REFLECTION POSITIVITY AND SPECTRAL THEORY

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Abstract. We consider reflection-positivity (Osterwalder-Schrader positivity, O.S.-p.) as it is used in the study of renormalization questions in physics. In concrete cases, this refers to specific Hilbert spaces that arise before and after the reflection. Our focus is a comparative study of the associated spectral theory, now referring to the canonical operators in these two Hilbert spaces. Indeed, the inner product which produces the respective Hilbert spaces of quantum states changes, and comparisons are subtle.

We analyze in detail a number of geometric and spectral theoretic properties connected with axiomatic reflection positivity, as well as their probabilistic counterparts; especially the role of the Markov property. This view also suggests two new theorems, which we prove. In rough outline: It is possible to express OS-positivity purely in terms of a triple of projections in a fixed Hilbert space, and a reflection operator. For such three projections, there is a related property, often referred to as the Markov property; and it is well known that the latter implies the former; i.e., when the reflection is given, then the Markov property implies O.S.-p., but not conversely. In this paper we shall prove two theorems which flesh out a much more precise relationship between the two. We show that for every OS-positive system \((E_+, \theta)\), the operator \(E_+ \theta E_+\) has a canonical and universal factorization.

Our second focus is a structure theory for all admissible reflections. Our theorems here are motivated by Phillips’ theory of dissipative extensions of unbounded operators. The word “Markov” traditionally makes reference to a random walk process where the Markov property in turn refers to past and future: Expectation of the future, conditioned by the past. By contrast, our present initial definitions only make reference to three prescribed projection operators, and associated reflections. Initially, there is not even mention of an underlying probability space. This in fact only comes later.

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The notion “reflection-positivity” came up first in a renormalization question in physics: “How to realize observables in relativistic quantum field theory (RQFT)?” This is part of the bigger picture of quantum field theory (QFT); and it is based on a certain analytic continuation (or reflection) of the Wightman distributions (from the Wightman axioms). In this analytic continuation, Osterwalder-Schrader (OS) axioms induce Euclidean random fields; and Euclidean covariance. (See, e.g., [OS73, OS75, GJ79, GJ87, Jor02, JP13, JJ17, JL17].) For the unitary representations of the respective symmetry groups, we therefore change these groups as well: OS-reflection applied to the Poincaré group of relativistic fields yields the Euclidean group as its reflection. The starting point of the OS-approach to QFT is a certain positivity condition called “reflection positivity.”

Now, when it is carried out in concrete cases, the initial function spaces change; but, more importantly, the inner product which produces the respective Hilbert spaces of quantum states changes as well. What is especially intriguing is that, before reflection we may have a Hilbert space of functions, but after the time-reflection is turned on, then, in the new inner product, the corresponding completion, magically becomes a Hilbert space of distributions.

The motivating example here is derived from a certain version of the Segal–Bargmann transform (see Example 4.2). For more detail on the background and the applications, we refer to two previous joint papers [JO98] and [JO00], as well as [Kle77, Kle78, KLS82, Jor86, Jor87, Nee94, Hal00, AJP07, JT17].

Our present purpose is to analyze in detail a number of geometric properties connected with the axioms of reflection positivity, as well as their probabilistic counterparts; especially the role of the Markov property. This view also suggests two new theorems, to follow in the rest of the paper.

In rough outline: It is possible to express Osterwalder-Schrader positivity (O.S.-p.) purely in terms of a triple of projections in a fixed Hilbert space, and a reflection operator. For such three projections, there is a related property, often referred to as the Markov property. It is well known that the latter implies the former; i.e., when the reflection is given, then the Markov property implies O.S.-p., but not conversely.

In this paper we shall prove two theorems which flesh out a much more precise relationship between the two. The word “Markov” traditionally makes reference to a random walk process where the Markov property in turn refers to past and future: Expectation of the future, conditioned by the past (details below). By contrast, our present initial definitions only make reference to three prescribed projection operators. Initially, there is not even mention of an underlying probability space. All this comes later. Now if we are in the context of a random walk process, then such a process may or may not have the Markov property; which is now
instead defined relative to notions of past, present, and future, and the associated conditional expectations.

While our discussion of the Markov property is couched here in an axiomatic framework; and is motivated by our particular aims, we stress that Markov properties, Markov processes, and Markov fields form an active and very diverse area. While there are links from those directions to our present results, the connections are not always direct. For the readers benefit we have included the following citations [Nel58, Nel73a, Nel73b, Nel75, BDS16, KA17, LR17] on Markov/random fields.

In order to make our paper accessible to non-specialists, we have chosen to begin by recalling the fundamentals in the subject. This choice in turn helps us outline the general framework in the form we need it for what is to follow.

2. The geometry of reflections and positivity

Let $\mathcal{H}$ be a given Hilbert space, and let $U, \theta : \mathcal{H} \to \mathcal{H}$ be two unitary operators, such that:

\begin{align*}
\theta^2 &= I_{\mathcal{H}}, \ 	heta^* = \theta, \text{ and} \\
\theta U \theta &= U^*. 
\end{align*}

(2.1)

(2.2)

Note that (2.1) states that $\theta$ has spectrum equal to the two point set $\{\pm 1\}$. We think of (2.2) as a reflection symmetry for the given operator $U$. In this case, (2.2) states that $U$ is unitarily equivalent to its adjoint $U^*$, and so $U$ and its adjoint $U^*$ have the same spectrum, but, except for trivial cases, $U$ is not selfadjoint.

We further assume that there exists a closed subspace $\mathcal{H}_+ \subset \mathcal{H}$ s.t.

\begin{align*}
U \mathcal{H}_+ &\subset \mathcal{H}_+, \ 	ext{and} \\
\langle h_+, \theta h_+ \rangle &\geq 0, \ \forall h_+ \in \mathcal{H}_+. 
\end{align*}

(2.3)

(2.4)

If $E_+$ is the projection onto $\mathcal{H}_+$, then (2.4) is equivalent to

\begin{equation}
E_+ \theta E_+ \geq 0
\end{equation}

(2.5)

with respect to the usual ordering of operators (see Definition 2.5).

Remark 2.1. In our discussion of (2.2)-(2.3), we state things in the simple case of just a single unitary operator $U$, but our conclusions will apply mutatis mutandis also to the case when $U$ is instead a strongly continuous unitary representation of a suitable non-commutative Lie group $G$ (see Section 7 and the papers cited there). In the Lie group case, there is a distinguished one-parameter subgroup of $G$ corresponding to a choice of time-direction. Hence the corresponding restriction will be a unitary one-parameter group, and the forward direction will be the positive half-line $\mathbb{R}_+$, viewed as a sub-semigroup. If $G$ is a Lie group, we shall also be concerned with sub-semigroups. Condition (2.3) will refer to invariance of $\mathcal{H}_+$ under this sub-semigroup. In all these cases, we shall simply refer to $U$ with regards to (2.2)-(2.3), even if it is not a single unitary operator. In case of a single unitary operator $U$, of course by iteration we will automatically have a representation of the group $\mathbb{Z}$ of integers, and in this case the sub-semigroup will be understood to be $\mathbb{N}_0$.

Note on terminology. Given a fixed Hilbert space $\mathcal{H}$, we shall make use of the following identification between projections $P$ in $\mathcal{H}$, on the one hand, and the corresponding closed subspaces $P \mathcal{H} \subset \mathcal{H}$ on the other. By projection $P$, we mean an operator $P$ in $\mathcal{H}$ satisfying...
\[ P^2 = P = P^* \]. Conversely, if \( \mathcal{L} \subset \mathcal{H} \) is a fixed closed subspace, then by general theory, we know that there is then a unique projection, say \( Q \), such that \( Q\mathcal{H} = \mathcal{L} = \{ h \in \mathcal{H} : Qh = h \} \).

In some of our discussions below, there will be more than one Hilbert space, say \( \mathcal{H} \) and \( \mathcal{K} \); and they may arise inside calculations. In those cases, it will be convenient to mark the inner products and norms with subscripts, \( \langle \cdot, \cdot \rangle_{\mathcal{K}}, \| \cdot \|_{\mathcal{K}} \) etc.

In the discussion of reflection positivity, there will typically be three projections \( E_0, E_\pm \) at the outset, and the corresponding closed subspaces will be denoted, \( \mathcal{H}_0 := E_0 \mathcal{H}, \mathcal{H}_\pm := E_\pm \mathcal{H} \).

We shall denote such a system of projections \( (E_\pm, E_0) \) by \( \varepsilon \). If a reflection \( \theta \) (see (2.1)) maps \( \mathcal{H}_+ \) to \( \mathcal{H}_- \) (plus minus parity), we say that \( \theta \in \mathcal{R}(\varepsilon) \). If also (2.2) and (2.3) hold, we shall say that \( \theta \in \mathcal{R}(\varepsilon, U) \). (See Section 5 and Definition 5.4.)

2.1. Definitions and Lemmas. In our study of reflections, and reflection positivity, we shall need a number of fundamental concepts from the theory of operators in Hilbert space. While they are in the literature, they are not collected in a single reference. For readers not in operator theory, we include below those basic facts in the form they will be needed inside the paper. A new feature is the notion of signed quadratic forms and subspaces which are positive with respect to such a given signed quadratic form; see Lemma 2.7.

Definition 2.2. When \( U, \theta, \) and \( E_\pm \) satisfy these conditions, i.e., (2.1)-(2.5), we then say that Osterwalder-Schrader reflection positivity holds, abbreviated O.S.P.

Below we discuss the standard ordering of projections. What will be important is that this ordering may be stated in terms of anyone of six equivalent properties. Each one will be relevant for the applications to follow; to geometry, to spectral theory, and to analysis of conditional expectations. For the latter, see e.g., Example 5.10, and Section 7.1.

Definition 2.3 (Order on projections).

(i) A projection in a Hilbert space \( \mathcal{H} \) is an operator \( P \) satisfying \( P = P^2 = P^* \).

(ii) If \( E \) and \( P \) are two projections, we say that \( E \leq P \) (Def.) one of the following equivalent conditions holds:

(a) \( E\mathcal{H} \subseteq P\mathcal{H} \);
(b) \( \| Eh \| \leq \| Ph \|, \forall h \in \mathcal{H} \);
(c) \( \langle h, Eh \rangle \leq \langle h, Ph \rangle, \forall h \in \mathcal{H} \);
(d) \( PE = E \);
(e) \( EP = E \);
(f) for vectors \( h \in \mathcal{H} \), the following implication holds: \( Eh = h \implies Ph = h \).

Proof. This is standard operator theory, and can be found in books. See e.g. [JT17].

We shall need this ordering in an analysis of system (2.1)-(2.5). From the conditions \( \theta^* = \theta, \theta^2 = I_\mathcal{H} \) (reflection) we conclude that \( \theta = 2P - I_\mathcal{H} \) where \( P \) is the projection onto \( \{ h \in \mathcal{H} \mid \theta h = h \} \).

Lemma 2.4. Let \( \theta \) be a reflection, and let \( P \) be the projection such that \( \theta = 2P - I_\mathcal{H} \), and let \( E_0 \) be a projection; then TFAE:

(i) \( \theta E_0 = E_0 \);
(ii) $E_0 \leq P$, i.e., $E_0 h = h \implies \theta h = h$.

Proof. We have the following equivalences:
\[ \theta E_0 = E_0 \iff (2P - I_{\mathcal{H}}) E_0 = E_0 \iff PE_0 = E_0, \]
and the result now follows from the equivalent statements in Definition 2.3.

\textbf{Definition 2.5.} Fix a Hilbert space $\mathcal{H}$, and let $A$ and $B$ be two selfadjoint operators in $\mathcal{H}$. We say that $A \leq B$ iff (Def.) $\langle h, Ah \rangle \leq \langle h, Bh \rangle$, for all $h \in \mathcal{H}$.

Note that in case $A$ and $B$ are projections, this order relation agrees with that in Definition 2.3. Also $A \geq 0$, i.e., $\langle h, Ah \rangle \geq 0$, for all $h \in \mathcal{H}$, states that the spectrum of $A$ is a closed subset of $[0, \infty)$.

\textbf{Definition 2.6.} Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}_+ \times \mathcal{L}_-$ be two subspaces. Equip $\mathcal{L}_+ \times \mathcal{L}_-$ with the following signed quadratic form,
\[ \langle x, y \rangle_{\text{sig}} := \langle k_+, l_+ \rangle_{\mathcal{H}} - \langle k_-, l_- \rangle_{\mathcal{H}}, \tag{2.6} \]
for all $x = (k_+, k_-), y = (l_+, l_-)$ in $\mathcal{L}_+ \times \mathcal{L}_-$.

