Exact solution of a time-reversal-invariant topological superconducting wire

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We consider a model proposed before for a time-reversal-invariant topological superconductor (TRITOPS) which contains a hopping term $t$, a chemical potential $\mu$, an extended $s$-wave pairing $\Delta$ and spin-orbit coupling $\lambda$. We show that for $|\Delta| = |\lambda|$, $\mu = t = 0$, the model can be solved exactly defining new fermion operators involving nearest-neighbor sites. The many-body ground state is four-fold degenerate due to the existence of two zero-energy modes localized exactly at the first and the last site of the chain. These four states show entanglement in the sense that creating or annihilating a zero-energy mode at the first site is proportional to a similar operation at the last site. By continuity, this property should persist for general parameters. Using these results we correct some statements related with the so called “time-reversal anomaly”. Addition of a small hopping term for a chain with an even number of sites breaks the degeneracy and the ground state becomes unique with an even number of particles. We also consider a small magnetic field applied to one end of the chain. We compare the many-body excitation energies and spin projection along the spin-orbit direction for both ends of the chains with numerical results obtaining good agreement.

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\section{Introduction}

In recent years, there is a lot of interest in topological superconductors. One of the main reasons of this interest is the fact that one-dimensional wires have Majorana fermions at the ends, which might be used in quantum computation exploiting their non-abelian nature\textsuperscript{12–13}. The first theoretical proposals\textsuperscript{13–14} and experimental research\textsuperscript{15} were focused on systems in which time-reversal symmetry is broken. More recently theoretical research on time-reversal-invariant topological superconductors (TRITOPS) has developed\textsuperscript{15–16}.

The TRITOPS belong to class DIII in the classification of topological superconductors\textsuperscript{16}. As such, they host a zero-energy fermionic excitation, or equivalently a Kramers pair of Majorana fermions at each end of the wire. This “Majorana Kramers Qubit” has been proposed as the basis of a universal gate set for quantum computing\textsuperscript{17–18}.

Zhang et al.\textsuperscript{19} proposed to construct TRITOPS wires via proximity effect between nodeless extended $s$-wave iron-based superconductors and semiconducting systems with large Rashba spin-orbit coupling. An extended $s$-wave superconducting gap $\Delta$ and spin-orbit coupling $\lambda$ are basic ingredients of the model we study.

An odd property of the TRITOPS wires is that the many-body ground state is characterized by fractional spin projection along the spin-orbit coupling (which we choose to be $z$) at each end of the wire. Specifically for the left and right ends $S_z^{\text{left}}$, $S_z^{\text{right}} = \pm 1/4$. The first argument to show this fractional spin was based on the so called “time-reversal anomaly”, first proposed for two- and three-dimensional TRITOPS\textsuperscript{20}. The argument can be summarized as follows. We denote as $a_{\text{left}}^{\dagger}$ the annihilation operator of the zero-energy mode at the left end of the chain. It commutes with the Hamiltonian $H$

\begin{equation}
K a_{\text{left}}^{\dagger} K^\dagger = a_{\text{left}}^{\dagger} \propto a_{\text{left}}^{\dagger} \end{equation}

with $K$ the time reversal operator, so that there is only one independent fermion at the left end, and the same happens at the right end. Let us assume that $|G_0\rangle$ is one of the degenerate ground states, with $a_{\text{left}}^{\dagger} |G_0\rangle = 0$. Then, using $[a_{\text{left}}^{\dagger}, H] = 0$, also $|G_1\rangle = a_{\text{left}}^{\dagger} |G_0\rangle$ belongs to the ground states (if it does not vanish). Due to time reversal invariance of the Hamiltonian, one expects that $|G_0\rangle$ and $|G_1\rangle$ are time reversal partners, and this implies $\langle G_1 | S_z^{\text{left}} | G_1 \rangle = -\langle G_0 | S_z^{\text{left}} | G_0 \rangle$, where $S_z^{\text{left}}$ is the spin projection at the left end of the chain (adding the projection of a few sites at the extreme left of the order of the localization length of the zero-energy mode). On the other hand, since $a_{\text{left}}^{\dagger}$ annihilates a spin up (or creates a spin down) at the left of the chain $\langle G_1 | S_z^{\text{left}} | G_1 \rangle - \langle G_0 | S_z^{\text{left}} | G_0 \rangle = -1/2$, and then $\langle G_0 | S_z^{\text{left}} | G_0 \rangle = 1/4$. In this argument, the right end is not considered. All this reasoning has a problem: it is expected that $|G_0\rangle$ and $|G_1\rangle$ differ in the fermion parity, and $K^2 = 1$ (-1) for even (odd) fermion parity, leading to a contradiction. This might be solved if both ends are included in the argument\textsuperscript{13}. Our results in which the many-body states are constructed explicitly shed light on the underlying physics (see Section\textsuperscript{11–13}).

