The General Solution of the Complex Monge-Ampère Equation in two dimensional space

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Abstract

The general solution to the Complex Monge-Ampère equation in a two dimensional space is constructed.
1 Introduction

The Complex Monge-Ampère equation in 2-dimensional space takes the form:

$$\det \begin{vmatrix} \frac{\partial^2 \phi}{\partial y_1 \partial \bar{y}_1} & \frac{\partial^2 \phi}{\partial y_1 \partial \bar{y}_2} \\ \frac{\partial^2 \phi}{\partial y_2 \partial y_1} & \frac{\partial^2 \phi}{\partial y_2 \partial y_2} \end{vmatrix} = 0. \quad (1)$$

Its real form, which arises from (1) under the assumption that the solution depends only upon 2 arguments $x_i = y_i + \bar{y}_i$ was solved before by different methods [1], [2]. But to the best of our knowledge the general solution of the complex M-A equation (1) is not known.

In the present paper we follow two different aims. On the one hand we want to fill the gap and using the method of our paper [2] (the solution of the real M-A equation in a space of arbitrary dimension) obtain and present the general solution of the complex version of M-A equation (1) in implicit form. On the other hand we want to prepare the reader to understand the methods for obtaining the general solution of the homogeneous M-A equation in a space of the arbitrary dimension. This work is now nearly finished.

2 Equivalent First Order Equations

The complex M-A equation, (1) is the eliminant of 2 + 2 linear equations (from the linear dependence between rows or columns of the determinant matrix), which may be written as:

$$\sum_{i=1}^{2} \alpha^i \phi_{y_i, \bar{y}_k} = 0, \quad \sum_{i=1}^{2} \beta^i \phi_{\bar{y}_i, y_k} = 0 \quad (2)$$

where $\phi_{y_i}$ denotes $\frac{\partial \phi}{\partial y_i}$ etc.

The next rows contain an obvious transformation (3) using only the rules of differentiation and some new definitions:

$$\left( \sum_{i=1}^{2} \alpha^i \phi_{y_i} \right)_{\bar{y}_k} = \sum_{i=1}^{2} \alpha^i_{y_k} \phi_{y_i}, \quad \left( \sum_{i=1}^{2} \beta^i \phi_{\bar{y}_i} \right)_{y_k} = \sum_{i=1}^{2} \beta^i_{y_k} \phi_{\bar{y}_i} \quad (3)$$

Introducing two new functions

$$R = \sum_{i=1}^{2} \alpha^i \phi_{y_i}, \quad \bar{R} = \sum_{i=1}^{2} \beta^i \phi_{\bar{y}_i}$$
and considering them as a functions of three arguments $R = R(\alpha, y), \bar{R} = \bar{R}(\beta, \bar{y})$ (under the assumption that $\det J(\alpha, \bar{y})$ and $\det J(\beta, y)$ are different from zero), we rewrite equations (3) in an equivalent form:

$$R_{\alpha_i} = \phi_{y_i}, \quad \bar{R}_{\beta_i} = \phi_{\bar{y}_i}. \quad (4)$$

Multiplying each equation (4) respectively by $\alpha_i, \beta_i$, summing the results (recalling the definitions of the functions $R, \bar{R}$, we come to the conclusion that they (with respect to $\alpha, \beta$ arguments) are homogeneous functions of degree 1. Introducing the notation

$$\frac{\alpha_1}{\alpha_2} = u, \quad \frac{\beta_1}{\beta_2} = v,$$

we can represent the dependence of the functions $R, \bar{R}$ in the following way:

$$R = \alpha_2 R(u; y_1, y_2), \quad \bar{R} = \beta_2 \bar{R}(v; \bar{y}_1, \bar{y}_2)$$

Substituting these expressions into equations (4) we arrive at the following relations which form the backbone of our further investigations

$$\phi_{y_1} = R_u, \quad \phi_{y_2} = R - uR_u \quad (5)$$

$$\phi_{\bar{y}_1} = \bar{R}_v, \quad \phi_{\bar{y}_2} = \bar{R} - v\bar{R}_v \quad (6)$$

3 Conditions of selfconsistency (part I)

Using the condition of the equivalence of second mixed partial derivatives taken in different orders, we shall be able to separate the main system (4), (5) and extract from it a very important system of equations connecting the functions $u, v$ only. To this end let us calculate and equate the second mixed derivatives of the following pairs of variables $(\bar{y}_1, y_1), (y_1, y_2), (\bar{y}_2, y_1), (y_2, \bar{y}_2)$.

