Abstract. We study the Hodge-Riemann relations (HRR) for graded Artinian Gorenstein (AG) algebras. We classify those AG algebras in codimension two satisfying the HRR in terms of their Macaulay dual generators. Our classification identifies classes of homogeneous polynomials that generalize the Lorentzian polynomials defined by Brändén and Huh.

1. Introduction

In this paper we introduce the Hodge Riemann relations and the Hodge-Riemann property on an arbitrary graded oriented Artinian Gorenstein (AG) algebra defined over \( \mathbb{R} \), and we give a criterion on the higher Hessian matrix of its Macaulay dual generator (Theorem 3.1). AG algebras can be regarded as algebraic analogues of cohomology rings (in even degrees) of compact complex manifolds, and, in case the manifold is Kähler, Hodge theory implies that its cohomology ring satisfies the HRP. Higher Hessians were introduced by J. Watanabe \([9]\) to study the strong Lefschetz property (SLP) of an AG algebra defined over an arbitrary field of characteristic zero (see also \([6]\)); over the real numbers, HRP implies SLP (Lemma 2.3).

In a recent paper \([2]\), Brändén and Huh introduced a remarkable class of real homogeneous polynomials, extending the class of (real) stable polynomials, which they called Lorentzian polynomials; polynomials lying in the interior of this set are called strictly Lorentzian. Among other things, they showed that the first Hessian matrix of a strictly Lorentzian polynomial is non-singular with exactly one positive eigenvalue, which, in terms of Macaulay duality, implies that its associated AG algebra satisfies the HRR in degree one; this can be considered an analogue of the Hodge index theorem for Kähler manifolds. In this paper, we focus on the two variable or codimension two case. We introduce the class of Lorentzian polynomials of order \( i \geq 0 \), which turn out to characterize, up to a linear change of coordinates, Macaulay dual generators of codimension two AG algebras satisfying HRR in degree \( i \).

Theorem A. Let \( F = F(X, Y) \) be a homogeneous polynomial of degree \( d \) in \( n = 2 \) variables, and let \( A_F = \mathbb{R}[x, y]/\text{Ann}(F) \) be its associated AG algebra.

(1) If \( F \) is strictly Lorentzian of order \( i \), then \( A_F \) satisfies HRR in degree \( i \).

MSC 2020 classification: Primary: 13H10; Secondary: 11B83, 13E10, 14F45.
Keywords: Hodge-Riemann property, strong Lefschetz property, Lorentzian polynomial, higher Hessian.
(2) If $A_F$ satisfies HRR in degree $i$, then there exists a linear change of coordinates $g \in \text{GL}(2, \mathbb{R})$ such that $g. F(X, Y) = F(g(X, Y))$ is Lorentzian of order $i$.

Our class of Lorentzian polynomials of order $i = 1$ agrees with the class of Lorentzian polynomials (in two variables) introduced by Brändén and Huh [2], characterized as those polynomials whose coefficient sequence is non-negative, ultra log concave, and has no internal zeros. For $i > 1$, Lorentzian polynomials of order $i$ are defined by determinantal inequalities that we regard as a higher (ultra) log concavity condition. Such determinants show up in our higher Hessian computations as indicated in the result below. Following [2], we define the normalization operator $N: \mathbb{R}[X, Y] \to \mathbb{R}[X, Y]$ as the $\mathbb{R}$-linear operator defined on monomials as

$$N(X^a Y^b) = \frac{X^a Y^b}{a! b!}.$$ 

**Theorem B.** Let $F = \sum_{d=0}^d a_{d-i}X^{d-i} Y^i$ be a homogeneous polynomial of degree $d$ in $n = 2$ variables. Then the determinant of the $i^{th}$ Hessian of its normalization, evaluated at $X = 0$ and $Y = 1$ satisfies

$$\det(\text{Hess}_i(N(F))|_{(0,1)}) = (-1)^{\lfloor \frac{d+1}{2} \rfloor} \cdot \left( \frac{1}{(d-2i)!} \right)^{i+1} \cdot \det(a_{i+p-q})_{0 \leq p, q \leq i}.$$ 

A stable polynomial (in two variables) is a real homogeneous polynomial $F(X, Y)$ whose dehomogenization $F(1, t)$ has only real non-positive roots. A classical result, which, according to [8], dates back to I. Newton and C. Maclaurin, implies that every stable polynomial (in two variables) is Lorentzian of order $i = 1$, but we give examples showing this fails for $i > 1$. On the other hand we show how Theorem B implies that the normalization of a stable polynomial (in two variables) is always Lorentzian of order $i$ for all $i \geq 0$. In fact, as we shall see our condition on the coefficient sequence of a Lorentzian polynomial of order $i$ is a weakening of the condition on a Pólya frequency (PF) sequence. In particular, by Theorem A, the normalization of a stable polynomial should always generate an AG algebra with the HRP. We offer another proof of this fact by interpreting the determinant on the RHS of Theorem B as a Schur polynomial.

**Theorem C.** If $F = F(X, Y) = \sum_{i=0}^d a_{d-i} X^{d-i} Y^i$ is stable with $a_0 \neq 0$, then $(A_{N(F)}, y)$ has the HRP.

This paper is organized as follows. In Section 2 we introduce the HRP and HRR in degree $i$ for AG algebras, and compare them to the more familiar strong Lefschetz property (SLP). We also give a matrix criterion HRR. In Section 3 we give necessary and sufficient conditions for an AG algebra to have HRP in terms of the higher Hessian matrices of its Macaulay dual generator. In Section 4 we restrict our attention to the two variable case, and introduce our notions of higher log concavity and higher Lorentzian polynomials. Then we prove Theorem B, Theorem A, and Theorem C.

