Hilbert Spaces Contractively Contained in Weighted Bergman Spaces on the Unit Disk

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Abstract. Sub-Bergman Hilbert spaces are analogues of de Branges-Rovnyak spaces in the Bergman space setting. They are reproducing kernel Hilbert spaces contractively contained in the Bergman space of the unit disk. K. Zhu analyzed sub-Bergman Hilbert spaces associated with finite Blaschke products, and proved that they are norm equivalent to the Hardy space. Later S. Sultanic found a different proof of Zhu’s result, which works in weighted Bergman space settings as well. In this paper, we give a new approach to this problem and obtain a stronger result. Our method relies on the theory of reproducing kernel Hilbert spaces.

1. Introduction

Let \( D \) denote the unit disk. Let \( L^2 \) denote the Lebesgue space of square integrable functions on the unit circle \( T \). The Hardy space \( H^2 \) is the subspace of analytic functions on \( D \) whose Taylor coefficients are square summable. Then it can also be identified with the subspace of \( L^2 \) of functions whose negative Fourier coefficients vanish. The space of bounded analytic functions on the unit disk is denoted by \( H^\infty \). The Toeplitz operator on the Hardy space \( H^2 \) with symbol \( f \) in \( L^\infty (D) \) is defined by

\[
T_f(h) = P(fh),
\]

for \( h \in H^2(D) \). Here \( P \) be the orthogonal projections from \( L^2 \) to \( H^2 \).

Let \( A \) be a bounded operator on a Hilbert space \( H \). We define the range space \( \mathcal{M}(A) = AH \), and endow it with the inner product

\[
\langle Af, Ag \rangle_{\mathcal{M}(A)} = \langle f, g \rangle_H, \quad f, g \in H \ominus \text{Ker}A.
\]

\( \mathcal{M}(A) \) has a Hilbert space structure that makes \( A \) a coisometry on \( H \). Let \( b \) be a function in \( H^\infty_\circ \), the closed unit ball of \( H^\infty \). The de Branges-Rovnyak space \( \mathcal{H}(b) \) is defined to be the space

\[
(I - T_bT_b^*)^{1/2}H^2.
\]

2010 Mathematics Subject Classification. Primary 47B32.
We also define the space $H(\bar{b})$ in the same way as $H(b)$, but with the roles of $b$ and $\bar{b}$ interchanged, i.e.

$$H(\bar{b}) = (I - T_{\bar{b}}T_b)^{1/2}H^2.$$  

The spaces $H(b)$ and $H(\bar{b})$ are also called sub-Hardy Hilbert spaces (the terminology comes from the title of Sarason’s book [11]).

The space $H(b)$ was introduced by de Branges and Rovnyak [3]. Sarason and several others made essential contributions to the theory [11]. A recent two-volume monograph [4], [5] presents most of the main developments in this area.

In this paper, we study analogues of sub-Hardy Hilbert spaces in a general setting. The Bergman space $A^2$ is the space of analytic functions on $\mathbb{D}$ that are square-integrable with respect to the normalized Lebesgue area measure $dA$. For $u \in L^\infty(\mathbb{D})$, the Bergman Toeplitz operator $T_u$ with symbol $u$ is the operator on $L^2_\mathbb{D}$ defined by

$$T_u h = \tilde{P}(uh).$$  

Here $\tilde{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2$. In [13], Zhu introduced the sub-Bergman Hilbert spaces. They are defined by

$$A(b) = (I - T_{\bar{b}}T_b)^{1/2}A^2$$  

and

$$A(\bar{b}) = (I - T_bT_{\bar{b}})^{1/2}A^2.$$  

Here $b$ is a function in $H_1^\infty$. It is easy to see from the definition that the spaces $A(b)$ and $A(\bar{b})$ are contractively contained in $A^2$, i.e. $A(b) \subset A^2$ and the inclusion map has norm at most 1. But in most cases they are not closed subspaces of $A^2$ (see [13, Corollary 3.13]).

