ON FINITE PRESENTABILITY OF GROUP $F/\langle M, N \rangle$

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Abstract. The characterization of normal subgroups $M$, $N$ of free group $F$ for which the quotient group $F/\langle M, N \rangle$ is finitely presented is given.

1. Introduction. Let $F$ be a free group and $M$, $N$ normal subgroups of $F$. When is the quotient group $F/\langle M, N \rangle$ finitely related? In other words: when is the mutual commutator subgroup $\langle M, N \rangle$ the normal closure of a finite set of elements?

The case $M = N$ has been already considered. For example, it follows from Theorem 5.3 of A.L. Shmelkin’s paper [10] that if $F$ is a finitely generated free group and $N$ is the normal closure of a finite set of words from $F$, then $F/\langle N, N \rangle$ is finitely presented if and only if $N$ has a finite index in $F$.

There is the following lemma in the paper [1] of J. Abarbanel, S. Rosset:

Lemma 1. If $M$ and $N$ are the normal closures of finite sets of elements in a finitely generated group $F$ and the group $MN$ has a finite index in $F$, then $\langle M, N \rangle$ is the normal closure of a finite set of elements.

For completeness, the proof of Lemma 1 will be given in Section 2.

The condition in Lemma 1 that $M$ and $N$ are normally finitely generated is essential. If $N$ is not finitely generated as normal subgroup, then $\langle F, N \rangle$ can be or not be finitely generated as normal subgroup depending on $N$. In [1] (referring to [2] and [4]), some examples showing that are given.

Also in [1] (and [3]), the authors consider the case, when the index of $MN$ in $F$ is infinite. They show that in this case under some restrictions on $M$ and $N$ the group $F/\langle M, N \rangle$ is not finitely related.

The following question remained open: suppose $M$ and $N$ are non-trivial normal subgroups of the free group $F$. Is it true that a finite index of $MN$ in $F$ is a necessary condition for $F/\langle M, N \rangle$ to be finitely related? The following theorem shows that the answer to this question is affirmative.

Key words and phrases. Free group, finitely presented group, mutual commutator subgroup, verbal wreath product.
Theorem 1. Let $M$ and $N$ be non-trivial normal subgroups in a free group $F$. Suppose that the group $MN$ has infinite index in $F$. Then the mutual commutator subgroup $[M,N]$ is not the normal closure of any finite set of elements.

So if $M$ and $N$ are the normal closures of finite sets of words in the finitely generated free group $F$, Lemma 1 and Theorem 1 lead to the following

Theorem 2. If $M$ and $N$ are the normal closures of non-empty finite sets of non-identical words in the finitely generated free group $F$, then $[M,N]$ is the normal closure of a finite set of elements if and only if the group $MN$ has a finite index in $F$.

Corollary 1. Let $F$ be a free group of infinite rank, and $M$, $N$ non-trivial normal subgroups in $F$. Then the mutual commutator subgroup $[M,N]$ is not the normal closure of any finite set of elements.

2. Proofs.

Proof of Lemma 1. The group $L = MN$ is finitely generated since it has a finite index in the finitely generated group $F$. Hence there is a finitely generated subgroups $M_1$ and $N_1$ in $M$ and $N$, respectively, such that $M_1$ and $N_1$ generate $L$, and $M$ and $N$ are the normal closures of $M_1$ and $N_1$ in $F$.

Let $\{x_i\}$ and $\{y_j\}$ be finite generating sets of $M_1$ and $N_1$. It is clear that the subgroup $K = [M, N]$ is the normal closure of all commutators $[x_i^f, y_j]$, where $f \in F$. So it suffices to show that the relation $[x_i^f, y_j] = 1$ of the group $F/K$ follows from relations $[x_i^{h_k}, y_j] = 1$ for all $h_k$ from some set of representatives of left cosets of $L$ in $F$. Let $f = h_k u_1 ... u_s$, where $u_1, ..., u_s \in \{x_i^{\pm 1}\} \cup \{y_j^{\pm 1}\}$. The proof will induct on $s$. If $s = 0$, there is nothing to proof. Let $s \geq 1$. If $u_s = y$, where $y \in \{y_j^{\pm 1}\}$, then the relation $[x_i^f, y_j] = 1$ follows, by assumption, from $x_i^{f'} = y^{-1} x_i^f y = x_i^f$ and $[x_i^{f'}, y_j] = 1$, where $f' = h_k u_1 ... u_{s-1}$. If $u_s = x$, where $x \in \{x_i^{\pm 1}\}$, then the relation $[x_i^f, y_j] = 1$ is equivalent to the relation $[x_i^{f'}, y_j x^{-1}] = 1$, which follows, by assumption, from $x y_j x^{-1} = y_j$ and $[x_i^{f'}, y_j] = 1$.

Proof of Theorem 1. Below we use the following notation:

$X$, a set of free generators of the free group $F$;

$L$, the product $MN$ of the non-trivial groups $M$ and $N$;

$\gamma_i(L)$, the $i$-th term of the lower center series for $L$;

$G$, the quotient group $F/L$.