A subspace $\mathcal{P} \subset \mathcal{L}_+ \times \mathcal{L}_-$ is said to be positive iff for all $x = (k_+, k_-) \in \mathcal{P}$, we have
\[ \langle x, x \rangle_{\text{sig}} = \|k_+\|^2_{\mathcal{H}} - \|k_-\|^2_{\mathcal{H}} \geq 0. \tag{2.7} \]

\textbf{Lemma 2.7.} Let $\mathcal{H}, \mathcal{L}_+,$ and $\langle \cdot, \cdot \rangle_{\text{sig}}$ be as in Definition 2.6. Then a subspace $\mathcal{P} \subset \mathcal{L}_+ \times \mathcal{L}_-$ is positive if and only if there is a contractive linear operator $\mathcal{L}_+ \xrightarrow{C} \mathcal{L}_-$ (w.r.t. the original norm from $\mathcal{H}$) such that $\mathcal{P}$ is the graph of $C$, and so $\mathcal{P} = \{(k_+, Ck_+); k_+ \in \mathcal{L}_+\}$.
\[ \langle x, x \rangle_{\text{sig}} = \|k_+\|^2_{\mathcal{H}} - \|Ck_+\|^2_{\mathcal{H}}. \tag{2.8} \]

Proof. It is clear that the graph of a contraction is a positive subspace in $\mathcal{L}_+ \times \mathcal{L}_-$.

Conversely, suppose $\mathcal{P}$ is a given positive subspace; then
\[ \|k_+\|^2_{\mathcal{H}} - \|k_-\|^2_{\mathcal{H}} \geq 0, \forall (k_+, k_-) \in \mathcal{P}. \tag{2.9} \]
Using (2.9), we see that if $(k_+, k_-)$ and $(k_+, k_-')$ are both in $\mathcal{P}$, then $k_- = k_-';$ and so $k_+ \xrightarrow{C} Ck_+ = k_-$ defines a unique contractive operator $\mathcal{L}_+ \xrightarrow{C} \mathcal{L}_-$. As a result, we get that $\mathcal{P}$ is then the graph of this contraction $C$. \qed

\subsection{Reflections with given spaces $\mathcal{H}_+$ and $\mathcal{H}_-$.}

The material in the previous subsection will serve to give a characterization of families of reflections; they will be computed from positive subspaces relative to certain signed quadratic forms; see especially Corollary 2.11. Signed quadratic forms in an infinite dimensional setting were first studied systematically by M. G. Krein et al [GKn62, KnvS66], and R. S. Phillips [Phi61].

\textbf{Lemma 2.8.} Let $\mathcal{H}, \mathcal{H}_+, \mathcal{H}_0,$ and $\theta$ be as in Lemma 2.4. Let $P$ be the projection onto $\{h \in \mathcal{H}; \theta h = h\}$. Then
\[ \mathcal{H} = P \mathcal{H} \oplus (1 - P) \mathcal{H}. \tag{2.10} \]

The decomposition is orthogonal and therefore unique,
\[ h = u + v, \quad Pu = u, \quad Pv = 0; \tag{2.11} \]
i.e., the $\pm 1$ eigenspaces for $\theta$. 

Fix a closed subspace $\mathcal{H}_+$. The O.S.-positivity $\langle h_+, \theta h_+ \rangle \geq 0$, $\forall h_+ \in \mathcal{H}_+$, holds iff $\mathcal{H}_+$ is contained in the graph of a contractive operator

$$C : P\mathcal{H} \rightarrow P^\perp \mathcal{H}, \quad (2.12)$$

i.e., $\mathcal{H}_+ \subseteq \{ u + Cu ; u \in P\mathcal{H} \}$. 

**Proof.** Decompose vectors $h_+ \in \mathcal{H}_+$ as in (2.10)-(2.11), and assume O.S.-positivity, then

$$\langle h_+, \theta h_+ \rangle = \|u\|^2 - \|v\|^2 \geq 0; \quad h_+ = u \oplus v \text{ as in (2.11)}.$$

But then the assignment $C : u \mapsto v$ will define a contractive operator $C$ as stated in the lemma. Indeed, suppose $h_+ = u \oplus v$ is as in (2.13). Since $\|u\|^2 - \|v\|^2 \geq 0$; if $u = 0$, it follows that $v = 0$; and so $Cu := v$ is well defined as a contractive operator (see Lemma 2.7).

When a contraction $C : P\mathcal{H} \rightarrow P^\perp \mathcal{H}$ is given, then the corresponding closed subspace $\mathcal{H}_+$ is $\mathcal{H}_+ = \{ u + Cu ; u \in P\mathcal{H} \}$; and the reflection $\theta = \theta_C$ is determined by $\theta (u + Cu) := u - Cu$, and $\langle h_+, \theta h_+ \rangle = \|u\|^2 - \|Cu\|^2 \geq 0$ follows. (See also Theorem 2.15 below.)

Since the converse implication is clear, the lemma is proved. \hfill \Box

**Corollary 2.9.** Given $\mathcal{H}$, $\mathcal{H}_+$, and $\mathcal{H}_0$, as stated in Lemma 2.8. Then there is a bijection between the admissible reflections $\theta$, on the one hand, and partially defined contractions defined as in (2.12), on the other $C : \mathcal{H}_+ (\theta) \rightarrow \mathcal{H}_- (\theta)$ where

$$\mathcal{H}_+ (\theta) = \{ h \in \mathcal{H} ; \theta h = h \},$$

$$\mathcal{H}_- (\theta) = \{ k \in \mathcal{H} ; \theta k = -k \}.$$

**Corollary 2.10.** Let $\theta$ be a reflection, and let $P = \text{proj} \{ x \in \mathcal{H} ; \theta x = x \}$ so that $\theta = 2P - \text{I}_\mathcal{H}$. Let $C$ be the corresponding contraction.

Given a projection $E_0$ such that $E_0 \leq P$, then TFAE:

(i) $E_0 \leq E_+$; and

(ii) $E_0 \leq \ker(C)$.

**Proof.** We shall identify closed subspaces in $\mathcal{H}$ with the corresponding projections; see Definition 2.3. By Corollary 2.9, $\theta = \theta_C$ has the form

$$\theta (u + Cu) = u - Cu, \quad u \in P\mathcal{H},$$

where $C : P\mathcal{H} \rightarrow P^\perp \mathcal{H}$, is a uniquely determined contraction.

Let $x_0 \in E_0$; then $x_0 \in \mathcal{H}_+$ iff $\exists (\exists !) u \in P\mathcal{H}$ such that $x_0 = u + Cu$. So

$$0 = (u - x_0) + \sum_{\mathcal{H}} Cu_+ \in P\mathcal{H}$$

and both terms are zero; i.e., $u = x_0$, and $Cu = Cx_0 = 0$. The equivalence (i) $\iff$ (ii) now follows. \hfill \Box

**Corollary 2.11.** Let $\theta$ be a reflection in a Hilbert space $\mathcal{H}$, and let $P := \text{proj} \{ x \in \mathcal{H} ; \theta x = x \}$. Let $C : P\mathcal{H} \rightarrow P^\perp \mathcal{H}$ be the corresponding contraction. Assume the subspaces $\mathcal{H}_\pm$ satisfy $\mathcal{H}_+ = \{ x + Cx ; x \in P\mathcal{H} \}$, and $\mathcal{H}_- = \theta (\mathcal{H}_+) = \{ x - Cx ; x \in P\mathcal{H} \}$. We now have:

$$\mathcal{H}_+ \cap \mathcal{H}_- = \ker(C) = \mathcal{H}_+ \cap P \quad (2.14)$$

where we identify subspaces with the corresponding projections.
Proof. The implication “⊃” is immediate from Corollary 2.10. Now, let \( h \in \mathcal{H}_+ \cap \mathcal{H}_- \). Hence, there are vectors \( x, y \in P \mathcal{H} \) such that \( h = x + Cx = y - Cy \). Hence,

\[
y - x = (x + Cx) - (y - Cy); \tag{2.15}
\]
so both sides of (2.15) must be zero. We get \( y = x \), and \( Cx = 0 \); so \( h = x \in \ker(C) \) which is the desired conclusion (2.14).

\[ \square \]

Remark 2.12. In Corollary 2.11, we assumed \( \mathcal{H}_+ = \{x + Cx ; x \in P \mathcal{H}\} \); but this is not necessarily satisfied in the general formulation (see (2.3)-(2.4)).

For example, let \( \mathcal{H} = \mathbb{C}^3 \) with the standard orthonormal basis \( \{e_j\}_{j=1}^3 \). Set

\[
\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and} \quad \mathcal{H}_+ = \text{span} \left\{ e_1 + \frac{1}{2} e_3 \right\}.
\]

So \( \mathcal{H}_+ \) is 1-dimensional. The contraction \( C \) is given by

\[
C : \text{span} \{e_1\} \longrightarrow \text{span} \{e_3\}
\]

\[
Ce_1 = \frac{1}{2} e_3;
\]
yields \( \mathcal{H}_+ = \text{span} \{e_1 + Ce_1\} \). Then we have \( \theta = 2P - I \), where

\[
P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_+ \theta E_+ \geq 0, \quad \text{where}
\]

\( E_+ \) denotes the projection onto \( \mathcal{H}_+ \). It is clear that

\[ \mathcal{H}_+ \subseteq \{x + Cx ; x \in P \mathcal{H}\}, \quad \text{proper containment}, \]
since \( \dim P = 2 \).

Now, extend the contraction to \( C : P \mathcal{H} \longrightarrow P^\perp \mathcal{H} \) via

\[
Ce_2 = 0;
\]
then \( \ker(C) = \text{span} \{e_2\} \). Thus, we get \( \mathcal{H}_\pm = \text{span} \left\{ e_1 \pm \frac{1}{2} e_3 \right\} \), but

\[ 0 = \mathcal{H}_+ \cap \mathcal{H}_- = \mathcal{H}_+ \cap P \neq \ker(C) = \text{span} \{e_2\}. \]

Remark 2.13. In the general configuration the two projections \( E_\pm \) from Corollary 2.11 can be more complicated. If it is only assumed that the system \( (E_\pm, \theta) \) satisfies the O.S.-condition in (2.5), \( \mathcal{H}_\pm := E_\pm \mathcal{H} \), then the best that can be said about \( \mathcal{H}_+ \cap \mathcal{H}_- \) is the following:

Let \( Q := E_+ \wedge E_- \) be the projection onto \( \mathcal{H}_+ \cap \mathcal{H}_- \); then the following limit holds (in the strong operator topology):

\[
Q = \lim_{n \to \infty} (E_+ E_-)^n. \tag{2.16}
\]

This conclusion follows from a general fact in operator theory, see e.g. [Aro50, sect.12], and also [JT17]. Moreover, the limit in (2.16) is known to be monotone (decreasing.)
2.3. Maximal Reflections. As we saw that the specification of reflections may be stated in terms of certain positive subspaces (Lemma 2.8), it is natural to ask for the corresponding notion of maximal subspaces. We address this in the theorem to follow. The significance of maximality will be further addressed in the subsequent section.

**Definition 2.14.** Let $\mathcal{H}$ be a Hilbert space and $\theta$ a reflection on $\mathcal{H}$, see (2.1). Let $P = \text{proj} \{ x \in \mathcal{H} : \theta x = x \}$, so that $\theta = 2P - I_\mathcal{H}$. Set

$$\text{Sub}_{\text{OS}}(\theta) = \{ E_+ : E_+ \text{ is a projection in } \mathcal{H} \text{ s.t. } E_+ \theta E_+ \geq 0 \}. \quad (2.17)$$

As usual properties for projections have equivalent formulation for closed subspaces: In this case, we may identify elements in $\text{Sub}_{\text{OS}}(\theta)$ with closed subspaces $\mathcal{H}_+$ such that

$$\langle h_+, \theta h_+ \rangle \geq 0, \text{ for } \forall h_+ \in \mathcal{H}_+. \quad (2.18)$$

Set $\mathcal{H}_+ := E_+ \mathcal{H}$.

Now, combining the results above, we arrive at the following conclusions:

**Theorem 2.15.** Let $\mathcal{H}$, $\theta$, and $P$ be as stated, and consider the corresponding $\text{Sub}_{\text{OS}}(\theta)$ as in (2.17), or equivalently (2.18).

Then $\text{Sub}_{\text{OS}}(\theta)$ is an ordered lattice of projections, and it has the following family of maximal elements: Let $C : P \mathcal{H} \rightarrow P^\perp \mathcal{H}$ be a contractive operator, and set

$$\mathcal{H}_+(P, C) := \{ x + Cx : x \in P \mathcal{H} \}. \quad (2.19)$$

Then $\mathcal{H}_+(P, C)$ is maximal in $\text{Sub}_{\text{OS}}(\theta)$, and every maximal element in $\text{Sub}_{\text{OS}}(\theta)$ has this form for some contraction $C : P \mathcal{H} \rightarrow P^\perp \mathcal{H}$.

**Proof.** (i) If $E_+ \mathcal{H} = \mathcal{H}_+$, and $E_+^2 \mathcal{H} = \mathcal{H}_+^2$ are in $\text{Sub}_{\text{OS}}(\theta)$, it is clear that then so is $(E_+ \wedge E_+^2)(\mathcal{H}) = \mathcal{H}_+ \cap \mathcal{H}_+^2$.

