General Solution of 7D Octonionic Top Equation

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ABSTRACT

The general solution of a 7D analogue of the 3D Euler top equation is shown to be given by an integration over a Riemann surface with genus 9. The 7D model is derived from the 8D Spin(7) invariant self-dual Yang-Mills equation depending only upon one variable and is regarded as a model describing self-dual membrane instantons. Several integrable reductions of the 7D top to lower target space dimensions are discussed and one of them gives 6, 5, 4D descendants and the 3D Euler top associated with Riemann surfaces with genus 6, 5, 2 and 1, respectively.
1 Introduction

The relevance and the importance of the algebra $\mathcal{O}$ of octonions in physics have been discussed by many authors, together with other three division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ of real, complex and quaternionic numbers. See, for example, [1][2][3] and references therein.

In Ref.[1], the 4D (anti-)self-dual Yang-Mills equation was generalized to higher-dimensional linear relations for the field strength $F_{\mu\nu}$, which lead to the full Yang-Mills equation, via the Bianchi identity. Among various examples noted in [1], an interesting case arises in eight dimensions, which is invariant under the transformation by a maximal subgroup $\text{Spin}(7)$ of the rotation group $SO(8)$ and can be expressed by using the octonionic structure constants $c_{ijk}$ ($i, j, k = 1, \cdots, 7$),

$$F_{8i} = \frac{1}{2} c_{ijk} F_{jk} .$$

(1)

Recalling the fact that the 4D self-dual equation can be cast in the form of (1) with the quaternionic structure constants $\varepsilon_{ijk}$ instead of $c_{ijk}$, we recognize the 8D equation (1) to be a natural generalization of the usual 4D self-dual one. Several solutions of (1) and its 7D relatives were found in [4][5][6] and were then used to construct string and membrane solitons [7][8][9]. In recent papers [8][9], the 8D equation has been applied to construct a topological Yang-Mills theory on Joyce manifolds as an 8D counterpart of the 4D Donaldson-Witten theory [10].

It is also recently discussed that (self-dual) Yang-Mills gauge fields depending only upon time play a role in the context of M-theory [11][12][13][14]. In the reduction to 1D world-sheet, (1) is modified to the form of the 7D Nahm equations, with the gauge condition $A_8 = 0$,

$$\frac{d}{dt} A_i(t) = \frac{1}{2} c_{ijk} [A_j(t), A_k(t)] .$$

(2)

The commutator in the right-hand-side (RHS) can be replaced by the Poisson bracket (P.B.) if we take the gauge group of the Yang-Mills to be the infinite-dimensional group of area-preserving diffeomorphisms $SDiff(M)$ on a 2D surface $M$,

$$\frac{d}{dt} A_i(t, \sigma, \tau) = \frac{1}{2} c_{ijk} \{A_j(t, \sigma, \tau), A_k(t, \sigma, \tau)\} ,$$

(3)

where matrix indices in (2) are Fourier-transformed to the coordinates $(\sigma, \tau)$ on $M$. It may be also worthwhile to consider the Nahm equations defined with the Moyal bracket [11][12], which is essentially equivalent to the commutator of $SU(N)$ matrices and reduces to the P.B. in the limit $N \to \infty$. 

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The equation (3) was proposed as an ansatz for self-dual membrane instantons embedded in seven space dimensions [11]. Iterating (3), with the identity for $c_{ijk}$,

$$c_{ikp}c_{jlp} = \delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj} + T_{ikjl},$$

(4)

where $T_{ikjl} = \frac{1}{3!}\varepsilon_{ikjhlmn}c_{hmn}$, we obtain a second order equation for $A_i$,

$$\frac{d^2}{dt^2}A_i = -\{\{A_i, A_j\}, A_j\}, \quad (j \text{ summed})$$

(5)

which arises from the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\dot{A}_i)^2 + \frac{1}{4}\{A_i, A_j\}^2.$$  

(6)

The Lagrangian is given by a sum of the square of the equation (3) up to a total derivative term and solutions of (3) satisfy the Bogomol'nyi bound in the theory. A similar Bogomol'nyi property holds for the commutator case (2).

