Fixed-Time Cooperative Tracking Control for Double-Integrator Multiagent Systems: A Time-Based Generator Approach

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Abstract—In this article, both the fixed-time distributed consensus tracking and the fixed-time distributed average tracking problems for double-integrator-type multiagent systems with bounded input disturbances are studied. First, a new practical robust fixed-time sliding-mode control method based on the time-based generator is proposed. Second, two fixed-time distributed consensus tracking observers for double-integrator-type multiagent systems are designed to estimate the state disagreement between the leader and the followers under undirected and directed communication, respectively. Third, a fixed-time distributed average tracking observer for double-integrator-type multiagent systems is designed to measure the average value of multiple reference signals under undirected communication. Note that all the proposed observers are constructed with time-based generators and can be trivially extended to that for high-order integrator-type multiagent systems. Furthermore, by combining the proposed fixed-time sliding-mode control method with the information provided by the fixed-time observers, the fixed-time controllers are designed to solve the fixed-time distributed consensus tracking and the distributed average tracking problems. Finally, a few numerical simulations are shown to verify the results.

Index Terms—Consensus tracking, distributed average tracking, distributed observer, fixed time, sliding-mode control, time-based generator.

I. INTRODUCTION

DISTRIBUTED cooperation control has been a popular research issue over the past decades due to its significant value in reality, such as distributed optimization [1], [2]; tracking control [3], [4], [5]; and flocking and containment control [6], [7], [8].

In distributed cooperation control of a flock of agents, an important research issue is to develop an algorithm that makes the agents achieve consensus. The consensus algorithm for single-integrator multiagent systems was first developed in [9], and then some sufficient and necessary conditions for the consensus of double-integrator multiagent systems were generalized in [10]. Distributed tracking control can be regarded as an extension of generalized consensus control, in which the followers not only have to reach consensus but also to follow with the specified trajectory. For example, in distributed consensus tracking and distributed average tracking, the target trajectories are the states of the leader and the average value of multiple reference signals, respectively. However, in distributed control systems, only a few of the agents can directly acquire the target information. Then, to obtain the target information, resorting to observers is a frequently used method. Some observer-based algorithms for solving the distributed consensus tracking problem were proposed in [11], [12], and [13]. For solving the distributed average tracking problem, some observers were developed in [14], [15], [16], and [17].

However, these above observers are asymptotically stable, which implies that the upper bounded convergence time is not guaranteed. Compared with the asymptotic control approach, the finite-time control technique usually has a faster response. Therefore, many finite-time control methods have been developed during the past few years [18], [19], [20], [21]. Moreover, some novel finite-time observers for multiagent systems were proposed in [22], [23], [24], and [25]. Nevertheless, the upper bounded convergence time of finite-time algorithms depends on the initial states of agents. In some engineering practices, the initial states are not available and hence the upper bounded convergence time can not be guaranteed as well. Motivated by this fact, the fixed-time stability strategy was first investigated in [26], in which the predefined convergence time does not depend on system initial conditions. Then, some novel fixed-time algorithms for single-integrator
multiagent systems were developed in [27]. For the double-integrator-type multiagent system under the undirected communication topology, a kind of fixed-time observer was designed to estimate the states of the leader in [28]. Then, Tian et al. developed a fixed-time method without relying on velocity measurements to solve the same problem in [29]. Under the directed communication, a kind of fixed-time attractive observer for higher-order integrator-type multiagent systems was developed in [30]. Note that this algorithm can only guarantee the system states to converge to a specific attractive region in the predefined time, and then the algorithm becomes asymptotically stable. To reduce energy consumption, Ni et al. [31] proposed to introduce the event-triggered strategy into the fixed-time consensus tracking algorithm for higher-order multiagent systems.

After the observer obtains the object’s trajectory in a fixed-time, the fixed-time controller will be used to drive the agent to track the object trajectory. For the double-integrator-type control system, sliding-mode control method is a classic nonlinear control protocol [32], which has the advantages of fast response, parameter change, insensitive to disturbance and simple physical implementation. Although some finite-time sliding-mode control methods have been proposed in [33] and [34], extending them to the fixed-time algorithm for double-integrator-type control systems is nontrivial. Zuo proposed a fixed-time sliding-mode control protocol by utilizing the sinusoid function to offset the singularity in the neighborhood of zero [35], but it lacks consideration to the disturbances. Moreover, a similar fixed-time convergent sliding surface algorithm can be found in [36].

The common ground of the above fixed-time protocol is that they incorporate power functions into their control inputs, which are prone to result in system input saturation. Recently, Ning et al. [37] proposed a novel fixed-time consensus strategy for single-integrator multiagent systems by using time-based generators. Compared with the power function-based methods, the time-based generator approaches need a less magnitude of control input and it could hence effectively avoid system input saturation. Some related references about time-based approaches can be found in [38], [39], and [40]. For the time synchronization between different agents, the clock synchronization device has been proposed to guarantee the time synchronization [41], [42], and it is hence not reiterated here.

Compared with the above related works, five main contributions are made in this article. First, a novel fixed-time sliding-mode control method is developed. Compared with the fixed-time algorithm in [35] and [36], its control input does not include the power function that might result in system input saturation. Moreover, the disturbance is taken into consideration. Second, a novel fixed-time distributed observer for second-order tracking problems under the undirected topology is proposed. Compared with the fixed-time observer in [28] and [29], the predefined convergence time of the observer via time-based generators can be adjusted more easily. Third, the fully fixed-time distributed consensus tracking observer under directed communication is proposed. Compared with the prior work in [30], the proposed method can precisely design the upper bounded convergence time independent of initial system states and the system disturbance is analyzed. Fourth, the fixed-time observer for distributed average tracking problems under the undirected topology graph is designed. Compared with the finite-time observer in [25], the proposed method can achieve fixed-time stability. Moreover, to the best of our knowledge, the second-order fixed-time observer for distributed average tracking under the undirected topology graph is first proposed in this article. Furthermore, all the proposed observers can be easily extended to that for higher-order integrator-type control systems. Finally, by combining the proposed sliding-mode control protocol with the proposed observers, three fixed-time controllers are developed to solve the fixed-time distributed consensus tracking problem under the undirected graph and directed graph, and the fixed-time distributed average tracking problem under the undirected graph, respectively.

Furthermore, the proposed algorithms in this article have a broad range of potential applications. The sliding-mode control algorithm has been used in the control of vehicles, spacecrafts, and robot manipulators [39], [43]. The distributed consensus tracking algorithms have been widely applied to aerospace systems [44], mechanical systems [45], and dynamic region following [46]. The remainder of this article is given as follows. In Section II, some mathematical preliminaries are given. In Section III, the fixed-time sliding-mode control protocol is proposed. Next, the fixed-time observers for distributed consensus tracking and distributed average tracking problems are designed. Then, the controllers for fixed-time distributed consensus tracking and fixed-time distributed average tracking problems are given. In Section IV, several simulations are shown. In Section V, a few conclusions are made.

II. Mathematical Preliminaries

A. Notations

The real number set and the N-dimensional real vector space are denoted by $\mathbb{R}$ and $\mathbb{R}^n$, respectively. The signum function is represented by $\text{sgn}(\cdot)$ and its vector form can be written as $\text{sgn}(\mathbf{z}) = [\text{sgn}(z_1), \text{sgn}(z_2), \ldots, \text{sgn}(z_n)]^T$, where $\mathbf{z} = [z_1, z_2, \ldots, z_n]^T$. Let $|\cdot|$ stand for the absolute value of a scalar. The q-norm of vector $\mathbf{z}$ can be written as $\|\mathbf{z}\|_q = (|z_1|^q + |z_2|^q + \cdots + |z_n|^q)^{1/q}$. Let $\lambda_1(\mathbf{Q})$ and $\lambda_2(\mathbf{Q})$ represent the smallest and the second-smallest eigenvalue of matrix $\mathbf{Q}$, respectively.

B. Graph Theory

The communication topology of a group of $n + 1$ agents can be represented by a graph $\mathcal{G}$. If there is a leader in them, the other $n$ agents can be expressed as a subgraph $\mathcal{G}$. The weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is constructed with a set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_{n+1}\}$ and a set of edges $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. A directed edge from $v_j$ to $v_i$ can be denoted as $(v_i, v_j)$, which means $v_i$ can receive information from $v_j$. A directed path from $v_j$ to $v_i$ consists of a sequence of edges in the form of $\mathcal{E}_i = \{(v_j, v_{j+1}), \ldots, (v_{i-1}, v_i)\}$, which means the information can flow from $v_j$ to $v_i$. When replacing the directed edges by the undirected, it becomes undirected.
path and the information flow is bidirectional. It is said to contain a spanning tree if at least there exists a node which has directed paths to all other nodes. The undirected graph is connected if and only if there exists an undirected path between any two notes. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times m}$ denote the adjacency matrix and the incidence matrix of the graph, respectively, and $a_{ij} \equiv 1$ if there exists a directed edge from $v_j$ to $v_i$, else $a_{ij} = 0$. With regard to undirected graphs, $a_{ij} = a_{ji}$, let $O = [o_{ij}] \in \mathbb{R}^{n \times n}$ denote the degree matrix, where $o_{ii} = \sum_{j=1}^{n} a_{ij}$ and other $a_{ij} = 0$. Then, the Laplacian matrix is written as $L \in \mathbb{R}^{n \times n} = O - A$. Set $a_{ii} = 1$ if the agent $i$ can acquire information from the leader, else $a_{ii} = 0$, and then set $B = \text{diag}(a_{i0}, a_{i1}, \ldots, a_{in})$.

C. Time-Based Generator

The time-based generator $\xi(t)$ is a kind of time-dependent function that can be seen as a termination function. Its general properties can be generalized as follows.

1) $\xi(t)$ is a nondecreasing and continuous function.
2) With time growing, $\xi(t)$ increases from the initial state $\xi(0) = 0$ to $\xi(t_s) = 1$, and when $t > t_s$, $\xi(t) \equiv 1$, where $t_s$ can be predesignated arbitrarily.
3) $\xi(0) = 0$ and when $t \geq t_s$, $\xi(t) \equiv 0$.

