PACKING AND COVERING IN HIGHER DIMENSIONS

G. FEJES TÓTH

Abstract. The present work surveys problems in \(n\)-dimensional space with \(n\) large. Early development in the study of packing and covering in high dimensions was motivated by the geometry of numbers. Subsequent results, such as the discovery of the Leech lattice and the linear programming bound, which culminated in the recent solution of the sphere packing problem in dimensions 8 and 24, were influenced by coding theory. After mentioning the known results concerning existence of economical packings and coverings we discuss the different methods yielding upper bounds for the density of packing congruent balls. We summarize the few results on upper bounds for the packing density of general convex bodies. The paper closes with some remarks on the structure of optimal arrangements.

1. Existence of economic packings and coverings

The celebrated Minkowski-Hlawka theorem (Hlawka [1943]) gives the lower bound

\[
\delta_L(K) \geq \frac{\zeta(n)}{2^{n-1}}
\]

for the packing density of an arbitrary \(n\)-dimensional centrally symmetric convex body. For sufficiently large \(n\), Schmidt [1963a, 1963b] improved the bound to \(c n^{2-n}\), provided \(c < \ln \sqrt{2}\). By the observation of Minkowski concerning translates of an arbitrary convex body \(K\) and \(\frac{1}{2}(K - K)\) and the inequality

\[
\frac{\text{vol}(K - K)}{\text{vol}(K)} \leq \left(\frac{2n}{n}\right)
\]

of Rogers and Shephard [1957], it follows that, for sufficiently large \(n\),

\[
\delta_L(K) \geq \sqrt{\pi c} n^{3/2}/4^{-n},
\]

where \(c < \ln \sqrt{2}\).

Slightly better lower bounds were proved for the packing density of the ball by Ball [1992], Vance [2011], and Venkatesh [2013]. Venkatesh proved that for any constant \(c > \sinh^2(\pi e)/\pi^2 e^3 = 65963.8\ldots\) there is a number \(n(c)\) such that for \(n > n(c)\) we have \(\delta_L(B^n) \geq c n 2^{-n}\). Moreover, there are infinitely many dimensions \(n\) for which \(\delta_L(B^n) \geq n \ln n 2^{-n-1}\).

Leech [1964, 1967] constructed lattice ball packings in dimensions \(n \leq 24\) and \(n = 2^m\). He found a remarkable packing in \(E^{24}\). This lattice, called after its

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inventor the *Leech lattice*, has density $\pi^{12}/12! = 0.001929\ldots$, and each ball in it is touched by 196560 others. As it was shown (see later), no other packing in $E^{24}$ has higher density and no ball can be touched by more than 196560 other balls of the same size. All the densest known lattice packings in dimensions less than 24 can be obtained as sections of the Leech lattice.

Leech did not mention, possibly he was not even aware of the fact, that all his packings were extensions of error-correcting codes, that is packings where the centers are vertices of the unit cube. A systematic study of constructions of ball packings were extensions of error-correcting codes, that is packings where the centers are within a fixed distance of each other. Their work inspired further research, which led to more constructive lower bounds for the packing density of special classes of convex bodies, approaching, and in some cases also improving the Minkowski-Hlawka bound.

Leech defined the *superballs*, that is the $l_p$-norm $(\sum_{i=1}^n|x_i|^p)^{1/p}$. *Rush* and *Sloane* 1987 adopted the constructions by Leech and Sloane for packings of superballs. They obtained an improvement of the Minkowski-Hlawka bound for all integers $p > 2$. As examples let us mention that $\delta_L(B^n_p) \geq 2^{-0.8262n+o(n)}$ and $\delta_L(B^n_4) \geq 2^{-0.6742n+o(n)}$. With further elaboration of the method *Rush* 1989 constructed lattice packings with density $2^{-n+o(n)}$ for every convex body which is symmetric through each of the coordinate hyperplanes. Moreover, *Elkies*, *Odlyzko* and *Rush* 1991 were also able to use the method for packings of centrally symmetric convex bodies. This made it possible to construct dense packings of *generalized superballs* defined as

$$f(x_1,\ldots,x_n) = f(x_{k+1},\ldots,x_{2k}) + \ldots + f(x_{n-k+1},\ldots,x_n) \leq 1 \quad \sigma \geq 1,$$

where $f(x_1,\ldots,x_n)$ is a distance function, that is $f(0) = 0$, $f(x) > 0$ for $x \neq 0$, and $f(tx) = tf(x)$ for $t > 0$. With a further generalization of the notion of superball *Rush* 1993 could also handle bodies like $\lVert x_1 \rVert + \lVert x_2 \rVert^3$ and $\max(\lVert x_1 \rVert, \lVert x_2 \rVert, \lVert x_3 \rVert)$. More results based on the construction of packings through error correcting codes can be found in *Rush* 1991, 1994, 1996 and *Liu* and *Xing* 2003.

For packing of balls in spherical space, independently of each other, *Chabauty* 1953, *Shannon* 1959, and *Wyner* 1965 proved the lower bound

$$M(n,\varphi) \geq (1+o(1))\sqrt{2\pi n} \frac{\cos \varphi}{\sin^{n-1}\varphi},$$

which was improved by *Jenssen*, *Joos* and *Perkins* 2016 by a linear factor.

Concerning coverings, *Rogers* 1957 gave the bound

$$\vartheta_T(K) \leq n\ln n + n\ln n + 5n,$$

for the translational packing density of an arbitrary $n$-dimensional convex body $K$. Rogers’ proof uses a mean value argument combined with saturation. There are several alternative arguments yielding an upper bound for $\vartheta_T(K)$ of the same order of magnitude. *Naszódi* 2016a gave a proof of the inequality $\vartheta_T(K) \leq n\ln n + n\ln n + 5n$ that relies on an algorithmic result of *Lovász* 1975. *Erdős* and *Rogers* 1961 showed that every $n$-dimensional convex body $K$ admits a covering by translates of $K$ with density $n\ln n + n\ln n + 4n$ so that no point is covered more than $e(n\ln n + n\ln n + n)$ times. *Füredi* and *Kang* 2005 used the *Lovász local lemma* to give a simple proof of a slightly weaker result: There exists a covering by translates of $K$ such that every point is covered at most $10n\ln n$ times. G. *Fejes Tóth* 2009 observed that with a modification of a proof of *Rogers* 1957.
one can show for every \( n \)-dimensional convex body \( K \) the existence of a lattice arrangement of \( K \) such that \( O(\ln n) \) translates of this arrangement form a covering of space with density not exceeding \( n \ln n \ln n + n \ln n + o(n) \). Thus, a low density covering can be achieved with an arrangement of relatively simple structure.

Rolfes and Vallentin [2018] suggest a greedy approach to constructing coverings of compact metric spaces by metric balls. Balls are iteratively chosen to cover the maximum measure of yet uncovered space. Their method is an extension of the argument of Chvátal [1979] for the finite set cover problem to the setting of compact metric spaces.

Rogers [1959] proved that

\[
\vartheta_L(K) \leq n^{\log_2 \ln n + c}
\]

for some suitable constant \( c \) and all \( n \)-dimensional convex bodies \( K \). The same bound was proved by Butler [1972] in a different way, providing a covering by translates of \( K \) with the above density such that the corresponding translates of \( \lambda K \), \( \lambda = \frac{\text{vol} K}{\text{vol}(K - K)} \), form a packing. For the covering density of balls Rogers [1959] proved the bound

\[
\vartheta_L(B^n) \leq cn(\ln n)^{1/2} \log_2 2\pi e.
\]

This result was generalized by Gritzmann [1985], who showed that

\[
\vartheta_L(K) \leq cn(\ln n)^{1 + \log_2 e}
\]

for every \( n \)-dimensional convex body \( K \) that has an affine image symmetric through at least \( \log_2 \ln n + 4 \) coordinate hyperplanes.

All the above mentioned results about lattice coverings have been superseded by Ordentlich, Regev and Weiss [2020], who proved that there is a constant \( c \) so that for any \( n \)-dimensional convex body \( K \)

\[
\vartheta_L(K) \leq cn^2.
\]

Using covering codes instead of error correcting codes, Rush [1992] adapted a construction by Leech and Sloane for the construction of thin lattice coverings by star-shaped bodies.

Rogers [1963], Böröczky and Wintsche [2003], Verger-Gaugry [2005], and Dumer [2007] showed the existence of reasonably thin coverings of the \( n \)-dimensional sphere by congruent spherical caps and an \( n \)-dimensional ball by congruent balls. Dumer’s result implies \( \vartheta(B^n) \leq (\frac{1}{2} + o(1))n \log n \) as \( n \to \infty \), improving the above bound of Rogers in the case of a ball by a factor 1/2. We note that the published version of Dumer’s paper contained a small error which he corrected in [2018]. Naszódi [2016a] established the existence of thin coverings of the \( n \)-dimensional sphere by congruent copies of a spherical convex body.

