(TE)-structures over the 2-dimensional globally nilpotent $F$-manifold

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Abstract: We find formal and holomorphic normal forms for a class of meromorphic connections (the so called (TE)-structures) over the germ $N_2$ at the origin of the 2-dimensional globally nilpotent $F$-manifold. In order to obtain the holomorphic normal forms we prove that the restriction of any (TE)-structure $\nabla$ over $N_2$ at the origin $0 \in N_2$ is either regular singular (in which case $\nabla$ is holomorphically isomorphic to its formal normal forms) or is holomorphically isomorphic to a Malgrange universal connection (in rank two, with pole of Poincaré rank one). We develop a careful treatment for such Malgrange universal connections. We find normal forms for Euler fields on $N_2$ and we use them to answer the question when a given Euler field on $N_2$ is induced by a (TE)-structure.

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1 Introduction

A (TE)-structure over a complex manifold $M$ is a meromorphic connection $\nabla$ on a holomorphic vector bundle over $\mathbb{C} \times M$, with poles of Poincaré rank one along $\{0\} \times M$. (TE)-structures have received much attention in the mathematical literature, owing to their relation with integrable systems and the theory of Frobenius manifolds. For the purpose of this paper it is particularly relevant the data induced by $\nabla$ on its parameter space $M$. Under a certain additional condition (the so called unfolding condition), $\nabla$ induces a multiplication $\circ$ on $TM$, with unit field $e \in T_M$, which makes $(M, \circ, e)$ into an $F$-manifold. We say that $\nabla$ lies over the $F$-manifold $(M, \circ, e)$. The (TE)-structure $\nabla$ also induces a vector field $E$ on $M$, which rescales the $F$-manifold multiplication $\circ$. Such a vector field is called an Euler field on $(M, \circ, e)$. 

1
A natural question which arises in this context is to classify (formally and holomorphically) the \((TE)\)-structures which lie over a given germ \(((M, 0), \circ, e)\) of an \(F\)-manifold. This question was answered in Theorem 8.5 of [4], when the germ \(((M, 0), \circ, e)\) is 2-dimensional, irreducible and generically semisimple (i.e. belongs to a well-known family of germs \(I_2(m)\), parameterized by \(m \in \mathbb{Z}_{\geq 3}\), see e.g. [4], Theorem 4.7). The classifications in this case were done by finding in a first stage explicit formal normal forms (which turned out to be unique, for any \((TE)\)-structure \(\nabla\)) and then by proving that \(\nabla\) is holomorphically isomorphic to its formal normal form. As a consequence of the holomorphic classification, any Euler field on \(N_2\) is induced by a \((TE)\)-structure (from Theorem 8.5 of [4] combined with Theorem 4.7 of [7]).

On the other hand, it is well-known that any irreducible germ of a 2-dimensional \(F\)-manifold is isomorphic either to a germ \(I_2(m)\) or to the germ \(N_2\) at the origin of the 2-dimensional globally nilpotent \(F\)-manifold (see Theorem 4.7 of [7]; see Section 2.3.2 for the definition of \(N_2\)). Our goal in this paper is to classify (formally and holomorphically) the \((TE)\)-structures over \(N_2\). Like for \(I_2(m)\), we do this by finding formal and holomorphic normal forms. As opposed to \(I_2(m)\), the classifications of \((TE)\)-structures over \(N_2\) have different features: there are \((TE)\)-structure over \(N_2\) which admit several formal normal forms; there are \((TE)\)-structures over \(N_2\) which are not holomorphically isomorphic to their formal normal form(s); finally, it turns out that there are Euler fields on \(N_2\) which are not induced by a \((TE)\)-structure.

The results of this paper together with those from [4] provide complete formal and holomorphic classifications of \((TE)\)-structures over arbitrary irreducible germs of 2-dimensional \(F\)-manifolds.

Formal and holomorphic classifications are important topics of research in the theory of meromorphic connections. The results of this paper add to the existing knowledge in this field, using down-to-earth arguments rather than the abstract, more commonly used theory of Stokes structures. This paper is also a natural continuation of [3], where a formal classification of \((T)\)-structures (rather than \((TE)\)-structures) over irreducible germs of 2-dimensional \(F\)-manifolds was developed.

**Structure of the paper.** In Section 2 we recall well-known facts we need on the theory of meromorphic connections. Our original contribution here is Subsection 2.1 where we prove various results from differential equations which will be useful in our treatment.

In Section 3 we find the normal forms for Euler fields on the globally nilpotent germ \(N_2\). This is analogous to Theorem 4.7 of [7]. These normal forms will be used in establishing which Euler fields on \(N_2\) are induced by \((TE)\)-structures.
In Section 4 we find the formal normal forms for \((TE)\)-structures over \(N^2\). We start with the formal normal forms for \((T)\)-structures over \(N^2\), we enrich them to \((TE)\)-structures and then we simplify the \(B\)-matrices of these \((TE)\)-structures using formal automorphisms of their underlying \((T)\)-structures. This leads to the formal normal forms, which are described in Theorem 20.

Section 5 is devoted to the holomorphic classification. In Subsection 5.1 we study the restriction \(\nabla^0\) of an arbitrary \((TE)\)-structure \(\nabla\) over \(N^2\) to the slice \(\Delta \times \{0\}\), where 0 is the origin of \(N^2\) and \(\Delta \subset \mathbb{C}\) is a small disc around the origin of \(\mathbb{C}\). We prove that either \(\nabla^0\) is regular singular (in which case \(\nabla\) is holomorphically isomorphic to its formal normal form(s)) or \(\nabla^0\) can be put into Birkhoff normal form with ‘residue’ a regular endomorphism (in which case \(\nabla\) is holomorphically isomorphic to a Malgrange universal connection). In Subsections 5.2 and 5.2.2 we find necessary and sufficient conditions for two such Malgrange universal connections to be isomorphic and we express them in local coordinates adapted to \(N^2\). This leads to the holomorphic classification we are looking for (see Definition 57 and Corollary 58).

As an application of the theory developed in the previous sections, in Section 6 we answer the question when an Euler field on \(N^2\) is induced by a \((TE)\)-structure.

2 Preliminary material

We preserve the notation used in [3], which we now recall.

**Notation 1.** For a complex manifold \(M\), we denote by \(\mathcal{O}_M\), \(\mathcal{T}_M\), \(\Omega^k_M\) the sheaves of holomorphic functions, holomorphic vector fields and holomorphic \(k\)-forms on \(M\) respectively. For an holomorphic vector bundle \(H\), we denote by \(\mathcal{O}(H)\) the sheaf of its holomorphic sections. We denote by \(\Omega^1_{\mathbb{C} \times M}(\log \{0\} \times M)\) the sheaf of meromorphic 1-forms on \(\mathbb{C} \times M\), which are logarithmic along \(\{0\} \times M\). Locally, in a neighborhood of \((0, p)\), where \(p \in M\), any \(\omega \in \Omega^1_{\mathbb{C} \times M}(\log \{0\} \times M)\) is of the form

\[
\omega = \frac{f(z, t)}{z} \, dz + \sum_i f_i(z, t) \, dt_i
\]

where \(t = (t_1, \ldots, t_m)\) is a coordinate system of \(M\) around \(p\) and \(f, f_i\) are holomorphic. The ring of holomorphic functions defined on a neighbourhood of 0 \(\in \mathbb{C}\) will be denoted by \(\mathbb{C}\{z\}\), the ring of formal power series \(\sum_{n \geq 0} a_n z^n\) will be denoted by \(\mathbb{C}[[z]]\), the subring of formal power series \(\sum_{n \geq 0} a_n z^n\) with \(a_n = 0\) for any \(n \leq k - 1\) will be denoted by \(\mathbb{C}[[z]]_{\geq k}\) and the vector space
of polynomials of degree at most \( k \) in the variables \((t_1, \cdots, t_m)\) will be denoted by \( \mathbb{C}[t]_{\leq k} \). Finally, we denote by \( \mathbb{C}[[z]] \) the ring of formal power series \( \sum_{n \geq 0} a_n z^n \) where all \( a_n = a_n(t) \) are holomorphic on the same neighbourhood of \( 0 \in \mathbb{C} \) and by \( \mathbb{C}[[z]][t]_{\leq k} \) the vector space of formal power series \( \sum_{n \geq 0} a_n z^n \) with \( a_n \) polynomials of degree at most \( k \) in \( t \). For a function \( f \in \mathbb{C}[[z]] \) and matrix \( A \in M_{k \times k} (\mathbb{C}[[z]]) \), we often write \( f = \sum_{n \geq 0} f(n) z^n \) and \( A = \sum_{n \geq 0} A(n) z^n \) where \( f(n) \in \mathbb{C}[[z]] \) and \( A(n) \in M_{k \times k} (\mathbb{C}[[z]]) \). The ring of meromorphic functions defined on a neighborhood of the origin \( 0 \in \mathbb{C} \), with pole at the origin only, will be denoted by \( k \).

### 2.1 Differential equations

Along this section \( t \in (\mathbb{C}, 0) \) is the standard coordinate. We shall use repeatedly the following well-known lemma (see e.g. [17] and the proof of Lemma 12 of [3]).

**Lemma 2.** Any formal solution \( u \in \mathbb{C}[[z]] \) of a differential equation of the form

\[
tu'(t) + A(t)u(t) = b(t),
\]

where \( A : (\mathbb{C}, 0) \to M_n(\mathbb{C}) \) and \( b : (\mathbb{C}, 0) \to \mathbb{C}^n \) are holomorphic, is holomorphic.

The next class of inequalities will be used in Lemma 4 below.

**Lemma 3.** Let \( C := 4 \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^2 = \frac{2}{3} \pi^2 \). For any \( b, l \in \mathbb{Z}_{\geq 2} \) with \( b \geq l \),

\[
\sum_{a_1, \cdots, a_l : (\ast)_{l,b}} (a_1 \cdots a_l)^{-2} \leq C^{l-1} b^{-2},
\]

where the condition \((\ast)_{l,b}\) on \((a_1, \cdots, a_l)\) means \( a_i \in \mathbb{Z}_{\geq 1} \) (for any \( 1 \leq i \leq l \)) and \( \sum_{i=1}^{l} a_i = b \).

**Proof.** We prove (2) by induction on \( l \). Consider first \( l = 2 \).

\[
\sum_{a_1, a_2 : (\ast)_{2,b}} (a_1 a_2)^{-2} = \sum_{a=1}^{b-1} \left( a^{-1} (b-a)^{-1} \right)^2 = \sum_{a=1}^{b-1} \left( (a^{-1} + (b-a)^{-1}) b^{-1} \right)^2
\]

\[
= b^{-2} \sum_{a=1}^{b-1} (a^{-1} + (b-a)^{-1})^2 \leq 2b^{-2} \sum_{a=1}^{b-1} (a^{-2} + (b-a)^{-2}) \leq b^{-2} C.
\]
Suppose that (2) holds for any $l \leq n - 1$. Using (2) for $l = 2$ and $l = n - 1$,

$$
\sum_{a_1, \ldots, a_n := (s)} (a_1 \cdots a_n)^{-2} = \sum_{b_1, b_2 := (s)} \sum_{a_1, \ldots, a_{n-1} := (s)} (a_1 \cdots a_{n-1} \cdot b_2)^{-2}
$$

\leq \sum_{b_1, b_2 := (s)} C^{n-2} (b_1 b_2)^{-2} \leq C^{n-1} b^{-2},

i.e. (2) holds for $l = n$ as well.

Lemma 4. Let $f \in \mathbb{C}\{t\}$ be a unit and $r \in \mathbb{Z}_{\geq 1}$. There is a unique $c \in \mathbb{C}$ such that the differential equation

$$
t \dot{\tau}(t) + r \tau(t) = \tau(t)^2 f(t) (1 + ct \tau(t))
$$

admits a formal solution $\tau = \sum_{n \geq 0} \tau_n t^n \in \mathbb{C}[[t]]$ with $\tau_0 \neq 0$. Any such solution $\tau$ is holomorphic. It is uniquely determined by $\tau_r \in \mathbb{C}$, which can be chosen arbitrarily.

Proof. We write $f = \sum_{n \geq 0} f_n t^n$. Identifying the coefficients in (3) (and using that $\tau_0 \neq 0$) we obtain that $\tau_n$, for $n \in \{1, \cdots, r\}$, are determined inductively by

$$
\tau_0 = r f_0, \quad \tau_n = \frac{1}{n-r} \sum_{j+k+p=n; k, p \leq n-1} f_j \tau_k \tau_p, \quad \forall n < r.
$$

Identifying the coefficients of $t'$ in (3) we obtain that $c$ is determined by

$$
c = -\frac{1}{\tau_0^2 f_0} \sum_{j+k+p=r; k, p \leq r-1} f_j \tau_k \tau_p.
$$

It follows that there is a unique $c \in \mathbb{C}$, namely the one defined by (5), for which (3) admits a formal solution $\tau$ with $\tau_0 \neq 0$: the coefficient $\tau_r$ of $\tau$ can be chosen arbitrarily and the remaining coefficients $\tau_n$, for $n \geq r + 1$, are determined inductively by

$$
\tau_n = \frac{1}{n-r} \left( \sum_{j+k+p=n; k, p \leq n-1} f_j \tau_k \tau_p + c \sum_{j+k+p+s=n-r} \tau_j \tau_k \tau_p f_s \right).
$$

It remains to prove that $\tau$ is holomorphic. Since $f$ is holomorphic, there is $M > 0$ and $\tilde{r} > 0$ such that

$$
|f_n| \leq \frac{M \tilde{r}^n}{(n+1)^2}, \quad \forall n \geq 0.
$$
The above relation for \( n = 0 \) implies that \( M \geq |f_0| \). We further assume that \( M \geq 1 \). We claim that for a suitable choice of \( M \) and \( \tilde{r} \) satisfying relations (7), the coefficients \( \tau_n \) of \( \tau \) satisfy

\[
|\tau_n| \leq \frac{M^{n+1} \tilde{r}^n}{(n+1)^2}, \quad \forall n \geq 0. \tag{8}
\]

Remark that for \( n = 0 \) relation (8) is equivalent to \( M \geq |\tau_0| = \frac{\tilde{r}}{|f_0|} \).

To prove the claim, let \( n \geq r + 1 \) be fixed. We assume that (8) holds for all \( \tau_0, \cdots, \tau_{n-1} \) and we study when it holds for \( \tau_n \). For this, we evaluate, using (6),

\[
|\tau_n| \leq \frac{1}{n-r} \left( \sum_{j+k+p=n; \ k,p \leq n-1} |f_j||\tau_k||\tau_p| + |c| \sum_{j+k+p+s=n-r} |\tau_j||\tau_k||\tau_p||f_s| \right)
\]

\[
\leq \frac{\tilde{r}^n}{n-r} \sum_{j+k+p=n; \ k,p \leq n-1} M^{n-j+3}(j+1)^{-2}(k+1)^{-2}(p+1)^{-2}
\]

\[
+ \frac{|c|\tilde{r}^{n-r}}{n-r} \sum_{j+k+p+s=n-r} M^{n-r-s+4}(j+1)^{-2}(k+1)^{-2}(p+1)^{-2}(s+1)^{-2}.
\]

Since \( M \geq 1 \), \( M^{n-j+3} \leq M^{n+3} \) and \( M^{n-r-s+4} \leq M^{n-r+4} \). From Lemma 3 we obtain that

\[
|\tau_n| \leq \frac{1}{n-r} \left( \tilde{r}^n M^{n+3}C^2(n+3)^{-2} + |c|M^{n-r+4}\tilde{r}^{n-r}C^3(n-r+4)^{-2} \right). \tag{9}
\]

We deduce that a sufficient condition for (8) to hold also for \( \tau_n \) is that

\[
M^2 + \frac{|c|M^{3-r}}{\tilde{r}^r} \left( \frac{n+3}{n-r+4} \right)^2 \leq \frac{1}{C^2} \left( \frac{n+3}{n+1} \right)^2. \tag{10}
\]

Consider now \( \epsilon_0 > 0 \) small, \( M_0 \geq \max\{\tilde{r}_0, M\} \) and \( n_0 > r \) such that

\[
M_0^2 \leq \frac{1}{C^2} \left( \frac{n+3}{n+1} \right)^2 - \epsilon_0, \quad \forall n \geq n_0. \tag{11}
\]

(This is possible since the right hand side of (11) tends to \(+\infty\) for \( n \to +\infty \)). With this choice of \( (M_0, n_0, \epsilon_0) \), we choose \( \tilde{r}_0 \geq \tilde{r} \) such that

\[
\tilde{r}_0^r \geq \frac{|c|M_0^{3-r}}{\epsilon_0} \left( \frac{n+3}{n-r+4} \right)^2, \quad \forall n \geq n_0. \tag{12}
\]
(This is possible since the right hand side of (12) is bounded when \( n \to +\infty \)).

Relations (11) and (12) imply
\[
M_0^2 + \frac{C|c|M_0^{3-r}}{\tilde{r}_0} \left( \frac{n + 3}{n - r + 4} \right)^2 \leq \frac{1}{C^2} \left( \frac{n + 3}{n + 1} \right)^2, \quad \forall n \geq n_0,
\]
i.e. relation (10) (with \( M \) and \( \tilde{r} \) replaced by \( M_0 \) and \( \tilde{r}_0 \) respectively) holds.

The above argument shows that the inequalities
\[
|\tau_n| \leq \frac{M_0^{n+1}\tilde{r}_0^n}{(n + 1)^2}, \quad \forall n \geq 0
\]
hold if they hold for any \( n \leq n_0 - 1 \) and
\[
|f_n| \leq \frac{M_0\tilde{r}_0^n}{(n + 1)^2}, \quad \forall n \geq 0.
\]

But (15) is obviously true from (7), since \( M_0 \geq M \) and \( \tilde{r}_0 \geq \tilde{r} \). Relations (14) for \( n \leq n_0 - 1 \) are satisfied as well, by imposing to \( \tilde{r}_0 \) (which can be chosen as large as needed) the additional conditions
\[
\tilde{r}_0^n \geq \frac{(n + 1)^2|\tau_n|}{M_0^{n+1}}, \quad \forall 0 \leq n \leq n_0 - 1.
\]

From (14), \( \tau \in \mathbb{C}\{t\} \).

The following lemma will be used in our formal classification of \((TE)\)-structures. Its proof is straightforward and will be omitted.

**Lemma 5.** Consider the system of equations
\[
mx + bx - b\dot{x} = g, \quad x^{(3)} = 0
\]
in the unknown function \( x \in \mathbb{C}\{t\} \), where \( m \in \mathbb{C}^* \) and \( g = g_2t^2 + g_1t + g_0 \), for \( g_i \in \mathbb{C} \).

i) Assume that \( b(t) = \lambda t, \) for \( \lambda \in \mathbb{C}^* \). If \( m \notin \{\pm \lambda \} \) then there is a unique solution \( x \) of \((17)\). If \( m = \lambda \), then there is a solution of \((17)\) if and only if \( g_2 = 0 \). If \( m = -\lambda \), then there is a solution of \((17)\) if and only if \( g_0 = 0 \).

ii) Assume that \( b(t) = \lambda t + 1, \) for \( \lambda \in \mathbb{C} \). If \( m \notin \{\pm \lambda \}, \) then there is a unique solution \( x \) of \((17)\). If \( m = \lambda \), then there is a solution of \((17)\) if and only if \( g_2 = 0 \). If \( m = -\lambda \), then there is a solution of \((17)\) if and only if \( m^2g_0 + mg_1 + g_2 = 0 \).

iii) Assume that \( b(t) = t^2 \). Then there is a unique solution of \((17)\).
2.2 Basic facts on \((TE)\)-structures

In order to keep the text self-contained, we recall basic facts on \((TE)\)-structures and the weaker notion of \((T)\)-structure. Let \(M\) be a complex manifold.

**Definition 6.**

i) A \((TE)\)-structure over \(M\) is a pair \((H \to \mathbb{C} \times M, \nabla)\) where \(H \to \mathbb{C} \times M\) is a holomorphic vector bundle and \(\nabla\) is a flat connection on \(H|_{\mathbb{C}^* \times M}\) with poles of Poincaré rank 1 along \(\{0\} \times M\):

\[
\nabla : \mathcal{O}(H) \to \frac{1}{z} \Omega^1_{\mathbb{C} \times M}(\log(\{0\} \times M)) \otimes \mathcal{O}(H). \quad (18)
\]

ii) A \((T)\)-structure over \(M\) is a pair \((H \to \mathbb{C} \times M, \nabla)\) where \(\nabla\) is a map

\[
\nabla : \mathcal{O}(H) \to \frac{1}{z} \Omega^1_{\mathbb{C} \times M} \cdot \Omega^1_M \otimes \mathcal{O}(H) \quad (19)
\]

such that, for any \(z \in \mathbb{C}^*\), the restriction of \(\nabla\) to \(H|_{\{z\} \times M}\) is a flat connection.

As we only consider \((TE)\)-structures over germs of \(F\)-manifolds, we will always assume in our computations that \(H = (\mathcal{O}(\mathbb{C}^m,0))^r\) is the trivial rank \(r\) vector bundle and \(M = (\mathbb{C}^m,0)\) with coordinates \((t_1, \ldots, t_m)\). For the same reason we shall sometimes refer to a \((T)\) or \((TE)\)-structure simply as a map \(\nabla\) with the properties from Definition 6 (without mentioning the vector bundle \(H\)).

