EULER-SYMMETRIC PROJECTIVE TORIC VARIETIES AND ADDITIVE ACTIONS

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Abstract. Let $G_a$ be the additive group of the field of complex numbers $\mathbb{C}$. We say that an irreducible algebraic variety $X$ of dimension $n$ admits an additive action if there is a regular action of the group $G_a^n = G_a \times \ldots \times G_a$ ($n$ times) on $X$ with an open orbit. In 2017 Baohua Fu and Jun-Muk Hwang introduced a class of Euler-symmetric varieties. They gave a classification of Euler-symmetric varieties and proved that any Euler-symmetric variety admits an additive action. In this paper we show that in the case of projective toric varieties the converse is also true. More precisely, a projective toric variety admitting an additive action is an Euler-symmetric variety with respect to any linearly normal embedding into a projective space. Also we discuss some properties of Euler-symmetric projective toric varieties.

1. Introduction

Let $X \subseteq \mathbb{P}^s$ be a projective variety over the field $\mathbb{C}$. Suppose that $X$ is nondegenerate, that is, $X$ is not contained in any hyperplane in $\mathbb{P}^s$. Denote by $G_m$ the group $\mathbb{C}^*$. The following definition was given in [16].

Definition 1. Let $x \in X$ be a smooth point. A $G_m$-action on $\mathbb{P}^s$ is said to be of Euler type at $x$, if the following conditions hold:

1. the variety $X$ is invariant with respect to this action;
2. the point $x$ is isolated fixed point in $X$ with respect to this action;
3. the induced action on the tangent space $T_xX$ acts by scalar operators.

We say that $x$ is an Euler point if there is a $G_m$-action on $\mathbb{P}^s$ of Euler type at $x$. The variety $X$ is called Euler-symmetric if there is an open subset $U \subseteq X$ such that every point of $U$ is an Euler point.

In [16] Baohua Fu and Jun-Muk Hwang gave a description of Euler-symmetric varieties using their fundamental forms at a general point. Also they proved that any Euler-symmetric variety admits an additive action [16, Theorem 3.7].

There are several results on additive actions on complete toric varieties. The first one is the work of Hassett and Tschinkel [17]. They established a correspondence between additive actions on the projective space $\mathbb{P}^n$ and local commutative associative algebras with unit of dimension $n + 1$. One can find in [19] a more general correspondence.

Additive actions on projective hypersurfaces are studying in [3] and [4]. One can find results on additive actions on flag varieties in [1], [10] and [14]. Also there are works on additive actions on singular del Pezzo surfaces [11], weighted projective planes [2] and Hirzebruch surfaces [17].

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We recall that a toric variety is a normal variety of dimension $n$ which admits an effective action of torus $T = \mathbb{G}_m^n = \mathbb{G}_m \times \ldots \times \mathbb{G}_m$ ($n$ times) with an open orbit. Due to a combinatorical description (see [15] or [8]) a lot of properties of toric varieties can be described in a nice and simple way. So it is always natural to consider the toric case while studying some general theory.

Toric varieties admitting additive actions are described in [5]. It was proven in [12] that a complete toric variety of dimension two which admits an additive action can have either one or two non-isomorphic additive actions. There is a description of complete toric varieties with a unique additive action [13]. In [21] projective toric hypersurfaces with additive actions are classified.

We study Euler-symmetric projective toric variety. It turns out that a linearly normal projective toric variety is Euler-symmetric if and only if it admits an additive action (Theorem 3). Also we study the set of Euler points on a projective toric variety (Proposition 3) and describe fundamental forms corresponding to linearly normal Euler-symmetric projective toric varieties (Proposition 5).

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2. Complete toric varieties admitting additive actions

Here we recall the description of complete toric varieties admitting additive actions obtained in [5].

Let $X$ be a complete toric variety of dimension $n$ with an acting torus $T$. Let $N$ be the lattice of one-parameter subgroups of $T$, $M$ be the dual lattice of characters and $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ be the natural pairing. We denote by $N_\mathbb{Q}$ and $M_\mathbb{Q}$ the vector spaces $N \otimes \mathbb{Z} \mathbb{Q}$ and $M \otimes \mathbb{Z} \mathbb{Q}$, respectively.

