On tunneling across horizons

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Abstract – The tunneling method for stationary black holes in the Hamilton-Jacobi variant is reconsidered in the light of some critiques that have been moved against. It is shown that once the tunneling trajectories have been correctly identified the method is free from internal inconsistencies, it is manifestly covariant, it allows for the extension to spinning particles and it can even be used without solving the Hamilton-Jacobi equation. These conclusions borrow support on a simple analytic continuation of the classical action of a pointlike particle, made possible by the unique assumption that it should be analytic in the complexified Schwarzschild or Kerr-Newman space-time. A more general version of the Parikh-Wilczek method will also be proposed along these lines.

Introduction. – When the tunneling method was first proposed [1–9] a certain discomfort appeared soon, having mainly to do with two aspects of the method. One was that even employing a coordinate system covering regularly the horizon, the action (or the radial momentum) exhibited a pole, thus demanding a proper treatment; in fact, a naïve integration across the singularity in Schwarzschild coordinates (or Boyer-Lindquist for Kerr) produced twice the correct Hawking’s temperature of the black hole (BH), as noted in [7]. The second had to do with the possibility that a complex contribution from the temporal part of the classical action I, namely the term \( \int \partial_t I \, dt \), could either cancel or double the relevant emission term, if not properly handled. Several proposal where soon advanced [7,10–15]. Although each one has its own merits, we shall see that no one is particularly compelling. Most of the derivations and the ensuing problems have to do with coordinates choices, or lack of manifest general covariance 1 (see [15] for a particular approach to this problem for weakly isolated horizons). To be completely clear it must be said that when Parikh and Wilczek [1] (see also [16]) introduced their method, successively dubbed “the null geodesics method”, the main motivation was to reveal the back reaction corrections to first order in \( \epsilon / M \), where \( M \) is the black-hole mass and \( \epsilon \) the energy scale of the process. No back reaction correction will be considered here, as it is not really needed to derive the Hawking effect. Our emphasis will be on covariance with the aim of resolving certain conflicts between different views, hence the preferred method should be invariant ab initio. To this aim we will work with the Hamilton-Jacobi (HJ) version of the method.

So after a brief review of this method we shall propose such a covariant derivation of the tunneling method. A “covariantized” Parikh-Wilczek method will also be proposed along these lines.

The tunneling path and the method. – According to a standard picture of the black-hole radiation, a pair is created somewhere near the horizon, one member escaping to infinity with positive energy, the other falling down the black hole and carrying negative energy. And this process continuously occurring here and there adds up to form the thermal streaming from the black hole. According to Hartle and Hawking [17], this can be described equivalently as the escape of a particle along the forbidden path, the right one in fig. 1. We shall call this the tunneling path. Once out of the horizon the particle cannot do anything but moving forward in time, eventually ending its journey reflected back into the black hole.

To see what motivated the perplexities, let us review the HJ method starting with the Schwarzschild metric

\[
d_s^2 = -\left(1 - \frac{2m}{r}\right) \, dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} \, dr^2 + r^2 d\Omega^2.
\] (1)

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1 Even the top cited paper by Parikh and Wilczek [1] used Painlevé-Gullstrand coordinates, thus avoiding problems with covariance.

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In the tunneling method one is interested in the imaginary part of the integral of the action along the tunneling path, say
\[ \Im \int dI, \]  
the arrow denoting the horizon crossing path and \( dI \) the differential of the action of a spinless massless particle. Ignoring for the time being that the portion of the path crossing the horizon is not covered by a single coordinate patch, the HJ equation gives
\[ I = -Et + \int_{r}^{r_+} \frac{E}{r - 2m} dr + J\phi. \]  
(3)

It is argued that the internal segment can only be reached through a journey into the complex \( r \)-space, and the right procedure to do so, as will be seen, is Feynman \( \imath \epsilon \)-prescription\(^2\): the substitution \( r - 2m \rightarrow r - 2m - \imath \epsilon \). Since \( (r - 2m - \imath \epsilon)^{-1} = \overline{P}[(r - 2m)^{-1}] + \imath \pi \delta(r - 2m) \), \( P \) denoting the principal part, we get
\[ \Im I = \frac{\pi E}{2\kappa}, \]  
(4)

