Quantum Accelerated Estimation of Algorithmic Information

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Abstract

In this research we present a quantum circuit for estimating algorithmic information metrics like the universal prior distribution. This accelerates inferring algorithmic structure in data for discovering causal generative models. The computation model is restricted in time and space resources to make it computable in approximating the target metrics. A classical exhaustive enumeration is shown for a few examples. The precise quantum circuit design that allows executing a superposition of automata is presented. As a use-case, an application framework for experimenting on DNA sequences for meta-biology is proposed. To our knowledge, this is the first time approximating algorithmic information is implemented for quantum computation. Our implementation on the OpenQL quantum programming language and the QX Simulator is copy-left and can be found on https://github.com/Advanced-Research-Centre/QuBio

Keywords: algorithmic information theory, quantum algorithms, Turing machines, meta-biology

1 Introduction

The evaluation of metrics like the algorithmic complexity and algorithmic probability of finite sequences is key in scientific inference. For example, in biological systems, a better understanding of the algorithmic structures in DNA sequences would greatly advance domains like personalized medication. Data-driven approaches like deep learning and lossless compression often fail in providing insights on the causal generative mechanism underlying the set of observations about the physical process under study. In this research, we follow the algorithmic information theoretic approach of enumerating the distribution of all generating automata (e.g. Turing machines). The uncomputable nature of these metrics is approximated in practice by restricting the resources available to the computational model like time/cycles and space/memory.

Such approaches however remain intractable except for the simplest of the cases. This is due to the exponential scaling of the number of possible automata that need to be enumerated for pragmatic observation data sizes. We explore the possibility of accelerating this technique on a (gate-based) circuit model of quantum computation. In this effect, we design the exact circuit required to simulate a superposition of resource-bounded stored-program automata. Thereafter, we present a full experimental framework for using this approach in genomics for inferring various metrics.

The interdisciplinary nature of this work is motivated by various seminal research in mathematics, computer science, physics and biology directing towards a general unified framework for discovering causal models and fundamental theories, as presented in the Appendix A. Our approach is rooted in various automata models, computability, algorithmic information theory, quantum computation models and resource complexity estimation techniques. The necessary backgrounds for these are presented in Section 2. In Section 3 we precisely define the computation model for our experimentation and translation into the corresponding quantum implementation. We also present an exhaustive enumeration of a few small examples of our model. The quantum implementation is presented in Section 4. The resources and circuit design of each functional block of the quantum algorithm are worked out for one of the example cases. In Section 5 we integrate the quantum circuit design into an application framework. Applications in meta-biology are presented as a case study. Section 6 concludes the paper.
2 Background and relevant work

The background concepts for this research are presented in this section. It includes the mathematical description of a Turing machine and algorithmic probability. Thereafter, the motivation for the exact model of a restricted Turing machine used in this work is explained. The translation to a quantum circuit and the advantage is discussed. We base our design on the fundamental principles to minimize the technical debt. The comparison with similar implementations is discussed.

2.1 Chomsky hierarchy of classical automata models

A Turing machine (TM) is a mathematical model of computation. It was invented by Alan Turing for proving the uncomputability of the Entscheidungsproblem. A TM manipulates symbols on an infinite memory strip of tape divided into discrete cells according to a table of transition rules (expressed as a finite state machine), as shown in Figure 1. The machine positions its head over a cell and reads the symbol there. Then, as per the symbol and its present place in a finite table of user-specified instructions, the machine (i) writes a symbol (e.g., a digit or a letter from a finite alphabet) in the cell then (ii) either moves the tape one cell left or right then (iii) (as determined by the observed symbol and the machine’s place in the table) either proceeds to a subsequent instruction or halts the computation.

![Figure 1: Computational model of a Turing machine](image)

Though TM can express arbitrary computations, their minimalistic design makes them unsuitable for computation in practice. Real-world computers are based on different but computationally equivalent designs like the random-access machine (RAM) and random-access stored-program (RASP) model. These, unlike sequential tape access in Turing machines, allow random index-based access to memory (e.g. a binary search is really slow on TM).

A finite state machine (FSM) is a mathematical model of computation that can be in exactly one of a finite number of states at any given time. It can transition from one state to another in response to some external inputs and/or a condition is satisfied. An FSM is defined by a list of its states, its initial state, and the conditions for each transition. Note, since it does not have a separate tape memory, an FSM by itself is computationally less capable than a TM (e.g. it cannot determine if an input string has the structure $a^i b^j$, like $aabb$, $aaaaabbbb$).

A universal Turing machine (UTM) simulates an arbitrary Turing machine on arbitrary input. It reads the description of the machine to be simulated as well as the input to that machine from its own tape, as shown in Figure 2. This is the idea behind the stored-program von Neumann architecture.

Given a UTM, every TM can, given its encoding on that machine, be assigned a number. This is the machine’s description number. They are very similar to Gödel numbers and play a key role in Alan Turing’s proof of the undecidability of the halting problem. The existence of this direct correspondence between natural numbers and TM shows that the set of all Turing machines is denumerable.

Given this set of possible programs, we can see some interesting properties. The busy beaver $BB(m, n)$ is a $n$-symbol alphabet TM that writes the most non-blank symbol on an initial blank tape, using $m$-states and halts. There is no way to know the value of this function (length of printed symbols or number of steps taken) without running each possible program for an infinite time. This is intimately related to the uncomputability of the halting problem. However, for small values of $m$ and $n$, it is possible to enumerate and run every possible machine to lower-bound the value of this function [1].

Various modifications of the TM results in equivalent computation power, like in the RAM model, RASP model, lambda calculus, Post machine, cyclic-tag system, etc. Just like multi-dimensional tapes are equivalent to a 1-D tape, 1-way infinite tapes are of equivalent power. Semi-weak and weak universal TM
applies to infinitely long tapes but with a repeated string (instead of blank character) written in it 1-way or both ways. For most practical applications we do not need an infinite tape. There exist automata models that restrict the tape features in various ways as shown in Figure 3.

Figure 3: The Chomsky hierarchy and formal language theory

These result in a reduced level in the Chomsky hierarchy of language and the corresponding automata model that accepts strings from the respective grammar.

- Type-0: Recursively enumerable language; Turing machine
- Type-1: Context-sensitive language; Linear-bounded non-deterministic Turing machine
- Type-2: Context-free language; Non-deterministic pushdown automaton
- Type-3: Regular languages; Finite state automaton

Each higher level (the highest being Type-0) can always simulate the lower level. At Type-0, universality is reached, thus, everything that is computable can be mapped to an algorithmic process on the TM. This includes quantum computation as well as it can be simulated on a classical TM (albeit using worst-case exponential resources).

2.1.1 Restricted Turing machine

The discrete nature of the physical universe is a central thesis in the computational approach to science. Digital physicists support that there is a fundamental limit to dividing space and time - be it the discrete cells and steps of a Turing machine or that of Planck units. However, a lot of the power of the Turing machine relies on the infinite length of the tape. It is not too hard to construct logical paradoxes like Zeno's
if we accept the concept of infinite but reject that of the infinitesimal.

