VACUUM DECAY VIA LORENTZIAN WORMHOLES

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Abstract
We speculate about the spacetime description due to the presence of Lorentzian wormholes (handles in spacetime joining two distant regions or other universes) in quantum gravity. The semiclassical rate of production of these Lorentzian wormholes in Reissner-Nordström spacetimes is calculated as a result of the spontaneous decay of vacuum due to a real tunneling configuration. In the magnetic case it only depends on the value of the field theoretical fine structure constant. We predict that the quantum probability corresponding to the nucleation of such geodesically complete spacetimes should be actually negligible in our physical Universe.

1. Introduction
This paper is devoted to the real tunneling configurations and spontaneous nucleation of vacuum into Lorentzian wormholes as a possibility to study the quantum gravity foam according to the seminal considerations of Wheeler 1 and Hawking 2. We will present an extension of the quantum mechanical analysis of a minisuperspace that describes wormholes in the Reissner-Nordström background spacetime; this model has been already studied in details by Visser 3 4. Wheeler-DeWitt equation corresponding to the quantisation of the wormhole dynamic variable, the throat radius, exibits, in this model, the form of an alternative finite differences relationship. Here we will develop the semiclassical approximation in order to construct from that relationship a Wheeler-DeWitt equation which is equivalent to the first one developed by Visser up to the smallest order in the WKB approximation. The solutions are quantum instantons and their amplitude of "probability" is dominated by the corresponding Euclidean signature action $P \sim \exp(-2I)$. As a result of this, we will

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develop an interpretation of the solutions in terms of the tunneling rate corresponding to
the nucleation "from nothing" of a lorentzian wormhole of a given charge and mass. The
calculation procedure is entirely equivalent to that from minisuperspace models in quantum
cosmology. There, the creation of the Universe "ex nihilo" is due to the presence of an
effective cosmological constant and the result may also be considered as a sort of classical
change of signature or a bounce in spacetime (i.e., the transition between two solutions with
the same boundary conditions having different actions). In quantum cosmology, a tunneling
solution of the Wheeler-DeWitt equation in minisuperspace with a positive cosmological
constant would represent the quantum rate of production corresponding to a spacetime of
the type of a round Euclidean sphere joined on an equator to the Lorentzian de Sitter space
at its radius of maximum contraction.

The spontaneous decay will be regulated by an arbitrarily small negative mass para-

mometer. This is required for any solution of general relativity which undertake a change of
topology. (actually, Tipler’s theorem is the negative of the previous sentence, i.e., given
the positiveness of the energy-momentum tensor, there exits no classical change in the
topology of spacetime). A change of topology could only exist as a quantum phenomenon
but still, since we require Einstein’s equations in order to formulate the ground theory, we
would need the violation of the energy condition, at least minimally. The important issue is
to notice that we obtain results according to the semiclassical theory but that they would,
very likely, be of value in order to investigate more profound properties of the expected
quantum theory of gravity.

In section 2, we review the dynamic of Lorentzian wormholes connecting two asymp-
totically flat Reissner-Nordström spacetimes after the approach of Visser. In section 3,
we derive, following semiclassical considerations, an alternative Wheeler-DeWitt equation
whose solutions exhibit a "tunneling from nothing" configuration allowing for a simpler
interpretation of Visser equations in the semiclassical limit. We also proof, in section 3, the
consistence of the quasiclassical expressions and the qualitative behaviour of the solutions.
The vacuum decay rate via Lorentzian wormholes is obtained from the tunneling configua-
rion; in the magnetic case, the rate can be written as an expression in terms of the field
theoretical fine structure constant. We summarize our conclusions in section 4.

2. Lorentzian wormholes

Wormholes are handles in the spacetime topology linking widely separated regions of the
Universe, or bridges joining two different spacetimes. Thus, a Lorentzian wormhole consists
on the union of two copies of identical asymptotically flat four dimensional regions \( \Omega_1 \) and
\( \Omega_2 \) representing those regions. Technically, one is left with two geodesically incomplete
manifolds with boundaries given by the hypersurfaces \( \partial \Omega_1 \) and \( \partial \Omega_2 \). Now, identify these
two hypersurfaces. The resulting spacetime \( M \) is geodesically complete and possesses two
asymptotically flat regions connected by a wormhole. The throat of the wormhole is at
\( \partial \Omega \). At the throat the Riemann curvature tensor is proportional to a delta function while
it vanishes exactly outside in \( \Omega_1 \) and \( \Omega_2 \). Since we are looking for solutions of the Einstein
equations with that topology, we require the "junction conditions" which were already
developed by Israel. The required junction conditions are most conveniently derived by
introducing gaussian normal coordinates in the neighbourhood of the hypersurface $\partial \Omega$:

