THE FIXATION LINE

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ABSTRACT. We define a Markov process in a forward population model with backward genealogy given by the $\Lambda$-coalescent. This Markov process, called the fixation line, is related to the block counting process through its hitting times. Two applications are discussed. The probability that the $n$-coalescent is deeper than the $(n-1)$-coalescent is studied. The distribution of the number of blocks in the last coalescence of $n$-Beta($2-\alpha,\alpha$)-coalescents is proved to converge as $n \to \infty$, and the generating function of the limiting random variable is computed. The probability for an integer to be in the range of the block counting process of $n$-Beta($2-\alpha,\alpha$)-coalescents is also given in the limit $n \to \infty$.

1. Introduction

A $n$-coalescent is a stochastic model for the genealogy of a (haploid) population of $n$ individuals backward in time. In this model, the individuals of the population are identified with the integers of the set $\{1, \ldots, n\}$, and the $n$-coalescent takes its values in the partitions of $\{1, \ldots, n\}$. A partition of $\{1, \ldots, n\}$ is composed of a certain number of blocks, between 1 and $n$. The initial state of the $n$-coalescent is the partition in $n$ blocks, that is the partition in singletons, $\{1\}, \ldots, \{n\}$. Any particular set of $k$ blocks then merges independently in one block at rate given by:

\begin{equation}
\int_{[0,1]} \Lambda(dx) x^{-2} x^k (1-x)^{n-k},
\end{equation}

where $\Lambda(dx)$ is a probability measure on $[0,1]$. After the first coalescence, again, any particular set of $k$ blocks merges independently in one block at rate given by (1), with $n$ replaced by the current numbers of blocks. The procedure is then repeated, until the process terminates at the partition in one single block, $\{1, \ldots, n\}$. The first motivation of this paper is to study the number of blocks involved in the last coalescence.

The interpretation of the model is the following: two integers are in the same block of the partition at time $t \geq 0$ in the $n$-coalescent if the corresponding individuals found their common ancestor at time $t$ backward in time. At the time of the last coalescence, all the individuals found their common ancestor.

When $\Lambda\{0\} = 0$, a construction of the $n$-coalescent is the following. We start with a Poisson point measure on $\mathbb{R}^+ \times (0,1]$, with intensity $dt \Lambda(dx)x^{-2}$. To each atom $(t,x)$ of this random measure, we associate a random subset

\[ J = \{j_1, j_2, j_3, \ldots\} \]
Figure 1. A realization of the 7-coalescent. The first coalescence arises at time $s$, and the values of $j_1, j_2, j_3$ associated to the coalescence event at time $s$ are 1, 2, 5. The set of records $T$ contains 2 and 3.

of the set of integers $\mathbb{N}$ by sampling each integer independently with the same probability $x$. Then, the blocks with labels in $J$ at time $t$ coalesce in one block. First, among the (possibly infinitely many) atoms $(t, x)$ on a finite time interval, only a finite number give rise to an effective merge in the $n$-coalescent, and so we may distinguish a first coalescence, a second one, ... Second, the construction requires to give labels to the blocks. While most definitions of the $n$-coalescent do not stress on the labelling of the blocks, the point of this paper is to emphasize a particular ordering.

Our choice is to label the blocks in the order of their smallest element. The $n$-coalescent is then described by the family $(B_i(t), t \geq 0, i \in \mathbb{N})$ of its ordered blocks at time $t$: $B_1(t)$ is the block containing 1, $B_2(t)$ is the block containing the smallest integer not in $B_1(t)$ (if any) ... The largest $i$ such that $B_i(t)$ is non-empty is then the number of blocks in the $n$-coalescent, denoted by $X^n(t)$. For instance, for the 7-coalescent depicted on figure 1 we have $X^7(s) = 5$ and:

$$B_1(s) = \{1, 2, 5\}, \ B_2(s) = \{3\}, \ B_3(s) = \{4\}, \ B_4(s) = \{6\} \ and \ B_5(s) = \{7\}. $$

This leads to the following alternative description of the coalescent as a family of (coalescing) maps. For each $i \in \{1, \ldots, n\}$, and $t \geq 0$, there exists a unique integer $j$ such that $i \in B_j(t)$ and we then set $i(t) = j$. The random map $t \mapsto i(t)$ is non-increasing, starts at $i(0) = i$ and terminates at 1. Furthermore, if two functions $i(t)$ started at different points meet, then they coincide at each further time. Figure 1 describes the collection of the maps $i(t)$ started at $i \in \{1, \ldots, 7\}$. Notice that a function $i(t)$ not only decreases when the block labelled $i(t)$ takes part to a coalescence, but also when at least two blocks with lower label take part to a coalescence.

Also, adding more and more functions $t \mapsto i(t)$ starting at 8, 9, 10, ... allows to define the $n$-coalescents in a coupled way, hereafter referred to as the natural coupling. This coupling allows to define the coalescent started from an infinite number of blocks, simply called the coalescent in the following, or the $\Lambda$-coalescent if we need to stress on the measure $\Lambda(dx)$. 
Let us argue on our first motivation: When adding more and more functions \( t \to i(t) \), the block counting process \( (X^n(t), t \geq 0) \) of the \( n \)-coalescent evolves, and we may think of it as a wave moving to the right. The motion to the right, measured by the depth \( \tau^n_1 = \inf \{ t \geq 0, X^n(t) = 1 \} \) of the \( n \)-coalescent, is either a.s. bounded, or a.s. unbounded - we come back to this fact in subsection 2.1. In both cases, we investigate the question of the existence of a limiting shape (in distribution in the second case) for the wave viewed from the right. It amounts to study the time-reversal of the block counting process for the coalescent started from \( n \) blocks as \( n \to \infty \): this is a slight elaboration on our first motivation.

The depth of the \( n \)-coalescent \( \tau^n_1 \) corresponds to the first time the \( n \) blocks have merged in 1 block. In the aforementioned natural coupling, we may consider the random subset of the integers

\[
\mathcal{T} = \{ n \geq 2, \tau^n_1 > \tau^n_{n-1} \},
\]

that will be called the set of records: the integer \( n \) belongs to the set of records \( \mathcal{T} \) when the function \( i(t) \) started at \( n \) reaches 1 at some later time than the functions \( i(t) \) started at lower values \( i \) for \( 2 \leq i \leq n - 1 \), thus establishing a new record. See figure 1 for an illustration. Since \( \tau^n_1 = 0 \) by definition, we have that \( 2 \in \mathcal{T} \) a.s. In terms of population genetics, the label of an individual corresponds to a record if its addition in the sample modifies the most recent common ancestor of the sample. The study of the set of records is our second motivation.

The paths \( i(t) \) started from \( n \in \mathcal{T} \) are good candidates to time-reversal as we now explain. To make this time-reversal clear, we introduce, after Donnelly and Kurtz [9], the following model, called the lookdown model. It is essentially constructed by time-reversal of the coalescent represented as a family of coalescing maps. The model consists in a family, indexed by the time \( t \geq 0 \), of countably many individuals distinguished by their levels. The level of an individual is an integer, and the individual at time \( t \) at level \( i \in \mathbb{N} \) is denoted by \( (t, i) \). The genealogical relationships between individuals are described as follows. We start with the same Poisson point measure on \( \mathbb{R}^+ \times (0,1] \), with intensity \( dt \Lambda(dx)x^{-2} \). A random set

\[
J = \{ j_1, j_2, j_3, \ldots \}
\]

is associated with each atom \((t, x)\) by sampling independently each integer with the same probability \( x \). To each atom \((t, x)\), there corresponds a reproduction event and:

- for \( j \) in \( J \), the individual \((t, j)\) is a child of the individual \((t-, \min J)\). Notice that \( J \), and therefore the set of children, is infinite.
- the other lineages are shifted upwards, keeping the order they had before the birth event: if \( k \notin J \), the individual \((t, k)\) is the (unique) child of the individual \((t-, k-k')\), where \( k' = \text{Card} \{ J \cap \{1, \ldots, k \} - 1 \} \lor 0 \), see figure 2.

