Stationary quantum stochastic processes from the cohomological point of view

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Abstract

Stationary quantum stochastic process \( j \) is introduced as a \(*\)-homomorphism embedding an involutive graded algebra \( \tilde{K} = \bigoplus_{i=1}^{\infty} K_i \) into a ring of (abelian) cohomologies of the one-parameter group \( \alpha \) consisting of \(*\)-automorphisms of certain operator algebra in a Hilbert space such that every \( x \) from \( K_i \) is translated into an additive \( i - \alpha \)-cocycle \( j(x) \). It is shown that (noncommutative) multiplicative markovian cocycle defines a perturbation of the stationary quantum stochastic process in the sense of such definition. The \( E_0 \)-semigroup \( \tilde{\beta} \) on the von Neumann algebra \( \mathcal{N} \) associated with the markovian perturbation of \( K \)-flow \( j \) posseses the restriction \( \tilde{\beta}|_{N_0}, N_0 \subset \mathcal{N} \), which is conjugate to the flow of Powers shifts \( \beta \) associated with \( j \). It yields for \( \tilde{\beta} \) an analogue of the Wold decomposition for classical stochastic process on completely nondeterministic and deterministic parts. The examples of quantum stationary stochastic processes on the algebras of canonical commutation, anticommutation and square of white noise relations are considered. In the model situation of the space \( L^2(\mathbb{R}) \) all markovian cocycles of the group of shifts are described up to unitary equivalence of perturbations.

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1 Introduction.

Let $\nu$ be a measure on the real axe $\mathbb{R}$. Denote $S = (S_t)_{t \in \mathbb{R}}$ the flow of shifts acting on the measurable functions $f$ on $\mathbb{R}$ by the formula $(S_tf)(x) = f(x - t), \; x, t \in \mathbb{R}$. Then given a number $r \in \mathbb{R}$ the function $I_r(t) = \nu([r, r + t]), \; t \in \mathbb{R}$, is a $1 - S$-cocycle, i.e. $I_r(t + s) = I_r(t) + S_t(I_r(s)), \; t, s \in \mathbb{R}$. Analogously fixing numbers $r_1, \ldots, r_k \in \mathbb{R}$ we get that the function $I_{r_1 \ldots r_k}(t_1, \ldots, t_k) = I_{r_1}(t_1)S_{t_1}(I_{r_2}(t_2)S_{t_2}(\ldots I_{r_k}(t_k) \ldots), \; t_i \in \mathbb{R}$, associated with the tensor product $\nu^{\otimes k}$ is a $k - S$-cocycle with the characteristic property $S_{t_1}(I(t_2, \ldots, t_{k+1})) - I(t_1 + t_2, t_3, \ldots, t_k) + \ldots + (-1)^jI(t_1, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{k+1}) + \ldots + (-1)^kI(t_1, \ldots, t_k) = 0, \; t_i \in \mathbb{R}$. One can consider a ring of cohomologies $H^* = \oplus H^i$ generated by the measure $\nu$ in this way. The canonical bilinear map $(x_i, x_j) \rightarrow x_i \cup x_j \in H^{i+j}, \; x_i \in H^i, x_j \in H^j$, defining a ring structure in $H^*$ can be obtained from the action of $S$ by the formula $(x_i \cup x_j)(t_1, \ldots, t_{i+j}) = x_i(t_1, \ldots, t_i)S_{t_1 + \ldots + t_i}(x_j(t_{i+1}, \ldots, t_{i+j}))$ (see [14]). We consider a graded algebra of cohomologies $A = \bigoplus_{i=1}^{\infty} A_i$ such that $A_i$ consists of additive $i$-cocycles of the group of automorphisms $\alpha$ associated with stationary quantum stochastic process $j$ over an involutive algebra $K$. In our construction $A_1$ is generated by the basic operator-valued stochastic measures which are the creation, annihilation and number of particles processes in the important applications. Given an involutive algebra $K$ we put $K_1 = K$ and construct the graded algebra $\tilde{K} = \bigoplus_{i=1}^{\infty} K_i$ with respect to a certain linear associative operation $\odot : K_i \times K_j \rightarrow K_{i+j}$. In the case when $K$ is a Lie algebra one can choose the universal enveloping algebra for $\tilde{K}$. We define a stationary quantum stochastic process as an $\ast$-homomorphism $j$ of $\tilde{K}$ into the standard ring of (abelian) cohomologies of the group $\alpha$ such that every $x \in K_i$ is mapped to an $i - \alpha$-cocycle $j(x)$. The homomorphism $j$ transfers the operation $\odot$ in $\tilde{K}$ to the cohomological multiplication $\cup$. We consider a class of cocycle perturbations of $j$ by markovian cocycles. The markovian cocycle perturbation of $K$-flow constructed through our procedure determines the associated $E_0$-semigroup $\tilde{\beta}$ on the von Neumann algebra $\mathcal{N}$ which possesses the restriction $\tilde{\beta}|_{\mathcal{N}_0}, \mathcal{N}_0 \subset \mathcal{N}$, conjugate to the flow of Powers shifts associated with the initial $K$-flow. Using the technics of [14], it is possible to extract an automorphic part of the $E_0$-semigroup. Hence our result defines an analogue of the Wold decomposition for classical stochastic process on completely nondeterministic and deterministic parts. We give several examples
where \( j \) determines quantum stochastic processes on the algebra of canonical commutation or anticommutation relations and the square of white noise correspondingly. The basic ideas are illustrated on the model of markovian perturbations for the group of shifts in \( L^2(\mathbb{R}) \). In this case the associated markovian cocycles are constructed in the explicit form by means of the inner function techniques.

