On the classification of non-aCM curves on quintic hypersurfaces in \( \mathbb{P}^3 \)

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Abstract

In this paper, we call a sub-scheme of dimension one in \( \mathbb{P}^3 \) a curve. It is well known that the arithmetic genus and the degree of an aCM curve \( D \) in \( \mathbb{P}^3 \) is computed by the \( h \)-vector of \( D \). However, for a given curve \( D \) in \( \mathbb{P}^3 \), the two invariants of \( D \) do not tell us whether \( D \) is aCM or not. In this paper, we give a classification of curves on a smooth quintic hypersurface in \( \mathbb{P}^3 \) which are not aCM.

Keywords ACM curve, \( h \)-vector, line bundle, quintic hypersurface,

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1 Introduction

We work over the complex number field \( \mathbb{C} \). Let \( C \) be a curve in \( \mathbb{P}^n \) which is not necessarily irreducible. Then we call \( C \) an arithmetically Cohen-Macaulay (aCM for short) curve if the homogeneous coordinate ring of it is Cohen-Macaulay. It is equivalent to the condition that \( h^1(\mathcal{I}_C(l)) = 0 \) for all \( l \in \mathbb{Z} \), where \( \mathcal{I}_C \) is the ideal sheaf of \( C \) in \( \mathbb{P}^n \). It seems that the projective normality of a smooth curve in \( \mathbb{P}^n \) is often defined by the equivalent condition that the natural map \( H^0(\mathcal{O}_{\mathbb{P}^n}(l)) \rightarrow H^0(\mathcal{O}_C(l)) \) is surjective for any \( l \geq 0 \). In general, it is difficult to discriminate whether a given curve \( C \) in \( \mathbb{P}^n \) is aCM or not. However, if \( C \) lies on a smooth projective surface, we can do it in several cases. For examples, aCM curves on a DelPezzo surface \( X \) with respect to the projective embedding induced by \( -K_X \) are classified by Pons-Llopis and Tonini ([4]). On the other hand, it is well known that any non-hyperelliptic curve \( C \) on a K3 surface has a canonical embedding and is projectively normal with respect to it (cf. [5]). A curve \( C \) on a smooth hypersurface \( X \) in \( \mathbb{P}^3 \) is aCM if and only if \( \mathcal{O}_X(C) \) is an aCM line bundle on \( X \). In the previous work [6,7], aCM line bundles on a smooth

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hypersurface of degree \( d = 4, 5 \) in \( \mathbb{P}^3 \) are characterized. In particular, a necessary and sufficient condition for a line bundle on a smooth quintic hypersurface \( X \) in \( \mathbb{P}^3 \) to be aCM is given as follows.

**Theorem 1.1** ([7, Theorem 1.1]) Let \( X \) be a smooth quintic hypersurface in \( \mathbb{P}^3 \), let \( H_X \) be the hyperplane class of \( X \), and let \( C \) be a smooth member of the linear system \( |H_X| \). Let \( D \) be a non-zero effective divisor on \( X \) of arithmetic genus \( P_a(D) \), and we set \( k = C.D + 1 - P_a(D) \). Then \( O_X(D) \) is aCM and initialized if and only if the following conditions are satisfied.

(i) \( 0 \leq k \leq 4 \).

(ii) If \( 0 \leq k \leq 1 \), then \( C.D = 10 - k \) and \( h^0(O_C(D - C)) = 0 \).

(iii) If \( k = 2 \), then the following conditions are satisfied.

\( a \) If \( C.D = 7 \), then \( h^0(O_X(2C - D)) = 1 \).

\( b \) If \( C.D = 8 \), then \( h^0(O_C(D - C)) = 0 \) and \( h^0(O_C(D)) = 3 \).

(iv) If \( 3 \leq k \leq 4 \), then the following conditions are satisfied.

\( a \) If \( 8 - k \leq C.D \leq 10 - k \), then \( h^0(O_C(D)) = 5 - k \).

\( b \) If \( k = 3 \), then \( C.D \neq 4 \).

Here, we say that a line bundle \( L \) on \( X \) is initialized if \( |L| \neq \emptyset \) and \( |L(-1)| = \emptyset \).

The Hilbert function of an aCM curve \( C \) in \( \mathbb{P}^3 \) is defined by \( H_C(l) = h^0(O_C(l)) \) \( (l \in \mathbb{N} \cup \{0\}) \), and the \( h \)-vector of \( C \) is defined as the second difference function of it. The degree \( \deg C \) and the arithmetic genus \( P_a(C) \) of \( C \) are denoted as follows respectively.

\[
\deg C = \sum_{l \geq 0} h_C(l) \quad P_a(C) = \sum_{l \geq 1} (l - 1)h_C(l).
\]

However, we can not discriminate whether a given curve in \( \mathbb{P}^n \) is aCM or not by using the two invariants of it. For example, any smooth irreducible aCM curve of degree 6 and genus 3 in \( \mathbb{P}^3 \) is not hyperelliptic and a smooth quartic hypersurface in \( \mathbb{P}^3 \) containing such an aCM curve is linear determinantal. However, a smooth curve of bidegree \((2, 4)\) on a smooth quadric hypersurface in \( \mathbb{P}^3 \) is a hyperelliptic curve which has the same degree and genus as it. Hence, it is natural and interesting to classify the pair \((d, g)\) consisting of integers \( d \) and \( g \) such that there exist an aCM curve of degree \( d \) and genus \( g \) and a non-aCM curve of the same degree and genus as it in \( \mathbb{P}^3 \). However, any concrete description concerning non-aCM curves on \( X \) is not given in Theorem 1.1. In this paper, we will give a best possible sufficient condition regarding the degree and the arithmetic genus, for a curve on a smooth quintic hypersurface \( X \) in \( \mathbb{P}^3 \) to be aCM, and a complete classification of non-aCM curves on \( X \). Our first main theorem is the following.
Theorem 1.2 Let the notation be as in Theorem 1.1. If $D$ satisfies one of the following conditions, then $D$ is an aCM curve.

