Superposed Hyperbolic Kink and Pulse Solutions of Coupled $\phi^4$, NLS and MKdV Equations

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Abstract:
We obtain novel solutions of a coupled $\phi^4$, a coupled nonlinear Schrödinger (NLS) and a coupled modified Korteweg de Vries (MKdV) model which can be re-expressed as a linear superposition of either the sum or the difference of two hyperbolic kink or two hyperbolic pulse solutions. These results demonstrate that the notion of superposed solutions extends to coupled nonlinear equations as well.

1 Introduction

Superposition principle is the hallmark of linear theories and it has helped us in understanding various features of such theories. In contrast, because of the nonlinear term, the nonlinear theories lack such a superposition principle. This we believe is one of the reasons why until now we only have a limited knowledge of such theories. Undoubtedly, during the last five decades substantial progress has been made in understanding several novel features of a number of nonlinear models but even at present new features of nonlinear theories are being uncovered. In a recent paper \[1\] we have shown that a large number of nonlinear equations, including both symmetric and asymmetric $\phi^4$, MKdV, NLS and several other nonlinear equations, many of which have found wide application in physics, admit superposed periodic kink and periodic pulse solutions.
By superposed here we mean that a solution is a linear combination of two kink solutions or two pulse solutions. However, only one hyperbolic superposed solution was obtained by us in some of these models as the difference of two kink solutions. In fact, even before us a number of previous researchers [2, 3, 4, 5, 6, 7] had obtained such a superposed kink solution in a variety of physical models. However, to the best of our knowledge, to date no one has been able to obtain solutions which can be re-expressed either as the sum or sum of two (hyperbolic) kink solutions or the sum or difference of two pulse solutions. The purpose of this paper is to partially fill this gap.

We present three well known coupled models, i.e. a coupled $\phi^4$, a coupled NLS and a coupled MKdV, which have found widespread applications in ferroelectric [8] and multiferroic [9] phase transitions, signal propagation in optical fibers [10], etc. We then obtain novel solutions which can be re-expressed either as the sum or the difference of two (hyperbolic) kink or two pulse solutions. Needless to say that these models also admit novel solutions which can be re-expressed as the sum or the difference of two periodic kink or two periodic pulse solutions. However, in this paper we only confine ourselves to the superposition of hyperbolic kink and pulse solutions. In a subsequent paper we hope to address the question of the superposition of periodic kink and pulse solutions in these models.

The plan of the paper is the following. In Sec. II, we discuss a coupled $\phi^4$ model which has received attention previously [11] and obtain seven solutions which can be re-expressed as either the sum or the difference of two kink or two pulse solutions. In Sec. III, we discuss a coupled NLS model, also popularly known as the Manakov model [12] and show that it admits seven solutions (which are similar to the corresponding solutions of the coupled $\phi^4$ model) which can be re-expressed as the sum or the difference of two kink or two pulse solutions. In Sec. IV we discuss a coupled MKdV model [13] and show that this model admits four solutions which can be re-expressed as the sum or the difference of two kink solutions. Finally, in Sec. V we summarize our main results and point out some of the open problems.

2 Novel Superposed Solutions of a Coupled $\phi^4$ Model

Let us consider the following coupled $\phi^4$ equations [11]

$$\phi_{1xx} = a_1 \phi_1 + [b_1 \phi_1^2 + d_1 \phi_2^2] \phi_1,$$  \hspace{1cm} (1)

$$\phi_{2xx} = a_2 \phi_2 + [b_2 \phi_1^2 + d_2 \phi_2^2] \phi_2.$$  \hspace{1cm} (2)
Before we discuss the superposed kink and pulse solutions, let us note that the coupled Eqs. (1) and (2) admit the following kink and pulse solutions.

It is well known that Eqs. (1) and (2) admit a kink solution in both $\phi_1$ and $\phi_2$, i.e.

$$\phi_1(x) = A_1 \tanh(\beta x), \quad \phi_2 = A_2 \tanh(\beta x),$$

provided

$$a_1 = -2 \beta^2, \quad b_1 A_1^2 + d_1 A_2^2 = b_2 A_1^2 + d_2 A_2^2 = 2 \beta^2.$$  \hspace{1cm} (4)

It is also known that Eqs. (1) and (2) admit a pulse solution in both $\phi_1$ and $\phi_2$, i.e.

$$\phi_1(x) = A_1 \text{sech}(\beta x), \quad \phi_2 = A_2 \text{sech}(\beta x),$$

provided

$$a_1 = -2 \beta^2, \quad b_1 A_1^2 + d_1 A_2^2 = b_2 A_1^2 + d_2 A_2^2 = -2 \beta^2.$$  \hspace{1cm} (6)

Finally, Eqs. (1) and (2) are also known to admit a mixed kink-pulse solution, i.e. say a kink solution in $\phi_1$ and a pulse solution in $\phi_2$ or vice versa. In particular, they admit

$$\phi_1(x) = A_1 \tanh(\beta x), \quad \phi_2 = A_2 \text{sech}(\beta x),$$

provided

$$b_1 A_1^2 - d_1 A_2^2 = b_2 A_1^2 - d_2 A_2^2 = 2 \beta^2,$$

$$b_2 A_1^2 = \beta^2 - a_2, \quad d_1 A_2^2 = -a_1 - 2 \beta^2.$$  \hspace{1cm} (8)

We now show that the coupled Eqs. (1) and (2) in fact also admit novel solutions which can be re-expressed as the sum or the difference of the above kink and pulse solutions.

