An improved bound on $\ell_q$ norms of noisy functions

Alex Samorodnitsky*

Abstract

Let $T_\epsilon$, $0 \leq \epsilon \leq 1/2$, be the noise operator acting on functions on the boolean cube $\{0,1\}^n$. Let $f$ be a nonnegative function on $\{0,1\}^n$ and let $q \geq 1$. In [4] the $\ell_q$ norm of $T_\epsilon f$ was upperbounded by the average $\ell_q$ norm of conditional expectations of $f$, given sets whose elements are chosen at random with probability $\lambda$, depending on $q$ and on $\epsilon$. In this note we prove this inequality for integer $q \geq 2$ with a better (smaller) parameter $\lambda$. The new inequality is tight for characteristic functions of subcubes.

As an application, following [5], we show that a Reed-Muller code $C$ of rate $R$ decodes errors on BSC($p$) with high probability if

$$R < 1 - \log_2 \left(1 + \sqrt{4p(1-p)}\right).$$

This is a (minor) improvement on the estimate in [5].

1 Introduction

We consider contractive properties of the noise operator acting on functions on the boolean cube $\{0,1\}^n$. This is an extensively investigated topic with numerous applications (see e.g., [3] for some background). One way to quantify the decrease in the $\ell_q$ norm of a function when this function is acted on by the noise operator was suggested in [4], where the $\ell_q$ norm of the ‘noisy version’ of $f$ was upperbounded by the average $\ell_q$ norm of conditional expectations of $f$, given sets whose elements are chosen at random with certain explicit probability $\lambda$, depending on $q$ and on $\epsilon$. Some applications of this inequality were described in [4, 5]. In this note we prove this inequality for integer $q \geq 2$ with a slightly better (smaller) parameter $\lambda$, which leads to corresponding improvement in the applications.

We introduce some relevant notions and notation. Given a noise parameter $0 \leq \epsilon \leq 1/2$, the noise operator $T_\epsilon$ acts on functions on the boolean cube as follows: for $f : \{0,1\}^n \to \mathbb{R}$, $T_\epsilon f$ at a point $x$ is the expected value of $f$ at $y$, where $y$ is a random binary vector whose $i^{th}$ coordinate is $x_i$ with probability $1-\epsilon$ and $1-x_i$ with probability $\epsilon$, independently for different coordinates. Namely, $(T_\epsilon f)(x) = \sum_{y \in \{0,1\}^n} \epsilon^{y-x}(1-\epsilon)^{n-|y-x|} f(y)$, where $|\cdot|$ denotes the Hamming distance. We will write $f_\epsilon$ for $T_\epsilon f$, for brevity.

*School of Engineering and Computer Science, The Hebrew University of Jerusalem, Jerusalem 91904, Israel. Research partially supported by ISF grant 1724/15.
For $0 \leq \lambda \leq 1$, let $T \sim \lambda$ denote a random subset $T$ of $[n]$ in which each element is chosen independently with probability $\lambda$. Let $E(f|T)$ be the conditional expectation of $f$ given $T$. This is a function on $\{0,1\}^n$ defined by $E(f|T)(x) = E_{y|T=x|T} f(y)$.

We prove the following claim.

**Theorem 1.1:** For any integer $q \geq 2$, and for any nonnegative function $f$ on $\{0,1\}^n$ holds

\[
\log \|T_\epsilon f\|_q \leq E_{T \sim \lambda} \log \|E(f|T)\|_q,
\]

with $\lambda = \lambda(q, \epsilon) = 1 + \frac{1}{q-1} \cdot \log_2 (\epsilon^q + (1-\epsilon)^q)$.

We also have

\[
\log \|T_\epsilon f\|_\infty \leq E_{T \sim \lambda} \log \|E(f|T)\|_\infty,
\]

with $\lambda = \lambda(\infty, \epsilon) = 1 + \log_2 (1-\epsilon)$.

These inequalities are tight if $f$ is a characteristic function of a subcube of $\{0,1\}^n$.

