Meson PVV Interactions are determined by Quark Loops

R. Delbourgo, Dongsheng Liu and M.D. Scadron

University of Tasmania, GPO Box 252-21, Hobart,
Australia 7001
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Abstract

We show that all abnormal parity three-body meson interactions can be adequately described by quark loops, evaluated at zero external momentum, with couplings determined by $U(N_f)$ symmetry. We focus primarily on radiative meson decays which involve one pseudoscalar. The agreement with experiment for non-rare decays is surprisingly good and requires very few parameters, namely the coupling constants $g_{\pi qq}$ and $g_{\rho qq}$ and some mixing angles. This agreement extends to some three-body decays that are dominated by pion pairs in a P-wave state.

I. INTRODUCTION

One can perfectly well view $\pi^o \rightarrow \gamma\gamma$ decay as arising from the $PVV$ quark-level linear $\sigma$ model ($\sigma M$) instead of through the $AVV$ chiral anomaly. In such a description it is straightforward to predict many other $PVV$ radiative decays through quark triangle graphs. The purpose of this paper is to investigate how well this quark loop picture accords with experimental data; in this way we avoid discussing anomaly predictions of these processes and notions of strong CP violation with consequential (low mass) axions, since these effects have yet to be discovered in the laboratory. We also consider the situation for rare $PVV$ decays that are associated with tiny quark mixings (such as $J/\psi \rightarrow \pi\rho, \eta'\phi$) and with isospin-changing corrections (such as $J/\psi \rightarrow \pi\omega, \eta\rho$).

In Section II we show that the dominant $\sigma M$ quark triangle $PVV$ predictions for $P \rightarrow \gamma\gamma$, $P \rightarrow V\gamma$ and $V \rightarrow P\gamma$ all fit the data, within experimental errors; the only parameters at our disposal are $g_{Pqq}$ and $g_{Vqq}$, the couplings of the pseudoscalar and vector mesons respectively to the quarks, the constituent quark masses $m_q$ and a couple of well-known mixing angles. We pursue this approach in Section III, for the rarer isospin-conserving interactions where the data gives valuable information about the small amount of admixture of light quarks in the heavy $q\bar{q}$ states, and vice versa; here the agreement is less good, but the experimental information is also sparser and may be expected to change with time. In section IV, we show that a simple picture of $\Delta I = 1$ meson-meson transitions, associated with the $u - d$ quark mass difference, nicely provides the rates for isospin changing decays. The last section contains some preliminary work on three-body decays in which at least one pair of pions is produced in a P-state; we show that $\rho$-meson poles essentially saturate the amplitudes for those processes, given the two-body amplitudes determined previously.
The reasonable agreement between naive theory and experiment suggests that this quark loop picture is a viable alternative to the field-theoretic anomaly-inspired evaluation of such amplitudes. However the tantalizing picture which emerges is that internal mass dependence of the box diagrams is largely negated by the form factors which must be present at the meson-quark vertices. Such are our conclusions in Section VI.

II. QUARK TRIANGLE GRAPHS FOR PVV INTERACTIONS

For the past thirty years, the $\pi^0 \rightarrow \gamma \gamma$ decay has been understood as an AVV chiral anomaly [2] combined with the partial conservation of axial current (PCAC). Writing this process and all other PVV amplitudes in the conventional form $M_{PVV} \epsilon_{\mu\nu\lambda\kappa} k^\mu e^\nu k^\kappa$, where $k$ and $e$ refer to vector particle momentum and polarization respectively, one derives

$$M_{\pi^0\gamma\gamma} = \frac{e^2 N_c}{12 \pi^2 f_\pi} = \frac{\alpha N_c}{3 \pi f_\pi} \approx 0.025 \text{ GeV}^{-1}. \quad (1)$$

With $f_\pi \approx 93$ MeV and a quark color number $N_c = 3$, the theoretical result is accurately consistent with data [3]:

$$|M_{\pi^0\gamma\gamma}^{\text{exp}}| = \sqrt{64 \pi \Gamma_{\pi^0\gamma\gamma}/m_\pi^3} = 0.025 \pm 0.001 \text{ GeV}^{-1}. \quad (2)$$

The question then arises as to which theoretical models also predict the same result while simultaneously demanding that $N_c = 3$.

One such favored model is the quark-level [1] linear $\sigma$ model (LSM). This model also provides $u$ and $d$ quark triangle graphs but with the meson interacting through the pseudoscalar $\gamma_5$ vertex, rather than through the divergence of an axial vector. No accompanying LSM $\pi^+$ loop is allowed because the $\pi^0\pi^+\pi^-$ coupling vanishes; consequently the AVV anomaly result immediately follows in the spirit of Steinberger [4]. (In carrying out the evaluation of $M$, we go to the soft limit, because the pion is essentially massless, relative to other hadronic masses and even the constituent quarks.) The fact that this LSM also demands $N_c = 3$ [4] is tied to the Lee null tadpole condition [5] and ensures dynamical generation of masses, in contrast to the original nucleon-level LSM [6]. The pion-quark coupling $g_{\pi qq}$ needed obeys the quark-level Goldberger-Treiman relation (GTR) $g_{\pi qq} = m_{ns}/f_\pi$ where the constituent (nonstrange) quark mass $m_{ns} \equiv \hat{m}$ is dynamically generated to the expected $M_N/3$ value in the chiral limit, since $N_c = 3$ is required now.

Our aim in this paper is to investigate how well the sigma model picture fits the data for PVV processes. We take the naive viewpoint that the PVV amplitudes, just like $\pi^0 \gamma \gamma$ can be estimated by working out $M_{PVV}$ in the limit of zero external momentum, even though this is a long way from the mass shell for the heavy mesons particularly. The reason why we do so is because many of these processes are quite close to particle thresholds and would otherwise be highly dependent on binding energies and the like — for which there is no evidence. In such a soft limit and including the crossed diagram, it is very simple to work out that

$$M_{P_1V_2V_3} = \frac{g_{P_1}g_{V_2}g_{V_3} N_c}{2 \pi^2} \int \frac{m_1 \alpha + m_2 \beta + m_3 \gamma}{m_1^2 \alpha + m_2^2 \beta + m_3^2 \gamma} \delta(1 - \alpha - \beta - \gamma) \, d\alpha \, d\beta \, d\gamma, \quad (3)$$
where $m_i$ are the internal quark masses, in an obvious cyclic notation. The Feynman parametric integral is especially easy to work out when all quark masses are equal to $m$, when

$$M_{P_{1}V_{2}V_{3}} \to \frac{g_{P_{1}}g_{V_{2}}g_{V_{3}}N_{c}}{4\pi^{2}m}.$$  

It is not much harder to determine the integral (3) if two masses are equal, say $m_{1} = m_{2} = m$, $m_{3} = m'$, when

$$M_{P_{1}V_{2}V_{3}} \to \frac{g_{P_{1}}g_{V_{2}}g_{V_{3}}N_{c}}{2\pi^{2}} \cdot \frac{J(r)}{m + m'}, \quad J(r) \equiv 1 + \frac{2r}{r^{2} - 1} - \frac{4r \ln r}{(r^{2} - 1)^{2}}; \quad r \equiv \frac{m}{m'}.$$  

Thus $J(1) = 2$ in the equal mass limit $m = m'$. See Figure 1 to see how $J(r)$ varies with $r$.

