Various notions of Best Approximation Property in Spaces of Bochner Integrable Functions

Tanmoy Paul

Abstract. We derive that for a separable proximinal subspace $Y$ of $X$, $Y$ is strongly proximinal (strongly ball proximinal) if and only if for $1 \leq p < \infty$, $L_p(I, Y)$ is strongly proximinal (strongly ball proximinal) in $L_p(I, X)$. Case for $p = \infty$ follows from stronger assumption on $Y$ in $X$ (uniform proximinality). It is observed that for a separable proximinal subspace $Y$ in $X$, $Y$ is ball proximinal in $X$ if and only if $L_p(I, Y)$ is ball proximinal in $L_p(I, X)$ for $1 \leq p \leq \infty$. Our observations also include the fact that for any (strongly) proximinal subspace $Y$ of $X$, if every separable subspace of $Y$ is ball (strongly) proximinal in $X$ then $L_p(I, Y)$ is ball (strongly) proximinal in $L_p(I, X)$ for $1 \leq p < \infty$. We introduce the notion of uniform proximinality of a closed convex set in a Banach space, which is wrongly defined in [10]. Several examples are given having this property, viz. any $U$-subspace of a Banach space, closed unit ball $B_X$ of a space with 3.2.I.P, closed unit ball of any M-ideal of a space with 3.2.I.P. are uniformly proximinal. A new class of examples are given having this property.

1. Preliminaries and Definitions

Let $X$ be a Banach space and $C$ be a closed convex subset of $X$. For $x \in X$, let $d(x, C) = \inf_{z \in C} \| x - z \|$ and $P_C(x) = \{ z \in C : \| x - z \| = d(x, C) \}$. The set valued mapping $P_C : X \rightarrow 2^C$ is called the metric projection of $C$ and the points in $P_C(x)$ are called the best approximation from $x$ in $C$. We call the subset $C$ proximinal (or it has best approximation property) if for every point $x \in X \setminus C$, $P_C(x) \neq \emptyset$.

2010 Mathematics Subject Classification. Primary 46B20, 41A50, 46E40; Secondary 46E15.

Key words and phrases. $L_p(I, X)$, proximinality, strong proximinality, ball proximinality, upper Hausdorff semi-continuity, 3.2.I.P, 1/2 ball property.

The research was supported by DST-SERB, India. Award No. MA/2013-14/003/DST/TPaul/0104.
Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. For a Banach space $X$ consider the Banach space of Bochner $p$-integrable (essentially bounded for $p = \infty$) functions on $\Omega$ with values in $X$, endowed with the usual $p$-norm viz. $L_p(\Omega, X)$. Let us recall any such function is essentially a strongly measurable function, separably valued and if $(s_n)$ is a sequence of simple functions such that $s_n(t) \to f(t)$ a.e. then $\lim_n \int_I \|s_n(t)\|^p dm(t) = \int_I \|f(t)\|^p dm(t)$.

In [8, 9, 16, 17] the authors discussed for a finite measure space how often the property of best approximation of $Y$ in $X$ is stable under the spaces of functions $L_p(\Omega, Y)$ in $L_p(\Omega, X)$. Let us recall the following Theorem in this context.

**Theorem 1.1.** Let $Y$ be a subspace of $X$ and $f \in L_p(\Omega, X)$ then,

(a) [12] Theorem 5 $d(f, L_p(\Omega, Y)) = \|d(f(.), Y)\|_p$ for $1 \leq p \leq \infty$.

(b) [17] Theorem 3.4 For a separable subspace $Y$ of $X$, $L_p(\Omega, Y)$ is proximinal in $L_p(\Omega, X)$ if and only if $Y$ is proximinal in $X$, for $1 \leq p \leq \infty$.

(c) [12] Corollary 2 $f \in P_{L_p(\Omega,Y)}(g)$ if and only if $f(t) \in P_Y(g(t))$ a.e. for $1 \leq p < \infty$.

(d) [17] Proposition 2.5 $L_\infty(\Omega,Y)$ is proximinal in $L_\infty(\Omega,X)$ if and only if for $f \in L_\infty(\Omega,X)$ there exists $g \in L_\infty(\Omega,Y)$ such that $f(t) \in P_Y(g(t))$ a.e.

Suppose $I = [0, 1]$, and $(I, \mathcal{B}, m)$ stands for the complete Lebesgue measure space over the Borel $\sigma$-field $\mathcal{B}$. After Saidi’s paper, [22], people find it is worth investigating about the proximinality of closed unit ball of a proximinal subspace. The authors in [1] investigate the proximinality of $L_p(I, B_Y)$ in $L_p(I, X)$ if $B_Y$ is proximinal in $X$. Recall the following results from [1, Pg 12].

**Theorem 1.2.** Let $Y$ be a separable ball proximinal subspace of $X$. Then

(a) $L_\infty(I,Y)$ is ball proximinal in $L_\infty(I,X)$.

(b) $L_p(I, B_Y)$ is proximinal in $L_p(I, X)$.

A latest article in this context is [16]. It is also relevant to mention here that for a proximinal subspace $Y$, $L_1(I, Y)$ is not necessarily proximinal in $L_1(I, X)$ if $Y$ is not separable [17]. Light and Cheney also discussed about this best approximation property in the function spaces of type $L_p(\Omega, X)$ in [13] Chapter 2. Discussion in [13] Chapter 10 is also relevant to the content.
of this paper. Our aim in this paper is to study various strengthenings of best approximation property, defined in Definition 1.3 of $L_p(I,Y)$ in $L_p(I,X)$. A concise presentation of this work is available in Section 2.

We now state few known Definitions from the literature which are relevant and also have impacts to the main theme of this paper. First recall from [1,5] the following stronger versions of proximinality.

**Definition 1.3.**

(a) A closed convex subset $C$ of $X$ is *Strongly proximinal* if it is proximinal and for a given $x \in X \setminus C$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $P_C(x,\delta) \subseteq P_C(x) + \varepsilon B_X$, where $P_C(x,\delta) = \{ z \in C : \| x - z \| \leq d(x,C) + \delta \}$.

(b) A subspace $Y$ is said to be *Ball proximinal* if $B_Y$ is proximinal in $X$.

(c) A subspace $Y$ is said to be *Strongly ball proximinal* if $B_Y$ is strongly proximinal.

Readers can come across the articles [1,3,5] for various examples of subspaces having these proximity properties.

Recall the following notions for a set valued map.

**Definition 1.4.** If $T$ is a topological space, then a set-valued map $\Gamma : T \to 2^X$ is said to be

(a) upper semi-continuous, abbreviated usc (resp. lower semi-continuous, abbreviated lsc) if for any neighborhood $U$ of $\Gamma(t)$ there exists a neighborhood $V$ of $t$ such that for all $s \in V, \Gamma(s) \subseteq U$ (if for $x \in \Gamma(t)$ any sequence $t_n \to t$ there exists a sequence $x_n \in \Gamma(t_n)$ converging to $x$).

(b) upper Hausdorff semi-continuous, abbreviated uHsc. (resp. lower Hausdorff semi-continuous, abbreviated lHsc) if for every $t \in T$ and every $\varepsilon > 0$, there is a neighborhood $N$ of $t$, such that $\Gamma(t) \subseteq \Gamma(t_0) + \varepsilon B_X$ (resp. $\Gamma(t_0) \subseteq \Gamma(t) + \varepsilon B_X$) for each $t \in N$.

(c) Hausdorff continuous, abbreviated H-continuous, if it is both uHsc and lHsc.

From the definition of strong proximinality, it is clear that if $Y$ is a strongly proximinal subspace then $P_Y$ is uHsc. In general we have usc $\Rightarrow$ uHsc and lHsc $\Rightarrow$ lsc and if the above $\Gamma$ is compact valued then usc $\Leftrightarrow$ uHsc and lHsc $\Leftrightarrow$ lsc.
The following notion was introduced by Yost in [24]. The author established some connections between the properties of best approximation and the following for a subspace of a Banach space.

