Time-varying cosmological term

J. Socorro¹, M. D’oleire¹ and Luis O. Pimentel²

¹Departamento de Física, DCEI, Universidad de Guanajuato-Campus León, C.P. 37150, León, Guanajuato, México
²Departamento de Física, Universidad Autónoma Metropolitana, Apartado Postal 55-534, C.P. 09340 México, DF, México

E-mail: socorro@fisica.ugto.mx
E-mail: marcodoleire@gmail.com
E-mail: lopr@xanum.uam.mx

Abstract. We present the case of time-varying cosmological term using the Lagrangian formalism characterized by a scalar field $\phi$ with standard kinetic energy and arbitrary potential $V(\phi)$. This model is applied to Friedmann-Robertson-Walker (FRW) cosmology. Exact solutions of the field equations are obtained by a special ansatz to solve the Einstein-Klein-Gordon equation and a particular potential for the scalar field and barotropic perfect fluid. We present the evolution on this cosmological term with different scenarios.

1. introduction

The present phase of an accelerated expansion of the universe stands as one of the most challenging open problems in modern cosmology and astrophysics. This acceleration is characterized by which is popularly known as dark energy. Among many possible alternatives, the simplest candidate for dark energy is the vacuum energy which is mathematically equivalent to the cosmological constant. Models with different decay laws for the variation of the cosmological term were investigated during last two decades in non covariant way, [1, 2, 3, 4, 5, 6, 7, 8, 9], in particular, in [10] appear several evolution relations for $\Lambda$ which many author have used. Anisotropic cosmological models, also has been treated in this formalism from different points of view [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

In this work we present an analysis in covariant way, using the Lagrangian density of the standard scalar field. The main idea arises by proposing that in the cosmological constant case, the scalar potential is identified as $V(\phi) = 2\Lambda$, when $\Lambda$ is a constant. So, we generalized this idea and suggest that this correspondence is valid yet when this cosmological term has a temporal dependence, i.e., $V(\phi(t)) = 2\Lambda(t)$. We include a barotropic equation state between the pressure and energy density of the scalar field, $p_\phi = \omega_\phi \rho_\phi$, quantities that we shall define in the following lines. In order to develop the study presented here, we consider a proportionality between the energy density of the scalar field and the energy density of the barotropic perfect fluid, with the proportionality constant $m_\phi$, that is, $\rho_\phi = m_\phi \rho$.

The corresponding Lagrangian density including a scalar field is

$$\mathcal{L}[g, \phi] = \sqrt{-g} \left( R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right) + \sqrt{-g} \mathcal{L}_{\text{matter}}$$ (1)
where $R$ is the Ricci scalar, $\mathcal{L}_{\text{matter}}$ correspond to barotropic perfect fluid, $p = \omega \rho$, $\rho$ is the energy density and $p$ is the pressure of the fluid in co-moving frame and $\omega$ is constant.

The corresponding variation of (1), with respect to the metric and the scalar field yields to the Einstein and Klein-Gordon equations, respectively

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{1}{2} \left( \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\mu \phi \nabla_\nu \phi \right) + \frac{1}{2} g_{\alpha\beta} V(\phi) - 8\pi G T_{\alpha\beta},$$

$$\Box \phi - \frac{\partial V}{\partial \phi} = 0.$$

From (2) it could be deduced that the energy-momentum tensor associated with the scalar field is

$$T_{\alpha\beta}^{(\phi)} = \frac{1}{2} \left( \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\mu \phi \nabla_\nu \phi \right) - \frac{1}{2} g_{\alpha\beta} V(\phi)$$

and the corresponding tensor for a barotropic perfect fluid is

$$T_{\alpha\beta} = (p + \rho) u_\alpha u_\beta + g_{\alpha\beta} p$$

here $u_\alpha$ is the four-velocity in comoving frame. The line element to be considered in this work is the FRW one,

$$ds^2 = -N^2(t)dt^2 + A^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

$$= -d\tau^2 + A^2(\tau) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

where we identify the time transformation $N(t)dt = d\tau$, this transformation will be used in the whole work, and in special gauge we recover directly the cosmic time $t$.

