Network Inference using Sinusoidal Probing

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Abstract: The aim of this manuscript is to present a non-invasive method to recover the network structure of a dynamical system. We propose to use a controlled probing input and to measure the response of the network, in the spirit of what is done to determine oscillation modes in large electrical networks. For a large class of dynamical systems, we show that this approach is analytically tractable and we confirm our findings by numerical simulations of networks of Kuramoto oscillators. Our approach also allows us to determine the number of agents in the network by probing and measuring a single one of them.

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1. INTRODUCTION

Complex networks are the medium for interactions in many natural and man-made systems. Such realizations range from the scale of people exchanging opinions on a social networks and power transmission on electrical grids to interacting molecules in chemical reactions and pacemaker cells [Boccaletti et al. (2006); Newman (2018)]. The way individual elements are coupled together primarily impacts the overall dynamics of network-coupled systems. However, in many cases, characteristics of the interaction network are not known exactly, or even not known at all. Inference techniques to uncover coupling between individual units and even units with themselves are therefore highly desirable. We distinguish mainly two types of such methods, (i) one can observe a dynamical system subjected to uncontrolled operational condition; or (ii) one can directly disturb the system and observe its reaction, which is what is done in this manuscript. In real applications, one has to be careful on the nature of the method. Introducing a disturbance into the system can have dramatic effect on its operation. For instance, Furtani et al. (2019) propose a method based on resonance of the network when subject to periodic disturbance, and Timme (2007) suggests to drive the system away from its operating state. Such (potentially invasive) methods could alter the operational state of the system.

To avoid a strong impact on the system, one can adopt the strategy of simply observing the system in its normal operation. This type of approach lead to successful and elegant results, e.g., leveraging the response of the system to external noise [Ren et al. (2010); Tyloo et al. (2020)]. But as there is no free lunch, these methods come with assumptions on the noise characteristics that are not necessarily met in general, namely on correlation time and uniformity over the system. Observing the relaxation of a system to its steady state, Mauroy and Hendrickx (2017) extract its spectral moments, but do not directly reconstruct the network. Furthermore, their method performs exactly for linear dynamics, but relies on Dynamical Mode Decomposition for nonlinear systems, which might require a very large number of measurements for an accurate estimation.

Half way between intrusive disturbances and passive observation, we will consider the injection of a small probing signal, which we will qualify as "non-invasive", in order to leave the operational state as unaffected as possible in the spirit of Pierre et al. (2010). To this day, most of such techniques, applied to large power grids, relied of very few measurement points and aimed only at identifying resonance modes of the network, and not the whole network structure [Dosiek et al. (2013)]. In this manuscript, we propose such a non-invasive inference technique. It relies on rather mild assumptions on the nature of the interaction between agents of the network, and does not need to know it, on the contrary to other methods [e.g., Yu et al. (2006)].

Our method applies generically to any network and does not need any knowledge of its characteristics, as for instance in Yeung et al. (2002). By adding a controlled sinusoidal input at single nodes, referred to as probing signal, we are able to reconstruct the interaction network by measuring the response to the probing at the other nodes of the network. The same approach allows us to determine the number of nodes in the network by probing and measuring the response of the network at a single node. This improves significantly on previous measurement-based methods to determine the number of nodes in a network [Haeene et al. (2019)]. In contrast to more exhaustive works [e.g.,
of an Erdős-Rényi graph with \( n = 120 \) vertices and \( m = 329 \) edges. The third one is a small-world graph realized according to the Watts-Strogatz process [Watts and Strogatz (1998)], with \( n = 120 \) vertices and \( m = 242 \) edges.

### 3. PROBING

In order to determine oscillation modes in large electrical networks, one method is to apply a probing signal at some points of the network and measure the system’s response at other points [e.g., Pierre et al. (2010)]. The probing is typically a sinusoidal signal with controlled amplitude and frequency.

In the same spirit, we propose here to inject a sinusoidal signal at agent \( i \) and to measure its impact at agent \( j \). Let

\[
\xi_i(t) = a_0 \sin(\omega_0 t),
\]

be the probing signal at agent \( i \). We do not inject a probing signal at other nodes. Note that any signal shape could do the job, but the advantage of sine is that its amplitude and frequency are easily identifiable.