We will consider a Linear-bounded deterministic Turing machine with a circular tape. The reason for considering a linear-bounded automata (LBA) is also driven by the quantum circuit implementation, as discussed later. There are 2 variables - the length of computation and the length of the tape. The latter is the dependent variable due to the region of influence (time cone). It is possible for a code to run for infinite time on a limited tape, but it is not possible to cover an infinite tape in a finite time. Thus, time-restricted automata normally further restrict the set of strings that are reachable by the computation process. In our model we do not consider halting states (though it is not too difficult to modify it). Recent research has shown that non-halting TM is statistically correlated in rank with LBA. Though the position of non-halting LBA has not been explored, we conjecture that it will be correlated to Type-1 that with lower levels in the Chomsky hierarchy. However, this research establishes that more that 60% of algorithmic features, as shown in Figure 4, can be empirically tested by output distributions produced by sub-universal computation models - a key component for our research. As discussed later, restricting the TM based on the number of cycles is an unavoidable technicality as we need to collapse the quantum superposition.

![Figure 4: Rank correlation of empirical output distributions of other computational models to TM(5,2), showing emergence of the universal distribution](image)

There is no definitive metric to compare the power of tape-restricted TM besides assigning them to the wide class in the Chomsky hierarchy. Most landmark research on the universality of TM has assumed infinite tape length, adding a few points on the pareto curve of universality for the number of symbols and states. For restricted TM, instead of universality, more important metrics are expressibility and reachability. Expressibility is the set of solutions/states that an automaton can compute given a non-universal set of operations but not restricted by memory/space or steps/time resources. Reachability is the set of solutions/states that an automaton can compute by restricting space and/or time resources. For our computational model of a time-bound LBA, the reachability metric will be more suitable, semantically being the set of non-zero probability strings in the universal distribution we empirically obtain. The precise computation model used will be further discussed in section 3.
2.2 Algorithmic information

Algorithmic information theory (AIT) \cite{3} allows studying the inherent structure of objects, and qualify randomness without reference to a generating distribution (often assumed erroneously as the uniform prior in statistical machine learning). The theory originated when Ray Solomonoff (1960) \cite{4}, Andrey Kolmogorov (1965) \cite{5}, and Gregory Chaitin (1966) \cite{6} looked at probability, statistics, and information through the algorithmic lens. Now the theory has become a central part of theoretical computer science \cite{9}. Our proposed computation platform allows approximating these metrics:

- **Kolmogorov complexity**: Kolmogorov complexity (KC), also called algorithmic complexity, is the length of the shortest program that produces a particular output and halts on a specific TM. Formally, it is defined as:

\[
KC_T(s) = \min\{ |p|, T(p) \rightarrow s \}
\]

where \( T \) is a TM, \( p \) is a program, \( |p| \) is the length of the program, \( s \) is a string \( T(p) \rightarrow s \) denotes the fact that \( p \) executed \( T \) outputs \( s \) and halts.

Algorithmic complexity is not a computable quantity in the general case due to fundamental limits of computations that arise from the halting problem (impossibility to determine whether any given program will ever halt without actually running this program, possibly for infinite time). However, it has bounded lower-approximate property, i.e. if we can find a program \( p_l \) in a language \( l \) (e.g. Java, Assembly), \( |p_l| \geq |p| \).

Since KC depends on the particular model of computation (i.e. the TM and the language), it is always possible to design a language where a particular string \( s \) will have a short encoding no matter how random. However, the invariance theorem guarantees that, there exists a constant additive factor \( c_{T_1\rightarrow T_2} \) independent of \( s \), such that

\[
KC_{T_2} \leq KC_{T_1} + c_{T_1\rightarrow T_2}
\]

This constant is the compiler length for translating any arbitrary program \( p_{T_1} \) for \( T_1 \) to \( p_{T_2} \) for \( T_2 \).

- **Algorithmic probability**: Algorithmic probability (AP) is the chance that a randomly selected program will output \( s \) when executed on \( T \). The probability of choosing such a program is inversely proportional to the length of the program.

\[
AP_T(s) = \sum_{p : T(p) \rightarrow s} 2^{-|p|}
\]

Thus, the largest contribution to the term comes from the shortest program that satisfies the condition. It is uncomputable for the same reasons as KC. AP is related to KC via the following law:

\[
KC_T(s) = -\log_2(AP_T(s)) + c
\]

i.e., if there are many long programs that generate a dataset, then there has to be also a shorter one.

The arbitrary constant is dependent on the choice of a programming language.

AP can be approximated by enumerating TM of a given type and count how many of them produce a given output and then divide by the total number of machines that halt. When exploring machines with \( n \) symbols and \( m \) states algorithmic probability of a string \( s \) can be approximated as follows:

\[
D(n,m)(s) = \frac{|T \in (n,m) : T \text{ outputs } s|}{|T \in (n,m) : T \text{ halts}|}
\]

The Coding Theorem Method (CTM) approximates the KC as:

\[
CTM(n,m)(s) = -\log_2 D(n,m)(s)
\]

Calculating CTM, although theoretically computable, is extremely expensive in terms of computation time. The space of possible Turing machines may span thousands of billions of instances.

- **Universal distribution**: The universal a priori probability distribution (universal distribution for short) is calculated for all computable sequences. It is the distribution of the algorithmic probability of all strings of a specific size. This mathematically formalizes the notion of Occam’s razor and Epicurus’ principle of multiple explanations using modern computing theory for the Bayesian prediction framework. It explains observations of the world by the smallest computer program that outputs those observations, thus, all computable theories which describe previous observations are used to calculate the probability of the next observation, with more weight put on the shorter computable theories. This is known as Solomonoff’s theory of inductive inference.
• **Levin search:** Although the universal probability is uncomputable, Levin search [10], also called universal search, converges to the distribution sequence when executed for longer periods of time. It solves inversion problems by interleaving the execution of all possible programs on a universal Turing machine, sharing computation time equally among them, until one of the executed programs manages to solve the given inversion problem.

This is related to the concept of Levin complexity, a computable (though often intractable), time-bounded version of KC which penalizes execution time $z$.

$$LC_T(s, z) = \min \{|p| + \log(z), T(p) \to s\}$$

The standard invariance theorem also holds for this.

• **Speed prior:** The universal distribution doesn’t take into account the computing resource (tape and time) required when assigning the probability of certain data. This does not match our intuitive notion of simplicity (Occam’s razor). Jurgen Schmidhuber proposed the measure speed prior [11], derived from the fastest way of computing data.

$$SP_T(s) = \sum_{z=1}^{\infty} \sum_{p: T(p, z) \to s} 2^{-|p|}$$

where, program $p$ generates an output with prefix $s$ after $2^{z-|p|}$ instructions.

• **AGI:** Levin search has inspired various artificial general intelligence (AGI) approaches that build upon this to calculate the expected value of an action. The more computing power that is available, the closer their predictions are to the predictions of inductive inference. Jurgen Schmidhuber and Marcus Hutter developed these AGI algorithms like Adaptive Levin Search, Probabilistic Incremental Program Evolution, Self-Modifying Probabilistic Learning Algorithms, Hutter Search, Optimal Ordered Problem Solver, AIXItl and Gödel Machine. This is beyond the scope of our current research.

