Fairing of Discrete Planar Curves by Discrete Euler’s Elasticae

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Abstract

After characterizing the integrable discrete analogue of the Euler’s elastica, we focus our attention on the problem of approximating a given discrete planar curve by an appropriate discrete Euler’s elastica. We carry out the fairing process via a $L^2$-distance minimization to avoid the numerical instabilities. The optimization problem is solved via a gradient-driven optimization method (IPOPT). This problem is non-convex and the result strongly depends on the initial guess, so that we use a discrete analogue of the algorithm provided by Brander et al., which gives an initial guess to the optimization method.

Keywords: Euler’s elastica, integrability, discrete curve, discrete differential geometry, fairing

1 Introduction

Recently, Brander et al. [1] have constructed an algorithm to fair a given planar curve segment by an Euler’s elastica, motivated mainly by the development of the robotic hot-blade cutting technology. In this work, we construct a discrete version of that algorithm, which would be used to fair a given discrete planar curve by an integrable discrete analogue of Euler’s elastica. The definition in this paper is a well-known expression appearing frequently in different forms in the literature on the discrete integrable systems [2, 3, 4]. Actually, the Euler’s elastica is known to be equivalent to the simple pendulum, and to have a close relationship with the Lagrange top, as shown in [2], for example.

2 Euler’s elastica

Let $\gamma(s) \in \mathbb{R}[2]$ be an arc length parameterized planar curve ($s$: arc length), and define the tangent and normal vectors as $T(s) = \gamma'(s)$ and $N(s) = R_{\pi/2} T(s)$, respectively, where $R_{\pi/2}$ is a $\pi/2$ rotation matrix and $\' = d/ds$. By definition, $\|T(s)\| = 1$, so the tangent vector can be parameterized as

$$T(s) = \begin{pmatrix} \cos \psi(s) \\ \sin \psi(s) \end{pmatrix},$$

where $\psi(s)$ is the angle function, namely, the angle of the tangent vector measured from the horizontal axis, so that $\kappa(s) = \psi'(s)$ is the curvature. With this notation, a plain curve satisfying

$$\psi'' + \mu \sin(\psi - \phi) = 0,$$  (2.1)
or equivalently
\[ \kappa'' + \frac{1}{2} \kappa^3 - \lambda \kappa = 0, \]  
(2.2)
where \( \mu > 0 \), and \( \phi, \lambda \in \mathbb{R} \) are constant parameters, is called an Euler’s elastica. These curves can also be characterized as critical points of the elastic energy
\[ E[\gamma] = \int_0^L \frac{1}{2} \psi'^2(s) \, ds = \int_0^L \frac{1}{2} \kappa^2(s) \, ds, \]  
(2.3)
when the total length \( L \) is fixed. Furthermore, (2.1) and (2.2) are reductions of the sine-Gordon equation and the modified KdV equation, respectively, where the latter describes the integrable deformation of planar curves.

