Structural health monitoring of constrained tapered beam-like structures using natural frequencies and nodal points

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Abstract. The integrity and safety of beam-like structures are dependent in part on their boundary conditions which can vary with time due to damage or aging. Structural health monitoring of such structures should therefore include attention to boundary conditions. Where the boundary conditions can be represented by a lumped spring then the identification of associated stiffness parameter values may be a means to quantifying the integrity of the support. This paper investigates such a method for identifying the equivalent translational and rotational stiffness of a constrained tapered beam-like structure. An analytical model of a beam of tapered width and thickness is adopted as a simplified representation of a tower-like structure. The model is used to explore in what scenarios natural frequencies and/or nodal points might be sufficiently sensitive to changes in support conditions to be measurable indicators of damage. The method is evaluated by Monte Carlo simulations for a numerical example where the severity of noise can be controlled.

1. Introduction

Structural health monitoring based on vibration measurements are hot topics that have received considerable attention in recent decades [1, 2]. The objective of many of structural health monitoring methods is to detect the damage occurring on the structure, for example, stiffness reductions in the damage location of the structure. However, the integrity and safety of some structures, for example, cantilever beam-like structures, can be seriously affected by damage affecting their boundary conditions. In the field of boundary conditions identification, two model updating methods, i.e. method based on the characteristic equation and method based on sensitivity analysis of natural frequency and mode shape, were utilized by Pabst and Hagedorn [3] for identifying boundary conditions of some simulative and experimental models. Similar to the method based on characteristic equation, Ahmadian et al [4] obtained a new characteristic equation which included the boundary conditions, the mass matrix and stiffness matrix, and then proposed a method to determine the boundary parameters based on the solution of reduced order characteristic equations. Besides these researches, Waters et al [5] proposed a boundary conditions identification method based on static stiffness measurements, and the beam is modelled as a uniform rigid beam that is constrained by

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collocated equivalent translational and rotational springs, and the root stiffnesses (i.e. boundary conditions) are identified by a quasi-static stiffness measurements obtained from impact tests.

This paper considers a flexible tapered beam which is constrained at one end by a translational and rotational spring. Similar to the method based on sensitivity analysis utilized by Pabst and Hagedorn [3], we deduced the theoretical sensitivity of natural frequencies and nodal point positions with respect to changes in boundary conditions based on the free vibration equation of the beam, and then an iterative method is utilized to obtain the boundary conditions using the sensitivities of natural frequencies and nodal point positions.

2. Modal analysis of tapered beam

2.1. The model

Figure 1 shows a tapered Euler-Bernoulli beam of rectangular cross section and length \( l \) which is constrained at its root by translational and rotational springs of stiffness \( k_t \) and \( k_r \). \( y(x,t) \) is the lateral deflection, \( E \) is the Young’s modulus, \( \rho \) is the density, \( A(x) \) and \( I(x) \) are the area and the second moment of the area of the cross section, respectively.

\[
\begin{align*}
E, \rho, A(x), I(x) \\
y(x,t)
\end{align*}
\]

(a)

\[
\begin{align*}
k_t \\
k_r
\end{align*}
\]

(b)

Figure 1. tapered Euler Bernoulli beam with flexible boundary conditions at one end

(a) front view ; (b) plan view

Suppose that the taper of the beam is such that both the width \( b(x) \) and depth \( d(x) \) of the beam vary linearly along its length, i.e.

\[
b(x) = b_0 \left[1 - \beta_b \frac{x}{l} \right] \quad (1)
\]

\[
d(x) = h_0 \left[1 - \beta_h \frac{x}{l} \right] \quad (2)
\]

where \( \beta_b = 1 - b_1 / b_0 \) and \( \beta_h = 1 - h_1 / h_0 \) is the degree of taper in each direction. The boundary conditions of the tapered beam should be [6]:

\[
\begin{align*}
\end{align*}
\]
\[ EI(x) \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right)_{t=0} = k_r \left( \frac{\partial y(x,t)}{\partial x} \right)_{t=0} \]  
\[ \frac{\partial}{\partial x} \left( EI(x) \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right) \right)_{t=0} = -k_r y(x,t) \]  
\[ EI(x) \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right)_{x=l} = 0 \]  
\[ \frac{\partial}{\partial x} \left( EI(x) \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right) \right)_{x=l} = 0 \]  