$$ds^2 = \sigma d\eta^2 + g_{ij}(\eta, x^i) dx^i dx^j,$$

where $\sigma = 1, (-1)$ if $\partial \Omega$ is spacelike (timelike). The gaussian coordinate $\eta$ parametrizes the proper distance measured perpendicularly through $\partial \Omega = \{(\eta, x^i) | \eta = 0\}$. Define the surface stress-tensor on $\partial \Omega$ by

$$S^\mu_\nu = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} d\eta T^\mu_\nu,$$

where $T^\mu_\nu$ contains all possible sources. We wish the geometry $(g_{ij})$ to be continuous at the boundary, but the metric need not be differentiable there. Then the metric junction conditions, derived by integrating the Einstein equations across $\partial \Omega$ are

$$S^0_\eta = 0,$$

$$S^0_i = 0,$$

and,

$$\sigma[\Delta K^i_j - \delta Tr(\Delta K)] = -8\pi S^i_j,$$

where

$$\Delta K^i_j = \lim_{\varepsilon \to 0} (K^i_j + - K^i_j -)$$

is the ”jump” in the extrinsic curvature of the layer in going from the $-\varepsilon$ to the $+\varepsilon$ side. The Ricci tensor at the junction can be calculated in terms of the extrinsic curvature

$$K^i_j = \frac{1}{2} g^{ik} \frac{\partial g_{kj}}{\partial \eta} |_{\eta = 0}.$$

Here $\eta$ also denotes the normal coordinate to the throat, while $\tau$ will hereafter denote the proper time along the throat, i.e., the geometry of the boundary is given by the metric

$$ds^2 = -d\tau^2 + a(\tau)^2 d\Omega^2_3,$$

where $a(\tau)$ is the throat radius and $d\Omega^2_3$ states for the metric of the unit three - sphere.

Let us consider the Ricci tensor from the Reissner-Nordström geometry almost everywhere but on the throat:

$$R^\mu_\nu = R^{(RN)}_\mu_\nu - 2 \left( \begin{array}{cc} K^i_j(x) & 0 \\ 0 & K(x) \end{array} \right) \delta(\eta);$$

So that the Einstein-Hilbert action reduces to

$$S_G = \frac{1}{16\pi} \int_M (-g_4)^{1/2} R = -\frac{1}{4\pi} \int_{\partial \Omega} (g_3)^{1/2} K.$$

By considering the joining conditions it is possible to write the extrinsic curvature in terms of the parameters $a$ and $\dot{a} = da/d\tau$ (the proper velocity of the throat). Upon taking into account the electromagnetic action within the regions $\Omega_1$ and $\Omega_2$ one finally obtains for the effective action of the throat

$$S_{eff} = 2 \int \{ a\dot{a} \sinh^{-1}(a\dot{a}/h(a)^{1/2}) - (h(a) + a^2\dot{a}^2)^{1/2} \} d\tau,$$
where,
\[ h(a) \equiv a^2 - 2ma + q^2, \]  
(12)

\( q \) and \( m \) being respectively the charge and the mass of the corresponding Reissner-Nordström metrics.

As stated above, let us now assume the presence of a certain amount of mass \( w \) on the throat, the mass of the dust shell. This is a constant of motion. The matter Lagrangian reduces to \( L_m = -w \), and the total lagrangian would be

\[ L = 2\{a\dot{a}\sinh^{-1}(a\dot{a}/h(a)^{1/2}) - (h(a) + a^2\dot{a}^2)^{1/2}\} - w. \]  
(13)

The Hamiltonian is given by

\[ H(p, a) = 2h(a)^{1/2} \cosh \frac{p}{2a} + w = 2(h(a) + a^2\dot{a}^2)^{1/2} + w, \]  
(14)

and the constraint \( H = 0 \) leads to the equation of motion for the throat radius

\[ h(a) + a^2\dot{a}^2 - \frac{w^2}{4} = 0. \]  
(15)

