Global Well Posedness for a Q-tensor Model of Nematic Liquid Crystals

Miho Murata\textsuperscript{*} and Yoshihiro Shibata

\textit{Communicated by T. Ozawa}

\textbf{Abstract.} In this paper, we prove the global well posedness and the decay estimates for a Q-tensor model of nematic liquid crystals in $\mathbb{R}^N$, $N \geq 3$. This system is a coupled system by the Navier–Stokes equations with a parabolic-type equation describing the evolution of the director fields $Q$. The proof is based on the maximal $L_p-L_q$ regularity and the $L_p-L_q$ decay estimates to the linearized problem.

\section{1. Introduction}

We consider the model for a viscous incompressible liquid crystal flow proposed by Beris and Edwards [4]. In this model, the molecular orientation of the liquid crystal is described by a tensor $Q = Q(x,t)$. More precisely, we consider the following system in the $N$ dimensional Euclidean space $\mathbb{R}^N$, $N \geq 3$.

\begin{equation}
\begin{aligned}
\partial_t U + (U \cdot \nabla)U + \nabla p &= \Delta U + \text{Div} (\tau(Q) + \sigma(Q)), \quad \text{div} U = 0 \quad \text{in} \ \mathbb{R}^N \text{ for } t > 0, \\
\partial_t Q + (U \cdot \nabla)Q - S(\nabla U, Q) &= \mathbb{H} \quad \text{in} \ \mathbb{R}^N \text{ for } t > 0, \\
(U, Q)|_{t=0} &= (U_0, Q_0) \quad \text{in} \ \mathbb{R}^N.
\end{aligned}
\end{equation}

Here, $\partial_t = \partial/\partial t$, $t$ is the time variable, $U(x,t) = (u_1(x,t), \ldots, u_N(x,t))^T$ is the fluid velocity, where $M^T$ denotes the transposed $M$, and $p = p(x,t)$ is the pressure. For vector of functions $v$, we set $\text{div} v = \sum_{j=1}^N \partial_j v_j$, and also for $N \times N$ matrix field $A$ with $(j,k)$th components $A_{jk}$, the quantity $\text{Div} A$ is an $N$-vector with $j$th component $\sum_{k=1}^N \partial_k A_{jk}$, where $\partial_k = \partial/\partial x_k$. The tensors $S(\nabla U, Q)$, $\tau(Q)$, and $\sigma(Q)$ are

\begin{align*}
S(\nabla U, Q) &= (\xi D(U) + W(U)) \left( Q + \frac{1}{N}I \right) + \left( Q + \frac{1}{N}I \right) \left( \xi D(U) - W(U) \right) - 2\xi \left( Q + \frac{1}{N}I \right) Q : \nabla U, \\
\tau(Q) &= 2\xi \mathbb{H} : Q \left( Q + \frac{1}{N}I \right) - \xi \left[ \mathbb{H} \left( Q + \frac{1}{N}I \right) + \left( Q + \frac{1}{N}I \right) \mathbb{H} \right] - \nabla Q \circ \nabla Q, \\
\sigma(Q) &= Q \mathbb{H} - \mathbb{H} Q,
\end{align*}

where $D(U) = (\nabla U + (\nabla U)^T)/2$ and $W(U) = (\nabla U - (\nabla U)^T)/2$ denote the symmetric and antisymmetric part of $\nabla U$, respectively, the $(i,j)$ component of $\nabla Q \circ \nabla Q$ is $\sum_{\alpha,\beta=1}^N \partial_i Q_{\alpha \beta} \partial_j Q_{\alpha \beta}, I$ is the $N \times N$ identity matrix. $S(\nabla U, Q)$ describes how the flow gradient rotates and stretches the order parameter $Q$. A scalar

\textsuperscript{*}M. Murata: Partially supported by JSPS Grant-in-aid for Young Scientists 21K13819.

Y. Shibata: Adjunct faculty member in the Department of Mechanical Engineering and Materials Science, University of Pittsburgh.

Partially support by JSPS Grant-in-aid for Scientific Research (A) 17H0109 and Top Global University Project.

This article is part of the topical collection “Yoshihiro Shibata” edited by Tohru Ozawa.
parameter $\xi \in \mathbb{R}$ denotes the ratio between the tumbling and the aligning effects that a shear flow would exert over the directors. The $(i, j)$ component of $\mathbb{H}$ is given by

$$H_{ij} = \Delta Q_{ij} - \mathcal{L} \left[ aQ_{ij} - b \sum_{k=1}^{N} Q_{jk}Q_{ki} + c|Q|^2 Q_{ij} \right],$$

where $\mathcal{L}[A]$ denotes the projection onto the space of traceless matrices, namely,

$$\mathcal{L}[A] = A - \frac{1}{N} \text{tr}[A]I$$

and $|Q|$ is the Frobenius norm of $Q$, namely, $|Q|^2 = \text{tr}(QQ^T) = \sum_{i,j=1}^{N} Q_{ij}Q_{ij}$. Note that $\mathbb{H}$ is derived from the variational derivative of free energy:

$$\mathcal{F}(Q) = \int_{\mathbb{R}^N} \left( \frac{L}{2} |\nabla Q|^2 + F(Q) \right) dx,$$

which is the one-constant approximation of the general Oseen-Frank energy (cf. [3]), where the gradient term corresponds to the elastic part of the free energy and $L > 0$ is the elastic constant. $F(Q)$ denotes the bulk energy of Landau-de Gennes type:

$$F(Q) = \frac{a}{2} |Q|^2 - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} |Q|^4$$

with material-dependent and temperature-dependent constants $a, b, c \in \mathbb{R}$. In our model, we set $L = 1$ for simplicity. Here and hereafter, we assume that $a, c > 0$.

The assumption $c > 0$ is derived from a modeling point of view to guarantee that the free energy $\mathcal{F}(Q)$ is bounded from below (cf. [17,18]). On the other hand, at least in the case $a > 0$, we can prove the existence of global strong solutions because an eigenvalue of the linear operator corresponding to (1.1) does not appear on the positive real axis.

The molecules of nematic liquid crystals flow as in a liquid phase, but they have long-range orientation order. Such a continuum theory for the hydrodynamics of nematic liquid crystals was first studied by Ericksen [10] in 1962. Based on his idea, Ericksen and Leslie proposed the so called Ericksen–Leslie model [11,14,15] in 1960s, in which the evolution of the unit director field is coupled with an evolution equation for the underlying flow field which is given by the Navier–Stokes equation with an additional forcing term. However, the Ericksen–Leslie model could not describe the biaxial nematic liquid crystals. In order to treat the biaxial nematic liquid crystals, P.G. de Gennes [6] introduced an $N \times N$ symmetric, traceless matrix $Q$ as the new parameter, which is called $Q$-tensor. After that, Beris and Edwards [4] proposed a model, in which the director field was replaced by a $Q$-tensor.

Mathematically the Beris–Edwards model has been studied by many authors recent years. Concerning weak solutions, Paicu and Zarnescu [20] obtained the first result for the simplified model with $\xi = 0$. They proved the existence of global weak solutions in $\mathbb{R}^N$ with $N = 2, 3$ as well as weak-strong uniqueness for $N = 2$. Here, $\xi = 0$ means that the molecules are such that they only tumble in a shear flow, but are not aligned by such a flow. In [19], they proved the same results as in [20] for the case that $\xi$ is sufficiently small. An improved result on weak solutions in $\mathbb{R}^2$ was established in [7]. Huang and Ding [13] proved the existence of global weak solutions with a more general energy functional and $\xi = 0$ in $\mathbb{R}^3$.

On the other hand, concerning strong solutions, Abels et al. [1] showed the existence of a strong local solution and global weak solutions with higher regularity in time in the case of inhomogeneous mixed Dirichlet/Neumann boundary conditions in a bounded domain without any smallness assumption on the parameter $\xi$. Liu and Wang [16] improved the spatial regularity of solutions obtained in [1] and generalized their result to the case of anisotropic elastic energy. Abels et al. [2] also proved the local well posedness with Dirichlet boundary condition for the classical Beris–Edwards model, which means that fluid viscosity depends on the $Q$-tensor, but for the case $\xi = 0$ only. Cavaterra et al. [5] showed the global well posedness in the two dimensional periodic case without any smallness assumption on the parameter.
ξ. Xiao [28] proved the global well posedness for the simplified model with ξ = 0 in a bounded domain. He constructed a solution in the maximal \(L_p-L_q\) regularity class.

In this paper, we treat the full model described by (1.1). Since the linear part of \(\xi[H(Q + \frac{1}{N}) + (Q + \frac{1}{N})]H\) is \(\frac{2\xi}{N}(\Delta Q - a(Q - \frac{1}{N}\text{tr}Q\mathbb{I}))\), we separate problem (1.1) into linear part and nonlinear part as follows:

\[
\begin{align*}
\partial_t U - \Delta U + \nabla p + \beta \text{Div} \left( \Delta Q - a \left( Q - \frac{1}{N}\text{tr}Q\mathbb{I} \right) \right) &= f(U, Q), \quad \text{div} U = 0 \quad \text{in } \mathbb{R}^N \text{ for } t > 0, \\
\partial_t Q - \beta D(U) - \Delta Q + a \left( Q - \frac{1}{N}\text{tr}Q\mathbb{I} \right) &= G(U, Q) \quad \text{in } \mathbb{R}^N \text{ for } t > 0, \\
(U, Q)|_{t=0} &= (U_0, Q_0) \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

where

\[
\begin{align*}
\beta &= 2\xi/N, \\
f(U, Q) &= -(U \cdot \nabla)U + \text{Div} [2\xi H : (Q + \mathbb{I}/N) - (\xi + 1)\text{tr}Q\mathbb{I} - \nabla \nabla Q] - \beta \text{Div} \mathcal{L}(F'(Q)), \\
G(U, Q) &= -(U \cdot \nabla)Q + \xi(D(U)Q + QD(U)) + W(U)Q - QW(U) - 2\xi(Q + \mathbb{I}/N) : \nabla U + \mathcal{L}[F'(Q)],
\end{align*}
\]

and the \((i, j)\) component of \(F'(Q) = b^{\sum_{k=1}^N Q_{jk}}Q_{ki} - c|Q|^2 Q_{ij}\). Recently, Schonbek and the second author in [23] considered the Beris–Edwards system removed \(\Delta Q\), that is \(H(Q + \frac{1}{N}) + (Q + \frac{1}{N})H\) is replaced by \(H((Q + \frac{1}{N}) - (Q + \frac{1}{N})\mathbb{I}\) in the tensor \(tau(Q)\) and proved the global well posedness for small initial data in the following solution class:

\[
\begin{align*}
U &\in \bigcap_{q=q_1, q_2} W^1_p((0, T), L_q(\mathbb{R}^N)) \cap L_p((0, T), W^2_q(\mathbb{R}^N)), \\
Q &\in \bigcap_{q=q_1, q_2} W^1_p((0, T), W^1_q(\mathbb{R}^N)) \cap L_p((0, T), W^3_q(\mathbb{R}^N)), \quad (1.3)
\end{align*}
\]

with certain \(p, q_1\), and \(q_2\). In [23], the linear part of the first equation in (1.2) is \(\partial_t U - \Delta U + \nabla p\), and so \(U\) part and \(Q\) part of linearized equations are essentially separated. On the other hand, in the present paper, equations for \(U\) and \(Q\) are completely coupled by third order term: \(\beta \text{Div} \Delta Q\). This is a big difference between [23] and the present paper.

In this paper, the global well posedness for small initial data shall be proved in the solution class (1.3) with the help of the maximal regularity theory and \(L_p-L_q\) decay properties of solutions to linearized equations. The spirit to use both of them is the same as in [23], but the idea how to use them is different, and we think that this approach here gives a general framework to prove the global well posedness for small initial data of quasilinear parabolic equations in unbounded domains [25]. To explain our idea more precisely, we write equations as \(\partial_t u - Au = f\) and \(u|_{t=0} = u_0\) symbolically, where \(u = (U, Q), f\) is the corresponding nonlinear term and \(A\) is a closed linear operator with domain \(D(A) = (W^2_{q_1} \cap W^2_{q_2}) \times (W^3_{q_1} \cap W^3_{q_2})\), where \(q_1\) and \(q_2\) are some exponents such that \(2 < q_1 < N < q_2\) chosen precisely in the statement of our main result below. We use the maximal \(L_p-L_q\) regularity for the time shifted equations \(\partial_t u + \lambda (u - Au) = f\) and \(u|_{t=0} = 0\), and \(L_p-L_q\) decay estimates of continuous analytic semigroup \(\{e^{At}\}_{t \geq 0}\) associated with the operator \(A\) for \(t > 1\) to prove the decay properties of solutions to the compensation equations: \(\partial_t v - Av = \lambda_1 u\) and \(v|_{t=0} = u_0\). The Duhamel’s principle implies that \(v = e^{At}u_0 + \lambda_1\int_0^t e^{A(t-s)}u(s)ds\).

To estimate \(\int_0^t e^{A(t-s)}u(s)ds\) we use the decay properties of \(e^{A(t-s)}\) for \(t - s > 1\), and to estimate \(\int_{t-1}^t e^{A(t-s)}u(s)ds\) we use a standard estimate: \(\|e^{A(t-s)}u(s)\|_{D(A)} \leq C\|u(s)\|_{D(A)}\) for \(0 < t - s < 1\). For the later part, what \(u(t) \in D(A)\) for \(t > 0\) is a key observation. Thanks to the standard estimate, the decay rates get better compared to the case that we apply Duhamel’s principle to \(\partial_t u - Au = f\) and \(u|_{t=0} = u_0\) directly. In fact, if we apply Duhamel’s principle to the original equations directly, we need an additional restriction for exponents \(p, q_1\), and \(q_2\) because we use the maximal \(L_p-L_q\) regularity in order to
estimate \( \int_{t-1}^t e^{A(t-s)} f(s) \, ds \), and so the decay rates become smaller. The key issues of this paper are the maximal \( L_p \)-\( D(A) \) regularity of the operator \( A \) and the \( L_p \)-\( L_q \) decay properties of \( \{e^{At}\}_{t \geq 0} \), both of which are new results and shall be proved respective in Sect. 2 and in Sect. 3 below. The \( L_p \)-\( L_q \) decay properties of \( \{e^{At}\}_{t \geq 0} \) are obtained by the estimates of the heat kernel because the linear system is parabolic type, as shown in the expansion of characteristic roots below. However, we apply the widely usable method, which is based on the idea presented in [22]. This method will also be applied to the parabolic-hyperbolic type system, for instance, the compressible Navier–Stokes equations in the half space with non-slip boundary conditions.

Before stating the main result of this paper, we summarize several symbols and functional spaces used throughout the paper. \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) denote the sets of all natural numbers, real numbers and complex numbers, respectively. We set \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( \mathbb{R}_+ = (0, \infty) \). Let \( q' \) be the dual exponent of \( q \) defined by \( q' = q/(q-1) \) for \( 1 < q < \infty \). For any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N \), we write \( |\alpha| = \alpha_1 + \cdots + \alpha_N \) and \( \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N} \) with \( x = (x_1, \ldots, x_N) \). For scalar function \( f \), \( N \)-vector of functions \( g \), and \( N \times N \) matrix fields \( G \), we set

\[
\nabla^k f = (\partial_x^\alpha f \mid |\alpha| = k), \quad \nabla^k g = (\partial_x^\alpha g_j \mid |\alpha| = k, \ j = 1, \ldots, N),
\]

\[
\nabla^k G = (\partial_x^\alpha G_{j\ell} \mid |\alpha| = k, \ j, \ell = 1, \ldots, N).
\]

For Banach spaces \( X \) and \( Y \), \( \mathcal{L}(X, Y) \) denotes the set of all bounded linear operators from \( X \) into \( Y \), \( \mathcal{L}(X) \) is the abbreviation of \( \mathcal{L}(X, X) \), and \( \text{Hol}(U, \mathcal{L}(X, Y)) \) the set of all \( \mathcal{L}(X, Y) \) valued holomorphic functions defined on a domain \( U \) in \( \mathbb{C} \). For any \( 1 \leq p, q \leq \infty \), \( L_q(\mathbb{R}^N) \), \( W^m_q(\mathbb{R}^N) \) and \( B^s_{q,p}(\mathbb{R}^N) \) denote the usual Lebesgue space, Sobolev space and Besov space, while \( \| \cdot \|_{L_q(\mathbb{R}^N)} \), \( \| \cdot \|_{W^m_q(\mathbb{R}^N)} \) and \( \| \cdot \|_{B^s_{q,p}(\mathbb{R}^N)} \) denote their norms, respectively. We set \( W^N_q(\mathbb{R}^N) = L_q(\mathbb{R}^N) \) and \( W^N_q(\mathbb{R}^N) = B^s_{q,q}(\mathbb{R}^N) \). \( C^\infty(\mathbb{R}^N) \) denotes the set of all \( C^\infty \) functions defined on \( \mathbb{R}^N \). \( L_p((a, b), X) \) and \( W^m_p((a, b), X) \) denote the standard Lebesgue space and Sobolev space of \( X \)-valued functions defined on an interval \((a, b)\), respectively. The \( d \)-product space of \( X \) is defined by \( X^d \), while its norm is denoted by \( \| \cdot \|_{X} \) instead of \( \| \cdot \|_{X^d} \) for the sake of simplicity. The space for a tensor is defined by

\[
X(\mathbb{R}^N; \mathbb{R}^{N^2}) = \left\{ G : \mathbb{R}^N \to \mathbb{R}^{N^2} \mid \|G\|_X = \sum_{i,j=1}^N \|G_{ij}\|_X < \infty \right\}
\]

for the Banach space \( X \). We set

\[
W^m_q,\ell(\mathbb{R}^N) = \{ (f, G) \mid f \in W^m_q(\mathbb{R}^N), \ G \in W^\ell_q(\mathbb{R}^N; \mathbb{R}^{N^2}) \},
\]

\[
\| (f, G) \|_{W^m_q,\ell(\mathbb{R}^N)} = \| f \|_{W^m_q(\mathbb{R}^N)} + \| G \|_{W^\ell_q(\mathbb{R}^N)}.
\]