It only remains to substitute the couplings of the mesons to the quarks in order to derive the theoretical values for $M_{P_{1}V_{2}V_{3}}$. To do so we will adopt the naive perspective that all strong couplings are related by $U(N_{f})$ symmetry. Therefore, in the absence of any isospin breaking and neglecting the third generation of quarks, let us see how far one can go with the effective Lagrangian (suppressing the $\gamma_{5}$ matrix between spinors),

$$\mathcal{L}_{P_{1}V_{2}V_{3}} = g_{P_{1}}[\pi^{0}(\bar{u}u - \bar{d}d)/\sqrt{2} + \pi^{+}\bar{u}d - \pi^{-}\bar{d}u + \eta_{ns}(\bar{u}u + \bar{d}d)/\sqrt{2}$$
$$+K^{+}\bar{s}s + K^{-}\bar{s}u + K^{0}\bar{d}s + K^{0}\bar{s}d + \eta_{c}\bar{c}s + \eta_{c}\bar{c}c$$
$$+D^{0}\bar{c}u + D^{-}\bar{c}d + D^{0}\bar{c}u + D^{+}\bar{c}d + D_{s}^{*}\bar{s}c + D_{s}^{*}\bar{s}c]. \quad (4)$$

The universal coupling $g_{P_{1}}$ above is fixed by the $\pi^{0}$ coupling, which in turn is determined by GTR; thus $g_{P_{1}} = (m_{u} + m_{d})/\sqrt{2}f_{\pi} \equiv \sqrt{2}\tilde{m}/f_{\pi} \simeq 5.13$. We must also relate the physical neutral pseudoscalar states to the isospin 0 combinations in Eq. (4) via certain mixings. For the purpose of this and the next section, we shall disregard electromagnetic effects in the strong sector and take (see Appendix)

$$\eta_{ns} = \eta \cos \phi_{P_{1}} + \eta' \sin \phi_{P_{1}}; \quad \eta_{c} = -\eta \sin \phi_{P_{1}} + \eta' \cos \phi_{P_{1}}; \quad \phi_{P_{1}} \simeq 42^\circ,$$

with $\eta_{c}$ unmixed. (Later on we will examine rare decays, where the small residual mixings become rather important.) The strong vector meson interactions are likewise written as

$$\mathcal{L}_{V_{1}V_{2}V_{3}} = g_{V_{1}}[\rho^{0}(\bar{u}u - \bar{d}d)/\sqrt{2} + \rho^{+}\bar{u}d + \rho^{-}\bar{d}u + \omega_{ns}(\bar{u}u + \bar{d}d)/\sqrt{2}$$
$$+K^{+}\bar{s}s + K^{-}\bar{s}u + K^{0}\bar{d}s + K^{0}\bar{s}d + \omega_{c}\bar{c}s + \omega_{c}\bar{c}c$$
$$+D^{0}\bar{c}u + D^{-}\bar{c}d + D^{0}\bar{c}u + D^{+}\bar{c}d + D_{s}^{*}\bar{s}c + D_{s}^{*}\bar{s}c], \quad (5)$$

suppressing vector indices and the $\gamma_{5}$ matrix between quark spinors. Here the coupling constant is fixed by that of the $\rho$-meson which is itself determined by the leptonic decay rate: $g_{V_{1}} = g_{\rho}/\sqrt{2} \simeq 3.56$. These vector states also undergo the semistrong mixing via

$$\omega_{ns} = \omega \cos \phi_{V_{1}} + \phi \sin \phi_{V_{1}}; \quad \omega_{c} = -\omega \sin \phi_{V_{1}} + \phi \cos \phi_{V_{1}}; \quad \phi_{V_{1}} \simeq 3.8^\circ.$$

(At this level we may identify $\omega_{c}$ with the $J/\psi$ particle.) Finally, it is trivial to consider radiative decays, by substituting $\gamma$ for the meson and including the proper electromagnetic coupling of the photon to the quarks, $e_{q}$.
The magnitudes of the $M_{PVV}$ can be found directly from experimental decay rates, as they occur [3]. For $P \to VV$ decays, the rate is given by

$$\Gamma_{PVV} = \Delta^3 |M_{PVV}|^2 / 32\pi m_P^3 = p_{VV}^3 |M_{PVV}|^2 / 4\pi,$$

(6)

where $\Delta = 2m_P p_{VV}$ and $p_{VV}$ is the magnitude of the three-momentum of one of the vector mesons in the rest frame of the decaying $P$. (We must be careful to multiply the right-hand-side of (6) by a factor of 1/2 when the two vector mesons are identical.) A very similar formula applies to the decay rate $V \to PV$,

$$\Gamma_{VPV} = \Delta^3 |M_{VPV}|^2 / 96\pi m_V^3 = p_{VP}^3 |M_{VPV}|^2 / 12\pi,$$

(7)

where $m_V$ refers to the mass of the initial vector and $p_{VP} = \Delta/2m_V$ is the momentum of one final particle in the decay rest frame. It is then a simple matter to extract the $|M|$ from the measured [3] rates $\Gamma$, for subsequent comparison with theory. We shall do this constantly, without further elaboration.

We mentioned at the start that the $M_{\pi^0\gamma\gamma}$ amplitude is theoretically determined in the sigma model by ing the $u$ and $d$ loop contributions and equals $e^2 g_P / 4\sqrt{2} \pi^2 m = \alpha / \pi f_\pi$, fitting experiment admirably. Indeed many of the processes involve neutral mesons and require the interaction terms,

$$g_P [\pi^0 (\bar{u}u - \bar{d}d) / \sqrt{2} + (\eta \cos \phi_P + \eta' \sin \phi_P)(\bar{u}u + \bar{d}d) / \sqrt{2} + (-\eta \sin \phi_P + \eta' \cos \phi_P)\bar{s}s + \eta_c \bar{c}c],$$

(8)

$$g_V [\rho^0 (\bar{u}u - \bar{d}d) / \sqrt{2} + (\omega \cos \phi_V + \phi \sin \phi_V)(\bar{u}u + \bar{d}d) / \sqrt{2} + (-\omega \sin \phi_V + \phi \cos \phi_V)\bar{s}s + \psi \bar{c}c].$$

(9)

Instead of treating each process one at a time, let us consider just three examples to explain what we are doing, before presenting our results for non-rare decays in a table.

