Dimension of Conformal Blocks in Five Dimensional Kähler-Chern-Simons Theory

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We briefly review the Kähler-Chern-Simon theory on 5-manifolds which are trivial circle bundles over 4-dimensional Kähler manifolds and present a detailed calculation of the path integral, using the method of Blau and Thompson.

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I. INTRODUCTION

Conformal Field Theories (CFTs) are powerful tools for investigating string theory and for addressing certain mathematical questions, for example, mirror pairs. However the technology of CFTs derives from 2D quantum field theory, and this limits applicability to situations that are, at some level, two dimensional.

There is evidence\textsuperscript{1,2} that theories involving objects with two or more degrees of freedom might be important for a full understanding of gravity and elementary particles. Thus it would seem reasonable to examine theories which are higher dimensional versions of CFTs. However, in general such theories do not have nearly as rich of a structure as 2D CFTs, and this is because the conformal group in three or more dimensions is finite dimensional, in contrast to the infinite dimensional Kac-Moody group which arises in 2D CFTs.

However there are higher dimensional theories that possess many of the important characteristics of 2D CFTs. These theories are not conformally invariant in the usual sense, but they are associated with infinite dimensional algebras that strongly resemble, indeed, generalize the Kac-Moody groups\textsuperscript{3-5}. The most directly relevant of these theories are the WZW\textsubscript{4} models\textsuperscript{6}. However, these are in turn associated (in the same manner that WZW\textsubscript{2} models are associated with Chern-Simons gauge theories in 3D) with higher dimensional gauge theories, namely 5D Chern-Simons and Kähler-Chern-Simons theories\textsuperscript{7-9}. What is especially intriguing here is that two dimensional integrable models can be described by these theories.

In the beautiful paper by Losev et. al.\textsuperscript{6}, WZW\textsubscript{4} models were fairly exhaustively studied. However, the closely related KCS models have not been as thoroughly studied. In this note, and in further work in progress, I intend to explore more deeply the quantum aspects of KCS theories.

Suppose that $P$ is a trivial principle $SU(r + 1)$ bundle over a four dimensional Kähler manifold $(M, \omega)$, where $\omega$ is the Kähler form and $A$ is an arbitrary connection on $P$. In\textsuperscript{9,10} Nair and Schiff introduced the so-called Kähler-Chern-Simons action:

$$S = \frac{1}{4\pi} \int_{M \times \mathbb{R}} \omega \wedge Tr[CS(A)] + dt \wedge Tr[(\Phi^{2,0} \wedge F + \Phi^{0,2} \wedge F)], \quad (1)$$

where $CS(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$ is the Chern-Simons 3-form, $t$ is the "time" coordinate on $\mathbb{R}$, $\Phi^{2,0}$ and $\Phi^{0,2}$ are two Lagrange multipliers that are Lie algebra valued $(2, 0)$ and $(0, 2)$
forms on $M$ respectively and $F$ is the curvature 2-form corresponding to a connection $A$. According to geometric quantization the physical Hilbert space, which is called the space of \textit{conformal blocks}, should be $\mathcal{H} = H^0(\mathcal{M}, \mathcal{L})$ where $\mathcal{M}$ is the phase space and $\mathcal{L}$ is the prequantum line bundle. In the author presented a formula for the dimension of the conformal blocks of the Kähler-Chern-Simons theory. The aim of this paper is to derive this formula by using Blau and Thompson’s method.

In section 2 we will analyze this system at the classical level. We show that the phase space is just the moduli space of Anti-Self-Dual(ASD) instantons. In section 3 and the appendix we will calculate the partition function of the Kähler-Chern-Simons theory on $M \times S^1$ based on the diagonalization assumption.

II. CLASSICAL KÄHLER-CHERN-SIMONS THEORY

First let us split the connection $A$ into spatial and "time" parts $A = A_0 + B$ where $A_0$ is a Lie-algebra valued 1-form on $\mathbb{R}$ and $B$ is a Lie-algebra valued 1-form on $M$. Substituting $A = A_0 + B$ into the Kähler-Chern-Simons action (1) we get

$$S = \frac{1}{4\pi} \int_{M \times \mathbb{R}} \omega \wedge Tr[2A_0 \wedge (dB + B \wedge B) + B \wedge d_0 B] + dt \wedge Tr(\Phi^{2,0} \wedge F_{B}^{0,2} + \Phi^{0,2} \wedge F_{B}^{2,0}),$$

where $d$ and $d_0$ are the differential operators on $M$ and $\mathbb{R}$, respectively, and $F_{B}^{0,2}(F_{B}^{2,0})$ is the $(0,2) (2,0)$ part of the curvature 2-form of connection $B$ on $M$. Varying the multipliers $\Phi^{2,0}$ and $\Phi^{0,2}$ in the Kähler-Chern-Simons action (1) gives equations

$$F_{B}^{0,2} = F_{B}^{2,0} = 0.$$  \hspace{1cm} (3)

Further, varying $A_0$ in the first term in (2) gives

$$\omega \wedge F_B = 0.$$  \hspace{1cm} (4)

These are the equations of motion. Then the phase space is the moduli space $\mathcal{M}$ of the solutions of (3,4), which is the moduli space of ASD instantons. On $\mathcal{M}$ there is a natural symplectic form

$$\Omega(a, b) := \frac{1}{2\pi} \int_{M} \omega \wedge a \wedge b.$$  \hspace{1cm} (5)

According to geometric quantization, the physical Hilbert space, which is called the space of \textit{conformal blocks}, should be $\mathcal{H} = H^0(\mathcal{M}, \mathcal{L})$ where $\mathcal{M}$ is the phase space and $\mathcal{L}$ is the
prequantum line bundle. Further, the Donaldson-Yau-Uhlenbeck theorem tells us that $\mathcal{M}$ is equivalent to the moduli space of semi-stable bundles\(^{13}\). As with three dimensional Chern-Simons theory\(^{12}\), we can borrow the statistical mechanics formula

$$Z_{M \times S^1} = \text{Tr} e^{-\beta H},$$

for a circle of radius $\beta$ to calculate the dimension of the physical Hilbert space. From now on we only consider the Kähler-Chern-Simons theory on $M \times S^1$.

