A Note on the Cohomology of Moduli of Rank Two Stable Bundles

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0 Introduction

In recent years the cohomology ring of the moduli space \( \mathcal{N}_g(2, d) \) of rank two and odd degree \( d \) stable bundles over a Riemann surface \( M \) of genus \( g \geq 2 \) has been extensively studied \cite{2, 5, 6, 10, 11}. The subring \( H^*(\mathcal{N}_g(2, d); \mathbb{Q})^F \) of \( H^*(\mathcal{N}_g(2, d); \mathbb{Q}) \) which is invariant under the induced action of the mapping class group \( \Gamma \) of \( M \) has also been much studied and has been shown to play a central role in the ring structure of \( H^*(\mathcal{N}_g(2, d); \mathbb{Q}) \).

In 1991 Zagier \cite{11} began a study of certain relations in the invariant cohomology ring. These are defined recursively in terms of Newstead’s generators \( \alpha, \beta, \gamma \) \cite{1} by

\[
(r + 1)\zeta_{r+1} = \alpha \zeta_r + r \beta \zeta_{r-1} + 2 \gamma \zeta_{r-2}
\]

with \( \zeta_0 = 1 \) and \( \zeta_r = 0 \) for \( r < 0 \). Each of the authors \cite{2, 5, 10, 11} showed that \( \zeta_r \) is a relation in \( H^*(\mathcal{N}_g(2, d); \mathbb{Q})^F \) for \( r \geq g \) and that \( \zeta_g, \zeta_{g+1}, \zeta_{g+2} \) generate the relation ideal of the invariant cohomology ring. King and Newstead further proved a decomposition theorem \cite[Prop.2.5]{5}, originally conjectured by Mumford, describing \( H^*(\mathcal{N}_g(2, d); \mathbb{Q}) \) in terms of \( H^*(\mathcal{N}_k(2, d); \mathbb{Q})^F \) \((k \leq g)\) and exterior powers of \( H^3(\mathcal{N}_g(2, d); \mathbb{Q}) \).

The methods employed by the authors \cite{2, 5, 6, 10, 11} differ greatly from Kirwan’s original proof \cite[§2]{3} of Mumford’s conjecture. Mumford introduced relations in \( H^*(\mathcal{N}_g(2, d); \mathbb{Q}) \) which are constructed from the vanishing Chern classes of a rank \( 2g-1 \) bundle \( \pi V \) over \( \mathcal{N}_g(2, d) \) and conjectured that these relations are complete. Zagier \cite[§6]{11} showed that the relations \( \zeta_r, r \geq g \) form a subset of the Mumford relations and for this reason we will refer to the vanishing \( \zeta_r \) as the Zagier–Mumford relations.

The purpose of this note is two-fold. We will firstly rederive the result that the first three Zagier–Mumford relations form a minimal complete set for the invariant cohomology. The second result is to prove a subsequent and stronger version of Mumford’s conjecture; namely we will show that the relations constructed solely from the first vanishing Chern class \( c_{2g}(\pi V) \) freely generate the relation ideal of \( H^*(\mathcal{N}_g(2, d); \mathbb{Q}) \) as a \( \mathbb{Q}[\alpha, \beta] \)-module. Both results follow easily from Kirwan’s calculations in \cite[§2]{3}. Partly the aim of this note is to demonstrate the power of the methods of \cite{3} which currently is the only approach to have generalised to the rank three case \cite{6}.

For ease of notation we will from now on write \( \mathcal{N} \) for \( \mathcal{N}_g(2, d) \) and write \( \mathcal{N}_0 \) for the moduli space of rank two odd degree stable bundles of fixed determinant. Also we write \( g \) for \( g - 1 \) and \([2g]\) for the set \( \{1, \ldots, 2g\} \).

