The Busemann-Petty problem on entropy of log-concave functions

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Abstract The Busemann-Petty problem asks whether symmetric convex bodies in the Euclidean space \(\mathbb{R}^n\) with smaller central hyperplane sections necessarily have smaller volumes. The solution has been completed and the answer is affirmative if \(n \leq 4\) and negative if \(n \geq 5\). In this paper, we investigate the Busemann-Petty problem on entropy of log-concave functions: for even log-concave functions \(f\) and \(g\) with finite positive integrals in \(\mathbb{R}^n\), if the marginal \(\int_{\mathbb{R}^n \cap H} f(x)\,dx\) of \(f\) is smaller than the marginal \(\int_{\mathbb{R}^n \cap H} g(x)\,dx\) of \(g\) for every hyperplane \(H\) passing through the origin, is the entropy \(\text{Ent}(f)\) of \(f\) bigger than the entropy \(\text{Ent}(g)\) of \(g\)? The Busemann-Petty problem on entropy of log-concave functions includes the Busemann-Petty problem, and hence its answer is negative when \(n \geq 5\). For \(2 \leq n \leq 4\), we give a positive answer to the Busemann-Petty problem on entropy of log-concave functions.

Keywords Busemann-Petty problem, entropy, intersection functions, log-concave functions

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1 Introduction

The dual Brunn-Minkowski theory introduced by Lutwak [39] is a milestone in convex geometry which has attracted extensive attention since the intersection body helped achieve a major breakthrough in the solution of the celebrated Busemann-Petty problem. Very recently, Huang et al. [30] studied the Minkowski problem in the dual Brunn-Minkowski theory, now known as the dual Minkowski problem. Since their outstanding work, the dual Brunn-Minkowski theory has gained renewed attention.

The Busemann-Petty problem and the dual Minkowski problem are among the most important problems in the dual Brunn-Minkowski theory. The main purpose of this paper is to study the Busemann-Petty problem for log-concave functions. Let us review the Busemann-Petty problem and its history. A subset in the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) is called a convex body if it is a compact convex set with non-empty interior. A convex body \(K\) is origin-symmetric if \(K = -K\), where \(-K = \{-x : x \in K\}\). Let \(V_{n-1}(\cdot)\) and \(V(\cdot)\) denote the \((n-1)\)-dimensional and the \(n\)-dimensional Hausdorff measures, respectively.
In 1956, Busemann and Petty [12] posed the following question. 

**Busemann-Petty problem.** Suppose that \( K \) and \( L \) are origin-symmetric convex bodies in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) such that

\[
V_{n-1}(K \cap H) \leq V_{n-1}(L \cap H)
\]

for every hyperplane \( H \) passing through the origin. Does it follow that

\[
V(K) \leq V(L).
\]

The Busemann-Petty problem has a long and dramatic history. It is not so difficult to see that the Busemann-Petty problem has a positive answer for \( n = 2 \). A negative answer to the problem for \( n \geq 5 \) was established in a series of papers by Larman and Rogers [38] (\( n \geq 12 \)), Ball [7] (\( n \geq 10 \)), Giannopoulos [27] and Bourgain [8] (independently; \( n \geq 7 \)), and Gardner [24] and Papadimitrakis [44] (independently; \( n \geq 5 \)). Intersection bodies introduced by Lutwak [40] help to completely solve the Busemann-Petty problem. The Busemann-Petty problem can be rephrased in terms of intersection bodies, i.e., the Busemann-Petty problem has an affirmative answer in \( \mathbb{R}^n \) if and only if every origin-symmetric convex body in \( \mathbb{R}^n \) is an intersection body. With the help of intersection bodies, the answer to the Busemann-Petty problem is affirmative for \( n \leq 4 \). Gardner [25] showed an affirmative answer to the Busemann-Petty problem for \( n = 3 \). In 1999, Zhang [51] provided an affirmative answer to the Busemann-Petty problem for \( n = 4 \), the last unsolved case of the Busemann-Petty problem. A unified solution to the Busemann-Petty problem for all the cases of \( n \) was provided by Gardner et al. [26].

In the past years, there are more generalizations of the Busemann-Petty problem (see, for example, [9,34–37,46–48] and [49,53,54]). Moreover, intersection bodies have received more attention and one can refer to [31–33,41,42,50,52] for more references.

