A Generalized Grover/Zeta Correspondence

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Abstract

We introduce a generalized Grover matrix of a graph and present an explicit formula for its characteristic polynomial. As a corollary, we give the spectra for the generalized Grover matrix of a regular graph. Next, we define a zeta function and a generalized zeta function of a graph $G$ with respect to its generalized Grover matrix as an analog of the Ihara zeta function and present explicit formulas for their zeta functions for a vertex-transitive graph. As applications, we express the limit on the generalized zeta functions of a family of finite vertex-transitive regular graphs by an integral. Furthermore, we give the limit on the generalized zeta functions of a family of finite tori as an integral expression.

1 Introduction

Starting from $p$-adic Selberg zeta functions, Ihara [5] introduced the Ihara zeta functions of graphs. Bass [2] generalized Ihara’s result on the Ihara zeta function of a regular graph to an irregular graph and showed that its reciprocal is a polynomial.

The Ihara zeta function of a finite graph was extended to an infinite graph. Clair [4] computed the Ihara zeta function for the infinite grid by using elliptic integrals and theta functions. Chinta, Jorgenson and Karlsson [3] gave a generalized version of the determinant formula for the Ihara zeta function associated to finite or infinite regular graphs.

There are exciting developments between quantum walk on a graph [1, 6, 7, 11, 16, 19] and the Ihara zeta function of a graph. Ren et al. [17] gave a relationship between the discrete-time quantum walk and the Ihara zeta function of a graph. Konno and Sato [13] obtained a formula of the characteristic polynomial of the Grover matrix by using the determinant expression for the second weighted zeta function of a graph. Komatsu, Konno and Sato [8] treated the generalized Ihara zeta function of $\mathbb{Z}$ as a limit of the Ihara zeta function of the cycle graph $C_n$ with $n$ vertices.

In Grover/Zeta Correspondence [9], Komatsu, Konno and Sato defined a zeta function and a generalized zeta function of a graph $G$ with respect to its Grover matrix, and presented the limits on the generalized zeta functions and the generalized Ihara zeta functions of a family of finite regular graphs as an integral expression by using the Konno-Sato theorem [13]. This result contained the result on the generalized Ihara zeta function in Chinta et al. [3]. Furthermore, they obtained the limit on the generalized Ihara zeta functions of a family of finite torus as an integral expression, and this result contained the result on the Ihara zeta function of the two-dimensional integer lattice $\mathbb{Z}^2$ in Clair [4].

In Walk/Zeta Correspondence [10], Komatsu, Konno and Sato defined a walk-type zeta function without use of the determinant expressions of the zeta function of a graph $G$, and presented various properties of walk-type zeta functions of random walk (RW), correlated random walk (CRW) quantum walk (QW) and open quantum walk (OQW) on $G$. Also, their limit formulas by using integral expressions were presented. Konno and Tamura [14] computed the walk-type zeta functions for the three- and four-state quantum walk, correlated random walk, the multi-state random walk on the one-dimensional torus, and the four-state quantum walk, correlated random walk on the two-dimensional torus. Furthermore, they gave an extension of the Konno-Sato theorem.

In this paper, we define a generalized Grover matrix of a graph and treat a walk-type zeta function of a vertex-transitive graph with respect to its generalized Grover matrix.

In Section 2, we review the Ihara zeta function of a finite graph and the generalized Ihara zeta function of a finite or infinite vertex-transitive graph. In Section 3, we state about the time evolution matrix, i.e., the Grover matrix of the Grover walk on a graph. In Section 4, we introduce a generalized Grover matrix of a graph and present an explicit formula for its characteristic polynomial. As a corollary, we give the spectra for the generalized Grover matrix of a regular graph. In Section 5, we define a zeta function and a generalized
zeta function of a graph $G$ with respect to its generalized Grover matrix as an analog of the Ihara zeta function and present explicit formulas for their zeta functions of a vertex-transitive graph. In Section 6, we express the limit on the generalized zeta functions of a family of finite vertex-transitive regular graphs by an integral. In Section 7, we give the limit on the generalized functions of a family of finite tori as an integral expression.

## 2 The Ihara zeta function of a graph

All graphs in this paper are assumed to be simple. Let $G = (V(G), E(G))$ be a connected graph (without multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges $uv$ joining two vertices $u$ and $v$. Furthermore, let $n = |V(G)|$ and $m = |E(G)|$ be the number of vertices and edges of $G$, respectively. For $uv \in E(G)$, an arc $(u, v)$ is the oriented edge (or the arc) from $u$ to $v$. Let $D_G$ the symmetric digraph corresponding to $G$. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$. For $v \in V(G)$, the degree $\deg_G v = \deg v = d_v$ of $v$ is the number of vertices adjacent to $v$ in $G$.