Remark 1: A typical time-based generator function $\xi(t)$ is presented as follows [37]:

$$\xi(t) = \begin{cases} \frac{10}{t} t^6 - \frac{24}{t} t^5 + \frac{15}{2} t^4, & 0 \leq t \leq t_s \\ 1, & t > t_s \end{cases}$$

where $h(t)$ is constructed as

$$h(t) = k \frac{\xi(t)}{1 - \xi(t) + \delta}$$

where $k \in \mathbb{R}$ and $\delta \in \mathbb{R}$ are two positive constants that satisfy $k > 1$ and $0 < \delta < 1$.

By solving (1), one has

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By solving (1), one has

$$z = \left( \frac{1 - \xi(t) + \delta}{1 + \delta} \right)^k z_0.$$  

With time $t$ grows from 0 to $t_s$, the time-based generator function $\xi(t)$ increases from 0 to 1 smoothly. Therefore, when $t \in [0, t_s)$, the system variable $z$ gradually approaches to $z_0(\delta/1 + \delta)^k$. When $t \geq t_s$, the system variable $z$ stays at the fixed point $z_0(\delta/1 + \delta)^k$. If let $\delta = 0.001$ and $k = 3$, when $t \geq t_s$, the solution of (1) will be $z \approx 10^{-9} z_0$. For this reason, we can nearly think that $z$ reaches to zero at $t_s$ and the initial state $z_0$ has no effect on the convergence time.

D. Problem Description

1) Fixed-Time Sliding-Mode Control: A typical double-integrator-type control system is given as follows:

$$\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= u + \varrho
\end{align*}$$

where $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}$ denote the position and velocity, respectively. Variable $\varrho \in \mathbb{R}$ is a disturbance bounded by a positive constant $\varrho_{\text{max}}$.

The objective of fixed-time sliding-mode control is to devise a control input $u$ that drives system (4) to the equilibrium point in a fixed time. In this article, the definition of fixed-time stability for system states (4) is given as follows.

**Definition 1:** For system (4), it is said to achieve fixed-time stability if the system satisfies

$$\lim_{t \to T_{\text{max}}} |z_1| + |z_2| \leq c$$

$$\lim_{t \to \infty} |z_1| + |z_2| = 0$$

where $T_{\text{max}}$ is the upper bounded convergence time independent of initial system states, and $c$ is a small positive constant close to zero.

2) Fixed-Time Distributed Consensus Tracking: Suppose that there is a double-integrator-type multiagent system with a leader and $n$ followers. The leader can be modeled by

$$\begin{align*}
\dot{x}_0 &= v_0 \\
\dot{v}_0 &= u_0
\end{align*}$$

where $x_0 \in \mathbb{R}$ and $v_0 \in \mathbb{R}$ represent the position and velocity of the leader, respectively, and $u_0 \in \mathbb{R}$ represents the control input that is bounded by a positive constant $u_{\text{max}}$.

Then, the followers can be modeled by:

$$\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= u_i + d_i, \quad i = 1, 2, \ldots, n
\end{align*}$$

where $x_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ denote the position and velocity of agent $i$, respectively. $u_i \in \mathbb{R}$ denotes the control input and $d_i \in \mathbb{R}$ denotes the disturbance satisfying $|d_i| \leq d_{\text{max}}$.

The objective of fixed-time distributed consensus tracking is to devise the control input for each follower that only depends on the local information. Meanwhile, the control inputs enable the followers to achieve consensus with the leader in a fixed time independent of initial system states.

**Definition 2 (Fixed-Time Distributed Consensus Tracking):** For the system described by (6) and (7), with the given observer and control input $u_i$, it is said to achieve fixed-time distributed consensus tracking if all the followers can achieve consensus with the leader in a fixed-time $T_{\text{max}}$ independent of initial system states, that is

$$\lim_{t \to T_{\text{max}}} |x_i - x_0| + |v_i - v_0| \leq c$$

$$\lim_{t \to \infty} |x_i - x_0| + |v_i - v_0| = 0$$

where $T_{\text{max}}$ can be predesignated arbitrarily independent of initial conditions and $c$ can be limited to the desired level.

3) Fixed-Time Distributed Average Tracking: Consider a double-integrator-type multiagent system with $n$ agents modeled by (7). Each agent $i$ has a reference signal $r_i \in \mathbb{R}$ described as follows:

$$\begin{align*}
\dot{r}_i &= f_i \\
\dot{f}_i &= a'_i, \quad i = 1, 2, \ldots, n
\end{align*}$$

where $f_i$ and $a'_i$ are the velocity and acceleration of reference signal $r_i$, respectively. Note that $a'_i$ is bounded by a positive constant $a'_{\text{max}}$. Let $\bar{r} = (1/n) \sum_{i=1}^{n} r_i$, $\bar{f} = (1/n) \sum_{i=1}^{n} f_i$, and $\bar{a} = (1/n) \sum_{i=1}^{n} a'_i$ denote the average values of the reference signals.
The objective of fixed-time distributed average tracking is to devise the control input for each agent that only depends on the local information. Meanwhile, the control inputs enables the agents to achieve consensus with the average value of the multiple reference signals in a fixed time without dependence on initial system states.

**Definition 3 (Fixed-Time Distributed Average Tracking):**
For the system described by (7) and (9), with the given observer and control input \( u_i \), it is said to achieve fixed-time distributed average tracking if all the agents can achieve consensus with the average value of the multiple reference signals in a fixed-time \( T_{\text{max}} \) independent of initial states, that is

\[
\lim_{t \rightarrow T_{\text{max}}} |x_i - \bar{x}| + |v_i - \bar{v}| \leq c, \\
\lim_{t \rightarrow \infty} |x_i - \bar{x}| + |v_i - \bar{v}| = 0. \quad (10)
\]

### III. Main Results

#### A. Fixed-Time Sliding-Mode Control

**Lemma 1 [30]:** Suppose that \( z(0) = z_0 \) and \( V(z) \) is a positive-definite Lyapunov candidate that satisfies the inequality as follows:

\[
\dot{V}(z) + \mu V^\nu(z) \leq 0 \quad (11)
\]

where \( \mu \geq 0 \) and \( \nu \in (0, 1) \). Then, \( z \) will converge to zero in a finite time \( T(z_0) \) such that

\[
T(z_0) \leq \frac{1}{\mu (1 - \nu)} V^{1 - \nu}(z_0). \quad (12)
\]

In this article, Lemma 1 is mainly used to demonstrate that the system states still stay at the equilibrium after the predefined convergence time.

The process of fixed-time double-integrator sliding-mode control is generally divided into two sections. In the first section, the control input forces the system states to arrive at the prescribed surface in a fixed time \( t_{s1} \); in the second section, the system states will slide along the surface to the equilibrium point in a fixed time \( t_{s2} \). Therefore, the entire convergence time is bounded by \( T_{\text{max}} = t_{s2} \). In order to force the system states to arrive at the specific positions in the fixed time during each stage, two time-based generators \( \xi_1(t) \) and \( \xi_2(t) \) are used sequentially. The detail of \( \xi_1(t) \) and \( \xi_2(t) \) is shown as follows:

\[
\xi(t) = \begin{cases} 
\xi_1(t), & 0 \leq t < t_{s1} \\
\xi_2(t), & t_{s1} \leq t < t_{s2} \\
1, & t_{s2} \leq t < \infty.
\end{cases} \quad (13)
\]

According to (2), the corresponding \( h(t) \) is constructed as follows:

\[
h(t) = \begin{cases} 
h_1(t) = k_1 \xi_1(t), & 0 \leq t < t_{s1} \\
h_2(t) = k_2 \xi_2(t), & t_{s1} \leq t < t_{s2} \\
0, & t_{s2} \leq t < \infty.
\end{cases} \quad (14)
\]

**Remark 2:** Since \( \dot{\xi}_1(0) = \ddot{\xi}_1(t_{s1}) = \ddot{\xi}_2(t_{s1}) = \ddot{\xi}_2(t_{s2}) = 0 \), one obtains \( h_1(0) = h_1(t_{s1}) = h_2(t_{s1}) = h_2(t_{s2}) = 0 \), which shows the continuity of \( h(t) \). Moreover, due to the nonnegativity of \( \dot{\xi}_1(t) \) and \( \ddot{\xi}_2(t) \), function \( h(t) \) is non-negative.

**Remark 3:** In order to clarify the idea clearly, we directly let \( \dot{\xi}_1(t) \) and \( \ddot{\xi}_2(t) \) have the same structure in the proof. But in simulation, due to the discontinuity of \( \dot{\xi}_1(t) \), that is, \( \dot{\xi}_1(t_{s1}) = 1 \) and \( \ddot{\xi}_2(t_{s1}) = 0 \), the sharp decrease may lead to the unavailability of the derivative of \( \dot{\xi}_2(t_{s1}) \). One method to solve this problem is to reset \( \ddot{\xi}_2(t) = \dot{\xi}_2(t) + 1 \) and \( h_2(t) = k_1 \frac{\ddot{\xi}_2(t)}{2} + \ddot{\xi}_2(t) + \delta_2 \), respectively.

For system (4), the fixed-time sliding-mode surface is designed as follows:

\[
s = \left( \frac{1}{2} h(t) + 1 \right) z_1 + z_2. \quad (15)
\]

If \( s = 0 \), the system states arrive at the sliding-mode surface and then can be rewritten as follows:

\[
z_2 = \dot{z}_1 = \left( \frac{1}{2} h(t) + 1 \right) z_1. \quad (16)
\]

The control input for system (4) is devised as follows:

\[
u = -\frac{1}{2} \dot{h}(t) z_1 - \left( \frac{1}{2} h(t) + 1 \right) z_2 - \frac{1}{2} h(t) s - \rho \text{sgn}(s) \quad (17)
\]

where \( \rho \) is a positive constant satisfying \( \rho \geq \varepsilon_{\text{max}} + 1 \).

**Theorem 1:** With the given control input (17), system (4) will achieve the fixed-time stability (see Definition 1) within the upper bounded convergence time \( T_{\text{max}} = t_{s2} \).

