Amplitude equations for SPDEs with quadratic nonlinearities forced by additive and multiplicative noise

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Abstract This article deals with stochastic partial differential equations with quadratic nonlinearities perturbed by small additive and multiplicative noise. We present the approximate solution of the original equation via the amplitude equation and give the rigorous error analysis. For illustration, we apply our main theorems to stochastic Burger’s equation.

Keywords: amplitude equations, stochastic partial differential equations, quadratic nonlinearities, additive noise, multiplicative noise.

1 Introduction

Stochastic partial differential equations (SPDEs) with quadratic nonlinearities of the type

\[ \frac{du}{dt} = Au + B(u, u) \, dt + G(u) \, dW(t), \]

are used to study some physical phenomenon such as hydrodynamic turbulence (Burgers’ equation) [11], surface erosion (Kuramoto-Sivashinsky equation) [18], amorphous thin-film growth [23], propagation of solitons (Korteweg-de Vries equation) [9, 10] and Rayleigh-Bénard convection [2].

In this paper, we consider (1.1) perturbed by small deterministic perturbation and small

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\[ \text{noise:} \]
\[ du = [Au + \varepsilon^2Lu + B(u, u)]dt + G(u, \varepsilon)dW(t), \quad u \in \mathcal{H}, \]
\[ u(0) = u_0, \]  
(1.2)

where \( \mathcal{H} \) is an infinite dimensional separate Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) and corresponding norm \( \| \cdot \| \). \( A \) is a self-adjoint and non-positive operator with finite dimensional kernel space called as dominated modes, \( L \) is a linear operator, \( B \) is a bilinear and symmetric operator, \( G(u, \varepsilon) \) is a Hilbert-Schmidt operator, \( W(t) \) is a cylindrical Wiener process with covariance operator \( I \) on some stochastic space, and \( \varepsilon \) is a small parameter characterizing the distance from bifurcation point and the strength of the noise.

This paper will answer two questions. One question is whether there exists a simplified system can characterize the limit behavior for the original system as \( \varepsilon \) tends to 0. This question arises naturally from the complexity of multi-scale SPDEs, which causes that it is not easy to analyze dynamical behaviors and provide numerical stimulation. After extracting effective information from the original system, we will present a reduced system, and rigorously prove that it is regarded as a good approximation of the original one. On the other hand, we will explain the extent to which stochastic forcing influences the dynamics near a change of stability. This argument is motivated by physics investigations [14, 15, 16, 25] in which scholars observe that noise has the potential to stabilize the dynamics. For specific systems, with the help of the simplified system, we will clearly give defined conditions under which the original system are stable or unstable.

The approach we rely on is use amplitude equation deriving from dominated pattern to captures the effective dynamics of the original system. We precisely interpret the procedure of the approach follows:

- Remove high order terms from dominated modes;
- Extract amplitude equations from dominated modes;
- Estimate the error between the amplitude equations and the original equations.

Amplitude equation not only contributes to the approximation for SPDEs, but also explains whether the noise could shift bifurcation point. The first rigorous result for SPDEs on bounded domain via amplitude equations was established by Blömker et al [5]. After that, there have been rapid progresses for SPDEs with additive noise, such as quadratic nonlinearities [1, 7], cubic nonlinearities [3, 6], as well as both quadratic and cubic nonlinearities [17, 20, 22]. Recently, Blömker and Fu [13] considered a class of SPDEs with cubic nonlinearities perturbed by multiplicative noise via amplitude equations. However, except results in [1, 8], amplitude equations for SPDEs with quadratic nonlinearities perturbed by multiplicative noise are unknown, let alone additive and multiplicative noise. The aim of this paper is to develop this research. We will consider (1.2) in two cases which are further investigations for the results in [7, 20].

In the first case, we are concerned about additive noise of order \( \varepsilon^2 \), and obtain amplitude equation without too many restrictions. Compared with previous work, our result underlines the important role that multiplicative noise plays in amplitude equation. Let us illustrate this point with stochastic Burger’s equation:

\[ du = [\partial_{xx}u + \varepsilon^2\nu u + u\partial_x u]dT + (\varepsilon^2 + \varepsilon u)dW(t) \]  
(1.3)

on \([0, \pi]\) subject to Dirichlet boundary condition. Under some assumptions, we obtain amplitude
equation:
\[ d\tilde{x} = \left( \nu \tilde{x} - \frac{1}{12} \tilde{x}^3 \right) dT + \alpha_1 d\beta_1(T) + \frac{8\sqrt{2}\alpha_1}{3\pi^2} \tilde{x} d\beta_1(T) - \frac{8\sqrt{2}\alpha_3}{15\pi^2} \tilde{x} d\beta_3(T), \quad (1.4) \]

where \( \beta_1(T) \) and \( \beta_3(T) \) are real-valued Brownian motion, \( \alpha_1 \) and \( \alpha_3 \) are coefficients from \( W(t) \).

However, if multiplicative noise does not involve in (1.3), under same assumptions, amplitude equation is
\[ d\tilde{x} = \left( \nu \tilde{x} - \frac{1}{12} \tilde{x}^3 \right) dT + \alpha_1 d\beta_1(T). \]

From the comparison of two amplitude equations, multiplicative noise makes a difference to amplitude equation. Our main theorem further states that
\[ u(t) = \tilde{x}(\varepsilon^2 t) \sin x + \mathcal{O}(\varepsilon^2), \]

with \( \tilde{x}(\varepsilon^{-2}) = \tilde{x}(T) \) from (1.4). Then, observing (1.4), we can clearly understand how multiplicative noise changes the stability of (1.3) if \( \alpha_1 = 0 \). In fact, the constant solution \( 0 \) is locally stable if \( \nu < \frac{64\alpha_2}{225\pi^2} \), and locally unstable if \( \nu > \frac{64\alpha_2}{225\pi^2} \). To our best knowledge, it is the first observation in quadratic nonlinearities with this field. More than that, such stability analysis allows us to explain that the bifurcation point of a deterministic system could change if it is perturbed by noise. Therefore, we think our result provides a new perspective for stochastic bifurcation theory.

In the second one, we take into account additive noise of order \( \varepsilon \), and assume that \( \mathcal{L} \) commute with the projection operators. Since multiplicative noise is studied in this case, fast fluctuation appears in diffusion terms. This problem causes that we can gain amplitude equation, but we can not show how fast the solution of the amplitude equation converges to that of the original one if the dimension of \( \ker A \) is more than one. Fortunately, we can give the explicit error between the approximation solution and the original one for one dimensional kernel space by martingale representation theorem.

The rest of the paper is organized as follows. In Section 2, we introduce some assumptions and notations. In Section 3, we study the amplitude equations of (1.2) forced by non-degenerate noise and multiplicative noise, and present the main theorem. In Section 4, we focus on (1.2) forced by degenerate noise and multiplicative noise, provide different amplitude equations in terms of different assumptions, and show rigorous convergence analysis and error estimate. In Section 4, as an application of the main results, we study the limit behavior of stochastic Burger’s equation, and discuss the relationship between the stability and noise.

### 2 Notations and Assumptions

Throughout the paper, notations \( C \) and \( C_i \) may denote different positive constants independent of \( \varepsilon \) in different occasions. In the followings, we provide some assumptions and notations.

**Assumption 2.1** Assume that \( A \) is a non-positive and self-adjoint operator on \( \mathcal{H} \) with eigenvalues \( 0 = \lambda_1 \leq \cdots \leq \lambda_k \cdots \), and \( \lambda_k \geq Ck^m \) holds for all sufficiently large \( k \), positive constants \( m \) and \( C \). Suppose that there is a complete orthonormal basis \( \{e_k\}_{k \in \mathbb{N}} \) such that \( Ae_k = -\lambda_k e_k \) and the kernel space of \( A \) is finite dimensional. Denote \( \ker A \) and the orthogonal complement of it by \( \mathcal{N} \) and \( \mathcal{S} \).
Assumption 2.2 Assume that the dimension of ker $A$ is 1.

Define projections $P_c : \mathcal{H} \rightarrow \mathcal{N}$ and $P_s = I - P_c$. For a map $L$, we use $L_c := P_c L$ and $L_s := P_s L$.

**Definition 2.1** For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^\alpha$ as

$$\mathcal{H}^\alpha = \left\{ \sum_{k=1}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} \gamma_k^2 (\lambda_k + 1)^\alpha < \infty \right\}$$

with the norm

$$\| \sum_{k=1}^{\infty} \gamma_k e_k \|_\alpha = \left( \sum_{k=1}^{\infty} \gamma_k^2 (\lambda_k + 1)^\alpha \right)^{\frac{1}{2}}.

If $A$ satisfies the Assumption 2.1, it can generate an analytic semi-group $\{e^{At}\}_{t \geq 0}$ on any space $\mathcal{H}^\alpha$, defined by

$$e^{At}(\sum_{k=1}^{\infty} \gamma_k e_k) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \gamma_k e_k, \quad t \geq 0.

Moreover, $A$ enjoys the following property.

**Lemma 2.1** Under Assumption 2.1, for $\forall \rho \in (\lambda_n, \lambda_{n+1}], \ t \geq 0, \ \beta \leq \alpha$, there is a constant $M > 0$, such that for $\forall u \in \mathcal{H}^\beta$,

$$\|e^{At} P_s u\|_\alpha \leq M t^{-\frac{\beta}{2}} e^{-\rho t} \|P_s u\|_{\alpha - \beta}.

**Assumption 2.3** Assume that $\mathcal{L} : \mathcal{H}^\alpha \rightarrow \mathcal{H}^{\alpha - \beta}$ is a linear continuous mapping, for some $\alpha \in \mathbb{R}$ and $\beta \in [0, m)$.

**Assumption 2.4** Assume that $B : \mathcal{H}^\alpha \times \mathcal{H}^\alpha \rightarrow \mathcal{H}^{\alpha - \beta}$ is a bounded bilinear and symmetric operator with $\alpha$ and $\beta$ given in Assumption 2.3. Moreover, suppose that $B_c(a) = 0$, for $a \in \mathcal{N}$, where we use the notation $B(a) := B(a,a)$.

**Assumption 2.5** Assume that $B_c(e_k,e_k) = 0$, for $k > n$.

**Definition 2.2** Define $\mathcal{F} : \mathcal{N} \times \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N}$ by

$$\mathcal{F}(u,v,w) = -B_c(u, A_s^{-1} B_s(v,w)), \quad u, v, w \in \mathcal{N}.

**Assumption 2.6** Assume that $\mathcal{F}$ is a trilinear, symmetric mapping and satisfies the following conditions: for positive constant $C_0$,

$$\|\mathcal{F}(u,v,w)\| < C_0 \|u\| \|v\| \|w\|, \quad \forall u, v, w \in \mathcal{N}, \quad (2.1)$$

and for positive constants $C_1, C_2, C_3$, for all $u, v, w \in \mathcal{N}$

$$\langle \mathcal{F}_c(u,v,w) - \mathcal{F}_c(v), u \rangle \leq -C_1 \|u\|^4 + C_2 \|w\|^4 + C_3 \|w\|^2 \|v\|^2, \quad (2.2)\]$$

where we use $\mathcal{F}(u) := \mathcal{F}(u,u,u)$ for short notation.

**Assumption 2.7** Let $U$ be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_U$. Assume that $W(t)$ is a $U$-valued cylindrical Wiener process on a stochastic base $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with covariance operator $I$, the identity operator.
Note that $W(t)$ has the expansion \[ W(t) = \sum_{k=1}^{\infty} \beta_j(t) f_j, \] where $\{\beta_j(t)\}_{j \in \mathbb{N}}$ are real valued Brownian motions mutually independent on the above stochastic basis and $\{f_j\}_{j \in \mathbb{N}}$ is a complete orthonormal system in $U$.

Since multiplicative noise runs through this paper, we recall Hilbert-Schmidt operator here. Suppose that $U, H$ are two separable Hilbert spaces with complete orthonormal basis $\{f_j\}_{j \in \mathbb{N}} \subset U$, $\{e_k\}_{k \in \mathbb{N}} \subset H$. A linear and bounded operator $L$ is said to be Hilbert-Schmidt if

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\langle Lf_j, e_k \rangle_H|^2 < \infty.$$ 

Denote the set of all Hilbert-Schmidt operators from $U$ to $H$ by $L_2(U,H)$. Note that $L_2(U,H)$ is separable Hilbert space, with the scalar product

$$\langle \cdot, \cdot \rangle_{L_2(U,H)} = \sum_{j=1}^{\infty} \langle f_j, f_j \rangle_H,$$

by which the norm of $L_2(U,H)$ is induced.

**Assumption 2.8** Suppose that $G : H^\alpha \times \mathbb{R}^+ \to L_2(U,H^\alpha)$ with $\alpha$ as in Assumption 2.3 and $G(u, \varepsilon) = \sigma \tilde{G} + \varepsilon \tilde{G}(u)$, where $\sigma$ is scaling parameter and $\tilde{G}(0) = 0$. We further assume

$$\tilde{G} \cdot f_k = \alpha_k e_k,$$

where $\alpha_k$ are real constants for $1 \leq k \leq N$, and $\alpha_k = 0$ for $k > N$. Moreover, assume that $\tilde{G}(u)$ is Fréchet differentiable up to order 2, and for $\forall u, v, w \in H^\alpha$ with $\|u\|_\alpha \leq r$, there exists a positive constant $l_r$ depending on $r$ such that

$$\|\tilde{G}(u)\|_{L_2(U,H^\alpha)} \leq l_r \|u\|_\alpha, \quad (2.3)$$

$$\|\tilde{G}'(u) \cdot v\|_{L_2(U,H^\alpha)} \leq l_r \|v\|_\alpha,$$

$$\|\tilde{G}''(u) \cdot (v, w)\|_{L_2(U,H^\alpha)} \leq l_r \|v\|_\alpha \|w\|_\alpha,$$

where $\tilde{G}'(u)$ and $\tilde{G}''(u)$ are the first and second Fréchet derivatives with respect to $u$ respectively.

**Remark 2.2** We can consider the case that $N = \infty$ and $\tilde{G}$ is non-diagonal operator, but addition assumptions need to be imposed to ensure the convergence of various infinite series. Therefore, we ignore the case for simplicity of presentation.

**Definition 2.3** An $H^\alpha$-valued process $u(t)$ is called the local mild solution of (1.2), if there exists some stopping time $\tau_{ex}$, on a full set in which $\tau_{ex} > 0$, we have $u \in C^0([0, \tau_{ex}), H^\alpha)$ and

$$u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)} [\varepsilon^2 L u + B(u)] ds + \int_0^t e^{A(t-s)} G(u, \varepsilon) dW(s),$$

$\forall t \in [0, \tau_{ex})$.