(i) $k = 2$ and $C.D \in \{1, 4\}$.
(ii) $k = 3$ and $C.D \in \{2, 3, 5, 6\}$.
(iii) $k = 4$ and $C.D \in \{3, 4\}$.

By the assertion of Theorem 1.1, the essence of Theorem 1.2 is that if $(k, C.D) = (3, 5), (3, 6)$ or $(4, 4)$, then $D$ is aCM. On the other hand, since the existence of an aCM curve $D$ on a smooth quintic hypersurface in $\mathbb{P}^3$ satisfying each condition in Theorem 1.1 is indicated in Section 5 of [7], we can classify the pair consisting of integers $d$ and $g$ such that the Hilbert scheme of curves of degree $d$ and arithmetic genus $g$ in $\mathbb{P}^3$ contains both an aCM curve and a non-aCM curve on a smooth quintic hypersurface in $\mathbb{P}^3$. Our second main theorem is the following.

Theorem 1.3 Let $d$ and $k$ be integers. If $d$ and $k$ satisfy one of the following conditions, then there exists a non-aCM curve $D$ of degree $d$ and arithmetic genus $P_a(D) = d - k + 1$ on a smooth quintic hypersurface in $\mathbb{P}^3$.

(i) $k = 0$ and $d = 10$.
(ii) $k = 1$ and $d = 9$.
(iii) $k = 2$ and $d \in \{7, 8\}$.
(iv) $k = 3$ and $d = 7$.
(v) $k = 4$ and $d \in \{5, 6\}$.

Non-aCM curves satisfying each condition in Theorem 1.3 can be concretely constructed, and we can obtain a complete classification of non-aCM curves on a smooth quintic hypersurface in $\mathbb{P}^3$ (see Section 4).

Our plan of this paper is the following. In section 2, we recall the previous work about the classification of aCM line bundles on a smooth quartic hypersurface in $\mathbb{P}^3$, and give a characterization and examples of non-aCM curves on such a surface. In section 3, we recall several facts concerning line bundles on smooth quintic hypersurfaces in $\mathbb{P}^3$. In section 4, we prove the main results and give a concrete description of non-aCM curves on smooth quintic hypersurfaces in $\mathbb{P}^3$.

Notations and conventions. In this paper, a surface is smooth and projective. Let $X$ be a smooth curve or a surface. Then we denote the canonical bundle of $X$ by $K_X$. For a divisor or a line bundle $L$ on $X$, we denote by $|L|$ the linear system of $L$, and denote the dual of a line bundle $L$ by $L^\vee$. For an irreducible curve $D$ which is not necessarily smooth, we denote by $P_a(D)$ the arithmetic genus of $D$. For irreducible divisors $D_1$ and $D_2$ on a surface $X$, the arithmetic genus of $D_1 + D_2$ is denoted as $P_a(D_1 + D_2) = P_a(D_1) + P_a(D_2) - D_1.D_2 - 1$. By induction, the arithmetic genus of a reducible effective divisor $D$ on $X$ is also defined. We denote
it by the same notation $P_a(D)$. It follows that $2P_a(D) - 2 = D.(K_X + D)$ by the
adjunction formula. Note that if $D$ is reduced and irreducible, then $P_a(D) \geq 0$.

The gonality of a smooth curve is the minimal degree of pencils on it. It is
well known that the gonality of a smooth plane curve of degree $d \geq 5$ is $d - 1$.

We denote the Néron-Severi lattice and the Picard lattice of a surface $X$ by
NS($X$) and Pic($X$) respectively. We call the rank of the Néron-Severi lattice
of $X$ the Picard number of $X$. If the Picard number of $X$ is $\rho$, then by the
Hodge index theorem, the signature of NS($X$) is $(1, \rho - 1)$. This implies that
$D_1^2D_2^2 \leq (D_1,D_2)^2$ for two divisors $D_1$ and $D_2$ on $X$ with $D_1^2 > 0$ and $D_2^2 > 0$.

Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^3$, and let $C$ be a hyperplane
section of $X$. For a non-zero effective divisor $D$ on $X$, we denote the degree of
$D$ by $\deg D$. In particular, we call a curve on $X$ of degree 1 and genus
0 a line. We denote the class of $C$ in Pic($X$) by $H_X$. For an integer $l$, $H_X^l$ is
often denoted as $\mathcal{O}_X(l)$. By the adjunction formula, $K_X \cong \mathcal{O}_X(d-4)$. For a line
bundle $L$ on $X$, we will write $L \otimes \mathcal{O}_X(l) = L(l)$.

2 ACM curves on quartic hypersurfaces in $\mathbb{P}^3$

In this section, we recall the result concerning aCM curves on smooth quartic
hypersurfaces in $\mathbb{P}^3$, and give a first example of a non-aCM curve as a toy model
of our work. First of all, we recall the relationship between aCM curves and
aCM line bundles on a hypersurface in $\mathbb{P}^3$.