Let $\Delta$ be the fan of polyhedral cones in $N_\mathbb{Q}$, which corresponds to $X$; see [8] or [15] for details. Let $\Delta(1) = \{\rho_1, \ldots, \rho_r\}$ be the set of one-dimensional cones in $\Delta$ and for a cone $\sigma \in \Delta$ by $\sigma(1)$ we mean the set of one-dimensional faces of $\sigma$. We denote the primitive lattice generator of a ray $\rho$ by $p_\rho$ and by $p_i$ we mean $p_{\rho_i}$.

**Definition 2.** A vector $e \in M$ is called a Demazure root of a complete fan $\Delta$ if there is a ray $\rho \in \Delta(1)$ such that $\langle p_\rho, e \rangle = -1$ and $\langle p_{\rho'}, e \rangle \geq 0$ for all $\rho' \in \Delta(1)$, $\rho' \neq \rho$.

A set of Demazure roots $e_1, \ldots, e_n$ of a complete fan $\Delta$ of dimension $n$ is called a complete collection if the rays in $\Delta(1)$ can be numbered in such a way that $\langle p_i, e_j \rangle = -\delta_{ij}$ for all $1 \leq i, j \leq n$ where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

The following result makes it possible to understand by the fan $\Delta$ whether $X$ admits an additive action.

**Theorem 1.** [5, Corollary 2] A complete toric variety $X$ admits an additive action if and only if there is a complete collection of Demazure roots of the fan $\Delta$.

It is easy to see that there is a complete collection of Demazure roots of the fan $\Delta$ if and only if one can order rays of the fan $\Delta$ in such a way that the primitive vectors on the first $n$ rays form a basis of the lattice $N$ and the remaining rays lie in the negative octant with respect to this basis (see Figure 1).

**Definition 3.** Let $X \subseteq \mathbb{P}^s$ be a normal nondegenerate projective variety. The embedding of $X$ into $\mathbb{P}^s$ defines a map of spaces $H^0(\mathbb{P}^s, \mathcal{O}(1)) \to H^0(X, \mathcal{O}_X(1))$. We say that $X$ is linearly normal projective variety if this map is surjective.
Every nondegenerate linearly normal projective toric variety can be given by a polytope in $M_\mathbb{Q}$. We recall that a lattice polytope in $M_\mathbb{Q}$ is a convex hull of a finite subset in $M$.

Let $m$ be a vertex of a lattice polytope $P$. Denote by $S_{P,m}$ the semigroup in $M$ generated by the set $(P \cap M)−m$. The semigroup $S_{P,m}$ is called saturated if for all $k \in \mathbb{N} \setminus \{0\}$ and for all $a \in M$ the condition $ka \in S_{P,m}$ implies $a \in S_{P,m}$. A lattice polytope $P$ is very ample if for every vertex $m \in P$, the semigroup $S_{P,m}$ is saturated.

Let $P \subseteq M_\mathbb{Q}$ be a full dimensional very ample lattice polytope and $P \cap M = \{m_0, m_1, \ldots, m_s\}$. Then one can consider the map

$$T \to \mathbb{P}^s, \quad t \mapsto [\chi_{m_0}^{m_0(t)} : \ldots : \chi_{m_s}^{m_s(t)}],$$

where $\chi_{m_i}$ is the character of $T$ corresponding to $m_i$. Denote by $X_P$ the closure of the image of this map. Then $X_P$ is a linearly normal nondegenerate projective toric variety.

Conversely, let $Y \subseteq \mathbb{P}^s$ be a linearly normal nondegenerate projective toric variety with an acting torus $T$. Then $Y$ coincides with $X_P$ for some very ample polytope $P$ in $M$ up to automorphism of $\mathbb{P}^s$.

**Definition 4.** A lattice polytope $P \subseteq M_\mathbb{Q}$ is inscribed in a rectangle if there is a vertex $v_0 \in P$ such that

1. the primitive vectors on the edges of $P$ containing $v_0$ form a basis $e_1, \ldots, e_n$ of the lattice $M$;
2. for every inequality $\langle p, x \rangle \leq a$ on $P$ that corresponds to a facet of $P$ not passing through $v_0$ we have $\langle p, e_i \rangle \geq 0$ for all $i = 1, \ldots, n$.

**Theorem 2.** [5, Theorem 4] Let $P \subseteq M_\mathbb{Q}$ be a very ample polytope and $X_P$ be the corresponding projective toric variety. Then $X_P$ admits an additive action if and only if the polytope $P$ is inscribed in a rectangle.
3. Fundamental forms

As we mentioned before an Euler-symmetric variety is uniquely determined by its fundamental form. We recall some definitions.

