On the Asymptotic $u_0$-Expected Flooding Time of Stationary Edge-Markovian Graphs

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Abstract

Consider that $u_0$ nodes are aware of some piece of data $d_0$. This note derives the expected time required for the data $d_0$ to be disseminated through-out a network of $n$ nodes, when communication between nodes evolves according to a graphical Markov model $\mathcal{G}_{n,\hat{p}}$ with probability parameter $\hat{p}$. In this model, an edge between two nodes exists at discrete time $k \in \mathbb{N}^+$ with probability $\hat{p}$ if this edge existed at $k-1$, and with probability $(1-\hat{p})$ if this edge did not exist at $k-1$. Each edge is interpreted as a bidirectional communication link over which data between neighbors is shared. The initial communication graph is assumed to be an Erdos-Renyi random graph with parameters $(n, \hat{p})$, hence we consider a stationary Markov model $\mathcal{G}_{n,\hat{p}}$. The asymptotic “$u_0$-expected flooding time” of $\mathcal{G}_{n,\hat{p}}$ is defined as the expected number of iterations required to transmit the data $d_0$ from $u_0$ nodes to $n$ nodes, in the limit as $n$ approaches infinity. Although most previous results on the asymptotic flooding time in graphical Markov models are either almost sure or with high probability, the bounds obtained here are in expectation. However, our bounds are tighter and can be more complete than previous results.

I. INTRODUCTION

The problem of quickly disseminating data through a highly dynamic network is one of the most basic questions to arise in a variety of fields, including epidemiology, database management, radio broadcasting, and wireless P2P networks [2], [3], [6], [8–11], [13], [14], [17]. An intuitively easy and robust solution to this problem is the flooding protocol, for which it is simply assumed that all available data is transferred between any two communicating nodes at all communication steps. In spite of its simplicity, the flooding protocol has been shown to be a relatively efficient method for spreading information for certain dynamic communication models [1], [4], [5], [7]. Due to the parsimonious nature of flooding, data can be spread through very large networks despite unpredictable link failures, provided that we have no synchronized set of links that remain absent for a prolonged period of time, such that one set of nodes remains cut off from another [5]. We can avoid the latter bottle neck type phenomenon by focusing our interest on the expected time at which the initial data is fully disseminated, and furthermore by considering a probabilistic model in which all communication links are treated identically and uniformly across time.

To frame our analysis of the flooding protocol’s performance we use a dynamic communication model known as the stationary Markov graph model $\mathcal{G}_{n,\hat{p}}$ [7]. For the $\mathcal{G}_{n,\hat{p}}$ model, the initial communication graph $G_0$ is an Erdos-Renyi random graph with parameters $(n, \hat{p})$. For all subsequent graphs $\{G_k : k \in \mathbb{N}^+\}$, the graph $G_k$ is obtained from $G_{k-1}$ according to the simple Markov rule: each edge $e \in \mathcal{E}(G_k)$ exists with probability $\hat{p}$ if $e \in \mathcal{E}(G_{k-1})$, and conversely $e \in \mathcal{E}(G_k)$ with probability $(1-\hat{p})$ if $e \notin \mathcal{E}(G_{k-1})$. The flooding time bounds that we obtain for the stationary Markov graph model $\mathcal{G}_{n,\hat{p}}$ are compared to the flooding time bounds obtained in previous works [1], [4], [7], which consider a more general Markov graph model referred to as the “birth-death rate” Markov model $G_{n,p,q}$. The birth-death rate Markov model is defined by a birth-rate $p$ and death-rate $q$, such that for an infinite sequence of graphs $(G_0,G_1,G_2,...)$ each graph $G_k$ is obtained from $G_{k-1}$ by two rules: if an edge $e \notin \mathcal{E}(G_{k-1})$ then $e \in \mathcal{E}(G_k)$ with probability $p$, and if $e \in \mathcal{E}(G_{k-1})$ then $e \notin \mathcal{E}(G_k)$ with probability $q$. Note that if $q = 1 - p$ and the initial graph $G_0$ is an Erdos-Renyi random graph with edge probability $p$, then we arrive at the stationary Markov graph $\mathcal{G}_{n,\hat{p}}$. Although most of the results in [1], [4], [7] are stronger than ours in a probabilistic sense, the results obtained here are tighter, often more complete, and also take into consideration the number of nodes that initially are aware of the data.