## 2 Markovian perturbations of the group of shifts in \( L^2(\mathbb{R}) \).

Let \( S = (S_t)_{t \in \mathbb{R}} \) be a strong continuous group of unitary operators on a Hilbert space \( \mathcal{H} \). A strong continuous family of unitaries \( W = (W_t)_{t \in \mathbb{R}} \) in \( \mathcal{H} \) is called a multiplicative \( 1 - S - \) cocycle if 
\[
W_{t+s} = W_t S_t W_s S_{-t}, \quad s, t \in \mathbb{R}, \quad W_0 = I.
\]
The cocycle \( W \) is said to be a multiplicative \( 1 - S - \) coboundary if there exists a unitary operator \( J \) defining \( W \) by the formula 
\[
W_t = JS_t J^* S_{-t}, \quad t \in \mathbb{R}.
\]
Every multiplicative cocycle \( W \) determines a new unitary group \( U = (U_t)_{t \in \mathbb{R}} \) in \( \mathcal{H} \) by the formula 
\[
U_t = W_t S_t, \quad t \in \mathbb{R}.
\]
This group can be named a cocycle perturbation of \( S \). Notice that if \( W \) is a coboundary, then 
\[
W_t S_t = JS_t J^* S_{-t}, \quad t \in \mathbb{R},
\]
i.e. the coboundary determines the perturbation which is unitary equivalent to the initial group. Consider the group of shifts \( S = (S_t)_{t \in \mathbb{R}} \) acting in the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}) \) by the formula \( (S_t f)(x) = f(x + t), \quad f \in \mathcal{H} \). The group of cohomologies \( H^1(S, L^2(\mathbb{R})) \) is generated by an additive \( 1 - S - \) cocycle \( \chi = (\chi_t)_{t \in \mathbb{R}} \) defined by \( \chi_t(x) = 1, \quad -t < x \leq 0, \quad \chi_t(x) = 0 \) otherwise. The cocycle \( \chi \) satisfies the characteristic properties 
\[
\chi_{t+s} = \chi_t + S_t \chi_s, \quad ||\chi_t - \chi_s|| = 2|t - s|^{1/2}, \quad t, s \in \mathbb{R}.
\]
Let \( \mathcal{H}_t \) be a subspace of \( \mathcal{H} \) generated by all functions with supports belonging to the segment \([-t, +\infty)\). Notice that \( \mathcal{H}_t = S_t \mathcal{H}_0, \quad t \in \mathbb{R} \). We shall call the multiplicative cocycle \( W \) by markovian if \( W_t|_{\mathcal{H} \ominus \mathcal{H}_t} = I, \quad t \geq 0 \). This property means that \( W \) doesn’t perturbe “a future” of the system. The Markov property for perturbations was introduced in [1]. Using the cocycle property 
\[
W_{-t} = S_t W_t^* S_{-t}, \quad t \in \mathbb{R},
\]
we can rewrite the condition of markovianity in the form 
\[
W_{-t}|_{\mathcal{H} \ominus \mathcal{H}_0} = I, \quad t \geq 0.
\]
It guarantees that linear operators 
\[
V_t = U_{-t}|_{\mathcal{H}_0}, \quad t \geq 0,
\]
are correctly defined and form a \( C_0 \)-semigroup of isometries \( V \) in the Hilbert space \( \mathcal{H}_0 \). We call \( V \) by a semigroup associated with the markovian cocycle perturbation of \( S \). Given a \( C_0 \)-semigroup of isometrical operators \( V \) in the Hilbert space \( \mathcal{H}_0 \), one can define the Wold decomposition \( \mathcal{H}_0 = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \) on the subspace \( \mathcal{H}^{(0)} \).
under the action of $U$ unitary operators. We get the Wold decomposition $H$ for the semigroup of isometries $V$ that the deficiency index of the generator of $V$ (see [17]). The semigroups $V|_{H(0)}$ and $V|_{H(1)}$ can be named a unitary part and a shift part of the semigroup $V$ correspondingly.

**Proposition 2.1.** The deficiency index of the generator of the semigroup $V$ associated with the markovian cocycle perturbation of $S$ equals 1.

