Linear Process Bootstrap Unit Root Test

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Abstract

One of the most widely applied unit root test, Phillips-Perron test, enjoys in general high
powers, but suffers from size distortions when moving average noise exists. As a remedy, this
paper proposes a nonparametric bootstrap unit root test that specifically targets moving aver-
age noise. Via a bootstrap functional central limit theorem, the consistency of this bootstrap
approach is established under general assumptions which allows a large family of non-linear time
series. In simulation, this bootstrap test alleviates the size distortions of the Phillips-Perron
test while preserving its high powers.

1 Introduction

Among extensive literature on unit root test, the Augmented Dickey-Fuller (ADF) test and the
Phillips-Perron (PP) test are perhaps the most renowned. When put into simulation, PP test has
been found to enjoy higher power than ADF test but suffers greater size distortion, especially under
negative Moving Average (MA) noises (Phillips and Perron [17], Nabeya and Perron [9], Cheung
and Lai [3], Leybourne and Newbold [7]). For a solution to this size distortion, see Perron and Ng [15].

We propose a bootstrap unit root test as a remedy. When the asymptotic distributions of the
test statistics involve unknown parameters, bootstrap circumvents the estimation of the unknown
parameters and as a result eases the hypothesis test. On the other hand, when the asymptotic
distributions are pivotal, bootstrap unit root test may enjoy second order efficiency, and may con-
sequently reduce the aforementioned size distortion (Park [13]). Variants of bootstrap unit root
test include AutoRegressive (AR) sieve bootstrap test (Psaradakis [22], Palm, Smeekes, and Urbain
[11]), block bootstrap test (Paparoditis and Politis [12]), stationary bootstrap test (Swensen [25],
Parker, Paparoditis, and Politis [14]), and wild bootstrap test (Cavaliere and Taylor [2]).

To target the size distortion with MA noise, we apply Linear Process Bootstrap (LPB) of Mc-
Murry and Politis [8] to unit root test. As the closest analogue of MA-sieve bootstrap, LPB first
estimates the autocovariance matrix of the noise, then pre-whitens the noise with the estimated
autocovariance matrix, then bootstraps from the pre-whitened noise, and finally post-colors
the bootstrap noise with the the estimated autocovariance matrix. In sample mean case, McMurry and
Politis [8], Jentsch, Politis, et al. [5] indicate good asymptotic and empirical performance of LPB,
particularly in the presence of MA noise.

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As a result, LPB unit root test becomes a promising solution to the size distortion under MA noise. To develop a large sample theory for LPB unit root test, we extend the bootstrap Central Limit Theorem (CLT) for LPB into a regression setting, establish a bootstrap Functional CLT (FCLT) for LPB, and prove the consistency of this bootstrap method. Despite its name, LPB unit root test turns out to be asymptotically valid under not only linear noises but also a large family of non-linear noises, i.e., the physical dependent process defined in Wu [26].

This paper proceeds as follows. Section 2 specifies the physical dependence assumption and recalls the popular Phillips-Perron test. Section 3 introduces LPB unit root test, details the estimation of the autocovariance matrix, and describes the adaptive bandwidth selection. Section 4 presents the empirical results of LPB unit root test. Appendix includes all technical proofs.

2 Phillips-Perron Test

Suppose \( \{Y_t\}_{t=1}^n \) is observable. For \( t \in \mathbb{N}^+ \), define \( \phi_t \) and \( V_t \) as the prediction coefficient and error, respectively, when predicting \( Y_t \) with \( Y_{t-1} \). Suppose \( \phi_t = \phi \) for all \( t \in \mathbb{N}^+ \). Then

\[
Y_t = \phi Y_{t-1} + V_t. \tag{2.1}
\]

Now we assume the noise sequence \( \{V_t\}_{t \in \mathbb{Z}} \) is strictly stationary, short-range dependent, and invertible. Specifically, consider the following assumptions on \( \{V_t\}_{t \in \mathbb{Z}} \).

**Assumption 2.1.** Let \( \{\epsilon_t\}_{t \in \mathbb{Z}} \) be a sequence of i.i.d. random variables. Let \( \epsilon_0' \) be identically distributed with \( \epsilon_0 \), and be independent of \( \{\epsilon_t\}_{t \in \mathbb{Z}} \). Suppose \( V_t = g(..., \epsilon_{t-1}, \epsilon_t) \). Let \( V_t^p = g(..., \epsilon_{t-1}, \epsilon_0', \epsilon_1, ..., \epsilon_t) \), and let \( \delta_p(t) = (E(V_t - V_t^p)^p)^{1/p} \) be the physical dependence measure of \( \{V_t\} \). Suppose \( \sum_{t=1}^{\infty} \delta_4(t) < \infty \). Let \( \gamma(h) = E(V_t V_{t-h}) \). Suppose \( \sum_{h \in \mathbb{Z}} \gamma(h) > 0 \), \( \sum_{h=0}^{\infty} h |\gamma(h)| < \infty \), \( E(V_t) = 0 \), and \( E(V_t^4) < \infty \).

**Assumption 2.2.** Recall Assumption 2.1. Further assume that for some \( p > 4 \), \( \sum_{t=1}^{\infty} \delta_p(t) < \infty \) and \( E(|V_t|^p) < \infty \); for some \( \beta > 2 \), \( |\gamma(h)| = o(h^{-\beta}) \); for some \( \alpha > 0 \), \( h^\alpha \sum_{k=h+1}^{\infty} |\gamma(k)| \) is non-increasing when \( h \) is large enough.