In [4] this inequality was proved for any real $q > 1$, but with a larger parameter $\lambda(q, \epsilon)$, given (for $q \geq 2$) by $(1 - 2^q (1+2q)) (2 \ln 2).$

**Remark 1.2:**

Both the arguments here and in [4] follow well-known proofs for the hypercontractive properties of the noise operator on the boolean cube. In [4] we followed the argument of [2], viewing both sides of the inequality as functions of $\epsilon$, and comparing the derivatives of these functions. In this note we follow the approach of [1], proving the inequality for the one-dimensional cube, and then extending it to any dimension, using the fact that the boolean cube is a product space. This allows for an improvement in the parameter. It should be mentioned that the one-dimensional claim turns out to be rather difficult, and we are only able to prove it for integer $q \geq 2$. On the other hand, all the applications which we mention here (and in [4, 5] as well) follow from the special case $q = 2$, which is much easier to prove (see Lemma 2.3).

Theorem 1.1 makes it possible to improve the parameters in the results in [4, 5] which use the inequality in [4]. We state some of these results, with the new parameters.

**Proposition 1.3:** Let $r_C(\cdot)$ be the rank function of the binary matroid on $\{1,\ldots,n\}$ defined by a generating matrix of a linear subspace $C$ of length $\{0,1\}^n$. Let $0 \leq p \leq 1$ and let $t = \log_2 (1+p)$. Then

\[
\log_2 E_{S \sim p} \left(2^{|S|} - r_C(S) \right) \leq E_{T \sim t} (|T| - r_C(T)).
\]

This inequality holds with equality if $C$ is a subcube.
Proposition 1.4:

Let $C$ be a doubly transitive binary linear code of rate $R$. Let $(a_0, \ldots, a_n)$ be the weight distribution of $C$. For $0 \leq i \leq n$, let $i^* = \min\{i, n-i\}$.

- For all $0 \leq i \leq n$ holds
  \[ a_i \leq 2^{o(n)} \cdot \left( \frac{1}{2^{1-R} - 1} \right)^{i^*}. \]

- For all $0 \leq i \leq n$ holds
  \[ a_i \leq 2^{o(n)} \cdot \left\{ \begin{array}{ll}
  \frac{|C|}{(2-2^R)(2^R)^{n-i}} & 0 \leq i^* \leq (1-2^{R-1}) \cdot n \\
  \frac{\binom{n}{i^*} \cdot |C|}{2^{n-i^*}} & \text{otherwise}
\end{array} \right. \]

Proposition 1.5:

Let $C$ be a binary Reed-Muller code of positive rate $0 < R < 1$. Then $C$ decodes errors on BSC$(p)$ with high probability (more precisely, a family of such codes $\{C_n\}_n$ with $\limsup_n R(C_n) \leq R$, attains vanishing error probability on BSC$(p)$ as $n \to \infty$) if

\[ R < 1 - \log_2 \left( 1 + \sqrt{4p(1-p)} \right). \]

This paper is organized as follows. We prove Theorem 1.1 in Section 2. Propositions 1.3-1.5 do not require new proofs since their claims are obtained by substituting the new value of $\lambda$ from Theorem 1.1 in the corresponding claims in [4, 5]. Note that Proposition 1.3 corresponds to Lemma 1.8 in [4], and Propositions 1.4-1.5 to Proposition 1.1 and Corollary 1.4 in [5]. The only new observation here is that Proposition 1.3 holds with equality for subcubes, and this follows immediately from the condition for equality in Theorem 1.1.

2 Proof of Theorem 1.1

We prove a more general claim. Consider a more general version of the noise operator. For a vector $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$, with $0 \leq \epsilon_1, \ldots, \epsilon_n \leq \frac{1}{2}$, the operator $T_{\vec{\epsilon}}$ acts on functions on the boolean cube as follows: for $f : \{0,1\}^n \to \mathbb{R}$, $T_{\vec{\epsilon}}f$ at a point $x$ is the expected value of $f$ at $y$, where $y$ is a random binary vector whose $i^{th}$ coordinate is $x_i$ with probability $1 - \epsilon_i$ and $1 - x_i$ with probability $\epsilon_i$, independently for different coordinates.

Theorem 2.1: For any integer $q \geq 2$, and for any nonnegative function $f$ on $\{0,1\}^n$ holds

\[ \log \|T_{\vec{\epsilon}}f\|_q \leq \mathbb{E}_{T \sim \vec{\lambda}} \log \| \mathbb{E}(f|T)\|_q, \]

with $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = \lambda_i(q, \epsilon) = 1 + \frac{1}{q-1} \cdot \log_2 (\epsilon^q_i + (1-\epsilon_i)^q)$.
We also have
\[ \log \|T_\varepsilon f\|_\infty \leq \mathbb{E}_{T \sim \lambda} \log \|\mathbb{E}(f|T)\|_\infty, \]
with \( \lambda_i = \lambda_i(\infty, \varepsilon) = 1 + \log_2 (1 - \varepsilon_i) \).

