On a local mass dimension one Fermi field of spin one-half and the theoretical crevice that allows it

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Since the 1928 seminal work of Dirac, and its subsequent development by Weinberg, a view is held that there is a unique Fermi field of spin one-half. It is endowed with mass dimension three-half. Combined, these characteristics profoundly affect the phenomenology of the high energy physics, astrophysics, and cosmology. We here present a counter example by providing a local, mass dimension one, Fermi field of spin one-half. The theory, inter alia, thus allows dimensionless quartic self interaction for the new fermions, and its only other dimensionless coupling is quadratic in the new fermions and in the standard-model scalar field. For these reasons, the immediate application of the new theory resides in the dark-matter sector of physical reality. The lowest-mass associated new particle may leave its unique signature at the Large Hadron Collider. We discuss in detail the theoretical crevice that allows the existence of the new quantum field.

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I. INTRODUCTION AND BACKGROUND

To report the existence of a local spin one-half fermion field with mass dimension one is tantamount to claiming an element of incompleteness in our knowledge of quantum fields at a basic level. If true, it would, for example, allow dimensionless couplings of the type $\tilde{f}(x)f(x)b^\dagger(x)b(x)$; where $f(x)$ and $b(x)$ are spin one-half fermionic and spin-zero bosonic fields respectively, and $\tilde{f}(x)$ is an appropriate adjoint. That the existence of such a field would have escaped even so careful an analysis as that of Weinberg’s indicates that there is either something non-trivial, or something non-trivially wrong, with such a construct. Tentatively, we assume the former and narrate the circumstances under which the claimed new quantum field came to exist and then proceed to
systematically present the construct. Because of the nature of the claim we make an effort to be explicit about every known element that may affect our results. This approach serves the dual purpose of making the presentation pedagogic and to make it more accessible to scrutiny.

The rest of this section is presented in the first person singular. This departure from the convention seems necessitated by the subject at hand.

During the years 1992-98 I was surrounded by experimental physicists at Los Alamos Meson Physics Facility (LAMPF) on the one hand and theoretical colleagues at the Theory Division of the Los Alamos National Laboratory on the other. Initially my interests were what I later called ‘mathematical science fiction’ (Ahluwalia, 1999) but because of the new results that were emerging from the neutrino experiments my colleagues informally encouraged me to explore Majorana neutrinos. The subject of Majorana field and Majorana spinors was confusing, and somewhat of a mess. And what had one to do with the other, I asked. This realization arose in a conversation with Peter Herczeg.

I asked the library to get Majorana’s 1937 paper (Majorana, 1937) translated into English, and two weeks later a professional translation was in my office. I found that in the standard language Majorana started with the Dirac field and then identified \( b^\dagger (p^\mu, \sigma) \) with \( a^\dagger (p^\mu, \sigma) \). An intrinsically neutral field was thus introduced for the first time. There was no mention of any new spinors.

My first exposure to Majorana spinors came through two papers (Case, 1957; McLennan, 1957) that appeared some twenty years after Majorana’s original paper on the subject. At the same time I found that in their Grassmannian incarnation, in Ramond’s primer (Ramond, 1988). My personal exploration of c-number Majorana spinors began with an observation of Ramond on the ‘magic of Pauli matrices’ and how it resulted in the existence of Majorana spinors (Ramond, 1988, Section 1.4). That ‘magic’ confined the Majorana spinors to spin one-half and concealed some of their real content. The realization that Ramond’s argument could be readily generalized to higher spins if the said magic was, instead, associated with the Wigner time reversal operator, led to the writing of two exploratory papers (Ahluwalia, 1996; Ahluwalia et al., 1994a) and other presentations; see, for example, (Ahluwalia, 2003; Ahluwalia et al., 1994a). The notion of a complete set of eigenspinors of the spin one-half charge conjugation operator, which was later dubbed Elko in (Ahluwalia-Khalilova and Grumiller, 2005a), originated in those early papers.

In the fall of 1998 I left Los Alamos National Laboratory to join the Universidad Autónoma de Zacatecas, México. There, on his way from MIT to Leipzig, Daniel Grumiller came for a short visit and it resulted in an unexpected collaboration. Preprint (Ahluwalia, 2003) was our starting point, and we now asked: What are the properties of a quantum field constructed with Elko as its expansion coefficients. Had any one of us fully appreciated Weinberg’s work of the sixties (Weinberg, 1964, 1965), or his later monograph (Weinberg, 1995), we would have never dared to ask such a question. So with certain element of innocence, and ignorance, we two wrote a paper which opened with the line (Ahluwalia-Khalilova and Grumiller, 2005a), “we report an unexpected theoretical discovery of a spin one-half matter field with mass dimension one,” soon to be followed by another paper that opened similarly (Ahluwalia-Khalilova and Grumiller, 2005b), “we provide the first details on the unexpected theoretical discovery of a spin-one-half matter field with mass dimension one.” Our excitement was quite apparent! It was also very clear that the new quantum field provided a very natural dark matter candidate with a quartic self interaction, and it coupled to EB-GHK-H scalar field (Englert and Brout, 1964; Guralnik et al., 1964; Higgs, 1964) but its interactions with other standard-model fields was suppressed.

However, when the locality-determining anticommutators were calculated this fermion field of spin one-half exhibited non-locality.

In the middle of 2006, I moved to the University of Canterbury in New Zealand. There, with my ever cheerful and hard working students, we learned, to our collective surprise, that the Dirac quantum field as presented in many, though not all, textbooks did not transform properly under Poincaré space-time transformations! Not only that, when we identified (in the usual notation), \( b^\dagger (p^\mu, \sigma) \) with \( a^\dagger (p^\mu, \sigma) \) a la Majorana’s 1937 paper we discovered that the resulting field exhibited nonlocal anticommutators! The fields presented in the monographs of Weinberg and Srednicki, on the other hand, were completely free from such inconsistencies (Srednicki, 2007; Weinberg, 1995).

With the insights gained from these monographs we discovered the culprits: there is a freedom of certain global phases associated with each of the expansion coefficients, and this freedom affects various properties of the quantum field; and designation also matters. Gradual understanding of these elements led to a much improved locality structure for the Elko-based

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1 Which since then has undergone several changes in its mission and its name.

2 Cheng-Yang Lee, Dimitri Schritt, Tom Watson, and later Sebastian Horvath

3 That is, what one calls \( v_+ (p^\mu) \) or \( v_- (p^\mu) \) matters. With a similar remark for Elko.
quantum field (Ahluwalia et al., 2010, 2011), but non-locality still remained stubborn and showed up in one additive integral. Then in the 2012-2013 period I reached Instituto de Matemática, Estatística e Computação Científica (IMECC), Brasil, for a sabbatical year, and the last hurdle evaporated. Under the IMECC expertise, the mischievous integral magically evaluated to zero (de Oliveira and Rodrigues, 2012).

It was also towards the end of my stay at University of Canterbury that a new connection with the Very Special Relativity (VSR) began to emerge (Ahluwalia and Horvath, 2010; Cohen and Glashow, 2006b). Again, the mathematical expertise at IMECC, and other Brazilian institutes, helped me to better understand the symmetry properties of the new field.

What follows is a detailed crystallization of these insights. The ‘Abstract’ above, and the opening paragraph of this section, places the mass dimension one Fermi field in the historical context and briefly describes the outcome of this research. The ‘Contents’ serve as a brief summary of the general flow of the paper: in II below resides the necessary background for constructing the local mass dimension one Fermi field of spin one-half in III.

II. COEFFICIENT FUNCTIONS FOR A LOCAL MASS DIMENSION ONE FERMI FIELD OF SPIN ONE-HALF

A. Parity, Charge conjugation, and Elko: A review

We begin our exposition by setting up the notation. The right- and left-handed Weyl spinors transform under Lorentz boost as

\[ \phi_R(p^\mu) = \exp\left(\frac{\sigma_2}{2} \cdot \phi \right) \phi_R(k^\mu) \] (1a)

\[ \phi_L(p^\mu) = \exp\left(-\frac{\sigma_2}{2} \cdot \phi \right) \phi_L(k^\mu) \] (1b)

The boost parameter \( \phi \) is defined so that \( \exp(iK \cdot \phi) \) acting on the standard four momentum

\[ k^\mu \overset{\text{def}}{=} (m, \lim_{p \to 0} \frac{p}{p}) \], \( p = \lvert p \rvert \).

equals the general four momentum

\[ p^\mu = (E, p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta) \]. (3)

This yields \( \cosh \phi = E/m \), \( \sinh \phi = p/m \) and \( \tilde{\phi} = \hat{p} \).

