Higher-order threshold resummation for semi-inclusive $e^+e^-$ annihilation

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Abstract

The complete soft-enhanced and virtual-gluon contributions are derived for the quark coefficient functions in semi-inclusive $e^+e^-$ annihilation to the third order in massless perturbative QCD. These terms enable us to extend the soft-gluon resummation for the fragmentation functions by two orders to the next-to-next-to-next-to-leading logarithmic (N$^3$LL) accuracy. The resummation exponent is found to be the same as for the structure functions in inclusive deep-inelastic scattering. This finding, together with known results on the higher-order quark form factor, facilitates the determination of all soft and virtual contributions of the fourth-order difference of the coefficient functions for these two processes. Unlike the previous (N$^2$LL) order in the exponentiation, the numerical effect of the N$^3$LL contributions turns out to be negligible at LEP energies.

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The electroweak prefactors $\sigma_{\text{ew}}$ can be found in Ref. [2]. The first- and second-order coefficient functions have been calculated long ago in Refs. [8] and [9], respectively. More recently the latter results have been confirmed (and some typos corrected) in two independent ways in Refs. [10,11]. The three-loop corrections $c_n^{(3)}(x)$ have not been derived so far.

The coefficient functions in Eq. (3) include large-$x$ (threshold) double-logarithmic enhancements of the form $a_n^{\text{sp}}(1-x)^{-1}\ln^k(1-x)$ with $k = 0, \ldots, 2n - 1$. Such contributions, which spoil the convergence of the perturbation series at sufficiently large values of $x$, can be resummed by the soft-gluon exponentiation [12,13]. For the process at hand this resummation has been worked out to next-to-leading logarithmic (NLL) accuracy in Ref. [14]. The inclusion of this resummation has led to improvements in the most recent global fit of fragmentation functions [5]. Hence an extension of the soft-gluon exponentiation for $e^+e^- \rightarrow \gamma Z \rightarrow h + X$ to a higher accuracy is not only of theoretical but also of phenomenological interest.

In this Letter we employ the analytic continuation approach of Ref. [10] to derive the soft and virtual contributions to the third-order coefficient functions in Eq. (3). These results are then used to extend the results of Ref. [14] to the next-to-next-to-next-to-leading logarithmic (N$^3$LL) accuracy reached before for inclusive deep-inelastic scattering [15] and the total cross sections for lepton-pair and Higgs-boson production in proton-(anti-)proton collisions [16,17]. A substantial intermediate step towards the present extension has been taken before in Ref. [18]. Recently progress on resummation in perturbative QCD has also been achieved in the framework of soft-collinear effective theory (SCET), see, e.g., Ref. [19] for applications of SCET to the massless-parton processes mentioned above.

Up to small contributions from higher-order group invariants entering at the third and higher orders, the soft plus virtual contributions are the same for the DIS quark coefficient functions for $F_1$, $F_2$ and $F_3$ [20,21]. The same holds for the corresponding (in this order) SIA coefficient functions for $F_T$, $F_I$ and $F_A$. Hence we will drop the index $a$ from now on, and refer to the coefficient functions collectively as $c_n^{(3)}(x)$, and the latter as $c_n^{(0)}(x)$.

In this limit the bare (unrenormalized and unfactorized) partonic DIS (spacelike) structure function $F^b_S(x)$ is given by [22,23]

$$F^b_S(x, Q^2) = \delta(1-x) + \sum_{i=1}^{\infty} \left( \frac{Q^2}{\mu^2} \right)^{-k_i} F^b_{SI,i}$$

with

$$F^b_{SI,1} = 2F_I \delta(1-x) + S_1,$$

$$F^b_{SI,2} = (2F_2 + (F_I)^2) \delta(1-x) + 2F_I S_1 + S_2,$$

$$F^b_{SI,3} = (2F_3 + 2F_I F_2) \delta(1-x) + (2F_2 + (F_I)^2) S_1 + 2F_I S_2 + S_3.$$  (5)

Here $\mu$ is the scale of dimensional regularization with $D = 4 - 2\epsilon$, and $\alpha^b_S$ the bare strong coupling. $F_I$ represents the $l$-loop quark form factor [22–27]. The $x$-dependence of the real-emission functions $S_k$ is given by the $D$-dimensional $+-$distributions

$$f_k(x) = \left[ (1-x)^{-1-k_\epsilon} \right]_+ = -\frac{1}{k_\epsilon} \delta(1-x) + \sum_{i=0}^{k_\epsilon} \frac{(-k_\epsilon)^i}{i!} D_i$$

where we have introduced the abbreviation $D_i = \{(1-x)^{-1} \ln^i(1-x)\}_+$.

