Low-rank Tensor Bandits

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Abstract

In recent years, multi-dimensional online decision making has been playing a crucial role in many practical applications such as online recommendation and digital marketing. To solve it, we introduce stochastic low-rank tensor bandits, a class of bandits whose mean rewards can be represented as a low-rank tensor. We propose two learning algorithms, tensor epoch-greedy and tensor elimination, and develop finite-time regret bounds for them. We observe that tensor elimination has an optimal dependency on the time horizon, while tensor epoch-greedy has a sharper dependency on tensor dimensions. Numerical experiments further back up these theoretical findings and show that our algorithms outperform various state-of-the-art approaches that ignore the tensor low-rank structure.

1 Introduction

The growing availability of tensor data provides an unique opportunity for decision-makers to efficiently develop multi-dimensional decisions for individuals [Ge et al., 2016, Frolov and Oseledets, 2017, Bi et al., 2018, Song et al., 2019]. For instance, consider a marketer who wants to design an advertising campaign for products with promotion offers across different marketing channels and user segments. This marketer needs to estimate the probability of user $i$ clicking offer $j$ in channel $k$ for any $(i, j, k)$ combination so that the most relevant users will be targeted for a chosen product and channel. See Figure 1 for the tensor formulation. Traditional static recommendation systems using tensor methods [Frolov and Oseledets, 2017, Bi et al., 2018, Song et al., 2019] do not interact with the environment to update the estimation and typically suffer the cold-start issue in the absence of information from new customers, new products or new contexts. Therefore, an interactive recommender system for multi-dimensional decisions is urgently needed.

In this paper, we introduce stochastic low-rank tensor bandits for multi-dimensional online decision-making problems. They are a class of bandits whose mean rewards can be represented as a low-rank tensor. This low-rank assumption greatly reduces the model complexity and is also well motivated from various practical problems in online recommendation and digital marketing. We develop two algorithms: tensor epoch-greedy and tensor elimination. The first algorithm gradually refines the low-rank tensor estimation that guides the switch between exploration phase...
1.1 Related work and our contributions

A line of related literature considers special rank-1 matrix bandits Katariya et al. [2017b,a], Trinh et al. [2019]. To find the largest entry of a non-negative rank-1 matrix, one just needs to identify the largest values of the left-singular and right-singular vectors. However, this is no longer applicable for higher-rank matrices. Thus, the algorithms and the theoretical results in Katariya et al. [2017b,a], Trinh et al. [2019] could not be easily extended to a general rank case.

For general low-rank matrix bandits, Jun et al. [2019] proposed a bilinear bandit that can be viewed as a contextual low-rank matrix bandit. However, their regret bound becomes trivial in the non-contextual setting due to the sub-optimal analysis of LinUCB [Abbasi-Yadkori et al., 2011] for linear bandits with finitely many arms. Kveton et al., 2017] also handled low-rank matrix bandits but assumed strong hot topic assumptions on the mean reward matrix and the algorithm was computationally intractable in practice. Sen et al. [2017] considered low-rank matrix bandits with one dimension

and exploitation phase. The second one estimates the singular vectors of the mean reward tensor and carries out a reduction to reshaped linear bandits that are solved by a elimination-based algorithm. As shown in the regret analysis and numerical experiments, each algorithm has its own advantages.

In theory, we establish cumulative regret bounds for both algorithms in a finite-time regime. We also compare our regret bounds with those of two competitive methods: vectorized UCB which unfolds the tensor into a long vector and then implements standard UCB1 [Auer, 2002] for multi-armed bandits, and matricized ESTR which unfolds the tensor into a matrix along one mode and implements Explore-Subspace-Then-Refine (ESTR) [Jun et al., 2019] for low-rank matrix bandits. Denote $n$ as the time horizon, $p = \max\{p_1, \ldots, p_d\}$ as the maximum tensor dimension with $d$ the order of the mean reward tensor. As shown in Table 1, in the data-rich regime where $n \geq \tilde{O}(p^{d-5})$, tensor elimination has the best overall regret bound, while tensor epoch-greedy performs the best in the data-poor regime due to its superior dependency on the tensor dimension. On the other hand, both vectorized UCB and matricized ESTR do not better utilize tensor low-rank structure so they suffer worse dimensional dependencies than our algorithms. These findings are further backed up by our numerical experiments.

Table 1: Regret bounds of different algorithms. Here $n$ is the time horizon and $p = \max\{p_1, \ldots, p_d\}$ is the maximum tensor dimension with $d$ the order of the mean reward tensor. We consider a higher-order tensor case where $d \geq 3$ and assume the maximum tensor rank $r$ is of constant order, and use $\tilde{O}$ to ignore logarithmic factors.

| Algorithm           | Regret bound                                                                 |
|---------------------|------------------------------------------------------------------------------|
| tensor epoch-greedy | $\tilde{O}(p^{d/2} + p^{(d+1)/3}n^{2/3})$                                   |
| tensor elimination  | $\tilde{O}(p^{d/2} + p^{(d-1)/2}n^{1/2})$                                   |
| vectorized UCB      | $\tilde{O}(p^d + p^{d/2}n^{1/2})$                                           |
| matricized ESTR     | $\tilde{O}(p^{d-1} + p^{(d-1)/2}n^{1/2})$                                   |

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| matricized ESTR     | $\tilde{O}(p^{d-1} + p^{(d-1)/2}n^{1/2})$                                   |
choosing by the nature and the other dimension choosing by the agent. They derived a logarithmic regret under a constant gap assumption. [Lu et al., 2018] utilized ensemble sampling for low-rank matrix bandits but did not provide any regret guarantee due to the theoretical challenges in handling sampling-based exploration.

In addition, Kotłowski and Neu [2019] studied bandit principle component analysis in an adversarial setting. Lale et al. [2019], Hamidi et al. [2019] also assumed low-rank structure but in a different stochastic linear contextual bandit framework. To the best of our knowledge, there is no existing work considering tensor bandits.

Our contributions. First, our low-rank tensor bandits model is the first one to address multi-dimensional online decision-making with tensor actions. Second, it is well known that many efficient methods for matrix data, such as nuclear norm minimization or singular value decomposition, are not directly applicable in the tensor framework [Richard and Montanari, 2014, Yuan and Zhang, 2016, 2017, Friedland and Lim, 2017, Zhang and Xia, 2018a], and thus a simple extension of matrix bandits to the tensor case may not be ideal. To address this, we propose two efficient algorithms tailored for low-rank tensor bandits. Third, we analyze the worse-case cumulative regrets of the proposed algorithms and show their superior performance over existing competitors. Our results also shed lights on the role of dimensionality in the high-dimensional bandit problem.

2 Preliminary

We first introduce some useful notations used throughout the paper and then review the relevant operations for tensors. Interested readers are referred to Kolda and Bader [2009] for more details.