**Proof.** Consider a family of functions $\zeta_t = W_{-t} \chi_{-t}$, $t \geq 0$, where $\chi$ is the additive $1 - S$-cocycle defined in the begin of the section. It follows from the definitions of $\chi$ and $W$ that the family $\zeta$ is continuous. Then $\zeta_{t+s} = W_{-t-s} \chi_{-t-s} = W_{-t} W_{-s} S_t(\chi_{-t} + S_t \chi_{-s}) = W_{-t} \chi_{-t} + W_{-t} S_t W_{-s} \chi_{-s} = \zeta_t + V_t \varsigma_s$, $t, s \geq 0$, where we have used the identity $W_{-s} S_t \chi_{-t} = S_t \chi_{-t}$, $s, t \geq 0$, following from the markovian property of $W$ in the form $W_{-s} |_{H \otimes H_0} = I$. Hence $\zeta = (\zeta_t)_{t \geq 0}$ is an additive $1 - V$-cocycle. Notice that $(V_{tn} \zeta_t, \zeta_t) = (W_{-t} S_{-t} V_{(n-1)t} \zeta_t, W_{-t} \chi_{-t}) = (S_{-t} V_{(n-1)t} \zeta_t, \chi_{-t}) = 0$, $n \in \mathbb{N}$, $t > 0$. Therefore the sum $\xi_t = \sum_{n=0}^{+\infty} e^{-tn} V_{tn} \zeta_t$ is well defined and we can write the integral sum $\xi_t = \lim_{t \to 0} \int_{0}^{+\infty} e^{-t} d\xi_t$. It follows that $V_t^* \xi = e^{-t} \xi$. So we have proved that the deficiency index of the generator of $V$ more or equal to one. Let $H_\zeta$ be a subspace of $H$ generated by $U_t \zeta_t$, $s, t \geq 0$. The cocycle property $\zeta_{t+s} = \zeta_t + V_t \varsigma_s = \zeta_t + U_t \varsigma_s$, $s, t \geq 0$, leads to the invariance of $H_\zeta$ under the action of $U_t$, $t \in \mathbb{R}$. Moreover $U_t \zeta_t = W_t S_t W_{-t} \chi_{-t} = W_t S_t W_{-t} S_t \chi_{-t} = S_t \chi_{-t} = - \chi_t$, $t \geq 0$. Hence $H \supset H_0 \subset H_\zeta$. Put $H^{(1)} = H_\zeta \cap H_0$, then the subspace $H^{(0)} = H_0 \cap H^{(1)}$ is invariant under the action of $U_t$, $t \in \mathbb{R}$. Hence the restriction $V|_{H(0)}$ consists of unitary operators. We get the Wold decomposition $H_0 = H^{(0)} \oplus H^{(1)}$ for the semigroup of isometries $V$. Therefore the index of $V$ equals the index of $V|_{H(1)}$ which is one.

**Theorem 2.2.** Let $W$ be a markovian cocycle, then there exists $s - \lim_{t \to +\infty} W_{-t} = W_{-\infty}$ and an isometrical operator $W_{-\infty}$ defines the Wold decomposition $H_0 = H^{(0)} \oplus H^{(1)}$ for the semigroup $V$ associated with the markovian perturbation by $W$ such that $H^{(1)} = W_{-\infty} H_0$.

**Proof.**
Firstly we shall apply the technics which is anlogues to one in the proof of theorem 2.5 in [8]. Notice that \( W_{-t-s}f = W_{-s}S_{-s}W_{-t}S_t f = W_{-s}f \) for all \( f \in \mathcal{H} \oplus \mathcal{H}_{-s}, \) \( s, t \geq 0 \) by the markovian property. Hence the sequence \( W_{-t}f \) converges when \( t \) tends to \(+\infty\) for the dense set of vectors \( f \). Therefore the limit exists by the Banach-Steinhaus theorem. One can see that the subspace \( \mathcal{H}^{(1)} = \cup_{t \geq 0} W_{-t}(\mathcal{H}_0 \ominus \mathcal{H}_{-t}) \) is generated by the functions \( \zeta_t = W_{-t} \chi_{-t}, \) \( t \geq 0 \), which form an additive \( 1 - \mathcal{V} \)-cocycle with the orthogonal increments (see the proof of Proposition 2.1). It follows that the restriction of \( V \) to \( \mathcal{H}^{(1)} \) is unitary equivalent to the semiflow of right shifts in \( L^2(\mathbb{R}) \).

Let \( \mathcal{V} \) be a subspace of \( \mathcal{H}_0 \) invariant under the action of the semigroup of right shifts \( S' = S'|_{\mathcal{H}_0} \). Then (see [8]) \( \mathcal{V} = M_\Theta \mathcal{H}_0 \), where \( F^{-1} M_\Theta F \) is the operator of multiplication by an inner function \( \Theta \), \( F \) is the Fourier transform. Notice that \( M_\Theta \) is an isometrical operator. Denote \( P_{[0,t]} \) and \( P_{[t,\infty)} \) the orthogonal projections on the spaces \( \mathcal{H}_0 \oplus \mathcal{H}_{-t} \) and \( \mathcal{H}_{-t}, \) \( t \geq 0 \), correspondingly. Given a unitary \( C_0 \)-group \( R = (R_t)_{t \in \mathbb{R}} \) in the Hilbert space \( \mathcal{H}_0 \ominus \mathcal{V} \), define a family of unitary operators in \( \mathcal{H}_0 \) by the formula

\[
W_t = (R_t P_{\mathcal{H}_0 \ominus \mathcal{V}} S_t + P_{\mathcal{V}}) P_{[t,\infty)} + M_\Theta P_{[0,t]}, \quad t \geq 0. \tag{2.1}
\]

In the following proposition we introduce the model describing all markovian cocyles up to unitary equivalence of perturbations. To be exact, for every markovian cocycle \( W \) there exists the markovian cocycle \( \tilde{W} \) of the form given in the proposition such that \( \tilde{W}_t = J_t W_t, \) \( t \in \mathbb{R} \), where the \( 1 - \mathcal{U} \)-coboundary \( J_t = J W_t S_t J^* W_{-t} S_{-t}, \) \( t \in \mathbb{R} \), of the group \( U = (W_t S_t)_{t \in \mathbb{R}} \) is defined by the unitary operator \( J \) satisfying the relation \( \tilde{W}_t S_t = J W_t S_t J^*, \) \( t \in \mathbb{R} \).