3 Discrete Euler’s elastica

Let \( \gamma_n \in \mathbb{R}^2 \) \( (n = 0, \ldots, N - 1) \) be a discrete planar curve with constant segment length \( h = \| \gamma_{n+1} - \gamma_n \| \), and define the tangent vector as \( T_n = (\gamma_{n+1} - \gamma_n)/h \), which we will parameterize as in [5]:
\[ T_n = \begin{pmatrix} \cos \psi_n \\ \sin \psi_n \end{pmatrix}, \]  
(3.1)
\[ \psi_n = \frac{\theta_n + \theta_{n+1}}{2}. \]  
(3.2)
Here, \( \psi_n \) is the discrete angle function and \( \theta_n \) plays the role of a potential function. Finally, noticing that \( \psi_n - \psi_{n-1} = (\theta_{n+1} - \theta_{n-1})/2 \) is the angle between \( T_{n-1} \) and \( T_n \), we introduce the discrete curvature \( \kappa_n \) by
\[ \kappa_n = \frac{2}{h} \tan \left( \frac{\theta_{n+1} - \theta_{n-1}}{4} \right). \]  
(3.3)
With these notations, a discrete planar curve satisfying
\[ \tan \left( \frac{\theta_{n+1} + \theta_{n-1}}{4} \right) = \frac{1 - h^2 \mu / 4}{1 + h^2 \mu / 4} \tan \left( \frac{\theta_n}{2} \right), \]  
(3.4)
or equivalently
\[ \sin \left( \frac{\theta_{n+1} - 2 \theta_n + \theta_{n-1}}{4} \right) + \frac{h^2 \mu}{4} \sin \left( \frac{\theta_{n+1} + 2 \theta_n + \theta_{n-1}}{4} \right) = 0, \]  
(3.5)
for some constant \( \mu > 0 \), will be referred to as an integrable discrete analogue of Euler’s elastica (or simply, as a discrete elastica); see, for example, equation (9) in [4]. These curves can also be characterized as critical points of the functional
\[ S[\theta] = \sum_{n=0}^{N-2} h \left[ \frac{\sin^2 \left( \frac{\theta_{n+1} - \theta_n}{4} \right)}{h^2/8} + \mu \cos \left( \frac{\theta_{n+1} + \theta_n}{2} \right) \right], \]  
(3.6)
when the endpoints \( \theta_0 \) and \( \theta_{N-1} \) are fixed. Furthermore, as well as in the continuous case, (3.4) can be seen as a reduction of the discrete sine-Gordon equation,
\[ \sin \left( \frac{\theta_{n+1} - \theta_n - \theta_{n+1} + \theta_n}{4} \right) = \frac{a_n}{b_m} \sin \left( \frac{\theta_{n+1} + \theta_{n+1} + \theta_{n+1} + \theta_{n+1}}{4} \right), \]  
(3.7)
where $a_n$ an $b_n$ are arbitrary functions in the indicated variables. Note that (3.7) can be rewritten as
\[\tan\left(\frac{\theta_{n+1}^m + \theta_n^m}{4}\right) = \frac{1 - \varepsilon_n^m}{1 + \varepsilon_n^m} \tan\left(\frac{\theta_{n+1}^m + \theta_n^m}{4}\right),\] (3.8)
with $\varepsilon_n^m = a_n/b_n$. Then, after regarding $\varepsilon = \varepsilon_n^m$ as a constant, and applying the change of variables $k = n - m$ and $l = n + m$, we get
\[\tan\left(\frac{\theta_{k+1}^l + \theta_{k-1}^l}{4}\right) = \frac{1 - \varepsilon}{1 + \varepsilon} \tan\left(\frac{\theta_{k+1}^l + \theta_{k-1}^l}{4}\right),\] (3.9)
which reduces to (3.4), if we neglect the $l$–dependence and replace $\varepsilon = h^2 \mu/4$. Lastly, it can be seen that the difference equation (3.4) possesses the conserved quantity
\[\cos\left(\frac{\theta_{n+1} - \theta_n}{2}\right) + \frac{h^2 \mu}{4} \cos\left(\frac{\theta_{n+1} + \theta_n}{2}\right) = A,\] (3.10)
where $A \in \mathbb{R}$ is a constant fixed by the initial condition.

**Remark 1.** Note that, in this context, the discrete angle function (3.2) has a simple expression, but it has imposed some difficulties when trying to find an energy expression for the discrete curve $\gamma_n$. In the simplest case, let us try to minimize the functional (3.6) with $\gamma_n$ as the independent variable. First, we need to establish a one-to-one relation between $\theta_n$ and $\gamma_n$, which can be done by fixing, for example, the end point $\theta_{N-1}^n = 0$. This allow us to write
\[\theta_n = \sum_{m=n}^{N-2} (-1)^{m-n} 2\psi_m.\] (3.11)
So, after solving the variational problem $\delta\mathcal{S}[\gamma_n] = 0$ we recover the equation (3.4) for the inner points $(n = 1, \ldots, N - 2)$, but we also need to satisfy
\[\sum_{n=0}^{N-2} (-1)^n 2\delta\psi_m = 0,\] (3.12)
or
\[\tan\left(\frac{\theta_1}{2}\right) = \frac{1 - h^2 \mu/4}{1 + h^2 \mu/4} \tan\left(\frac{\theta_0}{2}\right).\] (3.13)
The first condition (3.12) gives a non-local constraint for the perturbation, so we might want to satisfy the second condition (3.13). If we choose so, we will impose a boundary condition in the definition of the discrete elastica, which is undesired; among other reasons, the continuous limit will only reach a subset of the Euler’s elastica family. Because of this, we have chosen to describe a discrete elastica by the functional (3.6), defined over the potential function $\theta_n$.

