ESCAPE OF MASS IN HOMOGENEOUS DYNAMICS IN POSITIVE CHARACTERISTIC

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ABSTRACT. We show that in positive characteristic the homogeneous probability measure supported on a periodic orbit of the diagonal group in the space of 2-lattices, when varied along rays of Hecke trees, may behave in sharp contrast to the zero characteristic analogue: For a large set of rays, the measures fail to converge to the uniform probability measure on the space of 2-lattices. More precisely, we prove that when the ray is rational there is uniform escape of mass, that there are uncountably many rays giving rise to escape of mass, and that there are rays along which the measures accumulate on measures which are not absolutely continuous with respect to the uniform measure on the space of 2-lattices.

1. INTRODUCTION

This paper deals with the study of the positive characteristic analogue (which turns out to have a surprisingly different outcome) of the discussion in [1], thus we start this introduction by briefly recalling it.

The space $X = \text{PGL}_2(\mathbb{R})/\text{PGL}_2(\mathbb{Z})$ may be identified with the space of homothety classes of $(\mathbb{Z})$-lattices in the plane $\mathbb{R}^2$. If we denote by $A$ the diagonal subgroup of $\text{PGL}_2(\mathbb{R})$, then the $A$-orbits in $X$ constitute a very important object of study both from the dynamical point of view and because of their tight relation to geometry and number theory. Among these orbits, the periodic ones (recall that $A$ is isomorphic to $\mathbb{R}$), may be considered as most important. Given a periodic orbit $Ax$ in $X$, we denote by $\mu_x$ the unique $A$-invariant probability measure supported on it.

Fixing a prime $p \in \mathbb{Z}$ and a homothety class of a lattice $x \in X$, one looks at the countable subset $\mathcal{H}_p(x)$ of elements $y \in X$ such that there exist $k \in \mathbb{N}$ and $\Lambda_x \in x, \Lambda_y \in y$ with $\Lambda_y \subset \Lambda_x$ and $[\Lambda_x : \Lambda_y] = p^k$. Upon drawing an edge between $y, z \in \mathcal{H}_p(x)$ if and only if there exist representatives $\Lambda_y \in y$ and $\Lambda_z \in z$ such that $\Lambda_z \subset \Lambda_y$ and $[\Lambda_y : \Lambda_z] = p$, one endows $\mathcal{H}_p(x)$ with a graph structure which is in fact a $(p + 1)$-regular tree known as the $p$-Hecke tree through $x$. The edge structure (which is of arithmetic origin) allows one to talk about Hecke rays.

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starting from $x$ in $\mathcal{G}_p(x)$. These are simply sequences of homothety classes of lattices $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{G}_p(x)$ with no repetitions such that $x_0 = x$ and such that there is an edge between $x_n$ and $x_{n+1}$ for all $n \in \mathbb{N}$.

It turns out that if $Ax$ is periodic, then $Ay$ is periodic for any $y \in \mathcal{G}_p(x)$ and thus it is natural to consider the possible weak-star limits of the periodic measures $\mu_{x_n}$, where $(x_n)_{n \in \mathbb{N}}$ is a Hecke ray. Theorem 4.8 from [1] states that for all but potentially two special rays, the sequence $(\mu_{x_n})_{n \in \mathbb{N}}$ equidistributes in $\Omega$; that is, it weak-star converges to the unique $\text{PGL}_2(\mathbb{R})$-invariant probability measure on $X$. For the problematic two rays (which exist only when the prime $p$ splits in a quadratic extension of $\mathbb{Q}$ which corresponds to the periodic orbit $Ax$), the collection of measures $(\mu_{x_n})_{n \in \mathbb{N}}$ is finite and, so, nothing interesting dynamically is happening. In particular, what we wish to stress for our analogy with the positive characteristic case studied here are the following facts: (i) the sequences $(\mu_{x_n})_{n \in \mathbb{N}}$ cannot exhibit escape of mass (i.e., they cannot accumulate on a measure giving the space total mass strictly less than 1); (ii) there is a natural notion of rationality of Hecke rays and since the (potentially existing) two Hecke rays which do not give rise to equidistribution are never rational, one has that rational rays always give rise to equidistribution.

In the current paper which deals with the positive characteristic analogue of the above discussion, the picture is in sharp contrast. As will be demonstrated, all rational rays exhibit a uniform amount of escape of mass (conjectured to be full), there are uncountably many rays which exhibit escape of mass, and furthermore, there are uncountably many rays for which the periodic measures of the ray accumulate on measures which are singular to the uniform measure.

We now abandon the above notation and present the notation and concepts necessary for stating our results. Let $\mathbb{F}_q$ be a finite field of order a positive power $q$ of a prime $p$, and let $K = \mathbb{F}_q(Y)$ be the field of rational functions in one variable $Y$ over $\mathbb{F}_q$. Let $R_\infty = \mathbb{F}_q[Y]$ be the ring of polynomials in $Y$ over $\mathbb{F}_q$, let $K_\infty = \mathbb{F}_q((Y^{-1}))$ be the field of formal Laurent series in $Y^{-1}$ over $\mathbb{F}_q$ and let $X_\infty = \text{PGL}_2(K_\infty)/\text{PGL}_2(R_\infty)$ be the space of homothety classes of $R_\infty$-lattices in $K_\infty \times K_\infty$ (that is, of rank 2 free $R_\infty$-submodules spanning the vector plane $K_\infty \times K_\infty$ over $K_\infty$). A point $x \in X_\infty$ is called $A_\infty$-periodic if its orbit under the diagonal subgroup $A_\infty$ of $\text{PGL}_2(K_\infty)$ is compact. This orbit $A_\infty \cdot x$ then carries a unique $A_\infty$-invariant probability measure, denoted by $\mu_x$. The aim of this paper is to study the asymptotic behavior of these measures $\mu_x$ (and in particular to prove unexpected escape of mass phenomena) as $x$ varies in arithmetically defined subsets of $A_\infty$-periodic points.

Recall that for every $x_0 \in X_\infty$ and every prime polynomial $v$ in $R_\infty$, the Hecke tree $T_v(x_0)$ with root $x_0$ is the connected component of $x_0$ in the graph with vertex set $X_\infty$, with an edge between the homothety classes of two $R_\infty$-lattices $\Lambda$ and $\Lambda'$ when $\Lambda' \subset \Lambda$ and $\Lambda/\Lambda'$ is isomorphic to $R_\infty/vR_\infty$ as an $R_\infty$-module. The boundary at infinity $\Omega = \Omega_{x_0}$ of $T_v(x_0)$ identifies with the projective line $\mathbb{P}^1(K_v)$ over the completion $K_v$ of $K$ associated with $v$, and a point of $\Omega$ is called rational if it belongs to $\mathbb{P}^1(K)$. (Note that the identification of $\Omega$ with $\mathbb{P}^1(K_v)$ is
not canonical, but the notion of rationality is well defined.) For every \( \xi \in \Omega \), let \((x_n^\xi)_{n \in \mathbb{N}}\) be the vertices along the geodesic ray (called a Hecke ray) in \( T_v(x_0) \) from \( x_0 \) to \( \xi \).

In what follows, we fix an \( A_\infty \)-periodic point \( x_0 \) in \( X_\infty \). Note that the vertices of the Hecke-tree \( T_v(x_0) \) then also have periodic \( A_\infty \)-orbits. Our aim is to understand the possible sets \( \Theta_\xi \) of weak-star accumulation points of the sequences of measures \((\mu_{x_n^\xi})_{n \in \mathbb{N}}\) on \( X_\infty \) associated with the vertices of the Hecke ray with endpoint \( \xi \), when \( \xi \) varies in \( \Omega \). For all \( \xi \in \Omega \) and \( c > 0 \), we say that

- \( \xi \) has \( c \)-escape of mass if there exists \( \theta \in \Theta_\xi \) with \( \theta(X_\infty) \leq 1 - c \);
- \( \xi \) has uniform \( c \)-escape of mass if for every \( \theta \in \Theta_\xi \) we have \( \theta(X_\infty) \leq 1 - c \).

Here is a summary of our results.

**Theorem 1.** There exists \( c > 0 \) such that any rational \( \xi \in \Omega \) has uniform \( c \)-escape of mass.

The following result also exhibits full escape of mass phenomena along Hecke rays.

**Theorem 2.** There exists \((p, v, x_0)\) such that for every rational \( \xi \in \Omega \), the zero measure belongs to \( \Theta_\xi \).

The key approach to these results (proved in Section 4.1) is to use the geodesic flow on the quotient of the Bruhat-Tits tree of \((\text{PGL}_2, K_\infty)\) (see for instance [27] and Section 2.3) by the lattice \( \text{PGL}_2(R_\infty) \).

Theorem 1 proves an escape of mass phenomenon along only countably many Hecke rays. Using the remarkable fact that the above constant \( c \) is independent of the rational Hecke ray, we can strengthen this in the next result (see Section 4.2).

**Theorem 3.** There exists \( c > 0 \) such that the set of \( \xi \in \Omega \) having \( c \)-escape of mass is uncountable.

As guided by the analogy with \( \text{PGL}_2(\mathbb{R})/\text{PGL}_2(\mathbb{Z}) \), we could still wonder if the part of the measure which does not go to infinity still equidistributes in \( X_\infty \), that is, converges to a measure proportional to the probability measure on \( X_\infty \) invariant under \( \text{PGL}_2(K_\infty) \). The next result proves that this is also not always the case.

**Theorem 4.** There exists \( c' > 0 \) such that for every \( A_\infty \)-periodic point \( x \in X_\infty \), there exist \( \xi \in \Omega_x \) and \( \theta \in \Theta_\xi \) such that \( c' \mu_x \leq \theta \). In particular, \( \theta \) is not absolutely continuous with respect to the homogeneous measure on \( X_\infty \).

We give explicit constants \( c, c' \) in the above statements. We will actually prove a stronger result, Theorem 18 in Section 4.4, which mixes the behaviors in Theorems 3 and 4. For this, the main tool (proved in Section 4.3) is an effective equidistribution result of sectors of Hecke spheres in positive characteristic, which we prove using the known exponential decay of matrix coefficients, see
for instance [2]. We refer for instance to the works of Dani-Margulis [9], Clozel-Oh-Ullmo [7], Clozel-Ullmo [8], Eskin-Oh [13], Benoist-Oh [3] for equidistribution results of Hecke spheres in zero characteristic.

As we shall see in the main body of the text, Theorems 1, 3 and 4 are valid upon replacing $K$ by any global function field (see also Section 5 for further extensions). In this more general case, there are several (albeit finitely many) ways to go to infinity in $X_\infty$, and we will give more precise results towards which cusp of $X_\infty$ the escape of mass occurs.

Going back to the comparison between the zero and positive characteristic cases, the underlying phenomenon which changes drastically is as follows. While in zero characteristic the size of the orbit $A_\infty x_\xi^n$ is exponential in $n$, in positive characteristic, it is linear in $n$ due to the presence of the Frobenius automorphism (see Theorem 10). When this is combined with the fact that rational rays diverge in a linear speed, we get the results regarding the escape of mass.

Although the rigidity displayed in zero characteristic completely breaks down, as demonstrated by the above results, we still believe that the following conjecture holds. It implies in particular that the set of rays having uniform escape of mass (such as the rational rays) is a null set.

**Conjecture 5.** For almost any $\xi \in \Omega$ (with respect to the natural probability measure), the averages $\frac{1}{N+1} \sum_{n=0}^N \mu_{x_\xi^n}$ converge to the homogeneous probability measure on $X_\infty$.

Conjecture 5 reflects our belief that the behaviour along rational rays is far from generic. In fact, after some computer experiments, we suggest the following.

**Conjecture 6.** For any rational $\xi \in \Omega$, $\mu_{x_\xi^n}$ converges to the zero measure.

This work raises many other natural questions which we plan on studying in subsequent works. A few examples are: Is the rationality of the ray characterized by a uniform (or full) escape of mass? Can we find irrational rays exhibiting an escape of mass in average? Do we have a criterion for the convergence (or convergence in average) towards (a multiple of) the homogeneous measure? What is the Hausdorff dimension of the set of $\xi$ for which Theorem 3 holds?

Although we are working with the dynamics of rank 1 torus, it is interesting to compare our results with the huge corpus of works in dynamics on noncompact spaces, in particular locally homogeneous ones or moduli spaces, precisely devoted to prove that there is no escape of mass to infinity for nice sequences of probability measures on these spaces. This is in particular the case in homogeneous dynamics—with real Lie groups, thereby in zero characteristic—(see for instance [11, 4]) or in Teichmüller dynamics (see for instance [12, 15]).

Note that an escape of mass for the diagonal group is not a feature appearing only in positive characteristic. Over the reals, there are examples of escape of mass for the diagonal flow: for example, in [24, p. 232] the author arithmetically constructs a sequence of closed geodesics on the modular surface which
Hence vertical lines are included, for the zero measure (see also [28] for similar examples in higher dimensions). We stress, though, that these examples do not share the arithmetic relation between the measures along the sequence which is present in our results. Indeed, due to the results in [1], such an arithmetic relation cannot coexist with an escape of mass over the reals.

As another motivation for studying the limiting behaviour of \(\mu_{x_f}\) (also originating from the analogy with [1]), let us indicate a relation with the distribution properties of the periods of the continued fraction expansion of certain sequences of quadratic irrationals. We refer for instance to the surveys [16, 25] for background.

We denote by \(O_\infty = \mathbb{F}_q[[Y^{-1}]]\) the local ring of \(K_\infty\) (consisting of power series in \(Y^{-1}\) over \(\mathbb{F}_q\)). Any element \(f \in K_\infty\) may be uniquely written \(f = [f] + \{f\}\) with \([f]\) in the polynomial ring \(R_\infty = \mathbb{F}_q[Y]\) and \(\{f\} \in Y^{-1}O_\infty\). The Artin map \(\Psi : Y^{-1}O_\infty \setminus \{0\} \to Y^{-1}O_\infty\) is defined by \(f \mapsto \{\frac{1}{f}\}\). Any \(f \in K_\infty\) irrational (not in \(K = \mathbb{F}_q(Y)\)) has a unique continued fraction expansion

\[
f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},
\]

with \(a_0 = [f] \in R_\infty\) and \(a_n = \left[\frac{1}{\Psi^{-1}(f-a_0)}\right]\) a non constant polynomial, for \(n \geq 1\).

Let \(QI = \{f \in K_\infty : [K(f):K] = 2\}\) be the set of quadratic irrationals over \(K\) in \(K_\infty\). Given an irrational \(f \in Y^{-1}O_\infty\), we have \(f \in QI\) if and only if the continued fraction expansion of \(f\) is eventually periodic. We fix \(f \in QI\). Assume for simplicity that the characteristic \(p\) is different from 2, that \(f \in Y^{-1}O_\infty\) has purely periodic continued fraction expansion and that the Galois conjugate \(f^a\) of \(f\) over \(K\) belongs to \(K_\infty \setminus O_\infty\). Let \(g_f = \left[\begin{array}{cc} f^a & f \\ 1 & 1 \end{array}\right] \in \text{PGL}_2(K_\infty)\) and \(x_f = g_f^{-1}\text{PGL}_2(R_\infty) \in X_\infty\). It is then easy to prove that \(x_f\) is \(A_\infty\)-periodic. Using the main results of [6], we may construct a natural cross-section \(C\) for the action of \(A_\infty\) on (a full-measure subset of) \(X_\infty\) and a natural map from \(C\) onto (a full-measure subset of) \(Y^{-1}O_\infty\), sending the intersection with \(C\) of the \(A_\infty\)-orbit of \(x_f\) in \(X_\infty\) to the orbit of \(f\) under \(\Psi\) in \(Y^{-1}O_\infty\).

Given an irreducible polynomial \(P \in \mathbb{F}_q[Y]\), our results imply strong statements regarding the asymptotics of the periods of the continued fraction expansions of the quadratic irrationals \(P^n f\) as \(n \to +\infty\) (in terms of length of the period and the degrees of the polynomials composing it), with a strikingly different outcome than the ones in [1]. The exact statements and the detailed analysis of this translation are too long to be included here, and we refer to the future note [20]. We only mention here, as indicated by the referee, that our results imply in particular that if \((a_n;i)_{i \in \mathbb{N}}\) is the continued fraction expansion of \(P^n f\) for all \(n \in \mathbb{N}\), then \(\sup_{n,i \in \mathbb{N} \setminus \{0\}} \deg a_n,i = +\infty\), thus recovering a special case of [10, Theorem 4.5].
2. Global function fields and Bruhat-Tits trees

This section introduces the notation and preliminary results used in this paper. We refer the reader to the following commutative diagram for a global view of this notation.

\[
\begin{array}{c}
G_\infty = G(K_\infty) & \xleftarrow{\pi_\infty} & G_S = G_\infty \times G_\nu & \xrightarrow{\text{heg}_g} & G_\nu = G(K_\nu) \\
V_{T,\infty} = G_\infty / G(O_{\infty}) & & X_S = \Gamma_S \setminus G_S & & V_T = G_\nu / G(O_\nu) \\
\Gamma_\infty \setminus G_\infty = \Gamma_S \setminus G_S / G(O_\nu) = X_\infty \cap A_\infty & & & & \Gamma_\nu \setminus G_\infty / G(O_\nu) = \Gamma_\nu \setminus V_{T,\infty}
\end{array}
\]

2.1. Global function fields. We refer for instance to [23, 26] for the content of this section.

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements, where \( q \) is a positive power of a prime \( p \). Let \( K \) be a global function field over \( \mathbb{F}_q \), that is, the function field of a geometrically connected smooth projective curve \( C \) over \( \mathbb{F}_q \), or equivalently an extension of \( \mathbb{F}_q \) of transcendence degree 1, in which \( \mathbb{F}_q \) is algebraically closed. The set \( \mathcal{P} \) of primes of \( K \) is the set of closed points of \( C \), or equivalently the set of discrete valuations of \( K \), trivial on \( \mathbb{F}_q^* \), with value group exactly \( \mathbb{Z} \). We fix an element in \( \mathcal{P} \) that we denote by \( \infty \), and we denote by \( \mathcal{P}_f \) the set \( \mathcal{P} \setminus \{ \infty \} \).

For every \( \omega \in \mathcal{P} \), we denote by \( R_\omega \) the affine algebra of the affine curve \( C \setminus \{ \omega \} \) (which is a Dedekind ring), by \( v_\omega \) the discrete valuation of \( K \) associated with \( \omega \) (with the usual convention that \( v_\omega(0) = +\infty \)), by \( K_\omega \) the associated completion of \( K \) (and again by \( v_\omega \) the extension of \( v_\omega \) to \( K_\omega \)), by \( O_\omega \) its local ring, by \( \pi_\omega \) a uniformizer of \( O_\omega \), by \( k_\omega \) its residual field (that we identify with its canonical lift in \( O_\omega \)), and by \( \deg(\omega) \) the degree of \( k_\omega \) over \( \mathbb{F}_q \). We assume, as we may using for instance the Riemann-Roch theorem, that \( \pi_\nu \) belongs to \( R_\infty \) if \( \nu \in \mathcal{P}_f \). Note that \( R_\infty \subset O_\nu \) if \( \nu \in \mathcal{P}_f \) (since an element in \( R_\infty \) has no pole at the closed point \( \nu \neq \infty \) of \( C \)), and that \( R_\infty [\pi_\nu^{-1}] \cap O_\nu = R_\infty \).

We normalize the absolute value \( |\cdot|_\omega \) associated to \( v_\omega \) by \( |x|_\omega = |k_\omega|^{-v_\omega(x)} = q^{-\deg(\omega) v_\omega(x)} \) for every \( x \in K_\omega \). In particular, the product formula

\[
\forall \ x \in K, \quad \prod_{\omega \in \mathcal{P}} |x|_\omega = 1
\]

holds. Note that \( K_\omega \) is the field \( k_\omega ((\pi_\omega)) \) of Laurent series \( f = \sum_{i \in \mathbb{Z}} f_i (\pi_\omega)^i \) in the variable \( \pi_\omega \) over \( k_\omega \), where \( f_i \in k_\omega \) is zero for \( i \in \mathbb{Z} \) small enough. We have

\[
|f|_\omega = |k_\omega|^{-\sup \{ j \in \mathbb{Z} : \forall i < j, f_i = 0 \}},
\]
and \( O_\omega = k_\omega[[\pi_\omega]] \) is the local ring of power series \( f = \sum_{i \in \mathbb{N}} f_i(\pi_\omega)^i \) (where \( f_i \in k_\omega \)) in the variable \( \pi_\omega \) over \( k_\omega \).

