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To cite this article: A El-Shenawy and E A Shirokova 2019 J. Phys.: Conf. Ser. 1158 022040

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Approximate solution for 3D Dirichlet problem in a doubly connected arbitrary finite solid with smooth surface

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Abstract. In this paper we present the spline interpolation method for solving the 3D Laplace equation with Dirichlet boundary conditions for doubly connected solids with smooth surfaces. The solid is divided into $N$ layers and the spline solution construction at each layer for the 3D problem is reduced to the solution of a sequence of 2D Dirichlet problems. The 2D problem solution in each layer is restored via its boundary value with the help of Cauchy integral method. The Cauchy integral method is a boundary element method which reduces the Dirichlet problem to the Fredholm integral equation of the second type. The final spline solution of the 3D problem is continuous with respect to the three variables. Numerical examples are given to verify the efficiency of the method.

1. Introduction

An internal Dirichlet problem is the problem of finding a function which solves a specified partial differential equation in the interior of a given region and takes prescribed values on the boundary of the region. In this paper the three dimensional Dirichlet problem is solved for the Laplace equation in the interior of three dimensional doubly connected solid. The problem is formulated as follows:

Let $\Omega_0$ and $\Omega_1$ be a bounded three-dimensional simply connected solids with boundary smooth surfaces $\partial \Omega_0$ and $\partial \Omega_1$, respectively, and $\Omega_1 \subset \Omega_0$. We define the solution domain to be the doubly connected domain $\Omega = \Omega_0 \setminus \Omega_1$ with the boundary surface $\partial \Omega = \partial \Omega_0 \cup \partial \Omega_1$. See Figure(1) for an example of the solution domain.

The corresponding Dirichlet problem for the Laplace equation is formulated as follows: find the doubly differentiable function $u(x, y, h)$ in $\Omega$ which is continuous in $\Omega \cup \partial \Omega$ and satisfies the three dimensional Laplace equation

$$\Delta u(x, y, h) = \frac{\partial^2 u(x, y, h)}{\partial x^2} + \frac{\partial^2 u(x, y, h)}{\partial y^2} + \frac{\partial^2 u(x, y, h)}{\partial h^2} = 0, \quad (x, y, h) \in \Omega, \quad (1)$$

and the Dirichlet boundary conditions

$$u = \begin{cases} f_0, & \text{on } \partial \Omega_0; \\ f_1, & \text{on } \partial \Omega_1. \end{cases} \quad (2)$$

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Figure 1. Example of the doubly connected solution domain for 3D Dirichlet problem with $\Omega_0$ and $\Omega_1$ are the outer and inner domains, respectively.

Laplace equation describes many physical quantities, for example, gravitational potential, temperature potential, and electrical potential. A closed form solution of the three-dimensional Laplace equation can be found in certain solids with certain boundary conditions. It is important in engineering problems to approximate numerically the solution of Laplace equation in two-dimensional and three-dimensional regions. Many numerical methods give effective results to approximate the solution of the three-dimensional Laplace equation such as the finite difference method, boundary element method and finite element method. The boundary element method is particularly useful in the solution of problems involving irregular domains which are hard to mesh. Three-dimensional Poisson and Laplace equations with Dirichlet boundary conditions are solved numerically in simply connected solid by many approaches. For example, Atkinsons[1] converted the problem to the solution of a Fredholm integral equation of the second kind and solved this integral equation by using Galerkin’s method, applying spherical harmonics as the basis functions. Another approach proposes an extension of the two-dimensional method of fundamental solutions (MFS) to solve the three-dimensional Laplace equation as in [2]. An approach based on high-order quadrature and a high-order finite element method utilizing Taylor model methods was introduced in [3].

The proposed method based on the spline interpolation solution of 3D Dirichlet problem for Laplace equation. The method divides the solid into $N$ layers. The spline solution at each layer is a polynomial function of $h$ with coefficients which are poly-harmonic functions in the cross
sections of $\Omega$ with the planes $h = h_j, j = 1, 2, \cdots, N$. Define the spline solution at any layer in the form:

$$u(x, y, h) = \sum_{k=0}^{p} u_k(x, y)h^k.$$  

We obtain the linear spline $u(x, y, h) = u_0(x, y) + hu_1(x, y)$ by taking $p = 1$. If we put this form of solution into (1) we obtain the following relation:

$$\Delta_2 u_k(x, y) = 0, \quad k = 0, 1,$$

where $\Delta_2 = \partial_x^2 + \partial_y^2$. The coefficients $u_k(x, y), k = 0, 1$ are 2D harmonic functions of $x$ and $y$ which can be restored via their boundary values applying the Cauchy integral method [6].

