VERTEX ALGEBRAS AND TEICHMÜLLER MODULAR FORMS

GIULIO CODOGNI

Abstract. We associate to any holomorphic vertex algebra a collection of Teichmüller modular forms, one in each genus. In genus one we obtain the character of the vertex algebra, and we thus reprove Zhu’s modularity result. In higher genus, we prove that these forms have an expansion in terms of the correlation functions of the vertex algebra. We propose applications to the Schottky problem, to the study of the slope of the effective cone of the moduli space of curves, and to the classification of holomorphic vertex algebras. In particular, we prove a uniqueness result for high genera partition functions of the moonshine vertex algebra.

1. Introduction

The first goal of this paper, achieved in Section 4, is to prove the following theorem.

Theorem 1.1. Let $V$ be a holomorphic vertex algebra of central charge $c$. Let $\mathcal{M}_{g,n}$ be the moduli space of complex Deligne-Mumford stable curves of genus $g$ with $n$ marked points. Denote by $\lambda_{g,n}$ the Hodge line bundle over $\mathcal{M}_{g,n}$. Then, for every $g$ and $n$, we can associate a non-zero section $1 = 1_{g,n}(V)$ of $\lambda^{\otimes 2} \otimes c$ to $V$, called partition function. This section is preserved by all clutching and projection morphisms between moduli spaces.

From a physical point of view, the forms constructed in Theorem 1.1 should be the partition functions of the physical theory associated to the vertex algebra; it is denoted by 1 because it is associated to the vacuum vector of $V$, which is the replacement of the unit in the vertex algebras set-up. The partition function encodes many information about the vertex algebra $V$.

The construction of the partition function goes roughly speaking as follows. For every curve $C$, using the vertex operator, we define a map from $V$ tensored by a convenient space of meromorphic forms on $C$ to $\text{End}(V)$. We then consider the covariant $H$ of this action, namely $H$ is the quotient $V$ by $I_V$, where $I$ is the ideal in $\text{End}(V)$ generated by the image of the above map. Varying $C$, this construction gives a line bundle on the moduli space, which is canonically isomorphic to $\lambda^{\otimes 2}$; in the literature, $H$ is usually called the bundles of covacua. The image of the vacuum vector 1 in $H$ gives the partition function.

The second goal of this paper is to study expansion of the partition function around the boundary divisor $\delta_0$ of $\mathcal{M}_g$ (from now on, we take $n = 0$), and propose some applications. Before describing briefly this expansion, let us assure the reader that all notations will be properly explained in the main body of the paper. Given a point in $\delta_0$ representing a curve whose normalization is the projective line, the Schottky coordinates provide a natural local chart around it, see Section 2.6.

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Given $k$ vectors $u_i$ of $V$ and $k$ formal parameters $x_i$, we can define their correlation function as

$$Z(u_1, \ldots, u_k; x_1, \ldots, x_k) := \phi(Y(u_1, x_1) \cdots Y(u_k, x_k) 1),$$

where $Y$ is the vertex operator, $1$ is the vacuum vector and $\phi$ is its dual (in the bra-ket notations $1 = |1\rangle$ and $\phi = \langle 1|$).

Given a vector $k \in \mathbb{N}$, let

$$\chi_k = \sum_{(i_1, \ldots, i_g)} Z(u_1^{(1)}, u_2^{(1)}, \ldots, u_g^{(1)}, u_1^{(g)}, u_2^{(g)}, \ldots, u_g^{(g)}; w_1, z_1, \ldots, w_g, z_g),$$

where $u_i^{(j)}$ and $u_i^{(j)}$ are hortonormal basis for $V_{k_j}$, and the sum run over all values of $(i_1, \ldots, i_g)$ in $\times_{j=1}^{g} \{1, \ldots, \dim V_{k_j}\}$. The function $\chi_k$ depends on the variables $w_i$ and $z_i$. Then our expansion for the partition function is

$$1_g(V) = \sum_{k \in \mathbb{N}_g} \chi_k(w_1, \ldots, w_g, z_1, \ldots, z_g) q_1^{k_1} \cdots q_g^{k_g}$$

where $w_i$, $z_i$ and $q_i$ are the above mentioned Schottky coordinates.$^1$

We also prove the following theorem

**Corollary 1.2** (= Corollary 5.10). Let $V$ be a holomorphic vertex algebra of central charge $c$. For any $g \geq 1$, the right hand side of Equation 4 defines a holomorphic function on the Schottky space $S_g$; up to multiplying it by a holomorphic function which does not depend on $V$, its pull-back to the Teichmüller space $T_g$ transforms under the action of the mapping class group as a Teichmüller modular form of weight $\frac{1}{2}c$.

Specializing to the case $g = 1$, we obtain the following result, due in full generality to Zhu [Zhu96]. Our proof is different from the original one, as we do not analyze the explicit action of $SL(2, \mathbb{Z})$.

**Corollary 1.3** (= Corollary 5.11 ). Let $V$ be a holomorphic vertex algebra of central charge $c$, and $V = \oplus_{n \in \mathbb{N}} V_n$ its conformal grading. Then, the formal power series

$$\left( \prod_{n=1}^{\infty} (1 - q^n) \right)^{-c} \left( \sum_{n \geq 0} \dim(V_n)q^n \right)$$

converges in the unit disc $\Delta$ inside the complex plane, and it is a modular form of weight $\frac{1}{2}c$.

The following conjecture is very natural, especially from the point of view of physics, however its proof seems out of reach at the moment.

**Conjecture 1.4.** If two holomorphic vertex algebras of the same central charge have the same partition function in all genera, then they are isomorphic.

These results can be applied to the classification of vertex algebras; we illustrate our approach with two examples.

The moonshine vertex algebra $V^\natural$ is one of the most important examples of vertex algebras, it has been used by Richard Borcherds to prove the Conway-Norton moonshine conjecture; it has central

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$^1$The moduli space has dimension $3g - 3$, whereas this expansion depends on $3g - 2$ variables; we will see that this functions does not change if we scale the $w_i$ and $z_i$ by a common factor.
charge 24 and $V_1^2 = 0$, and it is a long-standing open conjecture that it is the unique holomorphic vertex algebras with these two properties.

**Corollary 1.5** (= Corollary 6.5). Let $V$ be a holomorphic vertex algebra of central charge 24 with $V_1 = 0$. Then the partition function of $V$ is equal to the partition function of the moonshine vertex algebra for all $g \leq 6$.

If there exists an integer $g_0$ such that the slope $s_g$ of the effective cone of the moduli space of genus $g$ curves is strictly greater than 6 for all $g \leq g_0$, then the partition function of $V$ is equal to the partition function of the moonshine vertex algebra for all $g \leq g_0$.

Let us recall that many experts believe, for reasons which are completely independent from vertex algebras, that $s_g > 6$ for all $g$. This fact, in conjunction with Conjecture 1.4, would prove the uniqueness of the moonshine vertex algebra.

Another important problem in the theory of vertex algebra is to understand if there are only finitely many holomorphic vertex algebra of a given central charge. In the following corollary, by partition function we mean the full collection $\{1_g(V)\}_{g \geq 1}$.

**Corollary 1.6** (= Corollary 6.6). Fix a central charge $c$, then there are countably many possibilities for the partition functions of holomorphic vertex algebras of central charge $c$.

If there exists a constant $C$ depending only on $c$ such that $h^0(\mathcal{M}_g, \lambda_g \otimes \frac{c}{2}) \leq C$ for all $g$, then there are at most $C$ possibilities for the partition functions of holomorphic vertex algebras of central charge $c$.

Our approach also gives a way to prove results about the moduli space of curves using the theory of vertex algebras. In Section 6.2 we explain how our approach can be applied to the Schottky problem. As an example, we give an alternative proof of the following result from [CSB14, GSM11].

**Corollary 1.7** (= Corollary 6.3). The Siegel modular form $\Theta_{E_8 \oplus E_8, 5} - \Theta_{D_4^*, 5}$ defined on $\mathcal{A}_5$ is not identically zero on $\mathcal{M}_5$.

The first part of Theorem 6.1 explains how our approach could be used to bound the slope of the effective cone of divisors of the moduli space of curves, but we do not give any concrete example. We think that these topics deserve further investigations.

**Remark 1.8** (Generalization to vector valued modular forms and non-holomorphic vertex algebras). The results of this paper could be extended to vertex algebras with more than one representation, i.e. to non-holomorphic vertex algebras. The key change should be the convenient generalization of the factorization theorem 3.7, see also Remark 3.8 and the main result of [DGTb]. In this case, the partition function would become a section of a vector bundle whose rank depends on $g$ and $n$.

1.1. **Relation with other works.** This project started in 2013, trying to understand the relation between the papers [GV09, GKV10] and the author’s Ph.D. thesis [Cod14]. Another important source of inspiration was [HvH16].

Partition functions of lattice vertex algebras have been studied systematically by Mason and Tuite, see [MT14, MT03, MT10]. However, their point of view is quite different from ours: they try to define it out if its expansion around the boundary and then they try to prove the convergence, rather than defining the partition function directly on the full moduli space and then studying its
expansion. After that the first version of this paper has appeared, Tuite and Welby published the paper \cite{TuiteWelby}, which is now used in the present work.

Our definition of conformal blocks and covacua is a mix of the ones given in \cite{FBZ}, and the ones given in \cite{Ueno, Ueno2}; as explained in \cite[Chapter 9.7]{FBZ}, they all agree on smooth curves, and on vertex algebras defined out of Kac-Moody groups. The main properties of these objects, which are the naturality \ref{prop:3.3}, the propagation of vacua \ref{def:3.6}, and the factorization \ref{prop:3.7} are expected to hold for any sort of conformal field theory. Our proofs are inspired by the arguments explained in \cite{Ueno}. After the first version of this paper has appeared, Gibney, Damiani and Tarasca published the series of papers \cite{Gibney, Damiani, Tarasca} where they generalize the results of our Section \ref{sec:3} to non-holomorphic vertex algebras. Their construction is similar in spirit to ours; they do not consider partition functions at all.

There is another approach to these topic which relies on chiral algebras rather than on vertex algebras, see \cite{BD, Gai89} and \cite[Chapter 19]{FBZ}. Chiral algebras are already defined on Riemann surfaces; on the other hand, vertex algebras are defined independently of Riemann surfaces and, to make the connection, we need to interpret the parameter $z$ appearing in the definition of vertex algebra as formal co-ordinate on a curve. We decided to focus on vertex algebras mainly because we wanted to translate some problems and features of the moduli space of curves into concrete and explicit traces of operators over vertex algebras, hoping that this could lead to further developments.

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2. Preliminaries

2.1. Holomorphic vertex algebras. The notion of vertex operator arose first physics, and was then formalized with the introduction of vertex operator algebras, or vertex algebras for short, by Richard Borcherds. In this paper we are interested in conformal vertex operator algebras, and their connection with Riemann surfaces. Let us recall the basic definition and establish some notations, the reader can look at \cite{FBZ, MT, FHL} for a more comprehensive exposition.

**Definition 2.1.** A conformal vertex operator algebra with central charge $c \in \mathbb{N}$, or vertex algebra for short, consist of

- An $\mathbb{N}$-graded complex vector space $V = \oplus_n V_n$ such that $\dim V_0 = 1$ and $\dim V_n < +\infty$ for all $n$; this is sometime called space of states, and the grading is called conformal degree;
- a linear map, called vertex operator or states-fields correspondence,
  $$Y : V \rightarrow \text{End}(V)[[z^\pm 1]]$$
  for each element $v$ in $V$, one writes $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$, so that $v_n = \text{Res}_0(z^n Y(v, z) dz)$. The endomorphisms $v_n$ are called modes of $v$;
- a non-zero vector $1 \in V_0$, called vacuum vector, such that $Y(1, z) = \text{Id}_V$ and satisfying the vacuum or creativity axiom:
  $$Y(v, z)1|_{z=0} = v,$$
  Sometime we will write $1_V$ rather than $1$ if we want to specify the vertex algebra;
• a non-zero vector \( \omega \in V_2 \) called conformal vector, whose mode are usually denoted as 
\[ Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \] rather than \( \omega_{n+1} \), such that 
\[ [L_n, L_m] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m,-n} \text{Id}_V \]
\[ [L_{-1}, Y(v, z)] = \frac{d}{dz} Y(v, z) \]
\[ V_n = \{ v \in V | L_0 v = n V \} \]

the operator \( L_0 \) is sometime called the grading operator, and \( L_{-1} \) is sometime denoted by \( T \) and called the translation operator. The field \( Y(\omega, z) \) is called the Virasoro field or the stress-energy tensor.

Moreover one asks for the following property

**Locality:** For any two vectors \( v \) and \( w \) in \( V \) there exists an integer \( k \) such that
\[ (z_1 - z_2)^k [Y(v, z_1), Y(w, z_2)] = 0 ; \]

**\( C_n \) cofiniteness:** Let \( C_n(V) = \{ v_{-n} w | v, w \in V \} \), we say that \( V \) is \( C_n \) cofinite if \( V/C_n(V) \) is finite dimensional; we will assume \( C_n \) cofiniteness for all \( n \geq 2 \);

**Rational:** A vertex algebra is called rational if all modules are semisimple (see Definition 2.5). In this paper we will consider just rational vertex algebras.

Some comments are in order. Vertex algebras can be thought as a quantization of the notion of algebra; indeed and algebra can be defined as a vector space \( A \) and a linear map \( m: A \to \text{End}(A) \) given by the multiplication, the vacuum vector replace the identity.

Locality is probably the most delicate axiom, in some sense replace associativity, we refer the reader to the [FBZ04] or [MT10] for extensive discussions.

The modes \( L_n \) spans inside \( \text{End}(V) \) a copy of the Virasoro algebra; \( V \) is thus a representation of the Virasoro algebra of central charge \( c \). In the application to Riemann surfaces, \( z \) will be a local co-ordinates and the Virasoro algebra will be the Lie algebra of the group of change of co-ordinates; we will give more details about this in Section 2.4.

The \( C_2 \) cofiniteness condition implies the \( C_n \) cofiniteness condition for all \( n \geq 2 \), see [MT10, Section 6.2] or [Buh02] and reference therein.

We will need also the following result, whose proof can be found in [Buh02, Li99, GN03].

**Theorem 2.2** (Finite generation). There exist a countable collection of elements \( \{ v^{(i)} \} \) in \( V \) such that the monomials \( \prod_{i=1}^{j} v^{(i)} \) span \( V \).

The monomials appearing in Theorem 2.2 could be thought as a replacement of the Poincaré-Birkhoff-Witt basis, see [FBZ04 Page 37 and Section 4.4]; let us stress that in this case the monomials are not meant to be linearly independent. It is convenient for us to give the following definition

**Definition 2.3.** A Poincaré-Birkhoff-Witt basis (PBW basis for short) is a basis of \( V \) obtained by taking a subset of the monomials introduced in Theorem 2.2

We finish this section by spelling out the following lemma for later uses.