In a previous publication, the excitations at both ends were studied\textsuperscript{15}. The contribution of each site to the operators $a_{\text{left}}^{\dagger}$ and $a_{\text{right}}^{\dagger}$ decays exponentially with the distance to the corresponding end, with a decay length $\lambda_c$ determined by solving a quartic equation. For the particular case of the chemical potential $\mu = 0$ and the length of the chain $L \to \infty$, an analytical form of the...
operators was given. Formally, one of the states $|e_1\rangle$ that is part of the ground state for $L \to \infty$ is constructed in a similar way as the ground state of the Bardeen-Cooper-Schrieffer (BCS) Hamiltonian, as the product of all annihilation operators $\Gamma_\nu$ satisfying $[\Gamma_\nu, H] = E_\nu \Gamma_\nu$, with positive $E_\nu$. For finite $L$, an exponentially small mixing of $a_{\text{left}}$ and $a_{\text{right}}$ takes place. Two new mixed annihilation operators $\gamma_\sigma$ with $[\gamma_\sigma, H] = E_\sigma \gamma_\sigma$ with positive $E_\sigma$ are found, so that the (now non-degenerate) ground state becomes $|g_e\rangle = \gamma \gamma L |e_1\rangle$. Although the explicit form of $|e_1\rangle$ is not known, the fact is that it is time reversal invariant in the absence of a magnetic field and that the operators $\Gamma_\nu$ correspond to finite energy have been used to calculate the spin projection $S^z_{\text{right}}$ for the ground state and the first excited states with odd number of particles, in particular for a magnetic field applied only to the right end, finding fractional values.

Some open questions regarding the nature of the ground state still remain. For example, for $L \to \infty$ there are two independent zero-energy modes, one at the left end of the chain and one at the right end of it. Then, one expects a four-fold degenerate ground state depending on the occupation number of these two fermions is zero or one. However, in principle there are 16 forms to apply between zero and four operators $\Gamma_{\text{left}\sigma}$ and $\Gamma_{\text{right}\sigma}$ to $|e_1\rangle$. How are they related? To answer this question the explicit form of $|e_1\rangle$ is needed.

In this work, we report on the exact solution of the model for particular parameters $|\Delta| = |\lambda| = \mu = t = 0$. This allows us to construct explicitly, not only the one-body operators that diagonalize the Hamiltonian, but also the four many-body states that are part of the ground state. We find that $\Gamma_{\text{right}\sigma}|e_1\rangle$ is proportional to $\Gamma_{\text{left}\sigma}|e_1\rangle$, where $|e_1\rangle$ is constructed as indicated above. By continuity, this property should be valid for general parameters inside the topological phase ($|\mu| < 2|\lambda|$). This indicates that although $|e_1\rangle$ does not seem to contain direct information about the zero-energy modes (it is constructed with operators that commute with the zero-energy ones), it is an entangled state and its ends are related.

We also discuss the effect of a small hopping $t$ and a magnetic field applied to one end on the exact solution. The analytical results for the many-body states are supported by numerical diagonalization of small systems.

The paper is organized as follows. In Sec. II we construct the model for $|\Delta| = |\lambda| = \mu = t = 0$ and describe the many-body ground state. We also calculate the expectation values of the spin projection at the ends of the states that compose the ground state, finding the fractional values $\pm 1/4$ expected from previous works. In Sec. III we analyze perturbatively the effect of a small hopping $t$ on the exact solution and compare with numerical results. In Sec. IV we present a summary and a brief discussion.

II. MODEL

The Hamiltonian describing the system is the following:

$$H = \sum_{\sigma} \left\{ \sum_{j=1}^{L-1} \left( -tc_{j+1\sigma}^\dagger c_{j\sigma} + is_\sigma \lambda c_{j+1\sigma}^\dagger c_{j\sigma} + s_\sigma \Delta e^{i\phi} c_{j+1\sigma}^\dagger c_{j\sigma}^\dagger + \text{H.c.} \right) - \mu \sum_{j=1}^{L} c_{j\sigma}^\dagger c_{j\sigma} \right\}, \quad (2)$$

where $s_{\uparrow, \downarrow} = \pm 1$ and $\uparrow = \downarrow, \downarrow = \uparrow$. The parameter $t$ corresponds to the nearest-neighbor hopping, $\mu$ is the chemical potential, $\lambda$ is the Rashba spin-orbit coupling and $\Delta$ is the strength of the extended s-wave pairing.

The exact solution corresponds to $|\Delta| = |\lambda|, t = \mu = 0$. In the following we take for simplicity $\Delta = \lambda = 1, \phi = t = \mu = 0$. The formalism used can be changed in a straightforward way to include a finite phase $\phi$ and other signs of $\Delta$ and $\lambda$. The effect of finite $t$ is treated in Section IV, and the general case is discussed in Section V.

For $\phi = 0$, the Hamiltonian is invariant under time reversal symmetry. In addition, the Hamiltonian conserves parity and the total spin projection $S_z$ (the total spin in the direction of the Rashba spin-orbit coupling). Below we will use these three symmetries.

III. CONSTRUCTION OF THE EXACT SOLUTION

A. One-particle operators

In this Section, we look for annihilation operators $\Gamma_\nu$ that satisfy

$$[\Gamma_\nu, H] = E_\nu \Gamma_\nu, \quad (3)$$

and therefore allow us to diagonalize the Hamiltonian Eq. (2) for $\Delta = \lambda = 1, \phi = t = \mu = 0$. Since Eq. (3) implies that $\Gamma_\nu$ satisfies the same equation with the opposite sign of $E_\nu$, we can redefine the operators so that $E_\nu \geq 0$.

