Lower Bound of Sectional Curvature of Fisher–Rao Manifold of Beta Distributions and Complete Monotonicity of Functions Involving Polygamma Functions

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Abstract. In the paper, by virtue of convolution theorem for the Laplace transforms and analytic techniques, the author finds necessary and sufficient conditions for complete monotonicity, monotonicity, and inequalities of several functions involving polygamma functions. By these results, the author derives a lower bound of a function related to the sectional curvature of the Fisher–Rao manifold of beta distributions.

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1. Motivations

In the literature [1, Section 6.4], the function

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0 \]

and its logarithmic derivative \( \psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)} \) are respectively called Euler’s gamma function and digamma function. Further, the functions \( \psi'(z) \),
\( \psi''(z), \psi'''(z), \) and \( \psi^{(k)}(z) \) are known as trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives \( \psi^{(k)}(z) \) for \( k \in \{0\} \cup \mathbb{N} \) are known as polygamma functions.

Recall from Chapter XIII in [7], Chapter 1 in [22], and Chapter IV in [23] that, if a function \( f(x) \) on an interval \( I \) has derivatives of all orders on \( I \) and satisfies \((-1)^n f^{(n)}(x) \geq 0 \) for \( x \in I \) and \( n \in \{0\} \cup \mathbb{N} \), where \( \mathbb{N} \) denotes the set of all positive integers, then we call \( f(x) \) a completely monotonic function on \( I \). Theorem 12b in [23, p. 161] characterized that a function \( f(x) \) is completely monotonic on \((0, \infty)\) if and only if

\[
f(x) = \int_{0}^{\infty} e^{-xt} d\sigma(t), \quad x \in (0, \infty),
\]

where \( \sigma(s) \) is non-decreasing and the integral in (1.1) converges for \( x \in (0, \infty) \). The integral representation (1.1) means that a function \( f(x) \) is completely monotonic on \((0, \infty)\) if and only if it is a Laplace transform of a non-decreasing measure \( \sigma(s) \) on \((0, \infty)\).

The Fisher–Rao manifold of beta distributions refers to the space of parameters of the family of beta distributions, equipped with the Fisher–Rao metric. It is of interest in information geometry, an expanding field that studies statistical objects from a geometrical point of view.

In [2, Proposition 3], [3, Proposition 13], and [4, Proposition 14], the sectional curvature \( K(x, y) \) of the Fisher–Rao manifold of beta distributions was given by

\[
K(x, y) = \frac{1}{4} \frac{\psi''(x)\psi''(y)\psi''(x + y)}{[\psi'(x)\psi'(x + y) + \psi'(y)\psi'(x + y) - \psi'(x)\psi'(y)]^2}.
\]

In [2, Proposition 4], [3, Proposition 14], and [4, Proposition 15], the limits

\[
\lim_{y \to 0^+} K(x, y) = \lim_{y \to 0^+} K(y, x) = \frac{3}{4} - \frac{1}{2} \frac{\psi''(x)}{[\psi''(x)]^2},
\]

\[
\lim_{y \to \infty} K(x, y) = \lim_{y \to \infty} K(y, x) = \frac{1}{4} \frac{\psi'(x) + x\psi''(x)}{[x\psi'(x) - 1]^2},
\]

and

\[
\lim_{(x, y) \to (0^+, \infty)} K(x, y) = \lim_{(x, y) \to (\infty, 0^+)} K(x, y) = -\frac{1}{4},
\]

were computed. In [2, Proposition 5] and [4, Theorem 6], the sectional curvature \( K(x, y) \) was proved to be negative and bounded from below.

**Conjecture 1.1** ([3, pp. 12–13] and [4, p. 14]). For \( x, y > 0 \), the sectional curvature \( K(x, y) \)

1. has a lower bound \(-\frac{1}{2}\), accurately, \( K(x, y) > -\frac{1}{2} \);
2. is decreasing in both \( x \) and \( y \).
In this paper, we consider the function
\[
K(x) = K(x, x) = \frac{1}{4} \frac{\psi''(x) 2\psi'(2x) - \psi'(2x)\psi''(x)}{[\psi'(x) - 2\psi'(2x)]^2}
\] (1.3)
on $(0, \infty)$ and prove the sharp double inequality
\[
0 > K(x) > -\frac{1}{2}
\] (1.4)
which verifies the first conjecture in Conjecture 1.1 along the half-line $x = y > 0$ in the first quadrant on $\mathbb{R}^2$.

2. Lemmas

The following lemmas are necessary in this paper.