**Lemma 2.4.** The vacuum vector is annihilated by the operators \( L_n \) for all \( n \geq 0 \).

**Proof.** As \( 1 \in V_0, L_0 1 = 0 \). The endomorphism \( L_n \) is homogeneous of degree \( -n \), see e.g. [MT10 Section 4.1], since \( V_n = \{ 0 \} \) for all \( n < 0 \), we obtain the lemma. \( \square \)
We recall the notion of module for a vertex algebra. We give directly the strongest definition, which is the one we are interested in.

**Definition 2.5.** A module $M$ of conformal weight $h \in \mathbb{N}$ for a vertex algebra $V$ is

- an $\mathbb{N}$-graded vector space $M = \oplus_n M_n$, with $\dim M_n < +\infty$; the subspace $M_0$ is called top level, or space of singular vectors; elements of $M_0$ are called singular vectors;
- a linear map
  \[ Y_M : V \to \text{End}(M)[[z^{\pm 1}]] \]
  which one writes out as
  \[ Y_M(v, z) = \sum_n v^M_n z^{-n-1} \]
  such that:
  - **Vacuum axiom:** $Y_M(1, z) = \text{Id}_M$;
  - **Grading:** we have
    \[ M_n = \{ m \in M \mid L_0^M m = (n + h)m \} \]
    where $L_0^M$ is the operator appearing in the expansion $Y_M(w, z) = \sum L_n^M z^{-n-2}$;
  - **Homogeneity:** if $v$ is in $V_k$, then $v_n^M$ maps $M_m$ to $M_{m+k-n-1}$;
  - **Locality:** For any two vectors $v$ and $w$ in $V$ there exists an integer $k$ such that
    \[ (z_1 - z_2)^k [Y_M(v, z_1), Y_M(w, z_2)] = 0; \]
  - **Associativity:** For any $v$ and $w$ in $V$ and $m$ in $M$ there exists an integer $k$ such that
    \[ (z_1 + z_2)^k Y_M(v, z_1 + z_2)Y_M(w, z_2)m = (z_1 + z_2)^k Y_M((v, z_1)w, z_2)m. \]

The elementary categorical notions such as morphisms of modules, sub-modules and simple modules are defined as expected, see for instance [FHL93] Section 4.3. Every vertex algebra is a module over itself. The restricted dual space
\[ V^\vee = \oplus_n V_n^\vee \]
has also a structure of $V$-module, see for instance [FRZ03] Proposition 10.4.6 or [FHL93] Section 5.2. In this paper, we will use the terms dual and restricted dual interchangeably. The module $V^\vee$ is also called the controgredient module of $V$. A map $\nabla$ of $V$-modules from $V$ to $V^\vee$ is equivalent to a bilinear $V$-invariant form $Q_\nabla$ on $V$; the grading of $V$ gives an orthogonal decomposition for such a form, see [FHL93] Remark 5.3.3. When $\nabla$ is an isomorphism, this quadratic form has to be symmetric, see [FHL93] Proposition 5.3.6. If the vertex algebra $V$ is simple, an isomorphism between $V$ and $V^\vee$, and hence the associated symmetric form, is necessarily unique up to a scalar; this because we are working over the complex numbers, and one can run the standard argument. We can normalize the quadratic form by requiring that the vacuum vector has norm one. If $V$ and $V^\vee$ are isomorphic, $V$ is said to be self-contragredient.

One of the main result about representation theory of vertex algebras is the following.

**Theorem 2.6.** There exists only finitely many simple representations up to isomorphism.

For every module $M$, one can take a countable collection of elements $\{v^{(i)}\}$ in $V$, a basis $m_i$ of the top level $M_0$, such that the monomials $\Pi_{i=1}^m v^{(i)}_{m_i} m_j$ span $M$.

Moreover, for every $n \geq 2$ let $S_n(M) = \{ v_{-n} m | v \in V \text{ and } m \in M_0 \}$, then the quotient $M/S_n(M)$ is finite dimensional.

The letter $S$ in the theorem stands for the initial of singular vectors. The proof relies on the $C_2$ cofiniteness condition of $V$, the reader can look at [MT10] Theorem 6.13 and subsequent remark or [Buih02, Li99, GN03] and references therein. It is convenient for us to generalize Definition 2.3 as follows.
Definition 2.7. A Poincaré-Birkhoff-Witt basis (PBW basis for short) for a \( V \) module \( M \) is a basis of \( M \) obtained by taking a subset of the monomials appearing in Theorem 2.6.

In this paper we are mainly interested in the following type of vertex algebras.

Definition 2.8 (Holomorphic vertex algebra). A vertex algebra is called holomorphic if it admits a unique module up to isomorphism.

Remark that holomorphic vertex algebras are in particular self-contragradient.

2.2. Lattice Vertex Algebras. Let \((\Lambda, Q)\) be an even, unimodular, positive definite lattice of rank \( d \). More explicitly, \( \Lambda \) is a group isomorphic to \( \mathbb{Z}^d \), and \( Q \) is symmetric quadratic form on \( \Lambda \) which is \( \mathbb{Z} \)-valued, unimodular, even (i.e. \( Q(v, v) \) is even for all \( v \in \Lambda \)) and positive definite.

These quadratic forms are widely studied. To start with, their rank is divisible by 8. Fixed the rank, there are finitely many quadratic form of that rank; this follow from the so called mass formula.

In rank 8 we have just one quadratic form \( E_8 \). In rank 16 we have two quadratic forms, \( E_8 \oplus E_8 \) and \( D_{16}^{+} \). In rank 24 we have exactly 24 quadratic forms. In rank 32, the forms are already many million, and there is no classifications, see [Kin03]

It is possible to attach a vertex algebra of central charge \( d \) to a rank \( d \) quadratic form; we refer the reader to [Don93], [FBZ04 Section 5.4.1] or [MT10, Section 7.3] for their definition and properties. These vertex algebras are called lattice vertex algebras, they are simple, and the number of representation is equal to determinant of the quadratic form. In particular, the vertex algebra associated to an unimodular quadratic form is holomorphic.

The vertex algebra associated to the lattice \( E_8 \) is isomorphic to the vertex algebra associated to the affine Kac-Moody group of \( E_8 \).

2.3. Stable curves. We work over the field of complex numbers and use the Euclidean topology. We recall the relevant definition for our paper; we refer the reader to [ACG11] for a complete discussion, and to [Uen08, Chapter 1] for a shorter introduction.

Definition 2.9. A genus \( g \) Deligne-Mumford stable curve over a base \( B \) with \( n \) marked points is a variety \( X \) together with a projective flat morphism \( \pi : X \to B \) and \( n \) sections \( x_1, \ldots, x_n \) of \( \pi \) such that

- for every \( b \in B \), the fibers of \( X_t = \pi^{-1}(b) \) has complex dimension one, genus \( g \), at worst nodal singularities and finite automorphism group;
- the images \( x_i(b) \) lie in the smooth locus of \( X_b \) and do not intersect.

The sections \( x_i \) are sometime denoted with letters \( a, b, \ldots \). Sometime we use the name of the section also for its image. Smooth curves are called Riemann surfaces.

We denote by \( \omega_\pi \) the relative canonical sheaf of the morphism \( \pi \).

Sometime, a curve over \( B \) is called a family of curves over \( B \), and co-ordinates on \( B \) are called the moduli of the family.

Given a vector of positive integers \( k = (k_1, \ldots, k_n) \), we can look at the line bundles \( \mathcal{O}_X(kx) := \mathcal{O}_X(\sum k_i x_i(B)) \) and \( \omega_\pi(kx) = \omega_\pi(\sum k_i x_i(B)) \) on \( X \). Given two such vectors \( k \) and \( j \), we say that \( k \geq j \) if \( k_i \geq j_i \) for every \( i \). If \( k \geq j \), we have a natural injective map from \( \mathcal{O}_X(jx) \) to \( \mathcal{O}_X(kx) \) and from \( \omega_\pi(jx) \) to \( \omega_\pi(kx) \); this defines a direct system and we can let

\[
\pi_* \mathcal{O}_X(*x) = \lim_k \pi_* \mathcal{O}_X(kx), \quad \pi_* \omega_\pi(*x) = \lim_k \pi_* \omega_\pi(kx).
\]
These are the sheaves of meromorphic functions and meromorphic forms with algebraic singularities at the marked points \( x_i \). We will need the following lemma.

**Lemma 2.10.** Given a morphism \( f : T \to B \), let \( \pi_T : X_T \to T \) be the fiber product, and denote still by \( x_i \) the pull-back of the sections. Then, the natural pull-back maps

\[
f^* : \pi_* \mathcal{O}_X(*x) \to (\pi_T)_* \mathcal{O}_{X_T}(*x) \quad \text{and} \quad f^* : \pi_* \omega_{\pi}(*x) \to (\pi_T)_* \omega_{\pi_T}(*x)
\]

are surjective.

**Proof.** We prove the statement just for \( \mathcal{O}_X(*x) \), the other proof being analogue. We have to show that for \( k \) big enough the map

\[
f_k^* : \pi_* \mathcal{O}_X(kx) \to (\pi_T)_* \mathcal{O}_{X_T}(kx)
\]

is surjective. If \( f \) is flat, \( f_k \) is surjective for all \( k \) by the standard base change theorem. In general, if \( \sum k_i \) is bigger than the genus \( g \), the cohomology groups appearing in the proof of the base change theorem do vanish, hence \( f_k \) is surjective.

\[\square\]

### 2.4. Local co-ordinate on a curve

The formal variable \( z \) appearing in the definition of vertex algebra will play the role of local co-ordinate on a Riemann surface; to make this precise, we need first to recall a few facts about the formal definition of local co-ordinate and discuss change of co-ordinates, we will follow [FBZ04 Chapter 6].

We recall the notion of standard disc, following mainly [FBZ04 Appendix A.1.1 and Section 6.2]. The standard disc \( D \) over the complex numbers is \( \text{Spec} \mathcal{O} \), where \( \mathcal{O} = \mathbb{C}[z] \); it contains two points, the closed point and the generic point. The maximal ideal of \( \mathbb{C}[[z]] \) is \( z \mathbb{C}[[z]] \), and \( z \) is his preferred generator. A local co-ordinate on \( D \) is the choice of a generator of the maximal ideal of \( \mathcal{O} \), or, equivalently, the choice of an isomorphism between \( \mathcal{O} \) and \( \mathbb{C}[[z]] \). For a \( \mathbb{C} \)-algebra \( R \), one can look at the disc \( \text{Spec}(\mathcal{O} \otimes R) \).

The group \( \text{Aut}(\mathcal{O}) \) of automorphism of \( D \) is a group scheme whose Lie algebra is isomorphic to \( z \mathbb{C}[[z]] \frac{\partial}{\partial z} \). The identification \( L_n = -z^{n+1} \frac{\partial}{\partial z} \) gives an isomorphism between this Lie algebra and the subalgebra of the Virasoro algebra spanned by the modes \( \{L_n\}_{n \geq 0} \). Let us stress for later uses that, as noticed in Lemma 2.4 this Lie algebra annihilates the vacuum vector.

Given a vertex algebra \( V \) and a module \( M \) (possibly equal to \( V \)), we would like to integrate the action of the Virasoro algebra obtained out of the modes of the conformal vector \( \omega \) to an action of \( \text{Aut}(\mathcal{O}) \). This is always possible because the action of \( L_0 \) is diagonal with integer eigenvalues, and, since \( \dim M_n = 0 \) for \( n < 0 \), the action of \( L_1 \) is locally nilpotent. We conclude that \( \text{Aut}(\mathcal{O}) \) acts naturally on both \( V \) and \( M \). The equivariance of the action of \( V \) on \( M \) is proven in [FBZ04 Lemma 6.5.6].

Let now \((X,x)\) be a stable pointed curve over a smooth base \( B \). A local co-ordinate at \( x \) is an isomorphism between the completion of the local ring \( \mathcal{O}_{X,x} \) and \( \mathcal{O}_B[[z]] \). The automorphism group \( \text{Aut}(\mathcal{O}) \) is a group scheme over \( B \), and the sheaf on \( B \) of local co-ordinate at \( x \) is denoted by \( \text{Co}_x \). When marked points are denoted by \( x_i \), then we use the notation \( \text{Co}_{x_i} \) for \( \text{Co}_{x_i} \), and \( \text{Aut}_{x_i} \cong \text{Aut}(\mathcal{O}) \) for the automorphism group of \( \text{Co}_{x_i} \).

We collect here some existence results that we will need later on.

**Theorem 2.11.** Let \((X,x_1,\ldots,x_n)\) be a curve; fix local co-ordinates \( z_i \) at \( x_i \) for every \( i \).

Fix strictly positive integers \( M \), \( t < n \) and \( m_i < M \) for \( i = 1,\ldots,t \). Up to shrinking \( B \), there exists a relative meromorphic differential \( \omega \) on \( X \) such that

1. \( \omega \) has poles along \( x_i \) for \( i > t \), and it is regular elsewhere;
Proof. Let $T_i(M)$ be the sheaf of Taylor expansion at $x_i$ of order up to $M$. For every positive integer $K$, we have an exact sequence

$$0 \to \omega_\pi \left( K \sum_{j > t} x_j - M \sum_{j \leq t} x_j \right) \to \omega_\pi \left( K \sum_{j > t} x_j \right) \to \bigoplus_{j \leq t} T_j(M) \to 0$$

If $K$ is big enough then $R^1\pi_*\omega_\pi \left( K \sum_{j > t} x_j - M \sum_{j \leq t} x_j \right) = 0$, so the map $H^0 \left( \omega_\pi \left( K \sum_{j > t} x_j \right) \right) \to H^0 \left( \bigoplus_{j \leq t} T_j(M) \right)$ is surjective. The requested form is any element in the preimage of $\oplus_{j \leq t} z_j^{m_j}$.

2.5. Kuranishi families and Schottky co-ordinates. A Kuranishi family is a stable curve $X \to B$ such that for every point $b$ in $B$, and every other stable curve $Y \to S$ such that there exists a point $s$ in $S$ with $Y_b$ isomorphic to $X_b$, there exists, up to shrinking $X$ around $b$ and $Y$ around $s$, a unique map $f: S \to B$ such that $Y$ is isomorphic to the fiber product $S \times_B X$. Here, we collect some of their properties, the reader can look at [ACG11 Section XI.4] or [Uen08 Section 1.2] for a more comprehensive introduction.

Given a point $b$ on the base $B$ of a Kuranishi family, there exists a canonical isomorphism, called Kodaira-Spencer map, between the tangent space of $B$ at $b$ and $H^1(X_b, T_{X_b}(-\sum x_i(b)))$, where $x_i$ are the marked points. By base change, we have that $H^1(X_b, T_{X_b}(-\sum x_i(b)))$ is isomorphic to the fiber at $b$ of $R^1\pi_* T_\pi X(-\sum x_i(B))$, and more generally, the tangent space $TB$ is isomorphic to $R^1\pi_* T_\pi X(-\sum x_i(B))$.

