Proper Kasparov Cycles and the Baum–Connes Conjecture

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Abstract

We introduce the notion of proper Kasparov cycles for Kasparov’s $G$-equivariant KK-theory for a general locally compact, second countable topological group $G$. We show that for any proper Kasparov cycle, its induced map on $K$-theory factors through the left-hand side of the Baum–Connes conjecture. This allows us to upgrade the direct splitting method, a recent new approach to the Baum–Connes conjecture which, in contrast to the standard gamma element method (the Dirac dual-Dirac method), avoids the need of constructing proper algebras and the Dirac and the dual-Dirac elements. We introduce the notion of Kasparov cycles with Property $(\gamma)$ removing the $G$-compact assumption on the universal space $E_G$ in the previous paper [Nis19]. We show that the existence of a cycle with Property $(\gamma)$ implies the split-injectivity of the Baum–Connes assembly map for all coefficients. We also obtain results concerning the surjectivity of the assembly map.

Introduction

In 1988, Kasparov [Kas88] proved the Strong Novikov conjecture, in particular the Novikov conjecture, for all groups which act properly and isometrically on a complete, simply connected Riemannian manifold of non-
positive sectional curvature, or on a homogeneous space \( G/K \) for an almost connected group \( G \) and its maximal compact subgroup \( K \), or more generally on what he called a special manifold. His method, which we currently call the \( \gamma \)-element method (or the Dirac and dual-Dirac method), became a powerful and versatile approach for attacking the Novikov conjecture and the Baum–Connes conjecture.

One of the striking hidden features of his method, as we now review below, is that it does not require any sort of cocompactness assumption for the group actions involved. We emphasize this point since in the study of isomorphism conjectures for K-theory or L-theory, a cocompactness assumption for the universal space (i.e. compactness assumption for the classifying space) has often been required and this is not a trivial issue: see for example, the decent principle [Roe96, Theorem 8.4] or [CP95], [CPV98].

The purpose of this paper is to introduce a simple concept, that of a proper \( KK \)-cycle, explain its relevance to Kasparov’s work, and streamline some of Kasparov’s arguments using it. The same notion also allows us to upgrade the “direct splitting method” for the Baum–Connes conjecture introduced in [Nis19].

To use the language of the Baum–Connes conjecture, Kasparov showed that the Baum–Connes assembly map (see [BCH94])

\[
(0.1) \quad \mu_G^A: \mathrm{RKK}_*(\mathcal{L}G, A) \to K_*(A \rtimes_r G)
\]

is split-injective for any coefficient \( G \)-\( C^* \)-algebra \( A \) and for any group \( G \) which acts properly and isometrically on a manifold \( M \) as above. Kasparov constructed and used a suitable proper \( G \)-\( C_0(M) \)-algebra \( A_M \) and morphisms \( \alpha \) in \( KK_*^G(A_M, \mathbb{C}) \) and \( \beta \) in \( KK_*^G(\mathbb{C}, A_M) \). The morphisms \( \alpha \) and \( \beta \) are called the Dirac element and the dual-Dirac element respectively. He defined the element \( \gamma \), which we call the gamma element for \( G \), as the composition \( \beta \otimes_{A_M} \alpha \) in \( KK^G(\mathbb{C}, \mathbb{C}) \).

Kasparov showed that the gamma element \( \gamma \) is an idempotent in the Kasparov ring \( R(G) = KK^G(\mathbb{C}, \mathbb{C}) \) which defines an endomorphism \( \gamma_* \) on the K-theory group \( K_*(A \rtimes_r G) \). He showed that the image of the assembly map \( \mu_A^G \) coincides with that of \( \gamma_* \), meaning that \( \mu_A^G \) is an isomorphism if and only if \( \gamma_* \) is the identity map on \( K_*(A \rtimes_r G) \). In this process, the assembly
map (0.1) is factored as \((A = \mathbb{C} \text{ for simplicity})\):

\[
\begin{array}{c}
\text{RK}^*_r(EG, \mathbb{C}) \xrightarrow{\mu^G} K^*_r(C^*_r(G)) \\
\mu^G_{AM} \circ \beta_* \xrightarrow{\cong} K_*(AM \rtimes_r G) \xrightarrow{\cong} K_*(A_M \rtimes_r G)
\end{array}
\]

where \(j^G_r(\alpha)_*\), \(j^G_r(\beta)_*\) are defined by Kasparov’s descent. The isomorphism \(\mu^G_{AM} \circ \beta_*\) is interpreted as an analogue of Poincaré duality in K-theory and K-homology. A left-inverse of the assembly map \(\mu^G_{AM}\) can be defined as the composition of \(j^G_r(\beta)_*\) and the inverse of the (duality) isomorphism (0.2)

\[
\mu^G_{AM} \circ \beta_*: \text{RK}^*_r(EG, \mathbb{C}) \cong K_*(A_M \rtimes_r G).
\]

It is the isomorphism (0.2) that miraculously tames the left-hand side group \(\text{RK}^*_r(EG, \mathbb{C})\), which is the inductive limit of a family of K-homology groups for all G-compact proper G-spaces, by identifying it with the single K-theory group \(K_*(AM \rtimes_r G)\), and the assembly map \(\mu^G\) with the map \(j^G_r(\alpha)_*\). We remark that, in an abstract level, a result by Meyer and Nest [MN06] says that for all groups \(G\), such a “miraculous” identification always exists in a canonical way: there is a suitable \(G-\mathbb{C}^*\)-algebra \(P\) built up from proper algebras and a morphism \(\alpha\) in \(\text{KK}^G(P, \mathbb{C})\) (called the Dirac morphism) both of which are canonical in a certain sense so that the assembly map (0.1) is factored as \((A = \mathbb{C} \text{ for simplicity})\):

\[
\begin{array}{c}
\text{RK}^*_r(EG, \mathbb{C}) \xrightarrow{\mu^G} K^*_r(C^*_r(G)) \\
\mu^G_{AM} \circ \beta_* \xrightarrow{\cong} K_*(P \rtimes_r G)
\end{array}
\]

identifying the assembly map \(\mu^G\) as the map \(j^G_r(\alpha)_*\).

Originating from work of Kasparov, one of the standard formulations of the gamma element and the gamma element method is as follows:

**Definition.** (see [Tu00]) An element \(x\) in the Kasparov ring \(R(G) = \text{KK}^G(\mathbb{C}, \mathbb{C})\) is called a gamma element for \(G\) and written as \(\gamma\) (or \(\gamma_G\)) if:

1. \(x = 1_K\) (the multiplicative identity) in \(R(K)\) for any compact subgroup \(K\) of \(G\).
2. there is a proper G-C*-algebra \( A, \alpha \) in \( KK^G(A, \mathbb{C}) \) and \( \beta \) in \( KK^G(\mathbb{C}, A) \) such that \( x = \beta \otimes_A \alpha \).

**Theorem.** (see [Tu00]) A gamma element for \( G \), if exists, is the unique idempotent in \( R(G) \) characterized by the listed properties. If a gamma element \( \gamma \) exists for \( G \), then:

1. the Strong Novikov conjecture holds for \( G \), i.e. the assembly map \( \mu^G \) is split-injective for any \( A \).

2. the assembly map \( \mu^G \) is an isomorphism if and only if \( \gamma_* \) is the identity map on \( K_*(A \rtimes_r G) \) where \( \gamma_* \) is defined via the composition

\[
KK^G(\mathbb{C}, \mathbb{C}) \xrightarrow{\sigma} KK^G(A, A) \xrightarrow{j_G} KK(A \rtimes_r G, A \rtimes_r G) \to \text{End}(K_*(A \rtimes_r G)).
\]

We remark that there is a different (a-priori, weaker) definition of a gamma element by Meyer and Nest [MN06]. For this definition, Emerson and Meyer [EM07] showed that, for a torsion free discrete group \( G \) with a finite dimensional classifying space \( BG \), the existence of the gamma element only depends on the coarse geometry of the group. They showed that the existence of a gamma element (in Meyer–Nest sense) is equivalent to isomorphism of a certain coarse co-assembly map.

In this paper, we introduce the following simple notion of proper Kasparov cycles. For a locally compact, (second countable) \( G \)-space \( X \), by a \( G \)-Hilbert space \( H \) over \( X \), we mean a \( G \)-Hilbert space \( H \) equipped with a \( G \)-equivariant, non-degenerate representation of \( C_0(X) \). Let \( (H, T) \) be a cycle defining an element \([H, T]\) in the Kasparov ring \( R(G) = KK^G(\mathbb{C}, \mathbb{C})\): the Hilbert space \( H \) is equipped with a grading and a unitary representation of \( G \); the odd, self-adjoint, bounded, \( G \)-continuous operator \( T \) is such that \( 1 - T^2 \) and \( g(T) - T \) are compact operators for any \( g \) in \( G \).

**Definition.** We say that a Kasparov cycle \((H, T)\) for \( KK^G(\mathbb{C}, \mathbb{C}) \) is proper if for some proper \( G \)-space \( X \), \( H \) is a \( G \)-Hilbert space over \( X \) such that for any \( \phi \) in \( C_0(X) \),

\[
\text{the function } g \mapsto [g(\phi), T] \text{ belongs to } C_0(G, \mathcal{R}(H))
\]

where \( \mathcal{R}(H) \) is the algebra of compact operators.

It turns out that for any proper Kasparov cycle \((H, T)\), the map \([H, T]_*\) on \( KK(\mathbb{C}, A \rtimes_r G) \) defined via the composition (0.3) factors through the left-hand side of the Baum–Connes conjecture:
Theorem A. (Theorem 2.1) For any proper Kasparov cycle \((H, T)\) for \(\text{KK}^G(C, \mathbb{C})\), there is a well-defined, natural homomorphism

\[ \nu^G_{A, (H, T)} : \text{KK}(C, A \rtimes_r G) \to \text{RKK}^G(E_G, A) \]

for any \(G\)-\(C^*\)-algebra \(A\) such that the composition \(\mu^G_A \circ \nu^G_{A, (H, T)}\) coincides with \([H, T]\) on \(\text{KK}(C, A \rtimes_r G)\) defined via the composition (0.3). Namely, we have:

\[
\begin{array}{ccc}
K_*(A \rtimes_r G) & \xymatrix{ \ar[r]^{[H, T]} & } & K_*(A \rtimes_r G) \\
\nu^G_{A, (H, T)} & \xymatrix{ \ar[r] & } & \mu^G_A
\end{array}
\]

An alternative approach to the Baum–Connes conjecture, which we call the direct splitting method, was introduced in [Nis19]. In this previous work, under the assumption that a group \(G\) admits a \(G\)-compact model of the universal proper \(G\)-space \(E_G\), we defined the notion of Property \((\gamma)\) for a cycle for the Kasparov ring \(R(G)\). It was shown that if there is such a cycle with Property \((\gamma)\), the Baum–Connes assembly map \(\mu^G_A\) is split-injective for any \(A\). A \((\gamma)\)-element was defined to be any element in \(R(G)\) represented by a cycle with Property \((\gamma)\). It was shown that a \((\gamma)\)-element, if exists, is the unique idempotent in \(R(G)\) characterized by the property. Moreover, it was shown that if a gamma element \(\gamma\) (as in Definition and Theorem above) exists, \(\gamma\) is a \((\gamma)\)-element, and hence the two notions coincide in this case.

Theorem A allows us to upgrade the direct splitting method for a general group \(G\) which may not admit a \(G\)-compact model of the universal proper \(G\)-space.

Definition. We say that a Kasparov cycle \((H, T)\) for \(\text{KK}^G(C, \mathbb{C})\) has Property \((\gamma)\) if:

1. \((H, T)\) is a proper Kasparov cycle.
2. \([H, T] = 1_K\) in \(R(K)\) for any compact subgroup \(K\) of \(G\).

We note that if there is a \(G\)-compact model \(E\) of the universal proper \(G\)-space \(E_G\), this definition gives an a-priori weaker notion of Property \((\gamma)\) compared to the one defined in [Nis19] which requires \((H, T)\) to be a.
proper cycle over the specific space, namely $E$ whereas the current definition allows us to use any (locally compact, second countable) proper $G$-space $X$, which is not necessarily $G$-compact or universal. For this reason, we prefer to call the version of Property $(\gamma)$ in [Nis19] as Property $(\gamma)^G$-compact.

The following is a simple consequence of Theorem A:

**Theorem B. (Theorem 3.3)** Suppose there is a Kasparov cycle $(H, T)$ for $\text{KK}^G(C, C)$ with Property $(\gamma)$. Then:

1. the Strong Novikov conjecture holds for $G$, i.e. the assembly map $\mu^G_A$ is split-injective for any $A$.
2. the morphism $\nu^G_A(H, T)$, which we call the $(\gamma)$-morphism, is a left-inverse of the assembly map $\mu^G_A$.
3. the assembly map $\mu^G_A$ is an isomorphism if and only if $[H, T]_*$ is the identity map on $\text{KK}(C, A \rtimes_r G)$ where $[H, T]_*$ is defined via the composition (0.3).

Property $(\gamma)$ is supposed to capture the essential property of the gamma element. Hence, the following may not be surprising:

**Theorem C.** Suppose there is a gamma element $\gamma$ for $G$. Then, $\gamma$ is represented by some Kasparov cycle $(H, T)$ with Property $(\gamma)$.

Theorem B and Theorem C explain the relevance of our approach to Kasparov work.

**Definition.** We define a $(\gamma)$-element for $G$ to be any element in $R(G) = \text{KK}^G(C, C)$ which is represented by a Kasparov cycle $(H, T)$ with Property $(\gamma)$.

Hence, if a gamma element $\gamma$ exists, $\gamma$ is a $(\gamma)$-element. We currently do not know if a $(\gamma)$-element is the unique idempotent in $R(G)$ characterized by the property.

