A PRIMAL FINITE ELEMENT SCHEME OF THE $H(d) \cap H(\delta)$ ELLIPTIC PROBLEM

SHUO ZHANG

Abstract. In this paper, a unified family, for $n \geq 2$ and $1 \leq k \leq n - 1$, of finite element schemes are presented for the primal weak formulation of the $n$-dimensional $H(d) \cap H(\delta)$ elliptic problem.

Contents

1. Introduction \hspace{1cm} 1
2. Finite element space for $H(d_k)(\Omega) \cap H(d_k)(\Omega)$ \hspace{1cm} 3
  2.1. Polynomial spaces on a simplex \hspace{1cm} 3
  2.2. Finite element space and global approximation \hspace{1cm} 8
3. A primal finite element scheme of the $H(d) \cap H(\delta)$ elliptic problem \hspace{1cm} 10
  3.1. Finite element scheme and error estimate \hspace{1cm} 10
  3.2. Implementation of the scheme: a set of locally supported basis functions \hspace{1cm} 11
Appendix A. Some detailed calculation \hspace{1cm} 15
References \hspace{1cm} 17

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipschitz boundary. In this paper, we study the $H(d) \cap H(\delta)$ elliptic problem: given $f \in L_2^d(\Omega)$, find $\omega \in H^d(\Omega) \cap H^0(\Omega)$, such that

\begin{equation}
\langle d^k \omega, \delta \mu \rangle_{L_2^d(\Omega)} + \langle \delta^k \omega, \delta \mu \rangle_{L_2^d(\Omega)} + \langle \omega, \mu \rangle_{L_2(\Omega)} = \langle f, \mu \rangle_{L_2(\Omega)}, \quad \forall \mu \in H^d(\Omega) \cap H^0(\Omega).
\end{equation}

Here, following [1], we denote by $\Lambda^k(\Xi)$ the space of differential $k$-forms on an $n$-dimensional domain $\Xi$, and $L^2_\Lambda(\Xi)$ consists of differential $k$-forms with coefficients in $L^2(\Xi)$ component by component, and $\langle \cdot, \cdot \rangle_{L^2_\Lambda(\Xi)}$ is the inner product of the Hilbert space $L^2_\Lambda(\Xi)$. In this paper, we will occasionally drop $\Omega$ for differential forms on $\Omega$. The exterior differential operator $d^k : \Lambda^k(\Xi) \to \Lambda^{k+1}(\Xi)$ is unbounded from $L^2_\Lambda(\Xi)$ to $\Lambda^{k+1}(\Xi)$. Denote,

$$H\Lambda^k(\Xi) = H(d^k, \Xi) := \{ \omega \in L^2_\Lambda(\Xi) : d^k \omega \in L^2_\Lambda(\Xi) \},$$

then $H\Lambda^k(\Xi)$ is a Hilbert space with the norm $\| \omega \|_{L^2_\Lambda(\Xi)} + \| d^k \omega \|_{L^2_\Lambda(\Xi)}$. Denote by $H_0\Lambda^k(\Xi)$ the closure of $C^0_\Omega \Lambda^k(\Xi)$ in $H\Lambda^k(\Xi)$. The Hodge star operator $\star$ maps $L^2_\Lambda(\Xi)$ isomorphically to

---

2010 Mathematics Subject Classification. Primary 47N40, 65N30.

Key words and phrases. $H(d) \cap H(\delta)$, primal formulation, nonconforming finite element method, non-Ciarlet type.

The research is partially supported by NSFC (11871465) and CAS (XDB 41000000).
The codifferential operator $\delta_k$ defined by $\delta_k \mu = (-1)^kn \star d^{n-k} \star \mu$ is unbounded from $L^2\Lambda^k(\Omega)$ to $L^2\Lambda^{k-1}(\Omega)$. Denote

$$H^*\Lambda^k(\Omega) = H(\delta_k, \Omega) := \{ \mu \in L^2\Lambda^k(\Omega) : \delta_k \mu \in L^2\Lambda^{k-1}(\Omega) \},$$

and $H_0^*\Lambda^k(\Omega)$ the closure of $C_0^\infty\Lambda^k(\Omega)$ in $H^*\Lambda^k(\Omega)$. Then $H^*\Lambda^k(\Omega) = \star H\Lambda^{n-k}(\Omega)$, and $H_0^*\Lambda^k(\Omega) = \star H_0\Lambda^{n-k}(\Omega)$. The model problem (1.1) corresponds to a strong form that

$$d^k \omega \in H_0^*\Lambda^{k+1}(\Omega), \quad \delta_k \omega \in H\Lambda^{k-1}(\Omega),$$

and

(1.2) \quad $$\delta_{k+1}d^k \omega + d^{k-1} \delta_k \omega + \omega = f.$$

The model problem arises in many applied sciences, including electromagnetics [18, 24], fluid-structure interaction [6, 7, 17], and others. Particularly, known as Hodge-Laplacian operator, the finite element methods associated with $\delta_{k+1}d^k + d^{k-1} \delta_k$ have been a central topic of the finite element exterior calculus (FEEC) discussed in many aspects, and we refer to [1-3] for a thorough introduction to FEEC.

It is well recognized that, the conforming finite element scheme for (1.1) may lead to a spurious solution that converges to a wrong limit when the exact solution $\omega$ is not smooth enough. Indeed, taking the $H\Lambda^1 \cap H^*\Lambda^1$ in two dimension for example, a piecewise polynomial subspace of $H\Lambda^1 \cap H_0^*\Lambda^1$ is contained in $H^1\Lambda^1 \cap H_0\Lambda^1$, while $H^1\Lambda^1 \cap H_0^*\Lambda^1$ is a closed subspace of $H\Lambda^1 \cap H_0\Lambda^1$, and thus the singular part of $\omega$ can not be captured by the conforming finite element space. To cope with this situation, a well-developed approach is to use mixed finite element method. Again, the main approach can be found in detail in [1-3]; some recent progress can be found in [14, 20] for a posteriori error estimation and adaptive methods, and in [19] for a detailed analysis of Discontinuous Galerkin (DG) methods in FEEC in the newly-presented eXtended Galerkin (XG) framework.

On the other hand, to discretize directly the primal formulation (1.1) has been still attracting research interests. Virtual element methods are designed for the three dimensional vector potential formulation of magnetostatic problems [13], with the major interests restricted to cases in which the solution is reasonably smooth and hence where higher order methods could be more profitable. Nonconforming element methods and discontinuous Galerkin methods are also designed which can lead to a correct approximation of the nonsmooth solution for the $H(\text{curl}) \cap H(\text{div})$ problem in two dimension; readers are referred to [10] for an interior penalty method, to [9] for a nonconforming finite element method, and to [8] for a nonconforming finite element used with inter-element penalties. Recent works also include [4, 5, 23].

In this paper, we present a unified family of nonconforming finite element schemes for the primal formulation (1.1) for any $n \geq 2$ and $1 \leq k \leq n - 1$, and on the subdivision of the domain by simplexes. The main feature of the finite element schemes is, all the finite element functions are defined by local shape function spaces and the continuity conditions, and no penalty term or stabilization is used in the schemes. The local shape function spaces are a slightly enrichment by $\mathcal{H}_d^2(T)$ (see (2.5)) based on $\mathcal{P}_0\Lambda^k(T) + \kappa(\mathcal{P}_0\Lambda^{k+1}(T)) + \star \kappa \star (\mathcal{P}_0\Lambda^{k-1}(T))$, namely the minimal local...
space for $\delta_{k+1} d^k + d^{k-1} \delta_k$, but they do not contain the complete linear polynomial spaces which are used in [4, 5, 8-10, 23]. Another difference from them, particularly [9, 23], is that the finite element functions in this present paper possess a different kind of inter-element continuity. As precisely described in (2.18), the continuity is imposed in a dual way, and the consistency error can this way be controlled. This dual way makes the finite element functions not correspond to a “finite element” in the sense of Ciarlet’s triple [12], and the analysis and implementation would rely on non-standard techniques. For one thing, a stable interpolator is given, which works for functions in $H^k(\Omega) \cap H^2(\Omega)$ with minimal regularity, and an optimal approximation error estimation can be proved. Different from classical interpolators discussed in [2, 11, 15, 21], the interpolator is cell-wise defined, namely for $\omega, \mu \in H^k \cap H^2$, if $\omega$ and $\mu$ are equal on a cell, then their interpolations are equal on this cell. Besides, a precise set of basis functions can be presented, the supports of which are each contained in a vertex patch, and the programming of the scheme can be done in a standard routine as for the standard “finite element” method. Indeed, locally supported basis functions have been also found and implemented for many specific non-Ciarlet type finite element spaces [16, 22, 25-29].

Finally we remark that, the construction of the finite element scheme is fit for quite general situations. For example, a quite the same scheme can be constructed where the local shape function space is a slight enrichment by $H^2$ situations. For example, a quite the same scheme can be constructed where the local shape function space is a slight enrichment by $H^2$ situations. For one thing, a stable interpolator is given, which works for functions in $H^k(\Omega) \cap H^2(\Omega)$ with minimal regularity, and an optimal approximation error estimation can be proved. Different from classical interpolators discussed in [2, 11, 15, 21], the interpolator is cell-wise defined, namely for $\omega, \mu \in H^k \cap H^2$, if $\omega$ and $\mu$ are equal on a cell, then their interpolations are equal on this cell. Besides, a precise set of basis functions can be presented, the supports of which are each contained in a vertex patch, and the programming of the scheme can be done in a standard routine as for the standard “finite element” method. Indeed, locally supported basis functions have been also found and implemented for many specific non-Ciarlet type finite element spaces [16, 22, 25-29].

