Approximating the covariance ellipsoid

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Abstract

We explore ways in which the covariance ellipsoid $\mathcal{B} = \{ v \in \mathbb{R}^d : \mathbb{E} \langle X, v \rangle^2 \leq 1 \}$ of a centred random vector $X$ in $\mathbb{R}^d$ can be approximated by a simple set. The data one is given for constructing the approximating set consists of $X_1, \ldots, X_N$ that are independent and distributed as $X$.

We present a general method that can be used to construct such approximations and implement it for two types of approximating sets. We first construct a (random) set $\mathcal{K}$ defined by a union of intersections of slabs $H_{z, \alpha} = \{ v \in \mathbb{R}^d : | \langle z, v \rangle | \leq \alpha \}$ (and therefore $\mathcal{K}$ is actually the output of a neural network with two hidden layers). The slabs are generated using $X_1, \ldots, X_N$, and under minimal assumptions on $X$ (e.g., $X$ can be heavy-tailed) it suffices that $N = c_1 d \eta^{-4} \log(2/\eta)$ to ensure that $(1 - \eta) \mathcal{K} \subset \mathcal{B} \subset (1 + \eta) \mathcal{K}$. In some cases (e.g., if $X$ is rotation invariant and has marginals that are well behaved in some weak sense), a smaller sample size suffices: $N = c_1 d \eta^{-2} \log(2/\eta)$.

We then show that if the slabs are replaced by randomly generated ellipsoids defined using $X_1, \ldots, X_N$, the same degree of approximation is true when $N \geq c_2 d \eta^{-2} \log(2/\eta)$.

The construction we use is based on the small-ball method.

1 Introduction

Identifying the covariance of a centred random vector using random data is of central importance in high-dimensional statistics and has been studied extensively in recent years. The hope is that by using a relatively small sample $X_1, \ldots, X_N$ of independent random vectors distributed as $X$, one can construct a good enough approximation of the covariance of $X$, and that such an approximation would be possible under minimal assumptions. The question is finding a ‘right way’ of generating an approximation and then estimating the resulting tradeoff between the given sample size $N$, the degree of approximation and the probability with which that degree of approximation can be guaranteed.

The random vector $X$ endows an $L_2$ norm on $\mathbb{R}^d$ by setting for $v \in \mathbb{R}^d$,

$$\|v\|_{L_2} \equiv \| \langle X, v \rangle \|_{L_2} = \left( \mathbb{E} (\langle X, v \rangle)^2 \right)^{1/2},$$

and the unit ball of that norm is

$$\mathcal{B} = \{ v \in \mathbb{R}^d : \|v\|_{L_2} \leq 1 \} = \{ v \in \mathbb{R}^d : \langle Tv, v \rangle^{1/2} \leq 1 \},$$

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where $T = \mathbb{E}(X \otimes X)$ is the covariance matrix of $X$. Throughout this note we assume without loss of generality that $T$ is invertible.

Given $X_1, \ldots, X_N$ that are independent and distributed as $X$, a natural option is to consider the empirical covariance matrix $\hat{T} = \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i$ and approximate $\mathcal{B}$ by the random ellipsoid

$$\hat{\mathcal{B}} = \left\{ v \in \mathbb{R}^d : \left( \hat{T} v, v \right)^{1/2} \leq 1 \right\}.$$  

Note that even if one selects $\hat{\mathcal{B}}$ as the approximating set, there are various notions of approximation that one may consider. For example, by ensuring that the operator norm $\| \hat{T} - T \|_{2 \to 2} \leq \eta$, it follows that

$$\mathcal{B} \subset \hat{\mathcal{B}} + \eta B_2^d \quad \text{and} \quad \hat{\mathcal{B}} \subset \mathcal{B} + \eta B_2^d,$$

where $B_2^d$ is the Euclidean unit ball and $A + B$ is the Minkowski sum $\{ a + b : a \in A, b \in B \}$.

A different notion of approximation, which is the one that we focus on here, is equivalence between sets:

**Definition 1.1.** The set $\mathcal{K} \subset \mathbb{R}^d$ is an $\eta$-approximation of $\mathcal{B}$ if

$$\mathcal{K} \subset \mathcal{B} \subset (1 + \eta) \mathcal{K}. \quad (1.1)$$

For the choice of $\mathcal{K} = \hat{\mathcal{B}}$ an equivalent formulation of $\eta$-approximation is that

$$\sup_{v \in \mathcal{B}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle X_i, v \rangle^2 - \mathbb{E} \langle X, v \rangle^2 \right| \leq \eta. \quad (1.2)$$

Observe that if $T = \mathbb{E}(X \otimes X)$ then $\mathcal{B} = T^{-1/2}B_2^d$; hence, the random vector $Y = T^{-1/2}X$ is *isotropic*: $\mathbb{E}(Y \otimes Y) = Id$, i.e., for every $v \in \mathbb{R}^d$, $\| \langle Y, v \rangle \|_{L_2} = \| v \|_2$. Moreover, denoting the Euclidean unit sphere by $S^{d-1}$, (1.2) becomes

$$\sup_{v \in S^{d-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle Y_i, v \rangle^2 - 1 \right| \leq \eta. \quad (1.3)$$

The behaviour of (1.3), the quadratic empirical process indexed by the unit sphere, is well understood (see e.g. [1, 14, 10]). It characterizes the extremal singular values of the random matrix $N^{-1/2} \sum_{i=1}^{N} \{ Y_i, \cdot \} e_i$, and is determined by two factors: the growth of moments of linear functionals $\langle Y, v \rangle$, and tail estimates on the Euclidean norm $\| Y \|_2$. The best known estimate on (1.3) in a heavy-tailed situation is due to Tikhomirov [16]:

**Theorem 1.2.** Let $Y$ be a centred, isotropic random vector in $\mathbb{R}^d$ and for $p > 2$ set $L = \sup_{v \in S^{d-1}} \| \langle Y, v \rangle \|_{L_p}$. Let $Y_1, \ldots, Y_N$ be independent, distributed according to $Y$. If $\hat{T} = N^{-1} \sum_{i=1}^{N} Y_i \otimes Y_i$ then with probability at least $1 - 1/d$,

$$C^{-1} \| Id - \hat{T} \|_{2 \to 2} \leq \frac{1}{N} \max_{1 \leq i \leq N} \| Y_i \|_2^2 + \left( \frac{d}{N} \right)^{1-2/p} \log^4 \left( \frac{eN}{d} \right) + \left( \frac{d}{N} \right)^{1-2/\min\{4,p\}},$$

for a constant $C$ that depends only on $L$ and $p$. 


If one believes that Theorem 1.2 is reasonably sharp, it casts a shadow on the choice of \( \hat{B} \) as an \( \eta \)-approximation of \( B \) in the sense of Definition 1.1. Indeed, when \( X \) is heavy-tailed it is likely that some of the vectors \( Y_i = T^{-1/2}X_i \) will have large Euclidean norms. In Section 3.3 we will give a concrete example of an isotropic random vector that satisfies an \( L_4 - L_2 \) norm equivalence, but still \( \hat{B} \) is very different from \( B \) with a non-negligible probability.

Of course, while \( \hat{B} \) is the natural choice for a data-dependent approximation of \( B \), it is certainly not the only choice. For one, there is no reason to restrict the approximating set to an ellipsoid, though it is not clear offhand how one may generate other approximating sets given the limited data at one’s disposal.

