500-th solution of 2D Ising model

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One more solution of 2D Ising model is found.

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I. FORMULATION OF THE PROBLEM

Let’s consider 2D Ising model on the simplest regular lattice with the partition function

$$Z = \sum_{\{\sigma=\pm 1\}} \exp \left\{ \theta \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} [\sigma_{m,n}(\sigma_{m+1,n} + \sigma_{m,n+1})] \right\},$$

$$m = 1, \ldots, M, \quad n = 1, \ldots, N. \quad (1.1)$$

A pair of numbers \((m, n)\) enumerate the vertex of the lattice which is at the intersection of \(m\)-th column and \(n\)-th line of the lattice. First the partition function \((1.1)\) was calculated by L. Onsager \([1]\), and then by several another authors and methods (see, for example, \([2]-[4]\), and others references can be found in \([5]\)). Taking into account that \(\sigma_{m,n}^2 = 1\), rewrite the partition function as

$$Z = \sum_{\{\sigma=\pm 1\}} \prod_{m=1}^{M-1} \prod_{n=1}^{N-1} \left[ (\text{ch} \theta + \text{sh} \theta \cdot \sigma_{m,n} \sigma_{m+1,n}) (\text{ch} \theta + \text{sh} \theta \cdot \sigma_{m,n} \sigma_{m,n+1}) \right]. \quad (1.2)$$

The right-hand side of the last equation is a polynomial in the variables \(\{\sigma_{m,n}\}\), each variable \(\sigma_{m,n}\) being in the degree not higher than 4. As a result of summation in \((1.2)\) all summands with odd degrees of each of these variables reduce to zero. Thus one should take into account only the summands which are proportional to \(\sigma_{m,n}^0 = \sigma_{m,n}^2 = \sigma_{m,n}^4 = 1\) for all \(m\) and \(n\). It follows from \((1.2)\) that the sum is equal to the sum of all closed contours (loops) on the lattice, generally with intersections and self-intersections. Let’s call the elementary parts of loops connecting the nearest vertexes (say, \((m, n)\) and \((m+1, n)\) or \((m, n)\) and \((m, n+1)\)) by connections. Thus the length of the loop is equal to the number of its connections. Number of

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connections coming to each vertex can be equal to 0, 2 or 4. Thus the partition function (1.2) can be expressed as

\[ Z = (\text{ch} \theta)^2(\text{M} - 1)(\text{N} - 1)2^{\text{MN}} \sum_{\text{loops}} (\text{th} \theta)^{\nu}, \quad (1.3) \]

where \( \nu \) is the total number of connections of the loop.

Here an another method of calculation of partition function (1.1) is proposed. Let’s consider the Clifford algebra with \( MN \) generatives:

\[ \gamma_x \gamma_y + \gamma_y \gamma_x = 2\delta_{xy}, \quad x, y = 1, \ldots, MN. \quad (1.4) \]

Matrices \( \{\gamma_x\} \) are Hermitian and I assume that their dimensions to be \( 2^{MN/2} \times 2^{MN/2} \), that is the minimal possible one at given number of matrices [7]. Algebra (1.4) implies that the trace of any odd product of \( \gamma \)-matrices is equal to zero, and

\[ \text{tr} \gamma_x \gamma_y = 2^{MN/2} \delta_{xy}, \quad \text{tr} \gamma_x \gamma_y \gamma_z \gamma_v = 2^{MN/2} (\delta_{xy} \delta_{zv} - \delta_{xz} \delta_{yv} + \delta_{xv} \delta_{yz}), \quad (1.5) \]

and so on. By virtue of algebra (1.4) any even product of \( \gamma \)-matrices is reduced to the \( \pm 1 \) (versus the number of their permutations) or to the product of two by two different \( \gamma \)-matrices. According to (1.5) the trace of the product is equal to \( \pm 2^{MN/2} \) in the first case and to zero in the last case.