Bacher [2009] considered the problem of finding universal covering sets. Let \( \mathcal{F} \) be a family of convex bodies all having the same area. A set \( U \) is a universal covering set for \( \mathcal{F} \) if the translates of any member of \( \mathcal{F} \) by the vectors of \( U \) cover the space. He showed the existence of such sets in all dimensions. In \( n \)-dimensional space he constructed a universal covering set \( U \) for convex bodies of unit volume such that \( rB^n \) contains at most \( \ln(r)^{n-1}r^nV(B^n) \) points of \( U \).
2. Upper bounds for $\delta (B^n)$ and lower bounds for $\vartheta (B^n)$

2.1. Blichfeldt’s bound. The first upper bound for the packing density of the $n$-dimensional ball was proved by Blichfeldt [1929]. Blichfeldt’s idea was the following. Consider a packing of unit balls in $E^n$. Replace each ball $S$ by a concentric ball of radius $\sqrt{2}$ such that the density at distance $d$ from the center of the ball is $2 - d^2$. Of course, the enlarged balls may overlap, however, it can be shown that their total density is at most 2 at every point of space. Comparing the volume of the unit ball to the mass of an enlarged ball, we get the bound

$$\delta (B^n) \leq \frac{n + 2}{2} 2^{-n/2}$$

for the packing density of the $n$-dimensional ball. With a modified density function Blichfeldt obtained a slightly better bound, and a further improvement was given by Rankin [1934]. Blichfeldt’s method was used by Rankin [1955] and Bloh [1956] to derive bounds for ball packings in $n$-dimensional spherical space.

2.2. The simplex bound. In an $n$-dimensional space of constant curvature let $d_n(r)$ be the density of $n + 1$ mutually touching balls of radius $r$ with respect to the simplex spanned by the centers of the balls. Coxeter [1964] and L. Fejes Tóth [1953b] conjectured that in an $n$-dimensional space of constant curvature the density of a packing of balls of radius $r$ cannot exceed the simplex bound $d_n(r)$. The corresponding simplex bound for coverings, formulated by L. Fejes Tóth [1959g, 1960d], reads as follows: In an $n$-dimensional space of constant curvature the density of a covering by balls of radius $r$ cannot be less than the density $D_n(r)$ of $n + 1$ balls of radius $r$ centered at the vertices of a regular simplex of circumradius $r$ relative to the simplex. In Euclidean space the bounds do not depend on $r$, so in this case we write them simply as $d_n$ and $D_n$. Concerning the notion of density and the interpretation of the simplex bound in hyperbolic space see Section 11.2.

For the Euclidean space, the conjecture about packings was verified by Rogers [1958] and independently by Baranovskii [1964], and the conjecture about coverings was proved by Coxeter, Few and Rogers [1959].

It can be seen that the simplex bound for $\delta (B^n)$ is sharper than Blichfeldt’s bound. However, the improvement is only a constant factor, namely we have

$$\delta (B^n) \leq d_n \approx \frac{n}{e} 2^{-n/2}, \quad n \to \infty.$$ 

The simplex bound

$$\vartheta (B^n) \geq D_n \approx \frac{n}{e \sqrt{e}}, \quad n \to \infty,$$

for the covering density of the ball compares quite favorably with the bound $\vartheta (B^n) \leq (\frac{1}{2} + o(1)) n \log n$.

The three-dimensional case of the conjecture about packing was settled by Böröczky in the paper by Böröczky and Florian [1964] and, finally, Böröczky [1976] proved the conjecture in full generality. More precisely, Böröczky proved that the density of each ball in its Dirichlet cell is at most $d_n(r)$. The conjecture about covering for spherical and hyperbolic space is still open.

Consider the balls inscribed in the cells of the 3-dimensional spherical tiling $\{2,3,3\}$, $\{3,3,3\}$, $\{4,3,3\}$ or $\{5,3,3\}$. The corresponding radii are $\frac{1}{2} \arccos(-\frac{1}{4})$, $\frac{1}{2} \arccos(-\frac{1}{3})$, $\pi/4$, and $\pi/10$, and the density of the balls in the Dirichlet cells is $0.61613 \ldots$, $0.68057 \ldots$, $0.72676 \ldots$, and $0.77412 \ldots$, respectively. These densities
agree with the corresponding tetrahedral density bound. Böröczky, Böröczky Jr, Glazyrin and Kovács [2018] proved the stability of the simplex bound in the cases mentioned here.

Recall that the ordinary sphere cannot be packed as densely, nor can be covered as thinly by at least two congruent circles as the Euclidean plane. Remarkably, in three dimension, the analogous statement does not hold for packings, and probably does not hold for coverings either. Namely, the density
\[
\frac{60}{\pi} \left( \frac{\pi}{5} - \sin \frac{\pi}{5} \right) = 0.77412 \ldots
\]
of the 120 balls inscribed in the cells of the tiling \(\{5, 3, 3\}\) is greater than \(\pi/\sqrt{18} = \delta(B^3)\). Similarly, the 120 balls circumscribed about the same cells form a covering of the spherical space, with density
\[
\frac{60}{\pi} (\omega - \sin \omega) = 1.44480 \ldots, \quad \omega = \frac{2\pi}{3} - \arccos \frac{1}{4},
\]
which is smaller than \(5\sqrt{5\pi}/24 = 1.46350 \ldots\), the conjectured value of \(\vartheta(B^3)\). It is conjectured (see L. Fejes Tóth [1959g]) that the inspheres and the circumspheres of \(\{5, 3, 3\}\) form the densest packing and the thinnest covering, respectively with at least 4 spheres in \(S^3\). Florian [2005] proved that \(d_3(r)\) is a strictly decreasing function for \(0 < r \leq \arctan \sqrt{2}\). It follows that the density of a packing of at least 4 equal spheres in \(S^3\) is at most \(\lim_{r \to 0} \sqrt{18}(\arctan \frac{1}{3} - \frac{\pi}{4}) = 0.77963 \ldots\). This bound is rather close to the conjectured minimum density.

Molnár [1963] defined a Segre-Mahler polytope as a convex \(n\)-dimensional polytope in a space of constant curvature, every dihedral angle of which is at most 120°. He conjectured that, when equal spheres of radius \(r\) are packed in such a region of \(n\)-space, the density cannot exceed the simplex bound \(d_n(r)\). He verified the conjecture for the 3-dimensional case.

G. Fejes Tóth [1980] observed that if a packing of \(n\)-dimensional balls arises as the intersection of a higher-dimensional packing of congruent balls with an \(n\)-dimensional subspace, then the simplex bound \(d_n\) for the density of the packing still holds. Obviously, no similar conclusion can be drawn for an arbitrary covering with balls. However, A. Bezdek [1984] proved that if a circle covering of the plane arises from a planar section of a lattice covering with balls in three dimensions, then the covering’s density cannot be smaller than \(2\pi/\sqrt{27} + 0.017 \ldots\). Moreover, equality occurs for exactly one lattice and only for certain cutting planes.

It should be mentioned that in general there is no connection between the density of an arrangement of bodies and the densities of the sections of the arrangement. Groemer [1966a] gave examples of packings \(\mathcal{P}_1\) and \(\mathcal{P}_2\) with density 0 and 1, respectively; such that in each plane parallel to a given plane the density of the intersection with the sets of \(\mathcal{P}_1\) is 1, while in each of these planes the density of the intersection with the sets of \(\mathcal{P}_2\) is 0. He gave similar examples for coverings.

Florian [2006] gave a nice survey on the simplex bound for packings of balls in spaces of constant curvature with emphasis on dimension 2 and 3.

2.3. The linear programming bound. It took more than 40 years until an improvement in exponential order was achieved for Blichfeldt’s bound of \(\delta(B^n)\). The first step was made by Sidelnikov [1973, 1974], who proved \(\delta(B^n) \leq 2^{-0.509619n+o(n)}\). Subsequently, Levenštejn [1975] improved the bound to \(\delta(B^n) \leq 2^{-0.5237n+o(n)}\),
and Kabatjanskii and Levensteĭn [1978] proved
\[ \delta(B^n) \leq 2^{-(0.599+o(1))n} \quad \text{(as } n \to \infty), \]
which remains the best known asymptotic upper bound for \( \delta(B^n) \). The gap between
this bound and the lower bound by Venkatesh [2013] remains considerable. It is
even unknown whether \( \lim_{n \to \infty} \frac{\ln \delta_L(B^n)}{n} \) and \( \lim_{n \to \infty} \frac{\ln \delta(B^n)}{n} \) exist, and if so, whether
they are equal.