A path $P$ of length $n$ in $G$ is a sequence $P = (e_1, \ldots, e_n)$ of arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leq i \leq n - 1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, n$, then we write $P = (v_0, v_1, \ldots, v_n)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, $P$ is called an $(o(P), t(P))$-path. We say that a path $P = (e_1, \ldots, e_n)$ has a backtracking if $e_{i+1}^{-1} = e_i$ for some $i(1 \leq i \leq n - 1)$. A $(v, w)$-path is called a $v$-cycle (or $v$-closed path) if $v = w$. The inverse cycle of a cycle $C = (e_1, \ldots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \ldots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \ldots, e_m)$ and $C_2 = (f_1, \ldots, f_m)$ are called equivalent if $f_j = e_{j+k}$ for all $j$. The inverse cycle of $C$ is in general not equivalent to $C$. Let $[C]$ be the equivalence class which contains a cycle $C$. Let $B^r$ be the cycle obtained by going $r$ times around a cycle $B$. Such a cycle is called a multiple of $B$. A cycle $C$ is reduced if both $C$ and $C^2$ have no backtracking. Furthermore, a cycle $C$ is prime if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph $G$ corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of $G$ at a vertex $v$ of $G$.

The Ihara zeta function of a graph $G$ is a function of a complex variable $u$ with $|u|$ sufficiently small, defined by

$$Z(G, u) = Z_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $|C|$ runs over all equivalence classes of prime, reduced cycles of $G$.

Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$. The adjacency matrix $A = A(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. If $\deg_G v = k$ (constant) for each $v \in V(G)$, then $G$ is called $k$-regular.

**Theorem 1 (Ihara; Bass)** Let $G$ be a connected graph. Then the reciprocal of the Ihara zeta function of $G$ is given by

$$Z(G, u)^{-1} = (1 - u^2)^{-r-1} \det(I - uA(G) + u^2(D - I)),$$

where $r$ is the Betti number of $G$, and $D = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ and $d_{ij} = 0, i \neq j$. $(V(G) = \{v_1, \ldots, v_n\})$.

Let $G = (V(G), E(G))$ be a connected graph and $x_0 \in V(G)$ a fixed vertex. Then the generalized Ihara zeta function $\zeta_G(u)$ of $G$ is defined by

$$\zeta_G(u) = \exp \left( \sum_{m=1}^{\infty} \frac{N_m^0}{m} u^m \right),$$

where $N^0_m$ is the number of closed paths $P = (e_1, \ldots, e_m)$ of length $m$ in $G$ at $x_0$.
where \( N^0_m \) is the number of reduced \( x_0 \)-cycles of length \( m \) in \( G \). A graph \( G \) is called \textit{vertex-transitive} if there exists an automorphism \( \phi \) of the automorphism group \( \text{Aut}(G) \) of \( G \) such that \( \phi(u) = v \) for each \( u, v \in V(G) \). Note that, for a finite vertex-transitive graph, the classical Ihara zeta function is just the above Ihara zeta function raised to the power equaling the number of vertices:

\[
\zeta_G(u) = Z(G, u)^{1/n}.
\]

Furthermore, the \textit{Laplacian} of \( G \) is given by

\[
\Delta = \Delta(G) = D - A(G).
\]

A formula for the generalized Ihara zeta function of a vertex-transitive graph is given as follows:

\textbf{Theorem 2 (Chinta, Jorgenson and Karlsson)} Let \( G \) be a vertex-transitive \((q + 1)\)-regular graph with spectral measure \( \mu_\Delta \) for the Laplacian \( \Delta \). Then

\[
\zeta_G(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left( \int \log(1 - ((q + 1 - \lambda)u + qu^2))d\mu_\Delta(\lambda) \right).
\]

Note, if \( G \) is a vertex-transitive graph with \( n \) vertices, then

\section{The Grover matrix of a graph}

We define the Grover matrix which is the time evolution matrix of the Grover walk on a graph.

Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then the \textit{Grover matrix} \( U = U(G) = (U_{ef})_{e,f \in D(G)} \) of \( G \) is defined by

\[
U_{ef} = \begin{cases} 
2/d_{t(f)} = 2/d_{o(e)} & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\
2/d_{t(f)} - 1 & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]

The discrete-time quantum walk with the matrix \( U \) as a time evolution matrix is called the \textit{Grover walk} on \( G \). Furthermore, we introduce the \textit{positive support} \( F^+ = (F^+_{ij}) \) of a real square matrix \( F = (F_{ij}) \) as follows:

\[
F^+_{ij} = \begin{cases} 
1 & \text{if } F_{ij} > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

In Konno and Sato \cite{13}, they presented the following result. The \( n \times n \) matrix \( P = P(G) = (P_{uv})_{u,v \in V(G)} \) is given as follows:

\[
P_{uv} = \begin{cases} 
1/(\deg_G u) & \text{if } (u,v) \in D(G), \\
0 & \text{otherwise}.
\end{cases}
\]

Note that the matrix \( P(G) \) is the transition probability matrix of the simple random walk on \( G \).

