FOURIER INTEGRATOR FOR PERIODIC NLS: LOW REGULARITY ESTIMATES VIA DISCRETE BOURGAINE SPACES

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Abstract. In this paper, we propose a new scheme for the integration of the periodic nonlinear Schrödinger equation and rigorously prove convergence rates at low regularity. The new integrator has decisive advantages over standard schemes at low regularity. In particular, it is able to handle initial data in $H^s$ for $\frac{1}{18} \leq s \leq 1$. The key feature of the integrator is its ability to distinguish between low and medium frequencies in the solution and to treat them differently in the discretization. This new approach requires a well-balanced filtering procedure which is carried out in Fourier space. The convergence analysis of the proposed scheme is based on discrete (in time) Bourgain space estimates which we introduce in this paper. A numerical experiment illustrates the superiority of the new integrator over standard schemes for rough initial data.

1. Introduction

We consider the cubic periodic Schrödinger equation (NLS)

$$i\partial_t u = -\partial_x^2 u + |u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{T} \quad (1)$$

which, together with its full space counterpart, has been extensively studied in the literature. In the last decades, Strichartz estimates and Bourgain spaces allowed various authors to establish well-posedness results for dispersive equations in low regularity spaces (see [1, 2, 16, 18]). The numerical theory of dispersive PDEs, on the other hand, is still restricted to smooth solutions, in general. In the case of the nonlinear Schrödinger equation (1) this stems from the following two reasons:

(A) Standard time stepping techniques, e.g., splitting methods [11] or exponential integrators [5], are based on freezing the free Schrödinger flow $S(t) = e^{it\partial_x^2}$ during a step of size $\tau$. Such freezing techniques, related to Taylor series expansion of the linear flow, however, produce derivatives in the local error terms restricting the approximation property to smooth solutions. More precisely, for first-order methods, the expansion of the free flow $S(t + \xi) = S(t) + O(\tau^2 \partial_x^2)$, $0 < \xi \leq \tau$ requires the boundedness of (at least) two additional derivatives, while higher order approximations increase the regularity requirements by two more derivatives for each additional order.

(B) Standard stability arguments in addition require smooth Sobolev spaces. Indeed, they rely on classical product estimates

$$\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}, \quad s > 1/2$$

to handle the nonlinear terms in the error analysis. This restricts the global error analysis to smooth Sobolev spaces $H^s$ with $s > 1/2$ leaving out the important class of $L^2$ spaces.

The standard local error structure introduced by the Schrödinger operator, i.e., the loss of two derivatives, together with a standard stability argument thus restricts global first-order convergence to $H^{2+1/2+\varepsilon}$ solutions (for any $\varepsilon > 0$). Using a refined global error analysis, by first proving fractional convergence of the scheme in a suitable higher order Sobolev space (which implies a priori the boundedness of the numerical solution in this space [11]), allows one to obtain stability in $L^2$ for $H^{1/2+\varepsilon}$ solutions. However, due to the standard local error structure $O(\tau^2 \partial_x^2)$, the first-order
convergence rate is nevertheless only retained for $H^2$ solutions. The latter is not only a technical formality. The order reduction in the case of non-smooth solutions is also observed numerically (see, e.g., the examples in [8, 12] and Fig. 1 in section 9 below). Only very little is known on how to overcome this problem.

Recently, the first obstacle (A) could be overcome partly by developing specifically tailored schemes which optimise the structure of the local error approximation. This has been achieved by employing Fourier based techniques that are able to discretize the central oscillations in an efficient and correct way (see [4, 12, 14, 17]). The second obstacle (B), on the other hand, is much harder to circumvent. The control of nonlinear terms in PDEs is an ongoing challenge in (computational) mathematics at large, and unlike in the parabolic setting no pointwise smoothing can be expected for dispersive PDEs. On the continuous level, however, important space time estimates featuring a gain in integrability can be used to extend well-posedness results to lower regularity spaces $H^s$ with $s < 1/2$. In the whole space, the Strichartz estimates

$$\left\| e^{it\Delta} u_0 \right\|_{L^s_t L^r_x} \leq c_{q, r} \left\| u_0 \right\|_2$$

for $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ (2)

can be used. In the periodic setting, though waves do not disperse, one can gain integrability by using Bourgain spaces (we shall give the definition of these spaces in section 2). For (1) on the torus, the crucial estimate used in the analysis is

$$\left\| u \right\|_{L^4(\mathbb{R} \times \mathbb{T})} \leq C \left\| u \right\|_{X^0, \frac{1}{2}}$$

which allows for global well-posedness for initial data in $L^2$. We refer for example to [1, 2, 16, 18].

The natural question therefore arises: In how far can we inherit this subtle smoothing property also on a discrete level? The critical issue thereby is twofold: the estimates (2) and (3) are not pointwise in time and, moreover, their gain lies in integrability and not regularity. Discrete versions of these estimates are therefore delicate to reproduce. At the same time they are essential to establish numerical stability in the same space where we have stability of the PDE. While discrete Strichartz-type estimates were successfully employed on the full space $\mathbb{R}^d$ (see, e.g., [6, 7, 13]) a global low regularity analysis on bounded domains $\Omega \subset \mathbb{R}^d$ remains an open problem. The step from the full space to the bounded setting is – as in the continuous setting – nontrivial due to the loss of dispersion. Strichartz estimates are weaker on bounded domains as the solution can not “disperse” to infinity in space. Nevertheless bounded domains are computationally very interesting as spatial discretizations of nonlinear PDEs are in general subjected to truncated domains.

In this work we introduce discrete Bourgain spaces for the periodic Schrödinger equation (1). This will allow us to break standard stability restrictions on the torus. In particular, we establish a discrete version of (3) permitting $L^2$ error estimates in $H^s$ also for $s \leq 1/2$. For the discretization of (1) we propose a new twice-filtered Fourier based technique that correctly discretizes the central oscillations of the problem. The novel discretisation approach can be applied to a larger class of dispersive equations. For simplicity we restrict our attention to the cubic Schrödinger equation. The stability analysis (based on discrete Bourgain space estimates) we develop can be extended to various numerical schemes, e.g., splitting methods. The benefit of the here introduced twice-filtered Fourier based approach it that it optimises the local error structure which allows for convergence under lower regularity than standard discretizations. The precise form of the scheme is given in (11) below. In particular, it involves a frequency localization through a filter $\Pi_K$ (see (8)) which projects on frequencies $|k| \leq K$ which will allow us to optimize the total (time and frequency) discretization error. Indeed, with the help of discrete Bourgain type estimates we can prove the following global error estimate for the new scheme.

**Theorem 1.1.** For every $T > 0$ and $u_0 \in H^{s_0}$, $0 \leq s_0 \leq 1$, let us denote by $u \in C([0, T], H^{s_0})$ the exact solution of (1) with initial datum $u_0$ and by $u^n$ the sequence defined by the scheme (11)
below. Then, there exists $\tau_0 > 0$ and $C_T > 0$ such that for every step size $\tau \in (0, \tau_0]$, we have the following error estimates:

(i) For $K = \tau^{-\frac{1}{2}}$ and $\frac{1}{18} < s_0 \leq \frac{1}{2} + \frac{1}{18}$, we have
\[
\|u^n_\tau - u(t_n)\|_{L^2} \leq C_T(\tau^{s_0} + \tau^{-s_0-\left(\frac{1}{18}\right)+}), \quad 0 \leq n\tau \leq T;
\]
(ii) For $\frac{1}{2} + \frac{1}{18} < s_0 \leq \frac{1}{2}$, with the choice $K = \tau^{-\frac{s_0}{2}}\frac{1}{s_0+\left(\frac{1}{18}\right)+}$, we have that
\[
\|u^n_\tau - u(t_n)\|_{L^2} \leq C_T\tau^{-s_0} (1 + \frac{1}{s_0+\left(\frac{11}{72}\right)+}), \quad 0 \leq n\tau \leq T;
\]
(iii) For $\frac{1}{2} < s_0 \leq 1$, we obtain with the choice $K = \tau^{-1+\frac{1}{s_0+\left(\frac{11}{72}\right)+}}$ that
\[
\|u^n_\tau - u(t_n)\|_{L^2} \leq C_T\tau^{-s_0-\left(\frac{1}{18}\right)+}, \quad 0 \leq n\tau \leq T.
\]

We have used the notation $x_+$ for any $y > x$.

In case (i), the error estimate we obtain for our new Fourier based discretization is not better than the one we would expect for standard schemes (based on classical Taylor series expansion techniques). The interesting feature, however, is that the analysis we develop is able to provide an error estimate even for data at this low level of regularity. So far error estimates (even with arbitrary low order of convergence) were restricted to solutions (at least) in $H^{\left(\frac{1}{2}\right)+}$. We note that our analysis can be employed for a large class of schemes, e.g., splitting methods or exponential integrators.

In the cases (ii) and (iii), we observe that we get a better estimate than $\tau^{\frac{s_0}{2}}$ which is the one we would expect for standard numerical schemes with a loss of two derivatives in the local error (cf. (A)). Observe for example that for $s_0 = 1$, we get an error estimate of order $\tau^{-\frac{3}{2}}$ which is much better than the standard $\tau^{-\frac{1}{2}}$. Note that it is even (slightly) better than the convergence order $\tau^{-\frac{3}{2}}$ which we obtained with the help of discrete Strichartz estimates on the full space in [13] though dispersive effects are much weaker in the periodic case. This comes from our improved Fourier based discretization with the use of the two different filters $\Pi_{\tau^{-1/2}}$ and $\Pi_K$. The favourable error behaviour is numerically underlined in Fig. 1 (see section 9 below).

The main idea in our discretization is the following. Instead of attacking directly (1), we discretize the projected equation
\[
i\partial_t u^K = -\partial_x^2 u^K + 2\Pi_K\left(\Pi_{\tau^{-1/2}} u^K \Pi_{\tau^{-1/2}} u^K \Pi_{K^+} u^K\right) + \Pi_K\left(\Pi_{\tau^{-1/2}} u^K \Pi_{\tau^{-1/2}} u^K \Pi_{K^+} u^K\right),
\]
where the projection operator $\Pi_L$ for $L > 0$ is defined by the Fourier multiplier
\[
\Pi_L = \hat{\chi}^2 \left(\frac{-i\hat{\partial}_x}{L}\right) = \Pi_L,
\]
and where $\Pi_{K^+}$ projects on the intermediate frequencies $\tau^{-1/2} \leq |k| \leq K$, i.e.,
\[
\Pi_{K^+} = \Pi_K - \Pi_{\tau^{-1/2}}.
\]
Here $\chi$ is a smooth nonnegative even function which is one on $[-1, 1]$ and supported in $[-2, 2]$. The number $K \geq 1$ is considered as a parameter that will later depend on the step size $\tau$. Note that the projection operator $\Pi_K$ in Fourier space reads
\[
\widehat{\Pi_K \hat{\phi}_\ell} = \hat{\phi}_\ell \hat{\chi}^2 \left(\frac{\ell}{K}\right), \quad \ell \in \mathbb{Z}.
\]
The reason why we base our discretization on (7) is twofold. First, we consider
\[
i\partial_t v^K = -\partial_x^2 v^K + \Pi_K(|\Pi_K v^K|^2 \Pi_K v^K), \quad v^K(0) = \Pi_K u_0
\]
as an intermediate problem the single-filtered equation where all high frequencies are truncated. The difference between solutions of (1) and (10) is estimated in Corollary 2.6 and easy to control. Second, we refine the truncated model (10) by considering a second projection $\Pi_{\tau-1/2}$ to low frequencies. Roughly speaking, each function with frequencies below $K$ is then decomposed into two parts: low frequencies for which $|k| \leq \tau^{-1/2}$ and the remaining intermediate frequencies. Since the original problem is cubic, these two projections lead to six terms in total. For our discretization, we only consider those terms in which two of the factors are of low frequencies. This motivates us to consider the twice-filtered equation (7) as an approximation to equation (1).

The discretization of the twice-filtered Schrödinger equation (7) is carried out in a way such that the terms with intermediate frequencies $\tau^{-1/2} < |k| \leq K$ are treated exactly while the lower order terms with frequencies $|k| \leq \tau^{-1/2}$ are approximated in a suitable manner. This approach allows for a low regularity approximation of solutions of (1). Motivated by our previous work [9], we thus propose the following numerical scheme:

$$
\begin{align*}
    u_{\tau}^{n+1} &=: \Phi^K_{\tau}(u_{\tau}^{n}) \\
    &= e^{i\tau \partial_x^2} u_{\tau}^{n} - 2i\Pi_K e^{i\tau \partial_x^2} J_1^\tau (\Pi_{\tau-1/2} \overline{u}_{\tau}^{n}, \Pi_K u_{\tau}^{n}, \Pi_{\tau-1/2} u_{\tau}^{n}) \\
    &
\quad - i\Pi_K e^{i\tau \partial_x^2} J_2^\tau (\Pi_K \overline{u}_{\tau}^{n}, \Pi_{\tau-1/2} u_{\tau}^{n}, \Pi_{\tau-1/2} u_{\tau}^{n}),
\end{align*}
$$

where

$$
\begin{align*}
    J_1^\tau(v_1, v_2, v_3) &= \frac{i}{2} e^{-i\tau \partial_x^2} \left[ \left( e^{i\tau \partial_x^2} \partial_x^{-1} v_2 \right) e^{i\tau \partial_x^2} \partial_x^{-1} v_1 v_3 \right] \\
    &\quad - \frac{i}{2} \left( \partial_x^{-1} v_2 \right) \partial_x^{-1} v_1 v_3 + \tau \overline{v_2} v_1 v_3 + \tau \overline{v_2 - \overline{v_2}} \overline{v_1 v_3}, \\
    J_2^\tau(v_1, v_2, v_3) &= \frac{i}{2} e^{-i\tau \partial_x^2} \left[ \left( e^{-i\tau \partial_x^2} \partial_x^{-1} v_1 \right) \left( e^{i\tau \partial_x^2} v_2 v_3 \right) \right] \\
    &\quad - \frac{i}{2} \partial_x^{-1} \left( v_2 v_3 \partial_x^{-1} v_1 \right) + \tau \overline{v_1 v_2 v_3} - \tau \overline{\overline{v_1} v_2 v_3}.
\end{align*}
$$

Here, we define for any function $f \in L^2(\mathbb{T})$ the operator $\partial_x^{-1}$ by $\partial_x^{-1} f(x) = \sum_{k \neq 0} (ik)^{-1} \hat{f}_k e^{ikx}$. Note that $u_0^n$ in (11) is considered as an approximation to the exact solution of the nonlinear Schrödinger equation (1) at time $t_n = n\tau$.

**Outline of the paper.** The paper is organized as follows. In section 2, we recall the main steps of the analysis of the Cauchy problem for (1) and we use them to estimate the difference between the exact solution of (1) and the solution of the projected equation (7). In particular, we prove that

$$
\sup_{[0,T]} \| u - u^K \|_{L^2} \leq C_T \left( \frac{1}{K^{s_0}} + \tau^{s_0} \right),
$$

see Corollary 2.6 and Proposition 2.7.

In section 3, we introduce a notion of discrete Bourgain spaces for sequences $(u_n)_n \in (L^2(\mathbb{T}))^N$ and prove their main properties. The crucial property for the error analysis is the following $L^4$ estimate,

$$
\| \Pi_K u_n \|_{L^4} \leq (K\tau^{1/2})^{1/2} \| u_n \|_{X^{s,1}},
$$

which holds uniformly for $K \geq \tau^{-3/2}$ and $0 < \tau \leq 1$. This estimate is proven in section 8. The $L^p$ norm for vector valued sequences is defined in (44). From this property, we see that the choice $K = \tau^{-3/2}$ allows one to get an estimate without loss similar to the continuous case (3). Nevertheless,
such a choice of $K$ yields a rather bad space discretization error (14). We shall thus optimize $K$ by taking it of the form $K = \tau^{-\frac{\alpha}{2}}$ for $\alpha \in [1, 2]$ to get the best possible total error.

In section 4, we establish embedding estimates between discrete and continuous Bourgain spaces. In section 5, we analyze the local error of our scheme and in section 6, we provide global error estimates. Finally in section 7, we prove the main error estimate of Theorem 1.1.

We conclude in section 9 with numerical experiments underlying the favorable error behavior of the new scheme for rough data.