### 2.2. Free vibration equation of motion

The free vibration equation of motion for lateral deflection of the tapered Euler-Bernoulli beam is given by

\[ \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} = 0, \quad 0 \leq x \leq l \]  

which can be solved to obtain the modal parameters of the beam. For any mode of vibration, the lateral deflection is time harmonic motion, and can be written as

\[ y(x,t) = W(x)e^{i\omega t} \]  

where \( W(x) \) is the mode shape and \( \omega \) is the circular frequency. Then, Eq.(7) can be written as the ordinary differential equation

\[ \frac{d^2}{dx^2} \left[ EI(x) \frac{dW(x)}{dx} \right] - \rho A(x) \omega^2 W(x) = 0, \quad 0 \leq x \leq l \]  

The solution of Eq.(9) can be solved by the Adomian modified decomposition method (AMDM), as follows [6]

\[ W(X) = \sum_{n=0}^{\infty} C_n X^n \]  

where, \( X = x/l \) is the dimensionless position along the beam, i.e. \( 0 \leq X \leq 1 \), and the coefficients \( C_n \) can be determined by the boundary conditions, which are

\[ C_0 = W(0), \quad C_1 = W'(0), \quad C_2 = \frac{W''(0)}{2}, \quad C_3 = \frac{W'''(0)}{6} \]  

and for \( n \geq 4 \)

\[ C_n = \frac{1}{n(n-1)(n-2)(n-3)} \sum_{m=0}^{n-4} \left[ 2\beta_b^{n-m-3} + 6\beta_h^{n-m-3} \right] (m+3)(m+2)(m+1) C_{m+3} \]

\[ \sum_{k=0}^{m+4} \beta_b^{k+1} \beta_h^{m-k-3} \right] (m+2)(m+1)C_{m+2} + \lambda \beta_h^{n-m-4} (n-m-3) C_m \]  

where, \( \lambda = \Omega^2 = \rho A_0 \omega^2 l^4 / (EI_0) \), and \( \Omega \) is the natural frequency expressed non-dimensionally. Usually, the solution in Eq.(10) has to be approximated by a truncated series of \( N+1 \) terms, i.e. \( W(X) = \sum_{n=0}^{N} C_n X^n \).

### 2.3. Calculation of natural frequencies and nodal points
Using the boundary conditions and the supposed solution of the free vibration equation, the eigenvalue problem for the tapered beam with flexible boundary conditions can be written as

\[
\begin{bmatrix}
  f^{[N]}_{10}(\beta, \lambda) & f^{[N]}_{11}(\beta, \lambda) \\
  f^{[N]}_{20}(\beta, \lambda) & f^{[N]}_{21}(\beta, \lambda)
\end{bmatrix}
\begin{bmatrix}
  C_0 \\
  C_1
\end{bmatrix} = 0
\]  

(13)

where

\[
f^{[N]}_{10}(\beta, \lambda) = \sum_{n=0}^{\text{fix}(N/4)} (c_{10,n} \lambda^n) + \beta_r \sum_{n=0}^{\text{fix}(N/4)} (c_{13,n} \lambda^n)
\]

(14)

\[
f^{[N]}_{11}(\beta, \lambda) = \sum_{n=0}^{\text{fix}((N-1)/4)} (c_{11,n} \lambda^n) + \beta_r \sum_{n=0}^{\text{fix}((N-1)/4)} (c_{12,n} \lambda^n)
\]

(15)

\[
f^{[N]}_{20}(\beta, \lambda) = \sum_{n=0}^{\text{fix}(N/4)} (c_{20,n} \lambda^n) + \beta_r \sum_{n=0}^{\text{fix}((N-3)/4)} (c_{23,n} \lambda^n)
\]

(16)

\[
f^{[N]}_{21}(\beta, \lambda) = \sum_{n=0}^{\text{fix}((N-1)/4)} (c_{21,n} \lambda^n) + \beta_r \sum_{n=0}^{\text{fix}((N-3)/4)} (c_{22,n} \lambda^n)
\]

(17)

where \( \beta_t = k \ell^3 / (EI_0) \) and \( \beta_r = k \ell / (EI_0) \) are the dimensionless translational and rotational root stiffnesses, the coefficients \( c_{i0,n}, c_{i1,n}, c_{i2,n}, c_{i3,n}, \ldots \) are functions of just the taper degree of the beam \( \beta_b, \beta_h \) and \( N \), \( \text{fix}() \) means rounding the value to the nearest integer towards zero.

