Quantum Computation Based on Photons with Three Degrees of Freedom

Ming-Xing Luo1, Hui-Ran Li1, Hong Lai2 & Xiaojun Wang3

Quantum systems are important resources for quantum computer. Different from previous encoding forms using quantum systems with one degree of freedom (DoF) or two DoFs, we investigate the possibility of photon systems encoding with three DoFs consisting of the polarization DoF and two spatial DoFs. By exploring the optical circular birefringence induced by an NV center in a diamond embedded in the photonic crystal cavity, we propose several hybrid controlled-NOT (hybrid CNOT) gates operating on the two-photon or one-photon system. These hybrid CNOT gates show that three DoFs may be encoded as independent qubits without auxiliary DoFs. Our result provides a useful way to reduce quantum simulation resources by exploring complex quantum systems for quantum applications requiring large qubit systems.

Quantum computer has shown its superiority for solving difficult problems such as the large integer decomposition1–3 and data searching4,5. Since its difficult the large integer decomposition is the mathematical foundation of the well-known RSA cryptography which may be used for classical cryptographic applications6–8. Most of these quantum computation tasks may be completed with evolutions of quantum systems and desired quantum measurements1–5,10. If the quantum circuit model11 is applied, these evolutions may be synthesized by series of local quantum gates. Exactly, proper small gates such as the controlled phase-flip (CZ) gate or controlled-not (CNOT) gate combined with single-qubit gates12–14 can be used to implement quantum tasks with multiple qubits. These small gates construct a universal quantum gate set for quantum computing. Up to now, the CNOT gate has been widely implemented using several quantum systems, such as the linear optics15,16, ion trap17,18, atom19,20, and nuclear magnetic resonance21,22.

The solid-state quantum system has also attracted much attention in quantum simulations because of its special optical property and scalability23–26. Moreover, electron-spin qubits associated with the nitrogen-vacancy (NV) defect centers are particularly useful. In fact, due to the long room-temperature coherent time27, the negatively charged NV defect center in the diamond lattice, consisting of a substitutional 14N atom and an adjacent vacancy, is an attractive candidate for quantum information processing. It has been used to prepare and detect optical sources28–33, generate hybrid quantum entanglements between the NV center and photon14, or electrons34,35, purify two-photonic hyperentanglement in both the polarization and spatial DoFs36, or implement the CZ gate between the NV centers assisted by the microsphere cavity38,39. The single-electron and nuclear-spin states can be faithfully detected even under ambient conditions40,41, when the electron spin of the NV defect center couples to nearby 13C nuclear spins. Another diamond NV− center is proposed with six electrons from the nitrogen and three carbons surrounding the vacancy42, which is confined in a microtoroidal resonator (MTR)43 with the quantized whispering-gallery mode (WGM). This useful system allows for an ultrahigh-Q and a small mode volume of WGM microresonators44–46, which has been applied to construct quantum gates on electron-spin qubits47,48 or remote qubits49,50. Furthermore, recent experiments have assembled several hybrid systems, where colour centers in diamond nanocrystals or bulk diamond are coupled to the evanescent fields of cavities, which are defined in non-diamond materials coupling to WGMs in a silica micro-sphere50–52, diamond-GaP micro-disk53, GaP micro-ring cavities54, or SiN photonic crystal55.

Most of previous quantum simulations focused on systems with single DoF15–22 or hybrid systems56–60. A few schemes have considered photons with two DoFs61,62. Our recent result63 presents the independence of two DoFs (polarization DoF and spatial DoF) of photonic system, and then is used to construct the ququart

