QUANTILES AND DEPTH FOR DIRECTIONAL DATA FROM ELLIPTICALLY SYMMETRIC DISTRIBUTIONS

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ABSTRACT

We present canonical quantiles and depths for directional data following a distribution which is el-
liptically symmetric about a direction \( \mu \) on the sphere \( S^{d-1} \). Our approach extends the concept of
Ley et al. [1], which provides valuable geometric properties of the depth contours (such as convex-
ity and rotational equivariance) and a Bahadur-type representation of the quantiles. Their concept
is canonical for rotationally symmetric depth contours. However, it also produces rotationally sym-
metric depth contours when the underlying distribution is not rotationally symmetric. We solve this
lack of flexibility for distributions with elliptical depth contours. The basic idea is to deform the
elliptic contours by a diffeomorphic mapping to rotationally symmetric contours, thus reverting to
the canonical case in Ley et al. [1]. A Monte Carlo simulation study confirms our results. We use
our method to evaluate the ellipticity of depth contours and for trimming of directional data. The
analysis of fibre directions in fibre-reinforced concrete underlines the practical relevance.

Keywords  directional statistics · contour · differential geometry · angular Mahalanobis depth · trimming

1 Introduction

The classes of rotationally symmetric distributions and elliptically symmetric distributions in \( \mathbb{R}^d \) have been well inves-
tigated by Kelker [2], Cambanis et al. [3] and Fang et al. [4]. A random vector \( V \in \mathbb{R}^d \) has a rotationally symmetric
distribution if \( V \overset{D}{=} O V \) for all \( O \in SO(d) \) where \( \overset{D}{=} \) refers to equality in distribution. Furthermore, every random vec-
tor \( V \in \mathbb{R}^d \) following a rotationally symmetric distribution can be represented as \( V \overset{D}{=} RU \), where \( U \sim \text{Unif}(S^{d-1}) \)
is independent of the real-valued random variable \( R \sim F_R \). \( U \) gives the direction of \( V \) and \( R \) is the length of \( V \). Rotational-
symmetric distributions are often regarded as the most natural non-uniform distributions in \( \mathbb{R}^d \). For instance,
the charge distribution of an electric field is rotationally symmetric around its source. However, not all phenomena
observed in practice can be represented by symmetric models.

Elliptically symmetric distributions extend the class of rotationally symmetric distributions. The distribution of a
random vector \( W \) is elliptically symmetric if and only if \( W \overset{D}{=} R \Sigma^{1/2}U \) with \( U \sim \text{Unif}(S^{d-1}) \), real-valued \( R \sim F_R \)
independent of \( U \), and \( \Sigma \in \mathbb{R}^{d \times d} \) a symmetric, positive definite matrix. A random vector \( W \) with an elliptically
symmetric distribution can be transformed into a random vector \( V = RU \) with a rotationally symmetric distribution via

\[
\Sigma^{-1/2}W \overset{D}{=} R \Sigma^{-1/2} \Sigma^{1/2}U = RU = V.
\] (1.1)

These concepts of symmetry transfer to the unit sphere \( S^{d-1} \), i.e., the case of directional data. Distributions on \( S^{d-1} \)
which are rotationally symmetric about a direction \( \mu \in S^{d-1} \) are also often regarded as the natural non-uniform
distributions on $S^{d-1}$ [5]. In most cases, rotationally symmetric distributions have tractable normalising constants. Note that the density of a rotationally symmetric distribution is proportional to a function $f(x^T \mu)$. Thus, a projection onto $\mu$ enables a one-dimensional analysis of the distribution, for example, its concentration around $\mu$. The class of distributions with rotational symmetry about $\mu \in S^{d-1}$ is denoted by $\mathcal{R}_\mu$.

In practice, symmetric models are often too restrictive. For instance, Leong and Carlile [6] illustrated that rotational symmetry about a direction is a too strong assumption for a directional data set from neurosciences. Kent [7] has fitted his elliptical model to a data set of 34 measurements of the directions of magnetisation for samples from the Great Whin Sill (Northumberland, England). As in $\mathbb{R}^d$, distributions that are elliptically symmetric about a direction $\mu$ on $S^{d-1}$ are an extension of the rotationally symmetric distributions. The contours are more flexible to form elliptical shapes. Due to the curved shape of the sphere, the transition from distributions which are rotationally symmetric about $\mu \in S^{d-1}$ to distributions which are elliptically symmetric about $\mu$ is not obvious. A remedy to this problem is to linearsie $S^{d-1}$ at some base point $\mu$ by considering the tangent space $T_\mu S^{d-1}$ at $\mu$. By using the theory for $\mathbb{R}^d$, a transformation between the two distributions can then be defined in the tangent space.

