SOME NON-HYPERBOLIC SYSTEMS WITH STRICTLY NON-ZERO
LYAPUNOV EXPONENTS FOR ALL INVARIANT MEASURES:
HORSESHOES WITH INTERNAL TANGENCIES

YONGLUO CAO, STEFANO LUZZATTO, AND ISABEL RIOS

Abstract. We study the hyperbolicity of a class of horseshoes exhibiting an internal tangency, i.e., a point of homoclinic tangency accumulated by periodic points. In particular these systems are strictly not uniformly hyperbolic. However we show that all the Lyapunov exponents of all invariant measures are uniformly bounded away from 0. This is the first known example of this kind.

1. Introduction

1.1. Hyperbolicity and tangencies. We consider $C^2$ diffeomorphisms on Riemannian surfaces. Our goal is to study the hyperbolic properties of a class of maps exhibiting a homoclinic tangency associated to a fixed saddle point $S$, as in Figure 1. We assume without loss of generality that we are working on $\mathbb{R}^2$ and in the standard Euclidean norm. We recall that a compact invariant set $\Lambda$ is uniformly hyperbolic if there exist constants $C > 0$, $\sigma > 1 > \lambda > 0$ and a continuous, $D\Phi$-invariant, decomposition $T_x\Lambda = E^s_x \oplus E^u_x$ of the tangent bundle over $\Lambda$ such that for all $x \in \Lambda$ and all $n \geq 1$ we have

\begin{equation}
\|D\Phi^n|_{E^s_x}\| \leq C\lambda^n \quad \text{and} \quad \|D\Phi^n|_{E^u_x}\| \geq C^{-1}\sigma^n.
\end{equation}

By standard hyperbolic theory, every point $x$ in $\Lambda$ has stable and unstable manifolds $W^s_x, W^u_x$ tangent to the subspaces $E^s_x$ and $E^u_x$ respectively, and thus in particular transversal to each

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other. The presence of the tangency therefore implies that the dynamics on $\Lambda$ cannot be uniformly hyperbolic.

We emphasize at this point that, in the case we are considering, the homoclinic tangency is accumulated by transverse homoclinic orbits which in turn are accumulated by periodic points. Thus it constitutes an intrinsic obstruction to uniform hyperbolicity which cannot be resolved by simply ignoring the orbit of tangency. Most of the classical theory of homoclinic bifurcations for diffeomorphisms (see [11] and references therein) considers the unfolding of homoclinic tangencies external to the set $\Lambda$ to which they are associated, thus causing no real issues with the hyperbolicity at the bifurcation parameter. The main goal of such a theory has often been to study the hyperbolicity and the occurrence of tangencies in a neighbourhood of the orbit of tangency after the bifurcation. The presence of an internal tangency gives rise to a more subtle situation and it has only recently been shown that this can actually occur as a first bifurcation [2,3,6,8,12]. Part of the motivation of the present paper is to study the global dynamics and hyperbolicity at this bifurcation parameter.

1.2. Hyperbolicity and Lyapunov exponents. Another part of the motivation for this result is to give an example of a compact invariant set which is as “uniformly” hyperbolic as possible in the ergodic theory sense, but still not uniformly hyperbolic. To formulate this result precisely, let $M(\Phi)$ denote the set of all $\Phi$-invariant probability measures $\mu$ on $\Lambda$. By the classical Multiplicative Ergodic Theorem of Oseledet’s there is a well defined set $L(\mu)$ of Lyapunov exponents associated to the measure $\mu$, we give the precise definitions below. We say that the measure $\mu \in M(\Phi)$ is hyperbolic if all the Lyapunov exponents are non-zero. The existence of an invariant measure with non-zero Lyapunov exponents indicates a minimum degree of hyperbolicity in the system. A stronger requirement is that all invariant measures $\mu$ are hyperbolic and of course an even stronger requirement is that they are all “uniformly” hyperbolic in the sense that all Lyapunov exponents are uniformly bounded away from 0. This condition is clearly satisfied for uniformly hyperbolic systems but, as we show in this paper, it is strictly weaker.