This defines a countable infinite population. The ancestral lineage of an arbitrary individual \((t, i)\) is the line composed of the individuals \((s, j(s)), 0 \leq s \leq t \) where \( j(s) \) is the level of the ancestor of the individual \((t, i)\) at time \( s \). Figure 2 displays the collection of the ancestral lineages. The map \( s \to j(t-s) \) has the same dynamics as the (restriction of the) map \( s \to i(s) \) in the coalescent model. More is true: for each \( t \in \mathbb{R}^+ \), the backward genealogy of the individuals at the \( n \) lowest levels at time \( t \) gives rise to a process valued in the partitions of \( \{1, \ldots, n\} \): \( i \) and \( j \) are in the same block at time \( s \), \( 0 \leq s \leq t \), if the individuals \((t, i)\) and \((t, j)\) share a common ancestor at time \( t-s \). This partition valued process has the law of (the restriction to \([0, t]\)) of a \( n \)-coalescent.
Figure 2. The ancestral lineages in a look-down graph restricted to its first 7 levels. The first reproduction event corresponds to $J = \{1, 3, 5, \ldots \}$. The fixation line started at level 1 at time 0 is blue. The dotted line above is the translation by one level of the fixation level.

A key ingredient in this paper is the following quantity $L_1(t)$ defined within the lookdown model. Instead of looking to the ancestors of an individual, the definition considers its offsprings. The levels of the offspring at time $t \geq 0$ of the individual $(0, 2)$, that is the individual at time 0 at level 2, form a subset of $\mathbb{N}$, the minimal element of which we define to be $L_1(t) + 1$. If the subset of $\mathbb{N}$ is empty, then we set $L_1(t) = \infty$. See the dotted line in figure 2 for an illustration. The shift by 1 in the definition is for technical reason. A direct, equivalent, but slightly more difficult definition of $L_1(t)$ (at least on a picture) is the following: The levels of the offspring at time $t \geq 0$ of the individual $(0, 1)$, that is the individual at time 0 at level 1, form a subset of $\mathbb{N}$, the connected component including 1 we denote by $\{1, \ldots, L_1(t)\}$. Again, we set $L_1(t) = +\infty$ in case this subset is $\mathbb{N}$. The collection of the random variables $(L_1(t), t \geq 0)$ builds a non-decreasing process called the fixation line. When the fixation line $L_1$ reaches level $n$, then the whole population of individuals at level $1, \ldots, n$ consists of offspring of individual $(0, 1)$, an event called fixation in population genetics: this explains the name fixation line. The link between the fixation line and the set of records $\mathcal{T}$ is as follows: for each $t \geq 0$, $L_1(t) + 1$ belongs to the set of records $\mathcal{T}$ for the coalescent describing the backward genealogy of the individuals at time $t$. See figure 2 for an example. This paper shows some applications of the fixation line.

This is a review of the literature: The origin of the fixation line may be traced back to Pfaffelhuber and Wakolbinger [20] in the Kingman case $\Lambda(dx) = \delta_0(dx)$. For general $\Lambda(dx)$, it appears in Labbé [17] and in [15]. The originality of the present work is to study the fixation line not for itself, but for understanding better the coalescent. The coalescent we presented is the $\Lambda$-coalescent, it was introduced independently by Pitman [21] and Sagitov [22], and the research area has been surveyed in Berestycki [4] and Bertoin [5]. We find the fixation line useful in studying two random quantities defined in the coalescent: the set of records $\mathcal{T}$ and the number of blocks implied in the last coalescence of the $n$-coalescent. The
set of records seems not to have been considered in the literature until now. An integer $n$ is a record when the corresponding external branch in the coalescent tree has depth equal to that of the $n$-coalescent tree. In that sense, our contribution may be seen as an atypical view on the intensively studied external branches, see Caliebe et. al. [7], Dhersin and Möhle [8] and the references therein. The numbers of blocks implied in the last coalescence, as well as the closely related hitting probabilities of the block counting processes, are quantities which relate to the coalescent tree near the root. This part of the tree is difficult to grasp from the standard construction of the coalescent. Original techniques have been developed in the papers [1, 2, 3, 12, 16, 18] to circumvent this difficulty. Among these papers, the ones with the closest objectives to ours are Abraham and Delmas [1, 2] and Goldschmidt and Martin [12], which both use connections to specific class of random trees. The article [12] and Möhle [18] are concerned with the Bolthausen-Sznitman coalescent, which is the $\text{Beta}(2-\alpha, \alpha)$-coalescent for $\alpha = 1$, whereas [1, 2] deal with the $\text{Beta}(2-\alpha, \alpha)$-coalescent for $\alpha \in (0, 1/2]$. We propose here a probabilistic approach based on the analysis of the fixation line. The main advantage with respect to previous methods is that it may be introduced for general $\Lambda$-coalescent, and conveniently specializes to the class of the $\text{Beta}(2-\alpha, \alpha)$-coalescent for the whole family of parameters $\alpha \in (0, 2)$. Short after the prepublication of this paper, Möhle extended in [19] the analytic method of [18] to the $\text{Beta}(2-\alpha, \alpha)$-coalescents, and obtained some of the results presented in this paper. We warmly invite the reader to consult this paper as a parallel reference.

This is the organization of the paper: Section 2 contains a key lemma relating the depth of the $n$-coalescent to the hitting times of the fixation line, see Lemma 2.1. Also, the transition rates of the fixation line are computed in Lemma 2.2 for a general probability measure $\Lambda(dx)$. They slightly differ from those of the block counting process, and they factorize in the special case of the $\text{Beta}(2-\alpha, \alpha)$-coalescents, as proved in Lemma 2.4. Two applications of the line, associated with our two motivations, are detailed in section 3. First, we compute in subsection 3.1 the probability for an integer to be in the random set of records $T$. Second, we present in subsection 3.2 the problem of the construction of the coalescent from the root, and propose, as a first and partial answer, to characterize the time-reversal of the block counting process. A simple observation reduces this problem to the computation of the law of the number of blocks implied in the last coalescence. Our main result, Theorem 3.5 is a limit theorem for the law of the number of blocks implied in the last coalescence in $n$-$\text{Beta}(2-\alpha, \alpha)$-coalescents. Our second main result, Corollary 3.6 reformulates Theorem 3.5 in term of hitting probabilities of an integer $j$ by the block counting process. We also give the asymptotics of these hitting probabilities as $j \to \infty$. Last, we take the opportunity to present a third, probabilistic derivation of the depth of the Bolthausen-Sznitman coalescent. We did our best to produce a paper as much self contained as possible.

2. The fixation line

Assumption: we will assume throughout that the probability measure $\Lambda(dx)$ gives no mass to the singletons 0 and 1,

$$\Lambda\{0\} = \Lambda\{1\} = 0.$$
The assumption on $\Lambda\{0\}$ allows to rely on the simple graphical construction mentioned in the introduction without struggling to include binary coagulations. This being said, most of the results are still valid for a probability measure $\Lambda(dx)$ with an atom at 0. The assumption on $\Lambda\{1\}$ avoids an uninteresting case.

2.1. A key lemma on hitting times, and coming down from infinity. We first come back to the definition of the fixation line and generalize it slightly by allowing the fixation line to be started at an arbitrary integer. Fix an integer $j$ and consider the set of individuals $(0, i)$ at time 0 at level $i$, for $i \in \{1, \ldots, j\}$. The offspring of this set of individuals at time $t \geq 0$ forms a subset of $\mathbb{N}$ in the lookdown model mentioned in the introduction, the connected component including 1 we denote by

$$\{1, \ldots, L_j(t)\},$$

with, again, $L_j(t) = \infty$ if this set is $\mathbb{N}$. Alternatively, the offspring at time $t$ of the single individual $(0, j+1)$ forms a subset of $\mathbb{N}$, the smallest element of which is $L_j(t) + 1$.

For $j$ and $n$ integers, we set

$$(2) \quad \tau^n_j = \inf\{t \geq 0, X^n(t) \leq j\}, \quad \alpha^n_j = \inf\{t \geq 0, L_j(t) \geq n\}$$

to denote the (partial) depth of the $n$-coalescent and the hitting time of the fixation line respectively. We now state our key lemma. In fact, the whole paper may be seen as a digression on this relation.