**Definition 2.1** Let $X$ be a hypersurface in $\mathbb{P}^3$, and let $L$ be a line bundle on
$X$. Then we say that $L$ is arithmetically Cohen-Macaulay (aCM for short) if
$h^1(L(l)) = 0$ for any $l \in \mathbb{Z}$.

**Remark 2.1** Let $D$ be a non-zero effective divisor on a hypersurface $X$ of degree
$d$ in $\mathbb{P}^3$. Then, by the exact sequence
\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{I}_D \to \mathcal{O}_X(-D) \to 0, \]
$\mathcal{O}_X(D)$ is aCM if and only if $D$ is aCM, where $\mathcal{I}_D$ is the ideal sheaf of $D$ in $\mathbb{P}^3$.

By Remark 2.1, our previous work as in [6] regarding the classification of aCM
line bundles on a quartic hypersurface $X$ in $\mathbb{P}^3$ implies the following assertion.

**Proposition 2.1** Let $X$ be a quartic hypersurface in $\mathbb{P}^3$, and let $D$ be a non-zero
effective divisor on $X$. Let $C$ be a hyperplane section of $X$. Then the following
conditions are equivalent.

1. $D$ is an aCM curve with $|D - C| = \emptyset$.
2. One of the following cases occurs.
   (a) $P_a(D) = 0$ and $1 \leq C.D \leq 3$. 
(b) \( P_a(D) = 1 \) and \( 3 \leq C.D \leq 4 \).
(c) \( P_a(D) = 2 \) and \( C.D = 5 \).
(d) \( P_a(D) = 3 \), \( C.D = 6 \), and \( |D - C| = |2C - D| = \emptyset \).

It turns out that if a curve \( D \) with \( P_a(D) \leq 2 \) belongs to a smooth quartic hypersurface in \( \mathbb{P}^3 \), we can discriminate whether or not it is aCM, by using the arithmetic genus and the degree of it. However, if a smooth curve \( D \) with \( \deg D = 6 \) and \( P_a(D) = 3 \) in \( \mathbb{P}^3 \) is hyperelliptic, then it is not aCM. In fact, if \( D \) is aCM, then there exists a quartic hypersurface \( X \) in \( \mathbb{P}^3 \) containing \( D \) which is linear determinantal, and \( |D| \) defines a birational map from \( X \) onto the image of \( X \). Since \( X \) is a K3 surface, this implies that \( K_D \) is very ample and hence, \( D \) is not hyperelliptic. This means that if we remove the condition that \( |D - C| = |2C - D| = \emptyset \) from Proposition 2.1 (ii) (d), the assertion is not correct.

**Proposition 2.2** Let \( X \), \( C \) and \( D \) be as in Proposition 2.1, and assume that \( C.D = 6 \) and \( P_a(D) = 3 \). Then the following conditions are equivalent.

(a) \( D \) is not an aCM curve.
(b) There exist two lines \( \Gamma_1 \) and \( \Gamma_2 \) on \( X \) with \( \Gamma_1.\Gamma_2 = 0 \) such that \( \Gamma_1 + \Gamma_2 \in |D - C| \) or \( \Gamma_1 + \Gamma_2 \in |2C - D| \).

**Proof.** (b) \( \Rightarrow \) (a) Let \( \Gamma_1 \) and \( \Gamma_2 \) be lines on \( X \) with \( \Gamma_1.\Gamma_2 = 0 \). If \( \Gamma_1 + \Gamma_2 \in |D - C| \), then \( D - C \) is not 1-connected. Since \( X \) is a K3 surface, we have \( h^1(\mathcal{O}_X(D)(-1)) = h^1(\mathcal{O}_X(1)(-D)) \neq 0 \). This implies that \( \mathcal{O}_X(D) \) is not aCM and hence, \( D \) is not an aCM curve. If \( \Gamma_1 + \Gamma_2 \in |2C - D| \), then \( 2C - D \) is not 1-connected. Hence, by the same reason as above, \( D \) is not aCM.

(a) \( \Rightarrow \) (b) We assume that \( D \) is not aCM. By Proposition 2.1 and Remark 2.1, we have \( |D - C| \neq \emptyset \) or \( |2C - D| \neq \emptyset \). Assume that \( |D - C| \neq \emptyset \). Since \( C.(D - C) = 2 \) and \( (D - C)^2 = -4 \), the arithmetic genus of the member of \( |D - C| \) is -1 and hence, there exist two lines \( \Gamma_1 \) and \( \Gamma_2 \) on \( X \) with \( \Gamma_1.\Gamma_2 = 0 \) such that \( \Gamma_1 + \Gamma_2 \in |D - C| \). If \( 2|C - D| \neq \emptyset \), then there exist two lines \( \Gamma_1 \) and \( \Gamma_2 \) on \( X \) with \( \Gamma_1.\Gamma_2 = 0 \) such that \( \Gamma_1 + \Gamma_2 \in |2C - D| \), by the same reason as above. Hence, we have the assertion. \( \square \)