### III. QUANTIZATION

According to the geometric quantization program\(^{6,11,14}\) the physical Hilbert space $\mathcal{H}$ should be the space of sections of the Quillen determinant bundle\(^{14-17}\). Our purpose is to try to calculate $\dim \mathcal{H}$.

#### A. Gauge Fixing

From now on we will focus on the following action and add the constraints (3) on the $B$ fields. In next subsection, I will give the definition of the path integral over the constraint surface by using Henneaux and Teitelboim’s formula\(^{18}\). Hence we write

$$S = \frac{1}{4\pi} \int_{M \times S^1} \omega \land \text{Tr} [2A_0 \land (dB + B \land B) + B \land d_0 B],$$

$$= \frac{1}{4\pi} \int_{M \times S^1} \omega \land \text{Tr} [2A_0 \land dB + B \land D_0 B],$$

where $D_0 := d_0 + [A_0, \cdot]$ is the covariant derivative operator on $S^1$. Since the Lie algebra $\mathfrak{su}(r+1)$ has the orthogonal decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$, the gauge fields $A_0$ and $B$ have the following decomposition:

$$A_0 = A_0^t + A_0^k,$$

$$B = B^t + B^k.$$

Substituting eq. (8, 9) into the action (7) we get

$$S = \frac{1}{4\pi} \int_{M \times S^1} \omega \land \text{Tr} [2A_0^t \land dB^t + 2A_0^k \land dB^k + \notag$$

$$B^t \land d_0 B^t + B^k \land D_0 B^k].$$

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2. \(^{12}\) Witten, E. (1988). Topological quantum field theory. Communications in Mathematical Physics, 117(3), 353-387.
3. \(^{11}\) Henneaux, M., & Teitelboim, C. (1992). Quantization of gauge systems. Vol. 1.
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7. \(^{11}\) Simons, J. (1983). G-structures and geometric quantization. Communications in Mathematical Physics, 87(4), 55-88.
8. \(^{17}\) Witten, E. (1988). Superconformal field theory on two-dimensional anti-de Sitter space. Communications in Mathematical Physics, 121(2), 551-567.
According to\cite{12} we can choose the following gauge fixing

\begin{align}
\partial_t A_{0t} &= 0, \\
A^k_0 &= 0.
\end{align}

Here $A_{0t}$ are the components of $A_0$, i.e. $A_0 = A_{0t}dt$. This was suggested by the corresponding insertion for the 3-dimensional Chern-Simons theory\cite{12}. Thus we insert the following term into the action:

\[
\frac{1}{4\pi} \int_{M \times S^1} \omega \wedge Tr(\Psi \wedge A_0 + 2i\omega \wedge cD_0c) \tag{13}
\]

where $c, \bar{c}$ are Grassmanian-valued functions and $\Psi$ is a Lie algebra valued 2-form on $M$. The constraints on $c, \bar{c}$ and $\Psi$ are

\[
\oint_{S^1} c^i = \oint_{S^1} \bar{c}^i = \oint_{S^1} \Psi^t \wedge dt = 0. \tag{14}
\]

The first term of eq. (13) is equivalent to adding the constraint (12). Then after choosing the gauge fixing the action becomes

\[
S = \frac{1}{4\pi} \int_{M \times S^1} \omega \wedge Tr(2A^t_0 \wedge dB^t + B^t \wedge d_0 B^t + B^k \wedge D_0 B^k \\
+ 2i\omega \wedge \bar{c}^i D_0 c^i + 2i\omega \wedge \bar{c}^k D_0 c^k), \tag{15}
\]

where the $B$ fields satisfy the constraints (3) and the $A_0$ fields satisfy the constraint (12). Before entering the next subsection, let us analyze the gauge fixing carefully. We notice that in the action (15) there are still residual gauge symmetries

\[
A^t_0 \rightarrow A^t_0, \\
B \rightarrow g^{-1}Bg + g^{-1}dg, \tag{16}
\]

where $g$ is a map from $M \times S^1$ to the maximum torus $T$ in $SU(r+1)$ satisfying $\partial_t g = 0$. Later we will use this residual gauge symmetry to regularize the functional determinant.