When \( \phi = 1 \), suppose \( Y_0 = 0 \); then \( \{Y_t\}_{t \in \mathbb{N}^+} \) is a unit root process starting at zero. When \( \phi < 0 \), suppose (2.1) holds for all \( t \in \mathbb{Z} \); then \( \{Y_t\}_{t \in \mathbb{N}^+} \) is a strictly stationary process. To separate these two cases, we test

\[
H_0 : \phi = 1 \ vs \ H_1 : \phi < 1. \tag{2.2}
\]

The famous PP test centers on the OLS estimator \( \hat{\phi} \) in \( Y_t = \hat{\phi} Y_{t-1} + \hat{V}_t \), and its t-statistic \( t \). Under Assumption 2.1, the asymptotic null distributions of \( \hat{\phi} \) and \( t \) results from the FCLT in Lemma 2.1

**Lemma 2.1** (Wu [26]). Suppose Assumption 2.1 holds. Let \( \sigma^2 = Var(n^{-1/2} \sum_{t=1}^{n} V_t) \), \( S(u) = n^{-1/2} \sigma^{-1} \sum_{i=1}^{[nu]} V_t \). Let \( W(u) \) be a standard Brownian motion. If \( \phi = 1 \), \( S \Rightarrow W \).

3 Linear Process Bootstrap Unit Root Test

As mentioned in introduction, PP test enjoys high empirical powers, but suffers from empirical size distortions under negative MA noise. To mitigate the size distortion while preserving the high power, we introduce LPB unit root test below. The name of LPB follows from the fact that the
bootstrapped noise is a linear process.

Let $V = (V_1, \ldots, V_n)'$, $\hat{V} = (\hat{V}_1, \ldots, \hat{V}_n)'$, $\hat{V} = (\hat{V}_1, \ldots, \hat{V}_n)'$, $V^* = (V_1^*, \ldots, V_n^*)'$, $\hat{\epsilon} = (\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n)'$, and $\epsilon^* = (\epsilon_1^*, \ldots, \epsilon_n^*)'$. Let $\Sigma = \text{Var}(V)$ and $\hat{\Sigma}_V$ be a positive definite estimator of $\Sigma$. In Algorithm 3.2 we will further specify $\hat{\Sigma}_V$. Let $\hat{\Sigma}_V^{1/2}$ be a lower triangular matrix that satisfies Cholesky decomposition $\hat{\Sigma}_V^{1/2} \hat{\Sigma}_V^{1/2} = \hat{\Sigma}_V$, and $\hat{\Sigma}_V^{-1/2}$ be the inverse of $\hat{\Sigma}_V^{1/2}$. Let $Y_t = n^{-1} \sum_{t=1}^n Y_t$, $\hat{V}_t = n^{-1} \sum_{t=1}^n \hat{V}_t$, $\hat{\epsilon}_t = n^{-1} \sum_{t=1}^n \hat{\epsilon}_t$, and $\hat{\sigma}^2_t = n^{-1} \sum_{t=1}^n (\hat{\epsilon}_t - \bar{\hat{\epsilon}})^2$. Let $P^*$, $E^*$, $\text{Var}^*$, $\text{Cov}^*$ be the probability, expectation, variance, and covariance, respectively, conditional on data $\{Y_t\}$.

Algorithm 3.1. [Linear process bootstrap unit root test]

Step 1: regress $Y_t = \hat{\phi} Y_{t-1} + \hat{V}_t$; record $\hat{\phi}$ and its t-statistic $t^*$.

Step 2: let $\hat{V}_t = \hat{V}_t - \bar{\hat{V}}$; let $\hat{\epsilon}_t = (\hat{\epsilon}_t - \bar{\hat{\epsilon}})/\hat{\sigma}$.

Step 3: let $V^* = \hat{\Sigma}_V^{1/2} \epsilon^*$; let $Y_t^* = Y_{t-1}^* + V_t^*$ and $Y_0^* = 0$.

Step 4: regress $Y_t^* = \hat{\phi}^* Y_{t-1}^* + \hat{V}_t^*$; record $\hat{\phi}^*$ and its t-statistic $t^*$.

Step 6: run Step 3-5 for $B$ times and get $\{\hat{\phi}_1^*, \ldots, \hat{\phi}_B^*\}$ and $\{t_1^*, \ldots, t_B^*\}$.

Step 7: reject the null if $B^{-1} \sum_{i=1}^B 1[\hat{\phi} > \hat{\phi}_i^*] < \text{size}$, or alternatively, $B^{-1} \sum_{i=1}^B 1[t > t_i^*] < \text{size}$.

Now we specify $\hat{\Sigma}_V$, the estimator of the autocovariance matrix $\Sigma$. Noticing the inconsistency of the sample autocovariance matrix, McMurry and Politis [8] propose a new autocovariance matrix estimator $\hat{\Sigma}_V$ detailed in Algorithm 3.2 below. By construction, $\hat{\Sigma}_V$ is positive definite and possesses a banded structure. By letting the bandwidth of this banded structure goes to infinity as sample size goes to infinity, $\hat{\Sigma}_V$ becomes a consistent estimator of $\Sigma$. Since the autocovariance matrix of a finite-order MA process has as well a banded nature, $\hat{\Sigma}_V$ constitutes a MA-sieve estimator of $\Sigma$ and hence performs especially well with MA noise.

Algorithm 3.2 (McMurry and Politis [8]). [Estimation of the autocovariance matrix]

Let $\hat{\gamma}_V(h) = n^{-1} \sum_{t=|h|+1}^n \hat{V}_t \hat{V}_{t-|h|}$. Define kernel function $\kappa(\cdot)$ by

$$\kappa(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ g(|x|), & \text{if } 1 < |x| \leq c_\kappa, \\ 0, & \text{if } |x| > c_\kappa. \end{cases}$$

where $g(\cdot)$ is a function satisfying $|g(x)| < 1$, and $c_\kappa$ is a constant satisfying $c_\kappa \geq 1$. An example of kernel function $\kappa(x)$ is the trapezoid kernel of Politis and Romano [24]:

$$\kappa(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 2 - |x|, & \text{if } 1 < |x| \leq 2, \\ 0, & \text{if } |x| > 2. \end{cases}$$ (3.1)

Let $\kappa_l = \kappa(x/l)$, where $l$ is a kernel bandwidth to be determined. Define tapered covariance matrix estimator

$$\hat{\Sigma}_V = [\kappa_l(i-j) \hat{\gamma}_V(i-j)]_{i,j=1}^n.$$

Suppose $\hat{\Sigma}_V = T D T'$, where $T$ is orthogonal and $D = \text{diag}(d_1, \ldots, d_n)$ is diagonal. Let $\hat{d}_j = \max(d_j, \hat{\gamma}_0 n^{-1})$, $D = \text{diag}(\hat{d}_1, \ldots, \hat{d}_n)$, and $\hat{\Sigma}_V = \hat{T} D \hat{T}'$. 