A. Previous Results

The amount of literature on broadcast and gossip algorithms in static and dynamic networks is vast (see, for example [12], [15], [16], [18], [19]). Despite this, there are only a few seminal papers that have been primarily...
concerned with the flooding time in edge-Markovian graphs. In this section we will review these few papers, namely [1], [4], [7]. Let us define \( T(G_n) \) as the flooding time for some graphical model \( G_n \), and \( \mathbb{E}(T(G_n)) \) denote the expected flooding time of \( G_n \). All the flooding time results in [1] hold almost surely (a.s.), whereas those in [4], [7] hold with high probability (w.h.p.). In fact, the only result we are aware of that explicitly addresses the expected flooding time of edge-Markovian graphs is Theorem 5.3 in [7]. This theorem obtains a lower bound on \( T(G_n) \) when assuming exactly the same stationary Markov model \( G_{n,p} \) that we consider in this note. Specifically, in Theorem 5.3 [7] obtains,

\[
\lim_{n \to \infty} \mathbb{E}(T(G_{n,p})) = \Omega\left( \frac{\log(n)}{\log(1 + np)} \right).
\]  

(1)

Other flooding time results are obtained w.h.p. in [4] for the stationary Markov model \( G_{n,p} \),

if \( \hat{p} \geq c \frac{\log(n)}{n} \) for sufficiently large \( c \), then w.h.p. \( \mathbb{E}(T(G_{n,p})) \leq O\left( \frac{\log(n)}{\log(np)} \right) + \log^2(n\hat{p}) \)

if \( c \frac{\log(n)}{n} \leq \hat{p} \leq c \frac{\log^2(n)}{n} \) for sufficiently large \( c \), then w.h.p. \( \mathbb{E}(T(G_{n,p})) = \Theta\left( \frac{\log(n)}{\log(np)} \right) \).

(2)

The flooding time results in [1] concern the stationary birth-death rate Markov model \( G_{n,p,q} \), where they define the “stationary” edge probability \( \hat{p} = \frac{p}{p+q} \) and assume (as we do) the initial graph \( G_0 \) is an Erdos-Renyi random graph with parameters \((n, \hat{p})\). Among the results obtained in [1] are the following complete set of a.s. bounds on the flooding time for \( G_{n,p,q} \):

\[
\text{if } \hat{p} \leq \frac{c}{n} \text{ for some constant } c > 0 \text{ then a.s. } T(G_{n,p,q}) = \Theta\left( \frac{\log(n)}{\log(np)} \right)
\]

\[
\text{if } \frac{1}{n} \ll \hat{p} \leq c \frac{\log(n)}{n} \text{ for some constant } c < 1 \text{ then a.s. } \begin{cases} \mathbb{E}(T(G_{n,p,q})) = \Theta\left( \frac{\log(n)}{\log(np)} \right) & \text{if } np \leq \log(n \hat{p}) \\ \mathbb{E}(T(G_{n,p,q})) = \Theta\left( \frac{\log^2(n)}{\log(np)} \right) & \text{if } np \geq \log(n \hat{p}) \end{cases}
\]

(3)

\[
\text{if } \hat{p} \geq c \frac{\log(n)}{n} \text{ for some constant } c > 1 \text{ then a.s. } T(G_{n,p,q}) = \Theta\left( \frac{\log(n)}{\log(np)} \right).
\]

Furthermore, in [1] the problem of redundant data transfers is partially alleviated by introducing a “k-active” flooding protocol, for which each node only transmits data that it has received in the past \( k \) communication steps. Note that the “1-active” flooding protocol is the standard protocol for static networks [1]. It is shown in [1] that if \( \hat{p} \geq c \frac{\log(n)}{n} \) for sufficiently large \( c \), then the “1-active” flooding protocol achieves the same a.s. flooding time as the traditional flooding protocol, that is the “\( \infty \)-active” flooding protocol. For all other \( \hat{p} \), the “\( \Theta\left( \frac{\log(n)}{np} \right) \)-active” implies the same a.s. flooding time as the “\( \infty \)-active” flooding protocol [1]. To our knowledge, the above results are the most precise flooding times when considering the stationary birth-death rate Markov graph model.

