A Generalized Ito Formula

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Abstract

An Ito formula is developed in a context consistent with the development of abstract existence and uniqueness theorems for nonlinear stochastic partial differential equations, which are singular or degenerate. This is a generalization of an earlier Ito formula for Gelfand triples. After this, an existence theorem is presented for some singular and degenerate stochastic equations followed by a few examples.

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1 Introduction

The Ito formula describes \( F(X) \) where

\[
X(t) = X_0 + \int_0^t \phi(s) \, ds + \int_0^t \Phi \, dW
\]

in which the last term is an Ito integral and \( W \) is a Wiener process. The above integral equation is the precise meaning for the stochastic differential equation

\[
dX = \phi dt + \Phi dW, \quad X(0) = X_0.
\]

There are various forms for the Ito formula depending on where \( X \) takes its values. When \( X \) has values in a separable Hilbert space and \( F \) is sufficiently smooth, the Ito formula takes the form

\[
F(t, X(t)) = F(0, X_0) + \int_0^t F_X(\cdot, X(\cdot)) \Phi \, dW + \int_0^t F_t(s, X(s)) + F_X(s, X(s)) \phi(s) \, ds + \frac{1}{2} \int_0^t (F_{XX}(s, X(s)) \Phi, \Phi)_{L_2(Q^{1/2}U, H)} \, ds
\]

In this formula, \( \Phi \) is a stochastically square integrable function having values in the Hilbert space of Hilbert Schmidt operators \( L_2(Q^{1/2}U, H) \) where \( Q \) is a nonnegative self adjoint operator defined on a Hilbert space \( U \).

In addition, there is a version of the Ito formula in the context of a Gelfand triple of spaces

\[
V \subseteq H = H' \subseteq V'
\]

in which

\[
X(t) = X_0 + \int_0^t Y(s) \, ds + \int_0^t Z(s) \, ds,
\]

(1)
the equation holding in $V'$ for $t \in [0, T]$ almost everywhere. In this case it is known that if for some $p > 1$
\[ X \in L^p ([0, T] \times \Omega, V) \cap L^2 ([0, T] \times \Omega, H), \quad Y \in L^p ([0, T] \times \Omega, V') \]
\[ Z \in L^2 ([0, T] \times \Omega, \mathcal{L}_2 (Q^{1/2} U, H)) \]
Then
\[ |X (t)|^2_H = |X_0|^2 + 2 \int_0^t \langle Y (s), \bar{X} (s) \rangle \, ds + \int_0^t \|Z (s)\|^2_{\mathcal{L}_2 (Q^{1/2} U, H)} \, ds + \mathcal{M} (t) \]
where $\mathcal{M} (t)$ is a local martingale defined as a stochastic integral, $\mathcal{M} (0) = 0$. Thus, one can obtain the important estimate
\[ E \left( |X (t)|^2_H \right) = E \left( |X_0|^2 \right) + 2E \left( \int_0^t \langle Y (s), \bar{X} (s) \rangle \, ds \right) \]
\[ + E \left( \int_0^t \|Z (s)\|^2_{\mathcal{L}_2 (Q^{1/2} U, H)} \, ds \right) \]
A discussion of this formula and its applications is found in [18]. It appears to be due to Krylov and is in Russian [19]. This is a much more difficult result. It is shown in this reference that this Ito formula is the fundamental idea in developing general existence and uniqueness theorems for nonlinear stochastic partial differential equations in the context of variational formulations involving Gelfand triples. The formula itself, without the stochastic terms, is fairly familiar to those who formulate partial differential equations in this way, but it is much more profound and difficult than the standard results for deterministic problems because the presence of the stochastic integral causes a loss of weak time derivatives. As is well known, the Wiener process is nowhere differentiable. There are other major technical difficulties related to the minimal assumption that $Z \in L^2 ([0, T] \times \Omega, \mathcal{L}_2 (Q^{1/2} U, H))$. These considerations require the use of the Burkholder Davis Gundy inequality.

For deterministic evolution equations, an interesting generalization was the step from evolution equations
\[ y' + Ay = f \]
to implicit or degenerate evolution equations
\[ (By)' + Ay = f \]
in which $B$ is an operator which may vanish. Since $B$ may fail to be one to one, it may be impossible to consider such an equation as an evolution equation. Instead it is called an implicit evolution equation or sometimes a degenerate evolution equation. It could also happen that $B$ comes from some sort of differential operator and may even be a Riesz map or as a special case, the identity map on a Hilbert space in the context of a Gelfand triple.

In the case of deterministic equations, this was a very natural generalization studied by many authors including Lions [15], Brezis [2], Showalter [4], Bardos [1], [10], and many others. It led to interesting theorems including abstract existence and uniqueness results for partial differential equations of mixed type, simple ways to include systems of equations which involved coupling an elliptic equation with a parabolic equation, and more transparent treatments of equations like the porous media equation. If a good theory of implicit stochastic equations can be obtained, many of the same interesting applications will also have an extension to stochastic problems. The Ito formula discussed above is a way to do integration by parts arguments for stochastic evolution equations, and the version in this paper will provide similar justification of integration by parts procedures for degenerate or implicit stochastic equations. Thus many of the interesting deterministic examples of the last forty years which are in terms of degenerate or partial differential equations of mixed type will have generalizations to stochastic versions.

In this paper, there will be a reflexive separable Banach space $V$ and a separable Hilbert space $W$, such that $V$ is dense in $W$. Thus it is possible to consider the following generalization of a Gelfand triple.

\[ V \subseteq W, \quad W' \subseteq V' \]
The usual pivot space $H$ is replaced with the pair $W, W'$. It is also assumed
\[ BX (t) = BX_0 + \int_0^t Y (s) \, ds + B \int_0^t Z \, dW, \quad (2) \]
where it is known that

\[ X \in L^p ([0, T] \times \Omega, V), \quad BX \in L^2 ([0, T] \times \Omega, W'), \quad Y \in L^{p'} ([0, T] \times \Omega, V') \]

\[ Z \in L^2 \left( [0, T] \times \Omega, L_2 \left( Q^{1/2} U, W \right) \right) \]

In terms of stochastic differential equations it is formally written as

\[ d(BX) = Y dt + BZdW, \quad BX(0) = BX_0. \]

It will be assumed \( B \) is a bounded nonnegative self adjoint operator which maps \( W \) to \( W' \). The case that \( B \) is not one to one is included. Then the Ito formula gives the justification for integration by parts manipulations commonly used in the study of evolution equations.

It is necessary to have the stochastic part of 2 to vanish in case \( B = 0 \), since otherwise, you might obtain an Ito integral equal to a deterministic integral. However, the Ito integral will likely be nowhere differentiable, due to this property which is possessed by the Wiener process, \[23\], \[22\] but the deterministic integral will have a derivative a.e. Thus the above formula for \( BX(t) \) is a reasonable generalization of the case of evolution equations \[1\].

When the formula for this more general situation is obtained, the more standard result like one obtained in \[13\] the context of a Gelfand triple is recovered by letting \( W = H \) and \( B = I \).

To begin with, the paper considers some preliminary results and then the proof of the Ito formula is presented. The techniques generalize those used in \[18\] to the situation where \( \Phi \) is not one to one is included. Then the Ito formula gives the justification for integration by parts manipulations commonly used in the study of evolution equations. All spaces will be assumed real and separable in the paper. Furthermore, there is the usual filtration determined from increments of the Wiener process with respect to which all martingale considerations are defined. This filtration is denoted by \( \mathcal{F}_t \) and it is assumed to be a normal filtration \[13\] so that each \( \mathcal{F}_t \) is complete and \( \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t \).

In Section 2 we give a brief discussion of background results. In Section 3 we give a fundamental equation which will serve as the basis for the proof of the Ito formula. In Section 4 a remarkable estimate is obtained along with some other assertions. Section 5 is devoted to obtaining a technical simplification. It is this which allows us to consider the most general initial conditions. Section 6 has the main result of the paper.

## 2 Preliminary results

The entire presentation is based on the following fundamental lemma \[9\].

**Lemma 1** Let \( \Phi : [0, T] \times \Omega \to E \), be \( \mathcal{B} ([0, T]) \times \mathcal{F} \) measurable and suppose

\[ \Phi \in K \equiv L^p ([0, T] \times \Omega; E), \quad p \geq 1 \]

Then there exists a sequence of nested partitions, \( \mathcal{P}_k \subseteq \mathcal{P}_{k+1}, \)

\[ \mathcal{P}_k \equiv \{ t_0^k, \ldots, t_m^k \} \]

such that the step functions given by

\[ \Phi_k^j (t) = \sum_{j=1}^{m_k} \Phi \left( t_j^k \right) \mathcal{X}_{[t_j-1, t_j]} \]

\[ \Phi_k (t) = \sum_{j=1}^{m_k} \Phi \left( t_j-1 \right) \mathcal{X} \]

both converge to \( \Phi \) in \( K \) as \( k \to \infty \) and

\[ \lim_{k \to \infty} \max \{ |t_j^k - t_{j+1}^k| : j \in \{0, \ldots, m_k \} \} = 0. \]

Also, each \( \Phi \left( t_j^k \right), \Phi \left( t_j-1 \right) \) is in \( L^p (\Omega; E) \). One can also assume that \( \Phi (0) = 0 \). The mesh points \( \{ t_j^k \}_{j=0}^{m_k} \) can be chosen to miss a given set of measure zero.
There is also a known result on quadratic variation which we use later.  

**Theorem 2** Let $H$ be a Hilbert space and suppose $(M, \mathcal{F}_t), t \in [0, T]$ is a uniformly bounded continuous martingale with values in $H$. Also let $\{t^n_k\}_{k=1}^{m_n}$ be a sequence of partitions satisfying

$$\lim_{n \to \infty} \max \{ |t^n_i - t^n_{i+1}|, i = 0, \ldots, m_n \} = 0,$$

Then

$$[M](t) = \lim_{n \to \infty} \sum_{i=0}^{m_n-1} |M(t \wedge t^n_k) - M(t \wedge t^n_{k+1})|^2_H$$

the limit taking place in $L^2(\Omega)$. In case $M$ is just a continuous local martingale, the above limit happens in probability.

In order to deal with the possibly degenerate operator $B$, we have the following interesting generalization of standard material involving inner products.

**Lemma 3** Suppose $V, W$ are separable Banach spaces, $W$ also a Hilbert space such that $V$ is dense in $W$ and $B \in \mathcal{L}(W; W')$ satisfies

$$\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle, B \neq 0.$$

Then there exists a countable set $\{e_i\}$ of vectors in $V$ such that

$$\langle Be_i, e_j \rangle = \delta_{ij}$$

and for each $x \in W$,

$$\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,$$

and also

$$Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i,$$

the series converging in $W'$.

**Proof:** Let $\{g_k\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Let $n_1$ be the first index such that $\langle Bg_{n_1}, g_{n_1} \rangle \neq 0$.

**Claim:** If there is no such index, then $B = 0$.

**Proof of claim:** First note that if $\langle Bg, g \rangle = 0$, then

$$|\langle Bg, x \rangle| \leq |\langle Bg, g \rangle|^{1/2} |\langle Bx, x \rangle|^{1/2} = 0$$

and so $Bg = 0$. Therefore, if $x$ is given, you could take $x_k$ in the span of $\{g_1, \ldots, g_k\}$ such that $\|x_k - x\|_W \to 0$.

Then

$$|\langle Bx, y \rangle| = \lim_{k \to \infty} |\langle Bx_k, y \rangle| \leq \lim_{k \to \infty} \langle Bx_k, x_k \rangle^{1/2} \langle By, y \rangle^{1/2} = 0$$

because $Bx_k$ is zero, being the sum of scalars times $Bg_i$ for finitely many $i$. Since $y$ is arbitrary, this shows $Bx = 0$.

Thus assume there is such a first index. Let

$$e_1 \equiv \frac{g_{n_1}}{\langle Bg_{n_1}, g_{n_1} \rangle^{1/2}}.$$

Then $\langle Be_1, e_1 \rangle = 1$. Now if you have constructed $e_j$ for $j \leq k$,

$$e_j \in \text{span} \{g_{n_1}, \ldots, g_{n_k}\}, \quad \langle Be_i, e_j \rangle = \delta_{ij},$$

$g_{n_{j+1}}$ being the first for which

$$\left\langle Bg_{n_{j+1}} - \sum_{i=1}^{j} \langle Bg_{n_{j+1}}, e_i \rangle Be_i, g_{n_{j+1}} - \sum_{i=1}^{j} \langle Bg_{n_{j}}, e_i \rangle e_i \right\rangle \neq 0,$$
and 
\[ \text{span} \left( g_{n_1}, \cdots, g_{n_k} \right) = \text{span} \left( e_1, \cdots, e_k \right), \]
let \( g_{n_{k+1}} \) be such that \( g_{n_{k+1}} \) is the first in the list \( \left\{ g_{n_j} \right\}_{j=1}^{\infty} \) such that
\[ \left\langle B g_{n_{k+1}} - \sum_{i=1}^{k} \left( B g_{n_{k+1}}, e_i \right) B e_i, g_{n_{k+1}} - \sum_{i=1}^{k} \left( B g_{n_{k+1}}, e_i \right) e_i \right\rangle \neq 0 \]

**Claim:** If there is no such first \( g_{n_{k+1}} \), then \( B \left( \text{span} \left( e_1, \cdots, e_k \right) \right) = BW \) so in this case, \( \left\{ B e_i \right\}_{i=1}^{k} \) is actually a basis for \( BW \).

**Proof:** Let \( x \in W \). Let \( x_r \in \text{span} \left( g_{1}, \cdots, g_{r} \right), r > n_k \) such that \( \lim_{r \to \infty} x_r = x \) in \( W \). Then
\[ x_r = \sum_{i=1}^{k} c_i^r e_i + \sum_{i \notin \{n_1, \cdots, n_k\}} d_i^r g_i \equiv y_r + z_r \]
(3)
If \( l \notin \{n_1, \cdots, n_k\} \), then by the construction and the above assumption, for some \( j \leq k \)
\[ \left\langle B g_l - \sum_{i=1}^{j} \left( B g_l, e_i \right) B e_i, g_l - \sum_{i=1}^{j} \left( B g_l, e_i \right) e_i \right\rangle = 0 \]
The reasoning is as follows. If \( l \leq k \) and if the above is nonzero for all \( j \leq k \), then \( l \) would have been chosen but it wasn’t. Thus in this case that \( l \leq k \), there exists \( j \) such that
\[ B g_l = \sum_{i=1}^{j} \left( B g_l, e_i \right) B e_i \]
If \( l > n_k \), then by assumption, the above is never nonzero for \( j = k \). Thus, in any case, it follows that for each \( l \notin \{n_1, \cdots, n_k\} \),
\[ B g_l \in B \left( \text{span} \left( e_1, \cdots, e_k \right) \right). \]
Now it follows from (3) that
\[ B x_r = \sum_{i=1}^{k} c_i^r B e_i + \sum_{i \notin \{n_1, \cdots, n_k\}} d_i^r B g_i \]
\[ = \sum_{i=1}^{k} c_i^r B e_i + \sum_{i \notin \{n_1, \cdots, n_k\}} d_i^r \sum_{j=1}^{k} c_j^r B e_j \]
and so \( B x_r \in B \left( \text{span} \left( e_1, \cdots, e_k \right) \right) \). Then \( B x = \lim_{r \to \infty} B x_r = \lim_{r \to \infty} B y_r \) where \( y_r \in \text{span} \left( e_1, \cdots, e_k \right) \). Say
\[ B x_r = \sum_{i=1}^{k} a_i^r B e_i \]
It follows easily that \( \left( B x_r, e_j \right) = a_j^r \). (Act on \( e_j \) by both sides and use \( \left( B e_i, e_j \right) = \delta_{ij} \).) Now since \( x_r \) is bounded, it follows that these \( a_j^r \) are also bounded. Hence, defining \( y_r \equiv \sum_{i=1}^{k} a_i^r e_i \), it follows that \( y_r \) is bounded in \( \text{span} \left( e_1, \cdots, e_k \right) \) and so, there exists a subsequence, still denoted by \( r \) such that \( y_r \to y \in \text{span} \left( e_1, \cdots, e_k \right) \). Therefore, \( B x = \lim_{r \to \infty} B y_r = B y \). In other words, \( BW = B \left( \text{span} \left( e_1, \cdots, e_k \right) \right) \) as claimed. This proves the claim.

If this happens, the process being described stops. You have found what is desired which has only finitely many vectors involved.

As long as the process does not stop, let
\[ e_{k+1} = \frac{g_{n_{k+1}} - \sum_{i=1}^{k} \left( B g_{n_{k+1}}, e_i \right) e_i}{\left\langle B \left( g_{n_{k+1}} - \sum_{i=1}^{k} \left( B g_{n_{k+1}}, e_i \right) e_i \right), g_{n_{k+1}} - \sum_{i=1}^{k} \left( B g_{n_{k+1}}, e_i \right) e_i \right\rangle^{1/2}} \]
Thus, as in the usual argument for the Gram Schmidt process, $\langle Be_i, e_j \rangle = \delta_{ij}$ for $i, j \leq k$.

Consider

$$\left\langle Bg_p - B \left( \sum_{i=1}^{k} \langle Bg_p, e_i \rangle e_i \right), g_p - \sum_{i=1}^{k} \langle Bg_p, e_i \rangle e_i \right\rangle$$

(4)

If $p$ is never one of the $n_k$, then there exists $k$ such that $p \in (n_k, n_{k+1})$ so 4 equals 0. If $p = n_k$ for some $k$, then from the construction, $g_{n_k} = g_p \in \text{span}(e_1, \cdots, e_k)$ and therefore,

$$g_p = \sum_{j=1}^{k} a_j e_j$$

which requires easily that

$$Bg_p = \sum_{i=1}^{k} \langle Bg_p, e_i \rangle Be_i,$$

and 4 equals 0, the above holding for all $k$ large enough. It follows that for any $x \in \text{span}(\{g_k\}_{k=1}^{\infty})$, (finite linear combination of vectors in $\{g_k\}_{k=1}^{\infty}$).

$$Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i$$

(5)

because for all $k$ large enough,

$$Bx = \sum_{i=1}^{k} \langle Bx, e_i \rangle Be_i$$

Also note that for such $x \in \text{span}(\{g_k\}_{k=1}^{\infty})$,

$$\langle Bx, x \rangle = \sum_{i=1}^{k} \langle Bx, e_i \rangle Be_i, x = \sum_{i=1}^{k} \langle Bx, e_i \rangle \langle Bx, e_i \rangle$$

$$= \sum_{i=1}^{k} |\langle Bx, e_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2$$

Now for $x$ arbitrary, let $x_k \to x$ in $W$ where $x_k \in \text{span}(\{g_k\}_{k=1}^{\infty})$. Then by Fatou’s lemma,

$$\sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \leq \lim \inf_{k \to \infty} \sum_{i=1}^{\infty} |\langle Bx_k, e_i \rangle|^2$$

$$= \lim \inf_{k \to \infty} \langle Bx_k, x_k \rangle = \langle Bx, x \rangle$$

(6)

$$\leq \|Bx\|_W, \|x\|_W \leq \|B\| \|x\|_W^2$$

Thus the series on the left converges. Then also, from the above inequality,

$$\left| \sum_{i=p}^{q} \langle Bx, e_i \rangle Be_i, y \right| \leq \sum_{i=p}^{q} |\langle Bx, e_i \rangle| |\langle Be_i, y \rangle|$$

$$\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=p}^{q} |\langle By, e_i \rangle|^2 \right)^{1/2}$$

$$\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |\langle By, e_i \rangle|^2 \right)^{1/2}$$
Consider the following diagram in which $J$ is a one to one Hilbert Schmidt operator and $Q$ is a nonnegative and self adjoint operator defined on the Hilbert space $U$.

\[
\begin{array}{c}
U_1 \supseteq JQ^{1/2}U \\
\downarrow Q^{1/2} \\
\Phi_n \downarrow \Phi \\
W
\end{array}
\]

The details of the definition of the stochastic integral are in [18], [6]. For completeness, here is a short summary.