The main aim of this paper is to study the Busemann-Petty problem on entropy, an important concept that is applied in physics, information theory, computer science, convex geometry, differential geometry, probability theory, analysis and other fields of applied mathematics. The motivation of this paper comes from the equivalence between the isoperimetric inequality and the Sobolev inequality. This equivalence provides a close link between functions and convex bodies. Numerous connections have been identified between functions and convex bodies over the past few years (see [1–5,13,14,16] and [15,18–23,43,45] for more detailed references).

A function \( f : \mathbb{R}^n \to [0, \infty) \) is log-concave if for any \( x, y \in \mathbb{R}^n \) and \( 0 < t < 1 \),

\[
f((1 - t)x + ty) \geq f^1-t(x)f^t(y).
\]

(1.1)

A typical example of log-concave functions is the characteristic function \( \chi_K \) of a convex body \( K \) in \( \mathbb{R}^n \):

\[
\chi_K(x) = \begin{cases} 
1, & \text{if } x \in K, \\
0, & \text{if } x \notin K.
\end{cases}
\]

We know that every log-concave function \( f : \mathbb{R}^n \to \mathbb{R} \) has the form

\[
f = e^{-\varphi},
\]

where \( \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is convex. The total mass functional \( J(f) \) of an integrable function \( f : \mathbb{R}^n \to \mathbb{R} \) is defined by

\[
J(f) = \int_{\mathbb{R}^n} f(x)dx.
\]

(1.2)

If \( f \) is an integrable log-concave function with \( J(f) > 0 \), the entropy \( \text{Ent}(f) \) of \( f \) is defined by

\[
\text{Ent}(f) = \int_{\mathbb{R}^n} f(x) \log f(x)dx - J(f) \log J(f).
\]

(1.3)
The entropy of log-concave functions provides multiple connections between convex bodies and log-concave functions. In [4], Artstein-Avidan et al. provided a functional version of the affine isoperimetric inequality for log-concave functions which turned out to be an inverse form of the logarithmic Sobolev inequality for entropy. Colesanti and Fragarà [17, Theorem 5.1] showed that the entropy is one part of the functional form of Minkowski’s first inequality. Inequalities on entropy were obtained in [13] by using a geometric inequality involving the $L_p$ affine surface area.

In the present paper, we study the following Busemann-Petty problem on entropy of log-concave functions.

**Problem 1.1.** Suppose that $f$ and $g$ are even log-concave functions with finite positive integrals in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n \cap H} f(x)dx \leq \int_{\mathbb{R}^n \cap H} g(x)dx$$

for each hyperplane $H$ passing through the origin. Does it follow that

$$\text{Ent}(f) \geq \text{Ent}(g)?$$

At first glance, the solution to Problem 1.1 is more difficult than the traditional Busemann-Petty problem since there is no information in $\mathbb{R}^2$. As in the traditional Busemann-Petty problem (see, e.g., [11]), the symmetry assumption on functions is necessary. The work of Colesanti and Fragarà [17, Theorem 5.1] shows that the entropy is an important part in the functional version of Minkowski’s first inequality and another part is the total mass functional. It is clear that the total mass functional is a more direct extension of the volume. We consider the following Busemann-Petty problem for the total mass functional of log-concave functions.

**Problem 1.2.** Suppose that $f$ and $g$ are even log-concave functions with finite positive integrals in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n \cap H} f(x)dx \leq \int_{\mathbb{R}^n \cap H} g(x)dx$$

for every hyperplane $H$ passing through the origin. Does it follow that

$$\int_{\mathbb{R}^n} f(x)dx \leq \int_{\mathbb{R}^n} g(x)dx?$$

When $f$ and $g$ are, respectively, the characteristic functions of origin-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$, Problem 1.2 deduces the Busemann-Petty problem. Hence Problem 1.2 has a negative answer for $n \geq 5$. Note that scalar multiplication does not affect Problem 1.2 and the function $t \log t$ is increasing when $t > e^{-1}$, so the crucial step to solve Problem 1.1 is the following problem.