The method we present does just that. Its starting point is identifying a random property that is satisfied only by points in a set that is ‘close enough’ to \( B \). To give an example of what we mean by a random property, assume, for example, that \( X \) is the standard gaussian vector in \( \mathbb{R}^d \). Then \( B = B_2^d \), and for each \( v \in \mathbb{R}^d \), \( \langle X, v \rangle \) is a centred gaussian random variable whose variance is \( \|v\|_2^2 \). Thus, using the values \( \langle X_1, v \rangle, \ldots, \langle X_N, v \rangle \) one may identify \( \|v\|_2 \) rather accurately and in particular pin-point the Euclidean sphere on which \( v \) is located. The difficulty lies in the fact that the accurate estimate has to hold uniformly for every \( v \in \mathbb{R}^d \), and how that can be achieved is not obvious. Our method leads to such uniform estimates, and as examples we obtain approximation results using two different types of sets.

The first example we consider has to do with approximations generated by slabs. For \( z \in \mathbb{R}^d \) and \( \alpha > 0 \) set \( H_{z,\alpha} = \{ v \in \mathbb{R}^d : |\langle z, v \rangle| \leq \alpha \} \). Given \( z_1, \ldots, z_n \in \mathbb{R}^d \) and \( \alpha_1, \ldots, \alpha_n > 0 \), define

\[
K = \{ v \in \mathbb{R}^d : v \in H_{z_j,\alpha_j} \text{ for at least } \beta n \text{ indices} \}.
\]

In other words, \( K \) is a union of all the intersections of \( \beta n \) slabs out of \( (H_{z_i,\alpha_i})_{i=1}^n \). Note that \( K \) need not be a convex set though it is star-shaped around 0: if \( v \in K \) then for any \( 0 \leq \theta \leq 1 \), \( \theta v \in K \).

This type of approximation has been studied in [4], where the authors attempted to approximate the characteristic function of the Euclidean unit ball in \( \mathbb{R}^d \) by the characteristic function of a simple set. It was well known that approximating the Euclidean unit ball by a polytope required the polytope to have at least \( \exp(c d) \) faces (see, e.g., [13, 7] for accurate statements), and the alternative studied in [4] was to approximate \( \mathbb{1}_{B_2^d} \) by the output of a neural network with two hidden layers; that is, by a characteristic function of a set of the form

\[
\left\{ v \in \mathbb{R}^d : \sum_{i=1}^n \gamma_i \mathbb{1}_{\{\langle z_i, v \rangle \geq \alpha_i \}} \geq k \right\}, \tag{1.4}
\]

It was shown in [4] that one may construct such a set \( K_1 \) using \( n = cd^2/\eta^2 \) points \( z_i \), and for the right choice of \( \alpha_i \) and \( \gamma_i \) one has

\[
(1 - \eta)B_2^d \subset K_1 \subset (1 + \eta)B_2^d.
\]

Unfortunately, although it is possible to derive a similar approximation for a general ellipsoid, that construction requires information on the ellipsoid’s principal axes, making it unhelpful for covariance approximation.

In [2] the authors considered similar approximating sets (which they called ‘zig-zag bodies’), but their approach for choosing the points \( z_i \) and thresholds \( \alpha_i \) was more promising from our perspective; moreover, it also led to a better estimate on the required number of slabs.
Theorem 1.3. [2] There exist absolute constants $c_1$ and $c_2$ for which the following holds. Let $Z$ be distributed according to the uniform measure on $S^{d-1}$ and let $Z_1, ..., Z_N$ be independent, distributed as $Z$. Set

$$K_2 = \left\{ v \in \mathbb{R}^d : \frac{1}{N} \sum_{i=1}^{N} |\langle v, Z_i \rangle| \leq \frac{\alpha_d}{2} \text{ for at least } N/2 \text{ indices} \right\}, \tag{1.5}$$

where $\alpha_d$ is the median of $|\langle Z, v \rangle|$ for $v \in S^{d-1}$. If $0 < \eta < 1$ and $N = c_1 d \eta^{-2} \log(2/\eta)$ then with probability at least $1 - 2 \exp(-c_2 d)$,

$$(1 - \eta) B_2^d \subset K_2 \subset (1 + \eta) B_2^d.$$

In other words, the Euclidean ball (which, up to a normalization factor of $c_d \sqrt{d}$, $\lim_{d \to \infty} c_d = 1$, is the covariance unit ball endowed by $Z$) can be approximated by the union of intersections generated by $c(\eta)d$ slabs, and this approximation holds with very high (exponential) probability.

Remark 1.4. Note that $K_2$ belongs to the family of sets (1.4). Indeed, this is evident because

$$K_2 = \left\{ v \in \mathbb{R}^d : \frac{1}{N} \sum_{i=1}^{N} 1\{\langle v, Z_i \rangle \leq \alpha_d \} \geq \frac{N}{2} \right\},$$

and for $\alpha > 0$, $1\{\langle v, z \rangle \leq \alpha \} = 1\{\langle v, z \rangle \geq -\alpha \} - 1\{\langle v, z \rangle \geq \alpha \}$.

The proof of Theorem 1.3 relies heavily on the fact that $Z_1, ..., Z_N$ are distributed according to the uniform measure on the sphere. However, it still opens the door to a possible way of addressing the problem at hand: one may try to select $K$ randomly, in a similar way to (1.5).

We will show that indeed Theorem 1.3 can be extended—with some necessary modifications—to an almost arbitrary centered random vector. The proof is based on a random property that allows one to check accurately whether $v \in \mathbb{R}^d$ actually belongs to $K$ or not. As we explain in what follows, that property is reflected by the ‘frequency’ with which the $X_i$’s belong to an appropriate slab defined by $v$ (see Section 2 for details).

To formulate our main results we need to introduce some additional notation. Throughout, absolute constants are denoted by $c, c_0, c_1, ...$; their values may change from line to line. $a \lesssim b$ means that there is an absolute constant $c$ such that $a \leq cb$, and $a \sim b$ implies that $ca \leq b \leq Ca$ for absolute constants $c$ and $C$. Finally, $a \sim_L b$ denotes that $ca \leq b \leq Ca$ for constants $c$ and $C$ that depend only on $L$.

Given integers $m$ and $n$ set $N = nm$. Let $\{X_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ be $N$ independent copies of $X$ and for $1 \leq j \leq n$ put

$$Z_j = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} X_{i,j}.$$  

Also, denote by $g$ the standard gaussian random variable and set $\alpha$ to be the median of $|g|$. For $\eta > 0$ define the random set

$$K_\eta = \left\{ v \in \mathbb{R}^d : |\langle Z_j, v \rangle| \leq \alpha + \eta \text{ for at least } \left(1 - \eta \right) n \text{ indices } j \right\}.$$
**Theorem 1.5.** Let $0 < \eta < 1/10$ and $L \geq 1$. Assume that for every $v \in \mathbb{R}^d$, $\| \langle X, v \rangle \|_{L_q} \leq L \| \langle X, v \rangle \|_{L_2}$ for some $q > 2$, set $m \geq c_0(\eta, L)$ and let $n \geq c_1(\eta)d$.

Then, with probability at least \(1 - 2 \exp(-c_2\eta^2 n)\),
\[
\mathcal{B} \subset K_\eta \subset (1 + c_3\eta)\mathcal{B},
\]
for absolute constants $c_2$ and $c_3$.

Moreover, if $q \geq 3$ one may take $c_0 \sim L\eta^{-2}$ and $c_1 \sim \eta^{-2}\log(2/\eta)$, implying that $N = c(L)d\eta^{-4}\log(2/\eta)$ points suffice.

As it happens, the superfluous factor of $\log(2/\eta)$ can be removed from Theorem 1.5 if one employs a different method of proof. However, the required argument is rather specific and holds only for approximation by slabs as in Theorem 1.5. Because the main point of this note is to advocate our method of constructing approximations, we chose to present the general argument and only outline the alternative proof of Theorem 1.5 (see Section 3.4).