**Proposition 2.3.** The family \( W \) given by (2.1) in \( \mathcal{H}_0 \) and acting identically in \( \mathcal{H} \oplus \mathcal{H}_0 \) defines a multiplicative markovian cocycle such that \( \lim_{t \to +\infty} W_{-t}f = M_\Theta f \) for \( f \in \mathcal{H}_0 \) and \( \lim_{t \to +\infty} W_{-t}f = f \) for \( f \in \mathcal{H} \ominus \mathcal{H}_0 \). The unitary part of the semigroup \( \tilde{V} \) associated with the markovian cocycle perturbation by \( W \) is \( R \).

**Proof.**

Extend the family \( W \) defined in (2.1) for \( t \geq 0 \) by the formula \( W_t = S_t W_{-t} S_{-t}, \) \( t \geq 0 \). Consider the set of unitary operators \( U_t = W_t S_t, \) \( t \in \mathbb{R} \). Notice that \( U_{-t} M_\Theta = M_\Theta S_{-t}, \) \( U_{-t} f \in \mathcal{V}, \) \( f \in \mathcal{H}_t, \) \( t \geq 0 \). Hence the subspace \( \mathcal{L} = \mathcal{V} \oplus (\mathcal{H} \ominus \mathcal{H}_0) \) is invariant under action of \( U_t, \) \( t \in \mathbb{R} \), and the restriction \( U|_\mathcal{L} \) is unitary equivalent to the group \( S \). To complete the proof notice that the restriction \( U \) to the subspace \( \mathcal{H}_0 \ominus \mathcal{V} = \mathcal{H} \ominus \mathcal{L} \) coincides with \( R^* \).
Theorem 2.4. Given a unitary group $\tilde{R}$ which is uniformly continuous or has a pure point spectrum, there exist the inner function $\Theta$ and the unitary group $R$ in the Hilbert space $H_0 \oplus M_0 H_0$ unitary equivalent to $\tilde{R}$ such that the markovian cocycle (2.1) satisfies the condition $W_t - I \in s_2, \ t \in \mathbb{R}$.

Theorem 2.4 for the case of uniformly continuous $\tilde{R}$ can be found in [4],[5]. The proof for $\tilde{R}$ with a pure point spectrum is also constructed in cited papers in the implicit form. One can compare the condition $W_t - I \in s_2, \ t \in \mathbb{R}$ on the unitary operator $W$ appearing in the theorem with the Feldman criterion on the equivalence of Gaussian measures (see [12],[13]) and the Araki criterion of the quasi-equivalence for quasifree states of the algebra of canonical commutation relations (see [7]). It seems that the unitary operators $W$ satisfying our condition translate equivalent states one to another. It is also useful to notice that the condition $W_t - I \in s_2, \ t \in \mathbb{R}$, is necessary and sufficient for the family $W = (W_t), t \in \mathbb{R}$ to define an inner cocycle on the hyperfinite factor generated by a quasifree representation of the algebra of canonical anticommutation relations (see [5],[16]).

3 Stationary quantum stochastic process as *-homomorphism into a ring of cohomologies.

Let $K$ be an involutive algebra and $K_i = K^\otimes i$. Supply the family $(K_i)_{i=1}^{\infty}$ with a linear associative operation $\circ$ defining left and right actions of every $x \in K_i$ on $K_j$ such that $x \circ y, \ y \circ x \in K_{i+j}, \ y \in K_j$. We assume that $K_i \circ K_j = K_{i+j}, \ i, j \in \mathbb{N}$. So we obtained the graded algebra $\tilde{K} = \oplus_{i=1}^{\infty} K_i$ with respect to the multiplication defined by the operation $\circ$. If $K$ is a Lie algebra, it is possible to take the universal enveloping algebra for $\tilde{K}$ and the multiplication in $\tilde{K}$ for $\circ$. Consider a one-parameter $\omega^*$-continuous group of *-automorphisms $\alpha = (\alpha_t)_{t \in \mathbb{R}}$ on the algebra $B(H)$ of all bounded operators in a Hilbert space $H$. Suppose that the action of $\alpha$ can be correctly defined on certain involutive algebra $\mathcal{M}$ consisting of linear operators (in general unbounded) in $H$. Consider the standard resolvent for $\alpha$ constructed from nonhomogeneous chains such that (see [14])

