AN ALTERNATIVE APPROACH ON THE SINGULAR METRIC ON A VECTOR BUNDLE

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Abstract. In this paper, we will provide an alternative definition for the singular Hermitian metric on a vector bundle. Moreover, we discuss the Griffiths and Nakano positivity under this circumstance and prove a generalised Griffiths’ vanishing theorem.

1. Introduction

The notion of singular Hermitian metric was first developed on a holomorphic line bundle, and it turns out to be a powerful tool in complex geometry. There are fruitful work in these aspects, such as [5, 6, 7, 8, 13, 24]. It is natural to discuss the singular metric on a vector bundle. The pioneering work dates back to [3], in which the global generation problems in the vector bundle setting was studied with the help of the singular metric introduced there. Then it is given a different definition for the singular metric in [2, 23], which also has good applications.

In their papers, there are mainly two questions when concerning a singular metric on a vector bundle. The first one is that how to define the associated curvature. Different from the situation on a line bundle, the example (Theorem 1.5) in [23] shows that the curvature associated with a singular metric on a vector bundle may not be a current with measure coefficients. Therefore it will lead to some trouble when talking about the positivity such as Griffiths and Nakano positivity. In [2, 23], a clever way is applied to approach Griffiths and Nakano semi-positivity without involving curvature. However, it seems hard to do the same thing for strict Griffiths positivity or Nakano positivity. There are several papers in this aspect, such as [23, 26]. Apart from that, it seems harder to talk about Chern form, which involves the wedge product of current. One could refer to [17, 25], in which they successfully defined the Chern form under some restrictions.

Another question is that how to define the multiplier ideal sheaf. There are some discussions in [3, 12], but it is not yet complete.

In this paper, we provide another way to describe the singular metric. More precisely, for a given vector bundle $E$ of rank $r + 1$ over a complex manifold $Y$, we consider its projectived bundle $X := \mathbb{P}(E^*)$ as well as the tautological line bundle $\mathcal{O}_E(1) := \mathcal{O}_X(1)$. Let $\pi : X \to Y$ be the natural projection. A basic observation in Finsler geometry says
that the complex Finsler metric on $E$ is one-one corresponds to the smooth Hermitian metric on $O_E(1)$. Fix a smooth Hermitian metric $h_0$ on $O_E(1)$. Let

$$H(X) := \{ \varphi \in L^1_{\text{loc}}(X); \varphi \text{ is smooth and } i\Theta_{O_X(1),h_0} + i\bar{\partial}\varphi \text{ is strictly positive on each fibre} \}.$$ 

Notice that for any $\varphi \in H(X)$, since $\varphi$ is smooth on each fibre,

$$\omega_{\varphi,y} := (i\Theta_{O_X(1),h_0} + i\bar{\partial}\varphi)|_{X_y}$$

is actually a Kähler metric on $X_y$. Now we define the singular Hermitian metric on $E$ to be the metric $H$ with form that

$$H(u,u) = \int_{X_y} |u|^2 e^{-\varphi} \omega_{\varphi,y}^r$$

with $\varphi \in H(X)$. Here we use the fact that $\pi_*O_E(1) = E$.

Let $M(E)$ be the set of all the smooth Hermitian metrics on $E$, let $M_H(E)$ be the set of all the singular Hermitian metrics in the sense of [23], and let $M_W(E)$ be the set of all the singular Hermitian metrics defined above. We will show that

**Theorem 1.1.** Let $Y$ be a complex manifold, and let $E$ be a holomorphic vector bundle over $Y$. Then we have

1. $M(E) \subset M_W(E) \subset M_H(E)$.
2. Let $M_H(E)$ be the set of all the singular Hermitian metrics $H$ in the sense of [23], such that $H$ is non-degenerated on each fibre. Then $M_W(E) = M_H(E)$.

In the rest part, we will focus on a vector bundle $E$ over a compact complex manifold $Y$. Fix the notations as before. Firstly, we will show that the singular metric defined above can be described more precisely. In fact, for any $H \in M_W(E)$, it naturally induces a singular metric $h_H$ on $O_E(1)$, hence an $L^1$-bounded function $\psi$ on $X$ with $h_H = h_0 e^{-\psi}$. Let $H_i(X)$ be the set of all these functions. then we have

**Theorem 1.2.** $H_i(X) \subset H(X)$. Moreover, for any $\varphi \in H(X)$ such that $i\Theta_{O_E(1),h_0} + i\bar{\partial}\varphi \geq 0$, it belongs to $H_i(X)$ iff it induces an isometry between the canonical isomorphism

$$K_{X/Y} \simeq O_X(-r - 1) + \pi^* \det E.$$

We will make an explicit explanation of this isometry in the text. Now let

$$H_{h_0}(X) := \{ \varphi \in H(X); i\Theta_{O_E(1),h_0} + i\bar{\partial}\varphi \geq 0 \}.$$ 

Then in the view of Theorem 1.2, the set $H_i(X) \cap H_{h_0}(X)$ equals to the set of all the $L^1$-bounded functions $\varphi$ on $X$ such that

1. $\varphi$ is smooth on each $X_y$,
2. $\omega_{\varphi,y} := (i\Theta_{O_E(1),h_0} + i\bar{\partial}\varphi)|_{X_y}$ is strictly positive.
(3) \( i\Theta_{O_{E}(1), h_0} + i\bar{\partial}\partial\varphi \geq 0. \)
(4) \( \omega_{\varphi, y} \) induces an isometry between
\[ K_{X/Y} \simeq O_{X}(-r - 1) + \pi^* \det E. \]

Later, we will see that the third property is assigned to the Griffiths positivity of the singular metric on \( E \) defined by \( \varphi \). We could rearrange the Griffiths positive singular Hermitian metric on \( E \) to be
\[ \varphi \in H_{i}(X) \cap H_{h_0}(X) \]
in the view of Theorem 1.2. Namely,

**Corollary 1.1.** There is a one-one correspondence between
\[ H \in M_{W}(E) \]
such that \( (E, H) \) is positive in the sense of Griffiths and
\[ \varphi \in H_{h_0}(X) \]
such that \( \varphi \) induces an isometry between (1).

The following part is devoted to discuss the Griffiths and Nakano positivity. Let \( (y_1, ..., y_n) \) be the local coordinate of \( Y \). Let \( H \) be a singular metric on \( E \), \( \varphi \) being the corresponding metric on \( O_{E}(1) \).
Now we say that \( (E, H) \) is (strictly) positive in the sense of Griffiths, if
\[ i\partial\bar{\partial}\varphi \geq (>)0 \text{ on } X. \]

Next we define the Nakano positivity. Suppose that \( (E, H) \) is positive in the sense of Griffiths. Let
\[ E \to Y \]
\[ (z, Z) \mapsto z \]
be a localization. Since \( H \) is a Hermitian metric on each fibre, it can be written as \( H(z, Z) = H_{\alpha\beta}(z) dZ_{\alpha} \land d\bar{Z}_{\beta} \). Moreover, since \( (E, H) \) is Griffiths positive by assumption, \( H_{\alpha\beta}(z) \) is well-defined at each point. It can be seen through a simple argument, see [22] for example. In particular, the matrix \( (H_{\alpha\beta}(z)) \) is definitely positive hence invertible.
Let \( (H_{\beta\alpha}(z)) \) be its inverse matrix, then it is well-defined for every \( z \).
Now we define
\[ \Gamma_{\alpha_{\beta i}} = \frac{\partial H_{\alpha_{\bar{\beta}}}}{\partial z_{i}} H_{\bar{\alpha}\beta}. \]

Notice that \( H_{\alpha\beta}(z) \) is not necessary to be differentiable, and \( \Gamma_{\alpha_{\bar{\beta} i}} \) is defined in the sense of current. Consider the local holomorphic section \( u = (u_0, ..., u_r) \) of \( E \) such that
\[ \frac{\partial u_\alpha}{\partial z_i} + \sum_{\beta} \Gamma_{\beta i}^\alpha u_\beta = 0 \]
at $x$ for any $\alpha$ and $i$. We say that $(E, H)$ is (strictly) positive in the sense of Nakano at $x$, if for any section $u_j, u_k$ with this property,

$$i\partial\bar{\partial}(\sum H(u_i, u_j)dz_i \wedge d\bar{z}_j)$$

is (strictly) negative at this point. Here $dz_j \wedge d\bar{z}_j$ denotes the wedge product of all $dz_k$ and $d\bar{z}_k$ except $dz_i$ and $d\bar{z}_j$, multiplied by a constant of absolute value 1, chosen so that $\sum H(u_i, u_j)dz_i \wedge d\bar{z}_j$ is a positive form. One could refer to [27] for another approach on the Nakano positivity based on the Griffiths positivity. We summarize some basic discussions as following theorem.

