Light propagation in \((2+1)\)-dimensional electrodynamics: the case of nonlinear constitutive laws

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Abstract
We scrutinize the geometrical properties of light propagation inside a hypothetical nonlinear medium modeled by a fully covariant electromagnetic theory in \((2+1)\)-dimensions. After setting the local and nonlinear constitutive relations, the phase velocity and the polarization of the waves are derived and three special cases are analyzed in detail. In particular, we show that this version of electromagnetism in low dimensions may present phenomena such as one-way propagation and controlled opacity, among others, for a large class of dielectric and magneto-electric parameters.

Keywords: electromagnetism in \((2+1)\)-spacetime, nonlinear optics, geometric optics limit

(Some figures may appear in colour only in the online journal)

1. Introduction

The last years have witnessed a great interest in the understanding of light propagation inside two-dimensional (2D) materials. The main reason behind this is the unique optical response such mono-layer structured media (of atomic thickness) present under excitation by external electromagnetic fields. This peculiar feature was already noticed from the first studies with graphene [1] and thenceforth, many other 2D materials have been discovered, generating a rapidly growing development of optoelectronic and photonics for these media. Examples of exciting applications include THz wave generation [2], passive optical power limiters [3, 4].

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electro-optic modulators [5, 6] among others (see [7] and references therein for a complete list of technological developments).

Nowadays, there is a reasonably large family of known materials presenting a relatively ultrafast and strong optical nonlinearity response like perovskites [8], gallium selenide [9], black phosphorus [10], transition metal chalcogenides [11] and others. The success of these media in paving the way for new optical devices is essentially due to their large optical and thermal damage threshold, with ultrafast recovery time and high chemical/mechanical stability. They also range the whole spectrum of electric conductivity, being (semi) metals, semiconductors, or insulators. At last, they are of low fabrication costs.

Inspired by the aforementioned technological achievements, we study some aspects of light propagation inside hypothetical nonlinear 2D materials in the limit of geometric optics, because, from the theoretical point of view, there is an open question in the literature regarding the correct formulation of electromagnetism to deal with physical 2D materials. According to Hadamard’s method of descent [12, 13], there are two non-equivalent possibilities: one in terms of a fundamental two-form \( F_{ab} \) (in which the electric field is a vector and the magnetic field is a pseudo-scalar) and another one in terms of a fundamental vector \( F_a \) (in which the magnetic field is a vector and the electric field is a pseudo-scalar). Some authors claim that the two formulations are simultaneously needed [14, 15] in order to cover all aspects of light propagation in 2D materials, while other authors prefer a more axiomatic-oriented formulation of the electromagnetic theory [16], dealing with a single two-form. Since the first approach often requires a limiting procedure with respect to the contracted dimension, we shall follow here the second one due to its mathematical simplicity.

Based mostly upon the results derived in [17], where the effective metric associated with a genuine \((2 + 1)\)-dimensional electrodynamics with a linear constitutive law was derived, we now turn our attention to quite general nonlinear constitutive laws. More specifically, we determine the general expressions for the phase velocities and the corresponding wave-polarization vectors for nonlinear dielectrics, nonlinear magnetoelectric media, and a combination thereof. Again, we leave for experimental physicists the task of deciding which approach better fits the increasing amount of data concerning these materials.

This paper is organized as follows. Section 2 starts with a discussion of the equations of motion and the corresponding nonlinear constitutive laws. Then, we construct the eigenvalue equation associated with the light propagation in this \((2 + 1)\)-dimensional framework. In section 3, the expression for the phase velocity is analyzed, showing the possibility of one-directional propagation for sufficiently high values of magneto-electric cross terms. In section 4, we characterize the eigenvectors of the original eigenvalue equation and derive the corresponding wave-polarization vectors. Finally, in section 5, we study the behavior of the phase speed and the wave-polarization vector for some special nonlinear systems: purely magnetic media, anisotropic non-magnetic dielectrics, and second-order magneto-electric materials. Throughout, we work with a Minkowski background metric expressed in a Cartesian coordinate system i.e. \( \eta_{ab} = \text{diag}(1, +1, +1) \). As usual, the quantities \( \varepsilon_0 \) and \( \mu_0 \) respectively denote the electric permittivity and magnetic permeability of the vacuum, and units are chosen such that \( c = 1 \), except where mentioned otherwise.

2. Mathematical setting

Here, we shall follow the steps of our previous work [17], dealing with the electromagnetic field being represented by a two-form field \( F_{ab} \). The main difference with respect to our previous analysis is the assumption of a timelike observer field that will decompose \( F_{ab} \), from the
very beginning, in terms of a vector electric field and a (pseudo) scalar magnetic field. Consequently, all the results derived thereafter are observer-dependent, although they are fully covariant. This is very convenient for the analysis of the special cases we shall handle and for further comparison with experiments.