According to the above assumptions, we could state that there exists local mild solution in (1.2) by cut-off technique and Theorem 7.2 in [12].

**Theorem 2.3** Under Assumption 2.1-2.3, for any given $u(0) \in H^\alpha$, there exists a unique local mild solution of (1.2), in the sense of Definition 2.3, such that $\tau_{ex} = \infty$ or $\lim_{t \to \tau_{ex}} \|u(t)\|_\alpha = \infty$. 

5
3 Non-degenerate additive noise

In this section, we will consider the case that the dominated modes are affected by additive noise, and assume $\sigma = \varepsilon^2$. The section is devoted to presenting the amplitude equation of $u$ and providing the approximation result. In short, we firstly bounded the fast modes. Next, we extract the amplitude equation after separating higher order terms from the slow modes in the drift and diffusion terms. Then, we show the error estimate between the solution of the amplitude equation and that of the slow modes. Finally, we prove that the solution of (1.2) can be approximated by that of the amplitude equations well in $[0, \varepsilon^{-2}T_0]$.

Split $u$ into slow modes $a$ and fast modes $b$:

$$\quad u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon^2 b(\varepsilon^2 t),$$

where $a \in \mathcal{N}$ and $b \in \mathcal{S}$. Introduce slow time scale $T = \varepsilon^2 t$. In what follows, we will study the behavior of (1.2) on such scale. Then, with the projections $P_c$ and $P_s$, we have

$$\begin{align*}
\dot{a} &= [\mathcal{L}_a + \varepsilon^2 b + 2B_s(a, b) + \varepsilon B_c(b)]dT + [\tilde{G}_c + \varepsilon^{-1} \bar{G}_c(\varepsilon a + \varepsilon^2 b)]d\tilde{W}, \\
\dot{b} &= \varepsilon^{-2} A_s b + \varepsilon^{-1} \mathcal{L}_a + L_a b + B_s(b) + 2\varepsilon^{-1} B_s(a, b)]dT \\
&\quad + \varepsilon^{-2} B_s(a) dT + [\varepsilon^{-1} \tilde{G}_s + \varepsilon^{-2} \bar{G}_s(\varepsilon a + \varepsilon^2 b)]d\tilde{W}.
\end{align*}$$

(3.1)

As stated in Theorem 2.3, the local mild solution of (1.2) may blow up in finite time, so we introduce some stopping time to ensure that $a(T)$ and $b(T)$ are not too large on some interval.

**Definition 3.1** For an $\mathcal{N} \times \mathcal{S}$-valued stochastic process $(a, b)$ given by (3.1) and (3.2), we define, for $T_0 > 0$ and $\kappa \in (0, \frac{2}{19})$, the stopping time $\tau^*$ as

$$\tau^* := T_0 \wedge \inf\{T > 0 \mid \| a(T) \|_{\alpha} > \varepsilon^{-\kappa} \text{ or } \| b(T) \|_{\alpha} > \varepsilon^{-3\kappa}\}.$$

3.1 Bounds of fast modes

Set $V(T) := \varepsilon^{-1} \mathcal{L}_a + L_a b + B_s(b) + 2\varepsilon^{-1} B_s(a, b) + \varepsilon^{-2} B_s(a)$. Writing the mild solution of $b(T)$:

$$\begin{align*}
b(T) &= \varepsilon^{-2} A_s b(0) + \int_0^T \varepsilon^{-2} A_s(T-s) V(s) ds + \varepsilon^{-1} \int_0^T \varepsilon^{-2} A_s(T-s) \tilde{G}_s d\tilde{W} \\
&\quad + \varepsilon^{-2} \int_0^T \varepsilon^{-2} A_s(T-s) \bar{G}s(\varepsilon a + \varepsilon^2 b) d\tilde{W}.
\end{align*}$$

In the following, we will bound $b(T)$ after estimating each part of it.

**Lemma 3.1** Under Assumption 2.4, 2.5, 2.6, 2.7, 2.8, for $p > 1$, there exists a positive constant $C$ such that

$$\mathbb{E}\left(\sup_{0 \leq T \leq \tau^*} \left\| \int_0^T \varepsilon^{-2} A_s(T-s) V(s) ds \right\|_{\alpha}^p \right) \leq C \varepsilon^{-2\kappa p}.$$

**Proof** For $p > 1$, by Lemma 2.4 we have

$$\mathbb{E}\left(\sup_{0 \leq T \leq \tau^*} \left\| \int_0^T \varepsilon^{-2} A_s(T-s) V(s) ds \right\|_{\alpha}^p \right).$$
exists a positive constant $C$

Lemma 3.2

Under Assumption 2.1, 2.7, 2.8, for

with $\tilde{\gamma} \in \mathbb{R}$

By Stochastic Fubini Theorem, we know

Then, based on Assumption 2.3-2.4, it is easy to obtain the desired result.

We complete the proof.

Let $Z(T) = \varepsilon^{-1} \int_0^T e^{-2A_z(T-s)} dW = \sum_{k=n+1}^N Z_k(T)\varepsilon_k$, where

$$Z_k(T) = \varepsilon^{-1}\alpha_k \int_0^T e^{-2\lambda_k(T-s)} d\beta_k(s)$$

with $\tilde{\beta}(s) := \varepsilon\beta(s)$. Then, according to Lemma 14 in [6], we have the following lemma.

**Lemma 3.2** Under Assumption 2.3 2.7 2.8 for $Z(T)$ and $Z_k(T)$ given by the above, there exists a positive constant $C$ depending on $p \geq 1$, $\lambda_k, \alpha_k, \kappa > 0$ and $T_0$, such that

$$\mathbb{E}\left( \sup_{0 \leq T \leq T_0} |Z_k(t)|^p \right) \leq C\varepsilon^{-\frac{p}{2}}.$$  (3.3)

$$\mathbb{E}\left( \sup_{0 \leq T \leq T_0} \|Z(t)\|^p \right) \leq C\varepsilon^{-\frac{p}{2}}.$$  (3.4)

**Lemma 3.3** Under Assumption 2.1 2.3 2.4 2.7 2.8 for $p > 1$, there exists a positive constant $C$ such that

$$\mathbb{E}\left( \sup_{0 \leq T \leq T_0} \|Z(T)\|^p \right) \leq C\varepsilon^{2-2\kappa p}.$$  (3.5)

**Proof** We complete the proof with the help of factorization method. For $p > 2$, we choose $\gamma \in (\frac{1}{p}, \frac{1}{2})$, and introduce

$$D(T) = \int_0^T (T-s)^{-\gamma} e^{A_z(T-s)} e^{-2b(s)} d\tilde{W}.$$  (3.5)

By Stochastic Fubini Theorem, we know

$$\int_0^T e^{A_z(T-s)} e^{-2b(s)} d\tilde{W} = C_\gamma \int_0^T (T-s)^{-\frac{\gamma}{2}} e^{A_z(T-s)} e^{-2b(s)} D(s) ds,$$

where $C_\gamma$ is a constant dependent of $\gamma$. By Lemma 2.7 and Hölder inequality, we further deduce

$$\left\| \int_0^T e^{A_z(T-s)} e^{-2b(s)} d\tilde{W} \right\|_\alpha^p \leq C \left( \int_0^T (T-s)^{-\frac{\gamma}{2}} e^{A_z(T-s)} e^{-2b(s)} D(s) ds \right)^p \leq C \left( \int_0^T (T-s)^{\frac{p-1}{2}} e\frac{e^{p(T-s)} e^{-2}}{e^{p(T-s)} e^{-2}} D(s) ds \right)^{p-1} \int_0^T \|D(s)\|_\alpha^p ds \leq C\varepsilon^{2\gamma p-2} \int_0^T \|D(s)\|_\alpha^p ds.$$  (3.5)
By Lemma 2.4, Burkholder-Davis-Gundy inequality and Assumption 2.7, we obtain the moments of $D(T)$ as follows:

$$
\mathbb{E}\|D(T)\|_p^p \leq C\mathbb{E}\left( \int_0^T (T-s)^{-2\gamma} \|e^{A_s(t-s)}\|^{-2} G_s(\varepsilon a + \varepsilon^2 b)^2 \rho_{\mathcal{D}(U, \mathcal{M})} ds \right)^{\frac{p}{2}}
$$

$$
\leq C\mathbb{E}\left( \int_0^T (T-s)^{-2\gamma} \|e^{-\rho(t-s)}\|^{-2} \|\varepsilon a + \varepsilon^2 b\|^2 ds \right)^{\frac{p}{2}}
$$

$$
\leq C\varepsilon^{-2p-2}\sup_{0 \leq t \leq T^*} \|\varepsilon a(T) + \varepsilon^2 b(T)\|_p^p
$$

We conclude

$$
\mathbb{E}\left( \sup_{0 \leq t \leq T^*} \| \int_0^t e^{A_s(T-s)} \|^{-2} G_s(\varepsilon a(s) + \varepsilon^2 b(s)) dW \right)_p^p \leq C\varepsilon^{-2-\kappa p}.
$$

Then, we can achieve this proof by Hölder inequality.

We complete the proof. 

Combining Lemma 3.1 and 3.3, we give the bound of $b(T)$ by triangle inequality.

**Lemma 3.4** Under Assumption 2.1, 2.3, 2.4, 2.7, 2.8, for $p > 1$, there exists a positive constant $C$ such that

$$
\mathbb{E}\left( \sup_{0 \leq t \leq T^*} \|b(T)\|_p^p \right) \leq C\|b(0)\|_p^p + C\varepsilon^{-2kp}.
$$

### 3.2 Amplitude equation

We now turn to obtain the amplitude equation by effective information from the dominant part $a(T)$. Observing (1.2), we notice that $A_s b = B_s(a, a) + \text{high order terms}$, so one attempt to replace $b$ by $A^{-1}_s B_s(a, a)$ in the drift part of (3.1). As for the diffusion term, we choose the linearization of $G$ and neglect information about $b(T)$. In what follows, we make this intuitive idea rigorous.

Applying Itô’s formula to $B_c(a, A_s^{-1} b)$, we rewrite $a(T)$ as

$$
a(T) = a(0) + \int_0^T [\mathcal{L}_c a + 2\mathcal{F}(a)] ds + \int_0^T [\overline{G_c} + \bar{G}_c'(0) a] d\overline{W} + R(T),
$$

where

$$
R(T) = 2\varepsilon^2 B_c(a(T), A_s^{-1} b(T)) - 2\varepsilon^2 B_c(a(0), A_s^{-1} b(0)) + \varepsilon \int_0^T \mathcal{L}_c b ds + \varepsilon \int_0^T B_c(b, b) ds
$$

$$
- 2\varepsilon^2 \int_0^T B_c(\mathcal{L}_c a, A_s^{-1} b) ds - 2\varepsilon^3 \int_0^T B_c(\mathcal{L}_c b, A_s^{-1} b) ds
$$

$$
- 2\varepsilon^3 \int_0^T B_c(b, b, A_s^{-1} b) ds - 4\varepsilon^2 \int_0^T B_c(b, b, A_s^{-1} b) ds
$$

$$
- 2\varepsilon \int_0^T (B_c(a, A_s^{-1} \mathcal{L}_c a)) ds - 2\varepsilon \int_0^T (B_c(a, A_s^{-1} \mathcal{L}_c b)) ds
$$

$$
- 2\varepsilon^2 \int_0^T B_c(a, A_s^{-1} B_s(b, b)) ds - 4\varepsilon \int_0^T B_c(a, A_s^{-1} B_s(b, b)) ds
$$

$$
- 2 \sum_{j=1}^\infty B_c(\bar{G}_c(\varepsilon a + \varepsilon^2 b) f_j, A_s^{-1} \bar{G}_s f_j)) ds
$$
Moreover, by Burkholder-Davis-Gundy inequality, we have

\[
-2\varepsilon^{-1} \int_0^T \sum_{j=1}^\infty B_c(\tilde{G}_c(\varepsilon a + \varepsilon^2 b)f_j, A_s^{-1}\tilde{G}_s(\varepsilon a + \varepsilon^2 b)f_j)\,ds
\]

\[
-2\int_0^T \sum_{j=1}^\infty B_c(\tilde{G}_c f_j, A_s^{-1}\tilde{G}_s(\varepsilon a + \varepsilon^2 b)f_j)\,ds - 2\varepsilon \int_0^T \sum_{j=1}^\infty B_c(\tilde{G}_c f_j, A_s^{-1}\tilde{G}_s f_j)\,ds
\]

\[
+ \int_0^T \frac{1}{\varepsilon} \tilde{G}_c(\varepsilon a + \varepsilon^2 b) - \tilde{G}_c'(0)a|d\tilde{W} - 2\varepsilon^2 \int_0^T B_c(\tilde{G}_c d\tilde{W}, A_s^{-1}b)
\]

\[
- 2\varepsilon \int_0^T B_c(\tilde{G}_c(\varepsilon a + \varepsilon^2 b)d\tilde{W}, A_s^{-1}b) - 2\varepsilon \int_0^T B_c(a, A_s^{-1}\tilde{G}_s d\tilde{W})
\]

\[
- 2\int_0^T B_c(a, A_s^{-1}\tilde{G}_s(\varepsilon a + \varepsilon^2 b)d\tilde{W}).
\]

Let us show the bounds of \(R(T)\).

**Lemma 3.5** Under Assumption 2.1, 2.3, 2.4, 2.7, 2.8, for \(p > 1\), there exists a positive constant \(C\) such that

\[
\mathbb{E}\left(\sup_{0 \leq T \leq \tau^*} \|R(T)\|_\alpha^p\right) \leq C\varepsilon^{p-3\alpha p}.
\]

**Proof** Notice that all \(\mathcal{H}\) norms are equivalent on \(\mathcal{N}\), and \(A_s^{-1}\) is a bounded linear operator from \(\mathcal{H}_{\alpha-1}\) to \(\mathcal{H}_\alpha\). Then, we obtain

\[
\mathbb{E}\left(\sup_{0 \leq T \leq \tau^*} \varepsilon^2 \int_0^T B_c(\mathcal{L}_c a, A_s^{-1}b)\,ds\right)^p
\]

\[
\leq C\mathbb{E}\left(\varepsilon^2 \int_0^T \|B_c(\mathcal{L}_c a, A_s^{-1}b)\|_\alpha\,ds\right)^p
\]

\[
\leq C\varepsilon^{2p}\|a\|_\alpha\|b\|_\alpha
\]

\[
\leq C\varepsilon^{2p-4\alpha}.
\]

We can estimate other drift terms by the similar calculation.