These inequalities are tight if \( f \) is a characteristic function of a subcube of \( \{0,1\}^n \).

We start with the one-dimensional case.

**Proposition 2.2:** Let \( q \geq 2 \) be an even integer. Let \( f \) be a nonnegative function on \( \{0,1\} \).
Then for any \( 0 \leq \varepsilon \leq \frac{1}{2} \) holds
\[ \|f_\varepsilon\|_q \leq \|f\|_1^{1-\lambda} \cdot \|f\|_q^\lambda, \]  
(1)

where \( \lambda = \lambda(q, \varepsilon) = 1 + \frac{1}{q-1} : \log_2 (\varepsilon^q + (1 - \varepsilon)^q) \).

We also have
\[ \|f_\varepsilon\|_\infty \leq \|f\|_1^{1-\lambda} \cdot \|f\|_\infty^\lambda, \]
with \( \lambda = \lambda(\infty, \varepsilon) = 1 + \log_2 (1 - \varepsilon) \).

These inequalities are tight if \( f \) is a characteristic function.

We will prove this claim in Section 2.1 below. For now we assume this claim to hold and proceed
with the proof of Theorem 2.1.

We introduce the following notation. Let \( T_{\varepsilon_i} \) denote the noise operator which applies noise \( \varepsilon_i \) on the \( i \)-th coordinate. Note that for \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) holds \( T_\varepsilon = \prod_{i=1}^n T_{\varepsilon_i} \). For a subset \( K \subseteq [n] \) of indices, let \( T_{\varepsilon,K} = \prod_{i \in K} T_{\varepsilon_i} \). Now, let \( S \) be the set of indices for which \( \varepsilon_i \neq 0 \). Observe that then \( T_\varepsilon = T_{\varepsilon,S} \).

Assume \( n \geq 2 \). The proof will be by induction on the cardinality of \( S \). Note that \( \varepsilon_i = 0 \) implies \( \lambda_i = 1 \). Hence for \( |S| = 0 \) the claim amounts to \( \|f\|_q \leq \|f\|_q \), which is trivial. Assume the claim holds for \( |S| = s - 1 \), and consider the case \( |S| = s \). Assume, w.l.o.g., that \( n \in S \). Then we have
\[ \|T_\varepsilon f\|_q = \|T_{\varepsilon,S} f\|_q = \left( \mathbb{E}_{x=(x_1 \ldots x_n)} (T_{\varepsilon,S} f(x))^q \right)^{\frac{1}{q}} = \left( \mathbb{E}_{x_1 \ldots x_{n-1}} \mathbb{E}_{x_n} (T_{\varepsilon_n} T_{\varepsilon,S \setminus n} f(x_1 \ldots x_n))^q \right)^{\frac{1}{q}}. \]

For each \( x_1, \ldots, x_{n-1} \), let \( g_{x_1 \ldots x_{n-1}} \) be the restriction of \( T_{\varepsilon,S \setminus n} f \) to the 1-dimensional cube \( (x_1, \ldots, x_{n-1}, 0), (x_1, \ldots, x_{n-1}, 1) \). Then the last expression is \( \left( \mathbb{E}_{x_1 \ldots x_{n-1}} \mathbb{E}_{x_n} (T_{\varepsilon_n} g_{x_1 \ldots x_{n-1}}(x_n))^q \right)^{\frac{1}{q}} \),
and we have
\[ \left( \mathbb{E}_{x_1 \ldots x_{n-1}} \mathbb{E}_{x_n} (T_{\varepsilon_n} g_{x_1 \ldots x_{n-1}}(x_n))^q \right)^{\frac{1}{q}} \leq \left( \mathbb{E}_{x_1 \ldots x_{n-1}} (\mathbb{E}_{x_n} g_{x_1 \ldots x_{n-1}}(x_n)^{(1-\lambda_n)q} \cdot (\mathbb{E}_{x_n} g_{x_1 \ldots x_{n-1}}(x_n)^q)^{\lambda_n})^{\frac{1}{q}} \right) \leq \]
\[
\left( \mathbb{E}_{x_1 \ldots x_{n-1}} \left( \mathbb{E}_{x_n} g_{x_1 \ldots x_{n-1}}(x_n)^q \right)^{\frac{1-\lambda_n}{q}} \cdot \left( \mathbb{E}_{x_1 \ldots x_n} g_{x_1 \ldots x_{n-1}}(x_n)^q \right)^{\frac{\lambda_n}{q}} \right),
\]