Lemma 2.1. For $k \in \mathbb{N}$, we have the limits
\[
\lim_{x \to 0^+} \left[ x^k \psi^{(k-1)}(x) \right] = (-1)^k (k - 1)!
\] (2.1)
and
\[
\lim_{x \to \infty} \left[ x^k \psi^{(k)}(x) \right] = (-1)^{k-1} (k - 1)!. 
\] (2.2)
Proof. These two limits can be found in [6, p. 9896, (13)], [9, p. 260, (2.2)], [18, p. 1689, (3.3)], [19, p. 286, (2.6)], [20, p. 81, (41)], and [24, p. 769], for example.

Lemma 2.2 (Convolution theorem for the Laplace transforms [23, pp. 91–92]). Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \geq 0$ such that $|f_k(t)| \leq M_k e^{c_k t}$ for $k = 1, 2$, then
\[
\int_0^\infty \left[ \int_0^t f_1(u) f_2(t - u) \, du \right] e^{-st} \, dt = \int_0^\infty f_1(u) e^{-su} \, du \int_0^\infty f_2(v) e^{-sv} \, dv.
\]

Lemma 2.3. For $t > u > 0$, the function
\[
W_t(u) = \frac{[e^{-(t-u)/2} - e^{-u/2}] (t - u)}{[1 - e^{-u/2}] [1 - e^{-(t-u)/2}]}
\]
is increasing in $u \in (0, t)$, with limits
\[
\lim_{u \to t^-} W_t(u) = 2 \quad \text{and} \quad \lim_{u \to 0^+} W_t(u) = -\infty.
\]
Proof. It is easy to see that
\[
\lim_{u \to t^-} W_t(u) = \frac{\lim_{u \to t^-} [e^{-(t-u)/2} - e^{-u/2}]}{1 - e^{-u/2}} \quad \lim_{u \to t^-} \frac{t - u}{1 - e^{-(t-u)/2}} = 2.
\]
Direct differentiation yields

\[
\frac{d}{du} W(2t, 2u) = \frac{2}{(e^u - 1)^2 (e^t - e^u)^2} \left[ e^{4u} + (e^t + 1)(t - u - 1) e^{3u} - 4(t - u) e^{t+2u} \\
+ (t - u + 1) e^{t+u} + (t - u + 1) e^{2t+u} - e^{2t} \right]
\]

\[\triangleq \frac{2W_1(t, u)}{(e^u - 1)^2 (e^t - e^u)^2},\]

\[\lim_{u \to t^-} W_1(t, u) = 0,\]

\[
\frac{d}{du} W_1(t, u) = e^u \left[ 4 e^{3u} + (3t - 3u - 4) e^{2u} (1 + e^t) \\
- 4(2t - 2u - 1) e^{t+u} + (t - u)(1 + e^t) e^t \right]
\]

\[\triangleq e^u W_2(t, u)\]

\[\to 0, \quad u \to t^- ,\]

\[
\frac{d}{du} W_2(t, u) = 12 e^{3u} + (6t - 6u - 11)(1 + e^t) e^{2u} \\
- 4(2t - 2u - 3) e^{t+u} - e^t(1 + e^t)
\]

\[\to (e^{2t} - 1) e^t\]

\[> 0,\]

\[
\frac{d^2}{du^2} W_2(t, u) = 4 e^u \left[ 9 e^{2u} + (3t - 3u - 7) (1 + e^t) e^u - e^t(2t - 2u - 5) \right]
\]

\[\triangleq 4 e^u W_3(t, u)\]

\[\to 8 e^{2t} (e^t - 1), \quad u \to t^-\]

\[> 0,\]

\[
\frac{d}{du} W_3(t, u) = 18 e^{2u} + (3t - 3u - 10)(1 + e^t) e^u + 2 e^t
\]

\[\to 8 e^t (e^t - 1), u \to t^-\]

\[> 0,\]

\[
\frac{d^2}{du^2} W_3(t, u) = e^u \left[ 36 e^u + (3t - 3u - 13) e^t + 3t - 3u - 13 \right]
\]

\[\triangleq e^u W_4(t, u)\]

\[\to e^t (23 e^t - 13), \quad u \to t^-\]

\[> 0,\]

\[
\frac{d}{du} W_4(t, u) = 3(12 e^u - e^t - 1)
\]

\[\to 3(11 e^t - 1), \quad u \to t^-\]

\[> 0,\]
Lemma 2.4. For \( k \in \{0\} \cup \mathbb{N} \) and \( a \geq 0 \), we have

\[
\lim_{x \to -\infty} \left( x^{k+1} \left[ \psi^{(k)}(x + a) - \psi^{(k)}(x) \right] \right) = (-1)^k k! a. \tag{2.3}
\]