Furthermore, we set

\[
L_{p,\gamma}(\mathbb{R}^+, X) = \{ f(t) \in L_{p,\text{loc}}(\mathbb{R}^+, X) \mid e^{-\gamma t} f(t) \in L_p(\mathbb{R}^+, X) \},
\]

\[
W^1_{p,\gamma}(\mathbb{R}^+, X) = \{ f(t) \in L_{p,\gamma}(\mathbb{R}^+, X) \mid e^{-\gamma t} \partial_t f(t) \in L_p(\mathbb{R}^+, X) \},
\]

\[
L_{p,\gamma}(\mathbb{R}^+, X(\mathbb{R}^N; \mathbb{R}^{N^2})) = \{ G(t) \in L_{p,\text{loc}}(\mathbb{R}^+, X(\mathbb{R}^N; \mathbb{R}^{N^2})) \mid e^{-\gamma t} G(t) \in L_p(\mathbb{R}^+, X(\mathbb{R}^N; \mathbb{R}^{N^2})) \},
\]

\[
W^1_{p,\gamma}(\mathbb{R}^+, X(\mathbb{R}^N; \mathbb{R}^{N^2})) = \{ G(t) \in L_{p,\gamma}(\mathbb{R}^+, X(\mathbb{R}^N; \mathbb{R}^{N^2})) \mid e^{-\gamma t} \partial_t G(t) \in L_p(\mathbb{R}^+, X(\mathbb{R}^N; \mathbb{R}^{N^2})) \}
\]

for \( 1 < p < \infty \) and \( \gamma > 0 \). Let \( \mathcal{F}_x = \mathcal{F} \) and \( \mathcal{F}_x^{-1} = \mathcal{F}^{-1} \) denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by setting

\[
\hat{f}(\xi) = \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} f(x) \, dx, \quad \mathcal{F}_x^{-1}[g](x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix \cdot \xi} g(\xi) \, d\xi.
\]

The letter \( C \) denotes generic constants and the constant \( C_{a,b,...} \) depends on \( a, b, \ldots \). The values of constants \( C \) and \( C_{a,b,...} \) may change from line to line. We use small boldface letters, e.g. \( f \) to denote vector-valued
functions and capital boldface letters, e.g. $G$ to denote matrix-valued functions, respectively. In order to state our main theorem, we introduce spaces and several norms:

$$J_q(\mathbb{R}^N) = \{U \in L_q(\mathbb{R}^N)^N | \text{div } U = 0 \text{ in } \mathbb{R}^N\},$$

$$D_{q,p}(\mathbb{R}^N) = \{(U, Q) | U \in (B_{q,p}^{2(1-1/p)}(\mathbb{R}^N)^N \cap J_q(\mathbb{R}^N)), \quad Q \in B_{q,p}^{1+2(1-1/p)}(\mathbb{R}^N; \mathbb{R}^{N^2})\},$$

$$X_{p,q,t} = \{(U, Q) | U \in L_p((0,t), W_q^2(\mathbb{R}^N)^N) \cap W_t^1((0,t), L_q(\mathbb{R}^N)^N), \quad Q \in L_p((0,t), W_q^3(\mathbb{R}^N; \mathbb{R}^{N^2})) \cap W_t^1((0,t), W_q^3(\mathbb{R}^N; \mathbb{R}^{N^2}))\},$$

$$\mathcal{N}(U, Q)(T) = \sum_{q=q_{1,2}} \left(\|<t >^b (U, Q)\|_{L_{\infty}((0,T), W_q^{0,1}(\mathbb{R}^N))} + \|<t >^b (\partial_t U, \partial_t Q)\|_{L_p((0,T), W_q^{0,1}(\mathbb{R}^N))}\right)$$

$$+ \|<t >^b (\nabla U, \nabla Q)\|_{L_p((0,T), W_q^{1,2}(\mathbb{R}^N))} + \|<t >^b (U, Q)\|_{L_p((0,T), W_q^{2,3}(\mathbb{R}^N))},$$

(1.4)

where $<t > = (1 + t^2)^{1/2}$, $b$ is given in Theorem 1.1 below.

The following theorem is our main result of this paper.

**Theorem 1.1.** Assume that $N \geq 3$. Let $0 < T < \infty$, $0 < \sigma < 1/2$, and let $p = 2$ or $p = 2 + \sigma$. Let $q_1$ and $q_2$ be numbers such that

$$q_1 = 2 + \sigma, \quad \begin{cases} q_2 \geq \frac{N(2 + \sigma)}{N - (2 + \sigma)} & \text{if } N = 3, 4, \\ q_2 > N & \text{if } N \geq 5. \end{cases}$$

(1.5)

Suppose that $b = (N - \sigma)/(2(2 + \sigma))$ if $p = 2$ and $b = N/(2(2 + \sigma))$ if $p = 2 + \sigma$. Then, there exists a small number $\epsilon > 0$ such that for any initial data $(U_0, Q_0) \in \bigcap_{i=1}^2 D_{q_{1,2}}(\mathbb{R}^N) \cap W_{q_{1,2}}^{0,1}(\mathbb{R}^N)$ with

$$\mathcal{T} := \sum_{i=1}^2 \|\langle U_0, Q_0 \rangle\|_{D_{q_{1,2}}(\mathbb{R}^N)} + \|\langle U_0, Q_0 \rangle\|_{W_{q_{1,2}}^{0,1}(\mathbb{R}^N)} < \epsilon^2,$$

problem (1.1) admits a unique solution $(U, Q)$ with

$$(U, Q) \in X_{p,q_1,T} \cap X_{p,q_2,T}$$

satisfying the estimate

$$\mathcal{N}(U, Q)(T) \leq \epsilon.$$

**Remark 1.2.**

1. $T > 0$ is taken arbitrarily and $\epsilon$ is chosen independent of $T$, therefore, Theorem 1.1 yields the global well posedness for (1.1).

2. $q_2$ satisfies $q_2 > N$ for any $N \geq 3$ because $N(2 + \sigma)/(N - (2 + \sigma)) > N$ when $N = 3, 4$. By (1.5), we can choose $q_2 = 15$ when $N = 3$, for instance.

3. To get a priori estimates, we need conditions $by' > 1$ and $(N/(2(2 + \sigma)) + 1/2 - b)p > 1$ for $0 < \sigma < 1/2$. We also need the condition $by > 1$ in order to estimate nonlinear term $<t > b Q^2$ in $L_p((0, T), L_{q_1/2}(\mathbb{R}^N))$. Thus, we choose $b = N/(2(2 + \sigma))$ if $p = 2 + \sigma$ and $b = (N - \sigma)/(2(2 + \sigma))$ if $p = 2$. For details see Sect. 4 below. In the case $N = 2$, our argument does not work. In fact, in the case $p = 2 + \sigma$, conditions $by' > 1$ and $(N/(2(2 + \sigma)) + 1/2 - b)p > 1$ furnish $(1 + \sigma)/(2 + \sigma) < b < (N + \sigma)/(2(2 + \sigma))$, which implies that $N > 2 + \sigma$. In the case $p = 2$, we also have $N > 2 + \sigma$ by the same way as $p = 2 + \sigma$.

4. If initial data $Q_0$ is a symmetric and traceless matrix, so is a solution $Q$, which can be proved by the uniqueness of heat equations.

This paper is organized as follows: Sect. 2 proves the maximal $L_p - L_q$ regularity by combining the existence of $R$-bounded solution operator families to the resolvent problem and the Weis operator valued Fourier multiplier theorem. Sect. 3 proves the $L_p - L_q$ decay estimates to the linearized problem. Sect. 4 proves the main theorem with the help of the maximal $L_p - L_q$ regularity and the $L_p - L_q$ decay estimates.
2. Maximal $L_p$–$L_q$ Regularity

In this section, we show the maximal $L_p$–$L_q$ regularity for problem:

$$
\begin{aligned}
\begin{cases}
\partial_t U - \Delta U + \nabla p + \beta \text{Div} \left( \Delta Q - a \left( Q - \frac{1}{N} \text{tr} Q \right) \right) = f, & \text{div } U = 0 & \text{in } \mathbb{R}^N \text{ for } t > 0, \\
\partial_t Q - \beta D(U) - \Delta Q + a \left( Q - \frac{1}{N} \text{tr} Q \right) = G & \text{in } \mathbb{R}^N \text{ for } t > 0, \\
(U,Q)|_{t=0} = (U_0,Q_0) & \text{in } \mathbb{R}^N.
\end{cases}
\end{aligned}
$$

We now state the maximal $L_p$–$L_q$ regularity theorem.

**Theorem 2.1.** Let $1 < p,q < \infty$. Then, there exists a constant $\gamma_1 \geq 1$ such that the following assertion holds: For any initial data $(U_0,Q_0) \in D_{q,p}(\mathbb{R}^N)$ and functions in the right-hand side $(f,G) \in L_{p,\gamma_1}(\mathbb{R}^+,W^{0,1}_q(\mathbb{R}^N))$, problem (2.1) admits unique solutions $U$ and $Q$ with

$$
U \in W^{1,1}_{p,\gamma_1}(\mathbb{R}^+,L_q(\mathbb{R}^N)) \cap L_{p,\gamma_1}(\mathbb{R}^+,W^2_q(\mathbb{R}^N)), \\
Q \in W^{1,1}_{p,\gamma_1}(\mathbb{R}^+,W^1_q(\mathbb{R}^N)) \cap L_{p,\gamma_1}(\mathbb{R}^+,W^3_q(\mathbb{R}^N)),
$$

possessing the estimate

$$
\|e^{-\gamma t} \partial_t U\|_{L_p(\mathbb{R}^+,L_q(\mathbb{R}^N))} + \|e^{-\gamma t} U\|_{L_p(\mathbb{R}^+,W^1_q(\mathbb{R}^N))} + \|e^{-\gamma t} \partial_t Q\|_{L_p(\mathbb{R}^+,W^2_q(\mathbb{R}^N))} + \|e^{-\gamma t} Q\|_{L_p(\mathbb{R}^+,W^3_q(\mathbb{R}^N))} 
\leq C_{p,q,N,\gamma_1} \left( \|U_0,Q_0\|_{D_{q,p}(\mathbb{R}^N)} + \|(e^{-\gamma t}f,e^{-\gamma t}G)\|_{L_p(\mathbb{R}^+,W^{0,1}_q(\mathbb{R}^N))} \right)
$$

for any $\gamma \geq \gamma_1$.

2.1. $\mathcal{R}$-boundedness of Solution Operators

In this subsection, we analyze the following resolvent problem in order to prove Theorem 2.1.

$$
\begin{aligned}
\begin{cases}
\lambda U - \Delta U + \nabla p + \beta \text{Div} \left( \Delta Q - a \left( Q - \frac{1}{N} \text{tr} Q \right) \right) = f, & \text{div } U = 0 & \text{in } \mathbb{R}^N, \\
\lambda Q - \beta D(U) - \Delta Q + a \left( Q - \frac{1}{N} \text{tr} Q \right) = G & \text{in } \mathbb{R}^N.
\end{cases}
\end{aligned}
$$

Here, $\lambda$ is the resolvent parameter varying in a sector

$$
\Sigma_{\epsilon,\lambda_0} = \{ \lambda \in \mathbb{C} \mid |\arg \lambda| < \pi - \epsilon, |\lambda| \geq \lambda_0 \}
$$

for $0 < \epsilon < \pi/2$ and $\lambda_0 \geq 1$. Moreover, we set an angle $\sigma_0 \in (0, \pi/2)$ by

$$
\sigma_0 = \begin{cases} 
0 & \text{if } \beta = 0, \\
\arg(1 + i/|\beta|) & \text{if } \beta \neq 0.
\end{cases}
$$

We introduce the definition of $\mathcal{R}$-boundedness of operator families.

**Definition 2.2.** A family of operators $T \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset T$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, we have the inequality:

$$
\left\{ \int_0^1 \left| \sum_{j=1}^n r_j(u)T_jf_j \right|^p_X du \right\}^{1/p} \leq C \left\{ \int_0^1 \left| \sum_{j=1}^n r_j(u)f_j \right|^p_X du \right\}^{1/p}.
$$

The smallest such $C$ is called $\mathcal{R}$-bound of $T$, which is denoted by $\mathcal{R}\mathcal{L}(X,Y)(T)$.

The following theorem is the main result of this subsection.
Theorem 2.3. Let $1 < q < \infty$ and $\lambda_0 > 0$. Then, for any $\sigma \in (\sigma_0, \pi/2)$, there exist operator families
\[
A(\lambda) \in \text{Hol}(\Sigma_{\sigma, \lambda_0}, \mathcal{L}(W^{0,1}_q(\mathbb{R}^N), W^{2}_q(\mathbb{R}^N)^N))
\]
\[
B(\lambda) \in \text{Hol}(\Sigma_{\sigma, \lambda_0}, \mathcal{L}(W^{0,1}_q(\mathbb{R}^N), W^{3}_q(\mathbb{R}^N; \mathbb{R}^{N^2})))
\]
such that for any $\lambda = \gamma + i\tau \in \Sigma_{\sigma, \lambda_0}$, $f \in L_q(\mathbb{R}^N)^N$, and $G \in W^1_q(\mathbb{R}^N; \mathbb{R}^{N^2})$,
\[
U = A(\lambda)(f, G), \quad Q = B(\lambda)(f, G)
\]
are unique solutions of problem (2.3), and
\[
\mathcal{R}_{\mathcal{L}(W^{0,1}_q(\mathbb{R}^N), A_q(\mathbb{R}^N))}((\tau \partial_\tau)^n S\lambda A(\lambda) \mid \lambda \in \Sigma_{\sigma, \lambda_0}) \leq r,
\]
\[
\mathcal{R}_{\mathcal{L}(W^{0,1}_q(\mathbb{R}^N), B_q(\mathbb{R}^N))}((\tau \partial_\tau)^n T\lambda B(\lambda) \mid \lambda \in \Sigma_{\sigma, \lambda_0}) \leq r
\]
for $n = 0, 1$, where $S\lambda U = (\nabla^2 U, \lambda^{1/2} \nabla U, \lambda U)$, $T\lambda Q = (\nabla^3 Q, \lambda^{1/2} \nabla^2 Q, \lambda Q)$, $A_q(\mathbb{R}^N) = L_q(\mathbb{R}^N)^{N^3 + N^2 + N}$, $B_q(\mathbb{R}^N) = L_q(\mathbb{R}^N; \mathbb{R}^{N^5}) \times L_q(\mathbb{R}^N; \mathbb{R}^{N^3}) \times W^1_q(\mathbb{R}^N; \mathbb{R}^{N^2})$, and $r = r_{N, q, \lambda_0}$ is a constant independent of $\lambda$.

Postponing the proof of Theorem 2.3, we are concerned with time dependent problem (2.1). Set
\[
X_q(\mathbb{R}^N) = J_q(\mathbb{R}^N) \times W^1_q(\mathbb{R}^N; \mathbb{R}^{N^2}).
\]
Let $A$ be a linear operator defined by
\[
\mathcal{A}(U, Q) = \left( P\Delta U - \beta P\text{Div} \left( \Delta Q - a \left( Q - \frac{1}{N} \text{trQI} \right) \right), \beta D(U) + \Delta Q - a \left( Q - \frac{1}{N} \text{trQI} \right) \right)
\]
for $(U, Q) \in D(A)$, where $P$ denotes solenoidal projection and
\[
D(A) = (W^{2}_q(\mathbb{R}^N)^N \cap J_q(\mathbb{R}^N)) \times W^3_q(\mathbb{R}^N; \mathbb{R}^{N^2}).
\]
Since Definition 2.2 with $n = 1$ implies the uniform boundedness of the operator family $T$, solutions $U$ and $Q$ of equations (2.3) satisfy the resolvent estimate:
\[
|\lambda|\|\mathcal{R}((U, Q))\|_{W^{0,1}_q(\mathbb{R}^N)} + |\lambda|^{1/2}\|\nabla(U, \nabla^2 Q)\|_{L_q(\mathbb{R}^N)} + \|\mathcal{R}(U, Q)\|_{W^2_q(\mathbb{R}^N)} \leq C_r\|\mathcal{S}(f, G)\|_{W^{0,1}_q(\mathbb{R}^N)}
\]
for any $\lambda \in \Sigma_{\sigma, \lambda_0}$ and $(f, G) \in X_q(\mathbb{R}^N)$. By (2.8), we have the following theorem.

Theorem 2.4. Let $1 < q < \infty$. Then, the operator $A$ generates an analytic semigroup $\{e^{At}\}_{t \geq 0}$ on $X_q(\mathbb{R}^N)$. Moreover, there exist constants $\gamma_1 \geq 1$ and $C_{q, N, \gamma_1} > 0$ such that $\{e^{At}\}_{t \geq 0}$ satisfies the estimates:
\[
\|e^{At}(U_0, Q_0)\|_{X_q(\mathbb{R}^N)} \leq C_{q, N, \gamma_1} e^{\gamma_1 t}\|\mathcal{R}(U_0, Q_0)\|_{X_q(\mathbb{R}^N)}
\]
\[
\|\partial_t e^{At}(U_0, Q_0)\|_{X_q(\mathbb{R}^N)} \leq C_{q, N, \gamma_1} e^{\gamma_1 t-1}\|\mathcal{R}(U_0, Q_0)\|_{X_q(\mathbb{R}^N)}
\]
\[
\|\partial_t e^{At}(U_0, Q_0)\|_{X_q(\mathbb{R}^N)} \leq C_{q, N, \gamma_1} e^{\gamma_1 t}\|\mathcal{R}(U_0, Q_0)\|_{D(A)}
\]
for any $t > 0$.

Combining Theorem 2.4 with a real interpolation method (cf. Shibata and Shimizu [26, Proof of Theorem 3.9]), we have the following result for Eq. (2.1) with $(f, G) = (0, O)$.

Theorem 2.5. Let $1 < p, q < \infty$. Then, for any $(U_0, Q_0) \in D_{q, p}(\mathbb{R}^N)$, problem (2.1) with $(f, G) = (0, O)$ admits a unique solution $(U, Q) = e^{At}(U_0, Q_0)$ possessing the estimate:
\[
\|e^{-\gamma t}\partial_t U\|_{L^p(\mathbb{R}^N)} + \|e^{-\gamma t}U\|_{L^p(\mathbb{R}^N; W^2_q(\mathbb{R}^N))} + \|e^{-\gamma t}\partial_t Q\|_{L^p(\mathbb{R}^N; W^1_q(\mathbb{R}^N))} + \|e^{-\gamma t}Q\|_{L^p(\mathbb{R}^N; W^2_q(\mathbb{R}^N))} \leq C_{p, q, \gamma_1}\|\mathcal{S}(f, G)\|_{D_{q, p}(\mathbb{R}^N)}
\]
for any $\gamma \geq \gamma_1$. 