\textbf{Theorem 3 (Konno and Sato)} Let \( G \) be a connected graph with \( n \) vertices \( v_1, \ldots, v_n \) and \( m \) edges. Then

\[
det(I_{2m} - uU) = (1 - u^2)^{m-n} \det((1 + u^2)I_n - 2uP(G))
\]

\[
= \frac{(1 - u^2)^{m-n}}{\prod_{i=1}^{n} \deg v_i} \det((1 + u^2)D - 2uA(G)).
\]
This theorem is called the Konno-Sato theorem (see \cite{12, 14}, for example).

Konno and Tamura \cite{14} extended the Grover matrix. Let $G$ be a connected graph with $m$ edges, and $a \in [0, 1]$. Then the extension $U_a = U_a(G) = (U_{ef})_{e,f \in D(G)}$ of the Grover matrix of $G$ is defined as follows:

$$U_{ef}^{(a)} = \begin{cases} (2/d_{e}(f) - 1)a + 1 & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ (2/d_{e}(f) - 1)a & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$

If $a = 1$, then $U_1 = U$ is the Grover matrix of $G$. In the case of $a = 0$, $U_0 = U^+$ is the positive support of the Grover matrix of $G$. Thus, the matrix $U_a$ is an extension of the Grover matrix $U$ of $G$.

Konno and Tamura \cite{14} presented the following result for the extension $U_a$ of the Grover matrix of a regular graph.

**Theorem 4 (Konno and Tamura)** Let $G$ be a connected $(q + 1)$-regular graph with $n$ vertices and $m$ edges. Then

$$\det(I_{2m} - uU_a) = (1 - u^2)^{m-n} \det((1 + (q + (1 - q)a)u^2)I_n - (1 + (1 - q)u)P(G)).$$

## 4 A generalized Grover matrix of a graph

We introduce a generalized Grover matrix of a graph.

Let $G$ be a connected graph with $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Then a generalized Grover matrix $\hat{U} = \hat{U}(G) = (\hat{U}_{ef})_{e,f \in D(G)}$ of $G$ is defined as follows:

$$\hat{U}_{ef} = \begin{cases} (2/d_{e}(f) - 1)a + b & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ (2/d_{e}(f) - 1)a & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$

If $a = b = 1$, then $\hat{U} = U$ is the Grover matrix of $G$. In the case of $a = 0$ and $b = 1$, $\hat{U} = U^+$ is the positive support of the Grover matrix of $G$. Thus, the generalized Grover matrix $\hat{U}$ is a generalization of the Grover matrix $U$ and the positive support of the Grover matrix $U^+$ of $G$.

We present a generalization of the Konno-Sato theorem as follows.

**Theorem 5 (A generalization of the Konno-Sato theorem)** Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Then

$$\det(I_{2m} - u\hat{U}) = \frac{(1 - b^2u^2)^{m-n}}{\prod_{i=1}^{n} \deg v_i} \det(D((1 + b(2a - b)u^2)I_n + b(b - a)u^2D) - uA_d),$$

where $A_d = (A_{uv}^{(d)})_{u,v \in V(G)}$ is given as follows:

$$A_{uv}^{(d)} = \begin{cases} (2 - \deg u)a + b \deg u & \text{if } (u, v) \in D(G), \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** At first, let

$$w(u, v) = \left(\frac{2}{\deg u} - 1\right)a + b \text{ if } (u, v) \in D(G),$$

and two $n \times n$ matrices $W = (w_{uv})_{u,v \in V(G)}$ and $D_w = (d_{uv}^{(w)})_{u,v \in V(G)}$ be defined as follows:

$$w_{uv} = \begin{cases} w(u, v) & \text{if } (u, v) \in D(G), \\ 0 & \text{otherwise,} \end{cases} \quad d_{uv}^{(w)} = \begin{cases} \sum_{o(e)=u} w(e) & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{1}{\deg u} = \frac{1}{\deg v}$$
Furthermore, we define two $2m \times 2m$ matrices $B_w = B_w(G) = (B_{ef})_{e,f \in D(G)}$ and $J_0 = (J_{ef})_{e,f \in D(G)}$ as follows:

$$B_{ef} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise}, \end{cases} \quad J_{ef} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$

Then we have

$$\tilde{U} = B_w - bJ_0.$$

Next, we introduce $2m \times n$ matrices $K = (K_{ev})_{e \in D(G); v \in V(G)}$ and $L = (L_{ev})_{e \in D(G); v \in V(G)}$ as follows:

$$K_{ev} = \begin{cases} 1 & \text{if } t(e) = v, \\ 0 & \text{otherwise}, \end{cases} \quad L_{ev} = \begin{cases} w(e) & \text{if } o(e) = v, \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore, a $2m \times n$ matrix $M = (M_{ev})_{e \in D(G); v \in V(G)}$ be given as follows:

$$M_{ev} = \begin{cases} 1 & \text{if } o(e) = v, \\ 0 & \text{otherwise}. \end{cases}$$