The Wheeler-DeWitt equation for the wave function of the dynamic variable \( a \) in this minisuperspace is written as follows

\[ \{2h(a)^{1/2-s} \cos \left[ \frac{1}{2aM^2} \frac{\partial}{\partial a} \right] h(a)^s + w \} \psi(a) = 0, \]  
(16)

where \( s \) is a constant factor representing part of the factor ordering ambiguity and \( M^2 \) denotes Planck’s mass squared. On the other hand, Visser [4] takes \( s = 1/4 \) and restricts the solutions to the space of integrable functions in \([0, \infty)\). Here, let us select simply \( s = 0 \). In this case, upon substituting the variable \( a^2 = x \), we have, instead, a finite differences equation [4], namely

\[ \psi(x + i/M^2) + \psi(x - i/M^2) + \frac{w}{h(x)^{1/2}} \psi(x) = 0; \]  
(17)

which has the remarkable property that it is complex conjugation invariant if and only if \( h(a)^{1/2} \) is real, i.e., in case that \( m = q \). In other words, the reality of the wave function implies that the asymptotically flat spacetime is just the extreme Reissner-Nordström black hole. Since time is absent from this quantum theory, such a complex conjugation invariance could be required by the consistence of the quantisation procedure (although, quantum gravity might not satisfy this requirement strictly [4]). On the other hand, the existence of a finite differences equation seems to be a fact related with the speculations of Wheeler [1], in the sense that spacetime might not be considered as classical on scales of the order of the Planck length \((a \sim M^{-1})\), i.e., in our terms, that the continuous character of spacetime can only be recovered, we will see, as an approximated phenomenon having a semiclassical meaning.
3. Semiclassical theory

Let us now obtain the semiclassical approximation to Eq. (17). Our hope will be, therefore, to recover a suitable continuous limit in case that \( Ma \gg 1 \). Notice that Eq. (17) may also be written as follows

\[
2xM^2\{\psi(x + i/M^2) + \psi(x - i/M^2) - 2\psi(x)\} + \frac{M^2V^0(x)}{2}\psi(x) = 0,
\]

where

\[
V^0(x) = 4(2x + \frac{wx}{\hbar(x)^{1/2}}) = 4(2a^2 + \frac{wa^2}{h(a)^{1/2}}),
\]

and that, for \( M^{-2}/x \ll 1 \) (radius of the throat much larger than the Planck length), we have as a first approximation

\[
2xM^2\{\psi(x + i/M^2) + \psi(x - i/M^2) - 2\psi(x)\} \approx -\frac{a}{2M^2} \frac{\partial}{\partial a} a^{-1} \frac{\partial}{\partial a} \hat{\psi}(a).
\]

Therefore, the wave function in Eq. (18) is approximated by the solution of

\[
-\frac{a}{2M^2} \frac{\partial}{\partial a} a^{-1} \frac{\partial}{\partial a} \hat{\psi}(a) + \frac{M^2V^0(a)}{2} \hat{\psi}(a) = 0.
\]

For \( w = -|w| = -\xi < 0 \), there exits a potential barrier between \( a = 0 \) and \( a = m - \xi/2 \) representing the nucleation of a wormhole "from nothing" in a way anagously to the cosmological case. This would change the causal structure of spacetime and, therefore, would also require the violation of the energy conditions according to Tipler’s theorem. In case that \( w > 0 \), there would exits no nucleation and there should be no causal violations in spacetime (the wave function is strongly peaked about \( a = 0 \)).

On the other hand, we may obtain a more rigurous approximation for the solutions of Eq. (18). To see this, let us write

\[
\psi(a) = \exp[M^2S(a)],
\]

then, the finite differences equation is approximated up to order \( O(M^{-4}) \) by

\[
-\frac{a}{2M^2} \frac{\partial}{\partial a} a^{-1} \frac{\partial}{\partial a} \hat{\psi}(a) + \frac{M^2V(a)}{2} \hat{\psi}(a) = 0,
\]

with \( p = 1 \) and \( V(a) \) given by the series

\[
V(a) = V^0(a) + 8a^2 \sum_{k=4}^{\infty} (-1)^k \frac{(\partial S}{\partial a})^k.
\]

Now, in the semiclassical approximation,

\[
\frac{\partial S}{\partial a} = V(a)^{1/2} + O(1/(M^2)),
\]


and, after replacing Eq. (25) in Eq. (24) and suming the series we get

\[ V(a) = V^0(a) + 8a^2[\cos(\frac{V(a)^{1/2}}{2a}) - 1 + \frac{1}{2}(\frac{V(a)^{1/2}}{2a})^2], \] (26)

or \(^4\)

\[ V(a)^{1/2} = 2a \cos^{-1}(\frac{\xi}{2h^{1/2}})). \] (27)

This is equivalent to taken

\[ V(a)^{1/2} = a\pi[1 - \frac{2}{\pi} \tan^{-1}(\frac{\xi}{(4h - \xi^2)^{1/2}})], \] (28)

if \( h(a) = (m - a)^2 \), the existence of a barrier between \( a = 0 \) ("no wormhole") and \( a = m - \xi/2 \) is now evident.