The remaining part of this subsection is devoted to proving Theorem 2.3. For this purpose, we first calculate a solution formula of (2.3). Substituting the formula: $\text{tr} \hat{Q} = (\lambda - \Delta)^{-1} \text{tr} G$, which obtained by the trace of the second equation of (2.3), into the first equation of (2.3), we have

$$\lambda U - \Delta U + \nabla p + \beta \text{div} (\Delta Q - aQ) = f - \frac{a\beta}{N} (\lambda - \Delta)^{-1} \nabla \text{tr} G. \quad (2.10)$$

Here, we used $\text{Div} \text{tr} \mathbb{G} = \nabla \text{tr} G$. Taking divergence of (2.10), we have

$$p = -\beta (\text{div} \text{Div} Q - a\Delta^{-1} \text{div} \text{Div} Q) + \Delta^{-1} \text{div} f - \frac{a\beta}{N} (\lambda - \Delta)^{-1} \text{tr} G. \quad (2.11)$$

Inserting (2.11) into (2.10), we have

$$(\lambda - \Delta)U - \beta \nabla (\text{div} \text{Div} Q - a\Delta^{-1} \text{div} \text{Div} Q) + \beta \text{div} (\Delta Q - aQ) = f - \nabla \Delta^{-1} \text{div} f. \quad (2.12)$$

Substituting the formula: $\text{tr} \hat{Q} = (\lambda - \Delta)^{-1} \text{tr} G$ into the second equation of (2.3) yields

$$(\lambda - (\Delta - a))Q - \beta \text{D}(U) = G + \frac{a}{N} (\lambda - \Delta)^{-1} \text{tr} \mathbb{G}. \quad (2.13)$$

Thus, setting $\mathbb{H} = \mathbb{G} + a(\lambda - \Delta)^{-1} \text{tr} \mathbb{G}/N$, we have

$$(\lambda - (\Delta - a))Q = \beta \text{D}(U) + \mathbb{H}. \quad (2.14)$$

Thanks to (2.13), $Q$ can be represented by $U$; therefore, we first consider a solution formula for $U$ below. Applying $(\lambda - (\Delta - a))$ to (2.12) and setting $g = f - \nabla \Delta^{-1} \text{div} f$, we have

$$(\lambda - \Delta)(\lambda - (\Delta - a))U - \beta \nabla (\lambda - (\Delta - a)) \text{div} \text{Div} Q - a\Delta^{-1} \text{div} \text{Div} Q)
+ \beta (\lambda - (\Delta - a)) \text{div} (\Delta Q - aQ) = (\lambda - (\Delta - a))g. \quad (2.14)$$

By (2.13) and $\text{div} U = 0$, the second and third terms of (2.14) can be calculated as follows:

$$\beta \nabla (\lambda - (\Delta - a)) \text{div} \text{Div} Q - a\Delta^{-1} \text{div} \text{Div} Q) = \beta \nabla (\text{div} \text{Div} \mathbb{H} - a\Delta^{-1} \text{div} \text{Div} \mathbb{H}),$$

$$\beta (\lambda - (\Delta - a)) \text{div} (\Delta Q - aQ) = \beta^2 (\Delta^2 - a\Delta) U + \beta \text{Div} (\Delta \mathbb{H} - a\mathbb{H}),$$

so that (2.14) is equivalent to

$$P_2(\lambda) U = (\lambda - (\Delta - a))g + \beta \nabla (\text{div} \text{Div} \mathbb{H} - a\Delta^{-1} \text{div} \text{Div} \mathbb{H}) - \beta \text{Div} (\Delta \mathbb{H} - a\mathbb{H}),$$

where $P_2(\lambda) = (\lambda - \Delta)(\lambda - (\Delta - a)) + \beta^2 (\Delta^2 - a\Delta)$. Noting that $\text{Div} \text{tr} \mathbb{G} = \nabla \text{tr} G$, we have

$$U = P_2(\lambda)^{-1} \{(\lambda - (\Delta - a))(f - \nabla \Delta^{-1} \text{div} f) + \beta \nabla (\text{div} \text{Div} \mathbb{G} - a\Delta^{-1} \text{div} \text{Div} \mathbb{G}) - \beta \text{Div} (\Delta \mathbb{G} - a\mathbb{G})\}. \quad (2.15)$$

Thus $U = (u_1, \ldots, u_N)$ has form:

$$u_j = A_j(\lambda)(f, \mathbb{G}) \quad (2.16)$$

with

$$A_j(\lambda)(f, \mathbb{G}) = \mathcal{F}^{-1} \left[ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \left( \hat{j}_j - \frac{\xi_j}{|\xi|^2} \hat{f} \right) \right]
- \mathcal{F}^{-1} \left[ \frac{\beta}{P_2(\xi, \lambda)} \left( i \xi_j \sum_{m=1}^{N} \xi_m \hat{G}_{\ell m} - \sum_{\ell=1}^{N} i \xi_{\ell} |\xi|^2 \hat{G}_{j\ell} \right) \right]
- a \mathcal{F}^{-1} \left[ \frac{\beta}{P_2(\xi, \lambda)} \left( i \xi_j \sum_{m=1}^{N} \frac{\xi_m}{|\xi|^2} \hat{G}_{\ell m} - \sum_{\ell, m=1}^{N} i \xi_{\ell} \hat{G}_{j\ell} \right) \right],$$

where

$$P_2(\xi, \lambda) = (\lambda + |\xi|^2)(\lambda + |\xi|^2 + a) + \beta^2 (|\xi|^4 + a|\xi|^2). \quad (2.17)$$
Moreover, using (2.13), $Q$ with $(j, k)$ component $Q_{jk}$ is represented as follows:

$$
Q_{jk} = \mathcal{F}^{-1} \left[ \frac{\beta}{\lambda + |\xi|^2 + a} (i \xi_j \hat{u}_k + i \xi_k \hat{u}_j) \right] + \mathcal{F}^{-1} \left[ \frac{1}{\lambda + |\xi|^2 + a} \hat{G}_{jk} \right] + \frac{a}{N} \mathcal{F}^{-1} \left[ \frac{1}{P_1(\xi, \lambda)} \text{tr} \hat{G} \delta_{jk} \right]
$$

= $B_{jk}(\lambda)(f, G)$

with

$$
B_{jk}(\lambda)(f, G) = \mathcal{F}^{-1} \left[ \frac{\beta}{P_2(\xi, \lambda)} (i \xi_j \hat{f}_k + i \xi_k \hat{f}_j) \right] - \mathcal{F}^{-1} \left[ \frac{2\beta}{P_2(\xi, \lambda)} \right. \\
+ \mathcal{F}^{-1} \left[ \frac{2|\xi|^2}{(\lambda + |\xi|^2 + a) P_2(\xi, \lambda)} \xi_j \xi_k \sum_{\ell, m=1}^{N} \xi_\ell \xi_m \hat{G}_{\ell m} \right] \\
- \mathcal{F}^{-1} \left[ \frac{|\xi|^2}{(\lambda + |\xi|^2 + a) P_2(\xi, \lambda)} \right. \\
+ 2a \mathcal{F}^{-1} \left[ \frac{\beta^2}{(\lambda + |\xi|^2 + a) P_2(\xi, \lambda)} \right. \\
- a \mathcal{F}^{-1} \left[ \frac{\beta^2}{(\lambda + |\xi|^2 + a) P_2(\xi, \lambda)} \right. \\
+ \mathcal{F}^{-1} \left[ \frac{1}{\lambda + |\xi|^2 + a} \hat{G}_{jk} \right] + \frac{a}{N} \mathcal{F}^{-1} \left[ \frac{1}{P_1(\xi, \lambda)} \text{tr} \hat{G} \delta_{jk} \right]
$$

where

$$
P_1(\xi, \lambda) = (\lambda + |\xi|^2)(\lambda + |\xi|^2 + a).
$$

Let $A(\lambda)(f, G)$ be a vector whose $j$th component is $A_j(\lambda)(f, G)$ and let $B(\lambda)(f, G)$ be a matrix whose $(j, k)$th component is by $B_{jk}(\lambda)(f, G)$, respectively. To prove the $R$-boundedness of $A(\lambda)$ and $B(\lambda)$, we introduce the following two lemmas. Lemma 2.6 was proved by [8, Proposition 3.4] and Lemma 2.7 proved by [9, Lemma 2.1], [12, Theorem 3.3], and [21, Lemma 2.5]

**Lemma 2.6.** (1) Let $X$ and $Y$ be Banach spaces, and let $T$ and $S$ be $R$-bounded families in $\mathcal{L}(X, Y)$. Then, $T + S = \{ T + S \mid T \in T, S \in S \}$ is also an $R$-bounded family in $\mathcal{L}(X, Y)$ and

$$
R_{\mathcal{L}(X,Y)}(T + S) \leq R_{\mathcal{L}(X,Y)}(T) + R_{\mathcal{L}(X,Y)}(S).
$$

(2) Let $X$, $Y$ and $Z$ be Banach spaces and let $T$ and $S$ be $R$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then, $ST = \{ ST \mid T \in T, S \in S \}$ is also an $R$-bounded family in $\mathcal{L}(X, Z)$ and

$$
R_{\mathcal{L}(X,Z)}(ST) \leq R_{\mathcal{L}(X,Y)}(T)R_{\mathcal{L}(Y,Z)}(S).
$$

**Lemma 2.7.** Let $1 < q < \infty$, $\delta > 0$. Assume that $k(\xi, \lambda)$, $\ell(\xi, \lambda)$, and $m(\xi, \lambda)$ are functions on $(\mathbb{R}^N \setminus \{0\}) \times \Sigma_{\sigma, 0}$ such that for any $\sigma \in (\sigma_0, \pi/2)$ and any multi-index $\alpha \in \mathbb{N}^N_0$ there exists a positive constant $M_{\alpha, \sigma}$ such that

$$
|\partial_\xi^\alpha k(\xi, \lambda)| \leq M_{\alpha, \sigma}|\xi|^{1-|\alpha|}, \quad |\partial_\xi^\alpha \ell(\xi, \lambda)| \leq M_{\alpha, \sigma}|\xi|^{-|\alpha|},
$$

$$
|\partial_\xi^\alpha m(\xi, \lambda)| \leq M_{\alpha, \sigma}(1^{|\lambda|/2 + |\xi|^{-1}|\xi|^{-|\alpha|}}
$$

for any $(\xi, \lambda) \in (\mathbb{R}^N \setminus \{0\}) \times \Sigma_{\sigma, 0}$. Let $K(\lambda)$, $L(\lambda)$, and $M(\lambda)$ be operators defined by

$$
[K(\lambda)f](x) = \mathcal{F}^{-1}[k(\xi, \lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\sigma, 0}),
$$

$$
[L(\lambda)f](x) = \mathcal{F}^{-1}[\ell(\xi, \lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\sigma, 0}),
$$

$$
[M(\lambda)f](x) = \mathcal{F}^{-1}[m(\xi, \lambda)\hat{f}(\xi)](x) \quad (\lambda \in \Sigma_{\sigma, \delta}).
$$
Then, the following assertions hold true:

1. The set \( \{ K(\lambda) \mid \lambda \in \Sigma_{\sigma,0} \} \) is \( \mathcal{R} \)-bounded on \( \mathcal{L}(W^1_q(R^N), L_q(R^N)) \) and there exists a positive constant \( C_{N,q} \) such that

\[
\mathcal{R}_{\mathcal{L}(W^1_q(R^N), L_q(R^N))}(\{ K(\lambda) \mid \lambda \in \Sigma_{\sigma,0} \}) \leq C_{N,q} \max_{|\alpha| \leq N+1} M_{\alpha,\sigma}.
\]

2. Let \( n = 0, 1 \). Then, the set \( \{ L(\lambda) \mid \lambda \in \Sigma_{\sigma,0} \} \) is \( \mathcal{R} \)-bounded on \( \mathcal{L}(W^n_q(R^N)) \) and there exists a positive constant \( C_{N,q} \) such that

\[
\mathcal{R}_{\mathcal{L}(W^n_q(R^N))}(\{ L(\lambda) \mid \lambda \in \Sigma_{\sigma,0} \}) \leq C_{N,q} \max_{|\alpha| \leq N+1} M_{\alpha,\sigma}.
\]

3. The set \( \{ M(\lambda) \mid \lambda \in \Sigma_{\sigma,\delta} \} \) is \( \mathcal{R} \)-bounded on \( \mathcal{L}(L_q(R^N), W^1_q(R^N)) \) and there exists a positive constant \( C_{N,q} \) such that

\[
\mathcal{R}_{\mathcal{L}(L_q(R^N), W^1_q(R^N))}(\{ M(\lambda) \mid \lambda \in \Sigma_{\sigma,\delta} \}) \leq C_{N,q,\delta} \max_{|\alpha| \leq N+1} M_{\alpha,\sigma}.
\]

To use Lemma 2.7, we prepare the following lemmas.

**Lemma 2.8.** (1) Let \( 0 < \epsilon < \pi/2 \). Then, for any \( \lambda \in \Sigma_{\epsilon,0} \) and \( \alpha \geq 0 \), we have

\[
|\lambda + \alpha| \geq \left( \sin \frac{\epsilon}{2} \right) (|\lambda| + \alpha).
\]  

(2) Let \( 0 < \epsilon < \pi/2 \) and \( a > 0 \). Then, for any \( (\xi, \lambda) \in R^N \times \Sigma_{\epsilon,0} \), we have

\[
|P_1(\xi, \lambda)| \geq \left( \frac{1}{2} \sin \frac{\epsilon}{2} \right)^2 (|\lambda|^{1/2} + |\xi|)^4.
\]

(3) Let \( \sigma_0 \) be the angle defined in (2.4). Then, for any \( \sigma \in (\sigma_0, \pi/2) \) and \( (\xi, \lambda) \in R^N \times \Sigma_{\sigma,0} \), we have

\[
|P_2(\xi, \lambda)| \geq C_{\sigma,\beta} (|\lambda|^{1/2} + |\xi|)^4
\]  

with some constant \( C_{\sigma,\beta} \) independent of \( \xi \) and \( \lambda \).

**Proof.** (1) is well-known (cf. e.g. [24, Lemma 3.5.2]). (2) follows from (1). We shall show (3). In the case \( \beta = 0 \), (2.22) is proved by (2) because \( P_2(\xi, \lambda) = P_1(\xi, \lambda) \). In the case \( \beta \neq 0 \), we rewrite \( P_2(\xi, \lambda) \) as follows:

\[
P_2(\xi, \lambda) = (\lambda - \lambda_+(|\xi|))(\lambda - \lambda_-(|\xi|)),
\]

where

\[
\lambda_\pm(|\xi|) = -\left( |\xi|^2 + \frac{a^2}{4} \right) \pm \sqrt{\frac{a^2}{4} - a^2|\xi|^2 - \beta^2|\xi|^4}
\]

which has the following the expansions:

\[
\begin{align*}
\lambda_+(|\xi|) &= -(1 + \beta^2)|\xi|^2 + O(|\xi|^4), \\
\lambda_-(|\xi|) &= -(1 - \beta^2)|\xi|^2 - a + O(|\xi|^4) \quad \text{as } |\xi| \to 0, \\
\lambda_\pm(|\xi|) &= (-1 \pm i|\beta|)|\xi|^2 + O(1) \quad \text{as } |\xi| \to \infty.
\end{align*}
\]

In the case of the low frequency parts, by (2.23), we can prove (2.22) for any \( \sigma \in (0, \pi/2) \) by the same way as \( \beta = 0 \). We consider the high frequency parts. Let \( \lambda = |\lambda| e^{i\theta} \) for \( |\theta| \leq \pi - \sigma, \sigma \in (\sigma_0, \pi/2) \). Noting that \( \lambda_\pm(|\xi|) = -\sqrt{1 + \beta^2}|\xi|^2 e^{\pm i\sigma_0} + O(1) \) for large \( |\xi| \), we have

\[
|\lambda - \lambda_\pm(|\xi|)|^2 \geq |\lambda|^2 + (1 + \beta^2)|\xi|^4 + 2\sqrt{1 + \beta^2}|\lambda||\xi|^2 \cos(\theta - \sigma_0) - (|\lambda| + |\xi|^2 + O(1))O(1) \\
\geq |\lambda|^2 + (1 + \beta^2)|\xi|^4 + 2\sqrt{1 + \beta^2}|\lambda||\xi|^2 \cos(\sigma - \sigma_0) - (|\lambda| + |\xi|^2 + O(1))O(1) \\
\geq (1 - \cos(\sigma - \sigma_0))(|\lambda|^2 + |\xi|^4) - (|\lambda| + |\xi|^2 + O(1))O(1).
\]

Here, we used that \( \cos(\theta - \sigma_0) \geq \cos|\{\pi - (\sigma - \sigma_0)\}| = -\cos(\sigma - \sigma_0) \). Therefore, we have (2.22) for any \( \sigma \in (\sigma_0, \pi/2) \) and \( \lambda \in \Sigma_{\sigma,0} \).

\( \Box \)
Using Lemma 2.8 and the following Bell’s formula for the derivatives of the composite functions:

\[ \partial^\alpha f(g(\xi)) = \sum_{k=1}^{[\alpha]} f^{(k)}(g(\xi)) \sum_{\alpha = \alpha_1 + \cdots + \alpha_k \atop |\alpha_i| \geq 1} \Gamma^\alpha_{\alpha_1, \ldots, \alpha_k}(\partial^\alpha_1 g(\xi)) \cdots (\partial^\alpha_k g(\xi)) \]

with \( f^{(k)}(t) = d^k f(t)/dt^k \) and suitable coefficients \( \Gamma^\alpha_{\alpha_1, \ldots, \alpha_k} \), we have the following estimates.

