CLASSIFYING COALGEBRA SPLIT EXTENSIONS OF HOPF ALGEBRAS

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Abstract. For a given Hopf algebra $A$ we classify all Hopf algebras $E$ that are coalgebra split extensions of $A$ by $H_4$, where $H_4$ is the Sweedler’s 4-dimensional Hopf algebra. Equivalently, we classify all crossed products of Hopf algebras $A \# H_4$ by computing explicitly two classifying objects: the cohomological ‘group’ $H^2(H_4, A)$ and $\text{Crp}(H_4, A) :=$ the set of types of isomorphisms of all crossed products $A \# H_4$. All crossed products $A \# H_4$ are described by generators and relations and classified: they are parameterized by the set $ZP(A)$ of all central primitive elements of $A$. Several examples are worked out in detail: in particular, over a field of characteristic $p \geq 3$ an infinite family of non-isomorphic Hopf algebras of dimension $4p$ is constructed. The groups of automorphisms of these Hopf algebras are also described.

Introduction

Let $A$ and $H$ be two given groups. The extension problem of Hölder asks for the classification of extensions of $A$ by $H$, i.e. of all groups $E$ that fit into an exact sequence

$$1 \rightarrow A \overset{i}{\rightarrow} E \overset{\pi}{\rightarrow} H \rightarrow 1 \quad (1)$$

The classical approach ([4], [21]) proves that any extension of $A$ by $H$ is equivalent to a crossed product extension and, if $A$ is an abelian group, the Schreier’s theorem shows that all extensions are classified by the second cohomology group $H^2(H, A)$ [21, Theorem 7.34]. The result remains valid in the non-abelian case: this time $H^2(H, A)$ is not a group anymore but only a pointed set [4, Exercise 8, pg. 86]. The first generalization of Schreier’s theorem from groups to Hopf algebras was given by Sweedler [25]: if $H$ is a cocommutative Hopf algebra and $A$ a commutative algebra the cohomology $H^i(H, A)$ was introduced such that $H^2(H, A)$ classifies all cleft extensions of $A$ by $H$ [25, Theorem 8.6]. The graded case was studied in [22]. The first obstacle in the way of generalizing the extension problem from groups to the level of Hopf algebras was overcome at the beginning of the 90’s by defining the notion of exact sequence of Hopf algebras. Nowadays the unanimously accepted definition for this concept is the one given in [6, Definition 1.2.0] (see also [15, Definition 3.1] and [24, Definition 1.5]). This is the context in which the extensions of Hopf algebras were studied in a series of papers [5], [6], [10], [11], [16], [17], [18], etc. The tool for studying the extension problem for Hopf algebras

2010 Mathematics Subject Classification. 16T10, 16T05, 16S40.

Key words and phrases. crossed product of Hopf algebras, split extension of Hopf algebras.

A.L. Agore is research fellow ”aspirant” of FWO-Vlaanderen. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, grant no. 88/05.10.2011.
is the so-called cocycle bicrossproduct \( A^\tau \#_\sigma H \) introduced in \[15\] Theorem 2.9 and independently in \[6\] Theorem 2.20. \( A^\tau \#_\sigma H \) is the vector space \( A \otimes H \) with a crossed product algebra structure and a crossed coproduct coalgebra structure. The datum that constructs the cocycle bicrossproduct \( A^\tau \#_\sigma H \) must satisfy several compatibility conditions, some of them very technical (\[6\] Theorem 2.20). \[5\] Proposition 3.12 shows that any cleft extension of Hopf algebras is equivalent to a cocycle bicrossproduct extension. The classification of all cleft extension was given in \[6\] Section 3] where it is shown that all cleft extensions of \( A \) by \( H \) are classified by a certain cohomological object denoted by \( H^1_\bullet (H, A) \) \[6\] Theorem 3.2.14]. This is probably the most general version of the classical Schreier theorem known for Hopf algebras. Unfortunately, its importance is rather a theoretical one: the explicit description and classification of all (cleft) extensions of \( A \) by \( H \) – or equivalently of all cocycle bicrossproducts \( A^\tau \#_\sigma H \) - is a very difficult task for two reasons (see \[5\] Section 5.2] for details). On the one hand the large number of compatibility conditions that need to be fulfilled for constructing all cocycle bicrossproducts \( A^\tau \#_\sigma H \) makes the problem very difficult for a computational approach. On the other hand, there is no efficient cohomology theory for arbitrary Hopf algebras, similar to the one from group theory, to make a direct description of \( H^1_\bullet (H, A) \) possible. One of the few examples known is \[14\] Lema 2.8] where it is proved that any extension of \( H_4 \) by \( H_4 \) is equivalent to the trivial extension \( H_4 \hookrightarrow H_4 \otimes H_4 \twoheadrightarrow H_4 \).

For this reason, in the present paper we deal with a special case of Hopf algebra extensions, namely the coalgebra split extensions. Let \( A \) and \( H \) be two given Hopf algebras. A coalgebra split extension of \( A \) by \( H \) is a pair \((E, \pi)\), where \( E \) is a Hopf algebra that fits into a sequence \( A \hookrightarrow E \xrightarrow{\pi} H \) such that the Hopf algebra map \( \pi : E \to H \) splits in the category of coalgebras and \( A \simeq E^{co(H)} \). Several other types of split extensions of Hopf algebras are studied in \[8\], \[9\], \[23\]. The coalgebra split extensions cover the extension problem from the theory of groups (Example \[13\]). Exactly as in the group case, any coalgebra split extension of \( A \) by \( H \) is equivalent to a crossed product extension \((A\# H, \pi_H)\) (Proposition \[13\]). Thus, the classification of all coalgebra split extensions of \( A \) by \( H \) is equivalent to the classification of all crossed products \( A\# f \) associated to all possible crossed systems of Hopf algebras \((A, H, \triangleright, f)\). The classification will be given in two ways: from the view point of the extension theory (that is, up to an isomorphism of Hopf algebras that stabilizes \( A \) and \( H \)) they will be classified after we explicitly compute the cohomological 'group' \( H^2(H, A) \) which is the counterpart for Hopf algebras of the second cohomology group from group theory. The second and more general way of classifying such extensions will be given by computing explicitly the second classifying object: \( Crp(H, A) := \) the set of types of isomorphisms of Hopf algebras of all crossed products \( A\# H \). There exists a canonical surjection \( H^2(H, A) \twoheadrightarrow Crp(H, A) \).

The paper is organized as follows: in Section 1] we recall the basic concepts related to crossed products of Hopf algebras. Section 2] contains some technical results: Theorem \[22\] describes the set of all morphisms between two arbitrary crossed products of Hopf algebras which is our tool in the classification problem as well as for computing the automorphisms group of a given crossed product of Hopf algebras. Section 3] provides an example of classification. More precisely, for a given Hopf algebra \( A \), the crossed systems \((A, H_4, \triangleright, f)\) are completely described in Theorem \[8\] they are parameterized by the set
\( \mathcal{ZP}(A) \) of all central primitive elements of \( A \). For a large class of Hopf algebras \( A \), including the enveloping algebras of Lie algebras, Theorem 3.7 classifies this new family of Hopf algebras by computing \( \mathcal{H}^2(H_4, A) \) and \( \text{CrP}(H_4, A) \). The group \( \text{Aut}_{\text{Hopf}}(A \# H_4) \) is explicitly described. In Section 3 we construct some explicit examples: we shall classify all crossed products of the form \( A \# H_4 \), for some specific Hopf algebras \( A \), namely for the polynomial Hopf algebra \( k[Y] \) and for two of its quotients in the case when \( \text{char}(k) = p \geq 3 \). Let \( k = \mathbb{F}_p(X_1, X_2, \cdots) \) be the field of rational functions in indeterminates \( \{X_i\}_{i \geq 1} \) over the finite field \( \mathbb{F}_p \). Corollary 4.7 proves that \( k(y \mid y^p = y) \# H_4 \) contains an infinite family of non-isomorphic 4-dimensional Hopf algebras. In particular, we construct an infinite number of types of Hopf algebras of dimension 12 over a field of characteristic 3.

1. Preliminaries

Unless specified otherwise, all algebras, coalgebras, Hopf algebras or tensor products are over an arbitrary field \( k \). For a coalgebra \( C \), we use Sweedler’s \( \Sigma \)-notation: \( \Delta(c) = c^{(1)} \otimes c^{(2)} \), \((I \otimes \Delta) \Delta(c) = c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \), etc. (summation understood). For a \( k \)-linear map \( f : H \otimes H \to A \) we denote \( f(g, h) = f(g \otimes h) \), for all \( g, h \in H \). For all unexplained notations we refer to [20]. Let \( A \) and \( H \) be two Hopf algebras. A linear map \( r : H \to A \) is called cocentral if \( r \) is a morphism of coalgebras and the following compatibility condition holds for any \( h \in H \):

\[
    r(h^{(1)}) \otimes h^{(2)} = r(h^{(2)}) \otimes h^{(1)}
\]  

(2)

The set \( \text{CoZ}(H, A) \) of all cocentral maps is a group with respect to the convolution product [3, pg. 338]. We denote by \( \text{CoZ}^1(H, A) \) the subgroup of \( \text{CoZ}(H, A) \) of all cocentral maps \( r : H \to A \) such that \( r(1) = 1 \). A \( k \)-linear map \( \triangleright : H \otimes A \to A \) is called a weak action of \( H \) on \( A \) if for any \( a, b \in A, h \in H \):

\[
    h \triangleright 1_A = \varepsilon_H(h)1_A, \quad 1_H \triangleright a = a
\]  

(3)

\[
    h \triangleright (ab) = (h^{(1)} \triangleright a)(h^{(2)} \triangleright b)
\]  

(4)

For a \( k \)-linear map \( f : H \otimes H \to A \) and a weak action \( \triangleright : H \otimes A \to A \) we shall denote by \( A\#^f H \) the \( k \)-vector space \( A \otimes H \) with the multiplication given by

\[
    (a \# h) \cdot (c \# g) := a(h^{(1)} \triangleright c)f(h^{(2)}, g^{(1)}) \# h^{(3)}g^{(2)}
\]  

(5)

for all \( a, c \in A, h, g \in H \), where we denoted \( a \otimes h \) by \( a \# h \). The object \( A\#^f H \) is called a crossed product of Hopf algebras if it is a Hopf algebra with the multiplication \( (5) \), the unit \( 1_A \# 1_H \) and the coalgebra structure given by the tensor product of coalgebras. In this case \((A, H, \triangleright, f)\) is called a crossed system of Hopf algebras [11, Definition 1.1]. If \( f \) is the trivial cocycle, that is \( f(g, h) = \varepsilon(g)\varepsilon(h)1_A \), for all \( g, h \in H \) then the associated crossed product \( A\#^f H = A \# H \) is the semi-direct (smash) product of Hopf algebras [19]. The following gives necessary and sufficient conditions for \( A\#^f H \) to be a crossed product of Hopf algebras.

