A BGG-TYPE RESOLUTION FOR TENSOR MODULES OVER GENERAL LINEAR SUPERAＬGEBRA

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ABSTRACT. We construct a Bernstein-Gelfand-Gelfand type resolution in terms of direct sums of Kac modules for the finite-dimensional irreducible tensor representations of the general linear superalgebra. As a consequence it follows that the unique maximal submodule of a corresponding reducible Kac module is generated by its proper singular vector.

Key words: Bernstein-Gelfand-Gelfand resolution, singular vector, Kac module, general linear superalgebra.

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1. Introduction

The classical result of Bernstein-Gelfand-Gelfand [BGG] resolves a finite-dimensional irreducible module over a finite-dimensional semi-simple Lie algebra in terms of direct sums of Verma modules. Such a resolution is sometimes called a strong BGG resolution. In [L, RC] it was shown that the finite-dimensional simple modules may also be resolved in terms of direct sums of generalized Verma modules.

While BGG resolutions have been known to exist for integrable representations over Kac-Moody algebras (see e.g. [RCW, K]), virtually nothing is known even for finite-dimensional simple Lie superalgebras. However, what seems to be known to experts is that, in general, the finite-dimensional simple modules over a finite-dimensional simple Lie superalgebra cannot be resolved in terms of Verma modules. For example, even the natural representation of the Lie superalgebra \( \mathfrak{sl}(1|2) \) (or \( \mathfrak{gl}(1|2) \)) cannot have a resolution in terms of Verma modules (see Example 5.1).

It is therefore surprising that resolutions for a large class of finite-dimensional representations of the general linear superalgebra \( \mathfrak{gl}(m|n) \) in terms of Kac modules do exist. The purpose of this article is to construct such a resolution for every irreducible tensor module (see Section 2.3) of \( \mathfrak{gl}(m|n) \).

Roughly the idea of the construction is to exploit the connection between the irreducible tensor representations of the Lie superalgebra \( \mathfrak{gl}(m|n) \) and the polynomial representations of the general linear algebra \( \mathfrak{gl}(m+n) \) in the limit \( n \to \infty \). This allows us to construct a “weak” resolution. The strong resolution is then obtained from the weak version using Brundan’s Kazhdan-Lusztig theory of \( \mathfrak{gl}(m|n) \) [B].

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All vector spaces, algebras and tensor products are over the complex number field \( \mathbb{C} \).

2. Preliminaries

Let \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{ \infty \} \), and set \( I(m|n) = \{-m, \cdots, -1, 1, \ldots, n\} \) for \( n \in \mathbb{N} \), and \( I(m|n) = \{-m, \cdots, -1\} \cup \mathbb{N} \) for \( n = \infty \). Let \( P_{m|n} \) denote the set of partitions \( \lambda = (\lambda_{-m}, \cdots, \lambda_{-1}, \lambda_{1}, \lambda_{2}, \cdots) \) with \( \lambda_{1} \leq n \). The set \( P_{m|\infty} \) is the set of all partitions. For a partition \( \lambda \), we use \( \lambda' \), \( \ell(\lambda) \), and \( |\lambda| \) to denote its conjugate, length, and size, respectively.

2.1. The Lie algebra \( \mathfrak{gl}(m+n) \). We let \( \mathbb{C}^{m+n} \) stand for the complex space of dimension \( m+n \) with the standard basis \( \{ e_{i} | i \in I(m|n) \} \). Let \( \mathfrak{g} = \mathfrak{gl}(m+n) \) be the general linear algebra which acts naturally on \( \mathbb{C}^{m+n} \). In the case of \( n = \infty \), we let \( \mathfrak{g} \) consist of linear transformations vanishing on all but finitely many \( e_{j} \)'s. Denote by \( \{ E_{ij} | i,j \in I(m|n) \} \) the set of elementary matrices in \( \mathfrak{g} \). Then \( \{ E_{ij} | j \in I(m|n) \} \) spans the standard Cartan subalgebra \( \mathfrak{h} = \mathfrak{h}_{n} \), while \( \{ E_{ij} | i \leq j \} \) spans the standard Borel subalgebra. For \( \lambda \in \mathfrak{h}^{*} \) we denote by \( L(\mathfrak{g}, \lambda) \) the irreducible highest weight \( \mathfrak{g} \)-module with highest weight \( \lambda \).

Let \( \epsilon_{j} \in \mathfrak{h}^{*} \) be determined by \( \langle \epsilon_{j}, E_{ii} \rangle = \delta_{ij} \) for \( i, j \in I(m|n) \). Let \( \alpha_{i} = \epsilon_{i} - \epsilon_{i+1} \), for \( i \in I(m|n) \) such that \( i+1 \in I(m|n) \), and \( \alpha_{-1} = \epsilon_{-1} - \epsilon_{1} \). Then the set \( \{ \alpha_{i} \} \) is a set of simple roots of \( \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \), and we denote the set of positive and negative roots by \( \Delta^{\pm} \), respectively. Let \( \Delta^{\pm}_{0} = \Delta^{\pm} \cap (\sum_{i \neq -1} \mathbb{Z} \alpha_{i}) \) and \( \Delta^{\pm}(0) = \Delta^{\pm} \setminus \Delta^{\pm}_{0} \).

Let \( \{ \alpha_{i}^{\vee} \} \) denote the corresponding simple coroots and let \( \{ e_{i}, f_{i}, \alpha_{i}^{\vee} \} \) be the corresponding Chevalley generators of \( \mathfrak{g}' \). Let \( \rho_{c} \in \mathfrak{h}^{*} \) be determined by \( \langle \rho_{c}, E_{jj} \rangle = -j \) for \( j < 0 \), and \( \langle \rho_{c}, E_{jj} \rangle = 1 - j \) for \( j > 0 \).

The Lie algebra \( \mathfrak{g} \) has a \( \mathbb{Z} \)-gradation determined by the eigenvalues of the operator \( \frac{1}{2} \left( \sum_{i<0} E_{ii} - \sum_{j>0} E_{jj} \right) \). We have

\[ \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g} \oplus \mathfrak{g}_{1} \).