Sub-Bergman Hilbert spaces share some common properties with sub-Hardy Hilbert spaces as the way those spaces are defined follows from a general theory on Hilbert space contractions [11]. For instance, both $H(b)$ and $A(b)$ are invariant under the corresponding Toeplitz operators with a co-analytic symbol [11, II-7]. One significant difference between the spaces $A(b)$ and $H(b)$ is the multipliers. The theory of $H(b)$ spaces is pervaded by a fundamental dichotomy, whether $b$ is an extreme point of the unit ball of $H^\infty(\mathbb{D})$. The multiplier structure of de Branges-Rovnyak spaces has been studied extensively by Lotto and Sarason in both the extreme and the nonextreme cases [7], [8], [9]. However, Zhu [13] showed that every function in $H^\infty$ is a multiplier of $A(b)$ and $A(\bar{b})$. As a consequence, the two sub-Bergman Hilbert spaces $A(b)$ and $A(\bar{b})$ are norm equivalent.

We are interested in the structure of sub-Bergman Hilbert spaces. For de Branges-Rovnyak spaces $H(b)$, there are two special cases: if $\|b\|_\infty < 1$, then $H(b)$ is just a renormed version $H^2$; if $b$ is an inner function (i.e. $|b| = 1$, a.e. on $\mathbb{T}$), then $H(b)$ is a closed subspace of $H^2$, called the model space. In the Bergman space setting, if $\|b\|_\infty < 1$, then $A(b)$ is norm equivalent to $A^2$. But it is not known what is the space $A(b)$ when $b$ is a general inner function. In [14], Zhu describes the space $A(b)$ when $b$ is a finite Blaschke product.

**Theorem 1.1.** *If $b$ is a finite Blaschke product, then $A(b) = H^2$.***
In [12], Sultanic reproved Theorem 1.1 using a different technique, and extend the result to the weighted Bergman space setting. The main purpose of this paper is to give a new approach to this problem. We use the theory of reproducing kernel Hilbert spaces (presented in Section 2) and obtain stronger version of Theorem 1.1.

**Theorem 1.2.** For every non-constant function \( b \) in \( H_1^\infty \), \( \mathcal{A}(b) \) always contain \( H^2 \). Moreover, 
\[
\mathcal{A}(b) = H^2
\]
if and only if \( b \) is a finite Blaschke product.

In Section 3, we shall prove Theorem 1.2 in the weighted Bergman space setting.

2. Reproducing kernel Hilbert spaces

In this section, we present some basic theory of reproducing kernel Hilbert spaces. For more information about reproducing kernels and their associated Hilbert spaces, see [2], [10].

Let \( X \subset \mathbb{C}^d \). We say a function \( K : X \times X \to \mathbb{C} \) is a positive kernel on \( X \) (written as \( K \preceq 0 \)) if it is self-adjoint (\( K(x, y) = \overline{K(y, x)} \)), and for all finite sets \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset X \), the matrix \( (K(\lambda_i, \lambda_j))_{i,j=1}^n \) is positive semi-definite, i.e., for all complex numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \),
\[
\sum_{i,j=1}^n \alpha_i \overline{\alpha_j} K(\lambda_i, \lambda_j) \geq 0.
\]

A reproducing kernel Hilbert space \( \mathcal{H} \) on \( X \) is a Hilbert space of complex valued functions on \( X \) such that every point evaluation is a continuous linear functional. Thus for every \( w \in X \), there exists an element \( K_w \in \mathcal{H} \) such that for each \( f \in \mathcal{H} \),
\[
\langle f, K_w \rangle_\mathcal{H} = f(w).
\]
Since \( K_w(z) = \langle K_w, K_z \rangle_\mathcal{H} \), \( K \) can be regarded as a function on \( X \times X \) and we write \( K(z, w) = K_w(z) \). Such \( K \) is a positive kernel and the Hilbert space \( \mathcal{H} \) with reproducing kernel \( K \) is denoted by \( \mathcal{H}(K) \).

