PERVERSE SHEAVES OF CATEGORIES AND SOME APPLICATIONS

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Abstract. We study perverse sheaves of categories their connections to classical algebraic and noncommutative geometry. We show how perverse sheaves of categories encode naturally derived categories of coherent sheaves on \(\mathbb{P}^1\) bundles, semiorthogonal decompositions, and a recent proof of Segal that all autoequivalences of triangulated categories are spherical twists. Furthermore, we show that perverse sheaves of categories can be used to represent certain degenerate Calabi-Yau varieties. We show that four dimensional quadratic Sklyanin algebras can be constructed from perverse sheaves of categories and finally we give a new proof of homological mirror symmetry using for \(\mathbb{P}^3\) and sketch a proof of homological mirror symmetry for four dimensional quadratic Sklyanin algebras.

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1. Introduction

1.1. Background. The theory of perverse sheaves arose in the 1980s, in groundbreaking work of Kashiwara, Schapira, Bernstein, Sato and others in order to deal with, concurrently, sheaves of solutions to D-modules under the Riemann-Hilbert correspondence, and to formalize the theory of intersection cohomology on singular spaces. It was observed early on by Galligo, Granger and Maisonobe [27] that if one specifies a stratification on \(\mathbb{C}^n\) that is given by a normal crossings union of hyperplanes, then the category of perverse sheaves with respect to this stratification is equivalent to the data of representations of certain quivers. The most famous example of this is the category of perverse sheaves on the disc with respect to the stratification given by a single point. The category of perverse sheaves with respect to this stratification is equivalent to the data of two vector spaces \((\phi, \psi, \text{can, var})\) where \(\phi, \psi\) are vector spaces, and \(\text{can}: \psi \to \phi\) and \(\text{var}: \phi \to \psi\) so that

\[
\text{id}_\phi - \text{can} \cdot \text{var}
\]

is invertible. More generally, according to Gelfand, MacPherson and Vilonen, [28], the category of perverse sheaves on a complex algebraic variety with characteristic variety contained in a fixed
conical Lagrangian subvariety of $T^*X$ is equivalent to the category of representations of a quiver. Certain nontrivial examples of this were studied by MacPherson and Vilonen in [51]. In [50], MacPherson and Vilonen also showed that perverse sheaves on the disc which are perverse with respect to the stratification given by a number of points $\{p_1, \ldots, p_k\}$ are equivalent to the data of a vector space $\phi_i$ for each point $p_i$, a vector space $\psi$ along with maps $\text{can}_i : \psi \rightarrow \phi_i$ and $\var_\psi : \phi_i \rightarrow \psi$ satisfying the same condition as in Equation [1]. The equivalence between this data and the category of perverse sheaves is fixed by a choice of non-intersecting branch cut for each point $p_i$.

In their seminal work [31], Kapranov and Schectman observed that if vector spaces are replaced by categories, and linear maps are replaced by functors then one may obtain a notion of perverse sheaves with values in dg categories. The correct analogue of Equation [1] and the existence of $\text{can}_i$ is that the functors $F_i$ be spherical in the sense of Anno and Logvinenko [4]. The data of a choice of points $\{p_1, \ldots, p_k\}$, pretriangulated dg categories $\mathcal{C}$ and $A_1, \ldots, A_k$, spherical functors $F_i : \mathcal{A}_i \rightarrow \mathcal{C}$ and a certain graph $K$ embedded into the disc is called a $(K$-coordinatized) perverse schober. Recently, Donovan [19] has applied perverse schobers to the study of GIT wallcrossings.

Despite the fact that perverse schobers are rather recent inventions, the structure that they describe has appeared in several guises over past few decades. Odeskii and Feigin [57], following Sklyanin [82] developed a theory of certain graded rings called Sklyanin algebras which are flat deformations of a polynomial ring in $n$ variables and whose Hilbert series coincide with those of the coordinate ring of $\mathbb{P}^n$. These algebras are often specified by the data of an elliptic curve $E$ along with a pair of line bundles on them of degree $n + 1$, and are denoted $A_n(E, \mathcal{L}_1, \mathcal{L}_2)$ in this introduction.

Bondal and Polishchuk studied the derived category of right graded noetherian modules of $A_2(E, \mathcal{L}_1, \mathcal{L}_2)$ modulo torsion objects. They showed that this category is equivalent to $D^b(\text{rep}(R))$ where

$$R = \text{End}_E(\mathcal{O}_E \oplus \mathcal{L}_1 \oplus (\mathcal{L}_1 \otimes \mathcal{L}_2)),$$

and furthermore that one can recover $A_2(E, \mathcal{L}_1, \mathcal{L}_2)$ from $R$. Attached to each object $\mathcal{O}_E, \mathcal{L}_1$ and $\mathcal{L}_1 \otimes \mathcal{L}_2$ there are functors

$$F_{\mathcal{O}_E} : D^b(\text{vect}_k) \rightarrow D^b(E), \quad F_{\mathcal{L}_1} : D^b(\text{vect}_k) \rightarrow D^b(E), \quad F_{\mathcal{L}_1 \otimes \mathcal{L}_2} : D^b(\text{vect}_k) \rightarrow D^b(E)$$

sending $k$ to $\mathcal{O}_E, \mathcal{L}_1$ and $\mathcal{L}_1 \otimes \mathcal{L}_2$ respectively. According to Seidel and Thomas [79], these functors are spherical. Therefore, the data defining the Sklyanin algebra $A_2(E, \mathcal{L}_2, \mathcal{L}_2)$ precisely defines a $K$-coordinatized perverse schober for an appropriate choice of $K$. There is a natural notion of the category of global sections of a perverse schober (Section 3.2). The category of global sections of the perverse schober associated to $(E, \mathcal{L}_1, \mathcal{L}_2)$ is exactly the category $D^b(\text{qgr}(A_2(E, \mathcal{L}_1, \mathcal{L}_2)))$. This construction only works when $n = 2$. In Section 7.3 we provide a similar but more complicated construction that produces noncommutative deformations of $\mathbb{P}^3$. This construction is intimately related to the quantization of certain Poisson structures on $\mathbb{P}^3$ which admit Poisson divisors [67]. In the process, we give a novel construction $D^b(\text{qgr}(A_3(E, \mathcal{L}_1, \mathcal{L}_2)))$ using perverse sheaves of categories.

Similar perverse schobers appear in the work of Seidel on Fukaya categories associated to Lefschetz fibrations [73][74]. If $\pi : E \rightarrow \mathbb{C}$ is a symplectic fibration with only Morse type singularities and $E$ is an exact symplectic manifold, then Seidel defines a category, often called the directed Fukaya category or the Fukaya-Seidel category, whose objects correspond to Lagrangian vanishing cycles of $\pi$. Particularly, if $\Sigma$ is the set of critical values of $\pi$, and $\gamma_i$ is a collection of counterclockwise oriented paths connecting a base point $s$ to each point $p_i$ in $\Sigma$ to $s$. We then obtain
a Lagrangian vanishing thimble for each $\gamma_i$ which defines an ordered collection $a_1, \ldots, a_k$ of Lagrangian spheres in the Fukaya category of the fiber over $s$. Each of these Lagrangian vanishing spheres defines a spherical functor from $D^b(\text{vect}_k)$ to the derived Fukaya category of the fiber over $s$. This canonically provides the data of a perverse schober whose category of global sections is what Seidel calls the directed Fukaya category associated to $(E, \pi)$. The analogy between Seidel’s construction and the work of Bondal and Kapranov lies at the root of Auroux, Katzarkov and Orlov’s proof of homological mirror symmetry for noncommutative del Pezzo surfaces [6].

There are in fact deeper and more fundamental relationships between perverse sheaves of categories and Fukaya categories, originating in unpublished work of Bondal and Wang [11]. It is known that the Fukaya category of a Riemann surface with several punctures is the category of global sections of a sheaf of categories on $S$ [22, 29, 62, 81]. Furthermore, according to work of Nadler and Zaslow [56] and Nadler [53–55], culminating in forthcoming work of Ganatra, Pardon and Shende, the Fukaya category of a Weinstein manifold $M$ can be recovered as the category of global sections of a perverse sheaf of categories on a singular Lagrangian skeleton of $M$. Similar results appear in work of Tamarkin [89] and Tsygan [93].

1.2. Outline. The purpose of this paper is to forge connections between classical geometry, category theory and the newly minted theory of perverse sheaves of categories. In the process, several new and interesting questions are raised.

The first two sections are devoted to developing in detail the theory of perverse sheaves of categories and their categories of global sections.

In Section 2 we outline briefly basic ideas in dg categories, including a number of ways to glue them together; we describe explicit models of homotopy fiber product and equalizers along with several constructions of semiorthogonal decompositions, coming from work of Tabuada [87], Orlov [61] and Kuznetsov and Lunts [41].

In Section 3 we will look at perverse sheaves of categories in the language of Kapranov and Schectman [31]. We give what seems to be a widely accepted definition of the category of global sections of a perverse schober as the homotopy fiber product over a certain diagram of categories. We show that it agrees with the gluing construction of Kuznetsov and Lunts, or Orlov. We give a general concept of $K$-coordinatized perverse sheaves of categories, whose definition was hinted at by Kapranov and Schectman [32] and by Kontsevich [35]. We define their monodromy representations and categories of global sections in the process.

Sections 4, 5 and 6 are dedicated to describing several situations in which perverse sheaves of categories appear naturally.

In Section 4 we study two interesting types of perverse sheaves of categories on $S^1 \times [0, 1]$, both of which are related to examples which appear in the prescient work of Kontsevich [36]. These sheaves of categories are both built from a pair of a category $\mathcal{C}$ and a monodromy autoequivalence $\Phi$ of $\mathcal{C}$. If we let $\mathcal{C} = \text{Perf}_k$ and $\Phi = \text{id}$, then the global sections of the corresponding perverse sheaves of categories are either $D^b(\mathbb{P}^1)$ or $\text{Perf}(\mathbb{A}^1)$. If we let $\mathcal{C}$ be $D^b_{\text{dg}}(X)$ for a smooth variety $X$, and we let monodromy be tensor product with a line bundle $L$ on $X$, then the global sections of these sheaves of categories recover the derived category of coherent sheaves on $\mathbb{P}_X(O_X \oplus L)$ (Theorem 4.6) or the total space of $L$. When monodromy is more general, the category of global sections should be regarded as the total space of a noncommutative $\mathbb{P}^1$ bundle or line bundle over $\mathcal{C}$. Recently Segal [72] proved that all autoequivalences of triangulated categories can be thought of as spherical twists by constructing a category which can be regarded as the total space of a noncommutative
line bundle. We show (Proposition 1.4) that our construction can be used to recover and give context to Segal’s.

From the constructions of Section 3 it follows that the category of global sections of a perverse schober admits a semiorthogonal decomposition. In Section 5 we note that the converse often holds; dg or triangulated categories which admit semiorthogonal decompositions appear as the global sections of a perverse schober whenever this category admits a spherical functor whose cotwist is the Serre functor up to twist.

**Theorem 1.1** (Theorem 5.6). Let $\mathcal{T}$ and $\mathcal{C}$ be dg enhanced triangulated categories and $F : \mathcal{T} \to \mathcal{C}$ be a spherical functor whose cotwist automorphism is equivalent to the Serre functor on $\mathcal{T}$ up to shift. If $\mathcal{T} = \langle A_1, \ldots, A_k \rangle$ is a semiorthogonal decomposition of $\mathcal{T}$ and if $\alpha_j : A_j \to \mathcal{C}$ is the natural embedding, then there is a perverse schober made up of the data of $A_i$ and functors $F_j = F\alpha_j$ whose category of global sections is $\mathcal{T}$.

For instance, if $X$ is a Fano variety whose bounded derived category of coherent sheaves admits a semiorthogonal decomposition and $Z$ a smooth or mildly singular anticanonical divisor, then there is a perverse schober whose generic fiber is $D^b(Z)$ and which encodes the semiorthogonal decomposition on $D^b(X)$. Conjecturally, the same relation holds between general Fukaya-Seidel categories and the Fukaya categories of their smooth fibers.

In Section 6 we show that mild degenerations of Calabi-Yau varieties, called Tyurin degenerations, or more generally simplified type II degenerations (Definition 6.6) can be represent as perverse sheaves of categories over $S^2$ with a number of boundary components. We show that Friedman’s d-semistability condition is intimately related to the structure of these perverse sheaves of categories. Perverse sheaves of categories are used to show that type I modifications of type II degenerations of K3 surfaces leave the category of perfect complexes on the degenerate fiber invariant (Theorem 6.19).

Sections 7 and 8 are devoted to showing how the perspective of perverse sheaves of categories can be used to gain new insights into classical problems. We will give a new construction of noncommutative $\mathbb{P}^3$ and a proof of homological mirror symmetry for $\mathbb{P}^3$ that have perverse sheaves of categories at their heart. The work in these sections can be immediately generalized to produce new noncommutative deformations of certain Fano varieties, and proofs of homological mirror symmetry in new cases.

We apply the ideas developed in Sections 5 and 6 to the construction of noncommutative deformations of certain categories. In [96], following Bondal and Polishchuk [15], Van den Bergh gives a construction of noncommutative deformations of quadric surfaces coming from the data of an elliptic curve $E$ along with a triple of line bundles $L_1, L_2$ and $L_3$ on $E$. Van den Bergh uses the algebra of homomorphisms of these line bundles to construct a $\mathbb{Z}$ algebra with attractive homological properties [90]. We note that Van den Bergh’s construction is remarkably similar to the construction in Section 5 and in fact can be recast in that language.

**Theorem 1.2** (Theorem 7.6). If $\text{qgr}(Q_V)$ is a noncommutative quadric in the sense of Van den Bergh [96], there is a perverse schober whose category of global sections is equivalent to $D^b(\text{qgr}(Q_V))$.

This is not hard to prove. Its utility is that it can be used to prove the following theorem:

**Theorem 1.3** (Theorem 7.14). There is a perverse schober over the disc whose generic fiber is the derived category of a noncommutative quartic surface (the homotopy fiber product of a pair of noncommutative quadrics), and whose global sections are equivalent to $D^b(\text{qgr}(A_3(E, L_1, L_2)))$. 
The algebras $A_3(E, L_1, L_2)$ are quantizations of Poisson structures which admit an anticanonical Poisson divisor [67], which explains why we have a noncommutative quartic surface appearing as the fiber category of this perverse schober.

Section 8 emphasizes the relationship between Fukaya category computations in [76] and perverse sheaves of categories in a specific example. We show that many of the structures in previous sections have analogues under mirror symmetry. After reviewing homological mirror symmetry for quadric surfaces, we use results of Seidel and the viewpoint of perverse sheaves of categories to prove (Theorem 8.9), homological mirror symmetry for the singular quartic surface

$$x^2y^2 - z^2w^2 = 0.$$  

We apply this to obtain a new proof of homological mirror symmetry for $\mathbb{P}^3$ (Theorem 8.14). We then describe how the results on noncommutative deformations of $\mathbb{P}^3$ in Section 7 can be used to produce mirrors of Sklyanin algebras of global dimension 4.

Section 9 discusses several applications of perverse sheaves of categories. First, we define a categorical version of the d-semistability criteria of Friedman [24] (also called log smoothness by Kawamata and Namikawa 34). If $X$ is a normal crossings variety, then $\mathcal{P}erf(X)$ can be obtained as a homotopy limit of the categories of perfect complexes of each component. This means that $\mathcal{P}erf(X)$ can be thought of as the category of global sections of a constructible sheaf of categories on the dual intersection complex of $X$. Friedman’s d-semistability, or log smoothness condition then has an interpretation as the triviality of a certain spherical functor (Definition 9.2). We formulate a conjecture analogous to the main result of Kawamata and Namikawa 34 saying that if a constructible sheaf of categories over a cell complex whose fibers are smooth and proper, and the category of global sections admits a Calabi-Yau structure and satisfies Definition 9.2 then there exist smooth deformations of the category of global sections.

We then mention that the construction in Section 4 can be generalized to sheaves of categories whose global sections are the category of coherent sheaves on smooth toric fiber bundles. We sketch a construction of noncommutative Fano varieties suggested by Section 7, and we conclude by describing future work on perverse sheaves of stability conditions on holomorphic sheaves of categories. These structures should be mirror dual to perverse sheaves of categories.

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2. Background on dg categories

Here we will provide background on dg categories, their model structures and gluing constructions. We will assume that we work over a field $k$ of characteristic 0, which one is free to assume is algebraically closed, or even $\mathbb{C}$. For us, a dg category will be a (small) category $\mathcal{C}$ whose homomorphisms have the following structure

(1) $\text{hom}_{\mathcal{C}}(a, b)$ is $\mathbb{Z}$-graded $k$-vector space, the graded components being denoted $\text{hom}^i_{\mathcal{C}}(a, b).$
(2) There is a $k$-linear differential $d : \hom^i_k(a, b) \to \hom^{i+1}_k(a, b)$ with $d^2 = 0$

(3) If $f \in \hom^i_k(a, b)$ and $g \in \hom^j_k(c, a)$ then $f \cdot g$ contained in $\hom^{i+j}_k(c, b)$

(4) Each $\hom^0_k(a, a)$ contains a closed identity element $\text{id}_a$ with the obvious properties.

(5) If $f \in \hom_k^i(b, c)$ and $g \in \hom_k^j(a, b)$ then $d(f \cdot g) = df \cdot g + (-1)^{\deg(f)} f \cdot dg$.

**Example 2.1.** The category of chain complexes over a field $k$ forms a dg category which we will denote $\text{Ch}_k$. Let $a^\bullet$ and $b^\bullet$ be $k$-chain complexes, and let $\hom^i_{\text{Ch}_k}(a, b) = \prod_{i \in \mathbb{Z}} \hom_k(a^i, b^{i+i})$ equipped with the differential which takes

$$f \in \hom_{\text{Ch}_k}(a^\bullet, b^\bullet) \mapsto d \cdot f + (-1)^{\deg(f)} f \cdot d$$

A dg functor between dg categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F$ between the underlying $k$-linear categories of $\mathcal{C}$ and $\mathcal{D}$ so that the gradings on $\hom_{\mathcal{C}}(a, b)$ and $\hom_{\mathcal{D}}(F(a), F(b))$ agree, and so that $F(df) = dF(f)$. The collection of small dg categories over $k$ themselves forms a category which we will denote $\text{dgcat}_k$.

To each dg category $\mathcal{C}$ there is an associated homotopy category denoted $\text{H}^0\mathcal{C}$ or sometimes $[\mathcal{C}]$ so that $\text{Ob}(\text{H}^0\mathcal{C}) = \text{Ob}(\mathcal{C})$. We denote the object in $\text{H}^0\mathcal{C}$ associated to $a \in \text{Ob}(\mathcal{C})$ by $[a]$. Homomorphisms in $\text{H}^0\mathcal{C}$ are given by $\hom_{\text{H}^0\mathcal{C}}([a], [b]) = \text{H}^0 \text{hom}_{\mathcal{C}}(a, b)$. Two objects $a$ and $b$ in $\mathcal{C}$ are quasi isomorphic if the objects $[a]$ and $[b]$ are isomorphic.

The set of dg functors from $\mathcal{C}$ to $\mathcal{D}$, denoted $\text{Fun}_{\text{dg}}(\mathcal{C}, \mathcal{D})$ forms a dg category whose homomorphisms are natural transformations. A (right) dg $\mathcal{C}$ module is a dg functor from $\mathcal{C}^{\text{op}}$ to $\text{Ch}_k$. The category of $\mathcal{C}$-modules will be denoted $\text{mod}_\mathcal{C}$. There is a triangulated structure on the category $\text{mod}_\mathcal{C}$ obtained from the triangulated structure on $\text{Ch}_k$. Specifically, if $M$ is a $\mathcal{C}$-module then we may define $M[1]$ to be the functor so that

$$M[1](a) = M(a)[1]$$

and similarly, if we have a homomorphism of degree 0 between a pair of $\mathcal{C}$-modules $f : M \to N$, then we may define the cone of $f$ to be the module

$$\text{Cone}(f)(a) = \text{Cone}(M(a)) \xrightarrow{f(a)} N(a)$$

There is a canonical functor from $\mathcal{C}$ to $\text{mod}_\mathcal{C}$ called the Yoneda embedding,

$$\mathcal{Y} : \mathcal{C} \to \text{mod}_\mathcal{C}, \quad a \mapsto \text{hom}_\mathcal{C}(-, a)$$

which is a full and faithful embedding a dg categories by a dg version of the Yoneda lemma. A $\mathcal{C}$-module is called representable if there is some $b$ so that $M \cong \mathcal{Y} b$. If there is some $b$ so that $M$ and $\mathcal{Y} b$ are quasi isomorphic then we will say that $M$ is quasi representable.

A functor $F$ between a pair of dg categories $\mathcal{C}$ and $\mathcal{D}$ is a quasi equivalence if for each pair of objects $a, b$ in $\mathcal{C}$, the map $\text{hom}_\mathcal{C}(a, b) \to \text{hom}_\mathcal{D}(F(a), F(b))$ is a quasi isomorphism of complexes and every object in $\mathcal{D}$ is quasi isomorphic to an object in the image of $F$. In other words, $F$ induces an equivalence between $\text{H}^0\mathcal{C}$ and $\text{H}^0\mathcal{D}$. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are quasi equivalent if there is a chain of quasi equivalences

$$\mathcal{C} \xleftarrow{\cong} \mathcal{C}_1 \xrightarrow{\cong} \ldots \xrightarrow{\cong} \mathcal{C}_n \xleftarrow{\cong} \mathcal{D}$$

between them. If $\mathcal{A}$ and $\mathcal{B}$ are a pair of dg categories, an $\mathcal{A}$-$\mathcal{B}$-bimodule is a right $\mathcal{A}^{\text{op}} \otimes \mathcal{B}$ dg module. For instance, if $F_1 : \mathcal{A} \to \mathcal{C}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ are dg functors, then there is a dg bimodule,

$$S_{F_1, F_2}(a, b) = \text{hom}_\mathcal{C}(F_2(b), F_1(a)).$$
A $A$-$B$ bimodule determines a functor from $A^{op}$ to $mod_{B^{op}}$ and from $B$ to $mod_A$. A quasi functor is a $A$-$B$ bimodule $S$ so that $S(a, -)$ is quasi representable for any $a$ in $A$. Such bimodules induce functors from $H^0A$ to $H^0B$. If we have a functor $F$ from $B$ to $C$ and $A$ is quasi equivalent to $B$ then there is a quasifunctor from $A$ to $B$ corresponding to $F$ obtained by taking tensor products of bimodules [4].

There is a model structure on $dgcat_k$ called the Dwyer-Kan (DK) model structure. This model structure has weak equivalences given by quasi equivalences. Fibrations given by functors for which $F : hom_C(a, b) \rightarrow hom_D(F(a), F(b))$ is a surjection of complexes and so that if $F(a)$ is quasi isomorphic to $c$ in $D$ then there is an object $c' \in C$ so that this isomorphism lifts to a quasi isomorphism between $a$ and $c'$. A happy consequence of this definition is that all dg categories are fibrant with respect to the DK model structure on $dgcat_k$ (see [SS] Remark 2.14).

A dg category $C$ is called idempotent complete or Karoubi complete if every idempotent morphism $f \in hom_{sepC}([a], [a])$ admits a splitting, that is, there are a pair of morphisms $r : [a] \rightarrow [b]$ and $s : [b] \rightarrow [a]$ so that $r \cdot s = \text{id}_b$ and $s \cdot r = f$. Here $b$ is called the direct image of $f$. A pair $(F, B)$ of a functor $F : C \rightarrow B$ which is full and faithful on the level of homotopy categories and so that every element of $H^0B$ is isomorphic to the direct image of some idempotent in $C$ is called a split closure of $C$. According to Seidel [76] Lemma 4.7, every dg (in fact every $A_\infty$) category admits a split closure which is unique up to quasi equivalence. This category is denoted $\Pi C$.

2.1. Pretriangulated dg categories. Bondal and Kapranov have introduced pretriangulated dg categories in order to amend some of the deficiencies of triangulated categories.

**Definition 2.2.** A dg category is pretriangulated if for every $Y^a \in mod_C$, the module $Y^a[1]$ is quasi representable, and if $f : a \rightarrow b$ is closed of degree 0, then

$$\text{Cone}(Y^a \xrightarrow{f} Y^b)$$

is quasi representable. In other words, the homotopy category of $C$ is triangulated and this triangulated structure is consistent with that of $mod_C$.

Every dg category can be minimally embedded into a pretriangulated dg category. This construction is described by Bondal and Kapranov.

**Definition 2.3.** Given a dg category $A$, the category $TwA$ is the category whose objects are given pairs of $(\bigoplus_{i=1}^k a_i[d_i], m)$ where $\bigoplus_{i=1}^k a_i[d_i]$ is a formal direct sum of formally shifted objects in $A$ and $d_i$ are integers. The second piece of data $m$ is a strictly upper triangular matrix so that $m_{i, j} \in hom_A(a_i, a_j)[d_j - d_i]$, is homogeneous of degree 1 and vanishes if $i \geq j$. These objects must satisfy the Maurer-Cartan relation, $dm + m^2 = 0$. For a pair of pairs $\alpha = (\bigoplus_{i=1}^k a_i[d_i], m)$ and $\beta = (\bigoplus_{i=1}^k b_i[e_i], n)$ we define $hom_{TwA}(\alpha, \beta)$ to be the space of matrices $g$ with entries $g_{i, j} \in hom_A(a_i, b_i)[e_i - d_j]$ and composition given by matrix multiplication. If $f$ is an element of $hom_{Tw(A)}(\alpha, \beta)$, then we define the differential acting on $f$ as follows,

$$d_Bf = d_Af + mf + (-1)^jf n$$

where $d_Af$ means we act element-wise on $f$ by the differential in $A$. There’s a full and faithful functor from $A$ to $TwA$ which sends an object $a \in A$ to the object $(a, 0)$ of $TwA$. Pretriangulated categories are precisely those for which the embedding of $A$ into $TwA$ is a quasi-equivalence. One can give explicit representatives of cones in $TwA$.

$\square$
If $F : \mathcal{A} \to \mathcal{B}$ is a dg functor, then there is an induced dg functor from $\text{Tw} \mathcal{A}$ to $\text{Tw} \mathcal{B}$ which we will denote $F_{\text{Tw}}$. 

**Remark 2.4.** Seidel [78, 2b] gives an analogous definition in the case where $\mathcal{C}$ is an $A_\infty$ category. 

**Remark 2.5.** One may also construct the pretriangulated envelope of a dg category $\mathcal{C}$ by taking the category of all semi free dg modules over $\mathcal{C}$. 

**Example 2.6.** If we let $k$ be the category with one object $e$ so that $\text{hom}_k(e, e) = k \cdot \text{id}_e$, then $\text{Tw} k$ is equivalent to the subcategory of $\text{Ch}_k$ whose objects are bounded complexes of $k$-vector spaces. We will denote this category $\text{Perf}_k$. 

**Definition 2.7.** A category $\mathcal{A}$ is said to be strongly pretriangulated if it is dg equivalent to $\text{Tw} \mathcal{A}$. 

**Definition 2.8.** A subcategory $\mathcal{U}$ of a pretriangulated dg category $\mathcal{C}$ is said to split generate $\Pi \mathcal{C}$ if the smallest split closed pretriangulated subcategory of $\Pi \mathcal{C}$ containing $\mathcal{U}$ is quasi equivalent to $\Pi \mathcal{C}$ itself. 

We will use the notation $D^\circ \mathcal{C}$ for $\Pi^0 \Pi \text{Tw} \mathcal{C}$. 

### 2.2. Categories of coherent sheaves

Here we make several remarks on dg extensions of triangulated categories of coherent sheaves. We will denote by $\text{coh}(X)$ the abelian category of coherent sheaves on $X$, and by $\text{qcoh}(X)$ the category of quasicoherent sheaves on $X$. To each of these, we have triangulated derived categories $D(\text{coh}(X))$ and $D(\text{qcoh}(X))$. Our goal is to describe dg extensions of such categories. Good overviews can be found in [61, Section 3.1] and [41, Section 3]. 

There is an obvious dg category $\mathcal{K}(\text{qcoh}(X))$ of unbounded complexes of quasicoherent sheaves on $X$. If we let $\text{Flat}(X)$ be the full subcategory of $\mathcal{K}(X)$ made up of h-flat complexes, and $\text{Acf}(X)$ is the full subcategory of acyclic h-flat complexes, then the category $\text{Flat}(X)/\text{Acf}(X)$ is a dg enhancement of $D(\text{qcoh}(X))$, which we will call $D_{\text{dg}}(\text{qcoh}(X))$. 

Since $D^b(\text{coh}(X))$ is equivalent to $D^b_{\text{coh}}(\text{qcoh}(X))$ of complexes with coherent cohomology if $X$ is noetherian and separated, we can define $D^b_{\text{dg}}(\text{coh}(X))$ to be the full subcategory of $D_{\text{dg}}(\text{qcoh}(X))$ whose class in $D(\text{qcoh}(X))$ has bounded coherent cohomology. For brevity, we will use the notation $D^b_{\text{dg}}(X)$ for $D^b_{\text{dg}}(\text{coh}(X))$. 

There are many different dg enhancements of the category of coherent sheaves, but for quasi projective schemes, all enhancements are quasi equivalent [49, Theorem 8.13]. 

The category $\text{Perf}(X)$ is a full triangulated subcategory of $D^b(X)$, so we may define $\text{Perf}(X)$ to be the full subcategory of objects in $D^b_{\text{dg}}(X)$ whose corresponding objects are in $\text{Perf}(X)$. Given a morphism $f : X \to Y$ of schemes, there is a functor $L f^* : D^b_{\text{dg}}(Y) \to D^b_{\text{dg}}(X)$. There is a right adjoint (quasi) functor $R f_* : D^b_{\text{dg}}(X) \to D^b_{\text{dg}}(Y)$. The functor $L f^*$ extends to a dg functor from $\text{Perf}(Y)$ to $\text{Perf}(X)$. Tensoring with any $F$ in $\mathcal{K}(\text{qcoh}(X))$ also defines a dg functor from $D^b_{\text{dg}}(X)$ to $D^b_{\text{dg}}(X)$ denoted $(-) \otimes^L F$ which extends to a functor from $\text{Perf}(X)$ to $\text{Perf}(X)$. If $f$ is proper and has finite Tor dimension, then $R f_*$ extends to a quasi functor from $\text{Perf}(X)$ to $\text{Perf}(Y)$. 

We will use the notation $R, L$ to denote derived quasi functors between dg enhancements and $R, L$ derived functors between their triangulated homotopy categories. 

It is known [41, Section 4.2] that for a separated scheme of finite type $Y$, the categories $\text{Perf}(Y)$ and $D^b_{\text{dg}}(Y)$ are idempotent closed.
2.3. **Various gluing constructions.** There are several related categorical constructions that we would like to discuss in this section. The first construction is due to Tabuada [87] and is expanded upon by Orlov [61]. Such a category begins with two dg categories \( \mathcal{A}_1, \mathcal{A}_2 \) and a \( \mathcal{A}_1 \sqcup \mathcal{A}_2 \) dg bimodule, which we will denote \( \phi \). We will denote the corresponding category as \( \mathcal{A}_1 \sqcup \phi \mathcal{A}_2 \). We have

\[
\text{Ob}(\mathcal{A}_1 \sqcup \phi \mathcal{A}_2) = \text{Ob}(\mathcal{A}_1) \coprod \text{Ob}(\mathcal{A}_2)
\]

and

\[
\text{hom}_{\mathcal{A}_1 \sqcup \phi \mathcal{A}_2}(a, b) = \begin{cases} 
\text{hom}_{\mathcal{A}_1}(a, b) & \text{if } a, b \in \text{Ob}(\mathcal{A}_1) \\
\text{hom}_{\mathcal{A}_2}(a, b) & \text{if } a, b \in \text{Ob}(\mathcal{A}_2) \\
\phi(b, a) & \text{if } a \in \text{Ob}(\mathcal{A}_1), b \in \text{Ob}(\mathcal{A}_2) \\
0 & \text{otherwise}
\end{cases}
\]

Composition is given by using the bimodule structure on \( \phi \). Usually, we will be interested just in the bimodule defined as \((b, a) \mapsto \text{home}(F_1(a), F_2(b))\) for functors \( F_i : \mathcal{A}_i \rightarrow \mathcal{C} \). In this case we will write \( \mathcal{A}_1 \sqcup_{F_1, F_2} \mathcal{A}_2 \).

We will consider the pretriangulated envelope of \( \mathcal{A}_1 \sqcup \phi \mathcal{A}_2 \). Kuznetsov and Lunts [41] give a concrete description of this category. Let \( \mathcal{A}_1 \times_{\phi} \mathcal{A}_2 \) be the dg category whose objects are \((a_1, a_2, \mu)\) where \( a_i \in \mathcal{A}_i \) and \( \mu \in \mathbb{Z}^0(\phi(a_1, a_2)) \) is a closed degree 0 element. Details of their construction can be found in [41 Section 4].

In the special case where there are two dg functors \( F_1 : \mathcal{A}_1 \rightarrow \mathcal{C} \) and \( F_2 : \mathcal{A}_2 \rightarrow \mathcal{C} \) and \( \phi = S_{F_1, F_2} \), we will give a precise definition.

**Definition 2.9.** If \( \mathcal{A}_1, \mathcal{A}_2 \) and \( \mathcal{C} \) are dg categories with \( F : \mathcal{A}_1 \rightarrow \mathcal{C} \) and \( G : \mathcal{A}_2 \rightarrow \mathcal{C} \) dg functors, then \( \mathcal{A}_1 \times_{F_1, F_2} \mathcal{A}_2 \) is the category with objects given by triples \((a, b, \mu)\) where \( a \in \text{Ob}(\mathcal{A}_1), b \in \text{Ob}(\mathcal{A}_2) \) and \( \mu \) is a closed degree element of \( \text{hom}_\mathcal{C}(F(a), G(b)) \). Homomorphisms between \((a, b, \mu)\) and \((c, d, \chi)\) are given by the cone (in \( \text{Ch}_k \) ) of the map

\[
\text{hom}_{\mathcal{A}_1}(a, c) \oplus \text{hom}_{\mathcal{A}_2}(b, d) \xrightarrow{\xi - \mu^*} \text{hom}_\mathcal{C}(F_1(a), F_2(b)).
\]

Here \( \mu^* \) means we compose the map \( \text{hom}_{\mathcal{A}_1}(b, d) \rightarrow \text{hom}_\mathcal{C}(F_1(b), F_2(d)) \) with \( F_1(\mu)^* \), and we define \( \xi \) similarly. Precisely, \( \text{hom}_{\mathcal{A}_1 \times_{F_1, F_2} \mathcal{A}_2}(a, b, \mu) \) is made up of triples

\[
(f, g, h) \in \text{hom}_{\mathcal{A}_1}(a, b) \oplus \text{hom}_{\mathcal{A}_2}(c, d) \oplus \text{hom}_\mathcal{C}^{1-1}(F_1(a), F_2(d))
\]

with differential,

\[
d(f, g, h) = (df, dg, dh + (F_2(g) \cdot \mu - \xi \cdot F_1(f)))
\]

and

\[
(f', g', h') \cdot (f, g, h) = (f' \cdot f, g' \cdot g, h' \cdot h + (-1)^\deg g' g' \cdot h).
\]

\[\square\]

There is always a functor which we denote \( c_{F_1, F_2} : \mathcal{A}_1 \times_{F_1, F_2} \mathcal{A}_2 \rightarrow \text{Tw} \mathcal{C} \) sending \((a, b, \mu)\) to \( \text{Cone}(\mu) \). We now list several properties of this construction.