All these improvements of the Blichfeldt bound were obtained as a corollary of
a bound for ball packings in spherical space. A spherical code in dimension \( n \) with
minimum angle \( \varphi \) is a set of points on the unit sphere in \( E^n \) with given minimum
angular distance \( \varphi \) among them. Let \( M(n, \varphi) \) denote the greatest size of such a
spherical code. Equivalently, \( M(n, \varphi) \) is the maximum number of balls of angular
diameter \( \varphi \) that can be packed on the sphere. Delsarte, Goethals and Seidel
[1977] and independently Kabatjanskii and Levensteĭn [1978] adopted the linear
programming bound developed by Delsarte [1972] for bounding the cardinality of
error-correcting codes of given minimum distance to spherical codes.

In the linear programming bound a basic role is played by a sequence of polynomials
\( P^k_n \), \( k = 0, 1, \ldots, \) called ultraspherical polynomials. They form a complete
orthogonal system on the interval \([-1, 1]\) with respect to the measure \((1-t^2)^{(n-3)/2} dt\).
In other words, for \( i \neq j \),
\[ \int_{-1}^{1} P^m_i(t)P^m_j(t)(1-t^2)^{(n-3)/2} dt = 0. \]
For the purpose of the linear programming bound, the normalization is irrelevant;
the sign should be chosen so that \( P^0_n(1) > 0 \). With different normalizations, they
are special Jacobi polynomials or Gegenbauer polynomials. The property of ultraspherical polynomials that is crucial for the application is that they are positive-definite kernels, that is for any \( N \) and any points \( x_1, \ldots, x_N \in S^{n-1} \), the \( N \times N \) matrix \((P^k_n(x_i, x_j))_{1 \leq i, j \leq N}\) is positive semidefinite. Schoenberg [1934] proved
that every continuous positive-definite kernel on \( S^{n-1} \) has an ultraspherical expansion
with nonnegative coefficients which converges absolutely and uniformly. Since
the sum of the entries of a positive-semidefinite matrix is nonnegative, it follows that
\[ \sum_{1 \leq i, j \leq N} P^m_k(x_i \cdot x_j) \geq 0 \]
for any points \( x_1, \ldots, x_N \in S^{n-1} \).

Now suppose that \( x_1, \ldots, x_N \) is a spherical code in dimension \( n \) with minimum
angle \( \varphi \) and let \( P(t) = \sum_{k=0}^{m} a_k P^k_n(t) \) be a real polynomial. Then
\[ \sum_{1 \leq i, j \leq N} P(x_i, x_j) = NP(1) + \sum_{1 \leq i, j \leq N, i \neq j} P(x_i, x_j) = N^2a_0 + \sum_{k=1}^{m} a_k \sum_{1 \leq i, j \leq N} P^k_n(x_i, x_j). \]
Suppose that \( P(t) \leq 0 \) for \(-1 \leq t \leq \cos \varphi\), \( a_0 > 0 \), and \( a_k \geq 0 \) for \( i = 1, 2, \ldots, k \).
Then \[ \sum_{1 \leq i, j \leq N, i \neq j} P(x_i, x_j) \leq 0 \]
and \[ \sum_{k=1}^{m} a_k \sum_{1 \leq i, j \leq N} P^k_n(x_i \cdot x_j) \geq 0. \] Hence we get
the linear programming bound for \( M(n, \varphi) \):
Let $P(t) = \sum_{k=0}^{m} a_k t^k$ be a real polynomial such that $P(t) \leq 0$ for $-1 \leq t \leq 0$, $a_0 > 0$, and $a_k \geq 0$ for $i = 1, 2, \ldots, m$. Then

$$M(n, \varphi) \leq P(1)/a_0.$$ 

Let $t_{1,k}^n$ denote the largest root of $P_k^n(t)$. With the choice of appropriate polynomials Kabatjanskiĭ and Levensteĭn [1978] proved that if $\cos \varphi \leq t_{1,k}^n$ then

$$M(n, \varphi) \leq 4\left(\frac{k+n-2}{k}\right)\left(1-t_{1,k+1}^n\right)^{-1}.$$ 

From this they derived the asymptotic bound

$$M(n, \varphi) \leq (1 - \cos \varphi)^{-n/2}2^{-n(0.099+o(1))} \quad \text{(as } n \to \infty \text{)}$$

for all $\varphi \leq \varphi^* = 62^\circ \ldots$.

Sardari and Zargar [2020] sharpened the above bound to

$$M(n, \varphi) \leq \left(\frac{k+n-2}{k}\right)\left(1 + \frac{2k}{n-1} + \frac{1}{1-t_{1,k+1}^n}\right).$$

The improvement is by a factor of at most 4 and it does not affect the exponent in the asymptotic bound.

The asymptotic upper bound for $M(n, \varphi)$ yields a similar bound for $\delta(B^n)$. Kabatjanskiĭ and Levensteĭn used the inequality $\delta(B^n) \leq \sin^n(\varphi/2)M(n+1, \varphi)$. However, there is a slightly better inequality using Blichfeldt’s method. For, it is easily seen that a ball of radius $r = 1/\sin(\varphi/2) \leq 2$ can contain at most $M(n, \varphi)$ centers of a packing of unit balls. Thus the concentric balls of radius $\sin(\varphi/2)$ form an $M(n, \varphi)$-fold packing, yielding $\delta(B^n) \leq \sin^n(\varphi/2)M(n, \varphi)$. This argument was well known in the community approaching problems of packing by geometric methods. It was first published by Levensteĭn [1983, p. 108]. It was rediscovered by Cohn and Zhao [2014], who also used it to derive an asymptotic improvement of the simplex bound in hyperbolic space.

The linear programming bound for spherical codes was used by Odlyzko and Sloane [1979] and Levensteĭn [1979] to solve the problem of densest packing of $N$ balls in $S^n$ for some special values of $n$ and $N$. The arrangements that were characterized as optimal spherical codes in this way are listed, with the exception of the set of 120 vertices of the 600 cell characterized as optimal spherical codes in this way are listed, with the exception of the set of 120 vertices of the 600 cell $\{3,3,5\}$, in Levensteĭn [1992, p. 72] and [1998, p. 621]. The optimality of the vertices of $\{3,3,5\}$ follows from the simplex bound, and was proved using the linear programming bound by Andreev [1999].

Elaborations of the linear programming bound by Musin [2003b, 2003a], Boyvalenkov, Pflender [2007], Bachoc and Vallentin [2008] and Mittelman and Vallentin [2010] yielded improved upper bounds for $M(n, \varphi)$ in some low dimensions. The surveys by Boyvalenkov, Dodunekov and Musin [2012, Cohn 2010, 2017a, and Viazovska 2018] describe these methods in detail.

Cohn and Elkies [2003] and Cohn [2002] modified the linear programming method, obtaining bounds for $\delta(B^n)$ directly. Although they did not improve on the asymptotic bound given above, their method proved to be exceptionally efficient for $n = 8$ and $n = 24$. The method enabled Cohn and Kumar [2004, 2005] to show that the Leech lattice is the unique densest lattice in 24 dimensions and...
to give an alternative proof of the result of Blichfeldt [1935] about the densest packing of balls in 8 dimensions. Furthermore they proved that no packing of congruent balls in 24 dimensions can exceed the Leech lattice’s density by a factor of more than $1 + 1.65 \times 10^{-30}$. A breakthrough was achieved by Viazovska [2017] who succeeded in proving that $\delta(B^8) = \delta_L(B^8) = \pi^4/384$. Some days later Cohn, Kumar, Miller, Radchenko and Viazovska [2017] showed that $\delta(B)^{24} = \delta_L(B^{24})\pi^{12}/12!$. The paper by Cohn [2017] explains the main ideas leading to this landmark achievement. The survey and interview by de Laat and Vallentin [2016] is also interesting reading on this subject.

2.4. Arrangements of points with minimum potential energy. Given a decreasing potential function $f$ defined on $(0,2]$ and an integer $N > 1$, we wish to place $N$ distinct points $\{x_1, x_2, \ldots, x_N\}$ on the unit sphere in $n$-dimensional space so that the potential energy $\sum_{i \neq j} f(|x_i - x_j|)$ is as small as possible. Yudin [1992] extended the linear programming bound to obtain an lower bound for the potential energy. His result contains the linear programming bound for $M(n, \phi)$ as a corollary. Using Yudin’s result the minimum potential energy of 240 points on $S^7$, of 196560 points on $S^{23}$, and of 120 points on $S^5$ with the potential of $f(x) = x^{n-1}$ on $S^n$ was determined by Kolushov and Yudin [1994] and Andreev [1997, 2000]. The corresponding optimal arrangements are the minimal vectors in the $E_8$ root lattice, the minimal vectors in the Leech lattice and the vertices of the 600-cell $\{3,3,5\}$.