(ii) Fix $E_+ \mathcal{H} = \mathcal{H}_+$, $E_+ \in \text{Sub}_{\text{OS}}(\theta)$. We have

$$\mathcal{H}_+ = P \mathcal{H}_+ + P^\perp \mathcal{H}_+, \quad (2.20)$$

and by the argument in the proof of Lemma 2.8, we conclude that there is a contractive operator $C : P \mathcal{H}_+ \rightarrow P^\perp \mathcal{H}_+$, and we get the representation

$$\mathcal{H}_+ = \{ x + Cx : x \in P \mathcal{H}_+ \}. \quad (2.21)$$

Let $\mathcal{H}_+^{(i)}$, $i = 1, 2$, be in $\text{Sub}_{\text{OS}}(\theta)$; and suppose $\mathcal{H}_+^{(1)} \subseteq \mathcal{H}_+^{(2)}$. Let $C_1$, $i = 1, 2$, be the corresponding contractions, i.e., $C_i : P \mathcal{H}_+^{(i)} \rightarrow P^\perp \mathcal{H}_+^{(i)}$, then it follows from (2.21) that the contraction $C_2$ is an extension of $C_1$.

(iii) By general theory, see e.g., [Phi61], any contraction $C$ as in (2.21) has contractive extensions $\tilde{C} : P \mathcal{H} \rightarrow P^\perp \mathcal{H}$. Setting $\mathcal{H}_+(P, \tilde{C})$ as in (2.19), we conclude that $\mathcal{H}_+ \subseteq \mathcal{H}_+(P, \tilde{C})$. Also see [JT17].

(iv) Converse, fix a contraction $D : P \mathcal{H} \rightarrow P^\perp \mathcal{H}$, and consider $\mathcal{H}_+(P, D)$, as in (2.19), the argument from the proofs of Lemma 2.8 and Corollary 2.9, shows that $\mathcal{H}_+(P, D)$ is maximal in $\text{Sub}_{\text{OS}}(\theta)$; and, conversely, every maximal element in $\text{Sub}_{\text{OS}}(\theta)$ has this form for some contraction $D : P \mathcal{H} \rightarrow P^\perp \mathcal{H}$. \(\square\)

**Example 2.16** (1-dimensional case of $\mathcal{H}_+$; see (2.21)). Fix $\theta = 2P - I_\mathcal{H}$; and consider $E_+ \in \text{Sub}_{\text{OS}}(\theta)$ with $\mathcal{H}_+ := E_+ \mathcal{H}$ as spanned by $h_+ = Pf + cP^\perp f$, $f \in \mathcal{H}$, $c \in \mathbb{C}$, $\| f \| = 1$;
and $\theta(h_+) = Pf - cP^f f$. Then

$$\|Pf\|_2^2 + \|P^f\|_2^2 = 1$$

so that $\langle h_+, \theta h_+ \rangle = \alpha - \|c\|^2 (1 - \alpha) \geq 0 \iff \|c\|^2 \leq \frac{\alpha}{1 - \alpha}$.

**Remark 2.17.** Our analysis of reflections $\theta$ and associated subspaces $\mathcal{H}_+$ is based on our Lemma 2.8 and Corollary 2.9 where we show that the admissible pairs $(\theta, \mathcal{H}_+)$ are determined by a certain family of partially defined contractive operators. This idea in turn is motivated by a parallel analysis of dissipative operators with dense domain, as pioneered by R.S. Phillips, see e.g., [Phi61]. In general, given $\theta$, there are many subspaces $\mathcal{H}_+$ which satisfy the O.S. positivity (2.5). In Corollaries 2.10-2.11 we concentrate on a particular case for $\mathcal{H}_+ = \mathcal{H}_+ (P)$ which is maximal; see the statement of Corollary 2.11.

Our present discussion is parallel to the theory of Phillips [Phi61] regarding dissipative extensions. Phillips’ theory is also formulated in the language of contractions. Since Phillips’ theory deals with unbounded operators with dense domain, the interesting statements are for infinite-dimensional Hilbert spaces. Our results in Section 5 below also deal with extensions, and there are many parallels between the arguments we use there, and those of Phillips in the case of Cayley transforms of dissipative operators.

### 3. New Hilbert Space from Reflection Positivity (renormalization)

Given a Hilbert space $\mathcal{H}$ and three closed subspaces (equivalently, systems of projections, $E$). In this very general setting, it is possible to give answers to the following questions: What are the conditions on a given system $E$ which admits reflections $\theta$? Suppose reflections exist, then fix $E$: What then is the variety of all compatible reflections $\theta$? Characterize the maximal reflections.

Given $E$, and an admissible reflection $(E, \theta)$, what are the unitary operators $U$ in $\mathcal{H}$ which define reflection symmetries with respect to $(E, \theta)$? Given $(E, \theta)$, what is the relationship between operator theory in $\mathcal{H}_+$, and that of the induced Hilbert space $\mathcal{K}$? Explore dichotomies at the two levels.

Let $\mathcal{H}$, $\mathcal{H}_+$, $\theta$, and $U$ be as above. In particular, we assume that $E_+ \theta E_+ \geq 0$. Set

$$\mathcal{N} = \ker (E_+ \theta E_+) = \{ h_+ \in \mathcal{H}_+ ; \langle h_+, \theta h_+ \rangle = 0 \}, \text{ and}$$

$$\mathcal{H}_+ = (\mathcal{H}_+/\mathcal{N})^\sim,$$

where “$^\sim$” in (3.2) means Hilbert completion with respect to the sesquilinear form: $\mathcal{H}_+ \times \mathcal{H}_+ \rightarrow \mathbb{C}$, given by

$$\langle h_+, h_+ \rangle_{\mathcal{K}} := \langle h_+, \theta h_+ \rangle,$$

a renormalized inner product; see (2.4)-(2.5).

Set $q(h_+) = \operatorname{class}(h_+) = h_+ + \mathcal{N}$, consider $q$ as a contractive operator,

$$\mathcal{H}_+ \xrightarrow{q} \mathcal{H}_+/\mathcal{N} \xrightarrow{(\mathcal{H}_+/\mathcal{N})^\sim} \text{ Hilbert completion} = \mathcal{K}.$$

**Remark.** Constructing physical Hilbert spaces entail completions, often a completion of a suitable space of functions. What can happen is that the completion may fail to be a Hilbert space of functions, but rather a suitable Hilbert space of distributions. Recall that a completion,
say $\mathcal{H}$, is defined axiomatically, and the “real” secret is revealed only when the elements in $\mathcal{H}$ are identified; see Example 4.2 below.

3.1. **Factorizations of $E_+\theta E_+$.** Given the basic framework of OS reflection positivity, the operator $E_+\theta E_+$ plays a crucial role since OS positivity is defined directly from this operator. We show that the operator $q$ from (3.4) offers a canonical factorization of $E_+\theta E_+ = q^* q$. But we further show that this factorization is universal; see Corollary 3.4.

**Theorem 3.1.** Let $\mathcal{H}, \theta, E_+$ be as above, $\mathcal{H}_+ := E_+\mathcal{H}$. Then TFAE:

(i) $E_+\theta E_+ \geq 0$, O.S.-positivity; and

(ii) there is a Hilbert space $\mathcal{L}$, and a bounded operator $B : \mathcal{H}_+ \to \mathcal{L}$ such that

$$E_+\theta E_+ = B^* B;$$

(3.5)

see Figure 3.1.

**Remark 3.2.** We show below that $\mathcal{H}_+ \xrightarrow{q} \mathcal{H}$ is a universal solution to the factorization problem (3.5) (see Corollary 3.4).

**Proof of Theorem 3.1.** The implication (i)$\implies$(ii) is contained in Lemma 3.3 below. Indeed, if (i) holds, then we may take $\mathcal{L} = \mathcal{H}$, and $B = q : \mathcal{H}_+ \to \mathcal{H}$; see (3.4).

Conversely; suppose (ii) holds (see Figure 3.1), then it is immediate that $E_+\theta E_+ = B^* B \geq 0$, by general theory; see Definition 2.5 above.

![Figure 3.1. A factorization of $E_+\theta E_+$.](image)

**Lemma 3.3.** Let $\mathcal{H}, \theta, E_+$ be as above. We assume further that $E_+\theta E_+ \geq 0$, i.e., O.S.-positivity holds. Set $\mathcal{H}_+ = E_+\mathcal{H}$. Let $\mathcal{H}$ be the induced Hilbert space

$$\mathcal{H} = (\mathcal{H}_+ / \{ h_+ : \langle h_+, \theta h_+ \rangle = 0 \})^\sim$$

(3.6)
as in (3.4), and let $q : \mathcal{H}_+ \to \mathcal{H}$ be the canonical contraction. Then the adjoint operator $q^* : \mathcal{H} \to \mathcal{H}_+$ is given by

$$q^* (q(h_+)) = E_+\theta h_+, \forall h_+ \in \mathcal{H}_+.$$  

(3.7)

In particular, the formula (3.7) defines $q^*$ unambiguously.

**Proof.** (i) We first show that the formula (3.7) defines an operator: We must show that if

$$\langle h_+, \theta h_+ \rangle = 0,$$

(3.8)

then $E_+\theta h_+ = 0$. But by Schwarz, for all $l_+ \in \mathcal{H}_+$, we have

$$|\langle l_+, \theta h_+ \rangle|^2 \leq \langle l_+, \theta l_+ \rangle \langle h_+, \theta h_+ \rangle = 0$$

by (3.8).
and so $E_+ \theta h_+ = 0$ as required in (3.7).

(ii) Since $q^*$ is contractive, it is determined uniquely by its values on a dense subspace of vectors in $\mathcal{K}$; in this case $\{q(h_+); h_+ \in \mathcal{H}_+\}$. 

(iii) It remains to verify that

$$\langle q^*(q(h_+)), l_+ \rangle_{\mathcal{K}} = \langle E_+ \theta h_+, l_+ \rangle_{\mathcal{K}} = \langle h_+, \theta l_+ \rangle_{\mathcal{K}} = \langle q(h_+), q(l_+) \rangle_{\mathcal{K}} ,$$

$\forall h_+, l_+ \in \mathcal{H}_+$. Details:

$$LHS_{(3.9)} = \langle E_+ \theta h_+, l_+ \rangle_{\mathcal{K}} = \langle \theta h_+, E_+ l_+ \rangle_{\mathcal{K}} = \langle h_+, \theta l_+ \rangle_{\mathcal{K}} = RHS_{(3.9)}$$

where we used the assumptions (2.1) and (2.5). In the computation, we omitted the subscript $\mathcal{H}$ in the inner products.

□

\[\text{Corollary 3.4.} \quad \text{The solution } q : \mathcal{H}_+ \to \mathcal{K} \text{ to the factorization problem } E_+ \theta E_+ = q^* q \text{ (see (3.5)), in the O.S.-p. case, is universal in the sense that if } \mathcal{H}_+ \xrightarrow{B} \mathcal{L} \text{ is any solution to (3.5) in Theorem 3.1, then there is a unique isomorphism } \mathcal{K} \xrightarrow{b} \mathcal{L} \text{ such that } b q = B, \text{ see Figure 3.2; and } b^* b = I_{\mathcal{K}}, \text{ so } b \text{ is isometric.}\]

\[\text{Proof.} \quad \text{Let } \mathcal{H}_+ \xrightarrow{B} \mathcal{L} \text{ be a solution to (3.5) in Theorem 3.1; we then define the isomorphism } b \text{ (so as to complete the diagram in Figure 3.2) as follow:}
$$b(q(h_+)) := B(h_+).$$

\[\text{Now this defines an operator } b : \mathcal{K} \to \mathcal{L}, \text{ since if } q(h_+) = 0, \text{ then } 0 = q^* q(h_+) = E_+ \theta E_+ = B^* B(h_+), \text{ so } 0 = \langle h_+, B^* B h_+ \rangle = \|B h_+\|^2, \text{ and so } B h_+ = 0 \text{ as required.}
$$

Now it is immediate from (3.10), that this operator $b : \mathcal{K} \to \mathcal{L}$ has the desired properties, in particular that the universality holds; see Figure 3.2. □

\[\text{Lemma 3.5.} \quad \text{Let } \mathcal{H} \text{ be a Hilbert space, and } \theta \text{ a reflection in } \mathcal{H} \text{ (see (2.1)). Let } P := \text{proj } \{x \in \mathcal{H} : \theta x = x\}, \text{ so } \theta = 2P - I_{\mathcal{K}}. \text{ Let } \mathcal{K} \text{ be the new Hilbert space in (3.4). Let } C : P \mathcal{H} \to P^\perp \mathcal{H} \text{ be the contraction, such that}
$$\mathcal{H}_+ = \{x + C x; x \in P \mathcal{H}\},$$

\[\text{and } \theta (x + C x) = x - C x; \text{ then for } h_+ = x + C x, \text{ we have}
$$\langle h_+, \theta h_+ \rangle_{\mathcal{K}} = \|h_+\|^2_{\mathcal{K}} = \left\|\left(I_{\mathcal{K}} - C^* C\right)^{1/2} x \right\|^2_{\mathcal{K}} .$$

\]
Proof. By $\mathcal{H}$ we refer here to the completion (3.4); see also Figure 3.3. For the LHS in (3.12), we have
\[
\langle h_+, \theta h_+ \rangle = \langle x + Cx, x + Cx \rangle = \left\| x \right\|^2 - \left\|Cx \right\|^2 = \left\| x \right\|^2 - \langle x, C^*Cx \rangle = \langle x, (I - C^*C)x \rangle = \left\| (I - C^*C)^{\frac{1}{2}} x \right\|^2 = \text{RHS (3.12)},
\]
where we have dropped the subscript $H$ in the computation. \hfill \square

Remark 3.6. The conclusion in Lemma 3.5 states that the range $\text{Ran} \left( (I - C^*C)^{\frac{1}{2}} \right)$ is a realization of the induced Hilbert space $\mathcal{K}$ in (3.4), so
\[
\| q (h_+) \|_{\mathcal{K}} = \left\| (I - C^*C)^{\frac{1}{2}} x \right\|_{\mathcal{H}}
\]
where $h_+ = x + Cx, x \in P\mathcal{H}.$

Lemma 3.7. Let the setting be as above, see (2.1)-(2.3). Then $\tilde{U} : \mathcal{H} \to \mathcal{H}$, given by
\[
\tilde{U} (\text{class } h_+) = \text{class} (Uh_+), h_+ \in \mathcal{H}_+
\]
where class $h_+$ refers to the quotient in (3.1), is selfadjoint and contractive (see Figure 3.3).

