Approximate formula for the ground state energy of anyons in 2D parabolic well

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Abstract

We determine approximate formula for the ground state energy of anyons in 2D parabolic well which is valid for the arbitrary anyonic factor $\nu$ and number of particles $N$ in the system. We assume that centre of mass motion energy is not excluded from the energy of the system. Formula for ground state energy calculated by variational principle contains logarithmic divergence at small distances between two anyons which is regularized by cut-off parameter. By equating this variational formula to the analogous formula of Wu near bosonic limit ($\nu \sim 0$) we determine the value of the cut-off and thus derive the approximate formula for the ground state energy for the any $\nu$ and $N$. We checked this formula at $\nu = 1$, when anyons become fermions, for the systems containing two to thirty particles. We find that our approximate formula has an accuracy within 6%. It turns out, at the big number $N$ limit the ground state energy has square root dependence on factor $\nu$.

I. INTRODUCTION

Anyons - two-dimensional particles obeying fractional statistics are one of possible quasi-particle excitations in fractional quantum Hall effect and High $T_c$ superconductors \cite{1,2}. The determination of the energy spectrum of anyons in 2D parabolic well is actual and interesting task of quantum mechanics. The analytical calculation of the energy spectrum of the two anyons in 2D parabolic well was performed in \cite{3,4}. The energy spectrum of systems containing more than two anyons in the same external field can not be obtained analytically.
Wu [8] generalized the class of exact two anyon solution into three anyons containing system and found it’s energy spectrum. There are the numerical calculations of the lowest part of the energy spectrum of three [9,10] and four [11] anyons in 2D parabolic well.

There is separate quantization of centre mass motion energy and relative motion energy to each other of anyons in 2D parabolic well [8]. The ground state of relative motion energy of the systems of two, three and four anyons can be linearly interpolated between bosonic and fermionic spectra [3–8] when anyonic factor $\nu$ changes from 0 to 1.

Linear dependence of full ground state energy (without separation of centre mass motion energy) from anyonic factor $\nu$ exists for arbitrary number of anyons only near the bosonic limit of spectra [8], i.e. at $\nu \to 0$. Beginning with the case of three anyons it deviates from linear character near the fermionic limit of spectra [8], i.e. at $\nu \to 1$.

By variational calculation we determine approximate formula for the ground state energy of anyons in 2D parabolic well which is valid for the arbitrary anyonic factor $\nu$ and number of particles $N$ in system. We assume that centre of mass motion energy is not excluded from energy of system and anyons represented by bosonic variational wave function. As in Laughlin’s variational calculation of infinite system of free anyons [12], our formula for the ground state energy calculated by variational principle contains logarithmic divergence at small distances between two anyons regularized by cut-off. We equate our variational ground state energy formula to the same formula of Wu [8] for anyons in 2D parabolic well near the bosonic limit on the factor $\nu$ to determine the cut-off and thus derive the approximate formula for the ground state energy at the any factor $\nu$ and number $N$. We check this formula at $\nu = 1$, when anyons become fermions, for the systems containing two to thirty particles. We find that our approximate formula has an accuracy within 6%. It turns out, at the big number $N$ limit the ground state energy has square root dependence on factor $\nu$.

The following section 2 describes the Hamiltonian of the system of anyons in 2D parabolic well and the choice of trial wave function for the variational calculation and physical argument for this choice. Section 3 contains results of variational calculation.

II. HAMILTONIAN AND WAVE FUNCTION OF SYSTEM

The Hamiltonian of $N$ anyons system in 2D parabolic well has a form:

$$\hat{H} = \frac{1}{2M} \sum_{i=1}^{N} (\vec{p}_i + \vec{A}(\vec{r}_i))^2 + \frac{M\omega_0^2}{2} \sum_{i=1}^{N} \vec{r}_i^2.$$

(1)

Here $M$ - mass of particle, $\vec{p} = -i\hbar \vec{\nabla}$, where $\vec{\nabla} = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}$, $\omega_0$ - eigen frequency of oscillatory motion of free particles in 2D parabolic well and $\vec{r}_i$ - radius vector of $i$ - th particle number.