Let us first note that in case $\phi_2(x) = \alpha \phi_1(x)$, where $\alpha$ is a real number, then Eq. (2) is identical to Eq. (1) provided

$$a_2 = a_1, \quad b_1 + \alpha^2 d_1 = b_2 + \alpha^2 d_2,$$  \hspace{1cm} (9)

and in this case we only need to solve Eq. (1) which as we have recently shown [1] does not admit superposed hyperbolic solutions.

We now show that instead when $\phi_2$ and $\phi_1$ are not proportional to each other then the coupled Eqs. (1) and (2) admit seven distinct solutions which can be re-expressed as the sum or the difference of either two kink or two pulse solutions.
**Superposed Solution I**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ \phi_1(x) = \frac{A}{B + \cosh^2(\beta x)}, \quad \phi_2(x) = \frac{D \cosh(\beta x) \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \tag{10} \]

provided

\[
\begin{align*}
a_1 &= a_2 = 2\beta^2, \\
b_1 &A^2 = -2B(\beta^2 + 1), \\
d_1 &D^2 = -6\beta^2,
\end{align*}
\]

\[
\begin{align*}
b_2 &A^2 = -6B(\beta^2 + 1), \\
d_2 &D^2 = -2\beta^2.
\end{align*}
\tag{11}
\]

Thus for this solution \(b_1, d_1, b_2, d_2\) are all negative while \(a_1, a_2\) are positive.

Now on making use of the addition theorem for \(\tanh(a + b)\), one can derive two novel identities

\[
\begin{align*}
tanh(x + \Delta) - \tanh(x - \Delta) &= \frac{\sinh(2\Delta)}{B + \cosh^2(x)}, \quad B = \sinh^2(\Delta), \tag{12} \\
tanh(x + \Delta) + \tanh(x - \Delta) &= \frac{2 \sinh(x) \cosh(x)}{B + \cosh^2(x)}, \quad B = \sinh^2(\Delta). \tag{13}
\end{align*}
\]

On comparing the solution (10) with the identities (12) and (13), solution I as given by Eq. (10) can be re-expressed as

\[
\begin{align*}
\phi_1(x) &= \frac{\beta}{\sqrt{2|b_1|}} [\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)], \\
\phi_2(x) &= \frac{\sqrt{3}\beta}{\sqrt{2|d_1|}} [\tanh(\beta x + \Delta) + \tanh(\beta x - \Delta)], \tag{14}
\end{align*}
\]

where \(\sinh^2(\Delta) = B\).

**Superposed Solution II**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ \phi_1(x) = \frac{A}{B + \cosh^2(\beta x)}, \quad \phi_2(x) = \frac{D \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \tag{15} \]

provided

\[
\begin{align*}
a_1 &= -4\beta^2, \\
b_1 &A^2 = 2(2B^2 + 5B + 3)\beta^2, \\
d_1 &D^2 = 6(2B + 1)\beta^2,
\end{align*}
\]

\[
\begin{align*}
a_2 &= -\beta^2, \\
b_2 &A^2 = 6(B + 1)\beta^2, \\
d_2 &B^2 = 2(3 + 4B)\beta^2. \tag{16}
\end{align*}
\]
Thus for this solution \( b_1, d_1, b_2, d_2 \) are all positive while \( a_1, a_2 \) are negative.

Now on making use of the addition theorem for \( \text{sech}(a + b) \), one can derive two novel identities

\[
\text{sech}(x - \Delta) - \text{sech}(x + \Delta) = \frac{2 \sinh(x) \sinh(\Delta)}{B + \cosh^2(x)}, \quad B = \sinh^2(\Delta), \quad (17)
\]

\[
\text{sech}(x + \Delta) + \text{sech}(x - \Delta) = \frac{2 \cosh(\Delta) \cosh(x)}{B + \cosh^2(x)}, \quad B = \sinh^2(\Delta). \quad (18)
\]

On comparing the solution (15) with the identities (12) and (17), solution II as given by Eq. (15) can be re-expressed as

\[
\phi_1(x) = \frac{\sqrt{3}\beta}{\sqrt{2|b_2| \cosh(\Delta)}} \left[ \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right],
\]

\[
\phi_2(x) = \frac{\sqrt{3} \cosh(2\Delta) \beta}{\sqrt{2d_1 \cosh(\Delta)}} \left[ \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \right], \quad (19)
\]

where \( B = \sinh^2(\Delta) \).

**Superposed Solution III**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[
\phi_1(x) = \frac{A}{B + \cosh^2(\beta x)}, \quad \phi_2(x) = \frac{D \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \quad (20)
\]

provided

\[
\begin{align*}
a_1 &= -4\beta^2, \quad b_1 A^2 = 2B(2B - 1)\beta^2, \quad d_1 D^2 = 6(2B + 1)\beta^2, \\
a_2 &= -\beta^2, \quad b_2 A^2 = -6B\beta^2, \quad d_2 \beta^2 = 2(1 + 4B)\beta^2. \quad (21)
\end{align*}
\]

Thus for this solution while \( d_1, d_2 < 0, \ a_1, a_2, b_2 < 0 \) while \( b_1 > \) (<) 0 depending on if \( B > \) (<) 1/2 while \( b_1 = 0 \) at \( B = 1/2 \).