The remaining of the paper is organized as follows. In Section 2, we define the finite element spaces and prove its optimal approximation to functions in $H^k \cap H^2$ by constructing a cell-wise defined interpolator. In Section 3, we define the finite element schemes, present the error estimation and illustrate that the scheme can practically implemented by presenting a set of locally supported basis functions.

2. Finite element space for $H^k(\Omega) \cap H^2(\Omega)$

2.1. Polynomial spaces on a simplex. Denote the set of $k$-indices as

$$\mathbb{I}^k_{k,n} := \{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n\}, \ \mathbb{N} \text{ the set of integers.}$$

Then

$$d^k(\kappa(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k})) = (k + 1) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}, \text{ for } \alpha \in \mathbb{I}^{k+1}_{k+1,n},$$

and

$$\delta_k(\kappa(\delta dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k})) = (-1)^{k-n-1}(n-k+1)(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}), \text{ for } \alpha \in \mathbb{I}^{k-1}_{k,n},$$

where $\kappa$ is the Koszul operator

$$\kappa(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) = \sum_{\alpha = 1}^{k} (-1)^{j+1} x^{\alpha_j} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_{j+1}} \wedge \cdots \wedge dx^{\alpha_k}, \text{ for } \alpha \in \mathbb{I}^{k}_{k,n}.$$
2.1.1. Structures of polynomials on a simplex. Given $T$ a simplex, denote on $T$ $\tilde{x}^j = x^j - c_j$ where $c_j$ is a constant such that $\int_T \tilde{x}^j = 0$. Denote a simplex dependent Koszul operator

$$
\kappa_T(\ dx^{a_1} \wedge \cdots \wedge dx^{a_k}) := \sum_{j=1}^{k} (-1)^{(j+1)} \nabla x^j dx^{a_1} \wedge \cdots \wedge dx^{a_{j-1}} \wedge dx^{a_{j+1}} \wedge \cdots \wedge dx^{a_k}, \quad \text{for } \alpha \in \mathbb{Z}_{k,n}.
$$

Then

$$
d^{k-1}\kappa_T(\ dx^{a_1} \wedge \cdots \wedge dx^{a_k}) = k dx^{a_1} \wedge \cdots \wedge dx^{a_k}.
$$

Lemma 2.1. There exists a constant $C_{k,n}$, depending on the regularity of $T$, such that

(2.1) \[ \|\mu\|_{L^2\Lambda^k(T)} \leq C_{k,n} h_T \|d^k \mu\|_{L^2\Lambda^{k+1}(T)}, \text{ for } \mu \in \kappa_T(\mathcal{P}_0 \Lambda^{k+1}(T)), \]

and

(2.2) \[ \|\mu\|_{L^2\Lambda^k(T)} \leq C_{k,n} h_T \|\delta_k \mu\|_{L^2\Lambda^{k+1}(T)} \text{ for } \mu \in \star \kappa_T \star (\mathcal{P}_0 \Lambda^{k+1}(T)). \]

Proof. Given $\mu = \sum_{\alpha \in \mathbb{X}_{k,n}} C_\alpha \left( \sum_{j=1}^{k+1} (-1)^{j+1} \nabla x^j dx^{a_1} \wedge \cdots \wedge dx^{a_{j-1}} \wedge dx^{a_{j+1}} \wedge \cdots \wedge dx^{a_{k+1}} \right)$,

$$
\|\mu\|^2_{H^1\Lambda^k(T)} = \left\| \sum_{\alpha \in \mathbb{X}_{k,n}} C_\alpha \sum_{j=1}^{k+1} (-1)^{j+1} \nabla x^j dx^{a_1} \wedge \cdots \wedge dx^{a_{j-1}} \wedge dx^{a_{j+1}} \wedge \cdots \wedge dx^{a_{k+1}} \right\|^2_{L^2\Lambda^k(T)}
$$

$$
= \left\| \sum_{\alpha \in \mathbb{X}_{k,n}} C_\alpha \sum_{j=1}^{k+1} \sum_{i=1}^{k+1} (-1)^{i+j} \nabla x^i dx^{a_1} \wedge \cdots \wedge dx^{a_{i-1}} \wedge dx^{a_{i+1}} \wedge \cdots \wedge dx^{a_{k+1}},
$$

$$
\sum_{\alpha' \in \mathbb{X}_{k,n}} C_{\alpha'} \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} (-1)^{i+j} e^{a_i} \cdot e^{a_j} \left( dx^{a_1} \wedge \cdots \wedge dx^{a_{i-1}} \wedge dx^{a_{i+1}} \wedge \cdots \wedge dx^{a_{k+1}} \right) \right\|_{L^2\Lambda^k(T)} = (k+1)|T| \sum_{\alpha} C_\alpha^2,
$$

and

$$
\|d^k \mu\|^2_{L^2\Lambda^{k+1}(T)} = (k+1)^2 \left\| \sum_{\alpha} C_\alpha dx^{a_1} \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_{k+1}} \right\|^2_{L^2\Lambda^{k+1}(T)} = (k+1)^2|T| \sum_{\alpha} C_\alpha^2.
$$

Namely

$$
\|d^k \mu\|_{L^2\Lambda^{k+1}(T)} = \sqrt{k+1} \|\mu\|_{H^1\Lambda^k(T)}.
$$

Therefore, by noting that $\int_T \tilde{x}^j = 0$, with a constant $C_n$ depending on the regularity of $T$, we obtain

$$
\|\mu\|_{L^2\Lambda^k(T)} \leq C_n h_T \|\mu\|_{H^1\Lambda^k(T)} = C_n (k+1)^{-1/2} h_T \|d^k \mu\|_{L^2\Lambda^{k+1}(T)}.
$$
This proves (2.1). Similarly can (2.2) be proved.

In this part and in the sequel, we make the convention that, for \( \alpha \in \mathbb{IX}_{k,n} \), we use \( \beta \) for one in \( \mathbb{IX}_{n-k,n} \), such that \( \alpha \) and \( \beta \) partition \([1,2,\ldots,n]\). For \( \alpha \in \mathbb{IX}_{k,n} \), denote

\[
\tilde{\mu}^\alpha_{\delta,T} = \sum_{j=1}^{k} [(\tilde{x}^{\alpha_j})^2 - c^{\alpha_j}] \, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k},
\]

and

\[
\tilde{\mu}^\alpha_{d,T} = \sum_{j=1}^{n-k} [(\tilde{x}^{\beta_j})^2 - c^{\beta_j}] \, dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k},
\]

where \( c^{\alpha_j} \) and \( c^{\beta_j} \) are constants such that \( \int_T [(\tilde{x}^{\alpha_j})^2 - c^{\alpha_j}] = 0 \) and \( \int_T [(\tilde{x}^{\beta_j})^2 - c^{\beta_j}] = 0 \). Then

\[
(2.3) \quad d^k \tilde{\mu}^\alpha_{\delta,T} = 0, \quad \delta_i \tilde{\mu}^\alpha_{\delta,T} = (-1)^i \cdot 2\kappa_T \left( dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k} \right),
\]

\[
(2.4) \quad \delta_i \tilde{\mu}^\alpha_{d,T} = 0, \quad \text{and,} \quad d^k \tilde{\mu}^\alpha_{d,T} = 2(-1)^{(1+k)+1} \star (\kappa_T \left( \star (dx^{\alpha_1} \wedge \cdots dx^{\alpha_k}) \right)).
\]

We refer to the appendix, particularly (A.1) and (A.2), for some detailed calculations.

Denote

\[
(2.5) \quad \mathcal{H}^2_\delta \Lambda^k(T) := \text{span}\{ \tilde{\mu}^\alpha_{\delta,T} : \alpha \in \mathbb{IX}_{k,n} \},
\]

and

\[
(2.6) \quad \mathcal{H}^2_\delta \Lambda^k(T) := \text{span}\{ \tilde{\mu}^\alpha_{d,T} : \alpha \in \mathbb{IX}_{k,n} \}.
\]

**Lemma 2.2.**

1. \( d^k \) is bijective from \( \kappa_T (P_0 \Lambda^{k+1}) \) onto \( P_0 \Lambda^{k+1} \), and bijective from \( \mathcal{H}^2_\delta \Lambda^k(T) \) onto \( \star \kappa_T \star (P_0 \Lambda^k(T)) \).

2. \( \delta_k \) is bijective from \( \star \kappa_T \star (P_0 \Lambda^{k-1}) \) onto \( P_0 \Lambda^{k-1} \), and bijective from \( \mathcal{H}^2_\delta \Lambda^k(T) \) onto \( \kappa_T (P_0 \Lambda^k(T)) \).