2. **Hodge-Riemann Bilinear Relations**

A graded Artinian Gorenstein (AG) $\mathbb{R}$-algebra $B$ of socle degree $b$ is a graded algebra with homogeneous graded components $B_i$ such that $b = \max \{ i : \dim_{\mathbb{R}}(B_i) > 0 \}, \dim_{\mathbb{R}}(B_b) =$
1, and for each \(0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor\) multiplication defines a nondegenerate pairing \(B_i \times B_{b-i} \to B_b\). A choice of an \(\mathbb{R}\)-linear isomorphism \(\int_B : B_b \to \mathbb{R}\) is called an orientation on \(B\) and the pair \((B, \int_B)\) is termed a graded oriented AG algebra over \(\mathbb{R}\) of socle degree \(b\). The element \(\sigma_B \in B_d\) that satisfies \(\int_B \sigma_B = 1\) will be called the distinguished socle generator of \(B\).

Fix a linear form \(\ell \in B_1\). For each degree \(i, 0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor\), define the \(i\)th primitive subspace to be the kernel of the multiplication map \(\times^\ell: B_i \to B_{b-i+1}\), i.e.

\[
P_{i,\ell} = \{ \beta \in B_i | \ell^{b-2i+1} \cdot \beta = 0 \} \subset B_i.
\]

For \(i > \left\lfloor \frac{b}{2} \right\rfloor\), set \(P_{i,\ell} = 0\).

**Definition 2.1** (Strong Lefschetz Property, Hodge-Riemann Property). (1) The linear form \(\ell \in B_1\) is strong Lefschetz (SL) for \(B\) if the multiplication maps

\[
\times^\ell^{b-2i}: B_i \to B_{b-i}
\]

are isomorphisms for each \(0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor\). We also say that the pair \((B, \ell)\) has the strong Lefschetz property (SLP).

(2) Let \(0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor\). The linear form \(\ell \in B_i\) is Hodge-Riemann (HR) in degree \(i\) if for every \(0 \leq j \leq i\) and every \(0 \neq \beta \in P_{j,\ell}\), we have

\[
(-1)^j \cdot \int_B \ell^{b-2j} \cdot \beta^2 > 0.
\]

We also say that the pair \((B, \ell)\) satisfies HRR in degree \(i\). We say that the pair \((B, \ell)\) has the Hodge-Riemann property (HRP) if it satisfies HRR in degree \(b_0 = \max \{ i | P_{i,\ell} \neq 0 \}\).

Since \(B\) is Gorenstein, multiplication defines a symmetric bilinear and non-degenerate pairing for each \(0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor\):

\[
B_i \times B_{b-i} \to \mathbb{R}
\]

\[
(\beta_1, \beta_2) \xrightarrow{\ell} \int_B \ell^{b-2i} \cdot \beta_1 \cdot \beta_2
\]

Using the linear form \(\ell \in B_1\) we get the \(i\)th Lefschetz pairing on \(B_i\) with respect to \(\ell\):

\[
B_i \times B_i \to \mathbb{R}
\]

\[
(\beta_1, \beta_2) \xrightarrow{\ell} \int_B \ell^{b-2i} \cdot \beta_1 \cdot \beta_2
\]

The \(i\)th Lefschetz pairing is clearly symmetric and bilinear. The following result is known as primitive decomposition.

**Lemma 2.2.** Assume that for some fixed degree \(0 \leq i - 1 \leq \left\lfloor \frac{b}{2} \right\rfloor\), the \((i - 1)\)th Lefschetz map \(\times^\ell^{b-2(i-1)}: B_{i-1} \to B_{b-i+1}\) is an isomorphism. Then in degree \(i\) there is vector space decomposition

\[
B_i = P_{i,\ell} \oplus \ell \cdot B_{i-1}
\]
which is orthogonal with respect to the $i^{th}$ Lefschetz pairing. In particular, if $\ell \in B_1$ is SL for $B$, then $B$ admits an orthogonal decomposition with respect to the Lefschetz pairing:

$$B = \bigoplus_{i=0}^{b} \bigoplus_{j=0}^{i} \ell^{i-j} \cdot P_{j,\ell}$$

called the primitive decomposition with respect to $\ell$.

**Proof.** There is a commutative diagram of linear maps

$$
\begin{array}{ccc}
B_{i-1} & \xrightarrow{\times \ell^{b-2(i-1)}} & B_{b-i+1} \\
\downarrow & & \downarrow \\
B_i & \xrightarrow{\times \ell^{b-2i+1}} & B_i
\end{array}
$$

Since the top horizontal map is an isomorphism, it follows that the diagonal map must map the image of the vertical map $(B_{i-1}) \subset B_i$ isomorphically onto its image, and since its kernel is $P_{i,\ell}$ the decomposition follows from linear algebra. The orthogonality of the decomposition follows directly from the definitions and its proof is left to the reader. If $\ell \in B_1$ is SL for $B$, its primitive decomposition follows from an easy inductive argument whose details are again left to the reader. $\square$

**Lemma 2.3.** A linear form $\ell \in B_1$ is SL for $B$ if and only if the $i^{th}$ Lefschetz pairing with respect to $\ell$ is non-degenerate on the primitive subspace $P_{i,\ell}$, for each $0 \leq i \leq \lfloor \frac{b}{2} \rfloor$.

**Proof.** Assume that $\ell \in B_1$ is SL for $B$. It follows that the $i^{th}$ Lefschetz pairing must be non-degenerate, and by orthogonality of the primitive decomposition, it follows that it must also be non-degenerate on the $i^{th}$ primitive subspace.

Conversely, assume that $\ell \in B_1$ is not SL for $B$, let $0 \leq i \leq \lfloor \frac{b}{2} \rfloor$ be any index for which $\times \ell^{b-2i} : B_i \rightarrow B_{b-i}$ is not an isomorphism, and suppose that $\alpha \in B_i$ is a non-zero element of its kernel. Then $\alpha \in P_{i,\ell}$ and we must have for every $\beta \in B_i$

$$(\alpha,\beta)_\ell = \int_A \ell^{b-2i} \alpha \beta = 0$$

which implies that the $i^{th}$ Lefschetz pairing is degenerate on $P_{i,\ell}$. $\square$

**Lemma 2.3** shows that $\ell \in B_1$ is HR for $B$ implies $\ell$ is SL for $B$. The converse does not hold however; see Example 4.13.

Next we derive some matrix conditions for verifying HRR.