where in the first inequality we applied the 1-dimensional inequality, and in the second inequality we have used Hölder’s inequality.

Consider the two terms above. Recalling that noise operators commute with conditional expectations, and using the induction hypothesis, the first term is

\[
\left( \|T_{E_S \setminus T} f \|_q \right)^{\lambda_n} \leq \exp \left\{ (1 - \lambda_n) \cdot \mathbb{E}_{T \sim (\lambda_1 \ldots \lambda_{n-1}, 1)} \ln \| \mathbb{E}_{f \cap \{1 \ldots n-1\}} \|_q \right\},
\]

and the second term is

\[
\left( \|T_{E_S \setminus T} f \|_q \right)^{\lambda_n} \leq \exp \left\{ \lambda_n \cdot \mathbb{E}_{T \sim (\lambda_1 \ldots \lambda_{n-1}, 1)} \ln \| \mathbb{E}_{f \cap \{1 \ldots n-1\}} \|_q \right\}.
\]

Combining both terms, we get

\[
\|T_{E_S \setminus T} f \|_q = \|T_{E_S} f \|_q \leq \exp \left\{ (1 - \lambda_n) \cdot \mathbb{E}_{T \sim (\lambda_1 \ldots \lambda_{n-1}, 1)} \ln \| \mathbb{E}_{f \cap \{1 \ldots n-1\}} \|_q \right\} + \lambda_n \cdot \mathbb{E}_{T \sim (\lambda_1 \ldots \lambda_{n-1}, 1)} \ln \| \mathbb{E}_{f \cap \{1 \ldots n-1\}} \|_q \}
= \exp \left\{ \mathbb{E}_{T \sim (\lambda_1 \ldots \lambda_{n-1}, 1)} \ln \| \mathbb{E}_{f \cap \{1 \ldots n-1\}} \|_q \right\}.
\]

It remains to show that the inequality in the theorem is tight for characteristic functions of subcubes. Let \( f \) be such a function. We may assume, by homogeneity, that the expectation of \( f \) is 1. Note that \( f \) is a product function, that is \( f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i) \), where each \( f_i \) is a function on \( \{0, 1\} \) which is either twice the characteristic function of 0 or the constant-1 function. In particular, by Proposition 2.2, the 1-dimensional inequality is tight for each \( f_i \).

Hence on one hand we have

\[
\|T_{E_S \setminus T} f \|_q = \prod_{i=1}^{n} \|T_{E_S} f_i \|_q = \prod_{i=1}^{n} \|T_{E_S} f_i \|_q = \prod_{i=1}^{n} \|f_i \|_{1 - \lambda_i} \cdot \|f_i \|_{\lambda_i} = \prod_{i=1}^{n} \|f_i \|_{\lambda_i},
\]

where in the last step we have used the fact that the expectation of each \( f_i \) is 1. On the other hand, note that for \( T \subseteq [n] \) we have \( E(f|T) = \prod_{i \in T} f_i \). Hence

\[
\exp \left\{ \mathbb{E}_{T \sim (\lambda_1 \ldots \lambda_{n-1}, 1)} \ln \| \mathbb{E}_{f \cap \{1 \ldots n-1\}} \|_q \right\} = \prod_{T \subseteq [n]} \| \mathbb{E}(f|T) \|_q E_{T \sim (\lambda_1 \ldots \lambda_{n-1}, 1)} \]
2.1 Proof of Proposition 2.2

First note that it suffices to prove the claim for finite values of \( q \), since the claim for \( \| \cdot \|_\infty \) follows by taking \( q \) to infinity. Note also that the inequalities in the proposition are easily seen to be tight for characteristic functions. In fact, they are trivially true for the constant function, and are easy to verify for a characteristic function of a point.