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1Information Security and National Computing Grid Laboratory, Southwest Jiaotong University, Chengdu 610031, China. 2School of Computer and Information Science, Southwest University, Chongqing 400715, China. 3School of Electronic Engineering, Dublin City University, Dublin 9, Ireland. Correspondence and requests for materials should be addressed to M.-X.L. (email: mxluo@home.swjtu.edu.cn)
(four-dimensional) quantum logic. Thus quantum simulation resources may be saved one half. In this paper, we further reduce quantum resources by considering photonic systems with three DoFs. Motivated by recent schemes, each photon may be encoded with two circularly polarized states and four modes, i.e., \(|l, E\rangle, |r, E\rangle, |l, I\rangle, |r, I\rangle\) and \(|l, E\rangle, |r, E\rangle\) for two crystal emissions. Here, \(l(r)\) refers to the left (right) side of each cone and \(l(E)\) denotes the internal (external) cone, as shown in Fig. 1. A general state is given by the product of one polarization state and two longitudinal momentum states. In the follow, we will investigate these photonic DoFs for the quantum simulation, without using auxiliary DoFs. From the quantum circuit model, CNOT gate will be schematically implemented on these DoFs of photonic states assisted by NV centers. For the symmetry of two spatial modes in each photonic system, fifteen CNOT gates are required to operate on the polarization DoFs and spatial-DoFs of the two-photon or one-photon system. Each gate is completed by interacting photons to auxiliary NV centers, disentangling NV centers, and correcting the emitting photons. These schemes are beyond to previous CNOT gates on the two-photon system (all combinations of three DoFs) and six CNOT gates on the one-photon system. It means that fifteen CNOT gates should be performed on photonic systems with three DoFs, where nine CNOT gates are on the two-photon system (all combinations of three DoFs) and six CNOT gates are on the one photon system. By exploring optical selection rules of the NV center in the crystal cavity, these CNOT gates may be realized without altering DoFs and auxiliary DoFs during implementations. In this case, each photonic DoF can be used as an independent qubits in one quantum task. Hence, two thirds of quantum resources may be saved for quantum simulations with large qubit systems, such as the Shor’s algorithm.

Results

To show the encoding independence of the polarization DoF and two spatial DoFs of each photon, it is necessary to prove that all quantum transformations in \(SU(2^3)\) may be implemented on these DoFs. Based on the theory of the universal logic gates, it is sufficient to consider the CNOT gate on any two DoFs of the photonic system. It means that fifteen CNOT gates should be performed on photonic systems with three DoFs, where nine CNOT gates are on the two-photon system (all combinations of three DoFs) and six CNOT gates are on the one photon system. By exploring optical selection rules of the NV center in the crystal cavity, these CNOT gates may be realized without altering DoFs and auxiliary DoFs during implementations. In this case, each photonic DoF can be encoded as an independent qubit in quantum applications.

Photon with three DoFs. Circularly polarized photon in the state \(\alpha_1|L\rangle + \alpha_2|R\rangle\) (left circularly polarized state \(|L\rangle\) and right circularly polarized state \(|R\rangle\)) is created at a degenerate wavelength \(\lambda = 2\lambda_p\) by each BBO crystal along two correlated directions belonging to the lateral surfaces of two SPDC cones, with full aperture angles \(\theta_l\) and \(\theta_r\), respectively. The dichotomy existing between the \(I\) cone and \(E\) cone is identified as an independent DoF. The corresponding mode emission as \(l(r)\) by referring to the left (right) side of each cone.

\[
|\phi\rangle = (\alpha_1|L\rangle + \alpha_2|R\rangle) \otimes (\beta_1|l, I\rangle + \beta_2|l, E\rangle + \beta_3|r, I\rangle + \beta_4|r, E\rangle)
\]

where \(|\alpha_1|^2 + |\alpha_2|^2 = 1\) and \(|\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 + |\beta_4|^2 = 1\). Here, \(\beta_i\) are dependent of aperture angles \(\theta_l\) and \(\theta_r\), and focal length \(f\), which are not goals in this paper.
The input pulse in the state $|m_i\rangle$ may decay into two ground states $|m_i\rangle$ and $|m_i\rangle$ by exciting the NV center with a polarized 2-ns $p$-pulse that is shorter than the emission timescale, and the reflection may be separated from fluorescence photons using detection timing\textsuperscript{34}. The normal boundary condition $\overline{A_{in}} = \overline{b_{out}}$ is derived by a right circularly polarized photon $\sigma_+$ ($|L\rangle$), and $|m_i\rangle$ is derived by a right circularly polarized photon $\sigma_-$ ($|R\rangle$).

A diamond NV center coupled to an MTR with a WGM. Schematic NV center in a diamond embedded in a photonic crystal cavity is shown in Fig. 2. The negatively charged NV center is consisted of a substitutional nitrogen atom and an adjacent vacancy with six electrons. The $\Lambda$-type three-level system is realized using specific excited state $|A_1\rangle = (|E_i\rangle|m_i\rangle + |E_f\rangle|m_i\rangle)$ as an ancillary state\textsuperscript{66,34}. Here, $|E_i\rangle$ are orbital states with angular momentum projection along the NV axis. The ground state is an electronic spin triplet with a splitting of 2.88 GHz by exciting the NV center with a polarized 2-ns $p$-pulse that is shorter than the emission timescale, and the reflection may be separated from fluorescence photons using detection timing\textsuperscript{34}. The normal boundary condition $\overline{b_{out}} = \overline{b_{in}} + \sqrt{\kappa} \overline{a}$ is used to derive the optical selection rule with the input field $\overline{b_{in}}$ output field $\overline{b_{out}}$ and cavity field operator $\overline{a}$. If spins stay in the ground states most of the time\textsuperscript{57}, the optical reflection coefficient may be approximately defined in the follow (see Methods)