**Theorem 1.3.** Let $Y$ be a complex manifold, and let $E$ be a vector bundle over $Y$ with a (singular) Hermitian metric $H$. Then

1. When $H$ is smooth, the notion of Griffiths (Nakano) positivity here coincides with the usual definition.
2. $(E, H)$ is positive (negative) in the sense of Griffiths iff it is positively (negatively) curved in the sense of Definition 1.2 in [23].
3. $(E, H)$ is negative in the sense of Nakano iff it is negatively curved in the sense of Definition 1.8 in [23].
4. Let $\varphi$ be the Finsler metric induced by $H$, then $(E, H)$ is strictly positive in the sense of Griffiths iff $(O_E(1), \varphi)$ is big.
5. If $(E, H)$ is positive in the sense of Griffiths, $(E \otimes \det E, H \otimes \det H)$ is positive in the sense of Nakano.

Finally we generalise Griffiths’s vanishing theorem.

**Theorem 1.4.** Let $Y$ be a compact Kähler manifold, and let $(E, H)$ be a strictly Griffiths positive vector bundle of rank $r + 1$ over $Y$. Let $\varphi$ be the corresponding metric on $O_E(1)$. Assume that $\varphi$ is well-defined along each fibre with $\mathcal{I}(\varphi) = O_X$. Then

$$H^q(Y, K_Y \otimes E \otimes \det E) = 0$$

for $q > 0$.

Here $\mathcal{I}(\varphi)$ refers to the multiplier ideal sheaf associated with $\varphi$. In [26], there is also an generalisation of Griffiths’s vanishing theorem (Corollary 1.4). We will make a precise computation on a stable vector bundle over a Riemann surface with genus $g > 1$ to show the relationship between his result and Theorem [14],

2. **Preliminary**

**2.1. Hermitian geometry revisit.** Let’s briefly recall the canonical Hermitian geometry first. This part is mostly taken from [13, 15]. Let $Y$ be a complex manifold, and let $f : E \to Y$ be a holomorphic vector bundle of rank $r + 1$ over $Y$. Let $z = (z_1, ..., z_n)$ be a system of local coordinate on $Y$, and let $Z = (Z_0, ..., Z_r)$ be the fibre coordinate
defined by a local holomorphic frame \( \{ u_0, \ldots, u_r \} \) of \( E \). Let \( H \) be a smooth Hermitian metric on \( E \), we write

\[
H_i = \frac{\partial H}{\partial z_i}, H_j = \frac{\partial H}{\partial \bar{z}_j}, H_{\alpha} = \frac{\partial H}{\partial Z_\alpha}, H_{\bar{\beta}} = \frac{\partial H}{\partial \bar{Z}_{\bar{\beta}}}
\]

to denote the derivative with respect to \( z_i, \bar{z}_j \) \((1 \leq i, j \leq n)\) and \( Z_\alpha, \bar{Z}_{\bar{\beta}} \)
\((0 \leq \alpha, \bar{\beta} \leq r)\). The higher order derivative is similar. Then locally \( H \) is represented as

\[
(H_{\alpha\bar{\beta}}(z)).
\]

Since \( (H_{\alpha\bar{\beta}}) \) is a positive Hermitian form, it is invertible. The inverse matrix is denoted by \( (H_{\gamma\bar{\delta}}) \). Then the associated curvature is

\[
\Theta_{E,H} = \sum \Theta_{\alpha\bar{\beta}i}dz^i \wedge d\bar{z}^j \otimes Z_\alpha \otimes Z_{\bar{\beta}}^*
\]

with

\[
\Theta_{\alpha\bar{\beta}ij} = -H_{\alpha\bar{\beta}ij} + H_{\gamma\bar{\delta}}H_{\alpha\bar{\gamma}}H_{\bar{\beta}\bar{\delta}}.
\]

The Griffiths and Nakano positivity is defined as follows:

**Definition 2.1.** Keep notations before,

(1) \( E \) is called (strictly) positive in the sense of Griffiths, if for any complex vector \( z = (z_1, \ldots, z_n) \) and section \( s = \sum s_\alpha u_\alpha \) of \( E \),

\[
\sum i\Theta_{\alpha\bar{\beta}i} s_\alpha \bar{s}_{\beta} z_i \bar{z}_j
\]

is (strictly) positive.

(2) \( E \) is called (strictly) positive in the sense of Nakano, if for any \( n \)-tuple \( (s^1 = \sum s^1_\alpha u_\alpha, \ldots, s^n = \sum s^n_\alpha u_\alpha) \) of sections of \( E \),

\[
\sum i\Theta_{\alpha\bar{\beta}ij} s^i_\alpha \bar{s}_j^\beta
\]

is (strictly) positive.

Let \( H \) be a smooth Hermitian metric on \( E \), and let \( u \) be a holomorphic section of \( u \). A direct computation implies the following well-known formula in [13]

\[
i\partial \bar{\partial} \log H(u) = -i \frac{\langle \Theta u, u \rangle}{H(u)} + i \frac{H(D^{1,0}u)H(u) - \langle u, D^{1,0}u \rangle \langle D^{1,0}u, u \rangle}{H(u)^2}.
\]

Note that the second term on the right hand side is positive by Cauchy–Schwarz inequality. Then we have

**Lemma 2.1.** \((E,H)\) is negative in the sense of Griffiths iff \( \log H(u) \) is plurisubharmonic for any holomorphic section \( u \).

Notice that there is no equivalence between strictly Griffiths positivity of \((E,H)\) and strictly plurisubharmonicity of \( \log H(u) \).

Fix a point \( y \in Y \). For any local holomorphic section \( s \) near \( y \), we can modify it by a linear combination \( \sum t_\alpha u_\alpha \) so that

\[
D^{1,0}s = 0
\]
at \( y \). Let
\[
\Gamma_{\beta i} = \frac{\partial H_{\alpha \bar{\gamma}}}{\partial z_i} H_{\beta \bar{\gamma}}^{\bar{\beta} \gamma},
\]
then for any section \( s = (s_0, \ldots, s_r) \), \( D^{1,0} s = 0 \) is equivalent to the fact that
\[
D^{1,0}_i s_\alpha := \frac{\partial s_\alpha}{\partial z_i} + \sum_\beta \Gamma_{\beta i}^\alpha s_\beta = 0
\]
for all \( i, \alpha \). Let \( s^i \) be an \( n \)-tuple of holomorphic sections of \( E \), satisfying this property. Put
\[
T_H = \sum H(s^i, s^j) \text{dz}_i \wedge \text{d\bar{z}}_j.
\]
Here \( \text{dz}_i \wedge \text{d\bar{z}}_j \) denotes the wedge product of all \( \text{dz}_k \) and \( \text{d\bar{z}}_k \) except \( \text{dz}_i \) and \( \text{d\bar{z}}_j \), multiplied by a constant of absolute value 1, chosen so that \( T_H \) is a positive form. The following lemma is well-known to experts.

Lemma 2.2. \((E, H)\) is (strictly) positive in the sense of Nakano at \( y \) iff
\[
i\partial \partial \bar{\partial} T_H
\]
is (strictly) negative for any choice of holomorphic sections \( s^i \) satisfying the property described above at this point.

