RELATING DIFFRACTION AND SPECTRAL DATA OF
APERIODIC TILINGS: TOWARDS A BLOCH THEOREM

ERIC AKKERMANS, YAROSLAV DON, JONATHAN ROSENBERG,
AND CLAUDE L. SCHOCHET

Abstract. The purpose of this paper is to show the relationship in all dimen-
sions between the structural (diffraction pattern) aspect of tilings (described by
Čech cohomology of the tiling space) and the spectral properties (of Hamiltoni-
ans defined on such tilings) defined by $K$-theory, and to show their equivalence
in dimensions $\leq 3$. A theorem makes precise the conditions for this relation-
ship to hold. It can be viewed as an extension of the “Bloch Theorem” to a
large class of aperiodic tilings. The idea underlying this result is based on the
relationship between cohomology and $K$-theory traces and their equivalence
in low dimensions.

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1. Introduction

Aperiodic tilings are structures obtained from the spatial arrangement of letters defining an alphabet, according to a set of deterministic rules \[7, 8, 34, 63\]. They constitute a rich playground to investigate features of physical systems in different contexts, e.g. condensed matter, statistical mechanics and dynamical systems.

This ubiquity is partly due to the existence of a large set of tiling families which includes periodic, nonperiodic (e.g. Wang tiles), quasiperiodic, and asymptotically periodic tilings. For periodic tilings, the Bloch theorem \[4\] provides a systematic and powerful relation between different aspects, such as diffraction and spectral data. For aperiodic tilings such as quasicrystals, despite having been thoroughly studied, these aspects remain as yet unrelated since the classical Bloch theorem is not applicable.

A celebrated family of tilings are quasicrystals or quasiperiodic tilings related to algebraic number theory and cut-and-project (C&P) sets \[26, 40\]. Despite their lack of periodicity, quasicrystals discovered by Shechtman \[64\], and predicted by Levine and Steinhardt \[49\], exhibit sharp Bragg peaks. Quasicrystals have been extensively investigated \[52, 63\], especially in one dimension \[18\].

Quasicrystals have also been studied from the viewpoint of the spectral characteristics of the waves (acoustic, optical, matter) they can sustain. Conveniently defined Laplacians (continuous or tight-binding) reveal a highly lacunar fractal energy spectrum, with an infinite set of energy gaps \[22, 51, 66, 67\].

Johnson and Moser \[35, 36\] studied the spectrum of self-adjoint linear differential operators, e.g. continuous Schrödinger or Helmholtz equations, with a potential being an almost periodic function, and presented a systematic way of enumerating the open intervals of the associated resolvent operator using the rotation number. This was an approach to gap labeling in the spirit of the Schwartzman winding number \[62\] and using cohomology ideas. A discrete version is in \[24\]. This description was based on the use of the rotation number, a quantity which from Sturm-Liouville theory equals half the counting function. The theorem of Johnson and Moser applies to one-dimensional systems with a quasiperiodic potential. The Gap Labelling Theorem (hereafter GLT) \[12, 14\] of Bellissard and coworkers provides a more general framework for the topological classification of these gaps and plays, for quasiperiodic tilings, a role similar to that of the Bloch theorem for periodic ones. The Bloch theorem makes it possible to label the eigenstates of a periodic system with a quasi-momentum and to identify topological (Chern) numbers \[68\]. This labelling is robust as long as the lattice translation symmetry is preserved.

Similarly, the GLT allows one to associate numbers to each gap in the spectrum of quasiperiodic tilings. Those numbers can be given both a topological meaning and invariance properties akin in nature to Chern numbers, but not expressible in terms of a [classical] curvature \[14, 47\].\(^1\) In both cases, topological invariants attached to the energy spectrum remain unchanged under a perturbation of the Hamiltonian, as long as gaps do not close. Fractal features often show up in the diffraction patterns of aperiodic tilings and in the spectral properties of related Laplacians, and have been suggested as a kind of generalization of the Bloch theorem for tilings \[51\]. A relation between the spectrum of dynamical systems and Bragg peaks at the basis of the GLT has also been advocated in \[27, 32, 33, 57\].

\(^1\) The Chern numbers are related to curvature in the noncommutative geometry sense of Connes \[20, 21\], however.
Current lore of topology of periodic and aperiodic tilings emphasizes the existence of different, even incompatible, classes of topological invariants. The band structure of periodic tilings, predicted by the Bloch theorem, naturally introduces an inherent torus topology of the Brillouin zone. Aperiodic tilings do not enjoy these benefits of a Bloch theorem. Nonetheless, as obtained from the GLT, their energy spectra present a ramified Cantor set gap structure, e.g. for 1D quasiperiodic C&P tilings, an infinite set of gaps which can be labeled using two integers [14]. The topological $K$-theoretical nature of these integers has been emphasized [12]. Nontrivial topology for quasiperiodic tilings has also been reported, and the resulting topological invariants have been related to the gap labeling integers and winding numbers. This is connected to scattering data and diffraction spectra of aperiodic tilings [10, 23]. The relevance of Čech cohomology as an important tool in the study of substitution tilings has been emphasized by Anderson and Putnam [2] and also in [3]. Given such a tiling space, they show that it is topologically conjugate to an inverse limit of explicit finite complexes, and hence the cohomology is readily computable. Anderson-Putnam remark: “Our point of view is that it is Čech cohomology which is really measuring the almost periodic structure of these tilings.”

Both characterisations of tilings, diffraction spectra (Bragg peaks) and spectral data for wave equations, are obtained by means of conveniently defined traces expressed either by a two-point correlation function or by the integrated density of states (a.k.a. counting function). At this point, there are two communities of mathematicians and physicists who build these traces from Ruelle-Sullivan currents. One group uses Čech cohomology and defines the cohomology trace $\tau^H_{\ast}$ there [2, 3]. The other group uses $K$-theory almost exclusively and defines the $K$-theory trace $\tau^K_{\ast}$ there [13, 15, 46, 48, 55, 65]. The goal of this paper is to unite the two groups of people, to show that for a large class of tilings, including cut-and-project (C&P) aperiodic tilings, these two traces are equivalent, at least in dimension $\leq 3$, a result which can be seen as an extension of the Bloch theorem to this family of aperiodic tilings. To that purpose, we will first show that the two approaches give exactly the same trace under very general circumstances and then we will show by way of examples how results from both groups fit under the same umbrella. In dimensions 4 and up, part of the GLT still seems to be in doubt since there is no convincing proof of integrality of the Chern character in the literature. But the equivalence between cohomology and $K$-theoretic traces is still valid up to perhaps an integral factor. We will explain this in Section 9.

This effort requires quite a bit of background knowledge. In order to keep the paper to a reasonable length, we will specify the tools that we need from foliated spaces, measure theory, cohomology and $K$-theory, and we will give precise references in the literature to theorems that we require.

In order to demonstrate the generality of our results, we will wait to introduce tiling terminology and conditions until needed. For now, we assume given a compact foliated space $X$ with oriented foliation bundle. Our basic reference for foliated spaces is the book of Moore and Schochet [54].

A foliated chart or foliated patch in a topological space is an open set homeomorphic to $L \times N$, where $L$ is a copy of $\mathbb{R}^d$ and $N$ is a separable locally compact metrizable space. A tangentially smooth function $f : L \times N \to \mathbb{R}$ is a continuous function such that $f(\bullet, n)$ is smooth for each $n \in N$ and the partial derivatives of
f in the $L$ direction are continuous on $L \times N$. A foliated space \[54, \text{p. 32}\] $X$ is a separable locally compact metrizable space with an open covering by foliated charts fitting together smoothly so that the local plaques $L \times \{n\}$ fit together to form $d$-dimensional smooth manifolds called leaves. Foliated manifolds are the classical examples of foliated spaces, but for us the relevant examples are tiling spaces, which are foliated spaces but typically are not foliated manifolds. We let $C^\infty_\tau(X)$ denote the tangentially smooth functions $f : X \to \mathbb{R}$; that is, they are tangentially smooth when restricted to every local patch. A foliated space has a natural $d$-dimensional real tangent bundle $F$ along the leaves. Its dual bundle is denoted $F^\ast$.

The foliated spaces relevant to tiling theory are quite special. In order to take advantage of that fact, we shall specialize at once.

**Definition 1.1.** A compact foliated space given by an $\mathbb{R}^d$-action is a compact foliated space $X$ with a locally free $\mathbb{R}^d$-action, such that the orbits of the action are the leaves of the foliated space, and $X \cong N \times_\Lambda \mathbb{R}^d$ for some compact totally disconnected space $N$ carrying an action of a lattice $\Lambda \subset \mathbb{R}^d$. There is a resulting fibre bundle

$$N \to X \overset{p}{\to} T^d,$$

where $T^d = \mathbb{R}^d / \Lambda$ is the $d$-torus, and the restriction of the projection to each leaf $\ell$ is a covering map $\ell \to T^d$.

Henceforth, all foliated spaces and in particular all tiling spaces will be assumed to be compact foliated spaces given by an $\mathbb{R}^d$-action. This implies that the foliation tangent bundle is orientable. Sadun and Williams [60] show that the hulls of most tilings are homeomorphic to spaces satisfying these conditions. In general, the homeomorphism $X \cong N \times_\Lambda \mathbb{R}^d$ is not equivariant for the $\mathbb{R}^d$-action on the tiling hull, but this won’t matter since all we need is the foliated space structure, not the group action, and the homeomorphism sends leaves to leaves. We comment on this result in detail below.

The remaining sections are organized as follows.

**Section 2** is a very quick introduction to Čech cohomology $\check{H}^\ast(X; F)$ for compact spaces.

**Section 3** introduces tangential cohomology $H^\ast_\tau(X)$ and homology, sculpted for foliated spaces. There is a natural map

$$s : \check{H}^k(X; \mathbb{R}) \to H^k_\tau(X)$$

which in our special context is shown by a spectral sequence comparison argument to be induced by the inclusion $C^\infty_\tau(N) \hookrightarrow C(N)$.

**Section 4** is devoted to coinvariants. We demonstrate that if $X = N \times_\Lambda \mathbb{R}^d$ is a compact foliated space given by an $\mathbb{R}^d$-action, then there is a natural isomorphism

$$\check{H}^0(N; \mathbb{R})_{2d} \cong \check{H}^d(X; \mathbb{R})$$

and the map $s : \check{H}^d(X; \mathbb{R}) \to H^d_\tau(X)$ has dense image in the Hausdorff quotient $\check{H}^d_\tau(X)$.

In **Section 5** we introduce the machinery of topological groupoids, tangential measures, invariant transverse measures $\nu$, and Ruelle-Sullivan currents $C_\nu$. We highlight the Riesz representation theorem, which identifies the group of signed Radon
invariant transverse measures with the top tangential homology group. This allows us to define the cohomology trace
\[ \tau^\mathring{H} : \mathring{H}^d(X; \mathbb{R}) \to \mathbb{R} \]
as the composition
\[ \mathring{H}^d(X; \mathbb{R}) \xrightarrow{s} H_d^+(X) \xrightarrow{\cap C^\nu} \mathbb{R} \]

Section 6 contains a very brief introduction to topological $K$-theory for $C^*$-algebras and the classical Chern character. Following Bellissard, we define the non-commutative Brillouin zone to be $\mathcal{B} = C^*(G(X))$, where $G(X)$ is the holonomy groupoid of the foliated space. Using Connes’ Thom isomorphism theorem
\[ \varphi : K_d(A) \xrightarrow{\cong} K_0(A \rtimes \mathbb{R}^d) \]
we record an isomorphism
\[ K^d(X) \xrightarrow{\cong} K_0(C(X) \rtimes \mathbb{R}^d) \cong K_0(C^*(G(X))) = K_0(\mathcal{B}) \]
which holds in our context; we denote it $\chi$.

Section 7 gives the analytical background for the $K$-theory trace.