Recall that a real symmetric $n \times n$ matrix $M$ has real eigenvalues $\lambda_1, \ldots, \lambda_n$. The signature of $M$ is then defined to be the difference between the number of positive eigenvalues and the number of negative ones:

$$\text{sgn}(M) = \#\{\lambda_i > 0\} - \#\{\lambda_i < 0\}.$$  

Sylvester’s Law of inertia implies that signature is invariant under the congruence relation, i.e. if $Q$ is a non-singular $n \times n$ matrix, and $M' = Q \cdot M \cdot Q^T$ where $Q^T$ is the transpose of $Q$, then $\text{sgn}(M) = \text{sgn}(M')$. 
Let \( \mathcal{E}_0 = \{1\} \), and for each \( 1 \leq i \leq b \), let \( \mathcal{E}_i = \{e_1, \ldots, e_i\} \) be a basis for \( B_i \), and let \( \mathcal{E}'_{b-i} = \{f_1, \ldots, f_{b-i}\} \) be its dual basis for \( B_{b-i} \) in the sense that \( \int_B e_i f_j = \delta_{ij} \). Accordingly, we set \( \mathcal{E}_b = \{e_b\} \) (the distinguished socle generator for \( B \)). Denote \( \mathcal{E} = \cup_{i=0}^b \mathcal{E}_i \), a homogeneous \( \mathbb{R} \)-basis for \( B \). For each \( 0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor \), let \( M'_i = M'_i(\mathcal{E}) \) denote the matrix for the multiplication map \( \times \ell^{b-2i} : B_i \to B_{b-i} \) with respect to \( \mathcal{E} \) and \( \mathcal{E}'_{b-i} \). Denote by \( E \) the distinguished socle generator for \( B \leq 2 \). Thus matrix \( M'_i(\mathcal{E}) \) will have a block diagonal form:

\[
M'_i(\mathcal{E}) = \begin{pmatrix}
A'_i & 0 \\
0 & M'_{i-1}(\mathcal{E})
\end{pmatrix}
\]

where \( A'_i \) is the matrix for the multiplication map \( \times \ell^{b-2i} : P_{i,\ell} \to \ell^{b-2i} \cdot P_{i,\ell} \). Since

\[
\text{sgn}(M'_i(\mathcal{E})) = \text{sgn}(A'_i) + \text{sgn}(M'_{i-1}(\mathcal{E}))
\]

the result follows. □

In the special case where the primitive subspace is one dimensional in each degree (as in the codimension two case), the HRP can be checked by the following determinantal condition.

**Lemma 2.5.** Assume that \( \ell \) is SL for \( B \), and assume that \( \dim_{\mathbb{R}}(P_{i,\ell}) \leq 1 \) for each \( 0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor \).
Then \( \ell \) is HR for \( B \) in degree \( i \) if and only if there exists a basis \( \mathcal{E} \) such that for each \( 0 \leq j \leq i \),

\[
(-1)^j \cdot \frac{\det(M'_j(\mathcal{E}))}{\det(M'_{j-1}(\mathcal{E}))} > 0.
\]
In particular, if \( \dim_{\mathbb{R}}(P_{i,\ell}) = 1 \) for \( 0 \leq i \leq r \) and \( P_{i,\ell} = 0 \) for \( i > r \), then \( \ell \) is HR for \( B \) in degree \( i \) if and only if for each \( 0 \leq j \leq i \leq r \),

\[
\sgn(\det(M'_i(\mathcal{E}))) = (-1)^{\binom{i+1}{2}}.
\]

**Proof.** Let \( \mathcal{E} \) be any basis respecting the primitive decomposition of \( B \) with respect to \( \ell \). Then for each \( 0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor \), the matrix for the \( i \)th Lefschetz map is a block matrix of the form

\[
M'_i(\mathcal{E}) = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & M'^{-1}(\mathcal{E}) \end{pmatrix},
\]

where \( \mathcal{B} \in \mathcal{E} \cap P_{i,\ell} \), and the first assertion follows. Note that if \( P_{i,\ell} \) is one dimensional for \( 0 \leq i \leq r \) and zero for \( i > r \), and if \( 0 \leq i \leq r \), then the matrix in Equation 2.5 is block diagonal with \( 1 \times 1 \) blocks, and hence the second assertion follows by induction on \( j \).

\[ \square \]

3. A HESSIAN CRITERION FOR HR

Let \( R = \mathbb{R}[x_1, \ldots, x_n] \) and \( Q = \mathbb{R}[X_1, \ldots, X_n] \) be polynomial rings where \( R \) acts on \( Q \) by differentiation, i.e.

\[ x_i \circ F = \frac{\partial F}{\partial x_i}, \quad 1 \leq i \leq n, \quad F \in Q. \]

A homogeneous polynomial \( F \in Q \) of degree \( d \) determines an oriented graded AG algebra \( A = R/\text{Ann}(F) \) of socle degree \( d \) with orientation given by \( \int_A \alpha = (\alpha \circ F)(0), \quad \forall \alpha \in A \).

Fix any homogeneous basis \( \mathcal{E} \) for \( A \), and for each degree \( 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor \) suppose that \( \mathcal{E}_i = \{e^i_1, \ldots, e^i_m\} \). Define the \( i \)th Hessian matrix of \( F \) with respect to \( \mathcal{E} \) as the \( m \times m \) polynomial matrix

\[
\text{Hess}_i(F) = \text{Hess}_i(F, \mathcal{E}) = (e^i_j \circ F)_{1 \leq j, k \leq m}.
\]

Note that the entries of \( \text{Hess}_i(F) \) are polynomials in the variables \( X_1, \ldots, X_n \). Given real numbers \( C_1, \ldots, C_n \), we shall write \( C = (C_1, \ldots, C_n) \in \mathbb{R}^n \) in vector notation and write \( \text{Hess}_i(F)|_C \) to mean the numerical matrix obtained by substituting the real number \( C_i \) for the variable \( X_i \) for each \( 1 \leq i \leq n \).