**Remark 1.6.** As we explain in what follows, if $X$ is a ‘nice’ random vector (and among these ‘nice’ random vectors are the standard gaussian vector or the vector distributed uniformly on the Euclidean unit sphere) then one may take $m = 1$ and $n \sim d\eta^{-2}\log(2/\eta)$ (or $n \sim d\eta^{-2}$ using the alternative proof). In particular, Theorem 1.5 improves Theorem 1.3.

In the other example we present we construct a more complex approximating set: it is the union of intersections of ellipsoids rather than the union of intersections of slabs. On the other hand, the required sample size is smaller and all that one needs is the following weak assumption on $X$:

**Assumption 1.1.** Assume that for every $\eta > 0$ there is some $m = m_0(\eta)$ for which the following holds: if $\| v \|_{L_2} = 1$ then
\[
\Pr \left( \left| \frac{1}{m} \sum_{i=1}^{m} \langle X_i, v \rangle^2 - 1 \right| \geq \frac{\eta}{10} \right) \leq 0.01.
\]

To see that Assumption 1.1 is rather minimal, note that under an $L_4-L_2$ norm equivalence (i.e., that for every $v \in \mathbb{R}^d$, $\| \langle X, v \rangle \|_{L_4} \leq L \| \langle X, v \rangle \|_{L_2}$), one has $m_0(\eta) \leq c(L)/\eta^2$. Naturally, nontrivial estimates on $m_0(\eta)$ are possible in more general situations than an $L_4-L_2$ norm equivalence.

The ‘ellipsoid approximation’ estimate is as follows:

**Theorem 1.7.** There exist absolute constants $c_0, c_1$ and $c_2$ for which the following holds. For $0 < \eta < 1/4$ let $m = m_0(\eta)$ and $n \geq c_0 \max\{d\log(m/\eta), m\}$. Put $N = nm$ and set $(X_{i,j})$, $1 \leq i \leq m$, $1 \leq j \leq n$ to be independent, distributed according to $X$. If
\[
\mathcal{D}_\eta = \left\{ v \in \mathbb{R}^d : \frac{1}{m} \sum_{i=1}^{m} \langle X_{i,j}, v \rangle^2 \leq 1 + \eta \text{ for at least } 0.9n \text{ indices } j \right\},
\]
then with probability at least $1 - 2 \exp(-c_1n/m)$,
\[
\mathcal{B} \subset \mathcal{D}_\eta \subset (1 + c_2\eta)\mathcal{B}.
\]
To put the outcome of Theorem 1.7 in some perspective, under an $L_4-L_2$ norm equivalence one has that $m_0(\eta) \leq c(L)/\eta^2$, implying that $n = c' \max\{d \log(L/\eta), \eta^{-2}\}$ suffices, and the resulting required sample size of $N \sim d\eta^{-3}\log(2/\eta)$ is better than the outcome of Theorem 1.5 by a factor of $1/\eta^2$ as long as $\eta \geq 1/d^{1/2}$.

In the next section we describe the general method and explain how it is used in the proofs of Theorem 1.5 and Theorem 1.7. The argument is actually a variant of the small-ball method introduced in [9]. The proofs of Theorem 1.5 and Theorem 1.7 are presented in Section 3.

2 The small-ball method

Let us begin by describing the argument used in the proof of Theorem 1.3. It is based on three crucial observations:

- **All the points on a centred sphere behave in the same way:** By rotation invariance, if $Z$ is distributed according to the uniform measure on $S^{d-1}$ then all the random variables $\langle Z, v/\|v\|_2 \rangle$ have the same distribution; therefore $|\langle Z, v/\|v\|_2 \rangle|$ all have the same quantiles, and in particular, the same median.

- **Quantiles can be used to ‘separate’ between different spheres:** If $\|u\|_2 \neq \|v\|_2$, that fact is reflected in a difference between $\Pr(|\langle Z, v \rangle| \leq \alpha)$ and $\Pr(|\langle Z, u \rangle| \leq \alpha)$.

- **Separation is visible through sampling:** For every $v \in \mathbb{R}^d$, the sum of independent indicators

$$\frac{1}{N} \sum_{i=1}^{N} 1_{\{|\langle Z, v \rangle| \leq \alpha\}}$$

exhibits sharp concentration around $\Pr(|\langle Z, v \rangle| \leq \alpha)$.

It follows that for every $v \in S^{d-1}$, the median $\alpha_d$ of $|\langle Z, v \rangle|$ is the same (and happens to be $c_d/\sqrt{d}$ with $\lim_{d \to \infty} c_d = 1$). Moreover, given $Z_1, ..., Z_N$ that are independent and distributed according to $Z$, $\{|j : |\langle Z_j, v \rangle| \leq \alpha_d\}$ is highly concentrated around $N/2$.

The heart of the proof is to show that a similar bound is true uniformly on $S^{d-1}$; that is, with high probability,

$$\sup_{v \in S^{d-1}} \left|\left| \{j : |\langle Z_j, v \rangle| \leq \alpha_d\} \right| \right| - \frac{N}{2} \right| 

(2.1)$$

is small provided that $N$ is large enough.

To establish (2.1), note that the high probability estimate that holds for every individual $v$ allows one to obtain uniform control on a fine enough net in $S^{d-1}$. And, if $\pi u$ denotes the best approximation to $u$ in the net, $|\langle Z_j, u \rangle|$ cannot be different from $|\langle Z_j, u - \pi u \rangle|$ by much; indeed, $|\langle Z, u - \pi u \rangle| \leq \|Z\|_2\|u - \pi u\|_2 = \|u - \pi u\|_2$ because $Z$ is supported on $S^{d-1}$.

Once (2.1) is established, the outcome of Theorem 1.3 follows immediately: the set

$$\mathcal{K}_2 = \left\{ v \in \mathbb{R}^d : |\langle v, z_i \rangle| \leq \alpha_d \text{ for at least } N/2 \text{ indices} \right\}$$

contains $(1 - \eta)S^{d-1}$, but does not contain any point on $(1 + \eta)S^{d-1}$. Therefore, since $\mathcal{K}_2$ is star-shaped around 0, $(1 - \eta)B_2^d \subset \mathcal{K}_2 \subset (1 + \eta)B_2^d.$
It is clear that when dealing with a general random vector, most of the features used in the proof of Theorem 1.3 are simply not true: quantiles $\Pr(|\langle X, v \rangle| \leq \alpha)$ may change on the $L_2$ unit sphere $\mathcal{S} = \{v \in \mathbb{R}^d : \mathbb{E}|\langle X, v \rangle|^2 = 1\}$; they need not ‘separate’ between two $L_2$ spheres; and ‘oscillations’ $|\langle X_i, u - \pi u \rangle|$ can be large, especially when $X$ is heavy-tailed rather than being bounded like in Theorem 1.3.

The analysis required for addressing these difficulties is based on the *small-ball method*, which was introduced in [9] to deal precisely with this sort of problem: obtaining high probability, uniform estimates in heavy-tailed situations. The path we take follows the main ideas of the method:

(a) Identify a property $\mathcal{P}$ that allows one to check whether a fixed $v \in \mathbb{R}^d$ belongs to $\mathcal{B}$ or not - using only the probability with which the property holds. Moreover, $\mathcal{P}$ should be defined using only on relatively small number of the independent copies of $X$ at one’s disposal.

For example, one may consider the functionals

$$\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \langle X_i, v \rangle \quad \text{and} \quad \frac{1}{m} \sum_{i=1}^{m} \langle X_i, v \rangle^2$$

where $m$ is relatively small. The former is close to a centred gaussian variable whose variance is $\mathbb{E}\langle X, v \rangle^2 = \|v\|_{L_2}^2$ while the latter concentrates around $\|v\|_{L_2}^2$. Therefore, if the goal is to check whether $\|v\|_{L_2} \leq 1$ one may define

$$\mathcal{P}_1 = \left\{ \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \langle X_i, v \rangle \leq \alpha + \eta \right\} \quad \text{and} \quad \mathcal{P}_2 = \left\{ \frac{1}{m} \sum_{i=1}^{m} \langle X_i, v \rangle^2 \leq 1 + \frac{\eta}{10} \right\} \quad (2.2)$$

respectively, where $\alpha$ appearing in $\mathcal{P}_1$ is the median of $|g|$, the absolute value of a standard gaussian, and $\eta$ is small.

