On $3+1$ Dimensional Scalar Field Cosmologies

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In this communication, we analyze the case of $3+1$ dimensional scalar field cosmologies in the presence, as well as in the absence of spatial curvature, in isotropic, as well as in anisotropic settings. Our results extend those of Hawkins and Lidsey [Phys. Rev. D 66, 023523 (2002)], by including the non-flat case. The Ermakov-Pinney methodology is developed in a general form, allowing through the converse results presented herein to use it as a tool for constructing new solutions to the original equations. As an example of this type a special blowup solution recently obtained in Christodoulakis et al. [gr-qc/0302120] is retrieved. Additional solutions of the $3+1$ dimensional gravity coupled with the scalar field are also obtained. To illustrate the generality of the approach, we extend it to the anisotropic case of Bianchi types I and V and present some related open problems.

I. INTRODUCTION

In the past few years, the ekpyrotic scenario has been proposed as an alternative to the standard inflationary cosmology [1]. In this interpretation, the big bang is viewed as the collision of two domain walls or branes described by Einstein’s equations coupled to a scalar field. The scalar in this case parametrizes the separation between the branes. Hence, in this dynamical setup, it is of interest to understand the coupling of scalar fields to gravity.

This role has been investigated rather extensively in 2+1 dimensional setups; see e.g., the earlier works of [2–4]. More recently the 3+1 dimensional case has become of interest; see e.g., [5–8].

These studies have in fact motivated the present work. In particular, in a significant recent paper [5], Hawkins and Lidsey have proposed a connection between the so-called Ermakov-Pinney (EP) equation and flat, 3+1 dimensional, scalar field cosmologies. This is of particular interest since the EP equation is a very special, linearizable, nonlinear ordinary differential equation (ODE) that can be solved exactly if the underlying linear Schrödinger equation can be solved. We note in passing that one of the authors of [5] went on to use this approach to illustrate analogies between scalar field cosmological models and the dynamics of moments of the wavefunction of Bose-Einstein condensates [9]. This point further highlights the importance of this ODE that has been recurring in a variety of different areas as diverse as nonlinear optics [10], elasticity [11], quantum field theory [12] or molecular physics [13]. For a recent review of the EP equation and its applications, see e.g., [14].

One of the purposes of the present short communication is to demonstrate how the methodology developed in [5] can be generalized in the case of non-zero curvature. We will also explicitly provide a converse result according to which, given a solution of the EP equation, the solution of a corresponding Einstein equation, coupled to a scalar field, can be derived. The use of the converse result will be explicitly demonstrated in a special case, that of a linear Klein-Gordon equation for the scalar field. The special solution for this case provided recently in [6] will be retrieved. More general solutions will also be constructed.

As our gravitational model, we will use the 3-dimensional Friedmann-Robertson-Walker (FRW) metric. While other models such as the Brans-Dicke metric, see e.g., [15], are also popular, the FRW metric is widely accepted in the early universe scenario that is of primary interest herein (see the references mentioned above). The relevant line element (in co-moving coordinates and with a “cosmological” time choice) will thus read:

$$ds^2 = -dt^2 + a^2(t) \left( \frac{1}{1 - cr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$

In the metric of Eq. (1), $a$ is the scale factor while $c$ describes the curvature of the spatial slice and can be normalized to the values $-1, 0, 1$ in the hyperbolic, flat and elliptic case respectively.

However, the above setting seems to restrict our considerations to an isotropic scenario. To illustrate the generality of the EP reduction and the usefulness of the corresponding technology, we also examine in the present work, an
anisotropic case, namely the one of Bianchi types I and V. In the latter, spatially homogeneous, yet anisotropic geometries, the line element is given by:

\[ ds^2 = -N(t)^2 dt^2 + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^idx^j \]  
\[ \text{(2)} \]

in the time gauge \( N(t) = \sqrt{\gamma(t)} \); \( \sigma^\alpha_i \) are the invariant basis one-forms of the homogeneous surfaces of simultaneity \( \Sigma_t \), while \( \gamma_{\alpha\beta} \) are the scale factors that constitute the (in principle) dynamical variables of the metric. We restrict ourselves in the present study to these anisotropic models, as these have been argued to have the desirable property of isotropizing at arbitrarily long times [16]. Additionally, type V is the simplest Bianchi model that admits velocities or tilted sources, hence it is natural to consider it as the simplest extension that would allow the universe to choose a reference frame at the exit from inflation (given that the de Sitter metric does not have a preferred frame).

