TOPOLOGICAL REPRESENTATIONS OF POSETS

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Abstract. In \cite{10} an arbitrary poset $P$ was proved to be isomorphic to the collection of subsets of a space $\mathcal{M}$ with two closures $\mathcal{C}_1$ and $\mathcal{C}_2$, which are closed in the first closure and open in other – $\mathcal{C}_1 \mathcal{O}_2(\mathcal{M}, \mathcal{C}_1, \mathcal{C}_2)$. As a space for this representation an algebraic dual space $P^*$ was used. Here we extend the theory of algebraic duality for posets generalizing the notion of an ideal. This approach yields a sufficient condition for the collection of $\mathcal{C}_1 \mathcal{O}_2$-subsets of $A \subset P^*$ (with respect to induced closures) to be isomorphic to $P$. Applying this result to certain classes of posets we prove some representation theorems and get a topological characterization of orthocomplementations.

1. Introduction

Since Stone introduced the topological representation of Boolean algebras \cite{4} there was a lot of attempts to generalize this result: the Stone-like representations of orthopsets by Mayet and Tkadlec \cite{5, 8}, different topological representations of distributive \cite{6, 7} and arbitrary (by Hartonas, Dunn and Urquhart) \cite{3, 9} lattices. We follow the construction introduced in \cite{10} where algebraic dual space $P^*$ is endowed with two closures $\mathcal{C}_1$ and $\mathcal{C}_2$ in such a way that the collection of all subsets of $P^*$ which are closed in $\mathcal{C}_1$ and open in $\mathcal{C}_2$ ordered by set inclusion (we denote this collection by $\mathcal{C}_1 \mathcal{O}_2(P^*, \mathcal{C}_1, \mathcal{C}_2)$) is isomorphic to the initial poset $P$:

\begin{equation}
\mathcal{C}_1 \mathcal{O}_2(A, \mathcal{C}_1, \mathcal{C}_2) \approx P
\end{equation}

The representation \cite{10} of $P$ works for arbitrary poset $P$. However, for particular classes of posets the ‘universal set’ $P^*$ can be contracted to a smaller one $A \subseteq P^*$ with the closures $\mathcal{C}_1$, $\mathcal{C}_2$ induced from $P^*$. In this paper we show that the representations of specific classes of posets mentioned above all have the form

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and differ only by the choice of \( A \subseteq P^* \).

1.1. **Spaces with two closures.** Mapping \( C : \exp(\mathcal{M}) \to \exp(\mathcal{M}) \) we call **closure** if

1. \( A \subset C(A) \);
2. \( C(C(A)) = C(A) \);
3. if \( A \subset B \) then \( C(A) \subset C(B) \).

A set \( A \subset \mathcal{M} \) is **closed** (or **C-closed**) if \( A = C(A) \), \( A \) is **open** if \( \overline{A} = \mathcal{M} \setminus A \) is closed and **clopen** if it is both closed and open. Note, that any intersection of closed sets is closed, and \( C(A) \) is the intersection of all closed sets which contain \( A \). \( \mathcal{K} \subset \exp(\mathcal{M}) \) is called the **base** of closure \( C \) (\( C = \text{clo} (\mathcal{K}) \)) if any closed set is an intersection of elements of \( \mathcal{K} \).

The closure \( C \) is **exact** if \( C(\emptyset) = \emptyset \), and **topological** if \( C(A \cup B) = C(A) \cup C(B) \). Note, that exact topological closure defines topology on \( \mathcal{M} \). For a closure \( C \) on \( \mathcal{M} \) define \( \text{CO}(\mathcal{M}, C) \) to be the collection of all clopen subsets of \( \mathcal{M} \). Obviously \( \text{CO}(\mathcal{M}, C) \) ordered by set inclusion is a bounded orthoposet. It was shown by Mayet and Tkadlec [5, 8], that for an arbitrary bounded orthoposet \( P \) there is a space \( \mathcal{M} \) with closure \( C \) such that \( P \approx \text{CO}(\mathcal{M}, C) \).

If we define two closures on \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \mathcal{M} \), then by \( C_1O_2(\mathcal{M}, \mathcal{C}_1, \mathcal{C}_2) \) we denote the collection of all subsets of \( \mathcal{M} \) which are both \( \mathcal{C}_1 \)-closed and \( \mathcal{C}_2 \)-open, ordered by set inclusion. We can say nothing about the structure of \( C_1O_2(\mathcal{M}, \mathcal{C}_1, \mathcal{C}_2) \) except it is a poset, moreover, as it was shown in [10] for an arbitrary poset \( P \) one can build a space with two closures such that \( P \approx C_1O_2(\mathcal{M}, \mathcal{C}_1, \mathcal{C}_2) \).