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1 Proper cycles for G-equivariant K-theory

This section consists of five subsections: subsection 1.1 defines a proper Kasparov cycle generalizing the same notion in [Nis19]; subsection 1.2 reviews the results in [Nis19] in a streamlined way; subsection 1.3 defines the notion of proper K-cycles for \( A \rtimes_r G \) with G-compact support which by the preceding discussions, is shown to be in the image of the assembly map \( \mu^G_A \); subsection 1.4 introduces the notion of G-completeness of a G-Hilbert space \( H \) for a bounded operator \( T \), which will be the key for us to extend the direct splitting method to non-cocompact setting; in the last subsection 1.5, we show that:

**Theorem.** (see Theorem 1.37) For any proper Kasparov cycle \((H, T)\) and for any G-C*-algebra \( A \), the image of \([H, T]_*\) on \( KK(C, A \rtimes_r G) \) lies in the image of the assembly map \( \mu^G_A \).

1.1 Proper Kasparov cycles

A G-Hilbert space is a (separable) Hilbert space equipped with a unitary representation of \( G \). A graded G-Hilbert space \( H \) is the direct sum of a pair of G-Hilbert spaces \( H^{(0)}, H^{(1)} \). Let \( X \) be a locally compact, second countable G-space. A (graded) G-Hilbert space \( H \) over \( X \) is a (graded) G-Hilbert space with a non-degenerate representation of the G-C*-algebra \( C_0(X) \). When \( H \) is graded, we understand that the representation is even: i.e. it is given by the direct sum of a pair of representations \( \pi^{(0)} \) on \( H^{(0)} \) and \( \pi^{(1)} \) on \( H^{(1)} \). We denote the graded commutator by the bracket \( [\ , \] \). The algebra of compact operators on \( H \) is denoted by \( \mathcal{K}(H) \).

**1.1 Definition.** A Kasparov cycle for \( KK^G(C, C) \) is a pair \((H, T)\) of a graded G-Hilbert space \( H \) and an odd, self-adjoint, bounded, G-continuous operator \( T \) on \( H \) such that \( 1 - T^2 \) is compact and that \( g(T) - T \) is compact for any \( g \in G \).

**1.2 Remark.** In [Nis19], the usual axiom of G-continuity of \( T \) was missing but it was implicitly used. This axiom is used, combined with the compactness of \( g(T) - T \), for knowing that the commutator \([a, T \rtimes_r 1]\) is compact for \( a \) in \( C^*_r(G) \) on the Hilbert \( C^*_r(G) \)-module \( H \rtimes_r G \), and hence for defining the descent map \( j^{G}_* \) for example.
The commutative ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$ is defined as the set of homotopy ([Kas88, Definition 2.3]) equivalence classes of Kasparov cycles. We write by $[H, T]$, the element in $R(G)$ defined by a Kasparov cycle $(H, T)$. The addition and the multiplication of the ring $R(G)$ are defined by the direct sum operation and by the Kasparov product. See [Kas88], [Bla98] for more details.

Any pair $(H^0, H^1)$ of finite-dimensional unitary representations of the group $G$ defines a Kasparov cycle $(H^0 \oplus H^1, 0)$ and hence an element in $R(G)$. We denote by $1_G$, the one $[C \oplus 0, 0]$ which corresponds to the trivial representation of $G$. The element $1_G$ is the multiplicative identity in the ring $R(G)$.

1.3 Definition. A Kasparov cycle $(H, T)$ for $KK^G(\mathbb{C}, \mathbb{C})$ is proper if for some proper $G$-space $X$, $H$ is a $G$-Hilbert space over $X$ such that for any $\phi$ in $C_0(X)$,

the function $g \mapsto [g(\phi), T]$ belongs to $C_0(G, \mathcal{R}(H))$.

We call such a cycle $(H, T)$ as a proper cycle for $KK^G(\mathbb{C}, \mathbb{C})$ over $X$.

1.4 Remark. Two remarks are in order when there is a $G$-compact model $E$ of the universal proper $G$-space $E_G$ ([BCH94]): Definition 1.3 gives an a-priori weaker notion of properness for Kasparov cycles for $KK^G(\mathbb{C}, \mathbb{C})$ compared to the one defined in [Nis19, Definition 3.1]. For example, it is not clear whether a cycle $(H, T)$ is proper over $E$ given that it is proper for some other not-necessarily $G$-compact, proper $G$-space $X$. The universal property of $E$ only provides us a map $C_0(E)$ to $C_b(X)$, not $C_0(X)$. For this reason, we prefer to distinguish the version of properness in [Nis19, Definition 3.1] by adding the extra words: proper “over $E$”. Secondly, the version [Nis19, Definition 3.1] required an extra condition which is that the Haar integral $\int_G g(c)Tg(c)d\mu_G(g)$ is a compact perturbation of $T$ for some cutoff function $c$ on $X$. Here, a cutoff function on a $G$-compact proper $G$-space $X$ is a compactly supported, nonnegative continuous function $c$ on $X$ satisfying $\int_{g \in G} g(c)^2d\mu_G(g) = 1$. As Proposition 1.6 below says, this extra condition is automatic.

1.5 Lemma. Suppose that a Kasparov cycle $(H, T)$ for $KK^G(\mathbb{C}, \mathbb{C})$ is proper over a proper $G$-space $X$. Then, for any increasing, exhausting sequence $K_n$ of compact subsets of $G$, there is a partition of unity $(\chi_n)_{n \geq 1}$ in $C_c(X)$ of $C_0(X)$ such that the following holds for $\bar{\chi}_n = (\chi_n - \chi_{n-1})^+ (\chi_0 = 0)$:
1. \(\|g(\bar{x}_n) - \bar{x}_n\| < 2^{-n}\) for \(g\) in \(K_n\).

2. \(\|T_n\bar{x}_n\| < 2^{-n}\).

3. \(\bar{x}_n\) has support contained in \(X_{n+1} - X_{n-1}\) for some increasing, exhausting sequence \(X_n\) of compact subsets of \(X\).

Proof. This follows by the standard quasi-central approximate unit argument (c.f. [Hig87], [Kas88, Lemma 1.4] [HR00, Theorem 3.2.6, Proposition 3.2.8]).

1.6 Proposition. Suppose that a Kasparov cycle \((H, T)\) for \(KK^G(\mathbb{C}, \mathbb{C})\) is proper over a \(G\)-compact, proper \(G\)-space \(X\). Then, for any cutoff function \(c\) on \(X\), we have

\[
\int_{g \in G} g(c)Tg(c)d\mu_G(g) - T \in \mathfrak{R}(H).
\]

Proof. Let \(x_n, \bar{x}_n\) in \(C_c(X)\) be as given by Lemma 1.5 for a proper Kasparov cycle \((H, T)\) over \(X\) (here, we only use the properties 2 and 3). Set

\[
T' = \sum_{n \geq 1} \bar{x}_n T \bar{x}_n.
\]

Note that \(T' - T\) is compact. Hence, by [Nis19, Lemma 2.7], it is enough to show the claim for \(T'\) in place of \(T\). We have (convergence is in SOT topology):

\[
\int_{g \in G} g(c)T'g(c)d\mu_G(g) - T' = \int_{g \in G} [g(c), T']g(c)d\mu_G(g)
\]

\[
= \int_{g \in G} \sum_{n \in F_g} \bar{x}_n [g(c), T]g(c) \bar{x}_n d\mu_G(g)
\]

\[
= \int_{g \in G} \sum_{n \in F_g} \bar{x}_n g(\chi)[g(c), T]g(c) \bar{x}_n d\mu_G(g)
\]

where \(F_g\) is a finite subset of \(\mathbb{N}\) consisting of \(n\) for which \(\bar{x}_n g(c)\) is nonzero and where \(\chi\) is any compactly supported function on \(X\) such that \(\chi c = c\). We see that the both families \(\bar{x}_n g(\chi)\) and \(g(c) \bar{x}_n\) of operators on \(H\) indexed by \(Z = \bigsqcup_{g \in G} (\{g\} \times F_g) \subset G \times \mathbb{N}\) are square-summable on \(H\) over \(Z\) in SOT topology with respect to the product measure of the Haar measure and
the counting measure. As in [Nis19, Lemma 2.4, 2.5], they define bounded linear maps

\[ V: v \mapsto (\bar{\chi}_n g(\chi) v)_{(g,n) \in \mathbb{Z}}, \quad V': v \mapsto (\bar{\chi}_n g(c) v)_{(g,n) \in \mathbb{Z}}, \]

from \( H \) to \( L^2(Z, H) \) and we have

\[ \int_{g \in G} \sum_{n \in F_g} \bar{\chi}_n g(\chi) [g(c), T] g(c) \bar{\chi}_n d\mu_G(g) = V^*([g(c), T])_{(g,n) \in \mathbb{Z}}. \]  

Here, \(([g(c), T])_{(g,n) \in \mathbb{Z}}\) is a bounded operator on \( L^2(Z, H) \) and it belongs to \( C_0(\mathbb{Z}, \mathbb{R}(H)) \) since \((H, T)\) is proper, \( c \) is in \( C_c(X) \subset C_0(X) \) and since the union \( \bigcup_{g \in K} F_g \) of \( F_g \) over any compact subset \( K \) of \( G \) is finite. It follows that the integral (1.7) is the norm-limit of a sequence of operators on \( H \) defined by the same type of integration

\[ \int_{g \in G} \sum_{n \in F_g} \bar{\chi}_n g(\chi) A_{g,n} g(c) \bar{\chi}_n d\mu_G(g) = V^*(A_{g,n})_{(g,n) \in \mathbb{Z}}. \]

where \( (A_{g,n})_{(g,n) \in \mathbb{Z}} \) belongs to \( C_c(Z, \mathbb{R}(H)) \) but this integration is absolutely convergent with compact integrands, converging to a compact operator on \( H \). It follows that their norm limit (1.7) is a compact operator on \( H \). \( \square \)

### 1.2 Proper K-cycles for \( A \rtimes_r G \)

For a graded \( C^* \)-algebra \( B \), the K-theory group \( K(B) = KK(C, B) \) is an abelian group of homotopy equivalence classes \([E, F]\) of pairs of the form \((E, F)\) where \( E \) is a countably generated, graded Hilbert \( B \)-module and \( F \) is an odd, self-adjoint, adjointable operator \( F \) on \( E \) such that \( 1 - F^2 \) is compact. In this paper, we call such a pair \((E, F)\) a K-cycle for \( B \).

We set \( H_0 = \ell^2([N])^{(0)} \oplus \ell^2([N])^{(1)} \), the standard separable graded Hilbert space. If \( B \) is trivially graded, the set of operator homotopy equivalence classes of odd, self-adjoint, adjointable operators \( F \) on a fixed graded Hilbert \( B \)-module \( H_0 \otimes B \) such that \( 1 - F^2 \) is compact is naturally identified with the group \( K(B) \). This is because this set can be identified with the set of homotopy equivalence classes of unitary elements in the Calkin algebra \( Q(B \otimes \mathbb{R}(\ell^2([N]))) \) which is properly infinite (c.f. [Bla98, Proposition 17.5.5]).

For a graded \( G \)-\( C^* \)-algebra \( A \) and a graded \( G \)-Hilbert \( A \)-module \( E \), we denote by \( E \rtimes_r G \), the graded \( G \)-Hilbert \( A \rtimes_r G \)-module as defined and
denoted as \( C^*_r(G, E) \) in [Kas88, Definition 3.8]. If \( G \) is a discrete group, \( E \rtimes_r G \) is the completion of \( C_c(G, E) \) which consists of the vectors of the form
\[
\sum_{g \in G} v_g \rtimes_r u_g \text{ for } v_g \text{ in } E, v_g = 0 \text{ a.e. } g \text{ in } G
\]
with the \( A \rtimes_r G \)-valued sesquilinear form \( \langle \ , \ \rangle \) on \( C_c(G, E) \) determined by
\[
\langle v_h \rtimes_r u_h, v_g \rtimes_r u_g \rangle = h^{-1}(\langle v_h, v_g \rangle_E)u_{h^{-1}g}
\]
where \( \langle \ , \ \rangle_E \) is the given \( A \)-valued sesquilinear form on \( E \). An element \( au_g \) in \( A \rtimes_r G \) acts from right on \( C_c(G, E) \) by
\[
(v_h \rtimes_r u_h)(au_g) = v_h h(a) \rtimes_r u_{hg}
\]
which extends and defines a right \( A \rtimes_r G \)-module structure on \( E \rtimes_r G \) compatible with \( \langle \ , \ \rangle \). For a general group \( G \), \( E \rtimes_r G \) is defined analogously as the completion of \( C_c(G, E) \) whose elements can be formally and practically expressed as
\[
\int_{g \in G} v_g \rtimes_r u_g d\mu_G(g)
\]
for a continuous, compactly supported function \( G \ni g \mapsto v_g \in E \). A sesquilinear form and a module structure can be analogously defined using this expression following the discrete case.

For trivially graded \( A \), we have a description of \( K(A \rtimes_r G) = KK(C, A \rtimes_r G) \) as the set of operator homotopy equivalence classes of odd, self-adjoint, adjointable operators \( F \) on a fixed graded \( A \rtimes_r G \)-module \( (H_0 \hat{\otimes} A) \rtimes_r G \) such that \( 1 - F^2 \) is compact.

Any Kasparov cycle \( (H, T) \) for \( KK^G(C, C) \) defines the following map on K-cycles for \( A \rtimes_r G \):
\[
((H_0 \hat{\otimes} A) \rtimes_r G, F) \mapsto ((H_0 \hat{\otimes} H \hat{\otimes} A) \rtimes_r G, F \sharp T),
\]
where we set
\[
F \sharp T = F + (1 - F^2)^{1/2}T(1 - F^2)^{1/2}.
\]
This descends to a ring homomorphism
\[
KK^G(C, C) \to \text{End}(K(A \rtimes_r G))
\]
which coincides with the composition

$$\text{(1.8)} \quad \text{KK}^G(\mathbb{C}, \mathbb{C}) \xrightarrow{\sigma^G} \text{KK}^G(A, A) \xrightarrow{j^G} \text{KK}(A \rtimes_r G, A \rtimes_r G) \rightarrow \text{End}(K(A \times_r G))$$

of the augmentation map $\sigma^G$, the descent map $j^G$ and the Kasparov product as in [Kas88].