**Definition 1.5.** [24] A subspace $Y$ of a Banach space $X$ is said to have the $1\frac{1}{2}$-ball property if, whenever $\|x - y\| < r + s$ where $y \in Y$ and $x \in X$ with $B[x, r] \cap Y \neq \emptyset$ then $B[x, r] \cap B[y, s] \cap Y \neq \emptyset$.

It is well known that a subspace $Y$ having $1\frac{1}{2}$ ball property is strongly proximinal. There are many function spaces and function algebras in the class of continuous functions having this property.

Let us recall the following notion from [14].

**Definition 1.6.** A Banach space $X$ is said to have 3.2.I.P. if for for any three closed balls in $X$ which are pairwise intersecting actually intersect in $X$.

Lindenstrauss monograph [15] was the first where the above property was appeared for the first time, although the article [14] by Lima encounters a systematic study of intersection properties of balls in Banach spaces.

2. Main results and subsequent discussion

The following problems are the origin of this investigation.

**Problem 2.1.** Let $Y$ be a subspace of $X$ which is strongly proximinal (ball proximinal). Is $L_p(\Omega, Y)$ strongly proximinal (ball proximinal) in $L_p(\Omega, X)$ for $1 \leq p \leq \infty$?

The above problem on ball proximinality is asked in [1, Pg 12].

**Problem 2.2.** Let $f \in L_p(\Omega, X)$ and $Y$ be a subspace of $X$. What is the numerical value of $d(f, B_{L_p(\Omega, Y)})$?

**Problem 2.3.** Let $Y$ be a subspace of $X$ having $1\frac{1}{2}$ ball property and $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. Does $L_p(\Omega, Y)$ has $1\frac{1}{2}$ ball property in $L_p(\Omega, X)$ for $p = 1, \infty$?

Remark 3.10 states if $L_\infty(\Omega, Y)$ is strongly proximinal in $L_\infty(I, X)$ then $P_Y$ must be lHsc, on the other $Y$ would be strongly proximinal in $X$ for the same. Hence $P_Y$ is Hausdorff continuous if $L_\infty(\Omega, Y)$ is strongly proximinal in $L_\infty(I, X)$. Hence it raises the following question.
**Problem 2.4.** Let $P_Y : X \to 2^Y$ be Hausdorff continuous. Then what is the appropriate condition on $Y$ in $X$ which makes $L_\infty(\Omega, Y)$ strongly proximinal in $L_\infty(\Omega, X)$ and vice versa?

We considered these problems for the measure space $(I, \mathcal{B}, m)$. The results in Section 5 only require that the measure space has to be positive with total variation 1, the other results can be derived for any finite measure space. The main results in this article are the following:

**Theorem 2.5 (Theorem 3.6, 5.7).** For a separable proximinal subspace $Y$ of $X$, $Y$ is strongly proximinal (strongly ball proximinal) in $X$ if and only if $L_p(I, Y)$ is strongly proximinal (strongly ball proximinal) in $L_p(I, X)$, for $1 \leq p < \infty$.

**Theorem 2.6 (Theorem 5.4).** For a separable proximinal subspace $Y$ of $X$, $Y$ is ball proximinal in $X$ if and only if $L_p(I, Y)$ is ball proximinal in $L_p(I, X)$, for $1 \leq p \leq \infty$.

And also,

**Theorem 2.7 (Theorem 4.9).** Let $Y$ be a separable proximinal subspace of $X$, then consider the following statements.

(a) $Y(B_Y)$ is uniformly proximinal in $X$.

(b) $L_\infty(I, Y)(B_{L_\infty(I, Y)})$ is uniformly proximinal in $L_\infty(I, X)$.

(c) $L_\infty(I, Y)(B_{L_\infty(I, Y)})$ is strongly proximinal in $L_\infty(I, X)$.

Then (a) $\iff$ (b) and (b) $\implies$ (c).

We couldn’t answer the Problem 2.4, the above Theorem is a partial answer of Problem 2.4. A section-wise illustration of this work is outlined in the next few paragraphs.

In Section 3 we discuss some distance formulas which enable us to conclude the strong proximinality of $L_p(I, Y)$ in $L_p(I, X)$. These distance formulas are proved with the help of pathologies of measurable set valued functions and their measurable selections. Problem 2.3 is answered in Theorem 3.12.

The non-availability of conclusion in Theorem 2.5 for $p = \infty$ invites a uniform version of strong proximinality of $Y$ in $X$, as discussed in Section 4. To begin with, the content of Section 4 we would like to thank the authors in [16] for drawing our attention towards the notion of ‘uniform proximinality’
in Banach space. However, a similar notion dates back to the paper by Pai and Nowroji ([19]) in the context of Property-(R2); nevertheless, the way used in [16, Pg 79] to define 'uniform proximinality' is wrong. A simple geometry in the Euclidean space \( \mathbb{R}^2 \) clarifies the flaw (Example 4.1).

We adopt the idea introduced in [19] in terms of Property-(R2) and define 'uniform proximinality' of a closed convex set. Section 4 is devoted to discussing this property. Strong proximinality can now be viewed as a local version of this 'uniform proximinality'. Several examples are given which satisfy this property; the list includes closed convex subsets of uniformly convex space, subspace with \( 1 \frac{1}{2} \)-ball property and any \( U \)-proximinal subspace (see [10]). An elegant observation in this context is that closed unit ball of a Banach space is not necessarily uniformly proximal (using Example in [11]), we derive that it is true if \( X \) has 3.2.I.P (see [14]). Finally, we prove the strong proximinality of \( L_\infty(I,Y) \) in \( L_\infty(I,X) \) as a necessary condition for uniform proximinality of \( Y \) in \( X \) (Theorem 2.7). A weaker version of [21, Theorem 15] is also proved here.

Section 5 is devoted to ball proximinality and strong ball proximinality of \( L_p(I,Y) \) in \( L_p(I,X) \). One can define \( L_p(I,B_X) \), similar to the space \( L_p(I,X) \), which represents the set of measurable functions from \( I \) to \( B_X \) which are \( p \)-integrable. It is proved for \( f \in L_p(I,X) \), \( d(f,L_p(I,B_Y)) = d(f,B_{L_p(I,Y)}) \) for \( 1 \leq p \leq \infty \) which answers Problem 2.2. This result together with Theorem 5.6 leads to some interesting observations. The main results in this Section are stated in Theorem 2.6. Our results answer the question raised in [1] after Theorem 4.10.

Since in a Banach space \( X \), \( B_X \) is not necessarily strongly proximinal in \( X \) we found it is meaningful to identify some cases when the answer is affirmative. From [4] it follows that \( B_{L_p(\mu)} \) is strongly proximinal in \( L_p(\mu) \) (spaces having reflexivity and Kadec-Klee property) for any positive measure \( \mu \) when \( 1 < p < \infty \). From our result it follows that the conclusion is still true for \( L_p(\mu) \) where \( p = 1, \infty \) (for real scalar); in fact the result holds true for \( B_{L_p(I,X)} \), \( 1 \leq p \leq \infty \) when and only when \( X \) has the similar property.

A new class of examples is given in Section 6 which are uniformly proximinal.

For a Banach space \( X \), \( B_X \), \( S_X \) and \( B[x,r] \) denote the closed unit ball, the closed unit sphere and closed ball with centre at \( x \) and radius \( r \) respectively. All Banach spaces are assumed to be complex unless otherwise stated. Those
spaces that have any intersection properties of balls like 3.2.I.P., 4.2.I.P. are assumed to be real. $X$ will always denote a Banach space and by a subspace we always mean a closed subspace.

3. Strong proximinality of $L_p(I, Y)$ in $L_p(I, X)$

Similar to the Theorem 1.1 we now approach towards a distance formula which is actually stated in Theorem 3.4. To this end we need the following pathologies related to the set valued functions which help us to derive Theorem 3.4.

**Lemma 3.1.**

(a) Let $X$ be a Banach space and $Y$ be a proximinal subspace of $X$ such that the metric projection $P_Y$ is uHsc. Then the mapping $G : X \times X \to \mathbb{R}$ defined by $G((x, z)) = d(x, P_Y(z))$ is upper semi-continuous in first variable and lower semi-continuous in second variable.