The cardinality of a set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. For integer $k \geq 1$, we denote $[k] = \{1, 2, \ldots, k\}$. We denote a basis vector as $e_j \in \mathbb{R}^p$ that has 1 on its $j$-th coordinate and 0 anywhere else. For a vector $a \in \mathbb{R}^d$, we denote $a_{1:s}$ as its first $s$ coordinate and $a_{(s+1):d}$ as its last $(d-s)$ coordinate. For a scalar $x$, we write $|x| = \min \{z \in \mathbb{N}^+: z \geq x\}.$

**Tensor operations.** For a order-$d$ tensor $X \in \mathbb{R}^{p_1 \times \cdots \times p_d}$, define its mode-$j$ fibers as the $p_j$ dimensional vectors $(X_{i_1,\ldots,i_{j-1},1,i_{j+1},\ldots,i_d})$, and mode-$j$ matricization of $X$ as $M_j(X) \in \mathbb{R}^{p_1 \times (p_1 \cdots \times p_{j-1} p_{j+1} \cdots p_d)}$, where column vectors of $M_j(X)$ are the mode-$j$ fiber of $X$. For instance, the mode-1 matricization $M_1(X) \in \mathbb{R}^{p_1 \times (p_2 p_3)}$ of a order-3 tensor $X$ is defined as

$$[M_1(X)]_{i,j,1:p_2+1} = X_{i,j,1:k}, \quad (2.1)$$

for all $i \in [p_1], j \in [p_2], k \in [p_3]$. For a tensor $X \in \mathbb{R}^{p_1 \times \cdots \times p_d}$ and a matrix $Y \in \mathbb{R}^{r_1 \times p_1}$, we define the marginal multiplication $X \times_1 Y : \mathbb{R}^{p_1 \times \cdots \times p_d} \times \mathbb{R}^{r_1 \times p_1} \rightarrow \mathbb{R}^{r_1 \times p_2 \times \cdots \times p_d}$ as

$$X \times_1 Y = \left( \sum_{t_1=1}^{p_1} X_{t_1, t_2, \ldots, t_d} Y_{t_1, t'_1} \right)_{1 \leq t_1 \leq r_1, 1 \leq t_2 \leq p_2, \ldots, 1 \leq t_d \leq p_d}. \quad (2.2)$$

Marginal multiplications $\times_2, \ldots, \times_d$ can be defined similarly. For two same-sized tensors $X, Y$, we define the tensor inner product as

$$\langle X, Y \rangle = \sum_{i_1=1}^{p_1} \cdots \sum_{i_d=1}^{p_d} X_{i_1, \ldots, i_d} Y_{i_1, \ldots, i_d}.$$

In addition, the tensor Frobenius norm and the element-wise max norm are defined as

$$||X||_F = \sqrt{\langle X, X \rangle}, \quad ||X||_\infty = \max_{i_1, \ldots, i_d} |X_{i_1, \ldots, i_d}|.$$ 

**Tensor decomposition.** Let $r_j$ be the rank of the matrix $M_j(X)$ for $j \in [d]$. The tensor Tucker rank of $X$ is the triplet: $(r_1, \ldots, r_d)$. Denote $r_{\text{max}} = \max(r_1, \ldots, r_d)$. Let $U_1 \in \mathbb{R}^{p_1 \times r_1}, \ldots, U_d \in \mathbb{R}^{p_d \times r_d}$ be the matrices whose columns are the left singular vectors of $M_1(X), \ldots, M_d(X)$, respectively. Then there exists a core tensor $S \in \mathbb{R}^{r_1 \times \cdots \times r_d}$ such that

$$X = S \times_1 U_1 \cdots \times_d U_d, \quad (2.3)$$
We propose two algorithms: tensor epoch-greedy with $P$.

As observed in various recommendation problems, the true reward tensor can usually be approximated well by a low-rank tensor [Frolov and Oseledets, 2017, Bi et al., 2018, Song et al., 2019]. In this case, we need to pull each arm at least once. Define an initial set of rounds:

$$
E_1 = \{ t | t \in [s_1] \},
$$

(3.3)

where $s_1 = \lceil C_0 P_\text{max}^{-2}(p_1 \ldots p_d)^{1/2} \rceil$ with some constant $C_0$. For simplicity, in the initialization phase we consider uniformly random sampling/exploration in tensor bandits. This is equivalent to the case with $\mathbb{P}(i_{jt} = k) = 1/p_j$, for any $k \in [p_j]$ in Eq. (3.1).

**Remark 3.1.** If some prior knowledge about the true reward tensor is available or some tensor modes are selected by the nature rather than by the agent, we could consider some more efficient non-uniform samplings [Klopp et al., 2014].

**Remark 3.2.** In theory, the choice of $s_1$ is the minimal sample complexity for exactly recovering a low-rank tensor in the noiseless case with polynomial-time algorithm [Xia and Yuan, 2019]. If one naively unfolds a tensor into a long vector, the tensor bandits problem will reduce to a standard multi-armed bandit problem. In this case, we need to pull $p_1 \ldots p_d$ times in the initialization which is significantly larger than $s_1$.

### 3.1 Tensor epoch-greedy

The tensor epoch-greedy proceeds in phases: initialization phase, exploration phase and exploitation phase. The initialization phase is only executed once in the beginning while the exploration
stage and exploitation stage alternatively switch. The switching time is guided by the accuracy of the estimation of the expected reward tensor, i.e., the more accurate we estimate the expected reward tensor, the more we exploit. We describe these two phases as follows.

- **Exploration phase.** We prescribe an exploration index set:

  \[ E_2 = \left\{ s_1 + l + 1 + \sum_{k=0}^{l} s_{2k} \mid l = 0, 1, \ldots \right\}. \tag{3.4} \]

  Here, \( s_{2k} \) is an increasing sequence. Theoretically, \( s_{2k} \) will be chosen inversely proportional to the tensor estimation error, see Eq. (4.2). For \( t \in E_2 \), we execute random explorations regardless of the past information.

- **Exploitation phase.** For \( t \notin E_1 \cup E_2 \), we query the optimal low-rank tensor completion algorithm (Algorithm 1 in [Xia et al., 2020], see Appendix C.2 for details) based on samples collected from the initialization phase and the exploration phase so far to construct an estimator \( \hat{X}_t \). Then we take the greedy action to select the arm

  \[(i_{1t}, \ldots, i_{dt}) = \arg\max_{i_1, \ldots, i_d} \langle \hat{X}_t, e_{i_1} \circ \cdots \circ e_{i_d} \rangle.\]

The full tensor epoch-greedy algorithm is described in Algorithm 1.