\( K \) are the \( 4 \times 4 \) matrices for the generators of boosts (in the vector representation) \( 6 \). Equations (1a) and (1b) follow from the fact that \( -i\sigma/2 \) are the generators of the boosts for the right-handed Weyl representation space, while \( +i\sigma/2 \) are for the left-handed Weyl representation space. For the direct sum of the right- and left-Weyl representation spaces, to be motivated below, the boost generator thus reads

\[ \kappa = \begin{pmatrix} -i\sigma/2 & 0 \\ 0 & +i\sigma/2 \end{pmatrix} \]. (4)

1. Parity

It is immediately clear from the transformations (1a) and (1b) that parity, which in the vector space corresponds to \( P : x^\mu = (x^0, \mathbf{x}) \rightarrow (x^0, -\mathbf{x}) \), interchanges the right- and left-handed Weyl representation spaces. Thus the operation of parity, up to a global phase, for the 4-component spinors

\[ \psi(p^\mu) = \begin{pmatrix} \phi_R(p^\mu) \\ \phi_L(p^\mu) \end{pmatrix} = \exp(iK \cdot \phi)\psi(k^\mu) \] (5)

must contain purely off-diagonal \( 2 \times 2 \) identity matrices 1, and in addition an operation that implements the action of \( P \) on \( p^\mu \). Up to a global phase, it is thus defined as

\[ \mathcal{P} \psi(p^\mu) \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi(p^\mu) = \gamma_0 \psi(p^\mu). \] (6)

Here, \( p^\mu \) is the \( P \) transformed \( p^\mu \); and 0 is a \( 2 \times 2 \) null matrix. Now \( \psi(p^\mu) \) may be related to \( \psi(p^\mu) \) as follows

\[ \psi(p^\mu) = \exp(-iK \cdot \phi)\psi(k^\mu) \]

\[ = \exp(-iK \cdot \phi) \exp(-iK \cdot \phi)\psi(p^\mu) \]

\[ = \exp(-2iK \cdot \phi)\psi(p^\mu). \] (7)

Substituting (4) in (6), and on using the anti-commutativity of \( \gamma_0 \) with each of the generators of the boost, \( \{\gamma_0, \kappa_i\} = 0 \), with \( i = x, y, z \), (6) becomes

\[ \mathcal{P}\psi(p^\mu) = \exp(2iK \cdot \phi)\gamma_0 \psi(p^\mu) \] (8)

A direct evaluation of the exponential in (8) gives

\[ \exp(2iK \cdot \phi) = m^{-1} \gamma_\mu p^\mu \gamma_0 \] (9)

where \( \gamma_\mu \) are the Dirac matrices in the Weyl representation. Finally, a substitution of the expansion (9) in (8), on combining with the identity \( \gamma_0^2 = 1_4 \) (where \( 1_4 \) is an

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4 The successes at IMECC, and post-earthquake academic turmoil of the University of Canterbury, made my sabbatical end in a transition to Brasil.

5 This definition allows us to introduce the helicity basis below. We have verified that the results reported here hold even if we do not work in the helicity basis. These alternate approaches require giving freedom of certain phases to \( \phi_L(k^\mu) \). A freedom that is ultimately constrained by working out the relevant equal-time anticommutators.

6 We shall use the conventions of Ryder, L H, 1985 with \( h \) and \( c \) set to unity.
identity matrix in the space of four-component spinors), results in
\[ \mathcal{P} \psi(p\mu) = m^{-1}\gamma_\mu p^\mu \psi(p\mu). \] (10)
Thus, we have the desired expression for the parity operator\[7\]
\[ \mathcal{P} = m^{-1}\gamma_\mu p^\mu. \] (11)
Its eigenvalues are \( \pm 1 \). Each of these has a two fold degeneracy
\[ \mathcal{P} \xi^S(p\mu) = +\xi^S(p\mu), \quad \mathcal{P} \xi^A(p\mu) = -\xi^A(p\mu). \] (12)
The superscripts refer to self and anti-self conjugacy of \( \xi(p\mu) \) under \( \mathcal{P} \). The Dirac’s \( u_\sigma(p\mu) \) and \( v_\sigma(p\mu) \) spinors are thus seen as the eigenspinors of the parity operator, \( \mathcal{P} \), with eigenvalues \( +1 \) and \( -1 \), respectively:
\[ \xi^S(p\mu) \rightarrow u_\sigma(p\mu), \quad \xi^A(p\mu) \rightarrow v_\sigma(p\mu), \] (13)
where the designation \( \sigma \) represents the degeneracy index. Seen from this perspective, with the help of (11), (12) translates to
\[ (\gamma_\mu p^\mu - m 1_4) u(p\mu) = 0, \quad (\gamma_\mu p^\mu + m 1_4) v(p\mu) = 0, \] (14)
which are the 1928 Dirac equations in the momentum space.

This stage still leaves the global phases associated with each of the Dirac spinors to be still free. They contain important elements of physics affecting locality and transformation properties of the single-particle states associated with the Dirac and the 1937 Majorana quantum fields \[8\] \[\text{Majorana, 1937}\] These can be read off from Weinberg’s construction of these fields \[\text{Weinberg, 1995}\].

Since the boost operator \( \exp(i\kappa \cdot \varphi) \) is not unitary, the Dirac spinors cannot in any sense be considered to represent quantum states. Instead, these enter as expansion coefficients in a quantum field which is specifically built to circumvent this problem.

In the configuration space the operator \( (i\gamma_\mu \partial^\mu - m 1_4) \) annihilates the Dirac and Majorana quantum fields, generically denoted by the same symbol \( \Psi(x) \). But, so does the spinorial Klein-Gordon operator. The question which determines that it is the former, and not the latter, that enters the Dirac/Majorana Lagrangian density is related to the structure of the \( \{ \xi(\{ x' \} \Psi(x)) \} \), where \( \xi \) is the usual time ordering operator. So the Lagrangian density is to be seen as a derived object after one has constructed a quantum field\[9\]. We shall work in this spirit.

Incorporating the four-component spinor \( \psi(p\mu) \) introduced in (10) is a necessary, but not sufficient, condition for preserving parity symmetry. We end this discussion of parity operator by noting that \( \mathcal{P}^2 = 1 \).

There exists an operator that transmutes the self-conjugate eigenspinors of the parity operator to the anti-self conjugates eigenspinors, and vice versa. This operator we discuss next, and arrive at the counterpart of (12) for this operator in \[23\] below. We shall discover that eigenspinors of this operator play a central role in constructing the new mass dimension one quantum field of spin one-half.

2. Charge conjugation operator and its eigenspinors (Elko)

Introduction of the four-component spinors \[5\], not necessarily as eigenspinors of the parity operator, doubles the degrees of freedom. In a quantum field theoretic formalism this doubling introduces the notion of antiparticles. The relative charges of the particles and antiparticles are then determined by the type of local gauge symmetries that the underlying kinematic framework supports. The particle-antiparticle symmetry enters via charge conjugation operator. For the four-component spinors, it may be similarly constructed as \( \mathcal{P} \) without first invoking a wave equation or a Lagrangian density. To see this we begin with the observation that the Wigner time reversal operator for spin one-half, \( \Theta \), acts on the Pauli matrices as follows\[10\]
\[ \Theta \sigma \Theta^{-1} = -\sigma^*, \] (15)
with
\[ \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \] (16)

It allows the following ‘magic’ to happen (cf. ‘magic of Pauli matrices’ in \[14\], Section 1.4). First complex conjugate \[13\] and \[14\], then multiply from the left by \( \Theta \), and use the above defining feature of the Wigner time reversal operator. This sequence of manipulations gives
\[ \Theta \phi^*_R(p\mu) = \exp \left( -\frac{\sigma}{2} \cdot \varphi \right) \Theta \phi^*_R(k\mu) \] (17a)
\[ \Theta \phi^*_L(p\mu) = \exp \left( -\frac{\sigma}{2} \cdot \varphi \right) \Theta \phi^*_L(k\mu). \] (17b)

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\[7\] In obtaining this result we have followed a recent work of Speranca \[\text{Speranca, 2013}\].

\[8\] Both of these fields carry Dirac’s \( u_\sigma(p\mu) \) and \( v_\sigma(p\mu) \) as their expansion coefficients.

\[9\] A reader who may not be familiar with this aspect of quantum field theory may wish to consult \[13\] below for an outline of this argument. This view is similar and consistent with the formalism presented in \[\text{Weinberg, 1995}\].

\[10\] For any spin, \( \Theta_{[j]} J \Theta_{[j]}^{-1} = -J^* \); with \( \Theta_{[j]} = (-1)^{1+i} \delta_{0} \cdot \sigma \) and \( \Theta_{[j]} \Theta_{[j]} = (-1)^2 \). For convenience, we abbreviate \( \Theta_{[1/2]} \) to \( \Theta \).
Note: multiplication by a phase does not affect the Lorentz transformation properties of the spinors.

That is, if \( \phi_L(p^\mu) \) transforms as a left-handed Weyl spinor then \( \zeta \Theta \phi_L^\ast(p^\mu) \) transforms as a right-handed Weyl spinor, where \( \zeta \) is an undetermined phase. Similarly, if \( \phi_R(p^\mu) \) transforms as a right-handed Weyl spinor then \( \zeta \rho \phi_R^\ast(p^\mu) \) transforms as a left-handed Weyl spinor, where \( \zeta \rho \) is an undetermined phase.

This crucial observation motivates the introduction of two sets of four-component spinors \cite{Ahluwalia1996}:

\[
\lambda(p^\mu) = \left( \begin{array}{c} \zeta \Theta \phi_L^\ast(p^\mu) \\ \phi_L(p^\mu) \end{array} \right)
\]

(18)

and

\[
\rho(p^\mu) = \left( \begin{array}{c} \phi_R(p^\mu) \\ \zeta \rho \phi_R^\ast(p^\mu) \end{array} \right).
\]

(19)

The \( \rho(p^\mu) \) do not provide an additional independent set of spinors from that in \cite{Ahluwalia1996} and for that reason we do not consider them further.

Generally, this result is introduced as a ‘magic of Pauli matrices’ \cite{Ramond1989}, Section 1.4) where \( \Theta \) gets concealed in Pauli’s \( \sigma_2 \), which equals \( i \Theta \). Our argument in terms of the Wigner time reversal operator \( \Theta \) has the advantage that it immediately generalizes to higher spins. Furthermore, the recognition that there is an element of freedom in the indicated phases, \( \zeta \), makes \( \lambda(p^\mu) \) escape their interpretation as Weyl spinors in a four-component disguise. It will become apparent below that we may now have four, rather than two, four-component spinors of the general form carried by \( \lambda(p^\mu) \).

With these observations at hand we are led to entertain the possibility that in addition to the symmetry operator \( P \), there may exist a second symmetry operator. Up to a global phase, it has the form

\[
\mathcal{C} \equiv \begin{pmatrix} 0 & \alpha \Theta \\ -\beta \Theta & 0 \end{pmatrix} K
\]

(20)

where \( K \) complex conjugates to its right. The arguments that lead to \cite{Ramond1989} are similar to the ones that give \cite{Ramond1989}. Requiring \( C^2 \) to be an identity operator determines \( \alpha = i, \beta = -i \); giving

\[
\mathcal{C} = \begin{pmatrix} 0 & i \Theta \\ -i \Theta & 0 \end{pmatrix} K = \gamma_2 K.
\]

(21)

There also exists a second solution with \( \alpha = -i, \beta = i \). But this does not result in a physically different operator and in any case the additional minus sign can be absorbed in the indicated global phase. This is the same operator that appears in the particle-antiparticle symmetry associated with the 1928 Dirac equation \cite{Dirac1928}.