(b) Let $Y$ be a subspace as defined in (a) and is also separable, then for any two measurable functions $f : I \to Y$ and $g : I \to X$ the mapping $\varphi : I \to \mathbb{R}$ defined by $\varphi(t) = d(f(t), P_Y(g(t)))$ is measurable.

**Proof.** (a). Upper semi-continuity of $G$ at it’s first variable follows from the fact that, for a closed set $A$ if $h(x) = d(x, A)$ then $h$ defines a continuous (and hence upper semi-continuous) mapping from $X$ to $\mathbb{R}$.

On the other hand let $\varepsilon > 0$. Since $P_Y$ is uHsc, there exists a $\delta > 0$ such that $P_Y(z) \subseteq P_Y(z_0) + \varepsilon B_Y$ whenever $\|z - z_0\| < \delta$. If $(z_n)$ converges to $z$, there exists an $N \in \mathbb{N}$ such that $\|z_n - z\| < \delta$ for all $n \geq N$. Hence for $n \geq N$ we get, $d(x, P_Y(z_n)) \geq d(x, P_Y(z) + \varepsilon B_Y) \geq d(x, P_Y(z)) - \varepsilon$.

Hence we have $\lim \inf_n d(x, P_Y(z_n)) \geq d(x, P_Y(z))$.

(b). Let $D \subseteq Y$ be a countable dense subset of $Y$. It is clear that the mapping $A : I \to Y \times X$ defined by $A(t) = (f(t), g(t))$ is measurable. We now show that $G : Y \times X \to \mathbb{R}$ defined by $G((y, x)) = d(y, P_Y(x))$ is measurable. Hence $\varphi(t) = G(A(t))$ will be measurable.

To this end we show that $G^{-1}((\alpha, \infty))$ is measurable for all real $\alpha$'s.

Now, $G((y, x)) \geq \alpha \iff$

$(\forall n \in \mathbb{N}) (\exists z_n \in D) [\|y - z_n\| < \frac{1}{n} \& G((z_n, x)) > \alpha - \frac{1}{n}] \iff$

$(y, x) \in \bigcap_n \bigcup_{z \in D} \left[\{y \in Y : \|y - z\| < \frac{1}{n}\} \times \{x \in X : G((z, x)) > \alpha - \frac{1}{n}\}\right]$.

Clearly if $(y, x) \in \text{RHS}$, then there exists a sequence $(z_n) \subseteq D$ such that $G((z_n, x)) > \alpha + \frac{1}{n}$ and $z_n \to y$ and hence $G((y, x)) \geq \lim \sup_n G((z_n, x)) \geq \alpha$. 


\[\delta\text{ defined in Lemma 3.1. Since all functions in } \Phi \text{ follow the representation for } \Phi_\delta \{\delta > 0\}, \text{ and the set of best approximation from a given point in } L_p(I, Y) \text{ to } L_p(I, Y). \]
Similar to Theorem 1.1, the distance function is an integral of the pointwise distance function.

**Theorem 3.4.** Let \( Y \) be a separable proximinal subspace of \( X \) such that \( P_Y \) is uHsc. Then for \( 1 \leq p < \infty \) and \( f \in L_p(I,Y), g \in L_p(I,X), \)
\[
d(f, P_{L_p(I,Y)}(g)) = \|d(f(\cdot), P_Y(g(\cdot)))\|_p.
\]

**Proof.** From Lemma 3.1 it follows that the map \( t \mapsto d(f(t), P_Y(g(t))) \) is measurable and hence the above integral is justified. Now for the given range of \( p, \)
\[
d(f, P_{L_p(I,Y)}(g)) = \inf_{h \in P_{L_p(I,Y)}(g)} \|f - h\|_p,
\]
\[
\geq \|d(f(\cdot), P_Y(g(\cdot)))\|_p, \text{ from Theorem 1.1(b).}
\]

Now for each \( n \) define \( \Phi_n : I \to 2^Y \) by \( \Phi_n(t) = P_{P_Y(g(t))}(f(t), \frac{1}{n}) \). From Lemma 3.3 it follows that the graph of \( \Phi_n \) is measurable and hence by Theorem 3.2 it has a measurable selection. Let \( h_n \) be such a selection. Clearly for all \( t, h_n(t) \in P_Y(g(t)) \) hence \( h_n \in P_{L_p(I,Y)}(g) \).
\[
d(f, P_{L_p(I,Y)}(g)) \leq \liminf_n \|f - h_n\|_p = \|d(f(\cdot), P_Y(g(\cdot)))\|_p. \]
The last equality follows from the Dominated convergence theorem for \( p < \infty \) and this establishes the other inequality.

**Remark 3.5.** For \( p = \infty, \) \( P_{L_\infty(I,Y)}(g) \supseteq \{ h \in L_\infty(I,Y) : h(t) \in P_Y(g(t)) \text{ a.e.} \} = Z, \) say. Hence \( d(f, P_{L_\infty(I,Y)}(g)) \leq \|d(f(\cdot), P_Y(g(\cdot)))\|_\infty : \)
In fact,
\[
d(f, P_{L_\infty(I,Y)}(g)) \leq d(f, Z)
\]
\[
= \inf_{h \in Z} \|f(t) - h(t)\|
\]
\[
= \text{esssup}_t d(f(t), P_Y(g(t)))
\]
\[
= \|d(f(\cdot), P_Y(g(\cdot)))\|_\infty.
\]

Our main results of this section are the following.

**Theorem 3.6.** Let \( Y \) be a separable proximinal subspace of \( X \). Then \( Y \) is strongly proximinal in \( X \) if and only if \( L_p(I,Y) \) is strongly proximinal in \( L_p(I,X) \) for \( 1 \leq p < \infty \).

**Proof.** Let \( Y \) be strongly proximinal in \( X \) and let for some \( p \in [1,\infty), \)
\( L_p(I,Y) \) be not strongly proximinal in \( L_p(I,X) \). Hence there exists \( f \in \)
$L_p(I, X)$, $\varepsilon > 0$ and $(g_n) \subseteq L_p(I, Y)$ such that $\|f - g_n\|_p \to d(f, L_p(I, Y))$

but $d(g_n, P_{L_p(I, Y)}(f)) \geq \varepsilon$.

Now $\|f - g_n\|_p \to d(f, L_p(Y))$

$$\implies \int_I \|f(t) - g_n(t)\|^p dm(t) \to \int_I d(f(t), Y)^p dm(t).$$

$$\implies \int_I \|f(t) - g_n(t)\|^p - d(f(t), Y)^p dm(t) \to 0.$$

A well known property of $L_p$ convergence ensures that there exists a subsequence $(g_{n_k})$ satisfying $\|f(t) - g_{n_k}(t)\|^p - d(f(t), Y)^p \to 0$ a.e.

Since $\|f(t) - g_{n_k}(t)\| \to d(f(t), Y)$ a.e. we have $d(g_{n_k}(t), P_Y(f(t))) \to 0$ a.e. Since $d(g_{n_k}(t), P_Y(f(t)))^p \leq 2\|f(t)\|^p$, a $L_1$ function. Hence by Dominated Converge Theorem, $\lim_{k \to 0} \int_I d(g_{n_k}(t), P_Y(f(t)))^p dm(t) = 0$, contradicting our assumption on $(g_n)$. Hence the result follows. \(\square\)

Since all $g_n$’s in the above proof are separably valued the above proof can be fitted with all such strongly proximinal $Y$ of which all its separable subspaces are also strongly proximinal.

**Corollary 3.7.** Let $Y$ be a strongly proximinal subspace of $X$. If every separable subspace of $Y$ is strongly proximinal in $X$ then $L_p(I, Y)$ is strongly proximinal in $L_p(I, X)$.

**Proof.** For such type of $(g_n)$ defined above get a separable subspace $Z \subseteq Y$ such that $d(f, L_p(I, Y)) = d(f, L_p(I, Z))$, $1 \leq p \leq \infty$. From our assumption and Theorem 3.6 it follows $d(g_n, P_{L_p(I, Z)}(f)) \to 0$ and hence $d(g_n, P_{L_p(I, Y)}(f)) \to 0$. \(\square\)

**Remark 3.8.** In general the conclusion of the Theorem 3.6 is not true for $p = \infty$. Example 3.9. In next Section we show that a stronger version of strong proximinality of $L_p(I, Y)$ in $L_p(I, X)$ can be achieved from the similar assumption of $Y$ in $X$ and also vice versa.