$$r(\omega) = \frac{\left(i\delta_0^2 - \frac{\kappa m_i^2}{2} + \frac{\omega^2}{2}\right)\left(i\delta_0^2 + \frac{\omega^2}{2}\right) + g^2}{\left(i\delta_0^2 + \frac{\omega^2}{2} + \frac{\omega^2}{2}\right)\left(i\delta_0^2 + \frac{\omega^2}{2} + \frac{\omega^2}{2}\right) + g^2} = |r(\omega)| e^{i\delta_0}$$

(2)

where $\delta_0$, and $\delta_0$, are frequency detunings satisfying $\delta_0 = \omega - \omega$ and $\delta_0 = \omega - \omega$, $\omega$, $\omega$ and $\omega$ are the frequencies of the cavity mode, input photon pulse, and NV center, respectively. $g$ is the coupling strength between the cavity and the NV center. $\kappa$, $\kappa$, and $\gamma$ are the damping rate of the cavity, cavity side leakage mode, and spontaneous decay rate of the NV center, respectively. If define the cooperativity $C = 2g^2/(\kappa\gamma)$, the photonic reflection probability\textsuperscript{68} is determined by the cooperativity $C$ and the cavity tuning as follow

$$P = 1 - \frac{1 + 4C + \left(\frac{\delta_0}{\gamma}\right)^2}{1 + 4C + 4\gamma^2 + \left(\frac{\delta_0}{\gamma}\right)^2}$$

(3)

Considering the coupling strength $g = 0$, an NV center is uncoupled from the cavity (the cold cavity), and the reflection coefficient $r(\omega)$ becomes

$$r_0(\omega) = \frac{i\delta_0\omega - \frac{\kappa m_i^2}{2} + \frac{\omega^2}{2}}{i\delta_0^2 + \frac{\omega^2}{2} + \frac{\omega^2}{2}} = |r_0(\omega)| e^{i\delta_0}$$

(4)

Thus the input pulse in the polarized state $|L\rangle$ gains a phase shift $\theta$ after reflecting from the hot cavity ($g > 0$) with the NV center $|m_i\rangle$, or a phase shift $\theta_0$ after reflecting from the cold cavity ($g = 0$) with the NV center $|m_i\rangle$. The input pulse in the state $|R\rangle$ gains a phase shift $\theta_0$ after reflecting from the cold cavity with the NV center $|m_i\rangle$, or a phase shift $\theta$ after reflecting from the hot cavity with the NV center $|m_i\rangle$. By choosing a proper frequency detuning $\delta_0 = 0\kappa$ and the cooperativity $C \gg 1$, the reflection coefficients may satisfy $|r(\omega)| \approx 1$ and $|r_0(\omega)| \approx 1$ when the cavity side leakage $\kappa_s$ is negligible. By adjusting the frequencies $\omega$ and $\omega$, such that $\delta_0/\kappa \rightarrow 0$ and $C \gg 1$, the phase shifts may be realized as $\theta = 0$ and $\theta_0 = \pi$. Hence, the following optical transition may be obtained as

$$|R\rangle |m_i\rangle \rightarrow - |R\rangle |m_i\rangle, \quad |R\rangle |m_i\rangle \rightarrow |R\rangle |m_i\rangle, \quad |L\rangle |m_i\rangle \rightarrow - |L\rangle |m_i\rangle, \quad |L\rangle |m_i\rangle \rightarrow - |L\rangle |m_i\rangle.$$  

(5)

From this optical transition, an NV center requires a polarization-degenerate cavity mode, which is also suitable in H1 photonic crystals\textsuperscript{69,70} and fiber-based cavities\textsuperscript{71}.
CNOT gate on the same DoF of the two-photon system. Schematic CNOT gate on the same DoF of the two-photon system is shown in Fig. 3. NV centers $e_i$ trapped in the photonic crystal NV are initially prepared in the superposition states $|+\rangle = |\rangle + |\rangle^{-}$.

