K-CONTINUITY IS EQUIVALENT TO K-EXACTNESS

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Abstract. It is well known that the functor of taking the minimal tensor product with a fixed C*-algebra preserves inductive limits if and only if it preserves extensions. In other words, tensor continuity is equivalent to tensor exactness. We consider a K-theoretic analogue of this result and show that K-continuity is equivalent to K-exactness, using a result of M. Dădărlat.

1. Introduction

We denote the spatial or minimal tensor product of C*-algebras by the symbol ⊗ (cf. [Tak02 Section IV.4], [BO08 Section 3.3]).

Let A be a C*-algebra. We say that A is ⊗-exact if for every extension (i.e. short exact sequence)

\[ 0 \rightarrow I \rightarrow D \rightarrow B \rightarrow 0 \]

of C*-algebras, the natural sequence

\[ 0 \rightarrow A \otimes I \rightarrow A \otimes D \rightarrow A \otimes B \rightarrow 0 \]

is exact. Let \( M_n \) denote the C*-algebra of \( n \times n \) complex matrices. Letting

\[ \prod_{n=0}^{\infty} M_n := \{ (a_n)_{n=0}^{\infty} \mid a_n \in M_n \text{ for all } n \text{ and } \sup_n ||a_n|| < \infty \} \]

and

\[ \bigoplus_{n=0}^{\infty} M_n := \{ (a_n)_{n=0}^{\infty} \mid a_n \in M_n \text{ for all } n \text{ and } \lim_{n \rightarrow \infty} ||a_n|| = 0 \}, \]

we get an extension

\[ 0 \rightarrow \bigoplus_{n=0}^{\infty} M_n \rightarrow \prod_{n=0}^{\infty} M_n \rightarrow \prod_{n=0}^{\infty} M_n / \bigoplus_{n=0}^{\infty} M_n \rightarrow 0. \]

E. Kirchberg proved the following fundamental result about ⊗-exactness. See [BO08] for more details.

Theorem (E. Kirchberg [Kir83, Kir95]). Let A be a C*-algebra. The following statements are equivalent.

(i) The algebra A is ⊗-exact.

(ii) The sequence

\[ 0 \rightarrow A \otimes \bigoplus_{n=0}^{\infty} M_n \rightarrow A \otimes \prod_{n=0}^{\infty} M_n \rightarrow A \otimes (\prod_{n=0}^{\infty} M_n / \bigoplus_{n=0}^{\infty} M_n) \rightarrow 0 \]

is exact.

(iii) The algebra A is nuclearly embeddable in the sense of [Vol90]. \( \square \)

We remark that the implication (iii) ⇒ (ii) was proved by S. Wassermann in [Was90].

We say that A is ⊗-continuous if for every inductive sequence

\[ B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow \ldots \]

of C*-algebras, the natural (surjective) map

\[ \lim_{\rightarrow} (A \otimes B_n) \rightarrow A \otimes \lim_{\rightarrow} B_n \]

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is an isomorphism, where \( \lim \) denotes the inductive limit functor.

The following result is well-known and follows from the equivalence (iii) \( \iff \) (i) in the theorem above. N. Ozawa attributes it to E. Kirchberg.

**Theorem 1.1.** A \( C^\ast \)-algebra is \( \otimes \)-exact if and only if it is \( \otimes \)-continuous.

In this paper, we consider a \( K \)-theoretic analogue of this result. See [WO93, Bla98, RLL00] for details about topological \( K \)-theory for \( C^\ast \)-algebras. We say that a \( C^\ast \)-algebra \( A \) is \( K \)-exact for every extension

\[
0 \to I \to D \to B \to 0
\]

of \( C^\ast \)-algebras, the sequence

\[
K_0(A \otimes I) \to K_0(A \otimes D) \to K_0(A \otimes B)
\]

is exact in the middle. We say that a \( C^\ast \)-algebra \( A \) is \( K \)-continuous if for every inductive sequence

\[
B_0 \to B_1 \to B_2 \to \ldots
\]

of \( C^\ast \)-algebras, the natural map

\[
\lim \to K_0(A \otimes B) \to K_0(A \otimes \lim \to B)
\]

is an isomorphism.

The following is our main result.