**Lemma 2.1.** Let $1 \leq j \leq n$. The two random variables $\tau^n_j$ and $\alpha^n_j$ have the same distribution.

**Proof.** Fix $t > 0$. It is enough to observe that the events

$$\{\tau^n_j > t\} \text{ and } \{\alpha^n_j > t\}$$

are equal in the coupling of a coalescent and of a fixation line provided by the lookdown model. More precisely, we compare the fixation line started at level $j$ at time 0 and the coalescent describing the backward genealogy of the $n$ lowest level individuals at time $t$. For the inclusion, observe that, if $\tau^n_j > t$ for the coalescent, then the fixation line started at $j$ at time 0 has not reached $n$ at time $t$, that is $\alpha^n_j > t$. For the reverse inclusion, if $\alpha^n_j > t$, then the coalescent presents more than $j$ blocks at time $t$, that is time 0 in the lookdown model, and $\tau^n_j > t$. □

Either the increasing sequence of the expected depth of the $n$-coalescent

$$(3) \quad (\mathbb{E}(\tau^n_1), n \geq 1)$$

is bounded, or it goes to $\infty$. In the first case, the coalescent is said to come down from infinity, whereas in the second case, it is said to stay infinite. The terminology is best explained by the following results of Schweinsberg [23]: For a coalescent which comes down from infinity, the increasing sequence $(\tau^n_1, n \geq 1)$ is a.s. bounded, and the coalescent started from an infinite number of blocks presents a finite number of blocks at each positive time a.s. For a coalescent which stays infinite however, the increasing sequence $(\tau^n_1, n \geq 1)$ a.s. goes to $\infty$, the coalescent presents an infinite number of blocks at each non-negative time a.s.

The link with the fixation line follows from Lemma 2.1: the coalescent comes down from infinity when the increasing sequence $(\alpha^n_1, n \geq 1)$ is a.s. bounded, that is when the fixation
line reaches $\infty$ in finite time a.s. Also, the coalescent stays infinite when the increasing sequence $(\alpha^n_n, n \geq 1)$ goes to $\infty$ a.s., that is when the the fixation line remains finite a.s.

2.2. The transition rates. Our next task is to determine the transition rates of the fixation line.

**Lemma 2.2.** For $1 \leq i < j$, the rate $\tilde{\Gamma}_{i,j}$ at which a fixation line $(L(t), t \geq 0)$ goes from $i$ to $j$ is:

$$\tilde{\Gamma}_{i,j} = \binom{j}{j-i+1} \int_{[0,1]} \Lambda(dx)x^{-2}x^{j-i+1}(1-x)^i, \quad 1 \leq i < j < \infty.$$  

**Proof.** A fixation line jumps from $i$ to $j$ when, at the time of a reproduction event, $j - i + 1$ levels exactly are chosen among the levels $1, 2, \ldots, j$, and the level $j + 1$ is not chosen. For a reproduction event with asymptotic frequency $x$, this has probability $x^{j-i+1}(1-x)^i$ for any unordered set with $j - i + 1$ elements in $\{1, \ldots, j\}$. Counting the number of such sets, and integrating with respect to the "law" of the asymptotic frequency $x$ gives the formula. □

The quantity $\tilde{\Gamma}_{i,j}$ should be compared with the rate $\Lambda_{j,i}$ at which the block counting process of the $n$-coalescent $(X^n(t), t \geq 0)$ jumps from $j$ to $i$:

$$\Lambda_{j,i} = \binom{j}{j-i+1} \int_{[0,1]} \Lambda(dx)x^{-2}x^{j-i+1}(1-x)^i, \quad 1 \leq i < j < \infty.$$  

Unlike the transitions of the block counting process, which involve (a mixture of) binomial distributions, the transitions of the fixation line involve (a mixture of) negative binomial distributions. The two quantities $\tilde{\Gamma}_{i,j}$ and $\Lambda_{j,i}$ differ in general, still we have the following relationship.

**Lemma 2.3.** For each $i < j$, the rate $\tilde{\Gamma}_{i,\geq j}$ at which a fixation line jumps from $i$ to a level $\geq j$ is equal to the rate $\Lambda_{j,\leq i}$ at which the block counting process jumps from $j$ to $\leq i$ blocks:

$$\tilde{\Gamma}_{i,\geq j} = \Lambda_{j,\leq i}.$$  

A computational proof is given in the appendix. For another instance of such a duality relationship, abstracted at the level of measure valued process, we refer the reader to Lemma 5 p. 282 of Bertoin and Le Gall [6].

The claim (6) may be justified directly as follows: a fixation line jumps from $i$ to a level $\geq j$ when, at the time of a reproduction event, at least $j - i + 1$ levels are chosen among the levels $1, 2, \ldots, j$, without any condition on the level $j + 1$. The same event backward corresponds to a coalescence from $j$ blocks to $\leq i$ blocks.

Setting $j = i + 1$ in (6), we obtain that the total rate at which the fixation lines jumps up from $i$ is equal to the total rate at which the block counting process jumps down from $i$:

$$\tilde{\Gamma}_{i,\geq i+1} = \Lambda_{i+1,\leq i},$$  

two quantities that we simply denote by $\Lambda_{i+1}$ in the following.
2.3. The Beta\((2−α, α)\) family. The Beta\((2−α, α)\) family of probability measures is given, for \(0 < α < 2\), by:

\[
Λ(dx) = \text{Beta}(2−α, α)(dx) = \frac{1}{Γ(2−α)Γ(α)} x^{1−α}(1−x)^{α−1} 1_{[0,1]}(x) \, dx.
\]

The associated Beta\((2−α, α)\)-coalescents interpolate between the star like coalescent, which corresponds to the limit case \(α = 0\), the Bolthausen-Sznitman coalescent, \(α = 1\), and the Kingman coalescent, which corresponds to the limit case \(α = 2\). Example 15 in Schweinsberg [23], or [19] and [20] in this paper, ensure that the Beta\((2−α, α)\)-coalescent comes down from infinity in the sense of [3] if and only if \(α \in (1, 2)\).

**Lemma 2.4.** When \(Λ(dx)\) is given by \((8)\) for some \(α \in (0, 2)\), the jump rates \(\tilde{Γ}_{i,i+j}\) of the fixation line \((L(t), t ≥ 0)\) factorize as follows:

\[
\tilde{Γ}_{i,i+j} = \frac{1}{α} \frac{Γ(i+α)}{Γ(i)} \frac{α}{Γ(2−α)} \frac{Γ(j−α+1)}{Γ(j+2)}.
\]

Conversely, it is not difficult to show that the Beta\((2−α, α)\) family contains all the probability measures \(Λ(dx)\) for which \(\tilde{Γ}_{i,i+j}\) factorizes as a product of a function of \(i\) and a function of \(j\). We stress that the transition rates \(Λ_{j−i}\) of the block counting process of the Beta\((2−α, α)\)-coalescents do not enjoy such a factorization property. To sum up, adopting a backward viewpoint results in a seemingly anecdotic change of the exponent of \(1−x\) in the rate \((4)\) with respect to the rate \((5)\), which in turn yields a factorization for Beta\((2−α, α)\)-coalescents. This factorization will be the key to exact computations.

**Proof.** The claim follows from the following elementary calculation:

\[
\tilde{Γ}_{i,i+j} = \frac{1}{Γ(α)Γ(2−α)} \frac{(i+j)}{(j+1)} \int_{(0,1)} dx \, x^{−α−1}(1−x)^{α−1} x^{j+1} (1−x)^i \, dx
\]

\[
= \frac{1}{Γ(α)Γ(2−α)} \frac{(i+j)!}{(j+1)!(i−1)!} \text{Beta}(j−α+1,i+α)
\]

\[
= \frac{1}{Γ(α)Γ(2−α)} \frac{Γ(i+α)}{Γ(i)} \frac{Γ(j−α+1)}{Γ(j+2)}.
\]

\(\square\)

Let \(S_j = \{L_j(t), t ≥ 0\}\) be the range of the fixation line started at \(j\). Lemma 2.4 entails that the law of the translated range \(S_j−j = \{L_j(t)−j, t ≥ 0\}\) does not depend on \(j\) in the Beta\((2−α, α)\) case. We shall simply use \(S\) to denote this random set. The set \(S\) is the range of a renewal process, and we compute its renewal measure. We set:

\[
φ_{η^*}(s) = \begin{cases} 
-\frac{s}{((1−s) \log(1−s))} & \text{if } α = 1, \\
-\frac{(α−1)s}{[(1−s)^α − (1−s)]]} & \text{if } α \in (0, 2) \backslash \{1\}.
\end{cases}
\]

**Proposition 2.5.** When \(Λ(dx)\) belongs to the Beta\((2−α, α)\) family given by \((8)\) for some \(α \in (0, 2)\), the generating function of the renewal measure is:

\[
\sum_{i≥0} P(i ∈ S) \, s^i = φ_{η^*}(s).
\]
Proof. The random set $S$ is a renewal point process on $\mathbb{Z}^+$ based on the interarrival measure:

$$\eta \{ j \} = \frac{\alpha}{\Gamma (2 - \alpha)} \frac{\Gamma (j - \alpha + 1)}{\Gamma (j + 2)}, \ j \geq 1.$$  