\[
0 \rightarrow \mathcal{M} \rightarrow Hom(\mathbb{R}, \mathcal{M}) \xrightarrow{d_1} \ldots \rightarrow Hom(\mathbb{R}^i, \mathcal{M}) \xrightarrow{d_i} \ldots
\]
where \(d_i(x(t_1, \ldots, t_{i+1}) = \alpha t_1(x(t_2, \ldots, t_{i+1})) - x(t_1 + t_2, \ldots, t_{i+1}) + \ldots + (-1)^i x(t_1, \ldots, t_i)\), \(x = x(t_1, \ldots, t_i) \in \text{Hom}(\mathbb{R}^i, \mathcal{M})\). Denote \(A_i = \ker d_i/\text{Im}d_{i-1}\), then \(A = \oplus_{i=1}^{+\infty} A_i\) is a ring with respect to the multiplication defined by a bilinear map \((x, y) \mapsto x(t_1, \ldots, t_i)\alpha t_1 + \ldots + t_i (y(t_{i+1}, \ldots, t_{i+j})) \in A_{i+j}, x \in A_i, y \in A_j\).

Every \(*\)-homomorphism \(j\) of the graded algebra \(\tilde{K}\) into the graded algebra \(A\) such that every \(x \in K_i\) is translated into an \(i-\alpha\)-cocycle \(j(x)\) we shall call by a stationary quantum stochastic process over the algebra \(K\). Notice that our definition is based upon the well-known one given in [3]. It is also useful to remark that we don’t need \(0-\alpha\)-cohomologies in our construction. Sometimes we can recognize two processes \(j\) and \(\tilde{j}\) determining the cocycles \(j(x)\) and \(\tilde{j}(x)\) differ on the coboundary for the fixed \(x\) as obtaining one from other by a shift in time. For example, given a stationary quantum stochastic process \(j\) the \(1-\alpha\)-cocycle \(j(x)(r)\) is differ on the coboundary \(\alpha j(x)(r) - j(x)(r)\) from the \(1-\alpha\)-cocycle \(j(x)(r + t) - j(x)(r)\) which is associated with the stationary quantum stochastic process \(\tilde{j}\) obtained from \(j\) by a shift in time on \(r\). In applications we claim that \(j\) keeps the basic algebraic structure in \(K\) and the multiplication \(\odot\) in \(\tilde{K}\) but we do not need to require for \(j\) the preserving of the multiplication in \(K\) if it is defined (see the next section). We shall suppose that the operators involved in the image of \(j\) generate whole \(\mathcal{M}\).

**Proposition 3.1.** Let \(j\) be defined on \(K = K_1\). Then there exists a unique extension of \(j\) to whole \(\tilde{K}\).

**Proof.**

Notice that \(j\) translates the operation \(\odot\) in the cohomological multiplication \(\cup\). Denote \(j^{(i)} = j|K_i\), then one can obtain the action of \(j\) by the induction,

\[
j^{(k+\ell)}_{t_1, \ldots, t_k, t_{k+\ell}}(x \odot y) = j^{(k)}_{t_1, \ldots, t_k}(x)\alpha t_1 + \ldots + t_k (j^{(\ell)}_{t_{k+1}, \ldots, t_{k+\ell}}(y)).
\]

for all \(x \in K_k, y \in K_{\ell}\). Here we use the property \(K_i \odot K_j = K_{i+j}\).
4 Stationary quantum stochastic processes on the algebras of canonical commutation relations, the square of white noise relations and canonical anticommutation relations.

Consider an involutive Lie algebra $K$ generated by the elements $B, B^+, \Lambda, 1$ satisfying the relations $[B, B^+] = 1, [\Lambda, B] = -B, [\Lambda, B^+] = B^+$. Let the operation $\odot$ is generated by the multiplication in the universal enveloping algebra of $K$. One can define a stationary quantum stochastic process $j$ over $K$ by the formula

$$j(B) = b_t, \quad j(B^+) = b_t^*, \quad j(\Lambda) = \Lambda_t, \quad j(1) = t1,$$

where $b_t, b_t^*, \Lambda_t$ are the bosonic annihilation, creation and number of particles processes (see [15]).

Analogously if $K$ is the Lie algebra generated by the elements $B^-, B^+, M$ satisfying the relations of $\mathfrak{sl}_2$ which are $[B^-, B^+] = M, [M, B^\pm] = \pm2B^\pm$, one can consider the universal enveloping algebra of $K$ and we obtain the quantum Levy process generated by the square of white noise (SWN) constructed in [3],

$$j(B^-) = b_t, \quad j(B^+) = b_t^*, \quad j(M) = \gamma t + n_t,$$

where the basic processes $b_t, b_t^*$ and $n_t$ satisfy the relations of SWN,

$$b_t b_t^+ - b_t^+ b_t = \gamma t + n_t, \quad n_t b_t - b_t n_t = -2b_t, \quad n_t b_t^+ - b_t^+ n_t = 2b_t^+, \quad (b_t)^* = b_t^+, \quad n_t^* = n_t,$$

with a fixed parameter $\gamma > 0$ and $t \in \mathbb{R}$.

