When is a quantum heat engine quantum?

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Abstract – Quantum thermodynamics studies quantum effects in thermal machines. Both quantum coherence and quantum correlations have been theoretically shown to be a physical resource able to boost their performance. But when is a heat engine, which cyclically interacts with external reservoirs that unavoidably destroy its phase coherence, really quantum? We here use the Leggett-Garg inequality to assess the nonclassical properties of a paradigmatic two-level Otto engine with quantum friction. We provide the complete phase diagram characterizing the quantumness of the engine as a function of its parameters and distinguish three different phases: a quantum phase separated from a classical phase by a transition regime. We further derive an analytical expression for the quantum-to-classical transition temperature.

Introduction. – The investigation of thermal machines has been a cornerstone of thermodynamics since its origins [1]. They are almost universally used to provide mechanical energy from thermal energy. Most engines in our modern industrialized society are heat engines. The performance of these devices is usually quantified with the help of two key parameters: efficiency, defined as the ratio of energy output and energy input, and power that characterizes the work production rate [1]. Macroscopic motors, refrigerators and heat pumps are notable examples of classical machines whose dynamics follow the laws of classical physics [1]. Quantum thermal machines, on the other hand, are microscopic engines that are described by dynamical equations of motion that obey the laws of quantum mechanics [2]. They have been extensively studied theoretically over the last fifty years [3]. Concrete proposals to experimentally build such quantum machines have been put forward using, for instance, trapped ions [4,5] and nanomechanical systems [6,7]. The realization of a classical single ion nanoengine with the potential to enter the quantum regime has been reported recently [8]. Remarkably, quantum coherence [9–11] and quantum correlations [12–14] have been predicted to enhance efficiency and power of thermal machines. Quantumness thus appears as a useful physical resource in thermodynamics. However, contrary to most quantum applications that aim at perfectly shielding systems from their environments [15], thermal engines, by their very nature, cyclically interact with heat reservoirs that unavoidably destroy their quantum coherence. A crucial issue, for theoretical and practical reasons, is therefore to assess the nonclassicality of heat engines and identify the regime where quantum resources might be effectively harnessed.

Characterizing nonclassicality is of fundamental importance from quantum computation [16] to quantum biology [17]. Assessing the nonclassical properties of a given state or dynamics is a nontrivial task, however, and, more than a century after the birth of quantum theory, the border between classical and quantum worlds remains fuzzy [18]. A common approach to identify quantum behavior is to impose classical constraints that are violated by quantum theory [19]. The assumptions of realism and locality thus lead to Bell’s inequality [20], while those of macroscopic realism and noninvasive measurements to the Leggett-Garg inequality (LGI) [21]. These two results allow to test whether a system has stronger-than-classical spatial or temporal correlations, respectively. The advantage of the LGI is that it applies to a single particle and not to a pair of particles as in Bell’s inequality [20], while those of macroscopic realism and noninvasive measurements to the Leggett-Garg inequality (LGI) [21]. These two results allow to test whether a system has stronger-than-classical spatial or temporal correlations, respectively. The advantage of the LGI is that it applies to a single particle and not to a pair of particles as in Bell’s inequality. It has hence become a valuable theoretical and experimental tool to detect quantumness of individual systems [22]. In the past years, experimental violations of the LGI have been observed in an increasing number of single systems, including a superconducting qubit [23], a spin in a diamond defect center [24], a nuclear spin [25,26], a photon [27], a phosphorus impurity in silicon [28], and a single diffusing atom [29]. The last two experiments have implemented ideal negative measurements, as originally discussed by...
Leggett and Garg. In the following, we employ the violation of the LGI as a criterion to assess the quantumness of a paradigmatic quantum thermal machine: a two-level Otto engine with quantum friction [30–34]. We identify the existence of three different phases: two regimes where the dynamics is either fully quantum or classical, and an intermediate domain where the engine appears quantum in some temperature intervals and classical in others. We further derive an explicit formula for the transition temperature between the quantum and classical phases. We finally reveal an unknown counterintuitive phenomenon by showing that trying to operate a conventional thermal machine faster, with the idea of keeping it coherent, actually leads to incoherent dynamics, owing to constraints imposed by thermodynamics.