In Ref.[15], Fairlie and the author investigated the integrability of the 7D Nahm equations (2) with an ansatz $A_i(t) = \omega_i(t)e_i$ ($i$ not summed), where $e_i$ are 7 unit imaginary octonions. With the use of the ansatz, (2) is reduced to a 7D analogue of the familiar Euler equation for a 3D top,

$$\dot{\omega}_i(t) = \frac{1}{2} c^2_{ijk} \omega_j(t)\omega_k(t).$$

(7)

The equation is also obtained from the P.B. case (3) by choosing an appropriate basis of functions on the surface $\mathcal{M}$ [13] and hence, gives self-dual membrane instantons. It was shown that the 7D top has 6 independent conservation laws to ensure the full integrability of the system [13], although it remained to find its explicit integral solutions.

In this paper, we give another simplified proof of the integrability of the 7D top and show that its general solution is given by an integration over a Riemann surface with genus 9. We discuss the procedure to reduce the 7D top to lower target space dimensions while keeping the integrability and obtain its lower-dimensional integrable descendants. A reduction is demonstrated to yield 6, 5, 4D models and the 3D Euler top corresponding to Riemann surfaces with genus 6, 5, 2 and 1, respectively.

### 2 3D Euler top equation

Before going to the 7D top, let us briefly review the 3D Nahm equations for self-dual fields in four dimensions,

$$\frac{dA_i}{dt} = \frac{1}{2}\varepsilon_{ijk}[A_j, A_k].$$

(8)
The set of non-denegerate linear transformations making (8) invariant is the group $SU(2) = Aut(\mathcal{H})$, the group of automorphisms on $\mathcal{H}$. Assuming an ansatz for a solution of (8), with a matrix representation of quaternions ($SU(2)$), e.g. the Pauli matrices $\sigma_i$, 

$$ A_i = \omega_i \sigma_i , \quad (i \text{ not summed}) $$

we obtain the standard 3D (Euclidean) Euler top equations,

$$ \dot{\omega}_1 = \omega_2 \omega_3 , \quad \dot{\omega}_2 = \omega_3 \omega_1 , \quad \dot{\omega}_3 = \omega_1 \omega_2 . $$

The system is integrable, for we have two independent conserved quantities,

$$ \omega_1^2 - \omega_2^2 = c_2 , \quad \omega_1^2 - \omega_3^2 = c_3 . $$

Solving (11) for $\omega_2$ and $\omega_3$ and substituting them into the equation for $\omega_1$, we have

$$ \dot{\omega}_1 = \sqrt{(\omega_1^2 - c_2)(\omega_1^2 - c_3)} . $$

As is well known, the Riemann surface associated with the integration of (12) is of genus $g = 1$ (torus) and the general solution of the 3D top is given by elliptic functions.

We note that the set of symmetry transformations of (8) becomes a finite group with 24 elements, unlike the case of (8). All the elements of the group are generated by the products of 3 generators changing the sign of two of $\omega_i$’s and 6 permutations of $\omega_i$’s. Since, in the latter, there appear 3 permutations with determinant $-1$, the finite group is not a subgroup of $SU(2)$.

### 3 7D top equation

The multiplication rule in the algebra $O$ is specified by the relation among the octonions,

$$ e_i e_j = -\delta_{ij} 1 + c_{ijk} e_k . $$

The exceptional group $G_2$ is nothing but the group $Aut(O)$ of automorphisms on $O$, which preserve the relation (13). The 7D Nahm equations (13) for 8D self-dual fields are invariant under the linear transformation by the group $G_2$.

In this paper, we take the following explicit realization of the totally anti-symmetric $c_{ijk}$,

$$ c_{127} = c_{631} = c_{541} = c_{532} = c_{246} = c_{734} = c_{567} = 1 , \quad \text{ (others zero)} $$
then the relation for $e_i$ can be read off diagrammatically from Fig.1; $e_1e_2 = e_7 = -e_2e_1$, $e_2e_4 = e_6 = -e_4e_2$ etc.. The diagram, the seven-point plane, arises from the projective geometry of a plane over a finite field of characteristic two; 3 points lie on every line and 3 lines pass through each point. The 7D top equation (15) becomes, with this choice of $c_{ijk}$,