The transition to the bare SIA (timelike) fragmentation functions $F^b_S(x)$ is performed as follows: In Eq. (5) the factors $2F_I$ are replaced everywhere by $2Re F^T_I$ and all products $F_x F_y$ by $|F^T_x F^T_y|^2$, where $F^T_I$ is the complex $l$-loop timelike form factor which can be obtained from the spacelike $F_I$ by Eq. (3.3) of Ref. [22]. The analytic continuation of the real-emission terms $S_k$ is carried out as discussed in Ref. [10]. In fact, these functions turn out to be the same for the spacelike and timelike cases (this holds only in the present large-$x$ limit, not for the full real emission contributions). Finally the standard renormalization and mass factorization is performed to the third order for the resulting timelike analogue of Eq. (5), yielding the $D_k$ and $\delta(1-x)$ terms of $c_n^{(3)}(x)$ in Eq. (3).

For the convenience of the reader, we include also the large-$x$ limits of the well-known first- and second-order MS coefficient functions [8,9]. As expected from the above discussion, these and the third-order coefficient function share all non-$c_2$ terms with their spacelike counterparts, hence we will present them via the corresponding differences $\delta c_n = c_n^{(0)} - c_n^{(0)}$. The results read

$$\delta c_1 = 12 c_2 C_F \delta(1-x),$$

$$\delta c_2 = 48 c_2 C_F^2 D_1 - 36 c_2 C_F^2 D_0 + \left\{ (-108 + 24 c_2) C_F^2 + \left( \frac{466}{3} - 24 c_2 \right) C_A C_F - \frac{76}{3} C_T n_f \right\} c_2 \delta(1-x),$$

$$\delta c_3 = 96 C_F^2 D_3 - \left[ 216 C_F^2 + 88 C_A C_F - 16 C_T n_f \right] c_2 D_2 - \left\{ (324 + 96 c_2) C_F^2 - \left( \frac{3332}{3} - 192 c_2 \right) C_A C_F^2 + \frac{536}{3} C_T n_f \right\} c_2 D_1$$

$$+ \left\{ \left( \frac{10504}{9} - 248 c_2 - 480 c_3 \right) C_A C_F + \left( \frac{1672}{9} - 32 c_2 \right) C_T n_f \right\} c_2 D_0$$

$$+ \left\{ \left( \frac{993}{2} + 180 c_2 - 936 c_3 + 72 c_2^2 \right) C_F^2 - \left( \frac{13457}{6} + \frac{220}{3} c_2 - 1616 c_3 + \frac{108}{5} c_2^2 \right) C_A C_F^2 \right\} c_2$$

$$+ \left\{ \left( \frac{74728}{27} - 196 c_2 - 1056 c_3 + \frac{528}{5} c_2^2 \right) C_F^2 + \left( \frac{667}{3} + \frac{136}{3} c_2 - 80 c_3 \right) C_T n_f \right\}$$

$$- \left( \frac{23504}{27} + \frac{16}{3} c_2 - 96 c_3 \right) C_A C_F n_f + \left( \frac{1624}{27} + \frac{16}{3} c_2 \right) C_T n_f^2 \right\} c_2 \delta(1-x).$$

(7)
Here $C_A$ and $C_F$ are the standard group invariants, with $C_A = 3$ and $C_F = 4/3$ in QCD, and $n_f$ the number of light flavours. $\zeta_k$ denotes Riemann’s $\zeta$-function. The third-order SIA coefficient functions can be obtained by adding the corresponding DIS results given in Eqs. (4.14)–(4.19) and Appendix B of Ref. [20], see also Eq. (3.8) of Ref. [21]. The first half of Eq. (9) agrees with the result of Ref. [18], the $\delta(1 - x)$ contribution in the second half has not been presented before.

Below we will need the $N$-independent parts $\delta g_{0k} \equiv \delta g_{k}(N)|_{N^0}$ of the Mellin transforms of Eqs. (7)–(9) obtained via

$$ d^N = \int_0^1 dx \left( x^{N-1} - 1 \right) a(x)_+. \tag{10} $$

(together with $\delta(1 - x) \to 1$. These contributions are given by ($\gamma_e$ is the Euler–Mascheroni constant)

$$ \zeta_2^{-1} \delta g_{01} = 12 C_F, \tag{11} $$

$$ \zeta_2^{-1} \delta g_{02} = C_A C_F \left( \frac{466}{3} - 24 \zeta_2 \right) - C_F^2 \left( 108 - 48 \zeta_2 - 36 \gamma_e - 24 \gamma_e^2 \right) - \frac{76}{3} C_F n_f, \tag{12} $$