• **Logical depth:** Bennett’s logical depth (LD) [12] is the number of time steps to run the shortest program $p_{KC}$ to output $s$ in a specific TM $T$. It captures the general notion of physical complexity and neither random nor simple objects are sophisticated.

$$LD_T(s) = \min \{t(p_{KC}) : T(p_{KC}) \to s\}$$

The choice of the TM has a multiplicative factor for LD is more crucial.

In general it is not a semi-computable measure but can be approximated based on the decompression time of compression algorithms. For a given significance level $d$, LD can be calculated as:

$$LD_T(s, d) = \min \{\{t(p) : T(p_{KC}) \to s\} \land (|p| - |p_{KC}| \leq d)\}$$

• **Omega number:** The Omega number, also called Chaitin constant or halting probability is a real number that, represents the probability that a randomly constructed program will halt.

$$\Omega_T = \sum_p 2^{-|p|}$$

where, $T$ denotes a prefix-free TM.

While many of the algorithmic metrics discussed here are uncomputable, most of them are either upper or lower semi-computable, i.e. it can be approached in the limit from above or below by enumerating the Turing machines [13]. Though time-bounded algorithmic metric has been researched before for various applications, pragmatically they remain intractable for real-world problem sizes for classical algorithms, motivating this research.

There are also methods to approximate these metrics by dividing the output into blocks. Block decomposition method (BDM) approximates the CTM value for an arbitrarily large object by decomposing into smaller slices of appropriate sizes for which CTM values are known and aggregated back to a global estimate. Quantum acceleration of such approaches forms the future aspect of this work.
### 2.3 Quantum circuit model

A quantum mechanical model of Turing machines was described by Paul Benioff using Hamiltonian models [13][15]. A computationally equivalent model using quantum gates (similar to classical binary logic gates) in a quantum circuit was proposed by David Deutsch [16][17][18]. Due to the more intuitive nature the circuit model has become the standard for quantum algorithm development. A generalized quantum Turing machine (GQTM) [19], which contains QTM as a special case and includes non-unitary dynamics (irreversible transition functions) as well, allow the representation of quantum measurements without classical outcomes [20]. From a more real-world quantum computer architecture perspective, QTM was compared [21] to quantum random-access machine (QRAM) and quantum random-access stored-program (QRASP) model, proving their equivalence in bounded-error polynomial-time formulations using the Solvay-Kitaev theorem. The QRASP considers that a program is stored in classical registers, and thus treated as classical data. A QRASP modeling a fully quantum computer can encode a program as quantum data, consequently generalizing the QRASP model to parallel quantum random-access stored-program machines (PQRASP) allowing a superposition of programs [22].

Though our computation model is more similar to a PQRASP, they are more suited for the formalization of quantum programming languages. We review the circuit implementations of QTM [23], quantum finite automata (QFA) [24][25] and other quantum equivalent automata models [26], which inspires our circuit architecture. Like its classical counterpart, a quantum Turing machine (QTM) is an abstract model capturing the power of quantum mechanical process for computation. Any quantum algorithm can be expressed formally as a particular QTM. QTM generalizes the classical Turing machine (TM) such that the internal states of a classical TM are replaced by pure or mixed states in a Hilbert space. The transition function is replaced by a collection of unitary matrices that map the Hilbert space to itself.

A classical TM is extended to the complex vector space for a QTM. For a three-tape QTM (one tape holding the input, a second tape holding intermediate calculation results, and a third tape holding output):

- The set of states is replaced by a Hilbert space.
- The tape alphabet symbols are replaced by a Hilbert space (usually a different Hilbert space than the set of states).
- The blank symbol corresponds to the zero-vector.
- The input and output symbols are usually taken as a discrete set, as in the classical system; thus, neither the input nor output to a quantum machine need be a quantum system itself.
- The transition function is a generalization of a transition monoid. It is a collection of unitary matrices that are automorphisms of the Hilbert space.
- The initial state may be either a mixed state or a pure state.
- The set of final or accepting states is a subspace of the Hilbert space.

It is important to realize that this is merely a sketch than a formal definition of a quantum Turing machine. Some important details like how often a measurement is performed are not defined. This is circumvented by formal proofs showing equivalence with an oblivious QTM, whose running time is a function of the size of the input (independent of the structure of the input). However, how to practically translate that to a priori knowledge of the number of steps the computation needs to be executed before collapsing the superposition needs further exploration [27][28].

$t \geq n$ steps of a QTM on an input of length $n$ can be simulated by a uniformly generated family of quantum circuits linear in $t$. The circuit architecture of a QTM is inspired by the solid-state implementation of the computational equivalent model to a TM of a 1-dimensional elementary cellular automata. Cellular automata is a discrete model of computation consisting of a regular grid of cells in any finite dimension (or type of tessellation). Each cell at any step in time is in one of a finite number of states, (such as on and off) (in contrast to a coupled map lattice). For each cell, a set of cells called its neighborhood is defined relative to the specified cell. The next step in time is created according to some fixed rule (typically same over the entire space) that determines the new state of each cell in terms of the current state of the cell and the states of the cells in its neighborhood. The whole grid typically updates simultaneously. The model with 1-dimensional tape over 2 symbols and a neighborhood size of 1 cell in either size along with the current cell, is called elementary cellular automata (ECA). These were studied extensively revealing interesting features like the Rule 110 (one of the 256 possible rules) capable of universal computation. Solid-state implementation of cellular automata involves expressing the update rule in terms of classical logic gates (e.g. AND, OR, NOT, NAND) as shown in Figure 5. For each cell, the cell and its two immediate neighbors are read, their values entered into the transition function, and the cell updated with the new value for the subsequent generation. The transition function (e.g. here for Rule 110) can be represented as a truth table, a logic function or a logic circuit. The result from this function is then written to the corresponding cell in a duplicate array. In this example, transition functions are being calculated from $C_L$ and written sequentially to $C_R$. Once a generation is complete, the origin array ($C_L$ here) will be cleared, and the results of rule
Figure 5: General execution of an elementary cellular automaton [29]

 applications to cells in $C_R$ written to $C_L$.

Figure 6: Universal computation in classical and quantum CA models as local quantum circuits [23]

The no-cloning principle prevents a direct translation of this classical ECA architecture on quantum circuits. This is because, when the state is sequentially updated (in practice), we lose the neighbor information to update the consecutive cell. The local simultaneous update is achieved by maintaining a local one-hot marker denoting the presence/absence of the tape head at the location, as shown in Figure 6a. This idea is extended for simulating a QTM in the circuit model. It defines a fixed unitary $G$ which encodes the rule, or the transition function and interleaved in a particular way as shown in Figure 6b. For the quantum version, the inputs are qubits, so they can be in a superposition of states, allowing simultaneous evolution of various inputs (initial Turing tapes).

Not all transition functions describe valid quantum Turing machines as it is imperative that the global evolution is unitary on the Hilbert space. QTM that exhibit local causal behavior can be mapped to such a transition function. Since the model is universal, there will be a transition function that will in effect read a stored program and the input from the tape as qubits and perform the computation entirely by local operations. However, it is not immediately clear how to construct such a unitary for an arbitrary transition function. Moreover, modeling the stored-program model on this local model would incur a high cost in
the number of operations to affect the tape, as at each step a TM head moves only by one step. This is alleviated by our research, where we drop the local structure on the transition function if read as input. Our quantum circuit intuitively provides a mechanistic model of a Turing machine, while allowing both the tape and the transition function to be in superposition. Thus, in effect it provides a scalable quantum circuit implementation for the PQRASP model of computation. The superposition of programs is the key feature allowing the framework to be applied for estimating algorithmic features, like the algorithmic probability and universal distribution.