Now for $x$ arbitrary, let $x_k \in \text{span} \left( \{g_j\}_{j=1}^\infty \right)$ and $x_k \to x$ in $W$. Then for a fixed $k$ large enough,

\[
\begin{align*}
\left\| Bx - \sum_{i=1}^{\infty} \langle Bx, e_i \rangle B e_i \right\| &\leq \left\| Bx - Bx_k \right\| \\
+ \left\| Bx_k - \sum_{i=1}^{\infty} \langle Bx_k, e_i \rangle B e_i \right\| + \sum_{i=1}^{\infty} \left| \langle Bx_k, e_i \rangle - \langle Bx, e_i \rangle \right| \left\| B e_i \right\| \\
&\leq \varepsilon + \sum_{i=1}^{\infty} \left| \langle B (x_k - x), e_i \rangle \right| \left\| B e_i \right\|
\end{align*}
\]

the middle term equaling 0 by $\S$ From $\S$ and $\S$

\[
\begin{align*}
&\leq \varepsilon + \|B\|^{1/2} \left( \sum_{i=1}^{\infty} |\langle B (x_k - x), e_i \rangle|^2 \right)^{1/2} \\
&\leq \varepsilon + \|B\|^{1/2} \langle B (x_k - x), x_k - x \rangle^{1/2} < 2\varepsilon
\end{align*}
\]

whenever $k$ is large enough. Therefore,

\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle B e_i
\]

in $W'$. It follows that

\[
\langle Bx, x \rangle = \lim_{k \to \infty} \left( \sum_{i=1}^{k} \langle Bx, e_i \rangle B e_i, x \right) = \lim_{k \to \infty} \sum_{i=1}^{k} |\langle Bx, e_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2
\]

It follows that

\[
\sum_{i=1}^{\infty} \langle Bx, e_i \rangle B e_i
\]
The idea is to define \( \int_{0}^{t} \Phi dW \) where \( \Phi \in L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2}U, W)) \), \( \mathcal{L}_2 (Q^{1/2}U, W) \) being the Hilbert Schmidt operators mapping \( Q^{1/2}U \) to \( W \) and \( J \) a Hilbert Schmidt operator. Here \( W(t) \) is the process

\[
W(t) = \sum_{i=1}^{\infty} \psi_i(t) Jg_i \text{ in } U_1,
\]

where the \( \psi_i(t) \) are real, independent Wiener processes. It is a \( Q_1 \) Wiener process on \( U_1 \) for \( Q_1 = JJ^* \). To get \( \int_{0}^{t} \Phi dW, \Phi \circ J^{-1} \) was approximated by a sequence of elementary functions, \( \{\Phi_n\} \), adapted step functions having finitely many values in \( \mathcal{L}(U_1, W) \). Then the stochastic integral was defined in the usual way. For

\[
\Phi_n(t) = \sum_{i=0}^{m-1} \phi_i \mathcal{X}_{[t_i, t_{i+1})} (t), \text{ \( \phi_i \) being } \mathcal{F}_{t_i} \text{ measurable,}
\]

\[
\int_{0}^{t} \Phi_n dW \equiv \sum_{i=0}^{m-1} \phi_i (W(t \wedge t_{i+1}) - W(t \wedge t_i)).
\]

Then it is shown that this sequence of processes converges in \( L^2 (\Omega, W) \) and

\[
\int_{0}^{t} \Phi dW \equiv \lim_{n \to \infty} \int_{0}^{t} \Phi_n dW
\]

It can be shown that this integral \( \int_{0}^{t} \Phi dW \) satisfies the Itô isometry and is independent of the choice of \( U_1 \) and \( J \).

In all that follows, \( Q \) will be a nonnegative self-adjoint operator defined on a separable Hilbert space \( U \). Also \( Z \) will be progressively measurable and in \( L^2 ([0, T] \times \Omega, \mathcal{L}_2 (Q^{1/2}U, W)) \) while \( J : Q^{1/2}U \to U_1 \) will be a one to one Hilbert Schmidt operator.

Now here is a technical result which will be needed. This is a technical application of the above description of the stochastic integral.

**Theorem 4** Let \( Z \) be progressively measurable and in

\[
L^2 ([0, T] \times \Omega, \mathcal{L}_2 (Q^{1/2}U, W))
\]

Also suppose \( P \) is progressively measurable and in \( L^2 ([0, T] \times \Omega, W') \). Let \( \{ t^n_j \}_{j=0}^{m_n} \) be a sequence of partitions of the sort in Lemma 7 such that if

\[
P_n(t) = \sum_{j=0}^{m_n-1} P \left( t^n_j \right) \mathcal{X}_{[t^n_j, t^n_{j+1})} (t) \equiv P_n^1(t)
\]

then \( P_n \to P \) in \( L^2 ([0, T] \times \Omega, W) \). Then the expression

\[
\sum_{j=0}^{m_n-1} \left< \int_{t^n_j}^{t^n_{j+1} \wedge t} ZdW \right>
\]

is a local martingale which can be written as a stochastic integral in the form

\[
\int_{0}^{t} (Z \circ J^{-1})^* P_n^1 \circ JdW
\]

**Proof:** Note that \( P_n \) is right continuous and progressively measurable. Thus one can define the stopping time

\[
\tau^n_p \equiv \inf \{ t : \| P_n(t) \|_W > p \},
\]

the first hitting time of an open set. We need the formula in \( [4] \) as a stochastic integral. First note that \( W \) has values in \( U_1 \).
Consider one of the terms of the sum more simply as
\[
\left\langle P(a), \int_a^b ZdW \right\rangle, \quad a = t^n_k \wedge t, \quad b = t^n_{k+1} \wedge t.
\]

Then from the definition of the integral, let \( Z \) be a sequence of elementary functions converging to \( Z \circ J^{-1} \) in \( L^2([a, b] \times \Omega, \mathcal{L}_2(JQ^{1/2}U, W)) \) and
\[
\left\| \int_a^t ZdW - \int_a^t Z_n dW \right\|_{L^2(\Omega, W)} \to 0.
\]

Using a maximal inequality and the fact that the two integrals are martingales along with the Borel Cantelli lemma, there exists a set of measure 0 \( N \) such that for \( \omega \notin N \), the convergence of a suitable subsequence of these integrals, still denoted by \( n_i \), is uniform for \( t \in [a, b] \). It follows that for such \( \omega \),
\[
\left\langle P(a), \int_a^t ZdW \right\rangle = \lim_{n \to \infty} \left\langle P(a), \int_a^t Z_n dW \right\rangle. \quad (10)
\]

Say
\[
Z_n(u) = \sum_{k=0}^{m_n-1} Z^n_k \mathcal{A}_{t^n_k, t^n_{k+1}}(u)
\]
where \( Z^n_k \) has finitely many values in \( \mathcal{L}(U_1, W) \), the restrictions of maps in \( \mathcal{L}(U_1, W) \) to \( JQ^{1/2}U \), and the \( t^n_k \) refer to a partition of \( [a, b] \). Then the product on the right in (10) is of the form
\[
\sum_{k=0}^{m_n-1} \left\langle P(a), Z^n_k \left( W(\tau \wedge t^n_{k+1}) - W(\tau \wedge t^n_k) \right) \right\rangle_{W', W}
\]
Note that it makes sense because \( Z^n_k \) is the restriction to \( J \left( Q^{1/2}U \right) \) of a map from \( U_1 \) to \( W \). Thus the above equals
\[
\sum_{k=0}^{m_n-1} \left\langle P(a), Z^n_k \left( W(\tau \wedge t^n_{k+1}) - W(\tau \wedge t^n_k) \right) \right\rangle_{W', W}
\]
\[
\sum_{k=0}^{m_n-1} \left\langle (Z^n_k)^* P(a), (W(\tau \wedge t^n_{k+1}) - W(\tau \wedge t^n_k)) \right\rangle_{U'_1, U_1}
\]
\[
\sum_{k=0}^{m_n-1} (Z^n_k)^* P(a) \left( W(\tau \wedge t^n_{k+1}) - W(\tau \wedge t^n_k) \right)
\]
\[
\int_a^t Z^n_k P(a) dW
\]
Note that the restriction of \( (Z^n_k)^* P(a) \) is in
\[
\mathcal{L}(U_1, \mathbb{R}) \subseteq \mathcal{L}_2 \left( JQ^{1/2}U, \mathbb{R} \right).
\]
Recall also that the space on the left is dense in the one on the right. Now let \( \{g_i\} \) be an orthonormal basis for \( Q^{1/2}U \), so that \( \{Jg_i\} \) is an orthonormal basis for \( JQ^{1/2}U \). Then
\[
\sum_{i=1}^{\infty} \left| \left( (Z^n_k)^* P(a) - (Z \circ J^{-1})^* P(a) \right) (Jg_i) \right|^2 = \sum_{i=1}^{\infty} \left| \left( P(a), (Z_n - Z \circ J^{-1}) (Jg_i) \right) \right|^2
\]
we need other assumptions.

Then it follows that, using the stopping time,

\[ P(1) = \|P(n)\|^2 \|Z_n - Z \circ J^{-1}(J(n))\|^2 \]

When integrated over \([a, b] \times \Omega\), it is given that this converges to 0 as \(n \to \infty\), assuming that \(\|P(n)\| \in L^\infty(\Omega)\), which is assumed for now. It follows that

\[ Z_n^*P(a) \to (Z \circ J^{-1})^*P(a) \]

in \(L^2([a, b] \times \Omega, L_2(JQ^{1/2}U, R))\). Writing this differently, it says

\[ Z_n^*P(a) \to \left((Z \circ J^{-1})^*P(a) \circ J\right) \circ J^{-1} \text{ in } L^2([a, b] \times \Omega, L_2(JQ^{1/2}U, R)) \]

It follows from the definition of the integral that the Ito integrals converge. Therefore,

\[ \left\langle P(a), \int_a^t ZdW \right\rangle = \int_a^t (Z \circ J^{-1})^*P(a) \circ JdW \]

The term on the right is a martingale.

Next it is necessary to drop the assumption that \(\|P(a)\| \in L^\infty(\Omega)\). This involves the above stopping time. From localization,

\[ \left\langle P(a), \int_{a \wedge \tau_p}^{t \wedge \tau_p} ZdW \right\rangle = \left\langle P(a), \int_a^t \mathcal{X}_{[0, \tau_p]}^ZdW \right\rangle \]

\[ = \int_a^t \mathcal{X}_{[0, \tau_p]}^Z \circ J^{-1} \circ JdW \]

\[ = \int_{a \wedge \tau_p}^{t \wedge \tau_p} (Z \circ J^{-1})^*P(a) \circ JdW \]

Then it follows that, using the stopping time,

\[ \sum_{j=0}^{m-1} \left\langle P(t_j^n), \int_{t_j^n \wedge \tau_p}^{t_{j+1} \wedge \tau_p} ZdW \right\rangle = \int_0^{t \wedge \tau_p} (Z \circ J^{-1})^*P_n^l \circ JdW \]

where \(P_n^l\) is the step function

\[ P_n^l(t) = \sum_{k=0}^{m_n-1} P(t_k^n) \mathcal{X}_{[t_k^n, t_{k+1}^n]}^Z(t) \]

Thus the given sum equals the local martingale

\[ \int_0^t (Z \circ J^{-1})^*P_n^l \circ JdW. \]

The original formula does not depend on \(J\) and so the same is true of this last expression although it does not look like it. The unaesthetic appearance of the above integral can be improved, but such an effort is of no significance in what follows.

The next question is whether the above stochastic integral converges as \(n \to \infty\) in some sense to an integral

\[ \int_0^t (Z \circ J^{-1})^*P \circ JdW. \quad (11) \]

The problem is that the integrand is not known to be in \(L^2([0, T] \times \Omega; L_2(Q^{1/2}U, R))\). It would be useful to define a stopping time

\[ \tau_n \equiv \inf\{t : \|P(t)\|_{W^l} > n\} \quad (12) \]

because then, you could localize and define the integral in \(11\) as a local martingale. However, to do this would require the stopping time to make sense. It is not known that \(P\) is continuous or even right continuous. Therefore, we need other assumptions.
Lemma 5 Suppose \( t \rightarrow P ( t ) \) is weakly continuous into \( W' \) for a.e. \( \omega \) and that \( P \) is adapted. Then \( \tau_n \) described in (12) is well defined. It also satisfies \( \lim_{n \to \infty} \tau_n = \infty \).

Proof: Let \( O \equiv \{ y \in W : \| y \|_{W'} > n \} \). Then the complement of \( O \) is a closed convex set. It follows that \( O^C \) is also weakly closed. Hence \( O \) must be weakly open. Now \( t \to P ( t ) \) is adapted as a function mapping into the topological space consisting of \( W' \) with the weak topology. Hence \( \tau_n \) is the first hitting time of an open set by a continuous process, so \( \tau_n \) is a stopping time. Also, by the assumption that \( t \to P ( t ) \) is weakly continuous, it follows from the uniform boundedness theorem that \( \| P ( t ) \| \) is bounded on \([0, T]\). Hence for a.e. \( \omega \), \( \tau_n = \infty \) for all \( n \) large enough. 

It follows that it is possible to define the stochastic integral of (11) as a local martingale when \( t \to P ( t ) \) is weakly continuous. In the derivation which follows, the computations will pertain to such a weakly continuous process.

It remains to consider the convergence of a suitable subsequence of

\[
\int_0^t (Z \circ J^{-1})^* P_k^t \circ J dW
\]

to the integral of (11). The desired result follows. The proof is similar to that given in [18] for a similar situation in the context of a Gelfand triple.

Lemma 6 In the above context, let \( P ( s ) - P_k^t ( s ) \equiv \Delta_k ( s ) \). Let

\[
Z \in L^2 \left( [a, b] \times \Omega, L_2 \left( JQ^{1/2} U, W \right) \right)
\]

and let \( P \in L^2 ([0, T] \times \Omega, W') \) with both \( P \) and \( Z \) progressively measurable. Also suppose \( t \to P ( t ) \) is weakly continuous. Then the integral

\[
\int_0^t (Z \circ J^{-1})^* P \circ J dW
\]

exists as a local martingale and the following limit is valid for a suitable subsequence, still denoted by \( k \)

\[
\lim_{k \to \infty} P \left( \left[ \sup_{t \in [0, T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right| \geq \varepsilon \right] \right) = 0.
\]

That is,

\[
\sup_{t \in [0, T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right|
\]

converges to 0 in probability.

Proof: The existence of the integral was dealt with earlier. Let \( k \) denote a subsequence for which for a.e. \( \omega \),

\[
P_k^t ( \cdot, \omega ) \to P ( \cdot, \omega )
\]
in \( L^p ([0, T], W') \) and also \( P_k^t ( t, \omega ) \to P ( t, \omega ) \) for a.e. \( t \). This is done as follows.

\[
P \left( \| P_k^t - P \|_{L^p(0,T,W')} > \lambda \right) \leq \frac{1}{\lambda} \int_\Omega \| P_k^t - P \|_{L^p(0,T,W')} dP
\]

and the integral on the right is small provided \( k \) is large. Therefore, there exists a subsequence still called \( k \) such that

\[
P \left( \| P_k^t - P \|_{L^p(0,T,W')} > 2^{-k} \right) < 2^{-k}
\]

Then this satisfies the desired conditions.

From the assumption of weak continuity, there exists for a.e. \( \omega \) a constant, \( C ( \omega ) \) such that

\[
\sup_{t \in [0,T]} \| P ( t ) \| \leq C ( \omega )
\]
For the first part of the argument, assume that \( C(\omega) \) is independent of \( \omega \) off a set of measure zero. Let \( \{e_i\} \) be an orthonormal basis of vectors in \( W \). Thus \( R(e_i) \) is an orthonormal basis of vectors in \( W' \) where \( R \) is the Riesz map. Hence

\[
P = \sum_{i=1}^{\infty} (P, R(e_i))_W, \quad R(e_i) = \sum_{i=1}^{\infty} (R^{-1} P, e_i)_W \quad R(e_i) = \sum_{i=1}^{\infty} (P, e_i) R(e_i)
\]

It follows that

\[
P_x = \sum_{i=1}^{\infty} \langle P, e_i \rangle \langle R(e_i), x \rangle
\]

Let

\[
\pi_n P = \sum_{i=1}^{n} (P, e_i) R(e_i)
\]

Thus

\[
\begin{align*}
P \left( \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* \Delta_k \circ JdW \right| \geq \varepsilon \right] \right) \\
&\leq P \left( \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* \pi_n \Delta_k \circ JdW \right| \geq \varepsilon/3 \right] \right) + \\
&\quad \quad P \left( \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* (I - \pi_n) P \circ JdW \right| \geq \varepsilon/3 \right] \right) + \\
&\quad \quad P \left( \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* (I - \pi_n) P_k^d \circ JdW \right| \geq \varepsilon/3 \right] \right)
\end{align*}
\]

Using the Burkholder Davis Gundy inequality on (13) along with the description of the quadratic variation given above,

\[
\begin{align*}
P \left( \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* (I - \pi_n) P \circ JdW \right| \geq \varepsilon/3 \right] \right) \\
&\leq \frac{3}{\varepsilon} \int_{\Omega} \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* (I - \pi_n) P \circ JdW \right| dP \\
&\leq \frac{3C}{\varepsilon} \int_{\Omega} \left( \int_{0}^{T} \|Z\|^2 \|(I - \pi_n) P\|^2 dt \right)^{1/2} dP \\
&\leq \frac{3C}{\varepsilon} \left( \int_{0}^{T} \|Z\|^2 \|(I - \pi_n) P\|^2 dt dP \right)^{1/2}
\end{align*}
\]

This integral converges to 0 as \( n \to \infty \) by the assumption that \( P \) is bounded along with the dominated convergence theorem applied to the finite measure \( \|Z\|^2 dt dP \). Letting \( \eta > 0 \) be given, choose \( n \) large enough that the above term is less than \( \eta \). From now on, use this \( n \). Thus \( \frac{3}{\varepsilon} \leq \eta \).

Next consider (14) By the Burkholder Davis Gundy inequality again,

\[
\begin{align*}
P \left( \left[ \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* (I - \pi_n) P_k^d \circ JdW \right| \geq \varepsilon/3 \right] \right) \\
&\leq \frac{3}{\varepsilon} \int_{\Omega} \sup_{t \in [0,T]} \left| \int_{0}^{t} (Z \circ J^{-1})^* (I - \pi_n) P_k^d \circ JdW \right| dP \\
&\leq \frac{3C}{\varepsilon} \int_{0}^{T} \|Z\|^2 \|(I - \pi_n) P_k^d\|^2 dt \right)^{1/2} dP
\end{align*}
\]
Implicit Ito Formula

\[ \lim_{k \to \infty} \pi_n \left( P \left( s - P_k^l (s) \right) = \lim_{k \to \infty} \sum_{i=1}^n \langle P \left( s - P_k^l (s), e_i \right) R (e_i) = 0 \ a.e \ \omega \right. \]

It follows that

\[ \lim_{k \to \infty} \int_0^T \| \pi_n \left( P \left( s - P_k^l (s) \right) \right) \right\|_{W^1}^2 \left( Z \left( s \right) \right)^2_{L_2} ds dP = 0 \]

because you can apply the dominated convergence theorem with respect to the measure \( \| Z \left( s \right) \|^2 dP \) along with the assumption that \( \| P \left( t \right) \| \) is bounded independent of \( \omega \).
Therefore, it follows that
\[ \lim_{k \to \infty} P \left( \left. \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \pi_n \Delta_k \circ J dW \right| \geq \varepsilon / 3 \right) = 0. \]

Here is why. By the Burkholder Davis Gundy theorem, and the description of the quadratic variation of a stochastic integral,
\[
\int_0^T \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \pi_n \Delta_k \circ J dW \right| dP \\
\leq C \int_0^T \int_0^T \| \pi_n (P(s) - P_k^l(s)) \|^2_{L_2} \| Z(s) \|^2_{L_2} ds dP \\
\leq C \int_0^T \int_0^T \| \pi_n (P(s) - P_k^l(s)) \|^2_{L_2} \| Z(s) \|^2_{L_2} ds dP \]

which converges to 0 as \( k \to \infty \).