The theory of \( \gamma \)-matrices in the spaces of large dimensions as well as in Hilbert space can be found in [6].

Further we shall consider that the indices \( x, y \) and so on enumerate the nodes of the lattice and thus they are integer vectors in the plane: \( \mathbf{x} = (m, n) \); it is assumed that the index \( m \) increases on the right while the index \( n \) increases up. Thus, the matrix \( \gamma_\mathbf{x} = \gamma_{m,n} \) is related to each node \( (m, n) \). We define also the elementary basic vectors \( \mathbf{e}_1 = (1, 0) \) and \( \mathbf{e}_2 = (0, 1) \), so that \( \mathbf{x} = m \mathbf{e}_1 + n \mathbf{e}_2 \).

The following statement is valid: the partition function (1.1) or (1.2) can be rewritten also as

\[ Z = 2^{MN/2}(\text{ch} \ 2\theta)^{(M-1)(N-1)} \text{tr} \left\{ \ldots \right\} \]

\[ \ldots \times \left[ (\lambda + \mu \gamma_x \gamma_{x+e_2})(\lambda + \mu \gamma_x \gamma_{x+e_1}) \right] \left[ (\lambda + \mu \gamma_{x+e_1} \gamma_{x+e_1+e_2})(\lambda + \mu \gamma_{x+e_1} \gamma_{x+2e_1}) \right] \times \ldots \]

\[ \ldots \times \left[ (\lambda + \mu \gamma_{x+e_2} \gamma_{x+2e_2})(\lambda + \mu \gamma_{x+e_2} \gamma_{x+e_1+e_2}) \right] \times \]

\[ \times \left[ (\lambda + \mu \gamma_{x+e_1+e_2} \gamma_{x+e_1+2e_2})(\lambda + \mu \gamma_{x+e_1+e_2} \gamma_{x+2e_1+e_2}) \right] \times \ldots \} \quad (1.6) \]

The expression in braces in (1.6) is some polynomial in \( \gamma \)-matrices. The statement is true if the traces of all monomials of \( \gamma \)-matrices in (1.6) are non-negative (the
positivity condition which is proved further) and
\[ \lambda \equiv \cos \frac{\psi}{2} = \frac{\text{ch} \theta}{\sqrt{\text{ch} 2 \theta}}, \quad \mu \equiv \sin \frac{\psi}{2} = \frac{\text{sh} \theta}{\sqrt{\text{ch} 2 \theta}} \rightarrow \sin \psi = \frac{2 \text{th} \theta}{1 + (\text{th} \theta)^2}. \] (1.7)

Note that the last expression (1.6) is obtained from (1.2) by substitutions \( \sigma_{m,n} \rightarrow \gamma_{m,n} \) and \( \sum_{\{\sigma=\pm 1\}} \rightarrow 2^{MN/2} \text{tr} \). But in contrast to (1.2) the sequence of multipliers arrangement as well as their form are crucial in the case (1.6). Draw attention to the fact that in the product under the sign of trace in (1.6) at first the brackets are multiplied in series along the lines, and then the results of multiplications along the lines are multiplied in series along the column.

To establish the coincidence of the right-hand sides of Eqs. (1.6) and (1.3), it is sufficient to prove the positivity condition and to take into account the fact that under the sign of trace in (1.6) only that polynomials in \( \gamma \)-matrixes ”survive” which contain each of \( \gamma \)-matrix in degrees 0, 2 or 4 (as well as in the case of variables \( \sigma \)) and use designations (1.7).

Let’s prove positivity condition with the help of mathematical induction method. During the process of calculations the numerical factors are ignored since only the question about the sign of the loop is significant here.

