ON THE GLOBAL WELL-POSEDNESS TO THE 3-D NAVIER-STOKES-MAXWELL SYSTEM

GAOCHENG YUE*
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 211106, China

CHENGKUI ZHONG
Department of Mathematics, Nanjing University
Nanjing 210093, China

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Abstract. The present paper is devoted to the well-posedness issue of solutions of a full system of the 3-D incompressible magnetohydrodynamic (MHD) equations. By means of Littlewood-Paley analysis we prove the global well-posedness of solutions in the Besov spaces \( \dot{B}^{1/2}_{2,1} \times \dot{B}^{2}_{2,1} \times \dot{B}^{3}_{2,1} \) provided the norm of initial data is small enough in the sense that

\[
(\|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|E_0\|_{\dot{B}^{2}_{2,1}} + \|B_0\|_{\dot{B}^{3}_{2,1}}) \exp\left\{\frac{C_0}{\nu^2} \|u_0\|_{\dot{B}^{1/2}_{2,1}}^2\right\} \leq c_0,
\]

for some sufficiently small constant \( c_0 \).

1. Introduction. In this paper, we study the global well-posedness of the following three-dimensional full Magnetohydrodynamics (MHD) system:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p - \nu \Delta u &= j \times B, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t E - \text{curl} B &= -j, \\
\partial_t B + \text{curl} E &= 0, \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0, \\
\sigma (E + u \times B) &= j, \\
u|_{t=0} &= u_0, \quad B|_{t=0} = B_0, \quad E|_{t=0} = E_0,
\end{align*}
\]

(1.1)

where \( u = u(t, x) \), \( B = B(t, x) \) and \( E = E(t, x) \) are the fluid velocity, magnetic and electric fields, depending on the spatial position \( x \) and the time \( t \). The scalar functions \( p = p(t, x) \) denote the pressure. \( j \) is the electric current which is given by Ohm’s law. The positive constants \( \nu \) is the viscosity coefficients and \( \sigma \) is the electric resistivity. Here, \( u, E, B, j \) are defined on \( \mathbb{R}^3 \) and take their values in \( \mathbb{R}^3 \).

The first equation in (1.1) is the Navier-Stokes equation for incompressible flows with a Lorentz force term \( j \times B \) under a quasi-neutrality assumption of the net...
charge carried by the fluid. The second equation is the Ampere-Maxwell equation which includes here the displacement current $\partial_t E$ for an electric field $E$. The third equation is the Faraday’s law. The fifth equation is the Ohm’s law which states that the electric current is proportional to the electric field measured in a frame moving with the local velocity of the conductor. One may refer to [6] for a physical review of the background to magnetohydrodynamics.

Similar to the Navier-Stokes equations, by using the standard $L^2$ energy estimate, we have the following formal energy identity:

$$\frac{1}{2} \frac{d}{dt} \left[ \|u\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2 \right] + \|j\|_{L^2}^2 + \nu \|
abla u\|_{L^2}^2 = 0. \quad (1.2)$$

This identity shows that the $L^2$ energy is dissipated by the viscosity and the electric resistivity. However, a global finite energy weak solution with initial data lying in $(L^2(\mathbb{R}^d))^3$ remains an interesting open problem for the Navier-Stokes-Maxwell equations (1.1) in both $d = 2, 3$.

For the 2-D Navier-Stokes-Maxwell system, the global existence of regular solutions as well as an exponential growth estimate of the solution has been obtained by Masmoudi[11]. For the 3-D case, Ibrahim and Yoneda[10] proved the existence of local in time unique solution and the loss of smoothness via Fujita-Katos method, and the authors[9] investigated the existence and uniqueness of a global weak solution for the 2-D and 3-D Navier-Stokes-Maxwell equations by means of the Fourier localization technique and Bonys para-product decomposition. For the 2-D case, the authors introduce the space $L^2_{log}$ and $L^2_H$ to deal with low and high frequencies, respectively, which defined by

$$\|u\|_{L^2_{log}} \overset{def}{=} \sum_{j \leq 0} \|\Delta_j u\|_{L^2}^2 + \sum_{j > 0} j \|\Delta_j u\|_{L^2}^2 < \infty;$$

$$\|u\|_{L^2_H}^2 \overset{def}{=} \sum_{j \leq 0} 2^{2j} \|\Delta_j u\|_{L^2}^2 + \sum_{j > 0} \|\Delta_j u\|_{L^2}^2 < \infty.$$
with
\[ u \in C([0, T]; \dot{B}^{\frac{1}{2}}_{2,1}) \cap \dot{L}^1([0, T]; \dot{B}^{\frac{3}{2}}_{2,1}), \]
\[ (B, E) \in \dot{L}^\infty([0, T]; \dot{B}^{\frac{3}{2}}_{2,1}) \times \dot{L}^\infty([0, T]; \dot{B}^{\frac{3}{2}}_{2,1}) \cap \dot{L}^1([0, T]; \dot{B}^{\frac{3}{2}}_{2,1}). \]
Moreover, if
\[ \|u_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|B_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|E_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \leq \varepsilon_0 \]
for \( \varepsilon_0 \) small enough, then \( T = \infty \).

We can now state the main result of this paper.

**Theorem 1.2.** There exist positive constants \( c_0 \) and \( C_0 \) such that, for any data \((B_0, E_0) \in \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \times \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \) and \( u_0 = (u^h_0, u^3_0) \in \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \) verifying
\[ \eta \stackrel{\text{def}}{=} \left( \|u^h_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|E_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|B_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \right) \exp \left\{ \frac{C_0}{\nu^2} \|u^3_0\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} \right\} \leq c_0, \quad (1.3) \]
the system (1.1) has a unique global solution \((B, E) \in C([0, \infty); \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \times C([0, \infty); \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \) and \( u \in C([0, \infty); \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \cap \dot{L}^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \) and \( B \in C([0, \infty); \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \cap \dot{L}^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \). Moreover, there holds
\[ \|B\|_{\dot{L}^\infty(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})} + \|E\|_{\dot{L}^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})} + \|B^h\|_{\dot{L}^\infty(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})} + \|u^h\|_{\dot{L}^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})} \leq C \eta, \quad (1.4) \]
and
\[ \|u^3\|_{\dot{L}^\infty(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})} + \nu \|u^3\|_{\dot{L}^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1})} \leq \|u^3_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \nu \tilde{c}^2 + \frac{c^3}{\sigma}, \quad (1.5) \]
where \( \tilde{c} = \min \{\nu, \sigma\} \).

The rest of the paper unfolds as follows. In the next section, we introduce the main tool for the proof—the Littlewood-Paley decomposition—and some related functional spaces. In Section 3, we focus on the proof of the existence and uniqueness of a solution of (1.1). In section 4, we shall present the estimate to the pressure function. Finally in the last section, we shall complete the proof of Theorem 1.2. Let us end this section with the notations we are going to use in this paper.

- Throughout this paper, \( C \) represents some “harmless” constant, which can be understood from the context. In some places, we shall alternate use the notation \( A \lesssim B \) instead of \( A \leq CB \), and \( A \approx B \) means \( A \lesssim B \) and \( B \lesssim A \).
- If \( X \) is a Banach space, \( T > 0 \) and \( p \in [1, +\infty] \) then \( L^p_T(X) \) stands for the set of Lebesgue measurable functions \( f \) on \([0, T]\) to \( X \) such that \( t \to \|f(t)\|_X \) belongs to \( L^p([0, T]) \). If \( T = +\infty \), then the space is merely denoted by \( L^p(X) \).
- Throughout this paper, \( (d_j)_{j \in \mathbb{Z}} \) denotes a generic element of the space of \( \ell^1(\mathbb{Z}) \) so that \( d_j \geq 0 \) and \( \sum_{j \in \mathbb{Z}} d_j = 1 \).

2. Basic results on Besov spaces. In order to define Besov space, we need the following a dyadic decomposition of the Fourier space in the case \( x \in \mathbb{R}^d \), see [1, 4].
Let $\varphi$ be a smooth function supported in the ring $C = \{ \xi \in \mathbb{R} | 3/4 \leq |\xi| \leq 8/3 \}$ and such that
$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \text{for } \xi \neq 0.$$  

For $a \in \mathcal{S}'(\mathbb{R}^3)$, we set
$$\forall j \in \mathbb{Z}, \quad \hat{\Delta}_j u := \varphi(2^{-j} D)u, \quad \text{and} \quad \hat{S}_j u := \sum_{\ell \leq j-1} \hat{\Delta}_\ell u.$$  

Then we have the formal decomposition
$$u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u, \quad \forall u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3],$$
where $\mathcal{P}[\mathbb{R}^3]$ is the set of polynomials. Moreover, the dyadic operators satisfy the property of almost orthogonality:
$$\hat{\Delta}_k \hat{\Delta}_j u = 0 \quad \text{if } |k - j| \geq 2 \quad \text{and} \quad \hat{\Delta}_k (\hat{S}_{j-1} u \hat{\Delta}_j u) = 0 \quad \text{if } |k - j| \geq 5. \quad (2.1)$$

We recall now the definition of homogeneous Besov spaces from [1].