On comparing the solution (21) with the identities (12) and (18), solution III as given by Eq. (20) can be re-expressed as

\[
\phi_1(x) = \frac{\sqrt{3}\beta}{\sqrt{2|b_2| \cosh(\Delta)}} \left[ \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right],
\]

\[
\phi_2(x) = \frac{\sqrt{3} \cosh(2\Delta) \beta}{\sqrt{2d_1 \cosh(\Delta)}} \left[ \text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta) \right], \quad (22)
\]

where \( B = \sinh^2(\Delta) \).
Superposed Solution IV

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ \phi_1(x) = \frac{A \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad \phi_2(x) = \frac{D \sinh(\beta x) \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \quad (23) \]

provided

\[ a_1 = \frac{(5 - B)\beta^2}{(1 + B)}, \quad b_1 A^2 = 2(2B^2 + 5B + 3)\beta^2, \quad d_1 D^2 = 6(2B + 1)\beta^2, \]
\[ a_2 = -\beta^2, \quad b_2 A^2 = 6(B + 1)\beta^2, \quad d_2 \beta^2 = 2(3 + 4B)\beta^2. \quad (24) \]

Thus for this solution \( b_1, d_1, b_2, d_2 \) are all positive while \( a_1, a_2 \) are negative.

On making use of the novel identities (13) and (17), one can then re-express solution IV as given by Eq. (23) as

\[ \phi_1(x) = \frac{\sqrt{3} \beta}{\sqrt{2b_2} \coth(\Delta)} \left[ \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \right], \]
\[ \phi_2(x) = \frac{\sqrt{3} \cosh(2\Delta) \beta}{\sqrt{2d_1}} \left[ \text{tanh}(\beta x + \Delta) + \text{tanh}(\beta x - \Delta) \right], \quad (25) \]

where \( B = \sinh^2(\Delta) \).

Superposed Solution V

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ \phi_1(x) = \frac{A \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad \phi_2(x) = \frac{D \sinh(\beta x) \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \quad (26) \]

provided

\[ a_1 = -\frac{(6 + B)\beta^2}{B}, \quad b_1 A^2 = \frac{2(B + 1)(4B + 3)\beta^2}{B}, \]
\[ d_1 D^2 = \frac{6\beta^2}{B}, \quad a_2 = \frac{2(2B + 3)\beta^2}{B}, \]
\[ b_2 A^2 = \frac{2(2B^2 + 13B + 3)\beta^2}{B}, \quad d_2 = 2(3 + 4B)\beta^2. \quad (27) \]

Thus for this solution \( b_1, d_1, b_2, d_2 \) are all positive while \( a_1, a_2 \) are negative.
On making use of the novel identities (13) and (18), one can then re-express solution V as given by Eq. (26) re-expressed as

\[
\phi_1(x) = \sqrt{\frac{4 \sinh^2(\Delta) + 3\beta}{2b_1 \sinh(\Delta)}} \left[ \text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta) \right],
\]

\[
\phi_2(x) = \sqrt{\frac{4 \sinh^2(\Delta) + 3\beta}{2d_1 \sinh(\Delta)}} \left[ \text{tanh}(\beta x + \Delta) + \text{tanh}(\beta x - \Delta) \right],
\]

(28)

where \( B = \sinh^2(\Delta) \).

**Superposed Solution VI**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[
\phi_1(x) = \frac{A \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad \phi_2(x) = \frac{D \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0,
\]

(29)

provided

\[
a_1 = -\beta^2, \quad b_1 A^2 = 2B \beta^2, \quad d_1 D^2 = 6(B + 1)\beta^2,
\]

\[
a_2 = -\beta^2, \quad b_2 A^2 = 6B \beta^2, \quad d_2 D^2 = 2(B + 1)\beta^2.
\]

(30)

Thus for this solution \( b_1, d_1, b_2, d_2 \) are all positive while \( a_1, a_2 \) are negative.

On making use of the novel identities (17) and (18), one can then re-express solution VI as given by Eq. (26) as

\[
\phi_1(x) = \frac{\beta}{\sqrt{2b_1}} \left[ \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \right],
\]

\[
\phi_2(x) = \frac{\beta}{\sqrt{2d_2}} \left[ \text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta) \right],
\]

(31)

where \( B = \sinh^2(\Delta) \).

**Superposed Solution VII**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[
\phi_1(x) = 1 - \frac{A}{B + \cosh^2(\beta x)}, \quad \phi_2(x) = \frac{D}{B + \cosh^2(\beta x)}, \quad A, B, D > 0,
\]

(32)

provided

\[
b_1 = -a_1 = 2\beta^2, \quad A = \frac{3(1 + 2B) + \sqrt{8 + (2B + 1)^2}}{4},
\]
\[ d_1 D^2 = \sqrt{8 + (2B + 1)^2 - (2B + 1)} \frac{3A\beta^2}{2}, \quad b_2 + a_2 = 4\beta^2, \]
\[ b_2 A = 3(2B + 1)\beta^2, \quad b_2 A^2 + d_2 D^2 = 8B(B + 1)\beta^2. \]  
(33)

Thus for this solution \( b_1, d_1, a_2, b_2 \) are all positive, \( a_1 < 0 \) while the sign of \( d_2 \) depends on the value of \( B \).