Proof. (See [Kle77, Jor86, Jor87, JO98, Jor02].) Despite the fact that proof details in one form or the other are in the literature, we feel that the spectral theoretic features of the argument have not been stressed; at least not in a form which we shall need below.

Denote the “new” inner product in $\mathcal{K}$ by $\langle \cdot, \cdot \rangle_{\mathcal{K}},$ and the initial inner product in $\mathcal{H}$ by $\langle \cdot, \cdot \rangle$.

$\tilde{U}$ is symmetric: Let $x, y \in \mathcal{H}_+$, then
\[
\langle x, \tilde{U} y \rangle_{\mathcal{K}} = \langle x, \theta U y \rangle = \langle x, U^* \theta y \rangle = \langle U x, \theta y \rangle = \langle \tilde{U} x, y \rangle_{\mathcal{K}}
\]
which is the desired conclusion.

$\tilde{U}$ is contractive: Let $x \in \mathcal{H}_+$, then
\[
\left\| \tilde{U} x \right\|^2_{\mathcal{K}} = \langle U x, \theta U x \rangle = \langle Ux, U^* \theta x \rangle = \left\langle \tilde{U}^2 x, x \right\rangle_{\mathcal{K}} \leq \left\| \tilde{U}^2 x \right\|_{\mathcal{K}} \cdot \left\| x \right\|_{\mathcal{K}} \quad \text{(by Schwarz in } \mathcal{K})
\]
\[
\leq \left\| \tilde{U}^4 x \right\|_{\mathcal{K}}^{\frac{1}{2}} \cdot \left\| x \right\|_{\mathcal{K}}^{1 + \frac{1}{2}} \quad \text{(by the first step)}
\]
\[
\leq \left\| \tilde{U}^{2^{n+1}} x \right\|_{\mathcal{K}}^{\frac{1}{2^n}} \cdot \left\| x \right\|_{\mathcal{K}}^{1 + \frac{1}{2} + \cdots + \frac{1}{2^n}}. \quad \text{(by iteration)}
\]
By the spectral-radius formula, $\lim_{n \to \infty} \left\| \tilde{U}^{2^n} x \right\|_{\mathcal{K}}^{\frac{1}{2^n}} = 1$; and we get $\left\| \tilde{U} x \right\|^2_{\mathcal{K}} \leq \left\| x \right\|^2_{\mathcal{K}}$, which is the desired contractivity. \hfill \square
Remark 3.8. In the proof of Lemma 3.7, we have made an identification:

\[ \mathcal{H}_+ \ni x \leftrightarrow q(x) \in \mathcal{K}, \]

see (3.4). So the precise vectors are as follows: \( \tilde{U} q(x) = q(Ux), \ (x \in \mathcal{H}_+) \); see Figure 3.3. The proof is in two steps:

Step 1. We verify the two conclusions for \( \tilde{U} \) (symmetry and contractivity) but only initially for the dense space of vectors in \( \mathcal{H} : \{ q(x) : x \in \mathcal{H}_+ \} \).

Step 2. Having the two properties verified on a dense subspace in \( \mathcal{K} \), it follows that the same conclusions will hold also on \( \mathcal{K} := \) completion of \( \{ q(x) : x \in \mathcal{H}_+ \} \). The reason is that the two properties are preserved by passing to limits; now limit in the \( \mathcal{K} \)-norm.

Lemma 3.9. Let \( \mathcal{H}, \mathcal{H}_+ \), and \( \theta \) be as above. Set

\[ \mathcal{A}_+ := \{ U \in \mathcal{H} \to \mathcal{H}, \text{ bounded operators,} \]

\[ U \mathcal{H}_+ \subset \mathcal{H}_+ \ (E_+ U E_+ = U E_+) \text{, and } \theta U = U^* \theta \}, \]

then \( U, V \in \mathcal{A}_+ \implies UV \in \mathcal{A}_+ \), and (UV)\( ^\sim = \tilde{U} \tilde{V} \), where \( \tilde{U} \) is determined by

\[ \tilde{U} (q(h_+)) = q(Uh_+), \ \forall h_+ \in \mathcal{H}_+. \]

Proof. Immediate from Lemma 3.7. \( \square \)

Lemma 3.10. Let \( \mathcal{H} \) be a fixed Hilbert space with subspaces \( \mathcal{H}_+ \) and \( \mathcal{H}_0 \). Let \( E_+ \) and \( E_0 \) denote the respective projections. Let \( \theta \) be a reflection, i.e., \( \theta^2 = 1_{\mathcal{H}}, \theta^* = \theta \). Assume

\[ E_- \theta E_+ = \theta E_+; \]

\[ E_+ \theta E_- = \theta E_-; \text{ and} \]

\[ \theta E_0 = E_0. \] \hspace{1cm} (3.14)

(i) Suppose \( \theta : \mathcal{H}_+ \to \mathcal{H}_- \) is onto. Then we have the following equivalence

\[ E_+ \theta E_+ \geq 0 \iff E_- \theta E_- \geq 0. \] \hspace{1cm} (3.15)
(ii) Suppose (i) holds, then we get two completions
\[ \mathcal{K}_\pm := (\mathcal{H}_\pm / \{h_\pm : (h_\pm, \theta h_\pm) = 0\})^\sim, \]  
see (3.4) above. Then \( \theta \) induces two isometries \( \tilde{\theta} : \mathcal{K}_+ \rightarrow \mathcal{K}_- \), \( \tilde{\theta} : \mathcal{K}_- \rightarrow \mathcal{K}_+ \),

(iii) In general, the isometries from (ii) are not onto. Indeed, \( \tilde{\theta} : \mathcal{K}_+ \rightarrow \mathcal{K}_- \) is onto iff \( H_- \ominus \theta H_+ = 0 \); and \( \tilde{\theta} : \mathcal{K}_- \rightarrow \mathcal{K}_+ \) is onto iff \( H_+ \ominus \theta H_- = 0 \).

Proof. The key step in the proof of the lemma is (3.15). Indeed we have the following:
\[ E_+ \theta E_+ \geq 0; \]
\[ \downarrow \]
\[ \langle h_+, \theta h_+ \rangle \geq 0, \forall h_+ \in \mathcal{H}_+; \]
\[ \downarrow \]
\[ \langle \theta h_+, \theta^2 h_+ \rangle \geq 0, \forall h_+ \in \mathcal{H}_+; \]
\[ \downarrow \]
\[ \langle h_-, \theta h_- \rangle \geq 0, \forall h_- = \theta (h_+) \in \mathcal{H}_-, \]
where we used assumption (3.14) above.

Moreover, for all \( h_+ \in \mathcal{H}_+ \), we have:
\[ \|\text{class} (\theta h_+)\|_{\mathcal{K}_-}^2 = \langle \theta h_+, \theta^2 h_+ \rangle \]
\[ = \langle h_+, \theta h_+ \rangle = \|\text{class} (h_+)\|_{\mathcal{K}_+}^2. \]

The remaining part of the proof is left to the reader. \( \square \)

We now turn to a closer examination of the unitary reflection operator \( U \) from (2.1)-(2.3). Given \( \theta \) as in (2.1), i.e., \( \theta = \theta^* \), \( \theta^2 = I_{\mathcal{H}} \); we assume that \( \mathcal{H}_\pm \) are two closed subspaces in \( \mathcal{H} \) such that \( \theta \mathcal{H}_+ \subset \mathcal{H}_- \); or, equivalently, \( E_- \theta E_+ = \theta E_+ \), where \( E_\pm \) denote the respective projection for the corresponding subspaces \( \mathcal{H}_\pm \); i.e.,
\[ \mathcal{H}_\pm = \{h_\pm \in \mathcal{H} : E_\pm h_\pm = h_\pm\}. \]  
(3.17)

Finally, we shall assume that the O.S.-positivity condition \( E_+ \theta E_+ \geq 0 \) holds; and so we are in a position to apply Lemma 2.8 and Corollary 2.9 above.

A given unitary operator \( U \) in \( \mathcal{H} \) is said to be a reflection-symmetry iff (Def.)
\[ \theta U \theta = U^*; \]
(3.18)
\[ U \mathcal{H}_+ \subseteq \mathcal{H}_+ \] (equivalently, \( E_+ U E_+ = U E_+ \))
(3.19)

**Theorem 3.11.** Let \( \mathcal{H} \), \( \mathcal{H}_\pm \), \( \theta \), and \( U \) be as above, i.e., we are assuming O.S.-positivity; and further that \( U \) satisfies (3.18)-(3.19). Let \( P \) be the projection onto \{\( h \in \mathcal{H} \) : \( \theta h = h \}\}, i.e., we have \( \theta = 2P - I_{\mathcal{H}} \).

(i) Then
\[ PUE_+ = PU^* \theta E_+. \]  
(3.20)

(ii) If \( C : P \mathcal{H} \rightarrow P_- \mathcal{H} \) denotes the contraction from Lemma 2.8 and Corollary 2.9, then there is a unique operator \( U_P : P \mathcal{H} \rightarrow P \mathcal{H} \) such that \( U_P = PUP \); and, if
\[ h_+ = x + Cx, \; x \in P\mathcal{H}, \quad \text{then} \]
\[ \left\| \tilde{U} q (h_+) \right\|_{\mathcal{H}}^2 = \left\| U_P x \right\|_{\mathcal{H}}^2 - \left\| CU_P x \right\|_{\mathcal{H}}^2. \]  

(iii) In particular, since \( \tilde{U} \) is contractive by Lemma 3.7, we have
\[ \left\| U_P x \right\|_{\mathcal{H}}^2 - \left\| CU_P x \right\|_{\mathcal{H}}^2 \leq \left\| x \right\|_{\mathcal{H}}^2 - \left\| Cx \right\|_{\mathcal{H}}^2, \quad \forall x \in P\mathcal{H}. \]

Proof. Note that (i) is immediate from (2.2) and Corollary 2.9.

The first half is immediate from definition of the contraction \( C \) from Lemma 2.8. For \( h_+ = x + Cx, \; x \in P\mathcal{H} \), we have
\[ \langle h_+, \theta h_+ \rangle_{\mathcal{H}} = \left\| q \left( h_+ \right) \right\|_{\mathcal{H}}^2 = \left\| x \right\|_{\mathcal{H}}^2 - \left\| Cx \right\|_{\mathcal{H}}^2, \]
and
\[ \left\| \tilde{U} (q (h_+)) \right\|_{\mathcal{H}}^2 = \left\| q (U h_+) \right\|_{\mathcal{H}}^2 = \left\| U_P x \right\|_{\mathcal{H}}^2 - \left\| CU_P x \right\|_{\mathcal{H}}^2; \]
and eq. (3.21) in (ii) follows.