For every finite extension \( \bar{k}_\omega \) of \( k_\omega \), we denote again by \( v_\omega \) the unique extension of \( v_\omega \) to a valuation on \( \bar{k}_\omega \), and by \( e(\bar{k}_\omega, k_\omega) = [v_\omega(\bar{k}_\omega^\times) : v_\omega(k_\omega^\times)] \) its ramification index (see for instance [26, II §2]).

For instance, if \( \mathbb{C} \) is the projective line \( \mathbb{P}^1 \) and if \( \infty = [1 : 0] \) is its usual point at infinity, then \( K = \mathbb{F}_q(Y), \pi_\infty = Y^{-1}, K_\infty = \mathbb{F}_q((Y^{-1})), O_\infty = \mathbb{F}_q[[Y^{-1}]], k_\infty = \mathbb{F}_q, \)
\( R_\infty = \mathbb{F}_q[Y] \) and the uniformizers \( \pi_\nu \) for \( \nu \in \mathcal{P}_f \) may be taken to be the monic prime polynomials in \( R_\infty \), with \( \deg \nu \) the degree of the polynomial \( \pi_\nu \). This is the example considered in the introduction.

2.2. The semi-simple group \( \text{PGL}_2 \). In this section, we give the group-theoretic notation we are going to use in this paper, except in Section 5. Let \( K \) be as in Section 2.1. We fix \( \nu \in \mathcal{P}_f \).

We denote by \( G = \text{PGL}_2 \) the (adjoint semi-simple absolutely simple) projective linear algebraic group over \( K \) in dimension 2. Whenever necessary, we embed \( \text{PGL}_2 \) in \( \text{GL}_3 \) by the adjoint representation on the vector space of traceless 2-by-2 matrices.

Let \( A \) be the diagonal subgroup of \( G \), that is, the algebraic subgroup of \( G \) consisting in the elements represented by diagonal matrices, which is a (split) maximal torus of \( G \) defined over \( K \).

For every \( \omega \in \mathcal{P} \) and every algebraic subgroup \( H \) of \( G \) defined over \( K_\omega \) (for instance if \( H \) is defined over \( K \)), we set \( H_\omega = H(K_\omega) \), which is a non-Archimedian Lie group (and in particular a locally compact group).

We define \( \Gamma_\infty = G(R_\infty) = \text{PGL}_2(R_\infty) \), which is a nonuniform lattice in \( G_\infty = \text{PGL}_2(K_\infty) \). For instance, when \( \mathbb{C} = \mathbb{P}^1 \), the lattice \( \Gamma_\infty \) is called Nagao’s lattice [18] (or Weil’s modular group [31]).

We denote by \( X_\infty \) the totally disconnected locally compact space \( \Gamma_\infty \backslash G_\infty \) (contrarily to the introduction, we consider the left quotient, since it makes the connection with Bruhat-Tits theory easier). The space \( X_\infty \) is non compact, and identifies by \( \Gamma_\infty g \rightarrow g^{-1}[R_\infty \times R_\infty] \) with the space of homothety classes \( [\Lambda] \) under \( K_\infty^\times \) of \( R_\infty \)-lattices \( \Lambda \) in \( K_\infty \times K_\infty \).

Let \( S = [\infty, \nu] \) and let \( \Gamma_S \) be the \( S \)-arithmetic group \( G(R_\infty[\pi_\nu^{-1}]) \), which embeds diagonally in the locally compact group \( G_S = G_\infty \times G_\nu \) as a nonuniform lattice, and let \( X_S = \Gamma_S \backslash G_S \). We identify \( G_\infty \) and \( G_\nu \), hence any subgroup of them, with their images in \( G_S \) by the maps \( x \mapsto (x, e) \) and \( y \mapsto (e, y) \). Note that \( \Gamma_S \cap G(O_\nu) = \Gamma_\infty \) since \( R_\infty[\pi_\nu^{-1}] \cap O_\nu = R_\infty \).

For every \( \omega \in S \), we denote by

- \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) the image in \( G_\omega = \text{PGL}_2(K_\omega) \) of \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(K_\omega) \),

- \( \nu_\omega \) the map from the abelian group \( A_\omega = A(K_\omega) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in K_\omega \right\} \)
  to \( \mathbb{Z} \) defined by \( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mapsto \nu_\omega(d/a) \), which is a group epimorphism with compact-open kernel \( A(O_\omega) = G(O_\omega) \cap A_\omega \).
The set of geodesic lines of $\Gamma$ induces a faithful, transitive left action of $\Gamma$ by the canonical basis of $K$. The pairs \{$(x, x')$\} of $(x, x')$, see for instance [30]. We use its description given in [27].

The tree of $\text{PGL}_2$ trees. Let $(K, v, G, A)$ and the associated notation be as in Section 2.2.

Trees. Let $T$ be a locally finite tree. Its set of vertices $VT$ is endowed with the maximal distance for which two adjacent distinct vertices are at distance 1. A geodesic ray or line in $T$ is an isometric map from $\mathbb{N}$ or $\mathbb{Z}$ to its set of vertices. The set of geodesic lines of $T$, endowed with the compact-open topology, is denoted by $\mathcal{G}T$.

An end of $T$ is an equivalence class of geodesic rays, when two geodesic rays are equivalent if the intersection of their images is the image of a geodesic ray. The set of ends of $T$, endowed with the (compact, totally disconnected) quotient topology of the compact-open topology, is denoted by $\partial T$, and called the boundary at infinity of $T$.

The translation length of an isometry $\gamma$ of $T$ is

$$\ell_T(\gamma) = \min_{x \in VT} d(x, \gamma x).$$

It is invariant under conjugation of $\gamma$ in the isometry group of $T$. We will say that $\gamma$ is loxodromic if $\ell_T(\gamma) > 0$, in which case there exists a unique image of a geodesic line in $T$ on which $\gamma$ translates a distance $\ell_T(\gamma)$, called the translation axis of $\gamma$.

The geodesic flow (with discrete times) $(\phi_m)_{m \in \mathbb{Z}}$ on the tree $T$ is the right action $(\mathcal{G}T \times \mathbb{Z}) \rightarrow \mathcal{G}T$ of $\mathbb{Z}$ on $\mathcal{G}T$ by translations at the source, defined by

$$(\ell, m) \rightarrow \{\phi_m \ell : n \mapsto \ell(n + m)\}$$

for all $m \in \mathbb{Z}$ and $\ell : Z \rightarrow VT$ in $\mathcal{G}T$. Given a group $\Gamma$ of automorphisms of $T$, the geodesic flow on $T$ induces a right action of $\mathbb{Z}$ on $\Gamma \backslash \mathcal{G}T$, also called the geodesic flow of $\Gamma \backslash T$, and again denoted by $(\phi_m)_{m \in \mathbb{Z}}$.

The tree of $\text{PGL}_2$ over local fields. For $\omega \in S = \{\infty, v\}$, let $T_\omega$ be the Bruhat-Tits tree of $(G, K_\omega)$, see for instance [30]. We use its description given in [27].

Recall that an $O_\omega$-lattice $\Lambda$ in the $K_\omega$-vector space $K_\omega \times K_\omega$ is a rank 2 free $O_\omega$-submodule of $K_\omega \times K_\omega$, generating $K_\omega \times K_\omega$ as a vector space. The Bruhat-Tits tree $T_\omega$ is the graph whose set of vertices $VT_\omega$ is the set of homothety classes (under $K_\omega^\times$) $[\Lambda]$ of $O_\omega$-lattices $\Lambda$ in $K_\omega \times K_\omega$, and whose non-oriented edges are the pairs $\{x, x'\}$ of vertices such that there exist representatives $\Lambda$ of $x$ and $\Lambda'$ of $x'$ such that $\Lambda < \Lambda'$ and $\Lambda'/\Lambda$ is isomorphic to $O_\omega/\pi_\omega O_\omega$. This graph is a regular tree of degree $|\mathcal{P}_1(k_\omega)| = |k_\omega| + 1$.

We denote by $\star_\omega$ the homothety class of the $O_\omega$-lattice $O_\omega \times O_\omega$ generated by the canonical basis of $K_\omega \times K_\omega$. The left linear action of $\text{GL}_2(K_\omega)$ on $K_\omega \times K_\omega$ induces a faithful, transitive left action of $G_\omega$ on $VT_\omega$. The stabilizer in $G_\omega$ of
where $A$ will hence identify $G\omega / G(O\omega)$ with $V\mathbb{T}_\omega$ by the map $gG(O\omega) \mapsto g^*\omega$.

We identify as usual the projective line $\mathbb{P}_1(K\omega)$ with $K\omega \cup \{\infty\}$ using the map $(x, y) \mapsto xy^{-1}$. There exists one and only one homeomorphism between the boundary at infinity $\partial \mathbb{T}_\omega$ of $\mathbb{T}_\omega$ and $\mathbb{P}_1(K\omega)$ such that the (continuous) extension to $\partial \mathbb{T}_\omega$ of the isometric action of $G\omega$ on $\mathbb{T}_\omega$ corresponds to the projective action of $G\omega$ on $\mathbb{P}_1(K\omega)$. From now on, we identify $\partial \mathbb{T}_\omega$ and $\mathbb{P}_1(K\omega)$ by this homeomorphism.

The group $G\omega$ hence acts simply transitively on the set of ordered triples of distinct points in $\partial \mathbb{T}_\omega$. In particular, the group $G\omega$ acts transitively on the space $\mathcal{G}\mathbb{T}_\omega$ of geodesic lines in $\mathbb{T}_\omega$. The stabilizer under this action of the geodesic line
\[
\ell_0 : n \mapsto [O\omega \times \pi^m_n O\omega] = a^n_\omega * \omega
\]
is the maximal compact-open subgroup $A(O\omega)$ of the diagonal group $A\omega$. We will hence identify $G\omega / A(O\omega)$ with $\mathcal{G}\mathbb{T}_\omega$ by $gA(O\omega) \mapsto g\ell_0$. Furthermore, the stabilizer in $G\omega$ of the ordered pair of endpoints $(\ell_0(-\infty) = 0, \ell_0(+\infty) = \infty)$ of $\ell_0$ in $\partial \mathbb{T}_\omega = \mathbb{P}_1(K\omega)$ is $A\omega$. Therefore any element $\gamma_0 \in G\omega$ which is loxodromic on $\mathbb{T}_\omega$ is diagonalisable over $K\omega$. Besides, by [27, page 108], the translation length on $\mathbb{T}_\omega$ of $\gamma_0 = \left[ \begin{array}{cc} \lambda_+ & 0 \\ 0 & \lambda_- \end{array} \right]$ is
\[
\ell_{\mathbb{T}_\omega}(\gamma_0) = |\nu_\omega(\lambda_+) - \nu_\omega(\lambda_-)|.
\]
Using the group morphism $\nu_\omega : A\omega \rightarrow \mathbb{Z}$, the action by translations on the right of $A\omega$ on $G\omega / A(O\omega)$ corresponds to the geodesic flow on $\mathcal{G}\mathbb{T}_\omega$: for all $a \in A\omega$ and $\ell \in \mathcal{G}\mathbb{T}_\omega = G\omega / A(O\omega)$, we have
\[
\ell \overset{a}{\circ} \ell = \phi_{\nu_\omega(a)} \ell.
\]
We denote by $\pi'_{\infty} : X_{\infty} = \Gamma_\infty \backslash G_{\infty} \rightarrow \Gamma_\infty \backslash \mathcal{G}\mathbb{T}_\infty = \Gamma_\infty \backslash G_{\infty} / A(O_{\infty})$ the canonical projection (see the diagram at the beginning of Section 2). The previous equation proves that $\pi'_{\infty}$ is equivariant with respect to the morphism $\nu_{\infty} : A_\infty \rightarrow \mathbb{Z}$, where $A_\infty$ acts by translation on the right on $X_{\infty}$ and $\mathbb{Z}$ by the (quotient) geodesic flow on $\Gamma_\infty \backslash \mathcal{G}\mathbb{T}_\infty$: for all $x \in X_{\infty}$ and $a \in A$,
\[
\pi'_{\infty}(x a) = \phi_{\nu_{\infty}(a)} \pi'_{\infty}(x).
\]

The principal bundle $\pi_{\infty} : X_{\mathbb{S}} \rightarrow X_{\infty}$. Since $\Gamma_{\mathbb{S}}$ is irreducible, the group $\Gamma_{\infty} = \Gamma_{\mathbb{S}} \cap G(O_v)$ is dense in the stabiliser $G(O_v)$ of the base point $*_{\mathbb{S}}$ of the Bruhat-Tits tree $\mathbb{T}_{\mathbb{S}}$. This stabilizer $G(O_v)$ acts transitively on the set of geodesic rays in $\mathbb{T}_{\mathbb{S}}$ starting from $*_{\mathbb{S}}$. Thus $\Gamma_{\infty}$ preserves and acts transitively on the sphere in $\mathbb{T}_{\mathbb{S}}$ of any given radius centered at $*_{\mathbb{S}}$. For every $g' \in G_v$, there hence exists $\gamma \in \Gamma_{\infty}$ and $n \in \mathbb{N}$ such that $\gamma^{-1}g' *_{\mathbb{S}} = [O_v \times \pi^m_n O_v] = a^n_o *_{\mathbb{S}}$. Therefore
\[
\Gamma_v = \bigcup_{n \in \mathbb{N}} a^n_o G(O_v).
\]
In particular, $\Gamma_v = \Gamma_{\mathbb{S}} G(O_v)$.

Therefore, every element $x$ of $X_{\mathbb{S}}$ may be written $\Gamma_{\mathbb{S}}(g, g')$ with $g \in G_{\infty}$ and $g' \in G(O_v)$. For all $g, h \in G_{\infty}$ and $g', h' \in G(O_v)$, we have $\Gamma_{\mathbb{S}}(g, g') = \Gamma_{\mathbb{S}}(h, h')$ if
and only if \( gh^{-1} = g'(h')^{-1} \in \Gamma_S \cap G(O_v) = \Gamma_\infty \). Hence the map \( \pi_\infty : X_S \to X_\infty \), where \( \pi_\infty(x) = \Gamma_\infty g \) if \( x = \Gamma_S (g, g') \) with \( g' \in G(O_v) \), is well defined and continuous. The action of \( G(O_v) \) by right translations on the second factor of \( G_\infty = G_\infty \times G_v \) induces an action of \( G(O_v) \) on \( X_\infty = \Gamma_\infty \backslash G_\infty \), which is transitive and free on the fibers of \( \pi_\infty \). Hence \( \pi_\infty : X_S \to X_\infty \) is a principal bundle under the group \( G(O_v) \), which gives an identification between \( X_\infty = \Gamma_\infty \backslash G_\infty \) and \( X_S / G(O_v) = \Gamma_\infty \backslash G_\infty / G(O_v) \) (see the diagram at the beginning of Section 2).

**Ends of the modular graph at the place \( \infty \) and heights.** The quotient graph \( \Gamma_\infty \backslash \mathcal{T}_\infty \) will be called the *modular graph at \( \infty \) of \( K \).* By for instance [27], the *set of cusps* \( \Gamma_\infty \backslash \mathcal{P}_1(K) \) is finite, and \( \Gamma_\infty \backslash \mathcal{T}_\infty \) is the disjoint union of a finite connected subgraph containing \( \Gamma_\infty \ast_\infty \) and of maximal open geodesic rays \( h_z([0, +\infty[) \), for \( z = \Gamma_\infty \tilde{z} \in \Gamma_\infty \backslash \mathcal{P}_1(K) \), where \( h_z \) (called a *cuspidal ray*) is the image by the canonical projection \( \mathcal{T}_\infty \to \Gamma_\infty \backslash \mathcal{T}_\infty \) of a geodesic ray whose point at infinity in \( \mathcal{P}_1(K) \subset \partial \mathcal{T}_\infty \) is equal to \( \tilde{z} \). Conversely, any geodesic ray whose point at infinity lies in \( \mathcal{P}_1(K) \subset \partial \mathcal{T}_\infty \) contains a subray that maps injectively by the canonical projection \( \mathcal{T}_\infty \to \Gamma_\infty \backslash \mathcal{T}_\infty \).

Let us denote by \( \bar{\Gamma_\infty \backslash \mathcal{T}_\infty} = (\Gamma_\infty \backslash \mathcal{T}_\infty) \cup \mathcal{E}_\infty \) Freudenthal's compactification (see [14]) of \( \Gamma_\infty \backslash \mathcal{T}_\infty \) by its finite set of ends \( \mathcal{E}_\infty \). This set of ends is indeed finite, in bijection with \( \Gamma_\infty \backslash \mathcal{P}_1(K) \) by the map which associates to \( z \in \Gamma_\infty \backslash \mathcal{P}_1(K) \) the end towards which the cuspidal ray \( h_z \) converges. See for instance [27] for a geometric interpretation of \( \mathcal{E}_\infty \) in terms of the curve \( C \).

Let \( \bar{X}_\infty = X_\infty \cup \mathcal{E}_\infty \) and let \( \bar{p}_\infty : \bar{X}_\infty \to \Gamma_\infty \backslash \mathcal{T}_\infty \) be the map equal to the identity map on \( \mathcal{E}_\infty \) and to the canonical projection \( p_\infty : X_\infty = \Gamma_\infty \backslash G_\infty \to \Gamma_\infty \backslash V \mathcal{T}_\infty = \Gamma_\infty \backslash G_\infty / G(O_v) \) on \( X_\infty \) (see the diagram at the beginning of Section 2). Since \( p_\infty \) is a proper map, this defines a compactification of \( X_\infty \), by endowing \( \bar{X}_\infty \) with the compact metrisable topology generated by the open subsets of \( U \) and the sets \( \bar{p}_\infty^{-1}(U) \) with \( U \) an open neighborhood of a point in \( \mathcal{E}_\infty \). We will say that \( \mathcal{E}_\infty \) is the *set of cusps* of \( X_\infty \), and we will indicate towards which cusp of \( X_\infty \) the escape of mass occurs.

For every \( x \in X_\infty \), define the *height of \( x \) in \( X_\infty \)* by

\[
ht_\infty(x) = d_{\Gamma_\infty \backslash \mathcal{T}_\infty}(p_\infty(x), \Gamma_\infty \ast_\infty).
\]

For every cusp \( z \in \mathcal{E}_\infty \) of \( X_\infty \), define the *height of \( x \) in \( X_\infty \) relative to the cusp \( z \)* by \( \ht_{\infty, z}(x) = 0 \) if \( p_\infty(x) \) does not belong to \( h_z([0, +\infty[) \), and

\[
ht_{\infty, z}(x) = d_{\Gamma_\infty \backslash \mathcal{T}_\infty}(p_\infty(x), h_z(0)),
\]

otherwise.