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Under certain conditions on the kernel bandwidth $l$, Algorithm 3.1 and Algorithm 3.2 together present a consistent bootstrap approach. First, a bootstrap FCLT with respect to bootstrap noise $\{V^*_t\}$ is established in Lemma 2.1. Based on this bootstrap FCLT, the conditional distributions of the bootstrap statistics $\hat{\phi}^*$ and $t^*$ converge to the asymptotic null distributions of $\hat{\phi}$ and $t$, respectively. Hence the justification of LPB unit root test, and as a byproduct the validity of LPB in regression.

**Condition 3.1.** Let $r_n = r_n(l) = \ln^{-1/2} + \sum_{h=l}^{\infty} |\gamma(h)|$. Suppose $l = l_n$ satisfies $r_n = O(n^{-1/4})$.

**Remark 3.1.** There exists $l = l_n$ such that Condition 3.1 holds, for example, $l = n^{1/4}$. Together with Assumption 2.1, Condition 3.1 guarantees that the operator norm of $\hat{\Sigma} - \Sigma$ decays at a rate faster or equal to $n^{-1/4}$, and as a result the measures of the partial sum processes $\{S^*(u)\}$ in Lemma 3.1 are tight.

**Lemma 3.1.** Suppose Assumption 2.1 and Condition 3.1 hold. Let $\hat{\sigma}^2 = \text{Var}^*(n^{-1/2} \sum_{t=1}^{n} V^*_t)$, $S^*(u) = n^{-1/2} \hat{\sigma}^{*-1} \sum_{t=1}^{[nu]} V^*_t$. Let $W(u)$ be a standard Brownian motion. Then no matter if $\phi = 1$ or $\phi < 1$, $S^* \Rightarrow W$ in probability.

**Theorem 3.1.** Suppose Assumption 2.1 and Condition 3.1 hold. Let $P_{H_0}$ be the probability measure corresponding to the null hypothesis. Then

$$
\sup_x |P^*(n(\hat{\phi}^* - 1) \leq x) - P_{H_0}(n(\hat{\phi} - 1) \leq x)| = o_p(1),
$$

$$
\sup_x |P^*(t^* \leq x) - P_{H_0}(t \leq x)| = o_p(1).
$$

To implement Algorithm 3.2, we choose bandwidth $l$ according to the adaptive bandwidth selection of Politis [18] in Algorithm 3.3. Lemma 3.2 shows the bandwidth selected by Algorithm 3.3 satisfies Condition 3.1. The validity of the bandwidth selection method follows immediately in Theorem 3.2. Notice that when validating the bandwidth selection method, Politis [18] and McMurry and Politis [8] require the autocovariance function $\gamma(h)$ to be either polynomial, exponential, or truncated. In contrary, our assumptions in Theorem 3.2 are much more general.

**Algorithm 3.3 (Politis [18]).** [Selection of the bandwidth]

Let $\hat{\gamma}(h) = \hat{\gamma}(h)/\hat{\gamma}(0)$. Select bandwidth $\hat{l}$ as the smallest positive integer satisfying

$$
|\hat{\gamma}(\hat{l} + k)| < c(\log n)^{1/2} n^{-1/2}, \quad k = 1, ..., K_n,
$$

where $K_n$ is a positive, non-decreasing sequence such that $K_n = o(\log n)$, and $c$ is a positive constant.

**Lemma 3.2.** Select bandwidth $\hat{l}$ by Algorithm 3.3 Under Assumption 2.1 and 2.2

$$
\hat{l} n^{-1/2} + \sum_{h=l+1}^{\infty} |\gamma(h)| = O_p(n^{-1/4}).
$$

**Theorem 3.2.** Select bandwidth $\hat{l}$ by Algorithm 3.3 Then under Assumption 2.1 and 2.2, the results in Theorem 3.1 hold.
4 Simulation

4.1 Data Generating Process

Let $X_t - X_{t-1} = \varphi X_{t-1} + V_t$. The values of $\varphi$ are set to be 0, −0.02, −0.04, −0.06, −0.08, −0.10 in order to generate the power curve. Let $\{V_t\}$ be generated by Table 1 below, and $\{\epsilon_t\} \sim iid N(0,1)$.

| Noises | $V_t = \epsilon_t$ |
|--------|------------------|
| $ma_{pos}$ | $V_t = \epsilon_t + 0.5\epsilon_{t-1}$ |
| $ma_{neg}$ | $V_t = \epsilon_t - 0.5\epsilon_{t-1}$ |
| $ar_{pos}$ | $V_t = \epsilon_t + 0.5\epsilon_{t-1}$ |
| $ar_{neg}$ | $V_t = \epsilon_t - 0.5\epsilon_{t-1}$ |
| arch | $V_t = \sigma_t^2 \epsilon_t$, $\sigma_t^2 = 10^{-6} + 0.25V_{t-1}^2$ |

4.2 Methods

In Table 2 we list the unit root tests we include in the simulation. In ADF and ARB-ADF we select lag order by Modified Akaike Information Criterion (MAIC) of Ng and Perron [10]. In FPP we harness the flat-top kernel spectral density estimator of Politis and Romano [20], and choose the kernel bandwidth according to the adaptive bandwidth selection of Politis [18]. The validity of FPP under Assumption 2.1 results from Shao, Wu, et al. [24]. In LPB-PP we harness the trapezoid kernel stated in (3.1). In CBB-PP we apply circular block bootstrap of Politis and Romano [19]. The block size of CBB-PP comes from the automatic block-length selection of Politis and White [21]. We conjecture the validity of CBB-PP on the basis of the validity of the block bootstrap PP test. The nominal sizes of all tests are set to be 0.05. Each sample has length 100. In bootstrap methods, 500 bootstrap replicates are generated. To estimate the powers of the tests, 600 tests are conducted. Tests based on both $\hat{\varphi}$ and its t-statistic $t$ are simulated. Unpublished simulation shows for each of the test listed in Table 2 the version based on $\hat{\varphi}$ are inferior to the version based on the t-statistic $t$. Therefore, we only report the results of the tests based on the t-statistic $t$.

