Entanglement thermodynamics

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Abstract. We investigate further the relationship between the entanglement spectrum of a composite many-body system and the energy spectrum of a subsystem making use of concepts of canonical thermodynamics. In many important cases the entanglement Hamiltonian is, in the limit of strong coupling between subsystems, proportional to the energy Hamiltonian of the subsystem. The proportionality factor is an appropriately defined coupling parameter, suggesting to interpret the latter as a inverse temperature. We identify a condition on the entanglement Hamiltonian which rigorously guarantees this interpretation to hold and removes any ambiguity in the definition of the entanglement Hamiltonian regarding contributions proportional to the unit operator. Illustrations of our findings are provided by spin ladders of arbitrary spin length, and by bilayer quantum Hall systems at total filling factor \( \nu = 2 \). Within mean-field description, the latter system realizes an entanglement spectrum of free fermions with just two levels of equal modulus where the analogies to canonical thermodynamics are particularly close.

Keywords: rigorous results in statistical mechanics, spin chains, ladders and planes (theory), entanglement in extended quantum systems (theory)
1. Introduction

Quantum entanglement is by now an established ingredient to the understanding and description of various phenomena in condensed matter and many body physics [1–3]. A more recent development represents the notion of the entanglement spectrum, i.e. the spectrum of the reduced density matrix obtained from the ground state of a composite system upon tracing out a subsystem [4]. Moreover, since this reduced density matrix does not have any negative eigenvalues, it can always be formulated as

$$\rho_{\text{red}} = \frac{e^{-H_{\text{ent}}}}{Z}$$

with a partition function $Z = \text{tr}(e^{-H_{\text{ent}}})$ and an entanglement Hamiltonian $H_{\text{ent}}$. The physical significance of the latter stems from the observation that in many important cases, the entanglement spectrum shows, in the regime of strong coupling of the constituent subsystems, a striking similarity to the energy spectrum of the subsystem itself or its complement [5–11],

$$H_{\text{ent}} \approx \lambda H_1/t.$$

Here $H_1$ is the energetic Hamiltonian of the subsystem and $t$ is a typical energy scale of it. The dimensionless parameter $\lambda$ describes the coupling between the subsystems and is small in the limit of strong coupling, and equations (1) and (2) clearly suggest to interpret it as an inverse temperature $\beta$.

As a typical example, in the case of spin ladders, $\lambda$ can be chosen as the ratio of the Heisenberg coupling parameters along the legs and rungs, while $t$ is the coupling strength along the legs [5–9]. Other recent work done in a similar spirit on coupled Luttinger liquids and spin ladders includes [12,13]. In the situation of quantum Hall bilayers at total filling factor of unity [10] a natural choice for $\lambda$ is the layer separation in units of the magnetic

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length $\ell$, and $t$ is (a multiple of) the Coulomb energy scale $e^2/\ell$. Moreover, for fermionic hopping models on lattices [7,11], $\lambda$ is naturally chosen to be the ratio of amplitudes for hopping within and between the subsystems, whereas $t$ is (proportional to) the former amplitude.

We note that equation (2) holds in prominent cases but not truly in general. For an instructive counterexample see [14] where a spin ladder of clearly nonidentical legs was considered. Besides, as the aforementioned studies show [5–11], a proportionality of the type (2) should in general not be expected to hold outside the strong-coupling regime. Moreover, a critical assessment of the information in general included in an entanglement spectrum was given very recently in [15]. These authors define the entanglement Hamiltonian by $H_{\text{ent}} = -\ln \rho_{\text{red}}$, implying $Z \equiv 1$, and argue that the thermal state described by $\rho_{\text{red}}$ should be viewed as being at an effective temperature $T = 1/\beta = 1$.