**Lemma 2.9.** (1) Let \( 0 < \epsilon < \pi/2, a > 0 \) and \( n = 0, 1 \). Then, for any multi-index \( \alpha \in \mathbb{N}_0^N \), there exists a positive constant \( C \) depending on at most \( \alpha, \epsilon \) and \( b \) such that for any \( (\xi, \lambda) \in \mathbb{R}^N \times \Sigma_{\epsilon, 0} \) with \( \lambda = \gamma + i\tau \),

\[
|\partial^\alpha_\xi (\tau \partial_\tau)^n (\lambda + |\xi|^2)^{-1}| \leq C(|\lambda|^{1/2} + |\xi|)^{-2-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n (\lambda + |\xi|^2 + a)^{-1}| \leq C(|\lambda|^{1/2} + |\xi|)^{-2-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n P_1(\xi, \lambda)^{-1}| \leq C(|\lambda|^{1/2} + |\xi|)^{-4-|\alpha|}.
\]

(2) Let \( a > 0 \) and \( n = 0, 1 \). Then, for any \( \sigma \in (\sigma_0, \pi/2) \) and any multi-index \( \alpha \in \mathbb{N}_0^N \), there exists a positive constant \( C \) depending on at most \( \alpha, \epsilon \) and \( b \) such that for any \( (\xi, \lambda) \in \mathbb{R}^N \times \Sigma_{\sigma, 0} \) with \( \lambda = \gamma + i\tau \),

\[
|\partial^\alpha_\xi (\tau \partial_\tau)^n P_2(\xi, \lambda)^{-1}| \leq C(|\lambda|^{1/2} + |\xi|)^{-4-|\alpha|}.
\]

**Proof of Theorem 2.3.** Let \( n = 0, 1, a, b, c, j, k, \ell, m = 1, \ldots, N \) and let \( \alpha \) be any multi-index of \( \mathbb{N}_0^N \). We first prove \( \mathcal{R} \)-boundedness of \( S_\lambda A(\lambda) \). For this purpose, we verify \( (\lambda, \lambda^{1/2} \xi_a, \xi_a \xi_b) A(\lambda) f \) satisfies the assumption of Lemma 2.7. Using Lemma 2.9 and Leibniz’s rule, for any \( (\xi, \lambda) \in (\mathbb{R}^N \setminus \{0\}) \times \Sigma_{\sigma, 0} \), we have

\[
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \left( \xi_a \xi_b, \lambda^{1/2} \xi_a, \lambda \right) \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \left( \xi_a \xi_b, \lambda^{1/2} \xi_a, \lambda \right) \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
\]

which combined with Lemma 2.7 yields (2.5) in Theorem 2.3.

We next consider \( \mathcal{R} \)-boundedness of \( T_\lambda B(\lambda) \).

Using Lemma 3.14 and Leibniz’s rule, for any \( (\xi, \lambda) \in (\mathbb{R}^N \setminus \{0\}) \times \Sigma_{\sigma, 0} \), we have

\[
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \xi_a \xi_c, \lambda^{1/2} \xi_a \xi_b \right\} \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
|\partial^\alpha_\xi (\tau \partial_\tau)^n \left\{ \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \right\}| \leq C_{\alpha, \sigma} |\xi|^{-|\alpha|},
\]
which combined with Lemma 2.7 and Lemma 2.6 gives

\[ \mathcal{R}_{\mathcal{L}(W^{2,1}_q(\mathbb{R}^N), L_q(\mathbb{R}^N)^N)} \left( \{ (\tau \partial_r)^n (\nabla^2 B(\lambda)) \mid \lambda \in \Sigma_{\sigma, \lambda_0} \} \right) \leq C_{N,q}. \]  

(2.25)

Analogously, for any \((\xi, \lambda) \in (\mathbb{R}^N \setminus \{0\}) \times \Sigma_{\sigma,0}\), we have

\[ \begin{aligned}
&\left| \partial^a(\tau \partial_r)^n \left\{ \frac{\lambda \xi_j}{P_2(\xi, \lambda)} \right\} \right| \leq C_{a,\sigma}(|\lambda|^{1/2} + |\xi|)^{-1}|\xi|^{-|\alpha|}, \\
&\left| \partial^a(\tau \partial_r)^n \left\{ \frac{\lambda \xi_j \xi_k \xi_l}{P_2(\xi, \lambda)} \right\} \right| \leq C_{a,\sigma}(|\lambda|^{1/2} + |\xi|)^{-1}|\xi|^{-|\alpha|}, \\
&\left| \partial^a(\tau \partial_r)^n \left\{ \frac{\lambda \xi_j \xi_k \xi_l}{(\lambda + |\xi|^2 + a) P_2(\xi, \lambda)} \right\} \right| \leq C_{a,\sigma}(|\lambda|^{1/2} + |\xi|)^{-1}|\xi|^{-|\alpha|}, \\
&\left| \partial^a(\tau \partial_r)^n \left\{ \frac{\lambda \xi_j \xi_k \xi_l}{(\lambda + |\xi|^2 + a) P_2(\xi, \lambda)} \right\} \right| \leq C_{a,\sigma}(|\lambda|^{1/2} + |\xi|)^{-1}|\xi|^{-|\alpha|}, \\
&\left| \partial^a(\tau \partial_r)^n \left\{ \frac{-\lambda}{(\lambda + |\xi|^2 + a) P_2(\xi, \lambda)} \right\} \right| \leq C_{a,\sigma}(|\lambda|^{1/2} + |\xi|)^{-1}|\xi|^{-|\alpha|}, \\
&\left| \partial^a(\tau \partial_r)^n \left\{ \frac{\lambda}{P_2(\xi, \lambda)} \right\} \right| \leq C_{a,\sigma}(|\lambda|^{1/2} + |\xi|)^{-1}|\xi|^{-|\alpha|},
\end{aligned} \]

which combined with Lemma 2.7 (2), (3), and Lemma 2.6 gives

\[ \mathcal{R}_{\mathcal{L}(W^{2,1}_q(\mathbb{R}^N), W^q(\mathbb{R}^N)^N)} \left( \{ (\tau \partial_r)^n (\lambda B(\lambda)) \mid \lambda \in \Sigma_{\sigma, \lambda_0} \} \right) \leq C_{N,q}. \]  

(2.26)

(2.25) and (2.26) imply (2.6) in Theorem 2.3.

We finally show the uniqueness of solutions. Let \((U, Q) \in W^{2,1}_q(\mathbb{R}^N) \times W^3_q(\mathbb{R}^N; \mathbb{R}^N)\) satisfy the homogeneous equations:

\[ \begin{aligned}
\begin{cases}
\lambda U - \Delta U + \nabla p + \beta \text{Div} \left( \Delta Q - a \left( Q - \frac{1}{N} \text{tr} QI \right) \right) = 0, & \text{div } U = 0 \quad \text{in } \mathbb{R}^N, \\
\lambda Q + \beta D(U) - \Delta Q + a \left( Q - \frac{1}{N} \text{tr} QI \right) = 0 & \text{in } \mathbb{R}^N.
\end{cases}
\end{aligned} \]

(2.27)

Applying formulas (2.11), (2.12), (2.13) and (2.14) with \(f = 0\) and \(G = 0\) yield that \(P_2(\lambda) U = 0\). If we set \(P_2(\xi, \lambda) = (\lambda + |\xi|^2)(\lambda + |\xi|^2 + a) + \beta^2(|\xi|^4 + a|\xi|^2)\), Fourier transformation \(\hat{U}\) of \(U\) satisfies \(P_2(\xi, \lambda) \hat{U} = 0\), which, combined with Lemma 2.8 (3), yields that \(U = 0\). Thus, by (2.13), \((\lambda + |\xi|^2 + a) \hat{Q} = 0\), where
\( \hat{Q} \) denotes Fourier transformation of \( Q \), which yields that \( Q = 0 \). By (2.11), we also have \( p = 0 \). This completes the proof of the uniqueness, and therefore we have proved Theorem 2.3.

### 2.2. A Proof of Theorem 2.1

To prove Theorem 2.1, the key tool is the Weis operator valued Fourier multiplier theorem. Let \( D(\mathbb{R}, X) \) and \( S(\mathbb{R}, X) \) be the set of all \( X \) valued \( C^\infty \) functions having compact support and the Schwartz space of rapidly decreasing \( X \) valued functions, respectively, while \( S'(\mathbb{R}, X) = \mathcal{L}(S(\mathbb{R}, X)) \). Given \( M \in L_{1,\text{loc}}(\mathbb{R}\setminus\{0\}, \mathcal{L}(X, Y)) \), we define the operator \( T_M : \mathcal{F}^{-1}D(\mathbb{R}, X) \to S'(\mathbb{R}, Y) \) by

\[
T_M \phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in D(\mathbb{R}, X)).
\] (2.28)

**Theorem 2.10** (Weis [27]) Let \( X \) and \( Y \) be two UMD Banach spaces and \( 1 < p < \infty \). Let \( M \) be a function in \( C^1(\mathbb{R}\setminus\{0\}, \mathcal{L}(X, Y)) \) such that

\[
\mathcal{R}_{\mathcal{L}(X,Y)} \left( \left\{ (\zeta \frac{d}{d\zeta})^\ell M(\zeta) \mid \zeta \in \mathbb{R}\setminus\{0\} \right\} \right) \leq \kappa < \infty \quad (\ell = 0, 1)
\]

with some constant \( \kappa \). Then, the operator \( T_M \) defined in (2.28) is extended to a bounded linear operator from \( L_p(\mathbb{R}, X) \) into \( L_p(\mathbb{R}, Y) \). Moreover, denoting this extension by \( T_M \), we have

\[
\|T_M\|_{\mathcal{L}(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))} \leq C\kappa
\]

for some positive constant \( C \) depending on \( p, X \) and \( Y \).

We now prove Theorem 2.1. Using the linear operator \( \mathcal{A} \) defined by (2.7), we can rewrite problem (2.1) as follows:

\[
\partial_t U - \mathcal{A} U = F \quad \text{in } \mathbb{R}^N \quad \text{for } t > 0, \quad U|_{t=0} = U_0,
\] (2.29)

where \( U = (U, Q) \), \( F = (PF, G) \), and \( U_0 = (PU_0, Q_0) \). Let \( F \in L_p(\mathbb{R}, W^{0,1}_q(\mathbb{R}^N)) \). Here, \( \gamma_1 \) is the same as in Theorems 2.4 and 2.5. To solve problem (2.29), we consider problem:

\[
\partial_t U_1 - \mathcal{A} U_1 = F_0 \quad \text{in } \mathbb{R}^N \quad \text{for } t \in \mathbb{R}.
\] (2.30)

Here, \( f_0 = (f_0, g_0) \) and \( f_0 \) and \( g_0 \) are zero extensions of \( f \) and \( G \) to \( t < 0 \). To solve equations (2.30), we introduce Laplace transformation \( \mathcal{L} \) and Laplace inverse transformation \( \mathcal{L}^{-1} \) defined by

\[
\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t) \, dt, \quad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\lambda \tau} g(\tau) \, d\tau
\]

where \( \lambda = \gamma + i\tau \in \mathbb{C} \), which are written by Fourier transformation \( \mathcal{F} \) and Fourier inverse transformation in \( \mathbb{R} \) as

\[
\mathcal{L}[f](\lambda) = \mathcal{F}[e^{-\gamma \ell} f(t)](\tau), \quad \mathcal{L}^{-1}[g](t) = e^{\gamma t} \mathcal{F}^{-1}[g](\tau).
\]

Applying Laplace transformation to equations (2.30) yields

\[
\lambda \mathcal{L}[U_1] - \mathcal{A} \mathcal{L}[U_1] = \mathcal{L}[F_0] \quad \text{in } \mathbb{R}^N.
\]

Applying the operators \( \mathcal{A}(\lambda) \) and \( \mathcal{B}(\lambda) \) given in Theorem 2.3 yields that

\[
\mathcal{L}[U_1](\lambda) = (\mathcal{A}(\lambda) \mathcal{L}[F_0](\lambda), \mathcal{B}(\lambda) \mathcal{L}[F_0](\lambda)).
\]

Since \( F_0 = 0 \) for \( t < 0 \), \( \mathcal{L}[F_0](\lambda) \) is holomorphic for \( \text{Re} \lambda > 0 \), and so by Cauchy’s theorem in theory of one complex variable we see that

\[
\mathcal{L}[U_1](\gamma_1 + i\tau) = \mathcal{L}[U_1](\gamma_1 + i\tau) \quad (\gamma > \gamma_1, \quad \tau \in \mathbb{R}).
\] (2.31)

Using Laplace inverse transformation, we define \( U_1 \) by

\[
U_1(\cdot, t) = \mathcal{L}^{-1}[(\mathcal{A}(\lambda) \mathcal{L}[F_0](\lambda), \mathcal{B}(\lambda) \mathcal{L}[F_0](\lambda))](t).
\]
Since we can write
\[ e^{-\gamma t}U_1 = F^{-1}[(A(\lambda)F[e^{-\gamma t}F_0](\tau), B(\lambda)F[e^{-\gamma t}F_0](\tau))] \]
using Theorem 2.3 and applying Theorem 2.10, we have
\[ \|e^{-\gamma t}U_1\|_{L_p(\mathbb{R}^2)} + \|e^{-\gamma t}\partial_t U_1\|_{L_p(\mathbb{R}^2)} \leq C\|e^{-\gamma t}F\|_{L_p(\mathbb{R}^2)} \]  \hspace{1cm} (2.32)
Since \(|(\tau\partial_t)^{\gamma}/\lambda| \leq C\) for \(\lambda \in \Sigma_{e,\lambda_0}\), we have
\[ \gamma\|e^{-\gamma t}U_1\|_{L_p(\mathbb{R}^2)} \leq C\|e^{-\gamma t}\partial_t U_1\|_{L_p(\mathbb{R}^2)} \]
as follows from [8, Proposition 3.6 and Corollary 3.7], which, combined with (2.32), yields that
\[ \gamma\|U_1\|_{L_p((-\infty,0), W^{0,1}_q(\mathbb{R}))} \leq \gamma\|e^{-\gamma t}U_1\|_{L_p(\mathbb{R}^2)} \leq \|e^{-\gamma t}\partial_t U_1\|_{L_p(\mathbb{R}^2)} \]
\[ \leq C\|e^{-\gamma t}\partial_t U_1\|_{L_p(\mathbb{R}^2)} \]
for any \(\gamma \geq \gamma_1 > \lambda_0\). Thus, letting \(\gamma \rightarrow \infty\) yields that \(\|U_1\|_{L_p((-\infty,0), W^{0,1}_q(\mathbb{R}))} = 0\), which implies that \(U_1(\cdot, t) = 0\) for \(t < 0\). By trace theorem in theory of real interpolation, \(U_1\) is a \(D_{q,p}(\mathbb{R}^2)\) valued continuous in \(\mathbb{R}\), we have \(U_1(\cdot, t) = 0\) for \(t \leq 0\).

In view of Theorem 2.5, we set \(U = U_1 + e^{At}(U_0, Q_0)\), and then \(U\) is a required solution of equations (2.1), which proves the existence part of Theorem 2.1.

We finally show the uniqueness of solutions. By Theorem 2.4 and Duhamel’s principle we can write \(U\) as follows:
\[ U = e^{At}U_0 + \int_0^t e^{A(t-s)}F(s) ds. \]
Thus, if \(U\) satisfies the Eq. (2.29) with \(F = 0\), \(U_0 = 0\), we have \(U = 0\). This completes the proof of Theorem 2.1.

3. Decay Property of Solutions to the Linearized Problem

In this section, we consider the following linearized problem:
\[
\begin{cases}
\partial_t U - \Delta U + \nabla p + \beta \text{Div} \left( \Delta Q - a \left( \frac{Q}{N} \right) \right) = 0, & \text{div} U = 0 \quad \text{in} \ \mathbb{R}^N \text{ for } t \in (0, T), \\
\partial_t Q - \beta \text{Div}(U) - \Delta Q + a \left( \frac{Q}{N} \right) = O & \quad \text{in} \ \mathbb{R}^N \text{ for } t \in (0, T), \\
(U, Q)|_{t=0} = (f, G) & \quad \text{in} \ \mathbb{R}^N.
\end{cases}
\]  \hspace{1cm} (3.1)
Let \(d = \text{tr}Q\). We consider the \(L_p-L_0\) decay estimates for \(d\), \(U\), and \(Q\) satisfying (3.1) below.

