Invariant Variational Schemes
for Ordinary Differential Equations

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Abstract

We propose a novel algorithmic method for constructing invariant variational schemes of systems of ordinary differential equations that are the Euler–Lagrange equations of a variational principle. The method is based on the invariantization of standard, non-invariant discrete Lagrangian functionals using equivariant moving frames. The invariant variational schemes are given by the Euler–Lagrange equations of the corresponding invariantized discrete Lagrangian functionals. We showcase this general method by constructing invariant variational schemes of ordinary differential equations that preserve variational and divergence symmetries of the associated continuous Lagrangians. Noether’s theorem automatically implies that the resulting schemes are exactly conservative. Numerical simulations are carried out and show that these invariant variational schemes outperform standard numerical discretizations.

1 Introduction

The aim of geometric numerical integration is to construct numerical schemes that preserve certain geometric features of differential equations. In doing so, geometric integrators typically provide better global and long term numerical results than comparable non-geometric methods. Typical examples include, amongst others, symplectic integrators, [3, 15, 20, 30], Lie–Poisson structure preserving schemes, [34], energy-preserving methods, [29], exactly conservative schemes, [32, 33], symmetry-preserving methods, [4–7, 10], and variational integrators, [25].
In this paper we use the method of moving frames, [22, 24, 27], to construct numerical schemes for ordinary differential equations that preserve variational symmetries of Euler–Lagrange equations. An application of Noether’s theorem implies that the resulting schemes are necessarily conservative and preserve the associated “constants of motion.” Such schemes are constructed as follows. Given the Euler–Lagrange equations \( E(L) = 0 \), with variational symmetry group \( G \), consider its Lagrangian functional \( \mathcal{L}[u] = \int L \, dx \). We note that this Lagrangian functional is not unique. It can always be scaled by a constant and one can always add a divergence term. Nevertheless, since \( G \) is a variational symmetry group, \( \mathcal{L} \) can be chosen so that it is invariant under the action of \( G \). Next, introduce a finite difference approximation \( \mathcal{L}^d = \sum_k L_k \) of \( \mathcal{L} \). In general, \( \mathcal{L}^d \) will not be invariant under the product action of \( G \). To obtain a symmetry-preserving Lagrangian, we follow the general procedure in [9, 18] and compute the invariantized Lagrangian \( \iota(L^d) \) using the method of equivariant moving frames. The discrete Euler–Lagrange equations \( E^d(\iota(L_k)) = 0 \) are then used to approximate the original equations \( E(L) = 0 \). Since \( E^d(\iota(L_k)) = 0 \) is invariant under the action of \( G \), Noether’s theorem implies that the scheme is conservative and preserves the constants of motion. The above procedure can be modified to deal with Lagrangian functionals that admit divergence (Bessel-Hagen) symmetries. In this case, it suffices to modify the Lagrangian in such a way that divergence symmetries become variational symmetries.

The proposed methodology is related to various other approaches used in geometric numerical integration. As the schemes developed in [25], the proposed methodology is variational, in that we discretize the Lagrangian rather than the associated Euler–Lagrange equations. Furthermore, as the schemes proposed in [4–7, 10], our schemes are invariant as well, due to the well-known fact that symmetries of a Lagrangian are also symmetries of the corresponding Euler–Lagrange equations, [28]. Lastly, similar to the exactly conservative schemes derived in [32, 33], our schemes will also be exactly conservative, thanks to Noether’s theorem. Therefore, our methodology combines several geometric features into one numerical integrator.

The organization of the paper is as follows. In Section 2 we begin by recalling standard results concerning variational problems and their symmetry groups. In particular, in Section 2.2 we explain how a divergence symmetry group can be made into a variational symmetry group by appropriately modifying the Lagrangian. In Section 3 we review the theory of discrete variational problems, their symmetries, and Noether’s Theorem. To construct symmetry-preserving discrete Lagrangians, and therefore invariant Euler–Lagrange equations, we use the method of equivariant moving frames, which is summarized in Section 4. In Section 5 we outline the procedure for constructing conservative schemes of Euler–Lagrange equations that preserve their variational/divergence symmetries. Finally, in Section 6 numerical simulations are carried out that verify numerically the exact conservative nature of the proposed invariant variational schemes. Also, when compared to “standard schemes,” invariant variational schemes provide better long term numerical results.

2 Invariant Lagrangians

In this section we recall standard results concerning invariant variational problems and, more generally, divergent invariant variational problems. For a detailed exposition, we refer the
reader to [28]. We begin by introducing some notation and terminology.

In this paper we consider ordinary differential equations and let \( x \in \mathbb{R} \) denote the independent variable. If \( u = (u^1, \ldots, u^q) \in \mathbb{R}^q \) represent the dependent variables, then the \( n^{th} \) order jet space of curves in \( \mathbb{R}^q \), denoted \( J^{(n)} = J^{(n)}(\mathbb{R}^q, 1) \), is locally parametrized by \( (x, u^{(n)}) \), where \( u^{(n)} = (u, u_x, \ldots, u_{x^n}) \) collects the derivatives \( u_{x^k} \) of order \( 0 \leq k \leq n \).

Let \( G \) be an \( r \)-dimensional Lie group acting on \( (x, u) \in \mathbb{R}^{q+1} \):
\[
X = g \cdot x, \quad U = g \cdot u.
\]
The induced action on the horizontal one-form \( dx \) yields the lifted horizontal form
\[
\omega = D_x(X) \, dx,
\]
where
\[
D_x = \frac{\partial}{\partial x} + \sum_{\alpha=1}^{q} \sum_{k \geq 0} \frac{u^\alpha_{x^{k+1}}}{\partial u^\alpha_{x^k}} \frac{\partial}{\partial u^\alpha_{x^k}}
\]
denotes the total derivative operator.

**Remark 2.1.** More precisely, the lift of \( dx \) should be
\[
\omega = D_x(X) \, dx + \sum_{\alpha=1}^{q} X_{u^\alpha} \theta^\alpha,
\]
where \( \theta^\alpha = du^\alpha - u_x^\alpha \, dx \) are the order zero basic contact one-forms, [19]. However, since our computations are performed modulo contact forms, these are omitted.

Dual to \( \omega \), we have the lifted derivative operator
\[
D_X = \frac{1}{D_x(X)} \, D_x.
\]
The prolonged action of \( G \) to the \( n^{th} \) order jet space \( J^{(n)} \) is given by
\[
U_{X^k}^\alpha = D_X^k(U^\alpha), \quad \alpha = 1, \ldots, q, \quad 0 \leq k \leq n.
\]
At the infinitesimal level, let
\[
v_\nu = \xi_\nu(x, u) \frac{\partial}{\partial x} + \sum_{\alpha=1}^{q} \phi_\nu^\alpha \frac{\partial}{\partial u^\alpha}, \quad \nu = 1, \ldots, r,
\]
denote a basis of infinitesimal generators. The prolongation formula for the infinitesimal generators is
\[
pr \, v_\nu = \xi_\nu(x, u) \frac{\partial}{\partial x} + \sum_{k \geq 0} \sum_{\alpha=1}^{q} \phi_\nu^{\alpha,k} \frac{\partial}{\partial u^\alpha_{x^k}}, \quad \nu = 1, \ldots, r,
\]
with the component \( \phi_\nu^{\alpha,k} \) given by the formula
\[
\phi_\nu^{\alpha,k} = D_x^k(Q_\nu^\alpha) + \xi_\nu u^\alpha_{x^{k+1}}, \quad \text{where} \quad Q_\nu^\alpha(x, u^{(1)}) = \phi_\nu^\alpha - \xi_\nu u^\alpha_x
\]
are the components of the characteristic \( Q_\nu = (Q_1^\nu, \ldots, Q_q^\nu) \).
Example 2.2. Consider the action of the special Euclidean group SE(2) on planar curves \{ (x, u(x)) \} given by

\[ X = x \cos \varphi - u \sin \varphi + a, \quad U = x \sin \varphi + u \cos \varphi + b, \]  

where \( a, b, \varphi \in \mathbb{R} \). Then the horizontal lifted one-form is

\[ \omega = D_x(X) \, dx = (\cos \varphi - u_x \sin \varphi) \, dx, \]

and the lifted derivative operator is

\[ D_X = \frac{1}{\cos \varphi - u_x \sin \varphi} D_x. \]

Therefore, the prolonged action is, up to order two,

\[ U_X = \frac{\sin \varphi + u_x \cos \varphi}{\cos \phi - u_x \sin \varphi}, \quad U_{XX} = \frac{u_{xx}}{(\cos \varphi - u_x \sin \varphi)^3}. \]

A basis of infinitesimal generators is given by the vector fields

\[ v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial u}, \quad v_3 = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}. \]

Up to order two, their prolongation is

\[ \text{pr}^{(2)} v_1 = \frac{\partial}{\partial x}, \quad \text{pr}^{(2)} v_2 = \frac{\partial}{\partial u}, \quad \text{pr}^{(2)} v_3 = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + 3u_x u_{xx} \frac{\partial}{\partial u_{xx}}. \]

2.1 Variational Symmetry

We now recall the notion of a variational symmetry group for a Lagrangian and the celebrated Noether’s theorem.

Definition 2.3. A connected Lie group of transformations \( G \) acting on \( \mathbb{R}^{q+1} \) is called a variational symmetry group of the functional \( L[u] = \int L(x, u^{(n)}(x)) \, dx \) if and only if

\[ \int g \cdot (L(x, u^{(n)}) \, dx = \int L(X, U^{(n)}) \omega = \int L(x, u^{(n)}) \, dx \]

for all \( g \in G \),

where the prolonged action is defined. At the infinitesimal level, if \( v_1, \ldots, v_r \) is a basis of infinitesimal generators, then \( G \) is a variational symmetry group of the functional if and only if

\[ \text{pr}^{(n)} v_\nu (L) + L D_x(\xi_\nu) = 0, \quad \nu = 1, \ldots, r. \]

Example 2.4. A classical example of SE(2)-invariant Lagrangian is given by the Euler elastica

\[ L = \int \frac{1}{2} \kappa^2 \, ds, \]
\[ \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \] is the curvature of a planar curve and \( \omega = ds = \sqrt{1 + u_x^2} \, dx \) is the arc-length element. In local coordinates, the functional (3) is

\[ \mathcal{L}[u] = \int \frac{u_{xx}^2}{2(1 + u_x^2)^{5/2}} \, dx. \] (4)

The elastica problem has a long history dating back to Euler, [13]. For a more modern account we refer to [21].