With respect to the standard basis \(s = (s_1, \ldots, s_r)\) of \(H\),

\[
\nabla s = s \cdot \Omega, \quad \Omega = \sum_{i=1}^{m} z^{-1} A_i(z, t) dt_i + z^{-2} B(z, t) dz, \quad (20)
\]

where \(A_i, B\) are holomorphic,

\[
A_i(z, t) = \sum_{k \geq 0} A_i^{(k)} z^k, \quad B(z, t) = \sum_{k \geq 0} B^{(k)} z^k \quad (21)
\]

and \(A_i^{(k)}\) and \(B^{(k)}\) depend only on \((t_i)\). The flatness of \(\nabla\) gives, for any \(i \neq j\),

\[
z \partial_i A_j - z \partial_j A_i + [A_i, A_j] = 0, \quad (22)
\]

\[
z \partial_i B - z^2 \partial_z A_i + z A_i + [A_i, B] = 0. \quad (23)
\]

(When \(\nabla\) is a \((T)\)-structure, the summand \(z^{-2} B(t, z) dz\) in \(\Omega\) and relations (23) are dropped). Relations (22), (23) split according to the powers of \(z\) as
follows: for any \( k \geq 0 \),

\[
\partial_i A_j^{(k-1)} - \partial_j A_i^{(k-1)} + \sum_{l=0}^{k} [A_i^{(l)}, A_j^{(k-l)}] = 0, \quad (24)
\]

\[
\partial_i B^{(k-1)} - (k-2)A_i^{(k-1)} + \sum_{l=0}^{k} [A_i^{(l)}, B^{(k-l)}] = 0, \quad (25)
\]

where \( A_i^{(-1)} = B^{(-1)} = 0 \).

Let \((H, \nabla)\) and \((\hat{H}, \hat{\nabla})\) be two \((TE)\)-structures over \((\mathbb{C}^m, 0)\), with underlying bundle \( H = (O_{(\mathbb{C}^m, 0)})^r \), defined by matrices \( A_i, B \) and \( \hat{A}_i, \hat{B} \) respectively. An isomorphism \( T \) between \((H, \nabla)\) and \((\hat{H}, \hat{\nabla})\) which covers \( h: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^m, 0) \), \( h = (h^1, \ldots, h^m) \), is given by a matrix \( (T_{ij}) = \sum_{r \geq 0} T^{(k)}z^k \in M_{r \times r}(O_{\Delta \times \mathbb{U}}) \) (where \( \Delta \subset \mathbb{C} \) is a small disc around the origin), with \( T^{(k)} \in M_{r \times r}(O_{\mathbb{U}}) \), \( T^{(0)} \) invertible, such that

\[
z\partial_i \hat{T} + \sum_{j=1}^{m} (\partial_i h^j)(A_j \circ h)\hat{T} - \hat{T} \hat{A}_i = 0, \quad \forall i \quad (26)
\]

\[
z^2 \partial_i \hat{T} + (B \circ h)\hat{T} - \hat{T} \hat{B} = 0, \quad (27)
\]

where \( \hat{T} := T \circ h \) (relation (27) has to be omitted when \( \hat{\nabla} \) and \( \nabla \) are \((T)\)-structures). Relations (26), (27) split according to the powers of \( z \) as

\[
\partial_i \hat{T}^{(r-1)} + \sum_{l=0}^{r} \sum_{j=1}^{m} (\partial_i h^j)(A_j^{(l)} \circ h)\hat{T}^{(r-l)} - \hat{T}^{(r-l)} \hat{A}_i^{(l)} = 0 \quad (28)
\]

\[
(r-1)\hat{T}^{(r-1)} + \sum_{l=0}^{r} ((B^{(l)} \circ h)\hat{T}^{(r-l)} - \hat{T}^{(r-l)} \hat{B}^{(l)}) = 0, \quad (29)
\]

for any \( r \geq 0 \), where \( \hat{T}^{(-1)} = 0 \). When \( h = \text{Id}_{(\mathbb{C}^m, 0)} \), the isomorphism \( T \) is called a gauge isomorphism. It satisfies

\[
z\partial_i T + A_i T - T \hat{A}_i = 0 \quad (30)
\]

\[
z^2 \partial_i T + BT - T \hat{B} = 0, \quad (31)
\]

or

\[
\partial_i T^{(r-1)} + \sum_{l=0}^{r} (A_i^{(l)}T^{(r-l)} - T^{(r-l)} \hat{A}_i^{(l)}) = 0, \quad (32)
\]

\[
(r-1)T^{(r-1)} + \sum_{l=0}^{r} (B^{(l)}T^{(r-l)} - T^{(r-l)} \hat{B}^{(l)}) = 0, \quad (33)
\]

for any \( r \geq 0 \).
Remark 7. i) A formal $(T)$ or $(TE)$-structure $\nabla$ over $(\mathbb{C}^m, 0)$ is given by a connection form (20), where $A_i$ and $B$ (the latter only when $\nabla$ is a $(TE)$-structure) are matrices with entries in $\mathbb{C}\{t, z\}$, satisfying relations (22), (23) or (24), (25) (relations (23), (25) only when $\nabla$ is a $(TE)$-structure).

ii) A formal isomorphism between two formal $(T)$ or $(TE)$-structures $\nabla$ and $\tilde{\nabla}$ which covers a biholomorphic map $h : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$, is given by a matrix $T = (T_{ij})$ with entries $T_{ij} \in \mathbb{C}\{t, z\}$, such that relations (26), (27) or (28), (29) are satisfied with $\tilde{T} = T \circ h$ (relations (27), (29) only when $\nabla$ and $\tilde{\nabla}$ are $(TE)$-structures). Formal gauge isomorphisms between $(T)$ or $(TE)$-structures are formal isomorphisms which cover the identity map. They are given by matrices $T = (T_{ij})$ with entries in $\mathbb{C}\{t, z\}$ such that relations (30), (31) or (32), (33) are satisfied.

2.3 $(TE)$-structures and $F$-manifolds

2.3.1 General results

Let $(H, \nabla)$ be a $(T)$-structure over a complex manifold $M$. It induces a Higgs field $C \in \Omega^1(M, \text{End}(K))$ on the restriction $K := H|_{\{0\} \times M}$, defined by

$$C_X[a] := [z\nabla_X a], \quad \forall X \in \mathcal{T}_M, a \in \mathcal{O}(H),$$

where $[ ]$ means the restriction to $\{0\} \times M$ and $X \in \mathcal{T}_M$ is lifted canonically to $\mathbb{C} \times M$. If $(H, \nabla)$ is a $(TE)$-structure then there is in addition an endomorphism $\mathcal{U} \in \text{End}(K)$,

$$\mathcal{U} := [z\nabla_z a] : \mathcal{O}(K) \to \mathcal{O}(K).$$

Definition 8. ([6]) The $(T)$-structure (or $(TE)$-structure) $(H, \nabla)$ satisfies the unfolding condition if there is an open cover $V$ of $M$ and for any $U \in V$ a section $\zeta_U \in \mathcal{O}(K|_U)$ (called a local primitive section) with the property that the map $T_U \ni X \to C_X \zeta_U \in K$ is an isomorphism.

When $(H \to \mathbb{C} \times M, \nabla)$ satisfies the unfolding condition the rank of $H$ coincides with the dimension of $M$. We now define the notion of $F$-manifold.

Definition 9. ([8]) A complex manifold $M$ with a fiber-preserving, commutative, associative multiplication $\circ$ on the holomorphic tangent bundle $TM$ and unit field $e \in \mathcal{T}_M$ is an $F$-manifold if

$$L_{X \circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ), \quad \forall X, Y \in \mathcal{T}_M.$$  

(36)

A vector field $E \in \mathcal{T}_M$ is called an Euler field (of weight 1) if

$$L_E(\circ) = \circ.$$  

(37)
The following theorem was proved in Theorem 3.3 of [10].

**Theorem 10.** A \((T)\)-structure \((H \to \mathbb{C} \times M, \nabla)\) with unfolding condition induces a multiplication \(\circ\) on \(TM\) which makes \(M\) an \(F\)-manifold. A \((TE)\)-structure \((H \to \mathbb{C} \times M, \nabla)\) with unfolding condition induces in addition a vector field \(E\) on \(M\), which, together with \(\circ\), makes \(M\) an \(F\)-manifold with Euler field. The multiplication \(\circ\), unit field \(e\) and Euler field \(E\) (the latter, in the case of a \((TE)\)-structure), are defined by

\[
C_{X \circ Y} = C_X C_Y, \quad C_e = \text{Id}, \quad C_E = -U
\]

where \(C\) is the Higgs field defined by \(\nabla\) and \(U\) is the endomorphism defined in \((52)\).

A \((T)\)- or \((TE)\)-structure as in Theorem [10] is said to lie over the \(F\)-manifold \((M, \circ, e)\) (or over the \(F\)-manifold with Euler field \((M, \circ, e, E)\) respectively). \(F\)-manifold isomorphisms lift naturally to isomorphisms between the spaces of \((T)\) or \((TE)\)-structures lying over the respective \(F\)-manifolds. In particular, the spaces of (formal or holomorphic) \((T)\)-structures over isomorphic germs of \(F\)-manifolds are isomorphic.

### 2.3.2 \((T)\)-structures over \(N_2\)

In dimension two, besides the family \(I_2(m)\) \((m \in \mathbb{Z}_{\geq 3})\) there is (up to isomorphism) a unique irreducible germ of \(F\)-manifold. It is the germ \(N_2\) at the origin of the 2-dimensional globally nilpotent \(F\)-manifold, which we are now going to describe. As a manifold germ, \(N_2\) is \((\mathbb{C}^2, 0)\). In the standard coordinates \((t_1, t_2)\) of \(\mathbb{C}^2\), \(N_2\) has \(\partial_1\) as unit field and \(\partial_2 \circ \partial_2 = 0\). The automorphism group \(\text{Aut}(N_2)\) is the group of all biholomorphic maps

\[
(t_1, t_2) \to (t_1, \lambda(t_2)),
\]

where \(\lambda \in \mathbb{C}\{t_2\}\), with \(\lambda(0) = 0\) and \(\dot{\lambda}(0) \neq 0\) (i.e. \(\lambda \in \text{Aut}(\mathbb{C}, 0))\)).

Following [3], we recall the formal normal forms of \((T)\)-structures over \(N_2\). They are the starting point in our treatment of \((TE)\)-structures over \(N_2\).

**Notation 11.** The formal normal forms are expressed in terms of matrices \(C_1, C_2, D\) and \(E\), defined by

\[
C_1 := \text{Id}_2, \quad C_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

\[(40)\]
Remark that
\[(C_2)^2 = 0, \quad D^2 = C_1, \quad E^2 = 0, \quad (41)\]
\[C_2 D = C_2 = -DC_2, \quad DE = E = -ED, \quad (42)\]
\[C_2 E = \frac{1}{2}(C_1 - D), \quad EC_2 = \frac{1}{2}(C_1 + D), \quad (43)\]
\[[C_2, D] = 2C_2, \quad [C_2, E] = -D, \quad [D, E] = 2E. \quad (44)\]

The formal normal forms are described in the next theorem.

**Theorem 12.** (3) Any \((T)\)-structure over \(\mathcal{N}_2\) is formally isomorphic to a \((T)\)-structure of the form
\[A_1 = C_1, \quad A_2 = C_2 + zE \quad (45)\]
\[A_1 = C_1, \quad A_2 = C_2 + zt_2E \quad (46)\]
\[A_1 = C_1, \quad A_2 = C_2 \quad (47)\]
or to a holomorphic or formal \((T)\)-structure of the form
\[A_1 = C_1, \quad A_2 = C_2 + z(t_2 + \sum_{k \geq 1} P_k z^k)E, \quad (48)\]
where \(r \in \mathbb{Z}_{\geq 2}\) and \(P_k \in \mathbb{C}[t_2]_{\leq r-2}\) are polynomials of degree at most \(r - 2\).

The formal normal forms are pairwise formally gauge non-isomorphic, but there exist (distinct) formal normal forms (of the form (48)) which are formally (non-gauge) isomorphic. For a precise statement on (non)-uniqueness of formal normal forms, see Theorem 21 of [3].

### 2.4 The Fuchs criterion

For the definition and properties of meromorphic connections with regular singularities, see e.g. [15], Chapter II. Consider a meromorphic connection \(\nabla\) on the germ \(\mathcal{M} = k^d\) of the meromorphic rank \(d\) trivial vector bundle over \((\mathbb{C}, 0)\), with pole at the origin only. The Fuchs criterion is an effective way to check if \(\nabla\) has a regular singularity at the origin. Namely, one considers a cyclic vector, i.e. a section \(v_0\) such that \(\{v_0, v_1 := \nabla_{\partial_z}(v_0), \ldots, v_{d-1} := \nabla_{\partial_z}^{d-1}(v_0)\}\) is a basis of \(\mathcal{M}\) (such a vector always exists). In this basis, \(\nabla\) has the expression
\[\nabla_{\partial_z}(v_i) = v_{i+1}, \quad 0 \leq i \leq d - 2 \quad (49)\]
\[\nabla_{\partial_z}(v_{d-1}) = a_0 v_0 + \cdots + a_{d-1} v_{d-1}, \quad (50)\]
for some \( a_i \in k \). We denote by \( v(f) \) the valuation of a function \( f \in k \), i.e. the unique integer such that \( f(z) = z^{v(f)}h(z) \), where \( h \in \mathbb{C}\{z\} \) and \( h(0) \neq 0 \).

The Fuchs criterion is stated as follows (see [14]):

**Theorem 13.** The connection \( \nabla \) has a regular singularity at the origin if and only if \( v(a_i) \geq i - d \), for any \( 0 \leq i \leq d - 1 \).

### 2.5 Irreducible bundles and Birkhoff normal form

Consider a meromorphic connection \( \nabla \) on the germ \( E = (\mathcal{O}(\mathbb{C},0))^d \) of the holomorphic rank \( d \) trivial vector bundle over \( (\mathbb{C},0) \), with pole of order \( r \geq 0 \) at the origin. In the standard basis of \( E \), the connection form of \( \nabla \) is given by \( A(z)dz \), where \( A \in M_{d \times d}(k) \) is such that \( z^{r+1}A(z) \) is holomorphic. We say that \( (E, \nabla) \) can be put in Birkhoff normal form if there is a holomorphic isomorphism \( T \in M_{d \times d}(\mathcal{O}(\mathbb{C},0)) \) such that the image \( T \cdot \nabla \) of \( \nabla \) by \( T \) has connection form

\[
\Omega := z^{-(r+1)}(B_0z^0 + \cdots + z^r B_r)\ dz
\]

where \( B_i \) are constant matrices.

We now recall the irreducibility criterion (see [1] and [15], Chapter IV) which provides a sufficient condition for the existence of \( T \). Given \( (E, \nabla) \) as above, let \( (\mathcal{M} = k^d, \nabla) \) be the germ of the meromorphic bundle with connection, with singularity at the origin only, for which \( (E, \nabla) \) is a lattice. From the Riemann-Hilbert correspondence (see e.g. [15], page 99), there is a unique (up to isomorphism) meromorphic bundle with connection \( (\hat{\mathcal{M}}, \hat{\nabla}) \) on \( \mathbb{P}^1 \), with poles at 0 and \( \infty \), whose germ at 0 is isomorphic to \( (\mathcal{M}, \nabla) \) and such that \( \infty \) is a regular singularity for \( \hat{\nabla} \). We say that \( (\hat{\mathcal{M}}, \hat{\nabla}) \) is irreducible if there is no proper meromorphic subbundle \( \mathcal{N} \rightarrow \mathbb{P}^1 \) of \( \hat{\mathcal{M}} \), which is preserved by \( \hat{\nabla} \), i.e. \( \hat{\nabla}(\mathcal{N}) \subset \Omega^1_{\mathbb{P}^1} \otimes \mathcal{N} \). The irreducibility criterion states that if \( (\hat{\mathcal{M}}, \hat{\nabla}) \) is irreducible, then \( (E, \nabla) \) can be put in Birkhoff normal form: one extends the lattice \( E \) around the origin to a globally defined lattice \( \hat{E} \) of \( (\hat{\mathcal{M}}, \hat{\nabla}) \), logarithmic at \( \infty \), and applies Corollary 2.6 of [15] (page 154) to obtain a new lattice \( \hat{E}' \) of \( \hat{\mathcal{M}} \), which also extends \( E \), is logarithmic at \( \infty \) and is trivial as a holomorphic vector bundle. A base change between the standard basis of \( E \) and a basis of \( \hat{E}' \) in a neighborhood of 0 \( \in \mathbb{C} \) gives the holomorphic isomorphism \( T \) above. We shall apply the irreducibility criterion in the following simple form, for rank 2 bundles.

**Lemma 14.** Assume that \( E \) is of rank two and let \( \{v_1, v_2\} \) be its standard basis. If there is no non-zero \( w = gv_1 + fv_2 \), with \( f, g \in k \), such that \( \nabla_{\partial_z}(w) = hw \) for a function \( h \in k \), then \( \nabla \) can be put in Birkhoff normal form.
Proof. In the above notation, we claim that \( (\hat{\mathcal{M}}, \hat{\nabla}) \) is irreducible: if it were reducible, then a basis of \( \mathcal{N} \) around the origin would provide a section \( w \) as in the statement of the lemma. We obtain a contradiction. \qed

2.6 Malgrange universal connections

Let \( (H \rightarrow \mathbb{C} \times (M, 0), \nabla) \) be a \((TE)\)-structure with unfolding condition over a germ \(( (M, 0), \circ, e, E) \) of an \( F \)-manifold with Euler field. Assume that the restriction \( \nabla^0 \) of \( \nabla \) to the slice at the origin \( \mathbb{C} \times \{0\} \) can be put in Birkhoff normal form. Let \( \tilde{v}_0 \) be a basis of \( H|_{(\mathbb{C}, 0) \times \{0\}} \) in which the connection form of \( \nabla^0 \) is given by

\[
\Omega^0 = \left( \frac{B_0}{z} + B_\infty \right) \frac{dz}{z},
\]

where \( B_0, B_\infty \in M_{n \times n}(\mathbb{C}) \) (and \( n = \text{rank}(H) = \text{dim}(M) \)). If \( B_0 \) is a regular matrix (i.e. distinct Jordan blocks in its Jordan normal form have distinct eigenvalues, or the vector space of matrices which commute with \( B_0 \) has dimension \( n \), with basis \{Id, \( B_0 \), \( \cdots \), \( (B_0)^{n-1} \)\}), then \( \nabla^0 \) has a universal deformation \( \nabla^{\text{univ}} := \nabla^{\text{univ}}, B_0, B_\infty \). In particular, \( \nabla^{\text{univ}} \) is isomorphic to the given \((TE)\)-structure \( \nabla \) and so are the parameter spaces of \( \nabla^{\text{univ}} \) and \( \nabla \) (as \( F \)-manifolds with Euler fields). The universal deformation \( \nabla^{\text{univ}} \) was constructed by Malgrange in [12, 13] (see also [15], Chapter VI, Section 3.a; see e.g. [15], page 199, for the definition of the universal deformation). We now recall its definition. Let \( \mathcal{D} \subset TM_{n \times n}(\mathbb{C}) \) be defined by

\[
\mathcal{D}_\Gamma := \text{Span}_\mathbb{C}\{\text{Id}, (B_0)_\Gamma, \cdots , (B_0)^{n-1}_\Gamma\} \subset T_{\Gamma}M_{n \times n}(\mathbb{C}) = M_{n \times n}(\mathbb{C}),
\]

where

\[
(B_0)_\Gamma := B_0 - \Gamma + [B_\infty, \Gamma].
\]

Because \( B_0^\prime \) is regular, so is \( (B_0)_\Gamma \), for any \( \Gamma \in W \), where \( W \) is a small open neighborhood of 0 in \( M_n(\mathbb{C}) \). For any \( \Gamma \in W \), \( \mathcal{D}_\Gamma \) is the \((n\text{-dimensional})\) vector space of polynomials in \( (B_0)_\Gamma \) and the distribution \( \mathcal{D} \rightarrow W \) is integrable. The parameter space \( M^{\text{univ}} \) of \( \nabla^{\text{univ}} \) is the maximal integral submanifold of \( \mathcal{D}|_W \), passing through \( 0 \in M_{n \times n}(\mathbb{C}) \) (the trivial matrix). Let \( \circ^{\text{univ}} \) be the multiplication on \( TM^{\text{univ}} \), which, on any tangent space \( T_{\Gamma}M^{\text{univ}} = \mathcal{D}_\Gamma \), is given by multiplication of matrices. It has unit field \( \epsilon^{\text{univ}} := \text{Id} \) (i.e. \( (\epsilon^{\text{univ}})_\Gamma := \text{Id} \in \mathcal{D}_\Gamma \), for any \( \Gamma \in M^{\text{univ}} \)). Let \( E^{\text{univ}} \) be the vector field on \( M^{\text{univ}} \) defined by \( E^{\text{univ}} := -B_0 \) (i.e. \( E_{\Gamma} := -(B_0)_\Gamma \), for any \( \Gamma \in M^{\text{univ}} \)). Then \( (M^{\text{univ}}, \circ^{\text{univ}}, \epsilon^{\text{univ}}, E^{\text{univ}}) \) is a regular \( F \)-manifold (see Definition 2 of [2]). The germ \( (M^{\text{univ}}, 0), \circ^{\text{univ}}, \epsilon^{\text{univ}}, E^{\text{univ}} \) is universal in the following sense: it is the unique (up to isomorphism) germ of \( F \)-manifold with Euler field \(( (M, 0), \circ, e, E) \) for which the endomorphism \( \mathcal{U}(X) := E \circ X \) of \( T_0M \) has the
same conjugacy class as $B_0^a$ (see [2]). Moreover, if $B_0^a$ has a unique eigenvalue (or a unique Jordan block) then the matrix $(B_0)_{\Gamma}$, for any $\Gamma \in M^{\text{univ}}$, has this property as well (see Proposition 15 of [2]).

The universal deformation $\nabla^{\text{univ}}$ of $\nabla^0$ is defined on the trivial bundle $E = (\mathbb{C} \times M^{\text{univ}}) \times \mathbb{C}^n \to \mathbb{C} \times M^{\text{univ}}$. Its connection form in the standard trivialization of $E$ is given by

$$\Omega^{\text{univ}} = \left( \frac{B_0}{z} + B_\infty \right) \frac{dz}{z} + \frac{C}{z}. \quad (54)$$

Here $B_0 : M^{\text{univ}} \to M_{n \times n}(\mathbb{C})$, $(B_0)(\Gamma) := (B_0)_{\Gamma}$ is given by (53) and $C_X := X$ is the action of the matrix $X$ on $\mathbb{C}^n$, for any $X \in T_\Gamma M^{\text{can}} \subset M_{n \times n}(\mathbb{C})$.