\[
\begin{align*}
\frac{d}{dt} \omega_1 &= \omega_2 \omega_7 + \omega_3 \omega_6 + \omega_4 \omega_5 , \\
\frac{d}{dt} \omega_2 &= \omega_7 \omega_1 + \omega_5 \omega_3 + \omega_4 \omega_6 , \\
\frac{d}{dt} \omega_3 &= \omega_1 \omega_6 + \omega_2 \omega_5 + \omega_4 \omega_7 , \\
\frac{d}{dt} \omega_4 &= \omega_1 \omega_5 + \omega_6 \omega_2 + \omega_7 \omega_3 , \\
\frac{d}{dt} \omega_5 &= \omega_4 \omega_1 + \omega_3 \omega_2 + \omega_6 \omega_7 , \\
\frac{d}{dt} \omega_6 &= \omega_3 \omega_1 + \omega_2 \omega_4 + \omega_7 \omega_5 , \\
\frac{d}{dt} \omega_7 &= \omega_1 \omega_2 + \omega_3 \omega_4 + \omega_5 \omega_6 .
\end{align*}
\]

The contributions to $\dot{\omega}_i$ come from the products of the pairs of $\omega$'s associated with the other points on each of the three lines through $i$ in Fig.1.

The $G_2$ invariance of the 7D Nahm equations (2) breaks down after the use of the ansatz $A_i(t) = \omega_i(t) e_i$ and as in the 3D case, the symmetry group of the 7D top (15) becomes finite with 7 generators changing the sign of all variables except for the three $\omega_i$'s which are associated with the points on each of seven lines and 168 permutations keeping the relative structure of points in Fig.1, that is, the permutations by which three points on a line are transformed to those on a line.

It is known that octonions $e_i$ do not have any matrix representation, due to the lack of associativity. However their adjoint-like representation given by $7 \times 7$ matrices

![Figure 1: 7 point plane.](image)
\((C^i)_{jk} = c_{ijk}\) with

\[ C^i = \frac{1}{6} c_{ijk} [C^j, C^k], \quad \text{Tr}C^iC^j = -6\delta_{ij}, \]

works to give the 7D top equation, up to a factor \(\frac{1}{2}\) in the RHS of (14), via the ansatz \(A_i = \omega_i C^i\). A P.B. equation similar to the first one in (16) was given for a basis of functions \(f_i(\sigma, \tau)\) on a surface \(M\) in the context of membrane instantons \([13]\).

A type of special solutions of the 7D top can be obtained by considering the embedding of the 3D Euler top into the 7D one. From the structure of (15), we can recognize that there are 7 ways of embedding, each of which takes three variables \((\omega_i, \omega_j, \omega_k)\) on a line in Fig.1 and makes the other four variables zero. Then the seven equations (15) reduce to the 3D top equations (14).

### 3.1 Integrability

In order to prove the integrability of the 7D top, it is convenient to define the following linear combinations \(a_i(t)\) of seven variables \(\omega_i(t)\),

\[ a_1 = \omega_3 + \omega_4 + \omega_5 + \omega_6, \quad a_2 = \omega_1 + \omega_2 + \omega_5 + \omega_6, \]
\[ a_3 = \omega_1 + \omega_3 + \omega_5 + \omega_7, \quad a_4 = \omega_2 + \omega_4 + \omega_5 + \omega_7, \]
\[ a_5 = \omega_2 + \omega_3 + \omega_6 + \omega_7, \quad a_6 = \omega_1 + \omega_4 + \omega_6 + \omega_7, \]
\[ a_7 = \omega_1 + \omega_2 + \omega_3 + \omega_4, \]

then the equations of motion (15) become

\[ \dot{a}_i = \frac{1}{4} a_i \left( \sum_{j=1}^{7} a_j - 4a_i \right), \quad (i = 1, \cdots, 7) \]

which give the time derivatives for the difference of \(a_i\)'s,

\[ (a_i - a_k) = \frac{1}{4} (a_i - a_k) \left( \sum_{j=1}^{7} a_j - 4a_i - 4a_k \right). \]