Cohn and Kumar [2007] succeeded in proving optimality of certain arrangements not just for a specific, but for a whole class of potential functions. A real function $f(x)$ is completely monotonic if it is decreasing, infinitely-differentiable and satisfy the inequalities $(-1)^j f^{(j)}(x) \geq 0$. Typical examples are the inverse power functions $f(x) = 1/x^s$ with $s > 0$ and exponential functions $f(x) = e^{-cx}$ with $c > 0$. In Cohn and Kumar’s investigation the argument of potential functions is the square of the distance, rather that the distance itself. They define an arrangement of points as universally optimal if it is optimal under every potential function $f(|x - y|^2)$ where $f(x^2)$ is completely monotonic. Note, that $f(x^2)$ is completely monotonic on an interval $(a, b)$, $a > 0$, then $f(x)$ is completely monotonic on $(a^2, b^2)$, but not vice versa.

Cohn and Kumar [2007] proved universal optimality of the sets of vertices of all regular simplicial polytopes in every dimension, as well as of several other arrangements in dimensions 2 - 8 and 21 - 24. These arrangements are listed in Table 1 of their paper. The list coincides with the list of optimal spherical codes mentioned above. For $f(x) = x^{-t}$ with $t$ large, the energy is asymptotically determined by the minimal distance, thus minimizing energy requires maximizing the minimal distance. Hence, universal optimality of an arrangement implies that it is an optimal spherical code. With the special choice of the potential function $f(x) = 2 - x^{1/2}$ and $f(x) = \ln(4/x)$, respectively, it also follows that these arrangements maximize the sum, as well as the product, of the distances between pairs of points. Earlier, Andreev [1996, 1997] proved that the vertices of the regular icosahedron maximize the product of the distances between 12 points on $S^2$ and the minimal vectors of the Leech lattice maximize the sum of the distances between 196560 points on $S^{23}$. Also, Kolushov and Yudin [1997] proved that the vertices of the regular simplex, the regular cross-polytope and the minimal vectors of the $E_8$ root lattice maximize
both the sum and the product of the distances between the corresponding number of points lying on the respective sphere.

On $S^2$, the only universal optima are a single point, two antipodal points, an equilateral triangle on the equator, and the vertices of a regular tetrahedron, octahedron, or icosahedron. That this list is complete follows from a result of Leech [1957] who enumerated all those configurations on $S^2$ that are in a (stable or unstable) equilibrium under any force that depends on the distance only. He showed that the only configurations of this kind are the vertices of the tilings \( \{p,q\} \) (\( p,q \geq 2, \frac{1}{p} + \frac{1}{q} > \frac{1}{2} \)) or the face-centers of a tiling \( \{2,q\} \) (\( q \geq 2 \)).

Our knowledge in higher dimensions is limited. It appears that universally optimal arrangements of points are rare. Through a computer search, Ballinger, Blekherman, Cohn, Giansiracusa, Kelly and Schürmann [2009] found two arrangements of points, one consisting of 40 points on $S^9$, and another of 64 points on $S^{13}$, which they conjecture to be universally optimal.

Besides the arrangements of points proved by Cohn and Kumar to be universal optimal, exact solutions are only known for a few special cases. Dragnev, Legg and Townsend [2002] investigated the problem of minimizing the energy under the logarithmic potential $\log(1/x)$. This is the same problem as finding the maximum of the product of the distances between $k$ points on $S^n$. They solved the problem for $k = 5$ points on $S^2$. The optimum is attained by the opposite poles of the sphere and three points distributed evenly on the equator, that is, by the vertices of a bipyramid. Dragnev [2016] investigated the problem in higher dimensions. He conjectured that the product of the distances between $n + 3$ points on $S^n$ attains its maximum for an arrangement of points that is the union of two mutually orthogonal regular simplices, one of dimension $\lfloor \frac{n+1}{2} \rfloor$, the other of dimension $\lfloor \frac{n+2}{2} \rfloor$. He proved the conjecture for $n = 3$ and $n = 4$. Dragnev and Musin [2019] verified the conjecture for all $n$ by enumerating all stationary configurations of $n + 2$ points on $S^n$ for the logarithmic potential.

Due to applications in stereochemistry, the potential sign $(s)/x^s$ received special attention. Through a computer search, Melnyk, Knop and Smith [1977] found conjecturally optimal solutions for up to 16 points for different values of $s$. For the case of 5 points they conjectured that there exists a phase transition constant $s_0 = 15.04808$ so that for $s < s_0$ the triangular bipyramid and for $s > s_0$ a quadrilateral pyramid is optimal. The conjecture was confirmed in three special cases. The triangular bipyramid was proved to be optimal by Hou and Shao [2011] if $s = -1$ and by Schwartz [2013] if $s = 1$ or $s = 2$. Subsequently, Schwartz [2018, 2020] determined the exact value of $s_0 = 15.0480773927797 \ldots$, proved that the triangular bipyramid is optimal for $-2 < s < s_0$ and the quadrilateral pyramid is optimal for $s_0 < s \leq 15 + 25/512 = 15.048828125$. For $s = s_0$ both the triangular bipyramid and the quadrilateral pyramid minimize the energy. For $s > s_0$ the problem is unsolved.

A further characterization of the triangular bipyramid is due to Tumanov [2013] who proved that this arrangement of points on the sphere constitutes the unique minimizer position under a potential of the form $f(r) = ar^4 - br^2 + c$, $a > 0$, $b > 8a$.

Remarkably, the solution of the sphere packing problem in 8 and 24 dimensions was not the end of the story: Cohn, Kumar, Miller, Radchenko and Viazovska [2019] proved that the $E_8$ root lattice and the Leech lattice are universally optimal among point arrangements in Euclidean spaces of dimensions 8 and 24.
respectively. In other words, they have minimum energy among all arrangements of points with given density for every potential function that is a completely monotonic function of the squared distance.

2.5. Lattice arrangements of balls. Besides dimensions 3, 8, and 24, where the maximum density of general packings of congruent balls is known, the value of $\delta_L(B^n)$ was determined for $n = 4$ and $n = 5$ by Korkine and Zolotareff [1872] [1877] and for $n = 6$ and 7 by Blichfeldt [1935]. The lattice covering density of the ball is known only in dimensions 4 and 5. The case $n = 4$ has been established by Delone and Ryškov [1963] and the case $n = 5$ by Ryškov and Baranovski˘ı [1975] [1976].

Schürmann and Vallentin [2006] designed an algorithm for approximating the values of the lattice covering density of the ball along with the critical lattice, with arbitrary accuracy. Implementing the algorithm in dimensions 6, 7, and 8 enabled them to find the best known lattices for thin sphere covering in these dimensions.

An $m$-periodic arrangement is the union of $m$ translates of a lattice arrangement.

Andreanov and Kallus [2020] presented an algorithm to enumerate all locally optimal 2-periodic sphere packings in any dimension, provided there are finitely many. They implemented the algorithm in 3, 4, and 5 dimensions and showed that no 2-periodic packing of balls surpasses the density of the optimal lattices in these dimensions.

3. Bounds for the packing and covering density of convex bodies

Finding the packing density $\delta(K)$ for a given convex body $K$, even finding a meaningful upper bound for it, is generally a very difficult task. Let $d(K)$ denote the density of the insphere of an $n$-dimensional body $K$ in $K$. Then, obviously, $\delta(K) \leq \frac{\delta(B^n)}{d(K)}$, which yields a non-trivial upper bound for $\delta(K)$ if $d(K) \geq \delta(B^n)$. Using this bound Torquato and Jiao [2009a] [2009b] gave non-trivial upper bounds for the packing density of the icosahedron and dodecahedron, as well as for several Archimedean solids and superballs. The bound obtained in this way for the octahedron is greater than 1, however we get a nontrivial bound for the packing density of the regular cross-polytope in dimensions greater than 23. Moreover, we get that the packing density of the $n$-dimensional regular cross-polytope approaches zero exponentially as $n$ tends to infinity.

The method of Blichfeldt was used to obtain upper bounds for the translational packing density of the superball $B^n_p = \left\{ (x_1, \ldots, x_n) \in E^n | (\sum_{i=1}^{n} |x_i|^p)^{1/p} \leq 1 \right\}$ by van der Corput and Schakee [1938], Hua [1945] and Rankin [1949a, 1949b] (see also Zong [1999b, Section 6.3]). For $p \geq 2$ their bound was recently improved by Sah, Sawhney, Stoner and Zhao [2020] based on the Kabatjanskii-Levenšteĭn bound for spherical codes.

