On the influence of linear blowing and constant
temperature factor on the values of functionals of
hypersonic aerodynamics

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\textbf{Abstract.} The problems of mathematical modeling of effective control of heat and mass transfer and friction on permeable cylindrical and spherical surfaces of hypersonic aircraft are considered. The systems of ordinary differential equations are obtained by A.A.Dorodnicyn generalized integral relations method to approximate the systems of partial differential equations describing laminar boundary layers on permeable cylindrical and spherical surfaces of hypersonic aircraft. The joint systems are applied in the mathematical model. The parameters of the mathematical model at the flow stagnation point are determined from the joint systems of nonlinear algebraic equations. The blowing into boundary layer, temperature factor and magnetic field are used as controls. Dependences of hypersonic aerodynamics functionals (the total heat flow, the total Newton friction force and total blowing system power) on controls (the linear blowing into boundary layer, the constant temperature factor, the constant magnetic field) are investigated. The domains of allowed values of functionals of hypersonic aerodynamics are obtained. The results of the computational experiments are presented: the dependences of total heat flow on controls; the dependences of total Newton friction force on controls; the dependences of blowing system power on controls; the mutual dependences of functionals (as the domains of allowed values “Heat and Friction”).

1. Introduction

Various physical and chemical processes such as vibrational excitation, dissociation, ionization characterize the hypersonic aircraft motion in dense layers of the atmosphere \cite{1}. The only one way to simulate all real conditions of hypersonic flight is to elaborate adequate mathematical models describing the phenomena. With the use of A. A. Dorodnitsyn \cite{2} generalized integral relations method the PDE systems describing laminar boundary layers can be reduced to the approximating ODE systems. This approach is very popular in engineering for compressible gas aerodynamic characteristics computation \cite{3–6}.

The approximating ODE systems describing laminar boundary layers on the control area for hypersonic aircraft permeable cylindrical and spherical surfaces are obtained in \cite{7,8}.

The influence of the following combination of controls: linear blowing and constant temperature factor and constant magnetic field on the mathematical model parameters, local heat and mass transfer and friction characteristics, local blowing system power is considered in \cite{9}.
This paper continues research of mathematical model [8–13] of electroconductive laminar boundary layer control on hypersonic aircraft permeable cylindrical and spherical surfaces. The following results concerning the hypersonic aerodynamics functionals are discussed:

1. the dependences of total heat flow on controls;
2. the dependences of total Newton friction force on controls;
3. the dependences of total blowing system power on controls;
4. the mutual dependences of functionals of hypersonic aerodynamics (presented in the form of domains of allowed values “Heat and Friction”).

The computational experiments results are compare with [12, 13].

2. Problem statement

Let’s consider the following direct problem (1) [13]:

\[(m, \tau_w, s) \rightarrow (q, f, \eta; Q, F, N). \]  \( (1) \)

According to preset given controls: \( m(x) \) is the blowing into boundary layer, \( \tau_w(x) \) is the temperature factor \( (\tau_w(x) = T_w(x)/T_{eq} \), where \( T_w(x) \) is the wall temperature, \( T_{eq} \) is the temperature in the flow stagnation point), \( s(x) = \sigma \Phi_B^2(x) \) is the magnetic field, where \( x \in X = [0; 1] \) (the axis \( x \) is directed along the body contour), it is necessary to compute the boundary layer mathematical model parameters \( \theta_0(x; m, \tau_w, s), \theta_1(\theta), \omega_0(\ldots), \omega_1(\ldots) \) [8] for permeable cylindrical and spherical surfaces of hypersonic aircraft.

