -w \\ -wt & -w^2 & t \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & -w^2 & 0 \\ -w & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where $w = e^{\frac{2\pi i}{3}}$, $t = \frac{1}{2}(1 + \sqrt{5})$.

All the reflection of $G_{27}$ are of order 2, but $G_{27}$ is not the complexification of a real group. It is well – generated irreducible 2-reflection group.

The corresponding reflection arrangement has "45" hyperplane and defined by:

$$\mathcal{A}(G_{27}) = Q(\mathcal{A}(G_{27})) = x_1x_2x_3x_1 \pm yx_j \pm y^2x_k(x_i \pm wy^2x_j \pm w^2yx_k)$$

Where $(i,j,k)$ is a cyclic permutation of $(123)$, and $y$ is a root of the equation ($t^2 + t - 1 = 0$), i.e. $y = \frac{1}{2}(1 + \sqrt{5})$.

The hyperplanes of $\mathcal{A}(G_{27})$ where $HP_i = \text{Ker}_{HP_i}$, $1 \leq i \leq 45$ are:
Table 3. The hyperplanes of $\mathcal{A}$ ($G_{27}$)

| $H_{P1}$ | $x_1=0$ |
|---------|---------|
| $H_{P2}$ | $x_2=0$ |
| $H_{P3}$ | $x_3=0$ |
| $H_{P4}$ | $x_2 + wx_1 = 0$ |
| $H_{P5}$ | $x_2 - wx_1 = 0$ |
| $H_{P6}$ | $x_3 + wx_2 = 0$ |
| $H_{P7}$ | $x_3 - wx_2 = 0$ |
| $H_{P8}$ | $x_1 + wx_3 = 0$ |
| $H_{P9}$ | $x_1 - wx_3 = 0$ |
| $H_{P10}$ | $x_1 + yx_2 + y^2x_3 = 0$ |
| $H_{P11}$ | $x_1 - yx_2 - y^2x_3 = 0$ |
| $H_{P12}$ | $x_3 + yx_2 + y^2x_3 = 0$ |
| $H_{P13}$ | $x_3 - yx_2 + y^2x_3 = 0$ |
| $H_{P14}$ | $x_4 + yx_3 + y^2x_1 = 0$ |
| $H_{P15}$ | $x_4 - yx_3 - y^2x_1 = 0$ |
| $H_{P16}$ | $x_4 + yx_3 - y^2x_1 = 0$ |
| $H_{P17}$ | $x_4 - yx_3 + y^2x_1 = 0$ |
| $H_{P18}$ | $x_4 + yx_1 - y^2x_2 = 0$ |
| $H_{P19}$ | $x_4 - yx_1 + y^2x_2 = 0$ |
| $H_{P20}$ | $x_4 + yx_1 + y^2x_2 = 0$ |
| $H_{P21}$ | $x_4 - yx_1 + y^2x_2 = 0$ |
| $H_{P22}$ | $x_4 + (1 - w^2y)x_2 + wx_3 = 0$ |
| $H_{P23}$ | $x_4 - (1 - w^2y)x_2 - wx_3 = 0$ |
| $H_{P24}$ | $x_4 + (1 - w^2y)x_2 - wx_3 = 0$ |
| $H_{P25}$ | $x_1 - (1 - w^2y)x_2 + wx_3 = 0$ |
| $H_{P26}$ | $x_2 + (1 - w^2y)x_3 + wx_1 = 0$ |
| $H_{P27}$ | $x_2 - (1 - w^2y)x_3 - wx_1 = 0$ |
| $H_{P28}$ | $x_2 + (1 - w^2y)x_3 + wx_1 = 0$ |
| $H_{P29}$ | $x_2 - (1 - w^2y)x_3 + wx_1 = 0$ |
| $H_{P30}$ | $x_3 + (1 - w^2y)x_1 + wx_2 = 0$ |
| $H_{P31}$ | $x_3 - (1 - w^2y)x_1 - wx_2 = 0$ |
| $H_{P32}$ | $x_3 + (1 - w^2y)x_1 - wx_2 = 0$ |
| $H_{P33}$ | $x_3 - (1 - w^2y)x_1 + wx_3 = 0$ |
| $H_{P34}$ | $x_1 + wy^2x_2 + w^2yx_3 = 0$ |
| $H_{P35}$ | $x_1 - wy^2x_2 - w^2yx_3 = 0$ |
| $H_{P36}$ | $x_1 + wy^2x_2 - w^2yx_3 = 0$ |
| $H_{P37}$ | $x_1 - wy^2x_2 + w^2yx_3 = 0$ |
| $H_{P38}$ | $x_2 + wy^2x_3 + w^2yx_1 = 0$ |
| $H_{P39}$ | $x_2 - wy^2x_3 - w^2yx_1 = 0$ |
| $H_{P40}$ | $x_2 + wy^2x_3 - w^2yx_1 = 0$ |
| $H_{P41}$ | $x_2 - wy^2x_3 + w^2yx_1 = 0$ |
| $H_{P42}$ | $x_3 + wy^2x_1 + w^2yx_2 = 0$ |
| $H_{P43}$ | $x_3 - wy^2x_1 - w^2yx_2 = 0$ |
| $H_{P44}$ | $x_3 + wy^2x_1 - w^2yx_2 = 0$ |
| $H_{P45}$ | $x_3 - wy^2x_1 + w^2yx_2 = 0$ |