1.2. **Algebraic duality for posets.** For a poset \( P \) its **algebraic dual space** \( P^* \) is the set of all isotone mappings from \( P \) to poset \( 2 = \{0, 1\} \) with \( 0 < 1 \). Here we develop the techniques needed to build the representation.

Consider \( A \subset P^* \). A set \( I \) we call an **ideal** (with respect to \( A \) or \( A \)-ideal) if \( I \) is an intersection of kernels of some mappings \( x \in A \) (i.e. \( I = \bigcap x^{-1}(0) \)). Dually, the intersection of co-kernels \( F \) we call a **filter** \( F = \bigcap x^{-1}(1) \).

For \( B \subset P^* \) we define an ideal \( I(B) \) (filter \( F(B) \)) to be the intersection of kernels (co-kernels, respectively) of \( x \in B \).

For \( Q \subset P \) define an ideal \( \overline{Q}_A \) (resp., filter \( Q]_A \) – the intersection of ideals (resp., filters) containing \( Q \).
Note that ideals with respect to $P^*$ coincide with order ideals ($I$ is an order ideal if $q \in I$ and $p \leq q$ implies $p \in I$). In general $A$-ideals are always order ideals, but the converse is not always true.

We say that $A \subset P^*$ is full if for all $p \not\leq q$ there exists $x \in A$ such that $x(p) = 1$, $x(q) = 0$.

$A \subset P^*$ is called separating if for any disjoint ideal $I$ and filter $F$ there exists $x \in A$ such that $x|_I = 0$ and $x|_F = 1$.

In some cases discussed in section 3 $\mathbf{3}(\mathbf{p})_A$ and $(\mathbf{p})_A$ coincide with lower and upper cones of $p$ respectively. Due to the following obvious lemma the separating set is full in this case.

**Lemma 1.** Let $A$ be a separating subset of $P^*$ and $[\mathbf{p}]_A \cap [\mathbf{q}]_A = \emptyset$ for all $q \not\leq p \in P$. Then $A$ is full.

2. **Topological representation: the general case**

Define two closures on $P^*$: For $p \in P$ consider two subsets of $P^*$:

$$\mathcal{U}p(p) = \{x \mid x(p) = 1\} \quad \mathcal{L}o(p) = \{x \mid x(p) = 0\}.$$  

Then define closures $C_1$, $C_2$ in the following way:

$$C_1 = \text{clos} \{\mathcal{U}p(p)\}_{p \in P} \quad \text{and} \quad C_2 = \text{clos} \{\mathcal{L}o(p)\}_{p \in P}.$$  

Note, that since $\overline{\mathcal{U}p(p)} = \mathcal{L}o(p)$ all $\mathcal{U}p(p)$ are $C_1O_2$-sets.

On $A \subset P^*$ consider closures $C_{1,A}$, $C_{2,A}$ induced by $C_1$ and $C_2$ (i.e. $C_{1,A}(X) = C(X) \cap A$). Let $\mathcal{U}p_A(p) = \mathcal{U}p(p) \cap A$ and $\mathcal{L}o_A(p) = \mathcal{L}o(p) \cap A$, then

$$C_{1,A} = \text{clos} \{\mathcal{U}p_A(p)\}_{p \in P} \quad \text{and} \quad C_{2,A} = \text{clos} \{\mathcal{L}o_A(p)\}_{p \in P}.$$  

We omit the index $A$ in $C_{i,A}$, $\mathcal{U}p_A$ etc. when it is clear which subspace is meant.

The following equations show the relation between the closures introduced on $A$ and $A$-ideals:

$$C_1(X) = \bigcap_{p \in \mathcal{F}(X)} \mathcal{U}p(p) \quad \text{and} \quad C_2(X) = \bigcap_{p \in \mathcal{I}(X)} \mathcal{L}o(p).$$  

**Theorem 2.** Let $A \subset P^*$. Consider $\sigma : P \to C_1O_2(A, C_{1,A}, C_{2,A})$ which maps $p$ to $\mathcal{U}p(p)$, then

1. $\sigma$ is isotone;
2. if $A$ is full then $\sigma$ is injective;
3. if $A$ is separating then $\sigma$ is surjective.
Proof. (1) Since \(p \leq q\) implies \(x(p) \leq x(q)\) for all \(x \in P^*\) then \(p \leq q\) implies \(U_P(p) \subseteq U_P(q)\), so \(\sigma\) is isotone.

(2) For \(p \neq q\) either \(p \not\leq q\) or \(q \not\leq p\), so there exists \(x \in A : x(p) \neq x(q)\), then exactly one of \(U_P(p), U_P(q)\) contains \(x\) and \(U_P(p) \neq U_P(q)\).