$$ \zeta_2^{-1} \delta g_{03} = C_F \left( \frac{993}{2} + 18 \zeta_2 - 792 \zeta_3 + \frac{768}{5} \zeta_2^2 - 306 \gamma_e + 288 \gamma_e \zeta_3 - 162 \gamma_e^2 + 96 \gamma_e^2 \zeta_2 + 72 \gamma_e^3 + 24 \gamma_e^4 \right) \left( \frac{13457}{6} + 482 \zeta_2 + \frac{5024}{3} \zeta_2^2 - \frac{588}{5} \zeta_2^3 - \frac{10504}{9} \gamma_e - 160 \gamma_e \zeta_2 - 480 \gamma_e \zeta_3 + \frac{1666}{3} \gamma_e^2 - 96 \gamma_e^2 \zeta_2 + \frac{88}{3} \gamma_e^3 \right) + C_A C_F \left( 74728 \frac{27}{27} - 196 \zeta_2 - 1056 \zeta_3 + \frac{528}{5} \zeta_2^2 \right) \zeta_n \left( \frac{667}{3} - 44 \zeta_2 - \frac{272}{9} \zeta_3 - 1672 \gamma_e + 16 \gamma_e \zeta_2 \right) - \frac{268}{3} \gamma_e^2 \left( \frac{16}{3} \gamma_e^3 \right) + C_A C_F n_f \left( - \frac{23504}{27} - \frac{16}{3} \zeta_2 + 96 \zeta_3 \right) + C_F n_f^2 \left( \frac{1624}{27} + \frac{16}{3} \zeta_3 \right). \tag{13} $$

The corresponding DIS coefficients can be found in Eqs. (4.6)–(4.8) of Ref. [15].

For processes such as DIS and SIA, the dominant large-$x$/large-$N$ contributions to the $\overline{\text{MS}}$ coefficient functions $C^N$ can be resummed by a single exponential in Mellin space [12]

$$ C^N(Q^2) = g_0(Q^2) \cdot \exp\left[ C^N(Q^2) \right] + O(N^{-1} \ln^N N). \tag{14} $$

The prefactor $g_0$ collects, order by order in the strong coupling constant $\alpha_s$, all $N$-independent contributions. The exponent $C^N$ contains terms of the form $\ln^N N$ to all orders in $\alpha_s$. Besides the physical hard scale $Q^2 (\equiv q^2)$ in DIS/SIA, with $q$ the four-momentum of the exchanged gauge boson, both functions depend on the renormalization scale $\mu_r$ and the mass-factorization scale $\mu_f$.

The exponential in Eq. (14) is build up from universal radiative factors for each initial- and final-state parton $p$, $\Delta_p$ and $J_p$, together with a process-dependent contribution $\Delta^{\text{int}}$. The resummation exponents for DIS and SIA [14] take the very similar form

$$ C_{\text{DIS}}^N = \ln \Delta_q + \ln J_q + \ln \Delta^{\text{int}}_{\text{DIS}}, \quad G_{\text{SIA}}^N = \ln \Delta_q + \ln J_q + \ln \Delta^{\text{int}}_{\text{SIA}}. \tag{15} $$

The radiation factors are given by integrals over functions of the running coupling. Specifically, the effects of collinear soft-gluon radiation off an initial-state or ‘observed’ final-state quark are collected by

$$ \ln \Delta_q(Q^2, \mu_f^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \int_{\mu_f^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A(\alpha_s(q^2)). \tag{16} $$

Collinear emissions from an ‘unobserved’ final-state quark lead to the so-called jet function,

$$ \ln J_q(Q^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left[ \int_{(1-z)^2 Q^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A(\alpha_s(q^2)) + B(\alpha_s((1-z)^2 Q^2)) \right]. \tag{17} $$

Finally the process-dependent contributions from large-angle soft gluons are resummed by

$$ \ln \Delta^{\text{int}}(Q^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} D(\alpha_s((1-z)^2 Q^2)). \tag{18} $$

The functions $g_0$ in Eq. (14) and $A$, $B$ and $D$ in Eqs. (16)–(18) are given by the expansions

$$ F(\alpha_s) = \sum_{i=0} F_i \alpha_s^i \equiv \sum_{i=0} F_i \alpha_s^i, \tag{19} $$

where $l_0 = 0$ with $g_{00} = 1$ for $F = g_0$, and $l_0 = 1$ else.