**Definition 2.5.** For \( 1 \leq \alpha \leq q \), the \( \alpha \)th Euler operator is the differential operator

\[ E_\alpha = \sum_{k=0}^{\infty} (-D_x)^k \frac{\partial}{\partial u_{x^k}^{\alpha}}. \] (5)

**Proposition 2.6.** If \( u = u(x) \) is a smooth extremal of the variational problem \( \mathcal{L}[u] = \int L(x, u^{(n)}) \, dx \), then it must be a solution of the Euler–Lagrange equations

\[ E(L) = (E_1(L), \ldots, E_q(L)) = 0. \]

**Example 2.7.** The Euler–Lagrange equation of the Euler elastica functional (4) is

\[ 2u_{xxxx}(1 + u_x^2)^2 + 5u_x^3(6u_x^2 - 1) - 20u_xu_{xx}u_{xxx}(1 + u_x^2) = 0. \] (6)

In terms of the curvature and its arc length derivative, the differential equation (6) simplifies to

\[ \kappa_{ss} + \frac{\kappa^3}{2} = 0. \] (7)

**Theorem 2.8.** If \( G \) is a variational symmetry group of \( \mathcal{L}[u] = \int L(x, u^{(n)}) \, dx \), then \( G \) is a symmetry group of the Euler–Lagrange equations \( E(L) = 0 \).

**Remark 2.9.** Since (6) is expressible in terms of the curvature and its arc length derivatives, the differential equation is immediately invariant under the prolonged action of the special Euclidean group \( \text{SE}(2) \).

We note that the converse of Theorem 2.8 is incorrect, [28]. In general, \( E(L) = 0 \) can admit a larger symmetry group than that of the functional \( \mathcal{L}[u] = \int L(x, u^{(n)}) \, dx \).

**Definition 2.10.** A conserved quantity (or constant of motion or first integral) for the system of ordinary differential equations \( \Delta(x, u^{(n)}) = 0 \) is a function \( C(x, u^{(m)}) \) such that

\[ D_x(C) = 0 \]

on the solution space of \( \Delta(x, u^{(n)}) = 0 \). In other words, \( C(x, u^{(m)}) \) is constant on solutions of \( \Delta(x, u^{(n)}) = 0 \).

We now state one of the simplest versions of Noether’s Theorem, [28].
Theorem 2.11. Let $G$ be a one-parameter group of variational symmetries for the functional $L[u] = \int L(x, u^{(n)}) \, dx$ with infinitesimal generator

$$v = \xi(x, u) \frac{\partial}{\partial x} + \sum_{\alpha=1}^{q} \phi_{\alpha}(x, u) \frac{\partial}{\partial u^{\alpha}}$$

and characteristic components $Q^{\alpha}(x, u^{(1)}) = \phi_{\alpha} - \xi u_{x}^{\alpha}$. Then, there exists a constant of motion $C = -(A + L\xi)$ where $A$ is a certain function depending on $Q = (Q^{1}, \ldots, Q^{q})$, $L$, and their derivatives.

Example 2.12. For a first order variational problem $L[u] = \int L(x, u^{(1)}) \, dx$, with infinitesimal variational symmetry generator (8),

$$C = - \left( \sum_{\alpha=1}^{q} Q_{\alpha} \frac{\partial L}{\partial u_{x}^{\alpha}} + \xi L \right)$$

is a conserved quantity of the Euler–Lagrange equations $E(L) = 0$.

Example 2.13. For a one-dimensional variational problem of order two, $L[u] = \int L(x, u^{(2)}) \, dx$, with infinitesimal variational symmetry generator (8),

$$C = - \left( \sum_{\alpha=1}^{q} Q_{\alpha} \frac{\partial L}{\partial u_{x}^{\alpha}} - D_{x} \left( \frac{\partial L}{\partial u_{xx}} \right) \right) + D_{x}(Q) \frac{\partial L}{\partial u_{xx}} + \xi L,$$

is a conserved quantity of the Euler–Lagrange equations $E(L) = 0$.

Example 2.14. For the Euler elastica functional (4), the conserved quantities that come from (10) are

$$C_{1} = \frac{2u_{x}u_{xxx}}{(1 + u_{x}^{2})^{5/2}} - \frac{(1 + 6u_{x}^{2})u_{xx}^{2}}{(1 + u_{x}^{2})^{7/2}}, \quad C_{2} = \frac{5u_{x}u_{xx}^{2}}{(1 + u_{x}^{2})^{7/2}} - \frac{2u_{xxx}}{(1 + u_{x}^{2})^{5/2}},$$

$$C_{3} = (x + uu_{x}) \left[ \frac{2u_{xxx}}{(1 + u_{x}^{2})^{5/2}} - \frac{5u_{x}u_{xx}^{2}}{(1 + u_{x}^{2})^{7/2}} \right] - \frac{uu_{xx}^{2}}{(1 + u_{x}^{2})^{5/2}} - \frac{2u_{xx}}{(1 + u_{x}^{2})^{3/2}}.$$

2.2 Divergence Symmetry

The notion of variational symmetry was extended by Bessel-Hagen, [2], to allow divergence symmetries of a variational functional, [28].

Definition 2.15. A connected Lie group of transformations $G$ acting on $\mathbb{R}^{q+1}$ is called a divergence symmetry group of the functional $L[u] = \int L(x, u^{(n)}) \, dx$ if and only if

$$\int g \cdot (L(x, u^{(n)}) \, dx) = \int [L(x, u^{(n)}) + D_{x}(P_{g}(x, u^{(n)}))] \, dx,$$

for some differential function $P_{g}(x, u^{(n)})$ depending on the group parameter $g \in G$. At the infinitesimal level, if $v_{1}, \ldots, v_{r}$ is a basis of infinitesimal generators of $G$, then $G$ is a divergence symmetry group if and only if

$$\text{pr}^{(n)}_{\nu}(L) + L D_{x}(\xi_{\nu}) = D_{x}(B_{\nu}), \quad \nu = 1, \ldots, r,$$

where $B_{\nu}(x, u^{(n)})$ are certain differential functions.
Since the kernel of the Euler–Lagrange operators (5) are total derivatives of differential functions, i.e. $D_x(B(x, u^{(n)})) = 0$, it follows that divergence symmetries produce symmetries of the corresponding Euler–Lagrange equations $\mathbf{E}(L) = 0$. Noether’s Theorem 2.11 still holds for divergence symmetries. Constants of motions are now given by $C = B - A - L\xi$.

We now show that any divergence symmetry group $G$ of a variational problem $\mathcal{L}[u] = \int L(x, u^{(n)}) \, dx$ can be made into the variational symmetry group of a modified Lagrangian with identical Euler–Lagrange equations. This observation will play an important role in Section 5.

**Theorem 2.16.** Let $\mathcal{L}[u] = \int L(x, u^{(n)}) \, dx$ be a functional with divergence symmetry group $G$ satisfying (11). Then $G$ is a variational symmetry group of the modified functional

$$
\mathcal{T}[u] = \int \mathcal{T} \, dx = \int (L + \zeta_x) \, dx,
$$

where $G$ acts on the new variable $\zeta$ according to

$$
g \cdot \zeta = \zeta - P_g.
$$

**Proof.** We first show that (13) induces a well-defined left group action on $D_x(\zeta) = \zeta_x$. To this end, let $h, g \in G$. We first note that

$$
D_x(P_{hg}) \, dx = (hg) \cdot (L \, dx) - L \, dx
= h \cdot (L + D_x(P_g)) \, dx - L \, dx
= (L + D_x(P_h)) \, dx + h \cdot [D_x(P_g) \, dx] - L \, dx
= D_x(P_h) \, dx + D_x(h \cdot P_g) \, \omega
= D_x(P_h) \, dx + D_x(h \cdot P_g) \, dx
= D_x(P_h + h \cdot P_g) \, dx.
$$

Thus

$$
(hg) \cdot [D_x(\zeta) \, dx] = D_x[(hg) \cdot \zeta] \, \omega
= D_x[(hg) \cdot \zeta] \, dx
= D_x(\zeta - P_{hg}) \, dx
= D_x(\zeta - P_h - h \cdot P_g) \, dx
= D_x(h \cdot (\zeta - P_g)) \, dx
= h \cdot (D_x(g \cdot \zeta) \, dx)
= h \cdot (g \cdot (D_x(\zeta) \, dx)),
$$

which shows that we have a well-defined left group action on $D_x(\zeta) = \zeta_x$.

It is now straightforward to show that $G$ is a variational symmetry group of the modified Lagrangian functional (12). For $g \in G$,

$$
\int g \cdot (\mathcal{T} \, dx) = \int g \cdot (L \, dx) + \int g \cdot (\zeta_x \, dx)
$$

7
\[
\begin{align*}
= & \int (L + D_x(P_g)) \, dx + \int D_x(g \cdot \zeta) \, \omega \\
= & \int (L + D_x(P_g)) \, dx + \int (D_x(X))^{-1}(\zeta_x - D_x(P_g)) \, (D_x X) \, dx \\
= & \int (L + D_x(P_g)) \, dx + \int (\zeta_x - D_x(P_g)) \, dx \\
= & \int (L + \zeta_x) \, dx \\
= & \int \mathcal{L} \, dx.
\end{align*}
\]

\[\square\]

**Remark 2.17.** By construction, we note that \( \mathcal{L} = \int L \, dx \) and \( \mathcal{L} = \int \mathcal{L} \, dx \) have the same conserved quantities.