**Proposition 2.10 (Kuznetsov-Lunts [41 Proposition 4.3]).** If \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are pretriangulated, then \( \mathcal{A}_1 \times_{\phi} \mathcal{A}_2 \) is also pretriangulated.

**Proposition 2.11 (Kuznetsov-Lunts [41 Proposition 4.9]).** If there are quasi isomorphisms of functors \( n_1 : F'_1 \rightarrow F_1 \) and \( n_2 : F'_2 \rightarrow F_2 \) then there is a quasi equivalence,

\[
\gamma_{n_1, n_2} : \mathcal{A}_1 \times_{F'_1, F'_2} \mathcal{A}_2 \rightarrow \mathcal{A}_1 \times_{F_1, F_2} \mathcal{A}_2
\]
so that $c_{F_1,F_2}$ is quasi equivalent to $c_{F_1,F_2} \cdot \gamma_{n_1,n_2}$ as a bimodule.

In Proposition 2.11 we say that two functors are quasi isomorphic if the corresponding bimodules are, or equivalently, if there is a natural transformation between them which induces isomorphisms in the homotopy category.

**Proposition 2.12** (Kuznetsov-Lunts [41 Proposition 4.14]). If $G : A \to A_1$ and $H : B \to A_2$ are quasi equivalences, then there is a quasi equivalence

$$\eta_{G,H} : A \times_{G,F_1,H,F_2} B \to A_1 \times_{F_1,F_2} A_2.$$ 

Furthermore, $c_{G,F_1,H,F_2}$ and $c_{F_1,F_2} \cdot \eta_{G,H}$ are quasi equivalent as bimodules.

The second statements in Propositions 2.11 and 2.12 are deduced from the proofs of [41, Proposition 4.9, Proposition 4.14].

**Remark 2.13.** According to [41, Corollary 4.5], if $A_1$ and $A_2$ are pretriangulated, then $H^0(A_1 \times_\phi A_2)$ admits a semiorthogonal decomposition $(H^0 A_1, H^0 A_2)$.

**Remark 2.14.** If $A_1$ and $A_2$ are pretriangulated dg categories, then $A_1 \times_\phi A_2$ is quasi equivalent to $\text{Tw}(A_1 \sqcup_\phi A_2)$.

There are two closely related constructions coming from Tabuada’s construction [87] of the path object in $\text{dgcat}_k$. Our description of the homotopy equalizer can also be found in [80] and our construction of the homotopy fiber product can be found in [3].

**Definition 2.15.** If we have two functors $F_1, F_2 : A \to \mathcal{C}$, the homotopy equalizer $\mathcal{E}q^h(F_1,F_2)$ is the category whose objects are pairs $(a, \mu)$ where $a \in \text{Ob}(A)$ and $\mu \in H^0 \text{hom}_C(F_1(a), F_2(a))$ so that $[\mu]$ is an isomorphism. Then the complexes $\hom_{\mathcal{E}q^h(F_1,F_2)}((a, \mu), (b, \xi))$ are given by pairs

$$(f, h) \in \hom^i_{\mathcal{A}}(a,b) \oplus \hom^{i-1}_{\mathcal{C}}(F_1(a), F_2(b))$$

equipped with the differential $d(f, h) = (df, dh + (\xi \cdot F_1(f) - F_2(f) \cdot \mu))$, and so that composition computed as

$$(f, h) \cdot (f', h') = (f \cdot f', F_2(f') \cdot h + (-1)^{\deg f'} h' \cdot F_1(f)).$$

**Definition 2.16.** The homotopy fiber product of a pair of functors $F_1 : A_1 \to \mathcal{C}$ and $F_2 : A_2 \to \mathcal{C}$, denoted $A_1 \times_{\mathcal{C}}^h A_2$ is the homotopy equalizer of the functors $F_1 \times 0 : A_1 \times A_2 \to \mathcal{C}$ and $0 \times F_2 : A_1 \times A_2 \to \mathcal{C}$.

**Remark 2.17.** We will also use the ordinary fiber product of categories. If $F_i : A_i \to \mathcal{C}$ are functors, then $\mathcal{A} \times_{\mathcal{C}} \mathcal{A}$ is the full subcategory of $\mathcal{A} \times^h_{\mathcal{C}} \mathcal{A}$ made up of objects $(a,b,\mu)$ where $\mu$ is an isomorphism in $\mathcal{C}$ between $F_1(a)$ and $F_2(b)$. If $F_1 : A_1 \to \mathcal{C}$ is a cofibration, then $A_1 \times^h_{\mathcal{C}} A_2$ is quasi isomorphic to $A_1 \times_{\mathcal{C}} A_2$.

It is important to note that the homotopy fiber product of the diagram

$$\mathcal{A} \xrightarrow{F_1} \mathcal{C} \xleftarrow{F_2} A_2$$

is a full subcategory of $\mathcal{A} \times_{F_1,F_2} A_2$ made up of objects $(a,b,\mu)$ so that $[\mu]$ is an isomorphism in $H^0 \mathcal{C}$, or equivalently, $\text{Cone}(\mu)$ is homotopically trivial. Therefore, Propositions 2.11 and 2.12 have analogues for homotopy fiber products, as one would expect.
Proposition 2.18. If $G : A'_1 \to A_1$ and $H : A'_2 \to A_2$ are quasi equivalences, then there is quasi equivalence

$$\eta_{G,H} : A'_1 \times^h_c A'_2 \to A_1 \times^h_c A_2.$$  

Proposition 2.19. If there are natural transformations $n_1 : F'_1 \to F_1$ and $n_2 : F'_2 \to F_2$ which are quasi isomorphisms of bimodules then there is a quasi equivalence $\gamma_{n_1,n_2}$ between the homotopy fiber products of $F'_1,F'_2$ and $F_1,F_2$.

Remark 2.20. One can check that the homotopy equalizer of a pair of functors between pretriangulated categories is a pretriangulated category by following the proof of [41, Lemma 4.3]. Details will be provided in Proposition 4.11.

We may then slightly generalize [8, Lemma 4.2] to determine when a pair of elements of $A_1 \times_{F_1,F_2} A_2$ are homotopy equivalent.

Lemma 2.21. A pair of objects $\alpha = (a_1,a_2,\mu)$ and $\beta = (b_1,b_2,\tau)$ are isomorphic in $H^0(A_1 \times_{F_1,F_2} A_2)$ if and only if there are morphisms $\xi_i : a_i \to b_i$ so that $[\xi_i]$ is an isomorphism and the diagram

$$\begin{array}{ccc}
F_1(a_1) & \xrightarrow{\mu} & F_2(a_2) \\
F_1(\xi_1) & \downarrow & F_2(\xi_2) \\
F_1(b_1) & \xrightarrow{\tau} & F_2(b_2)
\end{array}$$

commutes up to homotopy.

Proof. That the diagram above commutes up to homotopy means that there is a degree $-1$ element $g$ of $\mathcal{C}(F_1(a_1),F_2(b_2))$ so that

$$d_{\deg} = F_2(\xi_2) \cdot \mu - \tau \cdot F_1(b_2).$$

Therefore, by definition, $(\xi_1,\xi_2,g)$ is a closed, degree 0 homomorphism between $\alpha$ and $\beta$. Our claim is now that it is an isomorphism. Then one can follow the proof of [8, Lemma 4.2] to see that $(\xi_1,\xi_2,g)$ is a homotopy isomorphism if and only if $\xi_1$ and $\xi_2$ are homotopy isomorphisms.

2.4. Directed dg categories. Let us now look at categories which Seidel calls the directed category associated to a collection of objects in a dg category $A$. Properly, Seidel deals with $A_\infty$ categories, but the idea is the same for dg categories.

Let $U = \{a_1,\ldots,a_k\}$ be an ordered collection of objects in $A$. The directed category associated to $U$, denoted $A^\rightarrow_U$, is the category with objects elements of $U$ (we abuse notation and use the same symbols for the objects of $A^\rightarrow_U$ as we do for the objects of $A$), and whose morphisms are given by the following.

$$\hom_{A^\rightarrow_U}(a_i,a_j) = \begin{cases} 
  k \cdot \id_{a_i} & \text{if } i = j \\
  \hom_A(a_i,a_j) & \text{if } i < j \\
  0 & \text{if } i > j
\end{cases}$$

Composition is taken inside of $A$.

This construction can be recast in the language of categorical gluing in a very elementary way. Let us take $k$ to be the dg category with a single object $e$ so that $\hom_k(e,e) = k \cdot \id_e$. Then choosing an object $a_i$ in $A$ determines a functor from $k$ to $A$ which sends $e$ to $a_i$ and $\id_e$ to $\id_{a_i}$. For a pair of objects $a_1$ and $a_2$ in $A$, we obtain a $k$-$k$ bimodule which we will call $S(a_1,a_2)$ which is the bimodule which acts as

$$e_2^{op} \otimes e_1 \mapsto \hom_A(a_1,a_2).$$
We would like to show that if we have units of adjunction $R$ of coherent or quasi coherent sheaves satisfy descent properties. Particularly, Töen claims the between directed categories and the Kuznetsov-Lunts construction at will by taking categories of $Z$ then $k$ coherent sheaves on $X$.

Proposition 2.22. Let $A_1, A_2$ and $C$ be dg categories and let $F_i : A_i \to C$ be functors. Then there is a functor $\Phi$ of dg categories from $\text{Tw}(A_1) \times_{F_1, F_2} \text{Tw}(A_2)$ to $\text{Tw}(A_1 \cup F_1, F_2 A_2)$ which is an equivalence. Furthermore, $\pi^{\text{tw}} \cdot \Phi : \text{Tw}(A_1) \times_{F_1, F_2} \text{Tw}(A_2) \to \text{Tw} C$ sends $(\alpha, \beta, g)$ to $\text{Cone}(g)$.

Here $F_i^{\text{tw}}$ is the natural extension of $F_i$ to the category of twisted complexes. Therefore, Corollary 2.23. If $C$ is a pretriangulated dg category, and there is a set of objects $U \subseteq \text{Ob}(C)$, then

$$\text{Tw} C_U^* \cong \text{Perf}_k \times_{S(1, a_2)} \text{Tw}_k \text{Perf}_k \times_{S(1, a_2, a_3)} \text{Tw} \cdots \times_{S(1, \ldots, a_k)} \text{Tw} \text{Perf}_k.$$

This is proved by repeatedly applying Proposition 2.22. Therefore, we can switch back and forth between directed categories and the Kuznetsov-Lunts construction at will by taking categories of twisted complexes.

2.5. Gluing categories of perfect complexes. It is known that, in certain cases, categories of coherent or quasi coherent sheaves satisfy descent properties. Particularly, Töen claims the following result:

Proposition 2.24 (Töen, [91 Proposition 11]). If $U$ and $V$ are Zariski open subsets of $X$ so that $X = U \cup V$, then there is a quasi-equivalence

$$\text{Perf}(U) \times_{h \text{Perf}(U \cap V)} \text{Perf}(V) \cong \text{Perf}(X).$$

In [81 Proposition 2.11], Sibila, Treuman and Zaslow claim that the same is true if one replaces the Zariski topology with the étale topology. Finally, Kuwagaki [88 Proposition 7.1] shows that the same is true if one replaces the category $\text{Perf}(X)$ with the dg category $\text{D}_{dg}(\text{qcoh}(X))$ of quasi-coherent sheaves on $X$ and uses the Zariski topology. We remark that Töen’s and Sibila, Treuman and Zaslow’s results are with respect to the DK model structure, whereas Kuwagaki explicitly uses the Morita model structure on $\text{dgcat}_k$.

Proposition 2.25. Let $Z = Z_1 \cup Z_2$, $S = Z_1 \cap Z_2$ and assume that $i_k : Z_k \to Z$, $j_k : S \to Z_k$ and $i_{12} = i_1 \cdot j_1 = i_2 \cdot j_2 : S \to Z$ are proper morphisms of finite Tor dimension, and so that $Z, S, Z_1$ and $Z_2$ are projective. If the sequence

$$0 \to O_Z \to i_1^* O_{Z_1} \oplus i_2^* O_{Z_2} \to i_{12}^* O_{Z_1 \cap Z_2} \to 0$$

is a short exact sequence and $Ri_{j_1} O_{Z_j} \cong i_{j_1}^* O_{Z_j}$ for $j = 1, 2$ and $Ri_{12} O_{Z_1} \cong i_{12}^* O_{Z_1}$, then there is a full subcategory of $\text{Perf}(Z_1) \times_{h \text{Perf}(Z_1 \cap Z_2)} \text{Perf}(Z_2)$ which is quasi equivalent to $\text{Perf}(Z)$.

Proof. We would like to show that if we have units of adjunction $\epsilon_1 : id \to R\ell_1 \ell_1^*$ and $\tau_\ell : id \to Rj_\ell \ell_\ell^*$, then for every $F$ in $\text{Perf}(Z)$, the sequence

$$F \xrightarrow{\ell_{12} \ell_\ell^*} R i_{12} \ell_1^* F \oplus R i_{22} \ell_2^* F \xrightarrow{(Ri_{12} \tau_\ell \ell_1^* - Ri_{22} \tau_\ell \ell_2^*)^T} R i_{12} \ell_1^* F$$
forms a distinguished triangle. This sequence is equivalent, by the projection formula, to
\begin{equation}
\mathcal{F} \longrightarrow \mathcal{R}i_{1*}\mathcal{O}_{Z_1} \otimes^{L} \mathcal{F} \oplus \mathcal{R}i_{2*}\mathcal{O}_{Z_2} \otimes^{L} \mathcal{F} \longrightarrow \mathcal{R}i_{12*}\mathcal{O}_{Z_1 \cap Z_2} \otimes^{L} \mathcal{F}.
\end{equation}
This is the tensor product of the sequence in \( \text{Perf}(Z) \) given by
\[\mathcal{O}_Z \longrightarrow \mathcal{R}i_{1*}\mathcal{O}_{Z_1} \oplus \mathcal{R}i_{2*}\mathcal{O}_{Z_2} \longrightarrow \mathcal{R}i_{12*}\mathcal{O}_{Z_1 \cap Z_2}\]
with \( \mathcal{F} \). By the assumptions of the proposition, it follows that this sequence is a distinguished triangle, and in fact comes from a short exact sequence of sheaves. Therefore, Equation 2 corresponds to a distinguished triangle.

Now we can build a functor \( \Psi \) from \( \text{Perf}(Z) \) to \( \text{Perf}(Z_1) \times^h_{\text{Perf}(X)} \text{Perf}(Z_2) \). Take an object \( \mathcal{F} \) in \( \text{Perf}(Z) \), and let
\[\Psi : \mathcal{F} \mapsto (\mathcal{L}i_{1*}\mathcal{F}, \mathcal{L}i_{2*}\mathcal{F}, \text{id})\]
Let \( \Psi \) send \( f \in \text{hom}_{\text{Perf}(Z)}(\mathcal{F}, \mathcal{G}) \) to \( (\mathcal{L}i_{1*}f, \mathcal{L}i_{2*}f, 0) \). It is easy to verify that this is a functor. Our claim then is that this functor is quasi full and faithful. Let \( \mathcal{C}_\Psi \) be the full subcategory of \( \text{Perf}(Z_1) \times^h_{\text{Perf}(X)} \text{Perf}(Z_2) \) made up of objects in the image of \( \Psi \). Define another functor
\[\Theta : \mathcal{C}_\Psi \longrightarrow \text{Tw mod}_{\text{Perf}(Z)}\]
We use \( \text{Tw mod}_{\text{Perf}(Z)} \) as the image of this functor because \( \mathcal{R}i_{i*} \) are quasi functors, therefore they define functors from \( \text{Perf}(Z_i) \) to \( \text{mod}_{\text{Perf}(Z)} \), and we want to use an explicit definition of morphisms between cones, so we take twisted complexes over \( \text{mod}_{\text{Perf}(Z)} \), despite the fact that \( \text{Tw mod}_{\text{Perf}(Z)} \) and \( \text{mod}_{\text{Perf}(Z)} \) are quasi equivalent. This functor is given by
\[\Theta : (\mathcal{L}i_{1*}\mathcal{F}, \mathcal{L}i_{2*}\mathcal{F}, \text{id}) \mapsto \text{Cone}(\mathcal{R}i_{1*}\mathcal{L}i_{1*}\mathcal{F} \oplus \mathcal{R}i_{2*}\mathcal{L}i_{2*}\mathcal{F}, \mathcal{R}(\mathcal{R}i_{1*}\tilde{\tau}_{\mathcal{F}_1}\mathcal{L}i_{1*}\mathcal{F}, \mathcal{R}i_{2*}\tilde{\tau}_{\mathcal{F}_2}\mathcal{L}i_{2*}\mathcal{F})^T \rightarrow \mathcal{R}i_{12*}\mathcal{L}i_{12*}\mathcal{F} \rightarrow -1]\]
Here, the maps \( \tilde{\tau}_{\mathcal{F}_i} \) are obtained by choosing a bimodule lift of the natural transformations \( \tau_{\mathcal{F}_i} \). On homomorphisms we define
\[(f, g, h) \in \text{hom}_{\mathcal{C}_\Psi}((\mathcal{L}i_{1*}\mathcal{F}, \mathcal{L}i_{2*}\mathcal{F}, \text{id}), (\mathcal{L}i_{1*}\mathcal{G}, \mathcal{L}i_{2*}\mathcal{G}, \text{id})) \mapsto \left(\begin{array}{ccc}
\mathcal{R}i_{1*}f & 0 & 0 \\
0 & \mathcal{R}i_{2*}g & 0 \\
\mathcal{R}i_{12*}h & \mathcal{R}(\mathcal{R}i_{1*}\tilde{\tau}_{\mathcal{F}_1}\mathcal{L}i_{1*}\mathcal{F}, \mathcal{R}i_{2*}\tilde{\tau}_{\mathcal{F}_2}\mathcal{L}i_{2*}\mathcal{F})^T & \mathcal{R}i_{12*}\mathcal{L}i_{12*}\mathcal{F}
\end{array}\right)\]
Seeing that this defines a functor is a direct check making use of the fact that \( \tau_{\mathcal{F}_i} \) come from natural transformations. One then sees that \( \Theta \cdot \Psi \) is quasi isomorphic to \( \mathcal{Y}^\mathcal{F} \) in \( H^0 \text{Tw mod}_{\text{Perf}(Z)} \) by Equation 2 and that \( \Theta \cdot \Psi(f) \) is equivalent to \( \mathcal{Y}^f \). It follows that \( \Psi \) is homotopically full and faithful if we can show that the dimensions of \( \text{hom}_{H^0 \mathcal{C}_\Psi}([\Psi(\mathcal{F})], [\Psi(\mathcal{G})]) \) is equal to the dimension of \( \text{hom}_{H^0 \text{Perf}(Z)}([\mathcal{F}], [\mathcal{G}]) \). Both have finite dimension by the fact that everything is projective. This is easy to check. Using Equation 2 we know that the cone of
\[\text{hom}_{\text{Perf}(Z)}(\mathcal{F}, \mathcal{R}i_{i*}\mathcal{L}i_{i*}\mathcal{G}) \oplus \text{hom}_{\text{Perf}(Z)}(\mathcal{F}, \mathcal{R}i_{2*}\mathcal{L}i_{2*}\mathcal{G}) \rightarrow \text{hom}_{\text{Perf}(Z)}(\mathcal{F}, \mathcal{R}i_{12*}\mathcal{L}i_{12*}\mathcal{G}) \]
is quasi equivalent to \( \text{hom}_{\text{Perf}(Z)}(\mathcal{F}, \mathcal{G}) \). Applying adjunction, this cone is quasi equivalent to the cone of the map of complexes
\[\text{hom}_{\text{Perf}(Z_1)}(\mathcal{L}i_{1*}\mathcal{F}, \mathcal{L}i_{1*}\mathcal{G}) \oplus \text{hom}_{\text{Perf}(Z_2)}(\mathcal{L}i_{2*}\mathcal{F}, \mathcal{L}i_{2*}\mathcal{G}) \rightarrow \text{hom}_{\text{Perf}(Z)}(\mathcal{L}i_{12*}\mathcal{F}, \mathcal{L}i_{12*}\mathcal{G}) \]
By the definition of homomorphisms in \( \mathcal{C}_\Psi \), this is quasi equivalent to \( \text{hom}_{\mathcal{C}_\Psi}(\Psi(\mathcal{F}), \Psi(\mathcal{G})) \). This completes the proof. \( \square \)
Remark 2.26. If \( Z = Z_1 \cup Z_2 \) is a variety which has simple normal crossings and whose singular locus \( S \) is smooth of codimension 1, then one may use the proof of [69, Proposition 4.7] to see that \( \text{Perf}(X) \) is quasi equivalent to the homotopy fiber product \( \text{Perf}(Z_1) \times_{\text{Perf}(S)} \text{Perf}(Z_2) \). \( \square \)

Example 2.27. An important example for us is \( Z \), the union of pair of quadrics \( Q_1 \) and \( Q_2 \) in \( \mathbb{P}^3 \) determined by the vanishing of \( f_1 = xy - zw \) and \( f_2 = xy + zw \) respectively. We may deduce that \( \text{Perf}(Z) \) is a full subcategory of \( \text{Perf}(Q_1) \times^{h}_{\text{Perf}(E)} \text{Perf}(Q_2) \) where \( E = Q_1 \cap Q_2 \) by using Proposition 2.25.

The maps \( i_1, i_2 \) and \( i_{12} \) are closed immersions, therefore, we have that higher direct images \( R^ki_*\mathcal{O}_{Q_j} = 0 \) for \( k \neq 0 \) and \( R^ki_{12*}\mathcal{O}_E = 0 \) for \( k \neq 0 \). Hence the natural maps \( i_j_*\mathcal{O}_{Q_j} \to R^ki_{j*}\mathcal{O}_{Q_j} \) and \( i_{12*}\mathcal{O}_E \to R^k i_{12*}\mathcal{O}_E \) are quasi isomorphisms. It follows then that the conditions of Proposition 2.25 hold, and therefore, we have a quasi embedding of \( \text{Perf}(Z) \) into \( \text{Perf}(Q_1) \times^{h}_{\text{Perf}(E)} \text{Perf}(Q_2) \). \( \square \)

2.6. \( A_{\infty} \) categories. An \( A_{\infty} \) category is a collection of objects \( \text{Ob}(\mathcal{C}) \) along with \( Z \) graded \( k \)-vector spaces \( \text{hom}_\mathcal{C}(a, b) \) for every pair \( a, b \in \text{Ob}(\mathcal{C}) \). Composition is not associative but satisfies the \( A_{\infty} \) relations. For every \( d \geq 1 \) and \( (d + 1) \)-tuple of objects \( a_0, \ldots, a_d \), we have maps \( m_d \) for \( d > 0 \),

\[
m_d : \text{home}(a_{d-1}, a_d) \otimes \cdots \otimes \text{home}(a_0, a_1) \to \text{home}(a_0, a_d)[2 - d]
\]

which satisfy

\[
\sum_{m, n} (-1)^{\sigma_m} m_d(f_d, \ldots, f_{n+m+1}, m_m(f_{n+m}, \ldots, f_{n+1}), f_n, \ldots, f_1) = 0
\]

where \( \sigma_n = |f_1| + \cdots + |f_n| - n \). Here \( f_i \) are homomorphisms and \( |f_i| \) is the degree of \( f_i \). An \( A_{\infty} \) category satisfying \( m_i = 0 \) for \( i > 2 \) and which admits units is a dg category. Here \( m_1 \) is the differential and \( m_2 \) is composition of homomorphisms. As in the case of dg categories, one may define the homotopy category \( \mathcal{H}^0 \mathcal{C} \) of any \( A_{\infty} \) category \( \mathcal{C} \) to be the category with the same objects as \( \mathcal{C} \) and whose homomorphisms are \( \mathcal{H}^0 \text{home}(a, b) \) with cohomology taken with respect to \( m_1 \).

Here we will assume that all \( A_{\infty} \) categories are cohomologically unital, that is, that for each object \( a \in \text{Ob}(\mathcal{C}) \), there is an identity homomorphism in \( \text{hom}_{\mathcal{H}^0 \mathcal{C}}([a], [a]) \). In this case \( \mathcal{H}^0 \mathcal{C} \) is a category. A functor of \( A_{\infty} \) categories is a quasi equivalence if it induces an equivalence on the level of homotopy categories. The category of \( A_{\infty} \) functors from \( \mathcal{C}^{op} \) to the dg category \( \text{Ch}_k \) is called \( \text{mod}_\mathcal{C} \). This category is in fact a dg category. Again, one may define the Yoneda functor as before;

\[
Y : \mathcal{C} \to \text{mod}_\mathcal{C}, \quad a \mapsto \text{hom}_\mathcal{C}(-, a).
\]

This functor gives a quasi full and faithful embedding of \( \mathcal{C} \) into \( \text{mod}_\mathcal{C} \), hence [76, Corollary 2.14], every cohomologically unital \( A_{\infty} \) category is quasi equivalent to a dg category in a canonical way.

Seidel defines directed \( A_{\infty} \) categories just as we defined them in the dg case. Let \( U = \{c_1, \ldots, c_n\} \) be an ordered set of objects in an \( A_{\infty} \) category \( \mathcal{C} \) so that \( \text{home}_\mathcal{C}(c_i, c_j) \) are finite dimensional vector spaces.

The following proposition is straightforward, see also [76, pp. 82].

Proposition 2.28. Let \( G : \mathcal{A} \to \mathcal{B} \) be a functor of \( A_{\infty} \) categories which is full and faithful on the level of homotopy categories. Let \( U = \{c_1, \ldots, c_n\} \) be an ordered finite set of objects in \( \mathcal{A} \) and let \( G(U) = \{G(c_1), \ldots, G(c_n)\} \subseteq \mathcal{B} \). Then the directed categories \( \mathcal{A}_U \) and \( \mathcal{B}^{\to}_{G(U)} \) are quasi equivalent.

We can apply this to the Yoneda embedding to see that there is a quasi equivalence

\[
(4) \quad \text{Tw} \mathcal{C}_U \to \text{Tw} (\text{mod}_\mathcal{C})^{\to}_{Y(U)}
\]
On the left hand side, Tw denotes the $A_\infty$ version of the construction of twisted complexes given by [78, 2b]. By Corollary 2.23 the category of twisted complexes over a directed $A_\infty$ category of objects in $\mathcal{C}$ is quasi equivalent to a category constructed using the construction of Kuznetsov and Lunts.

3. Perverse schobers and perverse sheaves of categories

In this section, we will describe the work of Kapranov and Schectman [31] on perverse schobers and give the definition of the category of global sections. We will also describe a general construction of $K$-coordinatized perverse sheaves of categories on Riemann surfaces. We define their monodromy and categories of global sections.

3.1. Perverse sheaves on a disc.

Perverse sheaves in on a complex manifold $M$ are elements of the heart of a $t$-structure on the constructible bounded derived category. In the case where $M$ is the disc then a perverse sheaf $F$ is a (shifted) local system away from a finite set $\Sigma$ of points. Locally around a point in $\Sigma$, $F$ has a concrete linear algebraic description.

**Theorem 3.1** (Galligo, Grainger and Maisonobe [27]). Around a point $p \in \Sigma$, the data of a perverse sheaf is equivalent to the category of quadruples $(\phi, \psi, \text{var}, \text{can})$ where $\phi$ and $\psi$ are finite dimensional vector spaces and $\text{var}: \phi \to \psi$ are homomorphisms so that $\text{id}_\phi - \text{can} \cdot \text{var}$ and $\text{id}_\psi - \text{var} \cdot \text{can}$ are invertible. There is a similar description for perverse sheaves on $\mathbb{C}$ with stratification $\Lambda$ given by the coordinate axes and their intersections.

The method of proof of [27] gives us more than this. Let $K = \mathbb{R}_{\geq 0}$ be the intersection of the positive real axis in $\mathbb{C}$ with a disc $D$ centered around 0 and let $\Lambda_K$ be the stratification $\Lambda_{K,1} = \mathbb{R}_{>0}$ and $\Lambda_{K,0} = \{0\}$. Then one builds a functor $\mathcal{R}_K$ from $\text{Perv}(D, \Lambda)$ to $\text{Constr}(K, \Lambda_K)$ which is full and faithful, and whose image is the set of constructible sheaves on $K$ with fiber $\phi$ on $\Lambda_{K,0}$ and $\psi$ on $\Lambda_{K,1}$. The morphism $\text{var}$ is the variation map, and $\text{can}$ is the canonical map from the theory of perverse sheaves. If $F$ is a perverse sheaf, then the corresponding vector space $\phi$ is the vanishing cycles sheaf of $F$, $\phi$ is the nearby cycles sheaf of $F$.

Kapranov and Schectman then generalize this idea [31][32]. They show that if $X$ is a Riemann surface with boundary and with a set of points $\Sigma$, then for an appropriate choice of skeleton $K$ of $S$ there is a description of $\text{Perv}(S, \Sigma)$, in terms of the combinatorics of $K$.

The trickiest bit of their construction is to understand what the restriction map from $\text{Perv}(\Delta, \Sigma)$ to $\text{Constr}(K)$ around a point where $K$ looks like a wheel with $n$-spokes, as in Figure 1. We refer to [32] for details.

A perverse sheaf on $(D, \Sigma)$ which is a local system away from $\Sigma$ can be given by the following data. First, fix a skeleton $K$ given by:

1. A choice of a point $s \notin \Sigma$,
(2) A choice of a path $\gamma_\infty$ from $s$ to the boundary of $D$ which does not pass through any point $p \in \Sigma$

(3) A choice of ordering of the points $p \in \Sigma$. Label the points in $\Sigma$ as $p_1, \ldots, p_k$ based on this ordering.

(4) A choice of continuous paths $\gamma_i$ from $s$ to each $p_i \in \Sigma$ so that $\gamma_i$ near $s$ are ordered counterclockwise as $\gamma_1, \ldots, \gamma_k, \gamma_\infty$. We assume no paths intersect except at $s$. Then let $K = \gamma_\infty \cup \bigcup_{p \in \Sigma} \gamma_p$, with stratification given by points $\Sigma' = s \cup \Sigma$. A perverse sheaf on $(\Delta, \Sigma)$ is then equivalent to a constructible sheaf $S$ on $(K, \Sigma')$ with fiber $\psi$ over any point not in $\Sigma'$ and fibers $\phi_i$ at each point in $\Sigma$, along with maps $\text{var}_i : \phi_i \to \psi$ and $\text{can}_i : \psi \to \phi_i$.

3.2. Perverse schobers. Let $D$ be the closed unit disc in $\mathbb{C}$ and let $K = \mathbb{R}_{\geq 0}$. Kapranov and Schectman define a $K$-coordinatized perverse schober on $(D, 0)$ to be the data of two idempotent closed pretriangulated dg categories $A$ and $C$ along with a spherical functor $F : A \to C$.

A spherical functor $F$ is a quasi functor $F$ from $A$ to $B$ with right and left adjoint quasi functors $R$ and $L$. The quasi functors $F, R$ and $L$ are interpreted as $A$-$B$ and $B$-$A$ bimodules respectively. We define

$$T = \text{Cone}(FR \to \text{id}_B), \quad C' = \text{Cone}(LF \to \text{id}_A)$$

$$C = \text{Cone}(\text{id}_A \to RF)[-1], \quad T' = \text{Cone}(\text{id}_B \to FL)[-1]$$

Here composition of functors is interpreted as tensor product of bimodules and the cones are taken in categories of $B$-$B$ and $A$-$A$ bimodules where appropriate. The quasi functor $F$ is called spherical if $T$ induces an autoequivalence of $H^0 B$ and the composition

$$R \to RFL \to CL[1]$$

induces a quasi isomorphism of dg bimodules between $R$ and $CL[1]$. Under these conditions, $T, T'$ and $C, C'$ form mutually quasi inverse pairs.

The condition that $T$ be an autoequivalence is analogous to the condition that $\text{id}_\phi - \text{can} \cdot \text{var}$ be invertible. In the case of perverse sheaves, there is only one map between $\psi$ and $\phi$, whereas for a spherical functor, we require that there be two functors $R$ and $L$ from $C$ to $A$. The condition that $R \cong CL[1]$ should be read as saying that these two maps, while not identical, are at least compatible.

**Definition 3.2.** A $K$-coordinatized perverse schober on $(D, \Sigma)$ is a choice of a skeleton $K$ as above, along with the data of a category $\mathcal{C}$ and an ordered collection of categories $A_1, \ldots, A_k$ each
corresponding to each point $p_i \in \Sigma$ and spherical functors $F_i : A_i \to \mathcal{C}$. We will call this data $S_K(A_i, F_i)$. We will call the category $\mathcal{C}$ the fiber category of $S_K(A_i, F_i)$. □

This data will give rise to what is essentially a constructible sheaf of categories on $K$ if we can determine an appropriate category as a fiber over the point $s$. This will be described in the next section.

Remark 3.3. If we assume that $\mathcal{C}$ has the property that the Serre functor of $H^0 \mathcal{C}$ denoted $S_{\mathcal{C}}$ is equal to the shift functor $[k]$ for some integer $k$ and that $C$ induces the Serre functor on $H^0 A$ up to shift, then $R$ and $L$ are quasi equivalent up to shift. This occurs if $F$ admits a weak right Calabi-Yau structure [33]. An alternate definition of perverse schober assuming that $F$ admit a weak right Calabi-Yau structure may in fact be more appropriate. We will see similar structures in Section 5.1. □

3.3. The category of global sections of a perverse schober. Let $\mathcal{C}$ be a pretriangulated dg category, then define $A_2(\mathcal{C}) = \mathcal{C} \times \text{id} \mathcal{C}$, or quasi equivalently, we will often replace this with $\text{Tw} \mathcal{C} \times \text{id} \text{Tw} \mathcal{C}$. This category admits two obvious functors, $f_1, f_2$ to $\text{Tw} \mathcal{C}$ sending $(a, b, \mu)$ to the elements $a$ and $b$ of $\text{Tw} \mathcal{C}$ respectively. Since $\text{hom}_i A_2(\mathcal{C})((a, b, \mu), (c, d, \xi)) = \text{hom}_i^\mathcal{C}(a, c) \oplus \text{hom}_i^\mathcal{C}(b, d) \oplus \text{hom}_i^{\mathcal{C}-1}(a, d)$ we have natural projections onto $\text{hom}_i^\mathcal{C}(a, c)$ and $\text{hom}_i^\mathcal{C}(b, d)$ which describe how these functors act on homomorphisms. There’s another functor $f_3$ from $A_2(\mathcal{C})$ to $\text{Tw} \mathcal{C}$ given by sending $(a, b, \mu)$ to $\text{Cone}(\mu)$. The object $\text{Cone}(\mu)$ is represented as a twisted complex by $(a \oplus b[-1], \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix})$

A homomorphism $(r, s, t)$ of degree $i$ is then sent to $\begin{pmatrix} r & t \\ 0 & s \end{pmatrix}$.