**Lemma 2.3.** There exists a constant \( C_{k,n} \), depending on the regularity of \( T \), such that

\[
(2.7) \quad \| \mu \|_{L^2_\Lambda^k(T)} \leq C_{k,n} h_T \| \delta_k \mu \|_{L^2_\Lambda^{k-1}(T)}, \quad \text{for} \ \mu \in \mathcal{H}^2_\delta \Lambda^k(T),
\]

and

\[
(2.8) \quad \| \mu \|_{L^2_\Lambda^k(T)} \leq C_{k,n} h_T \| d^k \mu \|_{L^2_\Lambda^{k-1}(T)}, \quad \text{for} \ \mu \in \mathcal{H}^2_\delta \Lambda^k(T).
\]

**Proof.** For \( \mu = \sum_{\alpha \in \mathbb{IX}_{k,n}} C_\alpha \tilde{\mu}^\alpha_{\delta,T} \),

\[
\| \mu \|^2_{L^2_\Lambda^k(T)} \leq C_{k,n} h_T^2 \sum_{\alpha} C_\alpha \sum_{1 \leq i,j \leq k} \| \nabla (\tilde{x}^{\alpha_i}) \|^2 dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k} \|_{L^2_\Lambda^k(T)} \leq C_{k,n} h_T^2 |T| \sum_{\alpha} C_\alpha^2.
\]

Note that

\[
\delta_k \mu = 2(-1)^n \sum_{\alpha} C_\alpha \kappa_T \left( dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \right)
\]
and
\[ d^{k-1} \delta_k \mu = 2k(-1)^n \sum_{e \in \mathcal{E}_k} C_e \, dx_e \wedge \cdots \wedge dx_e. \]

Therefore, by the inverse inequality,
\[ 4k^2 h_T^2 |T| \sum_{e \in \mathcal{E}_k} C_e^2 = h_T^2 \|d^{k-1} \delta_k \mu\|_{L^2(\mathcal{A}^k(T))}^2 \leq C_{k,n} \|\delta_k \mu\|_{L^2(\mathcal{A}^{k-1}(T))}^2. \]

The proof of (2.7) is thus completed. Similarly can (2.8) be proved.

\[ \Box \]

**Lemma 2.4.** There exists a constant \( C_{k,n} \), depending on the regularity of \( T \), such that
\[ \|\mu\|_{L^2(\mathcal{A}^k(T))} \leq C_{k,n} h_T^2 \|\delta_k \mu\|_{L^2(\mathcal{A}^{k-1}(T))}, \quad \text{for } \mu \in \star \kappa_T \star (\mathcal{P}_0 \Lambda^{k-1}(T)) + \mathcal{H}_d^2 \Lambda^k(T), \]
and
\[ \|\mu\|_{L^2(\mathcal{A}^k(T))} \leq C_{k,n} h_T^2 \|d^k \mu\|_{L^2(\mathcal{A}^{k+1}(T))}, \quad \text{for } \mu \in \kappa_T(\mathcal{P}_0 \Lambda^{k+1}(T)) + \mathcal{H}_d^2 \Lambda^k(T). \]

**Proof.** Note that \( \delta_k(\star \kappa_T \star (\mathcal{P}_0 \Lambda^{k-1}(T))) \) and \( \delta_k(\mathcal{H}_d^2 \Lambda^k(T)) \) are orthogonal, and \( d^k(\kappa_T(\mathcal{P}_0 \Lambda^{k+1}(T))) \) and \( d^k(\mathcal{H}_d^2 \Lambda^k(T)) \) are orthogonal. The lemma follows by Lemmas 2.1 and 2.3.

\[ \Box \]

It is well known the lowest-degree trimmed space is
\[ \mathcal{P}_1 \Lambda^k(T) = \mathcal{P}_0 \Lambda^k(T) + \kappa(\mathcal{P}_0 \Lambda^{k+1}(T)), \]
and denote
\[ \mathcal{P}^{*\infty}_{-} \Lambda^k(T) = \star (\mathcal{P}_1 \Lambda^k(T)) = \mathcal{P}_0 \Lambda^k(T) + \star \kappa \star (\mathcal{P}_0 \Lambda^{k-1}(T)). \]

Here we introduce, for the \( HA^k \cap H^* \Lambda^k \) problem, an enriched trimmed space, defined by
\[ d^{m+d}_{\delta \in \delta, T} \Lambda^k(T) := \mathcal{P}_0 \Lambda^k(T) \oplus \kappa_T(\mathcal{P}_0 \Lambda^{k+1}(T)) \oplus \star \kappa_T \star (\mathcal{P}_0 \Lambda^{k-1}(T)) \oplus \mathcal{H}_d^2 \Lambda^k(T). \]

Then
\[ d^k(\mathcal{P}^{m+d}_{\delta \in \delta, T} \Lambda^k(T)) = \mathcal{P}^{*\infty}_{-} \Lambda^{k+1}(T), \quad \text{and } \delta_k(\mathcal{P}^{m+d}_{\delta \in \delta, T} \Lambda^k(T)) = \mathcal{P}_0 \Lambda^{k-1}(T). \]

### 2.1.2. A projective interpolator.

Define the interpolator
\[ \mathcal{I}^{m+d}_{d, \delta, T} : HA^k(T) \cap H^* \Lambda^k(T) \to \mathcal{P}^{m+d}_{d, \delta, T} \Lambda^k(T), \]
such that, for \( \mu \in HA^k(T) \cap H^* \Lambda^k(T), \)
\[ \langle d^{m+d}_{d, \delta, T} \mu, \eta \rangle_{L^2(\mathcal{A}^k(T))} = \langle \mathcal{I}^{m+d}_{d, \delta, T} \mu, \delta_k \eta \rangle_{L^2(\mathcal{A}^k(T))} = \langle d^k \mu, \eta \rangle_{L^2(\mathcal{A}^{k+1}(T))} - \langle \mu, \delta_k \eta \rangle_{L^2(\mathcal{A}^k(T))}, \quad \forall \, \eta \in \mathcal{P}^{*\infty}_{-} \Lambda^{k+1}(T), \]
and
\[ \langle d^{m+d}_{d, \delta, T} \mu, \tau \rangle_{L^2(\mathcal{A}^{k-1}(T))} = \langle \mathcal{I}^{m+d}_{d, \delta, T} \mu, d^{k-1} \tau \rangle_{L^2(\mathcal{A}^k(T))} = \langle \delta_k \mu, \tau \rangle_{L^2(\mathcal{A}^{k+1}(T))} - \langle \mu, d^{k-1} \tau \rangle_{L^2(\mathcal{A}^k(T))}, \quad \forall \, \tau \in \mathcal{P}_1 \Lambda^{k-1}(T). \]

**Lemma 2.5.** The interpolator \( \mathcal{I}^{m+d}_{d, \delta, T} \) is well defined, and \( \mathcal{I}^{m+d}_{d, \delta, T} \mu = \mu \) for \( \mu \in \mathcal{P}^{m+d}_{d, \delta, T} \Lambda^k(T). \)
Proof. Elementally, 

\[ \mathcal{P}_1^{\nu} \Lambda^{k+1}(T) = \mathcal{P}_0 \Lambda^{k+1}(T) \oplus ^{+2} * \kappa_T * (\mathcal{P}_0 \Lambda^k(T)), \]

and 

\[ \mathcal{P}_1^{-} \Lambda^{k-1}(T) = \mathcal{P}_0 \Lambda^{k-1}(T) \oplus ^{-2} \kappa_T (\mathcal{P}_0 \Lambda^k(T)). \]

Given \( \eta \in \mathcal{P}_1^{\nu} \Lambda^{k+1}(T) \), decompose it to \( \eta = \eta_0 + \eta_\delta \), with \( \eta_0 \in \mathcal{P}_0 \Lambda^{k+1}(T) \) and \( \eta_\delta \in * \kappa_T * (\mathcal{P}_0 \Lambda^k(T)) \), and decompose \( \tau \in \mathcal{P}_1^{-} \Lambda^{k-1}(T) \) to \( \tau = \tau_0 + \tau_d \) with \( \tau_0 \in \mathcal{P}_0 \Lambda^{k-1}(T) \) and \( \tau_d \in \kappa_T (\mathcal{P}_0 \Lambda^k(T)) \).

Now, given \( \mu \in H \Lambda^k(T) \cap H^\prime \Lambda^k(T) \), there exists a unique \( \mu_0 \in \mathcal{P}_0 \Lambda^k(T) \), such that 

\[ \langle \delta \mu_0, \tau_0 \rangle_{L^2 \Lambda^{k-1}(T)} = \langle \delta \mu, \tau_0 \rangle_{L^2 \Lambda^{k-1}(T)}, \forall \tau_0 \in \mathcal{P}_0 \Lambda^{k-1}(T), \]

and there exists a unique \( \mu_0 \in \mathcal{P}_0 \Lambda^k(T) \), such that 

\[ -\langle \mu_0, d^{k-1} \tau_d \rangle_{L^2 \Lambda^k(T)} = \langle \delta \mu, \tau_d \rangle_{L^2 \Lambda^{k-1}(T)} - \langle \mu, d^{k-1} \tau_d \rangle_{L^2 \Lambda^k(T)}, \forall \tau_d \in \kappa_T (\mathcal{P}_0 \Lambda^k(T)). \]

Then, there exists a unique \( \mu_d \in \kappa_T (\mathcal{P}_0 \Lambda^k(T)) \), such that 

\[ \langle d^k \mu_d, \eta_0 \rangle_{L^2 \Lambda^{k+1}(T)} = \langle d^k \mu, \eta_0 \rangle_{L^2 \Lambda^{k+1}(T)}, \forall \eta_0 \in \mathcal{P}_0 \Lambda^{k-1}(T), \]

and there exists a unique \( \mu_d \in \mathcal{H}_1 \Lambda^k(T) \), such that 

\[ \langle d^k \mu_d, \eta_\delta \rangle_{L^2 \Lambda^{k+1}(T)} = \langle d^k \mu, \eta_\delta \rangle_{L^2 \Lambda^{k+1}(T)} - \langle \mu, \delta^{k+1} \eta_\delta \rangle_{L^2 \Lambda^k(T)}, \forall \eta_\delta \in * \kappa_T * (\mathcal{P}_0 \Lambda^k(T)). \]

Now set \( \mu_k := \mu_0 + \mu_d + \mu_\delta + \mu_{d_\delta} \), and \( \mu_k \) satisfies all requirements (2.12) and (2.13) of \( \mathcal{P}_1^{d^{k+1}} \mu \); namely, \( \tau^{m+d_\delta}_d \mu = \mu_0 + \mu_d + \mu_\delta + \mu_{d_\delta} \), uniquely determined. Evidently, if \( \mu \in \mathcal{P}_1^{d^{k+1}} \mu \), \( \mu = \tau^{m+d_\delta}_d \mu \).