Two input photons $A_1$ and $A_2$ are in the states $\phi_{\alpha\beta} = \phi_{\alpha\beta} \otimes \phi_{\alpha\beta}$. Figure 3(a) is used to complete the CNOT gate on the polarization DoFs of two photons, i.e.,

$$CZ_{pA}(A_1, e_1) = \{R\} \otimes \{R\} + \{L\} \otimes \{R\} + \{L\} \otimes \{R\}$$

In detail, the photon $A_1$ from each spatial mode $(l_1I_1, l_1E_1, r_1I_1, r_1E_1)$ evolves as $CPBS \rightarrow NV_1 \rightarrow CPBS$ to complete the following controlled phase gate

$$CZ_{pA}(A_1, e_1) = \{R\} \otimes \{m^-\} \{m^-\} + \{m^-\} \{R\} + \{L\} \otimes \{R\} + \{m^-\} \{R\} + \{L\} \otimes \{m^-\} \{L\}$$

on the polarization DoF and the NV center $e_1$. And then, after one Hadamard operation $H^p$ on the NV center $e_1$ in the NV$_1$, the photon $A_2$ from each spatial mode evolves as $H^p \rightarrow CPBS \rightarrow NV_1 \rightarrow CPBS \rightarrow H^p$ to complete the following hybrid CNOT gate

$$C_{aA}(e_2, A_2) = \{m^-\} \otimes \{R\} \{R\} + |L\rangle \langle L| + \{m^-\} \otimes \{R\} \{L\}$$

on the NV center $e_1$ and the polarization DoF of the photon $A_2$.
the phase flip $Z_p = |R\rangle \langle R| - |L\rangle \langle L|$ is performed on the photon $A_1$ from each mode for the measurement outcome $|−\rangle_e$. Thus the CNOT gate $C_{pp}(A_1, A_2)$ has been realized on the photons $A_1$ and $A_2$. Figure 3(b) is a schematic circuit to complete the CNOT gate on the spatial DoF $\{l, r\}$s of two photons, i.e.,

$$C_{st_{12}}(A_1, A_2) = |l\rangle \langle l| \otimes |l\rangle \langle l| + \langle r| \langle r| + |r\rangle \langle r| \otimes |l\rangle \langle l| - |l\rangle \langle l| \otimes |r\rangle \langle r|$$

(9)

Here, the photon $A_1$, each from each spatial mode $r_1I_1$ or $r_1E_1$ evolves as $CPBS \rightarrow NV_2 \rightarrow (X \rightarrow NV_2 \rightarrow X) \rightarrow CPBS$ to complete the following controlled phase gate

$$CZ_{l,r}(A_1, e_1) = |m\rangle \langle m| \otimes |l\rangle \langle l| + \langle r| \langle r| + |r\rangle \langle r| \otimes |l\rangle \langle l| - |l\rangle \langle l| \otimes |r\rangle \langle r|$$

(10)

on the spatial DoF $\{l, r\}$ and the NV center $e_1$ in the state $|+\rangle$ (see Appendix B of Supplementary Information for details). Now, after a Hadamard gate $H^p$ performed on the NV center $e_1$ in the NV$_1$, the followed circuit $CBS \rightarrow CPBS \rightarrow NV_1 \rightarrow (X \rightarrow NV_1 \rightarrow X) \rightarrow CPBS$ is used to complete the hybrid CNOT gate on the NV center $e_2$ and the spatial DoF $\{l, r\}$ of the photon $A_2$ (see Supplementary Information for details), i.e.,

$$C_{al_{12}}(e_2, A_1) = |m\rangle \langle m| \otimes |l\rangle \langle l| + |r\rangle \langle r| + |r\rangle \langle r| \otimes |l\rangle \langle l| - |l\rangle \langle l| \otimes |r\rangle \langle r|$$

(11)

Now, the CNOT gate $C_{st_{12}}(A_1, A_2)$ may be realized by disentangling the NV center $e_2$ using the measurement under the basis $\{|\pm\rangle\}$, where $Z^p$ is performed on the photon $A_1$ from each spatial mode $r_1I_1$ and $r_1E_1$ for the measurement outcome $|−\rangle_e$.

A similar CNOT gate

$$C_{st_{12}}(A_1, A_2) = |l\rangle \langle l| \otimes |l\rangle \langle l| + |r\rangle \langle r| + |r\rangle \langle r| \otimes |l\rangle \langle l| - |l\rangle \langle l| \otimes |r\rangle \langle r|$$

(12)

holds for the spatial DoF $\{I, E\}$s of two photons using an NV center $e_3$ trapped in the optical cavity NV$_3$ (see Appendix C of Supplementary Information for details).