The following theorem, due to Moore, shows that there is a one-to-one correspondence between reproducing kernel Hilbert spaces and positive kernels (see e.g. [1, Theorem 2.23]).

**Theorem 2.1.** Let \( X \subset \mathbb{C}^d \) and let \( K : X \times X \to \mathbb{C} \) be a positive kernel. Then there exists a unique reproducing kernel Hilbert space \( \mathcal{H}(K) \) whose reproducing kernel is \( K \).

For two positive kernels \( K_1, K_2 \), we write
\[
K_1 \preceq K_2
\]
to mean that
\[
K_2 - K_1 \succeq 0.
\]
It is easy to check the sum of two positive kernels is still a positive kernel. The following result shows that the same holds for a product of two positive kernels, which generalizes the Schur product in matrix algebra (see e.g. [2]).
Proposition 2.1. Let \( K_1, K_2 \) be positive kernels on \( X \). Then
\[
K_1 \cdot K_2 \succeq 0.
\]
In particular, if
\[
K_1 \preceq K_2,
\]
then we can multiply another positive kernel \( K \) on both sides to get
\[
K_1 \cdot K \preceq K_2 \cdot K.
\]
The following proposition will be used in the proof of the main results.

Proposition 2.2. Let \( \mathcal{H}(K) \) be a reproducing kernel Hilbert space on \( X \) and let \( A \) be a bounded linear operator on \( \mathcal{H}(K) \).

1. If
\[
||A|| \leq C,
\]
then
\[
\langle AK_w, AK_z \rangle_{\mathcal{H}(K)} \leq C^2 \langle K_w, K_z \rangle_{\mathcal{H}(K)}.
\]
2. If
\[
||Af||_{\mathcal{H}(K)} \geq c||f||_{\mathcal{H}(K)}, \text{ for every } f \in \mathcal{H}(K),
\]
then
\[
c^2 \langle K_w, K_z \rangle_{\mathcal{H}(K)} \geq \langle AK_w, AK_z \rangle_{\mathcal{H}(K)}.
\]

Proof. We only prove (1) and (2) can be done in a similar way. For any distinct points \( w_1, w_2, \ldots, w_n \) in \( X \) and complex numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \), let
\[
f(z) = \sum_{j=1}^{n} \alpha_j K_{w_j}(z).
\]
We then have
\[
\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j C^2 \langle K_{w_i}, K_{w_j} \rangle_{\mathcal{H}(K)} - \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j \langle AK_{w_i}, AK_{w_j} \rangle_{\mathcal{H}(K)} = C^2 \langle f, f \rangle_{\mathcal{H}(K)} - \langle Af, Af \rangle_{\mathcal{H}(K)} = C^2 ||f||_{\mathcal{H}(K)}^2 - ||Af||_{\mathcal{H}(K)}^2 \geq 0.
\]
It then follows from the definition of a positive kernel.

We will need the next theorem that characterizes the functions that belong to a reproducing kernel Hilbert space in terms of the reproducing kernel.

Theorem 2.2. [10, Theorem 3.11] Let \( \mathcal{H}(K) \) be a reproducing kernel Hilbert space on \( X \) and let \( f : X \to \mathbb{C} \) be a function. Then \( f \in \mathcal{H}(K) \) with \( ||f||_{\mathcal{H}(K)} \leq c \) if and only if
\[
\overline{f(w)} f(z) \leq c^2 K(z, w),
\]
A function \( \varphi : X \to \mathbb{C} \) is called a multiplier of \( \mathcal{H}(K) \) on \( X \) if \( \varphi f \in \mathcal{H}(K) \) whenever \( f \in \mathcal{H}(K) \). If \( \varphi \) is a multiplier of \( \mathcal{H}(K) \), let \( M_{\varphi} : f \mapsto \varphi f \) be the multiplication operator on \( \mathcal{H}(K) \). In this case, it is well-known that the kernel functions are eigenvectors for the adjoints of multiplication operators:

\[
M_{\varphi}^* K_z = \overline{\varphi(z)} K_z.
\]

The following theorem characterizes multipliers of reproducing kernel Hilbert spaces (see e.g. [1, Corollary 2.37]).

