An exactly solvable record model for rainfall

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Abstract

Daily precipitation time series are composed of null entries corresponding to dry days and nonzero entries that describe the rainfall amounts on wet days. Assuming that wet days follow a Bernoulli process with success probability $p$, we show that the presence of dry days induces negative correlations between record-breaking precipitation events. The resulting non-monotonic behavior of the Fano factor of the record counting process is recovered in empirical data. We derive the full probability distribution $P(R, n)$ of the number of records $R_n$ up to time $n$, and show that for large $n$, its large deviation form coincides with that of a Poisson distribution with parameter $\ln(pn)$. We also study in detail the joint limit $p \to 0$, $n \to \infty$, which yields a random record model in continuous time $t = pn$. 


An important and widely recognized consequence of global climate change is an increase in the frequency of extreme weather conditions such as heat waves, droughts and heavy precipitation [1–5]. The public perception of weather extremes is particularly sensitive to record-breaking events, which often receive extensive media coverage. At the same time the analysis of records provides a useful tool for the distribution-free inference of trends in time series, because the temporal record statistics of sequences of independent random variables drawn from a continuous probability distribution is manifestly universal [6–15]. This observation has motivated a number of recent studies aimed at detecting and quantifying the effects of a warming climate on the frequency of temperature records [16–23].

In comparison, the effects of climatic trends on precipitation records are more complex and have generally received less attention [24–26]. In order to detect such trends, the null model describing a stationary climate has to account for the specific structure of precipitation time series. In contrast to temperature, which is well described as a continuous random variable with a Gaussian distribution [18, 23], the amount of daily rainfall at a specific location has a positive probability of being exactly zero. Stochastic precipitation models incorporate this basic feature by combining an occurrence process that determines whether a given day is dry (zero precipitation) or wet (nonzero precipitation) with an amount process that specifies the amount of rainfall on a wet day [27].

In this Letter we show that the presence of dry days has a profound effect on the occurrence statistics of precipitation records in a stationary climate. Assuming that the wet days follow a Bernoulli process with success probability $p$, we find that record events become negatively correlated when $p < 1$. This is in marked contrast to the well-known property of record events from sequences of independent, identically and continuously distributed (i.i.c.d.) random variables to be stochastically independent [8, 13–15]. As a consequence, the ratio of the variance and the mean of the record counting process, known as the Fano factor, displays a minimum at intermediate times when $q = 1 – p$ is sufficiently large. This minimum is an unequivocal signature of correlations between record events, and we demonstrate that it can be clearly identified in empirical data. For this comparison we use time series comprising rainfall amounts on a given calendar day over several decades, which justifies the assumption of uncorrelated occurrence and amount processes. We expect that the mechanism giving rise to correlations in the Bernoulli model is of broader relevance also beyond the specific context of precipitation records, and provide a detailed analysis of the
model including the full distribution of the number of records.

**Bernoulli model.** Within the Bernoulli model a dry day with zero precipitation occurs with probability \( q \), and a wet day with probability \( p = 1 - q \). For a wet day, the amount of precipitation \( x \) is a random variable drawn from a continuous probability density \( p_W(x) \) with support on the positive real axis. The full probability density of precipitation \( x_n \) on day \( n \) thus reads

\[
p(x) = q \delta(x) + (1 - q) p_W(x). \tag{1}
\]

The \( \delta \)-function at \( x = 0 \) implies that the corresponding cumulative distribution function

\[
P(x) = \int_0^x dx' p(x') = q \theta(x) + (1 - q) P_W(x) \tag{2}
\]

is discontinuous at the origin, as indicated by the Heaviside theta function. We are interested in the statistics of the number of record events \( R_n \) that have occurred up to time \( n \). It is convenient to introduce a binary indicator variable \( \sigma_m \) for the \( m \)-th day such that \( \sigma_m = 1 \) if a record occurs on the \( m \)-th day, and \( \sigma_m = 0 \) otherwise. Clearly

\[
R_n = \sum_{m=1}^n \sigma_m. \tag{3}
\]

We note one important point: If a record occurs on the \( m \)-th day, then the \( m \)-th day is necessarily wet.

The mean number of records is given by

\[
\langle R_n \rangle = \sum_{m=1}^n \langle \sigma_m \rangle = \sum_{m=1}^n r_m \tag{4}
\]

where the record rate \( r_m \) denotes the probability that a record occurs on the \( m \)-th day. The latter is given by

\[
r_m = (1 - q) \int_0^\infty dx p_W(x) P(x)^{m-1}, \tag{5}
\]

with the following interpretation: The probability that the \( m \)-th day is a wet day with precipitation \( x > 0 \) is \( (1 - q) p_W(x) \), and in order for this to be a record all the previous \( (m - 1) \) days must have precipitation less than \( x \). To perform the integral we make the substitution \( x \to u = P(x) \), noting that \( u \in [q, 1] \) and \( du = (1 - q) p_W(x) dx \) for \( x > 0 \). The resulting expression

\[
r_m = \int_q^1 du u^{n-1} = \frac{1 - q^m}{m} \tag{6}
\]
is independent of the distribution $p_W(x)$ and reduces to the classic result $r_m = 1/m$ for i.i.c.d. random variables when $q \to 0$. Correspondingly, the mean number of records up to day $n$ is given by

$$\langle R_n \rangle = \sum_{m=1}^{n} \frac{1 - q^m}{m}. \quad (7)$$

For large $n$ and fixed $q = 1 - p$, it is easy to show that $\langle R_n \rangle \approx \ln(pn) + \gamma_E$, where $\gamma_E = 0.57721...$ is the Euler constant (see [28] for details). Thus at late times the record sequence looks like a ‘diluted’ i.i.c.d. record process where the effective number of random variables that have been presented up to time $n$ is reduced by a factor $p$. We will see below that this observation applies also to the variance as well as to the full distribution of $R_n$.

To compute the second moment of $R_n$, we square and average Eq. (3), using that $\sigma_m^2 = \sigma_m$. This gives

$$\langle R_n^2 \rangle = \langle R_n \rangle + 2 \sum_{l_1=1}^{n-1} \sum_{l_2=1}^{n-l_1} \langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle, \quad (8)$$

where $\langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle$ is the joint probability of two records occurring on day $l_1$ and $l_1 + l_2$. To compute this, let the record at day $l_1$ have value $x_1$ and the one at $l_1 + l_2$ have value $x_2$ with $x_2 > x_1$. Evidently, both days have to be necessarily wet. All the days before $l_1$ must have precipitation values less than $x_1$, and all the days between $l_1$ and $l_1 + l_2$ must have precipitation values less than $x_2$. Writing down the corresponding probability in analogy to Eq. (5) and performing the substitution $x \to P(x)$ (see [28]) leads to the simple form

$$\langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle = \int_q^1 du_2 \int_q^{u_2} du_1 u_1^{l_1-1} u_2^{l_2-1} \quad (9)$$

with $l_2 \geq 1$. For $l_2 = 0$, $\langle \sigma_{l_1}^2 \rangle = r_{l_1}$. Combining Eq. (9) with the result (6) for the record rate yields the connected correlation function of record events,

$$g_{l_1,l_1+l_2} \equiv \langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle - r_{l_1} r_{l_1+l_2} = -\frac{q^{l_1}}{l_1} \int_q^1 du u^{l_2-1} (1 - u^{l_1}) = -\frac{q^{l_1}}{l_1} \left( \frac{1 - q^{l_2}}{l_2} - \frac{1 - q^{l_1+l_2}}{l_1 + l_2} \right) \quad (10)$$

which is universal (independent of $p_W(x)$) for all $l_1 \geq 1$ and $l_2 \geq 1$. The second equality in Eq. (10) manifestly shows that the correlation is negative for all $l_1, l_2$ and $0 < q < 1$. Thus the record events become anticorrelated when $q > 0$. The origin of these correlations ultimately lies in the discontinuity of the distribution function (2), which reduces the domain of integration in Eqs. (6) and (9) compared to the i.i.c.d. case. We are however not aware of
any intuitive explanation for why the correlations are negative. Moreover, for fixed \( l_1 \) and large \( l_2 \), the connected correlation function decays as a power law, \( g_{l_1, l_1+l_2} \sim -q^{l_1}/[l_2(l_1 + l_2)] \sim l_2^{-2} \). This indicates that the record breaking events are rather strongly correlated.