With an extension of Blichfeldt’s method, G. Fejes Tóth and W. Kuperberg [1993] was able to give non-trivial upper bounds for the packing density of other, suitable convex bodies, e.g., for “longish” bodies such as sufficiently long cylinders $B^{n-1} \times [0, h]$ and “sausage-like” solids $B^n + [0, h]$ in $R^n$. Applying this method G. Fejes Tóth, Fodor and Vígh [2015] gave a non-trivial upper bound for the packing density of the regular cross-polytope in all dimensions greater than 6.

Elaborating on the method used by A. Bezdek and W. Kuperberg [1990], W. Kusner [2014] obtained another bound for the packing density of finite-length
circular cylinders, namely $\delta(B^2 \times [0, h]) \leq \pi/\sqrt{12} + 10/h$. Although this bound is meaningful (smaller than 1) only if $h > 100$ and it improves upon the bound given in G. Fejes Tóth and W. Kuperberg only for $h$ greater than about 250, the advantage of Kusner’s bound is that it approaches the packing density of the circle as $h \to \infty$. In a further paper W. Kusner [2016] extended the result of A. Bezdek and W. Kuperberg by showing that the packing density of the set $B^2 \times \mathbb{R}^n$ is also $\pi/\sqrt{12}$.

We can get upper bounds for the translational packing density of convex bodies by the observation of Minkowski that a family of translates of a convex body $K$ forms a packing if and only if the corresponding translates of the centrally symmetric body $\frac{1}{2}(K - K)$ form a packing. Groemer [1961d] proved in this way that $\delta_T(C) \leq \frac{2^{n-1}}{2^{n-1}}$ for every convex cone $C$. For the $n$ dimensional simplex $S_n$ this argument yields $\delta_T(S_n) \leq 2^n (\frac{2n}{n})^{-n}$. In particular, we have $\delta_T(T) \leq 0.4$ for a tetrahedron $T$.

Zong [2014b] proposed a method based on the shadow regions introduced by L. Fejes Tóth [1983] to give upper bounds for the translative packing density of three-dimensional convex bodies. Applying the method for the tetrahedron $T$, he established the bound $\delta_T(T) \leq \frac{36}{\sqrt{10}/(95\sqrt{10} - 4)} = 0.3840610\ldots$

De Oliveira Filho and Vellentin [2018] extended the linear programming method to estimate the packing density of congruent copies of a convex body. Dostert, Guzmán, de Oliveira Filho and Vallentin [2017] exploited this method to obtain upper bounds for the translative packing density of some three-dimensional convex bodies with tetrahedral symmetry, such superballs and Platonic and Archimedean solids. They improved Zong’s upper bound for the translative packing density of the tetrahedron to 0.3745.

In some cases it can be proved that for a body $K$ we have $\delta(K) < 1$ or $\vartheta(K) > 1$ without establishing a concrete bound. Hlawka conjectured that the packing density of circular tori cannot be 1. Motivated by this conjecture Schmidt [1961] proved a general theorem which implies as a corollary, besides the positive answer to Hlawka’s conjecture, $\delta(K) < 1$ and $\vartheta(K) > 1$ for every smooth convex body. Schmidt’s theorem does not apply for packings of cones. W. Kuperberg proved that a packing consisting of translates of a cone $C$ and its images $-C$ in $E^3$ cannot have density 1. Bárány and Matoušek [2007] succeeded in proving an explicit bound smaller than 1 for the density of such a packing.

4. The Structure of Optimal Arrangements

In higher dimensions the occurrence of less organized arrangements among the optimal ones seems to be more frequent. It is likely that the equality $\delta_L(K) = \delta_T(K)$ fails in dimensions greater than 2, although no convex body is known for which $\delta_L(K) < \delta_T(K)$. Rogers [1964] page 15] conjectured that $\delta_L(B^n) < \delta_T(B^n)$ for all sufficiently large $n$. Best [1954] constructed non-lattice ball packings in dimensions 10, 11, and 13 that are denser than the densest known lattice packings. There is a special class of convex bodies for which the equality $\delta_L = \delta_T$ holds: Venkov [1954] and independently McMullen [1980] proved that parallelohedra, that is those polytopes whose translates tile space also admit a lattice tiling. We note, that according a theorem of Groemer [1964] parallelohedra are also characterized by the property that space can be tiled by positive homothetic copies of them.
The equality $\delta(K) = \delta_T(K)$, which holds in the plane for all centrally symmetric disks, fails already in dimension 3. There exist centrally symmetric convex bodies whose congruent copies can pack space perfectly (tile it without gaps), but whose maximum density attained in a packing of translates is smaller than 1. One such body is the right double-pyramid erected over and under the unit square, with height $1/2$. Moreover, A. Bezdek and W. Kuperberg \cite{BezdekKuperberg1991} observed that for $n \geq 3$, there exist ellipsoids in $E^n$ for which packing with congruent copies can exceed the maximum density by translates, that is, the ball packing density.

In their construction A. Bezdek and W. Kuperberg used the theorem of Heppes \cite{Heppes1960a} that one can place infinite circular cylinders in the void of every lattice packing of balls. Packing in the cylinders long ellipsoids of the same volume as the balls we get a mixed packing of balls and ellipsoids. With a suitable affinity, the balls and ellipsoids are then transformed into congruent ellipsoids. Refining this construction, Wills \cite{Wills2017} produced a packing of congruent ellipsoids with density $0.7549\ldots$ and Schürmann \cite{Schurmann2002} constructed dense ellipsoid packings in dimensions up to 8. The packing of congruent ellipsoids constructed by Schürmann in $E^8$ exceed the density of the densest packing of balls by more than 42.9%.

Motivated by the problem of understanding the structure of certain materials like crystals and glasses Donev, Stillinger Chaikin and Torquato \cite{Donev2004} constructed in $E^3$ packings of congruent copies of ellipsoids close to the ball with density grater than $\pi/\sqrt{18} = 0.74048\ldots$. If $a \leq b \leq c$ are the semiaxes of the ellipsoid and $c/a \geq \sqrt{3}$, then the construction has density 0.7707, exceeding $\pi/\sqrt{18}$ considerably. In the case $1.365 \leq a/b \leq 1.5625$ a packing constructed by Jin, Jiao, Liu, Yuan and Li \cite{Jin2020} has higher density.

The idea of the combination of a lattice arrangement with an arrangement in infinite cylinders was used by G. Fejes Tóth and W. Kuperberg \cite{FejesTóthKuperberg1995} for coverings. They proved that for every $n$-dimensional ($n \geq 3$) strictly convex body $K$ there is an affine-equivalent body $L$ whose congruent copies can cover space more thinly than any lattice covering. The assumption of strict convexity is essential: There exist convex polyhedra, e.g. a cube, which tile space in a lattice-like manner. In $E^3$ they presented an ellipsoid $E$ for which $\vartheta(E) \leq 1.394$. Since by the simplex bound of Coxeter, Few and Rogers \cite{CoxeterFewRogers1959} $\vartheta(B^3) \geq 1.35(3\arccos \frac{1}{3} - \pi) = 1.431\ldots$, it follows that $\vartheta(E) < \vartheta(B^3) = \vartheta_T(E)$.

The excellent book of Rogers \cite{Rogers1964} gives an exhaustive account of packing and covering in high dimensions. The book of Conway and Sloane \cite{ConwaySloane1999} is an encyclopedic source of information about sphere packing.

References

Andreanov, A. and Kallus, Y.
[2020] Locally Optimal 2-Periodic Sphere Packings. Discrete Comput. Geom. 63 (2020) no. 1, 182–208. MR405745, DOI 10.1007/s00454-019-00150-6

Andreev, N. N.
[1996] An extremal property of the icosahedron. East J. Approx. 2 (1996) no. 4, 459–462. MR1426716
[1997] Disposition points on the sphere with minimum energy (Russian). Tr. Mat. Inst. Steklova 219 (1997) Teor. Pribizh. Garmon. Anal., 27–21, translation in Proc. Steklov Inst. Math. 1997, no. 4 (219), 20–24. MR1642295
[1999] A spherical code. (Russian) Uspekhi Mat. Nauk 54 (1999) no. 1(325), 255–256; translation in Russian Math. Surveys 54 (1999) no. 1, 251–253. MR1706807, DOI 10.1070/RM1999v054n01ABEH000123
Bacher, R. [2009] Universal convex coverings. Bull. Lond. Math. Soc. 41 (2009) no. 6, 987–992. MR2575329, DOI 10.1112/blms/bdp076

Bachoc, C. and Vallentin, F. [2008] New upper bounds for kissing numbers from semidefinite programming. J. Amer. Math. Soc. 21 (2008) no. 3, 909–924. MR2393433, DOI 10.1090/S0894-0347-07-00589-9

Ball, K. M. [1992] A lower bound for the optimal density of lattice packings. Internat. Math. Res. Notices 1992, no. 10, 217–221. MR1191572, DOI 10.1155/S1073792892000242