### 2. Field equations

Making use of metric (5) and a comoving fluid, equations (2) y (3) are now

$$\frac{3A'^2}{A^2} + \frac{3\kappa}{A^2} - 8\pi G \rho - \frac{1}{4} \phi'^2 - \frac{V(\phi)}{2} = 0,$$

$$\frac{2A''}{A} + \frac{A'^2}{A^2} + \frac{\kappa}{A^2} + 8\pi G p + \frac{1}{4} \phi'^2 - \frac{1}{2} V(\phi) = 0,$$

$$\left[ 3 \frac{A'}{A} \phi'^2 + \phi' \phi'' \right] = -V',$$

the Klein-Gordon (8) can be rewritten as

$$\frac{d}{d\tau} \left[ \ln \left( A' \frac{\phi'}{2} \right) \right] = -\frac{V'}{\phi'^2}.$$

In the literature there are some articles where the authors try to solve these field equations in general way, for instance, in reference [23], the authors present an elaborated technique to solve the Klein-Gordon equation (8), and in [24] they use an algebraic method to obtain exact solutions taking as the basic variable the energy density of the scalar field.

In order to solve this set of equations, we introduce the ansatz, of considering that the energy density of the field $\phi$ is proportional to the energy density of the barotropic perfect fluid,
\( \rho_\phi = m_\phi \rho \), where \( m_\phi \) is a positive constant. The scaling behavior occurs when \( m_\phi < 1 \), otherwise, the quintessence field is dominant.

The energy density and pressure of the field \( \phi \) are given as

\[
16\pi G \rho_\phi = \frac{1}{2} \phi'^2 + V(\phi), \quad 16\pi G p_\phi = \frac{1}{2} \phi'^2 - V(\phi)
\]

now, equations (6,7) are rewritten as

\[
\frac{3A'^2}{A^2} + \frac{3\kappa}{A^2} - 8\pi G (\rho + \rho_\phi) = 0, \quad (10)
\]

\[
\frac{2A''}{A} + \frac{A'^2}{A^2} + \frac{\kappa}{A^2} + 8\pi G (p + p_\phi) = 0. \quad (11)
\]

We will make now the assumption that the scalar field is a barotropic fluid: \( p_\phi = \omega_\phi \rho_\phi \), where \( \omega_\phi \) is a constant that play the same role of the \( \omega \) parameter in the barotropic perfect fluid. Under this proposal, the field equations are

\[
\frac{3A'^2}{A^2} + \frac{3\kappa}{A^2} - 8\pi G \rho_T = 0, \quad (12)
\]

\[
\frac{2A''}{A} + \frac{A'^2}{A^2} + \frac{\kappa}{A^2} + 8\pi G p_T = 0, \quad (13)
\]

where the energy density \( \rho_T = \rho + \rho_\phi = \alpha_\phi \rho \) with \( \alpha_\phi = 1 + m_\phi > 1 \), and the pressure \( p_T = p + p_\phi = \omega_T \rho_T \) and the last barotropic index is given by

\[
\omega_T = \left( \frac{\omega + m_\phi \omega_\phi}{1 + m_\phi} \right). \quad (15)
\]

So far we have three barotropic indices, two assumed, \( \omega \) and \( \omega_\phi \), and one implied, \( \omega_T \). We show now that all three must be the same. Notice that he first two equations are the standard FRW equation for the total fluid with barotropic index \( \omega_T \). Also notice that under the assumption of the relation between the density of the scalar field and the density of the perfect fluid, and the proportionality of the \( p_\phi \) with \( \rho_\phi \) then the potential \( V, \phi'^2/2 \) and all the densities \( (\rho, \rho_\phi, \rho_T) \) are proportional to each other. From Eqs. (12,13) we now that

\[
\rho_T = c_T A^{-3(\omega_T+1)} \quad (16)
\]

Equation (14) can be written as

\[
\frac{d}{d\tau} \left[ \ln \left( A^6 V_T^{\frac{2}{\omega_\phi}} \right) \right] = 0, \quad (17)
\]

and the solution is

\[
V(\tau) = c_\omega A^{-3(1+\omega_\phi)}. \quad (18)
\]

Then as a consequence of the proportionality between \( V \) and \( \rho_T \) the exponents in Eqs. (16, 18) should be equal,

\[
\omega_T = \omega_\phi, \quad \Rightarrow \quad \omega = \omega_\phi = \omega_T. \quad (19)
\]
3. General solution for flat space