By using Program (2), it is found that

$$D_4(f) = \frac{\partial f}{\partial x}, D_5(f) = \frac{\partial f}{\partial y}, D_3(f) = \frac{\partial f}{\partial z}$$

of $\mathcal{A}$ ($G_{27}$) and found degree of $\mathcal{A}$($G_{27}$) is $\{4, 10, 28\}$. Thus, the exponent vector of $\mathcal{A}$ ($G_{27}$) is $\{5, 11, 29\}$ and the partition of this arrangement is $\pi = \{\Pi_1, \Pi_2, \Pi_3\}$ where

$$\begin{align*}
\Pi_1 &= \{H_{P_1}, H_{P_2}, H_{P_3}, H_{P_4}, H_{P_5}\}, \\
\Pi_2 &= \{H_{P_6}, H_{P_7}, H_{P_8}, H_{P_9}, H_{P_{10}}, H_{P_{11}}, H_{P_{12}}, H_{P_{13}}, H_{P_{14}}, H_{P_{15}}, H_{P_{16}}\}, \\
\Pi_3 &= \{H_{P_{17}}, H_{P_{18}}, H_{P_{19}}, H_{P_{20}}, H_{P_{21}}, H_{P_{22}}, H_{P_{23}}, H_{P_{24}}, H_{P_{25}}, H_{P_{26}}, H_{P_{27}}, H_{P_{28}}, H_{P_{29}}, H_{P_{30}}, H_{P_{31}}, H_{P_{32}}, H_{P_{33}}, H_{P_{34}}, H_{P_{35}}, H_{P_{36}}, H_{P_{37}}, H_{P_{38}}, H_{P_{39}}, H_{P_{40}}, H_{P_{41}}, H_{P_{42}}, H_{P_{43}}, H_{P_{44}}, H_{P_{45}}\}.
\end{align*}$$

The $\mathcal{A}_{x_i}$, for each $x_i \in \text{rk} 2$ has been found.

Theorem

i. The induced partition $\pi_x$ of $\mathcal{A}$ ($G_{27}$) has no singleton.

ii. $\mathcal{A}$ ($G_{27}$) is not factored arrangement.
Proof:

i. By the intersection of the partitions $\pi_i$, $i = 1,2,3$, with arrangement of $\text{rk} 2$ in (Table 4) the result is deduced.

ii. This part is direct result from Part i.

| $\mathcal{A}_{x_i}$ for each $x_i \in \text{rk} 2$ |
|-------------------------------------------------|
| $\mathcal{A}_1 = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_2 = \{HP_1, HP_2, HP_3, HP_6, HP_7\}$ |
| $\mathcal{A}_3 = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_4 = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_5 = \{HP_2, HP_3, HP_4, HP_5, HP_6\}$ |
| $\mathcal{A}_6 = \{HP_2, HP_3, HP_4, HP_5, HP_6\}$ |
| $\mathcal{A}_7 = \{HP_2, HP_3, HP_4, HP_5, HP_6\}$ |
| $\mathcal{A}_8 = \{HP_2, HP_3, HP_4, HP_5, HP_6\}$ |
| $\mathcal{A}_9 = \{HP_2, HP_3, HP_4, HP_5, HP_6\}$ |
| $\mathcal{A}_{10} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{11} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{12} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{13} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{14} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{15} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{16} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{17} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{18} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{19} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{20} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{21} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |
| $\mathcal{A}_{22} = \{HP_1, HP_2, HP_3, HP_4, HP_5\}$ |

i. Inductively Factored of $\mathcal{A} (G_{27})$

Let $\pi = \{\pi_1, \pi_2, \pi_3\}$

Recall that $HP_1$ be distinguished hyperplane then

$\pi' (\mathcal{A} (G_{27})) = \{HP_2, ..., HP_{45}\}.$

$\pi" (\mathcal{A} (G_{27})) = \{y_1, ..., y_{29}\}$

To show that $\delta: \mathcal{A} \setminus \pi'_1 \rightarrow \mathcal{A}''$ is injective