If \( k \) is large enough, this implies
\[
P \left( \left. \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right| \geq \varepsilon \right) \leq 3 \eta + \frac{3C}{\varepsilon} \left( \int_0^T \int_0^T \| Z \|^2 \| (P_k^l - P) \|^2 dt dP \right)^{1/2}
\]

The last integral also converges to 0 because of the assumption that \( P \), hence \( P_k^l \) is bounded, and the dominated convergence theorem. Thus
\[
\lim_{k \to \infty} \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right| = 0 \text{ in probability.}
\]

Now to finish the argument, define the stopping time
\[ \tau_m \equiv \inf \{ t > 0 : \| P(t) \| > m \}. \]

As discussed earlier, this is a valid stopping time by the weak continuity of \( t \to P(t) \). Then
\[ X^{\tau_m} - (X_k^l)^{\tau_m} \]
still converges pointwise to 0 as \( k \to \infty \). Let \( \Delta_k^{\tau_m} = (P(s) - P_k^l(s))^{\tau_m} \)

Now consider
\[ A_{k \varepsilon} \equiv \left[ \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right| \geq \varepsilon \right] \]

Then
\[
P \left( A_{k \varepsilon} \cap [\tau_m = \infty] \right) \leq P \left( \left. \left| \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k^{\tau_m} \circ J dW \right| \geq \varepsilon \right) \right)
\]

which converges to 0 as \( k \to \infty \) by the first part of the argument. This is because \( \| (P^{\tau_m})^l_k \| \) and \( \| P^{\tau_m} \| \) are both bounded by \( m \) uniformly off a set of measure zero. Now \( A_{k \varepsilon} \) can be partitioned in the following way.
\[ A_{k \varepsilon} = \bigcup_{m=1}^\infty A_{k \varepsilon} \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty]) \]

Thus
\[ P \left( A_{k \varepsilon} \right) = \sum_{m=1}^\infty P \left( A_{k \varepsilon} \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty]) \right) \]
and \(P(\{k \in \mathbb{N} \mid (\tau_m = \infty) \setminus (\tau_{m-1} < \infty)\}) \leq P(\{\tau_m = \infty \setminus (\tau_{m-1} < \infty)\})\) which is summable, since the sets are disjoint. Hence one can apply the dominated convergence theorem and conclude that

\[
\lim_{k \to \infty} P(A_k) = \sum_{m=1}^{\infty} \lim_{k \to \infty} P(A_k \cap (\tau_m = \infty \setminus (\tau_{m-1} < \infty))) = 0. \tag{15}
\]

The notation used in the above is inelegant. In fact it is more attractive to write this in the form

\[
\int_0^t (Z \circ J^{-1})^* P \circ JdW = \int_0^t PdM
\]

where \(M(t) = \int_0^t ZdW\). However, this is of no importance in what follows, and written in the above inelegant form, assertions about the quadratic variation are possibly more obvious.

### 3 The Integral Equation

For a set of measure zero \(N_0\) and \(\omega \notin N_0\),

\[
P(t) = BX_0 + \int_0^t Y(s) \, ds + B \int_0^t Z(s) \, dW(s), \tag{16}
\]

for all \(t\). Let \(X\) be progressively measurable into \(V\) such that for a set of measure zero \(S \subseteq [0,T]\) and \(t \notin S\),

\[
BX(t) = P(t) \text{ for a.e.} \omega.
\]

The exceptional set in the above may depend on \(t\). In short, for all \(t \notin S\),

\[
BX(t) = BX_0 + \int_0^t Y(s) \, ds + B \int_0^t Z(s) \, dW(s) \text{ a.e. } \omega, \tag{17}
\]

the exceptional set possibly depending on \(t\).

Also assume that \(X_0 \in L^2(\Omega;W)\) and is \(\mathcal{F}_0\) measurable, where \(Z\) is \(L_2\left(Q^{1/2}U,W\right)\) progressively measurable and

\[
\|Z\|_{L^2([0,T] \times \Omega;L_2(Q^{1/2}U,W))} < \infty.
\]

This is what is needed to define the stochastic integral in the above formula.

\(X, Y\) satisfy

\[
X \in K = L^p([0,T] \times \Omega;V) \cap L^2([0,T] \times \Omega;W), Y \in K' = L^{p'}([0,T] \times \Omega;V')
\]

where \(1/p + 1/p' = 1\), \(p > 1\), and \(X, Y\) are progressively measurable into \(V\) and \(V'\) respectively.

Also, by enlarging \(N_0\) if necessary, one can assume that off \(N_0\), the stochastic integral in \(16\) is continuous into \(W'\) and \(Y(., \omega) \in L^p(0,T,V')\) so that the deterministic integral in this equation is also continuous as a function with values in \(V'\), also that \(t \to X(t, \omega) \in L^p(0,T,V)\) for \(\omega \notin N_0\). From now on, let \(N_0\) be so enlarged.

The goal is to prove the following Ito formula for \(P(t)\) defined as the right side of \(16\)

\[
\langle P(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t (2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle) \, ds
\]

\[
+ \int_0^t (Z \circ J^{-1})^* P \circ JdW \tag{18}
\]

The most significant feature of the last term is that it is a local martingale. The term \(\langle BZ, Z \rangle\) will be discussed later. In the stochastic integral, \((Z \circ J^{-1})^* P \circ J\) has values in \(L_2\left(Q^{1/2}U,R\right)\) and so it makes sense to consider this stochastic integral.

The argument for the Ito formula will be based on a formula which follows in the next lemma.
Lemma 7 In the situation of the above integral equation, the following formula holds for a.e. $\omega$ for $0 < s < t$, where $s, t \in S$ where $M(t) \equiv \int_0^t Z(u) \, dW(u)$ which has values in $W$. In the following, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V, V'$.

\[
\langle BX(t), X(t) \rangle = \langle BX(s), X(s) \rangle + 2 \int_s^t \langle Y(u), X(t) \rangle \, du + \langle B(M(t) - M(s)), M(t) - M(s) \rangle
\]

\[
- \langle B(X(t) - X(s) - (M(t) - M(s))), X(t) - X(s) - (M(t) - M(s)) \rangle + 2 \langle BX(s), M(t) - M(s) \rangle
\]

Also for $t > 0$

\[
\langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + 2 \int_0^t \langle Y(u), X(t) \rangle \, du + 2 \langle BX_0, M(t) \rangle + \langle BM(t), M(t) \rangle - \langle B(X(t) - X_0 - M(t)), X(t) - X_0 - M(t) \rangle
\]

Proof: From the formula which is assumed to hold,

\[
BX(t) = BX_0 + \int_0^t Y(u) \, du + BM(t)
\]

\[
BX(s) = BX_0 + \int_0^s Y(u) \, du + BM(s)
\]

Then

\[
BM(t) - BM(s) + \int_s^t Y(u) \, du = BX(t) - BX(s)
\]

It follows that

\[
\langle B(M(t) - M(s)), M(t) - M(s) \rangle - \langle B(X(t) - X(s) - (M(t) - M(s))), X(t) - X(s) - (M(t) - M(s)) \rangle + 2 \langle BX(s), M(t) - M(s) \rangle
\]

\[
= \langle B(M(t) - M(s)), M(t) - M(s) \rangle - \langle B(X(t) - X(s)), X(t) - X(s) \rangle + 2 \langle B(X(t) - X(s)), M(t) - M(s) \rangle
\]

\[
- \langle B(M(t) - M(s)), M(t) - M(s) \rangle + 2 \langle BX(s), M(t) - M(s) \rangle
\]

Some terms cancel and this yields

\[
= - \langle B(X(t) - X(s)), X(t) - X(s) \rangle + 2 \langle BX(t), M(t) - M(s) \rangle
\]

\[
= - \langle B(X(t) - X(s)), X(t) - X(s) \rangle + 2 \langle B(M(t) - M(s)), X(t) \rangle
\]

\[
= - \langle B(X(t) - X(s)), X(t) - X(s) \rangle + 2 \left\langle B(X(t) - X(s) - \int_s^t Y(u) \, du), X(t) \right\rangle
\]

\[
= - \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle + 2 \langle BX(t), X(s) \rangle + 2 \langle BX(t), X(t) \rangle
\]

\[
- 2 \langle BX(s), X(t) \rangle - 2 \int_s^t \langle Y(u), X(t) \rangle \, du
\]

\[
= \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle - 2 \int_s^t \langle Y(u), X(t) \rangle \, du
\]
Therefore,
\[
\langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle = 2 \int_s^t \langle Y(u), X(t) \rangle du + \langle B(M(t) - M(s)) , M(t) - M(s) \rangle \\
- (B(X(t) - X(s)) - (M(t) - M(s))) , X(t) - X(s) - (M(t) - M(s))) \\
+ 2 \langle BX(s), M(t) - M(s) \rangle
\]

The case with \(X_0\) is similar. ■

The following lemma is what will be used. It says that you can replace \(BX(t)\) in Lemma\(^7\) with \(P(t)\) for all \(\omega\) off a set of measure zero. This just involves substituting \(P(t)\) for \(BX(t)\) in the above formula since these are equal.

**Lemma 8** For given \(t, s < t, s \notin S\), the following holds for a.e. \(\omega\)
\[
\langle P(t), X(t) \rangle = \langle P(s), X(s) \rangle + \\
+ 2 \int_s^t \langle Y(u), X(t) \rangle du + \langle B(M(t) - M(s)) , M(t) - M(s) \rangle \\
- ((P(t) - P(s) - B(M(t) - M(s))) , X(t) - X(s) - (M(t) - M(s))) \\
+ 2 \langle P(s) , M(t) - M(s) \rangle
\]
(21)

and for \(y \in W\),
\[
|\langle P(t), y \rangle| \leq \langle P(t), X(t) \rangle^{1/2} \langle By, y \rangle^{1/2}
\]
(22)

Let \(\{P_k\}\) denote a sequence of nested partitions of \([0, T]\) which satisfy the conditions needed in Lemma \(^1\) and also for \(X^t_k, X^r_k\) described there,
\[
X^t_k, X^r_k \to X \text{ in } K \equiv L^p([0, T] \times \Omega; V)
\]

Each \(P_k\) contains no points of \(S\). In what follows \(N\) will be a set of measure zero which includes \(N_0\). Each \(t \in P_k\) has the property that \(BX(t) = P(t)\) a.e., that is for \(\omega \in N_t\) a set of measure zero. Let \(N\) also include the set of measure zero
\[
\cup_k \cup \{N_t : t \in P_k\}
\]

Hence for each \(t \in P_k\), \(BX(t) = P(t)\) for all \(\omega \notin N\). Let
\[
\mathcal{D} \equiv \cup_k P_k.
\]

Thus \(\mathcal{D}\) is dense in \([0, T]\). Since \(BX(t) = P(t)\) a.e. \(\omega\) for \(t \notin S\), it follows that as \(k \to \infty\),
\[
P^t_k \equiv BX^t_k, \ P^r_k \equiv BX^r_k \text{ both converge to } P \text{ in } L^2([0, T] \times \Omega; W')
\]

For convenience, we only consider those points of \(P_k\) which are less than \(T\). These are the ones which are used in the left step functions.

## 4 The Main Estimate

The following estimate holds and it is this estimate which is the main idea in proving the Ito formula. The last assertion about continuity is like the well known result that if \(y \in L^p(0, T; V)\) and \(y' \in L^p(0, T; V')\), then \(y\) is actually continuous with values in \(H\), for \(V, H, V'\) a Gelfand triple.

In all that follows \(\{e_i\}\) will be the vectors of Lemma \(^3\)
\[
\sum_{i=1}^{\infty} \langle BX(t), e_i \rangle^2 = \langle BX(t), X(t) \rangle
\]
Lemma 9 In the situation of Section 3

\[ E \left( \sup_{t \in D} (P(t), X(t)) \right) \]
\[ < C \left( \|Y\|_{K'}, \|X\|_K, \|Z\|_J, \|BX_0, X_0\|_{L^1(\Omega)} \right) < \infty. \quad (23) \]

where \( D \) is a dense subset of \([0, T]\), such that for \( t \in D \), \( BX(t) = P(t) \) a.e. \( \omega \) where

\[ J = L^2 \left( [0, T] \times \Omega; \mathcal{L}_2 \left( Q^{1/2}U; W \right) \right), \quad K \equiv L^p \left( [0, T] \times \Omega; V \right), \]
\[ K' \equiv L^{p'} \left( [0, T] \times \Omega; V' \right), \]

the \( \sigma \) algebra being the progressively measurable sets. \( C \) is a continuous function of its arguments and \( C(0, 0, 0, 0) = 0 \). Also, for a.e. \( \omega \),

\[ \sup_{t \in [0, T]} \sum_{k} (P(t), e_k)^2 \leq C(\omega) < \infty \quad (24) \]

For \( t \in D \), then for all \( \omega \notin N \), a set of measure zero,

\[ \langle P(t), X(t) \rangle = \langle BX(t), X(t) \rangle \leq C(\omega) < \infty, \quad (25) \]

\[ \int_{\Omega} C(\omega) \, dP < \infty. \]

When each of \( \|Y\|_{K'}, \|X\|_K, \|Z\|_J, \|BX_0, X_0\|_{L^1(\Omega)} \) equal zero, \( C(\omega) \) can be taken to be 0 also. For a.e. \( \omega \), \( t \to P(t) \) is weakly continuous into \( W' \). In addition to this, \( P \) is progressively measurable into \( W' \).

**Proof:** For \( t_j > 0, X(t_j) \) is just the value of \( X \) at \( t_j \) but when \( t = 0 \), the definition of \( X(0) \) in this step function is \( X(0) \equiv 0 \). Consider the formula in Lemma 9. This is applied to \( P_k \) to obtain

\[ \langle P(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle = 2 \sum_{j=1}^{m-1} \int_{t_j}^{t_j+1} \langle Y(u), X_k(u) \rangle \, du + \]
\[ + 2 \sum_{j=1}^{m-1} \left( P(t_j), \int_{t_j}^{t_{j+1}} Z(u) \, dW \right) \]
\[ + \sum_{j=1}^{m-1} \langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle \]
\[ - \sum_{j=1}^{m-1} \langle P(t_{j+1}) - P(t_j) - B(M(t_{j+1}) - M(t_j)), X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j)) \rangle \]
\[ + 2 \int_0^{t_1} \langle Y(u), X(t_1) \rangle \, du + 2 \left( BX_0, \int_0^{t_1} Z(u) \, dW \right) + \left( BM(t_1), M(t_1) \right) \]
\[ - \langle P(t_1) - BX_0 - BM(t_1), X(t_1) - X_0 - M(t_1) \rangle \]
\[ \quad \quad \quad (26) \]

First consider the terms near to the end of the above expression,

\[ 2 \int_0^{t_1} \langle Y(u), X(t_1) \rangle \, du + 2 \left( BX_0, \int_0^{t_1} Z(u) \, dW \right) + \left( BM(t_1), M(t_1) \right) \]
Implicit Itô Formula

Each term of the above converges to 0 for a.e. \( \omega \) as \( k \to \infty \) and in \( L^1(\Omega) \) if a suitable subsequence is used. This follows right away for the second two terms from the Itô isometry and continuity properties of the stochastic integral. Consider the first term. This term is dominated by

\[
\left( \int_0^{t_1} \|Y(u)\|^{p'} \, du \right)^{1/p'} \left( \int_0^T \|X_k^{r}(u)\|^{p'} \, du \right)^{1/p} \\
\leq C(\omega) \left( \int_0^{t_1} \|Y(u)\|^{p'} \, du \right)^{1/p'} , \ C(\omega) < \infty
\]

By assumption, and Hölder’s inequality, the top expression converges to 0 in \( L^1(\Omega) \). Hence there is a further subsequence for which it converges pointwise.

At this time, not much is known about the last term in (26) but it is negative and is about to be neglected anyway. The reason it is negative is that it equals

\[- \langle B(X(t_1) - X_0 - M(t_1)) , X(t_1) - X_0 - M(t_1) \rangle\]

The term involving the stochastic integral equals

\[
2 \sum_{j=1}^{m-1} \left\langle P(t_j), \int_{t_j}^{t_{j+1}} Z(u) \, dW \right\rangle
\]

By Theorem 4, this equals

\[
2 \int_{t_1}^{t_m} (Z \circ J^{-1})^* P_k^l \circ JdW
\]

Also note that since \( \langle BM(t_1) , M(t_1) \rangle \) converges to 0 in \( L^1(\Omega) \) and for a.e. \( \omega \), the sum involving

\[
\langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle
\]

can be started at 0 rather than 1 at the expense of adding in a term which converges to 0 a.e. and in \( L^1(\Omega) \). Thus (26) is of the form

\[
\langle P(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle = e(k) + 2 \int_0^{t_m} \langle Y(u), X_k^{r}(u) \rangle \, du + \\
+2 \int_0^{t_m} (Z \circ J^{-1})^* P_k^l \circ JdW + \sum_{j=0}^{m-1} \langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle \\
- \sum_{j=1}^{m-1} \langle B(X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))), X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j)) \rangle \\
- \langle P(t_1) - BX_0 - BM(t_1), X(t_1) - X_0 - M(t_1) \rangle
\]

where \( e(k) \to 0 \) for a.e. \( \omega \) and also in \( L^1(\Omega) \).

By definition, \( M(t_{j+1}) - M(t_j) = \int_{t_j}^{t_{j+1}} ZdW \). Now it follows, on discarding the negative terms,

\[
\langle P(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle \leq e(k) + 2 \int_0^{t_m} \langle Y(u), X_k^{r}(u) \rangle \, du + \\
+2 \int_0^{t_m} (Z \circ J^{-1})^* P_k^l \circ JdW + \sum_{j=0}^{m-1} \left\langle B, \int_{t_j}^{t_{j+1}} ZdW, \int_{t_j}^{t_{j+1}} ZdW \right\rangle
\]
Thus also
\[
\langle P(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle \leq c(k) + 2 \int_0^T \langle Y(u), X_k^r(u) \rangle \, du + \\
+2 \int_0^{t_m} (Z \circ J^{-1})^* P_k^s \circ JdW + \sum_{j=0}^{m-1} \left( B \int_{t_j}^{t_{j+1}} ZdW, \int_{t_j}^{t_{j+1}} ZdW \right)
\]

The next task is to somehow take the expectation of both sides. The difficulty in doing this is that the stochastic integral is only a local martingale. Let
\[
\tau_p = \inf \{ t : \langle P_k(t), X_k(t) \rangle > p \}
\]
By right continuity of \(P_k^t\) and \(X_k^t\), this is a well defined stopping time. Then you obtain the above inequality stopped with \(\tau_p\). Take the expectation and use the Ito isometry to obtain
\[
\begin{aligned}
\int_\Omega \left( \sup_{t_m \in \mathcal{T}_K} \langle P(t_m \wedge \tau_p), X(t_m \wedge \tau_p) \rangle \right) \, dP \\
\leq E(\langle BX_0, X_0 \rangle) + 2 \|Y\|_{K^r} \|X_k^r\|_K \\
+ \|B\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|Z(u)\|^2 \, dP \, du \\
+ 2 \int_\Omega \left( \sup_{t \in [0,T]} \left| \int_0^{t \wedge \tau_p} (Z \circ J^{-1})^* (P_k^s)^{\tau_p} \circ JdW \right| \right) \, dP + E(\|e(k)\|)
\end{aligned}
\]
where the result of Lemma 1 that \(X_k^r\) converges to \(X\) in \(K\) shows the term \(2 \|Y\|_{K^r} \|X_k^r\|_K\) is bounded. Note that the constant \(C\) can be assumed to be a continuous function of
\[
\|Y\|_{K^r}, \|X\|_K, \|Z\|_J, \|BX_0, X_0\|_{L^1(\Omega)}
\]
which equals zero when all are equal to zero. (We can assume that \(\|X_k^r\|_K \leq 2 \|X\|_K\) by taking a suitable subsequence of the \(P_K\) if necessary.) The term involving the stochastic integral is next.