| Type of Tests |
|---------------|
| ADF           | ADF test          |
| ARB-ADF       | AR-sieve Bootstrap ADF test |
| FPP           | Flat-top pivoted PP test |
| LPB-PP        | Linear Process Bootstrap PP test |
| CBB-PP        | Circular Block Bootstrap PP test |

4.3 Results

The results in Table 3 separate the tests into two categories. ADF and ARB-ADF, as parametric tests, show better empirical sizes, particularly under negative moving average noise. On the other hand, the nonparametric tests, i.e., FPP, LPB-PP, and CBB-PP, attain higher powers. These high powers of the nonparametric tests not only stand out under conditional heteroscedastic noise, but also occur in other cases, e.g., when positive moving average noise occurs.
Now we focus on nonparametric tests, i.e., FPP, LPB-PP, and CBB-PP. First, recall that FPP estimates the spectral density with flat-top kernel and adaptive bandwidth selection. We found this estimation leads to much better empirical size, compared to other popular kernel-based spectral density estimations. See Kim and Schmidt [6], Perron and Ng [15] for evidences under the same or similar simulation settings.

Second, among FPP, LPB-PP, and CBB-PP, our LPB-PP achieves the best overall performance in empirical sizes and powers. More specifically, while these three tests have almost equally high powers, the LPB-PP distorts the size less under two of the least favorable noises, i.e., negative moving average noise and positive moving average noise. However, LPB-PP does not fully eradicate the size distortion problem.

| Table 3: Sizes and (unadjusted) Powers |
|----------------------------------------|
| $\varphi$ | iid | ma_{pos} | ma_{neg} | ar_{pos} | ar_{neg} | arch |
|-----------------|-----|---------|---------|--------|--------|-----|
| ADF              |     |         |         |        |        |     |
| 0.00            | 0.060 | 0.048 | 0.088 | 0.048 | 0.047 | 0.020 |
| -0.02           | 0.150 | 0.133 | 0.207 | 0.140 | 0.130 | 0.062 |
| -0.04           | 0.292 | 0.275 | 0.300 | 0.273 | 0.273 | 0.098 |
| -0.06           | 0.448 | 0.398 | 0.415 | 0.425 | 0.415 | 0.168 |
| -0.08           | 0.585 | 0.517 | 0.515 | 0.518 | 0.523 | 0.230 |
| -0.10           | 0.733 | 0.600 | 0.593 | 0.642 | 0.610 | 0.333 |
| ARB-ADF          |     |         |         |        |        |     |
| 0.00            | 0.057 | 0.050 | 0.065 | 0.043 | 0.050 | 0.045 |
| -0.02           | 0.153 | 0.157 | 0.182 | 0.158 | 0.118 | 0.127 |
| -0.04           | 0.248 | 0.245 | 0.290 | 0.268 | 0.258 | 0.270 |
| -0.06           | 0.428 | 0.332 | 0.428 | 0.372 | 0.388 | 0.340 |
| -0.08           | 0.533 | 0.475 | 0.495 | 0.505 | 0.498 | 0.400 |
| -0.10           | 0.673 | 0.557 | 0.530 | 0.582 | 0.613 | 0.512 |
| FPP              |     |         |         |        |        |     |
| 0.00            | 0.048 | 0.037 | 0.272 | 0.020 | 0.157 | 0.043 |
| -0.02           | 0.120 | 0.142 | 0.455 | 0.122 | 0.333 | 0.160 |
| -0.04           | 0.277 | 0.288 | 0.747 | 0.153 | 0.522 | 0.285 |
| -0.06           | 0.447 | 0.380 | 0.892 | 0.253 | 0.697 | 0.455 |
| -0.08           | 0.638 | 0.508 | 0.965 | 0.320 | 0.850 | 0.610 |
| -0.10           | 0.787 | 0.675 | 1.000 | 0.433 | 0.930 | 0.773 |
| LPB-PP           |     |         |         |        |        |     |
| 0.00            | 0.057 | 0.048 | 0.188 | 0.022 | 0.098 | 0.048 |
| -0.02           | 0.152 | 0.218 | 0.392 | 0.200 | 0.192 | 0.143 |
| -0.04           | 0.280 | 0.292 | 0.602 | 0.268 | 0.362 | 0.297 |
| -0.06           | 0.463 | 0.452 | 0.845 | 0.338 | 0.567 | 0.433 |
| -0.08           | 0.632 | 0.577 | 0.915 | 0.410 | 0.753 | 0.638 |
| -0.10           | 0.763 | 0.660 | 0.967 | 0.500 | 0.853 | 0.770 |
| CBB-PP           |     |         |         |        |        |     |
| 0.00            | 0.042 | 0.025 | 0.247 | 0.022 | 0.142 | 0.060 |
| -0.02           | 0.147 | 0.165 | 0.417 | 0.212 | 0.313 | 0.150 |
| -0.04           | 0.278 | 0.238 | 0.742 | 0.292 | 0.562 | 0.268 |
| -0.06           | 0.437 | 0.352 | 0.938 | 0.322 | 0.790 | 0.415 |
| -0.08           | 0.643 | 0.468 | 0.973 | 0.423 | 0.912 | 0.640 |
| -0.10           | 0.782 | 0.592 | 0.992 | 0.507 | 0.965 | 0.783 |
5 Conclusion

We propose LPB unit root test to smooth the size distortion of unit root test, in particular the PP test, with MA noises. Via a bootstrap functional central limit theorem, the validity of LPB unit root test is established under general assumptions which allow a large family of non-linear noises. Simulation shows LPB unit root test mitigates the size distortion of the PP test under moving average noises, while preserving its high powers.

Hence, LPB unit root test stands out as competitive alternative in testing unit root. Further study will be needed to compare the empirical and the (local) asymptotic efficiency of LPB unit root test and other variants of unit root tests, e.g., the modified test of Perron and Ng [13].