Note that \( \mathfrak{g}_{0} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \) and \( \mathfrak{g}_{-1} \cong \mathbb{C}^{m} \otimes \mathbb{C}^{n} \) as \( \mathfrak{g}_{0} \)-modules. Set \( \mathfrak{p} := \mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \) and let \( L^{0}(\lambda) \) be the irreducible representation of \( \mathfrak{g}_{0} \) with highest weight \( \lambda \in \mathfrak{h}^{*} \). We extend \( L^{0}(\lambda) \) trivially to a \( \mathfrak{p} \)-module, for which we also write \( L^{0}(\lambda) \). Denote the generalized Verma module by

\[ V(\mathfrak{g}, \lambda) := \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L^{0}(\lambda). \]

2.2. The Lie superalgebra \( \mathfrak{gl}(m|n) \). Now we let \( \mathbb{C}^{m|n} \) stand for the complex superspace of dimension \( (m|n) \) with the standard basis \( \{ \tau_{i} | i \in I(m|n) \} \). We assume that \( \deg \tau_{i} = 0 \) and \( 1 \) if \( i < 0 \) and \( i > 0 \), respectively. Let \( \mathfrak{g} = \mathfrak{gl}(m|n) \) be the general linear superalgebra acting naturally on \( \mathbb{C}^{m|n} \). For \( n = \infty \), we use a similar convention as before. Denote by \( \{ E_{ij} | i,j \in I(m|n) \} \) the set of elementary matrices in \( \mathfrak{g} \). Then \( \{ E_{ij} | j \in I(m|n) \} \) spans the standard Cartan subalgebra \( \mathfrak{h} = \mathfrak{h}_{n} \), while \( \{ E_{ij} | i \leq j \} \) spans the standard Borel subalgebra \( \mathfrak{b} \). For \( \lambda \in \mathfrak{h}^{*} \), we denote by \( L(\mathfrak{g}, \lambda) \) the irreducible highest weight \( \mathfrak{g} \)-module with highest weight \( \lambda \).
Let $\delta_i \in \overline{h}$ be determined by $\langle \delta_j, \overline{E}_{ii} \rangle = \delta_{ij}$ and let $\rho_s \in \overline{h}$ be determined by $\langle \rho_s, \overline{E}_{jj} \rangle = -j$ for $i, j \in I(\mathfrak{m|n})$. Let $\beta_i = \delta_i - \delta_{i+1}$ for $i \in I(\mathfrak{m|n})$ such that $i + 1 \in I(\mathfrak{m|n})$, and $\beta_{-1} = \delta_1 - \delta_1$. Then $\{ \beta_i \}$ is a set of simple roots of $\overline{g} = [\overline{h}, \overline{g}]$.

The Lie superalgebra $\overline{g}$ also has a $\mathbb{Z}$-gradation determined by the eigenvalues of the operator $\frac{1}{g} \left( \sum_{i < 0} E_{ii} - \sum_{j > 0} E_{jj} \right)$. We have

$$\overline{g} = \overline{g}_{-1} \oplus \overline{g}_0 \oplus \overline{g}_{+1}.$$  

Note that $\overline{g}_0 \cong \mathfrak{g}_0$ and $\overline{g}_{-1} \cong \mathbb{C}^{m \times n} \otimes \mathbb{C}^n$ as $\overline{g}_0$-modules. We set $\overline{g} := \overline{g}_0 \oplus \overline{g}_{+1}$. Given $\lambda \in \overline{h}$, we may extend $L^0(\lambda)$ trivially to a $\overline{g}$-module, which we also denote by $L^0(\lambda)$.

**Definition 1.** A $\overline{g}$-module $V$ is said to have a Kac flag if it has a filtration of $\overline{g}$-modules of the form

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{l-1} \subseteq V_l = V,$$

such that $V_j/V_{j-1}$ is isomorphic to a Kac module for $j = 1, \ldots, l$.

**Definition 2.** Let $n \in \mathbb{N} \cup \{ \infty \}$. Given a sequence of integers of the form

$$(2.1.a) \quad \mu = (\mu_{-m}, \cdots, \mu_{-1}, \mu_1, \mu_2, \cdots),$$

with $\mu_k = 0$ for $k \gg 0$ when $n = \infty$, and

$$(2.1.b) \quad \mu = (\mu_{-m}, \cdots, \mu_{-1}, \mu_1, \mu_2, \cdots, \mu_n),$$

when $n \in \mathbb{N}$, we may interpret it as $\sum_{i \geq -m, i \neq 0} \lambda_i e_i \in \overline{h}_n^*$ or $\sum_{i \geq -m, i \neq 0} \lambda_i \delta_i \in \overline{h}_n^*$. Suppose now that $\mu$ as in (2.1) such that $(\mu_1, \mu_2, \cdots)$ is a partition. We define $\mu^\circ$ to be the integer sequence

$$(2.2) \quad \mu^\circ := (\mu_{-m}, \cdots, \mu_{-1}, \mu_1', \mu_2', \cdots).$$

Let $\tilde{X}_{m|n}$ be the set of integer sequences of the form (2.1) with $\mu_j \geq \mu_{j+1}$, for all $j < n$ with $j \neq 0, -1$. Let $X_{m|n} \subseteq \tilde{X}_{m|n}$ consist of those $\mu$'s such that $(\mu_1, \mu_2, \cdots)$ is a partition. For $\mu \in X_{m|n}$, $\mu^\circ$ is well-defined, and the map $\mu \mapsto \mu^\circ$ is a bijection on $X_{m|n}$.

2.3. **Irreducible tensor $\mathfrak{gl}(m|n)$-modules.** The tensor powers of $\mathbb{C}^{m|n}$ are completely reducible as $\overline{g}$-modules. Indeed the irreducible representations that appear in these decompositions are as follows. An irreducible representation of $\overline{g}$ appears as a component of $(\mathbb{C}^{m|n})^\otimes k$ if and only if it is of the form $L(\overline{g}, \lambda^2)$, where $\lambda \in P_{m|n}$ with $|\lambda| = k$ [S, BR]. We call these irreducible $\overline{g}$-modules irreducible tensor $\overline{g}$-modules.

Let $\lambda \in P_{m|\infty}$. Clearly as $\mathfrak{g}_0$-modules $L(\mathfrak{g}, \lambda)$ and $L(\overline{g}, \lambda^2)$ are direct sums of $L^0(\eta)$ with $\eta \in X_{m|\infty}$. We have the following description of irreducible tensor $\overline{g}$-modules.