The class of examples we are interested in were first introduced in [12] and constitute perhaps the simplest situation in which an internal tangency can occur as a first bifurcation. In section 2 we give the precise definition of this class. For this class we shall then prove the following

**Theorem.** All Lyapunov exponents of all measures in $M(\Phi)$ are uniformly bounded away from zero.

As an immediate corollary we have the following statement which is in itself non-trivial and already remarkable.

**Corollary 1.** $\Phi$ is uniformly hyperbolic on periodic points.

We recall that uniform hyperbolicity on periodic points means that there exists constants $\sigma > 1 > \lambda > 0$ such that for each periodic point $p \in \Lambda$ of period $k$, the derivative $D\Phi^k_p$ has two distinct real eigenvalues $\tilde{\sigma}, \tilde{\lambda}$ with $|\tilde{\sigma}| > \sigma^k > 1 > |\tilde{\lambda}| > 0$. Notice that the bounds for the eigenvalues are exponential in $k$.

As far as we know this is the first known example of this kind, although it is possible, and indeed even likely, that such a property should hold for more complex examples such as Benedicks-Carleson and Mora-Viana parameters in Hénon-like families [4,9] and horseshoes at the boundary of the uniform hyperbolicity domain of the Hénon family. The weaker result
on the uniform hyperbolicity of periodic points has been proved recently for both cases in [13] and [3] respectively. Other known examples of non-uniformly hyperbolic diffeomorphisms include cases in which the lack of uniformity comes from the presence of “neutral” fixed or periodic points. In these cases, the Dirac-$\delta$ measures on such periodic orbits are invariant and have a zero Lyapunov exponent.

It is interesting to view our result in the light of some recent work which appears to go in the opposite direction: if a compact invariant set $\Lambda$ admits an invariant splitting $T_x\Lambda = E^1_x \oplus E^2_x$ such that, for a total measure set of points $x \in \Lambda$, the Lyapunov exponents are positive in $E^1_x$ and negative in $E^2_x$, then $\Lambda$ is uniformly hyperbolic. [1, 5, 7]. Here, the Lyapunov exponents are not even required to be uniformly bounded away from zero. Thus the existence of at least one orbit as which the splitting degenerates is a necessary condition for a situation such as the one we are considering, in which the Lyapunov exponents are all non-zero but $\Lambda$ is strictly not uniformly hyperbolic.

The concept of uniform hyperbolicity of the periodic points and of measures plays an important role in the general theory of one-dimensional dynamics. In some situations, such as for certain classes of smooth non-uniformly expanding unimodal maps these notions have been shown to be equivalent to each other and to various other properties usually associated to uniform hyperbolicity such as exponential decay of correlations [10]. Higher dimensional cases are generally much harder due to the complexity of the geometrical and dynamical structure of the examples and progress is only starting to be made.

1.3. Overview of the paper. After defining the class of systems of interest in section 2, in section 3 we construct a field of cones which are invariant under the derivative map. In Section 4 we show that some non-uniform hyperbolicity is satisfied off the orbit of tangency: the expanding and contracting directions of points which are very close to the orbit of tangency are almost aligned and therefore the contraction and/or expansion may take arbitrarily long time to become effective. We formalize this by saying that there exists a constant $C_x > 0$, depending on the point $x$ (compare with (1)), such that for all $x$ not in the orbit of tangency we have

$$\|D\Phi^n|_{E^2_x}\| \leq C_x \lambda^n \quad \text{and} \quad \|D\Phi^n|_{E^1_x}\| \geq C_x^{-1} \sigma^n.$$  

with $C_x$ arbitrarily small near the points of the orbit of tangency, i.e. $\inf_{x \in \Lambda} C_x = 0$. One way to show that the non-uniformity is not too extreme is to show that there is a uniformly hyperbolic core in the following sense: there exists a region $W$ containing one point of the orbit of tangency, and there exists constants $\sigma > 1 > \lambda > 0$ such that, for each point $p \in \Lambda \cap W$ that returns to $W$ at the $k$-th iterate, we have

$$\|D\Phi^k|_{E^2_p}\| \leq \lambda^n \quad \text{and} \quad \|D\Phi^k|_{E^1_p}\| \geq \sigma^n.$$  