Since $M$ is a Kähler manifold, the 1-forms $B^t$ and $B^k$ have the following decomposition:

\[
B^t = \sum_{i=1}^{2}(B^t_i \phi^i + B^t_i \bar{\phi}^i), \tag{17}
\]

\[
B^k = \sum_{i=1}^{2}(B^k_i \phi^i + B^k_i \bar{\phi}^i). \tag{18}
\]
where $\phi^i, \bar{\phi}^i$ are the holomorphic and antiholomorphic 1-form fields respectively, and satisfy

$$\omega = \frac{i}{2} \sum_{i=1}^{2} \phi^i \wedge \bar{\phi}^i.$$  

Plugging (17) and (18) into the action $S$ (15), we get

$$S = \frac{1}{2\pi} \int_{M \times S^1} \omega \wedge dt \wedge Tr(A^i_{0t} \wedge dB^i) + idV \wedge dt Tr(\sum_i B^i_t \partial_t B^i_t)$$
$$+ \sum_i B^i_t \nabla_t B^i_t + c^i \nabla_t c^i + \bar{c}^i \nabla_t \bar{c}^i),$$

where $dV = \omega \wedge \omega$ is the volume form of $M$ and $\nabla_t = \frac{d}{dt} + [A^i_{0t}, \cdot]$. $B^t$ is independent of time. This can be shown by using the Fourier expansion with respect to the time circle $S^1$:

$$B^t = B^t_0 e^{2\pi it}. \quad (20)$$

Thus

$$\int_{M \times S^1} \omega \wedge dt \wedge Tr(A^i_{0t} \wedge dB^i) = \int_{M \times S^1} \omega \wedge dt \wedge Tr(A^i_{0t} \wedge dB^i_0). \quad (21)$$

Here we already use the gauge fixing condition eq. (11). In other words, the $B^t$ in term $\int_{M \times S^1} \omega \wedge dt \wedge Tr(A^i_{0t} \wedge dB^i)$ only contains the time independent part.

\section*{B. Path Integral}

We know that Blau and Thompson’s method has not been established in higher dimensions\textsuperscript{20}. So here for simplicity we assume the diagonalization is applicable and assume that the complete set of obstructions to diagonalizing are all the $T-$bundle restrictions of the trivial $SU(r+1)$ bundle\textsuperscript{20}, where $T$ is the maximum torus of $SU(r+1)$. So after gauge fixing, the partition function $Z$ is defined as

$$Z := \int DA^i_{0t} dB^i dB^i DB^i DB^i \hat{\Omega}^{-1} \det \Omega e^{iS},$$

where the $B, \bar{B}$ fields are on the constraint surface $F^{(0,2)} = F^{(2,0)} = 0$. In order to avoid symbol confusion we use $B^i$ to replace $B^i_t$. According to the analysis of last section, $B^i$ is the time independent part of the full $B^i$. Here we borrow Henneaux and Teitelboim’s path integral definition on the constraint surface\textsuperscript{18}. We know in eq. (1) the equations $F^{(0,2)} = F^{(2,0)} = 0$ are the second class constraints.\textsuperscript{9} So according to Henneaux and Teitelboim’s definition we need to insert $\sqrt{\det \Omega}$ as the determinant of the symplectic form on the constraint surface. Following Henneaux and Teitelboim’s book\textsuperscript{18}, it is easy to show that this definition (22) is
\[
Z := \int DA_0^t DB^t Dc^t D\bar{c}^t Dc^k D\bar{c}^k \prod_{i=1}^2 DB^t_i DB^t_i DB^k_i D\bar{B}^k_i \cdot \sqrt{\text{Det}\left\{ F^{(2,0)}, F^{(0,2)} \right\}}_{P.B.} \delta(F^{(2,0)}) \delta(F^{(0,2)}) e^{iS},
\]

where \( \{F^{(2,0)}, F^{(0,2)}\}_{P.B.} \) is the Poisson bracket between \( F^{(2,0)} \) and \( F^{(0,2)} \), \( \delta(F^{(2,0)}) \), \( \delta(F^{(0,2)}) \) are the Dirac delta functions coming from the path integral over the Lagrangian multipliers \( \Phi, \bar{\Phi}, \) and \( B, \bar{B} \) fields are unconstrained.

Now we do the "background expansion". First choose a background \( B^t_c, \bar{B}^t_c \) which is time independent and satisfy \( F^{(0,2)}_{B^t_c} = F^{(2,0)}_{B^t_c} = 0 \). Then expand a connection in the small neighborhood of \( B^t_c, \bar{B}^t_c \) on the constraint surface as
\[
\begin{align*}
B &= B^t_c + B^t_q + B^k_q, \\
\bar{B} &= \bar{B}^t_c + \bar{B}^t_q + \bar{B}^k_q,
\end{align*}
\]

where \( B^t_q, \bar{B}^t_q \) is time dependent. In another word, for an arbitrary connection \( B \) using the Fourier expansion with respect to the time circle \( S^1 \), we have
\[
B = B^t_n e^{2n\pi it} + B^k_n e^{2n\pi it}.
\]

The background expansion means that \( B^t_c = B^t_0 \ll B - B_0^t \) and \( B^t_c = B^t_0 \) is on the constraint surface. Hence, we have the following equations for \( B^t_c, \bar{B}^t_c, B^t_q, \bar{B}^t_q, B^k_q, \bar{B}^k_q \):
\[
\begin{align*}
\partial B^t_c &= \partial \bar{B}^t_c = \partial B^t_q = \partial \bar{B}^t_q = 0 \\
\partial_{B^t_q} B^k_q &= \partial_{\bar{B}^t_q} \bar{B}^k_q = 0.
\end{align*}
\]

Thus substituting the background expansion (24) into the action eq. (19), we get
\[
S = \frac{1}{2\pi} \int_{M \times S^1} \omega \wedge dt \wedge Tr(A_0^t \wedge dB^t_c) + idV \wedge dt Tr(\sum_i B^t_q \partial_{B^t_q} \bar{B}^t_q + \sum_i B^k_q \nabla_{\bar{B}^t_q} + c^t \nabla_{c^t} + c^k \nabla_{c^k}),
\]

We define \( Z_{B^t_c} \) as
\[
Z_{B^t_c} := \int Dc^t D\bar{c}^t Dc^k D\bar{c}^k \prod_{i=1}^2 DB^t_q DB^t_q DB^k_q D\bar{B}^k_q \sqrt{\text{Det}_{B^t_c} e^{iS}}.
\]
Here we expand \( \sqrt{\text{Det} \Omega} \) around \( B^t_c = B^t_c + \tilde{B}^t_c \) and denote the leading term as \( \sqrt{\text{Det} \Omega_{B_c}} \).

