Dynamics, Circuit Design and Fractional-Order Form of a Modified Rucklidge Chaotic System

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Abstract. Rucklidge chaotic system is a nonlinear mechanical model of a double convection process. In this paper, we modify the dynamics of a Rucklidge chaotic system by adding a nonlinear term and derive a new chaotic system. The nonlinear dynamics of the proposed chaotic system is described through numerical simulations which include the stability analysis of equilibrium points, phase portraits, Lyapunov exponents, Kaplan-Yorke dimension, bifurcation diagram and a Poincaré map. For specific values of the parameters, the proposed system displays periodic and chaotic behaviour. In addition, a new circuit implementation of the modified Rucklidge chaotic system is reported and examined in MultiSIM. A good qualitative agreement is shown between the simulations and the MultiSIM results. Furthermore, the fractional-order form of the modified Rucklidge chaotic system is numerically studied. By tuning the commensurate fractional order, the new chaotic system displays chaotic and periodic attractors, respectively.

1. Introduction
Since Lorenz found the 3-D model for atmospheric convection in 1963 [1], chaos has been intensively studied and many chaotic systems have been discovered in the literature. In [2], Rössler constructed a seven-term chaotic system with just one quadratic nonlinearity. In [3-4], Rikitake discovered a new chaotic system to explain the irregular polarity switching of the earth’s geomagnetic field. A model to explain the irregular variability in the luminosity of stars was presented in 1976 [5] and this system is known as the Moore-Spiegel system.

In [6], Rucklidge discovered a double convection chaotic model in a horizontal layer of Boussinesq fluid with lateral constraints. In 1994, Sprott described 19 algebraically simple chaotic systems [7]. In 2000, Malasoma proposed the simplest dissipative jerk equation that is parity invariant [8].
In 2009, Sun and Sprott constructed a piecewise exponential jerk system [9]. In the last few years, several chaotic systems with an infinite number of equilibrium points have been proposed [10-14].

Chaos has been widely applied to many scientific disciplines such as physics [15], biology [16], ecology [17], economy [18], random bit generators [19], psychology [20], lasers [21], astronomy [22], chemical reactions [23], memristors [24], neural networks [25], robotics [26], encryption [27], secure communication systems [28-31], etc.

In this paper, we modify the dynamics of a Rucklidge chaotic system by adding a nonlinear term and derive a new chaotic system. Section 2 describes the nonlinear dynamics of the proposed chaotic system. We describe the phase portraits of the modified Rucklidge chaotic system using MATLAB and discuss the stability analysis of equilibrium points, Lyapunov exponents, Kaplan-Yorke dimension, bifurcation diagram and a Poincaré map for the new chaotic system. In Section 3, a new circuit implementation of the modified Rucklidge chaotic system is reported and examined in MultiSIM. A good qualitative agreement is shown between the simulations and the MultiSIM results. In Section 4, the fractional-order form of the modified Rucklidge chaotic system is numerically studied. By tuning the commensurate fractional order, the new chaotic system displays chaotic and periodic attractors, respectively. Section 5 contains the concluding remarks.

2. A modified Rucklidge chaotic system

Recently, Rucklidge has introduced a famous model for nonlinear double convection [6]. Rucklidge model has the following nonlinear dynamics

\[
\begin{align*}
\dot{x} &= -kx + \lambda y - yz \\
y &= x \\
\dot{z} &= -z + y^2
\end{align*}
\]  

where \( x, y, z \) are state variables and \( k, \lambda \) are system parameters.

In [6], it was shown that the Rucklidge system (1) is chaotic, when the system parameters take the values

\[
k = 2, \quad \lambda = 6.7
\]  

In this work, we derive a new chaotic system by modifying the Rucklidge dynamics (1) as

\[
\begin{align*}
\dot{x} &= -ax - b |z| + cy - yz \\
y &= x \\
\dot{z} &= -z + y^2
\end{align*}
\]  

where \( x, y, z \) are state variables and \( a, b, c \) are system parameters.