3 Normal forms for Euler fields on $N_2$

In this section we determine normal forms for Euler fields on $N_2$, as follows.

**Theorem 15.** i) Up to an automorphism, any Euler field on $N_2$ is of the form

$$E = (t_1 + c)\partial_1 + \partial_2 \quad (55)$$

$$E = (t_1 + c)\partial_1 \quad (56)$$

$$E = (t_1 + c)\partial_1 + c_0 t_2 \partial_2, \quad (57)$$

$$E = (t_1 + c)\partial_1 + t'_2(1 + c_1 t_2^{-1}) \partial_2, \quad (58)$$

where $c, c_1 \in \mathbb{C}$, $c_0 \in \mathbb{C}^*$ and $r \in \mathbb{Z}_{\geq 2}$.

ii) Any two (distinct) Euler fields from the above list belong to distinct orbits of the natural action of $\text{Aut}(N_2)$ on the space of Euler fields.

We divide the proof into several steps.

**Lemma 16.** i) A vector field on $N_2$ is an Euler field if and only if

$$E = (t_1 + c)\partial_1 + g(t_2)\partial_2, \quad (59)$$

for $c \in \mathbb{C}$ and $g \in \mathbb{C}\{t_2\}$.

ii) If $g \neq 0$, then $r := \text{ord}_0(g) \geq 0$ is $\text{Aut}(N_2)$-invariant. If $g \neq 0$ is constant, then up to an automorphism, $E$ is of the form (55).

iii) If $g = 0$ then $E$ is of the form (57) and the $\text{Aut}(N_2)$-orbit of $E$ reduces to $E$.  

Lemma 17. Let ̃f, formal solution ̃f be a vector field on \( N_2 \), where \( f, g \in \mathbb{C}\{t_1, t_2\} \).

Proof. i) Let \( \lambda \in \text{Aut}(\mathbb{C}, 0) \) and \( E \) an Euler field given by (59). Then

\[ (h_*E)_{(t_1, t_2)} = (t_1 + c)\partial_1 + (\lambda g) \circ \lambda^{-1}\partial_2. \]

Assume that \( g \neq 0 \) and let \( r := \text{ord}_0(g) \in \mathbb{Z}_{\geq 0} \). Relation (60) and \( \lambda \in \text{Aut}(\mathbb{C}, 0) \) implies that \( r \) is an invariant of the \( \text{Aut}(N_2) \)-action on Euler fields. If \( g = g_0 \) is a (non-zero) constant, let \( h(t_1, t_2) := (t_1, g_0^{-1}t_2) \). Then \( h_*E = (t_1 + c)\partial_1 + \partial_2 \) is of the form (55).

ii) Claim iii) is obvious from (60).

The next lemma concludes the proof of Theorem 15 i).

Lemma 17. Let \( E \) be an Euler field given by (59), with \( g \) non-constant and \( r = \text{ord}_0(g) \geq 0 \).

i) If \( r = 0 \) then, up to an automorphism of \( N_2 \), \( E \) is of the form (55).

ii) If \( r = 1 \) then, up to an automorphism of \( N_2 \), \( E \) is of the form (57).

iii) If \( r \geq 2 \) then, up to an automorphism of \( N_2 \), \( E \) is of the form (58).

Proof. i) From (60), we need to find \( \lambda \in \text{Aut}(\mathbb{C}, 0) \) such that \( (\lambda g)(\lambda^{-1}(t)) = 1 \), or \( \lambda(t)g(t) = 1 \). Writing \( \lambda(t) = t\lambda(t) \) with \( \lambda \in \mathbb{C}\{t\} \) a unit, the problem reduces to showing that the differential equation

\[ t\dot{\lambda}(t) + \dot{\lambda}(t) = \frac{1}{g(t)} \]

admits a holomorphic solution with \( \lambda(0) \neq 0 \). Equation (61) admits a (unique)
formal solution \( \lambda \), which is holomorphic from Lemma 2. Moreover, \( \lambda(0) = \frac{1}{g(0)} \in \mathbb{C}^* \). This proves claim i).

ii) From (60), we need to show that there is \( \lambda \in \text{Aut}(\mathbb{C}, 0) \) such that \( (\lambda g)(\lambda^{-1}(t)) = c_0t \) or

\[ \dot{\lambda}(t)g(t) = c_0\lambda(t), \]

for a suitable \( c_0 \in \mathbb{C}^* \). Writing as before \( \lambda(t) = t\lambda(t) \) and \( g(t) = \frac{1}{f(t)} \), with \( f, \lambda \in \mathbb{C}\{t\} \) units, we obtain that equation (62) is equivalent to

\[ t\dot{\lambda}(t) + (1 - c_0f(t))\dot{\lambda}(t) = 0. \]

It is easy to check that (63) has a formal solution \( \tilde{\lambda} = \sum_{n \geq 0} \tilde{\lambda}_n t^n \) (unique, when \( \lambda_0 \) is given) if and only if \( c_0 = \frac{1}{f(0)} \). Define \( c_0 := \frac{1}{f(0)} \) and let \( \lambda \) be a
formal solution of (63) with \( \tilde{\lambda}_0 \neq 0 \). From Lemma 2, \( \tilde{\lambda} \) is holomorphic. Let 
\[
h(t_1, t_2) := (t_1, t_2 \tilde{\lambda}(t_2))\]
Then \( h_\ast E \) is of the form (57).

iii) From (60), we need to find \( \lambda \in \text{Aut}(\mathbb{C}, 0) \) such that
\[
(\dot{\lambda} g)(\lambda^{-1}(t)) = t^r (1 + c_1 t^{r-1}) \quad \text{or} \quad (\dot{\lambda} g)(\lambda(t)) = \lambda(t) t^r (1 + c_1 \lambda(t)^{r-1}),
\]
for a suitably chosen \( c_1 \in \mathbb{C} \). Writing \( g(t) = t^r f(t), \lambda(t) = t \tilde{\lambda}(t) \) with \( f, \tilde{\lambda} \in \mathbb{C}\{t\} \) units, and \( \tau(t) = (1 - r)\tilde{\lambda}(t)^{r-1} \), equation (64) becomes
\[
t \dot{\tau}(t) + (r - 1) \tau(t) = -\frac{\tau(t)^2}{f} (1 + \frac{c_1}{1 - r} t^{r-1} \tau(t)).
\] (65)
This is an equation, in the unknown function \( \tau \), of type (53). Lemma 4 concludes claim iii).

It remains to prove Theorem 15 ii). From (60), the constant \( c \) from the Euler fields of Theorem 15 i) is \( \text{Aut}(\mathcal{N}_2) \)-invariant. From Lemma 16 we deduce that any two distinct Euler fields \( E \) and \( \tilde{E} \) from Theorem 15 i), which belong to the same orbit of \( \text{Aut}(\mathcal{N}_2) \), are necessarily either both of the form (57) (with the same constant \( c \) and distinct constants \( c_0, \tilde{c}_0 \)) or both of the form (58) (with the same constant \( c \) and distinct constants \( c_1, \tilde{c}_1 \)). But these cases cannot hold: assume, e.g. that \( E \) and \( \tilde{E} \) are of the form (58). Then there is a solution \( \tau \), with \( \tau_0 \neq 0 \), of the equation (65) with \( f := 1 + \tilde{c}_1 t^{r-1} \). From the uniqueness of the constant \( c \) in Lemma 4 we obtain that \( c_1 = \tilde{c}_1 \). The other case can be treated similarly.

4 Formal classification of \((TE)\)-structures

Our aim in this section is to prove the following two theorems, which provide the formal classification of \((TE)\)-structures over \( \mathcal{N}_2 \).

**Theorem 18.** Any formal \((TE)\)-structure over \( \mathcal{N}_2 \) is formally isomorphic to a \((TE)\)-structure of the following forms:

i) for \( c, \alpha, c_0 \in \mathbb{C} \),
\[
A_1 = C_1, \quad A_2 = C_2 + zE,
\]
\[
B = (-t_1 + c + \alpha z) C_1 + (-\frac{t_2}{2} + c_0) C_2 - \frac{z}{4} D + z(\frac{t_2}{2} + c_0) E; \quad (66)
\]

ii) for \( c, \alpha \in \mathbb{C} \) and \( r \in \mathbb{Z}_{\geq 1} \),
\[
A_1 = C_1, \quad A_2 = C_2 + z t_2 ^r E,
\]
\[
B = (-t_1 + c + \alpha z) C_1 - \frac{t_2}{r+2} C_2 - \frac{z(r+1)}{2(r+2)} D - \frac{z t_2 ^{r+1}}{r+2} E, \quad (67)
\]
iii) a \((TE)\)-structure with underlying \((T)\)-structure \(A_1 = C_1, A_2 = C_2\) and matrix \(B\) of one of the following forms:
\[
B = (-t_1 + c + \alpha z)C_1 - \frac{z}{2}D; \\
B = (-t_1 + c + \alpha z)C_1 + t_2^2C_2 - z(t_2 + \frac{1}{2})D - z^2E; \\
B = (-t_1 + c + \alpha z)C_1 + \lambda t_2C_2 - \frac{z}{2}(\lambda + 1)D, \ \lambda \notin \mathbb{Z}^*; \\
B = (-t_1 + c + \alpha z)C_1 + (\lambda t_2 + 1)C_2 - \frac{z}{2}(\lambda + 1)D, \ \lambda \notin \mathbb{Z}^*; \\
B = (-t_1 + c + \alpha z)C_1 + \lambda t_2C_2 - \frac{z}{2}(\lambda + 1 + 2t_2)D - z^{\lambda+2}E, \ \lambda \in \mathbb{Z}_{\geq 1}; \\
B = (-t_1 + c + \alpha z)C_1 + t_2(\lambda + t_2\lambda^2)C_2 - \frac{z}{2}(\lambda + 1 + 2t_2\lambda)D - z^{\lambda+2}E, \ \lambda \in \mathbb{Z}_{\geq 1}; \\
B = (-t_1 + c + \alpha z)C_1 + \lambda t_2C_2 - \frac{z(\lambda + 1)}{2}D, \ \lambda \in \mathbb{Z}_{\leq -1};
\]
where \(c, \alpha, \gamma \in \mathbb{C}\).

**Definition 19.** The \((TE)\)-structures from Theorem \(18\) are called formal normal forms.

The next theorem studies when two formal normal forms are formally isomorphic.

**Theorem 20.** Any two (distinct) \((TE)\)-structures \(\nabla\) and \(\tilde{\nabla}\) in formal normal form are formally non-isomorphic, except when

i) both \(\nabla\) and \(\tilde{\nabla}\) are of the form (66), with constants \(c, \alpha, c_0, \tilde{c}, \tilde{\alpha}, \tilde{c}_0\) and \(c_0\tilde{c}_0 \neq 0\). Then they are formally isomorphic if and only if \(\tilde{c} = c, \ \tilde{\alpha} = \alpha\) and \(\tilde{c}_0 = -c_0\) and they are formally gauge non-isomorphic;

ii) both \(\nabla\) and \(\tilde{\nabla}\) have underlying \((T)\)-structure \(A_1 = C_1, A_2 = C_2\) and their matrices \(B\) and \(\tilde{B}\) are of the fourth form in (68), with constants \(c, \alpha, \lambda, \tilde{c}, \tilde{\alpha}, \lambda\) and \(\lambda \tilde{\lambda} \neq 0\). Then they are formally isomorphic if and only if \(\tilde{c} = c, \ \tilde{\alpha} = \alpha\) and \(\tilde{\lambda} = -\lambda\) and they are formally gauge non-isomorphic.

In order to prove Theorems \(18\) and \(20\) we begin by determining the \((TE)\)-structures which extend the \((T)\)-structures from Theorem \(12\). This is done in Section 4.1 below.
4.1 \((TE)\)-structures with normal formal \((T)\)-structures

To present a unified argument for all \((T)\)-structures from Theorem 12, we consider the general situation of a formal \((TE)\)-structure \(\nabla\), with underlying formal \((T)\)-structure \(A_1 = C_1, A_2 = C_2 + zfE\) and \(f \in \mathbb{C}\{t_2, z\}\).

**Lemma 21.** The formal \((TE)\)-structure \(\nabla\) is formally isomorphic to a \((TE)\)-structure with \(A_1 = C_1, A_2 = C_2 + zfE\) and matrix \(B\) of the form

\[
B = (-t_t + c + az)C_1 + b_2C_2 + zb_3D + zb_4E, \tag{69}
\]

where \(a, c \in \mathbb{C}, b_2, b_3, b_4 \in \mathbb{C}\{t_2, z\}\),

\[
b_1 = -\frac{1}{2}(\partial_2b_2 + 1), \quad b_4 = -\frac{z}{2}\partial_2^2b_2 + fb_2 \tag{70}
\]

and

\[
-\frac{z}{2}\partial_2^2b_2 + (\partial_2f)b_2 + 2f\partial_3b_2 - z\partial_zf + f = 0. \tag{71}
\]

**Proof.** Relation (23) with \(i = 1\) gives \(\partial_1B = -C_1\). Relation (23) with \(i = 2\) gives \([C_2, B^{(0)}] = 0\) i.e. \(B^{(0)}\) is a linear combination of \(C_1\) and \(C_2\). We obtain

\[
B = (-t_1 + b_1)C_1 + b_2C_2 + zb_3D + zb_4E, \tag{72}
\]

where \(b_1 \in \mathbb{C}\{t_2, z\}, b_2, b_3, b_4 \in \mathbb{C}\{t_2, z\}\). With \(B\) of the form (72) and \(A_2 = C_2\), relation (23) for \(i = 2\) gives, after identifying coefficients, \(\partial_2b_1 = 0\), together with (70) and (71). It remains to show that \(b_1\) can be chosen of the form \(b_1(z, t_2) = c + az\), for \(c \in \mathbb{C}\). Since \(\partial_2b_1 = 0\) and \(b_1 \in \mathbb{C}\{t_2, z\}\), we can write \(b_1 = c + az + \sum_{k>2}b_1^{(k)}z^k\), for \(a, b_1^{(k)} \in \mathbb{C}\). Define

\[
T := \exp \left( -\sum_{k \geq 2} \frac{b_1^{(k)}}{k-1} z^{k-1} \right) C_1. \tag{73}
\]

The isomorphism \(T\) maps \(\nabla\) to a new \((TE)\)-structure which has the same underlying \((T)\)-structure \(A_1 = C_1, A_2 = C_2 + zfE\) and the only change in \(B\) is that \(b_1\) is replaced by \(b_1(z, t_2) = c + az\). \(\square\)

To simplify terminology we introduce the next definition.

**Definition 22.** Let \(\nabla\) be a formal \((TE)\)-structure as in Lemma 21. The functions \((f, b_2)\) are called associated to \(\nabla\).

In order to find all formal \((TE)\)-structures which extend the formal \((T)\)-structures from Theorem 12, we need to determine their associated functions \(b_2\), i.e. to solve equation (71) in the unknown function \(b_2\), for various classes of functions \(f\), which correspond to the various classes of formal \((T)\)-structures from Theorem 12. This is done in the next proposition.
Proposition 23. i) The first \((T)\)-structure from Theorem 12 extends to formal \((TE)\)-structures, with matrices \(B\) given by

\[
B = (-t_1 + c + \alpha z)C_1 + \left(-\frac{t_2}{2} + \sum_{k \geq 0} c_k z^k\right)C_2 - \frac{z}{4}D + z\left(-\frac{t_2}{2} + \sum_{k \geq 0} c_k z^k\right)E, \tag{74}
\]

where \(\alpha, c, c_k \in \mathbb{C}\).

ii) The second \((T)\)-structure from Theorem 12 extends to formal \((TE)\)-structures, with matrices \(B\) given by

\[
B = (-t_1 + c + \alpha z)C_1 - \frac{t_2}{3}C_2 - \frac{z}{3}D - \frac{zt_2^2}{3}E, \tag{75}
\]

where \(c, \alpha \in \mathbb{C}\).

iii) The third \((T)\)-structure from Theorem 12 extends to formal \((TE)\)-structures with matrices \(B\) as in (69), functions \(b_3\) and \(b_4\) given by (70) with \(f = 0\) and function \(b_2 = \sum_{n \geq 0} b_2^{(n)} z^n\), such that \(b_2^{(n)} \in \mathbb{C}\{t_2\}\) satisfies \(\partial_2^3 b_2^{(n)} = 0\), for any \(n \geq 0\).

iv) The fourth (formal or holomorphic) \((T)\)-structure from Theorem 12 extends to a formal \((TE)\)-structure if and only if \(P_k = 0\), for any \(k \geq 1\). When \(P_k = 0\) for any \(k \geq 1\), the extended formal \((TE)\)-structures have matrices \(B\) given by

\[
B = (-t_1 + c + \alpha z)C_1 - \frac{t_2}{r+2}C_2 - \frac{z(r+1)}{2(r+2)}D - \frac{zt_2^{r+1}}{r+2}E, \tag{76}
\]

where \(\alpha, c \in \mathbb{C}\) (and \(r \in \mathbb{Z}_{\geq 2}\)).

Proof. We only prove claim iv) (which is more involved), since the other claims can be proved similarly. Let

\[
f(z, t_2) = t_2^r + \sum_{k \geq 1} P_k(t_2)z^k, \tag{77}
\]

where \(P_k\) are polynomials of degree at most \(r - 2\). Equation (71) with \(f\) given by (77) becomes

\[
-\frac{z}{2} \partial_2^3 b_2 + (rt_2^{r-1} + \sum_{k \geq 1} \hat{P}_k(t_2)z^k)b_2 \\
+ 2(t_2^r + \sum_{k \geq 1} P_k(t_2)z^k)\partial_2 b_2 + \sum_{k \geq 1} (1 - k)P_k(t_2)z^k + t_2^r = 0. \tag{78}
\]
We write \( b_2 = \sum_{k \geq 0} b^{(k)}_2 z^k \) with \( b^{(k)}_2 \) independent on \( z \). Identifying the coefficients of \( z^0 \) in (78) we obtain

\[
rb_2^{(0)} + 2t_2 \partial_2 b_2^{(0)} + t_2 = 0,
\]

which implies

\[
b_2^{(0)} = -\frac{t_2}{r+2}.
\] (79)

Identifying the coefficients of \( z^1 \) in (78) and using (79) we obtain

\[
rt_2^{-1} b_2^{(1)} + 2t_2 \partial_2 b_2^{(1)} - \frac{1}{r+2} \left( \dot{P}_1(t_2) t_2 + 2P_1(t_2) \right) = 0.
\] (80)

The first two terms in (80) have degree at least \( r - 1 \) and the last two terms have degree at most \( r - 2 \). We obtain that (80) is equivalent to

\[
rb_2^{(1)} + 2t_2 \partial_2 (b_2^{(1)}) = 0
\]

\[
P_1(t_2) t_2 + 2P_1(t_2) = 0,
\]

which imply \( b_2^{(1)} = 0 \) and \( P_1 = 0 \). Identifying the coefficients of \( z^n \) for \( n \geq 2 \) in (78) and using an induction argument we obtain that \( b_2^{(k)} = 0 \) for any \( k \geq 2 \) and \( P_k = 0 \) for any \( k \geq 1 \). From (79) and (70) we obtain

\[
b_2 = -\frac{t_2}{r+2}, \quad b_3 = -\frac{r+1}{2(r+2)}, \quad b_4 = -\frac{t_2^{r+1}}{r+2},
\]

which implies claim iv).

**Remark 24.** The \((TE)\)-structures from Proposition 23 ii) and iv) can be unified as follows: for any \( r \in \mathbb{Z}_{\geq 1} \),

\[
A_1 = C_1, \quad A_2 = C_2 + z t_2^r E,
\]

\[
B = (-t_1 + c + \alpha z) C_1 - \frac{t_2}{r+2} C_2 - \frac{z(r+1)}{2(r+2)} D - \frac{z t_2^{r+1}}{r+2} E.
\] (81)

For \( r = 0 \), (81) is a \((TE)\)-structure of the first form in Proposition 23 with \( c_k = 0 \) for any \( k \geq 0 \).

## 4.2 Proof of Theorem 18

The existence of a formal isomorphism between an arbitrary \((TE)\)-structure and one from Theorem 18 will be proved by applying to the \((TE)\)-structures from Proposition 23 formal automorphisms of their underlying \((T)\)-structures.
We shall proceed in two steps: I) we start with the \((TE)\)-structures from Proposition 23 i) and we obtain the \((TE)\)-structures from Theorem 18 i); II) we start with the \((TE)\)-structures from Proposition 23 iii) and we obtain the \((TE)\)-structures from Theorem 18 iii). (The \((TE)\)-structures from Proposition 23 ii) and iii) are written in Theorem 18 ii) in a unified way).

4.2.1 The first step

The proof of the next lemma is straightforward and will be omitted.

**Lemma 25.** A formal automorphism of the \((T)\)-structure \(A = C_1, A_2 = C_2 + zE\) is either a formal gauge automorphism, given by

\[
T = \left( \sum_{n \geq 0} \tau_1^{(n)} z^n \right) C_1 + \left( \sum_{n \geq 0} \tau_2^{(n)} z^n \right) C_2 + \left( \sum_{n \geq 1} \tau_2^{(n-1)} z^n \right) E
\]

where \(\tau_1^{(n)}, \tau_2^{(n)} \in \mathbb{C}\) and \(\tau_1^{(0)} \neq 0\), or covers the map \(h(t_1, t_2) = (t_1, -t_2)\) and is given by

\[
T = \left( \sum_{n \geq 0} \tau_2^{(n)} z^n \right) C_2 + \left( \sum_{n \geq 0} \tau_3^{(n)} z^n \right) D - \left( \sum_{n \geq 1} \tau_2^{(n-1)} z^n \right) E,
\]

where \(\tau_2^{(n)}, \tau_3^{(n)} \in \mathbb{C}\) and \(\tau_3^{(0)} \neq 0\).