**Lemma 7.** For all \( g' \in G_\infty \) and \( x \in X_\infty \), we have

\[
|ht_\infty(x) - ht_\infty(xg')| \leq d_{\mathcal{T}_\infty}(\ast_\infty, g' \ast_\infty),
\]

and \( |ht_{\infty, z}(x) - ht_{\infty, z}(xg')| \leq d_{\mathcal{T}_\infty}(\ast_\infty, g' \ast_\infty) \) for every cusp \( z \in \mathcal{E}_\infty \) of \( X_\infty \).
Proof. Let \( g \in G_\infty \) be such that \( x = \Gamma_\infty g \). We have \( p_\infty(x) = \Gamma_\infty g *_\infty \) and \( p_\infty(xg') = \Gamma_\infty g g' *_\infty \). By the triangle inequality and since the projection map \( T_\infty \) does not increase the distances, we have
\[
|ht_\infty(x) - ht_\infty(xg')| \leq d_{T_\infty}|(\Gamma_\infty g *_\infty, \Gamma_\infty g g' *_\infty)\,
\leq d_{T_\infty}(g *_\infty, g g' *_\infty) = d_{T_\infty}(g *_\infty, g' *_\infty).
\]

The second assertion follows if \( p_\infty(x) \) and \( p_\infty(xg') \) simultaneously belong or do not belong to (the image of) \( h_\infty \). If for instance \( p_\infty(x) \) belongs to \( h_\infty \) and \( p_\infty(xg') \) does not belong to \( h_\infty \), then
\[
d_{T_\infty}(p_\infty(x), h_\infty(0)) \leq d_{T_\infty}(p_\infty(x), p_\infty(xg'))
\]
and the result holds as above. \( \square \)

**Example.** Assume that \( C \) is the projective line over \( F_q \) and that \( \infty \) is its usual point at infinity. Then the (image of the) geodesic ray in \( T_\infty \) starting from \(*_\infty \) with point at infinity \( \infty \in \mathbb{P}_1(K_\infty) \), which is
\[
n \in \mathbb{N} \mapsto [O_\infty \times \pi^\infty O_\infty] = a^n_\infty *_\infty \in V T_\infty,
\]
is a (weak) fundamental domain for the action of \( \Gamma_\infty \) on \( V T_\infty \): it injects onto \( \Gamma_\infty \setminus V T_\infty \) by the canonical map \( T_\infty \rightarrow \Gamma_\infty \setminus T_\infty \).

Hence \( G_\infty = \bigsqcup_{n \in \mathbb{N}} \Gamma_\infty a^n_\infty G(O_\infty) \). For every \( g \in G_\infty \), the height of \( x = \Gamma_\infty g \) is the unique \( n \in \mathbb{N} \) such that \( g \in \Gamma_\infty a^n_\infty G(O_\infty) \). Note that if one writes \( g \) in the Cartan decomposition of \( G_\infty \) as \( g \in G(O_\infty) a^n_\infty G(O_\infty) \) for some \( m \in \mathbb{N} \), then \( m = d_{T_\infty}(*_\infty, g *_\infty) \geq ht_\infty(x) \), with usually strict inequality.

The quotient graph of finite groups \( \Gamma_\infty \setminus T_\infty \), whose underlying graph is the geodesic ray \( \Gamma_\infty \setminus T_\infty \), is called the modular ray. With \( F_0 = G(k_\infty) \), \( F_0' = F_0 \cap F_1 \) and \( F_n = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_\infty : \nu_\infty(b) \geq -n \} \), the modular ray \( \Gamma_\infty \setminus T_\infty \) (which has only one end) is given by Figure 1.

![Figure 1. The modular ray PGL2(k∞[Y])\setminus T_\infty](image)

The full-down property in the modular graph (see for instance [27, 17]). If \( \rho \) is a geodesic ray in \( T_\infty \) whose image is a cuspidal ray in \( \Gamma_\infty \setminus T_\infty \), the stabilizers of the vertices of \( \rho \) different from the origin of \( \rho \) are strictly increasing along the ray. Hence the image in \( \Gamma_\infty \setminus T_\infty \) of a geodesic ray in \( T_\infty \) satisfies the following full-down property: if it starts to go down along the image of a cuspidal ray \( h_\infty \) for some \( z \in \mathcal{C}_\infty \), then it needs to go all the way down to \( h_\infty(0) \).

As explained in [27, 19], this full-down property has the following consequence: the image by the canonical map \( T_\infty \rightarrow \Gamma_\infty \setminus T_\infty \) of a geodesic ray \( \rho \)
in $\mathbb{T}_\infty$ starting from $*_{\infty}$ either is an infinite sequence $a_0b_0a_1b_1a_2b_2\ldots$ of concatenations of paths $a_i$ (possibly reduced to points) in the finite graph $\Gamma_\infty \setminus \mathbb{T}_\infty = \bigcup_{z \in \varepsilon_{\infty}} h_z[[0, +\infty[)$ and back and forth paths $b_i$ (of even lengths at least 2) from the origin $h_z(0)$ of the cuspidal ray $h_z$ to itself inside this ray, if $\rho$ ends in an irrational point at infinity (that is, in $\mathbb{P}_1(K_\infty) - \mathbb{P}_1(K)$), or starts by such a finite sequence and then follows some cuspidal ray to infinity, otherwise.

![Figure 2. Back and forth paths in cuspidal rays](image)

2.4. $A_\infty$-periodic orbits in $X_\infty$. Let $(K, \nu, G, A)$ and the associated notation be as in Section 2.2.

Let us give a description of the compact orbits for the action by translations on the right of the subgroup $A_\infty$ on $X_\infty = \Gamma_\infty \setminus G_\infty$.

**Proposition 8.** For every $g \in G_\infty$, the following assertions are equivalent, where $x = \Gamma_\infty g \in X_\infty$:

1. there exists a unique $A_\infty$-invariant probability measure on the orbit $xA_\infty$;
2. the subgroup $A_\infty \cap g^{-1}\Gamma_\infty g$ is a (uniform) lattice in $A_\infty$;
3. the orbit of $\pi'_\infty(x)$ under the geodesic flow $(\phi_n)_{n \in \mathbb{Z}}$ on $\Gamma_\infty \setminus \mathbb{G} \mathbb{T}_\infty$ is periodic;
4. there exists $\gamma_0 \in \Gamma_\infty$ and $t_0 \in K_\infty^*$ with $\nu_\infty(t_0)$ positive and minimal such that $\gamma_0 g = g \alpha_\infty(t_0)$.

If one of these conditions is satisfied, we say that $x$ is $A_\infty$-periodic, and the unique $A_\infty$-invariant probability measure on $xA_\infty$ is denoted by $\mu_x$.

The elements $\gamma_0 \in \Gamma_\infty$ and $t_0 \in K_\infty^*$ are said to be associated with $g$. Note that they depend on the choice of the representative $g$ of $x$: if $\gamma_0$ is associated with $g$, then $\gamma^{-1}\gamma_0\gamma$ is associated with $\gamma g$ for every $\gamma \in \Gamma_\infty$. Furthermore, $\gamma_0$ is primitive (not a proper power of an element of $\Gamma_\infty$) and loxodromic on $\mathbb{T}_\infty$. The period of $\pi'_\infty(x)$ under the geodesic flow $(\phi_n)_{n \in \mathbb{Z}}$ is the translation length of $\gamma_0$ on $\mathbb{T}_\infty$, which is equal to $\nu_\infty(t_0)$, and depends only on $x$.

**Proof.** The equivalence of (1) and (2) is well-known.

The equivalence of (2) and (3) follows from the equivariance of the canonical projection $\pi'_\infty : X_\infty = \Gamma_\infty \setminus G_\infty \to \Gamma_\infty \setminus \mathbb{G} \mathbb{T}_\infty = \Gamma_\infty \setminus G_\infty \setminus A(O_\infty)$ with respect to the morphism $\nu_\infty : A_\infty \to \mathbb{Z}$ (see equation (2)).

The image $\Gamma_\infty \ell = \Gamma_\infty g A(O_\infty)$ in $\Gamma_\infty \setminus \mathbb{G} \mathbb{T}_\infty$ of the geodesic line $\ell = g A(O_\infty) \in \mathbb{G} \mathbb{T}_\infty = G_\infty \setminus A(O_\infty)$ is periodic under the geodesic flow if and only if there exist $n > 0$ and $\gamma_0 \in \Gamma_\infty$ such that $\gamma_0 \ell = \phi^n(\ell) = \ell \alpha_\infty(\pi'_\infty)$, hence, since $\alpha_\infty : O_\infty^* \to A(O_\infty)$ is an isomorphism, if and only if there exist $n > 0$, $u_0 \in O_\infty$ and $\gamma_0 \in \Gamma_\infty$ such that $\gamma_0 g = g \alpha_\infty(\pi'_\infty)\alpha_\infty(\nu_\infty(t_0))$. With $t_0 = \pi'_\infty u_0$ so that $\nu_\infty(t_0) = n > 0$, this proves the equivalence of (3) and (4).
Let us now prove the additional properties of \((\gamma_0, t_0)\) and discuss its uniqueness. Assume that \(n\) in the above proof is minimal. Then \(\gamma_0\) is primitive and loxodromic, with translation axis the image of \(\ell\), translation length \(n\), which is the period of \(\Gamma_\infty \ell\) under the geodesic flow. Assume that \((\gamma_0', t_0') \in \Gamma_\infty \times K_\infty^\times\) satisfies \(\gamma_0' \ell = g \gamma_0 \alpha_\infty(t_0')\) with \(n' = \nu_\infty(t_0')\) positive and minimal. Then \(n' = n\) and \(\gamma_0' \ell = \phi_n(\ell)\). Hence \(\gamma_0^{-1} \gamma_0'\) belongs to the pointwise stabilizer in \(\Gamma_\infty\) of the image of \(\ell\), which is the finite group \(g A(O_\infty)g^{-1} \cap \Gamma_\infty\). Therefore, there exists \(u_0' \in \alpha_\infty^{-1}(g^{-1} \Gamma_\infty g \cap A(O_\infty)) \subseteq O_\infty^\times\) such that \(\gamma_0' = \gamma_0 g \alpha_\infty(u_0') g^{-1}\) and \(t_0' = t_0 u_0'\).

2.5. **Hecke trees.** Let \((K, \nu, G, A)\) and the associated notation be as in Section 2.2.

The set \(X_\infty\) of homothety classes of \(R_\infty\)-lattices in \(K_\infty \times K_\infty\) is the set of vertices of a graph, whose non-oriented edges are the pairs \(\{x, x'\}\) of vertices such that there exist representatives \(\Lambda\) of \(x\) and \(\Lambda'\) of \(x'\) such that \(\Lambda \subseteq \Lambda' \cap \Lambda\) is isomorphic to \(R_\infty/\pi\gamma R_\infty\). The action of \(G_\infty\) on \(X_\infty\) extends to an (isometric) action by graph automorphisms on this graph.

For every \(x \in X_\infty\), the connected component of the vertex \(x\) in this graph is a \((|k_v| + 1)\)-regular tree, called the \((\nu)\)-Hecke tree of \(x\), and denoted by \(T_\nu(x)\). We have \(T_\nu(x) g = T_\nu(x g)\) for all \(x \in X_\infty\) and \(g \in G_\infty\). A \((\nu)\)-Hecke ray from \(x\) is a geodesic ray in the Hecke tree \(T_\nu(x)\) starting from \(x\).

The following description of the \(\nu\)-Hecke trees in \(X_\infty\) is well known, and is given, besides in order to fix the notation, only for the sake of completeness.

**Lemma 9.** Let \(g \in G_\infty\) and \(x = \Gamma_\infty g\) its image in \(X_\infty\). The map from \(G_\nu\) to \(X_\infty\) defined by \(g' \mapsto \pi_\infty(\Gamma_S(g, g'))\) induces an isometric map \(\text{hec}_g\) from the vertex set \(V T_\nu = G_\nu \backslash G(O_\nu)\) of the Bruhat-Tits tree \(T_\nu\) onto the vertex set \(V T_\nu(x)\) of the Hecke tree \(T_\nu(x)\), sending \(*_\nu\) to \(x\). For every \(\gamma_0 \in \Gamma_\infty,\) the map \(\text{hec}_g\) conjugates the action of \(\gamma_0\) on \(T_\nu\) to the right action of \(g^{-1} \gamma_0 g \in \Gamma_\infty\) on \(V T_\nu(x)\): for every \(y \in V T_\nu,\) we have

\[
\text{hec}_g(\gamma_0 y) = \text{hec}_g(y) g^{-1} \gamma_0 g. 
\]

For all \(h \in G_\infty\) such that \(\Gamma_\infty h = x\), we have \(\text{hec}_g = \text{hec}_h\) if and only if \(g = h\); furthermore, the following diagram commutes:

\[
\begin{array}{ccc}
V T_\nu & \overset{g h^{-1}}{\longrightarrow} & V T_\nu \\
\text{hec}_g \downarrow & & \downarrow \text{hec}_g \\
V T_\nu(x) & & \\
\end{array}
\]

(6)

Note that \(\text{hec}_g\) depends on \(g\) and not only on \(x\). We will denote again by \(\text{hec}_g\) the (continuous) extension \(\partial T_\nu \rightarrow \partial T_\nu(x)\) of \(\text{hec}_g\) to the boundaries at infinity of the Bruhat-Tits and Hecke trees.

**Proof.** Since the action by translations on the right of \(G(O_\nu)\) on \(X_S\) preserves the fibers of the bundle map \(\pi_\infty: X_S \rightarrow X_\infty\), the map \(g' \mapsto \pi_\infty(\Gamma_S(g, g'))\) does induce a map \(\text{hec}_g: V T_\nu \rightarrow G_\nu \backslash G(O_\nu) \rightarrow X_\infty\).

By definition of the Hecke tree \(T_\nu(x)\) of \(x = \Gamma_\infty g = g^{-1} [R_\infty \times R_\infty]\), its vertices are the points \(g^{-1} \gamma [R_\infty \times \pi^n R_\infty]\) where \(\gamma \in \Gamma_\infty\) and \(n \in \mathbb{N}\). By equation (3), any element in \(G_\nu\) may be written \(\gamma a^n h\) for some \(\gamma \in \Gamma_\infty\), \(n \in \mathbb{N}\) and \(h' \in \mathbb{N}\).


\[ G(O_\nu) \]. Hence, the elements in \( \text{hec}_g(VT_\nu) \) are the points \( \pi_\infty(G,\gamma a^n g') = \Gamma_\infty a^n \gamma^{-1} g \) where \( g' \in G(O_\nu), \gamma \in \Gamma_\infty \) and \( n \in \mathbb{N} \). Therefore \( \text{hec}_g(VT_\nu) = VT_\nu(x) \).

If \( y, y' \in VT_\nu \) are joined by an edge in \( T_\nu \), then again by density of \( \Gamma_\infty \) in \( G(O_\nu) \), there exists an element in \( \Gamma_\infty \) mapping the edge between \( y \) and \( y' \) into the geodesic ray with vertices \( (a^n \gamma)_{n \in \mathbb{N}} \). Up to exchanging \( y \) and \( y' \), there exists \( n_1 \in \mathbb{N} \) and \( y_1 \in \Gamma_\infty \) such that \( \gamma^{-1} y = a^{n_1}_1 \gamma \) and \( \gamma^{-1} y' = a^{n_1+1}_1 \gamma \). In particular, \( \text{hec}_g(y) = \Gamma_\infty a^{-n_1}_1 \gamma^{-1} g \) is joined by an edge to \( \text{hec}_g(y') = \Gamma_\infty a^{-n_1-1}_1 \gamma^{-1} g \) in the Hecke tree \( T_\nu(x) \). Hence \( \text{hec}_g \) induces a surjective graph morphism between the trees \( T_\nu \) and \( T_\nu(x) \). Since both trees are regular of degree \( |k| + 1 \), the map \( \text{hec}_g \) is an isomorphism of trees.

Equation (5) follows by writing \( y \in VT_\nu = G_\nu / G(O_\nu) \) as \( y = g'G(O_\nu) \) for some \( g' \in \Gamma_S \) (see the line following equation (3)), and by using the following equalities:

\[
\pi_\infty(G(g, g')) g^{-1} y_0^{-1} g = \pi_\infty(G(g^{-1} g, e)) g^{-1} y_0^{-1} g = \Gamma_\infty (g^{-1} g) g^{-1} y_0^{-1} g = \pi_\infty(G(g, y_0 g')).
\]

Let \( h \) be another element in \( G_\infty \) such that \( \Gamma_\infty h = x \). Since \( G_\nu = \Gamma_S G(O_\nu) \) and by the definition of \( \pi_\infty \), we have \( \text{hec}_g = \text{hec}_h \) if and only if \( \Gamma_\infty y^{-1} g = \Gamma_\infty y^{-1} h \) for every \( y \in \Gamma_S \), that is \( y^{-1} (g h^{-1}) y \in \Gamma_\infty \) for every \( y \in \Gamma_S \). Writing \( g h^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and using \( y = e \), we may take \( a, b, c, d \in R_\infty \). Since the order of vanishing at a point \( \alpha \in (0, \infty) \) of the element \( \pi_\nu \) of \( R_\infty \) is nonnegative and \( v_\nu(\pi_\nu) = 1 \), we have \( v_\nu(\pi_\nu) \neq 0 \) by the product formula. Taking \( \gamma = \begin{bmatrix} \pi_\nu^n & 0 \\ 0 & 1 \end{bmatrix} \) gives \( \pi^n \nu c, \pi^n \nu b \in R_\infty \) for every \( n \in \mathbb{Z} \), that is \( c = b = 0 \). Taking \( \gamma = \begin{bmatrix} 1 & \pi_\nu^n \\ 0 & 1 \end{bmatrix} \) gives \( \pi^n \nu (a - d) \in R_\infty \) for every \( n \in \mathbb{Z} \), that is \( a = d \). Hence \( \text{hec}_g = \text{hec}_h \) if and only if \( g h^{-1} \) is the identity element in \( \Gamma_\infty = G(R_\infty) \).

The other claims are left to the reader. \( \square \)

3. Dynamics of the Modular Group at the Infinite Place on the Bruhat-Tits Tree at a Finite Place

Let \((K, \nu, G, \mathcal{A})\) and the associated notation be as in Section 2.2.

In this section, we study the dynamics of \( \Gamma_\infty \) on the Bruhat-Tits tree \( T_\nu \) of \((G, K_\nu)\). Since \( R_\infty \subset O_\nu \), the lattice \( \Gamma_\infty = \overline{G} \) is contained in the stabilizer \( G(O_\nu) \) in \( G_\nu \) of the base point \( *_\nu \) in \( T_\nu \). Hence \( \Gamma_\infty \) does act on \( T_\nu \), and for every \( n \in \mathbb{N} \), every \( y_0 \in \Gamma_\infty \) preserves the sphere

\[ S_\nu(n) = S_{T_\nu}(*_\nu, n) \]

of center \( *_\nu \) and radius \( n \) in \( T_\nu \). Since \( S_\nu(n) \) is finite, every orbit in \( S_\nu(n) \) of the cyclic group \( \gamma_0 \mathbb{Z} \) generated by \( \gamma_0 \) is periodic. The following linear growth property of these periodic orbits is a remarkable feature of the positive characteristic.
Theorem 10. Let $y_0$ be an element in $\Gamma_\infty$ which is loxodromic on $\mathbb{T}_\infty$. Let $\bar{K}_\nu = \bar{K}_\nu(\gamma_0)$ be the splitting field of $\gamma_0$ over $K_n$, with local ring $\bar{O}_\nu$, uniformizer $\bar{\pi}_\nu$ and residual field $\bar{k}_\nu$. Let $e_\nu = e_\nu(\gamma_0)$ be the ramification index $e(\bar{K}_\nu, K_\nu)$ of $\bar{K}_\nu$ over $K_\nu$. Let $d_\nu = d_\nu(\gamma_0)$ be the smallest positive integer such that the image of $\gamma_0^{d_\nu}$ in $G(\bar{k}_\nu)$ (by reduction modulo $\bar{\pi}_\nu \bar{O}_\nu$) is the identity. Let $r_\nu = r_\nu(\gamma_0)$ be the biggest positive integer such that the image of $\gamma_0^{r_\nu}$ in $G(\bar{O}_\nu/\bar{\pi}_\nu^{r_\nu+1} \bar{O}_\nu)$ is not the identity. Then there exists a constant $\kappa_\nu = \kappa_\nu(\gamma_0) \in \mathbb{N}$ such that for every big enough $n \in \mathbb{N}$, the maximal cardinality $m_n = m_n(\gamma_0)$ of an orbit of $\gamma_0^Z$ in $S_\nu(n)$ satisfies

$$m_n \leq d_\nu p^\lceil \log_p \frac{\nu_n^\infty}{r_\nu} \rceil.$$

This result implies that the sequence $(m_n)_{n \in \mathbb{N}}$ has linear growth: for every $n \in \mathbb{N}$ big enough, we have

$$(7) \quad m_n \leq \frac{d_\nu p}{r_\nu} (e_\nu n + \kappa_\nu),$$

and that if $y_0$ is diagonalisable over $K_\nu$, then for every $k \in \mathbb{N}$ big enough

$$m_{r_\nu p^k - \kappa_\nu} \leq d_\nu p^k.$$

Proof. We start the proof by the following lemma on the growth of the valuations of the powers of the elements of $O_\nu$ with their constant terms removed, which concentrates the positive characteristic feature.