**Proposition 1.1.** Let \( A, H \) be Hopf algebras, \( \triangleright : H \otimes A \to A \) a weak action and \( f : H \otimes H \to A \) a \( k \)-linear map. The following are equivalent:

1. \( A\#^f H \) is crossed product of Hopf algebras;
(2) \(\triangleright : H \otimes A \to A \) and \(\triangleright : H \otimes H \to A \) are morphisms of coalgebras satisfying the following compatibilities for any \(a \in A, \ g, \ h, \ l \in H\):

\[
f(h,1_H) = f(1_H, h) = \varepsilon_H(h)1_A
\]

\[
[g(1) \triangleright (h(1) \triangleright a)] f(g(2), h(2)) = f(g(1), h(1))((g(2)h(2)) \triangleright a)
\]

\[
(g(1) \triangleright f(h(1), l(1))) f(g(2), h(2)l(2)) = f(g(1), h(1))f(g(2)h(2), l)
\]

\[
g(1) \otimes g(2) \triangleright a = g(2) \otimes g(1) \triangleright a
\]

\[
g(1)h(1) \otimes f(g(2), h(2)) = g(2)h(2) \otimes f(g(1), h(1))
\]

In this case the antipode of \(A\#^\gamma H\) is given by

\[
S(a\#g) := (S_A[f(S_H(g(2)), g(3))])\#S_H(g(1)) \cdot (S_A(a)\#1_H)
\]

for all \(a \in A\) and \(g \in H\).

**Proof.** (2) \(\Rightarrow\) (1) is [7, Lemma 1.2.10], where the crossed product is viewed as a special case of the cocycle bicrossproduct [6, Theorem 2.20] if we let the cocycle cross-coproduct be the trivial one. It can be also obtained as a special case of the unified product of [8, Theorem 2.4, Examples 2.5(2)].

(1) \(\Rightarrow\) (2) Assume that \(A\#^\gamma H\) is a Hopf algebra with the above structures. In particular, it is an associutive algebra and hence [12, Proposition 6.1.10] the compatibility conditions (4), (7) and (8) holds. It remains to prove that \(f\) and \(\triangleright\) are morphisms of coalgebras satisfying (9) and (10). Indeed, it follows from \(\varepsilon((1_A\#h) \cdot (1_A\#g)) = \varepsilon(1_A\#h)\varepsilon(1_A\#g)\) that \(\varepsilon_A(f(h, g)) = \varepsilon_H(h)\varepsilon_H(g)\), for all \(h, g \in H\). On the other hand, the relation \(\varepsilon((1_A\#h) \cdot (a\#1_H)) = \varepsilon(1_A\#h)\varepsilon(a\#1_H)\) gives \(\varepsilon_A(h \triangleright a) = \varepsilon_H(h)\varepsilon_A(a)\). Now, applying \(I \otimes \varepsilon_H \otimes I \otimes \varepsilon_H\) to the relation \(\Delta((1_A\#g) \cdot (1_A\#h)) = \Delta(1_A\#g)\Delta(1_A\#h)\) we obtain that \(f\) is a coalgebra map and applying \(\varepsilon_A \otimes I \otimes I \otimes \varepsilon_H\) to the same relation we obtain (10). Similarly, applying \(I \otimes \varepsilon_H \otimes I \otimes \varepsilon_H\) to \(\Delta((1_A\#h) \cdot (a\#1_H)) = \Delta(1_A\#h)\Delta(a\#1_H)\) we obtain that \(\triangleright\) is a coalgebra map and applying \(\varepsilon_A \otimes I \otimes I \otimes \varepsilon_H\) to the same relation we obtain (9). \(\square\)

Let \((A, H, \triangleright, f)\) be a crossed system of Hopf algebras. There exist Hopf algebras morphisms

\[
i_A : A \to A\#^\gamma H, \quad i_A(a) = a\#1_H, \quad \pi_H : A\#^\gamma f H \to H, \quad \pi_H(a\#h) = \varepsilon_A(a)h
\]

The crossed product \(A\#^\gamma f H\) will be viewed as a left \(A\)-module (resp. right \(H\)-comodule) via the restriction of scalar through \(i_A\) (resp. \(\pi_H\), i.e. \(a' \cdot (a\#h) := a' a\#h\) and \(a\#h \mapsto a\#h(1) \otimes h(2)\), for all \(a, a' \in A\) and \(h \in H\). The crossed product of Hopf algebras is the construction responsible for the description of the following type of extensions of Hopf algebras:

**Definition 1.2.** Let \(A\) and \(H\) be two given Hopf algebras. A **coalgebra split extension** of \(A\) by \(H\) is a pair \((E, \pi)\) consisting of a Hopf algebra \(E\) that fits into a sequence

\[
A \hookrightarrow E \xrightarrow{\pi} H
\]
such that $\pi : E \to H$ is a morphism of Hopf algebras which has a section as a coalgebra map and $A \simeq E^{\text{co}(H)} := \{ x \in E \mid x_{(1)} \otimes \pi(x_{(2)}) = x \otimes 1 \}$.

Two coalgebra split extensions $(E, \pi)$, $(E', \pi')$ of $A$ by $H$ are called equivalent if there exists an isomorphism of Hopf algebras $\psi : E \to E'$ that stabilizes $A$ and co-stabilizes $H$, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
A & \overset{i}{\longrightarrow} & E \\
\downarrow & & \downarrow \psi \\
A & \overset{i'}{\longrightarrow} & E'
\end{array}
\]

Any crossed product $(A\#^\varphi_f H, \pi_H)$ is a coalgebra split extension of $A$ by $H$ via the canonical morphisms given by $\varphi_H$. Indeed, $i_H : H \to A\#^\varphi_f H$, $i_H(h) = 1_A \# h$ is a coalgebra map and a section of $\pi_H$. Now, $A\#^\varphi_f H$ is a right $H$-comodule algebra via $\pi_H$, i.e. the right $H$-coaction is given by $a \# h \mapsto a \# h_{(1)} \otimes h_{(2)}$. Then we can easily prove that

\[
(A\#^\varphi_f H)^{\text{co}(H)} = \{ x \in A\#^\varphi_f H \mid x_{(1)} \otimes \pi_H(x_{(2)}) = x \otimes 1 \} = A\#^1 H \cong A
\]

Conversely, using the theory of cleft extensions [13], we have the following result which reduces the classification of all coalgebra split extensions to the one of crossed products:

**Proposition 1.3.** Let $A$ and $H$ be two Hopf algebras. Then any coalgebra split extension $(E, \pi)$ of $A$ by $H$ is equivalent to a crossed product extension $(A\#^\varphi_f H, \pi_H)$ of $A$ by $H$.

**Proof.** Let $(E, \pi)$ be a coalgebra split extension of $A$ by $H$ and $\varphi : H \to E$ be a coalgebra map and a section for $\pi$. Without loss of generality, we can assume that $\varphi(1) = 1$ (otherwise we can replace $\varphi$ by $\varphi(1)^{-1}\varphi$). Then $\varphi$ is invertible in convolution with the inverse $\varphi^{-1} = S_E \circ \varphi$ and moreover, using that $\varphi : H \to E$ splits $\pi$, it is also a right $H$-comodule map. Indeed, for any $h \in H$

\[
\rho_E(\varphi(h)) = \varphi(h_{(1)}) \otimes \pi(\varphi(h_{(2)})) = \varphi(h_{(1)}) \otimes \pi(\varphi(h_{(2)})) = \varphi(h_{(1)}) \otimes h_{(2)} = (\varphi \otimes \text{Id}) \circ \Delta(h)
\]

Thus the right $H$-extension $E/A$ is cleft. It follows from [13 Theorem 11] that there exist well defined maps

\[
\triangleright = \triangleright_{\varphi} : H \otimes A \to A, \quad \triangleright \triangleright a := \varphi(h_{(1)}) a \varphi^{-1}(h_{(2)}),
\]

\[
f = f_{\varphi} : H \otimes H \to A, \quad f(g, h) := \varphi(g_{(1)}) \varphi(h_{(1)}) \varphi^{-1}(g_{(2)}h_{(2)})
\]

such that $\triangleright_{\varphi}$ is an weak action and

\[
\psi : A\#^\varphi_f H \to E, \quad \psi(a \# h) := a \varphi(h)
\]

is an isomorphism of associative unitary algebras. But there is more: $\psi : A \otimes H \to E$ is also a morphism of coalgebras, as a composition of such maps. Thus, $\psi : A\#^\varphi_f H \to E$ is an isomorphism of algebras and coalgebras between $A\#^\varphi_f H$ and $E$. Since $E$ is a Hopf algebra we obtain that $A\#^\varphi_f H$ is in fact a Hopf algebra and $\psi : A\#^\varphi_f H \to E$ is an isomorphism of Hopf algebras. The fact that $\psi$ stabilizes $A$ and co-stabilizes $H$ is straightforward. \qed
Example 1.4. Let $A$ and $H$ be two groups. Then a Hopf algebra $E$ is a coalgebra split extension of $k[A]$ by $k[H]$ if and only if $E \cong k[G]$, for a group $G$ which is an extension of $A$ by $H$. Indeed, using Proposition 1.3 we obtain that $E$ is a coalgebra split extension of $k[A]$ by $k[H]$ if and only if $E \cong k[A] \# k[H]$, for some crossed system of Hopf algebras $(k[A], k[H], \triangleright, f)$. Now, such crossed systems of Hopf algebras are in bijection to the usual crossed system of groups and the bijection is given such that there exists a canonical isomorphism of Hopf algebras $k[A] \# k[H] \cong k[A\# H]$ (see [1] Examples 1.2) for details), where $A\# H$ is a crossed product of groups, i.e. an extension of $A$ by $H$.