We shall show that the system (3) is chaotic when the system parameters take the values

\[
a = 1.5, \quad b = 0.2, \quad c = 5
\]  

For numerical simulations, we take the initial conditions as

\[
x(0) = 0.1, \quad y(0) = 0.1, \quad z(0) = 0.1
\]  

With the parameter values as in (4) and the initial conditions as in (5), the Lyapunov exponents of the new system (3) are calculated using Wolf algorithm [32] as

\[
L_1 = 0.1559, \quad L_2 = 0, \quad L_3 = -2.6559
\]  

This shows that the new modified Rucklidge system (3) is chaotic.

Also, the sum of the Lyapunov exponents of the system (3) is found as

\[
L_1 + L_2 + L_3 = -(a + 1) = -2.5 < 0
\]  

Hence, the system (3) is a dissipative chaotic system with a strange chaotic attractor.

The phase portraits of the new modified Rucklidge system are shown in Figure 1. The Lyapunov exponents of the new modified Rucklidge system are shown in Figure 2.
The Kaplan-Yorke dimension of the modified Rucklidge system (3) is calculated as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0587$$

(8)

Figure 1. Phase portraits of the modified Rucklidge chaotic system

Figure 2. Lyapunov exponents of the modified Rucklidge chaotic system
The equilibrium points of the new chaotic system (3) are obtained by solving the following system
\[
\begin{align*}
-ax - b|z| + cy - yz &= 0 \\
2x &= 0 \\
-z + y^2 &= 0
\end{align*}
\] (9)

We take the parameter values as in the chaotic case (4). A simple calculation shows that the new chaotic system (3) has three equilibrium points given by
\[
E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ 2.1383 \\ 4.5723 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ -2.3383 \\ 5.4677 \end{bmatrix}
\] (10)

To determine the stability type of the equilibrium points of the system (3), we calculate the Jacobian matrix of the system (3) at any point \( x \in \mathbb{R}^3 \):
\[
J(x) = \begin{bmatrix}
-1.5 & 5 - z & -y - 0.2 \text{ sign}(z) \\
1 & 0 & 0 \\
0 & 2y & -1
\end{bmatrix}
\] (11)

The eigenvalues of the matrix \( J_0 = J(E_0) \) are determined as
\[
\lambda_1 = -1, \quad \lambda_2 = -3.1085, \quad \lambda_3 = 1.6085
\] (12)

This shows that \( E_0 \) is a saddle point, which is unstable.

The eigenvalues of the matrix \( J_1 = J(E_1) \) are determined as
\[
\lambda_1 = -3.1330, \quad \lambda_{2,3} = 0.3165 \pm 1.7191i
\] (13)

This shows that \( E_1 \) is a saddle focus, which is unstable.

The eigenvalues of the matrix \( J_2 = J(E_2) \) are determined as
\[
\lambda_1 = -3.0046, \quad \lambda_{2,3} = 0.2523 \pm 1.8494i
\] (14)

This shows that \( E_2 \) is a saddle focus, which is unstable.

Thus, all three equilibrium points of the system (3) are unstable.

Next, to have a detailed view of the new system (3), which is a modified Rucklidge chaotic system, the behavior of the system with respect to the bifurcation parameter \( b \) is studied. For the chosen value of \( b \leq 0.28 \) the system (3) displays the expected chaotic behavior and for \( b > 0.28 \) periodic behavior is noted in the system. Figure 3 shows the bifurcation diagram for the new system (3). Figure 4 shows the Poincaré map of the new system (3). Figures 3 and 4 describe the chaotic properties of the modified Rucklidge chaotic system (3).
Figure 3. Bifurcation diagram of the new chaotic system (3) with $b$ as varying parameter

Poincare Section of the Modified Rucklidge Chaotic System

Figure 4. Poincarè map for the new chaotic system (3) when $a = 1.5$, $b = 0.2$ and $c = 5$
3. Circuit realization of the modified Rucklidge chaotic system

In this section, an electronic circuit which emulates the modified Rucklidge chaotic system (3) is described to show its feasibility. The circuit design in Figure 5 has been described following an approach based on operational amplifiers [33-35], where the state variables $x, y, z$ of the new system (3) are associated with the voltages across the capacitors $C_1, C_2$ and $C_3$ respectively.