Fix \( q \) and \( \epsilon \), which fixes the value of \( \lambda = 1 + \frac{1}{q-1} \cdot \log_2 \left( \epsilon^q + (1-\epsilon)^q \right) \). We may assume, by homogeneity, that \( \|f\|_1 = 1 \). Under this assumption, we need to show that

\[
\|f\|_q^\lambda \leq \|f\|_q
\]

which is equivalent to

\[
\ln \|f\|_q^\lambda \leq \ln \|f\|_q
\]

Note that under the assumption \( \|f\|_1 = 1 \), the function \( f \) is determined by its value in 0, which we denote by \( 1-x \), \( 0 \leq x \leq 1 \). Hence, for fixed \( q \) and \( \epsilon \), the ratio

\[
\ln \|f\|_q^\lambda \ln \|f\|_q
\]

is a univariate function of \( x \). It is easy to see that this function equals \( \lambda \) at \( x = 0 \), and we will claim that it indeed attains its maximum in \( x = 0 \).

It is convenient to introduce the following notation. Let

\[
F(y) = F_q(y) := \ln \left( \frac{(1-y)^q+(1+y)^q}{2} \right).
\]

It is easy to see that the function \( F \) is increasing for \( 0 \leq y \leq 1 \). Observe that for \( f \) given by values \( 1-x \) at 0 and \( 1+x \) at 1, we have \( \ln \|f\|_q = F(x) \), and \( \ln \|f\|_q = F((1-2\epsilon) \cdot x) \). So we want to show that for all \( 0 \leq \epsilon \leq \frac{1}{2} \) and for all \( 0 \leq x \leq 1 \) holds

\[
\frac{F((1-2\epsilon) \cdot x)}{F(x)} \leq \frac{F((1-2\epsilon))}{F(1)}.
\]

Let \( G(z) = \ln F(e^z) \). Then \( G \) is a function on \((-\infty, 0]\), and the inequality above is equivalent to

\[
G(0) + G(\ln(1-2\epsilon) + \ln(x)) \leq G(\ln(1-2\epsilon)) + G(\ln(x))
\]

which will follow if we show that \( G \) is concave. From now on we focus on proving the concavity of \( G \). We will show that \( G'' \leq 0 \). First, we deal separately with the simple special case \( q = 2 \), since this suffices for applications.

**Lemma 2.3**: The function \( G \) is concave if \( q = 2 \).

**Proof**: In this case, \( F(y) = \ln \left( 1 + y^2 \right) \), and hence \( G(z) = \ln \ln \left( 1 + e^{2z} \right) \). It is easy to see that, up to a positive factor, \( G''(z) \) is given by \( \ln \left( 1 + e^{2z} \right) - e^{2z} \), which is negative for all \( z \). □

We continue with the general case. It is easy to see that \( G'' \leq 0 \) is equivalent to

\[
(yF'' + F') \cdot F - y \left( F' \right)^2 \leq 0.
\]

Writing \( H = e^F = \frac{(1-y)^q+(1+y)^q}{2} \), the above is equivalent to, after some rearranging,

\[
\ln(H) \leq \frac{y (H')^2}{yH''H - y (H')^2 + H' H}.
\]
After some (tedious) simplification, we get that
\[ yH''H - y(H')^2 + H' = q(q - 1)y (1 - y^2)^{q-2} + \frac{q}{4} \cdot ((1 + y)^{2q-2} - (1 - y)^{2q-2}), \]
and hence that
\[ \frac{y (H')^2}{yH''H - y(H')^2 + H'} = \frac{qy ((1 + y)^{q-1} - (1 - y)^{q-1})^2}{4(q - 1)y (1 - y^2)^{q-2} + (1 + y)^{2q-2} - (1 - y)^{2q-2}}. \]

So, we need to prove that
\[ \ln(H) \leq \frac{qy ((1 + y)^{q-1} - (1 - y)^{q-1})^2}{4(q - 1)y (1 - y^2)^{q-2} + (1 + y)^{2q-2} - (1 - y)^{2q-2}}. \]

Since both sides vanish at 0, it suffices to prove the inequality for the derivatives, that is, show that
\[ \frac{H'}{H} \leq \frac{d}{dy} \frac{qy ((1 + y)^{q-1} - (1 - y)^{q-1})^2}{4(q - 1)y (1 - y^2)^{q-2} + (1 + y)^{2q-2} - (1 - y)^{2q-2}}. \]