**Theorem 3.1.** Let \( \mathcal{E} \) be any basis for \( A = R/\text{Ann}(F) \) and let \( \ell = \ell(C) = C_1 x_1 + \cdots + C_n x_n \in A_1 \) be any linear form in \( A \). Then for \( C = (C_1, \ldots, C_n) \in \mathbb{R}^n \)

\[
\text{Hess}_i(F, \mathcal{E})|_C = M'_{\ell(C)}(\mathcal{E}).
\]

In particular, the signature of \( \text{Hess}_i(F, \mathcal{E})|_C \) is independent of our choice of basis \( \mathcal{E} \).

**Proof.** The key observation here is the following formula: for any homogeneous form \( G \in Q \) of any degree \( a \), and any linear form \( \ell = C_1 x_1 + \cdots + C_n x_n \in R_1 \) as above, we have

\[
\ell^a \circ G = a! \cdot G(C_1, \ldots, C_n).
\]

To see this, note first that it holds for \( G = X_1^{e_1} \cdots X_n^{e_n} \) a monomial:

\[
\ell^a \circ G = \frac{a!}{e_1! \cdots e_n!} X_1^{e_1} \cdots X_n^{e_n} \circ X_1^{e_1} \cdots X_n^{e_n} = a! G(C_1, \ldots, C_n),
\]
and then it must hold for all homogeneous $G$ by linearity of the $R$-action on $Q$. Since the orientation on $A$ satisfies
\[ \int_A \alpha = (\alpha \circ F)(0), \forall \alpha \in A, \]
the $(j,k)$-entry of $M^i_{\ell(C)}(E)$ is
\[ m^i_{jk} = \int_A e_k^i \ell(C)^{d-2i} e_j^i = (\ell(C)^{d-2i} e_k^i \circ F)(0) = \ell(C)^{d-2i} \circ (e_k^i e_j^i \circ F) = (e_j^i e_k^i \circ F)|_C, \]
which is the $(j,k)$-entry of $\text{Hess}(F)|_C$ as claimed by the first assertion. The second assertion follows from the first.

Theorem 3.1 gives a criterion on the Macaulay dual generator for the associated AG algebra $A = R/\text{Ann}(F)$ to have HRP, via Lemma 2.4. We state this as a corollary below. Incidentally Theorem 3.1 also shows that the signature (and the determinant) of the $i^{th}$ Hessian matrix of a homogeneous form $F = F(X_1, \ldots, X_n)$ is invariant under the action of $\text{GL}(n, \mathbb{R})$.

**Corollary 3.2.** Given an oriented graded AG algebra $A = \mathbb{R}[x_1, \ldots, x_n]/\text{Ann}(F)$, a linear form $\ell = \ell(C) = C_1 x_1 + \cdots + C_n x_n \in A_1$ is HR for $A$ in degree $i$ if and only if $\det(\text{Hess}_j(F)|_C) \neq 0$ and
\[ \text{sgn}(\text{Hess}_j(F)|_C) = \text{sgn}(\text{Hess}_{j-1}(F)|_C) + (-1)^j(\dim_{\mathbb{R}}(A_i) - \dim_{\mathbb{R}}(A_{i-1})) \]
for all $j \leq i$.

### 4. The Two Variable Case

Let $A = A_F = \mathbb{R}[x, y]/\text{Ann}(F = F(X, Y))$ be an oriented AG algebra of socle degree $d$ and codimension two. Then the Hilbert function of $A$ has the form
\[ H(A) = (1, 2, 3, \ldots, r, r - 1, r - 1, \ldots, 3, 2, 1) \]
for some $0 \leq r, s \leq d$. The largest value of the Hilbert function, $r = r(A)$, is called the Sperner number of $A$, and, in codimension two, coincides with smallest degree of a homogeneous element of the ideal Ann$(F)$, called the order of the ideal Ann$(F)$. We say the Sperner number of $F$ to mean that of $A$. Note that for a generic $\ell \in A_1$, we have $r - 1 = \max \{ i \mid P_{i, \ell} \neq 0 \} = \max \{ i \mid h_i - h_{i-1} \neq 0 \}$. A direct application of Lemma 2.5 and Theorem 3.1 yields the following:

**Lemma 4.1.** Given $F = F(X, Y)$ as above, with Sperner number $r$, and given any $0 \leq i < r$, the pair $(A_F = \mathbb{R}[x, y]/\text{Ann}(F), \ell(C) = C_1 x + C_2 y)$ satisfies HRR in degree $i$ if and only if for each $0 \leq j \leq i$,
\[ \text{sgn}(\det(\text{Hess}_j(F)|_C)) = (-1)^{\lfloor \frac{j+1}{2} \rfloor}. \]

Following [2], we define the normalization operator $N: \mathbb{R}[X, Y] \to \mathbb{R}[X, Y]$ as the $\mathbb{R}$-linear map defined on monomials by
\[ N(X^a Y^b) = \frac{X^a Y^b}{a! b!}. \]
Theorem 4.2 (Theorem B). Let $F = \sum_{j=0}^d a_{d-j} X^{d-j} Y^j$ with $a_0 \neq 0$ and let $r = r(N(F))$ be the Sperner number of its normalization $N(F)$. Then for any $0 \leq i < r$, the $i$th Hessian of $N(F)$, evaluated at $X = 0$ and $Y = 1$ satisfies

$$\det(\text{Hess}_i(N(F))(0,1)) = (-1)^{\frac{i+1}{2}} \cdot \left(\frac{1}{(d-2i)!}\right)^{i+1} \cdot \det(a_{i+q-p})_{0 \leq p,q \leq i}.$$ 

Proof. The normalization of $F$ is $N(F) = \sum_{j=0}^d b_{d-j} X^{d-j} Y^j$, where $b_{d-j} = a_{d-j}/((d-j)!)$, and the $(p,q)$-entry of its $i$th Hessian matrix, with respect to the ordered monomial basis $\mathcal{E}_i = \{e_p^i = x^{i-p} y^p \mid 0 \leq p \leq i\}$ (which is a basis for $(A_{N(F)}), i = 0 \leq i < r$), evaluated at $X = 0$ and $Y = 1$ is

$$\frac{\partial^2 N(F)}{\partial X^{2i-p-q} \partial Y^{p+q}}(0,1) = (2i - p - q) \cdot (d - 2i + p + q) \cdots (d - 2i + 1) \cdot b_{2i-p-q} = \frac{1}{(d-2i)!} \cdot a_{2i-p-q}.$$ 