**Proof:** The Lyapunov candidate is constructed as \( V_1 = (1/2)s^2 \). Differentiate (15) against time, and then one has

\[
\dot{V}_1 = s \dot{s} = \left( \frac{1}{2} h(t) s^2 - \rho |s| + \varepsilon s \right) \leq \frac{1}{2} h(t) s^2 - (\rho - \varepsilon_{\text{max}}) |s| \leq \frac{1}{2} h(t) s^2 = -h(t) V_1 \quad (18)
\]

Substitute control input (17) into (18), and then one has

\[
\dot{V}_1 = s \dot{s} = \frac{1}{2} h(t) s^2 - \rho |s| + \varepsilon s \leq \frac{1}{2} h(t) s^2 - (\rho - \varepsilon_{\text{max}}) |s| \leq \frac{1}{2} h(t) s^2 \leq -h(t) V_1 \quad (19)
\]

Differentiate the Lyapunov candidate \( V_1 \) against time, and substitute (19) into \( V_1 \), and then one has

\[
\dot{V}_1 = s \dot{s} = \frac{1}{2} h(t) s^2 - \rho |s| + \varepsilon s \leq \frac{1}{2} h(t) s^2 - (\rho - \varepsilon_{\text{max}}) |s| \leq \frac{1}{2} h(t) s^2 \leq -h(t) V_1 \quad (20)
\]

where the second inequality holds due to the precondition \( \rho \geq \varepsilon_{\text{max}} + 1 \).

When \( t \in [0, t_{s1}] \), \( h(t) = h_1(t) \). According to the property of (1) one obtains

\[
\lim_{t \rightarrow t_{s1}} V_1 \leq \left( \frac{1 - \delta_1(t_{s1}) + \delta_1}{1 + \delta_1} \right)^k V_1(0) = \left( \frac{\delta_1}{1 + \delta_1} \right)^k V_1(0) \quad (21)
\]

where the value of \( (\delta_1/1 + \delta_1)^k V_1(0) \) is very small and close to zero, which also means that the system states are in the near region of the sliding-mode surface.
When \( t \geq t_{s1}, h(t) = h_2(t) \geq 0 \) and then one obtains
\[
\dot{V}_1 = -\frac{1}{2}h_2(t)s^2 - \rho|s| + gs \\
\leq -(\rho - \rho_{\text{max}})|s| \\
\leq -|s| \\
= -\sqrt{2V_1}, \tag{22}
\]

According to Lemma 1, one obtains that \( V_1 \) will converge to zero after \( t_{s2} \) within a finite time \( t_{s1} \), where \( t_{s1} \leq \sqrt{2V_1} \). Although \( V_1 \) may not converge to zero perfectly at \( t_{s1} \), which means that the system states are in the adjacent region of the sliding-mode surface \( s = 0 \), we will prove the system states will still converge to zero along the sliding surface in the fixed-time \( t_{s2} \). A Lyapunov candidate is constructed as \( V_2 = \left(1/2\right)z_1^2 \). The proof will be given in both cases: 1) \( V_1 \) perfectly converges to zero at \( t_{s1} \) and 2) \( V_1 \) converges to the adjacent region of zero at \( t_{s1} \).

Case 1: \( V_1(t_{s1}) = 0 \).
Since \( V_1(t_{s1}) = 0 \), one has \( s = 0 \). Differentiate \( V_2 \) along (16), and then one obtains
\[
\dot{V}_2 = z_1 \dot{z}_1 \\
= -\frac{1}{2}h(t)z_1^2 - z_1^2 \\
\leq -\frac{1}{2}h(t)z_1^2 \\
= -(h(t)V_2). \tag{23}
\]

When \( t \in [t_{s1}, t_{s2}] \), one has \( h(t) = h_2(t) \). Therefore, according to (1), one obtains \( \lim_{t \to t_{s2}} V_2 \leq \left(\delta_2/1 + \delta_2^k\right)V_2(t_{s1}) \), which means that the system states are in the adjacent region of zero, that is, \( \lim_{t \to t_{s2}} |z_1| \leq \sqrt{2(\delta_2/1 + \delta_2^k)V_2(t_{s1})} \). When \( t \geq t_{s2} \), due to \( h(t) = 0 \) and \( \dot{V}_2 = -z_1^2 = -2V_2 \), one can conclude that \( V_2 \) and \( z_1 \) will converge to zero exponentially. Moreover, since when \( t \geq t_{s2}, z_2 = -(h(t) + 1)z_1 = -z_1 \), system state \( z_2 \) will also converge to zero with the same rate of \( z_1 \).

Case 2: \( V_1(t_{s1}) \neq 0 \).
According to the relationship between \( V_1 \) and \( s \), suppose that the system states converge to the adjacent region of the sliding surface at \( t_{s1} \) and there exists an error \( e \), that is, \( s = \{e|t \geq t_{s1}\} \). When \( t \geq t_{s1} \), according to (22), one has \( \dot{V}_1 \leq -\sqrt{2V_1} \), which means \( V_1 \) as well as \( |e| \) is nonincreasing. Meanwhile, they are bounded by \( \dot{V}_1 = (\delta_1/1 + \delta_1^k)V_1(0) \) and \( |e| = \sqrt{2(\delta_1/1 + \delta_1^k)V_1(0)} \), respectively. Then, sliding-mode surface (15) can be rewritten as
\[
e = \left(\frac{1}{2}h(t) + 1\right)z_1 + z_2. \tag{24}
\]
The derivative of \( z_1 \) against time can be written as follows:
\[
\dot{z}_1 = -\left(\frac{1}{2}h(t) + 1\right)z_1 + e. \tag{25}
\]
Substitute (25) into \( \dot{V}_2 \), and then one has
\[
\dot{V}_2 = -\frac{1}{2}h(t)z_1^2 - z_1^2 + ez_1 \\
= -(h(t)V_2 - z_1^2 + ez_1). \tag{26}
\]

If \( |z_1| < |\dot{e}| \) at \( t_{s1} \), although we may have \(-z_1^2 + ez_1 \geq 0 \), the value of \( |z_1| \) has been in the adjacent region of zero. Meanwhile, we have \( |z_1| < |\dot{e}| \leq \sqrt{2(\delta_1/1 + \delta_1^k)V_1(0)} \), because \( z_1 \geq |e| \) leads to \( V_2 \leq 0 \), which means \( z_1 \) will move to the direction of zero. After \( t_{s1} + t_{s2} \), due to \( |e| = 0 \), \( |z_1| \) will at least converge exponentially.

If \( |z_1| \geq |e| \) at \( t_{s1} \), one has \( V_2 \leq -h(t)V_2 \). According to (1), one can obtain \( \lim_{t \to t_{s2}} V_2 \leq (\delta_2/1 + \delta_2^k)V_2(t_{s1}) \), which means \( V_2 \) converges to the adjacent region of zero at \( t_{s2} \), that is, \( \lim_{t \to t_{s2}} |z_1| \leq \sqrt{2(\delta_2/1 + \delta_2^k)V_2(t_{s1})} \). Therefore, one can conclude that the margin of \( |z_1| \) at \( t_{s2} \) is bounded by \( \sqrt{2(\delta_1/1 + \delta_1^k)V_1(0)} \). Therefore, \( |z_1| \leq -z_2 \), both \( z_1 \) and \( z_2 \) will converge to zero exponentially.

Therefore, in both cases, \( V_2 \) will converge nearly to zero in a fixed time \( T_{\text{max}} = t_{s2} \) and then converge quickly to zero. Therefore, the proof has been completed.

\section{B. Fixed-Time Distributed Consensus Tracking Observer Under Undirected Communication}

\textbf{Assumption 1:} The topology subgraph \( G_r \) for the followers is undirected and connected; there exist at least a follower which can acquire information from the leader.

\textbf{Lemma 2 [28]:} If \( L \in \mathbb{R}^{n \times n} \) is the Laplacian diagonal of a undirected connected graph, and the non-negative diagonal matrix \( B = \text{diag}(|a_1|, \ldots, |a_n|) \in \mathbb{R}^{n \times n} \) with at least one element greater than zero, then \( Q = L + B \) is a positive-definite matrix.

In the section, a fixed-time distributed observer under dynamics (6) and (7) is designed for each follower to measure the relative position and velocity disagreements between the leader and itself under undirected communication. Denote the real tracking errors as \( \tilde{x}_i = x_i - x_0 \) and \( \tilde{v}_i = v_i - v_0 \). Then, fixed-time distributed observers \( \tilde{a}_i \) and \( \tilde{b}_i \) are developed to estimate \( \tilde{x}_i \) and \( \tilde{v}_i \). The specific formulation is shown as follows:
\[
\dot{\tilde{a}}_i = \beta_i - \eta(t)\left\{ \sum_{j=0}^{n} a_{ij}\left(\alpha_i - \alpha_j\right) - (x_i - x_j) \right\} \\
- b_1 \text{sgn}\left\{ \sum_{j=0}^{n} a_{ij}\left(\alpha_i - \alpha_j\right) - (x_i - x_j) \right\} \\
\dot{\tilde{b}}_i = u_i - \eta(t)\left\{ \sum_{j=0}^{n} a_{ij}\left(\beta_i - \beta_j\right) - (v_i - v_j) \right\} \\
- c_1 \text{sgn}\left\{ \sum_{j=0}^{n} a_{ij}\left(\beta_i - \beta_j\right) - (v_i - v_j) \right\} \tag{27}
\]

where \( i = 1, \ldots, n \), \( \eta(t) = (1/[2a_1(Q)])h(t) \), \( a_0 = 0 \), and \( b_0 = 0 \). \( b_1 \) and \( c_1 \) are positive constants satisfying \( b_1 \geq 1 \) and \( c_1 > u_{\text{max}} + d_{\text{max}} \).