**Remark 2.** Let $\mathcal{S}[\theta_n]$ be the functional presented by Sogo in [4],
\[\mathcal{S}[\theta_n] = \sum_{n=0}^{N-1} \sin^2\left(\frac{\theta_{n+1} - \theta_n}{4}\right) - \varepsilon \sin^2\left(\frac{\theta_{n+1} + \theta_n}{4}\right),\] (3.14)
So, we can see that, after replacing $\varepsilon = h^2 \mu / 4$, the functionals (3.6) and (3.14) are related as
\[
S[\theta_*] = \frac{8}{h} \left( S[\theta_*] \right)_{\varepsilon = h^2 \mu / 4} + \frac{h \mu N (N - 1)}{2}.
\] (3.15)

Thus, both share the same critical points, and so they describe the same discrete curves. Moreover, we believe that this approach shows a more transparent connection between the continuous and discrete case; it can be seen that the functional (3.6) is a discrete approximation, of order one, of the continuous variational problem, i.e.
\[
\int_0^h \left[ \frac{1}{2} (\psi'(s))^2 + \mu \cos \psi(s) \right] \, ds = S[(\psi(0), \psi(h))] + O(h^2).
\] (3.16)

Actually, the summand of the discrete functional (3.6) yields by putting $s = nh$
\[
h \left[ \sin^2 \left( \frac{\psi(s + h) - \psi(s)}{4} \right) + \mu \cos \left( \frac{\psi(s + h) + \psi(s)}{2} \right) \right]
\]
\[
= h \left[ \left( \frac{h \psi'(s)}{4} \right)^2 + \mu \cos \psi(s) \right] + O(h^2)
\]
\[
= h \left[ \frac{1}{2} (\psi'(s))^2 + \mu \cos \psi(s) \right] + O(h^2).
\] (3.17)

### 3.1 General solution and characteristic parameters

Here we will just present the result without its derivation. We refer to [4] for a detailed explanation.

It can be shown that the general solution of (3.4) is
\[
\sin \frac{\theta_n}{2} = k \text{sn}(q + zhn, k),
\] (3.18)

along with
\[
\mu = \frac{4}{h^2} \left( \frac{1 - \text{cn}(zh, k)}{1 + \text{cn}(zh, k)} \right).
\] (3.19)

where $k \geq 0$, $q, z \in \mathbb{R}$ are parameters that can be determined from the initial condition $\gamma_0$, $\gamma_1$, and $\gamma_2$. Then, since we are going to implement the fairing process for arbitrary discrete curves, we incorporate the freedom of rotation via the parameter $\phi \in \mathbb{R}$. So, by applying the transformation $\theta_n \rightarrow \theta_n - \phi$, the general solution changes to
\[
\sin \frac{\theta_n - \phi}{2} = k \text{sn}(q + zhn, k).
\] (3.20)

Finally, to reconstruct the discrete elastica we also need the starting point $\gamma_0 = \gamma(x_0, y_0)$. So, the discrete elastica can be obtained via $\gamma_n = \gamma_{n-1} + h T_{n-1}$ $(n = 1, \ldots, N - 1)$, and parameterized by seven parameters:
\[
p = (x_0, y_0, h, \phi, z, q, k).
\] (3.21)
4 Fairing process