Our presentation will proceed as follows: in section 2 we will provide the general EP methodology for the FRW system in the presence of a scalar. In section 3, we will apply these results in the special case of a massive scalar. In section 4 we will give additional special solutions to these equations, while in section 5, we will apply the method to the anisotropic case of Bianchi types I and V. Finally, in section 6, we will summarize our findings and present our conclusions.

II. 3+1 ISOTROPIC SCALAR FIELD WITH CURVATURE: THE EP REDUCTION

We will follow the notation of [5] in what follows. In particular, in this setting, the Einstein equations of gravity and the Klein-Gordon equation for the scalar field can be respectively written as

\[ H^2 + \frac{c}{a^2} = \frac{\kappa^2}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{D}{a^n} \right] \]  
\[ \ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0. \]  
\[ \text{(3)} \]

\( H = \dot{a}/a \) represents the Hubble parameter. The first two terms in the bracketed expression of Eq. (3) denote the energy density of the scalar field with potential \( V(\phi) \). The last term is the density of matter for the barotropic fluid with equation of state \( p_{\text{mat}} = (n-3)\rho_{\text{mat}}/3; D \geq 0, 0 \leq n \leq 6, \kappa^2 = 8\pi/m_P^2 \), where \( m_P \) is the Planck mass. Finally, the dots will be used for differentiation with respect to the cosmological or coordinate time \( t \). Notice that with respect to the corresponding equations (2)-(3) of [5], the former is augmented by the spatial curvature term (recall that \( c \) can take the values \(-1, 0 \text{ and } 1 \) depending on the curvature of the hypersurface \( t = \text{const.} \)) in the left hand side.

Now, by using a further differentiation of Eq. (3) and the substitution \( b = a^{n/2} \) (cf. with Eq. (8) of [5]), we obtain:

\[ 2 - \frac{c}{b^{n/2}} = -\frac{\kappa^2}{2} \left[ \dot{\phi}^2 + \frac{nD}{3b^2} \right]. \]  
\[ \text{(5)} \]

If we now define a new comoving time \( \tau \) such that \( \dot{\tau} = b \), then Eq. (5) becomes

\[ \frac{db}{d\tau} + \frac{\kappa^2 n}{4} \left( \frac{db}{d\tau} \right)^2 b = -\frac{\kappa^2 n^2 D}{12b^3} + \frac{nc}{2b^{4+n}}, \]  
\[ \text{(6)} \]

where the chain rule has been used and the functions in eq. (5) have been assumed as \( b = b(t(\tau)) \) and \( \phi = \phi(t(\tau)) \). This equation can be compared with Eqs. (10)-(11) of [5], with the difference being evident in the inclusion of the curvature term (the last term on the right hand side of Eq. (6)).

This prompts us to (briefly) discuss the EP equation which naturally arises in this context not only in the flat case of \( c = 0 \) examined in [5], but also in the case of \( n = 2 \). The latter is a remarkable example of a nonlinear yet integrable ordinary differential equation (ODE) of the form:

\[ Y'' + Q(\tau)Y = \frac{\lambda}{Y^3}. \]  
\[ \text{(7)} \]

The particularly appealing feature of this nonlinear ODE is that its general solution can be obtained, provided that one is able to solve the linear Schrödinger (LS) problem \( Y'' + Q(\tau)Y = 0 \). For details on the properties of the EP equation, the interested reader is referred to [5,14] and references therein. Here we just mention its basic superposition
principle property. Namely, if the linearly independent solutions of the LS equation are $Y_1(\tau)$ and $Y_2(\tau)$, then the most general possible solution of the EP equation is given by

$$Y(\tau) = \left( AY_1^2 + BY_2^2 + 2CY_1Y_2 \right)^{1/2}$$  \hspace{1cm} (8)

where $A$, $B$ and $C$ are constants connected through

$$AB - C^2 = \frac{\lambda}{W^2}$$  \hspace{1cm} (9)

and the Wronskian $W = Y_1Y_2' - Y_2Y_1'$.  