The measure $\eta$ is a probability measure, as confirmed by setting $s = 1$ in the following computation of the generating function $\varphi_\eta (s)$ of $\eta$. We first do the computation for $\alpha \in (0, 2) \setminus \{ 1 \}$:

$$\varphi_\eta (s) = \sum_{j \geq 1} \frac{\alpha}{\Gamma (2 - \alpha)} \frac{\Gamma (j - \alpha + 1)}{\Gamma (j + 2)} s^j = \sum_{j \geq 1} \frac{1}{\Gamma (2 - \alpha)} \frac{1}{\Gamma (j + 2)} \frac{\Gamma (j - \alpha + 1)}{\Gamma (j + 1)} s^j = \frac{1}{(1 - \alpha)s} \sum_{j \geq 2} \left( \frac{\alpha}{j} \right) (-s)^j.$$ 

Using the binomial theorem, we deduce that:

$$\varphi_\eta (s) = \frac{1}{(1 - \alpha)s} \left[ (1 - s)^\alpha - 1 + \alpha s \right] = 1 + \frac{1}{(\alpha - 1)s} \left[ (1 - s)^\alpha - (1 - s) \right].$$ 

We now consider the case $\alpha = 1$:

$$\varphi_\eta (s) = \sum_{j \geq 1} \frac{\alpha}{\Gamma (2 - \alpha)} \frac{\Gamma (j - \alpha + 1)}{\Gamma (j + 2)} s^j = \frac{1}{s} \sum_{j \geq 1} \frac{1}{\Gamma (j + 2)} \frac{1}{\Gamma (j + 1)} s^j = 1 + \frac{(1 - s) \log (1 - s)}{s},$$

using for the last equality that the primitive of $s \mapsto -\log (1 - s)$ null at 0 is:

$$s \mapsto (1 - s) \log (1 - s) + s.$$ 

We deduce the generating function $\varphi_\eta^* (s)$ of the renewal measure using the renewal property. Let $S = \{ 0 = L^0 < L^1 < L^2 < \ldots \}$ be the enumeration of the elements of $S$ in increasing order. We have:

$$\varphi_\eta^* (s) = \sum_{i \in \mathbb{Z}^+} s^i \mathbb{P} (i \in S) = \mathbb{E} \left( \sum_{i \in \mathbb{Z}^+} s^{L^i} \right) = 1 + \mathbb{E} (s^{L^1}) \mathbb{E} \left( \sum_{i \in \mathbb{Z}^+} s^{L^i} \right) = 1 + \varphi_\eta (s) \varphi_\eta^* (s),$$

and the claim follows. \hfill \Box

In two particular cases, the renewal measure $\mathbb{P} (i \in S)$ is explicit: in the case $\alpha = 1/2$, we have:

$$\sum_{j \geq 0} \mathbb{P} (j \in S) s^j = \frac{1}{2} \left( \frac{1}{\sqrt{1 - s}} + 1 \right) = \frac{1}{2} \sum_{j \geq 0} \left( \frac{\Gamma (j + 1/2)}{\Gamma (1/2) \Gamma (j + 1)} + 1_{\{ j = 0 \}} \right) s^j.$$ 

and in the case $\alpha = 3/2$, we have:

$$\sum_{j \geq 0} \mathbb{P} (j \in S) s^j = \frac{1}{2} \left( \frac{1}{\sqrt{1 - s}} + 1 \right) = \frac{1}{2} \sum_{j \geq 0} \left( \frac{\Gamma (j + 1/2)}{\Gamma (1/2) \Gamma (j + 1)} + 1 \right) s^j.$$ 

The measure $\eta$ given by (12) being a probability measure, we have, using (9), that:

$$\Lambda_{i+1} = \hat{\Gamma}_{i+1} = \frac{1}{\alpha \Gamma (\alpha)} \frac{\Gamma (i + \alpha)}{\Gamma (i)} \sim \frac{1}{\alpha \Gamma (\alpha)} i^\alpha \text{ as } i \to \infty,$$
where \( a_n \sim b_n \) means that \( \lim_{n \to \infty} a_n/b_n = 1 \), using also the definition of \( \Lambda_{i+1} \) (short after (7)) at the first equality. We also notice, for future use, that the transition rate from \( i \) blocks to 1 block satisfies:



\[
\Lambda_{i,1} = \frac{1}{\Gamma(2-\alpha)} \frac{\Gamma(i-\alpha)}{\Gamma(i)} \sim \frac{1}{\Gamma(2-\alpha)} i^{-\alpha} \quad \text{as } i \to \infty.
\]

3. Applications

3.1. The fixation line and the set of records. The first application of the fixation line consists in the computation of the probability for an integer \( i \) to be a record. Recall that the set \( T \) of records is the set \( \{i \geq 2, \tau_{i-1}^1 > \tau_{i-1}^1\} \) where the sequence \( \tau_i^1 \) is defined in the natural coupling of the \( n \)-coalescents, see the introduction for the definition of this coupling. The Proposition uses the range \( S_1 = \{L_1(t), t \geq 0\} \) of the fixation line started at 1. We stress the Proposition is valid for a general probability measure \( \Lambda(dx) \).

**Proposition 3.1.** The marginal distribution of the set of records \( T \) satisfies:

\[
P(i \in T) = \frac{\mathbb{P}(i - 1 \in S_1)}{\Lambda_i}, \quad i \geq 2.
\]

**Proof.** Since

\[
\tau_i^1 = \tau_{i-1}^1 + 1_{i \in T} e, \quad i \geq 2
\]

for \( e \) an exponential random variable with parameter 1 (independent of \( \{i \in T\} \), but we do not use this fact), we deduce that:

\[
P(i \in T) = \mathbb{E}(\tau_i^1) - \mathbb{E}(\tau_{i-1}^1) = \mathbb{E}(\alpha_i^1) - \mathbb{E}(\alpha_{i-1}^1) = \mathbb{P}(i - 1 \in S_1)/\Lambda_i
\]

using Lemma 2.1 for the second equality, and relation (7) for the last equality. \( \square \)

Let \( (e_i)_{2 \leq i \leq n} \) be a collection of independent exponential random variables with parameter 1, also independent of \( T \). Iterating (17) yields

\[
\tau_i^n = \sum_{2 \leq i \leq n} 1_{i \in T} e_i.
\]

Combining with the discussion on coalescents which come down from infinity following Lemma 2, we deduce that the cardinality of the set \( T \) is a.s. infinite of a.s. finite. It is infinite when the coalescent stays infinite, and finite when the coalescent comes down from infinity.

Now, Proposition 3.1 (13) and (15) give the following expression for the record probabilities in the case \( \alpha = 1/2 \):

\[
P(i \in T) = \frac{1}{2} \left( \frac{1}{2i - 3} + 1_{i=2} \right), \quad i \geq 2.
\]

This result gains a clear interpretation in the representation of the \( n \)-Beta(3/2, 1/2)-coalescent found by Abraham and Delmas [1], which uses the pruning at nodes of a labelled binary tree with \( n \) leaves. In case \( \alpha = 3/2 \), we use (14) instead of (13) to obtain that:

\[
P(i \in T) = \frac{3}{2} \frac{1}{(2i - 1)(2i - 3)} + \frac{3}{4} \Gamma(3/2)\Gamma(i - 1)\Gamma(i + 1/2), \quad i \geq 2.
\]

For general \( \alpha \in (0, 2) \), we compute the generating function of the record probabilities.
Proposition 3.2. The marginal distribution of the set of records in Beta$(2-\alpha, \alpha)$-coalescents has the following generating function:

\[
\sum_{i \geq 2} \mathbb{P}(i \in T) s^i = s^3 \int_{(0,1)} dx \frac{-x}{(1-sx)\log(1-sx)} = s \int_{(0,s)} du \frac{-u}{(1-u)\log(1-u)}
\]

in the Bolthausen-Sznitman case $\alpha = 1$, and:

\[
\sum_{i \geq 2} \mathbb{P}(i \in T) s^i = \alpha (1-\alpha) s^3 \int_{(0,1)} dx \frac{x}{(1-x)^{1-\alpha}[(1-sx)^\alpha - (1-sx)]}
\]

in case $\alpha \in (1,2) \setminus \{1\}$.