**Example 2.18.** A simple example of Lagrangian admitting a divergence symmetry group is given by

\[
\mathcal{L} = \int L \, dx = \int \left( u_x^2 - \frac{1}{u^2} \right) \, dx,
\]

with Euler–Lagrange equation

\[ u_{xx} = \frac{1}{u^3}. \tag{14} \]

The corresponding divergence symmetry group action is

\[
X = \frac{\alpha x + \beta}{\delta x + \gamma}, \quad U = \frac{u}{\delta x + \gamma}, \quad \text{where} \quad \alpha \gamma - \beta \delta = 1.
\]

The associated infinitesimal generators are

\[
v_1 = \frac{\partial}{\partial x}, \quad v_2 = 2x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, \quad v_3 = x^2 \frac{\partial}{\partial x} + xu \frac{\partial}{\partial u}. \tag{15}\]

We note that the first two vector field generate variational symmetries since

\[
\text{pr} \, v_1(L) + LD_x(\xi_1) = 0, \quad \text{pr} \, v_2(L) + LD_x(\xi_2) = 0.
\]

On the other hand,

\[
\text{pr} \, v_3(L) + LD_x(\xi_3) = 2uu_x = D_x(u^2),
\]

which induces a divergence symmetry. Using (9), the corresponding conserved quantities are

\[
C_1 = u_x^2 + \frac{1}{u^2}, \quad C_2 = 2 \frac{x}{u^2} - 2(u - xu_x)u_x, \quad C_3 = \frac{x^2}{u^2} + (u - xu_x)^2.
\]

These constants of motion are not independent and satisfy the equation

\[
\frac{C_2^2}{4} - C_1 C_3 + 1 = 0. \tag{16}
\]
Since
\[ \int g \cdot (L \, dx) = \int \left[ ((\delta x + \gamma)u_x - \delta u)^2 - \frac{(\delta x + \gamma)^2}{u^2} \right] \frac{dx}{(\delta x + \gamma)^2} \]
\[ = \int \left[ u_x - \frac{2\delta uu_x}{\delta x + \gamma} + \frac{\delta^2 u^2}{(\delta x + \gamma)^2} - \frac{1}{u^2} \right] dx \]
\[ = \int \left[ L + D_x \left( -\frac{\delta u^2}{\delta x + \gamma} \right) \right] dx, \]
an invariant Lagrangian can be defined by introducing a new variable \( \zeta \) such that
\[ g \cdot \zeta = \zeta + \frac{\delta u^2}{\delta x + \beta}. \]
The induced prolonged action is
\[ g \cdot \zeta_x = D_x \left( \zeta + \frac{\delta u^2}{\delta x + \beta} \right) = (\delta x + \gamma)^2 D_x \left( \zeta + \frac{\delta u^2}{\delta x + \beta} \right) = (\delta x + \gamma)^2 \left[ \zeta_x + D_x \left( \frac{\delta u^2}{\delta x + \beta} \right) \right], \]
and the modified functional
\[ \int L \, dx = \int (L + \zeta_x) \, dx = \int \left( u_x^2 - \frac{1}{u^2} + \zeta_x \right) \, dx \quad (17) \]
is, by construction, invariant.

3 Discrete Lagrangians

We now adapt the results of the previous section to the discrete setting. Let \( z = (z^0, \ldots, z^q) = (x, u) \) be coordinates on \( \mathbb{R}^{q+1} \). In this section we are concerned with discrete \( \mathbb{R}^{q+1} \)-valued functions
\[ f: \mathbb{Z} \to \mathbb{R}^{q+1}, \quad k \mapsto f(k) = (f^0(k), \ldots, f^q(k)). \quad (18) \]
As it is customarily done, we use the index notation
\[ f_k = f(k) \]
to denote the value of \( f \) at \( k \in \mathbb{Z} \). Introducing the lattice variety
\[ \pi: \mathbb{Z} \times \mathbb{R}^{q+1} \to \mathbb{Z}, \]
the discrete map (18) defines a one-dimensional discrete submanifold
\[ \{(k, f_k) \mid k \in \mathbb{Z} \} \subset \mathbb{Z} \times \mathbb{R}^{q+1}. \]
The lattice space \( \mathbb{Z} \) does not admit a differentiable structure. Only the fibers \( \pi^{-1}(k) = \mathbb{R}^{q+1} \) are smooth manifolds. In the following, we use \( z_k = (z^0_k, \ldots, z^q_k) \) as coordinates on \( \pi^{-1}(k) = \mathbb{R}^{q+1} \). Natural operators on \( \mathbb{Z} \) are the forward shift
\[ S = S^+: k \mapsto k + 1, \quad (19) \]
and the backward shift

\[ S^- : k \mapsto k - 1. \]

The action of the shift maps \( S^\pm \) on the fiber coordinates \( z_k \) is

\[ S^\pm [z_k] = z_{k \pm 1}. \]

Using the forward shift (19) we define the forward difference operator

\[ \Delta = S - 1, \]

where \( \mathbb{I} : \mathbb{Z} \to \mathbb{Z} \) is the identity transformation.

**Definition 3.1.** Let \( n_1 \leq n_2 \) be two integers. The order \( n = n_2 - n_1 \) discrete jet space is the lattice variety

\[ J^{[n]}_{n_1, n_2} = \mathbb{Z} \times (\mathbb{R}^{q+1})^{(n+1)}, \]

with coordinates

\[ z^{[n]}_k = (k, \ldots, z_{k+\ell} \ldots) \in \mathbb{Z} \times (\mathbb{R}^{q+1})^{(n+1)}, \]

where \( n_1 \leq \ell \leq n_2 \). When \( n_1 = 0 \) and \( n_2 = n \in \mathbb{N}_0 \), we obtain what we call the \( n \)th order forward discrete jet space \( J^{[n]} = J^{[n]}_{0,n} \) and drop the subscript notation.

**Example 3.2.** For example, coordinates for \( J^{[2]} = J^{[2]}_{0,2} \) are given by

\[ z^{[2]}_k = (k, z_k, z_{k+1}, z_{k+2}), \]

while coordinates for \( J^{[4]}_{-2,2} \) are provided by \( z^{[4]}_k = (k, z_{k-2}, z_{k-1}, z_k, z_{k+1}, z_{k+2}) \).

**Definition 3.3.** Let \( F : J^{[n]} \to \mathbb{R} \) be a discrete function. A discrete functional is a formal sum

\[ \mathcal{L}^d[z] = \sum_{k \in \mathbb{Z}} L(z^{[n]}_k) = \sum_k L_k. \]

In the following we use the short-hand notation \( L_k \) to denote \( L(z^{[n]}_k) \) and omit the range of summation over the integer \( k \in \mathbb{Z} \).

**Definition 3.4.** Let \( \mathcal{F}(J^{[n]}) \) denote the space of real-valued discrete functions \( F : J^{[n]} \to \mathbb{R} \). For \( 0 \leq \alpha \leq q \), the \( \alpha \)th discrete Euler operator is the differential-difference operator

\[ E^d_\alpha : \mathcal{F}(J^{[n]}) \to \mathcal{F}(J^{[2n]}_{-n,n}) \]

given by

\[ E^d_\alpha = \sum_{0 \leq \ell \leq n} S^-\ell \frac{\partial}{\partial z^{\alpha}_{k+\ell}}. \]

**Theorem 3.5.** If \( z_k \) is an extremal of the discrete functional \( \mathcal{L}^d[z] = \sum_k L_k \), then it must be a solution of the discrete Euler–Lagrange equations

\[ E^d_\alpha(L_k) = 0, \quad \alpha = 0, \ldots, q. \]

Now let \( G \) be a Lie group acting on \( z_k \). The prolonged action to \( z^{[n]}_k \) is given by the product action

\[ g \cdot z^{[n]}_k = (k, \ldots, g \cdot z_{k+\ell} \ldots). \]

We note that the Lie group \( G \) does not act in the discrete variable \( k \in \mathbb{Z} \). Thus, the action is well-defined on each fiber \( \pi_n^{-1}(k) = J^{[n]}_{n_1,n_2}|_k \).
Definition 3.6. A Lie group of transformations $G$ is said to be a **variational symmetry group** of the discrete functional $L_d[z] = \sum_k L(z_k^{[n]})$ if and only if

$$g \cdot L(z_k^{[n]}) = L(g \cdot z_k^{[n]}) = L(z_k^{[n]}).$$

At the infinitesimal level, let

$$v_\nu = \sum_{\alpha, k} Q_{\nu, k}^\alpha \frac{\partial}{\partial z_k^\alpha} = \sum_{\alpha, k} Q^\alpha(k, z_k) \frac{\partial}{\partial z_k^\alpha}, \quad \nu = 1, \ldots, r,$$

be a basis for the Lie algebra of infinitesimal generators of the group action. Then $G$ is a variational symmetry group of $L_d[z] = \sum_k L_k$ if and only if

$$\text{pr} v(L_k) = \sum_{\alpha, \ell} Q_{\nu, k+\ell}^\alpha \frac{\partial L_k}{\partial z_{k+\ell}^\alpha} = 0.$$  

As in the continuous setting, Noether’s Theorem still holds in the discrete setting, and each infinitesimal generator yields a conserved quantity.

Definition 3.7. Let $F_k = F(z_k^{[n]}) = 0$ be a system of finite difference equations. A **conserved quantity** is a difference function $C_k = C(z_k^{[n]})$ such that

$$\Delta(C_k) = 0 \quad \text{on all solutions of} \quad F_k = 0.$$

Theorem 3.8. Let $L_d[z] = \sum_k L(z_k^{[n]})$ be a first order discrete Lagrangian with variational symmetry generator

$$v = \sum_{\alpha=0}^q Q^\alpha_k \frac{\partial}{\partial z_k^\alpha}.$$  

Then

$$C_k = \sum_{\alpha=0}^q Q^\alpha_k \frac{\partial L_{k-1}}{\partial z_k^\alpha}$$

is a conserved quantity.