In the light of the above developments we here further examine the concept of the entanglement temperature and entanglement thermodynamics [10,11]. The leading question of this work is: ‘When can a reduced density matrix of a many-body system be considered as a thermal state, and, if so, what is its temperature?’ As an important point, the definition (1) of the entanglement Hamiltonian is not unique as one can always add any multiple of the unit operator to $H_{\text{ent}}$ which is compensated by the partition function $Z$. If this additional term is independent of $\lambda$ (and therefore, as we shall see below in more detail, independent of the entanglement temperature), such a change has no significant effect. A nontrivial situation occurs if such contributions depend on $\lambda$. The latter situation generically arises since $H_{\text{ent}}$ will in general contain all orders in that coupling parameter. In the following section 2 we review the formalism developed in [11] and discuss how to eliminate the above ambiguity. The latter issue does essentially not occur for lattice models of free fermions [7,11,16] since the results given in [16,17] naturally lead to an entanglement Hamiltonian in terms of non-interacting fermions without any ‘artificial’ constant contribution. The canonical thermodynamics arising from such Hamiltonians are described in detail in section 3. An important special case occurs if the entanglement spectrum comprises only two levels just differing in sign. A nontrivial realization of this situation is, within mean-field approximation, given by quantum Hall bilayers at total filling factor $\nu = 2$ [18–21], as we discuss in section 4. We close with a summary and an outlook in section 5.

2. Entanglement thermodynamics

The entropy and the energy following from the reduced density matrix written in the form (1) read

$$S = \langle -\ln \rho_{\text{red}} \rangle,$$

$$\bar{E} = \langle H_{\text{ent}} \rangle$$

with $\langle \cdot \rangle = \text{tr}(\rho_{\text{red}} \cdot)$. The bar over $\bar{E}$ indicates that this quantity should receive some refinement as the derivative $\partial S/\partial \bar{E}$ will in general fail to be proportional to $\beta \propto \lambda$ at small $\lambda$, i.e. strong coupling [11]. To define an energy $E$ fulfilling the thermodynamic relation

$$\frac{\partial S}{\partial E} = \frac{\partial S}{\partial \lambda} \frac{\partial \lambda}{\partial E} = \beta(\lambda)$$

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we redefine the Hamiltonian as
\[ H_{\text{ent}}(\lambda) = \beta(\lambda) H_{\text{can}}(\lambda). \] (6)
in terms of the canonical entanglement Hamiltonian \( H_{\text{can}}(\lambda) \). The inverse thermodynamic
temperature \( \beta(\lambda) \) as a function of \( \lambda \) is now determined via equation (5): Introducing the
thermodynamic inner energy
\[ E(\lambda) = \langle H_{\text{can}}(\lambda) \rangle \] (7)
along with the free energy
\[ F(\lambda) = E(\lambda) - S(\lambda)/\beta(\lambda) \]
\[ = - \ln(Z(\lambda))/\beta(\lambda) \] (9)
it is easy to see that equation (5) is equivalent to
\[ \beta \frac{\partial \bar{F}}{\partial \beta} = \bar{E} \] (10)
with \( \bar{F} = \beta F = E - S \). Thus, one has
\[ \frac{\partial \ln \beta}{\partial \lambda} = \frac{1}{E} \frac{\partial \bar{F}}{\partial \lambda} = \frac{1}{E} \frac{\partial (E - S)}{\partial \lambda}, \] (11)
which is the desired equation connecting the phenomenological inverse temperature scale \( \lambda \) to thermodynamic one \( \beta(\lambda) \). Indeed, a very similar relation is found in standard
thermodynamics between different temperature scales describing the same equilibrium
states of a given body [22, 23].