Then, for non trivial solutions \( C_0 \) and \( C_1 \), the frequency equations are given as

\[
\begin{bmatrix}
  f^{[N]}_{10}(\beta, \lambda) & f^{[N]}_{11}(\beta, \lambda) \\
  f^{[N]}_{20}(\beta, \lambda) & f^{[N]}_{21}(\beta, \lambda)
\end{bmatrix}
\begin{bmatrix}
  C_0 \\
  C_1
\end{bmatrix} = 0
\]  

(18)

The \( \ell \)th estimated dimensionless natural frequency \( \Omega^{[N]}_{\ell} = \sqrt{\lambda^{[N]}_{\ell}} \) can be obtained by the solutions of Eq.(18), where \( \lambda^{[N]}_{\ell} \) is the \( \ell \)th eigenvalue estimated by truncating the expansion beyond the \( N+1 \)th term. \( N \) can be determined by requiring that \( \left| \Omega^{[N]}_{\ell} - \Omega^{[N-1]}_{\ell} \right| / \Omega^{[N]}_{\ell} \leq \epsilon \), where \( \epsilon \) is a small positive value.

Then, substituting the \( \ell \)th estimated eigenvalue \( \lambda^{[N]}_{\ell} \) in any of the equations in Eq.(13), we can obtain that

\[
C_{1,\ell} = -\frac{f^{[N]}_{10}(\beta, \lambda^{[N]}_{\ell})}{f^{[N]}_{11}(\beta, \lambda^{[N]}_{\ell})} C_{0,\ell}
\]

(19)

where \( C_{1,\ell} \) and \( C_{0,\ell} \) are the first two polynomial coefficients of the \( \ell \)th mode shape. Consequently, we can obtain all the coefficients \( C_{n,\ell} (1 \leq n \leq N) \) of the \( \ell \)th mode shape.

The nodal points of the \( \ell \)th mode can be estimated from the roots of the truncated polynomial series for the \( \ell \)th mode shape, i.e.

\[
\sum_{n=0}^{N} C_{n,\ell} X^n = 0
\]

(20)

The dimensionless position of the \( j \)th nodal point of the \( \ell \)th mode is denoted by \( X^{[N]}_{\ell, j} \), where the nodal points \( j = 1, 2, \ldots \), are ordered from the constrained end of the beam.
In this paper, all mode shape series were truncated to \(N = 31\) (corresponding to \(\epsilon = 5 \times 10^{-3}\) in solving Eq.(18)), and for clarity the superscript \([N] \) is dropped from the symbols \(\lambda_i\) and \(X_{i,j}\) in the following sections.

2.4. Sensitivities of eigenvalues and nodal points to boundary conditions

Substituting the \(i\)th estimated eigenvalue \(\lambda_i\) into Eq.(18), one can obtained the following identity

\[
\begin{bmatrix}
  f_{10}(\beta_i, \lambda_i) \\
  f_{11}(\beta_i, \lambda_i) \\
  f_{20}(\beta_i, \lambda_i) \\
  f_{21}(\beta_i, \lambda_i)
\end{bmatrix} = 0
\]  

(21)

Then, differentiating with respect to either \(\beta_i\) or \(\beta_r\) one can obtain the equations for calculating the sensitivity of eigenvalue \(\lambda_i\) with respect to changes in \(\beta_i\) or \(\beta_r\), i.e. \(\partial \lambda_i / \partial \beta_i\) or \(\partial \lambda_i / \partial \beta_r\).