We have in consequence for the pair $(\bar{y}_1, y_1)$:

$$(R_u)_{\bar{y}_1} = (\bar{R}_v)_{y_1}, \quad R_{u,u}u_{\bar{y}_1} = \bar{R}_{v,v}v_{y_1};$$

for the pair $(y_1, \bar{y}_2)$:

$$(R_u)_{\bar{y}_2} = (\bar{R} - \bar{R}_v\bar{R}_v)_{y_1}, \quad R_{u,u}u_{\bar{y}_2} = -\sum v\bar{R}_{v,v}v_{y_1};$$
for the pair \((\bar{y}_1, y_2)\):

\[ \bar{R}_{v,v} v_{y_2} = -uR_{u,u} u_{\bar{y}_1}; \]

and finally the pair \((y_2, \bar{y}_2)\), leads to equations

\[ uR_{u,u} u_{y_2} = v\bar{R}_{v,v} v_{y_2} \]

Multiplying first equations respectively by \(u, v\), summing the results and comparing with the second and the third sets of equations respectively we come to the following separate system of equations (assuming that \(R_{u,u} \neq 0, \bar{R}_{v,v} \neq 0\)), which the functions \((u, v)\) satisfy:

\[ u_{y_2} + vu_{\bar{y}_1} = 0, \quad v_{y_2} + u v_{\bar{y}_1} = 0 \quad (7) \]

The system (7) was solved before [2],[3] but for the convenience of the reader the next two sections will be devoted to its consideration.

The last comment is the following; the hydrodynamic type system (7) is the result of only 2 equations of second mixed derivatives. Namely it arises from combinations of the first, second and third. It is not difficult to check that the equation for the pair \((y_2, \bar{y}_2)\) automatically satisfies (7) also. Thus only one equation remains unsolved, that connecting the pairs with barred and unbarred index 1. The consequences will be considered in section 6.

### 4 System of hydrodynamic type

We understand by a system of hydrodynamic type the system of equations (7) rewritten below:

\[ v_{y_2} + uv_{\bar{y}_1} = 0, \quad u_{y_2} + v u_{\bar{y}_2} = 0 \quad (8) \]

Two properties of this system will be crucially important in what follows. Proposition 1.

The pair of operators:

\[ D = \frac{\partial}{\partial y_2} + u \frac{\partial}{\partial y_1}, \quad \bar{D} = \frac{\partial}{\partial \bar{y}_2} + v \frac{\partial}{\partial \bar{y}_1} \quad (9) \]

are mutually commutative if \((u, v)\) are solutions of the system (8).
Acting with the help of operators \((D, \bar{D})\) on the second and the first equations of (8) respectively we come to the conclusion that the two functions:

\[
\bar{D}(v) = v\bar{y}_2 + vv\bar{y}_1, \quad D(u) = u_y + uu_y,
\]

are also solutions of the first and the second system of equations (8).

As a corollary we obtain the following:

**Proposition 2.**

\[
v_y + vv = V(v; \bar{y}_1, \bar{y}_2), \quad u_y + uu = U(u; y_1, y_2)
\]

Indeed the 2 sets of variables \((1, u)\), and \((1, v)\) respectively satisfy a linear system of algebraic equations of 2 terms, the matrix of which coincides with the Jacobian matrix

\[
J = \det \begin{vmatrix} v & V \\ y_1 & y_2 \end{vmatrix}
\]

which in the case of a non-zero solution of the linear system must vanish. So Proposition 2 is proved.

In comparison with (8), (11) is an inhomogeneous system of hydrodynamic equations separated into functions \((u, v)\).

### 5 General solution of the hydrodynamic system

Suppose we have the equation defining implicitly unknown function \((\psi)\) in \((4)\) dimensional space \((y, \bar{y})\):

\[
Q(\psi; y) = P(\psi; \bar{y})
\]

where \(Q, P\) are arbitrary functions of its 3 arguments.

With the help of the usual rules of differentiation of implicit functions we find from (13):

\[
\psi_y = (P_\psi - Q_\psi)^{-1}Q_y, \quad \psi_{\bar{y}} = -(P_\psi - Q_\psi)^{-1}P_{\bar{y}}
\]

Let us assume, that between two derivatives with respect to the barred and unbarred variables there exists the following linear dependence:

\[
\sum_{1}^{2} c_i \psi^\alpha_{y_i} = 0, \quad \sum_{1}^{2} d_i \psi^\alpha_{\bar{y}_i} = 0
\]


and analyse the consequences following from these facts.

