Localization Transitions from Free Random Variables

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We motivate and use the concept of free random variables for the studies of the de-pinning transition of flux lines in superconductors as recently discussed by Hatano and Nelson. We derive analytical conditions for the critical points of the eigenvalue distribution in the complex plane for all values of the transverse magnetic field in one-dimension, in agreement with numerical calculations. We suggest a relation between reduced non-hermitean quantum mechanics in D-dimensions and non-hermitean random matrix models with weak non-hermiticity.

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1. Recently Hatano and Nelson [1] have shown that the de-pinning of flux lines from columnar defects in superconductors in D+1-dimensions, may be mapped onto a directed hopping, resulting into non-hermitean quantum mechanics. While it is generally accepted that all eigenstates are hop-conductors in D+1-dimensions, may be mapped onto

\[ H = H_0 + V = \]

\[ \frac{-t}{2} \sum_{A=1}^{N} (e^{i\hbar c_{A+1}^\dagger c_A} + e^{-i\hbar c_{A-1}^\dagger c_A}) + V_A c_A^\dagger c_A \]

where \( c_A^\dagger \) is a boson creation operator at site \( A \). Throughout the lattice spacing \( a = 1 \). The diagonal entries are random with elements distributed uniformly between \((-\Delta, \Delta)\), and the deterministic part \( H_0 \) is off-diagonal with hopping strengths \( t e^{\frac{i\hbar}{2}} \) \((t < 0)\). Here \( h \) is the typical strength of the ‘transverse magnetic field’ in units of the flux quantum \( \Phi_o \).

![Image](FIG. 1. Re \( \epsilon - \text{Im} \epsilon \) for \( t = \Delta = 1 \).)

The eigenvalues of \( H \) are complex valued for \( h \neq 0 \). Since \( H(-h) = H^T(h) \), the left-eigenvalues are equal to the right-eigenvalues. Their distribution in the z-plane is shown in Fig. 1 for \( h = 0.1, 0.3 \) and \( \Delta = t = 1 \). The results are for an ensemble of 100 matrices of size 100 \( \times \) 100. For \( h = 0.1 \) the eigenvalues are mostly real (localized). For \( h = 0.3 \) the eigenvalues around \( z = 0 \) are mostly complex (delocalized). For larger size matrices, the width of the rim shrinks to zero in agreement with the results discussed in [1].

An important question regarding the character of the spectrum is the occurrence of a critical \( h_c^{(1)} \) for which a gap near \( z = 0 \) sets in for the localized states. For increasing \( h > h_c^{(1)} \) the eigenvalues migrate from the real axis to the complex plane as also discussed by Feinberg and Zee [2]. The migration is total for \( h = h_c^{(2)} > h_c^{(1)} \). To try to quantify this and the bulk aspects of the spectrum in Fig. 1, we will analyze (1) using the addition law for free random variables [3].

2. Following Hatano and Nelson, we consider the non-hermitean tight-binding Hamiltonian in second quantized form for \( D = 1 \)

\[ \frac{-t}{2} \sum_{A=1}^{N} (e^{i\hbar c_{A+1}^\dagger c_A} + e^{-i\hbar c_{A-1}^\dagger c_A}) + V_A c_A^\dagger c_A \]

where \( c_A^\dagger \) is a boson creation operator at site \( A \). Throughout the lattice spacing \( a = 1 \). The diagonal entries are random with elements distributed uniformly between \((-\Delta, \Delta)\), and the deterministic part \( H_0 \) is off-diagonal with hopping strengths \( t e^{\frac{i\hbar}{2}} \) \((t < 0)\). Here \( h \) is the typical strength of the ‘transverse magnetic field’ in units of the flux quantum \( \Phi_o \).

3. In this section, we proceed to motivate the use of free random variables for the Hamiltonian (1). For that, we note that the diagrammatic expansion for (1) follows that of random matrix models [4] to the exception that the V-propagators are changed to...
with a uniform instead of Gaussian measure. Now, consider the resolvent $G(z) = \text{tr} G(z)$ for this model
\[
G(z) = \langle \frac{1}{z - H_0 - \mathcal{V}} \rangle = \frac{1}{z - H_0 - \Sigma(z)}
\] (3)
where $\Sigma(z)$ is the 1-particle irreducible (1PI) self-energy. Typical contributions to $\Sigma(z)$ are shown in Fig. 2. They may be decomposed into planar connected ($pl, c$) and nonplanar 1PI ones ($np$). Generically,
\[
\Sigma(z) = \sum_n \left( \langle \text{tr} \mathcal{V}(z) \mathcal{V} \ldots \mathcal{V}(z) \mathcal{V} \rangle_{pl,c} + \langle \text{tr} \mathcal{V}(z) \mathcal{V} \ldots \mathcal{V}(z) \mathcal{V} \rangle_{np} \right)
\] (4)
The planar piece is equal to
\[
\sum_{i=1}^{N} \langle V_i^n \rangle_{pl,c} \cdot (\mathcal{V}(z)_{ii})^{n-1} = \langle \text{tr} V^n \rangle_{pl,c} G(z)^{n-1}
\] (5)
In the last equality we made use of the invariance of $H_0$ under lattice translations. The nonplanar piece is involved. Here, we will approximate $G(z)$ by its diagonal part $G(z) \cdot 1$. The latter may be moved out of the trace, resulting into
\[
\Sigma(z) = \sum_n \left( \langle \text{tr} V^n \rangle_{pl,c} + \langle \text{tr} V^n \rangle_{np} \right) \cdot G(z)^{n-1}
\] (6)
This is our main approximation.