We begin the calculation with verification of the fact that the matrix factor of elementary cell, corresponding to elementary loop binding elementary cell with vertices

\[ \mathbf{x}, \quad \mathbf{x} + \mathbf{e}_1, \quad \mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{x} + \mathbf{e}_2, \] (1.8)

is equal to unit. Indeed, according to (1.6) this matrix factor is equal to

\[ (\gamma_x \gamma_{x+e_2}) (\gamma_x \gamma_{x+e_1}) (\gamma_{x+e_1} \gamma_{x+e_1+e_2}) (\gamma_{x+e_2} \gamma_{x+e_1+e_2}) = 1. \] (1.9)

The further calculation is based on the possibility of presentation of loop matrix factor as a product of matrix factor of smaller loop, binding smaller number of elementary cells as compared with initial loop, and the matrix factors of elementary cells supplying smaller loop up to the initial one. Each of the equalities in Figs. 1 a and 1 b represents graphically equality between loop matrix factor in the left-hand side and smaller loop matrix factor times elementary cells matrix factors. All loop matrix factors are calculated according to order in (1.6). In Fig. 1 solid lines describe loops while dotted lines divide loop interiors into elementary cells. Equations in Fig. 1 are established easily by direct testing. For example, for Figs. 1 a 1 and 2 we have:
Fig. 1a
Fig. 1 b

\[
\begin{align*}
\left[ (\gamma_1\gamma_4)(\gamma_1\gamma_2)(\gamma_2\gamma_3)(\gamma_3\gamma_6)(\gamma_4\gamma_5)(\gamma_5\gamma_6) \right] &= \\
&= \left[ (\gamma_1\gamma_4)(\gamma_1\gamma_2)(\gamma_2\gamma_5)(\gamma_4\gamma_5) \right] \left[ (\gamma_2\gamma_5)(\gamma_2\gamma_3)(\gamma_3\gamma_6)(\gamma_5\gamma_6) \right], \\
\left[ (\gamma_7\gamma_9)(\gamma_7\gamma_8)(\gamma_8\gamma_10)(\gamma_9\gamma_11)(\gamma_10\gamma_12)(\gamma_11\gamma_12) \right] &= \\
&= \left[ (\gamma_7\gamma_9)(\gamma_7\gamma_8)(\gamma_8\gamma_10)(\gamma_9\gamma_10) \right] \left[ (\gamma_9\gamma_11)(\gamma_9\gamma_10)(\gamma_10\gamma_12)(\gamma_11\gamma_12) \right],
\end{align*}
\]

and so on. Developing loops and their matrix factors following to the schemes in Fig. 1, one can realize matrix factors of any loops as the product of elementary cells
matrix factors which are equal to unit according to (1.9). Thus the equality of any loops matrix factors to unit is proved inductively. Hence the positivity condition is established.

It is evident that each multiplier \((\lambda + \mu \gamma_x \gamma_{x+e})\) in the braces in (1.6) is the matrix of orthogonal rotation by the angle \(\psi\) in spinor representation in \(MN\)-dimensional Euclidean space in the plane marked by the pair of \(\gamma\)-matrices in the multiplier. Therefore, all product in the braces in (1.6) is the matrix of an orthogonal rotation in spinor representation in \(MN\)-dimensional Euclidean space. It is known \([6]\) that this matrix can be expressed in the form

\[
\mathcal{U} = \exp \left( \frac{1}{4} \omega_{x,y} \gamma_x \gamma_y \right), \quad \omega_{x,y} = -\omega_{y,x}, \tag{1.11}
\]

and

\[
\mathcal{U}^\dagger \gamma_x \mathcal{U} = \mathcal{O}_{x,y} \gamma_y, \quad \mathcal{O}_{x,y} \equiv (e^\omega)_{x,y} = \delta_{x,y} + \omega_{x,y} + \frac{1}{2!} \omega_{x,z} \omega_{z,y} + \ldots. \tag{1.12}
\]