We introduce a quantity \(W\) with constants \(\rho_i\) and \(\chi_{ij}\),

\[ W = \sum_i \rho_i \ln a_i + \sum_{i<j} \chi_{ij} \ln (a_i - a_j), \]

then its time derivative is

\[ \dot{W} = \frac{1}{4} \left\{ \left( \sum_i \rho_i + \sum_{i<j} \chi_{ij} \right) \sum_k a_k - 4 \left( \sum_i \rho_i a_i + \sum_{i<j} \chi_{ij} (a_i + a_j) \right) \right\}. \]
The condition $W = 0$ requires that the coefficients of $a_i$’s in the RHS of (21) should be zero. These seven constraints for $\rho_i$ and $\chi_{ij}$ can be solved for $\rho_i$ and $W$ is expressed in terms of 21 constants of motion $N_{ij}$,

$$W = \sum_{i<j} \chi_{ij} \ln N_{ij}, \quad N_{ij} = (\prod_{k=1}^{7} a_k)^{1/7}(a_i - a_j)/a_ia_j,$$

(22)

although not all of $N_{ij}$ are independent,

$$N_{ij} = N_{1i} - N_{1j}. \quad (23)$$

This relation shows that the set of six $N_{1j}$ ($j \neq 1$) becomes a basis of conserved quantities in the 7D top.

We can prove $N_{1j}$ to be functionally independent, which ensures the full integrability of the system. In [13], we defined 7 conserved quantities $\gamma_i$, being quartic polynomials in $\omega_i$’s,

$$\gamma_i = N_{jki}N_{jlk}N_{jkl} = a_i(a_j - a_k)(a_{j1} - a_{k1})(a_{j2} - a_{k2}), \quad (24)$$

where $(j_p, k_p), (j_p < k_p, \ p = 1, 2, 3)$ lie on the respective three lines through the point $i$. We found that there exists only one constraint for the 7 $\gamma_i$’s, showing the independence of $N_{1j}$. Here, let us check the independence by solving the second equations in (22) for $a_j$ in terms of a variable, say, $a_1$. Defining the constants of motion $(\lambda_i, \xi_i)$, given by the initial values $a_{i0}$ of $a_i$,

$$\lambda_i = N_{1i}/N_{12} = a_{20}(a_{10} - a_{i0})/a_{i0}(a_{10} - a_{20}), \quad \xi_i = 1 - \lambda_i, \quad (25)$$

we can express $a_i$ in terms of $a_1$ and $a_2$,

$$a_i = a_1a_2/(\lambda_i a_1 + \xi_i a_2), \quad (26)$$

where $\lambda_1 = 0$ and $\lambda_2 = 1$. The variable $a_2$ is expressed implicitly as a function of $a_1$ through the relation obtained by substituting (26) into the equation for $N_{12}$ in (22),

$$\frac{(a_1a_2)^3(a_1 - a_2)^3}{(\lambda_3a_1 + \xi_3a_2)(\lambda_4a_1 + \xi_4a_2)(\lambda_5a_1 + \xi_5a_2)(\lambda_6a_1 + \xi_6a_2)(\lambda_7a_1 + \xi_7a_2)} = N, \quad (27)$$

where $N$ is a constant,

$$N = (N_{12})^3 = a_{30}a_{40}a_{50}a_{60}a_{70}(a_{10} - a_{20})^3/a_{10}^2a_{20}^2. \quad (28)$$
Figure 2: Riemann surface associated with the curve $y^4 = NR(R + 1) \prod_{p=3}^{7}(\xi_p R - \lambda_p)$ for the general solution of the 7D top. In (a), $x_p = \lambda_p/\xi_p$.

### 3.2 General Solution

We introduce the ratio of $a_2$ and $a_1$, $R(t) = -a_2(t)/a_1(t)$ and obtain the equation of motion for $R(t)$ using (19),

$$\dot{R} = a_1 R(R + 1) .$$  

(29)

Solving (27) for $a_1$ in terms of $R$, we have

$$a_1^4 = (a_1(R))^4 = N\frac{(\xi_3 R - \lambda_3)(\xi_4 R - \lambda_4)(\xi_5 R - \lambda_5)(\xi_6 R - \lambda_6)(\xi_7 R - \lambda_7)}{R^3(R + 1)^3} ,$$  

(30)

and the other variables are expressed in terms of $R$ as

$$a_i = a_1(R)R/(\xi_i R - \lambda_i) .$$  

(31)