The parameters \( \theta_0, \ldots, \omega_1 \) can be determined from the joint approximating ODE system (5)–(8) [13] obtained by A.A.Dorodnicyn generalized integral relations method [2]

\[ \theta_0' = 18m^2\pi^{4k_1} - 6\beta q \left[ \frac{9}{6} \theta_0 + \frac{7}{6} \theta_1 - \frac{4}{3} \omega_0 - \frac{5}{3} \omega_1 \right] + \frac{34b_0q}{\theta_0} - \frac{32b_1q}{\theta_1} + \]
\[ + 6AB_0^2 q \left\{ \theta_0 \left( \frac{1}{3} - \frac{1}{15} \alpha^2_\theta \right) + \theta_1 \left( \frac{1}{3} - \frac{1}{15} \alpha^2_\theta \right) - \frac{2}{15} \omega_0 - \frac{11}{30} \omega_1 \right\}; \]  \( (2) \)

\[ \theta_1' = 12m^2\pi^{4k_1} - 12\beta q \left[ \frac{1}{3} \theta_0 + \frac{1}{2} \theta_1 - \frac{1}{3} \omega_0 - \frac{2}{3} \omega_1 \right] + \frac{20b_0q}{\theta_0} - \frac{16b_1q}{\theta_1} + \]
\[ + 24AB_0^2 q \left\{ \frac{1}{60} \alpha^2_\theta \theta_0 + \theta_1 \left( \frac{1}{12} - \frac{1}{30} \alpha^2_\theta \right) - \frac{1}{60} \omega_0 - \frac{11}{15} \omega_1 \right\}; \]  \( (3) \)

\[ \omega_0' = (1 - \tau_w) \theta_0' - \tau_w \theta_0; \]
\[ \omega_1' = 6m\pi^{4k_1} \frac{\omega_0}{\theta_0} - 6\beta q \left[ \frac{1}{6} \omega_0 + \frac{1}{2} \omega_1 - \frac{1}{60} \alpha^2_\theta \frac{2}{3} \omega_1 \right] + \frac{6b_0\omega_0 q}{\theta_0^2} + \]
\[ + 6q \left[ \frac{1}{1 - Pr^k_1} + \right] \left\{ \frac{1}{6} \frac{b_0}{\theta_0} \left( -3 \frac{\omega_0}{\theta_0} + \frac{4}{3} \omega_1 \right) - \frac{2}{3} \frac{b_1}{\theta_1} \omega_1 \right\} - \]
\[ - \frac{6q b_0}{Pr \theta_0} \left( -3 \frac{\omega_0}{\theta_0} + \frac{4}{3} \omega_1 \right) + 4 \alpha^2_\theta q \left( \frac{1}{Pr} - 1 \right) \frac{b_1}{\theta_1} + \]
\[ + 6B_0^2 q \left\{ A \left( \frac{1}{60} \alpha^2_\theta \omega_0 + \omega_1 \left( \frac{1}{6} - \frac{1}{15} \alpha^2_\theta \right) - \frac{1}{60} \omega_0^2 - \frac{2}{3} \omega_1 \right) \right\} + \]
\[ + G \left\{ \theta_0 \left( \frac{1}{60} - \frac{1}{70} \alpha^2_\theta \right) + \theta_1 \left( -\frac{1}{20} + \frac{1}{42} \alpha^2_\theta \right) + \frac{1}{30} \omega_1 \right\}. \]  \( (5) \)

Here

\[ m = \left( \frac{\rho v}{\rho_{eq}} \right)_w \sqrt{\frac{\ell}{V_{max} \nu_{eq}}}, \quad \tau = \frac{r}{\ell}, \quad q = \alpha_e f \Phi^{k_1}, \quad \Phi = (1 - \alpha^2_\phi)^\gamma_1. \]
In (2)–(5) \(k_5 = 2 \cdot k_4\), so, \((k_4; k_5) = (0; 0)\) for the case of side surface of circular cylinder and \((k_4; k_5) = (1; 2)\) for the case of spherical nose (with current radius \(r(x)\)).

