Control design based on multi-index nonlinear control method

Zhaolong Zhao*, Minghui Huang, Yibao Li

School of Mechanical and Electrical Engineering, Central South University, Changsha, China

*Corresponding author e-mail:133701008@csu.edu.cn

Abstract. Firstly, a nonlinear control system based on the Riemannian manifold is defined from the perspective of holism, and the expression of the system's state equation in the local coordinate system of the Riemannian manifold is given. The geometric structure of the Riemannian manifold is discussed. The influence of nonlinear systems is studied. The controllability and observability of nonlinear systems is studied. Secondly, using the properties of involute distribution and fully geodesic submanifolds, the local parts of nonlinear systems based on Riemannian manifolds are given. Controllable structured decomposition, local observable structure decomposition and Kalman decomposition. Thirdly, using the properties of orthogonal involute distribution family, incremental involute distribution family and fully geodesic submanifold family, the research is based on Riemannian manifolds. The parallel decoupling problem and cascade decoupling problem of nonlinear control systems on the above, as well as the local disturbance decoupling problem of affine nonlinear control systems.

Keywords: Nonlinear system; Riemannian manifold; local coordinate representation; Kalman de-composition

1. Introduction.
The geometric theory of control systems is an important aspect of nonlinear control research. With the in-depth development of nonlinear control system research, people gradually realize that the geometric structure of the state space has an effect on nonlinear control. The precise control of the system has an important impact, such as a large-range aircraft system flying at a large angle of attack and a deep space detector system. In recent years, due to the wide application of differential geometry methods in the control theory of nonlinear systems, people can make in-depth discussions on the internal relationship between the geometric structure of the state space and the nonlinear control system. From the perspective of mechanical systems, related control problems, tracing problems of precise mechanical systems, and dynamics and control problems of non-precision mechanical systems are studied[1]. This discussion provides a clearer background to the actual problems and is easy to understand. However, due to mechanical Due to the limitation of the theorem, this research method is often trapped in a small local area to deal with problems, and the holistic thinking method of modern geometry is difficult to function in a systematic and in-depth manner. Some researchers have discussed nonlinear control systems based on general fiber bundles. The various characteristics of the system structure are discussed, but this method is not intuitive enough. Although the influence of the fiber space structure of the nonlinear
control system is discussed, the influence of the geometric structure of the bottom manifold on the nonlinear control system is not studied.

After the 1990s, the geometric method research on nonlinear control systems has weakened. Although differential algebraic methods are involved, the progress has not been obvious. Some well-known geometric cybernetics scholars have turned to the study of feedback stabilization and various quality designs of nonlinear systems. Some new control methods are involved, such as $H_{\infty}$ control, adaptive control, etc, and many research results have appeared, but there is still a lack of deep understanding and understanding of the overall and global problems of nonlinear control systems. With the development of this theory, we found that the control systems studied by this theory seldom directly discuss the influence of the geometric structure of the state space on the system. This is because the state equation of the system does not introduce quantities related to the geometric structure of the state space. In this way, on the one hand, the research on the control system is very inaccurate [2]. On the manifolds of different geometric structures, the equation of state of the system is the same; on the other hand, the powerful tools of geometry cannot fully play a role in the research of control systems. Therefore, we must The method establishes a direct connection between the geometric structure of the state space and the state equation of the control system.

In this paper, a smooth Riemannian manifold is selected as the state space of the system. In response to the above problems, according to the characteristics of the Riemannian manifold, we define a nonlinear control system based on the Riemannian manifold from the perspective of holism. Take the smooth tangent on the manifold. The vector field is the state vector field of the control system, and the Riemannian manifold is connected with the state vector field through the auxiliary curve. This has three advantages: First, the connection coefficient term related to the geometric structure of the Riemannian manifold is introduced in the state equation representation of the nonlinear system given in the local coordinate system of the manifold, and the flow can be directly studied through this connection. The influence of the geometric structure of the shape of the nonlinear control system [3]. Second, when the Riemannian manifold is a flat Euclidean space, the connection coefficients are all zero, and the nonlinear control system on the Riemannian manifold we define is different from the usual Euclidean space. The nonlinear control system is consistent. In addition, the Riemannian manifold has a metric structure, which can describe the orthogonality of the tangent vector field, so that the reasoning, demonstration, and calculation of the control system can be simplified. Third, there are various fields in physics and mechanics, such as flow fields, Electric field, magnetic field, gravitational field, gauge field, etc, to study nonlinear systems with smooth tangent vector field as the state vector field, not only has important theoretical significance in mathematics, but also can provide for the in-depth study of corresponding problems in mechanics and physics. Precise language and powerful tools[4].