2.1. Field equations

We start by considering a flat \((2+1)\)-dimensional spacetime and writing the Faraday tensor in Cartesian coordinates \(x^a = (t, x, y)\) as

\[
F_{ab} = \begin{pmatrix}
0 & -E_x & -E_y \\
E_x & 0 & B \\
E_y & -B & 0
\end{pmatrix}.
\]  

(1)

More generally, with the introduction of a future-directed, timelike, normalized field of inertial observers, henceforth denoted by \(\ell^a\), this tensor can be irreducibly decomposed as

\[
F_{ab} = \ell_a E_b - \ell_b E_a + \epsilon_{abc} \ell_c F^c,
\]  

(2)

where \(\epsilon_{abc}\) is the completely skew-symmetric Levi-Civita tensor with \(\epsilon_{012} = +1\). As usual, writing the Hodge dual as

\[
^* F^a = \frac{1}{2} \epsilon^{abc} F_{bc} = \epsilon^{abc} \ell^b E^c - B^a,
\]  

(3)

a direct calculation gives the projections

\[
E_a = F_{ab} \ell^b, \quad B = ^* F \ell^a,
\]  

(4)

which describe, respectively, the (vector) electric field and the (pseudo-scalar) magnetic induction as measured by the corresponding inertial observer. In addition, we write the squared norms of these quantities as

\[
||E||^2 = E_a E^a, \quad ||B||^2 = B^2,
\]  

(5)

and notice that the Faraday tensor and its dual do not share the same tensor rank, which is expected in any spacetime dimension different from \(3+1\).

Inside the medium, we define the ‘polarization’ tensor \(P_{ab}\) to account for its response to the external fields applied. This tensor is also irreducibly decomposed as

\[
P_{ab} = \ell_a D_b - \ell_b D_a + H \epsilon_{abc} \ell^c,
\]  

(6)

where the field excitations are the (vector) electric displacement \(D_a\) and the (pseudo-scalar) magnetic field \(H\). It is generally assumed that these field excitations and the field strengths are linked through local constitutive relations of the form

\[
D^a = \Xi^a_b E^b + \Omega^a B,
\]  

(7)

\[
H = \zeta_a E^a + \mathcal{D} B,
\]  

(8)

with \(\Xi^a_b\) denoting the electric permittivity tensor, \(\mathcal{D}\) the (inverse) magnetic permeability, and the other terms properly introduced in order to take into account possible magneto-electric effects, to wit, the ‘magnetic permittivity’ represented by \(\Omega^a\) and the ‘electric permeability’ represented by \(\zeta_a\). The above quantities are spacelike and orthogonal to the observer field by construction and are allowed to depend smoothly on spacetime position as well as on the external electromagnetic fields \(\{E^a(t, x, y), B(t, x, y)\}\).
The equations of motion are postulated to be

\[ p_{ab,\mu} = f^\mu \quad \text{and} \quad \ast F_{\alpha,\mu} = 0, \tag{9} \]

where ‘\(\ast\)’ means partial derivative and \(J^\mu\) is the three-vector current density. Projecting the latter along the observer field and onto the two-dimensional rest space orthogonal to it, through the projector \(h_{ab} = \delta_{ab} + t_a t_b\), yield

\[ D_{\alpha,\mu} = \rho, \tag{10} \]

\[ h_{ab} \dot{D}_{\mu} - \varepsilon_{\mu bc} t^c H_{ab} = h_{ab} f^\mu, \tag{11} \]

\[ \dot{B} + \varepsilon_{\alpha bc} t^c E_{\alpha,\mu} = 0, \tag{12} \]

with \(\dot{X} = X_{a,\mu} X_{b,\mu} + \dot{h}_{ab}\) and \(\rho = -J_{\alpha} F^\alpha\). Several interesting features of these equations and their main distinctions from the \(3+1\) case can be found in \([13, 18–21]\). From now on, we shall focus on the limit of geometric optics.

The partial derivatives of the constitutive relations \((7)\) and \((8)\) are easily calculated as

\[ D_{\alpha,\beta} = \tilde{A}_{\beta} E_{\alpha} + \tilde{B}_{\alpha} B_{\beta} + \ldots \tag{13} \]

\[ H_{\alpha,\beta} = \tilde{C}_{\beta} E_{\alpha} + \tilde{D}_{\alpha} B_{\beta} + \ldots \tag{14} \]

with the ellipsis standing for algebraic contributions whose explicit form is irrelevant to our discussion and

\[ \tilde{A}_{\beta} = A_{\beta} + \frac{\partial A_{\alpha}}{\partial E_{\alpha}} E_{\beta} + \frac{\partial A_{\alpha}}{\partial B_{\alpha}} B_{\beta}, \tag{15} \]

\[ \tilde{B}_{\alpha} = B_{\alpha} + \frac{\partial B_{\alpha}}{\partial E_{\alpha}} E_{\beta} + \frac{\partial B_{\alpha}}{\partial B_{\alpha}} B_{\beta}, \tag{16} \]

\[ \tilde{C}_{\alpha} = C_{\alpha} + \frac{\partial C_{\alpha}}{\partial E_{\alpha}} E_{\beta} + \frac{\partial C_{\alpha}}{\partial B_{\alpha}} B_{\beta}, \tag{17} \]

\[ \tilde{D} = D + \frac{\partial D}{\partial E_{\alpha}} E_{\beta} + \frac{\partial D}{\partial B_{\alpha}} B_{\beta}. \tag{18} \]

Henceforth, for the sake of terminology, we shall call the set \(\{\tilde{A}_{\beta}, \tilde{B}_{\alpha}, \tilde{C}_{\alpha}, \tilde{D}\}\) the constitutive tetrad of the nonlinear medium. Clearly, each element of the tetrad will also be a smooth function of position as well as field strengths.

