MAXIMUM PRINCIPLE FOR SEMI-ELLIPTIC TRACE OPERATORS AND GEOMETRIC APPLICATIONS

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ABSTRACT. Based on ideas of L. Alías, D. Impera and M. Rigoli developed in [13], we present a fairly general weak/Omori-Yau maximum principle for trace operators. We apply this version of maximum principle to generalize several higher order mean curvature estimates and to give an extension of Alias-Impera-Rigoli Slice Theorem of [13, Thms. 16 & 21], see Theorems 5, 6.

1. Introduction

The theory of minimal and constant mean curvature hypersurfaces of product spaces \( N \times \mathbb{R} \), where \( N \) is a complete Riemannian manifold, has been developed into a rich theory [30], [40], [41] yielding a wealth of examples and results, see for instance [1], [3], [4], [5], [6], [7], [11], [15], [17], [20], [21] [22], [24], [27], [32] [33], [34] and the references therein.

Recently, the theory minimal and constant mean curvature hypersurfaces started to be developed in more general spaces, as in the work of S. Montiel [31] and Alías-Dajczer [9], [10] where they studied constant mean curvature hypersurfaces in warped product manifolds \( M^{n+1} = \mathbb{R} \times_\rho \mathbb{P}^n \), where \( \mathbb{P}^n \) is a complete Riemannian manifold and \( \rho : \mathbb{R} \to \mathbb{R}_+ \) is a smooth warping function. Those studies were further extended by Alías, Dajczer and Rigoli [12] and Alías, Impera and Rigoli [13] to include constant higher order mean curvature hypersurfaces in warped product manifolds and in general setting by Albanese, Alías and Rigoli [2].

In this paper we give a small contribution to the theory proving appropriate extensions the results of [13]. We start in Section 2 presenting general conditions for the validity of the weak maximum principle for a fairly general class of semi-elliptic trace operators see Theorem 1. These operators and versions of Omori-Yau maximum principle were considered by Alias, Impera and Rigoli [13] and by Hong and Sung [25] under slightly more restrictive conditions. Then, we derive few geometric conditions on a manifold that guarantee that Theorem 1 applies, see Corollary 1 and Theorem 2.

In section 4 we consider the \( L^r \) operators and prove several higher order mean curvature estimates for hypersurfaces immersed into warped product spaces \( M^{n+1} = \mathbb{R} \times_\rho \mathbb{P}^n \), see Theorems 3, 4. In section 5 we extend the Slice Theorem of Alías-Impera-Rigoli [13, Thms. 16 & 21], see Theorems 5, 6.

2. Maximum Principle for Trace Operators

Following the terminology introduced in [37] we say that the Omori-Yau maximum principle for the Laplacian holds on \( M \) if for any given \( u \in C^2(M) \) with \( u^* = \sup_M u < +\infty \), there exists a...
sequence of points $x_k \in M$, depending on $M$ and $u$, such that
\[
\lim_{k \to +\infty} u(x_k) = u^*, \quad |\nabla u(x_k)| < \frac{1}{k}, \quad \Delta u(x_k) < \frac{1}{k}.
\]
Likewise, the Omori-Yau maximum principle for the Hessian is said to hold on $M$ if for any given $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $x_k \in M$, depending on $M$ and on $u$, such that
\[
\lim_{k \to +\infty} u(x_k) = u^*, \quad |\nabla u(x_k)| < \frac{1}{k}, \quad \text{Hess} u(x_k)(X, X) < \frac{1}{k} |X|^2.
\]
for every $X \in T_{x_k}M$. Accordingly, the classical results of Omori [35] and Yau [42] can be stated saying that the Omori-Yau maximum principle for the Laplacian holds on Riemannian manifold with Ricci curvature bounded from below. The importance of the Omori-Yau maximum principle lies on its wide range of applications in geometry and analysis. Applications that goes from the generalized Schwarz lemma [43] to the study of the group of conformal diffeomorphism of a manifold [35], from curvature estimates on submanifolds [2], [6], [7] to Calabi conjectures on minimal hypersurfaces [26], [35]. The essence of the Omori-Yau maximum principle was captured by Pigola, Rigoli and Setti in Theorem 1.9 of [37] whose corollary is the following result: The Omori-Yau maximum principle holds for every $C^2$ smooth function satisfying $\rho \geq -C^2G(\rho)$ where $\rho$ is the distance function on $M$ to a point, $C$ is positive constant and $G : [0, +\infty) \to [0, +\infty)$ is a smooth function satisfying $G(0) > 0$, $G'(t) \geq 0$, and $\lim_{t \to +\infty} \frac{ds}{\sqrt{G(s)}} = +\infty$ and $\limsup_{t \to +\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty$.

In most applications of the Omori-Yau maximum principle, the condition $|\nabla u(x_k)| < 1/k$ is redundant, which led to the following definition.

**Definition 1** (Pigola-Rigoli-Setti). The weak maximum principle holds on a Riemannian manifold $M$ if for every $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $x_k \in M$, such that
\[
\lim_{k \to +\infty} u(x_k) = u^*, \quad \Delta u(x_k) < \frac{1}{k}.
\]
This apparently simple minded definition proved to be surprisingly deep. For instance, it has been proven in [35], that the weak maximum principle for the Laplacian is equivalent to the stochastic completeness of the diffusion process associated to $\Delta$.

The Omori-Yau/weak maximum principle can be considered for differential elliptic operators other than the Laplacian, like the $\phi$-Laplacian [38], the weighted Laplacian $\Delta_{\ell} = \psi^\ell \text{div} (e^{-\ell} \nabla u)$ [13], [29] and semi-elliptic trace operators $L = \text{Tr}(P \circ \text{hess})$ considered in [13], [28] and in [16], where $P : TM \to TM$ is a positive semi-definite symmetric tensor on $TM$ and for each $u \in C^2(M)$, $\text{hess} u : TM \to TM$ is a symmetric operator defined by $\text{hess} u(X) = \nabla_X \nabla u$ for every $X \in TM$. Here $\nabla$ be the Levi-Civita connection of $M$. In this paper we are going to consider the trace operator
\[
Lu = \text{Tr}(P \circ \text{hess} u) + \langle V, \nabla u \rangle
\]
with $\limsup_{x \to \infty} |V|(x) < +\infty$, and prove an Omori-Yau maximum principle in the same spirit of Theorem 1.9 of [37]. Then we prove some geometric applications that extends those of [13]. In [25], K. Hong and C. Sung considered the same trace operator and proved an Omori-Yau maximum principle but their proof required the stronger condition $\sup_M \text{Tr}(P) + \sup_M |V| < \infty$. It should be pointed out that Albanese, Alias and Rigoli in [2] recently proved an all general Omori-Yau maximum principle for trace operators of the form $Lu = \text{Tr}(P \circ \text{hess} u) + \text{div} \text{P}(\text{grad} u) + \langle V, \text{grad} u \rangle$.
However, the main purpose of this paper is to extend the geometric applications involving the operator (4) proved in [13] and for the sake of completeness we keep the proof of our version of the Omori-Yau maximum principle, besides it is very simple.