**Problem 1.3.** If $f$ and $g$ are even log-concave functions in $\mathbb{R}^n$ with finite positive integrals such that

$$\int_{\mathbb{R}^n \cap H} f(x)dx \leq \int_{\mathbb{R}^n \cap H} g(x)dx$$

for every hyperplane $H$ passing through the origin. Does it follow that

$$\int_{\mathbb{R}^n} f \log f dx \geq \int_{\mathbb{R}^n} g \log g dx?$$

It is not hard to see that Problems 1.2 and 1.3 include the Busemann-Petty problem. In Section 5, we show that the Busemann-Petty problem can also be deduced from Problem 1.1. Therefore, Problems 1.1, 1.2 and 1.3 should be called the functional Busemann-Petty problems, and they have negative answers for $n \geq 5$. Hence, we just need to consider the functional Busemann-Petty problems in $\mathbb{R}^n$ for $2 \leq n \leq 4$. Obviously, if Problems 1.2 and 1.3 have positive answers for $2 \leq n \leq 4$, then Problem 1.1 has a positive answer for $2 \leq n \leq 4$. 
For our purposes, the functional version of intersection bodies (or the intersection function) is needed. The intersection function $If : \mathbb{R}^n \to [0, \infty)$, of a non-negative integrable function $f : \mathbb{R}^n \to \mathbb{R}$, is defined as
\[
If(x) = \exp \left\{ -\|x\| \left( \int_{\mathbb{R}^n \cap x^+} f(z)dz \right)^{-1} \right\},
\]
when $x \in \mathbb{R}^n \setminus \{0\}$ and $If(0) = 1$. Here, $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$ and $\bar{x} = \frac{x}{\|x\|}$ for $x \in \mathbb{R}^n \setminus \{0\}$. Moreover, for a continuous function $g : \mathbb{R}^n \to [0, \infty)$, if there exists a non-negative integrable function $f$ such that $g = If$, then $g$ is an intersection function. The intersection function not only contains the intersection body (of star bodies) but also plays a crucial role in solving the functional Busemann-Petty problems.

**Characteristic theorem.** Problems 1.2 and 1.3 have affirmative answers in $\mathbb{R}^n$ if and only if every even integrable log-concave function in $\mathbb{R}^n$ is an intersection function.

For a given continuous function (not necessary log-concave), Ball [6] introduced a body associated with it. Using Ball’s body, we will give a further connection between the intersection function and the intersection body in Lemma 4.4, which together with the solution to the Busemann-Petty problem assures that Problems 1.2 and 1.3 have affirmative answers when $2 \leq n \leq 4$. Therefore, we conclude the following theorem.

**Theorem 1.4.** Functional Busemann-Petty problems (i.e., Problems 1.1–1.3) have affirmative answers when $2 \leq n \leq 4$ and negative answers when $n \geq 5$.

## 2 Preliminaries

Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the $n$-dimensional volume of the compact body $K$ in $\mathbb{R}^n$. A subset $K \subset \mathbb{R}^n$ containing the origin in its interior is a star-shaped with respect to the origin if the intersection of every line through the origin with $K$ is a line segment. The radial function, $\rho_K(\cdot) = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact, star-shaped (with respect to the origin) $K \subset \mathbb{R}^n$, is defined for $x \neq 0$, by
\[
\rho(K, x) = \max \{t \geq 0 : tx \in K\}.
\]
If $\rho_K$ is positive and continuous, call $K$ a star body (with respect to the origin). Let $S^n_0$ denote the set of star bodies (with respect to the origin) in $\mathbb{R}^n$. Two star bodies $K$ and $L$ are said to be dilated (of one another) if $\frac{\rho_K(u)}{\rho_L(u)}$ is independent of $u \in S^{n-1}$. The radial function is positively homogeneous of degree $-1$, i.e.,
\[
\rho_K(tx) = t^{-1} \rho_K(x), \quad t > 0.
\]  
(2.1)

For a star body $K \in S^n_0$, we denote its Minkowski functional by
\[
\|x\|_K = \begin{cases} 
\rho_K^{-1}(x), & \text{if } x \neq 0, \\
0, & \text{if } x = 0.
\end{cases}
\]