It is well-known that the intersection of all terms of the lower center series $\gamma_i(L)$ in the free group is trivial. Therefore there is a number $c \geq 1$ such that the group $M$ is contained in $\gamma_c(L)$, but it is not
contained in $\gamma_{c+1}(L)$. Without loss of generality we can assume that $N$ is not contained in $\gamma_2(L)$, since $L = MN$. Then $[M, N]$ is contained in $\gamma_{c+1}(L)$.

To prove Theorem 1, it suffices to show that $[M, N]$ is not contained in the normal closure of any finite subset $Y$ from $\gamma_{c+1}(L)$. To show this, we use verbal wreath products, introduced by A.L. Shmelkin in [10]. More precisely, we use the $c$-nilpotent wreath product $S$ of a free $c$-nilpotent group $T(X')$ and $G = F/L$, where $X'$ is of the same cardinality as $X$.

By definition, $S$ is a semidirect product of a group $W$ and $G$, where $W$ is the $c$-nilpotent product of $|G|$ copies of the group $T(X')$. Moreover, $W$ is the free $c$-nilpotent group with basis $\{a_i|i = (x, h), x \in X', h \in G\}$. The group $G$ acts on $W$ by the rule $g^{-1}a_ia_j = a_j$, where $i = (x, h), j = (x, hg)$.

By Theorem 2.1 [10] (see also Addition 1 to [9]), the homomorphism $\psi$ from $F$ to $S$, mapping $x \in X$ to $\tilde{x}a_{(x, 1)}$, where $\tilde{x} = xL \in G = F/L$, has the kernel $\gamma_{c+1}(L)$, that is, the image $H$ of $F$ is isomorphic to the quotient group $F/\gamma_{c+1}(L)$. According to the properties of this homomorphism, the intersection of $H$ and $W$ is the image of the subgroup $L$, and $HW = S$.

Further we consider an arbitrary subset $Z$ in $G$ such that $Z = Z^{-1}$.

We say that two bi-indexes $i = (x', h)$ and $j = (x'', g)$, where $x', x'' \in X'$, $h, g \in G$, are $Z$-close, if the quotient $hg^{-1}$ of their second components belongs to $Z$, and are $Z$-distant otherwise.

For a given subset $Z$, let us construct an auxiliary semidirect product $S_Z$ of groups $W_Z$ and $G$ (so that $S_Z = S$ for $Z = G$). The group $W_Z$ is defined as follows. The generators for $W_Z$ are the same $a_i$ as in $W$ with the same bi-indexes $i$. The defining relations for $W_Z$ are left-normalized commutators $[a_{i_1}, \ldots, a_{i_{c+1}}]$ of generators such that any two bi-indexes $i_k$ and $i_l$ in such commutator are $Z$-close. It is clear that these defining relations are invariant under the action of the group $G$ on $W_Z$ by conjugation, which shifts the second components of bi-indexes of the generators $a_i$ by the rule $g^{-1}a_ig = a_j$, where $i = (x, h), j = (x, hg)$.

By such construction, if $Z = \emptyset$, then $W_\emptyset$ is an absolutely free group, and $S_\emptyset$ is the semidirect product of this free group $W_\emptyset$ and $G$. If $Z = G$, then $W_G = W$ and $S_G = S$. The other cases are intermediate between $W_\emptyset$ and $W$ and between $S_\emptyset$ and $S$, respectively.

Moreover, if $Z \subset Z' \subseteq G$, the group $S_Z$ is naturally mapped onto $S_{Z'}$ by the homomorphism $\phi_{Z \subseteq Z'}$ identical on $G$ and the set of generators $\{a_i\}$. 
For any subset \( Z \subseteq G \), one can define the homomorphism \( \psi_Z \) from the free group \( F \) to \( S_Z \), mapping \( x \in X \) to \( \tilde{x}a(x,1) \), where \( \tilde{x} = xL \in G = F/L \) and \( a(x,1) \) are generators of \( W_Z \). Since \( S_Z = GW_Z \) and \( \tilde{x} \) generate \( G \), we have that \( S_Z = HZW_Z \), where \( H = \psi_S(F) \). Besides, \( \psi = \psi_G \) and \( \psi = \phi_{Z \subseteq G} \psi_Z \) for any \( Z \subseteq G \).

Recall that \( \gamma_{c+1}(L) \) is the kernel of the homomorphism \( \psi : F \to S \). On the other hand, this kernel is the union of the kernels of the homomorphisms \( \psi_Z \) to \( S_Z \) for all finite subsets \( Z \subseteq G \). Hence any finite subset \( Y \) from \( \gamma_{c+1}(L) \) belongs to the kernel of a homomorphism \( \psi_Z : F \to S_Z \) for some finite subset \( Z \).