**Proposition 2.1.** Assume that $n = \infty$. For $\lambda \in P_{m|\infty}$ and $\eta \in X_{m|\infty}$, the $\mathfrak{g}_0$-module $L^0(\eta)$ is an irreducible component of $L(\mathfrak{g}, \lambda)$ if and only if the $\mathfrak{g}_0$-module $L^0(\eta^2)$ is an irreducible component of $L(\overline{g}, \lambda^2)$. Furthermore, their multiplicities coincide.
Proof. This is an immediate consequence of the well-known fact that the character of $L(\mathfrak{g}_0, \lambda^2)$ is given by the so-called Hook Schur function associated with $\lambda^2$ [BR, Theorem 6.10].

Remark 2.1. For a partition $\lambda$ with $\ell(\lambda) \leq m + n$ and $k \geq 0$, it is easy to see that $\Lambda^k(\mathfrak{g}_{-1}) \otimes L(\mathfrak{g}, \lambda)$ as a $\mathfrak{g}_0$-module decomposes into a direct sum of irreducible $\mathfrak{g}_0$-modules with highest weights belonging to $\mathfrak{X}_{m|n}$. Similarly, for $\lambda \in \mathfrak{P}_{m|n}$ and $k \geq 0$, $\Lambda^k(\mathfrak{P}_{-1}) \otimes L(\mathfrak{g}, \lambda^\natural)$ as a $\mathfrak{P}_0$-module decomposes into a direct sum of irreducible $\mathfrak{P}_0$-modules of the form $L^0(\mu)$ with $\mu \in \mathfrak{X}_{m|n}$.

3. Eigenvalues of Casimir operators

Throughout this section, we assume that $n = \infty$ unless otherwise specified.

We fix a symmetric bilinear form $(\cdot|\cdot)_c$ on $\mathfrak{h}^*$ satisfying

$$
(\lambda|\epsilon_i)_c = \langle \lambda, E_{ii} \rangle, \quad \lambda \in \mathfrak{h}^*, i \in I(m|n).
$$

By defining $(\alpha_i^\vee|\alpha_j^\vee)_c := (\alpha_i|\alpha_j)_c$ for simple coroots $\alpha_i^\vee$ and $\alpha_j^\vee$, we obtain a symmetric bilinear form on the Cartan subalgebra of $\mathfrak{g}'$, which can be extended to a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}'$ such that

$$
(e_i|f_j)_c = \delta_{ij}.
$$

Since every root space $\mathfrak{g}_\alpha$ is one-dimensional, we can choose a basis $\{u_\alpha\}$ of $\mathfrak{g}_\alpha$ for $\alpha \in \Delta^+$ and a dual basis $\{u^\alpha\}$ of $\mathfrak{g}_{-\alpha}$ with respect to $(\cdot|\cdot)_c$.

Let $V = \bigoplus_\mu V_\mu$ be a highest weight $\mathfrak{g}$-module, where $V_\mu$ denotes the $\mu$-weight space of $V$. Define $\Gamma_1 : V \to V$ to be the linear map that acts as the scalar $(\mu + 2\rho_c|\mu)_c$ on $V_\mu$. Let $\Gamma_2 := 2 \sum_{\alpha \in \Delta^+} u^\alpha u_\alpha$. The Casimir operator (cf. [J]) is defined to be

$$
\Omega := \Gamma_1 + \Gamma_2.
$$

It follows from (3.1) and (3.2) that $\Omega$ commutes with the action of $\mathfrak{g}$ on $V$ (cf. [J, Proposition 3.6]). Thus, if $V$ is generated by a highest weight vector with highest weight $\lambda$, then $\Omega$ acts on $V$ as the scalar $(\lambda + 2\rho_c|\lambda)_c$.

To produce the Casimir operator for $\mathfrak{g}$ we fix a symmetric bilinear form $(\cdot|\cdot)_s$ on $\mathfrak{h}^*$ satisfying

$$
(\lambda|\delta_i)_s = -\text{sign}(i) \langle \lambda, E_{ii} \rangle, \quad \lambda \in \mathfrak{h}^*, i \in I(m|n).
$$

An analogous argument allows us to generalize the construction above and define the Casimir operator $\Omega'$ of the Lie superalgebra $\mathfrak{g}$ that acts on a highest weight module with highest weight $\gamma \in \mathfrak{h}^*$ as the scalar $(\gamma + 2\rho_s|\gamma)_s$. We omit the details.

We will need the Weyl group of $\mathfrak{g}(m + \infty)$ in the sequel. For each $\alpha_j$, define simple reflection $\sigma_j$ by

$$
\sigma_j(\mu) := \mu - \langle \mu, \alpha_j^\vee \rangle \alpha_j,
$$

where $\mu \in \mathfrak{h}^*$. Let $W$ be the subgroup of $\text{Aut}(\mathfrak{h}^*)$ generated by the $\sigma_j$'s. For each $w \in W$, we let $l(w)$ denote the length of $w$. We have an action on $\mathfrak{h}$ given by $\sigma_j(h) = h - \langle \alpha_j, h \rangle \alpha_j^\vee$ for $h \in \mathfrak{h}$, so that $\langle w(\mu), w(h) \rangle = \langle \mu, h \rangle$ for $\mu \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. We also define

$$
w \circ \mu := w(\mu + \rho_c) - \rho_c, \quad w \in W, \quad \mu \in \mathfrak{h}^*.
$$
Consider $W_0$ the subgroup of $W$ generated by $\sigma_j$ with $j \neq -1$. Let

$$W^0 \ := \ \{ \ w \in W \mid w(\Delta^-) \cap \Delta^+ \subseteq \Delta^+(0) \ \}.$$ 

It is well-known that $W = W_0 W^0$ and $W^0$ is the set of the minimal length representatives of the right coset space $W_0/W$ (cf. [K, 1.3.17]). For $k \in \mathbb{Z}_+$, set

$$W^0_k := \{ \ w \in W^0 \mid l(w) = k \}.$$ 

Given $\lambda \in \mathcal{P}_{m|\infty}$, we have $\langle \lambda, \alpha_j^\vee \rangle \in \mathbb{Z}_+$ for all $j$. Since $w \in W^0$ implies that $w^{-1}(\Delta^+_0) \subseteq \Delta^+$, we obtain $\langle w \circ \lambda, \alpha_j^\vee \rangle \in \mathbb{Z}_+$, for all $j \neq -1$, and $w \circ \lambda \in \mathcal{X}_{m|\infty}$.