1 Kirwan’s Approach

Let \( \mathcal{C} \) denote the space of all holomorphic structures on a fixed \( C^\infty \) complex vector bundle \( \mathcal{E} \) over \( M \) of rank two and odd degree \( d \) and let \( \mathcal{G}_e \) denote the group of all \( C^\infty \) complex automorphisms of \( \mathcal{E} \). We may then identify \( \mathcal{N} \) with the quotient \( \mathcal{C}^e/\mathcal{G}_e \) where \( \mathcal{C}^e \subset \mathcal{C} \) is the open subset consisting of stable holomorphic structures. Let \( \mathcal{G} \) denote the gauge group of all \( C^\infty \) automorphisms of \( \mathcal{E} \) which are unitary with respect to a fixed Hermitian structure and let \( \mathcal{G}_e \) denote the quotient of \( \mathcal{G} \) by its \( U(1) \) centre.

Then \( H^*(\mathcal{N}; \mathbb{Q}) \) is naturally isomorphic to \( H^*_0(\mathcal{C}^e; \mathbb{Q}) \) \(\cite[9.1]{4}\), and Atiyah and Bott show further \(\cite[thm.7.14]{4}\) that the restriction map

\[
H^*(B\mathcal{G}_e, \mathbb{Q}) \cong H^*_0(\mathcal{C}^e; \mathbb{Q}) \rightarrow H^*_0(\mathcal{C}^e; \mathbb{Q}) \cong H^*(\mathcal{N}; \mathbb{Q})
\]

is surjective. They construct a rank two \( \mathcal{G} \)-equivariant holomorphic bundle \( \mathcal{V} \) over \( \mathcal{C} \times M \) and define generators

\[
a_1, a_2, f_2 \text{ and } b_1^s, b_2^s \quad (s \in [2g])
\]
for $H^*_G(C;\mathbb{Q}) = H^*(BG;\mathbb{Q})$ by taking the slant products

$$c_r(V) = a_r \otimes 1 + \sum_{s=1}^{2g} b^r_s \otimes e_s + f_r \otimes \omega$$

where $e_1, ..., e_{2g}$ is a fixed basis for $H^1(M;\mathbb{Q})$ and $\omega$ is the standard generator for $H^2(M;\mathbb{Q})$. The only relations amongst the generators $(\mathbb{B})$ are that $a_1, a_2, f_2$ commute with everything and that the $b^r_i$ anticommute amongst themselves.

Rather than $(\mathbb{B})$ we shall consider the surjection

$$H^*_G(C;\mathbb{Q}) \otimes \mathbb{Q}[a_1] \cong H^*_G(C;\mathbb{Q}) \to H^*_G(C^s;\mathbb{Q}) \cong H^s(N;\mathbb{Q}) \otimes \mathbb{Q}[a_1].$$

(4)

The images of $(\mathbb{B})$ under this map form generators for $H^*_G(C^s;\mathbb{Q})$ which we will also refer to as $a_1, a_2, f_2, b^1_1, b^2_2$ and the relations among these restrictions form the kernel of $(\mathbb{B})$. In order to study this kernel we introduce a $G$-perfect stratification of $C$ due to Shatz $(\mathbb{B})$.

Any unstable holomorphic bundle $E$ over $M$ of rank $n$ and degree $d$ has a canonical filtration (or flag) $(\mathbb{B})$ p.221 which in the rank $n = 2$ case is a line subbundle $L$ of $E$ of degree $d_1$ such that $d_1 > d/2$.

We define the type of $E$ to be $(d_1, d - d_1)$ and define the type of a stable bundle to be $\mu_0 = (d/2, d/2)$.

The stratum $\mathcal{C}_\mu \subseteq \mathcal{C}$ is the set of all holomorphic vector bundles of type $\mu$ and we construct a total order $\preceq$ on the set of types by writing $(\mu_1, \mu_2) \preceq (\nu_1, \nu_2)$ if $\mu_1 \leq \nu_1$.