Let \(M(t) = \int_0^t (Z \circ J^{-1})^* (P_k^s)^{\tau_p} \circ JdW\). Then from the description of the quadratic variation,
\[
d[M] = \left\| (Z \circ J^{-1})^* (P_k^s)^{\tau_p} \circ J \right\|^2 \, ds
\]
Applying the Burkholder Davis Gundy inequality, for \(F(r) = r\) in that stochastic integral,
\[
2 \int_\Omega \left( \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* (P_k^s)^{\tau_p} \circ J \right| \right) \, dP \\
\leq C \int_\Omega \left( \int_0^T \left\| (Z \circ J^{-1})^* (P_k^s)^{\tau_p} \circ J \right\|^2 \, ds \right)^{1/2} \, dP
\]
(29)
So let \(\{g_i\}\) be an orthonormal basis for \(Q^{1/2}U\) and consider the integrand in the above.
\[
\sum_{i=1}^\infty \left( \langle (Z \circ J^{-1})^* (P_k^s)^{\tau_p} \rangle (J(g_i)), \langle J(g_i) \rangle \right)^2 = \sum_{i=1}^\infty \left( \langle (P_k^s)^{\tau_p}, Z(g_i) \rangle \right)^2
\]
From (22)
\[
\leq \sum_{i=1}^\infty \left( \langle (P_k^s)^{\tau_p}, (X_k^s)^{\tau_p} \rangle \langle BZ(g_i), Z(g_i) \rangle \right)
\]
\[
\leq \left( \sup_{t_m \in \mathcal{P}_k} \left\langle \left( P^l_k \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle \right) \|B\| \|Z\|_{L_2}^2
\]

It follows that the integral in [29] is dominated by
\[
C \int_\Omega \sup_{t_m \in \mathcal{P}_k} \left\langle \left( P^l_k \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle^{1/2} \|B\|^{1/2} \left( \int_0^T \|Z\|_{L_2}^2 \, ds \right)^{1/2} \, dP
\]

Now return to [28] From what was just shown,
\[
E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle \left( P^l_k \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle \right) 
\leq C + E \left( |e\left( k \right)| \right) + 2 \int_\Omega \left( \sup_{t \in [0,T]} \left| \int_0^t \left( Z \circ J^{-1} \right)^* \left( P^l_k \right)^{\tau_p} \circ J dW \right| \right) \, dP
\]
\[
\leq C + C \int_\Omega \sup_{t_m \in \mathcal{P}_k} \left\langle \left( P^l_k \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle^{1/2} \|B\|^{1/2} \left( \int_0^T \|Z\|_{L_2}^2 \, ds \right)^{1/2} \, dP
\]
\[
+ E \left( |e\left( k \right)| \right)
\]

It follows that
\[
\frac{1}{2} E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle \left( P^l_k \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle \right) \leq C + E \left( |e\left( k \right)| \right)
\]

Now let \( p \to \infty \) and use the monotone convergence theorem to obtain
\[
E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle P^l_k \left( t_m \right), X_k^l \left( t_m \right) \right\rangle \right) = E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle P \left( t_m \right), X \left( t_m \right) \right\rangle \right) 
\leq C + E \left( |e\left( k \right)| \right)
\]

The monotone convergence theorem applies because \( \tau_p \) merely restricts the values \( t_m \in \mathcal{P}_k \) which can be considered in the above supremum. As mentioned above, this constant \( C \) is a continuous function of
\[
\|Y\|_{K'}, \|X\|_{K}, \|Z\|_{J}, \|BX_0, X_0\|_{L^1(\Omega, H)}
\]

and equals zero when all of these quantities equal 0. Also, for each \( \varepsilon > 0 \),
\[
E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle P \left( t_m \right), X \left( t_m \right) \right\rangle \right) < C + \varepsilon
\]

whenever \( k \) is large enough.

Let \( D \) denote the union of all the \( \mathcal{P}_k \). Thus \( D \) is a dense subset of \([0, T]\) and by the monotone convergence theorem, it has just been shown, since the \( \mathcal{P}_k \) are nested, that for a constant \( C \) dependent only on the above quantities,
\[
E \left( \sup_{t \in D} \left\langle P \left( t \right), X \left( t \right) \right\rangle \right) \leq C + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary,
\[
E \left( \sup_{t \in D} \left\langle P \left( t \right), X \left( t \right) \right\rangle \right) = E \left( \sup_{t \in D} \left\langle BX \left( t \right), X \left( t \right) \right\rangle \right) \leq C
\]

This establishes [28] Now it follows right away that
\[
\sup_{t \in D} \left\langle P \left( t \right), X \left( t \right) \right\rangle = \sup_{t \in D} \left\langle BX \left( t \right), X \left( t \right) \right\rangle \leq C \left( \omega \right) < \infty \text{ a.e. } \omega
\]

(30)
where \( C = \int_{\Omega} C(\omega) \, dP \).

The function \( t \to \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 \) is obviously lower semicontinuous on \([0,T]\). This is because \( t \to P(t) \) is continuous into \( V' \). Now also for \( t \in D \) and \( \omega \notin N \), a fixed set of measure zero which is defined in terms of \( D \),

\[
\langle P(t), X(t) \rangle = \langle BX(t), X(t) \rangle = \sum_{k=1}^{\infty} \langle BX(t), e_k \rangle^2 = \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2
\]

and so, for \( t \in D \) this infinite sum equals \( \langle P(t), X(t) \rangle \), and it was just shown that \( \sup_{t \in D} \langle P(t), X(t) \rangle \leq C(\omega) \). Hence, if \( t \) is arbitrary and \( t_n \to t \) for \( t_n \in D \), it follows from Fatou’s lemma that

\[
\sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 \leq \liminf_{n \to \infty} \sum_{k=1}^{\infty} \langle P(t_n), e_k \rangle^2 = \liminf_{n \to \infty} \langle P(t_n), X(t_n) \rangle \leq C(\omega)
\]

and so

\[
\sup_{t \in D} \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 = \sup_{t \in [0,T]} \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 \leq C(\omega)
\]

This establishes \( 24 \).

Finally, consider the claim about weak continuity of \( P(t) \). From the above estimate, \( 30 \)

\[
\sup_{t \in D} \|BX(t)\|_{V'} < C(\omega).
\]

Now let \( t \in [0,T] \). Then there exists a sequence \( t_n \to t \) where \( t_n \in D \). Then for \( \omega \notin N \), \( BX(t_n) \) is bounded in \( W' \). But also \( BX(t_n) \) equals \( P(t_n) \) for all \( \omega \) off a single set of measure zero, the one which came from \( D \), and so from the definition of \( P(t) \),

\[
P(t_n) \to P(t) \quad \text{in} \quad V'
\]

and also \( P(t_n) \) is bounded in \( W' \). Therefore, there is a subsequence such that \( P(t_n) \to \zeta \) weakly in \( W' \). It follows \( P(t) = \zeta \) and \( \|P(t)\|_{W'} \leq C(\omega) \). Thus \( t \to P(t) \) is continuous into \( V' \) and bounded in \( V' \). If \( t_n \to t \), then if \( P(t_n) \) fails to converge weakly to \( P(t) \) in \( V' \), there would exist a subsequence still called \( t_n \) such that \( P(t_n) \) converges weakly in \( V' \) to \( \zeta \neq P(t) \). However, \( P(t_n) \to P(t) \) in \( V' \) and so \( \zeta = P(t) \) after all.

For \( \omega \notin N \) the set of measure zero off which the above computations were considered which came from the points of \( D \), it was just shown that \( P(t, \omega) \in W' \). Also, the formula for \( P(t, \omega) \) implies that this function is progressively measurable into \( V' \). Therefore,

\[
(t, \omega) \to \langle P(t, \omega), v \rangle
\]

is progressively measurable if \( v \in V \). Thus also, if \( \omega \notin N \), and \( v_n \to v \in W \),

\[
\langle P(t, \omega), v_n \rangle \to \langle P(t, \omega), v \rangle.
\]

Since each \( F_t \) is assumed to be complete, this shows from the Pettis theorem that \( t, \omega) \to P(t, \omega) \) is progressively measurable. ■

Recall that for a.e. \( t \), \( P(t, \omega) = BX(t, \omega) \) for a.e. \( \omega \). Also \( BX \) is obviously progressively measurable because \( X \) is. However, it is not clear that \( t \to BX(t) \) is continuous into \( V' \) for all \( t \in [0,T] \) off a set of measure zero. In a sense, \( P(t, \omega) \) is filling in the missing values of \( t \) retaining both progressive measurability and continuity in \( t \).

Consider the case where \( B = I \) and \( W = H \) so that the situation is that of a Gelfand triple, \( V \subseteq H = H' \subseteq V' \). Then in this case, \( P(t) = X(t) \) and the vectors \( \{e_k\} \) reduce to an orthonormal basis for \( H \) such that each \( e_k \in V \). Then the above sum

\[
\sum_k \langle P(t), e_k \rangle^2 = \sum_k \langle X(t), e_k \rangle^2 = \|X(t)\|^2_H = \langle P(t), X(t) \rangle.
\]

Thus the main estimate in the above lemma would imply

\[
E \left( \sup_{t \in [0,T]} \|X(t)\|^2_H \right) < C.
\]

This is the estimate in this special case which is found in \( 18 \). In this special case, this is also the thing which will be of use in the study of variational formulations for stochastic equations. However, in the case considered
here in which there is a possibly degenerate operator $B$, it is not clear that $\sum_k \langle P (t) , e_k \rangle^2 = \langle P (t) , X (t) \rangle$ for all $t$. The vectors $\{ e_k \}$ are not necessarily an orthonormal basis for $W$.

If there is interest in a more general conclusion which avoids the explicit reference to $D$, one can do the following. Let $\{ f_i \}$ be an orthonormal basis for $(BW)^\perp$ in $W'$. Then consider the function

$$ F (t) = \sum_{k=1}^{\infty} \langle P (t) , e_k \rangle^2 + \sum_i \langle P (t) , f_i \rangle^2 $$

From the above lemma, $t \mapsto P (t)$ is weakly continuous into $W'$. Therefore, $F (t)$ is lower semicontinuous for each $\omega$ off a single set of measure zero. It follows that for such $\omega$,

$$ \sup_{t \in [0,T]} F (t) = \sup_{t \in D} F (t) $$

For $t \in D$ recall that $BX (t) = P (t)$ and so for such $t$,

$$ F (t) = \sum_{k=1}^{\infty} \langle P (t) , e_k \rangle^2 = \langle BX (t) , X (t) \rangle = \langle P (t) , X (t) \rangle $$

and this holds for all $\omega$ off a single set of measure zero, depending on $D$. Therefore, for such $\omega$ and the above lemma,

$$ \sup_{t \in [0,T]} F (t) = \sup_{t \in D} F (t) = \sup_{t \in \mathcal{D}} \langle P (t) , X (t) \rangle \leq C (\omega) , \int_{\Omega} C (\omega) dP < \infty. $$

It follows that

$$ E \left( \sup_{t \in [0,T]} F (t) \right) \leq C < \infty $$

For any particular $t \notin S$, we know that $BX (t) = P (t)$ a.e. $\omega$. Hence

$$ F (t) = \sum_{k=1}^{\infty} \langle P (t) , e_k \rangle^2 = \langle BX (t) , X (t) \rangle = \langle P (t) , X (t) \rangle \text{ a.e. } \omega $$

Therefore, for that $t \notin S$,

$$ E (\langle P (t) , X (t) \rangle) = E (F (t)) \leq E \left( \sup_{t \in [0,T]} F (t) \right) \leq C < \infty $$

**Corollary 10** For $C$ in the above lemma, and for any $t \notin S$,

$$ E (\langle P (t) , X (t) \rangle) \leq C. $$

## 5 A Simplification Of The Formula

This lemma also provides a way to simplify [27]. First suppose $X_0 \in L^p (\Omega , V)$ so that $X - X_0 \in L^p ([0,T] \times \Omega , V)$. Refer to [27]. One term there is

$$ \langle P (t_1) - BX_0 - BM (t_1) , X (t_1) - X_0 - M (t_1) \rangle $$

It equals

$$ \langle B (X (t_1) - X_0 - M (t_1)) , X (t_1) - X_0 - M (t_1) \rangle $$

$$ \leq 2 \langle B (X (t_1) - X_0) , X (t_1) - X_0 \rangle + 2 \langle BM (t_1) , M (t_1) \rangle $$

$$ = 2 \langle P (t_1) - BX_0 , X (t_1) - X_0 \rangle + 2 \langle BM (t_1) , M (t_1) \rangle $$

It was observed above that $2 \langle BM (t_1) , M (t_1) \rangle \to 0$ a.e. and also in $L^1 (\Omega)$ as $k \to \infty$. Apply the above lemma for $X$ replaced with $\bar{X} (t) \equiv X (t) - X_0$ and denote the resulting $P (t)$ by $\bar{P} (t)$. The new $X_0$ equals 0. Also use $[0, t_1]$ instead of $[0, T]$. Thus the above reduces with this new $X$ to

$$ 2 \langle \bar{P} (t_1) , \bar{X} (t_1) \rangle + 2 \langle BM (t_1) , M (t_1) \rangle $$
From the above lemma,
\[ E\left(\langle P, X \rangle \right) < C \left(||Y||_{K'_{l_1}}, ||X||_{K_1}, ||Z||_{J_{l_1}}\right) \]
where in the definitions of \(K, K', J\) replace \([0, T]\) with \([0, t]\) and let the resulting spaces be denoted by \(K_t, K_t', J_t\). Therefore, this term converges to 0 in \(L^1(\Omega)\) as \(k \to \infty\). In addition to this, the term converges to 0 pointwise for a.e. \(\omega\) after passing to a suitable subsequence. Thus we can enlarge \(e(k)\) and neglect the last term of (27).

Then, it would follow from (27)
\[ \langle P(t), X(t) \rangle - \langle BX_0, X \rangle = e(k) + 2 \int_0^t \langle Y(u), X'_k(u) \rangle du + \]
\[ + 2 \int_0^t (Z \circ J^{-1})^* P_k \circ JdW \]
\[ + \sum_{j=0}^{m-1} \langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle \]
\[ - \sum_{j=1}^{m-1} \langle \Delta P(t_j) - B \Delta M(t_j), \Delta X(t_j) - \Delta M(t_j) \rangle \]
(31)
where \(e(k) \to 0\) in \(L^1(\Omega)\) and a.e. \(\omega\) and
\[ \Delta X(t_j) = X(t_{j+1}) - X(t_j) \]
\(\Delta M(t_j)\) being defined similarly.

Can you obtain this equation even in case \(X_0\) is not assumed to be in \(L^p(\Omega, V)\)? Let \(X_{0k} \in L^p(\Omega, V) \cap L^2(\Omega, W), X_{0k} \to X_0\) in \(L^2(\Omega, W)\).

\(\langle P(t), BX_0, X(t) - X_0 \rangle^{1/2} \leq \langle P(t), BX_{0k}, X(t) - X_{0k} \rangle^{1/2} \]
\[ + \langle B(X_{0k} - X_0), X_{0k} - X_0 \rangle^{1/2} \]
Also, restoring the superscript to identify the partition,
\[ P(t_k^k) - BX_{0k} = B(X_0 - X_{0k}) + \int_0^{t_k^k} Y(s) ds + B \int_0^{t_k^k} Z(s) dW. \]

Of course \(\|X - X_{0k}\|_K\) is not bounded, but for each \(k\) it is finite. Let \(n_k\) denote a subsequence of \(\{k\}\) such that
\[ \|X - X_{0k}\|_{K'_{l_1}} < 1/k. \]
Then from the above Lemma \[E((P(t_{1_k}^k) - BX_{0k}, X(t_{1_k}^k) - X_{0k})) \leq C\left(||Y||_{K'_{l_1}}, ||X - X_{0k}||_{K'_{l_1}}, ||Z||_{J_{l_1}}, \langle B(X_0 - X_{0k}), X_0 - X_{0k} \rangle_{L^1(\Omega)}\right) \]
\[ \leq C\left(||Y||_{K'_{l_1}}, \frac{1}{K}, ||Z||_{J_{l_1}}, \langle B(X_0 - X_{0k}), X_0 - X_{0k} \rangle_{L^1(\Omega)}\right) \]
Hence
\[ E((P(t_{1_k}^k) - BX_{0k}, X(t_{1_k}^k) - X_0)) \leq 2E((P(t_{1_k}^k) - BX_{0k}, X(t_{1_k}^k) - X_{0k})) + 2E((B(X_{0k} - X_0), X_{0k} - X_0)) \]
\[ \leq 2C\left(||Y||_{K'_{l_1}}, \frac{1}{K}, ||Z||_{J_{l_1}}, \langle B(X_0 - X_{0k}), X_0 - X_{0k} \rangle_{L^1(\Omega)}\right) \]
\[ + 2\|B\| \|X_{0k} - X_0\|^2_{L^2(\Omega, W)} \]
which converges to 0 as $k \to \infty$. It follows that there exists a suitable subsequence such that (31) holds even in the case that $X_0$ is only known to be in $L^2(\Omega, W)$. From now on, assume this subsequence for the partitions $P_k$. Thus $k$ will really be $n_k$ and it suffices to consider the limit as $k \to \infty$ of the equation of (31). To emphasize this point again, the reason for the above observations is to neglect

$$\langle P(t_1) - BX_0 - BM(t_1), X(t_1) - X_0 - M(t_1) \rangle$$

in passing to the limit as $k \to \infty$ provided a suitable subsequence is used.

In order to eventually obtain the Ito formula (18), there is a technical result which will be needed. It was mostly proved in Lemma 6.

**Lemma 11** Let $P(s) - P_k^l(s) \equiv \Delta_k(s)$. Then the following limit occurs.

$$\lim_{k \to \infty} P \left( \left[ \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right| \geq \varepsilon \right] \right) = 0$$

The stochastic integral

$$\int_0^t (Z \circ J^{-1})^* P \circ J dW$$

makes sense because $BX = P$ is $W'$ progressively measurable. Also, there exists a further subsequence, still denoted as $k$ such that

$$\int_0^t (Z \circ J^{-1})^* P_k^l \circ J dW \to \int_0^t (Z \circ J^{-1})^* P \circ J dW$$

uniformly on $[0,T]$ for a.e. $\omega$.

**Proof:** This follows from Lemma 6. The last conclusion follows from the usual use of the Borel Cantelli lemma, Ito formula, and the maximal inequalities for submartingales. It was shown in this lemma that

$$\lim_{k \to \infty} P \left( \left[ \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right| > \varepsilon \right] \right) = 0$$

Thus, one can obtain the existence of a subsequence, still denoted as $k$ such that

$$P \left( \left[ \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* \Delta_k \circ J dW \right| > 2^{-k} \right] \right) < 2^{-k}$$

and then uniform convergence is obtained for this subsequence off a set of measure zero. ■

From now on, the sequence will either be this subsequence or a further subsequence. Also $N$ will be enlarged so that for $\omega \notin N$, the above uniform convergence of the stochastic integrals takes place in addition to the other items above.

### 6 The Ito Formula

The next lemma is the Ito formula for $t \in D$, the dense subset consisting of all the mesh points of all partitions $P_k$.

**Proposition 12** Let $X_0 \in L^2(\Omega; W)$ and be $F_0$ measurable. There exists a dense subset of $[0,T]$ denoted as $D$ such that for every $t \in D$,

$$\langle P(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t (2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle_{\mathcal{L}_2}) \, ds$$

$$\quad + 2 \int_0^t (Z \circ J^{-1})^* P \circ J dW$$

(32)
where in the above formula, 
\[
(BZ, Z)_{L_2} = (R^{-1}BZ, Z)_{L_2(Q^1/2U,W)}
\]
for \( R \) the Riesz map from \( W \) to \( W' \). In addition to this, for all such \( t \in D \), \( E((BX(t), X(t))) = 
E((P(t), X(t))) = E((BX_0, X_0)) + E \left( \int_0^t 2 \langle Y(s), X(s) \rangle + (BZ, Z)_{L_2} ds \right) 
\tag{33} \]
In addition to this,
\[
E \left( \sup_{t \in D} \langle P(t), X(t) \rangle \right) < C \left( \|Y\|_{K'}, \|X\|_K, \|Z\|_J, \|(BX_0, X_0)\|_{L^1(\Omega)} \right) < \infty. \tag{34} \]

Note first that for \( \{g_i\} \) an orthonormal basis for \( Q^{1/2}(U) \),
\[
(R^{-1}BZ, Z)_{L_2} = \sum_i (R^{-1}BZ(g_i), Z(g_i))_W = \sum_i (BZ(g_i), Z(g_i))_{W^*} \geq 0
\]

**Proof:** Inequality (34) follows from the earlier lemma. In the situation of (16) let \( D \) be the union of the \( P_k \) described above as the union of all positive mesh points less than \( T \) for all the \( P_k \). Then, since these \( P_k \) are nested, if \( t \in D \), then \( t \in P_k \) for all \( k \) large enough. Consider (31)
\[
\langle P(t), X(t) \rangle - (BX_0, X_0) = c(k) + 2 \int_0^t \langle Y(u), X_k(u) \rangle du
\]
\[+ 2 \int_0^t (Z \circ J^{-1})^* P_k \circ JdW + \sum_{j=0}^{q_k-1} (B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j))
\]
\[- \sum_{j=1}^{q_k-1} (\Delta P(t_j) - \Delta BM(t_j), \Delta X(t_j) - \Delta M(t_j)) \tag{35} \]
where \( t_{q_k} = t \), \( \Delta X(t_j) = X(t_{j+1}) - X(t_j) \) and \( c(k) \to 0 \) in probability. By Lemma (11) the stochastic integral on the right converges uniformly to
\[
2 \int_0^t (Z \circ J^{-1})^* P \circ JdW
\]
off a set of measure zero. The deterministic integral on the right converges to
\[
2 \int_0^t \langle Y(u), X(u) \rangle du
\]
in \( L^1(\Omega) \) because \( X_k \to X \) in \( K \). Then
\[
P \left( \left\{ \sup_{t \in [0,T]} \left| \int_0^t \langle Y(u), X(u) \rangle du - \int_0^t \langle Y(u), X_k(u) \rangle du \right| > \lambda \right\} \right)
\leq P \left( \left\{ \sup_{t \in [0,T]} \left| \langle Y(u), X(u) - X_k(u) \rangle \right| du > \lambda \right\} \right)
\leq P \left( \left\{ \int_0^T \left| \langle Y(u), X(u) - X_k(u) \rangle \right| du > \lambda \right\} \right)
\leq \frac{1}{\lambda} \int_0^T \int_0^T \|Y(u)\| \|X(u) - X_k(u)\| du du dP \leq \frac{1}{\lambda} \|Y\|_{K'} \|X - X_k\|_K
\]
Since \( \|X - X_k\|_K \to 0 \) as \( k \to \infty \), it follows that there is a further subsequence, \( n_k \) such that
\[
P \left( \left\{ \sup_{t \in [0,T]} \left| \int_0^t \langle Y(u), X(u) \rangle du - \int_0^t \langle Y(u), X_{n_k}(u) \rangle du \right| > 2^{-k} \right\} \right) \leq 2^{-k}.
To save notation, refer to the subsequence as \( k \). Thus, for a suitable subsequence,

\[
\lim_{k \to \infty} \int_0^t \langle Y(u), X_k'(u) \rangle \, du = \int_0^t \langle Y(u), X(u) \rangle \, du
\]

uniformly off some set of measure zero. Consider the fourth term. It equals

\[
\sum_{j=0}^{q_k-1} \left( R^{-1} B \left( M(t_{j+1}) - M(t_j) \right), M(t_{j+1}) - M(t_j) \right)_W
\]

where \( R \) is the Riesz map from \( W \) to \( W' \). This equals

\[
\frac{1}{4} \left( \sum_{j=0}^{q_k-1} \left\| R^{-1} BM(t_{j+1}) + M(t_{j+1}) - (R^{-1} BM(t_j) + M(t_j)) \right\|^2 - \sum_{j=0}^{q_k-1} \left\| R^{-1} BM(t_{j+1}) - M(t_{j+1}) - (R^{-1} BM(t_j) - M(t_j)) \right\|^2 \right)
\]

From Theorem 2, as \( k \to \infty \), the above converges in probability to \( (t_{q_k} = t) \)

\[
\frac{1}{4} \left[ \left\| R^{-1} BM + M \right\| \langle t \rangle - \left\| R^{-1} BM - M \right\| \langle t \rangle \right]
\]

However, from the well known description of the quadratic variation of a martingale, the above equals

\[
\frac{1}{4} \left( \int_0^t \left\| R^{-1} BZ + Z \right\|_{\mathcal{L}_2}^2 \, ds - \int_0^t \left\| R^{-1} BZ - Z \right\|_{\mathcal{L}_2}^2 \, ds \right)
\]

which equals

\[
\int_0^t (R^{-1} BZ, Z)_{\mathcal{L}_2} \, ds \equiv \int_0^t (BZ, Z)_{\mathcal{L}_2} \, ds
\]

This is what was desired.