Proof. Since $v$ is a variational symmetry of $L_d[z]$,

$$0 = \text{pr} v(L_k) = \sum_{\alpha=0}^q Q^\alpha_k \frac{\partial L_k}{\partial z_k^\alpha} + Q^{k+1}_k \frac{\partial L_k}{\partial z_{k+1}^\alpha}$$

$$= \sum_{\alpha=0}^q \left[ Q^\alpha_k \frac{\partial L_k}{\partial z_k^\alpha} + S(Q^\alpha_k \frac{\partial L_{k-1}}{\partial z_k^\alpha}) \right]$$

$$= \sum_{\alpha=0}^q \left[ Q^\alpha_k \frac{\partial L_k}{\partial z_k^\alpha} + \Delta \left( Q^\alpha_k \frac{\partial L_{k-1}}{\partial z_k^\alpha} + Q^\alpha_k \frac{\partial L_{k-1}}{\partial z_k^\alpha} \right) \right]$$

$$= \sum_{\alpha=0}^q Q^\alpha_k E^d_{\alpha}(L_k) + \Delta \left( \sum_{\alpha=0}^q Q^\alpha_k \frac{\partial L_{k-1}}{\partial z_k^\alpha} \right).$$

Since $E^d_{\alpha}(L_k) = 0$, $\alpha = 0, \ldots, q$, the result follows. □
Theorem 3.9. Let \( \mathcal{L}^d[z] = \sum_k L(z_k^{[2]}) \) be second order discrete Lagrangian with variational symmetry generator (20). Then

\[
C^d = \sum_{\alpha=0}^q \left[ Q^\alpha_k \frac{\partial L_{k-1}}{\partial z_k^\alpha} + Q^\alpha_k \frac{\partial L_{k-2}}{\partial z_k^\alpha} + Q^\alpha_{k+1} \frac{\partial L_{k-1}}{\partial z_{k+1}^\alpha} \right]
\]

is a conserved quantity.

Remark 3.10. As in Definition 2.15, we can also introduce the notion of divergence symmetry in the discrete setting. This more general notion of symmetry will not be used here since, as we have seen in the previous section, every divergence symmetry can be made into a variational symmetry by modifying the Lagrangian.

As outlined in Section 1, given a continuous Lagrangian functional \( \mathcal{L}[u] \) with variational symmetry group \( G \), our goal is to construct a discrete Lagrangian \( \mathcal{L}^d[u] \) that will remain invariant under the action of \( G \). As the next example shows, in general, a standard discretization of \( \mathcal{L}[u] \) will not preserve its symmetries.

Example 3.11. In an attempt to discretize the Euler elastica Lagrangian (4), consider the discrete Lagrangian

\[
\mathcal{L}^d = \sum_k L_k = \sum_k \left( \frac{(u_x^d)^2}{2(1 + (u_x^d)^2)^{5/2}} \cdot \sqrt{\Delta x_k \Delta x_{k+1}} \right),
\]

where

\[
u^d_x = \frac{\Delta u_k}{\Delta x_k} = \frac{u_{k+1} - u_k}{x_{k+1} - x_k}, \quad u^d_{xx} = \frac{1}{\sqrt{\Delta x_k \Delta x_{k+1}}} \left[ \frac{\Delta u_{k+1}}{\Delta x_{k+1}} - \frac{\Delta u_k}{\Delta x_k} \right].
\]

One can verify that this discrete functional is invariant under translations, but not under rotations.

To construct a discrete Lagrangian functional \( \mathcal{L}^d[z] \) that will preserve the variational symmetries of a continuous Lagrangian \( \mathcal{L}[u] \), we use the method of equivariant moving frames.

4 Discrete Moving Frames

In this section we review the method of equivariant moving frames in the discrete setting. We refer the reader to [22, 24, 27] for a complete exposition of the method.

Let \( G \) be an \( r \)-dimensional Lie group acting on \( \mathbb{R}^{q+1} \), which is extended to \( J^n \) via the product action. In the following, we assume that the action of \( G \) on each fiber \( \pi_n^{-1}(k) \) is (locally) free and regular, [14]. Recall that a Lie group \( G \) acts freely on \( J^n|_k = \pi_n^{-1}(k) \) if for all \( z^n_k \in \pi_n^{-1}(k) \) the isotropy subgroup \( G_{z^n_k} = \{ g \in G \mid g \cdot z^n_k = z^n_k \} \) is trivial, i.e. \( G_{z^n_k} = \{ e \} \). The action is locally free if the isotropy subgroup \( G_{z^n_k} \) is discrete for all \( z^n_k \in \pi_n^{-1}(k) \). This is equivalent to the fact that the orbits of the product group action have the same dimension as the group \( G \). By a result of Boutin, [11], when the action of \( G \) is (locally) effective on subsets of \( \mathbb{R}^{q+1} \), local freeness on an open subset of \( \pi_n^{-1}(k) \) can always be achieved for a sufficiently large and finite \( n \). Finally, the action is regular if the orbits form a regular foliation. When the action of \( G \) on each fiber \( \pi_n^{-1}(k) \) is (locally) free and regular, we say that \( G \) acts (locally) freely and regularly on \( J^n \).
Definition 4.1. Let $G$ act (locally) freely and regularly on $J^{[n]}$. A discrete (right) moving frame is a $G$-equivariant map $\rho : J^{[n]} \to G$ satisfying
\[
\rho(g \cdot z^{[n]}_k) = \rho(z^{[n]}_k) g^{-1},
\] (23)
for all $g \in G$ where the product action is defined.

To simplify the notation, we let $\rho_k = \rho(z^{[n]}_k)$ denote the moving frame $\rho$ evaluated at the discrete jet $z^{[n]}_k$. In applications the construction of a (discrete) moving frame relies on the choice of a (discrete) cross-section $K \subset J^{[n]}$ to the group orbits.

Definition 4.2. A subset $K \subset J^{[n]}$ is a discrete cross-section to the group orbits if for each $k \in \mathbb{Z}$, the restriction $K|_k \subset J^{[n]}|_k = \pi_{\mathbf{n}}^{-1}(k)$ is a submanifold of $J^{[n]}|_k$ transverse and of complementary dimension to the group orbits.

In general, a cross-section $K \subset J^{[n]}$ is specified by a system of $r = \dim G$ difference equations
\[
K = \{ E_\nu(z^{[n]}_n) = 0 \mid \nu = 1, \ldots, r \}.
\]

Once $K$ is fixed, the right moving frame at $z^{[n]}_k$ is the unique group element $g = \rho_k = \rho(z^{[n]}_k) \in G$ that sends $z^{[n]}_k$ onto $K|_k$. That is
\[
\rho_k \cdot z^{[n]}_k \in K|_k.
\]

The coordinate expressions for the moving frame $\rho_k$ are obtained by solving the normalization equations
\[
E_\nu(g \cdot z^{[n]}_k) = 0, \quad \nu = 1, \ldots, r,
\]
for the group parameters $g = (g^1, \ldots, g^r)$.

With a moving frame in hand, there is a systematic procedure, known as invariantization, for constructing joint invariants (also called discrete invariants or difference invariants).

Definition 4.3. The invariantization of the difference function $F(z^{[n]}_k)$ is the joint invariant
\[
\iota_k(F)(z^{[n]}_k) = F(\rho_k \cdot z^{[n]}_k).
\] (24)

The fact that the function in (24) is invariant follows from the $G$-equivariant property (23) that the right moving frame $\rho_k$ satisfies. The operator $\iota_k$ is called the invariantization map (with respect to $\rho_k$).

Thus, given a discrete Lagrangian functional $\mathcal{L}^d[z] = \sum_k L_k$ we can obtain a symmetry-preserving functional by invariantizing $\mathcal{L}^d[z]$: $\iota(\mathcal{L}^d[z]) = \sum_k \iota_k(L_k)$. 