Let $H$ be any graded $G$-Hilbert space. The right regular representation $\rho$ of $C^*_r(G)$ on the $G$-Hilbert space $L^2(G)$ where $G$-action on $L^2(G)$ is the left regular representation induces the following map

$$\rho: \mathcal{L}(H \rtimes_r G) \rightarrow \mathcal{L}^G(H \rtimes_r G \hat{\otimes}_{C^*_r(G)} L^2(G)) \cong \mathcal{L}^G(H \hat{\otimes} L^2(G))$$

where $\rho$ is used in the interior tensor product $\hat{\otimes}_{C^*_r(G)}$ here and $\mathcal{L}$ (resp. $\mathcal{L}^G$) stands for the algebra of adjointable (resp. $G$-equivariant) operators.

For a graded $G$-$C^*$-algebra $A$, in this paper, we shall think $A \hat{\otimes} L^2(G)$ as a $G$-Hilbert $A$-module where $G$-action on $L^2(G)$ is the left regular representation. The right regular representation of $A \rtimes_r G$ on the $G$-Hilbert $A$-module $A \hat{\otimes} L^2(G)$ is defined as

$$\rho_A: a \mapsto (g(a))_{g \in G}, \quad g \mapsto \rho_g$$

where $\rho_g$ is the right-translation by $g^{-1}$ on $L^2(G)$. We think of this as a representation of graded $G$-$C^*$-algebra $A \rtimes_r G$ equipped with the trivial $G$-action on the graded $G$-Hilbert $A$-module $A \hat{\otimes} L^2(G)$. As before, if $H$ is a graded $G$-Hilbert space, the right regular representation $\rho_A$ induces the following map

$$\rho_A: \mathcal{L}((H \hat{\otimes} A) \rtimes_r G) \rightarrow \mathcal{L}^G((H \hat{\otimes} A) \rtimes_r G \hat{\otimes}_{A \rtimes_r G} A \hat{\otimes} L^2(G)) \cong \mathcal{L}^G(H \hat{\otimes} A \hat{\otimes} L^2(G)).$$

For notational preference, we consider $H \hat{\otimes} A \hat{\otimes} L^2(G)$ as a graded $G$-Hilbert $A$-module where $G$-action is as given on $H$ and it is the left regular representation on $L^2(G)$. On the other hand, if we use the most natural isomorphism $(H \hat{\otimes} A) \rtimes_r G \hat{\otimes}_{A \rtimes_r G} A \hat{\otimes} L^2(G) \cong H \hat{\otimes} A \hat{\otimes} L^2(G)$, the $G$-action on $H$ would be trivial. Thus, the isomorphism used in (1.9) is the composition of this most natural isomorphism with the isomorphism

$$\int_{g \in G} v \otimes a_g \otimes \delta_g \text{d}\mu_G(g) \mapsto \int_{g \in G} g(v) \otimes a_g \otimes \delta_g \text{d}\mu_G(g)$$

of Hilbert $A$-module $H \hat{\otimes} A \hat{\otimes} L^2(G)$. 

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The only important thing to remember about the map (1.9) is that it just a \( G \)-equivariant representation of \( \mathcal{L}((H \hat{\otimes} A) \rtimes_r G) \) equipped with trivial \( G \)-action on a graded \( G \)-Hilbert \( A \)-module \( H \hat{\otimes} A \otimes L^2(G) \) where \( G \)-action is as given on \( H \) and is the left regular representation on \( L^2(G) \) which is determined by

\[
T \mapsto (g(T))_{g \in G} \quad \text{for } T \in \mathcal{L}(H \hat{\otimes} A) \subset \mathcal{L}((H \hat{\otimes} A) \rtimes_r G)
\]

and

\[
u_g \mapsto \rho_g \quad \text{for the left multiplication operator } u_g \text{ on } (H \hat{\otimes} A) \rtimes_r G.
\]

When \( H \) is a graded \( G \)-Hilbert space over \( X \) for a proper \( G \)-space \( X \), we shall think that \( H \hat{\otimes} A \otimes L^2(G) \) carry the \( G \)-equivariant non-degenerate representation \( \pi_X = \pi \hat{\otimes} 1 \hat{\otimes} 1 \) of \( C_0(X) \) where \( \pi \) is the given representation on \( H \). We will simply express this representation as

\[
\pi_X: \phi \mapsto (\phi)_{g \in G} \quad \text{or } (\phi \hat{\otimes} 1)_{g \in G}.
\]

1.10 Lemma. Let \( H \) be a graded \( G \)-Hilbert space over \( X \) for a proper \( G \)-space \( X \). For any graded \( G \)-\( C^* \)-algebra \( A \), the map \( \rho_A \) (1.9) sends any compact operator \( F \) in \( \mathcal{R}((H \hat{\otimes} A) \rtimes_r G) \) to a \( G \)-equivariant operator in \( \mathcal{L}^G(H \hat{\otimes} A \otimes L^2(G)) \) which is compact modulo \( \pi_X(C_0(X)) \).

Proof. Let us first see the easiest case when \( G \) is discrete. It follows from the non-degeneracy of the representation of \( C_0(X) \) on \( H \), for any \( \phi \) in \( C_0(X) \), the map

\[
g \mapsto \phi g(T)
\]

is a \( \mathcal{R}(H) \)-valued function on \( G \) which vanishes at infinity for any compact operator \( T \) on \( H \) (see [Nis19, Lemma 2.3]). The claim follows from this. For a general locally compact group \( G \), the claim follows from this and from

\[
C_0(G, \mathcal{R}(H)) \cdot C_r^*(G) \subset \mathcal{R}(H \hat{\otimes} L^2(G))
\]

where \( C_r^*(G) \) acts on \( L^2(G) \) by the right regular representation.

Let us quickly generalize the previous discussions. For a graded \( G \)-\( C^* \)-algebra \( A \), we say that a graded \( G \)-Hilbert \( A \)-module \( E \) is proper if there is a non-degenerate representation of \( G \)-\( C^* \)-algebra \( C_0(X) \) on \( E \) for some
proper G-space X. The right regular representation \( \rho_A \) induces the following map

\[
(1.11) \quad \rho_A : \mathcal{L}(E \rtimes_r G) \to \mathcal{L}^G(E \hat{\otimes} L^2(G)).
\]

We think \( E \hat{\otimes} L^2(G) \) as a G-Hilbert A-module where G-action is as given on E and is the left regular representation on \( L^2(G) \). The map \( \rho_A \) is determined by

\[
T \mapsto (g(T))_{g \in G} \quad \text{for } T \in \mathcal{L}(E) \subset \mathcal{L}(E \rtimes_r G)
\]

and

\[
u_g \mapsto \rho_g \quad \text{for the left multiplication operator } u_g \text{ on } E \rtimes_r G.
\]

We shall think \( E \hat{\otimes} L^2(G) \) carry the G-equivariant non-degenerate representation \( \pi_X = \pi \hat{\otimes} 1 \) of \( C_0(X) \) where \( \pi \) is the given representation on E. We will simply express this as \( \pi_X : \phi \mapsto (\phi)_g \in G \). We have:

**1.12 Lemma.** The map \( \rho_A \) sends any compact operator \( F \) in \( \mathfrak{K}(E \rtimes_r G) \) to a G-equivariant operator in \( \mathcal{L}^G(E \hat{\otimes} L^2(G)) \) which is compact modulo \( \pi_X(C_0(X)) \).

**1.13 Definition.** For a graded G-C\(^*\)-algebra A, a proper K-cycle for \( A \rtimes_r G \) is a K-cycle for \( A \rtimes_r G \) of the form \( (E \rtimes_r G, F) \) where E is proper over X for some proper G-space X and F in \( \mathcal{L}(E \rtimes_r G) \) satisfies

\[
[\pi_X(\phi), \rho_A(F)] \in \mathfrak{K}(E \hat{\otimes} L^2(G))
\]

for any \( \phi \) in \( C_0(X) \).

We recall that for graded G-C\(^*\)-algebras A and B, the triple \( (E, \pi, F) \) defines a cycle for \( \mathcal{K}^G(A, B) \) if E is a countably generated, graded G-Hilbert B-module equipped with the representation \( \pi \) of A and if F is an odd, self-adjoint, G-continuous adjointable operator in \( \mathcal{L}(E) \) such that \( 1 - F^2, g(F) - F \) for \( g \) in G are compact modulo \( \pi(A) \) and \( [\pi(a), F] \) is compact for any \( a \) in A (see [Kas88] or [Bla98, Chapter XIII] for more detail). We write by \( [E, \pi, F] \), the corresponding element in \( \mathcal{K}^G(A, B) \). The following is immediate from Definition 1.13 and Lemma 1.12:

**1.14 Lemma.** For any proper K-cycle \( (E \rtimes_r G, F) \) for \( A \rtimes_r G \), the triple

\[
(E \hat{\otimes} L^2(G), \pi_X, \rho_A(F))
\]

is a cycle for \( \mathcal{K}^G(C_0(X), A) \).
1.15 Definition. We call this cycle \((E \otimes L^2(G), \pi_X, \rho_A(F))\) for \(KK^G(C_0(X), A)\) as the right regular representation of a proper K-cycle \((E \rtimes_r G, F)\) for \(A \rtimes_r G\).

1.16 Proposition. Let \((H, T)\) be a Kasparov cycle for \(KK^G(C, C)\) which is proper over \(X\). For any K-cycle \(((H_0 \otimes A) \rtimes_r G, F)\) for \(A \rtimes_r G\), the K-cycle \(((H_0 \otimes H \otimes A) \rtimes_r G, F \sharp T)\) for \(A \rtimes_r G\) is proper over \(X\) with respect to the representation of \(C_0(X)\) on \(H_0 \otimes H \otimes A\) naturally induced from the given one on \(H\). The map

\[
((H_0 \otimes A) \rtimes_r G, F) \mapsto (H_0 \otimes H \otimes A \otimes L^2(G), \pi_X, \rho_A(F \sharp T))
\]

induces a homomorphism

\[
KK(C, A \rtimes_r G) \to KK^G(C_0(X), A).
\]

This map coincides with the Kasparov product by the element defined by the cycle

\[
(H \otimes A \otimes L^2(G), \pi_X \otimes \rho_A, \tilde{T} = (g(T))_{g \in G}).
\]

in \(KK^G(C_0(X) \otimes (A \rtimes_r G), A)\).

Proof. To see that the K-cycle \(((H_0 \otimes H \otimes A) \rtimes_r G, F \sharp T)\) for \(A \rtimes_r G\) is proper over \(X\), we just need to check

\[
[\pi_X(\phi), \rho_A(F \sharp T)] \in \mathfrak{K}(H_0 \otimes H \otimes A \otimes L^2(G))
\]

where we recall that

\[
F \sharp T = F + (1 - F^2) \frac{1}{2} T (1 - F^2) \frac{1}{2}
\]

on \((H_0 \otimes H \otimes A) \rtimes_r G\). Since the right regular representation \(\rho_A\) restricted to \(\mathfrak{L}((H_0 \otimes A) \rtimes_r G)\) commutes with the representation \(\pi_X\), we have

\[
[\pi_X(\phi), \rho_A(F)] = 0.
\]

On the other hand, \(\rho_A\) sends \(T\) in \(\mathfrak{L}(H)\) to \(\tilde{T} = (g(T))_{g \in G}\) in \(\mathfrak{L}(H \otimes L^2(G))\). Hence,

\[
[\pi_X(\phi), \rho_A(T)] = ([\phi, g(T)])_{g \in G}
\]

belongs to \(C_0(G, \mathfrak{K}(H))\). Therefore,

\[
[\pi_X(\phi), \rho_A((1 - F^2) \frac{1}{2} T (1 - F^2) \frac{1}{2})] = \rho_A((1 - F^2) \frac{1}{2}) [\pi_X(\phi), \rho_A(T)] \rho_A((1 - F^2) \frac{1}{2})
\]
belongs to $\mathfrak{a}(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} L^2(G))$.

The triple $(H \hat{\otimes} A \hat{\otimes} L^2(G), \pi_X \hat{\otimes} \rho_\lambda, \bar{T} = (g(T))_{g \in G})$ is indeed a cycle for $KK^G(C_0(X) \hat{\otimes} (A \rtimes_r G), A)$. When $G$ is discrete and $A = \mathbb{C}$, the similar statement is proved in [Nis19, Proposition 3.3]. The case for general $G$ and $A$ is analogous. There, $X$ was taken to be $E = \mathbb{E}G$, a cocompact model of universal proper $G$-space of $G$ but the only property of $E$ used in the proof is that it is a proper $G$-space.

Finally, we need to show that the element in $KK^G(C_0(X), A)$ defined by the triple

$$(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} L^2(G), \pi_X, \rho_\lambda[\mathbb{F}_G^\#T])$$

is equal to the Kasparov product of $((H_0 \hat{\otimes} A) \rtimes_r G, F)$ for $KK(C, A \rtimes_r G)$ and $(H \hat{\otimes} A \hat{\otimes} L^2(G), \pi_X \hat{\otimes} \rho_\lambda, \bar{T})$ for $KK^G(C_0(X) \hat{\otimes} (A \rtimes_r G), A)$. This can be checked as in [Bla98, Proposition 18.10.1].

When $X$ is a $G$-compact, proper $G$-space, the assembly map

$$\mu^G_{A,X}: KK^G(C_0(X), A) \rightarrow KK(C, A \rtimes_r G)$$

is defined as the composition of the decent map

$$j_r^G: KK^G(C_0(X), A) \rightarrow KK(C_0(X) \rtimes_r G, A \rtimes_r G)$$

and the Kasparov product with the element $[p_c]$ in $KK(C, C_0(X) \rtimes_r G)$ where $p_c$ is the projection in $C_c(G, C_0(X)) \subset \mathcal{C}(X) \rtimes G$ defined as

$$p_c: g \mapsto g(c)c$$

for some cutoff function $c$ in $C_c(X)$: a non-negative, compactly supported continuous function on $X$ such that the (left) Haar integral $\int_G g(c)^2 d\mu_G(g) = 1$. Let us call $p_c$, a cutoff projection. The element $[p_c]$ in $KK(C, C_0(X) \rtimes_r G)$ does not depend on the choice of a cutoff function. See [BCH94], [Val02], [HG04] for more details on the assembly map $\mu^G_{A,X}$ to name a few.