**Theorem 1.2.** A \( C^\ast \)-algebra is \( K \)-exact if and only if it is \( K \)-continuous.

In section 2, we give a proof of Theorem 1.1 as we couldn’t find a direct reference and the proof of the implication Theorem 1.1 \( \Rightarrow \) is used in proof of Theorem 1.2 \( \Rightarrow \). In section 3, we study the notions of \( K \)-exactness and \( K \)-continuity and prove Theorem 1.2. We note that our proof of the implication Theorem 1.2 \( \Leftarrow \) uses [Dăd94, Theorem 3.11] in a crucial way.

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## 2. Proof of Theorem 1.1

Let \( \mathbb{N} := \{0, 1, \ldots \} \) denote the set of positive integers. The following is obvious.

**Lemma 2.1.** Let \( A \) be a \( C^\ast \)-algebra and let

\[
B_0 \to B_1 \to B_2 \to \ldots
\]

be an inductive sequence of \( C^\ast \)-algebras. If the connecting maps are all injective, then the map

\[
\lim \to (A \otimes B) \to A \otimes \lim \to B
\]

is an isomorphism. \( \square \)

**Lemma 2.2.** Consider an inductive sequence of extensions of \( C^\ast \)-algebras

\[
\begin{array}{c}
0 \quad I_n \quad D_n \quad B_n \quad 0 \\
\downarrow \quad \delta_n \quad \downarrow \quad \beta_n \\
0 \quad I_{n+1} \quad D_{n+1} \quad B_{n+1} \quad 0
\end{array}
\]

(i) The limit sequence

\[
0 \to \lim \to I_n \to \lim \to D_n \to \lim \to B_n \to 0
\]

is exact.
(ii) Suppose that for every \( n \in \mathbb{N} \), the extension
\[
0 \to I_n \to D_n \to B_n \to 0
\]  
\hspace{1cm} (2.1)
is split (i.e. the quotient map admits a \(*\)-homomorphic section) and the connecting maps \( \iota_n \) and \( \delta_n \) are injective. Then for any \( C^\ast \)-algebra \( A \), the map
\[
\lim (A \otimes B_n) \to A \otimes \lim B_n
\]is an isomorphism if and only if the sequence
\[
0 \to A \otimes \lim I_n \to A \otimes \lim D_n \to A \otimes \lim B_n \to 0
\]is exact.

Proof. \( \Box \) is clear. For \( \square \), consider the diagram
\[
\begin{array}{c}
0 \to & \lim (A \otimes I_n) & \to & \lim (A \otimes D_n) & \to & \lim (A \otimes B_n) & \to & 0 \\
| \iota | \downarrow & & & \downarrow \delta & & \downarrow \beta & & \\
0 \to & A \otimes \lim I_n & \to & A \otimes \lim D_n & \to & A \otimes \lim B_n & \to & 0.
\end{array}
\]

Since the connecting maps \( \iota_n \) and \( \delta_n \) are injective, the maps \( \iota \) and \( \delta \) are isomorphisms by Lemma 2.1. For any \( n \in \mathbb{N} \), since (2.1) is split exact, the sequence
\[
0 \to A \otimes I_n \to A \otimes D_n \to A \otimes B_n \to 0
\]is exact. Thus the top row is exact by (i). It follows that \( \beta \) is an isomorphism if and only if the bottom row is exact by five-lemma. \( \Box \)

Proof of Theorem 1.1. (\( \Rightarrow \)): Let \( A \) be a \( \otimes \)-exact \( C^\ast \)-algebra and let \( B_0 \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \ldots \) be an inductive sequence.

Let \( n \in \mathbb{N} \) and let \( I_n := \bigoplus_{k=0}^{n-1} B_k \) and let \( D_n := \bigoplus_{k=0}^{n} B_k \). Then the obvious inclusion and projection maps give a split extension
\[
0 \to I_n \to D_n \to B_n \to 0.
\]

Let \( \iota_n : I_n \to I_{n+1} \) denote the natural inclusion and let \( \delta_n : D_n \to D_{n+1} \) denote the injective map given by
\[
\begin{array}{ccc}
B_0 & \overset{id}{\longrightarrow} & B_0 \\
\oplus & \quad \quad & \oplus \\
\vdots & \quad \quad & \vdots \\
B_n & \overset{id}{\longrightarrow} & B_n \\
\beta_n & \quad \quad & \beta_n \\
\oplus & \quad \quad & \oplus \\
\to & B_{n+1}.
\end{array}
\]

Then we get a map of extensions
\[
\begin{array}{c}
0 \to I_n \to D_n \to B_n \to 0 \\
\downarrow \iota_n & & \downarrow \delta_n & & \downarrow \beta_n \\
0 \to I_{n+1} \to D_{n+1} \to B_{n+1} \to 0.
\end{array}
\]