Now (iii) is immediate from (i)-(ii) combined with the fact that \( \tilde{U} \) is contractive in \( \mathcal{H} \); see Lemma 3.7.

\begin{corollary}
Let \( \mathcal{H}, \; \mathcal{H}_+ \), \( \mathcal{H}_0, \; E_\pm, \; E_0, \; \theta, \; \text{be as in the statement of Lemma 3.10}. \) Let \( \mathcal{H}_{\pm} \) be the corresponding induced Hilbert spaces, see (3.16). Now set
\[ \mathcal{H}_{\pm}^{ex} = \text{closed span of } \{ h_0 + h_+ : h_0 \in \mathcal{H}_0, \; h_+ \in \mathcal{H}_\pm \}, \]  
and let \( E_{\pm}^{ex} \) denote the corresponding projections, i.e., \( E_{\pm}^{ex} := E_0 \lor E_\pm \). Then the following analogies of (3.14) hold:
\[ E_{\pm}^{ex} \theta E_{\pm}^{ex} = \theta E_{\pm}^{ex}; \quad \text{and} \]
\[ E_{\pm}^{ex} \theta E_{\pm}^{ex} = \theta E_{\pm}^{ex}. \]  
Moreover, we have the implication
\[ E_+ \theta E_+ \geq 0 \implies E_+^{ex} \theta E_+^{ex} \geq 0, \]  
if and only if
\[ \langle h_+, h_0 \rangle \leq \langle h_+, \theta h_+ \rangle \left\| h_0 \right\|^2, \quad \forall h_+ \in \mathcal{H}_+, \forall h_0 \in \mathcal{H}_0. \]  

Proof. By Lemma 3.10, it is easy to prove one of the two formula (3.23)-(3.24).

In detail, we must show that if \( h_0 \in \mathcal{H}_0, \; h_+ \in \mathcal{H}_+, \) then \( \theta \langle h_0 + h_+ \rangle \in \mathcal{H}_{\pm}^{ex}; \) see (3.22). But this is clear since
\[ \langle h_0 + h_+, \theta h_+ \rangle = \langle h_0 + h_+, \theta h_+ \rangle = h_0 + \theta h_+, \]  
and \( \theta h_+ \in \mathcal{H}_+ \) by (3.14). We also used \( \theta h_0 = h_0 \) which is (ii) in Lemma 2.4.

The second conclusion follows from this, since if \( \langle h_+, \theta h_+ \rangle \geq 0, \; \forall h_+ \in \mathcal{H}_+ \); then
\[ \langle h_+ + h_0, \theta (h_+ + h_0) \rangle = \langle h_+ + h_0, \theta h_+ + h_0 \rangle \]  
by (3.27) = \langle h_+ + h_0, \theta h_+ \rangle + \langle h_+, h_0 \rangle + \langle h_0, \theta h_+ \rangle + \left\| h_0 \right\|^2.

Now use \( \langle h_0, \theta h_+ \rangle = \langle h_0 h_+ \rangle = \langle h_0, h_+ \rangle \), and the result follows.

Remark 3.13. In the statement of Corollary 3.12, we impose the technical assumption (3.26).

The following example shows that this restricting condition (3.26) does not always hold; i.e.,
that Corollary 3.12 cannot be strengthened.
Example 3.14 (Also see Remark 2.12). Let $\mathcal{H} = \mathbb{C}^3$ with standard orthonormal basis $\{e_j\}_{j=1}^3$. Consider the reflection

$$\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and set

$$\mathcal{H}_+ = \text{span}\left\{ e_1 + \frac{1}{2} e_3 \right\},$$

$$\mathcal{H}_- = \text{span}\left\{ e_1 - \frac{1}{2} e_3 \right\},$$

$$\mathcal{H}_0 = \text{span}\left\{ e_1 \right\}.$$

For $h_+ := e_1 + \frac{1}{2} e_3$, and $h_0 := e_1$, we get $|\langle h_+, h_0 \rangle|^2 = 1$, but

$$\langle h_+, \theta h_+ \rangle \|h_0\|^2 = \left\langle e_1 + \frac{1}{2} e_3, e_1 - \frac{1}{2} e_3 \right\rangle \|e_1\|^2 = \frac{3}{4}.$$

Hence condition (3.26) does not hold.

Note that $h_+ - h_0 \in \mathcal{H}_0^{\text{ex}}$, and

$$\langle h_+ - h_0, \theta (h_+ - h_0) \rangle = \left\langle \frac{1}{2} e_3, -\frac{1}{2} e_3 \right\rangle = -\frac{1}{4} < 0;$$

i.e., the positivity condition $E_+^{\text{ex}} \theta E_+^{\text{ex}} \geq 0$ in (3.25) is not satisfied.

Corollary 3.15. Let $\mathcal{H}$, $\theta$, and $\mathcal{H}_0$, $\mathcal{H}_\pm$ be as in Corollary 3.12, assume (3.26), and let $\mathcal{H}_\pm^{\text{ex}}$ be the corresponding induced Hilbert spaces; see (3.22) applied to $\mathcal{H}_\pm^{\text{ex}}$. Then the two quotient mappings $\mathcal{H}_0 \to \mathcal{H}_\pm^{\text{ex}}$ are isometric.

Proof. Immediate.

3.2. Contractive Inclusions. As sketched in Figure 4.2 below, there are three subspaces naturally associated with the geometry of a given reflection, $\mathcal{H}_0$, $\mathcal{H}_+$, and $\mathcal{H}_-$. The last two of these are determined naturally and directly from the given reflection $\theta$. The role of the subspace $\mathcal{H}_0$ is more subtle, and its role is more crucial in connection with the Markov property (see Definition 5.4 below). Below we specify the possibilities for $\mathcal{H}_0$; see especially the corollary to follow.

Corollary 3.16. Let $\mathcal{H}_+$, $\mathcal{H}_0$, and $\theta$ be as in Corollary 3.12, and assume $E_+ \theta E_+ \geq 0$ (i.e., O.S.-positivity). Then TFAE:

1. $|\langle h_+, h_0 \rangle|^2 \leq \langle h_+, \theta h_+ \rangle \|h_0\|^2$, $\forall h_0 \in \mathcal{H}_0$, $\forall h_+ \in \mathcal{H}_+$ (see (3.26));

2. $\exists l : \mathcal{H}_0 \to \mathcal{H} = (\mathcal{H}_+/\ker)^\sim$ which is linear bounded and contractive, i.e.,

$$\|l(h_0)\|_{\mathcal{H}} \leq \|h_0\|$, $\forall h_0 \in \mathcal{H}_0$;

(3.28)

we say that $\mathcal{H}_0$ is contractively contained in $\mathcal{H}$, and (3.29) holds.

Proof. (i)$\Rightarrow$(ii). Assume (i), then for $\forall h_0 \in \mathcal{H}_0$ fixed, the map $h_+ \mapsto \langle h_0, h_+ \rangle$ is a bounded linear functional, so by Riesz and (i) $\exists l : \mathcal{H}_0 \to \mathcal{H} = \mathcal{H}^*$ (the Hilbert space $\mathcal{H}$ is selfdual) s.t.

$$\langle h_0, h_+ \rangle = \langle l(h_0), q(h_+) \rangle_{\mathcal{H}}, \forall h_+ \in \mathcal{H}_+.$$

(3.29)
The inner product in $\mathcal{H}$ is denoted without subscript, but the $\mathcal{K}$-inner product is denoted $\langle \cdot, \cdot \rangle_{\mathcal{K}}$, so $\langle q(l_+), q(h_+) \rangle_{\mathcal{K}} = \langle l_+, \theta h_+ \rangle_{\mathcal{H}}$, $\forall l_+, h_+ \in \mathcal{H}_+$. 

By (i) and (3.29), we get $\|l(h_0)\|_{\mathcal{K}} \leq \|h_0\|$, $\forall h_0 \in \mathcal{H}_0$, which is the assertion in (ii).

(ii) $\implies$ (i). Assume (ii), and compute $|\langle h_+, h_0 \rangle|^2$ in (i). We have

\[
|\langle h_+, h_0 \rangle|^2 = |\langle l(h_0), q(h_+) \rangle_{\mathcal{K}}|^2 \\
\leq \|l(h_0)\|_{\mathcal{K}}^2 \|q(h_+)\|_{\mathcal{K}}^2 \\
\leq \|h_0\|^2 \langle h_+, \theta h_+ \rangle_{\mathcal{K}} \text{ which is (i).}
\]

Corollary 3.17. Let $\mathcal{H}$, $\mathcal{H}_\pm$, $\mathcal{H}_0$, and $\theta$ be as described above, and let $E_\pm$, $E_0$ be the corresponding projections. Introduce $\mathcal{E}_{\pm}^c$ as in Corollary 3.12. Then the following implication holds:

\[
E_+ E_0 E_- = E_+ E_- \text{ (the Markov property)} \\
\Downarrow \\
E_{\pm}^c E_0 = E_{\pm}^c E_{\pm}^c.
\]

Proof. We have

\[
E_+ E_0 E_- = E_+ E_- \implies E_{\pm}^c E_0 E_{\pm}^c = E_{\pm}^c E_{\pm}^c \iff E_+ E_0 = E_{\pm}^c E_{\pm}^c
\]

which proves the corollary. We used $E_0 \leq E_{\pm}^c$, so $E_0 E_{\pm}^c = E_0$. □

Remark 3.18. The purpose of Corollary 3.17 is a version (see (3.30)) of the Markov property which is closer to the one used for Markov processes; see Section 7.

4. Unitary operators, symmetries, and reflections

In this section we introduce certain unitary representations which are given to act on the fixed Hilbert space. So we consider a given Hilbert space $\mathcal{H}$ which carries a reflection symmetry (in the sense of Osterwalder-Schrader) as defined in Section 2. If the unitary representation under consideration, say $U$, is a representation of a group $G$, then reflection-symmetry will refer to a suitable semigroup $S$ in $G$, so a sub-semigroup. The setting is of interest even in the three cases when $G$ is $\mathbb{Z}$, $\mathbb{R}$, or some Lie group from quantum physics. In the cases $G = \mathbb{Z}$, or $\mathbb{R}$, the semigroups are obvious, and, in each case, they define a causality. (The case $G = \mathbb{Z}$ is simply the study of a single unitary operator.) Nonetheless, the choice of semigroup in the case when $G$ is a Lie group is more subtle; see Section 7 below. However, many of the important spectral theoretic properties may be developed initially in the cases $G = \mathbb{Z}$, or $\mathbb{R}$, where the essential structures are more transparent.

Lemma 4.1. Let $\{U_t\}_{t \in \mathbb{R}}$ be a unitary one-parameter group in $\mathcal{H}$, such that $\theta U_t \theta = U_{-t}$, $t \in \mathbb{R}$, and $U_t \mathcal{H}_+ \subset \mathcal{H}_+$, $t \in \mathbb{R}_+$; then

\[
S_t = \widetilde{U}_t : \mathcal{H} \rightarrow \mathcal{H},
\]

is a selfadjoint contraction semigroup, $t \in \mathbb{R}_+$, i.e., there is a selfadjoint generator $L$ in $\mathcal{H}$ (see Figure 4.1),

\[
\langle k, Lk \rangle_{\mathcal{H}} \geq 0, \forall k \in \text{dom}(L),
\]

(4.1)
where
\[ S_t(= \tilde{U}_t) = e^{-tL}, \quad t \in \mathbb{R}_+, \quad \text{and} \]
\[ S_{t_1}S_{t_2} = S_{t_1+t_2}, \quad t_1, t_2 \in \mathbb{R}_+. \quad (4.2) \]

\[ S_{t_1}S_{t_2} = S_{t_1} + t_2, \quad t_1, t_2 \in \mathbb{R}_+. \quad (4.3) \]

**Proof.** See [GJ79, GJ87, Jor87, JO00]. □

\[ A \xrightarrow{U_t=e^{-tA}} \mathcal{H} \quad A^* = -A \]
\[ L \xrightarrow{[S_t]e^{-tL}} \mathcal{H} \quad L^* = L, \quad L \geq 0 \]

**Figure 4.1.** Transformation of skew-adjoint \( A \) into selfadjoint semibounded \( L \).

### 4.1. Two Examples

We include details below (Example 4.2) to stress the distinction between an abstract Hilbert-norm completion on the one hand, and a concretely realized Hilbert space on the other.

**Example 4.2 ([JO98, JO00]).** Let \( 0 < s < 1 \) be given, and let \( \mathcal{H} = \mathcal{H}_s \) be the Hilbert space whose norm \( \|f\|_s \) is given by
\[ \|f\|_s^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \, |x-y|^{s-1} \, f(y) \, dx \, dy. \quad (4.4) \]

Let \( a \in \mathbb{R}_+ \) be given, and set
\[ (U(a)f)(x) = a^{s+1} f \left( \frac{1}{a} x \right). \quad (4.5) \]

It is clear that then \( a \mapsto U(a) \) is a unitary representation of the multiplicative group \( \mathbb{R}_+ \) acting on the Hilbert space \( \mathcal{H}_s \). It can be checked that \( \|f\|_s \) in (4.4) is finite for all \( f \in C_c(\mathbb{R}) \) (= the space of compactly supported functions on the line). Now let \( \mathcal{H}_s \) be the closure of \( C_c(-1, 1) \) relative to the norm \( \|\cdot\|_s \) of (4.4). It is then immediate that \( U(a) \), for \( a > 1 \), leaves \( \mathcal{H}_s \) invariant, i.e., it restricts to a semigroup of isometries \( \{U(a) : a > 1\} \) acting on \( \mathcal{H}_s \). Setting
\[ (\theta f)(x) = |x|^{s-1} f \left( \frac{1}{x} \right), \quad x \in \mathbb{R} \setminus \{0\}, \quad (4.6) \]
we check that \( \theta \) is then a period-2 unitary in \( \mathcal{H}_s \), and that
\[ \theta U(a) \theta = U(a)^* = U(a^{-1}) \quad (4.7) \]

and
\[ \langle f, \theta f \rangle_{\mathcal{H}_s} \geq 0, \quad \forall f \in \mathcal{H}_s, \quad (4.8) \]

where \( \langle \cdot, \cdot \rangle_{\mathcal{H}_s} \) is the inner product
\[ \langle f_1, f_2 \rangle_{\mathcal{H}_s} := \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(x) \, |x-y|^{s-1} \, f_2(y) \, dx \, dy. \quad (4.9) \]
In fact, if \( f \in C_c (-1, 1) \), the expression in (4.8) works out as the following reproducing kernel integral:

\[
\int_{-1}^{1} \int_{-1}^{1} f(x) (1 - xy)^{s-1} f(y) \, dx \, dy,
\]

and we refer to [Jor86, JO98, JO00, Jor02] for more details on this example.