**Algorithm 1 Tensor epoch-greedy**

1: **Input:** prescribed initial set \( E_1 \), exploration set \( E_2 \).
2: Initialize \( D = \emptyset \).
3: for \( t = 1, 2, \ldots, n \) do
4:  if \( t \in E_1 \cup E_2 \) then
5:      Randomly pull arm \( A_t \) and receive rewards \( y_t = \langle X, A_t \rangle + \epsilon_t \).
6:      Let \( D = D \cup \{(y_t, A_t)\} \).
7:  end if
8:  if \( t \notin E_1 \cup E_2 \) then
9:      Calculate the low-rank tensor estimator \( \hat{X}_t \) based on \( D \).
10:    Execute the greedy action: \( (i_{1t}, \ldots, i_{dt}) = \arg\max_{i_1, \ldots, i_d} \langle \hat{X}_t, e_{i_1} \circ \cdots \circ e_{i_d} \rangle \).
11:    Receive the reward: \( y_t = \langle X, e_{i_{1t}} \circ \cdots \circ e_{i_{dt}} \rangle + \epsilon_t \).
12:  end if
13: end for

3.2 Tensor elimination

Different from tensor epoch-greedy, the tensor elimination algorithm proceeds sequentially of three phases: initialization phase, exploration phase, and reduction phase. We describe the exploration phase and the reduction phase as follows:

- **Exploration phase.** This step executes random explorations with length \( n_1 \) that will be specified later. Based on the data collected in the initialization phase and exploration phase, we apply the low-rank tensor completion algorithm (Algorithm 1 in [Xia et al., 2020], see Appendix C.2 for details) to estimate the left singular vectors \( \hat{U}_1, \ldots, \hat{U}_d \) of the mean reward tensor.

- **Reduction phase.** We write \( \hat{U}_{1\perp}, \ldots, \hat{U}_{d\perp} \) as the orthogonal basis of the complement subspaces of \( \hat{U}_1, \ldots, \hat{U}_d \) and construct a rotated mean reward tensor as

  \[ Y = X \times_{1} [\hat{U}_1 \hat{U}_{1\perp}] \cdots \times_{d} [\hat{U}_d \hat{U}_{d\perp}] \in \mathbb{R}^{p_1 \times \cdots \times p_d}, \]

  where \( \times_1 \) is the marginal multiplication defined in Eq. (2.2). As shown in Appendix C.1, the tensor bandits in Eq. (3.1) is equivalent to

  \[ y_t = \langle Y, [\hat{U}_1 \hat{U}_{1\perp}]^\top e_{i_{1t}} \circ \cdots \circ [\hat{U}_d \hat{U}_{d\perp}]^\top e_{i_{dt}} \rangle + \epsilon_t. \]
Define $\text{vec}(\mathcal{X}) \in \mathbb{R}^{p_1 \cdots p_d}$ as a vectorized version of tensor $\mathcal{X}$, where the last $\Pi^d_{j=1}(p_j - r_j)$ dimensions of $\text{vec}(\mathcal{X})$ are $\mathcal{X}_{i_1, \ldots, i_d}$ for $i_j \in \{r_j + 1, \ldots, p_j\}$. We also define $\beta = \text{vec}(\mathcal{Y})$ and the corresponding vectorized action set

$$
\mathcal{A} := \left\{ \text{vec} \left( \left[ \mathcal{U}_1 \mathcal{U}_1 \right]^\top e_{i_1} \cdots \left[ \mathcal{U}_d \mathcal{U}_{d,\perp} \right] ^\top e_{i_d} \right), i_1 \in \{p_1\}, \ldots, i_d \in \{p_d\} \right\}.
$$

Therefore, we argue that the tensor bandits in Eq. (3.1) with mean reward tensor $\mathcal{X}$ and action set $\{e_{i_1}, \ldots, e_{i_d}\}$ are equivalent to the stochastic linear bandits with mean reward vector $\beta$ and action set $\mathcal{A}$ with finitely many arm: $y_t = \langle A_t, \beta \rangle + \epsilon_t$, for $A_t \in \mathcal{A}$. In the end, we adapt an elimination-based algorithm in [Valko et al., 2014] for stochastic linear bandits with finitely many arms.

The full tensor elimination algorithm is described in Algorithm 2.

**Algorithm 2 Tensor elimination**

1: **Input:** time horizon $n$, the exploration length $n_1$, regularization parameters $\lambda_1, \lambda_2$, length of confidence interval $\xi$;
2: **Initialize:** $D = \emptyset$. 
3: **for** $t = 1, \ldots, s_1 + n_1$ **do**
4: Randomly pull an arm $A_t$ and receive a reward $y_t = \langle \mathcal{X}, A_t \rangle + \epsilon_t$.
5: Let $D = D \cup \{(y_t, A_t)\}$.
6: **end for**
7: Obtain $\hat{U}_1, \ldots, \hat{U}_d$ based on $D$ and find $\hat{U}_1, \ldots, \hat{U}_d$ as the orthogonal basis of their complement subspaces.
8: **for** $k = 1$ to $\lfloor \log_2(n) \rfloor$ **do**
9: Set $V_{2^k-1} = \text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2)$ and $D = \emptyset$.
10: **for** $t = 2^{k-1}$ to $\min(2^k - 1, n - n_1 - s_1)$ **do**
11: Pull the arm $A_t \in \mathcal{A}_k$ such that $A_t = \arg\max_{a \in \mathcal{A}_k} \|a\|_{V_{t-1}}$.
12: Receive a reward $y_t = \langle A_t, \beta \rangle + \epsilon_t$ and update $V_{t+1} = V_t + A_t A_t^\top$.
13: Let $D = D \cup \{(y_t, A_t)\}$.
14: **end for**
15: Eliminate sub-optimal arms based on confidence intervals:
16: $$
\mathcal{A}_{k+1} = \left\{ a \in \mathcal{A}_k, \|\hat{\beta}_k, a\| + \|a\|_{V_{t-1}} \geq \max_{a \in \mathcal{A}_k} \left( \langle \hat{\beta}_k, a \rangle - \|a\|_{V_{t-1}}\xi \right) \right\},
$$
17: where
18: $$
\hat{\beta}_k = \arg\min_{\beta} \left\{ \sum_{(y_t, A_t) \in D} \left( y_t - \langle A_t, \beta \rangle \right)^2 + \lambda_1 \|\beta_{1:q}\|_2 + \lambda_2 \|\beta_{(q+1):d}\|_2 \right\}.
$$

**Remark 3.3.** It is worth to mention that tensor epoch-greedy is an any-time algorithm while tensor elimination requires the knowledge of total rounds $n$ to optimize the length of exploration phase. Moreover, we will discuss how to specify parameters $n_1, \lambda_1, \lambda_2, \xi$ in the next section.

**4 Regret Analysis**

For ease of presentation, we assume the tensor rank $r_1 = \ldots = r_d = r$ and the tensor dimension $p_1 = \ldots = p_d = p$. The general result can be obtained similarly. We first reproduce an estimation error bound for low-rank tensor completion with uniform random design.
Condition 4.1 (Sub-gaussian noise). The noise $\epsilon_t$ is assumed to follow a 1-sub-gaussian distribution such that all $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(\lambda \epsilon_t)] \leq \exp(\lambda^2/2).$$

Lemma 4.2 (Simplified version of Corollary 2 in Xia et al. [2020]). Assume Condition 4.1 holds. Suppose $\hat{X}_T$ is the low-rank tensor estimator constructed from $T$ uniformly random samples by Algorithm 1 in [Xia et al., 2020]. Then for any $\alpha > 1$, if the number of samples $T \geq C_0 \alpha^3 r p \gamma/2$ for sufficiently large constant $C_0$, the following holds with probability at least $1 - p^{-\alpha}$,

$$\frac{\|\hat{X}_T - X\|_F}{\|X\|_F} \leq C_1 \sqrt{\frac{\alpha r p \log(p)}{T}},$$

where $C_1$ is an absolute constant.