B. Main Results

In this note we introduce a parameter \( u_0 \) that is not considered [1], [4], [7]. The parameter \( u_0 \) can take on any value in the interval \([1, n-1]\) and represents the number nodes that are initially aware of the data. For any given \( u_0 \in [1, n-1] \) and \( \hat{p} \in (0, 1) \), the collection of results in this note are as follows,

\[
\lim_{n \to \infty} \mathbb{E}(T(G_{n,p}, u_0)) = \begin{cases} 1 & \iff u_0 > \frac{1 + \hat{p} - \sqrt{(np-1)^2 + 4p}}{2p} \approx \frac{1}{\hat{p}} \\ \frac{\log(n/u_0)}{\log(np)} + 1 & \iff np \to \infty \\ \frac{\log^2(n/u_0)}{np} + 1 & \iff np \leq c, \text{ for some constant } c > 0 \end{cases}
\]

(4)

We note that only the second line in [1] is non-trivial. Collectively [1] implies and builds upon the result [1]. Also, although [4] are flooding times in expectation, for the case \( u_0 = 1 \) they match the results [4] of [1] up to a multiplicative constant.

In addition to this, the results we obtain are “tight” in two ways: first, they are accurate and there is no arbitrary “multiplicative constant” that [1], [4], [7] require; and secondly, the flooding time bounds that we obtain are both necessary and sufficient. Moreover, our results are also “complete” in the sense that we have a specific bound on the flooding time for any \( \hat{p} \in (0, 1) \). Contrast this with, for example, the first line in [2], which is a flooding time bound that is restricted to \( \hat{p} \geq c \frac{\log(n)}{n} \) for sufficiently large \( c \).
II. Derivation of Results

Define \( k^*(u_0, G_{n,p}) \) as the asymptotic \( u_0 \)-expected flooding time \( \mathbb{E}(T(G_{n,p}, u_0)) \) in (1). Next, for simplicity let us parameterize the edge probability \( \tilde{p} = f_n/n \), for some function \( f_n \) of \( n \). Also, let \( x_k \) denote the probability that a node is aware of \( d_0 \) at time \( k \). Observe that \( x_0 = u_0/n \), and also that by the definition of \( k^*(u_0, G_{n,p}) \),

\[
x_{k^*(u_0, G_{n,p}) - 1} \leq 1 - \frac{1}{n} \quad \text{and} \quad x_{k^*(u_0, G_{n,p})} > 1 - \frac{1}{n} .
\]

The edge-Markovian model of \( G_{n,p} \) implies the following recursion,

\[
x_{k+1} = x_k + f_n x_k (1 - x_k) , \quad \forall k \in \mathbb{N}_0^+ .
\]

If the recursion (6) is written,

\[
x_{k+1} = x_0 f_n^{k+1} + g_{k+1}
\]

then it can be shown by induction that for all \( k \geq 0 \),

\[
g_{k+1} = Q(g_k) \triangleq -g_k^2 f_n + g_k (1 + f_n - 2x_0 f_n^{k+1}) + x_0 f_n^{k} - x_0^2 f_n^{2k+1} , \quad g_0 = 0 .
\]

We begin by proving the first line in (4).

**Theorem II.1.** If \( u_0 > 1 + f_n - \frac{(f_n - 1)^2 + 4f_n}{2f_n} \) then \( k^*(u_0, G_{n,p}) = 1 \).

**Proof.** Applying (6) directly yields the first line in (4),

\[
x_1 = x_0 + f_n x_0 (1 - x_0) > 1 - \frac{1}{n} \iff x_0 > x_0^\triangleq \frac{1 + f_n - \sqrt{(f_n - 1)^2 + 4f_n}}{2f_n} \approx \frac{1}{f_n} ,
\]

and thus \( k^*(u_0, G_{n,p}) = 1 \) for all \( u_0 > nx_0^\triangleq \).

Next we show the third line in (4).

**Theorem II.2.** If \( f_n \leq c \) for some constant \( c > 0 \), then \( k^*(u_0, G_{n,p}) = \left\lceil \frac{\log(n^2/u_0)}{f_n} \right\rceil + 1 \).

**Proof.** Let \( f_n \leq c \) for some constant \( c > 0 \). It follows that \( f_n \) is finite and thus the recursion (6) can be written,

\[
x_{k+1} - x_k = f_n x_k (1 - x_k) \quad \therefore \quad \frac{\partial}{\partial k} x_k = f_n (1 - x_k) \quad \therefore \quad x_k = \frac{1}{1 + (nu_0^{-1})e^{-f_n}} .
\]

By the last line in (10) it follows that \( x_k > 1 - \frac{1}{n} \) if and only if \( k \geq \left\lceil \frac{\log((n-1)(nu_0^{-1}) - 1)}{f_n} \right\rceil + 1 \).

