A STRUCTURE THEOREM FOR $SU_C(2)$ AND THE MODULI OF POINTED GENUS ZERO CURVES

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ABSTRACT. Let $SU_C(2)$ be the moduli space of rank 2 semistable vector bundles with trivial determinant on a smooth complex curve $C$ of genus $g > 1$, non-hyperelliptic if $g > 2$. In this paper we prove a birational structure theorem for $SU_C(2)$ that generalizes that of [Bol07] for genus 2. Notably we give a birational description of $SU_C(2)$ as a fibration over $\mathbb{P}^g$, where the fibers are GIT compactifications of the moduli space $\mathcal{M}_{0,2g}$ of $2g$-pointed genus zero curves. This is done by describing the classifying maps of extensions of the line bundles associated to some effective divisors. In particular, for $g = 3$ our construction shows that $SU_C(2)$ is birational to a fibration in Segre cubics over a $\mathbb{P}^3$.

1. Introduction

The first ideas about moduli of vector bundles on curves date back some eighty years, when in [Wei38] for the first time the author suggested the idea that an analogue of the Picard variety could be provided by higher rank bundles. Then, in the second half of last century, a more complete construction of these moduli spaces was carried out, mainly by Mumford, Newstead [MN68] and the mathematicians of the Tata institute, e.g. [NR69a]. Let us denote $SU_C(r)$ the moduli space of semistable vector bundles of rank $r$ and trivial determinant on a smooth complex curve $C$ of genus $g$. If $g \neq 2$ we will also assume through all the paper that $C$ is not hyperelliptic.

Some spectacular results have been obtained on the projective structure of these moduli spaces in low genus and rank, especially thanks to the relation with the work on theta functions and classical algebraic geometry of A.B.Coble [Cob82]. This interplay has produced a flourishing of beautiful results (see [Pau02], [Bea03], [Ort05], [Ngu07], or [DO88] for a survey) where both classical algebraic geometry and modern moduli theory come into play.

On the other hand, even if some important advances have been made in the Brill-Noether theory (for instance [Muk95], [TiB91] or [BV07]) and on the local structure of $SU_C(r)$ ([Ser07], [Las96]), the theory seem to lack general results in arbitrary genus about the structure of these moduli spaces and their birational geometry. For instance, even though it is known that $SU_C(r)$ is unirational, the only case where it is known whether it is rational is for $r = 2$ on a genus 2 curve, and the answer is positive, since $SU_C(2) \cong \mathbb{P}^3$.

This paper aims to start to fill this gap for rank 2 vector bundles. In fact our main theorem gives a description of the structure of $SU_C(2)$ for a curve $C$ of $g(C) > 1$. Here it is.
Theorem 1.1. Let $C$ be a smooth complex curve of genus $g > 1$, non-hyperelliptic if $g > 2$, then $SU_C(2)$ is birational to a fibration over $\mathbb{P}^g$, whose fibers are GIT compactifications of the moduli space $M_{0,2g}$ of $2g$-pointed genus zero curves.

The main tools of our proofs, apart from standard results about the geometry of the Jacobian and the theta series, are related to the results about the maps classifying extension classes described, for instance, in [Ber92] or [LN83]. In this paper we will generally deal with forgetful rational maps whose domain is $\mathbb{P}^{Ext^1(L^{-1}, L)}$, for some line bundle $L$. These maps send an extension equivalence class

$$0 \to L^{-1} \to E \to L \to 0$$

on the corresponding bundle $E \in SU_C(2)$, which in fact has clearly degree zero. Quite often these maps are not defined on all the projective space $\mathbb{P}^{Ext^1(L^{-1}, L)}$ since this contains also non semistable extensions. These extensions classes correspond to the points of certain varieties of secants of the projective model of the curve $C$ contained in $\mathbb{P}^{Ext^1(L^{-1}, L)} = |K + 2L|^*$.

Remark that the moduli spaces $M_{0, 2g}$ are rational themselves, thus our construction gives a fibration in rational varieties over a $\mathbb{P}^g$ Even if already the Brauer-Severi scheme associated to a conic bundle on $\mathbb{P}^2$ can be irrational, we hope that this result could be of help in answering the question of the rationality of $SU_C(2)$.

Description of contents. In Section 2 we give a brief account of the $2\Theta$ linear series on the Picard variety $\text{Pic}^{g-1}(C)$ and its relation with $SU_C(2)$ and a description of the map $\varphi_D : \mathbb{P}^{Ext^1(OC(D), OC(-D))} \to SU_C(2)$ classifying extension classes, for an effective divisor $D$ of degree $g$.

Section 3 will deal mainly the fibers of exceeding dimensions of the preceding map and with another analogue maps that classifies extension classes in $Ext^1(OC(B), OC(-B))$, where $B := K - D$ is the "Serre dual" divisor of $D$.

In Section 4 we prove the core theorem of the paper, giving an interpretation as a determinant to a projection in $|2\Theta|$ while in Section 5 we describe the generic fibers of $\varphi_D$ and discuss the genus 2 case.

In Section 6 we describe in detail the case of genus 3 by showing the birationality between the Coble quartic and a fibration in Segre cubics and finally in Section 7 we go through the relation of $SU_C(2)$ with $M_{0, 2g}$ for $g \geq 4$, completing the proof of the main theorem.

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2. Moduli of Vector Bundles and Classification of Extensions

2.1. Vector bundles and theta linear systems. Let $C$ be as usual a smooth genus $g \geq 2$ algebraic curve, non-hyperelliptic if $g > 2$. Let $\text{Pic}^d(C)$ be the Picard variety that parametrizes all degree $d$ line bundles on $C$. $\text{Pic}^0(C)$ will be often denoted as $\text{Jac}(C)$. We recall that there exists a canonical divisor $\Theta \subset \text{Pic}^{g-1}$ that set theoretically is defined as

$$\Theta := \{ L \in \text{Pic}^{g-1}(C) | h^0(C, L) \neq 0 \}.$$
Let moreover $SU_C(2)$ be the moduli space of semi-stable rank 2 vector bundles on $C$ with trivial determinant. It is well known that $SU_C(2)$ is locally factorial and that $\text{Pic}(SU_C(2)) = \mathbb{Z}$ [Bea88], generated by a line bundle $L$ called the determinant bundle. On the other hand, for $E \in SU_C(2)$ let us define

$$\theta(E) := \{ L \in \text{Pic}^{g-1} | h^0(C, E \otimes L) \neq 0 \}.$$

Luckily enough, while in higher rank there are examples of vector bundles $F$ s.t. $\theta(F)$ is the whole $\text{Pic}^{g-1}(C)$, in the rank two case $\theta(E)$ is a divisor in the linear system of $2\Theta$ for every $E \in SU_C(2)$. This gives the much celebrated theta map

$$\theta : SU_C(2) \longrightarrow |2\Theta| = \mathbb{P}^{2g-1}.$$

We recall that the linear system $|2\Theta|$ on $\text{Pic}^{g-1}$ is interesting because it contains the Kummer variety $Kum(C)$ of $C$. This is the quotient of the Jacobian of $C$ by the involution $x \mapsto -x$ and the map

$$k : \text{Jac}(C) \longrightarrow |2\Theta|, \quad x \mapsto \Theta_x + \Theta_{-x},$$

factors through an embedding $\kappa : Kum(C) \hookrightarrow |2\Theta|$. The geometry of the Kummer variety is intricately related to the geometry of $SU_C(2)$, in fact $Kum(C)$ coincides exactly with the non-stable part of $SU_C(2)$, which in fact consists of bundles of the form $L \oplus L^{-1}$, with $L \in \text{Jac}(C)$. One of the most striking properties of $\theta$ is the following.