Lemma 11. Let $a \in k_\nu^\times$, $\lambda \in a + \pi_\nu O_\nu$ and $n \in \mathbb{N}$. Define $m_n(\lambda) = \min\{k \in \mathbb{N} - \{0\} : \lambda^k \in a^k + \pi_\nu^n O_\nu\}$ and $r_\lambda = \nu_\nu(\lambda - a) > 0$. Then for every $n > r_\lambda$,

$$m_n(\lambda) = p^\lceil \log_p \frac{n}{r_\lambda} \rceil.$$

In particular, $m_n(\lambda) < \frac{n}{r_\lambda} n$ for every $n > r_\lambda$ and $m_{r_\lambda p^k}(\lambda) = p^k$ for every $k \in \mathbb{N} - \{0\}$.

Proof. Up to replacing $\lambda$ by $\frac{1}{\lambda}$, we may assume that $a = 1$. To simplify the notation, let $r = r_\lambda$. For every $k \in \mathbb{N} - \{0\}$, consider the expansion of $k$ in base $p$ given by $k = \sum_{i=0}^s a_ip^i$ where $s \in \mathbb{N}$ and $a_i \in \{0, \ldots, p-1\}$. Let

$$\nu_p(k) = \inf\{i \in \mathbb{N} : \forall j < i, \ a_j = 0\}$$

be the $p$-adic valuation of $k$. Then, using the Frobenius automorphism, and the fact that $a_i$ is invertible in the characteristic subfield $\mathbb{F}_p$, hence in $O_\nu$, if and only if $a_i$ is non zero, we have

$$(1 + \pi_\nu^r O_\nu^\times)^k \subset \bigcap_{i=0}^s (1 + \pi_\nu^{rp^i} O_\nu^\times)^{a_i} \subset \bigcap_{0 \leq i \leq s, a_i \neq 0} (1 + \pi_\nu^{rp^i} O_\nu^\times) \subset 1 + \pi_\nu^{rp_p(k)} O_\nu^\times.$$

Hence for every $n \in \mathbb{N}$, we have $\lambda^k \in 1 + \pi_\nu^n O_\nu$ if and only if $rp_p(k) \geq n$. Therefore, for every $n > r$, if $rp_p^{m-1} < n \leq rp_p^m$ (that is, if $m = \lceil \log_p \frac{n}{r} \rceil$), we have the equalities $m_n(\lambda) = \min\{k \in \mathbb{N} - \{0\} : \nu_p(k) = m\} = p^m$. The result follows. \qed
Now, let $\gamma_0 \in \Gamma_\infty$ be loxodromic on $\mathbb{R}_\infty$. Note that the constant $d_v$ is well defined since $R_\infty \subset O_v \subset \mathcal{O}_v$. As we have seen in Section 2.3, there exist $\lambda_\pm$ in a finite extension of $K$ such that the element $\gamma_0$ is conjugated to

$$
\begin{pmatrix}
\lambda_+ & 0 \\
0 & \lambda_-
\end{pmatrix}
$$

and $\tilde{K}_v = K_v(\tfrac{1}{\lambda_+})$. Note that $\lambda_-$ and $\lambda_+$ are distinct since $\gamma_0$ is not the identity element.

Let $\tilde{T}_v$ be the Bruhat-Tits tree of $(G, \tilde{K}_v)$, and $\tilde{e}_v = e G(\tilde{O}_v)$ its standard base point in $V \tilde{T}_v = G(\tilde{K}_v)/G(\tilde{O}_v)$. The value group of (the unique extension of) the valuation $v_\gamma$ on $\tilde{K}_v^*$ contains the value group $\mathbb{Z}$ of the valuation $v_\nu$ on $K_v^*$ with index $e_\nu$. By the correspondence between the action on the right of $\mathcal{A}(\tilde{K}_\omega)$ on $G(\tilde{K}_\omega)/\mathcal{A}(\tilde{O}_\omega)$ and the action of the geodesic flow on the geodesic lines in $\mathbb{T}_v$, the sphere $S_v(n)$ of center $\ast_v$ and radius $n$ in $\mathbb{T}_v$ is naturally contained in the sphere $S_{\tilde{e}_v}(\tilde{e}_v, e_\nu n)$ of center $\tilde{e}_v$ and radius $e_\nu n$ in $\tilde{T}_v$, for every $n \in \mathbb{N}$. Therefore, up to replacing $K_v$ by $\tilde{K}_v$, we may assume that $\gamma_0$ is diagonalisable over $K_v$, and we prove that the cardinality of every orbit of $\gamma_0^*Z$ in $S_v(n)$ is at most $d_v \nu_\gamma \log \frac{\nu_\gamma n^{k_v}}{r_v}$ for every $n \in \mathbb{N}$, for some $k_v \in \mathbb{N}$.

Note that the coefficients $\lambda_\pm$ have absolute value 1 in $K_v$. Indeed, $\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_-
\end{pmatrix}$ may be chosen to be conjugated to a representative of $\gamma_0$ in $GL_2(R_\infty)$. Hence $\lambda_\pm$ satisfy an equation $P(\lambda_\pm) = 0$ with $P$ a monic quadratic polynomial with coefficients in $R_\infty \subset O_v$. Therefore $|\lambda_\pm|v^2 \leq \max(|\lambda_\pm|_v, 1)$, so that $|\lambda_\pm|_v \leq 1$, and equality holds by replacing $\gamma_0$ by its inverse. Hence $\lambda_\pm \in a_\pm + \pi_v O_v$ with $a_\pm \in k_v^*$.

By the finiteness of $k_v^*$, there exists a smallest $d_v \in \mathbb{N} - \{0\}$ such that $a_\pm d_v = a_\pm^{d_v}$. Note that $d_v$ coincides with the notation introduced in the statement of Theorem 10. Let

$$
(8) \quad r_v = \nu_\gamma \left( \frac{\lambda_+}{\lambda_-} \right)^{a_\pm} - 1.
$$

Since $\gamma_0$ is loxodromic on $\mathbb{T}_\infty$, no power of $\gamma_0$ is the identity, hence $r_v > 0$. Note that $r_v$ coincides with the notation introduced in the statement of Theorem 10. Up to replacing $\gamma_0$ by $\gamma_0^d_v$, to modify $\lambda_\pm$ by a common multiple by an element of $k_v^*$, and to proving that $m_\nu(\gamma_0) \leq \nu_\gamma \log \frac{\nu_\gamma n^{k_v}}{r_v}$ for some $k_v \in \mathbb{N}$ and for $n$ big enough, we may assume that the constant terms in $k_v^*$ of $\lambda_\pm$ are equal to 1, so that $d_v = 1$.

Since $\gamma_0$ is diagonalisable over $K_v$, there exists $h \in G_v$ such that

$$
\gamma_0 = h \begin{pmatrix}
\lambda_- & 0 \\
0 & \lambda_+
\end{pmatrix} h^{-1}.
$$

Since $\lambda_- \neq \lambda_+$, the centralizer $Z_{G_v}(\gamma_0)$ of $\gamma_0$ in $G_v$ is the abelian group $h A_v h^{-1}$. Note that $h$ is well defined modulo multiplication on the right by an element of $A_v$.

Let $\ell_0 : n \mapsto a^n_\nu \ast_v$ be the geodesic line in $\mathbb{T}_v$ from $0 \in \mathbb{D}_v$ to $\infty \in \mathbb{D}_v$, through $\ast_v$ at time $n = 0$, which is pointwise fixed by $\mathcal{A}(O_v)$. The group $A_v$ preserves $\ell_0(Z)$ and acts transitively on it. Note that the projective action of $\mathcal{A}(O_v)$ on $\mathbb{P}(K_v)$ fixes 0 and $\infty$, and acts transitively on $\pi^{-k} O_v \subset \mathbb{P}(K_v)$ for every $k \in \mathbb{Z}$.
The geodesic line $\ell = h \ell_0$ is pointwise fixed by $h A(O_\nu) h^{-1}$. Up to multiplying $h$ on the right by an element of $A_\nu$, we may assume that the closest point to $*_\nu$ on (the image of) $\ell$ is $h *_\nu = \ell(0)$. Let $s_\nu = s_\nu(\gamma_0) \in \mathbb{N}$ be the distance between $*_\nu$ and $h *_\nu$ in $T_\nu$ (see Figure 3).

Let $\text{pr}_\ell : VT_\nu \to \ell(\mathbb{Z})$ be the closest point map on the geodesic line $\ell$. For all $n \in \mathbb{N}$ and $k \in \mathbb{N} - \{0\}$, define (see Figure 3)

$$E_{n,k} = \{x \in S_\nu(n) : \text{pr}_\ell(x) = \ell(k)\}$$

and

$$E_{n,0} = \{x \in S_\nu(n) : \text{pr}_\ell(x) = \ell(0), \ [h *_\nu*_\nu] \cap [h *_\nu, x] = [h *_\nu]\}.$$  

For all $n, k' \in \mathbb{N}$ with $0 \leq k' < s_\nu$, let $E'_{n,k'}$ be the set of $x \in S_\nu(n)$ such that the length of the common segment $[*_\nu, h *_\nu] \cap [*_\nu, x]$ is equal to $k'$. Then we have a partition

$$S_\nu(n) = \bigcup_{0 \leq k' < s_\nu} E'_{n,k'} \cup \bigcup_{-s_\nu \leq k \leq n} E_{n,k}.$$  

Since $\gamma_0$ fixes $*_\nu$, $h(0)$ and $h(\infty)$, it pointwise fixes $\ell(\mathbb{Z}) \cup [*_\nu, h *_\nu]$. Hence the above partition of $S_\nu(n)$ is invariant under $\gamma_0$.

Note that $E_{n,k}$ is contained in the set of points at distance $n - |k| - s_\nu$ from $h a_\nu^k *_\nu = \ell(k)$ on a geodesic ray from $h a_\nu^k *_\nu$ to a point in $h(\pi_\nu^k O_\nu) \subset \mathbb{P}^1(K_\nu)$. Hence for any two points in $E_{n,k}$ (with $n, k$ fixed), there exists an element in the centralizer of $\gamma_0$ mapping one to the other. In particular, the cardinality $c_{n,k} = \text{Card}(\gamma_0^Z y)$ is independent of $y \in E_{n,k}$.

Since $h a_\nu^{k-1} h^{-1}$ centralizes $\gamma_0$ and $h a_\nu^{-k+1} h^{-1} E_{n,k} \subset E_{n-|k|+1,1}$, we have $c_{n,k} = c_{n-|k|+1,1}$. For every $n' \in \mathbb{N}$, we have $c_{n',1} \leq c_{n'+1,1}$, since the closest point map $E_{n'+1,1} \to E_{n',1}$ is onto and equivariant under $\gamma_0$.
Every point of $E_{n',k}'$ is at distance $n+s_v-2k'$ from $h*y$. Hence $h\alpha_s h^{-1}(E_{n',k}') \subset E_{n+2s_v-2k'+1,1}$. Therefore $c'_{n,k'} = \text{Card}(\gamma_0^n y)$ is independent of $y \in E_{n',k}'$ and satisfies $c'_{n,k'} = c_{n+2s_v-2k'+1,1}$.

In particular, for every $n > s_v$, we have

$$m_n(\gamma_0) = \max \left\{ \max_{0 \leq k' < s_v} c'_{n,k'}, \max |c_{n,k}| \right\} = c_{n+2s_v-2k'+1,1}.$$

Note that $h(0)$ and $h(\infty)$ do not belong to $\mathbb{P}_1(K)$, since $\gamma_0$, being loxodromic on $\mathbb{T}_\infty$, fixes no point of $\mathbb{P}_1(K)$. The positive subray of $\ell_0$ hence has no subray whose image is entirely contained in the image of $\ell$. Therefore $\ell_0(\mathbb{N}) \cap \ell(\mathbb{Z})$ is either empty or the set of vertices of a compact interval $[\ell(0), \ell(k_0)]$ for some $k_0 \in \mathbb{Z}$.

Assume first that $\ell_0(\mathbb{N}) \cap \ell(\mathbb{Z})$ is empty. Then the segment $[\gamma_v, h*\gamma_v] \cap [\gamma_v, \infty]$ has length $k_0' \in [0, s_v] \cap \mathbb{N}$. Define $\kappa = 2k_0' \in \mathbb{N}$. Since the point $a_{\gamma_v}^{\kappa} \gamma_v$ belongs to $E_{n',k}_0'$ for every $n' \geq k_0'$, the number $m_n(\gamma_0) = c_{n+2s_v-1,1} = c_{n+2k_0'}$ is the cardinality of the orbit under $\gamma_0^n$ of $\alpha^{\kappa}_{\gamma_v} \gamma_v = [O_v \times \pi_v^{n+k} O_v]$, if $n$ is big enough.

Assume now that $\ell_0(\mathbb{N}) \cap \ell(\mathbb{Z}) = \{\ell(0), \ell(k_0)\} \cap V T_v$ for some $k_0 \in \mathbb{Z}$ (see Figure 3). Define $\kappa = |k_0| + 2s_v \in \mathbb{N}$. Since the point $a_{\gamma_v}^{\kappa} \gamma_v$ belongs to $E_{n',k_0}$ for every $n' \geq |k_0| + s_v$, the number $m_n(\gamma_0) = c_{n+2s_v,1} = c_{n+|k_0|+2s_v, k_0}$ is the cardinality of the orbit under $\gamma_0^n$ of $\alpha^{\kappa}_{\gamma_v} \gamma_v = [O_v \times \pi_v^{n+k} O_v]$, if $n$ is big enough.

For all $n \in \mathbb{N}$, an element of $GL_2(O_v)$ fixes $[O_v \times \pi_v^n O_v]$ if and only if its $(2,1)$-coefficient vanishes modulo $\pi_v^n$; that is, if it belongs to the Hecke congruence subgroup of $GL_2(O_v)$ modulo $\pi_v^n$. Let $\Gamma_\infty(\pi_v^n)$ be the kernel of the morphism $\Gamma_\infty \to G(R_\infty/\pi_v^n R_\infty)$ of reduction modulo $\pi_v^n$. Thus for every $k \in \mathbb{N}$, if $\gamma_k^n$ belongs to $\Gamma_\infty(\pi_v^{n+k})$, then it fixes $\ell_0(n+k)$. Therefore, by the proof of Lemma 11 applied with $\lambda = \frac{2}{\pi}$, since the constant $r_\lambda$ of Lemma 11 is equal to $r_v$ by equation (8) since $d_v = 1$, we have, if $n$ is big enough,

$$m_n(\gamma_0) = \min\{k \in \mathbb{N} - \{0\} : \gamma_0^n k \ell_0(n+k) = \ell_0(n+k)\}$$

$$\leq \min\{k \in \mathbb{N} - \{0\} : \gamma_k^n \in \Gamma_\infty(\pi_v^{n+k})\}$$

$$\leq \min\{k \in \mathbb{N} - \{0\} : \left(\frac{\lambda_+}{\lambda_-}\right)^k \in 1 + \pi_v^{n+k} O_v\}$$

$$= \min\{k \in \mathbb{N} - \{0\} : v_p(k) \geq \log_p \frac{n+k}{r_v}\} = p^{\log_p \frac{n+k}{r_v}}.$$

This concludes the proof of Theorem 10. \hfill $\Box$

4. Escape of mass along Hecke rays of $A_\infty$-periodic points

Let $(K, \nu, G, A)$ and the associated notation be as in Section 2.2. We fix from now on an $A_\infty$-periodic point $x_0$ in $X_\infty = \Gamma_\infty \backslash G_\infty$, as well as a representative $g_0$ of $x_0$ in $\Gamma_\infty$, so that $x_0 = \Gamma_\infty g_0$. In this section, we prove our main results on the asymptotic behavior of the $A_\infty$-invariant probability measures $\mu_x$ supported on the $A_\infty$-orbits in $X_\infty$ of the vertices $x$ of the $\nu$-Hecke tree $T_\nu(x_0)$ of $x_0$, as $x$ tends to infinity in this tree along rays. We will recall below a proof that every vertex of $T_\nu(x_0)$ is indeed $A_\infty$-periodic.
We denote by $\mathcal{P}(\overline{X}_\infty)$ the space of probability measures on the compactification $\overline{X}_\infty = X_\infty \cup \mathcal{E}_\infty$ of $X_\infty$ by its finite set of cusps $\mathcal{E}_\infty = \Gamma_\infty \setminus \mathbb{P}_1(K)$ (see Section 2.3). Let $\xi \in \partial T_v(x_0)$ be an end of the $v$-Hecke tree of $x_0$. Let $\Theta_\xi$ be the subset of $\mathcal{P}(\overline{X}_\infty)$ consisting of the weak-star accumulation points of the sequence $(\mu_{\xi,n})_{n \in \mathbb{N}}$ of $A_\infty$-invariant probability measures on the vertices $(x_\xi^n)_{n \in \mathbb{N}}$ along the geodesic ray in $T_v(x_0)$ from $x_0$ to $\xi$.

For all $c > 0$ and $z \in \mathcal{E}_\infty$, we say that

- $\xi$ has $c$-escape of mass if there exists $\theta \in \Theta_\xi$ with $\theta(\mathcal{E}_\infty) \geq c$.
- $\xi$ has $c$-escape of mass towards the cusp $z$ if there exists $\theta \in \Theta_\xi$ with $\theta([z]) \geq c$.
- $\xi$ has uniform $c$-escape of mass if for every $\theta \in \Theta_\xi$ we have $\theta(\mathcal{E}_\infty) \geq c$.
- $\xi$ has uniform $c$-escape of mass towards the cusp $z$ if for every $\theta \in \Theta_\xi$ we have $\theta([z]) \geq c$.

4.1. **Uniform escape of mass along rational Hecke rays.** We start this section by defining the rational Hecke rays in the $v$-Hecke tree $T_v(x_0)$ of $x_0$, and we will then prove Theorem 12, a uniform escape of mass phenomenon for the $A_\infty$-invariant probability measures $\mu_x$, as $x$ tends to infinity along these rays.

The group $G(K)$ acts transitively on $\mathbb{P}_1(K)$, but its subgroups $\Gamma_\infty = G(R_\infty)$ and $\Gamma_S = G(R_\infty[\pi_\infty^{-1}])$ do not in general. The sets $\mathcal{E}_\infty = \Gamma_\infty \setminus \mathbb{P}_1(K)$ (with order at most the class number of $R_\infty$) and $\Gamma_S \setminus \mathbb{P}_1(K)$ are finite and both canonical maps $\Gamma_\infty \setminus \mathbb{P}_1(K) \to \Gamma_S \setminus \mathbb{P}_1(K) \to G(K) \setminus \mathbb{P}_1(K)$ may be non injective. Note that for instance when $C$ is the projective line over $\mathbb{F}_q$ and $\infty$ its usual point at infinity, then $R_\infty$ is principal, and $\Gamma_\infty$ does act transitively on $\mathbb{P}_1(K)$.

Since $\Gamma_\infty$ preserves $\mathbb{P}_1(K)$ and by the commutativity of the diagram (6), the image $\text{hec}_{g_0}(\mathbb{P}_1(K)) \subset \partial T_v(x_0)$ by $\text{hec}_{g_0}$ of the set $\mathbb{P}_1(K)$ of rational points of $\partial \mathbb{P}_v = \mathbb{P}_1(K_v)$ does not depend on the choice of the representative $g_0$ of $x_0$, nor does the image by $\text{hec}_{g_0}$ of the orbit of $\infty$ by any subgroup of $G(K)$ containing $\Gamma_\infty$, as for instance $\text{hec}_{g_0}(\Gamma_S \infty)$.

A Hecke ray in $T_v(x_0)$, as well as its point at infinity, is said to be rational if its point at infinity belongs to $\text{hec}_{g_0}(\mathbb{P}_1(K))$, and $S$-rational if its point at infinity belongs to $\text{hec}_{g_0}(\Gamma_\infty \infty)$ (that is, when the graph $\Gamma_\infty \setminus \mathbb{T}_\infty$ has only one end, as for instance when $C$ is the projective line over $\mathbb{F}_q$ and $\infty$ its usual point at infinity), these two notions coincides. But there are examples of function fields when not all rational ends of $T_v(x_0)$ are $S$-rational (the two inclusions $\Gamma_S \infty \in \Gamma_\infty \infty < \mathbb{P}_1(K)$ may be strict).