Note that in the case of a Gelfand triple, when \( W = H = H' \), the term \( \langle BZ, Z \rangle_{\mathcal{L}_2} \) will end up reducing to nothing more than \( \|Z\|_{\mathcal{L}_2}^2 \).

Thus all the terms in (35) converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

\[
\langle P(t), X(t) \rangle - \langle BX_0, X_0 \rangle = 2 \int_0^t \langle Y(u), X(u) \rangle \, du
\]

\[
+ 2 \int_0^t (Z \circ J^{-1})^* P \circ J \, dW + \int_0^t (BZ, Z)_{\mathcal{L}_2} \, ds - a
\]

where \( a \) is the limit in probability of the term

\[
\sum_{j=1}^{q_k-1} (\Delta P(t_j) - \Delta BM(t_j), \Delta X(t_j) - \Delta M(t_j))
\]

(37)

Let \( \pi_n \) be the projection onto span \( (e_1, \cdots, e_n) \) where \( \{e_k\} \) is a complete orthonormal basis for \( W \) with each \( e_k \in V \). Then using

\[
P(t_{j+1}) - P(t_j) - (BM(t_{j+1}) - BM(t_j)) = \int_{t_j}^{t_{j+1}} Y(s) \, ds,
\]

the troublesome term in (35) above is of the form

\[
\sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y(s), \Delta X(t_j) - \Delta M(t_j) \rangle \, ds
\]
Since \( P \), Letting \( \{ \}
converges to 0 as \( n \rightarrow \infty \).

Now it is known that

\[
\sum_{t_j} Y (s), X (t_j) - X (t_j) - \pi_n (M (t) - M (t_j)) ds
\]

which equals

\[
\sum_{t_j} (\Delta P (t_j) - \Delta BM (t_j), (I - \pi_n) \Delta M (t_j))
\]

Since \( P (t) = BX (t) \) for the \( t \) of interest in the above, the Cauchy Schwarz inequality implies the term of \( 39 \) is dominated by

\[
\left( \sum_{j=1}^{q_k-1} (\Delta P (t_j) - \Delta BM (t_j), (I - \pi_n) \Delta M (t_j)) \right)^{1/2}
\]

\[
\left( \sum_{j=1}^{q_k-1} |(I - \pi_n) \Delta M (t_j)|^2 \right)^{1/2}
\]

Now it is known that \( \sum_{j=1}^{q_k-1} (\Delta P (t_j) - \Delta BM (t_j), (I - \pi_n) \Delta M (t_j)) \) converges in probability to \( a \geq 0 \). If you take the expectation of the square of the other factor above, it is no larger than

\[
\| B \| E \left( \sum_{j=1}^{q_k-1} \| (I - \pi_n) \Delta M (t_j) \|_W^2 \right)
\]

\[
= \| B \| E \left( \sum_{j=1}^{q_k-1} \left\| (I - \pi_n) \int_{t_j}^{t_j+1} Z (s) dW (s) \right\|_W^2 \right)
\]

\[
= \| B \| \sum_{j=1}^{q_k-1} E \left( \left\| \int_{t_j}^{t_j+1} (I - \pi_n) Z (s) dW (s) \right\|_W^2 \right)
\]

\[
= \| B \| \sum_{j=1}^{q_k-1} E \left( \left\| \int_{t_j}^{t_j+1} (I - \pi_n) Z (s) \right\|_{L^2 (Q^{1/2} U, W)}^2 \right)
\]

Letting \( \{ g_i \} \) be an orthonormal basis for \( Q^{1/2} U \),

\[
= \| B \| \int_{t_0}^T \sum_{i=1}^{t} \left\| (I - \pi_n) Z (s) (g_i) \right\|_W^2 \ dP
\]

The integrand \( \sum_{i=1}^{t} \left\| (I - \pi_n) Z (s) (g_i) \right\|_W^2 \) converges to 0. Also, it is dominated by

\[
\sum_{i=1}^{t} \left\| Z (s) (g_i) \right\|_W^2 \equiv \left\| Z \right\|_{L^2 (Q^{1/2} U, W)}^2
\]

which is given to be in \( L^1 ([0, T] \times \Omega) \). Therefore, from the dominated convergence theorem, the expression in \( 41 \) converges to 0 as \( n \rightarrow \infty \) independent of \( k \).
Thus the expression in (40) is of the form $f_k g_{nk}$ where $f_k$ converges in probability to $a^{1/2}$ as $k \to \infty$ and $g_{nk}$ converges in probability to 0 as $n \to \infty$ independent of $k$. Now this implies $f_k g_{nk}$ converges in probability to 0. Here is why.

\[
P(\{|f_k g_{nk}| > \varepsilon\}) \leq P(2\delta |f_k| > \varepsilon) + P(2C_\delta |g_{nk}| > \varepsilon) \\
\leq P(2\delta |f_k - a^{1/2}| + 2\delta |a^{1/2}| > \varepsilon) + P(2C_\delta |g_{nk}| > \varepsilon)
\]

where $\delta |f_k| + C_\delta |g_{nk}| > |f_k g_{nk}|$ and $\lim_{\delta \to 0} C_\delta = \infty$. Pick $\delta$ small enough that $\varepsilon - 2\delta a^{1/2} > \varepsilon/2$. Then this is dominated by

\[
P(2\delta |f_k - a^{1/2}| > \varepsilon/2) + P(2C_\delta |g_{nk}| > \varepsilon)
\]

Fix $n$ large enough that the second term is less than $\eta$ for all $k$. Now taking $k$ large enough, the above is less than $\eta$. It follows the expression in (40) and consequently in (39) converges to 0 in probability.

Now consider the other term (38) using the $n$ just determined. This term is of the form

\[
\sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y(s), X^i_k(s) - X^i_k(s) - \pi_n (M^i_k(s) - M^i_k(s)) \rangle ds \\
= \int_{t_1}^{t} \langle Y(s), X^i_k(s) - X^i_k(s) - \pi_n (M^i_k(s) - M^i_k(s)) \rangle ds
\]

where $M^i_k$ denotes the step function

\[
M^i_k(t) = \sum_{i=0}^{m_k-1} M(t_{i+1}) \mathcal{X}_{[t_i,t_{i+1}]}(t)
\]

with $M^i_k$ defined similarly as a step function featuring the value of $M$ at the left end of each interval. The term

\[
\int_{t_1}^{t} \langle Y(s), \pi_n (M^i_k(s) - M^i_k(s)) \rangle ds
\]

converges to 0 for a.e. $\omega$ as $k \to \infty$ thanks to continuity of $t \to M(t)$. However, more is needed than this. Define the stopping time

\[
\tau_p = \inf \{ t > 0 : \|M(t)\|_W > p \}.
\]

Then $\tau_p \to \infty$ a.e. $\omega$. Let

\[
A_k = \left[ \left\| \int_{t_1}^{t} \langle Y(s), \pi_n (M^i_k(s) - M^i_k(s)) \rangle ds \right\| > \varepsilon \right] \\
P(A_k) = \sum_{p=1}^{\infty} P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty]))
\]

(42)

Now

\[
P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) \leq P(A_k \cap ([\tau_p = \infty])) \\
\leq P \left( \left\| \int_{t_1}^{t} \langle Y(s), \pi_n (M^\tau_{r_p})(s) - (M^\tau_{r_p})^i_k(s) \rangle ds \right\| > \varepsilon \right)
\]

This is no larger than an expression of the form

\[
\frac{C_n}{\varepsilon} \int_{t_1}^{t} \int_0^T \|Y(s)\|_{V'} \left\| (M^\tau_{r_p})^i_k(s) - (M^\tau_{r_p})^i_k(s) \right\|_W d\sigma dP
\]

(43)

The inside integral converges to 0 by continuity of $M$. Also, thanks to the stopping time, the inside integral is dominated by an expression of the form

\[
\int_0^T \|Y(s)\|_{V'}, 2p d\sigma
\]
and this is a function in $L^1(\Omega)$ by assumption on $Y$. It follows that the integral in [43] converges to 0 as $k \to \infty$ by the dominated convergence theorem. Hence

$$\lim_{k \to \infty} P (A_k \cap ([\tau_p = \infty])) = 0.$$ 

Since the sets $[\tau_p = \infty] \setminus [\tau_{p-1} < \infty]$ are disjoint, the sum of their probabilities is finite. Hence by the dominated convergence theorem applied to the sum,

$$\lim_{k \to \infty} P (A_k) = \sum_{p=0}^{\infty} \lim_{k \to \infty} P (A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) = 0$$ 

Thus $\int_0^t \langle Y(s), \pi_n (M_k^r (s) - M_k^l (s)) \rangle \, ds$ converges to 0 in probability as $k \to \infty$.

Now consider the other part of this expression,

$$\int_0^t \langle Y(s), X_k^r (s) - X_k^l (s) \rangle \, ds.$$ 

This converges to 0 in $L^1(\Omega)$ because it is of the form

$$\int_0^t \langle Y(s), X_k^r (s) \rangle \, ds - \int_1^t \langle Y(s), X_k^l (s) \rangle \, ds$$

and both $X_k^r$ and $X_k^l$ converge to $X$ in $K$. Therefore, the expression

$$\sum_{j=1}^{q_k-1} \langle \Delta P (t_j) - \Delta BM (t_j), \Delta X (t_j) - \Delta M (t_j) \rangle$$

converges to 0 in probability. This establishes the desired formula for $t \in D$.

To verify the last formula, let $t \in D$. Then $t \in \mathcal{P}_k$ for some $k$. Define

$$\tau_p = \inf \{ t \in \mathcal{P}_k : \| P (t) \|_{W^*} > p \}$$

This is just the first hitting time of an adapted process so this is a well defined stopping time. Then stop both sides of [43]. Thus

$$\langle P (t \wedge \tau_p), X (t \wedge \tau_p) \rangle = \langle BX_0, X_0 \rangle + \int_0^{t \wedge \tau_p} \left( 2 \langle Y(s), X(s) \rangle + \langle B Z, Z \rangle \right) \, ds$$

$$+ 2 \int_0^{t \wedge \tau_p} X_{[a, \tau_p]} (Z \circ J^{-1})^* p^{\tau_p} \circ J dW$$

Now the last term is a martingale and you can take expectations of both sides. Then

$$E \langle P (t \wedge \tau_p), X (t \wedge \tau_p) \rangle = E \langle BX_0, X_0 \rangle + E \int_0^{t \wedge \tau_p} \left( 2 \langle Y(s), X(s) \rangle + \langle B Z, Z \rangle \right) \, ds$$

Then the integrands

$$\langle P (t \wedge \tau_p), X (t \wedge \tau_p) \rangle$$

are uniformly integrable because

$$\langle P (t \wedge \tau_p), X (t \wedge \tau_p) \rangle \leq \sup_{t \in D} \langle P (t), X (t) \rangle$$

which was shown to be in $L^1(\Omega)$. [44] Then apply the Vitali convergence theorem to the left and the dominated convergence theorem on the right to obtain the formula

$$E \langle P (t), X (t) \rangle = E \langle BX_0, X_0 \rangle + E \int_0^t \left( 2 \langle Y(s), X(s) \rangle + \langle B Z, Z \rangle \right) \, ds.$$
Also we have the following improved version of Lemma 9 in the case that the integral equation holds for all \( t \) off a set of measure zero. See [18] for a similar special case involving a Gelfand triple and \( B = I \). That is, for \( \omega \) off a set of measure zero,

\[
BX(t) = BX_0 + \int_0^t Y(s) \, ds + B \int_0^t Z(s) \, dW(s)
\]

for all \( t \in [0, T] \).

**Lemma 13** In the above situation where, off a set of measure zero, [17] the above integral equation holds for all \( t \in [0, T] \), and \( X \) is progressively measurable into \( V \),

\[
E \left( \sup_{t \in [0,T]} \langle BX(t), X(t) \rangle \right) < C \left( \|Y\|_{K'}, \|X\|_K, \|Z\|_J, \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right) < \infty.
\]

where

\[
J = L^2 \left( [0, T] \times \Omega; \mathbb{L}_2 \left( Q^{1/2} U; W \right) \right), K \equiv L^p \left( [0, T] \times \Omega; V \right),
\]

\[
K' \equiv L^p \left( [0, T] \times \Omega; V' \right).
\]

Also, \( C \) is a continuous function of its arguments and \( C(0, 0, 0, 0) = 0 \). Thus for a.e. \( \omega \),

\[
\sup_{t \in [0,T]} \langle BX(t, \omega), X(t, \omega) \rangle \leq C(\omega) < \infty.
\]

For a.e. \( \omega, t \to BX(t, \omega) \) is weakly continuous with values in \( \mathbb{W}' \). Also \( t \to \langle BX(t), X(t) \rangle \) is lower semicontinuous.

**Proof of Lemma 13** In the situation of this lemma, \( P(t) = BX(t) \) for all \( t \) provided \( \omega \) is off a single set of measure zero. Thus, there is a countable dense set \( \mathcal{D} \) such that

\[
E \left( \sup_{t \in \mathcal{D}} \langle BX(t), X(t) \rangle \right) = E \left( \sup_{t \in \mathcal{D}} \sum_{k=1}^{\infty} \langle BX(t), e_k \rangle^2 \right)
\]

\[
= E \left( \sup_{t \in \mathcal{D}} \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 \right)
\]

\[
< C \left( ||Y||_{K'}, ||X||_K, ||Z||_J, ||\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right) < \infty.
\]

(44)

Now the function \( t \to \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 \) is clearly lower semicontinuous. This is because the partial sums are all continuous. Therefore, off the exceptional set,

\[
\sup_{t \in \mathcal{D}} \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 = \sup_{t \in [0,T]} \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 = \sup_{t \in [0,T]} \langle BX(t), X(t) \rangle
\]

It follows that the desired estimates of Lemma 13 are valid. \( \blacksquare \)

**Theorem 14** Suppose that off a set of measure zero, [17] holds for all \( t \) so that \( BX(t) = P(t) \). Then off a set of measure zero, for every \( t \in [0, T] \),

\[
\langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_{0}^{t} \left( 2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle_{L^2} \right) \, ds
\]
Consider that last term. It equals

\[ + 2 \int_0^t (Z \circ J^{-1})^* BX \circ JdW \quad (45) \]

Also

\[ E (\langle BX (t), X (t) \rangle) = \]
\[ E (\langle BX_0, X_0 \rangle) + E \left( \int_0^t (2 \langle Y (s), X (s) \rangle + \langle BZ, Z \rangle_{L^2}) \, ds \right) \quad (46) \]

The quadratic variation of the stochastic integral is dominated by

\[ C \int_0^t \|Z\|^2_{L^2} \|BX\|^2_{W'} \, ds \quad (47) \]

for a suitable constant C. Also \( t \to BX \) is continuous into \( \mathcal{W}' \).

**Proof:** Let \( t \notin D \). For \( t > 0 \), let \( t (k) \) denote the largest point of \( \mathcal{P}_k \) which is less than \( t \). Suppose \( t (m) < t (k) \). Hence \( m \leq k \). Then

\[ P (t (m)) = BX_0 + \int_0^{t (m)} Y (s) \, ds + B \int_0^{t (m)} Z (s) \, dW (s), \]

Thus for \( t > t (m) \),

\[ P (t) - P (t (m)) = \int_{t (m)}^t Y (s) \, ds + B \int_{t (m)}^t Z (s) \, dW (s) \]

which is the same sort of thing studied so far except that it starts at \( t (m) \) rather than at 0 and \( BX_0 = 0 \). Therefore, from Proposition 12 it follows

\[ \langle P (t (k)) - P (t (m)), X (t (k)) - X (t (m)) \rangle = \int_{t (m)}^{t (k)} (2 \langle Y (s), X (s) - X (t (m)) \rangle + \langle BZ, Z \rangle_{L^2}) \, ds \]
\[ + 2 \int_{t (m)}^{t (k)} (Z \circ J^{-1})^* (P (s) - P (t (m))) \circ JdW \quad (48) \]

Consider that last term. It equals

\[ 2 \int_{t (m)}^{t (k)} (Z \circ J^{-1})^* (P (s) - P_m (s)) \circ JdW \quad (49) \]

This is dominated by

\[ 2 \left| \int_0^{t (k)} (Z \circ J^{-1})^* (P (s) - P_m (s)) \circ JdW \right| \]
\[ - \int_0^{t (m)} (Z \circ J^{-1})^* (P (s) - P_m (s)) \circ JdW \right| \]
\[ \leq 4 \sup_{t \in [0, T]} \left| \int_0^t (Z \circ J^{-1})^* (P (s) - P_m (s)) \circ JdW \right| \]

In Lemma 11, the above expression was shown to converge to 0 in probability. Therefore, by the usual appeal to the Borel Cantelli lemma, there is a subsequence still referred to as \( \{m\} \), such that the above expression converges to 0 pointwise in \( \omega \) for all \( \omega \) off some set of measure 0 as \( m \to \infty \). It follows there is a set of measure 0 such that for \( \omega \) not in that set, \( 49 \) converges to 0 in \( \mathbb{R} \). Similar reasoning shows the first term in the non stochastic integral of 48 is dominated by an expression of the form

\[ 4 \int_0^T \|Y (s), X (s) - X_m (s)\| \, ds \]
which clearly has a subsequence which converges to 0 for \( \omega \) not in some set of measure zero because \( X_{m}^{t} \) converges in \( K \) to \( X \). Finally, it is obvious that

\[
\lim_{m \to \infty} \int_{t(m)}^{t(k)} \langle BZ, Z \rangle_{\mathcal{L}_{2}} ds = 0 \text{ for a.e. } \omega
\]

due to the assumptions on \( Z \). For \( \{g_{i}\} \) an orthonormal basis of \( Q^{1/2}(U) \),

\[
\langle BZ, Z \rangle_{\mathcal{L}_{2}} = \sum_{i} (R^{-1}BZ(g_{i}), Z(g_{i})) = \sum_{i} \langle BZ(g_{i}), Z(g_{i}) \rangle
\]

\[
\leq \|B\| \sum_{i} \|Z(g_{i})\|_{W}^{2} \in L^{1}(0,T) \text{ a.e.}
\]

This shows that for \( \omega \) off a set of measure 0

\[
\lim_{m,k \to \infty} \langle P(t(k)) - P(t(m)), X(t(k)) - X(t(m)) \rangle = 0 \quad (50)
\]

Then for \( x \in W \),

\[
|\langle P(t(k)) - P(t(m)), x \rangle|\\
\leq |\langle P(t(k)) - P(t(m)), X(t(k)) - X(t(m)) \rangle|^{1/2} \langle Bx, x \rangle^{1/2}\\
\leq |\langle P(t(k)) - P(t(m)), X(t(k)) - X(t(m)) \rangle|^{1/2} \|B\|^{1/2} \|x\|_{W}
\]

and so

\[
\lim_{m,k \to \infty} \|P(t(k)) - P(t(m))\|_{W} = 0
\]

Recall \( t \) was arbitrary and \( \{t(k)\} \) is a sequence converging to \( t \). Then the above has shown that \( \{P(t(k))\}_{k=1}^{\infty} \) is a convergent sequence in \( W' \). Does it converge to \( P(t) \)? Let \( \xi(t) \in W' \) be what it converges to. Letting \( v \in V \) then, since \( t \to P(t) \) is continuous into \( V' \),

\[
\langle \xi(t), v \rangle = \lim_{k \to \infty} \langle P(t(k)), v \rangle = \langle P(t), v \rangle,
\]

and now, since \( V \) is dense in \( W \), this implies \( \xi(t) = P(t) \). Thus \( P(t) = \lim_{k \to \infty} P(t(k)) \). Next consider the product \( \langle P(t), X(t) \rangle \).