Hence our partition function eq. (22) is

\[
Z = \int DA^t_c DB^t_c Z_{B^t_c}
\]

So after integrating out all the modes of \( B^t_{q_1}, B^t_i, B^k_{q_i}, B^k_i, \alpha^t, \bar{\alpha}^t, \bar{\alpha}^k \), the partition function becomes

\[
Z = \int DA^t_c DB^t_c \sqrt{\text{Det} \Omega_{B^t_c}} \frac{\text{Det}_t(\partial_t)_{\Omega^{0,0}(M) \otimes \Omega^0(S^1)}}{\text{Det}_t(\partial_t)_{\Omega^{0,1}(M) \otimes \Omega^0(S^1)}} \cdot \frac{\text{Det}_k(\nabla_t)_{\Omega^{0,0}(M) \otimes \Omega^0(S^1)}}{\text{Det}_k(\nabla_t)_{\Omega^{0,1}(M) \otimes \Omega^0(S^1)}} \cdot \exp \left( \frac{i}{2\pi} \int_M \omega \wedge \text{Tr}(A^t_{0t} dB^t_c) \right),
\]

where \( \text{Det}' \) has no \( S^1 \) zero modes and \( \Omega^{0,1*}_i(M), \Omega^{0,1*}_k(M) \) are defined as follows:

\[
\Omega^{0,1*}_i(M) = \{ B_q \in \Omega^{0,1}(M; t) | \bar{\partial} B^t_q = 0 \},
\]

\[
\Omega^{0,1*}_k(M) = \{ B_q \in \Omega^{0,1}(M; k) | \bar{\partial} B^t_q + B^t_c \wedge B^t_k = 0 \}.
\]

C. Evaluation Of The Abelianized Partition Function

After substituting the values of the above functional determinants into the path integral (In the Appendix we show how to evaluate these functional determinants), we get

\[
Z = \int DA_0 DB_c \sqrt{\text{Det} \Omega_{B_c}} e^{\frac{i}{\pi} f_M \text{Tr}(A_0 F_{B_c}) c_1(M)} \times (-1)^{\frac{1}{2}} f_M \rho(F_{B_c}) c_1(M) \cdot \prod_{\alpha > 0} \left( 2 \sin \frac{\alpha(A_0)}{2} \right)^{\frac{1}{6}(c_1^2(M) + c_2(M)) + c_2^2(k_\alpha)} \cdot \exp \left\{ \frac{i}{2\pi} \int_M \omega \wedge \text{Tr}(A_0 F_{B_c}) \right\}
\]

\[
= \int DA_0 DB \sqrt{\text{Det} \Omega_{B_c}} e^{\frac{i}{\pi} f_M \text{Tr}(A_0 F_{B_c}) \wedge (\omega + h c_1(M))} \times (-1)^{\frac{1}{2}} f_M \rho(F_{B_c}) c_1(M) \cdot \prod_{\alpha > 0} \left( 2 \sin \frac{\alpha(A_0)}{2} \right)^{\frac{1}{6}(c_1^2(M) + c_2(M)) + c_2^2(k_\alpha)}.
\]

According to [12], we know after fixing the gauge, \( DB_c = DF_{B_c} \). So eq. (32) becomes

\[
Z = \int DA_0 DF_{B_c} \sqrt{\text{Det} \Omega_{B_c}} e^{\frac{i}{\pi} f_M \text{Tr}(A_0 F_{B_c}) \wedge (\omega + h c_1(M))} \times (-1)^{\frac{1}{2}} f_M \rho(F_{B_c}) c_1(M) \cdot \prod_{\alpha > 0} \left( 2 \sin \frac{\alpha(A_0)}{2} \right)^{\frac{1}{6}(c_1^2(M) + c_2(M)) + c_2^2(k_\alpha)}
\]

(33)

Denote \( \phi = A_0, [F_{B_c}] = \eta \in H^2(M, \mathbb{Z}^r) \) where \( \eta = (\eta_1, \cdots, \eta_{\alpha_r}) \in \text{Pic}(M)^r \). Then the above equation can be rewritten as
\[ Z = \sum_{\eta \in \text{Pic}(M)^r \atop c_2(\oplus \eta_\alpha) = 0} \sqrt{\text{Det} \Omega_\eta} \int_{t \cap \triangle_+} D\phi e^{\frac{i}{\hbar} \int_M \text{Tr}(\phi[\eta])} \wedge (\omega + c_1(M)) \times (-1)^{\frac{1}{2} \int_M \rho(\eta)c_1(M)} . \]

where \( t \cap \triangle_+ \) is the Weyl alcove\(^6\) and \( c_2(\oplus \eta_\alpha) \) is the 2nd Chern character of bundle \( \oplus \eta_\alpha \). The condition \( c_2(\oplus \eta_\alpha) = 0 \) comes from the assumption that the complete set of obstructions to diagonalizing are all the \( T \)-bundle restrictions of the trivial \( SU(r+1) \) bundle, where \( T \) is the maximum torus of \( SU(r+1) \). In\(^{20} \) Blau and Thompson discussed the condition for the existence of this restriction.