Consider first the decay $\eta' \to \omega\gamma$, which contains a charge coupling, one vector coupling and one scalar coupling. The sum of the loops of nonstrange quarks and the mixing angles implied by (8) and (9) gives ($e = \sqrt{4\pi\alpha} \simeq 0.3028$)

$$M_{\eta'\omega\gamma} = e N_c g_V g_P \sin \phi_P \cos \phi_V / 24\pi^2 \hat{m} \simeq \sqrt{2} e g_V \sin \phi_P / 8\pi^2 f_\pi.$$  

(10)

Using the values quoted previously, we obtain the rough theoretical magnitude, $M_{\eta'\omega\gamma} \simeq 0.139$ GeV$^{-1}$, which can be compared with the experimental result,

$$M_{\eta'\omega\gamma}^{\text{exp}} = \sqrt{4\pi \Gamma_{\eta'\omega\gamma} / p_{\omega\gamma}^2} \simeq 0.137 \text{ GeV}^{-1}.$$  

Our second example is the semistrong process $\phi \to \pi\rho$, which is governed by nonstrange quark loops but is rather sensitive to the vector meson mixing angle $\phi_V$. The theoretical result here is

$$M_{\phi\pi\eta} = \sqrt{2} N_c g_V^2 g_P \sin \phi_V / 8\pi^2 \hat{m} \simeq 3g_V^2 \sin \phi_V / 4\pi^2 f_\pi.$$  

(11)

Inserting the accepted value of $\phi_V \simeq 3.8^\circ$, the prediction is that $M_{\phi\pi\rho} \simeq 0.69$ GeV$^{-1}$, and this is in agreement with the upper bound obtained by experiment: $M_{\phi\pi\rho}^{\text{exp}} < 1.18 \text{ GeV}^{-1};$
in this connection we note that the 1996 PDG compilation \[7\] cites a specific value for this particular channel (which we prefer), but this has been retracted in the 1998 PDG compilation, resulting in less predictive power. Finally we take a look at the decay $\psi \to \eta_c\gamma$, which is governed by a charm loop and involves one photon. This time we get

$$M_{\psi\eta_c\gamma} = e N_c g_V g_p / 6\pi^2 m_c \simeq e g V \tilde{m} / \sqrt{2} \pi^2 f \pi m_c,$$

(12)

which we estimate to be about $0.19 \text{ GeV}^{-1}$, for a constituent quark mass ratio of $\tilde{m} / m_c \simeq 338 / 1500 \simeq 0.225$. This is close to average experimental value of $M_{\psi\eta_c\gamma}^{\text{exp}} \simeq 0.17 \text{ GeV}^{-1}$, obtained from the decay rate.

In this manner we can work out theoretical estimates of all dominant $PVV$ processes. We have tabulated the results and the experimental magnitudes for comparison in Table 1. The near agreement of naive theory with data is actually quite astonishing, because we are dealing with sizeable external masses in many instances, particularly for heavy charm quarks, and so are a very long way from the chiral limit. We will return to this point towards the end of the paper.

It should be noted that the last five decay channels in Table I correspond to unequal mass quarks running round the loop, in contrast to all the previous cases, and the absolute value of the amplitudes are perhaps not as impressively predicted are the others (although the ratio $|M_{K^+K^+\gamma}| / M_{K^0\eta_c\gamma}$ is reasonably close to experiment). On the whole the fit is quite satisfactory, considering the fact that we have no free parameters at our disposal.

### III. RARE ISOSPIN-CONSERVING DECAYS

There are many processes involving the heavier quarks which conserve isospin. They tend to be rarer than the cases considered above and measure the smaller admixture of lighter quarks in the meson composites. In fact these rare processes provide much valuable information about such small mixings and, emboldened by the success of the soft momentum predictions of $PVV$ amplitudes, we shall apply the same method to probe the mixing angles of neutral vectors and pseudoscalars. Let us write the meson which couples to the $\bar{c}c$ as the combination $\eta_c + \delta \eta + \delta' \eta'$, where the two $\delta$ are very small quantities (which is why we have not modified the normalization factor in front of $\eta_c$ in first approximation). It follows then that the non-strange and strange pseudoscalars are (in the same limit),

$$\eta_{ns} = \eta \cos \phi_P + \eta' \sin \phi_P - \eta_c (\delta_c \cos \phi_P + \delta'_c \sin \phi_P)$$
$$\eta_s = -\eta \sin \phi_P + \eta' \cos \phi_P + \eta_c (\delta_c \sin \phi_P - \delta'_c \cos \phi_P).$$

(13)

Consequently the interactions (8) of the neutral pseudoscalars get modified to

$$g_P[\pi^0(\bar{u}u - \bar{d}d)/\sqrt{2} + \eta \{\cos \phi_P(\bar{u}u + \bar{d}d)/\sqrt{2} - \sin \phi_P \bar{s}s + \delta_c \bar{c}c\}$$
$$+ \eta' \{\sin \phi_P(\bar{u}u + \bar{d}d)/\sqrt{2} + \cos \phi_P \bar{s}s + \delta'_c \bar{c}c\} + \eta_c \{-(\delta_c \cos \phi_P + \delta'_c \sin \phi_P)(\bar{u}u + \bar{d}d)/\sqrt{2} + (\delta_c \sin \phi_P - \delta'_c \cos \phi_P) \bar{s}s + \bar{c}c\}].$$

(14)

Much the same idea can be applied to the neutral vector interactions (with mixing parameters $\delta$ replaced by $\epsilon$):
\[ g_V[\rho^0(\bar{u}u - \bar{d}d)/\sqrt{2} + \omega\{\cos \phi_V(\bar{u}u + \bar{d}d)/\sqrt{2} - \sin \phi_V \bar{s}s + \epsilon_c \bar{c}c\} \\
+ \phi\{\sin \phi_V(\bar{u}u + \bar{d}d)/\sqrt{2} + \cos \phi_V \bar{s}s + \epsilon_c \bar{c}c\} \\
+ \psi\{-(\epsilon_c \cos \phi_V + \epsilon_c' \sin \phi_V)(\bar{u}u + \bar{d}d)/\sqrt{2} + (\epsilon_c \sin \phi_V - \epsilon_c' \cos \phi_V) \bar{s}s + \bar{c}c\}]. \tag{15} \]