The known expansion coefficients of the cusp anomalous dimension (the coefficients of $D_0 \equiv 1/(1-x)_+$ in the $\overline{\text{MS}}$ quark-quark splitting functions) read [28,29].
\[A_1 = 4C_F,\]
\[A_2 = 8C_F \left[ \frac{67}{18} - \zeta_2 \right] C_A - \frac{5}{9} n_f,\]
\[A_3 = 16C_F \left[ C_A^2 \left( \frac{245}{24} - \frac{67}{5} \zeta_2 + \frac{11}{6} \zeta_3 + \frac{11}{5} \zeta_2^2 \right) + C_F n_f \left( -\frac{55}{24} + 2 \zeta_2 \right) + C_{A F} n_f \left( -\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right) + n_f^2 \left( -\frac{1}{27} \right) \right].\]

(20)

The first three coefficients of the jet function (17) are given by [12,15,30]

\[B_1 = -3C_F, \]
\[B_2 = C_F^2 \left[ -\frac{3}{2} + 12\zeta_2 - 24\zeta_3 \right] + C_F C_A \left[ -\frac{3155}{54} + \frac{44}{3} \zeta_2 + 40 \zeta_3 \right] + C_F n_f \left[ \frac{247}{27} - \frac{8}{3} \zeta_2 \right], \]
\[B_3 = C_F^2 \left[ -\frac{29}{2} - 18\zeta_2 - 68\zeta_3 + \frac{288}{5} \zeta_2^2 + 32\zeta_2 \zeta_3 + 240 \zeta_5 \right] + C_A C_F^2 \left[ -46 + 287 \zeta_2 - \frac{712}{3} \zeta_3 - \frac{272}{5} \zeta_2^2 - 16 \zeta_2 \zeta_3 - 120 \zeta_5 \right]
+ C_A C_F \left[ -\frac{599375}{729} + \frac{32126}{81} \zeta_2 + \frac{21032}{27} \zeta_3 - \frac{652}{15} \zeta_2^2 - \frac{176}{3} \zeta_2 \zeta_3 - 232 \zeta_5 \right]
+ C_F n_f \left[ \frac{5501}{54} - 50 \zeta_2 + \frac{32}{9} \zeta_3 \right]
+ C_F n_f^2 \left[ -\frac{8714}{729} + \frac{232}{27} \zeta_2 - \frac{32}{27} \zeta_3 \right] + C_A C_F n_f \left[ \frac{1600906}{729} - \frac{9920}{81} \zeta_2 - \frac{776}{9} \zeta_3 + \frac{208}{15} \zeta_2^2 \right].\]

(23)

Together with Eqs. (11)–(13), all functions but \(D\) in Eqs. (16)–(18) are known to order \(\alpha_s^2\). Consequently the first three coefficients of \(D_{\text{SIA}}\) can by determined by comparing the \(\alpha_s\)-expansion of Eq. (14) with the fixed-order results (7)–(9). This procedure yields

\[D_{\text{SIA}} = 0\]

(24)

for \(k = 1, 2, 3\), hence \(\Delta_{\text{int}} = 1\) to at least \(N^3\text{LL}\) accuracy. \(D_1 = 0\) was of course, included in the NLL resummation of Ref. [14]. However, \(B_2\) was unknown at that time, and only \(B_2 + D_2\) could be extracted from the two-loop results of Ref. [9] alone.

As expected from the identity of the DIS and SIA soft-emission functions \(S_k\) in Eq. (5), there is a strong similarity between the respective coefficient functions also in the framework of the soft-gluon exponentiation — recall that

\[D_{\text{DIS}} = 0, \quad \Delta_{\text{DIS}} = 1\]

(25)

was proven to all orders in \(\alpha_s\) in Refs. [31,32]. We expect that such a proof can also be derived for SIA. For the time being assuming the all-order validity of Eq. (24), the difference between the SIA (timelike, \(T\)) and DIS (spacelike, \(S\)) large-\(N\) coefficient functions exponentiates as

\[\delta_{\text{TS}} G^N (Q^2) = \delta_{\text{TS}} g_0 (Q^2) \cdot \exp[G^N (Q^2)] + \mathcal{O}(N^{-1} \ln^2 N)\]

(26)

where, after performing the integrations in Eqs. (16)–(18), the function \(G^N\) takes the form

\[G^N (Q^2) = \ln N \cdot g_1 (\lambda) + g_2 (\lambda) + a_3 g_3 (\lambda) + a_4^2 g_4 (\lambda) + \cdots\]

(27)

with \(\lambda = \beta_0 a_1 \ln N\). The first three expansion coefficients of \(\delta_{\text{TS}} g_0\) for \(\mu_r = \mu_f = Q\) have been given above in Eqs. (11)–(13). We will address the fourth-order coefficient below.