Let $HP_6$ is a distinguished hyperplane then by Definition (2.7) $\delta$ is not injective since $\exists \alpha, \beta \in \mathcal{A} \setminus \pi'_1$ such that $\delta(\alpha) = \delta(\beta)$ and $\alpha \neq \beta$. Thus, $\mathcal{A} (G_{27})$ is not inductively factored.
Program (1)

syms x1 x2 x3 B
h1=x1
h2=x2
h3=x3
h4=x1+x2
h5=x1+x3
h6=x2+x3
h7=x1-x2
h8=x1-x3
h9=x2-x3
h10=B*x1+x2+x3
h11=B*x1-x2+x3
h12=B*x1+x2-x3
h13=B*x1-x2-x3
h14=B*x2+x1+x3
h15=B*x2-x1-x3
h16=B*x2+x1+x3
h17=B*x2+x1-x3
h18=B*x3+x1+x2
h19=B*x3-x1-x3
h20=B*x3+x1+x2
h21=B*x3+x1-x2
H=h1*h2*h3*h4*h5*h6*h7*h8*h9*h10*h11*h12*h13*h14*h15*h16*h17*h18*h19*h20*h21
L1=diff(H,x1)
L2=diff(H,x2)
L3=diff(H,x3)
L1=simplify(L1)
L2=simplify(L2)
L3=simplify(L2)

Program (2)

syms x1 x2 x3 y w
h1=x1
h2=x2
h3=x3
h4=x2+w*x1
h5=x2-w*x1
h6=x3+w*x2
h7=x3-w*x2
h8=x1+w*x3
h9=x1-w*x3
h10=x1+y*x2+y^2*x3
h11=x1-y*x2-y^2*x3
h12=x1+y*x2-y^2*x3
h13=x1-y*x2+y^2*x3
h14=x2+y*x3+y^2*x1
h15=x2-y*x3-y^2*x1
h16=x2+y*x3+y^2*x1
h17=x2-y*x3+y^2*x3
h18=x3+y*x1+y^2*x2
h19=x3-y*x1-y^2*x3
h20=x3+y*x1-y^2*x2
h21=x3-y*x1+y^2*x2
h22=x1+(1-w^2*y)*x2+w*x3
h23=x1-(1-w^2*y)*x2-w*x3
h24=x1+(1-w^2*y)*x2-w*x3
h25=x1-(1-w^2*y)*x2+w*x3
h26=x2+(1-w^2*y)*x3+w*x1
h27=x2-(1-w^2*y)*x3-w*x1
h28=x2+(1-w^2*y)*x3-w*x1
h29=x2-(1-w^2*y)*x3+w*x1
h30=x3+(1-w^2*y)*x1+w*x2
h31=x3-(1-w^2*y)*x1+w*x2
h32=x3+(1-w^2*y)*x1-w*x2
h33=x3-(1-w^2*y)*x1-w*x2
h34=x1+w*y^2*x2+w^2*y*x3
h35=x1-w*y^2*x2-w^2*y*x3
h36=x1+w*y^2*x2-w^2*y*x3
h37=x1-w*y^2*x2+w^2*y*x3
h38=x2+w*y^2*x3+w^2*y*x1
h39=x2-w*y^2*x3-w^2*y*x1
h40=x2+w*y^2*x3-w^2*y*x1
h41=x2-w*y^2*x3+w^2*y*x1
h42=x3+w*y^2*x1+w^2*y*x2
h43=x3-w*y^2*x1-w^2*y*x2
h44=x3+w*y^2*x1-w^2*y*x2
h45=x3-w*y^2*x1+w^2*y*x2
H=h1*h2*h3*h4*h5*h6*h7*h8*h9*h10*h11*h12*h13*h14*h15*h16*h17*h18*h19*h20*h21*h22*h23*h24*h25*h26*h27*h28*h29*h30*h31*h32*h33*h34*h35*h36*h37*h38*h39*h40*h41*h42*h43*h44*h45
L1=diff(H,x1)
L2=diff(H,x2)
L3=diff(H,x3)
L1=simplify(L1)
L2=simplify(L2)
L3=simplify(L2)

5. Conclusions

When studying the reflections arrangement $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$ we concluded that the reflections arrangement $\mathcal{A}(G_{24})$ and $\mathcal{A}(G_{27})$ are not nice and these arrangements are neither factored nor inductively factored.

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