4.1 Outline

In order to approximate a given discrete curve by a discrete elastica we must choose a suitable criteria. One possibility is to minimize the functional (3.6); however, having $N$ free parameters $(\theta_0, \theta_1, \ldots, \theta_{N-1})$, makes the fairing process numerically unstable. Furthermore, if we minimize it with respect to the discrete curve $\gamma_n$ instead of the potential function $\theta_n$, from Remark 1 we know that the constraints imposed on the starting points will restrict the domain of all compatible elasticae. Because of this, we have decided to do the fairing process via a $L^2$-distance minimization. Namely, we seek to find a set of parameters $p^*$ that minimize

$$\mathcal{L}(p) := \sum_{n=0}^{N-1} \frac{1}{2} \| \zeta_n(p) - \gamma_n \|^2,$$

(4.1)

where $\zeta_n(p)$ is the discrete elastica generated by $p$.

The optimization problem,

$$p^* = \arg \min_p \left\{ \sum_{n=0}^{N-1} \frac{1}{2} \| \zeta_n(p) - \gamma_n \|^2 \right\},$$

(4.2)

is non-convex and the result strongly depends on the initial guess. So, we have decided to implement a discrete analogue of the algorithm provided in [1], which provides an initial guess to the IPOPT method.

So, from here and after applying a discrete analogue of the algorithm shown in [1], we have observed that it is possible to recover the seven control parameters (3.21) that characterize a given discrete elastica segment, in a numerically stable manner.

Some examples are shown in Figure 1, where the optimization problem was solved via a gradient-driven optimization method (IPOPT [6]).

![Figure 1: Examples of discrete curves approximated by Euler’s elasticae, displayed for three different values of the parameter $k$. (a) left: $k < 0.9089$... (lemniscate-like curve), (b) center: $0.9089$... < $k$ < 1, (c) right: $k$ > 1. Red: given discrete curve, blue: output found with IPOPT optimization, green: initial guess.](image)
4.2 Some remarks on the algorithm

The algorithm shown in [1] is anchored on the following: in a continuous context, define \( u(s) \in \mathbb{R} \) as the projection of \( \gamma(s) \) onto the line spanned by \( (\sin \phi, -\cos \phi) \), and \( \psi_u(s) \) as the angle between the tangent \( T(s) \) and that same line, i.e.:

\[
\begin{align*}
    u(s) &= (\sin \phi, -\cos \phi) \cdot \gamma(s), \quad (4.3) \\
    \cos \psi_u(s) &= (\sin \phi, -\cos \phi) \cdot T(s), \quad (4.4) \\
    \sin \psi_u(s) &= (\cos \phi, \sin \phi) \cdot T(s). \quad (4.5)
\end{align*}
\]

Thus, using the fact that \( u' = \cos \psi_u \) and \( \psi'_u = \psi' \), we have

\[
\frac{d}{du} \sin \psi_u = \frac{1}{u'} \frac{d}{ds} \sin \psi_u = \psi' = \kappa. \quad (4.6)
\]

Then, from the differential equation (2.1) and the former expressions, it follows that

\[
\kappa(s) = \mu u(s) + \alpha, \quad \text{and} \quad (4.7) \\
\sin \psi_u = \frac{1}{2} \mu u^2 + \alpha u + \beta, \quad (4.8)
\]

where \( \alpha, \beta \in \mathbb{R} \) are constants of integration. From this point, all the parameters that characterize the smooth Euler’s elastica can be re-obtained, after solving several quadratic minimization sub-problems and using the explicit expressions for the curvature \( \kappa(s) \), the angle function \( \psi(s) \), and the curve itself \( \gamma(s) \).