Ley et al. [1] introduced a concept of quantiles and depth for directional data. They showed that their quantiles are absolutely continuous with respect to the Lebesgue measure on $S^{d-1}$, and summarise the findings of Ley et al. [1]. Section 3 contains our main contribution. The idea is to map the unit vectors into the tangent space $T_\mu S^{d-1}$ where $\mu$ is the median direction of the observed sample. The mapped vectors are elliptically symmetric around the origin in $T_\mu S^{d-1}$. The multivariate Mahalanobis transformation [10, 11] is then applied in $T_\mu S^{d-1}$ to obtain a rotationally symmetric sample in $T_\mu S^{d-1}$. Mapping it back to $S^{d-1}$, we obtain a sample of unit vectors which are rotationally symmetric about $\mu$. Thus, we can exploit the results from [1]. All transformations are diffeomorphic such that we can trace back the results to the original unit vectors. Section 4 affirms our findings by a Monte Carlo study. Furthermore, we apply our approach to a real-world data set from [12]: Directions of short steel fibres crossing a crack in ultra-high performance fibre-reinforced concrete (UHPFRC).

2 Basics

2.1 Rotational and elliptical symmetry about a direction on $S^{d-1}$

**Definition 2.1** (Rotational symmetry about a direction). Let $X \in S^{d-1}$ be a random vector and $\mu \in S^{d-1}$. The distribution of $X$ is rotationally symmetric about $\mu$ on $S^{d-1}$ if and only if $X \overset{D}{=} OX$ for every $O \in SO(d)$ fulfilling $O\mu = \mu$.

Let $\mathcal{R}_\mu$ be the class of distributions which are rotationally symmetric about $\mu \in S^{d-1}$. Projecting $X$ onto a vector space orthogonal to $\mu$ yields rotationally symmetric contours. Distributions $F_X \in \mathcal{R}_\mu$ are characterised by densities of the form

$$f_\mu(x) = c_d f(x^T \mu), \quad x \in S^{d-1},$$

where $f : [-1, 1] \rightarrow \mathbb{R}_{\geq 0}$ is absolutely continuous and $c_d$ a normalising constant [1]. The distribution of $X^T \mu$ is absolutely continuous w.r.t the Lebesgue measure on $[-1, 1]$ [13]. The density of $X^T \mu$ reads

$$f(t) = \omega_{d-1} c_d (1 - t^2)^{\frac{d-3}{2}} f(t),$$

where $\omega_{d-1}$ is the surface area of $S^{d-2}$ [5]. A widely known distribution in $\mathcal{R}_\mu$ is the von Mises-Fisher distribution where $f(t) = \exp(\kappa t)$.

**Definition 2.2** (von Mises-Fisher distribution $M_d(\mu, \kappa)$ [13]). The probability density function of the von Mises-Fisher distribution is given by

$$f_{VMF, \mu, \kappa}(x) = c_d \exp(\kappa x^T \mu),$$

where $\kappa \geq 0$ is a concentration parameter, $\mu \in S^{d-1}$ the mean direction, and $c_d$ the normalising constant.
The concentration around $\mu$ increases with $\kappa$. The von Mises-Fisher distribution is unimodal for $\kappa > 0$. For $\kappa = 0$, we get the uniform distribution on the sphere.

A generalisation of the von Mises-Fisher distribution is the Fisher-Bingham distribution [13], where a general quadratic equation is added in the exponent of the density in (2.1). An example is the Kent distribution [7].

**Definition 2.3** (Kent distribution $K(\mu, \kappa, A)$). The probability density function of the Kent distribution is given by

$$f_{K(\mu, \kappa, A)}(x) = c_d \exp(\kappa x^T \mu + x^T A x), \quad (2.4)$$

where $\kappa \geq 0$ is a concentration parameter, $\mu \in S^{d-1}$ the mean direction, $A \in \text{Sym}(d)$ with $A\mu = 0_d$ a shape parameter, and $c_d$ the normalising constant. The concentration around $\mu$ increases with $\kappa$, while $A \in \text{Sym}(d)$ controls the shape of the density contours.

For large $\kappa$, the Kent distribution has a mode at $\mu$ and density contours which are elliptical [14, p.177]. Let $F_\mu$ be the class of distributions on $S^{d-1}$ with a bounded density that admit a unique modal direction $\mu$. We further assume that $\mu$ coincides with the Fisher spherical median [15], that is

$$\mu = \arg\min_{\gamma \in S^{d-1}} E(\arccos(X^T \gamma)). \quad (2.5)$$

For i.i.d. random vectors $X, X_1, \ldots, X_n \in S^{d-1}$ with $X \sim F \in F_\mu$, we estimate $\mu$ by the root-$n$ consistent empirical Fisher spherical median [15]

$$\hat{\mu} = \arg\min_{\gamma \in S^{d}} \sum_{i=1}^{N} \arccos(X_i^T \gamma). \quad (2.6)$$

Note that the definition of the class $R_\mu$ does not include that $\mu$ is the unique modal direction. Here, we restrict attention to distributions in $R_\mu \cap F_\mu$ from now on. E.g., $M_d(\mu, \kappa) \in R_\mu \cap F_\mu$ for $\kappa > 0$, and $K(\mu, \kappa, A) \in F_\mu$ for $\kappa > 0$ under suitable conditions on $A \in \text{Sym}(d)$ given in the next section.