2. Construction of the linear spline solution of the 3D Dirichlet problem

Assume that the convex solids $\Omega_k, k \in \{0, 1\}$, possess two degenerate lower and upper ends $P_{A_k}$ and $P_{B_k}$ with the third coordinate $h = A_k$ and $h = B_k$, respectively. Let $H_j$ be the planes $h = h_j, j = 0, 1, \ldots, N$ parallel to $xy$ plane with $h_0 = A_0$ and $h_N = B_0$. We call $C_0$ the 2D simply connected plane section of $\Omega$ by the planes $h = \hat{h}$, where $A_0 < \hat{h} < A_1$ or $B_1 < \hat{h} < B_0$ with smooth boundary curves $\partial C_s, s = 1, 2, \ldots, n, n < N$. The boundary values given at $\partial C_s$ are denoted by $f^0_s, s = 1, 2, \ldots, n$. The other sections are the doubly connected domains $D_j, l = 1, 2, \ldots, m < N$, which are the sections of $\Omega$ by $h = \hat{h}, A_1 < \hat{h} < B_1$, with smooth boundary curves $\partial D_l = L_{0l} \cup L_{1l}$. The boundary values given at $\partial D_l$ are denoted by $f^1_l, f^2_l, l = 1, 2, \ldots, m$, for the outer and inner curves, respectively.

We begin the construction of the linear spline solution from the lowest layer which contains the vertex $A_0$. We will construct the linear spline solution by moving upward from the lower end $A_0$ to the upper end $B_0$. If $A_0 \leq h_j \leq A_1$ or $B_1 \leq h_j \leq B_0$ then the 2D harmonic functions are restored in the corresponding simply connected domains via the boundary values. If $A_1 \leq h_j \leq B_1$, then the 2D harmonic functions are restored in the corresponding doubly connected domains via the boundary values.

Assume that the initial layer $h \in [A_0, h_1]$ is a simply connected layer, then the linear spline solution takes the form[4, 5]:

$$u^1(x, y, h) = u_0^1(x, y) + (h - h_1)u_1^1,$$  \hspace{1cm} (3)

where $u_0^1(x, y)$ is a harmonic function in the domain $C_1$ and will be constructed via its boundary value $f^0_1$ using the Cauchy integral method as in [6, 7]. The constant $u_1^1$ can be calculated using the relation:

$$u_1^1 = \frac{u(P_{A_0}) - u_0^1(x_{A_0}, y_{A_0})}{A_0 - h_1}. \hspace{1cm} (4)$$

Let the second layer with $h \in [h_1, h_2]$ and $A_0 \leq h_2 \leq B_0$ be a simply connected layer. The constructed spline solution has the form

$$u^2(x, y, h) = u_0^2(x, y) + (h - h_1)u_1^2,$$

where the two coefficients $u_0^2(x, y)$ and $u_1^2(x, y)$ are harmonic functions in the union of the projection of the solid $\Omega$ intersections with the planes $h = h_1$ and $h = h_2$. By taking the value of $u_0^2(x, y)$ equal to $u_0^1(x, y)$, the boundary conditions at $C_1$ are met. The other harmonic function $u_1^2(x, y)$ is restored in the section of $\Omega$ with $h = h_2$ via its boundary value which is equal to:

$$u_1^2(x, y)|_{\partial C_2} = \frac{f^2_0 - u_0^2|_{\partial C_2}}{h_2 - h_1}. $$
For the other layers \( h \in [h_k, h_{k+1}] \), \( k = 2, 3, \cdots, N-2 \), the constructed spline solution is written as follows

\[
u_{k+1}^{k+1}(x, y, h) = u_0^{k+1}(x, y) + (h - h_k)u_1^{k+1}(x, y).
\]

To restore the harmonic functions \( u_0^{k+1}(x, y) \) and \( u_1^{k+1}(x, y) \) we have the following two cases.