**Hybrid CNOT gate on the different DoFs of the two-photon system.** Figure 4(a) is a schematic circuit to implement the CNOT gate on the polarization DoF of the photon $A_1$ and the spatial DoF $\{l, r\}$ of the photon $A_2$, i.e.,

$$C_{ps_{12}}(A_1, A_2) = |R\rangle \langle R| \otimes |l\rangle \langle l| + |r\rangle \langle r| + |r\rangle \langle r| \otimes |l\rangle \langle l| - |l\rangle \langle l| \otimes |r\rangle \langle r|$$

(13)

In fact, similar to the Fig. 3(a), the first controlled phase flip $CZ_{l,r}(A_1, e_1)$ in the equation (7) is used to change the photon $A_1$ and the NV center $e_1$ from $|φ\rangle_{A_1, e_1}$ to $CZ_{ps_{12}}(A_1, e_1) |φ\rangle_{A_1, e_1}$. And then, after one Hadamard operation $H^p$ performed on the NV center $e_1$ in the NV$_1$, the followed circuit $CBS \rightarrow CPBS \rightarrow NV_1 \rightarrow (X \rightarrow NV_1 \rightarrow X)$ is used to complete the hybrid CNOT gate on the NV center $e_2$ and the spatial DoF $\{l, r\}$ of the photon $A_2$. The subcircuits $H^p$, $CZ_{ps_{12}}, C_{al_{12}}$ and $C_{al_{12}}$ are shown in the Fig. 3.
Figure 5. Hybrid CNOT gate on different DoFs of one photon. $e_i$ denotes an auxiliary NV center in the NV-cavity $NV_1$. The subcircuits $C_{Ae}$ and $C_{ps}$ are shown in the Fig. 3.

$X \rightarrow$ CPBS $\rightarrow$ CBS for each spatial mode pair $(l_1, r_1)$ or $(l_2, r_2, E_2)$ is used to complete the CNOT gate $C_{ps}(e_2, A_1)$ in the equation (11) on the NV center $e_1$ and the spatial mode $[l, r]$ of the photon $A_1$ (similar to the Fig. 3(b)). After disentangling the NV center $e_1$ using the measurement under the basis $|\pm\rangle$, the hybrid CNOT gate $C_{ps}(A_1, A_2)$ is realized on the photons $A_1$ and $A_2$, where $Z^{\prime}$ is performed on the photon $A_1$ from each spatial mode $r_1 l_1$ and $r_2 E_2$ for the measurement outcome $|_c^-\rangle$, see Appendix D of Supplementary Information for details.

Similarly, after the controlled-phase flip $CZ_{ps}(A_1, e_1')$ on the photon $A_1$ and the NV center $e_1'$ in the state $|+\rangle$, a schematic circuit is applied to the photon $A_1$ from two spatial mode pairs $(l_1 l_2, l_1 E_2)$ and $(r_2 l_1, r_2 E_2)$ to complete the CNOT gate on the NV center $e_1'$ and the spatial DoF $[I, E]$ of the photon $A_1$ (see Appendix E of Supplementary Information for details). The hybrid CNOT gate

$$C_{ps,l} (A_1, A_2) = |R\rangle |R\rangle \otimes |E\rangle |E\rangle + |L\rangle |L\rangle \otimes |L\rangle |L\rangle$$

(14)
is implemented on the polarization DoF of the photon $A_1$ and the spatial DoF $[I, E]$ of the photon $A_2$ after disentangling the NV center $e_1$.

Figure 4(b) is used to implement the CNOT gate on the spatial DoF $[l, r]$ of the photon $A_1$ and the polarization DoF of the photon $A_2$, i.e.,

$$C_{ps,p} (A_1, A_2) = |l\rangle |l\rangle \otimes |R\rangle |R\rangle + |r\rangle |r\rangle \otimes |L\rangle |L\rangle$$

(15)