In what follows, we will take simply \( M = 1 \), using the semiclassical theory just in the sense that we expect \( m \gg \xi \sim 1 \).

On the other hand, Eq. (23) will exhibit the same qualitative behaviour than

\[ -\frac{a}{2} \frac{\partial}{\partial a} a^{-1} \frac{\partial}{\partial a} \psi(a) + \frac{1}{2} a^2 \pi^2[1 - \frac{\xi}{2(m - a)}] \psi(a) = 0; \] (29)

it has a pole at \( a = m^- \). In order to get consistence with the semiclassical approximations we should investigate the behaviour of the wave function in that limit; moreover, if we were interested on computing the tunneling rate between \( a = 0 \) and \( a = m^- \) from the known quasiclassical expressions, there should exist a suitable set of coordinates which contains the barrier and an asymptotically free region that map the pole to infinity within such an asymptotic region. In this case we require \( \lim_{a \to m^-} \psi(a) = 0. \)

In order to deal with the singularity let us use the variable \( \zeta = \frac{1}{m - a} \). We are interested on the qualitative behaviour of the solutions of Eq. (29) for \( m\zeta \gg 1 \).

Define

\[ \psi \equiv \frac{\phi((m\zeta - 1)^{1/2})}{\zeta^{3/2}}, \] (30)

so that,

\[ \phi''(\zeta) - \tilde{V}(\zeta)\phi(\zeta) = 0; \] (31)

where,

\[ \tilde{V}(\zeta) = \frac{3}{4\zeta^2(m\zeta - 1)^2} - \frac{\pi^2(\xi - 2)(m\zeta - 1)^2}{2\zeta^6}. \] (32)

Far from the turning point at \( \zeta \sim 2/\xi \) we may write the approximate asymptotic regime for \( \phi(\zeta) \),

\[ \phi_{m\zeta \sim 1} \sim (m\zeta - 1)^{-1/2}, \]

\(^4\)Notice that, by using Eq. (15), \( i\partial L/\partial ˙a = V(a)^{1/2} \), and, upon doing a Wick rotation, these are just the semiclassical instantons to which our solutions in Eq. (23) should be correlated about. This is consistent with the approximations.
\[
\phi_{m\zeta \gg 1} \sim \zeta^{3/4} \exp[\pm i \frac{\pi m (2\zeta)^{1/2}}{\zeta^{1/2}}],
\]

thus, from Eq. (30) we finally obtain
\[
\psi_{m\zeta \gg 1} \sim \frac{\phi(\zeta)}{\zeta} \sim \zeta^{-1/4} \cos(\frac{\pi m (2\zeta)^{1/2}}{\zeta^{1/2}} + \pi/4) \to 0,
\]

moreover,
\[
\psi_{m\zeta \sim 1} \sim (m\zeta - 1)^{1/2} \phi(\zeta) \sim O(1),
\]

which are the required results in order to apply the quasiclassical formulæ to the solutions of the Wheeler-DeWitt equation. The barrier (region \( R_I \)) stands for \( 1/m \leq \zeta < 2/\xi \) while the asymptotic region (namely, \( R_{II} \)) is the interval \( 2/\xi < \zeta < \infty \).

Moreover, WKB solutions of Eq. (23) may be expressed as
\[
\psi(a) = \frac{Ca^{p/2}}{V(a)^{1/4}} \exp(-\int V(a)^{1/2} da),
\]
or, for \( p = 1 \),
\[
\psi_I(a) \approx \frac{2^{1/2}}{\cos^{-1}[\xi/2(m - a)]^{1/2}} \exp(-\int V(a)^{1/2} da),
\]
\[
\psi_{II}(a) \approx \frac{2^{1/2}}{\cosh^{-1}[\xi/2(a - m)]^{1/2}} \cos(\int (-V(a))^{1/2} da + \pi/4),
\]