Assuming that $c_2 \neq 0, d_2 \neq 0$, dividing them into each equation of the left and right systems respectively and introducing the notation $u = \frac{c_1}{c_2}, v = \frac{d_1}{d_2}$ we rewrite the last systems in the form:

$$
\psi_{y_2} + u\psi_{y_1} = 0, \quad \psi_{\bar{y}_2} + v\psi_{\bar{y}_1} = 0
$$

(16)

Substituting the values of the derivatives from (14) and multiplying the result by $(P_\psi - Q_\psi)$ we obtain:

$$
Q_{y_2} + uQ_{y_1} = 0, \quad P_{\bar{y}_2} + vP_{\bar{y}_1} = 0
$$

(17)

From the last equations it immediately follows that:

$$
u = -\frac{Q_{y_2}}{Q_{y_1}}, \quad v = -\frac{P_{\bar{y}_2}}{P_{\bar{y}_1}}
$$

(18)

We see that if we augment the initial system (13), by two functions $(u, v)$ defined by (18) then the operators of differentiation $D, \bar{D}$ defined by (9) in connection with (16) annihilate each $\psi$ either as $Q, P$ functions:

$$
D\psi = \bar{D}\psi = DQ = DP = \bar{D}Q = \bar{D}P = 0
$$

(19)

This means that $Df(\psi, \bar{y}) = \bar{D}f(\psi, y) = 0$. And as a direct corollary of this fact $Dv = \bar{D}u = 0$ and so the generators $D, \bar{D}$ which have been constructed commute.

Thus we have found the general solution of the hydrodynamic system and a concrete realisation of the manifold with the properties of the previous section.

With respect to generators $D, \bar{D}$ all functions of 4 dimensional space $(y_i, \bar{y}_k)$ may be divided into the following subclasses: functions of general position $F, DF \neq 0, \bar{D}F \neq 0$, the holomorphic functions $f, \bar{D}f = 0, Df \neq 0$, antiholomorphic ones $\bar{f}, D\bar{f} = 0, \bar{D}\bar{f} \neq 0$ and $f^0$ ”central” holomorphic and antiholomorphic simultaneously; $Df^0 = \bar{D}f^0 = 0$.

Each central function may be represented in the form:

$$
f^0 = f^0(Q) = f^0(P) = g^0(\psi)
$$
6 Conditions of selfconsistency (part II)

Here we compare and equate second mixed derivatives of the pairs $(y_1, y_2), (\bar{y}_1, \bar{y}_2)$ and remaining from the section part I the pair $(y_1, \bar{y}_1)$. All calculations are straightforward.

The conditions of selfconsistency of the pairs $(y_1, y_2), (\bar{y}_1, \bar{y}_2)$ may be manipulated into the following compact form:

$$DR_u = R_{y_1}, \quad \bar{D}\bar{R}_v = \bar{R}_{\bar{y}_1}$$  (20)

There is only one equation of selfconsistency, connecting the barred and unbarred index 1:

$$(R_u)_{y_1} = (\bar{R}_v)_{\bar{y}_1}, \quad R_{uu}u_{\bar{y}_1} = \bar{R}_{vv}v_{y_1}, \quad R_{uu}u_{\psi}\bar{y}_1 = \bar{R}_{vv}v_{\psi}y_1.$$  (21)

7 Solution of selfconsistency equations

Substituting into (21) the known values of the derivatives of $\psi$ function (14) we pass to the final equation of interest:

$$\frac{R_{uu}u_{\psi}}{Q_{y_1}} = -\frac{\bar{R}_{vv}v_{\psi}}{P_{\bar{y}_1}} = A^0_{\psi}$$  (22)

Indeed the left hand side of the last equality is a holomorphic function, the right hand side an antiholomorphic one. Thus $A^0_\psi$ is a central function.