Below, for the nonlinear control system based on the Riemannian manifold, the relevant research results are introduced from three aspects: the local coordinate representation of the state equation, the local controllability, the decomposition of the observable structure, and various decoupling problems.

2. The local representation of nonlinear control system on Riemannian manifold

The concepts and definitions of mathematics and cybernetics cited in this article, let $(M, G)$ be a m-dimensional smooth Riemannian manifold, $G$ is the Riemannian metric on $M$, and $D$ is the Levi-Civita connection of the metric $G$ on $M[5]$. TM is the tangent bundle of $M$, and $\Gamma (TM)$ is the set of smooth sections of the tangent bundle TM. Consider the nonlinear control system on $M$.

$$\begin{align*}
\frac{dx(t)}{dt} &= f(X(t), u(t)) \\
Y(t) &= h(X(t))
\end{align*}$$

Among them, the state vector field $X(t)$ is the smooth tangent vector field on the Riemannian manifold $M$, $X(t) \in \Gamma(TM)$, the allowable control vector $u(t) \in R^n$. $R^n$ is a smooth vector function, $f: \Gamma(TM) \times R^n - \Gamma(TM)$ and $h: \Gamma(TM) - R^r$ are both smooth maps[5 6].

For $X(t)$ as the state vector field on $M$, if there is an embedded curve $\gamma$ on $M$; $[a, b] - M$, and its image set $\gamma([a, b])$ is contained in a coordinate of $M$ In the neighborhood $(U, x_i)$, such that $X(t) = X(\gamma$
system is shown. In order to further understand the nonlinear system, the study on the state vector field $X(t)$ and the study on the state curve $\gamma(t)$ of $X(t)$ can be equivalent. In fact, if the state vector field $X(t)$ is known, the integral curve of $X(t)$ is the state curve $\gamma(t)$. On the other hand, the overall The powerful tools of differential geometry can work through this connection[8].

Equations (3) and (4) are non-linear ordinary differential equations. According to the theory of ordinary differential equations, given the initial value, there will always be a unique solution that satisfies the initial value.

Equations (3) and (4) contain the connection coefficients of Levi-Civita $\gamma(t)$ is the state curve of the state vector field $X(t)$. Therefore, in the local coordinate system $(U, x_i)$, the state equation of the nonlinear system (1) can be expressed as:

$$\frac{dx(t)}{dt} + \sum_{i,j=1}^{m} L^j_i\gamma(t)X^i(t)\frac{dy^j}{dt} = f^k(X(t), \gamma(t), u(t)), \quad k = 1, 2, \ldots, m$$

This article focuses on the study of the restriction and influence of the geometric structure of the state manifold on the nonlinear control system based on it. In order to further understood the nonlinear system based on the Riemannian manifold is related to the internal structure of the Riemannian manifold. The representation of the nonlinear control system based on three spatial forms: Euclidean space, unit surface, and hyperbolic space in the local coordinate system is given, and the effect of the curvature structure of the Riemannian manifold on the nonlinear system is shown [9]. The representation of the nonlinear control system built on the cylindrical surface and the cone surface in the local coordinated system is presented, and the influence of the special curved surface structured and the singular structure of the nonlinear control system is shown.
3. The problems of controllability and observability decomposition for nonlinear control systems

3.1. The local controllability decomposition of structure

Below, we use the local orthogonality of the Riemannian metric on the Riemannian manifold and the properties of the involute distribution and the fully geodesic submanifolds to prove the local controllable structure decomposition theorem of the nonlinear system based on the Riemannian manifold. For convenience, the specified index change range is as follows: \( A, B, C = 1, 2, \ldots, m \); \( i, j, k = 1, 2, \ldots, d \); \( \alpha, \beta, \sigma = d + 1, d + 2, \ldots, m \).