2. Morphisms between crossed products

First of all we shall prove a technical result that will be our tool in the classification of all crossed products as well as for computing the automorphisms group of a given crossed product of Hopf algebras.

Theorem 2.1. Let $(A, H, \triangleright, f)$ and $(A', H', \triangleright', f')$ be two crossed systems of Hopf algebras. Then there exists a bijective correspondence between the set of all morphisms of Hopf algebras $\psi : A\#^p_f H \to A'\#^{p'}_{f'} H'$ and the set of all quadruples $(u, p, r, v)$, where $p : A \to H'$ is a morphism of Hopf algebras, $u : A \to A'$, $r : H \to A'$ and $v : H \to H'$ are unitary morphisms of coalgebras satisfying the following compatibility conditions:

(CP1) $u(a(1)) \otimes p(a(2)) = u(a(2)) \otimes p(a(1))$

(CP2) $r(h(1)) \otimes v(h(2)) = r(h(2)) \otimes v(h(1))$

(CP3) $u(ab) = u(a(1)) (p(a(2)) \triangleright u(b(1))) f' (p(a(3)), p(b(2)))$

(CP4) $v(h) v(g) = p(f(h(1), g(1))) v(h(2)g(2))$

(CP5) $v(h) p(a) = p(h(1) \triangleright a) v(h(2))$

(CP6) $r(h(1)) (u(h(2)) \triangleright' r(g(1))) f' (v(h(3)), v(g(2))) = u(f(h(1), g(1))) (p(f(h(2), g(2)) \triangleright' r(h(4)g(4))) f' (p(f(h(3), g(3)), v(h(5)g(5))))$

(CP7) $r(h(1)) (u(h(2)) \triangleright' u(a(1))) f' (v(h(3)), p(a(2))) = u(h(1) \triangleright a(1)) (p(h(2) \triangleright a(2)) \triangleright' r(h(4))) f' (p(h(3) \triangleright a(3)), v(h(5)))$

for all $a, b \in A, g, h \in H$. Under the above bijection the morphism of Hopf algebras $\psi : A\#^p_f H \to A'\#^{p'}_{f'} H'$ corresponding to $(u, p, r, v)$ is given by:

$$\psi(a\# h) = u(a(1)) (p(a(2)) \triangleright' r(h(1))) f' (p(a(3)), v(h(2))) \#' p(a(4)) v(h(3))$$

(14)

for all $a \in A$ and $h \in H$.

Proof. Let $\psi : A\#^p_f H \to A'\#^{p'}_{f'} H'$ be a morphism of Hopf algebras. We define

$$\alpha : A \to A'\#^p_f H', \quad \alpha(a) := \psi(a\# 1_H), \quad \beta : H \to A'\#^{p'}_{f'} H', \quad \beta(h) := \psi(1_A\# h)$$
Then $\alpha : A \to A' \otimes H'$ and $\beta : H \to A' \otimes H'$ are unitary morphisms of coalgebras as compositions of such maps and

$$\psi(a\# h) = \psi((a\# 1_H)(1_A\# h)) = \psi(a\# 1_H)\psi(1_A\# h) = \alpha(a)\beta(h)$$  

(15)

for all $a \in A$ and $h \in H$. It follows from [2] Lemma 2.1 that there exist four coalgebra maps $u : A \to A'$, $p : A \to H'$, $r : H \to A'$, $v : H \to H'$ such that

$$\alpha(a) = u(a(1)) \otimes p(a(2)), \quad \beta(h) = r(h(1)) \otimes v(h(2))$$  

(16)

and the pairs $(u, p)$ and $(r, v)$ satisfy the symmetry conditions (CP1) and (CP2). Explicitly $u, p, r$ and $v$ are defined by

$$u(a) = ((\text{Id} \otimes \varepsilon_{H'}) \circ \psi)(a\# 1_H), \quad p(a) = ((\varepsilon_{A'} \otimes \text{Id}) \circ \psi)(a\# 1_H)$$

$r(h) = ((\text{Id} \otimes \varepsilon_{H'}) \circ \psi)(1_A\# h), \quad v(h) = ((\varepsilon_{A'} \otimes \text{Id}) \circ \psi)(1_A\# h)$

for all $a \in A$ and $h \in H$. All these maps are unitary coalgebra maps. Now, for any $a \in A$ and $h \in H$ we have:

$$\psi(a\# h) = \alpha(a)\beta(h) = (u(a(1))\#' p(a(2)))(r(h(1))\#' v(h(2)))$$

$$= u(a(1)) (p(a(2)) \triangleright' r(h(1))) f'(p(a(3)), v(h(2))) \#' p(a(4)) v(h(3))$$

i.e. (14) also holds. Thus any bialgebra map $\psi : A\#^\triangleright H \to A'\#'^\triangleright H'$ is determined by the formula (14), for some unique quadruple of unitary coalgebra maps $(u, p, r, v)$.

Now, we prove that a map $\psi$ given by (14) is a morphism of algebras if and only if $\alpha : A \to A'\#'^\triangleright H'$ is an algebra map and the following compatibility conditions hold:

$$\beta(h) \beta(g) = \alpha(f(h(1), g(1))) \beta(h(2)g(2))$$  

(17)

$$\beta(h) \alpha(b) = \alpha(h(1) \triangleright b) \beta(h(2))$$  

(18)

for all $h \in H$ and $b \in A$. Indeed, if $\psi$ is an algebra map then $\alpha$ is an algebra map as a composition of algebra maps. On the other hand:

$$\psi(a\# h)\psi(b\# g) = \alpha(a)\beta(h)\alpha(b)\beta(g)$$

$$\psi((a\# h)(b\# g)) = \alpha(a)\alpha(h(1) \triangleright b)\alpha(f(h(2), g(1)))\beta(h(3)g(2))$$

Hence, the condition (17) (resp. (18)) follows by considering $a = b = 1_A$ (resp. $a = 1_A$ and $g = 1_H$) in the identity $\psi(a\# h)\psi(b\# g) = \psi((a\# h)(b\# g))$. The converse is obvious.

Now, we prove that $\alpha : A \to A'\#'^\triangleright H'$, $\alpha(a) = u(a(1))\#' p(a(2))$ is an algebra map if and only if $p : A \to H'$ is an algebra map and (CP3) holds. Indeed, $\alpha(ab) = \alpha(a)\alpha(b)$ is equivalent to:

$$u(a(1)b(1))\#' p(a(2)b(2)) = u(a(1))(p(a(2)) \triangleright' u(b(1))) f'(p(a(3)), p(b(2))) \#' p(a(4)) p(b(3))$$

If we apply $\text{Id} \otimes \varepsilon_{H'}$ to this equation we obtain (CP3), while if we apply $\varepsilon_{A'} \otimes \text{Id}$ to the same equation we obtain that $p : A \to H'$ is an algebra map, hence a morphism of Hopf algebras. The converse is obvious.
In a similar way we can show that the compatibility condition (17) holds if and only if (CP4) and (CP6) hold. Indeed, using the expressions of $\alpha$ and $\beta$ in terms of $(u,p)$ and respectively $(r,v)$, the equation (17) is equivalent to:

$$
\begin{align*}
\rho(h_1)(v(h_2) \trianglerightiv(g_1)) f'(v(h_3), v(g_2)) \#'v(h_4) v(g_3) &= \\
u(f(h_1), g_1)) p(f(h_2), g_2) \trianglerightiv r(h_5) g_5) f'(p(h_3), g_3)) v(h_6 g_6)) \#' \\
p(f(h_4), g_4) v(h_7 g_7)
\end{align*}
$$

If we apply $Id \otimes \varepsilon_{H'}$ to the above identity we obtain (CP6) while if we apply $\varepsilon_{A'} \otimes Id$ to it we get (CP4). Conversely, the compatibility condition (17) follows straightforward from (CP6) and (CP4).

Finally, we prove that the commutativity condition (18) is equivalent to (CP5) and (CP7). Indeed, (18) is equivalent to:

$$
\begin{align*}
\rho(h_1)(v(h_2) \trianglerightiv u(a_1)) f'(v(h_3), p(a_2)) \#'v(h_4) p(a_3) &= \\
u(h_1 \trianglerightiv a_1)(p(h_2) \trianglerightiv a_2) \trianglerightiv r(h_5) g_5) f'(p(h_3) \trianglerightiv a_3), v(h_6) \#' \\
p(h_4) \trianglerightiv a_4) v(h_7)
\end{align*}
$$

If we apply $Id \otimes \varepsilon_{H'}$ to the above identity we obtain (CP7) while if we apply $\varepsilon_{A'} \otimes Id$ to it we get (CP5). Conversely, the commutativity condition (18) follows straightforward from (CP7) and (CP5).

To conclude, we have proved that any bialgebra map $\psi : A \#^\varphi_f H \to A' \#^\varphi_{f'} H'$ is uniquely determined by a quadruple $(u,p,r,v)$, where $p : A \to H'$ is a morphism of Hopf algebras, $u : A \to A'$, $r : H \to A'$ and $v : H \to H'$ are unitary morphisms of coalgebras satisfying the compatibility conditions (CP1)-(CP7) such that $\psi : A \#^\varphi_f H \to A' \#^\varphi_{f'} H'$ is given by (14) and the proof is finished.