If $\xi$ is a rational end of $T_v(x_0)$, the cusp of $X_\infty$ associated with $\xi$ is $z_\xi = \Gamma_\infty \gamma \in \mathcal{E}_\infty$, where $\gamma \in G(K)$ is such that $\xi = \text{hec}_{g_0}(\gamma \infty)$. Note that $z_\xi$ does not depend on the choices of $g_0$ or $\gamma$. If $\xi$ is $S$-rational, we say that $z_\xi$ is an $S$-cusp of $X_\infty$.

**Theorem 12.** There exists $c = c(x_0) > 0$ such that every rational end $\xi$ of the Hecke tree of $x_0$ has uniform $c$-escape of mass, and if furthermore $\xi$ is $S$-rational, then $\xi$ has uniform $c$-escape of mass towards the cusp of $X_\infty$ associated with $\infty$. 
Proof: We start the proof by giving some notation. Let us fix elements \( \gamma_0 \in \Gamma_\infty \) and \( t_0 \in K_\infty^x \) associated with the chosen representative \( g_0 \) of \( x_0 \) (see Proposition 8 and its following comment): we have

\[ \gamma_0 g_0 = g_0 a_\infty(t_0) \]

and \( \rho_0 = \nu_\infty(t_0) > 0 \) is the translation distance of \( \gamma_0 \) on \( \mathbb{T}_\infty \).

Since \( G(K) \) acts transitively on \( \mathbb{P}_1(K) \) and \( \mathbb{E}_\infty^T = \Gamma_\infty \setminus \mathbb{P}_1(K) \) is finite, there exists a finite subset \( F_1 \) of \( G(K) \) such that \( \mathbb{P}_1(K) = \Gamma_\infty F_1 \mathbb{E}_{\infty} \), and we may assume that \( \Gamma_\infty F_1 = \Gamma_\infty (F_1 \cap \Gamma_S) \mathbb{E}_{\infty} \).

Since \( \overline{G}(K) \) commensurates \( \Gamma_S \), there exists a finite subset \( F_2 \) of \( \overline{G}(K) \) such that for all \( \gamma \in F_1 \) and \( n \in \mathbb{N} \), there exists \( b_{\gamma,n} \in F_2 \) such that

\[ \gamma a_\infty^n \gamma^{-1} \in \Gamma_S b_{\gamma,n}. \]

We assume that \( 1 \in F_2 \) and \( b_{\gamma,n} = 1 \) if \( \gamma \in \Gamma_S \).

For every \( b \in \overline{G}(K) \), let \( \overline{b} \in \Gamma_S \) be such that \( b \in \overline{b} G(O_{\nu}) \), which exists by Equation (3). We assume that \( \overline{b} = b \) if \( b \in \Gamma_S \).

Now that this notation has been given, we consider the rational ends \( \zeta \) of the Hecke tree \( T_v(x_0) \). Let \( \gamma' = \gamma_0' \in \Gamma_\infty \) and \( \gamma = \gamma_\zeta \in F_1 \) be such that \( \zeta = \text{hec}_{\gamma_0'}(\gamma' \gamma_\infty) \).

We assume that \( \gamma \in \Gamma_S \) if \( \zeta \) is \( S \)-rational.

\[ \begin{array}{c}
\gamma' \gamma^* \bullet \ldots \bullet \gamma' \gamma y_n \\
n_\zeta \\
\gamma' \gamma y_{n'n} = \text{hec}_{\gamma}^{-1}(x_\zeta) \\
\gamma' \gamma y_n = a_\nu^n \gamma^* \\
\gamma_\gamma \gamma' \infty \\
y_n = a_\nu^n \gamma^*
\end{array} \]

**Figure 4.** Rational Bruhat-Tits rays

For every \( n \in \mathbb{N} \), let \( y_n = a_\nu^n \gamma^* \), so that for every rational end \( \zeta \) of \( T_v(x_0) \), the point at infinity of the image by \( \text{hec}_{\gamma_0} \) of the geodesic ray \( n \to \gamma' \gamma y_n \) is \( \zeta \). Let \( n'_\zeta \in \mathbb{N} \) be the distance from \( \gamma \) to this ray, and let \( n_\zeta \in \mathbb{N} \) be such that

\[ [\gamma^{*}, \gamma' \gamma_\infty] \cap [\gamma' \gamma^{*}, \gamma' \gamma_\infty] = [\gamma' \gamma y_{n'}, \gamma' \gamma_\infty]. \]

Let \( r_\zeta = n_\zeta - n'_\zeta \in \mathbb{Z} \). Denote by \( (x_n = x_\zeta^n)_{n \in \mathbb{N}} \) the geodesic ray in the Hecke tree \( T_v(x_0) \) from \( x_0 \) to \( \zeta \). Using in the following sequence of equalities

- the definition of \( \text{hec}_{\gamma_0} \) for the third equality and
- the definition of \( \pi_\infty \) (since \( \gamma a_\nu^n \gamma^{-1} b_{\gamma,n}^{-1} \in \Gamma_S \) by equation (9)) for the fifth one,

we have, for every \( n \in \mathbb{N} \) with \( n \geq n_\zeta \),

\[ x_n - r_\zeta = \text{hec}_{\gamma_0}(\gamma' \gamma y_n) = \text{hec}_{\gamma_0}(\gamma' \gamma a_\nu^n \gamma^*) = \pi_\infty(\Gamma_S(g_0, \gamma' \gamma a_\nu^n)) \]

\[ = \pi_\infty(\Gamma_S(g_0, \gamma' \gamma a_\nu^n \gamma^{-1} b_{\gamma,n}^{-1} \gamma)) = \Gamma_\infty(\gamma' \gamma a_\nu^n \gamma^{-1} b_{\gamma,n}^{-1} \gamma)^{-1} g_0 \]

\[ = \Gamma_\infty((b_{\gamma,n} \gamma)^{-1} b_{\gamma,n} \gamma) a_\nu^n (\gamma' \gamma)^{-1} g_0. \]
Let $\mathcal{E}_\infty(\xi)$ be the subset of $\mathcal{E}_\infty$ consisting of the elements
\[ z_n = \Gamma_\infty(\hat{b}_{\gamma,n,\gamma}^{-1})b_{\gamma,n,\gamma}0 \]
as $n$ varies. When $\xi$ is $S$-rational, we have $x_{n-r_\xi} = \Gamma_\infty a_{\gamma}^{-n}(\gamma')^{-1}g_0$, and $\mathcal{E}_\infty(\xi)$ is the singleton of the cusp $z_\infty = \Gamma_\infty 0 = \Gamma_\infty \infty$ associated with $\infty$.

Let $\pi'_\infty : X_\infty = \Gamma_\infty \backslash G_\infty \to \Gamma_\infty \not\backslash T_\infty = \Gamma_\infty \backslash G_\infty / A(\infty)$ be the canonical projection, which is equivariant under $\nu_\infty : A_\infty \to Z$ (see Section 2.4). It extends to a continuous map from $\overline{X}_\infty = X_\infty \cup \hat{\mathcal{E}}_\infty$ to Freudenthal's compactification $\Gamma_\infty \not\backslash T_\infty = (\Gamma_\infty \not\backslash T_\infty) \sqcup \hat{\mathcal{E}}_\infty$, by the identity on $\hat{\mathcal{E}}_\infty$.

By equation (5), since $\gamma_0 g_0 = g_0 a_{\infty}(i_0)$ and by equation (2), for every $k \in \mathbb{N}$ and $y \in V T_v$, we have
\[ \pi'_\infty(\text{hec}_{\gamma_0}(y_0 k \gamma)) = \pi'_\infty(\text{hec}_{\gamma_0}(y)g_0^{-1} \gamma_0 k g_0) = \pi'_\infty(\text{hec}_{\gamma_0}(y)\alpha_{\infty}(i_0 k)) = \phi_{\rho_0 k}(\pi'_\infty(\text{hec}_{\gamma_0}(y))). \]

In particular, since the orbits of $\gamma_0$ on $V T_v$ are finite, every $x \in V T_v(x_0)$ is also $A_\infty$-periodic and the $A_\infty$-invariant probability measure $\mu_x$ on the compact orbit $x A_\infty$ is well defined. Furthermore, with the notation of Theorem 10, for every $n \geq n_0$, the orbit under the geodesic flow of $\pi'_\infty(x_n) = \pi'_\infty(\text{hec}_{\gamma_0}(\gamma')y_{n+r_\xi})$ is periodic, with period $\lambda_n$ bounded as follows:
\[ \lambda_n \leq \rho_0 \min\{k \in \mathbb{N} - \{0\} : \gamma_0 k \gamma' y_{n+r_\xi} = \gamma' y_{n+r_\xi}\} \leq \rho_0 m_n(\gamma_0). \]

Let $d$ be the distance in the graph $\Gamma_\infty \not\backslash T_\infty$. Recall that $p_\infty : X_\infty \to \Gamma_\infty \not\backslash T_\infty$ is the map $\Gamma_\infty g \to \Gamma_\infty g * \infty$ (see the diagram at the beginning of Section 2). Using

- Lemma 7 with $\kappa = \kappa_\xi = d_{\lambda_\xi}(*_v, (\gamma')^{-1}g_0 *_v)$ for the first inequality,
- the definition of the height (see equation (4)) and equation (10) with the notation $\beta_n = (\hat{b}_{\gamma,n+r_\xi,\gamma}^{-1})b_{\gamma,n+r_\xi,\gamma}$ for the second equality,

we have, for all $n \in \mathbb{N}$,
\[ \text{ht}_\infty(x_n) \geq \text{ht}_\infty(x_n(\gamma^{-1} \gamma' g_0^{-1})) - \kappa \]
\[ = d(p_\infty(\Gamma_\infty \beta_n a_{\gamma}^{-n-r_\xi}, \Gamma_\infty * \infty)) - \kappa \]
\[ = d(\Gamma_\infty \beta_n a_{\gamma}^{-n-r_\xi} * \infty, \Gamma_\infty * \infty) - \kappa. \]

Recall that $\beta_n$ belongs to $G(\Lambda)$, hence preserves the set of geodesic rays in $T_\infty$ ending in $\mathbb{P}_1(\Lambda) \subset \partial T_\infty$, and takes finitely many values as $n$ varies. Let
\[ \kappa' = \kappa' = \max_{n \in \mathbb{N}} d(\Gamma_\infty \beta_n *_\infty, \Gamma_\infty * \infty) + \kappa. \]

Recall that any geodesic ray in $T_\infty$ ending in $\mathbb{P}_1(\Lambda)$ has a subray that isometrically injects into $\Gamma_\infty \not\backslash T_\infty$. Hence, using equation (12) and the triangle inequality, there exist constants $n_\xi'' \geq n_\xi'$ and $\kappa_\xi'', \kappa_\xi''' \geq 0$ such that for every integer $n \geq n_\xi''$,
\[ \text{ht}_\infty(x_n) \geq d(\Gamma_\infty \beta_n a_{\gamma}^{-n-r_\xi} * \infty, \Gamma_\infty \beta_n * \infty) - \kappa' \]
\[ \geq d(\Gamma_\infty \beta_n a_{\gamma}^{-n-r_\xi} * \infty, \Gamma_\infty * \infty) - \kappa_\xi'' \]
\[ = (n + r_\xi)\nu_\infty(\pi_v) - \kappa_\xi'' = n \nu_\infty(\pi_v) - \kappa_\xi'''. \]
Since \( a^{-m}_v \ast \pi_\gamma \) tends to \( 0 \in \mathbb{P}^1(K) \) as \( m \to +\infty \), this argument in fact proves that \( \text{ht}_\infty z_n(x_n) \geq n |\nu_\infty(\pi_v)| - \kappa''_c \) for \( n \) big enough, where \( z_n \) is the cusp defined above.

For every \( n \in \mathbb{N} \), let \( \mu'_n = (\nu'_\infty)_* \mu_{x_n} \), which is the equiprobability on the finite orbit of \( \nu'_\infty(x_n) \) under the geodesic flow on \( \Gamma_\infty \backslash \mathcal{T}_\infty \). Recall that the pushforwards of measures by proper continuous maps preserve the total mass, and are weak-star continuous. The map \( \pi' \) towards of measures by proper continuous maps preserve the total mass, and are weak-star continuous. The map \( \pi'_n \) is a fibration with compact fiber, hence a proper map. Therefore \( \xi \) has uniform \( c \)-escape of mass (respectively uniform \( c \)-escape of mass towards the cusp \( z_\infty = \Gamma_\infty \infty \)) if and only if for every weak-star accumulation point \( \theta' \) of \( (\mu'_n)_{n \in \mathbb{N}} \) in the space of probability measures on \( \Gamma_\infty \backslash \mathcal{T}_\infty \), we have \( \theta'(\mathcal{E}_\infty) \geq c \) (respectively \( \theta'(\{z_\infty\}) \geq c \)).

Let \( o : \Gamma_\infty \backslash \mathcal{T}_\infty \to \Gamma_\infty \backslash \mathcal{V} \mathcal{T}_\infty \) be the origin map \( \Gamma_\infty \ell \to \Gamma_\infty \ell(0) \), which is a proper map. For all \( N \in \mathbb{N} \), let

\[
K_N = o^{-1}\{x \in \Gamma_\infty \backslash \mathcal{V} \mathcal{T}_\infty : x \in \bigcup_{\xi \in \mathcal{E}_\infty(\xi)} \mathcal{H}_\xi([0, +\infty]) \}
\]

which are open subsets of \( \Gamma_\infty \backslash \mathcal{T}_\infty \), which accumulate as \( N \to +\infty \) exactly to \( \mathcal{E}_\infty(\xi) < \Gamma_\infty \backslash \mathcal{G} \mathcal{T}_\infty \). By the full-down property (see Section 2.3), the orbit under the geodesic flow of \( \pi'_\infty(x_n) \) passes at a distance from \( \Gamma_\infty \infty \) which is bounded by the diameter \( N_0 \) of the finite graph \( \Gamma_\infty \backslash \mathcal{T}_\infty \cup \bigcup_{\xi \in \mathcal{E}_\infty} \mathcal{H}_\xi([0, +\infty]) \). Recall that this orbit is periodic, of period denoted by \( \lambda_n \). Hence, if \( N \geq N_0 \) and if \( \text{ht}_\infty(x_n) \geq N \), the origins of \( \varphi_t(\pi'_\infty(x_n)) \) for \( 0 < t \leq \lambda_n \) needs to range twice over all points at distance between \( N \) and \( \text{ht}_\infty(x_n) \) on a geodesic ray in \( \Gamma_\infty \backslash \mathcal{T}_\infty \) between \( \Gamma_\infty \infty \) and \( o(\rho_\infty(x_n)) \). Hence, if \( n \) is big enough, by the comment following equation (13) and by equation (11), we have

\[
\mu'_n(K_N) \geq \frac{2(\text{ht}_\infty(x_n) - N)}{\lambda_n} \geq \frac{2n |\nu_\infty(\pi_v)| - 2\kappa''_c - 2N}{\rho_0 m_n(\gamma_0)}.
\]

By the linear growth property of \( (m_n(\gamma_0))_{n \in \mathbb{N}} \) (see equation (7) and the notation of Theorem 10), the right hand side of Equation (14) has a limit as \( n \to +\infty \) at least

\[
c = \frac{2 r_v(\gamma_0) |\nu_\infty(\pi_v)|}{\rho_0 e_v(\gamma_0) d_v(\gamma_0) p}.
\]

Hence for every weak-star accumulation point \( \theta' \) of \( (\mu'_n)_{n \in \mathbb{N}} \), we have \( \theta'(\mathcal{E}_\infty(\xi)) \geq c \). This proves the result.

**Remark.** The aim of this remark is to give some estimations on the constant \( c \) appearing in this proof, and to give examples of full escape of mass along rational Hecke rays.

As above, let \( \gamma_0 \in \Gamma_\infty \) be a (primitive loxodromic on \( \mathcal{T}_\infty \)) element associated with \( x_0 \), and let us fix \( \overline{\gamma}_0 \in \text{GL}_2(R_\infty) \) whose image in \( \Gamma_\infty = \text{PGL}_2(R_\infty) \) is \( \gamma_0 \). Note that \( \det \overline{\gamma}_0 \in R_\infty = k_\infty \), hence \( \nu_\infty(\det \overline{\gamma}_0) = 0 \). We may denote by \( \lambda_{\pm} \) the eigenvalues of \( \overline{\gamma}_0 \) with \( \nu_\infty(\lambda_{\pm}) > 0 \), so that \( \nu_\infty(\lambda_{+}) = -\nu_\infty(\lambda_{-}) < \nu_\infty(\lambda_{+}) \) and, by equation (1),

\[
\rho_0 = 2 |\nu_\infty(\lambda_{-})| = 2 |\nu_\infty(\text{tr}(\overline{\gamma}_0))|.
\]
With the notation of Theorem 10, let us define

\[ \text{LOM}(\gamma_0) = \frac{r_v(\gamma_0) |\nu_\infty(\pi_v)|}{|\nu_\infty(\text{tr} \gamma_0)| e_v(\gamma_0) d_v(\gamma_0)}, \]

so that we chose \( c = \text{LOM}(\gamma_0) / p \) in the above proof.

Let us consider \( n_k = r_v(\gamma_0) p^k - \left[ \frac{\nu_v(\gamma_0)}{e_v(\gamma_0)} \right] \) for \( k \in \mathbb{N} \) big enough (again with the notation of Theorem 10), so that \( m_{n_k}(\gamma_0) \leq e_v(\gamma_0) d_v(\gamma_0) p^k \) by Theorem 10. Using this majoration on the denominator in equation (14), the above proof gives moreover that every weak-star accumulation point \( \theta' \) of \((\mu_{n_k})_{k \in \mathbb{N}}\) satisfies \( \theta'(\mathcal{E}_\infty(\xi)) \geq \text{LOM}(\gamma_0) \). In particular, the sequence \((\mu_{n_k})_{k \in \mathbb{N}}\) weak-star converges to 0 measure on \( X_\infty \) if \( \text{LOM}(\gamma_0) = 1 \). Let us give an example of this when \( C \) is the projective line and \( \infty \) its usual point at infinity. Let \( d = d_v(\gamma_0) \), \( e = e_v(\gamma_0) \) and \( r = r_v(\gamma_0) \), so that \( \text{LOM}(\gamma_0) = \frac{r |\nu_\infty(\pi_v)|}{e_d |\nu_\infty(\text{tr} \gamma_0)|} \). Let \( \tilde{k}_v \) be the residual field of the splitting field \( \tilde{K}_v \) of \( \gamma_0 \) over \( K_v \).

**Lemma 13.** Assume that the discriminant \( \Delta = (\text{tr} \gamma_0)^2 - 4 \det \gamma_0 \) of \( \gamma_0 \) is irreducible over \( \mathbb{F}_q \), and let \( \pi_v = \Delta \). Then \( \text{LOM}(\gamma_0) = 1 \).

This assumption is satisfied, for instance, if \(-1\) is not a square modulo \( p \) (as for \( p = 3 \), if \( p = q \) and if \( \gamma_0 = \begin{pmatrix} Y & 1 \\ 1 & 0 \end{pmatrix} \), where \( Y = \pi^{-1}_\infty \) is the indeterminate in \( K = \mathbb{F}_q(Y) \) since \( \Delta = Y^2 + 4 \). By the previous arguments, for every rational end \( \xi \in \Omega \), there exists an element \( \theta' \in \Theta_\xi \) which vanishes on \( X_\infty \). This proves Theorem 2 in the introduction. The above proof also gives a speed of escape of mass when \( \text{LOM}(\gamma_0) = 1 \): for every compact subset \( C \) of \( X_\infty \), we have \( \mu_{x_{n_k}}(C) = O(\frac{1}{n_k}) \) when \( n_k = r_v(\gamma_0) p^k - \left[ \frac{\nu_v(\gamma_0)}{e_v(\gamma_0)} \right] \).