We now show that strong proximinality of $L_\infty(I, Y)$ in $L_\infty(I, X)$ demands a stronger assumption on $Y$ in $X$.

From Michael’s selection theorem (see 18, Theorem 3.1’) it is clear that if $Y$ is a finite dimensional subspace of a normed linear space $X$ and the metric projection $P_Y$ is lsc then it has a continuous selection. Now in 2, Example 2.5] the author has shown that there exists a 1 dimensional subspace $Y$ in the 3 dimensional space $\mathbb{R}^3$ with a suitable norm where the metric projection $P_Y$ has no continuous selection. Hence it can not be lsc, and being a compact valued map $P_Y$ is not also lHsc. We now use these observations
in the following example for the subspace \( Y \) and the corresponding metric projection \( P_Y \) to derive the non stability behavior of \( L_\infty(I,Y) \) in \( L_\infty(I,X) \) in the context of strong proximinality.

Example 3.9. If \( Y \) is strongly proximinal in \( X \) then \( L_\infty(I,Y) \) not necessarily strongly proximinal in \( L_\infty(I,X) \) : Let \( X \) and \( Y \) be the spaces defined in [2, Example 2.5]. Then there exists a sequence \( (x_n) \subseteq X, x \in X \) such that \( x_n \to x \) but \( P_Y(x) \not\subseteq P_Y(x_n) + \varepsilon B_Y \) for some \( \varepsilon > 0 \). Define \( z_n = \frac{x_n}{d(x_n, Y)}, z_0 = \frac{2}{d(x,Y)} \). Then \( z_n \to z_0 \) and \( d(z_n, Y) = 1 = d(z_0, Y) \). Also we have,

\[
d(x, Y) P_Y(z_0) \not\subseteq d(x_n, Y) P_Y(z_n) + \varepsilon B_Y, \quad \text{for all } n \in \mathbb{N}.
\]

That is there exists \( y_n \in P_Y(z_0) \) such that \( d((d(x, Y), d(x_n, Y) P_Y(z_n))) \geq \varepsilon \) and hence \( d(y_n, \alpha_n P_Y(z_n)) \geq \eta \) where \( \alpha_n \to 1 \) and some \( \eta > 0 \).

It is clear that \( \| y_n - z_n \| - d(z_n, Y) \to 0 \). Let \( (I_n) \) be a sequence of pairwise disjoint intervals with \( \bigcup_n I_n = I \).

Define \( f \in L_\infty(I,X), g_k \in L_\infty(I,Y) \) with \( f|_{I_n} = z_n, g_k|_{I_n} = y_n \) if \( k = n \) otherwise \( g_k|_{I_n} \subseteq P_Y(z_k) \). Clearly we have \( \| f - g_k \|_\infty \to d(f, L_\infty(I,Y)) \) but \( d(g_k, P_{L_\infty(I,Y)}(f)) \geq \eta \), for all but finitely many \( k \)'s. The last inequality follows from the fact that,

\[
P_{L_\infty(I,Y)}(f) = \{ h \in L_\infty(I,Y) : h|_{I_n} \subseteq P_Y(z_n), \quad \text{for all } n \}.
\]

Remark 3.10. From above example it is clear if \( L_\infty(I,Y) \) is strongly proximinal in \( L_\infty(I,X) \) then \( P_Y \) must be Hausdorff continuous.

We conclude this Section by an application of Theorem 3.11. The scalar field for the Banach spaces considered in rest of this Section is \( \mathbb{R} \).

The following result, Theorem 3.12, concludes about strong proximinality of \( L_\infty(I,Y) \) in \( L_\infty(I,X) \). It is also a strengthening of [21, Theorem 15] which was proved for strong \( 1\frac{1}{2} \) ball property. Before we go for Theorem 3.12 here is a useful characterization of \( 1\frac{1}{2} \) ball property.

Theorem 3.11. [6] For a subspace \( Y \) of \( X \), the following are equivalent.

(a) \( Y \) has \( 1\frac{1}{2} \) ball property.
(b) \( \| x - y \| = d(x, Y) + d(y, P_Y(x)), \) for \( x \) in \( X \) and \( y \in Y \).
(c) \( \| x \| = d(x, Y) + d(0, P_Y(x)), \) for \( x \in X \).

Theorem 3.12. A separable subspace \( Y \) of \( X \) has \( 1\frac{1}{2} \) ball property if and only if \( L_1(I,Y)(L_\infty(I,Y)) \) has \( 1\frac{1}{2} \) ball property in \( L_1(I,X)(L_\infty(I,X)) \).
Proof. Suppose $Y$ has $1\frac{1}{2}$ ball property in $X$. We only show that the distance formula in Theorem 3.11(c) holds for any $f \in L_1(I, X)$. Now $\|f(t)\| = d(f(t), Y) + d(0, P_Y(f(t)))$ a.e. For $p = 1$, we get the result by integrating both sides and use the distance formulas discussed in Theorem 1.1, 3.4. For $p = \infty$ we take the essential supremum in both sides and use the Remark 3.5 and get $\|f\|_{\infty} \geq d(f, L_\infty(I, Y)) + d(0, P_{L_\infty(I, Y)}(f))$. The other inequality is obvious.

Conversely, for any $x \in X$ consider the constant function $f(t) = x$ for all $t \in I$. The result now follows from Theorem 3.11 and 3.4. □

4. Uniform proximinality of $L_p(I, Y)$ in $L_p(I, X)$

In a recent paper ([16]) the authors has introduced the notion uniform proximinality and it is claimed that closed unit ball of any uniformly convex space is uniformly proximinal. We first observe that the property does not holds even for the 2 dimensional Euclidean space.

Example 4.1. Let $C$ be the closed unit ball of $(\mathbb{R}^2, \|\cdot\|_2)$, $x = (2, 0)$. Then $P_C((2, 0)) = \{(1, 0)\}$. Let $\alpha = 2$ and $\varepsilon = 1/2$. Then there does not exist $\delta > 0$ satisfying the condition in [16], pg 79, which makes $C$ uniformly proximinal. In fact, if such a $\delta > 0$ exists then $\|(0, 0) - (2, 0)\| < \alpha + \delta$ but $\|(0, 0) - (1, 0)\| > \varepsilon$.

We now define a stronger version of proximinality, viz. uniform proximinality which is in fact stated in [19] in the context of centres of closed bounded sets.

Definition 4.2. Let $C$ be a closed convex subset of $X$. We call $C$ is uniformly proximinal if given $\varepsilon > 0$ and $R > 0$ there exists $\delta(\varepsilon, R) > 0$ such that for any $x \in X, d(x, C) \leq R$ and $y \in C$ with $\|x - y\| < R + \delta$, there exists $y' \in C$ with $\|y - y'\| < \varepsilon$ and $\|x - y'\| \leq R$.

Here are some examples of uniformly proximinal sets.

Example 4.3. (a) It is clear that a Banach space $X$ having 3.2.I.P., $B_X(B_{L_\infty(I, X)})$ is uniformly proximinal in $X(L_\infty(I, X))$.

(b) [19] Proposition 3.5] Any $w^*$-closed convex subset of $\ell_1$ is uniformly proximinal.

(c) [19] Proposition 3.7] Any closed convex proximinal subset of a LUR space is uniformly proximinal.
Any subspace $Y$ of $X$ having $1\frac{1}{2}$ ball property is uniformly proximinal: Let $R, \varepsilon > 0$ such that $d(x, Y) \leq R$ and $\|x - y\| < R + \varepsilon$ for some $y \in Y$, from the Definition 1.5 we have $B[x, R] \cap B[y, \varepsilon] \cap Y \neq \emptyset$. Any point from this intersection solve our purpose.