3.1. Decay Estimates for \(d\)

We first consider the decay estimates for \(d\). Taking trace of the second equation of (3.1) and using \(\text{div} U = 0\), \(d\) satisfies the heat equations:
\[ \partial_t d - \Delta d = 0 \quad \text{in} \ \mathbb{R}^N \text{ for } t \in (0, T), \quad d|_{t=0} = d_0 \quad \text{in} \ \mathbb{R}^N, \]  \hspace{1cm} (3.2)
where \(d_0 = \text{tr}Q_0\). By the \(L_p-L_q\) estimate for the heat semigroup we see that \(d\) satisfies
\[ \|\nabla^j d\|_{L_p(\mathbb{R}^N)} \leq C t^{-\frac{N}{q}(\frac{1}{q} - \frac{1}{p})} \|d_0\|_{L_q(\mathbb{R}^N)} \]  \hspace{1cm} (3.3)
for any \(t > 0\), where \(d_0 \in L_q(\mathbb{R}^N), j \in \mathbb{N}_0, 1 \leq q \leq p \leq \infty\).
3.2. Decay Estimates for $U$ and $Q$

We next consider the decay estimates for $U$ and $Q$. Theorem 2.5 implies that there exist operators
\[
S(t) \in \mathcal{L}(X_q(\mathbb{R}^n), W_q^2(\mathbb{R}^n)), \quad T(t) \in \mathcal{L}(X_q(\mathbb{R}^n), W_q^3(\mathbb{R}^n \setminus \mathbb{R}^{n^2}))
\] (3.4)
such that for any $(f, G) \in X_q(\mathbb{R}^n)$, $U = S(t)(f, G)$ and $Q = T(t)(f, G)$ satisfy (3.1). We shall prove the $L_p - L_q$ decay estimates of operators $S(t)$ and $T(t)$. For this purpose, we decompose the solution into low and high frequency parts. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $0 \leq \varphi(\xi) \leq 1$, $\varphi(\xi) = 1$ if $|\xi| \leq 1/3$ and $\varphi(\xi) = 0$ if $|\xi| \geq 2/3$. Let $\varphi_0$ and $\varphi_\infty$ be functions such that
\[
\varphi_0(\xi) = \varphi(\xi/A_0), \quad \varphi_\infty(\xi) = 1 - \varphi(\xi/A_0),
\]
where $A_0 \in (0, 1)$ is a sufficiently small number, which determined in Sect. 3.2.1. Moreover, we set
\[
S_n(t)(f, G) = \frac{1}{2\pi i} \sum_{j=1}^{2} \mathcal{F}^{-1} \left[ \int_{\Gamma} e^{\lambda t} \ell_j(\xi, \lambda) \varphi_n \hat{f} d\lambda \right](x) + \frac{1}{2\pi i} \sum_{j=1}^{4} \mathcal{F}^{-1} \left[ \int_{\Gamma} e^{\lambda t} m_j(\xi, \lambda) \varphi_n \text{Div} \hat{G} d\lambda \right](x),
\]
\[
T_n(t)(f, G) = \frac{1}{2\pi i} \sum_{j=1}^{2} \mathcal{F}^{-1} \left[ \int_{\Gamma} e^{\lambda t} \tilde{\ell}_j(\xi, \lambda) \varphi_n \hat{f} d\lambda \right](x) + \frac{1}{2\pi i} \sum_{j=1}^{4} \mathcal{F}^{-1} \left[ \int_{\Gamma} e^{\lambda t} \tilde{m}_j(\xi, \lambda) \varphi_n \text{Div} \hat{G} d\lambda \right](x)
\]
\[
+ \frac{1}{2\pi i} \mathcal{F}^{-1} \left[ \int_{\Gamma} e^{\lambda t} \tilde{m}_5(\xi, \lambda) \varphi_n \text{tr} \hat{Q} \hat{I} d\lambda \right](x),
\]
where $n = 0, \infty$,
\[
\ell_1(\xi, \lambda) \hat{f} = \frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \hat{f}, \quad \ell_2(\xi, \lambda) \hat{f} = -\frac{\lambda + |\xi|^2 + a}{P_2(\xi, \lambda)} \frac{\xi}{|\xi|^2} \cdot \hat{f},
\]
\[
m_1(\xi, \lambda) \text{Div} \hat{G} = -\frac{\beta \xi}{P_2(\xi, \lambda)} \xi \cdot \text{Div} \hat{G}, \quad m_2(\xi, \lambda) \text{Div} \hat{G} = \frac{\beta |\xi|^2}{P_2(\xi, \lambda)} \text{Div} \hat{G},
\]
\[
m_3(\xi, \lambda) \text{Div} \hat{G} = -\frac{a\beta}{P_2(\xi, \lambda)} \frac{\xi}{|\xi|^2} \xi \cdot \text{Div} \hat{G}, \quad m_4(\xi, \lambda) \text{Div} \hat{G} = \frac{a\beta}{P_2(\xi, \lambda)} \text{Div} \hat{G},
\]
\[
\tilde{\ell}_1(\xi, \lambda) \hat{f} = \frac{\beta}{P_2(\xi, \lambda)} (i \xi \otimes \hat{f} + \hat{f} \otimes i \xi), \quad \tilde{\ell}_2(\xi, \lambda) \hat{f} = -\frac{2\beta}{P_2(\xi, \lambda)} \frac{i \xi \otimes \xi}{|\xi|^2} \cdot \hat{f},
\]
\[
\tilde{m}_1(\xi, \lambda) \text{Div} \hat{G} = -\frac{2\beta^2 i \xi \otimes \xi}{(\lambda + |\xi|^2 + a)P_2(\xi, \lambda)} \xi \cdot \text{Div} \hat{G},
\]
\[
\tilde{m}_2(\xi, \lambda) \text{Div} \hat{G} = \frac{\beta^2 |\xi|^2 (i \xi \otimes \text{Div} \hat{G} + \text{Div} \hat{G} \otimes i \xi)}{(\lambda + |\xi|^2 + a)P_2(\xi, \lambda)},
\]
\[
\tilde{m}_3(\xi, \lambda) \text{Div} \hat{G} = -\frac{2a\beta^2 i \xi \otimes \xi}{(\lambda + |\xi|^2 + a)P_2(\xi, \lambda)} \frac{\xi}{|\xi|^2} \xi \cdot \text{Div} \hat{G},
\]
\[
\tilde{m}_4(\xi, \lambda) \text{Div} \hat{G} = \frac{a\beta^2 (i \xi \otimes \text{Div} \hat{G} + \text{Div} \hat{G} \otimes i \xi)}{(\lambda + |\xi|^2 + a)P_2(\xi, \lambda)}, \quad \tilde{m}_5(\xi, \lambda) \hat{G} = \frac{1}{\lambda + |\xi|^2 + a} \hat{G}.
\]
Here, we set the integral path $\Gamma = \Gamma^+ \cup \Gamma^-$ as follows:
\[
\Gamma^\pm = \{ \lambda \in \mathbb{C} \mid \lambda = \tilde{\lambda}_0(\sigma) + se^{\pm i(\pi - \sigma)}, s \geq 0 \to \infty \}
\] (3.5)
for some $\sigma_0 < \sigma < \pi/2$ with $\tilde{\lambda}_0(\sigma) = 2\lambda_0(\sigma)/\sin \sigma$, where $\lambda_0(\sigma)$ is the same as in Theorem 2.3. We then have the following $L_p - L_q$ decay estimates for $S(t)$ and $T(t)$.

**Theorem 3.1.** Let $S(t)$ and $T(t)$ be the solution operators of (3.1) such that $U = S(t)(f, G)$ and $Q = T(t)(f, G)$ for $(f, G) \in X_q(\mathbb{R}^n)$. Then, the following assertions hold.
(1) The operators $S(t)$ and $T(t)$ are decomposed as follows:
\[
S(t)(f, G) = S_0(t)(f, G) + S_\infty(t)(f, G), \quad T(t)(f, G) = T_0(t)(f, G) + T_\infty(t)(f, G)
\]
satisfying the following estimates for $t > 0$.
(a) Let $j, k \in \mathbb{N}_0$. Let $1 \leq q \leq 2 \leq p \leq \infty$ for $(j, k) \neq (0, 0)$ and let $1 \leq q < 2 \leq p \leq \infty$ for $(j, k) = (0, 0)$.
\[
\|\partial^j_t \nabla^i (S_0(t)(f, G), T_0(t)(f, G))\|_{W_{p, 1}^0(\mathbb{R}^N)} \leq Ct^{-\frac{\gamma}{2}}(1 - \frac{1}{p}) - \frac{1}{k} - \frac{1}{2} \| (f, G) \|_{W_{p, 1}^0(\mathbb{R}^N)}
\]  
(3.6)

for some positive constant $C$ depending on $j$, $k$, $p$, and $q$.
(b) Let $1 < p < \infty$.
\[
\|\partial_t (S_\infty(t)(f, G), T_\infty(t)(f, G))\|_{W_{p, 1}^0(\mathbb{R}^N)} \leq \| (S_\infty(t)(f, G), T_\infty(t)(f, G))\|_{W_{p, 1}^0(\mathbb{R}^N)}
\]

with some positive constants $C$ and $\gamma_{\infty}$.

(2) Let $1 < q \leq p \leq \infty$ and $(p, q) \neq (\infty, \infty)$. If $0 < t \leq 1$, $S(t)$ and $T(t)$ satisfy the following estimate:
\[
\|\nabla^j (S(t)(f, G), T(t)(f, G))\|_{W_{p, 1}^0(\mathbb{R}^N)} \leq Ct^{-\frac{\gamma}{2}}(1 - \frac{1}{p}) - \frac{1}{2} \| (f, G) \|_{W_{p, 1}^0(\mathbb{R}^N)},
\]

(3.7)

with $j = 0, 1$ and some positive constant $C$ depending on $p$ and $q$. Moreover, if $0 < t \leq 1$,
\[
\|\nabla^j (S(t)(f, G), T(t)(f, G))\|_{W_{p, 1}^0(\mathbb{R}^N)} \leq C\| (f, G)\|_{W_{p, 1}^{j+1}(\mathbb{R}^N)}, \quad (j = 0, 1, 2),
\]

(3.8)

with some positive constant $C$.

We shall prove Theorem 3.1. For Theorem 3.1 (2), by (2.8) and $0 < t \leq 1$, we see that $S(t)$ and $T(t)$ satisfy
\[
\|\nabla^j (S(t)(f, G), T(t)(f, G))\|_{W_{p, 1}^0(\mathbb{R}^N)} \leq C_\sigma e^{\tilde{\gamma}_0(\sigma)t} t^{-\frac{1}{2}} \| (f, G)\|_{W_{p, 1}^0(\mathbb{R}^N)}
\]

\[
\leq C_\sigma t^{-\frac{1}{2}} \| (f, G)\|_{W_{p, 1}^0(\mathbb{R}^N)}
\]

with $j = 0, 1, 2$. Moreover, we have
\[
\|\partial_t (S(t)(f, G), T(t)(f, G))\|_{W_{p, 1}^0(\mathbb{R}^N)} \leq C_\sigma t^{-1} \| (f, G)\|_{W_{p, 1}^0(\mathbb{R}^N)},
\]

which implies that (3.7). Furthermore, (3.8) follows from the fact that $\{e^{At}\}_{t \geq 0}$ is continuous analytic semigroup. Thus, we prove Theorem 3.1 (1).

3.2.1. Analysis of Low Frequency Parts. To prove Theorem 3.1 (1) (a), we prepare several lemmas. Recall that roots of $P_2(\xi, \lambda)$ have the following expansion formula:
\[
\lambda_+ = -(1 + \beta^2) |\xi|^2 + O(|\xi|^4), \quad \lambda_- = -(1 - \beta^2) |\xi|^2 - a + O(|\xi|^4).
\]

(3.9)

Let $\gamma_0 > 0$. We set $\tilde{\sigma}_0 = \tan^{-1}\{(|\xi|^2/8)/|\xi|^2\} = \tan^{-1} 1/8$ and
\[
\Gamma_1^\pm = \{\lambda \in \mathbb{C} | \lambda = \pm |\xi|^2 + (|\xi|^2/4)e^{\pm is}, \quad s : 0 \rightarrow \pi/2\},
\]
\[
\Gamma_2^\pm = \{\lambda \in \mathbb{C} | \lambda = \pm (|\xi|^2(1 - s) + \gamma_0 s) \pm i((|\xi|^2/4)(1 - s) + 2\gamma_0), \quad s : 0 \rightarrow 1\},
\]
\[
\Gamma_3^\pm = \{\lambda \in \mathbb{C} | \lambda = -\tilde{\gamma}_0 \pm \tilde{\sigma}_0, \quad s : 0 \rightarrow \infty\},
\]

where $\tilde{\gamma}_0 = \{(2\gamma_0/\sin \tilde{\sigma}_0) + \gamma_0\}/8 = \gamma_0(2\sqrt{65} + 1)/8$. By Cauchy’s integral theorem, $S_0(t)(f, G)$ and $T_0(t)(f, G)$ can be decomposed by
\[
S_0(t)(f, G) = \sum_{k=1}^{3} S_0^k(t)(f, G), \quad T_0(t)(f, G) = \sum_{k=1}^{3} T_0^k(t)(f, G),
\]
where

\[
S^k_0(t)(f, G) = \frac{1}{2\pi i} \sum_{j=1}^{\mathcal{F} - 1} \int_{\Gamma^+_k \cup \Gamma^-_k} e^{\lambda t} e_j(x, \lambda) \varphi_0 \hat{f} d\lambda(x) + \frac{1}{2\pi i} \sum_{j=1}^{\mathcal{F} - 1} \int_{\Gamma^+_k \cup \Gamma^-_k} e^{\lambda t} m_j(x, \lambda) \varphi_0 \hat{f} d\lambda(x),
\]

\[
T^k_0(t)(f, G) = \frac{1}{2\pi i} \sum_{j=1}^{\mathcal{F} - 1} \int_{\Gamma^+_k \cup \Gamma^-_k} e^{\lambda t} e_j(x, \lambda) \varphi_0 \hat{f} d\lambda(x) + \frac{1}{2\pi i} \sum_{j=1}^{\mathcal{F} - 1} \int_{\Gamma^+_k \cup \Gamma^-_k} e^{\lambda t} m_j(x, \lambda) \varphi_0 \hat{f} d\lambda(x)
\]

We only consider estimates on \( \Gamma^+_k (k = 1, 2, 3) \) below because the case \( \Gamma^-_k \) can be proved similarly.

**Estimates on \( \Gamma^+_1 \)**

First, we consider the case \( \Gamma^+_1 \).

**Lemma 3.2.** Let \( 1 \leq q \leq 2 \leq p \leq \infty \) and let \( f \in L_q(\mathbb{R}^N)^N \). Assume that there exist positive constants \( A_1 \in (0, 1) \) and \( C \) such that for any \( |\xi| \in (0, A_1) \)

\[
|\ell(x, \lambda)| \leq C|\xi|^{-2},
\]

Then, there exist positive constant \( A_0 \in (0, 1) \) and \( C \) such that for any \( t > 0 \)

\[
\left\| \partial^k \nabla^j \mathcal{F}^{-1} \left[ \int_{\Gamma^+_1} e^{\lambda t} \ell(x, \lambda) \varphi_0 \hat{f} d\lambda \right] \right\|_{L_p(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{j}{2} - k} \|f\|_{L_q(\mathbb{R}^N)} \tag{3.10}
\]

with \( j, k \in \mathbb{N}_0 \).

**Proof.** Choosing \( A_0 = A_1 \), by \( L_p - L_q \) estimates of heat kernel, and Parseval’s theorem, we have

\[
\left\| \partial^k \nabla^j \mathcal{F}^{-1} \left[ \int_{\Gamma^+_1} e^{\lambda t} \ell(x, \lambda) \varphi_0 \hat{f} d\lambda \right] \right\|_{L_p(\mathbb{R}^N)} \leq C \left\| \mathcal{F}^{-1} \left[ \int_0^{\pi/2} e^{-|\xi|^2 t + \frac{|\xi|^2}{4} e^{is}} \ell(x, \lambda)|\xi|^{2j+2k} e^{i\xi} \left( -1 + \frac{e^{is}}{4} \right)^k d\lambda \right] \right\|_{L_p(\mathbb{R}^N)}
\]

\[
\leq Ct^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p})} \left\| \mathcal{F}^{-1} \left[ \int_0^{\pi/2} e^{-|\xi|^2 t + \frac{|\xi|^2}{8} e^{is}} \ell(x, \lambda)|\xi|^{2j+2k} e^{i\xi} \left( -1 + \frac{e^{is}}{4} \right)^k d\lambda \right] \right\|_{L_2(\mathbb{R}^N)}
\]

\[
\leq Ct^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{j}{2} - k} \|f\|_{L_q(\mathbb{R}^N)}.
\]

**Estimates on \( \Gamma^+_2 \)**

Next, we consider the case \( \Gamma^+_2 \).

**Lemma 3.3.** Let \( 1 \leq q \leq 2 \leq p \leq \infty \) and let \( f \in L_q(\mathbb{R}^N)^N \). Assume that there exist positive constants \( A_1 \in (0, 1) \) and \( C \) such that for any \( |\xi| \in (0, A_1) \)

\[
|\ell(x, \lambda)| \leq C(|\lambda| + |\xi|^2)^{-1}.
\]

Then, there exist positive constant \( A_0 \in (0, A_1) \) and \( C \) such that for any \( t > 0 \)
First, we prove (3.11). We choose $A_0 \in (0, A_1)$ in such a way that
\begin{equation}
0 < \frac{\gamma_0 + \tilde{\gamma}_0 + |\xi|^2}{\gamma_0 - |\xi|^2} < C
\end{equation}
for any $|\xi| \in (0, A_0)$ with some positive constant $C$. Since $|\xi|$ and $\tilde{\gamma}_0$ satisfy $|\xi|^2 < 1 < (2\sqrt{65} + 1)/8 < \tilde{\gamma}_0$, we have
\begin{equation}
|\lambda + |\xi|^2| \geq C\{ |\xi|^2 + |\xi|^2(1-s) + \gamma_0 s \}.
\end{equation}
In fact,
\begin{align*}
|\lambda + |\xi|^2|^2 &= s(|\xi|^2 - \gamma_0) + t \left\{ \frac{|\xi|^2}{4} (1-s) + \tilde{\gamma}_0 s \right\}^2 \\
&= s^2(\gamma_0 - |\xi|^2)^2 + \left\{ \frac{|\xi|^2}{4} (1-s) + \gamma_0 s \right\}^2 \\
&\geq \left\{ \frac{|\xi|^2}{4} (1-s) + \gamma_0 s \right\}^2
\end{align*}
yields that
\begin{align*}
|\lambda + |\xi|^2| &\geq \frac{|\xi|^2}{4} (1-s) + \gamma_0 s \\
&= \frac{1}{2} \left\{ \frac{|\xi|^2}{4} (1-s) + \gamma_0 s \right\} + \frac{1}{2} \left\{ \frac{|\xi|^2}{4} (1-s) + \gamma_0 s \right\} \\
&\geq \frac{1}{8} |\xi|^2 + \frac{1}{8} (4 \tilde{\gamma}_0 - |\xi|^2) + \frac{1}{8} (\gamma_0 s) \\
&\geq \frac{1}{8} (|\xi|^2 + |\xi|^2(1-s) + \gamma_0 s).
\end{align*}
By (3.14), $L_p$-$L_q$ estimates of heat kernel, and Parseval’s theorem, we have
\begin{align*}
&\left\| \partial_{\xi}^{k} \mathcal{F}^{-1} \int_{\Gamma_{\zeta}} e^{\lambda t} \ell(\xi, \lambda) \phi_0 \mathcal{F} d\lambda \right\|_{L_p(R^N)} \\
&\leq C t^{-\frac{\nu}{2}(1-s) - \delta} \left\| e^{-\frac{|\xi|^2}{2} t} \int_{0}^{1} e^{-\frac{1}{2} (|\xi|^2(1-s) + \gamma_0 s) t} \big( \frac{\gamma_0 + \tilde{\gamma}_0 + |\xi|^2}{\gamma_0 - |\xi|^2} |\xi|^2(1-s) + \gamma_0 s \big)^k ds \phi_0 \mathcal{F} \right\|_{L_2(R^N)} \\
&\leq C t^{-\frac{\nu}{2}(1-s) - \delta} \left\| e^{-\frac{|\xi|^2}{2} t} \int_{0}^{1} e^{-\frac{1}{2} (|\xi|^2(1-s) + \gamma_0 s) t} (|\xi|^2(1-s) + \gamma_0 s)^{k-1+\frac{1}{2}} ds \phi_0 \right\|_{L_2(R^N)}
\end{align*}
for a sufficiently small $\delta > 0$. Here, by change of variables, we have
\begin{align*}
\int_{0}^{1} e^{-\frac{1}{2} (|\xi|^2(1-s) + \gamma_0 s) t} (|\xi|^2(1-s) + \gamma_0 s)^{k-1+\frac{1}{2}} ds &\leq C t^{-k-\frac{1}{2}} \frac{1}{\gamma_0 - |\xi|^2},
\end{align*}
which combined with (3.13) and Young’s inequality with $1 + 1/2 = 1/r + 1/q$ for $1 \leq q < 2$, we have
\[
\left\| \partial_t F^{-1} \left[ \int_{\Gamma_2^+} e^{\lambda \ell(t, \xi, \lambda)} \varphi_0 \hat{f} \, d\lambda \right] \right\|_{L_p(\mathbb{R}^N)} \\
\leq C t^{-\frac{N}{2} \left( \frac{1}{r} - \frac{1}{2} \right) - k t^\frac{3}{2} - \frac{N}{2} \left( \frac{1}{r} - \frac{1}{2} \right) t^{-\frac{3}{2}} \left\| \hat{f} \right\|_{L_q(\mathbb{R}^N)}.
\]

By a similar calculation, we can prove (3.12), but without using Young’s inequality. In fact, for $j \in \mathbb{N}$,
\[
\left\| \nabla^j F^{-1} \left[ \int_{\Gamma_2^+} e^{\lambda \ell(t, \xi, \lambda)} \varphi_0 \hat{f} \, d\lambda \right] \right\|_{L_p(\mathbb{R}^N)} \\
\leq C t^{-\frac{N}{2} \left( \frac{1}{r} - \frac{1}{2} \right) - k t^\frac{3}{2} - \frac{N}{2} \left( \frac{1}{r} - \frac{1}{2} \right) t^{-\frac{3}{2}} \left\| \hat{f} \right\|_{L_q(\mathbb{R}^N)}.
\]

\section*{Estimates on $\Gamma_3^+$}

Finally, we consider the case $\Gamma_3^+$.