For \( k, \ell \in \mathbb{N} \) and \( a \geq 0 \), we have

\[
\lim_{x \to -\infty} \left( x^{k+\ell+1} \left[ \psi^{(k)}(x) \psi^{(\ell+1)}(x) - \psi^{(k+1)}(x) \psi^{(\ell)}(x) \right] \right) = (-1)^{k+\ell} (k-1)! (\ell-1)! (k-\ell)
\]

and

\[
\lim_{x \to -\infty} \left( x^{k+\ell+1} \left[ \psi^{(k)}(x) \psi^{(\ell)}(x + a) - \psi^{(\ell)}(x) \psi^{(k)}(x + a) \right] \right) = (-1)^{k+\ell} (k-1)! (\ell-1)! (k-\ell) a. \tag{2.5}
\]

Proof. It is straightforward that

\[
\lim_{x \to -\infty} \left( x^{k+1} \left[ \psi^{(k)}(x + a) - \psi^{(k)}(x) \right] \right) = \lim_{x \to -\infty} \left[ x^{k+1} \int_{x}^{x+a} \psi^{(k+1)}(u) \, du \right] = \lim_{x \to -\infty} \left[ x^{k+1} \int_{0}^{a} \psi^{(k+1)}(x + u) \, du \right] = \int_{0}^{a} \lim_{x \to -\infty} \left[ x^{k+1} \psi^{(k+1)}(x + u) \right] \, du = \int_{0}^{a} (-1)^k k! \, du = (-1)^k k! a,
\]

where we used the limit (2.2).

It is also straightforward that

\[
\lim_{x \to -\infty} \left( x^{k+\ell+1} \left[ \psi^{(k)}(x) \psi^{(\ell+1)}(x) - \psi^{(k+1)}(x) \psi^{(\ell)}(x) \right] \right) = \lim_{x \to -\infty} \left( \left[ x^{k} \psi^{(k)}(x) \right] \left[ x^{\ell+1} \psi^{(\ell+1)}(x) \right] - \left[ x^{k+1} \psi^{(k+1)}(x) \right] \left[ x^{\ell} \psi^{(\ell)}(x) \right] \right) = (-1)^{k-1} (k-1)! (-1)^{\ell} \ell! - (-1)^k k! (-1)^{\ell-1} (\ell-1)! = (-1)^{k+\ell-1} [(k-1)! \ell! - k!(\ell-1)!]
\]
and
\[
\lim_{x \to \infty} (ax^{k+\ell+1}[\psi^{(k)}(x)\psi^{(\ell)}(x+a) - \psi^{(\ell)}(x)\psi^{(k)}(x+a)])
\]
\[
= \lim_{x \to \infty} \left(\frac{x^{\ell+1}}{(x+a)^k} [x^k\psi^{(k)}(x)] [(x+a)^k\psi^{(k)}(x+a)] \int_x^{x+a} \left[\frac{\psi^{(\ell)}(u)}{\psi^{(k)}(u)}\right]^' \, du \right)
\]
\[
= \lim_{x \to \infty} \left[\frac{x^{\ell+1}}{(x+a)^k} \int_0^a \frac{\psi^{(\ell+1)}(x+u)\psi^{(k)}(x+u) - \psi^{(\ell)}(x+u)\psi^{(k+1)}(x+u)}{[\psi^{(k)}(x+u)]^2} \, du \right]
\]
\[
= [((k-1)!)^2 \int_0^a \frac{(x+u)^{2k}}{(x+a)^k} \frac{(x+u)^{k+\ell+1}}{(x+u)^{k+\ell+1}} \frac{x^{\ell+1}}{(x+a)^k} \int_0^a \frac{\psi^{(\ell+1)}(x+u)\psi^{(k)}(x+u) - \psi^{(\ell)}(x+u)\psi^{(k+1)}(x+u)}{[\psi^{(k)}(x+u)]^2} \, du \right]
\]
\[
= (-1)^{k+\ell}(k-1)! (\ell-1)! (k-\ell),
\]
where we used the limits (2.2) and (2.4). The proof of Lemma 2.4 is complete.

3. Necessary and Sufficient Conditions of Complete Monotonicity

For verifying the lower bound in the double inequality (1.4), we find a lower bound for the second factor in (1.3) and more.