Let us now explore consequences of equation (11) related to the ambiguity of constant
contributions to the entanglement Hamiltonian mentioned already in the introduction. Expanding the the entanglement Hamiltonian in a power series in the coupling parameter \( \lambda \).
\[ H_{\text{ent}}(\lambda) = \lambda h_1 + \lambda^2 h_2 + \lambda^3 h_3 + \cdots, \] (12)
the leading order is, according to equation (2), determined by the energy Hamiltonian of
the subsystem,
\[ h_1 = H_1/t. \] (13)
For the pertaining thermodynamic quantities one finds
\[ Z(\lambda) = \text{tr}(1) \left( 1 - \lambda \frac{\text{tr}(h_1)}{\text{tr}(1)} - \lambda^2 \left( \frac{\text{tr}(h_2)}{\text{tr}(1)} - \frac{1}{2} \frac{\text{tr}(h_1^2)}{\text{tr}(1)} \right) \right. \]
\[ \left. - \lambda^3 \left( \frac{\text{tr}(h_3)}{\text{tr}(1)} - \frac{\text{tr}(h_1 h_2)}{\text{tr}(1)} + \frac{1}{6} \frac{\text{tr}(h_1^3)}{\text{tr}(1)} \right) \right) + \mathcal{O}(\lambda^4) \] (14)
and
\[ F(\lambda) = - \ln \text{tr}(1) + \lambda \frac{\text{tr}(h_1)}{\text{tr}(1)} + \lambda^2 \left( \frac{\text{tr}(h_2)}{\text{tr}(1)} - \frac{1}{2} \left( \frac{\text{tr}(h_1^2)}{\text{tr}(1)} - \left( \frac{\text{tr}(h_1)}{\text{tr}(1)} \right)^2 \right) \right) \]
\[ + \lambda^3 \left( \frac{\text{tr}(h_3)}{\text{tr}(1)} + \frac{1}{6} \left( \frac{\text{tr}(h_1^3)}{\text{tr}(1)} - 3 \frac{\text{tr}(h_1 h_2)}{\text{tr}(1)} + 2 \left( \frac{\text{tr}(h_1)}{\text{tr}(1)} \right)^3 \right) \right) \]
\[ - \frac{\text{tr}(h_1 h_2)}{\text{tr}(1)} + \frac{\text{tr}(h_1) \text{tr}(h_2)}{(\text{tr}(1))^2} \right) + \mathcal{O}(\lambda^4), \] (15)
\[
E(\lambda) = \frac{\lambda}{\text{tr}(1)} \text{tr}(h_1) + \lambda^2 \left( \frac{\text{tr}(h_2)}{\text{tr}(1)} - \frac{\text{tr}(h_1^2)}{\text{tr}(1)} + \left( \frac{\text{tr}(h_1)}{\text{tr}(1)} \right)^2 \right)
+ \lambda^3 \left( \frac{\text{tr}(h_3)}{\text{tr}(1)} + \frac{1}{2} \frac{\text{tr}(h_1^3)}{\text{tr}(1)} - \frac{3}{2} \frac{\text{tr}(h_1) \text{tr}(h_1^2)}{(\text{tr}(1))^2} \right)
- 2 \frac{\text{tr}(h_1 h_2)}{\text{tr}(1)} + \frac{2}{(\text{tr}(1))^2} \frac{\text{tr}(h_1) \text{tr}(h_2)}{(\text{tr}(1))^2} \right) + O(\lambda^4) .
\] (16)

In case \( \text{tr}(h_1) \neq 0 \), the r.h.s of equation (11) reads
\[
\frac{\partial \ln \beta}{\partial \lambda} = \frac{1}{\lambda} + O(1) ,
\] (17)
such that
\[
\beta(\lambda) = k\lambda + O(\lambda^2) ,
\] (18)
where the integration constant \( k \) reflects a unit chosen to measure \( \beta \) and has the same meaning as Boltzmann’s constant in standard thermodynamics. Thus, as expected, the inverse temperature is at strong coupling proportional to \( \lambda \). However, if \( \text{tr}(h_1) = 0 \) equation (17) holds only if additionally \( \text{tr}(h_2) = 0 \) leading to
\[
\frac{\partial \ln \beta}{\partial \lambda} = \frac{1}{\lambda} - 2 \frac{\text{tr}(h_3)}{\text{tr}(h_1^2)} + \frac{\text{tr}(h_1 h_2)}{\text{tr}(h_1^2)} + O(\lambda) .
\] (19)

Thus, demanding
\[
\text{tr}(\mathcal{H}_{\text{ent}}(\lambda) - \lambda h_1) = 0 ,
\] (20)
i.e.
\[
\text{tr}(h_2) = \text{tr}(h_3) = \cdots = 0 ,
\] (21)
guarantees equation (18) and completely removes the ambiguity in the definition of the entanglement Hamiltonian. In deriving equation (19) we have observed that \( \text{tr}(h_1^2) \) cannot be zero unless \( h_1 = 0 \), which, by assumption, should not be the case.