6 Appendix

We first introduce some extra notations. Let \( ||\cdot||_p \) be the \( L^p \) (induced) norm of vectors (or matrices). Let \( ||\cdot|| \) be the trace of matrices. Let \( \mathbf{1}_n \) be a \( n \)-dimensional column vector with first \([nu]\) entries one and the other entries zero. Let \( \mathbf{1}_{n,v} = \mathbf{1}_n - \mathbf{1}_v \). Define \( \tilde{\gamma}_V \), \( \bar{\rho}_V \), and \( \Sigma_V \) analogously as \( \tilde{\gamma}_V \), \( \bar{\rho}_V \), and \( \Sigma_V \).

Proof of Lemma 6.7 Let \( M^* \) be a random matrix with rows independently and uniformly selected from the standard basis vectors, e.g., \((1, 0, ..., 0)\), and \( e^* = \hat{M}^* \hat{e} \). Let \( \hat{e} = (\hat{e}_1, ..., \hat{e}_n)' \), \( \hat{\epsilon} = (\hat{\epsilon}_1, ..., \hat{\epsilon}_n)' \), \( \tilde{e} = (\tilde{e}_1, ..., \tilde{e}_n)' \), \( \tilde{\epsilon} = (\tilde{\epsilon}_1, ..., \tilde{\epsilon}_n)' \), \( e^* = (e^*_1, ..., e^*_n)' \), \( \epsilon^* = (\epsilon^*_1, ..., \epsilon^*_n)' \). Let \( \hat{\epsilon}_t = n^{-1} \sum_{t=1}^n \hat{\epsilon}_t \). Let \( \hat{\epsilon}_t \) be generated with the true autocovariance matrix \( \Sigma \), as follows:

Step 1: \( \hat{e} = \Sigma^{-1/2} \hat{V} \).

Step 2: \( \hat{\epsilon}_t = (\hat{\epsilon}_t - \hat{\epsilon}_1)/\bar{\sigma}_e \), \( \tilde{\epsilon}_t = (\tilde{\epsilon}_t - \tilde{\epsilon}_1)/\bar{\sigma}_e \)

Step 3: \( e^* = M^* \hat{e} \), \( \epsilon^* = M^* \hat{\epsilon} \).

Notice \( S^*(u) = R_1^*(u) + R_2^*(u) + R_3^*(u) \), where

\[
R_1^*(u) = \sigma^{*-1} n^{-1/2} \mathbf{1}_u \hat{\Sigma}_V^{1/2} e^* \quad R_2^*(u) = \sigma^{*-1} n^{-1/2} \mathbf{1}_u \hat{\Sigma}_V^{1/2} (\epsilon^* - e^*) \quad R_3^*(u) = \sigma^{*-1} n^{-1/2} \mathbf{1}_u \hat{\Sigma}_V^{1/2} (\epsilon^* - e^*)
\]

To show \( R_2^*(u) \) converges, that is, \( R_2^* \Rightarrow W \) in probability, \( R_2^* \Rightarrow 0 \) in probability, and \( R_3^* \Rightarrow 0 \) in probability, it suffices to prove the in-probability finite-dimensional convergence and the in-probability tightness of \( R_2^*(u) \). With Lemma 6.4 below, the proof of Theorem 5 of McMurry and Politis [8], and the Cramer-Wold device, it is straightforward to show the finite-dimensional convergence of \( R_1^*(u) \). The finite-dimensional convergence of \( R_2^*(u) \) and \( R_3^*(u) \) can be proven similarly. To establish the in-probability tightness of \( R_2^*(u) \), we apply Theorem 13.5 of Billingsley [1] (p. 142), and verify its conditions with Lemma 6.2 [6], Lemma 6.3 [6], Lemma 6.5 [6], and Lemma 6.6 [6] below. The in-probability convergence of \( S^*(u) \) follows from Slutsky’s Theorem on a metric space (see e.g. Billingsley [1], Theorem 3.1, p. 27).

Proof of Theorem 3.1. By Lemma 6.1 [3] and Lemma 6.7 the conditional asymptotic distributions of \( n(\hat{\phi} - 1) \) and the \( t \)-statistic \( t^* \) are both standard Phillips-Perron type distributions; see Theorem 3.1 of Phillips [10]. So do the unconditional asymptotic distributions of \( n(\hat{\phi} - 1) \) and the \( t \)-statistic \( t \).

Further, by Lemma 6.3 and 6.7 the parameters in the conditional asymptotic distributions converge to the parameters in the unconditional asymptotic distributions. The theorem follows.
Lemma 6.1. Under Assumption 2.4
\[
\hat{\phi} - \phi = \begin{cases} O_p(n^{-1/2}), & \text{if } \phi < 1, \\ O_p(n^{-1}), & \text{if } \phi = 1. \end{cases}
\]

Proof of Lemma 6.1 This result follows straightforwardly from Theorem 3 of Wu [20].

Lemma 6.2. Under Assumption 2.4 and Condition 3.1
\[
||\hat{\Sigma} - \Sigma|| = O_p(n^{-1/4}), \quad \text{and} \quad ||\hat{\Sigma}^{-1} - \Sigma^{-1}|| = O_p(n^{-1/4}).
\]

Proof of Lemma 6.2 By the proof of Theorem 2 and 3 of McMurry and Politis [8], it suffices to prove \(|\hat{\Sigma} - \Sigma| = O_p(r_n)\). By Theorem 1 of McMurry and Politis [8], \(|\hat{\Sigma} - \Sigma| = O_p(r_n)\), so it suffices to prove \(|\hat{\Sigma} - \hat{\Sigma}_V| = O_p(r_n)\), where \(\hat{\Sigma}_V\) is defined in Algorithm 3.2 and \(r_n\) in Condition 3.1 By H"{o}lder's inequality and the symmetry of \(\hat{\Sigma}_V\) and \(\hat{\Sigma}_V^t\),
\[
||\hat{\Sigma}_V - \hat{\Sigma}_V^t|| \leq ||\hat{\Sigma}_V - \hat{\Sigma}_V|| + \sum_{h=0}^{[c_kl]} |\hat{\gamma}_V(h) - \hat{\gamma}_V(h)|
\]
\[
\leq 2(l' + 1)(|\hat{\phi} - \phi|(C_1 + C_2) + (\hat{\phi} - \phi)^2C_3),
\]
where \(l' = [c_kl]\), and
\[
C_1 = \sup_{0 \leq h \leq l'} |n^{-1} \sum_{t=h+1}^{n} Y_{t-1}V_{t-h}|,
\]
\[
C_2 = \sup_{0 \leq h \leq l'} |n^{-1} \sum_{t=h+1}^{n} Y_{t-h-1}V_t|,
\]
\[
C_3 = \sup_{0 \leq h \leq l'} |n^{-1} \sum_{t=h+1}^{n} Y_{t-h-1}Y_{t-1}|.
\]