\textbf{Lemma 3.4.} Let $1 \leq q \leq 2 \leq p \leq \infty$ and let $\hat{f} \in L_q(\mathbb{R}^N)^N$. Assume that there exist positive constants $A_1 \in (0, 1)$ and $C$ such that for any $|\xi| \in (0, A_1)$
\[
|\ell(\xi, \lambda)| \leq C|\lambda|^{-1}.
\]

Then, there exist positive constant $A_0 \in (0, A_1)$ and $C$ such that for any $t > 0$
\[
\left\| \partial_t^{\frac{j}{2}} \nabla^j F^{-1} \left[ \int_{\Gamma_3^+} e^{\lambda \ell(t, \xi, \lambda)} \varphi_0 \hat{f} \, d\lambda \right] \right\|_{L_p(\mathbb{R}^N)} \\
\leq C t^{-\frac{N}{2} \left( \frac{1}{r} - \frac{1}{2} \right) - \frac{j}{2} - k} \left\| \hat{f} \right\|_{L_q(\mathbb{R}^N)} \tag{3.15}
\]
with $j, k \in \mathbb{N}_0$. 
Proof. We choose $A_0 \in (0, A_1)$ in such a way that $2|\xi|^2 \leq \gamma_0/2$ for any $|\xi| \in (0, A_0)$. By $L_p - L_q$ estimates of heat kernel, and Parseval’s theorem, we have

$$ \left\| \partial_t^k \nabla^j F^{-1} \left[ \int_{T_{\xi}} e^{\lambda t} \ell(\xi, \lambda) \varphi_0 \hat{f} \, d\lambda \right] \right\|_{L_q(\mathbb{R}^N)} $$

$$ \leq C t^{-\frac{j}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \left\| e^{-|\xi|^2 t} \int_0^\infty \left| e^{-\left( \frac{3}{2} + s \cos \tilde{\sigma}_0 \right) t} \frac{1}{|\lambda|^{1-k}} \, ds \right| |\xi|^j \varphi_0 \hat{f} \right\|_{L_2(\mathbb{R}^N)} $$

$$ \leq C t^{-\frac{j}{2} \left( \frac{1}{p} - \frac{1}{q} \right) - \frac{j}{2}} \int_0^\infty \frac{e^{-\left( \frac{3}{2} + s \cos \tilde{\sigma}_0 \right) t}}{|\lambda|^{1-k}} \, ds \|f\|_{L_q(\mathbb{R}^N)} $$

$$ \leq C \begin{cases} (\|\log t\| + 1) e^{\frac{3}{2} t} \|f\|_{L_q(\mathbb{R}^N)} & (k = 0), \\
-t^k e^{-\gamma_0 t} \|f\|_{L_q(\mathbb{R}^N)} & (k > 0). \end{cases} $$

Here, we used the following estimate

$$ \int_0^\infty \frac{e^{(\Re(\lambda)) t}}{|\lambda|^{1-k}} \, ds \leq C \begin{cases} (\|\log t\| + 1) e^{-\gamma_0 t} & (k = 0), \\
t^{-k} e^{-\gamma_0 t} & (k > 0). \end{cases} $$

Thus, we have (3.15).

Proof of Theorem 3.1 (1) (a). To obtain the estimates of $S_0^1(t)$, we first consider estimate of $P_2(\xi, \lambda)$ for $\lambda \in \Gamma^+_1$. By (3.9), there exists positive constant $A_1 \in (0, 1)$ and $C$ such that for any $\lambda \in \Gamma^+_1$, $|\xi| < A_1$, and $s \in [0, \pi/2]$

$$ |\lambda - \lambda_+| = |\beta^2|\xi|^2 + (|\xi|^2/4)e^{is} + O(|\xi|^4) \geq C|\xi|^2, $$

$$ |\lambda - \lambda_-| = |\beta^2|\xi|^2 + a + (|\xi|^2/4)e^{is} + O(|\xi|^4) \geq C(|\xi|^2 + a), $$

which yield

$$ |P_2(\xi, \lambda)| \leq C(|\xi|^4 + a|\xi|^2)^{-1}. $$

Using $|\lambda + |\xi|^2 + a| \leq C(|\xi|^2 + a)$ and (3.17), we have

$$ |\ell_j(\xi, \lambda)| \leq C|\xi|^{-2}, \quad |m_k(\xi, \lambda)| \leq C|\xi|^{-2} $$

for $\lambda \in \Gamma^+_1$, $j = 1, 2$, and $k = 1, \ldots, 4$.

We next consider the estimate of $S_0^2(t)$ and $S_0^3(t)$. Noting that $\Gamma^+_2, \Gamma^+_3 \subset \Sigma_{\sigma_0,0}$, by (2.21) and (2.22) for the low frequency parts, we know that

$$ |m_j(\xi, \lambda)| \leq C(|\lambda| + |\xi|^2)^{-1} \quad \text{for } \lambda \in \Gamma^+_2, $$

$$ |m_j(\xi, \lambda)| \leq C|\lambda|^{-1} \quad \text{for } \lambda \in \Gamma^+_3 $$

with $j = 1, 2$. Moreover, by the estimate

$$ |P_2(\xi, \lambda)| \leq C(|\lambda| + |\xi|^2)^{-1}(|\lambda| + |\xi|^2 + a)^{-1}, $$

which follows from (2.21) and the expansion formula (2.23), we have

$$ |m_k(\xi, \lambda)| \leq C|\lambda|^{-1}, \quad |\ell_j(\xi, \lambda)| \leq C(|\lambda| + |\xi|^2)^{-1} \quad \text{for } \lambda \in \Gamma^+_2, $$

$$ |m_k(\xi, \lambda)| \leq C|\lambda|^{-1}, \quad |\ell_j(\xi, \lambda)| \leq C|\lambda|^{-1} \quad \text{for } \lambda \in \Gamma^+_3 $$

with $j = 1, 2$, $k = 3, 4$.

By (3.18), (3.19), and (3.20), we can apply Lemma 3.2, Lemma 3.3, and Lemma 3.4, which gives (3.6) for $S_0(t)$.

Similarly, we have the estimates of $\nabla^j T_0(t)$ and $\nabla^{j+1} T_0(t)$ for $j \in \mathbb{N}_0$. In fact, $\tilde{\ell}_k(\xi, \lambda), \tilde{m}_k(\xi, \lambda), i\xi \tilde{\ell}_k(\xi, \lambda)$, and $i\xi \tilde{m}_k(\xi, \lambda)$ satisfy the assumption of Lemma 3.2, Lemma 3.3, and Lemma 3.4 for $\ell = 1, \ldots, N, k = 1, 2$, and $k' = 1, \ldots, 5$. Using (3.3), we have (3.6) for $T_0(t)$, which completes the proof of Theorem 3.1 (1) (a).
3.2.2. Analysis of High Frequency Parts. We shall prove Theorem 3.1 (1) (b). Recall that roots of $P_2(\xi, \lambda)$ have the following expansion formula:

$$\lambda_\pm = (-1 \pm i|\beta|)|\xi|^2 + O(1).$$

(3.21)

Let $\lambda_0 > 0$. Set $\gamma_\infty$ as follows:

$$0 < \gamma_\infty \leq 2^{-1}(A_0/6)^2,$$

where $A_0$ is the same as in Sect. 3.2.1. Moreover, we set integral paths:

$$\Gamma_+^\pm = \{\lambda \in \mathbb{C} | \lambda = -\gamma_\infty \pm is_0 + s e^{\pm i(\pi - \sigma_0)}, \ s : 0 \to \infty\},$$

$$\Gamma_{5}^\pm = \{\lambda \in \mathbb{C} | \lambda = -\gamma_\infty \pm is, \ s : 0 \to \gamma_\infty\},$$

where $\gamma_\infty = \{(2\lambda_0/\sin \sigma_0) + \gamma_\infty\} \tan \sigma_0$ and $\sigma_0$ is defined in (2.4). By Cauchy’s integral theorem, $S_\infty(t)(f, G)$ and $T_\infty(t)(f, G)$ can be decomposed by

$$S_\infty(t)(f, G) = \sum_{k=4}^{5} S_k^\infty(t)(f, G), \quad T_\infty(t)(f, G) = \sum_{k=4}^{5} T_k^\infty(t)(f, G),$$

where

$$S_k^\infty(t)(f, G) = \frac{1}{2\pi i} \int_{\Gamma_+^k \cup \Gamma_5^k} e^{\lambda t} U_\infty(\lambda, x) \, d\lambda,$$

$$U_\infty(\lambda, x) = F^{-1} \left[ \sum_{j=1}^{2} \ell_j(\xi, \lambda) \varphi_\infty \hat{f} \right] (x) + F^{-1} \left[ \sum_{j=1}^{4} m_j(\xi, \lambda) \varphi_\infty \text{Div} G \right] (x),$$

$$T_k^\infty(t)(f, G) = \frac{1}{2\pi i} \int_{\Gamma_+^k \cup \Gamma_5^k} e^{\lambda t} Q_\infty(\lambda, x) \, d\lambda,$$

$$Q_\infty(\lambda, x) = F^{-1} \left[ \sum_{j=1}^{2} \tilde{\ell}_j(\xi, \lambda) \varphi_\infty \hat{f} \right] (x) + F^{-1} \left[ \sum_{j=1}^{4} \tilde{m}_j(\xi, \lambda) \varphi_\infty \text{Div} G \right] (x),$$

$$+ F^{-1} \left[ \tilde{m}_5(\xi, \lambda) \varphi_\infty \hat{G} \right] (x) + \frac{a}{N} F^{-1} \left[ \tilde{m}_6(\xi, \lambda) \varphi_\infty \hat{G} \right] (x),$$

$$\tilde{m}_6(\xi, \lambda) \hat{G} = \frac{1}{P_1(\xi, \lambda)} \text{tr} \hat{G}.$$ We only consider estimates on $\Gamma_+^k$ ($k = 4, 5$) below because the case $\Gamma_5^-$ can be proved similarly.

Estimates on $\Gamma_+^k$. First, we consider the case $\Gamma_+^4$. Let $1 < q < \infty$. Noting that $\Gamma_+^k \subset \Sigma_{\sigma_0, \lambda_0(\sigma_0)}$, by (2.5) (2.6) in Theorem 2.3, there exists a positive constant $C$ such that for any $\lambda \in \Gamma_+^4$ and $(f, G) \in W^{q,1}_q(\mathbb{R}^N)^N$,

$$\|(\lambda U_\infty, \lambda Q_\infty)\|_{W^{q,1}_q(\mathbb{R}^N)} + \|(|\lambda|^{1/2} \nabla U_\infty, |\lambda|^{1/2} \nabla Q_\infty)\|_{W^{q,1}_q(\mathbb{R}^N)} + \|(|\nabla^2 U_\infty, \nabla^2 Q_\infty)\|_{W^{q,1}_q(\mathbb{R}^N)}$$

$$\leq C \|(f, G)\|_{W^{q,1}_q(\mathbb{R}^N)}.$$ (3.22)

By (3.22) and (3.16), there exists a positive constant $C$ such that for any $t > 0$

$$\||S_\infty^4(t)(f, G), T_\infty^4(t)(f, G)\|_{W^{q,1}_q(\mathbb{R}^N)} \leq C(\|\log t\| + 1)e^{-\gamma_\infty t}\|(f, G)\|_{W^{q,1}_q(\mathbb{R}^N)},$$

$$\||\nabla S_\infty^4(t)(f, G), \nabla T_\infty^4(t)(f, G)\|_{W^{q,1}_q(\mathbb{R}^N)} \leq C t^{-\frac{1}{2}} e^{-\gamma_\infty t}\|(f, G)\|_{W^{q,1}_q(\mathbb{R}^N)},$$

$$\||\partial_t, \nabla^2 S_\infty^4(t)(f, G), (\partial_t, \nabla^2 T_\infty^4(t)(f, G))\|_{W^{q,1}_q(\mathbb{R}^N)} \leq C t^{-\frac{1}{2}} e^{-\gamma_\infty t}\|(f, G)\|_{W^{q,1}_q(\mathbb{R}^N)},$$

(3.23)

Estimates on $\Gamma_5^+$. Next, we consider the case $\Gamma_5^+$. Let $1 < q < \infty$. We shall estimate $U_\infty$ by Fourier multiplier theorem. We consider estimates of $\lambda + |\xi|^2, P_1(\xi, \lambda)$, and $P_2(\xi, \lambda)$ for $\lambda \in \Gamma_5^+$ and $|\xi| \geq A_0/6$.
Lemma 3.5. Let $\alpha \in \mathbb{N}^N_0$. There exists a positive constant $C$ such that for any $\lambda \in \Gamma_5^+$ and $|\xi| \geq A_0/6$,

$$
|\partial_\xi^\alpha (\lambda L(\xi,\lambda), L(\xi,\lambda), i\xi_a L(\xi,\lambda), i\xi_a i\xi_b L(\xi,\lambda))| \leq C|\xi|^{-|\alpha|},
$$

$$
|\partial_\xi^\alpha (i\xi_a \lambda \tilde{\ell}_j (\xi,\lambda), i\xi_a i\xi_b i\xi_c \tilde{\ell}_j (\xi,\lambda))| \leq C|\xi|^{-|\alpha|},
$$

$$
|\partial_\xi^\alpha (i\xi_a \lambda \tilde{m}_k (\xi,\lambda), i\xi_a i\xi_b i\xi_c \tilde{m}_k (\xi,\lambda), i\xi_a \lambda \tilde{m}_6 (\xi,\lambda), i\xi_a i\xi_b i\xi_c \tilde{m}_6 (\xi,\lambda))| \leq C|\xi|^{-|\alpha|},
$$

$$
|\partial_\xi^\alpha (i\xi_a \lambda \tilde{m}_5 (\xi,\lambda), i\xi_a i\xi_b i\xi_c \tilde{m}_5 (\xi,\lambda))| \leq C|\xi|^{-|\alpha|},
$$

for any $\lambda \in \Gamma_5^+$ and $|\xi| \geq A_0/6$. Applying Fourier multiplier theorem, for any $(f, G) \in W^{0,1}_q(\mathbb{R}^N)$, we have

$$
\|\langle \lambda U, \lambda Q \rangle \|_{W^{0,1}_q(\mathbb{R}^N)} + \|U, Q \|_{W^{2,3}_q(\mathbb{R}^N)} \leq C\|f, G\|_{W^{0,1}_q(\mathbb{R}^N)},
$$

so that

$$
\|\partial_t (S^5(\tau)(t,f,G), T^5(\tau)(t,f,G))\|_{W^{0,1}_q(\mathbb{R}^N)} + \|S^5(\tau)(t,f,G), T^5(\tau)(t,f,G)\|_{W^{2,3}_q(\mathbb{R}^N)}
$$

$$
\leq C e^{-\gamma \tau} \|f, G\|_{W^{0,1}_q(\mathbb{R}^N)}
$$

for any $t > 0$ with some constant $C$, which combined with (3.23), we obtain Theorem 3.1 (1) (b).