Using (29) and (30), we obtain a first-order equation for the ratio $R$,

$$\dot{R} = 4\sqrt{NR(R + 1)(\xi_3 R - \lambda_3)(\xi_4 R - \lambda_4)(\xi_5 R - \lambda_5)(\xi_6 R - \lambda_6)(\xi_7 R - \lambda_7)} .$$  

(32)

The integral associated with this equation can be shown to correspond to a Riemann surface with $g = 9$ in Fig.2(b); the order $1/4$ in the RHS of (32) means that we need four complex surfaces, each of which has four cuts since the order of $R$ is 7 in the RHS. Picking up a cut on each surface in Fig.2(a), attaching the total of four cuts together and repeating it for other three sets of four cuts, we have a Riemann surface with 9 handles.
3.3 Integrable reduction to lower dimensions

Let us consider the reduction of the 7D top to lower target space dimensions which is compatible with the equations of motion and hence preserves the integrability. In the case that the variables $a_i$ and $a_j$ are not independent, that is, $a_i = r a_j$ with a constant $r$, only two cases $r = 0, 1$ are permitted from (19),

$$a_i = 0, \quad a_i = a_j. \quad (33)$$

Imposing the conditions to (18), we obtain lower-dimensional integrable descendants of the 7D top successively. In the section 3.2, it is assumed implicitly that $a_1$ and $a_2$ are independent and hence we impose (33) to the other five variables $a_p (p = 3, 4, 5, 6, 7)$. Then we encounter the following four cases;

(i) $a_p \to a_q$; from the definition of the constants $(\lambda_p, \xi_p)$, two factors in (32) coincide with each other,

$$(\xi_p R - \lambda_p)(\xi_q R - \lambda_q) \to (\xi_p R - \lambda_p)^2. \quad (34)$$

(ii) $a_p \to 0$; in this limit, $(\lambda_p, \xi_p)$ become singular but the products of them and the constant $N$ tend to

$$N (\xi_p R - \lambda_p) \to -M_p (R + 1), \quad M_p = \prod_{q=3,\neq p}^7 a_{q0} (a_{10} - a_{20})^2 / a_{10} a_{20}. \quad (35)$$

(iii) $a_p \to a_1$; the pair $(\lambda_p, \xi_p)$ tends to $(0, 1)$ and $(\xi_p R - \lambda_p) \to R$. 

(iv) $a_p \to a_2$; the pair $(\lambda_p, \xi_p)$ tends to $(1, 0)$ and $(\xi_p R - \lambda_p) \to -1$.

The procedures (i),(ii),(iii) give a square factor in the RHS of (32), while a factor in the RHS disappears under (iv).
We shall show a reduction of the 7D top to lower-dimensional descendants, whose corresponding Riemann surfaces are indicated in Fig.3.

(a) 6D case; in the limit (iv) \( a_3 \to a_2 \), the factor \((\xi_3 R - \lambda_3)\) in (32) goes to \(-1\). Then the four cuts on each surface in Fig.2(a) reduce to three cuts, and by the same procedure as in the 7D case, we find that the Riemann surface for the 6D model is of genus \( g = 6 \).

(b) 5D case; we take the limit (i) \( a_6 \to a_7 \), then (32) becomes

\[
\dot{R} = \sqrt[4]{-N R(R + 1)(\xi_4 R - \lambda_4)(\xi_5 R - \lambda_5)(\xi_7 R - \lambda_7)^2}.
\]

In Fig.2(a), we set three cuts between points \((-1, 0), (\lambda_4/\xi_4, \lambda_5/\xi_5)\) and \((\lambda_7/\xi_7, \infty)\). The third cut \((\lambda_7/\xi_7, \infty)\) has a different property from the others; an orbit passing the cut in the surface 1 goes to the surface 3, while the orbit in the surface 2 goes to 4. Copies of the cut in the surfaces 1 and 3 and those in 2 and 4 have to be identified separately, which produces two handles drawn as overlapped ones in Fig.3(b) and hence \( g = 5 \).