Let \( A(y) = A_q(y) = (1 + y)^{q-1} \), and similarly, \( B(y) = B_q(y) = (1 - y)^{q-1} \). Then, after some simplification, the RHS of the inequality above becomes
\[ \frac{q (A^2 - B^2) \cdot \left( (A - B)^2 + 8(q - 1)^2y^2 (1 - y^2)^{q-3} \right) + q(A - B)^2 \cdot 4(q - 1) (y - 3y^3) (1 - y^2)^{q-3}}{(4(q - 1)y (1 - y^2)^{q-2} + A^2 - B^2)^2}. \]

So, we need to verify
\[ \left( 4(q - 1)y (1 - y^2)^{q-2} + A^2 - B^2 \right)^2 \leq \frac{H}{H'} \cdot q (A^2 - B^2) \cdot \left( (A - B)^2 + 8(q - 1)^2y^2 (1 - y^2)^{q-3} \right) + q(A - B)^2 \cdot 4(q - 1) (y - 3y^3) (1 - y^2)^{q-3}. \]

Opening up and simplifying, the RHS is
\[ (A + B)^2 \cdot \left( (A - B)^2 + 8(q - 1)^2y^2 (1 - y^2)^{q-3} \right) + \]
\[ (A^2 - B^2) \cdot 4(q - 1) (y - 3y^3) (1 - y^2)^{q-3} + \]
\[ y (A^2 - B^2) \cdot \left( (A - B)^2 + 8(q - 1)^2y^2 (1 - y^2)^{q-3} \right) + \]
\[ y(A - B)^2 \cdot 4(q - 1) (y - 3y^3) (1 - y^2)^{q-3}. \]

After some simplification, the inequality becomes
\[ 4(q - 1)y (1 - y^2)^{q-3} \cdot \left( 2 (A^2 - B^2) (1 - y^2) + 4(q - 1)y (1 - y^2)^{q-1} \right) \leq \]
\[4(q-1)y(1-y^2)^q - 3(2(A + B)^2(q - 1)y + (A^2 - B^2)(1 - 3y^2) + 2(A^2 - B^2)(q - 1)y^2 + (A - B)^2(y - 3y^3)) + y(A^2 - B^2)(A - B)^2.\]

Clearly, it suffices to prove that
\[2(A^2 - B^2)(1 - y^2) + 4(q - 1)y(1 - y^2)^q \leq 2(A + B)^2(q - 1)y + (A^2 - B^2)(1 - 3y^2) + 2(A^2 - B^2)(q - 1)y^2 + (A - B)^2(y - 3y^3).\]

After rearranging, this is the same as
\[(2q - 4)y((1 + y)^{2q - 1} + (1 - y)^{2q - 1}) \geq \]
\[(1 - y^2)((1 - 3y)(1 + y)^{2q - 2} - (1 + 3y)(1 - y)^{2q - 2}) + 2y(1 - 3y^2)(1 - y^2)^{q - 1}.\]

Next, we change variables. Let \(t = \frac{1+y}{1-y}\). Then \(t \geq 1\). Substituting, dividing both sides of the above inequality by \((1 - y^2)^{q - 1}\), and multiplying by \((t + 1)^3\), the LHS becomes \((4q - 8)(t^2 - 1)(t^q + t^{-q+1})\), and the RHS becomes \(4((4 - 2t) \cdot t^q - (4t - 2) \cdot t^{-q+2} - (t - 1)(t^2 - 4t + 1))\).

Hence the inequality becomes
\[(q - 2)(t^2 - 1)(t^q + t^{-q+1}) \geq (4 - 2t) \cdot t^q - (4t - 2) \cdot t^{-q+2} - (t - 1)(t^2 - 4t + 1),\]
or, after multiplying by \(t^{q-1}\),
\[(q - 2)(t^2 - 1)(t^{q-1} + 1) \geq (4 - 2t) \cdot t^{2q-1} - (4t - 2) \cdot t - (t - 1)(t^2 - 4t + 1)t^{q-1}.\]

Let \(x = t - 1\), then \(x \geq 0\). Writing the above inequality in terms of \(x\), we get
\[(q-2)x(x+2)((1 + x)^{2q - 1} + 1) - (2-2x)(1+x)^{2q - 1} + (4x+2)(1+x) + x(x^2 - 2x - 2)(1+x)^{q-1} \geq 0.\]

From now on we use the assumption that \(q\) is an integer. If this is the case, the LHS is a polynomial of degree \(2q - 1\). We will show that all the coefficients of this polynomial are nonnegative, which will imply its nonnegativity for \(x \geq 0\).