As for the second, and only non-trivial, line in (4), we first will assume that only one node is initially aware of \( d_0 \), that is we set \( u_0 = 1 \).

**Theorem II.3.** If \( f_n \to \infty \), then \( k^*(1, G_{n,p}) = \left\lceil \frac{\log(n)}{\log(f_n)} \right\rceil + 1 \).

**Proof.** Let us define,

\[
k \triangleq \left\lceil \frac{\log(n - 1)}{\log(1 + f_n)} \right\rceil + 1 , \quad \overline{k} \triangleq \frac{\log(n - 1)}{\log(f_n)} .
\]

From (6) we have,

\[
x_{k+1} < x_k (1 + f_n) < x_0 (1 + f_n)^{k+1} \quad \Rightarrow \quad x_k \leq \frac{x_0 (1 + f_n)^{k+1}}{1 + f_n} \leq 1 - \frac{1}{n} ,
\]

thus \( k \geq \overline{k} \) is necessary for all nodes to be aware of \( d_0 \) in expectation, that is \( k^*(1, G_{n,p}) \geq \overline{k} \).

Let us assume that \( \overline{k} \in \mathbb{N}^+ \). Then from (11) we have \( x_0 f_n^{\overline{k}} = 1 - \frac{1}{n} \) and thus to show \( x_{\overline{k}} > 1 - \frac{1}{n} \) it will suffice that \( g_{\overline{k}} > 0 \). From (8) this is equivalent to \( g_{\overline{k} - 1} (f_n - 1) > g_{\overline{k} - 1} f_n \). In Lemma II.4 it is shown \( g_k > 0 \) for all \( k \in [1, \overline{k} - 1] \), thus it suffices to show \( g_{\overline{k} - 1} < 1 - f_n^{-1} \). Again from Lemma II.4 and (8) we have,

\[
g_{k+1} < g_k (1 + f_n) + f_n^{k-\overline{k}} < g_1 (1 + f_n)^k + f_n^{k-\overline{k}+1} (1 + f_n^{-1})^{k} - 1 ,
\]
and thus it suffices to show,
\[ g_1(1 + f_n)^{\overline{r} - 2} + f_n^{-1}(1 + f_n^{-1})^{\overline{r} - 2} - 1 < 1 - f_n^{-1}. \] (14)

Recall that \( g_1 = f_n^{-\overline{r}} - f_n^{1-2\overline{r}} < f_n^{-\overline{r}} \), thus it suffices to lastly show,
\[ f_n^{-\overline{r}}(1 + f_n)^{\overline{r} - 2} + f_n^{-1}(1 + f_n^{-1})^{\overline{r} - 2} - 1 < 1 - f_n^{-1}, \] (15)

which is verified in Lemma II.5. Combining (12) and Lemma II.5 yields \( k \leq k^*(1, G_{n,p}) \leq \lceil \overline{r} \rceil + 1 \), which then asymptotically implies \( k^*(1, G_{n,p}) = \left\lfloor \frac{\log(n)}{\log(f_n)} \right\rfloor + 1. \)

**Lemma II.4.** Given (8), the term \( g_k \) remains positive for all \( k \in [1, \overline{r} - 1] \).

*Proof.* From (8) it follows that \( g_1 = f_n^{-\overline{r}} - f_n^{1-2\overline{r}} > 0 \). For the induction we now assume that \( g_k > 0 \) for some \( k \in [1, \overline{r} - 2] \). Since \( \overline{r} \leq k + 1 \), by (12) we have \( g_k < 1 - f_n^{k-\overline{r}} \) for all \( k \leq \overline{r} - 2 \). From (8) we want to show \( Q(g_k) > 0 \), thus defining the roots \( g_k^{\{1,2\}} = \{g_k : Q(g_k) = 0\} \) yields the sufficient condition,
\[ 1 - f_n^{k-\overline{r}} \leq g_k^{+} \triangleq \frac{1 + f_n - 2f_n^{k+1-\overline{r}} + \sqrt{(1 + f_n)^2 - 4f_n^{k+2-\overline{r}}}}{2f_n}, \] (16)

which holds for all \( k \leq \overline{r} - 2 \).