Inserting Eq. (9) into Eq. (8) and performing the double sum yields, after a substantial amount of algebra (see [28] for details), the expression

\[
V_n(q) = \langle R_n^2 \rangle - \langle R_n \rangle^2 = \langle R_n \rangle + 2 \int_0^t du \int_v^1 \frac{1 - v^n}{1 - u} \left[ \int_q^u dv \frac{1 - v^n}{1 - v} - \int_q^{1/u} dv \frac{1 - v^n}{1 - v} \right]
\]

for the variance of the number of records up to day \( n \). Asymptotically for large \( n \) with fixed \( q = 1 - p \), it can be shown [28] that \( V_n(q) \rightarrow \langle R_n \rangle - \pi^2/6 \approx \ln(pn) + \gamma_E - \pi^2/6 \).

**Random record model.** In order to arrive at a more tractable expression for \( V_n \), we now analyze the problem in the scaling limit \( p \rightarrow 0, n \rightarrow \infty \) at fixed \( t = pn \). In this limit the Bernoulli sequence of wet days becomes a Poisson process of unit intensity in continuous time \( t \). In the mathematical literature this setting is known as the random record model [8, 30, 31], see also [32, 33]. For the mean number of records (7) the limit \( q \rightarrow 1, n \rightarrow \infty \) yields \( \langle R_n \rangle \rightarrow \mu(t) \) with

\[
\mu(t) = \int_0^t dy \frac{1 - e^{-y}}{y} = \ln t + \gamma_E + \int_t^\infty \frac{e^{-z}}{z} \, dz.
\]

The asymptotic behaviors of \( \mu(t) \) are \( \mu(t) \rightarrow t - t^2/4 \) as \( t \rightarrow 0 \) and \( \mu(t) \rightarrow \ln t + \gamma_E \) as \( t \rightarrow \infty \). Thus, the scaling function describes a crossover in the mean number of records from an early time linear growth \( \langle R_n \rangle \approx pn \) where the number of records is limited by the number of events, to a late time logarithmic growth \( \langle R_n \rangle \approx \ln(pn) + \gamma_E \). Taking the scaling limit of the expression (11) is not straightforward, but eventually leads to the relatively simple form (see [28])

\[
V_n(q) \rightarrow \mu(t) + 2 \int_0^t \frac{dz}{z} e^{-z} [\mu(t) - \mu(z) - \mu(t-z)]
\]

where \( \mu(t) \) is given in Eq. (12).

**Fano factor.** To quantify the correlations between record events, it is useful to introduce the Fano factor [29] defined as the ratio of the variance to the mean of the record counting process, \( F_n = \frac{V_n}{\langle R_n \rangle} \). We first prove that \( F_n \), for an arbitrary time-series, must be an increasing function of \( n \) if record events are uncorrelated. Let \( \langle \sigma_m \rangle = r_m \) denote the record rate at step \( m \) of the time-series. In the absence of correlations between record events, \( \langle \sigma_m \sigma_m \rangle = \ldots \)
\[ r_m \delta_{l,m} + r_l r_m (1 - \delta_{l,m}), \] which implies using (8) that
\[ V_n = \sum_{m=1}^{n} r_m (1 - r_m). \] (14)

As a consequence
\[ F_{n+1} - F_n = \frac{S_n}{\langle R_n \rangle} - \frac{S_{n+1}}{\langle R_{n+1} \rangle}, \] (15)
where \( S_n = \sum_{m=1}^{n} r_m^2 \). Based on this relation it is easy to show that \( F_{n+1} - F_n > 0 \) provided \( r_{n+1} < r_m \) for all \( m \leq n \), which only requires the record rate to be monotonically decreasing. Thus a non-monotonic behavior of \( F_n \) is an unambiguous signature of correlations.

![FIG. 1. (Color online) The Fano factor of the record process obtained from simulations (symbols) is compared to the analytic limit function \( F(t) \) in Eq. (16) (full line). Note that the numerical estimates start at \( F_1 = 1 - p \).](image)

Using the results from Eqs. (12) and (13), we find that in the scaling limit the Fano factor converges to the scaling form, \( F_n(q) \rightarrow F(t = p n) \) with
\[
F(t) = 1 + \frac{2}{\mu(t)} \int_0^t \frac{dz}{z} e^{-z} [\mu(t) - \mu(z) - \mu(t - z)].
\]
The scaling function \( F(t) \) is clearly non-monotonic, showing that the strong correlations between record events persist in the scaling limit (Fig. 1). It starts at \( F(0) = 1 \), decreases with increasing \( t \), reaches a minimum around \( t^* \approx 4.4 \), and converges slowly back to \( F = 1 \).
FIG. 2. (Color online) Blue squares show the Fano factor of precipitation records estimated from daily rainfall amounts at 144 German weather stations. For comparison, the full line shows simulation results obtained from the Bernoulli model with the average rainfall probability $p = 0.5$. 

as $t \to \infty$. Its asymptotic behaviors can be easily computed from the exact expression in Eq. (16), and we obtain $F(t) \to 1 - t/2 + O(t^2)$ as $t \to 0$ and $F(t) \to 1 - \pi^2/(6 \ln t)$ as $t \to \infty$. The figure also shows estimates for $F_n$ at finite $p > 0$ obtained from simulations. It can be seen that the minimum is even more pronounced at positive $p$, and the simulation results are indistinguishable from the asymptotic prediction (16) for $p = 0.02$.

**Comparison to precipitation data.** In order to test the predictions of the Bernoulli model we analyzed a large set of daily precipitation data compiled by the German weather service (DWD). The full data set comprises rainfall amounts from 5400 weather stations positioned throughout Germany. Out of these, 417 stations were selected which provided complete daily precipitation time series for the period 1974-2013 [34]. The average rainfall probability for this data set is close to $p = 0.5$ with some variability between stations. In order to minimize the effects of the variability in $p$, we further restricted the analysis to those stations where the time-averaged precipitation probability lies in the interval $p \in [0.48, 0.52]$. This leaves 144 stations covering the 40 year period. For each station we extracted 365 time series corresponding to precipitation amounts on a given calendar day.
Figure 2 shows the Fano factor of the number of precipitation records obtained from the empirical data, compared to simulations of the Bernoulli model with \( p = 0.5 \). The simulation data were averaged over \( 5 \times 10^4 \) runs, which is close to the total number of empirical time series \((144 \times 365 = 52560)\). We have checked that allowing \( p \) to vary over the interval \([0.48, 0.52]\) in the simulations does not significantly affect the results. The empirically determined Fano factor displays a pronounced minimum and the overall shape is in good agreement with the model. The remaining discrepancy at longer times is probably not of a statistical nature and could be related to features that are ignored in the model, such as spatial correlations between weather stations or trends in the model parameters.