2.4 Computability and computational complexity

Theoretical computer science strives to answer the capabilities and limitations of computing machines using 3 intertwined approaches:

- Computability theory: What problems are solvable for computers?
- Complexity theory: What makes some problems easy or hard for computers?
- Automata theory: What formal model of computation is required for a problem?

Computability theory works at a larger scale, often answering fundamental questions in mathematics and computation, like Gödel’s Incompleteness Theorem, the Entscheidungsproblem or the Halting problem. These results pose critical limits on the feasibility of any algorithms that are developed based on the language and automata models that accept strings based on the grammar of the language.

Complexity theory bridges the gap between practical algorithms running on computing hardware made out of simpler grammar of programming languages, and the hierarchy of languages in computability theory. The complexity of algorithms has been classified into a multitude of classes [30]. The boundaries and relationships between these classes are sometimes not proven but are based on current knowledge and popular conjectures in the scientific community as shown in Figure 7.

![Extended Chomsky Hierarchy](image)

(a) Extended Chomsky Hierarchy Reloaded

![Complexity Hierarchies](image)

(b) Complexity hierarchies [31]

Figure 7: Computability hierarchy and computational complexity classes

The most commonly referred complexity classes are of P and NP. Their relation to computability levels and quantum complexity classes are important for efficient algorithm development. Polynomial time (PTIME or P for short) refers to the class of algorithms that are efficiently solvable (or tractable) by a deterministic Turing Machine (DTM) in an amount of time that is polynomial in the size of the input problem. Non-deterministic polynomial time (NTIME or NP for short) refers to the set of problems that are tractable by a non-deterministic Turing Machine (NTM) in polynomial time. In contrast to Turing Machines (examples of DTM), in a NTM, the set of rules may prescribe more than one action to be performed for any given situation. The resolution of action is based on thought experiments. A way to look at it is to say that the machine is the ‘luckiest possible guesser’; it always picks a transition that eventually leads to an accepting state, if there is such a transition. Alternatively, it can be imagined as the machine ‘branches’ into many copies, each of which follows one of the possible transitions, thus, instead of a DTM’s single ‘computation path’, NTM follows a ‘computation tree’. If at least one branch of the tree halts with an ‘accept’ condition, the NTM is said to accept the input. P can easily be reasoned to be a subset of NP (might not be a proper subset). Given a certificate for a problem in P, we can ignore the certificate and just solve the problem in polynomial time, alternatively, a DTM is also trivially an NTM that just happens to not use any non-determinism. Whether an algorithm that can be efficiently checked for correctness can also
be efficiently solved is an open question \cite{32} (one of the Millennium problems). Another important concept here is of NP-completeness, which is the set of problems in NP such that every other problem in NP can be transformed (or reduced) to it in polynomial time.

Often the access to a genuine random-number source widens the class of feasible problems for a computer using probabilistic computation. Bounded-error probabilistic polynomial-time (BPP) class involves decision problems solvable by an NTM such that the ‘yes’ answers are achieved with \( \geq \frac{2}{3} \) of the computation paths, while the ‘no’ answers are achieved with \( \leq \frac{1}{3} \) paths. Bounded-error quantum polynomial-time (BQP) is the class of decision problems solvable by a quantum circuit whose length and number of qubits scale polynomially with respect to the instance size. Like its classical counterpart BPP, the error probability is bounded by at most \( \frac{1}{3} \) for all instances. Using Chernoff bound, the constant \( \frac{1}{3} \) can be reduced arbitrarily on repetition. These classes also have their exact versions, called, zero-error probabilistic polynomial-time (ZPP) and exact quantum polynomial-time (EQP). These constants for BPP \( (\frac{2}{3} \pm \frac{1}{3}) \), when modified to \( \frac{1}{2} \) gives the probabilistic polynomial-time (PP) class and to \( \frac{1}{3} \) gives the randomized polynomial-time (RP) class. Adding post-selection capabilities to these classes gives the class \( BPP_{path} \) and PostBQP (shown equal to PP). There are also many related classes based on verifiable proof systems (e.g. MA, IP, AM) and their quantum counterparts (e.g. QMA, QIP, QAM).

To compare the efficiency of quantum algorithms, it is crucial to compare them with the current best algorithm on classical computers and their complexity. The relationships of quantum computation complexity classes with the class hierarchy are not fully understood and are riddled with many open conjectures. P is a subset of BQP, but it is interesting to study algorithms that fall outside P, but in BQP. Quantum algorithms however does not allow super-Turing computation (e.g. solving the halting problem) thus placing them at the same Turing degree in the arithmetic hierarchy as shown in Figure \ref{fig:arithmetic_hierarchy}. However, in a technique similar to using quantum search, there are possibilities of accelerating classical techniques that approximate an uncomputable algorithmic metric.

The Bachmann–Landau notation (or asymptotic notations) are used to describe the limiting behavior of a function with the domain tending towards a particular value (often infinity). The big O notation is used to classify algorithms according to their running time or space requirements growth rate with input size. A description of a function in terms of big O notation provides an upper bound on the growth rate of the function. Formally, for real/complex valued functions \( f \) and \( g \) defined on some unbounded subset of the real positive numbers,

\[
  f(x) = O(g(x)) \quad \text{as } x \to \infty
\]

iff \( \forall \text{ sufficiently large } x \exists \ M \in \mathbb{R} \text{ and } x_0 \in \mathbb{R} \text{ s.t. } \left| f(x) \right| \leq M g(x) \quad \forall \ x > x_0 \) The O notation for a function \( f \) is derived by the simplification rules:

- If \( f(x) \) is a sum of several terms, term with largest growth rate is kept
- If \( f(x) \) is a product of several factors, constants are omitted

For example, \( f(x) = 4x^4 + 2x^3 + 100 \) has order \( O(x^4) \). Infinite asymptotics often looks over the lower order terms and constants, which might be the deciding factor for practical applications. Even for exponential problems in \( O(n^\omega) \) versus constant time \( O(1) \), there is a cross-over of applicability, where the preference shifts. It is important to estimate where the problem of interest lies on the horizontal axis for stricter comparison among algorithms. Other kinds of bounds on asymptotic growth rates use the symbols \( o, \Omega, \omega, \) and \( \Theta \). The \( \Theta \) notation is used to denote average-case complexity. While it is much more difficult to prove bound without guarantees on input data, it is of more practical significance and will be used later in this research.
3 Computation model

In our computation model, we will restrict a $m$ states, $n$ symbols, 1 dimension tape Turing machine by limiting the maximum time steps $z$ before a forced halt is imposed. This automatically bounds the causal cone on the tape to $[-z, +z]$ from the initial position of the tape head.

The tape is initialized to the blank character as this does not reduce the computational power of the TM (this can be thought of as, the initial part of the program prepares the input on the tape and then computes on it). The tape length, $c$ is ideally infinite. However, like the memory size of CPU, a specific size of the tape is enough for storing the program and as the work-memory for executing it. Specifically, since our model restricts the execution steps of the computation, the tape outside the causal cone of the computation can be trimmed. Thus, the range of pragmatic values for the tape length is $c \leq 2(z+1)$.