We have thus arrived at the charge conjugation operator from the analysis of the symmetries of the 4-component representation space of spinors. This perspective has the advantage of immediate generalization to any spin: if \( \Theta \) is taken as a spin \( j \) Wigner time reversal operator in footnote \cite{Ramond1989} then the resulting \( \mathcal{C} \) becomes the charge conjugation operator in the \( 2(2j + 1) \) dimensional representation space.

To construct the eigenspinors of \( \mathcal{C} \) for spin one-half we act it on \( \lambda(p^\mu) \) and re-write the result as

\[
\mathcal{C} \lambda(p^\mu) = (i \zeta \lambda)^{-1} \left( -\zeta \Theta \phi_L^\ast(p^\mu) \right). \phi_L(p^\mu).
\]

(22)

A comparison of the right-hand side of the above expression with the definition \cite{Ramond1989} shows that \( \lambda(p^\mu) \) become eigenspinors of \( \mathcal{C} \) with eigenvalues \( (i \zeta \lambda)^{-1} \) if we demand \( -\zeta = \zeta \lambda \). This requirement translates to, \( \Re \{\zeta \lambda\} = 0 \), resulting in \( \zeta = \pm i \alpha \) with \( \alpha \in \mathbb{R} \).

In order that both the right- and left-handed components of \( \lambda(p^\mu) \) remain on the same footing we shall here onwards study the case where we set \( \alpha = 1 \). Each of the signs provides a doubly degenerate set of \( \lambda(p^\mu) \); and, additionally, these ensure that \( \lambda(p^\mu) \) do not become Weyl spinors in disguise. This discussion adds new insights to the self and anti-self conjugate spinors first introduced in \cite{Ahluwalia1996, Ahluwalia1994a, Ahluwalia-Khalilova and Grumiller2005a,b}.

\[
\lambda(p^\mu) = \begin{cases} \lambda^S(p^\mu) & \text{for } \zeta = +i \\ \lambda^A(p^\mu) & \text{for } \zeta = -i \end{cases}
\]

(23a)

with

\[
\mathcal{C} \lambda^S(p^\mu) = +\lambda^S(p^\mu), \quad \mathcal{C} \lambda^A(p^\mu) = -\lambda^A(p^\mu).
\]

(23b)

The physics literature identifies \( \lambda^S(p^\mu) \) in \cite{Ahluwalia-Khalilova and Grumiller2005a,b} with ‘Majorana spinors’ (often, as Grassmann numbers). The \( \lambda^A(p^\mu) \) seem to have been entirely overlooked.

A failure to construct a Lagrangian density using Majorana spinors is noted in \cite{Aitchison and Hey2004} and Appendix P). We shall see below that this only reflects that the considered Majorana spinors do not satisfy Dirac equation.

As Weinberg has emphasized in his monograph on the subject and in his other writings \cite{Weinberg2012}, the modern version of Dirac equation in quantum field theory has a much richer physics and is to be separated from its original motivations and interpretations confined to the finite dimensional representation space. It is in this latter sense, as expansion coefficients of a quantum field,
that we treat the four-component spinors in this communication. In a quantum field theoretic setting the inevitability of antiparticles is elegantly argued by Feynman in [Feynman and Weinberg, 1987].

In order to avoid confusion with the folklore on Majorana spinors we refer to $\lambda(p^\mu)$ as Elko (German acronym for Eigenspinoren des Ladungskonjugationsoperators first introduced in (Ahluwalia-Khalilova and Grumiller, 2005a,b)).

B. Global phase transformations for Elko

In general, a global unitary transformation of the type

$$\lambda(p^\mu) \rightarrow \lambda'(p^\mu) = \exp(i\varphi)\lambda(p^\mu)$$

(24)

with $\varphi = a + \vartheta \in \mathbb{R}$, does not preserve the self/anti-self conjugacy of $\lambda(p^\mu)$ under $\mathcal{L}$ given in (23a) unless the matrix $a$ satisfies the condition

$$\gamma_2 a^\ast + a \gamma_2 = 0$$

(25)

This is due to the presence of the operator $K$ in (21). The general form of $a$ satisfying these requirements is found to be

$$a = \begin{pmatrix} \alpha & \beta & \lambda & 0 \\ \beta & \delta & 0 & \lambda \\ \lambda & 0 & -\delta & \beta \\ 0 & \lambda & \beta & -\alpha \end{pmatrix}$$

(26)

$$= \lambda \gamma_0 + i \frac{(\alpha - \delta)}{4} [\gamma_1, \gamma_2] + \frac{i}{4} \beta [\gamma_2, \gamma_3] - \frac{1}{2} (\alpha + \delta) \gamma_5$$

(27)

with $\alpha, \beta, \lambda, \delta \in \mathbb{R}$ (with no association with the same symbols used elsewhere in this work) and $\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

In (21) below we will discover that certain symmetries of the spin sums require $\beta = \lambda = 0$, and $\delta = \alpha$. With the scale factor absorbed in $\vartheta$, $a$ reduces to $\gamma_5$.

The counterpart of $a$ for the Dirac case that preserves the self/anti-self conjugacy under $\mathcal{P}$, (12) as opposed to (23a), is simply a $4 \times 4$ identity matrix.

C. Explicit construction of Elko and on the choice of certain phases

To obtain an explicit form of Elko calls for a choice of the ‘rest’ spinors $\lambda(k^\mu)$ with $k^\mu$ defined in (2). That done, one then has for an arbitrary $p^\mu$

$$\lambda(p^\mu) = \exp(i\kappa \cdot \varphi) \lambda(k^\mu).$$

(28)

In principle, the boosted spinors reside in the boosted frames. But since no frame is a preferred frame they must also exist in all frames (an argument originally due to E. P. Wigner). It is this interpretation that we attach to $\lambda(p^\mu)$. With the generator of the boost, $\kappa$, defined in (20), the boost operator in (28) can be readily evaluated using $(\sigma \cdot \mathbf{p})^2 = 1$, to the effect that

$$\exp(i\kappa \cdot \varphi) = \begin{pmatrix} e^{(\sigma/2) \cdot \varphi} & 0 \\ 0 & e^{-(\sigma/2) \cdot \varphi} \end{pmatrix}$$

$$= \sqrt{\frac{E + m}{2m}} \begin{pmatrix} 1 + \frac{\sigma \cdot \mathbf{p}}{E + m} & 0 \\ 0 & 1 - \frac{\sigma \cdot \mathbf{p}}{E + m} \end{pmatrix}.$$  

(29)

To provide a concrete example of a mass dimension one quantum field, we confine our attention to the $\lambda(k^\mu)$-defining $\phi_L(k^\mu)$ as eigenspinors of $\sigma \cdot \mathbf{p}$

$$\sigma \cdot \mathbf{p} \phi_L^+ (k^\mu) = \pm \phi_L^+ (k^\mu)$$

(30)

with $\mathbf{p} = (\sin \vartheta \cos \phi, \sin \vartheta \sin \phi, \cos \vartheta)$. Furthermore, we adopt the phases given below

$$\phi_L^+ (k^\mu) = \sqrt{m} \begin{pmatrix} \cos(\vartheta/2) \exp(-i\phi/2) \\ \sin(\vartheta/2) \exp(+i\phi/2) \end{pmatrix}$$

(31a)

$$= \phi_L^+ (0)$$

(32a)

$$\phi_L^- (k^\mu) = \sqrt{m} \begin{pmatrix} -\sin(\vartheta/2) \exp(-i\phi/2) \\ \cos(\vartheta/2) \exp(+i\phi/2) \end{pmatrix}$$

(31b)

$$= -\phi_L^- (0)$$

(32b)

The abbreviation AG stands for (Ahluwalia-Khalilova and Grumiller, 2005a,b).

In writing the above ansatz, we have explicitly noted the differences from the most-often used earlier work.

There is a second choice of phases, and designations (that is, the indices $\lambda(k^\mu)$ are assigned), which is invoked when Elko are used as expansion coefficients of a quantum field. This choice we make explicit below in defining the $\lambda(k^\mu)$

$$\lambda^+_S (k^\mu) = \lambda^+_S (0)$$

(32a)

$$\lambda^-_S (k^\mu) = \lambda^-_S (0)$$

(32b)

The dictionary of comparison for (Ahluwalia, 1996) is the same as for AG modulo a minor change of notation: AG’s $\{ -, + \} \rightarrow \uparrow$ and $\{ +, - \} \rightarrow \downarrow$. 

13 The dictionary of comparison for (Ahluwalia, 1996) is the same as for AG modulo a minor change of notation: AG’s $\{ -, + \} \rightarrow \uparrow$ and $\{ +, - \} \rightarrow \downarrow$. 

---
and
\[
\lambda^A(k^\mu) = + \left( -i \Theta \left[ \phi^\mu_L(k^\mu) \right]^* \right) \\
= - \left[ \lambda^A_{(+,+)}(0) \right]_{\text{of (3.10) of AG}} (32c) \\
\text{and not } \lambda^A_{(,,+)}(0) \\
\lambda^A(k^\mu) = - \left( -i \Theta \left[ \phi^\mu_L(k^\mu) \right]^* \right) \\
= - \left[ \lambda^A_{(-,-)}(0) \right]_{\text{of (3.10) of AG}} (32d) \\
\text{and not } \lambda^A_{(,-,+)}(0) 
\]

If one wishes one can keep the here-chosen phases free and
fix them later by demanding locality for the resulting
quantum field. It is worth noting that the freedom of
global phases associated with each of the four
\( \lambda(k^\mu) \) is restricted to \( \pm 1 \) as (32d) does not allow a general replacement, \( \lambda(p^\mu) \to e^{\gamma} \lambda(p^\mu), \gamma \in \mathbb{R} \) without affecting the self/ant-self conjugacy under
\( C \). This freedom, and its restriction to \( \pm 1 \), affects the locality properties of the quantum field we shall construct, and its judicious incorporation, in part, lies behind the removal of the non-locality encountered
in (Ahluwalia-Khalilova and Grumiller, 2005a,b).

