The Configuration of Fraternite-Egalite2-Egalite1 in the Neptune Ring Arches System

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ABSTRACT

By considering the finite mass of Fraternite, although small, it is shown that there are two time averaged stationary points in its neighborhood due to the reaction of the test body to the fields of Neptune, Galatea, and Fraternite. These two locations measuring 11.7 and 13.8 degrees from the center of Fraternite could correspond to the locations of Egalite 2 and Egalite 1. This model accounts for the 10 degree span of Fraternite and estimates its mass at $m_f = 6.4 \times 10^{16}$ Kg. The eccentricities of Egalite 2 and Egalite 1 are believed to be about $e = 5 \times 10^{-4}$.

Subject headings: Planets: Rings
1. Introduction

Ever since the discovery of the Neptune arcs [Hubbard et al 1986], the constant monitoring of their evolution have revealed much of their dynamic properties [Smith et al 1989, Sicardy et al 1999, Dumas et al 1999]. Nevertheless, a complete model to account for them is still not available. Currently, there are two models that attempt to explain their structures. The first is the two-satellite model consisting of Galatea for radial confinement of the arcs and an hypothetical Lagrange moon for their azimuthal confinement [Lissauer 1985, Sicardy and Lissauer 1992]. However, due to the Voyager data, a fair size Lagrange moon seems unlikely. The second is the one-satellite model where Galatea and the arcs are in the 42/43 corotation-Lindblad orbit-orbit resonance due to the finite eccentricities of Galatea and arcs respectively [Goldreich et al 1986, Porco 1991, Foryta and Sicardy 1996]. However, recent measurements have indicated that the arcs are off the corotation resonance location by 0.3 Km and the corotation velocity slightly differs from the arc velocity [Horanyi and Porco 1993, Sicardy et al 1999, Dumas et al 1999, Showalter 1999, Namouni and Porco 2002]. In order to close the mean motion mismatch, it is proposed to take into consideration the finite mass of the arcs that pulls on the epicycle frequency of Galatea [Namouni and Porco 2002]. This would constraint the eccentricity of Galatea to reside values which would almost eliminate the eccentricity corotation potential although the inclination type remains.

Further complications come from the angular spreading of and the spacing between the arcs. The two minor arcs Egalite 2 and Egalite 1 trail behind the main arc Fraternite by about 10 and 13 degrees respectively measuring from center to center. Fraternite has a span of about 10 degrees, Egalite 2 spans over 3 degrees while Egalite 1 spans about 1 degree only. Although Fraternite’s spreading appears to match a 42/43 corotation-Lindblad site of eccentricity type, the minor arcs and their spacing from the main arc call for inclination resonance or an eccentricity-inclination combination [Namouni and Porco 2002]. Here, we attempt to address this question of arc spacings. We take the standard eccentricity corotation resonance site model to set Fraternite location and spreading as a reference position. As for the minor
arcs, we examine their equations of motion for stationary points over long time average. Our model consists of the central body S (Neptune), the primary body X (Galatea), a minor body F (Fraternite), and a test body s (Egalite 2 and Egalite 1). The center of mass is given by SX pair. We consider F and X in corotation-Lindblad resonance with each other, and s is coorbital with F. We also take a finite mass for F although it is much smaller than that of X. We consider gravitational interactions of s and F. Due to the small mass of F, only close distance interactions are needed to be considered. By expanding in powers of eccentricity of s, and by taking a long time average, it is shown that the equations of motion have two time averaged stationary points near F that could correspond to the positions of the minor arcs.
2. Corotation-Lindblad Resonance