\textbf{Theorem 2:} With the given dynamics (6) and (7) and observer (27), under Assumption 1, \( \tilde{a}_i \) and \( \tilde{b}_i \) converge to \( \tilde{x}_i \) and \( \tilde{v}_i \) within a fixed-time \( T_{\text{max}} = t_{s2} \), that is
\[
\lim_{t \to t_{s2}} |\beta_i - \tilde{v}_i| \leq \alpha_1 \\
\lim_{t \to t_{s2}} |\alpha_i - \tilde{x}_i| \leq \alpha_2 \\
\lim_{t \to \infty} |\alpha_i - \tilde{x}_i| = 0 \\
\lim_{t \to \infty} |\beta_i - \tilde{v}_i| = 0 \tag{28}
\]
where \( t_{s1} \) and \( t_{s2} \) are the upper bounded convergence time [see (14)]. Parameters \( a_1 \) and \( a_2 \) are small positive constants close to zero and can be limited to a desired level.

**Proof:** Let \( \tilde{a}_i = a_i - \bar{x}_i \) and \( \tilde{b}_i = b_i - \bar{v}_i \) be the errors between the observed disagreements and the real disagreements. According to (27), \( \tilde{a}_i \) and \( \tilde{b}_i \) can be written as

\[
\begin{align*}
\dot{\tilde{a}}_i &= \tilde{b}_i - \eta(t) \sum_{j=0}^{n} a_{ij}(\tilde{a}_j - \tilde{a}_i) - b_1 \text{sgn}\left[ \sum_{j=0}^{n} a_{ij}(\tilde{a}_j - \tilde{a}_i) \right] \\
\dot{\tilde{b}}_i &= -\eta(t) \sum_{j=0}^{n} a_{ij}(\tilde{b}_j - \tilde{b}_i) - c_1 \text{sgn}\left[ \sum_{j=0}^{n} a_{ij}(\tilde{b}_j - \tilde{b}_i) \right] - d_i + u_0.
\end{align*}
\]

Let \( \tilde{\alpha} = [\tilde{a}_1, \ldots, \tilde{a}_n]^T, \tilde{\beta} = [\tilde{b}_1, \ldots, \tilde{b}_n]^T, d = [d_1, \ldots, d_n]^T \) and \( u \in \mathbb{R}^n = [u_0, \ldots, u_0]^T \). The vector form of (29) can be written as

\[
\begin{align*}
\dot{\tilde{\alpha}} &= \tilde{\beta} - \eta(t) Q \tilde{\alpha} - b_1 \text{sgn}(Q \tilde{\alpha}) \\
\dot{\tilde{\beta}} &= -\eta(t) Q \tilde{\beta} - c_1 \text{sgn}(Q \tilde{\beta}) - d + u.
\end{align*}
\]

where \( Q = L + B \) is a positive-definite matrix according to Lemma 2, matrix \( L \) is the Laplacian matrix for the followers, and matrix \( B = \text{diag}(a_{10}, \ldots, a_{0n}) \) represents the communication topology between the leader and the followers. For example, if the \( i \)-th agent can receive information from the leader, \( a_{i0} = 1; \) else, \( a_{i0} = 0 \).

Construct a Lyapunov candidate as \( V_3 = (1/2)\tilde{\beta}^T Q \tilde{\beta} \). Since \( Q \) is a positive-definite matrix, \( V_3 \) is well defined. Differentiate \( V_3 \) against time such that

\[
\dot{V}_3 = \tilde{\beta}^T Q \dot{\tilde{\beta}} = \tilde{\beta}^T Q \left[ -\eta(t) Q \tilde{\beta} - c_1 \text{sgn}(Q \tilde{\beta}) - d + u \right]
\]

\[
\leq -\eta(t) \left( Q^{1/2} \tilde{\beta} \right)^T \left( Q^{1/2} \tilde{\beta} \right) - (c_1 - u_{\text{max}} - d_{\text{max}}) ||Q\tilde{\beta}||^1_1.
\]

Due to the precondition \( c_1 > u_{\text{max}} + d_{\text{max}} \), one can obtain

\[
\dot{V}_3 \leq -\eta(t) \left( Q^{1/2} \tilde{\beta} \right)^T \left( Q^{1/2} \tilde{\beta} \right) - \lambda_1(Q) \eta(t) ||Q\tilde{\beta}||^2_2 = -h(t)V_3.
\]

When \( t \in (0, t_{s1}) \), \( h(t) = h_2(t) \). According to (1) one can conclude that

\[
\lim_{t \to t_{s1}} V_3 = \lim_{t \to t_{s1}} \left( \frac{1}{2} \tilde{\beta}^T (L + B) \tilde{\beta} \right)
\]

\[
= \lim_{t \to t_{s1}} \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\tilde{\beta}_i - \tilde{\beta}_j)^2 + \frac{1}{2} \sum_{i=1}^{n} a_{i0} \tilde{\beta}_i^2
\]

\[
\leq \left( \frac{\delta_1}{1 + \delta_1} \right)^k V_3(0),
\]

(33)

According to inequality \((1/2)\sum_{i=1}^{n} a_{i0} \tilde{\beta}_i^2 \leq (\delta_1/1 + \delta_1) V_3(0)\), one can obtain \( \lim_{t \to t_{s1}} ||\tilde{\beta}_i|| \leq \sqrt{2(\delta_1/1 + \delta_1) V_3(0)} \). Therefore, the velocity observer \( \beta_i \) successfully observe the relative velocity \( v_i - v_0 \) in the fixed time \( t_{s1} \).

When \( t \geq t_{s1} \), according to (31), one has

\[
V_3 \leq -(c_1 - u_{\text{max}} - d_{\text{max}}) ||Q\tilde{\beta}||^1_1.
\]

Due to

\[
||Q\tilde{\beta}||^1_1 \geq ||Q\tilde{\beta}||^2_2 = \sqrt{(Q\tilde{\beta})^T Q\tilde{\beta}} \geq \lambda_1(Q) \tilde{\beta}^T Q \tilde{\beta}
\]

one obtains

\[
V_3 \leq -(c_1 - u_{\text{max}} - d_{\text{max}}) \sqrt{2\lambda_1(Q)V_3} \leq 0.
\]

Following from Lemma 1, when \( t \geq t_{s1} \), \( V_3 \) will quickly converge to zero within an upper bounded convergence time \( t_{s2} \leq (1/2(\delta_1/1 + \delta_2)^k V_3(0))/\lambda_1(Q) \). Although \( V_3 \) may not converge perfectly to zero at \( t_{s1} \), the fixed-time stability of position observer \( \alpha_i \) is not affected. The proof of position observer \( \alpha_i \) is given as follows.

Construct a Lyapunov candidate as \( V_4 = (1/2)\tilde{\alpha}^T Q \tilde{\alpha} \). Differentiate it against time and then one has

\[
V_4 = \tilde{\alpha}^T Q \tilde{\alpha}
\]

\[
= \tilde{\alpha}^T Q \tilde{\beta} - \eta(t) \tilde{\alpha}^T Q \tilde{\alpha} - b_1 \text{sgn}(Q \tilde{\alpha})
\]

\[
= -\eta(t) \tilde{\alpha}^T Q \tilde{\beta} - b_1 ||Q\tilde{\alpha}||_1 + (Q\tilde{\alpha})^T \tilde{\beta}.
\]

Since when \( t \geq t_{s1} \), one has \( ||\tilde{\beta}|| \leq \sqrt{2(\delta_1/1 + \delta_2)^k V_3(0)} \). Then, due to the precondition \( b_1 \geq 1 \), one can conclude

\[
\dot{V}_4 \leq -\lambda_1(Q) \eta(t) ||Q\tilde{\alpha}||_1 - (b_1 - 1) ||Q\tilde{\alpha}||_1
\]

\[
= -h(t)V_4.
\]

(38)

When \( t \in [t_{s1}, t_{s2}) \), \( h(t) = h_2(t) \). According to (1), one concludes that

\[
\lim_{t \to t_{s2}} V_4 = \lim_{t \to t_{s2}} \frac{1}{2} \tilde{\alpha}^T (L + B) \tilde{\alpha}
\]

\[
= \lim_{t \to t_{s2}} \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\tilde{\alpha}_i - \tilde{\alpha}_j)^2 + \frac{1}{2} \sum_{i=1}^{n} a_{i0} \tilde{\alpha}_i^2
\]

\[
\leq \left( \frac{\delta_2}{1 + \delta_2} \right)^k V_4(t_{s1}).
\]

(39)

According to the inequality \((1/2)\sum_{i=1}^{n} a_{i0} \tilde{\alpha}_i^2 \leq (\delta_2/1 + \delta_2)^k V_4(t_{s1})\), one can obtain \( \lim_{t \to t_{s2}} ||\tilde{\alpha}|| \leq \sqrt{2(\delta_2/1 + \delta_2)^k V_4(t_{s1})} \). Therefore, position observer \( \alpha_i \) successfully observe the relative position \( x_i - x_0 \) in the fixed time \( t_{\text{max}} = t_{s2} \).

When \( t \geq t_{s2} \), according to (35) and (38), one has

\[
\dot{V}_4 \leq -(b_1 - 1) ||Q\tilde{\beta}||^1_1
\]

\[
\leq -(b_1 - 1) \sqrt{2\lambda_1(Q)V_4}.
\]

(40)

Following from Lemma 1, when \( t \geq t_{s2} \), \( V_4 \) will quickly converge to zero within an upper bounded convergence time \( t_{s2} \leq \frac{1}{b_1-1} \frac{1}{(\delta_2/1 + \delta_2)^k V_4(t_{s1})}/\lambda_1(Q) \). Note that \( t_{s2} \) is very small. Then, the entire proof has been completed.
C. Fixed-Time-Distributed Consensus Tracking Observer Under Directed Communication

Assumption 2: The topology subgraph $\mathcal{G}$ for the group of agents is directed and contains a spanning tree, where the leader is denoted as the root node. Note that, the subgraph $\mathcal{G}_i$ does not need to be strongly connected or contain a spanning tree.

Lemma 3 [49]: Under Assumption 2, $H = L + B$ is of full rank. Furthermore, define $p = [p_1, \ldots, p_n]^T = H^{-T}1_n$, $P = \text{diag}(p_i)$, and $Q = ((H^TP + PH)/2)$. Then, $P$ and $Q$ are both positive definite.