We now discuss the variation of singularities in a Kuranishi family $\pi: X \to B$, and their resolution. This should be compared with [ACG11 Chapter XI, Theorem 3.17] and [Uen08 Section 4.3.1].

Let $b$ be a point of $B$ such that the fiber $X_b$ is singular; we work locally around $b$, and in to perform the following construction one might have to shrink $B$. Let $n$ be a node of $X_b$, and assume for simplicity that it is the unique node. Then, there exist a local co-ordinate $q$ on $B$ such that around $n$ the curve $X$ is isomorphic to $\{xy = q\} \times \mathbb{C} \times Z$, where $Z = \{q = 0\} \subset B$. The locus $Z$ parametrizes deformations which are trivial around $n$, so that the node is preserved, and conversely, when $q \neq 0$, the node is smoothed out. We call $q$ a smoothing co-ordinate, or smoothing parameter, associated to $n$, and $Z$ the locus of equisingular curves.

Let $X_Z$ be the base change of $X$ to $Z$. Up to shrinking $Z$, we have a section, which by abuse of notation we still denote by $n$, whose image is the singular locus of $\pi$ around our original node $n$; in particular, for every $z$ in $Z$, $n(z)$ is a node for $X_z$. We can now look at the blow-up $\nu: \tilde{X}_Z \to X_Z$ of $n(Z)$; let $a$ and $b$ be the preimage of $n$, so $a$ and $b$ are section of $\tilde{X}_Z: \to Z$. In this way, we obtained either a family of smooth curves of genus $g - 1$, with two extra marked points, $a$ and $b$; or two families of smooth curves of genus respectively $h$ and $g - h$ for some $h$, each one with an extra marked point. Moreover, around $a$ and $b$ we have two distinguished local co-ordinates, which are the the pull-back of $x$ and $y$ from the local description of $X$ around $n$ given above.

If $X_b$ has more then one node, we can repeat the above construction enhancing the notation. So denote the nodes by $n_i$, the smoothing parameter by $q_i$. The locus $Z$ is given by $\{n_iq_i = 0\}$, so it is a a simple normal crossing divisor. If we base change to $Z$ and shrink $Z$, we blow-up all the
$n_i(Z)$, and we obtain new marked points $a_i$ and $b_i$ with distinguished local co-ordinates $z_i$ and $w_i$. Recall that in this case $X_Z$ can be the disjoint union of many different families of curves.

If $\pi: X \to B$ is not a Kuranishi family, the above construction still make sense, we just have to pull-back the $q_i$ from the Kuranishi family. However, it might be that a node $n$ is not be smoothed out on $B$, i.e. the pull-back of the corresponding smoothing co-ordinates is identically trivial; this is tautologically true in the case where $B$ is the locus $Z$ constructed above. We call such a node a horizontal node.

Let $\pi: X \to B$ be a Kuranishi family of nodal curves (with possibly one marked point), and assume that there exists a point $b$ in $B$ such that the normalization of $X_b$ is isomorphic to $\mathbb{P}^1$; in particular $X_b$ has $g$ nodes. We call such a family a Schottky family. Let $B^*$ be the locus parametrizing smooth curves, and $Z \subset B$ the equisingular locus of $X_b$. The locus $Z$ parametrizes the Schottky uniformization.

Theorem 2.12. Take $g \geq 2$. The space $S_g$ is an open subset of $\mathbb{C}^{3g-3}$. There exists an open subset $\Omega_g$ of $\mathbb{C}^{3g-2}$ with a projection $\pi: \Omega_g \to S_g$ and a $\pi$-invariant action of $F_g$ on $\Omega_g$ such that the quotient $\pi: \Omega_g/F_g =: \mathcal{C}_g \to S_g$ is a family of smooth Riemann surfaces and the associated moduli map to $\mathcal{M}_g$ is surjective with discrete infinite fibers. The universal cover of $S_g$ is canonically isomorphic to the Teichmüller space $T_g$.

Proof. We recall just the embedding of $S_g$ in $\mathbb{C}^{3g-3}$, because this defines meaningful co-ordinates on $S_g$.

First, we can realize $\hat{S}_g$ as an open subset of $(\mathbb{P}^1)^{\times g}$ by taking the fixed points and the translation length of each element $\gamma_i$ (which can be shown to be hyperbolic). The quotient $\hat{S}_g \to S_g$ a natural section obtained by choosing a representative $s = (\gamma_1, \ldots, \gamma_g)$ of a point $[s] \in S_g$ such that $a_1$ has
an attractive fixed point at 0 and a repelling fixed point at ∞, and 1 is an attractive fixed point for $a_2$. At first sight, this proves that $S_g$ can be realized as an opens subset of $(\mathbb{P}^1)^{3g-3}$; however, the translation length of an hyperbolic element can never be $\infty$, and the fixed point can neither be $\infty$ because this is already taken by $a_1$, hence the embedding is in $\mathbb{C}^{3g-3}$.

Taking the fiber product $C_g \times_{S_g} C_g$ over $C_g$, we obtain a family of curves with one marked point (the section is the diagonal). This gives a surjection from $C_g \rightarrow: S_{g,1}$ to $M_{g,1}$.

The following result proven in [GHSS] gives a compactification of the Schottky space called extended Schottky space.

**Theorem 2.13.** There exists a $3g - 3$ dimensional smooth complex manifold $\overline{S}_g$, called extended Schottky space, with a family of DM-stable curves $\pi: \overline{C}_g \rightarrow \overline{S}_g$, which contains $\pi: C_g \rightarrow S_g$ as an open subset, and such that the associated moduli map towards $\overline{M}_g$ is surjective with infinite discrete fibers. The complement of $S_g$ in $\overline{S}_g$ consists of divisors.

One can introduce the notion of Schottky structure for a family of Deligne-Mumford curves, and both $S_g$ and $\overline{S}_g$ are fine moduli spaces for this structure.

Let $\pi: C \rightarrow B$ be a Schottky family. Fix a point $b$ such that the normalization of $X_b$ is $\mathbb{P}^1$. Up to shrinking $B$ around $b$, fixing an ordering of the vanishing cycle, we obtain a Schottky structure, hence a map $B \rightarrow \overline{S}_g$. Restricting to the locus $B^*$ parametrizing smooth curves, the Schottky coordinates introduced at the end of Section 2.5 pull-back to the Schottky co-ordinates on $B^*$ (see also the proof of Theorem 2.12).

### 3. Conformal blocks and partition function

**3.1. Definitions.** Let $\pi: X \rightarrow B$ be a stable curve over a smooth base $B$ with $n$ marked points $x_1, \ldots, x_n$, and $V$ a vertex algebra. Let $M_1, \ldots, M_n$ be $V$ modules. The labeling of the modules correspond to the labeling of the marked points, and we will say that $M_i$ sits at $x_i$; if points are labeled by letters $a, b, \ldots$, then modules are labeled as $M_a, M_b, \ldots$. When $M_i$ is isomorphic to $V$ or $V^\vee$, we might write $V_i$ or $V_i^\vee$. The modules $M_i$ do not need to be non-isomorphic, indeed we will often take $M_i = V_i \cong V$ for every $i$.

Using the notation of Section 2.6 we define a bundle $\mathcal{M}_i$ over $B$ as $(M_i \times \text{Co}_i)/\text{Aut}_i$. Up to shrinking $B$, we can fix a global section of Co$_i$ and obtain an isomorphism between $\mathcal{M}_i$ and the trivial sheaf $M_i \otimes \mathcal{O}_B$. In particular, $\mathcal{M}_i$ is quasi-coherent, and locally it is a direct limit of locally free coherent sheaves $(\oplus_{k<N} M_k) \otimes \mathcal{O}_B$. Let $\mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$, and $\mathcal{V} = V_1 \otimes \cdots \otimes V_n$.

Let $f: T \rightarrow B$ be a map, and $X_T$ the fiber product of $X$ and $T$. The bundle $\mathcal{M}_i$ constructed on $T$ using the family $X_T$ is isomorphic to the pull-back of $\mathcal{M}_i$ via $f$.

For every $T \hookrightarrow B$ open, let $U_T$ the complement of $\{x_1(T), \ldots, x_n(T)\}$ in $X_T$. Take $T$ small enough such that one has local co-ordinates around each $x_i$ We have a map of sheaves

$$\mathcal{Y}: \left( \bigoplus_{i=1}^n V_i \right) \otimes ((\pi_T)_* \omega_{\pi_T}(*)x) \rightarrow \text{End}(\mathcal{M})$$

which is defined as

$$\mathcal{Y} \left( \sum_{i=1}^n v_i \otimes \omega \right) = \sum_{i=1}^n \text{Res}_{x_i} (f(z_i)Y_{M_i}(v_i, z_i)dz_i)$$

where $z_i$ is a local co-ordinate at $x_i$, and $\omega = f_i dz_i$ around $x_i(T)$. This does not depend on the choice of the local co-ordinate; see [FZ04] Theorem 6.5.4 and Lemma 6.5.6], remark that the quoted proof is given by a local argument. We will interchangeably use the notations $\mathcal{Y}(v, \omega)$ and
\[ \mathcal{Y}(v \otimes \omega). \] Loosely speaking, the main difference between the operators \( \mathcal{Y} \) and \( \mathcal{Y} \) is that \( \mathcal{Y} \) gives a power series whose coefficients are endomorphisms, instead \( \mathcal{Y} \) gives an actual endomorphism.

The image of \( \mathcal{Y} \) is a sub sheaf of \( \text{End}(\mathcal{M}) \) over \( B \), which we denote by \( U(V) \).

**Definition 3.1** (Covacua and conformal blocks). The sheaf of covacua, or coinvariant, \( H(X, x_1, \ldots, x_n) \) is the biggest quotient of \( \mathcal{M} \) where the action of \( U(V) \) is trivial. The sheaf of conformal blocks \( C(X, x_1, \ldots, x_n) \) is the sheaf of \( U(V) \)-invariant linear functional on \( \mathcal{M} \).

So far, these sheaves are quasi-coherent sheaves on \( B \). The space of coinvariants can be also seen as \( \mathcal{M}/I(U(V)) \), where \( I(U(V)) \) is the ideal generated by \( U(V) \) in \( \text{End}(\mathcal{M}) \). Let us also stress that these sheaves are dual sheaves.

**Lemma 3.2.** The quasi-coherent sheaves \( H(X, x_1, \ldots, x_n) \) and \( C(X, x_1, \ldots, x_n) \) are coherent.

**Proof.** Being dual to each other, it is enough to prove the claim for \( M = H(X, x_1, \ldots, x_n) \). The statement is local on \( B \), so we can shrink \( B \) and fix local co-ordinates \( z_i \) at \( x_i \); the sheaf \( H \) is now a quotient of \( M := \mathcal{O}_B \otimes_B M_i \otimes \cdots \otimes M_n \). The argument is similar to the proof of [Uem08 Theorem 3.5], we sketch it.

We have a filtration \( F_p M = \oplus_{i_1+\cdots+i_n=p} M_i \otimes \cdots \otimes M_n \). This induces a filtration of \( H \) by quasi-coherent sub-sheaves; we are going to show that the associated graded object \( G \) is coherent.

By Riemann-Roch, the associated graded object is a quotient of \( q: \mathcal{O}_B \otimes_B M/S_k(M) \rightarrow G \), for \( k \) big enough. The domain is a locally free sheaf of finite rank by Theorem 2.4. The kernel of \( q \) is quasi-coherent (see for instance [Sta17, Tag 01Y1]), so \( G \) is coherent.

The vacuum vector 1 is annihilated by the Lie algebra of the group of change of co-ordinate, see Lemma 2.4 and Section 2.4, hence it is invariant under the action of \( \text{Aut}(\mathcal{O}) \). We conclude that the vector \( 1 \otimes \cdots \otimes 1 \), which by abuse of notations we still denote by 1, is a well defined section of \( \mathcal{V} \). In the special case where \( \mathcal{M} = \mathcal{V} \), we give the flowing definition.

**Definition 3.3** (Partition Function). Let \( V \) be a vertex algebra, and take \( M_i \cong V \) for every \( i \). The partition function of the vertex algebra \( V \) on the stable curve \( \pi: X \rightarrow B \) is the image of the vacuum vector 1 in \( H(X, x_1, \ldots, x_n) \). In other words, the partition function is a section of the sheaf of coinvariants when \( \mathcal{M} = \mathcal{V} \). Sometime we will denote it by \( 1_g \), \( 1_V \), \( 1_g(V) \) or \( 1(V) \), if we want to stress the genus or the vertex algebra.

The partition function does not have to be a constant section, it can vanish somewhere on \( B \).

**Remark 3.4** (Genus zero case). Take \( X = \mathbb{P}^1 \times B, n = 1 \), and \( M_1 = V \). The space of covacua is canonically isomorphic to the trivial bundle associated to the vector space \( V_0 = \mathbb{C}1 \), the partition function is just 1. The space of conformal blocks is associated to \( \mathbb{C} \phi \), where \( \phi(1) = 1 \) and \( \text{Ker}(\phi) = \oplus_{n \geq 1} V_n \). (In the bra-ket notation, \( \phi \) is denoted by \( \langle 1 | \cdot \rangle \).

The following result is a consequence of Lemma 2.10.

**Lemma 3.5** (Naturality). Let \( f: T \rightarrow B \) be a morphism, then \( H(X_T, x_1, \ldots, x_n) \cong f^*H(X, x_1, \ldots, x_n) \), \( C(X_T, x_1, \ldots, x_n) \cong f^*C(X, x_1, \ldots, x_n) \) and, if \( \mathcal{M} = \mathcal{V} \), we have \( f^*1 = 1 \).

### 3.2. Propagation of vacua

We keep the notations of the previous section. Taking the tensor with 1, we have a natural map

\[ \rho: M_1 \otimes \cdots \otimes M_n \rightarrow M_1 \otimes \cdots \otimes M_n \otimes V \]
As explained in Lemma 2.3 and Section 2.4, the vacuum vector 1 is fixed by the change of co-ordinates, hence this induces a map $\rho: \mathcal{M} \to \mathcal{M} \otimes V_{n+1}$, where now $V$ is sitting at the $n+1$-th marked point $x_{n+1} = y$. We have the following result:

**Theorem 3.6 (Propagation of vacua).** The morphism $\rho$ defined above induces an isomorphism

$$\rho: H(X, x_1, \ldots, x_n) \to H(X, x_1, \ldots, x_n, y)$$

where $y$ is the $n+1$-th marked point on the curve. Moreover, when $\mathcal{M} = \mathcal{V}$, the morphism $\rho$ preserves the partition function.