Swapping columns $q$ and $i - q$ for $0 \leq q \leq \left\lfloor \frac{i}{2} \right\rfloor$ in the matrix $(a_{2i-p-q})_{0 \leq p,q \leq i}$ yields the matrix $(a_{i+q-p})_{0 \leq p,q \leq i}$ whose determinant differs by a sign $(-1)^{\frac{i+1}{2}}$. Therefore we get

$$\det(\text{Hess}_i(N(F))(0,1)) = (-1)^{\frac{i+1}{2}} \cdot \left(\frac{1}{(d-2i)!}\right)^{i+1} \cdot \det(a_{i+q-p})_{0 \leq p,q \leq i}$$

as claimed. \qed

Corollary 4.3. Let $F = \sum_{i=0}^d a_{d-i} X^{d-i} Y^i$ be as above, with $a_0 \neq 0$ and let $r = r(N(F))$ be the Sperner number of its normalization. Then the pair for any $0 \leq i < r$, the pair $(A = A_{N(F)}, \ell = y)$ satisfies HRR in degree $i$ if and only if for each $0 \leq j \leq i$,

$$\det(a_{i+q-p})_{0 \leq p,q \leq j} > 0.$$ 

Writing $F = a_0 \cdot \sum_{i=0}^d e_{d-i} X^{d-i} Y^i$, one can identify the coefficients $e_{d-i}$ as the elementary symmetric polynomials in the negative roots $\alpha = (\alpha_1, \ldots, \alpha_d)$ of the degenerated univariate polynomial of degree $d$, $F(1,t)$. In this case, the determinant is

$$\det(a_{i+q-p})_{0 \leq p,q \leq j} = a_0^{i+1} \cdot \det(e_{i+q-p}(\alpha))_{0 \leq p,q \leq j} = a_0^{i+1} \cdot s_{(i)}$$

where $s_{(i)}(\alpha)$ is the Schur polynomial of the rectangular partition $(i + 1)^j$ in the roots $\alpha$.

Corollary 4.4 (Theorem C). Let $F = \sum_{i=0}^d a_{d-i} X^{d-i} Y^i = a_0 \cdot \sum_{i=0}^d e_{d-i} X^{d-i} Y^i$ be a homogeneous polynomial of degree $d$ with $a_0 \neq 0$, and let $r = r(N(F))$ be the Sperner number of its normalization.

(1) If $F(1,t)$ has only real roots and $r > 1$, then $(A = A_{N(F)}, \ell = y)$ satisfies HRR in degree $i = 1$.

(2) If $F(1,t)$ has only real non-positive roots, then $(A = A_{N(F)}, \ell = y)$ satisfies the HRP.

Proof. To prove (1), note for $i = 1$, the Schur function for the rectangular partition $\lambda(1) = (2) = \square$ is

$$s_{\lambda(1)}(x) = \sum_{1 \leq i, j \leq d} x_i x_j = \frac{1}{2} (x_1^2 + \cdots + x_d^2) + \frac{1}{2} (x_1 + \cdots + x_d)^2$$

which is positive definite on $\mathbb{R}^d$. It follows that if $F(1,t)$ has only real roots, say $\alpha = (\alpha_1, \ldots, \alpha_d)$, then $s_{\lambda(1)}(\alpha) > 0$, hence by Corollary 4.3, the pair $(A_{N(F)}, y)$ satisfies HRR in degree one.
To prove (2), note first that if \( F(1, t) \) has only real negative roots, say \(-\alpha = (-\alpha_1, \ldots, -\alpha_d)\), where \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_{>0}^d \), then \( s_{d(i)}(\alpha) > 0 \) for each all \( i \) because Schur functions are sums of monomials with positive coefficients. Moreover since \( s_{d(i)}(x) \) is homogeneous of even degree, it follows that \( s_{d(i)}(-\alpha) = s_{d(i)}(\alpha) > 0 \) too. Therefore it follows from Corollary 4.3 that \((A_{N(F)}, y)\) has the HRP. In general some of the (negative) roots \( \alpha_1, \ldots, \alpha_d \) may be zero; we can label the roots so that \( \alpha_1 = \cdots = \alpha_k = 0 \) and \( \alpha_{k+1}, \ldots, \alpha_d \) are positive. In this case, \( F \) has the form \( F = Y^k \cdot G \), and hence also \( N(F) = Y^k \cdot H \) where \( G \) and \( H \) are not multiples of \( Y \). Since \( x^d-k+1 \circ N(F) = 0 \), it follows that the Sperner number of \( N(F) \) is at most the degree of \( H \), i.e. \( r \leq d - k + 1 \). In particular, for each \( 0 \leq i < r \leq d - k + 1 \), there is at least one semi-standard Young tableau on \( \lambda(i) \) with entries consisting only of the \( d - k \) indices \( \{k+1, \ldots, d\} \), e.g. each box in row 1 \( \leq j \leq i \) gets one number from the list (for \( i = 0 \), we should take \( s_{d(i)}(1) = 1 \)). It follows that \( s_{d(i)}(\alpha) = s_{d(i)}(-\alpha) > 0 \) and hence again by Corollary 4.4, \((A_{N(F)}, y)\) satisfies HRP.

Polynomials \( F = F(X, Y) \) such that \( F(1, t) \) has only real non-positive roots are called stable polynomials, and are discussed later in this section.