Our first result is the following extension of [13, Thm. 1].

**Theorem 1.** Let \((M, \langle \cdot, \cdot \rangle)\) be a complete Riemannian manifold. Consider a semi-elliptic operator \(L = \text{Tr}(P \circ \text{hess} (\cdot)) + (V, \text{grad} (\cdot)), \) where \(P : TM \to TM\) is a positive semi-definite symmetric tensor and \(V\) satisfies \(\lim \sup_{x \to \infty} |V(x)| < +\infty.\) Suppose that exists a non-negative function \(\gamma \in C^2(M)\) satisfying:

1. \(\gamma(x) \to +\infty\) as \(x \to \infty,\)
2. \(\exists A > 0\) such that \(|\nabla \gamma| \leq A \sqrt{G(\gamma)} \cdot \left( \int_0^\gamma \frac{1}{\sqrt{G(s)}} ds + 1 \right)\) off a compact set,
3. \(\exists B > 0\) such that \(\text{Tr}(P \circ \text{hess} \gamma) \leq B \sqrt{G(\gamma)} \cdot \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)\) off a compact set.
4. Where \(G : [0, \infty) \to [0, \infty)\) is such that: \(G(0) > 0\) \(G'(t) \geq 0\) and \(G(t)^{-\frac{1}{2}} \notin L^1(+\infty).\)

Then given any function \(u \in C^2(M)\) that satisfies

\[
\lim_{x \to \infty} \frac{u(x)}{\varphi(\gamma(x))} = 0,
\]

where

\[
\varphi(t) = \ln \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right),
\]

there exists a sequence \(\{x_k\}_{k \in \mathbb{N}} \subseteq M\) satisfying:

(a) \(|\nabla u|(x_k) < \frac{1}{j}\) and \((b)\) \(L u(x_k) < \frac{1}{j}\).

If instead of (6) we suppose that \(u\) is bounded above we have that

(c) \(\lim_{k \to +\infty} u(x_k) = \sup u.\)

2.1. **Proof of Theorem** [11] The proof of Theorem [1] we will follow the same steps of the proof of the maximum principle of Omori-Yau in the case of the Laplacian and the \(f\)-Laplacian presented in [29]. Thus, we only prove the item (b). Consider the following family of functions

\[ f_k(x) = u(x) - \varepsilon_k \varphi(\gamma(x)), \]

where \(\varphi\) is defined in (6) and \(\varepsilon_k \to 0^+\) when \(k \to +\infty.\) Observe that the condition (5) implies that \(f_k\) reaches a local maximum, say at \(x_k \in M.\) Suppose that the sequence \(\{x_k\}_{k \in \mathbb{N}}\) diverges, (leaves any compact subset of \(M\)) otherwise we have nothing to prove.

Using the fact that \(x_k\) be the point of maximum to \(f_k\) we infer that

\[
0 = \text{grad} u(x_k) - \varepsilon_k \varphi'(\gamma(x_k)) \text{grad} \gamma(x_k)
\]

and for all \(v \in T_{x_k}M\) we have

\[
0 \geq \text{Hess} u(x_k)(v, v) - \varepsilon_k \left[ \varphi'(\gamma(x_k)) \text{Hess} \gamma(x_k)(v, v) - \varphi''(\gamma(x_k)) \langle \text{grad} \gamma(x_k), v \rangle^2 \right].
\]
Calculating \( \varphi' \) and \( \varphi'' \) we obtain

(9) \[
\varphi'(t) = \left\{ \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(t)} \right\}^{-1}
\]

and

(10) \[
\varphi''(t) = -\left\{ \sqrt{G(t)} \left( \int_0^t \frac{1}{\sqrt{G(s)}} \, ds + 1 \right) \right\}^{-2} \left\{ \frac{G'(t)}{2 \sqrt{G(t)}} \left( \int_0^t \frac{1}{\sqrt{G(s)}} \, ds + 1 \right) + 1 \right\} \leq 0
\]

Taking (9) into (10), we have by the hypothesis ii)

(11) \[
|\text{grad } u(x_k)| \leq \epsilon_k |\varphi'(\gamma(x_k))| \text{grad } \gamma(x_k) \leq \epsilon_k.
\]

Taking (10) into (9), we get

(12) \[
\text{Hess } u(x_k)(v,v) \leq \epsilon_k \varphi'(\gamma(x_k)) \text{Hess } \gamma(x_k)(v,v).
\]

Choose a basis of eigenvectors \( \{v_1, \ldots, v_n\} \subset T_{x_k}M \) of \( P(x_k) \), corresponding to the eigenvalues \( \lambda_j(x_k) = \langle P(x_k)v_j, v_j \rangle \geq 0 \), with \( 1 \leq j \leq n = \dim(M) \). Therefore, by the inequality (12) it follows that

\[
\langle P(x_k) \text{hess } u(x_k)v_j, v_j \rangle = \lambda_j(x_k) \text{Hess } u(x_k)(v_j, v_j) \leq \epsilon_k \varphi'(\gamma(x_k))\langle P(x_k) \text{hess } \gamma(x_k)v_j, v_j \rangle
\]

Applying the trace on both sides of the inequality above and using (9) with the hypothesis iii), we have

(13) \[
\text{Tr}(P \circ \text{hess } u(x_k)) \leq \epsilon_k \varphi'(\gamma(x_k)) \text{Tr}(P \circ \text{hess } \gamma(x_k)).
\]

Therefore, by the item (a) and that \( \limsup_{x \to \infty} |V(x)| < +\infty \) we have

\[
L u(x_k) = \text{Tr}(P \circ \text{hess } u(x_k)) + \langle V(x_k), \text{grad } u(x_k) \rangle \leq \epsilon_k \varphi'(\gamma(x_k)) \text{Tr}(P \circ \text{hess } \gamma(x_k)) + \|\text{grad } u(x_k)\| \leq (D + 1) \epsilon_k.
\]

This finish the proof of Theorem [11]

**Remark 1.** The Theorem [11] can be proved substituting the conditions ii) and iii) by the apparently more general conditions:

i. \( |\nabla \gamma| \leq A \prod_{j=1}^\ell \left[ \ln^{(j)} \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right] \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(t)} \) and

ii. \( \text{Tr}(P \circ \text{hess } \gamma) \leq B \prod_{j=1}^\ell \left[ \ln^{(j)} \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right] \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(t)} \)

respectively. Just consider in the proof \( \varphi(t) = \log^{(t+1)} \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \).