To give a practical example, consider spin ladders described by the Hamiltonian
\[
\mathcal{H} = J_r \sum_i \vec{S}_i \vec{S}_{i+1} + J_l \sum_i \left( \vec{S}_i \vec{S}_{i+2} + \vec{S}_{i-1} \vec{S}_{i+1} \right)
\] (22)
for spins of arbitrary but uniform length \( S \) on \( L \) rungs, \( i \in \{0, \ldots, L-1\} \), with, for definiteness, periodic boundary conditions. \( J_r > 0 \) \( (J_l > 0) \) is the antiferromagnetic coupling along the rungs (legs). Defining \( \lambda = 2J_l/J_r \), the low-order contributions to the entanglement Hamiltonian after tracing out one leg are obtained via perturbation theory as [9]
\[
h_1 = \sum_i \vec{S}_i \vec{S}_{i+1} ,
\] (23)
\[
h_2 = -\frac{1}{5} S(S+1) \sum_i \vec{S}_i \vec{S}_{i+2} - \frac{1}{20} \sum_i \left( \left( \vec{S}_i \vec{S}_{i+1} \right) \left( \vec{S}_{i+1} \vec{S}_{i+2} \right) + \left( \vec{S}_{i+1} \vec{S}_{i+2} \right) \left( \vec{S}_i \vec{S}_{i+1} \right) \right)
- \frac{1}{12} \sum_i \left( \vec{S}_i \vec{S}_{i+1} \right)^2 + 2 \vec{S}_i \vec{S}_{i+1} + \frac{1}{30} (S(S+1))^2 L ,
\] (24)
where \( L \) is the number of rungs, and the constant term in equation (24) has been adjusted according to \( \text{tr}(h_2) = 0 \). We note that for the smallest non-trivial spin length \( S = 1/2 \) the second-order term simplifies to [8, 9]

\[
    h_2 = -\frac{1}{8} \sum_i \left( \vec{S}_i \vec{S}_{i+2} - \vec{S}_i \vec{S}_{i+1} \right),
\]

and no contribution proportional to the unit operator is required. We note that the first-order result (24) holds in fact not just for spin ladders but also in various more general geometries [9]; for a recent summary on spin systems and their entanglement spectra see also [24].

Now, using the condition \( \text{tr}(h_3) = 0 \) one finds from equation (19)

\[
    \frac{\partial \ln \beta}{\partial \lambda} = 1 - \frac{1}{8} \lambda + \mathcal{O}(\lambda)
\]

such that

\[
    \beta = k\lambda - \frac{k}{8} \lambda^2 + \mathcal{O}(\lambda^3).
\]

Remarkably, the second-order correction to the inverse temperature (27) is independent of the spin length \( S \). However, we expect such a dependence to occur in higher orders in \( \lambda \).

3. Free fermions

Given the ground state \( |\Psi(\lambda)\rangle \) of the full lattice system of free fermions, consider the correlation matrix

\[
    C_{\alpha\beta}(\lambda) = \langle \Psi(\lambda) | b^+_{\alpha} b_\beta | \Psi(\lambda) \rangle
\]

where \( b^+_{\alpha}, b_\alpha \) describe fermions in the remaining subsystem using some arbitrary basis. The correlation determines the entanglement Hamiltonian via [16, 17]

\[
    \mathcal{H}_\text{ent}(\lambda) = \ln \left( C^{-1}(\lambda) - 1 \right).
\]

More specifically, one has

\[
    \mathcal{H}_\text{ent} = \sum_n \xi_n a_n^+ a_n,
\]

where the entanglement levels \( \xi_n(\lambda) \) are given by

\[
    \xi_n(\lambda) = \ln \left( \frac{1 - \eta_n(\lambda)}{\eta_n(\lambda)} \right)
\]

with \( \eta_n \) being an eigenvalue of \( C \). The above fermionic operators read

\[
    a_n^+(\lambda) = \sum_\alpha U_{n\alpha}(\lambda) b^+_{\alpha}
\]

where the unitary matrix \( U \) diagonalizes \( C \),

\[
    U(\lambda) C(\lambda) U^+(\lambda) = \text{diag}(\eta_1(\lambda), \eta_2(\lambda), \ldots).
\]