The subject of Section 8 is the partial Chern character, which is a map
\[ c : K_0(C^*(G(X))) \to \mathring{H}^d_\tau(X) \]
defined for foliated spaces. Its existence depends upon the identification of the invariant transverse measures with homology classes via the Riesz theorem. Given an invariant transverse measure $\nu$, the $K$-theory trace, which we denote $\tau^K$, is simply the composition
\[ K_0(\mathcal{B}) = K_0(C^*(G(X))) \xrightarrow{\cong} \mathring{H}^d_\tau(X) \xrightarrow{\cap C^\nu} \mathbb{R}. \]

Section 9 contains our most general result, Theorem 9.1, relating the cohomology and $K$-theory traces.

**Theorem 9.1.** Suppose that $X$ is a compact foliated space given by an $\mathbb{R}^d$-action with invariant transverse measure $\nu$, and the holonomy cover of each leaf is simply connected. Then:

- **The diagram**
\[
\begin{array}{ccc}
K_0(\mathcal{B}) & \xrightarrow{(ch_d)s^{-1}} & K_0(C^*(G(X))) \\
\downarrow{c} & & \downarrow{\cong} \\
H^d_\tau(X) & \xrightarrow{id} & H^d_\tau(X)
\end{array}
\]

**commutes.**

- **Bloch Theorem:** For every invariant transverse measure $\nu$, the diagram
\[
\begin{array}{ccc}
K_0(\mathcal{B}) & \xrightarrow{(ch_d)s^{-1}} & \mathring{H}^d(X; \mathbb{R}) \\
\downarrow{\tau^K} & & \downarrow{\tau^\mathring{H}_\tau} \\
\mathbb{R} & \xrightarrow{id} & \mathbb{R}
\end{array}
\]

**commutes.**
We then have the following consequence for tilings. Suppose that \( T \) is a tiling satisfying the following conditions:

1. \( T \) satisfies the finite pattern condition (i.e., finite local complexity);
2. \( T \) has only finitely many tile orientations.

**Corollary 9.5.** Under the assumptions above, if the dimension is \( \leq 3 \), then for every invariant transverse measure the diagram

\[
K_0(\mathcal{B}_T) \xrightarrow{(ch_d) \circ (X^{-1})} \check{H}^d(\Omega_T; \mathbb{Z})
\]

commutes. In particular, the \( K \)-theory trace \( \tau^K_*: K_0(\mathcal{B}_T) \to \mathbb{R} \) and the cohomology trace \( \tau^\check{H}_*: \check{H}^d(\Omega_T; \mathbb{Z}) \to \mathbb{R} \) have the same image in \( \mathbb{R} \).

**Sections 10** gives a quick introduction to the building tools underlying basic families of tilings.

**Sections 11 and 12** present a comparison of the diffraction spectrum and the counting function (gap labeling) for three canonical families of one-dimensional tilings: periodic, quasiperiodic, and aperiodic. For the quasiperiodic and aperiodic cases, the diffraction spectrum involves three possible classes: pure-point (i.e., a discrete and countable set of Bragg peaks), absolutely continuous or singular continuous. The corresponding spectrum of Laplacians on these tilings involves also these three classes although the two spectra do not necessarily coincide. For the periodic case, Laplacian and diffraction spectra are pure-point and the Bloch theorem gives the equivalence between those two data sets.

**Section 13** gives some insights to a selection of earlier works closely related to ours. We emphasize the distinctions between the different definitions and results.

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### 2. Čech Cohomology

The classical reference for Čech cohomology for compact (Hausdorff) spaces is Eilenberg-Steenrod [28]. Suppose that \( X \) is a compact space. Then \( \check{H}^j(X; \mathbb{F}) \) is defined for \( \mathbb{F} \) any commutative ring. (This is the same as the sheaf cohomology of the constant sheaf defined by \( \mathbb{F} \).) If \( X = \varprojlim X_j \) is the inverse limit of finite CW-complexes (and every compact space may be written in such a manner), then the natural maps \( H^*(X_j; \mathbb{F}) \to \check{H}^*(X; \mathbb{F}) \) induce an isomorphism for each \( k \):

\[
\varprojlim H^k(X_j; \mathbb{F}) \xrightarrow{\cong} \check{H}^k(X; \mathbb{F}).
\]
Note that we do not have to specify the type of cohomology (Čech, singular, simplicial, ... ) that we use for the finite complexes, since they all agree for such spaces. If \( X \) is also separable metrizable then limits can be taken over sequences. It follows that if \( X \) is compact separable metrizable, then \( \hat{H}^k(X; \mathbb{Z}) \) is the direct limit of a sequence of finitely generated abelian groups, hence countable, and \( \hat{H}^k(X; \mathbb{R}) \) is the direct limit of a sequence of finite-dimensional vector spaces, hence a vector space of (at most) countable dimension. In addition, the natural map \( \hat{H}^k(X; \mathbb{Z}) \to \hat{H}^k(X; \mathbb{R}) \) induces an isomorphism

\[
\hat{H}^k(X; \mathbb{Z}) \otimes \mathbb{R} \cong \hat{H}^k(X; \mathbb{R}).
\]

If \( X \) is compact metric of dimension \( d \), then \( H^k(X; \mathbb{F}) \) is defined and vanishes for \( k > d \). The group \( \hat{H}^d(X; \mathbb{F}) \) is of special interest, and we will discuss it below. The transversal \( N \) in our applications is a Cantor set, zero-dimensional, so its cohomology vanishes in positive dimensions. Note that \( N \) is the inverse limit of a sequence of finite discrete spaces \( X_n, N = \lim_{\leftarrow} X_n. \) Thus \( \hat{H}^0(N; \mathbb{F}) \cong \lim_{\to} H^0(X_n; \mathbb{F}). \) Each \( H^0(X_n; \mathbb{F}) \) is a finitely generated free \( \mathbb{F} \)-module and the maps in the direct system all split, so \( \hat{H}^0(N; \mathbb{F}) \cong \oplus \mathbb{F}, \) where the sum is over countably many copies of \( \mathbb{F}. \)

### 3. Tangential Cohomology and Homology

A convenient reference for tangential cohomology is [54, Ch. III]. Tangential cohomology is referred to by various other names in the tiling literature.\(^2\) It is defined on foliated spaces \( \Gamma \) (say, of leaf dimension \( d \)). Recall that \( C^\infty_f(X) \) is the space of real-valued continuous functions on \( X \) that are smooth in the leaf directions. Let \( \Gamma_+(F^*) \) denote the tangentially smooth sections of \( F^* \), the dual of the tangent bundle to the leaves, and let \( \Omega^k_+ = \Gamma_+(\Lambda^k F^*) \) denote the tangential de Rham complex.\(^3\) Its cohomology groups are the **tangential cohomology** groups and are denoted \( H^k_+(X) \). These vanish for \( k > d \) because there are no forms in higher dimensions. There is a natural map [54, p. 58]

\[
s: \hat{H}^k(X; \mathbb{R}) \to H^k_+(X)
\]

from Čech cohomology to tangential cohomology, defined using sheaf theory. If \( X \) is a compact smooth foliated manifold \( M \), then this simply corresponds to the inclusion \( C^\infty(M) \subseteq C^\infty_f(M) \), since any smooth function on \( M \) is tangentially smooth.

The groups \( H^k_+(X) \) have a natural topology induced from the de Rham cochains, and the topology is not necessarily Hausdorff. We denote by \( \hat{H}^k_+(X) \) the Hausdorff quotient of \( H^k_+(X) \).

There is an associated **tangential homology** theory \( H^*_+(X) \) defined by taking the homology of \( [\Omega^*_+(X)]^* \), the (continuous) dual of the associated tangential de Rham

\(^2\) For example, Kellendonk-Putnam [43, p. 695] page call it *dynamical cohomology* and generalize it. Moustafa [55] calls it *longitudinal cohomology.*

\(^3\) Formally, the global sections of the graded sheaf \( \Lambda^k(F^*) \).
complex. Then there are natural isomorphisms
\[ \text{Hom}_{\text{cont}}(H^k(X), \mathbb{R}) \cong \text{Hom}_{\text{cont}}(\check{H}^k(X), \mathbb{R}) \cong H^k(X). \]

The comparison map \( s \) of (1) is for general foliated spaces neither injective nor surjective. But we have a substantial simplification when our foliated spaces satisfy Definition 1.1. To explain it, recall that when \( N \) is a totally disconnected compact metrizable space, \( C^\infty(N) \) is the algebra of locally constant functions on \( N \) (this notation figures prominently in analysis on \( p \)-adic groups), which is the algebra of functions that factor through some quotient map \( N \to F \) with \( F \) a discrete finite set. This is a dense subalgebra of \( C(N) \), of countable dimension, and there is a natural completion map with dense range \( i: C^\infty(N) \to C(N) \).

**Theorem 3.1.** Let \( X \) be a compact foliated space given by an \( \mathbb{R}^d \)-action in the sense of Definition 1.1, obtained by inducing a \( \mathbb{Z}^d \)-action on a totally disconnected compact space \( N \). Then there is a natural commuting diagram
\[
\begin{array}{ccc}
\check{H}^k(X; \mathbb{R}) & \xrightarrow{s} & H^k(X) \\
\downarrow \cong & & \downarrow \cong \\
H^k(\mathbb{Z}^d, C^\infty(N)) & \xrightarrow{i_*} & H^k(\mathbb{Z}^d, C(N)).
\end{array}
\]

The composite
\[ \check{s}: \check{H}^k(X; \mathbb{R}) \xrightarrow{s} H^k(X) \to H^k(X) \]
has dense image.

**Proof.** Recall that \( H^k(X) \) is the sheaf cohomology of the sheaf \( R_\tau \) of germs of continuous real-valued functions which are locally constant along the leaves, since the tangential de Rham complex is a fine resolution of this sheaf, whereas \( \check{H}^k(X; \mathbb{R}) \) is the sheaf cohomology of the sheaf \( \check{R} \) of germs of locally constant real-valued functions. These sheaves are not the same. But (1) is induced by the natural “inclusion” morphism \( i: \check{R} \to R_\tau \). The bundle projection \( p: X \to T^d \) gives rise to Leray spectral sequences of sheaves
\[ H^k(T^d, R^l p_* \check{R}) \Rightarrow H^{k+l}(X, \check{R}) \quad \text{and} \quad H^k(T^d, R^l p_* R_\tau) \Rightarrow H^{k+l}(X, R_\tau). \]

Here \( R^l p_* \check{R} \) is the \( l \)-th derived push-forward of a sheaf \( \check{R} \), defined by “sheafifying” the presheaf \( U \mapsto H^l(p^{-1}(U), \check{R}) \). The morphism \( i \) induces a morphism of spectral sequences from the first of the spectral sequences to the second. The first spectral sequence is just the familiar Serre spectral sequence \( H^k(T^d, \check{H}^l(N)) \Rightarrow \check{H}^{k+l}(X; \mathbb{R}) \), though note that we need cohomology with local coefficients here since \( \pi_1(T^d) = \mathbb{Z}^d \) acts nontrivially on \( N \) and on its only non-zero Čech cohomology group, \( H^0(N; \mathbb{R}) \).

Let’s examine the sheaf \( R^l p_* \check{R} \) which appears in the second spectral sequence (the one converging to tangential cohomology). For \( U \) a small connected open set in \( T^d \), \( p^{-1}(U) \) splits as \( N \times U \), and the sheaf \( R_\tau \) on this open set of \( X \) is just the sheaf of germs of continuous functions which are locally constant along the leaves, i.e., which depend only on the \( N \) factor in this product decomposition. Thus \( H^l(p^{-1}(U), R_\tau) \cong H^l(N, \mathbb{R}) \), where \( \mathbb{R} \) is the sheaf of germs of real-valued continuous functions, is actually independent of \( U \) (once it’s small enough) and is just \( C(N) \), continuous real-valued functions on \( N \), for \( l = 0 \), and 0 for \( l > 0 \). However,
\[ \check{H}^0(N; \mathbb{R}) \cong \check{H}^0(N; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong C(N, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong C^\infty(N), \]
the locally constant functions on \( N \), which is a dense subspace of \( C(N) \). So now it’s evident that

\[ \iota_* : H^k(T^d, R^\ell p_* R) \to H^k(T^d, R^\ell p_* R_\tau) \]

is just the completion map

\[ i_* : H^k_{\text{group}}(\pi_1(T^d), C^\infty(N)) \to H^k_{\text{group}}(\pi_1(T^d), C(N)) \]

for \( \ell = 0 \) and that both sides vanish identically when \( \ell > 0 \). So both spectral sequences collapse and \( \iota_* = i_* \).