\[
\begin{align*}
\frac{dx^k(t)}{dt} + \sum_{i,j=1}^{d} F^F_{ij}(y(t))x^i(t) \frac{dy^j}{dt} + \sum_{\alpha,\beta=d+1}^{m} \Gamma^*_{\alpha\beta}(y(t))x^\alpha(t) \frac{dy^\beta}{dt} = \\
f^k(X(t), y(t), u(t)), \quad k = 1, 2, \ldots, d, \\
\frac{dx^\sigma(t)}{dt} + \sum_{\alpha,\beta=d+1}^{m} \Gamma^\sigma_{\alpha\beta}(y(t))x^\alpha(t) \frac{dy^\beta}{dt} = 0
\end{align*}
\]

(5)

We noticed that if the Riemannian manifold \( M \) is regarded as a configuration manifold, \( y(t) = (y_1(t), y_2(t), \ldots, y_m(t)) \in M \) is regarded as a configuration, then the system of equations (5) describes a class of inexact mechanical systems. The equations (5) can be analyzed and discussed using relevant research methods of the literature. Inexact mechanical systems are often encountered in practice, such as flexible robot control systems, satellites attitude control systems, redundant mechanical control systems, etc.

The local observability decomposition of structure and local Kalman decompositions

Let \( \Delta c \) be a non-singular involute distribution on the Riemannian manifold \( M \), and the nonlinear system (2.1) for any allowable control \( u(t) \), has \( f(X(t), u(t)) \in \Delta c \). \( \Delta u \) is the largest non-singular involute distribution contained in \( \text{Ker}(h) \) on \( M \), and the nonlinear system is invariant to any allowable control \( u(t), \Delta u = f(X(t), u(t)) \). If for any point \( p \in M \), the integral submanifolds of \( \Delta c \cap \Delta u \) and \( (\Delta c + \Delta u) \) are all fully geodesic submanifolds of \( M \) under the induced connection of \( M \). Then, there exists \( M \) A local coordinate neighborhood \((W, zA)\) of the upper point \( p \) makes the nonlinear system have the following Kalman decomposition representation:

\[
\begin{align*}
&f^1_2(X(t), y(t), u(t)) \\
f^2_2(X(t), y(t), u(t)) \\
\frac{dx^1_k}{dt} + \sum_{\lambda=2}^{4} \sum_{\gamma,j} dx^\gamma_j + \sum_{l,j=2}^{d} F_{i,j\lambda} \\
y(t) = h(X(t))
\end{align*}
\]

(6)

4. The decoupling problems of nonlinear control systems on Riemannian manifold

For many years, the various decoupling problems of the system have been one of the important contents of the geometric theory research of linear and nonlinear control systems. This is not only because the method and design of the research system decoupling problem allow us to study the control system Turn the complex about simple, and make it easier. More importantly, the decoupling problem is the link between the structure of the control system and the state space structure of the system. The study of the decoupling problem reveals the essential structure of the control system. This section studies the foundation. The decoupling problem of nonlinear systems on Riemannian manifolds. Using the properties of orthogonal involute distribution families and fully geodesic submanifold families, the non-feedback parallel solutions to the state equations of nonlinear control systems based on Riemannian manifolds are studied [10]. The coupling problem and the state feedback parallel decoupling problem of the nonlinear control system. Using the properties of the increasing involute distribution family and the fully geodesic submanifold family, the non-feedback cascade of the state equations of the nonlinear
control system based on Riemannian manifolds is studied. The decoupling problem and the state feedback cascade decoupling problem of the nonlinear control system. As an application of the state feedback decoupling problem, the local disturbance decoupling problem of the affine nonlinear control system based on the Riemannian manifold is studied.

4.1. The cascade decomposition problem of state equation
Using the properties of the increasing involutive distribution family and the increasing full geodesic submanifold family, the problem of non-feedback cascade decoupling of the state equation of the nonlinear control system based on Riemannian manifolds is studied [11].