**Notations.** We close this section with some notation that will be used throughout the paper. For two expressions $a$ and $b$, we write $a \lesssim b$ whenever $a \leq Cb$ holds with some constant $C > 0$, uniformly in $\tau \in (0, 1]$ and $K \geq 1$. We further write $a \sim b$ if $b \lesssim a \lesssim b$. When we want to emphasize that $C$ depends on an additional parameter $\gamma$, we write $a \lesssim_{\gamma} b$. Further, we denote $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$.

2. Cauchy problem for (1)

Let us recall the definition of Bourgain spaces. A tempered distribution $u(t, x)$ on $\mathbb{R} \times \mathbb{T}$ belongs to the Bourgain space $X^{s,b}$ if its following norm is finite

$$
\|u\|_{X^{s,b}} = \left( \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} (1 + |\sigma - k^2|)^{2b} |\tilde{u}(\sigma, k)|^2 \, d\sigma \right)^{\frac{1}{2}},
$$

where $\tilde{u}$ is the space-time Fourier transform of $u$:

$$
\tilde{u}(\sigma, k) = \int_{\mathbb{R} \times \mathbb{T}} e^{-i\sigma t - ikx} u(t, x) \, dt \, dx.
$$

We shall also use a localized version of this space, $u \in X^{s,b}(I)$ where $I \subset \mathbb{R}$ is an open interval if $\|u\|_{X^{s,b}(I)} < \infty$, where

$$
\|u\|_{X^{s,b}(I)} = \inf\{\|\tilde{u}\|_{X^{s,b}}, \tilde{u}|_I = u\}.
$$

When $I = (0, T)$ we will often simply use the notation $X^{s,b}(T)$.

We shall recall now well-known properties of these spaces. For details, we refer for example to [1], and the books [10], [18].

**Lemma 2.1.** For $\eta \in C_c^\infty(\mathbb{R})$, we have that

$$
\|\eta(t) e^{it\partial_x^2} f\|_{X^{s,b}} \lesssim_{\eta,b} \|f\|_{H^s}, \quad s \in \mathbb{R}, \ b \in \mathbb{R}, \ f \in H^s(\mathbb{T}),
$$

$$
\|\eta(t) u\|_{X^{s,b}} \lesssim_{\eta,b} \|u\|_{X^{s,b}}, \quad s \in \mathbb{R}, \ b \in \mathbb{R},
$$

$$
\|\eta(t) u\|_{X^{s,b}} \lesssim_{\eta,b} T^{-b} \|u\|_{X^{s,b}}, \quad s \in \mathbb{R}, -\frac{1}{2} < b < \frac{1}{2}, \ b > 0, \ T \leq 1,
$$

$$
\left\| \eta(t) \int_{-\infty}^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{X^{s,b}} \lesssim_{\eta,b} \|F\|_{X^{s,b-1}}, \quad s \in \mathbb{R}, \ b > \frac{1}{2},
$$

$$
\|u\|_{L^\infty(\mathbb{R}, H^s)} \lesssim_{b} \|u\|_{X^{s,b}}, \quad b > 1/2, \ s \in \mathbb{R}.
$$

We actually have the continuous embedding $X^{s,b} \subset C(\mathbb{R}, H^s)$ for $b > 1/2$. Note that we shall discuss below an extension of the definition of the Bourgain spaces and of this lemma to a discrete setting suitable for the analysis of numerical schemes and give the proofs in this discrete setting.

The crucial estimate for the analysis of the cubic NLS on the torus $\mathbb{T}$ is the following:

**Lemma 2.2.** There exists a constant $C > 0$ such that for every $u \in X^{0,\frac{3}{2}}$, we have the estimate

$$
\|u\|_{L^4(\mathbb{R} \times \mathbb{T})} \leq C \|u\|_{X^{0,\frac{3}{2}}}.
$$
Again, we refer to [18] Proposition 2.13 for its proof. Note that, by duality, we also obtain that
\[ \|u\|_{X^0,-\frac{3}{\delta}} \lesssim \|u\|_{L^2((\mathbb{R} \times [0,T])).} \]
By combining the two estimates with Hölder, this further implies that
\[ \|u w v\|_{X^0,-\frac{3}{\delta}} \lesssim \|u\|_{X^0,\frac{3}{\delta}} \|v\|_{X^0,\frac{3}{\delta}} \|w\|_{X^0,\frac{3}{\delta}}. \] (20)

For (1), we have the following global well-posedness result.

**Theorem 2.3.** For every \( T > 0 \) and \( u_0 \in L^2 \), there exists a unique solution \( u \) of (1) such that \( u \in C([0,T],L^2) \cap X^{0,b}(T) \) for any \( b \in (1/2,5/8) \). Moreover, if \( u_0 \in H^{s_0}, s_0 > 0 \), then \( u \in C([0,T],H^{s_0}) \cap X^{s_0,b}(T) \).

**Proof.** Let us recall the main steps of the proof. The existence is proven by a fixed point argument on the following truncated problem:
\[ v \mapsto F(v) \]
such that
\[ F(v)(t) = \eta(t)e^{i\dot{\eta}^2 t}u_0 - i\eta(t) \int_0^t e^{i(t-s)\dot{\eta}^2} \left( \eta \left( \frac{s}{\delta} \right) |v(s)|^2 v(s) \right) \, ds, \] (21)
where \( \eta \in [0,1] \) is a smooth compactly supported function which is equal to 1 on \([-1,1]\) and supported in \([-2,2]\). For \( |t| \leq \delta \leq 1/2 \), a fixed point of the above equation gives a solution of the original Cauchy problem, denoted by \( u \).

Thanks to Lemma 2.1, there exists \( C > 0 \) which does not depend on \( u_0 \) such that
\[ \|\eta(t)e^{i\dot{\eta}^2} u_0\|_{X^{0,b}} \leq C \|u_0\|_{L^2}. \]
Moreover, by using Lemma 2.1 and (20), we can estimate the Duhamel term by
\[ \left\| \eta(t) \int_0^t e^{i(t-s)\Delta} \left( \eta \left( \frac{s}{\delta} \right) |v(s)|^2 v(s) \right) \, ds \right\|_{X^{0,b}} \leq C \left\| \eta \left( \frac{s}{\delta} \right) |v(s)|^2 v(s) \right\|_{X^{0,b-1}} \leq C \delta^{c_0} \|v\|^3_{X^{0,\frac{3}{\delta}}} \leq C \delta^{c_0} \|v\|^3_{X^{0,b}}, \]
where \( C > 0 \) is again a generic constant and \( c_0 = 5/8 - b > 0 \) by the choice of \( b \). Therefore, we have obtained that
\[ \|F(v)\|_{X^{0,b}} \leq C \|u_0\|_{L^2} + C \delta^{c_0} \|v\|^3_{X^{0,b}}. \]
In a similar way, we obtain that if \( v_1 \) and \( v_2 \) are such that \( \|v_1\|_{X^{0,b}} \leq R, \|v_2\|_{X^{0,b}} \leq R \), then
\[ \|F(v_1) - F(v_2)\|_{X^{0,b}} \leq 4C \delta^{c_0} R^2 \|v_1 - v_2\|_{X^{0,b}}. \]
Consequently, by taking \( R = 2C \|u_0\|_{L^2} \), we get that there exists \( \delta > 0 \) sufficiently small that depends only on \( \|u_0\|_{L^2} \) such that \( F \) is a contraction on the closed ball \( B(0,R) \) of \( X^{0,b} \). This proves the existence of a fixed point \( v \) for \( F \) and hence the existence of a solution \( u \) of (1) on \([0,\delta]\). By using Lemma 2.1, we actually get that \( u \in C([0,\delta],L^2) \). Since for \( s \geq 0 \),
\[ \|F(v)\|_{X^{s,b}} \leq C \|u_0\|_{H^s} + C \delta^{c_0} \|v\|^2_{X^{0,b}} \|v\|_{X^{s,b}}, \]
we also get that if \( u_0 \) is in \( H^s \) then \( u \in X^{s,b}([0,\delta]) \). Since the \( L^2 \) norm is conserved for (1), we can reiterate the construction on \([\delta,2\delta],..,\) to get a global solution. Moreover, since \( \delta \) depends only on the \( L^2 \) norm of \( u_0 \), we get that if \( u_0 \) is in \( H^s \), \( s \geq 0 \) then \( u \in X^{s,b}(T) \) and thus \( u \in C([0,T],H^s) \) for every \( T \).

Let us now consider \( v^K \) that solves the frequency truncated equation
\[ i\dot{\eta} v^K = -\partial_x^2 v^K + \Pi_K(|\Pi_K v^K|^2 \Pi_K v^K), \quad v^K(0) = \Pi_K u_0. \] (22)
As in Theorem 2.3, we can easily get:
Proposition 2.4. For \( u_0 \in H^{s_0}, s_0 \geq 0, \) and \( K \geq 1, \) there exists a unique solution \( v^K \) of (22) such that \( v^K \in X^{s_0,b}(T) \) for \( b \in (1/2, 5/8) \) and every \( T > 0. \) Moreover, for every \( T > 0, \) there exists \( M_T \) such that for every \( K \geq 1, \) we have the estimate

\[
\|v^K\|_{X^{s_0,b}(T)} \leq M_T.
\]

We shall not detail the proof of this proposition that follows exactly the lines of the proof of Theorem 2.3.

Remark 2.5. Since \( \Pi_{2K} \Pi_K = \Pi_K, \) we have that \( \Pi_{2K} v^K \) solves the same equation (22) with the same initial data. Hence, by uniqueness, we have that

\[
\Pi_{2K} v^K(t) = v^K(t) \quad \text{for all} \quad t \in [0,T].
\]

We can also easily get the following corollary.

Corollary 2.6. For \( u_0 \in H^{s_0}, s_0 \geq 0 \) and every \( T > 0, \) there exists \( C_T > 0 \) such that for every \( K \geq 1 \) we have the estimate

\[
\|u - v^K\|_{X^{s_0,b}(T)} \leq C_T K^{-s_0},
\]

where \( b \) is as in Theorem 2.3.

Proof. For appropriately chosen \( \delta > 0 \) and \( \eta \) as in (21), we observe that on \([0,\delta],\) \( v^K \) is the restriction of \( V^K \in X^{s_0,\frac{7}{8}}(\mathbb{R}) \) that solves

\[
V^K(t) = \eta(t) e^{it\partial_x^2} v^K(0) - i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left( \eta \left( \frac{s}{\delta} \right) \Pi_K \left( |\Pi_K V^K(s)|^2 \Pi_K V^K(s) \right) \right) ds.
\]

Consequently, by denoting \( U \in X^{s_0,\frac{7}{8}}(\mathbb{R}) \) the fixed point of \( F \) such that on \([0,\delta],\) \( U = u, \) we obtain that

\[
U(t) - V^K(t) = \eta(t) e^{it\partial_x^2} (1 - \Pi_K) u_0 - i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left( \eta \left( \frac{s}{\delta} \right) \Pi_K \left( |U(s)|^2 U(s) - |\Pi_K U(s)|^2 \Pi_K U(s) \right) \right) ds
- i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left( \eta \left( \frac{s}{\delta} \right) \Pi_K \left( |\Pi_K U(s)|^2 \Pi_K U(s) - |\Pi_K V^K(s)|^2 \Pi_K V^K(s) \right) \right) ds
- i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left( \eta \left( \frac{s}{\delta} \right) (1 - \Pi_K) \left( |U(s)|^2 U(s) \right) \right) ds.
\]

Now, let us fix \( M_T \) independent of \( K \geq 1 \) such that

\[
\|V^K\|_{X^{s_0,b}} + \|U\|_{X^{s_0,b}} \leq M_T.
\]

Note that for every \( f, \) we have the estimate

\[
\|f - \Pi_K f\|_{X^{0,b}} \lesssim \frac{1}{K^{s_0}} \|f\|_{X^{s_0,b}}.
\]

By employing the same estimates as before, we thus obtain that

\[
\|U - V^K\|_{X^{0,b}} \lesssim \frac{1}{K^{s_0}} \|u_0\|_{H^{s_0}} + \delta^{s_0} \|U - \Pi_K U\|_{X^{0,b}} \|U\|_{X^{s_0,b}}^2 + \delta^{s_0} \|U - V^K\|_{X^{0,b}} (\|U\|_{X^{s_0,b}}^2 + \|V^K\|_{X^{s_0,b}}^2)
+ \frac{1}{K^{s_0}} \|U\|_{X^{s_0,b}}^2 \|U\|_{X^{s_0,b} - 1}
\]

\[
\lesssim \frac{1}{K^{s_0}} (\|u_0\|_{H^{s_0}} + M_T^3) + \delta^{s_0} M_T^2 \|U - V^K\|_{X^{s_0,b}}.
\]

For \( \delta \) sufficiently small, this yields the desired estimate. We can then iterate in order to get the estimate on \([0,T].\) \( \square \)
Instead of performing directly a time discretization of equation (22), it will be convenient for the analysis to study a slightly modified equation. Let \( u^K \) be the solution of
\[
 i\partial_t u^K = -\partial^2_t u^K + 2\Pi_K \left( \Pi_{\tau-1/2} u^K \Pi_{\tau-1/2} \overline{u^K} \Pi_K u^K \right) + \Pi_K \left( \Pi_{\tau-1/2} u^K \Pi_{\tau-1/2} \overline{u^K} \Pi_K \overline{u^K} \right)
\]
(cf. (7)), again with the initial data \( u^K(0) = \Pi_K u_0 \). Note that the difference between this truncated equation and (22) is in the trilinear terms, where we can always project at least two factors on frequencies less than \( \tau^{-1/2} \). Further note that \( u^K \) depends also on \( \tau \) though we do not explicitly mention it in order to keep reasonable notation. However, we will henceforth link \( K \) and \( \tau \) by the relation
\[
 K = \tau^{-\frac{\alpha}{2}}, \quad \alpha \geq 1,
\]
with the (optimal) value of \( \alpha \) still to be determined.

Again, we have existence and uniqueness of the solution.

**Proposition 2.7.** For \( u_0 \in H^{s_0} \), \( s_0 \geq 0 \) and \( K \geq 1 \), there exists a unique solution \( u^K \) of (7) such that \( u^K \in X^{s_0,b}(T) \) (with \( b \) as before) for every \( T > 0 \). Moreover, for every \( T > 0 \), there exists \( M_T \) such that for every \( K \geq 1 \), we have the estimate
\[
 \|u^K\|_{X^{s_0,b}(T)} \leq M_T.
\]
Further, it holds that uniformly for \( K \geq 1 \),
\[
 \|u^K - v^K\|_{X^{0,b}(T)} \leq C_T T^{-s_0}.
\]

Observe that by combining the last estimate with the estimate of Corollary 2.6, we actually get that
\[
 \|u - u^K\|_{X^{0,b}(T)} \leq C_T T^{-\frac{s_0 + \alpha}{2}}
\]
for \( K = \tau^{-\frac{\alpha}{2}} \) and \( \alpha \) such that \( 1 \leq \alpha \leq 2 \).

**Proof.** The proof of the first part follows again the lines of the proof of Theorem 2.3. Let us explain how to prove the error estimate. Let us denote by \( G_{\tau,K}(u^K) \) the nonlinear term on the right-hand side of (7). We first observe that we can write
\[
 \Pi_K (\Pi_K u^K \Pi_K v^K) = G_{\tau,K}(u^K) + R_K(v^K),
\]
where the remainder is a sum of terms of the form
\[
 R_K(v^K) = \sum_{\{i_1,i_2,i_3\}} \Pi_K(Q_{i_1} u^K Q_{i_2} \overline{v^K} Q_{i_3} v^K)
\]
and where in the sum \( Q_i \) can be \( \Pi_{\tau-1/2} \) or \( (1 - \Pi_{\tau-1/2}) \Pi_K \) and at least two different \( Q_i \) are \( (1 - \Pi_{\tau-1/2}) \Pi_K \). Let us denote again by \( U^K \) and \( V^K \) the fixed points of the extended Duhamel formulation such that uniformly for \( K \geq 1 \), we have
\[
 \|V^K\|_{X^{s_0,b}} + \|U^K\|_{X^{s_0,b}} \leq M_T.
\]
Then we get that
\[
 U^K(t) - V^K(t) = -i\eta(t) \int_0^t e^{i(t-s)\Delta} \left( \eta \left( \frac{s}{\delta} \right) \left( G_{\tau,K}(U^K(s)) - G_{\tau,K}(V^K(s)) \right) \right) \, ds
\]
\[
 -i\eta(t) \int_0^t e^{i(t-s)\Delta} \left( \eta \left( \frac{s}{\delta} \right) R_K(V^K(s)) \right) \, ds.
\]
By using again the properties of Bourgain spaces and the estimate
\[
 \|f - \Pi_{\tau-1/2} f\|_{X^{0,b}} \lesssim \frac{T}{\delta} \|f\|_{X^{s_0,b}}, \quad \forall f \in X^{s_0,b},
\]
we obtain that
\[ \|U^K - V^K\|_{X^{0,b}} \lesssim \delta^4 \|U^K - V^K\|_{X^{0,b}} M_T^2 + \|V^K - \Pi_{t-1/2} V^K\|_{X^{0,b}}^2 \]
\[ \lesssim \delta^4 \|U^K - V^K\|_{X^{0,b}} M_T^2 + \tau^{s_0} M_T^3 \]
and we can conclude as we did before.