4.1 Regret bounds

First, we present the regret bound for tensor epoch-greedy algorithm. The detailed proof for Theorem 4.3 is deferred to Appendix A.1.

Theorem 4.3. Suppose Condition 4.1 holds and $\|X\|_\infty \leq 1$. Assume the mean reward tensor admits a tensor Tucker decomposition defined in Eq. (2.3). We choose $s_{2k}$ in Eq. (3.4) as

$$s_{2k} = \left[ C_2 p^{-d+1/2} r^{-1/2} (\log(p))^{1/2} (k + s_1)^{1/2} \right]$$

for some small constant $C_2$, where $s_1$ is the initialization parameter defined in Eq. (3.3). Suppose the number of rounds $n \geq C_0 r^{d+2} p^{d+2}$. We have the following upper bound on the cumulative regret of Algorithm 1,

$$R_n \leq C_0 r^{d+2} p^{d+2} + 8n^{2} p^{d+1/2} (r \log(p))^{1/2},$$

with probability at least $1 - p^{-10}$.

Remark 4.4. The first part of above bound characterizes the regret during the initialization phase and is independent of $n$. It clearly highlights the benefit of exploiting a tensor low-rank structure since unfolding the tensor into a vector or a matrix requires much longer initialization phase. Based on Lemma 4.2, the selection of $s_{2k}$ ensures the number of exploitation per phase is proportional to $1/\|\hat{X}_k - X\|_F$.

Second, we present the regret bound for tensor elimination algorithm. In order to achieve the optimal dependency on time horizon, we choose the length of exploration

$$n_1 = \left\lfloor n \frac{r d}{d^2 + d} \left( \frac{r d}{d^2 + d} \log d/2(p) \right) \right\rfloor,$$

where $\sigma_i$ is the $r$-th singular value of mode-$i$ matricization $M_i(X)$. And the length of confidence interval is chosen as $\xi = 2 \sqrt{\Lambda X_0} / (\beta_{2,d}) + \sqrt{\Lambda X_1} / (\beta_{2,d})^2 + \sqrt{\Lambda X_2} / (\beta_{k+1},p^d)^2$ with $0 < \lambda_1 \leq 1/p^d$ and $\lambda_2 = n/(q \log(1 + n/\lambda_1))$, where $q$ is defined in Eq. (3.6). This choice of $\xi$ is a special case of Lemma 7 in Valko et al. [2014].

Theorem 4.5. Suppose Condition 4.1 holds and $\|X\|_\infty \leq 1$. Assume the mean reward tensor admits a tensor Tucker decomposition defined in Eq. (2.3). The cumulative regret of Algorithm 2 with $n_1, \lambda_1, \lambda_2, \xi$ above satisfies,

$$R_n \leq C \left( \frac{d}{r} p^{d} + \left( \frac{r d}{d^2 + d} \log d/2(p) \right) \frac{2}{d^2 + d} p^{d+2} + \sqrt{(d \log(p) + \log(n)^2 p^{d-1} n) \right),$$

with probability at least $1 - dp^{-10} - 1/n$. Here $C > 0$ is some absolute constant. Ignoring any logarithmic and constant factor, the above bound can be simplified to

$$R_n \leq \tilde{O}(r^{d} p^{d} + r^{d+2} p^{d+2} + \frac{d}{d^2 + d} p^{d+2} + \frac{d}{d^2 + d} p^{d+2} n^{d+2} + p^{d-1} n^{d+2}).$$
The detailed proof of Theorem 4.5 is deferred to Appendix A.2.

**Remark 4.6.** It is worth to compare the leading term of regret bounds for high-order tensor bandits of tensor elimination in Eq. (4.4) and tensor epoch-greedy in Eq. (4.3). As summarized in Table 1, when \( d \geq 3 \) and \( r = \mathcal{O}(1) \), tensor elimination suffers \( \mathcal{O}(p^{(d-1)/2} \sqrt{n}) \) regret while tensor epoch-greedy suffers \( \mathcal{O}(p^{(d+1)/3} n^{2/3}) \) regret. Although the latter one has a sub-optimal dependency on the horizon due to the \( \varepsilon \)-greedy paradigm, it enjoys a better regret than the prior one in the high-dimensional regime (\( n \leq p^{d-5} \)).

**Remark 4.7.** One may wonder whether standard LinUCB [Abbasi-Yadkori et al., 2011] algorithm could be queried to handle the reshaped linear bandits as did in the matrix bandits Jun et al. [2019]. However, it is known that the analysis of LinUCB is suboptimal for linear bandits with finitely many arms and the sub-optimality will be amplified as the order of tensor grows. Hence, using LinUCB in the reduction phase results in \( \mathcal{O}(p^{d-1} n^{1/2}) \) for the leading term that is even worse than vectorized UCB.

5 Numerical Experiment

We carry out some preliminary experiments to compare the numerical performance of tensor epoch-greedy and tensor elimination with two competitive methods: vectorized UCB which unfolds the tensor into a long vector and then implements standard UCB1 [Auer, 2002] for multi-armed bandits, and matricized ESTR [Jun et al., 2019] which unfolds the tensor into a matrix along an arbitrary mode and implements ESTR for low-rank matrix bandits. We defer implementation details of four algorithms to Appendix E.

We first describe the way to generate an order-three mean reward tensor (\( d = 3 \)) according to Tucker decomposition in Eq. (2.3). The tensor dimensions are set to be same, i.e., \( p_1 = p_2 = p_3 = p \). The triplet of tensor Tucker rank is fixed to be \( r_1 = r_2 = r_3 = r = 2 \). Denote \( \tilde{U}_j \in \mathbb{R}^{p_j \times r_j} \) as i.i.d standard Gaussian matrices. Then we apply QR decomposition on \( \tilde{U}_j \), and assign the Q part as the singular vectors \( U_j \). The core tensor \( S \in \mathbb{R}^{r \times r \times r} \) is constructed as a diagonal tensor with \( S_{ii} = w p^{1.5} \), for \( 1 \leq i \leq r \). Here, \( wp^{1.5} \) indicates the signal strength [Zhang and Xia, 2018b]. The random noise \( \epsilon_t \) is generated i.i.d from a standard Gaussian distribution.

In Figure 2, we report the cumulative regrets of all four algorithms for the setting with \( w = 0.5 \). Additional experiments with \( w = 0.8 \) are included in Figure 3 of Appendix D. All the results are based on 30 replications. Figure 2 shows that tensor elimination outperforms all other methods for a long time horizon while tensor epoch-greedy is more competitive for small time horizon. When the tensor dimension \( p \) increases, the advantage of tensor epoch-greedy in early stage is more apparent. These results align with our theoretical findings.