We need some facts about the Radon transform. Let $f$ be a compactly supported continuous function in $\mathbb{R}^n$ and $\Sigma_n$ be the space of hyperplanes in $\mathbb{R}^n$. The Radon transform $Rf$ of $f$ is defined by (see [29])
\[
Rf(\xi) = \int_{\Sigma_n} f(x)dx, \quad \xi \in \Sigma_n.
\]
Here, the integral is taken with respect to the natural hypersurface measure $dx$. Observe that any element of $\Sigma_n$ is characterized as the solution locus of an equation
\[
y \cdot u = r,
\]
where \( u \in S^{n-1} \) is a unit vector and \( r \in \mathbb{R} \). Thus the \( n \)-dimensional Radon transform can be rewritten as a function on \( S^{n-1} \times \mathbb{R} \) via

\[
\mathcal{R}(u, r) = \int_{\{y \in \mathbb{R}^n : u \cdot y = r\}} f(y) dy.
\]

(2.2)

The Radon transform of \( f \) in the direction of \( x \) at \( r \) is defined by

\[
\mathcal{R}(x, r) = \|x\|^{-1} \mathcal{R}(\frac{x}{\|x\|}, \frac{r}{\|x\|}), \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

(2.3)

For a continuous function \( f \) on \( S^{n-1} \), the spherical Radon transform \( \mathcal{R} f \) of \( f \) is defined by

\[
\mathcal{R} f(u) = \int_{S^{n-1} \cap u^\perp} f(v) dv, \quad u \in S^{n-1},
\]

where \( u^\perp \) is the \((n-1)\)-dimensional subspace orthogonal to the unit vector \( u \).

### 3 Intersection bodies for functions

Let \( L \in S_n^1 \). We recall that the intersection body \( IL \) of the star body \( L \) is defined by

\[
\rho_{IL}(u) = V_{n-1}(L \cap u^\perp), \quad u \in S^{n-1},
\]

(3.1)

where \( u^\perp \) is the \((n-1)\)-dimensional subspace orthogonal to the unit vector \( u \). The radial function of \( IL \) equals the spherical Radon transform of \( \frac{1}{n-1} \rho_{L}^{n-1} \), i.e.,

\[
\rho_{IL}(u) = \mathcal{R} \left( \frac{1}{n-1} \rho_{L}^{n-1} \right)(u).
\]

A slightly more general notion was defined in [28] as follows. An origin-symmetric star body \( K \) in \( \mathbb{R}^n \) is said to be an intersection body if there exists a finite non-negative Borel measure \( \mu \) on \( S^{n-1} \) so that the radial function \( \rho_K \) of \( K \) equals the sphere Radon transform of \( \mu \).

In order to generalize the intersection body to its functional version, we define the parallel section function \( t \mapsto A_{f, x}(t) \), \( t \in \mathbb{R} \) of the integrable function \( f : \mathbb{R}^n \to \mathbb{R} \) by

\[
A_{f, x}(t) = \int_{\{z \in \mathbb{R}^n : x \cdot z = t\}} f(z) dz,
\]

(3.2)

where \( x \in \mathbb{R}^n \setminus \{0\} \) and \( \bar{x} = \frac{x}{\|x\|} \). Inspired by the definition of intersection bodies (3.1), we define the intersection function by using the parallel section functions.

**Definition 3.1.** Let \( f : \mathbb{R}^n \to [0, \infty) \) be an integrable function. The intersection function \( \mathcal{I} f : \mathbb{R}^n \to [0, \infty) \) of \( f \) is defined as

\[
\mathcal{I} f(x) = e^{-\|x\| A_{f, x}(0)^{-1}},
\]

(3.3)

when \( x \in \mathbb{R}^n \setminus \{0\} \) with \( \bar{x} = \frac{x}{\|x\|} \), and \( \mathcal{I} f(0) = 1 \). In addition, we say that \( g : \mathbb{R}^n \to [0, \infty) \) with \( g(0) > 0 \) is an intersection function if there exists a non-negative integrable function \( f \) such that \( g(x) = \mathcal{I} f(x) \).

If \( f : \mathbb{R}^n \to \mathbb{R} \) is a compactly supported continuous function, then the definition of the Radon transform (2.3) yields

\[
A_{f, x}(0) = \int_{\{z \in \mathbb{R}^n : x \cdot z = 0\}} f(z) dz = \|x\| \mathcal{R} f(x, 0).
\]

Hence, the intersection function can be rewritten as

\[
\mathcal{I} f(x) = e^{-\mathcal{R} f(x, 0)^{-1}}.
\]

(3.4)