By applying Kirchhoff’s circuit laws, the corresponding circuit equations of the designed circuit can be written as

$$\begin{cases} 
\dot{x} = -\frac{1}{C_1R_1} x - \frac{1}{C_1R_2} |z| + \frac{1}{C_1R_3} y - \frac{1}{10C_1R_4} yz \\
\dot{y} = \frac{1}{C_2R_5} x \\
\dot{z} = -\frac{1}{C_3R_6} z + \frac{1}{10C_3R_7} y^2 
\end{cases}$$

(15)

We choose the values of the circuit elements as

$$\begin{align*}
R_1 &= 6.66K\Omega, R_2 = 50K\Omega, R_3 = 2K\Omega, R_4 = R_7 = 1K\Omega \\
R_5 &= R_6 = R_9 = R_{10} = R_{11} = R_{12} = R_{13} = R_{14} = R_{15} = R_{16} = 10K\Omega \\
C_1 &= C_2 = C_3 = 10nF
\end{align*}$$

(16)

The circuit has three integrators by using Op-amp TL082CD in a feedback loop and two multipliers IC AD633. In addition, two diodes ($D_1, D_2$) are used, which provide the signal $|z|$. The supplies of all active devices are ±15 volt. With MultiSIM 10.0, we obtain the experimental observations of new system (3) as shown in Figures 6-8. The agreement between the experimental phase portraits with MultiSIM simulation (Figs. 6-8) and the numerical simulation with MATLAB (Fig. 1) confirms the feasibility of our new modified Rucklidge chaotic system (3).

![Figure 5 Schematic of the proposed new chaotic system by using MultiSIM](image-url)
Figure 6 2-D projection of the new chaotic system on the \((x, y)\) plane

Figure 7: 2-D projection of the new chaotic system on the \((x, z)\) plane
4. A fractional order model of the modified Rucklidge chaotic system
In this section, we describe a fractional order model of the modified Rucklidge chaotic system (3). Faraji and Tavazoei [36] describe the results of the simulation of a fractional capacitor in the model of chaotic circuit. The current flowing through a capacitor and the voltage to which it is subjected are related by the following fractional-order equation

\[ i = C_\alpha \frac{d^\alpha V}{dt^\alpha} \]  

where \( 0 < \alpha < 1 \) and \( \alpha \) tends to 1 (note that \( \alpha = 1 \) for an ideal capacitor) [37-38]. Also, [36] gives various values of \( \alpha \) and \( C_\alpha \) commonly used in circuitry implementation of chaotic systems.

The mathematical description of the commensurate fractional order model of the modified Rucklidge chaotic system (3) can be expressed as follows:

\[
\begin{align*}
\frac{d^\alpha x}{dt^\alpha} &= -ax - b | z | + cy - yz \\
\frac{d^\alpha y}{dt^\alpha} &= x \\
\frac{d^\alpha z}{dt^\alpha} &= -z + y^2
\end{align*}
\]  

(18)
We take the parameter values as in the chaotic case (4), i.e.
\[ a = 1.5, \ b = 0.2, \ c = 5 \]  \hspace{1cm} (19)

For numerical simulation of the fractional-order model (18) of the modified Rucklidge chaotic systems with parameter values as in (19), the Adams-Bashforth-Moulton predictor-corrector scheme [39-43] is used. This method is based on the Caputo definition of the fractional-order derivative given by [44] as follows:
\[ \frac{d^n X_i}{dt^n} = \frac{1}{\Gamma(\alpha - n)} \int_0^t (t - \tau)^{\alpha - n - 1} d\tau \] \hspace{1cm} (20)

where \( n - 1 < \alpha < n \), \( X_1 = x, X_2 = y, X_3 = z \) and \( \Gamma(.) \) is the Gamma function. We describe the numerical simulations of fractional-order model of the Rucklidge chaotic system (18) with parameter values as in (19) for different fractional-order \( \alpha (0 < \alpha < 1) \).