**Proof.** The map $\rho$ in the statement is defined as $\rho([v]) = [\rho(v)]$, so we first have to shows that it is well defined. To this end, remark that regular differentials on $U := X \setminus \{x_1(B), \ldots, x_n(B)\}$ are in particular regular differentials on $X,x_i$. Since in a local co-ordinate around $z \in Y$, regular function on $Z \subset Y$ is in the codomain of $\rho$ such that in a local co-ordinate around $y$ it looks like $(z^k + z^mg(z))dz$, we know nothing about the behavior of $\omega$ at $x_i$. This holds by inductive hypothesis on the conformal degree of $w'$.

To prove injectivity, we rather show the surjectivity of the dual map

$$\rho^\vee: C(X, x_1, \ldots, x_n, y) \to C(X, x_1, \ldots, x_n)$$

If the family is smooth, this is [FBZ04, Theorem 10.3.1], where an inverse of $\rho^\vee$ is exhibited. We now prove surjectivity of $\rho^\vee$ in the general case using the PBW basis introduced in Definition 2.4.1 rather than constructing an inverse; our argument is inspired by both the proofs of [FBZ04, Theorem 4.4.1] and [Uen08, Theorem 3.3.1]. Again, this is local on $B$, so we can fix local co-ordinates. Given an element $\phi \in C(X, x_1, \ldots, x_n)$, we want to define an element $\psi \in C(X, x_1, \ldots, x_n, y)$ such that $\rho^\vee(\psi) = \phi$. We let $\psi(v_1 \otimes \cdots \otimes v_n \otimes 1) := \phi(v_1 \otimes \cdots \otimes v_n)$. To define $\psi(v_1 \otimes \cdots \otimes v_n \otimes w)$, we choose a PBW basis, and by linearity we can assume that $w$ is of the form

$$w = A^{(i)}_{k_1} \cdots A^{(j)}_{k_{j-1}} 1$$

where $A^{(j)}$ is an element of $V$. Let $w_j$ be $A^{(j-1)}_{k_{j-1}} \cdots A^{(1)}_{k_1} 1$, so that $w_j = w$. Take integers $m_0$ such that $A^{(j)}_{m_j} w_j = 0$ for all $m > m_0$ and all $j$. Then, using Theorem 2.4.1 take abelian differentials $\omega_j$ on $U$ such that in a local co-ordinate around $y$ it looks like $(z^{k_j} + z^{s_j} g(z))dz$ for some $s_j > m_j$, where $g$ is a regular function on $z$. We let

$$\psi(v_1 \otimes \cdots \otimes v_n \otimes w) := (-1)^j \phi(\mathcal{Y}(\omega_1, A_1) \cdots \mathcal{Y}(\omega_j, A_j)(v_1 \otimes \cdots \otimes v_n))$$
This does not depend on the choice of \( \omega \) and gives the requested element. Invaraince can be checked following the argument of the proof of [FBZ04, Theorem 3.3.1].

We need to recall for later uses the expression of the inverse of \( \rho \) in the case where \( X \) is smooth of genus zero. These results are proven in [FBZ04, Section 10.4]. We will also slightly twist the notations for marked points, so that the statements are more suitable for later applications. Let \( X \) be \( \mathbb{P}^1 \times B \), with \( 2g + 1 \) marked points \( a_1, \ldots, a_g, b_1, \ldots, b_g, \) and \( x \). Let \( z \) be a co-ordinate on \( \mathbb{P}^1 \) such that \( z(x) = 1, z(b_g) = 0 \) and \( z(a_g) = \infty \); denote by \( z_i \) and \( w_i \) the value of \( z \) at \( a_i \) and \( b_i \). Let us put at the point \( a_g \) the module \( V^\vee \), for the others choose just \( V \). The conformal block \( C(X, a_g, b_g) \) has a canonical element given by the pairing \( \Phi_0 \) between \( V \) and \( V^\vee \); moreover, thanks to Theorem 3.6 if \( V \cong V^\vee \), this is also isomorphic to the space \( C(X, x) \) discussed in Remark 3.4 and the image of the pairing is a generator.

Under the isomorphism

\[
(\rho^\vee)^{-1} : C(\mathbb{P}^1, a_g, b_g) \to C(\mathbb{P}^1, a_1, \ldots, a_g, b_1, \ldots, b_g, x)
\]
given by Theorem 3.6, we have that

\[
(1) \quad \Phi_0(\psi \otimes A_1 \otimes \cdots A_{g-1} B_1 \otimes \cdots B_g \otimes 1) = \psi(Y(A_1, z_1) \cdots Y(A_{g-1}, z_{g-1}) Y(B_1, w_1) \cdots Y(B_{g-1}, w_{g-1}) B_g)
\]

where \( \psi \in V^\vee \), \( 1 \) is the vacuum vector sitting at \( x \), and \( A_i \) and \( B_i \) are element of \( V \). This equation is proven in [FBZ04, Subsection 10.4.10]; the terms appearing are rational function of the \( z_i \) and \( w_i \) with poles along the diagonal. We used that \( Y(1, z) = \text{Id} \), and \( Y(B_g, z) |_{z=0} = B_g \). In particular, following [FBZ04, Equation 10.4.10], for \( n = 1 \), we get

\[
(2) \quad \Phi_0(\psi \otimes B_1 \otimes 1) = \psi(B_1)
\]

3.3. Factorization. We assume that the vertex algebra is holomorphic; in particular, it is self-controgradient and there exist a unique isomorphism \( \nabla : V \to V^\vee \) of \( V \) module such that the norm of the vacuum vector for the associated symmetric quadratic form \( Q = Q_\nabla \) is one.

Theorem 3.7 (Factorisation). Assume that \( V \) is holomorphic, and let \( \pi : X \to B \) be a curve with one marked point \( x \) and a horizontal node \( n \) as in Section 2.2, denote by \( \nu : \hat{X} \to X \) the normalization of \( n \). Let \( F \) be the morphism \( F : V \to V \otimes V \otimes V \cong V \otimes V \otimes V^\vee \), mapping \( v \) to \( v \otimes 1 \otimes 1 \). If \( \hat{X} \) is irreducible, then \( F \) induces an isomorphism

\[
F : H(X, x) \to H(\hat{X}, x, a, b)
\]

and preserve the partition function. If \( \hat{X} \) has two connected components \( X_1 \) and \( X_2 \), and say \( x \) and \( a \) are on \( X_1 \) and \( b \) is on \( X_2 \), then \( F \) induces an isomorphism

\[
F : H(X, x) \to H(X_1, x, a) \otimes H(X_2, b)
\]

and the partition function is mapped to the tensor product of the partition functions.

Proof. We discuss just the connected case, the other been analogue. The map \( F \) induces a well-defined map on the sheaf of vacua because we can pull-back forms via the normalization morphism \( \nu \). It is also clear from the definition that \( F \) preserves the partition function.

First we show that \( F \) is surjective. Vectors of the form \( [v \otimes 1 \otimes 1] \) are of course in the image of \( F \). Take a homogeneous vector \( [v \otimes w_a \otimes w_b] \), we prove that this is in the image of \( F \) by induction on \( \deg(w_a) + \deg(w_b) \). Assume that \( \deg(w_a) \geq \deg(w_b) \), and write \( w_a = A_k w' \), for some \( A \in V \). Let \( m_0 \) be an integer such that \( A_m w_a = A_m w_b = 0 \) for all \( m > m_0 \). Then, by Theorem 2.11, take an abelian differential \( \omega \) on \( \hat{X} \setminus \{x, a, b\} \) such that in a local co-ordinate around \( a \) it looks
like \((z^k + z^m f(z)) dz\), and looks like \(z^n g(z)\) around \(b\), where \(f\) and \(g\) are regular functions and \(m,n > m_0\). Of course we can not control its principal part at \(x\). Now,

\[
\mathcal{Y}(A, \omega)v \otimes w' \otimes w_b = (\mathcal{Y}_{x}(A, \omega)v) \otimes w' \otimes w_b + v \otimes w_a \otimes w_b
\]

Hence in \(H(\tilde{X}, x, a, b)\) we have that \(v \otimes w_a \otimes w_b\) is equal to \(-(\mathcal{Y}_{x}(A, \omega)v)) \otimes w' \otimes w_b\), and the surjectivity follows by induction on the conformal degree.

To prove injectivity, we rather show the surjectivity of the dual map \(F^\vee : C(\tilde{X}, x, a, b) \rightarrow C(X, x)\). Given an element \(\phi \in C(X, x)\), we want to define an element \(\psi \in C(\tilde{X}, x, a, b)\) such that \(\rho^\vee(\psi) = \phi\). Of course, we let \(\psi(v \otimes 1 \otimes 1) := \phi(v)\). Arguing similarly to Theorem 3.6, we use a PBW basis to define \(\psi\) given a vector \(v \otimes w_a \otimes w_b\), we write out using a PBW basis, cf. Definition 2.11: \(w_a = A_1^{(1)} \cdots A_{k_1}^{(1)} w_a\) and \(w_b = B_1^{(1)} \cdots B_{k_2}^{(1)} w_b\). Take an integer \(m_0\) such that \(A_m^{(t)} v = B_m^{(t)} v = 0\) for all \(i, j\) and all \(m > m_0\). Using Theorem 2.11, we can find an \(m > m_0\) such that there exist meromorphic differentials \(\omega_a^{(t)}\), for \(t = 1, \ldots, \ell\), on \(\tilde{X}\) with a pole just at \(x\) such that, locally at \(a\), we have \(\omega_a^{(t)} = (z_a^{m_t} + z_a^m f(z_a)) dz_a\), where \(m_t > m_0\) and \(f\) is some regular function of \(z_a\), and locally around \(b\) we have \(\omega_a = z_b^m f(z_b) dz_b\), where \(m > m_0\) and \(f\) is some regular function of \(z_b\).

Analogously, we define differentials \(\omega_b^{(t)}\) for \(t = 1, \ldots, k\). We now let

\[
\psi(v \otimes w_a \otimes w_b) = \phi(\mathcal{Y}(A_1, \omega_a^{(1)}) \cdots \mathcal{Y}(A_{k_1}^{(1)} \omega_a) \mathcal{Y}(B_1, \omega_b^{(1)}) \cdots \mathcal{Y}(B_{k_2}^{(1)} \omega_b)v)
\]

Arguing as in the proof of [Uen08, Theorem 3.3.5], one shows that \(\psi\) is invariant, so it gives the requested element.

**Remark 3.8** (Factorization theorem in the general case). Dropping the hypothesis on \(V\) having a single representation, the natural generalization of Theorem 3.7 is an isomorphism

\[
F : H(X, x, M) \rightarrow \bigoplus_N H(\tilde{X}, x, a, b, M, N^\vee)
\]

where the sum run over all \(V\) module. The main twist in the proof of Theorem 3.7 is in the surjectivity of \(F^\vee\). This result has been proved in [DGTb], after the first version of this paper has appeared.

### 3.4. Sheaf of differential operators

In this section, we will use the notions of Kuranishi family and Kodaira-Spencer map introduced in Section 2.3. Our first goal is to prove the following result:

**Theorem 3.9.** Let \(\pi : X \rightarrow B\) be a smooth Kuranishi family of one pointed curves; let \(x\) be the marked point and choose as \(V\) module for this point the vertex algebra \(V\). Then, the sheaves of covacua \(H(X, x)\) and of conformal blocks \(C(X, x)\) are locally free on \(B\).

We will then compute the Chern class of \(H(X, x)\) in the line bundle case.

**Proof.** Being dual sheaves, it is enough to prove the claim for \(H(X, x)\). Up to shrinking \(B\), we can fix a vertical co-ordinate \(z\) at \(x(B)\). Denote with \(S\) the divisor on \(X\) given by \(x(B)\). We look at the exact sequence of sheaves on \(X\) given by

\[
0 \rightarrow T_{\pi} X(-S) \rightarrow T_{\pi} X(S) \rightarrow \mathcal{O}_B[z^{-1}] \frac{d}{dz} \rightarrow 0
\]

where \(*S\) is defined in Section 2.3. the last map takes the principal part of the vector field along \(S\).

The non-zero part of the associated long exact sequence in co-homology reads

\[
0 \rightarrow \pi_* T_{\pi} X(S) \rightarrow \mathcal{O}_B[z^{-1}] \frac{d}{dz} \delta \rightarrow R^1 \pi_* T_{\pi} X(-S) \rightarrow 0
\]
The Kodaira-Spencer map identifies $R^1\pi_*T\pi X(-S)$ with $TB$.

Denote by $\Lambda$ the sheaf $O_B[z^{-1}]\frac{d}{dz}$ on $B$. On $\Lambda$ we have a natural lie bracket say $[\cdot, \cdot]_0$, and we need to introduce a new bracket $[\ell, m] : = [\ell, m]_0 + \delta(\ell)(g) - \delta(m)(f)$, where $\ell = f\frac{d}{dz}$, $m = g\frac{d}{dz}$, $f$ and $g$ are in $O_B[z^{-1}]$, and hence, under the identification of $R^1\pi_*T\pi X(-S)$ with $TB$ given by the Kodaira-Spencer map, $\delta(\ell)$ acts on $g$ and $\delta(m)$ acts on $f$. With this new bracket, the above exact sequence becomes a sequence of Lie algebras.

Following the notations established in Section 2.1, let $\omega$ be the conformal vector of the vertex algebra and $c$ the central charge. For every $\ell = f\frac{d}{dz} \in \Lambda$, we let

$$\omega[\ell] = \text{Res}_{z=0}(Y(\omega, z)fdz)$$

so that $\omega[\ell]$ naturally act on $V$. The following formula can be found for instance in [Uen08, Lemma 2.13], and it is obtained by explicit computation

$$[\omega[\ell], \omega[m]] = -\omega[[\ell, m]_0] + \frac{c}{12} \text{Res}_{\{z=0\}} \left( \frac{d^3f}{dz^3} gdz \right)$$

The third derivative and the central charge appear because of the commutators of the modes $L_n$ of $\omega$ introduced in Section 2.1.

We now associate to an element $\ell$ of $\Lambda$ a differential operator $D(\ell)$ acting on $O_B \otimes V$ as

$$D(\ell)(F \otimes v) = \omega[\ell]v - \delta(\ell)(F)$$

Equation (4) implies the following cocycle relation

$$[D(\ell), D(m)] = D([\ell, m]) + \frac{c}{12} \text{Res}_{\{z=0\}} \left( \frac{d^3f}{dz^3} gdz \right) \text{Id}_V$$

where $\ell = f\frac{d}{dz}$ and $m = g\frac{d}{dz}$. So this exhibits a central extension $\hat{\Lambda}$ of $\Lambda$ as a sheaf of differential operators of order at most one acting on $O_B \otimes V$. The symbol map $\sigma: \hat{\Lambda} \to TB \cong R^1\pi_*T\pi X$ is obtained by first projecting to $\Lambda$ and then applying $\delta$, thus it is surjective.