For \(x_0 = 0\) let’s denote

\[
m_0 = m(x_0) ; \quad \tau_0 = \tau_w(x_0) ; \quad s_0 = s(x_0) ; \quad C_0 = \frac{\sigma V_{\text{max}} \ell}{\rho_c h_c} C.\]

For \(x_0 \approx x_0\) the initial conditions for system (2)–(5) according to [8,13] are

\[
\begin{align*}
\theta_0 &= \bar{\theta}_0 \cdot (x_0^*)^{k_3} ; \quad \theta_1 = \bar{\theta}_1 \cdot (x_0^*)^{k_3} ; \quad \omega_0 = \bar{\omega}_0 \cdot (x_0^*)^{k_3} ; \quad \omega_1 = \bar{\omega}_1 \cdot (x_0^*)^{k_3},
\end{align*}
\]

where \(\bar{\theta}_0, \bar{\theta}_1, \bar{\omega}_0, \bar{\omega}_1\) can be obtained for \(\tau_0 \neq 1\) from the following nonlinear algebraic system [13]

\[
\begin{align*}
18m_0 - k_0 \bar{\theta}_0 - 7 \bar{\theta}_1 + 8 \bar{w}_0 + 10 \bar{w}_1 + \frac{34b_0(0)C}{\bar{\theta}_0} - \frac{32b_1(0)C}{\bar{\theta}_1} + \\
+ \frac{1}{5} C_0 B_0^2 \cdot (5 \bar{\theta}_0 + 10 \bar{\theta}_1 - 4 \bar{w}_0 - 11 \bar{w}_1) &= 0; \quad (7) \\
12m_0 - 4 \bar{\theta}_0 - k_1 \bar{\theta}_1 - 4 \bar{w}_0 + 8 \bar{w}_1 + \frac{20b_0(0)C}{\bar{\theta}_0} - \frac{16b_1(0)C}{\bar{\theta}_1} + \\
+ \frac{2}{5} C_0 B_0^2 \cdot (5 \bar{\theta}_1 - 4 \bar{w}_0 + 4 \bar{w}_1) &= 0; \quad (8) \\
(1 - \tau_0) \cdot \bar{\theta}_0 - \bar{w}_0 &= 0; \quad (9) \\
6m_0 \bar{w}_0 \bar{\theta}_0 - \bar{w}_0 - k_2 \bar{w}_1 + \frac{\bar{w}_0^2}{\bar{\theta}_0} + 4 \frac{\bar{w}_0}{\bar{\theta}_1} + 6 \frac{\bar{w}_0 b_0(0)C}{\bar{\theta}_0} + \\
+ C \left( \frac{1}{Pr} + 1 \right) \left[ b_0(0) \left( \frac{1}{\bar{\theta}_0} - 3 \frac{\bar{w}_0}{\bar{\theta}_0} + 4 \frac{\bar{w}_1}{\bar{\theta}_1} \right) - 4 \frac{b_1(0) \bar{w}_0}{\bar{\theta}_1} \right] - \\
- \frac{6C b_0(0) \bar{w}_0}{Pr \bar{\theta}_0} \left( -3 \frac{\bar{w}_0}{\bar{\theta}_0} + 4 \frac{\bar{w}_1}{\bar{\theta}_1} \right) + \frac{1}{10} B_0^2 C_0 \left( 10 \bar{w}_1 - \frac{\bar{w}_0^2}{\bar{\theta}_0} - 8 \frac{\bar{w}_1^2}{\bar{\theta}_1} \right) &= 0. \quad (10)
\end{align*}
\]

For the case \(\tau_0 = 1\) (when \(\bar{w}_0 = 0, \bar{w}_1 = 0\)) system (7)-(10) can be reduced to the “shortened” form

\[
\begin{align*}
18m_0 - k_0 \bar{\theta}_0 - 7 \bar{\theta}_1 + \frac{34b_0(0)C}{\bar{\theta}_0} - \frac{32b_1(0)C}{\bar{\theta}_1} + C_0 B_0^2 \cdot (\bar{\theta}_0 + 2 \bar{\theta}_1) &= 0; \quad (11) \\
12m_0 - 4 \bar{\theta}_0 - k_1 \bar{\theta}_1 + \frac{20b_0(0)C}{\bar{\theta}_0} - \frac{16b_1(0)C}{\bar{\theta}_1} + 2 C_0 B_0^2 \bar{\theta}_1 &= 0, \quad (12)
\end{align*}
\]

In (6)–(12) \(k_0 = k_4 + 10, k_1 = k_4 + 7, k_2 = k_4 + 4, k_3 = k_4 + 1\), so, \((k_0; \ldots; k_4) = (10; 7; 4; 1; 0)\) for the case of side surface of circular cylinder and \((k_0; \ldots; k_4) = (11; 8; 5; 2; 1)\) for the case of spherical nose.

