Many-Body Localization of Symmetry Protected Topological States

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We address the following question: Which kinds of symmetry protected topological (SPT) Hamiltonians can be many-body localized? That is, which Hamiltonians with an SPT ground state have finite energy density excited states which are all localized by disorder? Based on the observation that a finite energy density state, if localized, can be viewed as the ground state of a local Hamiltonian, we propose a simple (though possibly incomplete) rule for many-body localization of SPT Hamiltonians: If the ground state and top state (highest energy state) belong to the same SPT phase, then it is possible to localize all the finite energy density states; If the ground and top state belong to different SPT phases, then most likely there are some finite energy density states which can not be fully localized. We will give concrete examples of both scenarios. In some of these examples, we argue that interaction can actually “assist” localization of finite energy density states, which is counter-intuitive to what is usually expected.

1. INTRODUCTION

Syntropy protected topological (SPT) states and many-body localization (MBL) are two striking phenomena of quantum many-body physics. A $d$-dimensional SPT state is the ground state of a local Hamiltonian whose $d$-dim bulk is fully gapped and nondegenerate, while its $(d-1)$-dim boundary is gapless or degenerate when and only when the system preserves a certain symmetry $G$. An SPT state must have “short range entanglement”; meaning that the entanglement entropy of its subsystems scales strictly with the area of the boundary of the subsystem: $S_A \sim L^{d-1-\delta}$, where $L$ is the linear size of the subsystem $A$. MBL refers to a phenomenon of the entire spectrum of a local Hamiltonian with disorder, including all of the highly excited states with finite energy density. Localization of single particle states under quenched disorder is well-understood [1], and recent studies suggest that localization can survive under interaction [5,9]. In our current work the phrase MBL refers to systems whose all many-body eigenstates are localized, namely the entanglement entropy of all finite energy density states obey the same area law as SPT states instead of the usual volume law typically obeyed by finite energy density states. These observations imply that in a many-body localized system, any finite energy density state actually behaves like the ground state of a local parent Hamiltonian. Indeed, it was proposed that phenomena such as stable edge states and spontaneous symmetry breaking [3,7–9], which usually occur at the ground state of a system, can actually occur in finite energy density states of MBL systems. In fact, we can define a MBL system as a system for which any finite energy density eigenstate is a short range entangled ground state of a local parent Hamiltonian. And if the system preserves a certain symmetry, then any finite energy density state of the MBL system should also obey the classification of SPT states. Then we can view energy density $\varepsilon$ as a tuning parameter between SPT states. Of course, in the thermodynamic limit, because there are infinite states in an infinitesimal energy density interval ($\varepsilon, \varepsilon + d\varepsilon$), we expect there exists many 1d curves in the spectrum parameterized by $\varepsilon$ with one state $|\psi\rangle_{\varepsilon}$ at each $\varepsilon$, which is the ground state of an effective SPT Hamiltonian $H_{\varepsilon}$. And on each such curve $|\psi\rangle_{\varepsilon}$ is (roughly speaking) continuous in the sense that $|\psi\rangle_{\varepsilon}$ and $|\psi\rangle_{\varepsilon+\varepsilon}$ are similar (despite being orthogonal), namely physical quantities averaged over the entire system change continuously with $\varepsilon$ on this curve. In an ergodic system, the eigenstate thermalization hypothesis [10] implies that most states with similar energy density $\varepsilon$ are similar (their reduced density matrices all behave like a thermal density matrix); in a MBL state, although states with the same energy density can in principle be very different, we still expect (assume) that the continuous curves mentioned above exist, although states in different curves can be very different.