On comparing the solution \((32)\) with the identity \((12)\), solution VII as given by Eq. \((32)\) can be re-expressed as

\[ \phi_1(x) = 1 - \frac{\beta}{\sqrt{2|h_1|}} [\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)], \]
\[ \phi_2(x) = \frac{\sqrt{3\beta}}{2\sqrt{d_1}} K [\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)], \]  
(34)

where \( \sinh^2(\Delta) = B \) and

\[ K = \left[ \cosh(2\Delta) \sqrt{8 + \cosh^2(2\Delta)} - [2 + \cosh^2(2\Delta)] \right]^{1/2}. \]  
(35)

3 Hyperbolic Superposed Solutions of the Coupled NLS Model

Let us consider the following coupled NLS equations \([12]\)

\[ iu_{1t} + u_{1xx} + [g_{11}|u_1|^2 + g_{12}|u_2|^2]u_1, \]  
(36)

\[ iu_{2t} + u_{2xx} + [g_{21}|u_1|^2 + g_{22}|u_2|^2]u_2. \]  
(37)

Before we discuss the superposed kink and pulse solutions, let us note that the coupled Eqs. \((36)\) and \((37)\) admit the following kink and pulse solutions. In particular, it is well known that Eqs. \((36)\) and \((37)\) admit a kink solution in both \( u_1 \) and \( u_2 \), i.e.

\[ u_1(x, t) = A_1 e^{i\omega_1 t} \tanh(\beta x), \quad u_2(x, t) = A_2 e^{i\omega_1 t} \tanh(\beta x), \]  
(38)

provided

\[ \omega_1 = \omega_2 = -2\beta^2, \quad g_{11}A_1^2 + g_{12}A_2^2 = g_{21}A_1^2 + g_{22}A_2^2 = 2\beta^2. \]  
(39)

It is also known that Eqs. \((36)\) and \((37)\) admit a pulse solution in both \( u_1 \) and \( u_2 \), i.e.

\[ u_1(x, t) = A_1 e^{i\omega_1 t} \text{sech}(\beta x), \quad u_2(x, t) = A_2 e^{i\omega_1 t} \text{sech}(\beta x), \]  
(40)
\[\omega_1 = \omega_2 = \beta^2, \quad g_{11}A_1^2 + g_{12}A_2^2 = g_{21}A_1^2 + g_{22}A_2^2 = -2\beta^2. \quad (41)\]

Finally, Eqs. (36) and (37) are also known to admit a mixed kink-pulse solution, i.e. say a kink solution in \(u_1\) and a pulse solution in \(u_2\) or vice versa. In particular, they admit
\[u_1(x, t) = A_1e^{i\omega_1 t} \tanh(\beta x)\], \[u_2(x, t) = A_2e^{i\omega_2 t} \sech(\beta x)\], \quad (42)

provided
\[g_{11}A_2^2 - g_{12}A_1^2 = g_{21}A_2^2 - g_{22}A_1^2 = 2\beta^2, \quad g_{21}A_1^2 = \beta^2 - \omega_2, \quad g_{12}A_2^2 = -\omega_1 - 2\beta^2. \quad (43)\]

We now show that the coupled Eqs. (36) and (37) in fact also admit novel solutions which can be re-expressed as the sum or the difference of the above kink and pulse solutions.

Let us first note that in case \(u_2(x, t) = \alpha u_1(x, t)\), where \(\alpha\) is a real number, then Eq. (37) is identical to Eq. (36) provided
\[g_{11} + \alpha^2 g_{12} = g_{21} + \alpha^2 g_{22}, \quad (44)\]

and in this case we only need to solve Eq. (36) which as we have recently shown \[1\] does not admit superposed hyperbolic solutions.

We now show that instead when \(u_2\) and \(u_1\) are not proportional to each other then the coupled Eqs. (36) and (37) admit seven distinct solutions which can be re-expressed as the sum or the difference of either two kink or two pulse solutions.

**Superposed Solution I**

It is easy to check that the coupled Eqs. (36) and (37) admit the superposed hyperbolic solution
\[u_1(x, t) = e^{i\omega_1 t} \frac{A}{B + \cosh^2(\beta x)}, \quad u_2(x, t) = e^{i\omega_2 t} \frac{D \cosh(\beta x) \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \quad (45)\]

provided
\[\omega_1 = \omega_2 = 2\beta^2, \quad g_{11}A_1^2 = -2B(B + 1)\beta^2, \quad g_{12}D^2 = -6\beta^2, \quad g_{21}A_1^2 = -6B(B + 1)\beta^2, \quad g_{22}D^2 = -2\beta^2. \quad (46)\]
Thus for this solution \( g_{11}, g_{12}, g_{21}, g_{22} \) are all negative while \( \omega_1 = \omega_2 \) are positive.

On comparing the solution (45) with the identities (12) and (13), the solution I as given by Eq. (45) can be re-expressed as

\[
\begin{align*}
    u_1(x, t) &= e^{i\omega_1 t} \frac{\beta}{\sqrt{2|g_{11}|}} \left[ \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right], \\
    u_2(x, t) &= e^{i\omega_2 t} \frac{\sqrt{3} \beta}{\sqrt{2|g_{12}|}} \left[ \tanh(\beta x + \Delta) + \tanh(\beta x - \Delta) \right],
\end{align*}
\]

(47)

where \( \sinh^2(\Delta) = B \).