Vector potential for anyons $\vec{A}(\vec{r}_i)$ [13] in (1) is

$$\vec{A}(\vec{r}_i) = \hbar \nu \sum_{j>i} \frac{\vec{z} \times \vec{r}_{ij}}{|\vec{r}_{ij}|^2}. \quad (2)$$

Here $\vec{z}$ is unit vector perpendicular to 2D plane where situated system, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ and anyonic factor $\nu$ characterizes the form of fractional statistics (spin of the anyon): it can be changed from $\nu = 0$ - noninteracting bosons to $\nu = 1$ - free fermions.
In the variational consideration the ground state energy of system determined as minimum of energy

\[ E = \frac{\int \psi_T^*(\vec{R})\hat{H}\psi_T(\vec{R})d\vec{R}}{\int \psi_T^*(\vec{R})\psi_T(\vec{R})d\vec{R}}. \]  

(3)

Here \( \hat{H} \) - the Hamiltonian, \( \psi_T(\vec{R}) \) - trial wave function of system that has a variational parameter and \( \vec{R} \) - the coordinates of all particles. Minimum energy of \( E \) (3) is reached by variation of variational parameter of trial wave function.

We suppose that trial wave function is provided by mean field approximation of Fetter, Hanna and Laughlin [13]. In this approximation particles move in the mean field \( \vec{A} \):

\[ \vec{A}(\vec{r}) = \frac{1}{2} \vec{B} \times \vec{r} \]  

(4)

induced by the average density \( \rho \) of particles in the system where \( \vec{B} = 2\pi\rho\nu z^2 \), \( \rho = \frac{1}{\pi r_o^2} \) and \( r_o \) - mean distance between particles. According magnetic field \( \vec{B} \) one can introduce corresponding magnetic length \( a_H = (h/B)^{1/2} \) and cyclotron frequency \( \omega_c = B/M \). Energetic spectrum of Hamiltonian (1) with \( \vec{A} \) and 2D parabolic well has the same energetic spectrum structure as a free particle in 2D parabolic well [14].

Thus assuming a bosonic representation of anyons

\[ \psi_T(\vec{R}) = \prod_{i=1}^{N} \psi_T(\vec{r}_i) \]  

(5)

for the ground state of anyons in 2D parabolic well one can bring wave function \( \psi_T(\vec{r}_i) \) in the following form:

\[ \psi_T(\vec{r}_i) = C \exp \left( -\left( \alpha + \nu \right) \frac{(x_i^2 + y_i^2)}{2r_o^2} \right). \]  

(6)

Here \( C \) - normalization constant, \( \alpha \) variational parameter and there assumed that mean distance between particles \( r_o \) equals of \( R_o = (h/M\omega_o)^{1/2} \) - radius localization for the wave function in the ground state of 2D oscillator. This equality follows from fact that interaction between two anyons at any not zero anyonic factor \( \nu \) has repulsive character and relation \( S_n = n\pi R_o^2 \) where \( S_n \) - area that occupies by free particle at \( n \) - th energetic level of 2D oscillator. Last relation for the area \( S_n \) one can easily get by using a virial theorem for 2D oscillator.

It is convenient to express all energetic quantities by energetic quanta of 2D oscillator \( \hbar\omega_o \) and length quantities by \( r_o \).

Normalized trial wave function of system has a form:

\[ \psi_T(\vec{R}) = \left( \frac{\alpha + \nu}{\pi} \right)^{N/2} \prod_{i=1}^{N} \exp \left( -\left( \alpha + \nu \right) \frac{(x_i^2 + y_i^2)}{2r_o^2} \right). \]  

(7)
III. VARIATIONAL CONSIDERATION. RESULT OF CALCULATION.

At the calculation of variational energy $E$ \( E_L(\vec{R}) = \Psi_T^{-1}(\vec{R}) \hat{H} \Psi_T(\vec{R}) \).