The trace of the braces in (1.6), i.e. the trace of the matrix \(\mathcal{U}\), is expressed simply through the eigenvalues of real orthogonal matrix \(\mathcal{O}_{x,y}\). Let the set of numbers

\[
(\rho_1, \overline{\rho}_1, \rho_2, \overline{\rho}_2, \ldots, \rho_{MN/2}, \overline{\rho}_{MN/2}) \tag{1.13}
\]

form the complete set of eigenvalues of the matrix \(\mathcal{O}_{x,y}\). Then (see Appendix)

\[
\text{tr} \mathcal{U} = \prod_{k=1}^{MN/2} \left[ 2 \text{ch} \left( \frac{\ln \rho_k}{2} \right) \right] = \prod_{k=1}^{MN/2} \left[ 2 \cos \left( \frac{\phi_k}{2} \right) \right] = \prod_{k=1}^{MN/2} \left( \sqrt{\rho_k} + \sqrt{\overline{\rho}_k} \right),
\]

\[
\rho_k = e^{i\phi_k}. \tag{1.14}
\]

In statistical limit, when \(M, N \to \infty\), the problem of matrix \(\mathcal{O}_{x,y}\) diagonalization simplifies radically since the matrices \(\mathcal{O}_{x,x+z}\) and \(\omega_{x,x+z}\) become dependent only on \(z\) at relatively small distances from boundary of the lattice. This property is known as translational invariance. Therefore the diagonalization of these matrices is performed by means of Fourier transformation, i.e. by passing to quasi-momentum representation. The following complete orthonormal set of functions on the lattice
is used for that purpose:

\[ |p\rangle \equiv \Psi_p(m) = \frac{1}{\sqrt{M}} e^{ipm}, \quad |q\rangle \equiv \Psi_q(n) = \frac{1}{\sqrt{N}} e^{iqn}, \]

\[ p = -\frac{\pi(M - 2)}{M}, -\frac{\pi(M - 4)}{M}, \ldots, 0, \frac{2\pi}{M}, \ldots, \pi, \]

\[ q = -\frac{\pi(N - 2)}{N}, -\frac{\pi(N - 4)}{N}, \ldots, 0, \frac{2\pi}{N}, \ldots, \pi, \]

\[ |k\rangle \equiv \Psi_k(x) = \Psi_p(m)\Psi_q(n), \quad k = (p, q), \]

\[ \sum_x \overline{\Psi}_k(x)\Psi_{k'}(x) = \delta_{kk'} \longleftrightarrow \sum_k \Psi_k(x)\overline{\Psi}_k(x') = \delta_{xx'}. \quad (1.15) \]

II. CALCULATION OF EIGENVALUES

At first let’s solve a particular problem: define for \( x = (m, n) \) the spinor rotation operator transforming only \( \gamma \)-matrices with indexes \( (x + m'e_1) \) and \( (x + m'e_1 + e_2) \), where \( m' = 0, \pm 1, \ldots \):

\[ \mathcal{U}^{(n)} \equiv \ldots \left[ (\lambda + \mu \gamma_x \gamma_{x+e_2}) (\lambda + \mu \gamma_x \gamma_{x+e_1}) \right] \times \]

\[ \times \left[ (\lambda + \mu \gamma_{x+e_1} \gamma_{x+e_1+e_2}) (\lambda + \mu \gamma_{x+e_1} \gamma_{x+2e_1}) \right] \ldots . \quad (2.1) \]

Here we have the ordered product of spinor rotation matrixes along only one line \( n \). The corresponding orthogonal matrix is defined just as in (1.12):

\[ \mathcal{U}^{(n)}\gamma_y \mathcal{U}^{(n)} = \mathcal{O}^{(n)}_{y,z} \gamma_z. \]

According to definitions (1.6), (1.12) and (2.1)

\[ \mathcal{U} = \ldots \mathcal{U}^{(n-1)}\mathcal{U}^{(n)}\mathcal{U}^{(n+1)} \ldots, \quad \mathcal{O} = \ldots \mathcal{O}^{(n-1)}\mathcal{O}^{(n)}\mathcal{O}^{(n+1)} \ldots . \quad (2.3) \]

In the right hand sides of Eqs. (2.3) the ordered products of matrixes corresponding to lines take place.

