Role of spatial higher order derivatives in momentum space entanglement

S. Santhosh Kumar* and S. Shankaranarayanan†

School of Physics, Indian Institute of Science Education and Research
Thiruvananthapuram (IISER-TVM), Trivandrum-695016, Kerala, India

We study the momentum space entanglement between different energy modes of interacting scalar fields propagating in general \((D+1)\)-dimensional flat space-time. As opposed to some of the recent works \cite{1}, we use Lorentz invariant normalized ground state to obtain the momentum space entanglement entropy. We show that the Lorentz invariant definition removes the spurious power-law behaviour obtained in the earlier works \cite{1}. More specifically, we show that the cubic interacting scalar field in \((1+1)\) dimensions leads to logarithmic divergence of the entanglement entropy and consistent with the results from real space entanglement calculations. We study the effects of the introduction of the Lorentz violating higher derivative terms in the presence of non-linear self-interacting scalar field potential and show that the divergence structure of the entanglement entropy is improved in the presence of spatial higher derivative terms.

PACS numbers: 03.65.Ud, 03.70.+k, 11.10.Hi

I. INTRODUCTION

Quantum entanglement depends on two properties — the superposition principle and the tensor product structure of the quantum states \cite{2}. Since the same quantum state has different tensor product structure in different Hilbert spaces, entanglement entropy is a partition dependent quantity \cite{3,4}. As of now, a large body of literature has investigated the robustness of the entanglement-area relation for the free quantum fields in the real space \cite{5-9}. It is natural to ask whether the entanglement entropy-area relation gets modified due to the presence of self interactions.

In the real space, however, the evaluation of entanglement entropy of self interacting field runs into difficulty: First, the modes functions can not be evaluated exactly and, second the evaluation of entanglement entropy is semi-analytical and the validity of the numerical results, in the perturbative regime, is opaque. Recently, Balasubramanian et al \cite{1} developed a technique to evaluate the entanglement entropy of the self-interacting scalar fields in the momentum space. The procedure of evaluating momentum space entanglement entropy is similar to the one used in the evaluation of the real-space entanglement entropy, i. e. the modes of different momenta are entangled in the ground state across a particular cut-off which act as an energy partition in the momentum space. More specifically, the low energy IR and high energy UV modes are entangled across the cut-off, say \(\mu\) \cite{1}.

Like the real space entanglement evaluation, there are unsettled issues in evaluating entanglement in the momentum space. First, the entanglement entropy is a cut-off dependent quantity and still we do not have the correct tool to renormalize the entropy. Second, the approach has a close resemblance to the Wilsonian effective low energy action theory as discussed in Refs. \cite{10,12}. In the Wilsonian renormalization, the UV degrees of freedom are integrated over and IR degrees of freedom are described by an effective density matrix. This is the case for every interacting field theory. Third, it is important to note that the momentum space entanglement is not a universal quantity and depends on what we are integrating out — UV or IR modes. The real space entanglement entropy for a pure bipartite system is symmetric w. r. t. the subsystems \cite{13}. However, it is not clear whether the momentum space entanglement entropy satisfies the symmetric property. The breakdown of symmetric property was reported in Ref. \cite{14} for the case of Boson-Fermion duality at high energy modes. Hence, the momentum space entanglement entropy is not useful to characterize theories in an invariant way though, real space entropy does \cite{15}. It depends on the partitioning of the UV and IR degrees of freedom. However, in 2-dimensional space-time field theories \cite{16}, it was shown that the entanglement entropy has UV-IR duality. Our analysis based on the Lorentz invariant definition of the ground state, show that the momentum space entanglement entropy is symmetric for 2-dimensional field theories.

As mentioned earlier, the evaluation of the momentum space entanglement was first reported in Ref. \cite{1}. While the ground states they used are not Lorentz invariant, the normalization of these states are not consistent with the ones used in the field theory literature \cite{17}. We show that their choice of normalization lead to spurious scaling behaviour for the entanglement entropy. More specifically, we show that the scaling behavior of the entropy \(S \propto \mu^{D(r-1)-2-r}\) used in Ref. \cite{1} go as \(S \propto \mu^{D(r-1)-2}\) where \(r\) is the index of power indicating the strength of the scalar field interaction and \(D\) is the number of space dimensions. We explicitly show that the extra power dependence in the entropy reported in Ref. \cite{1} is due to their choice of the normalization constant and show that using the standard normalization as in Ref. \cite{17}, the results are consistent with real space entanglement entropy. We study the effects of the introduction of the Lorentz

\* email: santhu@iisertvm.ac.in
† email: shanki@iisertvm.ac.in
invariance violating higher derivative terms in the presence of non-linear self-interacting scalar field potential and show that the divergence structure of the entanglement entropy improves when higher derivative terms are taken into account.

