ON THE RIGIDITY OF SYMBOLIC POWERS

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Abstract. We deal with the rigidity conjecture of symbolic powers over regular rings. This was asked by Huneke. Along with our investigation, we confirm a conjecture [4, Conjecture 3.8].

1. Introduction

In this note \((R, \mathfrak{m})\) is a regular local ring of dimension \(d\). Recall that the \(n\)-the symbolic power of an ideal \(I\) defined by \(I^{(n)} := \bigcap_{p \in \text{Ass}(I)} (I^n R_p \cap R)\). Recall from [12] Question 31 that:

Question 1.1. Let \(p \in \text{Spec}(R)\). If \(p^{(d)} = p^d\), does it follow that \(p^n = p^{(n)}\) for all \(n \geq 1\)?

Question 1.1 is true in dimension 3, see [10, Corollary 2.5]. This uses the intersection multiplicity due to Serre. Also, Question 1.1 is true for 1-dimensional Gorenstein prime ideals of a 4-dimensional regular ring. This was proved by Huneke, see [10, Corollary 2.6]. Huneke and Ulrich extended this to the class of 1-dimensional prime ideals of regular rings that are lieri, see [9]. In Theorem 3.6 we show that:

Theorem 1.2. Let \(I\) be a Cohen-Macaulay height-two ideal generated by exactly \(d\) elements in a regular local ring \(R\) of dimension \(d > 2\). Suppose \(I\) is locally complete intersection. Then \(I^i = I^{(i)}\) for all \(i < d - 1\) and \(I^n \neq I^{(n)}\) \(\forall n \geq d - 1\).

In the graded situation and over polynomial rings, the claims \(I^i = I^{(i)}\) for all \(i < d - 1\) and \(I^{d-1} \neq I^{(d-1)}\) are in the recent preprint [4]. One may try to drop the conditions forced over \(I\). Such a dream overlaps with the following recent conjecture based on "computer experiments":

Conjecture 1.3. (See [4]) Let \(X \subset \mathbb{P}^3_k\) be a subvariety of codimension 2. Assume that there is a point \(p \in X\) such that the localization of \(I_X\) at \(p\) is not a complete intersection. Then \(\forall m \geq 2\), the saturation of \(I_X^{(m)}\) has an embedded component at \(p\). In particular, \(I_X^{(m)} \neq I_X^{(m)}\) \(\forall m \geq 2\).

In Section 4 we show Conjecture 1.3 is true in the irreducible case. We say an ideal \(I\) is rigid, if \(I^n = I^{(n)}\) for all \(n\) provided \(I^{(i)} = I^i\) for all \(i \leq \dim R\). Recall from [12]: Is any prime ideal rigid? Here, we present a sample:

Corollary 1.4. Any Cohen-Macaulay prime ideal \(p\) of height 2 in a 4-dimensional regular local ring is rigid. In fact \(p^n = p^{(n)}\) for all \(n \geq 1\) provided \(p^{(3)} = p^3\).

In Section 5 we use the machinery of birational geometry to construct non-rigid ideals (the ideals may not be radical).

2010 Mathematics Subject Classification. Primary 13D40; 13F20.
Key words and phrases. Symbolic powers; regular rings; local cohomology; rigidity.
2. Towards the rigidity in codimension one

For the simplicity of the reader we collect some well-known results that we need:

**Subsection 2.A:** Preliminaries. Recall that an ideal $I$ is called complete intersection if $I$ is generated by a regular sequence of length equal to height of $I$.

**Lemma 2.1.** (See [10], Corollary 2.5) Let $(R, \mathfrak{m})$ be a regular local ring of dimension 3 and $\mathfrak{p}$ be a prime ideal of dimension one which is not a complete intersection. Then $\mathfrak{p}^i \neq \mathfrak{p}^{(i)}$ for all $i > 1$.

The grade of an ideal $\mathfrak{a}$ on a module $M$ is defined by $\text{grade}_A(\mathfrak{a}, M) := \inf \{i \in \mathbb{N}_0 | \text{Ext}^i_{R}(R/\mathfrak{a}, M) \neq 0 \}$. We use depth($M$), when we deal with the maximal ideal of *-local rings. Denote the minimal number of generators of $M$ by $\mu(M)$.

**Discussion 2.2.** An ideal $I \triangleleft R$ is called perfect if $p \cdot \text{dim}(R/I) = \text{grade}(I, R)$. Also, $I$ is called strongly Cohen-Macaulay, if its Koszul homologies $H_i(I, R)$ are all Cohen-Macaulay modules. The ideal $I$ satisfies the $G_\infty$ condition if for all $p \in V(I)$, one has $\mu(I_p) \leq \text{ht}(p)$.