Let \( X \subseteq \mathbb{P}^s \) be a \( n \)-dimensional nondegenerate irreducible projective subvariety of the projective space \( \mathbb{P}^s \) and \( x \in X \) be a smooth point. Denote by \( z_0, \ldots, z_s \) homogeneous coordinates on \( \mathbb{P}^s \) and by \( y_i = \frac{z_i}{z_0} \) the respective inhomogeneous coordinates on the affine chart \( U_0 = \{ z \in \mathbb{P}^s | z_0 \neq 0 \} \). We may assume that \( x = [1 : 0 : \ldots : 0] \) with respect to these coordinates and the tangent space \( T_x X \) is given by equations \( y_i = 0 \) for \( i = n + 1, \ldots, s \).

Then the functions \( y_1, \ldots, y_n \) are the system of local parameters on \( X \). Denote by \( L \) the set of linear combinations \( \{ l = \alpha_0 + \alpha_1 y_1 + \ldots + \alpha_n y_n : \alpha_i \in \mathbb{C} \} \).

For any function \( l \in L \) there is an open neighborhood of \( x \) such that \( l \) is given by the series
\[
l = \sum_{i=0}^{\infty} h_i^l(y_1, \ldots, y_n),
\]
where \( h_i^l \) is a homogeneous polynomial of degree \( i \). Denote by \( \overline{h}(l) \) the first non-zero term.

**Definition 5.** The \( k \)-th fundamental form of the variety \( X \) at the point \( x \) is a vector space
\[
F^k = \langle \overline{h}(l) | l \in L, \deg \overline{h}(l) = k \rangle \subseteq \text{Sym}^k(T_x X)^*.
\]

The fundamental form of the variety \( X \) at the point \( x \) is the vector space
\[
F = \bigoplus_{k=0}^{\infty} F^k \subseteq \bigoplus_{k=0}^{\infty} \text{Sym}^k(T_x X)^*.
\]

**Definition 6.** We say that the fundamental form is monomial if one can choose coordinates \( y_1, \ldots, y_n \) such that there is a basis in \( F \) consisting of monomials. Each monomial \( y_1^{a_1} \cdots y_n^{a_n} \) defines a point \( (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) so a monomial fundamental form defines a set \( D(F) = \{(a_1, \ldots, a_n) | y_1^{a_1} \cdots y_n^{a_n} \in F \} \).

**Remark 1.** One can find in [18, Chapter 3] a more generally accepted definition of fundamental form. It follows from [16, Lemma 2.5] that these definitions are equivalent.

**Definition 7.** Let \( V \) be a vector space. For any vector \( v \in V \) and a non-negative integer number \( k \) the contraction homomorphism is a linear map \( \iota_v : \text{Sym}^k V^* \rightarrow \text{Sym}^{k-1} V^* \) which sends \( \varphi \in \text{Sym}^k V^* \) to \( \iota_v(\varphi) \in \text{Sym}^{k-1} V^* \) defined by
\[
\iota_v(\varphi)(v_1, \ldots, v_k) = \varphi(v, v_1, \ldots, v_k),
\]
for any \( v_1, \ldots, v_k \in V \). By convention, we define \( \iota_v(\text{Sym}^0 V^*) = 0 \).

**Definition 8.** Consider a subspace \( W = \bigoplus_{k \geq 0} W^k \subseteq \bigoplus_{k \geq 0} \text{Sym}^k V^* \). Then the subspace \( W \) is called a symbol system if \( W^0 = \mathbb{C}, W^1 = V^* \), \( W^k \neq 0 \) only for finite number of \( k \) and \( \iota_v(W) \subseteq W \) for all \( v \in V \).

It is well known that the fundamental form at a general point is a symbol system (see [18, Chapter 3]). It follows from [16, Theorem 3.7] that an Euler-symmetric variety is uniquely defined by its fundamental form at a general point. Moreover, for any symbol system \( F \) there is an Euler-symmetric variety \( X \) such that a fundamental form of \( X \) at a general point is \( F \).
4. Euler-symmetric projective toric varieties

First of all, note that if a normal projective variety $X$ is an Euler-symmetric variety with respect to some nondegenerate embedding of $X$ into a projective space, then $X$ is Euler-symmetric with respect to any linearly normal embedding of $X$ into a projective space. More precisely, the following statement is true.