Let’s find the obvious expression for the matrix \( \mathcal{O}^{(n)}_{y,z} \) with the help of (2.2). This calculation is based on the simple relations

\[ (\lambda + \mu \gamma_x \gamma_{x+e})^\dagger \gamma_x (\lambda + \mu \gamma_x \gamma_{x+e}) = (\cos \psi) \gamma_x + (\sin \psi) \gamma_{x+e}, \quad (2.4) \]

\[ (\lambda + \mu \gamma_x \gamma_{x+e})^\dagger \gamma_{x+e} (\lambda + \mu \gamma_x \gamma_{x+e}) = (\cos \psi) \gamma_{x+e} - (\sin \psi) \gamma_x, \quad (2.5) \]
\[(\lambda + \mu \gamma x \gamma x + e)^\dagger \gamma_z (\lambda + \mu \gamma x \gamma x + e) = \gamma_z, \quad z \neq x, x + e, \quad (2.6)\]

following directly from Eqs. (1.4) and (1.7). Either vector \(e_1\) or vector \(e_2\) can be taken instead of a vector \(e\) in Eqs. (2.4)-(2.6).

Consider the first case when \(y = (m, n)\). Then in the left hand side of Eq. (2.2) all spinor rotation matrixes forming \(U(n)^\dagger\) placed to the right from \((\lambda + \mu \gamma y - e_1 \gamma y)^\dagger\) and all spinor rotation matrixes forming \(U(n)\) placed to the left from the matrix \((\lambda + \mu \gamma y - e_1 \gamma y)\) are cancelled mutually since they do not catch on matrix \(\gamma y\). But the "facings"

\[(\lambda + \mu \gamma y - e_1 \gamma y)^\dagger \ldots (\lambda + \mu \gamma y - e_1 \gamma y)\]

transform \(\gamma y\) in accordance with Eq. (2.5) in which one must make the identifications \(e = e_1, (x + e) = y\). So we obtain the matrix element

\[O_{y, y-e_1}^{(n)} = -(\sin \psi), \quad y = (m, n). \quad (2.7)\]

Further only Eq. (2.4) is used in which the identifications \(x = (y + m' e_1), m' = 0, 1, \ldots\) are made in series, and at each \(m'\) at first \(e = e_2\) and then \(e = e_1\) are put. This process is pictured symbolically in Fig. 2 where the bold point denotes the position of the initial \(\gamma\)-matrix placed in "facings" in the left hand side of Eq. (2.2), and the arrows show where its image comes under the action of the described linear transformations. Every time the result is reduced to multiplication by \((\sin \psi)\) if the matrix is transfered up or to the right, and to multiplication by \((\cos \psi)\) if the matrix is left on the former place. The matrix transfered up further remains unchanged. Thus the following result is obtained:

\[O_{y, y+m'e_1}^{(n)} = [\sin \psi \cos \psi]^{m'} (\cos \psi)^3, \quad O_{y, y+m'e_1+e_2}^{(n)} = [\sin \psi \cos \psi]^{m'}, \quad m' = 0, 1, \ldots, \quad y = (m, n). \quad (2.8)\]
The matrix elements as $O_{y+e_2,z}^{(n)}$ are calculated similarly. The process of this calculation is pictured in Fig. 3 where the bold point again denotes the position of $\gamma$-matrix at the beginning of the process, and the arrows show the subsequent movements of the images of initial matrix. At the first step the relation (2.5) is used in which the identifications $e = e_2, x = y$ are made. The relation (2.4) is used at all subsequent steps. As a result of simple calculation we obtain:

$$O_{y+e_2,y+e_2}^{(n)} = (\cos \psi), \quad O_{y+e_2,y+(m'+1)e_1+e_2}^{(n)} = -(\sin \psi)^3[\sin \psi \cos \psi]^{m'},$$

$$O_{y+e_2,y+m'e_1}^{(n)} = -[\sin \psi \cos \psi]^{m'+1}, \quad m' = 0, 1, \ldots, \quad y = (m, n).$$