Let $(H, T)$ be a Kasparov cycle for $KK^G(C, C)$ which is proper over a $G$-compact, proper $G$-space $X$. Let us write

$$\gamma^G_{A,(H,T)}: KK(C, A \rtimes_r G) \rightarrow KK^G(C_0(X), A),$$

the homomorphism defined in Proposition 1.16 for any $G$-$C^*$-algebra $A$. We can directly compute the composition

$$KK(C, A \rtimes_r G) \xrightarrow{\gamma^G_{A,(H,T)}} KK^G(C_0(X), A) \xrightarrow{\mu^G_{A,X}} KK(C, A \rtimes_r G).$$

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This kind of computations was done in the proof of [Nis19, Proposition 5.2] but let us explain this computation in a slightly different way for a later purpose.

Let \((\mathcal{H}_0 \hat{\otimes} \mathcal{A}) \rtimes_r G, F)\) be a K-cycle for \(A \rtimes_r G\). The map \(\nu^G_{A,(H,T)}\) sends this cycle to the cycle

\[
(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} L^2(G), \pi_X, \rho_A(F^*T))
\]

for \(\text{KK}^G(C_0(X), A)\) as in Proposition 1.16. Note that this cycle is the right regular representation of the proper K-cycle

\[
(H_0 \hat{\otimes} H \hat{\otimes} A \rtimes_r G, F^*T)
\]

for \(A \rtimes_r G\) which defines at the same time, the image of \((\mathcal{H}_0 \hat{\otimes} \mathcal{A}) \rtimes_r G, F)\) by the map \([H, T]_s\). In stead of computing the image by the assembly map \(\mu^G_{A,X}(g)\) of this cycle, we now present some general fact about the image by the assembly map \(\mu^G_{A,X}(g)\) of the right regular representation of a proper K-cycle over a G-compact, proper G-space \(X\). Let \(A\) be a graded \(G\)-C*-algebra and \((E \rtimes G, F)\) be a proper K-cycle for \(A \rtimes_r G\) over a G-compact proper G-space \(X\). Recall that the right regular representation \(\rho_A\) is the homomorphism

\[
(1.18) \quad \rho_A: \mathcal{L}(E \rtimes_r G) \to \mathcal{L}^G(E \hat{\otimes} L^2(G)).
\]

which sends an operator \(T\) in \(\mathcal{L}(E) \subset \mathcal{L}(E \rtimes_r G)\) to \((g(T))_{g \in G}\) in \(\mathcal{L}^G(E \hat{\otimes} L^2(G))\) and the left multiplication \(u_g\) in \(\mathcal{L}(E \rtimes_r G)\) to the right multiplication \(\rho_g\) in \(\mathcal{L}^G(E \hat{\otimes} L^2(G))\). The right regular representation of the proper K-cycle

\[
(E \hat{\otimes} L^2(G) \rtimes_r G, \pi_X \rtimes_r 1, \rho_A(F) \rtimes_r 1)
\]

for \(\text{KK}^G(C_0(X), A \rtimes_r G)\) where the representation \(\pi_X \rtimes_r 1\) of \(C_0(X) \rtimes_r G\) sends \(\phi\) in \(C_0(X)\) to \(\pi_X(\phi) \rtimes_r 1\) and \(u_g\) for \(g\) in \(G\) to the left multiplication by \(g\) on \(E \hat{\otimes} L^2(G) \rtimes_r G\):

\[
u_g: v \rtimes_r u_s \mapsto g(v) \rtimes_r u_{gs} \quad \text{for } v \in E \hat{\otimes} L^2(G) \text{ and } s \in G.
\]

The Kasparov product of this cycle with the cutoff projection \(p_c\) is simply the K-cycle

\[
(1.19) \quad (\pi_X \rtimes_r 1(p_c)(E \hat{\otimes} L^2(G) \rtimes_r G), \pi_X \rtimes_r 1(p_c) \rho_A(F) \rtimes_r 1(\pi_X \rtimes_r 1(p_c))
\]

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for $A \rtimes_r G$. We shall use the following natural isomorphism
\begin{equation}
E \rtimes_r G \cong \pi_X \rtimes_r \text{I}(p_c)(E \hat{\otimes} L^2(G)) \rtimes_r G
\end{equation}
defined as follows: if $G$ is a discrete group, it is given by
\[\xi \rtimes_r u_g \mapsto \sum_{h \in G} \text{ch}(\xi) \otimes \delta_h \rtimes_r u_{hg} \quad (\xi \in E, g \in G)\]
whose inverse is given by (the restriction of)
\[(\xi_h)_{h \in G} \rtimes_r u_g \mapsto \sum_{h \in G} h^{-1}(c \xi_h) \rtimes_r u_{h^{-1}g} \quad ((\xi_h)_{h \in G} \in E \hat{\otimes} L^2(G), g \in G).\]
For a general group $G$, it is given morally by the same formula. Via this isomorphism (1.20), the K-cycle (1.19) is identified as
\[(E \rtimes_r G, F')\]
where for any $F$ in $\mathcal{L}(E \rtimes_r G)$, we define $F'$ to be the operator in $\mathcal{L}(E \rtimes_r G)$ which corresponds to
\[\pi_X \rtimes_r \text{I}(p_c)\rho_A(F) \rtimes_r \text{I}\pi_X \rtimes_r \text{I}(p_c) \in \mathcal{L}(\pi_X \rtimes_r \text{I}(p_c)(E \hat{\otimes} L^2(G)) \rtimes_r G) \cong \mathcal{L}(E \rtimes_r G)\]
via the isomorphism (1.20). The strictly-continuous linear map $F \mapsto F'$ on $\mathcal{L}(E \rtimes_r G)$ is uniquely determined by the following:
\begin{equation}
T' = \int_G g(c)Tg(c)d\mu_G(g) \quad \text{for } T \in \mathcal{L}(E) \quad \text{and} \quad (u_g)' = u_g \quad \text{for } g \in G.
\end{equation}
Here, $c$ in $C_c(X)$ is a cutoff function on $X$ represented on $E \rtimes_r G$ as is given. Let us summarize our computation here:

**1.22 Proposition.** Let $X$ be a $G$-compact, proper $G$-space. For any proper K-cycle
\[(E \rtimes_r G, F)\]
for $A \rtimes_r G$ over $X$, its right regular representation
\[(E \hat{\otimes} L^2(G), \pi_X, \rho_A(F))\]
is a cycle for $\text{KK}^G(C_0(X), A)$ which is sent by the assembly map $\mu^G_{A,X}$ to the K-cycle
\[(E \rtimes_r G, F')\]
for $A \rtimes_r G$ up to isomorphism where $F'$ in $\mathcal{L}(E \rtimes_r G)$ is as determined by (1.21).
Clearly, we can use this fact to compute the composition (1.17). Recall that any Kasparov cycle \((H, T)\) in \(KK^G(C, C)\) defines a natural endomorphism \([H, T]_\ast\) on \(KK(C, A \rtimes_r G)\) via (1.8). Let us summarize here what we can say about the composition (1.17):

**1.23 Proposition.** Let \((H, T)\) be a Kasparov cycle for \(KK^G(C, C)\) which is proper over a \(G\)-compact, proper \(G\)-space \(X\). Let \(A\) be a \(G\)-C*-algebra. Then:

1. for any \(K\)-cycle \(((H_0 \otimes A) \rtimes_r G, F)\) for \(A \rtimes_r G\), its image by \([H, T]_\ast\) is represented by the \(K\)-cycle \(((H_0 \otimes H \otimes A) \rtimes_r G, F \circ T)\) for \(A \rtimes_r G\) which is proper over \(X\);

2. the image of \(((H_0 \otimes A) \rtimes_r G, F)\) by the map \(\nu_A^{G,(H,T)}\) is represented by the right regular representation of this proper \(K\)-cycle \(((H_0 \otimes H \otimes A) \rtimes_r G, F \circ T)\);

3. the image of \(((H_0 \otimes A) \rtimes_r G, F)\) by the composition (1.17) is represented by \(((H_0 \otimes H \otimes A) \rtimes_r G, F \circ T)\).

### 1.3 Proper \(K\)-cycles with \(G\)-compact support

**1.24 Definition.** Let \((E \rtimes_r G, F)\) be a \(K\)-cycle for \(A \rtimes_r G\) which is proper over a proper \(G\)-space \(X\). We say that the proper cycle \((E \rtimes_r G, F)\) has \(G\)-compact support if there is a \(G\)-compact, \(G\)-invariant closed subset \(Y \subset X\) and a \(G\)-equivariant projection \(P_Y\) on \(E\) which commutes with \(C_\infty(X)\) and \(F\) such that the following holds:

1. the \(K\)-cycle \((E_Y \rtimes_r G, F_Y)\) for \(A \rtimes_r G\) where \(E_Y = P_Y E\) and \(F_Y = P_Y F P_Y\) is naturally proper over \(Y\), i.e. the induced representation of \(C_\infty(X)\) on \(E_Y\) factors nondegenerately through \(C_\infty(Y)\) (this automatically implies that the cycle is proper with respect to this representation of \(C_\infty(Y)\)).

2. the complementary \(K\)-cycle \(((1 - P_Y)E \rtimes_r G, (1 - P_Y)F)\) for \(A \rtimes_r G\) is degenerate up to compact perturbation.

3. \(F'_Y - F_Y \in \mathcal{K}(E_Y \rtimes_r G)\) where \(F'_Y\) in \(\mathcal{L}(E_Y \rtimes_r G)\) is as determined by in (1.21) with respect to the proper structure on \(E_Y\) over \(Y\).

The following is an immediate consequence of previous discussions:
1.25 Proposition. Suppose \((E \rtimes_G F)\) is a proper \(K\)-cycle for \(A \rtimes_G G\) over \(X\) with \(G\)-compact support. Then, the corresponding element \([E \rtimes_G F]\) in \(KK(C, A \rtimes_G G)\) is in the image of the assembly map \(\mu_{A,Y}^{G,Y}\) for some \(G\)-compact, proper \(G\)-space \(Y \subset X\).

Proof. Let \((E_Y \rtimes_G F_Y)\) be as in Definition 1.24. By the second condition in Definition 1.24, we have \([E_Y \rtimes_G F_Y] = [E \rtimes_G F]\) in \(KK(C, A \rtimes_G G)\). The right regular representation \((E_Y \hat{\otimes} L^2(G), \rho_A(F_Y))\) of the proper \(K\)-cycle \((E_Y \rtimes_G G, F_Y)\) over \(Y\) is a cycle for \(KK^G(C_0(Y), A)\) which is sent by the assembly map \(\mu_{G,Y}^{G,Y}\) to the \(K\)-cycle \((E_Y \rtimes_G G, F'_Y)\) up to isomorphism by Proposition 1.22. By the third condition in Definition 1.24, we have \([E_Y \rtimes_G F, F'_Y] = [E_Y \rtimes_G G, F_Y]\). Overall, this shows that the cycle \((E \rtimes_G G, F)\) is in the image of \(\mu_{A,Y}^{G,Y}\) up to degenerate cycles, isomorphisms and compact perturbation. \(\square\)

In summary, any proper \(K\)-cycle with \(G\)-compact support is in the image of the assembly map at the level of cycles up to degenerate cycles, isomorphisms and compact perturbation.

1.4 \(G\)-completeness

The following definition is inspired from the notion of completeness of a manifold \(M\) for a differential operator \(D\) on \(M\) in [HR00, Definition 10.2.8].

1.26 Definition. Let \(T\) be a bounded operator on a \(G\)-Hilbert space \(H\) over \(X\). We say that \(H\) is \(G\)-complete for \(T\) if there is a measurable function

\[w: X \to \mathbb{R}\]

which is:

1. locally bounded, i.e. the image of compact sets are relatively compact.

2. proper, i.e. the pre-image of compact sets are relatively compact.

3. almost \(G\)-equivariant, i.e. \(g(w) - w\) is uniformly bounded in \(g\) over compact subsets of \(G\).

4. there is a bounded operator \(T_w\) such that \(T - T_w\) is compact and that \(T_w\) preserves the domain of the self-adjoint unbounded operator \(w\) on \(H\), and the commutator \([w, T_w]\) extends to a compact operator on \(H\).
An example of functions \( w \) satisfying the first three axioms 1, 2 and 3 in Definition 1.26 is a distance function \( w(x) = d_X(x_0, x) \) for any proper (bounded sets are relatively compact) metric space \((X, d_X)\) and for any fixed point \( x_0 \) in \( X \).

1.27 Proposition. For any proper Kasparov cycle \((H, T)\) for \( KK^G(C, C) \) over \( X \), \( H \) is \( G \)-complete for \( T \). Moreover, the function \( w \) witnessing the \( G \)-completeness of \( H \) for \( T \) can be taken to be a \( G \)-continuous function \( X \) which is continuous and positive. Here, \( G \)-continuous means that the locally bounded, bounded operator valued function \( g \mapsto g(w) - w \) on \( G \) is norm continuous.