For our experiments, we chose $c = z$ and a circular tape. This gives us similar final tape diversity (sometimes in a flipped manner) while reducing the tape resource (qubits). This shorter tape comes as the cost of not able to capture cases where part of the tape is unaffected from its initial state, which is not so interesting.

The generic model of this restricted Turing machine is represented by this 10-tuple.

$$T = \langle Q, \Gamma, b, \Sigma, \delta, q_0, F, z, d, c \rangle$$

- $Q$ is a finite, non-empty set of states.
- $\Gamma$ is a finite, non-empty set of symbols allowed on the tape, called the tape alphabet.
- $b \in \Gamma$ is the blank symbol.
- $\Sigma \subseteq \Gamma \setminus \{b\}$ is the set of input symbols, that is, the set of symbols allowed to appear in the initial tape contents.
- $\delta$ is a partial function called the transition function. It defines the next state, tape movement and write symbol based on the current state and the read symbol.
- $q_0 \in Q$ is the initial state.
- $F \subseteq Q$ is the set of final states or accepting states.
- $z$ is the number of steps the machine is executed before a forced halt is imposed.
- $d$ is the dimension of the tape. It also specifies if the tape is circular by a $\circ$ symbol for each dimension.
- $c$ is the length of the tape in each dimension.

For our computation model of $m = |Q|$ states, $n = |\Gamma|$ symbols, $d = 1^\circ$ dimension (circular) tape restricted Turing machine,

$$T = \langle \{Q_0, Q_1, \ldots, Q_{m-1}\}, \{s_0, s_1, \ldots, s_{n-1}\}, s_0, \emptyset, \delta, Q_0, \emptyset, z, 1^\circ, z \rangle$$

Note that, $\Sigma$ is empty, thus the tape is always initialized to the blank character $b = s_0$. The set of accepting states $F$ is also empty to prevent the machine from halting before $z$ steps. This includes machines that have defined halting states, by modifying the transition function to remain in the same state and write the same symbol that is read (in effect simulating a no-operation) once these states are reached.

Thus, the transition table exhaustively lists a transition for each combination of $(Q - F) = Q$ and $\Gamma$.

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{0, 1\}$$

where, 0 is left shift, 1 is right shift for the 1 dimensional tape. Each entry of the transition table requires $\log_2(m) + \log_2(n) + d$ bits and there are a total of $m \times n$ entries. Thus, the number of bits required to specify a single transition function is:

$$q_5 = (m \times n) \times (\log_2(m) + \log_2(n) + d)$$

The number of different machines (programs) that can be represented using $q_5$ bits is represented by:

$$P = 2^{q_5}$$

3.1 Examples

Here some examples of this computation model are enumerated. The cases are labeled as per the number of symbols, states and tape dimension (Case $m$-$n$-$d$). We start with the smallest natural number 1 and explore larger values of symbols and states (however, we will only consider 1 dimensional Turing tape).

For these experiments, the number of iterations is set equal to the size of the program, i.e. $z = q_5$. This allows us to compare the final tape and program to infer which programs can self-replicate, as will be explained further in the application section. The model as shown in Figure [8] is executed for $z$ iterations.
3.1.1 Case 1-1-1

For this case, the number of states $m = 1$ with the state set $Q : \{Q_0\}$. The alphabet is $\Gamma : \{0\}$, thus, $n = 1$ (the unary alphabet). This gives the values $q_0 = 1 \ast 1 \ast (0 + 0 + 1) = 1$ and $P = 2^1 = 2$ using the formula discussed before.

The machine is run for $z = q_0 = 1$ iteration. The initial tape is of length $c = z = 1$. To distinguish the blank character from a written 0, we will use $o$. Thus, the initial tape is: $o$.

The outputs for each program (description number) is shown on Table 1. $R_s/W_s$ refers to the read/write symbols and $M_{l/r}$ refers to the tape movement of left/right.

| P# | $Q_0 R_0$ | Final tape |
|----|------------|------------|
| 0  | $Q_0 M_0 W_0$ | 0          |
| 1  | $Q_0 M_1 W_0$ | 0          |

Table 1: Case 1-1-1 exhaustive UTMs

3.1.2 Case 2-1-1

For this case, the number of states $m = 2$ with the state set $Q : \{Q_0, Q_1\}$. The alphabet is again $\Gamma : \{0\}$, with, $n = 1$ (the unary alphabet). This gives the values $q_0 = 2 \ast 1 \ast (1 + 0 + 1) = 4$ and $P = 2^4 = 16$.

The machine is run for $z = q_0 = 4$ iteration. The initial tape is of length $c = z = 4$. The initial tape is: $o_000$, where the underline denotes the initial position.

The outputs for each program (description number) is shown on Table 2. It is easy to see all m-1-d cases will result in tapes with the blank/zero characters. The 0000 string has an algorithmic probability of 1. It is also the only string in this language. Thus, only the description number 0 case can be considered self-replicating.

3.1.3 Case 1-2-1

For this case, the number of states $m = 1$ with the state set $Q : \{Q_0\}$. The alphabet is $\Gamma : \{0, 1\}$, with, $n = 2$ (the binary alphabet). This gives the values $q_0 = 1 \ast 2 \ast (1 + 1 + 0) = 4$ and $P = 2^4 = 16$. The machine is run for $z = q_0 = 4$ iteration. The initial tape is of length $c = z = 4$. The initial tape is similar to the last case: $o_000$.

The outputs for each program (description number) is shown on Table 3. Machines 00 and 15 are self-replicating. This depends on how the state machine is encoded as the description number and where is the starting position of the tape. Strings 0000 and 1111 have a probability of 0.5 each. Note that the transitions for $R_1$ are never effectively used as the head never returns to a cell which was previously written by 1. If we run the experiment for more than 4 steps then we will see more variety in the output.
Let us consider the case where the number of states $m = 2$ (with the states set $Q \{Q_0, Q_1\}$) with the alphabet $\Gamma \{A, C, G, T\}$, with, $n = 4$. This is inspired by the DNA alphabet. This gives the values $q_3 = 2 \times 4 \times (1 + 2 + 1) = 32$ and $P = 2^{32} = 4294967296$. It is clear that even for the simple case of the DNA alphabet, an exhaustive search by enumeration is not possible on a classical algorithm (and maybe even on a quantum computer simulator running on classical hardware).

### 3.1.4 Case 2-4-1

Let us consider the case where the number of states $m = 2$ (with the states set $Q \{Q_0, Q_1\}$) with the alphabet $\Gamma \{A, C, G, T\}$, with, $n = 4$. This is inspired by the DNA alphabet. This gives the values $q_3 = 2 \times 4 \times (1 + 2 + 1) = 32$ and $P = 2^{32} = 4294967296$. It is clear that even for the simple case of the DNA alphabet, an exhaustive search by enumeration is not possible on a classical algorithm (and maybe even on a quantum computer simulator running on classical hardware).

