In a recent paper [1] Skoge et al. presented new accurate numerical data on amorphous packings of hard spheres in space dimension $d > 3$. It was shown (see e.g. Fig. 4 in [1]) that on compressing the low density liquid at a constant rate $\gamma$, the pressure of the system follows the equilibrium pressure of the liquid up to some density (often called glass transition density) above which the pressure starts to increase faster than in equilibrium, and diverges on approaching a value of density $\phi_J(\gamma)$, which is called jamming density. Crystallization seems to be strongly suppressed by kinetic effects in dimension $d > 3$ and can then be neglected in the following discussion. Values of $\phi_J(\gamma)$ have been accurately measured in [1] as a function of $\gamma$. On the contrary, $\phi_R(\gamma)$ is not precisely determined as long as $\gamma > 0$: the glass transition is smeared and happens in a crossover region $[\phi_R(\gamma), \phi_J(\gamma)]$. However, the amplitude of the crossover interval seems to decrease for $\gamma \to 0$ (see again Fig. 4 in [1], and see [2] for a recent theoretical discussion of these effects).

Recently, a theory of the glass transition of hard spheres, that can be applied in any space dimension $d$, was developed [3, 4]. The basic idea is that the phenomenology observed at finite $\gamma$ is due to an underlying thermodynamic glass transition [5–7]: i.e. that in the limit $\gamma \to 0$, $\phi_R(\gamma) \to \phi_K$, the Kauzmann density, at which an equilibrium phase transition to a glass phase happens. This is an idealized picture that neglects metastability effects due to the presence of the crystal: the consequences of this approximation might be important but cannot be discussed here, see [4] for a detailed discussion. However, as it seems that crystallization is negligible in $d > 3$, a comparison between the theory and the data of [1] is possible. In the equilibrium glass phase, the pressure increases faster than in the liquid phase (i.e. the compressibility is smaller), and diverges at a value of density $\phi_{RCP} = \lim_{\gamma \to 0} \phi_J(\gamma)$ that we call random close packing (RCP) density. This is the definition of random close packing that can be given within our theory. However it is affected by the metastability effects related to the presence of the crystal: see [4, 8] for a discussion. In particular, in [8] the notion of random close packing density has been criticized and an alternative notion of maximally random jammed (MRJ) packings has been proposed, see below.

Our theory is based on standard liquid theory and on the replica trick: it takes as input the equation of state of the liquid phase (in practice one has to choose an expression that describes well the liquid at low density and extrapolate to higher density), and gives as an output: i) the Kauzmann density $\phi_K$; ii) the random close packing density $\phi_{RCP}$; iii) the equation of state in the glass phase; iv) some properties of the pair correlation function in the glass phase, e.g. its shape close to contact; v) the equation of state of the metastable glass states that are reached for $\gamma > 0$ and their contribution to the entropy (the configurational entropy or complexity); see [3] for all the technical details. One of the most interesting predictions of the theory is that $2^d \phi_{RCP} \sim d \log d$ for large $d$. As far as I know, this scaling has been proposed in [3] for the first time. The aim of this paper is to show that the results of [1] are fully compatible with this prediction.

To this aim, an expression for the equation of state of the liquid in $d > 3$ is needed as input to the theory. The simplest choice is a generalization of the celebrated Carnahan-Starling equation of state to $d > 3$ [9]:

$$Y(\phi) = \frac{1 - \alpha \phi}{(1 - \phi)^2} , \quad \alpha = d - 2^{d-1}(B_3/b^2) ,$$

where $Y(\phi) = g(\sigma^+)$ is the value of the radial distribution function at contact, and $b$ and $B_3$ are the second and third virial coefficients, whose exact expression is known [9]. The entropy of the liquid $S(\phi)$ is obtained by integrating the exact expression

$$S(\phi) = \int_0^\phi \beta P d\phi \rho = -1 - b_0 Y(\phi) .$$

Given $S(\phi)$, the random close packing density is the so-

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TABLE I: Values of $\phi_K$ and $\phi_{RCP}$ from the theory (only values for $d \leq 8$ are reported for brevity, values for $d \geq 8$ are in Fig. 1) compared with the available measured values of $\phi_{MRJ}$ [1]. The last column gives the value of $\phi_{RCP}$ extrapolated from the data of [1] (see text).
The very nice data for $d = 4$ reported in Fig. 4 of [1] allow for a more precise comparison of the numerical and theoretical results: the value of $\phi_J(\gamma)$ is reported for 5 different values of $\gamma = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}$. A standard procedure to extrapolate to $\gamma \to 0$ is to fit the data to a Vogel-Fulcher law:

$$\gamma(\phi_J) = \gamma_0 10^{-\frac{D}{\log_{10}[\gamma/\gamma_0]}} \Leftrightarrow \phi_J(\gamma) = \phi_{RCP} + \frac{D}{\log_{10}[\gamma/\gamma_0]}.$$ 

Such extrapolations are often ambiguous; however the fit is good and gives $\phi_{RCP} = 0.473$, $D = 0.03$, $\gamma_0 = 0.45$. The final result for $\phi_{RCP}$ differs from $\sim 10\%$ from the theoretical value, see table I. This is a very good result given the ambiguities that are present both in the numerical data (numerical error and extrapolation) and in the theory (the choice of a particular expression for the equation of state of the liquid that is not exact, see [10] for recent contributions). Note that a similar extrapolation is not possible in $d = 3$ due to crystallization, and for $d > 4$ due to lack of numerical data. Hopefully new data for $d > 4$ will allow for a similar comparison also in this case. Note also that the value of $\phi_K \sim 0.43$ we obtain in $d = 4$ seems to agree very well with the extrapolation of $\phi_g(\gamma)$ (defined roughly as the point where the curves leave the liquid equation of state) to $\gamma = 0$ in Fig. 4 of [1].

A more accurate comparison of the theory with the numerical data is possible: for instance, the theory gives a prediction for the glass equation of state that is close to the measured pressure in the glass branch for very small $\gamma$, e.g. $\gamma = 10^{-7}$ in Fig. 4 of [1]; it also predicts that the amorphous packings are isostatic, i.e. the average number of contacts per sphere is $z = 2d$, in any dimension $d$, and this seems to be confirmed by the numerical data. Other properties of the packings such as the shape of the correlation function $g(r)$ close to contact are predicted to be independent of the dimension. Unfortunately we are still not able to give a prediction for the splitting of the second peak of $g(r)$, which seems to be strongly suppressed in $d > 3$. We hope that future work will address this and many other open questions [4]. The data reported in [1] provide a very good test of the theories aiming to understand the behavior of hard sphere in large space dimension [3, 11–13] and strongly call for further developments.

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