**Theorem 2.1.** [Bea88] There is a canonical isomorphism

$$H^0(SU_C(2), L) \cong H^0(\text{Pic}^{g-1}(C), 2\Theta)^*,$$

making the following diagram commutative

$\begin{tikzcd}
Kum(C)^\mathbf{c} \ar[r, \kappa] \ar[dr, \theta] & SU_C(2) \ar[d, \theta] \\
& |2\Theta| \ar[ur, |L|*]
\end{tikzcd}$

Furthermore, thanks to [BV96] and [vGI01] it is known that $\theta$ is an embedding for every $g \geq 2$.

The lower genus $SU_C(2)$ moduli spaces deserve a special mention for their significant and beautiful geometry.

- If $g = 2$ then $SU_C(2) \cong |2\Theta| = \mathbb{P}^3$ and its semi-stable boundary is the well known Kummer quartic surface. [NR69b]
- If $g = 3$ then Ramanan and Narashiman [NR69b] showed that $SU_C(2)$ is a quartic hypersurface in $\mathbb{P}^7 \cong |2\Theta|$ which is singular along $Kum(C)$. Several years before Coble [Cob82] had showed the existence of a unique such quartic hypersurface, that nowadays is named after him. Remarkably, this variety is also self-dual [Pau02].
2.2. The classifying maps. Let $D$ be a degree $g$ effective divisor on $C$ and let us introduce the $(3g - 2)$-dimensional projective space space

$$
P^{3g-2}_D := P\text{Ext}^1(\mathcal{O}_C(D), \mathcal{O}_C(-D)) = |K + 2D|^*.
$$

A point $e \in P^{3g-2}_D$ corresponds to an isomorphism class of extensions

$$
0 \to \mathcal{O}_C(-D) \to E_e \to \mathcal{O}_C(D) \to 0. \quad (e)
$$

There is a natural forgetful map $\varphi_D$ that is defined in the following way:

$$
\varphi_D : P^{3g-2}_D \rightarrow SU_C(2),
$$

$$
e \mapsto E_e.
$$

This kind of classifying maps have been described by Bertram in [Ber92]. In our case Theorem 2 of Bertram’s paper gives an isomorphism

$$
H^0(SU_C(2), \mathcal{L}) \cong H^0(P^{3g-2}_D, I_C\mathcal{L})^{2}.
$$

where $I_C$ is the ideal sheaf of the degree $4g - 2$ curve $C \subset P^{3g-2}$ embedded by $|K + 2D|^*$. This means that the classifying map is given by the full linear system of forms degree $g$ that vanish with multiplicity $(g - 1)$ on $C$. Let $\text{Sec}^n(C)$ be the variety of $n$-secant $(n-1)$-planes, it is easy to see that the base locus of the rational map $\varphi_D$ contains the base locus of the linear system $|I_{\text{Sec}^{g-1}(C)}(g)|$.

Throughout the paper, we will make massive use of the following remark (see for instance [LN83] Prop. 1.1).

**Remark 2.2.** Let $B$ be an effective divisor on $C$, $e \in P^{3g-2}_D$ and $E_e$ the corresponding rank 2 vector bundle. Then, if we denote $\overline{B}$ the linear span of $B$ in $P^{3g-2}_D$, $e \in \overline{B}$ if and only if $\mathcal{O}_C(D - B) \subset E_e$ as a sub-bundle.

The following Lemma is an easy consequence of Remark 2.2.

**Lemma 2.3.** Let $(e)$ be an extension class in $P^{3g-2}_D$, then the vector bundle $E_e$ is not semi-stable if and only if $e \in \text{Sec}^{g-1}(C)$ and it is not stable if $e \in \text{Sec}^{g}(C)$.

This also implies that the exceptional locus of the map $\varphi_D$ is exactly $\text{Sec}^{g-1}(C)$ (and then coincides with the base locus of $|I_{\text{Sec}^{g-1}(C)}(g)|$), since $SU_C(2)$ classifies only semi-stable bundles.

**Remark 2.4.** (See also [LN83]) In fact one can say something more precise. Let $L \oplus L^{-1}$ a general point (the locus where this description is not valid will be described later in Proposition 3.2) of $\text{Kum}(C)$ and let us suppose that $L = D(-x_1 \cdots - x_g)$ for some $x_1, \ldots, x_g \in C$. There exists only one effective divisor $x_1 + \cdots + x_g$ that accomplishes this. Moreover $L^{-1} = x_1 + \cdots + x_g - D$ and if we write $L^{-1}$ down as $D - y_1 \cdots - y_g$ we remark that there exits a unique effective divisor $y_1 + \cdots + y_g$ (linearly equivalent to $2D - x_1 - \cdots - x_g$) that realizes this equality as well. Then the fiber is the union of the two $P^{g-1} \subset P^{3g-2}$ spanned respectively by $\sum_{i=1}^{g} x_i$ and $\sum_{i=1}^{g} y_i$. Moreover, denote $\text{Jac}(C)[2]$ the group of 2-torsion points of the Jacobian, if $L \in \text{Jac}(C)[2]$ then $L \cong L^{-1}$ and the two effective divisors just described coincide, hence in this case the fiber is a double $P^{g-1}$.
Corollary 2.5. The image of the secant variety \( \text{Sec}^g(C) \) via the classifying map \( \varphi_D \) is the Kummer variety \( \text{Kum}(C) \).

In the next section we will describe the fibers of dimension bigger than expected out of the Kummer variety.

3. A Dual Classifying Map and the Exceptional Fibers

We recall that \( \dim(\text{SU}_C(2)) = 3g - 3 \), hence the generic fiber of \( \varphi_D \) has dimension one, accordingly to the fact that, for a general stable bundle \( E \), we have that \( \mathbb{P}(H^0(C, E \otimes \mathcal{O}_C(D))) \) has dimension one. However this is not true for all stable bundles. In order to check this we introduce the "Serre dual" divisor \( B \overset{\text{def}}{=} K - D \).