We are now in a position to tackle rare PVV processes involving \( \eta_c \) and \( \psi \). These are evaluated in the same way as before, but now using the neutral couplings (14) and (15). The results are summarized in Table II; theoretical values are only worked out to first order in \( \delta \) or \( \epsilon \), since these are already small quantities. We need to give some words of explanation as to what entry in the last column is being predicted and what is not, in contrast to Table I, which is essentially parameter-free. In our parametrizations (14) and (15), the cleanest predictor of the amount of nonstrange and strange quark components in \( \eta_c \) are the two decays \( \eta_c \rightarrow 2\rho \) and \( \eta_c \rightarrow 2\phi \); they determine separate combinations of the mixings \( \delta_c \) and \( \delta'_c \) (which represent the amount of charm quarks contained in \( \eta \) and \( \eta' \)). We may derive a reasonable fit by choosing the values \( \delta_c \approx 0.0054 \) and \( \delta'_c \approx -0.0008 \); this permits us to predict the decay rates for \( \eta_c \rightarrow 2\omega \) and \( \eta_c \rightarrow K^*\bar{K}^* \) in the table. While the latter prediction is good, the former is decidedly not – in fact the experimental value is slightly embarrassing and difficult to understand on any sensible theoretical basis; for U(2) symmetry between \( u \) and \( d \) entails that the process \( \eta_c\rho^0\rho^0 \) should be almost equal to \( \eta_c\omega\omega \), which is a far cry from what seems to be observed!

With respect to the \( \epsilon \) parameters (because \( \phi_V \) is so small) the process \( \psi \rightarrow \pi^0\rho^0 \) tells us about the nonstrange-charm mixing \( \epsilon_c \) almost at once, while \( \psi \rightarrow \eta\phi \) provides the mixing between strange and charm states, namely \( \epsilon'_c \). We find a fair fit (including sign) with the values, \( \epsilon_c \approx -0.0002, \epsilon'_c \approx 0.00009 \). This enables us to predict the amplitudes for the other entries in Table II involving \( \epsilon \). They are all roughly correct, except for \( \psi \rightarrow \eta'\gamma \) and \( \psi \rightarrow \eta'\omega \). We do not understand the reasons for this; blaming the discrepancy on the effect of mass extrapolation from the shell to zero-momentum would undermine the other reasonable answers; but in any case it should be noted that the vector meson dominance (VMD) prediction is a factor of 250 larger than the data which heightens our suspicions. It is possible that the experimental results may shift a little over time and bring the results into closer agreement with theoretical expectations, or possibly the experimental decay rate also includes the contribution from \( a_0(980) \).

**IV. RARE \( \Delta I = 1 \) VPV DECAYS**

Here we want to take a look at four processes,

\[ \psi \rightarrow \pi^0\omega, \quad \psi \rightarrow \pi^0\phi, \quad \psi \rightarrow \eta\rho^0, \quad \psi \rightarrow \eta'\rho^0, \]

because they involve isospin breaking interactions of mesons, without being accompanied by photons. (Interestingly, there seems to be no comparable data in the \( \eta_c \) decay sector.) All four processes are smaller by at least a factor of three compared to processes that conserve isospin; eg \( M_{\psi\eta^0\omega}^{\text{exp}} \approx 0.3 M_{\psi\eta^0\rho^0}^{\text{exp}} \).

We may ascribe the existence of these small amplitudes to electromagnetic effects and the \( u - d \) mass difference. With regard to the quark triangle loop, the \( u - d \) difference gives a correction of order \( \sim 5/350 \), which is insufficient to explain the observed magnitudes. We
shall find below that the experimental results can be reasonably well obtained by $\Delta I = 1$ mixings between mesons (including the well-known $\rho - \omega$ transition), which are themselves dominated by the $u - d$ difference in quark-loop self-energy contributions.

Before going further, let us estimate two strong $PVV$ amplitudes that cannot be directly measured from decays, because of the masses of the participating mesons; they are nonetheless important in a subsidiary role for what follows. Firstly, there is the interaction $\rho^0 \pi^0 \omega$ (which actually enters $\omega \rightarrow 3\pi$, via pole dominance), that we estimate to be

$$M_{\rho^0 \pi^0 \omega} = 3\sqrt{2} g_V^2 g_P \cos^2 \phi_V / 8\pi^2 \tilde{m} \simeq 3g_\rho^2 / 8\pi^2 f_\pi \equiv 3C \simeq 10.4 \text{ GeV}^{-1}. \quad (16)$$

Next we shall need the amplitude

$$M_{\rho \eta \rho} = 3C \cos \phi_P \simeq 7.85 \text{ GeV}^{-1}. \quad (17)$$

Finally we shall require amplitudes which we have estimated previously fairly well (or which we can take directly from experiment), namely $M_{\psi \pi^0 \rho} \simeq 0.0021 \text{ GeV}^{-1}$, $M_{\psi \omega \rho} \simeq 0.0016 \text{ GeV}^{-1}$ and $M_{\phi \pi^0 \rho} \simeq 0.69 \text{ GeV}^{-1}$.

Other quantities we will need are the $\rho - \omega$ amplitudes that cannot be directly obtained by $\Delta I = 1$ transitions. This arises from a quark self-energy bubble plus an $a_0$ tadpole term. The latter equals the former contribution because $g_{a_0 \eta \eta \pi} = (m_{a_0}^2 - m_{\eta \pi}^2) / f_\pi$ by SU(6) and chiral symmetry $\Gamma$. Noting the gap equation stemming from the GTR, the total result sums to

$$\langle \eta_{\pi^0} | H^{u-d} | \pi^0 \rangle = 4i N_c g_P^2 \int \frac{d^4 p}{(2\pi)^4} \left[ \frac{1}{p^2 - m_u^2} - \frac{1}{p^2 - m_d^2} \right]$$

$$\simeq 2(m_d^2 - m_u^2) \simeq -5400 \text{ MeV}^2 \quad \text{for} \quad m_d - m_u \simeq 4 \text{ MeV}. \quad (21)$$

We thereby estimate the two neutral pseudoscalar elements,
Actually the last element leads to very small corrections because each of the mesons is rather far from the mass shell of the other.