**Theorem 3.1.** Let \( p > m \geq n > q \geq 0 \) be integers such that \( m + n = p + q \) and let

\[
F_{p,m,n,q,c}(x) = \begin{cases} 
\left|\psi^{(m)}(x)\right| \left|\psi^{(n)}(x)\right| - c \left|\psi^{(p)}(x)\right|, & q = 0 \\
\left|\psi^{(m)}(x)\right| \left|\psi^{(n)}(x)\right| - c \left|\psi^{(p)}(x)\right| \left|\psi^{(q)}(x)\right|, & q \geq 1 
\end{cases}
\]

for \( c \in \mathbb{R} \) and \( x \in (0, \infty) \). Then

(1) for \( q \geq 0 \), if and only if

\[
c \leq \begin{cases} 
(m-1)!(n-1)! & q = 0 \\
\frac{(p-1)!}{(m-1)!(n-1)!} & q \geq 1
\end{cases} \tag{3.1}
\]

the function \( F_{p,m,n,q,c}(x) \) is completely monotonic in \( x \in (0, \infty) \);

(2) for \( q \geq 1 \), if and only if

\[
c \geq \frac{m!n!}{plq!}, \tag{3.2}
\]

the function \( -F_{p,m,n,q,c}(x) \) is completely monotonic in \( x \in (0, \infty) \);
(3) the double inequality
\[
- \frac{(m + n - 1)!}{(m - 1)!(n - 1)!} \leq \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} < 0
\]  
(3.3)
for \( m, n \in \mathbb{N} \) and the double inequality
\[
\frac{(m - 1)!(n - 1)!}{(p - 1)!(q - 1)!} \leq \frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} < \frac{m!n!}{p!q!}
\]  
(3.4)
for \( m, n, p, q \in \mathbb{N} \) with \( p > m \geq n > q \geq 1 \) and \( m + n = p + q \) are valid on \( (0, \infty) \) and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars respectively.

Proof. Let
\[
c_{p,m,n,q} = \frac{(m - 1)!(n - 1)!}{(p - 1)!(q - 1)!}, \quad q \geq 1,
\]
and
\[
c_{p,m,n,0} = \frac{(m - 1)!(n - 1)!}{(p - 1)!}, \quad d_{p,m,n,q} = \frac{m!n!}{p!q!}.
\]

Theorem 4.1 in [5] reads that
(1) the function \( F_{p,m,n,q;c_{p,m,n,q}}(x) \) for \( q \geq 1 \) or \( q = 0 \), and
(2) the function \( -F_{p,m,n,q;d_{p,m,n,q}}(x) \) for \( q \geq 1 \)
are both completely monotonic on \( (0, \infty) \). It is clear that
\[
F_{p,m,n,q;c}(x) = F_{p,m,n,q;c_{p,m,n,q}}(x) + \begin{cases} (c_{p,m,n,0} - c)|\psi^{(p)}(x)|, & q = 0 \\ (c_{p,m,n,q} - c)|\psi^{(p)}(x)||\psi^{(q)}(x)|, & q \geq 1 \end{cases}
\]
and
\[
-F_{p,m,n,q;c}(x) = -F_{p,m,n,q;d_{p,m,n,q}}(x) + (c - d_{p,m,n,q})|\psi^{(p)}(x)||\psi^{(q)}(x)|
\]
for \( q \geq 1 \). From the facts that
(1) the functions \(|\psi^{(p)}(x)|\) and \(|\psi^{(p)}(x)||\psi^{(q)}(x)|\) are both completely monotonic on \( (0, \infty) \), and
(2) the sum and the product of finitely many completely monotonic functions are still completely monotonic,
we conclude that the conditions in (3.1) and (3.2) are sufficient for the functions
\( \pm F_{p,m,n,q;c}(x) \) to be completely monotonic on \( (0, \infty) \) respectively.