Now it is necessary to make partial diagonalization of the orthogonal matrix using its translational invariance. For that purpose we make the partial Fourier transformation along the lines of $\gamma$-matrixes with the help of Eqs. (1.15):

$$\gamma_n(p) = \sum_m \bar{\Psi}_p(m)\gamma_{m,n} = \gamma_n^\dagger(-p), \quad \left[\gamma_n(p), \gamma_{n'}(p')\right]_+ = 2\delta_{n,n'}\delta_{p,p'}.$$  \hspace{1cm} (2.10)

Let’s pass to Fourier transformation along the lines in Eq. (2.2). Using translational invariance we rewrite this equation in the form

$$U^{(n)}\gamma_{n'}^\dagger(p)U^{(n)} = \left[\sum_{m'} O_{m,n';m+m',n'}^{(n)}\Psi_p(m')\right] \gamma_{n'}(p).$$ \hspace{1cm} (2.11)

The matrix in square bracket in the right hand side of Eq. (2.11) is designated as
and it is calculated with the help of Eqs. (2.7)-(2.9):

$$O_{p; n', n''}^{(n)} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 0 & 0 & 0 & 0 \\
\cdot & 0 & 1 & 0 & 0 & 0 \\
\cdot & 0 & 0 & a_p & b_p & 0 \\
\cdot & 0 & 0 & -b_p & c_p & 0 \\
\cdot & 0 & 0 & 0 & 0 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot 
\end{pmatrix}_{n'=n} \quad (2.12)
$$

Here

$$\left\{ \begin{array}{c}
a_p \\
c_p
\end{array} \right\} = \frac{\cos \psi - e^{\mp ip} (\sin \psi)}{1 - e^{ip} [\sin \psi \cos \psi]}, \quad b_p = \frac{[\sin \psi \cos \psi]}{1 - e^{ip} [\sin \psi \cos \psi]}, \quad (2.13)$$

and diagonal elements $a_p$ and $c_p$ are settled on $n$-th and $(n+1)$-th places, correspondingly. The direct check shows that the matrix (2.12) is unitary.

Thus, according to (2.3)

$$O_{p; n', n''} = \sum_{n'} O_{m,n'; m+m',n''} \Psi_p (m') = \left\{ \ldots O_{p}^{(n-1)} O_{p}^{(n)} O_{p}^{(n+1)} \ldots \right\}_{n',n''}. \quad (2.14)$$

The product (2.14) of matrix of the kind (2.12) is calculated easily:

$$O_{p; n, n+n'} = \left\{ \begin{array}{c}
0, \quad \text{for } n' < -1 \\
(-b_p), \quad \text{for } n' = -1 \\
(c_p b_p) a_p, \quad \text{for } n' \geq 0
\end{array} \right. \quad (2.15)$$

To find the eigenvalues of orthogonal matrix (1.12) one must calculate the Fourier components of (2.15) along the column:

$$\rho_{p,q} = O_{p,q} = \sum_{n'} O_{p; n, n+n'} \Psi_q (n') = \frac{a_p c_p + b_p^2 - e^{-iq} b_p}{1 - e^{iq} b_p} = \eta_{p,q},$$

$$\eta_{p,q} = 1 - (e^{-ip} + e^{-iq}) (\sin \psi \cos \psi). \quad (2.16)$$

Equation (2.16) is obtained with the help of (2.13).
III. THE PARTITION FUNCTION

It follows from Eqs. (1.14), (1.15) and (2.16) that free energy is proportional to the following integral

\[
\ln \left\{ \prod_{p,q} (\sqrt{\rho_{p,q}} + \sqrt{\rho_{p,q}}) \right\} = \frac{MN}{4\pi^2} \int_{-\pi}^{+\pi} dp \int_{-\pi}^{+\pi} dq \ln (\sqrt{\rho_{p,q}} + \sqrt{\rho_{p,q}}) = \\
= \frac{MN}{4\pi^2} \int_{-\pi}^{+\pi} dp \int_{-\pi}^{+\pi} dq \ln \left( \eta_{p,q} + \bar{\eta}_{p,q} \right) = \frac{MN}{4\pi^2} \int_{-\pi}^{+\pi} dp \int_{-\pi}^{+\pi} dq \ln (\eta_{p,q} + \bar{\eta}_{p,q}) .
\]