To obtain the explicit form of \( \lambda(p^\mu) \) we need one
last piece of information. It is deciphered by complex conjugating
(30), then replacing \( \sigma^* \) in accord with (15), using
\( \Theta^{-1} = -\Theta \), and finally multiplying from the left by \( \Theta \). This
exercise yields
\[
\sigma \cdot \hat{p} \left( \Theta [\phi^\mu_L(k^\mu)]^* \right) = \mp (\Theta[\phi^\mu_L(k^\mu)]^*)^* (33)
\]
and shows that the helicity of \( \Theta[\phi^\mu_L(k^\mu)]^* \) is opposite to that of \( \phi^\mu_L(k^\mu) \).

The interplay of the result (33) with the boost (29)
and the chosen form of \( \lambda(k^\mu) \) in (32c) to (32d) gives the
following analytically compact forms for \( \lambda(p^\mu) \)
\[
\lambda^S_+(p^\mu) = \sqrt{E + m} \left( 1 - \frac{p}{E + m} \right) \lambda^S_+(k^\mu) (34a) \\
\lambda^S_-(p^\mu) = \sqrt{E + m} \left( 1 + \frac{p}{E + m} \right) \lambda^S_-(k^\mu) (34b)
\]
and
\[
\lambda^A_+(p^\mu) = \sqrt{E + m} \left( 1 + \frac{p}{E + m} \right) \lambda^A_+(k^\mu) (34c) \\
\lambda^A_-(p^\mu) = \sqrt{E + m} \left( 1 - \frac{p}{E + m} \right) \lambda^A_-(k^\mu) (34d)
\]

These are the expansion coefficients of a quantum field
to be introduced below.

To construct an adjoint of the field we shall need the
spinorial duals of the \( \lambda(p^\mu) \) enumerated in (34a) to (34d).
This task is undertaken below in Section II.E after making
a few remarks in II.D on the Weinberg formalism for
the construction of quantum fields and a departure nec-
essitated by a circumstance encountered in constructing
a quantum field with Elko as its expansion coefficients.

D. Misconceptions in Literature and On a departure from
the Weinberg formalism

The choice for a set of \( \lambda(k^\mu) \)-like objects appears as a
trivial task in most textbooks: pick up a wave equation, like that of Dirac, set \( p^\mu \) to \( k^\mu \), and solve. This straightforward exercise immediately gives the required objects.
For the Dirac case, these are \( u_\gamma(k^\mu) \) and \( v_\gamma(k^\mu) \).

For the construction of quantum fields, this exercise
does not tell which of these coefficients, when appropriately boosted, shall accompany which one of the creation and destruction operators, nor does it tell us about the set of phases to be picked when making this pairing.
This is where one of the first errors occurs in many
of the modern textbooks on quantum fields. The other
important error resides, as already alluded to above, in
the lack of appreciation for the phases associated with each of the expansion coefficients. Our 2005 publications
(Ahluwalia-Khalilova and Grumiller, 2005a,b) were not immune to these errors.

Another question that escapes attention in these presenta-
tions is as to what tells us, for example, that the spin
one-half particles are described by the Dirac fermions,
and as to what is the deeper origin of the Dirac operator
\( (i\gamma_\mu \partial^\mu - m) \). The argument that it is the square root
of the Klein-Gordon operator is historically correct, but, in
the modern context it carries with it an element of triviality.

The answer to these questions emerges elegantly in
the Weinberg formalism (Weinberg, 1995). It, for example,
establishes the uniqueness of the Dirac field assuming
Lorentz symmetry along with symmetry under four
space-time translations, validity of the cluster decompo-
sition principle, and certain additional assumptions on
discrete symmetries. The quantum field \( \Psi(x) \) is obtained
first, and then by calculating the Feynman-Dyson propa-
gator through the evaluation of \( \left\langle \Theta [\Psi(x)] \right\rangle \) one arrives at the Dirac operator, and the Lagrangian
density. The Dirac spinors appear naturally as expansion
coefficients in \( \Psi(x) \) without any recourse, direct or indi-
rect (except certain symmetries), to Dirac or any other
wave equation.

15 A reader interested in specific examples of these errors in the
modern textbooks may correspond with the author in private;
or write to any of the former students of mine: Cheng-Yang Lee,
Sebastian Horvath, and Dimitri Schritt.
defined as
such that it yields a non-null Lorentz invariant norm, defined for massive particles. Call it \( \Xi \) (Ahluwalia, 1996). Under the Dirac dual \( \Xi = 1 \) (possibly, up to a phase). The requirement of a Lorentz invariant norm then translates to the statement that \( \eta \) in (37) must anti-commute with the generators of boosts, and commute with the generators of the rotations (Ahluwalia et al., 2014, 2011)

\[
\{ \kappa_i, \eta \} = 0, \quad [\zeta_i, \eta] = 0, \quad i = x, y, z. \quad (38)
\]

The three generators of boosts are given by (22). The three generators of rotation are

\[
\zeta = \begin{pmatrix} \sigma/2 & 0 \\
0 & -\sigma/2 \end{pmatrix}. \quad (39)
\]

A slightly lengthy but a straightforward calculation satisfying the constraints (38) shows \( \eta \) to have the form

\[
\eta = \begin{pmatrix} 0 & aI \\
bI & 0 \end{pmatrix}, \quad a, b \in \mathbb{R}. \quad (40)
\]

In order that the right and the left transforming components of a \( \xi(p^\mu) \) are treated symmetrically, we set \( a = b = 1 \) (21). This is an additional assumption that we explicitly note. The standard Dirac dual corresponds to \( \Xi = \mathbb{I}_4 \). Notationally, if \( \tilde{\omega}_\alpha(p^\mu) \) represents a Dirac spinor, \( \tilde{\omega}_\alpha(p^\mu) \to \tilde{\omega}_\alpha(p^\mu) \).

For Elko, the results (36a), (36b), and (36c) suggest that we define

\[
\Xi \equiv \frac{1}{2m} \left( \lambda_\alpha^A(p^\mu)\lambda_\alpha^S(p^\mu) + \lambda_\alpha^S(p^\mu)\lambda_\alpha^A(p^\mu) \right. \\
\left. -\lambda_\alpha^A(p^\mu)\lambda_\alpha^S(p^\mu) - \lambda_\alpha^S(p^\mu)\lambda_\alpha^A(p^\mu) \right). \quad (41)
\]

It is readily seen that \( \Xi^2 = \mathbb{I} \) and \( \Xi^{-1} \) indeed exists and equals \( \Xi \) itself. We may thus introduce a spinorial dual for Elko in accordance with the general definition (37) and an Elko-specific notation that distinguishes it from the Dirac dual

\[
\tilde{\lambda}_\alpha(p^\mu) = [\Xi \lambda_\alpha(p^\mu)]^\dagger \eta, \quad a = b = 1 \quad (42)
\]

with \( \Xi \) given by (41). This definition allows us to rewrite results (36a), (36b), and (36c) into the following orthonormality relations

\[
\lambda_\alpha^A(p^\mu)\lambda_\alpha^S(p^\mu) = 2m\delta_{\alpha\alpha'}, \quad (43a)
\]

\[
\lambda_\alpha^S(p^\mu)\lambda_\alpha^A(p^\mu) = -2m\delta_{\alpha\alpha'}, \quad (43b)
\]

\[
\lambda_\alpha^S(p^\mu)\lambda_\alpha^S(p^\mu) = 0, \quad \lambda_\alpha^A(p^\mu)\lambda_\alpha^A(p^\mu) = 0. \quad (43c)
\]

16 The last equality is unimportant. It simply sets a scale of the norms.

17 In obtaining this result we have followed a recent e-print of Speranca (Speranca, 2013).
The Elko dual, by construction, does precisely what it is intended to do. It provides a non-null Lorentz invariant norm. Mathematically, as well as physically, it encodes exactly the same information as that contained in \( g_{\alpha\beta} \), \( 36b \), and \( 36c \). As shall be seen below, in its present incarnation, as opposed to all earlier works since the 2005 publications, it is a much more powerful tool in investigating the symmetry structure that will appear for the mass dimension one quantum fields introduced below. We will discover that this structure is intrinsic to Elko, and to its Majorana cousin. The Elko dual simply makes it manifest through the here-introduced operator \( \Xi \). All results that we now obtain can be obtained in a much more cumbersome manner without invoking the Elko dual or the operator \( \Xi \).

As a consistency check, we find that the Elko dual defined using the operator \( \Xi \) yields exactly the same dual as in (Ahluwalia et al., 2010, Eq. 15) and (Ahluwalia et al., 2010, 2011). To see this we act \( 41 \) from the right by \( \lambda^S_\mu(p^\mu) \). Use of \( 36a \) and \( 36b \) then gives

\[
\Xi \lambda^S_\mu(p^\mu) = \frac{1}{2m} \lambda^S_\mu(p^\mu) \lambda^S_\mu(p^\mu) = i \lambda^S_\mu(p^\mu) \tag{44}
\]

where the last substitution is due to the relevant part of equations \( 36c \). Definition \( 42 \) thus yields

\[
\lambda^S_\mu(p^\mu) = -i \left[ \lambda^S_\mu(p^\mu) \right] \tag{45a}
\]

Repeating similar evaluations with \( \lambda^S_\mu(p^\mu) \), \( \lambda^A_\mu(p^\mu) \), and \( \lambda^A_\mu(p^\mu) \) in succession gives

\[
\lambda^S_\mu(p^\mu) = i \left[ \lambda^S_\mu(p^\mu) \right] \tag{45b}
\]
\[
\lambda^A_\mu(p^\mu) = -i \left[ \lambda^A_\mu(p^\mu) \right] \tag{45c}
\]
\[
\lambda^A_\mu(p^\mu) = i \left[ \lambda^A_\mu(p^\mu) \right] \tag{45d}
\]

A comparison of these results with those given in (Ahluwalia et al., 2010, Eq. 15) and (Ahluwalia et al., 2011, Eq. 22) establishes the equivalence of the Elko dual introduced here and the one introduced in the previous works. This, however, happens with the benefit of providing new insights (see below).