**Theorem 2.3.** Let \( \mathcal{H}(K) \) be a reproducing kernel Hilbert space on \( X \), and let \( \varphi : X \to \mathbb{C} \) be a function. Then \( \varphi \) is a multiplier of \( \mathcal{H}(K) \) with multiplier norm at most \( \delta \) if and only if

\[
(\delta^2 - \varphi(z) \overline{\varphi(w)}) \cdot K(z, w) \geq 0.
\]

If \( \delta \leq 1 \), then \( \varphi \) is called a contractive multiplier of \( \mathcal{H}(K) \).

As a corollary of Theorem 2.3, we have

**Theorem 2.4.** Let \( \mathcal{H}(K_1) \) and \( \mathcal{H}(K_2) \) be reproducing kernel Hilbert spaces on \( X \). Then

\[
\mathcal{H}(K_1) \subset \mathcal{H}(K_2)
\]

if and only if there is some constant \( \delta > 0 \) such that

\[
K_1 \preceq \delta K_2.
\]

The spaces we are concerned with in this paper are all reproducing kernel Hilbert spaces. One can calculate the reproducing kernel through an orthonormal basis of a reproducing kernel Hilbert space (see e.g. [10, Theorem 2.4]).

**Lemma 2.1.** Let \( \mathcal{H}(K) \) be a reproducing kernel Hilbert space on \( X \). If \( \{e_s| s \in S\} \) is an orthonormal basis for \( \mathcal{H}(K) \), then

\[
K(z, w) = \sum_{s \in S} e_s(w) e_s(z).
\]

The Hardy space \( H^2 \) has reproducing kernel

\[
k_w^S(z) = \frac{1}{1 - w z},
\]

which is called the Szegő kernel. The Bergman space \( A^2 \) has reproducing kernel

\[
k_w^0(z) = \frac{1}{(1 - w z)^2}.
\]

For a function \( b \in H^\infty_1 \), it is well-known that \( b \) is a multiplier of both \( H^2 \) and \( A^2 \) and has multiplier norm equals to 1. It follows from Theorem 2.3 that

\[
k_w^b(z) = \frac{1 - b(w) b(z)}{1 - w z}
\]

and

\[
k_w^{0,b}(z) = \frac{1 - b(w) b(z)}{(1 - w z)^2}.
\]
are positive kernels. In fact, (2.2) is the reproducing kernel for de Brange-Rovnyak space $\mathcal{H}(b)$ [11, II-3] and (2.3) is the reproducing kernel for sub-Bergman Hilbert space $\mathcal{A}(b)$ [13, Proposition 3.1].

3. Main Results

To state the main results, we define more general weighted sub-Bergman Hilbert spaces. For $\alpha \geq -1$, let $A^2_\alpha$ be the reproducing kernel Hilbert space on $\mathbb{D}$ with reproducing kernel

$$k^\alpha_w(z) = \frac{1}{(1 - \bar{w}z)^{\alpha+2}}.$$  

If $\alpha = -1$, it is just the Hardy space $H^2$. If $\alpha > -1$, $A^2_\alpha$ is the space of all analytic functions on $\mathbb{D}$ that are square-integrable with respect to the measure $dA_\alpha$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

In the latter case, the space $A^2_\alpha$ is called a weighted Bergman space (see e.g. [15, Chapter 4] for details).