Any subspace $Y$ of $X$ which is $U$-proximinal is also uniformly proximinal: Let $\eta, R > 0$, suppose $\varepsilon : \mathbb{R} \to \mathbb{R}$ be the continuous function corresponding to the subspace $Y$ in [10]. Get $\theta > 0$ satisfying $\varepsilon(\theta) < \eta/R$, let $\delta = R\theta$. Let $x \in X$ such that $d(x, Y) \leq R$ and $y \in Y$ be such that $\|x - y\| < R + \delta$.

Claim: There exists $y' \in Y$ such that $\|y - y'\| < \eta$ and $\|x - y'\| \leq R$.

Now $d(\vec{x}, Y) \leq 1$ and $\|\vec{x} - \vec{y}\| < 1 + \theta$, in other words $\vec{x} \in Y + B_X$ and $\vec{x} - \vec{y} \in (1 + \theta)B_X$ and hence $\vec{x} - \vec{y} \in Y + B_X$. And finally there exists $y_1 \in \varepsilon(\theta)B_Y$ such that $\|\vec{x} - \vec{y} - y_1\| \leq 1$. Define $y' = y + Ry_1$, this $y'$ satisfies the desired requirements.

We refer [19] to the reader for many other interesting uniformly proximinal subsets of Banach spaces.

**Remark 4.4.**

(a) In the Definition 4.4 if we demand to have $\delta = \varepsilon$ for all $R > 0$ we get back $1\frac{1}{2}$ ball property.

(b) From the Definition 4.4 it is clear that uniform proximinality of $C$ forces the set to be strongly proximinal.

(c) From the example by Godefroy in [11] it is clear that the closed unit ball of a Banach space not necessarily have uniformly proximinal property.

We now claim that converse of Remark 4.4(b) is not true. First observe the following.

**Proposition 4.5.** If a closed convex set $C$ in $X$ is uniformly proximinal then the metric projection $P_C : X \to 2^C$ is continuous in the Hausdorff metric.

**Proof.** Let $x_n \to x$ in $X$, without loss of generality we can assume $d(x, C) = 1, d(x_n, C) = 1$ for all $n$. Let $\delta(1, \varepsilon) > 0$ be the number corresponding to uniform proximinality of $C$. If possible let $P_C(x) \notin P_C(x_n) + \varepsilon B_Y$ for all but finitely many $n$’s, for some $\varepsilon > 0$. Hence there exists $y_n \in P_C(x)$ such that $d(y_n, P_C(x_n)) \geq \varepsilon$. Get a $N$ such that $\|x_n - y_n\| - d(x_n, C) < \delta$ for
all \( n > N \). Now using the property of uniform proximinality of \( C \) there exists \( y'_{n} \in P_{C}(x_{n}) \) such that \( \|y_{n} - y'_{n}\| < \varepsilon \), contradicting our hypothesis \( d(y_{n}, P_{C}(x_{n})) \geq \varepsilon \). This proves \( P_{C} \) is lHsc.

The uHsc of \( P_{C} \) follows from strong proximinality of \( C \).

From Proposition 4.5 and the arguments used before Example 3.9 it now follows that the subspace \( Y \) in [2] Example 2.5] can not be uniformly proximinal, while on the other hand being a finite dimensional subspace it is always strongly proximinal.

We now show that similar to proximinality and strong proximinality, the closed unit ball of a subspace by virtue of being uniformly proximinal forces the subspace to be uniformly proximinal.

**Proposition 4.6.** For a subspace \( Y \) of \( X \), if \( B_{Y} \) is uniformly proximinal then \( Y \) is also uniformly proximinal.

**Proof.** We use the technique used in [1] Lemma 2.3. If possible let \( B_{Y} \) is uniformly proximinal and \( Y \) is not. From the definition there exist \( R > 0, \varepsilon > 0, x \in X \) where \( d(x, Y) \leq R \) and also there exists \( (y_{n}) \subseteq Y \) such that \( \|x - y_{n}\| < R + \frac{1}{n} \) but for all \( y \in B(y_{n}, \varepsilon) \), \( \|x - y\| > R \).

Choose \( \lambda > \|x\| + R + 2\varepsilon \), then \( d(x, \lambda B_{Y}) = d(x, Y) \). From our assumption on \( y_{n} \) it follows that \( \|y_{n}\| < \|x\| + R + \frac{1}{n} \) and hence \( y_{n} \in \lambda B_{Y} \).

Uniform proximinality of \( \lambda B_{Y} \) (and hence \( B_{Y} \)) would be contradicted if we can show that \( B_{Y}(y_{n}, \varepsilon) \subseteq \lambda B_{Y} \), for all \( n \). And It follows from the following observation.

\[
\|y_{n}\| + \varepsilon < \|x\| + R + \varepsilon + \frac{1}{n} \leq \|x\| + R + 2\varepsilon < \lambda \text{, for large } n.
\]

This completes the proof.

We now propose the following problem which is relevant to the subsequent matter.

**Problem 4.7.** Let \( Y \) be a subspace of \( X \) which is uniformly proximinal. Is it necessary that \( B_{Y} \) is also uniformly proximinal in \( X \) ?

**Remark 4.8.** (a) It is clear from the Definition 4.2 that uniform proximinality of \( C \) is a uniform version of strong proximinality for the points which are of finite distance away from \( C \). Hence due to the Example by Godefroy in [11] it is clear that closed unit ball of a Banach space not necessarily uniformly proximinal.
(b) We do not know whether the converse of Example 4.3(e) is true or not.

(c) From Theorem 3.12 we have if $Y$ is separable and also has $1\frac{1}{2}$ ball property in $X$ then $L_p(I, Y)$ has $1\frac{1}{2}$ ball property (hence uniformly proximinal) in $L_p(I, X)$ for $p = 1, \infty$.

From the Definition 4.2 we now have the following.

**Theorem 4.9.** Let $Y$ be a separable proximinal subspace of $X$, Consider the following statements.

(a) $Y(B_Y)$ is uniformly proximinal in $X$.

(b) $L_\infty(I, Y)(B_{L_\infty(I,Y)})$ is uniformly proximinal in $L_\infty(I, X)$.

(c) $L_\infty(I, Y)(B_{L_\infty(I,Y)})$ is strongly proximinal in $L_\infty(I, X)$.

Then $(a) \iff (b)$ and $(b) \implies (c)$.

**Proof.** It is clear that $(b) \implies (a)$ and $(b) \implies (c)$. We only show that $(a) \implies (b)$. We prove the result for the subspace $Y$, case for $B_Y$ follows from that with obvious modifications.

Let us choose $R > 0$ and $\varepsilon > 0$. Choose $\delta(R, \varepsilon) > 0$ for the subspace $Y$. We claim that this $\delta$ will also work for $L_\infty(I, Y)$. Let $f \in L_\infty(I, X)$ with $d(f, L_\infty(I, Y)) \leq R$. Let $g \in L_\infty(I, Y)$ be such that $\|f - g\|_\infty < R + \delta$. Then from the property of uniform proximinality it follows that $B[f(t), R] \cap B[g(t), \varepsilon] \cap Y \neq \emptyset$ a.e. Consider the set valued map $\varphi : t \mapsto B[f(t), R] \cap B[g(t), \varepsilon] \cap Y$ from $[0, 1]$ to $2^Y$. It is clear that the graph of this map $\{(t, \varphi(t) : t \in I)\}$ is measurable and hence by Theorem 3.2 it follows it has a measurable selection, let us call it $h$. We have $h \in L_\infty(I, Y)$ and satisfies the requirements. □

Theorem 4.9 leads to the following problem.

**Problem 4.10.** Let $L_\infty(I, Y)$ is strongly proximinal in $L_\infty(I, X)$. Is it true that $Y$ is uniformly proximinal in $X$?

5. **Ball Proximinality of $L_p(I, Y)$ in $L_p(I, X)$**

We first prove the distance formula analogous to Theorem 3.4 for the closed unit ball of $L_p(I, Y)$, for $1 \leq p \leq \infty$.