As for the last statement about density of the image, it is obvious that density of the image of \( i_* : C^\infty(N) \to C(N) \) gives density of the image of \( i_* \) in the topology of \( H^k(\pi_1(T^d), C(N)) \) coming from convergence of (group) cocycles. However, this topology also agrees with the topology on \( H^k_{\tau}(X) \) coming from the tangential de Rham complex, i.e., given by convergence of differential forms, as one can see by restricting to a small open set of the form \( N \times U \), where the de Rham complex locally looks like \( C(N) \otimes \Omega^*(U) \).

\[ \square \]

Because of Theorem 3.1, we see that for tiling spaces, tangential and Čech cohomologies are close to being identical. In our application we will need the composition

\[ \bar{s} : \hat{H}^d(X; \mathbb{R}) \xrightarrow{s} H^d_\tau(X) \longrightarrow \hat{H}^d_\tau(X), \]

which under these assumptions will have dense image.

In this regard still an additional simplification occurs in the top degree, as explained in the following section.

\section{4. Coinvariants}

In the more recent tiling literature there is quite a bit of attention given to the Čech cohomology group \( \hat{H}^d_\tau(\Omega_T; \mathbb{F}) \), since that is the group that houses the cohomology information about the tiling. Whenever there is a fibration

\[ N \to X \to T^d \]

as is the case with \( X = \Omega_T \) usually, then \( \mathbb{Z}^d \cong \pi_1(T^d) \) acts on \( N \) and hence on \( \hat{H}^0(N; \mathbb{Z}) \). Let \( \hat{H}^0(N; \mathbb{Z})_{\mathbb{Z}^d} \) denote the coinvariants of the action, i.e., the quotient of \( \hat{H}^0(N; \mathbb{Z}) \) by the subgroup generated by all of the elements \( g : x - x \), for \( g \in \mathbb{Z}^d \) and \( x \in \hat{H}^0(N; \mathbb{Z}) \). The following result seems to be well-known to the cognoscenti.

\textbf{Theorem 4.1.} Fix some integer \( d > 0 \) and let \( T^d \) denote the d-torus. Suppose given a fibration

\[ N \to X \to T^d \]

with \( N \) and \( X \) compact and \( N \) totally disconnected. Then there is an isomorphism

\[ \hat{H}^d(X; \mathbb{Z}) \cong \hat{H}^0(N; \mathbb{Z})_{\mathbb{Z}^d}, \]

the coinvariants of the action of \( \mathbb{Z}^d \cong \pi_1(T^d) \) on \( \hat{H}^0(N; \mathbb{Z}) \). Similarly with \( \mathbb{R} \) or \( \mathbb{Q} \) coefficients.
Proof. We have a Leray-Serre spectral sequence
\[ \tilde{H}^k(T^d, \tilde{H}^\ell(N; \mathbb{Z})) \Rightarrow \tilde{H}^{k+\ell}(X; \mathbb{Z}). \]
Here \( T^d \) is the classifying space for \( \mathbb{Z}^d \) and the outer cohomology is just the same as group cohomology for the fundamental group \( \mathbb{Z}^d \) of \( T^d \). The spectral sequence collapses since \( \tilde{H}^\ell(N; \mathbb{Z}) = 0 \) for \( \ell > 0 \). So \[ \tilde{H}^d(X; \mathbb{Z}) \cong H^d_{\text{group}}(\mathbb{Z}^d, \tilde{H}^0(N; \mathbb{Z})). \]

By Poincaré Duality, \[ H^d_{\text{group}}(\mathbb{Z}^d, \tilde{H}^0(N; \mathbb{Z})) \cong H^0_{\text{group}}(\mathbb{Z}^d, \tilde{H}^0(N; \mathbb{Z})) = \tilde{H}^0(N; \mathbb{Z}) \times \mathbb{Z}^d. \]

The cases of \( \mathbb{R} \) or \( \mathbb{Q} \) coefficients follow immediately by the Universal Coefficient Theorem. \( \square \)

Corollary 4.2. Suppose that \( X \) is a compact foliated space given by an \( \mathbb{R}^d \)-action. Then there is a natural map \[ \tilde{H}^0(N; \mathbb{R}) \to H^d_{\tau}(X) \]
whose image is dense in the Hausdorff quotient \( \tilde{H}^d_{\tau}(X) \).

Proof. This is immediate from Theorems 3.1 and 4.1. \( \square \)

This corollary is important to us because of the relationship of \( \tilde{H}^d_{\tau}(X) \) to invariant transverse measures, as we will explain.

5. Groupoids and Measures

A compact foliated space \( X \) has an associated holonomy groupoid \( G(X) \), as in [54, p. 76, 77]. The holonomy group of a point \( x \in X \) is defined by \[ G^x_x = \{ h : hx = x \}, \]
which is, of course, the isotropy group of the action at the point \( x \). In general this group is countable. If the action of \( \mathbb{R}^d \) on \( X \) is free (which we do not assume), then each holonomy group will be trivial. Each point \( x \in X \) lies on a unique leaf, which is the orbit of the point \( x \) under the action. The leaf may be \( \mathbb{R}^d \), or some quotient torus, or something in between (\( \mathbb{R}^{d-k} \times T^k \)).

If the foliated space is given by an \( \mathbb{R}^d \)-action, as we assume throughout (see Definition 1.1), and if the holonomy cover of each leaf is simply connected, then \[ G(X) \cong \mathbb{R}^d \times X. \]

---

4 This isomorphism can be understood as follows. The group cohomology of the group \( \mathbb{Z}^d \) with coefficients in a module \( M \) can be computed from a standard cochain complex \( \left( \text{Hom}(\bigwedge^d \mathbb{Z}^d, M), \delta \right) \). Thus \( H^d(\mathbb{Z}^d, M) \) is the quotient of \( \text{Hom}\left(\bigwedge^d \mathbb{Z}^d, M\right) \cong \text{Hom}(\mathbb{Z}, M) = M \) by the image of \( \delta \) from \( \text{Hom}\left(\bigwedge^{d-1} \mathbb{Z}^d, M\right) \cong \text{Hom}(\mathbb{Z}^d, M) \). Examining the map shows that this is exactly the same as taking the coinvariants of the action on \( M \).
with unit space $X$, and associated map

$$\iota : X \to \mathbb{R}^d \times X \quad \iota(x) = (0,x),$$

range and source maps given by

$$s(g,x) = x \quad r(g,x) = gx,$$

and the inverse map given by

$$(g,x)^{-1} = (-g,gx).$$

Two elements $(g,y)$ and $(h,x)$ are multipliable if $y = hx$, and then

$$(g,y)(h,x) = (g+h,x).$$

Both $X$ and $G(X)$ are standard Borel spaces. A transversal $S$ is a Borel subset of $X$ that intersects each leaf in a countable (i.e., finite or countably infinite) set. It is complete if it intersects each leaf at least once. Foliated spaces always have complete transversals. Note that transversals need not be connected; for tilings they are typically Cantor sets. A transversal $N$ is open-regular if there is an open set $L \in \mathbb{R}^d$ and an isomorphism of foliated spaces of $L \times N$ onto an open subset of $X$ which is the identity on $N$. A transversal is regular if it is contained in an open regular transversal.

An invariant transverse measure on $X$ [54, p. 82] is a measure $\nu$ on the $\sigma$-ring of Borel transversals $\mathcal{S}$ such that $\nu|_S$ is $\sigma$-finite for each $S \in \mathcal{S}$ and $\nu|_S$ is invariant on $G^\mathcal{S}_S = \{(g,x) \in G(X) : x,gx \in S\}$. The key example for us will be the invariant transverse measure produced by a Ruelle-Sullivan current ([59], though Ruelle-Sullivan worked exclusively with foliated manifolds). It is important to note that not every foliated space has an invariant transverse measure, but the hull of a tiling space does. An invariant transverse measure $\nu$ is Radon if $\nu(N)$ is finite for each compact regular transversal $N$. Basically all of the transverse measures we will deal with in this paper are Radon.

A tangential measure is a Borel assignment $\ell \mapsto \lambda^\ell$ of a (nonnegative, $\sigma$-finite) measure $\lambda^\ell$ to each leaf $\ell$. For example, if $X = L \times N$ then we may simply take Lebesgue measure on each leaf $L \times \{n\} \cong \mathbb{R}^d$.

Given an oriented tangential measure $\lambda$ and a signed Radon invariant transverse measure $\nu$ on a compact foliated space $X$, we may define a new signed Radon measure $\mu$ on bounded Borel sets in $X$ by

$$\mu = \int \lambda \, d\nu.$$
called the \textit{Ruelle-Sullivan current} associated with the invariant transverse measure $\nu$.

Let $MT(X)$ denote the vector space of invariant Radon transverse measures on $X$.

\textbf{Theorem 5.1} (Riesz Representation Theorem [54, Theorem 4.27]). If $X$ is a compact oriented foliated space with leaf dimension $d$, then the continuous linear functionals on $H^d_\tau(X)$ can be identified as the invariant Radon transverse measures. In particular, the Ruelle-Sullivan map

\[ C: MT(X) \longrightarrow \text{Hom}_\text{cont}(H^d_\tau(X), \mathbb{R}) \cong H^d_\tau(X) \]

which takes an invariant transverse measure $\nu$ to its Ruelle-Sullivan current $C_\nu$ is an isomorphism of vector spaces.

\[ \square \]

\textbf{Corollary 5.2.} If $X$ is a compact foliated space given by an $\mathbb{R}^d$-action with associated fibration $N \rightarrow X \rightarrow T^d$, then every invariant transverse measure $\nu$ on $X$ is determined by the restriction of its Ruelle-Sullivan current to the image of $\hat{H}^d_\tau(N; \mathbb{R})_{\mathbb{Z}^d}$, or equivalently, by the associated $\mathbb{Z}^d$-invariant linear functional on $C^\infty(N)$.

\textit{Proof.} Apply Corollary 4.2. \[ \square \]

\textbf{Definition 5.3.} Let $K = \mathbb{Z}$ or $\mathbb{R}$. The \textit{cohomology trace}

\[ \tau^\hat{H}_* : \hat{H}^d(X; K) \longrightarrow \mathbb{R} \]

associated to an invariant transverse measure $\nu$ is defined to be the composition

\[ \hat{H}^d(X; K) \xrightarrow{\Delta} \hat{H}^d_\tau(X) \xrightarrow{\nu} \mathbb{R}. \]

\textbf{Remark 5.4.} One case of special interest is when the foliation is \textit{uniquely ergodic}, or in other words there is a unique (up to scaling) invariant transverse probability measure $\nu$. In the tiling space situation of Corollary 5.2, that means that $N$ has a unique $\mathbb{Z}^d$-invariant probability measure. This situation was studied in [9], where in the context of Delone multisets with finite local complexity, it was shown to be equivalent to \textit{uniform cluster frequencies}. In the uniquely ergodic case, since $MT(X)$ is one-dimensional, so is the Hausdorff quotient $\hat{H}^d_\tau(X)$ of $H^d_\tau(X) \cong C(N)_{\mathbb{Z}^d}$, and $C^\infty(N)$ must surject onto $\hat{H}^d_\tau(X)$.