### IV. CONCLUSION AND FURTHER STUDY

As in the three dimensional \( SU(N) \) Chern-Simons theory, the partition function \( Z \) is equal to the dimension of the physical Hilbert space, up to a renormalization\(^{12} \), i.e.

\[ \dim \mathcal{H} = N \cdot Z = N \sum_{\eta \in \text{Pic}(M)^r \atop c_2(\oplus \eta_\alpha) = 0} \sqrt{\text{Det} \Omega_\eta} \int_{t \cap \triangle_+} D\phi e^{\frac{i}{\hbar} \int_M \text{Tr}(\phi[\eta])} \wedge (\omega + c_1(M)) \times (-1)^{\frac{1}{2} \int_M \rho(\eta)c_1(M)} . \]

\[ \cdot \prod_{\alpha > 0} \left( 2 \sin \left( \frac{\alpha(A_0)}{2} \right) \right)^{\frac{1}{2}(c_1^2(M)+c_2(M))+[\eta_\alpha \cdot [\eta_\alpha]}. \]

(35)

The factor \( \sqrt{\text{Det} \Omega_\eta} \) is compatible with\(^{6,21,22} \) in the following sense: in\(^{6,21,22} \) the authors used the BRST (or equivariant cohomology) technique. In their results the path integral

\[ \int D\Psi D\bar{\Psi} \exp\{i \int_{X^1 \times S^1} \text{Tr}(\omega \wedge dt \wedge \Psi \wedge \bar{\Psi}) \} \]

(36)

corresponds to our factor \( \sqrt{\text{Det} \Omega_\eta} \). The difference is that our factor \( \sqrt{\text{Det} \Omega_\eta} \) is evaluated at certain "points" on the reduced surface. Further, it remains to determine the normalization \( N \). For three dimensional \( SU(N) \) Chern-Simons theory we can borrow the formula for the volume of the flat connection moduli space\(^{12,23} \). This suggests an investigation of the Kähler-Chern-Simons theory to see whether we can use the volume of the ASD connection moduli space\(^{24,25} \) to calculate the normalization \( N \). Work on this is in progress.

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Appendix A: APPENDIX: Evaluating The Functional Determinants

In this subsection we will focus on the calculation of the determinants

\[ \frac{\text{Det}'_t(\partial_t)_{\Omega^0,0(M)} \otimes \Omega^0(S^1)}}{\text{Det}'_t(\partial_t)_{\Omega^0,1^* (M)} \otimes \Omega^0(S^1)}} \]  \hspace{1cm} (A1)

and

\[ \frac{\text{Det}_k(\nabla_t)_{\Omega^0,0(M)} \otimes \Omega^0(S^1)}}{\text{Det}_k(\nabla_t)_{\Omega^0,1^* (M)} \otimes \Omega^0(S^1)}} \]  \hspace{1cm} (A2)

In order to make sense of these determinants, we need to regularize them. In Section 2 we already discussed the residual Abelian gauge symmetry. That implies that the regularization should not break the residual gauge symmetry. We will use the heat kernel regularization based on the t covariant Laplacian \( \triangle_{B_c} = -(\partial^*_B \partial_B + \partial_B \partial^*_B) \). Here we will use the same definition of the determinant as \(12,26\):

For an operator \( \mathcal{O} \) we define

\[ \log \text{Det} \mathcal{O} := Tr(e^{-t \triangle_{B_c}} \log \mathcal{O}) \]  \hspace{1cm} (A3)

Before calculating the determinants (A1) and (A2), let us analyze some exact sequences.

\( P \) is a trivial \( SU(r + 1) \) bundle on \( M \times S^1 \) and \( adP \) denotes its adjoint bundle which is still trivial. It is easy to find that the pullback bundles on \( M \) should also be trivial. From now on we denote \( P \) and \( adP \) as pullback bundles on \( M \).

**Lemma 0.1** If \( H^{0,1}(M, \mathbb{C}) = H^{0,2}(M, \mathbb{C}) = 0 \), then \( H^{0,1}(M, adP \otimes \mathbb{C}) = H^{0,2}(M, adP \otimes \mathbb{C}) = 0 \).

If \( t \) is a subbundle of \( adP \otimes \mathbb{C} \), then \( adP \otimes \mathbb{C} \) decomposes as \( adP \otimes \mathbb{C} = t \oplus k \) under the natural metric "Tr" of \( adP \otimes \mathbb{C} \). In other words, we have the following exact sequence:

\[ 0 \longrightarrow t \longrightarrow adP \otimes \mathbb{C} \longrightarrow k \longrightarrow 0. \]  \hspace{1cm} (A4)

Then we get the following long exact sequence:

\[ H^{0,1}(M, t) \longrightarrow H^{0,1}(M, adP \otimes \mathbb{C}) \longrightarrow H^{0,1}(M, k) \longrightarrow \]  \hspace{1cm} (A5)

\[ H^{0,2}(M, t) \longrightarrow H^{0,2}(M, adP \otimes \mathbb{C}) \longrightarrow H^{0,2}(M, k) \longrightarrow 0 \]
From the Lemma 0.1 we get

\[ H^{0,2}(M, k) = 0, \quad (A6) \]
\[ H^{0,1}(M, k) \cong H^{0,2}(M, t). \quad (A7) \]