**Proof.** Since \( \Delta \) is irreducible, we have \( p \neq 2 \). In particular, the roots of \( \gamma_0 \) are \( \lambda_\pm = \frac{1}{2}(\text{tr} \gamma_0 \pm \sqrt{\text{det} \gamma_0}) \). We have \( \tilde{k}_v = K_v(\sqrt{\pi_v}) \), and \( \tilde{k}_v = k_v \). In particular, the ramification index of the splitting field of \( \gamma_0 \) over \( K_v \) is \( e = 2 \). Since the constant terms in \( \tilde{k}_v \) (modulo \( \sqrt{\pi_v} \)) of \( \lambda_\pm \) are equal, we have \( d = 1 \). Since \( \deg(\det \gamma_0) = 0 \), we have

\[ |\nu_\infty(\pi_v)| = \deg((\text{tr} \gamma_0)^2 - 4 \det \gamma_0) = 2 \deg(\text{tr} \gamma_0) = 2 |\nu_\infty(\text{tr} \gamma_0)|. \]

Since \( \gamma_0 \) is not congruent to the identity modulo \( \sqrt{\pi_v} \), we have \( r = 1 \). Hence \( \text{LOM}(\gamma_0) = 1 \).

Let us give one more estimation on the constant \( \text{LOM}(\gamma_0) \) when \( p \neq 2 \) and \( \nu_v(\text{tr} \gamma_0) > 0 \). We then have

\[ \lambda_\pm = \frac{1}{2}(\text{tr} \gamma_0 \pm \sqrt{(\text{tr} \gamma_0)^2 - 4 \det \gamma_0}). \]

Since \( \det \gamma_0 \in k_\infty^* \subset O_v^* \), we have \( \nu_v(-4 \det \gamma_0) = 0 \), hence \( e = 1 \) (and \( |\tilde{k}_v : k_v| = 1 \) if \(- \det \gamma_0 \) is a square and 2 otherwise). The constant terms \( a_\pm = \pm \sqrt{-4 \det \gamma_0} \in \)
that \( \frac{\lambda_+}{\lambda_-} \) are opposite (and non zero), hence \( d = 2 \). By equation (8), we have \( r = v_\nu(\frac{\lambda_+}{\lambda_-} - 1) = v_\nu(\text{tr} \tilde{\gamma}_0) \). Furthermore
\[
|v_\nu(\text{tr} \tilde{\gamma}_0)| = \deg(\text{tr} \tilde{\gamma}_0) \geq v_\nu(\text{tr} \tilde{\gamma}_0) \deg \pi_\nu = r |v_\nu(\pi_\nu)|.
\]
Hence \( \text{LOM}(\gamma_0) \leq \frac{1}{2} \), with equality if and only if \( \text{tr} \tilde{\gamma}_0 \) is a constant multiple of a power of \( \pi_\nu \), as, for instance, when \( \pi_\nu = Y \) and \( \tilde{\gamma}_0 = \begin{pmatrix} Y & 1 \\ 1 & 0 \end{pmatrix} \). For these elements where equality holds, at least half the mass escapes to infinity along subsequences of every rational Hecke ray.

4.2. Escape of mass along uncountably many Hecke rays. In the previous section, we proved escape of mass phenomena along countably many Hecke rays, the rational ones. In this section, we use the uniformity of the escape of mass in Theorem 12 in order to prove that an escape of mass (towards a prescribed cusp of \( X_\infty \)) actually occurs along uncountably many Hecke rays. We first introduce some notation that we will use from now on in this paper.

We denote by \( \Omega = \partial T_\nu(x_0) \) the boundary at infinity of the Hecke tree \( T_\nu(x_0) \) of \( x_0 \). For every \( \xi \in \Omega \), we denote by \( [x_0, \xi] \) the geodesic ray in \( T_\nu(x_0) \) starting from \( x_0 \) and converging to \( \xi \). We denote by \( (x_\xi^n)_{n \in \mathbb{N}} \) the sequence of vertices of \( [x_0, \xi] \), in this order along this ray. In particular, \( x_\xi^0 = x_0 \) and \( d(x_\xi^k, x_\xi^n) = |k - n| \).

Let \( x \in VT_\nu(x_0) \). We define the sector of \( x \) by
\[
\Omega_x = \{ \xi \in \Omega : x \in [x_0, \xi] \},
\]
the cone of \( x \) by
\[
C_x = \{ y \in VT_\nu(x_0) : \exists \xi \in \Omega_x, y \in [x, \xi] \},
\]
and, for every \( n \in \mathbb{N} \), the sector-sphere of \( x \) of radius \( n \) by
\[
S^n_x = C_x \cap S_{T_\nu(x_0)}(x_0, n).
\]

The depth of the cone \( C_x \) or of the sector \( \Omega_x \) of \( x \) is defined to be the distance in the Hecke tree \( T_\nu(x_0) \) from \( x \) to \( x_0 \). The sector-sphere \( S^n_x \) is nonempty if and only if \( n \) is at least this depth. For every \( \xi \in \Omega \), the sequences \( (C_x^\xi)_{n \in \mathbb{N}} \) and \( (\Omega_x^n)_{n \in \mathbb{N}} \) are strictly decreasing, with \( \Omega_x^0 = \Omega \), \( C_x^0 = VT_\nu(x_0) \), \( \bigcap_{n \in \mathbb{N}} C_x^{x_\xi}_n = \emptyset \) and \( \bigcap_{n \in \mathbb{N}} \Omega_x^n = \{ \xi \} \).

Figure 5. Sector-spheres in Hecke trees
Note that if two cones (or sectors) intersect nontrivially, then one of them is contained in the other. Also, sectors are nonempty compact-open sets in \( \Omega \) and in particular contain infinitely many rational ends, and even infinitely many S-rational ends.

**Theorem 14.** There exists \( c = c(x_0) > 0 \) such that the set of \( \xi \in \Omega \) having \( c \)-escape of mass towards the cusp \( \Gamma_{c,\infty} \) is uncountable.

In particular, the set of \( \xi \in \Omega \) having \( c \)-escape of mass is uncountable. Theorem 3 in the introduction follows immediately, being the case when \( C \) is the projective line, in which case \( X_\infty \) has only one cusp.

**Proof.** Let \( c = c(x_0) \in [0, 1] \) be the constant introduced in Theorem 12. Let \( z = \Gamma_{c,\infty} \in \mathcal{E}_{c,\infty} \). We fix a fundamental system \((V_n)_{n \in \mathbb{N}}\) of open neighborhoods of the cusp \( z \) in \( X_\infty = X_{\infty} \cup \mathcal{E}_{\infty} \), so that \( \{z\} = \bigcap_{n \in \mathbb{N}} V_n \). For all \( n \in \mathbb{N} \), let \( \Sigma_n = \{0, 1\}^n \) be the set of words of length \( n \) in 0 and 1. Let \( \Sigma = \bigcup_{n \in \mathbb{N}} \Sigma_n \) be the set of finite words in 0 and 1.

We are going to define a map \( \psi : \Sigma \to V \Gamma_{c,\infty}(x_0) \) with the following properties: For all \( n \in \mathbb{N} \) and \( \alpha \in \Sigma_n \),

1. if \( \alpha \) is an initial subword of \( \alpha \), then \( \Omega_{\psi(\alpha)} \subseteq \Omega_{\psi(\beta)} \),
2. if \( \beta \) is an initial subword of \( \alpha \) with \( \beta \neq \alpha \), then the intersection \( \Omega_{\psi(\beta)} \cap \Omega_{\psi(\beta)} \) is empty,
3. the depth of the sector \( \Omega_{\psi(\alpha)} \) is at least \( n \),
4. we have \( \mu_{\psi(\alpha)}(V_n) \geq c - \frac{1}{n+1} \).

Assume for the moment that such a map \( \psi \) is constructed. Let \( \Sigma_\infty = \{0, 1\}^\mathbb{N} \), which is uncountable. For every \( w \in \Sigma_\infty \), let \( w_n \) be the initial subword of length \( n \) of \( w \). Note that by properties (1) and (3), for every \( w \in \Sigma_\infty \), the sequence of sectors \( \{\Omega_{\psi(w_n)}\}_{n \in \mathbb{N}} \) is strictly nested, and its intersection contains a single point, denoted by \( \xi_w \). Furthermore, for every \( n \in \mathbb{N} \), we have \( w_n \in [x_0, \xi_w] \). Note that by property (2), the map \( w \mapsto \xi_w \) from \( \Sigma_\infty \) to \( \Omega \) is injective. By property (4), for every \( w \in \Sigma_\infty \), if \( \theta_w \) is a weak-star accumulation point of \( \{\mu_{\psi(w_n)}\}_{n \in \mathbb{N}} \) in the space \( \mathcal{P}(X_\infty) \) of probability measures on the compact space \( X_\infty \), then \( \theta_w([z]) \geq c \). Hence \( \xi_w \) has \( c \)-escape of mass towards the cusp \( z \). This proves Theorem 14.

We now build \( \psi \mid \Sigma_n \) by induction on \( n \in \mathbb{N} \). Note that \( \Sigma_0 \) is reduced to the empty word \( \emptyset \), and define \( \psi(\emptyset) = x_0 \). Note that properties (1)–(4) with \( n = 0 \) are then satisfied. Let \( n \in \mathbb{N} \), assume that \( \psi \mid \Sigma_n \) is constructed, satisfying properties (1)–(4) for every \( \alpha \in \Sigma_n \). For every \( \alpha \in \Sigma_n \) and \( j \in \{0, 1\} \), let us now define \( \psi(\alpha j) \).

By density, there exist distinct points \( \xi_0 \) and \( \xi_1 \) in \( \Omega_{\psi(\alpha)} \) which are rational and whose associated cusps \( z_{\xi_0} \) and \( z_{\xi_1} \) of \( X_\infty \) respectively are both equal to \( z \). By Theorem 12, \( \xi_0 \) and \( \xi_1 \) both have uniform \( c \)-escape of mass towards the cusp \( z \).

For all \( j \in \{0, 1\} \) and \( m \geq d(x_0, \psi(\alpha)) + 1 \), the sector \( \Omega_{\xi_j} \) is strictly contained in \( \Omega_{\psi(\alpha)} \) and has depth at least \( n + 1 \) by induction. Since \( \xi_0 \neq \xi_1 \), there exists
\( m_0 \in \mathbb{N} \) such that \( x_{m_0}^{\xi_0} \neq x_{m_0}^{\xi_1} \), so that for every \( m, m' \geq m_0 \), the sectors \( \Omega_{x_{m}^{\xi_0}} \) and \( \Omega_{x_{m'}^{\xi_1}} \) are disjoint.

Let \( j \in [0, 1] \). We claim that there exists \( n_j \geq m_0 \) such that \( \mu_{x_{n_j}^{\xi_j}}(V_{n+1}) \geq c - \frac{1}{n+2} \).

Otherwise, for every accumulation point \( \theta \) of \( \{ \mu_{x_{n_j}^{\xi_j}} \}_{j \in \mathbb{N}} \), we have \( \theta([z]) \leq c - \frac{1}{n+2} \), which contradicts the fact that \( \xi_j \) has uniform \( c \)-escape of mass towards the cusp \( z \).

Defining \( \psi(0) = x_{n_0}^{\xi_0} \) and \( \psi(1) = x_{n_1}^{\xi_1} \) gives the result. \( \square \)

4.3. **Effective equidistribution of sector-spheres.** The aim of this section is to prove an effective statement regarding the equidistribution in \( X_\infty \) of the sectors of the vertices of the Hecke tree of \( x_0 \), Theorem 15, by using the effective decay of matrix coefficients for the action of \( G_S \) on \( L^2(X_S) \). This sectorial effective equidistribution result will be the main tool used in Section 4.4 in order to prove Theorem 4 and its improvements. We first introduce some notation.

We denote by \( |E| \) the cardinality of any finite set \( E \) and by \( \Delta_x \) the unit Dirac mass at any point \( x \) of any measurable space. For all \( x \in VT_v(x_0) \) and \( n \in \mathbb{N} \) with \( n \geq k \) where \( k = d_{T_v(x_0)}(x_0, x) \) is the depth of the sector \( C_x \), let \( \eta_{n, x} \) be the uniform probability measure on the (finite nonempty) sector-sphere \( S^n_x \):

\[
\eta_{n, x} = \frac{1}{|S^n_x|} \sum_{y \in S^n_x} \Delta_y,
\]

that we consider as a probability measure on the locally compact space \( X_\infty \) with support \( S^n_x \). Since the \( v \)-Hecke tree of \( x_0 \) (as is the Bruhat-Tits tree \( \mathbb{T}_v \)) is \( |P^1(k_v)| \)-regular, note that \( |S^n_x| = |k_v|^{-k} \) if \( x \neq x_0 \) and \( n \geq k \), and that \( |S^n_x| = (|k_v| + 1)|k_v|^{n-1} \) if \( x = x_0 \) and \( n > 0 \).

For every place \( \omega \in \mathcal{D} \), we define \( W_\omega = G(O_\omega) \), which is a maximal compact-open subgroup of \( G_\omega \), and \( W_S = W_\infty \times W_v \subset G_\infty \times G_v = G_S \), which is a maximal compact-open subgroup of \( G_S \).

We denote by \( m_\infty \) (respectively \( m_S \)) the Haar measure on \( G_\infty \) (respectively \( G_S \)), normalized so that \( m_\infty(W_\infty) = 1 \) (respectively \( m_S(W_S) = 1 \)). We again denote by \( m_\infty \) (respectively \( m_S \)) the measure on \( X_\infty \) (respectively \( X_S \)) such that the

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**Figure 6.** Iterated construction of nested sectors
covering map \( G_\infty \to X_\infty = \Gamma_\infty \backslash G_\infty \) (respectively \( G_S \to X_S = \Gamma_S \backslash G_S \)) locally preserves the measures. Note that this measure on \( X_\infty \) (respectively \( X_S \)) is nonzero and finite, but is not necessarily a probability measure, the above normalisation of the Haar measures will turn out to be more convenient. For every \( k \in [1, +\infty[ \), we define \( L^k(X_\infty) = L^k(X_\infty, m_\infty) \) (respectively \( L^k(X_S) = L^k(X_S, m_S) \)).

The group \( G = G_\infty \) (respectively \( G = G_S \)) acts (on the left) on the complex vector space of maps \( \psi \) from \( X = X_\infty \) (respectively \( X = X_S \)) to \( \mathbb{C} \), by right translation on the source: For every \( g \in G \), if \( R_g : X \to X \) is the right translation \( x \mapsto xg \), then \( g\psi = \psi \circ R_g : x \mapsto \psi(xg) \).

A map \( \psi \) from \( X \) to \( \mathbb{C} \) is \textit{locally constant} if there exists a compact-open subgroup \( U \) of \( W = W_\infty \) (respectively \( W = W_S \)) which leaves \( \psi \) invariant:

\[
\forall \ g \in U, \ g\psi = \psi,
\]
or equivalently, if \( \psi \) is constant on each orbit of \( U \) under the right action of \( G \) on \( X \). Note that \( \psi \) is continuous, since the orbits of \( U \) are compact-open subsets.

We define

\[
d_\psi = \dim(\text{Vect}_\mathbb{C} \ W \psi)
\]
as the dimension of the complex vector space generated by the images of \( \psi \) under the elements of \( W \), which is finite, and even satisfies \( d_\psi \leq [W : U] \). We define the \textit{lc-norm} of every bounded locally constant map \( \psi : X \to \mathbb{C} \) by

\[
\|\psi\|_{lc} = \sqrt{d_\psi \|\psi\|_\infty}.
\]

Though the lc-norm does not satisfy the triangle inequality, we have \( \|\lambda \psi\|_{lc} = |\lambda| \|\psi\|_{lc} \) for every \( \lambda \in \mathbb{C} \). We denote by \( lc(X) \) the vector space of bounded locally constant maps \( \psi \) from \( X \) to \( \mathbb{C} \).

Finally, given a set \( A \) and maps \( f, g : A \to [0, +\infty[ \), we will write \( f \ll g \) if there exists a constant \( c' > 0 \) such that \( f(a) \leq c'g(a) \) for all \( a \in A \). If \( f \) and \( g \) depend on a parameter \( p \), we write \( f \ll_p g \) if there exists a constant \( c' > 0 \), possibly depending on the parameter \( p \), such that \( f(a) \leq c'g(a) \) for all \( a \in A \).

The following result strengthens the well-known result of equidistribution of full Hecke spheres (see for instance the works of Dani-Margulis [9], Clozel-Oh-Ullmo [7], Clozel-Ullmo [8], Eskin-Oh [13], Benoist-Oh [3] in characteristic 0), to an equidistribution result of sector-spheres, which is furthermore effective. Taking \( x = x_0 \) gives as a particular case an effective equidistribution result of the full Hecke spheres.

**Theorem 15.** There exists \( \delta > 0 \) such that for every \( x \in VT_v(x_0) \), we have

\[
\left| \frac{m_\infty(\psi)}{m_\infty(X_\infty)} - \eta_n(x(\psi)) \right| \ll \|\psi\|_{lc} e^{-\delta n}
\]

for all \( n \gg x 1 \) and \( \psi \in lc(X_\infty) \).

**Proof:** Let us fix \( x \in VT_v(x_0) \) and \( \xi = \xi_x \in \Omega_x \), so that \( x = x_k^\xi \) for some fixed \( k = k_x \in \mathbb{N} \) (see Figure 5).
Step 1: Thickening the sector-spheres. Note that the sector-spheres are measure zero subsets of $X_\infty$. In order to be able to apply (effective) mixing arguments, we have to replace them by (regular) bump functions around them. In this step, we will define nice compact-open neighborhoods of the sector-spheres, whose characteristic functions will be our bump functions. By the construction of the sector-spheres, it is more natural to lift the sector-spheres in $X_S$ and to work in the bundle $X_S$ over $X_\infty$.

We will hence use a lot the $W_\nu$-bundle map $\pi_\infty$ (see Section 2.3) from $X_S = \Gamma_S \backslash G_S$ to $X_\infty = \Gamma_\infty \backslash G_\infty$, defined by $\Gamma_S(g, h) \mapsto \Gamma_\infty g$ whenever $h \in W_\nu$. Recall (see Section 2.5) that the map $\hec_{g_0}$ from the Bruhat-Tits tree $T_\nu$ to the Hecke tree $T_\nu(x_0)$, defined on $V T_\nu = G_\nu / W_\nu$ by $h W_\nu \mapsto \pi_\infty(\Gamma_S(g_0, h))$ is an isomorphism of trees, and we identify $\partial T_\nu = P_1(K_\nu)$ and $\Omega$ by (the extension to the boundary at infinity of) this map. We endow $T_\nu(x_0)$ with the (left) action of $G_\nu$ making $\hec_{g_0}$ equivariant. Since $W_\nu = G(O_\nu)$ acts transitively on $\Omega = P_1(O_\nu)$, we also fix $w = w_x \in W_\nu$ such that $w w_\infty = \check{\zeta}$, where $\infty = [1 : 0]$.

For all $n \in \mathbb{N}$, we denote by $B_\nu^n$ the stabiliser in $W_\nu$ of the point $x^n_\infty$ at distance $n$ from $x_0$ on the geodesic ray $[x_0, \infty[$ in the Hecke tree $T_\nu(x_0)$. The group $B_\nu = B_\nu^1$ acts transitively on the sector-spheres $S^n_x$ of $x^n_\infty$ for all $n \in \mathbb{N}$. As we have already seen, for all $n \in \mathbb{N}$, we have

$$x^n_\infty = \hec_{g_0}(a^n_\nu *) = \pi_\infty(\Gamma_S(g_0, a^n_\nu *)) .$$

Note that $x^n_\zeta = w x^n_\infty$ for all $n \in \mathbb{N}$. In particular, $x = w x^n_\zeta$, hence $w B_\nu w^{-1}$ is the stabilizer in $W_\nu$ of $x$. It acts transitively on the sector-spheres $S^n_x$ of $x$ for all $n \in \mathbb{N}$, with stabilizer of $x^n_\zeta$ equal to $w B^n_\nu w^{-1}$. Therefore, for all $n \in \mathbb{N}$,

$$S^n_x = w B_\nu w^{-1} x_n^\zeta = w B_\nu x^n_\infty = \pi_\infty(\Gamma_S(g_0, w B_\nu a^n_\nu *)) .$$

Now that we have this nice description of the sector-spheres, let us define nice neighborhoods of them.