Within one of these curves mentioned above, tuning $\varepsilon$ is just like tuning between the ground states of local Hamiltonians. Furthermore, by tuning $\varepsilon$ there may or may not be a phase transition. In particular, if all excited states belong to the same SPT phase for arbitrary energy density $\varepsilon$, then there does not have to be any quantum phase transition when tuning $\varepsilon$, which implies that all of the excited states have short range correlations and area-law entanglement entropy, i.e. all the finite energy density states are localized; on the other hand, if states with different energy density $\varepsilon$ on the same curve belong to different SPT phases, then there must be at least one phase transition at certain critical energy on this curve when tuning $\varepsilon$. This phase transition behaves just like an ordinary zero temperature quantum phase transition between different quantum ground states under disorder. For 1d systems this “critical” energy density state could be in the “infinite-randomness” phase [11, 12], whose entanglement entropy scales logarithmically with the subsystem size [13], hence it is not fully localized. The existence of the “infinite-randomness” states at finite energy density have already been observed in Ref. [16].
Due to the fact that in a generic nonintegrable Hamiltonian $H$, the ground state $|G\rangle$ and top state $|T\rangle$ (highest energy state of $H$ and also ground state of $-H$) are usually the easiest states to analyze, the most convenient way to determine the existence of “critical” states in the spectrum is to check whether the ground and top states belong to the same SPT phase or not. In summary, if $|G\rangle$ and $|T\rangle$ belong to different SPT phases, and if we understand that these two SPT states are separated by one or multiple continuous phase transitions (this will depend on the type of SPT phases $|G\rangle$ and $|T\rangle$ belong to), then there must be some “critical” excited states in the spectrum which cannot be fully localized \cite{46}. We will apply this rule to various examples in the next section.

2. EXAMPLES

2A. Kitaev’s chain: localization

We first apply our argument to the Kitaev’s chain:

$$H = \sum_j - (t + (-)^j \delta t + \Delta t_j) i \gamma_j \gamma_{j+1}, \quad (1)$$

where $\gamma_j$ are Majorana fermions and $\Delta t_j$ is a random hopping parameter with zero mean and standard deviation $\sigma_{\Delta t}$. The topological superconductor phase ($\delta t > 0$) and the trivial phase ($\delta t < 0$) can both be fully localized by disorder, because for either sign of $\delta t$, the ground state $|G\rangle$ and top state $|T\rangle$ both belong to the same phase (we choose the convention that $(2j-1, 2j)$ is a unit cell). This can be seen in the clean limit with $\Delta t = 0$. In momentum space $H = \sum_k d^\dagger(k) \tau^z + d(k) \tau^z$, and $d$ is a nonzero $O(2)$ vector in the entire 1d Brillouin zone with $\delta t \neq 0$. For either sign of $\delta t$, $H$ and $-H$ have the same topological winding number $n_1 = \frac{1}{2\pi} \int dk \ d^\dagger(k) \partial_k \partial_0 \epsilon_{ab}$, thus $|G\rangle$ and $|T\rangle$ belong to the same phase. Based on our argument, all the finite energy states with either sign of $\delta t$ can be fully localized by random hopping $\Delta t$. The only states not fully localized in the two dimensional phase diagram tuned by $\epsilon$ and $\delta t$ are located at the critical line $\delta t = 0$. The critical line $\delta t = 0$ is in an “infinite-randomness” fixed point, and it can be understood through the strong disordered real space renormalization group. \cite{13, 14, 15, 16}

Here we confirm the conclusions in Ref. \cite{3, 7, 9} that the finite energy density excited states of the Kitaev’s chain with $\delta t > 0$ are still “topological”. Since the energy level spacing between two eigenstates vanishes in the thermodynamic limit, the best way to determine if an excited state is topological or not is to compute its entanglement spectrum (the system is defined on a periodic 1d lattice). And because the system is noninteracting, we will compute the single-particle entanglement spectrum introduced in Ref. \cite{19} for each excited state. The single-particle entanglement spectrum for the topological phase ($\delta t > 0$) is shown in Fig.1(a), where two zero energy modes can be observed in the spectrum (corresponding to the Majorana zero modes at both entanglement cuts respectively). This topologically non-trivial feature persists for all energy eigenstates in the many-body spectrum, including the ground/top states and the finite energy density states in between. However at the critical line $\delta t = 0$, as shown in Fig.1(b), the zero energy modes are lifted by the long-range entanglement, and the single-particle entanglement levels become gapless around $\epsilon_E = 0$ which leads to the logarithmic scaling of the entanglement entropy.