In fact, we show that this uniform hyperbolicity occurs well before the return of the orbit of $p$ to $W$. Those exponential estimates are valid from the moment that the orbit of $p$ leaves a certain neighborhood of the fixed point $S$, and they keep working as long as the orbit wanders around $\Lambda \setminus W$. Finally, in section 5 we show that this implies the statement of our main theorem.

2. HORSESHOES WITH INTERNAL TANGENCIES

We start with a geometrical definition of the class of maps under consideration. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^2$ diffeomorphism and let $Q = [0,1] \times [0,1]$ denote the unit square in $\mathbb{R}^2$. 


We suppose that there exist 5 “horizontal” regions in $Q$ which are mapped as in Figure 2: $R_i = \Phi(R_i)$ for $i = 1, \ldots, 5$. We suppose that regions $R_1, R_3, R_5$ are mapped affinely to their images, with derivative

$$D\Phi(x, y) = \begin{pmatrix} \lambda & 0 \\ 0 & \sigma \end{pmatrix} \quad \text{and} \quad D\Phi(x, y) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\sigma \end{pmatrix}$$

in $R_1 \cup R_3$ and $R_5$ respectively, for two constants $\lambda, \sigma$ satisfying $\sigma > 3$ and $\lambda < 1/3$. More explicitly we suppose that $\Phi(x, y) = (\lambda x, \sigma y)$ in $R_1 \ (0 \leq y \leq \sigma^{-1})$, $\Phi(x, y) = (\lambda x + (1 - \lambda), \sigma y - (\sigma - 1))$ in $R_5 \ (1 - 2/3\sigma \leq y \leq 1)$ and that $R'_5$ is a vertical strip parallel to $R_5$. In particular there is a hyperbolic fixed point at the origin $(0, 0)$ whose stable and unstable manifolds contain the lower and left hand side of the square $Q$ respectively.

We emphasize that the explicit nature of the map in these regions is for simplicity only, and to allow us to concentrate on the strategy for dealing with the tangency. This could be weakened significantly, for example by assuming only some uniformly hyperbolic structure in regions $R_1, R_3, R_5$. Region $R_2$ is mapped outside the square $Q$ and thus we do not need to make any particular assumption on its form.

Region 4 is mapped to a “fold” $R'_4$ which contains a point of the orbit of tangency. Here we make some “non-degeneracy assumptions” essentially stating that vertical lines in $R_4$ are mapped to non-degenerate parabolas. More precisely, we suppose that there is a region $\hat{R}_4 \subset R_4$ bounded by two disjoint curves so that points in $R_4 \setminus \hat{R}_4$, are mapped inside region $R_2$ with second coordinate greater than $\sigma^{-1}$. Then, for each $x_0$ in $[0, 1]$ we have that $\Phi((x_0, y) : y \in \mathbb{R}) \cap \hat{R}_4)$ is contained in the graph of the map