The compatibility conditions of Theorem 2.1 are rather difficult to deal with. However, there are several special cases in which the two compatibilities simplify considerably. The first one, which will be used in Section 3 is the following:

**Corollary 2.2.** Let $(A,H,\trianglerightiv, f)$ and $(A',H,\trianglerightiv', f')$ be two crossed systems of Hopf algebras such that the only Hopf algebra map $p : A \to H$ is the trivial one. Then there exists a bijective correspondence between the set of all morphisms of Hopf algebras $\psi : A \#^\varphi_f H \to A' \#^\varphi_{f'} H$ and the set of all triples $(u,r,v)$, where $u : A \to A$, $v : H \to H$ are morphisms of Hopf algebras, $r : H \to A$ is a unitary coalgebra map satisfying the following compatibility conditions:

$$
\begin{align*}
\rho(h_1) \otimes v(h_2) &= \rho(h_2) \otimes v(h_1) \\
\rho(h_1)(v(h_2) \trianglerightiv r(g_1)) f'(v(h_3), v(g_2)) &= u(f(h_1), g_1) r(h_2) g_2 \\
\rho(h_1)(v(h_2) \trianglerightiv u(a)) &= u(h_1 \trianglerightiv a) r(h_2)
\end{align*}
$$

for all $a \in A$, $g, h \in H$. Under the above correspondence the morphism of Hopf algebras $\psi : A \#^\varphi_f H \to A' \#^\varphi_{f'} H$ corresponding to $(u,r,v)$ is given by:

$$
\psi(a \#^\varphi h) = u(a) \rho(h_1) \#'v(h_2)
$$
for all $a \in A$ and $h \in H$.

Furthermore, $\psi : A\#_{f} H \to A\#_{f'} H$ given by (22) is an isomorphism if and only if $u$ and $v$ are automorphisms of Hopf algebras.

Proof. The first part follows from Theorem 2.1 applied for $A' = A$, $H' = H$ and $p(a) = \varepsilon_{A}(a)1_{H}$, for all $a \in A$. Assume now that $\psi$ is an isomorphism and let $\psi^{-1}$ be its inverse associated to a triple $(u', r', v')$; that is $\psi^{-1}(a\# h) = u'(a) r'(h(1)) \#' v'(h(2))$. Then, we have:

\begin{align}
    a\# 1 &= (\psi^{-1} \circ \psi)(a\# 1) = u'(u(a))\# 1 \\
    1\# h &= (\psi^{-1} \circ \psi)(1\# h) = u'(r(h(1))) r'(v(h(2))) \#' v'(v(h(3)))
\end{align}

for all $a \in A$ and $h \in H$. Applying $Id_{A} \otimes \varepsilon$ in (22) (resp. $\varepsilon \otimes Id_{H}$ in (22)), that $u' \circ u = Id_{A}$ (resp. $v' \circ v = Id_{H}$). In a similar manner it follows from $\psi \circ \psi^{-1} = Id_{A\# H}$ that $u \circ u' = Id_{A}$ and $v \circ v' = Id_{H}$. Thus, $u$ and $v$ are isomorphisms.

Conversely, if $u$ and $v$ are automorphisms, then $\psi$ given by (22) is bijective with the inverse given by:

\[
    \varphi : A\# H \to A\# H, \quad \varphi(a\# h) = u^{-1}(a)(u^{-1} \circ S \circ r \circ v^{-1})(h(1)) \# v^{-1}(h(2))
\]

where $u^{-1}$ (resp. $v^{-1}$) is the composition inverse of $u$ (resp. $v$). \hfill \Box

Corollary 2.3. Let $\psi : A\#_{f} H \to A\#_{f'} H$ be a Hopf algebras map between two crossed products. Then $\psi$ stabilizes $A$ (resp. co-stabilizes $H$) if and only if $\psi$ is a left $A$-linear (reps. right $H$-co-linear) map.

Proof. A morphism $\psi : A\#_{f} H \to A\#_{f'} H$ stabilizes $A$ if and only if $\psi(a\# 1_{H}) = a\# 1_{H}$, for all $a \in A$. Taking into account the formula for $\psi$ given by (14), we obtain that $\psi$ stabilizes $A$ if and only if $u(\varepsilon_{A}(a_{1})1_{H}) = a\# 1_{H}$, i.e. $u(a) = a$ and $p(a) = \varepsilon_{A}(a)1_{H}$, for any $a \in A$. Thus, $\psi$ take the form $\psi(a\# h) = a r(h(1)) \#' v(h(2))$, for all $a \in A$ and $h \in H$ and such a map is obviously left $A$-linear. Conversely, if $\psi$ is left $A$-linear, then $\psi(a\# 1) = \psi(a \cdot (1\# 1_{H})) = a \cdot \psi(1\# 1_{H}) = a\# 1_{H}$, i.e. $\psi$ stabilizes $A$. The other statement can be proved in a dual manner. \hfill \Box

Remarks 2.4. 1. Using Proposition 1.3 and Corollary 2.3 the classification in the sense of Definition 1.2 of all coalgebra split extensions of $A$ by $H$ reduces to the classification of all crossed products $A\#_{f} H$. The classifying object is denoted by $\mathcal{H}^{2}(H, A)$ and was constructed in [1] Proposition 2.2] as a special case of [3] Theorem 3.4] as follows: let $\mathcal{CS}(A, H)$ be the set of all pairs $(\triangleright, f)$ such that $(A, H, \triangleright, f)$ is a crossed system of Hopf algebras. Two pairs $(\triangleright, f)$ and $(\triangleright', f') \in \mathcal{CS}(A, H)$ are called cohomologous and we denote this by $(\triangleright, f) \approx (\triangleright', f')$ if there exists an unitary cocentral map $r : H \to A$ such that:

\[
    h \triangleright a = r(h(1)) (h(2) \triangleright a) (S_{A} \circ r)(h(3)) \\
    f'(h, g) = r(h(1)) (h(2) \triangleright r(g(1))) f(h(3), g(2)) (S_{A} \circ r)(h(4)g(3))
\]

for all $a \in A$ and $h, g \in H$. Then [3 Theorem 3.4] proves that $(\triangleright, f) \approx (\triangleright', f')$ if and only if there exists a Hopf algebra isomorphism $A\#_{f} H \cong A\#_{f'} H$ that stabilizes.
Theorem 3.1. \( A \) and co-stabilizes \( H \). Thus, \( \approx \) is an equivalence relation on the set \( CS(A, H) \). The cohomological object \( H^2(H, A) \) is the pointed quotient set defined by

\[
H^2(H, A) := CS(A, H) / \approx
\]

2. A more general form of classification is the following: two coalgebra split extensions \((E, \pi)\) and \((E', \pi')\) of \( A \) by \( H \) are called isomorphic if there exists an isomorphism of Hopf algebras \( E \cong E' \). Using again Proposition \([13]\) this classification reduces also to classifying up to a Hopf algebra isomorphism all crossed products \( A \#_f^H \). We denote by \( Crp(H, A) \) the set of types of Hopf algebra isomorphisms of all crossed products \( A \#_f^H \) associated to all crossed systems \((A, H, \triangleright, f)\). It is obvious that two equivalent extensions are isomorphic and hence there exists a canonical surjection \( H^2(H, A) \to Crp(H, A) \).

3. Classifying crossed products with the Sweedler’s Hopf algebra

This section is devoted to the classification of all coalgebra split extensions of \( A \) by \( H_4 \), where \( A \) is a Hopf algebra and \( H_4 \) is the Sweedler’s 4-dimensional Hopf algebra. There are three steps that we have to go through. First of all we have to compute the set of all crossed systems \((A, H_4, \triangleright, f)\) between \( A \) and \( H_4 \). This is the computational part of our approach. Then we describe by generators and relations all crossed products \( A \#_f^H \) associated to these crossed systems. Finally, using Theorem \([21]\) we shall classify the above crossed products by computing the classifying objects \( H^2(H_4, A) \) and \( Crp(H_4, A) \).

As a bonus of our approach, the group of Hopf algebra automorphisms of these crossed products is computed.

For a Hopf algebra \( A \), \( G(A) \) is the set of group-like elements of \( A \) and for \( g, h \in G(A) \) we denote by \( P_{g,h}(A) \) the set of all \((g, h)\)-primitive elements, that is

\[
P_{g,h}(A) = \{ x \in A \mid \Delta_A(x) = x \otimes g + h \otimes x \}
\]

We denote by \( P(A) = P_{1,1}(A) \) the set of all primitive elements of \( A \) and by \( ZP(A) := P(A) \cap Z(A) \), where \( Z(A) \) is the center of \( A \).

Let \( k \) be a field of characteristic \( \neq 2 \) and \( H_4 \) the Sweedler’s 4-dimensional Hopf algebra having \( \{1, g, x, gx\} \) as a basis with the multiplication:

\[
g^2 = 1, \quad x^2 = 0, \quad xg = −gx
\]

and the coalgebra structure such that \( g \) is a group-like element and \( x \) is \((1, g)\)-primitive.

Our first classification result shows that for an arbitrary Hopf algebra \( A \) the set \( ZP(A) \) of central primitive elements of \( A \) parameterizes all crossed systems \((A, H_4, \triangleright, f)\).

Theorem 3.1. Let \( k \) be a field of characteristic \( \neq 2 \) and \( A \) a Hopf algebra. Then there exists a bijection between the set of all crossed systems \((A, H_4, \triangleright, f)\) and the set \( ZP(A) \) of all central primitive elements of \( A \) such that the crossed system \((A, H_4, \triangleright, f)\) corresponding to \( a \in ZP(A) \) is given as follows: the action \( \triangleright : H_4 \otimes A \rightarrow A \) is the trivial action \( h \triangleright b = \varepsilon(h)b \), for any \( h \in H_4, b \in A \) and the cocycle \( f = f_a : H_4 \otimes H_4 \rightarrow A \) is
given by the following formula:

\[
\begin{array}{c|cccc}
  f & 1 & g & x & gx \\
  \hline
  1 & 1 & 1 & 0 & 0 \\
  g & 1 & 1 & 0 & 0 \\
  x & 0 & 0 & a & -a \\
  gx & 0 & 0 & a & -a \\
\end{array}
\]  

(25)

In particular, if \( ZP(A) = \{0\} \), then there are no nontrivial crossed systems of Hopf algebras \((A, H_4, \triangleright, f)\) and thus the only crossed product \( A \#_{f} H_4 \) is the usual tensor product \( A \otimes H_4 \) of Hopf algebras, that is \( H^2(H, A) \cong \text{CRP}(H, A) = \{A \otimes H_4\} \).