The following proposition is well-known from the theory of standard modules over generalized Kac-Moody algebras (see e.g. [J, Proposition 3.11]).

**Proposition 3.1.** For $\lambda \in \mathcal{P}_{m|\infty}$ and $\eta \in \mathcal{X}_{m|\infty}$, the irreducible $\mathfrak{g}_0$-module $L^0(\eta)$ is a component of $\Lambda^k(\mathfrak{g}_{-1}) \otimes L(\mathfrak{g}, \lambda)$ with $(\eta + 2\rho_c|\eta)_c = (\lambda + 2\rho_c|\lambda)_c$ if and only if there exists $w \in W^0_k$ with $w \circ \lambda = \eta$. Furthermore each such $L^0(\eta)$ appears with multiplicity one.

**Lemma 3.1.** For $\lambda \in \mathcal{P}_{m|\infty}$ and $\eta \in \mathcal{X}_{m|\infty}$, $L^0(\eta)$ is an irreducible $\mathfrak{g}_0$-module in $\Lambda^k(\mathfrak{g}_{-1}) \otimes L(\mathfrak{g}, \lambda)$ if and only if $L^0(\eta^\vee)$ is an irreducible $\mathfrak{g}_0$-module in $\Lambda^k(\mathfrak{g}_{-1}) \otimes L(\mathfrak{g}, \lambda^\vee)$. Furthermore, the multiplicities are the same.

**Proof.** The symmetric [H, Theorem 2.1.2] and skew-symmetric [H, Theorem 4.1.4] $(\mathfrak{g}_1, \mathfrak{g}_1)$-Howe dualities give the precise decompositions of $\Lambda^k(\mathfrak{g}_{-1}) \cong S^k(C^m \otimes C^n)$ and $\Lambda^k(\mathfrak{g}_{-1}) \cong \Lambda^k(C^m \otimes C^n)$ as $\mathfrak{g}_0$-modules, respectively. From these decompositions one sees that $L^0(\eta)$ is an irreducible component in $\Lambda^k(\mathfrak{g}_{-1})$ if and only if $L^0(\eta^\vee)$ is an irreducible component in $\Lambda^k(\mathfrak{g}_{-1})$. This fact combined with Proposition 2.1 and the compatibility of $\vee$ under tensor products completes the proof.

We need the following combinatorial lemma.

**Lemma 3.2.** Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ be a partition with $\ell(\lambda) \leq N$. For $1 \leq i \leq N$ the sets $\{ \lambda'_i - i + \frac{1}{2} \mid \lambda'_i - i + \frac{1}{2} > 0 \}$ and $\{-\lambda_i + i - \frac{1}{2} \mid \lambda_i - i + \frac{1}{2} < 0 \}$ are disjoint.

Moreover, $\{ \lambda'_i - i + \frac{1}{2} \mid \lambda'_i - i + \frac{1}{2} > 0 \} \cup \{-\lambda_i + i - \frac{1}{2} \mid \lambda_i - i + \frac{1}{2} < 0 \}$ is a permutation of the set $\{ \frac{1}{2}, \frac{2}{2}, \cdots, N - \frac{1}{2} \}$.

**Proof.** The sets $\{ \lambda'_i - i + \frac{1}{2} \mid \lambda'_i - i + \frac{1}{2} > 0 \}$, $\{-\lambda_i + i - \frac{1}{2} \mid \lambda_i - i + \frac{1}{2} < 0 \}$, $\{ \frac{1}{2}, \frac{3}{2}, \cdots, N - \frac{1}{2} \}$ are denoted by $A$, $B$ and $C$, respectively. We first observe that the sequence $\{ \lambda'_i - i + \frac{1}{2} \}_{i=1}^N$ is strictly decreasing, while $\{-\lambda_i + i - \frac{1}{2} \}_{i=1}^N$ is strictly increasing. Also $A$ and $B$ are subsets of $C$. Since $\lambda'_i - i + \frac{1}{2} > 0$ if and only if $\lambda_i - i + \frac{1}{2} > 0$, we have $i < j$ for all $\lambda'_i - i + \frac{1}{2} \in A$ and $-\lambda_j - j + \frac{1}{2} \in B$. Furthermore, the sum of the cardinality of $A$ and the cardinality of $B$ equals the cardinality of $C$. So it is enough to show $A \cap B = \emptyset$.

Suppose that $\lambda'_i - i + \frac{1}{2} \in A$ and $-\lambda_j - j + \frac{1}{2} \in B$ with $\lambda'_i - i + \frac{1}{2} = -\lambda_j - j + \frac{1}{2}$. We have $i < j$ and $\lambda'_i + \lambda_j = i + j - 1$. If $\lambda'_i \geq j$, we have $\lambda'_i + \lambda_j \geq j + i > j + i - 1$. If $\lambda'_i < j$, we have $\lambda'_i + \lambda_j < j + (i - 1) = j + i - 1$. In either case, $\lambda'_i + \lambda_j \neq i + j - 1$. Thus we have $A \cap B = \emptyset$, which completes the proof.

**Lemma 3.3.** For $\mu \in \mathcal{X}_{m|\infty}$, we have $\langle \mu + 2\rho_c | \mu \rangle_c = (\mu^2 + 2\rho_c | \mu^2)_s$. 