The functions \(g_1\) to \(g_4\) have been derived in Refs. [12,15,33,34]. For completeness we include these functions, also here restricting ourselves to choose \(\mu_r = \mu_f = Q\) of the scales:

\[g_1^\text{DIS} (\lambda) = A_1 \left( 1 - \ln (1 - \lambda) + \lambda^{-1} \ln (1 - \lambda) \right),\]

\[g_2^\text{DIS} (\lambda) = (A_1 \beta_1 - A_2) \left( \lambda + \ln (1 - \lambda) \right) + \frac{1}{2} A_1 \beta_1 \ln^2 (1 - \lambda) - (A_1 \gamma_e - B_1) \ln (1 - \lambda),\]

\[g_3^\text{DIS} (\lambda) = \frac{1}{2} \left( A_1 \beta_2 - A_1 \beta_1^2 + A_2 \beta_1 - A_3 \right) \left( 1 + \frac{1}{1 - \lambda} \right) + A_1 \beta_1^2 \left( \frac{\ln (1 - \lambda)}{1 - \lambda} + \frac{1}{2} \ln^2 (1 - \lambda) \right) + (A_1 \beta_2 - A_1 \beta_1^2) \ln (1 - \lambda)
+ (A_1 \beta_1 \gamma_e + A_2 \beta_1 - B_1) \left( 1 - \frac{1}{1 - \lambda} \right) \ln (1 - \lambda)
- \left( A_1 \beta_2 + \frac{1}{2} A_1 \left( \gamma_e^2 + \zeta_2 \right) + A_2 \gamma_e - B_1 \gamma_e - B_2 \right) \left( 1 - \frac{1}{1 - \lambda} \right),\]

(30)

and
Factors of $\beta_0 = 11/3C_A - 2/3n_f$ have been suppressed in Eqs. (28)–(31) for brevity. The dependence on $\beta_0$ is recovered by $A_k \to A_k/\beta_0^k$ and multiplication of $g_3$ and $g_4$ by $\beta_0$ and $\beta_0^2$, respectively. Note that Eq. (31) includes all known coefficients of the beta function of QCD, see Ref. [35] and references therein.

All parameters entering Eqs. (28)–(31) are known except for the four-loop cusp anomalous dimension $A_c$. The small (see below) impact of this quantity – which first occurs in the $\alpha_s^3 \ln^3 N$ contribution to $\delta\Sigma^{C_N}$ – can be included by a Padé estimate as in Ref. [15], backed up by a recent calculation of one Mellin moment of the fourth-order quark–quark splitting function [36], cf. also Ref. [37]. E.g., for $n_f = 5$ one may use $A_c \approx 1550$ (recall our small expansion parameter $\alpha_s = \alpha_s/(4\pi)$) and assign a conservative uncertainty of 50% to this value.

Due to the vanishing of $\delta\Sigma^{SIO}_N$ the two highest logarithms, $\alpha_s^3 \ln^{21}/N$ and $\alpha_s^4 \ln^{21}/N$, are the same for the SIA and DIS structure functions to all orders in $\alpha_s$. The expansion of Eq. (26) with Eqs. (11)–(13) provides the six highest logarithms, cf. Ref. [15], of the coefficient-function difference $\delta\Sigma^{C_N}$, $\alpha_s^3 \ln^{21}/N$ with $a = 2, \ldots, 7$, at all orders from the fourth. In particular, all $\ln N$ enhanced terms are thus fixed at order $\alpha_s^4$. After transformation to $x$-space these contributions read