We note that both the eigenvalues \( \xi_n(\lambda) \) of the entanglement Hamiltonian and its eigenstates created by the operators \( a_n^+(\lambda) \) are in general functions of \( \lambda \).
The reduced density matrix can now be formulated as
\[ \rho_{\text{red}}(\lambda) = \frac{e^{-\mathcal{H}_{\text{ent}}(\lambda)}}{Z(\lambda)} = \prod_n \frac{e^{-\xi_n(\lambda)n^+_n a_n}}{1 + e^{-\xi_n(\lambda)}} , \] (34)
along with the related quantities governing the thermodynamics,
\[ S(\lambda) = \sum_n \left( \ln \left( \frac{1 + e^{-\xi_n(\lambda)}}{1 + e^{\xi_n(\lambda)}} \right) \right) = \sum_n \left( \ln \left( \frac{1 + e^{-\xi_n(\lambda)}}{1 + e^{\xi_n(\lambda)}} \right) + \frac{\xi_n(\lambda)}{1 + e^{\xi_n(\lambda)}} \right) , \] (35)
\[ \bar{E}(\lambda) = \sum_n \frac{\xi_n(\lambda)}{1 + e^{\xi_n(\lambda)}} , \] (36)
such that
\[ \bar{F} = \bar{E} - S = -\sum_n \ln (1 + e^{-\xi_n}) \] (37)
and
\[ \frac{\partial}{\partial \lambda} (\bar{E} - S) = -\sum_n \frac{\partial \xi_n}{1 + e^{\xi_n}} . \] (38)
A particularly simple but physically meaningful situation arises if the entanglement spectrum consists of just two levels differing in sign, \( \xi_\pm = \pm \xi \). Then one has
\[ \bar{E} = -\xi \tanh \left( \frac{\xi}{2} \right) , \quad \frac{\partial}{\partial \lambda} (\bar{E} - S) = -(\partial \lambda \xi) \tanh \left( \frac{\xi}{2} \right) \] (39)
leading to
\[ \frac{\partial \ln \beta}{\partial \lambda} = \frac{\partial \lambda \xi}{\xi} \] (40)
with the very simple solution
\[ \beta(\lambda) = k\xi(\lambda) , \] (41)
where the constant \( k \) is the same as in equation (18). Moreover, the spectrum of the canonical Hamiltonian is independent of \( \lambda \) (or \( \beta \)),
\[ \mathcal{H}_{\text{can}} = \frac{1}{k} (a^+_n a_n - a^+_n a^-_n) . \] (42)
The derivative of the entropy with respect to \( \lambda \) is given by
\[ \frac{\partial S}{\partial \lambda} = -\xi \frac{\partial \lambda \xi}{1 + \cosh \xi} , \] (43)
such that the specific heat can be formulated as \( (kT(\lambda) = 1/\xi(\lambda)) \)
\[ C = T \frac{\partial S}{\partial T} = \frac{\xi^2}{1 + \cosh \xi} = 2 \left( \frac{\xi/2}{\cosh (\xi/2)} \right)^2 \geq 0 . \] (44)
This quantity approaches zero for strong coupling, \( \xi \to 0 \), as well as in the limit of vanishing coupling, \( \xi \to \infty \). It attains a maximum at \( \xi = \xi_0 \) determined by
\[ \frac{\xi_0}{2} \tanh \left( \frac{\xi_0}{2} \right) = 1 , \] (45)
i.e. \( \xi_0 \approx 2.4 \), such that
\[ C (\xi_0) = \frac{2}{\sinh^2 (\xi_0/2)} = \frac{\xi_0^2}{2} - 2 \approx 0.88 \] (46)
and
\[ \bar{E} (\xi_0) = -2 . \] (47)

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4. Quantum Hall bilayers at $\nu = 2$

In quantum Hall bilayers at total filling factor $\nu = 2$, a canted antiferromagnetic phase occurs as a function of the Zeeman gap and the tunneling amplitude. Here the polarization directions of electron spins in both layers form a nontrivial angle with each other. This phase separates a spin-polarized phase and a spin-singlet phase. The theoretical prediction of this phase within Hartree–Fock approximation [18–20] was qualitatively (and, in part, also quantitatively) confirmed by an exact-diagonalization study [21] and has stimulated many further theoretical investigations; for most recent works see, e.g. [25, 26], older literature is summarized in [27]. Also from the experimental point of view, the existence of such a phase can by now be seen as established [28–32].

Here we shall follow the formalism of [20] where a large part of the ground state structure was obtained within Hartree–Fock theory in a fully analytical fashion. The Hartree–Fock approximation reduces the Hamiltonian to a system of free fermions such that our above findings are easily applied. In particular, due to the projection onto the lowest Landau level, the resulting entanglement spectrum will have just two distinct levels.