Proof. A direct calculation shows that the lemma is equivalent to the following identity for a partition \( \mu = (\mu_1, \mu_2, \ldots) \):
\[
\sum_{j>0} \mu_j^2 - \sum_{j>0} 2(j-1)\mu_j = \sum_{j>0} 2j\mu'_j - \sum_{j>0} (\mu'_j)^2.
\]
This identity is equivalent to
\[
(3.3) \quad \sum_{j=1}^N \left[ \left( \mu_j - \left( j - \frac{1}{2} \right) \right)^2 + \left( \mu'_j - \left( j - \frac{1}{2} \right) \right)^2 \right] = 2 \sum_{j=1}^N \left( j - \frac{1}{2} \right)^2,
\]
where \( N \geq \max(\ell(\mu), \ell(\mu')) \). However (3.3) follows readily from Lemma 3.2 applied to the partitions \( \mu \) and \( \mu' \). \( \square \)

**Proposition 3.2.** For \( \lambda \in P_{m|\infty} \) and \( \mu \in \bar{P} \), the irreducible \( \bar{\mathfrak{g}}_0 \)-module \( L^0(\mu) \) is a component of \( \Lambda^k(\bar{g}_{-1}) \otimes L(\bar{g}, \lambda^2) \) with \( (\mu + 2\rho_\lambda|_s) = (\lambda^2 + 2\rho_\lambda|_s) \) if and only if there exists \( w \in W_k^0 \) with \( \mu = (w \circ \lambda)^2 \). Furthermore, each such \( L^0(\mu) \) appears with multiplicity one.

Proof. Let \( L^0(\mu) \) be an irreducible \( \bar{\mathfrak{g}}_0 \)-module in \( \Lambda^k(\bar{g}_{-1}) \otimes L(\bar{g}, \lambda^2) \). By Remark 2.1, we have \( \mu = \eta^2 \) for some \( \eta \in X_{m|\infty} \). By Lemma 3.1, \( L^0(\eta) \) is an irreducible component of \( \Lambda^k(\bar{g}_{-1}) \otimes L(\bar{g}, \lambda) \) with the same multiplicity. By Lemma 3.3, if \( (\mu + 2\rho_\lambda|_s) = (\lambda^2 + 2\rho_\lambda|_s) \), then we have \( (\eta + 2\rho_c|_c) = (\lambda + 2\rho_c|_c) \). Furthermore by Proposition 3.1, \( \eta = w \circ \lambda \) for some \( w \in W_k^0 \), and the multiplicity of \( L^0(\mu) \) is one.

Conversely, if \( \mu = (w \circ \lambda)^2 \) for some \( w \in W_k^0 \), then by Lemma 3.3 we get
\[
(\mu + 2\rho_\lambda|_s) = (\lambda^2 + 2\rho_\lambda|_s).
\]
By Proposition 3.1, \( L^0(\mu) \) appears in \( \Lambda^k(\bar{g}_{-1}) \otimes L(\bar{g}, \lambda^2) \) with multiplicity one. Hence by Lemma 3.1 \( L^0(\mu) \) also appears in \( \Lambda^k(\bar{g}_{-1}) \otimes L(\bar{g}, \lambda^2) \) with multiplicity one. \( \square \)

4. Weak BGG-type resolutions for irreducible tensor \( \mathfrak{gl}(m|n) \)-modules

Since \( \bar{\mathfrak{g}}/\bar{\mathfrak{g}} \) is a \( \bar{\mathfrak{p}} \)-module, \( D_k := U(\bar{g}) \otimes U(\bar{g}) \Lambda^k(\bar{g}/\bar{g}) \) is a \( \bar{\mathfrak{g}} \)-module with \( \bar{\mathfrak{g}} \) acting on the first factor, for \( k \geq 0 \). Define the sequence
\[
(4.1) \quad \ldots \xrightarrow{\partial_{k+1}} D_k \xrightarrow{\partial_{k}} D_{k-1} \xrightarrow{\partial_{k-1}} \ldots \xrightarrow{\partial_1} D_0 \xrightarrow{\epsilon} \mathbb{C} \rightarrow 0,
\]
where \( \epsilon \) is the augmentation map from \( U(\bar{g}) \) to \( \mathbb{C} \) and
\[
\partial_k(a \otimes \bar{x}_1 \bar{x}_2 \cdots \bar{x}_k) := \sum_{j=1}^k ax_j \otimes \bar{x}_1 \cdots  \hat{x}_j \cdots \bar{x}_k,
\]
for \( a \in U(\bar{g}) \) and \( x_i \in \bar{g} \). Here \( \bar{x}_j \) denotes the image of \( x_j \) in \( \bar{g}/\bar{g} \) under the natural map. One easily checks that the \( \partial_k \)'s are well-defined \( U(\bar{g}) \)-maps and (4.1) is a chain complex. The exactness of (4.1) follows, for example, from the exactness of the dual of the Koszul complex [K, Appendix D.13] (see also [KK]).

For \( \lambda \in P_{m|n} \) and \( k \geq 0 \), \( Y_k := D_k \otimes L(\bar{g}, \lambda^2) \) is a \( \bar{\mathfrak{g}} \)-module. Tensoring (4.1) with \( L(\bar{g}, \lambda^2) \) we obtain an exact sequence [GL, K, J]
\[
(4.2) \quad \ldots \xrightarrow{d_{k+1}} Y_k \xrightarrow{d_k} Y_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_1} Y_0 \xrightarrow{d_0} L(\bar{g}, \lambda^2) \rightarrow 0,
\]
where $d_k := \partial_k \otimes 1$ for $k > 0$ and $d_0 := \epsilon \otimes 1$.

Let $V$ be a $\mathfrak{g}$-module, on which the action of $\mathfrak{g}_{-1}$ is locally nilpotent. We define

$$V^c := \{ v \in V \mid (\overline{\Omega} - c)^l v = 0 \text{ for } l \gg 0 \},$$

i.e. $V^c$ is the generalized $\overline{\Omega}$-eigenspace corresponding to the eigenvalue $c \in \mathbb{C}$. Clearly we have $V = \bigoplus_{c \in \mathbb{C}} V^c$. Put

$$c_\lambda = (\lambda^2 + 2\rho_s|\lambda^2)_s.$$

The restriction of (4.2) to the generalized $c_\lambda$-eigenspace of $\overline{\Omega}$ produces a resolution of $\mathfrak{g}$-modules

$$\cdots \to Y_{k+1}^{c_\lambda} \to Y_k^{c_\lambda} \to \cdots \to Y_1^{c_\lambda} \to Y_0^{c_\lambda} \to L(\mathfrak{g}, \lambda^2) \to 0. \quad (4.3)$$

**Proposition 4.1.** Assume that $n = \infty$. For $\lambda \in \mathcal{P}_m|\infty$, we have a resolution of $\mathfrak{g}$-modules of the form

$$\cdots \to Z_{k+1} \to Z_k \to \cdots \to Z_1 \to Z_0 \to L(\mathfrak{g}, \lambda^2) \to 0$$

such that each $Z_k$ has a Kac flag. Furthermore, $Z_k \cong \bigoplus_{w \in W_k} V(\mathfrak{g}, (w \circ \lambda)^2)$ as $\mathfrak{g}_{-1} + \mathfrak{g}_0$-modules.