Considering now $R_u = R_u(\psi; y_1, y_2)$ and $\bar{R}_v = \bar{R}_v(\psi; y_1, \bar{y}_2)$ we solve the equations containing the function $A^0_\psi$ in the form:

$$R_u = \Theta_{y_1}(A; y_1, y_2), \quad Q = \Theta_A(A; y_1, y_2)$$
$$\bar{R}_v = \bar{\Theta}_{\bar{y}_1}(A; \bar{y}_1, \bar{y}_2), \quad P = -\bar{\Theta}_A(A; \bar{y}_1, \bar{y}_2)$$  (23)

It remains only to check equalities (20). Let us distinguish by upper indices $u, A$ the corresponding derivatives \(\frac{\partial}{\partial y_i}, \frac{\partial^A}{\partial y_i}\) on the space coordinates $(y, \bar{y})$ taken keeping $u, (v)$ constant in first case and the function $A$ constant in the second. The equality which has to be checked in this notation has the form:

$$\frac{\partial^u}{\partial y_1} R_u = \frac{\partial}{\partial u} \frac{\partial^u}{\partial y_1} R$$  (24)
Keeping in mind that $D^n A = 0$ ($A$ is a central function) and the definition of all values involved in terms of the function $\Theta$ we obtain in consequence for the right hand side of the last equality

$$
(D^n R_u)_u = (D^n \frac{\partial A}{\partial y_1} \Theta)_u = (D^n \frac{\partial A}{\partial y_1} \Theta)_A A_u =
$$

$$
\left( \frac{\partial^2 A}{\partial y_1 \partial y_2} + u \frac{\partial^2 A}{\partial y_1 \partial y_1} \right) \Theta_A A_u = \Theta_{y_1, y_1} + (\Theta_{A, y_1, y_2} - \Theta_{A, y_1, y_1}) A_u =
$$

$$
\Theta_{y_1, y_1} + \Theta_{A, y_1, y_1} A_u \frac{\partial A}{\partial y_1}
$$

In all transformations above we have not written the upper index $A$ with respect to derivatives of the space coordinates $y$.

Similar calculations for the left hand side leads to:

$$
\frac{\partial^n}{\partial y_1^n} \Theta_{y_1} = \Theta_{y_1, y_1} + \Theta_{A, y_1, y_1} A_u \frac{\partial A}{\partial y_1}
$$

which shows that equalities (23) are satisfied. But (20) in its turn is an equation of second order with respect to the unknown function $R$. We rewrite it in explicit form substituting instead of $R_u$ its value from (23):

$$
(\frac{\partial^2}{\partial y_1 \partial y_2} + u \frac{\partial^2}{\partial y_1 \partial y_1}) \Theta = R_{y_1}^u = R_{y_1} + R_A A_{y_1} =
$$

$$
R_{y_1} - \frac{u_{y_1}}{u_A} R_A = R_{y_1} - u_{y_1} R_u = R_{y_1} - u_{y_1} \Theta_{y_1}
$$

In the process of evaluation of the last expression the crucial step was the calculation of $A_{y_1}$ keeping $u$ at a fixed value. It was achieved by direct differentiation of the definition of $u$ rewritten in the form:

$$
\Theta_{A, y_2} + u \Theta_{A, y_1} = 0
$$

with respect to the argument $y_1$ (with fixed $u$) and regrouping the terms arising.

Preserving in the last equality the first and the last terms we obtain the equations for the function $R$ in integrable form. The result of integration
determines the function $R$ in terms of the function $\Theta$ in a very attractive form:

$$R = D\Theta, \quad \bar{R} = \bar{D}\bar{\Theta}$$

Substituting these expressions in equations connecting derivatives of the solution of the M-A equation (5), (6) with the functions $R, \bar{R}$ we obtain finally:

$$\phi_{y_1} = \Theta_{y_1}, \quad \phi_{y_2} = \Theta_{y_2}, \quad \phi_{\bar{y}_1} = \bar{\Theta}_{\bar{y}_1}, \quad \phi_{\bar{y}_2} = \bar{\Theta}_{\bar{y}_2}$$

8 Concluding remarks

The main result of the present paper is in the theorem of the previous section. But not less important is the hydrodynamic like system of equations (13) solved in the middle of the calculations. It effectively defines the manifold of solutions of the complex M-A equation in two dimensions. In the final result it is encoded in the equation determining the function $A$. In the one dimensional limit it does not pass directly to the Monge equation but another new integrable system:

$$u_t + vu_x = 0, \quad v_t + uv_x = 0$$

The Monge equation is contained among special cases of this system under the reduction $u = v$.

In the near future using the scheme of the present paper we are going to publish a similar result in connection with the M-A equation in a space of arbitrary dimension.

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