\( \kappa = (4m)^{-1} \) being the horizon surface gravity. Identifying the emission probability with \( P_{\text{em}} \sim \exp(-2\Im I) = \exp(-\pi \kappa^{-1}E) \) gives then the temperature \( \kappa/\pi \), twice the Hawking result. This is in common with the Kerr metric, which in Boyer-Lindquist coordinates reads
\[ ds^2 = -\frac{\Delta \rho^2}{\sigma} dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma \sin^2 \theta}{\rho^2} (d\phi - \omega dt)^2 \]  
(5)

with the standard definitions
\[ \Delta = r^2 - 2mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \]  
(6)

\[ \Sigma = (r^2 + a^2) - \Delta a^2 \sin^2 \theta, \quad \omega = \frac{2mar}{\Sigma}. \]  
(7)

The parameter \( a \) is the angular momentum per unit mass, \( J = ma \), and \( m \) is the ADM mass. The horizon is the largest root of \( \Delta = 0 \): \( r_\pm = m \pm \sqrt{m^2 - a^2} \). The HJ equation is separable in the Kerr field \([19]\), i.e.
\[ I = -Et + J\phi + W(r) + S(\theta). \]  
Solving the equation gives
\[ W(r) = \int \sqrt{\mathcal{R}} \Delta^{-1} dr, \]  
(8)

where \( \mathcal{R} = [E(r^2 + a^2) - aJ]^{-2} - K \Delta \) and \( K \) is a constant. The integrand has a simple pole at \( r = r_+ \), where \( \Delta = 0 \). Therefore, as before,
\[ \Im I = \frac{\pi \mathcal{R}(r_+)}{r_+ - r_-} = \pi \frac{(\frac{r_+^2}{r_+} + a^2)}{r_+ - r_-} (E - \Omega J) = \frac{\pi (E - \Omega J)}{2\kappa}, \]  

where \( \kappa \) is the surface gravity and \( \Omega = a/2mr_+ \) is the angular velocity of the horizon. Again \( P_{\text{em}} \sim \exp(-2\Im I) = \exp(-\pi \kappa^{-1}(E - \Omega J)) \) gives \( T = \kappa/\pi \), twice the Hawking result.

A number of solutions were advanced. In \([7,11]\) it was proposed to use the proper distance from the horizon, on the ground of covariant requirements. In \([12]\) it was convincingly proved that a similar imaginary part would be produced by the temporal part of the action (see also \([13]\)). Others ventured to suggest that there is not even an imaginary part, the temporal part cancelling the simple pole on using the HJ equation \([20]\). Of course in general there is also an amplitude to cross the horizon inward \([10]\),
\[ \Im \int dI, \quad I = -Et + \int_{r}^{r_+} \frac{E}{r - 2m} dr + J\phi, \]  
(9)

because to an outside observer the particle never reaches the horizon in real Schwarzschild time. Hence deforming the contour as above would give
\[ \Im \int dI = -\frac{\pi E}{2\kappa}. \]  
Therefore
\[ 2\Im \int dI = 2\Im \int dI = \frac{2\pi E}{\kappa}. \]  
(10)

Taking the exponential gives
\[ P_{\text{em}} = P_{\text{abs}} e^{-2\pi E/\kappa}, \]  
(11)

which is recognized as the detailed balance condition for a system in thermal bath. This is certainly correct but the derivation looks very suspicious. On the one hand, the pole prescription seems contrived just to obtain the wanted result. On the other hand, the use of singular coordinates to describe horizon crossing trajectories is very awkward, a point which was made clear by the Parikh and Wilczek treatment of the problem. We cover this last point with an example. The metric (1) can be made regular across the horizon by passing to Eddington-Finkelstein advanced coordinates \((v, r, \theta, \phi)\)
\[ ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2dvdr + r^2 d\omega^2. \]  
(12)
The action is $I = -Ev + J\phi + W(r) + S(\theta)$ (note that $\partial_\theta I = \partial_t I$) and from the HJ equation, $g^{\mu\nu}\partial_\mu I\partial_\nu I = 0$, one gets

$$W(r) = 2E \int_r^\infty \frac{rdr}{r-2m}. \quad (13)$$

Notice the factor two coming from the $2\partial_\theta I\partial_t I$ term in the HJ equation. On the ingoing path, on the other hand, one can easily see that there is no pole on crossing the horizon and consequently no imaginary part. From (13) we obtain the correct result

$$\Im \int dI = \frac{\pi E}{k}. \quad (14)$$

Similar conclusions can be drawn using other regular coordinates, for example the Painlevé-Gullstrand coordinates employed by Parikh and Wilczek. The idea that by using coordinates which are regular across the horizon eliminates all sort of problems is one point made in [21].