### 2.2 Differential geometry

Differential geometry examines smooth manifolds using differential and integral calculus as well as linear and multi-linear algebra. It originates in studying spherical geometries related to astronomy and the geodesy of the earth. For an introduction to differential geometry, see e.g. [16].

We saw in (1.1) that a linear transformation $\Sigma$ transforms a random vector with a rotationally symmetric distribution into a random vector with an elliptically symmetric distribution in $\mathbb{R}^d$. We want to proceed analogously for distributions on the sphere. However, in general, the linear transformation $\Sigma$ does not necessarily map $S^{d-1}$ onto itself. A remedy is provided by linearising the sphere $S^{d-1}$ at a base point $\mu \in S^{d-1}$.

The tangent space $T_\mu S^{d-1}$ to $S^{d-1}$ at base point $\mu \in S^{d-1}$ is the collection of all tangent vectors to $S^{d-1}$ at $\mu$. It is a local Euclidean vector space with local origin in $\mu$. Given $\mu \in S^{d-1}$ and a tangent vector $v \in T_\mu S^{d-1}$, there is a unique geodesic from $\mu \in S^{d-1}$ to some $x \in S^{d-1}$ given as a mapping

$$e^{\mu, v} : [0, 1] \rightarrow S^{d-1}, \quad (2.7)$$

starting at $e^{\mu, v}(0) = \mu$ with initial velocity $\dot{e}^{\mu, v}(0) = v$ and ending in $e^{\mu, v}(1) = x$ [17].

In the following, we define mappings between the tangent space and the sphere.

**Definition 2.4** (Riemannian exponential map). The Riemannian exponential map

$$\text{Exp}_\mu : T_\mu S^{d-1} \rightarrow S^{d-1} \quad (2.8)$$

maps a vector $v \in T_\mu S^{d-1}$ to $S^{d-1}$ along the geodesic $e^{\mu, v}$ such that $x = \text{Exp}_\mu(v) = e^{\mu, v}(1)$.

The exponential map is locally diffeomorphic onto $V(\mu) = S^{d-1} \setminus \{-\mu\}$, where $-\mu$ is called cut point and the set $\{-\mu\}$ is called cut locus. Within $V(\mu)$ the exponential map $\text{Exp}_\mu$ has an inverse, the Riemannian logarithmic map.

**Definition 2.5** (Riemannian logarithmic map). The Riemannian logarithmic map

$$\text{Log}_\mu : V(\mu) \rightarrow T_\mu S^{d-1} \quad (2.9)$$

maps a vector $x \in S^{d-1}$ into $T_\mu S^{d-1}$ with $\text{Exp}_\mu(\text{Log}_\mu(x)) = x$. 

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3
The distance between \( \mu \) and a point on the sphere is described by the Riemannian distance function.

**Definition 2.6** (Riemannian distance function). For any point \( x \in V(\mu) \), the Riemannian distance function is given by

\[
d_{GD}(\mu, x) = ||\log_{\mu}(x)||_2 = \arccos (x^T \mu) \in [0, \pi).
\]

(2.10)

Consequently, \( \log_{\mu}(x) \in B_{d-1}(\pi) \subset T_{\mu}S^{d-1} \) with \( B_{d}(r) \) the \( d \)-dimensional open ball of radius \( r > 0 \) centred at the origin \( 0_d \).

### 2.2.1 Examples on \( S^2 \)

The locally diffeomorphic exponential map on \( S^2 \) reads

\[
\exp_{\mu} : T_{\mu}S^2 \to S^2 \setminus \{-\mu\},
\]

\[
v \mapsto \mu \cos (||v||_2) + \frac{v}{||v||_2} \sin (||v||_2),
\]

(2.11)

while the logarithmic map is given by

\[
\log_{\mu} : S^2 \setminus \{-\mu\} \to T_{\mu}S^2,
\]

\[
x \mapsto \frac{d_{GD}(x, \mu)}{\sin (d_{GD}(x, \mu))} z.
\]

(2.12)

Here, \( z = (I_d - \mu \mu^T)x \) is the tangential part of \( x \) and we use the convention \( \frac{0}{\sin(0)} = 1 \) [18, 19]. See Figure 1 for an illustration.

With \( \mu = (0, 0, 1)^T \) the logarithmic map in spherical coordinates \( \phi \in [0, 2\pi) \) and \( \theta \in [0, \pi] \) reads

\[
\log_{\mu}(x) = \begin{pmatrix}
\cos (\phi) \\
\sin (\phi) \sin (\theta)
\end{pmatrix} \cdot \frac{\theta}{\sin (\theta)} = \begin{pmatrix}
\cos (\phi) \\
\sin (\phi)
\end{pmatrix} \in T_{\mu}S^2,
\]

(2.13)

with \( d_{GD}(x, \mu) = \arccos (\cos(\theta)) \) = \( \theta \). Thus, the length of any vector \( \log_{\mu}(x) \in T_{\mu}S^2 \) coincides with the colatitude angle \( \theta \). For a random vector \( X \) with a distribution from \( \mathcal{R}_\mu \cap \mathcal{F}_\mu \), we have \( \Phi \sim Unif[0, 2\pi] \) and \( \Phi \) and \( \Theta \) are independent. Hence, \( \log_{\mu}(X) \) has circular density contours in \( T_{\mu}S^2 \).