**Distribution of the number of records.** Having derived the mean and the variance of the record number \( R_n \), one may naturally investigate its full distribution \( P(R, n) = \text{Prob.}[R_n = R] \). Exploiting the renewal structure of the record process in the Bernoulli model, we were able to derive a compact exact expression for the double generating function (see [28])

\[
\sum_{n=0}^{\infty} \sum_{R=0}^{\infty} P(R, n) \lambda^R z^n = \frac{(1 - qz)^{\lambda - 1}}{(1 - z)^\lambda}.
\]

For \( q = 0 \), the right hand side reduces to \((1 - z)^{-\lambda}\), a known result for the i.i.d case [35–39]. From Eq. (16), one can in principle compute all the moments. Moreover, by analysing Eq. (16) for large \( n \) (with fixed \( p = 1 - q \) and \( R \geq 1 \)), we can show (see [28]) that \( P(R, n) \) converges to the Poisson distribution

\[
P(R, n) \approx \frac{1}{pn} \frac{(\ln(pn))^{R-1}}{(R-1)!}.
\]

We conclude that the record occurrence events become a Poisson process in ‘time’ \( \ln(pn) \) for large \( n \), as was observed previously for the i.i.d. case \( q = 0 \) [38–44].

Interestingly, in the limit \( R \to \infty, n \to \infty \), but keeping the ratio \( x = R/\ln(pn) \) fixed, the Poisson distribution in Eq. (17) admits a large deviation form

\[
P(R, n) \sim e^{-\ln(pn)\Phi(\frac{R}{\ln(pn)})}
\]

with an explicit rate function

\[
\Phi(x) = 1 - x + x \ln x; \quad x \geq 0.
\]

Let us remark that the large deviation form in Eq. (18) may look a bit unfamiliar. Typically in statistical physics problems one finds a large deviation principle of the form \( \sim \)
exp[−L \Phi (\frac{R}{L})], where L represents the ‘size’ of the system. In the present problem, the effective size L is not n, but rather the average number of records \( \langle R_n \rangle \sim \ln(pn) \). Similar ‘anomalous’ large deviation forms appeared before in the context of the distribution of the number of zero crossings of smooth Gaussian fields in a certain time interval (or equivalently in the distribution of the number of real roots of a class of random polynomials of degree n) \[40\,42\], and more recently in the distribution of entanglement in random quantum spin chains \[43\].

The rate function \[19\] is independent of q. Typical fluctuations of \( R_n \) are described by the quadratic approximation of the rate function \( \Phi(x) \) around its unique minimum at \( x^* = 1 \). Substituting this quadratic form in Eq. \[18\], we find that the typical fluctuations are described by a Gaussian with mean and variance \( \ln(pn) \). Thus despite the power law correlations between the indicator variables \( \sigma_m \), their sum \( R_n = \sum_{m=1}^{n} \sigma_m \) satisfies a central limit theorem.

**Conclusions.** Motivated by the statistics of rainfall, we have investigated a simple extension of the classic i.i.c.d. record problem where the non-negative random variables forming the time series take on the value zero with a positive probability \( q > 0 \). Our key finding is that this induces long-ranged correlations between record events, which lead to a pronounced minimum in the Fano factor of the record counting process. The emergence of correlations between record events has been observed previously, e.g., for records drawn from distributions that broaden \[11\] or shift \[12, 45\] in time, or as a consequence of rounding effects \[46\]. Taken together, these results highlight the fact that the stochastic independence between record events in the standard i.i.c.d. setting is a highly non-generic and fragile feature.

The comparison with the empirical data in Fig. \[2\] shows that the Bernoulli model qualifies as a null model for precipitation time series comprising daily rainfall amounts on a given calendar day over a sequence of years. However, the model clearly fails to describe time series of rainfall amounts on consecutive days, which are characterized by strongly correlated spells of dry and wet days. This kind of data can be modeled by an alternating renewal process, where dry and wet spell lengths are drawn independently from two different probability distributions \[27\]. The record occurrence statistics is then again universal with respect to the amount distribution \( p_W(x) \) but depends explicitly on the spell length distributions. Detailed results for this model will be reported elsewhere, focusing in particular on the consequences of heavy-tailed distributions of dry spells \[47\].
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Supplemental material

Asymptotic behavior of the average number of records

The average number of records up to time \( n \) in the Bernoulli model is given in Eq. (7) of the main text that reads

\[
\langle R_n \rangle = \sum_{m=1}^{n} \frac{1 - q^m}{m}.
\] (S1)

To find the leading asymptotic behavior for large \( n \) and any \( 0 \leq q \leq 1 \), we first note that as \( n \to \infty \)

\[
\sum_{m=1}^{n} \frac{1}{m} = \ln(n) + \gamma_E + O\left(\frac{1}{n}\right)
\] (S2)

where \( \gamma_E = 0.57721 \ldots \) is the Euler constant. Furthermore, for \( 0 \leq q \leq 1 \) and \( n \to \infty \) we have

\[
\sum_{m=1}^{n} \frac{q^m}{m} = \sum_{m=1}^{\infty} \frac{q^m}{m} - \sum_{m=n+1}^{\infty} \frac{q^m}{m} = -\ln(1-q) + O\left(q^{n+1}\right).
\] (S3)

Subtracting Eq. (S3) from (S2), one gets using \( p = 1-q \), the following asymptotic behavior as \( n \to \infty \)

\[
\langle R_n \rangle = \ln(p\,n) + \gamma_E + O\left(\frac{1}{n}\right)
\] (S4)

as announced after Eq. (7) of the main text.

Derivation of the Variance of \( R_n \)

In this section, we provide a derivation of the main results for the variance in Eqs. (11) and (13) of the main text. On the way, we also give a derivation of Eq. (9) of the main text.

We start from Eq. (8) of the main text that reads

\[
\langle R_n^2 \rangle = \langle R_n \rangle + 2 \sum_{m_1 < m_2} \langle \sigma_{m_1} \sigma_{m_2} \rangle
\]

\[
= \langle R_n \rangle + 2 \sum_{l_1=1}^{n-1} \sum_{l_2=1}^{n-l_1} \langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle
\] (S5)

where \( \langle R_n \rangle \) is given by Eq. [S1]. To compute the correlation function \( \langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle \), we note that it is simply

\[
\langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle = \text{Prob.} \left[ \text{a record happens at day } l_1 \text{ and a record happens at day } l_1 + l_2 \right].
\] (S6)
To compute this joint probability, let the record at day $l_1$ have value $x_1$ and the one at $l_1 + l_2$ value $x_2$ with $x_2 > x_1$. Evidently, both days have to be necessarily wet. All the days before $l_1$ must have precipitation values less than $x_1$. In addition, all the days between $l_1$ and $l_1 + l_2$ must have precipitation values less than $x_2$ (this is needed if $x_2$ is a record). Hence, using the independence of days and knowing that the $l_1$-th day and the $l_1 + l_2$-th days are necessarily wet, we get

$$\langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle = \int_{\delta}^{\infty} dx_2 \int_{\delta}^{x_2} dx_1 [(1-q) p_W(x_2)] \left( \int_0^{x_2} p(x') dx' \right)^{l_2-1} [(1-q) p_W(x_1)] \left( \int_0^{x_1} p(x') dx' \right)^{l_1-1},$$

(S7)

where $p(x) = q \delta(x) + (1-q) p_W(x)$ is given in Eq. (1) of the main text. Note that we have introduced, for convenience, a lower cut-off $\delta$ in the integrals over $x_1$ and $x_2$. This is to indicate that a record occurs only on a wet day where the precipitation is strictly positive, i.e., the distribution $p_W(x)$ has support only over $x \in [\delta, \infty]$ with $\delta \to 0^+$. Thus we will keep this cut-off $\delta$ in the $x$-integrals and eventually take the limit $\delta \to 0^+$.