Figures 9-11 describe the phase portraits of the fractional-order model (18) of the new chaotic system in the planes \((x, y)\), \((x, z)\) and \((y, z)\) of significant results obtained for specific values of commensurate fractional-order \( \alpha \). For \( \alpha = 0.980 \), Figure 9 displays a chaotic attractor of fractional-order system (18) at \( a = 1.5, b = 0.2 \) and \( c = 5 \). Also, we derive periodic attractors at \( \alpha = 0.918 \) (see Figure 10) and \( \alpha = 0.910 \) (see Figure 11). The periodic behavior found in Figure 10 is confirmed in Figure 11.

**Figure 9** The phase portrait in the planes \((x, y)\), \((x, z)\) and \((y, z)\) of fractional-order system (18) at \( a = 1.5, b = 0.2 \) and \( c = 5 \) for specific values of commensurate fractional order \( \alpha = 0.980 \)

**Figure 10** The phase portrait in the planes \((x, y)\), \((x, z)\) and \((y, z)\) of fractional-order system (18) at \( a = 1.5, b = 0.2 \) and \( c = 5 \) for specific values of commensurate fractional order \( \alpha = 0.918 \)
Figure 11 The phase portrait in the planes \((x, y), (x, z)\) and \((y, z)\) of fractional-order system (18) at \(a = 1.5, b = 0.2\) and \(c = 5\) for specific values of commensurate fractional order \(\alpha = 0.910\).

5. Conclusions
In this paper, we derived a new chaotic system by adding a nonlinear term to Rucklidge chaotic system model for nonlinear double convection and by changing the values of the parameters in the Rucklidge system. The dynamical behaviors of the modified Rucklidge chaotic system are analyzed, both analytically and numerically, including some basic dynamical properties, phase portraits, Lyapunov exponents, Kaplan-Yorke dimension, bifurcation diagram and Poincaré map. The designed circuit has been implemented and examined using the MultiSIM and numerical results using MATLAB. Comparison of the MultiSIM result and MATLAB simulations showed good qualitative agreement between the chaotic system and its circuitry realization. Finally, a fractional-order model of the proposed system has been investigated. As the fractional order derivative order decreases, the chaotic attractor of the modified Rucklidge chaotic system exhibits chaotic behavior and periodic behavior respectively.