### 2.3 The Mahalanobis transformation on \( S^{d-1} \)

The idea of the Mahalanobis transformation in \( \mathbb{R}^d \) is to linearly transform a real-valued data matrix into a centred, standardised and uncorrelated data matrix, see e.g. [11].

The Mahalanobis transformation can be generalised to the Riemannian manifold \( S^{d-1} \), see [10]. Here, the row vectors \( y_1, \ldots, y_n \in S^{d-1} \setminus \{-\mu\} \) of an \( n \times d \)-data matrix \( Y \) are mapped onto \( T_{\mu}S^{d-1} \).
Figure 2: The Mahalanobis transformation of $y_1, \ldots, y_{500} \in S^2$. The realisations are i.i.d. Kent distributed with $\mu = (0, 0, 1)$, $\kappa = 12$ and $A = \text{diag}(\beta, -\beta, 0)$ with $\beta = 5$. The Z-direction points out of the page.

The empirical covariance matrix reads

$$\hat{\Sigma}_{T_\mu S_d^{-1}}(Y) = \frac{1}{n} \sum_{i=1}^{n} \log_\mu(y_i) \log_\mu(y_i)^T. \quad (2.14)$$

In analogy to the Mahalanobis transformation in $\mathbb{R}^d$ the transformed vector in $T_\mu S_d^{-1}$ reads

$$v_i = \hat{\Sigma}_{T_\mu S_d^{-1}}(Y)^{-1/2} \log_\mu(y_i). \quad (2.15)$$

Note that the condition $||v_i||_2 < \pi$, required for the application of the exponential map, may not be fulfilled. To ensure this, we normalise $\hat{\Sigma}_{T_\mu S_d^{-1}}(Y)^{-1/2}$ by

$$\hat{\Sigma}_{T_\mu S_d^{-1}}^*(Y)^{-1/2} = \frac{\hat{\Sigma}_{T_\mu S_d^{-1}}(Y)^{-1/2}}{||\hat{\Sigma}_{T_\mu S_d^{-1}}(Y)^{-1/2}||_2}, \quad (2.16)$$

where $|| \cdot ||_2$ is the spectral norm. Note that using (2.16) in (2.15) could increase the concentration of the points around $\mu$.

The Mahalanobis transformation of $y \in S^{d-1}$ reads

$$x = \text{Exp}_\mu\left(\hat{\Sigma}_{T_\mu S_d^{-1}}^*(Y)^{-1/2} \log_\mu(y)\right). \quad (2.17)$$

For $d = 3$, the Mahalanobis transformation is illustrated in Figure 2.

We will use (2.17) to transform realisations $y_1, \ldots, y_n \in S^{d-1}$ an elliptically symmetric distribution. If the Mahalanobis-transformed vectors are rotationally symmetric about $\mu$, we will use the results of Ley et al. [1] for the rotationally symmetric case. They are shortly summarised in the following section.

### 2.4 Quantiles for directional data and the angular Mahalanobis depth

The concept of quantiles for directional data and the angular Mahalanobis depth from Ley et al. [1] are summarised in the following.
variables and the scaled difference between \( \mu \).

Note that by (2.22), the rather complicated non-linear estimator as \( n \to \infty \) under the joint distribution of \( X_1, \ldots, X_n \).

**Proposition 2.1** (Proposition 3.1 in [1]). Let \( F \in \mathcal{F}_\mu \) and let \( f_{proj} \) denote the common density of the projections \( X_i^T \mu, i = 1, \ldots, n \). Set \( \Delta_{c,\tau} := f_{proj}(c_\tau) \). Then there exists a \( d \)-vector \( \Delta_{\mu,c,\tau} \) such that

\[
n^{1/2}(\hat{c}_\tau - c_\tau) = n^{1/2} \sum_{i=1}^N \left( \tau - 1[X_i^T \mu \leq c_\tau] \right) - \frac{\Delta_{\mu,c,\tau}}{\Delta_{c,\tau}} n^{1/2}(\hat{\mu} - \mu) + o_P(1) \tag{2.22}
\]

as \( n \to \infty \) under the joint distribution of \( X_1, \ldots, X_n \).

Note that by (2.22), the rather complicated non-linear estimator \( \hat{c}_\tau \) can be represented as a sum of i.i.d. random variables and the scaled difference between \( \mu \) and its estimator \( \hat{\mu} \). However, the calculation of the \( d \)-vector \( \Delta_{\mu,c,\tau} \) is not straightforward. In the rotationally symmetric case, the representation in Equation (2.22) simplifies.