In Proposition 2.2, if \( |D - C| \neq \emptyset \), then the two lines \( \Gamma_1 \) and \( \Gamma_2 \) form the fixed component of \( |D| \), and hence, we can not take such a divisor \( D \) on \( X \) to be smooth and irreducible. On the other hand, if \( |2C - D| \neq \emptyset \), then \( D \) is linearly equivalent to the union of two elliptic curves \( E_1 \) and \( E_2 \) of degree 3 on \( X \) with \( E_1.E_2 = 2 \) and hence, we can take such a curve \( D \) to be smooth and irreducible. Since the restrictions to \( D \) of the pencils \( |E_1| \) and \( |E_2| \) on \( X \) are pencils of degree 2 on \( D \), the curve \( D \) is hyperelliptic. We can construct a non-aCM curve lying on the Fermat quartic hypersurface in \( \mathbb{P}^3 \) satisfying the condition (b).
Example 2.1 Let $X$ be the quartic hypersurface in $\mathbb{P}^3$ defined by the equation $x_4^4 + x_1^4 + x_2^4 + x_3^4 = 0$, for a suitable homogeneous coordinate $(x_0 : x_1 : x_2 : x_3)$ on $\mathbb{P}^3$. Let $C_i$ be the hyperplane section of $X$ defined by the equation $x_0 + \omega x_i = 0$ ($i = 1, 2$), where $\omega$ is a primitive eighth root of unity. Let $\Gamma_1$ and $\Gamma_2$ be lines defined by the equations $x_0 + \omega x_1 = x_2 + \omega x_3 = 0$ and $x_0 + \omega x_2 = x_1 + \omega^2 x_3 = 0$, respectively. Since $\Gamma_1, \Gamma_2 = 0$, if we set $D_1 = C_1 + \Gamma_1 + \Gamma_2$ and $D_2 = C_1 + C_2 - \Gamma_1 - \Gamma_2$, then $P_0(D_i) = 3$ and $\deg D_i = 6$ for $i = 1, 2$. Moreover, the conditions that $|D_1 - C_1| \neq \emptyset$ and $|2C_1 - D_2| \neq \emptyset$ are also satisfied.

3 Linear systems on quintic hypersurfaces in $\mathbb{P}^3$

From now on let $X$ be a smooth quintic hypersurface in $\mathbb{P}^3$. In this section, we recall several basic facts regarding linear systems on $X$ and some useful propositions as in Section 2 of [7]. Let $D$ be a divisor on $X$ and let $C$ be a smooth hyperplane section of $X$. First of all, since $K_X \cong \mathcal{O}_X(1)$, the Riemann-Roch theorem for $\mathcal{O}_X(D)$ implies that

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D.(D - C) + 5,$$

where $\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D))$. Note that since $h^0(K_X) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, $h^1(\mathcal{O}_X) = 0$, and $h^0(\mathcal{O}_X) = 1$, we have $\chi(\mathcal{O}_X) = 5$. The Serre duality for $\mathcal{O}_X(D)$ is given by

$$h^i(\mathcal{O}_X(D)) = h^{2-i}(\mathcal{O}_X(1)(-D)) (0 \leq i \leq 2).$$

Remark 3.1 Since $C$ is a plane quintic curve, the gonality of $C$ is 4. Hence, a line bundle $L$ on $C$ with $h^0(L) \geq 2$ satisfies $\deg(L) \geq h^0(L) + 2$. In particular, if $|D - C| = \emptyset$ and $h^0(\mathcal{O}_X(D)) \geq 2$, then, by the exact sequence

$$(3.1) \quad 0 \rightarrow \mathcal{O}_X(D)(-1) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_C(D) \rightarrow 0,$$

we have $h^0(\mathcal{O}_X(D)) \leq h^0(\mathcal{O}_C(D)) \leq C.D - 2$.

By the Riemann-Roch theorem, the Serre duality, and Remark 3.1, a non-zero effective divisor on $X$ of degree $d \leq 2$ can be explicitly written as follows.

Lemma 3.1 ([7, Lemma 2.1]) Let $D$ be a divisor on $X$ satisfying $C.D = 1$. Then the following conditions are equivalent.

(a) $h^0(\mathcal{O}_X(D)) > 0$.
(b) $h^0(\mathcal{O}_X(D)) = 1$, $h^0(\mathcal{O}_X(C - D)) = 2$, and $h^1(\mathcal{O}_X(D)) = 0$.
(c) $D^2 = -3$. 
Lemma 3.2 ([7, Lemma 2.2]) Let $D$ be an effective divisor on $X$ with $C.D = 2$. If $D^2 \leq -6$, then one of the following cases occurs.

(a) There exists a line $D_1$ on $X$ with $D = 2D_1$.

(b) There exist two lines $D_1$ and $D_2$ such that $D = D_1 + D_2$ and $D_1.D_2 = 0$.

The arithmetic genus $P_a(D)$ of a non-zero effective divisor $D$ is given as follows.

$$P_a(D) = \frac{1}{2} D.(D + C) + 1.$$ 

Remark 3.2 Obviously, the arithmetic genus $P_a(D)$ of an effective divisor $D$ of degree one on $X$ is 0. On the other hand, if a divisor $D$ on $X$ satisfies $C.D = 2$, then by Remark 3.1, $h^0(O_X(D)) \leq 1$ and $h^0(O_X(1)(-D)) \leq 1$. Hence, by the Riemann-Roch theorem, we have $D^2 \leq -4$. If $D^2 = -4$, then the member of $|D|$ is a plane conic and hence, the arithmetic genus of it is zero.

By Lemma 3.1 and Lemma 3.2, we have the following assertion (cf. [7, Proposition 2.1]).

Proposition 3.1 Let $k$ be an integer with $0 \leq k \leq 4$, and let $D$ be a non-zero effective divisor on $X$ such that $P_a(D) = C.D + 1 - k$. If $C.D \geq 7 - k$, then $h^0(O_X(1)(-D)) = 0$.

Remark 3.3 By the Riemann-Roch theorem and the Serre duality, if $D$ is a non-zero effective divisor on $X$ with $h^1(O_X(-D)) = 0$, then $P_a(D) \geq 0$.