The algebra of canonical anticommutation relations (CAR) is generated by elements $a_t, a_t^*, t \in \mathbb{R}$, satisfying the relations $a_t a_t^* + a_t^* a_t = t \land s1, a_t a_s + a_s a_t = 0$. A graded algebra $K = \bigoplus_{i=1}^{\infty} K_i$ can be obtained in the following way, $K_i$ is a tensor product of i-th copies of the algebra of $2 \times 2$-matrix units. Let elements $a_k, a_k^*, k \in \mathbb{N}$, satisfy the canonical anticommutation relations $a_k a_l^* + a_l^* a_k = \delta_{kl}1, a_k a_l + a_l a_k = 0$. 

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0, \ k, \ l \in \mathbb{N}. \ Then \ K_i \ is \ generated \ by \ a_k, a_k^*, \ 1 \leq k \leq i. \ Determine \ a \ canonical \ operation \ \odot \ by \ the \ formula \ x_1 \odot x_2 \odot \ldots \odot x_n = y_1y_2\ldots y_n, \ where \ y_i = a_i, \ a_i^*, \ a_i^*a_i, \ a_i a_i^* \ if \ x_i = a, \ a^*, \ a^*a, \ aa^* \ correspondingly. \ Then \ a \ stationary \ quantum \ stochastic \ process \ j \ can \ be \ defined \ by \ the \ formula
\[
j(a) = a_t, \ j(a^*) = a_t^*,
\]
\[
j(a^*a) = \Lambda_t, \ j(1) = t1,
\]
where \(a_t, a_t^*, \Lambda_t\) \ are \ the \ basic \ Fermion \ processes \ (see \[1\]). \ Notice \ that \(j\) \ satisfies \ the \ relations \(j(a^*) = j(a)^*, \ j(a^*a + aa^*) = j(a)^*j(a) + j(a)j(a)^* = j(1), \ j([a^*a, a]) = [j(a^*a), j(a)] = -j(a) = a_t, \ j([a^*a, a^*]) = [j(a^*a), j(a^*)] = j(a^*) = a_t^*, \) \ but \ it \ is \ not \ the \ algebraic \ *-morphism \ because \(j(a^*a) \neq j(a)^*j(a)\).

5 \ Cocycle perturbations of \(K\)-flows and the Wold decomposition.

We \ shall \ use \ the \ notation \ of \ previous \ parts \ of \ this \ paper. \ Remember \ that \ strong \ continuous \ family \(W = (W_t)_{t \in \mathbb{R}}\) \ of \ unitary \ operators \ in \(\mathcal{H}\) \ is \ named \ a \ multiplicative \ \alpha\,-cocycle \ if \(W_{t+s} = W_t \alpha_t(W_s), \ s, t \in \mathbb{R}. \) \ Suppose \ that \ the \ action \ of \(W\) \ is \ correctly \ defined \ on \(\mathcal{M}, \ i.e. \ W_t x W_t^* \ are \ well \ defined \ for \ all \ t \in \mathbb{R}, \ x \in \mathcal{M}. \) \ Let \(\mathcal{M}_t, \mathcal{M}_s\) \ and \(\mathcal{M}_{[s,t]}\) \ be \ involutive \ subalgebras \ of \(\mathcal{M}\) \ generated \ by \ all \ increments \ of \ the \ form \(j(x)(r) - j(x)(l), \ x \in K_1, \ where \ l \leq r \leq t, \ s \leq l \leq r \) \ and \(s \leq l \leq r \leq t \ correspondingly. \) \ We \ shall \ call \ (see \ also \[2\]) \ a \ multiplicative \ \alpha\,-cocycle \ by \ markovian \ (with \ respect \ to \ the \ stationary \ quantum \ stochastic \ process \(j\)) \ if \(W_t \mathcal{M}_j W_t^* \subset \mathcal{M}_t\) \ and \(W_t x W_t^* = x \) \ for \ all \(x \in \mathcal{M}_t, \ t \geq 0. \)

**Theorem 5.1.** \ For \ any \ markovian \ cocycle \(W\) \ the \ formula
\[
\tilde{j}(x)(t) = j(x)(t), \ t \geq 0,
\]
\[
\tilde{j}(x)(t) = W_t j(x)(t)W_t^*, \ t \leq 0,
\]
\[
\tilde{\alpha}_t(\cdot) = W_t \alpha_t(\cdot)W_t^*, \ x \in K_1, \ t \in \mathbb{R},
\]
defines \ a \ new \ stationary \ quantum \ stochastic \ process \(\tilde{j}\) \ over \(K\) \ with \ an \ associated \ group \ of \ automorphisms \(\tilde{\alpha}. \)

**Proof.**

Analogously to the proof of Proposition 2.1 we obtain
\[
\tilde{j}(x)(t+s) = W_{t+s} j(x)(t+s)W_{t+s}^* = W_{t+s} (j(x)(t) + \alpha_t(j(x)(s)))W_{t+s}^* =
\]
\[
W_t \alpha_t(W_s)j(x)(t)\alpha_t(W_s^*)W_t^* + W_t \alpha_t(W_s)\tilde{j}(x)(s)W_s^*W_t^* = \\
W_t j(x)(t)W_t^* + \tilde{\alpha}_t(W_s j(x)(s)W_s^*) = \tilde{j}(x)(t) + \tilde{\alpha}_t(\tilde{j}(x)(s)),
\]
s, t \leq 0. Here we used the identity \(\alpha_t(W_s)j(x)(t)\alpha_t(W_s^*) = j(x)(t)\) due to the markovian property \(W_s \alpha_{-t}(j(x)(t))W_s^* = -\alpha_s(W_s^*j(x)(-t - s) - j(x)(-s))W_s = -\alpha_s(j(x)(-t - s) - j(x)(-s)) = -j(x)(-t) = \tilde{\alpha}_{-t}(j(x)(t)), s, t \leq 0\). One can extend \(\tilde{j}(x)(t)\) for \(t \geq 0\) using the cocycle condition for \(\tilde{j}(x)(t)\). It yields \(\tilde{j}(x)(t) = j(x)(t), t \geq 0\). To complete the proof we only need to apply Proposition 3.1.