In both cases the probability of the events in question are determined by $\|v\|_{L_2}$: the probability of $\mathcal{P}_1$ will be very close to $1/2$ if and only if $\|v\|_{L_2} = 1$, whereas $\mathcal{P}_2$ holds with probability that is close to 1 if $\|v\|_{L_2} \leq 1 + \eta$ and with probability that is close to 0 in $\|v\|_{L_2}$ is much larger.

In general, the idea in (a) is that the identity of $\|v\|_{L_2}$ is reflected by the probability with which $\mathcal{P}$ hold. The next step is to ‘detect’ that probability with very high confidence.

(b) Split $\{1, \ldots, N\}$ to $n$ coordinate blocks $I_j$, each one of cardinality $m$ and set $W_j(v) = 1_{\{v \text{ satisfies } \mathcal{P}\}}(X_i, \ i \in I_j)$. It is evident that $W(v) = n^{-1} \sum_{j=1}^{n} W_j(v)$ concentrates around its mean, i.e., the probability with which $\mathcal{P}$ holds. Therefore, the cardinality $|\{j : W_j(v) = 1\}|$ leads to a very good estimate of that probability, and in particular of $\|v\|_{L_2}$. Moreover, the resulting estimate is valid with confidence that is exponential in $n = N/m$, say $1 - 2\exp(-cn)$.

(c) Use (b) to define the random approximating set $\mathcal{K}$: $v \text{ belongs to the set if } W_j(v) = 1 \text{ for the ‘right number’ of indices } j$.
Now one needs to verify that the resulting set \( K \) is truly close to \( B \). If \( K \) happens to be star-shaped around 0, it suffices to ensure that \( S \subset K \), and at the same time that \( \{ v : \|v\|_{L^2} = 1 + \eta \} \subset K^c \). As a result, one has to obtain a uniform estimate on the cardinality \( |\{ j : W_j(v) = 1 \}| \) for \( v \)’s that belong to the two centred \( L^2 \) spheres: the unit one, and the one of radius \( 1 + \eta \):

\[(d)\] The high probability estimate with which (b) holds allows one to control a large collection of \( v \)’s uniformly. The obvious choice of such a set \( V \) is an appropriate \( L^2 \)-net in the sphere in question. This leads to an estimate that holds with high confidence but only for points in \( V \) rather than for the entire sphere.

\[(e)\] Finally, to pass from \( V \) to the entire sphere one must control the oscillations: show that if \( u \) is ‘close’ to \( v \), then the number of indices \( j \) on which \( W_j(u) = 1 \) is very close to the number of indices on which \( W_j(v) = 1 \).

Clearly, the key step is \((e)\): obtaining the required uniform control on random oscillations, a task that is nontrivial in heavy-tailed situations.

As this description indicates, the method is rather general and can be employed for a wide variety of choices of \( \mathcal{P} \). One may consider other alternatives beyond the two examples we present in what follows, and those would result in different approximating sets. The crucial point is that as long as \( \mathcal{P} \) is well chosen, those sets would all be good approximations of the covariance ellipsoid.

## 3 Proofs

Before we present the proofs of Theorem 1.5 and Theorem 1.7 we need the following standard observation:

**Lemma 3.1.** Let \( X \) be a centred random vector in \( \mathbb{R}^d \) and let \( X_1, \ldots, X_k \) be independent copies of \( X \). Then

\[
\mathbb{E} \sup_{v \in \mathcal{B}} \left| \sum_{i=1}^{k} \varepsilon_i \langle X_i, v \rangle \right| \leq \sqrt{k} \sqrt{d},
\]

where \( (\varepsilon_i)_{i=1}^{k} \) are independent, symmetric, \( \{-1, 1\} \)-valued random variables that are independent of \( X_1, \ldots, X_k \).

**Proof.** Let \( T = \mathbb{E}(X \otimes X) \), and recall that \( \mathcal{B} = T^{-1/2}B_2^d \) and that \( T^{-1/2}X \) is isotropic. Note that for an isotropic vector \( Y \),

\[
\mathbb{E}\|Y\|_2^2 = \mathbb{E} \sum_{i=1}^{d} \langle Y_i, e_i \rangle^2 = d.
\]

Therefore,

\[
\mathbb{E} \sup_{v \in \mathcal{B}} \left| \sum_{i=1}^{k} \varepsilon_i \langle X_i, v \rangle \right| = \mathbb{E} \sup_{w \in B_2^d} \left| \sum_{i=1}^{k} \varepsilon_i \langle X_i, T^{-1/2}v \rangle \right| = \mathbb{E} \sup_{w \in B_2^d} \left| \sum_{i=1}^{k} \varepsilon_i \langle T^{-1/2}X_i, v \rangle \right| \leq \mathbb{E} \left( \sum_{i=1}^{k} \|T^{-1/2}X_i\|_2^2 \right)^{1/2},
\]
and the claim follows from Jensen’s inequality and the fact that $T^{-1/2}X$ is isotropic. ■

3.1 Approximation by slabs

Recall that $S \subset \mathbb{R}^d$ is the $L_2$ unit sphere; that is, $S = \{v \in \mathbb{R}^d : \|\langle X, v \rangle\|_{L_2} = 1\}$.

As a starting point, let $Z$ be a random vector that has the same covariance as $X$, and therefore endows the same $L_2$ structure on $\mathbb{R}^d$—in particular, $Z$ endows the same unit ball $B$ and unit sphere $S$. Assume that there are $\alpha > 0$, $0 < \beta < 1$, $\eta < \beta/4$, $\varepsilon_0 < \alpha/2$ and $\gamma > 6\eta/\alpha$ such that for every $v \in S$ and every $\varepsilon_0 < \varepsilon < \alpha/2$,

1. $|Pr(|\langle Z, v \rangle| \leq \alpha) - \beta| \leq \eta$, and
2. $Pr(|\langle Z, v \rangle| \in [\alpha - \varepsilon, \alpha]) \geq \gamma \varepsilon$.

To explain this condition, one should think of $\eta$ as a small number (measuring the wanted degree of approximation), and that $\alpha$ and $\beta$ are just constants; thus, Condition (1) means that the function $\phi(v) = Pr(|\langle Z, v \rangle| \leq \alpha)$ is roughly a constant on the sphere $S$. Condition (2) means that the (marginal) mass of a small interval that ends at $\alpha$ is nontrivial; in other words, there is a noticeable difference between $Pr(|\langle Z, v \rangle| \leq \alpha)$ and $Pr(|\langle Z, v \rangle| \leq \alpha - \varepsilon)$ for every $v \in S$; the lower bound on $\gamma$ is there to ensure that the difference between the two is indeed noticeable.

Note that $G$, the standard gaussian vector in $\mathbb{R}^d$, satisfies (1) and (2): $S = S^{d-1}$; for every $v \in S^{d-1}$, $\langle G, v \rangle$ is distributed as a standard gaussian variable; and one may set $1/10 \leq \alpha \leq 10$, $\beta = Pr(|\langle g \rangle| \leq \alpha)$, $\gamma$ that is an absolute constant and $\varepsilon_0 = 0$. A similar argument shows that the uniform measure on $S^{d-1}$ also satisfies (1) and (2) for the right choice of constants.

As we explain in what follows, in general situations our choice of $Z$ will only have approximately gaussian one-dimensional marginals, and that would suffice to ensure that both (1) and (2) hold for $\alpha, \beta$ and $\gamma$ that are absolute constants.