Proof. We shall compute all crossed systems \((A, H_4, \triangleright, f)\): i.e. we have to describe all coalgebra maps \( \triangleright : H_4 \otimes A \to A, f : H_4 \otimes H_4 \to A \) satisfying the compatibility conditions \((3) - (10)\). Let \((\triangleright, f)\) be such a pair. We shall prove first that \( \triangleright : H_4 \otimes A \to A \) is necessarily the trivial action. Indeed, let \( a \in A \). If we apply the compatibility condition \((3)\) for \( g := x \) we obtain, taking into account \((3), \(1)\)

\[
x \otimes a + g \otimes (x \triangleright a) = 1 \otimes (x \triangleright a) + x \otimes (g \triangleright a)
\]

If we apply \( x^* \otimes \text{Id}_A \) and \( g^* \otimes \text{Id}_A \) to this equation (where \( x^* \) and \( g^* \in H_4^* \) are the elements of the dual basis of \( \{1, g, x, gx\} \)) we obtain that \( g \triangleright a = a \) and \( x \triangleright a = 0 \). Now, if we apply \((3)\) for \( g := gx \) we obtain, using \( g \triangleright a = a \), that \( 1 \otimes (gx) \triangleright a = g \otimes (gx) \triangleright a \) and hence \( (gx) \triangleright a = 0 \). Thus we have proved that \( \triangleright : H_4 \otimes A \to A \) acts trivially and hence the compatibility conditions \((3) - (10)\) are trivially fulfilled.

It remains to describe all the cocycles \( f : H_4 \otimes H_4 \to A \). Since the action \( \triangleright \) is trivial, the compatibility condition \((7)\) takes the form, \( b f(h, h') = f(h, h') b \), for all \( b \in A, h, h' \in H_4 \), i.e. \(7\) is equivalent to the fact that \( \text{Im}(f) \subseteq Z(A) \). Furthermore, the normalizing condition \((3)\) is equivalent to

\[
f(1, 1) = f(1, g) = f(g, 1) = 1, \quad f(1, x) = f(1, gx) = f(x, 1) = f(gx, 1) = 0
\]

From now on we assume that \( f \) is such a normalized map. The next step proves that the compatibility condition \((10)\) holds for \( f \) if and only if

\[
f(g, g) = 1, \quad f(x, g) = f(g, x) = f(gx, g) = f(g, gx) = 0
\]

(26)

Indeed, first of all we observe that \((10)\) is trivially fulfilled for \( g = 1 \) or \( h = 1 \). Now, the compatibility condition \((10)\) holds for \((x, g), (gx, g), (g, x)\) and respectively \((g, gx)\) if and only if:

\[
\begin{align*}
  xg \otimes 1 + 1 \otimes f(x, g) &= g \otimes f(x, g) + xg \otimes f(g, g) \\
  -x \otimes f(g, g) + g \otimes f(gx, g) &= 1 \otimes f(gx, g) - x \otimes 1 \\
  gx \otimes 1 + 1 \otimes f(g, x) &= g \otimes f(g, x) + gx \otimes f(g, g) \\
  x \otimes f(g, g) + g \otimes f(gx, g) &= 1 \otimes f(g, gx) + x \otimes 1
\end{align*}
\]

These four equations are equivalent to the fact that \((26)\) holds. Now, with the values of \( f \) given by \((26)\), it is just a straightforward computation to prove that \((10)\) holds for any other pair of \( \{g, x, gx\} \times \{g, x, gx\} \). More precisely, there are other five possibilities that need to be checked, namely \((g, g), (x, x), (x, gx), (gx, x)\) and \((gx, gx)\). In all these cases
we obtain compatibilities that are trivially fulfilled: in \((x, x)\) we obtain \(1 \otimes f(x, x) = 1 \otimes f(x, x)\), in \((x, gx)\) we obtain \(g \otimes f(x, gx) = g \otimes f(x, gx)\), in \((gx, x)\) we obtain \(g \otimes f(gx, x) = g \otimes f(gx, x)\) and finally in \((gx, gx)\) we obtain \(1 \otimes f(gx, gx) = 1 \otimes f(gx, gx)\).

To summarize, we have proved so far that a pair of maps \((\triangleright, f)\) satisfies all axioms of a crossed system \([3] - [10]\), except for the cocycle axiom \([8]\), if and only if \(\triangleright : H_4 \otimes A \to A\) is the trivial action and \(f : H_4 \otimes H_4 \to A\) is given by

| \(f\) | 1 | \(g\) | \(x\) | \(gx\) |
|-----|---|-----|-----|-----|
| 1   | 1 | 1   | 0   | 0   |
| \(g\)| 1 | 1   | 0   | 0   |
| \(x\)| 0 | 0   | \?  | \?  |
| \(gx\)| 0 | 0   | \?  | \?  |

In the final step of our investigation the above values of \(f\) marked with \? will be determined such that the cocycle condition \([8]\) holds. First of all, using the values of \(f\) that we have already obtained and the fact that \(f\) is a morphism of coalgebras one can prove that \(f(x, x), f(x, gx), f(gx, x)\) and \(f(gx, gx)\) are primitive elements of \(A\). Indeed, if we write down the condition \(\Delta_A(f(x, x)) = (f \otimes f) \circ \Delta_{H_4 \otimes H_4}(x \otimes x)\) we obtain that:

\[
\Delta_A(f(x, x)) = f(x, x) \otimes 1 + f(x, g) \otimes f(1, x) + f(g, x) \otimes f(x, 1) + f(g, g) \otimes f(x, x)
\]

that is \(f(x, x)\) is a primitive element of \(A\). In the same way we can prove that \(f(x, gx)\), \(f(gx, x)\) and \(f(gx, gx)\) are also primitive elements.

The only compatibility which remains to be fulfilled by \(f\) is the cocycle condition \([8]\) which, considering that the action \(\triangleright\) is trivial, takes the simplified form:

\[
f(h_{(1)}, l_{(1)}) f(y, h_{(2)}l_{(2)}) = f(y_{(1)}, h_{(1)}) f(y_{(2)}h_{(2)}, l) \tag{27}
\]

for all \(h, l, y \in H_4\). First we observe that the condition \(\Delta\) holds if one of the elements \(h, l\) or \(y\) is 1. Thus, \(\Delta\) holds if and only if it holds in all triples \((h, l, y) \in \{g, x, gx\}\). Thus, there are 27 equations that have to be fulfilled. However, this is a routinely check so we indicate only the main steps of the proof. First we observe that \(\Delta\) holds for the triple \((h, l, y)\) equal to \((g, x, x)\), \((g, x, gx)\) and respectively \((x, g, x)\) if and only if

\[
f(gx, x) = -f(x, gx), \quad f(gx, gx) = -f(x, x), \quad f(x, gx) = -f(x, x) \tag{28}
\]

Thus, if we denote \(f(x, x) = a \in Z(A) \cap P(A)\), we obtain that \(f(gx, x) = a\) and \(f(x, gx) = f(gx, gx) = -a\) i.e. \(\Delta\) holds. Now, by a long but straightforward computation one can see that for the other 24 possibilities of choosing the triple \((h, l, y)\), the cocycle condition \(\Delta\) is either trivially fulfilled or equivalent to one of the compatibilities \(\epsilon\). For instance, if we consider the triple \((h, l, y) := (x, g, gx)\), \(\epsilon\) holds if and only if \(f(gx, gx) = -f(gx, x)\), which can be obtained from \(\Delta\). On the other hand, using the values of \(f\) that we have already determined, it is easy to see that for the triple \((h, l, y) := (x, x, gx)\), \(\epsilon\) holds automatically. The last statement follows from the first part: if \(a = 0\), then \(f = f_0\) is just the trivial cocycle \(f(g, h) = \varepsilon(g) \varepsilon(h) 1_A\), for all \(g, h \in H_4\) and hence the proof is finished. □
Examples 3.2. (1) Let $A$ be a finite dimensional Hopf algebra over a field of characteristic zero. Then $P(A) = \{0\}$ [12, Exercise 4.2.16] and hence there are no nontrivial crossed systems of Hopf algebras $(A, H_4, \partial, f)$. The conclusion fails if char$(k) = p \geq 3$: in Corollary 3.7 we shall prove that there exists an infinite number of types of isomorphisms of crossed products $A \# H_4$.

(2) Let $A = U(L)$ be the enveloping algebra of a Lie algebra $L$ over a field $k$ of characteristic zero. Then, $P(U(L)) = L$ [20, Proposition 5.5.3]. Thus, the set of all primitive central elements of $U(L)$ if given by

$$ZP(U(L)) = \{l \in L \mid [l, y] = 0, \forall y \in L\} = Z(L)$$

where $Z(L)$ is the center of the Lie algebra $L$. Typical examples of Lie algebras with non-trivial center are the nilpotent Lie algebras. They will provide non-trivial examples of crossed products $U(L) \# H_4$.

For any $a \in ZP(A)$ we shall denote by $A_{(a)} := A \# H_4$, the crossed product associated to the crossed system constructed in Theorem 3.1. In what follows we will describe explicitly by generators and relations the Hopf algebras $A_{(a)}$. As the Hopf algebra map $A \to A \# H_4$, $z \mapsto z \# 1$, is injective we shall identify $z = z \# 1$, for any $z \in A$. Let $\{e_i\}_{i \in I}$ be a $k$-basis of $A$ and we denote

$$g = 1 \# g, \quad x := 1 \# x, \quad w := 1 \# gx \in A_{(a \mid g, x)}$$

We can easily show, by using formula (25), that $w = gx$ in $A_{(a)}$. Indeed,

$$gx = (1 \# g)(1 \# x) = f(g, x^{(1)}) \# g x^{(2)} = f(g, x) \# g + f(g, g) \# gx = 1 \# gx = w$$

Similarly, we can prove that the following relations hold in the Hopf algebra $A_{(a)}$:

$$g^2 = 1, \quad x^2 = a, \quad xg = -gx, \quad ge_i = e_i g, \quad xe_i = e_i x$$

for any $i \in I$. Indeed, for example (below we denote $x' = x$) we have:

\[
x^2 = (1 \# x)(1 \# x) = f(x^{(1)}, x'^{(1)}) \# x^{(2)} x^{(2)} = f(x, x) \# 1 + f(x, g) \# x + f(g, x) \# x + f(g, g) \# x^2 = a \# 1 = a \\
\]

and

$$xe_i = (1 \# x)(e_i \# 1) = e_i \# x = (e_i \# 1)(1 \# x) = e_i x$$

Now, as a vector space, $A_{(a)} := A \otimes H_4$, hence, the set $\{e_i, e_i g, e_i x, e_i gx \mid i \in I\}$ is a $k$-basis of $A_{(a)}$, where we identify $e_i = e_i \# 1$, $e_i \# g = (e_i \# 1)(1 \# g) = e_i g$ and so on.