The knowledge of the Elko dual will help us define an appropriate adjoint for the quantum fields constructed with Elko as expansion coefficients. The calculation of the Feynman-Dyson propagator associated with these fields would require spin sums for Elko. We, therefore, evaluate these next and study their symmetry properties. The latter lie behind the departure from the Weinberg formalism for the construction of quantum fields noted in Section 11.1.

F. Spin sums and projectors for Elko

The spin sums

\[
\sum_\alpha \lambda^S_\alpha(p^\mu) \lambda^S_\alpha(p^\mu) \quad \text{and} \quad \sum_\alpha \lambda^A_\alpha(p^\mu) \lambda^A_\alpha(p^\mu) \tag{46}
\]

can now be readily evaluated using \( 34a \) to \( 34d \) for the \( \lambda^S_\alpha(p^\mu) \) and \( \lambda^A_\alpha(p^\mu) \), and \( 40a \) to \( 44a \) for their duals. The first of the two spin sums evaluates to

\[
i \left[ \frac{E + m}{2m} \left( 1 - \frac{p^2}{(E + m)^2} \right) \right] \]
\[
\times \left( - \lambda^S_\mu(k^\mu) \left[ \lambda^S_\mu(k^\mu) \right]^\dagger + \lambda^S_\mu(k^\mu) \left[ \lambda^S_\mu(k^\mu) \right]^\dagger \right) \eta
\]
\[
= -im \begin{pmatrix}
1 & 0 & 0 & ie^{i\phi} \\
0 & 1 & ie^{i\phi} & 0 \\
0 & -ie^{-i\phi} & 1 & 0 \\
ie^{i\phi} & 0 & 0 & 1
\end{pmatrix}
\]

which suggests introducing \( G \)

\[
G(\phi) \overset{\text{def}}{=} \begin{pmatrix}
0 & 0 & 0 & ie^{i\phi} \\
0 & 0 & ie^{i\phi} & 0 \\
0 & -ie^{-i\phi} & 0 & 0 \\
ie^{i\phi} & 0 & 0 & 0
\end{pmatrix}. \tag{48}
\]

In the spherical polar coordinate system parity is implemented by \( \theta \to \pi - \theta, \phi \to \phi + \pi \). The definition \( 48 \) thus shows that \( G(\phi) \) is an odd function under parity

\[
G(\phi) = -G(\phi + \pi). \tag{49}
\]

The second of the spin sums can be evaluated in exactly the same manner. The combined result is

\[
\sum_\alpha \lambda^S_\alpha(p^\mu) \lambda^S_\alpha(p^\mu) = m \left[ G(\phi) + \mathbb{I}_4 \right] \tag{50a}
\]
\[
\sum_\alpha \lambda^A_\alpha(p^\mu) \lambda^A_\alpha(p^\mu) = m \left[ G(\phi) - \mathbb{I}_4 \right]. \tag{50b}
\]

These spin sums have the eigenvalues \( \{0, 0, 2m, 2m\} \), and \( \{0, 0, -2m, -2m\} \), respectively. Since eigenvalues of projectors must be either zero or one (Weinberg, 2012, Section 3.3), we define

\[
S \overset{\text{def}}{=} \frac{1}{2m} \sum_\alpha \lambda^S_\alpha(p^\mu) \lambda^S_\alpha(p^\mu) = \frac{1}{2} \left[ \mathbb{I}_4 + G(\phi) \right] \tag{51a}
\]
\[
A \overset{\text{def}}{=} \frac{1}{-2m} \sum_\alpha \lambda^A_\alpha(p^\mu) \lambda^A_\alpha(p^\mu) = \frac{1}{2} \left[ \mathbb{I}_4 - G(\phi) \right]. \tag{51b}
\]

\[\text{\footnotesize 18 Sometimes it may be convenient to use the functional dependence of } G \text{ on } k^\nu, \text{ and write it as } G(k^\nu).\]
and confirm that indeed they are projectors and furnish the completeness relation

\[ S^2 = S, \quad A^2 = A, \quad S + A = 1 \quad (52) \]

We have thus arrived at one of the most intriguing and subtle aspects of Elko: Manifestly, through \( G(\phi) \) the projectors break Lorentz symmetry. A detailed analysis found in (Ahluwalia and Horvath 2010), and reviewed afresh below, shows that \( G(\phi) \) respects symmetries of the theory of very special relativity (VSR). For reasons given in the seminal paper on VSR (Cohen and Glashow, 2006b), the theory thus immediately evades the usual sensitive searches devoted to look for departures from Lorentz invariance (that is, from the symmetries underlying the theory of special relativity (SR)).

G. The SR \( \rightarrow \) VSR breaking of the Lorentz symmetry

It is now necessary to first provide a brief summary of the Cohen-Glashow VSR which asserts, “invariance under HOM(2) \( ^{19} \) rather than (as is often taught) the Lorentz group, is both necessary and sufficient to ensure that the speed of light is the same for all observers, and inter alia, to explain the null result to the Michelson-Morley experiment and its more sensitive successors.” Besides the just mentioned constancy of speed light, VSR also shares with SR the same time dilation, the same law of velocity addition, the same existence of a center-of-mass frame, and the same universal and isotropic maximal attainable velocity. These observations have been explicitly made in the original VSR paper (Cohen and Glashow 2006b).

1. Cohen-Glashow VSR: a brief summary

The theory of VSR has its origin in the above-quoted observation on necessity and sufficiency of certain subgroups of Lorentz to accommodate the null result of Michelson-Morley experiment and its more sensitive successors. There are four avatars of VSR (Ahluwalia and Horvath 2010; Cohen and Glashow 2006b). They are defined through the associated Lie algebras as follows.

\( t(2) \): Generated by

\[ T_1 \stackrel{\text{def}}{=} K_x + J_y, \quad T_2 \stackrel{\text{def}}{=} K_y - J_x \quad (53) \]

where \( J \) and \( K \) are generators of rotations and boosts, respectively. These provide an Abelian Lie algebra which is isomorphic to the algebra associated with translations in a plane. The two generate the \( T(2) \) group transformations

\[ \exp (i T_1 \epsilon) \begin{cases} 1 & \text{for vectors} \\ 1 + i \tau_1 \epsilon & \text{for spinors} \end{cases} \quad (54) \]

where the parameter of transformation \( \epsilon \) is given by

\[ \epsilon = \frac{p_x}{E - p_z} \quad (55) \]

and \( T_1 \) and \( \tau_1 \) are the four-vector and the spinor representations of \( T_1 \), respectively; and similarly

\[ \exp (i T_2 \epsilon) \begin{cases} 1 & \text{for vectors} \\ 1 + i \tau_2 \epsilon & \text{for spinors} \end{cases} \quad (56) \]

where the parameter of transformation \( \epsilon \) is given by

\[ \epsilon = \frac{p_y}{E - p_z} \quad (57) \]

and \( T_2 \) and \( \tau_2 \) are the four-vector and the spinor representations of \( T_2 \), respectively.

To obtain the above group transformations the following identities were found helpful

\[ T_1^3 = T_2^3 = 0_4, \quad \tau_1^2 = \tau_2^2 = 0_4. \quad (58) \]

\( \epsilon(2), \hom(2), \) and \( \sim(2) \): Adjoining \( t(2) \) by

- \( J_z \) yields \( \epsilon(2) \), an algebra which is isomorphic to the 3-parameter algebra associated with the group of Euclidean motions \( E(2) \).
- \( K_z \) yields \( \hom(2) \), an algebra which is isomorphic to the 3-parameter algebra associated with the group of orientation preserving similarity transformations, or homotheties, of \( \text{HOM}(2) \).
- \( J_z \) and \( K_z \) simultaneously yields \( \sim(2) \), an algebra which is isomorphic to the algebra associated with the four-parameter similitude group, \( \sim(2) \).

The counterparts of (54) and (56) for rotation about, and boost along, the VSR preferred direction (taken here as the \( \hat{z} \)) are found to be

\[ \exp (i J_z \Phi) = \begin{cases} 1 & \text{for vectors} \\ 1 + i 2 \zeta \sin(\Phi/2) + 4 \zeta^2(\cos(\Phi/2) - 1) & \text{for spinors} \end{cases} \quad (59) \]

\[ \exp (i K_z \chi) = \begin{cases} 1 & \text{for vectors} \\ 1 + i 2 \zeta \sinh(\chi/2) + 4 \zeta^2(1 - \cosh(\chi/2)) & \text{for spinors} \end{cases} \quad (60) \]
where the boost parameter along the \( \hat{z} \) direction is defined as
\[
\zeta = -\ln \left( \frac{E - p_z}{m} \right) \tag{61}
\]
and \( J_z \) and \( \zeta_z \) are, respectively, the four-vector and the spinor representations of \( J_z \); while \( K_z \) and \( \kappa_z \) are, respectively, the four-vector and the spinor representations of \( K_z \) with the properties
\[
J_z^3 = J_z, \quad (2\zeta_z)^3 = 2\zeta_z \tag{62a}
\]
\[
K_z^3 = -K_z, \quad (2\kappa_z)^3 = -2\kappa_z. \tag{62b}
\]

For a general boost that takes the standard vector \( k^\mu \) to a general four-momentum \( p^\mu \), VSR has no single generator except for the boost along the \( \hat{z} \) direction. Instead, this task is assigned to a set of three successive VSR transformations
\[
B(k^\mu \to p^\mu) = \exp (iT_1 \epsilon) \exp (iT_2 \varsigma) \exp (iK_z \varsigma) \tag{63}
\]

While each of the VSR subgroups has a different character, they all share the property that incorporation of either \( P, T, CP \), or \( CT \) enlarges these subgroups to the full Lorentz group. The SR-defining Lorentz group is required by the dual demands of the null results of the Michelson-Morley experiment and the preservation of the indicated discrete symmetries. In the absence of the latter, the \( \text{HOM}(2) \), and not Lorentz, is both sufficient and necessary to accommodate the said null result.