Remark that \( \text{deg}(B) = g - 2 \). Then by Riemann-Roch and Serre duality we have that

\[
h^0(C, E \otimes \mathcal{O}_C(D)) = h^0(C, E \otimes \mathcal{O}_C(B)) + 6 - 4,
\]

which implies that \( h^0(C, E \otimes \mathcal{O}_C(D)) > 2 \) if and only if there exists a map \( \mathcal{O}_C(-B) \rightarrow E \). In turn this means that \( E \) is in the image of the map \( \varphi_B : \mathbb{P} \text{Ext}^1(\mathcal{O}_C(B), \mathcal{O}_C(-B)) \rightarrow \text{SU}_C(2) \) that classifies the extensions of the following type:

\[
0 \rightarrow \mathcal{O}_C(-B) \rightarrow E \rightarrow \mathcal{O}_C(B) \rightarrow 0.
\]

Remark that also in this case we have \( \mathbb{P}^{3g-6}_B \overset{\text{def}}{=} \mathbb{P} \text{Ext}^1(\mathcal{O}_C(B), \mathcal{O}_C(-B)) = |K + 2B|^* \). Obviously there is a copy of the curve \( C \subset \mathbb{P}^{3g-6}_B \) of degree \( 4g - 6 \) and of course even in this case the locus of non-semistable extensions is given by \( \text{Sec}^{g-3}(C) \).

By a conjecture of Oxbury and Pauly ([OP99], Conj. 10.3), subsequently proved by Pareschi and Popa ([PP03], Thm. 4.1), the map \( \varphi_B \) is given by the complete linear system \( |T_C^{g-3}(g - 2)| \) on \( \mathbb{P}^{3g-6}_B \). This linear system, by the Proof of the same conjecture has dimension \( \sum_{i=0}^{g-2} \binom{g}{i} - 1 \) and it is a linear subspace of \( |2\Theta| \).

Moreover he image of the open semistable locus of \( \mathbb{P}^{3g-6}_B \) is non degenerate and by definition contained in \( \text{SU}_C(2) \). Hence we have proved the following Proposition.

Proposition 3.1. Let \( E \in \text{SU}_C(2) \) a stable bundle, then \( \dim(\varphi^{-1}(E)) = 2 \) if and only if \( E \) is contained in the \((3g - 6)\)-dimensional image of \( \varphi_B \).

We described in Remark 2.4 what are the fibers over the general points of \( \text{Kum}(C) \). It turns out that \( \varphi_B \) influences the behaviour of the fibers over \( \text{Kum}(C) \) as well and over the points of \( \text{Kum}(C) \cap \varphi_B(\mathbb{P}_B) \) they have dimension bigger than over the generic decomposable bundle. Let us in fact consider the surjective Abel-Jacobi map:

\[
\begin{align*}
\sigma_g : \text{Sym}^g C &\rightarrow \text{Jac}(C), \\
p_1 + \cdots + p_g &\mapsto \mathcal{O}_C(D - p_1 - \cdots - p_g).
\end{align*}
\]

It is generically one to one and its fibers have positive dimension exactly over the \((g - 2)\)-dimensional subvariety of the Jacobian

\[
\text{Sym}^g_B C \overset{\text{def}}{=} \{ L \in \text{Jac}(C) | L \equiv \mathcal{O}_C(-B + q_1 + \cdots + q_{g-2}) \}, \quad q_i \in C \}
\]
Remark in fact that if \( D - \sum_{i=1}^{g} q_i \equiv \sum_{i=1}^{g-2} q_i - B \), then \( \sum_{i=1}^{g} p_i \equiv K - \sum_{i=1}^{g-2} q_i \) and by the geometric Riemann-Roch we get that \( \dim[K - \sum_{i=1}^{g-2} q_i] = 1 \).

**Proposition 3.2.** Let \( E \in \varphi_B(\mathbb{P}_B^{g-6}) \) be a semistable not stable bundle, then \( \varphi_D^{-1}(E) \) is a pencil of \( \mathbb{P}^{g-1} \).

*Proof.* In this proof \( Q \) will denote any effective divisor \( q_1 + \cdots + q_{g-2} \in \text{Sym}^{g-2} C \).

Let us consider now the fibers of \( \varphi_D \) over the decomposable bundles \( \mathcal{O}_{\mathcal{C}}(-B+Q) \oplus \mathcal{O}_{\mathcal{C}}(B-Q) \), letting \( Q \) vary in \( \text{Sym}^{g-2} C \). These bundles are exactly the points of the image of the variety \( \text{Sym}^{g-2} C \subset \text{Jac}(C) \) in \( \text{Kum}(C) \) via the usual quotient \( \pm I_d \). Remark moreover that these bundles are contained in \( \varphi_B(\mathbb{P}_B^{3g-6}) \) since \( \mathcal{O}_{\mathcal{C}}(B) \otimes \mathcal{O}_{\mathcal{C}}(Q-B) = \mathcal{O}_{\mathcal{C}}(Q) \) is effective. Via the Lange-Narashiman ([LN83], Prop. 1.1) construction explained in Remark 2.2 and recalling the dimension of the fibers of the map 3.1 one sees that the sub-bundle \( \mathcal{O}_{\mathcal{C}}(Q-B) \) forces the fiber over \( \mathcal{O}_{\mathcal{C}}(-B+Q) \oplus \mathcal{O}_{\mathcal{C}}(B-Q) \) to contain a one dimensional family of \( \mathbb{P}^{g-1} \). In fact the degree \( g \) divisor \( S \) s.t. \( D(-S) \equiv \mathcal{O}_{\mathcal{C}}(Q-B) \) is equivalent to \( K - Q \), for which we have \( h^0(C, K-Q) = 2 \). This gives the \( \mathbb{P}^1 \) of effective degree \( g \) divisors in \( \sigma_g^{-1}(Q-B) \), each of them spanning a \( \mathbb{P}^{g-1} \subset |K + 2D| \) contained in the fiber of \( \mathcal{O}_{\mathcal{C}}(Q-B) \oplus \mathcal{O}_{\mathcal{C}}(B-Q) \). This \( \mathbb{P}^1 \) of effective divisors is easily seen in the following way. By projecting off the linear envelope of the divisor \( 2D \) on the degree 2g curve in \( \mathbb{P}_B^{3g-2} \) one gets a \( \mathbb{P}^{g-1} \) which is \( |K| \) and the image of the curve is clearly its canonical model. Now the linear system \( |K-Q| \) is the image (a line) of the degree \( g \) projection of the curve off the \((g-3)\)-dimensional linear subspace spanned by the points of \( Q \). The image is a \( \mathbb{P}^1 \) and the effective divisors are then easily seen to be the fibers of these projections. \( \spadesuit \)

4. A Projection in \( \text{SU}_C(2) \) and its Determinantal Interpretation

As we have seen in the preceding section, the linear span of \( \varphi(\mathbb{P}_B^{3g-6}) \) is a linear subspace of \( |2\Theta| \) of dimension \( \sum_{i=0}^{g-2} \binom{g}{i} - 1 \), from now on it will be denoted \( \mathbb{P}_e \).