In fact, similar to the evolutions as shown in the Fig. 3(b), the controlled phase gate $CZ_{l_{ps}}(A_1, e_2)$ in the equation (10) is performed on the photon $A_1$ and the NV center $e_2$ in the state $|+\rangle$ to get $C_{Ae}(A_1, e_2)(|\phi\rangle _{A_1} |+\rangle _{e_2})$. And then, after one Hadamard operation $H^p$ on the NV center $e_2$ in the NV$_2$, the followed circuit for the photon $A_2$ from each spatial mode is used to complete the CNOT gate $C_{Ae}(e_2, A_2)$ on the NV center $e_2$ and the polarization DoF of the photon $A_2$ (see the Fig. 3(a)). The final joint state is $C_{Ae}(e_2, A_2)CZ_{l_{ps}}(A_1, e_2)(|\phi\rangle _{A_1} |+\rangle _{e_2})$. Finally, by disentangling the NV center $e_2$ using the measurement under the basis $|\pm\rangle$, the hybrid CNOT gate $C_{l_{ps}, A_{ps}}(A_1, A_2)$ is realized, where $-P$ will be performed on the photon $A_1$ from each spatial mode $r_1 l_1$ and $r_2 E_2$ for the measurement outcome $|_e^-\rangle$, see Appendix F of Supplementary Information for details. Moreover, if the second part of the present circuit above is applied to the photon $A_1$ from two spatial modes $l_1 l_1$ and $l_1 E_1$, the CNOT gate is implemented on the spatial DoF $[I, E]$ of the photon $A_1$ and the polarization DoF of the photon $A_2$, see Appendix G of Supplementary Information for details.

Figure 4(c) is used to implement the CNOT gate on the spatial DoF $[l, r]$ of the photon $A_1$ and the spatial DoF $[I, E]$ of the photon $A_2$, i.e.,

$$C_{l_{ps}, A_{ps}} (A_1, A_2) = |l\rangle |l\rangle \otimes |R\rangle |R\rangle + |r\rangle |r\rangle \otimes |L\rangle |L\rangle$$

(16)

In detail, similar to the evolutions as shown in the Fig. 3(b), the controlled phase gate $CZ_{l_{ps}}(e_2, A_1)$ in the equation (10) is performed for the photon $A_1$ and the NV center $e_2$ in the state $|+\rangle$ to get $C_{Ae}(A_1, e_2)(|\phi\rangle _{A_1} |+\rangle _{e_2})$. And then, after one Hadamard operation $H^p$ on the NV center $e_2$ the followed circuit for the photon $A_1$ from each spatial mode is used to realize the CNOT gate $C_{Ae}(e_2, A_2) = |m\rangle ^* (m| |(I) |E\rangle |E\rangle + |E\rangle |I\rangle ) |m\rangle |(I) |E\rangle |E\rangle + |E\rangle |I\rangle )$. Now, by disentangling the NV center $e_2$ using the measurement under the basis $|\pm\rangle$, $C_{l_{ps}, A_{ps}} (A_1, A_2)$ may be deterministically realized, where $-P$ will be performed on the photon $A_1$ from each spatial mode $l_1 E_1$ and $r_1 E_1$ for the measurement outcome $|_e^-\rangle$, see Appendix H of Supplementary Information for details. Similarly, the CNOT gate may be implemented on the spatial DoF $[I, E]$ of the photon $A_1$ and the spatial DoF $[l, r]$ of the photon $A_2$, see Appendix I of Supplementary Information for details.

**Hybrid CNOT gate on different DoFs of one photon.** Figure 5 is a schematic circuit to implement the CNOT gate $C_{ps,l}$ in the equation (13) on the polarization DoF and the spatial DoF $[l, r]$ of the photon $A_1$. In detail, similar to the Fig. 3(a), the controlled-phase flip $CZ_{ps}$ in the equation (7) is used to change the photon $A_1$ and the NV center $e_1$ from $|\phi\rangle _{A_1} |+\rangle _{e_1}$ to $CZ_{ps}(A_1, e_1)(|\phi\rangle _{A_1} |+\rangle _{e_1})$. And then, after one Hadamard operation $H^p$ performed on the NV center $e_1$, the followed CNOT gate $C_{Ae}$ in the equation (11) is performed on the NV center $e_1$ and the spatial DoF $[I, E]$ of the photon $A_1$ (similar to the Fig. 3(b)). After disentangling the NV center $e_1$ using the measurement under the basis $|\pm\rangle$, the hybrid CNOT gate $C_{ps,l}$ is realized on the photon $A_1$, where $Z'$ will be performed for the photon $A_1$ from each mode for the measurement outcome $|_e^-\rangle$, see Appendix J of Supplementary
Information for details. Moreover, if the CNOT gate $C_{\text{AE}}(e_1, A_1)$ is performed on the NV center $e_1$ and the photon $A_1$, after $C_{\text{E}A}(A_1, e_1)$, the hybrid CNOT gate $C_{\text{E}I}^{\text{IE}}$ in the equation (14) is realized on the photon $A_1$ after properly disentangling the auxiliary NV center, see Appendix K of Supplementary Information for details.