![Fig. 2. Planar and nonplanar contributions to $\Sigma(z)$. The dotted lines represent $G(z)$.](image)

Having said this, we note that the calculation of the resolvent would be simplified considerably if we could show that there exists a random matrix model $\mathcal{M}$ with a measure $P(\mathcal{M})$ such that
\[
\langle \text{tr} \mathcal{M}^n \rangle_{pl,c} = \left( \langle \text{tr} V^n \rangle_{pl,c} + \langle \text{tr} V^n \rangle_{np} \right)
\] (7)
Indeed, then $\Sigma(z)$ and hence $G(z)$ for our problem would simply follow from the analogue random matrix model $H_0 + \mathcal{M}$. The main advantage is that now only planar graphs in the random matrix model contribute, for which the addition law of free random variables [8] apply.

To show this, we note that all the nonplanar parts $\langle \text{tr} V^n \rangle_{np}$ can be absorbed into effective connected $i$-th moments. Hence, we may use the combinatorial relations discussed in [8], between connected and ordinary moments for planar diagrams to reduce the equality (7) just to the equality of moments
\[
\langle \text{tr} \mathcal{M}^n \rangle_{pl} = \langle \text{tr} V^n \rangle
\] (8)
with the pertinent measures for each averaging. This is equivalent to finding a probability distribution $P(\mathcal{M}) = \exp(-N \text{tr} \mathcal{V}(\mathcal{M}))$ such that the resolvents of $\mathcal{M}$ and $\mathcal{V}$ coincide. To this end we may solve the loop equation (technically with a single cut) of [7] to obtain
\[
\frac{d \mathcal{V}(z)}{dz} = G_V(z + i\epsilon) + G_V(z - i\epsilon)
\] (9)
A matrix model with such a potential will satisfy all our requirements. However to apply the methods of free random variables we need only to know that such a model exists, the explicit form of $\mathcal{V}$ through [8] is unnecessary. This concludes our proof of existence.

To summarize, our approximation scheme for (1) in the form of (6) includes all the planar graphs and also resums an infinite series of (diagonal) nonplanar contributions for the resolvent. It is easily implemented in the matrix analogue using the addition law for $H_0 + \mathcal{M}$. Although this procedure is not exact in large $N$, the results are close to the exact results for large $z$ and large $N$, as shown in the Table for the first few moments in the Laurent expansion of the resolvent, that is
\[
G(z) = \sum_{n=0}^{\infty} C_{2n} \frac{1}{z^{2n+1}}
\] (10)
for $h = 0$, $t = 2$ and $\Delta = 1$. The deviations from freeness start at the eight moment and are very small.

| $2n$ | $C_{2n}^{\text{exact}}$ | $C_{2n}$ | $2n$ | $C_{2n}^{\text{approx}}$ | $C_{2n}$ |
|-----|-----------------|-------|-----|-----------------|-------|
| 2   | 3               | 3     | 8   | 736             | 732   |
| 4   | 9               | 16    | 10  | 5644            | 5534  |
| 6   | 103             | 103   | 12  | 45634           | 43654 |

4. In this section we derive the resolvent $G(z)$ along the real axis using the addition law for ‘hermitean’ matrices. $\pi V(\lambda) = \text{Im} G(\lambda + i\epsilon)$ is the distribution of localized eigenvalues. With this in mind, the resolvent for the random part is simply given by
\[
G_V(z) = \frac{1}{N} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} N d\mathcal{V} = \frac{1}{2\Delta} \ln \frac{z + \Delta}{z - \Delta}
\] (11)
with an inverse (Blue’s function) given by $B_V = \Delta \coth \Delta z$, in agreement with [7]. The resolvent for the deterministic part is
\[
G_{H_0}(z) = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{z - t \cos (2\pi n/N) + ih}
\] (12)
In the large $N$ limit $G_{H_0}(z) = 1/\sqrt{z^2 - t^2}$ for $z$ outside the ellipse defined by
\[
\left( \frac{x}{\cosh h} \right)^2 + \left( \frac{y}{\sinh h} \right)^2 = t^2.
\] (13)
and zero inside. The inverse of the resolvent (Blue’s function) is just $B_{B_0} = \sqrt{1/z^2 + \tau^2}$.