Using Theorem 3.1 and the above computations we obtain:

Corollary 3.3. Let $k$ be a field of characteristic $\neq 2$, $A$ a Hopf algebra and consider $\{e_i\}_{i \in I}$ a $k$-basis of $A$. Then a Hopf algebra $E$ is isomorphic to a crossed product of Hopf algebras $A \# H_4$ if and only if $E \cong A_{(a)}$, for some $a \in ZP(A)$, where $A_{(a)}$ is the Hopf algebra having $\{e_i, e_i g, e_i x, e_i gx \mid i \in I\}$ as a $k$-basis and the multiplication is subject to the following relations for any $i \in I$:

\[
g^2 = 1, \quad x^2 = a, \quad xg = -gx, \quad ge_i = e_i g, \quad xe_i = e_i x \tag{29}
\]
Proof. We show that the bijection from the statement is given such that the isomorphism $u$ that satisfies Theorem 3.5.

The coalgebra structure and the antipode of $A_{(a)}$ are given by:

$$
\Delta(e_i) := \Delta_A(e_i), \quad \Delta(g) := g \otimes g, \quad \Delta(x) := x \otimes 1 + g \otimes x
$$

$$
\varepsilon(e_i) := \varepsilon_A(e_i), \quad \varepsilon(g) := 1, \quad \varepsilon(x) := 0
$$

$$
S(e_i) := S_A(e_i), \quad S(g) := g, \quad S(x) := -gx
$$

for all $i \in I$.

Example 3.4. Let $L := h(n, k)$ be the $2n + 1$ dimensional Heisenberg Lie algebra over a field $k$ of characteristic zero. That is, $h(n, k)$ has a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ and the only non-zero Lie brackets are $[x_i, y_i] = z$, for all $i = 1, \ldots, n$. Since $\text{char}(k) = 0$, it is well known that $Z(h(n, k)) = \{qz \mid q \in k\}$. Let $A := U(h(n, k))$. Then $ZP(A) = k^*$ and thus any primitive central element $l \in U(h(n, k))$ is of the form $l = qz$, for some scalar $q \in k$. If $q = 0$, then $U(h(n, k))(0) = U(h(n, k)) \otimes H_4$, the tensor product of the two Hopf algebras. Assume now that $q \neq 0$. Using Corollary 3.3 we can easily prove that $U(h(n, k))(qz) = k_q(x_i, y_i, g, x | i = 1, \ldots, n)$, where by $k_q(x_i, y_i, g, x | i = 1, \ldots, n)$ we denote the quantum group generated as an algebra by $\{x_i, y_i, g, x | i = 1, \ldots, n\}$ subject to the following relations for any $i = 1, \ldots, n$:

$$
g^2 = 1, \quad xg = -gx, \quad gx_i = x_i g, \quad gy_i = y_i g
$$

$$
xx_i = x_i x, \quad xy_i = y_i x, \quad x_i y_i - y_i x_i = q^{-1} x_i^2
$$

with the coalgebra structure given such that $g$ is a group-like element, $x$ is a $(1, g)$-primitive element and $x_i, y_i$ are primitive elements for any $i = 1, \ldots, n$. We observe that there exists a Hopf algebra isomorphism $k_q(x_i, y_i, g, x | i = 1, \ldots, n) \cong k_1(x_i, y_i, g, x | i = 1, \ldots, n)$, for any $q \in k^*$ and hence $\text{CRP}(U(h(n, k)), H_4)$ has two elements.

Now, we shall classify $A_{(a)}$ for those Hopf algebras $A$ such that the only Hopf algebra map $f : A \rightarrow H_4$ is the trivial one, namely $f(z) = \varepsilon(z)1$, for all $z \in A$. The typical example is again $A = U(L)$, for a Lie algebra $L$. We recall from [2, Lemma 4.6] that $\text{Aut}_{\text{Hopf}}(H_4) \cong k^*$: explicitly, any automorphism $v : H_4 \rightarrow H_4$ is of the form

$$
v(g) = g, \quad v(x) = \beta x, \quad v(gx) = \beta gx
$$

for some non-zero scalar $\beta \in k^*$. It what follows, the automorphism $v$ of $H_4$ implemented by $\beta \in k^*$ as in (30) will be denoted by $v = v_\beta$.

Theorem 3.5. Let $k$ be a field of characteristic $\neq 2$ and $A$ a Hopf algebra such that the only Hopf algebra map $A \rightarrow H_4$ is the trivial one. Let $a, b \in ZP(A)$ be two central primitive elements of $A$. Then there exists a bijection between the set of all Hopf algebra isomorphisms $\psi : A_{(a)} \rightarrow A_{(b)}$ and the set of all pairs $(u, \beta) \in \text{Aut}_{\text{Hopf}}(A) \times k^*$ such that $u(a) = \beta^2 b$.

Proof. We show that the bijection from the statement is given such that the isomorphism $\psi = \psi_{u, \beta} : A_{(a)} \rightarrow A_{(b)}$ corresponding to a pair $(u, \beta) \in \text{Aut}_{\text{Hopf}}(A) \times k^*$ satisfying $u(a) = \beta^2 b$, is given by

$$
\psi_{u, \beta}(z \# h) = u(z) \# v_\beta(h)
$$

(31)
Indeed, $A(a) = A#_a H_4$, where by $A#_a H_4$ we denoted the crossed product associated to the central primitive element $a$ from Theorem 3.1. It follows from Corollary 2.2 that the set of all Hopf algebra maps $\psi : A(a) \to A$ is in bijection with the set of all triples $(u, r, v)$, where $u : A \to A$, $v : H_4 \to H_4$ are Hopf algebra maps and $r : H_4 \to A$ is a unitary coalgebra map such that the compatibility conditions (19)-(21) are fulfilled. Under this bijection the morphism $\psi = \psi(u, r, v)$ corresponding to $(u, r, v)$ is given by (22).

We will prove now that, under this bijection, the isomorphisms $\psi = \psi(u, r, v)$ correspond precisely to the triples $(u, r, v)$ such that $u : A \to A$, $v : H_4 \to H_4$ are isomorphisms of Hopf algebras, $r : H_4 \to A$ is the trivial coalgebra map and $u(a) = \beta^2 b$, where $\beta \in k^*$ is the scalar that implements the isomorphism $v = v_\beta$ of $H_4$.

Indeed, it follows from Corollary 2.2 that $\psi = \psi(u, r, v)$ is an isomorphism if and only if $u : A \to A$ and $v : H_4 \to H_4$ are isomorphisms of Hopf algebras. Let $(u, v) \in \text{Aut}_{\text{Hopf}}(A) \times \text{Aut}_{\text{Hopf}}(H_4)$ and $\beta \in k^*$ such that $v = v_\beta$. We will show now that the compatibility conditions (19)-(21) are fulfilled for the triple $(u, r, v)$ if and only if $r : H_4 \to A$ is the trivial coalgebra map, i.e. $r(h) = \varepsilon(h) 1_A$, for all $h \in H_4$ and $u(a) = \beta^2 b$ and this will finish the proof. First we remark that if $r : H_4 \to A$ is the trivial coalgebra map then the compatibility condition (19) holds. Conversely, since $v$ is an isomorphism the compatibility condition (19) is equivalent to

$$r(h_{(1)}) \otimes h_{(2)} = r(h_{(2)}) \otimes h_{(1)}$$

for any $h \in H_4$, i.e. to the fact that $r : H_4 \to A$ is a cocentral map. Applying (32) for $h = x$ and then for $h = gx$ we obtain

$$r(x) \otimes 1 + r(g) \otimes x = 1 \otimes x + r(x) \otimes g$$
$$r(gx) \otimes g + 1 \otimes gx = r(g) \otimes gx + r(gx) \otimes 1$$

It follows from here that $r(x) = 0$, $r(g) = 1$ and $r(gx) = 0$. Thus $r$ is the trivial map. Thus (19) holds for a triple $(r, u, v)$ such that $\psi = \psi(u, r, v)$ is an isomorphism if and only if $r$ is a trivial map. In this case, the compatibility condition (21) also holds since $r : H_4 \to A$ is the trivial map and the actions $\triangleright$ and $\triangleright'$ of the crossed systems $(A, H_4, \triangleright, f = f_a)$ and $(A, H_4, \triangleright', f = f_b)$ are also the trivial actions according to Theorem 3.1.

Now, we look at the remaining compatibility condition (20) that has to be satisfied by a triple $(u, r, v)$ for which $\psi(u, r, v)$ is an isomorphism. Since $r$ is the trivial coalgebra map and $\triangleright$ and $\triangleright'$ are the trivial actions, the compatibility condition (20) is equivalent to

$$f_b(v_\beta(h), v_\beta(k)) = u(f_a(h, k))$$

for all $h, k \in H_4$. It is easy to see that this compatibility condition holds if and only if $\beta^2 b = u(a)$. For example, $f_b(v_\beta(x), v_\beta(x)) = \beta^2 f_b(x, x) = \beta^2 b$ and $u(f_a(x, x)) = u(a)$, so (20) holds for $(h, k) = (x, x)$ if and only if $\beta^2 b = u(a)$. The rest is straightforward and the proof is now complete.
Remark 3.6. Let $A$ be a Hopf algebra such that the only Hopf algebra map $A \to H_4$ is the trivial one. For any $a \in \mathcal{ZP}(A)$ we define

$$G(a) := \{ (u, \beta) \in \text{Aut}_{\text{Hopf}}(A) \times k^* \mid u(a) = \beta^2 a \}$$

Then $G(a)$ is a subgroup of $\text{Aut}_{\text{Hopf}}(A) \times k^*$ and it follows from Theorem 3.5 that there exists an isomorphism of groups

$$G(a) \cong \text{Aut}_{\text{Hopf}}(A_{(a)}), \quad (u, \beta) \mapsto \psi_{u,\beta}$$

where $\psi_{u,\beta}$ is given by (31).