(c) 4D case; in the limit (ii) \( a_5 \to 0 \), (36) reduces to

\[
\dot{R} = \sqrt[4]{M_5 R(R + 1)^2(\xi_4 R - \lambda_4)(\xi_7 R - \lambda_7)^2}.
\]

We set two cuts between points \((0, \lambda_4/\xi_4)\) and \((-1, \lambda_7/\xi_7)\) where copies of the latter in the surfaces 1 and 3 and those in 2 and 4 are identified separately, which gives a \( g = 2 \) surface.

(d) 3D Euler top; (37) becomes under the limit (iii) \( a_4 \to a_1 \),

\[
\dot{R} = \sqrt[4]{M_5 R^2(R + 1)^2(\xi_7 R - \lambda_7)^2} = \pm M_5^{\frac{1}{4}} \sqrt{R(R + 1)(\xi_7 R - \lambda_7)}.
\]

The power of the RHS reduces to 1/2 and the equation (38) describes the 3D Euler top in the section 2 defined with three variables \((\omega_1, \omega_4, \omega_5)\). The pairs of surfaces (1,3) and (2,4) are disconnected from each other and represented by two tori in Fig.3(d), each of which corresponds to the sign + or − in the RHS of (38).

(e) 2D case; taking \( a_7 \to 0 \) or \( a_1 \), (38) goes to a descendant equation associated with a torus in which one of two cycles is shrunk. Another limit \( a_7 \to a_2 \) gives a descendant corresponding to a sphere.

Adding to the above reduction, there are various other ways of reducing the 7D top by the successive use of the procedures (i) to (iv). Some of them may yield higher order factors in (32). For example, a reduction of the 7D top to a 5D model with \( a_5 = a_6 = a_7 \) gives a cubic factor,

\[
\dot{R} = \sqrt[4]{N R(R + 1)(\xi_3 R - \lambda_3)(\xi_4 R - \lambda_4)(\xi_7 R - \lambda_7)^3}.
\]
In this case, we can set three cuts \((-1, 0), (\lambda_3/\xi_3, \lambda_4/\xi_4) \) and \((\lambda_7/\xi_7, \infty)\) in each of the surfaces in Fig.2(a). The last cut arising from the cubic factor is different from the others in the sense that an orbit passing surfaces \(1 \rightarrow 4 \rightarrow 3 \rightarrow 2\) via one of the former two cuts goes to the opposite direction \(1 \rightarrow 2 \rightarrow 3 \rightarrow 4\) through the last cut. The last cut, however, plays the same role as the other two when we construct the surface for the 5D model, which is of genus \(g = 6\). It is also possible to make quartic or more than quartic factors in (32), although they generate singular points in the corresponding Riemann surfaces. A sort of singular points also arise from the case with lower order factors; for example, in a reduction of the 7D top to a 6D model by \(a_7 \rightarrow a_6\), we have to use the point \(\infty\) twice to make four cuts. The point \(\infty\) may become singular and would not produce a proper Riemann surface.

To complete the discussion in this section, we consider the 1D reduction of the 7D top. The equation (32) is not available for this case since \(a_1\) and \(a_2\) are now not independent. As noted around (33), in the 1D case, each of the variables \(a_i\) has to be zero or equal to the other variables.

1. All \(a_i\) are equal; substituting them into (17), (18) directly, we have

\[
\omega_i = -\frac{1}{3t}.
\]  

2. One of \(a_i\)'s is 0;

\[
\omega_i = \omega_j = \omega_k = -\frac{1}{t}, \quad \text{others 0},
\]  

where \((i, j, k)\) are on each line in Fig.1. These are solutions of the 3D top equation.

3. Two of \(a_i\)'s are 0;

\[
\omega_i = -\frac{3}{t}, \quad \omega_j = \omega_k = \frac{1}{t}, \quad \text{others } -\frac{1}{t},
\]  

where points \((j, k)\) lie on the respective three lines through the point \(i\).

4. Three of \(a_i\)'s are 0;

\[
\omega_n = c \text{ (const.)}, \quad \text{others 0},
\]  

and

\[
\omega_i = -\frac{c}{2}, \quad \omega_j = \omega_k = \omega_l = 0, \quad \text{others } \frac{c}{2},
\]  

where \((j, k, l)\) lie on each of the four lines which does not connect to the point \(i\).

All other solutions are obtained from the above ones (10), (11), (12) acted upon by the seven sign-changing transformations noted in the beginning of the section 3.
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