The rest of the paper is organized as follows: Sec. (II) discusses the approach used to evaluate entanglement entropy in momentum space and gives explicit formula for calculating the entanglement entropy of a scalar field in any dimension. In Sec. (III), we discuss the model action in any $(D + 1)$-dimensional space-time using the perturbative expansion. In Sec. (IV), we discuss the results for two specific cases and generalize to any space dimensions. We show that the entropy relation derived here is different from the ones obtained in Ref. II by an extra power factor which has quite compelling implications in the renormalization. It is shown that the divergence in the entropy is tunable by changing the dimension of the space-time and the power of the self interaction. Sec. (V), concludes with the discussion about our results and its possible connection to the renormalization of entanglement entropy. In this work, we set $c = \hbar = 1$.

**II. APPROACH TO EVALUATE MOMENTUM SPACE ENTANGLEMENT ENTROPY**

Let us start with the first protocol in the pure state entanglement — partition the total system into two parts. Let $H_A$ and $H_B$ be the Hamiltonian corresponding to two parts $A$ and $B$ with associated Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The total Hilbert space of the system is given by,

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

(1)

Let $|n\rangle$, $|N\rangle$ correspond to the complete energy eigenbasis of the subsystem $A$ and $B$, respectively. Furthermore, these represent the occupation numbers of the particle state in the Fock space. Before switching on the interaction, the Hamiltonian of the total system is given by,

$$H_0 = H_A \otimes I_B + I_A \otimes H_B$$

(2)

where $I$ is the unit operator and the energy eigenvalues are represented by $E_n$ and $E_N$. The ground state is the tensor product of the individual ground states of the Hamiltonian $H_A$ and $H_B$. Mathematically,

$$|0, 0\rangle \equiv |0_A\rangle \otimes |0_B\rangle$$

(3)

After turning on the interaction term, the total Hamiltonian can be written as,

$$H = H_0 + \lambda H_{AB}.$$

(4)

For the present discussion, we assume that strength of the interaction term $H_{AB}$ is weak in order to calculate the interacting ground state ($|\Omega\rangle$) perturbatively. Up to first order in $\lambda$, we have,

$$|\Omega\rangle = \frac{1}{\sqrt{N}} \left[ |0, 0\rangle + \lambda \left( \sum_{n \neq 0} \frac{\langle n, 0 | H_{AB} | 0, 0 \rangle}{E_0 - E_n} |n, 0\rangle + \sum_{N \neq 0} \frac{\langle 0, N | H_{AB} | 0, 0 \rangle}{E_0 - E_N} |0, N\rangle \right) + O(\lambda^2) \right]$$

$$= \frac{1}{\sqrt{N}} \left[ |0, 0\rangle + \lambda \left( \sum_{n \neq 0} A_n |n, 0\rangle + \sum_{N \neq 0} B_N |0, N\rangle \right) + \sum_{n \neq 0} \sum_{N \neq 0} C_{nN} |n, N\rangle \right] + O(\lambda^2)$$

(5)

where

$$A_n = \frac{\langle n, 0 | H_{AB} | 0, 0 \rangle}{E_0 - E_n},$$

$$B_N = \frac{\langle 0, N | H_{AB} | 0, 0 \rangle}{E_0 - E_N},$$

$$C_{nN} = \frac{\langle n, N | H_{AB} | 0, 0 \rangle}{E_0 + E_0 - E_n - E_N}$$

(6, 7, 8)

are the first order coefficients in the perturbative expansion and in general we can treat them as matrices and $N$ is the normalization constant.