In general, the vanishing condition of the cohomology of the sheaf as in Remark 3.3 can be characterized by the following notion.

Definition 3.1 Let $m$ be a positive integer. Then a non-zero effective divisor $D$ on $X$ is called $m$-connected if $D_1.D_2 \geq m$, for each non-trivial effective decomposition $D = D_1 + D_2$.

A non-zero effective divisor $D$ is 1-connected if and only if $h^0(O_D) = 1$. Moreover, by the exact sequence

$$0 \rightarrow O_X(-D) \rightarrow O_X \rightarrow O_D \rightarrow 0,$$

the condition that $h^0(O_D) = 1$ is equivalent to $h^1(O_X(-D)) = 0$. Hence, if $D$ is 1-connected, then we have $P_a(D) \geq 0$, by Remark 3.3. Conversely, the following fact is well known as a sufficient condition for a non-zero effective divisor $D$ on $X$ with $P_a(D) \geq 0$ to be 1-connected.

Proposition 3.2 ([1, p 179, Theorem 12.1]) Let $D$ be a numerical effective divisor on $X$ with $D^2 > 0$. Then $h^1(O_X(-D)) = 0$. 

The effective divisor $D$ consisting of two skew lines on $X$ as in Lemma 3.2 (b) is not 1-connected, and is not contained in any hyperplane of $\mathbb{P}^3$. Since such a divisor $D$ satisfies $|D - C| = \emptyset$, by the Riemann-Roch theorem and Remark 3.1, the conditions $h^0(\mathcal{O}_X(D)) = 1$ and $h^1(\mathcal{O}_X(D)) = 0$ are also satisfied. Conversely, any non-zero effective divisor $D$ which is not 1-connected is characterized as follows, under the condition that $|D - C| = \emptyset$ and $h^3(\mathcal{O}_X(D)) = 0$ (cf. [7, Proposition 2.3]).

**Proposition 3.3** Let $D$ be a non-zero effective divisor on $X$. If $|D - C| = \emptyset$ and $h^1(\mathcal{O}_X(D)) = 0$, then $h^1(\mathcal{O}_X(-D)) = 0$ or there exist two lines $D_1$ and $D_2$ on $X$ such that $D = D_1 + D_2$ and $D_1.D_2 = 0$.

We can construct an example of an effective divisor $D$ on a smooth quintic hypersurface $X$ in $\mathbb{P}^3$ satisfying the condition (b) as in Lemma 3.2.

**Example 3.1** Let $X$ be the quintic hypersurface in $\mathbb{P}^3$ defined by the equation $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$. Let $D_1$ and $D_2$ be divisors on $X$ which are defined by the equations $x_0 + x_1 = x_2 + x_3 = 0$ and $x_0 + x_2 = x_1 + \xi x_3 = 0$, respectively. Here, $\xi$ is a primitive fifth root of unity. Then $D_1$ and $D_2$ are skew lines on $X$. Hence, the divisor $D_1 + D_2$ on $X$ is not 1-connected.

By Proposition 3.3, we have the following assertion as a sufficient condition for a non-zero effective divisor $D$ on $X$ to be 1-connected.

**Corollary 3.1** Let $D$ be a non-zero effective divisor on $X$ satisfying the condition that $h^1(\mathcal{O}_X(D)) = 0$ and $|D - C| = \emptyset$. If $P_a(D) \geq 0$, then $h^1(\mathcal{O}_X(-D)) = 0$.

Assume that the linear system $|D|$ defined by a non-zero effective divisor $D$ is base point free and $\dim |D| = r$. Let $f : X \rightarrow \mathbb{P}^r$ be the map defined by $|D|$. The theorem of Bertini implies that if the dimension of the image of $f$ is two, then the general member of $|D|$ is smooth and irreducible; otherwise, there exists a Stein factorization $f = h \circ g$ which is the composition of a pencil $g : X \rightarrow \Gamma$ and a finite map $h : \Gamma \rightarrow \mathbb{P}^r$ onto the image, where $\Gamma$ is a smooth irreducible curve. Here, we have $\Gamma \cong \mathbb{P}^1$. Indeed, since $h^1(\mathcal{O}_X) = 0$, $\text{NS}(X) \cong \text{Pic}(X)$ and hence, the genus of the image of $g$ is 0. Hence, in the latter case, the general member of $|D|$ is the disjoint union of finitely many smooth curves $D_1, \cdots, D_m$ such that $D_i$ and $D_j$ are linearly equivalent for each $i$ and $j$. If $D$ is 1-connected, then this case does not occur. By Proposition 3.2, we have the following assertion.

**Proposition 3.4** Let $D$ be a base point free divisor on $X$ with $D^2 > 0$. Then the general member of $|D|$ is a smooth irreducible curve.

**Remark 3.4** For a non-zero effective divisor $D$ on $X$, if $|D|$ has no fixed component and $D^2 = 0$, then there exist a smooth irreducible curve $\tilde{D}$ and $m \in \mathbb{N}$ with $m\tilde{D} \in |D|$.
If $|D|$ has no fixed component, then the general member of it is smooth outside the set of base points of $|D|$. In particular, the following assertion follows.

**Proposition 3.5** Let $D$ be an effective divisor on $X$ with $0 < D^2 \leq 3$ such that $|D|$ has no fixed component. Then $|D|$ contains a smooth irreducible curve.