We shall need the following corollary about the propagation of higher regularity with respect to the \( b \) parameter.

**Corollary 2.8.** Let \( b \in (5/8,1) \) and set \( s'_0 = s_0 - \frac{8b-5}{36} \). Assuming that \( s'_0 \geq 0 \), we also have for every \( T > 0 \) and uniformly in \( \tau \),
\[ \|u^K\|_{X^{s'_0,b}(T)} \leq M_T. \]

**Proof.** Since \( u^K \) solves (7), we have that
\[ u^K(t) = e^{i\partial_t^2} \Pi_K u_0 + \int_0^t e^{i(t-s)\partial_t^2} F(u^K(s)) \, ds, \]
where we set for short
\[ iF(u^K) = 2\Pi_K \left( \Pi_{t-1/2} u^K \Pi_{t-1/2} \Pi_K + u^K \right) + \Pi_K \left( \Pi_{t-1/2} u^K \Pi_{t-1/2} u^K \Pi_K \right). \]
Let \( u^{K,\eta} \) denote the solution of the truncated Duhamel equation
\[ u^{K,\eta}(t) = \eta(t)e^{i\partial_t^2} u_0 + \eta(t) \int_0^t e^{i(t-s)\partial_t^2} \eta(s) F(u^{K,\eta}(s)) \, ds \]
that belongs to the global Bourgain space \( X^{s_0,b}(\mathbb{R} \times T) \) for \( b \in (1/2,5/8) \) as established in the previous proposition. From the same estimates as before, we obtain that
\[ \|u^{K,\eta}\|_{X^{s_0,b}} \lesssim \|u_0\|_{H^{s_0}} + \|F(u^{K,\eta})\|_{X^{s'_0,b-1}}. \]
In order to estimate \( F(u^{K,\eta}) \), we interpolate the estimate of Lemma 2.2 with the trivial estimate
\[ \|u\|_{L^2} \lesssim \|u\|_{X^{0,0}}. \]
This yields
\[ \|u\|_{L^2} \lesssim \|u\|_{X^{0,1-b}}, \quad \frac{1}{p} = \frac{4b-1}{6} \]
and hence by duality
\[ \|u\|_{X^{0,b-1}} \lesssim \|u\|_{L^{p'}}. \]
By using this last estimate together with the generalized Leibnitz rule, we get that
\[ \|F(u^{K,\eta})\|_{X^{s'_0,b-1}} \lesssim \|u^{K,\eta}\|_{W^{s'_0,3p'}}. \]
Since \( W^{s'_0,3,4} \subset L^{3p'} \) we finally obtain
\[ \|F(u^{K,\eta})\|_{X^{s'_0,b-1}} \lesssim \|u^{K,\eta}\|_{W^{s_0,4}}^3 \]
and hence, by using again Lemma 2.2 that
\[ \|F(u^{K,\eta})\|_{X^{s'_0,b-1}} \lesssim \|u^{K,\eta}\|_{X^{s_0,3}}^3. \]
This ends the proof. The right-hand side is already controlled thanks to Proposition 2.7. □
3. Discrete Bourgain spaces

For a sequence \((u_n)_{n \in \mathbb{Z}}\), we shall define its Fourier transform as
\[
\mathcal{F}_\tau(u_n)(\sigma) = \tau \sum_{m \in \mathbb{Z}} u_m e^{im\tau\sigma}.
\]
This defines a periodic function on \([−\pi/\tau, \pi/\tau]\) and we have the inverse Fourier transform formula
\[
u_m = \frac{1}{2\pi} \int_{-\pi/\tau}^{\pi/\tau} \mathcal{F}_\tau(u_n)(\sigma) e^{-im\tau\sigma} \, d\sigma.
\]
With these definitions the Parseval identity reads
\[
\|u_n\|_{L^2}^2 = \|\mathcal{F}_\tau(u_n)\|_{L^2(-\pi/\tau, \pi/\tau)}^2,
\]
where the norms are defined by
\[
\|u_n\|_{L^2}^2 = \tau \sum_{n \in \mathbb{Z}} |u_n|^2, \quad \|\mathcal{F}_\tau(u_n)\|_{L^2(-\pi/\tau, \pi/\tau)}^2 = \frac{1}{2\pi} \int_{-\pi/\tau}^{\pi/\tau} |\mathcal{F}_\tau(u_n)(\sigma)|^2 \, d\sigma.
\]

In this section, we write \(L^2\) instead of \(L^2(-\pi/\tau, \pi/\tau)\) for short. We stress the fact that this is not the standard way of normalizing the Fourier series.

We then define in a natural way Sobolev spaces \(H^b_\tau\) of sequences \((u_n)_{n \in \mathbb{Z}}\) by
\[
\|u_n\|_{H^b_\tau} = \|\langle d_\tau(\sigma) \rangle^b \mathcal{F}_\tau(u_n)\|_{L^2},
\]
with \(d_\tau(\sigma) = \frac{e^{i\tau\sigma} - 1}{\tau}\) so that we have equivalent norms
\[
\|u_n\|_{H^b_\tau} = \|\langle D_\tau \rangle^b u_n\|_{L^2},
\]
where the operator \(D_\tau\) is defined by \((D_\tau(u_n))_n = \left(\frac{u_n - u_{n-1}}{\tau}\right)_n\) since by definition of the Fourier transform
\[
\mathcal{F}_\tau(D_\tau u_n)(\sigma) = d_\tau(\sigma) \mathcal{F}_\tau(u_n)(\sigma).
\]
Note that \(d_\tau\) is \(2\pi/\tau\) periodic and that uniformly in \(\tau\), we have \(|d_\tau(\sigma)| \sim |\sigma|\) for \(|\tau\sigma| \leq \pi\).

For sequences of functions \((u_n(x))_{n \in \mathbb{Z}}\), we define the Fourier transform \(\widehat{u}_n(\sigma, k)\) by
\[
\mathcal{F}_{\tau,x}(u_n)(\sigma, k) = \widehat{u}_n(\sigma, k) = \tau \sum_{m \in \mathbb{Z}} \widehat{u}_m(k) e^{im\tau\sigma}, \quad \widehat{u}_m(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_m(x) e^{-ikx} \, dx.
\]
Parseval’s identity then reads
\[
\|\widehat{u}_n\|_{L^2(x)}^2 = \|u_n\|_{L^2}^2,
\]
where
\[
\|\widehat{u}_n\|_{L^2(x)}^2 = \int_{-\pi/\tau}^{\pi/\tau} \sum_{k \in \mathbb{Z}} |\widehat{u}_n(\sigma, k)|^2 \, d\sigma, \quad \|u_n\|_{L^2}^2 = \tau \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} |u_m(x)|^2 \, dx.
\]
We then define the discrete Bourgain spaces \(X^{s,b}_\tau\) for \(s \geq 0, b \in \mathbb{R}, \tau > 0\) by
\[
\|u_n\|_{X^{s,b}_\tau} = \|e^{-i\tau\partial^2_{x}} u_n\|_{H^s_b} = \|\langle D_\tau \rangle^b \langle \partial_x \rangle^s (e^{-i\tau\partial^2_{x}} u_n)\|_{L^2}.
\]
As in the continuous case, we obtain the following properties.
Lemma 3.1. With the above definition, we have that
\[ \|u_n\|_{X^{s,b}_\tau} \sim \| (k)^s (d_{\tau}(\sigma - k^2))^b \tilde{u}_n(\sigma, k) \|_{L^2}. \] (27)
Moreover, for \( s \in \mathbb{R} \) and \( b > 1/2 \), we have that \( X^{s,b}_\tau \subset l^\infty H^s \):
\[ \|u_n\|_{l^\infty H^s} \lesssim_b \|u_n\|_{X^{s,b}_\tau}. \] (28)

The weight \( d_{\tau}(\sigma - k^2) \) obviously vanishes if \( \tau(\sigma - k^2) = 2m\pi \) for \( m \in \mathbb{Z} \). For a localized function such that \( k \) is constrained to \( |k| \lesssim \tau^{-\frac{1}{2}} \) this will behave like in the continuous case with only a cancellation when \( \sigma = k^2 \). For larger frequencies, however, there are additional cancellations that will create some loss in the product estimates.

**Proof.** Let us set \( f_n(x) = e^{-in\tau \xi_2^2}u_n(x) \). From the definition of \( \mathcal{F}_\tau \), we get that
\[ \tilde{f}_n(\sigma, k) = \tau \sum_{m \in \mathbb{Z}} \tilde{u}_m(k) e^{inn(\sigma + k^2)} \]
so that
\[ \tilde{f}_n(\sigma, k) = \tilde{u}_n(\sigma + k^2, k). \] (29)
Therefore,
\[ \|u_n\|_{X^{s,b}_\tau}^2 = \sum_{k \in \mathbb{Z}} \left( \int_{-\pi}^{\pi} |d_{\tau}(\sigma)|^{2b} |\tilde{u}_n(\sigma + k^2, k)|^2 d\sigma \right) \]
and the result follows by a change of variables.

To prove the embedding (28), it suffices to prove that
\[ \|f_n\|_{l^\infty H^s} \lesssim \|f_n\|_{H^s_{\tau} H^s}. \]
Since
\[ \tilde{f}_n(k) = \int_{-\pi}^{\pi} \tilde{f}_m(\sigma, k) e^{-i\tau \sigma} d\sigma, \]
we get from Cauchy–Schwarz that
\[ |\tilde{f}_n(k)| \lesssim \left( \int_{-\pi}^{\pi} \frac{1}{|d_{\tau}(\sigma)|^{2b}} d\sigma \right)^{\frac{1}{2}} \|d_{\tau}(\sigma)|^b \tilde{f}_m(\sigma, k)\|_{L^2}. \]
The result then follows by multiplying the above inequality by \( (k)^s \) and taking the \( L^2 \) norm with respect to \( k \). \( \square \)

**Remark 3.2.** From Lemma 3.1, we can make the following useful observation
\[ \sup_{\delta \in [-4,4]} \|e^{in\tau \xi_2^2} u_n\|_{X^{s,b}_\tau} \lesssim_b \|u_n\|_{X^{s,b}_\tau}. \] (30)

Note that this follows at once from \( |d_{\tau}(\sigma - k^2 + \delta)| \lesssim |d_{\tau}(\sigma - k^2)| \).

**Remark 3.3.** Since \( (d_{\tau}(\sigma)) \lesssim \frac{1}{\tau} \), the discrete spaces satisfy the embedding
\[ \|u_n\|_{X^{0,b}_\tau} \lesssim \frac{1}{\tau^{b-b'}} \|u_n\|_{X^{0,b'}}, \quad b \geq b'. \] (31)

Indeed, from the above observation we get that the inequality is true for \( b \geq 0 \), \( b' = 0 \). Next, by interpolation we obtain the case \( b \geq b' \geq 0 \). The case \( 0 \geq b \geq b' \) then follows by duality and the general case by composition.

We shall now establish the counterpart of Lemma 2.1 at the discrete level.
Lemma 3.4. For \( \eta \in C^\infty_c(\mathbb{R}) \) and \( \tau \in (0, 1) \), we have that
\[
\| \eta(n \tau) e^{i n \tau \partial_x^2} f \|_{X^{s, b}_v} \lesssim_{n, b} \| f \|_{H^s}, \quad s \in \mathbb{R}, \ b \in \mathbb{R}, \ f \in H^s, \tag{32}
\]
\[
\| \eta(n \tau) u_n \|_{X^{s, b}_v} \lesssim_{n, b} \| u_n \|_{X^{s, b}_v}, \quad s \in \mathbb{R}, \ b \in \mathbb{R}, \ u_n \in X^{s, b}_v, \tag{33}
\]
\[
\left\| \eta \left( \frac{n \tau}{T} \right) u_n \right\|_{X^{s, b}_{v'}} \lesssim_{n, b, b'} T^{-1} \| u_n \|_{X^{s, b}_v}, \quad s \in \mathbb{R}, -\frac{1}{2} < b' \leq b < \frac{1}{2}, 0 < T = N \tau \leq 1, N \geq 1. \tag{34}
\]
In addition, for
\[
U_n(x) = \eta(n \tau) \sum_{m=0}^{\infty} e^{i (n-m) \tau \partial_x^2} u_m(x),
\]
we have
\[
\| U_n \|_{X^{s, b}_v} \lesssim_{n, b} \| u_n \|_{X^{s, b-1}_v}, \quad s \in \mathbb{R}, \ b > 1/2. \tag{35}
\]
We stress that all given estimates are uniform in \( \tau \).

Proof. We begin with (32). Let us set \( u_n(x) = \eta(n \tau) e^{i n \tau \partial_x^2} f(x) \). We first observe that
\[
\hat{u}_n(\sigma, k) = \mathcal{F}_{\tau}(\eta(n \tau))(\sigma - k^2) \hat{f}(k).
\]
The function \( g(\sigma) = \mathcal{F}_{\tau}(\eta(n \tau))(\sigma) \) is fastly decreasing in the sense that
\[
|d_{\tau}(\sigma) g(\sigma)| \lesssim 1, \tag{36}
\]
where the estimate is uniform in \( \tau \) and \( \sigma \) for every \( L \geq 1 \). Indeed we have that
\[
d_{\tau}(\sigma) g(\sigma) = \frac{\tau}{n} \sum_{n \in \mathbb{Z}} \frac{\eta((n-1) \tau) - \eta(n \tau)}{\tau} e^{i n \tau \sigma}
\]
and therefore
\[
|d_{\tau}(\sigma) g(\sigma)| \lesssim 1 \tag{37}
\]
follows from the smoothness of \( \eta \). We easily get the boundedness of higher powers by induction. The estimate then follows easily from Lemma 3.1.

Let us prove (33). We recall that
\[
\eta(n \tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\sigma) e^{-i n \tau \sigma} d\sigma.
\]
We deduce from (36) that for very \( L \geq 1 \), there exists \( C > 0 \) such that for every \( \tau \in (0, 1] \) and \( \sigma \) with \( \tau \sigma \in [-\pi, \pi] \),
\[
|g(\sigma)| \leq \frac{C}{\langle \sigma \rangle^L}. \tag{38}
\]
This yields, by using the fast decay of \( g(\sigma) \), that
\[
\| \eta(n \tau) u_n \|_{X^{s, b}_v} \lesssim \int_{-\pi}^{\pi} \frac{1}{\langle \sigma_0 \rangle^L} \| u_n e^{-i n \tau \sigma_0} \|_{X^{s, b}_v} d\sigma_0.
\]
Next, since \( \mathcal{F}_{\tau}(u_n e^{-i n \tau \sigma_0})(\sigma, k) = \hat{u}_n(\sigma - \sigma_0, k) \) and \( d_{\tau}(\sigma - \sigma_0 - k^2)^b \lesssim \langle \sigma_0 \rangle |b| \langle \sigma_0 \rangle (\sigma - k^2)^b \), we get that
\[
\| \eta(n \tau) u_n \|_{X^{s, b}_v} \lesssim \int_{-\pi}^{\pi} \frac{1}{\langle \sigma_0 \rangle^L} d\sigma_0 \| u_n \|_{X^{s, b}_v} \lesssim \| u_n \|_{X^{s, b}_v}
\]
by choosing \( L \) sufficiently large.