\begin{align}
\delta\Sigma^{C_k}(x) &= 96C_A^2C_F^2D_5 - \left\{360C_A^2 + \frac{880}{3}C_A C_F^2 - \frac{160}{3}C_F^3 n_f\right\}C_2 D_4 - \left\{(432 + 576\xi_2)C_F^2 - (3552 - 576\xi_2)C_A C_F^2 + 576C_F^3 n_f\right\}
- \frac{1936}{9}C_A^2 C_F^2 + \frac{704}{9}C_A C_F^3 n_f - \frac{64}{9}C_F^3 n_f^2 C_2 D_3 + \left\{(1674 + 2160\xi_2 + 192\xi_3)C_F^4\right\}
- \frac{25238}{3}C_A^2 C_F^2 + \left\{\frac{3248}{3} - 962\xi_2\right\}C_A^2 C_F^2 n_f - \frac{256}{3}C_F^2 n_f^2 C_2 D_2
- \left\{1122 + 936\xi_2 - 4320\xi_3 - \frac{1248}{5}\xi_2^2\right\}C_F^4 - \frac{22916}{3}C_2 D_1 - \frac{23120}{3}C_2 - 3584\xi_3 - \frac{4368}{5}\xi_2^2\right\}C_A C_F^2
+ \left\{\frac{488}{3} + \frac{4592}{3}\xi_2 + 64\xi_3\right\}C_F^3 n_f + \left\{\frac{224230}{9}C_2 D_1 - \frac{17176}{3}\xi_2 - 7392\xi_3 + \frac{5184}{5}\xi_2^2\right\}C_A C_F^2
- \left\{\frac{69728}{9} - \frac{3056}{3}\xi_2 - 576\xi_3\right\}C_A C_F^3 n_f + \left\{\frac{488}{9} - \frac{64}{3}\xi_2\right\}C_F^3 n_f^2 C_2 D_1
- \left\{\frac{3003}{2} + 3312\xi_2 - 3288\xi_3 + 792\xi_2^2 + 192\xi_2\xi_3 - 5184\xi_5\right\}C_A^3
- \left\{\frac{24507}{2} + 78428\xi_2 - 8816\xi_3 - 1452\xi_2^2 - 1728\xi_2\xi_3 - 1440\xi_5\right\}C_A C_F^3
+ \left\{\frac{6620501}{243} - \frac{243752}{27}\xi_2 - \frac{168560}{9}\xi_3 + \frac{5952}{5}\xi_2^2 + 1664\xi_2\xi_3 + 2784\xi_5\right\}C_A^2 C_F^2
+ \left\{\frac{3551}{9} + \frac{13568}{9}\xi_2 + \frac{688}{3}\xi_3\right\}C_F^3 n_f + \left\{\frac{1983208}{243} - \frac{66392}{27}\xi_2 - 2336\xi_3 + \frac{1152}{5}\xi_2^2\right\}C_A C_F^3 n_f
+ \left\{\frac{135020}{243} - \frac{464}{3}\xi_2 + \frac{128}{9}\xi_3\right\}C_F^3 n_f^2 C_2 D_0 + \cdots.
\end{align}
The fourth-order result (32) can be verified, and extended to the \(\delta(1-x)\) contribution, in the following manner. Eq. (5) is extended to the fourth order,

\[
P_{S,4}^D = (2F_4 + 2F_1F_3 + (F_3^2)\delta(1-x) + (2F_3 + 2F_1F_2)S_1 + (2F_2 + (F_1^2))S_2 + 2F_1S_3 + S_4,
\]

and is subtracted from its timelike counterpart obtained as discussed above. Assuming that also \(S_4\) is identical in the two cases, the only unknown in \(\delta_{TS}P_{S}^D\) to order \(\epsilon^0\) is the four-loop anomalous dimension \(A_4\). All other unknown quantities, such as the \(\epsilon^1\) and \(\epsilon^2\) contributions to the spacelike three-loop form factor [23,26,27] (also the latter new result is not needed in the present context), drop out in this difference. Also the four-loop form factor is known from its exponentiation [39] to a sufficient accuracy in \(\epsilon\) [23]. The soft and virtual contributions to \(\delta_{TS}C_4\) are then extracted from the fourth-order mass factorization formula (here given in terms of the bare coupling)

\[
\delta_{TS}P_{S}^D = \delta_{TS}C_4 + [3\beta_0 - \epsilon \beta_1] \delta_{TS}b_1 + \left[ \beta_1 - \frac{1}{2} \beta_0 \right] \delta_{TS}a_2 + \left[ 3\beta_0 - 5 \beta_0 P_0 + \frac{1}{2} \beta_0^2 \right] \delta_{TS}b_2 + [3\beta_0 - \epsilon \beta_1] \delta_{TS}a_3 + \epsilon \text{-terms.}
\]

For brevity we have suppressed the \(\epsilon^{-3}, \ldots \epsilon^{-1}\) terms which form a consistency check but do not provide new information. The functions \(\alpha_n, b_n\) and \(d_n\) are the \(\epsilon^1, \epsilon^2\) and \(\epsilon^3\) contributions, respectively, to the \(D\)-dimensional coefficient functions at order \(\alpha_s^4\), cf. Ref. [21], and \(P_n\) denotes the \(N^\text{LO}\) quark-quark splitting functions. In \(x\)-space obviously all products of these functions in Eq. (34) have to be read as Mellin-convolutions.