The main component in the proof of Theorem 3.2 is the next fact:

**Theorem 3.2.** There exist constants $c_0, c_1, c_2$ that depend only on $\alpha, \beta$ and $\gamma$ for which the following holds. Let $Z$ satisfy (1) and (2) for some $\varepsilon_0 \leq (3/\gamma)\eta$. Let $Z_1, \ldots, Z_n$ be independent copies of $Z$ and set

$$K = \{v \in \mathbb{R}^d : |\langle Z_j, v \rangle| \leq \alpha + \eta \text{ for at least } (\beta - \eta)n \text{ indices } j\}.$$

If $n \geq c_1 d \eta^{-2} \log(2/\eta)$ then with probability at least $1 - 2 \exp(-c_2 n \eta^2)$,

$$B \subset K \subset (1 + c_3 \eta)B$$

**Proof.** We follow the path outlined in Section 2. Thanks to (1) and (2) we have the wanted property using a single copy of $Z$. Indeed, as a preliminary step observe that $\{|\langle Z, v \rangle| \leq \alpha\}$ holds with probability that does not change much on $S$. At the same time, by the lower bound on $\gamma$, $\alpha/2 \leq \alpha - (3/\gamma)\eta < \alpha$, and fix $1 < \rho \leq 2$ such that $\alpha/\rho = \alpha - (3/\gamma)\eta$. Since $\varepsilon_0 \leq (3/\gamma)\eta$ it follows that (2) holds for $\varepsilon = (3/\gamma)\eta$ and one has

$$Pr(|\langle Z, v \rangle| \leq \alpha/\rho) \leq \beta - 3\eta.$$
Thus, there is a noticeable difference between $Pr(|\langle Z, v \rangle| \leq \alpha)$ and $Pr(|\langle Z, \rho v \rangle| \leq \alpha)$.

By Bernstein’s inequality, it follows that with probability at least $1 - 2\exp(-c_0(\beta)n\eta^2)$,

$$\left| \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{\{|\langle Z_j, v \rangle| \leq \alpha\}} - Pr(|\langle Z, v \rangle| \leq \alpha) \right| \leq \eta/2;$$

Therefore, on that event,

$$|\{j : |\langle Z_j, v \rangle| \leq \alpha\}| \geq n(\beta - \eta/2). \quad (3.1)$$

Applying Bernstein’s inequality again, with probability at least $1 - 2\exp(-c_0(\beta)\eta^2 n)$,

$$\left| \left\{ j : |\langle Z_j, v \rangle| \leq \frac{\alpha}{\rho} \right\} \right| \leq (\beta - 2\eta) n. \quad (3.2)$$

The heart of the proof is to show that slightly modified versions of (3.1) and (3.2) hold uniformly on $S$; that is, with high probability, for every $v \in S$,

$$|\{j : |\langle Z_j, v \rangle| \leq \alpha + \eta\}| \geq n(\beta - \eta), \quad (3.3)$$

and

$$\left| \left\{ j : |\langle Z_j, v \rangle| \leq \frac{\alpha + \eta}{\rho} \right\} \right| < n(\beta - \eta). \quad (3.4)$$

Let $c_1 = c_0/2$ and let $V \subset S$ be a maximal $r$-separated subset of $S$ with respect to the $L_2$ norm and of cardinality at most $\exp(c_1\eta^2 n)$. There is an event $A_1$ of probability at least $1 - 4\exp(-c_1\eta^2 n)$ on which (3.1) and (3.2) hold for every $v \in V$. Also, because $S$ is a convex, centrally-symmetric subset of $\mathbb{R}^d$, a standard volumetric estimate shows that

$$r \leq 5\exp(-c_1\eta^2 n/d). \quad (3.5)$$

For every $u \in S$ let $\pi u$ be the nearest in $V$ to $u$ with respect to the $L_2$ norm. Set

$$W = \sup_{u \in S} \sum_{j=1}^{n} \mathbb{1}_{\{|\langle Z_j, u - \pi u \rangle| \geq t\}}$$

for $t = \eta/2$ (which is smaller than $\eta/\rho$). Our aim is to ensure that with high probability $W \leq n\eta/2$, and to that end we first estimate $\mathbb{E}W$. Observe that

$$W \leq \sup_{u \in S} \frac{1}{t} \sum_{j=1}^{n} |\langle Z_j, u - \pi u \rangle|;$$

by the Giné-Zinn symmetrization theorem [6] followed by the contraction inequality for Bernoulli processes [8],

$$\mathbb{E}W \leq \frac{2}{t} \left( \mathbb{E} \sup_{u \in S} \left| \sum_{j=1}^{n} \varepsilon_j \langle Z_j, u - \pi u \rangle \right| + n \sup_{u \in S} \mathbb{E} |\langle Z, u - \pi u \rangle| \right) \leq \frac{2r}{t} \left( \mathbb{E} \sup_{u \in S} \left| \sum_{j=1}^{n} \varepsilon_j \langle Z_j, u \rangle \right| + n \right),$$
where we have used the fact that $\|u - \pi u\|_{L_1} \leq \|u - \pi u\|_{L_2} \leq r$. Moreover, by Lemma 3.1

$$E \sup_{u \in S} \sum_{j=1}^n \varepsilon_j \langle Z_j, u \rangle \leq \sqrt{n} \sqrt{d},$$

implying that if $n \geq d$ then

$$E W \leq \frac{c_2 n}{\ell^t} \exp(-c_1 \eta^2 n/d) = \frac{2c_2 n}{\eta} \exp(-c_1 \eta^2 n/d),$$

thanks to the estimate on $r$ from (3.5) and by the choice of $\ell$.

Now, by the bounded differences inequality (see, e.g., [3]), we have that for every $x > 0$, $Pr(W \geq EW + x) \leq \exp(-c_3 x^2/n)$. Setting $x = n\eta/4$, there is an event $A_2$ of probability at least $1 - 2 \exp(-c_4 \eta^2 n)$ on which

$$W \leq n \left(\frac{2c_2}{\eta} \exp(-c_1 \eta^2 n/d) + \frac{\eta}{4}\right) \leq \frac{\eta}{2} n,$$

where the last inequality holds if we set

$$n \geq \frac{d}{\eta^2} \log \left(\frac{2}{\eta}\right).$$

Combining the two estimates, on the event $A_1 \cap A_2$ one has that for any $u \in S$ both (3.3) and (3.4) hold. Indeed, for every $u \in S$ we have

- $|\langle Z_j, \pi u \rangle| \leq \alpha$ for at least $n(\beta - \eta/2)$ indices $j$; and
- $|\langle Z_j, u - \pi u \rangle| \geq \eta$ for at most $\eta/2$ indices $j$.

Therefore, there is a set of indices of cardinality at least $n(\beta - \eta)$ such that both $|\langle Z_j, \pi u \rangle| \leq \alpha$ and $|\langle Z_j, u - \pi u \rangle| \leq \eta$, and for those indices,

$$|\langle Z_j, u \rangle| \leq |\langle Z_j, \pi u \rangle| + |\langle Z_j, u - \pi u \rangle| \leq \alpha + \eta,$$

verifying (3.3). A similar argument may be used to confirm (3.4).

Setting

$$K = \{v \in \mathbb{R}^d : |\{i : |\langle Z_j, v \rangle| \leq \alpha + \eta\}| \geq (\beta - \eta)n\},$$

it follows from (3.3) that $S \subseteq K$; and, since $K$ is star-shaped around 0, $B \subseteq K$ as well.