Ballinger, B.; Blekherman, G.; Cohn, H.; Giansiracusa, N.; Kelly, E. and Schürmann, A. [2009] Experimental study of energy-minimizing point configurations on spheres. Experiment. Math. 18 (2009) no. 3, 257–283. MR2555698, DOI 10.1080/10586458.2009.10129052

Baranovsky, E. P. [1964] On packing n-dimensional Euclidean spaces by equal spheres. I. (Russian) Izv. Vysš. Učebn. Zaved. Matematika 1964 no. 2 (39), 14–24. MR0169142

Bárány, I. and Matoušek, J. [2007] Packing cones and their negatives in space. Discrete Comput. Geom. 38 (2007) no. 2, 177–187. MR2343302 DOI 10.1007/s00454-007-1332-9

Best, M. R. [1954] Binary codes with a minimum distance of four. IEEE Trans. Inform. Theory 26 (1980) no. 6, 738–742. MR0596287, DOI 10.1109/TIT.1980.1056269

Bezdek, A. [1984] On the section of a lattice-covering of balls. Proceedings of the 11th winter school on abstract analysis (Zelezná Ruda, 1983) Rend. Circ. Mat. Palermo (2) (1984) Suppl. no. 3, 23–45. MR0744363

Bezdek, A. and Kuperberg, W. [1990] Maximum density space packing with congruent circular cylinders of infinite length. Mathematika 37 (1990) no. 1, 74–80. MR1067888, DOI 10.1112/S0025579300012808

[1991b] Packing Euclidean space with congruent cylinders and with congruent ellipsoids. Applied geometry and discrete mathematics, 71–80, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Providence, RI, 1991. MR1116339

Blachman, N. M. and Few, L. [1960] Multiple packing of spherical caps. Mathematika 10 (1963) 84–88. MR0155324, DOI 10.1112/S0025579300003405

Blichfeldt, H. F. [1920] The minimum value of quadratic forms, and the closest packing of spheres. Math. Ann. 101 (1929) 605–608. MR1512555 DML, DOI 10.1007/BF01454863

[1935] The minimum values of positive quadratic forms in six, seven and eight variables. Math. Z. 39 (1935) no. 1, 1–15. MR1545485, DOI 10.1007/BF01201341

Blokh, É. L. [1956] On the most dense arrangement of spherical segments on a hypersphere. (Russian) Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956) 707–712. MR0082699, http://mi.mathnet.ru/eng/izv/3848

Böröczky, K. [1978] Packing of spheres in spaces of constant curvature. Acta Math. Acad. Sci. Hungar. 32 (1978) no. 3–4, 243–261. MR0512399, DOI 10.1007/BF01902361.

Böröczky, K.; Böröczky, K. Jr.; Glazyrin, A. and Kovács, Á. [2018] Stability of the simplex bound for packings by equal spherical caps determined by simplicial regular polytopes. Discrete geometry and symmetry, 31–60, Springer Proc. Math. Stat., 234, Springer, Cham, 2018. MR3816869, DOI 10.1007/978-3-319-78434-2_2

Böröczky, K. and Florian, A.
Anwendung einer Blichfeldtschen Beweismethode in der Geometrie der Zahlen. Acta Arithmetica 2 (1936) 152–160. DOI: 10.4064/aa-2-1-152-160

Coxeter, H. S. M.
[1964b] An upper bound for the number of equal nonoverlapping spheres that can touch another of the same size. 1963 Proc. Sympos. Pure Math., Vol. VII pp. 53–71 Amer. Math. Soc., Providence, R.I. 1964. MR0164283

Coxeter, H. S. M.; Few, L. and Rogers, C. A.
[1959] Covering space with equal spheres. Mathematika 6 (1959) 147–157. MR0124821, DOI 10.1112/S0025579300002059

Delone, B. N. and Ryškov, S. S.
[1963] Solution of the problem on the least dense lattice covering of a 4-dimensional space by equal spheres. (Russian) Dokl. Akad. Nauk SSSR 152 (1963) 523–524. English translation in Soviet Math. Dokl. 4 (1963) 1333–1334. MR0175850

Delsarte, P.
[1972] Bounds for unrestricted codes, by linear programming. Philips Res. Rep. 27 (1972) 272–289. MR0314545

Delsarte, P, Goethals, J.M. and Seidel, J.J.
[1977] Spherical codes and designs. Geom. Dedicata 6 (1977) no. 3, 363–388. MR0485471, DOI 10.1007/BF01232282

Donev, A.; Stillinger, F. H.; Chaikin, P. M. and Torquato, S.
[2004] Unusually dense crystal packings of ellipsoids. Phys. Rev. Lett. 92, 255506 (2004) [4 pages] DOI 10.1103/PhysRevLett.92.255506

Dostert, M.; Guzmán, C. and Vallentin, F.
[2017] New upper bounds for the density of translatively packings of three-dimensional convex bodies with tetrahedral symmetry. Discrete Comput. Geom. 58 (2017) no. 2, 449–481. MR3679945, DOI 10.1007/s00454-017-9882-y

Dragnev, P. D.
[2016] Log-optimal configurations on the sphere. Modern trends in constructive function theory, 41–55, Contemp. Math., 661, Amer. Math. Soc., Providence, RI, 2016. MR3489549, DOI 10.1090/conm/661

Dragnev, P. D.; Legg, D. A. and Townsend, D. W.
[2002] Discrete logarithmic energy on the sphere. Pacific J. Math. 207 (2002) no. 2, 345–358. MR1972249, DOI 10.2140/pjm.2002.207.345

Dragnev, P. D. and Musin, O. R.
[2019] Log–optimal $(d+2)$-configurations in $d$-dimensions. arXiv:1909.09909v1 [math.MG] 21 Sep 2019.

Dumer, I.
[2007] Covering spheres with spheres. Discrete Comput. Geom. 38 (2007) no. 4, 665–679. MR2365829, DOI 10.1007/s00454-007-9000-7
[2018] Covering spheres with spheres. arXiv:math/0606002v2 [math.MG] 20 May 2018.

Elkies, N. D.; Odlyzko, A. M. and Rush, J. A.
[1991] On the packing densities of superballs and other bodies. Invent. Math. 105 (1991) no. 3, 613–639. MR1117154, DOI 10.1007/BF01232282

Erdős, P. and Rogers, C. A.
[1961] Covering space with convex bodies. Acta Arith. 7 (1961/1962) 281–285. MR0149373, DOI 10.4064/aa-7-3-281-285

Fejes Tóth, G.
[1980] On the section of a packing of equal balls. Studia Sci. Math. Hungar. 15 (1980) no. 4, 487–489. MR608829
[2009] A note on covering by convex bodies. Canad. Math. Bull. 52 (2009) no. 3, 361–365. MR2547802, DOI 10.4153/CMB-2009-039-x

Fejes Tóth, G.; and Fodor, F. and Vígh, V.
[2015] The packing density of the $n$-dimensional cross-polytope. Discrete Comput. Geom. 54 (2015) no. 1, 182–194. MR3351762, DOI 10.1007/s00454-015-9699-5

Fejes Tóth, G. and Kuperberg, W.
[1993a] Blichfeldt’s density bound revisited. Math. Ann. 295 (1993) no. 4, 721–727. MR1214958, DOI 10.1007/BF01444913

[1995] Thin non-lattice covering with an affine image of a strictly convex body. Mathematika 42 (1995) no. 2, 239–250. MR1376725, DOI 10.1112/S002557930001456X

FEJES TÓTH, L.

[1953b] On close-packings of spheres in spaces of constant curvature. Publ. Math. Debrecen 3 (1953) 158–167. MR0061401

[1959g] Kugelunterdeckungen und Kugelüberdeckungen in Räumen konstanter Krümmung. Arch. Math. (Basel) 10 (1959) 307–313. MR0106437, DOI 10.1007/BF01240803

[1960d] Neuere Ergebnisse in der diskreten Geometrie. Elem. Math. 15 (1960) 25–36. MR0117655

[1983] On the densest packing of convex discs. Mathematika 30 (1983) no. 1, 1–3. MR0720944, DOI 10.1112/S0025579300010954

FLORIAN, A.

[2005] On the density of packings in spherical 3-space. Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 214 (2005) 89–100. MR2250830

[2006] On the density of packings of spheres in spaces of constant curvature. Rend. Circ. Mat. Palermo (2) Suppl. No. 77 (2006) 281–301. MR2245676

FÜREDI, Z. and KANG, J.-H.

[2008] Covering the n-space by convex bodies and its chromatic number. Discrete Math. 308 (2008) no. 19, 4495–4500. MR2433777

GRITZMANN, P.