Proof. Let \( \chi_n \) and \( \bar{\chi}_n \) in \( C_c(X) \) be as given by Lemma 1.5 for a proper Kasparov cycle \((H, T)\) over \( X \). Let

\[
\begin{align*}
w &= \sum_{n \geq 1} n \bar{\chi}_n, \\
T_w &= \sum_{n \geq 1} \bar{\chi}_n T \bar{\chi}_n.
\end{align*}
\]

Note that \( T_w \) is a compact perturbation of \( T \). We see that \( w \) is a continuous, locally bounded, proper positive function on \( X \). It is also easy to see that \( w \) is almost \( G \)-equivariant and \( G \)-continuous. Let \( H_c = C_c(X)H \) be the subspace of compactly supported vectors in \( H \). We see that \( T_w \) preserves \( H_c \) which is an essentially self-adjoint domain of \( w \). We compute \([w, T_w]\) on \( H_c \) by

\[
[w, T_w] = \sum_{n \geq 1} \bar{\chi}_n ((n-1)[\bar{\chi}_{n-1}, T] + n[\bar{\chi}_n, T] + (n+1)[\bar{\chi}_{n+1}, T]) \bar{\chi}_n
\]

which clearly extends as an absolutely summable sum of compact operators, which is a compact operator. \( \square \)

Let \( H \) be a \( G \)-Hilbert space over \( X \) which is \( G \)-complete for an operator \( T \) and \( w \) be a measurable function from \( X \) to \( \mathbb{R} \) witnessing the \( G \)-completeness of \( H \) for \( T \). We now define a family of bounded operators \( c_{w,t} \) for \( t \in [0, 1] \) on \( H \otimes \mathbb{C}_1(\mathbb{R}) \) where \( \mathbb{C}_1(\mathbb{R}) = C_0(\mathbb{R}, \mathbb{C}) \) is the Clifford algebra of \( \mathbb{R} \) with trivial \( G \)-action: \( \mathbb{C}_1 \) is the first complex Clifford algebra generated by a single odd, self-adjoint unitary \( e_1 = c(1) \). Before we define \( c_{w,t} \), let us set up notations. We define odd, real-valued functions \( f_0 \) and \( f_1 \) on \( \mathbb{R} \) as

\[
f_0(X) = \frac{X}{(1 + X^2)^{\frac{1}{2}}}, \quad f_1(X) = \begin{cases} 1 & (X \geq 1) \\ X & (-1 \leq X \leq 1) \\ -1 & (X \leq -1). \end{cases}
\]
We note that $f_0 - f_1$ is in $C_0(\mathbb{R})$. We denote by $c(y)$ the usual Bott operator on $\mathbb{R}$ which is an odd, self-adjoint unbounded operator on $C_\tau(\mathbb{R})$ defined as

$$c(y) : y \mapsto ye_1 \in C_1$$
on $\mathbb{R}$. In general, for any real number $y_0$, $c(y-y_0)$ is the Bott operator with origin $y_0$. It has the property

$$1 - f_1(c(y-y_0))^2 = 0$$

for any $y$ outside the unit ball $[y_0 - 1, y_0 + 1]$ of radius one with center $y_0$. Now we define the operators $c_{w,t}$:

1.28 Definition. Let $H$ be a $G$-Hilbert space over $X$ which is $G$-complete for an operator $T$ and $w$ be a measurable function from $X$ to $\mathbb{R}$ witnessing the $G$-completeness of $H$ for $T$. For $t$ in $[0, 1]$, we define an odd, self-adjoint, bounded operator $c_{w,t}$ on $H \otimes C_\tau(\mathbb{R})$ by:

$$c_{w,t} = f_1(c(y) - twe_1).$$

It is defined by functional calculus for an odd, self-adjoint, unbounded operator $C_{w,t} = c(y) - twe_1 = (y - tw)e_1$ on $H \otimes C_\tau(\mathbb{R})$.

Note that at $t = 0$, $c_{w,t}$ is simply the functional calculus $f_1(c(y))$ of the usual Bott operator $c(y)$ on $C_\tau(\mathbb{R})$. Note that $f_1(c(y))$ graded commutes with any $T$ on $H$ and it is $G$-equivariant on $H \otimes C_\tau(\mathbb{R})$. The operators $c_{w,t}$ can be thought of as perturbations of $c_{w,0} = f_1(c(y))$.

For $R > 0$, let $C_\tau((-R, R))$ be the subalgebra of $C_\tau(\mathbb{R})$ consisting of functions which vanish outside the interval $(-R, R)$. The $G$-completeness of $H$ implies the following:

1.29 Lemma. The family $c_{w,t}$ of odd, self-adjoint, bounded operators on $H \otimes C_\tau(\mathbb{R})$ for $t \in [0, 1]$ satisfies the following:

1. for any $t > 0$ and $R > 0$, the restriction of $(1 - c_{w,t}^2)$ to $H \otimes C_\tau((-R, R))$ has compact support on $H \otimes C_\tau((-R, R))$ with respect to $X$.

2. $\|g(c_{w,t}) - c_{w,t}\|$ goes to 0 as $t$ goes to 0 uniformly in $g$ over compact subsets of $G$. 

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3. The graded commutator $[T, (c_{w,t})_{t \in [0,1]}]$ belongs to $\mathcal{R}(H \hat{\otimes} C_\tau(\mathbb{R})[0,1])$.

Before proving this, we first note that for any compact operator $S$ on $H$, $(1 - c_{w,t}^2)S$ is in $\mathcal{R}(H \hat{\otimes} C_\tau(\mathbb{R})[0,1])$. Indeed, for any $f$ in $C_0(\mathbb{R})$ we have

$$f(c(y) - tw_1)S \in \mathcal{R}(H \hat{\otimes} C_\tau(\mathbb{R})[0,1]).$$

It is enough to check this for $f = (1 - f_0^2)$ and for $S$ which has compact support on $H$ with respect to $X$. In this case, on the relevant domain (support of $S$), $w$ is bounded and we have

$$f(c(y) - tw_1)S = \frac{1}{1 + |y - tw|^2}S$$

which is continuous in $t$, vanishes at infinity in $y$ in $\mathbb{R}$ and hence belongs to $\mathcal{R}(H \hat{\otimes} C_\tau(\mathbb{R})[0,1])$. Similarly, we have that

$$[S, (c_{w,t})_{t \in [0,1]}] \in \mathcal{R}(H \hat{\otimes} C_\tau(\mathbb{R})[0,1])$$

for any compact operator $S$ on $H$.

**Proof.**

1. We may write $c_{w,t}$ as $f_1(c(y - tw(x)))$. For any $t$ and $y$, $(1 - f_1^2)(c(y - tw(x))) \neq 0$ if and only if $y$ is in the ball of radius one with center $tw(x)$. If $y$ in $[-R, R]$, this means that $tw(x)$ is in $[-R - 1, R + 1]$. Thus, for $t > 0$, this means that $w(x)$ is in $\frac{1}{t}[-R - 1, R + 1]$ which means $x$ must sit inside a relatively compact subset $w^{-1}(\frac{1}{t}[-R - 1, R + 1])$. The claim follows from this.

2. It is enough to check this for $f_0$ instead of $f_1$. We first compute

$$g(f_0(c(y) - tw_1)) - f_0(c(y) - tw_1)$$

pointwisely in $y$ and $t$. Since $g$ preserves a domain of $w$, by the following formula for $C_{w,t} = c(y) - tw_1 = (y - tw)e_1$:

$$f_0(C_{w,t}) = \frac{C_{w,t}}{(1 + C_{w,t}^2)^2} = \frac{2}{\pi} \int_0^\infty \left( \frac{C_{w,t}}{1 + \lambda^2 + C_{w,t}^2} \right) d\lambda$$

$$= \pi^{-1} \int_0^\infty (C_{w,t} + \sqrt{1 + \lambda^2 i})^{-1} + (C_{w,t} - \sqrt{1 + \lambda^2 i})^{-1} d\lambda,$$

the following formula is valid:

$$g(f_0(C_{w,t})) - f_0(C_{w,t}) = \pi^{-1} \int_0^\infty (A_+ + A_-) d\lambda$$

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where
\[ A_\pm = \left( g(C_{w,t}) \pm \sqrt{1 + \lambda^2 i} \right)^{-1} - \left( C_{w,t} \pm \sqrt{1 + \lambda^2 i} \right)^{-1} \]
\[ = \left( g(C_{w,t}) \pm \sqrt{1 + \lambda^2 i} \right)^{-1} (C_{w,t} - g(C_{w,t})) \left( C_{w,t} \pm \sqrt{1 + \lambda^2 i} \right)^{-1}.\]

We have
\[ g(C_{w,t}) - C_{w,t} = t(g(w) - w)e_1 \]
which is independent of \( y \). It follows that, independently of \( y \), we have
\[ \|g(f_0(C_{w,t})) - f_0(C_{w,t})\| \leq tC_g \]
where \( C_g \) is some constant which is uniformly bounded in \( g \) over compact subsets of \( G \). The claim follows from this.

3. It is enough to check that
\[ [T_w, f_0(C_{w,t})] \in \mathcal{K}(H \otimes C_\tau(R)[0, 1]). \]
We compute the commutator pointwisely in \( y \) and \( t \). Since \( T_w \) preserves a domain of \( w \), using the same formula for \( f_0(C_{w,t}) \) as before, we get
\[ [T_w, f_0(C_{w,t})] = \pi^{-1} \left[ t \int_0^\infty (B_+ + B_-) d\lambda \right] \]
where
\[ B_\pm = \left( C_{w,t} \pm \sqrt{1 + \lambda^2 i} \right)^{-1} ([C_{w,t}, T_w]) \left( C_{w,t} \pm \sqrt{1 + \lambda^2 i} \right)^{-1}. \]
We have
\[ [C_{w,t}, T_w] = [(y - tw)e_1, T_w] = \left\{ \begin{array}{ll} t[T_w, w]e_1 & \text{if } T \text{ is even} \\ t[w, T_w]e_1 & \text{if } T \text{ is odd} \end{array} \right. \]
which is independent of \( y \). By the remark right before the proof of this proposition, we see that \( B_\pm \) is in \( \mathcal{K}(H \otimes C_\tau(R)[0, 1]) \) for all \( \lambda \) and the integral
\[ [T_w, f_0(C_{w,t})] = \pi^{-1} \int_0^\infty (B_+ + B_-) d\lambda \]
is absolutely convergent. It follows that \([T_w, f_0(C_{w,t})]\) belongs to \( H \otimes C_\tau(R)[0, 1] \).
The claim follows from this. \( \square \)

We also note that \( c_{w,t} \) on \( H \otimes C_\tau(R) \) commutes with any measurable function on \( X \) acting on \( H \).
1.5 Action of proper Kasparov cycles on $K$-cycles

Let $(H, T)$ be a proper Kasparov cycles over a proper $G$-space $X$ and $w$ be a measurable function from $X$ to $\mathbb{R}$ witnessing the $G$-completeness of $H$ for $T$. Let $c_{w,t}$ be the odd, self-adjoint, bounded operator on $H \otimes C_\tau(\mathbb{R})$ as in Definition 1.28. For any graded $C^*$-algebra $A$ and any $K$-cycle $((H_0 \hat{\otimes} A) \rtimes_r G, F)$ for $A \rtimes_r G$, we define an odd, self-adjoint, bounded operator $F_\sharp T_\sharp c_{w,t}$ for $t \in [0, 1]$ on $(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G$ as:

$$F_\sharp T_\sharp c_{w,t} = F + (1 - F^2) \frac{t}{2} \left( (1 - c_{w,t}^2) \frac{t}{2} t (1 - c_{w,t}^2) \frac{t}{2} \right) (1 - F^2) \frac{t}{2}.$$  

(1.30)

Note that at $t = 0$, the pair

$$(\{(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G, F_\sharp T_\sharp c_{w,0}\})$$

is a $K$-cycle for $(A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G$ which represents the Kasparov product of $K$-cycle $((H_0 \hat{\otimes} H \hat{\otimes} A) \rtimes_r G, F_\sharp T)$ for $A \rtimes_r G$ and the $K$-cycle (the Bott generator) $(C_\tau(\mathbb{R}), c_{w,0})$ for $C_\tau(\mathbb{R})$.

In general, for $t > 0$, $1 - (F_\sharp T_\sharp c_{w,t})^2$ is not compact, since, for example, the commutator $[(1 - F^2), c_{w,t}]$ is not necessarily compact. On the other hand, by the second property of $c_{w,t}$ in Lemma 1.29, for any $\epsilon > 0$, there is $0 < t_0 \leq 1$ (which depends on $F$) such that for any $0 \leq t \leq t_0$ we have

$$\|[(1 - F^2) \frac{t}{2}, c_{w,t}]\| < \frac{1}{100} \epsilon, \quad \|[F(1 - F^2) \frac{t}{2}, c_{w,t}]\| < \frac{1}{100} \epsilon,$$

and similarly for all the commutators (there are finitely many) involving $(1 - F^2) \frac{t}{2}$ or $F(1 - F^2) \frac{t}{2}$ and $c_{w,t}$ which appear when we compute $(F_\sharp T_\sharp c_{w,t})^2$ so that we have, combined with the third property of $c_{w,t}$ in Lemma 1.29,

$$\| (1 - (F_\sharp T_\sharp c_{w,t})^2) \| < \epsilon.$$

modulo $\mathcal{R}((H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G)$, i.e. the norm inequality is in the Calkin algebra $\mathcal{L}((H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G)/\mathcal{R}((H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G)$. In particular, taking $\epsilon = \frac{1}{2}$, the bounded, odd, self-adjoint operator $F_\sharp T_\sharp c_{w,t}$ has essential spectrum away from $[-\frac{1}{2}, \frac{1}{2}]$ for all $0 \leq t < t_0$. We shall fix and use any odd, continuous function $f_2$ which takes $+1$ on $[\frac{1}{2}, \infty)$ and $-1$ on $(-\infty, -\frac{1}{2}]$. We see that for all $0 \leq t \leq t_0$, $f_2((F_\sharp T_\sharp c_{w,t}))$ is an odd, self-adjoint, bounded operator on $(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G$ such that

$$1 - (f_2((F_\sharp T_\sharp c_{w,t})))^2 \in \mathcal{R}((H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau(\mathbb{R})) \rtimes_r G).$$

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Indeed, we have

\[ 1 - (f_2(F_r^* T^* c_{w,t})_{0 \leq t \leq t_0})^2 \in \mathcal{R} (\{(H_0 \otimes H \otimes A \otimes C_r(\mathbb{R})) \times_r G \} [0, t]) \]

which defines a homotopy from \( f_2(F_r^* T^* c_{w,0}) \) to \( f_2(F_r^* T^* c_{w,t_0}) \). This shows that the K-cycles \( \{(H_0 \otimes H \otimes A \otimes C_r(\mathbb{R})) \times_r G, f_2(F_r^* T^* c_{w,t})\} \) all define the same element as the one (1.30) in \( \mathcal{K}(\{(A \otimes C_r(\mathbb{R})) \times_r G) \). Let us pose and record our discussions so far:

**1.31 Lemma.** Let \((H, T)\) be a proper Kasparov cycle over a proper \(G\)-space \(X\) and \(w, c_{w,t}\) as above. For any K-cycle \(\{(H_0 \otimes A) \times_r G, F\}\) for \(A \rtimes_r G\), there is \(0 < t_0 \leq 1\) such that for any \(0 \leq t \leq t_0\), the pair \(\{(H_0 \otimes H \otimes A \otimes C_r(\mathbb{R})) \times_r G, f_2(F_r^* T^* c_{w,t})\}\) is a K-cycle for \(A \rtimes_r C_r(\mathbb{R}) \times_r G\) which defines the same element in \(KK(\mathbb{C}, A \otimes C_r(\mathbb{R}) \times_r G)\) as the Kasparov product of the K-cycle \(\{(H_0 \otimes H \otimes A) \times_r G, F_r^* T\}\) for \(A \rtimes_r G\) and the K-cycle (the Bott generator) \((C_r(\mathbb{R}), c_{w,0})\) for \(C_r(\mathbb{R})\).