It is of particular interest to note that the result of Eq. (6) can be used conversely for constructing solutions of scalar field cosmologies, using the EP equation structure and solutions. The converse result can be proved in the following form: Given $Q$ and $\lambda = -\kappa^2 n^2 D/12 < 0$, let $Y > 0$ be a solution of

$$\frac{d^2Y}{d\tau^2} + QY = \frac{\lambda}{Y^3} + \frac{nc}{2Y^{1+\frac{n}{2}}}$$  \hspace{1cm} (10)

Define a new time coordinate $t$ such that $\dot{\tau} = Y(\tau(t))$ and $a = Y^{2/n}$, as well as a new field $\phi$ satisfying:

$$\frac{n\kappa^2}{4} \left( \frac{d\phi}{d\tau} \right)^2 = Q,$$  \hspace{1cm} (11)

with $Q \neq 0$. Finally, define a potential:

$$V(\phi) = \frac{12}{\kappa^2 n^2} \left( \frac{dy}{d\tau} \right)^2 - \frac{Y^2}{2} \left( \frac{d\phi}{d\tau} \right)^2 - \frac{D}{Y^2} + \frac{3c}{\kappa^2 Y^{4/n}}.$$  \hspace{1cm} (12)

Then, if we consider the triplet $(a(\tau(t)), \phi(\tau(t)), V(\phi))$, the latter satisfies the Eqs. (3)-(4).

Let us now give a number of specific examples, where this construction scheme can be used to obtain solutions to the $3 + 1$ dimensional scalar field cosmology equations.

### III. Applications

#### A. Flat FRW Metric and Massless Scalar

In the absence of matter, we can set $D = 0 \Rightarrow \lambda = 0$, and consider for convenience the case of $n = 2$. For $\lambda = 0$, and for the flat FRW metric e.g., for $c = 0$, Eq. (10) becomes

$$\frac{d^2Y}{d\tau^2} + QY = 0$$  \hspace{1cm} (13)

Assuming the general case $V(\phi) = m^2\phi^2/2$, the particular scenario of a massless scalar yields $V(\phi) = 0$, hence Eq. (12) becomes:

$$\frac{d^2Y}{d\tau^2} + \frac{3}{Y} \left( \frac{dY}{d\tau} \right)^2 = 0$$  \hspace{1cm} (14)

whose solution yields:

$$Y(\tau) = A\tau^{1/4};$$  \hspace{1cm} (15)

hence from Eq. (13):

$$Q(\tau) = \frac{3}{16} \frac{1}{\tau^2}.$$  \hspace{1cm} (16)

This, in turn, through the definition of $\dot{\tau}$ yields:

$$\tau(t) = \left( \frac{3}{4} \right)^{\frac{4}{3}} A^{4/3} t^{\frac{1}{2}}$$  \hspace{1cm} (17)

and, thus, finally from Eq. (11):

$$\phi(t) = \sqrt{\frac{2}{3}} \log(\frac{3A}{4}) + \sqrt{\frac{2}{3}} \log(t - t_0).$$  \hspace{1cm} (18)

which is the same solution as obtained in Eq. (8) of [6].
B. A Generalization: Non Flat FRW Metric Coupled with Scalar

To demonstrate the generality of the technique also in non-flat cases with $c \neq 0$, we examine the example of $n = 2$, $\kappa = 1$ and $D > c/3$ (which implies that $\lambda + c < 0$).

In this case, Eq. (10) becomes:

$$
\frac{d^2 Y}{d\tau^2} + Q Y = \frac{\lambda + c}{Y^3}.
$$

Motivated by cases in which we are able to solve the EP (or equivalently the underlying linear Schrödinger) equation exactly, we choose $Q(\tau) = 3/(16\tau^2)$ (cf. Eq. (16)). In this case, using Eq. (8), we can find the general solution to Eq. (19) as:

$$
Y(\tau) = \left( A\tau^2 + B\tau + 2C\right)^{1/2},
$$

where $AB - C^2 = 4(\lambda + c)$. For convenience, we use our freedom of coefficients to set $A = B = 0$, hence $C = 2\sqrt{\lambda + c}$. We thus obtain:

$$
Y(\tau) = (2C\tau)^{1/2};
$$

From Eq. (11), we derive:

$$
\phi(\tau) = \frac{\sqrt{6}}{4} \log(\tau).
$$

From the definition of $\dot{\tau}$, it can be seen that:

$$
\tau = \frac{C}{2}(t - t_0)^2.
$$

Finally, from Eq. (12), it follows that:

$$
V(\phi) = \frac{S}{\tau} \equiv S e^{-\frac{4}{\kappa}\phi}
$$

where $S = 9c/8 - D/(2c) + 3/2$.