In this section we present solutions to the field equations for the flat case. Equation (14) is written as
\[
\frac{d}{d\tau} \left[ \ln \left( A^6 V^{\frac{2}{1+\omega}} \right) \right] = 0, \quad \Rightarrow \quad V(\tau) = c_\omega A^{-3(1+\omega)},
\]
using the well known time evolution of the scalar factor for the barotropic fluid, reported in different places, in particular in reference [25], that for future convenience we write as
\[
A_\omega(\tau) = \begin{cases} 
[a_\omega \tau]^{\frac{2}{1+\omega}}, & \omega \neq -1 \\
2^{\frac{2}{3}} \pi G_0 M_{-1} \tau, & \omega = -1.
\end{cases}
\]
Therefore, the temporal dependence of the potential and the scalar field are
\[
V(\tau) = c_\omega \frac{1}{(a_\omega \tau)^2}, \quad \Rightarrow \quad \Delta \phi = \ell_\omega \ln(\tau),
\]
where the constants value $c_\omega$ and $\ell_\omega$ are determined after substitution into the complete set of Einstein equation, with the scale factor solution (21), being the general solutions for any $\omega \neq \pm 1$ the following relations are obtained
\[
V(\phi) = 2m_\phi(1 - \omega) e^{-\sqrt{\frac{3(1+\omega)(1+\frac{1}{m_\phi})}\Delta \phi}}, \quad \Rightarrow \quad \Lambda(\phi) = \frac{m_\phi(1 - \omega)}{3(1+\omega)(1+m_\phi)} e^{-\sqrt{\frac{3(1+\omega)(1+\frac{1}{m_\phi})}\Delta \phi}}.
\]

We observe here that for all values of the barotropic parameter we have a decreasing cosmological function in time (remember the relation between the potential energy and the cosmological term, $V(\tau) = 2\Lambda(\tau)$), and that as function of the scalar field we have an exponential. In what follows we consider particular cases, divided into two branches of the barotropic parameter $0 \leq \omega < 1$, and $-1 < \omega < 0$.

3.1. Positive branch: $0 \leq \omega < 1$.

The relation between the kinetic term and the potential energy of the scalar field is
\[
\frac{1}{2} \dot{\phi}^2 = \frac{1+\omega}{1-\omega} V(\phi)
\]
and we have the following particular case for the barotropic parameter that are of interest in cosmology and astrophysics.

(i) Dust scenario, $\omega = 0$.

The set of equations (23,24,25) gives
\[
V(\phi) = \frac{2m_\phi}{3(1+m_\phi)} e^{-\sqrt{\frac{3(1+\omega)(1+\frac{1}{m_\phi})}\Delta \phi}}.
\]
this behavior emerges when the universe have the following scale factor
\[ A_0(\tau) = [a_0 \tau]^{\frac{2}{3}}, \quad a_0 = \sqrt{\frac{6\pi G\alpha \phi M_0}{3}}. \] (30)

Also this behavior is found using dynamical system and fitting that one critical point will be an attractor, obtaining that the corresponding factor \( \lambda \) in the exponential function \( V(\phi) \approx e^{\lambda \phi} \), will satisfy \( \lambda < -\sqrt{3} \) [26]. This value for the parameter \( \lambda \) was found using others techniques, in quantum solution and in supersymmetric quantum solutions in quantum cosmology, for the same flat FRW cosmological model [27, 28]. Chimento and Jakubi [23] found for inflationary era, solving the Einstein field equations with a power law expansion, that \( \lambda = -\sqrt{2} \).

Is common say that when the scalar potential have a exponential behavior as in this case, the scale factor of the universe must have a fast grow in time. Remembering that \( \alpha \phi = 1 + m_\phi \), we take as base that the percent to usual matter becomes as 4%, considering the dark matter scenario, we need to order of 23%, so the approximate value to \( \alpha \phi \) is 6. In the dark energy scenario, we need that \( \alpha \phi \) near to 18. In these two last cases, the scale factor have a fast growing.

(ii) Radiation, \( \omega = \frac{1}{3} \)

The set of equations (23,24,25), for this value of the barotropic parameter, have the following form
\[ V_{\omega}(\tau) = \frac{m_\phi}{4(1 + m_\phi)} \frac{1}{\tau^2}, \] (31)
\[ \Delta \phi(\tau) = \sqrt{\frac{m_\phi}{1 + m_\phi}} \ln(\tau), \] (32)
\[ V(\phi) = \frac{m_\phi}{4(1 + m_\phi)} e^{-\frac{2}{\sqrt{1 + m_\phi}} \Delta \phi}. \] (33)

and the corresponding scale factor
\[ A_{\frac{1}{3}}(\tau) = [a_{\frac{1}{3}} \tau]^{\frac{1}{2}}, \quad a_{\frac{1}{3}} = \frac{4}{3} \sqrt{\frac{6\pi G\alpha \phi M_{\frac{1}{3}}}{3}}. \]

The temporal dependence of the cosmological term goes to \( \frac{1}{\tau^2} \), result reported in all references that used an analogue relation between the energy density in the scalar field and the energy density to the barotropic perfect fluid, in non covariant theory.