Since \( A \) is \( \otimes \)-exact, the sequence
\[
0 \to A \otimes \lim I_n \to A \otimes \lim D_n \to A \otimes \lim B_n \to 0
\]is exact, hence the map
\[
\lim (A \otimes B_n) \to A \otimes \lim B_n
\]
is an isomorphism by Lemma 2.2(ii).

\[\footnote{Locally split is enough for our purposes \cite[Proposition 3.7.6]{BO08}. See \cite{EHS5}.}\]
\(\Leftrightarrow\): Conversely, let \(A\) be a \(\otimes\)-continuous \(C^*\)-algebra. Let \(M_n, n \in \mathbb{N}\), denote the \(C^*\)-algebra of \(n \times n\) complex matrices. Consider the following inductive system of split extensions

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{k=0}^{n} M_k & \longrightarrow & \prod_{k=0}^{\infty} M_k & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{k=0}^{n+1} M_k & \longrightarrow & \prod_{k=0}^{\infty} M_k & \longrightarrow & 0 \\
\end{array}
\]

given by the obvious injection and projection maps. Since \(A\) is \(\otimes\)-continuous, the map

\[
\lim_{\longrightarrow} (A \otimes \prod_{k=0}^{\infty} M_k) \longrightarrow A \otimes \left(\lim_{\longrightarrow} \prod_{k=0}^{\infty} M_k\right)
\]

is an isomorphism, hence the sequence

\[
0 \longrightarrow A \otimes \bigoplus_{k=0}^{\infty} M_k \longrightarrow A \otimes \prod_{k=0}^{\infty} M_k \longrightarrow A \otimes \left(\prod_{k=0}^{\infty} M_k / \bigoplus_{k=0}^{\infty} M_k\right) \longrightarrow 0
\]

is exact by Lemma 2.2(ii). It follows that \(A\) is \(\otimes\)-exact (cf. [Kir83]). □

3. \(K\)-exactness and \(K\)-continuity

3.1. \(K\)-exactness.

**Definition 3.1.** We say that a \(C^*\)-algebra \(A\) is \(K\)-exact if for every extension

\[
\begin{array}{ccccccc}
0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \\
\end{array}
\]

of \(C^*\)-algebras, the sequence

\[
K_0(A \otimes I) \longrightarrow K_0(A \otimes D) \longrightarrow K_0(A \otimes B)
\]

is exact in the middle.

**Remark 3.2.** A \(C^*\)-algebra \(A\) is \(K\)-exact if and only if for every extension

\[
\begin{array}{ccccccc}
0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \\
\end{array}
\]

of \(C^*\)-algebras, the natural six-term sequence

\[
\begin{array}{ccccccc}
K_0(A \otimes I) & \longrightarrow & K_0(A \otimes D) & \longrightarrow & K_0(A \otimes B) \\
\uparrow & & \uparrow & & \uparrow \\
K_1(A \otimes B) & \longleftrightarrow & K_1(A \otimes D) & \longleftrightarrow & K_1(A \otimes I) \\
\end{array}
\]

is exact (cf. [Bla98] Theorem 21.4.4]).

**Example 3.3.** \(\otimes\)-exact \(C^*\)-algebras are \(K\)-exact, by the half-exactness of \(K\)-theory (cf. [WO93] Theorem 6.3.2], [Bla98] Theorem 5.6.1]).

**Definition 3.4.** Let \(C_0[0,1]\) denote the commutative \(C^*\)-algebra of continuous functions on the interval \([0,1]\) vanishing at \(1 \in [0,1]\), and let

\[
ev_0 : C_0[0,1] \longrightarrow \mathbb{C}
\]

\[
f \longrightarrow f(0)
\]

denote the evaluation map at \(0 \in [0,1]\).