Similarly, using the exact sequence

\[ 0 \rightarrow k \rightarrow \text{ad}P \otimes \mathbb{C} \rightarrow t \rightarrow 0, \quad (A8) \]

we can show that

\[ H^{0,2}(M, t) = 0, \quad (A9) \]
\[ H^{0,1}(M, t) \cong H^{0,2}(M, k). \quad (A10) \]

So we have

\[ H^{0,2}(M, k) = H^{0,1}(M, k) = H^{0,1}(M, t) = H^{0,2}(M, t) = 0. \quad (A11) \]

Now for the \( \bar{B}_c^t \) satisfying \( \bar{\partial}\bar{B}_c^t = 0 \), we have the following complex

\[ 0 \rightarrow \Omega^{0,0}(M) \otimes t \rightarrow \Omega^{0,1}(M) \otimes t \rightarrow \Omega^{0,2}(M) \otimes t \rightarrow 0. \quad (A12) \]

For \( \forall f \in \Omega^{0,0}(M) \otimes t \) and \( \forall \alpha \in \Omega^{0,1}(M) \otimes t \),

\[ \bar{\partial}_B f = \bar{\partial}f + \bar{B}_c^t f, \quad (A13) \]
\[ \bar{\partial}_B \alpha = \bar{\alpha} + \bar{B}_c^t \alpha. \quad (A14) \]

Similarly for \( B_c^t \), which is the complex conjugate of \( \bar{B}_c^t \), we have

\[ 0 \rightarrow \Omega^{0,0}(M) \otimes t \rightarrow \Omega^{1,0}(M) \otimes t \rightarrow \Omega^{2,0}(M) \otimes t \rightarrow 0. \quad (A15) \]

Thus \( \forall h \in \Omega^{0,0}(M) \otimes t \) and \( \forall \beta \in \Omega^{1,0}(M) \otimes t \),

\[ \partial_B h = \partial h + B_c^t h, \quad (A16) \]
\[ \partial_B \beta = \partial \beta + B_c^t \beta. \quad (A17) \]

Denote

\[ \mathcal{B}_c^t := B_c^t + \bar{B}_c^t, \quad (A18) \]
\[ d_B := \partial_B + \bar{\partial}_B. \quad (A19) \]
Then for $\forall f \in \Omega^0(M) \otimes t$, we have the exact sequence

$$0 \rightarrow k_\alpha \rightarrow k \rightarrow \mathcal{N} \rightarrow 0$$

where $\mathcal{N}$ is the normal bundle of $k_\alpha$. Then we get the long exact sequence

$$\rightarrow H^{0,1}(M, k_\alpha) \rightarrow H^{0,1}(M, k) \rightarrow H^{0,1}(M, \mathcal{N}) \rightarrow H^{0,2}(M, k_\alpha) \rightarrow 0.$$

According to eq. (A11) we have

$$H^{0,2}(M, \mathcal{N}) = 0,$$

$$H^{0,1}(M, \mathcal{N}) \cong H^{0,2}(M, k_\alpha).$$

Next we will investigate the determinant (31). According to eq. (A11) we have

$$\log \frac{\text{Det}^t_1(\partial_t)_{\Omega^0,0(M) \otimes \Omega^0(S^1)}}{\text{Det}^t_1(\partial_t)_{\Omega^{0,1}(M) \otimes \Omega^0(S^1)}} = (\text{dim} H^{0,0}(M, t) - \text{dim} H^{0,1}(M, t)) \log \text{Det}^t_1(\partial_t)_{\Omega^0(S^1)}$$

$$= \text{Ind} \bar{\partial}_{B_t} \log \text{Det}^t_1(\partial_t)_{\Omega^0(S^1)}$$

$$= \int_M c_1(t)Td(M) \log \text{Det}^t_1(\partial_t)_{\Omega^0(S^1)}$$

$$= \int_M \left[r + c_1(t) + \frac{1}{2} c_1^2(t) - c_2(t)\right][1 + c_1(M) + \frac{1}{12}(c_1^2(M) + c_2(M)) + \ldots] \cdot \log \text{Det}^t_1(\partial_t)_{\Omega^0(S^1)}$$

$$= \int_M \left[r \frac{1}{12}(c_1^2(M) + c_2(M))\right] \cdot \log \text{Det}^t_1(\partial_t)_{\Omega^0(S^1)}, \quad (A20)$$

where $\text{Ind} \bar{\partial}_{B_t}$ denotes the index of operator $\bar{\partial}_{B_t}$ for the complex (A12), $ch(t)$ denotes the Chern character of bundle $t$, $Td(M)$ denotes the Todd genus of $M$ and $r$ denote the rank of bundle $t$. During the calculation we have used the index theorem. Furthermore, according to (A21), we know, up to a normalization,

$$\text{Det}^t_1(\partial_t)_{\Omega^0(S^1)} \sim 1. \quad (A21)$$

Then we have

$$\frac{\text{Det}^t_1(\partial_t)_{\Omega^0,0(M) \otimes \Omega^0(S^1)}}{\text{Det}^t_1(\partial_t)_{\Omega^{0,1}(M) \otimes \Omega^0(S^1)}} \sim 1. \quad (A22)$$