**Proof.** Assume that $|D|$ has a base point $P$ such that any member of $|D|$ is singular at $P$. Let $m$ be the minimum number of the multiplicity at $P$ of all members of $|D|$. Then $m \geq 2$. Let $\varphi : \hat{X} \to X$ be a blow up at $P$ and we set $E = \varphi^{-1}(P)$. If we set $\hat{D} = \varphi^*D - mE$, then $\hat{D}^2 \leq 3 - m^2 \leq -1$. However, this contradicts the fact that $|\hat{D}|$ has no fixed component. By the assumption, $D$ is not zero, and the cardinality of the set of the base points of $|D|$ is at most 3. Hence, the general member of $|D|$ is a smooth curve. Since $D$ is numerical effective and $D^2 > 0$, by Proposition 3.2, $h^1(O_X(-D)) = 0$. Hence, the general member of $|D|$ is irreducible. \qed

### 4 Proof and examples of main Theorems

In this section, we prove Theorem 1.2 and Theorem 1.3. Let $X$ be a quintic hypersurface in $\mathbb{P}^3$ and let $D$ be a non-zero effective divisor on $X$. Let $C$ be a smooth hyperplane section of $X$. We recall the following assertion which follows from Proposition 3.2 in [7] and Remark 2.1 to prove our main theorems.

**Proposition 4.1** Let $k$ be a positive integer with $C.D + 5 < 5k$. If $h^1(O_X(l)(-D)) = 0$ for $0 \leq l \leq k$, then $D$ is an aCM curve.

First of all, we investigate the case where $D$ is an aCM curve. By Theorem 1.1, it is sufficient to consider the case where $(C.D, P_a(D)) = (4,1), (5,3), \text{and}(6,4)$.

**Proposition 4.2** If $P_a(D) = 1$ and $C.D = 4$, then $D$ is aCM.

**Proof.** Since $(D - C).C = -1$, we have $|D - C| = \emptyset$. By Proposition 4.1, it is sufficient to show that $h^1(O_X(l)(-D)) = 0$, for $0 \leq l \leq 2$. We show that $h^1(O_X(1)(-D)) = 0$. First of all, by Proposition 3.1, we have $|C - D| = \emptyset$. Since $\chi(O_X(D)) = 1$, if $h^1(O_X(1)(-D)) \neq 0$, then $h^0(O_X(D)) \geq 2$. Let $\Delta$ be the fixed component of $|D|$ and we set $\hat{D} = D - \Delta$. Since $|D - C| = \emptyset$, we have $|\hat{D} - C| = \emptyset$. Since $\hat{D}^2 = -4 < 0$, $\Delta$ is not empty. By the ampleness of $C$, we have $C.D \leq 3$. This contradicts Remark 3.1. Therefore, we have $h^1(O_X(1)(-D)) = 0$. By the Serre duality and Corollary 3.1, we have $h^1(O_X(-D)) = 0$.

We show that $h^1(O_X(2)(-D)) = 0$. Let $\Delta$ be the fixed component of $|2C - D|$. We consider the case where $\Delta = \emptyset$. Since $(2C - D)^2 = 0$, $|2C - D|$ is base point free. By Remark 3.4, there exist a smooth irreducible curve $\hat{D}$ on $X$ and a positive integer $m$ such that $m\hat{D} \in |2C - D|$. Since $mC.\hat{D} = C.(2C - D) = 6$, [9]
if $m \geq 2$, then $(m, C, \tilde{D}) = (2, 3), (3, 2)$ or $(6, 1)$. Since $C, \tilde{D} = 2P_a(\tilde{D}) - 2 \equiv 0 \pmod{2}$, we have $m = 3$. By the exact sequence

\[(4.1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(\tilde{D}) \rightarrow \mathcal{O}_D(\tilde{D}) \rightarrow 0,\]

we have $h^0(\mathcal{O}_X(\tilde{D})) = 2$. Since $|\tilde{D} - C| = \emptyset$ and $C, \tilde{D} = 2$, this contradicts Remark 3.1, and hence, we have $m = 1$. By the exact sequence (4.1), we have $h^0(\mathcal{O}_X(2)(-D)) = 2$. Since $|D - C| = \emptyset$ and $\chi(\mathcal{O}_X(2)(-D)) = 2$, we have $h^1(\mathcal{O}_X(2)(-D)) = 0$.

We consider the case where $\Delta \neq \emptyset$. We set $\tilde{D} = 2C - D - \Delta$ and assume that $h^1(\mathcal{O}_X(2)(-D)) \neq 0$. Since $|D - C| = \emptyset$, we have $h^0(\mathcal{O}_X(2)(-D)) \geq 3$. By the ampleness of $C$, we have $C, \tilde{D} \leq 5$. By Remark 3.1, we have $C, \tilde{D} = 5$ and $h^0(\mathcal{O}_X(\tilde{D})) = 3$. Since $C, \Delta = 1$, by Lemma 3.1, $\Delta^2 = -3$. Since $(2C - D)^2 = 0$, we have $\tilde{D}^2 + 2\tilde{D}, \Delta = 3$. Since $\tilde{D}, \Delta \geq 0$ and $\tilde{D}^2$ is a positive odd number, $\tilde{D}^2 = 1$ or 3. Since $C, (C - \tilde{D}) = 0$, we have $|C - \tilde{D}| = \emptyset$, and hence, $\chi(\mathcal{O}_X(D)) \leq 3$. This means that $\tilde{D}^2 = 1$. By Proposition 3.5, we can assume that $\tilde{D}$ is smooth and irreducible. By the exact sequence

\[0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(\tilde{D}) \rightarrow \mathcal{O}_D(\tilde{D}) \rightarrow 0,\]

we have $h^0(\mathcal{O}_D(\tilde{D})) = 2$. However, since $\tilde{D}^2 = 1$ and $P_a(\tilde{D}) > 0$, this is a contradiction. Hence, $h^1(\mathcal{O}_X(2)(-D)) = 0$. Therefore, $D$ is an aCM curve.