**Example 2.3.** (See [11] Theorem 2.1(a) and [11] Supplement)

i) Perfect ideals of codimension two in a regular ring are strongly Cohen-Macaulay.

ii) If $\mu(I) \leq \text{ht}(I) + 2$, $R$ is Gorenstein and $R/I$ is Cohen-Macaulay, then $I$ is strongly Cohen-Macaulay.

**Lemma 2.4.** (See [9], Lemma 2.7) Let $R$ be a local Gorenstein ring, let $I$ be a perfect ideal which is strongly Cohen-Macaulay and $G_\infty$. We write $D := \text{dim}(R/I)$. Then for all $n \geq d := \mu(I) - \text{grade}(I, R)$, we have $\text{depth}(R/I^{n+1}) = D - d$.

An ideal $I$ is called almost complete intersection, if $\mu(I) = \text{ht}(I) + 1$. The definition presented in [10] is more general than this.

**Lemma 2.5.** (See [10], Theorem 3.1) Let $R$ be a Cohen-Macaulay ring and $\mathfrak{p}$ be a prime ideal which is an almost complete intersection. Then $\mathfrak{p}^{(2)} = \mathfrak{p}^2$ if and only if $\mathfrak{p}R_{\mathfrak{q}}$ is generated by an $R_{\mathfrak{q}}$-sequence $\forall \mathfrak{q} \in V(\mathfrak{p})$ such that $\text{dim}((R/\mathfrak{p})_{\mathfrak{q}}) \leq 1$.

**Subsection 2.B:** Towards the rigidity in codimension one. For an $R$-module $M$, the $i^{th}$ local cohomology of $M$ with respect to an ideal $\mathfrak{a}$ is defined by $H^i_{\mathfrak{a}}(M) := \lim_{\leftarrow n} \text{Ext}^i_{R}(R/\mathfrak{a}^n, M)$. By definition, $H^0_{\mathfrak{m}}(R/I^n) = \bigcup_i (I^n :_{R/I^n} \mathfrak{m}^i) =: (I^n)_{\text{ast}}$. Note that $\mathfrak{m} \notin \text{Ass}(R/I^n)$ if and only if $\text{depth}(R/I^n) > 0$ if and only if $(I^n)_{\text{ast}}$. In fact, our interest in 1-dimensional ideals coming from:

**Fact 2.6.** Let $I \triangleleft R$ be a radical ideal of dimension one. Then $I^{(n)}/I^n = H^0_{\mathfrak{m}}(R/I^n)$.

In the case $I^{(n)}/I^n = H^0_{\mathfrak{m}}(R/I^n)$, the equality of symbolic powers and ordinary powers translates to the positivity of the depth function $f(n) := \text{depth}(R/I^n)$.

**Proposition 2.7.** Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d$ and $\mathfrak{p}$ be a prime ideal of dimension one generated by at most $d$ elements. The following holds:

a) Suppose $\mathfrak{p}^i = \mathfrak{p}^{(i)}$ for some $i > 1$. Then

i) $\mathfrak{p}^i = \mathfrak{p}^{(i)}$ for all $i$.

ii) $\mathfrak{p}$ generated by exactly $d - 1$ elements.
b) Suppose $p^{i_0} \neq p^{(i_0)}$ for some $i_0 > 1$. Then $p^i \neq p^{(i)}$ for all $i > i_0$.

Proof. a): As $p$ is almost complete intersection and of dimension one, one can easily check that $p$ is perfect, strongly Cohen-Macaulay (see Example 2.3(ii)) and $G_{\infty}$.

i) In view of Lemma 2.4, $f(n) := \text{depth}(R/p^n)$ is constant for all $n > 1$. Having Fact 2.6 in mind, $f(i_0) \neq 0$. So, $f(n) \neq 0$ for all $n > 1$. As $f(n) \leq \dim R/p^n = 1$, we get $f(n) = 1$ for all $n > 1$. Again by Fact 2.6, $p^i = p^{(i)}$ for all $i$.

ii) By part i), $p^2 = p^{(2)}$. By Lemma 2.8, $p_q$ is generated by an $R_q$-sequence for all $q \in V(p)$ such that $\dim R_q/p_q \leq 1$. For example we can take $q := m$. The proof is now complete.

b): Every $d$-generated prime ideal of height $d - 1$ in a regular local ring of dimension $d$ is generated by a weak $d$-sequence, see [11, 1.3]. In view of [11, Corollary 2.5], for an ideal generated by a weak $d$-sequence over an integral domain, one has

$$\text{Ass}(R/p) \subset \text{Ass}(R/p^2) \subset \ldots \subset \text{Ass}(R/p^{i_0}) \subset \ldots \quad (*)$$

As $p^{i_0} \neq p^{(i_0)}$ and by Fact 2.6, $\text{depth}(R/p^{i_0}) = 0$. Thus $m \in \text{Ass}(R/p^{i_0})$. Applying $(*)$, we deduce that $m \in \text{Ass}(R/p^i)$ for all $i > i_0$. In view of Fact 2.6, $p^i \neq p^{(i)}$ for all $i > i_0$.