### 2.2. Field discontinuities

In order to study the main features of wave propagation, we consider a wavefront surface \(\Sigma(x) = \text{const}\) and define the normal co-vector as \(k_a = \partial \Sigma / \partial x^a\). Assuming the field strengths \(E^a\) and \(B\) are continuous through \(\Sigma\), but with a possible non-zero step in their first derivatives, according to Hadamard’s theorem \([22, 23]\), we determine such step as

\[ [E^a, B]_\Sigma = e^a k_b, \quad \text{and} \quad [B, E]_\Sigma = b k_a, \tag{19} \]

where \(e^a\) and \(b\) are amplitudes representing the wave-polarization vector and the wave-polarization scalar, respectively.

In order to proceed, it is convenient to introduce the following projections

\[ \omega \equiv k_a e^a, \quad q_a \equiv h^a_b k_b, \tag{20} \]
where $\omega$ is the wave frequency and $q_a$ is the corresponding co-vector orthogonal to the observer field. By writing the squared norm of the latter as $||q||^2 = h_{ab}q^a q^b$, we then define the space-like ortho-normal vectors
\begin{equation}
\mathbf{q}^a = q^a / ||q||, \quad \mathbf{p}^a = \epsilon^{abc} q_b q_c,
\end{equation}
and let the wave co-vector be schematically written as
\begin{equation}
k_a = ||q|| (v_\phi, q_a)
\end{equation}
where $v_\phi = \omega / ||q||$ is the usual phase velocity as measured by the observer. With these conventions, by first applying Hadamard’s step conditions to equation (12), and taking into account equation (14), gives
\begin{equation}
v_\phi b = \epsilon_{cde} p_c.
\end{equation}
Similarly, by applying the step conditions to equations (10) and (11) with equations (13) and (14), and using the above relation yields
\begin{equation}
\left( v_\phi \tilde{A}^a b + A^a_b \right) \mathbf{q}_a e^b = 0,
\end{equation}
\begin{equation}
\left( v_\phi^2 \tilde{A}^a b + v_\phi A^a_b - \tilde{D} \tilde{p}_b \tilde{p}_b \right) e^b = 0,
\end{equation}
where we have introduced the auxiliary magneto-electric mixed quantity $A^a_b = \tilde{B}^a \tilde{p}_b - \tilde{C}_b \tilde{p}^a$, for conciseness.

A closer inspection of the above relations reveals that equation (24) is actually a consequence of equation (25), due to the orthogonality of $\mathbf{q}_a$ and $\mathbf{p}_a$. Therefore, if we define the generalized Fresnel matrix as
\begin{equation}
Z^a_b \equiv \left( v_\phi^2 \tilde{A}^a b + v_\phi A^a_b - \tilde{D} \tilde{p}_b \tilde{p}_b \right),
\end{equation}
the problem of wave propagation reduces to the eigenvalue/eigenvector problem
\begin{equation}
Z^a_b e^b = 0 \rightarrow \operatorname{rank}(Z^a_b) \leq 2.
\end{equation}
However, since the observer field automatically belongs to the kernel of the Fresnel matrix, non-trivial solutions for the polarization will exist provided we require that $\operatorname{rank}(Z^a_b) = 1$.

3. Phase velocities

A well-known result of linear algebra implies that any real $3 \times 3$ matrix with rank one satisfies the algebraic relation
\begin{equation}
\frac{1}{2} \left( Z^a_a Z^b_b - Z^a_b Z^b_a \right) = 0,
\end{equation}
which is a direct consequence of the vanishing of the adjoint matrix. A direct calculation combining equation (28) with equation (26) shows that the corresponding fourth-order polynomial somehow factorizes, thus yielding a second-order polynomial equation for the phase velocity, as follows
\begin{equation}
\alpha v_\phi^2 - \beta v_\phi - \gamma = 0,
\end{equation}
where the coefficients are defined as
\begin{equation}
\alpha = \frac{1}{2} (\tilde{B}^a \tilde{A}_b - \tilde{C}_b \tilde{A}^a),
\end{equation}
\begin{equation}
\beta = \tilde{A}^a_a A^b_b - \tilde{A}^b_a A^a_b,
\end{equation}
\[ \gamma = \tilde{D} \tilde{q}_{ab} \tilde{q}^a \tilde{q}^b - \frac{1}{2} (A^a \tilde{A}_b - A^b \tilde{A}_a). \]  
(32)

It is worth mentioning that equation (29) gives rise to a homogeneous quadratic multivariate polynomial at the cotangent bundle. In the case of linear constitutive laws, the precise form of this polynomial was rigorously derived in [17], using the method of the effective metric. In principle, such an observer-independent and totally covariant approach could be carried out for the nonlinear case as well. However, since all relevant information about the characteristic surfaces as well as ray propagation is contained in equation (29), we shall concentrate on the study of phase velocities.