Proof. Notice that for \( s^k = \sum s_k^\alpha u_\alpha, s^l = \sum s_l^\alpha u_\alpha \) with above property,
\[
\bar{\partial}_i \partial z_k H(s^k, s^l) = \bar{\partial}_j \partial z_l H(D_z s^k, s^l) = \sum_{\alpha, \beta} \Theta_{\alpha \bar{\beta} i j} s_k^\alpha s_l^\bar{\beta} H(u_\alpha, u_\beta) d\text{z}_i \wedge d\text{z}_j.
\]
Then
\[
i\partial \partial \bar{\partial} T_H = - \sum_{\alpha, \beta, i, j} \Theta_{\alpha \bar{\beta} i j} s_i^\alpha s_j^\bar{\beta} H(u_\alpha, u_\beta) dV,
\]
hence the conclusion. \( \square \)

The famous Griffiths’ vanishing theorem says that

Proposition 2.1 (Griffiths’ vanishing theorem, [13]). Let \( Y \) be a compact Kähler manifold, and let \( E \) be a Griffiths positive vector bundle over \( Y \). Then
\[
H^q(Y, K_Y \otimes E \otimes \text{det } E) = 0 \text{ for } q > 0.
\]

This theorem can be seen as an application of Nakano’s vanishing theorem [19] in view of Demailly and Skoda’s work [9].

A quick proof of Proposition 2.1 Since \( E \) is positive in the sense of Griffiths, \( E \otimes \text{det } E \) is positive in the sense of Nakano by [9]. Therefore
\[
H^q(Y, K_Y \otimes E \otimes \text{det } E) = 0 \text{ for } q > 0
\]
by Nakano’s vanishing theorem. \( \square \)
2.2. Finsler geometry revisit. We present a few basic properties concerning the Finsler metric on a vector bundle here. This part is mainly based on [14, 16]. Under the same setting as above, we have

**Definition 2.2.** A Finsler metric $G$ on $E$ is a continuous function $G : E \to \mathbb{R}$ satisfying the following conditions:

1. $G$ is smooth on $E^0 := E - \{0\}$, where 0 denotes the zero section of $E$;
2. $G(z, Z) \geq 0$ for all $(z, Z) \in E$ with $z \in Y$ and $Z \in f^{-1}(z)$, and $G(z, Z) = 0$ iff $Z = 0$;
3. $G(z, \lambda Z) = |\lambda|^2 G(z, Z)$ for all $\lambda \in \mathbb{C}$.

In applications, one often requires that $G$ is strongly pseudo-convex, that is,

4. the Levi form $i\partial \bar{\partial} G$ on $E^0$ is positive-definite along fibres $E_z$ for $z \in Y$.

Now let $G$ be a strongly pseudo-convex Finsler metric. Using the same notation, $G$ can be rewritten as

$$(G_{\alpha\beta}(z, Z)).$$

Different from the Hermitian metric, $G_{\alpha\beta}(z, Z)$ does depend on the variable $Z$. Moreover, it is easy to verify that $G_{\alpha\beta}(z, Z)$ is homogenous of degree 0 with respect to $Z$. Since $G$ is strongly pseudo-convex, $\{G_{\alpha\beta}(z, Z)\}$ is a positive-definite matrix on $E^0$, hence a smooth metric. Moreover, it actually defines a Hermitian metric $g^G$ on $p : f^* E \to E^0$ by

$$g^G(z, Z, W) := G \circ p(z, Z, W) = G(z, Z).$$

Here $W$ is the coordinate of the fibre of $p$. As is pointed before, $G$ is homogenous of degree 0 with respect to $Z$, hence it lifts to a Hermitian metric on $f^* E \to \mathbb{P}(E)$ over the projectived bundle, which is still denoted by $g^G$. Also, as a subbundle of $f^* E$, the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$

$$\mathcal{O}_{\mathbb{P}(E)}(-1) := \{(z, [Z], W); W = \lambda Z, \lambda \in \mathbb{C}\}$$

inherits an Hermitian metric from $g^G$ on $f^* E$, which is denoted by $g$. One verifies easily that $g(z, [Z], \xi) = G(z, Z)$ for any $(z, [Z], \xi) \in \mathcal{O}_{\mathbb{P}(E)}(-1)$. Let $\varphi$ be the weight function of $g$, formally we have $\varphi = \log G$. Obviously, it is a one-one correspondence between the Finsler metrics on $E$ and the smooth Hermitian metrics on $\mathcal{O}_{\mathbb{P}(E)}(-1)$. Therefore we refer both of them to the Finsler metric on $E$, if nothing confused.

Let’s pause for a second here. When referring to the projectived bundle, it is preferable to consider $\pi : X := \mathbb{P}(E^*) \to Y$ in algebraic geometry. In this case, the tautological line bundle is defined as $\mathcal{O}_E(1) := \mathcal{O}_{\mathbb{P}(E^*)}(1)$. Both of them will appear in our paper for their
own advantage when dealing with different problems. We use the distinguished notations to emphasize that they are totally different line bundles.

The positivity in algebraic geometry is then defined as follows:

**Definition 2.3.** Keep notations before,

1. \( E \) is called ample, if \( \mathcal{O}_E(1) \) is ample.
2. \( E \) is called big, if \( \mathcal{O}_E(1) \) is big.

Next we should present a precise computation for the curvature of \( \varphi \) here. This part is mainly based on [14, 16]. Let \((z, W = (w_1, ..., w_r))\) be a system of local coordinate of \( X \). We expand \( i\partial \bar{\partial} \varphi \) on \( X \) as follows:

\[
i\partial \bar{\partial} \varphi = i(g_{ij}dz_i \wedge d\bar{z}_j + g_{i\beta}dz_i \wedge d\bar{w}_\beta + g_{\alpha j}dw_\alpha \wedge d\bar{z}_j + g_{\alpha \beta}dw_\alpha \wedge d\bar{w}_\beta).
\]

Since \( G \) is strongly pseudo-convex, \( \{G_{\alpha \beta}\} \) as well as \( \{(\log G)_{\alpha \beta}\} \) is invertible. Let \( \{G_{\bar{\beta} \alpha}\} \) and \( \{(\log G)_{\bar{\beta} \alpha}\} \) be their inverse matrices respectively. Then for the holomorphic vector field \( \frac{\partial}{\partial z_i} \), its horizontal lift is defined as

\[
\frac{\delta}{\delta z_i} := \frac{\partial}{\partial z_i} - (\log G)_{\bar{\beta} i}(\log G)_{\bar{\beta} \alpha}\frac{\partial}{\partial w_\alpha}.
\]

The dual basis of \( \{\frac{\delta}{\delta z_i}, \frac{\partial}{\partial w_\alpha}\} \) will be

\[
\{dz_i, \delta w_\alpha := dw_\alpha + (\log G)_{\bar{\beta} i}(\log G)_{\bar{\beta} \alpha}dz_i \}.
\]

Let

\[
\Psi := iK_{\alpha \bar{\beta} ij}\frac{w_\alpha \bar{w}_\beta}{G}dz_i \wedge d\bar{z}_j
\]

and

\[
\omega_{FS} := i\frac{\partial^2 \log G}{\partial w_\alpha \partial \bar{w}_\beta}\delta w_\alpha \wedge \delta \bar{w}_\beta,
\]

where

\[
K_{\alpha \bar{\beta} ij} := -G_{\alpha \bar{\beta} ij} + G_{\bar{\gamma} \bar{\delta}}G_{\alpha \bar{\delta}}G_{\gamma \bar{\beta} j}.
\]

It is easy to verify that they are globally defined \((1,1)\)-form on \( X \). Then the celebrated theorem given by Kobayashi says that

**Proposition 2.2** (Kobayashi, [14, 16]).

\( i\partial \bar{\partial} \varphi = -\Psi + \omega_{FS} \).

Notice that the definition of \( K_{\alpha \bar{\beta} ij} \) is formally the same as the curvature \( \Theta_{\alpha \bar{\beta} ij} \) in Sect. 2.1. Indeed, it is easy to verify that if \( G \) is a Hermitian metric, i.e. \( G_{\alpha \bar{\beta}} \) is independent of \( Z \), \( K_{\alpha \bar{\beta} ij} \) is just the curvature in the usual sense. So it is reasonable to consider \( \Psi \) as the "curvature" of the Finsler metric \( G \), which leads to the following two definitions.

**Definition 2.4.** The form is called the Kobayashi curvature associated with \( G \). \((E, G)\) is called (strictly) positive in the sense of Kobayashi if the matrix \((K_{\alpha \bar{\beta} ij}w_\alpha \bar{w}_\beta)\) is (strictly) positive for any \( w = (w_1, ..., w_r) \).
Definition 2.5 ([4, 10, 11]).

\[ c(\phi)_{ij} := (i\delta \delta z_i, i\delta \delta z_j) > i\bar{\partial} \partial \phi \]

is called the geodesic curvature of \( \phi \) in the direction of \( i, j \).