The following result shows that the assumption $\mu(p) = d$ is really important.

Lemma 2.8. (See [11]) Let $(R, m)$ be a regular local ring of dimension $d \geq 4$ and $p$ be a prime ideal of dimension one which is not a complete intersection and is in the linkage class of a complete intersection.

i) If $R/p$ is Gorenstein, then $p^2 = p^{(2)}$ and $p^i \neq p^{(i)}$ for all $i > 2$.

ii) If $R/p$ is not Gorenstein, then $p^i \neq p^{(i)}$ for all $i \geq 2$.

It was asked in [9] when is the natural map $\text{Ext}^d_R(R/a, R) \rightarrow H^n_a(R)$ is injective.

Corollary 2.9. Let $p$ be as Lemma 2.8. Then, the natural map $\text{Ext}^d_R(R/p^n, R) \rightarrow H^n_a(R)$ is not injective for all $n > 2$.

Proof. We are going to apply Hartshorne-Lichtenbaum vanishing theorem. As regular local rings are analytically irreducible and $\dim R/p = 1$ we get that $H^n_a(R) = 0$. Thus, we need to show $\text{Ext}^d_R(R/p^n, R) \neq 0$. Note that $\text{Ext}^d_R(R/p^n, R)$ is not if its Matlis dual $\text{Ext}^d_R(R/p^n, R)^v$ is nonzero. By local duality and Fact 2.6, we get that $\text{Ext}^d_R(R/p^n, R)^v \simeq H^n_{\mathfrak{m}}(R/p^n) = p^{(n)}/p^n$, which is nonzero.

3. Towards the rigidity in codimension two

The following well-known example (see e.g. [20]) illustrates our’s idea:

Example 3.1. Let $R := \mathbb{C}[x_0, \ldots, x_3]$ and let $p := I_2(A)$ where

$$A := \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$ 

Then $p$ is perfect of height two in a four-dimensional ring with three generators. Also, $p^n = p^{(n)}$ for all $n$. 

An ideal \( I \) is called \textit{generically complete intersection} if \( I_p \) is complete intersection for all \( p \in \min(I) \). Also, recall that \( I \) is called \textit{locally complete intersection} if \( I_p \) is complete intersection for all \( p \in V(I) \setminus \{m\} \). In the graded situation we assume in addition that \( p \) is homogeneous. Our interest on locally complete intersection comes from the following non-linear version of[4] Remark 2.2:

\textbf{Fact 3.2.} Let \( I \) be a locally complete intersection ideal. Then \( I^{(n)}/I^n = H^0_m(R/I^n) \).

\textbf{Sketch of proof:} Recall that symbolic powers and ordinary powers are the same if the ideal is complete-intersection. Also, localization behaves nicely with respect to symbolic power (see Lemma 4.3 below). It turns out that \( I^{(n)} = (I^n)^{\text{sat}} \). It remains to mention that \( H^n_m(R/I^n) = H^n_m(R/I^d) \) for all \( n > d - 1 \).

\textbf{Lemma 3.3.} Let \( I \) be a Cohen-Macaulay height two ideal generated by \( d \) elements in a regular local ring \( R \) of dimension \( d \). If \( I \) is locally complete intersection, then \( \text{depth}(R/I^n) = \text{depth}(R/I^d) \) for all \( n > d - 1 \).

\textbf{Proof.} It is easy to see that such an ideal is perfect and \( G_\infty \). Due to Example 2.3, \( I \) is strongly Cohen-Macaulay. It remains to apply Lemma 2.4. \( \square \)

The same argument suggests:

\textbf{Corollary 3.4.} Let \( p \) be a Cohen-Macaulay height two prime ideal generated by \( 3 \) elements in a regular local ring \( R \) of dimension \( 4 \). Then \( \text{depth}(R/p^n) = \text{depth}(R/p^4) \) for all \( n > 2 \).

For each \( X \subseteq \text{Spec}(R) \) and any \( \ell \), set \( X^{\geq \ell} := \{ p \in X : \text{ht}(p) \geq \ell \} \).

\textbf{Lemma 3.5.} Let \( I \) be a Cohen-Macaulay height two ideal locally complete intersection in a regular local ring \( R \) of dimension \( d = \dim R > 2 \). Then \( I^i = I^{(i)} \) for all \( i > 1 \) if and only if \( \mu(I) \leq d - 1 \).