![Figure 2: Cumulative regrets for four methods. The shaded areas represent the confidence bands.](image)
6 Conclusion

In this paper, we propose a novel stochastic low-rank tensor bandits model and solve it by two efficient algorithms. The regret bounds reflect the benefit of considering low-rank structure in different regimes. As a future work, it is interesting to see if a fully adaptive algorithm like UCB or Thompson sampling can solve tensor bandits. This may rely on a novel construction of confidence intervals or posterior distribution for tensor completion estimator based on adaptive data. Another direction is to study the minimax lower bound of tensor bandits, which is not covered in literature even for the matrix case.

Broader Impact Our work tends to provide a new solution for the multi-dimensional online decision making problem that appears frequently in modern online recommendation and digital marketing. Our algorithms are beneficial to marketing practitioners and our theoretical analysis pushes the boundaries of bandits theory. The ethical aspects may not be applicable for our work.

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In the appendix, we provide detailed proofs of Theorems 4.3 and 4.5 in Appendix A, proof of the main lemma in Appendix B, the equivalent formulation of tensor bandits in Appendix C.1, the algorithm for tensor completion in Appendix C.2, additional numerical experiments in Appendix D, and the implementation details in Appendix E.

A Proofs of Main Theorems

A.1 Proof of Theorem 4.3

The proof uses the trick that couples epoch-greedy algorithm with explore-then-commit algorithm with an optimal tuning.

Step 1. We decompose the pseudo regret defined in Eq. (3.2) as:

\[ R_n = \sum_{t=1}^{n} \langle A^* - A_t, \mathcal{X} \rangle \]

\[ = \sum_{t=1}^{s_1} \langle A^* - A_t, \mathcal{X} \rangle + \sum_{t=s_1+1}^{n} \langle A^* - A_t, \mathcal{X} \rangle, \]

where \( s_1 \) is the number of initialization steps. After initialization phase, from the definition of exploration time index set in Eq. (3.4), the algorithm actually proceeds in phases and each phase contains \((1 + \lceil s_2k \rceil)\) steps: 1 step random exploration plus \( \lceil s_2k \rceil \) steps greedy actions. When the greedy action \( A_t \) is taken at phase \( k \), it holds that

\[ \langle A^* - A_t, \mathcal{X} \rangle \leq \langle A^* - A_t, \mathcal{X} - \hat{\mathcal{X}}_{k+s_1} \rangle, \]

since \( \langle A_t - A^*, \hat{\mathcal{X}}_{k+s_1} \rangle \geq 0 \) and \( \hat{\mathcal{X}}_{k+s_1} \) is the low-rank tensor completion estimator at phase \( k \) based on \((k + s_1)\) random samples. By the choice of \( s_2k \) in Eq. (4.2), it is sufficient to guarantee

\[ \| \hat{\mathcal{X}}_{k+s_1} - \mathcal{X} \|_F \leq 1/s_2k, \]

holds with probability at least \( 1 - p^{-\alpha} \) from Lemma 4.2 for any \( \alpha > 1 \). By the Cauchy-Schwarz inequality, we have

\[ \langle A^* - A_t, \mathcal{X} \rangle \leq \| A^* - A_t \|_F \| \hat{\mathcal{X}}_{k+s_1} - \mathcal{X} \|_F \leq 2/s_2k, \]

where for the second inequality we use the fact that both tensors \( A^* \) and \( A_t \) have only one entry equal to 1 and others are 0. Denote \( n_2 = n - s_1 \) and \( K^* = \min\{K : \sum_{k=1}^{K} (1 + \lceil s_2k \rceil) \geq n_2\} \). Since we assume \( \| \mathcal{X} \|_{\infty} \leq 1 \), the maximum gap \( \Delta_{\max} \) is bounded by 2. Then we have

\[ R_n \leq s_1 \Delta_{\max} + \sum_{k=1}^{K^*} \left( 1 \cdot \Delta_{\max} + \lceil s_2k \rceil 2/s_2k \right) \leq (s_1 + K^*) \Delta_{\max} + 2K^* \leq 2s_1 + 4K^*, \]

(A1)

with probability at least \( 1 - K^*p^{-\alpha} \).

Step 2. We will derive an upper bound for \( K^* \). Let \( n_2^* = \arg\min_{u \in [0,n_2]} [u + (n_2 - u)/s_{2u}] \). Consider the following two cases.

1. If \( n_2^* \geq K^* \), it is obvious that

\[ K^* \leq n_2^* + (n_2 - n_2^*)/s_{2n_2^*}. \]

2. If \( n_2^* \leq K^* - 1 \), it holds that

\[ \sum_{k=1}^{K^*} s_{2k} \geq \sum_{k=n_2^*}^{K^* - 1} s_{2k} \geq (K^* - n_2^*)s_{2n_2^*}, \]
where the second inequality is from the fact that $s_{2k}$ is monotone increasing. By the definition of $K^*$, it holds that
\[
n_2 - 1 \geq \sum_{k=1}^{K^* - 1} (1 + [s_{2k}]) \geq \sum_{k=1}^{K^* - 1} (1 + s_{2k}) \geq K^* - 1 + (K^* - n_2^*)s_{2n_2^*},
\]
which implies
\[
K^* \leq n_2^* + (n_2 - n_2^*)/s_{2n_2^*}.
\]
Overall, $K^*$ is upper bounded by $n_2^* + (n_2 - n_2^*)/s_{2n_2^*}$.

**Step 3.** From Eq. (A.1), the cumulative regret can be bounded by
\[
R_n \leq 2s_1 + 4 \min_{u \in [0,n_2]} \left( u + (n_2 - u)/s_{2u} \right).
\]
The second term above is essentially the regret for explore-then-comment type algorithm with the optimal tuning for the length of exploration. Plugging the definition of $s_{2u}$ in Eq. (4.2) and letting $u = n/s_{2u}$, we have
\[
K^*/2 \leq n_2^* \leq n_2^{3/2} \frac{d+1}{3} (r \log(p))^{1/3}.
\]
Thus, we choose $\alpha = \log(2n_2^{3/2} p \frac{d+1}{3} (r \log(p))^{1/3})$ such that $K^* p^{-\alpha} \leq 1/p$. Plugging in $s_1 = C_0 r/d^2 p^{d/2}$, we have
\[
R_n \leq C_0 r/d^2 p^{d/2} + 8 \left( n_2^{3/2} p \frac{d+1}{3} (r \log(p))^{1/3} \right),
\]
with probability at least $1 - 1/p$. This ends the proof.