The next lemma shows that the intersection function includes the intersection body.
Lemma 3.2. Let $K \in S^n_0$. If $f(x) = e^{-c\|x\|}$, $x \in \mathbb{R}^n$ with $c > 0$, then
\[
\mathcal{I}f(x) = e^{-c^{(n-1)}\Gamma(n-1)\|x\|_K}.
\] (3.5)

Proof. Trivially, the equality in this lemma holds when $x = 0$. For $x \in \mathbb{R}^n \setminus \{0\}$, direct calculation shows that
\[
A_{f,0}(0) = \int_{\{z \in \mathbb{R}^n : z = 0\}} f(z)dz = \int_0^\infty \int_{S^{n-1}} t^{n-2}e^{-tc\|v\|}dvd\tau = c^{-(n-1)}\Gamma(n)\rho_K(x).
\]
Formula (3.5) comes from (2.1).

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function with $f(0) > 0$. For any $p > 0$, the set $K_p(f)$ was introduced by Ball [6], i.e.,
\[
K_p(f) = \left\{ x \in \mathbb{R}^n : \int_0^\infty f(rx)r^{p-1}dr \geq \frac{f(0)}{p} \right\},
\] (3.6)

From the definition, it follows that the radial function of $K_p(f)$ is given by
\[
\rho_{K_p(f)}(x) = \left( \frac{1}{f(0)} \int_0^\infty pr^{p-1}f(rx)dr \right)^{\frac{1}{p}}
\]
for $x \neq 0$.

The following properties were showed in [10].

Lemma 3.3. Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable function with $f(0) > 0$. For every $p > 0$, $K_p(f)$ has the following properties:

1. $0 \in K_p(f)$.
2. $K_p(f)$ is a star-shaped set.
3. $K_p(f)$ is symmetric if $f$ is even.
4. $K_p(f)$ is a convex body if $f$ is log-concave and has a finite positive integral.
5. $V(K_p(f)) = \frac{1}{f(0)}J(f)$.

In [10, Proposition 2.5.3], it has been proved that
\[
m^\frac{1}{p}K_p(g) \subseteq K_p(f) \subseteq M^\frac{1}{p}K_p(g)
\]
for $p > 0$ and any two integrable functions $f, g : \mathbb{R}^n \to [0, \infty)$ with $f(0) = g(0) > 0$, where $m = \inf\{\frac{f(x)}{g(x)} : g(x) > 0\}$ and $M = \inf\{\frac{g(x)}{f(x)} : f(x) > 0\}$. Using this observation, we have the following corollary.

Corollary 3.4. Let $f, g : \mathbb{R}^n \to [0, \infty)$ be two integrable functions with $f(0) = g(0) > 0$. For fixed $p > 0$ and $c > 0$, $K_p(f) = cK_p(g)$ if and only if $f(x) = g(c^{-1}x)$ for any $x \in \mathbb{R}^n$.

A connection of intersection functions and intersection bodies is presented in the following lemma.

Lemma 3.5. Let $f : \mathbb{R}^n \to [0, \infty)$ be a continuous integrable function with $f(0) > 0$. If $f$ has a finite positive integral, then
\[
\mathcal{I}f(x) = \exp \left\{ -\frac{1}{f(0)}\|x\|_1K_{n-1}(f) \right\}
\] (3.7)
for $x \in \mathbb{R}^n$.

Proof. It is trivial for $x = 0$. For $x \in \mathbb{R}^n \setminus \{0\}$, by Fubini’s theorem, we have
\[
A_{f,x}(0) = \int_{\{y \in \mathbb{R}^n : y = 0\}} f(y)dy
\]
The desired formula follows from the definition of intersection functions and (2.1).

4 The solutions to Problems 1.2 and 1.3

For convenience, for any non-negative function $f : \mathbb{R}^n \to \mathbb{R}$, we set

$$\hat{\delta J}(f, f) = -\int_{\mathbb{R}^n} f(x) \log f(x) dx. \quad (4.1)$$

Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $a > 0$, if $f(x) = e^{-\|x\|_a K}$, then direct calculation shows that

$$\int_{\mathbb{R}^n \cap u \perp} f(x) dx = a^{n-1} \Gamma(n) V_{n-1}(K \cap u \perp), \quad (4.2)$$

and

$$J(f) = a^n \Gamma(n+1) V(K) \quad (4.3)$$

and

$$\hat{\delta J}(f, f) = a^n n \Gamma(n+1) V(K). \quad (4.4)$$

In this section, we solve Problems 1.2 and 1.3. From (4.2)–(4.4), one can see that the “even” conditions in Problems 1.2 and 1.3 are necessary, since they all recover the geometric case where the functions $f$ and $g$ are limited to the characteristic function of convex bodies.