Hence up to a constant, the norm \( \| \cdot \|_s \) of (4.9) may be rewritten as

\[
\int_{\mathbb{R}} |\xi|^{-s} |\hat{f}(\xi)|^2 \, d\xi,
\]

and the inner product \( \langle \cdot, \cdot \rangle_s \) as

\[
\int_{\mathbb{R}} |\xi|^{-s} \hat{f}_1(\xi) \hat{f}_2(\xi) \, d\xi,
\]

where

\[
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx
\]

is the usual Fourier transform suitably extended to \( \mathcal{H}_s \), using Stein’s singular integrals. Intuitively, \( \mathcal{H}_s \) consists of functions on \( \mathbb{R} \) which arise as \( (d/dx)^s f \) for some \( f \in L^2 (\mathbb{R}) \). This also introduces a degree of “non-locality” into the theory, and the functions in \( \mathcal{H}_s \) cannot be viewed as locally integrable, although \( \mathcal{H}_s \) for each \( 0 < s < 1 \) contains \( C_c (\mathbb{R}) \) as a dense subspace.

A main conclusion in [Jor02] for this example is that, when \( \mathcal{H}_s \) and \( \mathcal{K}_s \) are as in (4.10), then the natural contractive operator \( q \) from (3.2)-(3.4) is automatically 1-1, i.e., its kernel is 0.

Remark 4.3. Note that, in general, the spectral type changes in passing from \( U \) to \( \tilde{U} \) in Lemma 3.7; see also Figure 3.3. For example, \( U \) from (4.5) above has absolutely continuous spectrum, while \( \tilde{U} \) has purely discrete (atomic) spectrum: When \( a > 1 \), one checks that the spectrum of \( \tilde{U} (a) \) is the set \( \{a^{-2n} : n \in \mathbb{N}\} \).

Example 4.4 (See [JKL89]: Reflection Positivity on a Schottky Double). Let \( S \) denote a compact Riemann surface which arises as a Schottky double of a bordered Riemann surface \( T \) with boundary \( \partial T \). A Schottky double \( S \) of \( T \) is defined as a mirror image \( \tilde{T} \), and \( S \) is the union of \( T \), with \( T \) glued on \( \partial T \). Thus, the double \( S \) of \( T \) has an antiholomorphic involution \( \theta : T \to \tilde{T} \), such that \( \partial T \) is the set of fixed points of \( \theta \). Let \( P_0 \in T \) and define \( P_\infty = \theta (P_0) \). The points \( P_0 \) and \( P_\infty \) then provide reference points on the Riemann surface, which are interchanged by \( \theta \), see Figure 4.2.

The standard case of a real, space-time \( S^1 \times \mathbb{R} \) can be understood as follows. For \( t \in \mathbb{R}, \ x \in S^1 \), let \( (t, x) \) denote a space-time point. The map

\[
z = \exp [i (x + it)]
\]

defines a Riemann sphere \( S \), with the half-space \( t \geq 0 \) mapping into the unit disc \( T \) around the origin. Time reflection \( (t, x) \to (-t, x) \) in space-time then maps into a reflection \( \theta \) on the Riemann sphere. In local coordinates,

\[
\theta (z) = \frac{1}{\bar{z}^*}.
\]
Figure 4.2. (a) The complex plane, inside and outside of the disk; see sect 6, and [Jor02]. (b) The Schottky double \( S \) of a bordered Riemann surface \( T \) with boundary \( \partial T \); see Example 4.4. (c) The real line, inside and outside of a fixed interval; see Example 4.2.

We identify \( T \) as the unit disc about infinity, and consider the Riemann sphere as a Schottky double of the unit disc, with \( \theta \) given by (4.15). A convenient choice for \( P_0 \) is \( P_0 = 0 \), so \( P_\infty = \infty \), and this comes from Euclidean space-time points at \( t = \mp \infty \). In particular, the operator \( \theta \) on the Riemann sphere can be thought of as a reflection through the unit circle \( |z| = 1 \).

The corresponding “infinite volume” space-time \((t,x) \in \mathbb{R}^2\) can also be studied. A compactification is given by the map

\[
z = \frac{x + i(t-1)}{x + i(t+1)}.
\]

5. A characterization of the Markov property: Markov vs O.-S. positivity

In the classical case of Gaussian processes (see [AD92, AD93, ABDD93, AJSV13, AJV14]), the question of reflection symmetry and reflection positivity is of great interest; see, e.g., [JP11a, JP11b, JP13, AJL13, JPT14, JP14, AJV14, Jaf15, JNO16, JJ17], and also [Kle77, Kle78, KLS82, AJP07].

Let \( \mathcal{H} \) be a given (fixed) Hilbert space; e.g., \( \mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P}) \), square integrable random variables, where \( \Omega \) is a set (sample space) with a \( \sigma \)-algebra of subsets \( \mathcal{F} \) (information), and \( \mathbb{P} \) a given probability measure on \((\Omega, \mathcal{F}) \). But the question may in fact be formulated for an arbitrary Hilbert space \( \mathcal{H} \), and possible inseparable generally.

Recall that \( \theta : \mathcal{H} \to \mathcal{H} \) is a reflection if it satisfies \( \theta^* = \theta \), and \( \theta^2 = I_\mathcal{H} \).

**Definition 5.1.** Given a Hilbert space \( \mathcal{H} \), let

\[
\operatorname{Ref}(\mathcal{H}) = \{ \theta : \mathcal{H} \to \mathcal{H} ; \theta^* = \theta, \theta^2 = I_\mathcal{H} \}\, ,
\]

i.e., all reflections in \( \mathcal{H} \); see (2.1).

**Lemma 5.2.** Let \( \theta : \mathcal{H} \to \mathcal{H} \) be linear, and let \( \mathcal{H}_\pm \) be a pair of closed subspaces of \( \mathcal{H} \) with respective projections \( E_\pm \); then TFAE:

(i) \( \theta(\mathcal{H}_+) \subseteq \mathcal{H}_- \), i.e., \( \theta \) maps \( \mathcal{H}_+ \) into \( \mathcal{H}_- \);

(ii) \( E_- \theta E_+ = \theta E_+ \).

**Proof.** Follows from the basic fact that \( E_+ \mathcal{H} = \{ h_+ \in \mathcal{H} ; E_+ h_+ = h_+ \} \). \(\square\)
We saw in Corollary 2.9 that reflection θ satisfying $E_+θE_+ ≥ 0$ are in 1-1 correspondence with contraction operators $C : \mathcal{H}_-(θ) \rightarrow \mathcal{H}_-(θ)$, where $\mathcal{H}_±(θ) = \{h_± ∈ \mathcal{H}_± ; \theta h_± = h_±\}$. We now fix $E_+$, and therefore the subspace $\mathcal{H}_+ := E_+\mathcal{H}$.

Let $θ, θ'$ be a pair of reflections (see Section 2 and Lemma 5.2 above), and assume they share the same pair $\mathcal{H}_±$, i.e.,

$$\theta\mathcal{H}_+ ⊆ \mathcal{H}_-, \quad \theta'\mathcal{H}_+ ⊆ \mathcal{H}_-.$$  \hspace{1cm} (5.2)

**Lemma 5.3.** Let $P$ and $P'$ be the projections onto $\{h ∈ \mathcal{H} ; θh = h\}$, and $\{h' ∈ \mathcal{H} ; θ'h' = h'\}$, i.e., we have $θ = 2P - I_\mathcal{H}$, and $θ' = 2P' - I_\mathcal{H}$. Let $C$ and $C'$ be the corresponding contractions: $C : P\mathcal{H} → P_\perp\mathcal{H}$, and $C' : P'\mathcal{H} → (P')_\perp\mathcal{H}$. Then

$$\mathcal{H}_+ = \text{Graph}(C) = \{x + Cx ; x ∈ P\mathcal{H}\}$$

$$= \text{Graph}(C') = \{x' + C'x' ; x' ∈ P'\mathcal{H}\};$$

and

$$θ(x + Cx) = x − Cx ∈ \mathcal{H}_-; \quad \text{and} \quad \theta'(x' + C'x') = x' − C'x' ∈ \mathcal{H}_-.$$

Moreover, $(I_{\mathcal{H}} + C')|_{P\mathcal{H}}$ has a one-sided inverse, and there is an operator $V : P\mathcal{H} → P'\mathcal{H}$ such that

$$V|_{P\mathcal{H}} = (I_{\mathcal{H}} + C')^{-1}(I_{\mathcal{H}} + C)|_{P\mathcal{H}}.$$  \hspace{1cm} (5.5)

**Proof.** This is essentially a consequence of the characterization in Lemma 2.8 and Corollary 2.9. Indeed, from this, we get the existence of the operator $V$ as specified in (5.4), and satisfying

$$\mathcal{H}_+ ∋ x + Cx = Vx + C'Vx,$$  \hspace{1cm} (5.6)

for all $x ∈ P\mathcal{H}$. But (5.6) may be rewritten as:

$$(I_{\mathcal{H}} + C)x = (I_{\mathcal{H}} + C')Vx;$$

and the desired conclusion (5.5) now follows. \hfill □

**Definition 5.4.** If $E_0, E_±$ are projections in $\mathcal{H}$, let $ε = (E_0, E_±)$, and set

$$δ(\text{Markov}) := \{(E_0, E_±) ; E_+E_0E_- = E_+E_-\},$$

$$\mathcal{R}(ε) := \{θ ∈ \text{Ref}(\mathcal{H}); \theta E_0 = E_0, θ E_+ = E_-θE_+, θ E_- = E_+θE_-\}. $$  \hspace{1cm} (5.7)

Fix $θ ∈ \text{Ref}(\mathcal{H})$, so that $θ^2 = I_{\mathcal{H}}$, $θ^* = θ$, set:

$$δ(θ) := \{(E_0, E_±) ; θ E_0 = E_0, θ E_+ = E_-θE_+, θ E_- = E_+θE_-\}. $$  \hspace{1cm} (5.8)

**Remark.** Recall that $E$ is a projection in $\mathcal{H}$ iff (Def.) $E = E^2 = E^*$; see Definition 2.3.

In (5.8) and (5.9), the conditions on $θ$ and the triple of projections $ε = (E_0, E_±)$ are as follows: $θ E_0 = E_0, θ(\mathcal{H}_+) ⊆ \mathcal{H}_-$ and $θ(\mathcal{H}_-) ⊆ \mathcal{H}_+$; see Lemma 5.2.

**Question.** (1) Given $ε$, what is $\mathcal{R}(ε)$? (2) Given $θ$, what is $δ(θ)$?

**Definition 5.5.** Suppose $ε = (E_0, E_±)$ is given, and $θ ∈ \mathcal{R}(ε)$.
(i) We say that reflection positivity holds iff (Def.)
\[ E_+ \theta E_+ \geq 0, \]  \hspace{1cm} (5.10)
also called Osterwalder-Schrader positivity (O.S.-p).

(ii) Given \( \varepsilon \), we say that it satisfies the Markov property iff (Def.)
\[ E_+ E_0 E_- = E_+ E_-, \]  \hspace{1cm} (5.11)

(iii) We set
\[ \mathcal{E}_{OS}(\theta) = \{(E_0, E_\pm) ; E_+ \theta E_+ \geq 0\}. \]  \hspace{1cm} (5.12)

**Lemma 5.6.** Suppose (5.11) holds (the Markov property), and \( \theta \in \mathcal{R}(\varepsilon) \), then
\[ E_+ \theta E_+ \geq 0, \]  \hspace{1cm} (5.13)
i.e., the O.S.-positivity condition (5.10) follows.

**Proof.** Using the properties in (5.8), we have
\[
E_+ \theta E_+ = E_+ E_- \theta E_+ = (E_+ E_0 E_-) \theta E_+
\]
\[= E_+ E_0 (E_- \theta E_+) \]
\[= E_+ E_0 \theta E_+ \]
\[= E_+ E_0 E_+ \geq 0, \]
where “\( \geq \)” is in the sense of ordering of selfadjoint operators.