It is not difficult to see that the complementary linear subspace (by this we mean the bigger linear subspace of \( |2\Theta| \) with no intersection with it) has dimension \( g \). For now we will denote it \( \mathbb{P}^{g} \). Before stating the next proposition we remark that if \( E \in \text{SU}_C(2) \) is stable then Riemann-Roch gives \( h^0(C, E(D)) = 2 \) and we denote \( s_1 \) and \( s_2 \) a basis of \( H^0(C, E(D)) \).

**Theorem 4.1.** The linear subspace \( \mathbb{P}^g \) can be identified with the linear system \( |2D| \) on the curve \( C \). The projection \( \pi_{ee} \) with center \( \mathbb{P}_e \) restricted to \( \text{SU}_C(2) - \text{Kum}(C) \) coincides with the following determinantal map

\[
\begin{align*}
(4.1) \quad \text{det} : \text{SU}_C(2) - (\text{Kum}(C) \cup \mathbb{P}_e) & \longrightarrow |2D|, \\
(4.2) \quad E & \mapsto \text{Zeros}(s_1 \wedge s_2). 
\end{align*}
\]

*Proof.* Recall how the theta map is defined: we have

\[
\begin{align*}
(4.3) \quad \theta : \text{SU}_C(2) & \longrightarrow |2\Theta| \\
(4.4) \quad E & \mapsto \theta(E) := \{ L \in \text{Pic}^{g-1}(C) : h^0(C, E \otimes L) \neq 0 \}.
\end{align*}
\]
Now, $\overline{\mathbb{P}}^g$ is the target space of the projection off $\mathbb{P}_e$ which is the linear span of $\varphi(\mathbb{P}^g_B)$, so one can interpret the projection in the following way. Let $\Xi$ be the locus in Pic$_g^{-1}(C)$ given by the line bundles $F$ s.t. $h^0(C, E \otimes F) \neq 0$ for every $E$ belonging to $Im(\varphi_B)$. We can then write

$$p_{\pi_e} : SU_C(2) \cap (Kum(C) \cup \mathbb{P}_e) \to \overline{\mathbb{P}}^g, \quad E \mapsto \Delta(E) := \{ L \in \Xi | h^0(C, E \otimes L) \neq 0 \}.$$ 

If $E \in Im(\varphi_B)$ then there exists one exact sequence

$$(4.5) \quad 0 \to O_C(-B) \to E \to O_C(B) \to 0 \quad (h).$$

Now suppose that we twist the sequence 4.5 by $F \in Pic^{-1}(C)$ and we take cohomology. We have

$$0 \to H^0(C, F \otimes O_C(-B)) \to H^0(C, F \otimes E) \to H^0(C, F \otimes O_C(B)) \to \cdots$$

One easily sees that $F \in \Xi$ if and only if $h^0(C, F \otimes O_C(-B)) \neq 0$. In fact this space injects in $H^0(C, F \otimes E)$, and whenever one has a morphism $F^{-1} \to E$ this factors through $O_C(-B)$. I claim that $h^0(C, F \otimes O_C(-B)) \neq 0$ if and only if

$$(4.6) \quad F = O_C(B + p), \text{ for some } p \in C.$$ 

It is obvious that $h^0(C, O_C(B + p) \otimes O_C(-B)) \neq 0$ and in fact we can see that there are no other line bundles $F$ s.t. $h^0(C, F \otimes O_C(-B)) \neq 0$ in the following way. We denote $C$ the curve contained in $Kum(C) \subset SU_C(2)$ which is image of the small diagonal of $Sym_B^g C$. The points of $C$ correspond to decomposable vector bundles of the form $O_C(B - (g - 2)t) \oplus O_C((g - 2)t - B)$ for some $t \in C$ and they are all contained in $Im(\varphi_B)$. Take 3 (only in genus 3 we need 4, but the proof is the same) different decomposable bundles $E_1, E_2, E_3 \in C; I$ claim that the only line bundles $F \in Pic^{-1}(C)$ that give $h^0(C, F \otimes E_i) \neq 0$ for every $i = 1, 2, 3$ are those defined by equation 4.6. The $E_i$ are all of the form $O_C(B - (g - 2)p_i) \oplus O_C((g - 2)p_i - B)$ for some $p_i \in C, i = 1, 2, 3$. Then we see that $h^0(O_C((g - 2)p_i - B) \otimes F) \neq 0$ iff $F = O_C(B + q)$ for some $q \in C$. This implies that if there are other line bundles $F$, s.t. $h^0(C, F \otimes E_i) \neq 0$, not of type $O_C(B + q)$ then they must realize the condition

$$(4.7) \quad h^0(O_C(B - (g - 2)p_i) \otimes F) \neq 0, \text{ i.e. } F = O_C(-B + (g - 2)p_i + s_1 + \cdots + s_{g-1})$$

for some $s_j \in C$. We see that one can find a bundle $F$ that realizes condition 4.7 for up to two different decomposable bundles $E_i, E_j$ (three in genus 3) in $\varphi_B(C)$, for instance $F = O_C(-B + (g - 2)p_i + (g - 2)p_j + s_1)$. However there’s no such bundle already for three different decomposable bundles (four in genus 3) since the points $p_i$ are different by hypothesis. So the only $F \in Pic^{-1}(C)$ that realize $h^0(C, F \otimes O_C(-B)) \neq 0$ are those contained in the curve $\tilde{C} \cong C \subset Pic^{-1}(C)$ made up of the line bundles $O_C(B + p)$ for $p \in C$, i.e. $\Xi = \tilde{C}$.

This means that the image of a stable bundle $E$ not contained in $\mathbb{P}_e$ via the projection off $\mathbb{P}_e$ is the divisor

$$\Delta(E) := \{ p \in C | h^0(E \otimes O_C(B + p) \neq 0) \}.$$
Remark the analogies with the less complicated but similar map in [Bol07]. Then we can identify $\mathbb{P}^g$ with a linear series on $C$. We use the fact that $|\Delta(E)| = |2\Theta|_C|$. Now, $\Theta|_C$ cuts (via RR and SD) out on $C$ the points $p$ s.t. $h^0(B + p) = h^0(K - B - p)) = h^0(D - p) \neq 0$ and $h^0(D - p) \neq 0$ iff $p \in D$. Thus $|2\Theta|_C| = |2D|$ and $\mathbb{P}^g$ can be seen as $|2D|$.