### A.2 Proof of Theorem 4.5

From Lemma 4.2 and the assumption $\|X\|_{\infty} \leq 1$, we know that with probability at least $1 - p^{-10}$,
\[
\|\hat{X}_{n_2} - X\|_F \leq C_1 \sqrt{\frac{p^{d+1} r \log(p)}{n_1}}.
\]

By definitions, $U_i, \hat{U}_i$ are left singular vectors of $M_i(X)$ and $M_i(\hat{X}_{n_2})$, respectively. Here, the matricization operator $M(\cdot)$ is defined in Eq. (2.1). Then we can verify
\[
U_i U_i^T M_i(X) = U_i U_i^T U_i \Sigma V_i^T = U_i \Sigma V_i^T = M_i(X).
\]

Let $\hat{U}_i \perp \in \mathbb{R}^{p \times (p-r)}$ be the orthogonal complement of $\hat{U}_i$ for $i \in [d]$. For an orthogonal matrix $U$ and an arbitrary matrix $X, Y$, we have $\|UX\|_F \leq \|U\|_2 \|X\|_F = \|X\|_F$ and $\|XY\|_F \geq \|X\|_F \sigma_{\min}(Y)$. Suppose $\sigma_i$ is the $i$-th singular value of $M_i(X)$. Using the above fact, we have
\[
\|M_i(\hat{X}_{n_2}) - M_i(X)\|_F \\
\geq \|\hat{U}_i^T (M_i(\hat{X}_{n_2}) - U_i U_i^T M_i(X))\|_F \\
= \|\hat{U}_i^T U_i \Sigma V_i^T - \hat{U}_i^T U_i \Sigma V_i^T\|_F \\
\geq \|\hat{U}_i^T U_i \Sigma V_i^T\|_F \sigma_i (U_i^T M_i(X)) = \|\hat{U}_i^T U_i \Sigma V_i^T\|_F \sigma_i.
\]

Therefore,
\[
\|\hat{U}_i^T U_i \Sigma V_i^T\|_F \leq \frac{\|M_i(X) - M_i(\hat{X}_{n_2})\|_F}{\sigma_i} = \frac{\|X - \hat{X}_{n_2}\|_F}{\sigma_i} \leq \frac{C_1}{\sigma_i} \sqrt{\frac{p^{d+1} r \log(p)}{n_1}}, \quad (A.2)
\]
with probability at least $1 - p^{-\alpha}$. As discussed in Section 3.2, we reformulate original tensor bandits into a stochastic linear bandits with finitely many arms. Recall that $\beta = \text{vec}(\mathcal{Y})$ with
\[
\mathcal{Y} = \mathcal{X} \times_1 [\hat{U}_1 \hat{U}_1 \perp] \cdots \times_d [\hat{U}_d \hat{U}_d \perp] \in \mathbb{R}^{p_1 \times \cdots \times p_d},
\]
and the corresponding action set
\[ \mathcal{A} := \{ \text{vec} \left( \left[ \hat{U}_1 \hat{U}_{1\perp}^\top \right] e_{i_1} \circ \cdots \circ \left[ \hat{U}_d \hat{U}_{d\perp}^\top \right] e_{i_d} \} , i_1 \in [p_1], \ldots, i_d \in [p_d] \} \].

From Eq. (A.2), we have
\[ \| \beta_{(q+1):p^d} \|_2 \leq \prod_{i=1}^{d} \| \hat{U}_{i\perp}^\top U_i \|_F \| S \|_F \]
\[ \leq \frac{\| \hat{X}_{n_1} - X \|_F^d}{\Pi_{i=1}^{d} \sigma_i} \| S \|_F \]
\[ \leq \frac{r_d/2}{\Pi_{i=1}^{d} \sigma_i} C_{1,d/2}^{d/2} p^{d/2+4} \log^{d/2}(p), \quad (A.3) \]
with probability at least \( 1 - dp^{-\alpha} \). Thus it is equivalent to consider the following linear bandit problem:
\[ y_t = (A_t, \beta) + \epsilon_t, \]
where \( \| \beta_{(q+1):p^d} \|_2 \) satisfies Eq. (A.3) and \( A_t \) is pulled from action set \( \mathcal{A} \). To better utilize the information coming from low-rank tensor completion, we present the following regret bound for the elimination-based algorithm for stochastic linear bandits with finitely-many arms. The detailed proof is deferred to Section B.

**Lemma A.1.** Consider the the elimination-based algorithm in Algorithm 2 with \( \lambda_2 = n/(k \log(1 + n/\lambda_1)) \) and \( \lambda_1 > 0 \). With the choice of \( \xi = 2 \sqrt{14 \log(2/\delta)} + \sqrt{\lambda_1} \| \beta_{1:p^d} \|_2 + \sqrt{\lambda_2} \| \beta_{(q+1):p^d} \|_2 \), the upper bound of cumulative regret of \( n \) rounds satisfies
\[ R_n \leq 8 \left( 2 \sqrt{14 \log(2n)p^{d/2}/\delta} + \sqrt{\lambda_1} \| \beta_{1:p^d} \|_2 \right) \left( 2qn \log(1 + \frac{n}{\lambda_1}) + 8 \sqrt{2n} \| \beta_{(q+1):p^d} \|_2 \right) \]
with probability at least \( 1 - \delta \), where \( q = p^d - (p - r)^d \).

Overall, we can decompose the pseudo regret Eq. (3.2) into two parts:
\[ R_n = R_{1n} + R_{2n} + R_{3n}, \]
where \( R_{1n} \) quantifies the regret during initialization phase, \( R_{2n} \) quantifies the regret during exploration phase and \( R_{3n} \) quantifies the regret during commit phase (linear bandits reduction). Note that \( q \leq C_1 p^{d-1} \) for sufficient large \( C_1 \). Denote
\[ \delta_{p,r} = \frac{r_d}{\Pi_{i=1}^{d} \sigma_i} p^{d/2+4} \log^{d/2}(p), \]
such that \( \| \beta_{(q+1):p^d} \|_2 \leq \delta_{p,r}/n_1^{d/2} \) from Eq. (A.3). Applying the result in Lemma A.1 to bound \( R_{3n} \) and properly choosing \( 0 < \lambda_1 \leq 1/p^d \), we have the following holds with probability at least \( 1 - dp^{-10} - 1/n, \)
\[ R_n \leq C \left( \sqrt{R_{1n}^{d/2} p^{d/2}} + n_1 + \delta_{p,r} n_2 / n_1 + \sqrt{\log(\log(n_2))} + \log(n_2) p^{d^2} \right) \]
\[ \leq \sqrt{R_{1n}^{d/2} p^{d/2}} + \sqrt{\log(\log(n_2))} + \log(n_2) p^{d^2} \]
where \( n_2 = n - n_1 - C p^{d/2} p^{d/2} \) and \( C > 0 \) is an universal constant. Here, \( R_{3n} \) is due to the fact that we run elimination-based algorithm for the rest \( n_2 \) rounds. For simplicity, we bound all \( n_2 \) by \( n \) as usually did for the proof of explore-then-commit type algorithm.