We define the following operators

$$a_{j\sigma} = (c_{j\sigma} + ic_{j\sigma}^\dagger)/\sqrt{2}, \quad b_{j\sigma} = (c_{j\sigma} - ic_{j\sigma}^\dagger)/\sqrt{2}. \quad (4)$$

Note that

$$a_{j\sigma}^\dagger = -ia_{j\sigma}, \quad b_{j\sigma}^\dagger = ib_{j\sigma}. \quad (5)$$
and under time reversal $K$, these operators transform as

$$K\alpha_j K^\dagger = \alpha_{j\uparrow}, \quad K\sigma_j K^\dagger = -\sigma_j\uparrow,$$

(6)

where $\alpha = a$ or $b$ [or $\Gamma$ defined by Eq. (8)].

Using Eq. (2), one obtains for a finite chain of $L$ sites with $j = 1, \ldots, L$

$$[a_j, H] = \left\{ \begin{array}{ll}
2s_\sigma ib_{j-1}\sigma & \text{if } j > 1 \\
0 & \text{otherwise}
\end{array} \right.$$

$$[b_j, H] = \left\{ \begin{array}{ll}
-2s_\sigma ia_{j+1}\sigma & \text{if } j < L \\
0 & \text{otherwise}
\end{array} \right.$$

(7)

Clearly $a_{1\sigma}$ and $b_{L\sigma}$, taking into account Eqs. (5), correspond to the zero-energy modes, one for each end of the chain, expected from the topological character of the TRITOPS phase.\textsuperscript{13,29} The remaining operators mix nearest-neighbor sites as sketched in Fig. 1, resembling the exact solution of the Kitaev model.\textsuperscript{19}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{(Color online) Sketch of the construction of the annihilation operators of finite energy, leaving the zero-energy modes $a_{1\sigma}$ and $b_{L\sigma}$ at the ends.}
\end{figure}

We define the operators for $j < L$

$$\Gamma_{j\sigma} = (b_j - is_\sigma a_{j+1\sigma})/\sqrt{2}.$$  

(8)

It is easy to see using Eqs. (7), that their commutators with the Hamiltonian are

$$[\Gamma_{j\sigma}, H] = 2\Gamma_{j\sigma}.$$ 

(9)

Therefore the desired annihilation operators are obtained. Note that the $\Gamma_{j\sigma}$ also satisfy the relations Eqs. (8).

### B. Many-body low-energy eigenstates

Since there are two independent fermionic modes with zero energy, one at each end of the chain, one expects a four-fold degenerate ground state depending on the occupancy of these modes. The analysis explained below as well as many-body calculations in chains with up to $L = 6$ sites confirm this expectation.

One of these four states, with even number of particles is obtained applying all annihilation operators with positive energy [those entering Eq. (9)] to the vacuum $|0\rangle$ of the $c_{j\sigma}$:

$$|e_1\rangle = N_L \prod_{j=1}^{L-1} \Gamma_{j\uparrow}\Gamma_{j\downarrow}|0\rangle,$$

(10)

where $N_L$ is a normalization factor. This state is invariant under time reversal [using Eqs. (8) one easily proves that $K|e_1\rangle = |e_1\rangle$], and also $S_z |e_1\rangle = 0$. Note that neither the Hamiltonian nor this state have inversion symmetry.

Two states with odd parity number can be written as

$$|o\sigma\rangle = N_o a_{1\uparrow} c_{1\uparrow} |e_1\rangle,$$

(11)

where $N_o$ is another normalization factor. These states are also eigenstates of the total spin projection with $S_z |o\sigma\rangle = (s_o/2)|o\sigma\rangle$.

Finally, there is another even state invariant under time reversal and with zero spin projection,

$$|e_2\rangle = N_e (a_{1\uparrow} a_{1\downarrow} - a_{1\downarrow} a_{1\uparrow}) |e_1\rangle.$$ 

(12)

This state is orthogonal to $|e_1\rangle$. Using Eqs. (5) one has, except for normalization factors, $\langle e_1|e_2\rangle \sim \langle o \downarrow |o \downarrow \rangle = 0$.

It might seem surprising that acting with the zero-mode operators at the right end $b_{L\sigma}$ on $|e_1\rangle$ does not lead to new states that are part of the ground state. Instead, we find

$$b_{L\sigma} |e_1\rangle = -(s_{\sigma})^{L-1} a_{1\sigma} |e_1\rangle,$$

$$a_{1\sigma} |e_1\rangle = -(s_{\sigma})^{L-1} b_{L\sigma} |e_1\rangle.$$ 

(13)

This is demonstrated below.

For the simplest chain with $L = 2$, the ground manifold is

$$|e_1\rangle = \frac{1}{2} [c_{1\uparrow} c_{1\downarrow} + c_{2\uparrow} c_{2\downarrow} + ic_{1\uparrow} c_{1\downarrow} + ic_{2\uparrow} c_{2\downarrow}] |0\rangle,$$

$$|e_2\rangle = \frac{1}{2} [1 - c_{1\uparrow} c_{1\downarrow} + c_{2\uparrow} c_{2\downarrow} - ic_{1\uparrow} c_{1\downarrow} + ic_{2\uparrow} c_{2\downarrow}] |0\rangle,$$

$$|o \downarrow\rangle = \frac{1}{2} [c_{1\downarrow} + ic_{2\uparrow} + ic_{1\downarrow} c_{2\downarrow} + c_{1\downarrow} c_{2\downarrow}] |0\rangle,$$

$$|o \uparrow\rangle = K|o \downarrow\rangle.$$ 

(14)