**Proposition 2.2** (Proposition 3.2 in [1]). Let \( F \in \mathcal{R}_\mu \). Then

\[
n^{1/2}(\hat{c}_\tau - c_\tau) = n^{1/2} \sum_{i=1}^N \left( \tau - 1[X_i^T \mu \leq c_\tau] \right) + o_P(1) \tag{2.23}
\]

as \( n \to \infty \) under the joint distribution of \( X_1, \ldots, X_n \). Therefore, letting \( f_{proj} \) stand for the density of \( X_i^T \mu \), we have that \( n^{1/2}(\hat{c}_\tau - c_\tau) \) is asymptotically normal with mean zero and variance \( \frac{(1-\tau)^2}{f_{proj}(c_\tau)} \).
The reason for the simplification in Equation (2.22) is that $\Delta_{1,\hat{c}}^T n^{1/2}(\hat{\mu} - \mu) \in o_P(1)$ for $F \in \mathcal{R}_\mu$. The absence of $\hat{\mu}$ in Equation (2.23) means that any root-$n$ consistent estimator (e.g., the empirical Fisher spherical median $\hat{\mu}$ or the spherical mean $\sum_{i=1}^n X_i / \|\sum_{i=1}^n X_i\|_2$) can substitute $\mu$ without changing the asymptotic distribution, independently of the dimension $d$. Furthermore, (2.22) is a Bahadur-type representation for univariate sample quantiles [8]. Hence, the directional quantiles of [1] have similar asymptotic properties as the quantiles in $\mathbb{R}$. Therefore, the directional quantiles of [1] can be regarded as canonical for $F \in \mathcal{R}_\mu$.

### 2.4.2 The angular Mahalanobis depth

The angular Mahalanobis depth (AMHD) is defined by [1]

$$AMHD_F(x) = \frac{1}{1 + 1/D_F(x)} \in [0, 1/2]$$

(2.24)

where

$$D_F(x) = \arg\min_{\tau \in [0,1]} \{ c_\tau \geq x^T \mu \}.$$

(2.25)

It provides a centre-outward ordering by assigning each $x \in S^{d-1}$ its depth value. The angular Mahalanobis depth is leaned on the classical Mahalanobis depth

$$MHD_F(x) = \frac{1}{1 + (x - \mu(F))^T (\Sigma(F))^{-1} (x - \mu(F))}, \quad x \in \mathbb{R}^d.$$

(2.26)

$\mu(F)$ and $\Sigma(F)$ are location and scatter functionals under $F$, respectively. The spherical centre $\mu$ corresponds to the centre $\mu(F)$. $MHD_F$ is suited for elliptically symmetric distributions on $\mathbb{R}^d$ since $\Sigma(F)$ contains all necessary information about the principal axes. In contrast, $AMHD_F$ is not suited for distributions which are elliptically symmetric about $\mu \in S^{d-1}$ since information about the shape of the distribution is lost due to the projection $X^T \mu$ in the definition of $D_F(x)$.

### 3 Quantiles for directional data from elliptically symmetric distributions and the elliptical Mahalanobis depth

Here, we present canonical quantiles and a depth for directional distributions which are elliptically symmetric about $\mu \in S^{d-1}$. The idea is to transform the elliptical contours in the tangent space to rotationally symmetric contours analogously to (1.1), such that we are again in the canonical case of Ley et al. [1].

In the following, we consider random vectors $Y, Y_1, \ldots, Y_n \in S^{d-1}$ i.i.d. following a distribution which is elliptically symmetric about $\mu$. Let

$$(\Sigma^*)^{-1/2} = \frac{\Sigma^{-1/2}}{||\Sigma^{-1/2}||_2}$$

(3.1)

and

$$(\Sigma^*)^{1/2} = ||\Sigma^{-1/2}||_2 \cdot \Sigma^{1/2}.$$  

(3.2)

Then, $||[(\Sigma^*)^{-1/2} \log_{\mu}(Y)]||_2 < \pi$ such that we can define

$$G(Y) = \exp_{\mu} \left( (\Sigma^*)^{-1/2} \log_{\mu}(Y) \right).$$

(3.3)

$G(Y)$ is locally diffeomorphic since $\exp_{\mu}$ and $\log_{\mu}$ are locally diffeomorphic, and $(\Sigma^*)^{-1/2}$ is invertible. Its inverse is

$$G^{-1}(Y) = \exp_{\mu} \left( (\Sigma^*)^{1/2} \log_{\mu}(Y) \right).$$

(3.4)

### 3.1 Quantiles for directional data from elliptically symmetric distributions

We define the elliptical projection quantile by

$$c^{G}_\tau = \arg\min_{c \in [-1,1]} E[\rho_{\tau}(G(Y)^T \mu - c)].$$

(3.5)
Figure 4: Illustration of minor $c_\tau^E$, major $c_\tau^E$ and the intrinsic small semi-axis $\arccos (\text{minor } c_\tau^E)$ and the intrinsic large semi-axis $\arccos (\text{major } c_\tau^E)$ of $C_{c_\tau^E \mu}$ for some $\tau$ and $d = 3$.