3.2. Negative branch: \(-1 < \omega < 0\).

In this case we write the relation between the field pressure and density as
\[ p_\phi = -|\omega| \rho_\phi \Rightarrow \frac{1}{2} \phi'^2 = \beta_\omega V(\phi), \quad \beta_\omega = \frac{1 - |\omega_\phi|}{1 + |\omega_\phi|} \] (34)
and we consider two particular values of \( \omega \)

(i) For instance, when we choose the case \( \omega_\phi = -\frac{2}{3} \), i.e., \( |\omega_\phi| = \frac{2}{3} \),

The set of equations (23,24,25) have the following form
\[ V_{\omega}(\tau) = \frac{10m_\phi}{1 + m_\phi} \frac{1}{\tau^2}, \] (35)
\[ \Delta \phi(\tau) = \frac{2\sqrt{m_\phi}}{1 + m_\phi} \ln(\tau), \] (36)
\[ V(\phi) = \frac{10m_\phi}{1 + m_\phi} e^{-\sqrt{1 + m_\phi} \Delta \phi}. \] (37)
with a quadratic time dependence for the scale factor

\[
A_{-\frac{2}{3}}(\tau) = \left[a_{-\frac{2}{3}} \right]^2 \tau^2, \quad a_{-\frac{2}{3}} = \frac{1}{3} \sqrt{\frac{6\pi G\alpha_m}{M}}.
\]  

(38)

We consider that in this phenomenon scenario, the values of the \( m_\phi \) yield between \((0, 1]\), for instance when \( m_\phi = 1 \), we recover the law to the potential field given by Chimento and Jakubi [23] with the corresponding scale factor.

(ii) When we choose \( \omega = -\frac{1}{3} \), i.e., \( |\omega_\phi| = \frac{1}{3} \), we have

\[
V(\tau) = \frac{2m_\phi}{1 + m_\phi} \frac{1}{\tau^2},
\]

(39)

so, the scalar field is

\[
\Delta \phi = \sqrt{\frac{2m_\phi}{1 + m_\phi}} \ln(\tau),
\]

(40)

thus, we can write \( V(\phi) \)

\[
V(\phi) = \frac{2m_\phi}{1 + m_\phi} e^{-\sqrt{\frac{2}{1 + m_\phi}} \Delta \phi},
\]

(41)

with a linear evolution for the scale factor

\[
A_{-\frac{1}{3}}(\tau) = a_{-\frac{1}{3}} \tau, \quad a_{-\frac{1}{3}} = \frac{2}{3} \sqrt{\frac{6\pi G\alpha_m}{M}}.
\]

(42)

4. Conclusions

In this work we have characterized the cosmological term \( \Lambda(\tau) \) as proportional the potential for the scalar field. Assuming a proportionality between the energy density of the scalar field and the the density of a barotropic fluid of the matter content, an also assuming that the pressure and density of the scalar field satisfy a barotropic law, so that the field equation in the case of the FRW metric reduces to the standard cosmology in term of a total energy density and pressure that also satisfy a barotropic law. We found that for consistency all the different barotropic parameters should be the same. In the case of flat space we were able to find general exact solutions. A common characteristic of all the solutions is that the dynamic cosmological "constant" is decreasing in time as \( \frac{1}{\tau^2} \) and that it is an exponential function of the scalar field. We also found the exponential behavior in the scalar field in the evolution of the universe, signal that the universe must have a fast growing scale factor. This fast growing correspond when we include the \( m_\phi > 1 \) parameter in the solution, that corresponds when the quintessence field is dominate in the universe, by instance we claim that if the percent to usual matter becomes as 4%, considering the dark matter scenario, we need to order of 23%, so the approximate value to \( \alpha_\phi \) is 6. In the dark energy scenario, we need that \( \alpha_\phi \) near to 18. In these two last cases, the scale factor have a fast growing. However the case \( m_\phi < 1 \) corresponds to scaling behavior with the usual matter. In all epochs analyzed in this work using the relation between the energy density to scalar field and the energy density of the ordinary matter, the behavior to the cosmological term goes as \( \frac{1}{\tau^2} \), these results were found by other authors in non covariant theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. We consider that this behavior is dependent on the relation between the energy densities assumed in this work and others.
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