**Proof.** Observe that $Y_k \cong U(\mathfrak{g}) \otimes U(\mathfrak{f}) \otimes L(\mathfrak{g}, \lambda^2)$. Suppose that as $\mathfrak{g}_0$-module, we have $\Lambda^k(\mathfrak{g}/\mathfrak{f}) \otimes L(\mathfrak{g}, \lambda^2) \cong \bigoplus_{\mu \in \mathcal{I}} L^0(\mu)$ for some multiset of weights $\mathcal{I}$. The $\mathfrak{g}$-module $\Lambda^k(\mathfrak{g}/\mathfrak{f}) \otimes L(\mathfrak{g}, \lambda^2)$ has a composition series, where the multiset of composition factors is precisely the multiset of $\mathfrak{g}$-module $L^0(\mu)$, $\mu \in \mathcal{I}$. Thus $Y_k$ has a Kac flag and $Y_k \cong \bigoplus_{\mu \in \mathcal{I}} V(\mathfrak{g}, \mu)$ as $\mathfrak{g}_{-1} + \mathfrak{g}_0$-modules. Now $\overline{\Omega}$ acts on $V(\mathfrak{g}, \mu)$ as the scalar $(\mu + 2\rho_s|\mu)_s$, and hence $Z_k = Y_k^{c_\lambda} \cong \bigoplus_{\mu} V(\mathfrak{g}, \mu)$, where the summation is over all $\mu \in \mathcal{I}$ such that $(\mu + 2\rho_s|\mu)_s = (\lambda^2 + 2\rho_s|\lambda^2)_s$. Proposition 3.2 now says that this set is precisely $\{ (w \circ \lambda)^2 \mid w \in W_k \}$. 

**Corollary 4.1.** Assume that $n \in \mathbb{N}$. For $\lambda \in \mathcal{P}_m|n$, we have a resolution of $\mathfrak{g}$-modules of the form

$$\cdots \to Z_{k+1,n} \to Z_k \to \cdots \to Z_0 \to L(\mathfrak{g}, \lambda^2) \to 0$$

such that each $Z_{k,n}$ has a Kac flag. Furthermore, $Z_{k,n} \cong \bigoplus_{w \in W_k} V(\mathfrak{g}, (w \circ \lambda)^2)$ as $\mathfrak{g}_{-1} + \mathfrak{g}_0$-modules. Here, by definition we have $V(\mathfrak{g}, \nu^2) = 0$ for $\nu \in \mathcal{X}_m|\infty$ with $\nu_1 > n$.

**Proof.** The corollary follows from applying the truncation functor $\text{tr}_n$ [CWZ, Definition 4.4] upon the resolution in Proposition 4.1 and using the facts that the truncation functor is an exact functor and is compatible with both irreducible and Kac modules [CWZ, Corollary 4.6]. 

5. **Strong BGG-type resolutions for irreducible tensor $\mathfrak{g}(m|n)$-modules**

For $n \in \mathbb{N}$ recall the definition of the super Bruhat ordering for $\mathfrak{g} = \mathfrak{g}(m|n)$ on $\mathcal{X}_m|n$ in [B, §2-b], which we denote by $\preceq$. This gives a partial ordering on $\mathcal{X}_m|n$. We can restrict $\preceq$ to $\mathcal{X}_m|n$, which can be defined for $\mathcal{X}_m|\infty$ as well (cf. [CWZ, Section 2.3]). Now we may also regard $\mathcal{X}_m|n$ as weights of $\mathfrak{g} = \mathfrak{g}(m + n)$. In doing so the usual Bruhat
ordering of $\mathfrak{g}$ determines a partial ordering $\leq$ on $\tilde{X}_{m/n}$ (see e.g. [CWZ, Section 2.2]), which restricts to $X_{m/n}$, and which in turn can be defined for $X_{m/\infty}$ as well. We have the following.

**Lemma 5.1.** [CWZ, Lemma 6.6] Let $\lambda, \mu \in X_{m/\infty}$. Then $\lambda \preceq \mu$ if and only if $\lambda \preceq \mu$.

In the remainder of this section we assume that $n \in \mathbb{N}$ unless otherwise specified.

**Lemma 5.2.** Let $n \in \mathbb{N}$ and $\lambda, \mu \in \tilde{X}_{m/n}$. Suppose that $\mu \not\preceq \lambda$. Then

$$\text{Hom}_{\mathfrak{g}}(V(\mathfrak{g}, \mu), V(\mathfrak{g}, \lambda)) = 0.$$ 

**Proof.** Suppose that $\text{Hom}_{\mathfrak{g}}(V(\mathfrak{g}, \mu), V(\mathfrak{g}, \lambda)) \neq 0$. Then $L(\mathfrak{g}, \mu)$ is a composition factor of the Kac module $V(\mathfrak{g}, \lambda)$. It follows from [B, Corollary 3.36 (i)] and [B, Theorem 4.37] that $\mu \preceq \lambda$. □

**Lemma 5.3.** Let $n \in \mathbb{N}$ and $\mu \in \tilde{X}_{m/n}$. Suppose that $M$ is a finite-dimensional $\mathfrak{g}$-module with a Kac flag

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M,$$

and $\text{Hom}_{\mathfrak{g}}(M_0/M_{i-1}, V(\mathfrak{g}, \mu)) = 0$ for all $i = 1, \ldots, l$. Then

$$\text{Hom}_{\mathfrak{g}}(M, V(\mathfrak{g}, \mu)) = 0.$$ 

**Proof.** Since $M$ is finite-dimensional we have $M_i/M_{i-1} \cong V(\mathfrak{g}, \mu_i)$ with $\mu_i \in \tilde{X}_{m/n}$ for all $i$. Consider the exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0.$$ 

Noting that $M/M_1$ has a Kac flag of length $l - 1$, the lemma follows easily from the long exact sequence and induction on $l$. □