The partition of the sphere induced by the hyperplane $H_{c_\tau^E \mu} = \{x \in \mathbb{R}^d | c_\tau^E = G(Y)^T \mu\}$ defines the $\tau$-depth contour $C_{c_\tau^E \mu}$ as in (2.20). Note that $c_\tau^E = c_\tau$ if $\Sigma^* = I_{d-1}$ since then $G(Y) = Y$.

An elliptical depth contour is obtained from $C_{c_\tau^E \mu}$ by inverting the transformation via the tangent space shown in Figure 2, that is

$$C_{\hat{c}_\tau^E \mu} = \{G^{-1}(Y) | Y \in C_{c_\tau^E \mu}\}. \quad (3.6)$$

Note that $C_{\hat{c}_\tau^E \mu}$ can be elliptically shaped which is not the case for $C_{c_\tau^E \mu}$.

To define an equivalent to the semi-axes lengths of an ellipse, we set

$$\text{minor } c_\tau^E = \max_{x \in C_{c_\tau^E \mu}} x^T \mu \quad \text{and} \quad \text{major } c_\tau^E = \min_{x \in C_{c_\tau^E \mu}} x^T \mu. \quad (3.7)$$

The intrinsic semi-minor axis of $C_{c_\tau^E \mu}$ is $\arccos (\text{minor } c_\tau^E)$ and the intrinsic semi-major axis of $C_{c_\tau^E \mu}$ is $\arccos (\text{major } c_\tau^E)$. Minor $c_\tau^E$ and major $c_\tau^E$ contain the main information about the concentration and shape of the distribution of $Y$ around $\mu$. A large difference between minor $c_\tau^E$ and major $c_\tau^E$ indicates a strong deviation from a rotationally symmetric distribution, whereas minor $c_\tau^E = \text{major } c_\tau^E$ in the rotationally symmetric case. See Figure 4 for an illustration for $d = 3$. Note that major $c_\tau^E \leq c_\tau \leq \text{minor } c_\tau^E$ by construction.

The empirical elliptical projection quantile reads

$$\hat{c}_\tau^G = \arg \min_{c \in [-1, 1]} \sum_{i=1}^N [\rho_\tau (\hat{G}(Y_i)^T \hat{\mu} - c)] \quad (3.8)$$

with

$$\hat{G}(Y) = \exp_m((\hat{\Sigma}^*)^{-1/2} (\log_m(Y))) \quad (3.9)$$

and $(\hat{\Sigma}^*)^{-1/2}$ given in (2.16). The empirical versions of (3.6) and (3.7) are denoted by $C_{\hat{c}_\tau^E \mu}$, minor $\hat{c}_\tau^E$ and major $\hat{c}_\tau^E$, respectively.

### 3.2 The elliptical Mahalanobis depth

The elliptical Mahalanobis depth (EMHD) is defined by

$$EMHD_F(y) = \frac{1}{1 + 1/\hat{D}_F(y)} = \frac{D_F^G(y)}{1 + D_F^G(y)} \in [0, 1/2], \quad (3.10)$$
Figure 5: Histograms of the longitudes $\phi_{y_{l,i}}$ and $\phi_{x_{l,i}}$, $l = 1, 2, 3, 4$. The red line corresponds to the density of the uniform distribution on $[-\pi, \pi]$.

where

$$D_G^\tau(y) = \arg\min_{\tau \in [0,1]} \{ c_G^\tau \geq \mathcal{G}(y)^T \mu \}.$$  \hspace{1cm} (3.11)

As the angular Mahalanobis depth, the elliptical Mahalanobis depth is leaned on the Classical Mahalanobis depth $\text{MHD}_F$ given in Equation (2.26). Our approach additionally yields a connection to $\Sigma(F)$ via $\hat{\Sigma}^*$ which contains all necessary information about the principal axes. Furthermore, $\text{EMHD}_F$ contains $\text{AMHD}_F$ as a special case: They are equal if the depth contours are rotationally symmetric.

4 Applications

To confirm our findings, we perform a Monte Carlo simulation study. We generated four independent samples $y_{l,i}$, $l = 1, 2, 3, 4$, $i = 1, \ldots, 200$, of Kent distributions with $\mu = (0, 0, 1)^T$, $A = \text{diag}(\beta, -\beta, 0)$, and $\kappa = 5$, $\beta = 2$ ($l = 1$), $\kappa = 7$, $\beta = 3$ ($l = 2$), $\kappa = 10$, $\beta = 4$ ($l = 3$), and $\kappa = 12$, $\beta = 5$ ($l = 4$). The Mahalanobis-transformed $y_{l,i}$ are denoted by $x_{l,i}$. The longitude of $y_{l,i}$ is denoted by $\phi_{y_{l,i}}$, and the longitudes of $x_{l,i}$ are $\phi_{x_{l,i}}$.