On the other hand, recalling that $\eta \leq \alpha \gamma/6$ then

$$\rho = 1 + \frac{3\eta}{\alpha \gamma - 3\eta} \leq 1 + c_5 \eta,$$

where $c_5 \sim 1/\alpha \gamma$. Thus, if $\|u\|_{L_2} = \rho > 1$, then

$$\{j : |\langle Z_j, u \rangle| \leq \alpha + \eta\} = \left\{j : |\langle Z_j, v \rangle| \leq \frac{\alpha + \eta}{\rho}\right\}$$

for some $v \in S$. Hence, by (3.4),

$$|\{j : |\langle Z_j, u \rangle| \leq \alpha + \eta\}| \leq (\beta - \eta)n,$$

and $u \not\in K$. It follows that $\{v : \|v\|_{L_2} = \rho\} \subseteq K^c$ and by homogeneity, $(\rho B)^c \subseteq K^c$, as required.
Once Theorem 3.2 is established, one may apply it to random vectors that satisfy (1) and (2) — for example, the standard gaussian vector or the vector distributed uniformly on $S^{d-1}$. It follows that for any $\eta \leq c_0$ and given more than $c_1d\eta^{-2}\log(2/\eta)$ random points, the random set $K$ is a $c_2\eta$-approximation of $B$ for an absolute constant $c_2$. In particular, Theorem 1.3 follows from Theorem 3.2.

Clearly, since a general random vector $X$ need not satisfy (1) and (2), the proof of Theorem 1.5 requires an additional step. To that end one may invoke the Berry-Esseen Theorem (see, e.g., [5]) to ‘smooth’ $X$ and construct a random vector $Z$ that does satisfy (1) and (2).

**Theorem 3.3.** Let $W$ be a mean-zero random variable and let $W_1, \ldots, W_m$ be independent copies of $W$. If

$$Y = \frac{1}{\sqrt{m}\|W\|_2} \sum_{i=1}^{m} W_i,$$

then

$$\sup_{t \in \mathbb{R}} |Pr(Y > t) - Pr(g > t)| \leq \psi(m),$$

where $\psi(m) = C(\|W\|_3^3/\|W\|_2^3)m^{-1/2}$. In particular, if $\|W\|_3 \leq L\|W\|_2$ then $\psi(m) = c(L)/\sqrt{m}$.

**Remark 3.4.** There are other versions of the Berry-Esseen Theorem with different conditions on $W$. For example, one may obtain nontrivial estimates on $\psi(m)$ as soon as $\|W\|_q \leq L\|W\|_2$ for some $q > 2$, although if $2 < q < 3$ then $\psi(m)$ tends to 0 at a slower (polynomial) rate than $1/\sqrt{m}$ (see [11]). Alternatively, if $Y \in L_{\psi_{\alpha}}$, one has better estimates on $\psi(m)$ (see, e.g., [12]).

For an integer $m \leq N$, set

$$Z = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} X_i,$$

and thus one has access to $n = N/m$ independent copies of $Z$. Clearly, $Z$ is centred and has the same covariance structure as $X$. Also, for any $v \in S$,

$$\sup_{t \in \mathbb{R}} |Pr(|\langle Z, v \rangle| \leq t) - Pr(|g| \leq t)| \leq 2\psi(m).$$

Therefore, if we set $\alpha$ to be the median of $|g|$, then for every $v \in S$,

$$\left| Pr(|\langle Z, v \rangle| \leq \alpha) - \frac{1}{2} \right| \leq 2\psi(m).$$

(3.7)

Moreover, if $\varepsilon \leq \alpha/2$, there is an absolute constant $c$ for which

$$Pr(|\langle Z, v \rangle| \leq \alpha - \varepsilon) \leq Pr(|g| \leq \alpha - \varepsilon) + 2\psi(m) = Pr(|g| \leq \alpha) - c\varepsilon + 2\psi(m) \leq Pr(|\langle Z, v \rangle| \leq \alpha) - c\varepsilon + 4\psi(m);$$

Hence, if $\varepsilon \geq 8\psi(m)/c$, it follows that

$$Pr(|\langle Z, v \rangle| \in [\alpha - \varepsilon, \alpha]) \geq c'\varepsilon$$
for an absolute constant $c'$.

Thus, Condition (2) holds for $\varepsilon_0 = 8\psi(m)/c$ and $\varepsilon$ that satisfies $\varepsilon_0 \leq \varepsilon \leq 1/8$; clearly, $\varepsilon_0$ can be made arbitrarily small by taking a large enough $m$.

**Proof of Theorem 1.5.** Given the wanted accuracy parameter $\eta$, let $m$ for which $8\psi(m)/c \leq \eta \leq 1/8$. By Theorem 3.3, if $q \geq 3$ and $\sup_{v \in S} \| \langle X, v \rangle \|_{L_q} \leq L$ then one may take $m = c(L)/\eta^2$, whereas by [11], if $2 < q < 3$ one may take $m = c(L)\text{poly}(1/\eta)$.

Define $Z$ as in (3.6) and take $Z_1, \ldots, Z_n$ to be $n$ independent copies of $Z$ for $n \geq c_1 \eta^{-2} \log(2/\eta)d$. Set $\alpha$ to be the median of $|g|$; by Theorem 3.2, with probability at least $1 - 2 \exp(-c_2 \eta^2 n)$, the random set $\mathcal{K}$ satisfies

$$\mathcal{B} \subset \mathcal{K} \subset (1 + c_3 \eta) \mathcal{B},$$

as required.

### 3.1.1 Isomorphic approximation

If one is interested in an isomorphic approximation, i.e., that $c \mathcal{B} \subset \mathcal{K} \subset C \mathcal{B}$ for constants $c$ and $C$ that need not be close to 1, the assumption required in Theorem 1.5 can be relaxed from norm equivalence to a small-ball condition: that there are $0 < \lambda, \delta < 1$ such that for every $v \in \mathbb{R}^d$,

$$\Pr(\| \langle X, v \rangle \| \geq \lambda \| v \|_{L_2}) \geq \delta. \quad (3.8)$$

By a similar argument to the one used in the proof of Theorem 1.5, it follows that for

$$N \gtrsim \max \left\{ \frac{d}{\delta} \log(1/\delta \lambda), \frac{d}{\lambda^2} \right\},$$

and setting

$$\mathcal{K} = \{ v \in \mathbb{R}^d : |\langle X_i, v \rangle| \leq \lambda/2 \text{ for at least } (1 - \delta/4)N \text{ indices } i \},$$

with probability at least $1 - 2 \exp(-c\delta N)$,

$$c' \lambda \sqrt{\delta} \mathcal{B} \subset \mathcal{K} \subset \mathcal{B}.$$  

The inclusion $\mathcal{K} \subset \mathcal{B}$ stems from the small-ball condition: for every $v \in S$, with probability at least $1 - 2 \exp(-cN)$, at least $\delta N/2$ of the values $|\langle X_i, v \rangle|$ are likely to be larger than $\lambda$. The reason behind the other inclusion, that $c' \lambda \sqrt{\delta} \mathcal{B} \subset \mathcal{K}$, is that $\Pr(\| \langle X, v \rangle \| \geq t \| v \|_{L_2}) \leq 1/t^2$; therefore, with probability at least $1 - 2 \exp(-cN)$, most of the values $|\langle X_i, v \rangle|$ cannot be ‘too large’. The high probability with which both properties hold allows one to control a fine enough net in the sphere, and the oscillation term is handled in a similar way to the proof of Theorem 3.2. We omit the straightforward details.

### 3.2 Approximation using ellipsoids

This section is devoted to the proof of Theorem 1.7. Let $m$ to be specified in what follows, set $n = N/m$ and let $I_1, \ldots, I_n$ be the natural decomposition of $\{1, \ldots, N\}$ to coordinate blocks of cardinality $m$. 