Let \( a_i^a := -(\log G)\bar{\partial}_a(\log G)\bar{\beta}_i \), we have

\[ c(\phi)_{ij} = (\log G)_{ij} - a_i^a a_{aj}. \]

Therefore the relationship between \( c(\phi)_{ij} \) and \( \Psi \) will be

\[ -\Psi = ic(\phi)_{ij} dz_i \wedge d\bar{z}_j. \]

The following result is a quick consequence of the formula (3).

Proposition 2.3 (Kobayashi, [14, 16]). \((E, \phi)\) is ample iff \((E^*, G)\) is negative in the sense of Kobayashi. In particular, if \( G \) is a Hermitian metric on \( E^* \), it is also equivalent to say that \((E, G^*)\) is positive in the sense of Griffiths.

Here \( \phi \) refers to the Hermitian metric on \( O_E(1) \) induced by \( G \).

3. THE SINGULAR HERMITIAN METRIC

3.1. The definition. Let \( Y \) be a complex manifold, and let \( E \) be a vector bundle of rank \( r + 1 \) over \( Y \). Let \( \pi : X := \mathbb{P}(E^*) \to Y \) be the projectived bundle of the dual bundle, and let \( O_E(1) \) be the tautological line bundle over \( X \). Using the same system of local coordinate as before, we have the following definition.

Definition 3.1. Fix a smooth Hermitian metric \( h_0 \) on \( O_E(1) \). Let

\[ H(X) := \{ \phi \in L^1_{\text{loc}}(X); \text{On each fibre, } \phi \text{ is smooth} \}
\]

and \( i\Theta_{O_X(1), h_0} + i\bar{\partial} \partial \phi \) is strictly positive\}.

Notice that for any \( \phi \in H(X) \), \( \phi \) is a priori measurable function. Therefore \( \phi|_{X_z} \neq \infty \) for almost every \( z \in Y \). By \( \phi \) being smooth along each fibre, we mean that \( \phi|_{X_z} \) is a smooth function as long as it is well-defined, namely \( \phi \) is not identically equal to \( \infty \) on \( X_z \). Let \( Z = \{ z \in Y; \phi|_{X_z} = \infty \} \). In the rest part, all the discussions are made on \( Y - Z \). In this situation,

\[ \omega_{\phi, z} := (i\Theta_{O_X(1), h_0} + i\bar{\partial} \partial \phi)|_{X_z} \]

is actually a Kähler metric on \( X_z \). Based on this observation we define the singular Hermitian metric on \( E \) as follows.

Definition 3.2. The singular Hermitian metric \( H \) on \( E \) is a measurable function on \( E \) with form that

\[ H(z, Z) = \int_{X_z} |Z|^2 \frac{e^{-\nu} \omega_{\phi, z}^r}{r!} \]

with \( \phi \in H(X) \). Here we use the fact that \( \pi_* O_E(1) \simeq E \).
Note that when $\varphi$ is smooth, it is just the traditional $L^2$-metric induced by the strongly pseudo-convex Finsler metric $\varphi$. When $\varphi$ is merely $L^1$-bounded, it is easy to see that $H(z, Z)$ is an a.e. well-defined and $L^1$-bounded function on $Y$.

### 3.2. Comparison with the former definition

Recall that in [3, 23], there are two types of singular Hermitian metric. In [23], the relationship between these two definitions have been discussed. Now we should prove Theorem 1.1, which says that our definition is compatible with the definition of a smooth Hermitian metric. Moreover, we will show that the singular metric in our case is exactly the one in Raufi’s definition that is not degenerated.

**Proof of Theorem 1.1**  
(1) The first inclusion is easy. Indeed, for any smooth Hermitian metric $G$ on $E$, it will induce a dual metric $G^*$ on $E^*$, hence a strongly pseudo-convex Finsler metric $\varphi$ on $O_E(1)$. Formally we could write $\varphi = \log G^*$. In this situation, $|Z|_h^2 e^{-\varphi}$ is actually the function $G(z, Z)$ pulled back through $\pi$. Here we use the isomorphism $\pi_*O_E(1) \simeq E$. It leads to

$$H(z, Z) := \int_{X_z} |Z|_h^2 e^{-\varphi} \frac{\omega_{\varphi, z}}{r!} = G(z, Z) \int_{X_z} \frac{\omega_{\varphi, z}}{r!}.$$  

Therefore the $L^2$-metric induced by $\varphi$ must be $G$ itself up to a constant.

Next we prove the second inclusion. We would like to provide the whole details for readers’ convenience. Recall that the singular Hermitian metric in [23] refers to a measurable map from $Y$ to the space of non-negative Hermitian forms on the fibres. Therefore it is enough to show that

$$\int_{X_z} |Z|_h^2 e^{-\varphi} \frac{\omega_{\varphi, z}}{r!}$$

is a non-negative Hermitian form at each $z$. We make a precise computation to illustrate this fact. First, we fix a local trivialization. For a coordinate ball $U \subset Y$, we denote the local trivialization of $E$ by

$$E|_U \simeq U \times \mathbb{C}^{r+1} \to U$$

$$(z, Z = (Z_0, ..., Z_r)) \mapsto z$$

and the natural fibration between $X = \mathbb{P}(E^*)$ and $Y$ by

$$f : X \to Y$$

Moreover, locally we have

$$X|_{f^{-1}(U)} \simeq U \times \mathbb{P}^r \to U$$

$$(z, [Z^*]) \mapsto z$$
where \((Z^*)\) means the dual basis. Let \(W_A = \{Z^*_A \neq 0\}\) with \(0 \leq A \leq r\), then we have
\[
X|_{U \times W_A} \to U
\]
\[(z, w_A) \mapsto z
\]
with \(w_A = (w_A^0, ..., 1^A, ..., w_A^r) = (\frac{Z^*_0}{Z^*_A}, ..., \frac{Z^*_r}{Z^*_A})\). Here \(1^A\) means the \(A\)-th component equals 1. In this setting the local trivialization of \(\mathcal{O}_E(1)\) is
\[
\mathcal{O}_E(1)|_{U \times W_A} \simeq U \times W_A \times \mathbb{C}
\]
with coordinate \((z, w_A, \xi)\). We remark here that \(\xi\) has the same transition function as \(Z_A\).

If we take a holomorphic basis \(\{u_0, ..., u_r\}\) of \(E\), then for an arbitrary section \(s = \sum s_i u_i\), we have
\[
s : U \to E|_U
\]
\[(z) \mapsto (z, (s_0(z), ..., s_r(z))).
\]
It can be pulled back to be a section of \(\pi^* E \to X\)
\[
f^* s : U \times W_A \to \pi^* E|_{U \times W_A}
\]
\[(z, w_A) \mapsto (z, w_A, (s_0, ..., s_r))
\]
Since there is a natural embedding:
\[
i : \mathcal{O}_E(-1) \to f^* E^*,
\]
we have the following surjection:
\[
\tilde{i} : f^* E \to \mathcal{O}_E(1)
\]
by taking the dual. Therefore
\[
\tilde{i} f^* s : U \times W_A \to \mathcal{O}_E(1)|_{U \times W_A}
\]
\[(z, w_A) \mapsto (z, w_A, \xi_A)
\]
which is a section of \(\mathcal{O}_E(1)\). In order to see how the \(\xi_A\) is determined by \(s = (s_0, ..., s_r)\), we take \(s = e_0\), i.e. \((s_0, ..., s_r) = (1, 0, ..., 0)\), for example. Obviously in \(U \times W_0\), \(\xi_0(1, 0, ..., 0) = 1\). In a general chart \(U \times W_A\) with \(A \neq 0\), the morphism \(\tilde{i} f^* z\) is
\[
(z, w_A) \mapsto (z, w_A, w_A^0).
\]
Remember that \(w_A^0 = \frac{Z^*_0}{Z^*_A}\) is the transition function of \(\mathcal{O}_E(1)\) from \(U \times W_0\) to \(U \times W_A\). It is also simple to verify that in this definition \(\xi_A\) is surely a section of \(\mathcal{O}_E(1)\). Indeed, let \(U_a\) and \(U_b\) be two coordinate balls such that \(U_a \cap U_b \neq \emptyset\), one only needs to verify that on \((U_a \cap U_b) \times W_{a,0}\), we have
\[
e^{-\varphi(z_a, w_{a,0})} = |\xi_{a,0}|^2 e^{-\varphi(z_a, w_{a,0})} = |\xi_{b,0}|^2 e^{-\varphi(z_b, w_{b,0})},
\]

i.e. \( |\xi_{a,0}|^{2} e^{-\varphi(z_a, w_{a,0})} \) is a global function of \( z_a \). Assume that \( e_{a,0} = \sum g_{ab}^{0} e_{b,1} \), then \( \xi_{b,0} = g_{ab}^{00} + \sum_{1 \leq i \leq r} w_{b,0}^{i} g_{ab}^{0i} \). We have

\[
|\xi_{b,0}|^{2} e^{-\varphi(z_b, w_{b,0})} = |g_{ab}^{00} + \sum_{1 \leq i \leq r} w_{b,0}^{i} g_{ab}^{0i}|^{2} e^{-\varphi(z_b, w_{b,0})}
\]