**Corollary 1.** Let \( (M, \langle , \rangle) \) be a complete, non-compact, Riemannian manifold with radial sectional curvature satisfying

(14) \[
K_M \geq -B^2 \prod_{j=1}^\ell \left[ \ln^{(j)} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) + 1 \right]^2 G(r), \text{ for } r(x) \gg 1,
\]
where \( G \in C^\infty([0, +\infty)) \) is even at the origin and satisfies iv), \( r(x) = \text{dist}_M(x_0, x) \) and \( B \in \mathbb{R} \). Then, the Omori-Yau maximum principle for any semi-elliptic operator \( L = \text{Tr}(P \circ \text{hess}(\cdot)) + \langle V, \text{grad}(\cdot) \rangle \) with \( \text{Tr} P \leq \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) \) and \( \limsup_{x \to \infty} |V| < +\infty \) holds on \( M \).

**Proof.** Following the same steps of the example [37, Example 1.13] one has that bound (14) implies

\[
\text{Hess} r \leq D \prod_{j=1}^\ell \left[ \ln^{(j)} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) \right] \sqrt{G(r)}
\]

and this implies

\[
\text{Tr}(P \circ \text{hess}) \leq D \prod_{j=1}^\ell \left[ \ln^{(j)} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) \right] \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(r)}.
\]

Hence Theorem 1 applies. \( \square \)

### 3. Immersions into Warped Products

By \( L^\ell \times_\rho P^n = N^{n+\ell} \) we denote the product manifold \( L^\ell \times P^n \) endowed with the warped product metric \( dL^2 + \rho^2(x)dN^2 \), where \( L^\ell \) and \( P^n \) are Riemannian manifolds and \( \rho : L \to \mathbb{R}_+ \) is a positive smooth function. We will need the following definition introduced in \([8]\).

**Definition 2.** Let \( M \) be a Riemannian manifold. We say that \((G, \tilde{\gamma})\) is an Omori-Yau pair for the Hessian in \( M \) if \( \tilde{\gamma} \in C^2(M) \) is proper and satisfies

\[
|\text{grad} \tilde{\gamma}| \leq \sqrt{G(\tilde{\gamma})} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right).
\]

(15)

\[
\text{Hess} \tilde{\gamma} \leq \sqrt{G(\tilde{\gamma})} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right).
\]

**Remark 2.** If a Riemannian manifold \( M \) has an Omori-Yau pair for the Hessian \((G, \tilde{\gamma})\) then the Omori-Yau maximum principle for the Hessian holds on \( M \), see \([8], [29]\).

The following result gives conditions for an isometric immersion \( f: M^m \to L^\ell \times_\rho P^n = N^{n+\ell} \) into a warped product, where \( P^n \) carries an Omori-Yau pair \((G, \tilde{\gamma})\) for the Hessian, to carry an Omori-Yau pair.

**Theorem 2.** Let \( f: M^m \to L^\ell \times_\rho P^n = N^{n+\ell} \) be an isometric immersion where \( P^n \) carries an Omori-Yau pair \((G, \tilde{\gamma})\) for the Hessian, \( \rho \in C^\infty(L) \) is a positive function, such that \( \inf \rho > 0 \) and \( \mathcal{H} = \text{grad} \log \rho \) satisfies

\[
|\mathcal{H}|(\pi_L(f)) \leq \ln \left( \int_0^{\pi_L(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right).
\]

(16)

If \( f \) is proper on the first entry and

\[
|\alpha| \leq \ln \left( \int_0^{\tilde{\gamma} \circ \pi_L(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right),
\]

(17)
where $\pi_L, \pi_P$ are the projections on $L^t$ and $P^n$ respectively, and $\alpha$ is the second fundamental form of the immersion $f$, then $M^m$ has an Omori-Yau pair for any semi-elliptic operator

$$L = \text{Tr}(P \circ \text{hess}(\cdot)) + \langle \nabla, \text{grad}(\cdot) \rangle \quad \text{with} \quad \text{Tr} P \leq \prod_{j=2}^{k} \left[ \ln(i) \left( \frac{1}{\sqrt{G(s)}} + 1 \right) \right].$$

**Proof.** By abuse of language, we will denote by $\langle \cdot, \cdot \rangle_N, \langle \cdot, \cdot \rangle_L, \langle \cdot, \cdot \rangle_P$ and $\langle \cdot, \cdot \rangle_M$ the Riemannian metrics on $N^{n+\ell}, L^t, M^m$ and on $P^n$ respectively. Let $\| \cdot \|_{N,L,M,P}$ be their respective norms. We will denote by $X, Y$, vector fields in $TL$ and by $W, Z$ vector fields in $TP$. Let $\nabla^N, \nabla^P$ and $\nabla^L$ denote the Riemannian connections on $N, P$ and $L$ respectively. We need few lemmas to prove Corollary 2.

**Lemma 1.** The proof of the following relations are straightforward.

(18) \[ \nabla^N W = \nabla^L X, \quad \nabla^N X = \nabla^N Z = X[\eta] Z, \quad \nabla^N W = \nabla^N Z - \langle Z, W \rangle \text{grad}^L \eta, \]

where $\eta = \log \rho$.

Recall that $P$ carries a Omori-Yau pair $(G, \tilde{\gamma})$. Letting $\pi_P : N^{n+\ell} \to P^n$ be the projection on the second factor we define $\beta : N \to \mathbb{R}$ by $\beta = \tilde{\gamma} \circ \pi_P \circ \gamma : M \to \mathbb{R}$ by $\gamma = \beta \circ f$. It is clear that

(19) \[ \langle \text{grad} \beta, X \rangle_N = 0 \quad \text{and} \quad \langle \text{grad} \tilde{\gamma}, Z \rangle_P = \langle \text{grad} \beta, Z \rangle_N = \rho^2 \langle \text{grad} \beta, Z \rangle_P. \]

Thus $\text{grad} \beta = \frac{1}{\rho^2} \text{grad} \tilde{\gamma}$, $\| \text{grad} \tilde{\gamma} \|_N = \rho \| \text{grad} \tilde{\gamma} \|_P$ and

$$\| \text{grad} \beta \|_M \leq \| \text{grad} \beta \|_N = \frac{1}{\rho^2} \| \text{grad} \tilde{\gamma} \|_N = \frac{1}{\rho} \| \text{grad} \tilde{\gamma} \|_P.$$ 