We will soon see that eq. (10) is correct (for Schwarzschild BH) although not always the ingoing term adds an imaginary part, except that when it does the coordinates as a rule fail to cover the horizon. Note that the left-hand side of (10) is a coordinate scalar, so it must be possible to obtain it from invariant arguments. This is provided by the analytic continuation of the classical action throughout complexified space-time.

**The analytic argument.** – To justify the above machinery, and in particular the Feynman $ic$-prescription, we shall now rotate the tunneling path away from the horizon, as shown in fig. 2.

We shall use in intermediate steps the well-known Kruskal coordinates $(U, V)$, such that $U < 0$, $V > 0$ in region I, $U > 0$, $V > 0$ in region $I'$, $U < 0$, $V < 0$ in region IV, and go to the complex $(U, V)$-plane putting $\tilde{U} = U \exp(-i\lambda)$, $\tilde{V} = V \exp(i\lambda)$, $0 \leq \lambda \leq \pi$. To understand this choice note that it corresponds to a Wick-like rotation of Schwarzschild time $t \rightarrow t - i\lambda/k$. Now

$$dI = \partial_U I d\tilde{U} + \partial_V I d\tilde{V} = \partial_U I dU + \partial_V I dV \quad (15)$$

$$+[U \partial_U I - V \partial_V I]id\lambda, \quad (16)$$

where a term $\partial_U I d\phi + \partial_V I d\theta$ has been omitted since it does not give contributions to the imaginary part (but see the Kerr solution below). From the property of Kruskal coordinates one has

$$-U \partial_U I + V \partial_V I = \kappa^{-1} \partial_t I = -\kappa^{-1} E. \quad (17)$$

Assuming analyticity, the integral over the segment $a \rightarrow b \rightarrow c$ is now equal to the integral over the semi-circle $[0, \pi]$ (over which $U$, $V$ are constant) plus the integral over the segment $a' \rightarrow b \rightarrow c$, over which $\lambda$ is constant (and equal to $\pi$); thus we obtain

$$\Im \int dI = \Im \int dI + \frac{\pi E}{k}. \quad (18)$$

where now the upward arrow refers to the path crossing the past horizon, which is classically allowed. Therefore $E$ is the conserved energy of the particle. The Feynman prescription is now clear, because it is the only one which is consistent with the analytic method. By time reversal invariance the amplitude to cross the past horizon outward is the same as the amplitude to cross the future horizon inward, therefore eq. (18) is just the same as eq. (10). If instead we choose to continue analytically the other way, say by putting $\tilde{U} = U \exp(i\lambda)$, $\tilde{V} = V \exp(-i\lambda)$, $0 \leq \lambda \leq \pi$, which correspond to a counter clockwise Wick rotation, $t \rightarrow t + i\lambda/k$ and a Feynman prescription $r \rightarrow r + i\epsilon$, we would obtain

$$P_{abs} = P_em e^{-2\pi E/k}. \quad (19)$$

This can be interpreted as the detailed balance condition for a white hole to absorb a quantum particle via the past horizon, a process that would be classically forbidden by causality.

**Spinning particles:** All we come to say should apply equally well to spinning particles. The known Lagrangian formulation of such systems does not modify the free action term, which is where the pole at the horizon resides.

Equivalently, the Hamilton-Jacobi equation for fermions is just the same as for spinless particles as it represents the phase of the spinor amplitude [22–24]. It is important that these expectations were recently extended to spin-1 bosons, and that the Hawking temperature will not receive higher-order corrections in $\hbar$ beyond the semi-classical ones [25] (see also [26] for a different view).