4.1. Higher Lorentzian Polynomials.

**Definition 4.5.** Given integers \( 0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor \), a sequence of real numbers \( (c_0, \ldots, c_d) \) is called *log concave of order* \( i \) if for each \( 0 \leq j \leq i \) and for every \( j \leq t \leq d - j \), the determinant of the shifted Hankel matrix \( (c_{i+q-p})_{0 \leq p, q \leq j} \) is non-negative, i.e.

\[
\begin{vmatrix}
    c_t & \cdots & c_{t+j} \\
    \vdots & \ddots & \vdots \\
    c_{t-j} & \cdots & c_j
\end{vmatrix} \geq 0
\]

(we count \( c_i = 0 \) if \( i < 0 \) or \( i > d \)). The sequence is called *ultra log concave of order* \( i \) if the sequence \( \left( \frac{c_0}{c_i}, \frac{c_1}{c_i}, \ldots, \frac{c_d}{c_i} \right) \) is log concave of order \( i \). The sequence is called *strictly log concave* (resp. *strictly ultra log concave*) of order \( i \) if the inequalities above are strict.

Note that log concave of order \( i = 0 \) is simply the condition that \( c_t \geq 0 \) for each \( 0 \leq t \leq d \). Log concave of order \( i = 1 \) means \( c_t \geq 0 \) for all \( 0 \leq t \leq d \) together with the usual log concave condition

\[
\det\begin{pmatrix}
    c_t & c_{t+1} \\
    c_{t-1} & c_t
\end{pmatrix} = c_{t}^{2} - c_{t-1}c_{t+1} \geq 0, \quad 1 \leq t \leq d - 1.
\]

Note also that the implication “log concave of order \( i \)” implies “log concave of order \( i - 1 \)” is built into this definition. Finally, note that the condition(s) “(strictly,ultra) log concave of order \( i \)” is symmetric meaning that \( (c_0, \ldots, c_d) \) is (strictly,ultra) log concave of order \( i \) if and only if \( (c_d, \ldots, c_0) \) is (strictly,ultra) log concave of order \( i \).

**Definition 4.6.** A homogeneous polynomial in two variables \( F = \sum_{i=0}^{d} c_i X^{d-i} Y^i \) is called *Lorentzian of order* \( i \) if its coefficient sequence \( (c_0, \ldots, c_d) \) is ultra log concave of order \( i \) with no internal zeros. It is called *strictly Lorentzian of order* \( i \) if its coefficient sequence is strictly ultra log concave of order \( i \). Let \( L_{d}(i) \subseteq L_{d}(i) \subseteq \mathbb{R}[X, Y]_d \) denote the set of strictly Lorentzian and Lorentzian polynomials of degree \( d \), respectively.
Note that class of Lorentzian polynomials of order \( i = 1 \) coincides with the class of Lorentzian polynomials defined by Brändén and Huh [2, Example 2.26].

4.2. **Classification of codimension two AG algebras satisfying HRR.** We would like to prove the following, which identifies Lorentzian polynomials of order \( i \) as a sort of class of normal forms for the Macaulay dual generators of codimension two AG algebras satisfying HRR of degree \( i \).

**Theorem 4.7 (Theorem A).** Let \( F = F(X, Y) = \sum_{i=0}^{d} c_i X^{d-i} Y^i \) be a homogeneous polynomial of degree \( d \) and Sperner number \( r \), and let \( A_F = \mathbb{R}[x, y]/\text{Ann}(F) \) be its corresponding AG algebra of socle degree \( d \), and fix an index \( 0 \leq i < r \).

1. If \( F \) is strictly Lorentzian of order \( i \), then \( A_F \) satisfies HRR in degree \( i \).
2. If \( A_F \) satisfies HRR in degree \( i \), then there exists a linear change of variables \( g \in \text{GL}(2, \mathbb{R}) \) such that \( g.F \) is strictly Lorentzian of order \( i \).

**Proof.** Assume that \( F \) is strictly Lorentzian of order \( i \), and define the polynomial

\[
G = \sum_{i=0}^{d} e_{d-i} \cdot X^{d-i} Y^i,
\]

where \( e_{d-i} = c_i (d-i)!(d)! / c_i / \binom{d}{i} \), so that \( N(G) = F = \sum_{i=0}^{d} c_i X^{d-i} Y^i \). Then by Theorem 4.2 we have

\[
\det(\text{Hess}_i(F)_{(0,1)}) = (-1)^{\frac{i+1}{2}} \cdot \left( \frac{1}{(d-2)!} \right)^{i+1} \cdot \det(e_{i+q-p})_{0 \leq p, q \leq i}.
\]

Since \( F \) is Lorentzian of order \( i \), the sequence \((c_0, \ldots, c_d)\) is strictly ultra log concave of order \( i \). It follows that the sequence \((e_d, \ldots, e_0) = (d! \cdot c_0 / \binom{d}{0}, \ldots, d! \cdot c_d / \binom{d}{d})\) is strictly log concave of order \( i \) and hence by symmetry so is the sequence \((e_0, \ldots, e_d)\). It therefore follows from the definition that the determinant \( \det(e_{i+q-p}) \) is positive for each \( j \leq t \leq d-j \) and each \( 0 \leq j \leq i \). Thus it follows that for each \( 0 \leq j \leq i \), we have \( \det(\text{Hess}_i(F)_{(0,1)}) \neq 0 \) and

\[
\operatorname{sgn}\left( \det(\text{Hess}_i(F)_{(0,1)}) \right) = (-1)^{\frac{i+1}{2}}
\]

which implies, by Lemma 4.1, that \( A_F \) satisfies HRR in degree \( i \).

Conversely assume that \( A = A_F \) satisfies HRR in degree \( i \), and fix a linear element \( \ell_1 \in A_1 \) that is HR of degree \( i \) for \( A \). Then we may choose another linear form \( \ell_2 \in A_1 \) satisfying the following conditions:

1. \( \ell_1, \ell_2 \) are linearly independent
2. \( \int_{A} \ell_1^{d-i} \cdot \ell_2 > 0, \forall 0 \leq i \leq d \)
3. \( \ell_2 \in A/(0 : A \ell_1^j) \) is HR in degree \( i \) for all \( 0 \leq j \leq i \).