The next protocol is the calculation of total density matrix, the matrix entries are written in the basis of $|0, 0\rangle$, $|n, 0\rangle$, $|0, N\rangle$ and $|n, N\rangle$ are given by

$$\rho = |\Omega\rangle \langle \Omega|$$

$$\begin{pmatrix}
|0, 0\rangle & |0, 0\rangle & |0, N\rangle & |n, N\rangle \\
|0, 0\rangle & \lambda A_A \lambda A_A^\dagger & \lambda B_B \lambda B_B^\dagger & \lambda C_C \lambda C_C^\dagger \\
|0, N\rangle & \lambda B_B \lambda B_B^\dagger & \lambda C_C \lambda C_C^\dagger & \lambda B_B \lambda B_B^\dagger + \lambda C_C \lambda C_C^\dagger \\
|n, N\rangle & \lambda C_C \lambda C_C^\dagger & \lambda B_B \lambda B_B^\dagger + \lambda C_C \lambda C_C^\dagger & \lambda C_C \lambda C_C^\dagger \\
\end{pmatrix}$$

(9)

where we fix the normalization constant as, $N = (1+|A|^2+|B|^2+|C|^2)^{-\frac{1}{2}}$ and here onwards drop the subscripts attached to the coefficients $A, B$ and $C$.

The reduced density matrix of the $A$ subsystem which is in the basis of $|0\rangle$ and $|n\rangle$ is obtained by tracing over the degrees of freedom of subsystem $B$. i. e.,

$$\rho_A = Tr_B \rho$$

$$= \frac{1}{N} \begin{pmatrix}
|0\rangle & |0\rangle \\
|n\rangle & \lambda A_A \lambda A_A^\dagger + \lambda B_B \lambda B_B^\dagger + \lambda C_C \lambda C_C^\dagger \\
\end{pmatrix}$$

(10)

For simplicity, we drop the summation, in the coefficients $A, B, C$, and different product combinations.
The leading contribution to entropy is with linear, higher spatial derivative terms: 

$$\rho_A = \left(1 - \lambda^2 |C|^2 \right) + \mathcal{O}(\lambda^3)$$  \hspace{1cm} (12)$$

The entanglement entropy of the $A$ subsystem is,

$$S_A = -\text{Tr}_A (\rho_A \log \rho_A) = -\text{Tr} \left[ (1 - \lambda^2 |C|^2) \log (1 - \lambda^2 |C|^2) \right]$$

$$+ \mathcal{O}(\lambda^2) \log \lambda^2 \text{Tr} [C C^\\dagger]$$

$$+ \mathcal{O}(\lambda^3)$$  \hspace{1cm} (13)$$

$$S_A \sim -\lambda^2 \log \lambda^2 \sum_{n \neq 0} \sum_{N \neq 0} \frac{|\langle n,N \rangle_{H_{AB}|0,0} \rangle|^2}{(E_0 + E_{n-N} - E_N)^2}$$

$$+ \lambda^2 \sum_{n \neq 0} \sum_{N \neq 0} \frac{|\langle n,N \rangle_{H_{AB}|0,0} \rangle (0,0 \rangle_{H_{AB}|n,N})}{(E_0 + E_{n-N} - E_N)^2}$$

$$\times \left[ 1 - \log \left( \frac{|\langle n,N \rangle_{H_{AB}|0,0} \rangle (0,0 \rangle_{H_{AB}|n,N})|}{(E_0 + E_{n-N} - E_N)^2} \right) \right]$$

$$+ \mathcal{O}(\lambda^3)$$  \hspace{1cm} (14)$$

The leading contribution to entropy is $\lambda^2 (\log \lambda^2)$ and vanishes as $\lambda \to 0$. This is consistent with the fact that the momentum space entanglement entropy of a free field is zero. In the next section, we apply the above procedure to calculate the momentum space entanglement entropy for massless interacting scalar field in different dimensions in the leading order of $\lambda^2 (\log \lambda^2)$.

### III. THE MODEL: MASSLESS SELF INTERACTING SCALAR FIELD

Let us consider the action for a massless scalar field $(\chi)$ propagating in $(D + 1)$-dimensional flat space-time with linear, higher spatial derivative terms,

$$S = \int dt' d^Dy \left[ \frac{1}{2} (\partial^2 \chi)^2 - \frac{c'}{2} (\nabla^2 \chi)^2 - \frac{\tau'}{2} (\nabla^3 \chi)^2 - \frac{g}{r!} \chi^r \right]$$

$$\hspace{1cm} (17)$$

where $c'$ and $\tau'$ are the constants with dimensions $[\text{Length}]^{-2}$ and $[\text{Length}]^{-4}$ respectively, $\nabla^2$, $\nabla^3$ are the higher order spatial derivatives, $g$ is a dimensionful tunable constant and $r$ refers to the index of interaction.