After that it is necessary to determine the local heat flow \(q(x; m, \tau_w, s)\); the local tangent friction \(f(x; m, \tau_w, s)\); the local blowing system power \(\eta(x; m, \tau_w, s)\).

Then it is necessary to obtain (for \(x_k = 1\)) the total heat flow

\[
Q(m, \tau_w, s) = \int_0^{x_k} (2 \pi r)^{k_1} \left( \frac{\lambda}{C_p} \frac{\partial H}{\partial y} \right)_{y=0} \cdot dx;
\]

\[
Q = \int_0^{x_k} \left( \frac{\lambda}{C_p} \frac{\partial H}{\partial y} \right)_{y=0} \cdot dx;
\]
the total Newton friction force
\[ F(m, \tau_w, s) = \int_0^{x_k} (2\pi r)^k \mu \frac{\partial u}{\partial y} \bigg|_{y=0} \cdot dx; \quad (14) \]

the total blowing system power
\[ N(m, \tau_w, s) = \int_0^{x_k} (2\pi r)^k av_w^2(x) \cdot dx \quad (15) \]
determined with the use of H. Darcy filtration law. In (13)–(15) \( k_4 = 0 \) for the case of side surface of circular cylinder and \( k_4 = 1 \) for the case of spherical nose (with current radius \( r(x) \)).

3. Computational Experiments
Let the unchangeable parameters values be fixed:

- the Mach number \( M_{\infty} \in [10; 40] \), \( (16) \)
- the flight altitude \( H \in [10; 30] \) [km], \( (17) \)
- the body radius \( R \in [0.1; 1] \) [m], \( (18) \)

The computational experiments are accomplished for the air in the atmosphere of Earth at \( H = 10 \) [km], \( M_{\infty} = 10 \), \( R = 0.1 \) [m] to compare with the results \([7, 10–13]\).

Let the controlling parameters changing ranges be restricted as follows:

- \( m \in M^c = [0; 1] \), \( (19) \)
- \( \tau_w \in T_{pr}^c = [0.15; 0.9] \), \( T_{pr}^c \subset T_{th}^c = [0; 1] \), \( (20) \)
- \( s = \sigma B_0^2 \in S^c = [0; 5 \cdot 10^4] \) [T/(Ohm·m)], \( (21) \)

Hereinafter the index of parameter \( \tau_w \) and the dimension [T/(Ohm·m)] of the control parameter \( s \) are omitted.

The blowing \( m(x) \) is given by linear law \([9]\)
\[ m(x) = m(x; m_0, m_1) = m_0 \cdot (1 - x) + m_1 \cdot x = m_0 + m' \cdot x \quad \text{for} \quad x \in [0; 1], \quad (22) \]

where \( m_0, m_1 \in M^c \), and \( m' = m_1 - m_0 \).

Let’s denote \([9, 13, 14]\)
\[ M_{05}^d = \{0; 0.05; \ldots; 0.95; 1\} \subset M^c, \quad (23) \]
\[ M_{25}^d = \{0; 0.25; 0.5; 0.75; 1\} \subset M_{05}^d, \quad (24) \]
\[ M_{50}^d = \{0; 0.5; 1\} \subset M_{25}^d, \quad (25) \]
\[ T_{05,th}^d = \{0; 0.05; \ldots; 0.95; 1\} \subset T_{th}^c, \quad (26) \]
\[ T_{05,pr}^d = \{0.15; 0.2; \ldots; 0.9\} = T_{05,th}^d \cap T_{pr}^c, \quad (27) \]
\[ T_{15}^d = \{0.15; 0.3; \ldots; 0.9\} \subset T_{05,pr}^d, \quad (28) \]
\[ T_{15}^d = \{0.15; 0.45; 0.9\} \subset T_{15}^c, \quad (29) \]
\[ S_{25}^d = \{0; 2.5 \cdot 10^4; 5 \cdot 10^4\} \subset S^c. \quad (30) \]
Let the letters of the alphabet from “a” to “u” be assigned [9, 14] to the elements of the sets \( M_{05}^d \) and \( T_{05,th}^d \), and the letters from “d” to “s” be mapped to the elements of \( T_{05,pr}^d \). The positions of \( Q, F, N \) and pairs \((Q,F)\) are shown in figures 1–13 for some combinations of controls \((m_0, m_1; \tau_0, \tau_1)\) using four-letter labels. For example, the label “akdd” corresponds to \((m_0 = 0.0, m_1 = 0.5; \tau_0 = 0.15, \tau_1 = 0.15)\).