Let \(H_r(\mathbb{R})\) be \(L^2(\mathbb{R}, \Lambda^*_C(\mathbb{R}))\), the graded Hilbert space of \(L^2\)-sections of exterior algebra bundles on the Euclidean space \(\mathbb{R}\) (with trivial \(G\)-action). The Clifford algebra \(C_r(\mathbb{R})\) is naturally represented on \(H_r(\mathbb{R})\). Denote this representation by \(\pi_{\mathbb{R}}\). The Dirac element \([d_{\mathbb{R}}]\) in \(KK(C_r(\mathbb{R}), \mathbb{C})\) is defined by the triple \((H_r(\mathbb{R}), \pi_{\mathbb{R}}, d_0)\) where \(d_0\) is the bounded transform of the Dirac operator

\[ d_{\mathbb{R}} = \begin{bmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix}. \]

It is a well-known fact that the Kasparov product of the Bott generator \([C_r(\mathbb{R}), c_{w,0}]\) in \(KK(\mathbb{C}, C_r(\mathbb{R}))\) with the Dirac element \([d_{\mathbb{R}}]\) in \(KK(C_r(\mathbb{R}), \mathbb{C})\) is the multiplicative identity in \(KK(\mathbb{C}, \mathbb{C})\). Thus, we see from this and Lemma 1.31, we obtain the following:

**1.32 Lemma.** Let \((H, T)\) be a proper Kasparov cycle over a proper \(G\)-space \(X\) and \(w, c_{w,t}\) as above. For any K-cycle \(\{(H_0 \otimes A) \times_r G, F\}\) for \(A \rtimes_r G\), there is \(0 < t_0 \leq 1\) such that for any \(0 \leq t \leq t_0\), the pair \(\{(H_0 \otimes H \otimes A \otimes C_r(\mathbb{R})) \times_r G, f_2(F_r^* T^* c_{w,t})\}\) defines an element in \(KK(\mathbb{C}, A \otimes C_r(\mathbb{R}) \times_r G)\) such that its product with the Dirac element \([d_{\mathbb{R}}]\) in \(KK(\mathbb{C}, C_r(\mathbb{R}))\) is equal to the one \(\{(H_0 \otimes H \otimes A) \times_r G, F_r^* T\}\) in \(KK(\mathbb{C}, A \rtimes_r G)\).

Now, for \(0 < t \leq t_0\), we examine the K-cycle \(\{(H_0 \otimes H \otimes A \otimes C_r(\mathbb{R})) \times_r G, f_2(F_r^* T^* c_{w,t})\}\) for \(A \rtimes_r C_r(\mathbb{R}) \times_r G\). Recall that

\[ 1 - (f_2(F_r^* T^* c_{w,t})_{0 \leq t \leq t_0})^2 \in \mathcal{R} (\{(H_0 \otimes H \otimes A \otimes C_r(\mathbb{R})) \times_r G\} [0, t_0]). \]
This means that if we view $f_2(F^\#_T z c_{w,t})$ as a family $f_2(F^\#_T z c_{w,t})_y$ of operators parametrized by $y$ in $\mathbb{R}$, $1 - (f_2(F^\#_T z c_{w,t})_y)^2$ are all compact whose norm vanish as $y$ in $\mathbb{R}$ goes to infinity uniformly in $0 \leq t \leq t_0$. We shall fix an odd, continuous function $f'_3$ on $\mathbb{R}$ which is $\pm 1$ near $\pm 1$ and we write

$$f_3(F^\#_T z c_{w,t}) = f'_3(f_2(F^\#_T z c_{w,t})).$$

We see that there is $R_0 > 0$ so that for any $0 \leq t \leq t_0$,

$$1 - (f_3(F^\#_T z c_{w,t}))^2 \in K(H_0 \otimes H \otimes C_\tau((-R_0, R_0)) \rtimes_r G) \subset K(H_0 \otimes H \otimes C_\tau(\mathbb{R}) \rtimes_r G).$$

Indeed, we have

$$1 - (f_3(F^\#_T z c_{w,t}))^2 \in K((H_0 \otimes H \otimes C_\tau((-R, R)) \rtimes_r G)[0, t_0]).$$

For any $R \geq R_0$, we consider the restriction $f_3(F^\#_T z c_{w,t})_R$ of $f_3(F^\#_T z c_{w,t})$ to $H_0 \otimes H \otimes C_\tau((-R, R)) \rtimes_r G$. We see that the pair

$$((H_0 \otimes H \otimes A \otimes C_\tau((-R, R))) \rtimes_r G, f_3(F^\#_T z c_{w,t})_R)$$

is a $K$-cycle for $A \otimes C_\tau((-R, R)) \rtimes_r G$ which by the inclusion $C_\tau((-R, R)) \subset C_\tau(\mathbb{R})$ is sent to the same element as $[(H_0 \otimes H \otimes A \otimes C_\tau(\mathbb{R})) \rtimes_r G, f_2(F^\#_T z c_{w,t})]$ in $KK(C, A \otimes C_\tau(\mathbb{R}) \rtimes_r G)$. Let us write $[d_R]$, the Dirac element for $(-R, R)$, i.e. the element in $KK(C_\tau((-R, R)), C)$ which is defined as the composition of the inclusion $C_\tau((-R, R)) \subset C_\tau(\mathbb{R})$ and the Dirac element $[d_{-R}]$ in $KK(C, C_\tau(\mathbb{R}), C)$. Here, we pose and summarize our discussions so far:

**1.33 Lemma.** Let $(H, T)$ be a proper Kasparov cycle over a proper $G$-space $X$ and $w, c_{w,t}$ as above. For any $K$-cycle $((H_0 \otimes A) \rtimes_r G, F)$ for $A \rtimes_r G$, there is $0 < t_0 \leq 1$ and $R_0 > 0$ such that for any $0 \leq t \leq t_0$ and $R \geq R_0$ the pair $((H_0 \otimes H \otimes A \otimes C_\tau((-R, R))) \rtimes_r G, f_3(F^\#_T z c_{w,t})_R)$ defines an element in $KK(C, A \otimes C_\tau((-R, R)) \rtimes_r G)$ such that its product with the Dirac element $[d_R]$ in $KK(C, C_\tau((-R, R)))$ is equal to the one $[(H_0 \otimes H \otimes A) \rtimes_r G, F^\#_T]$ in $KK(C, A \rtimes_r G)$.

Now, we shall show the following:

**1.34 Lemma.** For any $0 < t \leq t_0$ and $R \geq R_0$, the $K$-cycle

$$((H_0 \otimes H \otimes A \otimes C_\tau((-R, R))) \rtimes_r G, f_3(F^\#_T z c_{w,t})_R)$$

for $A \otimes C_\tau((-R, R)) \rtimes_r G$ is a proper $K$-cycle with $G$-compact support.
Before giving a proof of this Lemma, we give its main consequences:

**1.35 Theorem.** Let \((H, T)\) be a proper Kasparov cycle over proper \(G\)-space \(X\). For any graded \(G\)-C*-algebra \(A\), all elements in the image of \([H, T]_*\) on \(KK(C, A \rtimes_r G)\) via (1.8) are in the image of the assembly map \(\mu_{A^Y}^G\) for some \(G\)-compact, proper \(G\)-space \(Y \subset X\).

*Proof.* For any element \([(H_0 \hat{\otimes} A) \rtimes_r G, F]\) in \(KK(C, A \rtimes_r G)\), its image by \([H, T]_*\) is \([(H_0 \hat{\otimes} H \hat{\otimes} A) \rtimes_r G, F \otimes T]\). Lemma 1.33 says that this image is equal to the Kasparov product of \([(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_r((-R, R))) \rtimes_r G, f_3(F \otimes T \overline{c_{w,t}})_R]\) in \(KK(C, A \hat{\otimes} C_r((-R, R)) \rtimes_r G)\) and \([d_R]\) in \(KK(C, C_r((-R, R)))\). It follows from Proposition 1.25 and Lemma 1.34 that for some \(G\)-compact, proper \(G\)-space \(Y \subset X\), there is an element \(x\) in \(KK^G(C_0(Y), A \hat{\otimes} C_r((-R, R)))\) such that \([(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_r((-R, R))) \rtimes_r G, f_3(F \otimes T \overline{c_{w,t}})_R]\) is the image of \(x\) by the assembly map \(\mu_{A^Y}^G\). It follows \([(H_0 \hat{\otimes} H \hat{\otimes} A) \rtimes_r G, F \otimes T]\) is the image of \(x \hat{\otimes} C_r((-R, R))[d_R]\) in \(KK^G(C_0(Y), A)\) by the assembly map \(\mu_{A^Y}^G\).  

**1.36 Corollary.** Suppose there is a proper Kasparov cycle \((H, T)\) over a proper \(G\)-space \(X\) such that \([H, T]_*\) is surjective on \(KK(C, A \rtimes_r G)\) via (1.8). Then, all the elements of \(KK(C, A \rtimes_r G)\) are in the image of the assembly map \(\mu_{A^Y}^G\) for some \(G\)-compact, proper \(G\)-space \(Y \subset X\).

Recall that the Baum–Connes assembly map with coefficient \(A\):

\[
\mu_{A^G}^G : RK^G(EG, A) = \lim_{Y \subset EG} KK^G(C_0(Y), A) \to KK(C, A \rtimes_r G)
\]

is defined as the inductive limit of \(\mu_{A^Y}^G\) for all \(G\)-compact, proper \(G\)-space \(Y\). In practice, we may take this limit as the limit over \(G\)-compact \(G\)-invariant closed subsets \(Y\) in the fixed universal proper \(G\)-space \(EG\) of \(G\), hence the notation. The Baum–Connes conjecture with coefficients states that the assembly map \(\mu_{A^G}^G\) is an isomorphism of abelian groups for all \(A\). We refer to [BCH94] for more detail. We obtain:

**1.37 Theorem.** For any proper Kasparov cycle \((H, T)\) and for any \(G\)-C*-algebra \(A\), the image of \([H, T]_*\) on \(KK(C, A \rtimes_r G)\) via (1.8) lies in the image of the assembly map \(\mu_{A^G}^G\).

**1.38 Corollary.** Suppose there is a proper Kasparov cycle \((H, T)\) for \(KK^G(C, C)\) which acts surjectively on \(KK_* (C, A \rtimes_r G)\) via (1.8) for any \(A\). Then, the Baum–Connes conjecture with coefficients holds for \(G\).
Proof. The assumption implies that \( \mu^C \) is surjective for all \( A \). This automatically implies that the assembly map \( \mu^C \) is injective for all \( A \) as well. This “surjectivity implies injectivity” principle is explained in Remark [Nis19] for the case when the universal space \( \mathbb{E}G \) admits a \( G \)-compact model \( E \) but the explanation there generalizes verbatim.

1.39 Corollary. Suppose there is a proper Kasparov cycle \((H, T)\) for \( \text{KK}^G(\mathbb{C}, \mathbb{C}) \) which is homotopic to the multiplicative identity \( 1_G \). Then, the Baum–Connes conjecture with coefficients holds for \( G \).

Proof of Lemma 1.34. Let us fix \( 0 < t \leq t_0 \) and \( R \geq R_0 \) and consider the K-cycle

\[
(1.40) \quad (H_0 \otimes H \otimes A \otimes C_t((-R, R))) \rtimes_t G, f_3(F^t_\mathbb{R} C_{w,t})^R_R
\]

for \( A \otimes C_t((-R, R)) \rtimes_t G \). We recall that

\[
F^t_\mathbb{R} C_{w,t} = F + (1 - F^2)^{1/2} \left( c_{w,t} + (1 - c_{w,t}^2)^{1/2} \right) (1 - F^2)^{1/2}
\]

on \( H_0 \otimes H \otimes A \otimes C_t(\mathbb{R}) \) and that \( f_3(F^t_\mathbb{R} C_{w,t})^R_R \) is nothing but the restriction to \( C_t((-R, R)) \) of functional calculus \( f_3 = f_3' \circ f_2 \) applied to \( F^t_\mathbb{R} C_{w,t} \). We need show that the cycle \((1.40)\) is a proper K-cycle with \( G \)-compact support.