The importance of the higher derivative spatial terms were studied in Refs. [18][19] can be used to understand some quantum phase transitions. Unlike the Wilsonian type renormalization [10][12], the higher derivative terms introduce Next-to-Next-to-Next interaction in the lattice. It is important to note that this higher derivative terms appear in the effective Hamiltonian description of certain high temperature superconductors [20].

Rescaling these fields, coupling constants using the following scaling:

$$t' \to t = t'/L, y \to x = y/L,$$

$$\chi \to \phi = \frac{\chi}{L^{(1-D)/2}}, c' \to \epsilon = c'L^2,$$

$$\tau' \to \tau = \tau'L^4,$$

$$g \to \lambda = g L^{(1 + s/2) + D(1 - s/2)}$$

the Hamiltonian $H_{AB}$ corresponding to the action in eq. [17] is given by

$$H_{AB} = \int d^Dx \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 \right] + \frac{\tau}{2} (\nabla^3 \phi)^2 + \frac{\lambda}{r!} \phi^r$$

$$\hspace{1cm} (18)$$

where, $\pi$ is the canonical conjugate momentum corresponding to scalar field $\phi$ and satisfies the equal time commutation relation $[\phi(x), \pi(x')] = i\delta^{(D)}(x-x')$.

Expanding the field in terms of the bosonic creation and annihilation operators

$$\phi(x) = \frac{1}{\sqrt{2\omega_p}} \sum_p \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x})$$

the Lorentz invariant one particle excited states are [17][21],

$$|p\rangle = \sqrt{2\omega_p} a_p^\dagger |0\rangle \hspace{1cm} (24)$$

and satisfy Lorentz invariant orthogonality relation [17],

$$\langle p|q \rangle = 2\omega_p (2\pi)^D i^D_{p,q}$$

where $\omega_p = \sqrt{p^2 + \epsilon p^4 + \tau p^6}.

We would like to compare and contrast the above relations [24][25] to the ones used by Balasubramanian et al in Ref. [11]:

$$|p\rangle = a_p^\dagger |0\rangle \hspace{1cm} (26)$$

$$\langle p|q \rangle = (2\pi)^D i^D_{p,q}$$

It is important to note that authors have not used the normalization factor, $\sqrt{2\omega_p}$ and this, as we will show below, leads to interesting results for the momentum space entanglement entropy. To understand the effect of the normalization constant, let us calculate the expectation value of $H_{AB}$, i.e.,

$$\langle p_1, p_2, \ldots, p_r | H_{AB} | 0,0, \ldots, 0 \rangle =$$

$$\langle p_1, p_2, \ldots, p_r | \int d^Dx \frac{\phi(x)^r}{r!} | 0,0, \ldots, 0 \rangle$$

$$\hspace{1cm} = \langle p_1, p_2, \ldots, p_r | \int d^Dx \left( \frac{1}{(2\pi)^D} \right)$$
acting fields with higher spatial derivative terms. However, Eq. (31) of Ref. [1] is a dimensionful quantity. Second, the consideration of the Lorentz invariant ground state removes the factors, \( \omega_1, \ldots, \omega_p \), from the denominator of Eq. (32). In the next subsection, we obtain the momentum space entanglement of the interacting fields with higher spatial derivative terms.

IV. ROLE OF SPATIAL HIGHER DERIVATIVES IN MOMENTUM SPACE ENTANGLEMENT

Having shown the importance of the normalization constant in the evaluation of the momentum space entanglement entropy, in this section, we discuss the role the spatial higher derivatives play in the divergence of the entanglement entropy. We evaluate the entanglement entropy for two specific examples — \( \phi^3 \) theory in (1 + 1) and \( \phi^4 \) theory in (2 + 1) dimensions — and generalize to generic interaction in arbitrary dimensional space-time.