**Definition 2.1.** Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^3)$, we set
$$\|u\|_{\dot{B}^s_{p,r}} := \left(2^{js} \|\hat{\Delta}_j u\|_{L^p_r}\right)^{\frac{1}{r}}.$$  

For $s < \frac{3}{p}$ or $s = \frac{3}{p}$ if $r = 1$, we define $\dot{B}^s_{p,r}(\mathbb{R}^3) := \left\{ u \in \mathcal{S}'(\mathbb{R}^3) \mid \|u\|_{\dot{B}^s_{p,r}} < +\infty \right\}.$

From the above definition we know that for $s \in \mathbb{R}, 1 \leq p, r \leq \infty$, and $u \in \mathcal{S}'(\mathbb{R}^3),$ then $u$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^3)$ if and only if there exists $\{c_{j,r}\}_{j \in \mathbb{Z}}$ such that $c_{j,r} \geq 0,$ $\|c_{j,r}\|_{L^r} = 1$ and
$$\|\hat{\Delta}_j u\|_{L^p} \leq C_{j,r} 2^{-js} \|u\|_{\dot{B}^s_{p,r}} \quad \text{for all } j \in \mathbb{Z}.$$  

Similarly, we can also define the inhomogeneous Besov spaces. Indeed, let $\chi \in [0, 1]$ be a smooth function supported in the ball $\{\xi \in \mathbb{R}^3, \ |\xi| \leq \frac{3}{4}\}$ so that
$$\chi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) = 1 \quad \text{for } \forall \xi \in \mathbb{R}^3.$$  

For $u \in \mathcal{S}'(\mathbb{R}^3)$, we set
$$\forall j \geq 0, \quad \Delta_j u := \phi(2^{-j} D)u, \quad \Delta_{-1} = \chi(D)u; \quad \Delta_{-1} u = 0 \quad \text{if } j \leq -2; \quad \text{and} \quad S_j u := \sum_{\ell \leq j-1} \Delta_\ell u.$$  

Then for all $u \in \mathcal{S}'(\mathbb{R}^3)$, we have the inhomogeneous Littlewood-Paley decomposition $u = \sum_{j \in \mathbb{Z}} \Delta_j u,$ and for $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^3)$, we define the inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^3)$ as follows:
$$B^s_{p,r} := \left\{ u \in \mathcal{S}'(\mathbb{R}^3) \mid \|u\|_{B^s_{p,r}} = \left(2^{js} \|\Delta_j u\|_{L^p_r}\right)^{\frac{1}{r}} < \infty \right\}.$$  

We point out that if $s > 0$ then $B^s_{p,r}(\mathbb{R}^3) = \dot{B}^s_{p,r}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ and
$$\|u\|_{B^s_{p,r}} \approx \|u\|_{\dot{B}^s_{p,r}} + \|u\|_{L^p} \quad \text{and} \quad B^s_{p,r} \hookrightarrow \dot{B}^s_{p,r} \text{ with } p < \infty.$$  

We also define the following Chemin-Lerner type spaces, which were introduced by Chemin and Lerner [3].
Definition 2.2. (See [3].) Let \( s \leq \frac{3}{p} \) (or, in general, \( s \in \mathbb{R} \)), \((r, \lambda, p) \in [1, +\infty]^3\), and \( T \in (0, +\infty] \). We define the \( L^\lambda_T (\dot{B}^s_{p,r}(\mathbb{R}^3)) \)-norm by

\[
\|f\|_{L^\lambda_T (\dot{B}^s_{p,r})} \overset{\text{def}}{=} \left( \sum_{j \in \mathbb{Z}} 2^{qs} \left( \int_0^T \|\Delta_j f(t)\|_{L^p_{s}}^\lambda dt \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{q}} < \infty,
\]

With the usual change if \( r = \infty \). For short, we just denote this space by \( \dot{L}^\lambda_T (\dot{B}^s_{p,r}) \).

Similarly, we can define \( \dot{L}^\lambda_T (\dot{B}^s_{p,r}) \), which will be used in the subsequent sections.

By virtue of the Minkowski inequality, we have

\[
\|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})} \leq \|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})} \quad \text{if } r \geq \lambda,
\]

\[
\|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})} \geq \|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})} \quad \text{if } r \leq \lambda.
\]

In particular, we have

\[
\|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})} = \|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})} \quad \text{and } \|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})} \geq \|u\|_{\dot{L}^\lambda_T (\dot{B}^s_{p,r})}.
\]

As we shall repeatedly use the Littlewood-Paley theory in what follows, we list some basic facts here. Bernstein’s inequality is fundamental in the analysis involving Besov spaces. Please see the details in [2, 4].

Lemma 2.3. Let \( B \) be a ball and \( C \) a ring of \( \mathbb{R}^3 \). A constant \( C \) exists so that for any positive real number \( \lambda \), any nonnegative integer \( k \), any homogeneous function \( \sigma \) of degree \( m \) smooth outside of \( 0 \) and any couple of real numbers \((a, b)\) with \( b \geq a \geq 1 \), there hold

\[
\text{Supp} \hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^\lambda} \leq C^{k+1} \lambda^{k+3} (\frac{1}{a} - \frac{1}{b}) \|u\|_{L^\lambda},
\]

\[
\text{Supp} \hat{u} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^\lambda} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^\lambda} \leq C^{k+1} \lambda^k \|u\|_{L^\lambda},
\]

\[
\text{Supp} \hat{u} \subset \lambda C \Rightarrow ||\sigma(D)u||_{L^\lambda} \leq C_{\sigma,m} \lambda^{m+3} (\frac{1}{a} - \frac{1}{b}) \|u\|_{L^\lambda}.
\]

We shall use the following Bony’s decomposition in the homogeneous context:

\[
wv = Tu v + Tv u + Ru v,
\]

(2.2)

where

\[
T_u v = \sum_{k \in \mathbb{Z}} S_{k-1} u \Delta_k v, \quad R(u, v) = \sum_{k \in \mathbb{Z}} \Delta_k u \vec{\Delta}_k v, \quad \text{and } \vec{\Delta}_k v = \sum_{\ell=k-1}^{k+1} \Delta_{\ell} v.
\]

We first present the following property of Besov space, which is crucial to obtain the a priori estimate of the solution of (1.1).

Lemma 2.4. (See [13]) Let \( p_2 \geq p_1 \geq 1 \), and \( s_1 \leq \frac{3}{p_1}, s_2 \leq \frac{3}{p_2} \) with \( s_1 + s_2 > 3 \max(0, \frac{1}{p_1} + \frac{1}{p_2} - 1) \). Let \( a \in \dot{B}^{s_1}_{p_1,1}(\mathbb{R}^3), \) \( b \in \dot{B}^{s_2}_{p_2,1}(\mathbb{R}^3) \). Then \( ab \in \dot{B}^{s_1+s_2-\frac{3}{p_1}}{p_2,1} (\mathbb{R}^3) \), and

\[
\|ab\|_{\dot{B}^{s_1+s_2-\frac{3}{p_1}}{p_2,1}} \lesssim \|a\|_{\dot{B}^{s_1}_{p_1,1}} \|b\|_{\dot{B}^{s_2}_{p_2,1}}.
\]

In order to deal with the nonlinear coupling between the Navier-Stokes equations with a forcing induced by the magnetic field and the induction equation, we recall the following form of weighted Chemin-Lerner type norm from [12].
Definition 2.5. Let $f(t) \in L^1_{loc}(\mathbb{R}^+)$, $f(t) \geq 0$. We define

$$
\|u\|_{L^q_{T,f}(B^s_{r,p})} = \left( \sum_{q \in \mathbb{Z}} 2^{qs} \left( \int_0^T \|f(t)\|_{L^p} dt \right)^{\frac{r}{p}} \right)^{\frac{1}{s}}
$$

for $s \in \mathbb{Z}, p \in [1, \infty], q, r \in [1, \infty)$, and with the standard modification for $q = \infty$ or $r = \infty$.

We now state a standard estimate on the parabolic regularity that we will use in proving the existence of solutions of Navier-Stokes-Maxwell system (1.1).