**Lemma II.5.** The inequality (15) holds in the limit as \( n \to \infty \).

*Proof.* To show (15) we prove the following 2 results,
\[ f_n^{-\overline{r}}(1 + f_n)^{\overline{r} - 2} \to 0 \] (17)
\[ f_n^{-1}(1 + f_n^{-1})^{\overline{r} - 2} \to 0. \] (18)

Both results (17) – (18) are shown via the same recursive interpretation of L’Hopital’s rule. Specifically, we apply the following result which holds for any 4 continuous functions \( \{a_n, w_n, r_n, h_n\} \) of \( n \),
\[ \left[ \lim_{n \to \infty} \frac{a_n^{w_n}}{r_n^{h_n}} = \left( \frac{\partial}{\partial n} a_n^{w_n} \right) / \left( \frac{\partial}{\partial n} r_n^{h_n} \right) = \frac{a_n^{w_n}}{r_n^{h_n}} \left( \frac{q_n}{j_n} \right) \right. \text{ with } |q_n/j_n| < 1 \implies \left. \lim_{n \to \infty} \frac{a_n^{w_n}}{r_n^{h_n}} = 0 \right]. \] (19)

First we consider (17). In regard to (19) let us define the 4 functions \( \{a_n, w_n, r_n, h_n\} \),
\[ a_n = 1 + f_n, \quad w_n = \overline{r}, \quad r_n = f_n, \quad h_n = \overline{r} + 2. \] (20)

We can let \( (w_n, h_n) = (\overline{r}, \overline{r} + 2) \) rather than \( (w_n, h_n) = (\overline{r} - 2, \overline{r}) \), since \( (1 + f_n)^{\overline{r} - 2} / f_n^{\overline{r}} < (1 + f_n)^{\overline{r}} / f_n^{\overline{r} + 2} \), and this will simplify the remainder of the proof.

Note that,
\[ \frac{\partial}{\partial n} \overline{r} = \frac{\partial}{\partial n} \log(n - 1) = \frac{(\log(f_n)/(n - 1)) - (\log(n-1)f_n')/f_n}{\log(f_n^2)}, \] (21)

so taking the two respective derivatives in (19) yields,
\[ \frac{\partial}{\partial n}(1 + f_n)^{\overline{r}} = (1 + f_n)^{\overline{r}} \left( \frac{\overline{r} f_n'}{1 + f_n} + \log(1 + f_n) \frac{(\log(f_n)/(n - 1)) - (\log(n-1)f_n')/f_n}{\log(f_n^2)} \right) \] (22)
\[ \frac{\partial}{\partial n} f_n^{\overline{r} + 2} = f_n^{\overline{r} + 2} \left( \frac{(\overline{r} + 2)f_n'}{f_n} + \frac{(\log(f_n)/(n - 1)) - (\log(n-1)f_n')/f_n}{\log(f_n)} \right) = f_n^{\overline{r} + 2} \left( \frac{1}{n - 1} + \frac{2f_n'}{f_n} \right). \] (23)

Next observe that the ratio \( |q_n/j_n| \) in (19) is less than unity,
\[ \left( \frac{\overline{r} f_n'}{1 + f_n} + \log(1 + f_n) \frac{(\log(f_n)/(n - 1)) - (\log(n-1)f_n')/f_n}{\log(f_n^2)} \right) / \left( \frac{1}{n - 1} + \frac{2f_n'}{f_n} \right) < 1 \] (24)
as it can be simplified,
\[
\frac{\log(1 + f_n)}{(n - 1)\log(f_n)} < \frac{2f_n'}{f_n},
\]
which holds since \( \lim_{n \to \infty} f_n' = f_n/n \) by L’Hôpital’s rule.

Next we consider (18). Regarding (19) let us define the 4 functions \( \{a_n, w_n, r_n, h_n\} \),
\[
a_n = 1 + f_n^{-1}, \quad w_n = \overline{k} - 2, \quad r_n = f_n, \quad h_n = 1,
\]
It will suffice to let \( w_n = \overline{k} \). Taking the first derivative in (19) yields,
\[
\frac{\partial}{\partial n} (1 + f_n^{-1})\overline{k} = (1 + f_n^{-1})\overline{k} \left( -\overline{k} f_n' + \log(1 + f_n^{-1}) \right) + \log(1 + f_n^{-1}) (\log(f_n)/(n - 1)) - (\log(n - 1)f_n'/f_n) \left( \log(f_n)^2 \right).
\]
The ratio of interest \( |q_n/j_n| \) in (19) is,
\[
\left( \frac{-\overline{k} f_n'}{f_n^2(1 + f_n^{-1})} + \log(1 + f_n^{-1}) (\log(f_n)/(n - 1)) - (\log(n - 1)f_n'/f_n) \right) / \left( f_n'/f_n \right)
\]
which is less than unity since,
\[
\frac{\log(1 + f_n^{-1})}{\log(f_n)} < (n - 1)f_n'/f_n = \frac{n - 1}{n} \to 1.
\]
\[
\square
\]

Using the above results, we can now complete the proof of the second line (14) for arbitrary \( u_0 \in [1, n - 1] \).