Using these expressions, it is easy to check that Eq. (13) is valid for $L = 2$. Below we prove its validity for a chain of $L + 1$ sites assuming that Eq. (13) is valid for $L$ sites, allowing us to extend its validity to any $L$. For a chain of $L + 1$ sites, Eq. (10) can be written in the form

$$|e_1(L + 1)\rangle = N_{L+1} \hat{O}(L + 1)|0\rangle,$$

$$\hat{O}(L) = \prod_{j=1}^{L-1} \Gamma_{j\uparrow}\Gamma_{j\downarrow}.$$ 

(15)
Then using anticommutation rules and Eq. (13)

\[ a_{1\uparrow}|e_1(L + 1)\rangle = N_{L+1}\Gamma_{L\uparrow}\tilde{\Gamma}_{L\downarrow}a_{1\uparrow}\hat{O}(L)|0\rangle \]

\[ = (-i)^{L-1}N_{L+1}\Gamma_{L\uparrow}\Gamma_{L\downarrow}b_{L\downarrow}\hat{O}(L)|0\rangle \]

\[ = (-i)^{L-1}N_{L+1}\tilde{\hat{O}}(L)\Gamma_{L\downarrow}\Gamma_{L\uparrow}b_{L\uparrow}|0\rangle. \] (16)

From the definitions Eqs. (4) and (8) we find

\[ \Gamma_{L\uparrow}b_{L\downarrow}|0\rangle = ib_{L+1\uparrow}\Gamma_{L\downarrow}|0\rangle = \frac{-i}{2\sqrt{2}}c_{L+1\downarrow}^\dagger c_{L\downarrow}^\dagger|0\rangle, \] (17)

and replacing it in Eq. (16) we find

\[ a_{1\uparrow}|e_1(L + 1)\rangle = (-i)^{L}N_{L+1}\tilde{\hat{O}}(L)\Gamma_{L\downarrow}b_{L+1\uparrow}\Gamma_{L\uparrow}|0\rangle \]

\[ = (-i)^{L}N_{L+1}b_{L+1\uparrow}\Gamma_{L\uparrow}\Gamma_{L\downarrow}\hat{O}(L)|0\rangle \]

\[ = (-i)^{L}b_{L+1\uparrow}|e_1(L + 1)\rangle, \] (18)

in agreement with Eq. (13). The corresponding relation for the opposite spin of the end operators is obtained using the time-reversal operator \( K \).

1. Ground-state energy

The energy of the four-fold degenerate ground state \( E_g \) can be obtained from the following argument. Let us define the charge conjugation (or electron-hole transformation) \( C \) as the one which permutes annihilation and creation operators

\[ Cc_{j\sigma}^\dagger C = c_{j\sigma}, \quad Cc_{j\sigma} C = c_{j\sigma}^\dagger. \] (19)

Clearly \( C^2 = 1 \). For \( \phi = 0 \), the Hamiltonian Eq. (2) transforms as

\[ CHC = -2\mu L - H. \] (20)

In our case with \( \mu = 0 \), this implies that if a many-body state \( |i\rangle \) is an eigenstate with energy \( E_i \), its electron-hole partner \( C|i\rangle \) is also an eigenstate with energy \( -E_i \). Thus, the spectrum is symmetric around zero energy.

Since the system is non-interacting, the excited states are obtained applying creation operators \( \Gamma_{j\uparrow}^\dagger \) (each with an energy cost 2) and zero-energy operators (without energy cost) to \( |e_1\rangle \). Clearly one of the states of highest energy \( E_{\text{max}} \) is \( \prod_{j=1}^{L-1}\Gamma_{j\uparrow}^\dagger\Gamma_{j\downarrow}^\dagger|e_1\rangle \), and \( E_{\text{max}} - E_g = 4(L-1) \). In addition, since the total spectrum is symmetric \( E_{\text{max}} + E_g = 0 \). Thus, the ground-state energy is

\[ E_g = -2(L - 1). \] (21)

This has been confirmed by numerical calculations in small systems.

2. Expectation value of the spin at one end

Proceeding in a similar way as above, one can write the state \(|o \downarrow\rangle\) (except for a phase) as

\[ b_{L\downarrow}|e_1(L)\rangle = -N_L\tilde{\hat{O}}(L-1)\Gamma_{L-1\downarrow}b_{L\uparrow}\Gamma_{L-1\uparrow}|0\rangle \]

\[ = \frac{-1}{4\sqrt{2}}N_L\tilde{\hat{O}}(L-1) \left[ -c_{L\uparrow}^\dagger + ic_{L-1\uparrow}^\dagger + ic_{L-1\downarrow}^\dagger c_{L\downarrow}^\dagger \right]|0\rangle. \] (22)

From here it is easy to see that the expectation value of the spin projection at the right end \([\text{for the parameters of the exact solution}, S_{\text{right}}^z = S_L^z = (c_{L\uparrow}^\dagger c_{L\downarrow} - c_{L\downarrow}^\dagger c_{L\uparrow})/2]\) becomes

\[ \langle o \downarrow | S_{\text{right}}^z | o \downarrow \rangle = -1/4, \] (23)

since half of the terms of Eq. (22) contribute with \(-1/2\) and the other half do not contribute. Similarly

\[ \langle o \downarrow | S_{\text{left}}^z | o \downarrow \rangle = \langle o \uparrow | S_{\text{left}}^z | o \uparrow \rangle = -1/4, \]

\[ \langle o \uparrow | S_{\text{right}}^z | o \uparrow \rangle = \langle o \uparrow | S_{\text{right}}^z | o \uparrow \rangle = 1/4, \] (24)

in agreement with previous results derived for the general case.\(^{10}\)