**Definition 3.5.** Let \(\phi : D \rightarrow B\) be a \(*\)-homomorphism of \(C^*\)-algebras. The mapping cone \(C_\phi\) of \(\phi\) is given by the pullback

\[
\begin{array}{ccccccc}
C_\phi & \longrightarrow & C_0[0,1] \otimes B \\
\downarrow & & \downarrow \epsilon_{v_0 \otimes \text{id}_B} \\
D & \phi & \longrightarrow & B \\
\end{array}
\]
Remark 3.6. Let $\phi: D \to B$ be a $*$-homomorphism of $C^*$-algebras. Then for any $C^*$-algebra $A$, there is a natural isomorphism $C_{id_A \otimes \phi} \cong A \otimes C_\phi$. Indeed, since $ev_\phi: C_\phi[0,1) \to \mathbb{C}$ admits a completely positive section, we have a map of extensions

$$
\begin{array}{ccc}
0 & \longrightarrow & A \otimes C_\phi[0,1) \otimes B \\
\downarrow & & \downarrow \text{id}_A \otimes \phi \\
0 & \longrightarrow & A \otimes C_\phi[0,1) \otimes B \\
\end{array}
$$

where $C_\phi[0,1)$ denotes the kernel of $ev_\phi$. Now it is easy to see that the square on the right is a pullback square.

Lemma 3.7 (cf. [HLS02, p. 335-336]). A $C^*$-algebra $A$ is $K$-exact if and only if for every extension

$$
\begin{array}{ccc}
0 & \longrightarrow & I \longrightarrow & D \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow 0 \\
0 & \longrightarrow & I \longrightarrow & D \longrightarrow & 0 \\
\end{array}
$$

of separable $C^*$-algebras, the natural inclusion map $\iota: I \longrightarrow C_q$ induces an isomorphism

$$(\text{id}_A \otimes \iota)_\ast: K_0(A \otimes I) \cong K_0(A \otimes C_q).$$

Proof. Let $A$ be a $C^*$-algebra and let

$$
\begin{array}{ccc}
0 & \longrightarrow & I \longrightarrow & D \longrightarrow & 0 \\
\downarrow & & \downarrow q & & \downarrow q \\
0 & \longrightarrow & I \longrightarrow & D \longrightarrow & 0 \\
\end{array}
$$

be an extension of (not necessarily separable) $C^*$-algebras.

$(\Rightarrow)$: Suppose that $A$ is $K$-exact. By the homotopy invariance of $K$-theory, we have $K_\ast(A \otimes C_\phi[0,1) \otimes B) = 0$. Hence, applying Remark 3.6 to the pullback extension

$$
\begin{array}{ccc}
0 & \longrightarrow & I \longrightarrow & C_q \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow 0 \\
0 & \longrightarrow & I \longrightarrow & C_q \longrightarrow & 0 \\
\end{array}
$$

we see that $\text{id}_A \otimes \iota$ induces an isomorphism $K_0(A \otimes I) \cong K_0(A \otimes C_q)$.

$(\Leftarrow)$: Conversely, suppose that $A$ satisfies the necessary condition in the lemma. We prove that the sequence

$$
K_0(A \otimes I) \longrightarrow K_0(A \otimes D) \longrightarrow K_0(A \otimes B)
$$

is exact in the middle.

If $I$, $D$, and $B$ are separable, then the exactness follows from the Puppe exact sequence (cf. [Ros82] or [WO93, Lemma 6.4.8]) and the natural isomorphism $C_{id_A \otimes q} \cong A \otimes C_q$ of Remark 3.6.

The general case is reduced to the separable case as follows. Let $\Lambda$ denote the set of separable $C^*$-subalgebras of $D$, ordered by inclusion. Then $\Lambda$ is a directed set. For each $E \in \Lambda$, we associate a subextension

$$
\begin{array}{ccc}
0 & \longrightarrow & I \cap E \longrightarrow & E \longrightarrow & E/(I \cap E) \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow 0 \\
0 & \longrightarrow & I \longrightarrow & D \longrightarrow & B \longrightarrow & 0. \\
\end{array}
$$

It is clear that the inductive limit of the subextensions is the extension (3.1). Since all the connecting maps are injective, the proof is complete by the continuity of $K$-theory (cf. [WO93, Proposition 6.2.9]) and the exactness of the inductive limit functor for abelian groups (cf. [Wei94, Theorem 2.6.15]). \hfill \Box

Corollary 3.8. A $C^*$-algebra $A$ is $K$-exact if and only if the functor $B \mapsto K_0(A \otimes B)$, from the category of separable $C^*$-algebras to abelian groups, factors through the category $E$ of Higson (cf. [Hig90, CH90]).

Proof. Let $A$ be a $C^*$-algebra and let $F(B) := K_0(A \otimes B)$.