(3.1)

The last equality in (3.1) is true because

\[
\int_{-\pi}^{+\pi} dp \int_{-\pi}^{+\pi} dq \ln \eta_{p,q} = \int_{-\pi}^{+\pi} dp \int_{-\pi}^{+\pi} dq \ln \bar{\eta}_{p,q} = 0.
\]

(3.2)

Prove the equalities (3.2). According to (2.16)

\[
\eta_{p,q} = (1 - \alpha e^{-ip}) (1 - \beta e^{-iq}) ,
\]

\[
\alpha = \frac{1}{2} (\sin 2\psi) , \quad \beta = \frac{\sin 2\psi}{2 - (\sin 2\psi)e^{-ip}} , \quad |\alpha| \leq \frac{1}{2}, \quad |\beta| \leq 1 ,
\]

(3.3)

and \(\beta e^{-iq} = 1\) only in the case \((\sin 2\psi) = 1, \ p = q = 0\). Therefore

\[
\int_{-\pi}^{+\pi} dp \int_{-\pi}^{+\pi} dq \ln \eta_{p,q} = \\
= \int_{-\pi}^{+\pi} dq \left\{ \int_{-\pi}^{+\pi} dp \ln (1 - \alpha e^{-ip}) \right\} + \int_{-\pi}^{+\pi} dp \left\{ \int_{-\pi}^{+\pi} dq \ln (1 - \beta e^{-iq}) \right\} = \\
= -\sum_{n=1}^{\infty} \frac{1}{n} \left\{ \alpha^n \int_{-\pi}^{+\pi} dq \int_{-\pi}^{+\pi} dp e^{-inp} + \int_{-\pi}^{+\pi} dp \beta_p^n \int_{-\pi}^{+\pi} dq e^{-inq} \right\} = 0 .
\]

(3.4)

The second equality in (3.2) is proved similarly.

Eventually, we can write out the free energy with the help of formulas (1.6), (1.7),
(1.14), (2.16) and (3.1):

\[ F = -T \ln Z = \]

\[ = -MNT \left\{ \ln 2 + \frac{1}{8\pi^2} \int_{-\pi}^{+\pi} d p \int_{-\pi}^{+\pi} d q \ln \left[ (\text{ch} 2\theta)^2 - (\text{sh} 2\theta) (\cos p + \cos q) \right] \right\} = \]

\[ = -MNT \left\{ \ln 2 - \ln(1 - x^2) + \right. \]

\[ + \left. \frac{1}{8\pi^2} \int_{-\pi}^{+\pi} d p \int_{-\pi}^{+\pi} d q \ln \left[ (1 + x^2)^2 - 2x(1 - x^2)(\cos p + \cos q) \right] \right\}, \quad x = \text{th} \theta. \]  

(3.5)

The last expression in (3.5) coincides with the free energy of 2D Ising model given in [3].

The temperature of phase transition is obtained directly from (3.1) and (2.16). Indeed, free energy has a peculiarity at the phase transition point. It is seen from (3.1) that this takes place when \( (\eta_{p,q} + \overline{\eta}_{p,q}) \to 0 \) for a part of quasi-momenta, which occurs only if (see (2.16))

\[ \sin \psi_c = \cos \psi_c = \frac{1}{\sqrt{2}}, \quad p \to 0, \quad q \to 0. \]  

(3.6)

Thus with the help of (3.6) and (1.7) we find that critical temperature is found from the equation

\[ \text{th} \theta_c = \frac{1 - \cos \psi_c}{\sin \psi_c} = \sqrt{2} - 1. \]  

(3.7)

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APPENDIX A

Here the formula (1.14) is proved.