The determination of \(\delta_{TS}C_4\) from Eqs. (33) and (34) reproduces the result in Eq. (32) — hence \(D_{\text{SIA}}^k = D_{\text{DIS}}^k = 0\) in Eq. (18) corresponds to \(\delta_{TS}S_k = 0\) in Eqs. (5), (33) and their higher-order generalizations — and includes the final large-\(x\) coefficient,

\[
\zeta_2^{-1} \delta_{TS}C_4 \big|_{\delta(1-x)} = \left( \begin{array}{c}
\frac{7255}{2} - 3779\zeta_2 - 3816\zeta_3 - \frac{13896}{5}\zeta_2^2 + 4080\zeta_2\zeta_3 + 14880\zeta_5 + \frac{31856}{105}\zeta_3^2 - 1216\zeta_2^2 \\
+ \frac{191411}{12} + 153802\zeta_2 - 62452\zeta_2\zeta_3 + 8128\zeta_2\zeta_3 - \frac{67328}{3}\zeta_5 - \frac{102472}{105}\zeta_3^2 + 4064\zeta_2^2 \\
+ \frac{14817221}{324} - 63347\zeta_2 - 1856680\zeta_2\zeta_3 + 5306\zeta_2\zeta_3 - 2032\zeta_2\zeta_3 + 6256\zeta_5 + \frac{2584\zeta_3^2}{21} - 992\zeta_2^2 \\
+ \frac{13294462}{243} + 206162\zeta_2 - 416032\zeta_2\zeta_3 - \frac{11000\zeta_2^2}{9} + 1936\zeta_2\zeta_3 + 8976\zeta_5 \\
+ \frac{409}{6} - 23350\zeta_2 + 6840\zeta_3 - \frac{5592\zeta_2^2}{45} - 2272\zeta_2\zeta_3 + 6272\zeta_5 \\
+ \frac{706405}{81} - 187834\zeta_2 - 416384\zeta_2\zeta_3 - \frac{6932\zeta_2^2}{45} + 320\zeta_2\zeta_3 - 1408\zeta_5 \\
+ \frac{2109553}{81} + 106168\zeta_2 - 127000\zeta_5 \\
+ \frac{3233}{81} + 1482\zeta_2 - 20656\zeta_3 + \frac{2464\zeta_2^2}{45} \\
+ \frac{39352}{27} + 304\zeta_2 + 64\zeta_3 C_F n_f \left( 768 + 1920\zeta_2 + 896\zeta_5 - \frac{384\zeta_2}{5} - 512\zeta_5 \right) f l l_1 C_F d_{abc} d_{abc} / n_c \\
+ 3 A_4.
\end{array} \right)
\]

\[
\text{See Ref. [20] for the } f l l_1 \text{ diagram class leading to the term with } d_{abc} d_{abc} / n_c = 5/18n_f \text{ in QCD. The numerical effect of this contribution is very small and will be disregarded in the following.}
\]

The Mellin transform of these equations provides the \(\alpha_s^3\) prefactor \(\delta_{TS}g_{04}\) in Eq. (26), and hence (up to the residual uncertainty due to \(A_4\)) the seventh tower of large-\(x\) logarithms from order \(\alpha_s^2\) for this difference. For \(n_f = 5\) quark flavours, the numerical expansion of \(\delta_{TS}g_0\) is given by

\[
\delta_{TS}g_0(\alpha_s) \simeq 2.094\alpha_s(1 + 1.463\alpha_s + 2.749\alpha_s^2 + (6.659 + 0.094A_4/1000)\alpha_s^3 + \cdots).
\]

Thus the two new terms form a correction of almost 5% at \(\alpha_s = 0.12\), with a negligible uncertainty from the missing exact value of \(A_4\), and the fourth-order contribution is less than half of the previous term for \(\alpha_s < 0.2\). It is well known that the coefficients in Eq. (36) are due to \(\zeta_2\)-terms (i.e. powers of \(\pi^2\)) from the analytic continuation of the form factor which are subject to a separate exponentiation (see, e.g., Ref. [39]). The corresponding results for the SIA and DIS cases read

\[
\begin{align*}
\delta_{TS}g_0(\alpha_s) &= 1 + 1.045\alpha_s + 2.266\alpha_s^2 + 4.703\alpha_s^3 + \cdots, \\
\delta_{TS}g_0(\alpha_s) &= 1 - 1.050\alpha_s - 0.797\alpha_s^2 - 1.056\alpha_s^3 + \cdots.
\end{align*}
\]