To guarantee a minimal impact on the operation of the system, we need the amplitude \( a_0 \) to be sufficiently small. However, in most applications, the system under investigation will be subject to noise, and in order to be detectable, the probing amplitude should not be too small neither. Despite these contradicting requirements, we argue that, under our assumptions, an appropriate amplitude, satisfying both requirement simultaneously, can be chosen. Indeed, even though the system is subject to noise, we assume it is close to a steady state. If the noise amplitude was too large, it would push the system far away from its fixed point and it could not be considered as in (or close to) steady state. Under our assumptions, there is then a margin between the noise amplitude and the size of the basin of attraction of the steady state. The probing amplitude needs to be chosen in this margin. Determining the amplitude of the noise is rather straightforward from measurements. However, estimating the maximal tolerable disturbance magnitude preserving stability might represent a challenge and is beyong the scope of this manuscript [Menck et al. (2013)].

Also, the probing frequency \( \omega_0 \) needs to be kept small. Keeping a small probing frequency guarantees that the system can adapt to the input and follow the probing signal. More precisely, a probing frequency can be qualified as small as it is smaller than the smallest eigenvalue of the Jacobian matrix \( J_f \) in absolute value.

Introducing Eq. (5) into Eq. (4), and recombing the eigenmodes yields the following response measured at agent \( j \),

\[
x_j^*(t) = \sum_{\alpha} \frac{u_{\alpha,j} u_{\alpha,j} a_0}{\lambda_\alpha^2 + \omega_0^2} \left[ \lambda_\alpha \sin(\omega_0 t) + \omega_0 e^{-\lambda_\alpha t} - \omega_0 \cos(\omega_0 t) \right].
\]

To explicitly obtain an expression involving the Jacobian matrix, one should consider the long time limit \( \lambda_\alpha t \gg 1 \) with the asymptotic \( \omega_0 \ll \lambda_\alpha \) that yields,
Injecting a probing signal and varying its frequency, one responds together to the signal (i.e., adiabatic shift in the dynamical variable $x_i$). Note that even though one usually has access to the frequency of the probing signal, Eq. (5) at node $i$, with $\omega_0 \ll \lambda_\alpha$. Then for $\omega_0$ sufficiently small one has,

$$x_i^{\text{max}} = \max_t |x_i(t)| \approx \frac{2a_0}{n\omega_0},$$

from which one obtains an estimate for the number of nodes as,

$$\hat{n} = \frac{2a_0}{x_i^{\text{max}} \omega_0}.$$  

Note that we take the maximum to have a better accuracy on estimates. We see that the accuracy is very good, even for a small probing frequency.

5. NUMBER OF AGENTS

The number of agents in a coupled system is one of its primal properties. However, there are many physical examples where this number is not known exactly [Su et al. (2012), Haehne et al. (2019)]. In this section, we introduce a method that allows to accurately determine the number of agents in any system governed by the dynamics of Eq. (2). Moreover, it requires to probe and measure a single node, which makes the method very efficient. Let us inject a probing signal, Eq. (5) at node $i$, with $\omega_0 \ll \lambda_\alpha$. Then for $\omega_0$ sufficiently small one has,

$$x_i^{\text{max}} = \max_t |x_i(t)| \approx \frac{2a_0}{n\omega_0},$$

from which one obtains an estimate for the number of nodes as,

$$\hat{n} = \frac{2a_0}{x_i^{\text{max}} \omega_0}.$$  

Note that we take the maximum to have a better accuracy in the estimation. However one can choose a particular time step $t$, keeping in mind that $t$ too short leads to vanishing values for Eq. (7). We check the validity of the estimation of Eq. (11) in Fig. 2. Each cross corresponds to $\hat{n}$ obtained from a single node probing and measurement. One clearly sees that for $\omega_0$ small enough compared to $\lambda_\alpha$, the estimated number of agents precisely matches the real one.

Since the initial submission of this manuscript, this result has been developed and thoroughly discussed in Tyloo and Delabays (2020).

6. NETWORK INFERENCE

Assume now that each of the $[n(n - 1)]/2$ pairs $(i, j)$, $1 \leq i < j \leq n$, can be probed. For each pair $(i, j)$, we can measure either $\hat{x}_j^i(t)$ or $\hat{x}_i^j(t)$. According to Eq. (7) and by symmetry of $\mathcal{J}_f$, these two trajectories should be the same, at least for $t$ sufficiently large, in order for the influence of initial conditions to vanish.