13
Example 4.4. Consider the special Euclidean group action (1) acting on \( z_k = (x_k, u_k) \):
\[
X_k = x_k \cos \varphi - u_k \sin \varphi + a, \quad U_k = x_k \sin \varphi + u_k \cos \varphi + b.
\]
A moving frame is obtained by selecting the cross-section
\[
\mathcal{K} = \{ x_k = u_k = u_{k+1} = 0 \}.
\]
We observe that this cross-section is equivalent to
\[
\mathcal{K} = \{ x_k = u_k = u^d_x = 0 \}, \quad \text{where} \quad u^d_x = \frac{\Delta u_k}{\Delta x_k},
\]
the latter being a discrete approximation of the cross-section used in the continuous setting, [19]. Solving the normalization equations \( X_k = U_k = U_{k+1} = 0 \) for the group parameters \( a, b, \varphi \), we obtain
\[
a = -\frac{x_k \Delta x_k + u_k \Delta u_k}{\ell_k}, \quad b = \frac{x_k \Delta u_k - u_k \Delta x_k}{\ell_k}, \quad \varphi = -\tan^{-1} \left( \frac{\Delta u_k}{\Delta x_k} \right),
\]
where
\[
\ell_k = \sqrt{\Delta x_k^2 + \Delta u_k^2}.
\]
Using the invariantization map (24) we have that
\[
\iota_k(\Delta x_k) = \ell_k \quad \text{and} \quad \iota_k(\Delta u_{k+1}) = \frac{D_k}{\ell_k},
\]
\[
\text{where} \quad D_k = \det \begin{bmatrix} \Delta x_k & \Delta x_{k+1} \\ \Delta u_k & \Delta u_{k+1} \end{bmatrix}.
\]
In the literature, and as in Definition 4.3, it is customary to invariantize a discrete function \( F(z^i_k) \) with respect to \( \iota_k \) solely. In the following we expand this practice by using \( \iota_k \) and \( \iota_{k+1} \) simultaneously. For example, we invariantize \( u^d_{xx} \) given in (22) as follows
\[
\iota(u^d_{xx}) := \frac{1}{\sqrt{\ell_k(\Delta x_k)(\iota_{k+1}(\Delta x_{k+1}) - \iota_k(\Delta x_k))}} \left[ \frac{\iota_k(\Delta u_{k+1})}{\iota_{k+1}(\Delta x_{k+1})} - \frac{\iota_k(\Delta u_k)}{\iota_k(\Delta x_k)} \right] = \frac{1}{\sqrt{\ell_k \ell_{k+1}}} \frac{D_k}{\ell_k \ell_{k+1}} = \frac{D_k}{(\ell_k \ell_{k+1})^{3/2}}.
\]
We also invariantize \( \Delta x_{k+1} \) using \( \iota_{k+1}(\Delta x_{k+1}) = \ell_{k+1} \). Invariantizing the discrete Lagrangian functional (21), we obtain
\[
L^i_k = \iota(L_k) = \frac{D_k^2}{2(\ell_k \ell_{k+1})^{5/2}}.
\]
Computing the corresponding discrete Euler–Lagrange equations yields
\[
0 = E^d_x(L^i_k) = -\frac{D_{k-2} \Delta u_{k-2}}{(\ell_{k-2} \ell_{k-1})^{5/2}} + \frac{D_{k-1} \Delta u_{k-1}}{(\ell_{k-1} \ell_k)^{5/2}} - \frac{D_k \Delta u_k}{(\ell_k \ell_{k+1})^{5/2}} - \frac{5D_{k-2}^2 \ell_{k-2} \Delta x_{k-1}}{4 \ell_{k-1} (\ell_{k-2} \ell_{k-1})^{7/2}} + \frac{5D_{k-1}^2 \ell_{k-1} \Delta x_{k-1}}{4 \ell_{k-1} (\ell_{k-1} \ell_k)^{7/2}} + \frac{5D_k^2 \ell_{k+1} \Delta x_k}{4 \ell_k (\ell_k \ell_{k+1})^{7/2}},
\]
\[
0 = E^d_u(L^i_k) = \frac{D_{k-2} \Delta x_{k-2}}{(\ell_{k-2} \ell_{k-1})^{5/2}} - \frac{D_{k-1} \Delta x_{k-1}}{(\ell_{k-1} \ell_k)^{5/2}} + \frac{D_k \Delta x_k}{(\ell_k \ell_{k+1})^{5/2}} - \frac{5D_{k-2}^2 \ell_{k-2} \Delta u_{k-1}}{4 \ell_{k-1} (\ell_{k-2} \ell_{k-1})^{7/2}} + \frac{5D_{k-1}^2 \ell_{k-1} \Delta u_{k-1}}{4 \ell_{k-1} (\ell_{k-1} \ell_k)^{7/2}} + \frac{5D_k^2 \ell_{k+1} \Delta u_k}{4 \ell_k (\ell_k \ell_{k+1})^{7/2}}.
\]
Since the discrete Lagrangian (26) is invariant under the special Euclidean group action, Noether’s Theorem applies. Using Theorem 3.9, and recalling the infinitesimal generators (2), we obtain the conserved quantities

\[ C_1^d = \frac{\partial L_{k-1}}{\partial x_k} + \frac{\partial L_k}{\partial x_k} + \frac{\partial L_{k+1}}{\partial x_{k+1}} \]

\[ = \frac{D_{k-1} u_k}{(\ell_{k-1} \ell_k)^{5/2}} - \frac{D_{k-2} u_{k-2}}{(\ell_{k-2} \ell_{k-1})^{5/2}} - \frac{5D_{k-1}^2 \ell_k \Delta x_{k-1}}{4\ell_{k-1}(\ell_{k-1} \ell_k)^{7/2}} - \frac{5D_{k-2}^2 \ell_{k-2} \Delta x_{k-1}}{4\ell_{k-1}(\ell_{k-2} \ell_{k-1})^{7/2}} \]

\[ C_2^d = \frac{\partial L_{k-1}}{\partial u_k} + \frac{\partial L_k}{\partial u_k} + \frac{\partial L_{k+1}}{\partial u_{k+1}} \]

\[ = \frac{D_{k-2} \Delta x_{k-2}}{(\ell_{k-2} \ell_{k-1})^{5/2}} - \frac{D_{k-1} \Delta x_k}{(\ell_{k-1} \ell_k)^{5/2}} - \frac{5D_{k-1}^2 \ell_k \Delta u_{k-1}}{4\ell_{k-1}(\ell_{k-1} \ell_k)^{7/2}} - \frac{5D_{k-2}^2 \ell_{k-2} \Delta u_{k-1}}{4\ell_{k-1}(\ell_{k-2} \ell_{k-1})^{7/2}} \]

\[ C_3^d = -u_k \frac{\partial L_{k-1}}{\partial x_k} + x_k \frac{\partial L_k}{\partial u_k} - u_k \frac{\partial L_{k-2}}{\partial u_k} + x_k \frac{\partial L_{k-2}}{\partial u_k} - u_k \frac{\partial L_{k-1}}{\partial x_{k+1}} + x_k \frac{\partial L_{k-1}}{\partial u_{k+1}} \]

\[ = x_k C_2^d - u_k C_1^d + \frac{D_{k-1}}{(\ell_{k-1} \ell_k)^{5/2}} (\Delta x_{k-1} \Delta x_k + \Delta u_{k-1} \Delta u_k). \]

5 Invariant Variational Schemes

Given a continuous Lagrangian functional \( L[u] \), with Euler–Lagrange equations \( E(L) = 0 \), we now describe a procedure for constructing a numerical scheme that will preserve its variational symmetries and thereby be exactly conservative.

1. Let \( L[u] = \int L(x, u^{(n)}) \, dx \) be a Lagrangian functional with variational symmetry group \( G \), and let \( E(L) = 0 \) be the corresponding Euler–Lagrange equations.

2. Introduce a discrete Lagrangian functional \( L^d[z] = \sum_k L(z_k^{[n]}) \), whose continuous limit is \( L[u] \). In general \( L^d[z] \) will not be invariant under the product action of \( G \).

3. Assuming the product action is (locally) free and regular on \( J^{[n]} \), construct a discrete moving frame. As outlined in [9], and proved for curves in [26] and generalized in [24], for the discrete moving frame to have a well defined continuous limit, i.e. for the moving frame to converge to a differential moving frame and the discrete invariant Lagrangian and Euler–Lagrange equations to converge to their invariant differential counterparts, use a cross-section involving finite difference approximations of derivatives such as in (22).

4. Invariantize the discrete Lagrangian \( L^d[z] = \sum_k L_k \) introduced in step 2 using the moving frame constructed in step 3.

5. Compute the Euler–Lagrange equations \( E^d(\iota(L_k)) = 0 \) of the invariantized Lagrangian \( \iota(L_k) \). These provide a numerical scheme approximating \( E(L) = 0 \) that preserve the variational symmetry group \( G \). By Noether’s Theorem, the numerical scheme \( E^d(\iota(L_k)) = 0 \) also conserves the associated conserved quantities.
If the Lagrangian functional $\mathcal{L}[u] = \int L(x, u^{(n)}) \, dx$ admits a divergence symmetry group, the above steps still apply provided $\mathcal{L}[u]$ is replaced by the modified functional $\overline{\mathcal{L}}[u] = \int \overline{\mathcal{L}} \, dx = \int (L + \zeta_x) \, dx$ as described in Section 2.2.

**Example 5.1.** To show how the above procedure works for a Lagrangian admitting a divergence symmetry group, let us continue Example 2.18. Starting from the modified Lagrangian functional (17), a possible discretization of $\overline{\mathcal{L}}[u]$ is

$$\overline{\mathcal{L}}^d = \sum_k \left[ \left( \frac{\Delta u_k}{\Delta x_k} \right)^2 - \frac{1}{u_k^2} + \frac{\Delta \zeta_k}{\Delta x_k} \right] \Delta x_k = \sum_k \frac{(\Delta u_k)^2}{\Delta x_k} - \frac{\Delta x_k}{u_k^2} + \Delta \zeta_k. \tag{28}$$

This Lagrangian is not invariant under the product action

$$X_k = \frac{\alpha x_k + \beta}{\delta x_k + \gamma}, \quad U_k = \frac{u_k}{\delta x_k + \gamma}, \quad g \cdot \zeta_k = \zeta_k + \frac{\delta u_k^2}{\delta x_k + \gamma}, \quad \alpha \gamma - \beta \delta = 1.$$ 

To obtain a symmetry-preserving Lagrangian, we construct a moving frame. Consider the cross-section

$$\mathcal{K} = \{ x_k = 0, u_k = u_{k+1} = 1 \},$$

which is equivalent to $\mathcal{K} = \{ x_k = 0, u_k = 1, u^d_x = \frac{\Delta u_k}{\Delta x_k} = 0 \}$. Solving the normalization equations $X_k = 0, U_k = U_{k+1} = 1$, we obtain the moving frame

$$\alpha = \frac{1}{u_k}, \quad \beta = -\frac{x_k}{u_k}, \quad \delta = \frac{\Delta u_k}{\Delta x_k}, \quad \gamma = \frac{u_k \Delta x_k - x_k \Delta u_k}{\Delta x_k}. \tag{29}$$

Invariantizing (28)

$$\mathcal{L}_k(\overline{\mathcal{L}}^d) = u_k(\overline{\mathcal{L}}^d) = \sum_k \frac{\Delta u_k^2}{\Delta x_k} - \frac{\Delta x_k}{u_k u_{k+1}} + \Delta \zeta_k.$$ 