A. \( \phi^3 \) theory in (1+1)-dimensions

Setting \( r = 3 \) and \( D = 1 \) in Eq. (31), entanglement entropy is given by:

\[
S_A = -\lambda^2 \log \lambda^2 \sum_{\{p_1, \ldots, p_r\}} \frac{(2\pi)^{2D}}{(r!)^2} \frac{\delta^{D}(p_1 + p_2 + \ldots + p_r)}{\left(\omega_{p_1} + \omega_{p_2} + \ldots + \omega_{p_r}\right)^2} + \mathcal{O}(\lambda^2)
\]

Let us compare and contrast Eq. (31) with Eq. (31) of Ref. [1] that is reproduced here for easy comparison:

\[
S_A/L^D = -\lambda^2 \log \lambda^2 \sum_{\{\xi \}} \frac{(2\pi)^{D}}{(r!)^2} \frac{\delta^{D}(p_1 + p_2 + \ldots + p_r)}{\left(\omega_{p_1} + \omega_{p_2} + \ldots + \omega_{p_r}\right)^2} + \mathcal{O}(\lambda^2)
\]

First, the entanglement entropy is a dimensionless quantity. However, Eq. (31) of Ref. [1] is a dimensionfull quantity. Second, the consideration of the Lorentz invariant ground state removes the factors, \( \omega_{p_1}, \ldots, \omega_{p_r} \), from the denominator of Eq. (32). In the next subsection, we obtain the momentum space entanglement of the interacting fields with higher spatial derivative terms.
divergence of the two-point function [22]. Thus, our analysis is consistent with these results.

B. \( \phi^4 \) theory in (2+1)-dimensions

Let us now consider \( \phi^4 \) interacting, massless scalar field propagating in \((2 + 1)\)-dimension. Substituting \( D = 2 \) and \( r = 4 \), in Eq. (31), the momentum space entanglement entropy in the large \( \mu \) limit is given by

\[
S_A \propto -\lambda^2 \log \lambda^2 \begin{cases} 
\mu^4, & \text{for } \epsilon = \tau = 0; b = 1 \\
\mu^2, & \text{for } b = \tau = 0; \epsilon = 1 \\
\log \mu, & \text{for } b = \epsilon = 0; \tau = 1 
\end{cases} + \mathcal{O}(\lambda^2)
\]

(35)

As in the previous case, our results show that the introduction of the higher derivative makes the entanglement entropy less divergent and is consistent with the analysis of Ref. [22].

In general for any self-interacting scalar model of the type in Eq. (22) in \( D \)-dimensional space, the momentum space entanglement entropy, in the large \( \mu \) limit, is given by

\[
S_A \propto -\lambda^2 \log \lambda^2 \begin{cases} 
\frac{1}{\mu^{2-D(r-1)}}, & \text{for } \epsilon = \tau = 0; b = 1 \\
\frac{1}{\mu^{4-D(r-1)}}, & \text{for } b = \tau = 0; \epsilon = 1 \\
\frac{1}{\mu^{6-D(r-1)}}, & \text{for } b = \epsilon = 0; \tau = 1 
\end{cases} + \mathcal{O}(\lambda^2)
\]

(36)

The above results show that the interaction terms do not improve the divergence problem of the entanglement entropy, however, the higher derivative terms improve the divergence structure of the entanglement entropy.

V. CONCLUSIONS

In this work, we have evaluated the momentum space entanglement entropy of the interacting scalar fields in the presence of spatial higher derivative terms. We have explicitly shown that the correct choice of the normalization of the Lorentz invariant states does not improve the divergence problem of the entanglement entropy, however, the presence of higher derivative terms improves the divergence of the entanglement entropy.

Our analysis should be contrasted to the analysis reported in Ref. [1] where the authors claimed that the interaction terms help to improve the divergence problem of entanglement entropy. As we have shown here, this is due to the wrong choice of the normalization constant. Taking two specific examples, scalar fields in \((1+1)\)-dimensions and \((2+1)\)-dimensions, we have shown that the divergence structure of the entanglement entropy is not improved due to the presence of the interaction terms.

Real space entanglement entropy is symmetric, however, the momentum space entanglement defined in Ref. [1] is not and hence can not be considered as an universal quantity [15]. Our analysis in Sec. (IV-A), in the light of Ref. [16], shows that at least for the \((1+1)\)-dimensional field theory, the momentum space entanglement entropy can indeed be considered as an universal quantity. More specifically, since the UV and IR are related by a simple rescaling of the variables in \((1+1)\)-dimensions [16], the momentum space entanglement entropy evaluated by integrating over the UV modes or IR modes is identical. Our aim is to extend the analysis for higher dimensions. This is currently under investigation.