Substituting for the position of the \(j\)th nodal point of the \(i\)th mode \(X_{i,j}\) into Eq.(20), one can obtain the following identity

\[
\sum_{n=0}^{N} C_{n,j} X_n^i \equiv 0
\]  

(22)

Then the above equation can be differentiated to obtain the sensitivities of \(X_{i,j}\) to \(\beta_i\) and \(\beta_r\), i.e. \(\partial X_{i,j} / \partial \beta_i\) and \(\partial X_{i,j} / \partial \beta_r\).

3. Identification of boundary conditions

An initial estimate of the stiffness parameters \(a_0\) (i.e. flexural rigidity of the constrained end \(EI_0\), root stiffnesses \(k_i\) and \(k_r\)) gives rise to a vector of predicted responses (i.e. natural frequencies and/or positions of nodal points) which can be expected to differ from their measured values as quantified by the error vector \(\Delta b\). This can be assumed to be attributable to a linear combination of errors \(\Delta a\) in the initial estimates such that

\[
\Delta b = S \Delta a
\]  

(23)

where \(S\) is a Jacobian matrix representing the sensitivities of the response vector to the vector of stiffness parameters. When applied to the problem in hand, \(S\) is a \(p \times 3\) matrix where \(p\) is the number of response quantities (natural frequencies and nodal points) used, viz.

\[
S = \begin{bmatrix}
  \partial b_1 / \partial (EI_0) & \partial b_1 / \partial k_i & \partial b_1 / \partial k_r \\
  \partial b_2 / \partial (EI_0) & \partial b_2 / \partial k_i & \partial b_2 / \partial k_r \\
  \vdots & \vdots & \vdots \\
  \partial b_p / \partial (EI_0) & \partial b_p / \partial k_i & \partial b_p / \partial k_r
\end{bmatrix}
\]  

(24)

where expressions for the sensitivities of natural frequencies and nodal points to dimensional parameters \(EI_0\), \(k_i\) and \(k_r\) can be easily obtained from the preceding analysis.

Eq.(23) is potentially over-determined but can be solved by employing the Moore-Penrose inverse, or solved in a least squares sense by minimising the cost function

\[
J = \|W^{1/2} (S \Delta a - \Delta b)\|^2
\]  

(25)

where \(W\) is a weighting matrix which is chosen here so as to scale the rows of \(S\) to have a maximum element of unity. However, as with many other inverse problems, Eq.(23) is prone to ill-conditioning, and regularization may be required to improve robustness of the solution to measurement noise on \(\Delta b\). Tikhonov regularization [7] is employed here in which minimisation is sought of the cost function...
\[ J = \left\| W^{1/2}(S\Delta a - \Delta b) \right\|^2_2 + \gamma \left\| \Delta a \right\|^2_2 \]  \tag{26}

where \( \gamma \) is the regularization parameter, which if set to zero results in the least squares solution as above. The optimal regularization parameter \( \gamma \) can be calculated by the L-curve method (LCM) \[8\].

The solution of Eq.(26) requires that \( \partial J / \partial a = 0 \), and yields

\[ \Delta a = (S^T WS + \gamma I)^{-1} S^T W \Delta b \]  \tag{27}

where \( I \) is the identity matrix. The new estimate for the stiffness parameters, \( a_0 + \Delta a \) yields an improved model, if successful, with revised sensitivities which can be used to update the parameter values iteratively. i.e.

\[ a_{k+1} = a_k + (S_k^T WS_k + \gamma_k I)^{-1} S_k^T W \Delta b_k \]  \tag{28}

Various criteria for convergence exist. Here, convergence is deemed to have occurred if \( a_{k+1} - a_k \) is smaller than some threshold vector. Divergence can occur, especially when the initial estimate \( a_0 \) is not sufficiently accurate.