**Theorem 5.1.** Let $f \in L_p(I, X)$ be a strongly measurable function then

$$d(f, B_{L_p(I,Y)}) = \|d(f, .), B_Y\|_p, \text{ for } 1 \leq p \leq \infty.$$
Proof. Case for $p = \infty$ is already observed in [1], it remains to prove when $p < \infty$.

**Step 1:** Let $f(t) = x$ for all $t \in I$ and for some $x \in X$. Clearly
\[ d(f, B_{L_p(I,Y)}) \leq d(f, L_p(I, B_Y)) = d(x, B_Y). \]

Let $g \in B_{L_p(I,Y)}$ and $\varepsilon > 0$, then there is a sequence of simple functions $(s_n) \subseteq B_{L_p(I,Y)}$ such that $s_n \rightarrow g$ in $L_p(I,Y)$. Without loss of generality we may assume each $s_n$ has a following representation.

Now $d(x, B_Y)^p \leq \|x - z_n\|^p = \int_I \|f(t) - s_n(t)\|^p dm(t) = \|f - s_n\|^p \leq \|f - g\|^p + \varepsilon$ for all but finitely many $n$'s. Taking infimum over $g \in B_{L_p(I,Y)}$ we get the result.

**Step 2:** Let $f = \sum_{i=1}^n x_i \chi_{E_i}$, where $x_i \in X$, $\cup_i E_i = I$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.

Now,
\[
\begin{align*}
&d(f, B_{L_p(I,Y)})^p \\
&\leq \int_I d(f(t), B_Y) \, dm(t) \\
&= \sum_{i=1}^n d(x_i, B_Y)^p m(E_i) \\
&= \sum_{i=1}^n d(x_i, B_{L_p(I,Y)})^p m(E_i) \quad \text{follows from Step 1} \\
&= \inf_{g \in B_{L_p(I,Y)}} \sum_{i=1}^n \int_{E_i} \|x_i - g(t)\|^p dm(t) \\
&= \inf_{g \in B_{L_p(I,Y)}} \int_I \|f(t) - g(t)\|^p dm(t) = d(f, B_{L_p(I,Y)})^p
\end{align*}
\]

**Step 3:** Let $f \in L_p(I, X)$ and $\varepsilon > 0$. Get a sequence of simple functions $(s_n) \subseteq L_p(I, X)$ such that $s_n \rightarrow f$ in $L_p(I, X)$. Without loss of generality assume $s_n$ converges to $f$ pointwise and $\|s_n(t)\| \leq \|f(t)\|$ a.e.
Now,
\[ d(f, B_{L^p(I,Y)}) = \inf_{g \in B_{L^p(I,Y)}} \| f - g \|_p \]
\[ \geq \inf_{g \in B_{L^p(I,Y)}} \| s_n - g \|_p - \| s_n - f \|_p \]
\[ = d(s_n, B_{L^p(I,Y)}) - \| s_n - f \|_p \]
\[ = \left( \int_I d(s_n(t), B_Y)^p dm(t) \right)^{1/p} - \| s_n - f \|_p; \text{ from Step 2} \]
\[ \geq \left( \int_I d(f(t), B_Y)^p dm(t) \right)^{1/p} - \varepsilon; \text{ for large } n \]
\[ \geq \left( \int_I \| f(t) - s_n(t) \|^p dm(t) \right)^{1/p} - 2\varepsilon; \text{ for large } n \]

The last inequality follows from the following observation.

\[ \| d(\cdot, B_Y) \|_p \leq \| d(f(\cdot), B_Y) - d(s_n(\cdot), B_Y) \|_p + \| d(s_n(\cdot), B_Y) \|_p \]
\[ = \left( \int_I |d(f(t), B_Y) - d(s_n(t), B_Y)|^p dm(t) \right)^{1/p} + \| d(s_n(\cdot), B_Y) \|_p \]
\[ \leq \left( \int_I \| f(t) - s_n(t) \|^p dm(t) \right)^{1/p} + \| d(s_n(\cdot), B_Y) \|_p \]

Since \( \varepsilon > 0 \) is arbitrary, the result follows. \( \square \)

**Remark 5.2.**

(a) In [1] it is observed that for \( f \in L^p(I,X) \),
\( d(f, L^p(I,Y)) = \| d(f(\cdot), B_Y) \|_p \), hence from Theorem 5.1 it follows
\( P_{B_{L^p(I,Y)}}(f) \subseteq P_{B_{L^p(I,Y)}}(f) \) for \( 1 \leq p \leq \infty \).

(b) For a \( g \in L^p(I,Y) \) we have, \( g \in P_{B_{L^p(I,Y)}}(f) \) \( \iff \)
\( g(t) \in P_{B_Y}(f(t)) \) a.e. \( \iff g \in P_{L_p(I,Y)}(f) \) for \( 1 \leq p < \infty \).

Remark 5.2(a) leads to the following question.

**Problem 5.3.** For a subspace \( Y \) of \( X \) what are the functions \( f \in L^p(I,X) \)
for \( 1 \leq p < \infty \) for which \( P_{B_{L_p(I,Y)}}(f) = P_{L_p(I,Y)}(f) \)?

We now prove the main result of this Section.

**Theorem 5.4.** Let \( Y \) be a separable proximinal subspace of \( X \). Then the
following are equivalent.
(a) $Y$ is ball proximinal in $X$.
(b) $L_p(I,B_Y)$ is proximinal in $L_p(I,X)$, for $1 \leq p \leq \infty$.
(c) $L_p(I,Y)$ is ball proximinal in $L_p(I,X)$, for $1 \leq p \leq \infty$.

**Proof.** From [1] and Remark 5.2 it is now clear that (a) $\implies$ (b) and (b) $\implies$ (c). We now show that (c) $\implies$ (a). Now the Case for $p = \infty$ is already observed in [1], it remains to prove the result for $p < \infty$. Hence it is enough to prove that $Y$ is ball proximinal in $X$ if $L_p(I,Y)$ is same in $L_p(I,X)$ for some $p \in [1,\infty)$.

Let $x \in X$ and define $f(t) = x$ for all $t \in I$. Then $f \in L_p(I,X)$ and $d(f,B_{L_p(I,Y)}) = d(x,B_Y)$. Choose $g \in B_{L_p(I,Y)}$ satisfying $\|f - g\|_p = d(x,B_Y)$. Now choose a sequence of simple functions $(s_n)$ such that $\|s_n - g\|_p \to 0$ where $\|s_n\|_p \leq \|g\|_p$. Let $s_n = \sum_{i=1}^{k_n} x_i^m E_i^n$ where $x_i^m \in Y$ and $\cup_i E_i^n = I$. Let $y_n = \sum_{i=1}^{k_n} x_i^m m(E_i^n)$. Since $\sum_{i=1}^{k_n} ||x_i^m||^p m(E_i^n) \leq 1$ and $t \mapsto t^p$ is a convex function on $\mathbb{R}$ we have $y_n \in B_Y$. Now we have,

$$d(x,B_Y)^p \leq \|x - y_n\|^p$$
$$= \|x - \sum_{i=1}^{k_n} x_i^m m(E_i^n)\|^p$$
$$= \|\sum_{i=1}^{k_n} (x - x_i^m) m(E_i^n)\|^p$$
$$\leq \sum_{i=1}^{k_n} \|x - x_i^m\|^p m(E_i^n)$$
$$= \|x - s_n\|^p$$
$$\to d(x,B_Y)^p$$

Which ensures that $(y_n)$ is a minimizing sequence in $B_Y$ for $x$. Clearly $(y_n)$ is cauchy; in fact $\lim_{n} y_n = \int_I g(t) dm(t)$, and hence there exists $y_0 \in B_Y$ such that $\|x - y_0\| = d(x,B_Y)$.

The arguments involved in the proof of Corollary 3.7 lead to the following conclusion.

**Corollary 5.5.**
(a) Let $Y$ be a ball proximinal subspace of $X$, if every separable subspace of $Y$ is ball proximinal in $X$ then $L_p(I,Y)$ is ball proximinal in $L_p(I,X)$ for $1 \leq p \leq \infty$.
(b) Let $Y$ be a reflexive subspace of $X$ then $L_p(I,B_Y)$ (and hence $B_{L_p(I,Y)}$) is proximinal in $L_p(I,X)$ for $1 \leq p \leq \infty$. 