**Proposition 5.2.**

\[
\tilde{\alpha}_t(x) = \alpha_t(x), \ x \in \mathcal{M}_{[0, t]}, \ t \geq 0,
\]
\[
\tilde{\alpha}_{-t}(x) = \alpha_{-t}(x), \ x \in \mathcal{M}_{[t]}, \ t \geq 0.
\]

**Proof.**

It immediately follows from the markovian property of \(W\) that \(\tilde{\alpha}_t(x) = W_t \alpha_t(x)W_t^* = \alpha_t(x), \ x \in \mathcal{M}_{[0, t]}, \ t \geq 0\). The markovian property implies that \(W_t x W_t^* = x, \ x \in \mathcal{M}_{[t]}, \ t \geq 0\), which is equivalent to \(\alpha_t(W_{-t})^* x \alpha_t(W_{-t}) = x, \ x \in \mathcal{M}_{[t]}, \ t \geq 0\), or \(W_{-t}^* x W_{-t} = x, \ x \in \mathcal{M}_{[0, t]}, \ t \geq 0\) by the cocycle condition for \(W\). Hence \(\tilde{\alpha}_{-t}(x) = W_{-t} \alpha_{-t}(x)W_{-t}^* = \alpha_{-t}(x), \ x \in \mathcal{M}_{[t]}, \ t \geq 0\).

Denote \(\mathcal{N} = \mathcal{M}'' \cap \mathcal{B}(\mathcal{H}), \mathcal{N}_{[t]} = \mathcal{M}''_{[t]} \cap \mathcal{B}(\mathcal{H}), \mathcal{N}_{[t]}'' = \mathcal{M}''_{[t]} \cap \mathcal{B}(\mathcal{H})\) and \(\mathcal{N}_{[s, t]} = \mathcal{M}''_{[s, t]} \cap \mathcal{B}(\mathcal{H})\) the corresponding von Neumann algebras. Notice that \(\mathcal{N}_{t+s} = \alpha_t(\mathcal{N}_s), t, s \in \mathbb{R}\). We shall call a stationary quantum stochastic process \(j\) by a K-flow and the group \(\alpha\) associated with \(j\) by a group of automorphisms associated with K-flow if the following conditions hold,

\[
\mathcal{N}_s \subset \mathcal{N}_t, \ t > s,
\]
\[
\bigvee_{t \in \mathbb{R}} \mathcal{N}_t = \mathcal{N},
\]
\[
\bigwedge_{t \in \mathbb{R}} \mathcal{N}_t = C1
\]
(see [1]).

**Proposition 5.3.** If there exists a vector \(\Omega \in \mathcal{H}\) which is cyclic and separating with respect to \(\mathcal{N}\) and the increments of a stationary quantum stochastic process \(j\) are independent in the classical (commutative) sense that \(\phi(xy) = \phi(x)\phi(y), x \in \mathcal{N}_{[t]}, y \in \mathcal{N}_{[t]}\), \(t \in \mathbb{R}\), for the state \(\phi(\cdot) = (\Omega, \cdot)\Omega\), then \(j\) is a K-flow.
Proof.

Choose \( x \in \bigwedge_{t \in \mathbb{R}} N[t] \), then \( \phi((x - \phi(x)1)y) = \phi(x - \phi(x)1)\phi(y) = 0 \)
for all \( y \in \bigvee_{t \in \mathbb{R}} N[t] = N \). Hence \( (x - \phi(x)1)\Omega = 0 \) and \( x = \phi(x)1 \) as \( \Omega \) is cyclic and separating. The result follows from.

Let von Neumann algebras \( \tilde{N}[t], \tilde{N}_{[s,t]} \) and \( \tilde{N}[t] \) be associated with the perturbed process \( \tilde{j} \) in the same way as the algebras \( N[t], N_{[s,t]} \) and \( N[t] \) are associated with the process \( j \).

**Proposition 5.4.** Let \( j \) and \( W \) be a \( K \)-flow and a markovian cocycle correspondingly. Then the markovian perturbation \( \tilde{j} \) is also \( K \)-flow.

**Proof.**

One can see that the von Neumann algebras generated by the increments of \( \tilde{j} \) are \( \tilde{N}[t] = W_t N[t] W_t^* \subset N[t] \) by the markovian property. Hence \( \bigwedge_{t \in \mathbb{R}} \tilde{N}[t] = C1 \). The conditions \( \tilde{N}[s] \subset \tilde{N}[t] \), \( t > s \), \( \bigvee_{t \in \mathbb{R}} \tilde{N}[t] = \tilde{N} \)
are satisfied by the definition.