[1985] Lattice covering of space with symmetric convex bodies. Mathematika 32 (1985) no. 2, 311–315. MR0834499, DOI 10.1112/S0025579300011086

GROEMER, H.

[1961d] Eine Bemerkung über Lagerungen konvexer Kegel. Arch. Math. (Basel) 12 (1961) 78–80. MR0132457, DOI 10.1007/BF01650527

[1964b] Über die Zerlegung des Raumes in homothetische konvexe Körper. Monatsh. Math. 68 (1964) 21–32. MR0173983, DOI 10.1007/BF01298822

[1966a] Über ebene Schnitte von Lagerungen. Monatsh. Math. 70 (1966) 213–222. MR0199796 DOI 10.1007/BF01305301

HEPPEL, A.

[1960b] Ein Satz über gitterförmige Kugelpackungen. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 3–4 (1960/1961) 89–90. MR0133737

HLAWKA, E.

[1943] Zur Geometrie der Zahlen. Math. Z. 49 (1943) 285–312. MR0009782, DOI 10.1007/BF01174201

HOU, X. and SHAO, J.

[2011] Spherical distribution of 5 points with maximal distance sum. Discrete Comput. Geom. 46 (2011) no. 1, 156–174. MR2794362, DOI 10.1007/s00454-010-9307-7

HUA, L. K.

[1945] A remark on a result due to Blichfeldt Bull. Amer. Math. Soc. 51 (1945) 537–539. MR0012643 DOI 10.1090/S0002-9904-1945-08388-8 Errata, Volume 51: Bull. Amer. Math. Soc., Volume 52, Number 12 (1946), 1092–1092

JENSSEN, M., JOOS, F. and PERKINS, W.

[2018] On kissing numbers and spherical codes in high dimensions. Adv. Math. 335 (2018) 307–321. MR3836667, DOI 10.1016/j.aim.2018.07.001

JIN, W.; JIAO, Y.; LIU, L.; YUAN, Y. and LI, S.

[2020] Dense crystalline packings of ellipsoids. Phys. Rev. E 95, 033003 DOI 10.1103/PhysRevE.95.033003

KABATJANSKIĬ, G. A. and LEVENŠTEĬN, V. I.

[1978] Bounds for packings on the sphere and in space. (Russian) Problemy Peredachi Informacii 14 (1978) no. 1, 3–25. English translation: Problems of Information Transmission 14 (1978), no. 1, 1–17. MR0514023

KOLUSHOV, A. V. and YUDIN, V. A.
1994] On the Korkin-Zolotarev construction. (Russian) Diskret. Mat. 6 (1994) no. 1, 155–157; translation in Discrete Math. Appl. 4 (1994) no. 2, 143–146. MR1273240

1997] Extremal dispositions of points on the sphere. Anal. Math. 23 (1997) no. 1, 25–34. MR1630001, DOI 10.1007/BF02789828

Korkine, A. and Zolotareff, G.

1872] Sur les formes quadratiques positives quaternaires. Math. Ann. 5 (1872) no. 4, 581–583. MR1509795 DML, DOI 10.1007/BF01442912

1877] Sur les formes quadratiques positives. Math. Ann. 11 (1877) no. 2, 242–292. MR1509914 DML, DOI 10.1007/BF01442667

Kusner, W.

2014] Upper Bounds on packing density for circular cylinders with high aspect ratio. Discrete. Comput. Geom. 51 (2014) 964–978. MR3216672, DOI 10.1007/s00454-014-9593-6

2016] On the densest packing of polycylinders in any dimension. Discrete Comput. Geom. 55 (2016) no. 3, 638–641. MR3473671, DOI 10.1007/s00454-016-9766-6

De Laat, D. and Vallentin, F.

2016] A breakthrough in sphere packing: the search for magic functions. Nieuw Arch. Wiskd. (5) 17 (2016), no. 3, 184–192. MR3643686

Leech, J.

1964] Some sphere packings in higher space. Canadian J. Math. 16 (1964) 657–682. MR0167901, DOI 10.4153/CJM-1964-065-1

1967] Notes on sphere packings. Canadian J. Math. 19 (1967) 251–267. MR0209983, DOI 10.4153/CJM-1967-017-0

Leech, J. and Sloane, N. J. A.

1975] Sphere packing and error-correcting codes. Canadian J. Math. 23 (1971) no. 4, 718–745. MR0364368

Levenšteĭn, V. I.

1975] The maximal density of filling an n-dimensional Euclidean space with equal balls. (Russian) Mat. Zametki 18 (1975) no. 2, 301–311. English translation in Math. Notes 18 (1975) no. 1–2, 765–771. MR0397565, DOI 10.1007/BF01818046

1979] Bounds for packings in n-dimensional Euclidean space. (Russian) Dokl. Akad. Nauk SSSR 245 (1979) no. 6, 1299–1303. English translation in Soviet Math. Dokl. 20 (1979) 417–421. MR0529659

1983] Bounds for packings of metric spaces and some of their applications. (Russian) Problemy Kibernet. No. 40 (1983) 43–110. MR0717357

1992] Designs as maximum codes in polynomial metric spaces. Acta Appl. Math. 29 (1992) 1–82. MR1192833, DOI 10.1007/BF00053379

1998] Universal bounds for codes and designs. Handbook of coding theory, Vol. I, II, 499–648, North-Holland, Amsterdam, 1998. MR1667942

Leech, J.

1957] Equilibrium of sets of particles on a sphere. Math. Gaz. 41 (1957) 81–90. MR0086325, DOI 10.2307/3610579

Li, L. and Xing, C. P.

2008] Packing superballs from codes and algebraic curves. Acta Math. Sin. (Engl. Ser.) 24 (2008) no. 1, 1–6. MR2384226, DOI 10.1007/s10114-007-9213-3

Lovász, L.

1975] On the ratio of optimal integral and fractional covers, Discrete Math. 13 (1975) no. 4, 383–390. MR0384578, DOI 10.1016/0012-365X(75)90058-8

McMullen, P.

1980] Convex bodies which tile space by translation. Mathematika 27 (1980) no. 1, 113–121. MR0582003, DOI 10.1112/S0025579300010007

Melnikov, P.; Knop, O. and Smith, W. R.

1977] Extremal arrangements of points and unit charges on a sphere: equilibrium configurations revisited. Canadian Journal of Chemistry 55 (1977) no. 10, 1745–1761. MR0444497, DOI 10.1139/v77-246

Mittelmann, H. D. and Vallentin, F.
[2010] High-accuracy semidefinite programming bounds for kissing numbers. Experiment. Math. 19 (2010) no. 2, 175–179. MR2676746, DOI 10.1080/10586458.2010.10129070

Molnár, J.
[1963] Estensione del teorema di Segre-Mahler allo spazio. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 35 (1963) 166–168. MR0166688

Musin, O. R.
[2006b] The one-sided kissing number in four dimensions. Period. Math. Hungar. 53 (2006) no. 1-2, 209–225. MR2286472, DOI 10.1007/s10998-006-0033-0

Naszódi, M.
[2016a] On some covering problems in geometry. Proc. Amer. Math. Soc. 144 (2016), no. 8, 3555–3562. MR3503722, DOI 10.1090/proc/12992

[2018] Flavors of translative coverings. New trends in intuitive geometry, 335–358, Bolyai Soc. Math. Stud., 27, János Bolyai Math. Soc., Budapest, 2018. MR3889267, DOI 10.1007/978-3-662-57413-3_14

Odlyzko, A. M. and Sloane, N. J. A.
[1979] New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. J. Combin. Theory Ser. A 26 (1979) no. 2, 210–214. MR0530296, DOI 10.1016/0097-3165(79)90074-8

de Oliveira Filho, F. M. and Vallentin, F.
[2018] Computing upper bounds for the packing density of congruent copies of a convex body. New trends in intuitive geometry, 155–188, Bolyai Soc. Math. Stud., 27, János Bolyai Math. Soc., Budapest, 2018. MR3889260, DOI 10.1007/978-3-662-57413-3_7

Ordentlich, O.; Regev, O. and Weiss, B.
[2020] New bounds on the density of lattice coverings. J. Amer. Math. Soc. DOI 10.1090/jams/984

Pfender, F.
[2007] Improved Delsarte bounds for spherical codes in small dimensions. J. Combin. Theory Ser. A 114 (2007) no. 6, 1133–1147. MR2337242, DOI 10.1016/j.jcta.2006.12.001

Rankin, R. A.
[1947] On the closest packing of spheres in n dimensions. Ann. Math. 48 (1947) 1062–1081. MR0074013, DOI 10.2307/1969393

[1949a] On sums of powers of linear forms. I. Ann. of Math. (2) 50 (1949) no. 3, 691–698. MR0031499, DOI 10.2307/1969556