**Lemma 26.** Let \(\nabla\) and \(\tilde{\nabla}\) be two formal \((TE)\)-structures as in Proposition 23 i), with constants \(c, \alpha, c_k (k \geq 0)\) and, respectively, \(\tilde{c}, \tilde{\alpha}, \tilde{c}_k (k \geq 0)\). Then \(\nabla\) and \(\tilde{\nabla}\) are formally gauge isomorphic if and only if

\[
c = \tilde{c}, \quad \alpha = \tilde{\alpha}, \quad c_0 = \tilde{c}_0.
\]

In particular, \(\nabla\) is formally gauge isomorphic to the \((TE)\)-structure (66).

**Proof.** Let \(T\) be a formal gauge isomorphism between \(\nabla\) and \(\tilde{\nabla}\). As \(\nabla\) and \(\tilde{\nabla}\) have the same underlying \((T)\)-structure \(A_1 = C_1, A_2 = C_2 + zE\), \(T\) is a formal gauge automorphism of this \((T)\)-structure. From Lemma 25 \(T\) is of the form (82), and must satisfy relations (33), with \(B\) and \(\tilde{B}\) of the form (74), with constants \(c, \alpha, c_k\) and \(\tilde{c}, \tilde{\alpha}, \tilde{c}_k\).

Replacing \(T^{(l)}, B^{(l)}\) and \(\tilde{B}^{(l)}\) in (83) and identifying the coefficients of \(\{C_1, C_2, D, E\}\) we obtain, from a straightforward computation which uses
relations (41)-(44),
\[ c = \tilde{c}, \quad \alpha = \tilde{\alpha}, \quad c_0 = \tilde{c}_0; \]
\[ \frac{\tau_2^{(0)}}{2} + \tau_1^{(0)}(c_1 - \tilde{c}_1) = 0; \]
\[ (n - 1)\tau_1^{(n-1)} + \sum_{l=2}^{n} \tau_2^{(n-l)}(c_{l-1} - \tilde{c}_{l-1}) = 0; \]
\[ (n - \frac{1}{2})\tau_2^{(n-1)} + \sum_{l=1}^{n} \tau_1^{(n-l)}(c_l - \tilde{c}_l) = 0, \quad (85) \]
for any \( n \geq 2 \). (In all relations (43) the coefficients of \( D \) vanish; the coefficients of \( E \) in (43), with \( r = 0, 1 \), vanish as well and the coefficient of \( E \) in (43), with \( r \geq 2 \), coincides with the coefficient of \( C_2 \) in (43), with \( r \) replaced by \( r - 1 \). Thus, relations (43) are equivalent to the vanishing of their coefficients of \( C_1 \) and \( C_2 \), which is expressed by relations (85)). In particular, if \( \nabla \) and \( \tilde{\nabla} \) are formally gauge isomorphic, then (84) is satisfied. Conversely, assume that (84) is satisfied. We aim to construct a formal gauge isomorphism between \( \nabla \) and \( \tilde{\nabla} \), i.e. to find \( \tau_1^{(0)} \), \( \tau_2^{(0)} \) from (85) such that relations (85) hold. Let \( \tau_1^{(0)} \in C^* \) be arbitrary. The second relation (85) determines \( \tau_2^{(0)} \) and then, \( \tau_1^{(1)} \) is determined by the third relation (85) with \( n = 2 \):
\[ \tau_1^{(1)} = \tau_2^{(0)}(\tilde{c}_1 - c_1). \]
Knowing \( \tau_1^{(0)} \) and \( \tau_1^{(1)} \), the fourth relation (85) with \( n = 2 \) determines \( \tau_2^{(1)} \):
\[ \tau_2^{(1)} = \frac{2}{3} \left( \tau_1^{(1)}(\tilde{c}_1 - c_1) + \tau_1^{(0)}(\tilde{c}_2 - c_2) \right). \]
Repeating the argument we obtain inductively \( \tau_1^{(l)} \) and \( \tau_2^{(l)} \), for all \( l \geq 1 \).

4.2.2 The second step

We use a similar argument for the \((TE)\)-structures from Proposition 23 iv). As before, we begin by finding the automorphisms of their underlying \((T)\)-structure.

**Lemma 27.** i) Any formal automorphism \( T \) of the \((T)\)-structure \( A_1 = C_1, A_2 = C_2 \) covers an automorphism \( h \in \text{Aut}(N_2) \) of the form
\[ h(t_1, t_2) = (t_1, \frac{kt_2}{\tau_4^{(1)} t_2 + d}), \quad (86) \]
where \( \tau_4^{(1)} \in \mathbb{C}, \ k, d \in \mathbb{C}^* \) and

\[
\hat{T} := T \circ h = \sum_{n \geq 0} \hat{T}^{(n)} z^n, \quad \hat{T}^{(n)} = \tau_1^{(n)} C_1 + \tau_2^{(n)} C_2 + \tau_3^{(n)} D + \tau_4^{(n)} E
\]

is given by: \( \tau_4^{(0)} = 0, \ \tau_4^{(n)} \in \mathbb{C} \quad (n \geq 1) \) and, for any \( n \geq 0, \)

\[
\begin{align*}
\tau_1^{(n)} &= \frac{\tau_4^{(n+1)} t_2}{2} \left( \frac{\tau_4^{(1)} t_2 + d - k}{\tau_4^{(1)} t_2 + d} \right) + (\tau_1^{(n)})_0 \\
\tau_3^{(n)} &= \frac{\tau_4^{(n+1)} t_2}{2} \left( \frac{\tau_4^{(1)} t_2 + d + k}{\tau_4^{(1)} t_2 + d} \right) + (\tau_4^{(n)})_0 \\
\tau_2^{(n)} &= -\frac{\tau_4^{(n+2)} t_2}{\tau_4^{(1)} t_2 + d} + (\tau_1^{(n+1)})_0 t_2 \left( \frac{\tau_4^{(1)} t_2 + d - k}{\tau_4^{(1)} t_2 + d} \right) \\
&\quad - (\tau_3^{(n+1)})_0 t_2 \left( \frac{\tau_4^{(1)} t_2 + d + k}{\tau_4^{(1)} t_2 + d} \right) + (\tau_2^{(n)})_0,
\end{align*}
\]

(87)

where \( (\tau_1^{(n)})_0, (\tau_3^{(n)})_0, (\tau_2^{(n)})_0 \in \mathbb{C} \) and \( (\tau_1^{(0)})_0 = \frac{1}{2}(d + k), \ (\tau_3^{(0)})_0 = \frac{1}{2}(d - k). \)

ii) Any formal gauge automorphism of the \((T)\)-structure \( A_1 = C_1, \ A_2 = C_2 \) is of the form

\[
T = \left( \sum_{k \geq 0} \tau_1^{(n)} z^n \right) C_1 + \sum_{n \geq 0} (\tau_2^{(n)} z^n) C_2 + \sum_{n \geq 0} (\tau_3^{(n)} z^n) D + \sum_{n \geq 0} (\tau_4^{(n)} z^n) E
\]

where \( \tau_1^{(n)} \in \mathbb{C} \quad \text{with} \quad \tau_1^{(0)} \neq 0, \ \tau_2^{(n)}, \tau_3^{(n)}, \tau_4^{(n)} \in \mathbb{C}\{t_2\}, \ \tau_2^{(n)} \) satisfies \( \partial_2^2 \tau_2^{(n)} = 0 \)

for any \( n \geq 0 \) and

\[
\begin{align*}
\tau_3^{(n)} &= -\frac{1}{2} \partial_2 \tau_2^{(n-1)}, \quad \tau_4^{(n)} &= -\frac{1}{2} \partial_2^2 \tau_2^{(n-2)},
\end{align*}
\]

(88)

with the convention \( \tau_2^{(n)} = 0 \) for \( n < 0. \)

Proof. i) The formal automorphism \( T \) covers an automorphism \( h(t_1, t_2) = (t_1, \lambda(t_2)) \) of \( \mathcal{N}_2 \) and the functions \( \tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)}, \tau_4^{(n)} \) in the expression of \( \hat{T} \) are independent on \( t_1. \) This follows from relation (28) with \( i = 1 \) and \( A_1 = \hat{A}_1 = C_1. \) Relation (28) with \( i = 2 \) is

\[
\partial_2 \hat{T}^{(n-1)} + \hat{C}_2 \hat{T}^{(n)} - \hat{T}^{(n)} C_2 = 0, \quad n \geq 0.
\]

(89)

Using relations (41)-(44) we obtain that (89) is equivalent to

\[
\tau_4^{(0)} = 0, \quad \lambda = \frac{\tau_4^{(0)} - \tau_3^{(0)}}{\tau_1^{(0)} + \tau_3^{(0)}},
\]

(90)
\[ \tau_4^{(n)} \in \mathbb{C} \text{ (for } n \geq 1 \text{) and, for any } n \geq 0, \]
\[ \partial_2 \tau_1^{(n)} = \left( \frac{1 - \lambda}{2} \right) \tau_4^{(n+1)}, \]
\[ \partial_2 \tau_2^{(n)} = (1 - \lambda) \tau_1^{(n+1)} - (1 + \lambda) \tau_3^{(n+1)}, \]
\[ \partial_2 \tau_3^{(n)} = \left( \frac{1 + \lambda}{2} \right) \tau_4^{(n+1)}. \quad (91) \]

(Since \( \tilde{T}^{(0)} \) is invertible and \( \tau_4^{(0)} = 0 \), we obtain that \( \tau_1^{(0)} - \tau_3^{(0)}, \tau_1^{(0)} + \tau_3^{(0)} \) are units in \( \mathbb{C}\{t_2\} \)). Using that \( \tau_4^{(n)} \) are constant, we obtain from the first and third relation (91) that \( \tau_1^{(n)} \) and \( \tau_3^{(n)} \) are given by
\[ \tau_1^{(n)} = \frac{\tau_4^{(n+1)}}{2} (t_2 - \lambda) + (\tau_1^{(n)})_0 \]
\[ \tau_3^{(n)} = \frac{\tau_4^{(n+1)}}{2} (t_2 + \lambda) + (\tau_3^{(n)})_0. \quad (92) \]

where \((\tau_1^{(n)})_0, (\tau_3^{(n)})_0 \in \mathbb{C}\). From (92) and the second relation (91), we obtain
\[ \tau_2^{(n)} = -\tau_4^{(n+2)} t_2 \lambda - ((\tau_1^{(n+1)})_0 + (\tau_3^{(n+1)})_0) \lambda + ((\tau_1^{(n+1)})_0 - (\tau_3^{(n+1)})_0) t_2 + (\tau_2^{(n)})_0, \]

where \((\tau_2^{(n)})_0 \in \mathbb{C}\). Replacing the expressions of \( \tau_1^{(0)} \) and \( \tau_3^{(0)} \) provided by (92) in the second relation (90) we obtain that \( \lambda \) satisfies the differential equation
\[ \dot{\lambda} = \frac{-\tau_4^{(1)} \lambda + k}{\tau_4^{(1)} t_2 + d}. \]

Solving this differential equation for \( \lambda \) we obtain (86). Finally, replacing \( \lambda \) in the above expressions for \( \tau_1^{(n)}, \tau_2^{(n)} \) and \( \tau_3^{(n)} \) we conclude the proof of claim i).

Claim ii) follows from relations (91) with \( \dot{\lambda} = 1 \). The condition \( \partial_2^3 \tau_2^{(n)} = 0 \) follows from \( \partial_2^2 \tau^{(n)}_3 = 0 \) (from the third relation (91) and \( \tau_4^{(n+1)} \in \mathbb{C} \)). \qed

**Corollary 28.** i) Consider two formal \((TE)\)-structures \( \nabla \) and \( \tilde{\nabla} \) as in Proposition 23 iii), with associated functions \( b_2 \) and \( \tilde{b}_2 \) respectively. Assume that there is a formal isomorphism \( T \) between \( \nabla \) and \( \tilde{\nabla} \), given by Lemma 27 i). In the notation of that lemma,
\[ \tilde{b}_2^{(0)}(t_2) = b_2^{(0)} \left( \frac{kt_2}{\tau_4^{(1)} t_2 + d} \right) \left( \frac{\tau_4^{(1)} t_2 + d)^2}{kd} \right). \quad (93) \]
In particular, $b_2^{(0)}$ is a formal gauge invariant of $\nabla$.

ii) Let $\nabla$ be a formal $(TE)$-structure as in Proposition 23 iii). There is a formal isomorphism which maps $\nabla$ to a similar formal $(TE)$-structure, with associated function $b_2$, such that $\tilde{b}_2^{(0)}$ is of one of the following forms:

$$\tilde{b}_2^{(0)} = 0, \quad \tilde{b}_2^{(0)} = 1, \quad \tilde{b}_2^{(0)} = \lambda t_2 + 1, \quad \tilde{b}_2^{(0)} = \beta t_2, \quad \tilde{b}_2^{(0)} = t_2^2,$$

(94)

where $\lambda \in \mathbb{C}$ and $\beta \in \mathbb{C}^*$.

Proof.

i) Relation (93) follows by identifying the coefficients of $z^0$ in relation (27) with $B$, $\tilde{B}$ as in Proposition 23 iii) and $\tilde{T}$ given in Lemma 27 i).

ii) Let $b_2$ be the associated function of $\nabla$. Since $\partial_3 b_2^{(0)} = 0$ and $b_2^{(0)}$ is independent on $t_1$, we can write $b_2^{(0)} = at_2^2 + bt_2 + c$ for $a, b, c \in \mathbb{C}$. Let $T$ be any formal automorphism of the $(T)$-structure $A_1 = C_1$, $A_2 = C_2$, as in Lemma 27 i), and $\tilde{\nabla} := T \cdot \nabla$, with associated function $\tilde{b}_2$. From (93),

$$\tilde{b}_2^{(0)} = \left( a \frac{k}{d} + b \frac{\tau_4^{(1)}}{d} + c \frac{(\tau_4^{(1)})^2}{kd} \right) t_2^2 + (b + 2c \frac{\tau_4^{(1)}}{k}) t_2 + c \frac{d}{k}$$

(95)

Suitable choices of $k, d \in \mathbb{C}^*$ and $\tau_4^{(1)} \in \mathbb{C}$ in (95) show that $\tilde{b}_2^{(0)}$ can be reduced to one of the forms (94). Any formal automorphism of the $(T)$-structure $A_1 = C_1$, $A_2 = C_2$, as in Lemma 27 i), with such constants $k, d$ and $\tau_4^{(1)}$, maps $\nabla$ to a formal $(TE)$-structure with the required property.

Corollary 29.

i) Two formal $(TE)$-structures $\nabla$ and $\tilde{\nabla}$ as in Proposition 23 iii), with associated functions $b_2$ and $\tilde{b}_2$ respectively, such that $b_2^{(0)}$ and $\tilde{b}_2^{(0)}$ are distinct, of the form (94), are formally non-isomorphic, unless

$$b_2^{(0)} = \lambda t_2 + 1, \quad \tilde{b}_2^{(0)} = -\lambda t_2 + 1, \quad \lambda \in \mathbb{C}^*.$$

(96)

ii) A formal $(TE)$-structure $\nabla$ as in Proposition 23 iii), with associated function $b_2$ such that $b_2^{(0)} = \lambda t_2 + 1$, where $\lambda \in \mathbb{C}^*$, can be mapped by a formal isomorphism to a similar $(TE)$-structure, with associated function $\tilde{b}_2$, such that $\tilde{b}_2^{(0)} = -\lambda t_2 + 1$.

Proof. i) We consider the more general situation when $\nabla$ and $\tilde{\nabla}$ are as in Proposition 23 iii), with associated functions $b_2$ and $\tilde{b}_2$, but such that $b_2^{(0)}$ and $\tilde{b}_2^{(0)}$ are not necessarily of the form (94). As $\partial_3^2 b_2^{(0)} = \partial_3^2 \tilde{b}_2^{(0)} = 0$, $b_2^{(0)} = at_2^2 + bt_2 + c, \quad \tilde{b}_2^{(0)} = \hat{a} t_2^2 + \hat{b} t_2 + \hat{c}$,

(97)
for $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{C}$. From [103], if there is a formal isomorphism between $\nabla$ and $\hat{\nabla}$ then the system

$$
a \frac{k^2}{d} + b \frac{\tau_4^{(1)}}{d} + c \frac{(\mathfrak{r}_4^{(1)})^2}{kd} = \tilde{a}, \ b + 2c \frac{\tau_4^{(1)}}{k} = \tilde{b}, \ c \frac{\tau_4^{(1)}}{k} = \tilde{c}
$$

(98)

in the unknown constants $k, d \in \mathbb{C}^*$ and $\tau_4^{(1)} \in \mathbb{C}$, has a solution. When $\mathfrak{r}_2^{(0)}$ and $\mathfrak{r}_2^{(0)}$ are distinct, of the form [94], a solution of [98] exists (e.g. $k = d = 1$ and $\tau_4^{(1)} = -\lambda$) only when $\mathfrak{r}_2^{(0)}$ and $\mathfrak{r}_2^{(0)}$ are of the form [96].

ii) Consider the automorphism given by Lemma 27 i), with $k = d = 1$, $\tau_4^{(n)} = 0$ for any $n \neq 1$, $\tau_4^{(1)} = -\lambda$, $(\mathfrak{r}_1^{(n)})_0 = (\mathfrak{r}_3^{(n)})_0 = 0$ for any $n \geq 1$ and $(\mathfrak{r}_2^{(n)})_0 = 0$ for any $n \geq 0$. It maps $\nabla$ to a $(TE)$-structure $\hat{\nabla}$ with associated function $\hat{b}_2$, with the property $\hat{b}_2^{(0)} = -\lambda t_2 + 1$.

□

Lemma 30. Let $\nabla$ be a formal $(TE)$-structure as in Proposition 23 iii). Then $\nabla$ is formally isomorphic to a $(TE)$-structure with underlying $(T)$-structure $A_1 = C_1, A_2 = C_2$ and whose matrix $B$ either belongs to the first five lines in (68) or is of one of the forms

$$
B = \left((-t_1 + c + \alpha z)C_1 + t_2(\lambda + \gamma t_2z^\gamma)C_2 - \frac{z}{2}(\lambda + 1 + 2\gamma t_2z^\gamma)D - \gamma z^{\lambda+2}E, \ \lambda \in \mathbb{Z}_{\geq 1}\right)
$$

$$
B = \left((-t_1 + c + \alpha z)C_1 + (\lambda t_2 + \gamma z^{-\gamma})C_2 - \frac{z}{2}(\lambda + 1)D, \ \lambda \in \mathbb{Z}_{\leq -1}\right),
$$

(99)

where $c, \gamma \in \mathbb{C}$.

Proof. Let $\nabla$, $\hat{\nabla}$ be two formal $(TE)$-structures as in Proposition 23 iii), with constants $c, \alpha$ and associated function $b_2 = \sum_{n \geq 0} b_2^{(n)} z^n$, respectively constants $\tilde{c}, \tilde{\alpha}$ and associated function $\tilde{b}_2 = \sum_{n \geq 0} \tilde{b}_2^{(n)} z^n$. Recall that $b_2, \tilde{b}_2 \in \mathbb{C}\{t_2, z\}$ satisfy $\partial_2^2 b_2 = \partial_2^3 \tilde{b}_2 = 0$. The $(TE)$-structures $\nabla$ and $\hat{\nabla}$ are formally gauge isomorphic if and only if there is a formal gauge automorphism $T$ of their underlying $(T)$-structure $A_1 = C_1, A_2 = C_2$, such that (33), with matrices $B$ and $\tilde{B}$ of $\nabla$ and $\hat{\nabla}$, is satisfied. Let $T$ be given as in Lemma 27 ii). Relation (33) for $r = 0$ is equivalent to $b_2^{(0)} = \tilde{b}_2^{(0)}$ (which we already know, from Corollary 28) and $c = \tilde{c}$. For $r = 1$ it is equivalent to $\alpha = \tilde{\alpha}$ (by identifying the coefficients of $C_1$) together with

$$
b_2^{(0)} \partial_2^2 \tau_2^{(0)} - (\partial_2 b_2^{(0)} + 1) \tau_2^{(0)} + \tau_1^{(0)} (\tilde{b}_2^{(1)} - b_2^{(1)}) = 0
$$

(100)

(by identifying the coefficients of $C_2$). The coefficients of $D$ and $E$ give no relations in (33) with $r = 1$. 

27
We now consider relation (103) with \( r = n \geq 2 \). Identifying the coefficients of \( C_1 \) in this relation we obtain

\[
(n - 1)r_1^{(n-1)} - \frac{1}{4} \sum_{l=1}^{n} \partial_2^2 r_2^{(n-l-2)} (b_2^{(l)} - \tilde{b}_2^{(l)})
+ \frac{1}{4} \sum_{l=2}^{n-1} \partial_2 r_2^{(n-l-1)} \partial_2 (b_2^{(l-1)} - \tilde{b}_2^{(l-1)})
- \frac{1}{4} \sum_{l=2}^{n-1} \tau_2^{(n-l-1)} \partial_2 (b_2^{(l-1)} - \tilde{b}_2^{(l-1)}) = 0
\] (101)

(with the convention \( \tau_2^{(l)} := 0 \) for \( l \in \mathbb{Z}_{\leq -1} \)). Identifying the coefficients of \( C_2 \) in the same relation we obtain

\[
n \tau_2^{(n-1)} - b_2^{(0)} \partial_2 \tau_2^{(n-1)} + \tau_2^{(n-1)} \partial_2 b_2^{(0)} + \sum_{l=1}^{n} \tau_1^{(n-l)} (b_2^{(l)} - \tilde{b}_2^{(l)})
- \frac{1}{2} \sum_{l=1}^{n} \partial_2 \tau_2^{(n-l-1)} (b_2^{(l)} - \tilde{b}_2^{(l)})
+ \frac{1}{2} \sum_{l=1}^{n-1} \tau_2^{(n-l-1)} \partial_2 (b_2^{(l)} + \tilde{b}_2^{(l)}) = 0.
\] (102)

Identifying the coefficients of \( D \) in the same relation we obtain:

\[
(n - 1) \partial_2 r_2^{(n-2)} - b_2^{(0)} \partial_2^2 r_2^{(n-2)} - \frac{1}{2} \sum_{l=1}^{n} \partial_2^2 r_2^{(n-l-2)} (b_2^{(l)} + \tilde{b}_2^{(l)})
+ \sum_{l=2}^{n} \tau_1^{(n-l)} \partial_2 (b_2^{(l-1)} - \tilde{b}_2^{(l-1)})
+ \frac{1}{2} \sum_{l=0}^{n-2} \tau_2^{(n-l-2)} \partial_2^2 (b_2^{(l)} + \tilde{b}_2^{(l)}) = 0.
\] (103)

Identifying the coefficients of \( E \) in the same relation we obtain:

\[
(n - 2) \partial_2^2 r_2^{(n-3)} - \partial_2 b_2^{(0)} \partial_2^2 r_2^{(n-3)} - \frac{1}{2} \sum_{l=1}^{n-1} \partial_2^2 r_2^{(n-l-3)} \partial_2 (b_2^{(l)} + \tilde{b}_2^{(l)})
+ \frac{1}{2} \sum_{l=0}^{n-2} \partial_2 r_2^{(n-l-3)} \partial_2^2 (b_2^{(l)} + \tilde{b}_2^{(l)})
+ \sum_{l=2}^{n} \tau_1^{(n-l)} \partial_2^2 (b_2^{(l-2)} - \tilde{b}_2^{(l-2)})
= 0.
\] (104)

A long but straightforward computation shows that for \( n \geq 3 \), relation (103) is the derivative of relation (102) with \( n \) replaced by \( n - 1 \). Similarly, for
\( n \geq 4 \), relation (104) is the second derivative of (102) with \( n \) replaced by \( n - 2 \). Therefore, relations (101)-(104) are equivalent to relations (101), (102), together with relation (103) with \( n = 2 \) and relation (104) with \( n = 2, 3 \). But relation (103) with \( n = 2 \) is the derivative of (100), relation (104) with \( n = 2 \) follows from \( b_2^{(0)} = \tilde{b}_2^{(0)} \) and relation (104) with \( n = 3 \) is the second derivative of (100).