Moreover, for all $e \in TM$ we have that, (identifying $f_* e = e$),

(20) \[ \text{Hess}_M \gamma(e, e) = \text{Hess}_N \beta(e, e) + \langle \text{grad} \beta, \alpha(e, e) \rangle_N. \]

Here $\alpha$ is the second fundamental form of the immersion. Let us write $e = X + Z$ where $X \in TL$ and $Z \in TP$ and we have that $\text{Hess}_N(e, e) = \text{Hess}_N(X, X) + 2 \text{Hess}_N(X, Z) + \text{Hess}_N(Z, Z)$ and

(21) \[ \text{Hess}_N \beta(X, X) = \langle \nabla^N_X \text{grad} \beta, X \rangle_N = \langle X[\eta] \text{grad} \beta, X \rangle_N = 0. \]

(22) \[ \text{Hess}_N \beta(X, Z) = \langle \nabla^N_X \text{grad} \beta, Z \rangle_N = X[\eta] \langle \text{grad} \beta, Z \rangle_N = \langle \text{grad} \eta, X \rangle_L \langle \text{grad} \tilde{\gamma}, Z \rangle_P. \]

$$\text{Hess}_N \beta(Z, Z) = \langle \nabla^N_Z \text{grad} \beta, Z \rangle_N$$

(23) \[ = \langle \nabla^P_Z \text{grad} \tilde{\gamma}, Z \rangle_N - \langle \text{grad} \beta, Z \rangle_N \langle \text{grad} \eta, Z \rangle_N \to 0 
= \frac{1}{\rho^2} \langle \nabla^P_Z \text{grad} \tilde{\gamma}, Z \rangle_N = \text{Hess}_P \tilde{\gamma}(Z, Z). \]

Hence from (21, 22, 23) we have that

(24) \[ \text{Hess}_N \beta(e, e) = 2 \langle \text{grad} \eta, X \rangle_L \langle \text{grad} \tilde{\gamma}, Z \rangle_P + \text{Hess}_P \tilde{\gamma}(Z, Z). \]
Thus, from (20),
\[
\text{Hess}_M \gamma(e, e) = 2(\text{grad} \eta, X)\langle \text{grad} \hat{\gamma}, X \rangle_p + \text{Hess}_p \hat{\gamma}(Z, Z) + \langle \text{grad} \beta, \alpha(e, e) \rangle_N
\]
\[
\leq 2\|\mathcal{H}\|_L \cdot \|\text{grad} \hat{\gamma}\|_p \cdot \|X\|_L \cdot \|\alpha\|_p + \text{Hess}_p \hat{\gamma}(Z, Z) + \frac{1}{p^2} \langle \text{grad} \hat{\gamma}, \alpha(e, e) \rangle_N
\]
(25)
\[
\leq \ln \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \left( 3 + \frac{1}{p} \right) |e|^2
\]
off a compact set, since \(\ln \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) > 1\) there.

Let \(x \in M\) and choose a basis \(\{e_1, ..., e_n\}\) for \(T_xM\) formed by eigenvectors of \(P(x)\) with eigenvalues \(\lambda_i(x) = \langle P(x)e_i, e_i \rangle \geq 0\). Since
\[
\text{L} \gamma(x) = \sum_{i=1}^n \langle P(x)\text{hess} \gamma(x)(e_i), e_i \rangle = \sum_{i=1}^n \langle \text{hess} \gamma(x)(e_i), P(x)(e_i) \rangle = \sum_{i=1}^n \lambda_i(x) \text{Hess} \gamma(e_i, e_i)
\]
we have then
\[
\text{L} \gamma \leq \text{Tr} P \cdot \ln \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \left( 3 + \frac{1}{p} \right) + \frac{|\nabla \sqrt{G(\gamma)}|}{\sqrt{G(\gamma)}} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)
\]
\[
\leq \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \left( 3 + \frac{D}{\rho} \right) \prod_{j=1}^k \ln(j) \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)
\]
By Theorem \(\text{II}\) and Remark \(\text{II}\) the Omori-Yau maximum principle holds on \(M\).

4. Curvature estimates

4.1. The operators \(L_r\). Let \(f : M \to N\) be an isometric immersion of a connected \(n\)-dimensional Riemannian manifold \(M\) into the \((n+1)\)-dimensional Riemannian manifold \(N\). Let \(\alpha(X) = -\nabla_X \eta, X \in TM\) be the second fundamental form of the immersion with respect to a locally defined normal vector field \(\eta\), where \(\nabla\) is the Levi-Civita connection of \(N\). Its eigenvalues \(\kappa_1, ..., \kappa_n\) are the principal curvatures of the hypersurface \(M\). The elementary symmetric functions of the principal curvatures are defined by

\[
S_0 = 1, \quad S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \quad 1 \leq r \leq n.
\]

The elementary symmetric functions \(S_r\) define the \(r\)-mean curvature \(H_r\) of the immersion by
\[
H_r = \left( \begin{array}{c} n \\ r \end{array} \right)^{-1} S_r,
\]
so that \(H_1\) is the mean curvature and \(H_n\) is the Gauss-Kronecker curvature. The Newton tensors \(P_r : TM \to TM\), for \(r = 0, 1, ..., n\), are defined setting \(P_0 = I\) and \(P_r = S_r Id - \alpha P_{r-1}\) so that
If \( \mathcal{P}_r(x) : T_xM \to T_xM \) is a self-adjoint linear operator with the same eigenvectors as \( \alpha \). From here we will be following [13, p.3] closely. When \( r \) is even the sign of \( S_r \) does not depend on \( \eta \) which implies that the tensor \( \mathcal{P}_r \) is globally defined on TM. If \( r \) is odd we will assume that \( M \) is two sided, i.e., there exists a globally defined unit normal vector field \( \eta \) in \( f(M) \). When a hypersurface is two sided, a choice of \( \eta \) makes \( \mathcal{P}_r \) globally defined. To give an uniform treatment in what follows we shall assume from now on that \( M \) is two-sided.