The proof is completed. \( \square \)

Remark 2.6. Denote a set of quantities 

\[ \{ \langle d^k \mu, \eta \rangle_{L^2 \Lambda^{k+1}(T)} - \langle \mu, \delta^{k+1} \eta \rangle_{L^2 \Lambda^k(T)}, \langle \delta \mu, \tau \rangle_{L^2 \Lambda^{k-1}(T)} - \langle \mu, d^{k-1} \tau \rangle_{L^2 \Lambda^k(T)} \}. \]

Then, according to the proof of Lemma 2.5,

- given \( \mu \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \), \( \mu = 0 \) if and only if all quantities in (2.16) vanish for any \( (\eta, \tau) \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \times \mathcal{P}_1^{-} \Lambda^{k-1}(T) \);
- given \( (\eta, \tau) \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \times \mathcal{P}_1^{-} \Lambda^{k-1}(T) \), \( (\eta, \tau) = (0, 0) \), if and only if all quantities in (2.16) vanish for any \( \mu \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \).

This way, \( \mu \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \) is unsolvent by (2.16) with \( \mu \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \times \mathcal{P}_1^{-} \Lambda^{k-1}(T) \), and \( (\eta, \tau) \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \times \mathcal{P}_1^{-} \Lambda^{k-1}(T) \) is unsolvent by (2.16) with \( \mu \in \mathcal{P}_1^{d^{k+1}} \Lambda^k(T) \).

Lemma 2.7. There exists a constant \( C_{\kappa,n} \), depending on the regularity of the simplex \( T \), such that 

\[ \| d^{k+m+d_\delta}_d \mu \|_{L^2 \Lambda^k(T)} + \| d^k \mu \|_{L^2 \Lambda^{k+1}(T)} + \| \delta \mu \|_{L^2 \Lambda^{k-1}(T)} \leq C_{\kappa,n} (\| \mu \|_{L^2 \Lambda^k(T)} + \| d^k \mu \|_{L^2 \Lambda^{k+1}(T)} + \| \delta \mu \|_{L^2 \Lambda^{k-1}(T)}). \]
Proof. Given $\mu \in H^k(T) \cap H^k,T$, for the interpolation, we use $\mu_0$, $\mu_d$, $\mu_\delta$ and $\mu_{d\delta}$ as in the proof of Lemma 2.5, and $\mathcal{E}_{d\delta,T}^{m+d,k}\mu = \mu_0 + \mu_d + \mu_\delta + \mu_{d\delta}$. Then

$$\delta_k\mathcal{E}_{d\delta,T}^{m+d,k}\mu = \delta_k\mu = \mathcal{P}_0\delta_k\mu.$$ 

Further,

$$\langle \mu - \mu_0, d^{k-1}\tau_d \rangle_{L^2(A^k(T))} = \langle \delta_k\mu, \tau_d \rangle_{L^2(A^k(T))}, \forall \tau_d \in \kappa_T(\mathcal{P}_0\Lambda^k).$$

Then, by Lemmas 2.2 and 2.1, we have

$$\|\mathcal{P}_0\mu - \mu_0\|_{L^2(A^k(T))} \leq C_{k,n}h_T\|\delta_k\mu\|_{L^2(A^k(T))}.$$ 

By (2.14) and (2.15), we have

$$\| d^{k-1}\mathcal{E}_{d\delta,T}^{m+d,k}\mu - d^{k-1}\mu_0, \eta_0 \rangle_{L^2(A^k+1(T))} = 0, \forall \eta_0 \in \mathcal{P}_0\Lambda^{k+1}(T),$$

and

$$\langle d^{k-1}\mathcal{E}_{d\delta,T}^{m+d,k}\mu - d^{k-1}\mu, \eta_0 \rangle_{L^2(A^k+1(T))} = \langle \mu_0 - \mu, \delta_k\eta_0 \rangle_{L^2(A^k(T)), \forall \eta_0 \in \kappa_T(\mathcal{P}_0\Lambda^k(T)).$$

Therefore, by Lemma 2.1,

$$\| \mathcal{P}_{\mathcal{P}_0\Lambda^k}(T) d^{k}\mu - d^{k-1}\mathcal{E}_{d\delta,T}^{m+d,k}\mu \|_{L^2(A^k+1(T))} \leq C_{k,n}h_T^{-1}\| \mathcal{P}_0\mu - \mu_0\|_{L^2(A^k(T))} \leq C_{k,n}\| \delta_k\mu\|_{L^2(A^k+1(T)).}$$

Summing all above leads to the assertion and completed the proof. \hfill \Box

**Lemma 2.8.** There exists a constant $C_{k,n}$, depending on the regularity of the simplex $T$, such that,

$$\| \omega - \mathcal{E}_{d\delta,T}^{m+d,k}\omega \|_{L^2(A^k(T))} + \| d^{k}\omega - d^{k-1}\mathcal{E}_{d\delta,T}^{m+d,k}\omega \|_{L^2(A^k+1(T))} + \| \delta_k\omega - \delta_k\mathcal{E}_{d\delta,T}^{m+d,k}\omega \|_{L^2(A^k+1(T))}$$

$$\leq C_{k,n}\inf_{\mu \in \mathcal{P}_{\mathcal{P}_0\Lambda^k}(T)} \left[ \| \omega - \mu \|_{L^2(A^k(T))} + \| d^{k}(\omega - \mu) \|_{L^2(A^k+1(T))} + \| \delta_k(\omega - \mu) \|_{L^2(A^k+1(T))} \right].$$

**Proof.** The proof follows immediately by the projection and stability. \hfill \Box

**Remark 2.9.** It can be proved that

$$\inf_{\mu \in \mathcal{P}_{\mathcal{P}_0\Lambda^k}(T)} \left[ \| \omega - \mu \|_{L^2(A^k(T))} + \| d^{k}(\omega - \mu) \|_{L^2(A^k+1(T))} + \| \delta_k(\omega - \mu) \|_{L^2(A^k+1(T))} \right]$$

$$\leq \inf_{\mu \in \mathcal{P}_{\mathcal{P}_0\Lambda^k}(T)} \| \omega - \mu \|_{L^2(A^k(T))} + \inf_{\eta \in \mathcal{P}_{\mathcal{P}_0\Lambda^k+1}(T)} \| d^{k}\omega - \eta \|_{L^2(A^k+1(T))} + \inf_{\tau \in \mathcal{P}_{\mathcal{P}_0\Lambda^k+1}(T)} \| \delta_k\omega - \tau \|_{L^2(A^k+1(T)}$$

$$+ Ch_T(\| d^{k}\omega \|_{L^2(A^k+1(T))} + \| \delta_k\omega \|_{L^2(A^k+1(T))}).$$

2.2. **Finite element space and global approximation.** For $\Xi$ a subdomain of $\Omega$, we denote by $E_\Xi^\Omega$ the extension from $L^1_{\text{loc}}(\Xi)$ to $L^1_{\text{loc}}(\Omega)$, the spaces of locally integrable functions, respectively. Namely,

$$E_\Xi^\Omega : L^1_{\text{loc}}(\Xi) \to L^1_{\text{loc}}(\Omega), \quad E_\Xi^\Omega v = \begin{cases} v, & \text{on } \Xi, \\ 0, & \text{else,} \end{cases} \quad \text{for } v \in L^1_{\text{loc}}(\Xi).$$

We use the same notation $L^1_{\text{loc}}$ for both scalar and non-scalar locally integrable functions, and, here and in the sequel, use the same notation $E_\Xi^\Omega$ for both scalar and non-scalar functions.
Let $\mathcal{G}_\Omega = \{\mathcal{G}_h\}$ be a set of shape regular subdivisions of $\Omega$ by simplexes. For a subdivision $\mathcal{G}_h$, define formally the product of a set of function spaces $\{Y(T)\}_{T \in \mathcal{G}_h}$ defined cell by cell such that $E^\Omega_T Y(T)$ for all $T \in \mathcal{G}_h$ are compatible,

$$\prod_{K \in \mathcal{G}} Y(T) := \sum_{K \in \mathcal{G}} E^\Omega_T Y(T),$$

and the summation is direct. $\prod_{K \in \mathcal{G}} Y(T)$ defined this way is actually the tensor product of all $Y(T)$.