4.1. Hartree–Fock theory

The total Hamiltonian of the system is the sum of the Coulomb interaction projected onto the lowest Landau level and the single-particle Hamiltonian

$$h = -\frac{\Delta_v}{2} \tau^z - \frac{\Delta_t}{2} \tau^x - \frac{\Delta_z}{2} \sigma^x$$  \hspace{1cm} (48)

where the Pauli matrices $\vec{\tau}$ and $\vec{\sigma}$ act on the layer (pseudo-)spin and genuine electron spin, respectively. $\Delta_v$ is a bias voltage between the quantum wells, and $\Delta_t$ describes the tunneling between them. $\Delta_z$ is the Zeeman energy due to the perpendicular magnetic field which is chosen here (following [20]) to lie along the $x$-direction in electron spin space. Orbital states in the lowest Landau level are labeled by a quantum number $X$ which depends on the chosen gauge. Neglecting charge fluctuations within the layers, each mode $X$ is filled with two electrons leading to the following single Slater determinant as a variational ansatz for the ground state,

$$|\Psi[z]\rangle = \prod_X \sum_{k,l=1}^4 (z_k^1 z_l^2 b_k^X (X) b_l^X (X)) |0\rangle.$$  \hspace{1cm} (49)

Here $z^1$, $z^2$ are two orthonormal spinors, where $k = 1, 2$ describe a spin-up(down) electron in the upper layer, while $k = 3, 4$ refer to a spin-up(down) electron in the lower layer. [20] uses the parameterization

$$z^1 = \left(\begin{array}{c} \cos \frac{\theta}{2} \cos \frac{\chi_1}{2}, \sin \frac{\theta}{2} \sin \frac{\chi_1}{2}, \cos \frac{\theta}{2} \sin \frac{\chi_1}{2}, -\sin \frac{\theta}{2} \cos \frac{\chi_1}{2} \end{array}\right),$$  \hspace{1cm} (50)

$$z^2 = \left(\begin{array}{c} -\sin \frac{\theta}{2} \sin \frac{\chi_1}{2}, \cos \frac{\theta}{2} \cos \frac{\chi_1}{2}, \sin \frac{\theta}{2} \cos \frac{\chi_1}{2}, \cos \frac{\theta}{2} \sin \frac{\chi_1}{2} \end{array}\right)$$  \hspace{1cm} (51)

in terms of three angles $\theta$, $\chi_1$, and $\chi_1$, which turns out to be flexible enough to describe the Hartree–Fock ground state as a function of various system parameters.
\( \chi_\pm = (\chi_1 \pm \chi_1)/2 \), the variational energy per mode is
\[
\epsilon = -\Delta_v \cos \vartheta \cos \chi_+ \cos \chi_- - \Delta_t \sin \vartheta \sin \chi_+ \cos \chi_- - \Delta_z \sin \vartheta \sin \chi_-
- F_+ - 2H \cos^2 \vartheta \cos^2 \chi_+ \cos^2 \chi_-
- F_- \left( \sin^2 \vartheta \cos^2 \chi_+ \sin^2 \chi_- + \cos^2 \vartheta \cos^2 \chi_+ \cos^2 \chi_- + \sin^2 \chi_+ \sin^2 \chi_- \right)
\]
with
\[
F_\pm = \frac{1}{(2\pi)^2} \int d^2q e^{-q^2\ell^2/2V_\pm(\vec{q})},
\]
\[
H = \frac{1}{2\pi\ell^2} V_- (\vec{q} = 0).
\]
Here \( \ell \) is the magnetic length, and \( V_\pm(\vec{q}) \) describe the Coulomb interaction within and between the layers. Assuming quantum wells of negligible width, one has (using obvious notation)
\[
V_\pm(\vec{r}) = \frac{e^2}{2\epsilon} \left( \frac{1}{r} \pm \frac{1}{\sqrt{r^2 + d^2}} \right),
\]
where \( d \) is the layer separation.

For unbiased layers, \( \Delta_v = 0 \), minimization of (52) leads to \( \chi_+ = \pi/2 \) and
\[
\sin^2 \vartheta = -\frac{\Delta_t^2 \left( \Delta_t^2 - \Delta_z^2 \right)^2 - (2\Delta_t F_-)^2}{(\Delta_t^2 - \Delta_z^2)^3},
\]
\[
\sin^2 \chi_- = \frac{(2\Delta_t F_-)^2 - (\Delta_t^2 - \Delta_z^2)^2}{4F_-^2 (\Delta_t^2 - \Delta_z^2)},
\]
fulfilling
\[
\tan \vartheta = \frac{\Delta_z}{\Delta_t} \tan \chi_-,
\]
provided that the above r.h.s of equations (56) and (57) are nonnegative and do not exceed unity. This is the case if \( \Delta_z^{\min} \leq \Delta_z \leq \Delta_z^{\max} \) with
\[
\Delta_z^{\min} = \sqrt{\Delta_t (\Delta_t - 2F_-)},
\]
\[
\Delta_z^{\max} = \sqrt{F_-^2 + \Delta_t^2} - F_-.
\]
For \( \Delta_z^{\min} < \Delta_z < \Delta_z^{\max} \) the bilayer system is in the canted antiferromagnetic phase where the order parameter
\[
\langle \tau^z \sigma^z \rangle = 2 \cos \vartheta \sin \chi_+ \sin \chi_-
\]
is nonzero. This phase separates the spin-polarized phase \( \Delta_z > \Delta_z^{\max}, \vartheta = \chi_- = \pi/2 \) and the spin singlet phase \( \Delta_z < \Delta_z^{\min}, \vartheta = \chi_- = 0 \). A phase diagram for typical system parameters is given in figure 1.