Therefore, to show that \([M, N]\) is not contained in the normal closure of any finite subset \( Y \) from \( \gamma_{c+1}(L) \), it suffices to show that, for any finite subset \( Z \subseteq G \), \([M, N]\) is not mapped to the trivial subgroup by the homomorphism \( \psi_Z \) of the free group \( F \) to \( S_Z \). Below we use the following simple lemma.

**Lemma 2.** Let \( u \) and \( v \) be not identical elements of \( W_Z \) such that the bi-index of any generator in some writing of the element \( u \) is \( Z \)-distant to the bi-index of any generator in some writing of the element \( v \). Then \( w^{-1}uw \) and \( v \) do not commute for any element \( w \in W_Z \).

**Proof of Lemma 2.** Let \( A \) be a subgroup of \( W_Z \), generated by such \( a_i \) that occur in the writing of \( u \). Similarly, the subgroup \( B \) is defined for \( v \). Consider the free product \( A \ast B \). Let us define a homomorphism from the group \( W_Z \) to \( A \ast B \) as follows. It leaves fixed generators of \( A \) and \( B \) and maps the other generators to the identity. This definition is well-defined, since the defined homomorphism remains the defining relations of \( W_Z \). In addition the image of \( u \) belongs to \( A \) and the image of \( v \) belongs to \( B \). Now Lemma 2 follows from the well-known fact that if not identical elements are conjugated with elements from different factors of the free product, then they do not commute. 

Now we complete the proof of Theorem 1.

Since the groups \( M \) and \( N \) do not belong to \( \gamma_{c+1}(L) \), there exist elements \( u \in M \) and \( v \in N \), whose images in the group \( S \) are not equal to the identity. Moreover, their images under the homomorphism \( \psi \) belong to \( W \), since \( M \) and \( N \) belong to \( L \). Therefore, their images \( u' \) and \( v' \) under the homomorphism \( \psi_Z \) from \( F \) to \( S_Z \) are not equal to the identity in the group \( W_Z \), since \( \psi = \phi_{Z \subseteq G} \psi_Z \).

Recall that the second components of bi-indexes of generators \( a_i \) of the group \( W_Z \) are multiplied from the right by \( g \) under the conjugation by an element \( g \in G \). Since \( Z \) is finite and \( G \) is infinite, there exists an
element \( g \in G \) such that the bi-index of any generator in some writing of the following element

\[
u'' = g^{-1}u'g
\]  

is \( Z \)-distant from the bi-index of any generator in some writing of the element \( v' \).

Since \( G \leq S_Z = H_ZW_Z \), for the element \( g \), there exist \( h \in H_Z \) and \( w \in W_Z \) such that \( g = hw^{-1} \). According to (1), \( u'' = wh^{-1}u'hw^{-1} \).

Since the bi-index of any generator in the writing of the element \( u'' \) is \( Z \)-distant from the bi-index of any generator in the writing of the element \( v' \), it follows from Lemma 2 that the elements \( w^{-1}u''w \) and \( v' \) do not commute. Hence, taking into account that \( w^{-1}u''w \in \psi_Z(M) \) and \( v' \in \psi_Z(N) \), we have that the image of \([M,N]\) is non-trivial in \( S_Z \).

\[\Box\]

**Proof of Corollary 1.** If \( F \) has infinite rank and \( M, N \) are finitely generated as normal subgroups, then \( MN \) has infinite index in \( F \). Consequently, Corollary 1 follows from Theorem 1.

If \( M \) and (or) \( N \) are infinitely generated as normal subgroups, then the group \( M \) (respectively, \( N \)) can be presented as the union of subgroups \( M_k \) (respectively, \( N_k \)), where \( M_k \) (\( N_k \)) is finitely generated as normal subgroup. Similar to the previous paragraph, we have that the group \([M_k,N_k]\) is not generated by any finite set of elements as normal subgroup. Hence, the union of all \([M_k,N_k]\) can not be the normal closure of any finite set of elements. Since the union of all subgroups \([M_k,N_k]\) is equal to the group \([M,N]\), Corollary 1 is proved. \[\Box\]

**Remark.** Let \( F \) be a free group, and \( M, N \) non-trivial normal subgroups in \( F \) such that the group \( MN \) has infinite index in \( F \) or \( F \) has infinite rank. Suppose that \([M,N] = M \cap N \). For example, it is true when the union \( S \cup T \) satisfies the small cancellation condition \( C'(\lambda) \) (\( \lambda \leq 1/6 \)), where \( S \) (respectively, \( T \)) is a non-empty symmetrized set of cyclically reduced words from \( F \) such that \( S \) (respectively, \( T \)) generates \( M \) (respectively, \( N \)) as normal subgroup (see, for example, [5,6]). Then it follows from Theorem 1 (Corollary 1) that \( M \cap N \) is not the normal closure of any finite set of elements, even if the groups \( M \) and \( N \) are finitely generated as normal subgroups. That is contrasting to the well-known A.G.Howson’s Theorem (see, for example, Proposition 3.13 in [7]): if subgroups \( H \) and \( K \) in a free group are finitely generated, then their intersection \( H \cap K \) is finitely generated.
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