The Kitaev’s chain itself is just a free fermion model. But our argument indicates that under interaction, as long as $|G\rangle$ and $|T\rangle$ are still both in the topological superconductor phase, all of the excited states can still be localized. Such a generalization is justified given that the non-interacting Anderson localized states can be adiabatically connected to the many-body localized states under interaction, as proven in Ref. \cite{4].

![FIG. 1: Single-particle entanglement spectrum for many-body eigenstates of the random Kitaev’s chain, at (a) $\delta t = 0.5t$ and (b) $\delta t = 0$. In both cases $\sigma_{\Delta t} = 0.3t$. We take a 128-site system with periodic boundary condition, which is partitioned into two 64-site subsystems for the entanglement calculation. $\epsilon_E$ is the single-particle entanglement energy (s.t. the reduced density matrix $\rho_A = \exp(-\epsilon E \rho_{E})$, as shown in Ref. [19]. The spectrum of $\epsilon_E$ is shown as tanh $\epsilon E$, and is calculated for several many-body eigenstates: including the ground and the top states and other 5 randomly picked finite energy density states, which are arranged in order of their energy density $\epsilon$. The shading denotes the standard deviation of the entanglement energy levels under a disorder average over the system. Of note are the topologically non-trivial, two-fold degenerate, zero energy modes throughout the entire spectrum $\epsilon$ in the topological phase (a).]

2B. Modified Kitaev’s chain: critical states and interaction assisted localization

In this subsection we consider a modified Kitaev’s chain:

$$H = \sum_j - (t - (-)^j t' \sigma_j^z + \Delta t_j) i \gamma_j \gamma_{j+1} - h \sigma_j^z, \quad (2)$$
where again $\Delta t_j$ is random and $t, t', h > 0$. In this model $\sigma_j^z$ commutes with the Hamiltonian, which implies that any energy eigenstate will also be an eigenstate of $\sigma_j^z$. In the clean limit, the ground state $|G\rangle$ of the system has $\sigma_j^z = 1$ everywhere, and the fermions are in the trivial phase; in contrast, $|T\rangle$ must have $\sigma_j^z = -1$ everywhere, and hence $|T\rangle$ is in the topological superconductor phase. With disorder, both states can be localized, and their entanglement entropy shows the area-law scaling (i.e. $S \sim \text{const.}$ for 1$d$) as in Fig. 2(a). But since the ground state and the top state belong to different SPT phases, based on our argument, there must be some finite energy density states which cannot be fully localized. In this model it is easy to visualize these delocalized excited states. An excited state of the system has a static background configuration of $\sigma_j^z$ which does not satisfy $\sigma_j^z = 1$. If we consider a random configuration of $\sigma_j^z$ that has the average $\langle \sigma_j^z \rangle = 0$, then one can simply absorb $\sigma_j^z$ into the random numbers $\Delta t_j$, and the effective Hamiltonian for Majorana fermions $\gamma_j$ reads $H_{\text{eff}} = \sum_j - (t + \Delta t_j) i \gamma_j \gamma_{j+1}$, which is precisely the random hopping Majorana fermion model Eq. (1) tuned to the critical point $\delta t = 0$. And according to Ref. 14, 15 the ground state of $H_{\text{eff}}$ (which is a highly excited state of the original Hamiltonian Eq. (2) due to the $h$ term) has a power-law correlation after disorder average, and its entanglement entropy scales logarithmically with the subsystem size: $S \sim \log \ell$ [15]. So the delocalization happens right at the energy scale $E_\sigma \equiv -h \sum_j \sigma_j^z = 0$. In deed our numerical calculation shows that as long as $E_\sigma \neq 0$, the eigen states are all localized with area-law entanglement entropy as in Fig. 2(a,b); but for $E_\sigma = 0$, the eigen states are delocalized with logarithmically-scaled entanglement entropy as in Fig. 2(c). Thus the model Eq. (2) cannot be fully many-body localized, which is consistent with our statement made in the introduction.