If \( \pm F_{p,m,n,q;c}(x) \) are completely monotonic, then \( \pm F_{p,m,n,q;c}(x) \geq 0 \).
These are equivalent to the facts that,
(1) for \( q = 0 \),
\[
c \leq \frac{|\psi^{(m)}(x)||\psi^{(n)}(x)|}{|\psi^{(p)}(x)|} = \frac{x^m|\psi^{(m)}(x)|x^n|\psi^{(n)}(x)|}{x^p|\psi^{(p)}(x)|} \to \frac{(m - 1)!(n - 1)!}{(p - 1)!}
\]
as \( x \to \infty \), where we used \( p = m + n \) and the limit (2.2) in Lemma 2.1;
(2) for \( q \geq 1, \)
\[
\frac{c}{\lambda} \left| \frac{\psi^{(m)}(x)}{\psi^{(n)}(x)} \right| \leq \frac{x^m \psi^{(m)}(x) x^n \psi^{(n)}(x)}{x^p \psi^{(p)}(x) x^q \psi^{(q)}(x)} = \left\{ \begin{array}{ll}
x^m \psi^{(m)}(x) x^n \psi^{(n)}(x) \\
x^p \psi^{(p)}(x) x^q \psi^{(q)}(x)
\end{array} \right.
\]
\[
\rightarrow \left\{ \begin{array}{ll}
(m - 1)!(n - 1)! \\
(p - 1)!(q - 1)!
\end{array} \right., \quad x \to \infty;
\]
\[
\frac{m!n!}{p!q!}, \quad x \to 0^+,
\]
where we used \( p + q = m + n \) and the limits (2.1) and (2.2) in Lemma 2.1. Moreover, for \( p > m \geq n > q > 0 \) such that \( m + n = p + q \), we have
\[
\frac{p}{m} - 1 = \frac{q}{m} < 1 \iff \frac{mn}{pq} > 1 \iff \frac{m!n!}{p!q!} > \frac{(m - 1)!(n - 1)!}{(p - 1)!(q - 1)!}.
\]
Hence, necessary conditions are proved.

The double inequalities (3.3) and (3.4) come from the positivity of the functions \( \pm F_{p,m,n,q,c}(x) \) and their sharpness can be concluded from the limits
\[
\lim_{x \to 0^+} \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x) \psi^{(n)}(x)} = \lim_{x \to 0^+} \frac{x^{m+n+1} \psi^{(m+n)}(x)}{x^{m+1} \psi^{(m)}(x) x^n \psi^{(n)}(x)} \quad \lim_{x \to 0^+} x = 0,
\]
\[
\lim_{x \to \infty} \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x) \psi^{(n)}(x)} = \lim_{x \to \infty} \frac{x^{m+n} \psi^{(m+n)}(x)}{x^m \psi^{(m)}(x) x^n \psi^{(n)}(x)} = -\frac{(m + n - 1)!}{(m - 1)!(n - 1)!},
\]
\[
\lim_{x \to 0^+} \frac{\psi^{(m)}(x) \psi^{(n)}(x)}{\psi^{(p)}(x) \psi^{(q)}(x)} = \lim_{x \to 0^+} \frac{x^{m+1} \psi^{(m)}(x) x^n \psi^{(n)}(x)}{x^p \psi^{(p)}(x) x^q \psi^{(q)}(x)} = \frac{m!n!}{p!q!},
\]
\[
\lim_{x \to \infty} \frac{\psi^{(m)}(x) \psi^{(n)}(x)}{\psi^{(p)}(x) \psi^{(q)}(x)} = \lim_{x \to \infty} \frac{x^m \psi^{(m)}(x) x^n \psi^{(n)}(x)}{x^p \psi^{(p)}(x) x^q \psi^{(q)}(x)} = \frac{(m - 1)!(n - 1)!}{(p - 1)!(q - 1)!},
\]
where we used the limits (2.1) and (2.2) in Lemma 2.1 once again. The proof of Theorem 3.1 is complete. \( \square \)

**Theorem 3.2.** For \( k \in \mathbb{N} \) and \( x \in (0, \infty) \), let
\[
\mathcal{F}_{k,\eta_k}(x) = \psi^{(2k)}(x) + \eta_k \left[ \psi^{(k)}(x) \right]^2 \quad \text{and} \quad \mathfrak{F}_{k,\vartheta_k}(x) = \frac{\psi^{(2k)}(x)}{((-1)^{k+1} \psi^{(k)}(x))^{\vartheta_k}}.
\]
Then the following conclusions are true:

1. if and only if \( \eta_k \geq \frac{1}{2} \frac{(2k)!}{(k-1)!k!} \), the function \( \mathcal{F}_{k,\eta_k}(x) \) is completely monotonic on \( (0, \infty) \);
2. if and only if \( \eta_k \leq 0 \), the function \( -\mathcal{F}_{k,\eta_k}(x) \) is completely monotonic on \( (0, \infty) \);
3. if and only if \( \vartheta_k \geq 2 \), the function \( \mathfrak{F}_{k,\vartheta_k}(x) \) is decreasing on \( (0, \infty) \).
(4) if and only if \( \vartheta_k \leq \frac{2k+1}{k+1} \), the function \( \mathcal{F}_{k, \vartheta_k}(x) \) is increasing on \((0, \infty)\);