they have the expected behaviour for the values \( am \sim 1 \) and \( am \ll 1 \). The constant, \( C \approx 2 \), is given by the approximated normalisation of the wave function between the limits \( a = 0 \) and \( a = m \), we also used \( m \gg 1 \). In our real tunneling configuration we only have a free asymptotic region, \( R_{II} \). We obtain the WKB barrier penetration rate simply by
\[
\Gamma(m, \xi) = \exp\{-2 \int_0^{m-\xi/2} V(a)^{1/2} da\},
\]
in the limit \( \xi \to 0 \) we get
\[
\Gamma(m, 0) = \exp\{-2 \lim_{\xi \to 0} \int_0^{m-\xi/2} V(a)^{1/2} da\} = \exp(-\pi m^2) = \exp(-\pi q^2).
\]

On the other hand, we only expect magnetic charged wormholes to result from this tunneling configuration (the electric ones should decay, also spontaneously, due to the presence of virtual lower charged particles in vacuum).

More concerning is the fact that the presence of wormholes causes the structure of spacetime to fail since they represent geodesically complete spacetimes. This dangerous feature may originate doubts about the nature of the present calculation; on the other hand, we can also estimate how strong this violation of causality is upon suming the rates of production of all possible magnetic charges, i.e., recalling Dirac’s relation \( q_n = n/2\sqrt{\alpha} \), \( \alpha \) standing for the value of the electrodynamical fine structure constant,
\[
\Gamma(\alpha) = \sum_{n=1}^{\infty} \exp(-\frac{\pi n^2}{4\alpha});
\]

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this series is known in the theory of the Riemann’s Zeta function. It can be expressed as

\[ \Gamma(\alpha) = \sqrt{\alpha - \frac{1}{2}} + 2\sqrt{\alpha} \Gamma\left(\frac{1}{16\alpha}\right); \quad (42) \]

thus, it behaves as the squared root of the fine structure constant when \( \alpha \sim O(1) \) but it goes fairly to zero for small values of its argument, i.e.,

\[ \Gamma_{\alpha \ll 1} \approx \exp\left(-\frac{\pi}{4\alpha}\right) \ll 1, \quad (43) \]

but

\[ \Gamma_{\alpha \sim 1} \approx \sqrt{\alpha - \frac{1}{2}} \sim O(1). \quad (44) \]

Therefore, given the actual value of \( \alpha \), causal violations, though they might exist, will cause no real damage to the underlying spacetime description. In spite of this, wormholes could have been originated in a bigger proportion in the very early phases of the Universe (when we expect a sizable value of \( \alpha \)); this might well be of foundamental importance when considering other quantum physical properties during this phase.

4. Summary and conclusions

We have obtained a simpler interpretation of the quantum theory of Lorentzian wormholes connecting two Reissner-Nordström spacetimes. This is possible since the semiclassical approximations to the exact theory are in direct correspondence with the solutions of a more simpler theory whose Wheeler-DeWitt equation exhibits a potential barrier; an interpretation of the solutions can, therefore, be given in terms of real tunneling configuration of spacetimes (in a way enterely equivalent to the creation "ex nihilo" of the Universe in the cosmological case). Here, however, the semiclassical barrier is very close to a singularity in the potential. In spite of that, we have shown that the pole can be mapped to infinity within an asymptotically free region of the configuration space. The latter allows for the evaluation of the tunneling rate for a wormhole to be created from nothing in the extreme Reissner-Nordström case. Extreme Reissner-Nordström spacetime seems to be the favoured selection since it also allows for the complex conjugation invariance of the exact Visser’s theory. It might also be related with the existence of strong limitations for a four dimensional (Euclidean) topology to change.

The fact that the topology could change in this way has, however, two necessary requirements; the first one is that it must be regulated by a negative (even small) mass parameter. This agrees with Tipler’s theorem (the example that we have put forward represents, therefore, a very pedagogical model of the possible limitations of the classical theory of gravity). The second requirement is that the violations of causality arising from the existence of wormholes should not damage the underlying predictive spacetime description. However, we have also shown that the rate of creation of these structures is actually negligible, in the magnetic case, given the value of the electrodynamical fine structure constant (but that it might not be so small in some earlier phase of the Universe).

5. Acknowledgements
The author wants to thank the Department of Physics of the University of Freiburg for his hospitality. This work is supported by a postdoctoral grant from the Spanish Ministry of Education and Culture and the research project C.I.C. y T., PB 94-0194.

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