For each \( u \in C^2(M) \), define a symmetric operator \( \text{hess} \ u : TM \to TM \) by

\[
\text{hess} \ u(X) = \nabla_X \text{grad} \ u
\]

for every \( X \in TM \), where \( \nabla \) be the Levi-Civita connection of \( M \) and the symmetric bilinear form \( \text{Hess} \ u : TM \times TM \to C^\infty(M) \) by

\[
\text{Hess} \ u(X, Y) = \langle \text{hess} \ u(X), Y \rangle.
\]

Associated to each Newton operator \( \mathcal{P}_r : TM \to TM \) there is a second order self-adjoint differential operator \( \mathcal{L}_r : C^\infty(M) \to C^\infty(M) \) defined by

\[
\mathcal{L}_r(u) = \text{Tr}(\mathcal{P}_r \text{hess} \ u) = \text{div} (\mathcal{P}_r \text{grad} \ u) - \langle \text{grad} (\nabla \mathcal{P}_r), \text{grad} \ u \rangle
\]

However, these operators may be not elliptic. Regarding the ellipticity of the \( \mathcal{L}_r \)'s, one sees that the operator \( \mathcal{L}_r \) is elliptic if and only if \( \mathcal{P}_r \) is positive definite. There are geometric conditions implying the positiveness of the \( \mathcal{P}_r \) and thus the ellipticity of the \( \mathcal{L}_r \), e.g., \( H_2 > 0 \) implies that \( H_1 > 0 \) by the well known inequality \( H_1^2 \geq H_2 \). And that implies that all the eigenvalues of \( \mathcal{P}_1 \) are positive and the ellipticity of \( \mathcal{L}_1 \). For the ellipticity of \( \mathcal{L}_r \), \( r \geq 2 \), it is enough to assume that there exists an elliptic point \( p \in M \), i.e., a point where the second fundamental form \( \alpha \) is positive definite (with respect to an orientation) and \( H_{r+1} > 0 \). See details in [13, 19, 23, 39]. In this section we are going to apply Theorem 1 to the operators \( \mathcal{L}_r \) in order to derive curvature estimates.

Again, we denote by \( \mathbb{N}^{n+1} = I \times \mathbb{R}^n \) the product manifold \( I \times \mathbb{R}^n \) endowed with the warped product metric \( dt^2 + \rho^2(t)d^2\mathbb{R}^n \), where \( I \subset \mathbb{R} \) is an open interval, \( \mathbb{R}^n \) is a complete Riemannian manifold and \( \rho : I \to \mathbb{R}_+ \) is a smooth function. Given an isometrically immersed hypersurface \( f : \mathbb{M}^n \to \mathbb{N}^{n+1} \), define \( h : \mathbb{M}^n \to I \) the \( C^\infty(\mathbb{M}^n) \) height function by setting \( h = \pi_1 \circ f \), where \( \pi_1 : \mathbb{N}^{n+1} = I \times \mathbb{R}^n \to I \) is the projection on the first factor. We will need the following result similar to [13, Prop. 6].

**Lemma 2.** Let \( f : \mathbb{M}^n \to I \times \mathbb{R}^n = \mathbb{N}^{n+1} \) be an isometric immersion into a warped product space. Let \( h \) be the height function and define

\[
\sigma(t) = \int_{t_0}^{t} \rho(s)ds.
\]

Then

\[
\hat{L}_k \sigma(h) = c_k \rho(h) \left( \mathcal{H}(h) + \Theta \frac{H_{k+1}}{H_k} \right),
\]

where \( c_k = (n-k) \binom{n}{k} \), \( \Theta = \langle \eta, T \rangle \) is the angle function, \( \mathcal{H}(h) = (\rho'(h) / \rho(h) \right) \) and \( \hat{L}_k = \text{Tr}(\hat{P}_k \circ \text{hess}) \) with \( \hat{P}_k = \frac{P_k}{H_k} \).
Proof. We observe that \( \text{grad } \sigma (h) = \rho (h) \text{grad } h \) and consequently

\[
\text{hess } \sigma (h)(X) = \nabla_X \text{grad } \sigma (h)
\]

(33)

\[
= \rho (h) \nabla_X \text{grad } h + \langle \text{grad } (p \circ h), X \rangle \text{grad } h
\]

\[
= \rho (h) \text{hess } h(X) + \rho ' (h) \langle \text{grad } h, X \rangle \text{grad } h,
\]

for all \( X \in TM^n \). Therefore,

\[
\hat{L}_k \sigma (h) = \text{Tr} (\hat{P}_k \circ \text{hess } \sigma (h)) = \sum_i \frac{1}{H_k} \langle \hat{P}_k \circ \text{hess } \sigma (h)(e_i), e_i \rangle
\]

(34)

\[
= \frac{1}{H_k} \sum_i \langle \rho (h) \hat{P}_k \circ \text{hess } h(e_i), e_i \rangle + \rho ' (h) \langle \text{grad } h, e_i \rangle \langle \text{grad } h, \hat{P}_k \circ \text{grad } h \rangle
\]

\[
= \frac{1}{H_k} \left( \rho (h) \sum_i \langle \hat{P}_k \circ \text{hess } h(e_i), e_i \rangle + \rho ' (h) \langle \text{grad } h, \hat{P}_k \circ \text{grad } h \rangle \right).
\]

For the other hand, the gradient of \( \pi_1 \in C^\infty (M) \) is \( \text{grad } N \pi_1 = T \), where \( T \) stands for the lifting of \( \partial / \partial t \in T \) to the product \( I \times \rho P^n \). Then,

(35) \( \text{grad } h = (\text{grad } N \pi_1) ^\perp = T - \Theta \eta \).

Since the Levi-Civita connection of a warped product satisfies

\[
\nabla_X T = \mathcal{H}(X - \langle X, T \rangle T), \quad \forall X \in TN^{n+1}
\]

we have

\[
\nabla_X \text{grad } h = \mathcal{H}(h)(X - \langle X, \text{grad } h \rangle T) - X(\Theta \eta) + \Theta AX, \quad \forall X \in TM^n
\]

and thus

(36) \( \text{hess } h(X) = \mathcal{H}(h)(X - \langle X, \text{grad } h \rangle \text{grad } h) + \Theta AX \)

Taking \( \text{36} \) in account in \( \text{34} \) we get

\[
\hat{L}_k \sigma (h) = \frac{1}{H_k} [\rho (h) (\mathcal{H}(h)(\text{Tr } P_k - \langle \hat{P}_k \circ \text{grad } h, \text{grad } h \rangle) + \Theta \text{Tr } (\hat{P}_k A)) + \rho ' (h) \langle \text{grad } h, \hat{P}_k \circ \text{grad } h \rangle]
\]

\[
= \frac{1}{H_k} [\rho (h) (\mathcal{H}(h)(c_k H_k - \langle \hat{P}_k \circ \text{grad } h, \text{grad } h \rangle) + \Theta \text{Grad } H_k) + \rho ' (h) \langle \text{grad } h, \hat{P}_k \circ \text{grad } h \rangle]
\]

\[
= \frac{1}{H_k} [\rho ' (h) c_k H_k - \rho ' (h) \langle \hat{P}_k \circ \text{grad } h, \text{grad } h \rangle + \rho (h) c_k \Theta H_k + \rho ' (h) \langle \hat{P}_k \circ \text{grad } h, \text{grad } h \rangle]
\]

\[
= c_k \rho (h) \left( \mathcal{H}(h) + \Theta \frac{H_k + 1}{H_k} \right)
\]