**Lemma 16.** There exist $\sigma_1, \sigma_2 > 0$ and a nondecreasing family $(B^e_\infty)_{e>0}$ of compact-open subgroups of $W_\infty$, which is a fundamental system of neighborhoods of the identity element in $W_\infty$, and which satisfies

$$\forall \, \varepsilon > 0, \quad \sigma_1 \varepsilon^{-1} \leq [W_\infty : B^e_\infty] \leq \varepsilon^{-1} ,$$

and

$$\forall \, a \in A_\infty, \quad a^{-1} B^e_\infty a \subset B^e_\infty e^{\sigma_2 [W_\infty : Z_0]} .$$

**Proof.** For every $n \in \mathbb{N}$, let $Z_n$ be the kernel of the reduction modulo $\pi^{n+1}_\infty$ map from $W_\infty = G(O_\infty)$ to the finite group $G(O_\infty / \pi^{n+1}_\infty O_\infty)$. Note that $Z_{n+1} \subset Z_n$. Let us consider $B^n_\infty = Z_n$ with $n_\varepsilon = \lceil \frac{-\log(\varepsilon [W_\infty : Z_0])}{\log [Z_0 : Z_1]} \rceil$ and $\sigma_1 = \frac{1}{[Z_0 : Z_1]}$. Then $B^n_\infty$ is a compact-open subgroup of $W_\infty$, we have $B^n_\infty \subset B^e_\infty$ if $e \leq e'$ and $\bigcap_{e>0} B^e_\infty = \{1\}$. Equation (17) follows since the index $[Z_n : Z_{n+1}]$ is constant, hence $[W_\infty : Z_n] = [W_\infty : Z_0][Z_0 : Z_1]^n$. 


For all $a, b, c, d \in K_{\infty}$ and $t \in K^*_\infty$, we have
\[
\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix}.
\]
Hence, using the isomorphism $\alpha_\infty : K^*_\infty \to A_\infty$ defined in Section 2.2, we have $a^{-1} Z_n a \subset Z_{n-|v_\infty(a)|}$ for all $a \in A_\infty$ and $n \geq |v_\infty(a)|$ in $\mathbb{N}$. Equation (18) (which will only be used in Section 4.4) follows with $\sigma_2 = \log |Z_0 : Z_1|$.

For every $\epsilon > 0$, we finally define the following compact-open subset of $X_S$
\[
U_\epsilon = \Gamma_S(g_0 B^c_\infty, wB_v),
\]
so that, for all $n \in \mathbb{N}$, the image $\pi_\infty(U_\epsilon a^n)$ of its translate by $a^n$ is a (small when $\epsilon$ is small) neighborhood of the sector-sphere $S^n_\epsilon$ in $X_\infty$, by equation (16).

**Step 2: Using the decay of matrix coefficients.** In this step, we use the following theorem about effective decay of matrix coefficients for the action of $G_\infty$ on $\mathbb{L}^2(X_\infty)$ (see for instance [2]). For every $g = (g_\infty, g_v) \in G_\infty = G_\infty \times G_v$, we denote by $|g|_S$ the maximum of the norms of the adjoint representations of $g_\infty$, $g_v$ (for the operator norm on the $3 \times 3$ matrices with entries in $K_\infty, K_v$).

**Theorem 17.** There exists $\delta_1 > 0$ such that
\[
(19) \quad \left| m_S(\bar{\psi} \circ R_g) - \frac{1}{m_S(X_S)} m_S(\bar{\psi}) m_S(\bar{\psi}) \right| \ll \sqrt{d_\bar{\psi} d_\bar{\psi} \|\bar{\psi}\|_2 \|\bar{\psi}\|_2} |g|^{-\delta_1}_S
\]
for all locally constant maps $\bar{\psi}, \bar{\psi} \in \mathbb{L}^2(X_S)$ and for every $g \in G_S$. \hfill \square

Now, let us fix $\psi \in l c(X_\infty)$. We denote by $\bar{\psi} = \psi \circ \pi_\infty$ its lift to $X_S$, which is constant on each right $W_v$-orbit, hence is locally constant (since $g \in G_\infty \times G_v$ if $\psi$ is invariant under $U$). Note that $\bar{\psi} \in \mathbb{L}^2(X_S)$ since $m_S$ is finite and $\bar{\psi}$ is bounded. By the normalization of the Haar measures, we have
\[
m_S(\bar{\psi}) = m_\infty(\psi) \quad \text{and} \quad m_S(X_S) = m_\infty(X_\infty).
\]
Since $\sqrt{d_\bar{\psi}} = \sqrt{d_\psi}$ and $\|\bar{\psi}\|_2 \leq \sqrt{m_S(X_S)} \|\psi\|_\infty$, we have
\[
\sqrt{d_\bar{\psi}} \|\bar{\psi}\|_2 \ll \|\psi\|_{lc}.
\]
For every $\epsilon > 0$, let $\varphi_\epsilon = \frac{1}{m(S(U_\epsilon))} \chi_{U_\epsilon}$ be the normalized characteristic function of $U_\epsilon$, so that $m_S(\varphi_\epsilon) = 1$. The map $\varphi_\epsilon : X_S \to \mathbb{C}$ is locally constant, since it is invariant under the right action of the compact-open subgroup $B^c_\infty \times B_v$ of $W_S$. We have
\[
d_{\varphi_\epsilon} = \dim \text{Vect}_\mathbb{C} W_S \varphi_\epsilon \leq [W_S : B^c_\infty \times B_v] = [W_\infty : B^c_\infty][W_v : B_v].
\]
Since $W_v$ is compact and acts freely on each of its orbits on $X_S$, there exists $\epsilon_0 = \epsilon_0(x) > 0$ such that if $\epsilon \in [0, \epsilon_0]$, the map from $B^c_\infty \times B_v$ to $X_S$ defined by $(g, h) \mapsto \Gamma_S(g_0 g, w h)$ is injective, and measure preserving with image $U_\epsilon$. Hence, by the normalization of the Haar measures, we have, for every $\epsilon \in [0, \epsilon_0]$,
\[
(20) \quad \|\varphi_\epsilon\|_2 = m_S(U_\epsilon)^{-\frac{1}{2}} = (m_\infty(B^c_\infty) m_v(B_v))^{-\frac{1}{2}} = ([W_\infty : B^c_\infty][W_v : B_v])^{-\frac{1}{2}}.
\]
We therefore have $\sqrt{d_\phi} \| \varphi_* \|_2 \leq 1$. Note that for every $n \in \mathbb{N}$,
\[ |a_v^{-n}|_S = \max\{|a_v^{-n}|_\infty, |a_v^{-n}|_v\} = \max\{|\pi^+_v n|_\infty, |\pi^-_v n|_v\} = \max\{|k_\infty|^{n|\nu_\infty(\pi_v)|}, |k_v|^{-n}\}.
\]
Applying equation (19) to the functions $\tilde{\psi}$, $\tilde{\varphi} = \varphi_\epsilon$ and taking $g = a_v^{-n}$, we hence have, with $\delta_2 = \delta_1 \max\{|\nu_\infty(\pi_v)| \log |k_\infty|, \log |k_v| \} > 0$, for every $\epsilon \in [0, \epsilon_0]$,
\[ (21) \quad \left| \frac{1}{m_S(U_\epsilon)} \int_{U_\epsilon a_v^n} \tilde{\psi} \, dm_S - \frac{m_\infty(\psi)}{m_\infty(X_\infty)} \right| \leq \| \psi \|_{L^1} e^{-\delta_2 n}.
\]
Let us now relate, for $\epsilon$ small enough, the above quantity $\frac{1}{m_S(U_\epsilon)} \int_{U_\epsilon a_v^n} \tilde{\psi} \, dm_S$ to the average
\[ \eta_{n, \epsilon}(\psi) = \frac{1}{|S^n_x|} \sum_{y \in S^n_x} \psi(y)
\]
of $\psi$ on the sector-sphere $S^n_x$.
Let $w_1, \ldots, w_\ell$ be representatives of the right cosets in $B_v \setminus B^n_v$, so that $B_v$ is a disjoint union $B_v = \bigsqcup_{i=1}^\ell w_i B^n_v$ and $m_v(B_v) = [B_v : B^n_v] m_v(B^n_v)$. By the transitivity properties seen in Step 1, the map from $B_v \setminus B^n_v$ to $S^n_x$ defined by $[h] \mapsto whw^{-1} x_n^\epsilon$ is a bijection. For every $y \in S^n_x$, let $i_y \in \{1, \ldots, \ell\}$ be such that
\[ y = w w_{i_y} w^{-1} x_n^\epsilon = w w_{i_y} x_n^\infty = \text{hec}_{g_0}(w w_{i_y} a^n_{i_y} \ast \psi) = \pi_\infty(g_0, w w_{i_y} a^n_{i_y}).
\]
Let
\[ V_y = \Gamma_S(g_0B^c_\infty, w w_{i_y} B^n_v)
\]
so that $V_y a^n_{i_y} = \Gamma_S(g_0B^c_\infty, w w_{i_y} a^n_{i_y}(a_v^{-n} B^n_v a^n_{i_y}))$. Note that $a_v^{-n} B^n_v a^n_{i_y}$ is contained in $W_v$, since $B^n_v$ stabilizes $x_n^\infty = a^n_{i_y} x_0$, hence the restriction of $\pi_\infty$ to $V_y a^n_{i_y}$ has image $yB^c_\infty$ and its fibers are orbits of $a_v^{-n} B^n_v a^n_{i_y}$. For every $\epsilon \in [0, \epsilon_0]$, since the map $(g, h) \mapsto \Gamma_S(g_0 g, w h)$ from $B^c_\infty \times B_v$ to $X_S$ is injective, we hence have
\[ U_\epsilon = \bigsqcup_{i=1}^\ell \Gamma_S(g_0B^c_\infty, w w_{i_y} B^n_v) = \bigsqcup_{y \in S^n_x} V_y.
\]
Therefore, for every $\epsilon \in [0, \epsilon_0]$, using equation (20) and by desintegration of $m_S$, we have
\[ (22) \quad \frac{1}{m_S(U_\epsilon)} \int_{U_\epsilon a_v^n} \tilde{\psi} \, dm_S = \frac{1}{m_\infty(B^c_\infty)} m_v(B_v) \sum_{y \in S^n_x} \int_{V_y a^n_{i_y}} \tilde{\psi} \, dm_S
\]
\[ = \frac{1}{m_\infty(B^c_\infty)} m_v(B_v) |S^n_x| \sum_{y \in S^n_x} m_v(a_v^{-n} B^n_v a^n_{i_y}) \psi \, dm_\infty
\]
\[ = \frac{1}{|S^n_x|} \sum_{y \in S^n_x} \frac{1}{m_\infty(B^c_\infty)} \int_{yB^c_\infty} \psi \, dm_\infty
\]
Define the $\epsilon$-thin part $X^\epsilon_\infty$ of $X_\infty$ as the set of points $z \in X_\infty$ such that the map from $B^c_\infty$ to $X_\infty$ defined by $h \mapsto zh$ is not injective. Since $\psi$ is locally constant, there exists $\epsilon_1 = \epsilon_1(\psi) > 0$ such that if $\epsilon \in [0, \epsilon_1]$, then $\psi$ is $B^c_\infty$-invariant. If $y \in S^n_x - (S^n_x \cap X^\epsilon_\infty)$ and if $\epsilon \in [0, \epsilon_1]$, then
\[ (23) \quad \frac{1}{m_\infty(B^c_\infty)} \int_{yB^c_\infty} \psi \, dm_\infty = \psi(y).
\]
A trivial majoration gives

\begin{equation}
\left| \frac{1}{|S_x^n|} \sum_{y \in S_x^n \cap X_\infty} \left( \psi(y) - \frac{1}{m_\infty(B_\infty^n)} \int_{B_\infty^n} \psi \, dm_\infty \right) \right| \leq 2 \| \psi \|_\infty \left| \frac{|S_x^n \cap X_\infty^n|}{|S_x^n|} \right|.
\end{equation}

Separating the summation over $S_x^n$ on one hand over $S_x^n \cap X_\infty^n$ and on the other hand over $S_x^n - (S_x^n \cap X_\infty^n)$, for every $\epsilon \in ]0, \min\{\epsilon_0, \epsilon_1\}]$, by equations (21), (22), (23) and (24), we hence have

\begin{equation}
\left| \eta_{n, x}(\psi) - \frac{m_\infty(\psi)}{m_\infty(X_\infty)} \right| \leq \| \psi \|_C e^{-\delta_2 n} + \| \psi \|_\infty \left| \frac{|S_x^n \cap X_\infty^n|}{|S_x^n|} \right|.
\end{equation}

**Step 3: Estimating the thin part of sector-spheres.** The aim of this step is to prove that the part of the sector-spheres contained in the thin part of $X_\infty$ is negligible, if $\epsilon$ is well-chosen. More precisely, let us prove that there exists $\delta_4 > 0$ such that for every $n \gg x$, if $\epsilon = e^{-\delta_2 n}$, then

\begin{equation}
\frac{|S_x^n \cap X_\infty^n|}{|S_x^n|} \ll e^{-\delta_4 n}
\end{equation}

for every $n \in \mathbb{N}$ with $n > k$.

For this, we will apply the arguments of Step 2 to a particular map $\psi = \psi_\epsilon$, where $\psi_\epsilon$ is, for every $\epsilon > 0$, the characteristic function of the $\epsilon$-thick part $X_\infty - X_\infty^\epsilon$ of $X_\infty$. Note that $\psi_\epsilon$ is invariant under $B_\infty^\epsilon$, hence $\psi_\epsilon$ is bounded and locally constant, and $\epsilon_1(\psi_\epsilon) = +\infty$. Denoting by $\tilde{\psi}_\epsilon$ the lift of $\psi_\epsilon$ to $X_S$, by equations (22) and (23) applied with $\psi = \psi_\epsilon$, for every $\epsilon \in [0, \epsilon_0]$, we have

\begin{equation}
\frac{1}{m_S(U_S)} \int_{U_S} \tilde{\psi}_\epsilon \, dm_S = \frac{1}{|S_x^n|} \sum_{y \in S_x^n} \psi_\epsilon(y) = \frac{|S_x^n - (S_x^n \cap X_\infty^\epsilon)|}{|S_x^n|}.
\end{equation}

By equation (17), we have

\begin{equation}
\| \psi_\epsilon \|_C = \sqrt{d_{\psi_\epsilon} \| \psi_\epsilon \|_\infty} \leq \sqrt{[W_\infty : B_\infty^\epsilon]} \leq \epsilon^{-\frac{1}{2}}.
\end{equation}

By the exponential decay of the volumes in the cusps of the graph of groups $\Gamma_\infty \backslash \mathbb{T}_\infty$, hence in $X_\infty$, there exists $\delta_3 > 0$ such that

\begin{equation}
m_\infty(X_\infty^\epsilon) \ll \epsilon^{\delta_3}
\end{equation}

for every $\epsilon > 0$.

For every $n \in \mathbb{N}$, define $\epsilon = e^{-\delta_2 n}$. Note that if $n \gg x$, then $\epsilon \leq \epsilon_0$ (recall that $\epsilon_0$ depends on $x$). Therefore, using

- equation (29) for the second inequality,
- equation (27) and the definition of $\psi_\epsilon$ for the third line,
- equation (21) with $\psi = \psi_\epsilon$ for the fourth inequality,
- equation (28) for the fifth inequality,
- the definition of $\epsilon$ and the constant $\delta_4 = \delta_2 \min\{\delta_3, \frac{1}{2}\} > 0$ for the last inequality,
we have, for all $n \gg x$,
\[
\left| \frac{S^n_x \cap X^c_{\infty}}{|S^n_x|} \right| \leq \left| \frac{S^n_x \cap X^c_{\infty}}{|S^n_x|} - \frac{m_\infty(X^c_{\infty})}{m_\infty(X_{\infty})} \right| + \frac{m_\infty(X^c_{\infty})}{m_\infty(X_{\infty})} + e^{\delta_3}.
\]
\[
= \left| \frac{1}{m_S(U_\ell)} \int U_\ell a^\ell \tilde{\psi}_c \, dm_S - \frac{m_\infty(\psi_c)}{m_\infty(X_{\infty})} \right| + e^{\delta_3}.
\]
\[
\ll \| \psi_c \|_{\ell^c} e^{-\delta_2 n} + e^{\delta_3} \leq e^{-\frac{1}{2} n} e^{-\delta_2 n} + e^{\delta_3} \leq 2 e^{-\delta_4 n}.
\]

This proves claim (26) of Step 3.

**Step 4: Conclusion.** Since $\| \psi \|_\infty \leq \| \psi \|_{\ell^c}$, Theorem 15 now follows from equations (25) and (26), with $\delta = \min\{\delta_2, \delta_4\}$. 

### 4.4. Exotic behavior of $A_\infty$-periodic measures along Hecke rays

In this final section, we use the tools introduced in Sections 4.1 and 4.3 to construct even more exotic asymptotic behaviors of the $A_\infty$-periodic measures $\mu_x$ as $x$ varies along geodesic rays in the Hecke tree of $x_0$.

**Theorem 18.** Let $(\mu_i)_{i \in \mathbb{N}}$ be an enumeration of all periodic $A_\infty$-invariant probability measures on $X_{\infty}$. There exist $c, c' > 0$ such that the set of $\xi \in \Omega$ having $c$-escape of mass towards the cusp of $X_{\infty}$ associated with $\infty$ and verifying
\[
\forall i \in \mathbb{N}, \exists \theta_i \in \Theta_\xi, \ c' \mu_i \leq \theta_i
\]
is uncountable. In particular, $\{ \xi \in \Omega : |\Theta_\xi| = \infty \}$ is uncountable.

Note that there are indeed only countably many periodic $A_\infty$-orbits, and that this result immediately implies Theorem 4.

The proof of Theorem 18 relies on the following two lemmas. We consider again the family $(B^c_{\infty})_{c>0}$ of compact-open subgroups of $W_\infty$ constructed in Lemma 16.

**Lemma 19.** There exists $\delta' > 0$ such that for every $x \in VT_\ell(x_0)$, for every $A_\infty$-periodic point $y_0 \in X_{\infty}$, and for every $n \gg x, y_0, 1$, the intersection $S^n_x \cap (y_0 A_{\infty} B^c_{\infty})$ is nonempty.

**Proof.** Let $\delta' = \frac{\delta}{3}$ where $\delta$ is the constant given by our effective equidistribution result of sector-spheres, Theorem 15. For every $\epsilon > 0$, let $\psi_{c\epsilon} = \mathbb{1}_{y_0 A_{\infty} B^c_{\infty}}$ be the characteristic function of the $B^c_{\infty}$-thickening of the periodic orbit $y_0 A_{\infty}$, which is bounded and locally constant. We are going to use Theorem 15 applied to $\psi = \psi_{c\epsilon}$ for a suitably chosen $\epsilon$.