We turn to the proof of (34). We follow the steps of the proof of the continuous case in [18]. We observe that by composition it suffices to handle the cases \( 0 \leq b' \leq \left\lfloor b \right\rfloor \) or \( b' \leq b \leq 0 \). By duality, it
suffices then to establish the inequality in the case $0 \leq b' \leq b$. By standard interpolation, we have that
\[
\left\| \eta \left( \frac{n \tau}{T} \right) u_n \right\|_{X^{s,b}_T} \leq \left\| \eta \left( \frac{n \tau}{T} \right) u_n \right\|_{X^{s,0}_T}^{1-b'} \left\| \eta \left( \frac{n \tau}{T} \right) u_n \right\|_{X^{s,b}_T}^{b'}.
\]
It thus suffices to prove that
\[
\left\| \eta \left( \frac{n \tau}{T} \right) u_n \right\|_{X^{s,b}_T} \lesssim \left\| u_n \right\|_{X^{s,b}_T}
\] (39)
and that
\[
\left\| \eta \left( \frac{n \tau}{T} \right) u_n \right\|_{X^{s,0}_T} \leq T^b \left\| u_n \right\|_{X^{s,b}_T}
\] (40)
for $b < 1/2$, where the estimates are uniform for $T \in (0,1]$. We start with the first estimate. Note that we cannot use directly (33) to get an estimate uniform in $T$. Let us set $f_n = e^{-i \tau \sigma^2} u_n$ and $U_n = \eta(n \tau/T) f_n$. We want to estimate
\[
\left\| \eta \left( \frac{n \tau}{T} \right) f_n \right\|_{H^b_H} = \left\| U_n \right\|_{H^b_H}.
\]
We have that
\[
\bar{U}_m(\sigma, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_T(\sigma - \sigma') \tilde{f}_m(\sigma', k) d\sigma',
\]
where we have set
\[
g_T(\sigma) = \tau \sum_n \eta \left( \frac{n \tau}{T} \right) e^{i n \tau \sigma}.
\]
Using the same argument as above, we observe that for every $L \geq 0$
\[
\left\| g_T(\sigma) \right\| \lesssim_L \frac{T}{\langle T \sigma \rangle^L}. \tag{41}
\]
In particular, this yields
\[
\left\| g_T \right\|_{L^1(\pi/\tau, \pi/\tau)} \lesssim 1, \quad \left\| \langle \sigma \rangle^b g_T \right\|_{L^2(\pi/\tau, \pi/\tau)} \lesssim T^{b-1}, \quad \left\| \langle \sigma \rangle^b g_T \right\|_{L^1(\pi/\tau, \pi/\tau)} \lesssim T^{-b}. \tag{42}
\]
We can first write by using Young’s inequality for convolutions
\[
\left\| \langle d_T(\sigma) \rangle^b \bar{U}_m(\sigma, k) \right\|_{L^2} \lesssim \left\| g_T \right\|_{L^1} \left\| \langle d_T(\sigma') \rangle^b \tilde{f}_m(\sigma', k) \right\|_{L^2} + \int_{-\pi/\tau}^{\pi/\tau} \left\| \langle d_T(\sigma - \sigma') \rangle^b g_T(\sigma - \sigma') \right\| \left\| \tilde{f}_m(\sigma', k) \right\| d\sigma'.
\]
To estimate the last integral, we split $\tilde{f}_m(\sigma', k) = \tilde{f}_m(\sigma', k) 1_{|\sigma' T| \leq 1} + \tilde{f}_m(\sigma', k) 1_{|\sigma' T| \geq 1}$. For the first contribution, we write
\[
\left\| \int_{-\pi/\tau}^{\pi/\tau} \langle d_T(\sigma - \sigma') \rangle^b g_T(\sigma - \sigma') \tilde{f}_m(\sigma', k) 1_{|\sigma' T| \leq 1} d\sigma' \right\|_{L^2} \lesssim \left\| \langle \sigma \rangle^b g_T \right\|_{L^2(\pi/\tau, \pi/\tau)} \left\| 1_{|\sigma T| \leq 1} \tilde{f}_m(\sigma, k) \right\|_{L^1} \lesssim T^{b-\frac{b}{2}} T^{b-\frac{b}{2}} \left\| \tilde{f}_m(\cdot, k) \right\|_{L^2},
\]
where we have used Cauchy–Schwarz to get the last estimate. For the second contribution, we use
\[
\left\| \int_{-\pi/\tau}^{\pi/\tau} \langle d_T(\sigma - \sigma') \rangle^b g_T(\sigma - \sigma') \tilde{f}_m(\sigma', k) 1_{|\sigma' T| \geq 1} d\sigma' \right\|_{L^2} \lesssim \left\| \langle \sigma \rangle^b g_T \right\|_{L^1(\pi/\tau, \pi/\tau)} \left\| 1_{|\sigma T| \geq 1} \tilde{f}_m(\sigma, k) \right\|_{L^2} \lesssim T^{-b} \left\| 1_{|\sigma T| \geq 1} \tilde{f}_m(\cdot, k) \right\|_{L^2} \lesssim \left\| \langle \sigma \rangle^b \tilde{f}_m(\cdot, k) \right\|_{L^2}.
\]
where we have now used that for $|\sigma T| \geq 1$, $T^{-b} \lesssim \langle \sigma \rangle^b$. We have thus obtained that
\[
\left\| \langle d_T \rangle^b \bar{U}_m(\cdot, k) \right\|_{L^2} \lesssim \left\| \langle d_T \rangle^b \tilde{f}_m(\cdot, k) \right\|_{L^2}.
\]
It suffices to multiply by \( \langle k \rangle^s \) and to take the \( L^2 \) norm in \( k \) to get (39).

We next prove (40). Again, it suffices to prove that
\[
\| U_n \|_{2; H^s} \lesssim T^b \| f_n \|_{H^b_{-1} H^s}.
\]

To establish this estimate, we split \( f_n = f_{n,1} + f_{n,2} \) with
\[
\tilde{f}_{m,1}(\sigma, k) = \tilde{f}_m(\sigma, k) 1_{|T_\sigma| \geq 1}, \quad \tilde{f}_{m,2}(\sigma, k) = \tilde{f}_m(\sigma, k) 1_{|T_\sigma| \leq 1}.
\]

For the first part, we readily obtain from the definition of the norm that
\[
\| f_{n,1} \|_{2; H^s} \lesssim T^b \| f_n \|_{H^b_{-1} H^s}
\]

since \( 1 \lesssim T^b |\sigma|^b \) on the support of integration. Since \( \eta \) is bounded
\[
\left\| \eta \left( \frac{\eta \tau}{T} \right) f_{n,1} \right\|_{2; H^s} \lesssim T^b \| f_n \|_{H^b_{-1} H^s}.
\]

For the other part, we use that
\[
\tilde{f}_{n,2}(k) = \int_{-\pi}^{\pi} e^{-in\sigma |\sigma|^b 1_{|\sigma| \leq 1} |\sigma|^b} \tilde{f}_m(\sigma, k) \, d\sigma.
\]

This yields Cauchy–Schwarz that
\[
|\tilde{f}_{n,2}(k)|^2 \lesssim T^{2b-1} \left( \int_{-\pi}^{\pi} (d_\tau(\sigma))^2 |\tilde{f}_m(\sigma, k)|^2 \, d\sigma \right)
\]

and therefore for every \( n \), we have
\[
\| f_{n,2} \|_{H^s} = \| \langle k \rangle^s \tilde{f}_{n,2} \|_{2; H^s}^2 \lesssim T^{2b-1} \| f_m \|_{H^b_{-1} H^s}.
\]

This yields
\[
\left\| \eta \left( \frac{\eta \tau}{T} \right) f_{n,2} \right\|_{2; H^s}^2 = \tau \sum_n \eta \left( \frac{n \tau}{T} \right)^2 \| f_{n,2} \|_{H^s}^2 \lesssim \tau \sum_n \eta \left( \frac{n \tau}{T} \right)^2 T^{2b-1} \| f_m \|_{H^b_{-1} H^s} \lesssim T^{2b-1} \| f_m \|_{H^b_{-1} H^s}^2
\]

and we get (40), which concludes the proof of (34).

We finally prove (35). Let us set
\[
F_n(x) = e^{-in\tau \partial_x^2} U_n(x), \quad f_n(x) = e^{-in\tau \partial_x^2} u_n(x)
\]
so that
\[
F_n(x) = \eta(n\tau) \tau \sum_{m=0}^n f_m.
\]

It suffices to prove that
\[
\| F_n \|_{H^b_{-1} H^s} \lesssim \| f_n \|_{H^b_{-1} H^s}.
\]

We shall only prove the estimate for \( s = 0 \). The general case just follows by applying \( (\partial_x)^s \). Let us use again the function \( g(\sigma) = F_\tau(\eta(n\tau))(\sigma) \) as above. By direct computation, we find that
\[
\tilde{F}_n(k) = \eta(n\tau) \tau \int_{-\pi}^{\pi} \tilde{f}_m(\sigma_0, k) \frac{1 - e^{-i(n+1)\tau \sigma_0}}{1 - e^{-i\tau \sigma_0}} \, d\sigma_0
\]

and therefore
\[
\tilde{F}_m(\sigma, k) = \int_{-\pi}^{\pi} \frac{e^{i\tau \sigma_0}}{d_\tau(\sigma_0)} \tilde{f}_m(\sigma_0, k) \left( g(\sigma) - e^{-i\tau \sigma_0} g(\sigma - \sigma_0) \right) \, d\sigma_0.
\]

We then split
\[
\tilde{F}_m(\sigma, k) = \tilde{F}_{m,1}(\sigma, k) + \tilde{F}_{m,2}(\sigma, k),
\]
where we replace \( \tilde{f}_m \) by \( \tilde{f}_m(\sigma_0,k)1_{|\sigma_0|>1} \) in \( \tilde{F}_{m,1}(\sigma,k) \), and \( \tilde{f}_m \) by \( \tilde{f}_m(\sigma_0,k)1_{|\sigma_0|\leq 1} \) in \( \tilde{F}_{m,2}(\sigma,k) \). By using again that \( g \) has a fast decay \((41)\), this yields

\[
\langle d_\sigma \rangle^b |\tilde{F}_{m,1}(\sigma,k)| \lesssim \langle \sigma \rangle^{b-L} \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(d_\sigma(\sigma_0))^{2\nu}} d\sigma_0 \right)^{\frac{1}{2}} \| \langle d_\sigma \rangle^{b-1} \tilde{f}_m(\cdot,k) \|_{L^2} \\
+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \langle d_\sigma(\sigma_0) \rangle^{b-1} |\tilde{f}_m(\sigma_0,k)| \langle \sigma - \sigma_0 \rangle^{b-L} d\sigma_0.
\]

Therefore, by taking the \( L^2 \) norm in \( \sigma \), and by using Young’s inequality for convolutions for the second term, we obtain since \( b > 1/2 \),

\[
\| \langle d_\sigma \rangle^b \tilde{F}_{m,1}(\cdot,k) \|_{L^2} \lesssim \| \tilde{f}_m(k) \|_{H^{b-1}_T}.
\]

To estimate \( \tilde{F}_{m,2} \), we observe that for \( |\sigma_0| \leq 1 \), we can use Taylor’s formula to get that

\[
\left| \frac{\langle d_\sigma(\sigma_0) \rangle^b}{d_\sigma(\sigma_0)} (g(\sigma) - e^{-i\sigma_0}g(\sigma - \sigma_0)) \right| \lesssim \frac{1}{\langle \sigma - \sigma_0 \rangle^L}.
\]

The estimate then follows from the same arguments.

We have thus proven that

\[
\| \langle d_\sigma \rangle^b \tilde{F}_m(\cdot,k) \|_{L^2} \lesssim \| \tilde{f}_m(k) \|_{H^{b-1}_T}.
\]

To conclude, it suffices to take the \( L^2 \) norm with respect to \( k \). \( \square \)

**Remark 3.5.** Note that in the proof of \((33)\), we have also established a useful time translation invariance property of the discrete Bourgain spaces, that is to say

\[
\sup_{\delta \in [-4,4]} \| e^{in\tau_\delta} u_n \|_{X^{b}_{\tau}} \lesssim_b \| u_n \|_{X^{b}_{\tau}}.
\]

We shall finally study in this section the discrete counterpart of Lemma 2.2 which is crucial for the analysis of nonlinear problems.

In the discrete setting, for a sequence \((u_n) \in l^p(\mathbb{Z}, X)\), with \( X \) normed space we use the norm

\[
\| u_n \|_{l^p(X)} = \left( \tau \sum_{n \in \mathbb{Z}} \| u_n \|_X^p \right)^{\frac{1}{p}}.
\]

**Lemma 3.6.** For \( K \geq \tau^{-\frac{1}{2}} \), we have

\[
\| \Pi_K u_n \|_{l^4} \lesssim (K \tau^{\frac{1}{2}})^{\frac{1}{2}} \| u_n \|_{X^{0,\tau}}.
\]

The above inequality is an important result of the paper, but the understanding of its proof that requires tools not yet introduced is not necessary to continue reading the paper. For the convenience of the reader, we thus postpone it to section 8.

By duality, we also get from \((45)\) that

\[
\| \Pi_K u_n \|_{X^{\alpha,-\frac{a}{2}}} \lesssim (K \tau^{\frac{1}{2}})^{\frac{1}{2}} \| u_n \|_{l^2_\tau L^4}.
\]

As a consequence, we obtain the following crucial product estimates for sequences \( u_n, v_n, \) and \( w_n \).
Corollary 3.7. We have the following product estimate:

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim (K\tau^2)^{\frac{1}{8}}\|u_n\|_{X_\tau^{-0,\frac{3}{8}}}\|v_n\|_{X_\tau^{-0,\frac{3}{8}}}\|w_n\|_{X_\tau^{-0,\frac{3}{8}}} \]  \hspace{1cm} (47)

Moreover, for any \(s_1 > 1/4\), we have the estimates

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim (K\tau^2)^{\frac{1}{8}}\|u_n\|_{X_\tau^{-0,\frac{3}{8}}}\|v_n\|_{X_\tau^{-0,\frac{3}{8}}}\|w_n\|_{L_\tau^2H^{s_1}}, \]  \hspace{1cm} (48)

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim \|u_n\|_{X_\tau^{-s_1,\frac{3}{8}}}\|v_n\|_{X_\tau^{-s_1,\frac{3}{8}}}\|w_n\|_{L_\tau^2L^2}, \]  \hspace{1cm} (49)

and for \(s_2 > 1/2\), we have

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim \|u_n\|_{X_\tau^{-s_1,\frac{3}{8}}}\|v_n\|_{X_\tau^{-s_1,\frac{3}{8}}}\|w_n\|_{L_\tau^2H^{s_2}}, \]  \hspace{1cm} (50)

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim \|u_n\|_{X_\tau^{-s_1,\frac{3}{8}}}\|v_n\|_{X_\tau^{-s_1,\frac{3}{8}}}\|w_n\|_{L_\tau^2L^4}. \]  \hspace{1cm} (51)

Note that (49) is of particular interest if the two lower frequency factors have at least 1/4 regularity. Then we do not need the factor \(K\tau^{1/2}\) which is large if \(\alpha > 1\) (recall that \(K = \tau^{-\alpha/2}\)). This will be useful to prove the stability of the scheme for \(\alpha > 1\). The estimate (48) will turn out to be useful to optimize the convergence rate of the scheme when \(s_0\) is large enough.

**Proof.** We start with proving (47). We first obtain from the estimate (46) that

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim (K\tau^2)^{\frac{1}{8}}\|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{L_\tau^2L^\frac{3}{2}}. \]

From the continuity of \(P_K\) on \(L^p\) and the Hölder inequality, we next get that

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim (K\tau^2)^{\frac{1}{2}}\|\Pi_{\tau-\frac{1}{2}}u_n\|_{L_\tau^2L^4}\|\Pi_{\tau-\frac{1}{2}}v_n\|_{L_\tau^2L^4}\|P_\tau w_n\|_{L_\tau^2L^4}. \]  \hspace{1cm} (52)

By using again (45), we thus find

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim (K\tau^2)^{\frac{1}{2}}\|u_n\|_{X_\tau^{-0,\frac{3}{8}}}\|v_n\|_{X_\tau^{-0,\frac{3}{8}}}\|w_n\|_{X_\tau^{-0,\frac{3}{8}}}. \]

This proves (47).