To preserve the notion of mass, the VSR adjoins the four space-time translations generated by \( P^\mu \) to each of the VSR subgroups enumerated above.

As argued in (Cohen and Glashow, 2006b), the amplitude for the two body decay of a spinless particle at rest may depend on the direction of the decay products relative to the VSR preferred direction (nominally, \( \hat{z} \)). A similar VSR signature is expected to arise for the mass dimension one Fermi field (Dias et al. 2012).

2. VSR covariance of the Elko projectors and spin sums

The covariance of the Elko projectors and spins sums is completely determined by the covariance of \( \mathcal{G}(\phi) \). To examine this we first evaluate the four VSR transformations. Using the above-obtained expansions, a straight forward calculation yields the following results:

1. \( \mathcal{G}(\phi) \) is covariant under rotations about \( \hat{z} \), because (69) yields
\[
\exp (i\zeta_\phi \Phi) = \left( \begin{array}{cccc}
e^{i\frac{\Phi}{2}} & 0 & 0 & 0 \\
0 & e^{-i\frac{\Phi}{2}} & 0 & 0 \\
0 & 0 & e^{i\frac{\Phi}{2}} & 0 \\
0 & 0 & 0 & e^{-i\frac{\Phi}{2}} \\
\end{array} \right) \tag{64}
\]
and, consequently
\[
\exp (i\zeta_\phi \Phi) \mathcal{G}(\phi) \exp (-i\zeta_\phi \Phi) = \mathcal{G}(\phi - \Phi). \tag{65}
\]

2. \( \mathcal{G}(\phi) \) is invariant under boosts along the \( \hat{z} \), as (60) gives
\[
\exp (i\kappa_\phi \varsigma) \mathcal{G}(\phi) \exp (-i\kappa_\phi \varsigma) = \mathcal{G}(\phi). \tag{66}
\]

3. \( \mathcal{G}(\phi) \) is invariant under the VSR translations, because from (54) we obtain
\[
\exp (i\tau_1 \epsilon) = \left( \begin{array}{cccc}
1 & \epsilon & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -\epsilon & 1 \\
\end{array} \right) \tag{67}
\]
and from (54) follows
\[
\exp (i\tau_2 \varsigma) = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
\end{array} \right). \tag{68b}
\]

Additionally, on using the identity
\[
\epsilon \sin \phi = \varsigma \cos \phi \tag{69}
\]
and that for the pure \( T_1 \) translation induced by the parameter \( \epsilon \), we keep \( \varsigma = 0 \) (and vice versa for \( T_2 \)), we get
\[
\exp (i\tau_1 \epsilon) \mathcal{G}(\phi) \exp (-i\tau_1 \epsilon) = \mathcal{G}(\phi) \tag{70a}
\]
\[
\exp (i\tau_2 \varsigma) \mathcal{G}(\phi) \exp (-i\tau_2 \varsigma) = \mathcal{G}(\phi). \tag{70b}
\]

These results come about as follows. Take, as an example, the \( T_1 \) translation (68a) for spinors, then \( \exp (i\tau_1 \epsilon) \mathcal{G}(\phi) \exp (-i\tau_1 \epsilon) \) evaluates to
\[
\mathcal{G}(\phi) + \left( \begin{array}{cccc}
0 & 0 & -2\epsilon \sin \phi & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2\epsilon \sin \phi & 0 & 0 \\
\end{array} \right). \tag{71}
\]

Now because of definitions (55) and (57) imply the identity (69), we identify \( \epsilon \sin \phi \) with its equivalent \( \varsigma \cos \phi \) in the above equation, which for the pure \( T_1 \) translation identically vanishes. We thus obtain (70a), and similarly we arrive at (70b).

The Elko spin sums and projectors are thus covariant under the \( \text{SIM}(2) \) and \( \text{E}(2) \) subgroups, and invariant under \( \text{T}(2) \) and \( \text{HOM}(2) \).
3. Structure of $G(\phi)$

The right-transforming components of $\lambda^S(k^\mu)$ and $\lambda^A(k^\mu)$, respectively

$$i\Theta\phi^+_L(k^\mu), \quad -i\Theta\phi^+_{L\phantom{}}(k^\mu)$$  \hspace{1cm} (72)

and the left-transforming components for both

$$\phi_L(k^\mu)$$  \hspace{1cm} (73)

contain a $2 \times 2$ matrix that gives $G(\phi)$ its structure. To see this, we define a $2 \times 2$ matrix $Q(k^\mu)$ such that

$$i\Theta\phi^+_L(k^\mu) \overset{\text{def of } Q(k^\mu)}{=} Q(k^\mu)\phi_L(k^\mu),$$  \hspace{1cm} (74)

which determines $Q(k^\mu)$ to have the form

$$Q(k^\mu) = i \begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}.$$  \hspace{1cm} (75)

The $G(\phi)$ is thus entirely determined by $Q(k^\mu)$

$$G(\phi) = \begin{pmatrix} 0 & Q(k^\mu) \\ Q(k^\mu) & 0 \end{pmatrix}$$  \hspace{1cm} (76)

and is, therefore, intrinsic to the choice of the ‘rest’ spinors $\lambda(k^\mu)$.

A further relation exists between $G(\phi)$, $\Xi$, and Dirac’s $\gamma_\mu p^\mu$. To derive this we start with (41), and expand it as follows

$$\Xi = e^{i\kappa \cdot \phi} \left[ \frac{1}{2m} \left( \lambda^S_+(k^\mu) \left[ \lambda^S_+(k^\mu) \right]^\dagger + \lambda^S_-(k^\mu) \left[ \lambda^S_-(k^\mu) \right]^\dagger \right) + \lambda^A_+(k^\mu) \left[ \lambda^A_+(k^\mu) \right]^\dagger - \lambda^A_-(k^\mu) \left[ \lambda^A_-(k^\mu) \right]^\dagger \right] e^{i\kappa \cdot \phi} \gamma_0.$$  \hspace{1cm} (77)

Next we note that the terms inside the external square brackets evaluate to $G(\phi)$ and that the commutator of $e^{i\kappa \cdot \phi}$ with $G(\phi)$ vanishes:

$$[e^{i\kappa \cdot \phi}, G(\phi)] = 0.$$  \hspace{1cm} (78)

With these observations, the above equation takes the form

$$\Xi = G(\phi)e^{i2\kappa \cdot \phi} \gamma_0.$$  \hspace{1cm} (79)

Then recalling (43) to replace $e^{i2\kappa \cdot \phi}$ by $m^{-1}\gamma_\mu p^\mu \gamma_0$, (78) take the form

$$\Xi = m^{-1} G(\phi) \gamma_\mu p^\mu.$$  \hspace{1cm} (80)

Now since $(\gamma_\mu p^\mu)^{-1} = m^{-2} \gamma_\mu p^\mu$, (79) translates to

$$G(\phi) = m^{-1} \Xi \gamma_\mu p^\mu = m^{-1} \gamma_\mu p^\mu \Xi$$  \hspace{1cm} (81)

where the last equality follows from the commutativity of $\Xi$ and $\gamma_\mu p^\mu$:

$$[\Xi, \gamma_\mu p^\mu] = 0.$$  \hspace{1cm} (82)

H. Global phase transformations for Elko: VSR imposed restrictions

We now conclude the discussion started in §II.B on global phase transformations for Elko. The invariance under VSR dramatically reduces the form of $a$ arrived at in (27). The invariance of $a$ under rotations about $\hat{\varepsilon}$ requires that $[a - \exp(i\xi \Phi) a \exp(-i\xi \Phi)]$ vanishes. This requirement translates to $\beta = 0$. Similar calculation for the invariance under boosts along the $\tilde{\varepsilon}$, yields in addition $\lambda = 0$, while the invariance under each of the two the VSR translations requires, $\delta = \alpha$.

The invariance of $a$ under VSR transformations thus reduces the number of variables in (27) to a single parameter $\alpha$, whose magnitude can now be absorbed in the phase angle $\beta$; with the result: $a = \gamma_0$.

I. Dynamics for Elko: a hint towards mass dimension one
for a spin one-half field

A hint that we may be heading towards a mass dimension one quantum field resides in the result: The momentum space Dirac operator $(\gamma_\mu p^\mu \pm m \mathbb{1}_4)$ does not annihilate the $\lambda(p^\mu)$. To establish this, keeping (44) in mind we begin by operating $\gamma_\mu p^\mu$ on $\lambda^S_+(p^\mu)$:

$$\gamma_\mu p^\mu \lambda^S_+(p^\mu) = \sqrt{\frac{E + m}{2m}} \left( 1 - \frac{p}{E + m} \right) \times \left[ E\gamma_0 + p \begin{pmatrix} 0 & -\sigma \cdot \hat{p} \\ \sigma \cdot \hat{p} & 0 \end{pmatrix} \right] \lambda^S_+(k^\mu).$$  \hspace{1cm} (83)

A judicious use of (60) shows that

$$\begin{pmatrix} 0 & \sigma \cdot \hat{p} \\ -\sigma \cdot \hat{p} & 0 \end{pmatrix} \lambda^S_+(k^\mu) = \gamma_0 \lambda^S_+(k^\mu).$$  \hspace{1cm} (84)

and as a consequence (82) transforms to

$$\gamma_\mu p^\mu \lambda^S_+(p^\mu) = \sqrt{\frac{E + m}{2m}} \left( 1 - \frac{p}{E + m} \right) (E + p) \gamma_0 \lambda^S_+(k^\mu).$$  \hspace{1cm} (85)

Finally, invoking the standard dispersion relation and an easily-verified identity $\gamma_0 \lambda^S_+(k^\mu) = i\lambda^S(k^\mu)$ reduces (84) to

$$\gamma_\mu p^\mu \lambda^S_+(p^\mu) = \sqrt{E + m} \left( 1 + \frac{p}{E + m} \right) \lambda^S(k^\mu).$$  \hspace{1cm} (86)