Concerning these expectation values for the states \(|e_1\rangle\) and \(|e_2\rangle\), they are zero since both states are time-reversal invariant. However, this is no longer true for an arbitrary linear combination. For example for \( L = 2\) one can construct the state \(|e_3\rangle = (|e_2\rangle - i|e_1\rangle)/\sqrt{2}\), and using Eqs. (14) it is easy to see that

\[ \langle e_3 | S_{\text{left}}^z | e_3 \rangle = -\langle e_3 | S_{\text{right}}^z | e_3 \rangle = 1/4. \] (25)

States like this are favored when a magnetic field is applied to one end of the chain only.\(^{10}\) We have verified this fact in finite chains.

3. Discussion on the “time-reversal anomaly”

The explicit construction of the many-body eigenstates allows us to discuss in detail the arguments involved in the so-called “time-reversal anomaly.”\(^{12,13}\) Following for example Ref. 13, we can identify an independent zero-mode operator at the left side (\( a \) in the notation of Ref. 13 and \( a_{\text{left}} \) in the Introduction) as \( a = a_{1\uparrow} = ia_{1\downarrow} \), where we have used Eq. (3) in the last equality. Then, we can identify the ground state \(|G_0\rangle\) with the property \( a|G_0\rangle = 0 \), as \( |G_0\rangle = |o \downarrow\rangle \) [see Eqs. (10), (11), and (12)] . The time-reversal partner of \(|o \downarrow\rangle\) is clearly \(|o \uparrow\rangle\). A local fermion parity operator can be defined as \( P_{\text{left}} = 2a^\dagger a - 1\). Clearly \( P_{\text{left}}|G_0\rangle = -|G_0\rangle\). It has been proved that \( P_{\text{left}}\)
In the present case, this is proved using Eqs. (5) and (6). This implies that the time reversal partner of \(|G_0\rangle\) is even under time reversal
\[
P_{\text{left}} K |G_0\rangle = - KP_{\text{left}} |G_0\rangle = K |G_0\rangle.
\]
This agrees with previous arguments.

The assumption that is not correct is to associate the ground state \(|G_1\rangle = a^\dagger |G_0\rangle\) which is a mixture of the states \(|e_1\rangle\) and \(|e_2\rangle\), see Eqs. (5), (10) and (12) with the time-reversal partner of \(|G_0\rangle\), in spite of the fact that \(|G_0\rangle\) is even under \(P_{\text{left}}\). This is because the ground state is four-fold degenerate and contains more states than just \(|G_0\rangle\) and \(|G_1\rangle\). This implies that some arguments used to argue that there is a fractional spin projection at the ends of the chain are not valid although the result is still true (as shown in Section III B 2 in our case).

In addition, due to the property Eq. (13) displaying an entanglement of the ends, the actions of \(P_{\text{left}}\) and the corresponding local operator at the right on the ground state are closely related.

Note that the properties of the ground states under the discrete symmetries time reversal, total fermion parity and local fermion parities, although demonstrated for particular parameters, are valid by continuity inside the whole topological phase for \(L \to \infty\). For finite chains, in general there is a (very small) mixing of the end zero modes that split the ground state, as described in the next section.

**IV. EFFECT OF A SMALL HOPPING TERM**

In this section we discuss the effect of the term \(H_t = -t \sum_{j=1}^{L-1} \sum_{\sigma}(c_{j+1,\sigma}^\dagger c_{j,\sigma} + \text{H.c.})\) on the exact solution. To simplify the analysis we assume \(t \ll 1\). However the main results concerning the splitting of the degeneracy of the many-body states are general for a long enough chain in the topological phase, as discussed below.

**A. One-body operators**

For \(L \geq 4\), the commutator with the hopping term of the operators defined in Section III A satisfying Eq. (3) for \(t = 0\) are
\[
[a_{1\uparrow}, H_t] = \frac{t}{\sqrt{2}} \left( -\Gamma_{2\uparrow} + i \Gamma_{2\downarrow}^\dagger \right),
\]
\[
[b_{L\uparrow}, H_t] = \frac{t}{\sqrt{2}} \left( -i \Gamma_{L-2\uparrow} + \Gamma_{L-2\downarrow}^\dagger \right),
\]
\[
[\Gamma_{j\uparrow}, H_t] = t \Gamma_{j\downarrow}^\dagger + \frac{t}{2} (L_j + R_j)
\]
with
\[
L_j = \begin{cases} 0 & \text{if } j = 1 \\ -\sqrt{2} a_{1\uparrow}^\dagger & \text{if } j = 2 \\ -i \Gamma_{j-2\uparrow} + \Gamma_{j-2\downarrow}^\dagger & \text{if } j > 2 \end{cases}
\]
\[
R_j = \begin{cases} i \Gamma_{j+2\uparrow} + \Gamma_{j+2\downarrow}^\dagger & \text{if } j < L - 2 \\ i \sqrt{2} b_{L\uparrow} & \text{if } j = L - 2 \\ 0 & \text{if } j = L - 1 \end{cases}
\]
The corresponding relations for the operators with opposite spin are obtained applying the time-reversal operator [see Eqs. (6) and (8)]