To prove the characteristic theorem, we need the following two geometric results.

Lemma 4.1 (See [40, Theorem 10.1]). If $K$ is an intersection body and $L$ is an origin-symmetric star body, and for all $u \in S^{n-1}$,

$$V_{n-1}(K \cap u \perp) \leq V_{n-1}(L \cap u \perp),$$

then

$$V(K) \leq V(L)$$

with the equality holding if and only if $K = L$.

Lemma 4.2 (See [40, Theorem 12.2]). If $K$ is an origin-symmetric star body whose radial function is in $C^\infty_c(S^{n-1})$, then if $K$ is not an intersection body, there exists an origin-symmetric body $L$ such that for all $u \in S^{n-1}$,

$$V_{n-1}(L \cap u \perp) < V_{n-1}(K \cap u \perp),$$

but

$$V(L) > V(K).$$

We are ready to prove the characteristic theorem presented in Section 1.

Proposition 4.3. Problems 1.2 and 1.3 have affirmative answers in $\mathbb{R}^n$ if and only if every even integrable log-concave function in $\mathbb{R}^n$ is an intersection function.
Proof. The proofs of the statements for Problems 1.2 and 1.3 in this proposition are similar, and hence we only give a detailed proof for Problem 1.3 and the case of Problem 1.2 follows from the same line.

Assume that both $f$ and $g$ are even integrable concave functions and intersection functions. Let $f_0$ and $g_0$ be integrable continuous functions with $f_0(0) > 0$ and $g_0(0) > 0$ such that $\mathcal{I}f_0(x) = f(x)$ and $\mathcal{I}g_0(x) = g(x)$ for $x \in \mathbb{R}^n$. To prove the sufficiency, it suffices to show that for any $x \in \mathbb{R}^n$,

$$\mathcal{I}f(x) \leq \mathcal{I}g(x)$$

implies that

$$\hat{\delta} J(f, f) \leq \hat{\delta} J(g, g)$$

with the equality holding if and only if $f = g$.

Under our assumptions, Lemma 3.5 implies that for any $x \in \mathbb{R}^n$,

$$f(x) = \exp \left\{ -\frac{1}{f_0(0)} \|x\|^{1K_{n-1}(f_0)} \right\}$$

and

$$g(x) = \exp \left\{ -\frac{1}{g_0(0)} \|x\|^{1K_{n-1}(g_0)} \right\}.$$

By Lemma 3.2, the assumption

$$\mathcal{I}f(x) \leq \mathcal{I}g(x)$$

is equivalent to

$$f_0(0)^{n-1}I(1K_{n-1}(f_0)) \subseteq g_0(0)^{n-1}I(1K_{n-1}(g_0)).$$

By Lemma 4.1, we have

$$f_0(0)^nV(1K_{n-1}(f_0)) \subseteq g_0(0)^nV(1K_{n-1}(g_0))$$

(4.5)

with the equality holding if and only if $1K_{n-1}(f_0) = 1K_{n-1}(g_0)$, i.e., $f = g$. On the other hand, by (4.4) and (4.5), we have

$$\hat{\delta} J(f, f) = n\Gamma(n + 1)f_0(0)^nV(1K_{n-1}(f_0))$$

$$\leq n\Gamma(n + 1)g_0(0)^nV(1K_{n-1}(g_0))$$

$$= \hat{\delta} J(g, g)$$

with the equality holding if and only if $f = g$. This finishes the proof of sufficiency.

To prove the necessity part, it suffices to show that the existence of nonintersection functions implies a negative answer to Problem 1.3.

Let $K$ be an origin-symmetric star body whose radial function is in $C^\infty_c(S^{n-1})$, and let $K$ be not an intersection body. We claim that $f(x) = e^{-\|x\|^K} \in C^\infty_c(\mathbb{R}^n)$ and it is not an intersection function. In fact, if $e^{-\|x\|^K}$ is an intersection function, then there exists a non-negative integrable function $h$ in $\mathbb{R}^n$ such that $f(x) = e^{-\|x\|^K} = e^{-\|x\|^K}$. By Lemma 3.5, we have

$$e^{-\|x\|^K} = I_h(x) = \exp\{-h(0)\rho_{1K_{n-1}(h)}^{-1}(x, 0)\},$$

which implies that $K$ is an intersection body. This leads to a contradiction.