For large $z$ and in the planar approximation (matrix model), the resolvent $G(z)$ for $t$ along the real axis follows from its functional inverse $B[G(z)] = z$ through the ‘hermitean’ version of the addition law $[5]$, $B = B_{B_0} + B_Y + \frac{1}{z}$. Specifically

$$z = \sqrt{\frac{1}{G^2} + \Delta^2 + \Delta \coth \Delta G - \frac{1}{G}}$$

(14)

and is $h$-independent. Along the real axis, the end-points $z_c$ of the spectrum satisfy $z_c = B(z_c)$, with $dB(z_c)/dz = 0$, that is

$$-\frac{1}{2z^3} \frac{1}{\sqrt{1 + \Delta^2}} - \frac{\Delta^2}{\sinh^2 \Delta z} + \frac{1}{2z^2} = 0$$

(15)

For $\Delta = t = 1$ this yields $z_c = 1.5752$ or $z_c = 1.63915$.

5. In this section, we derive analytical conditions for the end-points of Fig. 4. For that, we analytically continue $G(z)$ to the $z$-plane minus the ellipse $[3]$, and use it to construct the potential $F(z) = \log Z/N$, where $Z(z) = (\det(z - H))$. In the planar approximation (matrix model) and to leading order in $1/N$, the potential follows from $G(z)$ through $F = \int dz G$, that is

$$F(z) = +\Delta G \coth \Delta G - 1 + \frac{1 + \sqrt{1 + \Delta^2}}{2} - \log \sinh \Delta G \Delta.$$  

(16)

The normalization $Z \sim z^N$ as $z \to \infty$ corresponds to

$$G(z) \to 1/z.$$  

For $z \to 0$, $Z(z) = t/2e^{N(h(1 + O(N^0))}$. This behavior is related to the other branch ($G(z) = 0$) of the resolvent, for which $F(z)$ with $z \sim 0$ is constant and equal to $h - \log 2/t$. Hence, a ‘cusp’ develops in $F(z)$ where the in and out potentials cross, that is

$$F[G_{out}(z)] = h - \log 2/t \equiv F[G_{in}(z)].$$

(17)

The locus of the cusp (7) coincides with the position of the complex eigenvalue distribution shown in Fig. 3 as $N \to \infty$. The real (localized) eigenvalues are the remnants of the hermitean addition law discussed above, and disappear at some critical value of $h = h_c^{(2)}$.

Fig. 3 shows the behavior of the critical cusp-line along the real axis, where the eigenvalue spectra have a branching point ($x_{branch}$), versus the strength $h$ for which (7) holds. At $x_{branch}(h_c^{(2)}) = z_c$ the localized states disappear from the spectrum. The dots are the numerically generated branching points, while the solid line corresponds to using (16)-(17). Fig. 3 shows the same along the imaginary axis for the topmost point of the spectrum $y_{max}$. For large $h$, the resolvent drops to zero in the outer region like $1/z$, and using (14) together with (16) we get for $y_{max}$

$$y_{max} \approx \sinh h - \frac{\Delta^2}{6 \sinh h}.$$  

(18)

The existence of the solution for given $h$ defines the critical value $h_c^{(2)}$ for which the cut starts to develop in the inner ellipse of Fig. 4. The dependence of $h_c^{(1)}$ on $\Delta$ is shown in Fig. 4 (solid line). The dotted lines are generated analytically by working out the leading contributions to $F$ in (16). For small values of $\Delta$ (dotted line)

$$F = \frac{\Delta^2}{6} + \frac{\Delta^4}{180} - \log 2 - \frac{i\pi}{2},$$

(19)

while for large values of $\Delta$ (long dashed line)

$$F = \log \Delta + \frac{\pi^2}{16 \Delta^2} - 1 - \frac{i\pi}{2}.$$  

(20)

These results are also reproduced analytically using a semi-circular distribution for the random part, that is

$$P(V_i) = \frac{1}{2\pi \tau} \sqrt{4\tau - V_i^2}$$

(21)

when $\tau = \Delta^2/3 \to \infty$, as indicated by the thick dashed line in Fig. 4.