Proof. We do the following computation:

\[
\sum_{i \geq 2} \mathbb{P}(i \in T) s^i = \sum_{i \geq 2} \alpha \frac{\Gamma(\alpha) \Gamma(i-1)}{\Gamma(i+1)} \mathbb{P}(i-2 \in S) s^i
\]

\[
= \int_{(0,1)} dx \alpha (1-x)^{\alpha-1} s^2 \sum_{i \geq 2} \mathbb{P}(i-2 \in S) (sx)^{i-2}
\]

\[
= \int_{(0,1)} dx \alpha (1-x)^{\alpha-1} s^2 \varphi_{\eta^*}(sx)
\]

using Proposition 3.1, the definition of $S = S_1 - 1$ and formula (15) at the first equality, the link between Gamma and Beta functions at the second equality as well as the Fubini Tonelli theorem. The claim now follows substituting $\varphi_{\eta^*}$ by its value given in (10), distinguishing weather $\alpha \in (0,2) \setminus \{1\}$ or $\alpha = 1$. □

Corollary 3.3. The depth $\tau_1^n$ of the $n$-Beta$(2-\alpha, \alpha)$-coalescent almost surely converges as $n \to \infty$ in the natural coupling to a random variable $\tau_1$ with expectation:

\[
\mathbb{E}(\tau_1) = \alpha(\alpha - 1) \int_{(0,1)} dx \frac{x}{(1-x)^{2-\alpha}[1-(1-x)^{\alpha-1}]}
\]

in case $\alpha \in (1,2)$.

Proof. The almost sure and monotone convergence of $\tau_1^n$ in the natural coupling of the $n$-coalescents follows form the definition of the natural coupling. For the expectation: set $s = 1$ in Proposition 3.2 and use the first equality in (18): this gives a telescopic sum with sum $\mathbb{E}(\tau_1)$. □

Since the sequence $(\mathbb{E}(\tau_1^n), n \geq 1)$ has limit $\mathbb{E}(\tau_1)$ by monotone convergence, which is finite according to (19), the Beta$(2-\alpha, \alpha)$-coalescent for $\alpha \in (1,2)$ comes down from infinity. On the other hand, when $\alpha \in (0,1]$, we have, using again Proposition 3.2 with $s = 1$, that:

\[
\lim_{n \to \infty} \mathbb{E}(\tau_1^n) = \infty,
\]

and the Beta$(2-\alpha, \alpha)$-coalescent with $\alpha \in (0,1]$ therefore stays infinite. In this case, the suitably rescaled random variables $\tau_1^n$, as $n \to \infty$, have been proved to converge in law. We refer to [11] and the references therein for the last improvements.
3.2. The fixation line and the last coalescence. We consider the block counting process $(X^n(t), t \geq 0)$ of the $n$-coalescent. Its embedded Markov chain starts at $n$, and has transitions probabilities:

\[(21) \quad P_{ji} = \frac{\Lambda_{ji}}{\Lambda_j}, \quad j > i \geq 1,\]

with $\Lambda_{ji}$ the transition rate from $j$ to $i$ blocks defined in (5) and $\Lambda_j = \sum_{i<j} \Lambda_{ji}$. In this subsection, we consider the problem of the convergence of the time-reversal of this Markov chain as the initial number of blocks $n$ goes to $\infty$. Unlike the transition probabilities of the original chain, the transition probabilities of the chain reversed in time depend on the starting point $n$, and we shall use $\tilde{P}_{ij}^n$ to denote the transition probability of the reversed chain from $i$ to $j$, $1 \leq i < j$, when the original chain is starting at $n$. This amounts to study the weak convergence of $\{(\tilde{P}_{ij}^n, 1 \leq i \leq j \leq n)\}$, viewed as a family, indexed by the integers $n$, of probability measures on $\mathbb{N}$, for each $i \geq 1$.

The question for an arbitrary $i \geq 1$ may be reduced to the case $i = 1$: If $\mathcal{R}^n = \{X^n(t), t \geq 0\}$ stands for the range of the block counting process of the $n$-coalescent, we have the key equality:

\[(22) \quad \mathbb{P}(i \in \mathcal{R}^n) \tilde{P}_{ij}^n = \mathbb{P}(j \in \mathcal{R}^n) P_{ji},\]

which entails that:

\[(23) \quad \tilde{P}_{ij}^n = \frac{P_{ji}}{P_{j1}} \tilde{P}_{1j}^n.\]

The following proposition gives an expression of the distribution $(\tilde{P}_{1j}^n, 2 \leq j \leq n)$ in term of quantities related to the fixation line. This is the essential conceptual step in the study of the last coalescence since the next steps, carried out in the case of the Beta$(2 - \alpha, \alpha)$-coalescent in the next subsection, reduce to the computation of $\mathbb{E}(\alpha^n_j)$.

**Proposition 3.4.** The distribution $(\tilde{P}_{1j}^n, 2 \leq j \leq n)$ of the number of blocks involved in the last coalescence of a $n$-coalescent satisfies:

\[(24) \quad \tilde{P}_{1j}^n = \Lambda_{j1} \left[ \mathbb{E}(\alpha^n_{j-1}) - \mathbb{E}(\alpha^n_j) \right], \quad 2 \leq j \leq n.\]

**Proof.** We compute:

\[\tilde{P}_{1j}^n = P_{j1} \mathbb{P}(j \in \mathcal{R}^n) = \Lambda_{j1} \mathbb{E} \left( \int_{(0,\infty)} ds \mathbf{1}_{\{X^n(s) = j\}} \right) = \Lambda_{j1} \left[ \mathbb{E}(\tau^n_{j-1}) - \mathbb{E}(\tau^n_j) \right],\]

setting $i = 1$ in (22) for the first equality, using the definition (21) of $P_{j1}$ and the fact that the block counting process spends an exponential time at $j$ with parameter $\Lambda_j$ when $j \in \mathcal{R}^n$ for the second equality, and the pathwise relation $\tau^n_{j-1} = \tau^n_j + \int_{(0,\infty)} ds \mathbf{1}_{\{X^n(s) = j\}}$ at the last equality. We conclude using Lemma (2.1). \qed

For each $j \geq 2$, we use $\alpha_j$ to denote the increasing limit of $(\alpha^n_j, n \geq j)$. Since the convergence is monotone, we have convergence of the expectations:

\[\lim_{n \to \infty} \mathbb{E}(\alpha^n_j) = \mathbb{E}(\alpha_j),\]
keeping in mind that the quantity on the right-hand side may be infinite. In case the coalescent comes down from infinity, \( \mathbb{E}(\alpha_j) < \infty \), and we have:

\[
\lim_{n \to \infty} \mathbb{E}(\alpha_{j-1}^n) - \mathbb{E}(\alpha_j^n) = \mathbb{E}(\alpha_{j-1}) - \mathbb{E}(\alpha_j)
\]

for each \( j \geq 1 \). The convergence of \((\tilde{P}_{ij}^n, n \geq 2)\) for arbitrary \( i < j \) follows from [23] and [24]. A much more interesting case is when the coalescent stays infinite. Then we cannot directly conclude to the convergence of the difference of the expectations \( \mathbb{E}(\alpha_{j-1}^n) - \mathbb{E}(\alpha_j^n) \). Setting \( S_j = \{L_j(t), t \geq 0\} \) for the range of the fixation line \( L_j \) started at \( j \), we have:

\[
(25) \quad \mathbb{E}(\alpha_{j-1}^n) - \mathbb{E}(\alpha_j^n) = \sum_{j-1 \leq i \leq n-1} \left[ \mathbb{P}(i \in S_{j-1}) - \mathbb{P}(i \in S_j) \right] \frac{1}{\Lambda_{i+1}}.
\]

using that the rate at which the fixation line leaves \( i \) is \( \Lambda_{i+1} \). If the coalescent stays infinite, the series with general term \( 1/\Lambda_i \) is easily seen to diverge: in fact, by the definition of the \( n \) coalescent, we have

\[
\mathbb{E}(\tau_1^n) = \sum_{2 \leq i \leq n} \frac{1}{\Lambda_i} \mathbb{P}(i \in \mathcal{R}^n) \leq \sum_{2 \leq i \leq n} \frac{1}{\Lambda_i},
\]

and the left-hand side goes to \( \infty \) a.s. for a coalescent which stays infinite. Proving convergence in (25) as \( n \to \infty \) therefore requires a further study of \([\mathbb{P}(i \in S_{j-1}) - \mathbb{P}(i \in S_j)]\), which is a difficult issue in general. The factorization property satisfied by the Beta\((2-\alpha, \alpha)\) coalescents, see Lemma [9] allows to circumvent the difficulty.