**Proposition 1.** Let $X$ be a normal projective variety and $x \in X$ be a smooth point. The following conditions are equivalent.

1. The point $x$ is an Euler point with respect to some nondegenerate embedding of $X$ into a projective space.
2. The point $x$ is an Euler point with respect to any nondegenerate linearly normal embedding of $X$ into a projective space.

**Proof.** $(1) \Rightarrow (2)$. Here we use reasoning similar to the proof of [7, Proposition 3.2.6].

Let $X \subseteq \mathbb{P}^s$ be a nondegenerate projective variety and $x \in X$ be a smooth Euler point. Then there is a $\mathbb{G}_m$-action of Euler type at $x$. Consider a nondegenerate linearly normal embedding $\rho : X \hookrightarrow \mathbb{P}^k$. Denote by $L$ the restriction on $X$ of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^k$. Since $\mathbb{G}_m$ is a factorial variety the line bundle $L$ can be linearized with respect to the action of $\mathbb{G}_m$ on $X$ [20, Proposition 2.4]. The linearization defines a rational linear $\mathbb{G}_m$-action on $H^0(X, L) \simeq H^0(\mathbb{P}^k, \mathcal{O}(1))$. It defines extended $\mathbb{G}_m$-action on $\mathbb{P}^k$. Therefore $x$ is an Euler point with respect to embedding $\rho : X \hookrightarrow \mathbb{P}^k$.

Any normal projective variety admits a nondegenerate linearly normal embedding into a projective space. So the implication $(2) \Rightarrow (1)$ is trivial. 

**Corollary 1.** Let $X$ be a normal projective variety. Then $X$ is Euler-symmetric with respect to some nondegenerate embedding into a projective space if and only if $X$ is Euler-symmetric with respect to any nondegenerate linearly normal embedding of $X$ into a projective space.

Now we consider the toric case.

**Proposition 2.** Let $X$ be a projective toric variety with an acting torus $T$ and $x \in X$ be a smooth $T$-fixed point. Then $x$ is an Euler point with respect to any linearly normal nondegenerate embedding of $X$ into a projective space.

**Proof.** Let $M$ be the lattice of characters of $T$. Any linearly normal nondegenerate embedding of $X$ into a projective space corresponds to some very ample lattice polytope $P \subseteq M_{\mathbb{Q}}$. Denote by $v_0$ the vertex of $P$ corresponding to $x$. We can assume that $v_0$ is the zero point of $M$.

Since $x$ is a smooth point, the primitive vectors on the edges of $P$ containing $v_0$ form a basis $e_1, \ldots, e_n$. All vertices of $P$ have non-negative coordinates with respect to this basis.

Denote by $\{m_0, m_1, \ldots, m_s\}$ the lattice points in $P$. We can assume that the first $n + 1$ points have the following coordinates in the basis $e_1, \ldots, e_n$:

$$m_0 = v_0 = (0, \ldots, 0), \quad m_1 = (1, 0, \ldots, 0), \quad m_2 = (0, 1, \ldots, 0), \ldots, \quad m_n = (0, 0, \ldots, 1).$$

We denote the coordinates of the other points as:

$$m_{n+i} = (a_{i,1}, \ldots, a_{i,n}), \quad i = 1, \ldots, s - n,$$

where $a_{i,j}$ are non-negative integer numbers.
Let \( t_1, \ldots, t_n \) be the coordinates on \( T \) corresponding to the basis \( e_1, \ldots, e_n \). For any character \( m \in M \) and for any \( t \in T \) we denote by \( t^m \) the value \( \chi^m(t) \).

Consider the map
\[
\varphi_D : T \to \mathbb{P}^s, \quad (t_1, \ldots, t_n) \to [t^{m_0} : \ldots : t^{m_s}] = [1 : t_1 : \ldots : t_n : t_1^{a_1,1} \cdots t_n^{a_1,n} : \ldots : t_1^{a_n,-n} \cdots t_n^{a_n,-n}].
\]

The variety \( X \) is the closure of the image of the map \( \varphi_D \). Note that the point \( x \) has coordinates \([1 : 0 : \ldots : 0]\).