In this section, a fixed-time distributed observer under dynamics (6) and (7) with directed communication is designed. Denote the real tracking errors as $\tilde{x}_i = x_i - x_0$ and $\tilde{v}_i = v_i - v_0$. Then, the fixed-time distributed observers $\alpha_i$ and $\beta_i$ are developed to estimate $\tilde{x}_i$ and $\tilde{v}_i$ in the fixed time. Let $\mathcal{A} = \max(\sum_{j=0}^n a_{ij}(d_i - d_j))$, $p_{\max} = \max(p_i)$, $p_{\min} = \min(p_i)$, $\eta(t) = (p_{\max}/|\mathcal{A}1(Q)|)h(t)$, and $d_0 = 0$. Then, the observer is designed as follows:

\[
\dot{\alpha}_i = \beta_i - 2[\eta(t) + 2]\sum_{j=0}^n a_{ij}\left[(\alpha_i - \alpha_j) - (x_i - x_j)\right]
- b_1[\eta(t) + 2]\text{sgn}\sum_{j=0}^n a_{ij}\left[(\alpha_i - \alpha_j) - (x_i - x_j)\right]
\]
\[
\dot{\beta}_i = u_i - 2[\eta(t) + 2]\sum_{j=0}^n a_{ij}\left[(\beta_i - \beta_j) - (v_i - v_j)\right]
- c_1[\eta(t) + 2]\text{sgn}\sum_{j=0}^n a_{ij}\left[(\beta_i - \beta_j) - (v_i - v_j)\right]
\] (41)

where $a_0 = \beta_0 = 0$, and parameters $b_1$ and $c_1$ are positive constants satisfying $b_1 \geq (p_{\max}/\lambda_1(Q))$ and $c_1 \geq (p_{\max}/|\mathcal{A}1(Q)|)$.

Theorem 3: With the given dynamics (6) and (7) and observer (41), under Assumption 2, $\alpha_i$ and $\beta_i$ converge to $\tilde{x}_i$ and $\tilde{v}_i$ within the fixed-time $T_{\text{max}} = t_{\text{max}}$, that is, [see (28)].

Proof: Let $\dot{\alpha}_i = \alpha_i - \tilde{\alpha}_i$ and $\dot{\beta}_i = \beta_i - \tilde{\beta}_i$, and then one has

\[
\dot{\alpha}_i = \dot{\beta}_i - 2[\eta(t) + 2]\sum_{j=0}^n a_{ij}\left[\alpha_i - \alpha_j\right]
- b_1[\eta(t) + 2]\text{sgn}\left[\sum_{j=0}^n a_{ij}\left[\alpha_i - \alpha_j\right]\right]
\]
\[
\dot{\beta}_i = -2[\eta(t) + 2]\sum_{j=0}^n a_{ij}\left[\beta_i - \beta_j\right]
- c_1[\eta(t) + 2]\text{sgn}\left[\sum_{j=0}^n a_{ij}\left[\beta_i - \beta_j\right]\right] - d_i + u_i.
\] (42)

In order to clarify the main idea clearly and simplify the proof, let $z_i = \sum_{j=0}^n a_{ij}(\beta_i - \beta_j)$, and then $\dot{\beta}_i$ in (42) can be rewritten as

\[
\dot{\beta}_i = -2[\eta(t) + 2]z_i - c_1[\eta(t) + 2]\text{sgn}(z_i) - d_i + u_i.
\] (43)

Differentiate $z_i$ against time, and then we can obtain the following equation:

\[
\dot{z}_i = \sum_{j=0}^n a_{ij}(\dot{\beta}_i - \dot{\beta}_j)
= -2[\eta(t) + 2]\sum_{j=0}^n a_{ij}(z_i - z_j)
- c_1[\eta(t) + 2]\left\{\sum_{j=0}^n a_{ij}[\text{sgn}(z_i) - \text{sgn}(z_j)]\right\}
- \sum_{j=0}^n a_{ij}(d_i - d_j) + a_{ij}u_i.
\] (44)

Let $z = [z_1, \ldots, z_n]^T$ and $\beta = [\beta_1, \ldots, \beta_n]^T$, and then one has $z = H\beta$, where $H$ is a nonsingular matrix. Therefore, if vector $z$ converges to zero, vector $\beta$ will converge to zero as well. Then, a Lyapunov candidate function is constructed as follows:

\[
V_5 = \sum_{i=1}^n p_i\left(z_i^2 + c_1|z_i|\right).
\] (45)

Since $P$ is a positive-definite diagonal matrix, one has each $p_i > 0$. If $V_5 = 0$, one can conclude each $z_i = 0$. Therefore, the Lyapunov candidate function is well constructed. Differentiate $V_5$ against time, and then one has

\[
\dot{V}_5 = \sum_{i=1}^n p_i\left[2z_i + c_1\text{sgn}(z_i)\right]z_i
= \sum_{i=1}^n p_i\left[2z_i + c_1\text{sgn}(z_i)\right]
\times \left\{-2[\eta(t) + 2]\sum_{j=0}^n a_{ij}(z_i - z_j)
- c_1[\eta(t) + 2]\sum_{j=0}^n a_{ij}[\text{sgn}(z_i) - \text{sgn}(z_j)]
- \sum_{j=0}^n a_{ij}(d_i - d_j) + a_{ij}u_i\right\}.
\] (46)

In order to give a better understanding of the proof, we separate (46) into two parts, where one part contains the term $-2[\eta(t) + 2]$ and the other part not. Then, we rewrite (46) as follows:

\[
\dot{V}_5 = -2[\eta(t) + 2]\sum_{i=1}^n p_i\left[2z_i + c_1\text{sgn}(z_i)\right]
\times \left\{2\sum_{j=0}^n a_{ij}(z_i - z_j) + c_1\sum_{j=0}^n a_{ij}[\text{sgn}(z_i) - \text{sgn}(z_j)]\right\}
- \sum_{i=1}^n p_i\left[2z_i + c_1\text{sgn}(z_i)\right]\left[\sum_{j=0}^n a_{ij}(d_i - d_j) - a_{ij}u_i\right].
\] (47)
In order to clarify the proof, we rewritten part of (47) into a vector form as follows:

\[
\sum_{i=1}^{n} p_i [2z_i + c_1 \operatorname{sgn}(z_i)]
\times \left\{ 2 \sum_{j=0}^{n} a_{ij}(z_i - z_j) + c_1 \sum_{j=0}^{n} a_{ij}[\operatorname{sgn}(z_i) - \operatorname{sgn}(z_j)] \right\}
= [2z + c_1 \operatorname{sgn}(z)]^T PH [2z + c_1 \operatorname{sgn}(z)].
\]

(48)

Then, (47) can be rewritten as

\[
\dot{V}_5 = -[\eta(t) + 2][2z + c_1 \operatorname{sgn}(z)]^T PH [2z + c_1 \operatorname{sgn}(z)]
- \sum_{i=1}^{n} p_i [2z_i + c_1 \operatorname{sgn}(z_i)] \left[ \sum_{j=0}^{n} a_{ij}(d_i - d_j) - a_{0} u_0 \right].
\]

(49)

Due to

\[
[2z + c_1 \operatorname{sgn}(z)]^T PH [2z + c_1 \operatorname{sgn}(z)]
= [2z + c_1 \operatorname{sgn}(z)]^T H^T P [2z + c_1 \operatorname{sgn}(z)].
\]

(50)

one can conclude

\[
[2z + c_1 \operatorname{sgn}(z)]^T PH [2z + c_1 \operatorname{sgn}(z)]
= [2z + c_1 \operatorname{sgn}(z)]^T H^T P [2z + c_1 \operatorname{sgn}(z)]
= [2z + c_1 \operatorname{sgn}(z)]^T Q [2z + c_1 \operatorname{sgn}(z)].
\]

(51)

Then, (49) can be rewritten as follows:

\[
\dot{V}_5 = -[\eta(t) + 2][2z + c_1 \operatorname{sgn}(z)]^T Q [2z + c_1 \operatorname{sgn}(z)]
- \sum_{i=1}^{n} p_i [2z_i + c_1 \operatorname{sgn}(z_i)] \left[ \sum_{j=0}^{n} a_{ij}(d_i - d_j) - a_{0} u_0 \right]
\leq -\lambda_1(Q)[\eta(t) + 2][2z + c_1 \operatorname{sgn}(z)]^T [2z + c_1 \operatorname{sgn}(z)]
- \sum_{i=1}^{n} p_i [2z_i + c_1 \operatorname{sgn}(z_i)] \left[ \sum_{j=0}^{n} a_{ij}(d_i - d_j) - a_{0} u_0 \right].
\]

(52)

Due to \([2z + c_1 \operatorname{sgn}(z)]^T [2z + c_1 \operatorname{sgn}(z)] = \sum_{i=1}^{n} (4z_i^2 + 4c_1 |z_i| + c_1^2), \delta = |\max_i(\sum_{j=0}^{n} a_{ij}(d_i - d_j))|, p_{\max} = \max_i(|p_i|), and |u_0| \leq u_{\max}, one can obtain

\[
\dot{V}_5 \leq -\lambda_1(Q)[\eta(t) + 1] \sum_{i=1}^{n} (4z_i^2 + 4c_1 |z_i| + c_1^2)
- \lambda_1(Q) \sum_{i=1}^{n} (4z_i^2 + 4c_1 |z_i| + c_1^2)
+ (\bar{d} + u_{\max}) p_{\max} \sum_{i=1}^{n} (2|z_i| + c_1)
\leq -4\lambda_1(Q)[\eta(t) + 1] \sum_{i=1}^{n} (z_i^2 + c_1 |z_i|)
- \lambda_1(Q) c_1 \sum_{i=1}^{n} (4|z_i| + c_1)
+ (\bar{d} + u_{\max}) p_{\max} \sum_{i=1}^{n} (2|z_i| + c_1).
\]

(53)

Then, according to the precondition \(c_1 \geq (\bar{d} + u_{\max})/\lambda_1(Q)|\eta(t)| \), one can conclude

\[
\dot{V}_5 \leq -4\lambda_1(Q)[\eta(t) + 1] \sum_{i=1}^{n} (z_i^2 + c_1 |z_i|).
\]

(54)