Proof. We only prove (a), (b) follows from (a). It remains to prove for a given \( f \in L_p(I, X), P_{L_p(I, B_Y)}(f) \neq \emptyset \). Choose \( (g_n) \subseteq L_p(I, B_Y) \) such that \( \|f - g_n\|_p \to d(f, L_p(I, B_Y)) \). Get a separable subspace \( Z \subseteq Y \) such that \( g_n(I) \subseteq Z \) for all \( n \). It is clear that \( d(f, L_p(I, B_Y)) = d(f, L_p(I, B_Z)) \). Since \( P_{L_p(I, B_Z)}(f) \neq \emptyset \) the result follows. \( \square \)

We now come to the strong proximinality of closed unit ball of \( L_p(I, Y) \).

A few routine modifications of Theorem 3.4 lead to the following result.

**Theorem 5.6.** Let \( Y \) be a strongly ball proximinal subspace of \( X \) and \( f \in L_p(I, X), g \in L_p(I, X) \) then, \( d(f, P_{B_{L_p(I,Y)}}(g)) = \|d(f, P_{B_Y}(g))\|_p, \) for \( 1 \leq p < \infty \).

Combining Theorem 5.6 and the routine modifications in Theorem 3.6, one can have the following.

**Theorem 5.7.** Let \( Y \) be a separable proximinal subspace of \( X \). Then the following are equivalent.

(a) \( Y \) is strongly ball proximinal subspace of \( X \).

(b) \( L_p(I, B_Y) \) is strongly proximinal in \( L_p(I, X) \), for \( 1 \leq p < \infty \).

(c) \( L_p(I, Y) \) is strongly ball proximinal in \( L_p(I, X) \), for \( 1 \leq p < \infty \).

Proof. It remains to prove (c) \( \implies \) (a). Choose \( p \in [1, \infty) \) arbitrarily. Let \( x \in X \) and \( (y_n) \subseteq B_Y \) be such that \( \|x - y_n\| \to d(x, B_Y) \). Define \( f(t) = x \) and \( g_n(t) = y_n \) for all \( t \in I \) then \( \|f - g_n\|_p \to d(f, B_{L_p(I,Y)}) = d(x, B_Y) \) and hence \( d(g_n, P_{B_{L_p(I,Y)}}(f)) \to 0 \). Choose \( h_n \in P_{B_{L_p(I,Y)}}(f) \) such that \( \|g_n - h_n\|_p \to 0 \). Hence there exists \( (z_n) \subseteq B_Y \) where \( z_n = \int_I h_n(t)dm(t) \).

**Claim:** \( z_n \in P_{B_Y}(x) \) and \( \|y_n - z_n\| \to 0 \).

\[
d(x, B_Y) \leq \|x - z_n\|^p \]
\[
= \|x - \int_I h_n(t)dm(t)\|^p
\]
\[
= \|\int_I (h_n(t) - x)dm(t)\|^p
\]
\[
\leq \int_I \|h_n(t) - x\|^pdm(t)
\]
\[
= \int_I d(x, B_Y)^pdm(t), \text{ follows from Theorem 1.1}
\]
\[
= d(x, B_Y)
\]
And finally,
\[ \|y_n - z_n\|^p = \|y_n - \int_I h_n(t) \, dm(t)\|^p = \|\int_I (y_n - h_n(t)) \, dm(t)\|^p \leq \int_I \|y_n - h_n(t)\|^p \, dm(t) \leq \|g_n - h_n\|_p^p \to 0 \]

This completes the proof. \[\square\]

For the case \( p = \infty \) the result follows under an additional assumption on \( B_Y \). The Banach spaces considered for rest of this Section are assumed to be Real.

Now it is clear from the above observations that,

**Corollary 5.8.** Let \( X \) be a separable Banach space.

(a) For \( 1 \leq p < \infty \), if \( B_X \) is strongly proximinal in \( X \) then \( B_{L_p(I,X)} \) is strongly proximinal in \( L_p(I,X) \).

(b) If \( X \) has 3.2.I.P. then \( B_{L_p(I,X)} \) is strongly proximinal in \( L_p(I,X) \) for \( 1 \leq p < \infty \).

**Proof.** Since \( X \) is separable, Theorem 5.7 is true for \( Y = X \) and hence (a) follows. If \( X \) has 3.2.I.P. then \( B_X \) is strongly proximinal in \( X \) (Example 6.7(a)). (b) is now follows from (a). \[\square\]

**Remark 5.9.**

(a) Uniform convexity of \( L_p(I,X) \) for \( 1 < p < \infty \) follows from uniform convexity of \( X \) and vice versa. Hence Corollary 5.8 ensures the strong ball proximinality of \( L_p(I,X) \) beyond the class of uniformly convex Banach space \( X \).

(b) It is not necessarily true that \( B_{L_{\infty}(I,Y)} \) is strongly proximinal in \( L_{\infty}(I,X) \) if \( B_Y \) is same in \( X \) (Example 5.9).

6. A NEW CLASS OF UNIFORMLY PROXIMAL SUBSETS

Motivated from the property defined in Definition 1.5 we define the following for a closed unit ball of a subspace but more generally it can be defined for a closed convex subset.
Definition 6.1. We call the closed unit ball $B_Y$ of a subspace $Y$ in $X$ has $1\frac{1}{2}$ ball property if for $x \in X, y \in B_Y$ and $r_1, r_2 > 0$ $B[x, r_1] \cap B_Y \neq \emptyset, \|x - y\| < r_1 + r_2$ implies $B[x, r_1] \cap B[y, r_2] \cap B_Y \neq \emptyset$.

Similar to our earlier observation Remark 4.3, the ball $B_Y$ having $1\frac{1}{2}$-ball property is uniformly proximinal for $\delta = \varepsilon$. Here are few immediate consequences of the above property.

Theorem 6.2. Let $Y$ be a subspace of $X$. Then,

(a) If $B_Y$ has $1\frac{1}{2}$ ball property then $Y$ has $1\frac{1}{2}$ ball property.

(b) If $B_Y$ has $1\frac{1}{2}$ ball property in $X$ then $Y$ is ball proximinal in $X$.

The proofs of the above Theorem follow from the similar arguments used to proof for a subspace for a similar claim. One can revisit the proofs in [1, Proposition 2.4] for (a) and [24, Lemma 1.1] for (b).

Remark 6.3. The converse of Theorem 6.2(a) is not necessarily true. It is clear that a $M$-ideal has $1\frac{1}{2}$ ball property but not necessarily ball proximinal as is observed in [7].

We now derive a characterization, similar to Theorem 3.11 for $1\frac{1}{2}$ ball property of $B_Y$ in $X$. An almost similar arguments can be used to prove the following, for the sake of completeness we briefly outline it here.

Notation. For a subset $C$ of $X$, define $C_\varepsilon = \{x \in X : d(x, B) \leq \varepsilon\}$.

Theorem 6.4. Let $Y$ be a subspace of $X$, then the following are equivalent.

(a) $B_Y$ has $1\frac{1}{2}$ ball property.

(b) $P_{B_Y}(x, \delta) = P_{B_Y}(x, \delta) \cap B_Y$. For all $x \in X$ and $\delta > 0$.

(c) $d(y, P_{B_Y}(x)) = \|y - x\| - d(x, B_Y)$. For all $x \in X, y \in B_Y$.

Proof. (a) $\implies$ (b) : Let $d = d(x, Y)$ and $\|x - y\| \leq d + \delta$ for some $y \in B_Y$. By (a), $B[x, d] \cap B[y, \delta] \cap B_Y \neq \emptyset$ for all $\delta' > \delta$. That is $B[y, \delta'] \cap P_{B_Y}(x) \neq \emptyset$ and hence $d(y, P_{B_Y}(x)) \leq \delta'$, true for all $\delta' > \delta$, thus $d(y, P_{B_Y}(x)) \leq \delta$. The other inclusion follows trivially from the definition of the sets involved in it.