Kirwan’s proof of Mumford’s conjecture is based upon a set of completeness criteria for a set $\mathcal{R}$ of relations in $H^*_G(C;\mathbb{Q})$ $(\mathbb{B})$ Prop.1. These criteria involve finding for each $\mu \neq \mu_0$ relations $\mathcal{R}_\mu \subseteq \mathcal{R}$ which in a technical sense correspond to the stratum $\mathcal{C}_\mu$. We introduce here similar completeness criteria for the invariant cohomology:

**PROPOSITION 1 (Invariant Completeness Criteria)** Let $\mathcal{R}$ be a subset of the kernel of the restriction map

$$H^*_G(C;\mathbb{Q})^\Gamma \to H^*_G(C^s;\mathbb{Q}).$$

(5)

Suppose that for each unstable type $\mu$ there is a subset $\mathcal{R}_\mu$ of the ideal generated by $\mathcal{R}$ in $H^*_G(C;\mathbb{Q})^\Gamma$ such that the restriction of $\mathcal{R}_\mu$ to $H^*_G(C_\nu;\mathbb{Q})$ is zero when $\nu \prec \mu$ and when $\nu = \mu$ equals the ideal generated by $e_\mu$ in $H^*_G(C_\mu;\mathbb{Q})^\Gamma$, where $e_\mu$ denotes the equivariant Euler class of the normal bundle to $C_\mu$ in $C$. Then $\mathcal{R}$ generates the kernel of the restriction map $(\mathbb{B})$ as an ideal of $H^*_G(C;\mathbb{Q})^\Gamma$.

**PROOF:** We include now the main points in the proof of the above proposition. However the only difference between this proof and the argument of $(\mathbb{B})$ prop.4 is to observe that $\mathcal{C}_\mu$ is $\Gamma$-invariant and hence $e_\mu \in H^*_G(C_\mu;\mathbb{Q})^\Gamma$.

For $\mu$ an unstable type let $\mu - 1$ denote the type previous to $\mu$ with respect to $\preceq$ and define $V_\mu = \bigcup_{\nu \preceq \mu} C_\nu$. Then $V_\mu$ is an open subset of $C$ which contains $C_\mu$ as a closed submanifold.

Let $d_\nu$ denote the complex codimension of $C_\nu$ in $C$. For any given $i \geq 0$ there are only finitely many $\nu \in \mathcal{M}$ such that $2d_\nu \leq i$ $(\mathbb{B})$ 7.16] and so for each $i \geq 0$ there exists some $\mu$ such that

$$H^*_G(C;\mathbb{Q}) = H^*_G(V_\mu;\mathbb{Q}).$$

Hence it is enough to show that for each $\mu$ the image in $H^*_G(V_\mu;\mathbb{Q})^\Gamma$ of the ideal generated by $\mathcal{R}$ contains the image in $H^*_G(V_\mu;\mathbb{Q})^\Gamma$ of the kernel of $(\mathbb{B})$. Note that the above is clearly true for $\mu = \mu_0$ as $V_{\mu_0} = C^s$.

We will proceed by induction with respect to $\preceq$.

Assume now that $\mu \neq \mu_0$ and that $\zeta \in H^*_G(C;\mathbb{Q})^\Gamma$ lies in the kernel of $(\mathbb{B})$. Suppose that the image of $\zeta$ in $H^*_G(V_{\mu-1};\mathbb{Q})$ is in the image of the ideal generated by $\mathcal{R}$. We may, without any loss of generality, assume that the image of $\zeta$ in $H^*_G(V_{\mu-1};\mathbb{Q})$ is zero. Then by the exactness of the Thom-Gysin sequence

$$\cdots \to H^{-2d_\nu}_{G}(C_\mu;\mathbb{Q}) \to H^*_G(V_\mu;\mathbb{Q}) \to H^*_G(V_{\mu-1};\mathbb{Q}) \to \cdots$$

there exists an element $\eta \in H^{-2d_\nu}_{G}(C_\mu;\mathbb{Q})$ which is mapped to the image of $\zeta$ in $H^*_G(V_\mu;\mathbb{Q})$ by the Thom-Gysin map. The composition

$$H^{-2d_\nu}_{G}(C_\mu;\mathbb{Q}) \to H^*_G(V_\mu;\mathbb{Q}) \to H^*_G(C_\mu;\mathbb{Q})$$

is given by multiplication by $e_\mu$ which is not a zero-divisor in $H^*_G(C_\mu;\mathbb{Q})$ $(\mathbb{B})$ p.569]. Hence the restriction of $\zeta$ in $H^*_G(C_\mu;\mathbb{Q})$ is $\eta e_\mu$ and by our initial observation $\eta \in H^{-2d_\nu}_{G}(C_\mu;\mathbb{Q})^\Gamma$. By hypothesis there exists
\[ H_\sigma^*(V_\mu; Q) \rightarrow \bigoplus_{\nu \leq \mu} H_\sigma^*(C_\nu; Q) \]  