From now on we will denote $\mathbb{P}^g$ as $\mathbb{P}^g_D$. Now showing that the projection off $\mathbb{P}^g$ coincides with the determinant map goes along the lines of [Bol07] (Lemma 1.2.3) that we repeat for convenience of the reader. Let $p \in C$, if $p \in \text{Zeroes}(s_1 \wedge s_2)$ then there exists $s_p \in H^0(C, E \otimes \mathcal{O}_C(D - p))$ and thus $h^0(C, E \otimes \mathcal{O}_C(D - p)) \neq 0$. Now via Riemann-Roch and Serre-Duality (recalling that $E \cong E^*$) one gets $h^0(C, E \otimes \mathcal{O}_C(D - p)) = h^0(C, E \otimes \mathcal{O}_C(B + p))$. This implies that, when $s_1, s_2 \in H^0(C, E(D))$ the divisor of zeroes of $s_1 \wedge s_2$ is $\Delta(E)$ and the two maps coincide.

Remark that we will often abuse notation by denoting $N$ both the point of $|2D|$ and the set of points of the divisor itself.

Let us consider now the linear subspace $< N > \subset \mathbb{P}^{3g-2}_D$ generated by the $2g$ points of a divisor $N \in |2D|$. We remark that the annihilator of $< N >$ is $H^0(C, K + 2D - N)$, that has dimension equal to $g$. This means that the linear span $< N > \subset \mathbb{P}^{3g-2}_D$ is a $\mathbb{P}^{2g-2}_N$, and we shall denote it as $\mathbb{P}^{2g-2}_N$.

**Lemma 4.2.** [LN83] Let $F$ be a divisor of $|2D|$ and $e \in \mathbb{P}^{2g-2}_N$ an extension

$$0 \longrightarrow \mathcal{O}_C(-D) \xrightarrow{i_e} E_e \xrightarrow{\pi_e} \mathcal{O}_C(D) \longrightarrow 0.$$ 

Then $e \in \mathbb{P}^{2g-2}_N$ if and only if there exists a section $\beta \in H^0(C, Hom(\mathcal{O}_C(-D), E))$ s.t. $\text{Zeroes}(\pi_e \circ \beta) = F$.

In the next lemma we go through the relation between the fibers of the projection $p_{e_D}$ and the classifying map $\varphi_D$.

**Proposition 4.3.** Let $N \in |2D|$ and $\mathbb{P}^{2g-2}_N \subset \mathbb{P}^{3g-2}_D$ the corresponding linear envelop. Then the image of

$$\varphi_D|\mathbb{P}^{2g-2}_N : \mathbb{P}^{2g-2}_N \longrightarrow \mathcal{SU}_C(2)$$

is the closure of the fiber over $N \in \mathbb{P}^g_D$ of the projection $p_{e_D}$.

**Proof.** First remark that $C \cap \mathbb{P}^{2g-2}_N = N$, since $h^0(C, K + 2D - N - c) = h^0(C, K - c) = g - 1$ for every $c \in C$. This by the way implies that the restriction $\varphi_D|\mathbb{P}^{2g-2}_N : \mathbb{P}^{2g-2}_N \longrightarrow \mathcal{SU}_C(2)$ is given by (possibly a linear subsystem of) the full linear system of forms of degree $g$ on $\mathbb{P}^{2g-2}_N \longrightarrow \mathcal{SU}_C(2)$ vanishing with multiplicity $g - 1$ at the points of $N$. Let $e$ be an extension class belonging to $\mathbb{P}^{2g-2}_N$ and $E$ its image in $\mathcal{SU}_C(2)$ via $\varphi_D$. Then, by Lemma 4.2, the extension class $e$ belongs to $\mathbb{P}^{2g-2}_N$ if and only if there exists a section $\alpha \in H^0(C, Hom(\mathcal{O}_C(-D), E))$ s.t., in the notation of Prop. 4.2, we have that $\text{Zeroes}(\pi_e \circ \alpha) = N$. This in turn implies that $\alpha$ and $i_e$ are 2 independent sections of $E \otimes \mathcal{O}_C(D)$ and that $\text{Zeroes}(i_e \wedge \alpha) = N$. Hence
Theorem 4.1 implies that $E$ is projected on $N \in \mathbb{P}^g_D$. More precisely the image of the restricted classifying map is the intersection of $SU_C(2)$ and the linear span of $\mathbb{P}_c$ and the point of $\mathbb{P}_D^g$ corresponding to $N$. ♠

5. GENERIC FIBERS, RATIONAL NORMAL CURVES AND POINTED GENUS 0 CURVES

Now let us consider the fiber of $\varphi_D$ over a general bundle. General, for what matters to us, will mean belonging neither to $Kum(C)$ nor to $\varphi_B(\mathbb{P}^{3g-6})$. Let us denote $C_E$ the closure of the fiber $\varphi_D^{-1}(E)$, and $\operatorname{Sec}^N := \operatorname{Sec}^{g-1}(C) \cap \mathbb{P}_N^{2g-2}$ for some $N \in |2D|$. Moreover we shall denote $\operatorname{Sec}^a(N)$ the configuration of $(n-1)$-linear spaces spanned in $\mathbb{P}_N^{2g-2}$ by $n$-ples of points of $N$. One would expect that $\operatorname{Sec}^N = \operatorname{Sec}^a(N)$ but this is not always true and, as the next Lemma shows, it depends on the gonality of the curve $C$.

**Lemma 5.1.** The restriction map

$$\varphi_D|_{\mathbb{P}^{2g-2}_N : \mathbb{P}^{2g-2}_N \longrightarrow SU_C(2)}$$

is given by the linear system $|T^{2g-1}_{\operatorname{Sec}^1}(g)|$. If $g < 4$ then $\operatorname{Sec}^N = \operatorname{Sec}^{g-1}(N)$, otherwise $\operatorname{Sec}^{g-1}(N) \subset \operatorname{Sec}^N$, i.e. there is some additional base locus.

**Proof.** By Remark 2.2 and Lemma 2.3, the first part is clear. It is also clear that $\operatorname{Sec}^{g-1}(N) \subset \operatorname{Sec}^N$. For the second, recall that $\mathbb{P}^{2g-2}_N \subset \mathbb{P}^{3g-2}_D$ is the annihilator of $|K + 2D - N| = |K|$; then there are $g$ sections of $H^0(C, K + 2D)$ that vanish on $\mathbb{P}^{2g-2}_N$. On the other hand, given a general effective degree $(g - 1)$ divisor $L_{g-1}$ via RR one sees that there are $2g$ sections of $H^0(C, K + 2D)$ vanishing on its points. Hence, since $h^0(C, K + 2D) = 3g - 1$ we see that $\mathbb{P}^{2g-2}_N$ has non-empty intersection with the span of $L_{g-1}$ if and only if $\dim(H^0(C, K) \cap H^0(C, K + 2D - L_{g-1})) \geq 2$ (remark that this is means exactly the condition we want to check, i.e. $\operatorname{Sec}^{g-1}(N) \subset \operatorname{Sec}^N$). This in turn means that in $\mathbb{P}^{g-1} = |K|^*$ there exists a linear subspace of codimension at least 2 that contains the points of $L_{g-1}$. Then the geometric form of RR gives

$$\dim(|L_{g-1}|) \geq g - 1 - 1 - (g - 3) = 1.$$

That is, we have $N \subseteq \operatorname{Sec}^N$ as long as the curve $C$ is $(g - 1)$-gonal. Finally, by the Existence Theorem of Brill-Noether theory (Thm. 1.1, page 206, [ACGH85]) we see that this is the case if $g \geq 4$. Furthermore the dimension of the variety $G_{g-1}$ of $g_{g-1}$ on $C$ is $g - 2r - 2$. ♠

In the following we will call **degenerate rational normal curve** a rational curve in $\mathbb{P}^n$ of degree smaller than $n$ that is a rational normal curve in its own linear span. For example a smooth conic in $\mathbb{P}^2 \subset \mathbb{P}^n$ or a twisted cubic in $\mathbb{P}^3 \subset \mathbb{P}^n$.