According to Taylor Formula, we have

\[
\tilde{G}(\varepsilon a + \varepsilon^2 b) = \tilde{G}(0) + \varepsilon \tilde{G}'(0)\alpha + \tilde{G}''(\theta)(\varepsilon a + \varepsilon^2 b, \varepsilon a + \varepsilon^2 b),
\]

where \(\theta\) is in the line connecting 0 and \(\varepsilon a + \varepsilon^2 b\).

Then, we apply Burkholder-Davis-Gundy inequality and 3.8 to derive

\[
\mathbb{E}\left(\sup_{0 \leq T \leq \tau^*} \left\| \frac{1}{\varepsilon} \tilde{G}_c(\varepsilon a + \varepsilon^2 b) - \tilde{G}_c'(0)a \right\|_\alpha^p\right)
\]

\[
\leq C\varepsilon^p \mathbb{E}\left(\sup_{0 \leq T \leq \tau^*} \left\| \frac{1}{\varepsilon} \tilde{G}_c(\varepsilon a + \varepsilon^2 b) - \tilde{G}_c'(0)a \right\|_{L_2(\mathcal{H}; \mathcal{U})}^p\right)
\]

\[
\leq C\varepsilon^{p-3\alpha p}.
\]

Moreover, by Burkholder-Davis-Gundy inequality, we have

\[
\mathbb{E}\left(\sup_{0 \leq T \leq \tau^*} \left\| \varepsilon \int_0^T B_c(\tilde{G}_c(\varepsilon a + \varepsilon^2 b)d\tilde{W}, A_s^{-1}b)\right\|_\alpha^p\right)
\]

\[
\leq C\varepsilon^{p} \mathbb{E}\left(\int_0^\tau \sum_{j=1}^\infty \|B_c(\tilde{G}_c(\varepsilon a + \varepsilon^2 b)f_j, A_s^{-1}b)\|_\alpha^2\,ds\right)^\frac{p}{2}.
\]
Similarly, we can estimate other diffusion terms, but the detail is not provided here.

Collecting all the estimates, we own (3.7).

Proof
Removing the \(R(T)\) from (3.6), we gain the amplitude equations as
\[
\begin{align*}
dx &= [L_c x + 2F(x)]dT + [\tilde{G}_c + \tilde{G}'_c(0)x]d\tilde{W} \\
x(0) &= a(0).
\end{align*}
\]

### 3.3 Rigorous error analysis

Now let us provide the bounds of \(x(T)\), and give the better estimate of \(a(T)\) via \(x(T)\).

**Lemma 3.6** Let Assumption \(2.1, 2.3, 2.4, 2.6-2.8\) hold. For \(p > 1\), there exists a positive constant \(C\) such that
\[
\mathbb{E} \left( \sup_{0 \leq T \leq T_0} \|x(T)\|^p \right) \leq C\|a(0)\|^p + C. \tag{3.10}
\]
Moreover, if \(\|a(0)\| \leq \epsilon^{-\frac{1}{p}}\),
\[
\begin{align*}
\mathbb{E} \left( \sup_{0 \leq T \leq \tau^*} \|a(T) - x(T)\|^p \right) &\leq C\epsilon^{-18\kappa p}, \\
\mathbb{E} \left( \sup_{0 \leq T \leq \tau^*} \|a(T)\|^p \right) &\leq C\epsilon^{-\frac{18}{p}}.
\end{align*}
\]

**Proof** Define some stopping time
\[
\tau_K := \inf\{T > 0, \|x(T)\| > K\}.
\]

For \(p \geq 2\) and \(0 \leq T \leq \tau_K\), Itô’s formula yields that
\[
\|x(T)\|^p \leq \|a(0)\|^p + p \int_0^T \|x\|^{p-2}\langle L_c, x \rangle ds + 2p \int_0^T \|x\|^{p-2}\langle F_c(x), x \rangle ds \\
+ p \int_0^T \|x\|^{p-2}\langle x, [\tilde{G}_c + \tilde{G}'_c(0)x]d\tilde{W} \rangle + Cp(p-1) \int_0^T (\|x\|^p + \|x\|^{p-2}) ds.
\]

For \(T_1 \in [0, T_0]\), by Cauchy-Schwarz inequality, Burkholder-Davis-Gundy inequality and Young’s inequality we deduce that
\[
\begin{align*}
\mathbb{E} \left( \sup_{0 \leq T \leq T_1 \wedge \tau_K} \|x(T)\|^p \right) \\
\leq C\|a(0)\|^p + C + C\mathbb{E} \left( \int_0^{T_1 \wedge \tau_K} \|x(s)\|^p ds \right) + C\mathbb{E} \left( \int_0^{T_1 \wedge \tau_K} \|x(s)\|^{2p} ds \right) \frac{1}{p} \\
\leq C\|a(0)\|^p + C + C \int_0^{T_1} \mathbb{E} \left( \sup_{0 \leq s_1 \leq s \wedge \tau_K} \|x(s)\|^p \right) ds \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq T \leq T_1 \wedge \tau_K} \|x(T)\|^p \right).
\end{align*}
\]
Using Gronwall’s lemma, we own

$$\mathbb{E}\left( \sup_{0 \leq T \leq T_0 \wedge \tau_K} \|x(T)\|^p \right) \leq C\|a(0)\|^p + C. \quad (3.11)$$

We claim (3.11) still holds if $T_0 \wedge \tau_K$ is replaced with $T_0$.

In fact, let the right side of (3.11) be $\bar{C}$. Then,

$$\mathbb{P}\left( \tau_K > T_0 \right) = \mathbb{P}\left( \sup_{T \leq T_0 \wedge \tau_K} \|x(T)\| < K \right)$$

$$= 1 - \mathbb{P}\left( \sup_{T \leq T_0 \wedge \tau_K} \|x(T)\| \geq K \right)$$

$$\geq 1 - \frac{C}{\bar{C}}.$$

Then, we know that $\sup_{0 \leq T \leq T_0 \wedge \tau_K} \|x(t)\|^p$ monotonously converges to $\sup_{0 \leq T \leq T_0} \|x(t)\|^p \text{ a.s.}$, as $K \to \infty$. Thus, monotone convergence theorem yields

$$\mathbb{E}\left( \sup_{0 \leq T \leq T_0} \|x(T)\|^p \right) = \lim_{K \to \infty} \mathbb{E}\left( \sup_{0 \leq T \leq T_0 \wedge \tau_K} \|x(T)\|^p \right) \leq \bar{C}.$$

For $1 \leq p < 2$, Hölder inequality yields (3.10).

Let $h(T) := x(T) - a(T) + R(T)$. Then, we have

$$h(T) = \int_0^T \mathcal{L}_c(h - R)ds + 2 \int_0^T \mathcal{F}_c(x)ds - 2 \int_0^T \mathcal{F}_c(x - h + R)ds$$

$$+ \int_0^T \varepsilon^{-1} \tilde{G}_c'(0)(h - R)d\tilde{W}(s). \quad (3.12)$$

For $p \geq 2$, using Itô’s Lemma, we obtain

$$\|h(T)\|^p \leq C \int_0^T \|h\|^{p-2} \langle \mathcal{L}_c(h - R), h \rangle ds + C \int_0^T \|h\|^{p-2} \langle \mathcal{F}_c(x) - \mathcal{F}_c(x - h + R), h \rangle ds$$

$$+ C \int_0^T \|h\|^{p-2} \langle h, \tilde{G}_c''(0)(h - R) d\tilde{W} \rangle + C \int_0^T \|h\|^{p-2} \|h - R\|^2 ds.$$

According to condition (2.2), Cauchy-Schwarz inequality and Young’s inequality, we obtain

$$\|h(T)\|^p \leq C \int_0^T (\|h\|^p + \|R\|^p + \|R\|^{2p} + \|R\|^p \|x\|^p) ds$$

$$+ C \int_0^T \|h\|^{p-2} \langle h, \tilde{G}_c''(0)(h - R) d\tilde{W} \rangle + C \int_0^T \|h\|^{p-2} \|h - R\|^2 ds.$$

Furthermore, for any $T^* \in [0, T_0]$, applying to Burkholder-Davis-Gundy inequality and Young’s inequality, we have

$$\mathbb{E}\left( \sup_{0 \leq T \leq T^* \wedge \tau^*} \|h(T)\|^p \right) \leq C\bar{E}\left( \sup_{0 \leq T \leq T^* \wedge \tau^*} \int_0^T \|h\|^p + \|R\|^p + \|R\|^{2p} + \|R\|^p \|x\|^p ds \right)$$

$$+ C\bar{E}\left( \int_0^{T^* \wedge T_0} (\|h\|^{2p} + \|R\|^{2p}) ds \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2} \mathbb{E}\left( \sup_{0 \leq T \leq T^* \wedge \tau^*} \|h(T)\|^p \right) + C \int_0^T \mathbb{E}\left( \sup_{0 \leq s \leq s \wedge \tau^*} \|h(s)\|^p \right) ds$$

$$+ C\varepsilon^{p-18sp}.$$
Gronwall’s lemma yields
\[ E\left( \sup_{0 \leq T \leq T^*} \|h(T)\|_p^p \right) \leq C\varepsilon^{p-18q}. \]

Then, by triangle inequality, we derive
\[ E\left( \sup_{0 \leq T \leq T^*} \|a(T) - x(T)\|_p^p \right) \leq E\left( \sup_{0 \leq T \leq T^*} \|h(T)\|_p^p \right) + E\left( \sup_{0 \leq T \leq T^*} \|R(T)\|_p^p \right) \leq C\varepsilon^{p-18q}, \]
and
\[ E\left( \sup_{0 \leq T \leq T^*} \|a(T)\|_p^p \right) \leq E\left( \sup_{0 \leq T \leq T^*} \|a(T) - x(T)\|_p^p \right) + E\left( \sup_{0 \leq T \leq T^*} \|x(T)\|_p^p \right) \leq C\varepsilon^{-\frac{q}{2p}}. \]

We complete the proof. \( \square \)

Although most of previous estimates hold on random interval \([0, \tau^*]\), we will show the main theorem holds on fixed interval \([0, T_0/\varepsilon^2]\).

**Lemma 3.7** Let \(2.1, 2.3, 2.4, 2.7, 2.8\) hold. For \(p > 1\) and \(\|u(0)\|_\alpha \leq -\frac{\alpha}{2}\), there exists a positive constant \(C\) such that
\[ E\left( \sup_{0 \leq T \leq T^*} \|R(T)\|_\alpha^p \right) \leq C\varepsilon^{2p-18q}, \]
where \(R(T) = u(\varepsilon^{-2}T) - \varepsilon x(T)\).

**Proof** Note that \(R(T) = u(\varepsilon^{-2}T) - \varepsilon x(T) = \varepsilon a(T) + \varepsilon^2 b(T) - \varepsilon x(T)\). Then, we can prove this Lemma by Lemma 3.4 and Lemma 3.6.

We complete the proof. \( \square \)

**Definition 3.2** For \(\kappa > 0\), define \(\Omega^* \subset \Omega\) of all \(\omega \subset \Omega\) satisfy that all the following estimations
\[ \sup_{0 \leq T \leq T^*} \|a(T)\| < \varepsilon^{-\frac{q}{2}}, \quad \sup_{0 \leq T \leq T^*} \|b(T)\|_\alpha < \varepsilon^{-\frac{5q}{2}}, \quad \sup_{0 \leq T \leq T^*} \|R(T)\|_\alpha < \varepsilon^{2-19\kappa}. \]

**Lemma 3.8** Under Assumption \(2.1, 2.3, 2.4, 2.7, 2.8\) for \(p > 1\), there exists a positive constant \(C\) such that
\[ \mathbb{P}(\Omega^*) \geq 1 - C\varepsilon^p. \]

**Proof** For fixed \(p > 1\), using Lemma 3.4, Lemma 3.6, Lemma 3.7, and Chebyshev inequality, we have
\[
\begin{align*}
\mathbb{P}(\Omega^*) & \geq 1 - \mathbb{P}\left( \sup_{0 \leq T \leq T^*} \|a(T)\| \geq \varepsilon^{-\frac{q}{2}} \right) - \mathbb{P}\left( \sup_{0 \leq T \leq T^*} \|b(T)\|_\alpha \geq \varepsilon^{-\frac{5q}{2}} \right) \\
& \quad - \mathbb{P}\left( \sup_{0 \leq T \leq T^*} \|R(T)\|_\alpha \geq \varepsilon^{2-19\kappa} \right) \\
& \geq 1 - \varepsilon^{-\frac{q}{2}}\mathbb{E}\left( \sup_{0 \leq T \leq T^*} \|a(T)\|^q \right) - \varepsilon^{-\frac{5q}{2}}\mathbb{E}\left( \sup_{0 \leq T \leq T^*} \|b(T)\|_\alpha^q \right) \\
& \quad - \varepsilon^{19\kappa - 2q}\mathbb{E}\left( \sup_{0 \leq T \leq T^*} \|R(T)\|_\alpha^q \right) \\
& \geq 1 - C\varepsilon^p,
\end{align*}
\]
where \(q\) is large enough.

We complete the proof. \( \square \)
Theorem 3.9 Let Assumption 2.1, 2.3, 2.4, 2.6-2.8 hold. Let $u(t)$ be the local mild solution of $\Omega$ with $\|u(0)\|_{\alpha} \leq \varepsilon^{1-\delta}$. Then, for any $p > 1$, there exists a positive constant $C$ such that

$$\mathbb{P}\left( \sup_{0 \leq t \leq \varepsilon^{-2} T_0} \|u(t) - \varepsilon x(\varepsilon^2 t)\|_{\alpha} > \varepsilon^{2-19\kappa} \right) \leq \varepsilon^p.$$  

**Proof** Note that

$$\Omega^* \subseteq \left\{ \omega \left| \sup_{0 \leq T \leq \tau^*} \|a(T)\| < \varepsilon^{-\kappa}, \sup_{0 \leq T \leq \tau^*} \|b(T)\|_{\alpha} < \varepsilon^{-3\kappa} \right\} \subseteq \{ \omega | \tau^* = T_0 \} \subseteq \Omega.$$  

Then,

$$\sup_{0 \leq T \leq T_0} \|\mathcal{R}(T)\|_{\alpha} = \sup_{0 \leq T \leq \tau^*} \|\mathcal{R}(T)\|_{\alpha} < \varepsilon^{2-19\kappa}, \omega \in \Omega^*.$$  

Lemma 3.8 implies that

$$\mathbb{P}\left( \sup_{0 \leq T \leq T_0} \|\mathcal{R}(T)\|_{\alpha} \geq \varepsilon^{2-19\kappa} \right) \leq 1 - \mathbb{P}(\Omega^*) \leq \varepsilon^p.$$  

We complete the proof.  

**Remark 3.10** We comment Theorem 3.9 covers the case that (1.2) is forced either additive noise or multiplicative one. In particular, if the amplitude equation is only with multiplicative noise, we can analyze the stability of the trivial solution, and further study how multiplicative noise changes the stability. We will give an example to illustrate it in Section 5.