Suppose that the constitutive tetrad \( \tilde{A}^a_b, \tilde{B}_a, \tilde{C}_a, \tilde{D} \) and a direction \( \hat{q}^a \) are given at a space-time point. Then, equation (29) will admit two roots, given by
\[ v_\phi^\pm = \frac{\beta \pm \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha}. \]  
(33)

Complex roots are, in general, associated with the propagation of evanescent waves, which we do not analyze here. Therefore, we assume that the condition
\[ \beta^2 + 4\alpha\gamma \geq 0 \]  
(34)
holds for all possible directions in the rest space. Clearly, the latter leads to algebraic constraints which must be fulfilled by the constitutive tetrad at the corresponding point. An interesting feature of equation (29) is that it is not, in general, invariant under spatial inversions. Indeed, a closer inspection of equations (30) and (31) reveals that
\[ \hat{q}^a \rightarrow -\hat{q}^a, \quad v_\phi^\pm \rightarrow -v_\phi^\pm. \]  
(35)

However, in the particular case where magneto-electric cross-terms are absent, the coefficient \( \beta \) vanishes identically and one recovers the usual situation, where the `effective light cone' is invariant under reflection about the rest space. The particular case of linear isotropic dielectrics, for which \( \tilde{A}^a_b = \varepsilon \tilde{h}^a_b, \tilde{B}^a = \zeta_a = 0 \) and \( \tilde{D} = 1/\mu \), with \( \varepsilon(x) \) and \( \mu(x) \) as the dielectric functions, clearly illustrates this phenomena. Indeed, in this case, we recover the symmetric expression \( v_\phi^2 = (\mu \varepsilon)^{-1} \), which shows the mirror invariance.

It is worth mentioning that there is no room for birefringence in this theory due to the quadratic form of the dispersion relation encoded in equation (29). However, for sufficiently high values (in modulus) of magneto-electric cross-terms, the theory predicts the existence of one-way propagation. Indeed, the latter will occur whenever the following inequalities hold
\[ -\beta^2 < 4\alpha\gamma < 0, \]  
(36)
for a particular direction. We shall see next how this situation emerges in concrete examples.

### 4. Polarization

In the last section we have seen that the eigenvalue/eigenvector problem equation (27) will admit nontrivial solutions provided the phase velocity is a root of the quadratic equation (29). Setting the wave-polarization vector \( e^a \) as a linear combination of the orthogonal space-like basis orthogonal to the observer \( \{ \hat{q}^a, \hat{p}^a \} \), it follows that:
\[ (v_\phi^2 + \varepsilon \tilde{A}^a_b A^a_b - \tilde{D} \hat{p}^a \hat{p}_b) (a_1 \hat{q}^b + a_2 \hat{p}^b) = 0, \]  
(37)
where \( a_1 \) and \( a_2 \) are direction-dependent coefficients to be determined. Contracting the above equation respectively with \( \hat{q}_{ab} \) and \( \hat{p}_{ab} \) we find a set of two algebraic equations for
$a_1$ and $a_2$. However, we need to solve only one of them because they are constrained to satisfy equation (29). In particular, choosing the equation corresponding to the Gauss law equation (24) and assuming $v_\phi \neq 0$, yields the relation

$$a_1 = - \left( \frac{v_\phi \tilde{\mathcal{A}}_{ab} \hat{q}^a \hat{q}^b + \tilde{\mathcal{B}}_{a} \hat{q}^a}{v_\phi \tilde{\mathcal{A}}_{ab} \hat{q}^a \hat{q}^b} \right) a_2. \quad (38)$$

A remarkable consequence of the latter is that neither the electric permeability term $\tilde{\mathcal{C}}_a$ nor the (inverse) permeability term $\tilde{\mathcal{D}}$ contribute to the polarization in a $(2 + 1)$-dimensional theory. Although one might think that each value of the phase velocity leads to a different wave-polarization vector, the quadratic dispersion relation deduced previously guarantees that both vectors lie actually on the same light cone, namely, there is no birefringence. From the examples below, it will be clear that this happens because the light cone is tilted with respect to the Cartesian coordinate system.

Furthermore, equation (38) reveals that the propagating wave will be transversal, i.e.

$$e^{\alpha} \perp \hat{q}^\alpha, \quad (39)$$

whenever $a_1 = 0$ or, equivalently

$$v_\phi \tilde{\mathcal{A}}_{ab} \hat{q}^a \hat{q}^b + \tilde{\mathcal{B}}_{a} \hat{q}^a = 0. \quad (40)$$

In particular, transversality is certainly guaranteed in the case of the above-mentioned linear isotropic dielectric, as expected. One might wonder whether the theory also admits longitudinal propagation. According to equation (38), this condition implies the relation $a_2 = 0$, which is the same as

$$\tilde{\mathcal{A}}_{ab} \hat{q}^a \hat{q}^b = 0. \quad (41)$$

Up to now, it is not entirely clear to us if the fulfillment of the latter would somehow imply an inconsistent propagation.