To proceed further, we make the change of variable

$$u = \int_0^{x} p(x') dx' = q \theta(x) + (1-q) \int_{\delta}^{x} p_W(x') dx'.$$

(S8)

Consequently, the complicated integral in Eq. (S7), upon taking $\delta \to 0^+$ limit, simplifies nicely to yield

$$\langle \sigma_{l_1} \sigma_{l_1+l_2} \rangle = \int_{u}^{1} du_2 \int_{u}^{u_2} du_1 u_1^{l_1-1} u_2^{l_2-1}$$

(S9)

valid for all $l_1 \geq 1$ and $l_2 \geq 1$. This then provides a derivation of Eq. (9) in the main text. As in the case of the mean $r_m = \langle \sigma_m \rangle$ in Eq. (6) of the main text, this two point correlation function is also universal, i.e., independent of $p_W(x)$.

Plugging Eq. (S9) into Eq. (S5) gives

$$\langle R_n^2 \rangle = \langle R_n \rangle + 2 \sum_{l_1=1}^{n-1} \sum_{l_2=1}^{n-l_1} \int_{q}^{1} du_2 \int_{q}^{u_2} du_1 u_1^{l_1-1} u_2^{l_2-1}.$$  

(S10)

We first perform the sum over $l_2$ which is a simple geometric series and obtain

$$\langle R_n^2 \rangle = \langle R_n \rangle + 2 \sum_{l_1=1}^{n-1} \int_{q}^{1} du_2 \left[ \frac{1 - u_2^{n-l_1}}{1 - u_2} \right] \int_{q}^{u_2} du_1 u_1^{l_1-1}.$$  

(S11)

Next we note that the sum over $l_1$ from 1 to $n-1$ can be extended up to $l_1 = n$, since the $l_1 = n$ term is identically 0. This step turns out to be rather convenient. Hence

$$\langle R_n^2 \rangle = \langle R_n \rangle + 2 \sum_{l_1=1}^{n} \int_{q}^{1} du_2 \left[ \frac{1 - u_2^{n-l_1}}{1 - u_2} \right] \int_{q}^{u_2} du_1 u_1^{l_1-1}. $$  

(S12)
Finally, performing the geometric sum over \( l_1 \) gives

\[
\langle R_n^2 \rangle = \langle R_n \rangle + 2 \int_q^1 \frac{du_2}{1-u_2} \int_q^{u_2} du_1 \left[ \frac{1-u_1^n}{1-u_1} - u_2^{n-1} \frac{1-(u_1/u_2)^n}{1-u_1/u_2} \right]
\]

\[
= \langle R_n \rangle + 2 \int_q^1 \frac{du_2}{1-u_2} \int_q^{u_2} du_1 \left[ \frac{1-u_1^n}{1-u_1} \right] - 2 \int_q^1 \frac{du_2}{1-u_2} \int_q^{u_2} du_1 \left[ \frac{1-(u_1/u_2)^n}{1-u_1/u_2} \right]
\]

\[
= \langle R_n \rangle + T_2 - T_3 \tag{S13}
\]

One can further simplify the term \( T_2 \) in Eq. \( \text{(S13)} \) in the following way

\[
T_2 = 2 \int_q^1 \frac{du_2}{1-u_2} \int_q^{u_2} du_1 \left[ \frac{1-u_1^n}{1-u_1} \right]
\]

\[
= 2 \int_q^1 \frac{du_2}{1-u_2} \left[ 1 - u_2^n + u_2^n \right] \int_q^{u_2} du_1 \left[ \frac{1-u_1^n}{1-u_1} \right]
\]

\[
= 2 \int_q^1 \frac{du_2}{1-u_2} (1 - u_2^n) \int_q^{u_2} du_1 \left[ \frac{1-u_1^n}{1-u_1} \right] + 2 \int_q^1 \frac{du_2}{1-u_2} \int_q^{u_2^n} du_1 \left[ \frac{1-u_1^n}{1-u_1} \right]
\]

\[
= T_{21} + T_{22} \tag{S14}
\]

The term \( T_{21} \) can be exactly integrated by a change of variable: \( z_2 = \int_q^{u_2} du_1 (1-u_1^n)/(1-u_1) \), yielding

\[
T_{21} = 2 \int_q^1 \frac{du_2}{1-u_2} (1 - u_2^n) \int_q^{u_2} du_1 \left[ \frac{1-u_1^n}{1-u_1} \right] = 2 \int_0^{\langle R_n \rangle} dz_2 z_2 = \langle R_n \rangle^2 \tag{S15}
\]

where we have used the following fact

\[
\int_q^1 du_1 \frac{1-u_1^n}{1-u_1} \int_q^1 du_1 \sum_{m=0}^{n-1} u_1^m = \sum_{m=1}^{n} \frac{1-q^n}{m} = \langle R_n \rangle . \tag{S16}
\]

In the final line, we have used the result for the mean number of records in Eq. (7) of the main text. Now, we consider the term \( T_3 \) in Eq. \( \text{(S13)} \). Making the change of variable \( u_1 = u_2 u_1' \), we get

\[
T_3 = 2 \int_q^1 \frac{du_2}{1-u_2} u_2^{n-1} \int_q^{u_2} du_1 \left[ \frac{1-(u_1/u_2)^n}{1-u_1/u_2} \right]
\]

\[
= 2 \int_q^1 \frac{du_2}{1-u_2} u_2^{n} \int_{u_2}^{u_2'} du_1' \left[ \frac{1-(u_1')^n}{1-u_1'} \right] \tag{S17}
\]

Putting all the terms together, we finally get a relatively compact expression for the variance

\[
V_n(q) = \langle R_n^2 \rangle - \langle R_n \rangle^2 = \langle R_n \rangle + 2 \int_q^1 \frac{du_2}{1-u_2} u_2^{n} \left[ \int_q^{u_2} du_1 \frac{1-u_1^n}{1-u_1} - \int_{q/u_2}^{1} du_1 \frac{1-u_1^n}{1-u_1} \right]
\]

\[
= \langle R_n \rangle + J_n(q) \tag{S18}
\]
Upon changing $u_2 \to u$ and $u_1 \to v$, we have
\[ J_n(q) = 2 \int_q^1 \frac{du}{1-u} \left[ \int_q^u dv \frac{1-v^n}{1-v} - \int_{q/u}^1 dv \frac{1-v^n}{1-v} \right] \tag{S19} \]
and Eq. (S18) reduces to Eq. (11) of the main text. Note that the result in Eq. (S18) is exact for any $n$ and any $0 \leq q \leq 1$.