By introducing the constant \(p_{\max} \) into (54), one has

\[
\dot{V}_5 \leq -4\lambda_1(Q)[\eta(t) + 1] \sum_{i=1}^{n} p_{\max} (z_i^2 + c_1 |z_i|)
\leq -4\lambda_1(Q)[\eta(t) + 1] \sum_{i=1}^{n} p_i (z_i^2 + c_1 |z_i|)
= -4\lambda_1(Q)[\eta(t) + 1] V_5
= -4\lambda_1(Q)[\eta(t) V_5 - \lambda_1(Q) V_5]
\leq -p_{\max} \eta(t) V_5
= -h(t) V_5.
\]

(55)

When \(t \in [0, t_{s_1}) \), \(h(t) = h_1(t) \). According to (1) one has

\[
\lim_{t \to t_{s_1}} V_5 = \lim_{t \to t_{s_1}} \sum_{i=1}^{n} p_i (z_i^2 + c_1 |z_i|)
= \lim_{t \to t_{s_1}} \sum_{i=1}^{n} p_i z_i^2 + \lim_{t \to t_{s_1}} \sum_{i=1}^{n} c_1 |p_i| |z_i|
\leq \left( \frac{\delta_1}{1 + \delta_1} \right)^k V_5(0).
\]

(56)

Then, one has

\[
\lim_{t \to t_{s_1}} \sum_{i=1}^{n} p_i z_i^2 \leq \left( \frac{\delta_1}{1 + \delta_1} \right)^k V_5(0).
\]

(57)

and

\[
\lim_{t \to t_{s_1}} \sum_{i=1}^{n} z_i^2 \leq \frac{1}{p_{\min}} \left( \frac{\delta_1}{1 + \delta_1} \right)^k V_5(0).
\]

(58)

Due to

\[
\sum_{i=1}^{n} z_i^2 = z^T z = \beta^T H^T \tilde{H} \beta
\]

and

\[
\beta^T H^T \tilde{H} \beta \geq \lambda_1(H^T H) \beta^T \beta = \sum_{i=1}^{n} \lambda_i(H^T H) \beta_i^2
\]

one can conclude

\[
\sum_{i=1}^{n} \lambda_i(H^T H) \beta_i^2 \leq \sum_{i=1}^{n} z_i^2 \leq \frac{1}{p_{\min}} \left( \frac{\delta_1}{1 + \delta_1} \right)^k V_5(0).
\]

(61)

According the above proof, one can make a conclusion that \(\lim_{t \to t_{s_1}} \sum_{i=1}^{n} \beta_i^2 \leq (1/\lambda_1(H^T H) p_{\min})(\delta_1/1 + \delta_1)^k V_5(0)\), and \(\lim_{t \to t_{s_1}} |\beta_i| \leq \sqrt{(1/\lambda_1(H^T H) p_{\min})(\delta_1/1 + \delta_1)^k V_5(0)}\). Therefore, velocity observer \(\beta_i \) successfully obtains the relative error \(V_i - V_0 \) in the fixed time \(t_{s_1}\). When \(t \geq t_{s_1} \), due to \(\dot{V}_5 \leq -4(\lambda_1(Q)/p_{\max}) V_5 \). The Lyapunov candidate function \(V_5 \) will continue to converge to zero exponentially. The proof
for velocity observer $\beta_i$ has been completed. Although $V_k$ does not perfectly converge to zero at $t_{s1}$, the stability of position observer $\alpha_i$ is not affected. The proof for position observer $\alpha_i$ is given as follows.

Let $w_i = \sum_{j=0}^{n} a_{ij}(\hat{\alpha}_i - \hat{\alpha}_j)$. Substitute $w_i$ into (42), and then one obtains
\[
\dot{\hat{\alpha}}_i = \tilde{\beta}_i - 2[\eta(t) + 2]w_i - b_1[\eta(t) + 2]sgn(w_i).
\]
Differentiate $w_i$ against time, and then one has
\[
\dot{w}_i = z_i - 2[\eta(t) + 2]w_i - b_1[\eta(t) + 2]sgn(w_i).
\]
Construct a Lyapunov candidate as follows:
\[
V_6 = \sum_{i=1}^{n} p_{i} \left[ w_i^2 + b_1 w_i \right].
\]
Let $w = [w_1, \ldots, w_n]^T$ and $\tilde{\alpha} = [\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n]^T$, and then one has $w = H\tilde{\alpha}$. Due to the full rank of matrix $H$, if vector $w$ converges to zero, vector $\tilde{\alpha}$ converges to zero as well. Moreover, if $V_6$ converges to zero, vector $w$ converges to zero as well. Therefore, the Lyapunov candidate function is well constructed. Differentiate $V_6$ against time, and then one has
\[
\dot{V}_6 = \sum_{i=1}^{n} p_{i} \left[ 2w_i + b_1 sgn(w_i) \right] \dot{w}_i.
\]
Substitute (63) into (65), and then one has
\[
\dot{V}_6 = -[\eta(t) + 2] \sum_{i=1}^{n} p_{i} \left[ 2w_i + b_1 sgn(w_i) \right] \times \left\{ 2 \sum_{j=0}^{n} a_{ij}(w_i - w_j) + b_1 \sum_{j=0}^{n} a_{ij}[sgn(w_i) - sgn(w_j)] \right\} - \sum_{i=1}^{n} p_{i} z_i \left[ 2w_i + b_1 sgn(w_i) \right].
\]
According to (48) and (51), one has
\[
\dot{V}_6 = -[\eta(t) + 2] \left[ 2w + b_1 sgn(w) \right]^T Q \left[ 2w + b_1 sgn(w) \right] - \sum_{i=1}^{n} p_{i} z_i \left[ 2w_i + b_1 sgn(w_i) \right] \leq \lambda_1(Q)[\eta(t) + 2] \left[ 2w + b_1 sgn(w) \right]^T \left[ 2w + b_1 sgn(w) \right] - \sum_{i=1}^{n} p_{i} z_i \left[ 2w_i + b_1 sgn(w_i) \right] \leq -4\lambda_1(Q)[\eta(t) + 1] \sum_{i=1}^{n} \left[ w_i^2 + b_1 w_i \right] - b_1 \lambda_1(Q) \sum_{i=1}^{n} \left[ 2w_i + b_1 sgn(w_i) \right].
\]

When $t \geq t_{s1}$, according to (58), one has $|z_i| < < 1$. Then, one can conclude
\[
\dot{V}_6 \leq -4\lambda_1(Q)[\eta(t) + 1] \sum_{i=1}^{n} \left[ w_i^2 + b_1 w_i \right] \leq -b_1\lambda_1(Q) \sum_{i=1}^{n} \left[ 2w_i + b_1 sgn(w_i) \right].
\]
\[ \dot{\beta}_i = -\eta(t) \sum_{j=1}^{n} a_{ij}(\beta_i - \beta_j) \]
\[ \quad - c_1 \sum_{j=1}^{n} a_{ij} \text{sgn}(\beta_i - \beta_j) + a_i' \]
(71)

where \( \eta(t) = (1/(2\lambda_2(L)))h(t) \), and parameters \( b_1 \) and \( c_1 \) are positive constants satisfying \( b_1 \geq (1/(\lambda_2(L))) \), \( c_1 > (2a_{\text{max}}/(\lambda_2(L))) \).

**Theorem 4:** With the given dynamics (7) and (9) and observer (71), under Assumptions 3 and 4, observers \( \alpha_i \) and \( \beta_i \) converge to \( \bar{r} \) and \( \bar{f} \) within a fixed-time \( T_{\text{max}} = t_{s2} \), that is
\[
\begin{align*}
\lim_{t \to t_{s1}} |\beta_i - \bar{f}| &\leq \alpha_1 \\
\lim_{t \to t_{s2}} |\beta_i - \bar{r}| &\leq \alpha_2 \\
\lim_{t \to \infty} |\beta_i - \bar{f}| &\leq 0 \\
\lim_{t \to \infty} |\beta_i - \bar{r}| &\leq 0
\end{align*}
\]
(72)

where the parameters \( \alpha_1 \) and \( \alpha_2 \) are small positive constants close to zero and can be limited to a desired level.

**Proof:** Since the communication topology graph is undirected, one has \( a_{ij} = a_{ji} \). According to (71), one can obtain
\[
\sum_{i=1}^{n} \dot{\beta}_i = \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} a_i' \]  
(73)

Due to \( \sum_{i=1}^{n} \dot{\beta}_i = \sum_{i=1}^{n} a_i' \) and \( \sum_{i=1}^{n} \beta_i(0) = \sum_{i=1}^{n} f_i(0) \), one can obtain \( \sum_{i=1}^{n} \dot{\beta}_i = \sum_{i=1}^{n} a_i' \) and \( \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} f_i \). Moreover, due to \( \sum_{i=1}^{n} \dot{\beta}_i = \sum_{i=1}^{n} a_i' \) and \( \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} f_i \), one can conclude \( \sum_{i=1}^{n} \dot{\beta}_i = \sum_{i=1}^{n} a_i' \) and \( \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} f_i \). Therefore, if observer \( \alpha_i \) and observer \( \beta_i \) achieve consensus in the fixed time \( t_{s2} \), they successfully observe the average position and velocity of reference signals.