As trial wave function $\Psi_T(\vec{R})$ is not true of the Hamiltonian \( H \), so

$E_L(\vec{R}) = ReE_L(\vec{R}) + iImE_L(\vec{R})$.  \( (8) \)

By the acting of the Hamiltonian $\hat{H}$ \( (\vec{R}) \) on $\Psi_T(\vec{R})$ \( (7) \) we find

$ReE_L(\vec{R}) = \sum_{i=1}^{N} [\alpha + \nu + \frac{x_i^2 + y_i^2}{2} - \frac{(\alpha + \nu)^2}{2}(x_i^2 + y_i^2) + \frac{\nu^2}{2}(A(\vec{r}_i))^2]$,  \( (9) \)

and

$ImE_L(\vec{R}) = -\nu(\alpha + \nu) \sum_{i=1}^{N} (A(\vec{r}_i)r_i)$.  \( (10) \)

In the expression \( (10) \) we have a scalar product of vector $A(\vec{r}_i)$ and $\vec{r}_i$.

A variational energy $E$ \( (3) \) expressed by $E_L(\vec{R})$ is

$E = \int \Psi_T(\vec{R}) E_L(\vec{R}) \Psi_T(\vec{R}) d\vec{R}$.  \( (11) \)

Direct calculation gives

$\int \Psi_T(\vec{R}) ImE_L(\vec{R}) \Psi_T(\vec{R}) d\vec{R} = 0$.  \( (12) \)

So, on energy $E$ contributes only $ReE_L(\vec{R})$.

Integrals \( (11) \) with $\Psi_T(\vec{R})$ \( (7) \) of first three terms in square brackets \( (9) \) are calculating elementary. Problems appear when we are calculating integrals with term proportional of $(A(\vec{r}_i))^2$. This term gives two kind integrals: integrals like $\int \Psi_T(\vec{R}) \frac{1}{|\vec{r}_{ij}|^2} \Psi_T(\vec{R}) d\vec{R}$ where $i \neq j$ number of which are $N(N-1)$ and integrals like $\int \Psi_T(\vec{R}) \frac{1}{|\vec{r}_{ij}| |\vec{r}_{ik}|} \Psi_T(\vec{R}) d\vec{R}$ where $i \neq j, i \neq k$ and $j \neq k$ number of which are $N(N-1)(N-2)$.

By taking account of integrals (see Gradshttein and Ryzik \[13\]):

$\int_{0}^{\infty} E_i(ax)e^{-\mu x} dx = -\frac{1}{\mu} \ln \left( \frac{\mu}{a} - 1 \right)$,  \( (13) \)

where constants $a > 0$, $Re\mu > 0$ and $\mu > a$ and

$\int_{0}^{\infty} E_i(-\beta x)e^{-\mu x} dx = \frac{1}{\mu} \ln \left( \frac{\mu}{\beta} + 1 \right)$,  \( (14) \)

where constants $Re(\beta + \mu) \geq 0, \mu > 0$ and $E_i(\gamma y) = -vp \int_{\gamma}^{\infty} e^{-\gamma z} dz/z$ - the exponential integral where $\gamma > 0$ and $vp \int$ indicates integral in main quantity mean one can get
\[ \int \Psi_T(\vec{R}) \frac{1}{|\vec{r}_{ij}|^2} \Psi_T(\vec{R}) d\vec{R} = (\alpha + \nu) \ln \left( \frac{2}{\delta} \right), \tag{15} \]

where cut-off parameter \( \delta \) regularize the integral (13).