Quantum Otto engine. – We consider a quantum Otto engine for a single two-level system [3] with time-dependent frequency $\omega_t$ and Hamiltonian $H = \omega_0 \sigma_z/2$, where $\sigma_z$ is the usual Pauli operator (we set $\hbar = k_B = 1$). The quantum Otto motor is a generalization of the familiar four-stroke car engine and its thermodynamic cycle consists of the following isentropic and isochoric steps, as shown in fig. 1: 1) Heating: the two-level system is weakly coupled to a hot reservoir at temperature $T_h$ during time $\tau_h$, while its frequency is fixed; 2) Expansion: the system is isolated and its frequency is unitarily changed from $\omega_2$ to $\omega_1$ during time $\tau_1$; 3) Cooling: the system weakly interacts with a cold reservoir at temperature $T_c$ during time $\tau_c$, while its frequency is again held constant; 4) Compression: the frequency is unitarily brought back to its initial value $\omega_2$ during time $\tau_2$.

Work and heat along the different steps may be determined from infinitesimal variations of the average energy of the system, $E = \langle H \rangle = \omega P$, where $P = \langle \sigma_z \rangle / 2$ is the polarization. We write accordingly $dE = P d\omega + \omega dP = \delta W + \delta Q$ and associate work $\delta W$ with changes of frequency and heat $\delta Q$ with changes of polarization. The Otto engine exchanges energy in the form of work during compression/expansion phases and in the form of heat during heating and cooling steps. In order to compute these quantities along the four branches of the cycle, we specify the finite-time dynamics of an arbitrary system observable $X$, with the Markovian quantum master equation in the Heisenberg picture [35],

$$
\frac{dX}{dt} = \frac{i}{\omega} [\sigma_z, X] + \frac{\partial X}{\partial t} + \frac{\sigma_0}{2} (\omega, T) (\sigma_- [X, \sigma_+] + [\sigma_-, X] \sigma_+) + \frac{\sigma_0}{2} (n(\omega, T) + 1) (\sigma_+ [X, \sigma_-] + [\sigma_+, X] \sigma_-),
$$

where $\sigma_\pm$ denote the raising and lowering spin operators and $n(\omega, T) = 1/[\exp(\beta \omega) - 1]$ the bosonic thermal occupation number at inverse temperature $\beta = 1/T$. The coupling constant $\gamma_0$ is assumed to instantaneously vanish during the unitary compression/expansion steps, when the engine is decoupled from the baths. In this limit eq. (1) reduces to the standard Heisenberg equation. We denote the total transition rate by $\gamma = \gamma_0 [2n(\omega, T) + 1]$.

When the two-level-system Hamiltonian does not commute with its external driving Hamiltonian, finite-time dynamics leads to nonadiabatic transitions and dissipative losses. This purely quantum phenomenon has been called quantum friction, as it reduces the performance of the engine [30–34]. Quantum friction arises, for instance, when the external driving field is not aligned with the field of the system [34], or in the presence of noncommuting interactions between pairs of two-level systems [31]. We here phenomenologically account for quantum friction by an increase of the polarization of the two-level system, $P_t = P_0 + (\sigma / \gamma)^2 t_i (i = 1, 2)$ [30]. The parameter $\sigma$ is the internal friction coefficient. It is directly related to the irreversible entropy production in the usual manner (see appendix). The advantage of this approach is that it generically applies to a large class of microscopic systems, without having to explicitly introduce the external driving Hamiltonian or the interacting two-level companion [3]. The linear time dependence corresponds to the first-order nonadiabatic correction [36]. In this framework, the quantum Otto cycle is characterized by the following ten parameters, $\omega_{1,2}, \tau_{1,2}, T_{h,c}, \gamma_0, \gamma$ and $\sigma$.