$$f_{x_0}(x) = c(x - q)^2 - \lambda x$$

for some fixed $c > 0$ and some $q \in (2/3, 1)$. Thus all vertical lines in $\hat{R}_4$ are mapped to parabolas with constant curvature $c$. Again, this could be weakened significantly although we do assume for less trivial reasons that the curvature $c$ is sufficiently large in relation to the constants $\lambda$ and $\sigma$. Notice that the point $(q, 0)$ is a point of tangency of the stable and unstable manifolds of the fixed point at the origin. To control the global properties of the
family of parabolas we also assume that
\[ \frac{\partial \Phi}{\partial y}(0, y) \geq \sigma \quad \text{and} \quad \frac{\partial \Phi}{\partial x}(\Phi^{-1}(q, 0)) = \lambda \]
and that, for every \((x, y) \in \hat{R}_4\),
\[ \left\langle \frac{\partial \Phi}{\partial y}(x, y), \frac{\partial \Phi}{\partial x}(x, y) \right\rangle = 0 \]
A completely explicit example of a map \(\Phi\) satisfying all these conditions is given in [12]. We now define the limit set
\[ \Lambda = \bigcap_{n \in \mathbb{Z}} \Phi^n(Q). \]
It follows from the construction that \(\Lambda\) is non empty and contains (at least) one orbit of tangency between stable and unstable manifolds. Our assumptions make it easy to control the dynamics in regions \(R_1, R_3, R_5\) and therefore our main objective is to control the dynamics of points returning to the fold \(\hat{R}_4\) and to a neighbourhood of the tangency. Thus we concentrate on the first return map from \(\hat{R}_4\) to itself and show that this map satisfies some strong hyperbolicity estimates from which we can deduce our main estimate on the hyperbolicity of periodic points.

### 3. The Conefield

Consider the foliation \(\mathcal{F}\), of \(\Phi(Q)\), whose leaves are images of vertical lines by the map \(\Phi\) (in \(\hat{R}_4\), the leaves are parabolas). To each point \((x, y)\) we associate the tangent direction, \(l(x, y)\), to the leaf of \(\mathcal{F}\) which passes through \((x, y)\), at this point \((l(x, y)\) is parallel to \(\frac{\partial \Phi}{\partial x}(\Phi^{-1}(x, y))\).

Let \(a(x, y)\) be the angle between \(l(x, y)\) and the horizontal. There is only one point for which \(a = 0\) and this is the point of tangency \((q, 0)\). At all other points we have with \(0 < a(x, y) \leq \pi/2\). Then, we divide \(\Lambda\) into 3 regions, according to the value of \(a\):
\[ L = \{a = \pi/2\}, \quad R = \{\pi/3 < a < \pi/2\}, \quad V = \{0 < a \leq \pi/3\}. \]

Notice that
\[ \Lambda \cap (R'_4 \cup R'_3 \cup R'_5) = L \quad \text{and} \quad V \cup R = \Lambda \cap (R'_4 \setminus (q, 0)) \]

For each point \(P \in R \cup V\) we define a cone \(C(P)\) of vectors in the tangent space \(T_P\mathbb{R}^2\) as follows: let \(P = (\xi, \eta)\), and \(P_i = \Phi^i(P) = (\xi_i, \eta_i)\) for any integer \(i\). All cones will be centered on the vertical and, for convenience, we define a standard cone
\[ C = \left\{ (v_1, v_2) \in \mathbb{R}^2 : \frac{|v_1|}{|v_2|} \leq \sqrt{3} \right\}. \]

For \(P \in R\) we simply let \(C(P) = C\). For \(P \in V\), we define cones which extend to each side of the vertical by three times the (cotangent of the) angle of the line \(l(P)\) defined above. More precisely, recall that \(l(P)\) is tangent to the graph of \(f(x) = c(x - q)^2 - \xi - 1\), see (4), and therefore \(l(P) = \{(v_1, v_2) \in \mathbb{R}^2 : v_2 = \pm 2v_1 \sqrt{c(\eta + \lambda \xi - 1)}\}.\) Then we let
\[ C(P) = \left\{ (v_1, v_2) \in \mathbb{R}^2 : \frac{|v_1|}{|v_2|} \leq \frac{3}{2 \sqrt{c(\eta + \lambda \xi - 1)}} \right\}. \]

Before defining the cones in the remaining points in \(L\) we show that the conefield defined above in \(R \cup V\) is invariant under the first return map to \(R \cup V\). As a first step towards proving this invariance we estimate the first return time.
Lemma 1. Consider $P = (\xi, \eta) \in R \cup V$, and $n$ the smallest positive integer such that $P_n \in R \cup V$. Then

$$n \geq \max \left\{ \frac{\ln 1/\eta}{\ln \sigma}, \frac{\ln 1/\xi}{\ln \lambda^{-1}} \right\}. \tag{6}$$