The corresponding Euler–Lagrange equations are

$$0 = \mathbf{E}_u^d(\mathcal{L}(\overline{\mathcal{L}}^d)) = -2 \left( \frac{\Delta u_k}{\Delta x_k} - \frac{\Delta u_{k-1}}{\Delta x_{k-1}} \right) + \frac{1}{u_k^2} \left( \frac{\Delta x_k}{u_{k+1}} + \frac{\Delta x_{k-1}}{u_{k-1}} \right),$$

$$0 = \mathbf{E}_x^d(\mathcal{L}(\overline{\mathcal{L}}^d)) = \left( \frac{\Delta u_k}{\Delta x_k} \right)^2 - \left( \frac{\Delta u_{k-1}}{\Delta x_{k-1}} \right)^2 + \frac{1}{u_k} \left( \frac{1}{u_{k+1}} - \frac{1}{u_{k-1}} \right). \tag{30}$$

Applying Noether’s Theorem 3.8, with the infinitesimal generators (15), we obtain the conserved quantities

$$C_1^d = (u_x^d)^2 + \frac{1}{u_k u_{k+1}},$$

$$C_2^d = \frac{x_k + x_{k+1}}{u_k u_{k+1}} + 2 u_x^d \cdot \frac{u_{k+1} x_k - x_{k+1} u_k}{\Delta x_k},$$

$$C_3^d = \frac{x_k x_{k+1}}{u_k u_{k+1}} + \frac{(u_{k+1} x_k - x_{k+1} u_k)^2}{(\Delta x_k)^2}. \tag{31}$$

These conserved quantities are independent and satisfy

$$\frac{(C_2^d)^2}{4} - C_1^d C_3^d + 1 = \frac{1}{4} \left( \frac{\Delta x_k}{u_k u_{k+1}} \right)^2. \tag{32}$$

We note that, in the continuous limit, the equality (32) converges to (16).
6 Numerical Simulations

In this section we conduct numerical tests for the invariant variational schemes (27) and (30). We also consider a version of (27) where the distance between points is constant. Computations were performed using nonlinear solvers from scipy’s optimize module. In particular, in Section 6.1 we use root with the Jacobian given analytically, while in Section 6.2 we use fsolve. All nonlinear systems are solved up to an absolute tolerance of $10^{-13}$. For a numerical approximation $u_k$, simulating an exact solution $u = u(x_k)$, we will examine the error in the $l_\infty$ norm using the formula

$$
\|u_k - u\|_{l_\infty} := \max_j \left( |u_j - u(x_j)| \right).
$$

(33)

In addition, when benchmarking our simulations we use the following definition.

Definition 6.1. Given two sequences $a(i), b(i)$, the experimental order of convergence (EOC) is described by

$$
\text{EOC}(a, b; i) = \frac{\log \left( \frac{a(i+1)}{a(i)} \right)}{\log \left( \frac{b(i+1)}{b(i)} \right)}.
$$

In the sequel $a(i)$ represents a sequence of $l_\infty$ errors given by (33), while $b(i)$ represents either a step size type parameter or the reciprocal of the number of steps taken.

6.1 Euler Elastica

From the perspective of symmetry, invariants, and moving frames, the free Euler elastica equation (7) was previously considered in [23]. Using an approach inspired by the group foliation method, [31], the SE(2) invariance of the Euler–Lagrange equations implies that these equations can be re-expressed in terms of discrete curvature, the arc-length function $\ell_k = \sqrt{\Delta x_k^2 + \Delta u_k^2}$ and their shifts. Solving the Euler–Lagrange equations for these two invariants, the solution $z_k = (x_k, u_k)$ to the original problem is found via a “reconstruction” process requiring the solution of a system of finite difference equations for the (left) moving frame. In this paper we omit this two step process and solve the Euler–Lagrange equations directly for $z_k = (x_k, u_k)$. From a numerical perspective, it is not a priori clear if the more involved approach used in [23] gives better results. On the other hand, the approach introduced in this paper is, we believe, more straightforward to implement.

The parametrized solution to the Euler elastica equation (7) is

$$
x(s) = \sqrt{\frac{2}{\alpha}} E\left( \text{am}\left( \sqrt{\frac{\alpha}{2}} s, -1 \right) - 1 \right) - s,
$$

$$
u(s) = \sqrt{\frac{2}{\alpha}} \text{sn}\left( \sqrt{\frac{\alpha}{2}} s, -1 \right),
$$

(34)

where sn($u, k$) is the Jacobian elliptic sine function, $E(u, k)$ is the incomplete elliptic integral of the second kind, and am($t, k$) is the Jacobian amplitude function. At the discrete level, the Euler–Lagrange equations (27) provide a nonlinear system of two equations for the unknown $z_{k+2} = (x_{k+2}, u_{k+2})$. Once the initial conditions $z_0, z_1, z_2, z_3$ are fixed using (34), the numerical solution evolves according to (27) and there is no way to control the distance between...
consecutive points, which is generally not numerically desirable. For small values of $\ell_{k-2}$, $\ldots$, $\ell_{k+1}$, the denominators occurring in the Euler–Lagrange equations are very small, to the point of round-off errors dominating the numerical solution when using standard double precision arithmetic. Therefore, to implement (27) we multiplied the Euler–Lagrange equations by $(\ell_{k-1}\ell_k)^{5/2}$ to obtain the scaled equations

$$
0 = \tilde{E}_x^d = -D_{k-2}\Delta u_{k-2} \left(\frac{\ell_k}{\ell_{k-2}}\right)^{5/2} + D_{k-1}(\Delta u_{k-1} + \Delta u_k) - D_k\Delta u_{k+1} \left(\frac{\ell_{k-1}}{\ell_{k+1}}\right)^{5/2},
$$

$$
0 = \tilde{E}_u^d = D_{k-2}\Delta x_{k-2} \left(\frac{\ell_k}{\ell_{k-2}}\right)^{5/2} - D_{k-1}(\Delta x_{k-1} + \Delta x_k) + D_k\Delta x_{k+1} \left(\frac{\ell_{k-1}}{\ell_{k+1}}\right)^{5/2},
$$

(35)

Supplying the Jacobian entries

$$
\frac{\partial \tilde{E}_x^d}{\partial x_{k+2}} = \left(\frac{\ell_{k-1}}{\ell_{k+1}}\right)^{5/2} \left(\frac{\Delta u_k \Delta u_{k+1}}{2\ell_k^2} - \frac{5D_k \Delta u_k \Delta x_k}{2\ell_k^2} + \frac{5D_k \Delta u_{k+1} \Delta x_{k+1}}{2\ell_{k+1}^2} - \frac{25D_k^2 \Delta x_k \Delta x_{k+1}}{8\ell_k^2 \ell_{k+1}^2}\right),
$$

$$
\frac{\partial \tilde{E}_x^d}{\partial u_{k+2}} = \left(\frac{\ell_{k-1}}{\ell_{k+1}}\right)^{5/2} \left(-D_k - \Delta u_{k+1} \Delta x_k + \frac{5D_k \Delta u_k^2}{2\ell_k^2} + \frac{5D_k \Delta u_{k+1}^2}{2\ell_{k+1}^2} - \frac{25D_k^2 \Delta x_k \Delta x_{k+1}}{8\ell_k^2 \ell_{k+1}^2}\right),
$$

$$
\frac{\partial \tilde{E}_u^d}{\partial x_{k+2}} = \left(\frac{\ell_{k-1}}{\ell_{k+1}}\right)^{5/2} \left(D_k - \Delta u_k \Delta x_{k+1} - \frac{5D_k \Delta u_k^2}{2\ell_k^2} - \frac{5D_k \Delta x_{k+1}^2}{2\ell_{k+1}^2} - \frac{25D_k^2 \Delta u_k \Delta x_{k+1}}{8\ell_k^2 \ell_{k+1}^2}\right),
$$

$$
\frac{\partial \tilde{E}_u^d}{\partial u_{k+2}} = \left(\frac{\ell_{k-1}}{\ell_{k+1}}\right)^{5/2} \left(\Delta x_k \Delta x_{k+1} + \frac{5D_k \Delta u_k \Delta x_k}{2\ell_k^2} - \frac{5D_k \Delta u_{k+1} \Delta x_{k+1}}{2\ell_{k+1}^2} - \frac{25D_k^2 \Delta u_k \Delta x_{k+1}}{8\ell_k^2 \ell_{k+1}^2}\right),
$$

to root in *scipy.optimize* yields an ill-conditioned problem. To improve the conditioning of the Jacobian matrix, we added to it a small constant multiple of the identity matrix. In our simulations this constant is $10^{-3}$, and we note that the specific choice of this constant depends heavily on $\ell_k$. To initialize the scheme we fixed

$$
\ell_0 = \ell_1 = \ell_2 = 0.01,
$$

(36)

and set $s_0 = -2$ in the exact solution (34). Substituting the exact solution in (36) we solved for $s_1 < s_2 < s_3$ in order to obtain the initial conditions $z_k = (x(s_k), u(s_k)), k = 0, 1, 2, 3$. Running the simulation for 500 steps we obtain Figure 1, which we compare against an exact solution where we assume that $\ell_k$ remains uniform for 500 steps. We observe that our numerical simulation is qualitatively accurate, although the numerical solution and the exact solution with uniform $\ell_k$ evolve at slightly different rates.