One has now to show that the action of $\hat{\Lambda}$ preserve $U(V)(O_B \otimes V)$, hence $\hat{\Lambda}$ acts on $H(X, x)$, one can argue as in [Uen08, Proposition 4.7], Theorem 3.9 now follows from the following general well-known proposition.

**Proposition 3.10.** Let $\mathcal{F}$ be a coherent sheaf on a smooth variety $B$, assume that $P$ is a sheaf of order at most one differential operator acting on $\mathcal{F}$, and and the symbol map is surjective, then $\mathcal{F}$ is locally free.

The above proof tells actually a bit more. We have a sequence

$$0 \to O_B \to \hat{\Lambda} \to TB \to 0$$

which is far from being exact; $\hat{\Lambda}$ act as a sheaf of differential operators of order at most one on $H(X, x)$, $O_B$ represents the identity and the last map is the symbol. If we mod out $\hat{\Lambda}$ by the operators which acts trivially, the sequence is still not exact. However, if $H(X, x)$ is a line bundle, then this quotient is exactly the Atiyah algebra of the line bundle and the sequence is exact.

Denote by $\mathcal{A}$ the sheaf of Lie algebras sitting in the exact sequence

$$0 \to O_B \to \mathcal{A} \to TB \to 0$$
define \(d\) by the cocycle
\[
(v, w) = \frac{1}{12} \text{Res}_x \left( \frac{d^3 f}{dz^3} \right)
\]
where \(v\) and \(w\) are sections of \(TB \cong \mathcal{O}_B[z^{-1}] \mathcal{T}_\pi X(-S)\), which admit \(f \frac{dz}{dz}^3\) and \(g \frac{dz}{dz}^3\) as lift in \(\Lambda = \mathcal{O}_B[z^{-1}] \mathcal{O}_\pi \mathcal{T}_\pi X(-S)\). This is an Atiyah algebra. We can also consider the algebras \(cA\) defined by scaling the cocycle by \(c\). We have the following result, which should be compared with and [Uen08 Theorem 5.6], see also [Tsu91] and [MOP15].

**Proposition 3.11.** Assume that \(H(X, x)\) is a line bundle. Then
1. the Atiyah algebra of \(H(X, x)\) is \(cA\);
2. let \(H'(X, x)\) be the bundle of covacua associated to another vertex algebra \(V'\) with the same central charge of \(V\); if \(\text{Pic}^0(B)\) is trivial then there is an isomorphism between \(H\) and \(H'\) which is equivariant for the action of \(cA\);
3. the Chern class of \(H(X, x)\) equals \(c/2\) times the Chern class of the Hodge line bundle.

**Proof.** First we prove item 1. Given a section of \(TB \cong \mathcal{O}_B[z^{-1}] \mathcal{T}_\pi X(-S)\), we can lift it to \(\Lambda = \mathcal{O}_B[z^{-1}] \mathcal{O}_\pi \mathcal{T}_\pi X(-S)\), and let it act on \(\mathcal{O}_B \otimes \mathcal{V}\) using Equation 5. This action passes to the quotient \(H(X, x)\). We conclude the proof observing that the cocycle described in Equation 6 matches up with the cocycle defining \(cA\).

For item 2, observe that the class of the extension defining the Atiyah algebra equals the first Chern class of the line bundle, hence \(H(X, x)\) and \(H'(X, x)\) have the same Chern class. Being \(\text{Pic}^0(B)\) trivial, the two line bundles are isomorphic. We can then choose an isomorphism which preserves the Atiyah algebras.

For item 3 it is enough to show that \(2A\) is the Atiyah algebra of the Hodge line bundle. This is well-known. A (not that direct) way to see it, is to look at the bundle of covacua for Kac-Moody groups. There, we both have an action of the Atiyah algebra of a convenient multiple of the Hodge line bundle (this is the main result of [Tsu91]), and of \(cA\), see e.g. [Tsu91] or [Uen08]. This allows us to identify the two algebras.

The relation between the Virasoro algebra and the Atiyah algebra of the Hodge line bundle has been also studied in [ADCKP88].

### 3.5. Local freeness at the boundary.

**Theorem 3.12.** Let \(\pi: X \to B\) a Kuranishi family of Deligne-Mumford stable curves with one marked point \(x\), and assume that the vertex algebra is holomorphic; then, the sheaves of covacua \(H(X, x)\) and conformal blocks \(C(X, x)\) are dual vector bundles.

**Proof.** It is enough to prove that \(C(X, x)\) is locally free. When the family is smooth, this theorem is equivalent to Theorem 3.9. Fix now a point \(b\) in \(B\), such that \(X_b\) is singular. The statement being local, we can shrink \(B\) if necessary, so, to start with, we fix a local vertical co-ordinate \(z\) around \(x(B)\) and identify \(V\) with \(V\).

**Lemma 3.13.** Let \(Z\) be the locus of equisingular curves introduced in Section 2.6, then the restriction of \(C(X, x)\) to \(Z\) is locally free.

**Proof.** We can apply the naturality of conformal blocks 3.5 and the factorization theorem 3.7 to translate the statement on the normalization \(X_Z \to Z\); this is now a smooth family, so the result follows from Theorem 5.8. \(\square\)
Let $\tilde{Z}$ be the formal completion of $Z$ in $B$. To prove Theorem 3.12 by semi-continuity of the rank of coherent sheaves, cf. [Har77] Exercise 5.8, it is enough to show that
\[ \operatorname{rk}(C(X, x)|_Z) \leq \operatorname{rk}(C(X, x)|_{\tilde{Z}}) \]
To this end, we will construct a lifting map of $O_{\tilde{Z}}$ modules
\[ \lambda: C(X, x)|_Z \to C(X, x)|_{\tilde{Z}} \]
This map will be the inverse of the natural restriction map $\rho: C(X, x)|_{\tilde{Z}} \to C(X, x)|_Z$, hence it will be injective. Later, $\lambda$ will also be used to compute the Taylor expansion of the partition function at the boundary.

Let $q_i$ be the smoothing co-ordinates associated to the nodes, as in Section 2.5. For a multindex $\vec{k} = (k_1, \ldots, k_t) \in \mathbb{N}^t$, set $q^{\vec{k}} = q_1^{k_1} \cdots q_t^{k_t}$.

Now, it comes a key construction. Take the identity $\Id_{n}$, where $\Id_{n}$ is a basis of $V_n$ and $b^i$ its dual). Let
\[ \tilde{v}_k := v \otimes \Id_{k_1} \otimes \cdots \otimes \Id_{k_t}. \]
This is an element of $V_Z := V \otimes (V_{a_i} \otimes V_{b_i})$; as usual, $V_{a_i}$ and $V_{b_i}$ are just copies of $V$ which we think sitting at $a_i$ and $b_i$. Let now
\[ \tilde{v} := \sum_{k \in \mathbb{N}^t} \tilde{v}_k q^{\vec{k}} \]
This belongs to $V_Z[q_1, \ldots, q_t]$.

Let $\Phi$ be an element of $C(X, x)|_Z$. Because of the Factorization Theorem 3.7, we can think at it as a linear functional on $(V_Z)^\vee$. For every $v \in V$, we can compute $\Phi$ on $\tilde{v}$; more precisely, we define
\[ \lambda(\Phi)(v) := \sum_{k \in \mathbb{N}^t} \Phi(\tilde{v}_k) q^{\vec{k}}. \]
We then extend $\lambda(\Phi)$ by $O_{\tilde{Z}}$-linearity to obtain a linear functional on $V \otimes O_{\tilde{Z}}$.

If we interpret $\tilde{v}$ as a section of $V \otimes O_{\tilde{Z}}$, its restriction to $Z$ is again $v$, hence $\rho(\lambda(\Phi)) = \Phi$.

The following is the so called formal Gauge condition, proven in [Uen08] Lemmas 4.16 and 4.17 by explicit computation.

**Lemma 3.14.** The linear functional $\lambda(\Phi)$ is invariant under the action of the meromorphic functions on $X$ with poles at $x(B)$.

The above lemma implies that $\lambda(\Phi)$ is in $C(X, x)|_{\tilde{Z}}$, and we have thus concluded the proof of Theorem 3.12.

### 3.6. Expansion of the partition function.

We want now to use the lifting map $\lambda$ appeared in the proof of Theorem 3.12 to compute the Taylor expansion of the partition function in term of the Schottky co-ordinates introduced in Section 2.5.

Let $X \to B$ a Schottky family with one marked point. Let $b_0$ be a point such that the normalization of $X_{b_0}$ is isomorphic to $\mathbb{P}^1$. Assume we can fix a local co-ordinate $z$ around the marked point. The partition function is a section of $H(X, x)$, in order to compute its Taylor expansion in term of the $q_i$ we need to trivialize $H(X, x)$ around $b_0$. To this end, remark that $C(X, x) \otimes H(X, x)$ has a natural trivialization given by the pairing, so we pair 1 with a non zero section of $C(X, x)$. We choose this section in a canonical way.

Recall that, thanks to the naturality 3.5, the propagation of vacua 3.6 and Remark 3.4, the restriction of $C(X, x)$ to $Z$ is canonically isomorphic to $(V_0)^\vee = \mathbb{C}\emptyset$. We can now lift $\emptyset$ via $\lambda$ to
a section of $C(X, x)$ on the formal completion of $Z$ in $B$, and pair it with 1. We state this as a definition for further reference.

**Definition 3.15.** With the above notation, we denote by $\lambda(\phi)$ the section of $C(X, x)$ defined on the formal completion of $Z$ in $B$ obtained by applying the lifting map $\lambda$ introduced in the proof of Theorem 3.12 to $\phi$.

Let us write out

$$\lambda(\phi)(1) = \sum_{k \in \mathbb{N}^g} \chi_k g^k,$$

where the $\chi_k$ are function on $Z$. We have

$$\chi_k(w_1, \ldots, w_g, z_1, \ldots, z_g) = \phi(Id_k \otimes 1)$$

Let us stress our abuse of notations: $\phi$ is a linear functional on $V$, we are computing it on a vector of $V^{\otimes 2g+1}$, to make sense of this we have to apply the propagation of vacua 3.6. Let

For $g = 1$, $k$ is just an integer $k$, and we have

$$\chi_k = \phi(Id_{k} \otimes 1) = \sum_{i=1}^{\dim V_k} \phi(Y(v_i, w)Y(v^i, z)) = \sum_{i=1}^{\dim V_k} Q(v_i, v^i) = \dim V_k ,$$

where $v_i$ and $v^i$ are hortonormal basis for $V_k$; $Q$ is the invariant bilinear form on $V$, we have used the propagation of vacua 3.6 (see also [FBZ04, Section 10.4.11] with $A = 1$), and $Y(1, z) = Id_V$.

To describe $\chi_k$ for general values of $g$, let us first recall the correlation function $Z$ of $k$ vectors $w_i$ of $V$ with formal parameters $x_i$:

$$Z(w_1, \ldots, w_k; x_1, \ldots, x_k) := \phi(Y(w_1, x_1) \cdots Y(w_k, x_k) 1),$$

(recall that in the bra-ket notation one writes $\phi = \langle 1 |$ and $1 = | 1 \rangle$) we then have

$$\chi_k = \sum_{(i_1, \ldots, i_g)} Z(v^{(1)}_{i_1}, v^{(1)}_{i_1}, \ldots, v^{(g)}_{i_g}, v^{(g)}_{i_g}; w_1, z_1, \ldots, w_g, z_g),$$

(8)

where $v^{(j)}_{i}$ and $v^{(j)}_{i}$ are hortonormal basis for $V_{k_j}$, and the sum run over all values of $(i_1, \ldots, i_g)$ in $\chi^g_{j=1} 1, \ldots, \dim V_{k_j}$).

The function $\chi_k$ is symmetric on the $w_i$ and $z_i$, this follows both from our construction (indeed, it has to be a holomorphic function on $Z$, and $w_i$ and $z_i$ are coordinates on $Z$), and, independently, from the theory vertex algebra, see e.g. [FHL93, Proposition 3.5.1].

Fixing a value $j_0$, and normalizing $z_{j_0} = 0$ ad $w_{j_0} = \infty$, we obtain

$$\chi_k = \text{Tr}_{V_{k_{j_0}}} \left( \prod_{j=1, j \neq j_0}^g Y(v^{(j)}_{i_j}, w_j) Y(v^{(j)}_{i_j}, z_i) \right).$$

Let us remark that the operator we are computing the trace of, does not preserves $V_{k_{j_0}}$, so we are implicitly composing it with the projection from $V = \oplus V_k$ to $V_{k_{j_0}}$.

The functions $\chi_k$ can be further Taylor expanded in the variables $w_j$ and $z_j$ if needed; in [GV09, Section 3], some of these coefficients have been explicitly computed in the case of low rank lattice vertex algebras.

The following lemma will be crucial for the application to the moonshine vertex algebra.

**Lemma 3.16.** If $V_1$ is trivial and one of the entry of $k$ is equal to 1, then $\chi_k = 0$. 


Proof. As $V_1$ is trivial, the identity in $\text{End}(V_1)$ is equal to zero, hence also $\text{Id}_{\mathbb{R}} = 0$ and $\phi(\text{Id}_{\mathbb{R}} \otimes 1) = \phi(0) = 0$. □

3.7. Differential equation near the boundary. In this section, we will show that the conformal block introduced in Definition 3.15 is annihilated by some Fuchsian differential operators. An analogue differential equation is given in [DGTb, Equation 38]

Notation 3.17. Let $\pi: X \to B$ be a Kuranishi family of stable curves with one marked point $x$, with a point $0 \in B$ such that the normalization of $X_0$ is $\mathbb{P}^1$. We assume that $X$ can be uniformized in the following way. Let $a_i$ and $b_i$, for $i = 1, \ldots, g$, be the preimage of the nodes $n_i$ of $X_0$ in $\mathbb{P}^1$. Choose a coordinate $z$ on $\mathbb{P}^1$ such that none of them is $\infty$ and the marked point is 1. Let $D_i$ (resp. $E_i$) be a disc around $a_i$ (resp. $b_i$) of radius 1. Assume none of these disc intersect each other and they do not contain 1. Let $q_i$ be a smoothing parameter for $n_i$, and assume that it is a coordinate defined on all $B$, and $|q_i| < 1$. Let $U$ be the open subset $\mathbb{P}^1 \times B$ obtained removing the sets $|z(p) - z(a_i)| < |q_i|$ and $|z(p) - z(b_i)| < |q_i|$ for all $i$. Then, we assume that $\pi: X \to B$ is the family obtained gluing the points $(a, s)$ and $(b, s)$ of $U$ such that, for some $i, p \in D_i, b \in E_i$ and $z(a)z(b) = q_i$. The section $x(s) = 1$, for all $s \in B$, and $z$ also gives a natural co-ordinate around 1.