Note for any pair of projections, we have:
\[
\langle h, E_+ E_0 E_+ h \rangle = (E_+ h, E_0 E_+ h) = \| E_0 E_+ h \|^2 \geq 0;
\]
where \( E_0 = E_0^* = E_0^2 \) by definition. \( \square \)

Recall the definition of \( \mathcal{R}(\varepsilon) \) and \( \mathcal{R}(\varepsilon, U) \). Lemma 5.6 can be reformulated as:

**Lemma 5.7.** For all \( \theta \in \mathcal{R}(\varepsilon) \), we have
\[ \mathcal{E}(\text{Markov}) \cap \mathcal{E}(\theta) \subseteq \mathcal{E}_{OS}(\theta). \]  \hspace{1cm} (5.14)

(See Definitions 5.1, 5.4, and eq. (5.19).)

**Question.** Let \( \varepsilon = (E_0, E_\pm) \) be given, and suppose \( E_+ \theta E_+ \geq 0 \), for all \( \theta \in \mathcal{R}(\varepsilon) \), then does it follow that \( E_+ E_0 E_- = E_+ E_- \) holds? (See Theorem 5.8 below for an affirmative answer.)

**Theorem 5.8.** Given an infinite-dimensional complex Hilbert space \( \mathcal{H} \), let the setting be as above, i.e., reflections, Markov property, and O.S.-positivity defined as stated. Then
\[
\bigcap_{\theta \in \mathcal{R}(\varepsilon)} \mathcal{E}_{OS}(\theta) = \mathcal{E}(\text{Markov}). \]  \hspace{1cm} (5.15)

**Remark 5.9.** If \( \varepsilon \), and \( U \) are given as in Section 2, and if \( (E_\pm, E_0, U) \) is Markov, then (5.15) also holds with \( \theta, U \) satisfying (2.2)-(2.3). The idea in (5.15) is that when a system \( \varepsilon \) of projections is fixed as specified on the RHS in the formula, then on the LHS, we intersect only over the subset of reflections \( \theta \) subordinated to this \( \varepsilon \)-system. And similarly when both \( \varepsilon \) and \( U \) are specified, we intersect over the smaller set of jointly \( \varepsilon, U \) subordinated reflections \( \theta \).
Proof. We must show that if \( \varepsilon := (E_0, E_\pm) \) is given to satisfy the Markov property, i.e., \( E_+ E_0 E_- = E_+ E_- \), then for all \( \theta \in \mathcal{R}(\varepsilon) \); see Lemma 3.10. Then by Lemma 5.6, the O.S. property will be automatic. Now given \( \varepsilon \), a reflection \( \theta \) may be constructed via an application of Zorn’s lemma to all reflections \( \theta \) from \( \mathcal{H}_+ \) to \( \mathcal{H}_- \), see (5.20) below. Note we can assume that both of the subspaces \( \mathcal{H}_\pm \) are infinite-dimensional. Hence, to show existence of \( \theta \in \mathcal{R}(\varepsilon) \) as asserted, we must show that, if \( \theta \) is initially only defined on closed subspaces \( \mathcal{H}_{\pm}^{\text{su}} \to \mathcal{H}_{\pm}^{\text{su}} \), then vectors \( h_\pm \in \mathcal{H}_\pm \cap \mathcal{H}_{\pm}^{\text{su}} \) may be chosen such that \( \theta h_+ = h_- \) offers a non-trivial extension. This is a contradiction since the two subspaces \( \mathcal{H}_{\pm}^{\text{su}} \) may be chosen maximal by Zorn’s lemma. (See also [Phi61, Jor79].)

In detail: By Lemma 5.7, we already have “\( \supseteq \)” in (5.15), and we now turn to the other inclusion:

Given \( (E_0, E_\pm) \), and suppose \( (E_0, E_\pm) \in \mathcal{E}_{\text{OS}}(\theta) \), \( \forall \theta \in \mathcal{R}(\varepsilon) \). We shall show that \( (E_0, E_\pm) \in \mathcal{E}(\text{Markov}) \), i.e.,

\[
\bigcap_{\theta \in \mathcal{R}(\varepsilon)} \mathcal{E}_{\text{OS}}(\theta) \subseteq \mathcal{E}(\text{Markov}).
\]

(5.16)

Indirect proof of (5.16):

We must prove that if \( (E_0, E_\pm) \notin \mathcal{E}(\text{Markov}) \) then \( \exists \theta \in \mathcal{R}(\varepsilon) \) s.t. \( (E_0, E_\pm) \notin \mathcal{E}_{\text{OS}}(\theta) \).

Suppose \( E_+ E_0 E_- \neq E_+ E_- \), then \( \exists \, h_\pm \neq 0, \, h_\pm \in \mathcal{H}_\pm \), where \( \mathcal{H}_\pm := E_\pm \mathcal{H} \), s.t.

\[
\langle h_+, E_0 h_- \rangle \neq \langle h_+, h_- \rangle,
\]

(5.17)

and we may choose these vectors s.t.

\[
\langle h_+, h_- \rangle \notin [0, \infty).
\]

(5.18)

See also Theorem 2.15.

Define \( \theta_0 \) on \( \text{Ch}_+ \to \text{Ch}_- \) (on 1-dimensional subspaces), \( \theta_0 h_+ := h_- \); and then extend it

\[
\theta_0 \to \theta : \mathcal{H}_+ \xrightarrow{\theta} \mathcal{H}_- \, \text{(extended)},
\]

(5.19)

to a reflection \( \theta \) with initial space \( \mathcal{H}_+ \) and final space \( \mathcal{H}_- \), (using again Theorem 2.15) s.t. the extension \( \theta \) satisfies

\[
\theta E_0 = E_0, \quad \theta E_+ = E_- \theta E_+,
\]

i.e., \( (E_0, E_\pm) \in \mathcal{E}(\theta) \), see (5.20).

\[
\begin{array}{c}
\mathcal{H}_+ \\
\text{Ch}_+
\end{array} \xrightarrow[\theta_0]{\theta} \begin{array}{c}
\mathcal{H}_-
\text{Ch}_-
\end{array}
\]

(5.20)

Then \( \langle h_+, \theta h_+ \rangle = \langle h_+, h_- \rangle \notin [0, \infty) \) by construction, see (5.19)-(5.20); and so, for this \( \theta \), \( (E_0, E_\pm) \notin \mathcal{E}_{\text{OS}}(\theta) \), and (5.16) is proved. \( \square \)

Example 5.10 (Markov property). Let \( \mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P}) \), where

- \( \Omega \): sample space;
- \( \mathcal{F} \): total information;
- \( \mathcal{F}_- \): information from the past (or inside);
- \( \mathcal{F}_+ \): information from the future (predictions), or from the outside;
- \( \mathcal{F}_0 \): information at the present.
Let $E(\cdot \mid \mathcal{F}_0)$, $E(\cdot \mid \mathcal{F}_\pm)$ be the corresponding conditional expectations, and the Markov property (5.11) then takes the form $E_0 \mathcal{H} = \mathcal{H}_0$, $E_+ \mathcal{H} = \mathcal{H}_+$.

The Markov process is a probability system:

$$E(\psi_+ \mid \mathcal{F}_- \mid \mathcal{F}_0) = E(\psi_+ \mid \mathcal{F}_0),$$

(5.21)

for $\forall \psi_+$ (random variables conditioned by $\mathcal{F}_- = \text{the future}$); or, if $\mathcal{F}_0 \subseteq \mathcal{F}_-$, it simplifies to:

$$E(\psi_+ \mid \mathcal{F}_- \mid \mathcal{F}_0) = E(\psi_+ \mid \mathcal{F}_0), \forall \psi_+ \in \mathcal{H}_+.$$

(5.22)

For more details on this point, see Section 7 below.

**Question.** Do we have analogies of O.S.-positivity (see (5.10)) in the free probability setting? That is, in the setting of free probability and non-commuting random variables.

### 6. A Model for Reflection Symmetry

This is a following up on a result in [Jor02], and we offer the following analysis as reflection operators $\theta$: Let $\mathcal{H}_i$, $i = 1, 2$, be a pair of Hilbert spaces; we shall assume that they are both separable and infinite dimensional. Set $\mathcal{H} = (\mathcal{H}_1 \oplus \mathcal{H}_2)$, i.e., column vectors. The Hilbert norm in $\mathcal{H}$ is the usual one:

$$\left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{\mathcal{H}}^2 = \|h_1\|^2_{\mathcal{H}_1} + \|h_2\|^2_{\mathcal{H}_2}.$$

(6.1)

Set

$$\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(6.2)

more precisely, $\theta (h_1 \oplus h_2) = h_1 \oplus (-h_2)$.

**Theorem 6.1.** Let $\mathcal{H}_i$, $i = 1, 2$, and $\theta$ be as above. A system $E_0$, $E_\pm$ with subspaces $\mathcal{H}_0$, $\mathcal{H}_\pm$ in $\mathcal{H}$, satisfies the O.S.-condition $\langle h_+, \theta h_+ \rangle_{\mathcal{H}} \geq 0$, $\forall h_+ \in \mathcal{H}_+$, if and only if there is a contractive linear operator $C : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $\mathcal{H}_1 = \text{Graph}(C)$, $\mathcal{H}_- = \text{Graph}(-C)$, and

$$\mathcal{H}_0 = \left( \begin{array}{c} \mathcal{H}_1 \\ \mathcal{H}_2 \end{array} \right) = \left\{ \begin{pmatrix} h_1 \\ 0 \end{pmatrix} : h_1 \in \mathcal{H}_1 \right\}.$$

(6.3)

**Proof.** We refer to [Jor02] for details, but the easy implication is as follows: Given $C : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, and $\theta$ be as in (6.2), then

$$\left\langle \begin{pmatrix} h_1 \\ Ch_1 \end{pmatrix}, \theta \begin{pmatrix} h_1 \\ Ch_1 \end{pmatrix} \right\rangle_{\mathcal{H}} = \|h_1\|^2_{\mathcal{H}_1} - \|Ch_1\|^2_{\mathcal{H}_2} \geq 0$$

iff $C$ is contractive. One checks that the converse implication holds as well. \qed

**Theorem 6.2.** Let $\mathcal{H} = \left( \begin{array}{c} \mathcal{H}_1 \\ \mathcal{H}_2 \end{array} \right)$ be as above, and let $C : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a contraction. Set

$$\mathcal{H}_+ = \text{Graph}(C), \quad \mathcal{H}_- = \text{Graph}(-C),$$

(6.4)

$$\mathcal{H}_0 = \left( \begin{array}{c} \mathcal{H}_1 \\ \mathcal{H}_2 \end{array} \right), \quad \text{and} \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(6.5)

Then the Markov property

$$E_+ E_0 E_- = E_+ E_-$$

(6.6)
holds if and only if \( C = 0 \).

Proof. Let \( E_\pm \) be the projections corresponding to the two subspaces \( \mathcal{H}_\pm \) in (6.4). One checks that

\[
E_+ = \begin{pmatrix} \left(1 + C^*C\right)^{-1} & (1 + C^*C)^{-1} C^* \\ C(1 + C^*C)^{-1} & 1 - (1 + CC^*)^{-1} \end{pmatrix}, \tag{6.7}
\]

and

\[
E_- = \begin{pmatrix} \left(1 + C^*C\right)^{-1} & -(1 + C^*C)^{-1} C^* \\ -C(1 + C^*C)^{-1} & 1 - (1 + CC^*)^{-1} \end{pmatrix}. \tag{6.8}
\]

In abbreviated form we have

\[
E_+ = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}
\]

with the operator entries as specified in (6.7), and so

\[
E_- = \begin{pmatrix} P_{11} & -P_{12} \\ -P_{21} & P_{22} \end{pmatrix}.
\]

A further computation yields

\[
E_+ E_0 E_- = \begin{pmatrix} P_{11} & -P_{12} \\ -P_{21} & P_{22} \end{pmatrix},
\]

and

\[
E_+ E_- = \begin{pmatrix} P_{11}^2 - P_{12} P_{21} & -P_{11} P_{12} + P_{12} P_{22} \\ P_{21} P_{11} - P_{22} P_{21} & -P_{21} P_{12} + P_{22}^2 \end{pmatrix}.
\]

Hence the Markov property (6.6) holds iff \( P_{12} P_{21} = 0 \). Note that \( P_{21} = P_{12}^* \). Using the operator entries from (6.7)-(6.8), we conclude that (6.6) holds iff \( C = 0 \), in which case \( E_+ = E_- = E_0 \), where \( E_0 \) is as in (6.5). \( \square \)

Remark 6.3. The matrix \( E_+ \) (i.e., the characteristic matrix of \( C \)) from (6.7) is obtained as follows:

Let \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \), then

\[
\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P_{11} x + P_{12} y \\ P_{21} x + P_{22} y \end{pmatrix} \in Graph(C),
\]

\( \downarrow \)

\[ C : (P_{11} x + P_{12} y) \mapsto P_{21} x + P_{22} y, \]

\( \downarrow \)

\[ CP_{11} = P_{21}, \quad CP_{12} = P_{22}. \tag{6.9} \]

On the other hand, \( E_+^\perp = I - E_+ \) is the projection from \( \mathcal{H} \) onto \( V(Graph(C^*)) \), where

\[
V := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

It follows that,

\[
C^* : P_{21} x - (1 - P_{22}) y \mapsto (1 - P_{11}) x - P_{12} y,
\]

\( \downarrow \)

\[ C^* P_{21} = 1 - P_{11}, \quad C^* (1 - P_{22}) = P_{12}. \tag{6.10} \]

Solving (6.9)-(6.10), we get \( E_+ \) as in (6.7).
7. Markov processes and Markov reflection positivity

In the above, we considered systems $\mathcal{H}$, $E_0$, $E_\pm$, $\theta$, and $U$, where $\mathcal{H}$ is a fixed Hilbert space; $E_0$, $E_\pm$ are then three given projections in $\mathcal{H}$, $\theta$ is a reflection, and $U$ is a unitary representation of a Lie group $G$.