(6)

is injective. The images of \( \theta \) and \( \zeta \) under (3) are equal and hence the restrictions of \( \theta \) and \( \zeta \) to \( H_\sigma^*(V_\mu; Q) \) are the same, completing the proof. \( \square \)

2 The Mumford and Zagier-Mumford Relations

The group \( \mathcal{G} \) acts freely on \( \mathcal{C}^* \) and the \( (1) \)-centre of \( \mathcal{G} \) acts as scalar multiplication on the fibres of \( \mathcal{V} \). The projective bundle of \( \mathcal{V} \) descends to a holomorphic projective bundle over \( \mathcal{N} \times M \) which is the projective bundle of a universal holomorphic bundle \( V \) of rank two and odd degree \([8, \text{p.}857, \text{p.}877]\) by requiring the relation

\[ f_2 = (d - 2\bar{g})a_1 + \sum_{s=1}^{g} b_1^s + g. \]

(7)

Let \( \pi : \mathcal{N} \times M \rightarrow \mathcal{N} \) be the first projection. When \( d = 4g - 3 \) then any \( E \in \mathcal{C}^* \) has slope \( \mu(E) = d/n > 2\bar{g} \) and thus \([8, \text{lemma} 5.2]\) \( H^1(M, E) = 0 \). Hence \( \pi_!V \) is a genuine vector bundle over \( \mathcal{N} \) of rank \( 2g - 1 \) with fibre \( H^0(M, E) \) over \( [E] \in \mathcal{N} \). We know from \([8, \text{prop.}9.7]\) that

\[ H_\sigma^*(\mathcal{C}; Q) \cong H^*(\mathcal{N}_0; Q) \otimes Q[a_1] \otimes \Lambda^*(b_1^1, \ldots, b_1^{2g}). \]

The Mumford relations \( c_{r,S} \) (\( r \geq 2g, S \subseteq [2g] \)) are then defined by writing

\[ c_r(\pi_!V) = \sum_{S \subseteq [2g]} c_{r,S} \prod_{s \in S} b_1^s, \]

(8)

where each \( c_{r,S} \) is written in terms of generators for \( H^*(\mathcal{N}_0; Q) \otimes Q[a_1] \), namely \( a_1 \) and Newstead’s generators \( \alpha, \beta, \psi_s \). In terms of the generators \([8]\) these are given by

\[ \alpha = 2f_2 - da_1, \quad \beta = (a_1)^2 - 4a_2, \quad \psi_s = 2b_2^s. \]

Kirwan’s proof of Mumford’s conjecture \([8, \text{§2}]\) shows that the Mumford relations together with the normalising relation \([8]\) form a complete set of relations for \( H^*(\mathcal{N}_0; Q) \). Following Kirwan \([8, \text{p.}871]\) we reformulate the definition \([8]\) and write

\[ \Psi(t) = \sum_{r=0}^{\infty} c_r(\pi_!V) t^{2g-1-r} = \sum_{r=-\infty}^{\bar{g}} (\sigma_r^0 + \sigma_r^1 t) (t^2 + a_1 t + a_2)^r, \quad \sigma_r^k = \sum_{S \subseteq [2g]} c_{r,S} \prod_{s \in S} b_1^s. \]

(9)