The model Eq. (2) has a time-reversal symmetry $T : \gamma_j \to (-)^j \gamma_j$ and $\sigma_j^z \to \sigma_j^z$. It is known that with this time-reversal symmetry and without interactions, the Kitaev’s chain has $\mathbb{Z}$ classification [20, 22]; that is with an arbitrary number of flavors of Eq. (2) $|T\rangle$ is always a nontrivial topological superconductor, while $|G\rangle$ is always a trivial phase. However under certain flavor mixing four-fermion interaction [23, 24], the classification of Kitaev’s chain with time-reversal symmetry reduces to $\mathbb{Z}_2$. Namely under this four-fermion interaction, for eight copies of Eq. (2) $|G\rangle$ and $|T\rangle$ become the same trivial phase, which implies that there does not have to be any phase transition when increasing $\varepsilon$, and all of the finite energy density excited states can be fully localized under the interplay between disorder and interaction.

In model Eq. (2) the logarithmic entanglement entropy at the critical excited state comes from the long range effective hopping under renormalization group [13, 15]. We can assume that the four-fermion interaction on each site is random, then when and only when there are $8k$ copies of Eq. (2) under interaction each site independently possesses a random set of many-body spectrum without degeneracy. Let $\delta V$ be the typical energy level spacing of the interaction Hamiltonian on each site. To create entangled pairs between distant sites, the effective long-range coupling $t_{\text{eff}}$ generated under RG must overcome the energy scale of $\delta V$ to hybridize the many-body states. However the effective coupling strength actually falls rapidly with the distance [13] as $t_{\text{eff}} \sim t e^{-\sqrt{t}}$, so the long-range coupling can only lead to exponentially small entanglement $\Delta S \sim (t_{\text{eff}}/\delta V)^2 \sim (t/\delta V)^2 e^{-2\sqrt{t}}$. Therefore even with weak interaction, all of the eigenstates are short-range entangled area-law states, and can

![FIG. 2: Entanglement entropy $S_\ell$ vs log subsystem length $\log_2 \ell$ vs fermion energy $E_\sigma$ (energy of the first term in Eq. (2)) for various boson energies $E_\rho \equiv -h \sum_j \sigma_j^z = -h L/2, 0$ (a,b,c) (second term in Eq. (2)). Calculations are done on a random Majorana chain with $L = 1024$ sites, and the standard deviation of $\Delta t_j$ is $\sigma_{\Delta t} = t$. States with $E_\rho = 0$ are the critical excited states which are delocalized. All states with different $E_\sigma$ at $E_\rho = 0$ have logarithmic entanglement entropy, and hence are delocalized.](image-url)
be fully localized. In contrast, without interaction, no matter what kind of fermion-bilinear perturbations we turn on in Eq. 2 as long as these terms preserve the time-reversal symmetry defined above and the topological nature of $|G\rangle$ and $|T\rangle$, there must necessarily be some finite energy density states which cannot be fully localized. Thus in this case interaction actually “assists” many-body localization, which is opposite from what is usually expected for weak interaction, in for example Ref. 25 and is also different from the strong interaction reinforced localization studied in Ref. 26–28.

Notice that this “interaction assisted localization” is only possible with 8$k$ copies of the Kitaev’s chain with time-reversal symmetry. With 4 copies of the Kitaev’s chain, the spectrum on each site contains two sets of two-fold degenerate states even under interaction that preserves time-reversal, then the effective long-range coupling $t_{\text{eff}}$ generated under RG will still lead to maximal entanglement between distant sites. A detailed RG analysis about this will be given in another paper 29.