**Proposition 5.2.** Let $E$ be a general vector bundle in $SU_C(2) - (Kum(C) \cup \varphi_B(\mathbb{P}^{3g-6}_B))$, then $C_E \subset \mathbb{P}^{2g-2}_{\Delta(E)}$ is a rational normal curve of degree $2g - 2$ passing through the 2g points of $\Delta(E) \subset \mathbb{P}^{2g-2}_{\Delta(E)}$.

**Proof.** First, recall that $SU_C(2) - (Kum(C) \cup \varphi_B(\mathbb{P}^{3g-6}_B))$ is the locus where $C_E$ has dimension one. Now remark that the same argument (in the opposite sense) of the proof of Proposition 4.3 implies that the curve $C_E$ is contained in $\mathbb{P}^{2g-2}_{\Delta(E)}$. Moreover, we have the obvious numerical equality
Remark 5.4. When $g(C) = 2$ and $D = K$ Proposition 5.2 coincides with the description given in [Bol07] where the closure of the fiber of the classifying map over a stable bundle $E$ is a plane conic passing by the four points of $\Delta(E) \in [2K]$. 

(5.1) \[(2g - 2)g = (g - 1)2g\]
that has some important implications. First, by Lemma 5.1 if $\varphi_D|_{\Delta(E)}$ has a base locus that contains strictly $\text{Sec}^{g-1}(\Delta(E))$ then the base locus is a family of rational normal curves (RNCs in the following) passing by the $2g$ points of $\Delta(E)$. In fact by Bezout Theorem and equation 5.1 if all forms of degree $g$ that define $\varphi_D|_{\Delta(E)}$ vanish on a point $p$ not belonging to $\Delta(E)$, they are forced to vanish on the unique RNC by $\Delta(E)$ and $p$. On the other hand the equality 5.1 implies that the classifying map $\varphi_D$ is constant along the RNC passing by the $2g$ points of $\Delta(E)$ that are not contained in its base locus. In fact by Lemma 5.1 the restriction $\varphi_D|_{\Delta(E)}$ is given by a linear subsystem of $|I^{g-1}(\Delta(E))|$ (if $g = 2, 3$ they coincide) thus the zero locuses of the forms of this linear system can not have further intersection with our RNCs than $\Delta(E)$. This means that $C_E$ is a finite collection of RNCs passing by $\Delta(E)$ and $\Delta(E)$ is the only intersection of $C_E$ with the base locus of $\varphi_D|_{\Delta(E)}$. On the other hand each point $e$ contained in the fiber over $E$ represents an exact sequence like the following

(5.2) \[
0 \to \mathcal{O}_C(-D) \to E \to \mathcal{O}_C(D) \to 0,
\]
then we can define a map

$$h : \varphi^{-1}(E) \to \mathbb{P}^0(C, E(D)) = \mathbb{P}^1$$

that sends the extension class $e \in \varphi^{-1}(E)$ on the point $h(e) \in \mathbb{P}^0(C, E(D))$ corresponding to the first morphism of the exact sequence 5.2. The map $h$ is birational and this implies that $C_E$ must be just one RNC because the arithmetic genus has to be 0. This concludes the proof.

\[\blacklozenge\]

We leave to the reader to check that there is a closed codimension one locus $\Omega$ in $\mathcal{SU}_C(2) - (\text{Kum}(C) \cup \varphi_B(\mathbb{P}^{2g-6}_B))$ s.t. the fibers of the bundles $E \in \Omega$ are degenerate rational curves contained in a $\mathbb{P}^n \subseteq \mathbb{P}^{2g-2}_\Delta$ spanned by $n + 1$ points of $\Delta(E)$ and passing by these $n + 1$ points plus the point given by the intersection of $\mathbb{P}^n$ and the $\mathbb{P}^{2g-n-2}$ spanned by the remaining $2g - n - 1$ points of $\Delta(E)$.

Now the space of rational normal curves in $\mathbb{P}^n$ passing by $n - 2$ points in general position is strictly related to the moduli space $\mathcal{M}_{0,n}$ of configurations of ordered distinct $n$ points on $\mathbb{P}^1$. This is quite well known and stated in the following theorem.

**Theorem 5.3.** ([Kap93], Thm. 0.1) Take $n$ points $q_1, \ldots, q_n$ of the projective space $\mathbb{P}^{n-2}$ which are in general position. Let $V_0(q_1, \ldots, q_n)$ be the space of all rational normal curves in $\mathbb{P}^{n-2}$ through the $q_i$. Consider it as a subvariety of the Hilbert scheme $\mathcal{H}$ parametrizing all subschemes of $\mathbb{P}^{n-2}$, then we have $V_0(q_1, \ldots, q_n) \cong \mathcal{M}_{0,n}$.

**Remark 5.4.** When $g(C) = 2$ and $D = K$ Proposition 5.2 coincides with the description given in [Bol07] where the closure of the fiber of the classifying map over a stable bundle $E$ is a plane conic passing by the four points of $\Delta(E) \in [2K]$. 

(5.3) \[
\text{Theorem 5.3.} \quad (\text{Kum}(C) \cup \varphi_B(\mathbb{P}^{2g-6}_B))
\]
In this case the map \( p_{\mathcal{P}_c} \) is the projection on \( |2K| = \mathbb{P}^2 \) with center the node \([O_C \oplus O_C] \) of the Kummer surface \( \text{Kum}(C) \subset |2\Theta| \) and the fiber of \( p_{\mathcal{P}_c} \) over a divisor \( \Delta(E) \in |2K| \) is a \( \mathbb{P}^1 \), that corresponds to the pencil of conics in \( \mathbb{P}^2_{\Delta(E)} \) passing by the four points. This can be seen as the base example of Theorem 5.3. Plane conics passing by 4 fixed points are in fact in bijection with configurations of four points on the projective line and \( \mathbb{P}^1 \) is in fact the GIT compactification of \( \mathcal{M}_{0,4} \). The semistable configurations correspond to the rank 2 reducible conics and to the three points of intersection of the projective line with \( \text{Kum}(C) \).