In the approximation-free exact-diagonalization study [21] the phase boundary between the spin-polarized phase and the canted antiferromagnetic phase was found to be exactly given by the Hartree–Fock result (60), whereas quantitative corrections occur to the lower phase boundary (59). As a result, the canted antiferromagnetic phase turns out to be smaller than predicted by mean-field theory, but definitely existing [21].
4.2. Entanglement spectrum

Tracing out, say, the top layer from the Hartree–Fock ground state, the entanglement spectrum is encoded in the correlation matrix $C_{kl} = \langle \Psi | b^+_k b_l | \Psi \rangle$, $k, l \in \{3, 4 \}$,

$$C = \frac{1}{2} \left( (1 - \cos \vartheta \cos \chi_+ \cos \chi_-) I_{2 \times 2} \right.$$

$$- \frac{1}{2} \sin \chi_- \left( \begin{array}{cc} \cos \vartheta \sin \chi_+ & - \sin \vartheta \\ - \sin \vartheta & - \cos \vartheta \sin \chi_+ \end{array} \right) \right) .$$

(62)

At vanishing bias voltage ($\chi_+ = \pi/2$) its eigenvalues

$$\eta_{\pm} = \frac{1}{2} \left( 1 \mp \sin \chi_- \right)$$

(63)

are independent of $\vartheta$ and lead to the entanglement levels

$$\xi_{\pm} = \ln \frac{1 - \eta_{\pm}}{\eta_{\pm}} = \ln \frac{1 \pm \sin \chi_-}{1 \mp \sin \chi_-} = \pm 2 \text{artanh} \left( \frac{\lambda}{2} \right) ,$$

(64)

where we have introduced the dimensionless coupling parameter $\lambda = 2 \sin \chi_-$. The entanglement Hamiltonian can be formulated as ($\xi = |\xi_{\pm}|$)

$$H_{\text{ent}} = \xi \sum_X \left( a^+_+(X) a_+(X) - a^+_-(X) a_-(X) \right)$$

(65)

with

$$a^+_+(X) = \cos \frac{\vartheta}{2} b^+_3(X) - \sin \frac{\vartheta}{2} b^+_4(X) ,$$

(66)

$$a^+_-(X) = \sin \frac{\vartheta}{2} b^+_3(X) + \cos \frac{\vartheta}{2} b^+_4(X) ,$$

(67)
where, as defined above, \(b_3^+(X)\) \(\left(b_4^+(X)\right)\) creates an electron with spin up (down) in the remaining bottom layer. At strong coupling \(\lambda \ll 1\) the entanglement Hamiltonian is given by (see equation (58))

\[
H_{\text{ent}} = \lambda \sum_X \left( b_3^+(X)b_3(X) - b_4^+(X)b_4(X) \right) + \mathcal{O}(\lambda^2).
\]

Thus, we have a relation of the type (2) with the energetic subsystem Hamiltonian

\[
H_1 = \frac{\Delta_z}{2} \sum_X \left( b_3^+(X)b_3(X) - b_4^+(X)b_4(X) \right)
\]

and \(t = \frac{\Delta_z}{2}\) describing the exchange-enhanced Zeeman splitting in a quantum Hall monolayer at filling factor \(\nu = 1\). In the latter system we have again neglected charge fluctuations, consistent with the above mean-field approximation. Interestingly the effective field in the Hamiltonian (69) points along the \(z\)-axis of spin space while the magnetic field in the original bilayer Hamiltonian (48) was in \(x\)-direction.