The histograms of the longitudes shown in Figure 5 indicate that the transformation leads to uniformly distributed longitudes. This supports that the $x_{l,i}$ are rotationally symmetric about $\mu$. To confirm this visual impression, we test the hypothesis of uniform longitudes. We use Watson’s test [22, p. 156] implemented in the R-package Directional [23]. Watson’s test applied on $\phi_{y_{l,i}}$ gave p-values less than 0.004 for all designs $l = 1, 2, 3, 4$. The p-values for the Mahalanobis-transformed angles $\phi_{x_{l,i}}$ were 0.7086 ($l = 1$), 0.5132 ($l = 2$), 0.3436 ($l = 3$), 0.5268 ($l = 4$) which supports an assumption of uniform longitudes.

For illustration, the empirical quartiles $\hat{c}_\tau$, $\tau = 0.25, 0.5$, and 0.75 as well as minor $\hat{c}_\tau^E$ and major $\hat{c}_\tau^E$ for $l = 4$ are given in Table 1. Figure 6 shows the corresponding depth contours.

4.1 Trimming of directional data

The angular Mahalanobis depth $\text{AMHD}_F$ is canonical for trimming of directional data from $F \in R_\mu$. The trimming corresponds to deleting the points on $S_d$ below the $\tau$-depth contour $C_{c_\tau, \mu}$ given in (2.20) with $\tau \in [0,1]$. If the
Quantiles and depth for directional data from elliptically symmetric distributions

| $\tau$ | 0.25 | 0.50 | 0.75 |
|--------|------|------|------|
| $c_{\tau}$ | 0.8110 | 0.9090 | 0.9660 |
| minor $c_{E}^{\tau}$ | 0.9370 | 0.9677 | 0.9870 |
| major $c_{E}^{\tau}$ | 0.6847 | 0.8347 | 0.9245 |

Table 1: The empirical quartiles minor $c_{E}^{\tau}$, major $c_{E}^{\tau}$ and $c_{\tau}$ of $y_{4,i}$, $i = 1, \ldots, n$. The sample is shown in Figure 6.

Figure 6: Realisations $y_{4,i}$, $i = 1, \ldots, n$, are given as points on $S^{2}$ with empirical $\tau$-depth contours, $\tau = 0.25, 0.5, 0.75$. The blue circle corresponds to the empirical $\tau$-depth contour $C_{c_{\tau}}$. The green ellipse corresponds to the empirical $\tau$-depth contour $C_{E}^{c_{E}^{\tau}}$. The Z-direction points out of the page.

The underlying distribution has elliptical contours, trimming results in circular contours when using $AMHD_{F}$. In contrast, trimming based on $EMHD_{F}$, which deletes points below $C_{E}^{c_{E}^{\tau}}$, preserves the elliptical shape of the contours. See Figure 7 for an illustration.

Figure 7: Realisations $y_{4,i}$, $i = 1, \ldots, n$, are given as points on $S^{2}$. The blue circle corresponds to $C_{c_{\tau}}$ (left) and the green ellipse corresponds to $C_{E}^{c_{E}^{\tau}}$ with $\tau = 0.25$ of $y_{4,i}$, $i = 1, \ldots, n$. Trimmed points are purple. The Z-direction points out of the page.
4.2 Analysis of fibre directions in ultra-high performance fibre-reinforced concrete

Ultra-high performance fibre-reinforced concrete (UHPFRC) is a relatively new material in civil engineering. If cracks appear in the concrete due to loading, fibres crossing the crack counteract the crack propagation. As fibres have no directional sense, our data are restricted to the upper hemisphere of $S^2$. We analyse a data set from [12] which consists of $n = 598$ measurements of fibre directions. The fibres crossed a crack in a UHPFRC-specimen subject to a bending test. The crack has a planar shape with normal direction corresponding to the Z-axis used in the analysis. The fibre directions are denoted by $y_i$, their Mahalanobis transforms are $x_i$, $i = 1, \ldots, n$. Furthermore, we denote by $\phi_{y_i}$ the longitudes of $y_i$ and by $\phi_{x_i}$ the longitudes of $x_i$. 

Figure 8: Fibre directions $y_i$ in UHPFRC before (a) and after (b) Mahalanobis transformation. The Z-direction points out of the page. Histograms of the longitudes $\phi_{y_i}$ (c) and $\phi_{x_i}$ (d). The red line corresponds to the density of the uniform distribution on $[-\pi, \pi]$. 

4.2.1 Visual inspection, rotational symmetry, and quartiles

As a first step, we inspect the data visually. Figure 8 shows the original and Mahalanobis-transformed fibre directions together with estimates of their densities. The distribution of the fibre directions was computed by the function `mediandir()` from the R-package `Directional` [23].