**Asymptotic behavior of the variance for large $n$ and fixed $q$.** To find the asymptotic large $n$ behavior of $V_n(q)$ in Eq. (S18) for fixed $q$, we can use the asymptotic behavior of $\langle R_n \rangle$ given in Eq. (S4). It remains to estimate the large $n$ behavior of $J_n(q)$ in Eq. (S19). We first show that $J_n(q) \to -\pi^2/6$ as $n \to \infty$, for any fixed $0 \leq q \leq 1$. To demonstrate this, it is first convenient to make a change of variable $u' = 1 - u$ and $v' = 1 - v$ in Eq. (S19), which then reads using $p = 1 - q$
\[ J_n(q) = 2 \int_0^p \frac{du'}{u'} (1 - u')^n \left[ \int_{u'}^p \frac{dv'}{v'} (1 - (1 - v')^n) - \int_0^{(p-u')/(1-u')} \frac{dv'}{v'} (1 - (1 - v')^n) \right]. \tag{S20} \]

Next, we make a rescaling $u' = u/n$ and $v' = v/n$ to rewrite $J_n(q)$ as
\[ J_n(q) = 2 \int_0^{pn} \frac{du}{u} \left( 1 - u/n \right)^n \left[ \int_{u/n}^{pn} \frac{dv}{v} (1 - (1 - v/n)^n) - \int_0^{(pn-u)/(1-u/n)} \frac{dv}{v} (1 - (1 - v/n)^n) \right]. \tag{S21} \]

It is now convenient to take the $n \to \infty$ limit in Eq. (S21) for fixed $q = 1 - p$, which then reduces to a constant independent of $q$
\[ J_n(q) \to -2 \int_{0}^{\infty} \frac{du}{u} e^{-u} \left[ \int_{u}^{\infty} \frac{dv}{v} (1 - e^{-v}) - \int_{0}^{\infty} \frac{dv}{v} (1 - e^{-v}) \right] \]
\[ = -2 \int_{0}^{\infty} \frac{du}{u} e^{-u} \int_{u}^{\infty} \frac{dv}{v} (1 - e^{-v}). \tag{S22} \]

To evaluate this constant, we use the power series expansion,
\[ \frac{1 - e^{-v}}{v} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{v^{k-1}}{k!}. \tag{S23} \]

Hence,
\[ \int_{u}^{\infty} \frac{dv}{v} (1 - e^{-v}) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{u^k}{k!} \tag{S24} \]

Substituting (S24) in Eq. (S22) and carrying out the integral over $u$ gives, using the identity $\int_{0}^{\infty} du e^{-u} u^{k-1} = \Gamma(k) = (k-1)!$
\[ J_n(q) \to -2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} u^k}{k!} = -\frac{\pi^2}{6}. \tag{S25} \]
Hence, using Eqs. (S18), (S4) and (S25), we obtain the two leading terms of the variance $V_n(q)$, for large $n$ and fixed $q$

$$V_n(q) = \ln(pn) + \gamma_E - \frac{\pi^2}{6} + O\left(\frac{1}{n}\right).$$

**(S26)**

**Asymptotic behavior of the variance in the random record limit.** We now analyse the variance $V_n(q)$ in Eq. (S18) in the ‘random record’ model, i.e., in the scaling limit, where $n \to \infty$, $p = 1 - q \to 0$, with the product $t = pn$ fixed. To derive this scaling behavior, it is convenient to first make a change of variables $u_1 = 1 - v_1$ and $u_2 = 1 - v_2$ in the integral $J_n(q)$ in Eq. (S18). This gives

$$J_n(q) = 2 \int_0^p \frac{dv_2}{v_2} (1 - v_2)^n \times \left[ \int_0^p \frac{dv_1}{v_1} (1 - (1 - v_1)^n) - \int_0^{(p-v_2)/(1-v_2)} \frac{dv_1}{v_1} (1 - (1 - v_1)^n) \right].$$

**(S27)**

Next, we rescale $v_1 = p y_1$ and $v_2 = p y_2$ and take the scaling limit $n \to \infty$, $p \to 0$ with the product $t = pn$ fixed. In this limit, Eq. (S27) reduces to

$$J_n(q = 1 - p) \to 2 \int_0^1 \frac{dy_2}{y_2} e^{-t y_2} \left[ \int_0^1 \frac{dy_1}{y_1} (1 - e^{-t y_1}) - \int_0^{1-y_2} \frac{dy_1}{y_1} (1 - e^{-t y_1}) \right]$$

$$= 2 \int_0^1 \frac{dy_2}{y_2} e^{-t y_2} \left[ \int_0^t \frac{dz_1}{z_1} (1 - e^{-z_1}) - \int_0^{t-y_2} \frac{dz_1}{z_1} (1 - e^{-z_1}) \right]$$

$$= 2 \int_0^t \frac{dz_2}{z_2} e^{-z_2} \left[ \int_0^t \frac{dz_1}{z_1} (1 - e^{-z_1}) - \int_0^{z_2} \frac{dz_1}{z_1} (1 - e^{-z_1}) - \int_0^{t-z_2} \frac{dz_1}{z_1} (1 - e^{-z_1}) \right]$$

$$= 2 \int_0^t \frac{dz}{z} e^{-z} [\mu(t) - \mu(z) - \mu(t-z)].$$

**(S28)**

where in the last line, we used the definition $\mu(t) = \int_0^t dz (1 - e^{-z})/z$ from Eq. (12) of the main text. Thus finally, the variance $V_n(q)$ in Eq. (S18) can be expressed, in the scaling limit as

$$V_n(q) \to \mu(t) + 2 \int_0^t \frac{dz}{z} e^{-z} [\mu(t) - \mu(z) - \mu(t-z)].$$

**(S29)**

where $\mu(t)$ is given in Eq. (12) of the main text. This then provides the detailed derivation of the result stated in Eq. (13) of the main text.
Derivation of the distribution of the record number

In this section we derive the exact double generating function of the record number distribution quoted in Eq. (17) of the main text. We would like to compute the full distribution of the record number $R_n = \sum_{m=0}^{n} \sigma_m$ (given in Eq. (3) of the main text), i.e., the probability

$$P(R, n) = \text{Prob. } [R_n = R].$$

(S30)

It turns out that, while this representation $R_n = \sum_{m=0}^{n} \sigma_m$ in terms of the binary variables $\sigma_m$’s is useful for the computation of the mean and variance of $R_n$, it quickly becomes cumbersome for higher moments. Thus, calculating the full distribution $P(R, n)$ by this method seems rather complicated. Hence to compute the full distribution $P(R, n)$, we will use a different strategy. It turns out that it is convenient to consider a more general set of observables, namely the record number $R$ as well as the set of ages $\{l_0, l_1, l_2, \cdots, l_R\}$ of the successive records (see Fig. [S1]). The age $l_k$ of the $k$-th record is the number of steps between the occurrence of the $k$-th record and the next $(k+1)$-th record. Note that a record can happen necessarily on a wet day. We denote by $l_0$ the number of dry days before the first record, and $l_0$ can take values in the range $l_0 = 0, 1, 2, \cdots, n$. Similarly, $l_R$ denotes the age of the last record till the $n$-th step and hence $l_R = 0, 1, 2, \cdots, n$. The ages of the intermediate records (i.e., excluding $l_0$ and $l_R$) can take values, $l_k = 1, 2, \cdots, n$ for $1 \leq k \leq (R-1)$. Note that the record ages satisfy a sum rule

$$l_0 + l_1 + l_2 + \cdots + l_R + 1 = n.$$  

(S31)

Our goal is to (i) first write down the joint distribution of the record number $R$ and the record ages $\{l_0, l_1, \cdots, l_R\}$ and (ii) then integrate out the record ages to finally obtain the marginal distribution of the record number $P(R, n)$ only.