To summarize: we proved that \( \nabla \) and \( \nabla' \) are formally gauge isomorphic if and only if \( c = \tilde{c}, \alpha = \tilde{\alpha}, b_2^{(0)} = \tilde{b}_2^{(0)} \) and relations (100), (101) and (102) are satisfied (the last two for any \( n \geq 2 \)).

Using the above considerations, we now prove our claim. From Corollary [28] ii) we can assume, without loss of generality, that \( b_2^{(0)} = \tilde{b}_2^{(0)} \) is of one of the forms (94). From Corollary 29 ii), we can further assume that the \( \lambda \) from (94) belongs to \( (\mathbb{C} \setminus \mathbb{Z}^*) \cup \mathbb{Z}_{\geq 1} \). Let \( \tau_1^{(0)} \in \mathbb{C}^* \). Using Lemma 5 when \( b_2 \neq 0 \), we choose \( \tilde{b}_2^{(1)} \), 'as simple as possible', such that relation (100), considered as an equation in the unknown function \( \tau_2^{(0)} \), has a solution with \( \partial_2 \tau_2^{(0)} = 0 \). The choice of \( \tilde{b}_2^{(1)} \) depends on the form of \( b_2^{(0)} \). (E.g. when \( b_2^{(0)} = t_2^{2} \) we choose \( \tilde{b}_2^{(1)} = 0 \).) When \( b_2^{(0)} = 0 \), Lemma 5 is not needed anymore and we choose \( \tilde{b}_2^{(1)} = 0 \) and \( \tau_2^{(0)} = -\tau_1^{(0)} b_2^{(1)} \). Suppose now that \( \tau_1^{(i)} \) and \( \tau_2^{(i)} \in \mathbb{C} \) (with \( i \leq n - 1 \) and \( \tilde{b}_2^{(i)} \) (with \( i \leq n \)) are known (and satisfy \( \partial_2 \tau_2^{(i)} = \partial_2^2 \tilde{b}_2^{(i)} = 0 \)). Relation (101) with \( n \) replaced by \( n + 1 \) determines \( \tau_1^{(n)} \):

\[
\begin{align*}
n \tau_1^{(n)} &= \frac{1}{4} \sum_{l=1}^{n+1} \partial_2^2 \tau_2^{(n-l+1)} (b_2^{(l)} - \tilde{b}_2^{(l)}) \\
- \frac{1}{4} \sum_{l=2}^{n} \partial_2 \tau_2^{(n-l)} (b_2^{(l-1)} - \tilde{b}_2^{(l-1)}) \\
+ \frac{1}{4} \sum_{l=2}^{n} \tau_2^{(n-l)} \partial_2^2 (b_2^{(l-1)} - \tilde{b}_2^{(l-1)}) &= 0. \\
\end{align*}
\]

We remark that \( \tau_1^{(n)} \in \mathbb{C} : \) straightforward computation, which uses \( \partial_2^2 b_2^{(l)} = \partial_2^3 \tilde{b}_2^{(l)} = \partial_2^3 \tau_2^{(i)} = 0 \) (for \( l \leq n - 1 \) and \( i \leq n - 2 \)) shows that the right hand side of (105) is constant. Relation (102) with \( n \) replaced by \( n + 1 \) is

\[
\begin{align*}
(n + 1) \tau_1^{(n)} + \partial_2 b_2^{(0)} \tau_2^{(n)} - b_2^{(0)} \partial_2 \tau_2^{(n)} + &\, \tau_1^{(0)} (b_2^{(n+1)} - \tilde{b}_2^{(n+1)}) \\
+ &\, \sum_{l=1}^{n} \tau_1^{(n+1-l)} (b_2^{(l)} - \tilde{b}_2^{(l)}) - \frac{1}{2} \sum_{l=1}^{n} \partial_2 \tau_2^{(n-l)} (b_2^{(l)} + \tilde{b}_2^{(l)}) \\
+ &\, \frac{1}{2} \sum_{l=1}^{n} \tau_2^{(n-l)} \partial_2 (b_2^{(l)} + \tilde{b}_2^{(l)}) &= 0.
\end{align*}
\]

29
The Lemma 31. The second and third lines from the left hand side of (106) are known. When \( b_2^{(0)} \neq 0 \) we choose as before (using Lemma 30) \( \tilde{b}_2^{(n+1)} \) "as simple as possible", such that (106) has a solution \( \tau_2^{(n)} \) and \( \tilde{b}_2^{(n+1)} b_2^{(n+1)} = \tilde{b}_2^{(n)} \tau_2^{(n)} = 0 \). When \( b_2^{(0)} = 0 \) we choose \( \tilde{b}_2^{(n+1)} = 0 \) and \( \tau_2^{(n)} \) to satisfy (106) (with this choice of \( \tilde{b}_2^{(n+1)} \)). Then \( \tilde{b}_2^{(n+1)} \tau_2^{(n)} = 0 \). Using an induction procedure we define an automorphism \( T \) of the (\( T \))-structure \( A_1 = C_1, A_2 = C_2 \), such that the associated function \( \tilde{b}_2 \) of \( \nabla := T \cdot \nabla \) is of one of the following forms:

\[
\tilde{b}_2 = 0, \quad \tilde{b}_2 = t_2^3, \quad \tilde{b}_2 = \lambda t_2 (\lambda \notin \mathbb{Z}^*), \quad \tilde{b}_2 = \lambda t_2 + 1 (\lambda \notin \mathbb{Z}^*), \quad \tilde{b}_2 = \lambda t_2 + 1 + \gamma t_2^2 z^\lambda (\lambda \in \mathbb{Z}_{\geq 1}, \gamma \in \mathbb{C}), \quad \tilde{b}_2 = \lambda t_2 + \gamma t_2^2 z^\lambda (\lambda \in \mathbb{Z}_{\leq -1}, \gamma \in \mathbb{C}).
\]

The first five forms above of \( \tilde{b}_2 \) give the (\( TE \))-structures with \( A_1 = C_1, A_2 = C_2 \) and matrix \( B \) as in the first five lines from (68). The last two forms of \( \tilde{b}_2 \) give the (\( TE \))-structures with \( A_1 = C_1, A_2 = C_2 \) and matrix \( B \) of either of the two forms (99).

The next lemma concludes the proof of Theorem 18. It shows that the (\( TE \))-structures from Lemma 30, with matrices \( B \) given in (99), are formally isomorphic to the last four classes of (\( TE \))-structures from Theorem 18 iii).

**Lemma 31.** The (\( TE \))-structures with \( A_1 = C_1, A_2 = C_2 \) and matrices \( B \) given by (99) are formally isomorphic to the (\( TE \))-structures of the same form (with the same constants \( c, \alpha, \lambda \)), but with \( \gamma \in \{0, 1\} \).

**Proof.** Let \( T^{[1]} \) be a formal automorphism of \( A_1 = C_1, A_2 = C_2 \) given by Lemma 27 i), with \( \tau_1^{(n)} = (\tau_2^{(n)})_0 = 0 \) (\( n \geq 0 \)) and \( (\tau_1^{(n)})_0 = (\tau_3^{(n)})_0 = 0 \) (\( n \geq 1 \)). It maps the (\( TE \))-structures from Lemma 30, with matrices \( B \) given by (99), to (\( TE \))-structures of the same form, with the same \( c, \alpha, \lambda \), but with \( \gamma \) replaced by \( \tilde{\gamma} := \frac{k}{d} \gamma \). When \( \gamma \neq 0 \) we can choose \( k, d \in \mathbb{C}^* \) such that \( \tilde{\gamma} = 1 \).

### 4.2.3 Proof of Theorem 20

Consider two distinct (\( TE \))-structures \( \nabla \) and \( \tilde{\nabla} \) from Theorem 18 and suppose that they are formally isomorphic. Then their underlying (\( T \))-structures are also formally isomorphic. From Theorem 21 ii) of [3], there are two possibilities, namely: a) \( \nabla \) and \( \tilde{\nabla} \) are of the form (66); b) \( \nabla \) and \( \tilde{\nabla} \) are as in Theorem 18 iii). The next lemma treats the first possibility.

**Lemma 32.** Consider two distinct (\( TE \))-structures of the form (66), with constants \( c, \alpha, c_0 \), respectively \( \tilde{c}, \tilde{\alpha}, \tilde{c}_0 \). Then \( \nabla \) and \( \tilde{\nabla} \) are formally isomorphic if and only if \( \tilde{c} = c, \tilde{\alpha} = \alpha \) and \( \tilde{c}_0 = -c_0 \).
Proof. Let $T$ be a formal automorphism the $(T)$-structure $A_1 = C_1, A_2 = C_2 + zE$. From Lemma 26 $T$ is not a formal gauge automorphism. From Lemma 25 $T$ covers the map $h(t_1, t_2) = (t_1, -t_2)$ and $\tilde{T} = T \circ h = T$ is of the form (83). As $\tilde{B} = \tilde{B} \circ h$, relation (27) becomes

$$z^2 \partial T + \tilde{B}T - TB = 0. \tag{107}$$

By identifying the coefficients of $z^0$ and $z$ in (107) we obtain that $\check{c} = c$, $\check{\alpha} = \alpha$ and $\check{c}_0 = -c_0$. Moreover, if these relations are satisfied then the isomorphism $T(0)(z, t_1, t_2) := D$ which covers $h$ maps $\nabla$ to $\tilde{\nabla}$. \hfill \square

It remains to study the second possibility. We begin with the next simple lemma.

Lemma 33. Let $\nabla$ be a $(TE)$-structure as in Theorem 18 iii). Then the constants $c$ and $\alpha$ are formal invariants.

Proof. We notice that $c$ and $\alpha$ remain unchanged under the particular class of automorphisms from Lemma 27 i), with $\tau_4^{(n)} = 0$ for any $n \neq 1$, ($\tau_4^{(n)})_0 = (\tau_3^{(n)})_0 = 0$ for any $n \geq 1$ and $n \geq 0$. Such particular automorphisms cover all automorphisms of $\mathcal{N}_2$ which lift to automorphisms of the $(T)$-structure $A_1 = C_1, A_2 = C_2$. They can be used to reduce the statement we need to prove to showing that $c$ and $\alpha$ are formal gauge invariant. This was shown in the proof of Lemma 30. \hfill \square

Lemma 34. Consider two distinct $(TE)$-structures $\nabla$ and $\tilde{\nabla}$ from Theorem 18 iii). They are formally isomorphic if and only if their matrices $B$ and $\tilde{B}$ are of the fourth form in (68), with constants $c$, $\alpha$, $\lambda$ and $\check{c}$, $\check{\alpha}$, $\check{\lambda}$, such that $\check{c} = c$, $\check{\alpha} = \alpha$ and $\check{\lambda} = -\lambda$.

Proof. Assume that $\nabla$ and $\tilde{\nabla}$ are formally isomorphic. From Lemma 33 $c = \check{c}$ and $\alpha = \check{\alpha}$. Using Corollary 29 i), we deduce (exchanging $\nabla$ with $\tilde{\nabla}$ if necessary) that one of the following cases holds: I) $\nabla$ and $\tilde{\nabla}$ belong to the fourth class in Theorem 20 iii) (and $\check{\lambda} = -\lambda$); II) $\nabla$ and $\tilde{\nabla}$ belong to the fifth class (and $\lambda = \check{\lambda}$, $\check{\gamma} \neq \gamma$); III) $\nabla$ belongs to the six class and $\tilde{\nabla}$ belongs to the seventh class (and $\check{\lambda} = \lambda$); IV) $\nabla$ belongs to the eight class and $\tilde{\nabla}$ to the nine class (and $\check{\lambda} = \lambda$). In case I) the automorphism $T$ used in Corollary 29 ii) maps $\nabla$ to $\tilde{\nabla}$. Case I) is therefore understood. It turns out that none of the remaining cases can hold. Let us sketch the argument for case III). Assume, by absurd, that $\nabla$ and $\tilde{\nabla}$ are formally isomorphic. Then the $B$-matrices of $\nabla$, $\tilde{\nabla}$ are of the first form (69), with $c = \check{c}$, $\alpha = \check{\alpha}$, $\lambda = \check{\lambda}$, $\gamma = 1$ and $\check{\gamma} = 0$. Since $\tilde{b}^{(0)} = b^{(0)} = \lambda t_2$, from (68) with $a = \check{a} = 0$, $b = \check{b} = \lambda$ and $c = \check{c} = 1$ we deduce that $T$ covers a map of the form $h(t_1, t_2) = (t_1, \frac{k}{d} t_2)$, with $k, d \in \mathbb{C}^*$. 31
The automorphism $T^{[1]}$ used in the proof of Lemma 31 maps $\nabla$ to a $(TE)$-structure $\nabla^{[1]}$ with $A_1^{[1]} = C_1$, $A_2^{[1]} = C_2$ and matrix $B^{[1]}$ of the first form (99) with $\gamma = \frac{k^2}{d}$. The $(TE)$-structures $\nabla^{[1]}$ and $\tilde{\nabla}$ are formally gauge isomorphic. Going through the argument of Lemma 30 and using Lemma 5 we obtain that $\nabla^{[1]} = \tilde{\nabla}$ which is a contradiction. 

5 Holomorphic classification of $(TE)$-structures

We begin by studying the restriction of $(TE)$-structures over $N_2$ at the origin of $N_2$.

5.1 Restriction of $(TE)$-structures at the origin; holomorphic classification in the non-elementary case

By an elementary model we mean a meromorphic connection $\nabla^0$ on the germ $(\mathcal{O}(\mathbb{C},0))^2$ with connection form $\Omega^0 = \frac{1}{z}B^0dz = \frac{1}{z^2} \sum_{k \geq 0} B_0^{(k)}z^kdz$ (where $B_0^{(k)} \in M_{2 \times 2}(\mathbb{C})$), such that $\nabla^1 := \mathcal{E}^{-\frac{\text{tr}B_0^{(0)}}{2z}} \otimes \nabla^0$ is regular singular. (This definition is a particular case, adapted to our situation, of the more general notion of elementary model defined in [15]; for any $\rho \in k$, we denote by $\mathcal{E}^\rho$ the connection in rank one with connection form $d\rho$). The property of a meromorphic connection to be an elementary model is invariant under holomorphic isomorphisms: if $\tilde{\nabla}^0$, with connection form $\tilde{\Omega}^0 = \frac{1}{z}\tilde{B}^0dz = \frac{1}{z^2} \sum_{k \geq 0} \tilde{B}_0^{(k)}z^kdz$ ($\tilde{B}_0^{(k)} \in M_{2 \times 2}(\mathbb{C})$) is isomorphic to $\nabla^0$ by means of an holomorphic isomorphism $T^0 = \sum_{k \geq 0} T_0^{(k)}z^k$, then $\text{tr} \tilde{B}_0^{(0)} = \text{tr} B_0^{(0)}$ and $T^0$ is an isomorphism also between $\nabla^1$ and $\tilde{\nabla}^1$ (the latter defined as $\nabla^1$ starting with $\tilde{\nabla}^0$ instead of $\nabla^0$). Obviously, $\nabla^0$ is an elementary model if and only if $\mathcal{E}^{-\frac{\text{tr}B_0^{(0)}}{2z}} \otimes \nabla^0$ is regular singular.

**Definition 35.** A $(TE)$-structure $\nabla$ over $N_2$ is called elementary if its restriction to the slice $(\mathbb{C},0) \times \{0\} \subset (\mathbb{C},0) \times N_2$ is an elementary model. A $(TE)$-structure which is not elementary is called non-elementary.

Our aim in this section is to prove the next proposition.

**Proposition 36.** Let $\nabla$ be a $(TE)$-structure over $N_2$, with associated functions $(f, b_2)$.

i) Then $\nabla$ is elementary if and only if $f(0,0)b_2(0,0) = 0$.

ii) If $\nabla$ is elementary then $\nabla$ is holomorphically isomorphic to its formal normal form(s).
iii) If $\nabla$ is non-elementary then $\nabla$ is holomorphically isomorphic to the Malgrange universal deformation of a meromorphic connection in Birkhoff normal form, with residue a regular endomorphism.

We divide the proof of the above proposition into several steps. The $(TE)$-structure $\nabla$ has the form described in Lemma 21. Let $\nabla^{\text{restr}}$ be the restriction of $\nabla$ to the slice $\Delta \times \{0\}$ (where $\Delta$ is a small disc around the origin in $\mathbb{C}$) and $\eta := b_2$, $\lambda := \partial_2 b_2$, $\beta := \partial_2^2 b_2$ and $\gamma := f$, all restricted to this slice. They are functions on $z$ only and are holomorphic. From (69), (70), the connection form of $\nabla^{\text{restr}}$ is

$$\Omega^{\text{restr}} = \frac{1}{z^2} \left( (c + z\alpha)C_1 + \eta C_2 - \frac{z(\lambda + 1)}{2} D + z\left(-\frac{z\beta}{2} + \gamma \eta\right) E \right) dz. \quad (108)$$

**Lemma 37.** The connection $\nabla^{\text{restr}}$ is an elementary model if and only if $\eta(0)\gamma(0) = 0$.

**Proof.** We need to show that $\nabla^{\text{restr}}$, with $c = \alpha = 0$, is regular singular. Assume from now on that $c = \alpha = 0$. When $\eta(0) = 0$, $\Omega^{\text{restr}}$ has a logarithmic pole and the regular singularity of $\nabla^{\text{restr}}$ is obvious. Assume assume that $\eta(0) \neq 0$. Let $\{v_1, v_2\}$ be the standard basis of $(\mathcal{O}(\mathbb{C},0))^2$, so that

$$\nabla^{\text{restr}}_{\partial z} (v_1) = (\Omega^{\text{restr}}_{\partial z})_{11} v_1 + (\Omega^{\text{restr}}_{\partial z})_{21} v_2$$
$$\nabla^{\text{restr}}_{\partial z} (v_2) = (\Omega^{\text{restr}}_{\partial z})_{12} v_1 + (\Omega^{\text{restr}}_{\partial z})_{22} v_2. \quad (109)$$

Using the definitions of the matrices $C_2$, $D$ and $E$, we rewrite (109) as

$$\nabla^{\text{restr}}_{\partial z} (v_1) = -\frac{\lambda + 1}{2z} v_1 + \frac{\eta}{z^2} v_2$$
$$\nabla^{\text{restr}}_{\partial z} (v_2) = \left(-\frac{\beta}{2} + \frac{\eta \gamma}{z}\right) v_1 + \frac{\lambda + 1}{2z} v_2. \quad (110)$$

Since $\eta(0) \neq 0$, $v_1$ is a cyclic vector. Let $\tilde{v}_2 := \nabla^{\text{restr}}_{\partial z} (v_1)$. Then

$$\nabla^{\text{restr}}_{\partial z} (\tilde{v}_2) = \left(-\partial_z \left(\frac{\lambda + 1}{2z}\right) + \frac{\eta}{z^2} \left(-\frac{\beta}{2} + \frac{\eta \gamma}{z}\right) + \frac{\lambda + 1}{2\eta} \left(\frac{\dot{\eta}}{z} - \frac{2\eta}{z^2}\right) + \frac{(\lambda + 1)^2}{4z^2}\right) v_1$$
$$+ \frac{1}{\eta} (\dot{\eta} - \frac{2\eta}{z}) \tilde{v}_2. \quad (111)$$

The valuation of the coefficient of $\tilde{v}_2$ in $\nabla^{\text{restr}}_{\partial z} (\tilde{v}_2)$ is equal to $-1$, while the valuation of the coefficient of $v_1$ is greater or equal to $-2$ if and only if $\gamma(0) = 0$ (we used that $\eta(0) \neq 0$). From the Fuchs criterion, we obtain our claim.
The second part of the next corollary solves the holomorphic classification of elementary \((TE)\)-structures.

**Corollary 38.** i) The property of a \((TE)\)-structure over \(N_2\) to be elementary is a formal invariant. Any formal isomorphism between elementary \((TE)\)-structures is holomorphic.

ii) Any elementary \((TE)\)-structure over \(N_2\) is holomorphically isomorphic to its formal normal form(s).