Next we will calculate the determinant (A22). We know the bundle $k$ has decomposition $k = \oplus_{\alpha > 0}(k_\alpha \oplus k_{-\alpha})$ where $\alpha > 0$ are the positive roots of the Lie algebra $su(r + 1)$. Then we have the exact sequence

$$H^{0,0}(M, k_\alpha) \rightarrow H^{0,1}(M, k_\alpha) \rightarrow H^{0,1}(M, \mathcal{N}) \rightarrow \cdots$$
Similarly using the exact sequence

\[ 0 \longrightarrow \mathcal{N} \longrightarrow k \longrightarrow k_\alpha \longrightarrow 0, \tag{A27} \]

we can get

\[ H^{0,2}(M, k_\alpha) = 0, \tag{A28} \]
\[ H^{0,1}(M, k_\alpha) \cong H^{0,2}(M, \mathcal{N}). \tag{A29} \]

So

\[ H^{0,1}(M, k_\alpha) = H^{0,2}(M, k_\alpha) = H^{0,1}(M, \mathcal{N}) = H^{0,2}(M, \mathcal{N}) = 0 \tag{A30} \]

Now we have the complex

\[ 0 \longrightarrow \Omega^{0,0}(M) \otimes k_\alpha \xrightarrow{\partial_{B^t_c}} \Omega^{0,1}(M) \otimes k_\alpha \xrightarrow{\partial_{B^t_c}} \Omega^{0,2}(M) \otimes k_\alpha \longrightarrow 0, \tag{A31} \]

where \( \partial_{B^t_c} \) is same as the previous one in (A12). For this complex, according to the index theorem, we have

\[ \text{Ind} \partial_{B^t_c} |_{k_\alpha} = \dim H^{0,0}(M, k_\alpha) - \dim H^{0,1}(M, k_\alpha) + \dim H^{0,2}(M, k_\alpha) \]
\[ = \int_M ch(k_\alpha) Td(M) \]
\[ = \int_M [1 + c_1(k_\alpha) + \frac{1}{2} c_1^2(k_\alpha)] [1 + c_1(M) + \frac{1}{12} (c_1^2(M) + c_2(M))] \]
\[ = c_1(k_\alpha) c_1(M) + \frac{1}{12} (c_1^2(M) + c_2(M)) + \frac{1}{2} c_2^2(k_\alpha). \tag{A32} \]

Now let us derive the \( c_1(k_\alpha) \). We have the following sequence

\[ 0 \longrightarrow \Omega^0(M) \otimes k_\alpha \xrightarrow{d_{B^t_c}} \Omega^1(M) \otimes k_\alpha \xrightarrow{d_{B^t_c}} \Omega^2(M) \otimes k_\alpha \longrightarrow \cdots, \tag{A33} \]

where \( d_{B^t_c} := \partial_{B^t_c} + \bar{\partial}_{B^t_c} \) and \( B^t_c \) is the complex conjugate of \( B^t_c \) and for \( \forall f \otimes k_\alpha \in \Omega^0(M) \otimes k_\alpha \),

\[ d_{B^t_c}(f \otimes k_\alpha) = df \otimes k_\alpha + f \otimes [B^t_c, k_\alpha] \]
\[ = df \otimes k_\alpha + f \otimes [B^t_i c^i, k_\alpha] \]
\[ = [df + i B^t_i c^i f] \otimes k_\alpha, \tag{A34} \]

where \( B^t_i c_i = B^t_i + \bar{B}^t_i \) and \( h^i, i = 1, \ldots, r \) are the basis of \( \mathfrak{t} \) satisfying \([h^i, k_\alpha] = i \alpha^i k_\alpha\). So we have

\[ c_1(k_\alpha) = -\frac{1}{2\pi} \alpha(F_{B^t_c}) = -\frac{1}{2\pi} F_{B^t_i c^i}(h^i) = -\frac{1}{2\pi} F_{B^t_i c^i} \alpha^i. \tag{A35} \]
As in \cite{12}, we have
\[
\log \frac{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,0(M) \otimes \Omega^0(S^1)}}{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,1(M) \otimes \Omega^0(S^1)}} = \left( \text{Tr}_{\Omega^0,0(M)} e^{-\epsilon \nabla_t \beta^\alpha} \right) \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}
\]
\[
= \left( \text{dim} H^0,0(M, k_\alpha) - \text{dim} H^0,1(M, k_\alpha) \right) \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}
\]
\[
= \text{Ind} \bar{\partial}_{\beta^\alpha} |_{k_\alpha} \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}
\]
\[
= c_1(k_\alpha) c_1(M) + \frac{1}{12} (c_1^2(M) + c_2(M)) + \frac{1}{2} c_1^2(k_\alpha)
\]
\[
= [c_1(k_\alpha) c_1(M) + \frac{1}{12} (c_1^2(M) + c_2(M)) + \frac{1}{2} c_1^2(k_\alpha)] \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}. \quad (A36)
\]

And similarly we have
\[
\log \frac{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,0(M) \otimes \Omega^0(S^1)}}{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,1(M) \otimes \Omega^0(S^1)}} = \left( \text{Tr}_{\Omega^0,0(M)} e^{-\epsilon \nabla_t \beta^\alpha} \right) \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}
\]
\[
= \left( \text{dim} H^0,0(M, k_\alpha) - \text{dim} H^0,1(M, k_\alpha) \right) \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}
\]
\[
= \text{Ind} \bar{\partial}_{\beta^\alpha} |_{k_\alpha} \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}
\]
\[
= [c_1(k_\alpha) c_1(M) + \frac{1}{12} (c_1^2(M) + c_2(M)) + \frac{1}{2} c_1^2(k_\alpha)] \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}. \quad (A37)
\]