Considering the relevant terms, we have that
\[
\text{coef}_{x^0} = -2 + 2 = 0, \\
\text{coef}_{x^1} = 4(q - 2) - 2(2q - 1) + 2 + 6 - 2 = 0, \\
\text{coef}_{x^2} = (q - 2)(2 + 2(2q - 1)) - 2\left(\frac{2q - 1}{2}\right) + 2(2q - 1) + 4 - 2 - 2(q - 1) = 0.
\]

For \(k \geq 3\) we have
\[
\text{coef}_{x^k} = -2\left(\frac{2q - 1}{k}\right) + 2q - 2\left(\frac{2q - 1}{k - 1}\right) + (q - 2)\left(\frac{2q - 1}{k - 2}\right) - 2\left(\frac{q - 1}{k - 1}\right) - 2\left(\frac{q - 1}{k - 2}\right) + \left(\frac{q - 1}{k - 3}\right) =
\]
\[
\frac{(2q-1)(2q-2)\cdots(2q-k+2)}{k!} \cdot (- (q + 2) \cdot k^2 + (4q^2 + 5q + 2) \cdot k - 4q(2q + 1)) + \\
\frac{(q-1)(q-2)\cdots(q-k+3)}{(k-1)!} \cdot (k^2 + (2q-3) \cdot k - (2q^2 + 4q - 2)).
\]

We claim that
\[
\text{coef}_{x^k} \geq 0 \quad \text{for} \quad 3 \leq k \leq 2q + 1.
\]

We consider two cases: \(3 \leq k \leq q + 2\) and \(q + 3 \leq k \leq 2q + 1\).

1. \(3 \leq k \leq q + 2\).
   
   Clearly \((2q-1)(2q-2)\cdots(2q-k+3) \geq (q-1)(q-2)\cdots(q-k+3)\), so it suffices to show
   
   \[(2q-k+2)\cdot(- (q + 2) \cdot k^2 + (4q^2 + 5q + 2) \cdot k - 4q(2q + 1)) \geq -k \cdot (k^2 + (2q-3) \cdot k - (4q^2 + 2q - 2)).\]
   
   and
   
   \[-(q + 2) \cdot k^2 + (4q^2 + 5q + 2) \cdot k - 4q(2q + 1) \geq 0.
   
   We start with the first of these inequalities. The difference between the LHS and the RHS is
   
   \[(q + 3) \cdot k^3 - 3 \cdot (2q^2 + 3q + 3) \cdot k^2 + 2 \cdot (4q^3 + 12q^2 + 7q + 3) \cdot k - 8q(q + 1)(2q + 1).
   
   We view this as a cubic \(P(k)\) in \(k\), and want to show that this cubic is nonnegative on \([3, q + 2]\). First, we check the endpoints of the interval. We have that \(P(3) = 2 \cdot (4q^3 - 3q^2 - 10q + 9)\). For \(q \geq 2\) this is at least \(2 \cdot (5q^2 - 10q + 9) \geq 18\). On the other end, we have that \(P(q + 2) = q \cdot (3q^3 - q - 2) > 0\).

Next, we claim that \(P\) either always increases on the interval, or first increases and the decreases. Since we have checked both endpoints, this will complete the proof. We have that \(P'(k) = 3(q + 3) \cdot k^2 - 6 \cdot (2q^2 + 3q + 3) \cdot k + 2 \cdot (4q^3 + 12q^2 + 7q + 3)\). It suffices to check that \(P'(3) > 0\) and that either \(P'\) is nonnegative throughout, or that the second root of \(P'\) is greater than \(q + 2\) (which means that \(P'\) is first positive and then negative).