Let $B^*$ be the locus parametrizing smooth curves

The previous uniformization is always possible either assuming that $\pi$ is a family of curves with Schottky structure, or up to shrinking $B$. We will need the following well-known lemma, whose proof can be found in [Uen08, Lemma 5.15 and Corollary 5.17]

Lemma 3.18. In the notation 3.17 restrict $\pi$ to the constructible subset $B_i := \{q_i = 0, q_j \neq 0 \text{ for } j \neq i\}$, let $\pi_i: X_i \to B_i$ be the normalization of the node $n_i$, then there exists a relative meromorphic vector field $\ell_i$ on $X_i$ with poles just around $x$, which locally around $a_i$ and $b_i$, for a convenient choice of coordinates $w_i$ and $z_i$, is of the form $\frac{1}{2} w_i \frac{dw_i}{d\ell_i}$ and $\frac{1}{2} z_i \frac{dz_i}{d\ell_i}$. Let $\ell_i$ be Laurent expansion of $\ell_i$ around $x$ in term of the coordinate $z$.

Under the Kodaira-Spencer isomorphisms, $\ell_i$ equals $q_i \frac{\partial}{\partial q_i}$.

Theorem 3.19. In the notation 3.17 let $\ell_i$ as in Lemma 3.18 and, for $i = 1, \ldots, g$, define the differential operator

$$D_i := D[\ell_i] + a_i(b) \text{Id}_V - q_i \frac{d}{dq_i},$$

where $a_i$ is a holomorphic function introduced during the proof which does not depend on the vertex algebra $V$, and $D[\ell_i] = \text{Res}_{z=0} (\ell_i(z) Y(\omega, z) dz)$. Assume one of the following two hypotheses

1. the coordinates coming from lemma 3.18 are related to the coordinates from notation 3.17 via projective transformations,

2. we can choose a non-singular odd theta characteristic for the family.

Then, over the locus $B^*$ parametrizing smooth curves, the section $\lambda(\phi)$ introduced in Definition 3.18 satisfies the differential equations

$$D_i \lambda(\phi) = 0 \quad i = 1, \ldots, g$$

In particular we have:

Convergence: the conformal block $\lambda(\phi)$ converges on all $B$;

Uniqueness: the section $\lambda(\phi)$ is the unique conformal block annihilated by the $g$ differential operators $D_i$ and equal to $\phi$ over the locus parametrizing curves whose normalization is $\mathbb{P}^1$. 20
Proof. Let \( T(z) = Y(\omega, z) = \sum L_n z^{-n-2} \). Let \( g(z) \) be a change of coordinates; \( g(z) \) is also an element of \( \text{Aut}(\mathcal{O}) \), which acts on \( V \), let \( R(g) \) be the action; we have the following well-known transformation rule is obtained by direct computation, see e.g. [FBZ04, Section 8.2.2]:

\[
T(z) = R(g)T(g(z))R(g)^{-1}(g'(z))^2 + \frac{c}{12}S(g)\text{Id}_V,
\]

where \( S(g) \) is the Schwarzian derivative of \( g \). Recall that projective transformation are exactly the functions \( g \) such that \( S(g) = 0 \); a projective atlas for a Riemann surfaces is an atlas such that the change of coordinates are projective transformation. If we are in the hypothesis 1, we can choose a projective atlas and then, up to the convergence issue, \( T(z) \) is a quadratic vertical differential valued in \( \text{End}(V) \).

If we are in the hypothesis 2, we can assume that there exists a projective connection \( \rho(z) \) of central charge \( c \) on \( X \), i.e. \( \rho \) is a collection of holomorphic functions on \( X \) which transform as

\[
\rho(z) = \rho(g(z))(g'(z))^2 + \frac{c}{12}S(g),
\]

see [Uen08, pages 15-17 and Theorem 1.15] or [FBZ04, Section 8.2.5]. We can then look at \( \tilde{T}(z) := T(z) - \rho(z)\text{Id}_V \). This, up to convergence issue, gives a two form valued in \( \text{End}(V) \).

Let

\[
\Omega := \lambda(\phi)\tilde{T}(z).
\]

Since \( \lambda(\phi) \) satisfies the formal Gauge condition, cf. [3.14] for all meromorphic functions on \( X \) with poles just around \( x \) and all \( v \in V \), we have

\[
\Omega \text{Res}_{z=1} (f(z)Y(z, v)dz) = 0.
\]

By the strong residue Theorem [FBZ04, Theorem 9.2.9] or [Uen08, Lemma 1.1.6], \( \Omega \) is the a relative two form on \( X \) valued in \( V^\vee \otimes \text{End}(V) \).

Let \( \ell(z) \) be a meromorphic vertical vector field on \( X \), or just the Laurent expansion of a vector field around \( x \). Pairing it with \( \Omega \) we obtain a one form, let us compute its residue at \( x \). Let \( \gamma \) be small cylinder around \( x(0) \). We have

\[
\int_{\gamma} \ell(z)\lambda(\phi)\tilde{T}(z)dz^2 = \lambda(\phi) \text{Res}_{z=0} \left( \ell(z)\tilde{T}(z)dz^2 \right) = D[\ell] \lambda(\phi) + a_i(b) \lambda(\phi),
\]

where

\[
a_i(b) = \int_{\gamma} \ell(z)\rho(z)(dz)^2
\]

Fix all coordinates on \( B \) except \( q_i \) at a generic values, and let \( \Delta_i \) be the resulting disc. Let \( X^{(i)} \) be the restriction of \( X \) to \( \Delta_i \); this is a family of smooth curves, except the curve over the origin which has a single node. Locally around the node \( n_i \), the family is of the form \( w_iz_i = q_i \), where \( w_i \), \( z_i \) and \( q_i \) are the coordinates from Lemma 3.18 outside the node \( n_i \), the family is trivial and consists of a constant smooth open subset \( U_i \). The vector field \( \tilde{\ell}_i \) from Lemma 3.18 is well defined on \( U_i \), and we can consider the differential form \( \tilde{\ell}_i\Omega \). Its only pole is at \( x \), and we know its residue is \( D[\tilde{\ell}_i] \). On the other hand, let \( \gamma_a \) be the boundary of \( U_i \) around \( a_i \) and \( \gamma_b \) the boundary around \( b_i \), integrating \( \tilde{\ell}_i\Omega \) over them we obtain

\[
\int_{\gamma_a} \tilde{\ell}_i\Omega = \frac{1}{2} \int_{\gamma_a} w_i \frac{d}{dw_i} \lambda(\phi)T(w_i)dw_i^2 = \frac{1}{2} \lambda(\phi)L_0 = \frac{1}{2} q_i \frac{d}{dq_i} \lambda(\phi)
\]

We get no contribution from \( \rho(z) \) as it extends regularly over \( w_i = 0 \).
A similar computation holds around \( b_i \), hence
\[
(D[\ell_i] + a_i(b)) \lambda(\phi) = q_i \frac{d}{dq_i} \lambda(\phi)
\]
This shows that \( D_i \lambda(\phi) = 0 \) for all \( i \). Both items 3.19 and 3.19 follows from the general theory of Fuchsian differential equations.

\[\square\]

Remark 3.20. The term \( a_i \) can be computed explicitly. If we can guarantee that the change of coordinate is projective, it vanishes. If the change of coordinate \( s \) is the exponential \( \exp(z) \) as in the Fuchsian uniformization, we have the familiar \( a_i = c q_i^2 \).

Remark 3.21 (Analogy with the heat equation). Recall that the classical theta function satisfies the heat equation, which reads \( \frac{\partial \theta}{\partial \tau}|_{z=0} = 2\pi i \frac{\partial^2 \theta}{(\partial z)^2}|_{z=0} \). In a fancy language, this equation relates the \( \tau \)-direction, which is the horizontal direction along the moduli space, with the \( z \)-direction, which is the vertical direction along the universal family. This interpretation has been developed in [Wel83]. The equations appearing in Theorem 3.19 are similar in spirit, indeed the conformal block \( \lambda(\phi) \) is a sort of modular form, the term \( q \frac{d}{dq} \) is in the horizontal direction, and the term \( D[\ell_i] \) in the vertical one.

4. Vertex Algebras and the moduli space of curves

In this section we start working with the moduli space of genus \( g \) smooth curves \( M_{g,n} \) and its Deligne-Mumford compactification \( \overline{M}_{g,n} \). The first space parametrizes smooth genus \( g \) curves with \( n \) marked points; the second parametrizes all Deligne-Mumford stable curves, as in Definition 2.9. They are both smooth Deligne-Mumford stacks; an atals is given by a disjoint union of finitely many Kuranishi families; in other words, Kuranishi families are local charts for \( \overline{M}_{g,n} \). A comprehensive description of these objects can be found in [ACG11].

Theorem 4.1. Let \( V \) be a holomorphic vertex algebra, then the sheaves of conformal blocks and covacua define dual line bundles \( C = C_{g,n} \) and \( H = H_{g,n} \) on \( \overline{M}_{g,n} \) for every \( g \geq 0 \) and every \( n \geq 1 \). The partition function \( 1 = 1_{g,n} \) is a non-zero regular section of \( H \) for every \( g \) and \( n \).

Proof. These two sheaves defines vector bundles on every Kuranishi family thanks to Theorem 3.12. The naturality 3.5 implies that these bundles glue to give vector bundles on \( \overline{M}_{g,n} \). To compute the rank, it is enough to look at the dimension of the fiber over one point. We choose a point which represent a curve \( C \) whose normalization is the projective line. Combining the naturality 3.5, the factorization 3.7, the propagation of vacua 3.6 and Remark 3.4, we obtain that the rank is one.

The partition function defines a section of \( H \) on every Kuranishi family, it glues to a global section thanks to the naturality 3.5. The above quoted results also show that the partition function restrict to a non-zero vector in the fiber over the curve \( C \) described above.

\[\square\]

We now consider a number of projections and clutching morphism between moduli spaces. For every \( g \), \( n \) and \( k \), we have a projection morphisms
\[
\pi: \overline{M}_{g,n+k} \to \overline{M}_{g,n}
\]
which forgets \( k \) sections. Moreover, by choosing a convenient gluing schemes, one has morphism
\[
G: \prod_{i=1}^{r} \overline{M}_{g_i,n_i} \to \overline{M}_{g,n}
\]
where \(g_i, n_i, g\) and \(n\) are appropriate, see [ACG11, Section XII.10] for details. Let us denote again by \(H\) and \(C\) the box products of line bundles on covacua and conformal blocks on \(\mathcal{M}_{g_i, n_i}\), and call partition function and denote by \(1\) the box product of the respective partition functions. The following theorem is a formal consequence of the propagation of vacua \(3.6\) as far as the projection morphisms are concerned, and of the factorization \(3.7\) as far as the clutchings morphisms are concerned.

**Theorem 4.2.** Let \(V\) be a holomorphic vertex algebra. The projections and clutchings morphisms preserve the bundles of conformal blocks and covacua, and the partition functions. With the above notations, \(\pi^* H = H\), \(\pi^* C = C\) and \(\pi^* 1 = 1\), and the identification are canonical; same statement hold if we replace \(\pi\) with \(G\).

The following result make these constructions descend to the moduli space of curves without marked points, so with \(n = 0\).

**Theorem 4.3.** Let \(\pi: \mathcal{M}_{g, 1} \rightarrow \mathcal{M}_g\) be the projection morphism, then there exist line bundles \(H = H_g\) and \(C = C_g\) on \(\mathcal{M}_g\) which pull-back to the line bundles of covacua and of conformal blocks; moreover, there exists a section \(1 = 1_g\) of \(H\) which pull-backs to the partition function. We again call these line bundles the line bundles of covacua and of conformal blocks, and we call \(1\) the partition function.

**Proof.** We have that \(\mathcal{M}_{g, 1} \times_\pi \mathcal{M}_{g, 1} \cong \mathcal{M}_{g, 2}\), and the propagation of vacua \(3.6\) gives descend data for \(H_{g, 1}, C_{g, 1}\) and \(1_{g, 1}\). We can now conclude using standard descend theory.

The Hodge bundle \(\mathcal{E}_g\) on \(\mathcal{M}_{g, n}\) is the rank \(g\) vector bundle whose fiber at \([C]\) is \(H^0(C, \omega_C)\). Its determinant is called the lambda class \(\lambda = \lambda_{g, n}\), it is semiample on \(\mathcal{M}_{g, n}\) and generates the Picard group of \(\mathcal{M}_g\). On \(\mathcal{M}_g\) we also have the irreducible boundary divisors \(\delta_i\), with \(i = 0, 1, \ldots, \lfloor \frac{g}{2}\rfloor\).

The generic point of \(\delta_0\) is a curve with just one node such that its normalization is an irreducible smooth curve of genus \(g - 1\). The generic point of \(\delta_i\) is a curve with just one node such that its normalization is the disjoint union of two curves of genus respectively \(i\) and \(g - i\). We denote by \(\delta_i\) also the line bundle associated to the boundary divisors.

The Picard group of \(\mathcal{M}_g\) is freely by \(\lambda\) and the \(\delta_i\) when \(g \geq 3\); moreover, the group of degree zero line bundle is trivial, so a line bundle can be recovered out of its Chern class. More details can be found in [ACG11] or [AC87].

**Theorem 4.4.** Let \(V\) be a holomorphic simple vertex algebra of central charge \(c\), then the line bundle \(H_{g, n}\) on \(\mathcal{M}_{g, n}\) associated to \(V\) is isomorphic to \(\frac{1}{c} \lambda_{g, n}\).

Moreover, when \(n = 0\), the partition function does not vanish on any boundary divisor of \(\mathcal{M}_g\).

**Proof.** First, we take \(n = 0\), and let \(H = H_g\). The line bundle \(H\) has a regular section, namely the partition function \(1\), so it is of the form \(a\lambda - \sum b_i\delta_i\).

To compute \(a\), we restrict \(H\) to the open subset \(\mathcal{M}_g\). There, the Picard group is generated by \(\lambda\) so \(H = a\lambda\) for some \(a\). Recall that \(\text{Pic}^0(\mathcal{M}_g) = 0\), so \(a\) is computed in Proposition \(3.11\).

We now focus on the \(b_i\). First, take \(i > 1\). Fix a genus \(i\) curve \(A\) and a genus \(g - i\) curve \(B\), and a marked point \(b\) on \(B\). We can embed \(A\) in \(\delta_i\) mapping a point \(a\) to the curve \(A \cup B/a \sim b\).

The intersection between \(A\) and \(H\) is \(b_i\). On the other hand, the restriction of \(H\) to \(A\) is trivial by the naturality of conformal blocks \(3.5\), the factorization \(3.7\) and the propagation of vacua \(3.6\). We conclude that \(b_i = 0\). The case \(i = 0\) or \(1\) can be treated similarly.