- Case 1. Simply connected layer, \( A_0 \leq h \leq A_1 \) or \( B_1 \leq h \leq B_0 \). The function \( u_0^{k+1}(x, y) \) is calculated by setting \( u_0^{k+1}(x, y) = u^k(x, y, h_k) \) and by restoring the harmonic function \( u_1^{k+1}(x, y) \) via its boundary value on the curve \( \partial C_{k+1} \) which is equal to

\[
u_1^{k+1}(x, y)|_{\partial C_{k+1}} = \frac{f_0^{k+1} - u_0^{k+1}|_{\partial C_{k+1}}}{h_{k+1} - h_k}.
\]

- Case 2. Doubly connected layer \( A_1 \leq h \leq B_1 \). The function \( u_0^{k+1}(x, y) \) is calculated by setting \( u_0^{k+1}(x, y) = u^k(x, y, h_k) \) on the doubly connected layer \( D_k \) and by restoring the harmonic function \( u_1^{k+1}(x, y) \) via its boundary value on the curves \( \partial D_{k+1} = L_0(k+1) \cup L_1(k+1) \) which is equal to

\[
u_1^{k+1}(x, y)|_{L_0(k+1)} = \frac{f_0^{k+1} - u_0^{k+1}|_{L_0(k+1)}}{h_{k+1} - h_k},
\]

\[
u_1^{k+1}(x, y)|_{L_1(k+1)} = \frac{f_1^{k+1} - u_0^{k+1}|_{L_1(k+1)}}{h_{k+1} - h_k}.
\]

For the last simply connected layer \( h \in [h_{N-1}, P_{B_0}] \) the spline solution is constructed by the same way as in the first layer:

\[
u^N(x, y, h) = u_0^N(x, y) + (h - h_{N-1})u_1^N(x, y),
\]

where \( u_0^N(x, y) = u^{N-1}(x, y, P_{B_0}) \) and the constant \( u_1^N(x, y) \) equal to

\[
u_1^N(x, y) = \frac{u(P_{B_0}) - u_0^N(x, y, P_{B_0})}{B_0 - h_{N-1}}.
\]

In the next section, we introduce a brief description of the method of Cauchy integral used in the above method for the construction of the two-dimension harmonic functions in simply and doubly connected domains.

3. Construction of the 2D harmonic function in simply and doubly connected domains

Here we explain the algorithm of the Cauchy integral method applied to restore the harmonic functions in each layer via the boundary conditions. The method was discussed in details in our previous works [6, 7].

3.1. Constructing the harmonic function in a simply connected domain:

Given the parametric function \( f_0(t) \), we have to find the harmonic function \( u(x, y) \) in the domain \( C \) with contour \( \partial C \) where

\[u|_{\partial C} = f_0(t).
\]

The algorithm of the method is summarized in the following steps:

- Consider the function \( u(x, y) \) as a real part of an analytic function \( B(x, y) \) in the domain \( C \).
• Define the function \( g_0(t) \) which is the imaginary part of the boundary value of the analytic function \( B(x, y) \) on \( \partial C \). Write the Fredholm integral equation of the second kind in the unknown function \( g_0(t) \).

• Apply the Hilbert formula in order to avoid the singularity in the Fredholm integral equation.

• Reduce the solution of the integral equation to the solution of a linear system of equations with the help of truncated Fourier series.

• Write the harmonic function as the real part of the Cauchy integral formula

\[
 u(x, y) = \text{Re} \left( \frac{1}{2\pi i} \int_0^{2\pi} \frac{f_0(t) + i g_0(t)}{z(t) - (x + iy)} z'(t) dt \right).
\]

3.2. Constructing the harmonic function in a doubly connected domain:

Given the parametric function, \( f_0(t) \) and \( f_1(t) \), we need to find the harmonic function \( u(x, y) \) in the domain \( D \) with the boundary \( \partial D = L_0 \cup L_1 \) where \( u|_{L_0} = f_0(t), L_0 = z_0(t) \) and \( u|_{L_1} = f_1(t), L_1 = z_1(t) \). The algorithm of the method is generalizes the above algorithm as follows.