5. Special cases

In this section, we investigate three interesting nonlinear systems representing hypothetical 2D material media: (i) purely magnetic material, (ii) anisotropic/non-magnetic dielectric, and (iii) second-order magneto-electric medium.

5.1. Purely magnetic media

We assume here that the (inverse) permeability tensor depends only on the magnetic field $\mathcal{D} = \mathcal{D} (B)$, while the other dielectric parameters are $\mathfrak{A}^a_b = \varepsilon_0 h^a_b$ and $\mathfrak{Y}^a = \zeta_a = 0$. Thus, the non-vanishing components of the constitutive tetrad, obtained from equations (15)–(18), reduce to

$$\tilde{\mathfrak{A}}^a_b = \varepsilon_0 h^a_b \quad \text{and} \quad \tilde{\mathcal{D}} = \mathcal{D} + \frac{\partial \mathcal{D}}{\partial B} B. \quad (42)$$

Under these conditions, the expression for the phase velocity, given by equation (33), becomes

$$v_\phi = \pm \sqrt{\frac{\gamma}{\alpha}} = \pm \sqrt{\frac{\left( \frac{\partial \mathcal{D}}{\partial B} \right)'}{\varepsilon_0}}, \quad (43)$$
Figure 1. Qualitative behavior of $v_\phi$ for $B$ fixed. The solid line corresponds to $\bar{\chi}^{(2)} > 0$ and the dashed line is for $\bar{\chi}^{(2)} < 0$. For a given order in the expansion of $D(B)$, a positive susceptibility coefficient will give a bigger value for the phase speed in comparison to the negative one. (For the plots, we use $\bar{\chi}^{(1)} = 0$, $\bar{\chi}^{(2)} = \pm 1$ and $B = 1/2$. The black circles indicate the values of the radial coordinate, with a step of 0.2 in dimensionless units.)

where ($'$) means here derivative with respect to $B$. This equation does not involve the spatial direction of the wave vector $\vec{q}$, indicating that $v_\phi$ is isotropic, for any given $B$. Furthermore, if $D(B)$ admits a Taylor expansion, for each order of the power series, the corresponding susceptibility coefficient will increase or decrease the phase velocity, depending if it is positive or negative. For a better illustration of this phenomenon, we refer the reader to figure (1), where we have depicted $v_\phi(B)$ taking into account corrections up to second-order susceptibility coefficients [24], i.e.

$$D(B) = \mu_0^{-1} \left( 1 + \bar{\chi}^{(1)} + \bar{\chi}^{(2)} B \right). \quad (44)$$

The wave-polarization vector $e^\alpha$ can be directly obtained from equation (38), which coincides with the wave-polarization vector of linear isotropic dielectrics, given by equation (39). Recall that it is orthogonal to the spatial direction of the wave vector.

5.2. Anisotropic and non-magnetic dielectric media

Now, we assume that the permittivity tensor depends only on the norm of the electric field, with independent behavior along each spatial direction, namely,

$$\mathbf{\Xi}^\alpha_b = \text{diag} [0, \varepsilon_x(||E||), \varepsilon_y(||E||)]. \quad (45)$$

The other dielectric and magneto-electric parameters are chosen as $\mathbf{D} = \mu_0^{-1}$ and $\mathbf{B}^\alpha = \mathbf{E}^\alpha = 0$. Thus, the only non-vanishing components of the constitutive tetrad are

$$\tilde{\Xi}^\alpha_b = \Xi^\alpha_b + \frac{\partial \Xi^\alpha_c}{\partial ||E||} \frac{E^b E^c}{||E||} \text{ and } \tilde{D} = \mu_0^{-1}. \quad (46)$$
Figure 2. Plot of $v_\phi$ as function of $\phi$. For the illustrative choice $\varepsilon_y = \varepsilon_0$, $\varepsilon_x = \varepsilon_0 ||E||$ with $||E|| = 1.0$, one sees that the phase speed decreases from its maximum value along the $x$-axis to its minimum along the $y$-axis. As $E$ enlarges, the curve is flattened along the $y$-direction, but the axis $x$ is unaltered, as shown by equation (47).

For the sake of simplicity, we can set the electric field along the $x$-direction and decompose the unit spatial wave vector as $\hat{q}_a = (0, \cos \phi, \sin \phi)$, where $\phi$ is measured with respect to the $x$-axis. Under these assumptions, the expression (33) for the phase speed reduces to

$$v_\phi = \pm \sqrt{\frac{\gamma}{\alpha}} = \pm \sqrt{\frac{\cos^2 \phi}{\mu_0 \varepsilon_y} + \frac{\sin^2 \phi}{\mu_0 (||E|| \varepsilon_x)'}}$$

(47)

where prime (’) means here derivative with respect to $||E||$. Note that we have only one possibility for the phase speed, in modulo, in addition to the fact the light ray is always extraordinary [25], that is, it depends on the wave vector direction. For $||E||$ fixed, equation (47) determines the surface normal of the waves (see figure 2).