We will use the measured value $\hat{x}_j^i(t)$ at large $t$ to estimate $\mathcal{J}_f$ via Eq. (7). To do this, we need to remember that
the dynamics of Eq. (1) are invariant under a constant shift of all variables. This implies that two trajectories whose initial conditions differ only by a constant shift of all variables are exactly parallel, i.e., the constant shift is preserved for all time. This means that, depending on the initial conditions (unknown a priori), the measured trajectory $\hat{x}^j(t)$ might be shifted with respect to the predicted trajectory $x^j(t)$ of Eq. (7). Denoting this constant shift as $c := \hat{x}^j(t) - x^j(t)$, one can rearrange Eq. (7) as

$$\hat{J}^{f}_{ij} := J^{f}_{ij} + \frac{c}{a_0 \sin(\omega_0 t)} = \left( \frac{\hat{x}^j(t) - \frac{a_0}{\omega_0} [1 - \cos(\omega_0 t)]}{a_0 \sin(\omega_0 t)} \right) .$$

(12)

Whereas we cannot determine the value of $c$, this is not an issue, as we show now. Indeed, the constant vector $u_1 = n^{-1/2} (1, ..., 1)^T$ is an eigenvector of $J^{f}_J$, and then an eigenvector of $J^{f}_{ij}$ as well. As the eigenbasis of $J^{f}_J$ (and $J^{f}_{ij}$) is orthonormal, adding a constant value to each component of $J^{f}_{ij}$ does not modify its eigenbasis, and modifies only one of its eigenvalues, $\lambda_1$. Diagonalizing $J^{f}_{ij}$, it is then straightforward to replace its eigenvalue associated to $u_1$ by 0 (as it should be for the exact $J^{f}_J$ and $J^{f}_{ij}$) and to invert all other eigenvalues to recover $\hat{J}^{f}_{ij}$. Remark that in order to avoid singularity in Eq. (12), one should choose a time step $t$ such that $\sin(\omega_0 t)$ is sufficiently different from zero.

The outcome of the procedure proposed above is illustrated in Fig. 3. Each dot corresponds to one element of the 120 $\times$ 120 Jacobian matrix for the Kuramoto model on our selected graphs. One can see that the matrix is reconstructed with very high accuracy. In Fig. 4, we show the accuracy of the Jacobian estimate with respect to the frequency of the probing, normalized by $\lambda_2$. The accuracy of the estimate is measured as the Frobenius norm of the difference between the real Jacobian matrix and its estimate. We see that the accuracy is very good, even for probing frequencies close to $\lambda_2$.

7. OUTLOOK

We showed that based on a sinusoidal probing signal, injected at a node of a networked dynamical system while measuring the response of the network allows to:

- Estimate the range of the spectrum of the Jacobian matrix of the system around its current fixed state;
- Estimate the number of agents in the system very efficiently: single node measurement;
- Recover the network structure of the system with high fidelity.

The main advantages of our method are that it is non-invasive (does not disturb the system far from its operating state), it applies to a large set of coupling functions, whereas it requires to probe the system for a sufficiently long time, it only needs measurement at one time step. Its main drawback is that, in order to recover the network using sinusoidal probing, one needs to be able to probe any pair of nodes in the network. The subsequent work would then investigate to what extent and with what confidence the network structure can be inferred if one has only probed a subset of the nodes.

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Fig. 3. Inferred vs. real values of the elements of the Jacobian matrix \( J_f \) for the three networks UK (left panel), Erdős-Renyi (middle panel), and Small-World (right panel).

Fig. 4. Relative error in the estimate of the Jacobian matrix \( \hat{J}_f \) with respect to the frequency of the probing signal, normalized by the smallest eigenvalue of the Jacobian, for the three networks UK (blue), Erdős-Renyi (orange), and Small-World (green). The relative error is computed as the Frobenius norm of the error of the Jacobian matrix normalized by the Frobenius norm of the real matrix. One sees that for probing frequencies smaller that 10% of \( \lambda_2 \), the estimate is very accurate.

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