(3) Let \(B \in C_1O_2(A, C_{1A}, C_{2A})\), then \(B = C_{1A}(B)\) and \(\overline{B} = C_{2A}(\overline{B})\).

Consider \(Q = I(\overline{B}) \cap F(B) = I \cap F\). If \(Q = \emptyset\) there exists \(x \in A : x|_I = 0\) and \(x|_F = 1\), so \(x \in U_P(p)\) for all \(p \in F\) and \(x \in LO(q)\) for all \(q \in I\). Thus \(x \in B\) and \(x \in \overline{B}\) simultaneously, so \(Q \neq \emptyset\). For \(p \in Q\) we have \(B \subset U_P(p), \overline{B} \subset LO(p) = \overline{U_P(p)}\) and \(B = U_P(p)\).

\(\square\)

**Corollary 2.1.** Let \(A\) be a full and separating subspace of \(P^*\), then \(P \approx C_{1O_2}(A, C_{1A}, C_{2A})\).

To get the topological representation of an arbitrary poset we prove

**Lemma 3.** \(P^*\) is full and separating.

**Proof.** For disjoint ideal \(I\) and filter \(F\), which are in this case order ideal and filter, consider \(x : x(p) = 0\) for \(p \in I\) and \(x(p) = 1\) otherwise. Obviously \(x \in P^*\) and separates \(I\) and \(F\). Applying lemma 1 we see that \(P^*\) is full.

\(\square\)

This leads us to the following theorem:

**Theorem 4.** Let \(P\) be an arbitrary poset, then \(P \approx C_{1O_2}(P^*, C_1, C_2)\).

Due to the following lemma in the case of bounded poset \(P\) subspaces \(A\) of \(P^*\) can be reduced:

**Lemma 5.** Let \(P\) be a bounded poset, \(A \subset P^*\) be full and separating, then \(A \setminus \{0, 1\}\), where \(0, 1 \in P^*\) are constant mappings, is also full and separating.

**Proof.** Note that the ideals (filters) with respect to \(A \setminus \{0, 1\}\) coincide with the proper \(A\)-ideals (\(A\)-filters) and for disjoint nonempty \(I\) and \(F\) the separating mapping \(x \in A\) is not constant.

\(\square\)

3. **Topological representations: special cases**

We apply the results of previous section to some special classes of posets.

3.1. **Orthoposets.** The bounded poset \(P\) is called an **orthoposet** if there exists an anti-isotone mapping \((\cdot)' : P \to P\) (orthocomplementation) such that \(p = (p)'\), \(p \lor p' = 1\) and \(p \land p' = 0\). For an orthoposet define its **orthodual** space \(P^{*\prime}\) to be the set of all \(x \in P^*\) such that \(x(p') = (x(p))'\).
Lemma 6. $P^*$ is full and separating.

Proof. For disjoint ideal $I$ and filter $F$ consider $x : x(p) = 0$ for $p \in I \cup F'$, $x(p) = 1$ for $p \in I' \cup F$, otherwise $x(p) = y(p)$ for some $y \in P^*$. Obviously $x \in P^*$ and separates $I$ and $F$, so $P^*$ is separating. As $[p]_{P^*}$ is the lower cone of $p$ for all $p \in P$ $P^*$ is full according to lemma 1.

Since $UP_{P^*}(p') = LO_{P^*}(p)$, the bases of closures $C_1$ and $C_2$ coincide and $C_1 = C_2$. Denote

$$C = C_1 = C_2$$

Then $C_1O_2$-sets are $C$-clopen. Applying theorem 2 we have

Theorem 7. Let $P$ be an orthoposet, then there exists a closure space $(\mathcal{M}, C)$ such that $P \approx CO(\mathcal{M}, C)$.

The representation obtained in previous theorem coicides with that described by Mayet [5] and Tkadlec [8].

Now we use the notion of full separating subspace to characterize all orthocomplementations which can be introduced on a bounded poset $P$. Any orthocomplementation $(\cdot)'$ defines a full separating subspace of $P^*$ on which the closures $C_1$ and $C_2$ coincide. Let $S$ be the collection of full separating subspaces of $P^*$ where $C_1 = C_2$. Consider $A \in S$ then the set complementation on $C_1O_2(A, C_1, C_2) \approx P$ is an orthocomplementation, so with every $A \in S$ we can associate an orthocomplementation $(\cdot)'_A$ on $P$.

Theorem 8. All orthocomplementations on $P$ are in one-to-one correspondence with maximal (with respect to set inclusion) elements of $S$.