We define the category $A_3(\mathcal{C})$ to be the homotopy fiber product of the diagram;

$$
\begin{array}{ccc}
A_3(\mathcal{C}) & \xrightarrow{g} & A_2(\mathcal{C}) \\
\downarrow h & & \downarrow f_3 \\
A_2(\mathcal{C}) & \xrightarrow{f_1} & \text{Tw} \mathcal{C}
\end{array}
$$

From this we obtain four functors $f_1, \ldots, f_4$ from $A_3(\mathcal{C})$ to $\text{Tw} \mathcal{C}$ as, respectively, $f_1 : g, f_2 : g, f_2 : h, f_3 : h$. Applying this construction recursively, we define categories $A_n(\mathcal{C})$ for any $n$ along with maps $f_i : A_n(\mathcal{C}) \to \text{Tw} \mathcal{C}$ for $i = 1, \ldots, n$.

Definition 3.4. The category of global sections of a perverse schober $S_K(\mathcal{C}, A_p, F_p)$, denoted by $\Gamma S_K(\mathcal{C}, A_p, F_p)$ is the homotopy fiber product of the following diagram,

$$
A_1 \times \cdots \times A_n \xrightarrow{F_1 \times \cdots \times F_n} \text{Tw} \mathcal{C} \times \cdots \times \text{Tw} \mathcal{C} \xleftarrow{f_1 \times \cdots \times f_n} A_n(\mathcal{C}).
$$

The functor $f_{n+1} : A_n(\text{Tw} \mathcal{C}) \to \text{Tw} \mathcal{C}$ induces a functor $\Gamma S_K(\mathcal{C}, A_i, F_i) \to \text{Tw} \mathcal{C}$ which we denote $s_{\infty}$.

The following proposition shows that for computational purposes, the word homotopy is not essential here. The essential fact is that
Lemma 3.5. for any dg category $\mathcal{C}$, the functor
\[ \mathcal{C} \times \text{id, id} \xrightarrow{f_1 \times f_2} \mathcal{C} \times \mathcal{C} \]
is a fibration in the DK model structure.

Proof. We have that $f_1 \times f_2$ acts on objects and homomorphisms as
\[ (a, b, \mu) \mapsto (a, b), \quad (r_1, r_2, g) \mapsto (r_1, r_2) \]
Since $\text{hom}_{\text{Tw} \mathcal{C}}^2((a, b), (c, d)) = \text{hom}_{\text{Tw} \mathcal{C}}^2(a, c) \times \text{hom}_{\text{Tw} \mathcal{C}}^2(b, d)$, any homomorphism can be lifted to a homomorphism of $A_2(\mathcal{C})$. We just need to show that if $(a, b, \mu) \in A_2(\mathcal{C})$ and there $(\xi, \tau) : (a, b) \to (a', b')$ for $a, b, a', b'$ in $\text{Tw} \mathcal{C}$ descends to an isomorphism in the homotopy category, then there is an object $(a', b', \mu')$ which is quasi isomorphic to $(a, b, \mu)$ in $A_2(\mathcal{C})$. If $\xi'$ is a quasi inverse of $\xi$, then we get a morphism $\mu' = \tau \cdot \mu \cdot \xi'$ from $a'$ to $b'$. We may apply Lemma 2.21 to see that $(a', b', \mu')$ is quasi isomorphic to $(a, b, \mu)$. \qed

Therefore, if there are only two categories $A_1, A_2$, the category $\Gamma_{S_K}(A_i, F_i)$ can be computed as the ordinary fiber product of the diagram in Equation 5 instead of the homotopy fiber product. It’s not hard to show that this just recovers $A_1 \times_{F_1, F_2} A_2$.

Proposition 3.6. If $|\Sigma| = 2$ and $S_K(A_i, F_i)$ is a perverse schober then $\Gamma_{S_K}(A_i, F_i)$ is quasi equivalent to $A_1 \times_{F_1, F_2} A_2$.

This quasi equivalence (which we denote $Q$) can be written as follows. We have that objects of $\Gamma_{S_K}(A_i, F_i)$ are given by
\[ ((a, b, \mu), (r, s), (g_{a,r}, g_{b,s})) \]
where $(a, b, \mu) \in \text{Ob}(A_2(\mathcal{C})), (r, s) \in \text{Ob}(A_1) \times \text{Ob}(A_2)$ and $g_{a,r} \in \text{hom}^0_{\mathcal{C}}(a, F_1(r))$ and $g_{b,s} \in \text{hom}^0_{\mathcal{C}}(b, F_2(s))$ are quasi isomorphisms. Objects $(c, d, \xi) \in \text{Ob}(A_1 \times_{F_1, F_2} A_2)$ are mapped by $Q$ to
\[ ((F_1(c), F_2(d), \xi), (c, d), (\text{id}_{F_1(c)}, \text{id}_{F_2(c)})) \]
The action of $Q$ on homomorphisms is defined in a straightforward way. We obtain the following diagram;

\[
\begin{array}{ccc}
A_1 \times_{F_1, F_2} A_2 & \xrightarrow{x} & A_1 \times A_2 \\
\downarrow{y} & & \downarrow{p} \\
\Gamma_{S_K}(\mathcal{C}, A_i, F_i) & \xrightarrow{q} & A_2(\mathcal{C}) \\
\end{array}
\]

where $x, y$ and $p, q$ are the natural maps. This diagram commutes except in the central square, where it only commutes up to homotopy. Composing $y$ with $f_3 : A_2(\text{Tw} \mathcal{C}) \to \text{Tw} \mathcal{C}$ recovers the map $c_{F_1, F_2}$. In other words, $s_\infty \cdot Q = c_{F_1, F_2}$.

We will need some notation for repeated gluings of dg categories.

Definition 3.7. Let us assume that we have $F_i : A_i \to \text{Tw} \mathcal{C}$ for $i = 1, \ldots, k$. Then we may construct the category $A_1 \times_{F_1, F_2} A_2$ which comes equipped with a functor $c_{F_1, F_2} : A_1 \times_{F_1, F_2} A_2 \to \text{Tw} \mathcal{C}$ sending $(a, b, \mu)$ to $\text{Cone}(\mu)$. We then have a pair of functors, $c_{F_3, F_2, F_3}$ to $\text{Tw} \mathcal{C}$, so we may construct the category $(A_1 \times_{F_1, F_2} A_2) \times c_{F_1, F_2, F_3} A_3$. Repeating this process, we obtain a category.
glued together from $A_1, \ldots, A_k$. We will denote this category $\mathcal{G}(A_i, F_i)$. Let $c_{F_1, \ldots, F_k}$ denote the final cone functor to $\text{Tw} \mathcal{C}$.

Now we will prove the following theorem.

**Theorem 3.8.** There is a quasi equivalence $Q_{F_1, \ldots, F_k} : \mathcal{G}(A_i, F_i) \to \Gamma \mathcal{S}_{K_k}(A_i, F_i)$ so that $s_{\infty} \cdot Q_{F_1, \ldots, F_k} = c_{F_1, \ldots, F_k}$.

**Proof.** We proceed by induction. Assume that for $A_1, \ldots, A_{k-1}$ and $F_1, \ldots, F_{k-1}$ there is a quasi equivalence $Q_{F_1, \ldots, F_{k-1}} : \mathcal{G}(A_i, F_i) \to \Gamma \mathcal{S}_{K_{k-1}}(A_i, F_i)$. The case where $n = 2$ is just Proposition 3.6

We now want to compute the homotopy limit of the following diagram:

\[
\begin{array}{ccc}
A_1 \times \cdots \times A_{k-1} & \rightarrow & A_1 \times \cdots \times A_{k-2} \\
\downarrow & & \downarrow \\
A_2 & \rightarrow & A_2 \\
\downarrow & & \downarrow \\
A_k & \rightarrow & \text{Tw} \mathcal{C}
\end{array}
\]

where the square is homotopy cartesian. By assumption, the homotopy fiber product of the top right span is $\mathcal{G}(A_i, F_i)$. Therefore, this reduces to finding the homotopy limit of the diagram:

\[
\begin{array}{ccc}
\mathcal{G}(A_i, F_i) & \rightarrow & \mathcal{G}(A_i, F_i) \\
\downarrow & & \downarrow \\
A_k & \rightarrow & \text{Tw} \mathcal{C}
\end{array}
\]

Equivalently, since the square is homotopy cartesian, we must find the homotopy limit of

\[
\begin{array}{ccc}
\mathcal{G}(A_i, F_i) \times A_k & \rightarrow & \mathcal{G}(A_i, F_i) \times A_k \\
\downarrow & & \downarrow \\
A_2 & \rightarrow & \text{Tw} \mathcal{C}
\end{array}
\]

By Proposition 3.6 there is a quasi equivalence $Q$ between $\mathcal{G}(A_i, F_i) \times c_{F_1, \ldots, F_{k-1}} A_k$ and the homotopy fiber product of this diagram. By definition, then the homotopy limit of (7) is equivalent to $\mathcal{G}(A_i, F_i)$ and the natural map to $A_2(\mathcal{C})$ composed with $f_3$ recovers $c_{F_1, \ldots, F_k}$ as required.

\[
\square
\]

**3.4. More general perverse sheaves of categories.** We now define perverse sheaves of categories in greater generality. Let $S$ be a connected, oriented, compact topological surface with $n$ boundary components and $k$ marked points $\Sigma$. In the following, by embedded graph, we mean a
finite collection of vertices $\text{Vert}(K)$ made up of points in the interior of $S$ and a finite set of edges $\text{Ed}(K)$. Edges will be embeddings $g_e$ of the closed interval $[0,1]$ into $S$ so that at least one of $g_e(0)$ or $g_e(1)$ is a vertex and the other is either a vertex or contained in a boundary component. We assume $g_e((0,1))$ is contained in the interior of $S$ and that $e_1$ and $e_2$ intersect only in elements of $\text{Vert}(K)$. A perverse sheaf of categories starts with the following data.

1. A spanning graph $K$ on $S$. That is, a graph $K$ embedded into $S$ to which $S$ is homotopic. We further require that that each $p_i \in \Sigma$ is a univalent vertex of $K$ and that there are no bivalent vertices.
2. A category $A_v$ for each vertex $v$ of $K$ and a category $\mathcal{C}_e$ for each edge of $K$. If $v$ is not in $\Sigma$ then $A_v = A_n(\mathcal{C})$, where $n+1$ is the valency of $v$. All $\mathcal{C}_e$ are equivalent to some fixed category $\mathcal{C}$.
3. If $e$ is incident to $v$, then there is a functor $F_{v,e} : A_v \to \mathcal{C}$. If $v$ is $(n+1)$-valent, then for some counterclockwise ordering of edges emanating from $v$, denoted $e_1, \ldots, e_{n+1}$, we have $F_{e_i,v} = \phi_{e_i,v} \cdot f_i$ for $\phi_{e_i,v}$ some quasi autoequivalence of $\mathcal{C}$.

**Remark 3.9.** In the case where $e$ has both ends adjacent to the same vertex, we must adjust (3) so that we have one functor $F^1_{v,e}$ and $F^2_{v,e}$ for each adjacency, with corresponding autoequivalences $\phi^1_{v,e}$ and $\phi^2_{v,e}$ of $\mathcal{C}$. If $v \in \Sigma$ then $F_{v,e}$ is spherical.

**Remark 3.10.** This definition is essentially a simplified restatement of the characterization of perverse sheaves on $S$ with singularities in $\Sigma$ as in [32] with vector spaces replaced by dg categories.

This allows us to define something resembling a constructible presheaf $S_K$ of dg categories on $K$ defined in terms of the following sets. If $v \in \text{Vert}(K)$, then let $U_v$ be the union of $v$ and the interior of the adjacent 1-cells. For $e \in \text{Ed}(K)$ let $U_e$ be the interior of the 1-cell corresponding to $e$. To the edges, we assign the categories $\Gamma(S_K, U_e) = \mathcal{C}_e$, and to vertices we let $\Gamma(S_K, U_v) = A_v$, the functors $F_{v,e}$ determine $\Gamma(S_K, U_v) \to \Gamma(S_K, U_e)$ for $e$ an edge adjacent to $v$.

A $K$-coordinatized perverse sheaf of categories should be thought of as a categorification of a perverse sheaf on $S$ which is a local system when restricted to $S \setminus \Sigma$. Therefore, we should have a $K$-coordinatized local system of categories on the complement of $\Sigma$. We define this as follows. For each $p \in \Sigma$ choose a small open disc centered at $p$ denoted $D_p$, then let $S^o = S \setminus \cup_{p \in \Sigma} D_p$. We may construct a skeleton $K^o$ of $S^o$ as follows. If $e_p$ is the edge adjacent to $p$ and $q$ its other endpoint, and $p'$ is a point in $e_p$ which is not in $D_p$, then we replace $e_p$ with an edge $e_{p'}$, the portion of $e_p$ between $q$ and $p'$. Then we attach one more edge $e_{F_p}$ to $p'$ with both endpoints at $p'$ and travelling in a small loop around $D_p$. Call the resulting skeleton $K^o$. We can equip $K^o$ with a $K^o$ coordinatized perverse sheaf of categories $S^o$ which has the same data as $S$ away from the new vertices $p'$. Each new vertex $p'$ is trivalent, and therefore, we must define three functors from $A_2(\mathcal{C})$ to $\mathcal{C}$. Let $F_{p',e_{p'}} = f_3$, let $F^1_{p',e_{F_p}} = T_{F_p} : f_1$ be the functor corresponding to the adjacency between $p'$ and $e_{F_p}$ to the immediate right of $e_{p'}$. Let $F^2_{p',e_{F_p}} = f_2$ be the functor corresponding to the second adjacency between $p'$ and $e_{F_p}$. Here $T_{F_p}$ is the spherical twist associated to $F_p$.

A cycle $C$ is a directed sequence of vertices $v_1, \ldots, v_n$ with $v_n = v_1$ and an edge $e_{i,i+1}$ adjacent to both of $v_i$ and $v_{i+1}$. To any cycle in $K^o$, we define the monodromy around $C$ to be

$$\text{Mon}(C) = \phi_{v_n,e_{n-1}}^{-1} \cdot \phi_{v_{n-1},e_{n-2}} \cdot \ldots \cdot \phi_{e_{i+1},e_i}^{-1} \cdot \phi_{v_{i+1},v_i}$$

which is a quasi autoequivalence of $\mathcal{C}$. Each cycle starting at a chosen vertex $v_1$ induces an element of $\pi_1(S^o, v_1)$, and since $S^o$ is homotopic to $K^o$, we see that every class in $\pi_1(S^o, v_1)$ can be represented.
this way. Our final requirement is that $\text{Mon}$ induce a representation of $\pi_1(S^0, v_1)$ in the category of autoequivalences of $H^0 \mathcal{C}$. The data of $\mathcal{C}, A_p, F_p, K$ satisfying (1), (2) and (3) and the monodromy condition in Equation $\S$ define a $K$-coordinatized perverse sheaf of categories.

Example 3.11. A $K$-coordinatized perverse schober gives rise to a $K$-coordinatized perverse sheaf of categories whose autoequivalences $\phi_{v,e}$ are all trivial. The monodromy around any cycle is a product of spherical twists. □

There is a diagram of categories associated to $S_K$, given by the categories $A_v$ for all $v \in \text{Vert}(K)$ and $\mathcal{C}_e$ for $e \in \text{Edge}(K)$ and the functors $F_{v,e}$ between them. The homotopy limit of this diagram is denoted $\Gamma S_K$, and called the category of global sections of $S_K$.

Remark 3.12. Such a structure should arise whenever we compute the Fukaya category of a symplectic fibration over $S$ with smooth fibers away from $\Sigma$. We allow that there be an indeterminate number of edges emanating towards each boundary component of $S$, which corresponds to allowing more general partial wrappings in the corresponding Fukaya category. □

4. Sheaves of categories on the cylinder, $\mathbb{P}^1$ bundles and a result of Segal

The goal of this section is to give two examples where sheaves of categories appear naturally. First, we will show that a construction of Segal [72] can be recast in terms of perverse sheaves of categories. Next we show that if $X$ is a smooth variety and $\Phi$ an autoequivalence of $\mathcal{D}_Y^h$ then it is clear that $(a[1], \mu)$ is isomorphic to $\mathcal{D}_Y^h((0_X \oplus \mathcal{L}))$. In both cases, we use perverse sheaves of categories on the cylinder $S^1 \times [0,1]$.

4.1. Perverse sheaves of categories and spherical functors. In [72], Segal shows that if $\Phi$ is an autoequivalence of a triangulated category $\mathcal{D}$ which admits a dg enhancement $\mathcal{D}_d$ then there is a triangulated category $\mathcal{D}_\Phi$ and a functor $j^*: \mathcal{D} \to \mathcal{D}_\Phi$ which is spherical and whose cotwist is the autoequivalence $\Phi$. In this section, we will show how this construction can be recast in terms of perverse sheaves of categories.

The first statement that we must prove is the following;

Proposition 4.1. If $A$ and $\mathcal{C}$ are pretriangulated dg categories and $F_1, F_2 : A \to \mathcal{C}$ are functors, then the homotopy equalizer $\mathcal{E}_\mathcal{C}^h(F_1, F_2)$ is pretriangulated.

Proof. Recall that objects of the homotopy equalizer of a dg category can be represented by pairs $(a, \mu)$ where $a \in A$ and $\mu : F_1(a) \to F_2(a)$ is degree 0 closed morphism whose induced morphism in $H^0 \mathcal{C}$ is an isomorphism. Our proof follows that of [41 Proposition 4.3].

If $a[1]$ is an element of $A$ isomorphic to $\mathcal{Y}^{a}[1]$ then it is clear that $(a[1], \mu)$ is isomorphic to $\mathcal{Y}^{(a, \mu)}[1]$. Hence the shift functor makes sense. Morphisms between $(a, \mu)$ and $(b, \xi)$ in $\mathcal{E}_\mathcal{C}^h(F_1, F_2)$ of degree $i$ are pairs of objects $(f, f') \in \text{hom}^i(a, b) \oplus \text{hom}^{i-1}(F_1(a), F_2(b))$, and $(f, f')$ is closed if and only if $df = 0$ and

$$df' = \xi \cdot F_1(f) + (-1)^i F_2(f) \cdot \mu.$$ 

Let $(f, f')$ be a closed degree 0 homomorphism from $(a, \mu)$ to $(b, \xi)$. Then we claim that the cone of $(f, f')$ is represented by the pair $c = \text{Cone}(f)$ along with a morphism that we construct as follows. According to [41 Remark 3.1] we have the following morphisms and identities;

$$a[1] \xrightarrow{i} c \xrightarrow{p} a[1], \quad b \xrightarrow{j} c \xrightarrow{s} b$$
Proposition 4.3. The category \( D \) spherical twist functor is \( \Phi \). Our goal in this section is to show that if \( j \) Segal shows that \( E \) There is a full subcategory of \( \text{trivalent vertex } v \) again pretriangulated, since the homotopy fiber product can be expressed as a homotopy equalizer.

Remark 4.2. This also implies that the homotopy fiber product of pretriangulated dg categories is again pretriangulated, since the homotopy fiber product can be expressed as a homotopy equalizer. \( \square \)

Segal \[72\] begins with a triangulated category \( D \) and an autoequivalence \( \Phi \) of \( D \). He then constructs a category \( D_\Phi \) which is generated by objects \( j^*a \) for all \( a \in D \), which satisfy

\[
\text{hom}_{D_\Phi}(j^*a, j^*b) = \text{hom}_D(x, y) \oplus \text{hom}_D(x, \Phi^{-1}y[1]).
\]

If \( (f, f') \in \text{hom}_{D_\Phi}(j^*a, j^*b) \) and \( (g, g') \in \text{hom}_{D_\Phi}(j^*b, j^*c) \), then

\[
(g, g') \cdot (f, f') = (g \cdot f, g' \cdot f + \Phi^{-1}(g) \cdot f').
\]

Segal shows that \( j \) has right and left adjoints \( j_* \) and \( j^! \) respectively, is spherical, and that its spherical twist functor is \( \Phi \). Our goal in this section is to show that if \( D \) has a dg extension \( D_{dg} \), then there is a perverse sheaf of categories \( S_p(D, \Phi) \) whose category of global sections has split closure which is quasi equivalent to the split closure of a dg extension of \( D_\Phi \). For simplicity, we will assume that \( \Phi \) is an autoequivalence of \( D_{dg} \) with inverse \( \Phi^{-1} \).

Then we let \( K_p \) be the skeleton of \( S^1 \times [0, 1] \) depicted in Figure 3 that is, a graph with one trivalent vertex \( v \) and two edges \( e_c \) and \( e_o \). The edge \( e_c \) has both ends equal to \( v \) and \( e_o \) has one end connected to \( v \) and the other is contained in one boundary component of \( S \times [0, 1] \). We let \( S_p(D, \Phi) \) be the perverse sheaf of categories on \( K_p \) so that \( A_v = A_2(D), C_e = D, F_{v,e_c} = f_1, F_{v,e_o} = \Phi \cdot f_2 \) (see Remark 3.9) and so that \( F_{v,e_o} = f_3 \). This means that the category of global sections \( \Gamma S_p(D, \Phi) \) is the category \( \text{Eq}^h(f_1, \Phi \cdot f_2) \) where,

\[
f_1 : A_2(D) \to D, \quad (a, b, \mu) \mapsto a, \quad \Phi \cdot f_2 : A_2(D) \to D, \quad (a, b, \mu) \mapsto \Phi(b).
\]

There is a full subcategory of \( \text{Eq}^h(f_1, \Phi \cdot f_2) \) made up of objects,

\[
\{(a, \Phi^{-1}(a), 0), \text{id}_a) : a \in D\}
\]

We will denote this category \( D_{\Phi,dg} \).

Proposition 4.3. The category \( D_{\Phi,dg} \) is a dg enhancement of \( D_\Phi \).
Proof. Let $\alpha = ((a, \Phi^{-1}(a), 0), \text{id}_a)$ and $\beta = ((b, \Phi^{-1}(b), 0), \text{id}_b)$. An element of $\text{hom}_{\mathcal{E}q^h(f_1, \Phi \cdot f_2)}(\alpha, \beta)$ is written as $f = ((s, r, t), u)$ where $s \in \text{hom}^i_D(a, b)$, $r \in \text{hom}^j_D(\Phi^{-1}(a), \Phi^{-1}(b))$, $t \in \text{hom}^{l-1}_D(a, \Phi^{-1}(b))$ and $u \in \text{hom}^{m-1}_D(a, b[1]) = \text{hom}^j_D(a, b)$. We check that
\[
d((s, t, r), u) = ((ds, dt, dr), du - (s - \Phi(t)))
\]
So if $f$ is closed of degree 0 then $s, r$ are closed, $t$ is closed and $du = s - \Phi(r)$. This means that $s$ and $\Phi(r)$ are homtopy equivalent, hence $r$ and $\Phi^{-1}(s)$ are homotopy equivalent, so that $d\Phi^{-1}(u) = \Phi^{-1}(s) - r$. Now we will check that
\[
d((0, \Phi^{-1}(u), 0), 0) = ((0, \Phi^{-1}(s) - r, 0), -u) = ((s, \Phi^{-1}(s), t), 0) - ((s, r, t), u)
\]
Therefore, every closed morphism is, up to sign, quasi equivalent to a morphism of the form $((s, \Phi^{-1}(s), t), 0)$. Such morphisms are automatically closed since $s$ and $t$ are closed and are exact if and only if $s$ and $t$ are exact. Therefore, $H^0 \mathcal{D}_{\Phi, dg}$ has morphisms represented by closed degree 0 morphisms $s : a \to b$, and $r : a \to \Phi^{-1}(b)[1]$. Composition can be computed inside of $\mathcal{D}_{\Phi, dg}$ to be;
\[
((s, \Phi^{-1}(s), t), 0) \cdot ((r, \Phi^{-1}(r), q), 0) = ((s \cdot r, \Phi^{-1}(s \cdot t), r \cdot t + \Phi^{-1}(s) \cdot q), 0).
\]
Therefore composition of morphisms agrees with that of $\mathcal{D}_{\Phi}$ so $\mathcal{D}_{\Phi, dg}$ is a dg enhancement of $\mathcal{D}_{\Phi}$.

Next we will show that any element of $\mathcal{E}q^h(f_1, \Phi \cdot f_2)$ is homotopic to a direct summand of the cone of a pair of elements in $\mathcal{D}_{\Phi, dg}$.

**Proposition 4.4.** If $\delta = ((a, b, \mu), \tau)$ is an element of $\mathcal{E}q^h(f_1, \Phi \cdot f_2)$, then $\delta$ is homotopic to a direct summand of the cone of a morphism of a pair of elements in $\mathcal{D}_{\Phi, dg}$. Therefore, $D^* \mathcal{E}q^h(f_1, \Phi \cdot f_2)$ is equivalent to the split closure of $\mathcal{D}_{\Phi}$.

**Proof.** Following the proof of Lemma 2.21, one sees that there is a triple $(a, \Phi^{-1}(a), \mu')$, $\text{id}_a$ in $\mathcal{E}q^h(f_1, \Phi \cdot f_2)$ which is quasi equivalent to $\delta$. Now one takes the cone of
\[
((a, \Phi^{-1}(a), 0), \text{id}_a) \xrightarrow{((0, 0, \mu'), 0)} ((a, \Phi^{-1}(a), 0), \text{id}_a)
\]
One can check using the description of cones in the proof of Proposition 4.1 that this cone is homotopy equivalent to
\[
((a \oplus a, \Phi^{-1}(a) \oplus \Phi^{-1}(a), \mu \oplus 0), \text{id}_a \oplus \text{id}_a)
\]
which is isomorphic to
\[
((a, \Phi^{-1}(a), \mu), \text{id}_a) \oplus ((a, \Phi^{-1}(a), 0), \text{id}_a).
\]
Therefore, we have that $D_{\Phi, dg}$ must split generate $\Pi \mathcal{E}q^h(f_1, \Phi \cdot f_2)$, so by [76] Corollary 4.9, we have $\Pi Tw D_{\Phi, dg}$ is quasi equivalent to $\Pi \mathcal{E}q^h(f_1, \Phi \cdot f_2)$. □

Two of Segal’s functors then have natural interpretations in terms of our construction. The functor $j^*$ is comes from the functor

\[ a \mapsto ((a, \Phi^{-1}(a), 0), \text{id}_a) \in D_{\Phi, dg}, \]

and its right adjoint $j_*$ comes from the functor inherited from $f_3$ from $A_2(D_{dg})$ to $Tw D$, sending

\[ ((a, b, \mu), \tau) \mapsto \text{Cone}(\mu) \]

This functor should be thought of as the pullback to a fiber functor, from the category of global sections of $S_p(D, \Phi)$ to the stalk along a fiber of the edge of $K_\Phi$ which intersects a boundary component of $S \times [0, 1]$.

### 4.2. $\mathbb{P}^1$ Bundles and Sheaves of Categories

In this section, we will deal with the bounded derived category of coherent sheaves on varieties of the form $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ for some line bundle $\mathcal{L}$ on a variety $X$. It is well known [36, 80, 81] that if $X = \text{pt}$ then there is a perverse sheaf of categories on the skeleton in Figure 4 whose category of global sections is $D^b(\mathbb{P}^1)$. We place the category $A_2(\text{Perf}_k)$ at the two vertices, and the category $\text{Perf}_k$ at each point along an edge. The category $\Gamma S_{\Phi}$ is the homotopy fiber product $A_2(\text{Perf}_k) \times_{\text{Perf}_k} \text{Perf}_k A_2(\text{Perf}_k)$ taken over the functors $f_1 \times f_2$ for both copies of $A_2(\text{Perf}_k)$.

The idea is now that we let a pretriangulated dg category $\mathbb{C}$ be the general fiber of our perverse sheaf of categories on the same complex $K_\Phi$, but this time, we will assume that there is no monodromy around the central $S^1$. This is encoded by defining the transition functor along one of the edges to be an auto equivalence $\Phi^{-1}$ of $\mathbb{C}$. We have four edges $g_1, g_2$ and $h_1, h_2$ and two vertices $v_1, v_2$. The edge $g_1$ is adjacent to $v_1$ and has its second vertex in one of the boundary components. Similarly, $g_2$ is adjacent to $v_2$ and has its second vertex in the other boundary component. The edges $f_1$ and $f_2$ are adjacent to both $v_1$ and $v_2$. Both vertices are trivalent, so $A_{v_1} = A_{v_2} = A_2(\mathbb{C})$ and we let $F_{v_1, g_1} = F_{v_2, g_2} = f_3$, $F_{v_1, h_1} = F_{v_2, h_1} = f_1$ and $F_{v_1, h_2} = \Phi \cdot f_2$ and $F_{v_2, h_2} = f_2$. Call the resulting sheaf of categories $S_{\Phi}(\mathbb{C}, \Phi)$. The category of global sections of $S_{\Phi}(\mathbb{C}, \Phi)$ is the homotopy fiber product of

\[ A_2(\mathbb{C}) \xrightarrow{f_1 \times \Phi \cdot f_2} \mathbb{C} \times \mathbb{C} \xrightarrow{f_1 \times f_2} A_2(\mathbb{C}). \]

There are two natural subsets of $\Gamma S_{\Phi}(\mathbb{C}, \Phi)$ written as

\[ S_1 = \{((a, 0, 0), (a, 0, 0), \text{id}_a) : a \in \mathbb{C}\}, \quad S_2\{((0, b, 0), (0, \Phi(b), 0), \text{id}_b) : b \in \mathbb{C}\} \]
It is not hard to check that if \( A, A' \in S_1 \) then \( \text{hom}_{\Gamma S_\phi(C, \Phi)}(A, A') \) is the cone of the map
\[
\text{hom}_{A_2(C)}((a, 0, 0) \times (a, 0, 0), (a', 0, 0) \times (a', 0, 0)) \to \text{hom}_{(T_w C)}(a \times 0, a' \times 0)
\]
\[
\text{hom}_C(a, a') \oplus \text{hom}_C(a, a') \xrightarrow{\text{id} \otimes \text{id}} \text{hom}_C(a, a')
\]
This cone is quasi isomorphic as a complex to the diagonal subcomplex of \( \text{hom}_C(a, a') \). Let us take then an object in \( \Gamma S_\phi(C, \Phi) \) subcategory \( S_\Phi(C, \Phi) \) of \( \Gamma S_\phi(C, \Phi) \) whose objects are \( S_1 \) and whose morphisms are given by the diagonal subcomplex. It is easy to see that this indeed forms a subcategory. Define similarly the subcategory \( S'_\phi(C, \Phi) \) of \( \Gamma S_\phi(C, \Phi) \) on objects \( S_1 \) and \( S_2 \). Clearly, \( S_1 \) is quasi equivalent to \( S'_1 \).

**Proposition 4.5.** The following statements are true.

1. \( \text{hom}_{\Gamma S_\phi(C, \Phi)}(B, A) = 0 \) if \( A \in C_1, B \in C_2 \).
2. \( \text{hom}_{\Gamma S_\phi(C, \Phi)}(A, B) \) is isomorphic to \( \text{hom}_C(a, b \oplus \Phi(b)) \) if \( A \in C_1 \) and \( B \in C_2 \).
3. Every element of \( \Gamma S_\phi(C, \Phi) \) is quasi isomorphic to \( \text{Cone}(A \to B) \) for \( A \in C_1 \) and \( B \in C_2 \).

Therefore, if \( G \) is the \( C \times C \) bimodule sending \( b \otimes a \) to \( \text{hom}_C(a, b \oplus \Phi(b)) \), there is a quasi equivalence between \( C \times C \) and \( \Gamma S_\phi(C, \Phi) \).

**Proof.** Seeing that condition (1) of the proposition holds is a direct check. To check condition (2), we suppose that \( A \in C_1 \) and \( B \in C_2 \). Then we see that:
\[
\text{hom}_{\Gamma S_\phi(C, \Phi)}(A, B) = \text{Cone}(\text{hom}_C(a, b) \oplus \text{hom}_C(a, \Phi(b)) \to \text{hom}_C(a, 0) \oplus \text{hom}_C(0, b)) = \text{hom}_C(a, b \oplus \Phi(b)).
\]
Therefore, we have a quasi full and faithful embedding of \( C \sqcup C \) into \( \Gamma S_\phi(C, \Phi) \) where \( G \) is the \( C \times C \) bimodule sending \( (a, b) \) to \( \text{hom}_C(a, b \oplus \Phi(b)) \).

If we can show that (3) holds, then we know that \( T_w(C \sqcup C) \) embeds quasi full and faithfully into \( T_w \Gamma S_\phi(C, \Phi) \), which is quasi equivalent to \( \Gamma S_\phi(C, \Phi) \). Therefore, (3) implies the final statement of the proposition. Let us take then an object in \( \Gamma S_\phi(C, \Phi) \),
\[
((a, b, \mu), (a', b', \mu'), \xi_a, \xi_b)
\]
where \( \xi_a : a \to a' \) and \( \xi_b : \Phi(b) \to b' \) are quasi isomorphisms in \( C \) and \( (a, b, \mu) \) and \( (a', b', \mu') \) are objects in \( A_2(C) \). We may construct a homotopy commutative diagram
\[
\begin{array}{ccc}
a & \xrightarrow{\tau} & \Phi(b) \\
\downarrow{\xi_a} & & \uparrow{\xi_b} \\
a' & \xrightarrow{\mu'} & b'
\end{array}
\]
where \( \xi_b' \) is a lift of \( [\xi_b]^{-1} \in Z^0 \text{hom}_C([b'], [\Phi(b)]) \) to \( \text{hom}_C(b', \Phi(b)) \). Therefore, \( (a, \Phi(b), \tau) \) is quasi isomorphic to \( (a', b', \mu') \) by \[8, \text{Lemma 4.2}\]. Now we have that \( (\mu, \tau, 0) \) is a closed degree 0 morphism in \( \text{hom}_{\Gamma S_\phi(C, \Phi)}(A, B) \) where \( A = ((a, 0, 0), (a, 0, 0), \text{id}) \) and \( B = ((0, b, 0), (0, \Phi(b), 0), \text{id}) \). Following the construction of cones in \[11, \text{Lemma 4.3}\], the cone of \( (\mu, \tau, 0) \) is represented by the object \((a, b, \mu), (a, \Phi(b), \tau), \text{id}_a, \text{id}_b \). Thus (3) follows and hence we have that \( S_1 \) and \( S_2 \) form a semiorthogonal decomposition of \( \Gamma S_\phi(C, \Phi) \).