3.2.1. Beta\((2-\alpha, \alpha)\)-coalescents. Before stating our main theorem, it is perhaps opportune to recall the statement of the problem of the last coalescence in a self contained manner: The block counting process \((X^n(t), t \geq 0)\) is a Markov chain started at \( n \) and a.s. absorbed at 1 in finite time. In the case of the Beta\((2-\alpha, \alpha)\)-coalescents, we recall from [5] that the transitions rates of the block counting process from \( j \) to \( i \) are given by:

\[
\Lambda_{j,i} = \frac{\Gamma(j+1)}{\Gamma(j)} \frac{\Gamma(j-i+1-\alpha)}{\Gamma(j-i+2)} \frac{\Gamma(i+\alpha-1)}{\Gamma(i)}, \quad 1 \leq i < j < \infty.
\]

What is the law of the last jump of \((X^n(t), t \geq 0)\) as \( n \to \infty \)? The following theorem answers the question.

**Theorem 3.5.** The distribution \((\tilde{P}_{ij}^n, j \geq 2)\) of the number of blocks implied in the last coalescence of the \( n \)-Beta\((2-\alpha, \alpha)\)-coalescent weakly converges as \( n \to \infty \) towards a distribution \((\tilde{P}_{ij}, j \geq 2)\) with generating function:

\[
(26) \quad \sum_{j \geq 2} \tilde{P}_{ij} s^j = s \int_{(0,1)} dx \frac{\log(1-sx)}{\log(1-x)}
\]

in the Bolthausen-Sznitman case \( \alpha = 1 \), and:

\[
(27) \quad \sum_{j \geq 2} \tilde{P}_{ij} s^j = \alpha s \int_{(0,1)} dx \frac{1}{1 - (1-x)^{1-\alpha}} \left[ \frac{1}{(1-sx)^{1-\alpha}} - 1 \right]
\]

in case \( \alpha \in (0,2) \setminus \{1\} \).
There relies on a connection with the pruning of Lévy trees. In the case from (26) that:

\[ \text{similar claim (including the limiting generating function) for } \infty \text{ in the previous subsection. Proposition 1.5 of Abraham and Delmas [2] establishes a similar claim (including the limiting generating function) for } \alpha \in (0, 1/2]. \] The proof given there relies on a connection with the pruning of Lévy trees. In the case \( \alpha = 1 \), we deduce from (26) that:

\[
\tilde{P}_{1j} = \frac{1}{j-1} \int_{(0,1)} dx \frac{x^{j-1}}{-(1-x)} = \frac{1}{j-1} \int_{0}^{\infty} \frac{du}{u} (1 - e^{-u})^{j-1} e^{-u} = \frac{1}{j-1} \sum_{1 \leq k \leq j-1} \binom{j-1}{k} (-1)^{k+1} \log(k+1)
\]

using the change of variable \( x = 1 - e^{-u} \) at the second equality, expanding \((1 - e^{-u})^{j-1}\) and using the Frullani integral at the third equality. This result has been obtained first by Goldschmidt and Martin [12], using a connection with the pruning of recursive trees. It is interesting to observe the diversity of the methods at work in [2, 12] and the present paper.

**Proof.** Recall \( S \) denotes the range of the renewal point process on \( \mathbb{Z}^+ \) containing 0 and with interarrival times with law \( \eta \) given by (12). We compute:

\[
\mathbb{E}(\alpha_{j-1}^n) - \mathbb{E}(\alpha_j^n) = \sum_{j-1 \leq i \leq n-1} \mathbb{P}(i \in S_{j-1}) \frac{1}{\Lambda_{i+1}} - \sum_{j \leq i \leq n-1} \mathbb{P}(i \in S_j) \frac{1}{\Lambda_{i+1}} = \sum_{j \leq i \leq n} \mathbb{P}(i-j \in S) \frac{1}{\Lambda_i} - \sum_{j \leq i \leq n-1} \mathbb{P}(i-j \in S) \frac{1}{\Lambda_{i+1}} = \sum_{j \leq i \leq n-1} \mathbb{P}(i-j \in S) \left( \frac{1}{\Lambda_i} - \frac{1}{\Lambda_{i+1}} \right) + \mathbb{P}(n-j \in S) \frac{1}{\Lambda_n}. \tag{28}
\]

beginning as in (25) for the first equality, using the definition of the translated range \( S = S_{j-} - j \), independent of \( j \) in the Beta\((2 - \alpha, \alpha)\) setting, and then changing the index in the first sum at the second equality. Bounding \( \mathbb{P}(i-j \in S) \) from above by 1, and using the positivity of \( \Lambda_n \), we obtain the following upper bound:

\[
\sum_{j \leq i \leq n-1} \mathbb{P}(i-j \in S) \left( \frac{1}{\Lambda_i} - \frac{1}{\Lambda_{i+1}} \right) \leq \sum_{j \leq i \leq n-1} \frac{1}{\Lambda_i} - \frac{1}{\Lambda_{i+1}} = \frac{1}{\Lambda_j} - \frac{1}{\Lambda_n} \leq \frac{1}{\Lambda_j}. \tag{29}
\]

The sequence \( (\Lambda_j, j \geq 2) \) is non-decreasing. Therefore, the serie on the left-hand side of (29) has non-negative terms. This serie is bounded, and therefore converges. Also, the sequence \( (\Lambda_j, j \geq 2) \) goes to \( \infty \) by (15). The second term in (28) then goes to 0. Using (24), we conclude that the limit as \( n \to \infty \) of the quantities \( \tilde{P}_{1j}^n \) exists, we denote it by \( \tilde{P}_{1j} \). Setting \( k = i-j \), we have that:

\[
\tilde{P}_{1j} = \Lambda_j \sum_{k \geq 0} \mathbb{P}(k \in S) \left( \frac{1}{\Lambda_{k+j}} - \frac{1}{\Lambda_{k+j+1}} \right) < \infty.
\]
Setting the explicit values (15) and (16) of $\Lambda_j$ and $\Lambda_{j_1}$ gives:

$$\tilde{P}_{1j} = \frac{1}{j-1} \sum_{k \geq 0} \mathbb{P}(k \in S) \left[ \frac{1}{k+j-1} - \frac{1}{k+j} \right]$$

in case $\alpha = 1$, and:

$$\tilde{P}_{1j} = \frac{\alpha^2 \Gamma(\alpha) \Gamma(j-\alpha)}{\Gamma(2-\alpha) \Gamma(j)} \sum_{k \geq 0} \mathbb{P}(k \in S) \frac{\Gamma(k+j-1)}{\Gamma(k+j+\alpha)}$$

in case $\alpha \in (0, 2) \setminus \{1\}$. Recall the expression (11) for the generating function of the numbers $\mathbb{P}(k \in S)$. Multiplying both sides of (11) by $s^{j-2}(1-s)^\alpha$, integrating with respect to $s \in (0,1)$ and using Fubini-Tonelli theorem, we deduce:

$$\sum_{k \geq 0} \mathbb{P}(k \in S) \left[ \frac{1}{k+j-1} - \frac{1}{k+j} \right] = - \int_{(0,1)} ds \frac{s^{j-1}}{\log(1-s)}$$

in case $\alpha = 1$, and

$$\sum_{k \geq 0} \mathbb{P}(k \in S) \frac{\Gamma(k+j-1)\Gamma(\alpha+1)}{\Gamma(k+j+\alpha)} = -(\alpha - 1) \int_{(0,1)} ds \frac{s^{j-1}}{1 - (1-s)^{1-\alpha}}$$

in case $\alpha \in (0, 2) \setminus \{1\}$, using also the expression of the Beta function in term of the Gamma function. From the last four equations displayed, we obtain:

$$\tilde{P}_{1j} = \frac{-1}{j-1} \int_{(0,1)} dx \frac{x^{j-1}}{\log(1-x)}$$

in case $\alpha = 1$, and

$$\tilde{P}_{1j} = (-1)^{j-1} \alpha \binom{\alpha - 1}{j-1} \int_{(0,1)} dx \frac{x^{j-1}}{1 - (1-x)^{1-\alpha}}$$

in case $\alpha \in (0, 2) \setminus \{1\}$, which imply respectively (26) and (27) by multiplying by $s^j$ and summing over $j \geq 2$.