(5) the following limits are valid:

\[
\lim_{x \to 0^+} \mathcal{F}_{k, \vartheta_k}(x) = \begin{cases} 
- \frac{(2k)!}{[(k+1)!!]^2}, & \vartheta_k = \frac{2k+1}{k+1} \\
0, & \vartheta_k > \frac{2k+1}{k+1} \\
-\infty, & \vartheta_k < \frac{2k+1}{k+1}
\end{cases}
\]

and

\[
\lim_{x \to \infty} \mathcal{F}_{k, \vartheta_k}(x) = \begin{cases} 
- \frac{(2k-1)!}{[(k-1)!!]^2}, & \vartheta_k = 2 \\
-\infty, & \vartheta_k > 2 \\
0, & \vartheta_k < 2
\end{cases}
\]

(6) the double inequality

\[
- \frac{1}{2} \frac{(2k)!}{(k-1)! k!} < \frac{\psi^{(2k)}(k)(x)}{((-1)^{k+1}\psi^{(k)}(x))^2} < 0
\]

is valid on \((0, \infty)\) and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.

**Proof.** Taking \( q = 0, m = n = k, \) and \( p = 2k \) in Theorem 3.1 leads to the fact that the function

\[
\left[ \frac{\psi^{(k)}(x)}{\psi^{(k)}(x)} \right]^2 + c\psi^{(2k)}(k)(x)
\]

is completely monotonic on \((0, \infty)\) if and only if \( c \leq \frac{[(k-1)!!]^2}{(2k-1)!} \). This result is equivalent to the fact that the function \( \mathcal{F}_{k, \eta_k}(x) \) is completely monotonic on \((0, \infty)\) if and only if \( \eta_k \geq \frac{(2k-1)!}{[(k-1)!!]^2} = \frac{1}{2} \frac{(2k)!}{(k-1)! k!} \).

If \( -\mathcal{F}_{k, \eta_k}(x) \) is completely monotonic on \((0, \infty)\), then \( -\mathcal{F}_{k, \eta_k}(x) \geq 0 \) on \((0, \infty)\), that is,

\[
\eta_k \leq -\frac{\psi^{(2k)}(k)(x)}{\psi^{(k)}(x)} = -\frac{x^{2k+1}\psi^{(2k)}(k)(x)}{[x^{k+1}\psi^{(k)}(x)]^2} \to 0, \quad x \to 0^+,
\]

where we used the limit (2.1).

The integral representation

\[
\psi^{(k)}(z) = (-1)^{k+1} \int_0^\infty \frac{t^k}{1 - e^{-t}} e^{-zt} \, dt
\]

for \( \Re(z) > 0 \) and \( k \in \mathbb{N} \), see [1, p. 260, 6.4.1], means that the functions \( |\psi^{(k)}(x)| \) for all \( k \in \mathbb{N} \) are completely monotonic on \((0, \infty)\). Further considering the fact that the sum of finitely many completely monotonic functions is also completely monotonic, we see that the necessary condition \( \eta_k \leq 0 \) is also sufficient for \( -\mathcal{F}_{k, \eta_k}(x) \) to be completely monotonic on \((0, \infty)\).

Direct computation gives

\[
\mathcal{F}'_{k, \vartheta_k}(x) = (-1)^{k+1} \frac{\psi^{(2k+1)}(k)(x)\psi^{(k)}(k)(x) - \vartheta_k \psi^{(2k)}(k)(x)\psi^{(k+1)}(x)}{[(-1)^{k+1}\psi^{(k)}(x)]^{\vartheta_k+1}}
\]
Taking \( p = 2k + 1 \), \( q = k \), \( m = 2k \), and \( n = k + 1 \) in Theorem 3.1 yields that the function

\[
(-1)^{k+1} \left[ \psi^{(2k)}(x) \psi^{(k+1)}(x) - c\psi^{(2k+1)}(x)\psi^{(k)}(x) \right]
\]

and its opposite is completely monotonic on \((0, \infty)\) if and only if

\[
c \leq \frac{(2k - 1)!k!}{(2k)!(k - 1)!} = \frac{1}{2}
\]

and

\[
c \geq \frac{(2k)!(k + 1)!}{(2k + 1)!k!} = \frac{k + 1}{2k + 1}
\]

respectively. Therefore, we conclude that,

(1) if and only if \( \vartheta_k \geq 2 \), the derivative \( \mathcal{F}'_{k, \vartheta_k}(x) \leq 0 \), and then the function \( \mathcal{F}_{k, \vartheta_k}(x) \) is decreasing, on \((0, \infty)\);

(2) if and only if \( \vartheta_k \leq 2 \frac{k + 1}{k + 1} \), the derivative \( \mathcal{F}'_{k, \vartheta_k}(x) \geq 0 \), and then the function \( \mathcal{F}_{k, \vartheta_k}(x) \) is increasing, on \((0, \infty)\).