For $1 \leq j \leq n$ and $v \in \mathbb{R}^d$ set

$$Z_j(v) = \frac{1}{m} \sum_{i \in I_j} \langle X_i, v \rangle^2$$

and recall that

$$\mathcal{D}_\eta = \{ v \in \mathbb{R}^d : \left| \{ j : Z_j(v) \leq 1 + \eta \} \right| \geq 0.9n \}.$$ 

Our aim is to show that if $m$ and $n$ are chosen properly, then with high probability,

$$\mathcal{B} \subset \mathcal{D}_\eta \subset (1 + c\eta)\mathcal{B}$$

for a suitable absolute constant $c$.

It is important to stress that the natural candidate for approximating $\mathcal{B}$, the empirical $L_2$ ball

$$\left\{ v \in \mathbb{R}^d : \frac{1}{N} \sum_{i=1}^{N} \langle X_i, v \rangle^2 \leq 1 \right\},$$

can be very different from $\mathcal{B}$ when $X$ is heavy-tailed; this will be illustrated in Section 3.3.

Again, we follow the general path outlined in Section 2. The property $\mathcal{P}$ is given by invoking Assumption 1.1—that if $m = m_0(\eta)$ then for every $v \in S$

$$\Pr\left(\left| \frac{1}{m} \sum_{i=1}^{m} \langle X_i, v \rangle^2 - 1 \right| \geq \frac{\eta}{10} \right) \leq 0.01.$$

Theorem 3.5. There are absolute constants $c_1$ and $c_2$ for which the following holds. If

$$n \geq c_1 \max\{d \log(2m_0(\eta)/\eta), m_0(\eta)\},$$

then with probability at least $1 - 2 \exp(-c_2n/m_0(\eta))$, for every $v \in \mathbb{R}^d$

$$\left| \{ j : Z_j(v) \in [(1 - \eta)\mathbb{E}Z(v), (1 + \eta)\mathbb{E}Z(v)] \} \right| \geq 0.96n.$$  (3.9)

In particular, if $m_0(\eta) \leq C\eta^{-k}$ then $n \geq c_1(k+1)d \log(2C/\eta)$ suffices.

Corollary 3.6. It is straightforward to verify that under an $L_4 - L_2$ norm equivalence with constant $L$ one has that $m_0(\eta) \leq c(L)/\eta^2$. Therefore, the required sample size is $N = m_0$ for

$$m_0 \leq c_1(L)\eta^{-2} \quad \text{and} \quad n = c'(L) \max\{d \log(L/\eta), \eta^{-2}\}$$

which is a better estimate than in Theorem 1.5 as long as $\eta \gtrsim 1/(d \log d)^{1/2}$.

Proof. Since the claim is homogeneous in $v$ it suffices to show that it holds for $v \in S$. By a binomial estimate, there is an absolute constant $c_0$ such that each $v \in \mathbb{R}^d$ satisfies

$$\left| \{ j : Z_j(v) \in [(1 - \eta/10)\mathbb{E}Z, (1 + \eta/10)\mathbb{E}Z] \} \right| \geq 0.98n$$  (3.10)

with probability at least $1 - 2 \exp(-c_0n)$.

Let $V \subset S$ be of cardinality at most $\exp(c_0n/2)$. Invoking the probability estimate with which (3.10) holds, there is an event $A_1$ of probability at least $1 - 2 \exp(-c_0n/2)$ such that (3.10) holds for every $v \in V$. As expected, our choice of $V$ is a maximal $r$-separated subset.
of $S$ with respect to the $L_2$ norm; and by a volumetric estimate, $r \leq 5 \exp(-c_1 n/d)$ for an absolute constant $c_1$.

To prove the wanted uniform estimate, for $u \in S$ let $\pi u \in V$ be the nearest element to $u$ with respect to the $L_2$ norm. Set

$$W = \sup_{u \in S} \{ i : |\langle X_i, u - \pi u \rangle | \geq \eta/10 \},$$

and the aim is to show that with high probability, $W \leq 0.02n$.

Just as in the proof of Theorem 3.2, let us first estimate $\mathbb{E} W$. By symmetrization and contraction, followed by the estimate on $r$ and Lemma 3.1,

$$\mathbb{E} W \leq \frac{10}{\eta} \mathbb{E} \sup_{u \in S} \sum_{i=1}^N \left( |\langle X_i, u - \pi u \rangle | - \mathbb{E} |\langle X_i, u - \pi u \rangle | \right) + \frac{10}{\eta} \sup_{u \in S} |\langle X, u - \pi u \rangle |$$

$$\leq \frac{20r}{\eta} \left( \mathbb{E} \sup_{u \in S} \sum_{i=1}^N \varepsilon_i \langle X_i, u \rangle \right. + N \left. \right) \leq \frac{20rN}{\eta} \left( \sqrt{dN} + N \right) \leq 0.01n,$$

provided that $n \geq c_3 d \log(m_0(\eta)/\eta)$. Therefore, by the bounded differences inequality, $W \leq 0.02n$ with probability at least $1 - 2 \exp(-c_4 n^2/N) = 1 - 2 \exp(-c_4 n/m)$ for a suitable absolute constant $c_4$.

Combining the two estimates, there is an event with probability at least $1 - 2 \exp(-c_5 n/m)$ on which:

- For every $v \in V$, $Z_j(v) \in [1 - \eta/10, 1 + \eta/10]$ for at least $0.98n$ indices $j$.
- For every $u \in S$, $|\langle X_i, u - \pi u \rangle | \geq \eta/10$ for at most $0.02n$ indices $i$; in particular, for every $u$ there could be at most $0.02n$ of the coordinate blocks $I_j$ that are ‘corrupted’ by such a large value of $|\langle X_i, u - \pi u \rangle | \geq \eta/10$. On all the other blocks, $\max_{i \in I_j} |\langle X_i, u - \pi, u \rangle | \leq \eta/10$.

Therefore, by the triangle inequality, for every $u \in S$ there are at least $0.96n$ indices $j$ for which $Z_j(u) \in [1 - \eta, 1 + \eta]$, as required.

**Proof of Theorem 1.7.** Consider the event from Theorem 3.3. If $u \in S$ then $Z_j(u) \leq 1 + \eta$ for more than $0.9n$ coordinate blocks, implying that $u \in D_\eta$. And, since $D_\eta$ is star-shaped around 0, it is evident that $B \subset D_\eta$.

At the same time, if $\|u\|_{L_2} = \rho$ then $Z_j(u) \geq (1 - \eta)\rho^2 > 1 + \eta$ provided that $\rho \geq 1 + c\eta$. Therefore, $(1 + c\eta)S \subset (D_\eta)^c$ and in particular, using the star-shape property again, $D_\eta \subset (1 + c\eta)B$.

### 3.3 Limitations of approximating using the empirical ellipsoid

Let us show that selecting $K = \{ v \in \mathbb{R}^d : N^{-1} \sum_{i=1}^N \langle X_i, v \rangle^2 \leq 1 \}$ as an approximation of $B$ is a poor choice when $X$ is heavy-tailed. To that end we construct a collection of random vectors that satisfy an $L_1 - L_2$ norm equivalence and for which $B$ is equivalent to $B_{2,2}^d$. At the same time, with a non-trivial probability there is $v \in S^{d-1}$ for which $N^{-1} \sum_{i=1}^N \langle X_i, v \rangle^2 \gg 1$. More accurately, for each $u \geq 1/\sqrt{d}$ we construct a centred random vector $X_u$ that satisfies:

- (a) For every $v \in S^{d-1}$, $1 \leq \| \langle X_u, v \rangle \|_{L_2} \leq 2$.
(b) $\sup_{v \in S^{d-1}} \| \langle X, v \rangle \|_{L^4} \leq L$ for an absolute constant $L$; and

(c) $\Pr(\|X_u\|^2 \geq ud) \geq 1/2u^2d$.