The reason for the last equality is that the section of \( \mathcal{O}_{E}(1) \) has the same transition function with \( Z_{b0} \) as is remarked above. Therefore formally

\[
\frac{|Z_{a0}^{*}|^{2} e^{-\varphi(z_a, w_{a,0})}}{Z_{b0}^{*}} = \frac{|Z_{a0}^{*}|^{2} e^{-\varphi(z_a, w_{a,0})}}{Z_{b0}^{*}}
\]

The explicit expression in chart \( U \times W_{A} \) of \( \xi \) corresponding to a general section \( s = \sum_{0 \leq i \leq r} s_{i} e_{i} \) is

\[
(z, w_{A}) \mapsto (z, w_{A}, \xi(s_0, ..., s_r) = s_A + \sum_{0 \leq i \leq r, i \neq A} s_{i} w_{A}^{i}).
\]

In summary, the canonical corresponding between the sections of \( E \) and \( \mathcal{O}_{E}(1) \) can be written down as above. Through this isomorphism, \( H \) is defined as

\[
H(z, Z) := \int_{X_z} |\xi|^{2} e^{-\varphi(z, w_{A})} \frac{\omega_{\varphi }^{r}}{r!}.
\]

We can also represent it locally in the form of matrix:

\[
\begin{pmatrix}
H_{\alpha \beta}
\end{pmatrix}.
\]

More precisely, on \( U \times W_{A} \) there are four cases:

1. \( \alpha = \beta = A \), then

\[
H_{\alpha \beta} = \int_{X_z} |s_A|^{2} e^{-\varphi(z, w_{A})} \frac{\omega_{\varphi }^{r}}{r!} = \int_{X_z} e^{-\varphi(z, w_{A})} \frac{\omega_{\varphi }^{r}}{r!};
\]

2. \( \alpha = A, \beta \neq A \), then

\[
H_{\alpha \beta} = \int_{X_z} s_{A} \bar{s}_{\beta} w_{A}^{\beta} e^{-\varphi(z, w_{A})} \frac{\omega_{\varphi }^{r}}{r!} = \int_{X_z} \bar{w}_{A}^{\beta} e^{-\varphi(z, w_{A})} \frac{\omega_{\varphi }^{r}}{r!}.
\]
3. \( \alpha \neq A, \beta = A \), then
\[
H_{\alpha \beta} = \int_{X_z} s_{\alpha} w_{\bar{A}} A \bar{s}_A e^{-\varphi_{\alpha \beta}} \frac{\omega_{\alpha \beta}^r}{r!} = \int_{X_z} w_{\bar{A}}^\alpha e^{-\varphi(z,w_A)} \frac{\omega_{\alpha \beta}^r}{r!}.
\]

4. \( \alpha \neq A, \beta \neq A \), then
\[
H_{\alpha \beta} = \int_{X_z} s_{\alpha} w_{\bar{A}} A \bar{s}_A \bar{w}_{\bar{A}}^\beta e^{-\varphi_{\alpha \beta}} \frac{\omega_{\alpha \beta}^r}{r!} = \int_{X_z} w_{\bar{A}}^\alpha \bar{w}_{\bar{A}}^\beta e^{-\varphi(z,w_A)} \frac{\omega_{\alpha \beta}^r}{r!}.
\]

It is obvious that \((H_{\alpha \beta})\) is a non-degenerated Hermitian matrix, and the second inclusion is proved.

(2) When \( h \in M_H(E) \), it is a Hermitian metric on each \( E_z = \mathbb{C}^{r+1} \). Moreover, it induces an \( h^* \in M_H(E^*) \) hence a \( \varphi \in L^1_{\text{loc}}(X) \) under the same procedure in Sect.2.2. One could refer Lemma 3.1 in [17] for an explicit representation of \( h^* \). Moreover, \( \varphi \) is smooth along \( X_z \) since so is \( h^* \) along \( E_z^* \). We claim that \( \varphi \in \mathcal{H}(X) \) and induces \( h \).

Now we prove the claim. The quasi-plurisubharmonicity of \( \varphi \) along \( X_z \) is equivalent to the fact that \( h^* \) is a Hermitian metric on \( E_z^* \). Next, following the same argument as the proof of the first inclusion in (1), the singular metric defined by \( \varphi \) is exactly \( h \) up to a constant. \( \square \)

Remember that in [23], there is an example (Theorem 1.5) showing that in general the curvature current cannot be defined for a singular metric \( h \) in their sense. Indeed, [23] defines a singular metric \( h \) on a trivial vector bundle \( \Delta \times \mathbb{C}^2 \) over the unit disc \( \Delta \). The main reason that \( \Theta_h := \bar{\partial}(h^{-1} \partial h) \) doesn’t have measure coefficients is that \( h \) degenerates at \( z = 0 \). This is one of the motivations for Definition 3.2, where this situation is excluded.

4. THE FINSLER GEOMETRY IN SINGULAR CASE

This section is devoted to prove Theorem 1.2. Note that in the following part, \( Y \) is assumed to be compact. As a warming up, we will make an explain of the isometry on (1). Keep the notations. Recall that for any \( \varphi \in \mathcal{H}(X) \), \( \omega_{\varphi,z} \) is a Kähler metric along each fibre. Therefore it induces a smooth Hermitian metric \( \omega_{\varphi,z}^r \) on \( K_{X_z} \). On the other hand, \( \varphi \) defines a singular Hermitian metric \( h_0 e^{-\varphi} \) on \( \mathcal{O}_E(1) \) and a singular Hermitian \( H \) on \( E \), hence a singular metric \( \det H \) on \( E \). Now we say that \( \varphi \) induces an isometry between the canonical isomorphism

\[
K_{X/Y} \simeq \mathcal{O}_E(-r - 1) + \pi^* \det E
\]

if

\[
\omega_{\varphi,z}^r = Ce^{-(r+1)\varphi} \cdot \pi^* \det Hdw \wedge d\bar{w}
\]

on each \( X_z \). Here we abuse the notation by letting \( \varphi \) be the weight function of \( h_0 e^{-\varphi} \) after local trivialization. The constant \( C \) can be determined as follows.