(b) $\implies$ (c) : Let $\varepsilon = \|y - x\| - d(x, B_Y)$, for $y \in B_Y$. Then $y \in P_{B_Y}(x, \varepsilon) = P_{B_Y}(x, \varepsilon) \cap B_Y$. Hence $d(y, P_{B_Y}(x)) \leq \varepsilon = \|y - x\| - d(x, B_Y)$. The other inequality is obvious.

(c) $\implies$ (a) : Let $B[x, r_1] \cap B_Y \neq \emptyset$ and $\|x - y\| < r_1 + r_2$ for some $y \in B_Y$. Then $r_1 = d + \delta$ for some $\delta > 0$, where $d = d(x, B_Y)$. If possible
let $B[x, r_1] \cap B[y, r_2] \cap B_Y = \emptyset$, that is $P_{B_Y}(x, \delta) \cap B[y, r_2] = \emptyset$. But then $P_{B_Y}(x, \delta) \cap B[y, r_2] = \emptyset$, that is $d(y, P_{B_Y}(x)) > r_2 + \delta$. By (c) $\|x - y\| - d > r_2 + \delta$ and finally $\|x - y\| > r_1 + r_2$, a contradiction. $\square$

We now show that the converse of Theorem 6.2(a) is not true.

**Example 6.5.** Consider the space $X = (\mathbb{R}^2, \|\cdot\|_2)$ and let $Z = X \oplus_\infty \mathbb{R}$. Then $X$ is an M-ideal in $Z$ but for $x = ((1, 1), 0) \in Z$, $\|x\| = \sqrt{2}$. Now for $y = ((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), 1) \in B_Z$. we have, $1 = \|x - y\| < d(x, B_X) + d(y, P_{B_X}(x)) = \sqrt{2}$ and hence from Theorem 6.4 it follows that $B_X$ can not have $1^{\frac{1}{2}}$ ball property in $Z$.

**Remark 6.6.**

(a) From the above characterizations it is clear that $1^{\frac{1}{2}}$ ball property of $B_Y$ forces the subspace $Y$ to be strongly ball proximinal.

(b) From the example by Godefroy in [11] it is clear that the closed unit ball of a Banach space not necessarily have $1^{\frac{1}{2}}$ ball property.

Remark 6.6(b) motivate us to investigate the class of Banach spaces and its subspaces whose closed unit balls are uniformly proximinal. The following examples are class of such spaces.

**Example 6.7.**

(a) If $X$ has 3.2.I.P. then $B_X$ has $1^{\frac{1}{2}}$ ball property in $X$, hence the closed unit ball of such a space is strongly proximinal. Hence for any real measure $\mu$, $L_1(\mu)$ or its isometric preduals have this property: Let $B[x, r] \cap B_X \neq \emptyset$ and $\|x - z\| < r + s$ for some $z \in B_Y$. The balls $B[x, r], B[z, s], B_X$ are pairwise intersecting and hence has non empty intersection.

(b) Let $Y$ be a M-ideal in a 3.2.I.P space $X$ then $B_Y$ has $1^{\frac{1}{2}}$-ball property in $X$: Let $B[x, r_1] \cap B_Y \neq \emptyset$ and $\|x - y\| < r_1 + r_2$ for some $y \in B_Y$. Hence we have 3 balls $B[x, r_1], B[y, r_2], B_X$ in $X$ intersect pairwise. From the property of 3.2.I.P. we have $B[x, r_1] \cap B[y, r_2] \cap B_X \neq \emptyset$. Now from [7] Theorem 4.7 it follows $Y$ has strong 3-ball property. Hence considering above 3 balls once again one can have $B[x, r_1] \cap B[y, r_2] \cap B_X \cap Y \neq \emptyset$ which in turn equivalent to $B[x, r_1] \cap B[y, r_2] \cap B_Y \neq \emptyset$.

From the Definition 6.1 Theorem 6.12 and the distance formulas proved in Theorem 6.1 [5.6] we have,
Corollary 6.8. Let $X$ be a separable Banach space. Then the following are equivalent.

(a) $B_X$ has $\frac{1}{2}$ ball property in $X$.
(b) $B_{L_1(I,X)}$ has $\frac{1}{2}$ ball property in $L_1(I,X)$.
(c) $B_{L_\infty(I,X)}$ has $\frac{1}{2}$ ball property in $L_\infty(I,X)$.

References

1. Pradipta Bandyopadhyay, Bor-Luh Lin, T. S. S. R. K. Rao, Ball proximinality in Banach spaces. Banach spaces and their applications in analysis, 251–264, Walter de Gruyter, Berlin, 2007.
2. A. L. Brown, Metric projections in spaces of integrable functions, J. Approx. Theory 81 (1995) no. 1, 78–103.
3. S. Dutta and Darapaneni Narayana, Strongly proximinal subspaces in Banach spaces, Contemp. Math. 435, Amer. Math. Soc., Providence, RI, 2007, 143–152.
4. S. Dutta and P. Shunmugaraj, Strong proximinality of closed convex sets, J. Approx. Theory 163 (2011), no. 4, 547-553.
5. G. Godefroy and V. Indumathi, Strong proximinality and polyhedral spaces, Rev. Mat. Complut. 14 (2001), no. 1, 105-125.
6. G. Godini, Best Approximation and Intersection of Balls, Banach space theory and its applications, Lecture Notes in Mathematics, Vol 991, 1983, 44–54.
7. C. R. Jayanarayanan and Tanmoy Paul, Strong proximinality and intersection properties of balls in Banach spaces, J. Math. Anal. Appl. 426 (2015), 2, 1217–1231.
8. R. Khalil, Best Approximation in $L^p(I,X)$, Math. Proc. Cambridge Phil. Soc. 94 (1983), 277–289.
9. R. Khalil and W. Deeb, Best Approximation in $L^p(I,X)$, II J. Approx. Theory (59) 1989, 296–299.
10. Ka-Sing Lau, On a sufficient condition of proximity, Trans. Amer. Math. Soc., 251 (1979) 343–356.
11. S. Lalithambigai, Ball proximinality of equable space, Collect Math. 60, (2001), 79–88.
12. W. A. Light, Proximinality in $L^p(S,Y)$, Rocky Mountain J. Math. 19, (1989), 251–259.
13. W. A. Light and E. W. Cheney, Approximation Theory in Tensor Product Spaces Lecture Notes in Mathematics 1169 Springer-Verlag
14. A. Lima, Intersection properties of balls and subspaces in Banach spaces, Trans. Amer. Math. Soc. 227 (1977), 1–62.
15. J. Lindenstrauss Extension of Compact Operators, Mem Amer. Math. Soc., vol 48, 1964.
16. Lin Pei-Kee, Zhang Wen, Zheng, Bentuo, Stability of ball proximinality, J. Approx. Theory, 183, (2014), 72–81.
17. J. Mendoza Proximinality in $L_p(\mu, X)$, J. Approx. Theory, 93, (1998), 331–343.
18. Ernest Michael, Continuous selection. I. Ann. of Math. (2) 63 (1956) 361–382.
19. D. V. Pai and P. T. Nowroji, *On restricted centers of sets*, J. Approx. Theory, 66, 1991, 2, 170–189.

20. R. Payá and D. Yost, *The two-ball property: transitivity and examples*, Mathematika 35 (1988), no. 2, 190–197.

21. T. S. S. R. K. Rao, *The one and half ball property in spaces of vector-valued functions*, J. Convex Anal., 20, (2013), 1, 13–23.

22. F. B. Saidi, *On the proximinality of the unit ball and of proximinal subspaces of Banach space: A counterexample*, Proc. Amer. Math. Soc., 133 (2005) no. 9, 2697–2703.

23. S. M. Srivastava, A Course on Borel Sets, Grad. Texts in Math. 180, Springer, New York, 1998.

24. D. Yost *Best approximation and intersection of balls in Banach spaces*, Bull. Austral. Math. Soc., 20, (1979), 285–300

(Tanmoy Paul) Department of Mathematics, Indian Institute of Technology Hyderabad, Kandi Campus, Sangareddy, Telangana 502285, India, E-mail: tanmoy@iith.ac.in