Finally, the dependence of the eigenvalue distribution shown in Fig. 4 on $1/N$, maybe qualitatively understood by using the circular version of (1). This amounts to trading $t/2e^{-h} \to r$ and $t/2e^{-h} \to 0$. For a uniform distribution of eigenvalues, the secular equation

$$\prod_i (\lambda - V_i) = (-r)^N.$$  

(22)

For small values of $\Delta$, hence $V_i$, (22) can be solved perturbatively,

$$\lambda^N - \left( \sum_i V_i \right) \lambda^{N-1} = (-r)^N,$$  

(23)
with the ansatz \( \lambda_j = r e^{i 2\pi j/N} + \epsilon_j \), giving \( \epsilon_j = -\sum_i V_i/N \). Typically, \( \langle N^2 \epsilon_i^2 \rangle = N \langle V_i^2 \rangle = N \Delta^2/3 \), showing the corrections to be \( \epsilon \sim 1/\sqrt{N} \) in large \( N \).

Numerically the width of the boundary between the outer \( (G(z) \) holomorphic) and the inner \( (G(z) = 0) \) maybe analyzed using

\[
\delta \lambda = \max (|\lambda_i| - |\lambda_j|) \tag{24}
\]

along the real axis. The dependence on the size of the matrix is indeed \( \sim 1/\sqrt{N} \) as shown in Fig. 5.

![Fig. 5. \( \delta \lambda \) versus \( N \) (matrix size) for \( r = \Delta = 1 \).](image)

6. The previous analysis borrows on some of the methods discussed in \([\ref{1}]\) for random matrix models. However, the Hatano-Nelson model differs in an important way from matrix models: it knows about the dimensionality of space. Recently Efetov \([\ref{2}]\) has used supersymmetric methods to argue that in 0-dimension the model reduces to a nonhermitian random matrix model with weak nonhermiticity \([\ref{3}]\), and suggested that the reduction may yield to new developments in the context of oriented quantum chaos.

The reduction is actually understandable from the point of view of continuum quantum mechanics of constant modes. The continuum version of \([\ref{1}]\) is \( H = (p + i b)^2 + V \), where \( p \) is the D-dimensional momentum and \( V \) the random site potential. The reduction to 0-dimension means that \( V \) is \( x \)-independent. Classically this would imply that \( p = 0 \). Quantum mechanically, however, \( p \) fluctuates. In a box of size \( N \), \( p \sim W/\sqrt{N} \) by the uncertainty principle. Hence, the reduction of \( H \) to \( H_* \) in 0-dimension amounts to

\[
H_* = V + \frac{i 2 b}{\sqrt{N}} W \tag{25}
\]

For sufficiently random hopping, the result is a random matrix model with small nonhermiticity in general. For \( D=2 \) this is just the case of \emph{weak} non-hermiticity discussed by Fyodorov et al. \([\ref{2}]\). For \( V, W \) chosen in the GOE ensemble as motivated by \( V \) real in the D-dimensional version of \([\ref{1}]\), our arguments suggest localization for \( D=2 \) in \([\ref{25}]\) as also noted in \([\ref{1}]\).

The present arguments may also extend to other models. For example, the (massless) QCD Dirac operator in a D-dimensional Euclidean box of volume \( N \) at finite chemical potential \( \mu \) is \( H = \gamma^{D+1}(i\gamma^a \nabla^a + i\mu \gamma^D) \), where \( \gamma^a \)'s are Dirac matrices with \( a = 1, ..., D \), and \( \nabla \) the covariant derivatives with external gauge fields. The squared operator,

\[
H_* = (i\nabla^a)(i\nabla^a) + \frac{i}{2} \sigma^{ab} [i\nabla^a, i\nabla^b] + 2i\mu(i\nabla^D) \tag{26}
\]

with \( \sigma^{ab} = i[\gamma^a, \gamma^b]/2 \), is analogous to \([\ref{25}]\). For sufficiently random hopping and \( i\nabla^b \sim 1/\sqrt{N} \), localization may take place for the GOE ensemble in \( D=2 \) dimensions at finite \( \mu \). Since two-dimensional QCD, in the limit of a large number of colors, exhibits quasi long-range order, this issue is worth investigating.

7. We have shown that the Hatano-Nelson model \([\ref{1}]\), may be understood in terms of the addition law for free random variables. We used the latter to find an analytical condition for the end-points of the complex eigenvalue distributions. The results compare well with the numerics. The size of the spread of the complex eigenvalue distribution is \( 1/\sqrt{N} \). We have presented generic arguments for how the constant modes of the model in \( D \)-dimensions and for sufficiently random hopping, relate to random matrix models with weak nonhermiticity. For \( D=2 \) our arguments confirm a recent observation by Efetov \([\ref{2}]\). In light of this, it would be interesting to repeat our analysis for \( D = 2 \) for the three Wigner’s ensembles.

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