For $\varphi \in L^\infty(\mathbb{D})$, the weighted Toeplitz operator on $A^2_\alpha$ ($\alpha > -1$) is defined by

$$T^\alpha_{\varphi} f = P_\alpha(\varphi f),$$

where $P_\alpha$ is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ to $A^2_\alpha$. When $\alpha = 0$, $A^2_0 = A^2$, the standard Bergman space, and $T^\alpha_{\varphi} = T_{\bar{\varphi}}$, the Bergman Toeplitz operator defined in 1.1. Let $b \in H^\infty_1$. We define a weighted sub-Bergman Hilbert space as

$$A^\alpha_\alpha (b) = (I - T^\alpha_{b} T^\alpha_{\bar{b}})^{\frac{1}{2}} A^2_\alpha.$$  

The reproducing kernel of $A^\alpha_\alpha (b)$ is

$$k^{b,\alpha}_w(z) = \frac{1 - b(w)b(z)}{(1 - \bar{w}z)^{\alpha+2}}.$$  

It is easy to generalize Theorem 1.1 to the weighted case (see e.g. [12, Theorem 3.5]). We strengthen Theorem 1.1 in the following two theorems.

**Theorem 3.1.** Let $b$ be a non-constant function in $H^\infty_1$. For every $\alpha \geq 0$, we have

$$A^{\alpha-1}_\alpha \subset A^\alpha_\alpha(b).$$

**Proof.** By (3.1), (3.2) and Theorem 2.4, it is sufficient to show that

$$\frac{1}{(1 - \bar{w}z)^{\alpha+1}} \leq C \cdot \frac{1 - b(w)b(z)}{(1 - \bar{w}z)^{\alpha+2}}$$

for some constant $C$. Since

$$\frac{1}{(1 - \bar{w}z)^{\alpha}} \geq 0,$$

we only need to show

$$\frac{1}{1 - \bar{w}z} \leq C \cdot \frac{1 - b(w)b(z)}{(1 - \bar{w}z)^2}.$$
Let $f_0$ be the normalized reproducing kernel of $\mathcal{H}(b)$ at 0, i.e.

$$f_0(z) = \frac{1 - \overline{b(0)}b(z)}{\sqrt{1 - |b(0)|^2}}.$$

Then $f_0 \in \mathcal{H}(b) \cap H^\infty$ and $||f_0||_{\mathcal{H}(b)} = 1$. By Theorem 2.2,

$$\overline{f_0(w)f_0(z)} \preceq \frac{1 - \overline{b(w)}b(z)}{1 - wz}.$$

Multiplying both sides by the Szegő kernel

$$\frac{1}{1 - wz} = \langle k^S_w, k^S_z \rangle_{H^2},$$

we have

$$\overline{f_0(w)f_0(z)}\langle k^S_w, k^S_z \rangle_{H^2} \preceq \frac{1 - \overline{b(w)}b(z)}{(1 - wz)^2}.$$

By (2.1),

$$T_{\overline{f_0}}k^S_w = f_0(w)k^S_w,$$

for every $w \in \mathbb{D}$. We get

(3.4)  $$\langle T_{\overline{f_0}}k^S_w, T_{\overline{f_0}}k^S_z \rangle_{H^2} \preceq \frac{1 - \overline{b(w)}b(z)}{(1 - wz)^2}.$$

Notice that for every $h \in H^2$,

$$||T_{\overline{f_0}}h||_{H^2} = \frac{1}{\sqrt{1 - |b(0)|^2}}||T_{1 - \overline{b(0)}h}h||_{H^2}$$

$$= \frac{1}{\sqrt{1 - |b(0)|^2}}||h - b(0)T_h||_{H^2}$$

$$\geq \frac{1}{\sqrt{1 - |b(0)|^2}}(||h||_{H^2} - |b(0)| \cdot ||T_hh||_{H^2})$$

$$\geq \frac{1}{\sqrt{1 - |b(0)|^2}}(||h||_{H^2} - |b(0)| \cdot ||h||_{H^2})$$

$$= \frac{1 - |b(0)|}{1 + |b(0)|}||h||_{H^2}.$$

So we have from Proposition 2.2 that,

$$C\langle k^S_w, k^S_z \rangle_{H^2} \preceq \langle T_{\overline{f_0}}k^S_w, T_{\overline{f_0}}k^S_z \rangle_{H^2}.$$