Figure 8d indicates that the Mahalanobis-transformed fibre directions have uniformly distributed longitudes. Using Watson’s test applied on the empirical projection quartiles, we obtained a p-value of less than $10^{-4}$ such that rotational symmetry is rejected at any meaningful nominal level. The p-value for $\phi_{x_i}$ is 0.2660 such that the assumption of rotational symmetry about $\mu$ is not rejected.

In Figure 9, we illustrate the empirical $\tau$-depth contours $C_{e_r, \mu}$ given in (2.20) and $C_{e^\tau, \mu}$ given in (3.6) for $\tau = 0.25, 0.5, 0.75$. The values minor $c_{e^\tau}$, major $c_{e^\tau}$, and $c_\tau$ are summarised in Table 2. We see that major $c_{e^\tau}$ is heavier than minor $c_{e^\tau}$ for all $\tau = 0.25, 0.5, 0.75$. Thus, the shape of the underlying density seems to be slightly better fitted by an elliptically symmetric distribution than by a rotationally symmetric distribution.

| $\tau$ | 0.25 | 0.5 | 0.75 |
|--------|------|------|------|
| $c_{e_\tau}$ | 0.8489 | 0.9349 | 0.9729 |
| minor $c_{e^\tau}$ | 0.8785 | 0.9507 | 0.9808 |
| major $c_{e^\tau}$ | 0.7986 | 0.9128 | 0.9629 |

Table 2: The empirical projection quartiles minor $c_{e^\tau}$, major $c_{e^\tau}$ and $c_\tau$ of the fibre directions $y_i$.

4.2.2 Goodness-of-fit test and trimming

In the next step, we fit a rotationally symmetric directional distribution to the Mahalanobis transformed data $x_i$. We use a von Mises-Fisher distribution $F_0 = M_5(\mu, \kappa)$. Maximum likelihood estimation by using the function `vmf.mle()` implemented in the R-package `Directional` yields $\hat{\kappa} = 6.97$. As the estimated mean direction is very close to the $Z$ axis, we consider $\mu = (0, 0, 1)^T$ and $\kappa = 6, 7, 8$. We then perform the goodness-of-fit test given in [1] based on the projection quartiles ($c_{0.25}, c_{0.5}, c_{0.75}$). The null hypothesis $H_0 : F = F_0$ against $H_1 : F \neq F_0$ is rejected for all three $\kappa$ values ($p < 10^{-4}$).

To analyze the reason for rejection, we convert the unit vectors $x_i$ in spherical coordinates $(\theta_i, \phi_i)$. A closer investigation of the co-latitude angle $\theta_i$ reveals that $1 - \cos(\theta_i)$ has a heavy tail (see Figure 10a). Under the hypothesis of a von Mises-Fisher distribution, we would expect $1 - \cos(\Theta) \sim Exp(\kappa)$ for large $\kappa$, see [14, Eq. (4.29)]. Trimming the directions $x_i$ below the $\tau$-depth contour $C_{e_r, \mu}, \tau = 0.15$, removes the heavy tail in the trimmed sample $x_i^{\text{trim}}$, see Figure 10b. We repeat the goodness-of-fit test with $F_0 = M_5(\mu, \kappa)$ with the trimmed data $x_i^{\text{trim}}$. We chose $\mu = (0, 0, 1)^T$ and $\kappa = 8, 9, 10$ because we expect a higher concentration parameter due to trimming. The asymptotic p-values are 0.0130 ($\kappa = 8$), 0.2854 ($\kappa = 9$) and 0.1288 ($\kappa = 10$). Thus, $\kappa = 9$ provides the best fit to the trimmed data.
Figure 10: Histograms of $1 - \cos(\theta_i)$, $i = 1, \ldots, n$, before (a) and after (b) truncation by trimming. The red line corresponds to the theoretical density of $Exp(\kappa)$, $\kappa = 7$.

In fact, the strong deviation of fibre directions from the tensile axis of almost 15% of the fibres may have favoured the cracking at this position.

5 Conclusion

We extended the concept of quantiles and depth for directional data from Ley et. al. [1]. Their concept provides useful geometric properties of the depth contours (such as convexity and rotational equivariance) and a Bahadur-type representation of the quantiles. However, a disadvantage is that rotationally symmetric depth contours are always produced, even if the underlying distribution is not rotationally symmetric [9]. Our extension solves this lack of flexibility for distributions with elliptical depth contours. The main idea was to transform the elliptical contours in the tangent space to rotationally symmetric contours, apply the results of Ley et al. [1] to those, and then transform back. In view of similarities with the classical Mahalanobis depth, our depth was called elliptical Mahalanobis depth ($EMHD_F$). Our results were confirmed by a Monte Carlo simulation study. Furthermore, we introduced tools to evaluate the ellipticity of depth contours and demonstrated that our approach is the obvious choice for trimming directional data from an elliptically symmetric distribution. We applied our quantiles and depth to analyse fibre directions in fibre-reinforced concrete.

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