Regarding \( \ell_1, \ell_2 \in R_1 \) as a linear basis for \( A_1 \) (and hence also a set of algebra generators), let \( L_1, L_2 \in Q_1 \) be the dual basis, meaning that \( \ell_1 \circ L_j = \delta_{i,j} \) for each \( i, j = 1, 2 \). Then \( L_1 \) and \( L_2 \) is a linear basis (hence also an algebra generating set) for \( Q_1 \), hence we may write \( F \) as
a polynomial in these generators, i.e.

\[ F = \sum_{i=0}^{d} b_i L_1^{d-i} L_2^i. \]

We want to show that \( F \) is strictly Lorentzian of order \( i \) in this new basis, or equivalently, that the coefficient sequence \((b_0, \ldots, b_d)\) is strictly ultra log concave of order \( i \). Note that we have

\[ a_i := \int_A \ell_1^{d-i} \cdot \ell_2^i = \ell_1^{d-i} \cdot \ell_2^i \circ F = b_i(d-i)! = d! \cdot \frac{b_i}{i!}. \]

Therefore it suffices to show that the sequence \((a_0, \ldots, a_d)\) is strictly log concave of order \( i \), where \( a_j = \int_A \ell_1^{d-j} \ell_2^j \). Note first that \( a_j > 0 \) for all \( j \) by condition (ii) above. Also note that if \( A' = A/(0 : A \ell_1) \), then for \( j < d \) we have

\[ a_j = \int_A \ell_1^{d-j} \ell_2^j = \int_{A'} \ell_1^{d-j} \ell_2^j = a'_j \]

by condition (iii) above.

Then inducting on the socle degree (the base case is trivial), we may assume that the sequence \((a'_0, \ldots, a'_{d-1}) = (a_0, \ldots, a_{d-1})\) is strictly log concave of order \( i \). To complete the induction we must show that for each \( 0 \leq j \leq i \), the determinant

\[ \det \begin{pmatrix} a_{d-j} & \cdots & a_d \\ \vdots & \ddots & \vdots \\ a_{d-2j} & \cdots & a_{d-j} \end{pmatrix} \]

is positive. Setting \( G = \sum_{i=0}^{d} a_i X^{d-i} Y^i \), then \( N(G) = F \), and hence by Theorem 4.2, it follows that

\[ \det(\text{Hess}(F)|_{(0,1)}) = (-1)^{\frac{j(j-1)}{2}} \cdot \left( \frac{1}{(d-2j)!} \right)^{j+1} \cdot \det(a_{d-j+q-p})_{0 \leq p, q \leq j}. \]

Since \( y \in A_1 \) is HR in degree \( i \) for \( A \), it follows that \( \det(a_{d-j+q-p}) > 0 \) for each \( 0 \leq j \leq i \), and the result follows.

\[ \square \]

4.3. Stable Polynomials and PF Sequences. Recall the definition of stable polynomial:

**Definition 4.8.** A homogeneous polynomial \( F = \sum_{i=0}^{d} c_i X^{d-i} Y^i \) is called stable if its coefficients are non-negative, and its dehomogenization \( F(1, t) \) has only real roots. Let \( S_d \subset Q_d \) denote the set of stable polynomials of degree \( d \).

**Definition 4.9.** A sequence of real numbers \((c_0, \ldots, c_d)\) is called a Pólya frequency (PF) sequence of order \( r \) if for every \( 1 \leq s \leq r \), every \( s \times s \) minor of the infinite matrix \( C = (c_{q-p})_{-\infty < p, q < \infty} \) is non-negative (count \( c_i = 0 \) if \( i < 0 \) or \( i > d \)). It is called a PF sequence if it is a PF sequence of order \( r \) for all \( r \geq 1 \).

Write \( C[p_0, \ldots, p_r; q_0, \ldots, q_r] \) for the \( r \times r \) submatrix of \( C \) obtained by taking rows \( p_0 < \cdots < p_r \) and columns \( q_0 < \cdots < q_r \) from \( C \).

**Lemma 4.10.** If \((c_0, \ldots, c_d)\) is a PF sequence of order \( i \), then it is log concave of order \( i \).
Proof. Assume that \((c_0, \ldots, c_d)\) is a PF sequence of order \(i\). Then for every \(0 \leq j \leq i\), every \(j \times j\) minor of the infinite matrix \(C = (c_{q-p})_{-\infty < p < \infty}\) is non-negative. Then in particular the minor

\[
C[0, \ldots, j; t, \ldots, t+j] = \det \begin{pmatrix} c_t & \cdots & c_{t+j} \\ \vdots & \ddots & \vdots \\ c_{t-j} & \cdots & c_t \end{pmatrix}
\]

is non-negative for each \(0 \leq j \leq i\) and each \(j \leq t \leq d - j\), and hence \((c_0, \ldots, c_d)\) is log concave of order \(i\).

The following result is due to A. Edrei [5, Proposition B], and we refer the reader there for its proof; see also [4, Theorem 2.2.4].

**Proposition 4.11.** A polynomial \(f(t) = \sum_{i=0}^d c_i t^i\) with non-negative coefficients has only real zeros if and only if \((c_0, \ldots, c_d)\) is a PF sequence.

In other words, \(F = \sum_{i=0}^d c_i X^{d-i} Y^i\) is stable if and only if \((c_0, \ldots, c_d)\) is a PF sequence.

The following is a direct consequence of Lemma 4.10 and Proposition 4.11.

**Corollary 4.12.** If \(F = F(X, Y)\) is stable, then its normalization \(N(F)\) is Lorentzian of order \(i\) for all \(0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor\).

Compare Corollary 4.12 with Corollary 4.4. One can show, e.g., [2, Proposition 2.2], that every stable polynomial itself is Lorentzian of order \(i = 1\), but the following example shows that such containment should not be expected for \(i > 1\).