In summary, we have demonstrated that the mean-field theory of bilayer quantum Hall systems at total filling factor \(\nu = 2\) constitutes a system of free fermions with an entanglement spectrum comprising only two different levels of the same modulus. As seen in equation (42), in such a case the spectrum of the redefined canonical entanglement Hamiltonian is independent of \(\lambda\) (and therefore independent of \(\beta\)). The redefined canonical entanglement Hamiltonian reads according to equation (42)

\[
H_{\text{can}} = \frac{1}{k} \sum_X \left( a_3^+(X)a_3(X) - a_4^+(X)a_4(X) \right)
\]

\[
= \frac{1}{k} \sum_X \left[ \left( b_3^+(X), b_4^+(X) \right) \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ -\sin \vartheta & -\cos \vartheta \end{pmatrix} \begin{pmatrix} b_3(X) \\ b_4(X) \end{pmatrix} \right].
\]

Moreover, \(\vartheta\) and \(\chi_-\) are determined via equations (56) and (57) by \(\Delta_z, \Delta_t\) which are independent parameters within the canted antiferromagnetic phase but bound to each other at the phase boundaries (59) and (60). Likewise, \(\vartheta\) and \(\chi_-\) are independent parameters within the canted antiferromagnetic phase outside the phase boundaries where they are identical. In this sense, the canonical entanglement Hamiltonian (71) is independent of \(\lambda = 2\sin \chi_-\) and therefore independent of the inverse temperature given in equation (41).

### 5. Summary and outlook

We have extended previous studies on the relationship between the entanglement spectrum of a composite many-body system and the energy spectrum of a subsystem. Inspired by the recent literature, the basic question investigated here is under which circumstances a reduced density matrix of a many-body system allows an interpretation as a thermal state, and, if so, how to determine its temperature. For strong coupling between the subsystems, the entanglement Hamiltonian is, in a variety of important cases, proportional to the energy Hamiltonian of the subsystem with the proportionality factor \(\lambda\) being an appropriately defined coupling parameter. It is suggestive to interpret this quantity as an
inverse temperature $\beta(\lambda)$. Indeed, for the regime away from strong coupling, a differential equation (11) for $\beta(\lambda)$ ensuring the fulfillment of standard thermodynamic relations has been given [11]. This construction is based on the redefinition (6) of the entanglement Hamiltonian. We note, however, that the redefined Hamiltonian $H_{\text{can}}(\lambda)$ will in general depend on the coupling parameter $\lambda$ and, in turn, on $\beta(\lambda)$, as illustrated in the present work on the example of spin ladders.

On the other hand, the definition of a Hamiltonian generating a given density matrix contains generally an ambiguity as one can always add a multiple of the unit operator to the former without changing the latter. Demanding that all contributions to the entanglement Hamiltonian of higher than linear order in the coupling parameter should have vanishing trace removes this ambiguity and guarantees the above interpretation as an inverse temperature to hold. In particular, this condition includes the case where the entanglement Hamiltonian is in total traceless. This case is realized, e.g. for spin ladders of arbitrary spin length, and for many fermionic hopping models on lattices. A more detailed analysis of the latter type of systems shows that the analogy to standard thermodynamics is closest if the entanglement spectrum consists of just two levels differing in sign. In this case the spectrum of the redefined entanglement Hamiltonian $H_{\text{can}}$ is independent of $\lambda$. A nontrivial example for such a situation is provided by the mean-field theory of bilayer quantum Hall systems at total filling factor $\nu = 2$. As an additional feature, here also the eigenstates (and not only the eigenvectors) of $H_{\text{can}}$ are independent of temperature.

Obviously, the results obtained here call for further study. For instance, it would be interesting to possibly identify other situations where the spectrum of $H_{\text{can}}$ (or $H_{\text{can}}$ in total) is independent of the entanglement temperature. Indeed, it is the fact that $H_{\text{ent}}(\lambda)$ in general fails to be linear in $\lambda$ which makes the entanglement thermodynamics developed here different from standard thermodynamics. On the other hand, a Hamiltonian depending on a temperature is not that unfamiliar to theoretical physics since many effective (Hartree–Fock type) descriptions of many-body systems involve, e.g. temperature-dependent occupation numbers.

Moreover, a deeper physical understanding of the condition (20) eliminating the abovementioned ambiguity in the definition of the entanglement Hamiltonian is desirable. This goal might be achieved by applying and testing the formalism developed here on other physical systems.

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