If \( a, b \in \text{Ob}(C) \) and \( \mu = \mu_1 \oplus \mu_2 \in \text{hom}_C(a, b) \oplus \text{hom}_C(a, \Phi(b)) \) then the quasi equivalence alluded to in the statement of Proposition \[4.5\] is given by
\[
\Xi : (a, b, \mu) \mapsto ((a, b, \mu_1), (a, \Phi(b), \mu_2), \text{id}_a, \text{id}_\Phi(b))
\]
Now we wish to show that if $\mathcal{E} = \text{D}^b_{\text{dg}}(X)$ and $\Phi_\mathcal{E} = (-) \otimes^L \mathcal{L}$ for some line bundle $\mathcal{L}$ on $X$ then $\Gamma_S(D^b_{\text{dg}}(X), \Phi_\mathcal{E})$ has homotopy category equivalent to $\text{D}^b(\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{L}))$. According to Orlov [61, Example 3.9], if $\mathcal{E}$ is a rank 2 vector bundle on a variety $X$, then $\text{D}^b_{\text{dg}}(\mathbb{P}_X(\mathcal{E}))$ is equivalent to the gluing of $\text{D}^b_{\text{dg}}(X)$ to itself along the bimodule

$$S_\mathcal{E}(b, a) \cong \text{hom}_{D^b_{\text{dg}}(X)}(a, b \otimes^L \mathcal{E}).$$

This can be proved as follows. One has that $\text{D}^b(\mathbb{P}_X(\mathcal{E}))$ admits a semiorthogonal decomposition

$$\langle Lp^* \text{D}^b(X), Lp^* \text{D}^b(X) \otimes^L \mathcal{O}_\mathbb{P}(1) \rangle$$

where $p : \mathbb{P}_X(\mathcal{E}) \to X$ is the natural projection map and $\mathcal{O}_\mathbb{P}(1)$ is the relative hyperplane bundle. Therefore, there is a quasi equivalence between $D^b_{\text{dg}}(X) \times_{S_\mathcal{E}} D^b_{\text{dg}}(X)$ and $\text{D}^b_{\text{dg}}(\mathbb{P}_X(\mathcal{E}))$ where

$$S_\mathcal{E}(b, a) = \text{hom}_{D^b_{\text{dg}}(\mathbb{P}_X(\mathcal{E}))}(Lp^*a, Lp^*b \otimes^L \mathcal{O}_\mathbb{P}(1)).$$

This follows from [41, Proposition 4.10]. Then, by [58], we have that the cohomology of $S_\mathcal{E}(b, a)$ is equal to that of $S_\mathcal{E}(b, a)$, and they are quasi isomorphic bimodules. Finally [41, Lemma 4.7] then says that $D^b_{\text{dg}}(X) \times_{S_\mathcal{E}} D^b_{\text{dg}}(X)$ is quasi equivalent to $D^b_{\text{dg}}(X) \times_{S_\mathcal{E}} D^b_{\text{dg}}(X)$.

If $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{L}$, then $S_\mathcal{E}$ is precisely the gluing functor $\mathcal{G}$ described in Proposition 4.5. It follows that:

**Theorem 4.6.** There categories $\Gamma_S(D^b_{\text{dg}}(X), \Phi_\mathcal{E})$ and $\text{D}^b_{\text{dg}}(\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{L}))$ are quasi equivalent.

**Remark 4.7.** More generally, if $\mathcal{E}$ is a dg category and $\Phi$ is an autoequivalence, then $\Gamma_S(\mathcal{E}, \Phi)$ should be thought of as a noncommutative $\mathbb{P}^1$ bundle over $\mathcal{E}$. These are analogues of holomorphic families of categories with fibers equivalent to $\text{D}^b(\mathbb{P}^1)$ over the base $\mathcal{E}$.

**Remark 4.8.** This notion of noncommutative $\mathbb{P}^1$ bundles does not coincide with that of Van den Bergh [97] in general. That being said, Van den Bergh constructs the noncommutative $\mathbb{P}^1$ bundle $Q_\mathcal{E}$ using a sheaf bimodule $\mathcal{E}$. One may extend $\mathcal{E}$ to a dg $D^b_{\text{dg}}(X)$-$D^b_{\text{dg}}(X)$ bimodule $\mathcal{E}$. We expect that $D^b_{\text{dg}}(\mathcal{gr}(Q_\mathcal{E}))$ is quasi equivalent to $D^b_{\text{dg}}(X) \times_{\mathcal{E}} D^b_{\text{dg}}(X)$.

**Remark 4.9.** The philosophy of perverse sheaves of categories is that their global sections represent partially wrapped Fukaya categories [86] of fibrations over, in our case, Riemann surfaces. Theorem 4.6 is a good demonstration of this philosophy. Assume that we have a symplectic manifold $(M, \omega)$ and a symplectomorphism $\Psi$ of $(M, \omega)$. Then we may build a symplectic fibration over $S^1 \times \mathbb{R}$ which is a symplectic fiber bundle $(V, \sigma)$ with fiber $(M, \omega)$ and whose symplectic monodromy around the meridian of $S^1 \times \mathbb{R}$ is $\Psi$. Proposition 4.5 suggests that the partially wrapped Fukaya category $\mathcal{W}_s(V, \sigma)$ with two stops, one at a fiber in each boundary component, satisfies

$$D^\sigma \mathcal{W}_s(V, \sigma) = D^\sigma \mathcal{F}(M, \omega) \times_{S_\Psi} D^\sigma \mathcal{F}(M, \omega)$$

where $S_\Psi$ is the $\mathcal{F}(M, \omega)$-$\mathcal{F}(M, \omega)$ bimodule which satisfies

$$S_\Psi(b, a) = \text{hom}_{\mathcal{F}(M, \omega)}(a, b \oplus \Psi(b)).$$

A consequence of this is that if $(M, \sigma)$ is homologically mirror to an algebraic variety $X$, and the symplectomorphism $\Psi$ acts on $\mathcal{W}_s(M, \omega)$ in a way that is mirror to $(-) \otimes^L \mathcal{L}$ for some line bundle $\mathcal{L}$ on $X$, then $(V, \sigma)$ should be the homological mirror to $\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{L})$. Similarly, the category $D^\sigma \mathcal{W}_s(V, \sigma)$ with $s$ denoting a stop at a fiber in one of the two boundary components should be equivalent to $D^b_X(L)$ where $L$ is the total space of $\mathcal{L}$. 
Remark 4.10 ([P^1 bundles and Segal’s construction]. Section 4.1 and Section 4.2 present very similar constructions. The category S_p(C, Φ) embeds into the category S_p(C, Φ) as the subcategory whose restriction to fibers along one of the two unbounded edges is homotopy trivial. This is a localization of S_p(C, Φ) along a functor. In the case of Theorem 1.6 this functor amounts to pullback to the derived category of one of the two sections of P_X(Ω_X ⊕ L), as we will show in Proposition 6.3. Localization of D^b(P_X(Ω_X ⊕ L)) along this functor gives us the category D^b_X(L) where L is the total space of L. Hence S_p(C, Φ) should be thought of as the category of coherent sheaves on a noncommutative line bundle over C.

Segal uses precisely the fact that D^b_X(L) can be thought of the category of sheaves of graded modules Ω_X ⊕ L algebras to deduce his construction in [72, Example 2.4]. This explains why the computations of Section 4.1 suffice to recover the category that Segal uses.

5. Perverse schobers and semiorthogonal decompositions

In this section, we will discuss how semiorthogonal decompositions and perverse schobers interact. In Section 5.1 we show that semiorthogonal decompositions combined with certain spherical functors give rise naturally to perverse schobers. In Section 5.2 we describe how mutation of semiorthogonal decompositions interacts with Kapranov and Schectman’s braid group action on perverse schobers.

5.1. Semiorthogonal decompositions and Serre functors. In this section, we will assume that all triangulated categories have finite dimensional spaces of homomorphisms, and that dg categories are enhancements of such triangulated categories. Let us recall a couple concepts. If T is a triangulated category, the Serre functor S is an autoequivalence of T which has the property that there are functorial isomorphisms \( \phi_{a,b} : \text{hom}_T(a,b) \to \text{hom}_T(b,S(a))^\vee \) and so that the pairing \( (\phi_{S(b),S(a)})^\vee \cdot \phi_{a,b} \) is the morphism \( \text{hom}_T(a,b) \to \text{hom}_T(S(a),S(b)) \) induced by S. Serre functors are unique if they exist [13]. We will write \( S_T \) if we want to emphasize the ambient category.

A triangulated subcategory \( \mathcal{B} \) of T is called admissible if the embedding functor \( \alpha : \mathcal{B} \to T \) has right and left adjoints \( \alpha^* \) and \( \alpha^! \). To an admissible subcategory \( \mathcal{B} \) there are admissible right and left orthogonal subcategories of T denoted \( \mathcal{B}^\perp \) and \( \perp \mathcal{B} \) respectively and defined

\[
\mathcal{B}^\perp = \{ t \in T : \text{hom}_T(b,t) = 0, \forall b \in \mathcal{B} \}, \quad \perp \mathcal{B} = \{ t \in T : \text{hom}_T(t,b) = 0, \forall b \in \mathcal{B} \}
\]

An ordered collection \( (\mathcal{B}_1, \ldots, \mathcal{B}_k) \) of admissible subcategories forms a semiorthogonal decomposition of T if \( \mathcal{B}_i \subseteq \mathcal{B}_j^\perp \) if \( i < j \) and every element of T can be obtained by repeatedly taking cones of morphisms between elements of \( \mathcal{B}_1, \ldots, \mathcal{B}_k \). If \( \mathcal{B} \) is an admissible subcategory of T, then there are semiorthogonal decompositions \( (\mathcal{B}, \perp \mathcal{B}) \) and \( (\mathcal{B}^\perp, \mathcal{B}) \).

If \( \mathcal{A} \) is a dg extension of T, and T admits a semiorthogonal decomposition, we have dg extensions \( \mathcal{A}_i \) of each \( \mathcal{B}_i \) defined to be the full subcategory of \( \mathcal{A} \) whose objects correspond to objects in \( \mathcal{B}_i \). By a slight abuse of notation, we will say that \( (\mathcal{A}_1, \ldots, \mathcal{A}_k) \) is a semiorthogonal decomposition of \( \mathcal{A} \) in this case. The following result of Addington tells us how spherical functors interact with semiorthogonal decompositions. This extends a result of Seidel and Thomas [79].

Proposition 5.1 (Addington [2 Proposition 1.1]). Let \( \mathcal{A} \) and \( \mathcal{C} \) be pretriangulated dg categories whose homotopy categories have Serre functors \( S_{[\mathcal{A}]} \) and \( S_{[\mathcal{C}]} \). Assume that there is a spherical functor \( F : \mathcal{A} \to \mathcal{C} \) with cotwist \( C \) which induces \( S_{[\mathcal{A}]}/k \) for some integer \( k \). If \( \alpha : \mathcal{B} \to \mathcal{A} \) is an admissible functor then \( F\alpha \) is spherical and its cotwist induces \( S_{[\mathcal{B}]}/k \).
Therefore if we have a functor $F$ satisfying the conditions of Proposition 5.1 and if $\mathcal{A}$ admits a semiorthogonal decomposition $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$, with admissible functors $\alpha_j : \mathcal{A}_j \to \mathcal{A}$ then we get an ordered sequence of spherical functors $F_j = F\alpha_j : \mathcal{A}_j \to \mathcal{C}$.

**Corollary 5.2.** In the situation of Proposition 5.1, there is a natural $K$-coordinatized perverse schober $S_K(\mathcal{C}, \mathcal{A}_i, F_i)$ for an appropriate choice of $K$.

We would like to check that the category of global sections of this $S(\mathcal{C}, \mathcal{A}_i, F_i)$ recovers $\mathcal{A}$. We will say a subcategory $\mathcal{B}$ of a pretriangulated dg category $\mathcal{A}$ is admissible if $H^0\mathcal{B}$ is an admissible subcategory of $H^0\mathcal{A}$. We will define $\perp \mathcal{B}$ and $\mathcal{B}^\perp$ as the full subcategories of $\mathcal{A}$ whose objects correspond to objects of $\perp H^0\mathcal{B}$ and $H^0\mathcal{B}^\perp$ respectively.

**Lemma 5.3.** Let $\mathcal{A}$ be a pretriangulated dg category and let $\mathcal{B}$ be an admissible subcategory. Let $a \in \mathcal{B}$ and $b \in \mathcal{B}^\perp$, and assume that there’s a spherical functor $F : \mathcal{A} \to \mathcal{C}$ with right adjoint $R$ so that the cotwist $C$ of $F$ on $\mathcal{A}$ is a shift of the Serre functor $S_{[\mathcal{A}]}$. Then we have a quasi isomorphism of complexes

$$\text{hom}_\mathcal{A}(a, b) \cong \text{hom}_\mathcal{C}(F(a), F(b)).$$

**Proof.** The condition that the cotwist $C$ of $F$ is a shift of the Serre functor says that we have a distinguished triangle,

$$b \to RF(b) \to C(b).$$

Applying the triangulated functor $\text{hom}_\mathcal{A}(a, -)$ from $\mathcal{A}$ to $\text{Ch}_k$ to this exact triangle, we get a distinguished triangle

$$\text{hom}_\mathcal{A}(a, b) \to \text{hom}_\mathcal{A}(a, RF(b)) \to \text{hom}_\mathcal{A}(a, C(b))$$

By adjunction, we have that the second term is quasi equivalent to $\text{hom}_\mathcal{C}(F(a), F(b))$ and the third term is quasi isomorphic to $\text{hom}_\mathcal{A}(b[k], a)$ by the fact that $C$ is equivalent to the Serre functor up to shift. Since $a \in \mathcal{B}$ and $b \in \mathcal{B}^\perp$, this vanishes up to homotopy and hence we get the required quasi isomorphism of complexes. \qed

Let us assume that there are dg categories $\mathcal{C}$ and $\mathcal{A}$ and that there is a spherical functor $F : \mathcal{A} \to \mathcal{C}$ with cotwist $C$ which is equivalent to $S_{[\mathcal{A}]}[k]$. Let us also assume that there is an admissible subcategory $\mathcal{B}$ of $\mathcal{A}$. Then we have an immediate corollary to Lemma 5.3.

**Corollary 5.4.** There is a quasi isomorphism between the $\mathcal{B}$-$\mathcal{B}^\perp$ dg bimodules defined as

$$\varphi : b \otimes a \in (\mathcal{B}^\perp)^\text{op} \otimes \mathcal{B} \mapsto \text{hom}_\mathcal{A}(a, b), \quad \varpi : b \otimes a \in (\mathcal{B}^\perp)^\text{op} \otimes \mathcal{B} \mapsto \text{hom}_\mathcal{C}(F(a), F(b)).$$

There is a quasi-equivalence, [11] Proposition 4.10] $c_\varphi : \mathcal{B} \times_\varphi \mathcal{B}^\perp \to \text{Tw} \mathcal{A}$ which sends a triple $(a, b, g)$ to $\text{Cone}(g)$ in $\text{Tw} \mathcal{A}$. There is also a natural functor $c_\varpi : \mathcal{B} \times_\varpi \mathcal{B}^\perp \to \text{Tw} \mathcal{C}$ which operates the same way, sending $(a, b, \mu)$ to $\text{Cone}(\mu)$. Finally, there is a functor $f : \mathcal{B} \times_\varpi \mathcal{B}^\perp \to \mathcal{B} \times_\varpi \mathcal{B}^\perp$ sending $(a, b, g)$ to $(a, b, F(g))$, which is a quasi-equivalence of categories [11] Lemma 4.7]. We obtain the following commutative diagram of categories

$$\begin{array}{cccc}
\mathcal{B} \times_\varphi \mathcal{B}^\perp & \xrightarrow{f} & \mathcal{B} \times_\varpi \mathcal{B}^\perp \\
\downarrow c_\varphi & & \downarrow c_\varpi \\
\text{Tw} \mathcal{A} & \xrightarrow{F^{\text{Tw}}} & \text{Tw} \mathcal{C} \\
\uparrow & & \uparrow \\
\mathcal{A} & \xrightarrow{F} & \mathcal{C}
\end{array}$$

(11)
The vertical maps on the bottom are the natural embeddings, which are quasi equivalences since we assume that $\mathcal{A}$ and $\mathcal{C}$ are pretriangulated. The chain of functors along the left hand side and top of (11) provide a quasi equivalence between $\mathcal{B} \times \phi \mathcal{B}^\perp$ and $\mathcal{A}$. This chain of quasi equivalences also identifies $c_\phi$ with $F$ as bimodules. We record this as a corollary;

**Corollary 5.5.** Keeping the notation of Corollary [5.4], the categories $\mathcal{B} \times \phi \mathcal{B}^\perp$ and $\mathcal{A}$ are quasi equivalent. Furthermore $F$ and $c_\infty$ are identified under the corresponding chain of quasi equivalences.

Using Propositions [2.11] and [2.12] we have that for any dg category $\mathcal{D}$ and functor $G : \mathcal{D} \to \mathcal{C}$, there is a chain of quasi equivalences between $\mathcal{D} \times_{G,F} \mathcal{A}$ and $\mathcal{D} \times_{G,c_\phi} (\mathcal{B} \times \phi \mathcal{B}^\perp)$ so that $c_{G,c_\phi}$ is quasi equivalent as a dg bimodule to $c_{G,F}$. We will now prove the following result.

**Theorem 5.6.** Let $F : \mathcal{A} \to \mathcal{C}$ and assume that $H^0 \mathcal{A}$ admits a semiorthogonal decomposition $\mathcal{A}_1, \ldots, \mathcal{A}_k$. Assume that if $a_i \in \mathcal{A}_i, a_j \in \mathcal{A}_j$ and $i < j$ then the map

$$\text{hom}_\mathcal{A}(a_i, a_j) \to \text{hom}_\mathcal{C}(F(a_i), F(a_j))$$

is a quasi isomorphism. Then the category $\mathcal{B}$ is quasi isomorphic to $\mathcal{B}(\mathcal{A}_i, F_i)$ (see Definition [3.4]). Furthermore, the natural functor $c_{F_1, \ldots, F_k} : \mathcal{B}(\mathcal{A}_i, F_i) \to \mathcal{C}$ is quasi isomorphic to $F$.

**Proof.** Let us proceed by induction. We have that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ forms a semiorthogonal decomposition for $\mathcal{A}$ and $\alpha_j : \mathcal{A}_j \to \mathcal{A}$ are the corresponding embeddings and $F : \mathcal{A} \to \mathcal{C}$ is the spherical functor to $\mathcal{C}$. We also have that the subcategories $\mathcal{A}_{1,j}$ of $\mathcal{A}$ which have semiorthogonal decompositions $\mathcal{A}_1, \ldots, \mathcal{A}_j$ are admissible, hence the composition of the embedding functor and the functor $F$ give a functor $F_{1,j} : \mathcal{A}_{1,j} \to \mathcal{C}$ which is spherical with cotwist $S_{\mathcal{A}_{1,j}}[\ell]$. It follows from Corollary [5.5] that $\mathcal{A}_{1,j+1}$ is quasiisomorphic to $\mathcal{A}_{1,j} \times F_{1,j}, F_{j+1}, \mathcal{A}_{j+1}$. Furthermore, the functor $c_\infty$ coincides with $F_{1,j+1}$ under this quasiisomorphism. The Theorem follows then by recursively applying this observation. \qed

As a corollary of Theorem [5.6] with Corollary [2.23] we have the following result.

**Corollary 5.7.** Let $\mathcal{A}$ be a pretriangulated category which admits a semiorthogonal decomposition $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and a spherical functor $F : \mathcal{A} \to \mathcal{C}$ whose cotwist induces the Serre functor on $H^0 \mathcal{A}$ up to shift. The perverse schober $S_{K_k}(\mathcal{C}, \mathcal{A}_i, F_i)$ has category of global sections quasi equivalent to $\mathcal{A}$.

The following example will be very important in Section 8.

**Example 5.8.** Let $G_4$ be the toric boundary divisor in $Q$ given by $x_0x_1y_0y_1 = 0$. This divisor is normal crossings and anticanonical in $Q$, therefore, if $i$ is the embedding of $G_4$ into $Q$ then the functor $Li^* : D^b_{\text{dg}}(Q) \to \text{Perf}(G_4)$ is a spherical functor with cotwist equal to the Serre functor on $D^b_{\text{dg}}(Q)$ up to a shift. The objects $\mathcal{O}_Q(0, 0), \mathcal{O}_Q(1, 0), \mathcal{O}_Q(0, 1)$ and $\mathcal{O}_Q(1, 1)$ form a strong exceptional collection on $Q$ and

(12) \[ Li^* \mathcal{O}_Q(0, 0) = \mathcal{O}_{G_4}, \quad Li^* \mathcal{O}_Q(1, 0) = \mathcal{O}_{G_4}(p_1 + p_3), \]

(13) \[ Li^* \mathcal{O}_Q(0, 1) = \mathcal{O}_{G_4}(p_2 + p_4), \quad Li^* \mathcal{O}_Q(1, 1) = \mathcal{O}_{G_4}(p_1 + p_2 + p_3 + p_4). \]

where $p_i$ denotes a smooth point on the $i$th copy of $\mathbb{P}^1$ in $G_4$, ordered cyclically. We may apply Corollary [5.7] and Proposition [2.22] to see that if $U$ is the set of objects in Equation [12] then the category $\text{Tw Perf}(G_4)_U^\to$ is quasi equivalent to $D^b_{\text{dg}}(Q)$. 

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According to [81], the category Perf$(G_4)$ can be constructed as a homotopy limit of four copies of $D^b_{dg}(\mathbb{P}^1)$.

5.2. Mutations and $K$-coordinatization. Kapranov and Schectman [31, Section 2.A] define an action of $\text{Br}_{|\Sigma}$ on the set of all $K$-coordinatized perverse schobers on $(\mathbb{C}, \Sigma)$. This acts on skeleta $K$, since $\text{Br}_{|\Sigma} = \pi_0\text{Diff}^+(\mathbb{C}, \Sigma, s)$ (see [31, 2.B] for details or Figure 5 for an illustration).

This group is generated by elements $\sigma_i$. We define an action of $\text{Br}_{|\Sigma}$ on the set of all perverse schobers by saying that $\sigma_i$ takes a schober $S_K(\mathbb{C}, A_j, F_j)$ and produces $S_{\sigma_i}(\mathbb{C}, A'_j, F'_j)$ so that $F'_j = F_j$ and $A'_j = A_j$ if $j \neq i, i + 1$, $A'_{i+1} = A_{i+1}$, $A'_i = A_i$, and that $F'_i = F_{i+1}, F'_{i+1} = T'_{i+1}F_i$ where $T_{i+1}$ is the spherical twist associated to $F_i$. The action of $\sigma_i$ on $K$ is depicted in Figure 5.

Assume we have a (dg enhanced) triangulated category $T$ is equipped with a semiorthogonal decomposition $\langle A_1, \ldots, A_n \rangle$ and a spherical functor $F : \mathcal{T} \to \mathcal{C}$, and $F_i : A_i \to \mathcal{C}$. Then we may mutate this semiorthogonal decomposition so as to change the order of $A_i$ and $A_{i+1}$. This then produces new spherical functors $F'_i : A_i \to \mathcal{C}$ and a new $K$-coordinatized perverse schober by Corollary 5.7. We will prove that the relation between $F_i$ and $F'_i$ is precisely as described by Kapranov and Schectman. For the rest of this section, we will operate exclusively with triangulated categories.

These mutations are defined as follows. Assume that we have a semiorthogonal decompositions $\mathcal{T} = \langle A, A^\perp \rangle$ and $\langle A^\perp, A \rangle$. Then there is an embedding functor $\alpha : A \to \mathcal{T}$ with right and left adjoints $\alpha^!$ and $\alpha^*$. The right and left mutation functors of $\mathcal{T}$ are given by

$$\alpha \alpha^!(a) \to a \to L_A(a), \quad R_A(a) \to a \to \alpha \alpha^*(a).$$

The functors $L_A$ and $R_A$ send $\mathcal{T}$ to $\mathcal{T}$, and specifically map onto the subcategories $A^\perp$ and $A^\perp$ respectively. Furthermore, they give mutually inverse equivalences between $A^\perp$ and $A^\perp$. If we have a semiorthogonal decomposition $\mathcal{T} = \langle A_1, \ldots, A_n \rangle$ we may perform mutations at a specific component of this semiorthogonal decomposition, by noting that we also have semiorthogonal decompositions

$$\langle A_1, \ldots, (A_i, A_{i+1}), \ldots, A_n \rangle$$

Therefore, we can define

$$L_{A_i} \langle A_1, \ldots, A_n \rangle$$

to be the semiorthogonal decomposition obtained by replacing the semiorthogonal summand $(A_i, A_{i+1})$ with $L_{A_i}A_{i+1}, A_i)$, and similarly, we define

$$R_{A_{i+1}} \langle A_1, \ldots, A_n \rangle$$
to be the semiorthogonal decomposition obtained by replacing the semiorthogonal summand $\langle A_i, A_{i+1} \rangle$ with the new semiorthogonal summand $\langle A_{i+1}, R_{A_{i+1}}(A_i) \rangle$.

**Proposition 5.9.** Assume that we have a semiorthogonal decomposition $\mathcal{F} = \langle A, B \rangle$ and a spherical functor $F : \mathcal{F} \to \mathcal{C}$ so that $C_F = S_{\mathcal{F}}[k]$ for some integer $k$. Let $\alpha : A \to \mathcal{F}$ and $\beta : B \to \mathcal{F}$ be the natural embeddings and let $F_\alpha : A \to \mathcal{C}$ and $F_\beta : B \to \mathcal{C}$ be the spherical functors from $A$ and $B$ obtained from Proposition 5.4. Define $T_\alpha, T_\beta, C_\alpha$ and $C_\beta$ as the corresponding twist and cotwist functors. Then

$$FLA_\beta \cong T_\alpha F_\beta, \quad FR_\beta \alpha \cong T_\beta F_\alpha.$$  

**Proof.** It is known [42, Remark 2.11] that there are isomorphisms of functors

$$L_\alpha \beta \cong S_{\mathcal{F}} \beta S_B^{-1}, \quad R_\beta \alpha \cong S_{\mathcal{F}}^{-1} \alpha S_A.$$

The rest of the proposition is provided by compatibility conditions between spherical functors, Serre functors and twist and cotwist functors. By construction, we have that $C_F \cong S_{\mathcal{F}}$. The fact that $F$ is spherical implies that $F C_F \cong T_F F[-2]$ (see [2, Section 1.3]). Therefore, we have that $F S_{\mathcal{F}} \cong F C_F[-k] \cong T_F F[k - 2]$. Therefore,

$$FL_\alpha \beta \cong T_F F \beta S_B^{-1}[k - 2] \cong T_F F_\beta S_B^{-1}[k - 2]$$

Since $F_\beta$ is also spherical with cotwist $S_B[k]$ by Proposition 5.11 it follows that $S_B^{-1}[k - 2] \cong C_\beta'[2]$. We also have that $F_\beta C_\beta' \cong T_\beta F_\beta[2]$ (see [2, Section 1.3]). Therefore,

$$F_\beta S_B^{-1}[k - 2] \cong F_\beta C_\beta'[-2] = T_\beta F_\beta.$$

Therefore, we see that $FL_\alpha \beta \cong T_F T_\beta F_\beta$. According to [3, Theorem 11], we have $T_F \cong T_\alpha T_\beta$. Hence $FL_\alpha \beta \cong T_\alpha F_\beta$.

Therefore, the second statement in the proposition is proved similarly. \[\Box\]

Therefore, it follows that

**Corollary 5.10.** If $S_K(\mathcal{C}, A_i, F_i)$ is the perverse schober obtained from the construction in Corollary 5.7, then $\sigma_j S_K(\mathcal{C}, A_i, F_i)$ and $S_K(\mathcal{C}, A_i, F_i)$ have quasi equivalent categories of global sections.

**Remark 5.11.** If $F_i : A_i \to \mathcal{C}$ is an ordered collection of spherical functors with a weak right relative Calabi-Yau structure then the category $\mathcal{C}$ is Calabi-Yau and the cotwist of $F_i$ is the Serre functor on $A_i$. Work in progress of Katzarkov, Pandit and Spaide shows that if $F_i$ have weak right Calabi-Yau structures, then the category of global sections $A = \Gamma S_K(\mathcal{C}, A_i, F_i)$ of the corresponding perverse schober also admits a functor $F : A \to \mathcal{C}$ which has a weak right Calabi-Yau structure. Therefore, we may apply Proposition 5.9 to see that Kapranov and Schectman’s action on a perverse schober constructed from functors with weak right relative Calabi-Yau structures leaves the category of global sections invariant. \[\Box\]

### 6. Type II degenerations of Calabi-Yau varieties

In this section, we draw a connection between certain perverse sheaves of categories on the punctured disc and categories of perfect complexes on type II degenerations of Calabi-Yau varieties. This incorporates information from Section 4 and 5 and is connected with our proof of homological mirror symmetry in Section 8.
6.1. Sheaves of categories and blowing up. In the next section, we will let $V$ be a smooth projective variety, $S$ a smooth anticanonical divisor in $V$ and $Z$ a smooth divisor in $S$. We let $\tilde{V}$ be the blow up of $V$ in $Z$. We would like to prove that there are perverse sheaves of categories which encode the data of $\tilde{V}$. Since $Z$ is contained in a smooth anticanonical divisor $S$, the proper transform of $S$ is isomorphic to $S$ and is still anticanonical in $\tilde{V}$. Let us take $j : Z \hookrightarrow S$ and let $k : S \hookrightarrow V$. We also let $\tilde{k}$ be the embedding $\tilde{S} \hookrightarrow \tilde{V}$, so if $\pi : \tilde{V} \to V$ is the blow up map, then $\pi \cdot \tilde{k} = k$.

**Proposition 6.1.** There is a quasi equivalence of dg categories between $D^b_{\text{dg}}(\tilde{V})$ and

$$D^b_{\text{dg}}(V) \times_{\mathbb{L}k^*, \mathbb{R}j_*} D^b_{\text{dg}}(Z).$$

so that the quasifunctor $c_{Lk^*, \mathbb{R}j_*}$ to $D^b_{\text{dg}}(S)$ is quasi isomorphic as a bimodule to $\mathbb{L}k^*$.

**Proof.** From [38], we have a semiorthogonal decomposition,

$$D^b(\tilde{V}) = (L\pi^* D^b(V), Rq_* Lp^* D^b(Z)).$$

Therefore, by [31] there is a quasi equivalence between $D^b_{\text{dg}}(\tilde{V})$ and $D^b_{\text{dg}}(V) \times_{\mathbb{L}p^* \mathbb{R}p^* \mathbb{L}q_*} D^b_{\text{dg}}(Z)$. We can use Corollary [39] to see that this category is quasi equivalent to

$$D^b(V) \times_{\mathbb{L}k^* \mathbb{L}p^* \mathbb{L}k^* \mathbb{L}Rq_* \mathbb{L}p^*} D^b_{\text{dg}}(Z),$$

since $S$ is anticanonical in $\tilde{V}$. Then we have that $\pi \cdot \tilde{k} = k$, so $\mathbb{L}k^* \mathbb{L}p^* = \mathbb{L}k^*$. Furthermore, we have a cartesian diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\sigma} & \mathbb{P}_Z(N_{Z/V}) \\
\downarrow & & \downarrow q \\
S & \xrightarrow{\tilde{k}} & \tilde{V}
\end{array}$$

In [39] Corollary 2.26 there are elements $K_{\tilde{k}, q}$ and $K^{j, \sigma}$ of $D^b(\mathbb{P}_Z(N_{Z/V}) \times S)$ constructed whose corresponding Fourier-Mukai transforms from $D^b(\mathbb{P}_Z(N_{Z/V})) \to D^b(Z)$ represent $\mathbb{L}k^* \mathbb{R}q_*$ and $\mathbb{R}j_* \mathbb{L}\sigma^*$ respectively. In loc. cit., Kuznetsov shows that there is an isomorphism $K_{\tilde{k}, q} \to K^{j, \sigma}$. This implies that there is a quasi isomorphism between the dg functors $\mathbb{R}j_* \mathbb{L}\sigma^*$ and $\mathbb{L}k^* \mathbb{R}q_*$ as dg bimodules. Therefore, since $p \cdot \sigma = \text{id}_X$, we have that $D^b_{\text{dg}}(\tilde{V})$ is quasi equivalent to

$$D^b_{\text{dg}}(V) \times_{\mathbb{L}k^*, \mathbb{R}j_*} D^b_{\text{dg}}(Z)$$

as expected. By construction and Propositions [21] and [22], the induced quasifunctor to $D^b_{\text{dg}}(S)$ is quasi isomorphic to $\mathbb{L}k^*$ as a bimodule. □

This means that the process of blowing up along a smooth anticanonical divisor not only can be recast as categorical gluing, but it can be recast as taking the category of global sections of a perverse schober, since the functors $\mathbb{L}k^*$ and $\mathbb{R}j_*$ are spherical.

**Remark 6.2.** We may also apply this proposition recursively to see that if $Z_1, \ldots, Z_n$ are smooth codimension 1 subvarieties in $S$, and if $\text{Bl}_{Z_1, \ldots, Z_n}(V)$ denotes the blow up of $V$ in $Z_1$, followed by the blow up in $Z_2$ etc., then there is a perverse schober on $K_{n+1}$ associated to the spherical functors $\mathbb{L}k^* : D^b_{\text{dg}}(V) \to D^b_{\text{dg}}(S)$ and $\mathbb{R}j^*_s : D^b_{\text{dg}}(Z) \to D^b_{\text{dg}}(S)$ whose category of global sections is quasi equivalent to $D^b_{\text{dg}}(\text{Bl}_{Z_1, \ldots, Z_n}(V))$.

□
Now we note that this construction can be simplified further in the case where $S$ decomposes into a pair of smooth connected components one of which contains $Z$.