Denote

- $\mathcal{P}^{-}_1 \Lambda^k(\mathcal{G}_h) := \prod_{T \in \mathcal{G}_h} \mathcal{P}^{-}_1 \Lambda^k(T)$;
- $\mathcal{P}^c_1 \Lambda^k(\mathcal{G}_h) := \prod_{T \in \mathcal{G}_h} \mathcal{P}^c_1 \Lambda^k(T)$;
- $\mathcal{P}^{m+d}_{d r, \delta} \Lambda^k(\mathcal{G}_h) := \prod_{T \in \mathcal{G}_h} \mathcal{P}^{m+d}_{d r, \delta} \Lambda^k(T)$.

Denote the conforming space by Whitney forms by

$$W^{k+1} = \mathcal{P}^c_1 \Lambda^k(\mathcal{G}_h) \cap H^{k+1}_0, \quad \text{and} \quad W^k \Lambda^k = \mathcal{P}^{-}_1 \Lambda^k(\mathcal{G}_h) \cap H^k\Lambda^k.$$  

On $\mathcal{G}_h$, define the finite element space:

\begin{equation}
V^{m+d}_{d r, \delta} \Lambda^k := \left\{ \mu_h \in \mathcal{P}^{m+d}_{d r, \delta} \Lambda^k(\mathcal{G}_h) : \langle d^k \mu_h, \eta_h \rangle_{L^2(\Omega^k)} - \langle \mu_h, \delta_{k+1} \eta_h \rangle_{L^2(\Omega^k)} = 0, \forall \eta_h \in W^{m+1}_0 \Lambda^k, \right. \\
\quad \left. \langle \delta_{k+1} \mu_h, \tau_h \rangle_{L^2(\Omega^k)} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2(\Omega^k)} = 0, \forall \tau_h \in W^{k-1}_h \Lambda^k \right\}.
\end{equation}

Here and in the sequel we use the subscript $\cdot_h$ to denote the piecewise operation on $\mathcal{G}_h$.

Define the interpolator

$$\mathbb{I}^{m+d, k}_{d r, \delta, h} : \prod_{T \in \mathcal{G}_h} H^k(T) \cap H^k \Lambda^k(T) \to \prod_{T \in \mathcal{G}_h} \mathcal{P}^{m+d}_{d r, \delta} \Lambda^k(T),$$

such that

$$\langle \mathbb{I}^{m+d, k}_{d r, \delta, h} \mu \rangle_T = \mathbb{I}^{m+d, k}_{d r, \delta, T} \mu, \forall T \in \mathcal{G}_h.$$

Lemma 2.10. If $\mu \in H^k \Lambda^k \cap H^k_0 \Lambda^k$, then $\mathbb{I}^{m+d, k}_{d r, \delta, h} \mu \in V^{m+d}_{d r, \delta} \Lambda^k$.

Proof. Provided $\mu \in H^k \Lambda^k \cap H^k_0 \Lambda^k$, it holds that

\begin{equation}
\begin{aligned}
\langle d^k \mu, \eta_h \rangle_{L^2(\Omega^k)} - \langle \mu, \delta_{k+1} \eta_h \rangle_{L^2(\Omega^k)} = 0, \forall \eta_h \in W^{m+1}_0 \Lambda^k, \\
\langle \delta_{k+1} \mu, \tau_h \rangle_{L^2(\Omega^k)} - \langle \mu, d^{k-1} \tau_h \rangle_{L^2(\Omega^k)} = 0, \forall \tau_h \in W^{k-1}_h \Lambda^k.
\end{aligned}
\end{equation}

By the definitions of $\mathbb{I}^{m+d, k}_{d r, \delta, h}$ and $\mathbb{I}^{m+d, k}_{d r, \delta, T}$, it holds that

\begin{equation}
\langle d^k \mathbb{I}^{m+d, k}_{d r, \delta, h} \mu, \eta_h \rangle_{L^2(\Omega^k)} - \langle \mathbb{I}^{m+d, k}_{d r, \delta, T} \mu, \delta_{k+1} \eta_h \rangle_{L^2(\Omega^k)} = \sum_{T \in \mathcal{G}_h} \langle d^k \mu, \eta_h \rangle_{L^2(\Omega^k)} - \langle \mu, \delta_{k+1} \eta_h \rangle_{L^2(\Omega^k)} = 0, \forall \eta_h \in W^{m+1}_0 \Lambda^k,
\end{equation}

\begin{equation}
\langle \delta_{k+1} \mathbb{I}^{m+d, k}_{d r, \delta, h} \mu, \tau_h \rangle_{L^2(\Omega^k)} - \langle \mathbb{I}^{m+d, k}_{d r, \delta, T} \mu, d^{k-1} \tau_h \rangle_{L^2(\Omega^k)} = 0, \forall \tau_h \in W^{k-1}_h \Lambda^k.
\end{equation}
and similarly
\[ \langle \delta_i, \omega \rangle_{L^2(A_i)} = \langle \delta_i, \omega \rangle_{L^2(A_i)} - \langle \omega \rangle_{L^2(A_i)} = 0, \forall \omega \in W_h \Lambda^{k-1}. \]
Namely, \( \delta_i \omega \in V^{m+d,h} \). The proof is completed.

By Lemma 2.8, we immediately obtain the lemma below.

**Lemma 2.11.** For \( \omega \in H^1(\Omega) \cap H^1_0(A_i) \), with a constant \( C_{k,h} \) depending on the shape regularity of \( \Omega \),
\[
(2.19) \quad ||\omega - \delta_i \omega ||_{L^2(A_i)} + ||\delta_i \omega ||_{L^2(A_i)} \leq C_{k,h} \inf_{\delta \in \delta_h} \left[ ||\omega - \mu_h ||_{L^2(A_i)} + ||\delta_i \omega ||_{L^2(A_i)} + ||\delta_i (\omega - \mu_h) ||_{L^2(A_i)} \right].
\]

3. **A primal finite element scheme of the \( H(d) \cap H(\delta) \) elliptic problem**

### 3.1. **Finite element scheme and error estimate.**

Given a simplicial subdivision \( \Gamma_h \) of \( \Omega \), we consider the finite element problem: find \( \omega_h \in V^{m+d,A} \), such that
\[
(3.1) \quad \langle d^k \omega, d^k \mu \rangle_{L^2(A_i)} + \langle \delta_i \omega, \delta_i \mu \rangle_{L^2(A_i)} + \langle \omega, \mu \rangle_{L^2(A_i)} = \langle f, \mu \rangle_{L^2(A_i)} \quad \forall \mu \in V^{m+d,A}.
\]

The well-posedness of (3.1) is immediate. The main theoretical result is the theorem below.

**Theorem 3.1.** Let \( \omega \) and \( \omega_h \) be the solutions of (1.1) and (3.1), respectively. Then
\[
(3.2) \quad ||\omega - \omega_h ||_{L^2(A_i)} + ||d^k(\omega - \omega_h) ||_{L^2(A_i)} + ||\delta_i (\omega - \omega_h) ||_{L^2(A_i)} \leq C \inf_{\mu \in \mathcal{P}_{m+d,A}(\Omega)} \left[ ||\omega - \mu ||_{L^2(A_i)} + ||d^k(\omega - \mu) ||_{L^2(A_i)} + ||\delta_i (\omega - \mu) ||_{L^2(A_i)} \right].
\]

**Proof.** According to Strang’s lemma, the consistency error part is
\[
CE(\mu) = \langle d^k \omega, d^k \mu \rangle_{L^2(A_i)} + \langle \delta_i \omega, \delta_i \mu \rangle_{L^2(A_i)} + \langle \omega, \mu \rangle_{L^2(A_i)} - \langle f, \mu \rangle_{L^2(A_i)}
\]
\[
= \langle d^k \omega, d^k \mu \rangle_{L^2(A_i)} - \langle \delta_i(\mu) \rangle_{L^2(A_i)} + \langle \delta_i \omega, \delta_i \mu \rangle_{L^2(A_i)} - \langle d^k(\omega - \mu) \rangle_{L^2(A_i)}.
\]
By the definition of \( V^{m+d,A} \), we have for any \( \eta \in W_h^* \) and \( \tau \in W_h \Lambda^{k-1} \),
\[
(3.3) \quad CE(\mu) = \langle d^k \omega - \eta, d^k \mu \rangle_{L^2(A_i)} - \langle \delta_i(\mu) \rangle_{L^2(A_i)} + \langle \delta_i \omega, \delta_i \mu \rangle_{L^2(A_i)} - \langle d^k(\mu) \rangle_{L^2(A_i)}
\]
\[
+ \langle \delta_i(\omega - \tau), \delta_i \mu \rangle_{L^2(A_i)} - \langle d^k(\omega - \tau) \rangle_{L^2(A_i)} \leq \langle ||d^k \omega - \eta ||_{L^2(A_i)} + ||\delta_i(\mu) \rangle_{L^2(A_i)} + ||\delta_i(\omega - \tau) \rangle_{L^2(A_i)}
\]
\[
+ ||\delta_i(\omega - \tau) \rangle_{L^2(A_i)} + ||d^k(\omega - \tau) \rangle_{L^2(A_i)} + ||d^k(\omega - \tau) \rangle_{L^2(A_i)}.
\]
This estimate, together with the approximation estimate Lemma 2.11, leads to the total error estimation, and completes the proof. \( \square \)
3.2. Implementation of the scheme: a set of locally supported basis functions. The finite element space \( V_{d\delta}^{m+d} \Lambda^k \) does not correspond to a “finite element” defined as Ciarlet’s triple [12]. Though, in this section, we present a set of basis functions of \( V_{d\delta}^{m+d} \Lambda^k \) which are tightly supported. Therefore, the finite element scheme can still be implemented by the standard routine.