Then we have

$$M = J_0 K, \quad K = J_0 M.$$

Furthermore, we have

$$K^t L = B_w, \quad L^t K = W, \quad L^t M = D_w, \quad M^t M = L^t L = D.$$

Now, we have

$$\det(I_{2m} - u \tilde{U}) = \det(I_{2m} - u \tilde{U})$$

$$= \det(I_{2m} - u (B_w - bJ_0))$$

$$= \det(I_{2m} - u (K^t L - bJ_0))$$

$$= \det(I_{2m} + buJ_0 - uK^t L)$$

$$= \det(I_{2m} - uK^t L(I_{2m} + buJ_0)^{-1}) \det(I_{2m} + buJ_0).$$

If $A$ and $B$ are an $m \times n$ and $n \times m$ matrices, respectively, then we have

$$\det(I_m - AB) = \det(I_n - BA).$$

Thus, we have

$$\det(I_{2m} - u \tilde{U}) = \det(I_n - u L(I_{2m} + buJ_0)^{-1}K) \det(I_{2m} + buJ_0).$$

But, we have

$$\det(I_{2m} + buJ_0) = \det \begin{bmatrix} I_m & buI_m \\ buI_m & I_m \end{bmatrix} \cdot \det \begin{bmatrix} I_m & -buI_m \\ 0_m & I_m \end{bmatrix}$$

$$= \det \begin{bmatrix} I_m & 0_m \\ buI_m & I_m - b^2u^2I_m \end{bmatrix}$$

$$= (1 - b^2u^2)^m.$$
Furthermore,

\[
(I_{2m} + buJ_0)^{-1} = \begin{bmatrix} I_m & buI_m \\ buI_m & I_m \end{bmatrix}^{-1} \\
\approx \begin{bmatrix} 1 & bu & \ldots & 0 \\ bu & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 1 & bu \end{bmatrix}^{-1} \\
= \frac{1}{1 - b^2u^2} \begin{bmatrix} 1 & -bu & \ldots & 0 \\ -bu & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -bu \\ 0 & \cdots & -bu & 1 \end{bmatrix} \\
= \frac{1}{1 - b^2u^2}(I_{2m} - buJ_0).
\]

Therefore, it follows that

\[
\det(I_{2m} - u\tilde{U}) = (1 - b^2u^2)^m \det\left( I_n - \frac{u}{1 - b^2u^2} tL(I_{2m} - buJ_0)K \right) \\
= (1 - b^2u^2)^{m-n} \det((1 - b^2u^2)I_n - u^tL^tK + bu^tL^tJ_0K) \\
= (1 - b^2u^2)^{m-n} \det((1 - b^2u^2)I_n - uW + bu^tLM) \\
= (1 - b^2u^2)^{m-n} \det((1 - b^2u^2)I_n - uW + bu^tD_w) \\
= (1 - b^2u^2)^{m-n} \det(I_n - uW + bu^t(D_w - bI_n)).
\]

The entries of two matrices \(W\) and \(D_w\) are given as follows:

\[
(W)_{uv} = \left(\frac{2}{\deg u} - 1\right) a + b = \frac{1}{\deg u} ((2 - \deg u)a + b \deg u) \text{ if } (u,v) \in D(G)
\]

and

\[
(D)_{uv} = \left\{\left(\frac{2}{\deg u} - 1\right) a + b\right\} \deg u = (b - a) \deg u + 2a.
\]

Thus, we have \(W = D^{-1}A_d\) and \(D_w = (b - a)D + 2aI_n\).

Therefore, it follows that

\[
\det(I_{2m} - u\tilde{U}) = (1 - b^2u^2)^{m-n} \det(I_n - uD^{-1}A_d + bu^2((b - a)D + (2a - b)I_n)) \\
= (1 - b^2u^2)^{m-n} \det(D^{-1}) \det(D - uA_d + bu^2((b - a)D^2 + (2a - b)D)) \\
= \prod_{i=1}^{m-n} \det(D((1 + b(2a - b))u^2)I_n + b(b - a)u^2D) - uA_d).
\]

\[\square\]

For a \((q + 1)\)-regular graph, we obtain the following result.

**Corollary 1** Let \(G\) be a connected \((q + 1)\)-regular graph with \(n\) vertices \(v_1, \ldots, v_n\) and \(m\) edges, \(a \in [0, 1]\) and \(b \in \mathbb{R}\). Then

\[
\det(I_{2m} - u\tilde{U}) = (1 - b^2u^2)^{m-n} \det\{(1 + b((1 - q)a + bq)u^2)I_n - u((1 - q)a + b(q + 1))P(G))\).
\]

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Thus, Corollary 3

Let $a$ edges, $\Delta = \{(1 - q)a + b(q + 1)\}$.