It is easy to see that the action of \( T \) on \( X \) can be extended to an action of \( T \) on \( \mathbb{P}^s \). Consider the subtorus
\[
T_1 = \{(t, \ldots, t)| t \in \mathbb{C}^\times \} \subseteq T.
\]

Denote by \( z_0, \ldots, z_s \) the homogeneous coordinates on \( \mathbb{P}^s \). Consider the affine chart \( U_0 = \{(z_0 : \ldots : z_s)| z_0 \neq 0\} \subseteq \mathbb{P}^s \). Let \( y_i = z_i / z_0 \) be the corresponding inhomogeneous coordinates on \( U_0 \). Then the variety \( X \cap U_0 \) satisfies equations \( y_{n+i} = y_1^{a_{1,i}} \cdots y_n^{a_{n,i}} \). We see that \( x = (0, \ldots, 0) \) is an isolated fixed point in \( X \) with respect to the action of \( T_1 \).

The tangent space at a point \( x \) on \( X \) is given by equations \( y_{n+i} = \ldots = y_n = 0 \). But \( T_1 \) acts by multiplication by \( t \) on the variables \( y_1, \ldots, y_n \). So \( T_1 \) acts by scalar operators on \( T_1X \). So \( x \) is an Euler point with respect to the action of \( T_1 \).

\[\square\]

Remark 2. It follows from the proof of Proposition 2 that the fundamental form \( F \) of \( X \) at the point \( x \) is monomial and the set \( D(F) \) coincides with the set \( \{m_0, m_1, \ldots, m_s\} \).

**Proposition 3.** Let \( X \) be a projective toric variety with an acting torus \( T \) and \( x \) be a smooth point in \( X \). The following conditions are equivalent.

1. The point \( x \) is an Euler point with respect to some nondegenerate embedding of \( X \) into a projective space.
2. The point \( x \) is an Euler point with respect to any nondegenerate embedding of \( X \) into a projective space.
3. There is an automorphism \( \varphi \) of \( X \) such that \( \varphi(x) \) is a \( T \)-fixed point.

**Proof.** We start with implication (1) \( \Rightarrow \) (3). Suppose \( x \in X \) is an Euler point with respect to an action of one-dimensional torus \( T_1 \) with respect to some nondegenerate embedding into a projective space. Denote by \( G \) the connected component of unit in the automorphism group \( \text{Aut}(X) \). Then \( G \) is an affine algebraic group (see [9, Theorem 4.2]). Hence there is a maximal subtorus \( T' \) in \( G \) such that \( T_1 \) is a subgroup of \( T' \). All points in the orbit \( T'x \) are \( T_1 \)-fixed. Since \( x \) is isolated \( T_1 \)-fixed point then \( x \) is a \( T' \)-fixed point.

All maximal subtorus in \( G \) are conjugated, so there is an automorphism \( \varphi \in G \) such that \( \varphi T' \varphi^{-1} = T \). Then \( \varphi(x) \) is a \( T \)-fixed point.

Now we prove (3) \( \Rightarrow \) (2). Suppose that there is an automorphism \( \varphi \) of \( X \) such that \( \varphi(x) \) is a \( T \)-fixed point. Then \( x \) is a fixed point with respect to the torus \( \varphi T \varphi^{-1} \). The variety \( X \) is a toric variety with respect to the action of the torus \( \varphi T \varphi^{-1} \). By Proposition 2 the point \( x \) is an Euler point with respect to any linearly normal nondegenerate embedding.

The equivalence of (1) and (2) follows from Proposition 1. \( \square \)

**Theorem 3.** Let \( X \) be a projective toric variety with an acting torus \( T \). The following conditions are equivalent:

1. \( X \) is Euler-symmetric with respect to some nondegenerate embedding into a projective space;
(2) $X$ is Euler-symmetric with respect to any nondegenerate linearly normal embedding into a projective space;

(3) $X$ admits an additive action.

Proof. (1) $\Rightarrow$ (3). Suppose that $X$ is Euler-symmetric with respect to some nondegenerate embedding into a projective space. Then $X$ admits an additive action by [16, Theorem 3.7] (see also Proposition 4 below).

(3) $\Rightarrow$ (2) Now suppose that $X$ admits an additive action. Then by [5, Theorem 3] $X$ admits an additive action normalized by $T$. Let $U$ be the open orbit with respect to this additive action. The orbit $U$ is isomorphic to the affine space and $T$-invariant. Therefore the variety $U$ is an affine toric variety with respect to the action of $T$. An affine toric variety has a $T$-fixed point if and only if it has no regular invertible functions except constants. So there is a $T$-fixed smooth point $p \in U$. By Proposition 3 all points in $U$ are Euler points with respect to any nondegenerate linearly normal embedding into a projective spaces.