The power of the Otto engine may be optimized with respect to the heating and cooling times $\tau_{h,c}$. The corresponding optimal times were determined in ref. [30] using the method of Lagrange multipliers (keeping the frequencies $\omega_{1,2}$ and times $\tau_{1,2}$ constant). Computing work and heat along the four branches of the cycle, assuming, for concreteness, a linear driving protocol, $\omega_t = \omega_0 + \omega_0 t$, the two optimal heating and cooling times are [30]:

$$
\tau_h = -\frac{1}{\gamma} \ln x_{\text{max}} - \frac{\sqrt {Rx_{\text{max}} (1 + R - x_{\text{max}})}}{x_{\text{max}}^2 (R + 1)},
$$

and

$$
\tau_c = -\frac{1}{\gamma} \ln x_{\text{max}} - \frac{\sqrt {Rx_{\text{max}} (1 + R - x_{\text{max}})}}{(R + 1)},
$$

Fig. 1: (Colour online) Four branches of the quantum Otto cycle for a two-level system: 1) constant frequency heating during $\tau_h$, 2) unitary expansion during $\tau_1$, 3) constant frequency cooling during $\tau_c$, 4) unitary compression during $\tau_2$. 

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for an engine with vanishing positive work. This is the minimal requirement for a thermal machine to be a heat engine. In that case, the total cycle duration \( \tau = t_1 + t_2 + t_3 + t_4 \) is the minimum time needed to obtain a nonnegative work output. We shall use this minimal time to determine the quantumness of the heat engine. The two quantities \( R \) and \( x_{\text{max}} \) are defined as [30]

\[
R = \frac{\sigma^2 \omega_1 (1/t_1 + 1/t_2)}{\Delta \omega (\Delta P_{\text{eq}} + \sigma^2/\tau_1)}, \quad x_{\text{max}} = \frac{\Delta P_{\text{eq}} - \sigma^2/\tau_2}{\Delta P_{\text{eq}} + \sigma^2/\tau_1},
\]

where \( \Delta P_{\text{eq}} = P_{b,\text{eq}} - P_{c,\text{eq}} = -\tanh(\omega_2/(2T_h))/2 + \tanh(\omega_1/(2T_c))/2 \) is the difference between the equilibrium polarizations after complete thermalization with the hot and cold reservoirs. The constraint of a closed cycle, that is, of a finite minimum cycle duration \( \tau \), leads to an additional condition on the friction coefficient \( \sigma \) [30],

\[
\frac{\sigma^2}{2} \leq \Delta P_{\text{eq}}.
\]

Heating and cooling times \( \tau_{h,c} \) diverge when the above bound is saturated, indicating that large nonadiabatic effects, generated by the finite-time driving, result in increased, and eventually infinite, thermalization times.

Expressions (2) and (5) form the basis of our investigation of the quantum nature of the Otto engine. Our strategy is to look for violations of the LGI during the heating phase, where decoherence is expected to be the strongest. Since coherence is preserved along the unitary compression/expansion steps, and decoherence is less adverse during cooling, this will ensure that the whole cycle remains quantum. This is important because the efficiency depends on all four steps of the cycle. Its properties thus depend on the quantumness of both work production and heat exchange. Our analysis could be readily extended to include the cooling phase, but calculations would be more involved, without affecting the physics.

**Leggett-Garg inequality.** – Under the classical assumptions of macroscopic realism and noninvasive measurability, the two-time correlation functions \( C_{ij} \) of a dichotomous observable, \( Q = \pm 1 \), measured at three distinct times \( t_i, i = (1, 2, 3) \), satisfy [21]

\[
K_3 = C_{21} + C_{32} - C_{31} \leq 1.
\]

Quantum mechanics violates the above inequality. A value of the Leggett-Garg function \( K_3 \) above one is therefore a clear signature of nonclassical behavior [22].