We designate $M, m_x, m_f$ as the masses of the central body $S$, the primary body $X$, and the minor body $F$ respectively. Also $\vec{r}_x = (r_x, \theta_x)$, $\vec{r}_f = (r_f, \theta_f)$, and $\vec{r}_s = (r_s, \theta_s)$ are the position vectors of $X$, $F$, and $s$ measured from $S$ with respect to a fixed reference axis in space. Furthermore, $\vec{R} = \vec{r}_s - \vec{r}_x$ and $\vec{R}' = \vec{r}_s - \vec{r}_f$ are the position vectors of $s$ measured from $X$ and $F$ respectively. We consider all the bodies moving on the ecliptic plane by neglecting the orbit inclinations. With respect to a coordinate system centered at the central body $S$, the equations of motion of $s$ are

\[
\frac{d^2 r_s}{dt^2} = +\{r_s \omega_s^2 - \frac{GM}{r_s^2}\} - \left\{\frac{Gm_x}{R^3} [r_s - r_x \cos(\Delta \theta_{sx})] + \frac{Gm_x}{r_x^3} r_x \cos(\Delta \theta_{sx})\right\} - \frac{Gm_f}{R'^3} [r_s - r_f \cos(\Delta \theta_{sf})] , \quad (1)
\]

\[
\frac{1}{r_s} \frac{d}{dt} (r_s^2 \omega_s) = -\left\{\frac{Gm_x}{R^3} - \frac{Gm_x}{r_x^3}\right\} r_x \sin(\Delta \theta_{sx}) - \frac{Gm_f}{R'^3} r_f \sin(\Delta \theta_{sf}) . \quad (2)
\]

Here, $\Delta \theta_{sx,sf} = (\theta_s - \theta_{x,f})$, whereas $\omega_s = d\theta_s/dt$ is the angular velocity of $s$ about the central body $S$ with respect to a reference axis. We expand the parameters of $s$ on the right sides of these equations of motion in powers of its eccentricity. Taking a time average over an interval long compared to the orbital period of $s$, we have

\[
\frac{d^2 r_s}{dt^2} = \left(\frac{GM}{L}\right)^2 \frac{1}{a^2} 2e^2 - \frac{m_x}{M} \left(\frac{GM}{L}\right)^2 a^2 b_{01} + \frac{m_x}{M} \left(\frac{GM}{L}\right)^2 a^2 e b_{11} \cos(\Phi_{sxL})
\]

\[
+ \frac{m_x}{M} \left(\frac{GM}{L}\right)^2 a r_x b_{02} - \frac{m_f}{M} \left(\frac{GM}{L}\right)^2 a^2 \frac{a^2}{R'^3} [1 - \cos(\Delta \theta_{sf})] ,
\]

\[
\frac{1}{r_s} \frac{d}{dt} (r_s^2 \omega_s) = - \frac{m_x}{M} \left(\frac{GM}{L}\right)^2 a r_x 2e b_{n2} \sin(\Phi_{sxL}) - \frac{m_f}{M} \left(\frac{GM}{L}\right)^2 a^2 \frac{a^2}{R'^3} \sin(\Delta \theta_{sf}) ,
\]
where \( L^2 = GMa \) with \( a \) as the semi-major axis of \( s \), and \( \Phi_{sxL} = [(n + 1)\theta_s - n\theta_x - \phi_s] \) is the Lindblad resonance variable of \( s \) with \( s \) and \( F \) outside of \( X \). The coefficients \( b_{01}, b_{n1}, b_{02}, \) and \( b_{n2} \) are defined through the Laplace coefficients as \( b_{01} = (1/2)(1/a^3)b_{3/2}^{(0)}, b_{n1} = (1/2)(1/a^3)b_{3/2}^{(n)}, b_{02} = (1/2)(1/a^3)b_{3/2}^{(1)}, b_{n2} = (1/4)(1/a^3)[b_{3/2}^{(n+1)} + b_{3/2}^{(n-1)}] \).