6. \textit{K-theory review}

\textit{K}-theory for $C^*$-algebras plays a central role in this story; in this section we review the properties that we will need (see Blackadar’s book [17] for a good reference).
The group $K_0(A)$ is defined for any $C^*$-algebra $A$. If $A$ is unital then we take the union of all of the self-adjoint projections $p$ living in finite-dimensional matrix rings over $A$, form the free abelian group on this set, and then quotient out by the subgroup generated by setting $[p] = [q]$ if $p$ and $q$ are unitarily equivalent, $[p + 0] = [p]$, and $[p + q] = [p] + [q]$. If $A$ is non-unital, then we form the unitization $A^+$, which is a unital algebra containing $A$ as an ideal of codimension 1, so that for example $C_0(X)^+ \cong C(X^+)$, when $X$ is a locally compact space with the one-point compactification $X^+$. We define

$$K_0(A) = \ker[K_0(A^+) \to K_0(A^+/A) = K_0(C) = \mathbb{Z}];$$

see [17, Chapter 5] for more details. It is easy to see that a $C^*$-map $f: A \to B$ induces a map $f_*: K_0(A) \to K_0(B)$ and that $K_0$ is a covariant functor from $C^*$-algebras to abelian groups. We can define $K_1(A)$ via an analogous procedure using unitaries instead of projections, or else just define

$$K_j(A) = K_0(A \otimes C_0(\mathbb{R}^j))$$

for all $j$ and note (fortunately) that Bott periodicity holds, so that

$$K_j(A) \cong K_{j+2}(A).$$

If $A$ is commutative and unital and hence of the form $A \cong C(X)$ then $K_j(A) \cong K^{-j}(X)$, which is the classical $K$-theory for compact topological spaces. (For $K$-theory for compact and locally compact spaces, see Atiyah [5].)

$K$-theory and cohomology are related via the classical Chern character map

$$ch: K^{-d}(X) \to \tilde{H}^d(X; \mathbb{Q})$$

where $\tilde{H}^d(X; \mathbb{Q})$ denotes the sum of the even or odd rational Čech cohomology groups of $X$, matching the parity of $d$. This was defined by Chern initially using differential forms but for us the simplest way is via characteristic classes as in Karoubi [39, pp. 280–284]. The Chern character becomes an isomorphism after tensoring with $\mathbb{Q}$:

$$ch: K^{-d}(X) \otimes \mathbb{Q} \xrightarrow{\cong} \tilde{H}^d(X; \mathbb{Q}),$$

and similarly, of course, for real coefficients.

Projecting to $\tilde{H}^d(X; \mathbb{R})$ gives a map that we denote

$$ch_d: K^{-d}(X) \to \tilde{H}^d(X; \mathbb{R}).$$

**Proposition 6.1.** Suppose that $X$ is a compact foliated space given by an $\mathbb{R}^d$-action, and assume the holonomy cover of each leaf is simply connected (automatic if the action is free). Then

$$C^*(G(X)) \cong C(X) \rtimes \mathbb{R}^d.$$  

**Proof.** This is immediate since $G(X) \cong \mathbb{R}^d \times X$ as topological groupoids. □

**Definition 6.2.** (Bellissard) The non-commutative Brillouin zone associated to a compact foliated space given by an $\mathbb{R}^d$-action is

$$\mathcal{B} = C^*(G(X)) \cong C(X) \rtimes \mathbb{R}^d.$$  

We use the letter $\mathcal{B}$ to honor Léon Brillouin, who introduced this concept for crystals. When $X = \Omega_T$ arises as the hull of a tiling, then we write $\mathcal{B}_T$ for the associated non-commutative Brillouin zone.
It follows immediately that
\[ K_0(B) = K_0(C^*(G(X))) \cong K_0(C(X) \rtimes \mathbb{R}^d). \]

Connes' Thom isomorphism theorem [19] implies that there is a canonical isomorphism
\[ \varphi : K_d(C(X)) \cong K_0(C(X) \rtimes \mathbb{R}^d) \]
and of course
\[ K_d(C(X)) \cong K_{-d}(X) \]
for any compact space \( X \).

Putting these isomorphisms together yields

**Proposition 6.3.** Suppose that \( X \) is a compact foliated space given by an \( \mathbb{R}^d \)-action, and assume the holonomy cover of each leaf is simply connected. Then there is a natural sequence of isomorphisms
\[
K_{-d}(X) \cong K_d(C(X)) \cong K_0(C(X) \rtimes \mathbb{R}^d) \cong K_0(C^*(G(X))) \cong K_0(B).
\]

We let \( \chi : K_{-d}(X) \to K_0(B) \) denote the composite isomorphism.

### 7. Traces on Groupoid \( C^* \)-algebras

Suppose given an invariant transverse measure \( \nu \) and a tangential measure \( \lambda \) on a foliated space \( X \). Then \( \mu = \int \lambda d\nu \) is a measure on \( X \) and turns \( G(X) \) into a measured groupoid (see [54, p. 142]) by defining a measure \( \tilde{\mu} \) on \( G^{}(X) \) by
\[
\tilde{\mu}(E) = \int_X \mu^*(E \cap G(X))^\mu d\mu(x).
\]

Form the Hilbert space with associated direct integral decomposition
\[
L^2(G^{}(X), \tilde{\mu}) = \int \oplus \ell^2(t, X^t) d\tilde{\mu}.
\]

Then [54, p. 142] we may form the \( * \)-algebra of integrable functions \( L^1(G(X), \tilde{\mu}) \) with natural \( * \)-representation
\[
\pi : L^1(G^{}(X), \tilde{\mu}) \to B(L^2(G^{}(X), \tilde{\mu})).
\]

Define the von Neumann algebra \( W^*(G^{}(X), \tilde{\mu}) \) to be the weak closure of \( \pi(L^1(G^{}(X), \tilde{\mu})) \) in \( B(L^2(G^{}(X), \tilde{\mu})) \). This is a Type II von Neumann algebra and will be a factor if \( C^*(G^{}(X)) \) is simple, which is the case if and only if the action of \( \mathbb{R}^d \) on \( X \) (or if the lattice \( \Lambda \) on \( \mathbb{N} \), in the situation of Definition 1.1) is minimal.\(^5\) If the action of \( \mathbb{R}^d \) on \( X \) is only topologically transitive (i.e., there is a dense orbit, but not every orbit need be dense), then \( C^*(G^{}(X)) \) will be primitive and \( W^*(G^{}(X), \tilde{\mu}) \) will still be a factor if the transverse measure has full support.

---

\(^5\) This fact was originally called the Effros-Hahn Conjecture and was proven in full generality in [31].
There is a natural map
\[ C^*(G(X)) \longrightarrow W^*(G(X), \tilde{\mu}). \]
For any leaf \( \ell \) we have the local representation
\[ \pi_\ell: C^*(G(X)) \longrightarrow \mathcal{B}(L^2(\ell, \lambda_\ell)). \]
Write \( m^x = \pi_\ell(m) \) for \( x \in \ell \). We wish to define
\[ \phi_\nu: C^*(G(X)) \longrightarrow \mathbb{R}. \]
This construction is described in detail in [54, pp. 149–154]. Let \( m \in C^*(G(X))^+ \).
As an element of the von Neumann algebra, think of \( m \equiv \{ m^x \} \) for \( m^x \in \mathcal{B}(L^2(\ell, \lambda_\ell)) \).
Then \( m^x \) is a positive operator on \( L^2(\ell, \lambda_\ell) \) and it is \textit{locally traceable} in the sense of [54, p. 18]. Define a measure \( \lambda_m(\ell) \) on \( \ell \) by the formula
\[ \int f d\lambda_m(\ell) = \text{Tr}(f^{1/2} m^x f^{1/2}) \quad \forall f \in L^2(\ell, \lambda_\ell) \]
for every positive \( f \) of bounded support. (Note that this is the same as the construction of D. Lenz, N. Peyerimhoff, and I. Veselić in [48, Theorem 4.2].)

Finally, define the trace itself by
\[ \phi_\nu(m) = \int \lambda_m(\ell) d\nu(\ell). \]

If the measure \( \mu = \lambda_m d\nu \) is finite on \( X \) and Radon on \( G(X) \) then for any \( g \in C^\infty(G(\Omega_T)) \) with compact support, we have
\[ \phi_\nu(g^* g) < +\infty. \]
The trace is thus densely defined. It is lower semi-continuous [54, pp. 149–154].

In our situation where \( G(X) \cong X \times \mathbb{R}^d \) as topological groupoids, we recall first that if \( \lambda \) is a tangential measure on \( X \) and \( \nu \) is an invariant transverse measure then
\[ \mu = \int \lambda_\ell d\nu \]
is a Radon measure on \( X \) which is \( \mathbb{R}^d \)-invariant, and \( \tilde{\mu} \) is a Radon measure on \( G(X) \), so that there is a natural trace
\[ \tau_\mu(f) = \int_{x \in \Omega_T} f(x, 0) \int \lambda_\ell d\nu. \]

To fit this into our framework, we would first represent \( C(X) \rtimes \mathbb{R}^d \) in the von Neumann algebra \( L^\infty(X, \mu) \rtimes \mathbb{R}^d \) associated to it via the direct integral procedure. Then the function \( f \) is sent to the family \( \{ f^x \} \), where \( f^x \) is a bounded operator on \( L^2 \) of the associated leaf \( \ell_x \). This produces a local trace \( \lambda_f^x \) on \( \ell_x \), the leaf of \( x \). Then we would define
\[ \phi_\nu: C(X) \rtimes \mathbb{R}^d \longrightarrow \mathbb{R} \]
by
\[ \phi_\nu(f) = \int \lambda_f d\nu \]
and then
\[ \tau_\nu(f) = \phi_\nu(f). \]
by [54, Prop. 6.25 and the discussion at the bottom of page 149].
8. The Partial Chern Character and $K$-theory Trace

There is a partial Chern character

$$c: K_0(C^*(G(X))) \to \bar{H}_d^\delta(X)$$

defined in [54, p. 161] as follows. Suppose that \([u] \in K_0(C^*(G(X)))\) is represented by \([e] - [f]\), where \(e, f \in M_n(C^*(G(X)^+))\) with common image in \(M_n(\mathbb{C})\). Let \(\nu\) denote a positive Radon invariant transverse measure on \(X\) and form the corresponding trace \(\phi_\nu\) on \(C^*(G(X))\). Extend \(\phi_\nu\) to \(\phi_\nu^\text{Tr} = \phi_\nu \otimes \text{Tr}\) on \(M_n(C^*(G(X)))\). Then \([u] \in \bar{H}_d^\delta(X)\) is the cohomology class of the tangentially smooth \(d\)-form \(\omega_u\) which (after identifying \(d\)-currents with Radon invariant transverse measures), is given by\(^6\)

\begin{equation}
\omega_u(\nu) = \phi_\nu^\text{Tr}(e - f)
\end{equation}

where \(\phi_\nu^\text{Tr}\) is the trace \(\phi_\nu\) on \(C^*(G(X))\) associated to the invariant transverse measure.

It is now easy to describe the $K$-theory trace.

**Definition 8.1.** Given an invariant transverse measure \(\nu\), the $K$-theory trace, denoted

$$\tau^K_\ast: K_0(G(X)) \to \mathbb{R}$$

is the composition

$$K_0(G(X)) \xrightarrow{\phi_\text{Tr}} \bar{H}_d^\delta(X) \xrightarrow{\phi_\nu} \mathbb{R}.$$ 

If \(X\) is a compact foliated space given by an \(\mathbb{R}^d\)-action with invariant transverse measure \(\nu\), then we may write

$$\tau^K_\ast: K_0(\mathcal{B}) \equiv K_0(G(X)) \to \mathbb{R}.$$ 

\(^6\) Here is the detail: We start by noting that

$$K_0(C^*(G(X))) \cong K_0(C^*(G_N^{\mathbb{N}}))$$

and so without loss of generality we may assume that \(e\) and \(f\) are in \(M_n(C^*(G_N^{\mathbb{N}})^+)\). Skipping some analysis (see [54, p. 162]), we may assume that any element in \(C^*(G_N^{\mathbb{N}})\) may be represented there by a kernel operator where the kernel is continuous and has compact support. Further, the kernel, when extended to \(G(X)\), is tangentially smooth. Following through with a little more analysis, we see that we may express the action of the trace on any element \(b \in C^*(G(X))\) as

$$\phi_\nu(b) = \int \omega_b \, d\nu$$

We see from this analysis that the partial Chern character is given by

$$c[u] = [\omega_{e-f}]$$

and hence

$$\phi_\nu(e - f) = \int \omega_{e-f} \, d\nu = \int c[u] \, d\nu.$$
9. Uniting the Traces

The goal of this paper, as explained in the introduction, is to demonstrate that the $K$-theory and cohomology approaches to the traces are related, and to show that they are equivalent in low dimensions. We can now state our main result.