For a better illustration of such an effect, we shall study two particular cases, assuming that the permittivity functions $\varepsilon_x$ and $\varepsilon_y$ admit a power-law expansion in $E$. Thus,

$$\varepsilon_x = \varepsilon_0 \left( 1 + \chi_x^{(1)} + \chi_x^{(2)} ||E|| + \chi_x^{(3)} ||E||^2 \right) + O(3),$$

(48)

$$\varepsilon_y = \varepsilon_0 \left( 1 + \chi_y^{(1)} + \chi_y^{(2)} ||E|| + \chi_y^{(3)} ||E||^2 \right) + O(3),$$

(49)

where $\chi_{x,y}^{(n)}$ are the $n$th order susceptibility coefficients. In principle, these coefficients can have both signs, but the change in the phase speed is qualitatively the same in a certain order. The linear electro-optic effect (‘Pockel-like’), due to $\chi^{(2)}$-corrections, makes the phase speed to increase ($\chi^{(2)} < 0$) or decrease ($\chi^{(2)} > 0$) for small values of $||E||$. But, for large $||E||$, both signs lead to a quadratic increasing in $v_\phi$ (see figure 3). The quadratic electro-optic effect (‘Kerr-like’) caused by the presence of $\chi^{(3)}$ does not allow such a change in the profile of $v_\phi(||E||)$, which always increases ($\chi^{(3)} < 0$) or decreases ($\chi^{(3)} > 0$) quadratically as $||E||$ grows (see figure 3).
Figure 3. Plots of $v_{\varphi}(||E||)$. On the left-hand side, the solid line corresponds to a positive $\chi_2$, with the phase speed having a minimum value as $||E||$ increases. The dashed line represents an always increasing phase speed for $\chi_2 < 0$. On the right-hand side, the solid line corresponds to a positive $\chi_3$, yielding a decreasing phase speed, while the dashed line indicates a monotonically increasing $v_{\varphi}$ for $\chi_3 < 0$. For convenience, we have defined $\bar{v}_{\varphi} = \sqrt{\varepsilon_0 (1 + \chi_1)} v_{\varphi}$ and $\tilde{E} = \frac{\chi_2}{\varepsilon_0 (1 + \chi_2)} ||E||$, correspondingly.

From equation (38), it is straightforward to determine the form of the polarization modes allowed for this case. Since the magneto-electric coefficients are absent, the wave-polarization vector $e^a$ is simply given by

$$e^a = \left(0, \frac{\sin \phi}{(E_{x})}, -\frac{\cos \phi}{E_y}\right)$$

(50)

5.3. Second order magneto-electric media

The magneto-electric media we shall study now is the 2D adaptation of the 3D medium analyzed in [26], where the electric displacement and the magnetic induction are obtained from first principles through a Taylor expansion of the Helmholtz free energy in terms of the electric and magnetic fields. The same is done here (see details in appendix ‘Polarization and magnetization vectors of a nonlinear medium at thermodynamic equilibrium’), leading to the following polarization vector and magnetization scalar

$$P_a = \varepsilon_0 \chi_{ab} E^b + \frac{1}{2} \chi^{(2)}_{ab} H^2,$$

(51)

$$M = \chi^{(1)} H + \mu_0^{-1} \chi^{(2)}_a E^a H,$$

(52)

where the susceptibility coefficients $\chi_{ab}$ and $\chi^{(1)}$ are first order contributions to the polarization and magnetization, respectively, while the susceptibility vector $\chi^{(2)}_a$ is a second order correction for both induced fields. Using the relations between $P_a$ and $M$ with the field excitations,
namely, \( D_a = \varepsilon_0 E_a + P_a \) and \( B = \mu_0 (H + M) \), one can directly read the constitutive tensor, vectors and scalar defined in equations (7) and (8), as follows
\[
\mathfrak{A}_{ab} = \varepsilon_0 \left[ h_{ab} + \chi^{(1)}_{ab} \right], \quad \mathfrak{B}_a = \frac{1}{2} B \mathfrak{D}^2 x_a^{(2)}, \quad \varepsilon^a = 0 \quad \text{and} \quad \mathfrak{D} = \left[ \mu_0 \left( 1 + \tilde{\chi}^{(1)} \right) + \chi^{(2)} E^a \right]^{-1}.
\] (53)

For this case, the elements of the constitutive tetrad are
\[
\tilde{\mathfrak{A}}_{ab} = \mathfrak{A}_{ab} - B^2 \mathfrak{D}^3 x_b^{(2)}, \quad \tilde{\mathfrak{B}}_a = 2 \mathfrak{B}^a,
\] (54)
\[
\tilde{\mathfrak{B}}_a = -2 \mathfrak{B}^a \quad \text{and} \quad \tilde{\mathfrak{D}} = \mathfrak{D}.
\] (55)

We shall assume again the electric field along the \( x \)-axis, decompose the unit spatial wave-vector in polar coordinates and write \( x_a^{(2)} = (0, \chi_1, \chi_2) \). Thus, the phase speed equation (33) becomes
\[
\sqrt{\mu \varepsilon} v_\psi^\pm = \frac{(E + 1) \left[ B (r \cos \phi - \sin \phi) \pm \Delta \right]}{(E + 1)^3 - (1 + r^2) B^2}
\] (57)
where \( \tilde{E} = \chi_1 E / \mu, \tilde{B} = B / \sqrt{\mu \varepsilon}, r = \chi_2 / \chi_1 \) and \( \Delta = (\tilde{E} + 1)^3 - \tilde{B}^2 (\cos \phi + r \sin \phi)^2 \), with \( \varepsilon = \varepsilon_0 [1 + \chi^{(1)}] \) and \( \mu = \mu_0 [1 + \tilde{\chi}^{(1)}] \).