The axioms for the system are as follows:

(i) $\theta E_0 = E_+;$
(ii) $E_+ \theta E_- = \theta E_-;
(iii) E_- \theta E_+ = \theta E_+$;
(iv) the O.S.-positivity holds, i.e.,
\[ E_+ \theta E_+ \geq 0; \] (7.1)
(v) $\theta U \theta = U^*$, or $\theta U (g) \theta = U(g^{-1})$.

It is further assumed that, for some sub-semigroup $S \subset G$, we have $U(s) \mathcal{H}_+ \subset \mathcal{H}_+$, $\forall s \in S$; or equivalently,
\[ E_+ U(s) E_+ = U(s) E_+, \ s \in S. \] (7.2)

From Section 6, it is clear that the additional Markov-restriction
\[ E_+ E_0 E_- = E_+ E_- \] (7.3)
is “very” strong. Moreover, if $\theta$ is fixed, we saw that (7.3) $\implies$ (7.1) (see Lemma 5.6).

Here we note that (7.3) holds in a natural setting of path space analysis:

7.1. Probability Spaces. By a probability space we mean a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is a set (the sample space), $\mathcal{F}$ is a $\sigma$-algebra of subsets (information), and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. Measurable functions $\psi$ on $(\Omega, \mathcal{F})$ are called random variables. If $\psi$ is a random variable in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, we say that it has finite second moment. An indexed family of random variables is called a stochastic process, or a random field.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. The expectation will be denoted
\[ E(\psi) = \int_\Omega \psi d\mathbb{P}, \] (7.4)
if $\psi$ is a given random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

We shall be primarily interested in the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ setting. If $\psi$ is a random variable (or a random field) then
\[ \psi^{-1}(\mathcal{B}) \subseteq \mathcal{F}, \] (7.5)
where $\mathcal{B}$ is the Borel $\sigma$-algebra of subsets of $\mathbb{R}$.

For every sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$, there is a unique conditional expectation
\[ E(\cdot | \mathcal{G}) : L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}). \] (7.6)
In fact $\mathcal{G}$ defines a closed subspace in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, the closed span of the indicator functions $\{\chi_S : S \in \mathcal{G}\}$, and $E(\cdot | \mathcal{G})$ in (7.6) will then be the projection onto this subspace.

If $\mathcal{G} \subset \mathcal{F}$ is as in (7.5) then, for random variables $\psi_1 \in L^2(\mathcal{G}, \mathbb{P})$, and $\psi_2 \in L^2(\mathcal{F}, \mathbb{P})$, we have
\[ E(\psi_1 \psi_2) = E(\psi_1 E(\psi_2 | \mathcal{G})). \] (7.7)
If \( \psi_1 \) is also in \( L^\infty (\mathcal{G}, \mathbb{P}) \), then
\[
E (\psi_1 \psi_2 \mid \mathcal{G}) = \psi_1 E (\psi_2 \mid \mathcal{G}).
\] (7.8)

The following property is immediate from this: If \( \mathcal{G}_i, i = 1, 2, \) are two sub-\( \sigma \)-algebras with \( \mathcal{G}_1 \subseteq \mathcal{G}_2 \), then for all \( \psi \in L^2 (\Omega, \mathcal{F}, \mathbb{P}) \) we have
\[
E (E (\psi \mid \mathcal{G}_2) \mid \mathcal{G}_1) = E (\psi \mid \mathcal{G}_1).
\] (7.9)

Indeed, this is immediate from the equivalences in Definition 2.3.

Let \( \{\psi_t\}_{t \in \mathbb{R}} \) be a random process in the given probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). For \( t \in \mathbb{R} \), set \( \mathcal{F}_t := \) the \( \sigma \)-algebra (\( \subseteq \mathcal{F} \)) generated by the random variables \( \{\psi_s ; s \leq t\} \). When \( t \) is fixed, we set \( \mathcal{B}_t := \) the \( \sigma \)-algebra generated by the random variable \( \psi_t \). We say that \( \{\psi_t\}_{t \in \mathbb{R}} \) is a Markov-process iff (Def.), for every \( t > s \), and every measurable function \( f \), we have
\[
E (f \circ \psi_t \mid \mathcal{F}_s) = E (f \circ \psi_t \mid \mathcal{B}_s)
\] (7.10)
where \( E (\cdot \mid \mathcal{F}_s) \), and \( E (\cdot \mid \mathcal{B}_s) \), refer to the corresponding conditional expectations. It is well known that the Markov property is equivalent to the following semigroup property:

Set
\[
(S_t f) (x) := E (f \circ \psi_t \mid \psi_0 = x),
\] (7.11)
then, for all \( t, s \geq 0 \), we have
\[
S_{t+s} = S_t S_s.
\] (7.12)

So the semigroup law (7.12) holds if and only if the Markov property (7.10) holds.

In order to make a direct comparison with the present Markov property from Corollary 3.17, it is convenient to restrict attention to stationary processes; and we now turn to the details of that below.

7.2. The covariance operator. Now let \( V \) be a real vector space; and assume that it is also a LCTVS, locally convex topological vector space. Let \( G \) be a Lie group, \( U \) a unitary representation of \( G \); and let \( \{\psi_{v,g}\}_{(v,g) \in V \times G} \) be a real valued stochastic process s.t. \( \psi_{v,g} \in \mathcal{H} = L^2 (\Omega, \mathcal{F}, \mathbb{P}) \), and
\[
E (\psi_{v,g}) = 0, \ (v, g) \in V \times G.
\] (7.13)

We further assume that a reflection \( \theta \) is given, and that
\[
\theta (\psi_{v,g}) = \psi_{\theta(g),-1}, \ (v, g) \in V \times G.
\] (7.14)

Let \((v_i, g_i), i = 1, 2,\) be given, and set
\[
E (\psi_{v_1,g_1} \psi_{v_2,g_2}) = \langle v_1, r (g_1, g_2) v_2 \rangle
\] (7.15)
where \( \langle \cdot, \cdot \rangle \) is a fixed positive definite Hermitian inner product on \( V \). Hence (7.15) determines a function \( r \) on \( G \times G \); it is operator valued, taking values in operators in \( V \). This function is called the covariance operator.

To sketch the setting for the Markov property (7.3), we shall make two specializations (these may be removed!):

(i) \( G = \mathbb{R}, \ S = \mathbb{R}_+ \cup \{0\} = [0, \infty) \), and

(ii) the process is stationary; i.e., referring to (7.15) we assume that the covariance operator \( r \) is as follows:
\[
E (\psi_{v_1,t_1} \psi_{v_2,t_2}) = \langle v_1, r (t_1 - t_2) v_2 \rangle,
\] (7.16)
\( \forall t_1, t_2 \in \mathbb{R}, \forall v_1, v_2 \in V \).
In this case, the O.S.-condition (7.1) is considered for the following three sub-\(\sigma\)-algebras \(\mathcal{A}_0\), \(\mathcal{A}_\pm\) in \(\mathcal{F}\):
\[
\mathcal{A}_0 = \text{the } \sigma\text{-algebra generated by } \{\psi_{v,0} \mid v \in V\},
\]
\[
\mathcal{A}_+ = \text{the } \sigma\text{-algebra generated by } \{\psi_{v,t} \mid v \in V, t \in [0,\infty)\}, \text{ and}
\]
\[
\mathcal{A}_- = \text{the } \sigma\text{-algebra generated by } \{\psi_{v,t} \mid v \in V, t \in (-\infty,0]\}.
\]
The corresponding conditional expectations will be denoted as follows:
\[
E_0(\psi) = E(\psi \mid \mathcal{A}_0), \text{ and }
\]
\[
E_\pm(\psi) = E(\psi \mid \mathcal{A}_\pm). \quad (7.17)
\]
The corresponding closed subspaces in \(\mathcal{H} = L^2(\Omega, \mathcal{F}, P)\) will be denoted \(\mathcal{H}_0, \mathcal{H}_\pm\), respectively, and we shall consider the positivity conditions (7.1) O.S.-p, and (7.3) Markov, in this context.

Translating a theorem in [Kle77], we arrive at the following:

**Theorem 7.1** (A. Klein [Kle77]). Let the stationary stochastic process \(\{\psi_{v,t}\}, (v, t) \in V \times \mathbb{R}\), be as specified above, and let \(\{r(t)\}_{t \in \mathbb{R}}\) be the covariance operator. Set \(\theta(\psi_{v,t}) := \psi_{v,-t}, t \in \mathbb{R}\). Assume \(\langle \psi_+, \theta \psi_+ \rangle \geq 0, \forall \psi_+ \in \mathcal{H}_+, \) then for \(\forall n \in \mathbb{N}, \forall \{v_i\}_{i=1}^n \subset V, \forall \{t_i\}_{i=1}^n \subset \mathbb{R}_+ \cup \{0\}, \) we have
\[
\sum_i \sum_j \langle v_i, r(t_i + t_j) v_j \rangle \geq 0; \quad (7.18)
\]
which is the O.S.-positivity condition.

Moreover, the Markov property \(E_+ E_0 E_- = E_+ E_-\) holds iff \(r(\cdot)\) is a semigroup, i.e.,
\[
r(t + s) = r(t) r(s), \quad (7.19)
\]
for \(\forall s, t \in [0,\infty)\).

In particular, in the case of stationary processes, when O.S.-positivity is assumed, then two conditions hold:

(i) the covariance function \(r(\cdot)\) is positive definite:
\[
\sum_i \sum_j \langle v_i, r(t_i - t_j) v_j \rangle \geq 0; \text{ and}
\]

(ii) condition (7.18) holds as well.

**Remark 7.2.** In the scalar case, a list of stationary positive definite, and Gaussian O.S.-positive, covariance functions \(\{r(t)\}_{t \in \mathbb{R}}\) includes:

- \(e^{-a|t|}, a > 0, \) fixed;
- \(\frac{1 - e^{-b|t|}}{b |t|}, b > 0, \) fixed;
- \(\frac{1}{1 + |t|}, \)
- \(\frac{1}{\sqrt{1 + |t|}} e^{-\frac{|t|}{1+|t|}}, t \in \mathbb{R}.\)
But of these, only the first one \( r(t) := e^{-a|t|} \) is also the generator of a Markov system; it is the Ornstein-Uhlenbeck process. The corresponding semigroup is called the Ornstein-Uhlenbeck semigroup and it is of independent interest in applications to stochastic analysis (Lévy processes) and to mathematical physics; see e.g., [Nel73a, Nel73b, Nel75, Kle78, App15, Che15, Jaf15, Teu16], and also [Kle77, Kle78, GJ79, Arv86, GJ87, JO98].

**Remark 7.3.** As outlined in recent papers by the first named author with Neeb and Olafsson ([JO98, JO00, JNO16]), the extension of the results also holds in the context of Lie groups \( G \), with semigroups \( S \subseteq G \). The above deals with the case \( G = \mathbb{R}, S = [0, \infty) \).

**Corollary 7.4.** Let \( \{\psi_{v,t}\} \) be as specified above, \( \theta \psi_{v,t} = \psi_{v,-t} \), and assume Osterwalder-Schrader positivity holds. Let \( K \) denote the Hilbert completion of \( \text{span} \{\psi_{v,t} : t \geq 0\} \) with respect to the induced inner product from (7.18). Then a selfadjoint and contractive semigroup \( \{R(s) : s \geq 0\} \) is well defined by \( R(s)\psi_{v,0} := \psi_{v,s} \); i.e., \( \{R(s)\} \) is a selfadjoint contractive semigroup of operators in \( \mathcal{K} \), \( R(s + s') = R(s)R(s') \).

**Proof.** Immediate. Note that

\[
\langle R(s)\psi_{v_1,t_1},\psi_{v_2,t_2}\rangle_{\mathcal{K}} = \langle \psi_{v_1,t_1}, R(s)\psi_{v_2,t_2}\rangle_{\mathcal{K}} = \langle v_1, r(t_1 + t_2 + s)v_2 \rangle_V.
\]

\( \square \)

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