The equivalence of (1) and (2) follows from Corollary 1. □

It is proved in [16] that any Euler-symmetric projective variety admits an additive action. However, in the case of toric varieties one can prove it using the combinatorial description of toric varieties. For this purpose we need a description of the orbits of the automorphism group of complete toric varieties obtained by Ivan Bazhon [6].

For a ray $\rho_i$ we denote by $D_i$ the $T$-invariant divisor corresponding to $\rho_i$ and by $[D_i]$ we mean the class of this divisor in the divisor class group. For a cone $\tau \in \Delta$ we define a monoid

$$\Gamma(\tau) = \sum_{\rho_i \in \Delta(1) \setminus \tau(1)} Z_{\geq 0}[D_i].$$

Denote by $\Upsilon(\Delta)$ the set of monoids $\{\Gamma(\tau) : \tau \in \Delta\}$. By $O_\tau$ we mean the torus orbit on $X$ corresponding to $\tau \in \Delta$.

**Theorem 4.** [6, Theorem 3.7] Let $X$ be a complete toric variety. Torus orbits $O_\sigma$ and $O_{\sigma'}$ on $X$ lie in the same $\text{Aut}(X)$-orbit if and only if there exists an automorphism $\phi : \text{Cl}(X) \to \text{Cl}(X)$ with the following properties:

- $\phi(\Gamma(\sigma)) = \Gamma(\sigma')$,
- $\phi(\Upsilon(\Delta)) = \Upsilon(\Delta)$,
- there exists a permutation $f$ of elements $\{1, \ldots, r\}$ such that $\phi([D_i]) = [D_{f(i)}]$.

**Proposition 4.** Let $X \subseteq \mathbb{P}^s$ be a nondegenerate Euler-symmetric projective toric variety. Then $X$ admits an additive action.

Proof. Let $X$ be an Euler-symmetric projective toric variety with an acting torus $T$. Then there is an open subset $U$ consisting of Euler points in $X$. Denote by $O$ the open $T$-orbit in $X$. It is clear that $O$ is a subset of $U$. By Proposition 3 there is a $T$-fixed Euler point $x$ which belongs to the same $\text{Aut}(X)$-orbit with points from $O$.

Denote by $\Delta$ the fan corresponding to $X$. Let $\sigma$ be the cone in $\Delta$ corresponding to the $T$-orbit $x$ and $\sigma_0$ be the cone corresponding to the $T$-orbit $O$. Note that $\sigma$ is a maximal cone in $\Delta$ and $\sigma_0$ is a vertex.

Let $\Delta(1) = \{\rho_1, \ldots, \rho_r\}$ be the set of rays in $\Delta$ and $p_1, \ldots, p_r$ be the primitive lattice generators on these rays. We suppose that the first $n$ rays belong to $\sigma$. Since $x$ is a smooth point the vectors $p_1, \ldots, p_n$ form a basis of the lattice $N$. Then the monoid $\Gamma(\sigma_0)$ is equal to $\sum_{i=1}^n Z_{\geq 0}[D_i]$ and the monoid $\Gamma(\sigma)$ is equal to $\sum_{i=n+1}^r Z_{\geq 0}[D_i]$. 


By [6, Theorem 3.7] there is an automorphism $\phi$ of the group $\text{Cl}(X)$ such that $\phi(\Gamma(\sigma_0)) = \Gamma(\sigma)$. The automorphism $\phi$ permutes the elements of the set $\{[D_1], \ldots, [D_r]\}$. Since the monoid $\Gamma(\sigma_0)$ is generated by the set $\{[D_1], \ldots, [D_r]\}$, we have $\phi(\Gamma(\sigma_0)) = \Gamma(\sigma_0)$. It implies that $\Gamma(\sigma_0) = \Gamma(\sigma)$. It follows that for any $i = 1, \ldots, n$ there are non-negative integers $a_{i,j}$ such that

$$[D_i] = \sum_{j=n+1}^r a_{i,j}[D_j].$$

There is an exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^r \rightarrow \text{Cl}(X) \rightarrow 0,$$

where the second arrow is given by the map

$$m \rightarrow (\langle p_1, m \rangle, \ldots, \langle p_r, m \rangle), \quad m \in M,$$

and the third arrow is given by the map

$$(b_1, \ldots, b_r) \rightarrow b_1[D_1] + \ldots + b_r[D_r],$$

where $b_j \in \mathbb{Z}$; see [8, Chapter 4] for details.