For the hybrid CNOT gate on the spatial DoF $l$, $r$ and the spatial DoF $I$, $E$ of the photon $A_1$, the photon $A_1$ from the spatial modes $r_I$, $r_E$ passes through CBS, $-I$, CBS, sequentially. The photon $A_1$ evolves as follows

$$\rho_{A_1}^{\text{CBS}} = \alpha_0|I\rangle\langle I| + \alpha_1|E\rangle\langle E| + \alpha_{1I}|I\rangle\langle I| + \alpha_{1E}|E\rangle\langle E| + \alpha_{2I}|I\rangle\langle E| + \alpha_{2E}|E\rangle\langle I|$$

where $\alpha_{1} = (\alpha_{1} + \alpha_{2})/\sqrt{2}$, $\alpha_{2} = (\alpha_{1} - \alpha_{2})/\sqrt{2}$, $\alpha_{1} = (\alpha_{1} + \alpha_{2})/\sqrt{2}$, and $\alpha_{2} = (\alpha_{1} - \alpha_{2})/\sqrt{2}$. Similar circuit may be used to realize the hybrid CNOT gate on the spatial DoF $l$, $E$ and the spatial DoF $I$, $r$ of the photon $A_1$, see Appendix K of Supplementary Information for details. Moreover, the CNOT gates on the spatial DoF and the polarization DoF of the photon $A_1$ are easily realized by two flip waveplates on two spatial modes $r_I$, $r_E$, and $r_I$, $r_E$, respectively.

**Discussions**

In ideal conditions, one may neglect the cavity side leakage, and the reflection coefficients satisfy $|r_0(\omega)| \approx 1$ and $|r_1(\omega)| \approx 1$. The corresponding fidelities of the present CNOT gates are nearly 100%. In experiment, the general fidelity is defined by $F = \int |\langle \Phi|r_0|\Phi\rangle|$, where $|\Phi\rangle$ is the ideal final state without side leakage while $r_0|\Phi\rangle$ is the final state under a real situation with side leakage. In the resonant condition $\delta\omega = 0$, if the cavity side leakage is considered, the optical selection rule for the NV-cavity system given by the equation (5) becomes

$$|R\rangle|m\rangle \rightarrow |R\rangle|m\rangle, |R\rangle|m\rangle \rightarrow |R\rangle|m\rangle$$

$$|L\rangle|m\rangle \rightarrow |L\rangle|m\rangle, |L\rangle|m\rangle \rightarrow |L\rangle|m\rangle$$

Due to the exchangeability of two spatial DoFs of one photon with respect to random initial photons, the fidelities and efficiencies are evaluated for four CNOT gates: CNOT gate on two polarization DoFs, CNOT gate on two spatial DoFs, CNOT gate on the polarization DoF of one photon and the spatial DoF of the other photon, and CNOT gate on the polarization and spatial DoFs of one photon system, as shown in Figs 6 and 7, respectively.

Generally, large cooperativity $C$ and low relative detuning $\delta\omega/\kappa$ are required for high fidelities and efficiencies. For the diamond NV centers, the photoluminescence is partially unpolarized, and the emission with ZPL is only 4% of the total emission. ZPL with zero phonon line is only 4% of $\gamma = 2 \times 15$ MHz31. For the diamond NV center in a MTR with WGM mode system, $|r_0(\omega)| \approx 0.95$ when $C \geq 1834$ with small detuning $\delta\omega/\kappa \approx 0$; $|r_0(\omega)| \approx 1$ when $C \geq 50$ with small detuning $\delta\omega/\kappa \approx 0$ for $\kappa \approx 1$ GHz or $\kappa \approx 10$ GHz. For our CNOT gates, if $C \geq 18$ and $\delta\omega/\kappa \approx 0.1$, their fidelities are greater than 82.6% and their efficiencies are greater than 75.4%. If $C \geq 50$ and $\delta\omega/\kappa \approx 0.1$, their fidelities are greater than 98.4% and their efficiencies are greater than 94.7%.

In conclusion, we have investigated the possibility of quantum simulations using photon systems with three DoFs. We have constructed fifteen schematic CNOT gates operating on the spatial and polarization DoFs of the two-photon system14,15,56,57. Our schemes are based on different DoFs of two photons or one photon.