Rewriting \( \Phi_{sxL} = [\Phi_{fxL} + (n + 1)\Delta\theta_{sf} - (\phi_s - \phi_f)] \) in terms of the \( FX \) corotation-Lindblad resonance variable \( \Phi_{fxL} \), and taking \( \phi_s = \phi_f \), the above equations become

\[
\frac{d^2 r_s}{dt^2} = \left( \frac{GM}{L} \right)^2 \frac{1}{a} \{ 2e^2 - \frac{M}{a} \left[ b_{01} - r_x b_{02} \right] + \frac{M}{a} \left[ b_{n1} \right] \cos(\Phi_{fxL} + (n + 1)\Delta\theta_{sf}) - \frac{M}{R^3} \sin(\Delta\theta_{sf}) \} ,
\]

(3)

\[
\frac{1}{r_s} \frac{d}{dt} \left( r_s^2 \omega_s \right) = - \left( \frac{GM}{L} \right)^2 \frac{1}{a} \left\{ \frac{M}{a} r_x b_{n2} \sin(\Phi_{fxL} + (n + 1)\Delta\theta_{sf}) + \frac{M}{R^3} \sin(\Delta\theta_{sf}) \right\} .
\]

(4)

The angular positions of \( s \) where the time averaged force acting on it vanishes are given by

\[
2e^2 + \frac{m_x}{M} a^3 e b_{n1} \cos(\Phi_{fxL} + (n + 1)\Delta\theta_{sf})
\]

\[
- \frac{m_x}{M} a^3 \left[ b_{01} - \frac{r_x}{a} b_{02} \right] - \frac{m_f}{M R^3} \left[ 1 - \cos(\Delta\theta_{sf}) \right] = 0 ,
\]

(5)

\[
e = - \frac{m_f}{m_x R^3 a^2 r_x 2b_{n2} \sin(\Phi_{fxL} + (n + 1)\Delta\theta_{sf})} \frac{\sin(\Delta\theta_{sf})}{\sin(\Delta\theta_{sf})} .
\]

(6)

These are the general conditions for vanishing time averaged force for two coorbital objects \( s \) and \( F \) that are in corotation orbital resonance with an interior \( X \).
3. Fraternite-Egalite2-Egalite1

Let us now apply these conditions to the Neptune ring arcs. With the Neptune system parameters, the last term on the left side of the first equation can be neglected unless \( \frac{R'}{a} \) is exactly zero which amounts to a collision. Also, considering the mass ratio \( m_x/M \) of the Neptune system much less than the expected eccentricity such that the linear eccentricity term can be neglected. We, therefore, keep only the quadratic term and the stationary locations are given by

\[
\left( \frac{R'}{a} \right)^3 \frac{\sin[\Phi_{fL} + (n + 1)\Delta\theta_{sf}]}{\sin(\Delta\theta_{sf})} = \frac{1}{2} \frac{m_f}{m_x} \frac{a}{r_x} \frac{1}{a^3} \frac{1}{b_{n2}} \left\{ \frac{M}{m_x} \frac{1}{a^3} \left[ b_{01} - \left( \frac{r_x}{a} \right) b_{02} \right] \right\}^{1/2}.
\]

Writing \( \left( \frac{R'}{a} \right) = 2 \sin(\Delta\theta_{sf}/2) \) Eq.(7) reads

\[
\left( 2 \sin\left( \frac{\Delta\theta_{sf}}{2} \right) \right)^3 \frac{\sin[\Phi_{fL} + (n + 1)\Delta\theta_{sf}]}{\sin(\Delta\theta_{sf})} = -\frac{1}{2} \frac{m_f}{m_x} \frac{a}{r_x} \frac{1}{a^3} \frac{1}{b_{n2}} \left\{ \frac{M}{m_x} \frac{1}{a^3} \left[ b_{01} - \left( \frac{r_x}{a} \right) b_{02} \right] \right\}^{1/2}.
\]