Since $[D_i] - \sum_{j=n+1}^r a_{i,j}[D_j] = 0$ for $i = 1, \ldots, n$ there are vectors $m_i \in M$ such that $\langle p_j, m_i \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$ and $\langle p_j, m_i \rangle = -a_{i,j}$, for $1 \leq i \leq n < j \leq r$.

Then $m_1, \ldots, m_n$ is a dual basis to $p_1, \ldots, p_n$ and the vectors $p_j$ with $j > n$ have coordinates $-a_{i,j}$. So rays $\rho_{n+1}, \ldots, \rho_r$ lie in the negative octant with respect to the basis $p_1, \ldots, p_n$. By Theorem 1 the variety $X$ admits an additive action.

It is easy to describe symbol systems which correspond to toric Euler-symmetric varieties.

**Proposition 5.** Let $X \subseteq \mathbb{P}^s$ be a linearly normal nondegenerate projective Euler-symmetric variety of dimension $n$ and $F$ be its fundamental form at a general point. Then $X$ is a toric variety if and only if $F$ is monomial and the set $D(F) \subseteq \mathbb{Z}_{\geq 0}^n$ coincides with the set of lattice points inside some very ample inscribed in a rectangle lattice polytope.

**Proof.** Let $X \subseteq \mathbb{P}^s$ be a linearly normal nondegenerate Euler-symmetric projective toric variety with an acting torus $T$. Then $X$ admits an additive action. The embedding of $X$ into the projective space $\mathbb{P}^s$ is given by some very ample lattice polytope $P \subseteq M_\mathbb{Q}$. By [5, Theorem 4] $P$ is inscribed in a rectangle.

Denote by $v_0$ the vertex of $P$ from Definition 4 and by $x \in X$ the smooth $T$-fixed point corresponding to $v_0$. We can assume that $v_0$ is the origin of $M$. The point $x$ belongs to the open $\mathbb{G}_a^n$-orbit in $X$. Since $\mathbb{G}_a^n$ is a factorial variety the action of $\mathbb{G}_a^n$ can be extended on $\mathbb{P}^s$. So at a general point in $X$ the fundamental form is isomorphic to $F$. By Remark 2 the fundamental form $F$ at the point $x$ is monomial and the set $D(F)$ coincides with the set of lattice points inside $P$.

Conversely, let $X \subseteq \mathbb{P}^s$ be a linearly normal nondegenerate projective Euler-symmetric variety and suppose that the fundamental form $F$ at a general point of $X$ is monomial and the set $D(F) \subseteq \mathbb{Z}^n$ coincides with the set of lattice points inside some very ample inscribed in a rectangle lattice polytope $P$. Consider the toric variety $X_P \subseteq \mathbb{P}^s$ corresponding to $P$. Then fundamental form at a general point of $X_P$ is $F$. Since Euler-symmetric variety is
uniquely determined by its fundamental form at a general point, the varieties $X$ and $X_P$ are isomorphic. So $X$ is a toric variety.

At the end we consider two examples.

**Example 1.** As follows from Proposition 3, each Euler point on projective toric varieties $X$ belongs to $\text{Aut}(X)$-orbit of a $T$-fixed point. At the same time two Euler points can lie in different $\text{Aut}(X)$-orbits.

Consider the Hirzebruch surface $H_s$ with $s \geq 1$. It is a toric variety with the fan $\Delta_{H_s}$ is shown in Figure 3.

![Figure 3](image.png)

**Figure 3.** The fan corresponding to Hirzerbruch surface $H_s$.

The Hirzebruch surface is smooth. It is well known that any complete toric surface is a projective variety; see [15, Chapter II.4].

By $\sigma_{ij}$ we mean a two-dimensional cone in $\Delta_{H_s}$ with rays $\rho_i$ and $\rho_j$. By $\sigma_i$ we mean a one-dimensional cone $\rho_i$ and by $\sigma_0$ the vertex of $\Delta_{H_s}$ respectively.