**Proposition 6.3.** Let $S_1 \coprod S_2$ be a pair of smooth disjoint divisors in a variety $V$ so that $S_1 + S_2$ is a section of $-K_V$, and let $k_{\ell} : S_\ell \hookrightarrow V$ be the embeddings. Let $i : Z \hookrightarrow S_1$ be a smooth divisor, and let $\tilde{V} = \text{Bl}_Z(V)$. Then there is a quasi equivalence between $D^b_{\text{dg}}(\tilde{V})$ and $D^b_{\text{dg}}(V) \times_S D^b_{\text{dg}}(Z)$ where

$$S(b,a) = \text{hom}_{\text{mod}_{D^b_{\text{dg}}(S_1)}}(\mathbb{L}k_1^*b, R_i^*a).$$

Furthermore, the functor $D^b_{\text{dg}}(V) \times_S D^b_{\text{dg}}(Z) \to \text{Tw mod}_{D^b_{\text{dg}}(S_1)}$ sending

$$(a,b,\mu) \mapsto \text{Cone}(\mu)$$

is quasi isomorphic to $\mathbb{L}k_1^*$ and $(a,b,\mu) \mapsto \mathbb{L}k_2^*b$ is equivalent to $\mathbb{L}k_2^*$, where $k_{\ell} : S_\ell \hookrightarrow \tilde{V}$ are the obvious embeddings.

**Proof.** The first part follows from Proposition 6.1; we have that $D^b_{\text{dg}}(\tilde{V})$ is quasi equivalent to $D^b_{\text{dg}}(V) \times_S D^b_{\text{dg}}(Z)$ where

$$S'(a,b) = \text{hom}_{D^b_{\text{dg}}(S_1) \times D^b_{\text{dg}}(S_2)}(\mathbb{L}k_1^*(b) \times \mathbb{L}k_2^*(b), R_i^*(a) \times 0),$$

which is naturally isomorphic to $S$. Since $Z \subseteq S_1$, the proper transforms of $S_1$ and $S_2$, which are isomorphic to $S_1$ and $S_2$ are anticanonical in $V$. According to Corollary 5.5 the functor

$$\mathbb{L}k_1^* \times \mathbb{L}k_2^* : D^b_{\text{dg}}(\tilde{V}) \to D^b_{\text{dg}}(S_1) \times D^b_{\text{dg}}(S_2)$$

sends $(a,b,\mu')$ for $\mu' \in S'(b,a)$ to Cone($\mu'$). Since $\mu'$ can be written as $\mu \times 0$ for $\mu \in S(b,a)$, it follows then from Propositions 2.11 and 2.12 that $\mathbb{L}k_1^*$ is quasi isomorphic to the functor sending $(a,b,\mu')$ to Cone($\mu$) and $\mathbb{L}k_2^*$ is quasi isomorphic to the functor sending $(a,b,\mu')$ to $\mathbb{L}k_2^*a$. □

We can represent this situation as the diagram of categories which commutes up to homotopy:

$$
\begin{array}{ccc}
D^b_{\text{dg}}(\tilde{V}) & \xrightarrow{Q} & D^b_{\text{dg}}(V) \\
\downarrow & & \downarrow \\
D^b_{\text{dg}}(V) \times \mathbb{L}k_1^* \times \mathbb{L}k_2^* & \xrightarrow{f_1 \times f_2} & D^b_{\text{dg}}(S_1) \times D^b_{\text{dg}}(S_1) \\
\downarrow & & \downarrow \\
A_2(D^b_{\text{dg}}(S_1)) & \xrightarrow{f_3} & D^b_{\text{dg}}(S_1) \\
\downarrow & & \downarrow \\
D^b_{\text{dg}}(S_1) & \xrightarrow{f_4} & D^b_{\text{dg}}(S_2) \\
\end{array}
$$

Where the square is homotopy cartesian, and arrows should be interpreted as quasifunctors and $Q$ is a quasi functor which induces an equivalence in the homotopy category.

6.2. **Sections of $\mathbb{P}^1$ bundles.** Recall in Section 4.2, when we constructed $\Gamma S_\phi(C, \Phi)$, we use $f_1$ and $f_2$ to glue $A_2(C)$ to $A_2(\mathbb{C})$. Thus we are left with a pair of functors $g_1$ and $g_2$ inherited from
Proposition 6.4. Assume that $S$ is compact, smooth and Calabi-Yau. The functors $L\sigma_S^+$ and $g_1$ are quasi isomorphic as bimodules, as are $L\sigma_L^+$ and $g_2$.

**Proof.** First, we remark that because of the conditions that we have placed on $S$, $S_0 \cup S_\mathcal{L}$ is a smooth anticanonical divisor on $\mathbb{P}_S(\mathcal{E})$. Let us denote by $p : \mathbb{P}_S(\mathcal{E}) \to X$ the natural projection map. We have a semi orthogonal decomposition,

$$D^b(\mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{L})) = \langle Lp^* D^b(S), Lp^* D^b(S) \otimes \mathcal{L} \mathcal{O}_S(1) \rangle.$$ 

Therefore, by the discussion preceding Corollary 5.5 we get that $D^b_{dg}(\mathbb{P}_S(\mathcal{O} \oplus \mathcal{L}))$ is quasi equivalent to

$$D^b_{dg}(S_\mathcal{L}) \times_{S_\mathcal{O}} D^b_{dg}(S_0).$$

The complex $\hom_{\Gamma_S(\mathcal{E},\Phi)}(A, A')$ is the diagonal subcomplex of $\hom_{\mathcal{E}}(a, a')^2$. One can check directly that $g_1$ and $g_2$ act on this subcomplex as the identity, sending $f$ to $f$. A similar statement is true for $C_2$.

Now we know that if $A$ is as above and $B \in C_2$ is written as $((0, b, 0), (0, b, 0), 0 \times \mathbb{id}_b)$, then by the proof of Proposition 4.5 $\hom_{\Gamma_S(\mathcal{E},\Phi)}(A, B)$ is equal to

$$\hom_{\mathcal{E}}(a, b \oplus \Phi(b)) = \hom_{\mathcal{E}}(a, b) \oplus \hom_{\mathcal{E}}(a, \Phi(b)).$$

The functors $g_1$ induces projection onto the first component, and $g_2$ induces projection onto the second component. Therefore, $f_3 \cdot m \cdot Q$ and $f_3 \cdot n \cdot Q$ are the functors sending $(a, b, \mu \oplus \xi)$ to

$$\text{Cone}(\mu : a \to b), \quad \text{Cone}(\xi : a \to \Phi(b)).$$

In the case considered in Theorem 4.6 we can ask whether $g_1$ and $g_2$ have any geometric significance. There are two filtrations of $\mathcal{E} = \mathcal{O}_S \oplus \mathcal{L}$, by sub-bundles coming from the obvious embedding of $\mathcal{O}_S$ and $\mathcal{L}$ into $\mathcal{E}$, thus there are two sections of $\mathbb{P}_S(\mathcal{E})$ which we call $S_0$ and $S_\mathcal{L}$, and we let $\sigma_S$ and $\sigma_\mathcal{L}$ be their embeddings into $\mathbb{P}_S(\mathcal{E})$.

**Proposition 6.4.** Assume that $S$ is compact, smooth and Calabi-Yau. The functors $L\sigma_S^+$ and $g_1$ are quasi isomorphic as bimodules, as are $L\sigma_L^+$ and $g_2$. 

**Proof.** First, we remark that because of the conditions that we have placed on $S$, $S_0 \cup S_\mathcal{L}$ is a smooth anticanonical divisor on $\mathbb{P}_S(\mathcal{E})$. Let us denote by $p : \mathbb{P}_S(\mathcal{E}) \to X$ the natural projection map. We have a semi orthogonal decomposition,

$$D^b(\mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{L})) = \langle Lp^* D^b(S), Lp^* D^b(S) \otimes \mathcal{L} \mathcal{O}_S(1) \rangle.$$ 

Therefore, by the discussion preceding Corollary 5.5 we get that $D^b_{dg}(\mathbb{P}_S(\mathcal{O} \oplus \mathcal{L}))$ is quasi equivalent to

$$D^b_{dg}(S_\mathcal{L}) \times_{S_\mathcal{O}} D^b_{dg}(S_0).$$
where
\[ S_p(b, a) = \text{hom}_{D^b_{\text{alg}}(\mathcal{P}_S(\xi))}(\mathbb{L}p^*a, \mathbb{L}p^*b \otimes^L \mathcal{O}_p(1)) \].

According to Corollary 6.5, this bimodule is isomorphic to
\[ S_{S_0 \cup S_L}(a, b) = \text{hom}_{D^b_{\text{alg}}(S_0)}(\mathbb{L}\sigma_0^* \mathbb{L}p^*(b), \mathbb{L}\sigma_0^*(\mathbb{L}p^*(a) \otimes^L \mathcal{O}_p(1))) \]
\[ \oplus \text{hom}_{D^b_{\text{alg}}(S_L)}(\mathbb{L}\sigma_L^* \mathbb{L}p^*(b), \mathbb{L}\sigma_L^*(\mathbb{L}p^*(a) \otimes^L \mathcal{O}_p(1))) \].

Since \( p \cdot \sigma_0 = p \cdot \sigma_L = \text{id}_X \) and \( \mathbb{L}\sigma_0^* \mathcal{O}_p(1) \cong \mathcal{O}_X \) and \( \mathbb{L}\sigma_0^* \mathcal{O}_p(1) \cong \mathcal{L} \) by [30], V, Proposition 2.6, it follows that this bimodule is isomorphic to
\[ S_{S_0 \cup S_L}(a, b) = \text{hom}_{D^b_{\text{alg}}(S)}(b, a) \oplus \text{hom}_{D^b_{\text{alg}}(S)}(b, a \otimes^L \mathcal{L}) \].

Using Corollary 6.5, the functor \( \mathbb{L}\sigma_0^* \times \mathcal{L} \) is quasi isomorphic as a bimodule to the functor sending \((a, b, \mu) \times \text{Cone(\xi)}\) to \( \text{Cone(\mu)} \times \text{Cone(\xi)} \). This proves the result.

6.3. Gluing components together. There are certain geometric situations in which one obtains normal crossings varieties whose components are either varieties with smooth anticanonical divisors or which are \( \mathbb{P}^1 \) bundles over a Calabi-Yau variety. Notably, these appear as particularly nice degenerations of Calabi-Yau varieties. For instance,

**Definition 6.5 (A. Tyurin [34], N.-H. Lee [44]).** A Tyurin degeneration of Calabi-Yau varieties is a smooth projective family \( g : X \rightarrow \Delta \) over an analytic disc \( \Delta \) centered at 0 with variable \( t \) which satisfies the following criteria.

1. The fiber over \( t \neq 0 \in \Delta \) is a smooth projective Calabi-Yau manifold with \( h^{i,0}(g^{-1}(t)) = 0 \) unless \( i = 0, \dim X - 1 \).
2. The fiber over 0 is normal crossings and consists of a pair of divisors \( Y_0 \) and \( Y_1 \) so that \( Y_0 \cap Y_1 = Z \) is an anticanonical subvariety in \( Y_0 \) and \( Y_1 \).

It goes back to work of Friedman [24] that this definition implies that \( N_{Z/Y_0} \otimes N_{Z/Y_1} \cong \mathcal{O}_Z \). If we perform base change along the map \( g : \Delta \rightarrow \Delta, g(s) = t^s \), then \( g^*X \) is no longer smooth but has a \( cA_n \) singularity along the preimage \( Z \) of \( Y_1 \cap Y_2 \). We may resolve \( g^*X \) by blowing up \((n - 1)\) times at \( Z \). The resulting variety has \( n + 1 \) components, \( Y'_0, \ldots, Y'_n \) so that \( Y'_0 = Y_0 \) and \( Y'_n = Y_1 \), and \( Y'_1, \ldots, Y'_{n-1} \cong \mathbb{P}(\mathcal{O}_Z \oplus N_{Z/Y_0}) \). If we let \( Z_i(\cong Z) = Y'_i \cap Y'_{i+1} \) then the fact that this is the fiber over 0 of a smooth family of varieties means that \( N_{Z_i/Y'_i} \otimes N_{Z_i/Y'_{i+1}} \cong \mathcal{O}_{Z_i} \). We generalized this in the following definition.

**Definition 6.6.** Let \( Y = Y_0 \cup \cdots \cup Y_n \) be a normal crossings union of smooth varieties so that

1. \( Y_i \cap Y_j = 0 \) if \( |i - j| \neq 0 \) and \( (Y_{i-1} \cap Y_i) \cup (Y_i \cap Y_{i+1}) \) are smooth and anticanonical in \( Y_i \) if \( i \neq 0, n \) and \( Y_0 \cap Y_1 \) and \( Y_{n-1} \cap Y_n \) are smooth anticanonical in \( Y_0 \) and \( Y_n \) respectively.
2. \( Y_i = \cdots = Y_{i-1} = \mathbb{P}(\mathcal{O} \oplus N_{Z_i/Y'_i}) \).
3. \( Y \) is \( d \)-semistable, in other words,

\[ N_{Y_i \cap Y_{i+1}/Y'_i} \otimes N_{Y_i \cap Y_{i+1}/Y'_{i+1}} \cong \mathcal{O}_{Y_i \cap Y_{i+1}}. \]

We call such a variety a simplified type II degeneration of Calabi-Yau varieties.

Calling this a degeneration of Calabi-Yau varieties is justified by a result of Kawamata and Namikawa [34] which says that such varieties can be smoothed to Calabi-Yau varieties. Such degenerations do not necessarily arise from base change of a Tyurin degeneration. According to
Remark 2.26 (which comes from [31] Proposition 4.7), there is a diagram of categories whose homotopy limit is quasi equivalent to \( \text{Perf}(Y) \)

\[
\begin{align*}
D^b_{dg}(Y_0) & \quad D^b_{dg}(Y_1) & \quad \cdots & \quad D^b_{dg}(Y_{n-1}) & \quad D^b_{dg}(Y_n) \\
D^b_{dg}(Z_0) & \quad D^b_{dg}(Z_1) & \quad \cdots & \quad D^b_{dg}(Z_{n-2}) & \quad D^b_{dg}(Z_{n-1})
\end{align*}
\]

The arrows in this diagram denote the pullbacks to \( Z_{i-1} \) or \( Z_i \) in \( Y_i \). Note that all of the varieties \( Z_i \) are equal. We will call such a variety \( Z \). We may use Theorem 4.6 and Proposition 6.4 to replace \( D^b_{dg}(Y_i) \) for \( i \neq 0, n \) by \( \Gamma S_{K_i}(D^b_{dg}(Z_i), (-) \otimes L N_{Z_i/Y_i}) \), and the restriction functors by \( g_1 \) and \( g_2 \) respectively. Denote this category \( \Gamma S_{K_\phi} \). Using Corollary 5.7 if \( D^b_{dg}(Y_0) \) and \( D^b_{dg}(Y_n) \) have semiorthogonal decompositions \( A_1, \ldots , A_{k_1} \) and \( B_1, \ldots , B_{k_2} \) and \( F_i : A_i \to D^b_{dg}(Z_0) \) and \( G_j : B_j \to D^b_{dg}(Z_{n-1}) \) are the spherical functors obtained by composing the embeddings of \( A_i \) into \( D^b_{dg}(Y_0) \) or of \( B_j \) into \( D^b_{dg}(Y_n) \) with the pullback map to \( Z \), then we may replace \( D^b_{dg}(Y_i) \) with \( \Gamma S_{K_{k_1}}(A_i, F_i) \) and the pullback to \( D^b_{dg}(Z_0) \) by \( s_\infty \), and similarly we can replace \( D^b_{dg}(Y_n) \) with \( \Gamma S_{K_{k_2}}(G_i, B_i) \) and the pullback to \( D^b_{dg}(Z_{n-1}) \) by \( s_\infty \) (see Section 3.3 for notation). We denote these categories \( \Gamma S_{K_{k_1}} \) and \( \Gamma S_{K_{k_2}} \) respectively. Therefore, we have the following diagram of categories whose homotopy limit is quasi equivalent to \( \text{Perf}(Y) \).

\[
\begin{array}{c}
\Gamma S_{k_1} \\
\bigg|_{s_\infty} \\
D^b_{dg}(Z)
\end{array}
\quad \begin{array}{c}
\Gamma S_{K_i} \\
\downarrow g_1 \\
D^b_{dg}(Z)
\end{array}
\quad \begin{array}{c}
\Gamma S_{K_{\phi}} \\
\downarrow g_2 \\
D^b_{dg}(Z)
\end{array}
\quad \begin{array}{c}
\Gamma S_{k_2} \\
\bigg|_{s_\infty} \\
D^b_{dg}(Z)
\end{array}
\]

We may then reinterpret this in the following way.

**Theorem 6.7.** To a simplified type II degeneration of Calabi-Yau varieties \( Y = Y_0 \cup \cdots \cup Y_n \) and a choice of semiorthogonal decompositions of \( Y_0 \) and \( Y_n \) there is a perverse sheaf of categories on \( S^2 \) with \( n \) boundary components whose category of global sections is \( \text{Perf}(Y) \).

This sheaf of categories is constructed by first gluing \( (n-1) \) copies of \( K_\phi \) together along their exterior edges to get a graph \( K_{(n-1)\phi} \). This graph is naturally a spanning graph on \( S^2 \) with \( n \) boundary components and we may equip it with a \( K_{(n-1)\phi} \)-coordinatized perverse sheaf of categories using precisely the same data as was used to describe \( S_{K_\phi}(D^b_{dg}(Z), (-) \otimes L N_{Z/Y_i}) \). This is given as follows. \( K_{(n-1)\phi} \) has \( 2(n-1) \) vertices called \( v_{1,j}, v_{2,j} \) for \( j = 1, \ldots , (n-1) \) and \( 3(n-1)+1 \) edges which we denote \( g_1, \ldots , g_n \) and \( f_{1,j}, f_{2,j} \) for \( j = 1, \ldots , (n-1) \). For \( i = 2, \ldots , n-1 \) let \( g_i \) be adjacent to both \( v_{2,i} \) and \( v_{1,i+1} \), and let \( g_1 \) be adjacent to \( v_{1,1} \) with its other end in a boundary component and \( g_n \) adjacent to \( v_{2,n} \) with its other end in another boundary component. Then \( f_{1,i} \) and \( f_{2,i} \) are adjacent to both \( v_{1,i} \) and \( v_{2,i} \). All of the vertices \( v_{j,\ell} \) are trivalent, so we assign to them the category \( A_2(D^b_{dg}(Z)) \). We have functors;

\[
F_{v_{1,1},g_1} = F_{v_{2,2},g_0} = F_{v_{2,2},g_1} = F_{v_{1,1},g_1} = f_3, \quad F_{v_{1,1},f_{1,1}} = N_{Z/Y_1} \otimes L f_2,
\]

\[
F_{v_{2,2},f_{1,1}} = f_1, \quad F_{v_{2,2},f_{1,1}} = f_2.
\]

The category of global sections of \( S_{K_{(n-1)\phi}} \) is equivalent to \( \text{Perf}(Y_1 \cup \cdots \cup Y_{n-1}) \). To \( S_{K_{k_1}}(A_i; F_i) \) and \( S_{K_{k_2}}(B_i; F_i) \), there are graphs with vertices \( c, u_1, \ldots , u_{k_1} \) and \( d, v_1, \ldots , v_{k_2} \) where \( c \) is \((k_1+1)\)-valent, \( d \) is \((k_2 + 1)\)-valent and \( u_i, w_i \) are univalent. There are edges \( r_1, \ldots , r_{k_1}, r_\infty \) and \( q_1, \ldots , q_{k_2}, q_\infty \) so that \( r_i \) connects \( c \) to \( u_i \) if \( i \neq \infty \) and \( q_i \) connects \( d \) to \( w_i \) if \( i \neq \infty \). We have that \( A_c = A_{k_1}(D^b_{dg}(Z)) \).
and $A_d = A_{k_2}(D^b_{iq}(Z))$, $A_{u_i} = A_i$ and $A_{w_i} = B_i$ and $F_{u_i,r_i} = F_i$ and $F_{w_i,q_i} = G_i$. Finally, $F_{c,r_i} = f_i$ and $F_{d,q_i} = f_i$ if $i \neq \infty$, and $F_{c,r_\infty} = f_{k_1+1}$ and $F_{d,q_\infty} = f_{k_2+1}$.

Attach $r_\infty$ to $g_1$ and $q_\infty$ to $g_n$ to obtain the graph $K_Y$. This comes equipped with a $K_Y$-coordinatized perverse sheaf of categories whose category of global sections is quasi equivalent to $\text{Perf}(Y)$ by the argument given above.

We may note that, until now, the $d$-semistability condition, that $N_{Z_i/Y_i} \otimes N_{Z_i/Y_{i+1}} = \mathcal{O}_{Z_i}$ played no role; indeed Theorem 6.7 carries through for any chain of varieties satisfying (1) and (2) of Definition 6.6. The $d$-semistability condition translates in terms of perverse sheaves of categories the the following statement.

To each boundary component of $S$, we have a small loop going around it. Therefore, for each boundary component, there is a cycle in $K^\circ$ (see Section 3.4 for definition) which descends to that loop in $\pi_1(S^\circ,v)$ for some vertex of $K^\circ$.

**Theorem 6.8.** If $Y$ is a normal crossings variety satisfying (1) and (2) of Definition 6.6, then the perverse sheaf of categories constructed in Theorem 6.7 has trivial monodromy around each boundary component of $S$ if and only if $Y$ satisfies (3), which is to say that it can be smoothed.

We first record a standard result. A section of a projective bundle $\mathbb{P}_X(E)$ is determined by a surjective map of vector bundles

$$E \rightarrow \mathcal{L}$$

where $\mathcal{L}$ is a line bundle.
Proposition 6.9. If $\mathcal{E}$ is a rank 2 vector bundle on a smooth projective variety $X$, and if $D$ is a section of $\mathbb{P}(\mathcal{E})$ determined by a short exact sequence of vector bundles
\[ 0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0. \]
then $N_{D/\mathbb{P}(\mathcal{E})} \cong \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$.

Therefore, if $D_1$ and $D_2$ are two sections of $P = \mathbb{P}(\mathcal{O} \oplus \mathcal{L})$ then $N_{D_1/P} = N_{D_2/P}^{-1}$. Now we proceed with the proof of Theorem 6.8.

Proof. There are three situations that we must consider. First, we look at the easiest case which is that of the interior punctures, that is, punctures 1, ..., $n-2$ in Figure 7. In this case, we have that our loop is made up of four vertices and five edges. We exhibit this as
\[ v_{1, j} \xrightarrow{f_{1, j}} v_{2, j} \xrightarrow{f_{2, j}} v_{3, j} \xrightarrow{g_j} v_{4, j+1} \xrightarrow{f_{2, j+1}} v_{3, j+1} \xrightarrow{f_{1, j+1}} v_{2, j+1} \xrightarrow{g_j} v_{1, j}. \]
Since the functor $\phi(v_{1, j}, f_{1, j}) = (\cdot) \otimes^L N_{Z/Y};$, and all other transition functors in this cycle are trivial, it follows that monodromy around this loop is trivial.

Now we make the following observation. Let $K_n$ be the skeleton associated to a perverse sheaber with $n$ marked points. Then we recall that $K_n^0$ is the skeleton $K_n$ with each univalent vertex $v_p$ for $p \in \Sigma$ replaced by a trivalent vertex $v_p'$ with a loop. See the definition of monodromy of a perverse sheber for details and notation. Counterclockwise monodromy around this loop is given by the spherical twist associated to $F_p : A_p \to C$. Now let $p_1, ..., p_n$ be the set $\Sigma$, so that the edges $q_i$ are oriented counterclockwise around $c$ and $q_\infty$ is directly counterclockwise to $q_1$. Then define cycles $C_i$
\[ c \xrightarrow{q_i} v_{p_i} \xrightarrow{e_{p_i}} v_{p'_i} \xrightarrow{q_i} c \]
and a cycle $C_\infty$ by the concatenation $C_n \cdots C_1$. Then it follows directly from the definition that monodromy around $C_\infty$ is simply
\[ T_{F_n} \cdots T_{F_1}. \]
If $V$ is a variety, $k : Z \hookrightarrow Y_0$ is a smooth anticanonical divisor, $\alpha_i : A_i \to D^b_k(Y_0)$ form a semiorthogonal decomposition and $F_i = \mathbb{L}k^* \cdot \alpha_i$, then it follows from work of Addington and Aspinwall [3, Theorem 11] that $T_{F_n} \cdots T_{F_1}$ is the spherical twist associated to $\mathbb{L}k^*$. Furthermore, this spherical twist is, up to shift, just $(\cdot) \otimes^L N_{Y_0/Z}$. Now if we have a simplified type II degeneration of Calabi-Yau varieties, then monodromy around boundary components at either end of Figure 7 is given by concatenation with $C_\infty$ and the cycle
\[ c \xrightarrow{q_\infty} v_{1, 1} \xrightarrow{f_{1, 1}} v_{1, 2} \xrightarrow{f_{2, 2}} v_{1, 1} \xrightarrow{q_\infty} c \]
Monodromy around this cycle is $(\cdot) \otimes^L N_{Z/Y_1}$. Therefore monodromy around the concatenation of these cycles is $\otimes^L (N_{Z/Y_1} \otimes^L N_{Z/Y_0}) = (\cdot) \otimes^L \mathcal{O}_Z$, which is trivial. Similarly we can deal with the monodromy around the final boundary component. The remaining case occurs when $n = 1$, that is, there are no ruled components of $Y$. In this case, a very similar argument suffices.

6.4. The case of K3 surfaces. When we assume that the dimension of $X$ is 3, hence the dimension of $X_0$ is 2, our results can be made more general. We have the following definition.

Definition 6.10. Let $X$ be a complex manifold of dimension 3 equipped with a projective morphism $\pi : X \to D$. We say that $X$ is a type II degeneration of K3 surfaces if
1. The bundle $\omega_X$ is trivial,
2. All fibers $\pi^{-1}(t)$ are smooth K3 surfaces if $t \neq 0$. 
Remark 6.12. The fiber over 0 is a union of surfaces $S_0 \cup \cdots \cup S_n$ so that if $i = 0, n$ then $S_i$ is a smooth rational surface, and if $i \neq 0, n$ then $S_i$ is a smooth ruled surface over an elliptic curve. Furthermore, $E_i = S_i \cap S_{i+1}$ is a smooth elliptic curve and $S_i \cap S_j = \emptyset$ if $|i - j| \neq 1$.

Note that we have changed notation slightly in order to emphasize that we are dealing with surfaces and curves instead of varieties of arbitrary dimension.

The fact that the total space is Calabi-Yau implies that $(S_i \cap S_{i+1}) \cup (S_{i-1} \cap S_i)$ is an anticanonical divisor in $S_i$ if $i \neq 0, 1$ and $S_0 \cap S_1$ and $S_n \cap S_{n-1}$ are anticanonical in $S_0$ and $S_n$ respectively. Generally \[37, 64\], there are three types of semistable degenerations of K3 surfaces, types I, II and III. Type I is essentially smooth K3 surfaces and type III degenerations are normal crossings.

Proposition 6.11. Let $S$ be an elliptically ruled surface with anticanonical divisor made up of a pair of smooth disjoint sections $D_1$ and $D_2$. Then there are two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on an elliptic curve $E$ so that $S$ can be obtained by blowing up $\mathbb{P}_E(\mathcal{L}_1 \oplus \mathcal{L}_2)$ repeatedly at (possibly infinitely near) points in either one of the two sections.

Remark 6.12. We would like to thank Alan Thompson for taking the time to explain the proof of this proposition to us.

Proof. The fibers of the ruling on $S$ must be of constant arithmetic genus 0. The arithmetic genus is greater than or equal to the sum of the arithmetic genera of the irreducible components of each fiber, thus all irreducible components of each fiber must have arithmetic genus 0. Any singular curve has arithmetic genus greater than 0, so it follows that all components of curves in fibers of the ruling on $S$ are smooth rational curves. The generic fiber of this ruling is a smooth curve of genus 0 whose intersection number with $-K_S$ is 2. Therefore, the genus formula for curves on surfaces says that the following situations occur for $C$ a smooth rational curve contained in the fiber of the ruling on $S$,

1. $C \cap (-K_S) = 0$ and $C^2 = -2$,
2. $C \cap (-K_S) = 1$ and $C^2 = -1$,
3. $C \cap (-K_S) = 2$ and $C^2 = 0$.

In the first case, $C$ is disjoint from $-K_S$, in the second case $C$ intersects just one component of $-K_S$ and in the third case, $C$ is a general fiber of the ruling. The minimal model of $S$ is obtained by repeatedly contracting $(-1)$ curves until none remain. Either the fiber over a point $p$ in $D_1$ is irreducible, in which case it contains no $(-1)$ curves or it is reducible. If it is reducible, then take a component which intersects $D_1$ in the point $p$. By the argument above, this curve is a $(-1)$ curve, which we may contract. Repeating this argument shows that $S$ is obtained by repeatedly blowing up a $\mathbb{P}^1$ bundle in points in a section. This bundle is equal to $\mathbb{P}_E(\mathcal{E})$ for some rank 2 bundle $\mathcal{E}$ on $E$. The same argument works for $D_2$.

The curves $D_1$ and $D_2$ descend to sections of this $\mathbb{P}^1$ bundle. Since $\mathbb{P}_E(\mathcal{E})$ has a pair of sections, we can use [30] Exercise V.2.2 to see that $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ for a pair of bundles $\mathcal{L}_1$ and $\mathcal{L}_2$.

There are well-known birational modifications of type II degenerations of K3 surfaces called type I modifications. See e.g. [25] for details.

Definition 6.13. Let $\pi : \mathcal{X} \to \Delta$ be a type II degeneration of K3 surfaces. Let $C$ be a smooth $(-1)$ curve in an irreducible component $S_i$ of $\pi^{-1}(0)$ which intersects $E_{i-1}$ or $E_i$ in a single point.
We deal with the case where $i$.

Therefore, the Euler characteristic of $\pi$. The cohomology of $p$ vanishing cycles functor, which are just the nearby and vanishing cycles functors composed with $\pi$ known blowing up just increases $h_p$ see [65, pp. 261]. Here $p$.

Proposition 6.11 shows that type I modifications can be applied repeatedly to move $(X, S)$. We deal with the case where $i$.

Proof. This follows from Proposition 6.11 directly along with the definition of a type I modification. Proposition 6.11 shows that type I modifications can be applied repeatedly to move $(-1)$ curves from $S_i$ to $S_{i+1}$ until $S_i$ becomes $\mathbb{P}_E(\mathcal{L}_1 \oplus \mathcal{L}_2)$. Repeat this until all $(-1)$ curves in $S_1, \ldots, S_{n-1}$ end up in $S_n$.

Now let’s make a numerical observation. Every rational surface $S$ has minimal model either $\mathbb{P}^2$ or $\mathbb{F}_n$ for some $n$. Both $\mathbb{P}^2$ and $\mathbb{F}_n$ admit full exceptional collections, therefore, by Orlov’s formula [58], $S$ admits a full exceptional collection. Let $k_0$ and $k_n$ be the number of exceptional objects in this exceptional collection on $\mathcal{D}^b(S_0)$ and $\mathcal{D}^b(S_n)$ respectively. If $i \neq 0, n$ we let $k_i$ be the number of points blown up in Proposition 6.11 to get from $\mathbb{P}_E(\mathcal{L}_1 \oplus \mathcal{L}_2)$ to $S_i$.

Proposition 6.15. If $S_0 \cup \cdots \cup S_n$ is a type II degeneration of K3 surfaces, then $k_0 + k_1 + \cdots + k_n = 24$.

Proof. The topological Euler characteristic of $S_0 \cup \cdots \cup S_n$ is 24. A. Thompson informs us that this can be proved for any semistable degeneration of K3 surfaces using the Clemens-Schmid exact sequence [52], but we opt for a quicker proof here. We have that there is a triangle of constructible sheaves on $\pi^{-1}(0)$,

$$p_{\psi_\pi} \mathbb{C} \to p_{\phi_{\pi}} \mathbb{C} \to \mathbb{C}$$

see [65] pp. 261]. Here $p_{\psi_\pi}$ denotes the perverse nearby cycles functor and $p_{\phi_{\pi}}$ the perverse vanishing cycles functor, which are just the nearby and vanishing cycles functors composed with [1]. The cohomology of $p_{\psi_\pi}$ is that of $\pi^{-1}(t)$ for $t \neq 0$, hence it has Euler characteristic 24, since $\pi^{-1}(t)$ is a K3 surface. It follows from [71] Theorem 10] that the Euler characteristic of the sheaf $p_{\phi_{\pi}} \mathbb{C}$ is the Euler characteristic of the critical locus of $\pi$ up to sign, which is a disjoint union of $n$ elliptic curves. Therefore, the Euler characteristic of $p_{\phi} \mathbb{C}$ is 0. It follows then that the topological Euler characteristic of $\pi^{-1}(0)$ is equal to that of $\pi^{-1}(t)$, which we know is 24.

Now applying the Mayer-Vietoris exact sequence along with the fact that the Euler characteristic of an elliptic curve is 0 implies that the sum of the topological Euler characteristics of $S_i$ is 24. We deal with the case where $i \neq 0, n$ first. We know that $\mathbb{P}_E(\mathcal{L}_1 \oplus \mathcal{L}_2)$ has topological Euler characteristic. One can then check that the topological Euler characteristic of $S_i$ is equal to the number of times we must blow up $\mathbb{P}_E(\mathcal{L}_1 \oplus \mathcal{L}_2)$ to get $S_i$. This number is precisely $k_i$, since we know blowing up just increases $h^{1,1}$ by 1 and does not affect any other Hodge numbers. Now we deal with the case where $i = 0, n$. Since $h^1(S_i) = h^3(S_i) = h^{0,2}(S_i) = 0$, the topological Euler characteristics of $S_i$ are equal to the rank of $\text{HH}_0(\mathcal{D}^b(S_i))$ by the Hochschild-Kostant-Rosenberg
isomorphism, which says that for a smooth projective variety $X$
\[ \text{HH}_n(D^b(X)) = \bigoplus_{q-p=n} \text{H}^q(X, \Omega^p_X). \]

Kuznetsov [40 Corollary 7.5] shows that Hochschild homology is additive with respect to semiorthogonal decompositions, so $\dim \text{HH}_0(D^b(S_i))$, which is equal to the topological Euler characteristic of $S_i$, is equal to $k_i$. Hence $k_0 + \cdots + k_n = 24$. \hfill \Box

Following Theorem 6.7 and Proposition 6.15 we have the following corollary.

**Corollary 6.16.** Let $\mathcal{X}$ be a type II degeneration of $K3$ surfaces $X$ with $n+1$ irreducible components. There is birational model $\mathcal{X}'$ of $\mathcal{X}$ of with central fiber $\mathcal{X}_0'$ and $K$-coordinatized perverse sheaf of categories $\mathcal{S}_K$ over $S^2$ with $n$ boundary components and with respect to a stratification given by a set of points $\Sigma$ so that $K, \Sigma$ and $\mathcal{S}_K$ have the following properties.

1. All edges of $K$ have both ends in $\text{Vert}(K)$.
2. The category of global sections is $\text{Perf}(\mathcal{X}_0')$.
3. The monodromy around all boundary components of $\mathcal{S}_K$ is trivial.
4. $|\Sigma| = 24$ and for each $p \in \Sigma$, the category $A_p = \text{Perf}_k$.