\( \square \)

The following result, (Theorem [3], generalizes [13, Thm.10]. We will assume that

(37) \( \lim_{x \to \infty} \frac{\sigma \circ h(x)}{\varphi (\gamma (x))} = 0 \)
where \( \sigma \) is given in (31) and \( \varphi \) is given by
\[
\varphi(t) = \ln\left(\int_0^t \frac{ds}{\sqrt{G(s)}} + 1\right)
\]
while \( (G, \gamma) \) is an Omori-Yau pair in the sense of Definition 2.

**Theorem 3.** Let \( f: M^n \to I \times_\rho P^n = N^{n+1} \) be a properly immersed hypersurface with second fundamental form \( \alpha \) satisfying (17) and \( H_2 > 0 \). Suppose that \( P^n \) carry an Omori-Yau pair \( (G, \gamma) \) for the Hessian, that \( \inf \rho > 0 \) and \( \mathcal{H} = \frac{\rho}{\rho'} \) satisfies (16). If \( \text{Tr}(\dot{P}_1) \leq \left(\int_0^{\Theta(x)} \frac{ds}{\sqrt{G(s)}} + 1\right) \) and if the height function \( \sigma \circ h \) satisfies (37) then
\[
\sup_M H_2^+ \geq \inf \mathcal{H}(h).
\]

**Proof.** By Theorem 2 the Omori-Yau maximum principle for \( \dot{L}_1 \) holds on \( M \) and then there exists a sequence \( x_j \in M \) such that
\[
\frac{1}{j} > \dot{L}_1 \sigma \circ h(x_j) = n(n-1)\rho(h(x_j)) \left(\mathcal{H}(h(x_j)) + \Theta(x_j) \frac{H_2}{H_1}(x_j)\right)
\]
(40)
\[
\geq n(n-1)\rho(h(x_j)) \left(\mathcal{H}(h(x_j)) - \frac{H_2}{H_1}(x_j)\right)
\]
\[
\geq n(n-1)\rho(h(x_j)) \left(\mathcal{H}(h(x_j)) - \sqrt{H_2}(x_j)\right)
\]
\[
\geq n(n-1)\rho(h(x_j)) \left(\inf_M \mathcal{H}(h) - \sup_M \sqrt{H_2}\right).
\]
Where we used Lemma 2 in the right hand side the first line of (40). Since \( \sigma \) is strictly increasing and \( h(x_j) \to \sup h \) as \( j \to +\infty \), we obtain that
\[
\sup_M \sqrt{H_2} \geq \inf_M \mathcal{H}(h).
\]

**Corollary 2.** Let \( P^n \) be a complete, non-compact Riemannian manifold whose radial sectional curvature satisfies
\[
K^r_{\text{rad}} \geq -C \cdot G(r),
\]
where \( G \in C^\infty([0, +\infty)) \) is even at the origin and satisfies iv) in Theorem 7 \( r(x) = \text{dist}_M(x_0, x) \). If \( f: M^n \to I \times_\rho P^n = N^{n+1} \) is a properly immersed hypersurface with \( H_2 > 0 \), satisfying (16), (17), (37) and \( \inf \rho > 0 \). If \( \text{Tr}(\dot{P}_1) \leq \left(\int_0^{\Theta(x)} \frac{ds}{\sqrt{G(s)}} + 1\right) \) then
\[
\sup_M H_2^+ \geq \inf \mathcal{H}(h).
\]

Our next result is just a version of Theorem 3 for higher order mean curvatures.

**Theorem 4.** Let \( f: M^n \to I \times_\rho P^n = N^{n+1} \) be a proper isometric immersion with an elliptic point and \( H_k > 0 \). Suppose that the second fundamental form \( \alpha \) satisfies (17) and that \( \inf \rho > 0 \). Moreover, assume that \( \mathcal{H} = \rho'/\rho \) satisfies (16) and that \( P^n \) carry an Omori-Yau pair \( (G, \gamma) \) for the
Hessian. If \( \text{Tr}(\hat{P}_{k-1}) \leq \left( \int_0^{\gamma \circ \pi P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right) \), \( 3 \leq k \leq n \) and if the height function \( \sigma \circ h \) satisfies (37) then
\[
\sup_M H_k^h \geq \inf_M \mathcal{H}(h).
\]

Likewise, we have the corollary

**Corollary 3.** Let \( P^n \) be a complete, non-compact Riemannian manifold whose radial sectional curvature satisfies
\[
K_{rad} \geq -C \cdot G(r)
\]
If \( f: M^n \to I \times \rho P^n = N^n + 1 \) is a properly immersed hypersurface with \( H_k > 0 \), satisfying (16), (17), (37) and \( \inf \rho > 0 \). If \( \text{Tr}(\hat{P}_1) \leq \left( \int_0^{\gamma \circ \pi P(f)} \frac{ds}{\sqrt{G(s)}} + 1 \right) \) then
\[
\sup_M H_k^h \geq \inf_M \mathcal{H}(h).
\]

5. Slice Theorem

In this section we will prove an extension of the Slice Theorem proved by L. Alias, D. Impera and M. Rigoli [13, Thms. 16 & 21]. We start with the following lemma.

**Lemma 3.** Let \( f: M^n \to I \times \rho P^n = N^{n+1} \) be a hypersurface with non-vanishing mean curvature and suppose that the height function \( h \) satisfies
\[
\lim_{x \to \infty} \frac{h(x)}{\varphi(\gamma(x))} = 0,
\]
where \( \varphi \) is given in (9) and \( \gamma \) in the statement of Theorem 2. Assume that \( \mathcal{H}' \geq 0 \) and that the angle function \( \Theta \) does not change sign. Choose on \( M^n \) the orientation so that \( H_1 > 0 \). Suppose the Omori-Yau maximum principle for the Laplacian holds on \( M^n \). Then we have that

i) If \( \Theta \leq 0 \) then \( \mathcal{H}(h) \geq 0 \),

ii) If \( \Theta \geq 0 \) then \( \mathcal{H}(h) \leq 0 \).