Now, summarizing our results, we obtain the following classification result:

Theorem 3.7. Let $k$ be a field of characteristic $\neq 2$ and $A$ a Hopf algebra such that the only Hopf algebra morphism $A \to H_4$ is the trivial one. Then:

1. There exists a bijection between the set of types of Hopf algebra isomorphisms of all crossed products $A \# H_4$ and the quotient pointed set $\mathcal{ZP}(A)/\equiv$, where $\equiv$ is the equivalence relation on $\mathcal{ZP}(A)$ defined by: $a \equiv b$ if and only if there exists a pair $(u, \beta) \in \text{Aut}_{\text{Hopf}}(A) \times k^*$ such that $u(a) = \beta^2 b$. Hence, $\text{CRP}(H_4, A) \cong \mathcal{ZP}(A)/\equiv$.

2. There exists a bijection $\mathcal{H}^2(H_4, A) \cong \mathcal{ZP}(A)$.

Proof. It follows from Theorem 3.5 and Theorem 3.1 once we observe that the isomorphism $\psi_{u,\beta}$ given by (31) stabilizes $A$ (resp. co-stabilizes $H_4$) if and only if $u = \text{Id}_A$ (resp. $\beta = 1$), i.e. two coalgebra split extensions of $A$ by $H_4$, $(A_{(a)}$, $\pi_{H_4})$ and $(A_{(b)}$, $\pi_{H_4})$ are equivalent if and only if $b = a$.

4. Examples

In this section we provide some explicit classification for all crossed products of the form $A \#^p H_4$, where $A$ is the polynomial Hopf algebra $k[Y]$ ($Y$ is a primitive element) or two of its quotients when $\text{char}(k) = p \geq 3$: the $p$-dimensional Hopf algebras $k(y \mid y^p = 0)$ and $k(y \mid y^p = y)$. To start with, we collect some technical results. First, if $A$ is a Hopf algebra generated by primitive elements then the only Hopf algebra morphism $p : A \to H_4$ is the trivial one, i.e. $p(a) = \varepsilon(a)1$, for all $a \in A$, since $P(H_4) = \{0\}$. We also need to know the set of primitive elements and the automorphism groups of these Hopf algebras. For $k[Y]$ the description of the primitive elements follows from a more general result for universal enveloping algebras [20 Proposition 5.5.3]: $P(k[Y]) = \{q Y \mid q \in k\}$ if $\text{char}(k) = 0$ and $P(k[Y])$ is the span of all $Y^p^i$, $i \geq 0$, if $\text{char}(k) = p \geq 2$. For the description of the primitive elements of the other Hopf algebras and of the groups of automorphisms we have the following:

Lemma 4.1. Let $k$ be a field of characteristic $\neq 2$. Then:

1. If $\text{char}(k) = p \geq 3$ then $P(k(y \mid y^p = 0)) = \{q y \mid q \in k\}$ and $P(k(y \mid y^p = y)) = \{q y \mid q \in k\}$;

2. $u : k[Y] \to k[Y]$ is a Hopf algebra automorphism if and only if there exists an $\alpha \in k^*$ such that $u(Y^i) = \alpha^i Y^i$ for all $i \geq 0$. In particular, $\text{Aut}_{\text{Hopf}}(k[Y]) \simeq k^*$;
(3) If \( \text{char}(k) = p \geq 3 \) then \( u : k\langle y \mid y^p = 0 \rangle \rightarrow k\langle y \mid y^p = 0 \rangle \) is a Hopf algebra automorphism if and only if there exists an \( \alpha \in k^* \) such that \( u(y^j) = \alpha^j y^j \), for all \( j \geq 0 \).
In particular, \( \text{Aut}_{\text{Hopf}}(k\langle y \mid y^p = 0 \rangle) \cong k^* \).

(4) If \( \text{char}(k) = p \geq 3 \) then \( u : k\langle y \mid y^p = y \rangle \rightarrow k\langle y \mid y^p = y \rangle \) is a Hopf algebra automorphism if and only if there exists an \( \alpha \in \mathbb{F}_p^* \) such that \( u(y^j) = \alpha^j y^j \), for all \( j \geq 0 \).
In particular, \( \text{Aut}_{\text{Hopf}}(k\langle y \mid y^p = y \rangle) \cong \mathbb{F}_p^* \), where \( \mathbb{F}_p \) is the field with \( p \) elements.

**Proof.** The proof is a straightforward computation. We only remark for (4) that if \( u : k\langle y \mid y^p = y \rangle \rightarrow k\langle y \mid y^p = y \rangle \) is an automorphism then \( u(y) = \alpha y \), for some \( \alpha \in k^* \) that must satisfy \( \alpha^p = \alpha \), since

\[
\alpha y = u(y) = u(y^p) = \alpha^p y = \alpha y
\]

As \( \alpha \neq 0 \), \( \alpha^{p-1} = 1 \). Taking into account that \( \{ \alpha \in k \mid \alpha^{p-1} = 1 \} = \mathbb{F}_p^* \), the conclusion follows. \( \square \)

In order to classify all crossed products \( k[Y] \# H_4 \) we distinguish two cases depending on the characteristic of the base field \( k \).

**Proposition 4.2.** Let \( k \) be a field of characteristic zero and \( A := k[Y] \) the polynomial Hopf algebra. Then:

(1) Up to an isomorphism of Hopf algebras there exist exactly two crossed products of Hopf algebras \( A \# H_4 : A \otimes H_4 \) and \( A_{(\infty)} \), where by \( A_{(\infty)} \) we denote the infinite dimensional Hopf algebra generated by \( g \) and \( x \) subject to the relations:

\[
g^2 = 1, \quad xg = -gx
\]

and with the coalgebra structure given such that \( g \) is a group-like element and \( x \) is \( (1, g) \)-primitive. Furthermore, we have the following isomorphisms of groups:

\[
\text{Aut}_{\text{Hopf}}(A \otimes H_4) \cong k^* \times k^* \quad \text{and} \quad \text{Aut}_{\text{Hopf}}(A_{(\infty)}) \cong k^*.
\]

(2) There exists a bijection \( H^2(H_4, A) \cong k \).

**Proof.** According to Corollary 3.3 a Hopf algebra is isomorphic to a crossed product of Hopf algebras \( A \# H_4 \) if and only if \( E \cong A_{(qY)} \) for some \( q \in k \), where \( A_{(qY)} \) is the infinite dimensional Hopf algebra generated by \( Y, g \) and \( x \), subject to the relations

\[
g^2 = 1, \quad x^2 = qY, \quad xg = -gx, \quad gY = Yg, \quad xY = Yx
\]

If \( q \neq 0 \) then, among the previous relations, the ones in \( (33) \) are independent. The final statement of (1) follows Remark 3.6 and Lemma 1.1. The classification part is a consequence of Theorem 3.7. \( \square \)

The case \( \text{char}(k) = p \geq 3 \) is more interesting. We denote by \( k^{(N)} \) the set of sequences with finitely many non-zero terms from \( k \). For \( (\alpha_i)_{i \geq 0} \in k^{(N)} \) for which there exists \( i \geq 0 \) such that \( \alpha_i \neq 0 \) we define:

\[
G((\alpha_i)_{i \geq 0}) := \{(\alpha, \beta) \in k^* \times k^* \mid \alpha^{p^i} = \beta^2, \text{for all } i \text{ such that } \alpha_i \neq 0\}
\]

It is easily seen that \( G((\alpha_i)_{i \geq 0}) \) is a subgroup of \( k^* \times k^* \).
Proposition 4.3. Let $k$ be a field of characteristic $p \geq 3$ and $A := k[Y]$ the polynomial Hopf algebra. Then:

1. A Hopf algebra $E$ is isomorphic to a crossed product $A \# H_4$ if and only if $E \cong A_{((\alpha_i)_{i \geq 0})}$, for some $(\alpha_i)_{i \geq 0} \in k^{(N)}$, where $A_{((\alpha_i)_{i \geq 0})}$ is the infinite dimensional quantum group generated by $Y$, $g$ and $x$, subject to the following relations

$$g^2 = 1, \quad x^2 = \sum_{i \geq 0} \alpha_i Y^{p^i}, \quad xg = -gx, \quad gY = Yg, \quad xY = Yx$$

and with the coalgebra structure given such that $Y$ is a primitive element, $g$ is a group-like element and $x$ is $(1, g)$-primitive. Furthermore, we have the following isomorphisms of groups:

$$\text{Aut}_{\text{Hopf}}(A \otimes H_4) \cong k^* \times k^* \quad \text{and} \quad \text{Aut}_{\text{Hopf}}(A_{((\alpha_i)_{i \geq 0})}) \cong G((\alpha_i)_{i \geq 0})$$

for all $(\alpha_i)_{i \geq 0} \in k^{(N)} \setminus \{0\}$.

2. There exists a bijection $\text{CRP}(H_4, k[Y]) \cong k^{(N)}/\sim$, where $\sim$ is the equivalence relation on $k^{(N)}$ defined by: $(\alpha_i)_{i \geq 0} \sim (\beta_i)_{i \geq 0}$ if and only if there exists $(\alpha, \beta) \in k^* \times k^*$ such that $\alpha^p \alpha_i = \beta^2 \beta_i$, for all $i \geq 0$.