This together with (3.4) implies (3.3).

\[\square\]

**Theorem 3.2.** Let $b$ be a non-constant function in $H^\infty_1$. For every $\alpha \geq 0$,

$$A^2_{\alpha - 1} = A_\alpha(b)$$

if and only if $b$ is a finite Blaschke product.
Proof. According to Theorem 3.1, 
\[ A^2_{\alpha-1} = \mathcal{A}_\alpha(b) \]
if and only if 
\[ \mathcal{A}_\alpha(b) \subset A^2_{\alpha-1}. \]

Suppose \( b \) is a finite Blaschke product of degree \( N \). Then \( \mathcal{H}(b) \) is a Hilbert space of finite dimension \( N \) (see e.g. [6]), and there exists a constant \( C \) such that 
\[ \frac{1 - |b(w)|^2}{1 - |w|^2} \leq C, \]
for all \( w \in \mathbb{D} \) ([14, Lemma 1]). Let \( \{f_n\}_{n=0}^{N-1} \) be an orthonormal basis for \( \mathcal{H}(b) \). Then by Lemma 2.1, we have 
\[ \frac{1 - b(w)b(z)}{1 - wz} = \sum_{n=0}^{N-1} f_n(w)f_n(z). \]

In particular, 
\[ \sum_{n=0}^{N-1} |f_n(w)|^2 = \frac{1 - |b(w)|^2}{1 - |w|^2} \leq C, \]
for every \( w \in \mathbb{D} \). Thus for every \( n \), 
\[ ||T_{f_n}|| = ||f_n||_\infty \leq \sqrt{C}. \]

By Proposition 2.2, 
\[ \langle T_{f_n} k^S_w, T_{f_n} k^S_z \rangle_{H^2} \leq C \langle k^S_w, k^S_z \rangle_{H^2}. \]

Thus 
\[ \frac{1 - b(w)b(z)}{1 - wz} = \sum_{n=0}^{N-1} \langle T_{f_n} k^S_w, T_{f_n} k^S_z \rangle_{H^2} \leq \sum_{n=0}^{N-1} \langle k^S_w, k^S_z \rangle_{H^2} = NC \cdot \frac{1}{1 - wz}. \]

Multiplying both sides by the positive kernel 
\[ \frac{1}{(1 - wz)^\alpha}, \]
we have 
\[ \frac{1 - b(w)b(z)}{(1 - wz)^{\alpha + 2}} \leq \frac{NC}{(1 - wz)^{\alpha + 1}}. \]

By Theorem 2.4, we get 
\[ \mathcal{A}_\alpha(b) \subset A^2_{\alpha-1}. \]

For the other direction, assume \( \mathcal{A}(b) \subset H^2 \). Using Theorem 2.4 again, we have 
\[ \frac{1 - b(w)b(z)}{(1 - wz)^{\alpha + 2}} \leq \frac{1}{(1 - wz)^{\alpha + 1}}. \]
for some constant $C$. Then
\[ \frac{1 - |b(z)|^2}{(1 - |z|^2)^{\alpha + 2}} \leq C \frac{1}{(1 - |z|^2)^{\alpha + 1}}, \]
which implies
\[ \frac{1 - |b(z)|^2}{1 - |z|^2} \leq C, \]
for all $z \in \mathbb{D}$. Therefore, $b$ is a finite Blaschke product (see e.g. [14, Lemma 1]). □

**Proof of Theorem 1.2.** This is just a special case of Theorem 3.1 and 3.2 when $\alpha = 0$. □

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