4 Degenerate noise

In this section, we are committed to (1.2) with $\sigma_x = \varepsilon$ and degenerate additive noise (i.e., the noise does not influence the dominant modes directly). Moreover, we suppose that $\mathcal{L}$ commute with $P_c$ and $P_s$.

Because the additive noise is of order $\varepsilon$ in this section, the same decomposition as Section 3 does not allow us to control the fast modes. In order to overcome this trouble, we will use another decomposition to consider (1.2).

Let $u(t) = \varepsilon \varphi(\varepsilon^2 t) + \varepsilon \psi(\varepsilon^2 t)$, where $\varphi \in \mathcal{N}$ and $\psi \in \mathcal{S}$. Introduce slow time scale $T = \varepsilon^2 t$.

By projections $P_c$ and $P_s$, we split $u$ into

$$d\varphi = [\mathcal{L}\varphi + 2 \varepsilon^{-1} B_c(\varphi, \psi) + \varepsilon^{-1} B_c(\psi)]dT + \varepsilon^{-1} \dot{G}_c(\varepsilon \varphi + \varepsilon \psi)d\bar{W}, \quad (4.1)$$

$$d\psi = [\varepsilon^{-2} A_s \psi + \mathcal{L}_s \psi + \varepsilon^{-1} B_s(\psi) + 2 \varepsilon^{-1} B_s(\varphi, \psi) + \varepsilon^{-1} B_s(\varphi)]dT + \varepsilon^{-1} \dot{G}_s(\varepsilon \varphi + \varepsilon \psi)d\bar{W}. \quad (4.2)$$

As Section 2, we also hope to replace $\psi(T)$ by $\varphi(T)$ in (4.1) and obtain amplitude solution. Although we can gain a reduced system after replacing $\psi(T)$ by $\varphi(T)$, there exists still O-U process in the diffusion part of it, which causes it is perturbed by fast fluctuation. We will respectively treat this problem in two cases. The first case is the dimension of $\ker A$ is more than one. In this case, we can deal with the O-U process and obtain the amplitude equation, but we just can show the law of the solution of the amplitude equation weakly converges to that of $\varphi(T)$ without explicit convergence rate. The other is $\ker A$ is one-dimensional space. In this case, we can not only obtain the amplitude equation, but also present the precise error between the approximation solution and the original one.
The section is divided into three subsections. In Subsection 4.1, we will estimate $\psi(T)$ and obtain a reduced system with O-U process. In Subsection 4.2, our aim is to study the amplitude equation in case that $\ker A$ is multi-dimensional space. In Subsection 4.3, we will show the amplitude equation for one-dimensional kernel space, and give rigorous error analysis.

4.1 Reduced system

In order to guarantee that $\varphi(T)$ and $\psi(T)$ do not get out of a bounded domain on some interval, we introduce a stopping time.

**Definition 4.1** For an $N \times S$-valued stochastic process $(\varphi, \psi)$ given by (4.1) and (4.2), we define, for $T_0 > 0$ and $\kappa \in (0, \frac{1}{13})$, the stopping time $\tau^*_1$ as

$$
\tau^*_1 := T_0 \wedge \inf \{T > 0 \mid \|\varphi(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-\kappa} \}.
$$

For simplicity of representation, we introduce a notation to express higher order term in the sense of probability.

**Definition 4.2** Let $\{X_{\varepsilon}(T)\}_{T \geq 0}$ be a family of real-valued processes, we say $X_{\varepsilon} = \mathcal{O}(f_{\varepsilon})$ with respect to stopping time $\tau^*_1$, if for every $p > 1$ there exists a positive constant $C$ such that

$$
E\left(\sup_{0 \leq T \leq \tau^*_1} |X_{\varepsilon}(T)|^p \right) \leq C f_{\varepsilon}^p.
$$

Letting $V_1(T) = \mathcal{L}_s \varepsilon + \varepsilon^{-1} B_s(\varepsilon)$, we present the mild solution of $\psi(T)$:

$$
\psi(T) = e^{\varepsilon^{-2} A T} \psi(0) + \int_0^T e^{\varepsilon^{-2} A (T-s)} V_1(s) ds + \varepsilon^{-1} \int_0^T e^{\varepsilon^{-2} A (T-s)} G_s d\tilde{W} + \varepsilon^{-1} \int_0^T e^{\varepsilon^{-2} A (T-s)} G_s (\varepsilon \varphi + \varepsilon \psi) d\tilde{W}.
$$

$$
:= Q(T) + J(T) + Z(T) + K(T).
$$

**Lemma 4.1** Under Assumption 2.1, 2.3, 2.4, 2.7, 2.8, for $p > 1$, there exists a constant positive $C$ such that

$$
E\left(\sup_{0 \leq T \leq \tau^*_1} \|J(T)\|_\alpha^p \right) \leq C \varepsilon^{-2p}. \tag{4.3}
$$

**Proof** This proof is similar to that of Lemma 3.1.

We complete the proof. □

**Lemma 4.2** Under Assumption 2.1, 2.3, 2.4, 2.7, 2.8, for $p > 1$, there exists a positive constant $C$ such that

$$
E\left(\sup_{0 \leq T \leq \tau^*_1} \|K(T)\|_\alpha^p \right) \leq C \varepsilon^{-2p}. \tag{4.3}
$$

**Proof** This proof is similar to that of Lemma 3.3.

We complete the proof. □

Combining Lemma 3.2, 4.1, 4.2 we can estimate $\psi(T)$ by triangle inequality.

**Lemma 4.3** Under Assumption 2.1, 2.3, 2.4, 2.7, 2.8, for $p > 1$, there exists a constant positive $C$ such that

$$
E\left(\sup_{0 \leq T \leq \tau^*_1} \|\psi(T)\|^p \right) \leq C \|\psi(0)\|^p + C \varepsilon^{-\frac{p}{p-1}}.
$$

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Now, we begin to replace $\psi(T)$ and $\varphi(T)$.

**Lemma 4.4** Under Assumption 2.3 2.4 2.7 2.8 we have

$$
\int_0^T B_c(\varphi, \psi)ds = -2\varepsilon \int_0^T B_c(B_c(\varphi, Z), A_s^{-1}Z)ds - \varepsilon \int_0^T B_c(B_c(Z), A_s^{-1}Z)ds
$$

Lemma 4.4

$$
- \varepsilon \int_0^T B_c(\varphi, A_s^{-1}B_s(\varphi))ds
$$

with $R_1(T) = \tilde{O}(\varepsilon^{2-6\kappa})$.

**Proof** Since the proof is fairly standard, we will omit some straightforward computations.

By Itô formula, we derive that

$$
dB_c(\varphi, A_s^{-1}\psi) = B_c(L_c\varphi, A_s^{-1}\psi)dt + 2\varepsilon^{-1}B_c(B_c(\varphi, \psi), A_s^{-1}\psi)dt + \varepsilon^{-2}B_c(\varphi, \psi)dt
$$

$$
+ B_c(\varphi, L_sA_s^{-1}\psi)dt + \varepsilon^{-2}\sum_{j=1}^{\infty}B_c(\bar{G}_c(\varepsilon \varphi + \varepsilon \psi)j, A_s^{-1}\bar{G}f_j)dt
$$

$$
+ \varepsilon^{-2}\sum_{j=1}^{\infty}B_c(\bar{G}_c(\varepsilon \varphi + \varepsilon \psi)j, A_s^{-1}\bar{G}s_j)dt
$$

Firstly, let us prove that

$$
\int_0^T B_c(B_c(\varphi, \psi), A_s^{-1}\psi)ds = \int_0^T B_c(B_c(\varphi, Z), A_s^{-1}Z)ds + \tilde{O}(\varepsilon^{1-5\kappa}). \quad (4.4)
$$

Recalling the mild solution of $\psi$, we have

$$
\int_0^T B_c(B_c(\varphi, \psi), A_s^{-1}\psi)ds - \int_0^T B_c(B_c(\varphi, Z), A_s^{-1}Z)ds
$$

$$
= \int_0^T B_c(B_c(\varphi, Q), A_s^{-1}Q)ds + \int_0^T B_c(B_c(\varphi, Q), A_s^{-1}(J + K))ds
$$

$$
+ \int_0^T B_c(B_c(\varphi, Q), A_s^{-1}Z)ds + \int_0^T B_c(B_c(\varphi, J + K), A_s^{-1}Q)ds
$$

$$
+ \int_0^T B_c(B_c(\varphi, J + K), A_s^{-1}(J + K))ds + \int_0^T B_c(B_c(\varphi, J + K), A_s^{-1}Z)ds
$$

Estimate the first term of the right hand of the above equation as follows:

$$
\mathbb{E}\left(\sup_{0 \leq T \leq \tau_T} \| \int_0^T B_c(B_c(\varphi, Q), A_s^{-1}Q)ds\|_p^p\right)
$$

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\[
\begin{align*}
&\leq C\mathbb{E}\left(\sup_{0 \leq t \leq \tau_1} \left( \int_0^T \| B_c(B_c(\varphi, Q), A_s^{-1}Q)\|_\alpha ds \right)^p \right) \\
&\leq C\varepsilon^{-\alpha p} \left( \int_0^{T_0} \| e^{-A_s\varepsilon^{-2}} \varphi(0) \|_2^2 ds \right) \\
&\leq C\varepsilon^{2p - 5n}\varepsilon^2.
\end{align*}
\]

By similar technique, it is easy to obtain (4.4) by Lemma 4.1-4.2. Furthermore, we can prove that

\[
\begin{align*}
B_c(B_c(\psi), A_s^{-1}\psi)ds &= \int_0^T B_c(B_c(Z), A_s^{-1}Z)ds + \bar{O}(\varepsilon^{1-6k}), \\
B_c(\varphi, A_s^{-1}B_s(\varphi + \psi))ds &= \int_0^T B_c(\varphi, A_s^{-1}B_s(\varphi + Z))ds + \bar{O}(\varepsilon^{1-5k}), \\
B_c(B_c(\psi), A_s^{-1}\psi)ds &= \int_0^T B_c(B_c(Z), A_s^{-1}Z)ds + \bar{O}(\varepsilon^{1-6k}).
\end{align*}
\]

Then, by Assumption 2.8 and Taylor formula, we obtain

\[
\begin{align*}
\int_0^T \sum_{j=1}^\infty [B_c(\tilde{G}_c(\varepsilon \varphi + \varepsilon \psi)f_j, A_s^{-1}\tilde{G} f_j) - B_c(\tilde{G}_c(0)(\varphi + Z)f_j, A_s^{-1}\tilde{G} f_j)]ds &= \bar{O}(\varepsilon^{2-2k}), \\
\int_0^T \sum_{j=1}^\infty B_c(\tilde{G}_c(\varepsilon \varphi + \varepsilon \psi)f_j, A_s^{-1}\tilde{G}_s(\varepsilon \varphi + \varepsilon \psi)f_j)ds &= \bar{O}(\varepsilon^{2-2k}),
\end{align*}
\]

by Assumption 2.8 and Burkholder-Davis-Gundy inequality, we obtain the stochastic terms are \(\tilde{O}(\varepsilon^{1-2k})\), and by simple calculation, we obtain other terms are \(\tilde{O}(\varepsilon^{2-2k})\).

We complete the proof. \(\blacksquare\)

To analyze \(B_c(\psi)\) precisely, we need to introduce several notations. For a Hilbert Space \(\mathcal{H}\), denote the tensor product of it by \(v_1 \otimes v_2\), and the symmetric tensor product of it by \(v_1 \mathbb{S} v_2 = \frac{1}{2}(v_1 \otimes v_2 + v_1 \mathbb{S} v_2)\), where \(v_1, v_2 \in \mathcal{H}\). Furthermore, define the scalar product in the tensor product space \(\mathcal{H}_a \otimes \mathcal{H}_b\) by \(\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{a,b} := \langle u_1, u_2 \rangle_a \langle v_1, v_2 \rangle_b\), where \(u_1, u_2 \in \mathcal{H}_a, v_1, v_2 \in \mathcal{H}_b\). For simplified notation, we adopt \(\langle \cdot \rangle_a\) instead of \(\langle \cdot \rangle_{a,a}\). The norm of \(\mathcal{H}_a \otimes \mathcal{H}_b\) is induced by such scalar product. For two linear operators \(\mathcal{L}_a\) and \(\mathcal{L}_b\) on \(\mathcal{H}\), define the symmetric tensor product of them by \((\mathcal{L}_a \mathbb{S} \mathcal{L}_b)(v_1 \otimes v_2) = \frac{1}{2}(\mathcal{L}_av_1 \otimes \mathcal{L}_bv_2 + \mathcal{L}_bv_2 \otimes \mathcal{L}_av_1)\).

**Lemma 4.5** Under Assumption 2.7, 2.9, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 we have

\[
\int_0^T B_c(\psi)ds = -\varepsilon \int_0^T B_c(I \mathbb{S} A_s)^{-1}(Z \mathbb{S} A_s(\varphi + Z))ds
\]

\[
= -\varepsilon \sum_{j=0+1}^N \alpha_j B_c(I \mathbb{S} A_s)^{-1}(e_i \mathbb{S} \tilde{G}_s(0)(\varphi + Z)f_j)ds
\]

\[
+ \varepsilon \int_0^T B_c(I \mathbb{S} A_s)^{-1}(Z \mathbb{S} \tilde{G} \tilde{W}) + R_2(T),
\]

with \(R_2(T) = \tilde{O}(\varepsilon^{2-6k})\).

**Proof** Applying Itô formula to \(\psi \otimes \psi\), we obtain

\[
\frac{1}{2}d(\psi \otimes \psi) = \varepsilon^{-2}(\psi \mathbb{S} A_s \psi)dT + (\psi \mathbb{S} \mathcal{L}_a \psi)dT + \varepsilon^{-1}(\psi \mathbb{S} B_s(\varphi + \psi))dT
\]

\[
+ \varepsilon^{-1}(\psi \mathbb{S} \tilde{G} \tilde{W}) + \varepsilon^{-1}(\psi \mathbb{S} \tilde{G}_s(\varepsilon \varphi + \varepsilon \psi))d\tilde{W}.
\]
\[
+ \varepsilon^{-2} \sum_{j=1}^{\infty} \tilde{G}_s(\varepsilon \varphi + \varepsilon \psi) f_j \otimes \tilde{G}_s(\varepsilon \varphi + \varepsilon \psi) f_j \,dT
\]
\[
+ \varepsilon^{-2} \sum_{j=n+1}^{N} a_j^2(e_j \otimes e_j) \,dT + \varepsilon^{-2} \sum_{j=n+1}^{N} \alpha_j(e_j \otimes \varepsilon \tilde{G}_s(\varepsilon \varphi + \varepsilon \psi) f_j) \,dT.
\]

Note that \((I \otimes_s A_s)^{-1}\) is bounded operator from \(\mathcal{H}_{\alpha-1} \otimes \mathcal{H}_{\alpha-1}\) to \(\mathcal{H}_{\alpha} \otimes \mathcal{H}_{\alpha}\). The remaining proof is similar to that of Lemma 4.4, so it is omitted.