3. There exists a bijection $\text{H}^2(H_4, k[Y]) \cong k^{(N)}$.

Proof. The set of all primitive elements of $A = k[Y]$ is $P(A) = \{ \sum_{i \geq 0} \alpha_i Y^{p^i} \mid (\alpha_i)_{i \geq 0} \in k^{(N)} \}$. Denoting by $A_{((\alpha_i)_{i \geq 0})}$ the Hopf algebra associated to the primitive element $\sum_{i \geq 0} \alpha_i Y^{i}$, where $(\alpha_i)_{i \geq 0} \in k^{(N)}$, we obtain the first part of (1). The description of the automorphism groups of $A_{((\alpha_i)_{i \geq 0})}$ follows from Remark 4.6 and Lemma 4.1, indeed, we have

$$\text{Aut}_{\text{Hopf}}(A \otimes H_4) \cong \text{Aut}_{\text{Hopf}}(A) \times k^* \cong k^* \times k^*$$

If $(\alpha_i)_{i \geq 0}$ is not the null sequence then, by Remark 4.6 $A_{((\alpha_i)_{i \geq 0})} \cong G(\sum_{i \geq 0} \alpha_i Y^{p^i})$, where

$$G(\sum_{i \geq 0} \alpha_i Y^{p^i}) = \left\{ (u, \beta) \in \text{Aut}_{\text{Hopf}}(A) \times k^* \mid u(\sum_{i \geq 0} \alpha_i Y^{p^i}) = \beta^2 \sum_{i \geq 0} \alpha_i Y^{p^i} \right\}$$

Taking into account the description of $\text{Aut}_{\text{Hopf}}(A)$ given in Lemma 4.1, we obtain

$$G(\sum_{i \geq 0} \alpha_i Y^{p^i}) \cong \left\{ (\alpha, \beta) \in k^* \times k^* \mid \alpha^p \alpha_i = \beta^2 \alpha_i \text{ for all } i \right\} = G((\alpha_i)_{i \geq 0})$$

(3) follows from Theorem 3.7 and the fact that $ZP(A) = P(A) \cong k^{(N)}$. For (2) we use Theorem 3.7 again and the fact that $\text{Aut}_{\text{Hopf}}(A) = \{ u_\alpha \mid \alpha \in k^* \}$, where $u_\alpha$, for $\alpha \in k^*$, is the automorphism given by $u_\alpha(Y) =\alpha Y$. Thus, if $(\alpha_i)_{i \geq 0}, (\beta_i)_{i \geq 0} \in k^{(N)}$ then $\sum_{i \geq 0} \alpha_i Y^{p^i} \equiv \sum_{i \geq 0} \beta_i Y^{p^i}$ if and only if there exists a pair $(\alpha, \beta) \in k^* \times k^*$ such that $u_\alpha(\sum_{i \geq 0} \alpha_i Y^{p^i}) = \beta^2 \sum_{i \geq 0} \beta_i Y^{p^i}$. Since the equality holds if and only if $\alpha^p \alpha_i = \beta^2 \beta_i$, for all $i \geq 0$, we obtain (2). \[\square\]
Corollary 4.4. Let $k$ be a field of characteristic $p \geq 3$ and $(e_j)_{j \geq 0}$ the canonical $k$-basis of $k^{(N)}$, i.e. $e_j = (\delta_{ij})_{i \geq 0}$, where $\delta_{ij}$ is Kronecker’s delta. Then $A(e_j)$, $j \geq 0$ defined by (35) is an infinite family of non-isomorphic Hopf algebras, i.e. $\text{CRP}(H_4, k[Y])$ is an infinite set.

We now consider $A := k\langle y \mid y^p = 0 \rangle$, the $p$-dimensional Hopf algebra generated by a primitive element $y$.

Proposition 4.5. Let $k$ be a field of characteristic $p \geq 3$ and $A := k\langle y \mid y^p = 0 \rangle$ the $p$-dimensional Hopf algebra generated by a primitive element $y$. Then:

1. Up to an isomorphism of Hopf algebras there exist exactly two crossed products of Hopf algebras $A\#H_4$: $A \otimes H_4$ and $A_{4p}$, where $A_{4p}$ is the $4p$-dimensional Hopf algebra generated by $g$ and $x$ subject to the relations:
   \[ x^{2p} = 0, \quad g^2 = 1, \quad xg = -gx \]  
   and with the coalgebra structure given such that $g$ is a group-like element and $x$ is $(1, g)$-primitive. Furthermore, we have the following isomorphisms of groups:
   \[ \text{Aut}_H(A \otimes H_4) \simeq k^* \times k^* \quad \text{and} \quad \text{Aut}_H(A_{4p}) \simeq k^* \]

2. There exists a bijection $\mathcal{H}^2(H_4, A) \cong k$.

Proof. The proofs of (2) follow exactly like the ones in Proposition 4.2. For (1) we only remark that, if $q \in k^*$ then $A(qy)$ is the $4p$-dimensional quantum group generated by $y$, $g$ and $x$, subject to the relations
   \[ y^p = 0, \quad g^2 = 1, \quad x^2 = qy, \quad xg = -gx, \quad gy = yg, \quad xy = yx \]
which can be reduced to the ones in (36). \qed

Finally, we consider $A := k\langle y \mid y^p = y \rangle$, the $p$-dimensional semi-simple Hopf algebra generated by a primitive element $y$. For a positive integer $d$ we denote by $U_d(k)$ the set of $d$-th roots of unity in $k$.

Proposition 4.6. Let $k$ be a field of characteristic $p \geq 3$ and $A := k\langle y \mid y^p = y \rangle$ the $p$-dimensional Hopf algebra generated by a primitive element $y$. Then:

1. A Hopf algebra $E$ is isomorphic to a crossed product $A\#H_4$ if and only if $E \cong A \otimes H_4$ or $E \cong A(q)$, for some $q \in k^*$, where $A(q)$ is the $4p$-dimensional quantum group generated by $g$ and $x$ subject to the following relations
   \[ g^2 = 1, \quad x^{2p} = q^{p-1}x^2, \quad xg = -gx \]  
   and with the coalgebra structure given such that $g$ is a group-like element and $x$ is $(1, g)$-primitive. Furthermore, we have the following isomorphisms of groups:
   \[ \text{Aut}_H(A \otimes H_4) \cong \mathbb{F}_p^* \times k^* \quad \text{and} \quad \text{Aut}_H(A(q)) \cong U_{2(p-1)}(k) \]
   for all $q \in k^*$.

2. There exists a bijection $\text{CRP}(H_4, A) \cong k/\sim$, where $\sim$ is the equivalence relation on $k$ defined by: $q \sim q'$ if and only if there exists $(\alpha, \beta) \in \mathbb{F}_p^* \times k^*$ such that $\alpha q = \beta^2 q'$.

3. There exists a bijection $\mathcal{H}^2(H_4, A) \cong k$. 
\textbf{Proof.} For (1) we apply Corollary \ref{cor:isomorphic}. If \( q \neq 0 \) then \( A_{(qy)} \) is generated by \( y, g \) and \( x \) subject to the relations
\begin{align*}
y^p &= y, \quad g^2 = 1, \quad x^2 = qy, \quad xy = -yx, \quad yx = yg
\end{align*}
Since \( y = q^{-1}x^2 \), the independent relations are the ones in (\ref{eq:relations}). For the automorphism groups we use Remark \ref{rem:isomorphic} and Lemma \ref{lem:isomorphic}. We have:
\[ \text{Aut}_{\text{Hopf}}(A \otimes H_4) \simeq \text{Aut}_{\text{Hopf}}(A) \times M \simeq \mathbb{F}_p^* \times \mathbb{Z} \]
and
\[ \text{Aut}_{\text{Hopf}}(A_{(q)}) \simeq G(qy) = \{(u, \beta) \in \text{Aut}_{\text{Hopf}}(A) \times \mathbb{F}_p^* \mid u(qy) = \beta^2 qy\} \simeq U_2(p-1)(k) \]
where the isomorphism \( G(y) \simeq U_2(p-1)(k) \) is given by \( G(qy) \equiv (u, \beta) \mapsto \beta \in U_2(p-1)(k) \).

We now use Theorem \ref{thm:classification} for the classification part. From Lemma \ref{lem:classification} we have \( \mathcal{Z}_P(A) = P(A) = \{qy \mid q \in \mathbb{K}\} \simeq k \), hence our claim in (3). We look now at the equivalence relations \( \equiv \) on \( \mathcal{Z}_P(A) \). Let \( q, q' \in k \). Then \( qy \equiv q'y \) if and only if there exists a pair \((u, \beta) \in \text{Aut}_{\text{Hopf}}(A) \times k^*\) such that \( u(qy) = \beta^2 q'y \). Following the description of \( \text{Aut}_{\text{Hopf}}(A) \) in Lemma \ref{lem:automorphisms} \( qy \equiv q'y \) if and only if there exists a pair \((\alpha, \beta) \in \mathbb{F}_p^* \times k^*\) such that \( \alpha q = \beta^2 q' \). This proves (2). \( \square \)

Proposition \ref{prop:classification} proves that the number of types of isomorphism of all crossed products \( k(y \mid y^p = y) \# H_4 \) depends heavily on the base field \( k \). We illustrate this by the following:

\textbf{Corollary 4.7.} Let \( p \geq 3 \) be a prime number and \( k = \mathbb{F}_p(X_1, X_2, \ldots, X_n, \ldots) \) the field of rational functions in indeterminates \( \{X_i\}_{i \geq 1} \) over the finite field \( \mathbb{F}_p \). Then \( A(X_i), i \geq 1 \), constructed in (\ref{eq:construction}) is an infinite family of non-isomorphic 4p-dimensional Hopf algebras.

\textbf{Proof.} Let \( i \) and \( j \) be two positive integers. Then \( A(X_i) \simeq A(X_j) \) as Hopf algebras if and only if \( X_i \sim X_j \). We claim that this is not the case if \( i \) and \( j \) are distinct. Indeed, suppose \( i \neq j \) and \( X_i \sim X_j \). Then, by Proposition \ref{prop:isomorphic} (2), there exists a pair \((\alpha, \beta) \in \mathbb{F}_p^* \times k^*\) such that \( \alpha X_i = \beta^2 X_j \). Let \( n \geq 1 \) and \( P, Q \in \mathbb{F}_p[X_1, \ldots, X_n] \) such that \( \beta = \frac{P}{Q} \). Then \( \alpha X_i Q^2 = X_j P^2 \). Considering both members of the equality as polynomials in \( X_i \) we obtain a contradiction, since the degree of \( \alpha X_i Q^2 \) is odd, while the degree of \( X_j P^2 \) is even. Thus, \( A(X_i) \not\simeq A(X_j), \) for any \( i \neq j \geq 1 \). \( \square \)

\textbf{Acknowledgment.} We would like to thank to the referee for his/her comments that substantially improved the first version of this paper.

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