We will also refer to \( \sigma_{r,S}^k \) (\( k = 0,1, r < 0, S \subseteq [2g] \)) as the Mumford relations. (Note \( \sigma_0^0 \) and \( \sigma_1^0 \) differ slightly from Kirwan’s terms \( \sigma_r \) and \( \tau_r \)) This new formulation will prove more convenient when we need to determine the restrictions of the Mumford relations to various strata. Atiyah and Bott define generators

\[ a_1^1, a_1^2, \text{ and } b_1^{1,s}, b_1^{2,s} \]

for \( H_\sigma^*(\mathcal{C}_\mu; Q) \) via the isomorphism \([8, \text{prop.}7.12]\)

\[ H_\sigma^*(\mathcal{C}_\mu; Q) \cong H^*_\sigma((1,d_1), (C(1,d_1)^{s*}; Q) \otimes H^*_\sigma((1,d_2),(C(1,d_2)^{s*}; Q)). \]

In terms of these generators the crucial calculation of Kirwan in her proof of Mumford’s conjecture is:

**Lemma 2 (Kirwan)** \([8, \text{pp.}871-873]\) Let \( \mu = (d_1, d_2) \) and write \( D = d_2 - 2g + 1 \). Then the restrictions \( \sigma_{D,S}^k \) of \( \sigma_{D,S}^0 \) (\( k = 0,1 \)) in \( H_\sigma^*(\mathcal{C}_\mu; Q) \) are given by

\[ \sigma_{D,S}^0 = \frac{(-1)^{g/2} g/2}{2^{2g} g!} \prod_{s \in S} (b_1^{2,s} - b_1^{1,s}) a_1^1 e_\mu, \quad \sigma_{D,S}^1 = \frac{(-1)^{g/2}}{2^{2g} g!} \prod_{s \in S} (b_1^{2,s} - b_1^{1,s}) b_1^{1,s} e_\mu. \]

(9)
This calculation plays a major role in the following two theorems.

**THEOREM 3** ([1], [3], [7], [9].) Each of the sets

\[
\{c_0, c_{g+1}, c_{g+2}\}, \quad (10) \\
\{c_{2g+1}, c_{2g+2}\} , \quad (11) \\
\{\sigma_{-1,2g}, \sigma_{0,1}, \sigma_{-2,2g}\}, \quad (12)
\]

forms a minimal complete set of relations for the invariant cohomology ring \(H^*(N_0; \mathbb{Q})^\Gamma\).

**PROOF:** From ([1], [6]) we know that the relations (10) and (11) generate the same ideal of \(H^*_G(C; \mathbb{Q})^\Gamma\). Since

\[
c_{2g+1} = \sigma_{-1,2g}, \\
c_{2g+2} = \sigma_{0,1} - a_1 \sigma_{1,2g}, \\
c_{2g+3} = \sigma_{-2,2g} + ((a_1)^2 - a_2) \sigma_{-1,2g},
\]

we can see that the relations (12) also generate the same ideal.

Let \(\mu = (d_1, d_2)\). Now \(H^*_G(C\mu; \mathbb{Q})^\Gamma\) is generated by

\[
a_1, a_2, c_{1,1}, c_{1,2} + c_{2,1}^2, \xi_{1,1}, \xi_{1,2}^2 + c_{2,1}^2,
\]

where \(\xi_{1,1}^i = \sum_{s=1}^{2g} b_1^{1,s} b_1^{2,s+g}\). On restriction to \(H^*_G(C\mu; \mathbb{Q})\)

\[
a_1 \mapsto a_1 + a_2, \quad b_1^s \mapsto b_1^{1,s} + b_1^{2,s}, \quad f_2 \mapsto d_1 a_1 + d_2 a_1 + \xi_{1,1}^2 + \xi_{1,2}^2.
\]