This result generalizes the corresponding result of Eq. (31) of [5] in the case in which curvature is present.

C. An Example of Quadratic Scalar Fields

As another example, we will use a case in which we use the EP approach in a reverse engineering way. In particular, in Eq. (10), we will postulate the solution and we will obtain the potential that is compatible with this solution. We assume that matter is absent, hence $D = \lambda = 0$, for $n = 4$ and $c \geq 0$. We now demand that $Y(\tau) = 2B\tau$. Then

$$
Q(\tau) = \frac{c}{4B^3\tau^3}.
$$

From the definition of $\dot{\tau}$, we obtain that

$$
\tau(t) = \frac{A^2}{2B} e^{2Bt},
$$

where $A$ is (without loss of generality) a positive constant. Then from Eq. (11), we obtain:

$$
\phi(t) = -\frac{\sqrt{2c}}{\kappa AB} e^{-Bt} + \alpha,
$$

where $\alpha$ is an arbitrary constant, while the potential

$$
V(\phi) = \frac{3B^2}{\kappa^2} + B^2(\phi - \alpha)^2
$$

is quadratic in $\phi$. Notice that in this example, the restriction of $n = 4$ can be lifted: in particular substituting $A \to A^{n/4}$ and $B \to B n/4$, we can perform the same calculation for any power $n$, but the expression for $V(\phi)$ is functionally the same as the one of Eq. (28).

We remark here that the solution found in this subsection was previously identified by means of a different (than the EP) approach in [17] (cf. Eqs. (24)-(28) therein).
D. An Example of Constant Scalar Fields

In the same inverse procedure spirit, another convenient choice of \( Q(\tau) \) is \( Q(\tau) = 0 \) (for \( n = 2 \)). Then the EP equation has the straightforward general solution:

\[
Y(\tau) = (A\tau^2 + B + 2C\tau)^{1/2},
\]

with

\[
AB - C^2 = \lambda + c = -\frac{\kappa^2 D}{3} + c \equiv \tilde{\lambda}.
\]

Consequently,

\[
\tau(t) = \frac{D}{4A} \exp \left( A^{1/2} t \right) + \frac{\tilde{\lambda}}{DA^{1/2}} \exp \left( -A^{1/2} t \right) - \frac{C}{A}.
\]

It can then be immediately seen that the scalar \( \phi \) and the potential \( V \) are constants, while the scale factor

\[
a(t) = a(0) \cosh \left( A^{1/2} t \right) + \sqrt{a(0)^2 - \frac{\tilde{\lambda}}{A}} \sinh \left( A^{1/2} t \right)
\]

with

\[
a(0) = \frac{D}{4A^{1/2}} + \frac{\tilde{\lambda}}{DA^{1/2}}.
\]

It should be noted that this is a generalization of the solution of Eq. (24) of [5] in the spatially non-flat case. Similar solutions have been obtained through direct calculations (i.e., instead of the EP methodology) in [3].

IV. ANISOTROPIC GENERALIZATIONS: EP REDUCTION FOR BIANCHI TYPES I AND V

A. Bianchi Type I

Since the anisotropic case was not previously discussed in the earlier work of [3–5], we discuss it here in more detail. In this case, the dynamical scale factors and the one-forms in the metric of Eq. (2) are given by:

\[
\gamma_{\alpha\beta}(t) = \begin{pmatrix}
A(t)^2 & 0 & 0 \\
0 & B(t)^2 & 0 \\
0 & 0 & \Gamma(t)^2
\end{pmatrix}, \quad \sigma_{i}(x) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

We can define the tensor: \( F_{\mu\nu} = G_{\mu\nu} - 8\pi T_{\mu\nu} \) where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor and \( T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}(\phi_{,\alpha}\phi_{,\alpha} + m^2 \phi^2) \) is the energy momentum tensor. Then, the quadratic constraint is the equation \( F^0_0 = 0 \), the kinematic equation is given by \( F^1_0 = 0 \), while the Klein-Gordon equation for the field is given by \( \phi_{,\mu} - m^2 \phi = 0 \). Notice additionally, that the two integrals of the motion, namely \( I_1 = F^1_1 - F^2_2 = 0 \) and \( I_2 = F^1_1 - F^3_3 = 0 \) yield \( B(t) = A(t) e^{\kappa t/2} \) and \( \Gamma(t) = A(t) e^{\lambda t/2} \).