4. Simulative Examples

A tapered beam is adopted as the simulative example to assess the effectiveness of the proposed method, and the properties of the beam are listed in Table 1. Three different sets of boundary conditions are selected for the beam, namely Boundary A, \( \beta_r = 1, \beta_t = 1 \); Boundary B, \( \beta_r = 10^3, \beta_t = 1 \) and Boundary C, \( \beta_r = 1, \beta_t = 10^4 \). Sensitivity analysis suggests that the first three natural frequencies might be utilized effectively for all of the three boundary conditions, and additionally the nodal points of the second and third modes might be utilized for Boundary A but not B or C.

| Young’s modulus \( E \) | Poisson ratio \( \nu \) | density \( \rho \) | length \( l \) | width \( b_0 \) | depth \( h_0 \) | taper degree \( \beta_b \) | taper degree \( \beta_h \) |
|------------------------|------------------|-------------|-------|-------|-------|---------------|---------------|
| 71GPa                  | 0.3              | 2700kg/m³  | 0.5m  | 0.03m | 0.003m| 0.5           | 0             |

In the simulations, the “measured” responses are simulated by

\[ \omega_{i, \text{true}} \]

\[ X_{i,j, \text{true}} \]

where \( \omega_{i, \text{true}} \) is the true value of the \( i \)th natural frequency, \( \eta_{\text{true}} \) is the noise level on the measured natural frequencies, \( X_{i,j, \text{true}} \) is the true value for the \( j \)th nodal point of the \( i \)th mode, \( \eta_{\text{np}} \) is the noise level on the measurement of the nodal point, \( \delta \) is a normally distributed random variable with zero mean and unit variance.

It is easy to verify that the precise stiffness values can be obtained when the noise levels \( \eta \) are zero. Thus, the simulations are performed for non-zero noise levels. Three noise levels are considered for natural frequency, i.e. \( \eta_{\text{true}} = 0.005, \eta_{\text{true}} = 0.01 \) and \( \eta_{\text{true}} = 0.02 \); and only one noise level is considered for the position of nodal points, i.e. \( \eta_{\text{np}} = 0.01 \). 100 simulations are performed for a particular noise level, and the relative error and the coefficient of variation of the identified parameter are utilized to evaluate the effectiveness of the methods. The relative error of the identified parameter is defined by
where $p$ indicates the true stiffness parameter value (translational root stiffness $k_t$, rotational root stiffness $k_r$, or flexural rigidity $EI_0$), $\mu_p$ indicates the estimate which is the mean value from all of the 100 simulations for which a converged solution was obtained. The coefficient of variation $|\sigma_p / \mu_p|$ is also calculated based on the converged solution, and $\sigma_p$ indicates the estimate which is the standard deviation from all of the 100 simulations for which a converged solution was obtained. The iteration is deemed to have converged if all the following three conditions are satisfied: 1) the relative change in 2-norm of the identified parameters $||a_{k+1} - a_k|| / ||a_k||$ is equal or smaller than $10^{-10}$; 2) all the identified parameters are positive real numbers in the iteration; 3) the maximum iterative step is smaller the 1000.

In each simulation, the initial values of $EI_0$, $k_t$ and $k_r$ are set as 1.25 times, 1.40 times and 1.45 times their true values, respectively, i.e. the initial values are significant over-estimates. Based on some preliminary analysis, the sensitivity matrix $S$ is expected to be well conditioned in the case of boundary A but ill conditioned for both Boundary B and Boundary C. Consequently, regularization is adopted for Boundary B and Boundary C but not Boundary A.

![Figure 2](image-url)  
Figure 2 shows the relative errors in identified parameters of the tapered beam with Boundary A using different responses, (a) updating using the first three natural frequencies only; (b) updating using the first three natural frequencies and the position of nodal point of the second mode ($\eta_{np} = 0.01$).

Figure 2 shows the relative errors in identified parameters of the tapered beam with Boundary A using different responses, and Table 2 lists the corresponding coefficient of variation of each identified parameter and the number of converged solution in the 100 simulations. It can be seen in Figure 2(a) that relatively good estimates of all the three parameters might be obtained. The addition of nodal point information, in Figure 2(b), only improves the robustness of estimates for $k_r$ when measurement noise level $\eta_{fs} = 0.02$, but is detrimental to the estimate of other parameters.