4. A Proof of Theorem 1.1

We prove Theorem 1.1 by the Banach fixed point argument. Let $p$, $q_1$, and $q_2$ be exponents given in Theorem 1.1. Let $\epsilon$ be a small positive number and let $\mathcal{N}(U, Q)$ be the norm defined in (1.4). We define the underlying space $\mathcal{I}_{T, \epsilon}$ as

$$
\mathcal{I}_{T, \epsilon} = \{ (U, Q) \in X_{p,q_1,T} \cap X_{p,q_2,T} \mid \|U, Q\|_{t=0} = (U_0, Q_0), \quad \mathcal{N}(U, Q)(T) \leq \epsilon, \quad \sup_{0 \leq t < T} \|Q(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq 1 \},
$$

(4.1)
Given \((U, Q) \in I_{T, \epsilon}\), let \((v, P)\) be a solution to the equation:

\[
\begin{cases}
\partial_t v - \Delta v + \nabla p + \beta \text{Div} \left( \Delta P - a \left( P - \frac{1}{N} \text{tr} P \right) \right) = f(U, Q) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\text{div} v = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\partial_t P - \beta \text{D}(v) - \Delta P + a \left( P - \frac{1}{N} \text{tr} P \right) = G(U, Q) & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
(v, P) = (U_0, Q_0) & \text{in } \mathbb{R}^N.
\end{cases}
\]

We shall prove the following inequality in several steps:

\[\mathcal{N}(v, P)(T) \leq C\epsilon^2.\]  

To prove (4.3), we divide \((v, P)\) into two parts: \(v = v_1 + v_2\) and \(P = P_1 + P_2\), where \((v_1, P_1)\) satisfies time shifted equations:

\[
\begin{cases}
\partial_t v_1 + \lambda_1 v_1 - \Delta v_1 + \nabla p + \beta \text{Div} \left( \Delta P_1 - a \left( P_1 - \frac{1}{N} \text{tr} P_1 \right) \right) = f(U, Q), & \text{div} v_1 = 0 \text{ in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\partial_t P_1 + \lambda_1 P_1 - \beta \text{D}(v_1) - \Delta P_1 + a \left( P_1 - \frac{1}{N} \text{tr} P_1 \right) = G(U, Q) \text{ in } \mathbb{R}^N \text{ for } t \in (0, T), \\
(v_1, P_1)|_{t=0} = (0, 0)
\end{cases}
\]

and \((v_2, P_2)\) satisfies compensation equations:

\[
\begin{cases}
\partial_t v_2 - P \Delta v_2 + \beta PD\text{iv} \left( \Delta P_2 - a \left( P_2 - \frac{1}{N} \text{tr} P_2 \right) \right) = \lambda_1 v_1 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\text{div} v_2 = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\partial_t P_2 - \beta \text{D}(v_2) - \Delta P_2 + a \left( P_2 - \frac{1}{N} \text{tr} P_2 \right) = \lambda_1 P_1 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
(v_2, P_2)|_{t=0} = (0, 0)
\end{cases}
\]

where \(P\) is solenoidal projection.

### 4.1. Analysis of Time Shifted Equations

We consider the following linearized problem for (4.4):

\[
\begin{cases}
\partial_t U + \lambda_1 U - \Delta U + \nabla p + \beta \text{Div} \left( \Delta Q - a \left( Q - \frac{1}{N} \text{tr} Q \right) \right) = f & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\text{div} U = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\partial_t Q + \lambda_1 Q - \beta \text{D}(U) - \Delta Q + a \left( Q - \frac{1}{N} \text{tr} Q \right) = G & \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
(U, Q)|_{t=0} = (0, 0)
\end{cases}
\]

By Theorem 2.3, we have the following theorem for (4.6).

**Theorem 4.1.** Let \(1 < p, q < \infty\). Let \(b \geq 0\). Then, there exists a constant \(\lambda_1 \geq 1\) such that the following assertion holds: For any \((f, G) \in L_p((0, T), W^{0,1}_q(\mathbb{R}^N))\), problem (4.6) admits unique solutions \((U, Q) \in X_{p,q,T}\) possessing the estimate

\[
\| < t >^b \partial_t(U, Q) \|_{L_p((0, T), W^{0,1}_q(\mathbb{R}^N))} + \| < t >^b (U, Q) \|_{L_p((0, T), W^{2,3}_q(\mathbb{R}^N))} \leq C \| < t >^b (f, G) \|_{L_p((0, T), W^{0,1}_q(\mathbb{R}^N))}.
\]  

(4.7)
Proof. Let \( f_0 \) and \( G_0 \) be the zero extension of \( f \) and \( G \) outside of \((0, T)\). Applying Laplace transform to equations (4.6) replaced \( \gamma \triangleright 34 \) yields that
\[
\begin{align*}
(\lambda + \lambda_1)\mathcal{L}[U] - \Delta \mathcal{L}[U] + \nabla \mathcal{L}[p] + \beta \text{Div} \left( \Delta \mathcal{L}[Q] - a \left( \mathcal{L}[Q] - \frac{1}{N} \text{tr} \mathcal{L}[Q] I \right) \right) &= \mathcal{L}[f_0] \quad \text{in } \mathbb{R}^N, \\
\text{div } \mathcal{L}[U] &= 0 \quad \text{in } \mathbb{R}^N, \\
(\lambda + \lambda_1)\mathcal{L}[Q] - \beta D(\mathcal{L}[U]) - \Delta \mathcal{L}[Q] + a \left( \mathcal{L}[Q] - \frac{1}{N} \text{tr} \mathcal{L}[Q] I \right) &= \mathcal{L}[G_0] \quad \text{in } \mathbb{R}^N.
\end{align*}
\]
Let \( \mathcal{A}(\lambda) \) and \( \mathcal{B}(\lambda) \) be \( \mathcal{R} \)-bounded solution operators given in Theorem 2.3 and using these operators we have
\[
\begin{align*}
U &= \mathcal{L}^{-1} [\mathcal{A}(\lambda + \lambda_1) (\mathcal{L}[f_0](\lambda), \mathcal{L}[G_0](\lambda))], \\
Q &= \mathcal{L}^{-1} [\mathcal{B}(\lambda + \lambda_1) (\mathcal{L}[f_0](\lambda), \mathcal{L}[G_0](\lambda))].
\end{align*}
\]
Here, choosing \( \lambda_1 > 0 \) so large that \( \lambda + \lambda_1 \in \Sigma_{\sigma, \lambda_0} \) for any \( \lambda = i\tau \in i\mathbb{R} \) and applying the Weis operator valued Fourier multiplier theorem [27], we have
\[
\begin{align*}
\| \partial_t (U, Q) \|_{L_p(\mathbb{R}, W^{0,1}_q(\mathbb{R}^N))} + \| (U, Q) \|_{L_p(\mathbb{R}, W^{2,3}_q(\mathbb{R}^N))} &\leq C \| (f_0, G_0) \|_{L_p(\mathbb{R}, W^{0,1}_q(\mathbb{R}^N))} = C \| (f, G) \|_{L_p((0,T), W^{0,1}_q(\mathbb{R}^N))}.
\end{align*}
\]
Noting that \( |(\tau \partial_{\tau})^\ell \gamma/(\lambda + \lambda_1)| \leq C_\ell \) for any \( \lambda = \gamma + i\tau \in \mathbb{R}_+ + i\mathbb{R} \) with some constant \( C_\ell \) depending on \( \ell = 0, 1, 2, \ldots \), and applying the Weis operator valued Fourier multiplier theorem to
\[
\begin{align*}
\gamma U &= \mathcal{L}^{-1} [\{\gamma/(\lambda + \lambda_1)\} (\lambda + \lambda_1) \mathcal{A}(\lambda + \lambda_1) (\mathcal{L}[f_0], \mathcal{L}[G_0])], \\
\gamma Q &= \mathcal{L}^{-1} [\{\gamma/(\lambda + \lambda_1)\} (\lambda + \lambda_1) \mathcal{B}(\lambda + \lambda_1) (\mathcal{L}[f_0], \mathcal{L}[G_0])].
\end{align*}
\]
yields that
\[
\gamma \| e^{-\gamma t} (U, Q) \|_{L_p(\mathbb{R}, W^{0,1}_q(\mathbb{R}^N))} \leq C \| e^{-\gamma t} (f_0, G_0) \|_{L_p(\mathbb{R}, W^{0,1}_q(\mathbb{R}^N))} = C \| (f, G) \|_{L_p((0,T), W^{0,1}_q(\mathbb{R}^N))}
\]
for any \( \gamma > 0 \). Since \( \mathcal{L}[(f_0, G_0)](\lambda) \) is holomorphic in \( \mathbb{R}_+ + i\mathbb{R} \) as follows from \( (f_0, G_0) = (0, O) \) for \( t < 0 \), by Cauchy’s theorem in theory of one complex variable, \( U \) and \( Q \) are independent of choice of \( \gamma > 0 \). In particular, we have
\[
\gamma \| (U, Q) \|_{L_p((-\infty, 0), W^{0,1}_q(\mathbb{R}^N))} \leq \gamma \| e^{-\gamma t} (U, Q) \|_{L_p(\mathbb{R}, W^{0,1}_q(\mathbb{R}^N))} \leq C \| (f, G) \|_{L_p((0,T), W^{0,1}_q(\mathbb{R}^N))}
\]
for any \( \gamma > 0 \), where \( C \) is a constant independent of \( \gamma \). Thus, letting \( \gamma \to \infty \) yields that \( (U, Q) \) vanishes for \( t < 0 \). But, \( (U, Q) \in C^0(\mathbb{R}, B^{1/2}_{q,p}(\mathbb{R}^N)) \times B^{1/2}_{q,p}(\mathbb{R}^N; \mathbb{R}^N) \) as follows from real interpolation theorem, and so \( (U, Q)|_{t=0} = (0, O) \).

For \( b \in (0, 1] \), setting \( t > b \) \( U = v \) and \( t > b \) \( Q = S \), we have
\[
\begin{align*}
\begin{cases}
\partial_t v + \lambda_1 v - \Delta v + \nabla(<t > b) p + \beta \text{Div} \left( \Delta S - a \left( S - \frac{1}{N} \text{tr} S I \right) \right) = <t > b f + (\partial_t < t > b) U, \\
\text{div } v = 0 \quad \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
\partial_t S + \lambda_1 S - \beta D(v) - \Delta S + a \left( S - \frac{1}{N} \text{tr} S I \right) = <t > b G + (\partial_t < t > b) Q \quad \text{in } \mathbb{R}^N \text{ for } t \in (0, T), \\
(v, S)|_{t=0} = (0, O). \quad \text{in } \mathbb{R}^N.
\end{cases}
\end{align*}
\]
Noting that \( |\partial_t (< t > b)| \leq 1 \) and applying the estimate for \( (U, Q) \) yields that
\[
\begin{align*}
\| < t > b \partial_t (v, S) \|_{L_p((0,T), W^{0,1}_q(\mathbb{R}^N))} + \| < t > b (v, S) \|_{L_p((0,T), W^{2,3}_q(\mathbb{R}^N))} &\leq C \| < t > b (f, G) \|_{L_p((0,T), W^{0,1}_q(\mathbb{R}^N))}.
\end{align*}
\]
If \( b > 1 \), the repeated use of the argument above yield estimates (4.7) for any \( b > 0 \), which completes the proof of Theorem 4.1. \[
\square
\]
We now estimate nonlinear terms \( f(U, Q) \) and \( G(f, Q) \) by using sup\(_{0<t<T}\) \( \|Q(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq 1 \) in (4.1) and Sobolev’s embedding theorem provided by \( q_2 > N \). For notational simplicity, we write
\[
\|f\|_{L^p((0,T),X(\mathbb{R}^N))} = \|f\|_{L^p(X)}, \quad \|t > b \|f\|_{L^p((0,T),X(\mathbb{R}^N))} = \|f\|_{L^p,b(X)},
\]
where \( X = L_q \) or \( X = \dot{W}^\ell_q \). Noting that \( b = N/(2(2 + \sigma)) \) if \( p = 2 + \sigma \) and \( b = (N - \sigma)/(2(2 + \sigma)) \) if \( p = 2 \), we see that \( 1 - bp < 0 \). Thus, we have the following estimates:
\[
\|f(U, Q)\|_{L^p,b(L_q)} \leq C(\|U\|_{L^p(L_q)} \|\nabla U\|_{L^p,b(W^1_{q_2})} + \|Q\|_{L^p(L_q)} \|\nabla Q\|_{L^p,b(W^2_{q_2})} + \|Q\|_{L^p(L_q)} \|\nabla^3 Q\|_{L^p,b(L_q)})
\]
where \( q = q_1 \) and \( q_2 \). Therefore, by (4.7) and (4.8), we have
\[
\sum_{q=q_1/2,q_2} \|< t > b \partial_t (v_1, P_1)\|_{L^p((0,T),W^1_{q_1}(\mathbb{R}^N))} + \|< t > b (v_1, P_1)\|_{L^p((0,T),W^2_{q_2}(\mathbb{R}^N))} \leq CN(U, Q)(T)^2.
\]

4.2. Analysis of Compensation Equations

In this subsection, we consider Problem (4.5). To get estimates of \( (v_2, P_2) \), we use the following estimates for \( \{e^{At}\}_{t\geq0} \) associated with (3.1). Recalling Theorem 3.1, we know that estimates for \( (U, Q) = e^{At}(f, G) = (S(t)(f, G), T(t)(f, G)) \) as follows:
\[
\|\nabla^j e^{At}(f, G)\|_{W^0_{p,j}(\mathbb{R}^N)} \leq Ct^{-\frac{n}{2}(\frac{1}{2} - \frac{j}{p}) - \frac{j}{2}} \|(f, G)\|_{W^0_{q_1}(\mathbb{R}^N)} + \|(f, G)\|_{W^0_{q_2}(\mathbb{R}^N)},
\]
\[
\|\partial_t e^{At}(f, G)\|_{W^0_{p,1}(\mathbb{R}^N)} \leq Ct^{-\frac{n}{2}(\frac{1}{2} - \frac{1}{p}) - 1} \|(f, G)\|_{W^0_{q_1}(\mathbb{R}^N)} + \|(f, G)\|_{W^0_{q_2}(\mathbb{R}^N)}
\]
for \( t \geq 1, 1 < q < 2 \leq p < \infty, j = 0, 1, 2 \). Moreover,
\[
\|e^{At}(f, G)\|_{W^2_{p,3}(\mathbb{R}^N)} + \|\partial_t e^{At}(f, G)\|_{W^1_{p,1}(\mathbb{R}^N)} \leq C\|(f, G)\|_{W^2_{q_1}(\mathbb{R}^N)}
\]
for \( 0 < t < 2 \). Applying Duhamel’s principle to (4.5) furnishes that
\[
(v_2, P_2) = e^{At}(U_0, Q_0) + \lambda_1 \int_0^t e^{A(t-s)}(v_1, P_1)(\cdot, s) \, ds.
\]
Let
\[
[(v_1, P_1)(\cdot, s)] = \|(v_1, P_1)(\cdot, s)\|_{W^0_{q_1,2}(\mathbb{R}^N)} + \sum_{q=q_1, q_2} \|(v_1, P_1)(\cdot, s)\|_{W^2_{q_1,3}(\mathbb{R}^N)}.
\]
Setting
\[
\tilde{N}(v_1, P_1)(T) = \left( \int_0^T (\int_0^t \|(v_1, P_1)(\cdot, t)\|)^p \, dt \right)^{1/p}
\]
and using (4.9), we have
\[ \tilde{N}(v_1, P_1)(T) \leq Ce^2. \]  
(4.13)

In what follows, we estimate \( N(v_2, P_2) \) with the help of \( \tilde{N}(v_1, P_1) \).

Using (4.10) to obtain decay estimates of \( e^{\lambda t}(U_0, Q_0) \) for \( t > 1 \) and Theorem 2.5 to estimate \( e^{\lambda t}(U_0, Q_0) \) for \( 0 < t < 1 \) we have
\[ N(e^{\lambda t}(U_0, Q_0)) \leq CT \]  
(4.14)

where we have set
\[ I = \|(U_0, Q_0)\|_{W^{0,1}_{q_1/2}(\mathbb{R}^N)} + \sum_{q=q_1,q_2} \|(U_0, Q_0)\|_{B^{2(1-1/p)}_{2/p}(\mathbb{R}^N) \times B^{1+2(1-1/p)}_{2/p}(\mathbb{R}^N)}. \]

Set \( (\tilde{v}_2, \tilde{P}_2) = \int_0^t e^{\lambda(t-s)}(v_1, P_1)(\cdot, s) \, ds \) and our main task is to estimate \( (\tilde{v}_2, \tilde{P}_2) \).