Proof. For $A \in S$ all $x \in A$ preserves $(\cdot)'_A$ because $x(p) = x(p') = 1$ implies $x \in UP(p)$ and $x \in UP(p'^A) = UP(p)$ (the similar contradiction holds for $x(p) = 0$). It means that $A \subseteq P'^A$, so all maximal elements of $S$ are of the form $P'^A$. Thus any orthodual space $P^*$ is a subspace of $P'^A$ for some $A$. Obviously, orthocomplementation associated with $P'^A$ is $(\cdot)'_A$ and the one associated with $P^*$ is $(\cdot)'$. Since $P^* \subseteq P'^A$ and orthocomplementations are induced by set complementation we get that $(\cdot)' = (\cdot)'_A$ and $P^* = P'^A$, so all orthodual spaces, defined by different orthocomplementations on $P$, are maximal in $S$.
orthocomplemented distributive lattice one can expect distributive lattice to be represented as the collection of $C_1O_2$-sets of some space with two topological closures. We are going to construct such a representation which follows from theorem 2 and is different from Priestley [6] and Rieger [7].

For a lattice $L$ let $L^{*\lor\land} \subset L^*$ be the set of all lattice morphisms (isotone mappings preserving lattice operations) from $L$ to 2. Note that $L^{*\lor\land}$-ideal is always lattice ideal (an order ideal $I$ is called lattice ideal if $a, b \in I$ implies $a \lor b \in I$).

**Lemma 9.** For any distributive lattice $L$ the ideals (filters) with respect to $L^{*\lor\land}$ coincide with the lattice ideals (filters). Besides that, $L^{*\lor\land}$ is full and separating.

**Proof.** First we prove that for disjoint lattice ideal $I$ and filter $F$ there exists $x \in L^{*\lor\land}$ such that $x|_I = 0; x|_F = 1$ (it means that $L^{*\lor\land}$ separates lattice ideals). Suppose $I_0$ to be the maximal lattice ideal containing $I$ which is disjoint with $F$. The set-complement of $I_0$ is a filter [2], thus the mapping $x$: $x|_{I_0} = 0, x|_{L \setminus I_0} = 1$ preserves $\lor$ and $\land$. For an arbitrary $p \in L$ the upper cone of $p$ is a lattice filter. Then we get every lattice ideal $I$ to be the intersection of kernels of all $x_p$, which separates $I$ and the upper cone of $p$, over all $p \not\in I$, so $I$ is an ideal with respect to $L^{*\lor\land}$ (recall the definition of ideal in section 1.2). Hence, the separating property for $L^{*\lor\land}$ is equivalent to the fact that $L^{*\lor\land}$ separates lattice ideals, which was proved above. $L^{*\lor\land}$ is full by lemma 1. □

**Theorem 10.** For any distributive lattice $L$ there exists a space with two topological closures $(\mathcal{M}, C_1, C_2)$ such that $L \approx C_1O_2(\mathcal{M}, C_1, C_2)$.

**Proof.** The only thing we need to prove is that the closures $C_1, C_2$ induced on $L^{*\lor\land}$ are topological. Since elements of $L^{*\lor\land}$ preserve both $\lor$ and $\land$ we have $\mathcal{U}\mathcal{P}(p \lor q) = \mathcal{U}\mathcal{P}(p) \cup \mathcal{U}\mathcal{P}(q)$ and $\mathcal{L}\mathcal{O}(p \land q) = \mathcal{L}\mathcal{O}(p) \cup \mathcal{L}\mathcal{O}(q)$, so the bases of $C_1$ and $C_2$ are closed under finite set union, therefore the closures themselves are topological. □

**Corollary 10.1.** A lattice $L$ is distributive iff $L^{*\lor\land}$ is a full separating subspace of $L^*$.  

**Proof.** This follows from lemma 2, the fact that for any lattice $L$ the closures induced on $L^{*\lor\land}$ are topological, and that for any space $\mathcal{M}$ with two topological closures $C_1O_2(\mathcal{M}, C_1, C_2)$ is a distributive lattice. □
3.3. Boolean algebras. Here we present a proof of the Stone representation theorem:

**Theorem 11** (Stone). *Any Boolean algebra $B$ is isomorphic to the collection of all clopen subsets of a topological space.*

**Proof.** Since $B$ is a bounded distributive lattice, $B^{\ast \vee \wedge} \setminus \{0, 1\}$ is full and separating. Every lattice morphism of Boolean algebras preserves orthocomplementation and, as in the case of orthoposets, the topological closures $C_1$ and $C_2$ do coincide.

Associating with every element of $B^{\ast \vee \wedge} \setminus \{0, 1\}$ its kernel (that is a maximal lattice ideal) one get the Stone space of Boolean algebra originally described in [4].

**Corollary 11.1.** Let $L$ be a distributive lattice, then $L$ is a Boolean algebra iff closures $C_1$ and $C_2$ coincide on $L^{\ast \vee \wedge} \setminus \{0, 1\}$.

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