By the first property of \( c_{w,t} \) in Lemma 1.29, we see that \( (1 - c_{w,t}^2) \) on \( H \otimes C_t(-R, R) \) has compact support \( X_{R,t} \subset X \) with respect to the given non-degenerate representation of \( C_0(X) \) on \( H \). Let \( Y \) be any \( G \)-compact, \( G \)-invariant closed subset of \( X \) containing \( X_{R,t} \) and \( P_Y \) be the corresponding \( G \)-equivariant projection which acts on \( H \). We see that \( P_Y \) commutes with the representation \( C_0(X) \) and the operator \( F^t_\mathbb{R} C_{w,t} \) on \( H_0 \otimes H \otimes A \otimes C_t(-R, R) \rtimes_t G \). Let \( E = H_0 \otimes H \otimes A \otimes C_t(-R, R) \) and \( E_Y = P_Y E \). We see that the representation of \( C_0(X) \) on \( E_Y \) factors through \( C_0(Y) \). To check that the cycle \((E_Y \rtimes_t G, P_Y f_3(F^t_\mathbb{R} C_{w,t})^R_R P_Y)\) is proper over \( Y \), we need to check that the commutator

\[
[\pi_Y(\phi), \rho_{A \otimes C_t((-R, R))}(P_Y f_3(F^t_\mathbb{R} C_{w,t})^R_R P_Y)]
\]

belongs to \( \mathfrak{H}(E_Y \otimes L^2(G)) \) for any \( \phi \) in \( C_0(Y) \). To see this, it is enough to check that

\[
[\pi_X(\phi), \rho_{A \otimes C_t((-R, R))}(F^t_\mathbb{R} C_{w,t})]
\]

belongs to \( \mathfrak{H}(E \otimes L^2(G)) \) for any \( \phi \) in \( C_0(X) \). This commutator is equal to

\[
\rho_{A \otimes C_t((-R, R))}((1 - F^2)^{1/2}(1 - c_{w,t}^2)^{1/2})([g(T), \phi])_{g \in G} \rho_{A \otimes C_t((-R, R))}((1 - F^2)^{1/2}(1 - c_{w,t}^2)^{1/2})
\]
which clearly belongs to \(\mathfrak{A}(E \otimes L^2(G))\) by the property of \(T\) in Definition 1.3.

Next, we need to show that the complementary cycle \(((1 - P_Y)E \times_r G, (1 - P_Y)f_3(F_\pi T_{\pi c_{w,t}})_R(1 - P_Y))\) is degenerate up to compact perturbation. Note that we have

\[
(1 - P_Y)f_3(F_\pi T_{\pi c_{w,t}})_R(1 - P_Y) = f_3((1 - P_Y)F_\pi T_{\pi c_{w,t}}(1 - P_Y))
\]

on \((1 - P_Y)E \times_r G\) and

\[
(1 - P_Y)F_\pi T_{\pi c_{w,t}}(1 - P_Y) = (1 - P_Y)(F + (1 - F^2)^\frac{1}{2}c_{w,t}(1 - F^2)^\frac{1}{2})
\]

on \((1 - P_Y)E \times_r G\). By the choice of \(t_0\), \(1 - (F + (1 - F^2)^\frac{1}{2}c_{w,t}(1 - F^2)^\frac{1}{2})^2\) is modulo \(\epsilon = \frac{1}{2}\),

\[
(1 - F^2)(1 - c_{w,t}^2)
\]

which is zero modulo \((1 - P_Y).\) It follows \((1 - P_Y)f_3(F_\pi T_{\pi c_{w,t}})_R(1 - P_Y)\) is an odd, self-adjoint unitary on \((1 - P_Y)E \times_r G\) showing that the degeneracy of the complementary cycle as required.

Finally, we need to compare the two operators \((P_Y f_3(F_\pi T_{\pi c_{w,t}})_R P_Y)\) and \((P_Y f_3(F_\pi T_{\pi c_{w,t}})_R P_Y)\)' on \(E_Y \times_r G\) where \(S'\) for \(S\) in \(\mathcal{L}(E_Y \times_r G)\) is as determined by (1.21) using the structure of \(E_Y\) proper over \(Y\). We have

\[
(P_Y F_\pi T_{\pi c_{w,t}} P_Y) - (P_Y F_\pi T_{\pi c_{w,t}} P_Y)' \in \mathfrak{A}(E_Y \times_r G).
\]

This follows since we can write this difference as

\[
P_Y(1 - F^2)^\frac{1}{2} \left( (1 - c_{w,t}^2)^\frac{1}{2} \int_G [g(c), T] g(c) d\mu_G(g)(1 - c_{w,t}^2)^\frac{1}{2} \right)(1 - F^2)^\frac{1}{2} P_Y
\]

\[
= P_Y(1 - F^2)^\frac{1}{2} \left( (1 - c_{w,t}^2)^\frac{1}{2} \int_G [g(c), T] g(c) d\mu_G(g)(1 - c_{w,t}^2)^\frac{1}{2} \right)(1 - F^2)^\frac{1}{2} P_Y.
\]

Here, integral is essentially taken over a compact subset of \(G\) since \((1 - c_{w,t}^2)\) has compact support on \(E_Y\). To show that

\[
(P_Y f_3(F_\pi T_{\pi c_{w,t}})_R P_Y) - (P_Y f_3(F_\pi T_{\pi c_{w,t}})_R P_Y)' \in \mathfrak{A}(E_Y \times_r G),
\]

we can do the same as above but lengthier computations for polynomials of \((F_\pi T_{\pi c_{w,t}})_R\) in stead of \(f_3(F_\pi T_{\pi c_{w,t}})_R\), and the continuity argument shows the claim. \(\square\)
2 Direct splitting

Our aim of this section is to prove the following:

2.1 Theorem. For any proper Kasparov cycle \((H, T)\) for \(KK^G(C, C)\), there is a well-defined natural homomorphism

\[
\nu^G_{A, (H, T)} : KK(C, A \rtimes_r G) \to RKK^G(EG, A)
\]

for any \(G\)-C*-algebra \(A\) such that the composition \(\mu^G_A \circ \nu^G_{A, (H, T)}\) coincides with \([H, T]_s\) on \(KK(C, A \rtimes_r G)\).

Here, naturality means that for any morphism \(\theta\) in \(KK^G(A, B)\), the following diagram is commutative:

\[
\begin{array}{ccc}
\nu^G_{A, (H, T)} : KK_s(C, A \rtimes_r G) & \longrightarrow & RKK^G(EG, A) \\
& \downarrow \theta_\ast & \\
\nu^G_{B, (H, T)} : KK_s(C, B \rtimes_r G) & \longrightarrow & RKK^G(EG, B)
\end{array}
\]

As explained in the proof of [Nis19, Proposition 4.5], to check this, it is enough to check this for \(G\)-equivariant \(*\)-homomorphisms.

Let us first briefly recall our previous discussions so that we get an idea of constructing the map \(\nu^G_{A, (H, T)}\) and what we need to check.

Let \((H, T)\) be a proper Kasparov cycle \((H, T)\) over a proper \(G\)-space \(X\) and \(w\) be a measurable function from \(X\) to \(R\) witnessing the \(G\)-completeness of \(H\) for \(T\). Let \(c_{w,t}\) be an odd, self-adjoint, bounded operator on \(H \otimes C_\tau([R, R])\) as in Definition 1.28. For any \(G\)-C*-algebra \(A\), the map \([H, T]_s\) on \(KK(C, A \rtimes_r G)\) sends an element represented by a \(K\)-cycle \(((H_0 \otimes A) \rtimes_r G, F)\) to the one represented by \(((H_0 \otimes H \otimes A) \rtimes_r G, F_\tau T)\) which is shown to be a proper \(K\)-cycle for \(A \rtimes_r G\) but it may not have \(G\)-compact support.

On the other hand, we saw in Lemma 1.33 that the element \([H_0 \otimes H \otimes A] \rtimes_r G, F_\tau T)\) in \(KK(C, A \rtimes_r G)\) can be recovered via Kasparov product by the Dirac element \([d_r]\) in \(KK(C, C_\tau((-R, R)))\) from the \(K\)-cycle

\[
\nu^G_{A, (H, T)} \circ [H_0 \otimes H \otimes A] \rtimes_r G, f_\ast (F_\tau T \otimes c_{w,t})
\]

for \(A \otimes C_\tau((-R, R)) \rtimes_r G\) for any \(0 \leq t \leq t_0\) and \(R \geq R_0\) and in Lemma 1.34 that if \(t > 0\) this cycle is a proper \(K\)-cycle with \(G\)-compact support. More
relevantly, for fixed $0 < t \leq t_0$ and $R \geq R_0$, we saw that in the proof of Lemma 1.34, there is a compact subset $X_{R,t}$ (which depends on $R$ and $t$) of $X$ so that for any $G$-compact, $G$-invariant closed subset $Y$ of $X$ containing $X_{R,t}$ and for $P_Y$, the corresponding $G$-equivariant projection in $H$, the cycle (2.2) is equal to the one

$$\text{(2.3)} \quad (P_Y(H_0 \otimes H \otimes A \otimes C_r((-R, R)))) \rtimes_t G, P_Y f_3(F^2 T^2 c_{w,t}) P_Y$$

up to the degenerate complementary cycle. We also saw that the cycle (2.3) is a proper $K$-cycle over $Y$ so that its right-regular representation (2.4)

$$\text{(2.4)} \quad (P_Y(H_0 \otimes H \otimes A \otimes C_r((-R, R)))) \otimes L^2(G), P_Y \rho_{A \otimes C_r((-R, R))} f_3(F^2 T^2 c_{w,t}) P_Y$$

defines an element in $KK^G(C_0(Y), A \otimes C_r((-R, R)))$. Let us denote this element by $x'_{f,R,t,Y}$ and let

$$x_{f,R,t,Y} = F'_{f,R,t,Y} \otimes C_r((-R, R))[d_R] \in KK^G(C_0(Y), A).$$

We saw that the image $\mu^{G,Y}_{A}(x'_{f,R,t,Y})$ of this element by the assembly map is equal to $\left(\mu_{f,R,t,Y}(H_0 \otimes H \otimes A) \rtimes_t G, F^2 T\right)$, the image of $\left(\mu_{f,R,t,Y}(H_0 \otimes A) \rtimes_t G, F\right)$ by the map $[H, T]$. Hence, we want to show that the assignment

$$\text{(2.5)} \quad ((H_0 \otimes A) \rtimes_t G, F) \mapsto x_{f,R,t,Y} \in \text{RKK}^G(EG, A)$$

descends to a well-defined, natural homomorphism

$$\text{(2.6)} \quad \nu^G_{A,(H,T)} : KK(C, A \rtimes_t G) \to \text{RKK}^G(EG, A).$$

Here, we are naturally identifying an element in $KK^G(C_0(Y), A)$ as its image in $\text{RKK}^G(EG, A)$ by a $G$-equivariant map from $X$ to $EG$ which exists and unique up to $G$-equivariant homotopy.

**Proof of Theorem 2.1.** We show that the assignment (2.5) descends to descends to a well-defined, natural homomorphism (2.6). First, for fixed $F$ and $0 < t \leq t_0$ and $R \geq R_0$, we show that the choice of $Y \supset X_{R,t}$ do not matter. This follows from the proof of Lemma 1.34. There, we showed that modulo $(1 - P_Y)$, $f_3(F^2 T^2 c_{w,t}) P_Y$ is

$$f_3(F + (1 - F^2)^{1/2} c_{w,t}(1 - F^2)^{1/2})$$

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on $H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau((-R, R)) \rtimes_r G$ which is self-adjoint unitary and commutes with all functions on $X$. From this, if $Y' \supset Y$, the difference of corresponding cycles defining $\chi_{F,R,t,Y}$ and $\chi_{F,R,t,Y'}$ in $KK^G(C_0(Y'), A)$ are clearly degenerate.

Next, we show that for fixed $F$ and $0 < t \leq t_0$, the choice of $R \geq R_0$ do not matter. Suppose $R' > R$, we take $Y \supset X_{R',t} \cup X_{R,t}$ and compare $\chi_{F,R,t,Y}$ and $\chi_{F,R',t,Y}$. We can easily see that the two elements

$$\chi_{F,R,t,Y}' \in KK^G(C_0(Y), A \hat{\otimes} C_\tau((-R, R))), \chi_{F,R',t,Y}' \in KK^G(C_0(Y), A \hat{\otimes} C_\tau((-R', R')))$$

are equal via the homotopy equivalence $C_\tau((-R, R)) \subset C_\tau((-R', R'))$ and hence their images by the Dirac elements $[d_R], [d_{R'}]$ also coincide.

To compare $\chi_{F,R,t,Y}$ and $\chi_{F,R',t,Y}$ for $0 < t' < t \leq t_0$, and for fixed $R \geq R_0$ and for fixed $Y \supset X_{R,t'} \cup X_{R,t}$ we recall that we have

$$1 - (f_3(F_\tau T_s c_{w,t})_{0 \leq t \leq t_0})^2 \in K(H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau((-R_0, R_0)) \rtimes_r G)[0, t_0]).$$

Thus, in particular, we have a homotopy (K-cycle for $(A \hat{\otimes} C_\tau((-R_0, R_0)) \rtimes_r G)[t, t']$

$$((H_0 \hat{\otimes} H \hat{\otimes} A \hat{\otimes} C_\tau((-R, R)) \rtimes_r G)[t', t], f_3(F_\tau T_s c_{w,s})_{t' \leq s \leq t}).$$

This homotopy itself is a proper K-cycle with $G$-compact support and it is not hard to see that the regular representation of its $P_Y$-part (which is proper K-cycle over $Y$) gives us a homotopy between $\chi_{F,R,t,Y}'$ and $\chi_{F,R',t,Y}'$, showing $\chi_{F,R,t,Y} = \chi_{F,R',t,Y}$.

Finally, given two $F$ and $F'$ defining the same element $KK(C, A \rtimes_r G)$, a homotopy $(F_s)_{s \in [0,1]}$ between the two may be regarded as a K-cycle for $(A \rtimes_r G)[0, 1] = A[0, 1] \rtimes_r G$. Our construction applied to this homotopy produces for some suitable $t > 0$, $R > 0$, and $Y \subset X$, a homotopy between $\chi_{F,t,R,t}$ and $\chi_{F',t,R,t}$.