### 3.1.5 Case 4-4-1

If we consider the number of states $m = 4$ (with the states set $Q \{Q_A, Q_C, Q_G, Q_T\}$) corresponding to the alphabet $\Gamma \{A, C, G, T\}$, with, $n = 4$ (the DNA alphabet), we can model the execution of the genetic code as a Markov process. This gives the values $q_3 = 4 \times 4 \times (2 + 2 + 1) = 80$ and $P = 2^{80} = 1208925819614629174706176$. This exponential growth in the number of cases shows the difficulty of estimating algorithmic metrics motivating some of our future research directions as will be discussed later.

### 3.1.6 Case 2-2-1

This case is both non-trivial as well as within the bounds of our current experimentation. The number of states $m = 2$ with the state set $Q \{Q_0, Q_1\}$. The alphabet is $\Gamma \{0, 1\}$, thus, $n = 2$ (the binary alphabet).
This gives the values \( q_0 = 2 \times 2 \times (1 + 1 + 1) = 12 \) and \( P = 2^{12} = 4096 \) using the formula discussed before.

The machine is run for \( z = q_0 = 12 \) iteration. The initial tape of length \( c = z = 12 \) is: \( \text{ooooo00000000} \)

The program (description number) is encoded as:

\[
[QMW]^{Q_1 R_1}[QMW]^{Q_1 R_0}[QMW]^{Q_0 R_1}[QMW]^{Q_0 R_0}
\]

There are too many cases to enumerate by hand, so a Python script (the classical kernel we want to accelerate) is written that emulates our restricted model of the Turing machine for all 4096 cases. The program is in the following link:

https://github.com/Advanced-Research-Centre/QuBio/blob/master/Project_01/classical/

We find machines 0 and 4095 are self-replicating. The tape output for all the 4096 machines is given in Figure 9.

![Figure 9: Tape output for Case 2-2-1](image)

The universal distribution is (for non-zero probabilities):

| Tape State | Probability |
|------------|-------------|
| 0000       | 0.420410    |
| 0001       | 0.002197    |
| 0002       | 0.001953    |
| 0003       | 0.000244    |
| 0004       | 0.000244    |
| 0007       | 0.000732    |
| 0008       | 0.003174    |
| 0015       | 0.000488    |
| 0021       | 0.002441    |
| 0024       | 0.000244    |
| 0027       | 0.000244    |
| 0065       | 0.000244    |
| 0128       | 0.003174    |
| 0192       | 0.000244    |
| 0193       | 0.000244    |
| 0256       | 0.000244    |
| 0257       | 0.000244    |
| 0512       | 0.001953    |
| 0517       | 0.000244    |

The universal distribution is (for non-zero probabilities):
4 Quantum implementation

The overall design of the quantum circuit to implement the computation model is presented here. This is a model of a QTM having the corresponding parts of a classical TM as qubits. As discussed in section 2, the circuit models a PQRASP more closely than a QTM. The circuit will require some ancilla qubits which will be introduced later. The test qubits for experimenting on the implementation are not part of the logic and thus not shown in Figure 10. The TM step needs to be repeated for the number of steps $z$ we intend to execute the machine before measuring out the qubits.

![Figure 10: Circuit blocks for the quantum implementation of a TM step in the stored-program model](image-url)

Figure 10: Circuit blocks for the quantum implementation of a TM step in the stored-program model.
4.1 Qubit complexity

The qubit complexities of the design elements are discussed here. The generic formula is derived before applying to the specific case of the 2-2-1 TM.

- **Alphabet**: Alphabet set cardinality \( n = |Γ| \) is the number of symbols in the alphabet. The number of bits/qubits required to represent a symbol, \( q_γ = \lceil \log_2(n) \rceil \)
- **Head position**: The current head position is represented either as binary or one-hot encoding. The one-hot encoding is more expensive in the number of qubits, but better in terms of gates. The number of bits/qubits required for one-hot encoding [23] is the same as the number of cells \( c \). Since the simulation bottleneck is the number of qubits instead of the number of gates, we prefer the binary encoding. For binary encoding, \( q_{head} = \lceil \log_2(c) \rceil \)
- **Read head**: The read head temporarily stores the content of the current head position, requiring bits/qubits, \( q_{read} = q_γ \)
- **Write head**: The write head temporarily stores the content to be written to the current head position, requiring bits/qubits, \( q_{write} = q_γ \)
- **Turing tape**: The number of bits/qubits required for the restricted tape size of \( c \) is, \( q_{tape} = c \times q_γ \)
- **Movement**: Specifying the movement of a \( d \) dimensional Turing tape requires, \( q_{move} = d \)
- **Current state**: The current state in binary encoding requires, \( q_{state} = \lceil \log_2(m) \rceil \). In a one-hot coded format, it would require \( m \) qubits.
- **Transition table**: The transition function is a unitary matrix that transforms the input and the current state to the output, next state and movement of the tape. Thus, for each combination of state and read character, we need to store the next state, write character and movement. The number of qubits required are, \( q_δ = (m \times n) \times (q_{state} + q_{write} + q_{move}) \)
- **Computation history**: Since the quantum circuit is reversible, the computation history for \( (z - 1) \) steps needs to be stored in ancilla qubits. The computation history is specified by the state and read symbol for each step, requiring, \( q_{ch} = (z - 1) \times (q_{state} + q_{read}) \)

Thus, the qubit complexity of the implementation (assuming \( q_a \) ancilla qubits) is:

\[
q_{QTM} = q_δ + q_{state} + q_{move} + q_{head} + q_{read} + q_{write} + q_{tape} + q_{ch} + q_a \\
= (m \times n \times \log(m) + \log(n) + d) + \log(m) + d + \log(c) + \log(n) + \log(n) + (c \times \log(n)) \\
+ ((z - 1) \times (\log(m) + \log(n)))) + q_a
\]

Considering the 2-2-1 TM example, the values of \( m = 2, n = 2, d = 1, c = 12, z = 12 \) are substituted in the above equation (all logarithms are base-2 and rounded up to be nearest integer) to yield,

\[
q_{221} = (2 \times 2 \times (\log(2) + \log(2) + 1)) + \log(2) + 1 + \log(12) + \log(2) + \log(2) + (12 \times \log(2)) \\
+ (11 \times (\log(2) + \log(2)))) + q_a \\
= 12 + 1 + 1 + 4 + 1 + 1 + 12 + 22 + q_a \\
= 54 + q_a
\]

Simulating in order of 50 qubits is near the quantum supremacy limits but the circuit is not always in full superposition. This allows using smart simulation techniques in the QX Simulator.
4.2 Initialize

The initialization circuit depends on the target application for this framework. For measuring the algorithmic probability or the universal distribution, all possible programs (represented by the transition table) needs to be evolved in a superposition. All other qubits are kept at the ground or default state of $|0\rangle$. The circuit is shown in Figure 11.

\[
\begin{align*}
|q_a\rangle & \xleftarrow{\text{f}^3} \text{PrepZ} \\
|q_{ch}\rangle & \xleftarrow{\text{f}^{22}} \text{PrepZ} \\
|q_{tape}\rangle & \xleftarrow{\text{f}^{12}} \text{PrepZ} \\
|q_{write}\rangle & \xleftarrow{\text{f}^1} \text{PrepZ} \\
|q_{read}\rangle & \xleftarrow{\text{f}^1} \text{PrepZ} \\
|q_{head}\rangle & \xleftarrow{\text{f}^4} \text{PrepZ} \\
|q_{move}\rangle & \xleftarrow{\text{f}^1} \text{PrepZ} \\
|q_{state}\rangle & \xleftarrow{\text{f}^1} \text{PrepZ} \\
|q_8\rangle & \xleftarrow{\text{f}^{12}} \text{PrepZ} H
\end{align*}
\]