In order to assess the quantumness of the Otto engine, we evaluate the symmetrized two-time correlation functions \( C_{ij} \) of the observable \( \sigma_x \) for the dynamics given by the master equation (1) with the help of the quantum regression theorem [35]. We find (see appendix)

\[
C_{ij} = \frac{1}{2} \langle \{ \sigma_x(t_i), \sigma_x(t_j) \} \rangle = e^{-\frac{1}{2}(t_i-t_j)} \cos [\omega_2(t_i-t_j)].
\]

By further considering equally spaced measurement times with time separation \( t \), as commonly done [22], we obtain the Leggett-Garg function

\[
K_3(t) = 2e^{-\frac{1}{2}t} \cos(\omega_2 t) - e^{-t} \cos(2\omega_2 t).
\]

The function \( K_3(t) \) is shown in fig. 2 for different values of the damping constant \( \gamma \). We observe that the LGI (6) is violated for unitary evolution with \( \gamma = 0 \). The amplitude of the violations decreases with increasing \( \gamma \) until the critical value, \( \gamma_q = 2\omega_2 \), at which the time scale of the coherent system dynamics coincides with that of the reservoir-induced decoherence, is reached (see appendix). Beyond that point there is no violation of the LGI for any value of \( t \) and the incoherent evolution imposed by the hot reservoir prevails. It should be emphasized that the LGI tests the quantumness of the dynamics and not that of the state. Violations of the LGI have indeed been experimentally observed after the decay of coherent Rabi oscillations [23]. The LGI thus proves whether the coherence induced by a first measurement can survive until a second measurement is performed [37]. In the sequel, we set the first measurement at the beginning of the heating phase and vary the time separation such that \( 2t \geq \tau_h \). We note that the LGI test does not affect the thermodynamics of the engine, as it is assumed to be applied only once to assess its nonclassicality and not continuously during each cycle.

We characterize the transition from coherent to incoherent dynamics by introducing a time \( \tau_q \) defined as the largest possible time for which the LGI can be violated:

\[
\tau_q = \max \{ 2t | K_3(t) > 1 \}.
\]

The factor 2 comes from the fact that the three measurements span a total time interval of \( 2t \). Owing to the oscillating feature of the Leggett-Garg function (8), the quantum time \( \tau_q \) is a step-like function; it decreases with increasing reservoir temperature, as expected, see fig. 3.

![Fig. 2: (Colour online) Leggett-Garg function \( K_3(t) \), eq. (8), as a function of time for four different values of the damping constant \( \gamma \). The shaded area indicates the classically forbidden region \( K_3 > 1 \). Above the critical value, \( \gamma_q = 2\omega_2 \), there are no violations of the LGI (6) for any value of \( t \).](image-url)
The engine behaves nonclassically when the LGI (6) is violated, \( \omega \) ated domain, iii) classical dynamics for large \( \bar{\tau}_q \) transition for small \( \bar{\tau}_h \) than the heating time, \( \tau \). Motor is nonclassical only if the quantum time is larger than the heating time, \( \tau_h \). b) We identify three regimes: i) single quantum-classical transition for small \( \bar{\sigma} \), ii) multiple transitions in an intermediate domain, iii) classical dynamics for large \( \bar{\sigma} \). Parameters are \( \omega_1 = 10 \), \( \omega_2 = 20 \), \( \tau_1 = 0.01 \), \( \tau_2 = 0.5 \) and \( T_c = 1 \).

The quantum time can be shown to be bounded by the coherence half-life of the two-level system [37]. However, while experimentally determining the engine coherence half-life might be involved, the LGI may be easily tested with the measurement of only three correlation functions.

The behavior of the engine depends decisively on the relationship between heating time \( \tau_h \) and quantum time \( \tau_q \), that is, on the one hand, on the internal dynamics of the machine, as dictated by thermodynamics, and, on the other hand, on the decoherence process induced by the coupling to the reservoir. The dynamics of the Otto motor is nonclassical only if the quantum time is larger than the heating time, \( \tau_q > \tau_h \), so that the LGI may be violated. As seen from eqs. (2) and (4), the heating time \( \tau_h \) depends on the two temperatures \( T_h \) and \( T_c \) and on the internal friction coefficient \( \sigma \); we will use in the following the reduced quantity \( \bar{\sigma} = \sigma / \tau_2 \).