And another hand the calculation gives

\[ \int \Psi_T(\vec{R}) \frac{1}{|\vec{r}_{ij}|} \frac{1}{|\vec{r}_{ik}|} \Psi_T(\vec{R}) d\vec{R} = (\alpha + \nu) A, \tag{16} \]

where numerical value \( A \approx 1. \)

So, term by term averaged quantity \( \text{Re} E_L(\vec{R}) \) (4) gives expression for energy

\[ E = N(\alpha + \nu) + \frac{N}{2(\alpha + \nu)} - \frac{N(\alpha + \nu)}{2} + \frac{1}{2} \nu^2 N(N - 1)(\alpha + \nu) \ln \left( \frac{2}{\delta} \right) - (N - 2). \tag{17} \]

The extremum condition \( \frac{dE}{d(\alpha + \nu)} = 0 \) gives a parameter

\[ (\alpha + \nu) = [1 + \nu^2(N - 1)\ln \left( \frac{2}{\delta} \right) - (N - 2)]^{-1/2} \tag{18} \]

at which minimum energy of \( E \) is

\[ E_v = N [1 + \nu^2(N - 1)\ln \left( \frac{2}{\delta} \right) - (N - 2)]^{1/2}. \tag{19} \]

As established Wu [8] near the bosonic limit of ground state energy of anyons in 2D parabolic well on the factor \( \nu \), i.e. \( \nu \to 0 \), ground state energy when centre mass motion not fixed has a following expression (here and below in this section we return into normal units of energy and length):

\[ E \approx \hbar \omega_o [N + N(N - 1)\nu/2] \tag{20} \]

for the arbitrary number of particles \( N \) in system. So, at \( \nu \to 0 \) we expand expression for the energy \( E_v \) (14) on the powers \( \nu^2 \) and by equating term proportional to \( \nu^2 \) of expansion to second term in square brackets of (20), we find that

\[ \delta = 2 \exp \left( \frac{1 + \nu(N - 2)}{\nu} \right) r_o \tag{21} \]

and finally analytical expression for the variational ground state energy is:

\[ E_v = \hbar \omega_o N [1 + \nu(N - 1)]^{1/2}. \tag{22} \]

At the Table 1 we bring exact energies for fermions in 2D parabolic well constructed by filling lowest energetic levels of 2D oscillator and energies calculated by formula (22). From this table one can see that deviation of variational energy \( E_v \) from exact energy \( E_{EXACT} \) is no more than 6%.
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TABLES

TABLE I. The ground state energy of $N$ fermions in 2D parabolic well.

$E_{\text{EXACT}}$ - the energies given by exact analytical formula from [14] (in $\hbar \omega_o$ units),

$E_v$ - the energies (in $\hbar \omega_o$ units) calculated by formula (22).

| N | $E_{\text{EXACT}}$ | $E_v$ | $(E_v - E_{\text{EXACT}})/E_{\text{EXACT}}$ in % |
|---|-----------------|------|---------------------------------|
| 2 | 3               | 2.83 | 6                               |
| 3 | 5               | 5.20 | 4                               |
| 4 | 8               | 8.00 | 0                               |
| 5 | 11              | 11.18| 2                               |
| 6 | 14              | 14.70| 5                               |
| 7 | 18              | 18.52| 3                               |
| 8 | 22              | 22.63| 3                               |
| 9 | 26              | 27.00| 4                               |
| 10| 30              | 31.62| 5                               |
| 11| 35              | 36.48| 4                               |
| 12| 40              | 41.57| 4                               |
| 13| 45              | 46.87| 4                               |
| 14| 50              | 52.38| 5                               |
| 15| 55              | 58.09| 6                               |
| 16| 61              | 64.00| 5                               |
| 17| 67              | 70.09| 5                               |
| 18| 73              | 76.37| 5                               |
| 19| 79              | 82.82| 5                               |
| 20| 85              | 89.44| 5                               |
| 21| 91              | 96.23| 6                               |
| 22| 98              | 103.19|                               |
| 23| 105             | 110.30|                               |
| 24| 112             | 117.58|                               |
| 25| 119             | 125.00|                               |
| 26| 126             | 132.57|                               |
| 27| 133             | 140.30|                               |
| 28| 140             | 148.16|                               |
| 29| 148             | 156.17|                               |
| 30| 156             | 164.32|                               |