**Proof.** By hypothesis, we have that Omori-Yau maximum principle for the Laplacian holds for the height function \( h \), therefore, there exists a sequences \( \{x_j\}, \{y_j\} \subset M^n \) such that
\[
\lim_{j \to +\infty} h(x_j) = \sup h = h^*, \quad |\text{grad} h|^2(x_j) < \left( \frac{1}{j} \right)^2 \quad \text{and} \quad \Delta h(x_j) < \frac{1}{j}
\]
and
\[
\lim_{j \to +\infty} h(y_j) = \inf h = h_*, \quad |\text{grad} h|^2(y_j) < \left( \frac{1}{j} \right)^2 \quad \text{and} \quad \Delta h(y_j) > -\frac{1}{j}.
\]
On the other hand, we know that \( \Delta h = \mathcal{H}(h)(n - |\text{grad} h|^2) + nH_1 \Theta \). Therefore, supposing that \( \Theta \geq 0 \) we get by (47) that
\[
0 \geq -nH_1(x_j) \Theta(x_j) > -\frac{1}{j} + \mathcal{H}(h(x_j))(n - |\text{grad} h|^2(x_j))
\]
and then, for \( j \gg 1 \) we obtain
\[
0 \geq \frac{-nH_1(x_j)\Theta(x_j)}{n - |\text{grad } h|^2(x_j)} > \frac{1}{j(n - |\text{grad } h|^2(x_j))} + \mathcal{H}(h(x_j))
\]
\[
\text{letting } j \to +\infty, \text{ we have}
\]
\[
0 \geq \limsup_{j \to +\infty} \mathcal{H}(h(x_j)) = \mathcal{H}(h^*) \geq \mathcal{H}(h).
\]
Similar proof gives the item i).

Define the operator \( \mathcal{L}_1 = \text{Tr}(\mathcal{P}_1 \circ \text{hess}) = (n - 1)\mathcal{H}(h)\Delta - \Theta L_1 \) where \( \mathcal{P}_1 = (n - 1)\mathcal{H}(h)I - \Theta \mathcal{P}_1 \).

Using that
\[
\Delta h = \mathcal{H}(h)(n - |\text{grad } h|^2) + nH_1\Theta
\]
and
\[
L_1(h) = n(n - 1)\mathcal{H}(h)(1 + \Theta H_2) - \mathcal{H}(h)(\mathcal{P}_1 \text{grad } h, \text{grad } h)
\]
we obtain
\[
\mathcal{L}_1(h) = n(n - 1)\mathcal{H}(h)(n - |\text{grad } h|^2) - (n - 1)\mathcal{H}(h)(\mathcal{P}_1 \text{grad } h, \text{grad } h)
\]
\[
\mathcal{L}_1(\sigma \circ h) = n(n - 1)\rho(h)(\mathcal{H}(h)(n - |\text{grad } h|^2) - (n - 1)\mathcal{H}(h)(\mathcal{P}_1 \text{grad } h, \text{grad } h))
\]
The following theorem extends [13, Thm. 16] which extends [9, Thm 2.9].

**Theorem 5.** Let \( f : M^n \to \mathbb{R}^n \) be a complete hypersurface of constant positive 2-mean curvature \( H_2 > 0 \) with radial sectional curvature \( K^{\text{rad}}_M \) satisfying

\[
K^{\text{rad}}_M \geq -B^2 \prod_{j=1}^f \left[ \ln^{(i)} \left( \int_0^r \frac{ds}{G(s)} + 1 \right) \right] G(r), \text{ for } r(x) \gg 1,
\]
where \( G \in C^\infty([0, +\infty)) \) is even at the origin and satisfies iv) in Theorem 2.5. \( r(x) = \text{dist}_M(x_0, x) \) and \( B \in \mathbb{R} \). Suppose also that height function satisfies the conditions (47), (46) and that

\[
|H_1|(r) \leq \frac{1}{n(n - 1)} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right),
\]
and

\[
|\mathcal{H}|(t) \leq \frac{1}{n(n - 1)} \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right)
\]

If \( \mathcal{H} \) > 0 almost everywhere and the angle function \( \Theta \) does not change sign, then \( f(M^n) \) is a slice.

**Proof.** Taking an orientation on \( M^n \) in which \( H_2 > 0 \) we have, by Lemma 3.2 that if \( \Theta \leq 0 \), then \( \mathcal{H}(h) > 0 \) and therefore the operator \( \mathcal{P}_1 \) is positive semi-definite. Therefore, which implies that \( \mathcal{L}_1 \) is semi-elliptic. Furthermore, by (41) and (42), we have
\[
\text{Tr } \mathcal{P}_1 = n(n - 1)(\mathcal{H}(h) - H_1\Theta) \leq \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right)
\]
By Corollary 4.1 the Omori-Yau maximum principle for the operator \( \mathcal{L}_1 \) holds on \( M^n \) with the functions \( h \) and \( \sigma \circ h \) (conditions (47) and (46)). Thus, there is a sequence \( \{x_j\} \subset M^n \) such that

i. \( \lim_{j \to +\infty} \sigma(h(x_j)) = (\sigma \circ h)^* \)

ii. \( |\text{grad } (\sigma \circ h)|(x_j) = \rho(h(x_j))|\text{grad } h|(x_j) < \frac{1}{j} \)

In particular, for \( x_j \) such that \( \mathcal{H}(h(x_j)) = \mathcal{H}(h^*) \) we have
\[
|\text{grad } (\sigma \circ h)|(x_j) = \rho(h(x_j))|\text{grad } h|(x_j) < \frac{1}{j}
\]
and
\[
\text{Tr } \mathcal{P}_1 = n(n - 1)(\mathcal{H}(h) - H_1\Theta) \leq \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right)
\]

By Corollary 4.1 the Omori-Yau maximum principle for the operator \( \mathcal{L}_1 \) holds on \( M^n \) with the functions \( h \) and \( \sigma \circ h \) (conditions (47) and (46)). Thus, there is a sequence \( \{x_j\} \subset M^n \) such that

i. \( \lim_{j \to +\infty} \sigma(h(x_j)) = (\sigma \circ h)^* \)

ii. \( |\text{grad } (\sigma \circ h)|(x_j) = \rho(h(x_j))|\text{grad } h|(x_j) < \frac{1}{j} \)
Thus, we know that $H \geq 2$.