Proposition 5.2 then gives us an interesting interpretation of the fibers of the projection \( p_{\mathcal{P}_c} \). We have just showed in fact that, if \( N \in |2D| \) the restriction

\[
\varphi_{D|\mathbb{P}^{2g-2}_N} : \mathbb{P}^{2g-2}_N \to SU_C(2)
\]

contracts every RNC passing through the \( 2g \) points of \( N \), when the RNC is not contained in the base locus. In particular, if \( g = 2, 3 \) then \( \varphi_{D|\mathbb{P}^{2g-2}_N} \) contracts every RNC passing through \( N \).

6. THE GENUS 3 CASE: A FIBRATION IN SEGRE CUBICS.

Let us now go through the details of the genus 3 case, notably we will assume that \( C \) is not hyperelliptic. As already stated, in this case \( SU_C(2) \) is embedded in \( \mathbb{P}^7 = |2\Theta| \) as a quartic hypersurface singular along the \( \text{Kum}(C) \), first discovered by Coble [Cob82]. Moreover it is invariant w.r.t. the action of \( H_3(2) \), the level 2 and genus 3 Heisenberg group, on \( \mathbb{P}^7 \).

In this case \( \text{deg}(D) = 3 \) and \( \varphi_B \) is a linear embedding of \( \mathbb{P}^3_B \) in \( \mathbb{P}^7 \). The image of the projection off \( \mathbb{P}^3_c = \mathbb{P}^3_B \) is a \( \mathbb{P}^3 \) as well, that is identified with \( |2D| \) by Theorem 4.1. On the other hand the extension classes belonging to \( \text{Ext}^1(O_C(D), O_C(-D)) \) are parametrized by a \( \mathbb{P}^7_D \) that contains a model of \( C \) and the classifying map \( \varphi_D \) is given by the complete linear system \( |I^2_{\text{Sec}^2(C)}(3)| \). This linear system can be identified with \( |\mathcal{I}_{\text{Sec}^2(C)}(3)| \).

**Remark 6.1.** The choice of a projective model of \( C \) in this case allows to do explicit calculations on this map, since they are still fairly simple and can be performed in a reasonable time by a computer. By computing the image of this map with Macaulay, we found some equations of Coble quartics in terms of the coefficients of a plane quartic model of \( C \). The same results, with methods coming from the context of integrable systems, were obtained by P. Vanhaecke in 2005 [Van05].

Let us now take as usual a divisor \( Q \in |2D| \) and consider the closure \( S_Q \) of its fiber \( p_{\mathcal{P}_c}^{-1}(Q) \). By Proposition 4.3 \( S_Q \) is the image of the \( \mathbb{P}^5_Q \) spanned by the six points of \( Q \).

**Proposition 6.2.** Let \( Q \in |2D| \), then \( S_Q \) is a Segre cubic.

The Segre cubic \( S_3 \) is a classical modular threefold variety (see for instance [DO88]). In \( \mathbb{P}^5 \) with homogeneous coordinates \( [x_0 : \cdots : x_5] \) we consider the complete intersection

\[
S_3 := \left\{ \sum_{i=0}^5 x_i = 0; \sum_{i=0}^5 x_i^3 = 0 \right\}.
\]
The first equation is linear, so $S_3$ is a hypersurface in the $\mathbb{P}^4 \overset{\text{def}}{=} \{ x \in \mathbb{P}^5 | \sum x_i = 0 \}$. Using $[x_0: \cdots : x_5]$ as projective coordinates, the relation $x_5 = -x_0 \cdots - x_4$ of course gives the equation of $S_3$ as a hypersurface but the equation in $\mathbb{P}^5$ has the advantage of showing that $S_3$ is invariant under the symmetry group $\Sigma_5$, acting on $\mathbb{P}^5$ by permuting coordinates, which is not so immediate from the hypersurface equation. It is also well known that $S_3$ is the GIT compactification of the moduli space $\mathcal{M}_{0,6}$ of ordered configurations of 6 points on $\mathbb{P}^1$. By considering these points as Weierstrass points, we can also see $S_3$ as a birational model of the Satake compactification of $\mathcal{A}_2(2)$, the moduli space of principally polarized abelian surfaces with a level 2 structure. Indeed $S_3$ is the dual variety of the Igusa quartic (for an account see [DO88] or [Hun96])

\textbf{Proof. (of Proposition 6.2)} Let $W \overset{\text{def}}{=} \bigcup_{i=1}^{15} L_i$ be the system of $\binom{6}{2} = 15$ lines each passing by two points of $Q$. Then the restriction of $\varphi_D$ to $\mathbb{P}_Q^4$ is the full linear system $|Z_W(3)|$. Moreover, by the general description of the fibers of the semistable boundary given in Remark 2.4, there are couples of $\mathbb{P}^2$ that are contracted to points of $\text{Sing}(SU_C(2)) = Kum(C)$. More precisely we have $10 = \binom{6}{3}/2$ couples of $\mathbb{P}^2$ that are contracted, each couple on a point of the intersection $S_Q \cap Kum(C)$, which is then 10 points. Remark now that the intersection of $SU_C(2)$ with the $\mathbb{P}^4$ spanned by $\mathbb{P}_B^3 = \mathbb{P}_r$ and the point in $\mathbb{P}_D^3$ corresponding to $Q$ is the union $S_Q \cup \mathbb{P}_B^3 \subset \mathbb{P}^4$. Since $\text{deg}(SU_C(2)) = 4$ this implies that $\text{deg}(S_Q) = 3$. Now it is known that for any degree $d$ there is an upper bound on the number of ordinary double points which a hypersurface of degree $d$ can have, the so-called Varchenko bound. For cubic threefolds this number is ten, and it has been known since the last century that the Segre cubic is the unique (up to isomorphism) cubic with ten nodes. Thus we realize immediately that $S_Q$ can not have other double points (for instance due to non transversal intersections of the $\mathbb{P}^4$ with $SU_C(2)$) and that it is a Segre cubic in $\mathbb{P}^4$.

There is also an interesting locus in the Segre cubic $S_3$, notably given by 15 2-planes contained in $S_3$ each containing 4 of the ten double points [Bak63]. In the case of the fiber $S_Q$ these 2-planes are the images of the fifteen 3-dimensional linear subspaces spanned by four of the six points of $Q$ in $\mathbb{P}_Q^6$. More precisely the fiber of $\varphi_D$ over each stable bundle in these 2-planes is a twisted cubic contained the 3-plane and passing by the 4 points and the intersection of the 3-plane with the line spanned by the remaining 2 points of $Q$.

Remark moreover that the rational map $\varphi_D$ is resolved to a morphism by first blowing up $\mathbb{P}^4$ in the 6 points of $Q$ and then along the proper transforms of the 15 lines. We leave to the reader the nice exercise to check that the images of 6 exceptional divisors over the points are the six ten-nodal Clebsch surfaces contained in $S_3$. On the other hand the images of the other 15 exceptional divisors are the 15 four-nodal Cayley surfaces contained in $S_3$. For an account of these surfaces see [Hun95].