Lemma 2.6. (See [1]) Let $u$ be a smooth divergence-free vector field solving

$$
\begin{cases}
\partial_t u - \Delta u + \nabla p = f, \\
u|_{t=0} = u_0,
\end{cases}
$$
on some time interval $[0, T]$. Then, for every $p \geq r \geq 1$ and $s \in \mathbb{R}$, we have

$$
\|u\|_{C([0,T];B^{s+\frac{3}{2}}_{r,p}) \cap L^2_t (L^r_x)} \lesssim \|u_0\|_{B^{s,1}_{r,p}} + \|f\|_{L^r_t L^r_x}.
$$

3. Proof of Theorem 1.1.

3.1. Local existence. We define by induction a sequence $\Gamma_n = (u_n, B_n, E_n)$ of smooth functions by solving the following linear equations:

$$
\begin{align*}
\partial_t u_{n+1} + \nabla p_{n+1} - \nu \Delta u_{n+1} &= -(u_n \cdot \nabla) u_n + j_n \times B_n, \\
\partial_t E_{n+1} - \text{curl } B_{n+1} &= j_n, \\
\partial_t B_{n+1} + \text{curl } E_{n+1} &= 0, \\
\nabla \cdot u_{n+1} &= 0, \\
\nabla \cdot B_{n+1} &= 0, \\
\sigma(E_n + u_n \times B_n) &= j_n,
\end{align*}
$$

with initial conditions

$$
u_{n+1}|_{t=0} = u_0, \quad B_{n+1}|_{t=0} = B_0, \quad E_{n+1}|_{t=0} = E_0.
$$

Let us set $u_1 = e^{\nu t}u_0$ and $B_1 = e^{\nu t}B_0$, $E_1 = e^{\nu t}E_0$, where $L$ is the operator define by $L(B, E) = (\text{curl } B, -\text{curl } E)$.

First Step. Uniform Bounds

We claim that the following estimates hold for some $T > 0$,

$$
\|u_{n+1}\|_{L^2_T(B^{\frac{1}{2}}_{2,1})} + \|B_{n+1}\|_{L^2_T(B^{\frac{1}{2}}_{2,1})} + \|E_{n+1}\|_{L^2_T(B^{\frac{3}{2}}_{2,1})} \leq (C + 1)\|\Gamma_0\|_{X_0},
$$

and

$$
\|u_{n+1}\|_{L^2_T(B^{\frac{1}{2}}_{2,1})} + \|E_{n+1}\|_{L^2_T(B^{\frac{3}{2}}_{2,1})} \leq (C + 1)\gamma_0, \quad n = 0, 1 \cdots .
$$

Indeed, if $\|u_0\|_{B^{\frac{1}{2}}_{2,1}} \leq C_0$, then for any small $\gamma_0$, there exists $T_0$, such that the following estimate holds

$$
\|e^{\nu t} u_0\|_{L^2_T(B^{\frac{1}{2}}_{2,1})} \leq \gamma_0, \quad \text{and } \|e^{\nu t} E_0\|_{L^2_T(B^{\frac{3}{2}}_{2,1})} \leq \gamma_0, \quad \text{for all } T \leq T_0.
$$
Thus, we find that (3.1) and (3.2) hold for $n = 0$. Assume (3.1) and (3.2) hold for $n - 1$. By using Lemma 2.4 and Lemma 2.6 we can choose $\gamma_0 \leq \frac{1}{(C + 1)^2 + (C + 1)}$ such that
\[
\|u_{n+1}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} + \|B_{n+1}\|_{L^p_T(B^{\frac{2}{3}}_{2,1})} + \|E_{n+1}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \\
\leq C\|\Gamma_0\|_B + 3(C + 1)^2 \gamma_0 \|\Gamma_0\|_B + (C + 1)^3 \gamma_0 \|\Gamma_0\|_B^2 \\
\leq (C + 1) \|\Gamma_0\|_B.
\]
Similarly, we have
\[
\|u_{n+1}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} + \|E_{n+1}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \\
\leq C\gamma_0 + t^2 \{3(C + 1)^2 \gamma_0 \|\Gamma_0\|_B + (C + 1)^3 \gamma_0 \|\Gamma_0\|_B^2 \} \\
\leq (C + 1) \gamma_0,
\]
by choosing $t \leq \min\{T_0, \frac{1}{(C + 1)^2 + (C + 1)} \|\Gamma_0\|_B \}$.

**Second Step. Convergence**

We are going to show that $\{\Gamma_n\}_{n \in \mathbb{N}} = \{(u_n, B_n, E_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; B^{\frac{1}{2}}_{2,1}) \times C([0, T]; B^{\frac{3}{2}}_{2,1}) \times C([0, T]; B^{\frac{3}{2}}_{2,1})$. For that purpose, we note that for all $(m, n) \in \mathbb{N}^2$, we have
\[
\begin{align*}
\{ \partial_t (u_{m+n+1} - u_{n+1}) + \nabla (p_{m+n+1} - p_{n+1}) - \nu \Delta (u_{m+n+1} - u_{n+1}) \\
= -u_{m+n} \cdot \nabla u_{m+n} + u_{m+n} \cdot \nabla u_{n} + j_{m+n} \times B_{m+n} - j_{n} \times B_{n} \} & \equiv I_1 + I_2, \\
\{ \partial_t (E_{m+n+1} - E_{n+1}) - \mbox{curl} (B_{m+n+1} - B_{n+1}) & = -(j_{m+n} - j_{n}), \\
\partial_t (B_{m+n+1} - B_{n+1}) + \mbox{curl} (E_{m+n+1} - E_{n+1}) & = 0, \\
\sigma (E_{n} + u_{n} \times B_{n}) & = j_{n}.
\end{align*}
\]
According to Lemma 2.6, we need to estimate $\|I_i\|_{L^p_T(B^{\frac{1}{2}}_{2,1})}, i = 1, 2, 3$.
\[
\|I_1\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} \leq C \|u_{m+n} - u_{n}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} (\|u_{m+n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} + \|u_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})}),
\]
and
\[
\|I_2\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} \leq C \sigma t \left\{ \|E_{m+n} - E_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \|B_{m+n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \\
+ \|B_{m+n} - B_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \|E_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} + \|u_{m+n} - u_{n}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} \|B_{m+n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \|B_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \|u_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \right\}.
\]
In addition, we perform a simple energy estimate for Maxwell’s equations and obtain
\[
\|E_{m+n+1} - E_{n+1}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} + \|B_{m+n+1} - B_{n+1}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} \leq C \sigma t \left\{ \|E_{m+n} - E_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} + \|u_{m+n} - u_{n}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} \|B_{m+n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \\
+ \|B_{m+n} - B_{n}\|_{L^p_T(B^{\frac{3}{2}}_{2,1})} \|u_{m+n}\|_{L^p_T(B^{\frac{1}{2}}_{2,1})} \right\}.
\]
Combining the above estimates with Lemma 2.6 and interpolation inequality \( \|a\|_{B^{\frac{3}{2}}_{2,1}} \), we obtain

\[
\begin{align*}
\|u_{m+n+1} - u_{n+1}\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} &+ \|E_{m+n+1} - E_{n+1}\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \|B_{m+n+1} - B_{n+1}\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} \\
\leq \frac{1}{2} \left( \|u_{m+n} - u_n\|_{L^\infty_t\left(B^{\frac{1}{2}}_{2,1}\right)} + \|E_{m+n} - E_n\|_{L^\infty_t\left(B^{\frac{3}{2}}_{2,1}\right)} + \|B_{m+n} - B_n\|_{L^\infty_t\left(B^{\frac{3}{2}}_{2,1}\right)} \right),
\end{align*}
\]

by choosing \( t \) and \( \varepsilon_0 \) small enough. Hence, \( \{\Gamma_n\}_{n\in\mathbb{N}} \) is a Cauchy sequence in \( C([0,T];\dot{B}^{\frac{5}{2}}_{2,1}) \times C([0,T];\dot{B}^{\frac{3}{2}}_{2,1}) \times C([0,T];\dot{B}^{\frac{3}{2}}_{2,1}) \) and converge to some limit function \( \Gamma = (u,B,E) \in C([0,T];\dot{B}^{\frac{5}{2}}_{2,1}) \times C([0,T];\dot{B}^{\frac{3}{2}}_{2,1}) \times C([0,T];\dot{B}^{\frac{3}{2}}_{2,1}) \).

**Third Step. Uniqueness**

Let \((u_1,B_1,E_1)\) be two solutions of \((1.1)\). Denote \((w_1,w_2,w_3) = (u_1 - u_2, B_1 - B_2, E_1 - E_2)\). By using the procedure of the second step on small time interval, we can prove \( w_1 = w_2 = w_3 = 0 \) in this interval. By repeating the procedure, we can obtain \( w_1 = w_2 = w_3 = 0 \) on \([0,T]\). We finish the proof of local existence.