**Corollary 3.6.** The probability for an integer $j \geq 2$ to be in the range $\mathcal{R}_n$ of the block counting process of the $n$-Beta($2-\alpha, \alpha$)-coalescent converges as $n \to \infty$ and:

$$\lim_{n \to \infty} \mathbb{P}(j \in \mathcal{R}_n) = (j-1) \int_{(0,1)} dx \frac{x^{j-1}}{\log(1-x)}$$

in the Bolthausen-Sznitman case $\alpha = 1$, and:

$$\lim_{n \to \infty} \mathbb{P}(j \in \mathcal{R}_n) = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(j-1+\alpha)}{\Gamma(j-1)} \int_{(0,1)} dx \frac{x^{j-1}}{1 - (1-x)^{1-\alpha}}$$

in case $\alpha \in (0, 2) \setminus \{1\}$.

Notice the integrands in both integral representations are non-negative whatever the value of $\alpha$. Also, the Bolthausen-Sznitman case in Corollary 3.6 corresponds to the statement of Theorem 1.1 of Möhle [18], and the case $\alpha \in (0, 2) \setminus \{1\}$ answers a question posted in the same paper, see Remark 3. The question has also been independently answered by Möhle in [19].
Proof. This is a consequence of the equation (22) together with formula (30) in the Bolthausen-Sznitman case \( \alpha = 1 \), and together with formula (31) in the case \( \alpha \in (0, 2) \setminus \{1\} \).

In case \( \alpha \in (1, 2) \), the coalescent comes down from infinity and it is possible to consider directly the range \( \mathcal{R} \) of the infinite coalescent: The range \( \mathcal{R} \) is the almost sure (local) limit of the \( \mathcal{R}_n \) in the sense that \( 1_{i \in \mathcal{R}} \) is the almost sure limit of \( 1_{i \in \mathcal{R}_n} \). Dominated convergence theorem then ensures that the right-hand side of (32) corresponds to \( \mathbb{P}(j \in \mathcal{R}) \). We propose to write

\[
\mathbb{P}(j \in \mathcal{R}) := \lim_{n \to \infty} \mathbb{P}(j \in \mathcal{R}_n)
\]

whatever the value of \( \alpha \in (0, 2) \): this is however an abuse of notation since we do not give a meaning to \( \mathcal{R} \) when \( \alpha \in (0, 1] \).

Corollary 3.7. The limiting probability for an integer \( j \geq 2 \) to be in the range of the block counting process of the \( n \)-Beta\((2 - \alpha, \alpha)\)-coalescent satisfies:

\[
\lim_{j \to \infty} \mathbb{P}(j \in \mathcal{R}) = \alpha - 1 \text{ as } j \to \infty,
\]

in case \( \alpha \in (1, 2) \), and:

\[
\mathbb{P}(j \in \mathcal{R}) \sim \frac{1 - \alpha}{\Gamma(\alpha)} j^{\alpha - 1} \text{ as } j \to \infty,
\]

in case \( \alpha \in (0, 1) \).

The asymptotics for \( \alpha \in (1, 2) \) have been previously derived in Berestycki et al. using a connection with \( \alpha \)-stable branching processes, see Theorem 1.8 of [3], and the Bolthausen-Sznitman case \( \alpha = 1 \) has been covered in Möhle [18], see Corollary 1.3; in this case,

\[
\mathbb{P}(j \in \mathcal{R}) \sim \frac{1}{\log(j)} \text{ as } j \to \infty.
\]

Proof. We first consider the case \( \alpha \in (0, 1) \). Estimating the left factor in (32) is easy:

\[
\Gamma(j - 1 + \alpha) \sim j^{\alpha - 1} \Gamma(j - 1) \text{ as } j \to \infty.
\]

For the remaining integral factor in (32), we write:

\[
\int_{(0,1)} dx \; x^{j-1} \frac{1}{1 - (1 - x)^{1-\alpha}} = \int_{(0,1)} dx \; x^{j-2} h(x)
\]

for \( h(x) = x/ [1 - (1 - x)^{1-\alpha}] \). Then we decompose as follows:

\[
\int_{(0,1)} dx \; x^{j-2} h(x) = \frac{1}{j-1} \left[ h(1) + \int_{(0,1)} dx \; (j-1) \; x^{j-2} (h(x) - h(1)) \right].
\]

Fix \( \epsilon > 0 \). From the continuity of \( h \) at 1, there exists \( \eta > 0 \) such that such \( |h(x) - h(1)| \leq \epsilon/2 \)

for \( x \in (1 - \eta, 1] \), and:

\[
\int_{(0,1)} dx \; (j-1) \; x^{j-2} |h(x) - h(1)| \leq \frac{\epsilon}{2} + 2 \|h\|_{\infty} (j-1)(1-\eta)^{j-2} \leq \epsilon
\]
for \( j \) large enough. Therefore, the left-hand side of \((31)\) is equivalent to \(1/j\), and the claim follows in the case \( \alpha \in (0, 1) \). For the case \( \alpha \in (1, 2) \), it is more convenient to rewrite the integral factor as follows:

\[
\int_{(0,1)} dx \frac{x^{-1} - \alpha}{1 - (1-x)^{\alpha-1}} = (\alpha - 1) \int_{(0,1)} dx \frac{x^{-1}(1-x)^{\alpha-1}}{1 - (1-x)^{\alpha-1}}.
\]

Then, we write:

\[
\int_{(0,1)} dx \frac{x^{-1}(1-x)^{\alpha-1}}{1 - (1-x)^{\alpha-1}} = \int_{(0,1)} dx \frac{x^{-2}(1-x)^{\alpha-1} h(x)}{1 - (1-x)^{\alpha-1}}
\]

for \( h(x) = x/[1 - (1-x)^{\alpha-1}] \) this time. The same reasoning as before allows to conclude that:

\[
\int_{(0,1)} dx \frac{x^{-2}(1-x)^{\alpha-1} h(x)}{1 - (1-x)^{\alpha-1}} \sim_{j \to \infty} h(1) \int_{(0,1)} dx \frac{x^{-2}(1-x)^{\alpha-1}}{1 - (1-x)^{\alpha-1}} = \Gamma(\alpha) \frac{\Gamma(j-1)}{\Gamma(j-1+\alpha)}.
\]

This term is compensated by the first factor in \((32)\), and the claim follows for \( \alpha \in (1, 2) \). □

The definition of the block counting process of the \( n \)-coalescent entails:

\[
E(\tau^n_t) = \sum_{2 \leq j \leq n} \frac{P(j \in \mathbb{R}^n)}{\Lambda_j},
\]

Taking the limit \( n \to \infty \) in this formula gives an alternative proof of Corollary \(3.3\) on the expected depth of the Beta\((2-\alpha, \alpha)\)-coalescent for \( \alpha \in (1, 2) \).

### 3.3. Depth of the Beta\((1, 1)\)-coalescent.

We propose to investigate further the Bolthausen-Sznitman coalescent associated with \( \Lambda(dx) = 1_{[0,1]}(x) \, dx \). This coalescent stays infinite. In fact, it plays a special role in the class of the Beta\((2-\alpha, \alpha)\)-coalescents, since it separates those coalescents which come down from infinity, \( \alpha > 1 \), from those which stay infinite, \( \alpha \leq 1 \). The case \( \alpha = 1 \) in \((9)\) gives:

\[
\tilde{\Gamma}_{i,i+j} = \frac{i}{j(j+1)}.
\]

Therefore, the fixation line \((L(t), t \geq 0)\) is a continuous time discrete state space branching process (and this is the only coalescent for which it is the case). The reproduction law \( \mu \) of the branching process is defined by:

\[
(35) \quad \mu\{j\} = \eta\{j-1\} = \frac{1}{j(j-1)}, \quad j \geq 2.
\]

since a jump of \( j - 1 \) for the fixation line corresponds to the arrival of \( j \) children together with the death of the father. The generating function associated with the reproduction law \( \mu \) is:

\[
\varphi_\mu(s) = s \varphi_\eta(s) = s + (1-s) \log(1-s), \quad 0 \leq s < 1.
\]

The reproduction law \( \mu \) has infinite mean, but the branching process is conservative, meaning it does not reach infinity in finite time. This agrees with our observation (after Lemma \(2.1\)) that the fixation line \((L(t), t \geq 0)\) remains finite for coalescents which stay infinite, and the Bolthausen-Sznitman coalescent stays infinite. The rate of increase of \((L(t), t \geq 0)\) is well-known, see Grey \(13\) for instance, we nevertheless include a proof for the ease of reference.
Proposition 3.8. In the Bolthausen-Sznitman case \( \Lambda(dx) = 1_{[0,1]}(x) \, dx \), we have
\[
e^{-t} \log L_1(t) \to e, \text{ a.s. as } t \to \infty,
\]
with \( e \) an exponential random variable with parameter 1.