Hence \(e_\mu H^*_G(C\mu; \mathbb{Q})^\Gamma\) is generated by

\[
\sum_{S \subseteq \{g\} \atop |S| = k} \left( \prod_{s \in S} b_1^{1,s} b_1^{2,s+g} - b_1^{2,s} b_1^{2,s+g} \right) (a_1)^i e_\mu 
\]

for \(i = 0, 1, 0 \leq k \leq g\) and the restrictions of \(a_1, f_2\) and \(\xi_{1,1} = \sum_{s=1}^{2g} b_1^s b_1^{s+g}\). Let \(P(S)\) denote the set of partitions \(S\) into two sets \(S_1, S_2\). We then see from lemma 2 that (13) above is the restriction of

\[
\frac{1}{2^k} \sum_{S \subseteq \{g\} \atop |S| = k} \sum_{P(S)} (\pm) \left( \prod_{s \in S_1} b_1^s \right) \left( \prod_{s \in S_2} \xi_{1,1}^s \right) \sigma_{D,2g}^{i-1} w_{S_2 \cup (S_1+g)}
\]

where the sign \((\pm)\) depends on the particular partition of \(S\). Thus by proposition [1] the relations (14) above for \(D < 0, i = 0, 1, 0 \leq k \leq g\) generate the invariant relation ideal of

\[
H^*_G(C^*; \mathbb{Q}) \cong H^*(N_0; \mathbb{Q}) \otimes \Lambda^* \{b_1^{1,0}, \ldots, b_1^{2,g}\} \otimes \mathbb{Q}[a_1].
\]

In particular the relations

\[
\{\sigma_{D,2g}^i : D < 0, i = 0, 1\}
\]

generate \(H^*(N_0; \mathbb{Q})^\Gamma\). It follows from (1) that the sets (10), (11) and (12) each form a complete set of relations for the invariant cohomology ring of \(N_0\).

Minimality then follows easily. Suppose that for some \(\eta, \theta \in H^*(N_0; \mathbb{Q})^\Gamma\) we have

\[
\sigma_{-2,2g} - \eta \sigma_{-1,2g} + \theta \sigma_{-1,2g} = 0.
\]

So \(\eta\) has degree 2 and \(\theta\) has degree 4.

Let \(\mu = (2g+1, 2g)\). Restricting equation (13) to \(H^*_G(C\mu; \mathbb{Q})\) we find from (1) that

\[
\eta \mu a_1 + \theta \mu = 0
\]

since \(e_\mu\) is not a zero-divisor in \(H^*_G(C\mu; \mathbb{Q})\). The restriction map

\[
H^*_G(C; \mathbb{Q}) \rightarrow H^*_G(C\mu; \mathbb{Q})
\]
is given by
\[ a_1 \mapsto a_1^2 + a_1^3, \quad a_2 \mapsto a_1^2 a_1, \quad f_2 \mapsto 2g a_1 + (2g + 1)a_2^2 + \xi_{1,1}^2 + \xi_{1,1}^2. \]
\[ b_1^s \mapsto b_1^{s+1} + b_1^{s+2}, \quad b_2^s \mapsto a_1^2 b_1^{s+1} + a_1 b_1^{s+2}. \]

From (14) we see that \( \eta_\mu \) and \( \theta_\mu \) are both zero. Since the restriction map (17) is injective in degrees 4 and less, we have that \( \eta \) and \( \theta \) are both zero – which contradicts (13).

Similarly the equation
\[ \sigma_{-1, [2g]}^0 + \eta \sigma_{-1, [2g]}^1 = 0 \]
has no solutions for \( \eta \in H^2_\mathbb{C}(C; \mathbb{Q}). \)

3 Mumford’s Conjecture

The final result of this note is a stronger version of Mumford’s conjecture as proven by Kirwan [6, §2] and which is confusingly also referred to as Mumford’s conjecture.

The Poincaré polynomial of the relation ideal of \( H^*(N_0; \mathbb{Q}) \) equals [4, p.593]
\[ \frac{t^{2g}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)}. \]
Now there are \( \binom{2g}{2} \) relations of the form \( c_{2g,S} \) of degree \( 2g + r \); \( \alpha \) has degree two, \( \beta \) has degree four and neither are nilpotent in \( H^*_\mathbb{C}(C; \mathbb{Q}) \). This strongly suggests:

**THEOREM 4** The relation ideal of \( H^*(N_0; \mathbb{Q}) \) is freely generated as a \( \mathbb{Q}[\alpha, \beta] \)-module by the Mumford relations \( c_{2g,S} \) for \( S \subseteq [2g] \).