To improve on the previous results, and to control the length between neighboring points, we now consider the constrained invariant Lagrangian

$$
L_k = \ell(L_k) + \lambda\left(\sqrt{\Delta x_k^2 + \Delta u_k^2} - \ell\right),
$$

(37)
where \( \ell > 0 \) is a positive constant, \( \lambda \) is a Lagrange multiplier, and \( \ell(L_k) \) is given in (26). We note that since \( \sqrt{\Delta x_k^2 + \Delta u_k^2} - \ell \) is invariant under translations and rotations, the constrained Lagrangian (37) is \( \text{SE}(2) \) invariant. After the multiplication by \( \ell^5 \), to avoid small denominators, the resulting Euler–Lagrange equations are

\[
0 = \ell^5 E^d_x(L_k) = -D_{k-2} \Delta u_{k-2} + D_{k-1} (\Delta u_{k-1} + \Delta u_k) - D_k \Delta u_{k+1} \\
+ \frac{5}{4\ell^2} \left[ -D_{k-2}^2 \Delta x_{k-1} + D_{k-1}^2 (\Delta x_k - \Delta x_{k-1}) + D_k^2 \Delta x_k \right] - \alpha \mu \ell^4 (\Delta x_{k-1} - \Delta x_k),
\]

(38)

where we made the substitution \( \lambda = -\alpha \mu \). In the continuous limit, the equations (38) converge to

\[
-\mu_s \left( \kappa_{ss} + \frac{\kappa^3}{2} + \alpha \mu \kappa \right) = 0, \quad x_s \left( \kappa_{ss} + \frac{\kappa^3}{2} + \alpha \mu \kappa \right) = 0,
\]

(39)

respectively. Therefore, the difference equations (38) provide an approximation of the general Euler elastica equation

\[
\kappa_{ss} + \frac{\kappa^3}{2} + \alpha \mu \kappa = 0.
\]

(40)

The solution to this ordinary differential equation depends on the value of \( \mu \), [12]. Some of our solutions differ from those appearing in [12], but have been checked with Mathematica to indeed satisfy the Euler elastica equation:

\[\mu \in (-1, 1): \text{Let } a = \sqrt{\frac{2(1 - \mu)}{\alpha}} \text{ and } c = \sqrt{\frac{2(1 + \mu)}{\alpha}}, \text{ then}
\]

\[
x(s) = c E \left( \text{am} \left( \frac{c \alpha}{2} s, -\frac{a^2}{c^2} \right), -\frac{a^2}{c^2} \right) - s, \quad u(s) = a \text{sn} \left( \frac{c \alpha}{2} s, -\frac{a^2}{c^2} \right),
\]

(41)
where \( \text{sn}(u, k) \) is the Jacobian elliptic sine function, \( E(u, k) \) is the incomplete elliptic integral of the second kind, and \( \text{am}(t, k) \) is the Jacobian amplitude function.

\[ \mu = -1: \] The solution is
\[
x(s) = \frac{2 \tanh(\sqrt{\alpha}s)}{\sqrt{\alpha}} - s, \quad u(s) = \frac{2 \text{sech}(\sqrt{\alpha}s)}{\sqrt{\alpha}}.
\] \hfill (42)

\[ \mu < -1: \] Let \( a = \sqrt{\frac{2(1-\mu)}{\alpha}} \) and \( c = \sqrt{\frac{-2(1+\mu)}{\alpha}} \), then
\[
x(s) = cE\left(\text{am}\left(\frac{c\alpha}{2}, 1 - \frac{a^2}{c^2}\right), 1 - \frac{a^2}{c^2}\right) + \mu s, \quad u(s) = c \text{dn}\left(\frac{c\alpha}{2}, 1 - \frac{a^2}{c^2}\right),
\] \hfill (43)
where \( \text{dn} \) is the delta amplitude function.

For the numerical implementation of (38), we note that the two equations are equivalent. This can be seen by expressing the two equations in the polar coordinates
\[
\Delta x_k = \ell \cos \theta_k, \quad \Delta u_k = -\ell \sin \theta_k.
\]
One then finds that
\[
E_x^d(L_k^c)[\theta_k + \frac{\pi}{2}] = E_u^d(L_k^c)[\theta_k].
\]
To decide which equation from (38) to choose, we consider their continuous limit (39) and note that when \( \Delta u_k \approx u_s \) is close to zero, the first equation in (39) almost vanishes. Similarly, when \( \Delta x_k \approx x_s \) is close to zero, the second equation in (39) almost vanishes. Thus, our code for the implementation of the scheme follows Algorithm 1.

**Algorithm 1** Constrained Lagrangian Implementation

```
if \(|\Delta x_k| < |\Delta u_k|\) then
    Solve \( \{\ell^5 E_x^d(L_k^c) = 0, \Delta x_k^2 + \Delta u_k^2 - \ell^2 = 0\} \) for \((x_{k+2}, u_{k+2})\).
else
    Solve \( \{\ell^5 E_u^d(L_k^c) = 0, \Delta x_k^2 + \Delta u_k^2 - \ell^2 = 0\} \) for \((x_{k+2}, u_{k+2})\).
end if
```

Notice that once the Euler–Lagrange equation is selected, the second equation used is always \( \Delta x_k^2 + \Delta u_k^2 - \ell^2 = 0 \) to guarantee that the distance between points is constant. Before sharing our numerical results, we note that the conserved quantities for the Euler–Lagrange equations (38) are

\[
C_1^d = -\alpha \mu \ell^4 \Delta x_{k-1} + D_{k-1} \Delta u_k - D_{k-2} \Delta u_{k-2} - \frac{5D_{k-1}^2 \Delta x_{k-1}}{4\ell^2} - \frac{5D_{k-2}^2 \Delta x_{k-1}}{4\ell^2},
\]
\[
C_2^d = -\alpha \mu \ell^4 \Delta u_{k-1} + D_{k-2} \Delta x_{k-2} - D_{k-1} \Delta x_k - \frac{5D_{k-1}^2 \Delta u_{k-1}}{4\ell^2} - \frac{5D_{k-2}^2 \Delta u_{k-1}}{4\ell^2},
\]
\[
C_3^d = x_k C_2^d - u_k C_1^d + D_{k-1}(\Delta x_k \Delta x_{k-1} + \Delta u_k \Delta u_{k-1}).
\] \hfill (44)
We begin by benchmarking our scheme against the exact solution where \( \mu = -1 \) and \( \alpha = 4 \). In this case the exact solution is given by (42) and it forms a single loop centered at \( x = 0 \) with \( u \to 0 \) as \( x \to \pm \infty \). We initialize the scheme the same way we did for (35) using \( s_0 = -2 \) and the fixed value of \( \ell \) to determine the initial data \( z_0, z_1, z_2, z_3 \) through the exact solution. While benchmarking we decrease the length \( \ell \) and increase the number of steps proportionally to fix the domain of the simulation. Furthermore, we measure the \( l_\infty \) error of both of \( x \)- and \( u \)-components of the solution. We observe in Table 1 that, experimentally, our scheme is second order. This result is interesting as the moving frame (25) used to construct the invariant variational scheme is a first order approximation of its continuous counterpart, [19], and the non-invariant Lagrangian (21) is also a first order approximation of (4). This gain in the order of convergence obtained by invariantizing a numerical scheme has also been observed in [16,17].

| \( i \) | \( \ell \) | steps | \( \| x_k - x \|_\infty =: e_x \) | EOC \((e_x, \ell; i - 1)\) | \( \| u_k - u \|_\infty =: e_u \) | EOC \((e_u, \ell; i - 1)\) |
|---|---|---|---|---|---|---|
| 1 | 0.02 | 200 | 2.98e-3 | 9.21 | 1.40e-2 | 6.15 |
| 2 | 0.01 | 400 | 7.14e-4 | 2.03 | 3.41e-3 | 1.98 |
| 3 | 0.005 | 800 | 1.74e-4 | 8.33e-4 | 1.98 |
| 4 | 0.0025 | 1600 | 4.24e-5 | 2.02 | 2.05e-4 | 1.97 |

Table 1: The \( l_\infty \) error and order of convergence for the invariant variational scheme (38) for various values of \( \ell \), and where the number of steps is \( 2^i \), subject to the exact solution (42) with \( \mu = -1 \) and \( \alpha = 4 \).

Remark 6.2. One might observe that the error for \( i = 0 \) is not displayed in Table 1. If desired, this quantity can be recovered through Definition 6.1 after noting that for \( i = 0 \) we fixed \( \ell = 0.04 \).

For sake of comparison, we also consider the invariant scheme

\[
\kappa_{ss}^d + \frac{(\kappa_{ss}^d)^3}{2\ell^4} + \mu \alpha \ell^2 \kappa^d = 0, \tag{45}
\]

where, up to factors of \( \ell \),

\[
\kappa^d = \Delta^2 u_k \Delta x_k - \Delta^2 x_k \Delta u_k, \quad \kappa_{ss}^d = \Delta^4 u_{k-2} \Delta x_{k} + \Delta^3 u_{k-1} \Delta^2 x_{k-1} \Delta^2 u_{k-1} - \Delta^3 x_{k-2} \Delta u_k - \Delta^3 x_{k-1} \Delta^2 u_k,
\]

are approximations of the curvature and its second arc-length derivative with

\[
\Delta z_k = z_{k+1} - z_k, \quad \Delta^2 z_k = z_{k+2} - 2z_{k+1} + z_k, \\
\Delta^3 z_{k-1} = z_{k+2} - 3z_{k+1} + 3z_k - z_{k-1}, \quad \Delta^4 z_{k-2} = z_{k+2} - 4z_{k+1} + 6z_k - 4z_{k-1} + z_{k-2}.
\]

We note that (45) is a straightforward discretization of the Euler elastica equation (40) where the variational nature of the equation is omitted. Finally, we supplement (45) with the equation

\[
\Delta x_{k+1}^2 + \Delta u_{k+1}^2 = \ell^2
\]

to ensure the distance between points is constant.
In Figure 2 we observe that the invariant variational scheme (38) successfully completes the loop when $\mu = -1, \alpha = 4$, however, the invariant numerical scheme (45) fails to decay as $x$ decreases to $-\infty$.

![Figure 2: Numerical simulation of solution (42), where $\mu = -1$ and $\alpha = 4$, using both the invariant variational scheme (IVS) (38) and the invariant scheme (IS) (45) with $\ell = 0.01$ and the number of steps equal to 500.](image)

In Figure 3 we plot the deviation of the conserved quantities (44).