**Proof.** Let \(\nabla\) and \(\tilde{\nabla}\) be two formally isomorphic \((TE)\)-structures over \(N_2\), with associated functions \((f, b_2)\) and \((\tilde{f}, \tilde{b}_2)\) and matrices \(B\) and \(\tilde{B}\) respectively, where \(B\) and \(\tilde{B}\) are defined in (69). From Theorem 19 i) of [3] we know that \(f(0,0) = 0\) if and only if \(\tilde{f}(0,0) = 0\). Since \(\nabla\) and \(\tilde{\nabla}\) are formally isomorphic, \(B(0)\) and \(\tilde{B}(0)\) are conjugated, which implies that \(b_2(0,0) = 0\) if and only if \(\tilde{b}_2(0,0) = 0\). We proved that \(\nabla\) is elementary if and only if \(\tilde{\nabla}\) is elementary, i.e. being elementary is a formal invariant. Assume now that \(\nabla\) and \(\tilde{\nabla}\) are elementary. Let \(T = \sum_{k \geq 0} T^{(k)} z^k\) be a formal isomorphism between them and \(T^0\) be the restriction of \(T\) to \(\Delta \times \{0\}\). As explained at the beginning of this section, \(T^0\) is an isomorphism between \((\nabla^{\text{restr}})^1\) and \((\tilde{\nabla}^{\text{restr}})^1\). Since these are regular singular, we deduce that \(T^0\) is holomorphic. Claim i) is concluded by Theorem 5.6 of [4]. Claim ii) follows trivially from claim i).

Claims i) and ii) from Proposition 36 are proved. It remains to prove claim iii). We begin with the next lemma (recall the definition of \(\eta\) and \(\gamma\) stated before Lemma 37).

**Lemma 39.** Let \(\nabla\) be a \((TE)\)-structure with associated functions \((f, b_2)\). If \(\eta(0)\gamma(0) \neq 0\), then \(\nabla^{\text{restr}}\) can be put in Birkhoff normal form.

**Proof.** In the standard basis \(\{v_1, v_2\}\) of \((O(\mathbb{C}, 0))^2\), \(\nabla^{\text{restr}}\) is given by

\[
\nabla_{\partial z}^{\text{restr}}(v_1) = \left(\frac{c + \alpha z}{z^2} - \frac{\lambda + 1}{2z}\right)v_1 + \frac{\eta}{z^2}v_2
\]

\[
\nabla_{\partial z}^{\text{restr}}(v_2) = \left(-\frac{\beta}{2} + \frac{\eta \gamma}{z}\right)v_1 + \left(\frac{c + \alpha z}{z^2} + \frac{\lambda + 1}{2z}\right)v_2.
\]

(Remark that these relations with \(c = \alpha = 0\) reduce to relations (110)). We apply the irreducibility criterion as stated in Lemma [14]. Suppose, by absurd, that there is a section \(w\) such that

\[
\nabla_{\partial z}^{\text{restr}}(w) = hw,
\]

34
for a function \( h \in k \). Since \( \eta(0) \neq 0 \), the first relation (112) shows that \( w \) cannot be a multiple of \( v_1 \). Rescaling \( w \) if necessary, we can assume that \( w = gv_1 + v_2 \), where \( g \in k \). A straightforward computation which uses (112) shows that (113) is equivalent to

\[
\begin{align*}
    h &= \frac{\eta}{z^2} + \frac{\lambda + 1}{2z} + \frac{c + \alpha z}{z^2} \\
     \lambda^2 \dot{g} - ((\lambda + 1)z + \eta g)g - \frac{\beta z^2}{2} + \eta \gamma z &= 0.
\end{align*}
\]

We will show that (114) leads to a contradiction. Since \( g \in k \), we can write it as \( g(z) = z^k r(z) \) for \( k \in \mathbb{Z} \) and \( r \in \mathbb{C}\{z\} \) a unit. Relation (114) is equivalent to

\[
kz^{k+1} r(z) + z^{k+2} \dot{r}(z) - z^{k+1} r(z)(\lambda(z) + 1) - z^{2k} \eta(z) r(z)^2 - \frac{\beta(z) z^2}{2} + \eta(z) \gamma(z) z = 0.
\]

If \( k \leq 0 \) then, multiplying the above relation by \( z^{-2k} \), we obtain

\[
kz^{-k+1} r(z) + z^{-k+2} \dot{r}(z) - z^{-k+1} r(z)(\lambda(z) + 1) - \eta(z) r(z)^2 - \frac{\beta(z) z^{-2k+2}}{2} + \eta(z) \gamma(z) z^{-2k+1} = 0.
\]

All terms, except \( \eta(z) r(z)^2 \), contain \( z \) as a factor. Since \( r, \lambda, \eta, \beta, \gamma \in \mathbb{C}\{z\} \) and \( \eta, r \), are units we obtain a contradiction. If \( k \geq 1 \) the argument is similar: we multiply (115) by \( z^{-1} \) and we use that \( \eta, \gamma \) are units in \( \mathbb{C}\{z\} \).

To conclude Proposition 36 we notice that if \( b_2(0,0) \neq 0 \) then the ‘residue’ \( cC_1 + \eta(0) C_2 \) of the restriction \( \nabla^{rest} \) of \( \nabla \) to \( \Delta \times \{0\} \) is a regular endomorphism. Therefore, the Birkhoff normal form provided by Lemma 39 also has a regular residue and admits a (unique, up to holomorphic isomorphisms) Malgrange universal deformation. The latter is (holomorphically) isomorphic to \( \nabla \).

5.2 Holomorphic classification: non-elementary case

The holomorphic classification of elementary \((TE)\)-structures follows from Corollary 38 ii): the formal normal forms for elementary \((TE)\)-structures coincide with the holomorphic normal forms. It remains to determine the holomorphic normal forms for non-elementary \((TE)\)-structures. This will be done in the next sections.
5.2.1 Classification of non-elementary models in Birkhoff normal form

Lemma 40. i) Any non-elementary \((TE)\)-structure \(\nabla\) is isomorphic to the Malgrange universal deformation of a connection \(\nabla^{B_0, B_{\infty}}\) in Birkhoff normal form, with connection form

\[
\Omega^{B_0, B_{\infty}} = \frac{1}{z^2} (B'_0 + B_{\infty} z) dz,
\]

(116)

where

\[
B'_0 = \begin{pmatrix} c & 0 \\ c_0 & c \end{pmatrix}, \quad B_{\infty} = \begin{pmatrix} B_{11}^\infty & B_{12}^\infty \\ B_{21}^\infty & B_{22}^\infty \end{pmatrix}
\]

(117)

with \(B_{ij}^\infty \in \mathbb{C}\), \(c, c_0 \in \mathbb{C}\) and \(c_0 B_{12}^\infty \neq 0\).

ii) Two non-elementary \((TE)\)-structures \(\nabla, \tilde{\nabla}\) are isomorphic if and only if the associated connections in Birkhoff normal form \(\nabla^{B_0, B_{\infty}}\) and \(\nabla^{\tilde{B}_0, \tilde{B}_{\infty}}\) are isomorphic.

Proof. From Lemma 39 we know that the restriction \(\nabla^{\text{restr}}\) of \(\nabla\) to the origin of \(\mathcal{N}_2\) can be put in Birkhoff normal form. Let \(\nabla^{B_0, B_{\infty}}\) be a connection in Birkhoff normal form, with connection form given by (116) (for some matrices \(B'_0, B_{\infty} \in M_{2 \times 2}(\mathbb{C})\)), isomorphic to \(\nabla^{\text{restr}}\). The connection \(\nabla^{B_0, B_{\infty}}\) has two properties: it is not an elementary model and its ‘residue’ \(B'_0\) is a regular endomorphism, with only one eigenvalue (these two properties are satisfied by \(\nabla^{\text{restr}}\) and are invariant under holomorphic isomorphisms). From the second property, the ‘residue’ \(B'_0\) is as in (117), with \(c_0 \neq 0\). A direct check (using e.g. the Fuchs criterion), shows that \(\nabla^{B_0, B_{\infty}}\) is not an elementary model if and only if \(B_{12}^\infty \neq 0\). This proves claim i). Claim ii) follows from the unicity of Malgrange universal deformations.

Next, we establish when two meromorphic connections in Birkhoff normal form, as in Lemma 40, are isomorphic. We start with the next lemma.

Lemma 41. i) Any meromorphic connection \(\nabla^{B_0, B_{\infty}}\) in Birkhoff normal form, with connection form given by (116) where

\[
B'_0 = c C_1 + c_0 C_2, \quad B_{\infty} = \alpha C_1 + c_1 C_2 + y D + f E,
\]

(118)

and \(c_0 f \neq 0\), can be mapped, by means of a constant isomorphism, to a connection \(\nabla^{\tilde{B}_0, \tilde{B}_{\infty}}\) in Birkhoff normal form with

\[
\tilde{B}'_0 = c C_1 + c_0 C_2, \quad \tilde{B}_{\infty} = \alpha C_1 + c_1 C_2 - \frac{1}{4} D + c_0 E,
\]

(119)
where the constants $c$ and $\alpha$ are the same as in (118), $c_0$ and $c_1$ are possibly different from those in (118) and $c_0 \neq 0$.

ii) The constant isomorphism $T := \text{diag}(1, -1)$ maps the connection in Birkhoff normal form $\nabla^{B_0, B_\infty}$, with matrices $B_0^o$, $B_\infty$ given in (119), to a connection of the same form, with constants $(\tilde{c}, \tilde{c}_0, \tilde{\alpha}, \tilde{c}_1)$ satisfying $\tilde{c} = c$, $\tilde{\alpha} = \alpha$, $\tilde{c}_0 = -c_0$ and $\tilde{c}_1 = -c_1$.

**Proof.** i) First we map $\nabla^{B_0, B_\infty}$ to a connection of the same form (116), (118), with the coefficient of $D$ equal to $-\frac{1}{4}$. This is realized using the constant isomorphism $T_1 := C_1 - \frac{1}{y + \frac{1}{4}}C_2$. Therefore, without loss of generality we may (and will) assume that $\nabla^{B_0, B_\infty}$ is given by (116), (118), with $y = -\frac{1}{4}$.

Under this assumption, if $c_0 \neq f$ in (118), let $\tilde{c}_0$ such that $(\tilde{c}_0)^2 = c_0f$ and $T_2 := -2\text{diag}(\frac{\tilde{c}_0 - c_0}{c_0 - c_0}, \frac{\tilde{c}_0 - c_0}{c_0 - c_0})$. The isomorphism $T_2$ maps $\nabla^{B_0, B_\infty}$ to the connection $\nabla^{\tilde{B}_0, \tilde{B}_\infty}$ with

$$\tilde{B}_0^o = cC_1 + \tilde{c}_0C_2, \quad \tilde{B}_\infty = \alpha C_1 + \frac{c_1\tilde{c}_0}{c_0}C_2 - \frac{1}{4}D + \tilde{c}_0E.$$

This proves claim i). Claim ii) follows from a direct check. \qed

The next lemma is a particular case of Exercise 3.10 of [15] (page 100). For completeness of our exposition we include its proof.

**Lemma 42.** Let $\nabla$ and $\tilde{\nabla}$ be two meromorphic connections in Birkhoff normal form, with pole of Poincaré rank one at the origin, defined on the trivial rank $r$ holomorphic vector bundle $\mathbb{C}^r \times \mathbb{C} \to \mathbb{C}$. Any holomorphic isomorphism $T : ((\mathcal{O}_{\mathbb{C}^r})^r, \nabla) \to ((\mathcal{O}_{\mathbb{C}^r})^r, \tilde{\nabla})$ is a polynomial.

**Proof.** We extend $T$ to a $(\nabla \otimes \tilde{\nabla}^*)$-flat section, also denoted by $T$, defined on $\mathbb{C}^r$. As $\nabla$, $\tilde{\nabla}$ are regular singular at infinity, so is $\nabla \otimes \tilde{\nabla}^*$ and $T$ is meromorphic there. Therefore, $T$ is a vector-valued function on $\mathbb{C}P^1$, holomorphic on $\mathbb{C}$ and meromorphic at infinity. We obtain that $T$ is a polynomial. \qed

**Proposition 43.** Consider two distinct connections $\nabla$ and $\tilde{\nabla}$ in Birkhoff normal form (116), with matrices $B_0^o$, $B_\infty$, respectively $\tilde{B}_0^o$, $\tilde{B}_\infty$ as in (119), with constants $c, \alpha, c_0, c_1$ and, respectively $\tilde{c}, \tilde{\alpha}, \tilde{c}_0, \tilde{c}_1$. Assume that $c_0\tilde{c}_0 \neq 0$.

i) If $\nabla$ and $\tilde{\nabla}$ are formally isomorphic, then $c = \tilde{c}$, $\alpha = \tilde{\alpha}$ and $c_0 = \epsilon \tilde{c}_0$ where $\epsilon \in \{\pm 1\}$.

ii) Assume that the conditions from i) are satisfied. Then $\nabla$ is isomorphic to $\tilde{\nabla}$ if and only if there is $n \in \mathbb{N}_{\geq 2}$ such that

$$4(c_0)^2(c_1 - \epsilon \tilde{c}_1)^2 - 8(n - 1)^2c_0(c_1 + \epsilon \tilde{c}_1) + (2n - 1)(2n - 3)(n - 1)^2 = 0 \quad (120)$$

and, for any $2 \leq r \leq n - 1$, $r \in \mathbb{N}$,

$$c_0(c_1 + \epsilon \tilde{c}_1) \neq \frac{(2n - 1)(2n - 3)(n - 1)^2 - (2r - 1)(2r - 3)(r - 1)^2}{8(n - r)(n - 2 + r)} \quad (121)$$

37
Proof. i) We consider a formal isomorphism $T := \sum_{n\geq 0} T(n) z^n$ which maps $\nabla$ to $\tilde{\nabla}$. We write $T(n) = \tau_1(n) C_1 + \tau_2(n) C_2 + \tau_3(n) D + \tau_4(n) E$, where $\tau_i(n) \in \mathbb{C}$. Relation (33) with $r = 0$, applied to $\nabla$, $\tilde{\nabla}$ and $T$, gives $c = \tilde{c}$, $\tau_4^{(0)} = 0$ and

$$
\tau_1^{(0)} (c_0 - \tilde{c}_0) + \tau_3^{(0)} (c_0 + \tilde{c}_0) = 0.
$$

(122)

Using that $\tau_4^{(0)} = 0$, relation (33) with $r = 1$ becomes

$$
\begin{align*}
\frac{\tau_4^{(1)}}{2} (c_0 - \tilde{c}_0) + (\alpha - \tilde{\alpha}) \tau_1^{(0)} + (c_0 - \tilde{c}_0) \tau_2^{(0)} &= 0 \\
\tau_1^{(1)} (c_0 - \tilde{c}_0) + \tau_3^{(1)} (c_0 + \tilde{c}_0) + (\alpha - \tilde{\alpha} + \frac{1}{2}) \tau_2^{(0)} + (c_1 - \tilde{c}_1) \tau_1^{(0)} + (c_1 + \tilde{c}_1) \tau_3^{(0)} &= 0 \\
\frac{\tau_4^{(1)}}{2} (c_0 + \tilde{c}_0) - (\alpha - \tilde{\alpha}) \tau_3^{(0)} - \frac{\tau_2^{(0)}}{2} (c_0 + \tilde{c}_0) &= 0 \\
(c_0 - \tilde{c}_0) \tau_1^{(0)} - (c_0 + \tilde{c}_0) \tau_3^{(0)} &= 0.
\end{align*}
$$

(123)

Suppose that $\tilde{c}_0 \neq c_0$. Then, from (122), $\tau_1^{(0)} = -(\frac{\tilde{c}_0 + c_0}{c_0 - \tilde{c}_0}) \tau_3^{(0)}$ and $\tau_3^{(0)} \neq 0$ (if $\tau_3^{(0)} = 0$ then $\tau_1^{(0)} = 0$; since $\tau_4^{(0)} = 0$ we obtain that $T^{(0)}$ is not invertible, which is a contradiction). The last relation (123) implies $c_0 = -\tilde{c}_0$. We proved that $c = \tilde{c}$ and $c_0 = \epsilon \tilde{c}_0$ where $\epsilon \in \{\pm 1\}$. The first and third relations (123) (together with $T^{(0)}$-invertible and (122)) imply that $\alpha = \tilde{\alpha}$ in both cases (i.e. $\epsilon = 1$ or $\epsilon = -1$). The first claim follows.

ii) Using, if necessary, the isomorphism $T := \text{diag}(1, -1)$ (see Lemma 44 ii)), we assume that $c_0 = \tilde{c}_0$. Under this assumption, relation (33) with $r = 0, 1$ is equivalent to $c = \tilde{c}$, $\alpha = \tilde{\alpha}$, $\tau_4^{(0)} = \tau_3^{(0)} = 0$, $\tau_1^{(0)} \neq 0$, $\tau_4^{(1)} = \tau_2^{(0)}$, and

$$
\frac{\tau_2^{(0)}}{2} + (c_1 - \tilde{c}_1) \tau_1^{(0)} = -2c_0 \tau_3^{(1)}.
$$

(124)

If $\nabla$ and $\tilde{\nabla}$ are isomorphic, any isomorphism $T$ which maps $\nabla$ to $\tilde{\nabla}$ is a polynomial (from Lemma 44). Therefore, there is $n \in \mathbb{Z}^*$ such that $T^{(k)} = 0$ for any $k \geq n$ and $T^{(n-1)} \neq 0$. We remark that $n \geq 2$: if, by absurd, $T^{(1)} = 0$, then $\tau_2^{(0)} = \tau_4^{(1)} = \tau_3^{(1)} = 0$, and, from (124), $\tau_1^{(0)} = 0$ (since $c_1 \neq \tilde{c}_1$, as $\nabla \neq \tilde{\nabla}$). We obtain that $T^{(0)} = 0$, which is a contradiction. Thus $n \geq 2$.

Next, we consider relations (33) with $r \geq 2$. They are equivalent to

$$
\tau_1^{(r-1)} = \frac{\tilde{c}_1 - c_1}{2(r - 1)} \tau_4^{(r-1)}, \quad \tau_3^{(r-1)} = \frac{2r - 3}{4c_0} \tau_4^{(r-1)}, \quad r \in \mathbb{Z}_{\geq 2},
$$

(125)

together with

$$
\begin{align*}
(r - \frac{1}{2}) \tau_2^{(r-1)} + (c_1 - \tilde{c}_1) \tau_1^{(r-1)} + (c_1 + \tilde{c}_1) \tau_3^{(r-1)} &= -2c_0 \tau_3^{(r)} \\
(r - 1) \tau_3^{(r-1)} - (\frac{c_1 + \tilde{c}_1}{2}) \tau_4^{(r-1)} + c_0 \tau_2^{(r-1)} &= c_0 \tau_4^{(r)}.
\end{align*}
$$

(126)
Replacing the expressions \((125)\) of \(\tau_1^{(r-1)}\) and \(\tau_3^{(r-1)}\) in \((120)\), we obtain
\[
(r - \frac{1}{2})\tau_2^{(r-1)} + \frac{1}{2}\left( (c_1 + \tilde{c}_1)\left(\frac{2r - 3}{2c_0} - \frac{(c_1 - \tilde{c}_1)^2}{r - 1}\right) \right) \tau_4^{(r-1)} = -2c_0\tau_3^{(r)}
\]
\[
c_0\tau_2^{(r-1)} + \frac{1}{2}\left( (r - 1)(2r - 3) - (c_1 + \tilde{c}_1) \right) \tau_4^{(r-1)} = c_0\tau_4^{(r)}.
\]
(127)

Relations \((127)\) with \(r = n\) form a system in the unknown constants \(\tau_2^{(n-1)}\) and \(\tau_4^{(n-1)}\) (since \(\tau_3^{(n)} = \tau_4^{(n)} = 0\)). As \(T^{(n-1)} \neq 0\), this system has a non-zero solution (if \(\tau_2^{(n-1)} = \tau_4^{(n-1)} = 0\) then, from \((125)\), \(T^{(n-1)} = 0\) which is a contradiction). We deduce that discriminant \(\Delta_n\) of \((127)\) with \(r = n\) vanishes. The condition \(\Delta_n = 0\) is equivalent to \((120)\) with \(\epsilon = 1\).

Assume that \((120)\) with \(\epsilon = 1\) holds and let \((\tau_2^{(n-1)}, \tau_4^{(n-1)})\) be a solution of \((127)\) with \(r = n\), such that \(\tau_4^{(n-1)} \neq 0\). Then \(T^{(n-1)}\) is determined completely, as \(\tau_1^{(n-1)}\) and \(\tau_3^{(n-1)}\) can be recovered from \((125)\) with \(r = n\). With the same argument we construct inductively the remaining \(T^{(r)}\), for \(r = n - 2, n - 3, \ldots, 1\). The idea is to consider \((127)\) at each step \((2 \leq r \leq n - 1)\) as a system of equations in the unknown constants \(\tau_2^{(r-1)}\) and \(\tau_4^{(r-1)}\), with known \(\tau_3^{(r)}\) and \(\tau_4^{(r)}\) (determined by the previous step). The two equations which form this system are not proportional, since
\[
\frac{r - \frac{1}{2}}{c_0} \neq -2\frac{\tau_3^{(r)}}{\tau_4^{(r)}}
\]
(from the second relation \((125)\)). Thus, the system admits a solution \((\tau_2^{(n-1)}, \tau_4^{(r-1)})\) if and only if its discriminant \(\Delta_r\) is non-zero, i.e.
\[
4(c_0)^2(c_1 - \tilde{c}_1)^2 - 8(r - 1)^2c_0(c_1 + \tilde{c}_1) + (2r - 1)(2r - 3)(r - 1)^2 \neq 0, \quad (128)
\]
(for any \(2 \leq r \leq n - 1\)). The coefficients \(\tau_1^{(r-1)}\) and \(\tau_3^{(r-1)}\) (and the entire \(T^{(r-1)}\)) are determined as before by \((125)\). Replacing in \((128)\) \((c_0)^2(c_1 - \tilde{c}_1)^2\) with its expression in terms of \(c_0(c_1 + \tilde{c}_1)\) provided by \((120)\) (with \(\epsilon = 1\)), we obtain \((121)\). An induction argument also shows that \(\tau_4^{(r)} \neq 0\) for any \(r \geq 1\).