Proof. Note that, since $V \subset R_1$, we have $\Phi^i(P) = (\lambda^i \xi, \sigma^i \eta)$, provided that $\sigma^{i-1} \eta \leq \sigma^{-1}$. In fact, the images $P_i$ of the point $P$ will stay in region $R_1$ while $i$ is such that $\eta \sigma^{-i} < \sigma^{-1}$. Since $P_{n-1} \in R_4$, there must be at least one more linear iterate of $P_i$ before its orbit visits the pre-image of $R \cup V$. So, we have $\eta \sigma^{n-1} > \sigma^{-1}$. Analogously, using the inverse map, we have $n \geq \frac{\ln 1/\xi}{\ln \lambda^{-1}}$. \hfill \Box

We shall show below that these estimates mean that cones are sufficiently contracted before returning, guaranteeing that they are mapped back strictly into existing cones even if they started out very wide. In particular we shall use the following simple

Corollary 2. Consider $P = (\xi, \eta) \in R \cup V$, and $n$ the smaller positive integer such that $P_n \in R \cup V$. Then for this $n$ we have

$$\frac{\lambda^n}{\sigma^n} < \eta^{1-\ln \lambda \ln \sigma} \tag{7}$$

and

$$\frac{\lambda^n}{\sigma^n} < (\lambda \xi)^{-1-\ln \eta \ln \sigma}. \tag{8}$$

Proof. The first inequality follows from (3) which gives

$$\frac{\lambda^n}{\sigma^n} < \left( \frac{\lambda}{\sigma} \right)^{\ln 1/\eta \ln \sigma} = \left( \frac{1}{\eta} \right)^{\ln \lambda \ln \sigma} \left( \frac{1}{\sigma} \right)^{\ln 1/\xi \ln \sigma} = \eta^{1-\ln \lambda \ln \sigma}. $$

The second follows also by (4) by a similar straightforward calculation. \hfill \Box

We are now ready to prove the invariance of the conefield for the first return map.

Lemma 2. There exists $c_1 > 0$ such that, if $c > c_1$ in the definition of $\Phi$, $P \in V \cup R$ and $P_n$ is the first positive iterate of $P$ in $R \cup V$, then

$$D\Phi^n(P)(C(P)) \subset C(P_n).$$

Proof. Since $C(P)$ is centered in the vertical line, we have, after applying

$$D\Phi^{n-1}(P) = \begin{pmatrix} \pm \lambda^{n-1} & 0 \\ 0 & \pm \sigma^{n-1} \end{pmatrix}$$

to the vectors of $C(P)$, that $D\Phi^n(P)(C(P))$ is a cone centered in the vertical line at the point $P_{n-1}$ whose width is $\frac{\lambda^{n-1}}{\sigma^{n-1}}$ times the width of $C(P)$. After applying $D\Phi$ to this cone, since $P_{n-1} \in R_4$, we obtain a cone centered in $l(P_n)$ with width smaller than $\frac{\lambda^n}{\sigma^n}$ times the original width.

There are many cases to be considered depending on the location of $P$ and $P_n$ in $R \cup V$. Here we present the case where $P$ and $P_n$ are both in $V$. The other cases are made following the same steps. Let $P$ and $P_n$ be in $V$. Then

$$C(P) = \left\{ (v_1, v_2) \in \mathbb{R}^2 : \left| \frac{v_1}{v_2} \right| \leq \frac{3}{2\sqrt{c(\eta + \lambda \xi^{-1})}} \right\}.$$
and
\[ C(P_n) = \left\{ (v_1, v_2) \in \mathbb{R}^2 : \frac{|v_1|}{|v_2|} \leq \frac{3}{2\sqrt{c(\eta_n + \lambda \xi_{n-1})}} \right\}. \]