To proceed, we define $P(\vec{l}, R, n)$ as the joint distribution of the record ages $\vec{l} \equiv \{l_0, l_1, l_2 \cdots, l_R\}$ and the record number $R$ in $n$ steps. The marginal distribution of the record number only, i.e., $P(R, n)$ can then be obtained from this joint distribution by summing over the record ages

$$P(R, n) = \sum_{\vec{l}} P(\vec{l}, R, n).$$

(S32)

It turns out one can explicitly write down the joint PDF $P(\vec{l}, R, n)$ as follows. Let $\{x_1, x_2, \cdots, x_R\}$ denote the precipitation amounts on the record days, i.e., the record values.
FIG. S1. A typical configuration of the sequence, with black dots denoting dry days (no rainfall), blue vertical lines denoting the amount of rainfall on a ‘wet’ day and the red filled circles (at the top of a blue vertical line) denoting the record precipitation amounts. Let $R$ be the number of records in a sequence of $n$ steps. The sequence $\{l_0, l_1, l_2, \ldots, l_R\}$ denotes the ages of the records. Note that a record happens necessarily on a wet day. $l_0$ denotes the number of dry days before the first wet day, hence the range of $l_0$ is $l_0 = 0, 1, 2, \ldots, n$. Similarly, $l_R$ denotes the age of the last record before the step $n$ and the range of is $l_R = 0, 1, 2, \ldots, n$. For all intermediate ages (i.e., excluding $l_0$ and $l_R$), the range is $l_k = 1, 2, \ldots, n$ for for $k \neq 0$ and $k \neq R$.

Since they are successive records, we must have $x_1 < x_2 < x_3 < \cdots < x_R$. Also, these record occurrences must be wet days. Hence we can write the joint distribution $P(\vec{l}, R, n)$ as a nested integral

$$P(\vec{l}, R, n) = \int_{\delta}^{\infty} dx_R \left[ (1 - q)p_W(x_R) \left( \int_0^{x_R} p(x')dx' \right)^{l_R} \right] \times \int_{\delta}^{x_R} dx_{R-1} \left[ (1 - q)p_W(x_{R-1}) \left( \int_0^{x_R} p(x')dx' \right)^{l_{R-1}-1} \right] \cdots \times \cdots \int_{\delta}^{x_2} dx_1 \left[ (1 - q)p_W(x_1) \left( \int_0^{x_2} p(x')dx' \right)^{l_1-1} \right] q^{l_0} \delta_{l_0+l_1+\ldots+l_R+n} \quad (S33)$$

where $p(x) = q\delta(x) + (1 - q)p_W(x)$ is the effective PDF of precipitation given in Eq. (1) of the main text. Note again that the lower limit $\delta$ in each integration refers to the fact that $p_W(x)$ has support only over $x \in [\delta, \infty]$. Eventually we take the limit $\delta \to 0$. 

The result in Eq. (S33) can be understood as follows. Consider first the value \( x_R \) of the \( R \)-th record (the last one) (see Fig. S1). If \( x_R \) is a record, it has to be a wet day and hence the probability of its occurrence is \( (1 - q) p_W(x_R) \). Now, given that this is the last record, all \( l_R \) days following this must have values less that \( x_R \). The probability of this event is \( (\int_0^{x_R} p(x')dx')^{l_R} \), with \( l_R = 0, 1, 2, \cdots n \). Hence, the product \( \left[ (1 - q) p_W(x_R) \left( \int_0^{x_R} p(x')dx' \right)^{l_R} \right] \) explains the first factor in the first line of Eq. (S33).

Now consider the last but one record, i.e., \( x_{R-1} \). The probability of its occurrence is again \( (1 - q) p_W(x_{R-1}) \) and all the days between the \((R-1)\)-th record and the \( R \)-th record (and there are \((l_{R-1} - 1)\) such days) must have values less than \( x_{R-1} \) if \( x_R \) is a record. Hence, the product \( \left[ (1 - q) p_W(x_{R-1}) \left( \int_0^{x_{R-1}} p(x')dx' \right)^{l_{R-1}-1} \right] \) explains the second factor in the first line of Eq. (S33). Similarly one can proceed in a nested way. Finally, one needs to integrate over the record values \( \{x_1, x_2, \cdots, x_R\} \), but respecting the constraint \( x_1 < x_2 < x_3 < \cdots < x_R \). This explains the limits of the integrations. The last factor \( q^{l_0} \) denotes the probability that there are exactly \( l_0 \) dry days (each occurs with probability \( q \) independently) before the first wet day occurs (the first wet day is necessarily a record day with value \( x_1 \)). Finally, the record ages must satisfy the sum rule in Eq. (S31), explaining the Kronecker delta function in Eq. (S33).

To proceed, we first make the customary change of variables as in Eq. (S8), namely

\[
  u = \int_0^x p(x')dx' = q \theta(x) + (1 - q) \int_\delta^x p_W(x')\,dx'.
\]

With this change of variable and taking \( \delta \to 0^+ \) limit, the explicit dependence on \( p_W(x) \) disappears and Eq. (S33) transforms into

\[
  P(\vec{l}, R, n) = q^{l_0} \int_q^{u_R} du_k u_R^{l_R} \int_q^{u_{R-1}} du_{R-1} u_R^{l_{R-1}-1} \cdots \int_q^{u_2} du_1 u_1^{l_1-1} \delta_{l_0+l_1+\cdots+l_R+1,n}.
\]

Now, to get rid of the delta function constraint, we consider the generating function, i.e., we multiply both sides of Eq. (S35) by \( z^n \) and sum over \( n \), as well as over \( \vec{l} \). When we sum over \( \vec{l} \), we recall that while \( l_0 = 0, 1, 2, \cdots \), and \( l_R = 0, 1, 2, \cdots \), all other \( l_k = 1, 2, 3, \cdots \) (for \( k \neq 0 \) and \( k \neq R \)). This gives

\[
  \sum_{\vec{l}} \sum_{n=1}^{\infty} P(\vec{l}, R, n) z^n = \frac{1}{1 - qz} \int_q^z \frac{du_R}{1 - u_R z} \int_q^{u_R} \frac{du_{R-1}}{1 - u_{R-1} z} \cdots \int_q^{u_2} \frac{du_1}{1 - u_1 z}.
\]

This can be further simplified by making the change of variables, \( u_k z = v_k \), to give