2C. Bosonic SPT states, Haldane phase

Many bosonic SPT parent Hamiltonians can be written as a sum of mutually commuting local terms. For example, the “cluster model” for the 1d SPT with $Z_2 \times Z_2$ symmetry 9, the Levin-Gu model 30 and the CZX model 31 for the 2d SPT states with $Z_2$ symmetry, and the 3d bosonic SPT state with time-reversal symmetry 32 are all a sum of commuting local operators; thus their ground states are a product of eigenstates of local operators 17. SPT Hamiltonians written in this form are very similar to the “universal” Hamiltonian of MBL state proposed in Ref. 43 which is also a sum of mutually commuting local terms, because a MBL system has an infinite number of local conserved quantities.

All of the idealized SPT models mentioned above have a $Z_2$ classification, and their ground and top states belong to the same SPT phase. Obviously there should be no phase transition while increasing energy density $\epsilon$. This statement is still valid with small perturbations which make these models nonintegrable as long as the nature of $|G\rangle$ and $|T\rangle$ are not affected by the perturbations. Thus these models (and their nonintegrable versions) can all be fully localized by disorder.

However, some other bosonic SPT models can not be fully localized. In the following we will give one such example for the Haldane phase 34, 35:

$$H = \sum_j (-1)^j (J + \Delta J_j) S_j \cdot S_{j+1} + \cdots$$

$S_j$ are spin-1/2 operators. The ellipsis includes perturbations that break the system’s symmetry down to a smaller symmetry (such as time-reversal or $Z_2 \times Z_2$) that is sufficient to protect the Haldane phase, but do not lead to degeneracy in the bulk spectrum, namely only the boundary transforms nontrivially under symmetry. If the random coupling $\Delta J_j$ is not strong enough to change the sign of $J$, then the ground state and top state of this model correspond to two opposite dimerization patterns of the spin-1/2s. Thus one of them is equivalent to the Haldane’s phase while the other is a trivial phase as long as we pick a convention of boundary. If we assume the random Heisenberg coupling $\Delta J$ is sufficient to localize most of the excited states, then there must be an unavoidable phase transition while increasing energy density $\epsilon$. According to our argument in the introduction, this phase transition should behave just like an ordinary quantum phase transition at zero temperature. It is known that the quantum phase transition between a Haldane phase and a trivial phase is a conformal field theory, and it is equivalent to a spin-1/2 chain without dimerization. With strong disorder, this quantum critical point will be driven into the infinite-randomness spin singlet phase 11, 13 with a power-law decaying disorder averaged spin-spin correlation function and a logarithmic entanglement entropy 14.

2D. 2d interacting topological superconductor: critical states and interaction assisted localization

In this subsection we will discuss the nonchiral 2d $p \pm ip$ topological superconductor, i.e. $p + ip$ pairing for spin-up fermions, and $p - ip$ pairing for spin-down fermions. On a square lattice this TSC can be written in the Majorana fermion basis:

$$H = \sum_k \chi_k^\dagger (\tau^x \sin k_x + \tau^y \sigma^z \sin k_y) \chi_k$$

$$+ \chi_k^\dagger \tau^y (e - \cos k_x - \cos k_y) \chi_k,$$

where $\sigma^z = \pm 1$ represents spin-up and down, while $\tau^z = \pm 1$ represents the real and imaginary parts of the electron operator. Without any symmetry, this system is equivalent to the trivial state, i.e. its boundary can be gapped out without degeneracy. However, when $0 < e < 2$, with a $Z_2$ symmetry which acts as $Z_2 : \chi \rightarrow \sigma^z \chi$, the system is a nontrivial TSC. This system can also have another time-reversal symmetry, which is unimportant to our analysis. The boundary of this system reads: $H = \int dx \chi^\dagger(-i\partial_x \sigma^z) \chi$, $Z_2 : \chi \rightarrow \sigma^z \chi$. The $Z_2$ symmetry forbids any single particle backscattering at the boundary for arbitrary copies of the system, thus the $p \pm ip$ TSC with the $Z_2$ symmetry has a $Z$ classification without interaction.