Our main results are summarized in the phase diagram shown in fig. 3. For a fixed cold temperature \( T_c \), we identify three distinct regimes: i) \( \bar{\sigma} < \bar{\sigma}_{C1} \): for a reduced friction coefficient \( \bar{\sigma} \) smaller than a critical value \( \bar{\sigma}_{C1} \), the behavior of the engine is quantum for temperatures below a threshold quantum temperature \( T_q \) (see appendix),

\[
T_h < T_q = \frac{\omega_2}{2 \coth^{-1}(2\omega_2/\gamma_0)}. \tag{10}
\]

Owing to the almost vertical shape of \( \tau_q \), the threshold temperature \( T_q \) is mostly independent of \( \bar{\sigma} \) in that domain; ii) \( \bar{\sigma}_{C1} < \bar{\sigma} < \bar{\sigma}_{C2} \): for intermediate values of \( \bar{\sigma} \), the engine appears quantum in some temperature intervals and classical in some others. This corresponds to a transition regime where the boundary between classical and quantum worlds is blurry. Such behavior follows from the oscillating nature of the Leggett-Garg function \( K_3(t) \), eq. (8); iii) \( \bar{\sigma}_{C2} < \bar{\sigma} \): finally for a reduced friction \( \bar{\sigma} \) larger than a second critical value \( \bar{\sigma}_{C2} \), the dynamics of the Otto engine is classical for all temperatures \( T_h \).

The phase diagram in fig. 3 reveals some subtle interplay between thermodynamics and quantum physics. We first remark that in the low \( \bar{\sigma} \) phase, the threshold temperature \( T_q \) is equivalent to the condition \( \gamma_q = 2\omega_2 \) for a two-level system coupled to a single reservoir at temperature \( T_2 \) (see appendix). Here the internal dynamics of the engine which is set by thermodynamics does not play any role, and the quantumness of the Otto cycle is solely controlled by the decoherence process generated by the reservoir. However, the situation changes dramatically in the two other phases as the heating time \( \tau_h \) increases significantly with increasing \( \bar{\sigma} \). In the high \( \bar{\sigma} \) regime, thermodynamics thus imposes a purely classical behavior, although decoherence alone would predict the existence of a quantum sector for low enough temperatures. In other words, the classical or quantum nature of the engine is not just governed by the coupling to the reservoirs, but also by its internal dynamics, in particular the friction \( \bar{\sigma} \). A second, nonintuitive, observation is that trying to operate the engine faster in order to keep it coherent is bound to fail. We first note that large nonadiabatic effects characterized by a small driving time \( \tau_2 \) are equivalent to a large reduced friction \( \bar{\sigma} = \sigma / \tau_2 \). As a consequence, shortening the compression/expansion steps results in increased nonadiabatic excitations induced by a large \( \bar{\sigma} \), and, in turn, to longer thermalization times, as seen from eq. (2). These longer reservoir couplings eventually lead to stronger decoherence and, therefore, to incoherent dynamics. The rule “the faster, the better” which might be true for most quantum applications, hence does not apply to conventional quantum heat engines that we here consider. It will be interesting to investigate how this is modified when shortcut-to-adiabaticity techniques are used to suppress nonadiabatic transitions [38–40].

Figure 4 finally presents the influence of the cold reservoir temperature \( T_c \) (for fixed \( \bar{\sigma} \)) on the quantumness of the engine. We note that it is largely similar to that of the internal friction \( \sigma \) displayed in fig. 3, as both parameters decrease the ability of the system to draw heat from the hot reservoir. For \( T_c \) below the critical value \( T_{C1} \), the LGI is violated and the behavior of the engine is quantum up to the threshold temperature \( T_q \) (the discussion in the previous section was based in this \( T_c \) regime). In an intermediate domain, the dynamics of the engine alternates between quantum and classical as \( T_h \) is increased. Ultimately, for \( T_c \) larger that the critical value \( T_{C2} \), the LGI is never violated and the engine is purely classical.
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Fig. 4: (Colour online) Phase diagram characterizing the quantumness of the Otto engine as a function of the hot and cold temperatures $T_h$ and $T_c$ for constant reduced friction coefficient $\sigma$. The engine displays an analogous behavior as in fig. 3, the cold temperature $T_c$ having a similar effect as the internal friction coefficient $\sigma$. Same parameters as in fig. 3 with $\sigma = 2/3$.