The divisor class group $\text{Cl}(X)$ is freely generated by $[D_{\rho_3}]$ and $[D_{\rho_4}]$. We have $[D_{\rho_1}] = [D_{\rho_3}]$ and $[D_{\rho_2}] = [D_{\rho_2}] + s[D_{\rho_3}]$. Then all monoids in $\Upsilon(\Delta_{H_s})$ except $\Gamma(\sigma_{12})$, $\Gamma(\sigma_{23})$ and $\Gamma(\sigma_2)$ are equal to $\langle [D_{\rho_2}], [D_{\rho_3}] \rangle$, where by $\langle S \rangle$ we mean a monoid generated by a set $S$. We denote the monoid $\langle [D_{\rho_2}], [D_{\rho_3}] \rangle$ by $A$.

Monoids $\Gamma(\sigma_{12})$, $\Gamma(\sigma_{23})$ and $\Gamma(\sigma_2)$ are equal to the monoid $B = \langle [D_{\rho_3}], [D_{\rho_2}] + s[D_{\rho_3}] \rangle$. The element $[D_{\rho_2}]$ does not belong to $B$, so $A \neq B$.

Any automorphism $\phi$ from [6, Theorem 3.7] permutes monoids in $\Upsilon(\Delta_{H_s})$. But in $\Upsilon(\Delta_{H_s})$ there are 5 monoids equal to $A$ and 3 monoids equal to $B$. So there is no such $\phi$ that $\phi(A) = B$.

Then all points in $H_s$ belong to $\text{Aut}(X)$-orbit of some smooth $T$-fixed point and all points are Euler points with respect to any nondegenerate linearly normal embedding into a projective space. But there are two different $\text{Aut}(X)$-orbits.

**Example 2.** Not all smooth points on a projective toric variety are necessarily Euler points. Any projective toric variety that does not admit an additive action is suitable as a counterexample. Indeed, points from the open $T$-orbit are smooth but not Euler. However it is also interesting to consider an example of Euler-symmetric variety, in which not all smooth points are Euler.

Consider a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point. It is a smooth projective toric variety and the corresponding fan is shown in Figure 4.

![Figure 4](image.png)

**Figure 4.** The fan corresponding to Hirzerbruch surface $H_s$. 

The divisor class group $\text{Cl}(X)$ is freely generated by classes $[D_{\rho_3}]$, $[D_{\rho_4}]$ and $[D_{\rho_5}]$. Also we have $[D_{\rho_1}] = [D_{\rho_3}] + [D_{\rho_4}]$ and $[D_{\rho_2}] = [D_{\rho_4}] + [D_{\rho_5}]$. Let us consider $[D_{\rho_3}]$, $[D_{\rho_4}]$, $[D_{\rho_5}]$ as a basis of $\text{Cl}(X)$. We will show that the points from the orbit corresponding to $\sigma_4$ are
not Euler. Indeed, the monoid $\Gamma(\sigma_4) = \langle (1, 0, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1) \rangle$ is not equal to any of monoids corresponding to fixed points listed below:

$$\Gamma(\sigma_{23}) = \langle (0, 1, 0), (0, 0, 1), (1, 1, 0) \rangle, \quad \Gamma(\sigma_{15}) = \langle (0, 1, 0), (1, 0, 0), (0, 1, 1) \rangle,$$
$$\Gamma(\sigma_{34}) = \langle (0, 0, 1), (1, 1, 0), (0, 1, 1) \rangle, \quad \Gamma(\sigma_{45}) = \langle (1, 0, 0), (1, 1, 0), (0, 1, 1) \rangle,$$
$$\Gamma(\sigma_{12}) = \langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle.$$ 

Any automorphism $\phi$ of the group $\text{Cl}(X)$ satisfying the condition of Theorem 4 permutes the elements $[D_{\rho_3}]$. So $\phi$ is an automorphism of monoid $\langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle$. The elements $[D_{\rho_3}], [D_{\rho_4}], [D_{\rho_5}]$ are irreducible in this monoid but $[D_{\rho_1}]$ and $[D_{\rho_2}]$ are not. So $\phi$ permutes the elements $[D_{\rho_3}], [D_{\rho_4}], [D_{\rho_5}]$. The elements $[D_{\rho_1}]$ and $[D_{\rho_2}]$ are both divisible by $[D_{\rho_4}]$. So $\phi$ is either identical or $\phi$ permutes $[D_{\rho_4}]$ and $[D_{\rho_5}]$ and preserves $[D_{\rho_3}]$. In both cases $\Gamma(\sigma_4)$ is preserved.

Therefore $\text{Aut}(X)$-orbit of any $T$-fixed point does not meet points in $O_{\sigma_4}$. So points in $O_{\sigma_4}$ are not Euler but smooth.

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