Solving the Klein-Gordon equation for \( \phi'(t) \) and substituting the result into \( \partial_0 F^0_0 = 0 \) (as well as solving \( F^0_0 = 0 \) for \( \phi \) and using the resulting expression in \( \partial_0 F^0_0 = 0 \)), one is led to a dynamical equation for the remaining scale factor \( A(t) \) in the form:

\[
\frac{\kappa \lambda}{4} + \frac{\kappa A'(t)}{A(t)} + \frac{\lambda A'(t)}{A(t)} + \frac{4 A'(t)^2}{A(t)^2} - \frac{\phi'(t)^2}{A(t)} = 0
\]

Using now: \( A(t) = Y(t)^{2/n} \) and a change of variable \( \tau = \int \Omega(t')dt' \), we obtain:

\[
\ddot{Y}(\tau) + \dot{Y}(\tau) \frac{\Omega'(t)}{\Omega(t)^2} - \frac{\dot{Y}(\tau)^2}{Y(\tau)} (6 + n) - (\kappa + \lambda) \frac{\ddot{Y}(\tau)}{8 \Omega(t)^2} - \frac{n \kappa \lambda Y(\tau)}{8 \Omega(t)^2} + \frac{n Y(\tau) \phi'(\tau)^2}{4} = 0
\]
Hence, a choice of time reparametrization according to:

$$\frac{\Omega'(t)}{\Omega(t)} = \kappa + \lambda + \frac{(6 + n) Y'(t)}{n Y(t)}$$  \hspace{1cm} (36)

(which leads to \(\Omega(t) = \theta e^{(\kappa + \lambda)t} Y(t)^{(6+n)/n}\), where \(\theta > 0\) is a constant of integration), results in the form:

$$\ddot{Y}(\tau) + QY(\tau) = \frac{\Gamma}{Y(\tau)^{1+12/n}}, \quad Q = n \frac{\dot{\phi}(\tau)^2}{4}, \quad \Gamma = \frac{n \kappa \lambda e^{-(\kappa + \lambda)t}}{8 \theta^2}.$$  \hspace{1cm} (37)

Eq. (37) is of the form of Eq. (6) and becomes an EP equation for the choice of \(n = 6\), for \(\kappa = -\lambda\).

In the more general case, the problem becomes extremely complex as \(\tau\) depends on \(\tau\) through \(d\tau/dt = \Omega(t)\) and \(\Omega\) is itself a function of the solution. Typically, this problem will not be analytically tractable (if combined, it leads to an integro-differential equation for \(\dot{Y}(\tau)\)). However, this can be used as an inverse problem: we can postulate \(\tau(t)\), derive from it \(\Omega(t)\) and \(Y(t)\) and use them in Eq. (37) to derive the form of \(\phi(t)\).

A simple example of the above methodology can be given as follows: let us assume that \(\tau = e^{(\kappa + \lambda)t}/(\kappa + \lambda)\), then \(\Omega(t) = e^{(\kappa + \lambda)t}Y(\tau)\). Hence, choosing \(\theta = 1\), for \(n = 6\) (the EP-like case for Eq. (37)) \(Y(t) = Y(\tau) = 1\). Then, using the Eq. (37), we find that

$$\phi(\tau) = C + \frac{\kappa \lambda}{2(\kappa + \lambda)^2} \log(\tau)$$  \hspace{1cm} (38)

and hence (e.g., choosing the integration constant \(C = \log(\kappa + \lambda)\sqrt{\kappa \lambda/(2(\kappa + \lambda)^2)}\), \(\phi(t) = \sqrt{\frac{2\kappa \lambda}{\kappa + \lambda}}t\).