Let $\Gamma = N^{-1/2} \sum_{i=1}^N \langle X_i, \cdot \rangle e_i$ and observe that

$$\sup_{v \in S^{d-1}} \frac{1}{N} \sum_{i=1}^N \langle X_i, v \rangle^2 = \| \Gamma \|_{L^2}^2 - \| \Gamma \|_{L^2}^2 \geq \max_{1 \leq i \leq N} \| \Gamma e_i \|_2^2 \geq \frac{1}{N} \max_{1 \leq i \leq N} \| X_i \|_2^2.$$  

**Lemma 3.7.** Let $0 < \delta < 1/4$ and set $X_u$ as above for $u = (N/4\delta)^{1/2}$. Then with probability at least $\delta$,

$$\frac{1}{N} \max_{1 \leq i \leq N} \| X_i \|_2^2 \geq \sqrt{\frac{d}{4\delta N}}.$$  

In particular, with probability at least $\delta$, $B_{d}^d \not\subset C_{\mathbb{K}}$ unless $C \geq (d/4N\delta)^{1/4}$, making even an isomorphic approximation impossible if one would like it to hold with probability $1 - \delta$ for a small $\delta$ (corresponding to a large $u$), particularly taking into account that we would like $N$ to scale linearly in $d$.

**Proof.** Recall that $\Pr(\|X_u\|^2 \geq ud) \geq 1/2u^2d = 2\delta/N \equiv \rho$. Therefore, given $N$ independent copies of $X_u$ denoted by $Y_1, ..., Y_N$,

$$\Pr(\text{there exists } 1 \leq i \leq N, \| Y_i \|^2 \geq ud) \geq N\rho(1 - \rho)^{N-1} = 2\delta(1 - 2\delta/N)^N \geq \delta.$$  

On that event,

$$\frac{1}{N} \max_{1 \leq i \leq N} \| Y_i \|_2^2 \geq \frac{ud}{N} = \left( \frac{d}{4N\delta} \right)^{1/2},$$  

as claimed. □

All that is left now is to construct the random vectors $X_u$. To that end, let $\eta_1, ..., \eta_d$ be independent $\{0, 1\}$-valued random variables with mean $1/u^2d^2$ and set $\varepsilon_1, ..., \varepsilon_d$ to be independent, symmetric $\{-1, 1\}$-valued random variables that are independent of $\eta_1, ..., \eta_d$. Let $z_i = \varepsilon_i \max \{ \eta_i R, 1 \}$ where $R = \sqrt{ud}$, and set $X_u = (z_1, ..., z_d)$.

Clearly, $\mathbb{E}z_i = 0$ and

$$\mathbb{E}z_i^2 = \frac{R^2}{u^2d^2} + \left( 1 - \frac{1}{u^2d^2} \right);$$  

hence, $1 \leq \| z_i \|_{L^2} \leq 2$ if $u \geq 1/d$ as was assumed. Moreover,

$$\mathbb{E}z_i^4 \leq \frac{R^4}{u^2d^2} + \left( 1 - \frac{1}{u^2d^2} \right) \leq 2.$$  

Now, for $v \in \mathbb{R}^d$ we have that $\mathbb{E} \langle X_u, v \rangle^2 = \sum_{i=1}^d v_i^2 \mathbb{E}z_i^2$ and (a) follows from the estimate on $\mathbb{E}z_i^2$. As for (b), it is straightforward to verify that since $\mathbb{E}z_i^4 \leq 2$, $\| \sum_{i=1}^d v_i z_i \|_{L^4} \leq L\| v \|_2$ for an absolute constant $L$. Finally, to prove (c), consider $u \geq 1/\sqrt{d}$ and observe that $\| X_u \|_2^2 = \sum_{i=1}^d z_i^2$. Note that with probability at least $d \cdot (1/u^2d^2) \cdot (1 - 1/u^2d^2)^{d-1} \geq 1/2u^2d$, there is at least one index $i$ for which $z_i^2 \geq R^2 = ud$; hence, on that event, $\| X_u \|_2^2 \geq ud$, as required. □
3.4 Improving Theorem 1.5

Let us sketch an alternative proof of Theorem 1.5. On the one hand, it leads to a better estimate on the required sample size; on the other, it is based on a special property of slabs. The components of the proof are well understood so we will only sketch the argument.

In what follows we consider $Z_1, ..., Z_n$ that are distributed as $m^{-1/2} \sum_{i=1}^{m} X_i$ and satisfy (3.7); specifically we assume that $m$ is large enough to ensure that for $v \in S$,  

$$ \left| \Pr(\langle Z, v \rangle \leq \alpha) - \frac{1}{2} \right| \leq \frac{\eta}{2} \quad (3.11) $$

where $\alpha$ is the median of $|g|$. Here, the approximating body will be  

$$ K = \left\{ v \in \mathbb{R}^d : |\langle Z_j, v \rangle| \leq \alpha \text{ for at least } \left(\frac{1}{2} - \eta\right)n \text{ indices } j \right\}. $$

To show that indeed $K$ is an $\eta$-approximation of $B$, let us estimate the supremum of the empirical process  

$$ W = \sup_{v \in S} \left| \frac{1}{n} \sum_{j=1}^{n} 1_{\{|\langle Z_j, v \rangle| \leq \alpha\}} - \Pr(|\langle Z,v \rangle| \leq \alpha) \right|. \quad (3.12) $$

This is an empirical process indexed by a collection $U$ of subsets of $\mathbb{R}^d$—the slabs $\{x \in \mathbb{R}^d : |\langle x, v \rangle| \leq \alpha\}$. It is standard to verify that the VC dimension of $U$ is at most $cd$: each set is generated by the intersection of two halfspaces, and the VC dimension of the collection of halfspaces in $\mathbb{R}^d$ is at most $c'd$ (see, for example, [17] for more information on VC classes).

By Talagrand’s concentration inequality for empirical processes indexed by a class of bounded functions ([15], see also [3]), it follows that with probability at least $1 - \exp(-c_1t)$,  

$$ W \leq c_1 \left( \mathbb{E}W + \sqrt{\frac{d}{n} + \frac{t}{n}} \right). $$

And, by a standard argument,  

$$ \mathbb{E}W \leq c_2 \sqrt{\frac{d}{n}}. $$

Thus, with probability at least $1 - \exp(-c_3\eta^2n)$, $W \leq \eta/2$ provided that $n \gtrsim d/\eta^2$.

Therefore, on that event  

$$ \sup_{v \in S} \left| \left\{ j : |\langle Z_j, v \rangle| \leq \alpha \right\} - n\Pr(|\langle Z,v \rangle| \leq \alpha) \right| \leq \frac{n\eta}{2}, \quad (3.13) $$

Combining (3.13) and (3.11) it follows that with probability at least $1 - 2\exp(-c\eta^2n)$, for every $v \in S$,  

$$ \left| \left\{ j : |\langle Z_j, v \rangle| \leq \alpha \right\} \right| \geq n \left( \frac{1}{2} - \eta \right). \quad (3.14) $$

In particular we have that $S \subset K$, and since $K$ is star-shaped around $0$ then also $B \subset K$.

A similar estimate to (3.14) leads to the fact that $(1+\eta)S \subset K^c$ and completes the proof.}

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1The proof is based on symmetrization, the fact that a Bernoulli process is subgaussian with respect to the $\ell_2$ metric, a Dudley entropy integral bound and well-known estimates on the covering numbers of VC-classes.
The feature that makes this proof simple is that the class of indicators one is interested in happens to be a VC class. In general, there is no reason to expect such a happy coincidence when choosing a property $P$, and controlling the resulting empirical process can be a nontrivial problem. In contrast, the method presented here allows one by bypass this difficulty for rather general choices of $P$ and at a price of a slightly suboptimal dependency on $\eta$.

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