In first-order perturbation theory, \(H_t\) lifts the degeneracy of the \(\Gamma_{j\sigma}\), introducing a hopping to next-nearest neighbors. The operators with even and odd \(j\) remain at independent Hilbert subspaces. For each subspace, diagonalization of \(H_t\) is equivalent to solve an open chain with \(M\) sites and nearest-neighbor hopping. For \(L\) odd, \(M = (L - 1)/2\) for both subspaces. For \(L\) even, \(M = L/2 - 1\) for the sites with even \(j\) and \(M = L/2\) for the subspace of the \(\Gamma_{j\sigma}\) with odd \(j\).

By solving these equivalent problems, we obtain that, for any \(L\), one can define states such that
\[
[\Gamma_{k\sigma}, H_t] = (2 + t \cos k) \Gamma_{k\sigma}.
\]
In the subspace of the operators with odd \(j\) they have the form
\[
\Gamma_{k\uparrow} = \sqrt{\frac{2}{M + 1}} \sum_{l=1}^{M} \sin(kl)(i)^{l-1} \Gamma_{2l-1\uparrow},
\]
while for even \(j\)
\[
\Gamma_{k\downarrow} = \sqrt{\frac{2}{M + 1}} \sum_{l=1}^{M} \sin(kl)(i)^{l-1} \Gamma_{2l\downarrow},
\]
The relations for spin down operators can be obtained replacing \(\uparrow\) by \(\downarrow\) and the imaginary unit \(i\) by \(-i\).

For a finite chain of an even number of sites \(L\), the zero-energy end modes \(a_{1\sigma}, b_{L\sigma}\), are also split by a perturbative process of order \(tL/2\) that involves the \(\Gamma_{j\sigma}\) and \(\Gamma_{j\uparrow}\) with even \(j\), as it can be seen from Eqs. (26). For odd \(L\), \(H_t\) mixes \(b_{L\sigma}\) with the subspace of \(\Gamma_{j\sigma}\) and \(\Gamma_{j\uparrow}\) with odd \(j\) and \(b_{L\sigma}\) remains decoupled from \(a_{1\sigma}\) at finite \(t\).

To calculate the perturbative effective coupling between the end modes it is easier to map the problem introducing kets associated with the annihilation (a) and creation (c) operators
\[
c_\alpha \leftrightarrow |\alpha\rangle, \quad c_\alpha^\dagger \leftrightarrow |\alpha\rangle,
\]
and introduce the Hamiltonian
\[
\hat{H} = \sum_{\beta} A^{\beta}_{a} |\beta \rangle \langle a | + B^{\beta}_{a} |\beta | \langle a | - T^{\beta}_{a} |\beta | \langle c | + \bar{B}^{\beta}_{a} |\beta | \langle c | ,
\]
with coefficients defined from the equations
\[
[e_{\alpha}, \hat{H}] = \sum_{\beta} (A^{\beta}_{a} c_{\beta} + B^{\beta}_{a} c_{\beta}^{\dagger}),
\]
so that solving \(\hat{H} |\Gamma_{\nu} a \rangle = E_{\nu} |\Gamma_{\nu} a \rangle\) is equivalent to solve Eq. [3].

In this new language the effective perturbative mixing of the end modes for spin up can be written in the form
\[
\hat{H}_{m} = V |b_{L+} a \rangle \langle a_{1} \uparrow | + \text{H.c.},
\]
where
\[
V = \sum_{M} \frac{\langle b_{L+} | H_{\Gamma} | e_{M} \rangle}{\Pi_{i=1}^{M} (-E_{i})}
\]
and \(|e_{i}\rangle, E_{i}\rangle\) label the two possible intermediate states at each of the \(M = L/2 - 1\) sites with even \(j\) and the corresponding energies. The sum runs over all possible \(2^{M}\) combinations of intermediate states. The state \(|e_{i}\rangle\) is either \(|\Gamma_{2i} a \rangle\) with energy \(E_{i} = 2\) or \(|\Gamma_{2i+1} a \rangle\) with energy \(E_{i} = -2\). Using Eq. (25), it is easy to see that the contribution of the sum when all intermediate state correspond to annihilation operators \(|\langle \Gamma_{2i+1} a \rangle\rangle\) is \(-t(-it/4)^{M}.\) In addition, each time \(|\Gamma_{2i} a \rangle\) is replaced by \(|\Gamma_{2i+1} a \rangle\), a factor \((-i)^{2}\) appears because of the change in two matrix elements which is compensated by a change of sign in \(E_{i}\). Therefore, the \(2^{M}\) possibilities of choosing the intermediate states lead to the same contribution. Thus
\[
V = -E e^{i\theta},
\]
\[
E = t (t/2)^{M},
\]
\[
e^{i\theta} = (-i)^{M}.
\]

The eigenstate of Eq. (35) with positive energy \(E\) is \(|\langle a_{1} \uparrow a \rangle + e^{i\theta} |b_{L+} a \rangle\rangle/\sqrt{2}\) which corresponds to the annihilation operator
\[
\gamma_{\uparrow} = \frac{1}{\sqrt{2}} (a_{1} \uparrow + e^{i\theta} b_{L+} \uparrow) .
\]

From time reversal symmetry, one has for spin down
\[
\gamma_{\downarrow} = \frac{1}{\sqrt{2}} (a_{1} \downarrow + e^{-i\theta} b_{L+} \downarrow) .
\]

These results agree with previous ones obtained for a long chain with \(\mu = 0\) but otherwise arbitrary parameters using an algebraic approach. Here, the nature of the coupling between the end modes becomes more transparent.