From Lemma 4.2 and (4.4), it follows that there exists an even function $g(x) = e^{-\|x\|^L}$ (where $L$ is an origin-symmetric star body) such that

$$\mathcal{I}f(x) < \mathcal{I}g(x) \quad \text{but} \quad \hat{\delta} J(f, f) > \hat{\delta} J(g, g).$$

We complete the proof. □
Next, we investigate the equivalence between intersection functions and intersection bodies.

**Lemma 4.4.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a non-negative continuous integrable function with \( f(0) = 1 \). Then \( f \) is an intersection function if and only if \( K_{n-1}(f) \) is an intersection body.

**Proof.** Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a non-negative, continuous, integrable function with \( g(0) > 0 \). Assume that \( f \) is an intersection function of \( g \). By Lemma 3.5, we have

\[
f(x) = I g(x) = \exp \left\{ -\frac{1}{g(0)} \|x\|_1 K_{n-1}(g) \right\}.
\]

Moreover, from the definition of the body \( K_{n-1}(f) \) for \( x \in \mathbb{R}^n \setminus \{0\} \), we have

\[
\rho_{K_{n-1}(f)}(x) = \frac{1}{f(0)} \int_0^\infty (n-1)r^{n-2} \exp \left\{ -\frac{1}{g(0)} \|r x\|_1 K_{n-1}(g) \right\} dr = \Gamma(n)^{-\frac{n}{n-1}} f(0)^{-\frac{n}{n-1}} g(0)^{\frac{n}{n-1}} \rho_{K_{n-1}(g)}(x).
\]

This means that \( K_{n-1}(f) \) is an intersection body.

If \( K_{n-1}(f) \) is an intersection body, there exists a star body \( L \) in \( \mathbb{R}^n \) such that \( \rho_{K_{n-1}(f)}(x) = \rho_L(x) \).

Let \( g(x) = e^{-\|x\|_L} \). It is clear that \( K_{n-1}(g) = \Gamma(n)^{-\frac{n}{n-1}} L \). By Lemma 3.5, we have

\[
\rho_{K_{n-1}(f)}(x) = \Gamma(n)^{-1} \rho_{K_{n-1}(g)}(x) = \Gamma(n)^{-1} \mathcal{R} g(x, 0).
\]

Direct calculation shows that

\[
\rho_{K_{n-1}(I g)}^{n-1}(x) = (n-1) \int_0^\infty r^{n-2} e^{-\mathcal{R} g(r x, 0)^{-1}} dr = (n-1) \int_0^\infty r^{n-2} e^{-\mathcal{R} g(r x, 0)^{-1}} dr = \Gamma(n) \mathcal{R} g(x, 0)^{n-1}.
\]

Therefore,

\[
\rho_{K_{n-1}(f)}(x) = \Gamma(n)^{-\frac{n}{n-1}} \rho_{K_{n-1}(I g)}(x),
\]

i.e.,

\[
K_{n-1}(f) = \Gamma(n)^{-\frac{n}{n-1}} K_{n-1}(I g).
\]

Corollary 3.4 deduces that \( f \) is an intersection function.

Now, we give a positive answer to Problems 1.2 and 1.3 when \( 2 \leq n \leq 4 \).

**Theorem 4.5.** Problems 1.2 and 1.3 have affirmative answers when \( 2 \leq n \leq 4 \).

**Proof.** In [26], Gardner et al. proved that every origin-symmetric convex body in \( \mathbb{R}^n \) is an intersection body when \( 2 \leq n \leq 4 \). Therefore, By Proposition 4.3 and Lemma 4.4, Problems 1.2 and 1.3 have affirmative answers when \( 2 \leq n \leq 4 \).



5 Proof of Theorem 1.4

In this section, we finish the proof of Theorem 1.4. The following result is the key reason why Problems 1.1–1.3 can be considered the functional Busemann-Petty problems.