### 3.2. Global existence

We construct approximate solutions to \((1.1)\) which are smooth solutions. Let \((u_0,B_0,E_0) = (0,0,0)\) and let the sequence \( \Gamma_n = (u_n,B_n,E_n) \) be solutions of the following system by induction

\[
\begin{align*}
\partial_t u_{n+1} + (u_n \cdot \nabla)u_n + \nabla p_{n+1} - \nu \Delta u_{n+1} &= j_n \times B_n, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t E_{n+1} - \text{curl} \ B_{n+1} &= -j_n, \\
\partial_t B_{n+1} + \text{curl} \ E_{n+1} &= 0, \\
\nabla \cdot u_{n+1} &= 0, \quad \nabla \cdot B_{n+1} = 0, \\
\sigma(E_n + u_n \times B_n) &= j_n, \\
\theta_{n+1} &= S_{\theta_{n+1}} u_0, \quad B_{n+1}|_{t=0} = S_{\theta_{n+1}} B_0, \quad E_{n+1}|_{t=0} = S_{\theta_{n+1}} E_0.
\end{align*}
\]

The sequence \( \{\theta_n\} \subset \mathbb{N} \) is chosen so that

\[\|(S_{\theta_{n+1}} - S_{\theta_n})(u_0,B_0,E_0)\|_{\mathcal{E}} \leq 2^{-n}.\]

Before stating our local existence result, we introduce the following function space:

\[\mathcal{E} \overset{\text{def}}{=} \dot{L}^\infty(B^{\frac{1}{2}}_{2,1}) \cap \dot{L}^1(B^{\frac{3}{2}}_{2,1}) \times \dot{L}^\infty(B^{\frac{3}{2}}_{2,1}) \times \dot{L}^\infty(B^{\frac{3}{2}}_{2,1}) \cap \dot{L}^1(B^{\frac{3}{2}}_{2,1}).\]

We are going to show that \( \{(u_n,B_n,E_n)\}_{n\in\mathbb{N}} \) is a Cauchy sequence in \( \mathcal{E} \). According to Lemma 2.6, we have the following inequality for all \( n \in \mathbb{N} \):

\[
\|u_{n+1}\|_{\dot{L}^\infty(B^{\frac{1}{2}}_{2,1}) \cap \dot{L}^1(B^{\frac{3}{2}}_{2,1})} \leq \|u_{n+1}\|_{B^{\frac{1}{2}}_{2,1}} + \|u_n \cdot \nabla u_n\|_{\dot{L}^1(B^{\frac{3}{2}}_{2,1})} + \|E_n \times B_n\|_{\dot{L}^1(B^{\frac{3}{2}}_{2,1})} + \|E_n \times B_n\|_{\dot{L}^1(B^{\frac{3}{2}}_{2,1})}.
\]

Applying Lemma 2.4 yields

\[
\|u_{n+1}\|_{\dot{L}^\infty(B^{\frac{1}{2}}_{2,1}) \cap \dot{L}^1(B^{\frac{3}{2}}_{2,1})} \leq \|u_{0_{n+1}}\|_{B^{\frac{1}{2}}_{2,1}} + \|u_n\|_{\dot{L}^1(B^{\frac{3}{2}}_{2,1})} \|u_n\|_{\dot{L}^\infty(B^{\frac{3}{2}}_{2,1})} + \|E_n\|_{\dot{L}^1(B^{\frac{3}{2}}_{2,1})} \|B_n\|_{\dot{L}^\infty(B^{\frac{3}{2}}_{2,1})} + \|u_n\|_{\dot{L}^1(B^{\frac{3}{2}}_{2,1})} \|B_n\|_{\dot{L}^\infty(B^{\frac{3}{2}}_{2,1})} (3.3)
\]
On the other hand, a simple energy estimate for Maxwell’s equations gives
\[
\|B_{n+1}\|_{L^\infty(\mathcal{B}_{2,1}^3)} + \|E_{n+1}\|_{L^\infty(\mathcal{B}_{2,1}^3) \cap L^1(\mathcal{B}_{2,1}^3)} \\
\lesssim \|E_{0_{n+1}}\|_{\mathcal{B}_{2,1}^3} + \|B_{0_{n+1}}\|_{\mathcal{B}_{2,1}^3} + \|u_n\|_{L^3(\mathcal{B}_{2,1}^3)} \|B_n\|_{L^\infty(\mathcal{B}_{2,1}^3)},
\]
which together with \(L^2(\mathbb{R}^3)\) energy estimate implies that
\[
\|B_{n+1}\|_{L^\infty(\mathcal{B}_{2,1}^3)} + \|E_{n+1}\|_{L^\infty(\mathcal{B}_{2,1}^3) \cap L^1(\mathcal{B}_{2,1}^3)} \\
\lesssim \|E_{0_{n+1}}\|_{\mathcal{B}_{2,1}^3} + \|B_{0_{n+1}}\|_{\mathcal{B}_{2,1}^3} + \|u_n\|_{L^3(\mathcal{B}_{2,1}^3)} \|B_n\|_{L^\infty(\mathcal{B}_{2,1}^3)}.
\]
(3.4)
Combining (3.3) and (3.4), we obtain
\[
\|\Gamma_{n+1}\| \lesssim \|\Gamma_{0_{n+1}}\|_{E} + \|\Gamma_{n}\|_{E}^{3} + \|\Gamma_{n}\|_{E}^{3}
\lesssim \varepsilon_{0} + \|\Gamma_{n}\|_{E}^{3} + \|\Gamma_{n}\|_{E}^{3}.
\]
Bearing in mind that \(\Gamma_{0} = 0\), we end up with
\[
\|\Gamma_{n}\| \lesssim \varepsilon_{0} \quad \text{for all } n \in \mathbb{N}.
\]
Similarly, applying the above estimate to \(\Gamma_{n+1} - \Gamma_{n}\), we obtain
\[
\|\Gamma_{n+1} - \Gamma_{n}\| \lesssim \|\Gamma_{0_{n+1}}\|_{E} + \|\Gamma_{n} - \Gamma_{n-1}\| \lesssim \|\Gamma_{n-1}\|_{E} + \|\Gamma_{n-1}\|_{E}^{3} + \|\Gamma_{n}\|_{E} + \|\Gamma_{n}\|_{E}
\lesssim 2^{-n} + \varepsilon_{0} \|\Gamma_{n} - \Gamma_{n-1}\|_{E}.
\]
It is now obvious that \(\{\Gamma_{n}\}_{n \in \mathbb{N}}\) is Cauchy sequences in \(E\), hence converge to some functions \((u, B, E) \in E\), which resolves the system (1.1).

Using a similar argument as that in [9], one can easily obtain the uniqueness of the solutions \((u, B, E)\). Thus, we finish the proof of Theorem 1.1. \(\square\)

4. The estimate of the pressure. The goal of this section is to provide the pressure estimates in the framework of weighted Chemin- Lerner type norms. We first get by taking div to the momentum equation of (1.1) that
\[
-\Delta P = \text{div}_h(\text{div}_h(u^h \otimes u^h) + 2\partial_3 \text{div}_h(u^3u^h) + 2\partial_3(u^3 \text{div}_h u^h)) \quad (4.1)
- \sigma \text{div} \left( \left( \frac{E^2}{-E^1} \right) B^3 + \left( \frac{-B^2}{B^1} \right) E^3 + \left( \frac{u^2}{u^1} \right) B^1 B^2 - \left( \frac{u^1}{u^2} \right) (B^3)^2 \right)
+ \left( \frac{B^1}{B^2} \right) u^3 B^3 - \left( \frac{u^1}{u^2} (B^3)^2 \right),
\]
where, for a vector field \(u = (u^h, u^3)\) we denote \(\text{div}_h u^h = \partial_1 u^1 + \partial_2 u^2\).

The following proposition concerning the estimate of the pressure will be the main ingredient used in the estimate of \(u^h\).

**Proposition 4.1.** Let \(u \in \hat{L}^\infty_T(\mathcal{B}_{2,1}^3) \cap L^1_T(\mathcal{B}_{2,1}^3)\). We denote
\[
f_1(t) \overset{def}{=} \|u^3(t)\|_{\mathcal{B}_{2,1}^3}, \quad f_2(t) \overset{def}{=} \max \left\{ \|u^3(t)\|_{\mathcal{B}_{2,1}^3}, \|u^3(t)\|_{\mathcal{B}_{2,1}^3} \right\}, \quad \text{and} \quad (4.2)
P_{\lambda_1} \overset{def}{=} \int_0^t f_1(t') dt' - \lambda_2 \int_0^t f_2(t') dt'
\]
for \(\lambda_1, \lambda_2 > 0\).
and similar notations for $B_{\chi}, E_{\chi}$ and $u_{\chi}$. Then (4.1) has a unique solution $\nabla P \in L_t^1(B_t^{2,1})$ which decays to zero when $|x| \to \infty$ so that for all $t \in [0,T]$, there holds

$$
\|\nabla P\|_{L_t^1(B_t^{2,1})} \leq \|u_{h}\|_{H_t^\frac{3}{2}}^{\frac{1}{2}} \|u_{h}\|_{L_t^\infty(B_t^{\frac{1}{2},1})}^{\frac{1}{2}} + \|u_{h}\|_{L_t^\infty(B_t^{\frac{1}{2},1})} \|u_{h}\|_{L_t^1(B_t^{2,1})} (4.3)
$$

Proof. Under the assumptions of Proposition 4.1 and $f_1, f_2$ being given by (4.2), one has

$$
\|\tilde{\Delta}_j(u^3\cdot h)\|_{L_t^1(L_t^2)} \lesssim d_j 2^{-\frac{2j}{3}} \|u^3\cdot h\|_{L_t^1(B_t^{\frac{3}{2},1})}^{\frac{1}{2}} \|u^3\cdot h\|_{L_t^1(B_t^{\frac{3}{2},1})} \quad \text{and}
$$

$$
\|\tilde{\Delta}_j(u^3\cdot h)\|_{L_t^1(L_t^2)} \lesssim d_j 2^{-\frac{2j}{3}} \left( \|u^3\|_{L_t^1(B_t^{\frac{3}{2},1})} \|u^3\|_{L_t^1(B_t^{\frac{3}{2},1})} \right)^{\frac{1}{2}} \left( \int_0^t \|u^3(t')\|_{L_t^1(B_t^{\frac{3}{2},1})} \right)^{\frac{1}{2}} \left( \int_0^t \|u^3(t')\|_{L_t^1(B_t^{\frac{3}{2},1})} \right)^{\frac{1}{2}}
$$

This proves the first inequality of Lemma 4.2.