Conclusions. – We have used the Leggett-Garg inequality to characterize the quantum properties of a two-level Otto engine with quantum friction. We have obtained a phase diagram that allows to identify the parameter regimes where the engine behaves nonclassically. We have further shown that trying to run a conventional thermal machine faster, with the hope of beating decoherence, is not a winning strategy, as thermodynamics will make the machine faster, with the hope of beating decoherence, is not a winning strategy, as thermodynamics will make the dynamics classical. The Leggett-Garg inequality thus appears to be a powerful tool to assess the quantum properties of thermal engines. It may be successfully employed in theoretical studies like the present one on the two-level Otto engine, but also extended to many-level motors [41]. At the same time, it may be efficiently applied experimentally to a large number of platforms for which violations of the LGI have already been observed. In addition to its fundamental importance in highlighting the interplay of thermodynamics and quantum theory, we expect it to play a central role in future theoretical and experimental investigations of quantum machines in determining nonclassical regimes where quantum resources, such as coherence and correlations, might be advantageously exploited.

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Appendix. – Derivation of the threshold temperature. There is a threshold value of the damping coefficient $\gamma_q$, and therefore a threshold temperature $T_q$, above which the Leggett-Garg function $K_3(t)$ given in eq. (8) can never, even for arbitrary small measurement spacings $t$, assume classically forbidden values $K_3(t) > 1$. These quantities may be determined in the following way. From eq. (8), it is apparent that $K_3(0) = 1$, independently of all other parameters. Accordingly, we demand that the slope of $K_3(t)$ be nonpositive for arbitrary small values of the measurement spacing $t$. A Taylor expansion of the time derivative of $K_3(t)$ to first order in $t$ yields

$$\frac{d}{dt}K_3(t) = t \left( 2\omega^2 - \frac{\gamma_q^2}{2} \right) + O(t^2). \quad (A.1)$$

The condition of a vanishing slope leads to the threshold value $\gamma_q = 2\omega$. The threshold temperature $T_q$ (10) directly follows from the definition of the transition rate $\gamma$.

Relation to entropy production. We here establish a connection between the internal friction coefficient $\sigma$ and the irreversible entropy production $\Sigma$. We begin by evaluating the heat exchanged during the two isochoric steps. Since the frequency is constant, we have in any time interval $[t_1, t_2]$

$$Q = \int_{t_1}^{t_2} \omega_i \dot{P}_i \, dt = \omega(P_{B2} - P_{t_1}). \quad (A.2)$$

As a result, we find for the heating and cooling steps,

$$Q_h = \omega_2(P_{B} - P_A)$$
$$Q_c = \omega_1(P_D - P_C) = -\omega_1 \left[ P_B - P_A + \sigma^2 \left( \frac{1}{\tau_2} + \frac{1}{\tau_1} \right) \right], \quad (A.3)$$

where $A, B, C, D$ denote the four corners of the thermodynamic cycle [30]. The entropy variation of the engine vanishes for a full cycle. The overall entropy change is then given by the entropy difference in the two reservoirs:

$$\Delta S = -\left( \frac{Q_c}{T_c} + \frac{Q_h}{T_h} \right)$$
$$= \left( \frac{\omega_1}{T_c} - \frac{\omega_2}{T_h} \right) (P_B - P_A) + \omega_1 \sigma^2 \frac{1}{T_c} \left( \frac{1}{\tau_2} + \frac{1}{\tau_1} \right). \quad (A.4)$$