We written (71) in the vector form as follows:
\[ \dot{\alpha} = -\eta(t)La - b_1 \text{Dsgn}(D^T \alpha) + \beta \]
\[ \dot{\beta} = -\eta(t)L\beta - c_1 \text{Dsgn}(D^T \beta) + a' \]
(74)

where \( a' = [a_1', \ldots, a_n']^T \). Then, we construct a Lyapunov candidate as \( V_7 = (1/2)b^T L \beta \). Note that \( (1/2)b^T L \beta = \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_i - \beta_j)^2 \). Differentiate \( V_7 \) against time and then one has
\[ V_7 = b^T L \dot{\beta} \]
\[ = b^T (\eta(t)L\beta - c_1 \text{Dsgn}(D^T \beta) + a') \]
\[ \leq -\eta(t)\lambda_2(L)b^T L \beta - c_1 \text{Dsgn}(D^T \beta) \]
\[ + b^T L a' \]
(75)

According to Lemma 4 and the precondition \( c_1 > (2a_{\text{max}}/(\lambda_2(L))) \), one has
\[ \dot{V}_7 \leq -\eta(t)\lambda_2(L)b^T L \beta - c_1 \lambda_2(L)b^T \text{Dsgn}(D^T \beta) + b^T L a' \]
\[ = -h(t)\dot{V}_7 - c_1 \lambda_2(L)||D^T \beta||_1 + ||D^T \beta||^T D^T a' \]
\[ \leq -h(t)\dot{V}_7 - (c_1 \lambda_2(L) - 2a_{\text{max}})||D^T \beta||_1 \]
\[ \leq -h(t)\dot{V}_7. \]
(76)

When \( t \in [0, t_{s1}) \), \( h(t) = h_1(t) \). According to (1), one has \( \lim_{t \to t_{s1}} V_7 = (1/2)\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\beta_i - \beta_j)^2 \leq (\delta_1/1 + \delta_1)^T V_7(0) \). Then, one can conclude
\[ \lim_{t \to t_{s1}} \left| \beta_i - \frac{1}{n} \sum_{j=1}^{n} \beta_j \right| = \lim_{t \to t_{s1}} \left| \beta_i - \frac{1}{n} \sum_{j=1}^{n} \beta_j \right| \]
\[ \leq \lim_{t \to t_{s1}} \max \{|\beta_i - \beta_j| \forall i, j \in n \} \]
\[ \leq \sqrt{2} (\frac{\delta_1}{1 + \delta_1}) V_7(0). \]
(77)

When \( t \geq t_{s1}, V_7 \leq -(c_1 \lambda_2(L) - 2a_{\text{max}})||D^T \beta||_1 - c_1 \lambda_2(L) - 2a_{\text{max}} \sqrt{\lambda_2(L)} V_7 \). Then, one has \( V_7 \) will converge quickly to zero in the finite time after \( t_{s1} \) [see the proof of (36)].

Construct the Lyapunov candidate as \( V_8 = (1/2)a^T L a \) and differentiate it against time
\[ \dot{V}_8 = a^T L \dot{a} \]
\[ = a^T L [-\eta(t)La - b_1 \text{Dsgn}(D^T \alpha) + \beta] \]
\[ = -\eta(t)a^T LLa - b_1 a^T LDsgn(D^T \alpha) + aL \beta] \]
\[ \leq -\lambda_2(L)\eta(t)a^T L \alpha - b_1 ||D^T \alpha|| + (D^T \alpha)^T D^T \beta \]
\[ \leq -h(t)V_8 \]
(78)

where the second inequality holds due to \( |\beta_i - \beta_j| < 1 \) and the precondition \( b_1 \geq 1 \). The following proof is the same as that of (76) and hence omitted. Then, the proof of Theorem 4 has been finished.

**Remark 4:** All the observers proposed in this article can be extended to that for high-order multiagent systems by using more time-based generators and more integrators.

### E. Distributed Consensus Tracking and Distributed Average Tracking Control

After designing the observers, we will show how to design the fixed-time controllers by using the information provided by the observers. Two new time-based generators \( \xi_1(t) \) and \( \xi_2(t) \) are designed as follows:
\[ \xi(t) = \begin{cases} 
\xi_1(t) & t_b \leq t < t_3 \\
\xi_2(t) & t_3 \leq t < t_{s4} \\
1 & t_{s4} \leq t < \infty 
\end{cases} \]
(79)

where \( t_b \geq t_{s2} \). According to (2), the corresponding \( h(t) \) is constructed as follows:
\[ h(t) = \begin{cases} 
h_3(t) = k_1 \frac{\xi_1(t)}{\xi_1(t) + s^2}, & t_b \leq t < t_3 \\
h_4(t) = k_2 \frac{\xi_2(t)}{\xi_2(t) + s^2}, & t_3 \leq t < t_{s4} \\
0, & t_{s4} \leq t < \infty 
\end{cases} \]
(80)

First, the detail of the controller for the distributed consensus tracking problem is shown follows.

**Theorem 5:** Under dynamics (6) and (7) and Assumption 1 (Assumption 2), with given observer (27) [observer (41)] and controller
\[ u_i = \begin{cases} 
0, & t \in [0, t_b) \\
-\frac{1}{2} h(t) \alpha_i - \left( \frac{1}{2} h(t) + 1 \right) \beta_i, & t \geq t_b 
\end{cases} \]
(81)
Differentiate the Lyapunov candidate function as $V$ the proof of (20). Let $\rho \geq d_{\text{max}} + u_{\text{max}} + 1$; the fixed-time distributed consensus tracking problem for double-integrator-type multiagent systems is solved. Furthermore, the upper bounded convergence time is $T_{\text{max}} = t_{44}$.

**Proof:** When $t \geq t_b$, we consider $\alpha_i = x_i - x_0$ and $\beta_i = v_i - v_0$. Differentiate $s_i$ against time and then one has

$$
\dot{s}_i = \frac{1}{2} h(t) \alpha_i + \left(\frac{1}{2} h(t) + 1\right) \dot{\alpha}_i + \beta_i
$$

$$
= \frac{1}{2} h(t) \alpha_i + \left(\frac{1}{2} h(t) + 1\right) \beta_i + v_i - \dot{v}_0
$$

$$
= \frac{1}{2} h(t) \alpha_i + \left(\frac{1}{2} h(t) + 1\right) \beta_i + u_i + d_i - u_0. \quad (82)
$$

Substitute controller (81) into (82), and then one has

$$
\dot{s}_i = -\frac{1}{2} h(t) s_i - \rho \text{sgn}(s_i) + d_i - u_0. \quad (83)
$$

Choose $V_1 = (1/2)s_i^2$ as the Lyapunov candidate functions. Differentiate $V_1$ against time and then one has

$$
\dot{V}_1 = s_i \dot{s}_i
$$

$$
= s_i \left[ -\frac{1}{2} h(t) s_i - \rho \text{sgn}(s_i) + d_i - u_0 \right]
$$

$$
= -h(t)V_1 - \rho |s_i| + (d_i - u_0)s_i
$$

$$
\leq -h(t)V_1 - (\rho - d_{\text{max}} - u_{\text{max}})|s_i|. \quad (84)
$$

Due to precondition $\rho \geq d_{\text{max}} + u_{\text{max}} + 1$, one has

$$
\dot{V}_1 \leq -h(t)V_1 - |s_i|
$$

$$
\leq -h(t)V_1. \quad (85)
$$

According to the property of (1), one can come to the conclusion that $\lim_{t \to t_b} V_1 \leq (\delta_3/1 + \delta_3)h(t_0) \leq \sqrt{2(\delta_3/1 + \delta_3)}h(t_0)$. Let $z_1 = \alpha_i$ and $z_2 = \beta_i$, and construct the Lyapunov candidate function as $V_2 = (1/2)c_i^2$. Then, the following proof is the same as Theorem 1 and omitted [see the proof of (20)].

Next, the detail of the controller for the fixed-time distributed average tracking problem is shown as follows.

**Theorem 6:** Under dynamics (7) and (9) and Assumption 3, with given observer (71) and controller

$$
\dot{u}_i = \left\{ \begin{array}{ll}
0, & i \in [0, t_b) \\
-\frac{1}{2} h(t)(x_i - \alpha_i) - \left(\frac{1}{2} h(t) + 1\right)(v_i - \beta_i) & i \geq t_b
\end{array} \right. \quad (86)
$$

where $s_i$ is the sliding-mode surface, and $s_i = ((1/2)h(t) + 1)(x_i - \alpha_i) + (v_i - \beta_i)$; the fixed-time distributed average tracking problem for double-integrator-type multiagent systems is solved. Furthermore, the upper bounded convergence time is $T_{\text{max}} = t_{44}$.

**Proof:** When $t \geq t_b$, we consider $\alpha_i = \bar{r}$ and $\beta_i = \bar{f}$. Differentiate $s_i$ against time and then one has

$$
\dot{s}_i = \frac{1}{2} h(t) \alpha_i + \left(\frac{1}{2} h(t) + 1\right) (v_i - \beta_i)
$$

$$
+ u_i + d_i - \alpha_\bar{r}. \quad (87)
$$

IV. NUMERICAL SIMULATIONS

Three kinds of communication topology graphs for multiagent systems are shown in Fig. 1, which will be used in the following experiments.

**Example 1:** A simulation for Theorem 1 is given here and the results are shown in Fig. 2. The experiment is divided into two comparison groups ($c_1$ and $c_2$) and the parameter settings are shown in Table I. The initial states in $c_1(a)$ and $c_2(a)$ are $z_1(0) = 100$ and $z_2(0) = 200$, and those in $c_1(b)$ and $c_2(b)$ are $z_1(0) = 300$ and $z_2(0) = 600$, and disturbance $\varphi = \sin(t)$. By comparing the results between $c_1(a)$ and $c_1(b)$ or $c_2(a)$ and $c_2(b)$ in Fig. 2, we conclude the upper bounded convergence time is not influenced by the initial states. Comparing the results between $c_1(a)$ and $c_2(a)$ or $c_1(b)$ and $c_2(b)$ in Fig. 2, we conclude that the upper bounded convergence time $T_{\text{max}}$ can be easily adjusted by modifying the parameter $t_{44}$.

**Example 2:** A simulation for Theorems 2 and 5 under undirected communication is shown in Figs. 3 and 4. Consider
a multiagent system described by (6) and (7) with the communication topology depicted in the first subgraph of Fig. 1. The simulation contains two comparison groups (c1 and c2), and the parameter settings are shown in Table I. In c1, we set leader control input $u_0 = 1 + 5\sin(t)$, disturbance $d_1 = \sin(t)$, and system initial states $x_0(0) = v_0(0) = 0$, $x_1(0) = v_1(0) = -200$, $x_2(0) = v_2(0) = -100$, $x_3(0) = v_3(0) = 0$, $x_4(0) = v_4(0) = 100$, $x_5(0) = v_5(0) = 200$. In c2, we reset the initial states as $x_1(0) = v_1(0) = -600$, $x_2(0) = v_2(0) = -400$, $x_3(0) = v_3(0) = 0$, $x_4(0) = v_4(0) = 400$, and $x_5(0) = v_5(0) = 600$. By comparing the results of the two comparison groups in Figs. 3 and 4, we conclude the observer successfully observe the object value in $t_{s2}$ and the controller drive the multiagent system to achieve consensus with the leader in $T_{max} = t_{s4}$. Moreover, the upper bounded convergence time is not influenced by system initial states.