We are now in a position to estimate these rare amplitudes. (Remember that we may ignore the negligible correction induced by \( u - d \) mass differences within the triangle loop.) The first case is

\[
M_{\psi\pi\pi} = \frac{M_{\psi\rho\pi}M_{\rho\pi\pi}}{m_\psi^2 - m_\rho^2} + \frac{M_{\rho\omega}M_{\pi\rho\pi}}{m_\omega^2 - m_\rho^2} + \frac{M_{\pi\eta}M_{\psi\pi\eta}}{m_\pi^2 - m_\eta^2} \simeq 0.0007 \text{ GeV}^{-1},
\]

upon substituting the \( \Delta I = 1 \) transition elements found above and the vertex amplitudes determined previously. Similarly, we find

\[
M_{\psi\pi\pi} = \frac{M_{\psi\rho\pi}M_{\rho\pi\pi}}{m_\psi^2 - m_\rho^2} + \frac{M_{\rho\omega}M_{\pi\rho\pi}}{m_\omega^2 - m_\rho^2} + \frac{M_{\pi\eta}M_{\psi\pi\eta}}{m_\pi^2 - m_\eta^2} \simeq 0.0009 \text{ GeV}^{-1},
\]

\[
M_{\psi\eta\rho} = \frac{M_{\psi\rho\eta}M_{\rho\eta\rho}}{m_\psi^2 - m_\rho^2} + \frac{M_{\rho\omega}M_{\pi\rho\pi}}{m_\omega^2 - m_\rho^2} + \frac{M_{\pi\eta}M_{\psi\pi\eta}}{m_\pi^2 - m_\eta^2} \simeq 0.0005 \text{ GeV}^{-1},
\]

\[
M_{\psi\eta'\rho} = \frac{M_{\psi\rho\eta'}M_{\rho\eta'\rho}}{m_\psi^2 - m_\rho^2} + \frac{M_{\rho\omega}M_{\pi\rho\pi}}{m_\omega^2 - m_\rho^2} + \frac{M_{\pi\eta}M_{\psi\pi\eta}}{m_\pi^2 - m_\eta^2} \simeq 0.0006 \text{ GeV}^{-1}.
\]

The experimental values of these amplitudes, deduced from the measured decay rates, are, in units of GeV\(^{-1}\),

\[
|M_{\psi\pi\pi}^{\text{exp}}| = 0.00067 \pm 0.00004, \quad |M_{\psi\pi\pi}^{\text{exp}}| < 0.0009,
\]

\[
|M_{\psi\pi\pi}^{\text{exp}}| = 0.00048 \pm 0.00003, \quad |M_{\psi\pi\pi}^{\text{exp}}| = 0.00040 \pm 0.00004.
\]

All of these answers are in the “right ball park” and we venture to conclude that this \( \Delta I = 1 \), pole-dominated mechanism captures the main features of such feeble decays.

**V. SOME THREE-BODY DECAYS**

Here we shall have a look at a few three-body decays, in which pions are produced in a P-state and are known to be dominated by \( \rho \)-meson poles; specifically we shall examine the three processes \( \omega \rightarrow 3\pi \) and \( \eta, \eta' \rightarrow 2\pi\gamma \) to see how well they tie up with the earlier \( I \) amplitudes. Before doing so we shall require more accurate values of the on-shell couplings \( g_{\rho\pi\pi}, g_{\omega\rho\pi} \). We recall that the quark loop alone gives \( g_{\rho\pi\pi} = \sqrt{2} g_V \approx 5.03 \) and \( g_{\omega\rho\pi} = 3g_V^2/4\pi^2 f_\pi \approx 10.3 \text{ MeV}^{-1} \). However, in a sigma model, besides the non-strange quark loop we have to add a meson loop associated with sigma exchange. The effect of this is to enhance \( g_{\rho\pi\pi} \) by a factor of \( 6/5 \), to about 6.1, which is very close to the experimental value coming from the \( \rho \) decay width. Correspondingly, the \( \rho - \omega - \pi \) coupling is enhanced \( g_{\omega\rho\pi} \) by a factor of \( (6/5)^2 \) to \( 3g_V^2/8\pi^2 f_\pi \approx 15 \text{ GeV}^{-1} \).
The amplitude for the $VPPP$ process $\omega \rightarrow 3\pi$ may be written in the covariant form

$$A_{\omega \pi \pi \pi} = \epsilon_{\kappa \lambda \mu \nu} \epsilon_{\pi} p^\kappa p^\lambda p^\mu p^\nu M_{\omega \pi \pi \pi},$$

where the scalar amplitude $M$ is dominated by $\rho$ mesons [12] in each of the three possible two-body channels:

$$M_{\omega \pi \pi \pi} = 2g_{\omega \rho \pi}g_{\rho \pi \pi} \left[ \frac{1}{m_\rho^2 - s} + \frac{1}{m_\rho^2 - t} + \frac{1}{m_\rho^2 - u} \right] \simeq 1480 \text{ GeV}^{-3},$$

(28)

In deriving this result, we have averaged over the Mandelstam variables: $\langle s \rangle = \langle t \rangle = \langle u \rangle = m_\omega^2/3 + m_\pi^2 \simeq 0.223 \text{ GeV}^2$. If we then follow Thews’ phase space analysis [13], we get

$$\Gamma(\omega \rightarrow 3\pi) = |M_{\omega \pi \pi \pi}|^2 m_\omega^7 Y_\omega/(2\pi)^3$$

(29)

with the constant matrix element giving $Y_\omega = 4.57 \times 10^{-6}$. In this way, we predict $\Gamma(\omega \rightarrow 3\pi) \simeq 7.3$ MeV, very close to the observed rate of $7.5 \pm 0.1$ MeV.

Our next object of study, $\eta \rightarrow \pi^+\pi^-\gamma$, is dominated by a $\rho^0$ meson pole, but just in the two-pion channel [14]. The process also takes the same covariant form as the previous case. Here we estimate

$$M_{\eta \pi \pi \gamma} = 2g_{\rho \pi \pi}g_{\rho \gamma} / (m_\rho^2 - s) \simeq 9.74 \text{ GeV}^{-3},$$

(30)

upon substituting the $\eta\rho\gamma$ amplitude found earlier (Table I) and going to the soft pion limit. Once more, we find

$$\Gamma(\eta \rightarrow \pi\pi\gamma) = |M_{\eta \pi \pi \gamma}|^2 m_\eta^7 Y_\eta/(2\pi)^3 \simeq 55 \text{ eV},$$

(31)

for a phase space factor [13] of $Y_\eta = 0.98 \times 10^{-5}$. The prediction (31) compares well with the observed rate of $56 \pm 5$ eV.

Finally we consider the $\eta' \rightarrow \pi^+\pi^-\gamma$ decay. Unlike the previous two examples, the $\rho^0$ pole is now well within the physical region. The latest particle data tables state that $\Gamma(\eta' \rightarrow 2\pi\gamma)$ is $61 \pm 5$ keV, whereas our prediction (see Table I) is that $\Gamma(\eta' \rightarrow \rho\gamma) = |M_{\eta' \rho \gamma}|^2 p_\rho^3/4\pi \simeq 65$ keV. This strongly suggests that any nonresonant P-wave contact contributions are quite small and that the $\rho$ poles accurately match the data. Incidentally it also confirms that the pseudoscalar mixing angle is $\phi_P \simeq 42^\circ$, rather than $\phi \sim 35^\circ$, coupled with the anomaly, as the latter angle gives a rate for $\eta' \rightarrow \pi\pi\gamma$ which is a factor of 20 less than the observed rate [15].