We optimize with respect to \( n_1 \) such that
\[ n_1 = \left( n \delta_{p,r} \right)^{2}. \]
It implies the following bound holds with probability at least $1 - dp^{-10} - 1/n,$

$$R_n \leq C \left( r^{d/2} p^{d/2} + \frac{r^d}{\prod_{i=1}^d \sigma_i} \frac{d^{d/2} \log(d/2)(p)}{d^{d/2} \sigma_i} \right)^{2} \frac{2 n^{2} d^{2}}{n^{d/2}} + \sqrt{\log(\log(n)) + \log(np^d) \sqrt{p^{d/2} \log(n)}}$$

$$\leq C \left( r^{d/2} p^{d/2} + \frac{r^d}{\prod_{i=1}^d \sigma_i} \frac{d^{d/2} \log(d/2)(p)}{d^{d/2} \sigma_i} \right)^{2} \frac{2 n^{2} d^{2}}{n^{d/2}} + \sqrt{(d \log(p) + \log(n)) p^{d/2}}.$$ 

This ends the proof. 

\section*{B Proof of Lemma A.1}

The proof mainly follows the proof of Theorem 2 in \cite{Valko et al., 2014} with a special choice of spectral basis. We introduce some notations first. For a vector $x$ and matrix $V,$ we define $\|x\|_V = \sqrt{x^\top V x}$ as the weighted $\ell_2$-norm and $\det(V)$ as its determinant. Let $K = \lfloor \log_2(n) \rfloor$ and $t_k = 2^{k-1}.$ Denote $x^* = \arg\max_{a \in A} \langle a, \beta \rangle.$

We have the following regret decomposition by phases:

$$R_n = \sum_{t=1}^n \langle x^* - A_t, \beta \rangle = \sum_{k=0}^{K} \sum_{t=k}^{t_k-1} \langle x^* - A_t, \beta \rangle$$

$$= \sum_{k=0}^{K} \sum_{t=k}^{t_k-1} \left( \langle x^* - A_t, \hat{\beta}_k \rangle - \langle x^* - A_t, \beta_k - \beta \rangle \right),$$

where $\hat{\beta}_k$ is the ridge estimator only based on the sample collected in the current phase, defined in Eq. (3.8). According to Lemma 7 in \cite{Valko et al., 2014}, for any fixed $x \in \mathbb{R}^p$ and any $\delta > 0,$ we have, at phase $k$,

$$\mathbb{P} \left( |x^\top (\hat{\beta}_k - \beta)| \leq \|x\|_{V_k^{-1}, \xi} \right) \geq 1 - \delta,$$

where $\xi = 2 \sqrt{14 \log(2/\delta) + \sqrt{\lambda_1 \|\beta_1\|_2} + \sqrt{\lambda_2 \|\beta_{q+1}\|_2}}.$ Applying Eq. (B.1) for $x^*$ and $A_t,$ we have with probability at least $1 - K p^d \delta,$

$$R_n \leq \sum_{k=0}^{K} \sum_{t=k}^{t_k-1} \langle x^* - A_t, \hat{\beta}_k \rangle + \sum_{k=0}^{K} \langle x^* - A_t, \beta_k - \beta \rangle \left( \|x^*\|_{V_k^{-1}} + \|A_t\|_{V_k^{-1}} \right) \xi,$$

By step (3.7) in Algorithm 2, we have

$$\langle x^* - A_t, \hat{\beta}_k \rangle \leq \left( \|x^*\|_{V_k^{-1}} + \|A_t\|_{V_k^{-1}} \right) \xi.$$ 

According to Lemma 8 in \cite{Valko et al., 2014}, for all the actions $x \in \mathcal{A}_k$ defined in Eq. (3.7),

$$\|x\|_{V_k^{-1}}^2 \leq \frac{1}{t_k - t_{k-1}} \sum_{t=t_{k-1}+1}^{t_k} \|x_t\|_{V_{t-1}^{-1}}^2.$$ 

Then using the elliptical potential lemma (Lemma 19.4 in \cite{Lattimore and Szepesvári [2018]}), with probability at least $1 - K p^d \delta,$ we have

$$R_n \leq 2 \sum_{k=0}^{K} (t_{k+1} - t_k) \left( \|x^*\|_{V_k^{-1}} + \|A_t\|_{V_k^{-1}} \right) \xi$$

$$\leq 4 \sum_{k=0}^{K} (t_{k+1} - t_k) \sqrt{\frac{1}{t_k - t_{k-1}} \log \left( \frac{\det(V_k)}{\det(A)} \right)} \xi,$$
where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2)$. According to Lemma 5 in [Valko et al., 2014], we have

$$
\log \left( \frac{\det(V_k)}{\det(A)} \right) \leq k \log(1 + \frac{n}{\lambda_1}) + \sum_{i=k+1}^{p^d} \log(1 + \frac{t_i}{\lambda_2}),
$$

where $\sum_{i=k+1}^{p^d} t_i \leq T$. With the choice of $\lambda_2$,

$$
\log \left( \frac{\det(V_k)}{\det(A)} \right) \leq k \log(1 + \frac{n}{\lambda_1}) + \sum_{i=k+1}^{p^d} \frac{t_i}{\lambda_2} \leq 2k \log(1 + \frac{n}{\lambda_1}).
$$

We know that $t_{k+1} - t_k = 2^{k-1}$ and $t_k - t_{k-1} = 2^{k-2}$. Then one can have

$$
\sum_{k=0}^{K} \frac{1}{\sqrt{t_{k+1} - t_k}} = \sum_{k=0}^{K} 2^{k/2} \leq \sqrt{n}.
$$

Overall, with probability at least $1 - Kp^d\delta$, we have

$$
R_n \leq 8\sqrt{2kn \log(1 + \frac{n}{\lambda_1}) \left( 2\sqrt{14\log(2/\delta)} + \sqrt{\lambda_1}\|\theta_{1:k}\|_2 + \sqrt{\lambda_2}\|\theta_{(k+1):p^d}\|_2 \right)}
$$

$$
= 8 \left( 2\sqrt{14\log(2\log(n)p^d/\delta)} + \sqrt{\lambda_1}\|\theta_{1:k}\|_2 \right) \sqrt{2kn \log(1 + \frac{n}{\lambda_1})} + 8\sqrt{2n\|\theta_{(k+1):p^d}\|_2}.
$$

This ends the proof. 

C Auxiliary Results

C.1 An equivalent formulation of tensor bandits

We write $\hat{U}_{1,\perp}, \ldots, \hat{U}_{d,\perp}$ as the orthogonal basis of the complement subspaces of $\hat{U}_1, \ldots, \hat{U}_d$. By definitions, $[\hat{U}_j \hat{U}_{j,\perp}]$ is an orthogonal matrix for all $j \in [d]$ such that $[\hat{U}_j \hat{U}_{j,\perp}] [\hat{U}_j \hat{U}_{j,\perp}]^\top = [\hat{U}_j \hat{U}_{j,\perp}]^\top [\hat{U}_j \hat{U}_{j,\perp}] = I_{d \times d}$.

Denote a rotated mean reward tensor as $Y = \mathcal{X} \times_1 [\hat{U}_1 \hat{U}_{1,\perp}] \cdots \times_d [\hat{U}_d \hat{U}_{d,\perp}] \in \mathbb{R}^{p_1 \times \cdots \times p_d}$, where $\times_1$ is the marginal multiplication defined in Eq. (2.2). Denote $\mathcal{E}_1 = [\hat{U}_1 \hat{U}_{1,\perp}]^\top e_{i_1} \odot \cdots \odot [\hat{U}_d \hat{U}_{d,\perp}]^\top e_{i_d}, \mathcal{E}_2 = e_{i_1} \odot \cdots \odot e_{i_d}$.