**Lemma 4.1.** The normalization constant \( C \) in (6) equals to \( \frac{1}{((r+1))^{(r+1)}} \).
Proof. The proof is a continuation of the computation in the proof of Theorem 1.1. Using the same notation there, the determination
\[ \det H = \det G_{ij} u^* \wedge \bar{u}^* \]
defines a metric on \( \det E \). Here
\[ u^* \wedge \bar{u}^* := u_1^* \wedge ... \wedge u_r^* \wedge \bar{u}_1^* \wedge ... \wedge \bar{u}_r^* \]
is a basis of \( \det E^* \). Then the right hand side of (6) equals
\[ C(|\xi_A|^2 e^{-\varphi})^{r+1} \pi^*(\det G_{ij}(z)u^* \wedge \bar{u}^*)^{-1}. \]
We claim that in the canonical isomorphism
\[ \mathcal{O}_E(r + 1) \simeq -K_{X/Y} \otimes \pi^* \det E, \]
\( |\xi_A|^{2(r+1)} \) maps to
\[ dw_A \wedge d\bar{w}_A \otimes u^* \wedge \bar{u}^* \]
in \( U \times W_A \). In fact, one just need to compare the transition function. Remember that by the canonical surjection
\[ f^* E \rightarrow \mathcal{O}_E(1) , \]
\( \xi \in \mathcal{O}_E(1) \) is the preimage of a section \( f^* z \in f^* E \). Taking the \((r+1)\)-th exterior product, we get
\[ f^* \det E \rightarrow \mathcal{O}_E(r + 1), \]
which means that in the horizontal direction, \( \xi^{\otimes (r+1)} \in \mathcal{O}_E(r + 1) \) can be seen as a section of \( \pi^* \det E \). In the vertical direction it is obviously the same as a section of \( K_{X/Y} \). Therefore in \( U \times W_A \) its transition function is the same as
\[ dw_A \wedge d\bar{w}_A \otimes u^* \wedge \bar{u}^*. \]
It means the right hand side of (6) equals
\[ Ce^{-(r+1)\varphi(z,w_A)} \pi^* \det G_{ij}(z)dw_A \wedge d\bar{w}_A. \]
Remember that it’s a local representative in \( U \times W_A \) of a global form on \( X_z \). Thus the equation (6) can be reformulated as
\[ (i\partial \bar{\partial} w_A \varphi)^r = Ce^{-(r+1)\varphi(z,w_A)} \pi^* \det G_{ij}(z)dw_A \wedge d\bar{w}_A. \]
Now the equation restricted on each fibre is
\[ (i\partial \bar{\partial} w_A \varphi)^r = C \cdot C_z e^{-(r+1)\varphi(z,w_A)} dw_A \wedge d\bar{w}_A, \]
with \( C_z = \pi^* \det G_{ij}^{-1}(z) \). It is just the Kähler–Einstein equation on \( X_z = \mathbb{P}^r \). Thus the solution of the equation (7) must be written as follows:
\[ \varphi = \log(1 + \sum_A |w_A|^2) - \frac{1}{r+1} \log(C_z + C) \]
on $X_z$. Now we replace $\varphi$ by $\log(1 + \sum_A |w_A|^2) - \frac{1}{r+1} \log(C_z + C)$ in the explicit expression of $\det G_{ij}$, we get

$$C_z = \pi^r \det G_{ij}^{-1} (z)$$

$$= C_z \cdot C \det(\int_{\mathbb{P}^r} \frac{Z_i^* Z_j^* \omega^{FS}}{|Z^*|^2 r!})^{-1}$$

$$= ((r + 1)!)^C C_z \cdot C.$$

It means that $C = \frac{1}{((r+1)!)^r}$.

Now we are ready to prove Theorem 1.2. Recall that $H_i(X) := \{ \varphi \in L^1(X); \varphi \text{ is induced by a singular Hermitian metric } H \text{ on } E \}$.

Proof of Theorem 1.2. One direction is easy. If $\varphi \in H_i(X)$, it is induced by a singular Hermitian metric $H$ on $E$. In particular, $\varphi$, formally written as $\varphi = \log H^*$, is the Fubini–Study metric on $X_z = \mathbb{P}^r$. Therefore it solves the Kähler–Einstein equation on $X_z$. Then $\varphi$ solves the equation (7) in the view of Lemma 4.1. It exactly means that $\varphi$ induces an isometry between $K_{X/Y} \simeq O_E(-r - 1) + \pi^* \det E$.

The opposite direction is mainly based on the technique developed in [21]. Let’s explain the result in [21] first. In fact, it is proved in [21] that given a metric $\varphi$ on $O_E(1)$ with positive curvature, if it induces an isometry on $\mathbb{P}^1$, then the $L^2$-metric $h$ on $E$ defined by $\varphi$ is positive in the sense of Griffiths. More precisely, [21] actually indicates that (8)

$$i \Theta_{\alpha\bar{\beta}ij} w_{\alpha} \bar{w}_{\beta} dz_i \wedge d\bar{z}_j = -\Psi^T h$$

up to a constant. Hence the Griffiths positivity of $h$ is due to the Kobayashi negativity of $\Psi$, which is equivalent to the ampleness of $\varphi$ by formula (2). Here $\Psi^T$ means to take the conjugate transpose with respect to indexes $\alpha, \beta$ of the Kobayashi curvature associated to $\varphi$. Namely, if we write $\Psi = i \Theta_{\alpha\bar{\beta}ij} \frac{w_{\alpha} \bar{w}_{\beta}}{H} dz_i \wedge d\bar{z}_j$ on $E^*$, then

$$\Psi^T = i \Theta_{\beta\bar{\alpha}ij} \frac{w_{\alpha} \bar{w}_{\beta}}{H} dz_i \wedge d\bar{z}_j.$$

$H$ is the Finsler metric on $E^*$ corresponding to $\varphi$. Therefore the formula (8) actually says that the associated curvature of $h^*$ is the same as the Kobayashi curvature of $\varphi$. It is equivalent to say that $\varphi$ is induced by $h^*$, which is a Hermitian metric.

In order to complete the proof under the same scheme as [21], we only need to prove that the formula (8) is suitable for our situation. It is a byproduct of [21] except that we prefer to use Berndtsson’s curvature formula in [1] other than the To–Wang formula [28]. The former one seems to be more suitable for our situation. The proof is divided into three parts.

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1. By \( [1] \), we know that for any section \( u_\alpha, u_\beta \in E \) and complex vector \( v = (v_1, \ldots, v_n) \), the curvature of the \( L^2 \)-metric \( H \) defined by a smooth metric \( \varphi \) on \( O_E(1) \) equals to:

\[
(i \Theta_H^{ij} u_\alpha, \bar{u}_\beta) v_i \bar{v}_j = C_{r} \int_{X_z} (i \partial \bar{\partial} \psi)_{ij} w_\alpha \wedge \bar{w}_\beta v_i \bar{v}_j e^{-\psi} + i \int_{X_z} \eta_\alpha \wedge \bar{\eta}_\beta dV_z
\]

Notice that \( e^{-\psi} := e^{-r \omega_\varphi^r} \), and \( u_\alpha \in E \) maps to \( w_\alpha \in K_{X/Y} \otimes O_E(r + 2) \otimes \pi^* \det E^* \) through the canonical isomorphism. Here \( \eta_\alpha \) is given by

\[
\eta_\alpha = \frac{\bar{\partial} w_\alpha \wedge \sum v_i \hat{dz}_i}{dz}.
\]

\( \hat{dz}_i \) denotes the wedge product of all \( dz_j \) except \( dz_i \). Formally,

\[
\eta_\alpha = \kappa_\varphi^z w_\alpha,
\]

where \( \kappa_\varphi^z \) is the Kodaira–Spencer class. We extend the formula (9) here. The first term of the right hand side of (9) is well-defined for \( \psi \), since \( \psi = (r + 2) \varphi - \pi^* \det H \) which is quasi-plurisubharmonic along each fibre. The second term is zero, which will be illustrated in Step 2.

Now we approximate \( \varphi \) by a family of smooth functions \( \{ \varphi_\varepsilon \} \). Apply (9) on \( \varphi_\varepsilon \) and take the limit, we have successfully extend (9) to \( \varphi \).

2. It is proved in \([21]\) (Proposition 1) that \( \kappa_\varphi^z = 0 \), and (Corollary 1)\n
\[
\frac{\pi^* R^{\det E}_{H,ij}}{r!} = \int_{X_z} c(\varphi)_{ij} \frac{\omega_\varphi^r}{r!} = - \int_{pr} \Theta_{\sigma \tau ij} w_\sigma \bar{w}_\tau \frac{\omega_\varphi^{FS}}{r!}
\]

when \( \varphi \) is a smooth function that induces the isometry \([1]\).