By Corollary 1, we obtain the following result.

$$
\det(I_{2m} - u\tilde{U})
= (1 - b^2u^2)^{m-n} \det((q + 1)(1 + b(2a - b)u^2) + (q + 1)(b - a)bu^2)I_n - u((1 - q)a + b(q + 1))A)
= (1 - b^2u^2)^{m-n} \det((1 + b((1 - q)a + bq)u^2)I_n - u((1 - q)a + b(q + 1)))P(G)).
$$

□

If $G$ is a $(q + 1)$-regular graph with $n$ vertices, then we have

$$
P = \frac{1}{q + 1}A = \frac{1}{q + 1}(D - \Delta) = I_n - \frac{1}{q + 1} \Delta.
$$

By Corollary 1, we obtain the following result.

**Corollary 2** Let $G$ be a connected $(q + 1)$-regular graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Then

$$
\det(I_{2m} - u\tilde{U})
= (1 - b^2u^2)^{m-n} \det((1 - u((1 - q)a + b(q + 1))) + b((1 - q)a + bq)u^2)I_n
+ u\left(b - \frac{q - 1}{q + 1}a\right)\Delta).

\textbf{Proof.} By Corollary 1, we have

$$
\det(I_{2m} - u\tilde{U})
= (1 - b^2u^2)^{m-n} \det((1 + b((1 - q)a + bq)u^2)I_n
- u((1 - q)a + b(q + 1))(I_n - \frac{1}{q + 1} \Delta))
= (1 - b^2u^2)^{m-n} \det((1 - u((1 - q)a + b(q + 1))) + b((1 - q)a + bq)u^2)I_n
+ u\left(b - \frac{q - 1}{q + 1}a\right)\Delta).

□

Substituting $u = 1/\lambda$ to Corollaries 1 and 2, we obtain the following result.

**Corollary 3** Let $G$ be a connected $(q + 1)$-regular graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Then

$$
\det(\lambda I_{2m} - \tilde{U})
= (\lambda - b^2)^{m-n} \det((\lambda^2 + b((1 - q)a + bq))I_n - \lambda((1 - q)a + b(q + 1)))P(G))
= (\lambda - b^2)^{m-n} \det((\lambda^2 - \lambda((1 - q)a + b(q + 1))) + b((1 - q)a + bq))I_n
+ \lambda\left(b - \frac{q - 1}{q + 1}a\right)\Delta).
$$
By Corollary 3, the following result holds. Let $\text{Spec}(F)$ be the set of eigenvalues of a square matrix $F$.

**Corollary 4** Let $G$ be a connected $(q + 1)$-regular graph with $n$ vertices $v_1, \ldots, v_n$ and $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Set $\eta = (1 - q)a + b(q + 1)$ and $\sigma = b((1 - q)a + bq)$. Then the spectra of the generalized Grover matrix $\tilde{U}$ are given as follows:

1. $2n$ eigenvalues:
   \[
   \lambda = \frac{\mu \eta \pm \sqrt{\mu^2 \eta^2 - 4\sigma}}{2}, \quad \mu \in \text{Spec}(P);
   \]

2. $2(m - n)$ eigenvalues: $\pm b$ with multiplicities $m - n$.

**Proof.** By Corollary 3, we have
\[
\det(\lambda I_{2m} - \tilde{U}) = (\lambda - b^2)^{m-n} \prod_{\mu \in \text{Spec}(P)} (\lambda^2 + \sigma - \mu \eta \lambda).
\]
Solving $\lambda^2 - \mu \eta \lambda + \sigma = 0$, we obtain
\[
\lambda = \frac{\mu \eta \pm \sqrt{\mu^2 \eta^2 - 4\sigma}}{2}.
\]
The result follows. $\square$

## 5 A generalized Grover/Zeta Correspondence

Now, we propose a new zeta function of a graph. Let $G$ be a connected graph with $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Then we define the $(a, b)$-zeta function $Z_{a,b}(G, u)$ of $G$ is defined as follows:

\[
Z_{a,b}(G, u)^{-1} = Z_{a,b}(u)^{-1} = \det(I_{2m} - u\tilde{U}).
\]

By Corollaries 1 and 2, we obtain the following result.