**Example 3:** A simulation for Theorems 3 and 5 under directed communication topology is shown in Figs. 5 and 6. Consider a multiagent system described by (6) and (7) with the communication topology depicted in the second subgraph of Fig. 1. The experiment contains two comparison groups (c1 and c2), and the parameter settings are shown in Table I. In c1, we set leader control input $u_0 = 18\sin(10t) + 2$, disturbance $d_1 = \sin(t)$, and system initial states $x_0(0) = v_0(0) = 300$, $x_1(0) = v_1(0) = -100$, $x_2(0) = v_2(0) = -50$, $x_3(0) = v_3(0) = 0$, $x_4(0) = v_4(0) = 50$, and $x_5(0) = v_5(0) = 100$. In c2, we reset the initial states as $x_1(0) = v_1(0) = -500$, $x_2(0) = v_2(0) = -400$, $x_3(0) = v_3(0) = 0$, $x_4(0) = v_4(0) = 400$, and $x_5(0) = v_5(0) = 500$. By comparing the results of two comparison groups in Figs. 5 and 6, we conclude the observer successfully observe the object value and the controller drive the multiagent system to achieve consensus with the leader in the fixed time. Moreover, the upper bounded convergence time is not influenced by system initial states.

**Example 4:** A simulation for Theorems 4 and 6 is given in Figs. 7 and 8. Consider a multiagent system described by (7) and (9) with the communication topology depicted in the third subgraph of Fig. 1. The experiment contains two comparison groups (c1 and c2). In c1, we set the acceleration of the $i$th reference signal as $a_i = 20\sin(5\pi t) + 10i$, the reference signal initial states as $r_1(0) = f_1(0) = 200$, $r_2(0) = f_2(0) = 100$, $r_3(0) = f_3(0) = -100$, and $r_4(0) = f_4(0) = -200$, and the system initial states as $x_1(0) = v_1(0) = -100$, $x_2(0) = v_2(0) = -200$, $x_3(0) = v_3(0) = 100$, and $x_4(0) = v_4(0) = 300$. In c2, we reset the reference signal initial states as $r_1(0) = f_1(0) = 600$, $r_2(0) = f_2(0) = 400$, $r_3(0) = f_3(0) = -400$, and $r_4(0) = f_4(0) = -600$, and the system initial states $x_1(0) = v_1(0) = 600$, $x_2(0) = v_2(0) = 500$, $x_3(0) = v_3(0) = 400$, and $x_4(0) = v_4(0) = 300$. By comparing the results of the two comparison groups in Figs. 7 and 8, we conclude the observers successfully observe the mean value of multiple reference signals in $t_{s2}$ and the controllers drive the multiagent system to achieve consensus with the mean value of multiple reference signals in $T_{max} = t_{s4}$. Moreover,
Fig. 5. Experimental results of fixed-time distributed consensus tracking under directed communication (comparison group $c_1$). (a) Position observer. (b) Velocity observer. (c) Position controller. (d) Velocity controller.

Fig. 6. Experimental results of fixed-time distributed consensus tracking under directed communication (comparison group $c_2$). (a) Position observer. (b) Velocity observer. (c) Position controller. (d) Velocity controller.

The upper bounded convergence time is not changed with the variation system initial states.

**Remark 5:** Increasing the value of parameter $k$ or decreasing that of parameter $\delta$ can improve the convergence accuracy at the predefined convergence time, but it will also increase the accuracy demand of the control system. The predefined convergence time $t_s$ can be designed arbitrarily, but a small $t_s$ has to demand a large control input. The main effect of other parameters is to guarantee the proof correct, and have a limited influence on the real convergence time.

**V. CONCLUSION**

In this article, both the fixed-time distributed consensus tracking and the fixed-time distributed average tracking problems for double-integrator-type multiagent systems were solved by using time-based generators. Different from traditional fixed-time methods, the time-based generator approach can avoid control input saturation and determine the fixed time without complicated computation. In the future, we try to extend the proposed fixed-time sliding-mode control method to Euler–Lagrange systems. By combining the sliding-mode control method of Euler–Lagrange systems and the proposed observers, the fixed-time distributed consensus tracking and distributed average tracking problems for multiple Euler–Lagrange systems can be solved.

**REFERENCES**

[1] Z. Li, Z. Wu, Z. Li, and Z. Ding, “Distributed optimal coordination for heterogeneous linear multiagent systems with event-triggered mechanisms,” IEEE Trans. Autom. Control, vol. 65, no. 4, pp. 1763–1770, Apr. 2020.
Y. Zhao, Y. Liu, G. Wen, and G. Chen, “Distributed optimization for linear multiagent systems: Edge-and node-based adaptive designs,” *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3602–3609, Jul. 2017.

Z. Li, X. Liu, W. Ren, and L. Xie, “Distributed tracking control for linear multiagent systems with a leader of bounded unknown input,” *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 518–523, Feb. 2013.

G. Wen, T. Huang, W. Yu, Y. Xia, and Z. Liu, “Cooperative tracking of networked agents with a high-dimensional leader: Qualitative analysis and performance evaluation,” *IEEE Trans. Cybern.*, vol. 48, no. 7, pp. 2060–2073, Jul. 2018.

W. Ren and J. Xiong, “Tracking control of nonlinear networked and quantized control systems with communication delays,” *IEEE Trans. Autom. Control*, vol. 65, no. 8, pp. 3685–3692, Aug. 2020.

H. Su, X. Wang, and Z. Lin, “Flocking of multi-agents with a virtual leader,” *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 293–307, Feb. 2009.

Q. Xiong, P. Lin, W. Ren, C. Yang, and W. Gui, “Containment control for discrete-time multiagent systems with communication delays and switching topologies,” *IEEE Trans. Cybern.*, vol. 49, no. 10, pp. 3827–3830, Oct. 2019.

G. Wen, Y. Zhao, Z. Duan, W. Yu, and G. Chen, “Containment of higher-order multi-leader multi-agent systems: A dynamic output approach,” *IEEE Trans. Autom. Control*, vol. 61, no. 4, pp. 1135–1140, Apr. 2016.

R. Olfati-Saber and R. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

W. Yu, G. Chen, and M. Cao, “Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems,” *IEEE Trans. Autom. Control*, vol. 66, no. 6, pp. 1089–1090, 2019.

Y. Cao, L. Zhang, C. Li, and M. Z.-Q. Chen, “Observer-based consensus tracking of nonlinear agents in hybrid varying directed topology,” *IEEE Trans. Cybern.*, vol. 47, no. 8, pp. 2212–2222, Aug. 2017.

W. Yu, Y. Li, G. Wen, X. Yu, and J. Cao, “Observer design for tracking consensus in second-order multi-agent systems: Fractional order less than two,” *IEEE Trans. Autom. Control*, vol. 62, no. 2, pp. 894–900, Feb. 2017.

G. Wen, P. Wang, Y. Lv, G. Chen, and J. Zhou, “Secure consensus of multi-agent systems under denial-of-service attacks,” *Asian J. Control*, to be published, doi: 10.1002/asjc.2953.

F. Chen, W. Ren, W. Lan, and G. Chen, “Distributed average tracking for reference signals with bounded accelerations,” *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 863–869, Mar. 2015.

Y. Zhao, Y. Liu, Z. Li, and Z. Duan, “Distributed average tracking for multi-signal signals generated by linear dynamical systems: An edge-based framework,” *Automatica*, vol. 75, pp. 158–166, Jan. 2017.

Y. Zhao, Y. Liu, G. Wen, X. Yu, and G. Chen, “Distributed average tracking for Lipschitz-type of nonlinear dynamical systems,” *IEEE Trans. Cybern.*, vol. 49, no. 12, pp. 4140–4152, Dec. 2019.

C. Xian, Y. Zhao, Z.-G. Wu, G. Wen, and J.-A. Pan, “Event-triggered distributed average tracking control for Lipschitz-type non-linear multi-agent systems,” *IEEE Trans. Cybern.*, early access, Apr. 12, 2022, doi: 10.1109/TCYB.2022.3159250.

J. Yu, P. Shi, and L. Zhao, “Finite-time command filtered backstepping control for a class of nonlinear systems,” *Automatica*, vol. 92, pp. 173–180, Jun. 2018.

J. Yu, L. Zhao, H. Yu, and C. Lin, “Barrier lyapunov functions-based command filtered output feedback control for full-state constrained nonlinear systems,” *Automatica*, vol. 105, pp. 71–79, Jul. 2019.

G. Cui, J. Yu, and Q.-G. Wang, “Finite-time adaptive fuzzy control for MIMO nonlinear systems with input saturation via improved command-filtered backstepping,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 52, no. 2, pp. 980–989, Feb. 2022, doi: 10.1109/TSMC.2020.3010642.

C. Fu, Q.-G. Wang, J. Yu, and C. Lin, “Neural network-based finite-time command filtering control for switched nonlinear systems with backlash-like hysteresis,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 7, pp. 3268–3273, Jul. 2021.

T. Li, R. Zhao, C. L. P. Chen, L. Fang, and C. Liu, “Finite-time formation control of under-actuated ships using nonlinear sliding mode control,” *IEEE Trans. Cybern.*, vol. 48, no. 11, pp. 3243–3253, Nov. 2018.

Y. Cao, W. Ren, and Z. Meng, “Decentralized finite-time sliding mode estimators and their applications in decentralized finite-time formation control,” *IEEE Trans. Autom. Control*, vol. 59, no. 5, pp. 522–529, May 2014.

Y. Zhao, Z. Duan, G. Wen, and Y. Zhang, “Distributed finite-time tracking control for multi-agent systems: An observer-based approach,” *Syst. Control Lett.*, vol. 62, no. 1, pp. 22–28, 2013.