The difference in polarization $P_B - P_A$ is further [30],

$$P_B - P_A = \Delta P^{eq} \frac{(1 - x)(1 - y)}{1 - xy} - \sigma^2 (1 - y) \frac{(\frac{x}{\tau_2} + \frac{1}{\tau_1})}{1 - xy} \quad (A.5)$$

and we can thus cast the entropy variation in the form,

$$\Delta S = \left( \frac{\omega_1}{T_c} - \frac{\omega_2}{T_h} \right) \Delta P^{eq} \frac{(1 - x)(1 - y)}{1 - xy}$$
$$- \left( \frac{\omega_1}{T_c} - \frac{\omega_2}{T_h} \right) \sigma^2 (1 - y) \left( \frac{x}{\tau_2} + \frac{1}{\tau_1} \right)$$
$$+ \frac{\omega_1}{T_c} \sigma^2 \left( \frac{1}{\tau_2} + \frac{1}{\tau_1} \right). \quad (A.6)$$

We here identify three sources of irreversibility: partial thermalization ($x, y$ vanish after complete equilibration), the internal friction $\sigma$, and the intrinsic irreversibility of the engine given by the term $(\omega_2/T_h - \omega_1/T_c)$. The efficiency of the Otto engine is indeed only equal to the
Carnot efficiency, when \( \omega_1 / \omega_2 = T_c / T_h \). In that quasistatic limit, the first two terms in eq. (A.6) vanish and the total entropy change simplifies to

\[
\Delta S = \frac{\omega_1 \sigma^2}{T_c} \left( \frac{1}{\tau_2} + \frac{1}{\tau_1} \right) = \frac{\omega_1 \sigma^2}{T_c \tau_2 \tau_1} + \frac{\omega_2 \sigma^2}{T_h \tau_1}. \tag{A.7}
\]

We recognize the usual expression for the entropy production in the long-time limit, \( \Delta S = \sum \Delta S_i \) [42] with \( \Delta S_i = \omega_i \sigma^2 / T_c \) and \( \Sigma_i = \omega_i \sigma^2 / T_h \). A finite friction coefficient \( \sigma \) therefore leads to a finite entropy production.

**Total work produced by the engine.** For completeness, we summarize in this section the derivation of the work produced by the engine [30]. The adiabatic work during either compression/expansion is

\[
W_i = \int_0^{\tau_i} \dot{W} dt = (\omega_f - \omega_i) \left[ \frac{\sigma^2}{2 \tau_i} + P_0 \right], \tag{A.8}
\]

where \( \omega_f, \omega \) denotes the initial and final frequencies \( \omega_1, \omega_2 \). Additionally, the irreversible work associated with the nonadiabatic driving (and the corresponding increase in polarization) along these branches is given by

\[
W_{irr,i} = \int_0^{\tau_i} \omega P dt = \frac{\omega_f + \omega_i}{2} \sigma^2 / \tau_i. \tag{A.9}
\]

The total work done during a compression or expansion step is therefore the sum

\[
W_i^{tot} = (\omega_f - \omega_i) \left[ \frac{\sigma^2}{2 \tau_i} + P_0 \right] + \frac{\omega_f + \omega_i}{2} \sigma^2 / \tau_i. \tag{A.10}
\]

The work produced by the engine during one cycle, eq. (2), is the sum over the compression and expansion steps (with proper time matching) [30].

\[
W = -(\omega_2 - \omega_1) \left[ \Delta P^q (1-x)(1-y) \right. \left. (1-xy) \right] - \left[ \frac{\sigma^2}{2 \tau_i} \right] \left( \frac{1}{\tau_2} + \frac{1}{\tau_1} \right) (1-y) + \sigma^2 \omega_1 \left( \frac{1}{\tau_2} + \frac{1}{\tau_1} \right). \tag{A.11}
\]