**Methods**

**A diamond NV center coupled to an MTR with a WGM.** The master equation of the whole system may be expressed by a Lindblad form as follows

$$\dot{\rho} = i[H, \rho] + \mathcal{L}_r \rho + \mathcal{L}_s \rho$$

where $H = H_1 + H_2 + H_3 + H_4 + H_5$,$ H_1 = g(\hat{a} \sigma_x + \hat{a}^\dagger \sigma_x)$ is the standard Jaynes-Cummings Hamiltonian for a two-level system interacting with a single electromagnetic
mode by applying the rotating wave approximation and dropping the energy nonconserving terms. \( \sigma_- \) and \( \sigma_+ \) are the Pauli raising and lowering operators, respectively. \( g \) is the coupling strength between the cavity and \( X \). \( \sigma_\gamma \) is the Hamiltonian of the dipole. \( \sigma_z \) is the Pauli operator for the population inversion. \( \kappa \) is the interaction between the excitation field and system. \( \mathcal{H}_0 \) accounts for the damping of the input photon pulse. \( \mathcal{H}_2 \) accounts for spontaneous emission of the dipole. The input-output optical relation of the NV center system may be calculated from the Heisenberg equations in terms of the cavity field operator \( \hat{a} \), input pulse field \( \hat{b} \) and dipole operator \( \sigma_- \),

\[
\begin{align*}
\frac{d\hat{a}}{dt} &= -i(\delta \omega - \frac{\kappa}{2})\hat{a} - g\hat{\sigma}_- - \sqrt{\kappa}\hat{b}_{in}, \\
\frac{d\sigma_-}{dt} &= -i(\delta \omega + \frac{\gamma}{2})\hat{\sigma}_- - g\sigma_+ \hat{a} + \sqrt{\gamma}\hat{b}_{in},
\end{align*}
\]

If spins stay in the ground states most of the time \( \langle \sigma_- \rangle = -1 \), the cavity output \( \hat{b}_{out} \) is connected with the input field by the standard input-output relation by a reflection coefficient \( r(\omega) \).

**Measurement of the NV center e in cavity.** To measure the NV center \( e \) of an entangled system \( \alpha|m\rangle_1 |\Omega_1\rangle + \beta|m\rangle_1 |\Omega_2\rangle \), an auxiliary photon \( c \) in the state \( \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) \) may be used as follows. Let the photon \( c \) pass through one CPBS to split the circular polarizations \( |R\rangle \) and \( |L\rangle \), and the right-circular polarization \( |R\rangle \) interact with the cavity system, and its output combine with \( |L\rangle \) of the photon \( c \) using the other CPBS. Thus, the joint system evolves

\[
\begin{align*}
\frac{\alpha}{\sqrt{2}}(|R\rangle + |L\rangle) |m\rangle_1 |\Omega_1\rangle + \frac{\beta}{\sqrt{2}}(|R\rangle + |L\rangle) |m\rangle_1 |\Omega_2\rangle \\
\implies \frac{\alpha}{\sqrt{2}}(|R\rangle - |L\rangle) |m\rangle_1 |\Omega_1\rangle + \frac{\beta}{\sqrt{2}}(|R\rangle + |L\rangle) |m\rangle_1 |\Omega_2\rangle.
\end{align*}
\]

---

**Figure 6.** Average fidelities of the present CNOT gates via the cooperativity \( C \) and relative detuning \( \delta \omega / \kappa \). (a) The average fidelity of the CNOT gate on the polarization DoFs of two photons; (b) The average fidelity of the CNOT gate on the spatial DoFs of two photons; (c) The average fidelity of the hybrid CNOT gate on the polarization and spatial DoFs of the two-photon system; (d) The average fidelity of the hybrid CNOT gate on the polarization and spatial DoFs of the one photon system. The average fidelity is computed as the expectation of random input photons.
Hence, the NV center can be determined by measuring the photon in the orthogonal basis $\pm \{R \}$.

The NV center is $|m^-\rangle$ or $|m^+\rangle$ for the measurement outcome $\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$ or $\frac{1}{\sqrt{2}}(|R\rangle - |L\rangle)$, respectively.

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Author Contributions
M.-X.L. proposed the theoretical method. M.-X.L. and H.-R.L. wrote the main manuscript text. M.-X.L. and H.L. and X.W. reviewed the manuscript.

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