Notice that the proof of Proposition 1 in \([21]\) is a clever calculation involving the covariant derivatives along each fibre. When \( \varphi \in \mathcal{H}(X) \), it is smooth along each fibre. Hence the same computation implies that \( \kappa_\varphi^z = 0 \) almost everywhere, which means that the function (of \( z \))

\[
i \int_{X_z} \eta_\alpha \wedge \bar{\eta}_\beta dV_z = 0
\]

almost everywhere. It is enough for our purpose. Similarly,

\[
\frac{\pi^* R^{\det E}_{\det H,ij}}{r!} = \int_{X_z} c(\varphi)_{ij} \frac{\omega_\varphi^r}{r!}
\]

almost everywhere for \( \varphi \in \mathcal{H}(X) \) which induces the isometry \([1]\).
3. Now we can calculate \((i\Theta_{ij}^H w_\alpha, \bar{w}_\beta) v_i \bar{v}_j\). By (9) it actually equals to

\[
= iC_r v_i \bar{v}_j \int_{X_z} c(\psi)_{\bar{\imath}j} w_\alpha \wedge \bar{w}_\beta e^{-\psi}
\]

\[
= iC_r v_i \bar{v}_j \int_{X_z} c((r + 2) \varphi - \pi^* \log \det H)_{\bar{\imath}j} w_\alpha \wedge \bar{w}_\beta e^{-\varphi'}
\]

\[
= - r! C_r v_i \bar{v}_j \int_{\mathbb{P}_r} ((r + 2) \Theta_{\sigma \tau ij}^\varphi \frac{w_{\sigma} \bar{w}_r}{|w|^2} + \pi^* R_{H,ij}) \frac{w_\alpha \bar{w}_\beta \omega_{FS}^r}{|w|^2} r!
\]

\[
= - r! C_r v_i \bar{v}_j \int_{\mathbb{P}_r} w_{\sigma} \bar{w}_r \omega_{FS}^r \frac{\omega_{FS}^r}{r!}
\]

\[
= - \delta_{\alpha \beta} \Theta_{\sigma \tau ij}^\varphi \frac{\Theta_{\sigma \tau \bar{i}j}^\varphi}{(r + 1)!} \left((\delta_{\alpha \beta} \delta_{\sigma \tau} + \delta_{\alpha \tau} \delta_{\sigma \beta} - \delta_{\alpha \beta} \delta_{\sigma \tau})
\]

\[
= - C_r v_i \bar{v}_j \frac{\Theta_{\beta \alpha \bar{i}j}^\varphi}{r + 1}
\]

The first equality is due to the equality that

\((i\partial \bar{\partial} \psi)_{\bar{\imath}j} w_\alpha \wedge \bar{w}_\beta = c(\psi)_{\bar{\imath}j} w_\alpha \wedge \bar{w}_\beta\).

The second equality comes from the discussion in Step 2. (Hence the equality should be understood to hold almost everywhere.) Then apply the same polarization as before, we get the third equality.

Since \(\{w_\alpha\}\) is an orthogonal basis on \(X_y\), \((i\Theta_{ij}^H w_\alpha, \bar{w}_\beta) = i\Theta_{ij}^H w_\alpha \bar{w}_\beta\) at \(z\). It gives the formula (9) almost everywhere. As a result, \(\varphi\) and the corresponding Finsler metric of \(H\) have the same Kobayashi curvature (current). Therefore they are equal up to a constant.

In summary, for any \(\varphi \in \mathcal{H}(X)\) that induces the isometry (11), it defines a singular Hermitian metric \(H\) on \(E\). Moreover, the singular Finsler metric on \(\mathcal{O}_E(1)\) induced by \(H\) is just \(\varphi\) itself up to a constant. Therefore we have successfully proved that \(\varphi \in \mathcal{H}_i(X)\).

Let

\[
\mathcal{H}_{ho}(X) := \{\varphi \in \mathcal{H}(X); i\Theta_{\mathcal{O}_E(1),ho} + i\partial \bar{\partial} \varphi \geq 0\}
\]

Then \(\mathcal{H}_i(X) \cap \mathcal{H}_{ho}(X)\) equals to the set

\[
\{\varphi \in \mathcal{H}(X); \varphi \text{ induces an isometry between (11) and} \ i\Theta_{\mathcal{O}_E(1),ho} + i\partial \bar{\partial} \varphi \geq 0\}
\]

by Theorem 1.2. On the other hand, there is one-one correspondence between \(\mathcal{H}_i(X)\) and \(M_W(E)\) with the same argument in Sect.2.2. Therefore we have proved Corollary 1.1. After we finish the definition of the Griffiths positivity in the next section, the singular Hermitian metric on \(E\) with Griffiths positive curvature can be arranged as
Definition 4.1. Let $Y$ be a compact complex manifold, and let $E$ be a holomorphic vector bundle of rank $r+1$. Let $\pi : X = \mathbb{P}(E^*) \to Y$ be the projectived bundle and let $\mathcal{O}_E(1)$ be the tautological line bundle over $X$. Then a singular metric on $E$ whose associated curvature is positive in the sense of Griffiths is defined as $\varphi \in \mathcal{H}(X)$ that induces an isometry between

$$K_{X/Y} \simeq \mathcal{O}_E(-r-1) + \pi^* \det E$$

and

$$i\Theta_{\mathcal{O}_E(1), h_0} + i\partial \bar{\partial} \varphi \geq 0.$$

5. The positivity

5.1. The Griffiths and Nakano positivity. In this section, we will talk about the positivity defined by the singular Hermitian metric. Keep notations before, we have

Definition 5.1. Let $H$ be a (singular) Hermitian metric on $E$ (Definition 3.2), and let $\varphi$ be the corresponding metric on $\mathcal{O}_E(1)$. Then

1. $(E, H)$ is (strictly) positive in the sense of Griffiths, if $i\partial \bar{\partial} \varphi$ is (strictly) positive on $X$.
2. $(E, H)$ is (strictly) negative in the sense of Griffiths, if $(E^*, H^*)$ is positive in the sense of Griffiths.
3. Assume that $(E, H)$ is positive in the sense of Griffiths, and that $\det H \leq C$. Let

$$\Gamma^\alpha_{\beta i} := \frac{\partial H_{\alpha \bar{\beta}}}{\partial z_i} H^{\gamma \beta}.$$ 

Consider the local holomorphic section $u = (u_0, ..., u_r)$ of $E$ such that

$$\frac{\partial u_\alpha}{\partial z_i} + \sum_{\beta} \Gamma^\alpha_{\beta i} u_\beta = 0$$

at a given point $z \in Y$ for any section $\alpha$ and $i$. $(E, H)$ is (strictly) positive in the sense of Nakano at $z$, if for any sections $u_i, u_j$ with this property,

$$i\partial \bar{\partial} (\sum H(u_i, u_j) dz_i \wedge d\bar{z}_j)$$

is (strictly) negative at this point.
4. Assume that $(E, H)$ is negative in the sense of Griffiths. $(E, H)$ is (strictly) negative in the sense of Nakano at $z$, if for any sections $u_i, u_j$ with the property above,

$$i\partial \bar{\partial} (\sum H(u_i, u_j) dz_i \wedge d\bar{z}_j)$$

is (strictly) positive at this point.
Let’s give a short explanation for $\Gamma_{\beta i}^\alpha$ here. By [23], the associated connection and curvature of $H$ can be defined through approximation in the circumstance of (3) and (4), hence

$$\frac{\partial u_\alpha}{\partial z_i} + \sum_\beta \Gamma_{\beta i}^\alpha u_\beta$$

is an $L^1$-bounded function on $Y$. In particular, it is meaningful to ask this function to vanish at one point. Notice that in (4) we do not need to furthermore assume that $\det H > 0$, since it is automatically satisfied in our definition for a singular metric. We remark here that whether it is necessary to assume the Griffiths positivity before talking about the Nakano positivity is unknown currently.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. (1) Assume that $H$ is a smooth Hermitian metric, then the corresponding metric $\varphi$ on $\mathcal{O}_E(1)$ is also smooth. By Proposition 2.3, $(E, H)$ is positive in the sense of Griffiths iff $(\mathcal{O}_E(1), \varphi)$ is ample, hence the conclusion for the Griffiths positivity. The Nakano positivity part is a direct consequence of Lemma 2.2.

(2) When $(E, H)$ is positive in the sense of Griffiths, $\varphi = \log H^*$ is plurisubharmonic by definition. It exactly means that $(E^*, H^*)$ is negatively curved in the sense of [23]. Therefore $(E, H)$ is positively curved by definition.

When $(E, H)$ is positively curved, then locally there exists an increasing regularising sequence $\{H_\nu\}$ such that $(E, H_\nu)$ are positively curved by Proposition 1.3 in [23]. Let $\varphi_\nu$ be the corresponding metric on $\mathcal{O}_E(1)$, then $\varphi_\nu$ is convergent to $\varphi$. Moreover, apply Proposition 2.2 to $H_\nu$, we get that

$$i\partial \bar{\partial} \varphi_\nu = -\Psi_\nu + \Omega'_{FS}.$$

Since $(E, H_\nu)$ is positively curved, $\Psi_\nu$ is negative by Proposition 2.3. It means that $\varphi_\nu$ is plurisubharmonic, so will be $\varphi$. Hence $(E, H)$ is positive in the sense of Griffiths by definition.