\textbf{Proof.} Suppose first that \( I^i = I^{(i)} \) for all \( i > 1 \). The ideal \( I \) is perfect, because \( \text{p.dim}(I) < \infty \). Clearly, a locally complete intersection ideal is generically complete intersection. In particular, we are in the situation of[13] Corollary 3.4. We note that the proof of[13] Corollary 3.4 uses the existence of a prime ideal of height \( \geq 3 \). Here, is the place that we use the assumption \( d > 2 \). Now, [13] Corollary 3.4] shows that \( \mu(I) \leq d - 1 \). Now, suppose \( \mu(I) \leq d - 1 \). Recall that a locally complete intersection ideal is generically complete intersection. In view of Example 2.3 I is strongly Cohen-Macaulay. Since \( I \) is locally complete intersection, \( \mu(IR_q) = \text{grade}(IR_q, R_q) = \text{ht}(IR_q) = 2 \leq \text{ht}(Q) \) for all \( q \in V(I) \setminus \{m\} \) with \( 3 \leq \text{ht}(Q) \). Also, \( \mu(I) \leq d - 1 = \text{ht}(m) - 1 \). In particular, \( \mu(I_q) \leq \text{ht}(Q) - 1 \) for all \( q \in V(I)^{\geq 3} \).

By 15 Lemma 3.1, \( I^i = I^{(i)} \) for all \( i \).

The assumption \( d > 2 \) is important: Let \( R := k[[x, y]] \) and \( I := (x, y) \). Then \( I^n = I^{(n)} \) for all \( n \). But, \( \mu(I) > d - 1 \) (in the case of projective geometry there is no restriction on the dimension, because in the construction of \( I^n \) we disregard the irrelevant ideal).

\textbf{Theorem 3.6.} Let \( I \) be a Cohen-Macaulay height two ideal generated by exactly \( d \) elements in a regular local ring \( R \) of dimension \( d > 2 \). Suppose \( I \) is locally complete intersection. Then \( I^i = I^{(i)} \) for all \( i < d - 1 \) and \( I^n \neq I^{(n)} \) \( \forall n \geq d - 1 \).
We barrow some lines from [4].

Proof. Since $I$ is locally complete intersection and in view of Fact 3.2 we observe that $I^n/I^n = H^n_{R}(R/I^n)$. This means that $I' = I^{(i)}$ if and only if $\text{depth}(R/I')} > 0$. It is easy to see that $I$ is $G_{\infty}$ and strongly Cohen-Macaulay. Combining these along with Lemma 3.3 we observe

$$I' = I^{(i)} \iff I' = I^{(i)} \forall i \geq d - 1. \quad (*)$$

Since $I$ is perfect, $p. \dim(R/I) = \text{grade}(I, R) = \text{ht}(I) = 2$. So, $p. \dim(I) = 1$, and recall that $\mu(I_p) \leq \text{ht}(p) = \text{depth}(R_p)$, because $I$ is locally complete intersection assumption. These assumptions imply that $\text{Sym}^i(I) \simeq I^i$ for all $i \leq d$ (see [19, Theorem 5.1]). Due to Hilbert-Burch, $0 \rightarrow R^{d-1} \xrightarrow{(a_{ij})} R^d \rightarrow I \rightarrow 0$ is exact. In the light of [22], the following (not necessarily exact) complex

$$F_i : 0 \rightarrow \bigwedge^i (R^{d-1}) \rightarrow \bigwedge^{i-1} (R^{d-1}) \otimes R^d \rightarrow \cdots \rightarrow R^{d-1} \otimes \text{Sym}^{i-1}(R^d) \rightarrow \text{Sym}^i(R^d)$$

approximates $\text{Sym}^i(I) \simeq I^i$.

Let $C := \ker \left( \text{Sym}(R^d) \rightarrow \text{Sym}(I) \right) = \text{im} \left( \text{Sym}(R^{d-1}) \rightarrow \text{Sym}(R^d) \right)$. Note that $\mathcal{R} := \text{Sym}(R^d) = R[X_1, \ldots, X_d]$. Thus, $C = \left( \sum a_{ij} X_i : 1 \leq j \leq d - 1 \right)$ when we view $C$ as an ideal of $\mathcal{R}$. Denote the Koszul complex with respect to a generating set of $C$ in $\mathcal{R}$ by $K$. We should remark that $\mathcal{R}$ is a graded polynomial ring. The $i$-th spot of $K$ is the above complex $F_i$. Let $p \in V(I) \setminus \{m\}$. To prove $(F_i)_p$ is exact, we need to show $K_p[x_1, \ldots, x_d]$ is exact. Such a thing is the case if $C_{p[x_1, \ldots, x_d]}$ is complete intersection. Let us check this. There are regular elements $f, g$ such that $I_p = (f, g)$. Then $0 \rightarrow R_p \rightarrow R_p^2$ approximates $I_p$. Denote the identity matrix by $\text{Id}$. Thus

$$0 \rightarrow R_p^{d-1} \xrightarrow{(a_{ij})} R_p^d \rightarrow I_p \rightarrow 0 \simeq (0 \rightarrow R_p \xrightarrow{(f, g)} R_p^2 \rightarrow I_p \rightarrow 0) \oplus (0 \rightarrow R_p^{d-2} \xrightarrow{\text{Id}} R_p^{d-2} \rightarrow 0 \rightarrow 0).$$