**Superposed Solution II**

It is easy to check that the coupled Eqs. (36) and (37) admit the superposed hyperbolic solution

\[
\begin{align*}
    u_1(x, t) &= e^{i\omega t} \frac{A}{B + \cosh^2(\beta x)}, \\
    u_2(x, t) &= e^{i\omega_2 t} \frac{D \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0,
\end{align*}
\]

(48)

provided

\[
\begin{align*}
    \omega_1 &= -4\beta^2, \quad g_{11} A^2 = 2(2B^2 + 5B + 3)\beta^2, \quad g_{12} D^2 = 6(2B + 1)\beta^2, \\
    \omega_2 &= -\beta^2, \quad g_{21} A^2 = 6(B + 1)\beta^2, \quad g_{22} \beta^2 = 2(3 + 4B)\beta^2.
\end{align*}
\]

(49)

Thus for this solution \( g_{11}, g_{12}, g_{21}, g_{22} \) are all positive while \( \omega_1, \omega_2 \) are negative.

On comparing the solution (48) with the identities (12) and (17), solution II as given by Eq. (48) can be re-expressed as

\[
\begin{align*}
    u_1(x, t) &= e^{i\omega_1 t} \frac{\sqrt{3} \beta}{\sqrt{2g_{21} \sinh(\Delta)}} \left[ \tanh(\beta x + \Delta) - \tanh(\beta x - \Delta) \right], \\
    u_2(x, t) &= e^{i\omega_2 t} \frac{\sqrt{3} \cosh(2\Delta) \beta}{\sqrt{2g_{12} \sinh(\Delta)}} \left[ \sech(\beta x - \Delta) - \sech(\beta x + \Delta) \right],
\end{align*}
\]

(50)

where \( B = \sinh^2(\Delta) \).

**Superposed Solution III**

It is easy to check that the coupled Eqs. (36) and (37) admit the superposed hyperbolic solution

\[
\begin{align*}
    u_1(x, t) &= e^{i\omega_1 t} \frac{A}{B + \cosh^2(\beta x)}, \\
    u_2(x, t) &= e^{i\omega_2 t} \frac{D \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0,
\end{align*}
\]

(51)
provided

\[ \omega_1 = -4\beta^2, \quad g_{11}A^2 = 2B(2B - 1)\beta^2, \quad g_{12}D^2 = 6(2B + 1)\beta^2, \]

\[ \omega_2 = -\beta^2, \quad g_{21}A^2 = 6B\beta^2, \quad g_{22}\beta^2 = 2(1 + 4B)\beta^2. \]  

(52)

Thus for this solution while \( g_{12}, g_{22} < 0, \omega_1, \omega_2, g_{21} < 0 \) while \( g_{11} > 0 \) depending on if \( B > (\leq) \frac{1}{2} \) while \( g_{11} = 0 \) at \( B = \frac{1}{2} \).

On comparing the solution (52) with the identities (12) and (18), solution III as given by Eq. (51) can be re-expressed as

\[ u_1(x, t) = e^{i\omega_1 t} \frac{\sqrt{3}\beta}{\sqrt{2|g_{21}|} \cosh(\Delta)} \left[ \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \right], \]

\[ u_2(x, t) = e^{i\omega_2 t} \frac{\sqrt{3}\cosh(2\Delta)\beta}{\sqrt{2|g_{12}|} \cosh(\Delta)} \left[ \text{tanh}(\beta x + \Delta) + \text{tanh}(\beta x - \Delta) \right], \]  

(53)

where \( B = \sinh^2(\Delta) \).

**Superposed Solution IV**

It is easy to check that the coupled Eqs. (36) and (37) admit the superposed hyperbolic solution

\[ u_1(x, t) = e^{i\omega_1 t} \frac{A \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad u_2(x, t) = e^{i\omega_2 t} \frac{D \sinh(\beta x) \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \]  

(54)

provided

\[ \omega_1 = \frac{(5 - B)\beta^2}{(1 + B)}, \quad g_{11}A^2 = 2(2B^2 + 5B + 3)\beta^2, \quad g_{12}D^2 = 6(2B + 1)\beta^2, \]

\[ \omega_2 = -\beta^2, \quad g_{21}A^2 = 6(B + 1)\beta^2, \quad g_{22}\beta^2 = 2(3 + 4B)\beta^2. \]  

(55)

Thus for this solution \( g_{11}, g_{21}, g_{21}, g_{22} \) are all positive while \( \omega_1, \omega_2 \) are negative.