**Lemma 5.4.** Let $n \in \mathbb{N}$ and $\lambda, \mu \in X_{m/n}$. Suppose that $\lambda$ and $\mu$ are not comparable under the super Bruhat ordering. Then

$$\text{Ext}^1(V(\mathfrak{g}, \lambda), V(\mathfrak{g}, \mu)) = 0.$$ 

**Proof.** Consider $P(\lambda)$ the projective cover (in the category of finite-dimensional $\mathfrak{g}$-modules) of $L(\mathfrak{g}, \lambda)$. We have an exact sequence

$$(5.1) \quad 0 \to K \to P(\lambda) \to V(\mathfrak{g}, \lambda) \to 0.$$ 

Now $P(\lambda)$ has a Kac flag [Z, Proposition 2.5] and hence so has $K$. By [B, Theorem 4.37], $P(\lambda)$ is a tilting module and if $V(\mathfrak{g}, \gamma)$ with $\gamma \neq \lambda$ appears in a Kac flag of $P(\lambda)$, then $\gamma \in \tilde{X}_{m/n}$ and $\gamma \succ \lambda$.

Now the induced long exact sequence from (5.1) gives rise to the following exact sequence

$$\text{Hom}_{\mathfrak{g}}(K, V(\mathfrak{g}, \mu)) \to \text{Ext}^1(V(\mathfrak{g}, \lambda), V(\mathfrak{g}, \mu)) \to 0.$$ 

Since all $V(\mathfrak{g}, \gamma)$ that appears in the Kac flag of $K$ are such that $\gamma \succ \lambda$, we see that $\gamma \not\preceq \mu$ by hypothesis. Thus by Lemmas 5.2 and 5.3, $\text{Hom}_{\mathfrak{g}}(K, V(\mathfrak{g}, \mu)) = 0$, and the lemma follows. □
Theorem 5.1. For \( n \in \mathbb{N} \) and \( \lambda \in P_{m|n} \), we have a resolution of \( \mathfrak{g} \)-modules of the form
\[
\cdots \xrightarrow{d_{k+1}} Z_{k,n} \xrightarrow{d_k} Z_{k-1,n} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} Z_{0,n} \xrightarrow{d_0} L(\mathfrak{g}, \lambda^2) \to 0,
\]
where \( Z_{k,n} \cong \bigoplus_{w \in W_k^n} V(\mathfrak{g}, (w \circ \lambda)^2) \) as \( \mathfrak{g} \)-modules. As before, by definition, we have \( V(\mathfrak{g}, \nu^2) = 0 \) for \( \nu \in X_{m|\infty} \) with \( \nu_1 > n \).

Proof. We have a natural embedding of \( X_{m|N} \) into \( X_{m|N+1} \) for any \( N \in \mathbb{N} \). Also we have the truncation map \( X_{m|N+1} \xrightarrow{\text{Tr}_{N+1,N}} X_{m|N} \) [CWZ, Section 6.6] that sends an element \( \lambda = (\lambda_m, \ldots, \lambda_{N+1}) \) to \( \lambda = (\lambda_m, \ldots, \lambda_N) \), if \( \lambda_{N+1} = 0 \), and to \( \emptyset \) otherwise. The usual Bruhat orderings of \( X_{m|N} \) and \( X_{m|N+1} \) are compatible in the following sense:

(i) For \( \lambda, \mu \in X_{m|N} \), one has \( \lambda \leq \mu \) if and only if \( \text{Tr}_{N,N+1}(\lambda) \leq \text{Tr}_{N,N+1}(\mu) \).

(ii) For \( \lambda, \mu \in X_{m|N+1} \) with \( \text{Tr}_{N,N+1}(\lambda) \neq \emptyset, \text{Tr}_{N,N+1}(\mu) \neq \emptyset \), one has \( \lambda \leq \mu \) if and only if \( \text{Tr}_{N,N+1}(\lambda) \leq \text{Tr}_{N,N+1}(\mu) \).

Thus the Bruhat ordering of \( X_{m|N} \) is compatible with that of \( X_{m|\infty} \).

We view \( \lambda \) as a weight of \( \mathfrak{g}(m + \infty) \) and so as an element in \( X_{m|\infty} \). For a fixed \( j \in \mathbb{N} \), it is not hard to see that the weights \( \{ w \circ \lambda \mid w \in W_j^0 \} \) form a finite set and they all may be regarded as lying in the same \( X_{m|N} \), for \( N \gg 0 \). Thus we may regard them all as weights of \( \mathfrak{g}(m + N) \) for some \( N \gg 0 \). But for such weights, it is well-known from classical theory of semi-simple Lie algebras that they are not comparable under the usual Bruhat ordering (see e.g. [K, Lemma 1.3.16]). Thus viewing them as weights of \( \mathfrak{g}(m + \infty) \), they are not comparable under the Bruhat ordering, either. Hence, by Lemma 5.1, the weights \( (w \circ \lambda)^2 \) are not comparable under the super Bruhat ordering of \( \mathfrak{g}(m | \infty) \). The theorem now follows from a similar compatibility of the super Bruhat orderings of \( \mathfrak{g}(m | \infty) \) and of \( \mathfrak{g}(m | n) \), Lemma 5.4 and Corollary 4.1. \( \square \)

Remark 5.1. Note that \( W \) above is the infinite Weyl group of \( \mathfrak{g}(m + \infty) \), even though we are considering the finite-dimensional Lie superalgebra \( \mathfrak{g}(m | n) \).

Remark 5.2. Theorem 5.1 has the counterpart in the case of \( n = \infty \) as well.

Recall that for \( \lambda, \mu \in X_{m|n} \) with \( \lambda \succ \mu \) there is a relative length function defined in [B, §3-g], which we denote by \( \overline{\ell}(\mu, \lambda) \). Fix \( \lambda \in P_{m|n} \) so that \( \lambda^2 \in X_{m|n} \). For \( \mu \in X_{m|n} \) with \( \lambda^2 \succ \mu \) define an absolute length function by
\[
\overline{\ell}(\mu) := \overline{\ell}(\mu, \lambda^2).
\]
We can now formulate Theorem 5.1 intrinsically without referring to the infinite Weyl group of \( \mathfrak{g}(m + \infty) \) as follows.