With \( \alpha = r_x/a = 0.98444 \), the Laplace coefficients are \( b_{3/2}^{(0)} = 0.26487 \times 10^4 \), \( b_{3/2}^{(1)} = 0.26470 \times 10^4 \), \( b_{3/2}^{(41)} = 0.20168 \times 10^4 \), \( b_{3/2}^{(42)} = 0.19975 \times 10^4 \), \( b_{3/2}^{(43)} = 0.19782 \times 10^4 \). The right side of the above equation can be calculated to give

\[
\left( 2 \sin\left( \frac{\Delta\theta_{sf}}{2} \right) \right)^3 \frac{\sin[\Phi_{fL} + (n + 1)\Delta\theta_{sf}]}{\sin(\Delta\theta_{sf})} = -1.5528 \times 10^{-4} \frac{m_f}{m_x} \left( \frac{M}{m_x} \right)^{1/2} = -1.1 \frac{m_f}{m_x}.
\]

The second equality is reached by taking \( M = 1 \times 10^{26} \) Kg for the central body Neptune, and \( m_x = 2 \times 10^{18} \) Kg for the primary body Galatea. This leaves only one parameter \( m_f/m_x \) in the equation.

Considering the center of Fraternite be at the maximum of the corotation site with \( \Phi_{fL} = \pi/2 \), the positions where the time averaged force vanishes are given by

\[
\left( 2 \sin\left( \frac{\Delta\theta_{sf}}{2} \right) \right)^3 \frac{\sin[\Phi_{fL} + (n + 1)\Delta\theta_{sf}]}{\sin(\Delta\theta_{sf})} = -1.1 \frac{m_f}{m_x}.
\]
On the left side of Eq.(9), the cosine function starts with a central maximum at $(n+1)\Delta \theta_s = 0$ and reaches its first minimum at $(n+1)\Delta \theta_s = \pm \pi$ on each side forming a complete site of 8.4 degrees with $n = 42$. However, due to the other factor, the central maximum is replaced by a null and a nearby maximum on each side of it. Numerical solution of Eq.(9) in Fig.1 shows the first minimum slightly shifted outwards to 4.85 degrees on each side spanning an angular width of 9.7 degrees which corresponds to the observed extension of Fraternite. The second minimum is located at 12.8 degrees from the center. The roots of Eq.(9) are given by the intercepts of the left side with the right side. There are either two intersects around this second minimum or none of them. Taking the mass ratio $m_f/m_x = 3.2 \times 10^{-2}$ gives two intercepts at 11.7 and 13.8 degrees which are approximately where the minor arcs are observed. This intercept corresponds to a mass ratio $m_f/m_x = 3.2 \times 10^{-2}$ which gives $m_f = 6.4 \times 10^{16}$ Kg for Fraternite. The slight difference of one degree or so between the calculated and observed positions is probably because we have represented the elongated distribution of Fraternite’s mass by a point mass at its center. Besides giving a mass estimate of Fraternite and the positions of Egalite 2 and Egalite 1, we can also estimate the eccentricity of the two minor arcs by using Eq.(5) or Eq.(6). Simple calculations from both equations give $e = 5 \times 10^{-4}$ approximately.
4. Conclusions

To conclude, we have studied a four-body system where the center of mass is set by the central and primary bodies. A minor body is in corotation-Lindblad resonance with the primary body, and a test body is coorbital with the minor body at close distances. Through the equations of motion, we have shown that there are stationary locations where the time averaged force vanishes. These points are located behind the minor body as well as in front of it. They differ from the Lagrangian points of a restricted three-body system in that the averaged force is zero, and that they are dynamically self-generated by the test body’s reaction to the fields of the minor and primary bodies in orbit-orbit resonance plus the field of the central body. We have applied these points to account for the arcs’ configuration in the Neptune-Galatea system. Using this model, it is able to explain the 10 degree extension of Fraternite. By requiring Fraternite’s mass be $6.4 \times 10^{16}$ Kg, two locations with vanishing time averaged force exist at 11.7 and 13.8 degrees from the center of Fraternite which seem to be compatible with the observed positions of Egalite 1 and Egalite 2. It also estimates the eccentricity of Egalite 2 and Egalite 1 at $5 \times 10^{-4}$.

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Fig. 1.— Denoted by the y label, the left and right sides of Eq. (9) are plotted as a function of $\Delta \theta_{sf}$ in rad/s, and the intercepts define the locations where the time averaged force vanishes.