**Proposition 5.1.** Problems 1.1–1.3 all include the Busemann-Petty problem.

**Proof.** Let \( K \) be an origin-symmetric convex body in \( \mathbb{R}^n \). For \( x \in \mathbb{R}^n \) and \( a > 0 \), if \( f(x) = e^{-\|x\|_a K} \), from (4.3) and (4.4), we have

\[
\text{Ent}(f) = -a \cdot J(f, f) - J(f) \log J(f) = -a^n \Gamma(n+1) V(K)[n + \log \Gamma(n+1) + \log(a^n V(K))].
\]

(5.1)
Then (4.2) and (4.3) show that Problem 1.2 includes the Busemann-Petty problem, and (4.2) and (4.4) show that Problem 1.3 includes the Busemann-Petty problem.

Let $K$ and $L$ be any origin-symmetric convex bodies in $\mathbb{R}^n$. There exists a constant $a > 0$ (for example, $a^{-n} = \min\{V(K), V(L)\}$) such that $V(aK) \geq 1$ and $V(aL) \geq 1$. If $f(x) = e^{-\|x\|_a K}$ and $g(x) = e^{-\|x\|_a L}$, $x \in \mathbb{R}^n$, then (4.2) deduces that
\[
\int_{\mathbb{R}^n \cap u^+} f(x) dx \leq \int_{\mathbb{R}^n \cap u^+} g(x) dx \Rightarrow V_n-1(K \cap u^+) \leq V_n-1(L \cap u^+)
\]
for every $u \in S^{n-1}$. By (5.1) and the facts $V(aK) \geq 1$ and $V(aL) \geq 1$ (i.e., $\log V(aK) > 0$ and $\log V(aL) \geq 0$), we have
\[
\text{Ent}(f) \geq \text{Ent}(g) \Rightarrow V(K) \leq V(L).
\]
This completes our proof. \hfill \Box

Proposition 5.1 tells us that Problems 1.1–1.3 have negative answers when $n \geq 5$. Hence, we only need to prove Theorem 1.4 for $2 \leq n \leq 4$.

Proof of Theorem 1.4. Problems 1.2 and 1.3 have been proved in Theorem 4.5. We only need to solve Problem 1.1. Assume that $J(f) > e^{-1}$ and $J(g) > e^{-1}$. If $2 \leq n \leq 4$, then Problems 1.2 and 1.3 have positive answers. Namely, for integrable even log-concave functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with finite positive integrals, if $2 \leq n \leq 4$ and
\[
\int_{\mathbb{R}^n \cap H} f(x) dx \leq \int_{\mathbb{R}^n \cap H} g(x) dx
\]
for every hyperplane $H$ passing through the origin, then
\[
J(f) \leq J(g) \quad \text{and} \quad \delta \tilde{J}(f,f) \leq \delta \tilde{J}(g,g). \tag{5.2}
\]
Since the function $t \log t$ (where $t > e^{-1}$) is strictly increasing, by (5.2) we have
\[
\text{Ent}(f) = -\tilde{\delta} \tilde{J}(f,f) - J(f) \log J(f) \geq -\tilde{\delta} \tilde{J}(g,g) - J(g) \log J(g) = \text{Ent}(g).
\]
Problem 1.1 has a positive answer when $2 \leq n \leq 4$.

In general cases where the assumptions $J(f) > e^{-1}$ and $J(g) > e^{-1}$ are removed, we consider the functions $\bar{f} = cf$ and $\bar{g} = cg$ with $c = \max\{J(f)^{-1}, J(g)^{-1}\}$. Therefore, $J(\bar{f}) \geq 1 > e^{-1}$, $J(\bar{g}) \geq 1 > e^{-1}$ and
\[
\text{Ent}(\bar{f}) = c \int_{\mathbb{R}^n} f \log(cf) dx - cJ(f) \log(cJ(f)) = c \text{Ent}(f).
\]
Since $c > 0$,
\[
\int_{\mathbb{R}^n \cap H} f(x) dx \leq \int_{\mathbb{R}^n \cap H} g(x) dx
\]
is equivalent to
\[
\int_{\mathbb{R}^n \cap H} \bar{f}(x) dx \leq \int_{\mathbb{R}^n \cap H} \bar{g}(x) dx.
\]
Hence, we conclude that Problem 1.1 has an affirmative answer for $2 \leq n \leq 4$ when the assumptions $J(f) > e^{-1}$ and $J(g) > e^{-1}$ are removed. \hfill \Box

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