For a stationary quantum stochastic process \( j \) one can name the \( E_0 \)-semigroup \( \beta_t = \alpha_{-t}|N[0] \), \( t \geq 0 \), on the von Neumann algebra \( N[0] \) by associated with \( j \). In the case when \( j \) is a \( K \)-flow, the semigroup \( \beta = (\beta_t)_{t \geq 0} \) is a flow of Powers shifts \( [3] \), i.e. \( \bigwedge_{n \in \mathbb{N}} \beta_n (N[0]) = C1, t > 0 \) (see \( [3] \)). Fix a stationary quantum stochastic process \( j \) with respect to \( j \) we shall call the \( E_0 \)-semigroup \( \tilde{\beta}_t(\cdot) = W_{-t} \beta_t(\cdot) W_{-t}^* \), \( t \geq 0 \), on \( N[0] \) by associated with the markovian perturbation of the initial process by \( W \). Two \( E_0 \)-semigroups \( \beta \) and \( \tilde{\beta} \) on the von Neumann algebras \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) correspondingly are called to be conjugate if there exist two injective *-homomorphisms \( \theta : \mathcal{A} \to \tilde{\mathcal{A}} \) and \( \theta^+ : \tilde{\mathcal{A}} \to \mathcal{A} \) such that \( \beta_t(x) = \theta^+ \beta_t(\theta(x)) \), \( \theta^+ \theta(x) = x, \theta \theta^+(y) = y, x \in \mathcal{A}, y \in \tilde{\mathcal{A}}, t \geq 0 \).

**Theorem 5.5.** Given a markovian perturbation of the \( K \)-flow \( j \) with the associated flow of Powers shifts \( \beta \) on the von Neumann algebra \( N[0] \) acting in the Hilbert space \( \mathcal{H} \) with a ciclyc vector \( \Omega \), there exists the von Neumann algebra \( \tilde{N}[0] \subset N[0] \) such that the restriction \( \tilde{\beta}|\tilde{N}[0] \) of the \( E_0 \)-semigroup \( \tilde{\beta} \) associated with the markovian perturbation is conjugate to \( \beta \).

**Proof.**

Put \( \mathcal{H}_t = [N[-t] \Omega], t \in \mathbb{R}, \) then the set \( \mathcal{H}_t, t \geq 0, \) is dense in the Hilbert space \( \mathcal{H} \). Arguing similarly to the proof of theorem 2.2.
one can obtain that there exists \( s - \lim_{t \to +\infty} W_{-t} = W_{-\infty} \). Then the injective *-endomorphism \( \theta : \mathcal{N}_{0} \to \tilde{\mathcal{N}}_{0} \subset \mathcal{N}_{0} \) given by the formula \( \theta(x) = W_{-t} x W_{-t}^{*} \), \( x \in \mathcal{N}_{[-t,0]} \), \( t \geq 0 \), is well defined because

\[
W_{-t-s} x W_{-t-s}^{*} = W_{-t} \alpha_{-t}(W_{-s}) x \alpha_{-t}(W_{-s}^{*}) W_{-t} = W_{-t} x W_{-t}^{*}
\]

for all \( x \in \mathcal{N}_{[-t,0]} \) by the markovian property \( W_{-s} y W_{-s}^{*} = y \), \( y \in \mathcal{N}_{0} \), which implies that \( W_{-s} \alpha_{t}(x) W_{-s}^{*} = \alpha_{t}(x) \) for all \( x \in \mathcal{N}_{[-t,0]} \). It follows that

\[
\lim_{t \to +\infty} W_{-t} x W_{-t}^{*} f = \lim_{t \to +\infty} W_{-t} x W_{-s} f = W_{-\infty} x W_{-s} f
\]

for all \( f \in \mathcal{H}_{-s} \), \( x \in \mathcal{N}_{0} \), \( s \geq 0 \). Hence the sequence \( W_{-t} x W_{-t}^{*} f \) converges when \( t \) tends to \( +\infty \) for all \( f \in \mathcal{H}_{0} \) by the Banach-Steinhaus theorem.

Analogously it is possible to define the injective *-homomorphism \( \theta^{+} : \tilde{\mathcal{N}}_{0} \to \mathcal{N}_{0} \) by the formula \( \theta^{+}(x) = W_{-t}^{*} x W_{-t} \), \( x \in \tilde{\mathcal{N}}_{[-t,0]} \), \( t \geq 0 \). Notice that \( \theta^{+}(x) = W_{-\infty}^{*} x W_{-\infty} \), \( x \in \tilde{\mathcal{N}}_{0} \). One can see that

\[
\beta_{t}(x) = \theta_{0} \theta^{+}(x), \quad x \in \tilde{\mathcal{N}}_{0} \]

This proves the theorem.

Earlier it was investigated the existence of “an automorphic part” in the quantum dynamical semigroup which is completely compatible with the faithful normal state (see [10]). Theorem 5.5 allows to obtain “a shift part” of the \( E_{0} \)-semigroup obtained by a markovian cocycle perturbation from the flow of Powers shifts. So it can be considered as some analogue of the picking out a completely nondeterministic part in the Wold decomposition for the classical stochastic processes.

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