To illustrate the general procedure in Section 3.2.1, a two-dimensional example is given in Section 3.2.2, where we particularly refer to Figures 2 and 3 for the illustration of the local supports of the basis functions.

3.2.1. A general procedure. Denote

\[
P_{d\delta}^{m+d} \Lambda^k(T) := \left\{ (\delta_k \mu, \eta)_{L^2 V_{d\delta}^{k+1}(T)} - \langle \mu, d^{k-1} \tau \rangle_{L^2 V_{d\delta}^k(T)} = 0, \ \forall \tau \in \mathcal{P}_1 \Lambda^k(T) \right\},
\]

and

\[
P_{d\delta}^{m+d} \Lambda^k(T) := \left\{ (d^k \mu, \eta)_{L^2 V_{d\delta}^{k+1}(T)} - \langle \mu, \delta_{k+1} \eta \rangle_{L^2 V_{d\delta}^k(T)} = 0, \ \forall \eta \in \mathcal{P}_1 \Lambda^k(T) \right\}.
\]

Then \( P_{d\delta}^{m+d} \Lambda^k(T) \) is unisolvent by \( \left\{ (d^k \mu, \eta)_{L^2 V_{d\delta}^{k+1}(T)} - \langle \mu, d^{k-1} \tau \rangle_{L^2 V_{d\delta}^k(T)}, \ \forall \eta \in \mathcal{P}_1 \Lambda^k(T) \right\} \), \( P_{d\delta}^{m+d} \Lambda^k(T) \) is unisolvent by \( \left\{ (\delta_k \mu, \tau)_{L^2 V_{d\delta}^{k+1}(T)} - \langle \mu, d^{k-1} \tau \rangle_{L^2 V_{d\delta}^k(T)}, \ \tau \in \mathcal{P}_1 \Lambda^k(T) \right\} \). Denote

\[
P_{d\delta}^{m+d} \Lambda^k(G) = \bigcap_{T \in G_h} P_{d\delta}^{m+d} \Lambda^k(T), \ \text{ and } \ P_{d\delta}^{m+d} \Lambda^k(G) = \bigcap_{T \in G_h} P_{d\delta}^{m+d} \Lambda^k(T).
\]

Then

\[
P_{d\delta}^{m+d} \Lambda^k(T) = P_{d\delta}^{m+d} \Lambda^k(T) \oplus P_{d\delta}^{m+d} \Lambda^k(T) \text{ and } \ P_{d\delta}^{m+d} \Lambda^k(G) = P_{d\delta}^{m+d} \Lambda^k(G) \oplus P_{d\delta}^{m+d} \Lambda^k(G).
\]

It follows that

\[
V_{d\delta}^{m+d} \Lambda^k = \left\{ \mu_h \in P_{d\delta}^{m+d} \Lambda^k(G) : (d^k \mu_h, \eta_h)_{L^2 V_{d\delta}^{k+1}} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2 V_{d\delta}^k} = 0, \ \forall \eta_h \in \mathcal{P} V_{d\delta}^k \right\},
\]

and

\[
V_{d\delta}^{m+d} \Lambda^k = \left\{ \mu_h \in P_{d\delta}^{m+d} \Lambda^k(G) : (\delta_k \mu_h, \eta_h)_{L^2 V_{d\delta}^{k+1}} - \langle \mu_h, \delta_{k+1} \eta_h \rangle_{L^2 V_{d\delta}^k} = 0, \ \forall \eta_h \in \mathcal{P} V_{d\delta}^k \right\}.
\]

Now we figure out the basis functions of \( V_{d\delta}^k \) and \( V_{d\delta}^k \), respectively. They two form the whole set of basis functions of \( V_{d\delta}^{m+d} \Lambda^k \).

Basis functions of \( V_{d\delta}^k \). The basis functions of \( V_{d\delta}^k \) are determined by a 3-step procedure.

**Step 1:** Let \( B_{W}^{-1} = \{ \psi_j \}_{j=1}^{\dim W \Lambda^{k-1}} \) be the set of nodal basis functions of \( W \Lambda^{k-1} \); then on every simplex \( T \), the restrictions \( \psi_j|_T \) of those \( \psi_j \) that are nonzero on \( T \) are linearly independent, and such \( \psi_j|_T \) expand \( \mathcal{P}_1 \Lambda^{k-1}(T) \);
Step 2: for any $T \in \mathcal{G}_h$, set $I^T := \{ 1 \leq i \leq \dim(W_h \Lambda^{k-1}) : \hat{T} \cap \text{supp}(\psi_j) \neq \emptyset \}$, and there exist a set of functions $\{ \mu^T_i : i \in I^T \} \subset \mathbf{P}^{m+d}_{d, \delta, \delta} \Lambda^k(T)$, such that $\langle \delta_{i,j} \mu^T_i, \psi_j \rangle_{L^2 \Lambda^{k-1}} - \langle \mu^T_i, d^{k-1} \psi_j \rangle_{L^2 \Lambda^k} = \delta_{ij}$. Then $\mathbf{P}^{m+d}_{d, \delta, \delta} \Lambda^k(T) = \text{span}\{ \mu^T_i : i \in I^T \}$, and all $\mu^T_i$ are linearly independent.

Step 3: Now, by definition,

$$V_{\delta} = \left\{ \mu_h \in \mathbf{P}^{m+d}_{d, \delta, \delta} \Lambda^k(\mathcal{G}_h) : \langle \delta_{k,h} \mu_h, \tau_h \rangle_{L^2 \Lambda^{k-1}} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2 \Lambda^k} = 0, \quad \forall \tau_h \in W_h \Lambda^{k-1} \right\}$$

$$= \left\{ \mu_h \in \sum_{T \in \mathcal{G}_h} \sum_{i \in I^T} \text{span}\{ E_{\tilde{T}}^2 \mu^T_i \} : \langle \delta_{k,h} \mu_h, \psi_j \rangle_{L^2 \Lambda^{k-1}} - \langle \mu_h, d^{k-1} \psi_j \rangle_{L^2 \Lambda^k} = 0, \quad \forall \psi_j \in \mathbf{B}_W^{k-1} \right\}$$

$$= \sum_{1 \leq j \leq \dim(W_h \Lambda^{k-1})} \left\{ \mu_h \in \sum_{T \cap \text{supp}(\psi_j) \neq \emptyset} \text{span}\{ E_{\tilde{T}}^2 \mu^T_i \} : \langle \delta_{k,h} \mu_h, \psi_j \rangle_{L^2 \Lambda^{k-1}} - \langle \mu_h, d^{k-1} \psi_j \rangle_{L^2 \Lambda^k} = 0 \right\}.$$

Namely, a set of basis functions of $V_{\delta}$ consists of, for $1 \leq j \leq \dim(W_h \Lambda^{k-1})$, functions

$$\mu_h \in \sum_{T \cap \text{supp}(\psi_j) \neq \emptyset} \text{span}\{ E_{\tilde{T}}^2 \mu^T_i \}, \quad \text{such that } \langle \delta_{k,h} \mu_h, \psi_j \rangle_{L^2 \Lambda^{k-1}} - \langle \mu_h, d^{k-1} \psi_j \rangle_{L^2 \Lambda^k} = 0.$$

Note that the supports of these functions are contained in the support of $\psi_j$.

Basis functions of $V_{\delta}$. The basis functions of $V_{\delta}$ are determined by a 3-step procedure, slightly different from the procedure for $V_{\delta}$. They basis functions have basically two categories.

Step 1: Let $\mathbf{B}_{W, 0}^{k+1} = \{ \psi_j \}_{j=1}^{\dim(W_{h0} \Lambda^{k+1})}$ be the set of nodal basis functions of $W_{h0} \Lambda^{k+1}$; then on every simplex $T$, the restrictions $\psi_j|_T$ of those $\psi_j$ that are nonzero on $T$ are linearly independent, and for interior $T$, such $\psi_j|_T$ expand $\mathcal{P}_{1,T}^r \Lambda^{k+1}(T)$.