Our goal now is to show that $\text{Perf}(\mathcal{X}_0)$ is equivalent to $\text{Perf}(\mathcal{X}_0')$. Assume that $X = S_1 \cup S_2$ is normal crossings of dimension 2 and $Z = S_1 \cap S_2$ is smooth, and so that there are smooth divisors $Z_1$ and $Z_2$ in $S_1$ and $S_2$ which are disjoint from $Z$ and so that $Z \cup Z_1$ is anticanonical in $S_1$ and $Z \cup Z_2$ is anticanonical in $S_2$. Assume that $p$ is a point contained in $Z$. Then we can blow up either $S_1$ or $S_2$ in $p$ to get varieties $\tilde{S}_1$ and $\tilde{S}_2$. This produces, geometrically, two different objects $X_1 = \tilde{S}_1 \cup S_2$ and $X_2 = S_1 \cup \tilde{S}_2$. First, we record a proposition;

**Proposition 6.17.** Let $S$ be a surface, and let $Z_1$ and $Z_2$ be disjoint smooth divisors in $S$ so that $Z_1 + Z_2$ is anticanonical, and let $\mathfrak{pt}$ be a point in $Z_1$, with $k_i : Z_i \hookrightarrow S$ and $i : \mathfrak{pt} \hookrightarrow Z_1$ the embedding maps. Let $\tilde{S}$ be the blow up of $S$ in $\mathfrak{pt}$ and $\tilde{k}_i : Z_i \hookrightarrow \tilde{S}$ the embedding maps. The category $D^b_{\text{dg}}(\mathfrak{pt}) \times_{\text{Ri}, \text{Li}_k} D^b_{\text{dg}}(S)$ is quasi equivalent to $D^b_{\text{dg}}(\tilde{S})$. The functor $c_{\text{Ri}, \text{Li}_k}$ is quasi isomorphic as a dg bimodule to $\mathbb{L}^{k_1}$, and $(a, b, \mu) \mapsto \mathbb{L}^{k_2}(b)$ is quasi isomorphic to $\mathbb{L}^{k_2}$ as a dg bimodule.

**Proof.** We have that there is a semiorthogonal decomposition of a blow up of a surface $S$ at a point given by
\[ (\mathbb{R}q_* (L\mathfrak{p}^* D^b(\mathfrak{pt}) \otimes \mathcal{O}_{\mathfrak{pt}}(-1)), L\pi^* D^b(S)). \]

as described in [58]. Following the arguments of Propositions 6.1 and 6.2 precisely, we obtain that there is a quasi equivalence between $D^b_{\text{dg}}(\tilde{S})$ and
\[ D^b_{\text{dg}}(\mathfrak{pt}) \times_{\text{Ri}, \text{Li}_k} \mathbb{L}^{k_1} D^b_{\text{dg}}(S). \]

Therefore, since $\text{Li}_k N_{\mathfrak{pt}/S}$ is clearly trivial, the result follows. \hfill \Box

The germ of a type II degeneration $\mathcal{X}$ of K3 surfaces near 0 gives rise to a Calabi-Yau variety over $\mathbb{C}[[t]]$ by the fact that $\omega_\mathcal{X} = 0$. Bridgeland [16], followingBondal and Orlov [13] showed that birational modifications of compact Calabi-Yau varieties should not affect the derived category of coherent sheaves. We expect that the same is true of $\mathcal{X}$. If this is true then the fibers of $\mathcal{X}$ must also be derived invariant under this birational modification. The following theorem uses the formalism of perverse sheaves of categories to show that $\text{Perf}(\mathcal{X}_0)$ is not changed by a type I modifications.

**Proposition 6.18.** With notation as above, $\text{Perf}(X_1)$ and $\text{Perf}(X_2)$ are quasi equivalent categories.
Proof. Before we proceed, note that the category $A_2(\mathcal{C})$ always admits a quasi autoequivalence

$$\sigma_2 : (a, b, \mu) \mapsto (b, \text{Cone}(\mu), \xi)$$

where $\xi : b \to \text{Cone}(\mu)$ is the natural closed map of degree 0. Clearly, $f_i \cdot \sigma_2$ is quasi equivalent to $f_{i+1}$ where $i$ is taken mod 3.

Using the fact that $D^b(V_1)$ is quasi equivalent to $D^b_{dg}(S_1) \times_{\mathbb{A}^3} D^b_{dg}(Z)$, we see that $D^b_{dg}(\widetilde{S}_1)$ is quasi equivalent to the homotopy limit over the diagram,

$$D^b_{dg}(S_1) \xrightarrow{Lk^*_{1}} \text{Tw} D^b_{dg}(Z) \xleftarrow{f_1} A_2(D^b_{dg}(Z)) \xrightarrow{f_2} D^b_{dg}(Z) \xleftarrow{\mathbb{R}Z} D^b_{dg}(pt)$$

There’s a functor from this homotopy limit to $A_2(D^b_{dg}(Z))$ by the universal property of homotopy limits, and composing this functor with $f_3$ we get a functor to $D^b_{dg}(Z)$, which is equivalent to $\mathbb{L}k^*_1$ where $k_1$ is the embedding of $Z$ into $S_1$. Therefore, $\text{Perf}(\widetilde{X}_1)$ is quasi equivalent to the homotopy limit over the diagram

$$D^b_{dg}(S_1) \xrightarrow{Lk^*_1} D^b_{dg}(Z) \xleftarrow{f_1} A_2(D^b_{dg}(Z)) \xrightarrow{f_2} D^b_{dg}(Z) \xleftarrow{\mathbb{R}Z} D^b_{dg}(pt) \xrightarrow{f_3} D^b_{dg}(Z)$$

(14)

However, applying Proposition 6.17 along with the symmetry of $A_2(D^b_{dg}(Z))$ mentioned in the beginning of the proof, the homotopy limit over the diagram

$$A_2(D^b_{dg}(Z)) \xrightarrow{f_2} D^b_{dg}(Z) \xleftarrow{\mathbb{R}Z} D^b_{dg}(pt) \xrightarrow{f_3} D^b_{dg}(Z)$$

is equivalent to $D^b_{dg}(\widetilde{S}_2)$ and the functor from the limit to $D^b_{dg}(Z)$ induced by $f_1$ is equivalent to $\mathbb{L}k^*_2$. Therefore, the homotopy limit over the diagram in Equation 14 is also quasi equivalent to $\text{Perf}(X_2)$. \hfill \Box

Therefore, as a corollary, we have that

Theorem 6.19. If $\mathcal{X}$ and $\mathcal{X}'$ are type II degenerations of K3 surfaces related by type I modifications, then $\text{Perf}(\mathcal{X}_0)$ and $\text{Perf}(\mathcal{X}'_0)$ are quasi equivalent.

Proof. Apply Proposition 6.18 along with Proposition 6.14. \hfill \Box

It follows then, by Proposition 6.14 and Corollary 6.16 that if $\mathcal{X}$ is a type II degeneration of K3 surfaces, there is a perverse schober whose category of global sections is $\text{Perf}(X_0)$.

6.5. Connection with homological mirror symmetry. The reader should interpret this in the following way. The global sections of the perverse sheaf of categories that we are studying should be equivalent to the Fukaya category of some symplectic fibration over $S^2 \setminus \{D_0, \ldots, D_n\}$ whose fibers
are mirror to $D^b(E)$, for instance symplectic 2-tori. The singular fibers of this fibration should be nodal elliptic curves whose vanishing cycles correspond to spherical objects in $E$ under mirror symmetry. Proposition 6.14 says that this fibration should have trivial symplectic monodromy around the boundary components. This should be interpreted as saying that we can compactify this fibration by adding a smooth symplectic torus at each boundary component. Proposition 6.15 says that this fibration should have 24 degenerate fibers. Therefore, if it is a complex fibration, then it must be in fact a K3 surface. This dovetails nicely with work done by the first author along with C. Doran and A. Thompson [20], where they conjecture that there is a bijection between certain fibration structures on a Calabi-Yau variety $X$ and Tyurin degenerations on the mirror Calabi-Yau variety $X^\vee$. In [21], Doran and Thompson extend this to type II degenerations of K3 surfaces. In homological mirror symmetry, this conjecture takes the following form.

**Conjecture 6.20.** Let $X$ be a $d$-dimensional Calabi-Yau variety and let $X^\vee$ be its mirror. Assume there is a fibration $f : X \to \mathbb{P}^1$ on $X$ by Calabi-Yau $(d - 1)$-folds. Let $Z_1, \ldots, Z_n$ be smooth fibers of $\pi$. Then there is a type II degeneration of $X^\vee$ to $Y_0 \cup \cdots \cup Y_n$ so that

$$D^b \mathcal{F}(X \setminus \{Z_1, \ldots, Z_n\}) \cong \text{Perf}(Y_0 \cup \cdots \cup Y_n).$$

According to Seidel [75], the exact Fukaya category of $X \setminus \{Z_1, \ldots, Z_n\}$ deforms to $\mathcal{F}(X)$. We conjecture that this deformation agrees with the deformation of $Y$ to $X^\vee$ given by Friedman [23].

Proposition 6.14 gives a direct relationship between the fact that the divisor $Z_1, \ldots, Z_n$ are smooth and the fact that $\text{Perf}(Y_0 \cup \cdots \cup Y_n)$ admits a smooth deformation with smooth total space. We will prove a degenerate version of this conjecture in Theorem 8.9 in which the fiber $Z$ removed is not smooth, and we will see that the mirror is not d-semistable or even normal crossings. On the B-side, this example takes the following form.

**Example 6.21.** Look at the Gorenstein anticanonical hypersurface $Z$ in $\mathbb{P}^3$ cut out by the equation

$$x^2y^2 - z^2w^2 = 0$$

This hypersurface is the union of two smooth quadrics which we denote $Q_1$ and $Q_2$, respectively given by

$$xy + zw = 0, \quad xy - zw = 0.$$ 

The intersection of $Q_1$ and $Q_2$ is the union of four lines which we denote $G_4$ as in Example 5.8

$$x = z = 0, \quad x = w = 0, \quad y = z = 0, \quad y = w = 0.$$ 

Therefore, according to Example 2.27, $\text{Perf}(Z)$ is quasi equivalent to a full subcategory of $D^b_{dg}(Q_1) \times_{\text{Perf}(G_4)} D^b_{dg}(Q_2)$.

Using the same reasoning as in Theorem 6.7, we see that $\text{Perf}(Z)$ is quasi equivalent to a full subcategory of the category of global sections of the perverse sheaf of categories on the skeleton in Figure 8 with fiber at a point on an edge given by $\text{Perf}(G_4)$, spherical functors at vertices determined by the following spherical objects

$$p_1, q_1 \mapsto \mathbb{L}i^* \mathcal{O}_Q(0, 0), \quad p_2, q_2 \mapsto \mathbb{L}i^* \mathcal{O}_Q(1, 0), \quad p_3, q_3 \mapsto \mathbb{L}i^* \mathcal{O}_Q(0, 1), \quad p_4, q_4 \mapsto \mathbb{L}i^* \mathcal{O}_Q(1, 1).$$

and with trivial transition data.

We will use this point of view to show that there is a quasi equivalence between $\text{Perf}(Z)$ and $D^b(\mathcal{F}(U))$ where $U$ is described as follows. Let $W$ be the mirror quartic. It admits an elliptic fibration over $\mathbb{P}^1$ with four sections $\sigma_1, \ldots, \sigma_4$ and a fiber over $\infty$ made up of a cycle of 16 rational
Figure 8.

curves. Then $U$ is the complement of the union of the four sections and 16 rational curves over $\infty$. In this mirror correspondence, both sides are degenerate. The B-side is degenerate since $Q_1 \cap Q_2$ is neither simple normal crossings nor semistable, and the A-side is degenerate since the curve removed is not smooth but nodal. Regardless, we view this as evidence for Conjecture 6.20.

Remark 6.22. In the next two sections, we will see that perverse schobers whose fibers are represented as global sections of perverse sheaves appear in both noncommutative geometry and homological mirror symmetry. Before we proceed, we will explain this idea in the example of $\mathbb{P}^3$.

Let us take our notation to be as in Example 6.21. Since $j : Z \hookrightarrow \mathbb{P}^3$ is of degree 4, the functor $Lj^* : D_{dg}(\mathbb{P}^3) \to \mathcal{P}(Z)$ is spherical with cotwist $S_{\mathbb{P}^3}$.

Therefore, we can take the set of spherical objects $W = \{Lj^*\mathcal{O}_{\mathbb{P}^3}(i)\}_{i=1}^4$ in $\mathcal{P}(Z)$ and use Theorem 5.6 to see that the category of global sections of the corresponding perverse schober is quasi equivalent to $D_{dg}(\mathbb{P}^3)$. In fact, since $\mathcal{P}(Z)$ is quasi isomorphic to a full subcategory of the homotopy fiber product, $\mathcal{E} := D_{dg}(Q_1) \times_{\mathcal{P}(G_4)}^h D_{dg}(Q_2)$, Theorem 5.6 shows that the perverse schober associated to the objects

$$M = \{M_i = (\mathcal{O}_{Q_1}(i,i), \mathcal{O}_{Q_2}(i,i), \text{id}_{\mathcal{O}_{G_4}(i)})\}_{i=1}^4$$

in $D_{dg}(Q_1) \times_{\mathcal{P}(Z)}^h D_{dg}(Q_2)$ is quasi equivalent to $D_{dg}(\mathbb{P}^3)$.

This will be used in Section 8 for our proof of homological mirror symmetry for $\mathbb{P}^3$. □

7. Noncommutative deformations of projective spaces

Here we will discuss the relationship between perverse schobers and specific noncommutative deformations of varieties. First we will examine the case of $\mathbb{P}^1 \times \mathbb{P}^1$, which will be instructive, but its novelty lies more in the language used than the actual results. The case of $\mathbb{P}^3$ is more complicated, and will highlight the intimate relationship between noncommutative quadrics and noncommutative $\mathbb{P}^3$.

7.1. Z-algebras. The following results fit well in the context of Z-algebras. We will use definitions and constructions of Van den Bergh [96 Sections 3, 4] and Polishchuk [67 Section 1] in this section.

Definition 7.1. A Z-algebra is a $k$-algebra $A = \bigoplus_{(m,n) \in \mathbb{Z}^2} A_{m,n}$ so that the composition $A_{n,t}A_{m,n} \subseteq A_{m,t}$ and compositions are 0 otherwise. We also assume there exists an identity element $e_n \in A_{n,n}$ for all $n$ so that $e_n x = x = xe_m$ for any $x \in A_{m,n}$.

□
Remark 7.2. This uses Polishchuk’s conventions in order so that multiplication more closely resembles composition of homomorphisms.

We will always assume that $A_{m,n} = 0$ if $m > n$. We will be concerned with categories graded right $A$-modules. Such modules are $\mathbb{Z}$-graded and admit right actions of $A$ which satisfy

$$M_n A_{m,n} \subseteq M_m, \quad \text{and} \quad M_j A_{m,n} = 0 \text{ if } j \neq n.$$ 

There are natural projective graded right $A$-modules which we denote $P_n$, which are simply $e_n A = \bigoplus_{j \in \mathbb{Z}} A_{j,n}$. The category of graded right $A$-modules is denoted $\text{Gr}(A)$. We follow the notation of [96] and say that $\text{gr}(A)$ is the subcategory of $\text{Gr}(A)$ of noetherian objects. The category $\text{gr}(A)$ is abelian. Denote by $qgr(A)$ the quotient of $\text{gr}(A)$ by the subcategory of torsion modules. We will be interested in the category $D^b(\text{qgr}(A))$. Let $\pi$ be the quotient functor from $\text{gr}(A)$ to $\text{qgr}(A)$.

The objects $P_i$ form a set of projective generators for $\text{Gr}(A)$ and we have that $\text{hom}_A(P_i, P_j) = A_{i,j}$ and composition of homomorphisms agrees with composition in $A$.

To any graded algebra $R$, we may canonically associate a $\mathbb{Z}$ algebra $\bar{R}$ by letting

$$\bar{R}_{i,j} = R_{j-i}.$$ 

We have, in the same way, categories of right $\bar{R}$-modules and categories $\text{Gr}(\bar{R})$, $\text{Qgr}(\bar{R})$, $\text{gr}(\bar{R})$ and $\text{qgr}(\bar{R})$. There are equivalences between $\text{Gr}(R)$ and $\text{Gr}(\bar{R})$ and respectively, for the rest of the categories above.

To a $\mathbb{Z}$ algebra $A$ and a positive integer $n$ there is an $n$-Veronese algebra $\mathbb{Z} A$, which is defined to be the subalgebra of $A$ so that

$$n A_{i,j} = A_{ni,nj}.$$ 

There are equivalences between categories of modules over a $\mathbb{Z}$ algebra and its Veronese algebras, specifically, $qgr(\mathbb{Z} A) \cong qgr(A)$. See [96] for details.

7.2. Noncommutative quadrics. In the following definition, $E$ denotes a smooth genus 1 curve, but any variety would do.

Definition 7.3. Let $\mathcal{L}_i$ ($i \geq 1$) be a sequence of line bundles on $E$ so that $\deg \mathcal{L}_i = c$ is constant and let $\mathcal{M}_i = \bigotimes_{j=1}^{i} \mathcal{L}_j$. We define the twisted coordinate ring of $E$ associated to $\mathcal{L}_i$ to be the $\mathbb{Z}$-algebra $C = \bigoplus_{i,j} C_{i,j}$ where $C_{i,j} = \text{hom}_E(\mathcal{M}_i, \mathcal{M}_j)$ and with multiplication given by composition of homomorphisms.

This is called the twisted homogeneous coordinate ring of $E$ associated to the sequence $\mathcal{L}_i$. The category $qgr(C)$ is equivalent to $\text{coh}(E)$ by [67] Theorem 2.4 if $\mathcal{L}_i$ are general enough.

Let us take the quadruple $V = (E, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ for $E$ a smooth elliptic curve and $\mathcal{L}_i$ line bundles on $E$ of degree 2. Define a sequence of degree two line bundles starting with $V$ as follows;

$$\mathcal{L}_i = \mathcal{L}_{i-3}^{-1} \otimes \mathcal{L}_{i-2} \otimes \mathcal{L}_{i-1}. \quad (15)$$

Let $C_V$ be the twisted homogeneous coordinate ring built from this sequence of line bundles. The ring $C_V$ is generated by $C_{V,i,i+1}$. This allows us to define a noncommutative quadric.

Definition 7.4. Let $V = (E, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ and let $C_V$ be the $\mathbb{Z}$-algebra associated to $V$. We let $T$ be the tensor $\mathbb{Z}$-algebra generated by $C_{V,i,i+1}$, so that $T_{i,i+n} = \bigotimes_{k=0}^{n-1} C_{V,k+i,k+i+1}$. There is a surjective map of $\mathbb{Z}$-algebras from $T$ to $C_V$ since $C_V$ is generated by $C_{V,i,i+1}$. The intersection of the kernel of this map with $T_{i,i+n}$ is 0 if $n \leq 2$ and has dimension 2 if $n = 3$ (see [96] Lemma 5.5.2)). Let $J_V$ be the two-sided ideal in $T$ generated by the kernel of the maps $T_{i,i+3} \to C_{V,i,i+3}$. Then we define $Q_V = T/J_V$. 

\[\square\]
There is a quotient morphism $q_V$ from $Q_V$ to $C_V$. Its kernel is generated by a single element in $C_{V,i,i+4}$ for each $i$. In the following, we will often write $C$ and $Q$ instead of $C_V$ and $Q_V$.

**Proposition 7.5.** There is a functor $\Phi$ from $\text{qgr}(Q)$ to $\text{coh}(E)$ which sends $\pi P_i$ to $M_i$. The map

$$\Phi : \text{hom}_{\text{qgr}(Q)}(\pi P_i, \pi P_j) \to \text{hom}_{\text{coh}(E)}(M_i, M_j)$$

is an isomorphism if $0 \leq j - i \leq 3$.

**Proof.** First, we note that there is a functor $\Theta$ from the category $\text{gr}(Q)$ to the category $\text{gr}(C)$ coming from the pullback. The functor $\Theta$ assigns to an $Q$-module $M$ the module $C \otimes_Q M$ where $C$ is viewed as an $Q_V$-module. If $P_{i,Q}$ is the projective generator of $Q_V$ described above, then $C \otimes_Q P_{i,Q} \cong P_{i,C}$ since the map from $Q$ to $C$ is surjective. Therefore, the induced functor from $\text{qgr}(Q)$ to $\text{qgr}(C)$ sends $\pi P_{i,Q}$ to $\pi P_{i,C}$.

The sequence of line bundles $L_i$ on $E$ is an ample coherent sequence in the language of Polishchuk [68]. Therefore, there is an isomorphism between $\text{qgr}(C)$ and $\text{coh}(E)$. There are functors $\Gamma_{\leq m} : \text{coh}(E) \to \text{qgr}(C)$ for any $m$ (see [68 Theorem 2.4]), and for large enough $m$ this functor is an equivalence. This functor assigns to a coherent sheaf $F$ on $E$ the module

$$\Gamma_{\leq m} F = \bigoplus_{i \leq m} \text{hom}_{\text{coh}(E)}(M_i, F)$$

For $m \geq j$, we see that

$$\Gamma_{\leq m} M_j = \Gamma_* M_j = \bigoplus_{i \in \mathbb{Z}} \text{hom}_{\text{coh}(E)}(M_i, M_j).$$

Recall that the projective modules $\pi P_{j,C}$ are given by

$$\bigoplus_{i \in \mathbb{Z}} C_{i,j} = \bigoplus_{i \in \mathbb{Z}} \text{hom}_{\text{coh}(E)}(M_i, M_j).$$

Therefore $\Gamma_* M_j = \pi P_{j,C}$ and the inverse of $\Gamma_{\leq m}$, which Polishchuk calls $\Psi$, satisfies $\Psi(\pi P_{j,C}) \cong M_j$. Therefore, the induced functor

$$\text{qgr}(Q) \to \text{coh}(E)$$

sends $\pi P_{i,Q}$ to $M_i$. The functor described here recovers the morphism $f : Q \to C$ as

$$Q \cong \bigoplus_{i,j} \text{hom}_{\text{qgr}(Q)}(\pi P_{i,Q}, \pi P_{j,Q}) \overset{\Theta}{\to} \bigoplus_{i,j} \text{hom}_{\text{qgr}(C)}(\pi P_{i,C}, \pi P_{j,C}) \cong C \cong \bigoplus_{i,j} \text{hom}_{\text{coh}(E)}(M_i, M_j).$$

Which is surjective when $0 \leq j - i \leq 3$ since $C = Q/I_Q$ and $I_Q$ is generated by elements in $Q_{i,i+n}$ for $n \geq 4$. \hfill \square

We may let $D^{b}_{\text{dg}}(\text{qgr}(C))$ be a dg enhancement of $D^b(\text{qgr}(C))$. This category can be chosen to coincide with $D^{b}_{\text{dg}}(E)$, since $\text{qgr}(C) \cong \text{coh}(E)$, though may be more useful to take the dg enhancement made up of h-injective complexes for both $\text{coh}(E)$ and $\text{qgr}(C)$, which also coincide. Let us take $U$ to be the collection $\{\pi P_{0,C}, \pi P_{1,C}, \pi P_{2,C}, \pi P_{3,C}\}$ of objects in $D^{b}_{\text{dg}}(\text{qgr}(C))$ and $\Theta(U)$ the corresponding collection in $D^{b}_{\text{dg}}(E)$. The pullback functor $q^*$ from $\text{qgr}(Q)$ to $\text{qgr}(C)$ then extends to a dg functor $L q^* : D^{b}_{\text{dg}}(\text{qgr}(Q)) \to D^{b}_{\text{dg}}(\text{qgr}(C))$. Since we have chosen $D^{b}_{\text{dg}}(\text{qgr}(C))$ to be equivalent to $D^{b}_{\text{dg}}(E)$, we will conflate $L q^*$ and $L \Phi^*$.

We have the following theorem.

**Theorem 7.6.** There is a quasi equivalence,

$$\text{Tw} \left( D^{b}_{\text{dg}}(E)_{\Theta(U)} \right) \cong D^{b}_{\text{dg}}(\text{qgr}(Q))$$
The natural functor from $\text{Tw}(\mathcal{D}_{\text{dg}}^b(E)_{\Phi(U)})$ to $\mathcal{D}_{\text{dg}}^b(E)$ is quasi equivalent to $\mathbb{L}q^*_V$ as a dg bimodule.

Proof. The objects \{\(\pi P_{1,Q}, \pi P_{2,Q}, \pi P_{3,Q}, \pi P_{4,Q}\)\} form a full, strong exceptional collection in the category $\mathcal{D}_{\text{dg}}^b(\text{qgr}(Q))$ [7]. Therefore, by [76 Lemma 5.2], we have that $\mathcal{D}_{\text{dg}}^b(\text{qgr}(Q))$ is quasi equivalent to $\text{Tw}(\mathcal{D}_{\text{dg}}^b(\text{qgr}(Q)))^\tau$.

Because \{\(\pi P_{1,Q}, \pi P_{2,Q}, \pi P_{3,Q}, \pi P_{4,Q}\)\} forms a full, strong exceptional collection in $\mathcal{D}_{\text{dg}}^b(\text{qgr}(Q))$, we know that $\operatorname{Ext}^q(\pi P_{Q,i}, \pi P_{Q,j}) = 0$ if $i \neq 0$. Furthermore, $i < j$ then $\operatorname{Ext}^q(M_i, M_j) = 0$. Therefore, Proposition [7.5] implies that $\mathbb{L}q^*$ induces quasi isomorphisms

$$\operatorname{hom}_{\mathcal{D}_{\text{dg}}^b(\text{qcoh}(Q))}(\pi P_{i,Q}, \pi P_{j,Q}) \to \operatorname{hom}_{\mathcal{D}_{\text{dg}}^b(E)}(M_i, M_j)$$

whenever $0 < j - i < 4$, and composition agrees on the level of cohomology. Therefore,

$$\text{Tw}(\mathcal{D}_{\text{dg}}^b(\text{qgr}(Q)))^\tau \cong \text{Tw}(\mathcal{D}_{\text{dg}}^b(E)_{\Phi(U)}).$$

The proposition then follows. \qed

### 7.3. Noncommutative $\mathbb{P}^3$.

The goal of this section is to show that if $A$ is the coordinate ring of a quadratic Sklyanin algebra of dimension 4, then there is a perverse schober with generic fiber the fiber product of a pair of noncommutative quadrics whose category of global sections is quasi equivalent to $\mathcal{D}_{\text{dg}}^b(\text{qgr}(A))$.

First, let us review what exactly we will mean when we talk about noncommutative $\mathbb{P}^3$. There are many reasonable noncommutative deformations of $\mathbb{P}^3$ (see [70]), but only some of them show up as perverse schobers, and only one of them, the most well known, will be discussed in this section. The quadratic Sklyanin algebra of global dimension 4 is a graded algebra which is a quotient of $\mathbb{C}(x_0, x_1, x_2, x_3)$ by the ideal generated by elements

\begin{align*}
(16) & \quad x_0 x_1 - x_1 x_0 - a_1(x_2 x_3 + x_3 x_2), & x_0 x_1 + x_1 x_0 - (x_2 x_3 - x_3 x_2), \\
(17) & \quad x_0 x_2 - x_2 x_0 - a_2(x_1 x_3 + x_3 x_1), & x_0 x_2 + x_2 x_0 - (x_1 x_3 - x_3 x_1), \\
(18) & \quad x_0 x_3 - x_3 x_0 - a_3(x_1 x_2 + x_2 x_1), & x_0 x_3 + x_3 x_0 - (x_1 x_2 - x_2 x_1),
\end{align*}

where the parameters $a_1, a_2, a_3$ satisfy the equation $a_1 + a_2 + a_3 + a_1 a_2 a_3 = 0$. One can also define four dimensional quadratic Sklyanin algebras starting from a triple $V = (E, \mathcal{L}_1, \mathcal{L}_2)$ where $E$ is a smooth elliptic curve and $\mathcal{L}_1, \mathcal{L}_2$ are line bundles of degree 4 on $E$. This formulation can be found in work of Smith and Smith and Levasseur [17,18], or in work of Tate and Van den Bergh [90]. In their formulation $a_1, a_2$ and $a_3$ are given by values of theta functions. We will denote the corresponding graded algebra $A_V$. As usual, we define the category $\text{qgr}(A_V)$ to be the abelian category $\text{gr}(A_V)$ of noetherian graded right $A_V$ algebras modulo the category of right torsion modules.

According to work of Smith and Tate [81] (see also [18 Theorem 6.5]), if $a_1, a_2, a_3$ are generic enough, then the center of $(A_V)_2$ is of dimension 2. If $J_V$ is the ideal generated by this subset of $(A_V)_2$ then $A_V/J_V$ is a twisted coordinate algebra of $E$ associated to a pair of line bundles $\mathcal{L}_1, \mathcal{L}_2$. This is defined as follows. If $\sigma$ is the automorphism of $E$ which acts on line bundles as the tensor product with $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, then $A_V/J_V$ is isomorphic to the graded ring

$$B_V = \bigoplus_{i=0}^\infty H^0(E, \mathcal{L}_1 \otimes \sigma^* \mathcal{L}_1 \otimes \cdots \sigma^{(i-1)*} \mathcal{L}_1)$$

and composition is given by twisted multiplication of sections. If $a_i \in H^0(E, \mathcal{L}_1 \otimes \sigma^* \mathcal{L}_1 \otimes \cdots \otimes \sigma^{(i-1)*} \mathcal{L}_1)$, then multiplication sends $a_i \otimes a_j$ to $a_i \cdot \sigma^* a_j$ (see [90] Section 4.1). We may also define
a $\mathbb{Z}$ algebra starting with the data $(E, \mathcal{L}_1, \mathcal{L}_2)$. Let
\[ \mathcal{L}_i = (\mathcal{L}_2 \otimes \mathcal{L}_1^{-1})^{i-1} \otimes \mathcal{L}_2, \quad \mathcal{M}_i = \bigotimes_{j=1}^{i} \mathcal{L}_i \]
Then we define $D_V$ so that
\[ D_{V,i,j} = \text{hom}_E(\mathcal{M}_i, \mathcal{M}_j) \]
and multiplication is given by straightforward composition. By \[67\], $\text{gcr}(D)$ is equivalent to $\text{coh}(E)$. We can compare the algebra $D_V$ to the algebra $B_V$.

**Lemma 7.7.** Let $V = (E, \mathcal{L}_1, \mathcal{L}_2)$. Then if $D_V$ is the $\mathbb{Z}$-algebra associated $V$ and $B_V$ is the graded algebra associated to $V$ then $B_V \cong C_V$.

**Proof.** By construction, $\mathcal{L}_i = \sigma^{i-1} \ast \mathcal{L}_1$, thus $B_{V,m,n} = B_{V,n-m} = H^0(E, \mathcal{M}_{n-m})$. This is isomorphic to \[(19) \quad \text{hom}_E(\mathcal{O}_E, \mathcal{M}_{n-m}) \cong \text{hom}_E(\mathcal{O}_E, \sigma^{m\ast} \mathcal{M}_{n-m}) \cong \text{hom}_E(\mathcal{M}_m, \mathcal{M}_m \otimes \sigma^{m\ast} \mathcal{M}_{n-m}) \cong \text{hom}_E(\mathcal{M}_m, \mathcal{M}_n). \]
Multiplication of cohomology classes can be interpreted as composition of homomorphisms. The isomorphism between $H^0(E, \mathcal{M}_{n-m})$ and $\text{hom}_E(\mathcal{O}_E, \mathcal{M}_{n-m})$ is given by sending a local section $a$ to the homomorphism obtained by multiplying sections of $\mathcal{O}_E$ by $a$. Denote this homomorphism $\phi_a$. The second isomorphism in Equation \[19\] sends $\sigma^m \ast \phi_a$ to $\sigma^{m\ast} \phi_a = \phi_{\sigma^m \ast a}$. The third isomorphism sends $\phi_{\sigma^m \ast a}$ to the homomorphism which multiplies local sections by $\sigma^{m\ast} a$. Therefore, the multiplication map
\[ a_m \otimes a_n \in H^0(E, \mathcal{M}_m) \otimes H^0(E, \mathcal{M}_{n-m}) \mapsto a_m \otimes \sigma^{m\ast} a_n \in H^0(E, \mathcal{M}_n) \]
is identified with the composition of homomorphisms after the isomorphisms in Equation \[19\].

The next proposition shows that there is a compatibility between the twisted coordinate algebra construction starting from a pair of line bundles on $E$ of degree 2 and the twisted coordinate algebra construction starting from a pair of line bundles on $E$ of degree 4.

**Proposition 7.8.** Let $C_W$ be the $\mathbb{Z}$-algebra twisted coordinate ring associated to the data $W = (E, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$. Then $2C_W$ is isomorphic to $D_V$ for $V = (E, \mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{L}_2 \otimes \mathcal{L}_3)$.

**Proof.** Let us take the algebra $C_W$ to begin with and let us compute $2C_W$. We know that
\[ 2C_{W,i,j} = \text{C}_{W,2i,2j} = \text{hom}_E(\mathcal{M}_{2i}, \mathcal{M}_{2j}) \]
where $\mathcal{L}_i = \mathcal{L}_i^{-1} \otimes \mathcal{L}_{i-2} \otimes \mathcal{L}_{i-1}$ and $\mathcal{M}_n = \bigotimes_{i=1}^{n} \mathcal{L}_i$. We can compute that
\[ \mathcal{L}_{2n+1} = \mathcal{L}_1 \otimes (\mathcal{L}_1^{-1} \otimes \mathcal{L}_3)^n, \quad \mathcal{L}_{2n} = \mathcal{L}_2 \otimes (\mathcal{L}_1^{-1} \otimes \mathcal{L}_3)^{n-1}. \]
Therefore,
\[ \mathcal{M}_{2n} = (\mathcal{L}_1 \otimes \mathcal{L}_2)^n \otimes (\mathcal{L}_1^{-1} \otimes \mathcal{L}_3)^2 \sum_{i=1}^{n} (i-1). \]
It follows then that if we let $\sigma$ be the automorphism of $E$ so that $\sigma^\ast \mathcal{L} = (\mathcal{L}_1^{-1} \otimes \mathcal{L}_3)^2 \otimes \mathcal{L}$, then
\[ \mathcal{M}_{2n} = \bigotimes_{i=1}^{n} \sigma^{(i-1)^\ast} (\mathcal{L}_1 \otimes \mathcal{L}_2) \]
Hence $2C_W$ is the twisted (Z-algebra) coordinate ring of $E$ associated to the data $(E, \mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{L}_2 \otimes \mathcal{L}_3)$, which is $D_V$.