We showed that the assignment (2.5) is well-defined, respects homotopy and by construction, is clearly seen to be a homomorphism of abelian groups. To see it is natural with respect to a $G$-equivariant $\ast$-homomorphism $\theta$ from $A$ to $B$, we note that we can quite straightforwardly generalize our construction by considering all the K-cycles of the form $(E \rtimes_r G, F)$ for $A \rtimes_r G$ where $E$ is a graded $G$-Hilbert $A$-module in stead of the standard form $(H_0 \hat{\otimes} A \rtimes_r G, F)$. For example, in this case, $F_\tau T_s c_{w,t}$ acts on $(E \hat{\otimes} H \hat{\otimes} C_\tau(R)) \rtimes_r G$ and all the constructions generalize verbatim. After this remark, the
naturality with respect to $\theta$ can be checked step by step: for example, for a $K$-cycle $(E, r, G, F)$ for $A \rtimes_r G$, and its image $(E_\theta, r, G, F_\theta)$ by $\theta_*$, where $E_\theta = E \hat{\otimes} A B$ and $F_\theta = F \hat{\otimes} A 1$, the cycle $(E_\theta \hat{\otimes} H \hat{\otimes} C \rtimes_r G, F_\theta \hat{\otimes} T c_{w,t})$ is nothing but the image of $(E \hat{\otimes} H \hat{\otimes} C \rtimes_r G, F \hat{\otimes} T c_{w,t})$ by $\theta_*$. We conclude that the homomorphism $\mu_{A, G, (H, T)}$ is natural and by construction, the composition $\mu^G_A \circ \nu_{A, G, (H, T)}$ coincides with $[H, T]_*$ on $KK(C, A \rtimes_r G)$. 

\section{Property $(\gamma)$}

\subsection{Definition.} We say that a Kasparov cycle $(H, T)$ for $KK^G(C, C)$ has Property $(\gamma)$ if:

1. $(H, T)$ is a proper Kasparov cycle.
2. $[H, T] = 1_K$ in $R(K)$ for any compact subgroup $K$ of $G$.

\subsection{Remark.} If there is a $G$-compact model $E$ of the universal proper $G$-space $EG$, this definition gives an a-priori weaker notion of Property $(\gamma)$ compared to the one defined in [Nis19] which requires $(H, T)$ to be a proper cycle over the specific space, namely $E$ whereas the current definition allows us to use any (locally compact, second countable) proper $G$-space $X$, which is not necessarily $G$-compact or universal. For this reason, we prefer to call the version of Property $(\gamma)$ in [Nis19] as Property $(\gamma)^{G\text{-compact}}$.

The following is a simple consequence of Theorem 2.1:

\subsection{Theorem.} Suppose there is a Kasparov cycle $(H, T)$ for $KK^G(C, C)$ with Property $(\gamma)$. Then:

1. the Strong Novikov conjecture holds for $G$, i.e. the assembly map $\mu^G_A$ is split-injective for any $A$.
2. the map $\nu_{A, G, (H, T)}$ in Theorem 2.1, which we call the $(\gamma)$-morphism, is a left-inverse of the assembly map $\mu^G_A$.
3. the assembly map $\mu^G_A$ is an isomorphism if and only if $[H, T]_*$ is the identity map on $KK(C, A \rtimes_r G)$ where $[H, T]_*$ is defined by the composition (1.8).
Proof. Let \( \nu_A^{G,(H,T)} \) be the map as in Theorem 2.1 defined by \((H, T)\). By Theorem 2.1, we know that the composition \( \mu_A^G \circ \nu_A^{G,(H,T)} \) coincides with \([H, T]_*\) on \(\text{KK}(\mathbb{C}, A \rtimes_r G)\) for all \(A\). Using naturality of \(\nu_A^{G,(H,T)}\), as in the proof of [Nis19, Proposition 5.3], it follows that the other composition \(\nu_A^{G,(H,T)} \circ \mu_A^G\) is the identity on \(\text{KK}(C, A \rtimes_r G)\) for all \(A\). Showing the first and the second claims. It follows that \(\mu_A^G \circ \nu_A^{G,(H,T)} = [H, T]_*\) is an idempotent on \(\text{KK}(\mathbb{C}, A \rtimes_r G)\). Hence, the assembly map \(\mu_A^G\) is an isomorphism if and only if \([H, T]_*\) is the identity on \(\text{KK}(\mathbb{C}, A \rtimes_r G)\).

3.4 Corollary. Suppose there is a Kasparov cycle \((H, T)\) for \(\text{KK}^G(\mathbb{C}, \mathbb{C})\) with Property \((\gamma)\) which acts surjectively on \(\text{KK}(\mathbb{C}, A \rtimes_r G)\) by the composition (1.8) for any \(A\). Then, the Baum–Connes conjecture with coefficients holds for \(G\). The \((\gamma)\)-morphism \(\nu_A^{G,(H,T)}\) is the inverse of the assembly map \(\mu_A^G\) for any \(A\).

3.5 Corollary. Suppose there is a Kasparov cycle \((H, T)\) for \(\text{KK}^G(\mathbb{C}, \mathbb{C})\) with Property \((\gamma)\) which is homotopic to \(1_G\). Then, the Baum–Connes conjecture with coefficients holds for \(G\). The \((\gamma)\)-morphism \(\nu_A^{G,(H,T)}\) is the inverse of the assembly map \(\mu_A^G\) for any \(A\).

3.6 Definition. A \((\gamma)\)-element for \(G\) is any element in \(\mathbb{R}(G)\) which is represented by \((H, T)\) with Property \((\gamma)\).

We currently do not know if a \((\gamma)\)-element is the unique idempotent in \(\mathbb{R}(G)\) characterized by the property. On the other hand, if a gamma element \(\gamma\) exists, \(\gamma\) is a \((\gamma)\)-element:

3.7 Theorem. Suppose there is a gamma element \(\gamma\) for \(G\). Then, \(\gamma\) is represented by some Kasparov cycle \((H, T)\) with Property \((\gamma)\).

Proof. The proof of [Nis19, Theorem 2.10] generalizes straightforwardly and partly in a simpler way, so we shall be brief here. We just need to show that in general, for any graded proper \(G\)-\(C_0(X)\)-algebra \(P\), the Kasparov product \(x \otimes_p y\) of elements \(x\) in \(\text{KK}^G(\mathbb{C}, P)\) and \(y\) in \(\text{KK}^G(P, \mathbb{C})\) is represented by a proper Kasparov cycle. Without loss of generality by stabilizing \(P\), we can assume that \(x\) is of the form \([P, b]\) where \(b\) is an odd, self-adjoint, \(G\)-continuous element in the multiplier algebra \(M(P)\) satisfying \(1 - b^2 \in P\) and \(g(b) - b \in P\) for any \(g\) in \(G\) and that \(y\) is of the form \([H, \pi_P, F]\) where \(F\) is an odd, self-adjoint, bounded, \(G\)-continuous operator on a graded \(G\)-Hilbert space \(H\) equipped with a non-degenerate representation \(\pi_P\) of \(P\).
satisfying \( a(1 - F^2) \in K(H) \), \( a(g(F) - F) \in K(H) \) and \( \{a, F\} \in K(H) \) for any \( g \) in \( G \) and for any \( a \in P \). We just need to find a cycle \((H, T)\) where \( T \) is of the form

\[
T = M_1^\dagger b M_1^\dagger + M_2^\dagger F M_2^\dagger
\]

which defines a Kasparov product of \((P, b)\) and \((H, \pi_P, F)\) and at the same time satisfying

\[
(g \mapsto [g(\phi), T]) \in C_0(G, K(H))
\]

for \( \phi \) in \( C_0(X) \) naturally represented on \( H \) through \( P \). Here \( M_1, M_2 = 1 - M_1 \) are suitable operators on \( H \) which can be constructed as in the proof of [Nis19, Theorem 2.8]. This time, we do not need the condition (VII) in [Nis19, Theorem 2.8] about a cutoff function. On the other hand, in [Nis19, Theorem 2.8], \( X \) was \( E \) which is a \( G \)-compact proper \( G \)-space. Hence, a little modification is necessary. We just describe this modification very briefly and leave the rest of details to the reader. To prove Theorem 2.8 without the condition (VII) where we replace \( E \) by an arbitrary proper \( G \)-space \( X \), we just need the following modifications in the proof. In stead of a compact subset \( Y \) in \( C_0(E) \), we simply use an increasing sequence of relatively compact open subsets \( X_n \) of \( X \) and an increasing sequence \( Y_n \subset C_0(X_n) \) of compact subsets which generates \( C_0(X) \). After constructing an approximate unit \( a_n \) in \( P_c \) as in the proof, for each \( n \), we set a compact subset \( K_n \subset G \) to be so that \( a_n g(\phi) = 0 \) for all \( \phi \) in \( Y_n \) unless \( g \) is in \( K_n \). Finally, we construct an approximate unit \( u_n \) in \( J \) as in the proof, but the condition (f) is unnecessary and we modify the condition “for \( \phi \) in \( Y \)” to “for \( \phi \) in \( Y_n \)” in items (d), (e). That is all we need to modify the proof.

\[\square\]

4 Unbounded proper cycles

4.1 Definition. An unbounded Kasparov cycle for \( KK^G(C, C) \) is a pair \((H, D)\) of a graded \( G \)-Hilbert space \( H \) and an odd, (essentially) self-adjoint, unbounded operator \( D \) on \( H \) such that

1. \( D \) has compact resolvent.

2. The \( G \)-action on \( H \) preserves the domain of \( D \) and \( g(D) - D \) extends to a bounded operator on \( H \) for any \( g \) in \( G \) and defines a strongly continuous, locally bounded, bounded operator valued function \( g \mapsto g(D) - D \) of \( G \).
Let \((H, D)\) be an unbounded Kasparov cycle. It defines a (bounded) Kasparov cycle \((H, T)\) for \(KK^G(\mathbb{C}, \mathbb{C})\) where \(T\) is the bounded transform of \(D\):

\[
T = \frac{D}{(1 + D^2)^{\frac{1}{2}}}.
\]

Now, suppose that \(H\) is a \(G\)-Hilbert space over \(X\) for some proper \(G\)-space \(X\). The proof of [Nis19, Theorem 6.1] goes verbatim to show the following:

**4.2 Theorem.** Let \((H, D)\) be an unbounded Kasparov cycle for \(KK^G(\mathbb{C}, \mathbb{C})\) where \(H\) is a \(G\)-Hilbert space over \(X\) for some proper \(G\)-space \(X\). Suppose that there is a dense \(G\)-subalgebra \(B\) of \(C_c(X)\) which preserves the domain of \(D\), such that for any \(b\) in \(B\):

1. the commutator \([D, g(b)]\) extends to a bounded operator on \(H\) for any \(g\) in \(G\).
2. \([D, g(b)]\) is uniformly bounded in \(g\) in \(G\).
3. the operator \([D, g(b)]\) on \(H\) has compact support \(g(K_b)\) where \(K_b \subset X\) only depends on \(b\).

Then, the corresponding bounded Kasparov cycle \((H, T)\) is proper over \(X\).

**4.3 Example.** Generalizing [Nis19, Example 6.2] (see [NP19, Proposition 29] for more explanations), for any group \(G\) which acts properly, isometrically on a simply connected, complete Riemannian manifold \(M\) of nonpositive sectional curvature which is bounded from below, the unbounded Kasparov cycle \((H_M, D_M)\) satisfies the assumption in Theorem 4.2 where

\[
H_M = L^2(M, \Lambda^* T^*_C M)
\]

is the Hilbert space of \(L^2\)-sections of the complexified exterior algebra bundles on \(M\) and

\[
D_M = d_f + d^*_f
\]

is the self-adjoint operator on \(H_M\) where

\[
d_f = d + df^\Lambda
\]

is the Witten type perturbation of the exterior derivative \(d\) with respect to the function \(f = d_M^2(x_0, x)\), the squared distance function on \(M\) for some
fixed point \( x_0 \) of \( M \). The corresponding bounded Kasparov cycle \((H_M, F_M)\) where \( F_M = \frac{D_M}{1 + D_M^2} \) has Property \((\gamma)\). On the other hand, it is well-known that this cycle represents the gamma element for \( G \) (see [Kas88], [Val02]). We remark that a function \( w \) for the \( G \)-completeness of \( H_M \) for \( F_M \) can be taken as \( w = \sqrt{1 + d_M^2(x, x_0)} \) or any suitable smooth approximation of \( d_M(x, x_0) \).

4.4 Example. For any group \( G \) which acts properly and co-compactly on a locally finite tree \( Y \), a concrete unbounded Kasparov cycle \((H_Y, D_Y)\) satisfying the assumption in Theorem 4.2 is described in [NP19, Section 2]. The corresponding bounded cycle \((H_Y, F_Y)\) has Property \((\gamma)\). This construction generalizes to the general, not necessary cocompact situations. We remark that the construction generalizes from groups acting on a tree to groups acting on a Euclidean building in a sense of [KS91].

4.5 Example. For any group \( G \) which acts properly on a bounded geometry \( \text{CAT}(0) \)-cubical space \( X \), a concrete cycle \((\Omega^*_L(X), D_{dR})\) with Property \((\gamma)\) is constructed in [BGHN19]. Furthermore, this cycle is shown to be homotopic to \( 1_G \). In this way, a new proof of the Baum–Connes conjecture for such groups is obtained in [BGHN19].

4.6 Example. The Baumslag–Solitar group \( \text{BS}(m, n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle \) for a natural number \( m \neq n \) does not admit any proper action on \( \text{CAT}(0) \)-cubical spaces but it acts properly on the product of its (locally finite) Bass–Serre tree \( T_{m,n} \) and the real hyperbolic space \( \mathbb{H}^2 \) (see [Hag07]). If we denote by \((H_{T_{m,n}}, D_{T_{m,n}})\) and \((H_{\mathbb{H}^2}, D_{\mathbb{H}^2})\), the unbounded Kasparov cycles for a tree and for a hyperbolic space as mentioned above, their graded tensor product \((H_{T_{m,n}} \hat{\otimes} H_{\mathbb{H}^2}, D_{T_{m,n}} \hat{\otimes} 1 + 1 \hat{\otimes} D_{\mathbb{H}^2})\) readily satisfies the assumption in Theorem 4.2 for \( G = \text{BS}(m, n) \). It is now easy to see that it has Property \((\gamma)\). One can show that this cycle is homotopic to \( 1_G \) to get a new proof the Baum–Connes conjecture. We remark that the classical gamma element method can be similarly used here, let alone invoking the Higson–Kasparov Theorem [HK01]: \( \text{BS}(m, n) \) is \( \alpha \)-T-menable. The author would like to thank Erik Guentner for informing him about this example.
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