When \(\phi < 1\), by Theorem 1 of Hannan [3],
\[
C_1 = \sup_{0 \leq h \leq l'} |n^{-1} \sum_{t=h+1}^{n} Y_{t-1}(Y_{t-h} - \phi Y_{t-h-1})| = O_p(1).
\]

When \(\phi = 1\),
\[
C_1 = \sup_{0 \leq h \leq l'} |n^{-1} \sum_{t=h+1}^{n} Y_{t-h-1}V_{t-h} + n^{-1} \sum_{t=h+1}^{n} \sum_{k=1}^{h} V_{t-k}V_{t-h}|
\]
\[
= \sup_{0 \leq h \leq l'} |(2n)^{-1}(\sum_{t=h+1}^{n} (Y_{t-h-1} + V_{t-h})^2 - Y_{t-h-1}^2 - V_{t-h}^2) + n^{-1} \sum_{t=h+1}^{n} \sum_{k=1}^{h} V_{t-k}V_{t-h}|
\]
\[
= \sup_{0 \leq h \leq l'} |(2n)^{-1}Y_{n-h}^2 - (2n)^{-1} \sum_{t=h+1}^{n} V_{t-h}^2 + n^{-1} \sum_{t=h+1}^{n} \sum_{k=1}^{h} V_{t-k}V_{t-h}|
\]
\[
= O_p(1) + O_p(l) + O_p(l^3n^{-1}) = O_p(l + l^3n^{-1}),
\]
Hence, by Lemma 6.3, for all $0 \leq u \leq 1$,

\[
\sup_{0 \leq h \leq l'} n^{-1} Y_{n-h}^2 \leq (\sup_{0 \leq u \leq 1} n^{-1/2} Y_{[nu]}^2)^2 = O_p(1),
\]

and

\[
\sup_{0 \leq h \leq l'} n^{-1} \sum_{t=h+1}^n V_{t-h}^2 = O_p(1),
\]

so

\[
\sup_{0 \leq h \leq l'} \sum_{t=h+1}^n V_{t-h} = n^{-1} \sum_{t=1}^n V_t^2 = O_p(1),
\]

and

\[
\sup_{0 \leq h \leq l'} n^{-1} \sum_{t=h+1}^n \sum_{k=1}^h V_{t-k} V_{t-h} \leq (l'+1) \sup_{0 \leq h \leq l'} \sum_{0 < k < h} \sup_{0 < j \leq l'} n^{-1} \sum_{t=1}^n V_t V_{t-j} \leq (l'+1)^2 (2n)^{-1} \sum_{n-l < t \leq n, 0 < j < l'} (V_t^2 + V_{t-j}^2) + O_p(l) = O_p(l^2 n^{-1} + l).
\]

Similarly, it can be shown that when $\phi < 1$, $C_2 = O_p(1)$, and $C_3 = O_p(1)$, and when $\phi = 1$, $C_2 = O_p(l + l^3 n^{-1})$, and $C_3 = O_p(n + l^2 + l^4 n^{-2})$. By Lemma 6.4,

\[
||\hat{\Sigma}_V - \hat{\Sigma}_V|| = \begin{cases} O_p(ln^{-1/2}), & \text{if } \phi < 1, \\
O_p(l^2 n^{-1} + l^4 n^{-2}), & \text{if } \phi = 1. \end{cases}
\]

Lemma 6.3. Under Assumption 2.1 and Condition 3.1

\[
Var^*(n^{-1/2} \sum_{t=1}^n V_t^*) - Var(n^{-1/2} \sum_{t=1}^n V_t) = o_p(1).
\]

Proof of Lemma 6.3. The result follows from Lemma 6.2 and Lemma 3 and 4 of McMurry and Politis [3].

Lemma 6.4. Suppose Assumption 2.1 and Condition 3.1 hold. For all $0 \leq u \leq v \leq 1$,

\[
\text{Cov}^*(R_1^*(u), R_1^*(v)) = u + o_p(1).
\]

Proof of Lemma 6.4. By Lemma 6.3, for all $0 \leq w \leq 1$,

\[
n^{-1} T_w^* \hat{\Sigma}_V 1_w = n^{-1} [nw] Var^*([nw]^{-1/2} \sum_{t=1}^{[nw]} V_t^*) = \sigma^2 w + o_p(1).
\]

Hence,

\[
\text{Cov}^*(R_1^*(u), R_1^*(v)) = \sigma^2 - n^{-1} T_w^* \hat{\Sigma}_V 1_w = \sigma^2 - (2n)^{-1} (1_w^* \hat{\Sigma}_V 1_w + 1_u^* \hat{\Sigma}_V 1_u - 1_{w-u}^* \hat{\Sigma}_V 1_{w-u}) = u + o_p(1).
\]

Lemma 6.5. Suppose $A$ is an $n \times n$-dimensional symmetric positive semi-definite matrix, $A = A^{1/2} A^{1/2}$, and $A^{1/2} = \{a_{ij} \}_{i,j=1}^n$. Suppose $\xi_1, ..., \xi_n$ are $P^*$-i.i.d random variables with $E^* \xi^*_i = 0$ and $E^* (\xi^*_i)^2 = \sigma^2$. Let $R^*(u) = \sigma^* n^{-1/2} 1_u^* A^{1/2} \xi^*$. Then, for all $0 \leq u \leq v \leq w \leq 1$,

\[
E^*((R^*(u) - R^*(u))(R^*(w) - R^*(v)) \leq 4\sigma^* \sigma E^* (\xi^*_i)^2 ||A||^2 (w-u)^2.
\]
Proof of Lemma 6.6

\[ E^*((R_1^*(v) - R_1^*(u))^2(R_1^*(w) - R_1^*(v))^2) = \sigma^s - 4n^{-2}E^*((1'_{u,v}A^{1/2}\xi^*)^2(1'_{u,v}A^{1/2}\xi^*)^2) = \sigma^s - 4(B_1 + B_2 + B_3), \]  