The next result is standard
Lemma 7.9. Let $U = (E, \mathcal{L}_0, \mathcal{L}_1)$ and $U' = (E', \mathcal{L}_0', \mathcal{L}_1')$, and let $B_U$ and $B_{U'}$ be the corresponding twisted coordinate rings. Then $B_U \cong B_{U'}$ if and only if $E = E'$ and there is some automorphism of $E$ so that $\sigma^* \mathcal{L}_0 = \mathcal{L}_0'$ and $\sigma^* \mathcal{L}_1 = \mathcal{L}_1'$.

Proof. The proof of this relies on a standard trick which goes back at least to Bondal and Polishchuk [15]. Assume we have an algebra $B_U$, then we may recover geometric data from this algebra. We have that $B_U$ has $B_{U,1} = V$ a dimension 4 vector space. We have that the kernel of the multiplication map

$$B_{U,1} \otimes B_{U,1} \to B_{U,2}$$

determines a subscheme of $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$. This subscheme has two projections onto $\mathbb{P}(V^*)$. The two pullbacks of $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ to $E$ determine a pair of line bundles $\mathcal{K}_0$ and $\mathcal{K}_1$ on $E$, which are equal to $\mathcal{L}_0$ and $\mathcal{L}_1$ up to automorphism of $E$.

The following result then follows without much difficulty from the propositions above.

Proposition 7.10. Let $V = (E', \mathcal{L}'_1, \mathcal{L}'_2)$ Let $A_V$ be a generic quadratic Sklyanin algebra of global dimension 4. Assume that $\omega$ is a central quadratic element of $A_{V,2}$ so that $R_\omega = A_V/\langle \omega \rangle$ and $R_\omega = 2Q_W$ for some data $W = (E, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$. Then,

$$E' = E, \quad \mathcal{L}'_1 = \mathcal{L}_1 \otimes \mathcal{L}_2, \quad \mathcal{L}'_2 = \mathcal{L}_2 \otimes \mathcal{L}_3.$$

Remark 7.11. According to Van den Bergh [95] Section 6], for every noncommutative quadric $Q_W$ there is a four dimensional quadratic Sklyanin algebra $A_V$ and a central quadratic element $\omega$ of $A_V$ so that $R_\omega = A_V / \langle \omega \rangle$ satisfies $R_\omega$.

Remark 7.12. For a given pair of line bundles $\mathcal{K}_1$ and $\mathcal{K}_2$ on $E$, there is a 1-dimensional family of triples $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ so that $\mathcal{K}_1 = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $\mathcal{K}_2 = \mathcal{L}_2 \otimes \mathcal{L}_3$. Therefore, there is a 1 parameter family of noncommutative quadrics in each noncommutative $\mathbb{P}^3$. This coincides with the pencil of quadric hypersurfaces in quadratic Sklyanin algebras of global dimension 4 studied by Smith and Van den Bergh [85].

Then we have the following;

Proposition 7.13. Let $\omega_1$ and $\omega_2$ be generic linearly independent generators of the center of $A_{V,2}$ and let $Q_{V_1}$ and $Q_{V_2}$ be cubic Sklyanin algebras of dimension 3 so that $R_{\omega_1} \cong 2Q_{V_1}$ where $R_{\omega_1} = A_V / \langle \omega_1 \rangle$. Then there is a short exact sequence

$$0 \to A / \langle \omega_1, \omega_2 \rangle \to R_{\omega_1} \times R_{\omega_2} \to A_V / \langle \omega_1, \omega_2 \rangle = B_V \to 0$$

of noncommutative algebras where $q_i$ is the natural quotient map and $B_V$ is the twisted coordinate ring of $E$ associated to $V$.

Therefore, since $\omega_1 \omega_2$ is in $A_{V,4}$, the structure of $A$ in lower degrees can be determined from the structure of $R_{\omega_1}$ and $R_{\omega_2}$ and the maps $q_1$ and $q_2$. The information in $R_{\omega_1}$ and $R_{\omega_2}$ is also contained in that of $Q_{V_1}$ and $Q_{V_2}$, which comes from the exceptional collection

$$\{ \pi P_{1,Q_{V_j}}, \pi P_{2,Q_{V_j}}, \pi P_{3,Q_{V_j}}, \pi P_{4,Q_{V_j}} \}.$$

Our goal now is to describe the categorical implications of Proposition 7.13. According to Van den Bergh, there is a functor $\text{Res} : \text{Gr}(Q_j) \to \text{Gr}(2Q_j)$ which sends a module $\bigoplus_{i \in \mathbb{Z}} M_i$ to $\bigoplus_{i \in \mathbb{Z}} M_{2i}$ and which is an equivalence of categories. In particular, the projective modules $P_{i,Q_j}$ map to
The twisted coordinate ring $P_{i,2}Q_j$, and $\text{hom}_{\text{qgr}(C)}(\pi P_{i,2}Q_j, \pi P_{k,2}Q_l) = (2Q_j)_{2i,2k}$. The following theorem says that one can reconstruct the category $D^b(\text{qgr}(A_V))$ as the category of global sections of a perverse schobers constructed from the data of an elliptic curve along with line bundles satisfying certain conditions. Specifically, we will take pairs of noncommutative quadrics $R_{\omega_1}$ and $R_{\omega_2}$ represented as quotients of $A_V$ by degree 2 central elements $\omega_1$ and $\omega_2$. Then if $\pi P_{i,A_V}$ in $\text{qgr}(A_V)$ are the standard projective generators and $f_j : A_V \to R_{\omega_j}$ is the quotient map, the derived pullback functor satisfies $\mathbb{L}f_j^*(\pi P_{i,A_V}) = \pi P_{i,R_{\omega_j}}$. Therefore if $q_j : R_{\omega_j} \to B$ is the quotient map as before, then we have $\mathbb{L}f_1^*\mathbb{L}q_1^*(\pi P_{i,R_{\omega_1}}) = \mathbb{L}f_2^*\mathbb{L}q_2^*(\pi P_{i,R_{\omega_2}})$ so that there are objects $c_i = (\mathbb{L}q_1^*\pi P_{i,R_{\omega_1}}, \mathbb{L}q_2^*\pi P_{i,R_{\omega_2}}, \text{id})$ in $D_{dg}^b(\text{qgr}(Q_1)) \otimes_{D_{dg}^b(\text{qgr}(B))} D_{dg}^b(\text{qgr}(Q_2))$.

**Theorem 7.14.** Let $\omega_1$ and $\omega_2$ be generic linearly independent generators of the center of $A_{V,2}$ and let $Q_1$ and $Q_2$ be cubic Sklyanin algebras of dimension 3 so that $\tilde{R}_{\omega_1} \cong 2Q_1$. Let $c_i$ be the objects in $D = D_{dg}^b(\text{qgr}(Q_1)) \otimes_{D_{dg}^b(\text{qgr}(B))} D_{dg}^b(\text{qgr}(Q_2))$ defined above. The directed dg category $D_{C}^\rightarrow$ corresponding to $C = \{c_0, c_1, c_2, c_3\}$ satisfies $H^0 \text{Tw} D_{C}^\rightarrow \cong D^b(\text{qgr}(A_V))$.

**Proof.** The category $D^b(\text{qgr}(A_V))$ admits a full, strong exceptional collection of objects $\pi P_{i,A_V}$ for $i = 0, 1, 2, 3$, according to [59]. The endomorphism algebra $E$ of the direct sum of these objects is isomorphic to $\bigoplus_{0 \leq i \leq j \leq 3} A_{V,i,j}$. Therefore, by a theorem of Bondal [10], $D^b(\text{qgr}(A_V)) \cong D^b(\text{rep}(E_V))$. The goal now is to show that the homomorphisms between $c_i$ and $c_j$ in $D_{C}^\rightarrow$ recover $E_V$.

The objects $c_i$ satisfy $\text{hom}_{D_{C}^\rightarrow}(c_i, c_i) = k \cdot \text{id}_{c_i}$ and $\text{hom}_{D_{C}^\rightarrow}(c_i, c_j) = 0$ if $j < i$. Now, $\text{hom}_{D_{C}^\rightarrow}(c_i, c_j) = \text{hom}_{D}(c_i, c_j)$ for $j > i$, which is the cone of the morphism

$$\text{hom}_{D_{dg}^b(\text{qgr}(Q_1))}(\pi P_{i,Q_1}, \pi P_{j,Q_1}) \otimes \text{hom}_{D_{dg}^b(\text{qgr}(Q_2))}(\pi P_{i,Q_2}, \pi P_{j,Q_2}) \xrightarrow{\mathbb{L}q_1^* - \mathbb{L}q_2^*} \text{hom}_{D_{dg}^b(\text{qgr}(B))}(\pi P_{i,B}, \pi P_{j,B})$$

We have that $\text{Ext}^k(\pi P_{i,Q_1}, \pi P_{j,Q_1}) = 0$ for $k \neq 0$ and any pair $i, j$. Therefore the cohomology sequence associated to (20), along with the vanishing of higher Ext groups says that $H^0 \text{hom}_{D_{C}^\rightarrow}(c_i, c_j)$ is the kernel of

$$\text{hom}_{\text{qgr}(Q_1)}(\pi P_{2i,Q_1}, \pi P_{2j,Q_1}) \otimes \text{hom}_{\text{qgr}(Q_2)}(\pi P_{2i,Q_2}, \pi P_{2j,Q_2}) \xrightarrow{q_1^* - q_2^*} \text{hom}_{\text{qgr}(B)}(\pi P_{2i,B}, \pi P_{2j,B})$$

According to Proposition 7.5, the maps $q_k^*$ agree with the quotient map from $Q_k$ to $B$, we may apply Proposition 7.7 to see that $\text{Hom}_{H^0 D \to}(c_i, c_j)$ is isomorphic to $A_{i,j}$ if $0 \leq i \leq j \leq 3$. Furthermore, looking at the definition of composition of homomorphisms in the homotopy fiber product, it follows that it agrees with composition of homomorphisms in $\text{qgr}(Q_{V_1}) \times \text{qgr}(Q_{V_2})$, which in turn is multiplication in $Q_{1,i,j} \times Q_{2,i,j}$. Therefore, the endomorphism algebra of $\bigoplus_{i=0}^3 c_i$ agrees with that of $\bigoplus_{i=0}^3 \pi P_{i,A_V}$. It follows then that $H^0 \text{Tw} D_{C}^\rightarrow \cong D^b(\text{rep}(E))$ is equivalent to $D^b(\text{qgr}(A_V))$. \[\square\]

**Remark 7.15.** This approach will generally fail, as was noted by Tate and Van den Bergh [90], since it depends upon the existence of a regular sequence of normal elements in $A$ which cuts out the point scheme of $A$. For all $n$ the $n$-dimensional Sklyanin algebra admits as a quotient the twisted coordinate ring $B$ of an elliptic curve corresponding to the line bundles used to construct $A$. 
however in dimension greater than 4, there is no regular sequence of elements in \( A \) whose quotient algebra is \( B \).

**Remark 7.16.** As in Example 6.21 and Remark 6.22 we should view the fiber of the perverse schober that we allude to in Theorem 7.14 as the category of global sections of a schober itself.

### 7.4. Relation to quantization

A Poisson bracket on \( \mathbb{P}(V^*) \) is an element \( \wedge^2 V \) which satisfies the condition that

\[
\{x, yz\} = y\{x, z\} + z\{x, y\}
\]

for any \( x, y, z \in V \). Such a pairing determines a section of \( H^0(\mathbb{P}(V^*), \wedge^2 T_{\mathbb{P}(V^*)}) \). As in Equation 16 a noncommutative deformation of \( \mathbb{P}(V^*) \) can be viewed as a deformation of the subspace \( \wedge^2 V \) inside of \( V \otimes V \). A family of such relations determines a family of products \( \star_h \) on the graded vector space underlying \( \mathbb{C}[V] \) parametrized by \( h \) which deforms to the product structure on the homogeneous coordinate ring of \( \mathbb{P}(V^*) \) when \( h = 0 \). This determines a Poisson bracket on \( \mathbb{P}(V^*) \) by the formula,

\[
\{f, g\} = \left. \frac{d}{dh} \right|_{h=0} (f \star_h g + g \star_h f).
\]

This correspondence identifies the tangent space of noncommutative deformations of \( \mathbb{P}(V^*) \) with the space of Poisson brackets on \( \mathbb{P}(V^*) \).

By work of Pym, it is known that the space of Poisson structures on \( \mathbb{P}^3 \) has six irreducible components \[70\]. Two of these were discussed by Polishchuk \[66\], and are Poisson structures which admit Poisson divisors \( D_\omega \). These are divisors in \( \mathbb{P}^3 \) so that the Poisson bracket \( \omega \) induces a Poisson bracket on \( D_\omega \). These Poisson divisors are either

1. A union of two quadric surfaces in \( \mathbb{P}^3 \),
2. A union of a hyperplane and a cubic hypersurface in \( \mathbb{P}^3 \).

According to Pym \[70\], the noncommutative deformation of \( \mathbb{P}^3 \) associated to the first is the quadratic Sklyanin algebra of global dimension 4, and the second is a central extension of the 3-dimensional quadric Artin-Schelter algebra \[43\]. The existence of a Poisson divisor implies that the corresponding noncommutative deformation comes with a corresponding noncommutative subscheme.

In the first case, the Poisson divisor explains why we are able to find a pair of noncommutative quadrics in the elliptic Sklyanin algebra. The second case should be very similar. It seems like the literature on noncommutative cubic surfaces is more sparse than the literature on noncommutative quadrics \[95\] but we expect that Theorem 7.14 has an analogue in the case of the central extension of the three dimensional quadratic Sklyanin algebra. In particular, we expect that if we take an appropriate noncommutative cubic and noncommutative \( \mathbb{P}^2 \), then there’s a collection of spherical objects in their homotopy fiber product whose corresponding directed category is equivalent to the derived category of noetherian graded right modules on the central extension of the quadratic Sklyanin algebra of dimension 3.

### 8. Mirror symmetry for \( \mathbb{P}^3 \) via perverse sheaves of categories

Here we will give a simple proof of a version of mirror symmetry for \( \mathbb{P}^3 \) that makes explicit the role that perverse sheaves of categories play. Except for several geometric computations all of the symplectic work needed to prove mirror symmetry for \( \mathbb{P}^3 \), and similar examples, is contained in either \[76\] or work of Lekili and Polishchuk \[46\] and Lekili and Perutz \[45\] on mirror symmetry for the elliptic curve. Seidel’s work shows that computation of directed Fukaya categories of exact
Lefschetz fibrations can be reduced to Fukaya category computations in the fibers of the fibration, which allows us to reduce our computation of the Fukaya category of the Landau-Ginzburg model of \( \mathbb{P}^3 \) to computations in the exact Fukaya category of an \( n \)-punctured elliptic curve. Lekili and Polishchuk [46] show that the exact Fukaya category of a punctured elliptic curve corresponds to the category of perfect complexes on chain of \( n \) rational curves.

We use work in the previous sections and the geometric observation about the fibers of the LG model of \( \mathbb{P}^3 \) to see that the directed Fukaya category of the LG model of \( \mathbb{P}^3 \) is computed using data that is exactly the data appearing in Example [6.21].

**Remark 8.1.** The main result of this section is known; a sketch appears in [6], and a proof in [26]. Our purpose is to illustrate that the constructions in the previous sections have direct analogues in the context of Fukaya categories. Our approach is similar to [26].

### 8.1. Fukaya categories

We will begin with a symplectic manifold \( M \) of dimension \( 2n \) equipped with a symplectic form \( \omega \) and a compatible almost complex structure \( J \) and so that \( 2c_1(M) = 0 \). Let \( L \) be a Lagrangian submanifold of \( M \), then \( \omega \) and \( J \) allow us to define a grading \( \alpha \) on \( L \). We will denote a grading on \( L \) by \( \alpha \). We will also equip our objects with a spin structure \( P \). In the case where \( \dim \mathbb{R} \mathbb{M} = 2 \), we will deal exclusively with Lagrangians which are diffeomorphic to \( S^1 \) and whose spin structures are nontrivial. This nontrivial spin structure can be kept track of by marking a point on \( L \). The triple \( L^\# = (L, \alpha, P) \) is called a Lagrangian brane.

This data allows us to define Floer intersection complexes associated to pairs of Lagrangian branes \( L_1^\# \) and \( L_2^\# \). These complexes are given by \( \mathbb{Z} \)-graded vector spaces over the Novikov field with the points \( L_1 \cap L_2 \), assuming \( L_1 \) and \( L_2 \) meet transversally. Furthermore, there is a (not necessarily unital) \( A_\infty \) product on Lagrangian branes

\[
\text{CF}(L_{n-1}^\#, L_n^\#) \otimes \cdots \otimes \text{CF}(L_1^\#, L_2^\#) \to \text{CF}(L_1^\#, L_n^\#).
\]

The Fukaya \( A_\infty \) category \( \mathcal{F}(M) \) of \( M \) is the category whose objects are given by Lagrangian branes up to Hamiltonian isotopy and so that \( \text{hom}_{\mathcal{F}(M)}(L^\#, L^\#) = \text{CF}(L_0^\#, L_1^\#) \). Composition comes from the product in Equation (22). Dependence on \( \omega \) is suppressed in our notation. In the case where \( \omega \) is an exact symplectic form, one can perform these computations over \( \mathbb{C} \) instead of the Novikov field.

In all of our computations, \( \omega \) will be exact. In this case, we will deal only with exact Lagrangians, that is, Lagrangians \( i_L : L \hookrightarrow M \) that satisfy the condition \( \int_L i_L^* \omega = 0 \).

### 8.2. The elliptic curve

Lekili and Polishchuk have shown that there is a quasi equivalence between \( \text{Perf}(G_4) \) and \( \Pi \text{Tw}\mathcal{F}(\mathbb{T}_0) \) where \( \mathbb{T}_0 \) is the 2-torus with four points removed and the corresponding exact symplectic form and \( G_4 \) denotes the union of four copies of \( \mathbb{P}^1 \) meeting transversally and whose dual intersection complex is a circle. In other words there are four rational curves \( R_i \), so that \( G_4 = R_1 \cup R_2 \cup R_3 \cup R_4 \) so that \( R_i \) meets \( R_j \) transversally in a single point if and only if \( j = i - 1, i + 1 \mod 4 \).

Let us denote by \( p_i \) a point on \( R_i \) which is not a singular point of \( G_4 \). The objects

\[
\mathcal{O}_{G_4}, \mathcal{O}_{p_1}, \mathcal{O}_{p_2}, \mathcal{O}_{p_3}, \mathcal{O}_{p_4}
\]

split generate \( \text{Perf}(G_4) \). In [46, Corollary 3.4.1] it is shown that there is a quasi equivalence between \( \Pi \text{Tw}\mathcal{F}(\mathbb{T}_0) \) and \( \text{Perf}(G_4) \), so that the objects in Equation (22) correspond to the exact Lagrangians in Figure 9. Here we have identified the elliptic curve \( \mathbb{R}^2/\mathbb{Z}^2 \) with the the fundamental domain \([0,1] \times [0,1] \) in its universal cover, \( \mathbb{R}^2 \). The Lagrangian branes drawn vertically in the figure above are denoted \( L_1^\#, L_2^\#, L_3^\# \) and \( L_4^\# \), in that order, and the horizontal brane is denoted \( L_0^\# \).
According to [46], the Lagrangian corresponding to the horizontal line is sent to \( \mathcal{O}_G \) in \( \text{Perf}(G_4) \) and the vertical lines are sent to \( \mathcal{O}_{p_i} \) (given in their cyclic orderings on both sides). The spherical twist functor along a Lagrangian brane \( L^\# \) corresponds, on the level of underlying Lagrangian submanifolds, to a Dehn twist \( \tau_L \) and preserves nontrivial spin structure. We define objects:

\[
\begin{align*}
m_0 &= L_0^\# , & m_1 &= \tau_{L_3}^\# \tau_{L_1}^\# L_0^\# , & m_2 &= \tau_{L_4}^\# \tau_{L_2}^\# , & m_3 &= \tau_{L_4}^\# \tau_{L_3}^\# \tau_{L_2}^\# \tau_{L_1}^\# L_0^\# ,
\end{align*}
\]

which are drawn in in Figure 10. There the blue line denotes \( L_1 \) the red line denotes \( L_2 \), the green line denotes \( L_3 \) and the purple denotes \( L_4 \). Under the correspondence given by [46], these objects correspond to repeated spherical twists of \( \mathcal{O}_G \) by \( \mathcal{O}_{p_i} \) for some sequence of points \( p_i \). Spherical twist by \( \mathcal{O}_{p_i} \) is the tensor product with \( \mathcal{O}_G(p_i) \) therefore under the mirror correspondence of [46], we have,

\[
\begin{align*}
m_0 &\mapsto \mathcal{O}_G, & m_1 &\mapsto \mathcal{O}_G(p_1 + p_3), & m_2 &\mapsto \mathcal{O}_G(p_2 + p_4), & m_3 &\mapsto \mathcal{O}_G(p_1 + p_2 + p_3 + p_3).
\end{align*}
\]
There is a natural embedding $\iota : G_4 \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ as the toric boundary divisor. Under this embedding, one sees that
\[
\mathcal{O}_{G_4} \cong \mathcal{L}^*\mathcal{O}_Q, \quad \mathcal{O}_{G_4}(p_1 + p_3) \cong \mathcal{L}^*\mathcal{O}_Q(1,0), \quad \mathcal{O}_{G_4}(p_2 + p_4) \cong \mathcal{L}^*\mathcal{O}_Q(0,1), \quad \mathcal{O}_{G_4}(p_1 + p_2 + p_3 + p_4) \cong \mathcal{L}^*\mathcal{O}_Q(1,1).
\]
Denote these objects $n_0, n_1, n_2$ and $n_3$ respectively. The following result then follows;

**Proposition 8.2.** If $U = \{m_0, m_1, m_2, m_3\}$ and $W = \{n_0, n_1, n_2, n_3\}$. The directed categories $(\text{mod}_{\mathcal{F}^{(T_0)\mathbb{C}}_U})_U$ and $\text{Perf}(G_4)_W$ are quasi equivalent dg categories. Therefore, $\text{Tw}(\text{mod}_{\mathcal{F}^{(T_0)\mathbb{C}}_U})_U$ is quasi equivalent to $D^b_{\text{dg}}(Q)$ (see Example 5.8).

### 8.3. Lefschetz fibrations

We will now begin to study exact Lefschetz fibrations, following Seidel. We refer to [76] for the precise definition, since it is technical and we do not make use of the details. Roughly, an exact Lefschetz fibration is a quadruple of objects $(E, \pi, \theta, J)$ for $E$ a symplectic manifold, $\theta$ a 1-form so that $d\theta = \omega$ is a symplectic form on $E$ and $J$ is a compatible almost complex structure. We require that $\pi : E \to D$ be a map to the disc $D$, and which has connected fibers and at worst holomorphic Morse critical points. Furthermore, Seidel imposes a set of conditions regarding the behaviour of $\pi$ near the boundary in every fiber.

**Remark 8.3.** Seidel requires that there is at most one singular point in each fiber of $\pi$, though according to [76, 15d, paragraph 1], this is purely for the sake of notational convenience. The example that we are now interested in will have a single fiber with two holomorphic Morse critical points.

Let $\Sigma = \{p_1, \ldots, p_k\}$ be the set of critical values of $\pi$ and let $s$ be a point outside of $\Sigma$. Let $M_s = \pi^{-1}(s)$. We may choose a set of nonintersecting paths $\gamma = \{\gamma_1, \ldots, \gamma_k\}$ from $s$ to the points $p_1, \ldots, p_k$, which are cyclically oriented counterclockwise at $s$. Along each $\gamma_i$, one may use $\omega$ to produce a Lagrangian thimble which is a Lagrangian $S^{n-1}$ in the fiber over a point in $\gamma_i$ and which is a point in the fiber over $p_i$. Therefore, each $\gamma_i$ determines a Lagrangian vanishing sphere $L_i$ in $M_s$. If $\dim_{\mathbb{R}} M \neq 4$, then the only spin structure on $L_i$ is trivial. In the case where $\dim_{\mathbb{R}} M = 4$, we choose the nontrivial spin structure on $L_i$. Making a choice of grading, we obtain a collection of Lagrangian branes $L_i^\#$. We define the category $\mathcal{F}(E, \pi, \gamma)$ to be the directed category associated to the collection $V_{\gamma} = \{L_1^\#, \ldots, L_k^\#\}$.

### 8.4. Mirror symmetry for quadrics in $\mathbb{P}^3$

Here we discuss a well known fact (see [26]), that homological mirror symmetry holds for quadric surfaces $Q$ in $\mathbb{P}^3$. We will begin with the following model for the LG mirror of $Q$. We take
\[
\pi_Q(x, y) = x + \frac{1}{x} + y + \frac{1}{y} : E_Q = (\mathbb{C}^2)^* \to \mathbb{C}
\]
In order to apply the machinery developed by Seidel [76] to compute $\mathcal{F}(E_Q, \pi_Q, \gamma)$, we need to know that this is an exact Lefschetz fibration.

**Proposition 8.4.** The LG model $(\mathbb{C}^*)^2$ equipped with the function $\pi_Q$ is an exact Lefschetz fibration with the standard exact symplectic form.

**Proof.** In [77, Example 6.1], Seidel notes that if a Laurent polynomial $f$ is nondegenerate with respect to its toric boundary, then the pair $((\mathbb{C}^*)^n, \pi_Q)$ is an exact Lefschetz fibration. The Laurent polynomial $\pi_Q$ has this property, as can be checked directly, therefore that it is an exact Lefschetz fibration follows. \qed
For the sake of future reference we can record the following fact. The map $\pi_Q$ is a fibration whose generic fiber is $T_0$, the 2-torus with four points removed. This fibration is birational to an elliptic fibration over $\mathbb{C}$, written in Weierstrass form as

$$(24) \quad y^2 = x^3 + (-1/3)(t^4 - 16t^2 + 16)xz^4 + \frac{2}{27}(t^2 - 8)(t^2 - 16t^2 - 8)z^6$$

as a hypersurface in $\mathbb{C} \times \mathbb{P}(1, 2, 3)$. There is a simple node in this surface in the fiber over $t = 0$ which may be resolved by blowing up once. Call the total space of this $E$. Furthermore, one may deduce from [63] and Kodaira’s classification of elliptic surfaces that this fibration has group of sections which is isomorphic to $\mathbb{Z}/4$. The torus $(\mathbb{C}^\times)^2$ is the complement of these four sections in $\overline{E}$, and the potential $\pi_Q$ is just the projection onto $\mathbb{C}$ of the resulting quasiprojective variety.

According to Proposition 8.4, we may apply the results of [76, Chapter III] to this situation. First, we will compute the classes of the Lagrangian vanishing cycles $L_i^#$ of $\pi_Q$ in $\mathcal{F}(M_s)$. Here $M_s = T_0$.

It will then be enough to compute the topological classes of the vanishing cycles in $\mathcal{F}(T_0)$. This computation also appears in [5, Proposition 5.1]. Projecting onto the $x$ plane defines a degree 2 map from each fiber of $\pi$ to $\mathbb{C}^\times$. Specifically, we have that this map ramifies over the four points

$$(25) \quad (x^2 - \lambda x + 2x + 1)(x^2 - \lambda x - 2x + 1)$$

We choose the collection of paths $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ which are straight line paths from $\sqrt{-1}$ to $4, 0, 0, -4$ respectively. More generally, we can let $\gamma_2, \gamma_3$ curve slightly so that they are distinct paths. The Lagrangians $L_1^#$ and $L_4^#$ are obtained canonically from the paths $\gamma_1$ and $\gamma_4$. To determine the Lagrangian branes $L_2^#$ and $L_3^#$, we need to make a choice of vanishing cycle along each path. The vanishing cycles associated to the two critical points in the fiber over 0 do not intersect, so the choice of order does not affect the resulting category.

At both $\lambda = 4$ and $\lambda = -4$, a pair of the points in Equation 25 come together. The paths in Figure 11 contract at the points 4, 0 and -4 respectively. These paths can then be lifted to the double cover ramified along the four points in the diagram to give the vanishing cycles in $T_0$ depicted in Figure 14.
We note that Figure 14 is equivalent to Figure 10 after a coordinate change. Therefore, by Proposition 8.2 the derived directed category of objects in 𝒫(𝑻₀) associated to the vanishing cycles of 𝜋𝑄 is equivalent to 𝒫_b(𝑄).

**Proposition 8.5.** There is a quasi equivalence between 𝑇wstring(E𝑍, 𝜋𝑍, 𝜈) and 𝒫_b dg(𝑄) so that the natural restriction functors to 𝑇wstring(𝐹(𝑻₀)) agrees with ℴf*: 𝒫_b dg(𝑄) → Perf(G₄) for f: G₄ ↪ Q the embedding map.

**8.5. Double covers and the mirror of a degenerate quartic.** Let 𝑃 be a double cover of (ℂ×)² ramified along a smooth fiber 𝜋⁻¹(𝜇) of 𝜋𝑄 for some 𝜇 /∈ {4, 0, −4}. Then 𝑃 is naturally equipped with the structure of an exact Lefschetz fibration 𝜋𝑍: 𝑃 → ℂ where the new base ℂ is identified with a double cover of the target of 𝜋𝑄 ramified over 𝜇.

We will show that 𝑇wstring(𝐹(𝑃)) is quasi equivalent to the category Perf(Z) where Z is the quartic

\[ x^2 y^2 - z^2 w^2 = 0. \]

We may present 𝑃 as a Zariski open piece of a Weierstrass fibration, just as we did the LG model of 𝑄 in Equation 25. Let s be the coordinate on the image of 𝜋𝑍. Then we may represent 𝑃 in terms of the hypersurface

\[
\begin{align*}
y^2 &= x^3 + (-1/3)(s(s - \lambda))^4 - 16(s(s - \lambda))^2 + 16)x \\
&\quad + (2/27)((s(s - \lambda))^4 - 8)((s(s - \lambda))^2 - 16(s(s - \lambda))^2 - 8)
\end{align*}
\]

in ℂ × ℂP(1, 2, 3).

**Remark 8.6.** This hypersurface can be compactified to the mirror quartic. □

This fibration has a pair of A₁ singularities in fibers over s = 0 and s = λ and a Mordell-Weil group isomorphic to ℤ/4 (for a generic choice of μ). Blowing up and resolving the two singularities and then excising all sections produces a variety biregular to the double cover of (ℂ×)² ramified along a smooth fiber of 𝜋𝑄 and so that the projection onto ℂ is 𝜋𝑍. We may choose λ so that this map ramifies over μ. According to Seidel [76 Chapter III], we may equip (𝑃, 𝜋𝑍) with an exact symplectic structure so as to make it an exact Lefschetz fibration. Let s₀ be the preimage of μ under this map and choose it to be our basepoint in the base of 𝜋𝑍.
Seidel [76] Corollary 18.13] has proven that if \( E \) is an exact symplectic manifold with an exact Lefschetz fibration \( \pi \), and an ordered basis of paths \( \gamma \) from a base point \( s \in \mathbb{C} \) to each critical point of \( \pi \), then there is a quasi full and faithful embedding of \( A_\infty \) categories

\[
(28) \quad \text{Tw} \mathcal{F}(E) \hookrightarrow \text{Tw} \mathcal{F}(E, \pi, \gamma).
\]

The category \( \mathcal{F}(E) \), as usual, denotes the exact Fukaya category of \( E \).

In fact, Seidel shows something much more precise. Let \( \eta \) be a matching path \( p \) between two critical values \( q_i \) and \( q_{i+1} \) of \( \pi \), and let \( a \) be the corresponding exact Lagrangian matching sphere. Precisely, \( \eta \) is a path between two critical points of \( \pi_Z \) along which the Lagrangian vanishing cycles at each end coincide and patch together to form a Lagrangian sphere in \( E \). Then under Seidel’s embedding, the Lagrangian matching sphere \( a \) can be represented as a cone. We may choose a basis of paths \( \gamma \) of \( \pi \) so that there are adjacent paths \( \gamma_i \) and \( \gamma_{i+1} \) and so that the matching path sits in a triangle with edges \( p, \gamma_i \) and \( \gamma_{i+1} \) and with no critical points on its interior. Since \( p \) is a matching path, the objects \( L_{\#}^i \) and \( L_{\#}^{i+1} \) in \( \mathcal{F}(M_s) \) are isomorphic, which gives a closed degree 0 homomorphism \( r \) between them in \( \mathcal{F}(E, \pi, \gamma) \). Then under the functor in Equation \( 28 \) the matching sphere \( a \) is represented by

\[
(29) \quad \text{Cone}(L_{\#}^{i} \xrightarrow{r} L_{\#}^{i+1}).
\]

In our case, there is a distinguished class of matching paths. Since \( E_Z \) is a double cover of \( E_Q \) along the smooth fiber \( \pi_Q^{-1}(\mu) \), we obtain matching spheres in \( E_Z \) that Seidel calls Lagrangians of type (B). If \( \mu \) is the point over which the ramification divisor of the cover \( E_Z \) lives, let us choose an ordered basis of paths \( \gamma \) from \( \lambda \) to \( 4, 0, -4 \) whose vanishing cycles are the same as those in the previous section. If \( \mu = \sqrt{-1} \), then this can be the basis \( \gamma \) in the previous section.

Under the double cover of the base of \( \pi_Q \), these vanishing paths lift to a basis of vanishing paths for \( \pi_Z \) to the base point \( s_0 \). If we choose a branch cut in \( \mathbb{C} \) from \( \mu \) to \( \infty \) not intersecting \( 4, 0, -4 \), then get two distinguished preimages \( \gamma_{i, \pm} \) of each \( \gamma_i \in \gamma \) in such a way that if

\[
\gamma_+ = \{ \gamma_{1, +}, \ldots, \gamma_{4, +} \}, \quad \gamma_- = \{ \gamma_{1, -}, \ldots, \gamma_{4, -} \}
\]

then \( \gamma_{\pm} = \gamma_+ \cup \gamma_- \) is a counterclockwise cyclically ordered basis of vanishing paths for \( \pi_Z \) from the base point \( s_0 \). The paths \( \gamma_{i,+} \cup \gamma_{i,-} \) form matching paths between the two preimages \( p_{i, \pm}^{\pm} \) of \{4, 0, -4\}. Therefore, in \( E_Z \), there are four distinguished Lagrangian matching spheres (equipped with brane structures) \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) in \( \mathcal{F}(E_Z) \) for any chosen set of Lagrangian vanishing thimbles on \( E_Q \).