For the proof of (48), we use again (52). However, we only estimate \(\|\Pi_{\tau-\frac{1}{2}}u_n\|_{L_\tau^2L^4}\) and \(\|\Pi_{\tau-\frac{1}{2}}v_n\|_{L_\tau^2L^4}\) with the help of (45). For the last term, we use the Sobolev embedding \(H^{s_1} \subset L^4\) to get that

\[ \|P_\tau w_n\|_{L_\tau^2H^{s_1}} \lesssim \|P_K w_n\|_{L_\tau^2H^{s_1}}. \]

To get (49), we just use that

\[ \|P_K(\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n)\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim \|\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n\|_{X_\tau^{-0,0}} \]  \hspace{1cm} (53)

and employ Hölder’s inequality to get

\[ \|\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n\|_{X_\tau^{-0,-\frac{3}{8}}} \lesssim \|\Pi_{\tau-\frac{1}{2}}u_n\|_{L_\tau^2L^\infty}\|\Pi_{\tau-\frac{1}{2}}v_n\|_{L_\tau^2L^\infty}\|P_\tau w_n\|_{L_\tau^2L^2}. \]

We then use (28) to write

\[ \|P_\tau w_n\|_{L_\tau^2L^\infty} \lesssim \|w_n\|_{X_\tau^{0,\frac{3}{8}}}, \quad \|\Pi_{\tau-\frac{1}{2}}v_n\|_{L_\tau^2L^\infty} \lesssim \|v_n\|_{X_\tau^{-s_1,\frac{3}{8}}}. \]

This concludes the proof of (49).

For (50), (51), we just use again (53), and then that

\[ \|\Pi_{\tau-\frac{1}{2}}u_n\Pi_{\tau-\frac{1}{2}}v_nP_\tau w_n\|_{X_\tau^{0,0}} \lesssim \|\Pi_{\tau-\frac{1}{2}}u_n\|_{L_\tau^2L^4}\|\Pi_{\tau-\frac{1}{2}}v_n\|_{L_\tau^2L^4}\|P_\tau w_n\|_{L_\tau^2L^\infty}. \]
To conclude, we use again Lemma 3.6 and the Sobolev embedding $H^{s_2} \subset L^\infty$ or Lemma 3.6 and the Sobolev embeddings $H^{s_1} \subset L^4$, $H^{s_2} \subset L^4$.

**Remark 3.8.** Another version intermediate between (48) and (49) will be also useful. We have that

$$\|\Pi_K(\Pi_{\tau^{-\frac{1}{2}}} u_n \Pi_{\tau^{-\frac{1}{2}}} v_n \Pi K w_n)\|_{X_{\tau^{-\frac{1}{2}}}} \lesssim (K \tau^{\frac{1}{2}})^{\frac{s}{2}} \|u_n\|_{\dot{X}_{\tau^{-\frac{1}{2}}}} \|v_n\|_{\dot{X}_{\tau^{-\frac{1}{2}}}} \|w_n\|_{L^1 L^2}$$

(54)

with $s_1 > 1/4$. Indeed, we first use (46) to get that

$$\|\Pi_K(\Pi_{\tau^{-\frac{1}{2}}} u_n \Pi_{\tau^{-\frac{1}{2}}} v_n \Pi K w_n)\|_{X_{\tau^{-\frac{1}{2}}}} \lesssim (K \tau^{\frac{1}{2}})^{\frac{s}{2}} \|\Pi_K(\Pi_{\tau^{-\frac{1}{2}}} u_n \Pi_{\tau^{-\frac{1}{2}}} v_n \Pi K w_n)\|_{L^4 L^2}$$

and we employ Hölder’s inequality to get

$$\|\Pi_K(\Pi_{\tau^{-\frac{1}{2}}} u_n \Pi_{\tau^{-\frac{1}{2}}} v_n \Pi K w_n)\|_{L^4 L^2} \leq \|\Pi_K w_n\|_{L^1 L^2} \|\Pi_{\tau^{-\frac{1}{2}}} v_n\|_{L^4 L^\infty} \|\Pi_{\tau^{-\frac{1}{2}}} u_n\|_{L^4 L^4}.$$

We conclude by using the Sobolev embedding $W^{s_1,4} \subset L^\infty$ and (45).

**4. Estimates of the exact solution in discrete Bourgain spaces**

In this section, we shall prove that the sequence $u^K(t_n)$ is an element of $X^{s,b}_\tau$ for suitable $s$. It will be convenient to use the following general lemma.

**Lemma 4.1.** For any $b \geq 0$, $b' > 1/2$ and $s \in \mathbb{R}$, let us consider a sequence of functions $(u_n(x))_{n \in \mathbb{Z}}$ of the form $u_n(x) = u(n\tau, x)$. Then

$$\|u_n\|_{X^{s,b}_\tau} \lesssim b \|u\|_{X^{s,b + b'}_\tau}.$$

Proof. By setting $f = e^{-it\partial_x^b} u$ and $f_n(x) = f(n\tau, x)$, it suffices to prove that

$$\|f_n\|_{H^{b}_{\tau L^2}} \lesssim \|f\|_{H^{b + b'}_{\tau L^2}},$$

the extension to general $s$ being straightforward. Since we have by definition that

$$\widetilde{f}_n(\sigma, k) = \tau \sum_{n \in \mathbb{Z}} \hat{f}(n\tau, k) e^{i\tau\sigma},$$

we have by Poisson’s summation formula that

$$\hat{f}_n(\sigma, k) = \sum_{m \in \mathbb{Z}} \hat{f}(\sigma + \frac{2\pi}{\tau} m, k).$$

Therefore,

$$\langle d_\tau(\sigma) \rangle^b \hat{f}_n(\sigma, k) = \sum_{m \in \mathbb{Z}} \langle d_\tau(\sigma + \frac{2\pi}{\tau} m) \rangle^b \hat{f}(\sigma + \frac{2\pi}{\tau} m, k),$$

since $d_\tau$ is also a $2\pi/\tau$ periodic function. Since, we always have that $|d_\tau(\sigma)| \lesssim \langle \sigma \rangle$, this yields by Cauchy–Schwarz,

$$|\langle d_\tau(\sigma) \rangle^b \hat{f}_n(\sigma, k)|^2 \lesssim \sum_{\mu} \langle \sigma + \frac{2\pi}{\tau} m \rangle^{2b'} \sum_{m \in \mathbb{Z}} \langle \sigma + \frac{2\pi}{\tau} m \rangle^{2b + 2b'} \left| \hat{f}(\sigma + \frac{2\pi}{\tau} m, k) \right|^2 \lesssim \sum_{m \in \mathbb{Z}} \langle \sigma + \frac{2\pi}{\tau} m \rangle^{2b + 2b'} \left| \hat{f}(\sigma + \frac{2\pi}{\tau} m, k) \right|^2$$

since $2b' > 1$. By integrating with respect to $\sigma$, we obtain that

$$\|\langle d_\tau \rangle^b \hat{f}_n(\cdot, k)\|_{L^2(-\pi/\tau, \pi/\tau)} \lesssim \|\langle \sigma \rangle^{b + b'} \hat{f}(\cdot, k)\|_{L^2(\mathbb{R})}^2.$$
As a consequence of the previous lemma, we obtain the following result.

**Proposition 4.2.** Let $u^K$ be the solution of (7) and define the sequence $u^K_n(x) = u^K(n\tau + t', x)$. Assume that $u_0 \in H^{s_0}$, $s_0 > \frac{1}{18}$. Then, for every $s_1$, such that $0 \leq s_1 < s_0 - \frac{1}{18}$, we have that

$$\sup_{t' \in [0, 4\tau]} \|\eta(n\tau)u^K_n\|_{X^{s_1+\frac{1}{18}}} \leq C_T.$$ **Proof.** It suffices to combine Lemma 4.1 and Corollary 2.8 by taking $b'$ arbitrarily close to $1/2$. □

5. LOCAL ERROR OF TIME DISCRETIZATION

In this section we analyse the time discretization error which is introduced when discretising the twice-filtered Schrödinger equation (7) with the scheme (11).

Setting

$$A = \left\{ \kappa = (\kappa_1, \kappa_2, \kappa_3); \kappa_1, \kappa_2, \kappa_3 \in \left\{ \tau^{-1/2}, K^+ \right\} \exists i \neq j : \kappa_i = \kappa_j = \tau^{-1/2} \right\}$$

$$= \left\{ (\tau^{-1/2}, \tau^{-1/2}, \tau^{-1/2}), (\tau^{-1/2}, \tau^{-1/2}, \tau^{-1/2}), (\tau^{-1/2}, K^+, \tau^{-1/2}), (K^+, \tau^{-1/2}, \tau^{-1/2}) \right\} \right.$$ (55)

allows us to express the filtered Schrödinger equation (7) as follows

$$i \partial_t u^K = -\partial_x^2 u^K + \sum_{\kappa \in A} \Pi_K \left( \Pi_{\kappa_1} u^K \Pi_{\kappa_2} \Pi_{\kappa_3} u^K \right)$$ (56)

and Duhamel’s formula (with step size $\tau$) takes the form

$$u^K(t_n + \tau) = e^{i\tau^2 u^K(t_n)} - i \overline{\Pi_K} e^{i\tau^2 u^K(t_n)} \sum_{\kappa \in A} T_\kappa(u^K)(\tau, t_n),$$ (57)

where

$$T_\kappa(u^K)(\tau, t_n) = \int_0^\tau e^{-is\partial_x^2} (\Pi_{\kappa_1} u^K(t_n + s)) \Pi_{\kappa_2} \Pi_{\kappa_3} u^K(t_n + s)) ds.$$ (58)

Henceforth, we will use the following notation:

$$V^K_{\kappa_1}(s, t) = e^{is\partial_x^2} \Pi_{\kappa_1} u^K(t), \quad W^K_{\kappa_1}(s, t) = e^{is\partial_x^2} \Pi_{\kappa_1} \sum_{\sigma \in A} T_\sigma(u^K)(s, t).$$

Iterating Duhamel’s formula (57), i.e., plugging the expansion

$$\Pi_{\kappa_1} u^K(t_n + s) = V^K_{\kappa_1}(s, t_n) - i W^K_{\kappa_1}(s, t_n)$$

into (57), yields the representation

$$u^K(t_n + \tau) = e^{i\tau^2 u^K(t_n)} - i \overline{\Pi_K} e^{i\tau^2 u^K(t_n)} \sum_{\kappa \in A} \int_0^\tau e^{-is\partial_x^2} \left[ V^K_{\kappa_1}(s, t_n) W^K_{\kappa_2}(s, t_n) V^K_{\kappa_3}(s, t_n) \right] ds$$

$$- i \overline{\Pi_K} e^{i\tau^2 u^K(t_n)} \sum_{\kappa \in A} E_\kappa(\tau, t_n)$$ (59)

with the remainder

$$E_\kappa(\tau, t_n) = \int_0^\tau e^{-is\partial_x^2} (E_{\kappa,1} + E_{\kappa,2} + E_{\kappa,3} + E_{\kappa,4} + E_{\kappa,5} + E_{\kappa,6} + E_{\kappa,7})(s, t_n) ds,$$ (60)
defined by
\[ E_{\kappa,1}(s, t_n) = iV^K_{\kappa_1}(s, t_n)\overline{W^K_{\kappa_2}(s, t_n)}V^K_{\kappa_3}(s, t_n) \]
\[ E_{\kappa,2}(s, t_n) = -iV^K_{\kappa_1}(s, t_n)\overline{W^K_{\kappa_2}(s, t_n)}W^K_{\kappa_3}(s, t_n) \]
\[ E_{\kappa,3}(s, t_n) = -iW^K_{\kappa_1}(s, t_n)\overline{W^K_{\kappa_2}(s, t_n)}V^K_{\kappa_3}(s, t_n) \]
\[ E_{\kappa,4}(s, t_n) = W^K_{\kappa_1}(s, t_n)\overline{W^K_{\kappa_2}(s, t_n)}V^K_{\kappa_3}(s, t_n) \]
\[ E_{\kappa,5}(s, t_n) = -W^K_{\kappa_1}(s, t_n)\overline{W^K_{\kappa_2}(s, t_n)}W^K_{\kappa_3}(s, t_n) \]
\[ E_{\kappa,6}(s, t_n) = V^K_{\kappa_1}(s, t_n)\overline{W^K_{\kappa_2}(s, t_n)}W^K_{\kappa_3}(s, t_n) \]
\[ E_{\kappa,7}(s, t_n) = -iW^K_{\kappa_1}(s, t_n)\overline{W^K_{\kappa_2}(s, t_n)}W^K_{\kappa_3}(s, t_n). \]

It remains to analyse the error introduced by the time discretization of the integrals in (59), where the discretization is carried out in such a way that the dominant terms in (59), i.e., the intermediate frequency terms \( \tau^{-1/2} < |k| \leq K \), are solved exactly while the lower order frequency terms \( |k| \leq \tau^{-1/2} \) are approximated in a suitable manner.

**Lemma 5.1.** For sufficiently smooth functions \( v, w \) it holds that
\[ \int_0^\tau e^{-is\partial_x^2} \left[ (e^{is\partial_x^2}v) \overline{(e^{is\partial_x^2}w)} \right] ds = \mathcal{J}_1^*(w, v, w) + R_1(w, v, w) \] with \( \mathcal{J}_1^* \) defined in (12) and the remainder given by
\[ R_1(v_1, v_2, v_3) = -2i \int_0^\tau e^{-is\partial_x^2} \left[ (e^{is\partial_x^2}v_2) \int_0^s e^{i(s-s_1)\partial_x^2} \left[ (e^{-is_1\partial_x^2}v_1) \left( e^{is_1\partial_x^2}v_3 \right) \right. \right. \\
+ \left( e^{-is_1\partial_x^2}\partial_x v_1 \right) \left( e^{is_1\partial_x^2}\partial_x v_3 \right) \left. \right] ds \right] ds. \]

**Proof.** The proof follows two steps. First we will show that in fact
\[ \mathcal{J}_1^*(w, v, w) = \int_0^\tau e^{-is\partial_x^2} \left[ (e^{is\partial_x^2}v) e^{is\partial_x^2}|w|^2 \right] ds. \]
The Fourier expansion of the above integral together with the relation
\[ (-k_1 + k_2 + k_3)^2 - (k_2^2 + (-k_1 + k_3)^2) = 2k_2(-k_1 + k_3) \]
yields that
\[ \int_0^\tau e^{-is\partial_x^2} \left[ (e^{is\partial_x^2}v) e^{is\partial_x^2}|w|^2 \right] ds = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} \overline{w}_{k_1} \hat{v}_{k_2} \hat{w}_{k_3} e^{ikx} \int_0^\tau e^{is(-k_1+k_2+k_3)^2} e^{-is(k_2^2+(-k_1+k_3)^2)} ds \]
\[ = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} \overline{w}_{k_1} \hat{v}_{k_2} \hat{w}_{k_3} e^{ikx} \int_0^\tau e^{2isk_2(-k_1+k_3)} ds \]
\[ = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} \overline{w}_{k_1} \hat{v}_{k_2} \hat{w}_{k_3} e^{ikx} \int_0^\tau e^{2isk_2(-k_1+k_3)} ds + \tau \hat{v}_0 |w|^2 + \tau \left( |w|^2 \right)_0 (v - \hat{v}_0) \]
\[ + \sum_{k_1,k_2,k_3 \in \mathbb{Z}} \overline{w}_{k_1} \hat{v}_{k_2} \hat{w}_{k_3} e^{ikx} \frac{2\tau k_2(-k_1+k_3) - 1}{2i k_2(-k_1+k_3)} + \tau \hat{v}_0 |w|^2 + \tau \left( |w|^2 \right)_0 (v - \hat{v}_0) \]
which implies (64). Thanks to (64) we can furthermore conclude by (62) that
\[
R_1(\overline{w},v,w) = \int_0^\tau e^{-is\partial_x^2} \left[ \left( e^{is\partial_x^2} v \right) \left( e^{is\partial_x^2} w \right) - e^{is\partial_x^2} w^2 \right] \, ds. \tag{65}
\]

We note that
\[
-2i \int_0^s e^{i(s_1-s_2)\partial_x^2} \left[ \left( e^{-is_1\partial_x^2} (\partial_x^2 - \tau) \left( e^{is_1\partial_x^2} v \right) \right) \left( e^{is_2\partial_x^2} w \right) + \left( e^{is_2\partial_x^2} \partial_x v \right) \right] \, ds_2
\]
\[
= -2i \sum_{\ell_1, \ell_2 \in \mathbb{Z}} \overline{w}_{\ell_1} \hat{w}_{\ell_2} e^{i(-\ell_1+\ell_2)x} e^{-is(-\ell_1+\ell_2)^2} \int_0^s e^{is(-\ell_1+\ell_2)^2} e^{is(\ell_1^2-\ell_2^2)} (-\ell_1^2 + \ell_1 \ell_2) \, ds_1
\]
\[
= \sum_{\ell_1, \ell_2 \in \mathbb{Z}} \overline{w}_{\ell_1} \hat{w}_{\ell_2} e^{i(-\ell_1+\ell_2)x} \left( e^{is(\ell_1^2-\ell_2^2)} - e^{-is(-\ell_1+\ell_2)^2} \right) = \left| e^{is\partial_x^2} w \right|^2 - e^{is\partial_x^2} w^2.
\]