**PROOF:** Define
\[ \bar{\alpha} = \alpha - \sum_{s=1}^g b_1^s b_1^{s+g} = 2f_2 - da_1 - \sum_{s=1}^g b_1^s b_1^{s+g}. \]
We will show that the relations \( c_{2g,S} \) generate the relation ideal of \( H^*_\mathbb{C}(C; \mathbb{Q}) \) as a \( \mathbb{Q}[\bar{\alpha}, \beta] \)-module. As \( \bar{\alpha} \) restricts to \( \alpha \) in \( H^*(N_0; \mathbb{Q}) \) then this is equivalent to the above result. It will be sufficient to prove that
\[ \sum_{S \subseteq [2g]} \lambda_S(\bar{\alpha}, \beta)c_{2g,S} = 0 \quad \lambda_S(\bar{\alpha}, \beta) \in \mathbb{Q}[\bar{\alpha}, \beta] \] (18)
in \( H^*_\mathbb{C}(C; \mathbb{Q}) \) if and only if \( \lambda_S(\bar{\alpha}, \beta) = 0 \) for each \( S \subseteq [2g] \).

Let \( \mu = (2g + 1, 2g) \). Then from lemma 2 we know that the restriction of \( c_{2g,S} = \sigma_{1-S,S}^1 \) in \( H^*_\mathbb{C}(C^{\mu}; \mathbb{Q}) \) equals
\[ (-1)^{g/2} \frac{2^{2g}g!}{2^{2g}g!} \left( \prod_{s \in S} (b_1^{2s} - b_1^{1s}) \right) e_{\mu}. \]
If we restrict equation (18) to \( H^*_\mathbb{C}(C^{\mu}; \mathbb{Q}) \) and recall that \( e_{\mu} \) is not a zero-divisor in \( H^*_\mathbb{C}(C^{\mu}; \mathbb{Q}) \) [4, p.569] we obtain
\[ \sum_{S \subseteq [2g]} \lambda_S(\bar{\alpha}_\mu, \beta_\mu) \left( \prod_{s \in S} (b_1^{2s} - b_1^{1s}) \right) = 0. \]
Now the restrictions of \( \bar{\alpha} \) and \( \beta \) in \( H^*_\mathbb{C}(C^{\mu}; \mathbb{Q}) \) equal
\[ \bar{\alpha}_\mu = (a_1^2 - a_1^1) - \sum_{s=1}^g (b_1^{1s} - b_1^{2s})(b_1^{1s+g} - b_1^{2s+g}), \quad \beta_\mu = (a_1^2 - a_1^1)^2. \]
By comparing the coefficients of \( \prod_{s \in S} (b_1^{2s} - b_1^{1s}) \) for each \( S \subseteq [2g] \) we see that
\[ \lambda_S(\bar{\alpha}_\mu, \beta_\mu) = 0 \quad S \subseteq [2g]. \]
Consider the restriction map
\[ \mathbb{Q}[\bar{\alpha}_\mu, \beta_\mu] \rightarrow \mathbb{Q}[\bar{\alpha}_\mu, \beta_\mu]. \] (20)
From the expressions (13) of $\bar{\alpha}_\mu$ and $\beta_\mu$, we can see that the kernel of the restriction map (20) is the ideal of $\mathbb{Q}[\bar{\alpha}, \beta]$ generated by 

$$(\bar{\alpha}^2 - \beta)^{g+1}.$$  

However $\bar{\alpha}^2 - \beta$ is not a zero-divisor in $H^*_G(C; \mathbb{Q})$. So we can assume without any loss of generality that for some $S$, $\lambda_S$ is either zero or not in the ideal generated by $\bar{\alpha}^2 - \beta$. For this $S$ we have $\lambda_S = 0$ since $\lambda_S(\bar{\alpha}_\mu, \beta_\mu) = 0$. Inductively we can see that 

$$\lambda_S(\bar{\alpha}, \beta) = 0$$  

for $S \subseteq [2g]$. □

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