**Theorem 9.1.** Suppose that $X$ is a compact foliated space given by an $\mathbb{R}^d$-action with invariant transverse measure $\nu$, and the holonomy cover of each leaf is simply connected. Then:

- The diagram
  \[
  \begin{array}{ccc}
  K_0(B) & \xrightarrow{\chi^{-1}} & K^{-d}(X) \\
  c \downarrow & & \downarrow s \\
  H^d_\tau(X) & \xrightarrow{id} & H^d_\tau(X)
  \end{array}
  \]
  commutes.

- Bloch Theorem: For every invariant transverse measure $\nu$, the diagram
  \[
  \begin{array}{ccc}
  K_0(B) & \xrightarrow{\text{ch}_d \circ (\chi^{-1})} & \tilde{H}^d(X; \mathbb{R}) \\
  \downarrow & & \downarrow s \\
  \mathbb{R}^K & \xrightarrow{id} & \mathbb{R}
  \end{array}
  \]
  commutes.

**Proof.** We gratefully acknowledge Kaminker-Putnam [38, Prop. 2.4] for putting us on the right track for this theorem.

Starting at $K^{-d}(X)$ and moving left one obtains composition

\[ K_d(C(X)) \xrightarrow{\varphi} K_0(C(X) \times \mathbb{R}^d) \cong K_0(C^*(G(X))) \xrightarrow{\nu} \tilde{H}^d(X) \]

and this is the abstract analytic index map $a$ of A. Connes as described in [20]. In the other direction, the composition

\[ K_d(C(X)) \cong K^{-d}(X) \xrightarrow{\text{ch}_d} \tilde{H}^d(X; \mathbb{R}) \xrightarrow{\nu} \tilde{H}^d_\tau(X) \]

is the abstract topological index $\text{index}_t$. Connes shows [20, Theorem 9] that

\[ \text{index}_a = \text{index}_t \in \tilde{H}^d_\tau(X) \]

This is an early version of the abstract foliation index theorem of Connes and Skandalis.

The commutativity of the second diagram follows at once from applying the definitions of the traces. We expand the diagram

\[
\begin{array}{ccc}
K_0(B) & \xrightarrow{\chi^{-1}} & K^{-d}(X) \\
\downarrow c & & \downarrow s \\
H^d_\tau(X) & \xrightarrow{id} & H^d_\tau(X)
\end{array}
\]
and then observe that
\[ \tau^K_* = (\cap C_\nu) \circ c \]
and
\[ \tau^R_* = (\cap C_\nu) \circ s \]
This, then, is essentially a special case of the Index Theorem for foliated spaces [54]. \( \square \)

Now we apply this result to tilings. Given a tiling \( T \), we denote its continuous hull by \( \Omega_T \). Suppose that for any \( R > 0 \) that there are, up to translation, only finitely many patches in \( T \) (i.e., subsets of \( T \)) whose union has diameter less than \( R \). Then by [58, Lemma 2], \( \Omega_T \) is compact. This condition is called the \textit{finite pattern condition} or \textit{finite local complexity}.

We assume the following conditions:

1. \( T \) satisfies the finite pattern condition (i.e., finite local complexity);
2. \( T \) has only finitely many tile orientations

Then by Sadun-Williams [60, Theorem 1], under these conditions, \( \Omega_T \) is homeomorphic to the total space of a fibre bundle of the form

\[ N \to N \times \mathbb{Z}_d \to T^d, \]

obtained by suspending a \( \mathbb{Z}_d \)-action on a totally disconnected space \( N \). The proof of this result depends on [60, Lemma 4], which asserts that this holds for rational tiling spaces. While the proof of this Lemma in [60] is somewhat condensed, Ian Putnam has explained it to us as follows.

Start with a rational tiling space and rescale so that a translate of the tiling has the property that all vertices of tiles are in \( \mathbb{Z}_d \) (i.e., are at points with integral coordinates). We have a space of tilings such that the vectors joining any two adjacent vertices are in \( \mathbb{Z}_d \). In consequence, if one vertex of a tiling is on an integer point, then they all are. Let \( N \) be the set of all tilings in the space whose vertices lie in \( \mathbb{Z}_d \). First, it is compact. Second, because of finite local complexity, it is totally disconnected. (Fix a radius \( R \). Look at all possible patches of radius \( R \). There are only finitely many. This partitions \( N \) into a finite number of closed disjoint sets. Let \( R \) get bigger.) It also has an action of \( \mathbb{Z}_d \) by translation. The map from \( N \times \mathbb{R}_d \) to the tiling space \( \Omega_T \) which sends \( (T, x) \) to \( T - x \) induces a homeomorphism

\[ h: N \times \mathbb{Z}_d \mathbb{R}_d \cong \Omega_T. \]

The properties claimed should be clear on \( N \times \mathbb{Z}_d \mathbb{R}_d \). The map given by Sadun and Williams is just projection onto the second component.

The fact that \( \Omega_T \) is the total space of a fibre bundle as described implies that the hull \( \Omega_T \) is a compact foliated space given by an \( \mathbb{R}_d \)-action. So we can specialize the theorem above as follows.

**Theorem 9.2.** Suppose that \( T \) is a tiling satisfying the following conditions:

1. \( T \) satisfies the finite pattern condition (i.e., finite local complexity);
2. \( T \) has only finitely many tile orientations

Then
• The diagram

\[
\begin{array}{c}
K_0(B_T) \xrightarrow{\phi_0(C^*(G(\Omega_T)))} K_0(G(\Omega_T)) \xrightarrow{\chi^{-1}} K^{-d}(\Omega_T) \xrightarrow{ch_d} \tilde{H}^d(\Omega_T; \mathbb{R}) \\
\text{id} \downarrow \hspace{5cm} \downarrow \text{id} \\
H^d(\Omega_T) \xrightarrow{c} \tilde{H}^d(\Omega_T; \mathbb{R}) \\
\end{array}
\]

commutes.

• Bloch Theorem: For every invariant transverse measure \( \nu \), the diagram

\[
\begin{array}{c}
K_0(B_T) \xrightarrow{(ch_d) \circ (\chi^{-1})} \tilde{H}^d(\Omega_T; \mathbb{R}) \\
R \xrightarrow{\alpha} \mathbb{R} \\
\end{array}
\]

commutes.

Remark 9.3. Note that aperiodicity of \( T \) is not needed for the commutativity. But it is useful since it implies that \( \mathbb{R}^d \) acts freely on \( \Omega_T \).

Remark 9.4. Up to this point we have used \( \check{C}ech \) cohomology with real coefficients. However, to understand the precise values of the trace, and thus gap labelling, we need finer information, based on \( \check{C}ech \) cohomology with integer coefficients. These are related, of course, since \( \check{H}^*(X; \mathbb{Z}) \otimes \mathbb{R} \cong \check{H}^*(X; \mathbb{R}) \). The problem arises because of the Chern character. It is a map

\[ ch: K^*(X) \longrightarrow \check{H}^{**}(X; \mathbb{Q}) \]

which induces an isomorphism

\[ ch: K^*(X) \otimes \mathbb{Q} \longrightarrow \check{H}^{**}(X; \mathbb{Q}) \]

and thus also over the real numbers. For arbitrary compact spaces the Chern character does NOT take only integral values. So in general there is no reason to think, for instance, that

\[ ch_d: K^d(X) \longrightarrow \check{H}^d(X; \mathbb{Q}) \]

factors through \( \check{H}^d(X; \mathbb{Z}) \).

The good news is that this is the case in dimensions \( \leq 3 \). (See for example [2, Prop. 6.2], though this is well known in the topology literature. It was also observed indirectly in [69].) The explanation of this is as follows. Complex line bundles over a compact space \( X \) are classified by a single invariant, the first Chern class in \( \check{H}^2(X; \mathbb{Z}) \). If \( \text{dim } X \leq 3 \), then every complex vector bundle over \( X \) is a direct sum of line bundles, and one can define the integral Chern character \( ch: K^0(X) \longrightarrow \check{H}^0(X; \mathbb{Z}) \oplus \check{H}^2(X; \mathbb{Z}) \) by sending a virtual bundle (i.e., \( \mathbb{Z} \)-linear combination of line bundles) to the “rank” in \( \check{H}^0(X; \mathbb{Z}) \) and the first Chern class \( c_1 \) in \( \check{H}^2(X; \mathbb{Z}) \). This makes sense even if \( c_1 \) is torsion, and defines a ring isomorphism

\[ ch: K^0(X) \rightarrow \check{H}^0(X; \mathbb{Z}) \oplus \check{H}^2(X; \mathbb{Z}) \].

The case of \( K^{-1} \) and spaces of dimension \( \leq 3 \) is similar. If \( \text{dim } X \leq 3 \), then \( K^{-1} \) can be identified with the homotopy classes of maps \( X \rightarrow U(2) \). Subtracting off the class of a map \( X \rightarrow U(1) \), which can be viewed as the classifying map of a class in
\( \tilde{\mathbf{H}}^1(X; \mathbb{Z}) \), we can assume we have a map \( X \to SU(2) = S^3 \), which for \( \dim X \leq 3 \) is classified by an element of \( \tilde{\mathbf{H}}^3(X; \mathbb{Z}) \). So we get an isomorphism
\[
\text{ch}: K^{-1}(X) \to \tilde{\mathbf{H}}^1(X; \mathbb{Z}) \oplus \tilde{\mathbf{H}}^3(X; \mathbb{Z}).
\]
But for \( X \) of dimension 4 and up, the Chern character involves \( \frac{1}{2}c_1^2 \) (and higher-order terms in higher dimension) and so is only defined in rational cohomology. When \( \dim X = 4 \), the differentials in the Atiyah-Hirzebruch spectral sequence (for computing \( K \)-theory from Čech cohomology) vanish, so that there is a filtration of \( K^0(X) \) with quotients \( \tilde{\mathbf{H}}^0(X; \mathbb{Z}) \), \( \tilde{\mathbf{H}}^2(X; \mathbb{Z}) \), and \( \tilde{\mathbf{H}}^4(X; \mathbb{Z}) \), which is just a bit weaker than what happens in lower dimensions. However, the extension in recovering \( K^0(X) \) from the cohomology can be non-trivial. For example, \( K^0(\mathbb{R}P^4) \cong \mathbb{Z}/4 \) while
\[
\tilde{\mathbf{H}}^2(\mathbb{R}P^4; \mathbb{Z}) \cong \tilde{\mathbf{H}}^4(\mathbb{R}P^4; \mathbb{Z}) \cong \mathbb{Z}/2.
\]
In this dimension the only denominator needed to define the Chern character is 2, so the Chern character can be viewed as a ring homomorphism
\[
K^0(X) \to \tilde{\mathbf{H}}^0(X; \mathbb{Z}) \oplus \tilde{\mathbf{H}}^2(X; \mathbb{Z}) \oplus \tilde{\mathbf{H}}^4(X; \mathbb{Z}[\frac{1}{2}]),
\]
which is an isomorphism after inverting 2. Anyway, for \( \dim X \leq 3 \), we have isomorphisms
\[
\text{ch}: K^0(X) \cong \tilde{\mathbf{H}}^1(X; \mathbb{Z}) \oplus \tilde{\mathbf{H}}^3(X; \mathbb{Z}), \quad K^1(X) \cong \tilde{\mathbf{H}}^1(X; \mathbb{Z}) \oplus \tilde{\mathbf{H}}^3(X; \mathbb{Z}).
\]
This will be used below.