Plugging the above relation into (65) yields (63). This concludes the proof. \[\square\]

**Lemma 5.2.** For sufficiently smooth functions \(v, w\) it holds that
\[
\int_0^\tau e^{-is\partial_x^2} \left[ \left( e^{-is\partial_x^2} \overline{v} \right) \left( e^{is\partial_x^2} w \right) \right] \, ds = J_2^*(\overline{v},w,w) + R_2(\overline{v},w,w) \tag{66}
\]
with \(J_2^*\) defined in (13) and the remainder given by
\[
R_2(v_1,v_2,v_3) = -2i \int_0^\tau e^{-is\partial_x^2} \left[ \left( e^{-is\partial_x^2} v_1 \right) \int_0^s e^{i(s_1-s_2)\partial_x^2} \left( e^{is_1\partial_x^2} \partial_x v_2 \right) \left( e^{is_2\partial_x^2} \partial_x v_3 \right) \, ds_1 \right] \, ds. \tag{67}
\]

**Proof.** Again we prove the assertion in two steps. First we show that in fact
\[
J_2^*(\overline{v},w,w) = \int_0^\tau e^{-is\partial_x^2} \left[ \left( e^{-is\partial_x^2} \overline{v} \right) \left( e^{is\partial_x^2} w \right)^2 \right] \, ds. \tag{68}
\]
The above assertion follows by Fourier expansion of the integral together with the relation
\[
(-k_1 + k_2 + k_3)^2 + k_1^2 - (k_2 + k_3)^2 = -2k_1(-k_1 + k_2 + k_3)
\]
which implies that
\[
\int_0^\tau e^{-is\partial_x^2} \left[ \left( e^{-is\partial_x^2} \overline{v} \right) \left( e^{is\partial_x^2} w \right)^2 \right] \, ds = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} \overline{v}_{k_1} \hat{w}_{k_2} \hat{w}_{k_3} e^{ikx} \int_0^\tau e^{is(-k_1+k_2+k_3)} e^{is(k_1^2-(k_2+k_3)^2)} \, ds
\]
\[
= \sum_{k_1,k_2,k_3 \in \mathbb{Z}} \overline{v}_{k_1} \hat{w}_{k_2} \hat{w}_{k_3} e^{ikx} \int_0^\tau e^{-2isk_{1,k}} \, ds = \sum_{k_1,k_2,k_3 \in \mathbb{Z}} \overline{v}_{k_1} \hat{w}_{k_2} \hat{w}_{k_3} e^{ikx} \frac{e^{-2ik_{1,k} \tau} - 1}{-2ik_{1,k}}
\]
\[
= \frac{i}{2} e^{-is\partial_x^2} \partial_x^{-1} \left[ \left( e^{-is\partial_x^2} \partial_x^{-1} \overline{v} \right) \left( e^{is\partial_x^2} w \right) \right] - \frac{i}{2} \partial_x^{-1} \left( w^2 \partial_x^{-1} \overline{v} \right) + \tau(\overline{vw})_0 + \tau(\overline{w^2-v_0}_0) \tag{69}
\]

Thanks to (68) we can furthermore conclude by (66) that
\[
R_2(\overline{v},w,w) = \int_0^\tau e^{-is\partial_x^2} \left[ \left( e^{-is\partial_x^2} \overline{v} \right) \left( e^{is\partial_x^2} w \right)^2 - e^{is\partial_x^2} w^2 \right] \, ds. \tag{70}
\]
We note that
\[
-2i \int_0^s e^{i(s-s_1)\Phi} \left( e^{is_1\Phi} \partial_x^2 \right)^2 ds_1 \\
= 2i \sum_{\ell_1, \ell_2 \in \mathbb{Z}} \hat{w}_{\ell_1} \hat{w}_{\ell_2} e^{i(\ell_1+\ell_2)x} e^{-is(\ell_1+\ell_2)^2} \int_0^s e^{is_1(\ell_1+\ell_2)^2} e^{-is_1(\ell_1^2+\ell_2^2)} \ell_1 \ell_2 ds_1 \\
= 2i \sum_{\ell_1, \ell_2 \in \mathbb{Z}} \hat{w}_{\ell_1} \hat{w}_{\ell_2} e^{i(\ell_1+\ell_2)x} e^{-is(\ell_1+\ell_2)^2} \int_0^s e^{2is_1\ell_1\ell_2} \ell_1 \ell_2 ds_1 \\
= \sum_{\ell_1, \ell_2 \in \mathbb{Z}} \hat{w}_{\ell_1} \hat{w}_{\ell_2} e^{i(\ell_1+\ell_2)x} e^{-is(\ell_1+\ell_2)^2} \left( e^{is(\ell_1+\ell_2)^2} e^{-is(\ell_1^2+\ell_2^2)} - 1 \right) = \left( e^{is\Phi} \right)^2 - e^{is\Phi^2} \Phi^2.
\]

Plugging the above relation into (70) proves the assertion.

\( \square \)

**Lemma 5.3** (Local error). *The local error*

\[ \mathcal{E}(\tau, t_n) := u^K(t_n + \tau) - \Phi^K_\tau(u^K(t_n)) \]

*of the time discretization scheme* (11) *applied to the filtered Schrödinger equation* (7) *reads*

\[ \mathcal{E}(\tau, t_n) = -2i\Pi K e^{i\tau\Phi^2} R_1(\Pi_{\tau-1/2} t^K(t_n), \Pi_{\tau-1/2} u^K(t_n)) \]

\[ - i\Pi K e^{i\tau\Phi^2} R_2(\Pi_{\tau-1} t^K(t_n), \Pi_{\tau-1} u^K(t_n)) \]

\[ - i\Pi K e^{i\tau\Phi^2} \sum_{\kappa \in A} E_\kappa(\tau, t_n), \]

*where* \( E_\kappa(\tau, t_n) \) *is defined in* (60) *and the remainders* \( R_1 \) *and* \( R_2 \) *are given in* (63) *and* (67), *respectively.*

\[ \]

**Proof.** The assertion follows by the expansion of the exact solution \( u^K(t_n + \tau) \) given in (59) together with Lemmas 5.1 and 5.2.

More precisely, employing Lemma 5.1 to approximate the integral arising for \( \kappa_1 = K^+ \) or \( \kappa_3 = K^+ \) in (59) and Lemma 5.2 to approximate the integrals arising for \( \kappa_2 = K^+ \) or \( \kappa_1 = \kappa_2 = \kappa_3 = \tau^{-1/2} \) in (59) yields that

\[ u^K(t_n + \tau) = e^{i\tau\Phi^2} u^K(t_n) \]

\[ - 2i\Pi K e^{i\tau\Phi^2} \mathcal{J}_1(\Pi_{\tau-1/2} t^K(t_n), \Pi_{\tau-1/2} u^K(t_n)) \]

\[ - i\Pi K e^{i\tau\Phi^2} \mathcal{J}_2(\Pi_{\tau-1/2} t^K(t_n), \Pi_{\tau-1/2} u^K(t_n)) \]

\[ - 2i\Pi K e^{i\tau\Phi^2} R_1(\Pi_{\tau-1} t^K(t_n), \Pi_{\tau-1} u^K(t_n)) \]

\[ - i\Pi K e^{i\tau\Phi^2} R_2(\Pi_{\tau-1} t^K(t_n), \Pi_{\tau-1} u^K(t_n)) \]

\[ - i\Pi K e^{i\tau\Phi^2} \sum_{\kappa \in A} E_\kappa(\tau, t_n). \]

The assertion thus follows by taking the difference of the expansion of the exact solution given in (72) and the numerical flow defined in (11). \( \square \)

6. Global Error Analysis

Let \( e_{n+1} = u^K(t_{n+1}) - u^K_n \) denote the time discretization error, i.e., the difference between the numerical solution \( u^K_{n+1} = \Phi^K_\tau(u^K_n) \) defined in (11) and the exact solution of the filtered Schrödinger equation (7). Inserting a zero in terms of \( \pm \Phi^K_\tau(u^K(t_n)) \), i.e., using that

\[ e_{n+1} = u^K(t_{n+1}) - \Phi^K_\tau(u^K(t_n)) + \Phi^K_\tau(u^K(t_n)) - \Phi^K_\tau u^n \]

\[ = u^K(t_{n+1}) - \Phi^K_\tau(u^K(t_n)) + 2i\Pi K e^{i\tau\Phi^2} R_1(\Pi_{\tau-1} t^K(t_n), \Pi_{\tau-1} u^K(t_n)) \]

\[ - i\Pi K e^{i\tau\Phi^2} R_2(\Pi_{\tau-1} t^K(t_n), \Pi_{\tau-1} u^K(t_n)) \]

\[ - i\Pi K e^{i\tau\Phi^2} \sum_{\kappa \in A} E_\kappa(\tau, t_n). \]
we obtain by the definition of the numerical flow $\Phi^K_t$ in (11) that
\[
e^{n+1} = e^{i\tau\partial_x^2}e^n - 2i\Pi Ke^{i\tau\partial_x^2} [J_1^T (\Pi_{\tau-1/2}^\omega u^K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau-1/2}^\omega u^K(t_n)) \\
- J_1^T (\Pi_{\tau-1/2}^\omega u^n_t, \Pi_{K+}u^n_t, \Pi_{\tau-1/2}^\omega u^n_t)] \\
- i\Pi Ke^{i\tau\partial_x^2} [J_2^T (\Pi_{K}^\omega u^K(t_n), \Pi_{\tau-1/2}^\omega u^K(t_n)) \\
- J_2^T (\Pi_{K}^\omega u^n_t, \Pi_{\tau-1/2}^\omega u^n_t)] + E(\tau, t_n),
\]
where $J_1^T$ and $J_2^T$ are defined in (12) and (13) and the local error (71) is given in Lemma 5.3.

By solving the above recursion, we get that for $0 \leq n \leq N_1 = \lfloor \frac{T_1}{\tau} \rfloor$ with $T_1 \leq T$, the global error $e^n$ satisfies
\[
e^n = \tau\eta(t_n) \sum_{k=0}^{n-1} e^{i(n-k)\tau\partial_x^2}\eta(k)g_k + R_{1,n} + R_{2,n},
\]
where we have set
\[
G_n = \frac{2\Pi}{\tau} \left[ J_1^T (\Pi_{\tau-1/2}^\omega u^K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau-1/2}^\omega u^K(t_n)) - J_1^T (\Pi_{\tau-1/2}^\omega u^n_t, \Pi_{K+}u^n_t, \Pi_{\tau-1/2}^\omega u^n_t) \right] \\
- \frac{i}{\tau} \left[ J_2^T (\Pi_{K}^\omega u^K(t_n), \Pi_{\tau-1/2}^\omega u^K(t_n)) - J_2^T (\Pi_{K}^\omega u^n_t, \Pi_{\tau-1/2}^\omega u^n_t) \right],
\]
and the remainders
\[
R_{i,n} = \tau\eta(t_n) \sum_{k=0}^{n-1} e^{i(n-k)\tau\partial_x^2}\eta(k)g_i(t_k), \quad i = 1, 2,
\]
with
\[
F_1(t_n) = \frac{1}{\tau} (\frac{2i}{\tau} e^{i\tau\partial_x^2}R_1(\Pi_{\tau-1/2}^\omega u^K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau-1/2}^\omega u^K(t_n))) \\
- i\Pi Ke^{i\tau\partial_x^2}R_2(\Pi_{K}^\omega u^K(t_n), \Pi_{\tau-1/2}^\omega u^K(t_n), \Pi_{\tau-1/2}^\omega u^K(t_n))
\]
and
\[
F_2(t_n) = \frac{i}{\tau} \sum_{k \in A} E_{\kappa}(\tau, t_n).
\]

Note that $E_{\kappa}$ is defined in (60) and $R_1$, $R_2$ in (63) and (67). We have introduced the truncation function $\eta$ in order to work with global Bourgain spaces. As before we will assume that $u^n_t$ and $u^K$ are globally defined though they coincide with the actual solutions of the scheme and the PDE on a finite interval of time. We will choose $T_1$ sufficiently small later.

We shall first estimate $R_{1,n}$, which gives the dominant contribution to the error.

**Lemma 6.1.** Let $s_0 \in (0, 1]$ and $b \in (1/2, 5/8)$. For $s_0 > 1/18$ we, have the estimate
\[
\|R_{1,n}\|_{X^{s,b}} \leq C_T \tau^{\frac{1}{2} - \frac{s_0}{2}} -(\frac{1}{b})^+.
\]
Moreover, if $s_0 > 1/4 + 1/18$, we have
\[
\|R_{1,n}\|_{X^{s,b}} \leq C_T (K\tau)^{-\frac{1}{2} + \frac{1}{2} - \frac{s_0}{2}} -(\frac{1}{b})^+,
\]
and if $s_0 > \frac{1}{2}$, we have
\[
\|R_{1,n}\|_{X^{s,b}} \leq C_T \tau^{s_0} -(\frac{1}{b})^+.
\]

**Proof.** By using again (34) and (35), we get that
\[
\|R_{1,n}\|_{X^{s,b}} \leq \frac{1}{\tau} \|J_1,n\|_{X^{s,b}}.
\]
By using (76) this amounts to estimate

\[ I_1 = \frac{1}{\tau} \| R_1(\Pi_{\tau-1/2}K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau-1/2}u^K(t_n)) \|_{\mathcal{X}_\tau^{0,\frac{1}{4}}} \]

\[ I_2 = \frac{1}{\tau} \| R_2(\Pi_K\Pi_{\tau-1/2}K(t_n), \Pi_{\tau-1/2}u^K(t_n), \Pi_{\tau-1/2}u^K(t_n)) \|_{\mathcal{X}_\tau^{0,\frac{1}{4}}} \]

We first prove (78). We start with the estimate of \( I_2 \). We use (67), (47) and Remark 3.2 to obtain that

\[ I_2 \lesssim K\tau^{\frac{1}{2}} \| u^K(t_n) \|_{\mathcal{X}_\tau^{\frac{1}{2}}} \| \tau^{\frac{1}{2}} \partial_x \Pi_{\tau-1/2}u^K(t_n) \|^2_{\mathcal{X}_\tau^{0,\frac{3}{4}}} \]

By using Proposition 4.2, this yields

\[ I_2 \lesssim K\tau^{\frac{3}{2}} \| u^K \|_{\mathcal{X}_{\tau_0,b}} \| \tau^{\frac{3}{2}} \partial_x \Pi_{\tau-1/2}u^K \|^2_{\mathcal{X}_{\tau_2,b}} \]

with \( s_0 > s_2 > 1/18, s_2 \) arbitrarily close to \( 1/18 \) and \( b \in (7/8, 1) \). Consequently, from the frequency localization, we find that

\[ I_2 \lesssim \tau^{\frac{5}{2} - s_2}(K\tau^{\frac{3}{2}}) \| u^K \|_{\mathcal{X}_{\tau_0,b}} \| \Pi_{\tau-1/2}u^K \|^2_{\mathcal{X}_{\tau_2,b}} \leq C_T \tau^{\frac{5}{2} - s_2} (K\tau^{\frac{3}{2}}). \]

It remains to estimate \( I_1 \). By using the definition (63) and the same arguments, we get that

\[ I_1 \lesssim K\tau^{\frac{1}{2}} \| (\frac{1}{2}\tau) \partial_x \Pi_{\tau-1/2}u \|_{\mathcal{X}_{\tau_2,b}} \| \Pi_{K+}u^K \|_{\mathcal{X}_{\tau_2,b}} \| u^K \|_{\mathcal{X}_{\tau_2,b}} + K\tau^{\frac{1}{2}} \| u^K \|_{\mathcal{X}_{\tau_2,b}} \| \tau^{\frac{1}{2}} \partial_x \Pi_{\tau-1/2}u^K \|^2_{\mathcal{X}_{\tau_2,b}} \]

again with \( s_0 > s_2 > 1/18 \). The second term is similar as before. For the first term, by using the frequency localization, in particular the fact that on the support of \( \Pi_{K+}, \tau^{\frac{1}{2}} |\xi| \geq 1 \), we then obtain that

\[ \| (\frac{1}{2}\tau) \partial_x \Pi_{\tau-1/2}u \|_{\mathcal{X}_{\tau_2,b}} \lesssim \tau^{(s_0 - s_2)/2} \| u \|_{\mathcal{X}_{\tau_0,b}}, \quad \| \Pi_{K+}u^K \|_{\mathcal{X}_{\tau_2,b}} \lesssim \tau^{(s_0 - s_2)/2} \| u^K \|_{\mathcal{X}_{\tau_0,b}}. \]

This also yields

\[ I_1 \leq C_T \tau^{s_0 - s_2} (K\tau^{\frac{3}{2}}), \]

which concludes the proof of (78).