• Consider the function \( u(x, y) \) as the real part of an analytic function \( B(x, y) \) in the interior of the doubly connected domain \( D \) in the form:

\[
 u(x, y) = \text{Re} (B(x, y)) + A \log |x + iy|, \quad \text{where } A \text{ is a constant.}
\]

• Define the function \( g_0(t) \) and \( g_1(t) \) to be the imaginary parts of the boundary values of the analytic function \( B(x, y) \) on \( \partial D \). Write the system of Fredholm integral equations of the second kind in the unknown functions \( g_0(t), g_1(t) \).

• Apply the Hilbert formula in order to avoid the singularities in the Fredholm integral equations.

• Reduce the solution of the integral equation to the solution of a linear system of equations with the help of truncated Fourier series.

• Calculate the constant \( A \) due to the special property of the Cauchy integral and write the harmonic function as the real part of the Cauchy integral formula

\[
 u(x, y) = \text{Re} \left( \sum_{j=0}^{\kappa} \frac{1}{2\pi i} \int_0^{2\pi} \frac{f_j(t) - A \log |z_j(t)| + i g_j(t)}{z_j(t) - (x + iy)} z_j'(t) dt \right) + \frac{A}{2} \log \left( x^2 + y^2 \right).
\]

3.3. Analytic continuation

When we move through the layers along the h-axis and construct the spline solutions from one layer to adjacent one successively, it can be necessary to move from a domain \( C_j, j < N \), to such a domain \( C_{j+1} \) that \( C_j \cup C_{j+1} \neq C_j \). In this case the technique of analytic continuation is used to extend the domain of the analytic function [8]. Let \( z_0 = (x_0 + iy_0) \) be a point in the simply connected domain \( C \) with the boundary curve \( \partial C = z(t) \). As mentioned above, the harmonic function solution at this point is approximated by the real part of the following Cauchy integral formula

\[
 u(x_0, y_0) = \text{Re} \left( \frac{1}{2\pi i} \int_0^{2\pi} \frac{B(z(t)) z'(t)}{z(t) - (x_0 + iy_0)} dt \right),
\]

where \( B(z(t)) = f_0(t) + i g_0(t) \) is the boundary value of the analytic function \( B(z) \) at the points of the curve \( \partial C \). By taking arbitrary disk with radius \( R \) and center \( \sigma \) in \( C \) we can write the
Figure 2. (a) The doubly connected solid in Ex. (1). (b) The corresponding 2D cross section for Ex. (1).

analytic in \( C \) function at any point \( z \) inside this disk using the Cauchy formula as follows:

\[
B(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{B(Re^{it} + \sigma)iRe^{it}}{(Re^{it} + \sigma) - z} dt,
\]

where \( B(Re^{it} + \sigma) \) is the boundary value of the analytic function in this disk. After applying the Taylor expansion we get

\[
B(z) = \sum_{k=0}^{\infty} \frac{1}{2\pi R^k} \left[ \int_0^{2\pi} B(Re^{it} + \sigma)e^{-ikt} dt \right] (z - \sigma)^k.
\]

The domain of the harmonic function can be extended by applying the analytic continuation of the Taylor expansion (6) of \( B(z) \) to the disk of a radius more that \( R \). This relation is used to calculate the value of analytical function at the points outside the domain \( C \) in the neighborhood of the boundary curve \( \partial C \).

For the case of the doubly connected domain \( D \) with boundary \( \partial D = L_0 \cup L_1 \), the analytical function can be extended outside of \( L_0 \) or inside of \( L_1 \) by the same method applied for the simply connected case.

4. Numerical Examples

Example 1: Let \( \Omega \) be the solid shown in Figure (2) which consists of the external and internal finite domains \( \Omega_0 \) and \( \Omega_1 \), respectively. Let \( \Omega_0 \) be the sphere with center at the origin and radius \( r_0 = 2 \) and

\[
\Omega_1 = (1 - h^2)\sqrt{(0.6 \cos t + 0.3 \cos 2t - 0.2)^2 + (0.6 \sin t)^2}, \quad -1 \leq h \leq 1, \quad t \in [0, 2\pi].
\]

The boundary values at the surface \( \partial \Omega \) are calculated using the exact solution \( u(x, y, h) = h + e^x \cos y \). By using \( N = 15 \), the spline solution of Laplace equation (1-2) was calculated at some arbitrary layers and the numerical absolute error is shown in Table (1).