For every \( t \in \mathcal{D} \),

\[
\langle P(t), X(t) \rangle = \langle BX_{0}, X_{0} \rangle + \int_{0}^{t} \langle 2(Y(s), X(s)) + \langle BZ, Z \rangle_{\mathcal{L}_{2}} \rangle ds ds
\]

\[
+ 2 \int_{0}^{t} (Z \circ J^{-1})^{*} P \circ JdW \quad (51)
\]

Does this formula hold for all \( t \in [0,T] \)?

\[
|\langle P(t(k)), X(t(k)) \rangle - \langle P(t), X(t) \rangle|
\]

\[
\leq |\langle P(t(k)), X(t(k)) \rangle - \langle P(t), X(t(k)) \rangle| + |\langle P(t), X(t(k)) \rangle - \langle P(t), X(t) \rangle|
\]

\[
= |\langle P(t(k)) - P(t), X(t(k)) \rangle| + |\langle P(t), X(t(k)) - X(t) \rangle|
\]

Since \( BX(t) = P(t) \), the Cauchy Schwarz inequality implies that the above expression is dominated by

\[
\leq \langle P(t(k)) - P(t), X(t(k)) - X(t) \rangle^{1/2}
\]

\[
\cdot \left( \langle BX(t(k)), X(t(k)) \rangle^{1/2} + \langle BX(t), X(t) \rangle^{1/2} \right)
\]
Also,
\[ \langle P(t(k)) - P(t), X(t(k)) - X(t) \rangle = \langle BX(t) - P(t(k)), X(t) - X(t(k)) \rangle \]
\[ = \langle BX(t), X(t) \rangle - \langle P(t), X(t(k)) \rangle - \langle P(t(k)), X(t) \rangle + \langle P(t(k)), X(t(k)) \rangle \]

The expression above simplifies to
\[ \langle BX(t), X(t) \rangle - 2 \langle P(t), X(t(k)) \rangle + \langle P(t(k)), X(t(k)) \rangle \]
which is clearly lower semicontinuous in \( t \) due to the continuity of \( P(t) \) into \( V' \) and the equation
\[ \langle BX(t), X(t) \rangle = \sum_{k=1}^{\infty} \langle BX(t), e_k \rangle^2 = \sum_{k=1}^{\infty} \langle P(t), e_k \rangle^2 \]

Summarizing the above, this has shown that
\[ |\langle P(t(k)), X(t(k)) \rangle - \langle P(t), X(t) \rangle| \leq \langle P(t(k)) - P(t), X(t(k)) - X(t) \rangle^{1/2}, \]
\[ \left( \langle BX(t(k)), X(t(k)) \rangle^{1/2} + \langle BX(t), X(t) \rangle^{1/2} \right) \]
and also that \( t \to \langle P(t(k)) - P(t), X(t(k)) - X(t) \rangle^{1/2} \) is lower semicontinuous.

Consider the right side of the above.
\[ t \to \langle P(t(k)) - BX(t), X(t(k)) - X(t) \rangle \]
Since \( \langle BX(t), X(t) \rangle = \langle P(t), X(t) \rangle \) is bounded, it follows that
\[ |\langle P(t(k)), X(t(k)) \rangle - \langle P(t), X(t) \rangle| \leq C \langle P(t(k)) - P(t), X(t(k)) - X(t) \rangle^{1/2} \]
From the above, the right side equals a lower semicontinuous function. Therefore, passing to a limit and using the lower semicontinuity,
\[ |\langle P(t(k)), X(t(k)) \rangle - \langle P(t), X(t) \rangle| \leq C \lim_{m \to \infty} \inf \langle P(t(k)) - P(t(m)), X(t(k)) - X(t(m)) \rangle^{1/2} < \varepsilon \]
provided \( k \) is sufficiently large (by (50)). Since \( \varepsilon \) is arbitrary,
\[ \lim_{k \to \infty} \langle P(t(k)), X(t(k)) \rangle = \langle BX(t), X(t) \rangle. \]

It follows that for \( \omega \) off the set of measure zero \( N \), the formula (51) is valid for all \( t \). Now this formula shows that off a set of measure zero, \( t \to \langle P(t), X(t) \rangle \) is continuous.

This implies that \( t \to P(t) = BX(t) \) is continuous with values in \( W' \). Here is why. The fact that the formula (51) holds for all \( t \) implies that \( t \to \langle BX(t), X(t) \rangle \) is continuous. Then for \( x \in W' \),
\[ |\langle BX(t) - BX(s), x \rangle| \leq \langle B(X(t) - X(s)), X(t) - X(s) \rangle^{1/2} \|B\|^{1/2} \|x\|_{W'} . \]

Also
\[ \langle B(X(t) - X(s)), X(t) - X(s) \rangle \]
\[ = \langle BX(t), X(t) \rangle + \langle BX(s), X(s) \rangle - 2 \langle BX(t), X(s) \rangle \]
By weak continuity of \( t \to BX(t) \) shown earlier,
\[ \lim_{t \to s} \langle BX(t), X(s) \rangle = \langle BX(s), X(s) \rangle . \]
Therefore,
\[ \lim_{t \to s} \langle B(X(t) - X(s)), X(t) - X(s) \rangle = 0 \]
and so the inequality (53) implies the continuity of \( t \to BX(t) \) into \( W' \).

Now consider the claim about the expectation. Since the stochastic integral
\[
2 \int_0^t (Z \circ J^{-1})^* P \circ JdW
\]
is only a local martingale, it is necessary to employ a stopping time. Since \( t \to \langle BX(t), X(t) \rangle \) is continuous, one can define a stopping time
\[
\tau_p \equiv \inf \{ t > 0 : \langle BX(t), X(t) \rangle > p \}
\]
Then use the stopping time in both sides of (53) and take the expectation. The stopped local martingale has expectation equal to 0. Thus
\[
E(\langle BX^{\tau_p}(t), X^{\tau_p}(t) \rangle) = E(\langle BX_0, X_0 \rangle)
\]
\[
+ E \left( \int_0^t \mathcal{X}_{[0,\tau_p]} \left( 2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle_{L_2} \right) ds \right)
\]
Next use the dominated convergence theorem on the right and the monotone convergence theorem on the left to let \( p \to \infty \) and obtain the desired result. The claim about the quadratic variation follows from the description of the quadratic variation for a stochastic integral.

Another interesting observation is that \( t \to BX(t) \) is continuous into \( W' \).
\[
\langle BX(t) - BX(s), X(t) - X(s) \rangle = \langle BX(t), X(t) \rangle + \langle BX(s), X(s) \rangle - 2 \langle BX(t), X(s) \rangle
\]
From the above formula, it is known that \( t \to \langle BX(t), X(t) \rangle \) is continuous. It was also shown above that \( t \to BX(t) \) is weakly continuous into \( W' \). Therefore, you could let \( t \to s \) and conclude that
\[
\lim_{t \to s} \langle BX(t) - BX(s), X(t) - X(s) \rangle = 0
\]
It follows that for \( w \in W \),
\[
\langle BX(t) - BX(s), w \rangle \leq \langle BX(t) - BX(s), X(t) - X(s) \rangle^{1/2} \langle Bw, w \rangle^{1/2}
\]
\[
\leq \langle BX(t) - BX(s), X(t) - X(s) \rangle^{1/2} \| B \|^{1/2} \| w \|
\]
and so
\[
\| BX(t) - BX(s) \|_{W'} \leq \langle BX(t) - BX(s), X(t) - X(s) \rangle^{1/2} \| B \|^{1/2}
\]
which converges to 0 as \( t \to s \).

7 An application to evolution equations

First we consider the case of a stochastic equation in a single Hilbert space. Here we give an example of how the Ito formula can be used to obtain theorems of existence and uniqueness. This begins with an introductory result on evolution equations in a single Hilbert space which is included for the sake of completeness. In what follows, \( H \) is a separable Hilbert space. It will be assumed that for each \( t, \omega \),
\[
u \to A(t, u, \omega)
\]
is a mapping from \( H \) to \( H \). Assume also that
\[
(t, u, \omega) \to A(t, u, \omega)
\]
is progressively measurable.

It is possible to assume only that \( u \to A(t, u, \omega) \) is continuous and base the theory on this. It is more troublesome because you end up having to consider finite dimensional subspaces and it would distract attention from the issue of interest in this paper. Therefore, it will be assumed here that for each \( B(0, r) \) the restriction of \( A(t, \cdot, \omega) \) to \( B(0, r) \) is Lipschitz continuous. Thus
\[
| A(t, u, \omega) - A(t, v, \omega) | \leq K_r | u - v |
\]
(54)
It is important to get an estimate now. From the standard Ito formula or Theorem 14, letting the boundedness of \( u \)

Thus from localization as described in [18] and [6],

That a unique progressively measurable solution exists follows readily from showing that a high enough power of \( u \) is given. It is routine to generalize this to the case where \( \Phi \) depends on the unkown function \( u \).

Then under these conditions, we can prove the following theorem.

**Theorem 15** Let \( u \rightarrow A(t,u,\omega) \) be locally Lipschitz in the sense that for each \( B(0,r) \), \( A \) restricted to \( B(0,r) \) is Lipschitz. Also suppose \( (t,u,\omega) \rightarrow A(t,u,\omega) \) is progressively measurable. Then if \( u_0 \in L^2(\Omega,H) \) with \( u_0 \) measurable in \( F_0 \),

where \( C \in L^1([0,T] \times \Omega), C \geq 0 \). It is also assumed that

Then under these conditions, we can prove the following theorem.

**Proof:** Let \( P_n \) denote the projection onto \( B(0,\varrho^n) \) and let \( u_n \) be the solution to

That a unique progressively measurable solution exists follows readily from showing that a high enough power of an operator is a contraction map, just as in the deterministic case. The solution holds for all \( t \in [0,T] \) for \( \omega \) off a set of measure zero.

Next let

Thus from localization as described in [18] and [6],

\[
\frac{1}{2} |u_n^\tau(t)|^2 - \frac{1}{2} |u_0|^2 + \int_0^t \mathcal{L}_{[0,\tau_n]} [A(u_n^\tau), u_n^\tau] ds = \int_0^t \mathcal{L}_{[0,\tau_n]} \|\Phi\|^2 ds + M(t)
\]

where \( M(t) \) is a local martingale with

Then from maximal estimates and Burkholder Davis Gundy inequality,

\[
P \left( \left\{ \sup_{t \in [0,T]} \left| \int_0^t \mathcal{L}_{[0,\tau_n]} [A(u_n^\tau), u_n^\tau] ds - \int_0^t \mathcal{L}_{[0,\tau_n]} \|\Phi\|^2 ds \right| > \lambda \right\} \right) \\
\leq \frac{1}{\lambda} \int_{\Omega} \sup \{|M(t)|, t \in [0,T]\} \, dP \leq C \frac{1}{\lambda} \int_{\Omega} [M(T)]^{1/2} \, dP
\]

Now from the description of the quadratic variation for stochastic integrals and using the stopping time,

\[
\leq C \frac{1}{\lambda} \int_{\Omega} \left( \int_0^T \|\Phi\|^2 \, 2^n \right)^{1/2} dP = \frac{C(\Phi)}{\lambda} 2^{n/2}
\]
The above holds for each \( n \). Let \( \lambda = \left(\frac{3}{2}\right)^n \). Then the above implies
\[
P \left( \sup_{t \in [0, T]} \left| \frac{1}{2} |u_n^{\tau_n} (t)|^2 - \frac{1}{2} |u_0|^2 - k \int_0^t |u_n^{\tau_n} (s)|^2 \right| ds - \int_0^t \| \Phi \|^2 ds - \int_0^t Cds \right) > \left( \frac{3}{2} \right)^n \leq C (\Phi) \frac{2^n/2}{(3/2)^n} \leq C (\Phi) (.96)^n
\]

By the Borel Cantelli lemma, it follows that there exists a set of measure zero \( N \) such that for \( \omega \notin N \), all \( n \) large enough, say \( n \geq M (\omega) \) and \( t \),
\[
\frac{1}{2} |u_n^{\tau_n} (t)|^2 - \frac{1}{2} |u_0|^2 - k \int_0^t |u_n^{\tau_n} (s)|^2 ds - \int_0^t \| \Phi \|^2 ds - \int_0^t Cds \leq \left( \frac{3}{2} \right)^n
\]
for some \( \omega \) and \( m \),
\[
|u_n^{\tau_n} (t)|^2 \leq 2 \left( \frac{3}{2} \right)^n + |u_0|^2 + 2 \int_0^T \| \Phi \|^2 ds + 2 \int_0^T Cds + 2k \int_0^t |u_n^{\tau_n} (s)|^2 ds
\]
Apply Gronwall’s inequality to conclude that for all \( t \in [0, T] \),
\[
|u_n^{\tau_n} (t)|^2 \leq 2 \left( \frac{3}{2} \right)^n + |u_0|^2 + 2 \int_0^T \| \Phi \|^2 ds + 2 \int_0^T Cds + 2k \int_0^t |u_n^{\tau_n} (s)|^2 ds \leq e^{2kT}
\]
If \( n \) is sufficiently large, the right side is smaller than \( 2^n \) for all \( t \in [0, T] \). Thus for such \( \omega \) and \( n \),
\[
|u_n (t \wedge \tau_n)|^2 < 2^n
\]
It follows that \( t < \tau_n \) for all \( t \in [0, T] \) because if not, then you would have
\[
2^n = |u_n (\tau_n)|^2 < 2^n.
\]
Hence for all \( n \) large enough, \( \tau_n = \infty \). For \( n \) and \( \omega \) as just described,
\[
u_n (t) - u_0 + \int_0^t A (P_n u_n) ds = \int_0^t \Phi dW.
\]
(57)

Of course the problem here is that \( n \) depends on \( \omega \) and we need a single function \( u \). Suppose then that \( \omega \notin N \) and both \( m, n \) are so large that \( \tau_m (\omega) = \tau_n (\omega) = \infty \). Say \( n > m \). Then
\[
u_n (t) - u_0 + \int_0^t A (P_n u_n) ds = \int_0^t \Phi dW
\]
\[
u_m (t) - u_0 + \int_0^t A (P_m u_m) ds = \int_0^t \Phi dW
\]
However, \( |u_n (t)|^2 < 3^n, |u_m (t)|^2 < 3^m \) and so the \( P_n u_n \) and \( P_m u_m \) equal \( u_n \) and \( u_m \) respectively. Hence
\[
u_n (t) - u_m (t) + \int_0^t A (u_n) - A (u_m) ds = 0.
\]
Furthermore, all the values of these two functions are in \( B (0, 3^n) \). Note that this is a deterministic integral, not one of those stochastic integrals. Therefore, by the local Lipschitz assumption, there exists a \( K \) such that
\[
|u_n (t) - u_m (t)| \leq \int_0^t |A (u_n) - A (u_m)| ds \leq K \int_0^t |u_n - u_m| ds
\]
and so, by Gronwall’s inequality, \( u_n (t) = u_m (t) \) for that \( \omega \).
Because of this, define for $\omega \notin N$,
\[ u(t) \equiv u_n(t) \text{ where } \tau_n(\omega) = \infty. \]

It was just shown that this is well defined. Also from [57] it follows that
\[ u(t) - u_0 + \int_0^t A(u) \, ds = \int_0^t \Phi dW. \]

This proves existence.

It only remains to verify uniqueness. If $v$ is another such solution, then taking the union of the two exceptional sets, it follows that for $\omega$ not in this union,
\[ u(t) - v(t) + \int_0^t (A - A) \, ds = 0 \]

Thus, since both $u$ and $v$ are bounded, there is a Lipschitz constant $K$ such that
\[ |u(t) - v(t)| \leq \int_0^t |A - A| \, ds \leq K \int_0^t |u(t) - v(t)| \, dt \]

and so, by Gronwall's inequality, $u(t) = v(t)$. $\blacksquare$

Note that there is no monotonicity required on $A$ in order to obtain existence.

## 8 Multiple Spaces

Next we consider the case of variational evolution equations in infinite dimensional spaces. Consider the case of a reflexive separable Banach space $V$ and a Hilbert space $W$ such that $V \subseteq W$ with $V$ dense in $W$. Thus $W' \subseteq V'$. Suppose there exists a Hilbert space $E$ which is dense in $V$. Thus
\[ E \subseteq V \subseteq W, \quad W' \subseteq V' \subseteq E' \quad \text{(58)} \]

and let $R : E \to E'$ be the Riesz map. Let $(t, u, \omega) \to A(t, u, \omega)$ where $A(t, u, \omega) \in V'$. Suppose
\[ (t, u, \omega) \to A(t, u, \omega) \quad \text{(59)} \]

is progressively measurable. Also assume the coercivity condition
\[ (A(t, u, \omega), u) \geq k\|u\|_{V'}^2 - C(t, \omega) \quad \text{(60)} \]

where $C \in L^1([0, T] \times \Omega)$. Let $V = L^p([0, T] \times \Omega, V)$ and let $V'$ be its dual space $L^{p'}([0, T] \times \Omega, V')$. We assume the operator $A : V \to V'$ is type $M$ [15].

This is a more general condition than monotone and hemicontinuous. However, the question whether there exist meaningful examples which are type $M$ on $V$ which are not also monotone and hemicontinuous is being left open for now. We have no such examples. However, the type $M$ condition is convenient to use and so this is why we make this theoretically more general assumption.

Also let $B : W \to W'$ be nonnegative and self adjoint. In all of the above, the $\sigma$ algebra will be the product measurable sets $\mathcal{B}([0, T]) \times \mathcal{F}_T$. Then we need some sort of continuity condition on $u \to A(t, u, \omega)$. In general, is suffices to assume this map is demicontinuous, possibly even less. However, here we will assume more for the sake of convenience. Assume
\[ u \to A(t, u, \omega) \text{ is locally Lipschitz} \quad \text{(61)} \]

as a map from $E$ to $E'$. We note that this condition is often true in many applications of interest thanks to the Sobolev embedding theorem. One takes $E$ to be a suitable closed subspace of $H^k(U)$ for $k$ sufficiently large.

Also let $\Phi \in L^2([0, T] \times \Omega, \mathcal{L}_2 \{ JQ^{1/2}U, W \})$ so we can consider $\int_0^t \Phi dW$, and it has values in the space $W$.

Then the main result to be proved is Theorem [17] and its corollaries stated below. They give existence for a solution to the integral equation
\[ Bu(t) - Bu_0 + \int_0^t A(u) \, ds = \int_0^t B \Phi dW \]
in the sense that for a.e. \( \omega \), the equation holds for all \( t \in [0, T] \).

This theorem is proved by using Theorem \[\text{15}\] to obtain existence for a regularized problem. The Ito formula is then used to obtain estimates on these solutions. After this, weakly convergent subsequences are obtained which are then shown to converge to the desired solution through the use of the Ito formula presented above, along with the assumption that \( A \) is type \( M \).