**Proposition 1** Let $G$ be a connected $(q + 1)$-regular graph with $n$ vertices and $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Set $\eta = (1 - q)a + b(q + 1)$ and $\sigma = b((1 - q)a + bq)$. Then
\[
Z_{a,b}(G, u)^{-1} = (1 - b^2 u^2)^{m-n} \det\left((1 + \sigma u^2)I_n - \eta u P(G)\right)
\]
\[
= (1 - b^2 u^2)^{m-n} \det\left((1 - \eta u + \sigma u^2)I_n + \frac{\eta u}{q+1} \Delta\right).
\]

By Theorem 3, we obtain the exponential expression for $Z_{a,b}(u)$. We give a weight functions $w : D(G) \times D(G) \rightarrow \mathbb{C}$ as follows:

\[
w(f, e) = \begin{cases} 
\frac{2}{\deg(t(f)) - 1}a + b & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\
\frac{2}{\deg(t(f)) - 1}a & \text{if } f = e^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\]

For a cycle $C = (e_1, e_2, \ldots, e_r)$, let
\[
w(C) = w(e_1, e_2) \cdots w(e_{r-1}, e_r)w(e_r, e_1).
\]

**Theorem 6** Let $G$ be a connected graph with $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Then
\[
Z_{a,b}(u) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r}{r} u^r\right),
\]
where $N_r$ is defined by
\[
N_r = \sum \{w(C) \mid C : a \text{ cycle of length } r \text{ in } G\}.
\]
Proof. By the definition of $Z_{a,b}(u)$, we have

$$\log Z_{a,b}(u) = \log \{ \det(I_{2m} - u\tilde{U})^{-1} \} = - \text{Tr}[\log(I_{2m} - u\tilde{U})] = \sum_{r=1}^{\infty} \frac{1}{r} \text{Tr}[\tilde{U}^r]u^r.$$  

Since

$$w(f, e) = (\tilde{U})_{ef}, \ e, f \in D(G),$$

we have

$$\text{Tr}[\tilde{U}^r] = \sum_{\{C \mid C: a \text{ cycle of length } r \text{ in } G\}} N_r.$$ 

Hence,

$$\log Z_{a,b}(u) = \sum_{r=1}^{\infty} \frac{N_r}{r} u^r.$$ 

Thus,

$$Z_{a,b}(u) = \exp \left( \sum_{r=1}^{\infty} \frac{N_r}{r} u^r \right).$$ 

\[\square\]

Next, we define a generalized zeta function with respect to the generalized Grover matrix of a graph. Let $G = (V(G), E(G))$ be a connected graph, $x_0 \in V(G)$ a fixed vertex, $a \in [0, 1]$ and $b \in \mathbb{R}$. Then the generalized $(a, b)$-zeta function $\zeta_{a,b}(G, u)$ of $G$ is defined by

$$\zeta_{a,b}(G, u) = \exp \left( \sum_{r=1}^{\infty} \frac{N^0_r}{r} u^r \right),$$

where

$$N^0_r = \sum_{\{C \mid C: an \ x_0 - \text{cycle of length } r \text{ in } G\}}.$$ 

Note, if $G$ is a vertex-transitive graph with $n$ vertices, then

$$\zeta_{a,b}(u) = Z_{a,b}(G, u)^{1/n}. \quad (1)$$

If $a = b = 1$, then $\zeta_{1,1}(G, u) = \zeta(G, u)$ is the generalized zeta function of $G$ (see [9]). In the case of $a = 0$ and $b = 1$, $\zeta_{0,1}(G, u) = \zeta(G, u)$ is the generalized Ihara zeta function of $G$. Thus, the generalized $(a, b)$-zeta function $\zeta_{a,b}(G, u)$ is a generalization of the generalized zeta function and the generalized Ihara zeta function of $G$.

Now, we present an explicit formula for the generalized $(a, b)$-zeta function for a vertex-transitive graph.

Let $G$ be a vertex-transitive $(q + 1)$-regular graph with $n$ vertices and $m$ edges. Then, since $m = (q + 1)n/2$, we have

$$\frac{m - n}{n} = \frac{q - 1}{2}.$$ 

By Proposition 1, we obtain the following result.

**Theorem 7 (Generalized Grover/Zeta Correspondence)** Let $G$ be a connected vertex-transitive $(q + 1)$-regular graph with $n$ vertices and $m$ edges, $a \in [0, 1]$ and $b \in \mathbb{R}$. Set $\eta = (1 - q)a + b(q + 1)$ and $\sigma = b((1 - q)a + bq)$. Then

$$\zeta_{a,b}(G, u)^{-1} = (1 - b^2u^2)^{(q - 1)/2} \exp \left[ \frac{1}{n \sum_{\lambda \in \text{Spec}(\Delta)} \log \{(1 + \sigma u^2) - \eta u\lambda\}} \right], \quad (2)$$

$$\zeta_{a,b}(G, u)^{-1} = (1 - b^2u^2)^{(q - 1)/2} \exp \left[ \frac{1}{n \sum_{\lambda \in \text{Spec}(\Delta)} \log \left\{ (1 - \eta u + \sigma u^2) + \frac{\eta u}{q + 1} \lambda \right\}} \right]. \quad (3)$$
Theorem 8

Let $\{G_n\}_{n=1}^{\infty}$ be a series of finite vertex-transitive $(q+1)$-regular graphs such that

$$\lim_{n \to \infty} |V(G_n)| = \infty.$$ 

Then we have

$$\frac{|E(G_n)| - |V(G_n)|}{|V(G_n)|} = \frac{(q-1)|V(G_n)|}{2|V(G_n)|} = \frac{q-1}{2}.$$ 

Set

$$\nu_n = |V(G_n)|, \ m_n = |E(G_n)|.$$ 

Then the following result holds.