From the equation above, one can read the condition for the one-way propagation as
\[
\Delta > 0 \quad \text{and} \quad (1 + \tilde{E})^3 < \tilde{B}^2 (1 + r^2).
\] (58)

Note that the first inequality has an angular dependence on the direction \( \phi \) of the wave vector, while the second inequality involves only the field strengths. The former provides a window for the one-way propagation from the roots of the quadratic equation
\[
[r^2 \tilde{B}^2 - (1 + \tilde{E})^3] \tan^2 \phi + 2 r \tilde{B}^2 \tan \phi + \tilde{B}^2 - (1 + \tilde{E})^3 = 0,
\] (59)
which are
\[
\tan \phi^\pm = \frac{-r \tilde{B}^2 \pm \sqrt{[\tilde{B}^2 (r^2 + 1) - (1 + \tilde{E})^3] (1 + \tilde{E})^3}}{\left[ r^2 \tilde{B}^2 - (1 + \tilde{E})^3 \right]}.
\] (60)

Therefore, the phenomenon of the one-way propagation exists for \( \tan \phi^- < \tan \phi < \tan \phi^+ \), which is depicted in figure 4. Note that out of this window, the medium is opaque. In figure 5, we show that it is possible to have light propagating for all spatial directions by changing only the magnitude of the electric field \( E \), with \( r \) and \( B \) fixed, but the propagation is anisotropic. For both figures, the increase of the ratio \( r \) basically rotates the circles counterclockwise, while the increase of \( B \) diminishes the radius of the circles. Moreover, it is evident the symmetry \( v_\psi^+ \) into \(-v_\psi^-\) under the transformation \( \phi \to \phi + \pi \), as mentioned before, recalling that in polar plots, the radius represents the magnitude of the quantity, without taking into account its sign.

Concerning the wave-polarization vector in this case, equation (38) can be written down as
\[
a_1 = \frac{(1 + \tilde{E})^3}{\varepsilon \Delta} \left( \frac{\tilde{B}_a \tilde{q}_a^\pm}{v_\pm^\psi} - \tilde{B}_a \tilde{q}_a^\mp \tilde{B}_b \tilde{p}_b^\mp \right) a_2.
\] (61)

Since the explicit form of this equation is cumbersome, we depict the behavior of the ratio \( a_1/a_2 \) in figure 6. There, one can see once again, the spatial symmetry in flipping the direction of \( \tilde{q}_a^\pm \) and also the distinction with respect to the phenomenon of birefringence: when \( \phi \to \phi + \pi \) the wave-polarization vector of \( v_\psi^+ \) is mapped onto the wave-polarization vector of \( v_\psi^- \).
Figure 4. Phase speeds $v_+^\phi$ (solid red line) and $v_-^\phi$ (dashed blue line), in modulo, as a function of $\phi$. It is possible to see the complementary windows of one-way propagation and opacity. For this, we choose $\tilde{B} = 1.0$, $\tilde{E} = -0.5$ and $r = 0.5$.

Figure 5. Phase speeds $v_+^\phi$ (solid red line) and $|v_-^\phi|$ (dashed blue line), in modulo, as a function of $\phi$. For the choice $B = 1.0$, $\tilde{E} = 0.5$ and $r = 0.5$, one of the conditions for the one-directional propagation is not satisfied and then we have $v_+^\phi$ and $v_-^\phi$ for all directions.
Figure 6. The ratio $a_1/a_2$ as function of $\phi$. This is an alternative way to see windows of one-way propagation and opacity. The transformation $\phi \rightarrow \phi + \pi$ evinces now the map between the wave-polarization vectors of $v_+^\phi$ (solid red line) and $v_-^\phi$ (dashed blue line). We choose here $B = 1.0$, $E = -0.5$ and $r = 0.5$.

and vice versa, rendering impossible the configuration of two independent phase velocities with independent wave-polarization vectors for a given direction.

6. Concluding remarks

Starting with a $(2+1)$ formulation of electromagnetism with nonlinear constitutive relations, the geometric optics limit is achieved by applying Hadamard’s theorem on the characteristic surface. We then obtain the dispersion relation, the phase speed formulae, and the general form of the wave-polarization vector. Thus, we examined three distinct cases of interest, calculating the phase speed and the wave-polarization vector for each of them. Interestingly, we proved the existence of one-way propagation and controlled opacity in such media, particularly for the magneto-electric ones.