To prove (79), we follow the same lines, but we use (48) instead of (47) since \( s_0 > 1/4 \). This yields

\[ \| \mathcal{R}_{1,n} \|_{\mathcal{X}_{\tau_0,b}} \lesssim (K\tau^{\frac{1}{2}}) \| u^K(t_n) \|_{\mathcal{I}_{\tau}H^\frac{1}{4}} \| \tau^{\frac{1}{2}} \partial_x \Pi_{\tau-\frac{1}{2}}u^K(t_n) \|^2_{\mathcal{X}_\tau^{0,\frac{1}{4}}} + (K\tau^{\frac{1}{2}}) \| (\frac{1}{2}\tau) \partial_x \Pi_{\tau-\frac{1}{2}}u^K(t_n) \|_{\mathcal{X}_{\tau_0,b}} \| \Pi_{K+}u^K(t_n) \|_{\mathcal{I}_{\tau}H^\frac{1}{4}} \| \Pi_{\tau-\frac{1}{2}}u^K(t_n) \|_{\mathcal{X}_\tau^{0,\frac{3}{4}}}. \]

By using again the same estimates as above, it thus only remains to estimate \( \| u^K(t_n) \|_{\mathcal{I}_{\tau}H^\frac{1}{4}} \) and \( \| \Pi_{K+}(u^K(t_n)) \|_{\mathcal{I}_{\tau}H^\frac{3}{4}} \). We can just use that

\[ \| u^K(t_n) \|_{\mathcal{I}_{\tau}H^\frac{1}{4}} \lesssim T^\frac{1}{4} \| u^K \|_{L_t^\infty L_x^8} \lesssim T^\frac{1}{4} \| u^K \|_{\mathcal{X}_{\tau_0,b}}, \quad b > 1/2 \]

and, by frequency localization for \( |\xi| \geq \tau^{-\frac{1}{2}} \), that

\[ \| \Pi_{K+}(u^K(t_n)) \|_{\mathcal{I}_{\tau}H^\frac{3}{4}} \lesssim \tau^{\frac{1}{2}((s_0 - 1)/4)} \| u^K \|_{L_t^\infty L_x^8}. \]

This yields (79).

Finally to get (80), we follow the same lines but now use (50) and (51). This yields

\[ \| \mathcal{R}_{1,n} \|_{\mathcal{X}_{\tau_0,b}} \lesssim \tau^{\frac{1}{2}} \partial_x \Pi_{\tau-\frac{1}{2}}u^K(t_n) \|_{\mathcal{X}_\tau^{0,\frac{1}{4}}} \| u^K(t_n) \|_{\mathcal{I}_{\tau}H^\frac{3}{4}} + \| (\tau^{\frac{3}{2}} \partial_x) \|_{\mathcal{X}_\tau^{0,\frac{3}{4}}} \| \Pi_{K+}u^K(t_n) \|_{\mathcal{I}_{\tau}H^\frac{3}{4}} \| u^K(t_n) \|_{\mathcal{I}_{\tau}H^\frac{3}{4}}. \]
We then use the same estimates, in particular (81) and the fact that
\[ \|u^K(t_n)\|_{L^p_{
abla}H^{(1)}_+} \leq \|u^K\|_{L^p_{
abla}H^{(1)}_+}. \]
This ends the proof. \hfill \Box

We shall next estimate \( R_{2,n} \).

**Lemma 6.2.** For \( s_0 > \frac{1}{18} \) and \( b \in (1/2, 5/8) \), we have the estimate
\[ \|R_{2,n}\|_{X^{0,b}_r} \leq C_T \left( \tau^{\frac{1}{3}}(K\tau^{\frac{1}{2}})^2 + \tau^{\frac{1}{2}}(K\tau^{\frac{1}{2}})^3 + \tau^{\frac{3}{2}}(K\tau^{\frac{1}{2}})^4 \right). \] (82)
Moreover if \( s_0 > \frac{1}{4} + \frac{1}{18} \), we have
\[ \|R_{2,n}\|_{X^{0,b}_r} \leq C_T(K\tau^{\frac{1}{2}})\tau^{\frac{5}{32}} \] (83)
and if \( s_0 > 1/2 \), we have
\[ \|R_{2,n}\|_{X^{0,b}_r} \leq C_T\tau. \] (84)

**Proof.** We first use (35) and Remark 3.2 to estimate
\[ \|R_{2,n}\|_{X^{0,b}_r} \Rightarrow \|\mathcal{F}_{2,n}\|_{X^{0,-\frac{3}{16}}_r}. \]
Next, by using (60) and the product estimate (47), we get
\[ \|\mathcal{F}_{2,n}\|_{X^{0,-\frac{3}{16}}_r} \leq K\tau^{\frac{1}{2}} \left( \|u^K(t_n)\|_{X^{0,b}_r}^2 \|\mathcal{T}_n(u^K(t_n))\|_{X^{0,-\frac{3}{16}}_r} \right) \]
\[ + \|u^K(t_n)\|_{X^{0,b}_r} \|\mathcal{T}_n(u^K(t_n))\|_{X^{0,-\frac{3}{16}}_r}^2 + \|\mathcal{T}_n(u^K(t_n))\|_{X^{0,-\frac{3}{16}}_r}^3. \]
Next, by using (58), we get that
\[ \|\mathcal{T}_n(u^K(t_n))\|_{X^{0,-\frac{3}{16}}_r} \leq \tau^{\frac{1}{3}} \sup_{t' \in [0,\tau]} \sum_{\kappa \in A} \|\Pi_{\kappa_1}u^K(t_n + t')\Pi_{\kappa_2}\mathcal{U}^{\mathcal{K}}(t_n + t')\Pi_{\kappa_3}\mathcal{U}^{\mathcal{K}}(t_n + t')\|_{X^{0,-\frac{3}{16}}_r}. \]
By using (31), we thus obtain that
\[ \|\mathcal{T}_n(u^K(t_n))\|_{X^{0,-\frac{3}{16}}_r} \leq \tau^{\frac{1}{3}} \sup_{t' \in [0,\tau]} \sum_{\kappa \in A} \|\Pi_{\kappa_1}u^K(t_n + t')\Pi_{\kappa_2}\mathcal{U}^{\mathcal{K}}(t_n + t')\Pi_{\kappa_3}\mathcal{U}^{\mathcal{K}}(t_n + t')\|_{X^{0,-\frac{3}{16}}_r}. \]
Consequently, by using again the product estimate (47), we find that
\[ \|\mathcal{T}_n(u^K(t_n))\|_{X^{0,-\frac{3}{16}}_r} \leq \tau^{\frac{1}{3}}(K\tau^{\frac{1}{2}}) \sup_{t' \in [0,\tau]} \|u^K(t_n + t')\|_{X^{0,-\frac{3}{16}}_r}^3. \]
Then, by using Proposition 4.2, we get
\[ \|\mathcal{T}_n(u^K(t_n))\|_{X^{0,-\frac{3}{16}}_r} \leq \tau^{\frac{1}{3}}(K\tau^{\frac{1}{2}})C_T \]
and hence
\[ \|R_{2,n}\|_{X^{0,b}_r} \leq K\tau^{\frac{1}{2}} \left( \tau^{\frac{1}{3}}(K\tau^{\frac{1}{2}}) + (\tau^{\frac{1}{3}}(K\tau^{\frac{1}{2}}))^2 + (\tau^{\frac{1}{3}}(K\tau^{\frac{1}{2}}))^3 \right) C_T. \]
This proves (82).
To get (83), we use (46) and (48) to get this yields
\[
\|R_{2,n}\|_{\mathcal{X}^{0,0}} \lesssim \|F_{2,n}\|_{\mathcal{X}^{0,0}} - \frac{1}{3}
\]
\[
\lesssim (K^\frac{1}{2})^\frac{1}{2} \left( \|u^K(t_n)\|_{2}^{2} \|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} + \|u^K(t_n)\|_{0}\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} + \|\mathcal{T}_n(u^K(t_n))\|_{0}\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} + \|\mathcal{T}_n(u^K(t_n))\|_{0}\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} + \|\mathcal{T}_n(u^K(t_n))\|_{0}\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} + \|\mathcal{T}_n(u^K(t_n))\|_{0}\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} \right).
\]
We can use again (31) and the trivial estimate
\[
\|u^K(t_n)\|_{\mathcal{J}^0 H^\frac{1}{2}} \lesssim T^\frac{1}{2} \|u^K\|_{L^\infty H^\frac{1}{2}}
\]
so that it only remains to estimate \(\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{X}^{0,0}}\) and \(\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}}\). For the first one, we use again (31) to write
\[
\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{X}^{0,0}} \lesssim \tau \sup_{t' \in [0,\tau]} \sum_{n \in \mathcal{A}} \|\Pi_{\kappa_1} u^K(t_n + t')\Pi_{\kappa_2} \Pi_{\kappa_3} u^K(t_n + t')\|_{\mathcal{X}^{0,0}}.
\]
Next, from Hölder’s inequality and the Sobolev embedding \(W^{\frac{1}{2},4} \subset L^\infty\), we get that
\[
\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{X}^{0,0}} \lesssim \tau \sup_{t' \in [0,\tau]} \|\Pi_{\kappa_1} u^K(t_n + t')\|_{\mathcal{J}^0 W^{\frac{1}{2},4}}^2 \leq \|\Pi_{\kappa_1} u^K(t_n + t')\|_{\mathcal{J}^0 W^{\frac{1}{2},4}}^2
\]
and hence by using again (45), Proposition 4.2 and (30) we get that for \(s_0 > 1/4 + 1/18\),
\[
\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} \lesssim \tau \frac{3}{2} \|u^K\|_{\mathcal{X}^{s_0,0}} \lesssim C_{T} T^\frac{3}{2}.
\]
Next, we estimate \(\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}}\). We begin with
\[
\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} \lesssim \tau \sup_{t' \in [0,\tau]} \sum_{n \in \mathcal{A}} \|\Pi_{\kappa_1} u^K(t_n + t')\Pi_{\kappa_2} \Pi_{\kappa_3} u^K(t_n + t')\|_{\mathcal{J}^0 H^\frac{1}{2}}.
\]
By using the fact that the derivatives act only on the function that has the highest frequencies, that is to say that for all sequences \((u_n), (v_n), (w_n)\), and \(s > 0\) we have
\[
\|\Pi_{\kappa_1} u_n \Pi_{\kappa_2} \Pi_{\kappa_3} v_n\|_{H^s} \lesssim \|\Pi_{\kappa_1} u_n\|_{H^s} \|\Pi_{\kappa_2} v_n\|_{L^\infty} \|\Pi_{\kappa_3} w_n\|_{L^\infty},
\]
we get by using also Sobolev embeddings that
\[
\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} \lesssim \tau \sup_{t' \in [0,\tau]} \|u^K(t_n + t')\|_{\mathcal{J}^0 H^\frac{1}{2}} \|\Pi_{\kappa_1} u^K(t_n + t')\|_{\mathcal{J}^0 W^{\frac{1}{2},4}}^2 \lesssim \tau \sup_{t' \in [0,\tau]} \|u^K(t_n + t')\|_{\mathcal{J}^0 H^\frac{1}{2}} \|\Pi_{\kappa_1} u^K(t_n + t')\|_{\mathcal{J}^0 W^{\frac{1}{2},4}}^2.
\]
Hence by using again (45), Proposition 4.2 and (30) we finally get that
\[
\|\mathcal{T}_n(u^K(t_n))\|_{\mathcal{J}^0 H^\frac{1}{2}} \lesssim C_{T} T^\frac{3}{4}
\]
if \(s_0 > 1/4 + 1/18\). We thus deduce (83).
It remains to prove (84). We now use (50) and (51) to get that
\[ \| R_{2,n} \|_{X^b} \lesssim \| F_{2,n} \|_{X^0} \]
\[ \lesssim \| u^K(t_n) \|_{X^0} \| T_\kappa(u^K(t_n)) \|_{L^\infty H^\frac{1}{2} +} + \| u^K(t_n) \|_{X^0} \| T_\kappa(u^K(t_n)) \|_{L^\infty H^1 +} + \| T_\kappa(u^K(t_n)) \|_{L^\infty H^\frac{1}{2} +} \cdot \]

Note that, for the last term in the above right-hand side, we have used that in the estimate (50), we can replace in the right-hand side the norm \( \| u_n \|_{X^0} \) by the norm \( \| u_n \|_{L^\infty H^\frac{1}{2} +} \) by using the Sobolev embedding in space instead of the Bourgain estimate (45). Since we have the obvious estimate \( \| u^K(t_n) \|_{L^\infty H^\frac{1}{2} +} \lesssim \| u^K \|_{L^\infty H^\frac{1}{2} +} \) and since
\[ \| T_\kappa(u^K(t_n)) \|_{L^\infty H^\frac{1}{2} +} \lesssim \| T_\kappa(u^K(t_n)) \|_{L^\infty H^\frac{1}{2} +} \],

it only remains to estimate \( \| T_\kappa(u^K(t_n)) \|_{L^\infty H^\frac{1}{2} +} \). From standard product estimates since \( H^\frac{1}{2} + \) is an algebra, we get that
\[ \| T_\kappa(u^K(t_n)) \|_{L^\infty H^\frac{1}{2} +} \lesssim \tau \| u^K(t_n) \|_{L^\infty H^\frac{1}{2} +} \lesssim C_T \tau. \]

This concludes the proof.

7. Proof of Theorem 1.1

We first observe that thanks to (23), we have from the triangle inequality that
\[ \| u(t_n) - u^0 \|_{L^2} \leq C_T \tau^{s_0} + \| u^K(t_n) - u^0 \|_{L^2} \leq C_T \tau^{s_0} + \| u^n \|_{L^2}, \]
where \( e^n \) solves (74). To get the error estimates of Theorem 1.1, it thus suffices to estimate \( \| e^n \|_{X^0} \) for some \( b \in (1/2, 5/8) \) thanks to (28). Note that there are two parts in the total error, the space discretization part above and the time discretization error on the right-hand side of (74) which is estimated in Lemma 6.1 and Lemma 6.2. We shall optimize the total error by choosing the best possible \( \alpha \) as regularity allows.

We first prove (4). For very rough data, when \( 1/4 < s_0 \leq 1/4 + 1/18 \), we need the estimate (45) without loss. This forces us to choose \( K = \tau^{-\frac{1}{4}} \), hence \( \alpha = 1 \) without allowing us to optimize the error. We thus obtain from Lemma 6.1 and Lemma 6.2 that
\[ \| R_{1,n} \|_{X^b} + \| R_{2,n} \|_{X^0} \leq C_T (\tau^{s_0}(\frac{1}{18}) + \tau^{\frac{1}{4}}) \leq C_T \tau^{s_0}(\frac{1}{18}) \].