This yields that $C_{p[x_1, \ldots, x_d]} = (fX_1 - gX_2, X_3 \ldots, X_d)$ which is complete intersection. In sum, we observed that $F_i$ has finite length homologies.

Suppose $i < d$. Then, in view of the new intersection theorem [17], $F_i$ is acyclic. Conclude from Auslander-Buchsbaum formula that $\text{depth}(R/I') > 0$ for all $i < d - 1$. This implies that

$$I' = I^{(i)} \forall i < d - 1.$$

Suppose on the contradiction that $I^n = I^{(n)}$ for some $n \geq d - 1$. Via the last displayed item and $(*)$, we deduce that $I^n = I^{(n)}$ for all $n$. Then by Lemma 3.3 $\mu(I) \leq d - 1$ which is a contradiction. □

Corollary 3.7. Let $I$ be a Cohen-Macaulay height two ideal generated by at least $d$ elements in a regular local ring $R$ of dimension $d > 2$. Suppose $I$ is locally complete intersection. Then $I' = I^{(i)}$ for all $i < d - 1$ and $I^{d-1} \neq I^{(d-1)}$.

Proof. Recall that $\mu(I_p) \leq \text{ht}(p) = \text{depth}(R_p)$ for all $p \in V(I)$ of height less than $d$ and that $p. \dim(I) = 1$. Therefore, $\text{Sym}^i(I) \simeq I^i$ for all $i \leq d - 1$ (see [19, Theorem 5.1]). This follows by above proof that $I' = I^{(i)}$ for all $i < d - 1$. Suppose on the contradiction that $I^{d-1} = I^{(d-1)}$. Let us summarize things: $I' = I^{(i)}$ for all $i < d - 1$, $I$ is perfect of height-two and $I$ is generically complete intersection. Under these assumptions [15, Theorem 3.2] implies that $I' = I^{(i)}$ for all $i$.
In the light of Lemma 5.5 \( \mu(I) \leq d - 1 \). This is excluded by the assumption. This contradiction implies that \( I^{d-1} \neq I^{(d-1)} \). 

The assumption \( d > 2 \) is important: Let \( R := k[[x, y]] \) and \( I := (x, y) \). Then \( \mu(I) = 2 > d - 1 \). Clearly, \( I^2 = I^{(2)} \). The locally complete intersection assumption is important:

**Example 3.8.** Look at \( I := (yzw, xzw, xyw, xyz) \) as an ideal in \( R := k[x, y, z, w] \). Then \( I \) is a Cohen-Macaulay height two ideal generated by exactly 4 elements in a *-regular ring \( R \) of dimension 4. It is easy to observe that \( x^n yzw \in I^{(2)} \setminus I^2 \) for all large \( n \). Thus \( I^{(2)} \neq I^2 \).

The height-two assumption is important:

**Example 3.9.** Set \( R := k[x_1, \ldots, x_5] \). We look at the pentagon as a simplicial complex. Its Stanley-Reisner ring is \( R_\Delta := k_\Delta/I_\Delta := R/(x_1 x_3, x_1 x_4, x_2 x_4, x_2 x_5, x_3 x_5) \). Note that \( I_\Delta \) is a height three ideal. So, \( \text{dim}(\Delta) = \text{dim} R_\Delta - 1 = 1 \). By the help of Macaulay 2, the projective resolution of \( R_\Delta \) over \( R \) is \( 0 \rightarrow R \rightarrow R^4 \rightarrow R^5 \rightarrow R \rightarrow R_\Delta \rightarrow 0 \). Due to Auslander-Buchsbaum formula, \( \text{depth}_R(R_\Delta) = 2 = \text{dim}(R_\Delta) \). Thus \( I_\Delta \) is perfect and of codimension 3 generated minimally by 5 elements. In view of \( [18 \text{ Proposition 1.11}] \), \( I_\Delta \) is locally complete intersection. Thus, for all \( p \in V(I) \), one has \( \mu(I_p) \leq \text{ht}(p) \). By definition, \( I \) is \( G_\infty \). Thanks to Macaulay2, \( p, \text{dim}(R/I_\Delta^1) = 5 \). By Auslander-Buchsbaum formula, \( \text{depth}(R/I_\Delta^1) = 0 \). In the light of \( \text{Lemma 2.3} \), \( \text{depth}(R/I_\Delta^2) = 0 \) for all \( n \geq 3 \). Consequently, \( I_\Delta^2 \neq I_\Delta^{(n)} \) for all \( n \geq 3 \), because \( I_\Delta \) is locally complete intersection (see Fact 3.2). By \( [18 \text{ Example 2.8}] \), \( p, \text{dim}(R/I_\Delta^1) = 3 \) and consequently \( I_\Delta^3 = I_\Delta^{(2)} \).