If a curve $D$ is linked to an aCM curve by a complete intersection of $X$ and a hypersurface in $\mathbb{P}^3$, $D$ is also aCM. Hence, we have the following assertion.

**Corollary 4.1** If $P_a(D) = 4$ and $C, D = 6$, then $D$ is an aCM curve.

**Proof.** Since $(D - C)^2 = -7$ and $C, (D - C) = 1$, by Lemma 3.1, we have $|D - C| = \emptyset$. Since $\chi(\mathcal{O}_X(2)(-D)) = 1$, we have $|2C - D| \neq \emptyset$. If we take $\tilde{D} \in |2C - D|$, $P_a(\tilde{D}) = 1$ and $C, D = 4$. By Proposition 4.2, $\tilde{D}$ is an aCM curve. Hence, $D$ is also an aCM curve.

**Proposition 4.3** If $P_a(D) = 3$ and $C, D = 5$, then $D$ is an aCM.

**Proof.** Since $C, (D - C) = 0$ and $D$ is not linearly equivalent to $C$, we have $|D - C| = \emptyset$ and $|C - D| = \emptyset$. Hence, $h^0(\mathcal{O}_X(D)) \leq \chi(\mathcal{O}_X(D)) = 2$. Since $D^2 = -1$, $|D|$ has a fixed component. We denote it by $\Delta$, and set $\tilde{D} = D - \Delta$. Then we have $C, \tilde{D} \leq 4$. By Remark 3.1, we have $C, \tilde{D} = 4$ and $h^0(\mathcal{O}_X(D)) = 2$, and hence, $h^1(\mathcal{O}_X(1)(-D)) = 0$. Since $|D - C| = \emptyset$, by Corollary 3.1, we have $h^1(\mathcal{O}_X(-D)) = 0$.

On the other hand, since $C, (2C - D) = 5$, $(2C - D)^2 = -1$, and $|D - C| = \emptyset$, we have $|2C - D| \neq \emptyset$. By the same reason as above, we have $h^1(\mathcal{O}_X(2)(-D)) = 0$ and $h^1(\mathcal{O}_X(3)(-D)) = h^1(\mathcal{O}_X(D)(-2)) = 0$. Since $C, D + 5 < 15$, by Proposition 4.1, $D$ is aCM.
Proposition 4.4 Assume that $P_a(D) = 6$ and $C.D = 7$. Then the following conditions are equivalent.

(a) $D$ is not an aCM curve.
(b) There exist two lines $\Gamma_1$ and $\Gamma_2$ on $X$ with $\Gamma_1.\Gamma_2 = 0$ and $C + \Gamma_1 + \Gamma_2 \in |D|$.

Proof. (b) $\implies$ (a). Since the member of $|D - C|$ is not 1-connected, $D$ is not an aCM curve.

(a) $\implies$ (b). Since $(C - D).C = -2$, we have $|C - D| = \emptyset$. Since $(2C - D).C = 3 < 4$, by Remark 3.1, if $|2C - D| \neq \emptyset$, we have $h^0(\mathcal{O}_X(2)(-D)) = 1$. In this case, by Theorem 1.1, $D$ is aCM. Hence, we have $|2C - D| = \emptyset$. By the Serre duality and the Riemann-Roch theorem, $h^0(\mathcal{O}_X(D)(-1)) \geq \chi(\mathcal{O}_X(2)(-D)) = 1$.

By Example 3.1, the non-aCM curve as in Proposition 4.4 can be constructed on the Fermat quintic hypersurface in $\mathbb{P}^3$.

Proposition 4.5 Assume that $P_a(D) = 11$ and $C.D = 10$. Then the following conditions are equivalent.

(a) $D$ is not an aCM curve.
(b) $|D - C| \neq \emptyset$ or $|3C - D| \neq \emptyset$.

Remark 4.1 In Proposition 4.5, since $(2C - D).C = 0$ and $D$ is not linearly equivalent to $2C$, we have $|2C - D| = \emptyset$ and $|D - 2C| = \emptyset$. Since $\chi(\mathcal{O}_X(2)(-D)) = 0$, the condition that $h^1(\mathcal{O}_X(2)(-D)) = 0$ is equivalent to $|D - C| = \emptyset$. Moreover, since $\chi(\mathcal{O}_X(3)(-D)) = 0$, $h^1(\mathcal{O}_X(3)(-D)) = 0$ if and only if $|3C - D| = \emptyset$.

Proof of Proposition 4.5. (b) $\implies$ (a). By Remark 4.1, the assertion is clear.

(a) $\implies$ (b). Assume that $|D - C| = \emptyset$ and $|3C - D| = \emptyset$. By Remark 4.1, we have $h^1(\mathcal{O}_X(D)(-2)) = h^1(\mathcal{O}_X(D)(-3)) = 0$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D)(-2) \longrightarrow \mathcal{O}_X(D)(-1) \longrightarrow \mathcal{O}_C(D)(-1) \longrightarrow 0,$$

we have $h^0(\mathcal{O}_C(D)(-1)) = 0$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D)(-3) \longrightarrow \mathcal{O}_X(D)(-2) \longrightarrow \mathcal{O}_C(D)(-2) \longrightarrow 0,$$

we have $h^1(\mathcal{O}_X(4)(-D)) = h^1(\mathcal{O}_X(D)(-3)) = 0$. Since $C.(2C - D) = 0$, if $|\mathcal{O}_C(2)(-D)| \neq \emptyset$, then we have $\mathcal{O}_C(D) \cong \mathcal{O}_C(2)$. This contradicts the fact that $h^0(\mathcal{O}_C(D)(-1)) = 0$. Hence, we have $h^0(\mathcal{O}_C(2)(-D)) = 0$. 