(6.1)

where

\[ B_1 = s^4n^{-2}(\sum_{j=1}^{n} \sum_{i=\lceil nu \rceil + 1}^{\lceil nv \rceil} a_{ij})^2(\sum_{j=1}^{n} \sum_{i=\lceil nv \rceil + 1}^{\lceil nu \rceil} a_{ij})^2 \]

\[ B_2 = 2s^4n^{-2}(\sum_{j=1}^{n} [\sum_{i=\lceil nu \rceil + 1}^{\lceil nv \rceil} a_{ij}]^2(\sum_{j=1}^{n} [\sum_{i=\lceil nv \rceil + 1}^{\lceil nu \rceil} a_{ij}]^2) \]

\[ B_3 = E^*(\xi_j^4 - 3s^4)n^{-2}(\sum_{j=1}^{n} [\sum_{i=\lceil nu \rceil + 1}^{\lceil nv \rceil} a_{ij}]^2(\sum_{j=1}^{n} [\sum_{i=\lceil nv \rceil + 1}^{\lceil nu \rceil} a_{ij})^2) \]

Notice \( B_1 = s^4n^{-2}1'_{u,v}A1'_{u,v}1'_{v,w}A1'_{v,w}, \) and \( B_2 = 2s^4n^{-2}(1'_{u,v}A1'_{v,w})^2. \) Since

\[ 1'_{u,v}A1'_{v,w} = 2^{-1}(1'_{u,w}A1'_{u,w} - 1'_{v,w}A1'_{v,w}), \]

and for \( 0 \leq r \leq s \leq 1, \)

\[ 1'_{r,s}A1'_{r,s} \leq ||A||([nr] - [ns]), \]

we have

\[ B_1 \leq s^4||A||^2((\lceil nv \rceil - \lceil nu \rceil)/n)(\lceil nu \rceil - \lceil nv \rceil)/n) \leq 4s^4||A||^2(w - u)^2, \]

\[ B_2 \leq 2s^4||A||^2((\lceil nu \rceil - \lceil nv \rceil)/n)^2 \leq 8s^4||A||^2(w - u)^2, \]

\[ B_3 \leq E^*(\xi_j^4 - 3s^4)s^4 - 4E^*(\xi_j^4 - 3s^4)||A||^2(w - u)^2. \]

The lemma follows from (6.1).

Lemma 6.6. Under Assumption 2.1 and Condition 3.1,

(i) \( E^*(\epsilon_j^4) = O_p(1), \) (ii) \( E^*(\epsilon_j^4 - \epsilon_j^4)^2 = O_p(1), \) and (iii) \( E^*(\epsilon_j^4 - \epsilon_j^4) = O_p(1). \)

Proof of Lemma 6.6. The proof applies Lemma 6.1 through out. Notice that

\[ n^{-1}\sum_{t=1}^{n} \tilde{e}_t^2 = n^{-1}V'\Sigma^{-1}V \geq ||\Sigma||^{-1}n^{-1}\sum_{t=1}^{n} \tilde{e}_t^2 = ||\Sigma||^{-1}v_0 + o_p(1). \]  

(6.2)

By Chebyshev’s Inequality,

\[ \tilde{e}_t^2 \leq 2(n^{-1}||\Sigma||^{-1/2}(V - V)^2 + 2(n^{-1}||\Sigma||^{-1/2}V)^2) \leq 2||\Sigma^{-1}||n^{-1}||V - V||^2 + o_p(1) = o_p(1). \]  

(6.3)

By (6.2) and (6.3), \( \hat{\sigma}^2 = O_p(1). \) Similarly, \( \hat{\sigma}^2 = O_p(1). \) Further,

\[ n^{-1}\sum_{t=1}^{n} \tilde{e}_t^4 = n^{-1}||\Sigma||^{-2}V^4 \leq 8n^{-1}(||\Sigma||^{-1/2}(V - V)^4 + ||\Sigma||^{-1/2}V||^4) = O_p(1), \]

where

\[ V = \frac{1}{n}\sum_{t=1}^{n} \Sigma^{-1/2}e_t. \]
since by Lemma 5 of McMurry and Politis [8],
\[ n^{-1}||\Sigma^{-1/2}V||^4_1 = O_p(1), \]
and
\[ n^{-1}||\Sigma^{-1/2}(V - \hat{V})||^4_1 \leq n^{-1}||\Sigma^{-1/2}(V - \hat{V})||^4 \leq 8||\Sigma^{-1}||^2 n^{-1}||\hat{V} - V||^4 + ||\hat{V}||^4 \]
\[ \leq 64||\Sigma^{-1}||^2((\hat{\phi} - \phi)^4 n^{-1}\sum_{t=1}^{n} Y^4_t + (\hat{\phi} - \phi)^4 Y^4_t) = O_p(1). \]

For (i) and (ii), therefore,
\[ E^*(\epsilon_j^4) = \hat{\sigma}_e^4 n^{-1}(\hat{e}_t - \hat{\epsilon}_t)^4 = O_p(1), \]
\[ E^*((\epsilon_j - \epsilon_j^*)^4) = (\hat{\sigma}_e - \hat{\sigma}_e^*)^4 n^{-1}(\hat{e}_t - \hat{\epsilon}_t)^4 = O_p(1). \]