To prove that \(1\) does not vanish along \(\delta_i\), let \([C_i]\) be point of \(\delta_i\) parametrizing a curve whose normalization is a disjoint union of copies \(\mathbb{P}^1\). Combining the naturally \(3.5\) with the factorization
the propagation of vacua and the description of the covacua on a genus zero curve given in Remark 3.4, we see that 1 is not zero at \([C]\).

For the case \(n > 0\), look at the projection \(\pi: \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g\). We have \(\pi^*H_g = H_{g,n}\) and \(\pi^*\lambda_g = \lambda_{g,n}\) (cf. [ACG11] and [Cor93 page 2]), so the conclusion.

\[\square\]

**Remark 4.5.** Let \(V\) and \(V'\) be two holomorphic vertex algebras with the same central charge \(c\); take \(g \geq 3\) and let \(H\) and \(H'\) be the sheaves of covacua associated to \(V\) and \(V'\). Theorem 4.4 shows that \(H\) and \(H'\) are isomorphic. The isomorphism can be chosen in a canonical way. Indeed, the automorphism group of \(\overline{\mathcal{M}}_g\) is trivial for \(g \geq 3\), so the isomorphism is unique up to a scalar. To fix this scalar, let \(Z \subset \delta_0 \subset \overline{\mathcal{M}}_g\) be the locus parametrizing curves whose normalization is \(\mathbb{P}^1\). On \(Z\), the line bundles \(H\) and \(H'\) are canonically isomorphic respectively to \(V_0 \otimes O_Z\) and \(V'_0 \otimes O_Z\). We can now normalize the isomorphism by requiring that it maps the vacuum vector to the vacuum vector.

5. **Vertex Algebras and Teichmüller modular form**

5.1. **Stable Siegel modular forms and theta series.** Here, we recall some well known fact about Siegel modular forms. Let \(\mathcal{H}_g\) be the Siegel upper half space. The quotient stack \(\mathcal{H}_g/Sp(2g,\mathbb{Z})\) is the fine moduli space \(\mathcal{A}_g\) of principally dimension abelian \(g\)-fold. Let \(L_g\) be the Hodge bundle on \(\mathcal{A}_g\). Modular forms of weigh \(k\) and degree \(g\) are elements of \(H^0(\mathcal{A}_g, L_g^\otimes k)\). They can be identified with holomorphic functions \(f\) on \(\mathcal{H}_g\) such that

\[f(g\tau) = \rho(\tau, g)^k f(\tau)\]

for all \(\tau \in \mathcal{H}_g\) and all \(g \in Sp(2g, \mathbb{Z})\), where \(\rho\) is a convenient explicit cocycle.

There is a restriction morphism

\[\Phi: H^0(\mathcal{A}_g, L_g^\otimes k) \to H^0(\mathcal{A}_{g-1}, L_{g-1}^\otimes k)\]

**Definition 5.1** (Stable Siegel modular forms ). A weight \(k\) stable Siegel modular forms is a collection \((s_g)_{g\geq 1}\) of weight \(k\) and degree \(g\) Siegel modular forms such that \(\Phi(s_g) = s_{g-1}\). Denote by \(H^0(\mathcal{A}_\infty, L_\infty^\otimes k)\) the space of weight \(k\) stable Siegel modular forms, and by \(\mathcal{R}(\mathcal{A}_\infty, L_\infty) := \oplus_{k\geq 0} H^0(\mathcal{A}_\infty, L_\infty^\otimes k)\) the ring of stable Siegel modular forms.

Let \((\Lambda, Q)\) be a quadratic form of rank \(c\), which is positive definite, unimodular and even. The following function is the theta series associated to \(Q\)

\[\Theta_{Q,g}(\tau) = \sum_{v_1, \ldots, v_g \in \Lambda} \exp(\pi \sqrt{-1} \sum_{i,j} Q(v_i, v_j) \tau_{i,j})\]

This is well known to be modular form of weight \(\frac{1}{2}c\). Moreover, in [Pre77] it is shown that if \(g\) is big enough with respect to the weight, the space of modular forms is spanned by theta series.

We have \(\Phi(\Theta_{Q,g}) = \Theta_{Q,g-1}\), we can thus define the stable Siegel modular form \(\Theta_Q := (\Theta_{Q,g})_{g\geq 1}\).

As explained in [Cor10], the ring \(\mathcal{R}(\mathcal{A}_\infty, L_\infty)\) is the polynomial ring in the \(\Theta_Q\) with \(Q\) irreducible, it carries a co-product (i.e. it is a Hopf algebra without the inverse), and the theta series are precisely the characters.

We can consider the jacobian morphism

\[j: \mathcal{M}_g \to \mathcal{A}_g,\]

the pull-back of \(L_g\) is the lambda class \(\lambda_g\) introduced in Section 3. Let \(\mathcal{J}_g\) be the pre image of \(\mathcal{M}_g\) in \(\mathcal{H}_g\), so the locus of matrices which are period matrices of a smooth curve.
5.2. Stable Teichmüller modular forms.

Definition 5.2 (Teichmüller modular forms). Teichmüller modular forms of weight \( k \) and degree \( g \geq 2 \) are the elements of \( H^0(\overline{M}_g, \lambda^\otimes k) \). For \( g = 1 \), they are elements of \( H^0(\overline{M}_{1,1}, \lambda^\otimes k) \).

The universal cover of \( \mathcal{M}_g \) is the Teichmüller space \( T_g \), and \( \mathcal{M}_g \) is the quotient stack of \( T_g \) by the mapping class group \( \Gamma_g \). The Jacobian morphism lift to a map

\[
j : T_g \rightarrow J_g.
\]

The group \( \Gamma_g \) has a morphism \( \chi : \Gamma_g \rightarrow Sp(2g, \mathbb{Z}) \) obtained taking the action of the diffeomorphism on the homology of the surface. Teichmüller modular forms can be identified with holomorphic functions \( f \) on \( T_g \) such that

\[
f(gp) = \rho(j(p), \chi(g))^k f(p)
\]

for all \( p \in T_g \), all \( g \in \Gamma_g \), and where \( \rho \) is the same cocycle which defines Siegel modular forms; see also [Ich94, Proposition 1.4].

Remark 5.3. It is shown in [OSS80], that \( j \) induces an isomorphism between the coarse moduli space \( M_g \) and the quotient variety \( J_g/Sp(2g, \mathbb{Z}) \). However, the quotient stack \( [J_g/Sp(2g, \mathbb{Z})] \) is not isomorphic to the fine moduli space \( \mathcal{M}_g \); the problem is due to the hyperelliptic involution. This implies that Teichmüller modular forms can not be identified with holomorphic function on \( J_g \) which transform appropriately under the action of \( Sp(2g, \mathbb{Z}) \). On the other hand, if we take convenient finite index subgroups \( G \subset Sp(2g, \mathbb{Z}) \), the quotient variety \( J_g/G \) is a fine moduli space for smooth curves with level structure, and forgetting the level structure one obtains a fine étale map \( J_g/G \rightarrow \mathcal{M}_g \).

This shows that Teichmüller modular forms are a subspace of the holomorphic function on \( J_g \) which transforms appropriately under the action of \( G \).

There is a restriction map

\[
\Phi : H^0(\overline{M}_g, \lambda^\otimes k) \rightarrow H^0(\overline{M}_{g-1}, \lambda^\otimes k)
\]

as explained for instance in [Ich94]. A way to define it, is to recall that the clutching morphism gives a surjective finite map \( c : \overline{M}_{g-1,2} \rightarrow \delta_g \subset \overline{M}_g \), and the pull-back of degree \( g \) Teichmüller modular forms via this map descends to degree \( g - 1 \) Teichmüller modular forms via the forgetful morphism \( f : \overline{M}_{g-1,2} \rightarrow \overline{M}_{g-1} \).

Definition 5.4 (Stable Teichmüller modular forms). A weight \( k \) stable Teichmüller modular forms is a collection \((s_g)_{g \geq 1}\) of weight \( k \) and degree \( g \) Teichmüller modular forms such that \( \Phi(s_g) = s_{g-1} \). Denote by \( H^0(\overline{M}_\infty, \lambda^\otimes k) \) the space of weight \( k \) stable Teichmüller modular forms, and by \( R(\overline{M}_\infty, \lambda_\infty) := \oplus_{g \geq 0} H^0(\overline{M}_\infty, \lambda_\infty^\otimes k) \) the ring of stable Teichmüller modular forms.

Stable Siegel modular forms restrict to stable Teichmüller modular forms. The ring of stable Teichmüller modular forms carries the following co-product. Given a stable Teichmüller modular form \( s \) of weight \( k \), its co-product is a collection of elements \( s_g \otimes s_h \) in \( H^0(\overline{M}_g, \lambda_g^\otimes k) \otimes H^0(\overline{M}_h, \lambda_h^\otimes k) \) for all possible values of \( g \) and \( h \); fix a pair \((g,h)\), restrict \( s_{g+h} \) to the boundary divisor \( \delta_g \), this divisor is isomorphic to \( \overline{M}_{g,1} \times \overline{M}_{h,1} \); the restriction of \( \lambda_{g+h} \) to \( \overline{M}_{g,1} \times \overline{M}_{h,1} \) equals \( \lambda_g \boxtimes \lambda_h \), hence we can descend the restriction of \( s_{g+h} \) to an element \( s_g \otimes s_h \).

Theorems 4.2 and 3.7 allow us to give the following key theorem.

Theorem 5.5. Given a holomorphic vertex algebra \( V \) of central charge \( c \), the partition function

\[
1(V) := \{ 1_{g(V)} \}_{g \geq 0}
\]

is a weight \( \frac{1}{2} c \) stable Teichmüller modular form; moreover it is a character for the co-product described above.
5.3. **Satake compactification.** Given a partition function $\chi_g(V)$ in $H^0(\overline{M}_g, g^k)$ it is natural to ask if it is the restriction of a Siegel modular form. In Section 5.3 we will see that this is always the case for lattice vertex algebra. In general, we think that this question should be related to the normality of the Satake compactification, as explained below.

The Satake compactification of $A_g$ of $A_g$ is a normal projective variety, which can be defined as the proj of the ring of modular form. The boundary $A_g \setminus A_g$ is isomorphic to $A_{g-1}^S$, and $L_g$ restricts to $L_{g-1}$. The jacobian map can be extended to a morphism

$$j: \overline{M}_g \rightarrow A_g^S$$

mapping a curve to the Jacobian of its normalization, and again $j^* L_g = \lambda_g$. We define the Satake compactification $M_g^S$ of $M_g$ to be the schematic image of $\overline{M}_g$ via $j$, or, equivalently, the schematic closure of $j(M_g)$ in $A_g^S$. This is a variety smooth in codimension one; we do not know if it is normal.

The line bundle $\lambda_g$ is semiample on $M_g$, and it gives a morphism $f$ towards a variety $\Sigma_g$; the variety $\Sigma_g$ is normal just because $\overline{M}_g$ is normal and $\lambda$ is semiample. The jacobian map factors as the composition of $f$ and a morphism

$$\nu: \Sigma_g \rightarrow M_g^S$$

This second map is a homeomorphism, see [ACG11] Pages 436-437 (in [ACG11], $\Sigma_g$ rather than $M_g^S$ is called the Satake compactification). Because of Zariski main theorem, $\nu$ has to be the normalization of $M_g^S$. In particular, $M_g^S$ is normal if and only if $\nu$ is an isomorphism, if and only if the map

$$j^*: H^0(A_g^S, L_g^k) \rightarrow H^0(\overline{M}_g, \lambda^k)$$

is surjective for $k$ big and divisible enough.

**Remark 5.6.** The morphism $j^*$ is not surjective for all $k$. For instance, the square root of the product of theta constant is not in its image, as shown in [Tsu91]. In [MV13, Section 3], it is shown that this modular form, when $g = 4$, cuts out the locus of curves in $M_4$ whose canonical model lies on a singular quadric. We could imagine that the cokernel of $j^*$ consists of modular forms which detect the projective geometry of the canonical curve.

5.4. **Theta series and lattice vertex algebras.** Let $Q$ be an even, unimodular, positive definite quadratic form of rank $c$, and $V(Q)$ the associated vertex algebra, as in Section 2.2. Let also

$$F_1(q) := \prod_{n=1}^{+\infty} (1 - q^n).$$

This function is equal, up to the factor $q^{c/24}$ to the Dedekin eta function, so it is convergent; it is equal to the inverse of the graded dimension of the Heisenberg vertex algebra (also known as free boson). The following formula is well-known, see e.g. [MT10, Section 7.3]

$$\sum_{n \geq 0} \dim V_n^{(Q)} q^n = F_1^{-c}(q) \Theta_{Q,1}(q)$$

In the recent paper [MP11, Section 6.5], the following formula has been proven

$$\sum_{k \in \mathbb{N}^c} \chi_k q^k = F_g^{-c}(q) \Theta_{Q,g},$$

(10)
where the left hand side is as in Equation 7, the function \( F_\lambda \) is a holomorphic function on the Schottky space independent of \( V \), and actually defined using the Heisenberg vertex algebra, and the identity is an identity of function on the Schottky space. (The theta series is defined on the Siegel space, but it is possible to pull-it back on the Schottky space.)

**Remark 5.7** (Eichler’s trick). There is another approach which generalize the so called Eichler’s trick. Let \( A \) be the universal ppav over the Siegel upper half space, and \( A_c \), the cartesian product of \( c \) copies of \( A \) over \( \mathcal{H}_g \). We can use \( Q \) to define a principal polarization on \( A_c \). The Eichler’s trick is to notice that the classical theta function of \( A_c \) evaluated at the identity is the theta series of \( Q \). This can be used to reduce the modularity of \( \Theta_{Q,g} \) to the modularity of the classical theta function. We can generalize the Eichler trick as follows. Let \( A \rightarrow \mathcal{M}_g \) be the \( d \) fold product of the universal jacobian. We can use \( Q \) to define a line bundle on \( A \), while gives a principal polarization on each fibre. As shown [Ga99] and [Le93], the space of covacua of \( V^{(Q)} \) is just the push-forward of this line bundle to \( \mathcal{M}_g \). Now, the claimed result follows from the standard Eichler’s trick and a careful analysis of the trivialization \( \lambda(\phi) \), as explained in Remark 5.12

5.5. Expansion of the partition functions on the Teichmüller and Schottky spaces. We are going to use the Schottky space introduced in Section 2.4.

**Lemma 5.8.** The section \( \lambda(\phi) \) of \( C = \text{Hom}(H, \mathcal{O}_{S_{g,1}}) \), introduced in Definition 5.16 on the formal completion of an irreducible component \( Z \) of the locus parametrizing curve whose normalization is a projective line, converges on all the Schottky space \( S_{g,1} \). Moreover, it descends along the projection \( S_{g,1} \rightarrow S_g \).

**Proof.** Consider the universal family \( \pi : X \rightarrow S_{g,1} \). Fix a point \( p \in S_{g,1} \). The key remark is that, as this is a family of curves with Schottky structure, we can always find an open subset \( B \) of \( \overline{S}_{g,1} \) containing \( p \) where the uniformization described in Notation 5.17 holds and we can apply Theorem 5.19.