Next, we decompose
\[ G_n = L_n - Q_n + C_n \]
with
\[ L_n = \frac{1}{\tau} \left\{ -2i \left[ J_1^n (\Pi_{\tau^{-\frac{1}{2}}} - \bar{e}^n, \Pi_K + u^K(t_n), \Pi_{\tau^{-\frac{1}{2}}} u^K(t_n)) + J_1^n (\Pi_{\tau^{-\frac{1}{2}}} - \bar{u}^K(t_n), \Pi_K + e^n, \Pi_{\tau^{-\frac{1}{2}}} u^K(t_n)) \right. \\
\left. + J_1^n (\Pi_{\tau^{-\frac{1}{2}}} \bar{u}^K(t_n), \Pi_K + u^K(t_n), \Pi_{\tau^{-\frac{1}{2}}} e^n) \right] \\
- i \left[ J_2^n (\Pi_K \bar{e}^n, \Pi_{\tau^{-\frac{1}{2}}} u^K(t_n), \Pi_{\tau^{-\frac{1}{2}}} u^K(t_n)) + J_2^n (\Pi_K \bar{u}^K(t_n), \Pi_{\tau^{-\frac{1}{2}}} e^n, \Pi_{\tau^{-\frac{1}{2}}} u^K(t_n)) \right. \\
\left. + J_2^n (\Pi_K \bar{u}^K(t_n), \Pi_{\tau^{-\frac{1}{2}}} u^K(t_n), \Pi_{\tau^{-\frac{1}{2}}} e^n) \right] \right\} \]
and \( \frac{\varepsilon}{2} \) and hence by using (85), we get from (74) that
\[
\|e^n\|_{X_r^{0, b}} \leq C_{T_1} T_1^s n \|G_n\|_{X_r^{0, -\frac{3s}{2}}} + C_T T^s n \eta, \quad n \eta \leq T_1,
\]
where \( \varepsilon = 5/8 - b > 0 \). Next, we have that
\[
\|G_n\|_{X_r^{0, -\frac{3s}{2}}} \leq \|L_n\|_{X_r^{0, -\frac{3s}{2}}} + \|Q_n\|_{X_r^{0, -\frac{3s}{2}}} + \|C_n\|_{X_r^{0, -\frac{3s}{2}}}.
\]
To estimate the right-hand side, we use the equivalent definitions (64), (68) and again (47) and (43) (we recall that for this case we choose \( K = 1 \)). This yields
\[
\|e^n\|_{X_r^{0, b}} \leq C_{T_1} T_1^s \left( \|e^n\|_{X_r^{0, b}} + \|e^n\|_{X_r^{0, b}}^2 + \|e^n\|_{X_r^{0, b}}^3 \right) + C_T T^s n \eta.
\]
By choosing \( T_1 \) sufficiently small we thus get that
\[
\|e^n\|_{X_r^{0, b}} \leq C_T T^s n \eta.
\]
This proves the desired estimate (4) for \( 0 \leq n \leq N_1 = T_1/\tau \). We can then iterate the argument on \( T_1/\tau \leq n \leq 2T_1/\tau \) and so on to get the final estimate.

Next we prove (5). We follow the same lines, but we can now optimize the total error. From Lemma 6.1 and Lemma 6.2, we get that
\[
\|R_{1, n}\|_{X_r^{0, b}} + \|R_{2, n}\|_{X_r^{0, b}} \leq C_T \left( (K\tau^2)^{\frac{1}{2}} T^{s_0} \frac{1}{\sqrt{\tau}} + (K\tau^2)^{\frac{1}{2}} T^{\frac{1}{2}} \right) \leq C_T (K\tau^2)^{\frac{1}{2}} T^{s_0} \frac{1}{\sqrt{\tau}}.
\]
We thus choose \( K \) such that \( (K\tau^2)^{\frac{1}{2}} T^{s_0} \frac{1}{\sqrt{\tau}} = \frac{1}{\tau} \), which gives
\[
K = \tau^{-\alpha/2} = \tau^{-\frac{1}{\tau^2} + s_0 + \frac{1}{2}} = 2 \left( 1 - \frac{1}{2s_0 + 1} \left( \frac{2s_0 + 1}{2s_0 + s_0 + \frac{1}{2}} \right) \right).
\]
Note that we have \( \alpha \in [1, 2] \) since \( 1/4 + 1/18 < s_0 \leq 1/2 \), and further
\[
\|R_{1, n}\|_{X_r^{0, b}} + \|R_{2, n}\|_{X_r^{0, b}} \leq C_T T^{s_0} \frac{1}{\sqrt{\tau}}. \quad (91)
\]
By using Lemma 3.4, we get from (74) that
\[
\|e^n\|_{X_r^{0, b}} \leq C_T T_1^s (\|L_n\|_{X_r^{0, -\frac{s}{2}}} + \|Q_n\|_{X_r^{0, -\frac{s}{2}}} + \|C_n\|_{X_r^{0, -\frac{s}{2}}}) + C_T T^{s_0} \frac{1}{\sqrt{\tau}}. \quad (92)
\]
To estimate \( L_n \), we use the product estimates (48), (49) to get
\[
\|L_n\|_{X_r^{0, -\frac{s}{2}}} \lesssim \|u^n\|_{X_r^{0, -\frac{s}{2}}} \|e^n\|_{L^2} + \|e^n\|_{X_r^{0, -\frac{s}{2}}} \|u^n\|_{X_r^{0, -\frac{s}{2}}} + \|e^n\|_{X_r^{0, -\frac{s}{2}}} \|u^n\|_{X_r^{0, -\frac{s}{2}}} \|u^n\|_{X_r^{0, -\frac{s}{2}}} \|u^n\|_{X_r^{0, -\frac{s}{2}}} \|u^n\|_{X_r^{0, -\frac{s}{2}}} \|u^n\|_{X_r^{0, -\frac{s}{2}}}
\]
and hence by using again Proposition 4.2 and (28), we obtain that
\[
\|L_n\|_{X_r^{0, -\frac{s}{2}}} \lesssim C_T \|e^n\|_{X_r^{0, b}}. \quad (93)
\]
To estimate $C_n$, we use again (47) and (43). This yields
\[
\|C_n\|_{X^{0,-\frac{5}{8}}} \leq C_T K \tau^{\frac{5}{2}} \|e^n\|_{X^{0,\frac{9}{8}}}. \tag{94}
\]
To estimate $Q_n$, we use (48) and (54) and again Proposition 4.2. This yields
\[
\|Q_n\|_{X^{0,-\frac{5}{8}}} \leq C_T (K \tau)^{\frac{5}{2}} \|e^n\|_{X^{0,b}}^2 \tag{95}
\]
since $s_0 > 1/4 + 1/18$. By setting $Y = \|e^n\|_{X^{0,b}/\tau^{s_0}}(1 - \frac{1}{2s_0+1}(\frac{29}{39})^+)\), we deduce from the above estimates and (92) that
\[
Y \leq C_T T_1^{s_0} \left( Y + (K \tau)^{\frac{5}{2}} \tau^{s_0}(1 - \frac{1}{2s_0+1}(\frac{29}{39})^+) Y^2 + \left( (K \tau)^{\frac{5}{2}} \tau^{s_0}(1 - \frac{1}{2s_0+1}(\frac{29}{39})^+) \right)^2 Y^3 \right) + C_T.
\]
We can then check that with the choice (90), for $1/4 + 1/18 < s_0 \leq 1/2$, the exponent $\beta$ of
\[
\tau^\beta = (K \tau)^{\frac{5}{2}} \tau^{s_0(1 - \frac{1}{2s_0+1}(\frac{29}{39})^+)}
\]
is positive. Hence, we can conclude as before to get (5).

It remains to prove (6). From Lemma 6.1 and Lemma 6.2, we now get that
\[
\|R_{1,n}\|_{X^{0,b}} + \|R_{2,n}\|_{X^{0,b}} \leq C_T \tau^{s_0-(\frac{11}{22})^+}. \]
We thus choose $K$ such that $\tau^{s_0-(\frac{11}{22})^+} = \frac{1}{2s_0}$ in order to optimize the total error. We find
\[
K = \tau^{\frac{1}{2s_0}(\frac{11}{22})^+}, \quad \alpha = 2 - \frac{1}{s_0}(\frac{11}{22})^+. \tag{96}
\]
We can use again (94), (95) and (93) to obtain from (74)
\[
\|e^n\|_{X^{0,b}/\tau^{s_0}} \leq C_T T_1^{s_0} \left( \|e^n\|_{X^{0,b}} + (K \tau)^{\frac{5}{2}} \|e^n\|_{X^{0,b}}^2 + K \tau^{\frac{5}{2}} \|e^n\|_{X^{0,b}}^3 \right) + C_T \tau^{s_0-(\frac{11}{22})^+}. \]
Again, by setting $Y = \|e^n\|_{X^{0,b}/\tau^{s_0}}$, we get that
\[
Y \leq C_T T_1^{s_0} \left( Y + (K \tau)^{\frac{5}{2}} \tau^{s_0-(\frac{11}{22})^+} Y^2 + \left( (K \tau)^{\frac{5}{2}} \tau^{s_0-(\frac{11}{22})^+} \right)^2 Y^3 \right) + C_T
\]
and we conclude as before, by observing that the exponent of $\tau$ in
\[
(K \tau)^{\frac{5}{2}} \tau^{s_0-(\frac{11}{22})^+}
\]
is positive with before, by observing that the choice (96).

8. PROOF OF Lemma 3.6

We have to prove (45). For this purpose, we adapt the proof in [18] (which is attributed to N. Tzvetkov). We first observe that
\[
\|\Pi_K u_n\|_{L^4}^2 = \|(\Pi_K u_n)^2\|_{L^2}^2. \tag{97}
\]
By the definition of the $X^{0,b}$ norm and by setting $f_n = e^{-imx\partial^2} \Pi_K u_n$, it is equivalent to prove that
\[
\left\| (e^{imx\partial^2} f_n)^2 \right\|_{L^2} \leq K \tau \|f_n\|_{H^{\frac{3}{2}} L^2}^2.
\]
By using the space-time Fourier transform we shall decompose $\tilde{f}_n(\sigma, k)$ by using a Littlewood–Paley decomposition with respect to $\sigma$. Note that since $\sigma \in [-\pi/\tau, \pi/\tau]$, there is actually a finite number of terms. We write
\[
f_n = \sum_{l \geq 0} f_{n,l}.
\]
where \( \widetilde{f}_{m,l}(\cdot,k) \) is supported in \( 2^{l-1} \leq \langle \sigma \rangle \leq 2^{l+1} \) for every \( k \). By symmetry and the triangle inequality, it is sufficient to prove that
\[
\sum_{p \leq q} \| e^{i(n\sigma)} f_{n,p} e^{i(n\sigma)} f_{n,q} \|_{L^2} \lesssim K \tau^{\frac{1}{2}} \| f_n \|^2_{H^s_{\mathcal{F} L^2}}.
\]
We shall actually prove that there exists \( \epsilon > 0 \) (we shall see that we can take \( \epsilon = 1/8 \)) such that for every \( p, q \) with \( p \leq q \),
\[
\| e^{i(n\sigma)} f_{n,p} e^{i(n\sigma)} f_{n,q} \|_{L^2} \lesssim K \tau^{\frac{1}{2}} 2^p (p^q \| f_{n,p} \|_{L^2}) (1 + \| f_{n,q} \|_{L^2}). \tag{98}
\]
Once this inequality is proven, the result follows easily. Indeed, let us set \( a_m = 2^3 m \| f_{n,m} \|_{L^2}, \quad b_m = 2^{m1} m \| f_{n,m} \|_{L^2}, \quad b_m = 2^{m1} m \| f_{n,m} \|_{L^2}. \) By Parseval, we have that \( \| a_m \| \leq \| f_n \|_{H^s_{\mathcal{F} L^2}} \) and that
\[
\| a_m \|_{L^2} \lesssim \| f_n \|_{H^s_{\mathcal{F} L^2}}.
\]
Moreover, assuming that (98) is proven we obtain that
\[
\sum_{p \leq q} \| e^{i(n\sigma)} f_{n,p} e^{i(n\sigma)} f_{n,q} \|_{L^2} \lesssim K \tau^{\frac{1}{2}} \| (b * a)_m a_m \|_{L^2} \lesssim K \tau^{\frac{1}{2}} \| a_m \|^2_{L^2}
\]
from Cauchy–Schwarz and Young’s inequality for sequences (observe that \( b \in l^1 \)) which is the desired estimate.

We shall now prove (98). From Parseval and by using (29), we have that
\[
\| e^{i(n\sigma)} f_{n,p} e^{i(n\sigma)} f_{n,q} \|_{L^2}^2 = \sum_k \int_{\tau}^\pi \sum_{k_1 + k_2 = k} \int_{\tau}^\pi \widetilde{f}_{n,p}(\sigma_1 - k_1, k_1) \widetilde{f}_{n,q}(\sigma_2 - k_2, k_2) \, d\sigma_1 \, d\sigma_2 \, d\sigma.
\]
Now let us notice that we have a nontrivial contribution if \( \sigma_1 - k_1^2 \) is in the support of \( \widetilde{f}_{n,q}(\cdot, k_2) \) and \( \sigma_2 - k_2^2 \) in the one of \( \widetilde{f}_{n,q}(\cdot, k_2) \). By periodicity in the \( \sigma \) variable, this means that there exist \( m_1, m_2 \in \mathbb{Z} \) such that
\[
\left| \sigma_1 - k_1^2 - \frac{2m_1\pi}{\tau} \right| \lesssim 2^p, \quad \left| \sigma_2 - k_2^2 - \frac{2m_2\pi}{\tau} \right| \lesssim 2^q.
\]
In other words, we have that \( \sigma_1 - k_1^2 \in E_p, \sigma_2 - k_2^2 \in E_q \) where \( E_l = \bigcup_{|m| \leq N} [2m \pi / \tau - 2^l, 2m \pi / \tau + 2^l] \). Note that since the frequencies \( k_1^2, k_2^2 \) are smaller than \( K^2 \), we can take \( N \lesssim \tau K^2 \).

By using again Cauchy–Schwarz, we thus get that
\[
\| e^{i(n\sigma)} f_{n,p} e^{i(n\sigma)} f_{n,q} \|_{L^2} \lesssim M_{p,q} \| \widetilde{f}_{n,p} \|_{L^2} \| \widetilde{f}_{n,q} \|_{L^2}, \tag{99}
\]
where
\[
M_{p,q} = \sup_{k, \sigma} \sum_{k_1 + k_2 = k} \int_{\sigma_1 + \sigma_2 = \sigma, \sigma_1 - k_1^2 \in E_p, \sigma_2 - k_2^2 \in E_q} d\sigma_1.
\]
To estimate \( M_{p,q} \), we observe that only \( \sigma \in k_1^2 + k_2^2 + E_p + E_q \subset k_1^2 + k_2^2 + 2E_q \) gives a nonzero contribution and that the integral is bounded by a constant times \( 2^p \). Since \( k_1 + k_2 = k \), we have
\[
k_1^2 + k_2^2 = \frac{1}{2} (k^2 + (k_1 - k_2)^2)
\]
and hence
\[
(k_1 - k_2)^2 \in 2\sigma - k^2 - 4E_q.
\]
Therefore, \( k_1 - k_2 \) is constrained in intervals of length \( \lesssim 2^\frac{p}{2} \) and there are at most \( 2\tau K^2 \) intervals. As a consequence, we obtain that
\[
M_{p,q} \lesssim \tau K^2 2^\frac{p}{2} = \tau K^2 2^\frac{3p}{4} 2^\frac{3p}{4}.
\]
Taking the square root, we thus deduce (98) from (99). This concludes the proof of (45).

9. Numerical experiment

In this section we illustrate our main result (Theorem 1.1) on the $L^2$ error estimate by a numerical experiment. For this purpose, we solve the periodic Schrödinger equation (1) with initial value

$$u(0) = f_{H^1} + \frac{2 \sin x}{2 - \cos x}$$

on the torus. Here $f_{H^1}$ is a randomized $H^1$ function normalised in $L^2$ (see [9] for details on the construction of $f_{H^1}$). We compare our new integrator (11) with the previously introduced single-filtered Fourier based method [12, 13] and two standard integration schemes for periodic Schrödinger equations: a Lie splitting and exponential integrator method (see, e.g., [3, 11]). For the latter, we employ a standard Fourier pseudospectral method for the discretization in space and we choose as largest Fourier mode $K = 2^{10}$ (i.e., the spatial mesh size $\Delta x = 0.0061$). On the other hand, for our twice-filtered Fourier based integrator, we have to use the relation $K = \tau^{-\alpha/2}$ with $\alpha$ given in (96), as we are in case (iii) of Theorem 1.1 (recall that $s_0 = 1$). This results in $K \approx \tau^{-5/6}$.

We observe from the experiment that the new twice-filtered Fourier integrator is convergent of order one for rough solutions in $H^1$ whereas the standard discretization techniques as well as our previously introduced single-filtered Fourier based method all suffer from order reduction, see Figure 1. In particular, the numerically obtained order for the exponential integrator is reduced down to 0.3, whereas the results for the standard Lie splitting scheme are highly irregular. Both integrators are thus unreliable and inefficient for such low regularity initial data. The single-filtered Fourier based integrator shows a more regular error behaviour for the considered example, however, its order is reduced to $3/4$. The only method that is able to integrate the considered low regularity problem appropriately is the twice-filtered Fourier based scheme (11) proposed in this paper.

![Figure 1](image1.png)

**Figure 1.** $L^2$ error of the new twice-filtered Fourier based scheme (11) (purple), the Lie splitting scheme (yellow), the exponential integrator (blue), and the original single-filtered Fourier based scheme (red) proposed in [12, 13]. Left picture: the slope of the (black) reference lines is 1 and $\frac{3}{4}$, respectively. Right picture: zoom into the region of the left lower corner; the slope of the (black) reference line is 1.

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