**Corollary 9.5** (Bloch theorem in low dimensions). Under the assumptions above, if the dimension is \( \leq 3 \), then for every invariant transverse measure the diagram
\[
\begin{array}{c}
K_0(\mathcal{B}_T) \xrightarrow{(\text{ch}_d) \circ (\chi^{-1})} \tilde{\mathbf{H}}^d(\Omega_T; \mathbb{Z}) \\
\downarrow \tau^K \quad \downarrow \tau^R \\
\mathbb{R} \xrightarrow{id} \mathbb{R}
\end{array}
\]
commutes. In particular, the \( K \)-theory trace \( \tau^K: K_0(\mathcal{B}_T) \to \mathbb{R} \) and the cohomology trace \( \tau^R: \tilde{\mathbf{H}}^d(\Omega_T; \mathbb{Z}) \to \mathbb{R} \) have the same image in \( \mathbb{R} \).

**Proof.** In (5), we can replace real cohomology with integral cohomology in the upper right of the diagram. \( \square \)

**Remark 9.6.** In higher dimensions, the top degree part of the Chern character \( \text{ch}_d: K^{-d}(\Omega_T) \to \tilde{\mathbf{H}}^d(\Omega_T; \mathbb{Q}) \) does not obviously factor through \( \tilde{\mathbf{H}}^d(\Omega_T; \mathbb{Z}) \), but it does after multiplying by \( [\frac{d}{2}]! \). So the images of the \( K \)-theory trace and the cohomology trace agree at least up to this factor.

**Remark 9.7.** We take this opportunity to explain the connection between these results and the “Gap Labeling Theorem” (GLT) of [13, 16, 38]. In the situation of Theorem 9.2, the GLT asserts that the image of the \( K \)-theory trace on \( K_0(\mathcal{B}_T) \) is equal to the image of this trace on \( K_0 \) of the subalgebra \( C(\Omega_T) \) of \( C(\Omega_T) \times \mathbb{R}^d \). This is equivalent to the equality of the images of the \( K \)-theory and cohomology traces, for the following reason. As pointed out in [38, §2], we can use the equivalence between the groupoids \( G(\Omega_T) \) and \( N \times \mathbb{Z}^d \) (where \( N \) is the totally disconnected transversal) to convert the statement of the GLT to equality of the image of the
K-theory trace on $K_0(C(N) \times \mathbb{Z}^d)$ with the image of the trace on $K_0(C(N)) = K^0(N) = \hat{H}^0(N; \mathbb{Z})$ (since $N$ is totally disconnected). Since the transverse measure is assumed invariant, the measure on $N$ is $\mathbb{Z}^d$-invariant, and this factors through the coinvariants $\hat{H}^0(N; \mathbb{Z})$, which is $\hat{H}^d(\Omega_T; \mathbb{Z})$ by Theorem 4.1. In other words, the GLT asserts the equality of the image of the K-theory trace with the image of the cohomology trace on $\hat{H}^d(\Omega_T; \mathbb{Z})$. By Theorem 9.2, the image of the K-theory trace is equal to the image of the cohomology trace on the subgroup of $\hat{H}^d(\Omega_T; \mathbb{Q})$ given by the image of $ch_d$ on $K^d(\Omega_T)$. So the images of the two traces are equal if the image of $ch_d$ is contained in $\hat{H}^d(\Omega_T; \mathbb{Z})$. Inclusion of the image of $ch_d$ in $\hat{H}^d(\Omega_T; \mathbb{Z})$ is used in all of the proofs of the GLT in [13, 16, 38].

10. Building tilings: A quick introduction

Having stated our main result, Theorems 9.1 and 9.2 and specialized to tilings, we wish to apply it to show the relationship between structural (diffraction) data and spectral data. For the sake of self-completeness, we now present a quick introduction to some popular approaches used to build tilings. Further details can be found in [7, 8, 34, 40, 52, 56, 63].

Cut & Project – Characteristic function – Phason

A commonly accepted view of a quasicrystal in dimension $d$ is modeled as a section of a periodic structure (crystal lattice $\mathbb{Z}^n$) in an $n$-dimensional ambient space $\mathbb{R}^n$, with $n > d$. We have the decomposition $\mathbb{R}^n = E^\parallel \oplus E^\perp$, where $E^\parallel$ is the $d$-dimensional physical space in which the structure is embedded, whereas $E^\perp$ is an $(n-d)$-dimensional internal space. This setting is usually implemented for the Cut & Project algorithm (hereafter C&P), very useful and popular for the building of quasicrystals [26, 40, 52, 63].

A simple example is given by the quasiperiodic tiling of a line ($d = 1$) with a Fourier module $\mathcal{F}$ with two generators $n = 2$. The ambient space is $\mathbb{R}^2$ with the square lattice $\mathbb{Z}^2$; the physical space is the line $E^\parallel$, which makes a tilt angle $\theta$ with the horizontal axis. If the slope $s \equiv 1/(1 + \cot \theta)$ is irrational, the structure thus obtained is quasiperiodic; it is a one-dimensional ($d = 1$) quasicrystal. If the slope is an irreducible rational $s = p/q$, the structure is periodic with $q$ atoms in a cell. A celebrated example of one-dimensional quasicrystal is the Fibonacci sequence obtained for the irrational slope $s = \tau^{-1} = 2/(1 + \sqrt{5})$.

For $n = 2$, we end up with a deterministic arrangement of two types of tiles, i.e., a two-letter alphabet $\{a, b\}$ which generally represents a piecewise modulation of a physical parameter (e.g. density, potential, dielectric constant, etc.). The offset of the line cut fixes the first letter of the iteration. It is thus immaterial for the infinite tiling but not for finite chains. Since all choices are equivalent, the offset appears as a gauge freedom known as a phason and obtained by sliding the cut along the internal space $E^\perp$. 
A characteristic function, \( \chi(n, \phi) \equiv \text{sign } [\cos (2\pi n s + \phi) - \cos (\pi s)] \) with \( n \in \mathbb{N} \), equivalent to the C&P algorithm can be defined, which takes the two values \( \pm 1 \) respectively identified to the two letters \( \{a, b\} \). The parameter \( \phi \in [0, 2\pi) \) is the aforementioned phason serving as an extra gauge degree of freedom. \( \chi(n, \phi) \) has been successfully used in the determination of both the diffraction spectrum [51, 52, 63] and spectral properties of Schrödinger and of wave operators [44, 45, 50].

**Substitutions**

Aperiodic tilings can also be generated by inflation rules known as substitutions [8, 52, 53, 63]. For a two-letter alphabet \( \{a, b\} \), the substitution rule is defined by its action \( \sigma \) on a word \( w = l_1 l_2 \ldots l_k \) by the concatenation \( \sigma(w) = \sigma(l_1)\sigma(l_2)\ldots\sigma(l_k) \). An occurrence primitive matrix \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) defined by \( \sigma(a) = a^\alpha b^\beta \) and \( \sigma(b) = a^\gamma b^\delta \) (ignoring the order of letters) is associated to \( \sigma \). It allows us to define a sequence of numbers \( F_N \) from the recurrence \( F_{N+1} = tF_N - pF_{N-1} \), where \( t = \text{Tr } M \), \( p = \text{det } M \) and \( F_{0,1} = 0, 1 \). The largest eigenvalue \( \lambda_1 \) of \( M \) is larger than 1 (Frobenius-Perron theorem). For the Fibonacci substitution \( M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \), \( \lambda_1 = \tau = \frac{1+\sqrt{5}}{2} \) and \( F_N \) are the Fibonacci numbers. The left eigenvector

\[
\mathbf{v}_1 = (\rho_a, \rho_b),
\]

normalised to \( \rho_a + \rho_b = 1 \), with \( \rho_a = \frac{\gamma}{\lambda_1 + \gamma - \alpha} \), \( \rho_b = \frac{\beta}{\lambda_1 + \beta - \gamma} \), portrays the frequencies or densities of the letters \( a \) and \( b \) in the infinite word. The right eigenvector \( \mathbf{w}_1 = (d_a, d_b)^T \), normalized such that \( \frac{d_a}{d_b} = \frac{d}{\lambda_1 - \alpha} = \frac{\lambda_1 - \delta}{\gamma} \), expresses the lengths of the corresponding tiles.

The C&P and substitution algorithms are not equivalent; e.g. no substitution is associated to the C&P slope \( s = 1/\pi \) since \( \pi \) is a transcendental and not an algebraic irrational. Conversely, the substitution \( M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) has no C&P counterpart.

Substitutions in 1d, which are equivalent to C&P, are quasiperiodic and can be identified as follows. Let \( x_n \) be atomic positions on the boundary between tiles, \( d = \lim_{n \to \infty} x_n/n = \rho_a d_a + \rho_b d_b \) the mean tile length, and \( u_n = x_n - dn \) fluctuations about the mean. Then

\[
\Delta_u = \lim_{n \to \infty} \sup_{n} u_n - \lim_{n \to \infty} \inf_{n} u_n,
\]

is the extension of the atomic surface [53]. Then a substitution is quasiperiodic if \( \Delta_u = 1 \). Furthermore, a substitution \( \sigma \) is common unimodular if it is primitive, irreducible, Pisot, unimodular (\( \text{det } M = \pm 1 \)), and has a common prefix (or suffix) [3]. A quasiperiodic substitution is necessarily common unimodular.

### 11. Diffraction spectrum

In this section we discuss the diffraction spectrum information obtained from our examples of \( (d = 1) \) one-dimensional tilings using Čech cohomology and Ruelle-Sullivan currents.

For a two-letter alphabet \( \{a, b\} \), an atomic density \( \rho(x) = \sum_n \delta(x-x_n) \) is defined by placing identical atoms at boundaries \( x_n \) between \( a \) and \( b \) tiles. The structure
factor or two-point correlation for a tiling of length (number of tiles) \( N \) is

\[
S(k) = \frac{1}{N} |G(k)|^2 = \frac{1}{N} \sum_{m,n} e^{ik(x_m-x_n)},
\]

where \( G(k) = \sum_n e^{-ikx_n} \) is the Fourier transform of \( \rho(x) \) and \( k \) is the 1d wave vector in units of an inverse mean lattice spacing.

Bragg peaks are essential in the definition of quasiperiodicity [18, 27, 32]. There is a Bragg peak at \( k_0 \) if \( G(k_0) \propto N \) for large \( N \), namely if a macroscopic fraction of atoms diffracts coherently and \( S(k) \approx \delta(k-k_0) \). For a 1d quasicrystal described by the C&P algorithm,

\[
G(k) = \sum_{p,q} C_{pq} \delta \left( \frac{1}{\pi} k - p + q s^{-1} \right),
\]

hence the corresponding Fourier transform consists of Bragg peaks located at

\[
k_{pq} = 2\pi (p + q s^{-1}),
\]

where \((p,q)\) are integers (see Fig. 1B for a Fibonacci quasicrystal). The diffraction spectrum is pure-point, and the corresponding Fourier module \( \mathbb{Z} + s\mathbb{Z} \) is the projection onto the physical space \( E_\parallel \) of the reciprocal ambient space lattice [53].

The diffraction spectrum of tilings generated by substitutions is obtained from the solutions of

\[
k \lambda_1^n \rightarrow 0 \pmod{1}, \quad \lambda_1 > 1
\]

with integer coefficients, with \( \lambda > 1 \) and such that all other roots of \( P(x) \) are less than 1 in absolute value. For example, the golden ratio \( \phi \sim 1.618 \) is a root of \( P(x) = x^2 - x - 1 \). The other root of \( P(x) \) is \(-\phi^{-1} \sim -0.618 \) which has absolute value less than 1, so \( \phi \) is a Pisot number.

We identify the following cases, which all fulfill conditions of Theorem 9.2:

**\( \lambda_1 \) Pisot and \( \det M = \pm 1 \).** The pure-point structure factor consists of Bragg peaks supported by a Fourier module with a finite number \( p \) of generators. The structure is a quasicrystal and it can be described in the ambient space formalism.