Theorem 5.2. For \( n \in \mathbb{N} \) and \( \lambda \in P_{m|n} \), we have a resolution of \( \mathfrak{g} \)-modules of the form
\[
\cdots \xrightarrow{d_{k+1}} Z_{k,n} \xrightarrow{d_k} Z_{k-1,n} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} Z_{0,n} \xrightarrow{d_0} L(\mathfrak{g}, \lambda^2) \to 0,
\]
where \( Z_{k,n} \cong \bigoplus_{\overline{\ell}(\mu)=k} V(\mathfrak{g}, \mu) \) as \( \mathfrak{g} \)-modules.
Proof. For \( \nu, \mu \in \tilde{X}_{m|n} \) recall Brundan’s Kazhdan-Lusztig polynomials \( l_{\mu}(q) \) of [B, (2.18)]. By [B, Theorem 4.51] and [Z, Theorem 5.1] we have the following cohomological interpretation:
\[
l_{\mu}(q) = \sum_{i=0}^{\infty} \dim \left[ \text{Hom}_{\mathfrak{g}} \left( L^0(\mu), H^i \left( \mathfrak{g}_{-1}; L(\mathfrak{g}, \nu) \right) \right) \right] q^i.
\]

The calculation of the \( \mathfrak{g}_{-1} \)-cohomology groups in [CZ, Corollary 4.14] now implies that
\[
l_{\mu}(q) = \begin{cases} q^k, & \text{if there exists } w \in W_\mu \text{ with } \mu = (w \circ \lambda)^2 \text{ and } (w \circ \lambda)_1 \leq n, \\ 0, & \text{otherwise.} \end{cases}
\]

From [B, Corollary 3.45] we conclude that for such \( \mu \) we have \( k = \tilde{\ell}(\mu) \). On the other hand if \( \mu \in \tilde{X}_{m|n} \) with \( \tilde{\ell}(\mu) = k \), then [B, Corollary 3.45] implies that \( l_{\mu}(q) \neq 0 \) and hence \( \mu \) is of the form \((w \circ \lambda)^2\) with \( w \in W_\mu \). Thus for \( \mu \in \tilde{X}_{m|n} \) the condition that there exists \( w \in W_\mu \) with \( \mu = (w \circ \lambda)^2 \) is equivalent to the condition that \( \tilde{\ell}(\mu) = k \). This together with Theorem 5.1 completes the proof.

We record the following corollary of the proof of Theorem 5.2.

**Corollary 5.1.** Let \( \lambda \in P_{m|n} \). As a \( \mathfrak{g}_0 \)-module we have, for all \( k \in \mathbb{Z}_+ \),
\[
H^k \left( \mathfrak{g}_{-1}; L(\mathfrak{g}, \lambda) \right) \cong \bigoplus_{\tilde{\tau}(\mu) = k} L^0(\mu).
\]

We conclude with an example, which shows that finite-dimensional irreducible representations over a simple Lie superalgebra cannot be resolved in terms of direct sums Verma modules in general.

**Example 5.1.** Let \( \lambda \in \mathfrak{g}^* \) and let \( C_\lambda \) denote the one-dimensional \( \mathfrak{g} \)-module that transforms by \( \lambda \). We extend \( C_\lambda \) trivially to a \( \mathfrak{g} \)-module and denote by \( M(\mathfrak{g}, \lambda) = \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} C_\lambda \) the Verma module of highest weight \( \lambda \). Suppose that \( L(\mathfrak{g}, \lambda) \) can be resolved in terms of Verma modules. Then we have an exact sequence of \( \mathfrak{g} \)-modules of the form
\[
\cdots \to \bigoplus_{i \in I} M(\mathfrak{g}, \mu_i) \to M(\mathfrak{g}, \lambda) \to L(\mathfrak{g}, \lambda) \to 0,
\]
and \( \text{Hom}(M(\mathfrak{g}, \mu_i), M(\mathfrak{g}, \lambda)) \neq 0 \), for all \( i \in I \). It follows that there exist singular vectors \( v_i \) of weight \( \mu_i \) in \( M(\mathfrak{g}, \lambda) \). \( \text{Im} \psi = \text{Ker} \phi \) implies that the unique maximal submodule of \( M(\mathfrak{g}, \lambda) \) must be generated by the proper singular vectors of \( M(\mathfrak{g}, \lambda) \).

Now consider \( \lambda = \delta_{-1} \) and \( \mathfrak{g} = \mathfrak{gl}(1|2) \) or \( \mathfrak{g} = \mathfrak{sl}(1|2) \). In the sequel we will suppress \( \mathfrak{g} \). One can show by a direct calculation that the only proper singular vectors in the Verma module \( M(\delta_{-1}) \) are scalar multiples of either \( E_{2,1}v \) or \( E_{1,-1}E_{2,1}v \), where \( v \) is a highest weight vector of \( M(\delta_{-1}) \). If \( M_1 \) is the submodule of \( M(\delta_{-1}) \) generated by \( E_{2,1}v \), then \( M_1 \) is the submodule generated by all proper singular vectors of \( M(\delta_{-1}) \). But \( \text{dim}(M(\delta_{-1})/M_1) = 4 \) by the PBW Theorem and, since \( \text{dim}L(\delta_{-1}) = 3 \), it follows that \( M_1 \) cannot be the unique maximal submodule of \( M(\delta_{-1}) \). Thus \( L(\delta_{-1}) \) cannot have a resolution in terms of Verma modules. We note that \( M(\delta_{-1})/M_1 \) is isomorphic to the Kac module of highest weight \( \delta_{-1} \) and \( E_{1,-1}E_{2,-1}v \) is a singular vector in \( M(\delta_{-1})/M_1 \).
Theorem 5.1, the fact that $H^1(\mathfrak{T}_{n+1}; L(\mathfrak{g}, \lambda^3))$ is irreducible, and the discussion in Example 5.1 imply the following.

**Corollary 5.2.** Let $n \in \mathbb{N}$ and $\lambda \in \mathcal{P}_{m|n}$. The unique maximal submodule of a reducible $V(\mathfrak{g}, \lambda^3)$ is generated by the proper singular vector of $V(\mathfrak{g}, \lambda^3)$.

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