Since \( 0 < a(P_n) < \pi/3 \), it satisfies
\[ \arctan \left( \frac{\tan a(P_n)}{3} \right) < \frac{a(P_n)}{2}. \]
This implies that a cone centered in \( l(P_n) \) with width \( 2\sqrt{c(\eta_n + \lambda \xi_{n-1})}/3 \) will be properly contained in \( C(P_n) \). Then, in order to obtain \( D\Phi^n(P)(C(P)) \subset C(P_n) \) we need
\[ \frac{\lambda^n}{\sigma^n} \frac{3}{2\sqrt{c(\eta_n + \lambda \xi_{n-1})}} < \frac{2\sqrt{c(\eta_n + \lambda \xi_{n-1})}}{3} \]
or, equivalently,
\[ \frac{\lambda^n}{\sigma^n} < \frac{4c(\eta_n + \lambda \xi_{n-1})}{9}. \]

We now distinguish two cases. In the case where \( \eta \leq \lambda \xi_{n-1} \) we have \( \eta < \sqrt{(\eta_n + \lambda \xi_{n-1})} \sqrt{(\eta + \lambda \xi_{-1})} \) and therefore it is enough to show that \( \lambda^n/\sigma^n < 4c\eta/9 \). By (7) this follows as long as we have
\[ \eta^{\frac{1 - \ln \lambda}{m\sigma}} \leq \frac{4c\eta}{9}, \]
for all \( \eta \leq 1 \) (remember that we are working with points of \([0, 1] \times [0, 1]\)). Since \( \ln \lambda/\ln \sigma \) is fixed and negative, the condition (10) holds if \( c \) is big enough.

In the case where \( \eta > \lambda \xi_{n-1} \) we argue in the same way and, using (5), reduce the problem to showing that
\[ (\lambda \xi_{n-1})^{\frac{1 - \ln \lambda}{m\sigma}} < \frac{4c\lambda \xi_{n-1}}{9}, \]
for any \( \xi_{n-1} \leq 1 \). Again this follows as long as \( c \) is sufficiently large. The other cases follow analogously. \( \square \)

We can now extend the conefield to the set \( \Lambda \setminus \{ \Phi^i(q,0), i \in \mathbb{Z} \} \), in a natural way by considering the images of all cones in \( R \cup V \) and taking slightly wider cones at each point. In this way we obtain a conefield defined at every point outside the orbit of tangency such that
\[ D\Phi^n(P)(C(P)) \subset C(\Phi(P)) \]
for every point \( P \). Notice that these cones can be arbitrarily wide close to the orbit of tangency. For points \( P \) which do not enter \( R \cup V \) in either forward or backward time, we simply let \( C(P) \) be the standard cone \( C \), see (5). For points \( P \) which intersect \( R \cup V \) only in forward time, we define
\[ C(P) = C \cap D\Phi^{-i}(C(\Phi(P))) \]
where \( i > 0 \) is the first time for which \( \Phi(P) \in R \cup V \) and \( C \) is that standard cone.

The stable cone field is defined assigning to each \( P \in \Lambda \setminus \{ \Phi^i(q,0), i \in \mathbb{Z} \} \), the closure of the complement of \( C(P) \), and satisfies the inclusion condition for the inverse map.
4. Uniform hyperbolicity for the escape time

We fix a neighbourhood $W$ of the tangency point $(q, 0)$ of radius $1/c$. Let $P = (\xi, \eta) \in W$. We start with a simple estimate of the number of iterations $n$ it takes before $P_n$ falls outside the set $[0, 1] \times [0, 1/3]$ ($n$ is the escape time of $P$).

**Lemma 3.** We have $n \geq -\log 3\eta / \log \sigma$.

**Proof.** Note that $W$ is contained in the domain where $\Phi$ is linear, and, as long as the second coordinate of the image of a point of $\Lambda$ is less than $1/3$, it is contained in $R_1$, the second coordinate of $P$ being multiplied by $\sigma$ at each iteration. □

For $P = (\xi, \eta)$ as above, let $v$ be a vector contained in $C(P)$ and $v^n = D\Phi^n_P v$.