On making use of the novel identities (13) and (17), one can then re-express solution IV as given by Eq. (54) as

\[ u_1(x, t) = e^{i\omega_1 t} \frac{\sqrt{3}\beta}{\sqrt{2|g_{21}|} \coth(\Delta)} \left[ \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \right], \]

\[ u_2(x, t) = e^{i\omega_2 t} \frac{\sqrt{3}\cosh(2\Delta)\beta}{\sqrt{2|g_{12}|} \cosh(\Delta)} \left[ \text{tanh}(\beta x + \Delta) + \text{tanh}(\beta x - \Delta) \right], \]  

(56)

where \( B = \sinh^2(\Delta) \).
Superposed Solution V

It is easy to check that the coupled Eqs. (36) and (37) admit the superposed hyperbolic solution

\[ u_1(x, t) = e^{i\omega_1 t} \frac{A \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad u_2(x, t) = e^{i\omega_2 t} \frac{D \sinh(\beta x) \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \]

provided

\[ \omega_1 = -\frac{(6 + B)\beta^2}{B}, \quad g_{11} A^2 = \frac{2(B + 1)(4B + 3)\beta^2}{B}, \]
\[ g_{12} D^2 = \frac{6\beta^2}{B}, \quad \omega_2 = -\frac{2(2B + 3)\beta^2}{B}, \]
\[ g_{21} A^2 = \frac{2(2B^2 + 13B + 3)\beta^2}{B}, \quad g_{22} = \frac{2(3 + 4B)\beta^2}{B}. \]

Thus for this solution \( g_{11}, g_{12}, g_{21}, g_{22} \) are all positive while \( \omega_1, \omega_2 \) are negative.

On making use of the novel identities (13) and (18), one can then re-express solution V as given by Eq. (57) as

\[ u_1(x, t) = \sqrt{\frac{4 \sinh^2(\Delta) + 3\beta}{2g_{11} \sinh(\Delta)}} \left[ \text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta) \right], \]
\[ u_2(x, t) = \sqrt{\frac{4 \sinh^2(\Delta) + 3\beta}{2g_{22} \sinh(\Delta)}} \left[ \tanh(\beta x + \Delta) + \tanh(\beta x - \Delta) \right], \]

where \( B = \sinh^2(\Delta) \).

Superposed Solution VI

It is easy to check that the coupled Eqs. (36) and (37) admit the superposed hyperbolic solution

\[ u_1(x, t) = e^{i\omega_1 t} \frac{A \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad u_2(x, t) = e^{i\omega_2 t} \frac{D \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \]

provided

\[ \omega_1 = -\beta^2, \quad g_{11} A^2 = 2B\beta^2, \quad g_{12} D^2 = 6(B + 1)\beta^2, \]
\[ \omega_2 = -\beta^2, \quad g_{21} A^2 = 6B\beta^2, \quad g_{22} = \frac{2(B + 1)\beta^2}{\beta}. \]

Thus for this solution \( g_{11}, g_{12}, g_{21}, g_{22} \) are all positive while \( \omega_1, \omega_2 \) are negative.
On making use of the novel identities (17) and (18), one can then re-express solution VI as given by Eq. (60) as
\[
\begin{align*}
  u_1(x,t) &= \frac{\beta}{\sqrt{2}g_{11}}[\text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta)], \\
  u_2(x,t) &= \frac{\beta}{\sqrt{2}g_{22}}[\text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta)],
\end{align*}
\] (62)
where \( B = \sinh^2(\Delta) \).

**Superposed Solution VII**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution
\[
\begin{align*}
  u_1(x,t) &= e^{i\omega_1 t}[1 - \frac{A}{B + \cosh^2(\beta x)}], \\
  \phi_2(x) &= e^{i\omega_2 t} \frac{D}{B + \cosh^2(\beta x)}, 
\end{align*}
\] (63)
provided
\[
\begin{align*}
  g_{11} &= -\omega_1 = 2\beta^2, \\
  A &= \frac{3(1 + 2B) + \sqrt{8 + (2B + 1)^2}}{4}, \\
  g_{12}D^2 &= \left[\sqrt{8 + (2B + 1)^2} - (2B + 1)\right] \frac{3A\beta^2}{2}, \\
  g_{21}A &= 3(2B + 1)\beta^2, \\
  g_{21}A^2 + g_{22}D^2 &= 8B(B + 1)\beta^2.
\end{align*}
\] (64)
Thus for this solution \( g_{11}, g_{12}, \omega_2, g_{21} \) are all positive, \( \omega_1 < 0 \) while the sign of \( g_{22} \) depends on the value of \( B \).

On comparing the solution (63) with the identity (12), solution VII as given by Eq. (63) can be re-expressed as
\[
\begin{align*}
  u_1(x,t) &= e^{i\omega_1 t}[1 - \frac{\beta}{\sqrt{2}g_{11}}[\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)]], \\
  u_2(x,t) &= \frac{\sqrt{3}\beta}{2\sqrt{g_{12}}}K[\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)],
\end{align*}
\] (65)
where \( \sinh^2(\Delta) = B \) and
\[
K = \left[\cosh(2\Delta)\sqrt{8 + \cosh^2(2\Delta)} - [2 + \cosh^2(2\Delta)]\right]^{1/2}.
\] (66)

**4 Hyperbolic Superposed Solutions of the Coupled MKdV Model**

Let us consider the following coupled MKdV equations (13)
\[
u_{1t} + u_{1xxx} + 6[g_{11}u_1^2 + g_{12}u_2^2]u_{1x} = 0,
\] (67)
Before we discuss the superposed kink and pulse solutions, let us note that the coupled Eqs. (67) and (68) admit the following kink and pulse solutions. In particular, it is well known that Eqs. (67) and (68) admit a kink solution in both $u_1$ and $u_2$, i.e.