**B. Bianchi Type V**

In this case, the scale factors and one-forms are, in turn, given by:

$$\gamma_{\alpha \beta}(t) = \begin{pmatrix} B(t)^2 & 0 & 0 \\ 0 & \Gamma(t)^2 & 0 \\ 0 & 0 & A(t)^2 \end{pmatrix}, \quad \sigma^\alpha_i(x) = \begin{pmatrix} 0 & e^{-x} & 0 \\ 0 & 0 & e^{-x} \\ 1 & 0 & 0 \end{pmatrix}.$$

In type V, there exist three integrals of the motion, namely: \(I_1 = F_1^1 - F_2^2 = 0\), \(I_2 = F_1^1 - F_3^3 = 0\) and \(I_3 = F_1^3 = 0\), which yield: \(B(t) = A(t)e^{\kappa t/2}\) as well as: \(\Gamma(t) = A(t)e^{-\kappa t/2}\).

Once again following the same path as before and substituting into \(\partial_t F_0^0 = 0\), we obtain a single ODE for \(A(t)\) of the form:

$$-6 \kappa^2 A(t) A'(t) - 24 A(t)^5 A'(t) + \frac{96 A'(t)^3}{A(t)} - 12 A(t) A'(t) \Phi'(t)^2 - 24 A'(t) A''(t) = 0$$  \hspace{1cm} (39)

Through a similar motivation as in the previous subsection, we use the change of variables: \(A(t) = Y(t)^{2/n}\), alongside the time reparametrization \(\tau = \int^t \Omega(t')dt'\), with \(\Omega(t) = \theta Y^{(6+n)/n}\). This results in the final form:

$$\ddot{Y}(\tau) + QY(\tau) = -\frac{n \kappa^2}{8 \theta^2} \frac{1}{Y(\tau)^{1+12/n}} - \frac{n}{2 \theta^2} \frac{1}{Y(\tau)^{1+4/n}}.$$  \hspace{1cm} (40)

where \(Q = n \frac{\dot{\phi}(\tau)^2}{4}\), which is again a generalized form of an EP equation. This statement is made in the sense that while there is no choice of the exponent that yields a mere inverse cubic dependence on \(Y(t)\) in the right hand side of Eq. (40), the typical scenario involves inverse power dependence in a manner similar to the EP equation.

While the resulting Eq. (40) is not directly of the form of the EP equation, its dynamics can be controllably tuned to be close to those of the EP solutions. In particular, it is clear that the asymptotic behavior of the equation will be similar to that of \(\dot{Y}(\tau) + QY(\tau) = -\frac{\kappa^2}{8 \theta^2} \frac{1}{Y(\tau)^{1+12/n}}\) for small initial data, while it will be close to that of \(\dot{Y}(\tau) + QY(\tau) = -\frac{n \kappa^2}{2 \theta^2} \frac{1}{Y(\tau)^{1+4/n}}\) for the case of large initial data. Hence, the choice of \(n = 6\) for small initial data and the one of \(n = 2\) for large initial data yields EP behavior (this point has also been verified in numerical investigations of the equation not shown here). In the intermediate regime where there is competition between the two terms the EP results will not be immediately applicable and one should resort to numerical simulations of Eq. (40).
V. CONCLUSIONS

In this short communication, we have generalized the earlier work of [5] and of [4] in the direction of obtaining solutions to 3+1 dimensional cosmological models for the Friedmann-Robertson-Walker case in a systematic way. The method re-casts the relevant ordinary differential equations into one of the Ermakov-Pinney type which is explicitly solvable. From the solutions of the resulting EP equation one can re-construct the solutions to the original cosmological model in a step-by-step inverse process that has been detailed. The present study generalizes that of [5] in that cases of non-zero curvature of the FRW metric can be also considered. It also extends the results of [4] in the 3+1 dimensional context. The method can be used to derive a variety of solutions in the latter context, including ones considered previously in [2,3,6].

On the other hand, we have also illustrated the generality of our method, by exploring the potential of such Ermakov-Pinney reductions to systems with anisotropy. In particular, we have demonstrated that the case examples of Bianchi types I and V can be reduced to Ermakov-Pinney-like equations. These Bianchi types were chosen as prototypical examples where a velocity/tilt can be included and as examples of Bianchi types that may eventually isotropize. The EP reduction of these cases highlights the generality and usefulness of the procedure. However, by the same token and since the EP is the only one of these inverse power, nonlinear ordinary differential equations for which an explicit solution exists, it underlines the importance of understanding the behavior of such classes of equations. In particular, examining numerically their temporal evolution for a number of physically motivated cases would be a natural next step. Such studies are currently in progress and will be reported in a future work.

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