We complete the proof.

**Lemma 4.6** Under Assumption 2.1, 2.3, 2.4, 2.7, 2.8, we have
\[
\mathbb{E} \left\| \int_0^T \varepsilon^{-1} \tilde{G}_c(\varepsilon \varphi + \varepsilon \psi) d\tilde{W} - \int_0^T \tilde{G}'_c(0)(\varphi + Z) d\tilde{W} + R_3(T) \right\|^{2\alpha} \leq C(T-s)^p \varepsilon^{2p},
\]
with \(R_3(T) = \mathcal{O}(\varepsilon^{1-3\alpha})\)

**Proof** The proof mainly relies on Taylor formula, and is similar to Lemma 4.4, so we do not present the detail.

We complete the proof.

Although we can replace \(\psi(T)\) by \(\varphi(T)\) in slow modes by previous lemmas, one still need to eliminate the fast Ornstein-Uhlenbeck process \(Z(T)\) appearing in the drift and diffusion terms. Therefore, we will present some useful lemmas to deal with such problem. We introduce some notations before showing them.

Set \(\hat{Z}(T) := \sum_{k=n+1}^{N} \hat{Z}_k(T)\), where \(\hat{Z}_k(T) = e^{-\lambda_k \varepsilon^{-2}T} \hat{Z}_k(0) + Z_k(T)\), with \(\hat{Z}_k(0)\) is the normal distribution \(N(0, \frac{\alpha_k^2}{2\lambda_k})\). Set \(\hat{G} := \sum_{k=n+1}^{N} \frac{\alpha_k^2}{2\lambda_k} (e_k \otimes e_k)\).

**Lemma 4.7** For every \(p > 0\) and \(\alpha > 0\), there exists a positive constant \(C\) such that the bounds
\[
\mathbb{E} \left\| \int_s^T (\hat{Z}(r)) \,dr \right\|_{2\alpha}^{2p} \leq C(T-s)^p \varepsilon^{2p},
\]
\[
\mathbb{E} \left\| \int_s^T (\hat{Z}(r) \otimes \hat{Z}(r) - \hat{G}) \,dr \right\|_{2\alpha}^{2p} \leq C(T-s)^p \varepsilon^{2p},
\]
\[
\mathbb{E} \left\| \int_s^T (\hat{Z}(r) \otimes \hat{Z}(r) \otimes \hat{Z}(r)) \,dr \right\|_{2\alpha}^{2p} \leq C(T-s)^p \varepsilon^{2p},
\]
hold for every \(T > s > 0\).

**Lemma 4.8** Let \(\alpha\) be as in Assumption 2.3. Let \(f_i\) with \(i \in \{1, 2, 3\}\) be \(\alpha\)-Hölder continuous functions on \([0, \tau]\) with values in \(((\mathcal{H}^{\alpha})^{\otimes 1})^*\), respectively. Let \(F_\varepsilon\) be given by
\[
F_\varepsilon(t) := \int_0^t \left( (f_1(s)) (\hat{Z}) + (f_2(s)) (\hat{Z} \otimes \hat{Z} - \hat{G}) + (f_3(s)) (\hat{Z} \otimes \hat{Z} \otimes \hat{Z}) \right) \,ds.
\]

Then, for \(p > 0\) and every \(0 < \gamma < \frac{2\alpha}{1+2\alpha}\) there exists a positive constant \(C\) depending only on \(p\) and \(\gamma\) such that
\[
\mathbb{E} \sup_{t \in [0, \tau]} |F_\varepsilon(t)|^p \leq C \varepsilon^{\gamma p} \left( \mathbb{E} \left( \|f_1\|_{C^\alpha} + \|f_2\|_{C^\alpha} + \|f_3\|_{C^\alpha} \right)^2 p \right)^{\frac{1}{2}},
\]
where \(\| \cdot \|_{C^\alpha}\) denotes the \(\alpha\)-Hölder norm for \(((\mathcal{H}^{\alpha})^{\otimes 1})^*\)-valued functions on \([0, \tau]\).
With the help of Lemma 4.8, we can average the fast O-U process in the drift term of \( \varphi(T) \), and give the next lemma.

**Lemma 4.9** Under Assumption 2.1, 2.3, 2.4, 2.5, 2.7, 2.8, we have

\[
\varphi(T) = \varphi(0) + \tilde{\mathcal{L}}\varphi + 2 \int_0^T \mathcal{F}(\varphi) ds + \int_0^T \Gamma(\varphi, \hat{Z}) d\tilde{W} + R_\varphi(T),
\]

where \( \mathcal{L} \) is a linear operator from \( \mathcal{H}^{\alpha} \) to \( \mathcal{H}^{\alpha - \beta} \) satisfying

\[
\tilde{\mathcal{L}}\varphi = \mathcal{L}_c \varphi - \sum_{i=n+1}^{\infty} \frac{2\alpha_i^2}{\lambda_i} B_c(B_c(\varphi, e_i), e_i) - \sum_{i=n+1}^{\infty} \frac{\alpha_i^2}{\lambda_i} B_c(\varphi, A_s^{-1} B_s(e_i, e_i))
\]

\[\quad - 2 \sum_{j=1}^{\infty} B_c(\hat{G}'(0)(\varphi) f_j, A_s^{-1} \hat{G}_f) - \sum_{i=n+1}^{\infty} \frac{\alpha_i^2}{\lambda_i} B_c(I \otimes_s A_s)^{-1}(e_i \otimes_s B_s(\varphi, e_i))
\]

\[\quad - \sum_{j=n+1}^{N} \alpha_j B_c(I \otimes_s A_s)^{-1}(e_j \otimes_s \hat{G}'(0)(\varphi) f_j),
\]

\( \Gamma(\cdot, \hat{Z}) \) is an operator from \( \mathcal{H}^{\alpha} \) to \( \mathbb{L}_2(U, \mathcal{H}^\alpha) \) satisfying

\[
\Gamma(\varphi, \hat{Z}) = \hat{G}_c'(0)(\varphi + \hat{Z}) \cdot -2 B_c(\varphi, A_s^{-1} \hat{G}_c) - B_c(I \otimes_s A_s)^{-1}(\hat{Z} \otimes_s \hat{G}_c).
\]

and \( R_\varphi(T) = O(\varepsilon^{-6\alpha}). \)

Moreover, for any \( T_1, T_2 \in [0, T_0] \), there exists a positive constant \( C \) such that

\[
E\|R_\varphi(T_1 \wedge \tau^*) - R_\varphi(T_2 \wedge \tau^*)\|^{2p} \leq C|T_1 - T_2|^p. \tag{4.5}
\]

**Proof** Firstly, according to Lemma 4.4, 4.9 we have

\[
\varphi(T) = \varphi(0) + \int_0^T \tilde{\mathcal{F}}(\varphi, Z) ds + \int_0^T \Gamma(\varphi, Z) d\tilde{W} + \sum_{k=1}^{3} R_k(T), \tag{4.6}
\]

where

\[
\tilde{\mathcal{F}}(\varphi, Z) = \mathcal{L}_c \varphi + 2 \mathcal{F}(\varphi) - 4 B_c(B_c(\varphi, Z), A_s^{-1} Z) - 2 B_c(B_c(Z, Z), A_s^{-1} Z)
\]

\[\quad - 4 B_c(\varphi, A_s^{-1} B_s(\varphi, Z)) - 2 B_c(\varphi, A_s^{-1} B_s(Z, Z))
\]

\[\quad - 2 \sum_{j=1}^{\infty} B_c(\hat{G}'(0)(\varphi + Z) f_j, A_s^{-1} \hat{G}_f)
\]

\[\quad - \sum_{j=n+1}^{N} \alpha_j B_c(I \otimes_s A_s)^{-1}(e_j \otimes_s \hat{G}'(0)(\varphi + Z) f_j).
\]

Based on Hölder inequality and Burkholder-Davis-Gundy inequality, we note that there exists a positive constant \( C_1 \) such that

\[
E\left( \sup_{0 \leq s < T \leq \tau^*_1} \left\| \sum_{k=1}^{3} (R_k(T) - R_k(s)) \right\|_\alpha \right) \leq C_2 \varepsilon^{1 - 6\alpha}, \tag{4.7}
\]

with \( 0 < \alpha < \frac{1}{2} \). For simplify, we choose \( \hat{\alpha} = \frac{1}{2} \) in the following part of this paper. Thus, there exists a positive constant \( C_2 \), such that

\[
E\left( \sup_{0 \leq s < T \leq \tau^*_1} \frac{\|\varphi(T) - \varphi(s)\|_\alpha}{(T - s)^{\hat{\alpha}}} \right) \leq C_2 \varepsilon^{-6\alpha}. \tag{4.8}
\]
Combining $\varphi(T) = \mathcal{O}(\varepsilon^{-3\kappa})$ with (3.3), we get $\|\varphi(T)\|_{c_\varepsilon^4} = \mathcal{O}(\varepsilon^{-6\kappa})$.

Next, replacing $Z$ with $\hat{Z}$ in (4.6), we obtain

$$\varphi(T) = \varphi(0) + \int_0^T \bar{F}(\varphi, \hat{Z})ds + \int_0^T \Gamma(\varphi, \hat{Z})d\tilde{W} + \sum_{k=1}^4 R_k(T). \quad (4.9)$$

Here we just estimate one term in $R_4(T)$ as follow:

$$\mathbb{E}\left(\sup_{0 \leq T \leq \tau^*_i} \| \int_0^T B_c(\varphi, A_s^{-1}B_s(\varphi, Z - \hat{Z}))ds \| p \right)$$

$$\leq C\varepsilon^{-2\kappa p} \sum_{k=n+1}^N \mathbb{E}\left(\| \int_0^{T_1} e^{-\lambda_k \varepsilon^{-2}\hat{s}} \hat{Z}_k(0)ds \| p \right)$$

$$\leq C\varepsilon^{2p-2\kappa p}.$$ 

We can estimate other terms by similar calculation, and obtain $R_6(T) = \mathcal{O}(\varepsilon^{2-2\kappa})$. Then, applying Lemma 4.8 to (4.9) and choosing $\gamma = \frac{1}{\delta}$, we can obtain $R_\varphi = \mathcal{O}(\varepsilon^{\frac{1}{\delta} - 6\kappa})$.

Obviously, (4.5) is a consequence of simple calculation. In fact, we can prove that for $p > 0$,

$$\mathbb{E}\left(\| \int_{T_1 \land \tau^*_1}^{T_1 \land \tau^*_i} B_c(\varphi, A_s^{-1}B_s(\varphi, Z - \hat{Z}))ds \| 2p \right)$$

$$\leq C\varepsilon^{-2\kappa} \sum_{k=n+1}^N \mathbb{E}\left(\| \int_{T_2}^{T_1} e^{-\lambda_k \varepsilon^{-2}\hat{s}} \hat{Z}_k(0)ds \| 2p \right)$$

$$\leq C|T_1 - T_2|^p,$$

and

$$\mathbb{E}\left(\| \int_{T_2 \land \tau^*_1}^{T_2 \land \tau^*_i} B_c(\varphi, A_s^{-1}B_s(\varphi, \hat{Z}))ds \| 2p \right)$$

$$\leq C\varepsilon^{-2\kappa} \sum_{k=n+1}^N \mathbb{E}\left(\| \int_{T_2}^{T_1} \hat{Z}_k(s)ds \| 2p \right)$$

$$\leq C|T_1 - T_2|^p,$$

We can also estimate other terms in $R_\varphi$ by similar technique, but the detail is omitted.

We complete the proof. □

Remove $R_\varphi(T)$ from $\varphi(T)$, we gain the reduced system:

$$dy_1 = [\bar{L}y_1 + 2\mathcal{F}(y_1)]dT + \Gamma(y_1, \hat{Z})d\tilde{W},$$

$$y_1(0) = \varphi(0). \quad (4.10)$$

Observing (4.10), there exists $\hat{Z}$ in the diffusion term. We will get rid of it in next two subsections. Our next task is to give the bound of $y_1(T)$ and estimate the error between $\varphi(T)$ and $y_1(T)$. This work will play an important role in later analysis.

**Lemma 4.10** Let Assumption 2.1, 2.3-2.8 hold. Suppose that $\varphi(T)$ and $y_1(T)$ are defined in (4.7) and (4.10) with the same initial value $\varphi(0)$. For $p > 1$, there exists a positive constant $C$, such that

$$\mathbb{E}\left(\sup_{0 \leq T \leq T_0} \| y_1(T) \| p \right) \leq C\|\varphi(0)\| p + C\left(\mathbb{E}\left(\sup_{0 \leq T \leq T_0} \| \hat{Z}(T) \|^{2p}\right)\right)^{\frac{1}{2}}. \quad (4.11)$$
Moreover, if \( \|\varphi(0)\| \leq \varepsilon^{-\frac{\gamma}{p}} \),

\[
E\left( \sup_{0 \leq T \leq \tau^*} \|\varphi(T) - y_1(T)\|^p \right) \leq C\varepsilon^{-\frac{p}{2} - 12\varepsilon p}, \tag{4.12}
\]

\[
E\left( \sup_{0 \leq T \leq \tau^*} \|\varphi(T)\|^p \right) \leq C\varepsilon^{-\frac{p}{2}}. \tag{4.13}
\]

**Proof** The proof of this lemma is similar to that of Lemma 3.6, so we just show the key procedures. Define some stopping time

\[ \tau_{K_1} := \inf\{T > 0, \|y_1(T)\| > K_1\} \]

For \( p \geq 2 \) and \( 0 \leq T \leq \tau_{K_1} \), Itô’s formula yields that

\[
\|y_1(T)\|^p \leq \|\varphi(0)\|^p + p\int_0^T \|y_1\|^{p-2}\langle \mathcal{L}y_1, y_1 \rangle ds + 2p\int_0^T \|y_1\|^{p-2}\langle \mathcal{F}(y_1), y_1 \rangle ds
\]

\[
+ p\int_0^T \|y_1\|^{p-2}\langle y_1, \Gamma(y_1, \hat{Z})d\hat{W}(s) \rangle
\]

\[
+ Cp(p-1)\int_0^T (\|y_1\|^p + \|y_1\|^{p-2}\|\hat{Z}\|^2) ds.
\]

For \( T_1 \in [0, T_0] \), Utilizing Cauchy-Schwarz inequality, Burkholder-Davis-Gundy inequality and Young’s inequality, we derive that

\[
E\left( \sup_{0 \leq T \leq T_1 \wedge \tau_K} \|y_1(T)\|^p \right)
\]

\[
\leq C\|\varphi(0)\|^p + CE\left( \sup_{0 \leq T \leq T_0} \|\hat{Z}(T)\|^p \right) + CE\left( \int_0^{T_1 \wedge \tau_K} \|y_1(s)\|^p ds \right)
\]

\[
+ CE\left( \int_0^{T_1 \wedge \tau_K} \|y_1(s)\|^{2p} ds \right)^{\frac{1}{2}}
\]

\[
\leq C\|\varphi(0)\|^p + CE\left( \sup_{0 \leq T \leq T_0} \|\hat{Z}(T)\|^p \right) + C \int_0^{T_1} E\left( \sup_{0 \leq s_1 \leq s \wedge \tau_K} \|y_1(s_1)\|^p \right) ds
\]

\[
+ \frac{1}{2} E\left( \sup_{0 \leq T \leq T_1 \wedge \tau_K} \|y_1(T)\|^p \right).
\]

Gronwall’s lemma yields that

\[
E\left( \sup_{0 \leq T \leq T_0 \wedge \tau_K} \|y_1(T)\|^p \right) \leq C\|\varphi(0)\|^p + CE\left( \sup_{0 \leq T \leq T_0} \|\hat{Z}(T)\|^p \right).
\]

By analogous technique in Lemma 3.6 we obtain (4.11).