$$u_1(x,t) = A_1 \tanh(\xi), \quad u_2(x,t) = A_2 \tanh(\xi), \quad \xi = \beta(x - vt),$$  \hspace{1cm} (69)

provided

$$v = -2\beta^2, \quad g_{11}A_1^2 + g_{12}A_2^2 = g_{21}A_1^2 + g_{22}A_2^2 = -\beta^2.$$  \hspace{1cm} (70)

It is also known that Eqs. (67) and (68) admit a pulse solution in both $u_1$ and $u_2$, i.e.

$$u_1(x,t) = A_1 \sech(\xi), \quad u_2(x,t) = A_2 \sech(\xi),$$  \hspace{1cm} (71)

provided

$$v = \beta^2, \quad g_{11}A_1^2 + g_{12}A_2^2 = g_{21}A_1^2 + g_{22}A_2^2 = 2\beta^2.$$  \hspace{1cm} (72)

Finally, Eqs. (67) and (68) are also known to admit a mixed kink-pulse solution, i.e. say a kink solution in $u_1$ and a pulse solution in $u_2$ or vice versa. In particular, they admit

$$u_1(x,t) = A_1 \tanh(\xi), \quad u_2(x,t) = A_2 \sech(\xi),$$  \hspace{1cm} (73)

provided

$$g_{12}A_2^2 - g_{11}A_1^2 = g_{22}A_2^2 - g_{21}A_1^2 = \beta^2, \quad v = 6g_{21}A_1^2 + \beta^2 = 6g_{12}A_2^2 - 2\beta^2.$$  \hspace{1cm} (74)

We now show that the coupled Eqs. (67) and (68) admit four novel solutions which can be re-expressed as the sum or the difference of the above kink solution.

We start with the ansatz

$$u_1(x,t) = \phi_1(\xi), \quad u_2(x,t) = \phi_2(\xi), \quad \xi = \beta(x - vt).$$  \hspace{1cm} (75)

On substituting the ansatz (75) in the coupled Eqs. (67) and (68) we get

$$\beta^2 \phi_{1\xi\xi\xi} = v\phi_1 - 6[g_{11}\phi_1^2 + g_{12}\phi_2^2]\phi_{1x},$$  \hspace{1cm} (76)

$$\beta^2 \phi_{2\xi\xi\xi} = v\phi_2 - 6[g_{21}\phi_1^2 + g_{22}\phi_2^2]\phi_{2x}.$$  \hspace{1cm} (77)
We now show that the coupled Eqs. (76) and (77) and hence coupled Eqs. (67) and (68) admit four hyperbolic superposed solutions in terms of \( \tanh(x + \Delta) \pm \tanh(x - \Delta) \). Out of these four solutions, while the first solution is obtained when \( u_2(x, t) \propto u_1(x, t) \), for the other three solutions, \( u_1(x, t) \) and \( u_2(x, t) \) are not proportional to each other.

**Solution I**

It is easy to check that

\[
 u_1(x, t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad u_2(x, t) = \alpha u_1(x, t), \quad A, B > 0, \quad (78)
\]

is an exact solution to the coupled Eqs. (67) and (68), where \( \alpha \) is any real number provided

\[
 g_{11} + \alpha^2 g_{12} = g_{21} + g_{22} \alpha^2 = \frac{(2B + 1)^2}{4B(B + 1)} \beta^2, \\
 A = \frac{4B(B + 1)}{2B + 1}, \quad v = -\frac{(4B^2 + 4B + 3)}{2B(B + 1)} \beta. \quad (79)
\]

On comparing it with the identity (12), we can re-express solution (78) as

\[
 u_1(x, t) = 1 - \frac{\beta \tanh(2\Delta)}{\sqrt{|g_{11} + \alpha^2 g_{12}|}} \left[ \tanh(\xi + \Delta) - \tanh(\xi - \Delta) \right], \quad B = \sinh^2(\Delta), \quad (80)
\]

with \( \xi = \beta(x - vt) \) and \( u_2(x, t) = \alpha u_1(x, t) \).

**Solution II**

Yet another solution to the coupled Eqs. (67) and (68) is

\[
 u_1(x, t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad u_2(x, t) = \frac{D \cosh(\xi) \sinh(\xi)}{B + \cosh^2(\xi)}, \quad A, B, D > 0, \quad (81)
\]

provided

\[
 v = 4\beta^2 + 6g_{11} + 6g_{12} D^2, \quad g_{11} + g_{12} D^2 = g_{21} + g_{22} D^2, \\
 g_{11} A^2 - B(2 + B) g_{12} D^2 = 4B(B + 1) \beta^2, \quad (2B + 1) g_{11} A^2 + 2B(B + 1) A g_{11} = 2B(B + 1) (2B + 1) \beta^2, \\
 g_{22} D^2 = -\frac{(10B^2 + 12B + 3)}{(2B + 1)^2} \beta^2, \\
 g_{21} A^2 = -\frac{B(14B^2 + 12B + 3)}{(2B + 1)^2}, \quad A = \frac{B(14B^2 + 12B + 3)}{(2B + 1)(B + 1)} \beta^2. \quad (82)
\]

Note that for this solution \( g_{21}, g_{22}, g_{12} < 0 \) while \( g_{11} > 0 \).
On comparing it with the novel identities (12) and (13) we can re-express solution (81) as the superposed solution
\[ u_1(x,t) = 1 - \frac{\beta[\tanh(\xi + \Delta) - \tanh(\xi - \Delta)]}{2\sinh(\Delta)\sqrt{|g_{21} \cosh(2\Delta)|}}, \] (83)

\[ u_2(x,t) = \frac{\beta \sqrt{\cosh(\Delta)}}{\sqrt{|g_{22}| \cosh(2\delta)}} [\tanh(\xi + \Delta) + \tanh(\xi - \Delta)], \] (84)

where \( B = \sinh^2(\Delta) \).