In fact, the above example suggests:

**Corollary 3.10.** Let \( I \) be a perfect ideal of height \( d - 2 \) with minimally \( d \) generators in a \( d \)-dimensional regular local ring \( R \). If \( I \) is locally complete intersection, then \( I^n \neq I^{(n)} \) for all \( n \geq 3 \).

**Proof.** As \( \mu(I) - \text{grade}(I, R) \leq 2 \), \( I \) is strongly Cohen-Macaulay, because of Example 2.3. Since \( \mu(I_p) \leq \text{ht}(p) \), we see \( I \) is \( G_\infty \). By \( \text{Lemma 2.3} \), \( \text{depth}(R/I^{(n)}) = 0 \) for all \( n \geq 3 \). Since \( I \) is locally complete intersection and in view of Fact 3.2 we observe that \( I^{(n)}/I^n = H^0_m(R/I^n) \). So, \( I^n \neq I^{(n)} \) for all \( n \geq 3 \). 

The Cohen-Macaulay assumption is important:

**Example 3.11.** Let \( C \) be the curve in \( \mathbb{P}^3 \) parameterized by \( \{s^4, s^3 t, s t^3, t^4\} \). This is the Macaulay’s curve. Denote the ideal of definition of \( C \) by \( p \) which is a prime ideal in \( k[x, y, z, w] \). This is well-known that \( \mu(p) = 4 \), \( p^n = p^{(n)} \) for all \( n \) and that \( R/p \) is not Cohen-Macaulay, see [20 Example]. One can show that \( p \) is locally complete intersection (for a quick proof please see Theorem 4.3 below). Also, \( \text{ht}(p) = 2 \). In particular, the Cohen-Macaulay assumption in Theorem 3.6 is really important.

4. Towards the rigidity in dimension four

**Conjecture 4.1.** (See [4 Conjecture 3.8]) Let \( X \subset \mathbb{P}^3_k \) be a subvariety (reduced and unmixed) of codimension 2. Assume that there is a point \( p \in X \) such that the localization of \( I_X \) at \( p \) is not a complete intersection. Then \( \forall m \geq 2 \), the saturation of \( I_X^m \) has an embedded component at \( p \). In particular, \( I_X^m \neq I_X^{(m)} \) for all \( m \geq 2 \).
Observation 4.2. The monomial-situation rarely happens: Recall that $I$ is an ideal in the ring $R := k[x_1, \ldots, x_4]$. As $I$ is radical unmixed monomial and 2-dimensional we have

$$I = \text{rad}(I) = \bigcap_{(i \neq j)} (x_i, x_j).$$

Let $G$ be the graph with the vertex set $\{1, \ldots, 4\}$ where $\{i, j\}$ is an edge, if such a pair does not appear in the above intersection. Suppose on the contradiction that $I^m = I^{(m)}$ for some $m \geq 2$. Thanks to [18] Lemma 3.1, $G$ is a path or a cycle or the union of two disjoint edges. In the case of union of two disjoint edges, $I$ is locally complete intersection (see [18] Example 1.18) which is excluded by the conjecture. The corresponding ideal of the paw-graph

![Paw Graph](image)

is $(x_1x_4, x_2x_4)$ which is of height one. This is excluded. The ideal of the diamond

![Diamond Graph](image)

is $(x_1x_3)$ which is of height one. This is excluded. The corresponding ideal of

![Tetrahedral Graph](image)

is $(x_1x_4, x_2x_4, x_3x_4)$ which is of height one. This is excluded. Also, the corresponding ideal of tetrahedral graph excluded. Therefore, $G$ is either 4-gon (square graph) or 4-pointed path. In particular, it is connected. Deduce from [18] Proposition 1.11] that $I$ is locally complete intersection which is excluded by the conjecture. Therefore, $G$ is either 4-gon (square graph) or 4-pointed path. In particular, it is connected. Deduce from [18] Proposition 1.11] that $I$ is locally complete intersection which is excluded by the conjecture. Therefore, $G$ is either 4-gon (square graph) or 4-pointed path. In particular, it is connected. Deduce from [18] Proposition 1.11] that $I$ is locally complete intersection which is excluded by the conjecture.