By Lemma 2.1, we obtain

\[
\lim_{x \to 0^+} \mathcal{F}_{k, \vartheta_k}(x) = \lim_{x \to 0^+} \left[ \frac{x^{2k+1}\psi^{(2k)}(x)}{((-1)^{k+1}x^{k+1}\psi^{(k)}(x))} \right] \vartheta_k \lim_{x \to 0^+} x^{(k+1)\vartheta_k - (2k+1)}
\]

\[
= (-1)^{k+1}(2k)! \frac{2k + 1}{k + 1}, \quad \vartheta_k = \frac{2k + 1}{k + 1};
\]

\[
0, \quad \vartheta_k > \frac{2k + 1}{k + 1};
\]

\[
-\infty, \quad \vartheta_k < \frac{2k + 1}{k + 1}
\]

and

\[
\lim_{x \to \infty} \mathcal{F}_{k, \vartheta_k}(x) = \lim_{x \to \infty} \left[ \frac{x^{2k}\psi^{(2k)}(x)}{((-1)^{k+1}x^{k}\psi^{(k)}(x))} \right] \vartheta_k \lim_{x \to \infty} x^{(\vartheta_k - 2)k}
\]

\[
= (-1)^{k-1}(2k - 1)! \frac{2k - 1}{k - 1}, \quad \vartheta_k = 2;
\]

\[
-\infty, \quad \vartheta_k > 2;
\]

\[
0, \quad \vartheta_k < 2.
\]

The double inequality (3.7) follows from the decreasing property of the function \( \mathcal{F}_{k, 2}(x) \) on \((0, \infty)\) and the limits in (3.5) and (3.6) for \( \vartheta_k = 2 \). The proof of Theorem 3.2 is complete. \( \square \)
4. A Completely Monotonic Function Involving Tetragamma Function

For verifying the lower bound in the double inequality (1.4), we establish an upper bound for the third factor in (1.3) and more.

Theorem 4.1. If and only if $\nu \geq 2$, the function

$$I_\nu(x) = \nu[\psi'(x) - 2\psi'(2x)]^2 - 2\psi'(x)\psi''(2x) + \psi'(2x)\psi''(x)$$

(4.1)

is completely monotonic on $(0, \infty)$. Consequently, the double inequality

$$0 < \frac{2\psi'(x)\psi''(2x) - \psi'(2x)\psi''(x)}{[\psi'(x) - 2\psi'(2x)]^2} < 2$$

(4.2)

is valid on $(0, \infty)$ and sharp in the sense that the lower bound 0 and the upper bound 2 cannot be replaced by any greater number and any less number.

Proof. Utilizing the duplication formula

$$\psi(2z) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi\left(z + \frac{1}{2}\right) + \ln 2$$

in [1, p. 259, 6.3.8] gives

$$\psi'(2z) = \frac{1}{4}\left[\psi'(z) + \psi\left(z + \frac{1}{2}\right)\right]$$

(4.3)

and

$$\psi''(2z) = \frac{1}{8}\left[\psi''(z) + \psi''\left(z + \frac{1}{2}\right)\right].$$

(4.4)

Then

$$4I_\nu(x) = \nu\left[\psi'(x) - \psi\left(x + \frac{1}{2}\right)\right]^2 - \psi'(x)\psi''\left(x + \frac{1}{2}\right) + \psi\left(x + \frac{1}{2}\right)\psi''(x).$$

Let

$$g(t) = \begin{cases} 
\frac{t}{1 - e^{-t}}, & t \neq 0; \\
1, & t = 0.
\end{cases}$$

Then, by the integral representation (3.8) and Lemma 2.2, we obtain

$$4I_\nu(x) = \nu\left[\int_0^\infty g(t)(1 - e^{-t/2}) e^{-xt} \, dt\right]^2$$

$$- \int_0^\infty g(t) e^{-xt} \, dt \int_0^\infty t e^{-t/2} g(t) e^{-xt} \, dt$$

$$+ \int_0^\infty e^{-t/2} g(t) e^{-xt} \, dt \int_0^\infty t g(t) e^{-xt} \, dt$$

$$= \nu\int_0^\infty \left[\int_0^t g(u)(1 - e^{-u/2})g(t-u)[1 - e^{-(t-u)/2}] \, du\right] e^{-xt} \, dt.$$
Employing Lemma 2.3 and the positivity of \( g(t) \) yields that, when \( \nu \geq 2 \), the function \( 4I_\nu(x) \) is completely monotonic on \((0, \infty)\).