**\( \lambda_1 \) Pisot and \( \det M \neq \pm 1 \).** The pure-point structure factor consists of Bragg peaks supported by a Fourier module not finitely generated. For 1d chains, it contains at least the infinite family of Bragg diffractions \( \{2\pi\lambda_1^n, n \geq 0\} \). An example is provided by the Thue-Morse tiling whose occurrence primitive matrix is \( M = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \). The diffraction spectrum has both pure-point and singular continuous components. The corresponding Bragg peaks are computed in [29, Theorem 3.9]. The corresponding Fourier module is \( \mathbb{Z} + \mathbb{Z}\left[\frac{1}{2}\right] \). The full diffraction spectrum is represented in Fig. 1C and detailed in Table 1. Another example is the period doubling substitution, whose occurrence matrix is \( M = \left( \begin{array}{cc} 1 & 1 \\ 2 & 0 \end{array} \right) \) with a pure-point diffraction spectrum of Fourier module \( \mathbb{Z}\left[\frac{1}{2}\right] \).

**Non-Pisot substitutions** correspond to the case where the second eigenvalue of the occurrence matrix \( M \) is \( |\lambda_2| > 1 \). This property has several consequences, among them the occurrence of unbounded density fluctuations [25, 30]. It has been

\[7\] There appears to be a slight misprint there; the \( 4\pi \) factor should apply to both summands in the cohomology.
shown that both the fluctuation, denoted by \( u_n \), of the atomic positions in \( E^\parallel \) and the extension of the tiling in the internal space \( E^\perp \), scale with the power law \( u_n \approx n^\beta \) where \( \beta = \ln |\lambda_2|/\ln \lambda_1 \). Such unbounded density fluctuations destroy the coherence of any would-be Bragg diffraction so that the Fourier diffraction spectrum of non-Pisot tilings is generically continuous. The Rudin-Shapiro tiling provides such an example as displayed in Fig. 1e.

An alternative description of the Bragg spectrum for one-dimensional tilings is based on the \( \check{\text{C}} \text{ech} \) cohomology group \( \check{H}^1(\Omega_T;\mathbb{Z}) \) associated to the hull \( \Omega_T = \{T - x | x \in E^\parallel \} \).

A known result [37] is that for C&P (quasiperiodic) tilings, \( \check{H}^1(\Omega_T;\mathbb{Z}) \cong \mathbb{Z}^a \), where \( a \) counts the number of letters of the tiling. The same applies to quasiperiodic substitutions. The diffraction spectrum is obtained using the Ruelle-Sullivan map \( C_\nu \) which projects \( \check{H}^1(\Omega_T;\mathbb{Z}) \) into \( \mathbb{R} \),

\[
(11) \quad C_\nu(\check{H}^1(\Omega_T;\mathbb{Z})) = \mathbb{Z} + \rho_b \mathbb{Z},
\]

so that Bragg peaks are labeled using the two integer coordinates of \( \check{H}^1(\Omega_T;\mathbb{Z}) \cong \mathbb{Z}^2 \).

When \( \det M \neq \pm 1 \), then the \( \check{\text{C}} \text{ech} \) cohomology often is not free abelian. For example, an analysis [2, p. 531] or [11] shows that for the Thue-Morse substitution tiling \( \check{H}^1(\Omega_T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}] \).

By Cor. 9.5, we can calculate the range of the cohomology trace as well. So here is a summary of the results:

**Proposition 11.1.**

1. **For the Fibonacci tiling,**

\[
\check{H}^1(\Omega_T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad \tau^\check{H}_\ast(\check{H}^1(\Omega_T;\mathbb{Z})) = (\mathbb{Z} + \lambda^{-1}\mathbb{Z}),
\]

using \( \rho_b = 1 - \lambda^{-1} \) in (9).

2. **For the Thue-Morse tiling,**

\[
\check{H}^1(\Omega_T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}], \quad \tau^\check{H}_\ast(\check{H}^1(\Omega_T;\mathbb{Z})) = \frac{1}{2} \mathbb{Z}[\frac{1}{2}].
\]

12. **Spectral properties and gap labeling theorem**

This section is devoted to spectral information extracted from the Laplacian operators conveniently defined on the previously discussed representative classes of one-dimensional \( d = 1 \) tilings, using \( K \)-theory and the partial Chern character.

The band structure for non-interacting excitations (e.g. electronic, electromagnetic, acoustic or mechanical waves) propagating in a tiling is modeled either by a

---

*We have defined the Ruelle-Sullivan class \([C_\nu]\) as lying in tangential homology \( H^\tau_d(X) \), so that it naturally pairs with tangential cohomology \( H^d_\tau(X) \). We have constructed a natural map \( s : \check{H}^d(X;\mathbb{Z}) \rightarrow H^\tau_d(X) \) and this induces a pairing of the Ruelle-Sullivan class with \( \check{H}^d(X;\mathbb{Z}) \). This composition is exactly the cohomology trace \( \tau^\check{H}_\ast : \check{H}^d(X;\mathbb{Z}) \rightarrow \mathbb{R} \) as defined in 5.3.*
Figure 1. Diffraction spectra of representatives of five families of tilings. (A) describes a periodic tiling with pure-point (PP) diffraction spectrum and a finite number of Bragg peaks, (B) a quasiperiodic (Fibonacci) tiling. The PP diffraction spectrum displays an infinite countable number of Bragg peaks, (C) a non-quasiperiodic tiling (Thue-Morse) with both PP Bragg peaks and a singular continuous (SC) component made of localised but not Bragg diffraction peaks (see Fig. 2 for details), (D) a limit-quasiperiodic tiling (Period Doubling) with only PP Bragg peaks and (E) a non-quasiperiodic and non-Pisot tiling (Rudin-Shapiro) with an absolutely continuous diffraction spectrum.

"tight binding" model, where the tiles \{a, b\} represent atomic locations with particles hopping from tile to tile, or by a continuous wave equation. Periodic tilings model traditional crystalline structures. The quantum/wave mechanical model of this motion is a certain self adjoint operator on the space of square-summable functions in the set of tiles. We are interested in the spectrum of this operator (spectral data).

The continuous versions of the Schrödinger and Helmholtz equations,

\[
\frac{1}{2} \frac{d^2 \psi}{dx^2} - v(x) \psi = -k^2 \psi
\]

where \(v(x)\) accounts for the tilings, have their advantages. The numerically more tractable (discrete) tight-binding version

\[
\phi_{n+1}(e) + \phi_{n-1}(e) + v_n \phi_n = 2e \phi_n
\]

is extremely well documented in the condensed-matter physics literature [1]. It is obtained from (12) by defining the dimensionless quantities \(e = 1 - k^2 \varepsilon^2\), \(\phi_n(e) = e^{\varepsilon^2 v_n/2} \psi_n\) and \(t_{n,n+1} = \exp \left( -\varepsilon^2 (v_n + v_{n+1}) \right)\). For a tiling of length (number of tiles) \(N\), (13) can be rewritten in a matrix form \(H_N \Phi = e \Phi\). The energy spectrum
thus comprises $N$ eigenenergies denoted by $e_i$, $1 \leq i \leq N$. The counting function $\mathcal{N}(e)$ or integrated density of states is defined as the fraction of eigenenergies which are smaller than a given energy $e$, namely,

\begin{equation}
\mathcal{N}(e) = \frac{1}{N} \sum_{i=1}^{N} \theta(e - e_i)
\end{equation}

where $\theta(x)$ is the Heaviside function. For large enough $N$, the counting function is independent of the choice of boundary conditions and it is usually a well defined and continuous function of energy. The counting function $\mathcal{N}(e)$ is represented in Fig. 3 for different types of tilings.

For periodic atomic arrangements, the Bloch theorem indicates that the spectrum of (13) consists of bands and hence gaps whose locations are directly related to the (pure-point) Bragg diffraction spectrum as displayed in Fig. 4a.

For aperiodic tilings, the gap labeling theorem (GLT) is an important and elegant result valid in space dimensions $d \leq 3$, that allows to calculate systematically the counting function at gap values [12–14, 41, 42, 61]. The GLT states that possible values of $\mathcal{N}(e)$ in the gaps are given by all possible letter frequencies of all possible words in the infinite tiling generated by a substitution. Those frequencies can be expressed as linear combinations of the frequencies of one and two letters words only, namely using the left eigenvector $\mathbf{v}_1 = (\rho_a, \rho_b)$ previously defined in (9). The GLT makes use of $K$-theory, identifies the $K_0$ group and it allows to systematically build the gap labeling group as a trace, $\tau^K : K_0(B) \to \mathbb{R}$. For quasicrystals, the
Figure 3. Counting Functions of representatives of five families of tilings whose diffraction spectra are displayed in Fig. 1. (A) describes a periodic tiling with two gaps corresponding to the Bragg peaks in the Brillouin zone, (B) a quasiperiodic tiling (Fibonacci), (C) a non-quasiperiodic tiling (Thue-Morse), (D) a limit quasiperiodic tiling (Period Doubling) and (E) a non-quasiperiodic and non-Pisot tiling (Rudin-Shapiro). In contrast to the periodic case, note that for the other examples of aperiodic tilings, there is an infinite number of spectral gaps. This fractal structure (discrete scaling symmetry) is typical of aperiodic tilings.

A systematic calculation [12] gives (see Table 1 for more examples):

**Proposition 12.1.**

1. For the quasiperiodic Fibonacci tiling,
   $$ \tau^K_s(K_0(B)) = (\mathbb{Z} + \rho_b \mathbb{Z}) \cap [0, 1], $$
   using \( \rho_b = 1 - \lambda^{-1} \) in (9).

2. For the aperiodic but non-quasiperiodic Thue-Morse tiling,
   $$ \tau^K_s(K_0(B)) = \left( \mathbb{Z} \left\lceil \frac{1}{2} \right\rceil \right) \cap [0, 1]. $$

Comparing (11) and (15), it cannot escape our attention that for the Fibonacci tiling, Bragg peaks and spectral gaps locations are in one-to-one correspondence,
Figure 4. Comparison between diffraction and spectral data for the five representative families of one-dimensional tilings considered previously and to which Theorem 9.2 applies. For the periodic (A), quasiperiodic (B) and aperiodic (limit-quasiperiodic) (D) tilings, there is a direct correspondence between the two sets of data. This can be viewed as an extension of the Bloch theorem. Note that for these three cases, the diffraction spectrum is PP, a result to be contrasted with the non-quasiperiodic Pisot Thue-Morse (C) and the aperiodic Rudin-Shapiro (E) tilings for which the diffraction spectrum is respectively SC and AC, while the spectral counting function accounts for infinitely countable gaps well described by the GLT.

a result strongly reminiscent of the Bloch theorem for periodic structures. Its extension to classes of aperiodic tilings, asserting the commutativity of the diagram

\[
\begin{array}{c}
K_0(B_T) \xrightarrow{(ch_d) \circ (\chi^{-1})} \hat{H}^d(\Omega_T; \mathbb{Z}) \\
\downarrow \tau^K \quad \downarrow \tau^R \\
\mathbb{R} \xrightarrow{id} \mathbb{R}
\end{array}
\]

with \((ch_d) \circ (\chi^{-1})\) onto provides an explanation for this empirical result. Applying this general result to the Fibonacci example yields the commuting diagram

\[
\begin{array}{c}
\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{id} \mathbb{Z} \oplus \mathbb{Z} \\
\downarrow \tau^K \quad \downarrow \tau^R \\
\mathbb{R} \xrightarrow{id} \mathbb{R}
\end{array}
\]
DIFFRACTION AND SPECTRAL DATA OF APERIODIC TILINGS

with

\[
\begin{align*}
(p, q) & \xrightarrow{id} (p, q) \\
\tau^K & \downarrow \\
p + \rho_b q & \xrightarrow{id} p + \rho_b q
\end{align*}
\]

which can indeed be viewed as a generalisation of Bloch theorem to quasicrystals. At a general level, it is not too surprising a result, since we have observed that structural and spectral data of quasiperiodic substitutions have been deduced from the appearance frequencies of the single and double letters tiles by means of the left eigenvector \(v_1 = (\rho_a, \rho_b)\). For the \(K_0(B)\) group, it is by construction. On the other hand, the Čech cohomology \(\check{H}^1(\Omega_T; \mathbb{Z})\) contains additional information about the order of the tiles. But for quasiperiodic substitutions, the order of the tiles is irrelevant, thus leading to the equivalence between the two groups. In more complicated cases, e.g., the Thue-Morse tiling, the order of the tiles plays a role. It is easy to see this from the corresponding substitution matrix \(M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\) which is identical to the periodic substitution \(a \mapsto ab\) and \(b \mapsto ab\) but with very different structural (order) and spectral (two letter frequencies) properties.