The following result is well-known (see e.g. the stacks project).

**Lemma 4.3.** Let $R \to S$ be a flat ring map (e.g. localization with respect to a multiplicative closed set). Let $q \subseteq R$ be a prime ideal such that $p = qS$ is a prime ideal of $S$. Then $q^{(n)}S = p^{(n)}$.

**Theorem 4.4.** Conjecture 4.1 is true for irreducible varieties.

**Proof.** Let $q$ be the defining ideal of the variety $X \subset \mathbb{P}^3$. Then, $q$ is a height-two prime ideal in $R := k[x_1, \ldots, x_4]$ which is not locally complete intersection. By definition, there is a homogeneous prime ideal $p \in V(q) \setminus \{m\}$ such that $qR_p$ can not be generated by a regular sequence. Note that $qR_q$ is generated by a regular sequence, because it is a maximal ideal of a regular local ring. Deduce from this that $p \supsetneq q$. One may find easily that $p \supsetneq m$. We conclude by this that $\text{ht}(p) = 3$. We
summarize things as follows: $R_p$ is a regular local ring of dimension 3 and $qR_p$ is a prime ideal of height two which is not a complete intersection ideal. Set $S := R \setminus p$. In the light of Lemma 4.3
\[(S^{-1}q)^m \neq (S^{-1}q)^{(m)} \quad \forall m > 1 \quad (*)\]
One can find easily that $(S^{-1}q)^m = S^{-1}(q^m)$.
If $q^m = q^{(m)}$ were be the case, in view of Lemma 4.3 we should have
\[(S^{-1}q)^m = S^{-1}q^m = S^{-1}(q^m) = (S^{-1}q)^{(m)},\]
which is a contradiction via $(*)$. So, $q^m \neq q^{(m)}$ for all $m > 1$.

**Corollary 4.5.** Let $p$ be a 2-dimensional Cohen-Macaulay prime ideal in a regular local ring of dimension four. If $p^{(3)} = p^3$, then $p^n = p^{(n)}$ for all $n \geq 1$.

**Proof.** By the proof of Theorem 4.4 we may assume that $p$ is locally complete intersection. If $\mu(p) \leq 3$, by the help of Lemma 3.5, we observe that $p^n = p^{(n)}$ for all $n \geq 1$. Suppose $\mu(p) \geq 4$. Then we are in the situation of Corollary 3.7. In view of Corollary 3.7, $p^{(3)} \neq p^3$. This is excluded by the assumptions. The proof is now complete.

5. **Towards non-rigidity**

The rings in this section are of zero characteristic.

**Discussion 5.1.** Let $X \subset \mathbb{P}^n$ be a projective variety. Given a rational map $F : X \dashrightarrow \mathbb{P}^n$ into another projective space, $Y \subset \mathbb{P}^n$ denote its image. Recall that $F$ is called birational onto its image if there exists a rational map $G : Y \dashrightarrow \mathbb{P}^n$ whose image is $X$. When such a thing happens we say $F$ is a Cremona transformation. Note that $F$ (resp. $G$) determines by forms $\underline{f} := f_0, \ldots, f_n$ (resp. $\underline{g}$) of same degree. By $d$ (resp. $d'$) we mean $\deg(f_i)$ (resp. $\deg(g_j)$). Also, $\underline{g}$ is called representatives of the inverse. The ideal generated by $\underline{f}$ is called the base ideal. By a result of Gabber, $d' \leq d^{n-1}$, see [2]. In particular, when $n = 2$ one can recover the classical result $d = d'$. The following is a method to construct non-rigid ideals.

**Proposition 5.2.** Adopt the above notation and let $I$ be the base ideal generated by forms of degree $d \geq 2$. Assume the following conditions hold:

i) $\text{depth}(R/I) > 0$,

ii) $I^{(\ell)}/I^\ell$ is either zero or $\mathfrak{m}$-primary for all $\ell$, and

iii) The Rees algebra $R(I)$ satisfies Serre’s condition $S_2$.

Then $I^\ell = I^{(\ell)}$ for all $\ell < d'$ and $I^d \neq I^{(d')}$, where $d'$ is the degree of representatives of the inverse.

**Proof.** In the light of [3] Theorem 2.1 we observe that the symbolic Rees algebra $R(I) := \bigoplus_s f^{(s)}_s$ is equal to $R[I, Dt^{d'}]$, where $D$ is called the source inversion. We note that $D$ is defined by the equation
\[g_i(f_0, \ldots, f_n) = Dx_i \quad \forall i \quad (* \star)\]
In particular, for all $i < d'$, one has
\[I^{(i)} = R_t(I) = R(I)_i = I_i\]
By (⋆), we have $\deg(D) = dd' - 1$. Recall that $I^{d'}$ has no element of degree less than $dd'$. Thus, $D \in \mathbb{R}_{d'}^{(d')} \setminus I^{d'} = I^{(d')} \setminus I^{d'}$. 