In fact, \(P'(3) = 8q^2 - 12q^2 - 13q + 33\). For \(q \geq 2\) this is at least \(4q^2 - 13q + 33 > 0\). Next, the discriminant of the quadratic \(P'\) is

\[D(q) = 36 \cdot (2q^2 + 3q + 3)^2 - 24 \cdot (q + 3) \cdot (4q^3 + 12q^2 + 7q + 3).\]

There are two cases. First, \(D(q) < 0\), in which case \(P'\) is always positive. It is not hard to check that this is the case for \(q = 2, 3, 4\). The other case is \(D(q) \geq 0\), in which case the second root of \(P'\) is given by \(\frac{6(2q^2 + 3q + 3) + \sqrt{D(q)}}{6(q + 3)}\). We claim that this is larger than \(q + 2\).

In fact, we claim that \(\frac{2q^2 + 3q + 3}{q+3} > q + 2\), which is easily seen to be true for \(q \geq 3\).
Next, we verify that

\[(q + 2) \cdot k^2 - (4q^2 + 5q + 2) \cdot k + 4q(2q + 1) \leq 0\]

Let \(Q(k) = (q + 2)k^2 - (4q^2 + 5q + 2) \cdot k + 4q(2q + 1)\). We need to show that \(Q \leq 0\) for \(3 \leq k \leq q + 2\). With forethought we show \(Q\) to be nonpositive on a larger interval, that is for \(3 \leq k \leq 2q + 1\).

**Lemma 2.4:**

\[Q \leq 0, \quad \text{for} \quad 3 \leq k \leq 2q + 1.\]

**Proof:** We investigate \(Q\) as a quadratic in \(k\). Let \(A(q) = q + 2\), \(B(q) = 4q^2 + 5q + 2\), \(C(q) = 4q(2q + 1)\), and \(D(q) = B^2 - 4AC\). Simplifying, we get that \(D(q) = 16q^4 + 8q^3 - 39q^2 - 12q + 4\), which is easily seen to be positive for \(q \geq 2\), since in this case \(16q^4 + 8q^3 \geq 80q^2\). The roots of \(Q\) are \(k_{1,2} = \frac{B \pm \sqrt{D}}{2A}\). We claim that \(k_1 \leq 3\) and that \(k_2 \geq 2q + 1\), which will prove what we need.

We start with \(\frac{B - \sqrt{D}}{2A} \leq 3\), which is the same as \(B - 6A \leq \sqrt{D}\). It suffices to verify \(B^2 - 4AC \geq (B - 6A)^2\), which is equivalent to \(9A + C \leq 3B\). This is the same as \(4q^2 + 2q \geq 12\), which is clearly true for \(q \geq 2\).

We proceed with \(\frac{B + \sqrt{D}}{2A} \geq 2q + 1\), which is the same as \(\sqrt{D} \geq 2(2q + 1)A - B\). It suffices to verify \(B^2 - 4AC \geq (2(2q + 1)A - B)^2\), which is equivalent to \((2q + 1)^2A + C \leq (2q + 1)B\). This is the same as \(2q^2 \geq 4q\), which is clearly true for \(q \geq 2\). This concludes the case \(3 \leq k \leq q + 2\).

2. \(q + 3 \leq k \leq 2q + 1\).

In this case, recalling the assumption that \(q\) is integer, we have that \(2\binom{q-1}{k-1} + 2\binom{q-1}{k-2} - \binom{q-1}{k-3} = 0\) and hence, using Lemma 2.4 in the last step, we have

\[
\operatorname{coef}_{x^k} = -2 \binom{2q-1}{k} + (2q-2) \binom{2q-1}{k-1} + (q-2) \binom{2q-1}{k-2} = \\
\frac{(2q-1)(2q-2) \cdots (2q-k+2)}{k!} \cdot (-Q(k)) \geq 0.
\]

Acknowledgments

We would like to thank Ori Sberlo for a very helpful discussion.
References

[1] W. Beckner, *Inequalities in Fourier Analysis*, Annals of Math., 102(1975), pp. 159-182.

[2] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. of Math., 97, 1975, pp. 1061-1083.

[3] R. O’Donnell, *Analysis of Boolean functions*, Cambridge University Press, 2014.

[4] A. Samorodnitsky, *An upper bound on ℓ_q norms of noisy functions*, IEEE Transactions on Information Theory, 66(2) 742-748, (2020).

[5] A. Samorodnitsky and O. Sberlo, *On codes decoding a constant fraction of errors on the BSC*, arXiv:2008.07236, 2020.