In order to further clarify the content of Theorem 9.2 and its conditions of applicability, we wish to first discuss features of the Thue-Morse aperiodic tiling. It is not a C&P quasicrystal, yet it is a Pisot substitution. The possible values of \(\mathcal{N}(e)\) in the gaps are obtained from the gap labeling group \(\tau^e_\check{K}(K_0(B))\). This is to be compared to the diffraction spectrum composed of Bragg peaks (PP) and of a SC broad range contribution which does not appear in the cohomology trace \(\tau^e_\check{H} \check{H}^1(\Omega_T; \mathbb{Z})\). This lack of equivalence between diffraction and (Laplacian) spectral data is not a limitation of Theorem 9.2 since the Thue-Morse tiling abides its conditions of applicability. It is the expression of a discrepancy between \(\tau^e_\check{H} \check{H}^1(\Omega_T; \mathbb{Z})\) and the structure factor \(S(k)\) which contains additional information not accessible from the cohomology description. It is interesting to note though that in a detailed experimental measurement of the Thue-Morse diffraction spectrum [6], it was effectively challenging to observe diffraction peaks other than those predicted by the cohomology trace \(\frac{1}{2} \mathbb{Z}[\frac{1}{2}]\). Furthermore, the aforementioned lack of equivalence is unrelated to the lack of periodicity or quasiperiodicity (e.g. Period Doubling tiling) but rather a consequence of the nature of the diffraction spectrum, a quantity which, unlike spectral data, is sensitive to both local symmetries of the tiles, a condition of applicability of Theorem 9.2, and to long-range correlations driven e.g. by the order of the letters (inmaterial for periodic or C&P quasicrystals). For instance, the Rudin-Shapiro tiling has an absolutely continuous and structureless diffraction spectrum but a fractal spectral gap distribution well accounted by \(\tau^H_\check{H} \check{H}^1(\Omega_T; \mathbb{Z}) = \mathbb{Z}[\frac{1}{2}]\). These features are summarised in Table 1.
Table 1. Summary of our results applied to main representatives of 1d tilings. For each of them, we have indicated the Čech cohomology \(^\check{\text{H}}^1(\Omega_T; \mathbb{Z})\), the nature of the diffraction spectrum, pure-point (PP), absolutely continuous (AC) and singular continuous (SC). Theorem 9.2 applies to all cases so that the cohomology trace \(\tau_{\check{\text{H}}}^*\left(\check{\text{H}}^1(\Omega_T; \mathbb{Z})\right)\) is calculated using the trace of the \(K_0\) group. Here, \(\lambda = (\sqrt{5} + 1)/2\) is the golden ratio with \(\rho = 1 - \lambda - 1\), and \(\mathbb{N}, \mathbb{Z}, \mathbb{N} \in \mathbb{Z}\).

| Family | Period Doubling | Thue-Morse | Fibonacci | Rudin-Shapiro |
|--------|----------------|------------|-----------|--------------|
| \(\frac{\sqrt{5} + 1}{2} = \rho\) | \(\mathbb{N}_1\) | \(\mathbb{N} \cup \mathbb{Z}\) | \(\mathbb{N}_2\) | \(\mathbb{N} \cup \mathbb{Z}\) |
| \(\frac{\sqrt{5} + 1}{2} = \rho\) | \(\mathbb{N}_1\) | \(\mathbb{N} \cup \mathbb{Z}\) | \(\mathbb{N}_2\) | \(\mathbb{N} \cup \mathbb{Z}\) |
| \(\frac{\sqrt{5} + 1}{2} = \rho\) | \(\mathbb{N}_1\) | \(\mathbb{N} \cup \mathbb{Z}\) | \(\mathbb{N}_2\) | \(\mathbb{N} \cup \mathbb{Z}\) |
| \(\frac{\sqrt{5} + 1}{2} = \rho\) | \(\mathbb{N}_1\) | \(\mathbb{N} \cup \mathbb{Z}\) | \(\mathbb{N}_2\) | \(\mathbb{N} \cup \mathbb{Z}\) |
13. Insights into prior work

This section is designed to link up the discussion of traces in the previous sections with the so-called Shubin trace used in the mathematical-physics literature. We discuss the papers of Shubin [65], Lenz, Peyerimhoff, and Veselić [48], Moustafa [55], Kriesel [46], and Benameur-Mathai [15].

The equivalence found for quasicrystals between the structural Fourier module and the counting function

$$C_\nu (\hat{\mathbf{H}}^1(\Omega_T;\mathbb{Z})) = \tau^K_\nu (K_0(\Omega_T;\mathbb{Z})) = \mathbb{Z} + \rho_0 \mathbb{Z},$$

respectively described by the two traces $C_\nu (\hat{\mathbf{H}}^1(\Omega_T;\mathbb{Z}))$ and $\tau^K_\nu (K_0(\Omega_T;\mathbb{Z}))$ has been first noticed in R. Johnson and J. Moser [35, 36] (1982). These works studied the spectrum of self-adjoint linear differential operators and presented a systematic way of enumerating the open intervals of the associated resolvent operator (GLT), using the rotation number. This was an alternative treatment of gap labeling also in the spirit of the Schwartzman winding number [62] and using cohomology ideas. A discrete version is in [24]. Yet, this interpretation was based on the use of the rotation number, a quantity neither related to the Čech cohomology nor to $C_\nu (\hat{\mathbf{H}}^1(\Omega_T;\mathbb{Z}))$. Moreover, it is not obviously generalizable to higher dimensions. Note that formula (19) is just a special case of our Corollary 9.5.

Shubin’s paper [65, formula (2.3)] (1994) is all about the irrational rotation $C^*$-algebra $A_\alpha$. For our purposes, we consider the Kronecker flow on the torus. Let $N$ be a circle that is transverse to the foliation with its natural $\mathbb{Z}$-action given by the foliation. Its natural Lebesgue measure is an invariant transverse measure $\nu$ for the foliation. Then the $C^*$-algebra $C(N) \rtimes \mathbb{Z}$ sits in the von Neumann algebra $L^\infty(N) \rtimes \mathbb{Z}$, which is a II$_1$ factor with trace associated to $\nu$. It is denoted $W_\alpha$ by Shubin. Its trace, given as (2.8) in Shubin’s paper, is exactly the discrete version of the canonical trace on $A_\alpha$, which sits in the II$_\infty$ factor $W_\alpha \otimes \mathcal{B}(\mathcal{H})$. The situation is very much like that for tiling spaces, except that in this example $N$ is a circle instead of being 0-dimensional.

J. Bellissard, R. Benedetti, and J.-M. Gambaudo [13] (2006) defined the trace that they use initially by taking advantage of the fact that the groupoid $C^*$-algebra is a crossed product. Let $A$ be the dense subalgebra of $C(\Omega_T) \rtimes \mathbb{R}^d$ consisting of continuous compactly supported functions $\Omega_T \times \mathbb{R}^d \to \mathbb{C}$ and let $\mu$ be an invariant probability measure on $\Omega_T$. The trace $\tau_\mu$ is defined by

$$\tau_\mu(f) = \int_{x \in \Omega_T} f(x,0) \, d\mu(x).$$

This is exactly as we have described the $K$-theory trace.

To see the connection with the paper of Lenz, Peyerimhoff, and Veselić [48] (2007) we have to do some translation. They use the terminology of A. Connes in his original treatment of the foliation index theorem. We have been using instead the Moore-Schochet terminology, which we prefer. In [48, Section 4] the authors define the canonical trace on the von Neumann algebra of the foliation which we denote $W^*(G(\Omega_T), \bar{\mu}))$. Their Theorem 4.2 demonstrates the use of locally traceable operators in the construction of the $K$-theory trace.
Andress and Robinson [3] (2010) explicitly use cohomology to study tilings. We convert to our terminology to state their results. Let \( N \) be a Cantor set and \( T \) a homeomorphism which is strictly ergodic (minimal and uniquely ergodic.) Let \( \mu \) denote the invariant measure. Then we may form the suspension \( \Omega_T = N \times \mathbb{Z} \) with associated measure \( \mu' \) on \( \Omega_T \). Andress and Robinson define the first Čech group \( \tilde{H}^1(\Omega_T; \mathbb{Z}) \), the coinvariant group \( \tilde{H}^0(N; \mathbb{Z}) \mathbb{Z} \) (which they call the dynamic cohomology group), and prove that
\[
\tilde{H}^0(N; \mathbb{Z}) \mathbb{Z} \cong \tilde{H}^1(\Omega_T; \mathbb{Z}),
\]
anticipating Theorem 4.1.

Next, they recall the Schwartzman winding number [62], which is a real-valued functional on \( \tilde{H}^1(\Omega_T; \mathbb{Z}) \cong [\Omega_T, S^1] \) defined on continuous functions which are continuously differentiable along the leaves (which are dense) \( f : \Omega_T \to S^1 \) by
\[
W(f) = \frac{1}{2\pi i} \int_{\Omega_T} \frac{f'(y)}{f(y)} d\mu'(y).
\]
(One could just as well work with continuous functions smooth along the leaves.) Let \( W(\Omega_T) \) denote the set of values of the Schwartzman function. This is a countable subgroup of \( \mathbb{R} \). Regarding elements of \( \tilde{H}^1(\Omega_T; \mathbb{Z}) \) as eigenfunctions, then \( W(\Omega_T) \) is the set of all eigenvalues associated with the tiling. Now it is possible that \( \ker(W) \neq 0 \); such classes are called in [3] cohomologically invariant. They are represented by continuous functions \( f : \Omega_T \to S^1 \) with \( \int d\log f d\mu' = 0 \). Coboundaries have this property but possibly other functions do as well. The usual situation is for \( \ker(W) = 0 \) on \( \tilde{H}^1(\Omega_T; \mathbb{Z}) \), in which case the tiling is said to be saturated. Of course this is equivalent to the statement that the winding map \( W : \tilde{H}^1(\Omega_T; \mathbb{Z}) \to W(\Omega_T) \) is an isomorphism. When this fails there are “invisible” Čech cohomology classes not detected by the trace.

To further fit this work into our framework, we note that the unique ergodicity assumption forces
\[
\tilde{H}^1(\Omega_T) \cong \mathbb{R}
\]
(via the unique trace) and the canonical map \( s : \tilde{H}^1(\Omega_T; \mathbb{Z}) \to \tilde{H}^1(\Omega_T) \cong \mathbb{R} \) coincides with the Schwartzman winding number, essentially by uniqueness of the ergodicity.

The connection with the paper of Moustafa [55] (2010) is the easiest to make, since he explicitly uses the partial Chern character (which he calls the longitudinal Chern character) and tangential cohomology (longitudinal cohomology).

Kriesel [46] (2016) makes use of Gabor frames to generalize the gap labeling theorem to the situation of non-trivial magnetic fields. His treatment is entirely \( K \)-theoretic, and includes precise results on gap labelling in dimension 2.

Benameur and Mathai [15] (2020) further pursue the topic of \( K \)-theoretic traces in the presence of a magnetic field. They give [15, Appendix B] a history of gap-labelling theorems and present two gap-labelling conjectures, which to our knowledge are still open in general, though they prove a number of special cases.
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Department of Physics, Technion, Haifa 3200003, Israel

E-mail address: eric@physics.technion.ac.il

E-mail address: yarosd@campus.technion.ac.il

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

E-mail address: jmr@math.umd.edu

E-mail address: clsmath@gmail.com