![Figure 3: The deviation in the conserved quantities $C^d_i$, $i = 1, 2, 3$, as described by (44), for the invariant variational scheme (38) and invariant scheme (45) simulating (42) with $\mu = -1$, $\alpha = 4$, and where $\ell = 0.01$.](image)

We observe that over time the deviation in the conservative scheme (38) propagates to $10^{-12}$ while the invariant scheme (45) propagates to $10^{-10}$. We note that this deviation remains small due to the order of magnitude of the conserved quantities themselves, however, we do not look at the relative deviation here as the errors propagating below solver precision become significant.
While the schemes (38) and (45) are both invariant under the special Euclidean group action, the above simulation shows that the scheme which is also variational, and therefore preserves the constants of motion, provides better long term numerical results.

For completeness, we also consider a non-invariant variational scheme obtained by computing the Euler–Lagrange equations of the non-invariant Lagrangian (21) subject to the constraint \[ \sqrt{\Delta x_k^2 + \Delta u_k^2} = \ell. \] The resulting equations are

\[
0 = \ell^5 E_u(L_k^c) = \left( D_k \Delta x_{k+1} + \frac{5D_k^2 \Delta u_k}{2\ell^2} \right) \left( \frac{\Delta x_k}{\Delta x_{k+1}} \right)^{5/2} + (D_{k-2} \Delta x_{k-2}) \left( \frac{\Delta x_{k-2}}{\Delta x_{k-1}} \right)^{5/2} \\
- \left( D_{k-1}(\Delta x_k + \Delta x_{k-1}) + \frac{5D_{k-1}^2 \Delta u_{k-1}}{2\ell^2} \right) \left( \frac{\Delta x_{k-1}}{\Delta x_k} \right)^{5/2} - \ell^4 \alpha \mu (\Delta u_{k-1} - \Delta u_k),
\]

\[
0 = \ell^5 E_x(L_k^c) = \left( -D_k \Delta u_{k+1} + \frac{5D_k^2 \Delta x_k}{2\ell^2} \right) \left( \frac{\Delta x_k}{\Delta x_{k+1}} \right)^{5/2} + \left( D_{k-1}(\Delta u_k + \Delta u_{k-1}) - \frac{5D_{k-1}^2 \Delta x_{k-1}}{2\ell^2} \right) \left( \frac{\Delta x_{k-1}}{\Delta x_k} \right)^{5/2} + \left( \frac{5D_{k-1}^2}{4\Delta x_{k-1}} \right) \left( \frac{\Delta x_{k-1}}{\Delta x_k} \right)^{5/2} \\
+ \left( \frac{5D_{k-2}^2}{4\Delta x_{k-2}} \right) \left( \frac{\Delta x_{k-2}}{\Delta x_{k-1}} \right)^{5/2} - \ell^4 \alpha \mu (\Delta x_{k-1} - \Delta x_k),
\]

to which we add the constraint equation \( \Delta x_{k+1}^2 + \Delta u_{k+1}^2 - \ell^2 = 0. \) As for the invariant variational scheme (38), we employ Algorithm 1 to choose the optimal combination of the equations to solve. Replicating the experiments showcased in Figure 2 and Figure 3, we obtain Figure 4.

Figure 4: (a) Numerical simulation of solution (42), where \( \mu = -1 \) and \( \alpha = 4 \), using the non-invariant scheme (46) compared against the exact solution. (b) The deviation in the conserved quantities \( C_{d_i}^q \), \( i = 1, 2, 3 \), as described by (44), for the non-invariant scheme (46). For both simulations we fix \( \ell = 0.01 \) and the number of steps = 500.

We observe that the solution to the non-invariant scheme diverges when the tangent line to the curve becomes vertical. This was to be expected since in (46) the Euler–Lagrange equa-
tions are divided by $\Delta x_k$ (and $\Delta x_{k+1}, \Delta x_{k-1}$). Furthermore the deviation in the conserved quantities is orders of magnitude greater than the invariant variational scheme (38).

There are a multitude of interesting dynamics exhibited by the Euler elastica equation which may be simulated by our model. Fixing $\ell = 0.01$ and iterating for 1000 steps, we obtain Figure 5 for different values of $\mu$.

Figure 5: The invariant variational scheme (38) with $\alpha = 4$ for various $\mu$ values simulating the solutions described in (41) and (43). We initialize all simulations at $s_0 = -2$ and iterate 1,000 steps with $\ell = 0.01$. We note that all dynamics presented in this figure are accurate, when compared to the exact solutions.

6.2 Divergence Invariant Lagrangian

We now shift our focus to the divergence invariant scheme (30). For comparison, we compare our invariant approximation against a standard approximation of (14) given by

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} = \frac{1}{u_k'^2},$$  \hspace{1cm} (47)

where $\Delta x_k = h$ is constant. We note that the general solution to (14) is

$$u(x) = \sqrt{\frac{(Ax + B)^2 + 1}{A}},$$  \hspace{1cm} (48)

where $A$ and $B$ are constants. For our simulations, we consider the case where $A = 1$ and $B = 0$. As the numerical solution $(x_k, u_k)$ evolves according to (30), the component $u_k$ will
provide an approximation of exact solution \( u(x_k) \). Therefore, when benchmarking our approximation we only consider the error in the \( u \) component. We initialize our simulation by setting \( x_0 = -1, x_1 = -1 + \frac{2}{\text{steps} - 1} \) and \( u_0 = u(x_0), u_1 = u(x_1) \). Benchmarking our numerical approximation (30) in Table 2 we obtain a quadratic experimental order of convergence. Additionally, by design, the non-invariant scheme (47) also converges to second order. We note that the quadratic convergence of the invariant variational scheme is better than expected, as the modified Lagrangian (28) is first order accurate and the discrete moving frame (29) is also a first order approximation of its continuous counterpart. This indicates that, in this example, an order of accuracy has been gained through the invariantization procedure.

| \( i \) | \( \text{steps} \) | \( \| u_k - u \|_{\infty} =: e_u \) | EOC \((e_u, \text{steps}^{-1}; i - 1)\) |
|---|---|---|---|
| 1 | 200 | 2.32e-6 | 2.04 |
| 2 | 400 | 5.73e-7 | 2.02 |
| 3 | 800 | 1.42e-7 | 2.01 |
| 4 | 1600 | 3.55e-8 | 2.00 |

| \( i \) | \( \text{steps} \) | \( e_u \) | EOC \((e_u, \text{steps}^{-1}; i - 1)\) |
|---|---|---|---|
| 1 | 200 | 8.50e-6 | 2.01 |
| 2 | 400 | 2.11e-6 | 2.00 |
| 3 | 800 | 5.27e-7 | 2.00 |
| 4 | 1600 | 1.32e-7 | 2.00 |

Table 2: The \( l_{\infty} \) error and the order of convergence in the \( u \) component for the invariant variational scheme (30) (left) and the standard numerical approximation (47) (right) approximating solution (48) with \( A = 1 \) and \( B = 0 \).

Fixing the number of steps to 100, we simulate both the invariant and standard schemes and compute the conserved quantities in Figure 6. We note that deviation in the conserved quantities changes slowly for the invariant variational scheme remaining on the order of the solver precision \((10^{-13})\), while for the standard scheme all quantities deviate significantly above machine precision with \( C^d_{\text{cd}} \) reaching \( O(10^{-4}) \) by the end of the simulation.

![Figure 6](image)

(a) Invariant variational scheme

(b) Standard scheme

Figure 6: The deviation in the conserved quantities \( C^d_{\text{cd}}, i = 1, 2, 3 \), as described by (31), for the invariant variational scheme (30) and the non-invariant scheme (47). The simulations are initialized with the exact solution at \( x_0 = -1 \) and \( x_1 = -1 - 2/99 \), and 100 steps are implemented. We observe that the deviations for the invariant variational scheme remain on the order of solver precision while for the non-invariant scheme the deviations quickly propagate.
7 Conclusion

Given a system of ordinary differential equations, one can use the Helmholtz conditions to determine whether or not these coincide with the Euler–Lagrange equations of some Lagrangian, [1]. For ordinary differential equations that originate from a variational problem, we introduced a procedure for discretizing the equations so as to preserve both its variational (and divergence) symmetries and its conserved quantities. This is done in a three step process where we first discretize the continuous Lagrangian to obtain a discrete variational problem. During this discretization procedure, Lie point symmetries are usually lost. To recover the lost symmetries we implement the moving frame method and invariantize the discrete Lagrangian. The numerical scheme is then obtained by computing the Euler–Lagrange equations of the invariantized Lagrangian.

The invariant variational approach outlined in the previous paragraph offers several advantages over other related geometric integrators. First, compared to invariant integrators, [4–7,10], that only focus on preserving the symmetries of the Euler–Lagrange equations, without consideration to its variational origin, the invariant variational schemes constructed in this paper have the additional benefit of preserving the conserved quantities of the problem. By preserving first integrals, the schemes should be more stable and produce better long term numerical results, which is one of the main appealing properties of geometric numerical integrators. Next, compared to the conservative method introduced in [32,33], our construction is simpler to implement and avoids the use of divided difference calculus, which can become challenging at times. Similarly, the discrete gradient method introduced in [29], which requires recasting the system in a skew-gradient form, is nontrivial to implement, in particular for large dynamical systems with many first integrals. On the other hand, in our approach one can naively discretize a Lagrangian and recover a suitable symmetry-preserving Lagrangian via the algorithmic process of invariantization.

Finally, we note that the methodology developed in this paper can also be applied to partial differential equations. As for ordinary differential equations, the discrete Euler–Lagrange equations will simultaneously approximate the differential equation and provide equations for the mesh. Though, as with any symmetry-preserving integrators, the mesh equations might lead to mesh entangle and poor numerical results. To alleviate these issues one could possibly use invariant r-adaptive meshes, [8] or evolution–projections techniques, [6], adapted to the variational framework. Doing so would require more attention, and we therefore reserve this problem for future considerations elsewhere.

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