Figure 11: Initialization quantum circuit for TM 2-2-1
4.3 TM step

Each iteration of the TM undergoes the following transforms:

1. \( \{q_{\text{read}}\} \leftarrow U_{\text{read}}(\{q_{\text{head}}, q_{\text{tape}}\}) \)
2. \( \{q_{\text{write}}, q_{\text{ch}}, q_{\text{move}}\} \leftarrow U_{\delta}(\{q_{\text{read}}, q_{\text{state}}, q_{\delta}\}) \)
3. \( \{q_{\text{tape}}\} \leftarrow U_{\text{write}}(\{q_{\text{head}}, q_{\text{write}}\}) \)
4. \( \{q_{\text{head}}\} \leftarrow U_{\text{move}}(\{q_{\text{head}}, q_{\text{move}}\}) \)

These sub-steps in each iteration (i.e. read, transition evaluation, write, move and reset) are shown in the circuit in Figure 12.

Figure 12: Circuit elements in a single TM step (special symbols are explained in text)

Few special circuit symbols are used. These are:

- 1-hot control: this type of control on a qubit bus represents for each qubit on the bus being high, some operation happens. It is different from dual-control as when one line of the bus is considered, the other lines are don’t-care.

- Dual control: where when the control qubit is high some operation happens on the target, while when the control qubit is low, some other operation takes place. This is similar to the if-elseif construct in programming.

- Binary coded control: this type of control on a qubit bus represents for each qubit encoding on the bus some operation happens. It is the bus-from of dual-control.

- Bus target: it is the bus-form of target.

We will now discuss each of these TM steps in detail.
4.3.1 Read tape

The quantum circuit implements a multiplexer with the tape as the input signals and the binary coded head position as the selector lines. The read head is the output.

\[ |q_{tape}\rangle \quad |q_{read}\rangle \quad |q_{head}\rangle \]

The read circuit for the TM 2-2-1 is shown in Figure 13.

![Figure 13: Read tape quantum circuit for TM 2-2-1](image-url)
4.3.2 Transition table lookup

The transition table encoding is: \([Q_t|R_T] \rightarrow [Q_{t+1}|M_{l,r}|W_T]\)

Note that we use \(q_{\text{state}}\) instead of \(q^+_{\text{state}}\) though we are storing the next state in the qubit. This is corrected by the reset circuit. The transition function circuit for the TM 2-2-1 is shown in Figure 14.

Figure 14: Transition function quantum circuit for TM 2-2-1
4.3.3 Write tape

The quantum circuit implements a de-multiplexer with the tape as the output signals and the head position as the selector lines. The write head is the input.

\[ |q_{\text{tape}}\rangle / \square \]
\[ |q_{\text{write}}\rangle \bullet \]
\[ |q_{\text{head}}\rangle \]

The write circuit for the TM 2-2-1 is shown in Figure 15. The qubit elements not involved are not shown.

![Figure 15: Write tape quantum circuit for TM 2-2-1](image-url)
4.3.4 Move

There are many choices for implementing the move, e.g. a looped tape (overflow/underflow is ignored and trimmed), error flag is raised and halts, overflow/underflow is ignored, etc. Here a looped tape is implemented.

The head is incremented/decremented using the move qubit as control.

\[
\begin{align*}
|q_{\text{head}}\rangle & \rightarrow \text{Inc} \quad \text{Dec} \quad X \\
|q_{\text{move}}\rangle & \quad X \\
\end{align*}
\]

The increment/decrement circuit is a special case of the quantum full adder \[35\] with the first register, \(a\) set to 1. For the TM 2-2-1 case, the length of the circular tape is 12, thus, the increment and decrement needs to be modulo 12. For increment, when the \(q_{\text{head}}\) equals 11, it should increment to \((11 + 1) \mod 12 = 0\), while for decrement, \((0 - 1) \mod 12 = 11\). Thus, for these edge cases, we need to increment/decrement by 5 instead of 1. We set the \(a_2\) bit to change the effective value of \(a\) from \(1 = 0001_2\) to \(5 = 0101_2\) for the addition/subtraction. This operation is conditioned on the head value and move bit, and denoted as the overflow/underflow qubit \(|\text{ovfw}\rangle/|\text{udfw}\rangle\). The \(a_2\) bit is uncomputed based on the incremented value being 0 or the decremented value being 11.

The carry (\(C\)), sum (\(S\)) and reverse carry (\(C^\dagger\)) blocks are defined as follows:

\[
\begin{align*}
\text{.sum} & : \text{cnot A0,S0} \quad \text{toffoli A0,B0,C1} \\
\text{.carry} & : \text{cnot B0,S0} \quad \text{cnot A0,B0} \quad \text{toffoli C0,B0,C1} \\
\text{.reverse_carry} & : \text{cnot A0,S0} \quad \text{cnot A0,B0} \quad \text{toffoli C0,B0,C1} \\
\end{align*}
\]
**Increment**

In this design, \( c_3c_2c_1 = 000 \) are 3 ancilla (\( c_0 \) is not required), \( a_0 = q_{move}, a_3a_2a_1 = 000, b_3b_2b_1b_0 = q_{head}q_{head}q_{head}q_{head} \) and \( b_4 \) is ignored. The circuit in Figure 16 shows the 4-bit modulo-12 quantum increment circuit using the quantum adder blocks.

![Figure 16: Modulo-12 quantum adder for implementing move tape head for TM 2-2-1](image)

The circuit can be simplified considering some of the control qubits are always in the 0 state, so those CNOT/Toffoli gates can be ignored. The optimized increment circuit for the TM 2-2-1 is shown in Figure 17.

![Figure 17: Modulo-12 increment quantum circuit for TM 2-2-1](image)
Decrement

In this design, \(c_3c_2c_1 = 000\) are 3 ancilla (\(c_0\) is not required), \(a_0 = q_{move}\), \(a_3a_2a_1 = 000\), \(b_3b_2b_1b_0 = q_{head}q_{head}q_{head}q_{head}\) and \(b_4\) is ignored. The circuit in Figure 18 shows the 4-bit modulo-12 quantum decrement circuit using the reverse order of blocks as the standard quantum adder.

![Figure 18: Modulo-12 quantum subtractor for implementing move tape head for TM 2-2-1](image)

The circuit can be simplified considering some of the control qubits are always in the 0 state, so those CNOT/Toffoli gates can be ignored. The optimized increment circuit for the TM 2-2-1 is shown in Figure 19.

![Figure 19: Modulo-12 decrement quantum circuit for TM 2-2-1](image)
4.3.5 Reset

Is it possible to implement this quantum circuit without an ancilla? In general, in quantum, we want to implement the following logic:

\[
\text{if } a == b, \text{then } a = c
\]

This is a violation of reversibility as the old value of \( a \) is getting erased. Both the state and the read together preserve the evolution history of the computation. Thus, we need ancilla qubits in each step of the computation that would hold the transition history for the TM.