Introduce \( h_1(T) := y_1(T) - \varphi(T) + R_\varphi(T) \).

Note that

\[
h_1(T) = \int_0^T \mathcal{L}(h_1 - R_\varphi) ds + 2\int_0^T \mathcal{F}(y_1) ds - 2\int_0^T \mathcal{F}(y_1 - h_1 + R_\varphi) ds
\]

\[
+ \int_0^T \varepsilon^{-1}\tilde{G}_s(0)(h_1 - R_\varphi)d\tilde{W} + 2\int_0^T B_s(R_\varphi - h_1, \mathcal{A}_s^{-1} \tilde{G}d\tilde{W}).
\]

Then, we can obtain (4.21) and (4.13) by similar procedure in Lemma 3.6

We complete the proof.
4.2 Weak convergence for amplitude equations

In Subsection 4.1, we do not eliminate the fast O-U process \( \hat{Z} \) in diffusion term, which causes that the reduced system (4.10) is still affected by \( \varepsilon \). In next two Subsections, we attempt to remove \( \hat{Z} \) from (4.10), obtain amplitude equations, and give rigorous error analysis. This Subsection is devoted to the case that the dimension of kernel space \( \mathcal{N} \) is more than one. Briefly, we will provide the amplitude equations after averaging \( \hat{Z} \) in such case, and states that the law of \( \varphi(T) \) weakly converge to that of the solution of amplitude equation as \( \varepsilon \) tends to 0. However, the convergence rate we present is not precise as that of Theorem 3.4.

Now let us start with some notations and a key lemma.

Set

\[
g_1(T) := \int_0^T \| \hat{G}_c'(0)(y_1 + \hat{Z}) - 2B_c(y_1, A_s^{-1}\tilde{G}) - B_c(I \otimes A_s)^{-1}(\hat{Z} \otimes \tilde{G}) \|^2_{L_2(U, H^\alpha)} ds,
\]

\[
g_2(T) := \int_0^T \| \hat{G}_c'(0)(y_1) - 2B_c(y_1, A_s^{-1}\tilde{G}) \|^2_{L_2(U, H^\alpha)} ds + \sum_{k=n+1}^{N} \frac{\alpha_k^2}{2\beta_k} \int_0^T \| \hat{G}_c'(0)\varepsilon_k - B_c(I \otimes A_s)^{-1}(\varepsilon_k \otimes \tilde{G}) \|^2_{L_2(U, H^\alpha)} ds,
\]

\[
g_3(T) := \int_0^T \| \hat{G}_c'(0)(\varphi + \hat{Z}) - 2B_c(\varphi, A_s^{-1}\tilde{G}) - B_c(I \otimes A_s)^{-1}(\hat{Z} \otimes \tilde{G}) \|^2_{L_2(U, H^\alpha)} ds,
\]

\[
g_4(T) := \int_0^T \| \hat{G}_c'(0)(\varphi) - 2B_c(\varphi, A_s^{-1}\tilde{G}) \|^2_{L_2(U, H^\alpha)} ds + \sum_{k=n+1}^{N} \frac{\alpha_k^2}{2\beta_k} \int_0^T \| \hat{G}_c'(0)\varepsilon_k - B_c(I \otimes A_s)^{-1}(\varepsilon_k \otimes \tilde{G}) \|^2_{L_2(U, H^\alpha)} ds.
\]

Lemma 4.11 Under Assumption 2.1, 2.3-2.8, we obtain \( g_1(T) = g_2(T) + O(\varepsilon^{1-3\kappa}) \) and \( g_3(T) = g_4(T) + O(\varepsilon^{1-6\kappa}) \).

Proof Note that there exists a positive constant \( C \), such that \( \| y_1 \|_{\mathcal{L}_{2,1}} \leq C \varepsilon^{-3\kappa} \). Recalling the definition of \( \mathcal{L}_{2,1} \) and applying Lemma 4.8, we can prove the first result:

\[
\int_0^T \| \hat{G}_c'(0)(y_1 + \hat{Z}) - 2B_c(y_1, A_s^{-1}\tilde{G}) - B_c(I \otimes A_s)^{-1}(\hat{Z} \otimes \tilde{G}) \|^2_{L_2(U, H^\alpha)} ds
\]

\[
= \sum_{i=1}^{\infty} \int_0^T \| \hat{G}_c'(0)(y_1 + \hat{Z}) f_i - 2B_c(y_1, A_s^{-1}\tilde{G} f_i) - B_c(I \otimes A_s)^{-1}(\hat{Z} \otimes \tilde{G} f_i) \|^2_{H^\alpha} ds
\]

\[
= \sum_{i=1}^{\infty} \int_0^T \| \hat{G}_c'(0)(y_1) f_i - 2B_c(y_1, A_s^{-1}\tilde{G} f_i) \|^2_{H^\alpha} ds + 2 \sum_{i=1}^{\infty} \int_0^T \langle \hat{G}_c'(0)(y_1) f_i - 2B_c(y_1, A_s^{-1}\tilde{G} f_i), \hat{G}_c'(0)(\hat{Z}) f_i \rangle_{H^\alpha} ds
\]

\[
+ 2 \sum_{i=1}^{\infty} \int_0^T \langle \hat{G}_c'(0)(y_1) f_i - 2B_c(y_1, A_s^{-1}\tilde{G} f_i), B_c(I \otimes A_s)^{-1}(\hat{Z} \otimes \tilde{G} f_i) \rangle_{H^\alpha} ds
\]

\[
+ \sum_{i=1}^{\infty} \int_0^T \| \hat{G}_c'(0)(\hat{Z}) f_i \|^2_{H^\alpha} + \| B_c(I \otimes A_s)^{-1}(\hat{Z} \otimes \tilde{G} f_i) \|^2_{H^\alpha} ds
\]

\[
- 2 \sum_{i=1}^{\infty} \int_0^T \langle \hat{G}_c'(0)(\hat{Z}) f_i, B_c(I \otimes A_s)^{-1}(\hat{Z} \otimes \tilde{G} f_i) \rangle_{H^\alpha} ds
\]
Theorem 4.13
Let Assumption 2.1, 2.3-2.8 hold. If $C_\pi$ averaging technique, $\overline{\epsilon}$, where $W$ is constant.

Remark 4.12
Although we can obtain better precise error between $g_1(T)$ and $g_2(T)$ by another averaging technique, $\mathcal{O}(\epsilon^{\frac{1}{b} - \delta})$ is enough for us in this paper.

Let operator $\Sigma : \mathcal{N} \to L(\mathcal{N}, \mathcal{N})$ be defined by

\[
\langle \Sigma(\varphi_1)\varphi_2, \varphi_2 \rangle := \sum_{j=1}^{\infty} \langle \tilde{G}'(0)(\varphi_1) - 2B_c(\varphi_1, A^{-1}_s \tilde{G}) \rangle f_j, \varphi_2 \rangle^2
\]

\[
+ \sum_{i=n+1}^{N} \frac{\alpha_i^2}{2\lambda_i} \sum_{j=1}^{\infty} \langle \tilde{G}'(0)\epsilon_i - B_c(I \otimes_s A_s)^{-1}(\epsilon_i \otimes_s \tilde{G}) \rangle f_j, \varphi_2 \rangle^2,
\]

where $\varphi_1, \varphi_2 \in \mathcal{N}$.

According to operator $\Sigma$, we present amplitude equation:

\[
dy = [\tilde{\mathcal{L}}y + 2\mathcal{F}(y)]dT + \Sigma(\varphi(y))dW_N,
\]

\[y(0) = a(0), \]

where $W_N$ is $\mathcal{N}$-valued Wiener process.

Denote the law of $(\varphi(T), \psi(T))$ stopped at $\tau_1^*$ by $\mathbb{P}_\alpha$, and the projection : $C([0, T_0], \mathcal{H}^\alpha) \to C([0, T_0], \mathcal{N})$ by $\pi$.

Theorem 4.13
Let Assumption 2.1, 2.3, 2.8 hold. If $\|\varphi(0)\|_\alpha \leq C$, the sequence of measures $\pi \mathbb{P}_\alpha$ converges weakly to the measure $\mathbb{P}$, the law of $y(T)$.

Proof
The proof of Theorem 4.13 is divided into two steps.

Step 1. Tightness of the sequence of the probability measures $\pi \mathbb{P}_\alpha$.

Set $T^* := T \wedge \tau^*$ and introduce $\varphi_R(T) := \varphi(T^*) - R_\varphi(T^*)$.

We derive that

\[
\varphi_R(T) = \varphi_R(0) + \int_0^{T^*} \tilde{L}\varphi ds + 2\int_0^{T^*} \mathcal{F}(\varphi) ds + \int_0^{T^*} \Gamma(\varphi, \tilde{Z}) d\tilde{W}.
\]

According to Itô Lemma, we obtain that

\[
E\|\varphi_R(T)\|_4^4 \leq \|\varphi(0)\|_4^4 + CE\left( \int_0^{T^*} \|\varphi_R(s)\|_4^4 + \|R_\varphi(s)\|_4^4 + \|R_\varphi(s)\|_8^8 ds \right)
\]

\[+ CE\left( \int_0^{T^*} \|\varphi_R(s)\|_2^2 \|\varphi_R(s)\|_4^2 + \|R_\varphi(s)\|_2^2 + \|\tilde{Z}(s)\|_8^2 ds \right)
\]

\[\leq \|\varphi(0)\|_4^4 + C + \int_0^{T^*} E\|\varphi_R(s)\|_4^4 ds,
\]

where the last inequality is a consequence of Assumption 2.4. Lemma 4.9 and the fact that $E(\|\tilde{Z}(T)\|_p^p)$ is uniformly bounded in $[0, T_0]$. Thus, by Gronwall's Lemma, there exists a positive constant $C_0$, such that

\[
\sup_{0 \leq T \leq T_0} E\|\varphi_R(T)\|_4^4 \leq C_1.
\]

(4.15)
where $C_1$ is a positive constant. It follows from (4.16) that there exists a positive constant $C_2$, such that
\[
\mathbb{E}\|\varphi_R(T_1) - \varphi_R(T_2)\|^4 \leq C(T_1 - T_2)^2, \forall T_1, T_2 \in [0, T_0].
\] (4.16)

By (4.5) and (4.16), we conclude $\mathbb{E}\|\varphi(T_1^\varepsilon) - \varphi(T_2^\varepsilon)\|^4 \leq C_3(T_1 - T_2)^2$, with a positive constant $C_3$. Then, we achieve this step by Kolmogorov’s criterion for weak compactness [24].

**Step 2. Every accumulation point of $\pi^*\mathbb{P}_\varepsilon$ is a solution to the martingale problem associated with (4.14).**

Denote smooth and compactly supported function defined in $\mathcal{N}$ by $C_0^\infty(\mathcal{N})$. Considering $\Theta(\varphi_R(T))$ with $\Theta \in C_0^\infty(\mathcal{N})$, by Itô formula, we derive
\[
\Theta(\varphi_R(T)) - \Theta(\varphi(0)) = M^\Theta(T) + \int_0^T \langle D\Theta(\varphi), \tilde{L}\varphi + 2\mathcal{F}(\varphi) \rangle ds \\
+ \frac{1}{2} \int_0^T \sum_{j=1}^\infty \langle D^2\Theta(\varphi) \rangle (\Gamma(\varphi, \tilde{Z}) f_j, \Gamma(\varphi, \tilde{Z}) f_j) ds \quad a.s.,
\] (4.17)

where $M^\Theta(T)$ is a martingale with respect to $\pi^*\mathbb{P}_\varepsilon$.

Due to Lemma 4.9, Lemma 4.11 and smooth enough function $\Theta$, we deduce that the third term in the right hand of (4.17) converges to
\[
\frac{1}{2} \int_0^T \text{Tr}[\langle D^2\Theta(\varphi) \rangle \Sigma(\varphi)] ds,
\]
as $\varepsilon$ tends to 0. Moreover, recalling Lemma 4.3 and Lemma 4.10, we can easily obtain
\[
\mathbb{P}(\tau^* = T_0) \geq 1 - \varepsilon^p, \text{ for } p > 1,
\]
which implies
\[
\lim_{\varepsilon \to 0} \mathbb{P}(\tau^* = T_0) = 1.
\] (4.18)

Then, replacing $T^*$ with $T$, we derive that
\[
\Theta(\varphi(T)) - \Theta(\varphi(0)) = \dot{M}^\Theta(T) + \int_0^T \langle D\Theta(\varphi), \tilde{L}\varphi + 2\mathcal{F}(\varphi) \rangle ds \\
+ \frac{1}{2} \int_0^T \text{Tr}[\langle D^2\Theta(\varphi) \rangle \Sigma(\varphi)] ds + \dot{R}_\varepsilon(T) \quad a.s.,
\]
where $\dot{M}^\Theta(T)$ is a martingale with respect to $\pi^*\mathbb{P}_\varepsilon$ and $\dot{R}_\varepsilon(T)$ is an error term. Since $\dot{M}^\Theta(T)$ dependent of $\varphi(T)$ stops at $\tau^*$ and $\Theta$ is smooth enough, we own that
\[
\lim_{\varepsilon \to 0} \mathbb{E}_\varepsilon \left( \sup_{0 \leq T \leq T_0} |\dot{R}_\varepsilon(T)| \right) = 0.
\] (4.19)

Define the continuous function $\dot{M}^\Theta : \mathcal{C}([0, T_0], \mathcal{N}) \to \mathcal{C}([0, T_0], \mathbb{R})$ by
\[
(\dot{M}^\Theta(\varphi))(T) = \Theta(\varphi(T)) - \Theta(\varphi(0)) - \int_0^T (\mathcal{L}\Theta)(\varphi) ds,
\]
where $\mathcal{L}$ is the infinitesimal generator of (4.14).