**Lemma 4.** We have $\|v^n\| \geq \sigma^n|v_2|$.

**Proof.** As we saw in the last section, since $\Phi^n$ is linear in $W$, we have $\|v^n\| = \|\ell^n(v_1, \sigma^n v_2)\| \geq \sigma^n|v_2|$. □

**Lemma 5.** With the notation above, if $c$ is sufficiently big, then $\frac{\|v^n\|}{\|v\|} \geq \frac{\sigma^{n/2}}{\sqrt{3 + 4c\eta}}$.

**Proof.** Since $v \in C(P)$, we have that

\[(11) \quad \|v\| = \sqrt{v_1^2 + v_2^2} \leq \sqrt{\left(\frac{3}{2\sqrt{c\eta}}\right)^2 + 1|v_2|}
\]

Using lemmas 4 and 3 we obtain

$$\frac{\|v^n\|}{\|v\|} \geq \frac{\sigma^n}{\sqrt{\frac{9}{4c\eta} + 1}} = \frac{\sigma^n2\sqrt{c\eta}}{\sqrt{9 + 4c\eta}}.$$  

By lemma 3, we have that $\sigma^{n/2} \geq \frac{1}{\sqrt{c\eta}}$ and therefore

$$\frac{\|v^n\|}{\|v\|} \geq \frac{2\sqrt{c}\sigma^{n/2}}{\sqrt{3\sqrt{9 + 4c\eta}}}.$$ 

Since $|\eta| < 1/c$, we can write

$$\frac{\|v^n\|}{\|v\|} \geq \frac{2\sqrt{c}\sigma^{n/2}}{\sqrt{39}}.$$ 

The result follows if $c$ is sufficiently large (greater than $\sqrt{39}/4$ in this case). □

5. Lyapunov exponents

Let $W_j$ be the connected component of $\Phi^j(W) \cap R_1$ containing $\Phi^j(q, 0)$, and $\tilde{W} = \bigcup_{j \geq 0} W_j$. In this section we estimate the growth of the unstable vectors of points in $\Lambda$ outside $\tilde{W}$, under the action of $D\Phi$. We also compute bounds for the Lyapunov exponents of $\Phi$, and prove the main result.

**Lemma 6.** There exists $\sigma_1 > 1$ such that, if $P \in \Lambda \setminus \tilde{W}$, and $v \in C(P)$, then $\|D\Phi_P v\| > \sigma_1\|v\|$. 

Proof. First we claim that, if $P \in \Lambda \setminus \tilde{W}$, we have $\mathcal{C}(P) \subset \mathcal{C}$. Indeed, by construction, if $P \in R_\delta$ and at distance at least $1/c$ from the tangency point $(q, 0)$, then we have that $\eta + \lambda \xi_{-1} \geq 1/c$, and the width of $\mathcal{C}(P)$ is less than $3/2 < \sqrt{3}$. Notice that vectors in $\mathcal{C}$ grow by at least a constant factor $\sigma_1 \geq 1$ at each iteration where $\sigma_1 > 1$ is the rate of growth of the vector $(\sqrt{3}, 1)$ by the linear map $(x, y) \mapsto (\lambda x, \sigma y)$. If $P$ is a point outside the set $\tilde{W}$, then we have already $\mathcal{C}(P) = \mathcal{C}$, and the estimate applies to the unstable vectors of $P$. □

Let $\tilde{\sigma} = \min\{\sigma_{n/2}, \sigma_1\}$. Until now we proved that, if the orbit of $P$ visits the set $W$, then when it leaves the set $\tilde{W}$, it has accumulated exponential growth to the unstable vectors by a factor $\sigma_{1/2} \geq \tilde{\sigma}$ as we have seen in lemma 6. As shown above, iterates outside $\tilde{W}$ contribute as well with the same factor. Similar calculations hold for the stable vectors, and we define $\lambda$ through the analogous estimates for $\Phi^{-1}$. We remark that in particular these estimates improve significantly on the growth estimates of [12], which already imply the existence of a non-uniformly hyperbolic splitting as defined in the introduction.