The pattern of the corrections in Eq. (37) and the size of the \(\alpha_s^2\)-term in Eq. (36) strongly suggests that the fourth-order contribution to \(\delta_{TS}\) amounts to less than 0.5% for \(\alpha_s = 0.12\).
of the DIS coefficients, cf. Table 1 of Ref. [15]. Indeed, the coefficient for the two cases are very similar for the size of its resummed coefficient function (14) is illustrated in Fig. 1 for a value of $t$. It turns out that the resummation exponents and the relative impact of the new N2LL and N3LL corrections, increases towards lower CM energies. Nevertheless one can conclude from Fig. 1 that the accuracy now reached for the dominant large-$\alpha_s$ contributions should be sufficient for the foreseeable future. To summarize, we have first employed the close relation between the perturbative corrections to the structure functions in deep-inelastic scattering (DIS) and the fragmentation functions in semi-inclusive $e^+e^-$ annihilation (SIA), see also Ref. [40], to derive the complete soft and virtual corrections to the third-order quark coefficient functions for the latter observables. This result then made it possible to extend the soft-gluon exponentiation in SIA from the next-to-leading logarithmic (NLL) contributions [14] by two orders to N3LL accuracy (we confirm the intermediate results in Ref. [18]). It turns out that the resummation exponents are the same, presumably to all orders, for the DIS and SIA coefficient functions. Hence the threshold enhancement is structurally identical for the N3LL exponentiation. The coefficients of the known ln $N$ terms are given in Table 1 to the tenth order in $\alpha_s$, using the notation $c_{\alpha_0}$ for the coefficient of $a^2_\alpha$ ln $N$ in $C_{\alpha_0}^N$. Hence, as in Ref. [15] for the DIS case, the coefficients of the leading (next-to-leading, etc.) logarithms are denoted by $c_{\alpha_1}$ ($c_{\alpha_2}$, etc.). The qualitative pattern of these coefficients is similar to the DIS case (where all numbers $c_{\alpha_2>2}$ are smaller). The higher-order corrections rise very rapidly, by about an order of magnitude or more, with $N$ (increasing $a$), but the SIA coefficient are more than double their DIS counterparts at $a > k$ where the numbers are large. Consequently the higher-order soft plus virtual contributions are qualitatively similar, but larger in the timelike case. The numerical size of its resummed coefficient function (14) is illustrated in Fig. 1 for a value of $\alpha_s$ corresponding to LEPI, $s = M_Z^2$. Obviously the size of the coefficient function, as well as the relative impact of the new N2LL and N3LL corrections, increases towards lower CM energies. Nevertheless one can conclude from Fig. 1 that the accuracy now reached for the dominant large-$x$/large-$N$ contributions should be sufficient for the foreseeable future.

Table 1
Numerical values of the five-flavour coefficients $c_{\alpha_0}$ of the $a^2_\alpha$ ln $N$ contributions to the coefficient function $C_{\alpha_0}^N$. The first six columns are exact up to the numerical truncation, and the same for $F_1$, $F_T$ and $F_A$. The seventh column neglects the tiny (and non-universal) $f_{11}$ contributions, and uses the estimate $A_4 = 1550$ for the four-loop cusp anomalous dimension.

| $k$ | $c_{\alpha_1}$ | $c_{\alpha_2}$ | $c_{\alpha_1}$ | $c_{\alpha_3}$ | $c_{\alpha_5}$ | $c_{\alpha_6}$ | $c_{\alpha_7}/10$ |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|----------------|
| 1   | 2.68667       | 7.0785        | -             | -             | -             | -             | -              |
| 2   | 3.55556       | 25.6908       | 105.621       | 104.34        | -             | -             | -              |
| 3   | 3.16040       | 43.3408       | 309.335       | 1016.50       | 2306.0        | -             | -              |
| 4   | 2.10700       | 46.6020       | 514.068       | 3125.96       | 11774.1       | 23741         | 4664           |
| 5   | 1.12373       | 36.4525       | 577.143       | 5393.82       | 32365.2       | 110255        | 29009          |
| 6   | 0.49944       | 22.3131       | 481.110       | 6314.54       | 55037.7       | 293931        | 119399         |
| 7   | 0.19026       | 11.1933       | 315.972       | 5515.83       | 65426.2       | 506294        | 294105         |
| 8   | 0.00342       | 4.7503        | 170.251       | 3808.07       | 58765.0       | 618949        | 487177         |
| 9   | 0.001879      | 1.7455        | 77.500        | 2160.26       | 41980.1       | 574684        | 589591         |
| 10  | 0.000501      | 0.5652        | 30.470        | 1035.7        | 24725.4       | 425171        | 551698         |

Fig. 1. Left: the LL, NLL, N2LL and N3LL results for the threshold resummation (14) of the SIA coefficient functions (3) in N-space. Terms to order $a^0_\alpha$ are included in $g_t^0$, for the N2LL curves. Right: the convolutions of these results with a schematic large-$x$ shape for the quark fragmentation functions, using the standard ‘minimal prescription’ contour [13] for the Mellin inversion.
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