$(\Rightarrow)$: Suppose that $A$ is $K$-exact. Then $F$ is half-exact and since $F$ is homotopy invariant and stable (under tensoring with the compacts), it factors through the category $E$ by the universal property (cf. [CH90, Théorème 7]).
(⇐): Suppose that $F$ factors through $E$. For any extension $0 \to I \to D \xrightarrow{q} B \to 0$ of separable $C^*$-algebras, the inclusion $i: I \to C_q$ is an equivalence in $E$ (cf. [CH90] Lemma 12)). Now Lemma 3.7 completes the proof. □

**Example 3.9.** $C^*$-algebras satisfying the Künneth formula of Schochet (cf. [Sch82], [Bla98, Theorem 23.1.3]) are $K$-exact by Lemma 3.7 (cf. [CEO04, Remark 4.3]). In particular, the full group $C^*$-algebra $C^*(F_2)$ of the free group $F_2$ on two generators is $K$-exact (but not $⊗$-exact).

Needless to say, not all $C^*$-algebras are $K$-exact.

**Example 3.10.** (1) Let $Γ$ be an infinite countable discrete group with Khazdan property (T), Kirchberg property (F) and Akemann-Ostrand property (AO), such as a lattice in $Sp(n, 1)$ (cf. [AD09]). The full group $C^*$-algebra $C^*(Γ)$ is not $K$-exact (G. Skandalis [Ska91]).

(2) The product $\prod_{n=0}^\infty M_n$ is not $K$-exact (N. Ozawa [Oza03, Theorem A.1]).

### 3.2. $K$-continuity.

**Definition 3.11.** We say that a $C^*$-algebra $A$ is $K$-continuous if for every inductive sequence

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \ldots$$

of $C^*$-algebras, the natural map

$$\lim K_0(A \otimes B_i) \longrightarrow K_0(A \otimes \lim B_i)$$

is an isomorphism.

**Remark 3.12.** In Definition 3.11 we could use $K_1$ instead of $K_0$.

**Example 3.13.** $⊗$-continuous $C^*$-algebras are $K$-continuous, by the continuity of $K$-theory (cf. [WO93, Proposition 6.2.9], [Bla98, 5.2.4, 8.1.5]).

**Example 3.14.** $C^*$-algebras satisfying the Künneth formula of Schochet (cf. [Sch82], [Bla98, Theorem 23.1.3]) are $K$-continuous. Indeed, let $A$ be a $C^*$-algebra satisfying the Künneth formula and let

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \ldots$$

be an inductive sequence of $C^*$-algebras. Then the top row in the diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & \lim K_*(A) \otimes K_*(B_i) & \longrightarrow & \lim K_*(A \otimes B_i) & \longrightarrow & \lim \text{Tor}(K_*(A), K_*(B_i)) & \longrightarrow & 0 \\
0 \longrightarrow K_*(A) \otimes K_*(\lim B_i) \longrightarrow K_*(A \otimes \lim B_i) \longrightarrow \text{Tor}(K_*(A), K_*(\lim B_i)) & \longrightarrow & 0
\end{array}$$

is an extension of abelian groups by [Wei94, Theorem 2.6.15] and the second row is an extension by the Künneth formula. The left and right vertical maps are isomorphisms by [Wei94, Corollary 2.6.17] and thus the middle vertical map is also an isomorphism by five-lemma. Hence $A$ is $K$-continuous.

### 3.3. Proof of Main Theorem 1.2.

**Proof of Theorem 1.2 (⇒):** Let $A$ be a $K$-exact $C^*$-algebra and let

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \ldots$$

be an inductive sequence. We use the notations of the proof of Theorem 1.1 (⇒). Applying $K$-theory to the diagram (2.2), we get a map of exact sequences

$$\begin{align*}
K_0(\lim A \otimes I_n) & \to K_0(\lim A \otimes D_n) \to K_1(\lim A \otimes I_n) \to K_1(\lim A \otimes D_n) \\
K_0(A \otimes \lim I_n) & \to K_0(A \otimes \lim D_n) \to K_1(A \otimes \lim I_n) \to K_1(A \otimes \lim D_n)
\end{align*}$$
By five-lemma, the map $\beta_*$ is an isomorphism.

$(\Leftarrow)$: Conversely, let $A$ be a $K$-continuous $C^*$-algebra. Then the functor $F(B) := K_0(A \otimes B)$, considered on the category of separable $C^*$-algebras, factors through the asymptotic homotopy category of Connes-Higson by [Dăd94, Theorem 3.11]. Since $F$ is stable and satisfies Bott periodicity, it in fact factors through the category $E$. Hence by Corollary 3.8 $A$ is $K$-exact. 

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