Step 2: For any $T \in \mathcal{G}_h$, set $I^T := \{ 1 \leq i \leq \dim(W_h \Lambda^{k-1}) : \hat{T} \cap \text{supp}(\psi_j) \neq \emptyset \}$, and there exist, not necessarily unique, a set of functions $\{ \mu^T_i : i \in I^T \} \subset \mathbf{P}^{m+d}_{d, \delta, \delta} \Lambda^k(T)$, such that $\langle d^i \mu^T_i, \psi_j \rangle_{L^2 \Lambda^{k-1}} - \langle \mu^T_i, \delta_{k+1} \psi_j \rangle_{L^2 \Lambda^k} = \delta_{ij}$. Denote $\langle \text{span}(\mu^T_i : i \in I^T) \rangle^c = \{ \mu \in \mathbf{P}^{m+d}_{d, \delta, \delta} \Lambda^k(T) : \langle d^k \mu, \psi_j \rangle_{L^2 \Lambda^{k-1}} - \langle \mu, \delta_{k+1} \psi_j \rangle_{L^2 \Lambda^k} = 0, \quad j \in I^T \}$, not necessarily non-empty. Then

$$\mathbf{P}^{m+d}_{d, \delta, \delta} \Lambda^k(T) = \langle \text{span}(\mu^T_i : i \in I^T) \rangle^c + \langle \text{span}(\mu^T_i : i \in I^T) \rangle^c.$$
Step 3: By definition,

\[ V_d = \left\{ \mu_h \in P_{d_{\text{rot},d}}^{n+\text{rot}}(G_h) : \langle d_h^k \mu_h, \eta_h \rangle_{L^2} - \langle \mu_h, \delta_{k+1} \eta_h \rangle_{L^2} = 0, \forall \eta_h \in W_{h0}^{r+1} \right\} \]

\[ = \prod_{T \in G_h} (\text{span}(\mu_i^T : i \in I^T))^c \]

\[ \oplus \left\{ \mu_h \in \sum_{T \in G_h} \sum_{i \in I^T} \text{span}(E_T \mu_i^T) : \langle d_h^k \mu_h, \psi^*_h \rangle_{L^2} = 0, \forall \psi^*_h \in B_{W,\theta}^{k-1} \right\} \]

\[ = \prod_{T \in G_h} (\text{span}(\mu_i^T : i \in I^T))^c \]

\[ \oplus \sum_{1 \leq j \leq \dim(W_{h0}^{r+1})} \left\{ \mu_h \in \sum_{T \cap \text{supp} \psi^*_h \neq \emptyset} \text{span}(E_T \mu_j^T) : \langle d_h^k \mu_h, \psi^*_h \rangle_{L^2} = 0 \right\}. \]

Namely, a set of basis functions of \( V_d \) consists of two parts:

1. Part I: \( \prod_{T \in G_h} (\text{span}(\mu_i^T : i \in I^T))^c \). The supports of these functions are each contained in a single simplex.

2. Part II: for \( 1 \leq j \leq \dim(W_{h0}^{r+1}) \), functions

\[ \mu_h \in \sum_{T \cap \text{supp} \psi^*_h \neq \emptyset} \text{span}(E_T \mu_j^T) \text{ such that } \langle d_h^k \mu_h, \psi^*_h \rangle_{L^2} = 0. \]

Note that the supports of these functions are contained in the support of \( \psi^*_j \).

3.2.2. Examples. We take the two-dimensional \( H^1 \cap H^1_0 \) problem for example. Let \( \Omega \) be a polygon. The problem reads: find \( \omega \in H(\text{rot}, \Omega) \cap H_0(\text{div}, \Omega) \), such that

\[ (\text{rot} \omega, \text{rot} \mu) + (\text{div} \omega, \text{div} \mu) + (\omega, \mu) = (f, \mu), \forall \mu \in H(\text{rot}, \Omega) \cap H_0(\text{div}, \Omega). \]

The corresponding spaces are \( H^1(\Omega) = H(\text{grad}, \Omega) \) for \( H^1 \) and \( H_0(\text{curl}, \Omega) = H_0(\text{curl}, \Omega) \) for \( H^1 \), respectively. We use for the conforming spaces of Whitney forms the linear element space \( V_h^1 \) for \( H(\text{grad}, \Omega) \) and \( V_{h0}^1 = V_h^1 \cap H_0(\text{curl}, \Omega) \) for \( H_0(\text{curl}, \Omega) \).

Let \( T_h \) be a shape-regular triangular subdivision of \( \Omega \) with mesh size \( h \), such that \( \overline{\Omega} = \cup_{T \in T_h} \overline{T} \), and every boundary vertex is connected to at least one interior vertex. Denote by \( \mathcal{E}_h, \mathcal{E}_h^i, \mathcal{E}_h^b, \mathcal{X}_h, \mathcal{X}_h^i, \mathcal{X}_h^b \) and \( \mathcal{X}_h^c \) the set of edges, interior edges, boundary edges, vertices, interior vertices, boundary vertices and corners, respectively.

In the setting, the shape function space is

\[ P_{d_{\text{rot},d}}^{n+\text{rot}}(T) = P_{\text{rot},\text{div}}^{n+\text{rot}}(T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ -\tilde{x} \end{pmatrix}, \begin{pmatrix} \tilde{y}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{x}^2 \end{pmatrix} \right\}. \]
Step 1. Both interior and boundary vertices, and also the supports of \( \phi \) each one of the barycentric coordinates on the triangle. We refer to Figure 1 for an illustration of Step 2.

The global basis functions of \( V \) here) and \( B \), respectively.

Step 3. On a cell \( T \), with \( a_i \in X \) being its vertices, let \( \lambda_{ij}^k \) be the barycentric coordinates of \( T \), and find \( \mu_{a_i,T}^{\text{div}}, \mu_{a_i,T}^{\text{rot}} \in P_{m+d} \Lambda^1(T) \), \( 1 \leq i \leq 3 \), such that

\[
(\text{rot} \mu_{a_i,T}^{\text{rot}}, \lambda_T^{ij})_T - (\mu_{a_i,T}^{\text{rot}}, \text{curl} \lambda_T^{ij}) = 0, \quad (\text{div} \mu_{a_i,T}^{\text{div}}, \lambda_T^{ij})_T - (\mu_{a_i,T}^{\text{div}}, \text{grad} \lambda_T^{ij}) = \delta_{ij}, \quad 1 \leq i, j \leq 3,
\]

and

\[
(\text{rot} \mu_{a_i,T}^{\text{rot}}, \lambda_T^{ij})_T - (\mu_{a_i,T}^{\text{rot}}, \text{curl} \lambda_T^{ij}) = \delta_{ij}, \quad (\text{div} \mu_{a_i,T}^{\text{rot}}, \lambda_T^{ij})_T - (\mu_{a_i,T}^{\text{div}}, \text{grad} \lambda_T^{ij}) = 0, \quad 1 \leq i, j \leq 3.
\]

Step 3. The global basis functions of \( V_{m+d} \Lambda^k \) fall into two categories, corresponding to \( V_{\delta} (V_{\text{div}} \text{ here}) \) and \( V_{\text{rot}} \) (as 

Category I: All these functions in \( \text{span} \{ \mu_{a_i,T}^{\text{div}} : a \in X_h, \ T \in \mathcal{G}_h \} \) such that conditions in (3.6) are satisfied with respect to every \( \mu_{a_i,T}^{\text{div}} \) a basis function. Particularly, with respect to any vertex \( a \), the associated basis functions of \( V_{m+d} \Lambda^k \) are all these functions in \( \text{span} \{ \mu_{a_i,T}^{\text{div}} : T \cap \text{supp}(\phi_a) \neq \emptyset \} \), namely, functions

\[
\omega_h = \sum_{\partial T \ni a} c_T E_{\Omega} \mu_{a_i,T}^{\text{div}}, \text{ such that } \sum_{\partial T \ni a} (\text{div} \omega_h, \phi_{a_i,T})_T + (\omega_h, \nabla \phi_{a_i,T})_T = 0.
\]
A PRIMAL FINITE ELEMENT SCHEME OF THE $H(d) \cap H(\delta)$ ELLIPTIC PROBLEM

**Figure 2.** A is an interior vertex; cf. Figure 1. Five basis functions associated with the interior vertex $A$. The shadowed parts are respectively the supports of the basis functions.

Those $\omega_h$ in this category are each supported in a two-successive-cell patch. We refer to Figure 2 for the case $a \in X^i_h$, and to Figure 3 (right) for an illustration that $a \in X^b_h$.

**Category II:** All these functions in $\text{span}\{\mu_{a,T}^{\text{rot}} : a \in X_h, T \in G_h\}$ such that conditions in (3.6) are satisfied with respect to every $\psi_a \in V^1_{h0}$ a basis function. Particularly, with respect to an interior vertex $a$, the associated basis functions of $V^{m+d}_{d\cap\delta} \Lambda^k$ are all these functions in $\text{span}\{\mu_{a,T}^{\text{rot}} : \hat{T} \cap \text{supp}(\phi_a) \neq \emptyset\}$, namely, functions

$$\omega_h = \sum_{\partial T \ni a} c_T E^{G_T} \mu_{a,T}^{\text{rot}}, \text{ such that } \sum_{\partial T \ni a} (\text{rot}\omega_h, \phi_a|_T) - (\omega_h, \text{curl}\phi_a|_T) = 0.$$  

Those $\omega_h$ in this category are each supported in a two-successive-cell patch. We refer to Figure 2 for the case $a \in X^i_h$.

With respect to a boundary vertex $a$, the associated basis functions of $V^{m+d}_{d\cap\delta} \Lambda^k$ are all these functions in $\text{span}\{\mu_{a,T}^{\text{rot}} : \hat{T} \cap \text{supp}(\phi_a) \neq \emptyset\}$. The supports of the functions are each a single triangle. We refer to Figure 3, left, for an illustration.