VI. CONCLUSIONS

Taking stock of the results, we appear to have succeeded in estimating some 34 $VVP$ amplitudes with reasonable accuracy (see the Tables) barring two or three rare decays involving $\eta'$ and $\omega$ mesons—which are themselves extracted from one or two experiments that surely deserve repeating in the future. The success extended to a few abnormal parity three body decays which are known to be dominated by P-wave pion pairs. We do not regard the coupling constants $g_P$ and $g_V$ which provide the overall scales to be ‘parameters’, because they are predicted in [1] or are derived from other processes and have well-established magnitudes; nor do we think that the quark masses are variables at our disposal, since they are also fixed by other means. The mixing angles $\phi_P$ and $\phi_V$ are parameters but they too can be
found through processes other than the ones which we have considered in this paper. The only (four) variables which we were truly able to adjust were the mixing angles describing the non-$c\bar{c}$ content of $\eta_c$ and $\psi$.

Perhaps the most surprising aspect of the analysis is the fact that the amplitudes, which we estimated in the soft limit, have reasonable magnitudes even for external heavy meson $c\bar{c}$ composites. We really did not expect this and we think that the conclusions are rather deep and point to some underlying heavy mass scale, way beyond the usual QCD scale of about 300 MeV. To put this deduction into perspective, consider some field-theoretic model which includes meson interactions with spinors; assume it is renormalizable for definiteness. Then the equation for the proper vertex function $\Gamma(p,q)$ (where $p$ is the meson momentum and $q$ is the relative fermion momentum) will take the form

$$\Gamma(p,q) = Z\gamma + g^2 \int d^4q' S(p/2 + q')\Gamma(p,q') S(-p/2 + q')^* K_p(q,q'),$$

where $K$ is the appropriate interaction kernel for the model in question, $S$ stands for the spinor propagator and $Z\gamma$ is the renormalized contact term. If one specialises to pseudoscalar mesons and ignores fermion dressing for simplicity, the equation has the generic scalar form,

$$A(p,q) = Z_p + g^2 \int d^4q' A(p,q') S(p, q')^* K_p(q', q')/[p^2/4 - q'^2 - m^2].$$

The subscript $p$ has been attached to the renormalization constant $Z$ to remind ourselves that the renormalization procedure is often momentum dependent, although the infinite part of $Z$ (or a $1/(D - 4)$ pole term in the dimensional method) is of course insensitive to $p$. The limit as $p \to 0$ of the equation above is readily taken and describes the meson vertex in the (off-shell) soft limit.

If the pseudoscalar field is actually a composite of the fermions, then $Z$ must vanish on the meson mass shell ($p^2 = M^2$) in order that the vertex equation reduces to a homogeneous Bethe-Salpeter equation. The results which we found indicate that the products of vertex functions and quark propagators running round the loop are rather insensitive to external momentum since they suggest that an amplitude like $A(p,q)$ above depends very little on $p$, i.e. the $q$-dependence on the right hand side has compensating effects from form factors and propagators. (It is striking that the propagator denominators would appear to depend sensitively on the binding energy near the constituent threshold $p^2 = 4m^2$ in the infrared region of $q'$, and there is no sign of this.) Therefore it indicates that $Z_p$ depends very little on $p$ and that the homogeneous Bethe-Salpeter equation can very likely be extrapolated to $p = 0$ with relatively little change. Our guess is that such loop integrals are really dominated by the ultraviolet region and a mass scale $\Lambda$ of at least 10 GeV—much higher than the standard QCD scale—associated with the ratio $p^2/\Lambda^2$. Indeed, the seemingly weak dependence of our results on $p$ are the most extraordinary part of this work and trying to understand the issue properly is an exciting avenue of future research; our discussion above does scant justice to the problem.

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APPENDIX: PSEUDOSCALAR MIXING ANGLE

Since we are using quark loops to predict $PVV$ amplitudes involving $\eta$ or $\eta'$ pseudoscalar mesons, it is imperative that we determine the most important $\eta - \eta'$ mixing angle $\phi_P$ from processes other than $PVV$ decays. Here we shall consider two other ways of fixing this angle.

First we consider tensor ($T$) $\to PP$ decays, $a_2(1320) \to K\bar{K}, \eta\pi, \eta'\pi$. Given the respective percentage branching ratios, from the PDG tables, 4.9$\pm$0.8, 14.5$\pm$1.2, 0.53$\pm$0.09, and remembering the $p^5$ phase space, we may deduce the two independent ratios,

$$B(a_2 \to \eta\pi/K\bar{K}) = (p_{\eta\pi}/p_K)^5 \cdot 2\cos^2 \phi_P = 2.96 \pm 0.53$$ \hspace{1cm} (A1)

$$B(a_2 \to \eta'\pi/\eta\pi) = (p_{\eta'\pi}/p_{\eta\pi})^5 \tan^2 \phi_P = 0.037 \pm 0.007.$$ \hspace{1cm} (A2)

Inserting the known magnitudes $(p_{\eta\pi}, p_{\eta'\pi}, p_K) = (535, 287, 437)$ MeV, into this pair of equations, we may conclude from the first that $\phi_P = 43 \pm 5^0$, and from the second that $\phi_P = 42 \pm 3^0$. Likewise the measured $K_2^* \to K\eta/K\pi$ branching ratio of 0.003 requires $\phi_P = 41 \pm 4^0$. As well, there are the tensor decays $f, f' \to \pi\pi, K\bar{K}, \eta\eta$, but then the tensor angle $\phi_T$ enters the analysis; yet $\phi_P \sim 42^0$ survives the results of the analysis.

A second way of extracting $\phi_P$ is through the QCD quark-annihilation diagram, involving at least two gluon exchange. Diagonalization of the NS-S meson mass matrix leads \[18\] to the $\eta - \eta'$ mixing angle,

$$\phi_P = \arctan \left[ \frac{(m_{\eta'}^2 - 2m_K^2 + m_\pi^2)(m_{\eta}^2 - m_\pi^2)}{(2m_K^2 - m_\pi^2 - m_{\eta'}^2)(m_{\eta'}^2 - m_\pi^2)} \right]^{1/2} = 41.9^0. \hspace{1cm} (A3)$$

The close agreement between $\phi_P$ determined by $T \to PP$ data and QCD theory justifies our use of $\phi_P \approx 42^0$ in Tables I and II.