VI. ACKNOWLEDGMENTS

SSK is financially supported by Senior Research Fellowship of the Council of Scientific & Industrial Research (CSIR), Government of India. SSK is also thankful to the Condensed Matter and Statistical Physics section at ICTP, Italy for the hospitality during the final stages of the manuscript. This work is supported by Max Planck-India Partner Group on Gravity and Cosmology.

[1] V. Balasubramanian, M. B. McDermott, and M. Van Raamsdonk, “Momentum-space entanglement and renormalization in quantum field theory,” Phys. Rev. D 86 045014 (2012).
[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, “Quantum entanglement,” Rev. Mod. Phys. 81 865–942 (2009).
[3] P. Zanardi, “Virtual quantum subsystems,” Phys. Rev. Lett. 87 037901 (2001).
[4] W. Thirring, R. A. Bertlmann, P. Köhler, and H. Narnhofer, “Entanglement or separability: the choice of how to factorize the algebra of a density matrix,” The European Physical Journal D 64 2, 181–196 (2011).
[5] L. Bombelli, R. R. Koul, J. Lee, and R. D. Sorkin, “Quantum source of entropy for black holes,” Phys. Rev. D 34 373–383 (1986).
[6] M. Srednicki, “Entropy and area,” Phys. Rev. Lett. 71 666–669 (1993).
[7] P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory,” Journal of Statistical Mechanics: Theory and Experiment 0406 P06002 (2004).
[8] S. Das and S. Shankaranarayanan, “How robust is the entanglement entropy-area relation?,” *Phys. Rev. D* **73**, 121701 (2006)

[9] J. Eisert, M. Cramer, and M. B. Plenio, “Colloquium: Area laws for the entanglement entropy,” *Rev. Mod. Phys.* **82**, 277–306 (2010)

[10] K. G. Wilson and J. Kogut, “The renormalization group and the $\epsilon$ expansion,” *Physics Reports* **12** no. 2, 75 – 199 (1974)

[11] J. Zinn-Justin, “Quantum Field Theory and Critical Phenomena,” International series of monographs on physics, Clarendon Press, (2002).

[12] N. Goldenfeld, “Lectures on Phase Transitions and the Renormalization Group,” Frontiers in physics, Addison-Wesley, Advanced Book Program, (1992).

[13] M. A. Nielsen and I. L. Chuang, “Quantum Computation and Quantum Information,” Cambridge University Press, 10th anniversary edition ed., (2011).

[14] D. C. Mattis and E. H. Lieb, “Exact solution of a many-fermion system and its associated boson field,” *Journal of Mathematical Physics* **6** (1965)

[15] M. Headrick, A. Lawrence, and M. Roberts, “Bose-fermi duality and entanglement entropies,” *Journal of Statistical Mechanics: Theory and Experiment* **2013** no. 02, P02022 (2013)

[16] K. Mallayya, R. Tibrewala, S. Shankaranarayanan, and T. Padmanabhan, “Zero modes and divergence of entanglement entropy,” *Phys. Rev. D* **90**, 044058 (2014)

[17] M. Peskin and D. Schroeder, “An Introduction to Quantum Field Theory,” Advanced Book Program, Addison-Wesley Publishing Company, (1995).

[18] S. Ghosh and S. Shankaranarayanan, “Entanglement signatures of phase transition in higher-derivative quantum field theories,” *Phys. Rev. D* **86**, 125011 (2012)

[19] S. S. Kumar and S. Shankaranarayanan, “Evidence for phase transition in vacuum entanglement of higher derivative scalar quantum field theories,” *arXiv:1606.05472 [cond-mat.stat-mech]* (2016)

[20] A. Tagliacozzo, Private communication.

[21] Y. Shi, “Entanglement in relativistic quantum field theory,” *Phys. Rev. D* **70**, 105001 (2004)

[22] N. H. Barth and S. M. Christensen, “Quantizing fourth order gravity theories. 1. the functional integral,” *Phys. Rev. D* **28**, 1876 (1983)