The descends follows from the propagation of vacua 3.6

Let \( F_g \) be the holomorphic function on the Schottky space introduced in Section 5.1. Abusing notation, denote always by \( \lambda(\phi) \) the descended section from Lemma 5.8.

**Proposition 5.9.** Let \( \pi : T_g \rightarrow \mathcal{M}_g \) be the canonical projection. Then the image of \( \pi^* H^0(\overline{\mathcal{M}}_g, H) \) via \( F_g \lambda(\phi) \) in \( H^0(T_g, \mathcal{O}_{T_g}) \) is the space of weight \( c/2 \) Teichmüller modular forms.

**Proof.** As \( \overline{\mathcal{M}}_g \) is the stack theoretic quotient of \( T_g \) by \( \Gamma_g \), the line bundle \( \pi^* H \) carries a \( \Gamma_g \)-action, and the sections \( \pi^* H^0(\overline{\mathcal{M}}_g, H) \) are \( \Gamma_g \)-invariant. In particular, for every \( s \) in \( H^0(\overline{\mathcal{M}}_g, H) \), \( p \in T_g \) and \( g \in \Gamma_g \), we have

\[
\lambda(\phi)(\pi^* s)(g p) = \rho_V(g, p)\lambda(\phi)(\pi^* s)(p)
\]

for some cocycle \( \rho_V \) independent of \( s \).

For lattice vertex algebra, as \( \lambda(\phi)(1_V) \) is explicitely computed in Equation 10 the cocycle \( F_g^* \rho_{V^{(Q)}} \) is the cocycle \( \rho^{c/2} \) defining Teichmüller modular forms of weight \( \frac{c}{2} \).

For any other vertex algebra \( V \), fix \( V^{(Q)} \) a lattice vertex algebra of the same central charge. We have an isomorphism of the line bundles \( H_V \) and \( H_{V^{(Q)}} \) which is equivariant for the action of the extension of the Virasoro algebra, see Proposition 5.11. This isomorphism preserves the first order differential operator introduced in Section 5.3. By Theorem 5.19 the isomorphism also preserve \( \lambda(\phi) \). This means that the cocycle \( \rho \) is independent of the vertex algebra.

□
Corollary 5.10. Let $V$ be a holomorphic vertex algebra of central charge $c$. For any $g \geq 2$, the expression appearing in Equations (7) and (8) define a holomorphic function on the Schottky space $S_g$; its pull-back to the Teichmüller space $T_g$ multiplied by the function $F_g^c$ discussed above is a Teichmüller modular form of weight $\frac{1}{2}c$.

Let us also spell out the case $(g,n) = (1,1)$. Let $\Delta$ be the unit disc inside the complex plane, with coordinate $q$. Recall that $\Delta$ is the basis of a universal family of elliptic curves, and the associated map $\pi: \Delta \to \mathcal{M}_{1,1}$ is a Galois cover with group of deck transformation $SL(2,\mathbb{Z})$.

Corollary 5.11. Let $V$ be a holomorphic vertex algebra of central charge $c$. The formal power series

$$f_V(q) := \left(\prod_{n \geq 0} (1 - q^n)\right)^c \left(\sum_{n \geq 0} \dim(V_n)q^n\right)$$

converges on all $\Delta$ and it is a modular form of weight $\frac{1}{2}c$.

Remark 5.12. We conjecture that the function $F_g$ is equal to the function $F$ appearing in [MT06, Theorem 1], see also [ZS7, Zog89]. The idea is that if we take the Fuchsian uniformization of the moduli space of curves, the natural trivialization of the Hodge bundle gives Teichmüller modular form, if we take the Schottky uniformization the natural trivialization differs from the previous one by the factor $F_g$. The expansion of a Teichmüller modular forms in term of the Schottky co-ordinates is also studied in [Ich94], where Schottky co-ordinates are called Koebe coordinates.

Our $\lambda(\phi)$ should correspond to the Schottky trivialization of the Hodge line bundle. This is hinted also in [GKV10, Equation B.3].

6. Classification of vertex algebras, the slope of the effective cone of the moduli space of curves and the Schottky problem

6.1. The slope of the effective cone of the moduli space of curves. Let $\Pi$ be the plane in $\text{Pic}^0(M_g) \otimes \mathbb{Q}$ spanned by $\lambda$ and $\delta_0$. In this plane, we can look at the cone of effective line bundles. One ray of the cone is given by $\lambda$. To determine the other ray is one of the main open problem in the theory of the moduli space of curves. The slope of this cone, computed using as Cartesian co-ordinates on $\Pi$ the line bundles $\lambda$ and $-\delta_0$, is usually called the slope of $\mathcal{M}_g$, and we will denote it by $s_g$. More explicitly

$$s_g := \inf\left\{ \frac{a}{b} \text{ such that } a\lambda - b\delta_0 \text{ is effective} \right\}$$

The value of $s_g$ is known for low values of $g$: known lower and upper bounds for $s_g$ are asymptotically respectively of the form $O\left(\frac{1}{g}\right)$ and $6 + O\left(\frac{1}{g}\right)$. It is conjectured that the sequence $s_g$ is decreasing, or at least convergent, and the two main guesses for its limit are 0 and 6. Let us mention that sometime one consider the plane spanned by $\lambda$ and $\delta = \sum \delta_i$ rather than $\delta_0$, and the resulting slope is closely related to the one we are considering here. We refer to [FT05, Far09] for a comprehensive discussion and bibliography about this problem.

In this section, we denote by $1_g(V)$ the partition function associated to a holomorphic vertex algebra $V$ of central charge $c$. This can be seen both as an element of $H^0(\mathcal{M}_g, \lambda \otimes \mathcal{F})$ (cf. Theorems 4.3, 4.4), or as a holomorphic function on the Schottky space $g$ whose pull-back to the Teichmüller space transform appropriately under the action of the mapping class group (cf. Proposition 5.9 and Corollary 5.10). We will use the functions $\chi_k$ introduced in Equation (8); we will write $\chi_k(V)$ if we want to stress the vertex algebra used to define $\chi_k$. 

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Ideally, the two statements of the following theorem should be used in combination, to bootstrap information from low to high values of $g$.

**Theorem 6.1.** Fix a central charge $c$. Let $V_i$ be a finite collection of holomorphic vertex algebras of central charge $c$, and $a_i$ a collection of complex numbers; let $P_g := \sum a_i \chi_g(V_i)$ be the associated genus $g$ Teichmüller modular form.

Given a constant $b$, let $B_b := \{ k \in \mathbb{N}^g | k_i \leq b i = 1, \ldots, g \}$ be the box of size $b$ in $\mathbb{N}^g$.

If

1. there exist a $g_0$ such that $P_g$ vanishes identically on $\overline{M}_g$ for all $g < g_0$;
2. there exists a constant $b$ such that
   \[ \sum_i a_i \chi_k^{(g_0)}(V_i) = 0 \quad \forall k \in B_b; \]
3. there exist at least a value of $k$ such that
   \[ \sum_i a_i \chi_k^{(g_0)}(V_i) \neq 0; \]

then
\[ s_{g_0} \leq \frac{c}{2b}. \]

On the other hand, if

1. there exists a $g_0$ such that $P_g$ vanishes identically on $\overline{M}_g$ for all $g < g_0$;
2. there exists a constant $b$ such that
   \[ \sum_i a_i \chi_k^{(g_0)}(V_i) = 0 \quad \forall k \in B_b; \]
3. we know that
   \[ s_{g_0} > \frac{c}{2b} \]

then $P_{g_0}$ vanishes identically on $\overline{M}_{g_0}$.

**Proof.** Using the trivialization introduced in Proposition 5.9, we can identify $P_g$ with an holomorphic function on the Schottky space; with this trivialization, its expansion in Schottky coordinates is the one given in Equations 7 and 8; in particular the coefficient of $q_k$ is $\sum_i a_i \chi_k^{(g)}(V_i)$. The equation $q_1 \cdots q_g = 0$ defines the preimage of the boundary divisor $\delta_0$ inside the extended Schottky space. This shows that $P_g$ is a section of $\frac{c}{2}c\gamma - b\delta_0$ if and only if \( \sum_i a_i \chi_k^{(g_0)}(V_i) = 0 \) for all $k \in B_b$.

The statement now follows from the definition of the slope $s_g$. \qed

The following example should be thought as a baby case of the second part of Theorem 6.1.

**Example 6.2.** Let $V$ and $V'$ two vertex algebras of central charge $c = 2k$. The slope of the effective cone of $\overline{M}_{1,1}$ is well-known; more concretely, it is well-known that if a modular form of weight $k$ on $\overline{M}_{1,1}$ vanishes with order at least $\frac{k}{12} + 1$ at the boundary, then it is identically zero. Applying this to the difference of the partition functions of $V$ and $V'$ on $\overline{M}_{1,1}$, and using the expansion given in Corollary 5.11, we get that if
\[ \dim V_n = \dim V'_n \quad \forall n \leq \frac{k}{12} + 1 \]
then the two partition functions are equal, and we can deduce that
\[ \dim V_n = \dim V'_n \quad \forall n \geq 0. \]

6.2. Application to the Schottky problem. The Schottky problem asks for a characterization of \( \mathcal{M}_g \) inside \( \mathcal{A}_g \), a classical approach is to look for Siegel modular forms vanishing on \( \mathcal{A}_g \). We can prove the following negative result. Recall that \( E_8 \oplus E_8 \) and \( D_{16}^+ \) are the unique unimodular, positive definite, even lattices of rank 16.

**Corollary 6.3.** The modular form \( \Theta_{E_8 \oplus E_8, 5} - \Theta_{D_{16}^+, 5} \) is not identically zero on \( \mathcal{M}_5 \).

**Proof.** Let \( V \) be the lattice vertex algebra associated to \( E_8 \oplus E_8 \), and \( V' \) the one associated to \( D_{16}^+ \). In [GV09] Section 3.1 it is shown that some of the functions \( \chi^{(5)}_k (V) - \chi^{(5)}_k (V') \) is not zero for \( k = (1, 1, 1, 1, 1) \). (More specifically, using residues, they compute the coefficient of \( \prod_{i=1}^4 (zw_i)^{-1} \) in \( \chi_k \) as eigenvalue of a convenient Casimir operator.) The result follows from Equations 7, 8 and 10. \( \square \)

Let us recall that the above corollary was proven in a completely different way in [CSB14] and [GSM11]. The modular form under analysis is the so called Schottky form, it is well-known to vanish on \( \mathcal{A}_g \) for \( g \leq 3 \), and to be the equation of \( \mathcal{M}_4 \) in \( \mathcal{A}_4 \).

More generally, Theorem 6.1 can be applied to the Schottky problem in the following way. Take a non-zero Siegel modular form of degree \( g \) which can be written as linear combination of the theta series introduced in Section 5.4; then Equation 10 and Proposition 5.9 combined with the second part of Theorem 6.1 can be used to determine if it is zero on \( \mathcal{M}_g \).

6.3. Application to the classification of vertex algebra. First, we prove the following corollary, likely well-know to expert in vertex algebras.

**Corollary 6.4.** Let \( P \) and \( Q \) two different quadratic forms of the same rank. Then, the two associated vertex algebras are not isomorphic.

**Proof.** It is enough to show that the partition functions of the associated vertex algebras are not equal on \( \mathcal{M}_g \) for \( g \) big enough. These partition functions are the theta series associated the quadratic forms by Equations 7, 8 and 10. The requested result is proved in [CSB14]. \( \square \)

Using instead the slope of the effective cone of the moduli space of curves, we can prove the following result about the moonshine vertex algebra. Recall that the moonshine vertex algebra \( V^3 \) has central charge 24 and \( V^3_1 = 0 \), it is an open conjecture if it is the unique vertex algebra with these two properties.

**Corollary 6.5.** Let \( V \) be a holomorphic vertex algebra of central charge 24 with \( V_1 = 0 \). Then the partition function of \( V \) is equal to the partition function of the moonshine vertex algebra for \( g \leq 6 \).

If there exists an integer \( g_0 \) such that the slope \( s_g \) of the effective cone of moduli space of genus \( g \) curves is strictly greater than 6 for all \( g \leq g_0 \), then the partition function of \( V \) is equal to the partition function of the moonshine vertex algebra for all \( g \leq g_0 \).

**Proof.** It is well-known that \( s_g > 6 \) for \( g \leq 6 \), see e.g. [Far09] Theorem 4.9, so the second claim implies the first. The second claim is a combination of the second part of Theorem 6.1 and Lemma 8.10. \( \square \)
Another important question about the classification of vertex algebra is if, in analogy with the case of positive definite, unimodular even quadratic forms of a given rank, the number of holomorphic vertex algebras with a fixed central charge is finite. In the following result, by partition function we mean the stable Teichmüller modular forms $\{1_g(V)\}_{g \geq 1}$.

**Corollary 6.6.** Fix a central charge $c$, then there are countably many possibilities for the partition functions of holomorphic vertex algebras of central charge $c$.

If there exists a constant $C$ depending only on $c$ such that $h^0(\mathcal{M}_g, \lambda_g^{\otimes \frac{c}{2}}) \leq C$ for all $g$, then there are at most $C$ possibilities for the partition functions of holomorphic vertex algebras of central charge $c$.

**Proof.** The statement follows from Theorem 5.5 and the independence of characters [Cod16, Lemma 2.3].

### 6.4. Reconstruction of the vertex algebra from its partition function.

The following conjecture is very natural, especially from the point of view of physics.

**Conjecture 6.7.** Let $V$ and $V'$ be two vertex algebra of the same central charge. If their partition functions are equal in all genera, then they are isomorphic.

Let us make the following comments

1. Conjecture 6.7 is true if one restricts the attention to lattice vertex algebras, see Corollary 6.4.
2. Conjecture 6.7 is false if one restricts the partition functions to the moduli space of hyperelliptic curves, as in [Poo96] and [Cod16], it is shown that theta series associated to non-isomorphic quadratic forms can become equal in all genera after restriction to the hyperelliptic locus;
3. In [SB13] it is shown that on the locus of $n$-gonal curves, for $n \geq 3$, theta series are distinct if the genus is big enough. One could ask Conjecture 6.7 after restricting the partition functions to the locus of $n$-gonal curves for $n \geq 3$, or other geometrically meaningful loci.

We can propose the following effective version of Conjecture 6.7.

**Conjecture 6.8.** Fix a central charge $c$, then there exists an integer $g(c)$ such that two holomorphic vertex algebras of central charge $c$ are isomorphic if and only if they have the same partition function in genus $g(c)$.

Both conjectures seems out of reach at the moment. A proof of these conjectures would be useful for the classification of vertex algebras, for instance applied in combination with our results 6.5 and 6.6.

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