It is interesting to notice that the existence of a relation between the figure of six points in general position in $\mathbb{P}^4$, rational normal quartics and the Segre cubic was already known to J.Bronowski [Bro43].
7. The fibration in $\mathcal{M}_{0,2g}$ for $g(C) \geq 4$.

7.1. The moduli spaces $\mathcal{M}_{0,n}$. Before going through the main results of this section we give a brief account of some well known results on the moduli spaces of pointed curves. Set-theoretically, $\mathcal{M}_{0,n}$ is the set of projective equivalence classes of $n$-tuples of distinct points on $\mathbb{P}^1$. Moreover, if $n \geq 3$ the moduli spaces $\mathcal{M}_{0,n}$ are fine and have a structure of quasi-projective algebraic variety. There exist different compactifications of these spaces. The most ancient historically is probably the GIT compactification $\mathcal{M}_{0,n}^{GIT}$ (the Segre cubic is $\mathcal{M}_{0,6}^{GIT}$), whose glorious construction dates back to [MFK82] and has known some more recent interest thanks to [HMSV09] where the equations for $\mathcal{M}_{0,n}^{GIT}$ are computed. Another well known compactification is the Mumford-Knudsen $\mathcal{M}_{0,n}$ obtained via stable marked curves [Knu83].

These two compactifications differ on the boundary while they both contain $\mathcal{M}_{0,n}$ as an open subset. Their relation with rational normal curves contained in $\mathbb{P}^{n-2}$ is very tight and a large amount of results on $\mathcal{M}_{0,n}$ can be obtained by studying the geometry of these curves and the birational transformations of $\mathbb{P}^{n-2}$: the first example of this interplay was Theorem 5.3.

The Mumford-Knudsen space $\mathcal{M}_{0,n+1}$ has a few different realizations as a blow-up of the projective space. We are particularly interested in the following.

**Theorem 7.1. ([Has03] Sect. 6.2)**

The Mumford-Knudsen compactification $\mathcal{M}_{0,n+1}$ has the following realization as a sequence of blow ups of $\mathbb{P}^{n-2}$, let $q_1, \ldots, q_n$ be general points in $\mathbb{P}^{n-2}$:

1: blow up the points $q_1, \ldots, q_n$;
2: blow up proper transforms of lines spanned by pairs of the points $q_1, \ldots, q_n$;
3: blow up proper transforms of 2-planes spanned by triples of the point; . . .
n-3: blow up proper transforms of $(n-4)$-planes spanned by $(n-3)$-tuples of the points.

The corresponding blow-down map $b : \mathcal{M}_{0,n+1} \rightarrow \mathbb{P}^{n-2}$ was first described by Kapranov [Kap93] and it has the following property. Let $\pi : \mathcal{M}_{0,n+1} \rightarrow \mathcal{M}_{0,n}$ be the forgetful morphism that drops the $(n+1)^{st}$ point, then the fibers of $\pi$ constitute the universal curve over $\mathcal{M}_{0,n}$. Now, by [KM96] Prop. 3.1., the images via $b$ of the fibers over points of $\mathcal{M}_{0,n} \subset \mathcal{M}_{0,n}$ are the rational normal curves in $\mathbb{P}^{n-2}$ passing by the $n$ general points.

7.2. Birational geometry of the fibers of $p_{\mathcal{P}_c}$. By lemma 5.1, if $g(C) \geq 4$ we are not in the ideal situation of the lower genera any more. If $Q \in \vert 2D \vert$, then the base locus of the restricted map $\varphi_{D|\mathcal{P}_c^{2g-2}}$ contains strictly the configuration of linear spaces $\text{Sec}^{g-1}(Q)$ and the further base locus is a family of rational normal curves.

**Theorem 7.2.** Let $N \in \vert 2D \vert$ and $g \geq 4$. Then the closure of every fiber $p_{\mathcal{P}_c}^{-1}(N)$ of the projection

$$p_{\mathcal{P}_c} : \mathcal{SU}_C(2) - \mathbb{P}_c \longrightarrow \mathbb{P}^g_D$$

is birational to the moduli space $\mathcal{M}_{0,2g}$. 
Proof. Let us denote $S_N$ the closure of $p^{-1}_N(N)$. By theorem 5.3 there is a bijection between the space $V_0(p_1, \ldots, p_n)$ of rational normal curves in $\mathbb{P}^{n-2}$ passing by $n$ points $p_i$ in general position and $\mathcal{M}_{0,n}$, which is the set of projective equivalence classes of ordered $n$-tuples of distinct points on $\mathbb{P}^1$. We fix again a divisor $N = q_1 + \cdots + q_2g \in [2D]$ and remark that each RNC, not contained in the base locus of $\varphi_D|_{\mathbb{P}^{2g-2}}$, belonging to $V_0(q_1, \ldots, q_{2g})$ is contracted by $\varphi_D$ to the $s$-equivalence class of a stable bundle in $p^{-1}_N(N)$. These stable bundles are an open set $U$ in $S_N$.

Now we blow up $\mathbb{P}^{2g-2}$ recursively, as described in Theorem 7.1, until we obtain $\mathcal{M}_{0,2g+1}$. Remember that this is accomplished by blowing up recursively the proper transforms of linear spans of points of $N$ until the codimension two ones. Remark that the map induced by pull back of $\varphi_D$ on the iterate blow-ups of $\mathbb{P}^{2g-2}$ becomes a morphism $\tilde{\varphi}_D$ (that resolves the rational map $\varphi_D$) starting from from the $(g-1)^{th}$ blow up. Let us denote $Bl\mathbb{P}^{2g-2}$ this intermediate blow-up. For ease of the reader we will factor the blow-down map

$$b: \mathcal{M}_{0,2g+1} \longrightarrow Bl\mathbb{P}^{2g-2} \xrightarrow{\xi} \mathbb{P}^{2g-2}$$

into two different blow-downs $\sigma$ and $\xi$. Moreover by [KM96] the fibers of the induced map $\tilde{\varphi}_D: \mathcal{M}_{0,2g+1} \rightarrow S_N$ give a sub-family, defined over the open set $U$, of the universal curve. This in turn, by the universal property of the fine moduli space $\mathcal{M}_{0,2g}$, induces an embedding of the open set $U \subset p^{-1}_N(N)$ in $\mathcal{M}_{0,2g+1}$ thus yielding a birational map between $S_N$ and $\mathcal{M}_{0,2g}$. The situation is then resumed in the following commutative diagram.

Now, the proof of the main Theorem (Theorem 1.1) of this paper is just the collection of Remark 5.4 (see also [Bol07]), Proposition 6.2 and Theorem 7.2.

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