It remains to construct \(T^{(0)}\) and show that it is invertible. Since \(T^{(1)}\) is known, \(\tau_2^{(0)} = \tau_4^{(1)}\) is also known and from \((124)\) we obtain \(\tau_1^{(0)}\) (since \(c_1 \neq \tilde{c}_1\)). Recall also that \(\tau_3^{(0)} = \tau_4^{(0)} = 0\), so \(T^{(0)}\) is known. It is easy to see that \(\tau_2^{(0)} \neq 0\) (otherwise, from \((124)\), \(\tau_2^{(0)} = -4c_0\tau_3^{(1)}\); since \(\tau_2^{(0)} = \tau_4^{(1)}\) we obtain \(\tau_4^{(1)} = -4c_0\tau_3^{(1)}\) which contradicts the second relation \((125)\) with \(r = 2\) and \(\tau_4^{(1)} \neq 0\). Hence \(T^{(0)}\) is invertible, as required. \(\Box\)
Corollary 44. In the setting of Proposition 43, assume that \( \tilde{c}_1 = 0 \). Then \( \nabla \) is isomorphic to \( \nabla \) if and only if \( c = \tilde{c}, \alpha = \tilde{\alpha}, (c_0)^2 = (\tilde{c}_0)^2 \) and there is \( n \in \mathbb{Z}_{\geq 2} \) such that

\[
\begin{align*}
c_0c_1 & \in \left\{ \frac{(n-1)(2n-1)}{2}, \frac{(n-1)(2n-3)}{2} \right\}. 
\end{align*}
\] (129)

Proof. When \( \tilde{c}_1 = 0 \), relation (120) is an equation in \( c_0c_1 \), with solutions given by the right hand side of (129). If \( \tilde{c}_1 = 0 \) then (129) implies (121).

Remark 45. i) Lemmas 40 and 41, combined with Proposition 43, provide a criterion to decide when two non-elementary \((TE)\)-structures are isomorphic, using their restriction at the origin of \( \mathcal{N}_2 \).

ii) The only non-elementary formal normal forms from Theorem 18 are those from Theorem 18 i) with \( c_0 \neq 0 \). They represent the formal normal forms of non-elementary \((TE)\)-structures. Their restriction at the origin are in Birkhoff normal form, with

\[
B^\circ_0 = cC_1 + c_0C_2, \quad B_\infty = \alpha C_1 - \frac{1}{4}D + c_0E, 
\] (130)

(where \( c_0 \neq 0 \)).

iii) There are non-elementary \((TE)\)-structures which are not (holomorphically) isomorphic to their formal normal form(s): from Corollary 43 they coincide (up to isomorphism) with the Malgrange universal deformations of connections \( \nabla^{B^\circ_0, B_\infty} \) in Birkhoff normal form, with matrices \( B^\circ_0, B_\infty \) as in (119), such that \( c_0c_1 \) does not satisfy (129).

In order to obtain a list of holomorphic normal forms for non-elementary \((TE)\)-structures we will express the Malgrange universal deformations of the meromorphic connections \( \nabla^{B^\circ_0, B_\infty} \) in Birkhoff normal form (116), with matrices \( B^\circ_0, B_\infty \) given by (119), with \( c_0B^{\infty}_{12} \neq 0 \), in local coordinates \((t_1, t_2)\) of \( \mathcal{N}_2 \). This will be done in the next sections.

5.2.2 Malgrange universal connections

Let \( B^\circ_0, B_\infty \in M_{2 \times 2}(\mathbb{C}) \) be two matrices, where \( B^\circ_0 \) is regular, with one Jordan block, i.e. \( B^\circ_0 = cC_1 + c_0C_2 \), and \( c_0 \neq 0 \). We are interested in the case \( B^{\infty}_{12} \neq 0 \) but for the moment we don’t make this assumption. We denote by \( \nabla^{\text{univ}} = \nabla^{\text{univ}, B^\circ_0, B_\infty} \) the Malgrange universal deformation of the meromorphic connection \( \nabla^{B^\circ_0, B_\infty} \) with connection form (51). We consider \((M^{\text{univ}}, \circ_{\text{univ}}, e_{\text{univ}}, E_{\text{univ}})\) the parameter space of \( \nabla^{\text{univ}} \). The germs \((M^{\text{univ}}, 0), \circ_{\text{univ}}, e_{\text{univ}})\) and \( \mathcal{N}_2 \) are isomorphic.
For any $\Gamma \in M_{2 \times 2}(\mathbb{C})$, we identify $T_{\Gamma}M_{2 \times 2}(\mathbb{C})$ with $M_{2 \times 2}(\mathbb{C})$ in the natural way. Therefore, vector fields on $M_{2 \times 2}(\mathbb{C})$ or on the submanifold $\text{univ}$ will be viewed as $M_{2 \times 2}(\mathbb{C})$-valued functions (defined on $M_{2 \times 2}(\mathbb{C})$ or $\text{univ}$ respectively).

Let $X_0, X_1$ be vector fields on $M_{2 \times 2}(\mathbb{C})$ defined by $(X_0)_{\Gamma} = C_1$ and $(X_1)_{\Gamma} = B_0^0 - \Gamma + [B_\infty, \Gamma]$, for any $\Gamma \in M_{2 \times 2}(\mathbb{C})$. In the standard coordinates $(\Gamma_{ij})$ of $M_{2 \times 2}(\mathbb{C})$ (where $\Gamma_{ij} : M_{2 \times 2}(\mathbb{C}) \to \mathbb{C}$ is the function which assigns to $\Gamma \in M_{2 \times 2}(\mathbb{C})$ its $(i, j)$-entry),

$$X_0 = \sum_{i,j=1}^2 \frac{\partial}{\partial \Gamma_{ij}}, \quad X_1 = \sum_{i,j=1}^2 ((B_0^0)_{ij} - \Gamma_{ij} + (B_\infty)_{ik}\Gamma_{kj} - \Gamma_{ik}(B_\infty)_{kj}) \frac{\partial}{\partial \Gamma_{ij}}. \tag{131}$$

(To simplify notation, we omitted the summation sign over $k \in \{1, 2\}$). Let

$$\tilde{k} : M_{2 \times 2}(\mathbb{C}) \to \mathbb{C}, \quad \tilde{k}(\Gamma) := -\frac{1}{2}\text{trace}(X_1)_{\Gamma} = \frac{1}{2} \sum_{i=1}^2 \Gamma_{ii} - c. \tag{132}$$

Viewing a vector field $X$ on $M_{2 \times 2}(\mathbb{C})$ as an $M_{2 \times 2}(\mathbb{C})$-valued function, we can consider its derivative along any other vector field $Y$ on $M_{2 \times 2}(\mathbb{C})$. The result is a function $Y(X) : M_{2 \times 2}(\mathbb{C}) \to M_{2 \times 2}(\mathbb{C})$, whose $(i, j)$-entry is the function $Y(X_{ij})$. Various such derivatives are computed in the next lemma (below $C_1$ denotes the constant function on $M_{2 \times 2}(\mathbb{C})$ equal to $C_1$).

**Lemma 46.** The following relations hold:

$$X_0(X_1) = -C_1, \quad X_0(\tilde{k}) = 1, \quad X_0(\tilde{k}X_0 + X_1) = 0; \quad X_1(X_1) = -X_1 + [B_\infty, X_1], \quad X_1(\tilde{k}) = -\tilde{k}. \tag{133}$$

**Proof.** As $X_0(\Gamma_{ij}) = \delta_{ij}$ for any $i, j \in \{1, 2\}$ we obtain

$$X_0([B_\infty, \Gamma]_{ik}) = \sum_{j=1}^2 X_0((B_\infty)_{ij}\Gamma_{jk} - \Gamma_{ij}(B_\infty)_{jk})$$

$$= \sum_{j=1}^2 ((B_\infty)_{ij}\delta_{jk} - (B_\infty)_{jk}\delta_{ij}) = (B_\infty)_{ik} - (B_\infty)_{ik} = 0$$

and

$$X_0((B_0^0)_{ik} - \Gamma_{ik} + [B_\infty, \Gamma]_{ik}) = -\delta_{ik}.$$

We proved that $X_0(X_1) = -C_1$. Since $X_0(\Gamma_{ii}) = 1$ we obtain $X_0(\tilde{k}) = 1$. Obviously, $X_0(X_0) = 0$ (since $(X_0)_{ij} = \delta_{ij}$ are constants) and so

$$X_0(\tilde{k}X_0 + X_1) = X_0(\tilde{k})X_0 + X_0(X_1) = C_1 - C_1 = 0.$$

The first line of \(133\) follows. The second line can be proved similarly. \(\square\)
Using the expression (131) of $X_0$ and $X_1$ we compute the Lie derivative of $X_1$ in the direction of $X_0$: $\mathcal{L}_{X_0}X_1 = -X_0$. Using $X_0(\tilde{k}) = 1$ we obtain that the vector fields $X_0$ and $\tilde{k}X_0 + X_1$ commute. Their restriction to $M^{univ}$ are the fundamental vector fields of a coordinate system $(t_1, t_2)$ on $M^{univ}$, which we choose to be centred at the origin of $M^{univ}$ (recall that $X_0$ and $X_1$ are sections of the defining distribution $D \subset TM_{2\times 2}(\mathbb{C})$ of $M^{univ}$, see Section [2.6]). As $X_0(\tilde{k}) = 1$ we obtain that the vector fields $X_0$ and $\tilde{k}X_0 + X_1$ commute. Their restriction to $M^{univ}$ are the fundamental vector fields of a coordinate system $(t_1, t_2)$ on $M^{univ}$, which we choose to be centred at the origin of $M^{univ}$ (recall that $X_0$ and $X_1$ are sections of the defining distribution $D \subset TM_{2\times 2}(\mathbb{C})$ of $M^{univ}$, see Section [2.6]). As $X_0(\tilde{k}) = 1$ and $(\tilde{k}X_0 + X_1)(\tilde{k}) = 0$ (from Lemma 46) and $\tilde{k}(0) = -c$ (from the definition (132) of $\tilde{k}$) we obtain that $\tilde{k}(t_1, t_2) = t_1 - c$.

**Remark 47.** For any $\Gamma \in M^{univ}$, the matrix $(\tilde{k}X_0 + X_1)(\Gamma)$ has the following properties: it is trace free (from the definition of $\tilde{k}$); it is regular (since $(X_1)(\Gamma)$ is regular, being regular at $\Gamma = 0$); it has only one Jordan block (since $\text{Bo}^0$ has this property; see our comments from Section [2.6]). We obtain that $(\tilde{k}X_0 + X_1)(\Gamma)$ is conjugated to a matrix with all entries zero except the $(2,1)$-entry which is non-zero. In particular, $(\tilde{k}X_0 + X_1)(\Gamma)$ equals zero. Also, $(\tilde{k}X_0 + X_1)_{21} = (X_1)_{21}$ at $\Gamma = 0$ is equal to $c_0 \neq 0$. The function $y := (X_1)_{21}$ on $M^{univ} \to \mathbb{C}$ is non-vanishing in a neighborhood of the origin in $M^{univ}$ and the function $\tilde{k}X_0 + X_1$ on $M^{univ} \to M_{2\times 2}(\mathbb{C})$ can be written as

\[
\tilde{k}X_0 + X_1 = y \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix},
\]

where

\[
x = \frac{1}{(X_1)_{21}}(\tilde{k} + (X_1)_{11}) = \frac{1}{2(X_1)_{21}}((X_1)_{11} - (X_1)_{22}) : M^{univ} \to \mathbb{C}.
\]

**Proposition 48.** In the coordinate system $(t_1, t_2)$, the Malgrange universal connection $\nabla^{univ}$ is given by the matrices $A_1, A_2$ and $B = B^{(0)} + B^{(1)}z$, where

\[
A_1 = C_1, \quad A_2 = y \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix},
\]

and

\[
B^{(0)} = \begin{pmatrix} xy - t_1 + c & -x^2y \\ y & -xy - t_1 + c \end{pmatrix}, \quad B^{(1)} = B_\infty.
\]

**Proof.** Recall the definition (54) of the Malgrange universal connection. The equality $A_1 = C_1$ follows from the fact that $A_1 = \frac{\partial}{\partial t_1}$ is the matrix valued function on $M^{univ}$ identified with the vector field $\frac{\partial}{\partial t_1} = X_0$, which is $C_1$. The expression of $A_2$ follows similarly, from $A_2 = C_2 \frac{\partial}{\partial t_2}$, together with $\frac{\partial}{\partial t_2} = \tilde{k}X_0 + X_1$ and (134). The expression of $B^{(0)}$ is obtained as follows: from (53),
\[B^{(0)}(\Gamma) = (X_1)(\Gamma)\] for any \(\Gamma \in M^{\text{univ}}\). Therefore,

\[
B^{(0)} = (\hat{k}X_0 + X_1) - \hat{k}X_0 = y \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} - (t_1 - c)\text{Id}
\]

\[= \begin{pmatrix} xy - t_1 + c & -x^2y \\ y & -xy - t_1 + c \end{pmatrix}
\]

where we used (134) and \(\hat{k}(t_1, t_2) = t_1 - c\). \[\square\]

To simplify notation, in the proof of the next lemma instead of the vector field \(X_1\) we simply write \(X\). The \((i, j)\)-entry of \(X_1\) (viewed as a \(M_{2 \times 2}(\mathbb{C})\) -valued function) will be denoted by \(X_{ij}\).

**Lemma 49.** The functions \(x\) and \(y\) are independent on \(t_1\) and their derivatives with respect to \(t_2\) are given by

\[
\dot{x} = -B_{21}^\infty x^2 + (B_{11}^\infty - B_{22}^\infty)x + B_{12}^\infty
\]

\[
\dot{y} = y(2B_{21}^\infty x + B_{22}^\infty - B_{11}^\infty - 1),
\]

(138)

where \(B_{ij}^\infty\) denotes the \((i, j)\)-entry of the matrix \(B^\infty\). They satisfy the initial conditions \(x(0) = 0\) and \(y(0) = c_0\).

**Proof.** From the first line of (133) and the definition of \(x\) and \(y\), \(X_0(x) = X_0(y) = 0\), i.e. \(x\) and \(y\) are independent on \(t_1\). From the second line of (133),

\[X(X_{11}) = -X_{11} + [B^\infty, X]_{11} = -X_{11} + B_{12}^\infty X_{21} - B_{21}^\infty X_{12} \quad (139)
\]

and similarly

\[X(X_{21}) = -X_{21} + B_{21}^\infty (X_{11} - X_{22}) + (B_{22}^\infty - B_{11}^\infty) X_{21} \quad (140)
\]

Using (139), (140), \(X(\hat{k}) = -\hat{k}\) and \(\det(X + \hat{k}\text{Id}) = 0\), we obtain

\[X \left(\frac{\hat{k} + X_{11}}{X_{21}}\right) = B_{12}^\infty + \frac{1}{2}(B_{11}^\infty - B_{22}^\infty) \left(\frac{X_{11} - X_{12}}{X_{21}}\right) - \frac{B_{21}^\infty}{4} \left(\frac{X_{11} - X_{22}}{X_{21}}\right)^2,
\]

which implies the first relation (138) (we use the definition (135) of \(x\) and \(\dot{x} = X(x)\), since \(\frac{\partial}{\partial t_2} = \hat{k}X_0 + X\) and \(X_0(x) = 0\)). The second relation (138) can be obtained similarly. \[\square\]

**Remark 50.** When \(B_{12}^\infty = 0\), the system (138) is solved by \(x = 0\) and \(y(t_1, t_2) = c_0 e^{kt_2}\) where \(c_0 \in \mathbb{C}\) and \(k := B_{22}^\infty - B_{11}^\infty - 1\). Assume that \(k \neq 0\). Replacing the expressions of \(x\) and \(y\) in (136), (137) we obtain that \(\nabla^{\text{univ}}\) is
the pull-back by \( \mu(t_1, t_2) = (t_1, \frac{c_0}{k}(e^{kt_2} - 1)) \) of the \((TE)\)-structure \( \tilde{\nabla} \) given by
\[
\tilde{A}_1 = C_1, \quad \tilde{A}_2 = C_2, \quad \tilde{B} = (-t_1 + c)C_1 + (kt_2 + c_0)C_2 + zB_{\infty}.
\]
When \( k = 0 \) the same statement holds with \( \mu(t_1, t_2) = (t_1, c_0t_2) \). We obtain that \( \nabla^{\text{univ}} \) is isomorphic to \( \tilde{\nabla} \). Remark that \( \tilde{\nabla} \) is of the third type in Theorem 18.

We now turn to the Malgrange universal deformations \( \nabla^{\text{univ}} \) we are interested in, namely those which are non-elementary. Therefore, we assume that \( B_{12}^{\infty} \neq 0 \). From Lemma 13 we may (and will) assume, without loss of generality, that \( B_{11}^{\infty} - B_{22}^{\infty} = -\frac{1}{2} \) and \( B_{12}^{\infty} = c_0 \). We distinguish two subcases, namely \( B_{21}^{\infty} = 0 \) and \( B_{21}^{\infty} \neq 0 \). In the first subcase, \( \nabla^{\text{univ}} \) is isomorphic to a \((TE)\)-structure of the first type in Theorem 18.

**Corollary 51.** If \( B_{12}^{\infty} = c_0, B_{21}^{\infty} = 0 \) and \( B_{11}^{\infty} - B_{22}^{\infty} = -\frac{1}{2} \), then \( \nabla^{\text{univ}} \) is isomorphic to the \((TE)\)-structure \( \tilde{\nabla} \) given by
\[
\tilde{A}_1 = C_1, \quad \tilde{A}_2 = C_2 + zE
\]
\[
\tilde{B} = (-t_1 + c + \alpha z)C_1 + (\frac{-t_2}{2} + c_0)C_2 - \frac{z}{4}D + z(\frac{-t_2}{4} + c_0)E,
\]
where \( \alpha := \frac{1}{2}(B_{11}^{\infty} + B_{22}^{\infty}) \).

**Proof.** The functions \( x(t_1, t_2) = 2c_0(1 - e^{-\frac{t_2}{2}}) \) and \( y(t_1, t_2) = c_0e^{-\frac{t_2}{2}} \) solve the system (138). From \( y = \dot{x} \) we obtain that \( \nabla^{\text{univ}} \) is the pull-back, by the function \( \mu(t_1, t_2) = (t_1, x(t_1, t_2)) \), of the \((TE)\)-structure \( \nabla^{[1]} \) with matrices \( A^{[1]}_1, A^{[1]}_2, B^{[1]} = \sum_{k \geq 0} B^{[1],(k)} z^k \) given by
\[
A^{[1]}_1 = C_1, \quad A^{[1]}_2 = C_2 + t_2 D - t_2^2 E
\]
\[
B^{[1],(0)} = (-t_1 + c)C_1 + (\frac{-t_2}{2} + c_0)C_2 + t_2(-\frac{t_2}{2} + c_0)D + t_2^2(\frac{t_2}{2} - c_0)E
\]
\[
B^{[1],(1)} = \alpha C_1 - \frac{1}{4} D + c_0 E
\]
\[
B^{[1],(k)} = 0, \quad k \geq 2.
\]
\]
(142)
The gauge isomorphism \( T := C_1 + t_2 E \) maps \( \nabla^{[1]} \) to \( \tilde{\nabla} \).

It remains to consider the case when both \( B_{12}^{\infty} \) and \( B_{21}^{\infty} \) are non-zero.

**Corollary 52.** Assume that \( B_{12}^{\infty} = c_0, B_{21}^{\infty} \neq 0 \) and \( B_{11}^{\infty} - B_{22}^{\infty} = -\frac{1}{2} \). Define \( a, b \in \mathbb{C} \) by \( a + b = -\frac{1}{2B_{21}^{\infty}} \) and \( ab = -\frac{c_0}{B_{21}^{\infty}} \). When \( B_{21}^{\infty} \neq -\frac{1}{16c_0} \), the system
(138) is solved by
\[ x(t_1, t_2) = ab(1 - e^{(b-a)B_{21}^\infty t_2})(b - ae^{(b-a)B_{21}^\infty t_2})^{-1}, \]
\[ y(t_1, t_2) = \frac{c_0}{(b - a)^2} (b - ae^{(b-a)B_{21}^\infty t_2})^2 e^{(B_{21}^\infty (a-b)-1) t_2}. \] (143)

When \( B_{21}^\infty = -\frac{1}{16c_0} \), it is solved by
\[ x(t_1, t_2) = \frac{4c_0 t_2}{t_2 + 4}, \quad y(t_1, t_2) = \frac{c_0}{16} e^{-t_2(t_2 + 4)^2}. \] (144)

**Proof.** We write equation (138) as \( \dot{x} = -B_{21}^\infty (x-a)(x-b) \). Since \( x(0) = 0 \) this determines \( x \) as stated in the lemma. Then \( y \) is determined from the second equation (138). Remark that \( B_{21}^\infty = -\frac{1}{16c_0} \) if and only if \( a = b \).

To simplify terminology we introduce the following definition.

**Definition 53.** A non-elementary Malgrange normal form of the first (respectively second) type is a Malgrange universal deformation \( \nabla^{\text{univ}} \) as in Proposition 48, with functions \( x \) and \( y \) satisfying (138), \( B_{12}^\infty = c_0 \neq 0 \), \( B_{11}^\infty - B_{22}^\infty = -\frac{1}{2} \) and \( B_{21}^\infty = 0 \) (respectively, \( B_{21}^\infty \neq 0 \)).

Non-elementary \((TE)\)-structures coincide (up to isomorphisms) to non-elementary Malgrange normal forms. From Corollary 51, those which are of the first type are isomorphic to the \((TE)\)-structures (66) from Theorem 18, with \( c_0 \neq 0 \). In the next section we study the non-elementary Malgrange normal forms of the second type.