We proceed to prove that $\dot{M}^\Theta$ is $\mathbb{P}$-martingale where $\mathbb{P}$ is an arbitrary accumulation point of

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Thus, \( \hat{a}_{\text{m}} \) amplitude equation. With this convenience, we can show explicit convergence rate of the error. Then, for any \( 0 \leq \varepsilon \leq 4 \), we claim

\[
\text{Lemma 4.14}
\]

Denote the quadratic variation of \( \tilde{\varepsilon}_{\varepsilon} \) by averaging \( \hat{\varepsilon}_{\varepsilon} \). We complete the proof.

\[
\text{Lemma 4.11}
\]

Let \( \hat{\varepsilon}_{\varepsilon} \) be a sequence of \( \hat{\varepsilon}_{\varepsilon} \) such that \( \hat{\varepsilon}_{\varepsilon} \) weakly converges to \( \hat{\varepsilon} \) as \( n \to \infty \).

Then, for any continuous function \( f : C([0, T_0], \mathcal{N}) \to \mathbb{R} \) and \( T \in [0, T_0] \), we obtain

\[
\hat{E}(\hat{M}_t^\varepsilon(\varphi))(T)f(\varphi)
= \lim_{n \to 0} \mathbb{E}_{\varepsilon_n}(\hat{M}_t(\varphi))(T)f(\varphi)
= \lim_{n \to 0} \mathbb{E}_{\varepsilon_n}(\hat{M}_t(\varphi) - \hat{\varepsilon}_{\varepsilon_n})(T)f(\varphi),
\]

where the first equality holds since \( \hat{M}_t^\varepsilon \) and \( f \) are continuous functions, and the last equality is evident from (4.19).

Note that \( \hat{M}_t^\varepsilon(T) = (\hat{M}_t^\varepsilon(\varphi) - \hat{\varepsilon}_{\varepsilon_n})(T) \) is a martingale with respect to \( \pi^* \mathbb{P}_{\varepsilon_n} \).

Then, for any \( 0 \leq T_a < T_b \leq T_0 \), we get

\[
\lim_{n \to 0} \mathbb{E}_{\varepsilon_n}(\hat{M}_t^\varepsilon(\varphi) - \hat{\varepsilon}_{\varepsilon_n})(T_a)f(\varphi) = \lim_{n \to 0} \mathbb{E}_{\varepsilon_n}(\hat{M}_t^\varepsilon(\varphi) - \hat{\varepsilon}_{\varepsilon_n})(T_b)f(\varphi),
\]

which and (4.20) imply that

\[
\hat{E}(\hat{M}_t^\varepsilon(\varphi))(T_a)f(\varphi) = \hat{E}(\hat{M}_t^\varepsilon(\varphi))(T_b)f(\varphi).
\]

Thus, \( \hat{\varepsilon} \) is a solution to the martingale problem associated with (4.11). In addition, since there is a unique solution of (4.11) due to (26), we obtain \( \hat{\varepsilon} = \varepsilon \).

We complete the proof. \( \square \)

Based on Theorem 4.13, we know \( \varphi(t) \approx y(t) \) in the law. Then, by Lemma 4.11 and Lemma 4.12, we claim

\[
u(t) \approx \varepsilon y(\varepsilon^2 t) + \varepsilon Q(\varepsilon^2 t) + \varepsilon Z(\varepsilon^2 t).
\]

4.3 Amplitude equation for one dimensional kernel space

In this subsection, our aim is to investigate the case that \( \dim \mathcal{N} = 1 \). Compared with multi-dimensional kernel space, we can apply martingale representation theorem to one construct amplitude equation. With this convenience, we can show explicit convergence rate of the error.

As the amplitude equation is just one dimensional SDE in this subsection, we introduce some notations for convenience of understanding.

Set:

\[
\hat{y}_1(T) := (y_1(T), e_1),
\]

\[
\sigma_1 := \langle \tilde{L}e_1, e_1 \rangle, \sigma_2 := 2 \langle \mathcal{F}(e_1), e_1 \rangle, \sigma_3 = \| \tilde{G} e_1(0) e_1 - 2 B_1(e_1, A e_1^{-1} \tilde{G}) \|^2_{\mathcal{L}_2(U, \mu)}.
\]

\[
\sigma_4 := \sum_{k=2}^{N} \frac{\sigma_2}{2 \lambda_k} \| \tilde{G}_k(0) e_k - B_1(I \otimes e_1^{-1} (e_k \otimes A) \|_{\mathcal{L}_2(U, \mu)}^2,
\]

\[
M_1(T) := \langle \int_0^T \Gamma(y_1, \hat{Z}) d\hat{W}(s), e_1 \rangle.
\]

Obviously, \( M_1 \) is a real-valued martingale with the quadratic variation \( g_1 \) given in Lemma 4.11. Now we further remove \( \hat{Z} \) from \( M_1 \) by averaging \( \hat{Z} \) in \( g_1 \).

**Lemma 4.14** \[4\] Let \( M_1(T) \) be a continuous martingale with respect to filtration \( (\mathcal{F}_T)_{T \geq 0} \). Denote the quadratic variation of \( M_1(T) \) by \( \check{g}_1(T) \) and let \( \check{g}_2(T) \) be an arbitrary \( \mathcal{F}_T \)-adapted
increasing process with \( \tilde{g}_2(0) = 0 \). Then, there exists a filtration \( \mathcal{F}_T \) with \( \mathcal{F}_T \subset \mathcal{F}_T \) and a continuous \( \mathcal{F}_T \) martingale \( M_2(T) \) with quadratic variation \( \tilde{g}_2(T) \) such that, for every \( r_0 < \frac{1}{2} \), there exists a positive constant \( C \) with

\[
E \sup_{0 \leq T \leq T_0} \left| M_1(T) - M_2(T) \right|^p \leq C(E|\tilde{g}_2(T_0)|^{2p})^{\frac{1}{2}} \left( E \sup_{0 \leq T \leq T_0} |\tilde{g}_1(T) - \tilde{g}_2(T)|^p \right)^{r_0} + C E \sup_{0 \leq T \leq T_0} |\tilde{g}_1(T) - \tilde{g}_2(T)|^{\frac{2p}{p}}.
\]

Lemma 4.15 Suppose Assumptions 2.7-2.8 hold. Let \( M_1(T) \) be given in (4.22). Then, for \( p > 1 \) and \( \|\tilde{g}(0)\| \leq \varepsilon^{-\frac{1}{2}} \), there exists a continuous \( \mathcal{F}_T \) martingale \( M_2(T) \) with the quadratic variation \( g_2(T) \) and a positive constant \( C \), such that

\[
E \left( \sup_{0 \leq T \leq T_0} |M_1(T) - M_2(T)|^p \right) \leq C \varepsilon^{\frac{1}{p}} - \frac{2p}{p}.
\]

Moreover, there exists a Brownian motion \( B(T) \) with respect to the filtration \( \mathcal{F}_T \), such that

\[
M_2(T) = \int_0^T (\sigma_3 \tilde{y}_1 + \sigma_4)dB.
\]

Proof Lemma 4.11 leads to

\[
E \left( \sup_{0 \leq T \leq T_0} |g_2(T)|^p \right) \leq C E \left( \sup_{0 \leq T \leq T_0} \|\tilde{g}_1(T)\|^{2p} \right) + C \leq C \varepsilon^{\frac{1}{p}} - \frac{2p}{p}.
\]

Choosing \( r_0 = \frac{1}{3} \), it is easy to show (4.228) by (4.229), Lemma 4.4 and Lemma 4.14. In view of martingale representation theorem, we obtain (4.224).

Our next task is to provide amplitude equation and complete the approximation result. Firstly, we introduce some equations on stochastic basis \( (\Omega, \mathcal{F}_T, \mathbb{P}) \):

\[
d\tilde{y}_1 = \sigma_1 \tilde{y}_1 dT + \sigma_2 \tilde{y}_1^2 dT + dM_1(T), \quad \tilde{y}_1(0) = \tilde{\phi}(0),
\]

\[
d\tilde{y}_2 = \sigma_1 \tilde{y}_2 dT + \sigma_2 \tilde{y}_2^2 dT + dM_2(T), \quad \tilde{y}_2(0) = \tilde{\phi}(0),
\]

\[
d\tilde{y}_3 = \sigma_1 \tilde{y}_3 dT + \sigma_2 \tilde{y}_3^2 dT + (\sigma_3 \tilde{y}_3^2 + \sigma_4)dB, \quad \tilde{y}_3(0) = \tilde{\phi}(0),
\]

where \( \tilde{\phi}(0) := (\varphi(0), e_1) \) and \( B(T) \) is the Brownian motion given in (4.21). We note that (4.28) is the amplitude equation one desire.

Lemma 4.16 Suppose Assumption 2.1-2.8 hold. Then, for \( p > 1 \), there exists a positive constant \( C \), such that

\[
E \left( \sup_{0 \leq T \leq T_0} |\tilde{y}_2(T)|^p \right) \leq C |\tilde{\phi}(0)|^p + C \left( E \left( \sup_{0 \leq T \leq T_0} \|\tilde{Z}(T)\|^{2p} \right) \right)^{\frac{1}{2}} + C.
\]

Moreover, if \( |\tilde{a}(0)| \leq \varepsilon^{-\frac{1}{2}} \),

\[
E \left( \sup_{0 \leq T \leq T_0} |\tilde{y}_1(T) - \tilde{y}_2(T)|^p \right) \leq C \varepsilon^{\frac{1}{p}} - \frac{2p}{p}.
\]

Proof Recalling \( \sigma_2 \leq 0 \) and \( M_1 - M_2 = \bar{O}(\varepsilon^{\frac{1}{p}} - \frac{2p}{p}) \), we can get (4.29) and (4.30) by similar deduction of the proof of Lemma 4.6 and Lemma 4.10.

We complete the proof.
Lemma 4.17 Suppose Assumption 2.1-2.8 hold. Then, for $p > 1$, there exists a positive constant $C$, such that

$$
\mathbb{E}\left(\sup_{0 \leq T \leq T_0} |\tilde{y}_3(T)|^p\right) \leq C|\check{\varphi}(0)|^p + C. \tag{4.31}
$$

Moreover, if $|\check{\varphi}(0)| \leq \varepsilon^{-\frac{1}{4}}$,

$$
\mathbb{E}\left(\sup_{0 \leq T \leq T_0} |\tilde{y}_2(T) - \tilde{y}_3(T)|^p\right) \leq C\varepsilon^{\frac{p}{4} - \frac{2p}{5}}. \tag{4.32}
$$

Proof Since the proof of (4.31) is similar to that of (4.11), we do not present the detail. Let us start to prove (4.32).

Introduce a function $g(x) = (\sigma_3 x^2 + \sigma_4)^{\frac{3}{2}}$, and a notation $R_5(T) = \tilde{y}_2(T) - \tilde{y}_3(T)$.

Then,

$$
R_5(T) = \sigma_1 \int_0^T R_5 \, ds + \sigma_2 \int_0^T \tilde{b}_3^2 - \tilde{b}_3^3 \, ds + \int_0^T g(\tilde{y}_1) - g(\tilde{y}_3) \, dB.
$$

For $p \geq 2$, thanks to Itô’s formula, we derive

$$
|R_5(T)|^p = \sigma_1 p \int_0^T |R_5|^p \, ds + \sigma_2 p \int_0^T |R_5|^{p-2} R_5 (\tilde{b}_3^2 - \tilde{b}_3^3) \, ds
$$

$$
+ p \int_0^T |R_5|^{p-2} R_5 |g(\tilde{y}_1) - g(\tilde{y}_3)| \, dB
$$

$$
+ p(p-1) \int_0^T |R_5|^{p-2} |g(\tilde{y}_1) - g(\tilde{y}_3)|^2 \, ds
$$

$$
\leq C \int_0^T |R_5|^p \, ds + C \int_0^T |R_5|^{p-2} R_5 |g(\tilde{y}_1) - g(\tilde{y}_3)| \, dB
$$

$$
+ C \int_0^T |R_5|^p + |\tilde{y}_1 - \tilde{y}_2|^p \, ds, \tag{4.33}
$$

where we use the globally Lipschitz property of $g(x)$.

Since the stochastic integral is a martingale, so we get

$$
\mathbb{E}\left(|R_5(T)|^p\right) \leq C \int_0^T \mathbb{E}\left(|R_5(s)|^p\right) \, ds + C \int_0^T \mathbb{E}\left(|\tilde{y}_1(s) - \tilde{y}_2(s)|^p\right) \, ds.
$$

By Gronwall’s lemma and (4.30), we have

$$
\mathbb{E}\left(|R_5(T)|^p\right) \leq C\varepsilon^{\frac{p}{4} - \frac{2p}{5}}, \quad T \in [0, T_0]. \tag{4.34}
$$

Taking the expectations of the supremum of (4.33) on both sides, by Burkholder-Davis-Gundy inequality, we get

$$
\mathbb{E}\left(\sup_{0 \leq T \leq T_0} |R_5(T)|^p\right) \leq \mathbb{E}\left(\int_0^{T_0} |R_5(s)|^p \, ds\right)
$$

$$
+ \mathbb{E}\left(\int_0^{T_0} |R_5(s)|^{2p} + |\tilde{y}_1(s) - \tilde{y}_2(s)|^{2p} \, ds\right)^{\frac{1}{p}}. \tag{4.35}
$$

Then, combining (4.30), (4.34) and (4.35), we obtain (4.32).