(3) Note that for any holomorphic section $u = (u_\alpha)$, we can always arrange things that

$$\frac{\partial u_\alpha}{\partial z_i} + \sum_\beta \Gamma_{\beta i}^\alpha u_\beta = 0$$

at a given point by modifying $u$ by a suitable linear combination. The equivalence then follows.

(4) is by definition.

(5) Since $(E, H)$ is positive in the sense of Griffiths, $\varphi$ is plurisubharmonic on $X$ by definition. Remember that the curvature formula in [1] has been extended to a formula suitable for $\varphi$ in the proof of Theorem 1.2. Now apply it to $(\mathcal{O}_E(r + 1), (r + 1)\varphi)$, we have that

$$\pi_*(K_{X/Y} + \mathcal{O}_E(r + 2)) = E \otimes \det E.$$
equipped with \((H \otimes \det H)\) is positive in the sense of Nakano. \(\square\)

5.2. The vanishing theorem. Now the generalised Griffiths’ vanishing theorem is rather intuitive.

Proof of Theorem 1.4 Let \(\varphi\) be the corresponding metric on \(O_E(1)\). Since \((E, H)\) is positive in the sense of Griffiths, \((O_E(1), \varphi)\) is big by Theorem 1.3. Apply Nadel’s vanishing theorem to \((O_E(r+2), (r+2)\varphi)\), and remember that \(\mathcal{I}((r+2)\varphi) = O_X\) by assumption, we have

\[ H^q(X, K_X \otimes O_E(r+2)) = 0 \text{ for all } q > 0. \]

Using Nadel’s vanishing theorem along each fibre, we get that

\[ R^i\pi_*(K_X \otimes O_E(r+2)) = 0 \text{ for all } i > 0. \]

Notice that \(\varphi|_{X_z}\) is well-defined for every \(z\) by assumption. By Larey spectral sequence we obtain that

\[ H^q(X, K_X \otimes O_E(r+2)) = H^q(Y, \pi_*(K_X \otimes O_E(r+2))). \]

Combine with the fact that

\[ \pi_*(K_X \otimes O_E(r+2)) = K_Y \otimes E \otimes \det E, \]

we immediately get the desired result. \(\square\)

In [26], there is also a generalised Griffiths’ vanishing theorem (Corollary 1.4). Currently, the relationship between their work and Theorem 1.4 is not clearly to me. However, in some special situation, we will see more generality on Theorem 1.4. Let’s precise compute the following example. This example has been presented in [29].

Example 5.1. Let \(E\) be a stable vector bundle of rank \(r+1\) over a smooth projective curve \(Y\) of genus \(\geq 2\). Then it is proved in [20] that there exists a coordinate chart \(\{V_i\}\) of \(Y\) such that the transition matrices of \(E\) can be written in the form \(g_{ij} = f_{ij}U_{ij}\), where \(f_{ij}\) is a scalar function and \(U_{ij}\) is a unitary matrix on \(V_i \cap V_j\). In this situation, there is a one-one correspondence between the singular Hermitian metric on \(E\) and \(\det E\).

Assume that \((\det E, \{h_i\})\) is big. Then \(\{h_i^{1/(r+1)}I_{r+1}\}\) defines a singular Hermitian metric \(H\) on \(E\). Here \(I_{r+1}\) is the identity matrix of rank \(r+1\). If we denote the weight function of \(\{h_i\}\) by \(\phi\), the associated curvature of \(H\) is given by \(\frac{1}{r+1}H_i^{1/(r+1)}I_{r+1}i\partial\bar{\partial}\phi\), which is strictly positive in the sense of Griffiths. Take \(\varphi\) to be the weight function of the metric on \(O_E(1)\) induced by \(h\). Let

\[ j : E^* - \{0\} \rightarrow \mathbb{P}(E^*) \]

\[(z, (Z^*_\alpha)) \mapsto (z, [Z^*_\alpha]) = (z, (\frac{Z^*_\alpha}{Z^*_A})) = (z, (w^*_\alpha)) \]

be the natural morphism. Here we represent it on an open subset \(\{Z^*_A \neq 0\}\).
of $\mathbb{P}(E^*)$. In this setting, we denote the local coordinate of $\mathcal{O}_E(1)$ by $(z, (w^*_a, \xi))$. Let $\pi : X := \mathbb{P}(E^*) \to Y$ be the projection. The formula [3] then gives that
\[
\nu = -\Psi + \omega_{FS} + \{(\xi = 0)\}
\]
\[
\begin{align*}
\nu &= -i\Theta_{E^*} w^*_A \bar{w}_A \partial z_i \wedge \partial z_j + i \frac{\partial^2 (\log H^*)}{\partial w^*_A \partial \bar{w}_A} \delta w^*_A \wedge \delta \bar{w}_A + \{(\xi = 0)\} \\
&= \frac{1}{r+1} \pi^* (h_i^{-1/(r+1)} i \partial \bar{\partial} \phi) \sum_{\alpha} \frac{|w^*_\alpha|^2}{h_i^{-1/(r+1)} \sum_{\alpha} |w^*_\alpha|^2} + i \frac{dw^*_A \wedge d\bar{w}_A}{1 + \sum_{\alpha} |w^*_\alpha|^2} \\
&= \frac{1}{r+1} i \partial \bar{\partial} \pi^* \phi + i \frac{dw^*_A \wedge d\bar{w}_A}{(1 + \sum_{\alpha} |w^*_\alpha|^2)^2}.
\end{align*}
\]
Here $\{(\xi = 0)\}$ is the current of integration of the zero section. We use the fact that $\xi$ has the same transition function as $\frac{1}{Z_A}$ to get the forth equality. Obviously $i \partial \bar{\partial} \varphi$ is a closed strictly positive $(1,1)$-current. It means that $(\mathcal{O}_E(r+2), (r+2) \varphi)$ is big. Moreover, by (the proof of) Theorem [12], the $L^2$-metric defined by $(r+2)\varphi$ on $\pi_* \mathcal{O}_E(r+2) = S^{r+2} E$ is just $S^{r+2} H$ up to a constant. Now let $R = (2r+2)$, which is the rank of $S^{r+2} E$. Let $\Xi = \pi$ be the section of $\mathcal{O}_E(r+2)$ and let $U_\alpha$ be the corresponding section of $S^{r+2} E$ under $\pi_* \mathcal{O}_E(r+2) \simeq S^{r+2} E$. Based on the discussion before, we have
\[
\int_X |\Xi|^2 e^{-(r+2) \varphi} \omega^{-1}_{\varphi} = h_i^{(r+2)(r+3)} \sum |U_\alpha|^2.
\]
Here $\omega_\varphi := i \partial \bar{\partial} w_A \varphi$.

Assume that $H((r+2) \varphi) = \mathcal{O}_X$, then by Theorem [1, 4]
\[
H^q(Y, K_Y \otimes E \otimes \det E) = 0 \text{ for all } q > 0.
\]
On the other hand, by formula (10), $\int_X |\Xi|^2 e^{-(r+2) \varphi} \omega^{-1}_{\varphi}$ is integrable on $Y$ if the integral on the right hand side against $\frac{r+2}{2R} \phi$ is finite. A sufficient condition is $\nu(\phi) < \frac{2R}{(r+2)(r+3)}$. As a conclusion, we have

**Conclusion:** Let $E$ be a stable vector bundle of rank $r+1$ over a smooth projective curve $Y$ of genus $\geq 2$. Let $R = (2r+2)$. Assume that $(\det E, \phi)$ is big with $\nu(\phi) < \frac{2R}{(r+2)(r+3)}$. Then
\[
H^q(Y, K_Y \otimes E \otimes \det E) = 0 \text{ for all } q > 0.
\]

Notice that in [26] (Corollary 1.4), they obtain the same vanishing result under the assumption that $\nu(\phi) < 1$. It is easy to see that $\frac{2R}{(r+2)(r+3)} > 1$ when $r > 1$.

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