### 5.2.3 Non-elementary Malgrange normal forms of second type

We are looking for (holomorphic) isomorphisms \( T \) which map an arbitrary non-elementary Malgrange normal form of the second type \( \nabla := \nabla^{\text{univ}, B_0^\infty, B_\infty} \) to a \((TE)\)-structure which is 'as close as possible' to the formal normal forms from Theorem 18. Recall that the underlying \((T)\)-structures of the formal normal forms are given by \( A_1 = C_1, A_2 = C_2 + zfE, \) where \( f = t^r_2 \) \((r \in \mathbb{Z}_{\geq 0})\) or \( f = 0 \). Moreover, their matrices \( B \) are of the form \( B = B^{(0)} + B^{(1)} z \) with \( B^{(0)} \) and \( B^{(1)} \) independent on \( z \).

With this motivation we are looking for isomorphisms \( T \) which map \( \nabla \) to a \((TE)\)-structure with these two features. Let \( A_1, A_2, B = B^{(0)} + zB^{(1)} \) be the matrices of \( \nabla \), described in Proposition 48 in terms of functions \( x, y \) determined in Corollary 52. Recall that \( B_0^\infty = cC_1 + c_0 C_2 \) (with \( c_0 \neq 0 \)), \( B_{22}^\infty = c_0 \) and \( B_{11}^\infty - B_{22}^\infty = -\frac{1}{2} \). From Remark 47, \( A_2 = A_2^{(0)} \) is conjugated to a matrix of the form \( FC_2 \), for a function \( F = F(t_2) \).
Therefore, there is a gauge isomorphism \( T = T^{(0)} \), which depends only on \( t_2 \), such that the underlying \((T)\)-structure of \( \nabla^{[1]} := T \cdot \nabla \) is
\[
A_1^{[1]} = C_1, \quad A_2^{[1]} = FC_2 + zT^{-1}\partial_2 T.
\]
As \( T \) is independent on \( z \), the matrix \( B^{[1]} \) of \( \nabla^{[1]} \) is given by
\[
B^{[1]} = T^{-1}B^{[1],(0)}T + zT^{-1}B^{[1],(1)}T,
\]
where \( B^{[1],(0)} \) and \( B^{[1],(1)} \) are constant.

**Lemma 54.** The gauge isomorphisms \( T \), which depend only on \( t_2 \), and map \( \nabla \) to a \((TE)\)-structure \( \nabla^{[1]} := T \cdot \nabla \), whose underlying \((T)\)-structure satisfies
\[
A_1^{[1]} = C_1, \quad A_2^{[1]} = FC_2 + zGE
\]
for functions \( F = F(t_2) \) and \( G = G(t_2) \), are of the form
\[
T = \begin{pmatrix}
k_0 & k_1 x \\
k_1 & k_0 x
\end{pmatrix}, \quad T = \begin{pmatrix}
k_1 & k_0 x \\
0 & k_0
\end{pmatrix}
\]
where \( k_0, k_1, k \in \mathbb{C}^* \). If \( T \) is given by the first formula \((148)\), then \( F = \frac{k_0}{k_1}(k - x)^2 y \), \( G = \frac{k_1 x}{k_0 (k - x)^2} \). If \( T \) is given by the second formula \((148)\), then \( F = \frac{k_0}{k_1} y \), \( G = \frac{k_1 x}{k_0} \). In both cases, \( F(0) \neq 0 \).

**Proof.** By a straightforward computation, the matrices \( T = T(t_2) \) which satisfy \( T^{-1}A_2 T = FC_2 \) and \( T^{-1}\partial_2 T = GE \) are of the form
\[
T = \begin{pmatrix}
qx + \tilde{q} \frac{F}{y} & x\tilde{q} \\
q & \tilde{q}
\end{pmatrix}, \quad T = \begin{pmatrix}
q & \tilde{q} \\
\frac{F}{y} & \frac{\tilde{F}}{y}
\end{pmatrix}
\]
where \( q, \tilde{q} \in \mathbb{C}\{t_2\} \) and
\[
\frac{d}{dt_2} \left( \frac{\tilde{q} F}{y} \right) + q \dot{x} = 0
\]
\[
\tilde{F} = q\dot{\tilde{F}}
\]
\[
q^2 \ddot{x} + q \frac{d}{dt_2} \left( \frac{\tilde{q} F}{y} \right) - \tilde{q} \frac{\tilde{F}}{y} = 0.
\]
Moreover, if \((150)\) are satisfied, then \( G = \frac{\tilde{q} F}{y} \). If \( q(0) \neq 0 \), we divide the third relation \((150)\) by \( q^2 \) and we obtain \( \frac{d}{dt_2} \left( \frac{\tilde{q} F}{y} \right) = -\dot{x} \), which implies that \( \tilde{q} F = qy(k - x) \) for \( k \in \mathbb{C} \). Using \( \tilde{q} F = q(k - x) \), the first relation \((150)\) implies
that $q = k_0$ is constant. The second relation (150) determines $\tilde{q}$ as $\tilde{q} = \frac{k_1}{k-x}$, for $k_1 \in \mathbb{C}^*$. The expressions for $T$, $F$, and $G$ follow. As $T$ is invertible and $x(0) = 0$, $k, k_0 \in \mathbb{C}^*$. If $q(0) = 0$ then $q = 0$ (otherwise $q(z) = z^r \eta(z)$ for $r \in \mathbb{Z}_{\geq 1}$ and $\eta \in \mathbb{C}\{t_2\}$ non-trivial. But writing $q$ in this way we obtain a contradiction in the third relation (150)). The case $q = 0$ can be treated similarly and leads to the second expression in (148) for $T$ and to $F, G$ as required. Since $x(0) = 0, k, k_0 \in \mathbb{C}^*$, we obtain that $F(0) \neq 0$ (in both cases).

Let $T$ be a gauge isomorphism as in Lemma 54. The underlying $(T)$-structure of $\nabla^{[1]} = T \cdot \nabla$ is of the form

$$A_1^{[1]} = C_1, \quad A_2^{[1]} = F(C_2 + \frac{G}{F} E) = \mu_2(C_2 + zfE)$$

where $\mu_2 \in \text{Aut}(\mathbb{C}, 0)$ satisfies $\dot{\mu}_2 = F$ and in the second expression for $A_2^{[1]}$ the function $\frac{G}{F}$ is written in terms of $\mu_2$, i.e. $\frac{G}{F} = f(\mu_2)$. (Remark that $\dot{\mu}_2(0) \neq 0$ since $F(0) \neq 0$). We obtain that $\nabla^{[1]}$ is the pull-back by $\mu(t_1, t_2) = (t_1, \mu_2(t_2))$ of the $(T)$-structure

$$\tilde{A}_1 = C_1, \quad \tilde{A}_2 = C_2 + zfE.$$  \hspace{1cm} (151)

Therefore, the underlying $(T)$-structure of $\nabla$ is isomorphic to the $(T)$-structure (151). In the following we will make suitable choices in Lemma 54 which lead to ’simplest’ expressions for the function $f$.

**Proposition 55.** i) If $B^\infty_{12} B^\infty_{21} = -\frac{1}{16}$, then the underlying $(T)$-structure of $\nabla$ is isomorphic to the $(T)$-structure given by

$$\tilde{A}_1 = C_1, \quad \tilde{A}_2 = C_2 + z\frac{(c_0)^2}{1-t_2} E.$$  \hspace{1cm} (152)

ii) If $B^\infty_{12} B^\infty_{21} \notin \{-\frac{1}{16}, \frac{3}{16}\}$, then the underlying $(T)$-structure of $\nabla$ is isomorphic to the $(T)$-structure given by

$$\tilde{A}_1 = C_1, \quad \tilde{A}_2 = C_2 + z\left(\frac{\lambda}{c_0}t_2 + 1\right)^2 - \frac{4}{1} E,$$  \hspace{1cm} (153)

where $\lambda := B^\infty_{21}(b-a) - 1$ and $a, b \in \mathbb{C}$ are defined in Lemma 52.

iii) If $B^\infty_{12} B^\infty_{21} = \frac{3}{16}$ then the underlying $(T)$-structure of $\nabla$ is isomorphic to the $(T)$-structure given by

$$\tilde{A}_1 = C_1, \quad \tilde{A}_2 = C_2 + z(c_0)^2 e^{-t_2} E.$$  \hspace{1cm} (154)
Proof. i) Let $T$ be the gauge isomorphism given by the first expression (148), with $k := 4c_0$ (and $k_0, k_1 \in \mathbb{C}^*$ arbitrary). Define $\gamma := \frac{b}{3}$. Using $F = \gamma(k - x)^2y$ (see Lemma 54), and the expressions of $x, y$ from (144), we obtain that

$$F = \gamma(4c_0 - x)^2y = 16\gamma(c_0)^3e^{-t_2} = \mu_2(t_2)$$

where $\mu_2 \in \text{Aut}(\mathbb{C}, 0)$ is given by $\mu_2(t_2) = 16\gamma(c_0)^3(1 - e^{-t_2})$. Then $e^{t_2} = \frac{1}{16(c_0)^3\gamma\mu_2}$ and

$$G = \frac{x}{\gamma^2(4c_0 - x)^3y} = \left(\frac{1}{16\gamma(c_0)^2}\right)^2e^{t_2} = \frac{1}{16\gamma c_0(16\gamma(c_0)^3 - \mu_2)}.$$

We obtain that the underlying $(T)$-structure of $\nabla$ is isomorphic to the $(T)$-structure $\hat{\nabla}$ given by

$$\hat{A}_1 = C_1, \hat{A}_2 = C_2 + \frac{z}{16\gamma c_0(16\gamma(c_0)^3 - t_2)}E.$$

For $\gamma = \frac{1}{16(c_0)^3}\gamma\mu_2$, we obtain the $(T)$-structure (152).

ii) Since $B_{21}^\infty B_{21}^\infty \not\in \{-\frac{1}{16}\frac{3}{16}\}$, $a \neq b$ and $B_{21}^\infty(b - a) \neq \pm 1$. Let $T$ be given by the first expression (148), with $k := a$ (and $k_0, k_1 \in \mathbb{C}^*$ arbitrary). Define $\gamma := \frac{b}{3}$. With functions $x, y$ given by (143), we obtain, by similar computations,

$$F = \gamma a^2 c_0 e^{(B_{21}^\infty(b - a) - 1)t_2}, \quad G = \frac{b B_{21}^\infty}{c_0 \gamma^2 a^3} e^{(1 + 2B_{21}^\infty(a - b))t_2}.$$

Define $\mu_2 \in \text{Aut}(\mathbb{C}, 0)$ by

$$\mu_2(t_2) := \frac{\gamma a^2 c_0}{B_{21}^\infty(b - a) - 1} e^{(B_{21}^\infty(b - a) - 1)t_2} - 1.$$

Then $F = \mu_2$ and

$$G = \frac{1}{\gamma a^2} \left(\frac{1}{\gamma a^2} \frac{B_{21}^\infty(b - a) - 1}{B_{21}^\infty(b - a) - 1}\right)^{1 + 2B_{21}^\infty(a - b)}.$$

We obtain that the underlying $(T)$-structure of $\nabla$ is isomorphic to the $(T)$-structure

$$\hat{A}_1 = C_1, \hat{A}_2 = C_2 + \frac{1}{\gamma a^2} \left(\frac{1}{\gamma a^2} \frac{B_{21}^\infty(b - a) - 1}{B_{21}^\infty(b - a) - 1}\right)^{1 + 2B_{21}^\infty(a - b)}.$$

(155)

For $\gamma := \frac{1}{a^2}$, the $(T)$-structure (155) coincides with (153).

iii) When $B_{12}^\infty B_{21}^\infty = \frac{3}{16}$, $B_{21}(b - a) = \pm 1$. Without loss of generality, we assume that $B_{21}^\infty(b - a) = 1$. The claim follows by a similar argument, by taking $T$ given by the first formula in (148), with $k = a$ and $k_1 = a^2 c_0$. □
We arrive at our main result from this section.

**Theorem 56.** Let $\nabla = \nabla^{\text{univ}, B_0, B_\infty}$ be a non-elementary Malgrange normal form of the second type, with $B_0^\alpha = cC_1 + c_0 C_2$ and $B_\infty = (B_{ij})$ (thus $B_{12}^\infty = c_0 \neq 0$ and $B_{11}^\infty - B_{22}^\infty = -\frac{1}{2}$). Let $\alpha := \frac{1}{2}(B_{11}^\infty + B_{22}^\infty)$.

i) If $B_{12}^\infty B_{21}^\infty = -\frac{1}{16}$, then $\nabla$ is isomorphic to the $(T E)$-structure $\hat{\nabla}$ given by

$$
\hat{A}_1 = C_1, \quad \hat{A}_2 = C_2 + \frac{z(c_0)^2}{1 - t_2} E
$$

$$
\hat{B} = (-t_1 + c + \alpha z)C_1 + (1 - t_2)C_2 + z(c_0)^2 E.
$$

(156)

ii) If $B_{12}^\infty B_{21}^\infty \notin \left\{-\frac{1}{16}, \frac{3}{16}\right\}$, then $\nabla$ is isomorphic to the $(T E)$-structure $\hat{\nabla}$ given by

$$
\hat{A}_1 = C_1, \quad \hat{A}_2 = C_2 + z\left(\frac{\lambda}{c_0} t_2 + 1\right)^{-2 + \frac{1}{4}} E
$$

$$
\hat{B} = (-t_1 + c + \alpha z)C_1 + (\lambda t_2 + c_0)C_2 - \frac{z}{2}(\lambda + 1)D + z c_0 \left(\frac{\lambda}{c_0} t_2 + 1\right)^{-1 - \frac{1}{4}} E.
$$

(157)

where $\lambda := B_{21}^\infty(b - a) - 1$ and $a, b \in \mathbb{C}$ are defined in Lemma 52.

iii) If $B_{12}^\infty B_{21}^\infty = \frac{3}{16}$, then $\nabla$ is isomorphic to the $(T E)$-structure $\hat{\nabla}$ given by

$$
\hat{A}_1 = C_1, \quad \hat{A}_2 = C_2 + z(c_0)^2 e^{-t_2} E
$$

$$
\hat{B} = (-t_1 + c + \alpha z)C_1 + C_2 - \frac{z}{2} D + z(c_0)^2 e^{-t_2} E.
$$

(158)

**Proof.** The idea of the proof is common to the three cases. Recall that the matrix $B$ of $\nabla$ is given by (147), with functions $x$, $y$ as in Lemma 52. Let $T$ be the gauge isomorphism used in the proof of Proposition 55. Recall that $T$ is given by the first formula in (148) (with various choices of constants $k, k_0, k_1$, according to the three cases of Proposition 55). The $(T E)$-structure $\nabla^{[1]} = T \cdot \nabla$ has matrix $B^{[1]}$ given by $B^{[1]} = T^{-1} B T$ and a straightforward computation shows that

$$
B^{[1]} = (-t_1 + c + \alpha z)C_1 + \frac{k_0}{k_1} \left( y(k - x)^2 + z \left( \frac{k}{2} + k^2 B_{21}^\infty - c_0 \right) \right) C_2
$$

$$
+ \frac{z}{2(k - x)} \left( -\frac{k}{2} + 2 c_0 \right) - x (2k B_{21}^\infty + \frac{1}{2}) \right) D
$$

$$
+ \frac{zk_1}{k_0(k - x)^2} \left( c_0 - x \left( \frac{1}{2} + B_{21}^\infty \right) \right) E.
$$

(159)
The choice of $k$ in the proof of Proposition 55 (in all three cases), implies that $\frac{k_0}{k_1} y (k-x)^2 = c_0 = 0$. Therefore, the coefficient of $C_2$ in $B^{[1]}$ reduces to $\frac{k_0}{k_1} y k \geq 1$. Define $\tilde{B}$ by $\tilde{B} := (\mu^{-1} \ast B^{[1]})$, where $\mu(t_1, t_2) = (t_1, \mu_2(t_2))$ and $\mu_2$ is the function (for each case) constructed in the proof of Proposition 55. The matrix $\tilde{B}$ is obtained by writing $B^{[1]}$ in terms of $\mu_2$ (and $t_1$). Together with the $(T)$-structures from Proposition 55, the matrices $\tilde{B}$ form $(TE)$-structures $\tilde{\nabla}$ isomorphic to $\nabla$ (in all three cases). Their associated functions $\tilde{b}_2$ are obtained by writing the coefficient $b_2^{[1]} = \frac{k_0}{k_1} y (k-x)^2$ of $C_2$ in the expression of $B^{[1]}$ in terms of $\mu_2$. Making the computations explicit we obtain that $\tilde{\nabla}$ have the expressions stated in Theorem 56.

To illustrate our argument we consider the case $B_{12}^{[1]} B_{21}^{[1]} = -\frac{1}{16}$. Then, from the proof of Proposition 52 $k = 4c_0$, $\frac{k_0}{k_1} = \gamma = \frac{1}{16(c_0)^2}$, $\mu_2(t_2) = 1 - e^{-t_2}$ and the underlying $(T)$-structure of $\tilde{\nabla}$ is given by the first line of (156). Using the expressions for $x$, $y$ given by (144), we obtain that

$$b_2^{[1]} = \frac{1}{16(c_0)^2} y (4c_0 - x)^2 = e^{-t_2} = 1 - \mu_2(t_2).$$

Thus, $\tilde{b}_2(t_2) = 1 - t_2$, which leads to the matrix $\tilde{B}$ given in (156).

**Definition 57.** A holomorphic normal form for $(TE)$-structures over $\mathcal{N}_2$ is a $(TE)$-structure which belongs either to the list of $(TE)$-structures from Theorem 18 or to the list of $(TE)$-structures from Theorem 56.

The next corollary summarises our discussion on the holomorphic classification.

**Corollary 58.** Any $(TE)$-structure over $\mathcal{N}_2$ is isomorphic to a holomorphic normal form.

It remains to establish when two holomorphic normal forms $\nabla$ and $\tilde{\nabla}$ are isomorphic. If $\nabla$ and $\tilde{\nabla}$ are isomorphic (and distinct), they are both elementary or both non-elementary. In the first case, $\nabla$ and $\tilde{\nabla}$ are as in Theorem 18 i) with $c_0 = 0$, ii) or iii). They are isomorphic if and only if they are formally isomorphic and this happens if and only if the conditions from Theorem 20 ii) are satisfied. In the second case, $\nabla$ and $\tilde{\nabla}$ are as in Theorem 18 i) with $c_0 \neq 0$ or as in Theorem 56. We shall associate to $\nabla$ (and $\tilde{\nabla}$) a constant $c_1$ (respectively, $\tilde{c}_1$) which will be used to establish when $\nabla$ and $\tilde{\nabla}$ are isomorphic. If $\nabla$ is of the form (153), we define $c_1 := -\frac{1}{16c_0}$; if $\nabla$ is of the form (157), we define $c_1 := \frac{1}{16c_0}$ (4$\lambda^2 + 8\lambda + 3$); if $\nabla$ is of the form (158), we define $c_1 := \frac{2}{16c_0}$. Finally, if $\nabla$ is as in Theorem 18 i) with $c_0 \neq 0$, we define $c_1 := 0$. In a similar way, we assign to $\tilde{\nabla}$ a constant $\tilde{c}_1$. 

50
Corollary 59. Let $\nabla$, $\tilde{\nabla}$ be two holomorphic normal forms, as in Theorem 18 i) or Theorem 56 (and $c_0 \tilde{c}_0 \neq 0$ when $\nabla$ and $\tilde{\nabla}$ belong to Theorem 18 i)). Let $c_1$ and $\tilde{c}_1$ be the constants associated to $\nabla$ and $\tilde{\nabla}$, as described above. Then $\nabla$ and $\tilde{\nabla}$ are isomorphic if and only if the constants $(c, \alpha, c_0)$ and $(\tilde{c}, \tilde{\alpha}, \tilde{c}_0)$ involved in their expressions satisfy $c = \tilde{c}$, $\alpha = \tilde{\alpha}$, $c_0 = \epsilon \tilde{c}_0$ where $\epsilon \in \{ \pm 1 \}$ and relations (120) and (121), with $c_1$ and $\tilde{c}_1$ defined above, are satisfied as well.

Proof. The constant $c_1$ associated to $\nabla$ using the above procedure coincides with the $(2,1)$-entry $B_{21}^\infty$ of the matrix $B_\infty$ from the Malgrange universal deformation $\nabla^{\text{univ}}, B_0, B_\infty$ isomorphic to $\nabla$. The same is true for $\tilde{c}_1$ and $\tilde{\nabla}$. The claim follows from Proposition 43.

6 Euler fields and $(TE)$-structures

As an application of the theory developed in the previous sections, we characterise the Euler fields on $N_2$ which are induced by $(TE)$-structures over $N_2$. As any isomorphism $f : (M_1, o_1, e_1, E_1) \rightarrow (M_2, o_2, e_2, E_2)$ between $F$-manifolds with Euler fields defines (by the pull-back $(\text{Id} \times f)^*$) an isomorphism between the spaces of $(TE)$-structures over $(M_2, o_2, e_2)$ and $(M_1, o_1, e_1)$, which induce $E_2$ and $E_1$ respectively, we may assume, without loss of generality, that the Euler fields are in the normal form provided by Theorem 15.

Proposition 60. All Euler fields on $N_2$, in the normal form provided by Theorem 15, are induced by a $(TE)$-structure over $N_2$, except those of the form

\[
E = (t_1 + c) \partial_1 + t_2^r(1 + c_1 t_2^{-1}) \partial_2, \quad r \in \mathbb{Z}_{\geq 3}, \quad c_1 \in \mathbb{C}
\]

\[
E = (t_1 + c) \partial_1 + t_2^2(1 + c_1 t_2) \partial_2, \quad c_1 \in \mathbb{C}^*.
\]

(160)

Proof. From the explicit expressions of the holomorphic normal forms, we obtain that the Euler fields induced by them are given by $E = (t_1 - c) \partial_1 + g(t_2) \partial_2$, where $c \in \mathbb{C}$ and $g$ takes one of the following forms: $g = 0$, $g = -\lambda t_2$ ($\lambda \in \mathbb{C}$), $g = -1$, $g = -t_2^2$, or $g \in \mathbb{C}\{t_2\}$ is a non-constant unit. Up to the action of $\text{Aut}(N_2)$, they cover all normal forms of Euler fields on $N_2$, except those stated in (160).

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