Here \( \Delta P^q \equiv P_{h,c} - P_{e,c} = -\tanh(\omega_2 / (2T_h)) / 2 + \tanh(\omega_1 / (2T_c)) / 2 \) is the difference between the equilibrium polarizations after complete thermalization with the hot and cold reservoirs, \( x = \exp(−\gamma_c \tau_c) \) and \( y = \exp(−\gamma_h \tau_h) \). \( \gamma_c, \gamma_h \) coth(\( \omega_2 / (2T_h) \)). Optimizing the work (A.11) with the constraint of a fixed total cycle time \( \tau = \tau_1 + \tau_2 + \tau_1 \) leads to the Lagrange function

\[
L(x, y, \lambda) = W + \lambda \left( \frac{1}{\gamma_c} \ln x + \frac{1}{\gamma_h} \ln y - \tau_1 - \tau_2 \right), \tag{A.12}
\]

with Lagrange multiplier \( \lambda \). Equating the partial derivatives of \( L(x, y, \lambda) \) with respect to \( x \) and \( y \) to zero and setting for simplicity \( \gamma_c = \gamma_h = \gamma \) eventually yields (keeping frequencies \( \omega_1, \omega_2 \) and times \( \tau_1, \tau_2 \) constant) the optimal hot and cold thermalization times \( \tau_c, \tau_h \) (2), (3).

**Correlation function of the two-level system.** The two-time correlation functions of the two-level system may be obtained in the following way. We introduce the vector \( \vec{A} \) having for components the three Pauli operators and the unit operator. The latter quantities form a complete set of observables for the two-level system. The master equation can then be rewritten as a matrix differential equation, \( d\vec{A}/dt = M \cdot \vec{A} \), or explicitly,

\[
\frac{d}{dt} \left( \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_z \\ \end{array} \right) = \left( \begin{array}{cccc} -\gamma & -\omega & 0 & 0 \\ \omega & -\gamma & 0 & 0 \\ 0 & 0 & -\gamma & -\gamma_0 \\ 0 & 0 & 0 & \gamma_0 \\ \end{array} \right) \left( \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_z \\ \end{array} \right). \tag{A.13}
\]

According to the quantum regression theorem [35], the equation of motion for the two-time correlation functions is the same as that for the operators, and we thus have

\[
\frac{d}{d\tau} \langle \vec{A}(t) \vec{A}(t + \tau) \rangle = \langle M(t) \vec{A}(t + \tau) \rangle. \tag{A.14}
\]

Since we are only interested in the correlation functions of the operator \( \sigma_x \), we can restrict ourselves to the upper left \( 2 \times 2 \) submatrix of \( M \). We find

\[
\frac{d}{d\tau} \vec{C} = \left( \begin{array}{cc} -\frac{\gamma}{2} & -\omega \\ \omega & -\frac{\gamma}{2} \end{array} \right) \vec{C}(\tau), \tag{A.15}
\]

with the correlation vector

\[
\vec{C}(\tau) = \left( \begin{array}{c} \langle \sigma_x(t) \sigma_x(t + \tau) \rangle \\ \langle \sigma_x(t) \sigma_y(t + \tau) \rangle \\ \langle \sigma_x(t) \sigma_z(t + \tau) \rangle \end{array} \right). \tag{A.16}
\]

The solution to eq. (A.15) is given by

\[
\vec{C}(\tau) = \left( \begin{array}{c} e^{-\frac{\gamma}{2} \tau} \cos(\omega \tau) - e^{-\frac{\gamma}{2} \tau} \sin(\omega \tau) \\ e^{-\frac{\gamma}{2} \tau} \sin(\omega \tau) & e^{-\frac{\gamma}{2} \tau} \cos(\omega \tau) \end{array} \right) \vec{C}(\tau = 0), \tag{A.17}
\]

with the initial condition

\[
\vec{C}(\tau = 0) = \left( \begin{array}{c} \langle \sigma_x(t) \rangle \\ \langle \sigma_x(t) \sigma_y(t) \rangle \end{array} \right) = \left( \begin{array}{c} 1 \\ i \langle \sigma_z(t) \rangle \end{array} \right). \tag{A.18}
\]

The last equality follows from the algebraic properties of the Pauli matrices. The symmetrized correlation function is equal to the real part of the above correlation function and reads

\[
C_{ij} = \frac{1}{2} \langle \sigma_x(t + \tau), \sigma_x(t) \rangle = e^{-\frac{\gamma}{2} \tau} \cos(\omega \tau). \tag{A.19}
\]

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