Making $j \to +\infty$, we get

\begin{equation}
\mathcal{H}^2(h^*) \leq H_2.
\end{equation}

On the other hand, there is a sequence $\{y_j\} \subset M^n$ such that

\begin{enumerate}
  \item $\lim_{j \to +\infty} h(y_j) = h^*$
  \item $|\text{grad} h(y_j)| = |\text{grad} h(y_j)| < \frac{1}{j}$
  \item $L_1(h)(y_j) > -\frac{1}{j}$.
\end{enumerate}

Since $P_1$ is positive semi-definite, there exists a constant $\beta \geq 0$ such that

$$
\langle P_1 \text{grad} h, \text{grad} h \rangle \geq \beta |\text{grad} h|^2.
$$

Thus,

\begin{align*}
-\frac{1}{j} &< n(n-1) \left( (\mathcal{H}^2(h(y_j))) - \Theta^2(y_j) H_2(y_j) \right) - \mathcal{H}(h(y_j)) \langle P_1 \text{grad} h(y_j), \text{grad} h(y_j) \rangle \\
&\leq n(n-1) \left( (\mathcal{H}^2(h(y_j))) - \Theta^2(y_j) H_2(y_j) \right) - \beta |\text{grad} h|^2(y_j) \\
&\leq n(n-1) \left( (\mathcal{H}^2(h(y_j))) - \Theta^2(y_j) H_2(y_j) \right) \cdot
\end{align*}

Since $1 \geq \Theta^2(y_j) = 1 - |\text{grad} h|^2(y_j) \geq 1 - \frac{1}{j}$, by the item ii, we have $\lim_{j \to +\infty} \Theta^2(y_j) = 1$. Doing $j \to +\infty$, we get

\begin{equation}
H_2 \leq \mathcal{H}^2(h^*).
\end{equation}

Combining (53) with (54), we get that

$$
\mathcal{H}(h^*) \leq H_2 \leq \mathcal{H}(h^*).
$$

and as $\mathcal{H}$ is an increasing function, conclude that $h^* = h^* < \infty$.

If $\Theta \geq 0$, we applied in a manner entirely analogous the Omori-Yau maximum principle to the semi-elliptic operator $-L_1$.

To extend the previous findings in the case of higher order curvatures, let us define for each $2 \leq k \leq n$, the operators

\begin{equation}
L_{k-1} = \text{Tr} (P_{k-1} \circ \text{hess}),
\end{equation}

where

\begin{equation}
P_{k-1} = \sum_{j=0}^{k-1} (-1)^j \left( \frac{\mathcal{H}^{k-1-j}}{\mathcal{H}} \right)(h) \Theta^j P_j.
\end{equation}
Observe that
\[ \mathcal{P}_{k-1} = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \mathcal{P}_{k-2} + (-1)^{k-1} \Theta^k \mathcal{P}_{k-1} \]
and consequently
\[ (57) \quad \lambda_{k-1} = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \lambda_{k-2} + (-1)^{k-1} \Theta^k \lambda_{k-1}. \]

By induction, we see that
\[ (58) \quad \lambda_{k-1} = c_{k-1} \left( \mathcal{H}^k(h) - (-1)^k \Theta^k H_k \right) - \mathcal{H}(h) \langle \mathcal{P}_{k-1} \text{grad} \ h, \text{grad} \ h \rangle \]
and
\[ (59) \quad \lambda_{k-1} \sigma(h) = c_{k-1} \rho(h) \left( \mathcal{H}^k(h) - (-1)^k \Theta^k H_k \right). \]

In fact, assuming that the expression \((58)\) is valid for \(k - 2\), we have:
\[
\begin{align*}
\lambda_{k-1} & = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \lambda_{k-2} + (-1)^{k-1} \Theta^k \lambda_{k-1} \\
& = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \left[ c_{k-2} \left( \mathcal{H}^{k-1}(h) - (-1)^{k-1} \Theta^{k-1} H_{k-1} \right) - \mathcal{H}(h) \langle \mathcal{P}_{k-2} \text{grad} \ h, \text{grad} \ h \rangle \right] + \\
& \quad + (-1)^{k-1} \Theta^k \left[ \mathcal{H}(h) \left( c_{k-1} H_{k-1} - \langle \mathcal{P}_{k-1} \text{grad} \ h, \text{grad} \ h \rangle \right) + c_{k-1} \Theta H_k \right] \\
& = c_{k-1} \mathcal{H}^k(h) - (-1)^k c_{k-1} \mathcal{H}(h) \Theta^{k-1} H_{k-1} - \frac{c_{k-1}}{c_{k-2}} \mathcal{H}^2(h) \langle \mathcal{P}_{k-2} \text{grad} \ h, \text{grad} \ h \rangle + \\
& \quad + (-1)^{k-1} c_{k-1} \mathcal{H}(h) \Theta^{k-1} H_{k-1} + (-1)^{k-1} c_{k-1} \Theta^k H_k - \\
& \quad - (-1)^{k-1} \mathcal{H}(h) \Theta^k \langle \mathcal{P}_{k-1} \text{grad} \ h, \text{grad} \ h \rangle \\
& = c_{k-1} \left( \mathcal{H}^k(h) - (-1)^k \Theta^k H_k \right) - \mathcal{H}(h) \langle \mathcal{P}_{k-1} \text{grad} \ h, \text{grad} \ h \rangle.
\end{align*}
\]

The proof of the expression \((59)\) follows similarly.

The next theorem extends the Theorem \ref{thm:6} for the case of higher order curvatures and your proof is analogous and use only the expressions \((58)\) and \((59)\).

**Theorem 6.** Let \( f : M^n \to I \times \rho P^n = N^{n+1} \) be a complete hypersurface of constant positive \(k\)-mean curvature \( H_k > 0 \), \( 3 \leq k \leq n \) with radial sectional curvature \( K^\text{rad}_M \) satisfying
\[
(60) \quad K^\text{rad}_M \geq -B^2 \prod_{j=1}^t \left( \ln^{(j)} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} \right) + 1 \right)^2 G(r), \text{ for } r(x) \gg 1,
\]
where \( G \in C^\infty([0, +\infty)) \) is even at the origin and satisfies iv) in Theorem \ref{thm:7} \( r(x) = \text{dist}_M(x_0, x) \) and \( B \in \mathbb{R} \). Suppose also that height function satisfies the conditions \((37)\), \((40)\) and that
\[
(61) \quad |H^2| : |\mathcal{H}^k(H)| \leq \frac{1}{k c_{k-1}} \left( \int_0^r \frac{ds}{\sqrt{G(s)}} + 1 \right), \quad \forall j = 0, \ldots, k - 1.
\]

Assume that there exists an elliptic point in \( M^n \). If \( \mathcal{H}' > 0 \) almost everywhere and the angle function \( \Theta \) does not change sign, then \( f(M^n) \) is a slice.
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