We want to prove

$$
\langle Y, \mathcal{E}_1 \rangle = \langle \mathcal{X}, \mathcal{E}_2 \rangle.
$$

To see this, we use a fact of the Kronecker product (see details in Section 2.6 in [Kolda and Bader, 2009]). Let $Z_1 \in \mathbb{R}^{l_1 \times \cdots \times l_N}$ and $A^{(n)} \in \mathbb{R}^{l_n \times l_n}$ for all $n \in [N]$. Then, for any $n \in [N]$, we have

$$
\mathcal{M}_n(Z_2) = A^{(n)} \mathcal{M}_n(Z_1) \left( A^{(N)} \otimes \cdots \otimes A^{(n+1)} \otimes A^{(n-1)} \otimes \cdots \otimes A^{(1)} \right)^\top,
$$

where $\mathcal{M}_n(Z)$ is the mode-$n$ matricization and $\otimes$ is a Kronecker product. Denote $H = [\hat{U}_2 \hat{U}_{2,\perp}] \otimes \cdots \otimes [\hat{U}_d \hat{U}_{d,\perp}]$. By a matricization of $Y, \mathcal{E}$ along the first mode, we have

$$
\langle Y, \mathcal{E}_1 \rangle = \langle \mathcal{M}_1(Y), \mathcal{M}_1(\mathcal{E}_1) \rangle
$$

$$
= \langle [\hat{U}_1 \hat{U}_{1,\perp}], \mathcal{M}_1(X)H^\top, [\hat{U}_1 \hat{U}_{1,\perp}], \mathcal{M}_1(\mathcal{E}_2)H^\top \rangle
$$

$$
= \text{trace} \left( H \mathcal{M}_1(\mathcal{X})^\top [\hat{U}_1 \hat{U}_{1,\perp}]^\top [\hat{U}_1 \hat{U}_{1,\perp}], \mathcal{M}_1(\mathcal{E}_2)H^\top \right)
$$

$$
= \text{trace} \left( H \mathcal{M}_1(\mathcal{X})^\top \mathcal{M}_1(\mathcal{E}_2)H^\top \right)
$$

$$
= \langle \mathcal{X} \times_1 I_{d \times d} \times_2 [\hat{U}_2 \hat{U}_{2,\perp}] \cdots \times_d [\hat{U}_d \hat{U}_{d,\perp}], e_{i_1} \odot \cdots \odot [\hat{U}_d \hat{U}_{d,\perp}]^\top e_{i_1} \rangle.
$$

Recursively using the above arguments along each mode, we reach our conclusion.
C.2 Algorithm for tensor completion

For the sake of completeness, we replicate the tensor completion algorithm (Algorithm 1 in [Xia et al., 2020]). The goal is to estimate the true tensor $X \in \mathbb{R}^{p_1 \times \ldots \times p_d}$ from

$$y_t = \langle X, A_t \rangle + \epsilon_t, \quad t = 1, \ldots, T,$$

where $A_t = e_{i_1} \circ \ldots \circ e_{i_d}$. This is a standard tensor completion formulation with uniformly random missing data. The algorithm consists of two stages: spectral initialization and power iteration.

**Spectral initialization.** We first construct an unbiased estimator $X_{ini}$ for $X$ as follows:

$$X_{ini} = \frac{1}{p_1 \cdots p_d} \sum_{t=1}^{n} y_t A_t.$$

For each $j \in [d]$, we construct the following $U$-statistic:

$$\hat{R}_j = \frac{(p_1 \cdots p_d)^2}{T(T - 1)} \sum_{1 \leq t \neq t' \leq T} y_t y_{t'} M_j(A_t) M_j(A_{t'})^\top,$$

where $M_j$ is the mode-$j$ matricization defined in Eq. (2.1). Compute the eigenvectors of $\{\hat{R}_j\}_{j=1}^d$ with eigenvalues greater than $\delta$, where $\delta$ is a tuning parameter, and denote them by $\{\hat{U}^{(0)}_j\}_{j=1}^d$.

**Power iteration.** Given $\{\hat{U}^{(l-1)}_j\}_{j=1}^d$, $X_{ini}$ can be denoised via projections to $j$-th mode. For $l = 1, 2, \ldots$, we alternatively update $\{\hat{U}^{(l)}_j\}_{j=1}^d$ as follows,

$$\hat{U}^{(l)}_j = \text{first } r_j \text{ left singular vectors of } M_j \left( X_{ini} \times_{j'} (\hat{U}^{(l-1)}_{j'})^\top \times_{j''} (\hat{U}^{(l-1)}_{j''})^\top \right).$$

The iteration is stopped when either the increment is no more than the tolerance $\epsilon$, i.e.,

$$\|X_{ini} \times_1 (\hat{U}^{(l)}_1)^\top \times_d (\hat{U}^{(l)}_d)^\top\|_F - \|X_{ini} \times_1 (\hat{U}^{(l-1)}_1)^\top \times_d (\hat{U}^{(l-1)}_d)^\top\|_F \leq \epsilon,$$

or the maximum number of iterations is reached. With the final estimates $\hat{U}_1, \ldots, \hat{U}_d$, it is natural to estimate $S$ and $X$ as

$$\hat{S} = X_{ini} \times_1 \hat{U}_1^\top \times_d \hat{U}_d^\top, \hat{X} = \hat{S} \times_1 \hat{U}_1 \cdots \times_d \hat{U}_d.$$

D Additional Experiments

![Figure 3: Additional simulations with a stronger signal strength $w = 0.8$.](image)
E Implementation Details

Note that all algorithms involve some hyper-parameters, such as the length of confidence interval and the round of pure exploration. In this section, we discuss the choice of hyper-parameters for tensor elimination, tensor epoch-greedy, matricized ESTR and vectorized UCB. For a fair comparison, we do a grid search of some hyper-parameters and report the ones with lowest cumulative regret for each algorithm.

Both matricized ESTR and tensor elimination are optimism-based algorithms that utilize confidence interval. In practice, the theoretically suggested confidence interval, i.e. Theorem 4.5, may be quite conservative. So in this paper, we set a base $\xi$ according to its theoretical result and find the best multiplier $c$ by grid search from $\{0.01, 0.05, 0.1, 0.5\}$. For both algorithms, the initial exploration length is set to be $c_0 n_1$ where $n_1$ follows the theoretical value. We tune the unknown constant by varying $c_0 \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. For tensor epoch-greedy, we set $C_0 = 1$ for the initialization phase length $s_1$ defined in (3.3) and tune $C_2 \in \{1, 5, 10, 20\}$ for the exploitation parameter $s_2k$ defined in (4.2). We set the ridge regularization parameter $\lambda_1 = 0.1$ for both matricized ESTR and tensor elimination.