So
\[
\log \frac{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,0(M) \otimes \Omega^0(S^1)}}{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,1(M) \otimes \Omega^0(S^1)}} = \sum_{\alpha > 0} \left\{ \left[ \frac{1}{12} (c_1^2(M) + c_2(M)) + \frac{1}{2} c_1^2(k_\alpha) \right] \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)} \cdot \log \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)} - c_1(k_\alpha) c_1(M) \log \frac{\text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}}{\text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}} \right\}. \quad (A38)
\]

Finally
\[
\frac{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,0(M) \otimes \Omega^0(S^1)}}{\text{Det}_{k_\alpha}(\nabla_t)_{\Omega^0,1(M) \otimes \Omega^0(S^1)}} = \prod_{\alpha > 0} \exp \left\{ - c_1(k_\alpha) c_1(M) \log \frac{\text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}}{\text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)}} \right\} \cdot \prod_{\alpha > 0} \left( \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)} \text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)} \right)^{\frac{1}{4} (c_1^2(M) + c_2(M)) + \frac{1}{2} c_1^2(k_\alpha)}. \quad (A39)
\]

From \cite{12, 23} we know, up to a normalization,
\[
\text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)} \sim \text{Det} \left[ \text{Id} - \text{Ad}_{k_\alpha} (\exp iA_0) \right] = \text{Det} \left[ \text{Id} - \text{Ad}_{k_\alpha} (e^{i\alpha(A_0)}) \right], \quad (A40)
\]
\[
\text{Det}_{k_\alpha} \nabla_t |_{\Omega^0(S^1)} \sim \text{Det} \left[ \text{Id} - \text{Ad}_{k_\alpha} (\exp A_0) \right] = \text{Det} \left[ \text{Id} - \text{Ad}_{k_\alpha} (e^{-i\alpha(A_0)}) \right]. \quad (A41)
\]
We denote
\[ M_\alpha := \text{Id} - \text{Ad}_{k_\alpha}(e^{i\alpha(A_0)}), \]  
\[ M_{-\alpha} := \text{Id} - \text{Ad}_{k_{-\alpha}}(e^{-i\alpha(A_0)}). \]

According to (A35), we know
\[ \rho = \prod \exp \left\{ -c_1(k_\alpha)c_1(M) \log \frac{\text{Det}_{k_\alpha}\nabla_{t|\Omega^0(S^1)}}{\text{Det}_{k_{-\alpha}}\nabla_{t|\Omega^0(S^1)}} \right\} \]
\[ = \prod_{\alpha > 0} \left\{ e^{-i\alpha(A_0)c_1(k_\alpha)c_1(M)} \times (-1)^{-c_1(k_\alpha)c_1(M)} \right\} \]
\[ = e^\sum_{\alpha > 0} \left\{ -i\alpha(A_0)c_1(k_\alpha)c_1(M) \right\} \times (-1)^{c_1(k_\alpha)c_1(M)}. \]  
(A45)

According to (A35) we know
\[ -\sum_{\alpha > 0} \alpha(A_0)c_1(k_\alpha) = \sum_{\alpha > 0} \frac{1}{2\pi} \int_M \alpha(A_0)\alpha(F_{B_c}) = \frac{h}{2\pi} \int_M \text{Tr}(A_0F_{B_c}), \]  
(A46)
where \( h \) is the Coxeter number of \( SU(r+1) \). So
\[ \prod_{\alpha > 0} \exp \left\{ -c_1(k_\alpha)c_1(M) \log \frac{\text{Det}_{k_\alpha}\nabla_{t|\Omega^0(S^1)}}{\text{Det}_{k_{-\alpha}}\nabla_{t|\Omega^0(S^1)}} \right\} = e^{\frac{ih}{2\pi} \int_M \text{Tr}(A_0F_{B_c})c_1(M)} \times \left( -1 \right)^{c_1(k_\alpha)c_1(M)}, \]  
(A47)
where \( \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \). Finally from (A35) we know that
\[ \text{Det}_{k_\alpha}\nabla_{t|\Omega^0(S^1)}\text{Det}_{k_{-\alpha}}\nabla_{t|\Omega^0(S^1)} = 4\sin^2 \frac{\alpha(A_0)}{2}. \]  
(A48)
So we have
\[ \prod_{\alpha > 0} \left( \text{Det}_{k_\alpha}\nabla_{t|\Omega^0(S^1)}\text{Det}_{k_{-\alpha}}\nabla_{t|\Omega^0(S^1)} \right)^{\frac{1}{2} \left( c_1^2(M) + c_2(M) \right) + \frac{1}{2} c_1^2(k_\alpha)} \]
\[ = \prod_{\alpha > 0} \left( 2\sin \frac{\alpha(A_0)}{2} \right)^{\frac{1}{2} \left( c_1^2(M) + c_2(M) \right) + c_1^2(k_\alpha)}. \]  
(A49)
At last, we have
\[ \frac{\text{Det}_{k}(\nabla_{t|\Omega^{0,0}(M)\otimes\Omega^0(S^1)})}{\text{Det}_{k}(\nabla_{t|\Omega^{0,1}(M)\otimes\Omega^0(S^1)})} = e^{\frac{ih}{2\pi} \int_M \text{Tr}(A_0F_{B_c})c_1(M)} \times \left( -1 \right)^{c_1(k_\alpha)c_1(M)} \]
\[ \prod_{\alpha > 0} \left( 2\sin \frac{\alpha(A_0)}{2} \right)^{\frac{1}{2} \left( c_1^2(M) + c_2(M) \right) + c_1^2(k_\alpha)}. \]  
(A50)
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