By the expression in (4.1), if \( I_\nu(x) \) is completely monotonic on \((0, \infty)\), then \( I_\nu(x) \geq 0 \), which is equivalent to

\[
\nu \geq \frac{2\psi'(x)\psi''(2x) - \psi'(2x)\psi''(x)}{\left[\psi'(x) - 2\psi'(2x)\right]^2} = \frac{\psi'(x)\psi''(x + \frac{1}{2}) - \psi'(x + \frac{1}{2})\psi''(x)}{\left[\psi'(x) - \psi'(x + \frac{1}{2})\right]^2} = \frac{x^4\left[\psi'(x)\psi''(x + \frac{1}{2}) - \psi'(x + \frac{1}{2})\psi''(x)\right]}{\left[-(-1)^{1+2}(1-1)!(2-1)!(1-2)^\frac{1}{2}\right]} = 2
\]

as \( x \to \infty \), where we used the formulas (4.3) and (4.4), and the limits (2.3) and (2.5) in Lemma 2.4. Accordingly, the condition \( \nu \geq 2 \) is necessary.

Since \( K(x, y) < 0 \) was proved in [2, Proposition 5] and [4, Theorem 6], by the expression (1.3), the lower bound in (4.2) is immediate. The upper bound of (4.2) comes from the complete monotonicity of the function \( I_\nu(x) \).

The sharpness of the double inequality (4.2) can be deduced from the limit in (4.5) and the limit

\[
\lim_{x \to 0^+} \frac{2\psi'(x)\psi''(2x) - \psi'(2x)\psi''(x)}{\left[\psi'(x) - 2\psi'(2x)\right]^2} = \lim_{x \to 0^+} \frac{\psi'(x)\psi''(x + \frac{1}{2}) - \psi'(x + \frac{1}{2})\psi''(x)}{\left[\psi'(x) - \psi'(x + \frac{1}{2})\right]^2} = \lim_{x \to 0^+} \frac{x^4\left[\psi'(x)\psi''(x + \frac{1}{2}) - \psi'(x + \frac{1}{2})\psi''(x)\right]}{\left[\psi'(x) - \psi'(x + \frac{1}{2})\right]^2} = 0
\]

\[
\lim_{x \to 0^+} \frac{0(-1)^21!\psi''(\frac{1}{2}) - \psi'(\frac{1}{2})(-1)^32!}{\left[(-1)^21! - 0\right]^2} = 0
\]
where we used the formulas (4.3) and (4.4), the limit (2.1) in Lemma 2.1, and the limit (2.3) in Lemma 2.4. The proof of Theorem 4.1 is complete.

\[ \Box \]

5. The Lower Bound of Sectional Curvature

In this section, we prove the double inequality (1.4) and its sharpness.

**Theorem 5.1.** For \( x > 0 \), the double inequality \( 0 > \mathcal{K}(x) > -\frac{1}{2} \) is valid on \((0, \infty)\) and sharp in the sense that the lower bound \(-\frac{1}{2}\) and the upper bound 0 cannot be replaced by any larger scalar and any smaller scalar respectively.

**Proof.** By the double inequality (3.7) for \( k = 1 \) in Theorem 3.2, we obtain

\[
-1 < \frac{\psi''(x)}{[\psi'(x)]^2} < 0
\]
on \((0, \infty)\). Combining this double inequality with the double inequality (4.2) gives

\[
-2 < \frac{\psi''(x) 2\psi'(x)\psi''(2x) - \psi'(2x)\psi''(x)}{[\psi'(x) - 2\psi'(2x)]^2} < 0
\]
which is equivalent to

\[
-\frac{1}{2} < \frac{1}{4} \frac{\psi''(x) 2\psi'(x)\psi''(2x) - \psi'(2x)\psi''(x)}{[\psi'(x) - 2\psi'(2x)]^2} = \mathcal{K}(x) < 0
\]
on \((0, \infty)\), where we used the expression (1.3) for \( \mathcal{K}(x) \).

From the limits in (1.2), we immediately deduce

\[
\lim_{x \to 0^+} \mathcal{K}(x) = 0 \quad \text{and} \quad \lim_{x \to 0^+} \mathcal{K}(x) = -\frac{1}{2}.
\]
The sharpness follows. The proof of Theorem 5.1 is complete. \( \Box \)

**Remark 5.1.** This paper is a shortened version of the preprint [13] and the fifth one in a series of articles including [8,10–12,14–17,21].

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