[1949b] On sums of powers of linear forms. II. Ann. of Math. (2) 50 (1949) no. 3, 699-704. MR0031500, DOI 10.2307/1969557

[1955] The closest packing of spherical caps in n dimensions. Proc. Glasgow Math. Assoc. 2 (1955) 139–144. MR0074013, DOI 10.1017/S00255793000033219

Rogers, C. A.
[1957] A note on coverings. Mathematika 4 (1957) 1–6. MR0090824, DOI 10.1112/S0025579300001030

[1959] Lattice coverings of space. Mathematika 6 (1959) 33–39. MR0124820, DOI 10.1112/S0025579300000190X

[1963] Covering a sphere with spheres. Mathematika 10 (1963) 157–164. MR0166687, DOI 10.1112/S0025579300004083

[1964] Packing and covering. Cambridge Tracts in Mathematics and Mathematical Physics, No. 54 Cambridge University Press, New York 1964 viii+111 pp. MR0172183

Rogers, C. A. and Shephard, G. C.
[1957] The difference body of a convex body. Arch. Math. (Basel) 8 (1957) 220–233. MR0092172, DOI 10.1007/BF01899997

Rolfes, J. H. and Vallentin, F.
[2018] Covering compact metric spaces greedily. Acta Math. Hungar. 155 (2018) no. 1, 130–140. MR3813630, DOI 10.1007/s10474-018-0829-4

Rush, J. A.
[1989] A lower bound on packing density. Invent. Math. 98 (1989) no. 3, 499–509. MR1022304, DOI 10.1007/BF01393834
[1991] Constructive packings of cross polytopes. Mathematika 38 (1991) no. 2, 376–380. MR1147836 DOI 10.1112/S0025579300006719

[1992] Thin lattice coverings. J. London Math. Soc. (2) 45 (1992) no. 2, 193–200. MR1171547, DOI 10.1112/jlms/s2-45.2.193

[1993] A bound, and a conjecture, on the maximum lattice-packing density of a superball. Mathematika 40 (1993) no. 1, 137–143. MR1239136, DOI 10.1112/S0025579300013760

[1994] An indexed set of density bounds on lattice packings. Geom. Dedicata 53 (1994) no. 2, 217–221. MR1307294, DOI 10.1007/BF01264023

[1996] Lattice packing of nearly-euclidean balls in spaces of even dimension. Proc. Edinburgh Math. Soc. (2) 39 (1996) no. 1, 163–169. MR1375676, DOI 10.1017/S0013091500022872

Rush, J. and Sloane, N. J. A.

[1987] An improvement to the Minkowski-Hlawka bound for packing superballs. Mathematika 34 (1987) no. 1, 8–18. MR0908835, DOI 10.1112/S0025579300013231

Ryškov, S. S. and Baranovskii, E. P.

[1975] Solution of the problem of the least dense lattice covering of five-dimensional space by equal spheres. (Russian) Dokl. Akad. Nauk SSSR 222 (1975) no. 1, 39–42. translation in: Soviet Math. Dokl. 16 (1975) 586–590. MR0427238

[1976] C-types of n-dimensional lattices and 5-dimensional primitive parallelohedra (with application to the theory of coverings). (Russian) Trudy Mat. Inst. Steklov. 137 (1976) 3–131. MR0469874; English translation by R. M. Erdahl in Proc. Steklov Inst. Math. 1978, no. 4, 140 pp. MR0535314

Sah, A.; Sawhney, M.; Stoner, D. and Zhao, Y.

[2020] Exponential improvements for superball packing upper bounds. Adv. Math. 365 (2020) 107056, 9 pp. MR4064778, DOI 10.1016/j.aim.2020.107056

Sardari, N. T. and Zargar, M.

[2020] A new upper bound for spherical codes. arXiv:2001.00185v1 [math.MG] 1 Jan 2020.

Schürmann, A.

[2002b] Dense ellipsoid packings. Discrete Mathematics 247 (2002) 243–249. MR1893033, DOI 10.1016/S0012-365X(01)00313-2

Schönenberg, I. J.

[1993] Positive definite functions on spheres. Duke Math. J. 9 (1942) 96–107. MR0005922, DOI 10.1215/S0012-7094-42-00908-6

Schürmann, A. and Vallentin, F.

[2006] Computational approaches to lattice packing and covering problems. Discrete Comput. Geom. 35 (2006) no. 1, 73–116. MR2183491, DOI 10.1007/s00454-006-01202-2

Schwartz, R. E.

[2013] The five-electron case of Thomson’s problem. Exp. Math. 22 (2013) no. 2, 157–186. MR3047910, DOI 10.1080/10586458.2013.765670

[2018] The phase transition in five point energy minimization. arXiv:1610.03303v3 [math.OC] 21 Nov 2016

[2020] Five point energy minimization: A synopsis. Constr. Approx. 51 (2020) no. 3, 537–564. MR4093114, DOI 10.1007/s00365-020-09500-7

Shannon, C. E.

[1959] Probability of error for optimal codes in a Gaussian channel. Bell System Tech. J. 38 (1959) 611–656. MR0103137, DOI 10.1002/j.1538-7305.1959.tb03905.x

Sidelnikov, V. M.

[1973] The densest packing of balls on the surface of the n-dimensional Euclidean sphere, and the number of vectors of a binary code with prescribed code distance. (Russian) Dokl. Akad.
20 G. FEJES TÓTH

Nauk SSSR 213 (1973) 1029–1032. English translation in Soviet Math. Dokl. 14 (1973) 1851–1855. MR0334000

1974 New estimates for the closest packing of spheres in $n$-dimensional Euclidean space. (Russian) Mat. Sb. (N.S.) 95 (137) (1974) 148–158, 160. English translation in Sb. Math. 24 (1974), no. 1, 147–157. MR0362060

TORQUATO, S. and JIAO, Y.

2009a Dense packings of the Platonic and Archimedean solids. Nature 460 (2009) 876–879. DOI 10.1038/nature08239

2009b Dense packings of polyhedra: Platonic and Archimedean solids. Phys. Rev. E (3) 80 (2009) no. 4, 041104, 21 pp. MR2607418, DOI 10.1103/PhysRevE.80.041104

TUMANOV, A.

2013 Minimal biquadratic energy of five particles on a 2-sphere. Indiana Univ. Math. J. 62 (2013) no. 6, 1717–1731. MR3205529, DOI 10.1512/iumj.2013.62.5148

VANCE, S.

2011 Improved sphere packing lower bounds from Hurwitz lattices. Advances in Mathematics 227 (2011) 2144–2156. MR2803798, DOI 10.1016/j.aim.2011.04.016.

VENKATESH, A.

2013 A note on sphere packings in high dimension. Int. Math. Res. Not. IMRN 2013, no. 7, 1628–1642. MR3044452, DOI 10.1093/imrn/rns096.

VENKOV, B. A.

1954 On a class of Euclidean polyhedra. (Russian) Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. 9 (1954) no. 2, 11–31. MR0094790

VERGER-GAUGRY, J.-L.

2005 Covering a ball with smaller equal balls in $R^n$. Discrete Comput. Geom. 33 (2005) no. 1, 143–155. DOI 10.1007/s00454-004-2916-2

VIAZOVSKA, M.

2017 The sphere packing problem in dimension 8. Ann. of Math. (2) 185 (2017) no. 3, 991–1015. MR3664816, DOI 10.4007/annals.2017.185.3.7

2018 Sharp sphere packings. Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, 455–466, World Sci. Publ., Hackensack, NJ, 2018. MR3966775 DOI 10.1142/9789813272880_0063

WILLS, J. M.

2017 An ellipsoid packing in $E^3$ of unexpected high density. Mathematika 38 (1991) no. 2, 318–326. DOI 10.1112/S00255793000006653

WYNER, A. D.

1965 Capabilities of bounded discrepancy decoding. Bell Systems Tech. J. 44 (1965) 1061–1122. MR0180417, DOI 10.1002/j.1538-7305.1965.tb04170.x

YUDIN, V. A.

1992 Minimum potential energy of a point system of charges. (Russian) Diskret. Mat. 4 (1992) no. 2, 115–121; translation in Discrete Math. Appl. 3 (1993) no. 1, 75–81. MR1181534 DOI 10.1515/dma.1993.3.1.75

ZONG, C.

1999b Sphere packings. Universitext. Springer-Verlag, New York, 1999. xiv+241 pp. ISBN: 0-387-98794-0. MR1707318, DOI 10.1007/b98975

2014b On the translative packing densities of tetrahedra and cuboctahedra. Adv. Math. 260 (2014) 130–190. MR3209351, DOI 10.1016/j.aim.2014.04.009

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, RÉÁLTANODA U. 13-15., H-1053, BUDAPEST, HUNGARY

Email address: gfejes@renyi.hu