\[
  \sum_{\vec{l}} \sum_{n=1}^{\infty} P(\vec{l}, R, n) z^n = \frac{1}{1 - qz} \int_{qz}^z \frac{dv_R}{1 - v_R} \int_{qz}^{v_R} \frac{dv_{R-1}}{1 - v_{R-1}} \cdots \int_{qz}^{v_2} \frac{dv_1}{1 - v_1}.
\]
This last nested integral can be computed explicitly as follows. Let us first rewrite Eq. (S37) as
\[
\sum_{\vec{l}} \sum_{n} P(\vec{l}, R, n) z^n = \frac{1}{1 - qz} W_R(z, z) \tag{S38}
\]
where we define the following nested integral
\[
W_R(x, z) = \int_{qz}^{x} \frac{dv}{1 - v} \int_{qz}^{v} \frac{dv_{R-1}}{1 - v_{R-1}} \cdots \int_{qz}^{v_2} \frac{dv_1}{1 - v_1}. \tag{S39}
\]
To evaluate \(W_R(x, z)\), we take the derivative of Eq. (S39) with respect to \(x\) for fixed \(z\). For simplicity of notation, we denote this derivative by an ordinary derivative and not a partial derivative (\(z\) can be thought of just a parameter in \(W_R(x, z)\)). We find that \(W_R(x, z)\) satisfies the recursion relation
\[
\frac{dW_R(x, z)}{dx} = \frac{1}{1 - x} W_{R-1}(x, z); \quad \text{for } R \geq 2 \tag{S40}
\]
starting from
\[
W_1(x, z) = \int_{qz}^{x} \frac{dv}{1 - v} = -\ln \left( \frac{1 - x}{1 - qz} \right). \tag{S41}
\]
We can now check easily that the solution of the recursion relation (S40), satisfying the initial condition in (S41) is given by
\[
W_R(x, z) = \frac{1}{R!} \left[ -\ln \left( \frac{1 - x}{1 - qz} \right) \right]^R. \tag{S42}
\]
Substituting this result (S42) for \(W_R(x = z, z)\) in Eq. (S38), we obtain our final result
\[
\sum_{n=1}^{\infty} P(R, n) z^n = \sum_{\vec{l}} \sum_{n=1}^{\infty} P(\vec{l}, R, n) z^n = \frac{W_R(z, z)}{1 - qz} = \frac{1}{1 - qz} \frac{1}{R!} \left[ -\ln \left( \frac{1 - z}{1 - qz} \right) \right]^R; \quad R \geq 1. \tag{S43}
\]
For \(R = 0\), we have \(P(0, n) = q^n\) since the probability of having no records is the same as the probability that all \(n\) days are dry. Hence,
\[
\sum_{n=0}^{\infty} P(0, n) z^n = \frac{1}{1 - qz}; \quad R = 0. \tag{S44}
\]
As a nontrivial check one can verify that \(P(R, n)\) is normalized to unity. Summing Eq. (S43) over all \(R = 1, 2 \ldots \) and Eq. (S44) for \(R = 0\), one obtains (using \(P(R, 0) = \delta_{R, 0}\) and a few minor steps of algebra)
\[
\sum_{n=0}^{\infty} \sum_{R=0}^{\infty} P(R, n) z^n = \frac{1}{1 - z}. \tag{S45}
\]
indicating that \( \sum_{R=0}^{\infty} P(R, n) = 1 \). Furthermore, by taking the derivative of Eq. (S43) with respect to \( z \) and setting \( z = 1 \), one can show that one recovers the result for the mean given in Eq. (7) of the main text. Similarly, taking derivatives twice with respect to \( z \) and setting \( z = 1 \), one recovers, after straightforward algebra, the result for the second moment in Eq. (11) of the main text, obtained by a different method (using correlations of the \( \sigma_m \)’s).

Furthermore, multiplying Eq. (S43) by \( \lambda^R \) and summing over \( R = 1, 2, 3, \ldots \), one gets

\[
\sum_{R=1}^{\infty} \sum_{n=1}^{\infty} P(R, n) \lambda^R z^n = \frac{1}{1 - qz} \sum_{R=1}^{\infty} \frac{\lambda^R}{R!} \left[-\ln \left(\frac{1 - z}{1 - qz}\right)\right]^R = \frac{(1 - qz)^{\lambda-1}}{(1 - z)^\lambda} - \frac{1}{1 - qz}.
\] (S46)

Including the terms corresponding to \( n = 0 \) and \( R = 0 \) (using \( P(0, n) = q^n \) and \( P(R, 0) = \delta_{R,0} \)) on the left hand side of Eq. (S46), we can finally write a compact expression for the double generating function

\[
\sum_{n=0}^{\infty} \sum_{R=0}^{\infty} P(R, n) \lambda^R z^n = \frac{(1 - qz)^{\lambda-1}}{(1 - z)^\lambda}.
\] (S47)

This completes the derivation of Eq. (17) in the main text.

Note that for \( q = 0 \), Eq. (S47) reduces to

\[
\sum_{n=0}^{\infty} \sum_{R=0}^{\infty} P_{q=0}(R, n) \lambda^R z^n = (1 - z)^{-\lambda}.
\] (S48)

This result for \( q = 0 \) was already known in the literature in a slightly different disguise. In fact, it is well known that the number of records \( R \) of \( n \) independent and identically (and continuously) distributed (i.i.c.d.) variables has the same statistical law as the number of cycles \( R \) in a random permutation of \( n \) elements [35,37]. This connection has also appeared in various statistical physics problems, such as in growth processes on networks [38] and in a class of one dimensional ballistic aggregation models [39]. The double generating function for the distribution of the number of cycles in random permutation of \( n \) elements was known to have the form in Eq. (S48) with \( R \) denoting the number of cycles. Thus our result for arbitrary \( 0 \leq q \leq 1 \) in Eq. (S47) provides a generalization of the \( q = 0 \) result in Eq. (S48). There is a precise combinatorial interpretation of our formula for general \( 0 \leq q \leq 1 \) in terms of the number of cycles in a random permutation of \( n \) elements. Indeed, consider a dilute version of the permutation problem, where each of the \( n \) elements is either present with probability \( p = 1 - q \), or absent with probability \( q \). Then, the number of ‘present’ elements
FIG. S2. The large deviation rate function $\Phi(x) = 1 - x - x \ln x$ plotted as a function of $x$. The rate function has a unique minimum at $x = x^* = 1$ around which it has a quadratic behavior, $\Phi(x) \approx (x - 1)^2/2$.

becomes a random variable with binomial distribution, and consequently the number of cycles $R$ of the random permutation of the 'present' elements is precisely our $P(R, n)$ with a general binomial parameter $p = 1 - q$ (see also Sect. ).

**Asymptotic behavior of $P(R, n)$ for large $n$**

In this section, we perform an asymptotic analysis of the double generating function in Eq. (S47) to derive the large $n$ behavior of $P(R, n)$, for general $q$. To proceed, it is first convenient to set $z = e^{-\mu}$ in Eq. (S47). Now, for large $N$, the most important contribution comes from the vicinity of $\mu = 0$ (or $z = 1$). Expanding the r.h.s. of Eq. (S47) for small $\mu$, one gets, to leading order in $\mu$,

$$
\sum_{n=1}^{\infty} \sum_{R=1}^{\infty} P(R, n) \lambda^R e^{-\mu n} \approx \frac{p^{\lambda-1}}{\mu^{\lambda}},
$$

where we used $q = 1 - p$. Note that since, to this leading order, the contribution from the pole at $z = 1/q$ in Eq. (S47) is neglected, we do not include the $R = 0$ term in the sum.
on the left hand side of Eq. (S47). Using the identity, \( \int_0^\infty n^{\lambda-1} e^{-\mu n} dn = \Gamma(\lambda) \mu^{-\lambda} \), one can then invert the Laplace transform with respect to \( n \) in Eq. (S49). This gives, for large \( n \),

\[
\sum_{R=1}^\infty P(R, n) \lambda^R \approx \frac{(pn)^{\lambda-1}}{\Gamma(\lambda)} = \frac{1}{pm} \frac{\lambda (pn)^\lambda}{\Gamma(1+\lambda)},
\]

(S50)

where we used the identity \( \Gamma(\lambda) = \Gamma(1+\lambda)/\lambda \). The next step is to expand the right hand side (rhs) of Eq. (S50) in a power series in \( \lambda \) and identify the coefficient of \( \lambda^R \). For this, we use

\[
(pn)^\lambda = \sum_{k=0}^\infty \frac{(\ln(pn))^k}{k!} \lambda^k
\]