Example 2: We convert to another solid which is shown in Figure (3) and is consisted of the ellipsoid

\[
\Omega_0 : \frac{x^2}{9} + \frac{y^2}{16} + \frac{h^2}{4} = 1,
\]
| $h$  | $Error = \max |u(\text{Exact}) - u(\text{Approximate})|$ |
|------|--------------------------------------------------|
| -1.714 | 1.304 E-08                                     |
| -1.143 | 2.565 E-08                                     |
| -0.571 | 2.123 E-05                                     |
| -0.286 | 5.937 E-05                                     |
| 0.286  | 6.995 E-05                                     |
| 0.857  | 1.713 E-05                                     |
| 1.429  | 1.743 E-08                                     |
| 1.714  | 2.643 E-08                                     |

Table 1. The numerical absolute error for Example (1).

**Figure 3.** (a) The doubly connected solid in Ex. (2). (b) The corresponding 2D cross section for Ex. (2).

and the internal domain defined by the equation

$$\Omega_1 = (1 - h^2) \left(-0.5 + e^t + 0.5e^{i2t} + 0.2ie^{-i2t}\right), \quad -1 \leq h \leq 1, \quad t \in [0, 2\pi].$$

The boundary values at the surface $\partial \Omega$ are calculated using the exact solution $u(x, y, h) = h + x^3 - 3xy^2$. By using $N = 15$, the spline solution of the 3D Laplace equation (1-2) was calculated at some arbitrary layers and the numerical absolute error is shown in Table (2).

**Example 3:** Let $\Omega$ be the region shown in Figure (4) bounded by the external and internal non-spherical surfaces $\Omega_0$ and $\Omega_1$, respectively, as follows:

$$\Omega_0 = (9 - h^2)\sqrt{(0.6 \cos t + 0.3 \cos 2t - 0.2)^2 + (0.6 \sin t)^2}, \quad -3 \leq h \leq 3, \quad t \in [0, 2\pi].$$

And the internal domain defined as:

$$\Omega_1 = (1 - h^2)\frac{2.5 + \cos t + 0.5 \sin 2t}{1.5 + 0.7 \cos t}, \quad -1 \leq h \leq 1, \quad t \in [0, 2\pi].$$
\[
\text{Error} = \max |u_{(\text{Exact})} - u_{(\text{Approximate})}|
\]

| \(h\)  | Error       |
|-------|-------------|
| -1.714 | 1.264 E-07  |
| -1.143 | 3.098 E-07  |
| -0.857 | 8.802 E-07  |
| -0.286 | 1.513 E-05  |
| 0.286  | 7.758 E-07  |
| 0.857  | 9.012 E-07  |
| 1.143  | 6.098 E-07  |
| 1.714  | 6.098 E-07  |

**Table 2.** The numerical absolute error for Example (2).

![3D solid](a) ![2D cross section](b)

**Figure 4.** (a) The doubly connected solid in Ex. (3). (b) The corresponding 2D cross section for Ex. (3).

| \(h\)  | Error       |
|-------|-------------|
| -2.571 | 1.141 E-04  |
| -1.714 | 3.433 E-04  |
| -0.857 | 1.973 E-04  |
| 0.000  | 4.231 E-04  |
| 0.429  | 4.231 E-04  |
| 1.286  | 1.973 E-04  |
| 2.143  | 2.039 E-04  |

**Table 3.** The numerical absolute error for Example (3).

The boundary values at the surface \(\partial \Omega\) are calculated using the exact solution \(u(x, y, h) = h - x^2 + y^2\). The numerical absolute error shown in Table (3) is the absolute value between the exact and spline approximate solution at different layers. The results show the efficiency of the method even for solid with non-spherical shapes.
5. Conclusion
The method gives a continuous solution of 3D Laplace equations with Dirichlet boundary conditions. The method is applicable for arbitrary and complex doubly connected solid with smooth boundaries. The solution is continuous with respect to the three variables. The results shows the high accuracy of the method.

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