In the singlet-octet basis, the mixing angle $\theta$ is instead given by \[17\] \[18\]

$$\theta = \phi - \arctan \sqrt{2} = \phi - 54.7^0, \text{ or}$$

$$\cos \phi = \frac{1}{\sqrt{3}} \cos \theta - \sqrt{\frac{2}{3}} \sin \theta, \quad \sin \phi = \sqrt{\frac{2}{3}} \cos \theta + \frac{1}{\sqrt{3}} \sin \theta.$$ 

Thus for the pseudoscalar mesons, $\theta_P \approx -13^0$, which is midway between the original Gell-Mann-Okubo value of $-10^0$ and most recent determinations of $-20^0$. In this connection, it is interesting to compare the quark triangle predictions,

$$|M_{\eta\gamma\gamma}| = \frac{\alpha}{3\sqrt{3} \pi f_\pi} \left[ (5 - \frac{2m}{m_s}) \cos \theta_P - \sqrt{2}(5 + \frac{m}{m_s}) \sin \theta_P \right], \hspace{1cm} (A4a)$$

$$|M_{\eta'\gamma\gamma}| = \frac{\alpha}{3\sqrt{3} \pi f_\pi} \left[ (5 - \frac{2m}{m_s}) \sin \theta_P + \sqrt{2}(5 + \frac{m}{m_s}) \cos \theta_P \right], \hspace{1cm} (A4b)$$

with the predictions obtained via the AVV anomaly \[13\].
\[ |M_{\eta'\gamma\gamma}^{\text{anom}}| = \frac{\alpha}{\sqrt{3} f_\pi} \left[ \frac{f_\pi}{f_8} \cos \theta_P - 2\sqrt{2} \frac{f_\pi}{f_0} \sin \theta_P \right], \quad (A5a) \]

\[ |M_{\eta'\gamma\gamma}^{\text{anom}}| = \frac{\alpha}{\sqrt{3} f_\pi} \left[ \frac{f_\pi}{f_8} \sin \theta_P + 2\sqrt{2} \frac{f_\pi}{f_0} \cos \theta_P \right]. \quad (A5b) \]

Comparing (A.4) with (A.5), we see that

\[ f_8/f_\pi = 3/(5 - 2\hat{m}/m_s) \simeq 0.83, \quad (A6a) \]

\[ f_0/f_\pi = 6/(5 + \hat{m}/m_s) \simeq 1.05. \quad (A6b) \]

Both (A.6ab) exhibit U(3) symmetry in the limit \( m_s \to \hat{m} \). However for the actual constituent quark mass ratio (obtained from the GTR or magnetic moments) of

\[ m_s/\hat{m} = (2f_K/f_\pi - 1) \simeq 1.44, \]

the numerical values in (A.6) follow. Note that the anomaly version \(^{[15]}\) estimates \( f_8/f_\pi \simeq 1.3 \) via chiral perturbation theory but also finds \( f_0/f_\pi \simeq 1.04 \), which is close to our prediction; we agree with their justification that \( f_0/f_\pi \) being near unity is what one expects if the singlet state and the pion have the same wave function.
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∗ E-mail: Bob.Delbourgo@utas.edu.au
† E-Mail: D.Liu@utas.edu.au
‡ Permanent address: Physics Department, University of Arizona, Tucson, AZ 85721; E-Mail: scadron@physics.arizona.edu;

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TABLE I. Dominant strong and radiative decays that are somewhat insensitive to mixing angles. Under the column of theoretical formulae, $M_{PV}$, we use the abbreviations, $A = e^2/4\pi f_\pi = a/\pi f_\pi \approx 0.025$, $B = e g_V g_P/8\pi^2 \hat{m} \approx 0.207$, $C = g_V^2/4\pi f_\pi = \sqrt{2} g_V g_P/8\pi^2 \hat{m} \approx 3.47$, in units of GeV$^{-1}$, and $\tau_q = \hat{m}/m_q$ for the constituent quark mass ratio. The latter are determined by the assumed values (in GeV) $\hat{m} = 0.34$, $m_s = 0.49$, $m_c = 1.5$, $m_b = 4.8$.