We complete the proof.
Lemma 4.18 Let Assumption 2.1–2.8 hold. For $p > 1$, $\|u\|_\alpha \leq -\frac{\kappa}{2}$, there exists a positive constant $C$, such that

$$E \left( \sup_{0 \leq T \leq T^*} \|R_1(T)\|_\alpha^p \right) \leq C\varepsilon^{\frac{\kappa}{p} - 12\kappa p},$$

where

$$R_1(T) = u(\varepsilon^{-2}T) - \varepsilon \tilde{y}_3(T)e_1 - \varepsilon Q(T) - \varepsilon Z(T).$$

Proof Rewriting $R_1$ as follows:

$$R_1 = \varepsilon[\varphi(T) + \psi(T) - \tilde{y}_3(T)e_1 - Q(T) - Z(T)]$$

$$= \varepsilon[\varphi(T) - \tilde{y}_3(T)e_1 + J(T) + K(T)]$$

$$= \varepsilon[\varphi(T) - \tilde{y}_1(T)e_1 + \tilde{y}_1(T)e_1 - \tilde{y}_2(T)e_1 + \tilde{y}_2(T)e_1 - \tilde{y}_3(T)e_1]$$

$$+ \varepsilon[J(T) + K(T)].$$

Thanks to Lemma 4.1, 4.10, 4.16 and 4.17 we can easily prove this lemma by triangle inequality. \[\blacksquare\]

Definition 4.3 Let Assumption 2.1–2.8 hold. For $\kappa > 0$, define $\bar{\Omega}^* \subset \Omega$ of all $\omega \subset \Omega$ such that all these estimations

$$\sup_{0 \leq T \leq T^*} \|\varphi(T)\| < \varepsilon^{-\frac{\kappa}{2}}, \quad \sup_{0 \leq T \leq T^*} \|\psi(T)\|_\alpha < \varepsilon^{-\frac{\kappa}{2}}, \quad \sup_{0 \leq T \leq T^*} \|R_1(T)\|_\alpha < \varepsilon^{\frac{16\kappa}{13}\kappa}.$$

hold.

Lemma 4.19 For $p > 1$, there exists a positive constant $C$, such that

$$P(\bar{\Omega}^*) \geq 1 - C\varepsilon^p.$$

Proof The proof follows from Chebyshev inequality and simple calculation, so it is omitted here.

We complete the proof. \[\blacksquare\]

Now, we give the main result of this subsection.

Theorem 4.20 Let Assumption 2.1–2.8 hold and $\|u(0)\|_\alpha \leq \varepsilon^{1-\frac{\kappa}{2}}$. Then, for $p > 1$, there exists a positive constant $C$, such that

$$P \left( \sup_{0 \leq t \leq \varepsilon^{-2}T_0} \|u(t) - \varepsilon \tilde{y}_3(\varepsilon^2 t)e_1 - \varepsilon Q(\varepsilon^2 t) - \varepsilon Z(\varepsilon^2 t)\|_\alpha > \varepsilon^{\frac{16\kappa}{13}\kappa} \right) \leq \varepsilon^p.$$

Proof Note that

$$\bar{\Omega}^* \subseteq \left\{ \omega \left| \sup_{0 \leq T \leq T^*_\omega} \|\varphi(T)\| < \varepsilon^{-\kappa}, \sup_{0 \leq T \leq T^*_\omega} \|\psi(T)\|_\alpha < \varepsilon^{-\kappa} \right\} \subseteq \left\{ \omega \left| T^*_\omega = T_0 \right\} \subseteq \Omega.$$

Then,

$$\sup_{0 \leq T \leq T_0} \|R_1(T)\|_\alpha = \sup_{0 \leq T \leq T^*_\omega} \|R_1(T)\|_\alpha < \varepsilon^{\frac{16\kappa}{13}\kappa}, \omega \in \bar{\Omega}^*.$$

By Lemma 4.19

$$P \left( \sup_{0 \leq T \leq T_0} \|R_1(T)\|_\alpha \geq \varepsilon^{\frac{16\kappa}{13}\kappa} \right) \leq 1 - P(\bar{\Omega}^*) \leq \varepsilon^p.$$

We complete the proof. \[\blacksquare\]

We would like to give additional remarks before closing this section.
Remark 4.21

(1). For the case that there is only multiplicative noise in (1.2), we can easily obtain the amplitude equation, and prove that the approximation solution converges to the original one with the rate $O(\varepsilon^{-13\kappa})$.

(3). For the case that $\tilde{O}(0)(\tilde{Z}), -B_0(I \otimes A_0)^{-1}(\tilde{Z} \otimes \tilde{G}) = 0$, we just need to deal with the O-U process in drift terms, then obtain the amplitude equation and further prove the error between the approximation solution and the original one is $O(\varepsilon^{-13\kappa})$.

(3). For any $1 < h < \frac{5}{4}$, we can choose suitable $\tilde{a}$ in Lemma 4.9 and $\gamma_0$ in Lemma 4.17 such that the convergence rate is $O(h^{-\kappa})$ to $O(\varepsilon^{-13\kappa})$.

(4). If the amplitude equation is autonomous, we can analyze the stability of the original system via it.

5 Example

In the section, we will apply our main results to establish the amplitude equation for the following Burgers’ equation with additive and multiplicative noise on the spatial domain $D = [0, \pi]$ subject to Dirichlet boundary condition:

$$du = [(\partial_{xx} + 1)u + \varepsilon^2 ru + u\partial_x u]dT + (\sigma_x + \varepsilon u)dW(t), \quad (5.1)$$

where $W(t)$ is introduced later.

Set $\mathcal{H} := L^2[0, \pi]$, $A := \partial_{xx} + 1$, $L := \nu I$ and $B(u, v) := \frac{1}{2}u\partial_x v + \frac{1}{2}v\partial_x u$.

Let us check Assumption 2.6.

Assumption 2.3 holds for the case $\alpha = \frac{1}{4}$, $\beta = \frac{5}{4}$. Note $\mathcal{H}^{\frac{1}{4}}$ is with base $f_k := k^{-\frac{1}{4}}e_k$.

Obviously,

$$P_\mathcal{H}B(\sin x, \sin x) = 0,$$

and Hölder inequality and Sobolev embedding theorem yield

$$2\|B(u, v)\|_{\mathcal{H}^{-1}} = \|\partial_x(uv)\|_{\mathcal{H}^{-1}} \leq \|uv\|_{L^2} \leq C\|u\|_{L^4}\|v\|_{L^4} \leq C\|u\|^{\frac{1}{2}}_{\mathcal{H}^{\frac{1}{4}}}\|v\|^{\frac{1}{2}}_{\mathcal{H}^{\frac{1}{4}}}.$$ 

Thus Assumption 2.4 is true.

Set $\mathcal{F}(u, v, w) := -B_0(u, A_0^{-1}B_0(v, w))$, $u, v, w \in \mathcal{N}$. Let us check Assumption 2.6

$$\mathcal{F}(u_1 \sin x, u_2 \sin x, u_3 \sin x) = \frac{1}{24}u_1u_2u_3 \sin(x)$$

implies that $\mathcal{F}$ is a trilinear, symmetric and continuous map. For $u_1, u_2, u_3 \neq 0$, there exists $C_0 > 0$ such that

$$\|\mathcal{F}(u_1 \sin x, u_2 \sin x, u_3 \sin x)\| \leq C_0\|u_1\|\|u_2\|\|u_3\|.$$

Moreover, it is easy to find $C_1, C_2, C_3$ such that 2.2 holds, so $\mathcal{F}$ satisfies Assumption 2.6.

$W(t)$ is standard cylindrical $\mathcal{H}$-valued Wiener process with covariance operator $Q$ on a stochastic base $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Define $Q$ by $Qe_k = \alpha_k^2 e_k$ with $\alpha_k = 0$, for $k = 4, \cdots$. 

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Define $G(u)$ by $G(u) \cdot v := u \sqrt{\bar{Q}^\dagger} v$. Clearly, $G(\cdot) : \mathcal{H}^\dagger \to \mathcal{L}^2(\mathcal{H}, \mathcal{H}^\dagger)$ is a Hilbert-Schmidt operator such that all conditions of Assumption 2.8 are satisfied.

**Case I:** $\sigma_c = \varepsilon^2$. We consider the amplitude equation of (1.3). Based on (3.9), we derive the amplitude equation:

$$d\tilde{x} = (\nu \tilde{x} - \frac{1}{12} \tilde{x}^3) dT + \alpha_1 d\tilde{\beta}_1(T) + \frac{8\sqrt{2\alpha_1}}{3\pi} \tilde{x} d\tilde{\beta}_1(T) - \frac{8\sqrt{2\alpha_3}}{15\pi} \tilde{x} d\tilde{\beta}_3(T). \quad (5.2)$$

According to Theorem 3.9, we state

$$u(t) \approx \varepsilon \tilde{x}(\varepsilon^2 t) \sin x.$$ 

If $\alpha_1 = 0$, we note that the Stratonovich version of (5.2) is

$$d\tilde{x} = [(\nu - \frac{64\alpha_3^2}{225\pi^3}) \tilde{x} - \frac{1}{12} \tilde{x}^3] dT - \frac{8\sqrt{2\alpha_3}}{225\pi} \tilde{x} \circ d\tilde{\beta}_3(T).$$

Then according to [19], the constant solution 0 is locally stable if $\nu < \frac{64\alpha_3^2}{225\pi^3}$ and locally unstable if $\nu < \frac{64\alpha_3^2}{225\pi^3}$. By Theorem 3.9, we conclude that if $\alpha_3$ is large enough, the multiplicative noise could stabilize the dynamics of (1.3) with high probability.

**Case II:** $\sigma_c = \varepsilon$. We consider the amplitude equation of

$$d\tilde{u} = [\langle \partial_{xx} + 1 \rangle u + \varepsilon^2 \nu u + u \partial_x u] dT + (\varepsilon + \varepsilon u) dW(t). \quad (5.3)$$

Note that Assumption 2.5 is satisfied due to $B_c(\sin kx, \sin kx) = 0$, for $k > n$. We further assume $\alpha_1 = \alpha_2 = 0$.

Then, we obtain the amplitude equation by (4.28):

$$d\tilde{y} = [(\nu - \frac{\alpha_3^2}{4048\pi}) \tilde{y} - \frac{1}{12} \tilde{y}^3] dT + \frac{128\alpha_2^2}{225\pi^3} \tilde{y}^2 + \frac{5184\alpha_3^2}{1225\pi^3} \tilde{y} dB(T),$$

where $B(T)$ is a real-valued Brownian motion. Furthermore, according to Theorem 4.20, we state

$$u(t) \approx \varepsilon \tilde{y}(\varepsilon^2 t) \sin x + e^{-\frac{8\alpha_3^2}{\pi}} \langle u(0), \sin 3x \rangle \sin 3x + \varepsilon \int_0^t e^{-8(t-s)} \alpha_3 d\beta_3(t) \sin 3x.$$ 

**Reference**

**References**

[1] D. Blömker, Amplitude equations for stochastic partial differential equations, Interdisciplinary Mathematical Sciences, 3. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[2] D. Blömker, Approximation of the stochastic Rayleigh-Rénard problem near the onset of convection and related problems, Stoch.Dyn. 5 (2005) 441-474.

[3] D. Blömker, M. Hairer, Multiscale expansion of invariant measures for SPDEs, Comm. Math. Phys. 251 (2004) 515-555.

[4] D. Blömker, M. Hairer, G. A. Pavliotis, Multiscale analysis for stochastic partial differential equations with quadratic nonlinearities, Nonlinearity 20 (2007) 1-25.
[5] D. Blömker, S. Maier-Paape, G. Schneider, The stochastic Landau equation as an amplitude equation, Discrete Contin. Dyn. Syst. Ser. B 1 (2001) 527-541.

[6] D. Blömker, W. W. Mohammed, Amplitude equations for SPDEs with cubic nonlinearities, Stochastics 85 (2013) 181-215.

[7] D. Blömker, W. W. Mohammed, Amplitude equations for SPDEs with quadratic nonlinearities, Electron. J. Probab. 14 (2009) 2527-2550.

[8] D. Blömker, W. Wang, Qualitative properties of local random invariant manifolds for SPDEs with quadratic nonlinearity, J. Dynam. Differential Equations 22 (2010) 677-695.

[9] A. de Bouard, A. Debussche, Random modulation of solitons for the stochastic Korteweg–de Vries equation, Ann. Henri Poincaré 24 (2009) 251-278.

[10] A. de Bouard, A. Debussche, Soliton dynamics for the Korteweg-de Vries equation with multiplicative homogeneous noise, 14 (2009) 1727-1744.

[11] A. Chekhlov, V. Yakhot, Kolmogorov turbulence in a random-force-driven Burgers equation: anomalous scaling and probability density functions Phys. Rev. E. 52 (1995) 5681-5684.

[12] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, second ed., Cambridge University Press, Cambridge, 2014.

[13] H. Fu, D. Blömker, The impact of multiplicative noise in SPDEs close to bifurcation via amplitude equations, Nonlinearity 33 (2020) 3905-3927.

[14] A. Hutt, Additive noise may change the stability of nonlinear systems, Europhys. Lett. 84 (2008) 34003.

[15] A. Hutt, A. Longtin, L. Schimansky-Geier, Additive global noise delays Turing bifurcations, Phys. Rev. Lett. 98 (2007), 230601.

[16] A. Hutt, A. Longtin, L. Schimansky-Geier, Additive noise-induced turing transitions in spatial systems with application to neural fields and the Swift-Hohenberg equation, Phys.D 237 (2008) 755-773.

[17] K. Klepel, D. Blömker, W. W. Mohammed, Amplitude equation for the generalized Swift-Hohenberg equation with noise, Z. Angew. Math. Phys. 65 (2014) 1107-1126.

[18] K. B. Lauritsen, R. Cuerno, H. A. Makse, Noisy Kuramoto-Sivashinsky equation for an erosion model. Phys. Rev. E.,54, 3577-3580, 1996.

[19] X. Mao, Stochastic differential equations and applications, second ed., 2008 Horwood Publishing Limited, Chichester, 2008.

[20] W. W. Mohammed, Amplitude equation with quintic nonlinearities for the generalized Swift-Hohenberg equation with additive degenerate noise, Adv. Difference Equ. 1 (2016) 1-18.

[21] W. W. Mohammed, Approximate solutions for stochastic time fractional reaction diffusion equations with multiplicative noise, Math. Methods Appl. Sci. 44 (2021) 2140-2157.
[22] W. W. Mohammed, D. Blömker, K. Klepel, Multi-scale analysis of SPDEs with degenerate additive noise, J. Evol. Equ. 14 (2014), 273-298.

[23] M. Raible, S. Mayr, S. Linz, M. Moske, P. Hänggi and K. Samwer, Amorphous thin film growth: theory compared with experiment, Europhys. Lett. 2000 (50) 61-67

[24] Revuz, M. Yor, Continuous martingales and Brownian motion, third ed., Grundlehren der Mathematischen Wissenschaften, vol. 293, Springer-Verlag, Berlin, 1999.

[25] A. J. Roberts, A step towards holistic discretisation of stochastic partial differential equation, ANZIAM J. 45 (2003), C1-C15.

[26] D. Stroock, S. Varadhan Multidimensional Diffusion Processes, Springer-Verlag, Berlin, 2006.