Where again thanks to the standard product law in Besov space and the interpolation inequality we get that

$$
\|\tilde{\Delta}_j(u^3\cdot h)\|_{L_t^1(L_t^2)} \lesssim d_j 2^{-\frac{2j}{3}} \int_0^t d_j(t')\|u^3(t')\|_{B_t^{\frac{3}{2},1}} \|h(t')\|_{B_t^{\frac{3}{2},1}} \, dt'
$$

$$
\lesssim 2^{-\frac{2j}{3}} \int_0^t d_j(t')\|u^3(t')\|_{B_t^{\frac{3}{2},1}}^{\frac{1}{2}} \|u^3(t')\|_{B_t^{\frac{3}{2},1}}^{\frac{1}{2}} \|u^3(t')\|_{B_t^{\frac{3}{2},1}}^{\frac{1}{2}} \|h(t')\|_{B_t^{\frac{3}{2},1}}^{\frac{1}{2}} \, dt'
$$

$$
\lesssim d_j 2^{-\frac{2j}{3}} \left( \|u^3\|_{L_t^1(B_t^{\frac{3}{2},1})} \|u^3\|_{L_t^1(B_t^{\frac{3}{2},1})} \right) \|u^3\|_{L_t^1(B_t^{\frac{3}{2},1})} \|h\|_{L_t^1(B_t^{\frac{3}{2},1})} \|u^3\|_{L_t^1(B_t^{\frac{3}{2},1})}.
$$

This completes the proof of the lemma.

Lemma 4.3. Under the assumptions of Proposition 4.1 and $f_1$ being given by (4.2), one has

$$
\|\tilde{\Delta}_j(u^4\cdot B^3\cdot h)\|_{L_t^1(L_t^2)} \lesssim d_j 2^{-\frac{2j}{3}} \|B^3\|_{L_t^\infty(B_t^{\frac{3}{2},1})} \|u^h\|_{L_t^1(B_t^{\frac{3}{2},1})} \quad \text{and}
$$

$$
\|\tilde{\Delta}_j(u^4\cdot B^3\cdot h)\|_{L_t^1(L_t^2)} \lesssim d_j 2^{-\frac{2j}{3}} \left( \|u^4\|_{L_t^1(B_t^{\frac{3}{2},1})} \|B^3\|_{L_t^\infty(B_t^{\frac{3}{2},1})} \|u^h\|_{L_t^1(B_t^{\frac{3}{2},1})} \right).
$$
Proof. Using the standard product law in Besov space and the interpolation inequality ensure that
\[ \| \hat{\Delta}_j(u^h B^3 u^3) \|_{L^1_t(L^2)} \lesssim d_j 2^{-\frac{j}{2}} \| u^h B^3 u^3 \|_{L^1_t(B^\frac{5}{2})} = d_j 2^{-\frac{j}{2}} \| u^h B^3 u^3 \|_{L^1_t(B^\frac{5}{2})} \]
\[ \lesssim d_j 2^{-\frac{j}{2}} \int_0^t \| B^3(t') \|_{B^\frac{5}{2}} \| u^3(t') \|_{B^\frac{3}{2}} \| u^h(t') \|_{B^\frac{1}{2}} dt' \]
\[ \lesssim d_j 2^{-\frac{j}{2}} \| B^3 \|_{L^\infty_t(B^\frac{3}{2})} \| u^h \|_{L^1_t(B^\frac{5}{2})} . \]
This proves the first inequality of Lemma 4.2.

Where again thanks to the standard product law in Besov space and the interpolation inequality we get that
\[ \| \hat{\Delta}_j(u^h B^3 u^3) \|_{L^1_t(L^2)} \lesssim d_j 2^{-\frac{j}{2}} \int_0^t \| B^3(t') \|_{B^\frac{5}{2}} \| u^3(t') \|_{B^\frac{3}{2}} \| u^h(t') \|_{B^\frac{1}{2}} dt' \]
\[ \lesssim 2^{-\frac{j}{2}} \int_0^t d_j(t') \| B^3(t') \|_{B^\frac{5}{2}} \| u^3(t') \|_{B^\frac{3}{2}} \| u^h(t') \|_{B^\frac{1}{2}} dt' \]
\[ \lesssim d_j 2^{-\frac{j}{2}} \| B^3 \|_{L^\infty_t(B^\frac{3}{2})} \| u^h \|_{L^1_t(B^\frac{5}{2})} . \]
This completes the proof of the lemma. \qed

Lemma 4.4. Under the assumptions of Proposition 4.1 and \( f_1, f_2 \) being given by (4.2), one has
\[ \| \hat{\Delta}_j(u^3 \text{div}_h u^h) \|_{L^1_t(L^2)} \leq d_j 2^{-\frac{j}{2}} \| u^h \|_{L^1_t(B^\frac{5}{2})} \| u^h \|_{L^1_t(B^\frac{5}{2})} \]
and
\[ \| \hat{\Delta}_j(u^3 \text{div}_h u^h) \|_{L^1_t(L^2)} \leq d_j 2^{-\frac{j}{2}} \| u^3 \|_{L^1_t(B^\frac{5}{2})} + \| u^h \|_{L^1_t(B^\frac{5}{2})} \| u^3 \|_{L^1_t(B^\frac{5}{2})} . \]

Proof. Using the standard product law in Besov space and the interpolation inequality ensure that
\[ \| \hat{\Delta}_j(u^3 \text{div}_h u^h) \|_{L^1_t(L^2)} \leq d_j 2^{-\frac{j}{2}} \| u^3 \text{div}_h u^h \|_{L^1_t(B^\frac{5}{2})} = d_j 2^{-\frac{j}{2}} \| u^3 \text{div}_h u^h \|_{L^1_t(B^\frac{5}{2})} \]
\[ \lesssim d_j 2^{-\frac{j}{2}} \int_0^t \| u^3(t') \|_{B^\frac{3}{2}} \| \text{div}_h u^h(t') \|_{B^\frac{1}{2}} dt' \]
\[ \lesssim d_j 2^{-\frac{j}{2}} \int_0^t \| u^3(t') \|_{B^\frac{3}{2}} \| u^h(t') \|_{B^\frac{1}{2}} dt' \]
\[ \lesssim d_j 2^{-\frac{j}{2}} \left( \int_0^t \| u^3(t') \|_{B^\frac{3}{2}}^2 \| u^h(t') \|_{B^\frac{1}{2}} dt' \right)^{\frac{1}{2}} \left( \int_0^t \| u^h(t') \|_{B^\frac{1}{2}}^2 dt' \right)^{\frac{1}{2}} \]
\[ \lesssim d_j 2^{-\frac{j}{2}} \| u^h \|_{L^1_t(B^\frac{5}{2})} \| u^h \|_{L^1_t(B^\frac{5}{2})} . \]
This proves the first inequality of Lemma 4.4.

Where again thanks to the standard product law in Besov space and the interpolation inequality we get that
\[ \| \hat{\Delta}_j(u^3 \text{div}_h u^h) \|_{L^1_t(L^2)} \leq 2^{-\frac{j}{2}} \int_0^t \| \text{div}_h u^h(t') \|_{B^\frac{5}{2}} dt' \]
which along with Lemmas 4.2-4.4 and (4.5) implied that
\[
\|\tilde{\Delta}_j(\nabla P_x)\|_{L^1_t(L^2_x)} \lesssim d_j 2^{-2j} \left\{ \|u^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|u^h\|_{L^1_t(\dot{B}_{2,1}^{2})} + \|u^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|u^h\|_{L^1_t(\dot{B}_{2,1}^{2})} + \|B^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|B^h\|_{L^1_t(\dot{B}_{2,1}^{2})} + \|B^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|B^h\|_{L^1_t(\dot{B}_{2,1}^{2})} \right\},
\]
This finishes the proof of Proposition 4.1. □

To deal with the estimate of \(u^3\), we also need the following proposition:

**Proposition 4.5.** Under the assumptions of Proposition 4.1, one has
\[
\|\nabla P\|_{L^1_t(\dot{B}_{1,1}^{2})} \leq 2^{-2} \int_0^t d_j(t') \|u^3(t')\|_{B_{1,1}^{2}} \|u^h(t')\|_{B_{2,1}^{2}} \, dt',
\]
This completes the proof of the lemma.