4.2.1. Estimates of Spatial Derivatives in \( \mathbb{L}_p \). First, we consider the case \( 2 \leq t \leq T \). Let \( (v_3, P_3) = (\nabla^1 v_2, \nabla^2 \nabla P_2) \) when \( q = q_1 \) and \( (v_3, P_3) = (\nabla^2 v_2, \nabla^3 P_2) \) when \( q = q_2 \). Here, \( \nabla^m f = (\partial_x^m f \mid |\alpha| \leq m). \)

\[ \|(v_3, P_3)(\cdot, t)\|_{L_q(\mathbb{R}^N)} \leq C \left( \int_0^{t/2} + \int_{t/2}^{t-1} + \int_t^{t-1} \|(\nabla^1 \nabla, \nabla^2 \nabla) \) or \( (\nabla^3) e^{\lambda(t-s)}(v_1, P_1)(\cdot, s)\|_{L_q(\mathbb{R}^N)} ds \right) \]
\[ =: I_q(t) + II_q(t) + III_q(t). \]

We shall consider estimates of \( I_q(t) \), \( II_q(t) \), and \( III_q(t) \) by (4.10). Setting \( \ell = N/(2(2+\sigma)) + 1/2 \), we see that all the decay rates used below, which are obtained by (4.10), are greater than or equal to \( \ell \). In fact, by (1.5) and (4.10) with \( (p, q) = (q_1, q_1/2), (q_2, q_1/2) \), we have the following decay rates:
\[ \frac{N}{2} \left( \frac{2}{q_1} - \frac{1}{q_1} \right) + \frac{j}{2} = \frac{N}{2(2+\sigma)} + \frac{j}{2} \geq \ell \quad (j = 1, 2), \]
\[ \frac{N}{2} \left( \frac{2}{q_1} - \frac{1}{q_2} \right) + \frac{j}{2} \geq \frac{2N}{2(2+\sigma)} - \frac{N - (2 + \sigma)}{2(2 + \sigma)} + \frac{j}{2} \geq \ell \quad (j = 0, 1, 2) \]

Using (4.10) with \( (p, q) = (q, q_1/2) \), we have
\[ I_q(t) \leq C \int_0^{t/2} (\cdot - s)^{-\ell} ([v_1, P_1](\cdot, s)] ds \]
\[ \leq C(t/2)^{-\ell} \left( \int_0^\infty < s >^{-p' b} \, ds \right)^{1/p'} \left( \int_0^T < s >^b \left[ [v_1, P_1](\cdot, s)] \right]^p \, ds \right)^{1/p} \]
\[ \leq C t^{-\ell} N(v_1, P_1)(T). \]

Noting that \( b = (N - \sigma)/(2(2 + \sigma)) \) when \( p = 2 \) and \( b = N/(2(2 + \sigma)) \) when \( p = 2 + \sigma \), we see that \( 1 - (\ell - b)p < 0 \) for \( p = 2 \) and \( p = 2 + \sigma \). Thus, we have
\[ \int_2^T < s >^b I_q(t)^p \, dt \leq C \int_2^T < t >^{-(\ell-b)p} \, dt \tilde{N}(v_1, P_1)(T)^p \]
\[ \leq C \tilde{N}(v_1, P_1)(T)^p. \]

(4.15)

Using \( < t >^b \leq C < s >^b \) for \( t/2 < s < t - 1 \) and Hölder’s inequality, we have
\[ < t >^b II_q(t) \]
\[ \leq C \left( \int_{t/2}^{t-1} (\cdot - s)^{-\ell} ds \right)^{1/p'} \left( \int_{t/2}^{t-1} (\cdot - s)^{-\ell} \left[ [v_1, P_1](\cdot, s)] \right]^p ds \right)^{1/p} \].
By Fubini’s theorem, we have
\[
\int_2^T (t > b) II_q(t)^P dt \leq C \int_1^T \int_{s+1}^T (t-s)^{-\ell} dt (s > b) \left[\left\| (v_1, p_1) (\cdot, s) \right\|_{W_{2,q}^2(\mathbb{R}^N)}^P \right] ds
\]
\[
\leq C \tilde{N}(v_1, p_1)(T)^P.
\]  
(4.16)

By (4.11), we have
\[
III_q(t) \leq C \int_{t-1}^t \left\| (v_1, p_1)(\cdot, s) \right\|_{W_{2,q}^2(\mathbb{R}^N)} ds \leq C \int_{t-1}^t \left[\left\| (v_1, p_1)(\cdot, s) \right\|_{W_{2,q}^2(\mathbb{R}^N)}^P \right] ds.
\]

Employing the same method as in the estimate of II_q(t), we have
\[
\int_2^T (t > b) III_q(t)^P dt \leq C \tilde{N}(v_1, p_1)(T)^P.
\]  
(4.17)

Combining (4.15), (4.16), and (4.17), we have
\[
\int_2^T (t > b) \left\| (v_3, p_3)(\cdot, t) \right\|_{L_q(\mathbb{R}^N)}^P dt \leq C \tilde{N}(v_1, p_1)(T)^P.
\]  
(4.18)

Next, we consider the case 0 < t < min(2, T). Employing the same method as in the estimate of III_q(t), we have
\[
\int_0^{\min(T, 2)} (t > b) \left\| (v_3, p_3)(\cdot, t) \right\|_{L_q(\mathbb{R}^N)}^P dt \leq C \tilde{N}(v_1, p_1)(T)^P,
\]

which combined (4.18), we have
\[
\int_0^T (t > b) \left\| (v_3, p_3)(\cdot, t) \right\|_{L_q(\mathbb{R}^N)}^P dt \leq C \tilde{N}(v_1, p_1)(T)^P,
\]

that is,
\[
\| < t >^b \nabla (v_2, p_2) \|_{L_p(0,T), W_{p,1}^2(\mathbb{R}^N)} + \| < t >^b (v_2, p_2) \|_{L_p((0,T), W_{p,2}^2(\mathbb{R}^N))} \leq C \tilde{N}(v_1, p_1)(T). \]  
(4.19)

\subsection*{4.2.2. Estimates of Time Derivatives in $L_p$–$L_q$}  
Let $q = q_1, q_2$. By (4.12), we have
\[
(\partial_t \tilde{v}_2, \partial_t \tilde{p}_2) = -\lambda_1 (v_1, p_1) - \lambda_3 \int_0^T \partial_t e^{At}(v_1, p_1)(\cdot, s) ds.
\]

Setting \((v_4, p_4) = \int_0^T \partial_t e^{At}(v_1, p_1)(\cdot, s) ds\) and employing the same calculation as in the proof of (4.19), we have
\[
\int_0^T \left( < t >^b \left\| (v_4, p_4)(\cdot, t) \right\|_{W_{q,1}^{0,1}(\mathbb{R}^N)}^P \right) dt \leq C \tilde{N}(v_1, p_1)(T)^P,
\]

which yields that
\[
\| < t >^b \partial_t (v_2, p_2)(\cdot, t) \|_{L_p((0,T), W_{q,1}^{0,1}(\mathbb{R}^N))} \leq C \tilde{N}(v_1, p_1)(T). \]  
(4.20)

\subsection*{4.2.3. Estimates of the Lower Order Term in $L_{\infty}$–$L_q$}  
Let $q = q_1, q_2$. First, we consider the case $2 < t < T$. By (4.12), we divide three parts as follows:
\[
\left\| (v_2, p_2)(\cdot, t) \right\|_{W_{q,1}^{0,1}(\mathbb{R}^N)}
\]
\[
= C \left( \int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t \right) \left\| e^{A(t-s)}(v_1, p_1)(\cdot, s) \right\|_{W_{q,1}^{0,1}(\mathbb{R}^N)} ds =: I_{q,0}(t) + II_{q,0}(t) + III_{q,0}(t).
\]
Using (4.10) with \((p, q) = (q, q_1/2)\) and noting that \(1 - bp' < 0\) provided by \(0 < \sigma < 1/2\) and \(\frac{N}{2} \left( \frac{2}{q_1} - \frac{1}{q} \right) \geq b\) for \(q = q_1, q_2\), we have

\[
I_{q,0}(t) \leq C \int_0^{t/2} (t - s)^{-\frac{N}{2} \left( \frac{2}{q} - \frac{1}{q} \right)} ||(v_1, P_1)(\cdot, s)|| ds
\]

(4.21)

\[
\leq C(t/2)^{-b} \left( \int_0^{t/2} (t - s)^{-bp'} ds \right)^{1/p'} \widetilde{N}(v_1, P_1)(T)
\]

(4.22)

\[
II_{q,0}(t) \leq C \int_{t/2}^{t-1} (t - s)^{-\frac{N}{2} \left( \frac{2}{q} - \frac{1}{q} \right)} ||(v_1, P_1)(\cdot, s)|| ds
\]

(4.23)

\[
\leq C \left( \int_{t/2}^{t-1} (t - s)^{-b} ||(v_1, P_1)(\cdot, s)|| ds \right)^{1/p'} \left( \int_{t/2}^{t-1} (s > b ||(v_1, P_1)(\cdot, s)||)^{p} ds \right)^{1/p}
\]

(4.24)

By (4.11) and \(t > b \leq C < s > b\) for \(t - 1 < s < t\), we have

\[
III_{q,0}(t) \leq C \int_0^t ||(v_1, P_1)(\cdot, s)|| ds \leq C \int_0^t (s > b ||(v_1, P_1)(\cdot, s)||)^{p} ds
\]

(4.25)

\[
\leq C \left( \int_0^t (s > b ||(v_1, P_1)(\cdot, s)||)^{p} ds \right)^{1/p}
\]

Combining (4.23), (4.24), and (4.25), we have

\[
\sup_{2 < t < T} \| (\tilde{v}_2, \tilde{P}_2)(\cdot, t) \|_{W^{0,1}_p(\mathbb{R}^N)} \leq C\widetilde{N}(v_1, P_1)(T).
\]

(4.26)

Next, we consider the case \(0 < t < \min(2, T)\). By (4.11) we have

\[
\sup_{0 < t < \min(2, T)} \| (\tilde{v}_2, \tilde{P}_2)(\cdot, t) \|_{W^{0,1}_q(\mathbb{R}^N)} \leq C \sup_{0 < t < \min(2, T)} \| (v_1, P_1)(\cdot, t) \|_{W^{0,1}_q(\mathbb{R}^N)}
\]

\[
\leq C\widetilde{N}(v_1, P_1)(T),
\]

which combined (4.26), yields that

\[
\| (\tilde{v}_2, \tilde{P}_2) \|_{L^\infty((0,T),W^{0,1}_q(\mathbb{R}^N))} \leq C\widetilde{N}(v_1, P_1)(T).
\]

(4.27)

Therefore, by (4.14), (4.19), (4.20), and (4.27), we have

\[
\mathcal{N}(v_2, P_2)(T) \leq C(T + \mathcal{N}(v_1, P_1)(T)).
\]

(4.28)

### 4.3. Conclusion

Recall \(v = v_1 + v_2\) and \(P = P_1 + P_2\). We know that there exists a constant \(C\) independent of \(T\) such that for any \(f \in L_p((0, T), W^2_q(\mathbb{R}^N)) \cap W^1_p((0, T), L_q(\mathbb{R}^N))\)

\[
\sup_{0 < t < T} \| f(\cdot, t) \|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} \leq C(\| f(\cdot, 0) \|_{B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)} + \| t > b f \|_{L_p((0,T),L_q(\mathbb{R}^N))}) + \| t > b \partial_t f \|_{L_p((0,T),L_q(\mathbb{R}^N))}
\]

(4.29)
as follows from the trace method of real interpolation theorem, and so using (4.29) and the fact that $\langle v_1, P_1 \rangle \mid_{t=0} = (0, O)$ we have
\[
\| t > b \ (v_1, P_1)\|_{L^\infty((0,T), W^{3,1}_{p}(R^N))} \\
\leq C(\| t > b \ (v_1, P_1)\|_{L^p((0,T), W^{2,1}_{q}(R^N))} + \| t > b \ \partial_t (v_1, P_1)\|_{L^p((0,T), W^{0,1}_{q}(R^N))}).
\] (4.30)
Combining (4.30), (4.9), and (4.28) yields that
\[
\mathcal{N}(v, P)(T) \leq C(\mathcal{I} + \mathcal{N}(U, Q)(T)^2).
\]
Assuming that $\mathcal{I} \leq \epsilon^2$ and recalling that $\mathcal{N}(U, Q)(T) < \epsilon$, we have $\mathcal{N}(v, P)(T) \leq C\epsilon^2$ with some constant $C$ independent of $\epsilon$. Thus, choosing $\epsilon > 0$ so small that $C\epsilon < 1$, we have
\[
\mathcal{N}(v, P)(T) \leq \epsilon. \tag{4.31}
\]
Moreover, by the form $P = Q_0 + \int_0^T \partial_s P \, ds$ and Sobolev’s embedding theorem, we have
\[
\|P(\cdot, t)\|_{L^\infty(R^N)} \leq C\left(\|Q_0\|_{W^{1,2}_{q^2}(R^N)} + \int_0^T \|\partial_t P(\cdot, t)\|_{W^{1,2}_{q^2}(R^N)} \, dt\right) \\
\leq C\left(\|Q_0\|_{W^{1,2}_{q^2}(R^N)} + \left(\int_0^\infty \| t > b \ \partial_t P(\cdot, t)\|_{L^p((0,T), W^{1,2}_{q^2}(R^N))} \, dt\right)^{1/p'} \right) \\
\leq C\epsilon^2.
\]
Choosing $\epsilon > 0$ so small that $C\epsilon^2 < 1$ if necessary, we have $\sup_{0 < t < T} \|P(\cdot, t)\|_{L^\infty(R^N)} \leq 1$. Thus, we have $(v, P) \in I_{T, \epsilon}$. Therefore, we define a map $\Phi$ acting on $(U, Q) \in I_{T, \epsilon}$ by $\Phi(U, Q) = (v, P)$, and then $\Phi$ is the map from $I_{T, \epsilon}$ into itself. Considering the difference $\Phi(U_1, Q_1) - \Phi(U_2, Q_2)$ for $(U_i, Q_i) \in I_{T, \epsilon}$ $(i = 1, 2)$, employing the same argument as in the proof of (4.31) and choosing $\epsilon > 0$ smaller if necessary, we see that $\Phi$ is a contraction map on $I_{T, \epsilon}$, and therefore there exists a fixed point $(U, Q) \in I_{T, \epsilon}$ which solves (1.2). Since the existence of solutions to (1.2) is proved by the contraction mapping principle, the uniqueness of solutions belonging to $I_{T, \epsilon}$ follows immediately, which completes the proof of Theorem 1.1.

Declarations
Conflict of interest The author states that there is no conflict of interest.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
[1] Abels, H., Dolzmann, G., Liu, Y.: Well-posedness of a fully coupled Navier–Stokes/Q-tensor system with inhomogeneous boundary data SIAM. J. Math. Anal. 46(4), 3050–3077 (2014)
[2] Abels, H., Dolzmann, G., Liu, Y.: Strong solutions for the Beris–Edwards model for nematic liquid crystals with homogeneous Dirichlet boundary conditions. Adv. Differ. Equ. 21(1–2), 109–152 (2016)
[3] Ball, J., Majumdar, A.: Nematic liquid crystals: from Maier–Saupe to a continuum theory. Mol. Cryst. Liq. Cryst. 525, 1–11 (2010)
[4] Beris, A.N., Edwards, B.J.: Thermodynamics of Flowing Systems with Internal Microstructure, Oxford Engineering Science Series, vol. 36. Oxford University Press, Oxford (1994)
[5] Cavaterra, C., Rocca, E., Wu, H., Xu, X.: Global strong solutions of the full Navier–Stokes and Q-tensor system for nematic liquid crystal flows in two dimensions. SIAM J. Math. Anal. 48(2), 1368–1399 (2016)
[6] de Gennes, P.G.: The Physics of Liquid Crystals. Clarendon Press, Oxford (1974)
[7] De Anna, F.: A global 2D well-posedness result on the order tensor liquid crystal theory. J. Differ. Equ. 262(7), 3932–3979 (2017)
[8] Denk, R., Hieber, M., Prüss, J.: $\mathcal{R}$-boundedness, Fourier multiplier and problems of elliptic and parabolic type. Mem. AMS 166(788) (2003)
[9] Denk, R., Schnaubelt, R.: A structurally damped plate equation with Dirichlet–Neumann boundary conditions. J. Differ. Equ. 259, 1323–1353 (2015)
[10] Ericksen, J.L.: Hydrostatic theory of liquid crystals. Arch. Ration. Mech. Anal. 9, 371–378 (1962)
[11] Ericksen, J.L.: Equilibrium theory of liquid crystals. Adv. Liq. Cryst. 2, 233–298 (1976)
[12] Enomoto, Y., Shibata, Y.: On the $\mathcal{R}$-sectoriality and its application to some mathematical study of the viscous compressible fluids. Funk. Ekvac. 56, 441–505 (2013)
[13] Huang, J., Ding, S.: Global well-posedness for the dynamical Q-tensor model of liquid crystals. Sci. China Math. 58, 1349–1366 (2015)
[14] Leslie, F.M.: Some constitutive equations for anisotropic fluids. Q. J. Mech. Appl. Math. 19, 357–370 (1966)
[15] Leslie, F.M.: Some constitutive equations for liquid crystals. Arch. Ration. Mech. Anal. 28(4), 265–283 (1968)
[16] Liu, Y., Wang, W.: On the initial boundary value problem of a Navier–Stokes/Q-tensor model for liquid crystals. Discrete Contin. Dyn. Syst. Ser. B 23(9), 3879–3899 (2018)
[17] Majumdar, A.: Equilibrium order parameters of liquid crystals in the Landau-de Gennes theory. Eur. J. Appl. Math. 21, 181–203 (2010)
[18] Majumdar, A., Zarnescu, A.: Landau-de Gennes theory of nematic liquid crystals: the Oseen-Frank limit and beyond. Arch. Ration. Mech. Anal. 196, 227–280 (2010)
[19] Paicu, M., Zarnescu, A.: Global existence and regularity for the full coupled Navier–Stokes and Q-tensor system. SIAM J. Math. Anal. 43(5), 2009–2049 (2011)
[20] Paicu, M., Zarnescu, A.: Energy dissipation and regularity for a coupled Navier–Stokes and Q-tensor system. Arch. Ration. Mech. Anal. 203(1), 45–67 (2012)
[21] Saito, H.: Compressible fluid model of Korteweg type with free boundary condition: model problem. Funk. Ekvac. 62, 337–386 (2019)
[22] Saito, H., Shibata, Y.: On decay properties of solutions to the Stokes equations with surface tension and gravity in the half space. J. Math. Soc. Jpn. 68(4), 1559–1614 (2016)
[23] Schonbek, M., Shibata, Y.: Global well-posedness and decay for a Q tensor model of incompressible nematic liquid crystals in $\mathbb{R}^N$. J. Differ. Equ. 266(6), 3034–3065 (2019)
[24] Shibata, Y.: $\mathcal{R}$ Boundedness Maximal Regularity and Free Boundary Problems for the Navier Stokes Equations. Mathematical Analysis of the Navier–Stokes Equations, Lecture Notes in Mathematics, Fondazione CIME/CIME Foundation Subseries, vol. 2254, pp. 193–462. Springer, Berlin (2020)
[25] Shibata, Y.: New thought on Matsumura–Nishida theory in the $L_p\text{-}L_q$ maximal regularity framework, preprint
[26] Shibata, Y., Shimizu, S.: On the $L_p\text{-}L_q$ maximal regularity of the Neumann problem for the Stokes equations in a bounded domain. J. Reine Angew. Math. 615, 157–209 (2008)
[27] Weis, L.: Operator-valued Fourier multiplier theorems and maximal $L_p$-regularity. Math. Ann. 319, 735–758 (2001)
[28] Xiao, Y.: Global strong solution to the three-dimensional liquid crystal flows of Q-tensor model. J. Differ. Equ. 262(3), 1291–1316 (2017)

Miho Murata
Department of Mathematical and System Engineering, Faculty of Engineering
Shizuoka University
3-5-1 Johoku, Naka-ku
Hamamatsu-shi Shizuoka 432-8561
Japan
e-mail: murata.miho@shizuoka.ac.jp

Yoshihiro Shibata
Department of Mathematics
Waseda University
Ohkubo 3-4-1, Shinjuku-ku
Tokyo 169-8555
Japan
e-mail: yshibata@waseda.jp

(accepted: February 14, 2022; published online: March 16, 2022)