4.2. The state feedback decoupling problem of nonlinear control system
By discussing the decoupling problem of nonlinear systems. If the nonlinear system has parallel (cascade) decoupling, then the nonlinear system has state feedback parallel (cascade) decoupling [12]. It can be proved that the nonlinear system state feedback parallel decoupling theorem.

4.3. The local disturbance decoupling problem of affine nonlinear control system
As an application of state feedback decoupling of nonlinear systems based on Riemannian manifolds, we will discuss the problem of local disturbance decoupling of affine nonlinear control systems based on Riemannian manifolds.

5. Conclusion
This paper studies the geometric theory of nonlinear control systems based on Riemannian manifolds. In order to study the influence of the geometric structure of Riemannian manifolds on nonlinear control systems, we define a kind of nonlinear control system based on Riemannian manifolds from the point of view of integration. For nonlinear control system, the expression of the state equation of the nonlinear control system in the local coordinatized system of the Riemannian manifold is given. This establishes a direct connection between the state equation of the nonlinear control system and the geometric structure of the Riemannian manifold. Starting from this, we have studied the Riemannian manifold structure and connection structure of the state space from several aspects such as the controllability, observability, controllability, observable structure decomposition of the nonlinear system, and various decoupling problems of the control system. The influence of, curvature structure and submanifold structure on nonlinear control system.

These theoretical studies enable us to have a deeper understanding and understanding of the overall nonlinear control system. We know that general finite-dimensional differential manifolds are always established on the background of Euclidean space. The famous Whitney theorem tells us that a differential manifold can always be embedded in a sufficiently high-dimensional Euclidean space as a submanifold. This kind of differential manifold that starts in Euclidean space and converges on Euclidean space are the overall differential geometry and large-scale analytical research provides a wide range of markets. Therefore, for the theoretical framework of nonlinear control systems based on general differential manifolds, in-depth research on various algorithms and designs will be an important aspect of the development of modern control theory one.

References
[1] Mixed H ∞ and passive control for a class of nonlinear switched systems with average dwell time via hybrid control approach[J]. Qunxian Zheng, Hongbin Zhang, Youzhu Ling, Xingzhong Guo. Journal of the Franklin Institute. 2018 (3)
[2] Nonlinear Controllability and Optimal Control. Sussmann H J. Marcel Dekker. 1990
[3] Control and Decoupling of Nonlinear Systems. Xia X H and Gao W B. Science Press. 1993
[4] Some developments and perspective of modern control theory. Chen H F and Guo L. Science Bulletin. 1998
[5] Dynamics and control of a class of underactuated mechanical systems. Reyhanoglu M, Van der
Schaft A J, Mc Clamroch N H and Kolmanovsky I. IEEE Transactions on Automatic Control. 1999

[6] Nonlinear systems and differential geometry. Brockett R W. Proc. of IEEE. 1976
[7] Local disturbance decoupling with stability for nonlinear systems. Van der Wegen L L M.. 1990
[8] Analysis and Design of Nonlinear Control Systems (2nd). Feng B C and Fei S M. Electronic Industrial Press. 1998
[9] Configuration flatness of lagrangian systems underactuated by one control. Rathinam M and Murray R M. The SIAM Journal on Control and Optimization. 1998

[10] Introduction of Differential Manifold. Chen W H. Advanced Education Press. 1998

[11] Adaptive Neural State-Feedback Tracking Control of Stochastic Nonlinear Switched Systems: An Average Dwell-Time Method. [J]. Niu Ben, Wang Ding,Alotaibi Naif D,Alsaadi Fuad E. IEEE transactions on neural networks and learning systems. 2019 (4)

[12] Stability Analysis of Switched Positive Nonlinear Systems by Mode-Dependent Average Dwell Time Method.[J]. Yazhou Tian, Yuangong Sun. IEEE Access. 2019

[13] Multiple Lyapunov Functions for Adaptive Neural Tracking Control of Switched Nonlinear Nonlower-Triangular Systems[J]. Niu Ben, Liu Yanjun,Zhou Wanlu,Li Haitao,Duan Peiyong,Li Junqing. IEEE Transactions on Cybernetics. 2019 (5)