Besides the ancilla in State and Read, there are other qubits that needs to be reset for the next cycle. These are the write qubit and move qubit. The move and write qubits can be reset by calling the FSM transition function once again with the previous state and the read.

The entire circuit can now be compiled from these parts. This was implemented on OpenQL and tested on the QX Simulator. The copyleft implementation can be found at [https://github.com/Advanced-Research-Centre/QuBio](https://github.com/Advanced-Research-Centre/QuBio)
5 Applications

The model of computation discussed in this research can estimate the universal distribution, and thereby the other algorithmic metrics in a time-bound complexity. Estimation of algorithmic properties of a dataset can be very useful as it points to mechanistic connections between elements of a system, even such that do not yield any regular statistical patterns that can be captured with more traditional tools based on probability theory and information theory.

Here we present a framework for empirical experimentation on a quantum accelerator. This is shown in Figure 20. The implementations for the initial two blocks were presented in detail in this paper. Measuring out the tape of this system would by default approximate the time-bound universal distribution for the linear bounded automaton. These experiments can be augmented to discover other properties, like the shortest FSM producing the data (the Kolmogorov complexity) or cases where the FSM and tape matches or has minimum Hamming distance (as a model of self-replication). It can also be used to accelerate the meta-biology experiment of open-ended evolution [36] as Busy Beaver functions originally proposed in [37].

The specific target solutions can be amplified using quantum algorithms of Grover search or variational approaches like the quantum approximate optimization algorithm (QAOA). Experimenting on the framework forms our current research direction. However, an accelerated estimation of the universal distribution itself is a valuable contribution from this research that can have various applications.

| FSM | Tape |
|-----|------|
| $H|0\rangle\otimes|\psi\rangle$ | $|0\rangle\otimes|\delta\rangle$ |
| measure | target DNA sequence |

Figure 20: Proposed application development platform

5.1 Quantum accelerated meta-biology

Meta-biology, as introduced by Gregory Chaitin, provides DNA linguistics [38, 39, 40] with an algorithmic information theory perspective, allowing exhaustive enumeration as an experimental method to understand the DNA as a code. In recent years, the field of algorithmic bioinformatics [41] has been pioneered by multi-disciplinary research in Hector Zenil’s group. This group has achieved impressive results [42] using the coding Theorem method (CTM) as a method to approximate the algorithmic probability and thereby the universal distribution. While larger problem sizes are intractable in classical computation, the block decomposition method (BDM) was used to approximate the CTM. Our research will complement most of the work [43] in this field that uses classical algorithms, by considering a quantum accelerator extending to the same wide range of applications. Our implementations are currently executed on quantum computing simulators like the QX due to the unavailability of quantum processors that meet the requirements of the multiplicity, connectivity and quality of qubits. While this limits our problem size due to the exponential overhead of simulation, the high-level OpenQX is generic and can target real quantum processors once they reach a better technology readiness level. This is facilitated by the abstraction layers of the quantum accelerator stack as part of this research.

In our past research, we have developed quantum algorithms for accelerating DNA sequence reconstruction using both alignment [44] and assembly [45] based approaches. This research targets the analysis phase, i.e. after the genome has been sequenced. Thus from pattern matching, this work extends into the domain of pattern recognition and generation (e.g. synthetic biology, xenobiology, minimal genome). It remains to be seen how recent results in quantum machine learning and quantum neural networks fit in this framework.
6 Conclusion

In this research, we presented a framework for empirically evaluating algorithmic metrics on a quantum accelerator. The estimation of the universal prior distribution (and thereby the algorithmic complexity and algorithmic probability) of finite sequences is theoretically the most optimal technique for inferring algorithmic structure in data for discovering causal generative models. These metrics are uncomputable but can be approximated in practice by restricting the time and memory resources available to the computational model. Nevertheless, due to the exponential scaling of the number of possible automata that need to be enumerated they are intractable except for the simplest of the cases on classical computation. In this work we propose a quantum algorithm to accelerate estimation of the universal distribution by simulating a superposition of programs (or transition function) for a resource-bounded automata. The quantum circuit implementation scales linearly in the qubit and gate complexity with the data or automata size.

The algorithmic information theoretic approach of causal generative mechanism discovery is more theoretically sound than data-driven approaches like deep learning, lossless compression and Shannon entropy based correlation, allowing it to find causal insights missed by these approaches. As a use-case, we present a full experimental framework for using this approach on DNA sequences for meta-biology and genome analysis. This is the first time a quantum computation approach is implemented for approximating algorithmic information. We implemented our copy-left design on the OpenQL quantum programming language and tested using the QX Simulator. The availability of better quantum processors would allow this algorithm to be readily ported on a quantum accelerator with our quantum computing stack [46]. With quantum-accelerated genome analysis, a better understanding of the algorithmic structures in DNA sequences would greatly advance domains like personalized medication.
Appendices

A Motivation

Over the last century the boundaries between the various branches of science is noticeably fading. The fundamental ideas of emergence and unification has shown glimpse in fields like mathematics, computer science, physics and biology. These advances has allowed applying theories and tools developed in one discipline to reinforce conjectures in another.

Kurt Gödel’s work [47] on the incompleteness of mathematics, and thereby of an automatic theorem prover is well complemented by the Alan Turing’s work [48] defining an universal mechanistic process. This work was further broadened by the work of John von Neumann’s self-replication cellular automata [49] and later by Stephan Wolfram’s elemental cellular automata. The link between computation and physics was also hinted by Konrad Zuse [50]. Eventually, the theory of computation was formally connected to physics with the work of David Deutsch’s Quantum Turing Machine [16] as the only known violation of the extended Church-Turing thesis. This has been endorsed by John Archibald Wheeler as the ‘it from bit’ principle [51] and the principle of computational equivalence of Stephan Wolfram [52] in digital physics. The idea was extended to include quantum computation by Seth Lloyd [53]. More recently, similar links between physics and theoretical computer science has been found via the holographic principle [54, 55, 56], tensor networks [57] and a code-theoretic [58] approach to physics.

Pancomputationalism was eventually extended to include artificial general intelligence [60, 61]. This was extended to the quantum domain by Roger Penrose [62]. These developments are surveyed and discussed extensively by Scott Aaronson [63]. The new field of quantum machine learning [64] also employs these tools for its development.

Besides the universe and intelligence, the third dimension in emergence is of life, the theme we focus in this research. The idea of using lambda calculus [65] and computer programs to describe genetic code [66] is pioneered by Gregory Chaitin [37] as the field meta-biology. This is in line with the constructor theory [67] approach towards scientific developments.

We develop on these ideas of a computational description of physically occurring phenomena. The second inspiration of our work stems from the recent realization in the machine learning community of the limitation of statistical approaches [68] and a shift towards causal models [69]. While this has always been the motive of the founding fathers of AI (Marvin Minisky, Ray Solomonoff, John McCarthy), the relative difficulty with respect to black-box data driven models had set it on a back foot in the last decade. Genetic programming [70, 71], neuro-evolution [72], eXplainable AI [73] and more broadly the field of AutoML and meta-learning are some advancements in this direction.

In this research, we use quantum computation to accelerate the discovery of causal models in string. This method is generic and can be applied to fields like image processing, financial time series, genomics, etc. We will present a case study of the latter.
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