Using (61) in the right-hand side of (85) gives

$$\gamma_\mu p^\mu \lambda^S_+(p^\mu) = \sqrt{E + m} \left( 1 + \frac{p}{E + m} \right) \lambda^S_+(p^\mu).$$  \hspace{1cm} (87)

An exactly similar exercise complements (86) with

$$\gamma_\mu p^\mu \lambda^S_+(p^\mu) = -i\lambda^S_+(p^\mu)$$  \hspace{1cm} (87a)

and

$$\gamma_\mu p^\mu \lambda^A_+(p^\mu) = i\lambda^A_+(p^\mu).$$  \hspace{1cm} (87b)

(87c)
Combined (86a) to (86d) establish the result: \((\gamma_{\mu}p^{\mu} \pm m\mathbb{1}_4)\) does not annihilate the \(\lambda(p^{\mu})\). An alternate, though somewhat cumbersome, derivation of this result can be traced back to Dvoeglazov [1995a]. Equations (86a) to (86d) encode the dynamical content by implying annihilation of Elko by the spinorial Klein-Gordon operator (in the momentum space), and not the Dirac operator. This follows on multiplication of these equations from the left by \(\gamma_{\nu}p^{\nu}\), using them to reinsert the effect of \(\gamma_{\nu}p^{\nu}\) on the relevant \(\lambda(p^{\mu})\), and then using \(\{\gamma_{\mu},\gamma_{\nu}\} = 2\eta_{\mu\nu}\mathbb{1}_4\), where \(\eta_{\mu\nu}\) is the space-time metric with signature \((+1,-1,-1,-1)\):

\[
(\eta_{\mu\nu}p^{\mu}p^{\nu}\mathbb{1}_4 - m^2\mathbb{1}_4)\lambda(p^{\mu}) = 0. \tag{87}
\]

This wave equation is Lorentz covariant. In contrast, the spin sums carry lower symmetries: that is, those of SIM(2) of VSR.

Thus for a quantum field with Elko as its expansion coefficients, to be introduced below, we shall encounter a scenario where we may have two options:

a. Have a Lorentz covariant Lagrangian density and a Feynman-Dyson propagator with a SIM(2) symmetry of VSR, or

\[
\Gamma = \frac{1}{2m} \begin{pmatrix}
0 & 0 & -i e^{i\phi}(E + p \cos \theta) & m + ip \sin \theta \\
-i e^{i\phi}(E - p \cos \theta) & m + ip \sin \theta & 0 & 0 \\
i e^{i\phi}(E - p \cos \theta) & m - ip \sin \theta & 0 & 0 \\
0 & 0 & -i e^{i\phi}(E + p \cos \theta) & m - ip \sin \theta
\end{pmatrix} \tag{90}
\]

which diagonalises \(\Xi\)

\[
\Xi \Gamma \Xi^{-1} \rightarrow \text{diag}\{-1, -1, 1, 1\} \tag{91}
\]

and gives (88) the form

\[
\begin{pmatrix}
\pm m & m(m - ip \sin \theta) & 0 & 0 \\
m(m + ip \sin \theta) & \mp m & m(m + ip \sin \theta) & 0 \\
0 & 0 & \mp m & m(m - ip \sin \theta) \\
0 & 0 & 0 & \mp m
\end{pmatrix} \lambda'(p^{\mu}) = 0 \tag{92}
\]

where \(\lambda'(p^{\mu}) = \Gamma \lambda(p^{\mu})\). These equations are non-local. On introducing two \(E(2)\) invariant null vectors \(n_{\pm}^\mu = (1, 0, 0, 1)\) and \(n_{-}^\mu = (1, 0, 0, -1)\), the non-locality inducing factors

\[
\frac{1}{E \mp p \cos \theta} \tag{93}
\]

in (92) become \((n_{\pm}^\mu p_{\mu})^{-1}\). This is precisely the same type of non-locality as is encountered in Cohen and Glashow [2006a]. The nonlocal equations arrived here, and contrary to the assertion in Ahluwalia and Horváth [2010, Footnote 5], the ones proposed in Cohen and Glashow [2006a], are free from violation of the dispersion relation, \(E^2 = p^2 + m^2\). This is immediately verified by studying the determinant of the SIM(2) covariant Cohen-Glashow operator

\[
\gamma_{\mu}p^{\mu} - \frac{m^2 \gamma_{\mu}n_{\pm}^\mu}{2 n_{\pm}^\mu p_{\mu}} \tag{94}
\]

b. Have a SIM(2) covariant theory at the cost of non-locality suggested in Cohen and Glashow [2006a].

The first of these options is akin to a situation where solutions of a differential equation carry symmetries that are lower than that of the associated differential equation.

We now attempt to make the second of the two remarks more transparent by working out a simple exercise.

J. A way out of Cohen-Glashow non-locality

The exercise begins with the observation that the definition of \(\Xi\) given in (11) when combined with the results (36a) to (36c) yields

\[
\lambda_{\pm}^{\mathcal{S}}(p^{\mu}) = \pm i \Xi \lambda_{\mp}^{\mathcal{S}}(p^{\mu}), \quad \lambda_{\pm}^{\mathcal{A}}(p^{\mu}) = \pm i \Xi \lambda_{\mp}^{\mathcal{A}}(p^{\mu}). \tag{88}
\]

Next, when the \(\lambda(p^{\mu})\)s, that appear on the right hand sides of (86a) to (86d) are replaced in accordance with the above action of \(\Xi\) on the indicated \(\lambda(p^{\mu})\)s, we obtain

\[
(\gamma_{\mu}p^{\mu} \mp m \Xi) \lambda(p^{\mu}) = 0 \tag{89}
\]

where the upper sign holds for the self conjugate, and the lower sign for the anti-self conjugate, \(\lambda(p^{\mu})\). To obtain a diagonal mass term, we use
and the operator that acts on $\lambda'(p^\mu)$ in (92).

So, for Elko at least, non-locality may be avoided if one opts for option ‘a’ mentioned at the end of Section II.I.

1. A parenthetic remark

Is $\lambda'(p^\mu)$ in (92) an Elko? The answer is found by writing $\Gamma$ as $\exp(i\theta)$, where

$$\epsilon \equiv \frac{1}{i} \frac{\partial \Gamma}{\partial \theta} \bigg|_{\theta \to 0}$$

$$= \frac{p}{2m} \left( \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 \end{array} \right).$$

(95)

For the answer to the asked question to be in the affirmative the discussion in II.I requires that $\epsilon$ be proportional to $\gamma_5$. Since it is not, the $\Gamma$ transformed $\lambda(p^\mu)$ is no longer an Elko.

III. LOCAL MASS DIMENSION ONE FERMI FIELD OF SPIN ONE-HALF

A. The new quantum field and its adjoint

Having developed the necessary background we now define a new quantum field with self and anti-self conjugate Elko as its expansion coefficients

$$f(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(p)}} \sum_a \left[ a_\alpha(p) \lambda^S(p) \exp(-ip_\mu x^\mu) + \delta^{\alpha\alpha'}(p) \lambda^A(p) \exp(ip_\mu x^\mu) \right]$$

(96)

with $\lambda^S(p)$ and $\lambda^A(p)$ given by (34a) to (34d). The creation and annihilation operators satisfy Fermi statistics (Ahluwalia-Khalilova and Grumiller, 2005b), Section 7)

$$\{a_\alpha(p), a_\beta(p')\} = (2\pi)^3 \delta^3(p-p') \delta_{\alpha\beta} \quad \{a_\alpha(p), a_{\alpha'}(p')\} = 0$$

$$\{a_\alpha(p), a_{\alpha'}(p')\} = 0, \quad \{a_\alpha^+(p), a_{\alpha'}^+(p')\} = 0$$

(97a)

(97b)

with similar anticommutators for $b_\alpha(p)$ and $b_\alpha^+(p)$. The $f(x)$ differs from its 2005 counterpart $\eta(x)$ through its expansion coefficients (see II.C).

To calculate the mass dimensionality of $f(x)$, we define the adjoint

$$\tilde{f}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(p)}} \sum_a \left[ a_\alpha^+(p) \bar{\lambda}^S(p) \exp(ip_\mu x^\mu) + \delta^{\alpha\alpha'}(p) \bar{\lambda}^A(p) \exp(-ip_\mu x^\mu) \right]$$

(98)

where $\bar{\lambda}^S(p)$ and $\bar{\lambda}^A(p)$ are defined in (45a) to (45d).

B. Mass dimension of the new field

The mass dimension of the new field is determined by the Feynman-Dyson propagator (Weinberg, 1995)

$$S_{FD}(x' - x) = i \langle \Sigma (f(x') \tilde{f}(x)) \rangle$$

(99)

where $\Sigma$ is the canonical time-ordering operator. Using the above definitions of $f(x)$ and its adjoint $\tilde{f}(x)$, the spin sums (50a) and (50b), along with the property (49), a straightforward calculation shows

$$S_{FD}(x' - x) =$$

$$- \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu(x'_\mu - x_\mu)} \left[ \frac{\begin{pmatrix} \gamma_4 & G(\phi) \\ 0 & \gamma_4 \end{pmatrix}}{p_\mu p_\mu - m^2 + i\epsilon} \right]$$

(100)

with $\epsilon = 0^+$. As a consequence of (100), and following the foundational discussion on the determination of mass dimensionality of quantum fields given in (Weinberg, 1995, Section 12.1), the mass dimension of the field $f(x)$ is thus one

$$D_f = 1$$

(101)

and not three-half, as is the case for the Dirac field. The discussion of section II.C.2 also implies that the associated Feynman-Dyson propagator (99) is SIM(2) covariant.

The 2005 field $\eta(x)$ mentioned above also yields precisely the same Feynman-Dyson propagator. The difference, for reasons discussed at length above, resides in the locality anticommutators (see II.D and II.E below).