### B. Effect of a magnetic field at one end

Since the total spin projection in the direction \(z\) of the spin orbit coupling \(S_{z}\) is a good quantum number, the effect of a uniform magnetic field in the \(z\) direction is trivial and does not modify the eigenstates, just changing the energies. Instead, a magnetic field applied to only one end of the chain leads to non-trivial results. It is easy to generalize Eqs. (40) and (41) to this case, adding to the Hamiltonian the term \(-\Delta Z S_{z}^{\text{right}},\) with \(\Delta Z = g \mu_{B} B\) and \(S_{z}^{\text{right}} = \sum_{j=L/2}^{L} S_{z}^{j}\) (the total spin projection at the right half of the chain). In fact, only the terms within a distance to the end of the chain less or of the order of the localization length of the zero-energy mode contribute to the sum, because for the other sites the singlet character of the superconductor tends to decrease \(|S_{z}^{j}\rangle\). The expectation value of \(S_{z}^{j}\) as a function of lattice site \(j\) has been studied numerically in Ref. [30].

The result for the annihilation operators and energies is
\[
\gamma_{\uparrow} = \frac{1}{\sqrt{2}} (\alpha a_{1} \uparrow + \beta e^{i\theta} b_{L+} \uparrow), \quad E_{\uparrow} = r - \frac{\Delta Z}{4},
\]
\[
\gamma_{\downarrow} = \frac{1}{\sqrt{2}} (\beta a_{1} \downarrow + \alpha e^{i\theta} b_{L+} \downarrow), \quad E_{\downarrow} = r + \frac{\Delta Z}{4},
\]
with
\[
r = \sqrt{\frac{(\Delta Z/4)^{2} + E^{2}},}
\]
\[
\alpha^{2} = \frac{1}{2} + \frac{\Delta Z}{4r}, \quad \beta^{2} = 1 - \alpha^{2}, \quad \text{and} \quad \alpha, \beta > 0,
\]
\[
E = \begin{cases} t^{L/2}/2L^{L/2-1} & \text{if } L \text{ even} \\ 0 & \text{if } L \text{ odd} \end{cases}
\]

### C. Low-energy many-body eigenstates

Let us discuss first the case without any magnetic field and odd or infinite \(L,\) so that the end zero modes are not mixed. In this case, following a similar reasoning that lead to Eq. (10), one of the states that are part of the ground state is for small \(t\)
\[
|e_{1}\rangle_{t} = N t \big(\prod_{k} r_{k} \Gamma_{k}^{0} \Gamma_{k}^{0} \big) \big(\prod_{k} \Gamma_{k}^{0} \Gamma_{k}^{0} \big) |0\rangle,
\]
where the annihilation operators are given above [see Eqs. (29), (30), (31)]. It is important to note that for
small $t$ as we assume, the energies corresponding to all these operators are positive [Eqs. (29)]. For each spin, the operators $\Gamma^\sigma_k$ and $\Gamma^\sigma_{-k}$ are related with the local ones $\Gamma^\sigma_0$ by a unitary matrix $U_\sigma$ with coefficients given by Eqs. (30) and (31) and similarly for spin down. Using this transformation and Eq. (10), it is easy to see that

$$|e_1\rangle = \det(U_F) \det(U_i)|e_1\rangle.$$  \hspace{1cm} (45)

Thus, $|e_1\rangle$ corresponds to the same physical state as $|e_1\rangle$.

In the general case, one of the states that is part of the ground state (non degenerate for finite even $L$) has an even number of particles and is given by

$$|g_e\rangle = N \gamma_1 \gamma_2 |e_1\rangle,$$  \hspace{1cm} (46)

where $N$ is a normalization factor and $\gamma_\sigma$ are given by Eqs. (40) and (41).

The other three low-energy states and their excitation energies with respect to the ground state for small enough $t$ and $B$ are

$$|l \uparrow\rangle = \gamma_1^{|g_e\rangle} |e_1\rangle, \quad E_\uparrow,$$

$$|l \downarrow\rangle = \gamma_1^{|g_e\rangle} |e_1\rangle, \quad E_\downarrow,$$

$$|l_e\rangle = \gamma_1^{|g_e\rangle} |e_1\rangle, \quad 2r.$$ \hspace{1cm} (47)