**Solution III**

Yet another hyperbolic superposed solution is
\[ \phi_1(\xi) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad \phi_2(\xi) = \frac{\alpha A}{B + \cosh^2(\xi)}, A, B > 0, \] (85)

provided
\[ v = 4\beta^2 + 6g_{11}, \quad g_{21} = g_{11} < 0, \quad g_{22} = g_{12}, \quad Ag_{11} = -(2B + 1)\beta^2, \]
\[ (g_{11} + \alpha^2g_{12})A^2 = -4B(2B + 1)\beta^2. \] (86)

On comparing it with the identity (12), we can re-express solution (85) as
\[ u_1(x,t) = 1 - \frac{\beta}{\sqrt{|g_{11} + \alpha^2g_{22}|}} [\tanh(\xi + \Delta) - \tanh(\xi - \Delta)], \] (87)

and
\[ u_2(x,t) = \frac{\alpha \beta}{\sqrt{|g_{11} + \alpha^2g_{22}|}} [\tanh(\xi + \Delta) - \tanh(\xi - \Delta)], \quad B = \sinh^2(\Delta), \] (88)

where \( B = \sinh^2(\Delta) \).

**Solution IV**

Yet another hyperbolic superposed solution to the coupled Eqs. (67) and (68) is
\[ u_1(x,t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad \phi_2(\xi) = -b - \frac{\alpha A}{B + \cosh^2(\xi)}, A, B, b > 0 \] (89)

provided
\[ v - 4\beta^2 = g_{12} + b^2g_{22} = g_{11} + b^2g_{12}, \]
\[ A(g_{11} - g_{12}\alpha b) = A(g_{21} - g_{22}\alpha b) = -(2B + 1)\beta^2, \]
\[ (g_{11} + \alpha^2g_{12})A^2 = (g_{21} + \alpha^2g_{22}) = -4B(2B + 1)\beta^2. \] (90)
On comparing it with the novel identity, we can re-express solution (88) as
\[ u_1(x, t) = 1 - \frac{\beta}{\sqrt{|g_{11} + \alpha^2 g_{22}|}} [\tanh(\xi + \Delta) - \tanh(\xi - \Delta)], \quad (91) \]
and
\[ u_2(x, t) = -b - \frac{\alpha \beta}{\sqrt{|g_{11} + \alpha^2 g_{22}|}} [\tanh(\xi + \Delta) - \tanh(\xi - \Delta)], \quad (92) \]
where \( B = \sinh^2(\Delta) \).

5 Conclusion and Open Problems

In this paper we have considered a coupled \( \phi^4 \) [11], a coupled NLS [12] and a coupled MKdV [13] model and demonstrated that all of them not only admit single kink and single pulse solutions but also admit novel solutions which can be re-expressed in terms of the sum or the difference of two kink or two pulse solutions. For the coupled \( \phi^4 \) and the coupled NLS models, we have obtained six superposed solutions essentially covering all possible cases among \( \tanh(x + \Delta) \pm \tanh(x - \Delta) \) and \( \text{sech}(x - \Delta) \pm \text{sech}(x + \Delta) \) with the two coupled members (or fields) being distinct. On the other hand, for the coupled MKdV we have only one such solution involving \( \tanh(x + \Delta) \pm \tanh(x - \Delta) \). Besides, we have obtained one superposed solution each of the coupled \( \phi^4 \) and the coupled NLS models and three superposed solutions of the coupled MKdV model where both the coupled members (or fields) involve only \( \tanh(x + \Delta) - \tanh(x - \Delta) \).

This paper raises several questions some of which are:

1. How many of these superposed solutions are stable in each coupled model? If stable, where can one look for such structures experimentally, e.g. in photonics [10]? What is the connection of such superposed solutions vis a vis a single kink or a single pulse solution?

2. So far, except for \( \tanh(x + \Delta) - \tanh(x - \Delta) \), no one has been able to find superposed hyperbolic solutions of the form \( \tanh(x + \Delta) + \tanh(x - \Delta) \) or of the form \( \text{sech}(x + \Delta) \pm \text{sech}(x - \Delta) \) in any uncoupled model. It would be worthwhile finding such solutions in uncoupled models. This will enable the physical interpretation of such superposed solutions in comparison to the corresponding single kink or single pulse solution.
3. Presumably there are other coupled systems, e.g. coupled KdV arising in two-layer fluids [13], which might admit solutions similar to those presented in this paper. It would be desirable to find such models and the corresponding superposed solutions.

4. The three coupled models considered in this paper will obviously also admit superposed periodic kink and pulse solutions in terms of Jacobi elliptic functions. It would be interesting to find such solutions and see how many of them smoothly go over to the hyperbolic superposed solutions obtained in this paper. We hope to address this issue in the near future.

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