Figure 3 shows the relative errors in identified parameters of the tapered beam with Boundary B and Boundary C using the first three natural frequencies, and Table 3 lists the corresponding coefficient of variation of each identified parameter and the number of convergence in the 100 simulations. It can be seen in Figure 3 that $EI_0$ is estimated well for both boundaries, $k_r$ is estimated...
well for Boundary B (meanwhile $k_r$ is unchanged, see Figure 3(a)) and $k_r$ is estimated well for Boundary C (meanwhile $k_t$ is unchanged, see Figure 3(b)).

It can be seen in Table 2 and Table 3 that the coefficients of variation are generally not too much larger than the noise imposed on the simulated measurements, especially in Table 3 where regularization is employed. This suggests that the estimates are reasonably robust to measurement noise, at least for the noise model and the regularization adopted here.

![Figure 3. the relative errors in identified parameters of the tapered beam with Boundary B and Boundary C using the first three natural frequencies, (a) Boundary B; (b) Boundary C](image)

| noise level $\eta_{fr}$ | number of converged solutions | $|\sigma_{Eh}/\mu_{Eh}|$ | $|\sigma_{k_r}/\mu_{k_r}|$ | $|\sigma_{k_t}/\mu_{k_t}|$ |
|-------------------------|-----------------------------|------------------|------------------|------------------|
| 0.005                   | 100                         | 0.0107           | 0.0223           | 0.0148           |
| 0.01                    | 100                         | 0.0215           | 0.0446           | 0.0297           |
| 0.02                    | 100                         | 0.0397           | 0.0930           | 0.0770           |

| noise level $\eta_{fr}$ | number of converged solutions | $|\sigma_{Eh}/\mu_{Eh}|$ | $|\sigma_{k_r}/\mu_{k_r}|$ | $|\sigma_{k_t}/\mu_{k_t}|$ |
|-------------------------|-----------------------------|------------------|------------------|------------------|
| 0.005                   | 100                         | 0.0366           | 0.0142           | 0.0150           |
| 0.01                    | 100                         | 0.0348           | 0.0278           | 0.0212           |
| 0.02                    | 99                          | 0.0343           | 0.0472           | 0.0486           |

### 5. Conclusions

The natural frequencies and nodal positions of cantilever-like beams can depend on the exact translational and rotational stiffness at the constrained end. This paper has explored the inverse problem of identifying constraint stiffnesses from modal parameters. The equations of motion have been solved for free vibration of a linearly tapered beam, and the sensitivity of natural frequencies and nodal points to constraint stiffnesses has been considered. These sensitivities have then been used in a non-linear least squares estimation to estimate the constraint stiffnesses from modal data.
examples have been presented to illustrate the method, and to demonstrate the modest potential benefit of using nodal points as an additional source of information for some particular cases.

Table 3(a) the coefficient of variation of the identified parameters and the number of converged solutions in 100 simulations for the tapered beam with Boundary B using the first three natural frequencies

| noise level $\eta_{fs}$ | number of converged solutions | $|\sigma_{Eh} / \mu_{Eh}|$ | $|\sigma_{kh} / \mu_{kh}|$ | $|\sigma_{tk} / \mu_{tk}|$ |
|------------------------|-------------------------------|-----------------|-----------------|-----------------|
| 0.005                  | 100                           | 0.0071          | 0               | 0.0099          |
| 0.01                   | 100                           | 0.0141          | 0               | 0.0198          |
| 0.02                   | 100                           | 0.0168          | 0               | 0.0264          |

Table 3(b) the coefficient of variation of the identified parameters and the number of converged solutions in 100 simulations for the tapered beam with Boundary C using the first three natural frequencies

| noise level $\eta_{fs}$ | number of converged solutions | $|\sigma_{Eh} / \mu_{Eh}|$ | $|\sigma_{kh} / \mu_{kh}|$ | $|\sigma_{tk} / \mu_{tk}|$ |
|------------------------|-------------------------------|-----------------|-----------------|-----------------|
| 0.005                  | 100                           | 0.0077          | 0.0123          | 0               |
| 0.01                   | 100                           | 0.0153          | 0.0247          | 0               |
| 0.02                   | 100                           | 0.0306          | 0.0494          | 0               |

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