**Lemma 16** Let \( u_0 \in L^q (\Omega, E) \) where \( q = \max (p, 2) \). Also let \( R \) be the Riesz map from \( E \) to \( E' \). Then there exists a solution to the integral equation

\[
u (t) - u_0 + \int_0^t (B + \varepsilon R)^{-1} (A(u) + \varepsilon R(u)) \, ds = \int_0^t (B + \varepsilon R)^{-1} B\Phi dW + (B + \varepsilon R)^{-1} \int_0^t f \, ds
\]

in the sense that off a set of measure zero, the equation holds for all \( t \). This solution satisfies the estimate

\[
\frac{1}{2} E \langle (B + \varepsilon R) u(t), u(t) \rangle - \frac{1}{2} E \langle (B + \varepsilon R) u_0, u_0 \rangle + E \int_0^t \langle A(u), u \rangle + \varepsilon \langle Ru, u \rangle \, ds
\]

\[
\leq \frac{1}{2} E \int_0^t (R^{-1} B\Phi, \Phi)_{L^2(Q^{1/2} U, W)} \, ds + E \int_0^t \langle f, u \rangle \, ds
\]

where \( R \) is the Riesz map from \( W \) to \( W' \).

**Proof:** Let \( R \) be the Riesz map from \( E \) to \( E' \). Then there exists an equivalent Hilbert space norm on \( E \) such that for fixed \( \varepsilon > 0 \), the Riesz map is \( B + \varepsilon R \). To simplify the notation, let

\[
A_\varepsilon (u) \equiv A(u) + \varepsilon R(u)
\]

By Theorem \[\text{15}\] for \( u_0 \in L^2 (\Omega, E) \) with \( u_0 \) an \( \mathcal{F}_0 \) measurable function, there exists a unique progressively measurable function \( u \) having values in \( E \) such that

\[
u (t) - u_0 + \int_0^t (B + \varepsilon R)^{-1} A_\varepsilon (u) \, ds = \int_0^t (B + \varepsilon R)^{-1} B\Phi dW + (B + \varepsilon R)^{-1} \int_0^t f \, ds \tag{62}
\]

This is because the integrand is locally Lipschitz and it satisfies

\[
(B + \varepsilon R)^{-1} A_\varepsilon (t, u, \omega) , u \rangle_{E} = \langle A_\varepsilon (t, u, \omega) , u \rangle_{V', V} \geq C_\varepsilon \|u\|^2_E + \|u\|^p_{V'} - C(t, \omega)
\]

Multiplying through by \( (B + \varepsilon R) \), this shows that there exists a unique progressively measurable \( u \) which is the solution to

\[
(B + \varepsilon R) u(t) - (B + \varepsilon R) u_0 + \int_0^t A_\varepsilon (u) \, ds = \int_0^t B\Phi dW + \int_0^t f \, ds \tag{63}
\]

That stochastic integral on the right equals

\[
(B + \varepsilon R) \int_0^t (B + \varepsilon R)^{-1} B\Phi dW
\]

From now on, we use the usual norm on \( E \) and usual Riesz map \( R \) mapping \( E \) to \( E' \). At this point, use the above implicit Ito formula \[\text{14}\] on \[\text{62}\] to obtain

\[
\frac{1}{2} E \langle (B + \varepsilon R) u(t), u(t) \rangle - \frac{1}{2} E \langle (B + \varepsilon R) u_0, u_0 \rangle + E \int_0^t \langle A_\varepsilon u, u \rangle \, ds
\]

\[
= \frac{1}{2} E \int_0^t \langle (B + \varepsilon R) (B + \varepsilon R)^{-1} B\Phi, (B + \varepsilon R)^{-1} B\Phi \rangle \, ds + E \int_0^t \langle f, u \rangle \, ds
\]

\[
= \frac{1}{2} E \int_0^t \langle R^{-1} B\Phi, (B + \varepsilon R)^{-1} B\Phi \rangle_{L^2} \, ds + E \int_0^t \langle f, u \rangle \, ds \tag{64}
\]
where the symbol $\mathcal{L}_2$ signifies $\mathcal{L}_2(Q^{1/2}U, E)$. Letting $\{g_j\}$ be an orthonormal basis in $Q^{1/2}U$,

$$
\left(R^{-1}B\Phi, (B + \varepsilon R)^{-1} B\Phi\right)_{\mathcal{L}_2(Q^{1/2}U, E)} = \sum_j \left(R^{-1}B\Phi g_j, (B + \varepsilon R)^{-1} B\Phi g_j\right)_E
$$

$$
= \sum_j \left\langle B\Phi g_j, (B + \varepsilon R)^{-1} B\Phi g_j \right\rangle_{E', E} = \sum_j \left\langle B\Phi g_j, (B + \varepsilon R)^{-1} B\Phi g_j \right\rangle_{W', W}
$$

Consider

$$
\sum_j \left\langle B\Phi g_j, (B + \varepsilon R)^{-1} B\Phi g_j \right\rangle_{W', W} - \sum_j \left\langle B\Phi g_j, \Phi g_j \right\rangle_{W', W}
$$

$$
= \sum_j \left\langle B\Phi g_j, (B + \varepsilon R)^{-1} B\Phi g_j - \Phi g_j \right\rangle
$$

$$
\left\langle \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle = \left\langle (B + \varepsilon R)(B + \varepsilon R)^{-1} \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle
$$

$$
\geq \left\langle B(B + \varepsilon R)^{-1} \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle
$$

Hence

$$
\left(\Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right)^{1/2} \leq \left\langle B(B + \varepsilon R)^{-1} \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle^{1/2}
$$

and so

$$
\left\langle \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle^{1/2} \geq \left\langle B(B + \varepsilon R)^{-1} \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle^{1/2}
$$

Therefore,

$$
\left\langle \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle \leq \left\langle \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle^{1/2}
$$

$$
\leq \left\langle \Phi g_j, (B + \varepsilon R)^{-1} \Phi g_j \right\rangle^{1/2} = \left\langle \Phi g_j, \Phi g_j \right\rangle
$$

Therefore, $\mathbf{65}$ is non positive.

Return to $\mathbf{64}$ The above has shown that

$$
\frac{1}{2}E \int_0^t \left\langle R^{-1}B\Phi, (B + \varepsilon R)^{-1} B\Phi \right\rangle_{\mathcal{L}_2} ds
$$

$$
= \frac{1}{2}E \int_0^t \sum_j \left\langle B\Phi g_j, (B + \varepsilon R)^{-1} B\Phi g_j \right\rangle_{W', W}
$$

$$
\leq \frac{1}{2}E \int_0^t \sum_j \left\langle B\Phi g_j, \Phi g_j \right\rangle_{W', W} = \frac{1}{2}E \int_0^t (R^{-1}B\Phi, \Phi)_{\mathcal{L}_2(Q^{1/2}U, W)} ds
$$

where $R$ is the Riesz map from $W$ to $W'$, distinct from $R$ the Riesz map from $E$ to $E'$. Summarizing this, the following inequality has been established.

$$
\frac{1}{2}E \left\langle (B + \varepsilon R) u(t), u(t) \right\rangle - \frac{1}{2}E \left\langle (B + \varepsilon R) u_0, u_0 \right\rangle + E \int_0^t \left\langle A\varepsilon u\varepsilon, u\varepsilon \right\rangle ds
$$

$$
\leq \frac{1}{2}E \int_0^t \left\langle R^{-1}B\Phi, \Phi \right\rangle_{\mathcal{L}_2(Q^{1/2}U, W)} ds + E \int_0^t \left\langle f, u \right\rangle ds \quad \blacksquare
$$

(66)

From now on, we will use a subscript of $\varepsilon$ on $u$ because we are about to take limits as $\varepsilon \to 0$. From the coercivity condition $\mathbf{64}$ the following inequality is obtained.

$$
E \left\langle (B + \varepsilon R) u\varepsilon(t), u\varepsilon(t) \right\rangle + E \int_0^t \|u\varepsilon\|_V^p ds + \varepsilon E \int_0^t \|u\varepsilon\|_E^2 ds \leq C (\Phi, u_0, f) + CE \left\langle (B + \varepsilon R) u_0, u_0 \right\rangle
$$
Since $u_0$ is in $L^2(\Omega, E)$, the right side is bounded independent of $t \leq T$ and $\varepsilon$. In particular, for some constant $C$ independent of $\varepsilon$,

$$E\langle Bu_\varepsilon(t), u_\varepsilon(t)\rangle$$

is bounded independent of $\varepsilon$. Therefore, if $v \in L^2(\Omega, W)$,

$$|E\langle Bu_\varepsilon(t), v \rangle| \leq (E\langle Bu_\varepsilon(t), u_\varepsilon(t)\rangle)^{1/2} (E\langle Bv, v \rangle)^{1/2} \leq C \|B\|^{1/2} \|v\|_{L^2(\Omega, W)}$$

(67)

It follows that there exists a subsequence still called $\varepsilon$ such that

$$\varepsilon Ru_\varepsilon(t) \to 0 \text{ in } L^2(\Omega, E') \text{ uniformly in } t$$

(68)

$$u_\varepsilon \to u \text{ weakly in } V$$

$$\varepsilon Ru_\varepsilon \to 0 \text{ in } E'$$

where $E' \equiv L^{q'}([0,T] \times \Omega, E')$

$$Au_\varepsilon \to \xi \text{ weakly in } V'$$

This last convergence implies that

$$\int_0^t Au_\varepsilon ds \to \int_0^t \xi ds \text{ weakly in } L^{q'}(\Omega, V')$$

From the integral equation (68) and boundedness of $A$, it also follows that a further subsequence satisfies

$$\left( (B + \varepsilon R) u_\varepsilon - (B + \varepsilon R) u_0 - B \int_0(\cdot) \Phi dW \right)' \to \zeta \text{ weakly in } E'$$

where $E' \equiv L^{q'}([0,T] \times \Omega, E')$ for $q = \max(p, 2)$ and as usual, $1/q' + 1/q = 1$. Thus

$$\zeta + \xi = f \text{ in } E'$$

However, both $f$ and $\xi$ are in $V'$ so in fact $\zeta \in V'$ also from the fact that $E$ is dense in $V$. Thus the equation actually holds in $V'$.

Consider $\zeta$. Let $g \in L^q(\Omega, E), q = \max(2, p)$, and let $\psi$ be infinitely differentiable and equal to 0 near $T$. Then since $Bu_\varepsilon(0) \equiv Bu_0$ a.e. $\omega$,

$$\int_0^T \int_\Omega \langle \zeta, \psi g \rangle dPdt = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left\langle \left( (B + \varepsilon R) u_\varepsilon - B \int_0(\cdot) \Phi dW - (B + \varepsilon R) u_0 \right)', \psi g \right\rangle dPdt$$

Then using (68) and $u_0 \in L^2(\Omega, E)$,

$$= - \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left\langle (B + \varepsilon R) u_\varepsilon - B \int_0(\cdot) \Phi dW - (B + \varepsilon R) u_0, \psi' g \right\rangle dPdt$$

$$= - \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \left\langle Bu_\varepsilon - B \int_0(\cdot) \Phi dW - Bu_0, \psi' g \right\rangle dPdt$$

$$= \int_0^T \int_\Omega \left\langle \psi' Bg, u_\varepsilon - \int_0(\cdot) \Phi dW - u_0 \right\rangle dPdt$$

$$= - \int_0^T \int_\Omega \left\langle \psi' Bg, u - \int_0(\cdot) \Phi dW - u_0 \right\rangle dPdt$$

$$= - \int_0^T \int_\Omega \left\langle Bu - B \int_0(\cdot) \Phi dW - Bu_0, \psi' g \right\rangle dPdt$$
Since \( g \) is arbitrary, this shows that \( \zeta = \left( Bu - B \int_0^t \Phi dW - Bu_0 \right)' \) in \( \mathcal{E}' \). Also, it shows that, on integrating by parts,

\[
\int_0^T \int_\Omega \langle \zeta, \psi g \rangle \, dP \, dt = \int_\Omega \langle Bu(0) - Bu_0, \psi(0) g \rangle \, dP + \\
\int_0^T \int_\Omega \left( Bu - B \int_0^t \Phi dW - Bu_0 \right)' \, \psi g \, dP \, dt
\]

\[
= \int_0^T \int_\Omega \langle \zeta, \psi g \rangle \, dP \, dt
\]

and so

\[
\int_\Omega \langle Bu(0) - Bu_0, \psi(0) g \rangle \, dP = 0
\]

which shows that in \( L^q' (\Omega, E') \), you have

\[
Bu(0) = Bu_0.
\tag{69}
\]

By density considerations, this implies the equation also holds in \( L^2 (\Omega, W') \). In particular, off a set of measure zero, for all \( t \),

\[
\int_0^t \zeta ds = Bu(t) - B \int_0^t \Phi dW - Bu_0
\]

Also from 68 and the above weak convergence in \( \mathcal{E}' \) of

\[
\left( B + \varepsilon R \right) u_\varepsilon - \left( B + \varepsilon R \right) u_0 - B \int_0^t \Phi dW \right)'
\]

\[Bu_\varepsilon(t) \to Bu(t) \text{ weakly in } L^q' (\Omega, E')\]

for each \( t \). By 67 there is a further subsequence such that

\[Bu_\varepsilon(T) \to Bu(T) \text{ weakly in } L^2 (\Omega, W')\]

Now if \( e \in W' \), consider the functional defined on \( L^2 (\Omega, W') \) which is given by

\[v \to \int_\Omega \langle v, e \rangle^2 \, dP\]

This is clearly convex and lower semicontinuous. Therefore, it is also weakly lower semicontinuous. It follows that

\[
\liminf_{\varepsilon \to 0} \int_\Omega \langle Bu_\varepsilon(T), e \rangle^2 \, dP \geq \int_\Omega \langle Bu(T), e \rangle^2 \, dP
\]

Letting \( \{ e_i \} \) be the vectors of Lemma 3, it follows from the above observation and Fatou’s lemma,

\[
\liminf_{\varepsilon \to 0} E \langle Bu_\varepsilon(T), u_\varepsilon(T) \rangle = \liminf_{\varepsilon \to 0} \sum_{i=1}^\infty E \langle Bu_\varepsilon(T), e_i \rangle^2
\]

\[
\geq \sum_{i=1}^\infty \liminf_{\varepsilon \to 0} E \langle Bu_\varepsilon(T), e_i \rangle^2
\]

\[
\geq \sum_{i=1}^\infty E \langle Bu(T), e_i \rangle^2 = E \langle Bu(T), u(T) \rangle
\]

\[
(70)
\]

As explained above,

\[
\zeta + \xi = f \text{ in } \mathcal{V}'
\tag{71}
\]

It follows that there is a set of measure zero \( N \) such that for \( \omega \notin N \),

\[
\zeta(t) + \xi(t) = f(t) \text{ a.e. } t,
\]
the equation holding in $V'$. In addition to this, we have also obtained

$$\zeta = \left( B u - B \int_0^t \Phi dW - B u_0 \right)'$$

(72)

Enlarging $N$ if necessary, it follows that for $\omega \notin N$,

$$B u(t) - B u_0 + \int_0^t \xi(s) ds = \int_0^t f ds + B \int_0^t \Phi dW$$

(73)

It follows, since $A$ is progressively measurable, each $A_{\varepsilon} u_{\varepsilon}$ is progressively measurable and so an application of the Pettis theorem implies $\xi$ is also progressively measurable. Therefore, we can apply the implicit Ito formula to the above integral equation of (73) and obtain

$$\frac{1}{2} E \langle B u(t) , u(t) \rangle - \frac{1}{2} E \langle B u_0 , u_0 \rangle + E \int_0^t \langle \xi , u \rangle - \frac{1}{2} \langle B \Phi , \Phi \rangle ds = \int_0^t \langle f , u \rangle ds$$

Thus, letting $t = T$,

$$E \int_0^T \langle \xi , u \rangle ds = \int_0^T \frac{1}{2} \langle B \Phi , \Phi \rangle + \langle f , u \rangle ds + \frac{1}{2} \langle B u_0 , u_0 \rangle - \frac{1}{2} E \langle B u(T) , u(T) \rangle$$

Recall the integral equation for the approximate solution,

$$B u_{\varepsilon}(t) - B u_0 + \int_0^t A_{\varepsilon}(u_{\varepsilon}) ds = \int_0^t B \Phi dW + \int_0^t f ds - \varepsilon R u_{\varepsilon}(t) + \varepsilon R u_0$$

and Lemma 16. Using the result of this lemma, when $t = T$,

$$\frac{1}{2} E \langle (B + \varepsilon R) u_{\varepsilon}(T) , u_{\varepsilon}(T) \rangle - \frac{1}{2} E \langle (B + \varepsilon R) u_0 , u_0 \rangle$$

$$+ E \int_0^T \langle A_{\varepsilon} u_{\varepsilon} , u_{\varepsilon} \rangle - \frac{1}{2} \langle B \Phi , \Phi \rangle ds = \int_0^T \langle f , u_{\varepsilon} \rangle ds$$

Then, dropping the term $\langle \varepsilon R u_{\varepsilon} , u_{\varepsilon} \rangle$ from $A_{\varepsilon}$,

$$E \int_0^T \langle A u_{\varepsilon} , u_{\varepsilon} \rangle ds \leq \frac{1}{2} E \int_0^T \langle B \Phi , \Phi \rangle + \langle f , u_{\varepsilon} \rangle ds + \frac{1}{2} E \langle (B + \varepsilon R) u_0 , u_0 \rangle$$

$$- \frac{1}{2} E \langle B u_{\varepsilon}(T) , u_{\varepsilon}(T) \rangle$$

Now take lim sup of both sides and use (70) to write

$$\limsup_{\varepsilon \to 0} \left( - \frac{1}{2} E \langle B u_{\varepsilon}(T) , u_{\varepsilon}(T) \rangle \right) = - \liminf_{\varepsilon \to 0} \frac{1}{2} E \langle B u_{\varepsilon}(T) , u_{\varepsilon}(T) \rangle$$

$$\leq - \frac{1}{2} E \langle B u(T) , u(T) \rangle$$

Thus

$$\limsup_{\varepsilon \to 0} E \int_0^T \langle A u_{\varepsilon} , u_{\varepsilon} \rangle ds \leq \frac{1}{2} E \int_0^T \langle B \Phi , \Phi \rangle + \langle f , u \rangle ds$$

$$+ \frac{1}{2} E \langle B u_0 , u_0 \rangle - \frac{1}{2} E \langle B u(T) , u(T) \rangle$$

$$= E \int_0^T \langle \xi , u \rangle ds$$

Since $A$ is assumed type $M$ on $V$, it follows that $\xi = Au$. Then referring to (73) this has proved the following theorem which is the main result.
Theorem 17 Let the spaces $E, V, W$ be as described in 58 and suppose

$$u \to A(t, u, \omega)$$

is locally Lipschitz as a map from $E$ to $E'$,

$$(t, u, \omega) \to A(t, u, \omega)$$

is progressively measurable. Also suppose that the map $A : V \to V'$ is type $M$ where

$$V \equiv L^p([0, T] \times \Omega ; V)$$

with the $\sigma$ algebra equal to $B([0, T]) \times F_T$ and there is a coercivity condition

$$\langle A(t, u, \omega), u \rangle \geq k \|u\|_p^p - C(t, \omega)$$

where $C \in L^1([0, T] \times \Omega)$. Also let $u_0 \in L^q(\Omega, E)$, $u_0$ being $F_0$ measurable, where $q = \max(p, 2)$ and let $f \in V'$. Then there exists a progressively measurable function $u \in V$ which is a solution to the integral equation

$$Bu(t) = Bu_0 + \int_0^t Au(s) \, ds = \int_0^t f \, ds + B \int_0^t \Phi \, dW, \ t \in [0, T]$$

in the sense that the equation holds in $V'$ for all $\omega \notin N$ where $N$ is a set of measure zero. In terms of the weak derivative, this solution is of the form

$$\left( Bu - B \int_0^t \Phi \, dW \right)' + Au = f \text{ in } V'$$

$$Bu(0) = Bu_0 \text{ in } L^2(\Omega, W)$$

It is easy to generalize to assume only that $u_0 \in L^2(\Omega, W)$.