**Theorem 8** Let $\{G_n\}_{n=1}^{\infty}$ be a series of finite vertex-transitive $(q+1)$-regular graphs such that

$$\lim_{n \to \infty} |V(G_n)| = \infty.$$ 

Furthermore let $a \in [0, 1], \ b \in \mathbb{R}, \ \eta = (1-q)a + b(q+1)$ and $\sigma = b((1-q)a + bq)$. Then

1. $\lim_{n \to \infty} \zeta_{a,b}(G_n, u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left( (1 + \sigma u^2) - \eta u \right) d\mu_P(\lambda) \right]$;

2. $\lim_{n \to \infty} \zeta_{a,b}(G_n, u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left( (1 - \eta u + \sigma u^2) + \frac{\eta u}{q+1} \right) d\mu_\Delta(\lambda) \right],$ 

where $d\mu_P(\lambda)$ and $d\mu_\Delta(\lambda)$ are the spectral measures for the transition operator $P$ and the Laplacian $\Delta$.

**Proof.** By Theorem 7, we have

$$\lim_{n \to \infty} \zeta_{a,b}(G_n, u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left( (1 + \sigma u^2) - \eta u \right) d\mu(\lambda) \right].$$ 

Similarly, the second formula follows. $\square$

If $a = b = 1$, then we obtain the Grover/Zeta Correspondence (see [9]).
Corollary 5 (Grover/Zeta Correspondence) Let \( \{G_n\}_{n=1}^{\infty} \) be a series of finite vertex-transitive \((q+1)\)-regular graphs such that

\[
\lim_{n \to \infty} |V(G_n)| = \infty.
\]

Then

1. \( \lim_{n \to \infty} \zeta_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left( (1 + u^2) - 2u\lambda \right) d\mu_\Delta(\lambda) \right] \);

2. \( \lim_{n \to \infty} \zeta_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left( (1 - 2u + u^2) + \frac{2u}{q+1}\lambda \right) d\mu_\Delta(\lambda) \right] \),

where \( d\mu_\Delta(\lambda) \) and \( d\mu_\Delta(\lambda) \) are the spectral measures for the transition operator \( P \) and the Laplacian \( \Delta \).

In the case of \( a = 0 \) and \( b = 1 \), we obtain the Grover(Positive Support)/Ihara Zeta Correspondence (see [9]).

Theorem 9 (Grover(Positive Support)/Ihara Zeta Correspondence) Let \( \{G_n\}_{n=1}^{\infty} \) be a series of finite vertex-transitive \((q+1)\)-regular graphs such that

\[
\lim_{n \to \infty} |V(G_n)| = \infty.
\]

Then

1. \( \lim_{n \to \infty} \zeta_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left( (1 + qu^2) - (q + 1)u\lambda \right) d\mu_\Delta(\lambda) \right] \);

2. \( \lim_{n \to \infty} \zeta_{G_n}(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log \left( (1 + qu^2) - (q + 1 - \lambda)u \right) d\mu_\Delta(\lambda) \right] \),

where \( d\mu_\Delta(\lambda) \) and \( d\mu_\Delta(\lambda) \) are the spectral measures for the transition operator \( P \) and the Laplacian \( \Delta \).

The second formula is Theorem 1.3 in Chinta et al. [3].

7 Torus cases

We consider the generalized \((a, b)\)-zeta function of the \(d\)-dimensional integer lattice \( \mathbb{Z}^d \) \((d \geq 2)\).

Let \( T_N^d \) \((d \geq 2)\) be the \(d\)-dimensional torus (graph) with \( N^d \) vertices. Its vertices are located in coordinates \( i_1, i_2, \ldots, i_d \) of a \(d\)-dimensional Euclidian space \( \mathbb{R}^d \), where \( i_j \in \{0, 1, \ldots, N - 1\} \) for any \( j \) from 0 to \( d - 1 \). A vertex \( v \) is adjacent to a vertex \( w \) if and only if they have \( d - 1 \) coordinates that are the same, and for the remaining coordinate \( k \), we have \( |i_k^v - i_k^w| = 1 \), where \( i_k^v \) and \( i_k^w \) are the \( k \)-th coordinate of \( v \) and \( w \), respectively. Then we have

\[
|E(T_N^d)| = dN^d,
\]

and \( T_N^d \) is a vertex-transitive \(2d\)-regular graph.