(S51)

and the power series expansion

\[
\frac{1}{\Gamma(1+\lambda)} = \sum_{m=0}^\infty d_m \lambda^m
\]

(S52)

where \( d_0 = 1 \). Expanding the rhs of Eq. (S50) using (S51) and (S52) and identifying the power of \( \lambda^R \) gives, for fixed \( R \geq 1 \)

\[
P(R, n) \approx \frac{1}{pm} \sum_{m=0}^{R-1} \frac{\ln(pn)^{R-1-m}}{(R-1-m)!} d_m.
\]

(S53)

Finally, noticing that for large \( n \), the dominant contribution comes from the \( m = 0 \) term in the rhs of Eq. (S53), we get for large \( n \) and fixed \( R \geq 1 \)

\[
P(R, n) \approx \frac{1}{pn} \frac{(\ln(pn))^{R-1}}{(R-1)!}
\]

(S54)

which is just a Poisson distribution with parameter \( \ln(pn) \). This provides the derivation of Eq. (18) of the main text. Finally, in the limit when both \( R \to \infty \) and \( \ln(pn) \to \infty \), but with the ratio \( x = R/\ln(pn) \) fixed, we can use Stirling formula to express the rhs of Eq. (S54) in a large deviation form

\[
P(R, n) \sim e^{-\ln(pn) \Phi(x)} = e^{-\ln(pn) \Phi\left(\frac{R}{\ln(pn)}\right)}
\]

(S55)

where the rate function \( \Phi(x) \) is given by

\[
\Phi(x) = 1 - x + x \ln x,
\]

(S56)

as reported in Eqs. (19) and (20) of the main text. Interestingly, the rate function \( \Phi(x) \) is independent of \( q \). The \( q \) dependence appears only in renormalizing \( n \) to \( pn = (1-q)n \).
Indeed, even for the case \( q = 0 \) (i.i.c.d.), we are not aware of any result in the literature pointing out this explicit large deviation form. The rate function, plotted in Fig. S2, has a unique minimum at \( x^* = 1 \), where it has a quadratic behavior, \( \Phi(x) \approx (x - 1)^2/2 \). This means, from Eq. (S55), that \( P(R, n) \) is maximal near \( x = 1 \), i.e., at \( R = \ln(p n) \). Indeed, using the quadratic behavior near \( x = 1 \), we see that the typical fluctuations of \( R \) are described by a Gaussian form

\[
P(R, n) \approx \frac{1}{\sqrt{2\pi \ln(p n)}} e^{-\frac{(R-\ln(p n))^2}{2\ln(p n)}}
\]

with mean \( \ln(p n) \) and variance \( \ln(p n) \).

**An alternative derivation of Eq. (S47)**

There is an alternative way to compute the distribution of the record number \( P(R, n) \) in the Bernoulli model for arbitrary \( 0 \leq q \leq 1 \) (\( q \) being the probability that a dry day occurs), knowing already the result for the \( q = 0 \) case. Consider a sequence of \( n \) days, and let \( n_W \) denote the number of wet days, while \( n - n_W \) denotes the number of dry days. Given that a wet day occurs with probability \( p = 1 - q \), it follows that the number of wet days \( n_W \) has a binomial distribution

\[
Q(n_W, n) = \binom{n}{n_W} p^{n_W} q^{n-n_W} \quad \text{where} \quad n_W = 0, 1, 2, \ldots, n.
\]

(S58)

Now, a record can happen only on a wet day. Let \( \text{Prob}(R, n_W) \) denote the probability of having \( R \) records among \( n_W \) wet days. Thus, \( \text{Prob}(R, n_W) \) is just the record number distribution of the pure i.i.c.d. case (i.e., \( q = 0 \)). Hence we have

\[
\text{Prob}(R, n_W) = P_{q=0}(R, n_W)
\]

(S59)

where the double generating function of \( P_{q=0}(R, n_W) \) satisfies Eq. (S48), i.e.,

\[
\sum_{n_W=0}^{\infty} \sum_{R=0}^{\infty} P_{q=0}(R, n_W) \lambda^R \alpha^{n_W} = (1 - \alpha)^{-\lambda}.
\]

(S60)

Now, knowing \( P_{q=0}(R, n_W) \), it is clear that \( P(R, n) \) for fixed \( n \) and arbitrary \( q \) can be written simply as

\[
P(R, n) = \sum_{n_W=0}^{n} \text{Prob}(R, n_W) Q(n_W, n)
\]

\[
= \sum_{n_W=0}^{n} P_{q=0}(R, n_W) \binom{n}{n_W} p^{n_W} q^{n-n_W}.
\]

(S61)
Thus, basically it amounts to studying the record number distribution of just the i.i.d. case, albeit with a random number of \( n_W \) entries and one needs to average over \( n_W \).

To compute the double generating function of \( P(R, n) \) using the exact formula in Eq. (S61), it is useful to first formally invert Eq. (S60) with respect to \( u \) using Cauchy’s theorem. This gives

\[
\sum_{R=0}^{\infty} P_q = 0 (R, n_W) \lambda^R = \int_{C_0} \frac{du}{2\pi i} \frac{1}{u^{n_W+1}} (1 - u)^{-\lambda}
\]

where \( C_0 \) is any contour encircling the origin in the complex \( u \) plane. Now, multiplying Eq. (S62) by the binomial distribution \( Q(n_W, n) \) in Eq. (S58) and summing over \( n_W \), we get

\[
\sum_{n_W=0}^{n} Q(n_W, n) \sum_{R=0}^{\infty} P_q = 0 (R, n_W) \lambda^R = \int_{C_0} \frac{du}{2\pi i} \frac{1}{u} (1 - u)^{-\lambda} \sum_{n_W=0}^{n} \binom{n}{n_W} \left( \frac{p}{u} \right)^{n_W} q^{n-n_W}
\]

\[
= \int_{C_0} \frac{du}{2\pi i} \frac{1}{u} (1 - u)^{-\lambda} \left( \frac{p}{u} + q \right)^n .
\]

We next multiply Eq. (S63) by \( z^n \) and sum over \( n \). To ensure the convergence of the geometric series, we need to assume \( u > p z / (1 - q z) \) for a given \( z \). Indeed, we can do this by deforming the original contour \( C_0 \), such that it includes \( u = p z / (1 - q z) \) inside it. Once ensured of the convergence, summing over \( n \) we get, upon using Eq. (S61), the following identity

\[
\sum_{n=0}^{\infty} \sum_{R=0}^{\infty} P(R, n) \lambda^R z^n = \int_{C_0} \frac{du}{2\pi i} \frac{1}{u} (1 - u)^{-\lambda} \frac{1}{1 - z \left( \frac{p}{u} + q \right)}
\]

\[
= \frac{1}{1 - qz} \int_{C_0} \frac{du}{2\pi i} (1 - u)^{-\lambda} \frac{1}{u - \frac{p z}{1 - q z}} .
\]

Finally, noting that there is a simple pole at \( u = p z / (1 - q z) \), and since our deformed contour \( C_0 \) contains this pole inside it, the integral is just given by the residue at the pole \( u = p z / (1 - q z) \). This gives, using \( p + q = 1 \), the desired result

\[
\sum_{n=0}^{\infty} \sum_{R=0}^{\infty} P(R, n) \lambda^R z^n = \frac{(1 - qz)^{\lambda-1}}{(1 - z)^{\lambda}}
\]

which was derived before in Eq. (S47) using a completely different method exploiting the renewal structure of the underlying record process.