References
[1] Lorenz E N 1963 J. Atmospheric Sciences 20 130-41
[2] Rössler O 1976 Physics Letters A 57 397-8
[3] Ito K 1980 Sci. Lett. 51 451-6
[4] Rikitake T 1998 Proc. Cambridge Philos. Soc. 54 89-105
[5] Moore D W and Spiegel E A 1986 Astrophys. J. 143 871-87
[6] Rucklidge A M 1992 J. Fluid Mechanics 237 209-29
[7] Sprott J 1994 Phys. Lett. E 50 647-50
[8] Malasoma J. M. 2000 Phys. Lett. A 264 383-9
[9] Sun K H and Sprott J H 2009 Internat. J. Nonlinear Sci. Num. Simulat. 10 1443-50
[10] Gotthans T and Petrzelka J 2015 Nonlinear Dynamics 73 429-36
[11] Zhou P and Yang F 2014 Nonlinear Dynamics 76 473-80
[12] Pham V T, Jafari S, Volos C and Kapitaniak T 2016 Chaos Solit. Fract. 93 58-63
[13] Pham V T, Volos C, Vaidyanathan S and Wang X 2016 Advances Math Phys. 2016 4024836
[14] Pham V T, Jafari S, Volos C, Vaidyanathan S and Kapitaniak T 2016 Optik 127 9111-7
[15] Awrejcewicz J, Supel B, Lamarque C H, Kudra G, Wasilewski G and Olejnik P 2008 Int. J. Bifur. Chaos 18 2883-915
[16] Hannon B and Ruth M 2014 Modeling Dynamic Biological Systems (Berlin: Springer)
[17] Mamat M, Sanjaya W M, Salleh Z and Ahmad M F 2011 J. Sustainability Science and Management 6 44-50
[18] Bouali S, Buscarino A, Fortuna L, Fresca M and Gambuzza L V 2012 Nonlinear Analysis: RWA 13 2459-65
[19] Akgul A, Moroz, I, Pehlivan I and Vaidyanathan S 2016 Optik 127 5491-99
[20] Jafari S, Sprott J C and Golpayegani S M R H 2016 Nonlinear Dynamics 83 615-22
[21] Sciamanna M and Shore, K A 2015 Nature Photonics 9 151-62
[22] Daquin J, Rosengren A J, Alessi E M, Feleffi F, Valsecchi G B and Rossi A 2016 Celestical Mechanics and Dynamical Astronomy 124 335-66
[23] Bringer M R, Gerdts C J, Song H, Tice J D and Ismagilov R F 2004 Phil. Trans. Royal Society of London A 362 1087-104
[24] Pham V T, Vaidyanathan S, Volos C K, Jafari S, Kuznetsov N V and Hoang T M 2016 European Physical Journal: Special Topics 225 127-36
[25] Lu, H 2002 Physics Letters A 298 109-116
[26] Sambas A, Vaidyanathan S, Mamat M, Sanjaya W M and Rahayu D S 2016 Studies in Computational Intelligence 636 283-310
[27] Volos C K, Kyprianidis I M and Stouboulos I N 2013 J. Engineering Science and Technology Review 6 9-14
[28] Sambas A, Sanjaya W S, Mamat M and Prastio R P 2016 Studies in Fuzziness and Soft Computing 337 133-53
[29] Sambas A, Sanjaya W S and Mamat M, 2015 J. Engineering Science and Technology Review 8 89-95
[30] Sambas A, Sanjaya W S and Halimaturessadiyah 2012 WSEAS Transactions on Systems 9 506-15
[31] Xu G, Xu J, Xiu C, Liu F and Zang Y 2017 Neurocomputing 227 108-12
[32] Wolf A, Swift J B, Swinney H L and Vastano J 1985 Physica D 16 285-317
[33] Li X F, Chu Y D, Zhang J G and Chang Y X 2009 Chaos Solitons & Fractals 41 2360-70
[34] Li C, Sprott J C and Thio W 2014 IEEE Trans. Circuits Sys.-II: Exp. Briefs 61 977-81
[35] Li C, Pehlivan I, Sprott J C and Akgul A 2015 IEICE Electronics Express 12 1-12
[36] Faraji S and Tavazoei M 2013 Open Physics 11 836-44
[37] Kingni S T, Pham V T, Jafari S, Kol G R and Woafo P 2016 Circuits Systems and Signal Processing 35 1933-48
[38] Kingni S T, Pham V T, Jafari S and Woafo P 2017 Chaos Solitons & Fractals 99 209-18
[39] Deng W 2007 J. Comp. Applied Math. 206 174-88
[40] Herrmann R 2014 Fractional Calculus: An Introduction to Physicists (Singapore: World Scientific)
[41] Petras I, Chen Y and Vinagre, B M 2004 Problem 6 208-10.
[42] Petras I 2011 Fractional-Order Nonlinear Systems: Modelling, Analysis and Simulation (Berlin: Springer)
[43] Caponetto R 2010 Fractional Order Systems: Modelling and Control Applications (Singapore: World Scientific)