**Proof of Proposition 4.1.** Again as both the proof of the existence and uniqueness of solutions to (4.1) is essentially followed by the estimates (4.3) for some appropriate approximate solutions of (4.1). For simplicity, we just prove (4.3) for smooth enough solutions of (4.1). Indeed, thanks to (4.1) and \(\text{div} u = 0\), we have
\[
\nabla P_x = \nabla (\Delta)^{-1} \left[ \text{div}_h \text{div}_h (u^h \otimes u^h_X) + 2\partial_3 \text{div}_h (u^3 u^h_X) - 2\partial_3 (u^3 \text{div}_h u^h_X) \right]
\]
Applying \(\Delta_j\) to the above equation and using Lemma 2.3 leads to
\[
\|\tilde{\Delta}_j(\nabla P_x)\|_{L^1_t(L^2)} \leq 2^j (\|\tilde{\Delta}_j(u^h \otimes u^h_X)\|_{L^1_t(L^2)} + \|\tilde{\Delta}_j(u^3 u^h_X)\|_{L^1_t(L^2)}) \|\tilde{\Delta}_j(\text{div}_h u^h_X)\|_{L^1_t(L^2)}
\]
Applying Lemma 2.4 and standard product laws in Besov space gives rise to
\[
\|\tilde{\Delta}_j(u^h \otimes u^h_X)\|_{L^1_t(L^2)} \lesssim d_j 2^{-2j} \|u^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|u^h\|_{L^1_t(\dot{B}_{2,1}^{2})},
\]
\[
\|\tilde{\Delta}_j(u^3 u^h_X)\|_{L^1_t(L^2)} \lesssim d_j 2^{-2j} \|u^3\|_{L^1_t(\dot{B}_{1,1}^{2})} \|u^h\|_{L^1_t(\dot{B}_{2,1}^{2})},
\]
which along with Lemmas 4.2-4.4 and (4.5) implied that
\[
\|\tilde{\Delta}_j(\nabla P_x)\|_{L^1_t(L^2)} \lesssim d_j 2^{-2j} \left\{ \|u^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|u^h\|_{L^1_t(\dot{B}_{2,1}^{2})} + \|u^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|u^h\|_{L^1_t(\dot{B}_{2,1}^{2})} + \|B^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|B^h\|_{L^1_t(\dot{B}_{2,1}^{2})} + \|B^h\|_{L^1_t(\dot{B}_{1,1}^{2})} \|B^h\|_{L^1_t(\dot{B}_{2,1}^{2})} \right\}.
\]
Toward this and motivated by [8, 12, 14], we first rewrite (1.1) as follows:

\[
\leq ||u^3||_{L^1_t(B^\frac{5}{2}_2)}||u^h||_{L^\infty_t(B^\frac{5}{2}_2)} + ||u^h||_{L^1_t(B^\frac{5}{2}_1)} + ||u^3||_{L^\infty_t(B^\frac{5}{2}_1)} \\
+ ||u^h||_{L^1_t(B^\frac{1}{2}_1)}||B^h||_{L^\infty_t(B^\frac{1}{2}_1)} + ||B^h||_{L^1_t(B^\frac{1}{2}_1)}||B^3||_{L^\infty_t(B^\frac{1}{2}_1)} ||u^3||_{L^\infty_t(B^\frac{1}{2}_1)} \\
+ ||E^h||_{L^\frac{1}{2}_t(B^\frac{1}{2}_1)}||B^3||_{L^\infty_t(B^\frac{1}{2}_1)} + ||B^h||_{L^1_t(B^\frac{3}{2}_1)}||E^3||_{L^\infty_t(B^\frac{3}{2}_1)} \\
+ ||u^h||_{L^1_t(B^\frac{3}{2}_1)}||B^3||_{L^\infty_t(B^\frac{3}{2}_1)},
\]

for all \( t \in (0, T) \).

**Proof.** The proof of this proposition follows exactly the same lines as that of Proposition 4.1. We first applying the standard product laws in Besov space gives rise to

\[
\|\hat{\Delta}_j \text{div}(u^h \otimes u^h)\|_{L^1_t(L^2)} \lesssim d_j 2^{-\frac{t}{2}} \int_0^t d_j(t') ||u^h \otimes u^h||_{B^\frac{3}{2}_2} dt',
\]

from which and Lemma 4.2-Lemma 4.4, we deduce that

\[
\|\nabla P\|_{L^1_t(L^2)} \leq C d_j 2^{-\frac{t}{2}} \left\{ ||u^3||_{L^1_t(B^\frac{5}{2}_2)} ||u^h||_{L^\infty_t(B^\frac{5}{2}_1)} + ||u^h||_{L^1_t(B^\frac{1}{2}_1)} + ||u^3||_{L^\infty_t(B^\frac{1}{2}_1)} \\
+ ||u^h||_{L^1_t(B^\frac{3}{2}_1)}||B^h||_{L^\infty_t(B^\frac{1}{2}_1)} + ||B^h||_{L^1_t(B^\frac{1}{2}_1)}||B^3||_{L^\infty_t(B^\frac{1}{2}_1)} ||u^3||_{L^\infty_t(B^\frac{3}{2}_1)} \\
+ ||E^h||_{L^\frac{1}{2}_t(B^\frac{1}{2}_1)}||B^3||_{L^\infty_t(B^\frac{1}{2}_1)} + ||B^h||_{L^1_t(B^\frac{3}{2}_1)}||E^3||_{L^\infty_t(B^\frac{3}{2}_1)} \\
+ ||u^h||_{L^1_t(B^\frac{3}{2}_1)}||B^3||_{L^\infty_t(B^\frac{3}{2}_1)} \right\},
\]

for all \( t \leq T \), from which we conclude the proof of (4.6). \( \square \)

5. The proof of Theorem 1.2. According to Theorem 1.1, there exist a positive time \( T \) so that (1.1) has a unique solution \((u, B, E)\) with

\[
u \in C([0, T]; B^\frac{5}{2}_1) \cap \tilde{L}^1([0, T]; B^\frac{5}{2}_1), \tag{5.1}
\]

\((B, E) \in \tilde{L}^\infty([0, T]; B^\frac{3}{2}_1) \times \tilde{L}^\infty([0, T]; B^\frac{3}{2}_1)). \tag{5.1}
\]

We denote \( T^* \) to be the largest time so that there holds (5.1). Hence to prove Theorem 1.2, we only need to prove that \( T^* = \infty \) and there holds (1.4)-(1.5). Toward this and motivated by [8, 12, 14], we first rewrite (1.1) as follows:

\[
\begin{cases}
\partial_t u^h + u \cdot \nabla u^h - \nu \Delta u^h + \nabla p \\
= B^2 + (-B^2) \cdot E_3 + \left( u^3 \right) \cdot B^1 B^2 - \left( u^3 \right) \cdot \left( u^3 \right)^2 \\
+ \left( u^3 \right)^2 B^3 - \left( u^3 \right)^2 B^3, \\
\partial_t u^3 + u \cdot \nabla u^3 - \nu \Delta u^3 \tag{5.2}
+ \partial_t u^3 = 0, \\
(\nu^h, \nu^3)|_{t=0} = (\nu^h_0, \nu^h_0),
\end{cases}
\]

\[
\begin{cases}
\partial_t u^h + u \cdot \nabla u^h - \nu \Delta u^h + \nabla p \\
= E^2 B^2 + u^2 B^2 B^3 - u^3 (B^2)^2 - \nu \Delta u^3 \tag{5.2}
+ \partial_t u^3 = 0, \\
(\nu^h, \nu^3)|_{t=0} = (\nu^h_0, \nu^h_0),
\end{cases}
\]

\[
\begin{cases}
\partial_t u^h + u \cdot \nabla u^h - \nu \Delta u^h + \nabla p \\
= u^3 B^2 - u^3 B^2 - u^3 (\nu^h)^2 + \partial_t u^3 \tag{5.2} \\
+ \partial_t u^3 = 0, \\
(\nu^h, \nu^3)|_{t=0} = (\nu^h_0, \nu^h_0),
\end{cases}
\]

\[
\begin{cases}
\partial_t u^h + u \cdot \nabla u^h - \nu \Delta u^h + \nabla p \\
= E^2 B^2 + u^2 B^2 B^3 - u^3 (B^2)^2 - \nu \Delta u^3 \tag{5.2}
+ \partial_t u^3 = 0, \\
(\nu^h, \nu^3)|_{t=0} = (\nu^h_0, \nu^h_0),
\end{cases}
\]
and

\[
\begin{cases}
\partial_t E - \text{curl} \, B = -j, \\
\partial_t B + \text{curl} \, E = 0, \\
\text{div} \, B = 0, \\
\sigma(E + v \times B) = j, \\
|B|_{t=0} = B_0, \ E|_{t=0} = E_0.
\end{cases}
\] (5.3)