The dependences \( Q, F \) and \( N \) for constant \( m = x \) are shown in figures 1–13 for some combinations of \((Q,F)\) presented in this paper for \( s = 0 \mid 12, 13 \mid 25 \).

The symbols “∗” and “□” are used for the cases \( m_0 < m_1 \) and \( m_0 = m_1 \), respectively, when \( m' \geq 0 \). The symbols “∇” and “O” are used for the cases \( m_0 > m_1 \) and \( m_0 = m_1 \), respectively, when \( m' \leq 0 \).

![Figure 1](image.png)

**Figure 1.** The dependence \( Q \) on \( m \) for constant \( m(x), \tau(x) \) and \( s = 0 \)

Remarks.

1) The influence of another combination of controls: linear temperature factor and constant blowing and constant magnetic field on the total heat flow, the total Newton friction force and total blowing system power is investigated in [15].

2) In addition to the dependences \( Q, F, N \) and \((Q,F)\) presented in this paper for \( s = 0 \), the effect of the proposed combinations of \( m \) and \( \tau \) on \( Q, F, N \) and \((Q,F)\) was studied at various constant values of magnetic fields \( s \in S_{25}^d \).

3) The results of computational experiments, as well as the results of [12, 13] can be used as models of restrictions \((29)\)–\((31)\) [16] and \((28_1)\)–\((28_r)\) [17] in the effective control synthesis problems, both for entire control segment and for its fragments.
Figure 2. The dependence $Q$ on $m_0$ at $m_1 \in M_{50}^d$ for constant $\tau(x)$ and $s \equiv 0$.

Figure 3. The dependence $Q$ on $m_1$ at $m_0 \in M_{50}^d$ and constant $\tau(x)$ and $s \equiv 0$.

Figure 4. The dependence $F$ on $m$ for constant $m(x)$, $\tau(x)$ and $s \equiv 0$. 
Figure 5. The dependence $F$ on $m_0$ at $m_1 \in M_{d0}^d$ for constant $\tau(x)$ and $s \equiv 0$

Figure 6. The dependence $F$ on $m_1$ at $m_0 \in M_{d0}^d$ and constant $\tau(x)$ and $s \equiv 0$

Figure 7. The dependence $N$ on $m$ for constant $m(x)$, $\tau(x)$ and $s \equiv 0$
Figure 8. The dependence $N$ on $m_0$ at $m_1 \in M_{50}^d$ for constant $\tau(x)$ and $s \equiv 0$

Figure 9. The dependence $N$ on $m_1$ at $m_0 \in M_{50}^d$ and constant $\tau(x)$ and $s \equiv 0$

Figure 10. The domain $\Omega = \{(Q, F)\}$ for constant $m(x)$, $\tau(x)$ and $s \equiv 0$
Figure 11. The dependence $(Q, F)$ on $m_0$ at $m_1 \in M^d_{50}$ and constant $\tau(x)$ and $s \equiv 0$

Figure 12. The dependence $(Q, F)$ on $m_1$ at $m_0 \in M^d_{50}$ and constant $\tau(x)$ and $s \equiv 0$

Figure 13. The dependence $(Q, F)$ for constant $\tau(x)$ and $s \equiv 0$
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