By Corollary 1, we obtain the following result.

\[
Z_{a,b}(T_N^d, u)^{-1} = \det(\mathbf{I}_{2dN^d} - u \mathbf{U}(T_N^d)) = (1 - u^2)^{(d-1)N^d} \det((1 + \sigma u^2)\mathbf{I}_{N^d} - \eta u \mathbf{P}^{(s)}(T_N^d)). \tag{4}
\]

Here, it is known that \( \text{Spec} (\mathbf{P}^{(s)}(T_N^d)) \) is given as follows (see [15]):

\[
\text{Spec} (\mathbf{P}^{(s)}(T_N^d)) = \left\{ \frac{1}{d} \sum_{j=1}^{d} \cos \left( \frac{2\pi k_j}{N} \right) \mid k_1, \ldots, k_d \in \{0, 1, \ldots, N - 1\} \right\}.
\]
Corollary 6 (Grover/Zeta Correspondence (Tₙ case)) Let $Tₙ^d$ be the $d$-dimensional torus with $N^d$ vertices. Then
\[
\lim_{n \to \infty} \zeta(Tₙ^d, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \frac{1}{N^d} \sum_{j=1, k_j=0}^d \sum_{j=1}^{N-1} \log \left\{ (1 + \sigma u^2) - \frac{\eta u}{d} \sum_{j=1}^{d} \cos \left( \frac{2\pi k_j}{N} \right) \right\} \right].
\]

Therefore, we obtain the following theorem.

Theorem 10 (Generalized Grover/Zeta Correspondence ($Tₙ^d$ case)) Let $Tₙ^d$ be the $d$-dimensional torus with $N^d$ vertices. Furthermore, let $a \in [0, 1]$, $b \in \mathbb{R}$, $\eta = 2(1 - d) a + 2db$ and $\sigma = b(2a - b) + 2db - a$. Then
\[
\lim_{n \to \infty} \zeta(Tₙ^d, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \log \left\{ (1 + \sigma u^2) - \frac{\eta u}{d} \sum_{j=1}^{d} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi} \right],
\]
where $\int_{0}^{2\pi} \cdots \int_{0}^{2\pi}$ is the $d$-th multiple integral and $\frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}$ is the uniform measure on $[0, 2\pi)^d$.

If $a = b = 1$, then we obtain the Grover/Zeta Correspondence ($Tₙ^d$ case) (see [9]).

Corollary 7 (Grover(Positive Support)/Ihara Zeta Correspondence ($Tₙ^d$ case)) Let $Tₙ^d$ be the $d$-dimensional torus with $N^d$ vertices. Then
\[
\lim_{n \to \infty} \zeta(Tₙ^d, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \log \left\{ (1 + \sigma u^2) - 2u \sum_{j=1}^{d} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi} \right],
\]
where $\int_{0}^{2\pi} \cdots \int_{0}^{2\pi}$ is the $d$-th multiple integral and $\frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}$ is the uniform measure on $[0, 2\pi)^d$.

In the case of $a = 0$ and $b = 1$, we obtain the Grover(Positive Support)/Ihara Zeta Correspondence ($Tₙ^d$ case) (see [9]).

Corollary 8 (Grover(Positive Support)/Ihara Zeta Correspondence ($Tₙ^d$ case)) Let $Tₙ^d$ be the $d$-dimensional torus with $N^d$ vertices. Then
\[
\lim_{n \to \infty} \zeta(Tₙ^d, u)^{-1} = (1 - u^2)^{d-1} \exp \left[ \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \log \left\{ (1 + (2d - 1)u^2) - 2u \sum_{j=1}^{d} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi} \right],
\]
where $\int_{0}^{2\pi} \cdots \int_{0}^{2\pi}$ is the $d$-th multiple integral and $\frac{d\theta_1}{2\pi} \cdots \frac{d\theta_d}{2\pi}$ is the uniform measure on $[0, 2\pi)^d$.

Specially, in the case of $d = 2$, we obtain the following result.

Corollary 8 Let $Tⁿ^2$ be the 2-dimensional torus with $N^2$ vertices. Then
\[
\lim_{n \to \infty} \zeta(Tⁿ^2, u)^{-1} = (1 - u^2) \exp \left[ \int_{0}^{2\pi} \int_{0}^{2\pi} \log \left\{ (1 + 3u^2) - 2u \sum_{j=1}^{2} \cos \theta_j \right\} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \right].
\]

This result corresponds to Equation (10) in Clair [4].

Finally, we should remark $d = 1$ case studied in Komatsu, Konno and Sato [8]. In this case, we easily check $U = U^+$. So we can apply both of our results (Corollaries 6 and 7) and get the same result given by Komatsu, Konno and Sato [8].
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