Notice this growth rate deviates from the exponential growth rate satisfied by supercritical branching processes with a finite mean reproduction law, cf. the Seneta-Heyde theorem.

Proof. We begin with general considerations on continuous-time branching processes. The generating function \( f_t(s) = \mathbb{E}(s^{L_1(t)}) \) of \( L_1(t) \) may be computed from the infinitesimal generating function \( \phi(s) = \varphi_\mu(s) - s \) using the partial differential equation:
\[
\begin{align*}
\partial_t f_t(s) &= \phi(f_t(s)), \\
\f_0(s) &= s,
\end{align*}
\]
see Harris [14] on chapter V for instance. The function \( f_t \) is a bijection from \([0,1] \) into itself, with inverse function \( g_t \). The process \((g_t(s)^{L_1(t)}, t \geq 0)\) is Markov and has constant expectation since:
\[
\mathbb{E}(g_t(s)^{L_1(t)}) = f_t(g_t(s)) = s.
\]
Therefore it is a \([0,1]\)-valued martingale, almost surely converging towards a limiting random variable \( V \) as \( t \to \infty \). At this point, we take advantage of the explicit formulas available in our case:
\[
\phi(s) = \varphi_\mu(s) - s = (1 - s) \log(1 - s),
\]
which entails:
\[
f_t(s) = 1 - (1 - s)e^{-t} \text{ and } g_t(s) = 1 - (1 - s)e^t.
\]
Using the dominated convergence theorem, we deduce that for each \( \alpha > 0 \):
\[
\mathbb{E}(V^\alpha) = \lim_{t \to \infty} \mathbb{E}(g_t(s)^{\alpha L_1(t)}) = \lim_{t \to \infty} f_t(g_t(s)^\alpha) = \lim_{t \to \infty} 1 - \left(1 - (1 - s)e^t\right)^\alpha e^{-t} = s.
\]
This is possible only if \( V \) is \([0,1]\)-valued, equal to 1 with probability \( s \). Now, since \( g_t(s) \) is increasing in \( s \), there is a.s. a threshold random variable:
\[
U = \inf\{s \in \mathbb{Q} \cap [0,1], \lim_{t \to \infty} g_t(s)^{L_1(t)} = 1\},
\]
which is uniformly distributed on \([0,1]\) since \( \mathbb{P}(U < s) = \mathbb{P}(V = 1) = s \). Then, we form the logarithm of the expression \( g_t(1 - s)^{L_1(t)} \) and use that \( \log g_t(1 - s) \) is equivalent as \( t \to \infty \) to \( g_t(1 - s) - 1 \), itself equal to \(-se^t\) from the previous computation, to deduce that:
\[
\text{for } s < 1 - U, \lim_{t \to \infty} se^t L_1(t) = 0 \text{ and for } s > 1 - U, \lim_{t \to \infty} se^t L_1(t) = \infty.
\]
Set \( V = 1 - U \). The random variable \( V \) is again uniformly distributed on \([0,1]\). Taking again logarithm, for \( \epsilon > 0 \), we have:
\[
-\log(V + \epsilon) \leq \liminf_{t \to \infty} e^{-t} \log(L_1(t)) \leq \limsup_{t \to \infty} e^{-t} \log(L_1(t)) \leq -\log(V - \epsilon).
\]
Now, the random variable \(-\log(V)\) is exponentially distributed with parameter 1. This concludes the proof. \( \square \)
This a.s. growth rate is the key to the following (third) proof of the depth of the Bolthausen-Sznitman coalescent, after the one by Goldschmidt and Martin [12] based on a connection with recursive trees, and the one by Freund and Möhle [10] based on the analysis of a recurrence equation.

**Theorem 3.9.** In the Bolthausen-Sznitman case \( \Lambda(dx) = 1_{[0,1]}(x) \, dx \), we have the following convergence in distribution for the depth of the \( n \)-coalescent:

\[
\tau_n^1 - \log \log(n) \Rightarrow -\log e, \quad \text{as } n \to \infty
\]

where \( e \) is an exponential random variable with parameter 1.

The sequence \( (\tau^n_n, n \geq 1) \) evolves by independent exponential jumps at the moments of records in the natural coupling, see equation (17). The convergence in distribution can therefore not be reinforced in an a.s. convergence.

**Proof.** From the a.s. growth rate (36) and the definition (2) of the hitting time \( \alpha_1^n \), we deduce:

\[
e^{-\alpha_1^n} \log n \leq e^{-\alpha_1^n} \log L_1(\alpha_1^n) \to e \text{ a.s. as } n \to \infty,
\]

using the definition of \( \alpha_1^n \) for the inequality and (36) for the almost sure convergence. Therefore,

\[
\limsup_{n \to \infty} \log \log n - \alpha_1^n \leq \log e.
\]

Similarly, taking the left limit at \( \alpha_1^n \) this time,

\[
e^{-\alpha_1^n} \log n \geq e^{-\alpha_1^n} \log L_1(\alpha_1^n) \to e \text{ a.s. as } n \to \infty,
\]

and this implies

\[
\liminf_{n \to \infty} \log \log n - \alpha_1^n \geq \log e.
\]

This proves (37) with \( \alpha_1^n \) instead of \( \tau_1^n \) and with a.s. convergence instead of weak convergence. We conclude using Lemma 2.1.

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4. Appendix

Proof of Lemma (2.3). We perform the following calculations:

\[
\tilde{\Gamma}_{i,j} = \sum_{k \geq j-i} \binom{k+i}{k+1} \int_{[0,1]} \Lambda(dx)x^{-2} x^{k+1}(1-x)^i
\]

\[
= \int_{[0,1]} \Lambda(dx)x^{-2} \left[ \sum_{k \geq j-i} \binom{k+i}{k+1} x^{k+1} \right] (1-x)^i
\]

\[
= \int_{[0,1]} \Lambda(dx)x^{-2} \left[ \frac{1}{(1-x)^i} - \sum_{0 \leq k \leq j-i} \binom{k+i-1}{k} x^k \right] (1-x)^i
\]

\[
= \int_{[0,1]} \Lambda(dx)x^{-2} \left[ 1 - \sum_{0 \leq k \leq j-i} \binom{k+i-1}{k} x^k(1-x)^i \right]
\]
using the binomial theorem at the third equality. We also compute:

\[ \Lambda_{j,i} = \sum_{j-i \leq k \leq j-1} \binom{j}{k+1} \int_{[0,1]} \Lambda(dx) x^{-2} x^{k+1}(1-x)^{j-(k+1)} \]

\[ = \int_{[0,1]} \Lambda(dx) x^{-2} \sum_{j-i \leq k \leq j-1} \binom{j}{k+1} x^{k+1}(1-x)^{j-(k+1)} \]

\[ = \int_{[0,1]} \Lambda(dx) x^{-2} \left[ 1 - \sum_{0 \leq k \leq j-i} \binom{j}{k} x^k(1-x)^{j-k} \right] \]

using the same theorem at the third equality again. It is enough to prove that the two integrands are equal, which amounts to verify:

\[ \sum_{0 \leq k \leq j-i} \binom{k+i-1}{k} x^k = \sum_{0 \leq k \leq j-i} \binom{j}{k} x^k(1-x)^{j-i-k}. \]

But setting \( \ell = j - i \) in the right-hand side, we obtain:

\[ \sum_{0 \leq k \leq j-i} \binom{j}{k} x^k(1-x)^{j-i-k} = \sum_{0 \leq k \leq \ell} \binom{\ell+i}{k} x^k(1-x)^{\ell-k} \]

\[ = \sum_{0 \leq k+k' \leq \ell} \binom{\ell+i}{k} \binom{\ell-k}{k'} (-1)^{k'} x^{k+k'}. \]

The claim therefore reduces to the following combinatorial statement:

(38) \[ \binom{n+i-1}{n} = \sum_{k+k'=n} \binom{\ell+i}{k} \binom{\ell-k}{k'} (-1)^{k'} \] for \( \ell \geq n. \)

If \( k+k' = n \) however, we have:

\[ \binom{\ell-k}{k'} = (-1)^{k'} \binom{n-\ell-1}{k'}, \]

and (38) reduces to:

\[ \binom{n+i-1}{n} = \sum_{k+k'=n} \binom{\ell+i}{k} \binom{n-\ell-1}{k'} \] for \( \ell \geq n, \)

a simple identity (also known as the Vandermonde identity). \( \square \)

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