**Example 4.13.** Let \(f(t) = t^4 - t^2\) so that \(F = Y^4 - X^2 Y^2\) and hence \(N(F) = \frac{1}{24} Y^4 - \frac{1}{4} X^2 Y^2 = \frac{1}{24}(Y^4 - 6X^2 Y^2)\). Then we have

\[
\det(\text{Hess}_1(N(F))|_{(0,1)}) = \det \frac{1}{24} \begin{pmatrix} -12 & 0 \\ 0 & 12 \end{pmatrix} = -\frac{1}{4} < 0,
\]

and it follows that \(\ell = y\) is HR in degree one for \(A\). However upon computing the second Hessian, we see that \(A\) cannot have any linear forms satisfying HRR in degree two:

\[
\det(\text{Hess}_2(N(F))) = \det \frac{1}{24} \begin{pmatrix} 0 & 0 & -24 \\ 0 & -24 & 0 \\ -24 & 0 & 0 \end{pmatrix} = 1 > 0.
\]

Note however that \(\ell = y\) is SL for \(A\). Also note that we can change coordinates, to get say \(G(X, Y) = g.F = F(X, 2X + Y) = Y^4 + 8Y^3 X + 23Y^2 X^2 + 28YX^3 + 12X^4\) which is stable, with roots \(\alpha = (-2, -2, -3, -1)\); on the other hand, note that \(G\) is not Lorentzian of order \(i = 2\).

5. **Open Problems**

In their paper [2], P. Brändén and J. Huh have defined Lorentzian (and stable) polynomials in \(n\)-variables, and we think it would be interesting to extend our results to this more general framework. We give their definitions below, and end with some open problems.

Let \(Q = \mathbb{R}[X_1, \ldots, X_n]\) be the standard graded polynomial ring in \(n\)-variables with real coefficients, let \(Q_d \subset Q\) be the degree \(d\) graded piece, i.e., the set of homogeneous polynomials of degree \(d\), and let \(P_d \subset Q_d\) denote the subset of homogeneous polynomials whose
coefficients are positive. The following definitions are taken directly from [2, Definition 2.1].

**Definition 5.1.** Define the set of strictly Lorentzian polynomials $\mathcal{L}_d \subset P_d$ inductively as follows: 
\[ \mathcal{L}_0 = P_0, \mathcal{L}_1 = P_1, \text{ and } \]
\[ \mathcal{L}_d = \{ F \in P_d \mid \text{Hess}_t(F) \text{ is non-singular and has exactly one positive eigenvalue} \}. \]

Then for $d > 2$ define
\[ \mathcal{L}_d = \{ F \in P_d \mid x_i \circ F \in \mathcal{L}_{d-1}, \text{ for all } i \}. \]

**Definition 5.2.** The set of stable polynomials $S_d \subset Q_d$ consists of homogeneous polynomials $F$ with non-negative coefficients satisfying the following condition: For some $U = (U_1, \ldots, U_n) \in \mathbb{R}_{\geq 0}^n$, $F(U) > 0$ and for every $V = (V_1, \ldots, V_n) \in \mathbb{R}^n$, the univariate polynomial $f(t) = F(tU - V) \in \mathbb{R}[t]$ has only real roots. Denote the set of stable polynomials by $S_d \subset Q_d$.

**Definition 5.3.** A subset $J \subset \mathbb{N}^n$ is called $M$-convex if it satisfies the following exchange property: For each $\alpha, \beta \in J$ and each index $i$ satisfying $\alpha_i > \beta_i$, there exists an index $j$ satisfying $\alpha_j < \beta_j$ and $\alpha - \epsilon_i + \epsilon_j \in J$, where $\epsilon_i \in \mathbb{N}^n$ is the $i^{th}$ standard basis vector.

A homogeneous polynomial $F = \sum_{\alpha \in J} c_{\alpha} X^\alpha \in Q_d$ is called $M$-convex if its support $\text{supp}(F) = \{ \alpha \mid c_{\alpha} \neq 0 \} \subset \mathbb{N}^n$ is $M$-convex. Denote the set of $M$-convex polynomials by $M_d \subset Q_d$.

As a consequence of [3, Theorem 3.2], every stable polynomial is $M$-convex, i.e. $S_d \subset M_d$.

**Definition 5.4.** Define the set of Lorentzian polynomials $L_d \subset Q_d$ inductively as follows: 
Set $L_1 = S_1$, $L_2 = S_2$, and for $d > 2$ define
\[ L_d = \{ F \in M_d \mid x_i \circ F \in L_{d-1}, \forall 1 \leq i \leq n \}. \]

According to [2, Theorem 2.25], the set of Lorentzian polynomials $L_d$ is equal to the closure of the set of strictly Lorentzian polynomials $\mathcal{L}_d$ with respect to the Euclidean topology on $Q_d$. As in the two variable case, the normalization operator $N : Q \to Q$ is the $\mathbb{R}$-linear map defined on monomials by
\[ N(X_1^{a_1} \cdots X_n^{a_n}) = \frac{X_1^{a_1} \cdots X_n^{a_n}}{a_1! \cdots a_n!}. \]

The following is [2, Theorem 2.16].

**Fact 5.5.** If $F$ is strictly Lorentzian, then for any $C = (C_1, \ldots, C_n) \in \mathbb{R}_{\geq 0}^n$, the pair $(A_F, \ell(C))$ satisfies HRR in degree $i = 1$.

Interestingly, Murai-Nagaoka-Yazawa [7, Theorem 3.8] have shown that Fact 5.5 holds for all Lorentzian polynomials:

**Fact 5.6.** If $F$ is Lorentzian, then for any $C = (C_1, \ldots, C_n) \in \mathbb{R}_{\geq 0}^n$, the pair $(A_F, \ell(C))$ satisfies HRR in degree $i = 1$. 
Here are some open problems.

**Problem 5.7.** Prove or disprove: If $F = F(X, Y)$ is Lorentzian of order $i$, then $A_F$ satisfies HRR in degree $i$.

**Problem 5.8.** Prove or disprove: If $A_F = \mathbb{R}[x_1, \ldots, x_n]/\text{Ann}(F = F(X_1, \ldots, X_n))$ satisfies HRR in degree $i = 1$, then there exists a linear change of coordinates $g \in \text{GL}(2, \mathbb{R})$ such that $g.F$ is Lorentzian.

**Problem 5.9.** What are the higher Lorentzian polynomials in $n > 2$ variables?

**Problem 5.10.** Prove or disprove: If $F = F(X_1, \ldots, X_n)$ is stable, then $A_{N(F)}$ satisfies the HRP.

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