For (iii), by Lemma 6.2
\[ E^*((\epsilon_j^* - \epsilon_j^*)^4) = \hat{\sigma}_e^4 n^{-1}(\hat{e}_t - \hat{\epsilon}_t)^4 - \hat{\sigma}_e^4 n^{-1}(\hat{e}_t - \hat{\epsilon}_t)^4)^2 \]
\[ = \hat{\sigma}_e^4 n^{-1}||V - 11'||(\hat{\Sigma}^{-1/2} - \Sigma^{-1/2})\hat{V}||^4 \leq \hat{\sigma}_e^4 n||\hat{\Sigma}^{-1/2} - \Sigma^{-1/2}||^4(n^{-1}\sum_{t=1}^{n} V^2_t)^2 = O_p(1). \]

**Lemma 6.7.** Suppose Assumption [2.1] and Condition [3.1] hold. Then
\[ Var^*(n^{-1}\sum_{t=1}^{n} V^2_t) = o_p(1), \text{ and } E^*(n^{-1}\sum_{t=1}^{n} V^2_t) = \gamma_0 + o_p(1). \]

**Proof of Lemma 6.7.** Notice that \( \sum_{t=1}^{n} V^2_t = \epsilon^* \hat{\Sigma} \epsilon^* \). By Lemma 6.2 and 6.6 above and Seber and Lee [23] (Theorem 1.5 and 1.6, pp. 9-10),
\[ E^*(n^{-1}\sum_{t=1}^{n} V^2_t - \gamma_0) = n^{-1}tr(\hat{\Sigma} - \Sigma) \leq ||\hat{\Sigma} - \Sigma|| = o_p(1), \]
\[ Var^*(n^{-1}\sum_{t=1}^{n} V^2_t) \leq n^{-2}E(\epsilon^*_1^4)tr(\hat{\Sigma}^2) \leq n^{-1}E(\epsilon^*_1^4)||\hat{\Sigma}||^2 = o_p(1). \]

**Proof of Lemma 6.6.** We now prove
\[ \sum_{h=l+1}^{\infty} |\gamma(h)| = O_p(n^{-1/4}). \]
Assume at this stage $\gamma(h) \neq 0$ for infinitely many $h$. Let $g_h = |\gamma(h)|$, $G_h = \sum_{k=0}^{\infty} g_h$, $G^{-1}(x) = \min\{h \geq 0 : G_h \leq x\}$, and $a = G^{-1}(n^{-1/4})$. Then

$$P\left(\sum_{h=1}^{\infty} |\gamma(h)| > n^{-1/4}\right) = P(\tilde{l} < a) = 1 - P(\forall l = 1, \ldots, a - 1, \sup_{1 \leq k \leq K_n} |\hat{\rho}_V(l + k)| \geq c(\log n/n)^{1/2})$$

$$= 1 - (D_1 - D_2 - D_3), \quad (6.5)$$

where

$$D_1 = P(\forall l = 1, \ldots, a - 1, \sup_{1 \leq k \leq K_n} |\rho(l + k)| \geq 3c(\log n/n)^{1/2}),$$

$$D_2 = P(\exists l = 1, \ldots, a - 1, \sup_{1 \leq k \leq K_n} |\hat{\rho}_V(l + k) - \rho(l + k)| > c(\log n/n)^{1/2}),$$

$$D_3 = P(\exists l = 1, \ldots, a - 1, \sup_{1 \leq k \leq K_n} |\hat{\rho}_V(l + k) - \hat{\rho}_V(l + k) - \rho(l + k)| > c(\log n/n)^{1/2}).$$

By the proof of Lemma 6.2 and Theorem 1 of Xiao and Wu [27],

$$D_2 = o(1) \text{ and } D_3 = o(1). \quad (6.6)$$

Now we show $D_1 = 1 + o(1)$. Let $f(l) = \sup_{1 \leq k \leq K_n} g_{l+k}$ and $f_n(l) = \sup_{1 \leq k \leq K_n} g_{l+k}$. For some $0 < D < 1$,

$$\inf_{1 \leq l < a} f_n(l) \geq \inf_{1 \leq l < a} f(l) - \sup_{1 \leq l < a} |f_n(l) - f(l)| \geq \sup_{k \geq a} g_k - \sup_{k \geq a} g_k \geq D \sup_{k \geq a} g_k \geq Dg_a.$$

Hence, for some $C > 0$,

$$D_1 \geq P\left(\inf_{1 \leq l < a} f_n(l) \geq C(\log n/n)^{1/2}\right) \geq P(g_a \geq (C/D)(\log n/n)^{1/2})$$

$$= P(g_a \geq (C/D)(\log n/n)^{1/2}) = 1 + o(1), \quad (6.7)$$

where the last equation results from Lemma 6.8 below. A combination of (6.3), (6.6), and (6.7) gives (6.4) when $\gamma(h) \neq 0$ for infinitely many $h$. When $\gamma(h) \neq 0$ only for finitely many $h$, (6.4) follows analogously. It can be similarly derived that

$$\tilde{h}n^{-1/2} = O_p(n^{-1/4}).$$

**Lemma 6.8.** Suppose Assumption [2.7] and Assumption [2.2] hold. Suppose $\gamma(h) \neq 0$ for infinitely many $h$. Let $g_h = |\gamma(h)|$, $G_h = \sum_{k=0}^{\infty} g_h$, $G^{-1}(x) = \min\{h \geq 0 : G_h \leq x\}$. Then for a small enough positive number $x$,

$$g_{G^{-1}(x)} > (a/4)x^{\beta/(\beta-1)}.$$

**Proof of Lemma 6.8.** If $g_h = o(h^{-\beta})$, then $G_h = o(h^{1-\beta})$, and then for a small enough positive number $x$,

$$G^{-1}(x) < x^{1/(1-\beta)}. \quad (6.8)$$

By Assumption [2.2] for large enough $h$, $h^{\alpha}G_{h}$ is non-increasing. It follows straightforwardly that for large enough $h$,

$$hg_h \geq (a/2)G_h. \quad (6.9)$$
Hence, by (6.8) and (6.9), for a small enough positive number $x$,

$$
g_{G^{-1}(x)} = \frac{G^{-1}(x)g_{G^{-1}(x)}}{(\alpha/2)G_{G^{-1}(x)}^{-1}} > (\alpha/4)x^{\beta/(\beta-1)}.
$$

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