Note that for odd or infinite $L$, $E = 0$, which implies $E_\uparrow = 0$ and a two-fold degeneracy remains in the ground state and in the other two low-energy states. In addition, for the two states with even number of particles, $|g_e\rangle$ and $|l_e\rangle$, the total spin projection $S_z = 0$, while $S^z_{\uparrow\downarrow} = 0$. Defining $S^z_{\text{left}} = S^z \uparrow - S^z \downarrow$, using previous results, and the vanishing of the trace of $S_z$ in the low-energy subspace one obtains

$$\langle g_e | S^z_{\text{right}} | g_e \rangle = -\langle l_e | S^z_{\text{right}} | l_e \rangle = \langle l_e | S^z_{\text{left}} | l_e \rangle = \frac{\Delta z}{4\sqrt{(\Delta z)^2 + 16E^2}}$$

$$= \frac{1}{4} \text{ for the left or right part of the chain, Eqs. (48), agree very well with our numerical results for finite chain. In Table I we list some of these numerical results for $L = 4$. The numerical excitation energies presented in the table coincide with the analytical results to the precision of the former. For the spin projection, there is a small discrepancy. For example, for $t = 0.04$ and $B = 0.002$, the analytical result for $S^z_{\text{right}}$ in the ground state is larger by $1.0 \times 10^{-4}$. For the excited state in the even subspace, the discrepancy is near $3 \times 10^{-4}$.}

\hspace{1cm}

V. SUMMARY AND DISCUSSION

We have solved exactly a particular case of a model for a chain of a time-reversal-invariant topological superconductor. This allows us to construct the degenerate ground state of the system which consists of two states with even fermion parity and total spin projection $S_z = 0$, and two states with odd fermion parity and $S_z = \pm 1/2$. The latter two states have spin projections $S^z = \pm 1/4$ at the ends. If a magnetic field is applied to one end of the chain, the former two states are split in two states having expectation values $1/4$ at one end and $-1/4$ at the other. In addition, creating a zero-mode at one end or at the other one give related results. This property might be used for teleportation of Majorana fermions.

Since the ground state is separated from the excited states by a finite gap, by continuity these properties remain for a chain of infinite length and general values of the parameters. The coefficients of proportionality in some relations [like Eq. (13)] change, but not the proportionality itself.

Concerning the discussion of the “time-reversal anomaly,” although the concept was useful to explain different properties of the system, some statements should be corrected (see Section II B 3).

The main effect of a finite $t$ (or a finite chain of length $L$ in a general case) is to split the two states of even parity in the ground state by a quantity of order $\exp(-L/\lambda_c)$, where $\lambda_c$ is the localization length of the end modes. This changes the fractional expectation values of the spin projection at the ends of the even-parity many-body states under the application of a magnetic field at one of the ends of the superconducting wire. This might be experimentally detectable.

\hspace{1cm}

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| Cases       | Energy | $\langle S_z \rangle$ | $\langle S_{1z} \rangle$ | $\langle S_{2z} \rangle$ | $\langle S_{3z} \rangle$ | $\langle S_{4z} \rangle$ |
|------------|--------|-----------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| $t = 0$    |        |                      |                           |                           |                           |                           |
| $B = 0$    | Even   | -6 0 0 0              | 0                         | 0                         | 0                         | 0                         |
|           | Odd    | -6 1/2 1/4 0         | 0                         | 0                         | 0                         | 0                         |
| $t = 0.02$ |        |                      |                           |                           |                           |                           |
| $B = 0$    | Even   | -6.00075 0 0 0       | 0                         | 0                         | 0                         | 0                         |
|           | Odd    | -6.00055 1/2 0.249975| 0.000022499               | 0.000022499               | 0.249975                  |
| $t = 0$    |        |                      |                           |                           |                           |                           |
| $B = 2 \times 10^{-4}$ | Even   | -6.001 0 -0.25 -0.0000125 | 0.0000125 | 0.25                       |                           |                           |
|           | Odd    | -6.001 -1/2 -0.25 -0.0000125 | 0.0000125 | -0.25                      |                           |                           |
| $t = 0.02$ |        |                      |                           |                           |                           |                           |
| $B = 2 \times 10^{-4}$ | Even   | -6.00077 0 -0.111788 -1.31948 x 10^{-6} | 1.31948 x 10^{-6} | 0.111788                  |                           |                           |
|           | Odd    | -6.00033 0 0.111788 -0.00002236793 | 0.00002236793 | -0.111788                  |                           |                           |
| $t = 0.04$ |        |                      |                           |                           |                           |                           |
| $B = 0$    | Even   | -6.003 0 0 0         | 0                         | 0                         | 0                         | 0                         |
|           | Odd    | -6.0022 -1/2 -0.2499 -0.0000998403 | -0.0000998403 | -0.2499                   |                           |                           |
| $t = 0$    |        |                      |                           |                           |                           |                           |
| $B = 0.002$ | Even   | -6.001 0 -0.25 -0.0000125 | 0.0000125 | 0.25                      |                           |                           |
|           | Odd    | -6.001 -1/2 -0.25 -0.0000125 | 0.0000125 | -0.25                      |                           |                           |
| $t = 0.04$ |        |                      |                           |                           |                           |                           |
| $B = 0.002$ | Even   | -6.00348 0 -0.195124 -0.0000468943 | 0.0000468943 | 0.195124                  |                           |                           |
|           | Odd    | -6.00092 0 0.195124 -0.000203056 | -0.000203056 | -0.195124                  |                           |                           |
| $t = 0.04$ |        |                      |                           |                           |                           |                           |
| $B = 0.002$ | Even   | -6.0032 -1/2 -0.2499 -0.0000224815 | -0.0000224815 | -0.2499                   |                           |                           |
|           | Odd    | -6.0012 1/2 0.2499 -0.0000251346 | 0.0000251346 | 0.2499                    |                           |                           |

TABLE I: Numerical results for a chain of $L = 4$ sites.
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