Corollary 18 Let the spaces $E, V, W$ be as described in 58 and suppose

$$u \to A(t, u, \omega)$$

is locally Lipschitz as a map from $E$ to $E'$,

$$(t, u, \omega) \to A(t, u, \omega)$$

is progressively measurable. Also suppose that the map $A : V \to V'$ is type $M$ where

$$V \equiv L^p([0, T] \times \Omega ; V)$$

with the $\sigma$ algebra equal to $B([0, T]) \times F_T$ and there is a coercivity condition

$$\langle A(t, u, \omega), u \rangle \geq k \|u\|_p^p - C(t, \omega)$$

where $C \in L^1([0, T] \times \Omega)$. Also let $u_0 \in L^2(\Omega, W)$, $u_0$ being $F_0$ measurable, and let $f \in V'$. Then there exists a progressively measurable function $u \in V$ which is a solution to the integral equation

$$Bu(t) = Bu_0 + \int_0^t Au(s) \, ds = \int_0^t f \, ds + B \int_0^t \Phi \, dW, \ t \in [0, T]$$

in the sense that the equation holds in $V'$ for all $\omega \notin N$ where $N$ is a set of measure zero. In terms of the weak derivative, this solution is of the form

$$\left( Bu - B \int_0^t \Phi \, dW \right)' + Au = f \text{ in } V'$$

$$Bu(0) = Bu_0 \text{ in } L^2(\Omega, W)$$
**Proof:** Let $u_n$ be the solution to the above theorem satisfying the integral equation

$$Bu_n(t) - Bu_0 + \int_0^t Au_n ds = \int_0^t f ds + B \int_0^t \Phi dW$$

as described there, where $u_0 \in L^q(\Omega, E)$ and

$$u_0 \to u \text{ in } L^2(\Omega, W).$$

Then by the implicit Ito formula,

$$\frac{1}{2} E \langle Bu_n(t), u_n(t) \rangle - \frac{1}{2} E \langle Bu_0, u_0 \rangle + \int_0^t \langle Au_n, u_n \rangle ds - \frac{1}{2} \left< B \Phi, \Phi \right> dt = \int_0^t \langle f, u_n \rangle ds$$

(74)

Then, as in the above argument, there is a subsequence, still denoted by $n$ such that

$$u_n \to u \text{ weakly in } \mathcal{V}$$

$$Au_n \to \xi \text{ weakly in } \mathcal{V}'$$

$$\left( Bu_n - Bu_0 - B \int_0^t \Phi dW \right)' \to \left( Bu - Bu_0 - B \int_0^t \Phi dW \right)' \text{ in } \mathcal{V}'$$

$$Bu(0) = Bu_0 \text{ in } L^2(\Omega, W)$$

As before, the integral equation implies

$$Bu_n(t) \to Bu(t) \text{ weakly in } L^q(\Omega, E')$$

and there is a subsequence such that also

$$Bu_n(T) \to Bu(T) \text{ weakly in } L^2(\Omega, W')$$

(75)

Then passing to a limit,

$$Bu(t) - Bu_0 + \int_0^t \xi ds = \int_0^t f ds + B \int_0^t \Phi dW$$

(76)

By the implicit Ito formula,

$$E \int_0^t \langle \xi, u \rangle ds = E \int_0^t \langle f, u \rangle ds + E \int_0^t \frac{1}{2} \left< B \Phi, \Phi \right> ds + \frac{1}{2} E \langle Bu_0, u_0 \rangle - \frac{1}{2} E \langle Bu(t), u(t) \rangle$$

Then from (74) and (75) along with similar arguments given in the above theorem,

$$\limsup_{n \to \infty} \int_0^T \langle Au_n, u_n \rangle \leq \frac{1}{2} E \langle Bu_0, u_0 \rangle - \frac{1}{2} E \langle Bu(T), u(T) \rangle$$

$$+ \int_0^T \frac{1}{2} \left< B \Phi, \Phi \right> ds + \int_0^T \langle f, u \rangle ds = \int_0^T \langle \xi, u \rangle ds$$

and so $Au = \xi$. With (76) this proves the corollary. ■

Note that there is no conclusion of uniqueness in the above theorem and corollary.

One can make the assumptions on $AB + A$ rather than $A$ and get the same conclusions. Also, there is a uniqueness result available under an assumption of weak monotonicity.
Corollary 19 Suppose the situation of the above corollary but replace the coercivity, and type M conditions, with the following weaker conditions

\[ \lambda \langle Bu, u \rangle + \langle A(t, u, \omega), u \rangle_V \geq \delta \| u \|_V^p - C(t, \omega) \] (77)

for all \( \lambda \) large enough where \( C \in L^1([0, T] \times \Omega) \). Also

\[ \lambda B + A : V \to V' \text{ is type M} \]

Then the conclusion of Theorem \([72]\) is still valid. There exists a progressively measurable function \( u \in V \) which is a solution to the integral equation

\[ Bu(t) - Bu_0 + \int_0^t A u(s) \, ds = \int_0^t f \, ds + B \int_0^t \Phi dW, \ t \in [0, T] \] (78)

in the sense that the equation holds in \( V' \) for all \( \omega \not\in N \) where \( N \) is a set of measure zero. If the weak monotonicity condition

\[ \langle \lambda Bu + A(t, u, \omega) - (\lambda Bv + A(t, v, \omega)), u - v \rangle \geq 0 \]

is valid, then if \( u, v \) are two solutions to (78) it follows that off a set of measure zero, \( Bu(t) = Bv(t) \).

Proof: Define \( A_\lambda \) by

\[ (A_\lambda(t, w, \omega), v)_{V', V} = \langle e^{-\lambda t} A(t, e^\lambda w, \omega), v \rangle_{V', V} \]

Then

\[
\begin{align*}
\lambda \langle Bu, u \rangle + \langle A_\lambda(t, u, \omega), u \rangle_V & \geq e^{-2\lambda t} \left( \lambda \langle B(e^\lambda u), e^\lambda u \rangle + \langle A(t, e^\lambda u, \omega), e^\lambda u \rangle \right) \\
& \geq e^{-2\lambda t} \left( \delta \| e^\lambda u \|_V^p - C(t, \omega) \right) \\
& \geq e^{-2\lambda t} \left( \delta \| u \|_V^p - e^{-\lambda \delta t} C(t, \omega) \right)
\end{align*}
\]

which is of the right form. By Corollary \([18]\) there exists a solution in \( V' \) to

\[
Bu(t) = Bu_0 \text{ in } L^2(\Omega, W)
\]

Now let \( e^\lambda u(t) = w(t) \). Writing in terms of \( w \),

\[
\left( Be^{-\lambda t} w - Be^{-\lambda t} \int_0^t \Phi dW \right)' + \lambda Be^{-\lambda t} w + e^{-\lambda t} Aw
\]

\[ = e^{-\lambda t} f + \lambda e^{-\lambda t} B \int_0^t \Phi dW \text{ in } V' \]

\[ Bw(0) = Bu_0 \text{ in } L^2(\Omega, W) \]

It follows that

\[
\begin{align*}
e^{-\lambda t} \left( Bw - B \int_0^t \Phi dW \right)' - \lambda e^{-\lambda t} \left( Bw - B \int_0^t \Phi dW \right) \\
+ \lambda Be^{-\lambda t} w + e^{-\lambda t} Aw = e^{-\lambda t} f + \lambda e^{-\lambda t} B \int_0^t \Phi dW
\end{align*}
\]

After cancelling terms and multiplying by \( e^{-\lambda t} \), this yields

\[
\left( Bw - B \int_0^t \Phi dW \right)' + Aw = f \text{ in } V'
\]
along with the initial condition

\[ Bw(0) = Bu_0 \]

Then integrating, one obtains that for \( \omega \) not in a suitable set of measure zero,

\[ Bw(t) - Bu_0 - B \int_0^t \Phi dW + \int_0^t Awds = \int_0^t fds \]

Next suppose \( u, v \) are two solutions to the above integral equation as described above. Then off the union of the two exceptional sets,

\[ B(u(t) - v(t)) + \int_0^t Au - Avds = 0 \]

and so by the implicit Ito formula,

\[ \frac{1}{2} \langle Bu(t) - v(t), u(t) - v(t) \rangle + \int_0^t \langle Au - Av, u - v \rangle ds = 0 \]

and so from the monotonicity condition,

\[ \frac{1}{2} \langle Bu(t) - v(t), u(t) - v(t) \rangle \leq \lambda \int_0^t \langle B(u - v), u - v \rangle ds \]

Apply Gronwall’s inequality. \( \Box \)

There is an assumption that \( A : V \to \mathcal{V}' \) is type \( M \). This is made because this is what will be used. Recall that monotone and hemicontinuous implies type \( M \) [20]. What are some easy to check conditions which will imply \( A \) is hemicontinuous on \( \mathcal{V}' \)? Make the following specific assumption involving an estimate.

There exists \( c \in [0, \infty) \) and \( g \in L^p'([0, T] \times \Omega) \) which is \( B([0, T]) \times \mathcal{F} \) measurable such that for all \( v \in V, t \in [0, T] \)

\[ \|A(t, v)\|_V \leq g(t) + c \|v\|_V^{-1}. \quad (79) \]

Here \( 1/p + 1/p' = 1 \).

If \( \lim_{\lambda \to 0} \langle A(t, u + \lambda v, \omega), w \rangle = \langle A(t, u, \omega), w \rangle \), then it will also follow that \( A \) is hemicontinuous on \( V \). You need to verify that

\[ \lim_{\lambda \to 0} \int_0^T \int_\Omega \langle A(t, u + \lambda v, \omega), w \rangle dtdP = \int_\Omega \int_0^T \langle A(t, u, \omega), w \rangle dtdP \]

for \( w, u, v \in V \). But this follows from the growth condition above, Vitali’s convergence theorem, and showing that the integrand is uniformly integrable. Letting \( Q \subseteq \Omega \times [0, T] \),

\[ \int_Q \|\langle A(t, u + \lambda v, \omega), w \rangle\| dtdP \leq \int_Q \left( g(t, \omega) + c \|u + \lambda v\|_V^{-1} \right) \|w\|_V dtdP \]

\[ \leq \left( \int_Q \|g\|_V^{p'} dtdP \right)^{1/p'} \left( \int_Q \|w\|_V^p dtdP \right)^{1/p} \]

\[ + c \left( \int_Q \|u + \lambda v\|_V^p dtdP \right)^{1/p'} \left( \int_Q \|w\|_V^p dtdP \right)^{1/p} \]

which is small independent of \( \lambda \) for \( |\lambda| < 1 \) provided \( Q \) has sufficiently small measure.

It would be very interesting to get interesting examples where \( \lambda B + A \) is type \( M \) on \( V \) without being monotone for any \( \lambda \). It would also be interesting to obtain theorems which involve \( \lambda B + A \) being type \( M \) on \( L^p(0, T, V) \) rather than \( V \) or on a suitable space of solutions. The difficulty in doing this latter problem is obtaining the appropriate measurability which seems to require the need for the solution to the non stochastic evolution problem to have a unique solution. Of course this is usually achieved by having some combination of the operators \( B, A \) being monotone. However, one can obtain more general conditions which do include the case of non monotone operators by using the theory presented here as a starting point and adding various nonlinear operators as compact perturbations. This will be explored later.
9 Some examples

Here we give a few examples. The first is a standard example, the porous media equation, which is discussed well in 20. For stochastic versions of this example, see 19 and 18. The generalization to stochastic equations does not require the theory of this paper. We will show, however, that it can be considered in terms of the theory of this paper without much difficulty using an approach proposed in 2.

Example 20 The stochastic porous media equation is

\[ u_t - \Delta \left( u |u|^{p-2} \right) = f, \quad u(0) = u_0, \quad u = 0 \text{ on } \partial U \]

where here \( U \) is a bounded open set in \( \mathbb{R}^n, n \leq 3 \) having Lipschitz boundary. One can consider a stochastic version of this as a solution to the following integral equation

\[ u(t) - u_0 + \int_0^t (-\Delta) \left( u |u|^{p-2} \right) ds = \int_0^t \Phi dW \]

where here \( \Phi \in L^2 \left( [0, T] \times \Omega, L_2 \left( Q^{1/2} U, H) \right), H = L^2(U) \right) \) and the equation holds in the manner described above in \( H^{-1}(U) \). Assume \( p \geq 2 \).

One can consider this as an implicit integral equation of the form

\[ (-\Delta)^{-1} u(t) - (-\Delta)^{-1} u_0 + \int_0^t u |u|^{p-2} ds = (-\Delta)^{-1} \int_0^t \Phi dW \]

where \(-\Delta\) is the Riesz map of \( H^1_0(U) \) to \( H^{-1}(U) \). Then we can also consider \((-\Delta)^{-1}\) as a map from \( L^2(U) \) to \( L^2(U) \) as follows:

\[ (-\Delta)^{-1} f = u \text{ where } -\Delta u = f, \quad u = 0 \text{ on } \partial U. \]

Thus we let \( W = L^2(U) \) and \( V = L^p(U) \). Let \( B \equiv (-\Delta)^{-1} \) on \( L^2(U) \) as just described. Let \( A(u) = u |u|^{p-2} \).

Then clearly there exists a Hilbert space \( E \) dense in \( V \) which has the property that \( A: E \to E' \) is locally Lipschitz. Just pick \( E = H^2(U) \) for example. By the Sobolev embedding theorem, \( H^2(U) \) embeds continuously into \( C \left( \overline{U} \right) \).

Furthermore, the function \( F(x) = x |x|^{p-2} \) is differentiable as a map from \( \mathbb{R} \) to \( \mathbb{R} \). Thus for \( u, v \in E \),

\[ F(u + v) = F(u) + F'(u) v + o(v) \]

If \( w \in E \), then if \( \|v\|_{H^2(U)} \) is small enough, it follows that \( \|v\|_{C(\overline{U})} \) is also small enough that

\[ |\langle o(v), w \rangle| \leq \varepsilon \|w\|_{C(\overline{U})} \leq \varepsilon \|w\|_{H^2} \]

which shows that this function is differentiable as a map from \( E \) to \( E' \). Similarly, the derivative is continuous. Hence it is locally Lipschitz as a map from \( E \) to \( E' \). Thus \( A: E \to E' \) is locally Lipschitz on \( E \). It is obvious that the necessary coercivity condition holds. In addition, there is a monotonicity condition which holds. In fact, \( A \) is monotone and hemicontinuous so it also is type \( M \). Therefore, if \( u_0 \in L^2 \left( \Omega, L^2(U) \right) \) and \( F_0 \) measurable, Corollary 19 applies and we can conclude that there exists a unique solution to the integral equation 81 in the sense described there. Here \( u \in L^p \left( [0, T] \times \Omega, L^p(U) \right) \) and is progressively measurable, the integral equation holding for all \( t \) for \( a.e.w \). Since \( A \) satisfies for some \( \delta > 0 \) an inequality of the form

\[ (Au - Av, u - v) \geq \delta \|u - v\|_{L^p(U)}^2, \]

it follows easily from the above methods that the solution is also unique.

Next we give a simple example which is a singular and degenerate equation.

Example 21 Suppose \( U \) is a bounded open set in \( \mathbb{R}^3 \) and \( b(x) \geq 0, b \in L^p(U), p \geq 4 \) for simplicity. Consider the degenerate stochastic initial boundary value problem

\[ b(\cdot) u(t, \cdot) - b(\cdot) u_0(\cdot) - \int_0^t \nabla \cdot \left( \nabla u |u|^{p-2} \nabla u \right) = b \int_0^t \Phi dW \]

\[ u = 0 \text{ on } \partial U \]

where \( \Phi \in L^2 \left( [0, T] \times \Omega, L_2 \left( Q^{1/2} W, W) \right) \right) \) for \( W = H^1_0(U) \).
To consider this equation and initial condition, it suffices to let $W = H^1_0(U)$, $V = W^{1,p}_0(U)$,

$$ A : V \to V', \langle Au, v \rangle = \int_U |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, $$

$$ B : W \to W', \langle Bu, v \rangle = \int_U b(x) u(x) v(x) dx $$

Then by the Sobolev embedding theorem, $B$ is obviously self adjoint, bounded and nonnegative. This follows from a short computation:

$$ \left| \int_U b(x) u(x) v(x) dx \right| \leq \|v\|_{L^4(U)} \left( \int_U |b(x)|^{4/3} |u(x)|^{4/3} dx \right)^{3/4} $$

$$ \leq \|v\|_{H^1_0(U)} \left( \left( \int_U |b(x)|^4 dx \right)^{1/3} \left( \int \left( |u(x)|^{4/3} \right)^{3/2} dx \right)^{2/3} \right)^{3/4} $$

$$ = \|v\|_{H^1_0(U)} \|b\|_{L^4(U)} \|u\|_{L^2(U)} \leq C \|b\|_{L^4} \|u\|_{H^1_0} \|v\|_{H^1_0} $$

The nonlinear operator is obviously monotone and hemi-continuous, so it is pseudomonotone. The technical requirement that the operator $A$ be locally Lipschitz on some Hilbert space $E$ which is dense in $V$ and embeds continuously into $V$ is easily satisfied by taking $E = H^{m+1}(U)$ for $2m > 3$ then using the Sobolev embedding theorem, similar to the above example. As for $u_0$, it is only necessary to assume $u_0 \in L^2(\Omega, W)$ and $\mathcal{F}_0$ measurable. Then Corollary 19 gives the existence of a solution. Note that $b$ can be unbounded and may also vanish. Thus the equation can degenerate to the case of a non-stochastic nonlinear elliptic equation.

The existence theorems can easily be extended to include the situation where $\Phi$ is replaced with a function of the unknown function $u$. This is done by splitting the time interval into small sub intervals of length $h$ and retarding the function in the stochastic integral, like a standard proof of the Peano existence theorem. Then the Ito formula is applied to obtain estimates and a limit is taken. However, this will be done later.

**References**

[1] Bardos and Brezis, Sur une classe de problemes d’evolution non lineaires, *J. Differential Equations* 6 (1969).

[2] Brezis, On some degenerate Nonlinear Parabolic Equations, *Proceedings of the Symposium in Pure Mathematics*, Vol. 18, (1968).

[3] Browder, F.E. Pseudomonotone operators and nonlinear elliptic boundary value problems on unbounded domains. Proc. Nat. Acad. Sci. USA 74 (1977) 2659-2661.

[4] Carroll R. W. and Showalter R. E. Singular and degenerate Cauchy problems, Mathematics in science and engineering, vol. 127, Academic Press, New York, 1976.

[5] Chung and Williams, *Introduction to Stochastic Integration*, Birkhauser, 1983.

[6] Da Prato, G. and Zabczyk J., *Stochastic Equations in Infinite Dimensions*, Cambridge 1992.

[7] Gross L. Abstract Wiener Spaces, Proc. fifth Berkeley Sym. Math. Stat. Prob. 1965.

[8] Kallenberg O., *Foundations Of Modern Probability*, Springer 2003.

[9] Karatzas and Shreve, *Brownian Motion and Stochastic Calculus*, Springer Verlag, 1991.

[10] Kuttler K. L., *Time dependent implicit evolution equations*, Nonlinear Analysis 10(5) (1986), 447-463.

[11] Kuttler K. L. and Shillor M., *Set-valued pseudomonotone maps and degenerate evolution equations*, Comm. Contemp. Math. 1(1)(1999), 87–123.
[12] Kuttler K. Quasistatic evolution of damage in an elastic-viscoplastic material, Electron. J. Diff. Eqns., Vol. 2005(2005), No. 147, pp. 1-25.

[13] Krylov N.V. and Rozowskii B.L., Stochastic evolution equations, Current problems in mathematics, Vol. 14, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchin. i Tekhn. Informatsii, Moscow, 1979, PP. 71-147, 256. MR MR570795(81m:60116)

[14] Krylov N. V., On Kolmogorov’s equations for finite dimensional diffusions, Stochastic PDE’s and Kolmogorov equations in infinite dimensions (Cetraro, 1998), Lecture Notes in Math, vol. 1715, Springer, Berlin, 1999, pp. 1-63. MR MR1731794 (2000k:60155)

[15] Lions, J. L. “Quelques Methods de Resolution des Problemes aux Limites Non Lineaires,” Dunod, Paris, 1969.

[16] Pardoux E., Sur des Equations aux derivees partielles stochastiques monotones, C. R. Acad. Sci. Paris Ser. A-B 275 (1972)

[17] Pardoux E., Equations aux derivees partielles stochastiques de type monotone, Seminaire sur les Equations aux Derivees Partielles (1974-1975), III, Exp. No. 2, College de France, Paris, 1975, p10. MR MR0651582 (58#31406)

[18] Prévôt C. and Röckner, A Concise Course on Stochastic Partial Differential Equations, Lecture notes in Mathematics, Springer 2007.

[19] Ren J., Rockner M., and Wang F.Y., Stochastic porous media and fast diffusion equations

[20] Showalter R. E., Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Mathematical Surveys and Monographs Vol. 49, AMS, 1996.

[21] Simon, J, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura. Appl. 146(1987), 65-96.

[22] Stroock D. W. Probability Theory An Analytic View, Second edition, Cambridge University Press, 2011

[23] Stromberg, K. R. Probability for analysts, Chapman and Hall, 1994.