Contrarily to the linear case in vacuum [19, 20, 27], it is possible to see again [17] the role of the constitutive relations in the non-equivalence between a two-form formulation of electrodynamics in $(3+1)$-dimensions and the $(2+1)$-dimensional one developed here. In particular, our constitutive relations (7) and (8) involve objects with different tensor ranks and, obviously, the number of degrees of freedom is distinct in each case. This is a crucial point in distinguishing both formulations and possibly favoring one over the other in the description of nonlinear 2D materials. In other words, we hope our formalism may shed light on the debate concerning the behavior of electromagnetic fields propagating inside physical thin layers.

Finally, in virtue of the observer-independent formulation developed in our previous work, the effective optical metric of a nonlinear 2D material can be directly obtained from equation (37) of [17], by replacing the constitutive tetrad basis of a linear medium by the one associated with a nonlinear medium, given by equations (15)–(18). In this vein, the next step will be the investigation of analogue models of gravity in this context.

Data availability statement

No new data were created or analyzed in this study.
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Appendix. Polarization and magnetization vectors of a nonlinear medium at thermodynamic equilibrium

Following the standard thermodynamic approach, the free-energy density $\mathcal{F}$ of a 2D material in the presence of electromagnetic fields can be written as a function of the electric field $E^a$ and the magnetic excitation $H$, at a fixed temperature $T$; that is

$$\mathcal{F} = \mathcal{F}(E^a, H; T).$$  \hspace{1cm} (A.1)

Needless to say, this quantity is not covariant, in the sense that its definition is frame-dependent since it mixes field strengths with field excitations.

As long the fields $E^a$ and $H$ are bounded, for practical reasons, we can expand $\mathcal{F}$ in terms of those fields (up to third order) as

$$\mathcal{F} = \mathcal{F}_0 - P_a^{(1)} E^a - M^{(2)} H - \frac{1}{2} \varepsilon_0 \chi^{(1)}_{ab} E^a E^b - \frac{1}{2} \mu_0 \chi^{(1)} a H^2 - \frac{1}{2} \varepsilon_0 \chi^{(2)}_{abc} E^a E^b E^c - \frac{1}{2} \mu_0 \chi^{(2)}_{abc} H^3 - \frac{1}{4} \chi^{(2)}_{ab} E^a E^b H^2 - \frac{1}{4} \chi^{(2)}_{abc} E^a H^3 E^b H^2,$$

where $P^{(1)}$ and $M^{(2)}$ stand for spontaneous polarization and magnetization effects, respectively. All the other tensor coefficients are the susceptibilities of the material, with the subscript number in the parentheses indicating the corresponding order in the power expansion of $\mathcal{F}$.

Hence, the polarization vector can be calculated and it is written down as

$$P^a = -\frac{\partial \mathcal{F}}{\partial E^a} = P_a^{(1)} + \varepsilon_0 \chi^{(1)}_{ab} E^b + \alpha^{(1)}_{ab} H + \varepsilon_0 \chi^{(2)}_{abc} E^b E^c + \alpha^{(2)}_{abc} E^b H + \frac{1}{2} \beta^{(2)}_{ab} H^2.$$  \hspace{1cm} (A.3)

Analogously, the magnetization is given by

$$\mu_0 M = -\frac{\partial \mathcal{F}}{\partial H} = M^{(2)} + \mu_0 \chi^{(1)} a H + \alpha^{(1)}_{ab} E^a + \mu_0 \chi^{(2)}_{abc} E^b E^c + \alpha^{(2)}_{abc} E^b H + \frac{1}{2} \beta^{(2)}_{ab} E^a,$$  \hspace{1cm} (A.4)

Thus, the electric displacement $D_a = \varepsilon_0 E_a + P_a$ is obtained through

$$D_a = \varepsilon_0 \left( \delta_{ab} + \chi^{(1)}_{ab} + \chi^{(2)}_{abc} E^c + \frac{1}{2} \alpha^{(2)}_{abc} H \right) E^b + \left( \alpha^{(1)}_{ab} + \frac{1}{2} \beta^{(2)}_{ab} H \right) H,$$  \hspace{1cm} (A.5)

while the magnetic strength $B = \mu_0 (H + M)$ can be written as

$$B = \mu_0 \left( 1 + \chi^{(1)} a + \chi^{(2)} b H + \frac{1}{2} \mu_0 \beta^{(2)}_{ab} E^a \right) H + \left( \alpha^{(1)}_{ab} + \frac{1}{2} \alpha^{(2)}_{abc} E^b \right) E^a.$$  \hspace{1cm} (A.6)

From this, one sees that equation (A.6) is a polynomial equation in $H$, which should be solved in order to find the constitutive relation $H = H(E^a, B)$. In possession of it, we can substitute
\[ H = H(E^a, B) \] in equation (A.5) to find the other constitutive relation \( D^a = D^a(E^b, B) \). Only after such manipulation, the dielectric and magneto-electric coefficients can be determined. For the sake of illustration, we study the case \( \chi^{(2)} = 0 \) in section 5.3, such that \( B \) and \( H \) are linearly related.

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