Without any interaction, for $n$–copies of the $p \pm ip$ TSC, $|G\rangle$ and $|T\rangle$ belong to different SPT phases. This is because for either spin-up or down fermions, the Chern number of $|G\rangle$ and $|T\rangle$ are opposite. And because the system has a $Z$ classification, $|G\rangle$ and $|T\rangle$ must belong to
different SPT states. Using our argument in the introduction, this implies that under disorder that preserves the $Z_2$ symmetry, there must be some finite energy density states which cannot be fully localized. This is not surprising, considering that even at the single particle level there are likely extended single particle states under disorder. The existence of extended single particle states is well-known in integer quantum Hall state [30] and recently generalized to quantum spin Hall insulator [31], and an extended discussion on single particle states which cannot be fully localized.

The situation will be very different with interactions. Once again a well-designed interaction will reduce the classification of this $p \pm ip$ TSC from $Z$ to $Z_8$ [32] [33]. Namely $n$--copies of $G$ is topologically equivalent to $(n+8k)$--copies. This implies that under interaction $|G\rangle$ and $|T\rangle$ actually belong to the same phase when $n = 4k$. Thus when $n = 4k$, the phase transition in the noninteracting limit will be circumvented by interaction above a certain critical value. Thus once again interaction assists MBL in this case. When $n = 8$, $|G\rangle$ and $|T\rangle$ are both trivialized by interaction, namely interaction can adiabatically connect both states to a direct product of local states. When $n = 4$, Ref. [33] showed that interaction can confine the fermionic degrees of freedom, and drive four copies of the $p \pm ip$ TSC into a 2$d$ bosonic SPT state with $Z_2$ symmetry, which as we discussed in the previous section, can also be fully many-body localized.

Please note that in the noninteracting limit the quantum phase transition between 2$d$ TSC and trivial state is described by gapless $(2 + 1)d$ Majorana fermions, and since a weak short range four-fermion interaction is irrelevant for gapless $(2 + 1)d$ Dirac/Majorana fermions, only strong enough interaction can gap out the quantum phase transition. Thus unlike the 1$d$ analogue discussed in section 2B, we expect that in this 2$d$ system only strong enough interaction can “assist” disorder and localize all the excited states even for $n = 4k$.

3. SUMMARY

In this work we propose a simple rule to determine whether a local Hamiltonian with symmetry can be many-body localized. Since MBL is a phenomenon for the entire spectrum, we need to start with a lattice Hamiltonian for our analysis. Therefore the low energy field theory description and classification of SPT states such as the Chern-Simons field theory [34] and the non-linear sigma model field theory [35] will not be able to address this question. Instead, our argument is based on the nature of the ground and top states of the same lattice Hamiltonian. Our argument is general enough, that it can be applied to both free and interacting systems, bosonic and fermionic SPT systems. And counterintuitively, we found that because interactions change the classification of fermionic topological insulators and topological superconductors, in some cases interactions actually assists localization, rather than delocalization.

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In this work, the phrase “SPT states” also include direct product states, we view direct product states as “trivial” SPT states. So far not all quantum phase transitions between SPT states have been completely studied, and it is possible that some SPT states are separated by a first order transition. Our statement only applies to the cases that $|G\rangle$ and $|T\rangle$ belong to two different SPT phases that we know are separate by a continuous phase transition, for example the transition between the topological superconductor and the trivial state of the Kitaev’s chain (section 2B).

Ref. [1] actually proposed a general way of constructing parent Hamiltonians for all bosonic SPT states within the group cohomology classification. However, in Ref. [1] the local Hilbert space is labeled by group elements, which implies that for a system with continuous symmetry the local Hilbert space in Ref. [1]’s construction already has infinite dimension, and hence its excited states can also have infinite local energy density. In this work we only discuss systems with a finite dimensional Hilbert space and finite energy density.