Let \( \{v^{(k)}_x, \overline{v}^{(k)}_x\}, \quad k = 1, \ldots, MN/2, \) be the complete orthonormal set of eigenvectors of the matrix \( \mathcal{O}_{x,y} \), so that the eigenvalue \( \rho_k (\overline{\rho}_k) \) corresponds to the eigenvector \( v^{(k)}_x (\overline{v}^{(k)}_x) \). Further also the designation

\[ \{v^{(k)}_x, \overline{v}^{(k)}_x\} \equiv \{v^a_x\}, \quad a = 1, \ldots, MN \]
is used. We shall consider the introduced vectors as vector-columns and the upper indices $^T$ and $^\dagger$ denote the transposition and Hermitian conjugation of vectors and matrices. By definition

$$v^{(k)}T v^{(k')} \equiv \sum_x v^{(k)}_x v^{(k')}_x, \quad v^{(k)}\dagger v^{(k')} \equiv \sum_x \overline{v^{(k)}_x} \overline{v^{(k')}_x}. \quad (A1)$$

The given definitions imply the following formulas:

$$v^{(k)}T v^{(k')} = 0, \quad v^{(k)}\dagger v^{(k')} = \delta_{kk'}, \quad (A2)$$

$$U_{xa} \equiv v^a_x \text{ or } U \equiv \left( v^{(1)}, \overline{v^{(1)}}, v^{(2)}, \overline{v^{(2)}}, \ldots \right), \quad (U\dagger U)_{ab} = \delta_{ab}, \quad (A3)$$

$$(U\dagger \omega U)_{ab} = \text{diag} \left( \ln \rho_1, -\ln \rho_1, \ln \rho_2, -\ln \rho_2, \ldots \right) \equiv \Delta_{ab}. \quad (A5)$$

Due to (A3) and (A5) we have

$$\frac{1}{4} \gamma_{x\omega x,y} \gamma_y = \frac{1}{4} \left( \gamma_x U_{xa} \right) \Delta_{ab} \left( U_{by} \gamma_y \right). \quad (A6)$$

$2^{MN/2} \times 2^{MN/2}$-matrixes

$$c^\dagger_k = \gamma_x v^{(k)}_x, \quad c_k = \overline{v^{(k)}_x}. \quad (A7)$$

possess all properties of fermion creation and annihilation operators. Indeed, in consequence of (1.4) and (A2)

$$[c_k, c^\dagger_{k'}]_+ = \delta_{kk'}, \quad [c_k, c_{k'}]_+ = [c^\dagger_k, c^\dagger_{k'}]_+ = 0. \quad (A8)$$

According to the definitions (A3) and (A7) we have

$$\gamma_x U_{xa} = \left( c^\dagger_1, c_1, \ldots, c^\dagger_{MN/2}, c_{MN/2} \right). \quad (A9)$$

With the help of Eqs. (A5), (A8) and (A9) the quantity (A6) is rewritten as

$$\frac{1}{4} \gamma_{x\omega x,y} \gamma_y = \frac{1}{2} \sum_{k=1}^{MN/2} \left[ \ln \rho_k \left( c^\dagger_k c_k - c_k c^\dagger_k \right) \right] = \sum_{k=1}^{MN/2} \left[ \left( \ln \rho_k \right) c^\dagger_k c_k - \frac{1}{2} \ln \rho_k \right]. \quad (A10)$$

Equality (1.14) follows immediately from (A10) since the calculation of the trace in terms of $\gamma$-matrixes is equivalent to the calculation of trace in terms of the corresponding fermionic operators (A7).

[1] L. Onsager, Phys. Rev. 65 (1944) 117.
For simplicity it is assumed that the numbers $M$ and $N$ are even.