There exists $\epsilon_2 = \epsilon_2(y_0) > 0$ such that if $\epsilon \in [0, \epsilon_2]$, then the orbit map $B^c_{\infty} \to y_0 B^c_{\infty}$ is injective. Hence by the normalisation of the Haar measure and by equation (17), we have, for every $\epsilon \in [0, \epsilon_2]$,
\[
m_\infty(\psi_{c\epsilon}) = m_\infty(y_0 A_{\infty} B^c_{\infty}) \geq m_\infty(y_0 B^c_{\infty}) = m_\infty(B^c_{\infty}) = \frac{1}{[W_\infty : B^c_{\infty}]}.
\]
Furthermore,
\[ \| \psi_e \|_1 = \sqrt{d \psi_e \| \psi_e \|_\infty} \leq \sqrt{|W: B_{\infty}^0|} \leq \epsilon^{-\frac{1}{2}}. \]
By Theorem 15, there exists \( \kappa > 0 \) such that if \( n \gg_x 1 \), we have, for every \( \epsilon \in [0, \epsilon_2] \),
\[ \eta_{n,x}(\psi_e) = \frac{m_\infty(\psi_e)}{m_\infty(X_\infty)} - \kappa \| \psi_e \|_1 e^{-\delta n} \geq \frac{\epsilon}{m_\infty(X_\infty)} - \kappa \epsilon^{-\frac{1}{2}} e^{-\delta n}. \]
Let us now consider \( \epsilon = 2(\kappa m_\infty(X_\infty) e^{-\delta n})^{\frac{1}{2}} \). By the definition of \( \delta' \), we have \( \epsilon \leq e^{-\delta n} \) if \( n \gg 1 \). If \( n \gg y_0, 1 \), then \( \epsilon \) belongs to \( [0, \epsilon_2] \). The previous centered formula then gives \( \eta_{n,x}(\psi_e) > 0 \) if \( n \gg_x y_0 \). Hence if \( n \gg_x y_0 \), the support of the measure \( \eta_{n,x} \), which is \( S_n^\sigma \), meets the support of the function \( \psi_e \), which is \( y_0 A_{\infty} B_{\infty}^\epsilon < y_0 A_{\infty} B_{\infty}^e \), as wanted.

**Lemma 20.** Let \( y \) and \( x_k \), for \( k \in \mathbb{N} \), be \( A_{\infty} \)-periodic points in \( X_\infty \). Suppose that there exist \( \sigma, \delta' > 0 \) and an increasing sequence \( (n_k)_{k \in \mathbb{N}} \) of positive integers such that:

1. For every \( k \in \mathbb{N} \), the period, under the geodesic flow in \( \Gamma_\infty \mathcal{M}_{\infty} \), of \( \pi_{\infty}'(x_k) \) is at most \( \sigma n_k \).
2. There are infinitely many \( k \in \mathbb{N} \) such that \( x_k \in y A_{\infty} B_{\infty}^e - \delta n_k \).

Then there exists a weak-star accumulation point \( \theta \) of \( (\mu_{x_k})_{k \in \mathbb{N}} \) such that
\[ \frac{\delta'}{2 \sigma_2} \sigma \mu_y \leq \theta. \]

**Proof.** Recall that \( \sigma_2 \) is given by Lemma 16. Up to extracting a subsequence, we may assume that \( x_k \in y A_{\infty} B_{\infty}^e - \delta n_k \) for every \( k \in \mathbb{N} \).

By assumption (1) and by the equivariance of the canonical bundle map \( \pi_{\infty}' : X_\infty = \Gamma_\infty \mathcal{M}_{\infty} \to \Gamma_\infty \mathcal{M}_{\infty} = \Gamma_\infty \mathcal{M}_{\infty} / A(\infty) \) with respect to the epimorphism \( \nu_\infty : A_{\infty} \to Z \) where \( Z \) acts by the geodesic flow (see Section 2.3 and in particular equation (2)), we have
\[ x_k A_{\infty} = \{ x_k a : a \in A_{\infty} \text{ and } |\nu_\infty(a)| \leq \sigma n_k \}. \]
For every \( a \in A_{\infty} \) such that \( |\nu_\infty(a)| \leq \delta n_k \), we have, by assumption (2) and by equation (18),
\[ x_k a \in y A_{\infty} a^{-\delta n_k} A_{\infty} B_{\infty}^e - n_k \epsilon^2 |\nu_\infty(a)| \subseteq y A_{\infty} B_{\infty}^e - \delta n_k \epsilon^2 |\nu_\infty(a)|. \]
If \( m_{A_{\infty}} \) is the Haar measure of \( A_{\infty} \) normalized so that \( m_{A_{\infty}}(A(\infty)) = 1 \), then the pushforward of \( m_{A_{\infty}} \) by \( \nu_\infty \) is the counting measure of \( Z \) and \( \mu_{x_n} \) is the measure on \( x_n A_{\infty} \) induced by \( m_{A_{\infty}} \), normalized to be a probability measure. Hence \( \{ x_k a : |\nu_\infty(a)| \leq \delta \} \) occupies at least \( \frac{\delta'}{2 \sigma_2} \) of the total mass of \( x_k A_{\infty} \), and accumulates on \( y A_{\infty} \). Hence at least \( \frac{\delta'}{2 \sigma_2} \) of the total mass of any weak-star accumulation point of \( (\mu_{x_k})_{k \in \mathbb{N}} \) is accumulated on \( \mu_y \). \( \square \)
Proof of Theorem 18. Let \((\mu_i)_{i \in \mathbb{N}}\) be as in this statement and let \(z \in \mathcal{E}_\infty^0\) be the cusp of \(X_\infty\) associated with \(\infty\).

Let us denote by \((\eta_n)_{n \in \mathbb{N}}\) a sequence of measures on \(X_\infty\) which contains the zero measure as well as all the \(A_\infty\)-invariant probability measures of the \(A_\infty\)-periodic points of \(X_\infty\), in such a way that each measure appears infinitely many times. Using Lemma 19 and arguing similarly to the proof of Theorem 14, with \((V_n)_{n \in \mathbb{N}}\) a fundamental system of open neighborhoods of \(z\) in \(X_\infty^0\), we can build inductively uncountably many sequences \((x_k)_{k \in \mathbb{N}}\) in \(VT^0_\infty(x_0)\) such that the following holds.

1. The sequence of cones \((C_{x_k})_{k \in \mathbb{N}}\) is strictly nested, so that if \(\bigcap_{k \in \mathbb{N}} \Omega_{x_k} = \{\xi\}\), then \((x_k)_{k \in \mathbb{N}}\) is a subsequence of the sequence \((x'_k)_{n \in \mathbb{N}}\) of vertices of the Hecke ray from \(x_0\) to \(\xi\).
2. If \((x_k)_{k \in \mathbb{N}} \neq (x'_k)_{k \in \mathbb{N}}\) are two of these sequences, then the sectors \(\Omega_{x_k}\) and \(\Omega_{x'_k}\) are disjoint for \(k\) big enough. In particular, the map \((x_k)_{k \in \mathbb{N}} \rightarrow \xi = \lim_{k \to \infty} x_k\) is injective, and there are uncountably many such \(\xi\)'s.
3. For every \(k \in \mathbb{N}\), denoting by \(n_k\) the depth of \(x_k\) which we may assume to be at least 1,
   a) if \(\eta_k = 0\) then \(\mu_{x_k}(V_k) \geq c - \frac{1}{k^{1+\varepsilon}}\), where \(c = c(x_0) > 0\) is the constant introduced in Theorem 12,
   b) if \(\eta_k\) is the \(A_\infty\)-invariant probability measure on the orbit of an \(A_\infty\)-periodic point \(y'_k\), then \(x_k \in y'_k A_\infty B_\infty^{e^{-\varepsilon} n_k}\).

Since case (a) occurs infinitely many times, the set \(\Theta_\xi\) contains a weak-star accumulation point \(\theta\) of \((\mu_{x_k})_{k \in \mathbb{N}}\) such that \(\theta(\{z\}) \geq c\).

Let \(i \in \mathbb{N}\), and let \(y_i\) be in the support of \(\mu_i\). By case (b), there are infinitely many \(k \in \mathbb{N}\) such that \(\eta_k = \mu_i\), there are infinitely many \(k \in \mathbb{N}\) such that \(x_k \in y_i A_\infty B_\infty^{e^{-\varepsilon} n_k}\). With the terminology of Section 2.4, we use Theorem 10 applied to a loxodromic element \(\gamma_0\) associated with the chosen representative \(g_0\) of the \(A_\infty\)-periodic point \(x_0\). This result gives that the period, under the geodesic flow in \(\Gamma_\infty \backslash \mathcal{H}\), of \(\pi'_{\infty}(x_k)\) (since \(C_{x_k}\) has depth \(n_k \geq 1\) in \(T^0_\nu(x_0)\)) is at most \(\sigma n_k\) for some \(\sigma > 0\) (depending only on \(p, \gamma_0, \nu\)). Applying Lemma 20 with \(y = y_i\), the set \(\Theta_\xi\) contains a weak-star accumulation point \(\theta_i\) of \((\mu_{x_k})_{k \in \mathbb{N}}\) such that \(\frac{\delta'}{2\sigma^2} \mu_i \leq \theta_i\).

This proves the result, with \(c' = \frac{\delta'}{2\sigma^2}\) (which does not depend on \(i\)).

5. Generalisation to rank-one semi-simple groups

The aim of this final section is to explain to which rank-one groups the tools introduced in this paper are applying besides \(\text{PGL}_2\). For the readability of this paper, we had restricted ourselves to the case of \(\text{PGL}_2\) in Section 2.2. We refer for instance to [29, 30] for the already known content of this section.

Let \(K\) be as in Section 2.1. Let \(G\) be a connected semi-simple linear algebraic group defined over \(K\), with \(K_\infty\)-rank one. We fix an embedding \(G \rightarrow \text{GL}_N\) for some \(N \in \mathbb{N}\). The example considered before Section 5 (and in particular in the introduction) is \(G = \text{PGL}_2\) (which is adjoint and absolutely simple).
For every \( \omega \in \mathcal{P} \) and every algebraic group \( H \) defined over \( K_\omega \) (for instance if \( H \) is defined over \( K \)), we set \( H_\omega = \overline{H(K_\omega)} \), which is a non-Archimedean Lie group.

Note that when \( \nu \in \mathcal{P}_f \), the \( K_\nu \)-rank of \( G \) may be 1 (as in the case \( G = \text{PGL}_2 \)) or not. For instance, let \( D \) be a (finite dimensional) central simple algebra over \( K \) which is ramified at \( \infty \) (that is, \( D_\infty = D \otimes_K \mathbb{K}_\infty \) is a division algebra). Then the algebraic group \( G \) with \( \overline{G(L)} = \text{PGL}_2(D \otimes_K \mathbb{K}_\infty) \) for every \( K \)-algebra \( L \) is an (adjoint absolutely quasi-simple) connected semi-simple linear algebraic group defined over \( K \), with \( \mathbb{K}_\infty \)-rank one. For all \( \nu \in \mathcal{P}_f \), the group \( G \) has \( K_\nu \)-rank 1 if and only if \( D \) ramifies at \( \nu \) (that is, when \( D_\nu = D \otimes_K K_\nu \) is a division algebra).

The next two results will be used in Remark 23 to explain the restrictions on the considered algebraic groups. The first one follows from a well-known argument of weak approximation.

**Lemma 21.** Let \( \nu \in \mathcal{P}_f \), if the \( K_\nu \)-rank of \( G \) is 1, then there exist tori \( A \) in \( G \) defined over \( K \), which splits over both \( K_\infty \) and \( K_\nu \) (hence is a maximal \( K_\infty \)-split and \( K_\nu \)-split torus).

**Proof.** By [21, Theorem 2] applied to the semisimple connected algebraic group \( G \) defined over the infinite field \( K \), there exists \( m \in \mathbb{N} \) such that the closure of the image of the diagonal embedding of \( G(K) \) in \( G_\infty \times G_\nu \) contains the subgroup of \( G_\infty \times G_\nu \) generated by \( m \)-th powers. Let \( \gamma_\infty \) and \( \gamma_\nu \) be nontrivial elements in \( G(K) \) which split over \( K_\infty \) and \( K_\nu \) respectively. Hence there exists an element in \( G(K) \) arbitrarily close to both \( \gamma_\infty^m \) and \( \gamma_\nu^n \), which therefore splits simultaneously over \( K_\infty \) and \( K_\nu \). \( \square \)

**Proposition 22.** Let \( H \) be an adjoint, absolutely quasi-simple, connected, semi-simple algebraic group over a local field \( F \) of \( F \)-rank one. Let \( T \) be a maximal \( F \)-split torus, \( Z \) its centralizer, \( P \) a minimal parabolic subgroup of \( H \) over \( F \), and \( U \) its unipotent radical. If \( H \) is isomorphic over \( F \) to the algebraic group \( L \hookrightarrow \text{PGL}_2(D \otimes_F L) \) for every \( F \)-algebra \( L \), where \( D \) is a central division algebra over \( F \), then \( Z(F) \) acts transitively on \( U(F) \setminus \{0\} \) by conjugation. If \( H \) is isomorphic over \( F \) to the algebraic group \( L \hookrightarrow \text{PU}_{1,1}(D \otimes_F L) \) for every \( F \)-algebra \( L \), where \( D \) is a quaternion division algebra over \( F \), then \( Z(F) \) acts with finitely many orbits on \( U(F) \setminus \{0\} \) by conjugation. Otherwise, \( Z(F) \) acts with infinitely many orbits on \( U(F) \setminus \{0\} \) by conjugation.

In particular, by the classification theorem [29], if furthermore \( F = K_\nu \) for some \( \nu \in \mathcal{P} \) and \( H \) is defined and isotropic over \( K \), then \( Z(F) \) acts transitively on \( U(F) \setminus \{0\} \) by conjugation if \( H \) is isomorphic over \( K \) to \( \text{PGL}_2(D) \), where \( D \) is a central division algebra over \( K \), and \( Z(F) \) acts with finitely many orbits on \( U(F) \setminus \{0\} \) by conjugation if \( H \) is isomorphic over \( K \) to \( \text{PU}_{1,1}(D) \), where \( D \) is a quaternion division algebra over \( K \).

**Proof.** Let \( H = H(F) \), \( T = T(F) \), \( Z = Z(F) \) and \( U = U(F) \). Let us denote by \([a_{ij}]\) the image in \( \text{PGL}_2 \) of a matrix \( (a_{ij}) \) in \( \text{GL}_2 \).
If $\mathcal{U}$ is non abelian (or equivalently if the (relative) root system of $H$ is not reduced), it is easy to see that the action of $\mathcal{Z}(F)$ on $\mathcal{U}(F) - \{0\}$ by conjugation has infinitely many orbits. Conversely, assume that $\mathcal{U}$ is abelian. When $F = \mathbb{C}$ (then $H = \text{PGL}_2(\mathbb{C})$) or $F = \mathbb{R}$ (then $H = \text{PO}(1, n)$), the action of $\mathcal{Z}(F)$ on $\mathcal{U}(F) - \{0\}$ by conjugation is transitive. Hence assume that $F$ is non Archimedian. By the classification theorem [30], up to isomorphism, $H$ is either $\text{PGL}_2(D)$ for a central division algebra $D$ over $F$, or $\text{PU}_{1,1}(D)$ for a quaternion division algebra $D$ over $F$ and the Hermitian form $h(z_1, z_2) = \overline{z_1}z_2 + \overline{z_2}z_1$.

In the first of the above two cases, we may take

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in F^* \right\}, \quad Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in D^* \right\}, \quad U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in D \right\}. $$

The transitivity of the action by conjugation of $Z$ on $U - \{0\}$ follows hence from the transitivity of the action of $D^* \times D^*$ on $D - \{0\}$ by $(a, d) \cdot b = abd^{-1}$, which is immediate.

In the second case, we denote by $z \mapsto \overline{z}$ the canonical involutory in the quaternion division algebra $D$ over $F$, by $N : x \mapsto x\overline{x}$ and Tr : $x \mapsto x + \overline{x}$ its (reduced) norm and trace, and by $(1, i, j, k)$ a standard basis of $D$ over $F$. Recall that $F^*/(F^*)^2$ is finite and non trivial. Indeed, this group is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (f^*/(f^*)^2)$ where $f$ is the (finite) residue field of $F$, since if $\mathcal{O}_F$ is the local ring and $\pi_F$ is a uniformizer in $F$, the map $(n, x) \mapsto \pi^n_F x$ from $\mathbb{Z} \times \mathcal{O}_F^\times$ to $F^\times$ is an isomorphism.

Let $\text{Im } D = \{x \in D : \text{Tr}(x) = 0\}$ be the $K$-vector space of purely imaginary elements of $D$, endowed with the action of the orthogonal group $O(N_{\text{Im } D})$ of the restriction to $\text{Im } D$ of the norm. Since $F^*/(F^*)^2$ is finite and $N(F^*) = (F^*)^2$, there exists a finite subset $A$ of $F^*$ such that every line in $\text{Im } D$ contains a vector whose norm lies in $A$. By Witt’s theorem, the group $O(N_{\text{Im } D})$ hence acts with finitely many orbits on the lines of $\text{Im } D$.

The group $\text{SL}_2(D)$ acts by $g \cdot M = g^T Mg$ on the 6-dimensional $F$-vector space $E = \left\{ M = \begin{pmatrix} a & b \\ b & d \end{pmatrix} : a, d \in F, b \in D \right\}$, by preserving the Dieudonné determinant $\det M = ad - N(b)$, which is a quadratic form $Q$ on $E$. Let $M_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is a $Q$-anisotropic element of $E$. The group $\text{SU}_{1,1}(D)$ is the stabilizer of $M_0$ in $\text{SL}_2(D)$ for the above action. Let $M_0^\perp$ be the 5-dimensional orthogonal of $M_0$ in $E$ for $Q$, which is invariant under $\text{SU}_{1,1}(D)$, and note that the restriction $Q_{M_0^\perp}$ is non degenerate. We consider the basis

$$\left( e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

of $M_0^\perp$, and we sometimes write matrices by blocks in the decomposition

$$(e_1, (e_2, e_3, e_4), e_5).$$
The group $T = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} : \lambda \in F^\times \right\}$ is a maximal $F$-split torus in $\text{PO}(Q_{M_0^+})$, whose centralizer $Z$ contains $Z' = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix} : A \in \text{O}(N_{\text{Im}D}) \right\}$. The projective upper triangular subgroup $P$ of $\text{PO}(Q_{M_0^+})$ is a minimal $F$-parabolic subgroup of $\text{PO}(Q_{M_0^+})$, whose unipotent radical is, by an easy computation,

$$U = \left\{ \begin{bmatrix} 1 & xN(i) & yN(j) & zN(k) & N(xi + yj + zk) \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} : x, y, z \in F \right\}.$$  

The action of $Z'$ on $U$ by conjugation thus identifies with the linear action of $\text{O}(N_{\text{Im}D})$ on $\text{Im} D$. The natural map $\text{SU}_{1,1}(D) \to \text{SO}(Q_{M_0^+})$ induced by the action of $\text{SU}_{1,1}(D)$ on $M_0^+$ is an isogeny, by the semi-simplicity of $\text{SU}_{1,1}(D)$. Hence the adjoint groups $\text{PU}_{1,1}(D)$ and $\text{PO}(Q_{M_0^+})$ are isomorphic.

For every $\lambda \in F$, the action by conjugation of $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \in T$ on each line in $U$ is the multiplication by $\lambda$. Hence the action of $T$ on each line in $U$ is transitive on its non zero vectors. The fact that the action of $Z(F)$ on $U(F) - \{0\}$ by conjugation has finitely many orbits hence follows from the fact that the action of $\text{O}(N_{\text{Im}D})$ on the lines of $\text{Im} D$ has finitely many orbits.

**Remark 23.** An appropriate version of this paper (including loss of mass phenomena of the homogeneous probability measures on the periodic orbits of the points along appropriate rays of the Hecke tree of any given periodic point of $X_\infty$) is valid when we replace $\Gamma$ by the linear algebraic group over $K$ defined

- either by $G(L) = \text{PGL}_2(D \otimes_K L)$ for every $K$-algebra $L$, where $D$ is a (finite dimensional) central division algebra over $K$ which ramifies at the places $\infty$ and $v$, and we endow the algebraic group $G$ with a $R_\infty$-structure such that $G(R_\infty) = \text{PGL}_2(\mathcal{R}_\infty)$ where $\mathcal{R}_\infty$ is a $R_\infty$-order in $D$ (see [22] for any information on orders),
- or by $G(L) = \text{PU}_{1,1}(D \otimes_K L)$ for every $K$-algebra $L$, where $D$ is a quaternion algebra over $K$ (and the underlying Hermitian form is $(z_1, z_2) \mapsto \bar{z}_1 z_2 + \bar{z}_2 z_1)$, allowing, thanks to the transitivity properties described in Proposition 22, to prove a modified version of Theorem 10, when we replace $\Gamma_\infty$ by a congruence subgroup and when we replace $A$ by any torus over $K$ in $G$ which splits over both $K_\infty$ and $K_v$ (which exists by Lemma 21).

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