5.1. The estimate of \(u^h\). Thanks to (5.2), we have

\[
\begin{align*}
\frac{d}{dt}\|\dot{\Delta}_j u^h_\lambda\|_{L^2}^2 + \sum_{i=1}^6 \lambda_i f_i(t) |\dot{\Delta}_j u^h_\lambda|_{L^2}^2 - \nu \Delta u^h_\lambda - \nabla_h P_\lambda + \left\{ \left( \frac{E^2}{-E_1} \right) B_3 + \left( \frac{-B_3^2}{B_1^2} \right) E_3 \right\} \\
+ \left( \frac{u^2}{u^1} \right) B_1 B_2^2 - \left( \frac{u^1}{u^2} \right) (B^3)^2 + \left( \frac{B_1^2}{B_2^2} \right) u^3 B^3 - \left( \frac{u^1 (B^2)^2}{u_2 (B^1)^2} \right) \\
\times \exp \left( -\sum_{i=1}^2 \lambda_i \int_{0}^{t} f_i'(t') dt' \right) \\
\equiv -u \cdot \nabla u^h_\lambda - \nabla_h P_\lambda + P_j.
\end{align*}
\]

Applying the operator \(\dot{\Delta}_j\) to the above equation and taking the \(L^2\) inner product of the resulting equation with \(\dot{\Delta}_j u^h_\lambda\), we obtain

\[
\left( \frac{1}{2} \right) \frac{d}{dt}\|\dot{\Delta}_j u^h_\lambda\|_{L^2}^2 + \sum_{i=1}^6 \lambda_i f_i(t) |\dot{\Delta}_j u^h_\lambda|_{L^2}^2 - \nu \int_{\mathbb{R}^3} \Delta \dot{\Delta}_j u^h_\lambda \dot{\Delta}_j u^h_\lambda \, dx \\
= \int_{\mathbb{R}^3} \left( \Delta \dot{\Delta}_j (u \cdot \nabla u^h_\lambda) - \dot{\Delta}_j \nabla_h P_\lambda + \dot{\Delta}_j P_j \right) \dot{\Delta}_j u^h_\lambda \, dx.
\]

However thanks to [5], there exists a positive constant \(c\) such that

\[-\nu \int_{\mathbb{R}^3} \Delta \dot{\Delta}_j u^h_\lambda \dot{\Delta}_j u^h_\lambda \, dx \geq c 2^{2}\|\dot{\Delta}_j u^h_\lambda\|_{L^2}^2,
\]

from which, we deduce that

\[
\left( \frac{d}{dt}\right) \|\dot{\Delta}_j u^h_\lambda\|_{L^2}^2 + \sum_{i=1}^6 \lambda_i f_i(t) |\dot{\Delta}_j u^h_\lambda|_{L^2}^2 + \nu 2^{2}\|\dot{\Delta}_j u^h_\lambda\|_{L^2}^2 \\
\leq \|\dot{\Delta}_j (u \cdot \nabla u^h_\lambda)\|_{L^2}^2 + \|\dot{\Delta}_j \nabla_h P_\lambda\|_{L^2}^2 + \|\dot{\Delta}_j P_j\|_{L^2}^2.
\]

(5.4)

Applying Lemma 2.4 and Lemma 4.2, we obtain

\[
\|\dot{\Delta}_j (u \cdot \nabla u^h_\lambda)\|_{L^2(L^2)} \leq 2^{2} \left( \|\dot{\Delta}_j (u^h \otimes u^h_\lambda)\|_{L^2(L^2)} + \|\dot{\Delta}_j (u^3 u^h_\lambda)\|_{L^2(L^2)} \right) \\
\leq C d 2^{-\frac{1}{2}} \left( \|u^h\|_{L^\infty(B^\frac{1}{2}_{2,1})} \|u^h_\lambda\|_{L^1(B^\frac{3}{2}_{2,1})} \right) + \|u^h_\lambda\|_{L^1(B^\frac{1}{2}_{2,1})} \|u^h_\lambda\|_{L^1(B^\frac{1}{2}_{2,1})} \|u^h_\lambda\|_{L^1(B^\frac{3}{2}_{2,1})}.
\]

While applying Lemma 2.4, Lemma 4.2 and Lemma 4.3 leads to

\[
\|\dot{\Delta}_j P_j\|_{L^1(L^2)} \leq d 2^{-\frac{1}{2}} \left\{ \|u^h_\lambda\|_{L^1(B^\frac{1}{2}_{2,1})} \|B^h\|_{L^2(L^\infty(B^\frac{1}{2}_{2,1}))}^2 + \|E^h_\lambda\|_{L^2(L^1(B^\frac{3}{2}_{2,1}))} \|B^h\|_{L^2(L^\infty(B^\frac{1}{2}_{2,1}))}^2 \\
+ \|B^h_\lambda\|_{L^1(B^\frac{1}{2}_{2,1})} \|E^h_\lambda\|_{L^1(B^\frac{3}{2}_{2,1})} + \|u^h_\lambda\|_{L^1(B^\frac{1}{2}_{2,1})} \|B^h\|_{L^2(L^\infty(B^\frac{1}{2}_{2,1}))}^2 \\
+ \|B^h\|_{L^1(B^\frac{1}{2}_{2,1})} \|B^h\|_{L^1(B^\frac{1}{2}_{2,1})} \|E^h\|_{L^1(B^\frac{3}{2}_{2,1})} \|B^h\|_{L^2(L^\infty(B^\frac{1}{2}_{2,1}))}^2 \right\}.
\]
Integrating (5.4) over [0, t] and substituting the above estimates and (4.3) into the resulting inequality, we obtain

\[ \|u^h\|_{L^\infty_t(B^1_{2,1})} + \sum_{j=1}^{\infty} \lambda_j \|u^h\|_{L^1_t(B^1_{2,1})} + \nu \|u^h\|_{L^1_t(B^1_{2,2})} \]

\[ \leq \|u^h\|_{B^2_{2,1}} + \frac{C^2}{2} \|u^h\|_{L^1_t(B^1_{2,1})} + \frac{1}{\nu} \|u^h\|_{L^1_t(B^1_{2,2})} + \|E^h\|_{L^1_t(B^{1/2}_{2,2})} + \|B^h\|_{L^1_t(B^{1/2}_{2,2})} + \|\Delta_j u^h\|_{L^1_t(B^{1/2}_{2,2})}, \]

for all \( t \leq T \).

5.2. The estimate of \( u^3 \). We use that the equation on the vertical component of the velocity is a linear equation with coefficients depending on the horizontal components. Thanks to the \( u^3 \) equation of (5.2), we get by a similar derivation of (5.4) that

\[ \|\Delta_j u^3\|_{L^\infty_t(L^2)} + \nu \|\Delta_j u^3\|_{L^1_t(L^2)} \]

\[ \leq \|\Delta_j u^3\|_{L^2} + C\left(\|\Delta_j (u \cdot \nabla u^3)\|_{L^1_t(L^2)} + \|\Delta_j \partial_t P\|_{L^1_t(L^2)} + \|\Delta_j Q_j\|_{L^1_t(L^2)}\right), \]

where

\[ Q_j = E^1 B^2 - E^2 B^1 + u^2 B^2 - u^3 (B^2)^2 - u^3 (B^1)^2 + u^1 B^3. \]

Applying Lemma 4.2 and Lemma 4.4 ensures that

\[ \|\Delta_j (u \cdot \nabla u^3)\|_{L^1_t(L^2)} \]

\[ \lesssim 2^j \|\Delta_j (u^3)\|_{L^1_t(L^2)} + \|\Delta_j (u^3 \text{div} u^h)\|_{L^1_t(L^2)} \]

\[ \lesssim 2^j \|u^3\|_{L^1_t(B^{1/2}_{2,1})} + \|u^h\|_{L^1_t(B^{1/2}_{2,1})} + \|u^h\|_{L^1_t(B^{1/2}_{2,2})}. \]

While applying Lemma 2.4 and standard product laws in Besov space yields

\[ \|\Delta_j Q_j\|_{L^1_t(L^2)} \lesssim 2^j \|u^3\|_{L^1_t(B^{1/2}_{2,1})} + \|u^h\|_{L^1_t(B^{1/2}_{2,2})} + \|u^h\|_{L^1_t(B^{1/2}_{2,2})} \]