Lemma 7. If $z$ does not belong to the orbit of homoclinic tangency, $v \in E^u_z$, and $w \in E^s_z$, then there exist sequences $n_k \to \infty$ and $n_1 \to \infty$, and a positive number $c(z)$ such that $\|D\Phi^{k}v\| > c(z)\tilde{\sigma}^{n_k}\|v\|$ and $\|D\Phi^{-n_k}w\| > c(z)\tilde{\sigma}^{-n_k}\|w\|$.

In particular, $\limsup_{n \to \infty} \frac{\log \|D\Phi^n\|}{n} \geq \log \tilde{\sigma}$ and $\liminf_{n \to \infty} \frac{\log \|D\Phi^n\|}{n} \leq \log \lambda$.

Proof. Let $z$ be a point in $\Lambda$ outside the orbit of homoclinic tangency. For $i \in \mathbb{N}$, let $z_i = \Phi^i(z)$ and $v_i = D\Phi^iv$. Let $N(z) = \{i \in \mathbb{N} : z_i \in W\}$. If $N(z)$ is non-empty and infinite, let $N(z) = \{i_1, i_2, i_3, \ldots\}$ and consider $n_k$ as the smaller natural number bigger than $i_k$ such that $z_{n_k} \notin R_1$. In this case, we have $i_k < n_k < i_{k+1}$, for any $k \in \mathbb{Z}$. As a consequence of the estimates in lemmas 6 and 6, we have that

$$\|v_{n_k}\| > \tilde{\sigma}^{n_k-i_k}\|v_k\|,$$

and hence

$$\|v_{n_k}\| > \tilde{\sigma}^{n_k}\|v_{i_k}\|.$$  

Taking $c(z) = \|v\|/\|v_{i_k}\|$, the result follows in this case.

If $N(z)$ is finite, $i_1 = \sup N(z)$, then for all $n \in \mathbb{N}$ larger than $i_1$, we have $\|v_{n}\| > \tilde{\sigma}^{i_1-i}\|v_{i_1}\|$, and the lemma is true for $z$, choosing $c(z) = \|v\|/\|v_{i_1}\|$.

If $N(z)$ is empty, then, there are three cases to consider. If the orbit of $z$ is not in $\tilde{W}$, then we have exponential growth beginning at the first iterate. If $z \in \tilde{W}$ and leaves it eventually, we wait until it does so to have exponential growth, and argue as before. If $z_i \in \tilde{W}$ and never leaves it, then $z_i \in R_1$ for all $i \in \mathbb{N}$, meaning that $z$ is in the local stable manifold of $P = (0, 0)$. In this case, any non-horizontal vector based in $z$ will eventually grow exponentially by a factor $\sigma - \varepsilon > \tilde{\sigma}$.

Similar arguments prove the statement for the stable vectors. □

Now, define $\Lambda' = \Lambda \setminus \{\Phi^n(q,o) : n \in \mathbb{Z}\}$.

Lemma 8. If $\mu$ is an invariant probability measure supported on $\Lambda$, then $\mu(\Lambda') = 1$

Proof. Let $q$ be a point in the orbit of homoclinic tangency, and suppose that $\mu(\{q_i, i \in \mathbb{Z}\}) > 0$. Then, for some $i_0$, we must have $\mu\{q_{i_0}\} > 0$, and the entire orbit would have infinite measure. □
Proof of the theorem. Let $\mu$ be an invariant measure, and $B$ the subset of $\Lambda'$ for which the lyapunov exponents exist. Then, by the Oseledet’s theorem $\mu(B \cap \Lambda') = 1$. By lemma (7) all Lyapunov exponents are outside the interval $(\log \tilde{\lambda}, \log \tilde{\sigma})$, for all $z \in B$. □

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