| Process | $\Gamma^{\text{exp}}$(MeV) | $|M^{\text{exp}}|$ (GeV$^{-1}$) | $M_{PV}$ | $|M^{\text{thy}}|$ (GeV$^{-1}$) |
|---------|----------------|----------------|--------|----------------|
| $\pi^0 \rightarrow \gamma\gamma$ | $(7.74 \pm 0.52) \times 10^{-6}$ | $0.0251 \pm 0.008$ | $A$ | 0.0250 |
| $\eta \rightarrow \gamma\gamma$ | $(4.6 \pm 0.4) \times 10^{-4}$ | $0.024 \pm 0.001$ | $A[5 \cos \phi_P - \sqrt{2} r_s \sin \phi_P]/3$ | 0.0255 |
| $\eta' \rightarrow \gamma\gamma$ | $(4.3 \pm 0.4) \times 10^{-3}$ | $0.032 \pm 0.001$ | $A[5 \sin \phi_P + \sqrt{2} r_s \cos \phi_P]/3$ | 0.035 |
| $\eta_c \rightarrow \gamma\gamma$ | $(7.5 \pm 2.5) \times 10^{-3}$ | $0.0075 \pm 0.0013$ | $4\sqrt{2} Ar_c/3$ | 0.0107 |
| $\eta' \rightarrow \rho\gamma$ | $0.061 \pm 0.005$ | $0.40 \pm 0.02$ | $3B \sin \phi_P$ | 0.42 |
| $\eta' \rightarrow \omega\gamma$ | $(6.1 \pm 0.8) \times 10^{-3}$ | $0.14 \pm 0.01$ | $B[\cos \phi_V \sin \phi_P + 2r_s \sin \phi_V \cos \phi_P]$ | 0.15 |
| $\rho^\pm \rightarrow \pi^\pm\gamma$ | $0.068 \pm 0.007$ | $0.22 \pm 0.01$ | $B$ | 0.21 |
| $\rho^0 \rightarrow \eta\gamma$ | $0.036 \pm 0.002$ | $0.45 \pm 0.07$ | $3B \cos \phi_P$ | 0.46 |
| $\omega \rightarrow \pi^0\gamma$ | $0.71 \pm 0.04$ | $0.70 \pm 0.02$ | $3B \cos \phi_V$ | 0.62 |
| $\omega \rightarrow \eta\gamma$ | $(5.5 \pm 1) \times 10^{-3}$ | $0.16 \pm 0.01$ | $B[\cos \phi_P \cos \phi_V - 2r_s \sin \phi_P \sin \phi_V]$ | 0.14 |
| $\phi \rightarrow \pi^0\gamma$ | $(5.8 \pm 0.5) \times 10^{-3}$ | $0.042 \pm 0.002$ | $3B \sin \phi_V$ | 0.04 |
| $\phi \rightarrow \eta\gamma$ | $0.056 \pm 0.002$ | $0.21 \pm 0.01$ | $B[2r_s \sin \phi_P \cos \phi_V + \sin \phi_V \cos \phi_P]$ | 0.20 |
| $\phi \rightarrow \eta'\gamma$ | $(5.3 \pm 3) \times 10^{-4}$ | $0.30 \pm 0.08$ | $B[\sin \phi_V \phi_P - 2r_s \cos \phi_V \cos \phi_P]$ | 0.21 |
| $\phi \rightarrow \pi^0\rho^0$ | $< 0.23 \pm 0.01$ | $< 1.18 \pm 0.02$ | $3C \sin \phi_V$ | 0.69 |
| $\psi \rightarrow \eta_c\gamma$ | $(1.1 \pm 0.4) \times 10^{-3}$ | $0.17 \pm 0.03$ | $4 Br_c$ | 0.19 |
| $K^{*+} \rightarrow K^{+}\gamma$ | $0.050 \pm 0.006$ | $0.252 \pm 0.015$ | $\frac{Br_c [2J(r_c) - J(1/r_c)]}{(1+r_c)}$ | 0.20 |
| $K^{*0} \rightarrow K^{0}\gamma$ | $0.116 \pm 0.011$ | $0.39 \pm 0.02$ | $\frac{Br_c [J(r_c) + J(1/r_c)]}{(1+r_c)}$ | 0.34 |
| $D^{*+} \rightarrow D^{+}\gamma$ | $< 0.0014$ | $< 0.14 \pm 0.10$ | $\frac{Br_c [-J(r_c) + 2J(1/r_c)]}{(1+r_c)}$ | 0.03 |
| $D^{*0} \rightarrow D^{0}\gamma$ | $< 0.8$ | $< 3.4$ | $\frac{2Br_c [J(r_c) + J(1/r_c)]}{(1+r_c)}$ | 0.26 |
| $D_s^{*+} \rightarrow D_s^{+}\gamma$ | $< 1.8$ | $< 5.0$ | $\frac{Br_{s c} [J(r_c/r_s) + 2J(r_c/r_s)]}{(r_s + r_c)}$ | 0.04 |
TABLE II. Rare decays which are sensitive to the four mixing parameters $\delta$ and $\epsilon$ in (14) and (15). $A$, $B$ and $C$ are the same abbreviations as in Table I, and again $r_q = m_q / m_\ell$ stands for the constituent quark mass ratio. In writing down the $\psi$ amplitudes, we have set $\cos \phi_V = 1$ as a first approximation and ignored terms of order $\epsilon \delta$, which can be justified a posteriori.

| Process         | $\Gamma^{exp}$(keV) | $|M^{exp}|$ (TeV$^{-1}$) | $M_{PV\psi}$ | $|M^{thy}|$ (TeV$^{-1}$) |
|-----------------|-----------------------|---------------------------|--------------|-------------------------|
| $\eta_c \to \rho^0\rho^0$ | (113 ± 40)            | 37 ± 6                    | $-3C(\delta_c \cos \phi_P + \delta'_c \sin \phi_P)$ | 37                      |
| $\eta_c \to \omega\omega$  | < 84                  | < 32                      | $-3C(\delta_c \cos \phi_P + \delta'_c \sin \phi_P)$ | 37                      |
| $\eta_c \to \phi\phi$      | 94 ± 30               | 43 ± 6                    | $3\sqrt{2}C(\delta_c \sin \phi_P - \delta'_c \cos \phi_P)r_s$ | 43                      |
| $\eta_c \to K^*0K^+0$       | 58 ± 30               | 17 ± 4                    | $\frac{3Cr_s}{1+r_s} \left[ - (\delta_c \cos \phi_P + \delta'_c \sin \phi_P)J(r_s) + \frac{r_s}{\sqrt{2}}(\delta_c \sin \phi_P - \delta'_c \cos \phi_P)J(1/r_s) \right]$ | 16                      |
| $\psi \to \pi^0\gamma$     | 0.003 ± 0.001         | 0.19 ± 0.02               | $-3B\epsilon_c$ | 0.13                    |
| $\psi \to \eta\gamma$      | 0.075 ± 0.008         | 0.92 ± 0.05               | $B[-\epsilon_c \cos \phi_P - 2r_s \epsilon'_c \sin \phi_P + 4r_c \delta_c]$ | 1.1                      |
| $\psi \to \eta'\gamma$     | 0.37 ± 0.04           | 2.3 ± 0.1                 | $B[-\epsilon_c \sin \phi_P + 2r_s \epsilon'_c \cos \phi_P + 4r_c \delta'_c]$ | 0.1                      |
| $\psi \to \pi^0\rho^0$     | 0.37 ± 0.02           | 2.1 ± 0.1                 | $-3C\epsilon_c$ | 2.1                      |
| $\psi \to \eta\omega$      | 0.14 ± 0.02           | 1.4 ± 0.1                 | $-3C[\epsilon_c \cos \phi_P + \sqrt{2}r_s \epsilon'_c \sin \phi_V \sin \phi_P]$ | 1.6                      |
| $\psi \to \eta\phi$        | 0.057 ± 0.008         | 0.97 ± 0.07               | $-3C[\epsilon_c \cos \phi_P \sin \phi_V - \sqrt{2}r_s \epsilon'_c \sin \phi_P]$ | 0.62                     |
| $\psi \to \eta'\omega$     | 0.015 ± 0.002         | 0.52 ± 0.04               | $-3C[\epsilon_c \sin \phi_P - \sqrt{2}r_s \epsilon'_c \sin \phi_V \cos \phi_P]$ | 1.4                      |
| $\psi \to \eta'\phi$       | 0.029 ± 0.003         | 0.80 ± 0.04               | $-3C[\epsilon_c \sin \phi_P \sin \phi_V + \sqrt{2}r_s \epsilon'_c \cos \phi_P]$ | 0.68                     |
| $\psi \to K^0\bar{K}^{*0}$  | 0.37 ± 0.03           | 2.3 ± 0.1                 | $-\frac{6Cr_s[\epsilon_c J(r_s) + \epsilon'_c J(1/r_s)]}{(1+r_s)}$ | 2.2                      |
| $\psi \to K^+\bar{K}^{*-}$  | 0.43 ± 0.07           | 2.5 ± 0.2                 | $-\frac{6Cr_s[\epsilon_c J(r_s) + \epsilon'_c J(1/r_s)]}{(1+r_s)}$ | 2.2                      |
FIG. 1. Plot of the function $J(r)$ from $r = 0$ to 6.