**Corollary II.6.** For an arbitrary \( u_0 \in [1, n - 1] \), if \( f_n \to \infty \), then \( k^*(u_0, G_n,p) = \left\lfloor \frac{\log(n/u_0)}{\log(f_n)} \right\rfloor + 1 \).

**Proof.** From Theorem II.1 we need only consider \( u_0 < \frac{n}{f_n} \). First let us assume \( u_0 \to \infty \). Define \( k^*(G_n,p)_{u_0,0} \) as the expected number of iterations it takes data \( d_0 \) from \( u_0 \) nodes to reach \( u_1 \gg u_0 \) nodes. By Theorem II.3 it takes approximately \( k^*(G_n,p)_{1,0} = \frac{\log(u_0)}{\log(f_n)} \) iterations for data \( d_0 \) from a single node to reach all other nodes in expectation. Thus, for \( u_0 \to \infty \) it takes approximately \( k^*(G_n,p)_{1,u_0} = \frac{\log(u_0)}{\log(f_n)} \) iterations for data \( d_0 \) from a single node to reach \( u_0 \) other nodes in expectation. Now, since \( u_0/n < \frac{1}{f_n} \to 0 \), the quantity of interest \( k^*(G_n,p)_{u_0,n} \) can be computed by the difference of the latter two approximations, that is,
\[
\lim_{n \to \infty} k^*(u_0, G_n,p) \triangleq k^*(G_n,p)_{0,0} = k^*(G_n,p)_{1,n} - k^*(G_n,p)_{1,u_0} + 1 = \left\lfloor \frac{\log(n/u_0)}{\log(f_n)} \right\rfloor + 1.
\]

Conversely, if \( \lim_{n \to \infty} u_0 = m \) for some constant integer \( m > 0 \), then \( 1/n \approx u_0/n \to 0 \) and thus the flooding time \( k^*(G_n,p)_{u_0,n} \) is asymptotically identical to \( k^*(G_n,p)_{1,n} \), that is, \( \lim_{n \to \infty} \left\lfloor \frac{\log(n)}{\log(f_n)} \right\rfloor + 1 \approx \left\lfloor \frac{\log(n/u_0)}{\log(f_n)} \right\rfloor + 1. \) \( \square \)

**III. Conclusion**

This note derives tight and exhaustive bounds on the the asymptotic \( u_0 \)-expected flooding time for a dynamic communication network modeled by the stationary Markov graph \( \mathcal{G}_{n,p} \). The summary of our results are presented in (14). Although the only non-trivial line in (14) is the second line, collectively all three lines in (14) are comparable to some previous results in (11), (4), (7). More specifically, the results of (14) imply and improve upon (11), and furthermore the results in (14) match, up to a multiplicative constant, those of (2) – (3). To the extent these results can inform the various applications to which they apply, there still is motivation to pursue a multitude of questions that remain within view.
A. Open Questions

There are a number of possible extensions to the results obtained in this note. For example, instead of assuming the “stationary” initial graph, that is the \( G_0 = [n] \times [n] \) or the empty graph \( G_0 = \emptyset \). Furthermore, in regard to the “k-active” flooding protocol proposed in [1], some numerical results have indicated that indeed, the “1-active” flooding protocol yields different expected flooding times for \( \hat{p} > \frac{\log(n)}{n} \) than for \( \hat{p} < \frac{\log(n)}{n} \). It is incidentally also the case that \( \hat{p} = \frac{\log(n)}{n} \) is the connectivity threshold of an \( (n, \hat{p}) \) Erdos-Renyi random graph [1]. Another open question is whether the \( u_0 \) parameter can be absorbed into the flooding time expression simply by substituting \( n'(u_0) = n/u_0 \) for \( n \). This does not appear to be the case for [1], as can be discerned by comparing the second and third lines. However, this may not be the case when considering the a.s., or w.h.p., flooding time for the general birth-death rate Markov graph model \( G_{n,p,q} \). In any case, it appears obvious that any further results regarding the interplay between the various input parameters, such as interplay between \( u_0 \) and the k-active protocol, should help shed light on a deeper understanding of the most efficient, reliable, and fastest method to spread data through-out dynamic communication networks.

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