For our case, define \( u_n \) and \( \psi_{u_n} \) in the same way as the smooth case,

\[
\begin{align*}
    u_n &= (\sin \phi, -\cos \phi) \cdot \gamma_n, \quad (4.9) \\
    \sin \psi_{u_n} &= (\cos \phi, \sin \phi) \cdot T_n. \quad (4.10)
\end{align*}
\]

We can show that (see Section 5) the affine relation (4.7) between \( \kappa(s) \) and \( u(s) \) holds also in the discrete framework, i.e.:

\[
\kappa_n = \frac{1}{A} \mu u_n + \alpha, \quad (4.11)
\]

where \( A \) is the conserved quantity (3.10), which can be easily checked that it goes to 1 in the limit \( h \to 0 \). Finally, for the discrete version of the equation (4.8) we have chosen to consider its approximation up to order one,

\[
\sin \psi_{u_n} = \frac{1}{2} \mu u^2_n + \alpha u_n + \beta + O(h^2), \quad (4.12)
\]

and similarly for the rest of the quadratic minimization sub-problems that follows.

5 Affine relation for the discrete curvature

In this last section we will see how the affine relation for the continuous curvature (4.7) has an equivalent expression (4.11) in the discrete framework.
First, and in order to simplify the notation, let us consider the following quantities:

\( \varphi_n = \frac{\theta_{n+1} - \theta_n}{2} \), \hspace{1cm} (5.1)

\( K_n = \frac{\theta_{n+1} - \theta_{n-1}}{2} = \varphi_n + \varphi_{n-1} = \psi_n - \psi_{n-1} \). \hspace{1cm} (5.2)

With this notation, the discrete Euler’s elastica equation (3.4) reads as

\[ \sin \left( \frac{\varphi_n - \varphi_{n-1}}{2} \right) = -\frac{h^2 \mu}{4} \sin \left( \frac{\psi_n + \psi_{n-1}}{2} \right), \] \hspace{1cm} (5.3)

and the discrete curvature as

\[ \kappa_n = \frac{2}{h} \tan \left( \frac{K_n}{2} \right). \] \hspace{1cm} (5.4)

We can reorder the terms in (5.3), in two different ways, to obtain

\[ \sin \left( \frac{K_n}{2} - \varphi_{n-1} \right) = -\frac{h^2 \mu}{4} \sin \left( \frac{K_n}{2} + \psi_{n-1} \right), \] \hspace{1cm} (5.5)

\[ \sin \left( \varphi_n - \frac{K_n}{2} \right) = -\frac{h^2 \mu}{4} \sin \left( \psi_n - \frac{K_n}{2} \right). \] \hspace{1cm} (5.6)

Then, after using the addition formula for the sine function, and the expression for the conserved quantity \( A \), equation (3.10), the previous two equations can be rearranged as

\[ A \tan \left( \frac{K_n}{2} \right) = \sin \varphi_{n-1} - \frac{h^2 \mu}{4} \sin \psi_{n-1}, \] \hspace{1cm} (5.7)

\[ A \tan \left( \frac{K_n}{2} \right) = \sin \varphi_n + \frac{h^2 \mu}{4} \sin \psi_n \cdots \] \hspace{1cm} (5.8)

So, if we compute the difference between (5.7), with \( n \mapsto n + 1 \), and (5.8), we get

\[ A \left( \tan \left( \frac{K_{n+1}}{2} \right) - \tan \left( \frac{K_n}{2} \right) \right) = -\frac{h^2 \mu}{2} \sin \psi_n, \] \hspace{1cm} (5.9)

which can be rewritten in terms of the discrete curvature \( \kappa_n \) given in (5.4) as

\[ \kappa_{n+1} - \kappa_n = -\frac{h \mu}{A} \sin \psi_n. \] \hspace{1cm} (5.10)

Finally, introducing the rotation angle \( \phi \) to replace \( \theta_n \mapsto \theta_n - \phi \), the last equation is modified as

\[ \kappa_{n+1} - \kappa_n = -\frac{1}{A} \mu h \sin(\psi_n - \phi). \] \hspace{1cm} (5.11)

By using the definition (4.9) of \( u_n \) we get

\[ \kappa_{n+1} - \kappa_n = \frac{1}{A} \mu (u_{n+1} - u_n), \] \hspace{1cm} (5.12)

which implies that

\[ \kappa_n - \frac{1}{A} \mu u_n = \alpha, \] \hspace{1cm} (5.13)
with \( \alpha \in \mathbb{R} \) constant.

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