Then we get by substituting the above estimates and (4.6) into (5.6) that

\[ \|u^3\|_{L^\infty_t(B^1_{2,2})} + \nu \|u^3\|_{L^1_t(B^1_{2,2})} \]

\[ \leq \|u^3\|_{B^1_{2,1}} + \frac{C^2}{2} \|u^3\|_{L^1_t(B^1_{2,1})} + \|B^h\|_{L^1_t(B^{1/2}_{2,2})} + \|B^h\|_{L^1_t(B^{1/2}_{2,2})} + \|u^3\|_{L^1_t(B^{1/2}_{2,2})} + \|u^3\|_{L^1_t(B^{1/2}_{2,2})} \]

\[ + \|u^h\|_{L^1_t(B^{1/2}_{2,1})} + \|B^h\|_{L^1_t(B^{1/2}_{2,2})} + \|B^h\|_{L^1_t(B^{1/2}_{2,2})} + \|u^3\|_{L^1_t(B^{1/2}_{2,2})} + \|u^3\|_{L^1_t(B^{1/2}_{2,2})} \]

\[ + \|E^h\|_{L^1_t(B^{1/2}_{2,2})} + \|E^3\|_{L^1_t(B^{1/2}_{2,2})} + \|E^3\|_{L^1_t(B^{1/2}_{2,2})} + \|u^3\|_{L^1_t(B^{1/2}_{2,2})} + \|u^3\|_{L^1_t(B^{1/2}_{2,2})} \]
5.3. The estimate of $B_\lambda$ and $E_\lambda$. The existence and uniqueness of solutions to Maxwell’s equations is essentially followed by the estimates (4.3) for some appropriate approximate solutions of (4.1). For simplicity, we just prove (4.3) for smooth enough solutions of Maxwell’s equations. Indeed thanks to (1.1), we have

\[
\begin{aligned}
\partial_t E_\lambda - \text{curl } B_\lambda + \sigma (E_\lambda + u \times B_\lambda) &= 0, \\
\partial_t B_\lambda + \text{curl } E_\lambda &= 0, \\
\nabla \cdot u_\lambda &= 0, \quad \nabla \cdot B_\lambda &= 0, \\
B|_{t=0} &= B_0, \quad E|_{t=0} = E_0,
\end{aligned}
\]

(5.8)

Applying the operator $\hat{\Delta}_j$ to the above two equations and then taking the $L^2$ inner product of the resulting equation with $\hat{\Delta}_j E_\lambda$ and $\hat{\Delta}_j B_\lambda$, respectively, we have

\[
\frac{1}{2} \frac{d}{dt} (\|\hat{\Delta}_j E_\lambda\|_{L^2}^2 + \|\hat{\Delta}_j B_\lambda\|_{L^2}^2) + \sum_{i=1}^2 \lambda_i f_i(t) (\|\hat{\Delta}_j E_\lambda^i\|_{L^2}^2 + \|\hat{\Delta}_j B_\lambda^i\|_{L^2}^2) + \sigma \|\hat{\Delta}_j E_\lambda\|_{L^2}^2
\leq \sigma \|\hat{\Delta}_j (u \times B_\lambda)\|_{L^2} \|\hat{\Delta}_j E_\lambda\|_{L^2}.
\]

which implies that

\[
\frac{d}{dt} (\|\hat{\Delta}_j E_\lambda\|_{L^2}^2 + \|\hat{\Delta}_j B_\lambda\|_{L^2}^2) + \sum_{i=1}^2 \lambda_i f_i(t) (\|\hat{\Delta}_j E_\lambda^i\|_{L^2}^2 + \|\hat{\Delta}_j B_\lambda^i\|_{L^2}^2)
\]

(5.9)

\[
\leq \sigma^2 \|\hat{\Delta}_j (u \times B_\lambda)\|_{L^2}^2 \leq \sigma \|\hat{\Delta}_j (u \times B_\lambda)\|_{L^2}^2 \leq \sigma^2 \|\hat{\Delta}_j (u \times B_\lambda)\|_{L^2}^2 \leq \sigma \|\hat{\Delta}_j (u \times B_\lambda)\|_{L^2}^2 \leq \sigma \|\hat{\Delta}_j (u \times B_\lambda)\|_{L^2}^2,
\]

Integrating (5.9) over $[0, t]$, we obtain

\[
\|E_\lambda\|_{L^\infty_t(L^2_x)}^2 + \|B_\lambda\|_{L^\infty_t(L^2_x)}^2
\]

(5.10)

\[
+ \sum_{i=1}^2 \lambda_i \|B_\lambda^i\|_{L^2_t(B^2_x)} + \|E_\lambda^i\|_{L^2_t(B^2_x)} + \sigma \|E_\lambda\|_{L^2_t(B^2_x)}
\]

\[
\leq \|E_0\|_{B^2_x}^2 + \|B_0\|_{B^2_x}^2 + \sigma \|B_\lambda\|_{L^\infty_t(B^2_x)} \|u\|_{L^2_t(B^2_x)} + \int_0^t \|u^3(t')\|_{B^2_x} \|B_\lambda\|_{B^2_x} dt'
\]

(5.10)

\[
= \|E_0\|_{B^2_x}^2 + \|B_0\|_{B^2_x}^2 + \sigma \|B_\lambda\|_{L^\infty_t(B^2_x)} \|u\|_{L^2_t(B^2_x)} + \|B_\lambda\|_{L^2_t(B^2_x)}.
\]

Whereas a standard energy estimate applied to the Maxwell’s equations gives

\[
\|E_\lambda\|_{L^\infty_t(L^2_x)}^2 + \|B_\lambda\|_{L^\infty_t(L^2_x)}^2
\]

(5.11)
on, we define.

Now let \( \lambda_1 \geq 4C \) and \( \lambda_2 \geq \max\{\frac{2C}{\nu}, 2C\} \), we get by summing up (5.5) and (5.10)-(5.11) that

\[
\|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + \sigma \|B_\lambda\|_{L^\infty(B^{3/2}_{2,1})}^2 \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|B_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2,
\]

where we have used that \( B^{3/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \).

Then taking \( \lambda_1 \geq 4C \) and \( \lambda_2 \geq \max\{\frac{2C}{\nu}, 2C\} \), we get by summing up (5.5) and (5.10)-(5.11) that

\[
\|E_0\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \|B_\lambda\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \sigma \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2
\]

\[
+ \frac{c_\nu}{2} \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2 + \sum_{i=1}^2 \lambda_i \left( \|B_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 \right) + \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2
\]

\[
\leq \|u_0^h\|_{B^{3/2}_{2,1}}^2 + \|E_0\|_{B^{3/2}_{2,1}}^2 + \|B_0\|_{B^{3/2}_{2,1}}^2 + C \left\{ \|u^h\|_{L^\infty_t(B^{3/2}_{2,1})} \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})} \right\}
\]

\[
+ \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|B_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2 \}
\]

\[
\leq \|u_0^h\|_{B^{3/2}_{2,1}}^2 + \|E_0\|_{B^{3/2}_{2,1}}^2 + \|B_0\|_{B^{3/2}_{2,1}}^2 + C \left\{ \|u^h\|_{L^\infty_t(B^{3/2}_{2,1})} \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})} \right\}
\]

\[
+ \|B_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|B_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 \}
\]

\[
\text{for all } t \leq T.
\]

Now let \( c_2 \) be a small enough positive constant, which will be determined later on, we define \( T \) by

\[
T \overset{def}{=} \max\{t \in [0, T^*] : \|u^h\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \|B\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \sigma \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|E\|_{L^1_t(B^{3/2}_{2,1})}^2 \leq c_2 \}
\]

In what follows, we shall prove that \( T = \infty \) under the assumption of (1.3). Otherwise, taking \( c_2 \leq \min\{\frac{1}{3C_1}, \frac{c\nu C_1}{4}\} \), we deduce from (5.12) that

\[
\|E_\lambda\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \|B_\lambda\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \sigma \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2
\]

\[
+ \|u^h\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 \}
\]

\[
\text{for } t \leq T.
\]

On the other hand, it is easy to observe from (4.2) that

\[
\left( \|E\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \|E\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|u^h\|_{L^\infty_t(B^{3/2}_{2,1})}^2 \right)
\]

\[
\text{for } t \leq T.
\]

where we have used that \( B^{3/2}_{2,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \).

Then taking \( \lambda_1 \geq 4C \) and \( \lambda_2 \geq \max\{\frac{2C}{\nu}, 2C\} \), we get by summing up (5.5) and (5.10)-(5.11) that

\[
\|E_0\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \|B_\lambda\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \sigma \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2
\]

\[
\leq \|u_0^h\|_{B^{3/2}_{2,1}}^2 + \|E_0\|_{B^{3/2}_{2,1}}^2 + \|B_0\|_{B^{3/2}_{2,1}}^2 + C \left\{ \|u^h\|_{L^\infty_t(B^{3/2}_{2,1})} \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})} \right\}
\]

\[
+ \|E_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|B_\lambda\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2 \}
\]

\[T \overset{def}{=} \max\{t \in [0, T^*] : \|u^h\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \|B\|_{L^\infty_t(B^{3/2}_{2,1})}^2 + \sigma \|u^h\|_{L^1_t(B^{3/2}_{2,1})}^2 + \|E\|_{L^1_t(B^{3/2}_{2,1})}^2 \leq c_2 \}.
\]
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E-mail address: yuegch@nuaa.edu.cn
E-mail address: ckzhong@nju.edu.cn