We also have pairs of vanishing thimbles associated to each path \( \gamma_i \) in the definition of the category \( \mathcal{F}(E_Z, \pi_Z, \gamma_{\pm}) \). Let \( \{ \gamma_i \}_{i=1}^{4} \) be the basis of paths from the previous section. Let \( L_{i, \pm}^{\#} \) be Lagrangian branes in \( \mathcal{F}(T_0) \) which correspond to the vanishing paths \( \gamma_{i, \pm} \). Let

\[
V_+ = \{ L_{1, +}^{\#}, \ldots, L_{4, +}^{\#} \}, \quad V_- = \{ L_{1, -}^{\#}, \ldots, L_{4, -}^{\#} \}.
\]

We then have two categories \( \mathcal{F}(T_0)_{V_+} \) and \( \mathcal{F}(T_0)_{V_-} \) which are equivalent, and both have categories of twisted complexes which are quasi equivalent to \( D_{b}^{\text{rg}}(Q) \). The objects \( \Delta_i^{\#} \) correspond to

\[
\text{Cone}(L_{i,+}^{\#} \xrightarrow{\text{id}} L_{i,-}^{\#})
\]

in \( \text{Tw} \mathcal{F}(E_Z, \pi_Z, \gamma_{\pm}) \).

Looking carefully at the proof of [76 Lemma 18.15], we obtain the following result.
Lemma 8.7. Let $T_{\Delta_i^\#}$ be the spherical twist functors on $\mathcal{D}^\pi(\mathcal{F}(E_Z))$ associated to $\Delta_i^\#$. Then
\[ (T_{\Delta_1^\#} \ldots T_{\Delta_4^\#})^2 = [2]. \]
Therefore, $\Delta_1^\#, \ldots, \Delta_4^\#$ split generate $\mathcal{D}^\pi \mathcal{F}(E_Z)$.

Proof. According to Seidel, the action of $(T_{\Delta_1^\#} \ldots T_{\Delta_4^\#})^2$ agrees with the monodromy automorphism of $E_Z$ obtained by letting the ramification fiber travel in a large circle around $C$ twice. This automorphism is the identity diffeomorphism of $E_Z$, but acts on gradings by $\alpha_i \mapsto \alpha_i - 2$, and on spin structure by $((- \otimes \mathcal{E}))^2$ for some spin structure $\mathcal{E}$. Therefore the action on spin structures is trivial. One notes that by the definition of the shift functor acting on Lagrangian branes, the result follows.

Then according to [76 Corollary 5.8] this implies that $\Delta_i^\#$ are split generators for the category $\mathcal{D}^\pi \mathcal{F}(E_Z)$.

It follows from Lemma [8.7] that $\mathcal{F}(E_Z)$ is quasi equivalent to a full subcategory of $\mathcal{F}(E_Z, \pi_Z, \gamma_{\pm})$ split generated by matching spheres. The first step towards the mirror symmetry statement alluded to at the beginning of this section is to identify $\mathcal{F}(E_Z, \pi_Z, \gamma_{\pm})$ with a category that we understand in terms of gluing constructions.

Proposition 8.8. The category $\text{Tw} \mathcal{F}(E_Z, \pi_Z, \gamma_{\pm})$ is quasi equivalent to a subcategory of $D^b_{\text{dg}}(Q) \times_F D^b_{\text{dg}}(Q)$ where $F$ is the bimodule
\[ b \otimes a \mapsto \text{hom}_{\text{perf}(G_4)}(F(a), F(b)). \]

Proof. Let us take a set of paths $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$ in $D$ from a given base point to the critical points of $\pi$. Under the double cover $g : \tilde{D} \to D$, these paths lift to eight non-intersecting paths $\gamma_{1,+}, \ldots, \gamma_{4,+}$ and $\gamma_{1,-}, \ldots, \gamma_{4,-}$ so that $\gamma_{1,\pm}$ are lifts of $\gamma_i$. By choosing a branch cut in $D$ this can be done so that these paths are cyclically ordered $\gamma_{1,+}, \ldots, \gamma_{4,+}, \gamma_{1,-}, \ldots, \gamma_{4,-}$. Therefore, the vanishing cycles $V_{n,\pm}$ along the paths $\gamma_{i,\pm}$ agree, and correspond to the same Lagrangian branes in $\mathcal{F}(\mathbb{T}_0)$.

Specifically, we can use the same arguments as in the proof of Proposition [8.2] to see that $\mathcal{F}(E_Z, \pi_Z, \{V_{i,\pm}^\#\}_{i=1}^4)$ is quasi equivalent to the directed category $\text{perf}(G_4)_{\mathcal{U}}$ where
\[ U = \{n_0, n_1, n_2, n_3, n_0, n_1, n_2, n_3\}. \]

By construction then we have that
\[ \text{perf}(G_4)_{\mathcal{U}} \cong \text{perf}(G_4)_{W^+} \cup_G \text{perf}(G_4)_{W^-}, \]
where $W_{\pm} = \{n_0, n_1, n_2, n_3\}$. If $\tilde{n}_{i,\pm}^\pm$ is object in $D^b_{\text{dg}}(G_4)_{W_{\pm}}$ then $G$ is the functor sending a pair of objects $(\tilde{n}_{i,\pm}^+)$ to $\text{hom}_{\text{perf}(G_4)}(n_j, n_i)$. Therefore, by Proposition [2.22] along with the fact that $\text{Tw} \text{perf}(G_4)_{W_{\pm}} \cong D^b_{\text{dg}}(Q)$ shows that
\[ \text{Tw} \text{perf}(G_4)_{\mathcal{U}} \cong D^b_{\text{dg}}(Q) \times_F D^b_{\text{dg}}(Q) \]
as required.

There are now full and faithful quasi embeddings,
\[ \Pi \text{Tw} \mathcal{F}(E_Z) \subseteq \text{Tw} \mathcal{F}(E_Z, \pi_Z, \{V_{i,\pm}^\#\}_{i=1}^4) \subseteq D^b_{\text{dg}}(Q) \times_F D^b_{\text{dg}}(Q) \]
\[ \text{perf}(Z) \subseteq D^b_{\text{dg}}(Q) \times_{\text{perf}(G_4)} D^b_{\text{dg}}(Q) \subseteq D^b_{\text{dg}}(Q) \times_F D^b_{\text{dg}}(Q) \]
where the second line comes from Proposition \ref{1.25} and the first line comes from Proposition \ref{8.8}. Our claim now is that the images of these two embeddings coincide. According to Lemma \ref{8.7}, the objects $\Delta^\#_i$ split generate the category $\text{D}^\pi \mathcal{F}(E_Z)$. In $\text{Tw} \mathcal{F}(E_Z, \pi_Z, \gamma_\pm)$, they are represented by

$$\text{Cone}(L^\#_{i,+} \xrightarrow{\xi_i} L^\#_{i,-}).$$

The quasiisomorphism between $\text{Tw} \mathcal{F}(E_Q, \pi_Q, \gamma_\pm)$ and $\text{Perf}(Q)$ identifies the objects $L^\#_{i,+}$ with $\mathcal{L}_i$ (up to shift) where

$$\mathcal{L}_1 = \mathcal{O}_Q, \quad \mathcal{L}_2 = \mathcal{O}_Q(1,0), \quad \mathcal{L}_3 = \mathcal{O}_Q(0,1), \quad \mathcal{L}_4 = \mathcal{O}_Q(1,1).$$

In $\text{D}^b_{\text{dg}}(Q) \times_F \text{D}^b_{\text{dg}}(Q)$, the objects $\Delta^\#_i$ have image quasi isomorphic to $\text{Cone}((\mathcal{L}_i, 0, 0) \xrightarrow{(0,0,\text{id})} (0, \mathcal{L}_i, 0)) = (\mathcal{L}_i, \mathcal{L}_i, \text{id})$. Since $\text{id}$ is an isomorphism, this object corresponds to an object in $\text{D}^b_{\text{dg}}(Q) \times_{\text{Perf}(\mathbb{G}_a)}^h \text{D}^b_{\text{dg}}(Q)$ since the cone of $\text{id}$ is homotopically trivial. More precisely, looking at the proof of Proposition \ref{1.25} these are objects which come from pullbacks of line bundles on $Z$, hence are quasi isomorphic to objects in the image of the embedding $\text{Perf}(Z) \hookrightarrow \text{D}^b_{\text{dg}}(Q) \times_{\text{Perf}(\mathbb{G}_a)}^h \text{D}^b_{\text{dg}}(Q)$. Therefore, since $\text{Perf}(Z)$ is split-closed, it follows that $\Pi \text{Tw} \mathcal{F}(E_Z)$ is quasi equivalent to a subcategory of $\text{Perf}(Z)$.

In the proof of Theorem \ref{8.13}, we will show that there are matching spheres in $\mathcal{F}(E_Z)$ corresponding to $\mathcal{O}_Z, \mathcal{O}_Z(1)$ and $\mathcal{O}_Z(2)$. According to Orlov \cite{Orlov} Theorem 4, for any projective surface $X$ with ample line bundle $\mathcal{L}$, the bundles $\mathcal{O}_X, \mathcal{L}$ and $\mathcal{L}^2$ generate $\text{Perf}(Z)$. Therefore,

**Theorem 8.9.** There is an equivalence of categories,

$$\text{D}^\pi \mathcal{F}(E_Z) \cong \text{Perf}(Z).$$

**Proof.** We observed that $\text{D}^\pi \mathcal{F}(E_Z)$ is equivalent to a full subcategory of $\text{H}^0 \text{Perf}(Z)$ and that the image contains objects which split generate $\text{H}^0 \text{Perf}(Z)$. Since both categories are split-closed, this must be an equivalence of categories. \hfill $\square$

The most important point in this section that this identification gives us a very clean dictionary between objects in $\text{Perf}(Z)$ and $\text{D}^\pi \mathcal{F}(E_Z)$. Since $\text{D}^b(\text{coh} \mathbb{P}^3)$ can be recovered as the directed category associated to certain objects in $\text{Perf}(Z)$ and the directed Fukaya category of the mirror LG model is nothing but the directed category associated to certain objects in $\mathcal{F}(E_Z)$, homological mirror symmetry for $\mathbb{P}^3$ reduces to identifying vanishing cycles of the LG model of $\mathbb{P}^3$ in $\mathcal{F}(E_Z)$ with the corresponding objects in $\text{Perf}(Z)$ and showing that they are quasi equivalent the same under the isomorphism in Theorem \ref{8.9}.

**Remark 8.10.** This section should be reinterpreted as saying that the derived Fukaya category of $E_Z$ is naturally embedded into the category of global sections of the $K$-coordinatized perverse sheaf of categories associated to the graph made up of paths $\gamma_\pm$ with fiber at any point on an edge given by $\mathcal{F}(T_0)$ and spherical functors at each vertex $\Sigma$ given by the vanishing cycles at each point in $\Sigma$ at $s_0$. \hfill $\square$

### 8.6. The geometry of the LG model of $\mathbb{P}^3$

We begin with the standard LG mirror of $\mathbb{P}^3$ which is is given by the pair $(\mathbb{C}^\times)^3$ and the potential

$$\pi_{\mathbb{P}^3}(x, y, z) = x + y + z + \frac{1}{xyz} \quad (30)$$

Our first goal is to show that there is a partial compactification of $(\mathbb{C}^\times)^3$ to which this potential extends, so that the fibers of this new potential are equivalent to $E_Z$. One may combinatorially
decompose the potential function $\pi_{P^3}$ into the sum of two potential functions

$$w_1(x, y, z) = x + y, \quad w_2(x, y, z) = z + \frac{1}{xyz}.$$  

This decomposition induces a morphism $(w_1, w_2) : (\mathbb{C}^*)^3 \to \mathbb{C}^2$ which sends

$$(x, y, z) \mapsto (w_1(x, y, z), w_2(x, y, z)).$$

Composition of $(w_1, w_2)$ with the diagonal map $(t_1, t_2) \mapsto t_1 + t_2$ recovers $\pi_{P^3}$.

**Remark 8.11.** There is a direct correspondence between the toric boundary divisors of a crepant resolution of a Gorenstein toric Fano variety $X_\Delta$ and the monomials in the dual potential. The fact that we have chosen to decompose into two pairs of two monomials corresponds to degenerating the anticanonical divisor in $\mathbb{P}^3$ into the union of a pair of quadrics. As we will see, there’s a partial compactification of $(\mathbb{C}^*)^3$ which is natural with respect to $w_1$ and $w_2$ which exhibits a general fiber of $\mathcal{W}$ as the symplectic manifold $E_Z$ in the previous section. \qed

This is a fibration over $\mathbb{C}$ whose generic fiber is an elliptic curve with four punctures. This may be compactified to fibration Weierstrass form, which may be written as:

\begin{align}
(31) \quad & y^2 = x^3 + (-1/3)((t_1t_2)^4 - 16(t_1t_2)^2 + 16)x \\
(32) \quad & \quad + (2/27)((t_1t_2)^2 - 8)((t_1t_2)^4 - 16(t_1t_2)^2 - 8).
\end{align}

There are two lines of $A_1$ singularities in this hypersurface in fibers over $t_1 = 0$ and $t_2 = 0$, which may be resolved by blowing up. This fibration admits four sections $s_1, \ldots, s_4$. We will let $E_{P^3}$ be the resolution of the hypersurface blown up these two lines, with $s_i = 0$ removed for $i = 1, 2, 3, 4$. This fibration is degenerate along the discriminant curve

$$t_1t_2(t_1t_2 + 4)(t_1t_2 - 4) = 0.$$  

We will let $\pi_{P^3}$ be the composition of the fibration of the hypersurface in Equation (31) onto the $(t_1, t_2)$ plane composed with the map $(t_1, t_2) \mapsto t_1 + t_2$. Comparing this to Equation (26) we obtain the following result.

**Proposition 8.12.** The smooth fibers of $\pi_{P^3}$ are symplectomorphic to $E_Z$.

An alternate approach is to take $\mathcal{E}$ be the subset of $E_Q \times \mathbb{C}^2$ which is cut out by the equation $\pi_Q - t_1t_2 = 0$. We must be a little bit careful in the fiber over the point $t_1t_2 = 0$, since this curve has two isolated singularities in $\mathcal{E}$, but both singularities admit small resolutions to a variety $\tilde{\mathcal{E}}$. The variety $\tilde{\mathcal{E}}$ is just the threefold described above, and projection onto $\mathbb{C}^2$ is the elliptic fibration $(w_1, w_2)$.

Therefore, the mirror of of $\mathbb{P}^3$ is precisely a bifibration in the sense of Seidel [76, Chapter III], so the vanishing cycles of $\pi_{P^3}$ can be identified with matching cycles of $(E_Z, \pi_Z)$ by looking at how the critical points of $\pi_Y$ behave as we approach critical points of $\pi_{P^3}$.

The preimages of the lines parametrized by $t_1 + t_2 = \lambda$ are tangent to the curve in Equation (33) at the points $\lambda = 4(\sqrt{-1})^i$ for $i = 1, 2, 3, 4$. These values of $\lambda$ are the locations of the singular fibers of $\pi_{P^3}$. Let us take $\pi_{P^3,u}$ to be the natural Lefschetz fibration on $\pi_{P^3}^{-1}(u)$ induced by the bifibration, or alternately, $(w_1, w_2)$ for some generic value of $u$. The map from the base of the Lefschetz fibration $\pi_{P^3,u}$ (with parameter $\lambda$) to the base of the Lefschetz fibration $\pi_Q$ (with parameter $s$) is given by

$$s = \lambda(\lambda - u)$$
where $u$ is the parameter in the base of $\pi_{p^3}$. This map is quadratic and simply ramified over the point

$$s = \frac{-u^2}{4}.$$ 

Then we may choose paths in the $s$-plane from $s = 1 + \sqrt{-1}$ to each critical value $s = 4(\sqrt{1})$ of $\pi$, which we will denote $\gamma_1, \ldots, \gamma_4$. These paths are shown in Figure 15. Along each of the paths $\gamma_i$, we can determine how the branch points in the $s$-plane behaves in order to determine the vanishing paths and vanishing cycles of $\pi_{p^3}$. Beginning at the point $\rho$ in the $t$-plane and following the paths in the above diagram, we see that the matching paths which determine the vanishing cycles of $\pi_{p^3}$ are given by the paths in Figure 16. The red and blue paths in Figure 16 are simply the matching paths which come from the vanishing paths in the base of the Lefschetz fibration on $E_Q$. We know quite a bit about the category $\Pi \mathcal{F}(E_Z)$ and its embedding into with $\mathcal{P}$erf($Z$), which is enough to identify the vanishing cycles with objects in $\mathcal{P}$erf($Z$).

**Theorem 8.13.** The matching cycles in the diagram above correspond to

$$\mathcal{O}_Z(-1), \mathcal{O}_Z, \mathcal{O}_Z(1), \text{ and } \mathcal{O}_Z(2)$$

in $\mathcal{P}$erf($Z$) under the quasi embedding of $\Pi \mathcal{F}(E_Z)$ into $\mathcal{P}$erf($Z$) described in the discussion preceding Theorem 8.9.

**Proof.** All of the vanishing spheres in question are elements of the category $\Pi \mathcal{F}(E_Z, \pi_Z, \gamma_{\pm})$ under the embedding mentioned in Equation 28. There is a quasi equivalence of categories,

$$\Pi \mathcal{F}(E_Z, \pi_Z, \gamma_{\pm}) \cong \Pi \mathcal{F}(E_Q, \pi_Q, \gamma_{\pm}) \times_{\mathcal{F}} \Pi \mathcal{F}(E_Q, \pi_Q, \gamma)$$

where $\mathcal{F}$ denotes the bimodule which sends

$$(a, b) \in \Pi \mathcal{F}(E_Q, \pi_Q, \gamma) \otimes \Pi \mathcal{F}(E_Q, \pi_Q, \gamma)^{\text{op}} \mapsto \text{hom}_{\Pi \mathcal{F}(\mathcal{T}_0)}(b, a).$$
The blue and red vanishing cycles are, up to choice of grading, given by

\[
\text{Cone} \left( L_{1,+}^\# \xrightarrow{id} L_{1,-}^\# \right), \quad \text{Cone} \left( L_{4,+}^\# \xrightarrow{id} L_{4,-}^\# \right).
\]

The halves of the orange and green cycles correspond to objects

\[
R_{L_{4,\pm}}^\# R_{L_{2,\pm}}^\# L_{1,\pm}^\#; \quad R_{L_{1,\pm}}^{-1} R_{L_{2,\pm}}^{-1} R_{L_{3,\pm}}^{-1} L_{4,\pm}^\#
\]

in \( \text{Tw}(E_Q, \pi_Q, \gamma) \). Therefore, the orange and green vanishing cycles correspond to objects

\[
\text{Cone} \left( R_{L_{4,+}}^\# R_{L_{3,+}}^\# R_{L_{2,+}}^\# L_{1,+}^\# \xrightarrow{id} R_{L_{4,-}}^\# R_{L_{3,-}}^\# R_{L_{2,-}}^\# L_{1,-}^\# \right),
\]

\[
\text{Cone} \left( R_{L_{1,+}}^{-1} R_{L_{2,+}}^{-1} R_{L_{3,+}}^{-1} L_{4,+}^\# \xrightarrow{id} R_{L_{1,-}}^{-1} R_{L_{2,-}}^{-1} R_{L_{3,-}}^{-1} L_{4,-}^\# \right)
\]

The mirror correspondence which equates \( D_{\text{dg}}^b(Q) \) and \( \text{Tw}(\pi_Q) \) in Proposition \ref{prop:mirror} sends

\[
L_{1,\pm}^\# \mapsto \mathcal{O}_Q, \quad L_{2,\pm}^\# \mapsto \mathcal{O}_Q(1,0), \quad L_{3,\pm}^\# \mapsto \mathcal{O}_Q(0,1), \quad L_{4,\pm}^\# \mapsto \mathcal{O}_Q(1,1),
\]

The mutations in Equation \ref{eq:mutations} correspond to the action of Serre functor and inverse Serre functor in \( D_{\text{dg}}^b(Q) \) respectively, \cite{12}. Up to shift, this is just the tensor product with \( \mathcal{O}_Q(2,2) \) and \( \mathcal{O}_Q(-2,-2) \) respectively. Therefore the orange and green vanishing cycles correspond to \( \mathcal{O}_Q(2,2) \) and \( \mathcal{O}_Q(-1,-1) \) respectively. Thus these four vanishing cycles correspond to objects

\[
(\mathcal{O}_Q(-1,-1), \mathcal{O}_Q(-1,-1), \text{id}), \quad (\mathcal{O}_Q, \mathcal{O}_Q, \text{id}),
\]

\[
(\mathcal{O}_Q(1,1), \mathcal{O}_Q(1,1), \text{id}), \quad (\mathcal{O}_Q(2,2), \mathcal{O}_Q(2,2), \text{id}),
\]

up to shift. These objects are isomorphic to objects in \( D_{\text{dg}}^b(Q) \times_{\text{perf}(G_4)} D_{\text{dg}}^b(Q) \), and in fact are objects in the image of \( \text{perf}(Z) \) under the embedding in Example \ref{example:embedding}. Using the proof of Proposition \ref{prop:embedding} these come from \( \mathcal{O}_Z(-1), \mathcal{O}_Z, \mathcal{O}_Z(1), \mathcal{O}_Z(2) \) up to shift, as required. \( \square \)

The following theorem is contained in \cite{6,26} but is also a corollary to Theorem \ref{thm:main} and Example \ref{example:embedding}.
Theorem 8.14. Homological mirror symmetry holds for $\mathbb{P}^3$.

8.7. Homological mirror symmetry for quantum $\mathbb{P}^3$. The mirror symmetry theorems in Section 8.6 are based off of the construction in Example 6.21. The computations in Section 7.3 generalize this, replacing quadrics with noncommutative quadrics. Therefore one should prove homological mirror symmetry for noncommutative versions of $\mathbb{P}^3$ as well. In this section, we will give a sketch of this. The Fukaya categories used here will be different from those above, but are all still quite standard.

The basic idea is to follow the steps of the proof of Theorem 8.14 with the addition of B-fields and with LG models and symplectic forms changed slightly. Since all of our work in Section 8 starts with homological mirror symmetry for $G_4$, we must have an analogue of this for a smooth elliptic curve. In [69], Polishchuk and Zaslow prove a large part of homological mirror symmetry on the level of homotopy categories, and this was extended to $A_\infty$ structures by Polishchuk [67] later. Their proof shows that, if one ignores endomorphisms of objects, there is an equivalence between the category $\Pi \text{Tw}F(E')$ and $D^b_\infty(E)$ for $E$ and $E'$ equipped with particular complex and symplectic structures respectively. Here $D^b_\infty(E)$ is the minimal $A_\infty$ category which is homotopy equivalent to $D^b_{dg}(E)$.

In [1], Abouzaid and Smith suggest that their proof may be completed using techniques of Seidel [76], though to our knowledge, the details of this do not appear literature. Abouzaid and Smith also prove a version of homological mirror symmetry for the elliptic curve which holds over the Novikov field. Furthermore, Lekili and Perutz [46] prove another version of homological mirror symmetry for elliptic curves, which is again quite close to completing Polishchuk and Zaslow’s computation.

Then, one must prove homological mirror symmetry for noncommutative quadrics. This should follow easily by imitating the computations of Auroux, Katzarkov and Orlov [6]. To do this, we use a relative compactification of the LG model of $E_Q$ previously discussed. Particularly, let $\overline{E}_Q$ be the Weierstrass equation in Equation 24 blown up at its singular point and let $\overline{\pi}_Q$ be its natural fibration over $\mathbb{C}$.

Then we allow the symplectic form $\omega$ on $\overline{E}_Q$ to vary and we add a B-field. The corresponding directed Fukaya category will be equivalent to $D^b(qgr(Q_V))$ for some data $V = (E, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$, where $E$ is mirror to a fiber of $\overline{\pi}_Q$ under the mirror correspondence of Polishchuk and Zaslow.

At this point, one can construct an LG model for noncommutative $\mathbb{P}^3$ by taking the Weierstrass equation in Equation 24 and resolving its lines of $A_1$ singularities by blowing up. The resulting quasiprojective threefold is relatively compact over $\mathbb{C}^2$, and admits a fibration over $\mathbb{C}$ by composing the fibration over $\mathbb{C}^2$ with the map $(t_1, t_2) \mapsto t_1 + t_2$. Call this threefold $\overline{E}_{\mathbb{P}^3}$ and let this fibration over $\mathbb{C}$ be called $\overline{\pi}_{\mathbb{P}^3}$. The fibers of $\overline{\pi}_{\mathbb{P}^3}$ are double covers of $\overline{E}_Q$ ramified along a fiber of $\overline{\pi}_Q$. We will let $\omega$ be a generic symplectic form on $\overline{E}_{\mathbb{P}^3}$.

These fibers will have Fukaya category equivalent to the category $D^b(qgr(Q_V)) \times_{D^b(E)} D^b(qgr(Q_{V_2}))$ where $E$ is mirror to a smooth fiber of $\overline{\pi}_Q$, depending on the restriction of the symplectic form $\omega$ on $\overline{E}_{\mathbb{P}^3}$. The vanishing cycles of $\overline{\pi}_{\mathbb{P}^3}$ will still be expressed as matching spheres with respect to the same paths in the base of the natural fibration of the fibers of $\overline{\pi}_{\mathbb{P}^3}$ over $\mathbb{C}$. The fact that such matching spheres exist as Lagrangian submanifolds should be equivalent to a consistency condition on $V_1$ and $V_2$ saying that there is some $A_V$ so that $qgr(Q_{V_1})$ and $qgr(Q_{V_2})$ appear as hypersurfaces in $qgr(A_V)$. At this point one can follow the proof of Theorems 7.13 and 8.13 to show that the Fukaya-Seidel category of the pair $(\overline{E}_{\mathbb{P}^3}, \overline{\pi}_{\mathbb{P}^3})$ is equivalent to $D^b(qgr(A_V))$ where the data $V$ is determined by the data of $V_1$ and $V_2$. 
The most difficult part of implementing this is perhaps the fact that it seems like no version of [76, Proposition 18.13] exists in the literature for Lefschetz fibrations which are not exact.

9. Future directions

There are many avenues of research suggested by our work here. In the next section, we will outline in some detail several different problems.

9.1. A categorical version of results of Kawamata and Namikawa. If a variety $X$ is normal crossings of dimension $d$, then there is a corresponding constructible sheaf of categories on its dual intersection complex. Let $X_1, \ldots, X_n$ be the irreducible components of $X$, then for each $I \subseteq [1, n]$, we have a smooth variety

$$C_I = \bigcap_{i \in I} X_i$$

The dual intersection complex $|\Gamma(X)|$ of $X$ is a cell complex which has an $|I| - 1$ dimensional stratum $s_I$ for each non empty $C_I$, and $C_I \subseteq C_J$ if and only if $J \subseteq I$. There is a constructible sheaf of categories on $|\Gamma(X)|$ whose stalk at any point in $s_I$ is the category $\text{Perf}(C_I)$. There are restriction functors $\text{Perf}(C_I) \to \text{Perf}(C_J)$ for any $I \subseteq J$ given by $\mathbb{L}f^*_{J,I}$ where $f_{J,I} : C_J \hookrightarrow C_I$ is the natural embedding. These functors satisfy the appropriate consistency conditions because they come from geometry.

According to Section 2.5, the category of global sections of this perverse sheaf of categories (which is simply the homotopy limit of the corresponding diagram of categories) should be equivalent to $\text{Perf}(X)$.

Friedman [24] has given necessary conditions in order for a deformation of a normal crossings variety $X$ with smooth total space to exist, called d-semistability. He proved that in the case where $K_X \cong 0, h^1(O_X) = 0$ and $\dim X = 2$, this is in fact a sufficient condition. Kawamata and Namikawa [34] show that this d-semistability condition is equivalent to the existence of a smooth log structure on $X$. They extend Friedman’s result to show that for any $n$-dimensional, d-semistable normal crossings $X$ with $K_X \cong 0, h^{i-1}(O_X) = 0$ for $i \neq 0, n$, there is a smooth deformation of $X$.

**Definition 9.1.** If $D$ is the singular locus of $X$, which is the union of all strata of $X$ of codimension $\geq 1$, and $X_1, \ldots, X_n$ are the irreducible components of $X$, then $X$ is called d-semistable if

$$(I_{X_1}/I_{X_1}I_D) \otimes \cdots \otimes (I_{X_n}/I_{X_n}I_D) \cong O_D.$$
One would expect that d-semistability allows one to simplify the deformation theory of $\text{Perf}(S_\Gamma)$, however, little is known about deformation theory of dg categories $[9, 48, 78]$.

**Remark 9.3.** A normal crossings variety should be d-semistable if and only if the corresponding constructible sheaf of categories is d-semistable.

We conjecture that the main result of [34] can be extended to the noncommutative case.

**Conjecture 9.4.** If $\text{Perf}(S_\Gamma)$ is Calabi-Yau and d-semistable then $\text{Perf}(S_\Gamma)$ can be smoothly deformed to a smooth dg category $S_\Gamma^\mu$.

9.2. **Sheaves of categories and toric fiber bundles.** The proof of Theorem 4.6 which identifies the derived category of coherent sheaves on a $\mathbb{P}^1$ bundle with the category of global sections on a perverse schober is very much specific to the case of $\mathbb{P}^1$ bundles, but a more general result should hold for much more general fiber bundles. Specifically, let us assume that we have a fiber bundle $\pi: Z \to X$ whose fiber is a smooth toric variety $Y_{\Sigma}$. To the toric variety $Y_{\Sigma}$, there is a Lagrangian skeleton $K_{\Sigma}$ of $(\mathbb{C}^*)^{\dim \Sigma}$ associated to $\Sigma$ by Fang, Liu, Treumann and Zaslow [23]. In the case where $Y_{\Sigma}$ is $\mathbb{P}^1$, this skeleton is, after deformation, the skeleton in Figure 4.

One may equip $K_{\Sigma}$ with a constructible sheaf of categories in such a way that the category of global sections is equivalent to $D_{dg}(Z)$. When these coefficients are the category $\text{Perf}_k$, this construction should recover a version of the non equivariant coherent constructible correspondence suggested in unpublished work of Bondal and Wang formulated by Fang, Liu, Treumann and Zaslow [23], and proved by Kuwagaki [38] and D. Vaintrob. We will return to this problem in future work.

This of course should hold for more general types of bundles, such as Grassmannian bundles or flag bundles, however, the difficulty in those situations is determining the correct skeleton. This all reflects the philosophy that mirror symmetry exchanges the fiber and base of fiber bundles.

9.3. **Noncommutative deformations of more general Fano varieties.** Our study of the four dimensional Sklyanin algebra in Section 7.3 suggests an alternate construction of this Sklyanin algebra starting from a pair of noncommutative quadrics. One takes a pair $Q_{V_1}$ and $Q_{V_2}$ of noncommutative quadrics so that

$$V_1 = (E, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3), \quad V_2 = (E, \mathcal{L}_1', \mathcal{L}_2', \mathcal{L}_3'),$$

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L}_1' \otimes \mathcal{L}_2', \quad \mathcal{L}_3 \otimes \mathcal{L}_2 \cong \mathcal{L}_3' \otimes \mathcal{L}_2'.$$

Then we let $R_1$ and $R_2$ be graded algebras so that $\tilde{R}_i = 2Q_i$. Then we have a pair of maps $q_i: R_i \to B$ for $B$ the (graded) twisted coordinate ring of $E$ associated to $\mathcal{L}_1 \otimes \mathcal{L}_2$ and $\mathcal{L}_2 \otimes \mathcal{L}_3$. If we truncate the kernel of the map

$$R_1 \times R_2 \xrightarrow{q_1 - q_2} B$$

in degree 4, we obtain an algebra $S$ so that $D^b(\text{rep}(S))$ is equivalent to the category $D^b(\text{gr}(A))$ for some Sklyanin algebra $A$. One may do the same thing with different algebras. For instance, take $\text{WP}(1, 1, 1, 3)$. We have a divisor $Z$ of degree 6 comprised of a pair of copies of $\mathbb{P}^2$ meeting along a cubic elliptic curve. There are four spherical objects in $\text{Perf}(Z)$ whose corresponding directed category is equivalent to $D^b(\text{WP}(1, 1, 1, 3))$. If we deform $\mathbb{P}^2$ noncommutatively, there are identifiable analogues of these spherical objects and the associated directed category should be a noncommutative deformation of $\text{WP}(1, 1, 1, 3)$.

The resulting dg category $\mathcal{C}$ will have homotopy category which admits a Serre functor $S$. The inverse of $S$ can be lifted to an object $S^{-1} \in \mathcal{C}$-bimod, and let $S^{-i}$ denote the $i$-fold tensor product.
of \( \mathcal{S} \) with itself. Let \( \text{id} \) denote the object in \( \mathcal{C}-\mathcal{C} \) corresponding to the identity functor. We denote the homotopy category of \( \mathcal{C}-\mathcal{C} \) bimodules as \( \text{D}(\mathcal{C}-\mathcal{C}) \). Then the algebra

\[
A_{\mathcal{C}} = \bigoplus_{i \geq 0} \text{hom}_{\text{D}(\mathcal{C}-\mathcal{C})}(\text{id}, [\mathcal{S}^{-i}])
\]

with twisted multiplication should then satisfy;

\[
\text{D}^b(\text{qgr}(A_{\mathcal{C}})) \cong H^0 \mathcal{C}.
\]

**Remark 9.5.** Note that in this case, all of the results of Section 8 can be imitated by replacing quadrics with \( \mathbb{P}^3 \) and replacing the hypersurface

\[
x^2y^2 - z^2w^2 = 0
\]

in \( \mathbb{P}^3 \) with the hypersurface

\[
w^2 - x^2y^2z^2 = 0
\]

in \( \mathbb{WP}(1, 1, 1, 3) \) with variables \( x, y, z, w \) of weights 1, 1, 1, 3 respectively. Then, taking the technical parts of Section 9 for granted, the proof of homological mirror symmetry \( \mathbb{WP}(1, 1, 1, 3) \) reduces to an easy computation of vanishing cycles. \( \square \)

### 9.4. Perverse sheaves of stability conditions.

Given a proper map of varieties \( f : X \to B \), one may associate an algebraic sheaf of categories \([92]\). To this data, one should have a special class of limiting stability conditions constructed from stability conditions on the generic fiber of \( f \) and \( X \) itself whose restriction to each fiber of \( f \) is constant in precise way. We call such stability conditions perverse stability conditions. On the other hand, on a perverse sheaf of categories, one should be able to ascribe stability conditions by using metric properties of the surface on which the perverse sheaf of categories lives along with metric properties of the fiber categories. The second approach to stability conditions is part of work in progress of the second author along with F. Haiden, M. Kontsevich and P. Pandit, and has already appeared to a certain extent in work of Haiden, Katzarkov and Kontsevich \([29]\) and Bridgeland and Smith \([17]\).

As we have seen in Section 9.1, simple perverse sheaves of categories encode \( \mathbb{P}^1 \) and more general fiber spaces according to our speculation in Section 9.2. This correspondence should relate stability conditions on perverse sheaves of categories to perverse stability conditions. We intend to study perverse stability conditions in more depth in order to understand this relationship.

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