**APPENDIX A. SOME DETAILED CALCULATION**

Let $\alpha \in \mathbb{X}_{k,n}$ and $\beta \in \mathbb{X}_{n-k,n}$ be such that $\alpha$ and $\beta$ partition $\{1, \ldots, n\}$. Elementary calculation leads to that

$$\star (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) = (-1)^{pm} (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-k}}), \quad pm = \sum_{j=1}^k \alpha_j - \frac{k(k+1)}{2};$$
\[ \star (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_n}) = (-1)^{pm} (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}), \quad pm = \sum_{j=1}^{n-k} \beta_j - \frac{(n-k)(n-k+1)}{2}; \]

\[ dx^{\alpha_1} \wedge dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-k}} = (-1)^{pm} \ dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-1}} \wedge dx^{\alpha_1} \wedge dx^{\beta_m} \wedge \cdots \wedge dx^{\beta_{n-k}}, \]

such that \( \beta_{m-1} < \alpha_i < \beta_m, \ pm = \alpha_i - i. \)

\[ (-1)^{\bar{m}} \delta_3 \tilde{\mu}^\alpha_{\delta,T} = \star d^{n-k} \star \mu^\alpha_{\delta,T} = \star d^{n-k} (\sum_{j=1}^{k} (\tilde{x}^{\alpha_j})^2 \star (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k})) \]

\[ = \star d^{n-k} \left[ \sum_{j=1}^{k} (\tilde{x}^{\alpha_j})^2 (-1)^{pm_1} (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-k}}) \right] = \star \sum_{j=1}^{k} 2 \tilde{x}^{\alpha_j} (-1)^{pm_1} dx^{\alpha_j} \wedge dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-k}} \]

\[ = \sum_{j=1}^{k} (-1)^{pm_1+pm_2+pm_3} 2 \tilde{x}^{\alpha_j} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_j+1} \wedge dx^{\alpha_k} \]

\[ = 2 \cdot (-1)^{(k-1)(n-1)} \sum_{j=1}^{k} (-1)^{j+1} \tilde{x}^{\alpha_j} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_j+1} \wedge dx^{\alpha_k} \]

\[ = 2 \cdot (-1)^{(k-1)(n-1)} k T (dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k}), \]

where \( pm_1 = \sum_{j=1}^{k} \alpha_j - \frac{k(k+1)}{2}, \ pm_2 = \alpha_j - j, \ pm_3 = \alpha_j + \sum_{j=1}^{n-k} \beta_j - \frac{(n-k)(n-k+2)}{2}, \) and \( pm_1 + pm_2 + pm_3 = 2\alpha_j - (k-1)(n-1) - (j+1). \)

Namely

\[ (A.1) \quad \delta_3 \tilde{\mu}^\alpha_{\delta,T} = (-1)^n \cdot 2k T (dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k}). \]
\[ d^k \tilde{\mu}_{d,T} = \sum_{j=1}^{n-k} 2\tilde{x}^{\beta_j} \, dx^{\beta_j} \wedge dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \]

\[ = \sum_{j=1}^{n-k} 2\tilde{x}^{\beta_j}(-1)^{\text{pm}_1} \, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{m-1}} \wedge dx^{\beta_j} \wedge dx^{\alpha_m} \wedge \cdots \wedge dx^{\alpha_k} \quad (\alpha_{m-1} < \beta_j < \alpha_m) \]

\[ = \sum_{j=1}^{n-k} 2\tilde{x}^{\beta_j}(-1)^{\text{pm}_1}(-1)^{\text{pm}_2} \star (dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{j-1}} \wedge dx^{\beta_{j+1}} \wedge \cdots \wedge dx^{\beta_{n-k}}) \]

\[ = 2 \cdot (-1)^{\text{pm}_1} \star (\sum_{j=1}^{n-k} \tilde{x}^{\beta_j}(-1)^{j+1} \, dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{j-1}} \wedge dx^{\beta_{j+1}} \wedge \cdots \wedge dx^{\beta_{n-k}}) \]

\[ = 2 \cdot (-1)^{\text{pm}_1} \star (\kappa_T(dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{n-k}})) = 2 \cdot (-1)^{\text{pm}_1}(-1)^{\text{pm}_4} \star (\kappa_T(\star(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}))) \]

\[ = -2(-1)^{(n-k)(k+1)} \star (\kappa_T(\star(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}))), \]

where \( \text{pm}_1 = \beta_j - j, \text{pm}_2 = \sum_{i=j}^{n-k} \beta_i + \sum_{i=j}^{n-k} \beta_i - \frac{1}{2}(n-k-1)(n-k) \), \( \text{pm}_3 = \sum_{i=1}^{n-k} \beta_i - \frac{1}{2}(n-k-1)(n-k)+1 \), and \( \text{pm}_4 = \sum_{i=1}^{k} \alpha_i - \frac{1}{2}k(k+1) \). Namely

\[ (A.2) \quad d^k \tilde{\mu}_{d,T} = 2(-1)^{(n+1)(k+1)} \star (\kappa_T(\star(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}))). \]

**References**

[1] Douglas Arnold, *Finite element exterior calculus*. SIAM, 2018.

[2] Douglas Arnold, Richard Falk, and Ragnar Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica*, 15:1–155, 2006.

[3] Douglas Arnold, Richard Falk, and Ragnar Winther. Finite element exterior calculus: from hodge theory to numerical stability. *Bulletin of the American Mathematical Society*, 47(2):281–354, 2010.

[4] Mary Barker, *A Nonconforming Finite Element Method for the 2D Vector Laplacian*. PhD thesis, Washington University in St. Louis, 2022.

[5] Mary Barker, Shuhao Cao, and Ari Stern. A nonconforming primal hybrid finite element method for the two-dimensional vector Laplacian. *arXiv preprint arXiv:2206.10567*, 2022.

[6] KJ Bathe, C Nitikitpaiboon, and X Wang. A mixed displacement-based finite element formulation for acoustic fluid-structure interaction. *Computers & Structures*, 56(2-3):225–237, 1995.

[7] Alfredo Bermúdez and Rodolfo Rodríguez. Finite element computation of the vibration modes of a fluid–solid system. *Computer Methods in Applied Mechanics and Engineering*, 119(3-4):355–370, 1994.

[8] Susanne Brenner, Jintao Cui, Fengyan Li, and Li-Yeng Sung. A nonconforming finite element method for a two-dimensional curl-curl and grad-div problem. *Numerische Mathematik*, 109(4):509–533, 2008.

[9] Susanne Brenner and Li-Yeng Sung. A quadratic nonconforming vector finite element for \( H(curl; \Omega) \cap H(div; \Omega) \). *Applied Mathematics Letters*, 22:892–896, 2009.

[10] Susanne Brenner, Li-Yeng Sung, and Jintao Cui. An interior penalty method for a two dimensional curl-curl and grad-div problem. *ANZIAM Journal*, 50:C947–C975, 2008.

[11] Snorre Christiansen and Ragnar Winther. Smoothed projections in finite element exterior calculus. *Mathematics of Computation*, 77(262):813–829, 2008.

[12] Philippe G Ciarlet. *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.

[13] Lourenço Beirão da Veiga, Franco Brezzi, L Donatella Marini, and Alessandro Russo. Virtual element approximations of the vector potential formulation of magnetostatic problems. *The SMAI Journal of Computational Mathematics*, 4:399–416, 2018.
[14] Alan Demlow and Anil N. Hirani. A posteriori error estimates for finite element exterior calculus: The de Rham complex. *Foundations of Computational Mathematics*, 14(6):1337–1371, 2014.
[15] Richard Falk and Ragnar Winther. Local bounded cochain projections. *Mathematics of Computation*, 83(290):2631–2656, 2014.
[16] M. Fortin and M. Soulie. A non-conforming piecewise quadratic finite element on triangles. *International Journal for Numerical Methods in Engineering*, 19(4):505–520, 1983.
[17] Mohamed Ali Hamdi, Yves Ousset, and Georges Verchery. A displacement method for the analysis of vibrations of coupled fluid-structure systems. *International Journal for Numerical Methods in Engineering*, 13(1):139–150, 1978.
[18] Ralf Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237–339, 2002.
[19] Qingguo Hong, Yuwen Li, and Jinchao Xu. An extended Galerkin analysis in finite element exterior calculus. *Mathematics of Computation*, 91(335):1077–1106, 2022.
[20] Yuwen Li. Some convergence and optimality results of adaptive mixed methods in finite element exterior calculus. *SIAM Journal on Numerical Analysis*, 57(4):2019–2042, 2019.
[21] Martin Licht. Smoothed projections and mixed boundary conditions. *Mathematics of Computation*, 88(316):607–635, 2019.
[22] Wenjia Liu and Shuo Zhang. A lowest-degree strictly conservative finite element scheme for incompressible stokes problem on general triangulations. *arXiv preprint*, 2108.10522, 2021.
[23] Jean-Marie Mirebeau. Nonconforming vector finite elements for $H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$. *Applied Mathematics Letters*, 25(3):369–373, 2012.
[24] Peter Monk. *Finite element methods for Maxwell’s equations*. Oxford University Press, 2003.
[25] Chunjae Park and Dongwoo Sheen. $P_1$-nonconforming quadrilateral finite element methods for second-order elliptic problems. *SIAM Journal on Numerical Analysis*, 41(2):624–640, 2003.
[26] Yingxia Xi, Xia Ji, and Shuo Zhang. A high accuracy nonconforming finite element scheme for Helmholtz transmission eigenvalue problem. *Journal of Scientific Computing*, 83, 2020.
[27] Yingxia Xi, Xia Ji, and Shuo Zhang. A simple low-degree optimal finite element scheme for the elastic transmission eigenvalue problem. *Communications in Computational Physics*, 30:1061–1082, 2021.
[28] Shuo Zhang. Minimal consistent finite element space for the biharmonic equation on quadrilateral grids. *IMA Journal of Numerical Analysis*, 40(2):1390–1406, 2020.
[29] Shuo Zhang. An optimal piecewise cubic nonconforming finite element scheme for the planar biharmonic equation on general triangulations. *Science China Mathematics*, 64(11):2579–2602, 2021.

LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190; University of Chinese Academy of Sciences, Beijing, 100049; People’s Republic of China

*Email address*: szhang@lsec.cc.ac.cn