**Kerr black hole:** We can extend the calculation to the Kerr solution by noticing that throughout the complex manifold the azimuthal angle must also be rotated to keep the metric regular, more precisely $\phi \rightarrow \phi - i\Omega \lambda/k$. Then adding the term $\partial_\phi I d\phi$ to the differential $dI$ would produce an imaginary term after a $\pi$-rotation, equal to $-i\pi \Omega \partial_\phi I/k$. The outgoing trajectory from the past horizon is a classical solution so $\partial_\phi I = J$, the conserved angular momentum. We obtain in this way the result

$$\Im \int dI = \Im \int dI + \frac{\pi}{k}(E - \Omega J). \quad (20)$$
or
\[ P_{\text{em}} = P_{\text{abs}} e^{-2\pi (E - \Omega j)/\kappa}. \] (21)

Of course in quantum theory the angular momentum is quantized. As is well known the emission and absorption probabilities for particles with energy \( E \) and angular momentum \( j \) are related to the Bogoliubov \( \beta \)-coefficients, whose computation is a classical problem involving the relevant field equations. Unitarity in the space of classical solutions relates them to the transmission coefficient \( \Gamma_{E,j}\) through the potential barrier surrounding the horizon

\[ P_{\text{abs}} \pm P_{\text{em}} = \Gamma_{E,j}. \] (22)

where the \((+\) is for fermions and the \((-\) for bosons. Together with (11) it gives the Bose-Einstein or Fermi-Dirac spectrum.

Charged black holes: The prototypical charged solution is the Reissner-Nordström metric
\[ ds^2 = -V(r)dt^2 + \frac{1}{V(r)} dr^2 + r^2 d\omega^2, \] (23)

where
\[ V(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} = \frac{1}{r^2} (r-r_+)(r-r_-) \] (24)

and the electromagnetic field has potential \( A = r^{-1}Q dt \) (a one-form). The action to be integrated on the tunneling path is \( I_0 + eA \), where \( I_0 \) is the free action and \( e \) the electric charge. The form \( A \) is ill-defined at the horizon, for this reason one usually makes a gauge transformation to a form \( A = A + df \) which is regular there. In our case the analytic continuation takes \( A \) away from the horizon so this will actually be unnecessary. Using as above complex \((U, V)\) coordinates we obtain
\[ \Im \int dI = \Im \int dI + \frac{\pi}{\kappa} (E - e\Phi), \] (25)

where \( E = -\partial_t I_0 \) is the mechanical energy and \( \Phi = q/r_+ \). The quantity \( E - e\Phi \) is gauge invariant and conserved along the outgoing path from the past horizon. The extension to cover the Kerr-Newmann solution should now be obvious.

A generalized Parikh-Wilczek method. – The previous considerations suggest a simple generalization of the null geodesics method. The authors manage to compute the imaginary part of the integral of the radial momentum in Painlevé-Gullstrand coordinates, namely
\[ \Im \int p_r dr. \] (26)

This looks non-covariant, but we may substitute the full Liouville differential one-form, \( \omega = p_r dx^\mu \), in place of the radial momentum, which is nothing but the reduced action. We can now analytically continue as explained above, first by writing \( \omega = p_r dU + p_V dV \), then rotating \((U, V)\) from zero to \( \pi \) and finally integrating along the rotated curve. As in eq. (15) the imaginary part will be \( i\pi (U p_V - V p_U) \): but, as in eq. (17), this is \( -i\pi p_I /\kappa = i\pi E /\kappa \), where \( E \) is the Killing energy as measured at infinity. In all we get
\[ \Im \int \omega = \frac{\pi E}{\kappa}. \] (27)

Incidentally this shows that the null geodesic method and the HJ method are completely equivalent as far as stationary black holes are concerned. A confirmation of this fact based on specific coordinate systems was presented in [27]. This is true because the Hawking effect is an energy conserving process, so that the reduced action is all one needs in a static geometry.

Conclusions. – The main result of this work, namely eqs. (18), (20), (25) and (27), show that it is possible to formulate a coordinate-invariant statement about semi-classical horizon tunneling. It is not even necessary to use the Hamilton-Jacobi equation or the Hamiltonian equations of motion, although one needs to know which paths are forbidden and which ones are not. Nor it is necessary to prescribe some special coordinate system, as sometime it is rumored in relation to the Painlevé-Gullstrand frame. In particular the imaginary temporal contributions can be present or not, depending on the chosen time, but they will never cancel the pole part. In fact the formalism appears covariant and therefore independent of which particle concept (or time) one employs. It is hoped that this will contribute to a better understanding of the tunneling mechanism.

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