C. Lagrangian density for the new field

As a next step in developing the formalism for $f(x)$, we act (100) from the left by the spinorial Klein-Gordon operator and obtain

$$\left( \partial_{\mu'} \partial^\mu - m^2 \right) S_{FD}(x' - x) = \delta^4(x' - x)$$

$$+ \int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu(x'_\mu - x_\mu)} \bar{G}(\phi)$$

(102)

where ‘deR’ denotes (de Oliveira and Rodrigues, 2012). The free field Lagrangian density for the field under consideration is, therefore

$$\mathcal{L}_0(x) = \partial^\mu f^\ast \partial_\mu f(x) - m^2 f(x) f(x)$$

(103)

to which may be added two very natural interaction terms

$$\mathcal{L}_{int}(x) = g_{ff} \left( f(x) f(x) \right)^2 + g_{bf} \left( f(x) b(x) b(x) \right)$$

(104)

where $g_{ff}$ and $g_{bf}$ are dimensionless coupling constants.
The first of these is a quartic self interaction, and the other is the interaction of the new field with spin-zero bosonic fields (such as EB-GHK-II scalar (Englert and Brout, 1964; Guralnik et al., 1964; Higgs, 1964) generically represented here by \( b(x) \)).

For the mass dimension three-half Dirac field, similar interactions are suppressed, respectively, by two and one powers of the unification/Planck scale.

There is also a possibility of a dimensionless coupling of the form

\[ g_{b2} \tilde{f}(x) [\gamma_{\mu}, \gamma_{\nu}] f(x) \tilde{g}^{\mu\nu}(x) \]  

(105)

where \( \tilde{g}^{\mu\nu}(x) \) is a gauge field strength tensor associated with a local U(1) gauge symmetry of \( f(x) \) under \( \exp[i\gamma_5 \partial(x)] \).

This opens up the possibility that the new field provides a natural self-interacting dark matter candidate that is self-referentially luminous. With the notable exception of scalar fields and gravity, its interactions with the standard model matter and gauge fields is suppressed by at least one power of unification/Planck scale.

D. Locality structure of the new field

Now that we have \( \Sigma_0(x) \) we can calculate the momentum conjugate to \( f(x) \)

\[ p(x) = \frac{\partial \Sigma_0(x)}{\partial f(x)} = \frac{\partial}{\partial t} \tilde{f}(x). \]  

(106)

To establish that the new field is local we calculate the standard equal-time anticommutators. The first of the three anticommutators we calculate is the ‘\( \bar{f} \cdot p \)’ anticommutator

\[ \{ f(t, x), p(t, x') \}. \]  

(107)

It evaluates to

\[ i \int \frac{d^3p}{(2\pi)^3} \frac{e^{i p \cdot (x-x')}}{2m} \sum_\alpha \left[ \lambda^S_\alpha(p) \tilde{\lambda}^S_\alpha(p) - \lambda^A_\alpha(-p) \tilde{\lambda}^A_\alpha(-p) \right]. \]  

(108)

The spin sum (50b) in conjunction with (49) yield

\[ \sum_\alpha \lambda^A_\alpha(-p) \tilde{\lambda}^A_\alpha(-p) = -m [\mathcal{G}(\phi) + \mathbb{1}_4]. \]  

(109)

Therefore the spin sum that appears in (108) equals \( 2m [\mathcal{G}(\phi) + \mathbb{1}_4] \), giving

\[ \{ f(t, x), p(t, x') \} = i \int \frac{d^3p}{(2\pi)^3} e^{i p \cdot (x-x')} [\mathcal{G}(\phi) + \mathbb{1}_4]. \]  

(110)

The ‘\( \mathcal{G} \)’ integration has recently been shown to vanish (de Oliveira and Rodrigues, 2012), and we thus obtain

\[ \{ f(t, x), p(t, x') \} = i\delta^3(x - x') \mathbb{1}_4. \]  

(111)

A still simpler calculation shows that the remaining two, that is, ‘\( \bar{f} \cdot \bar{p} \)’ and ‘\( \bar{p} \cdot \bar{p} \)’, equal time anticommutators vanish

\[ \{ f(t, x), f(t, x') \} = 0, \quad \{ p(t, x), p(t, x') \} = 0. \]  

(112)

E. Majorana-isation of the new field

Even though field \( f(x) \) is uncharged under local U(1) supported by the Dirac fields of the SM, it may carry a charge under a different local U(1) gauge symmetry such as the one suggested in the remark around (105). This gives rise to the possibility of having a fundamentally neutral field in the sense of Majorana (Majorana, 1937)

\[ m(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2mE(p)}} \sum_\alpha \left[ a_\alpha(p) \lambda^S(p) \exp(-ip_\mu x^\mu) \right. \]

\[ + a_\alpha^*(p) \lambda^A(p) \exp(ip_\mu x^\mu) \]. \]  

(113)

with momentum conjugate

\[ q = \frac{\partial}{\partial t} \tilde{m}(x). \]  

(114)

The calculation for the ‘\( m-q \)’ equal time anticommutator goes through exactly as before and one gets

\[ \{ m(t, x), q(t, x') \} = i\delta^3(x - x') \mathbb{1}_4. \]  

(115)

The calculation of the remaining two anticommutators requires knowledge of the following ‘twisted’ spin sums

\[ \sum_\alpha \left[ \lambda^S_\alpha(p) \lambda^A_\alpha(-p) \right] + \lambda^A_\alpha(-p) \lambda^S_\alpha(-p) \]  

(116a)

\[ \sum_\alpha \left[ \lambda^S_\alpha(p) \lambda^A_\alpha(-p) \right] + \lambda^A_\alpha(-p) \lambda^S_\alpha(-p) \]  

(116b)

Using (34a) to (34d) one finds that each of these vanish. With this result at hand, we immediately decipher vanishing of the ‘\( m-m \)’ and ‘\( q-q \)’, equal time anticommutators

\[ \{ m(t, x), m(t, x') \} = 0, \quad \{ q(t, x), q(t, x') \} = 0. \]  

(117)

IV. GENERALIZATIONS, AND IMPACT ON THE EXISTING LITERATURE

To provide a concrete example of a mass dimension one quantum field, we confined our attention to the \( \lambda(k^\nu) \)-defining \( \phi_L(k^\nu) \) as eigenspinors of \( \sigma \cdot \hat{p} \) (see, (39)). Our claim to having constructed a local mass dimension Fermi field of spin one-half is confined to this specific ansatz.

We have explored various alternatives to this assumption and found the general result to be quite robust as long as one leaves free various phases; and later constrains them by the demand that the resulting fields satisfy canonical equal time anticommutators. So, for example, one may start with

\[ \phi^+_L(p^\mu) = e^{ia} \begin{pmatrix} \sqrt{m} & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi^-_L(p^\mu) = e^{ib} \begin{pmatrix} 0 & \sqrt{m} \\ \sqrt{m} & 0 \end{pmatrix} \]  

(118)
with $a, b \in \mathbb{R}$ and proceed with the rest of the construction. Or, choose for $\phi_i^L(p^\mu)$ a linear combination of the $\phi^L_x(p^\mu)$ and $\phi^R_x(p^\mu)$ and examine the resulting construct.

In any such analysis counterpart of $G(\phi)$ may not necessarily be VSR or SR covariant. The variants that violate SR or VSR in this manner are more likely to be ruled out by the existing limits on violations of Lorentz invariance (Cohen and Glashow, 2006). However, as of yet, no systematic and general analysis on the alternative possibilities has been done in its entirety.

It is also clear that higher spins are likely to support similar constructs. A preliminary exercise towards this generalization, without incorporating recent developments reported here, can be found in (Lee, 2012a).

Since the publication of two 2005 papers on mass dimension one field of spin one-half there has come into existence a significant literature entirely devoted to Elko but the emphasis on the quantum field theoretic aspects has been limited. These publications can be divided in roughly two categories: those which depend on the bilinear invariants of Elko, and those which depend on the non-local counterpart of the $f(x)$ and $m(x)$ from previous publications starting with (Ahluwalia-Khalilova and Grumiller, 2005a,b). An example of the former is Elko cosmology (Basak et al., 2013) which uses Elko condensate to drive inflation. The results reported there remain unaffected unless one considers its quantum field theoretic extension. An example of the latter is the work of (Dias et al., 2012) suggesting a LHC signature of the non-local antecedents of the new local quantum fields constructed here. The results of such works remain qualitatively intact through their dependence on traces of the products of $[1 \pm G(\phi_i)]$, where $\phi_i$ refers to various particles in a process, but the precise predictions shall suffer significant changes.

V. CONCLUDING REMARKS

In the framework of quantum theory, and Poincaré space-time symmetries of the theory of special relativity, there is a unique Fermi field of spin one-half. It is endowed with mass dimension three-half. It forms the kinematic structure of the standard model of high energy physics. This structure invites interactions through a class of local gauge fields to erect one of the most successful descriptions of physical reality with two exceptions: A quantum theory of gravity escapes its confines, and the standard model of cosmology further challenges its completeness by requiring the dark sector. On the observational side, the strong evidence for dark energy, and dark matter, confirm this incompleteness. Whether this reflects the need for a fundamentally new set of symmetries or even a revision of the very foundations of the relativistic quantum framework shall be decided only by time.

The present paper establishes an element of incompleteness in the theory of quantum fields by suggesting an entirely new class of matter and gauge fields. The crevice that allows the local mass dimension one Fermi field of spin one-half to emerge has several layers: First, the idea that we consider a quantum field with c-number eigenspinors of the charge conjugation operator (beyond Majorana spinors), Elko, as its expansion coefficients. Second, that certain phases be assigned to these to obtain the canonical equal time anticommutators. Third, at the interpretational level, we exploit the Cohen-Glashow insight on the breaking of Lorentz symmetry and evade stringent tests of Lorentz invariance. This last observation results in a quantum field whose Lagrangian density is Lorentz covariant but the Feynman-Dyson propagator carries symmetries of a four-parameter SIM(2) subgroup of Very Special Relativity.

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