We give a non-rigid ideal:

**Example 5.3.** The primeness of the ideal is important. Let $R := \mathbb{k}[x,y,z]$ and let $d$ be any integer. Take $I := (x^d, x^{d-1}y, y^{d-1}z)$. This is the base ideal of $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and is of degree $d$. The base ideal is one-dimensional and saturated. In particular, it is Cohen-Macaulay. Under this assumption it is shown in [3, Proposition 3.4] that the conclusion of Proposition 5.2 holds true. Recall from Discussion 5.4 that $d = d'$. It follows that $I^i = I^{(i)}$ for all $i < d$ and $I^d \neq I^{(d)}$.

**Discussion 5.4.** Let us revisit $I := (yzw, xzw, xyw, xyz)$ as an ideal in $A := \mathbb{k}[x,y,z,w]$. Then $I$ is square-free and is 3-Veronese. Such an ideal is perfect, of height 2, 4-generated and the depth of its powers are computed by the following table (see [3, Corollary 10.3.7])

$$\text{depth}(R/I^n) = \begin{cases} 
2 & \text{if } n = 1 \\
1 & \text{if } n = 2 \\
0 & \text{if } n > 2
\end{cases}$$

Let us show that the assumption ii) in Proposition 5.2 is really needed.

**Example 5.5.** Look at $I := (yzw, xzw, xyw, xyz)$ as an ideal in $R := \mathbb{k}[x,y,z,w]$. We note that $I$ is the base ideal of the Cremona map $F : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

i) In view of Example 3.8, the module $I^{(2)}/I^2$ is not zero. By Discussion 5.4, $\text{depth}(R/I^2) = 1$. Thus, $m \nsubseteq \text{Ass}(R/I^2)$. Therefore, $I^2$ has no $m$-primary component. So, $I^{(2)}/I^2$ is not $m$-primary.

ii) Here, we show that $I$ respects the conditions i) and iii) in Proposition 5.2. In the light of [21], $I^n = I^0$ for all $n$. This means that $R/I$ is normal. Normal rings are S2. Also, $\text{depth}(R/I) = 2 > 0$.

iii) Suppose on the contradiction that Proposition 5.2 is true without its second assumption. Recall that $F$ defines via the partial derivations of $f := xyzw$. This is well-known and classical from birational geometry that $F$ is standard involution, i.e., it is self-inverse as a rational map. In particular, $d = d' = 3$. In view of Proposition 5.2 we should have $I^2 = I^{(2)}$ which is a contradiction with Example 3.8.

**Example 5.6.** Let us apply things in an example due to Dolgachev: Let $f := x(xz + y^2)$. In view of [5, Page 192], the partial derivations $\{\partial f/\partial x, \partial f/\partial y, \partial f/\partial z\}$ define a plane Cremona transformation. Such a transformation is called polar transformation. It is easy to find that the base ideal $I = (2xz + y^2, xy, x^2)$ is perfect and of codimension 1. Recall that $d' = d = 2$. By Proposition 5.2, $I^2 \neq I^{(2)}$.

**Remark 5.7.** Let us look at the system $\{X^2, XY, WX + Y^2, ZX + W^2, UX + Z^2\}$. This defines a birational map $\mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ such that $d' = d^{n-1}$. Let $A := \mathbb{k}[X,Y,W,Z,U]$ and let $I := (X^2, XY, WX + Y^2, ZX + W^2, UX + Z^2)$. Set $R := \frac{A}{I}$. Here, we only check that $\text{depth}(R) > 0$: We use lowercase letters to represent the elements in $R$. We show $u$ is a regular element. It turns out that $u$ is a homogeneous parameter element. Look at $R$ as a $k[u]$-module. The set $\Gamma := \{1, x, y, w, z, y^2, yw, w^2, wz, w^3, yzw\}$ is a generating set for $R$ as a $k[u]$-module. Since $\Gamma$ is...
linearly independent over $k[u]$, we observe that $R$ is free as a $k[u]$-module. Clearly, $u$ is regular over $k[u]$. So, $u$ is regular, as claimed.

**Conjecture 5.8.** Assume that $R$ is a $d$-dimensional polynomial ring over a field and $I \triangleleft$ be an ideal. Suppose there is a polynomial function $f$ of degree at most $d$ with coefficients depend only on the degree of generators of $I$ and $d$ such that $I^{(i)} = I^i$ for all $i < f$. Then $I^n = I^{(n)}$ for all $n \geq 1$.

**Acknowledgement**. I thank Prof. Simis for his comments on the very earlier version of this note.

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