Non-aCM curves on $X$ can be explicitly written. From now on we give a necessary and sufficient condition for $D$ to be a non-aCM curve.
On the other hand, since $|D - C| = \emptyset$, by Remark 4.1, $h^1(\mathcal{O}_X(2)(-D)) = 0$. By the exact sequence 

$$0 \rightarrow \mathcal{O}_X(1)(-D) \rightarrow \mathcal{O}_X(2)(-D) \rightarrow \mathcal{O}_C(2)(-D) \rightarrow 0,$$

we have $h^1(\mathcal{O}_X(1)(-D)) = 0$. By Corollary 3.1, we have $h^1(\mathcal{O}_X(-D)) = 0$. Since $C.D + 5 < 20$, by Proposition 4.1, $D$ is an aCM curve. □

**Definition 4.1** Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^3$. We say that $\mathcal{F}$ is $m$-regular, if $h^i(\mathcal{F}(m - i)) = 0$ for each $i > 0$. Moreover, we call the minimum number $m$ such that $\mathcal{F}$ is $m$-regular the regularity of $\mathcal{F}$.

There exist non-aCM curves $D$ on $X$ satisfying the condition (b) as in Proposition 4.5. We prepare the following theorems to construct such curves lying on smooth hypersurfaces of degree $d \leq 4$ in $\mathbb{P}^3$.

**Theorem 4.1** ([3, p.100]) If a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$ is $m$-regular, then $\mathcal{F}(m)$ is globally generated.

The regularity of a curve in $\mathbb{P}^3$ is defined as the regularity of the ideal sheaf of it in $\mathbb{P}^3$. By the theorem of Bertini, we have the following assertion.

**Theorem 4.2** ([2, Proposition 5.1 and Remark 5.2]) If the regularity of a curve $D$ in $\mathbb{P}^3$ is $m$, then there exists a reduced and irreducible hypersurface $Y \in |I_D(m)|$ which is smooth at any point of $Y \setminus \text{Supp}(D)$. In particular, if $D$ is reduced and the embedding dimension at any point of $D$ is at most 2, then we can take such a surface $Y$ to be smooth.

In Theorem 4.2, the embedding dimension of $D$ at $P$ is defined as the dimension of the Zariski tangent space of $D$ at $P$. If $D$ is a reduced divisor on a smooth hypersurface in $\mathbb{P}^3$, then the embedding dimension at any point of $D$ is at most 2.

**Example 4.1** Let $Y$ be a smooth cubic surface in $\mathbb{P}^3$. Let $\pi : Y \rightarrow \mathbb{P}^2$ be a blow up at six points $P_1, \ldots, P_6$ on $\mathbb{P}^2$ in general position. We set $E_i = \pi^{-1}(P_i)$ $(1 \leq i \leq 6)$, and let $l$ be the total transform of a line on $\mathbb{P}^2$ by $\pi$. Let $H_Y$ be the hyperplane class of $Y$. Then $-K_Y = H_Y$ and $H_Y \in |3l - \sum_{i=1}^{6} E_i|$.

We take a reduced divisor $\tilde{D} \in |H_Y + E_1 + E_2|$. Then $P_6(\tilde{D}) = 1$ and $H_Y.\tilde{D} = 5$. Moreover, $\tilde{D}$ has regularity $\leq 5$, in the sense of Castelnuovo-Mumford. Indeed, since $|H_Y - E_i|$ is an base point free pencil on $Y$ for $1 \leq i \leq 6$, the linear system $|4H_Y - E_1 - E_2|$ is base point free and ample. Since $h^1(O_Y(4)(-\tilde{D})) = h^1(K_Y(4)(-E_1 - E_2)) = 0$, by the exact sequence 

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow I_{\tilde{D}}(4) \rightarrow \mathcal{O}_Y(4)(-\tilde{D}) \rightarrow 0,$$

we have
we have $h^1(\mathcal{I}_D(4)) = 0$. Since $h^2(\mathcal{O}_Y(3)(-\hat{D})) = h^0(\mathcal{O}_Y(-3)(E_1 + E_2)) = 0$, by the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{I}_D(3) \rightarrow \mathcal{O}_Y(3)(-\hat{D}) \rightarrow 0,$$

we have $h^2(\mathcal{I}_D(3)) = 0$. Since $h^3(\mathcal{O}_{\mathbb{P}^3}(-1)) = h^3(K_{\mathbb{P}^3}(3)) = 0$, by the similar reason as above, we have $h^3(\mathcal{I}_D(2)) = 0$. Hence, by Theorem 4.1, $\mathcal{I}_D(5)$ is generated by the global sections of it. Since $\hat{D}$ has embedding dimension $\leq 2$ at any point, by Theorem 4.2, there exists a smooth quintic hypersurface $X$ in $\mathbb{P}^3$ containing $\hat{D}$. Let $D_1$ be a curve on $X$ which is linked to $\hat{D}$ by the complete intersection $X \cap Y$. Then the degree of $D_1$ is 10 and $P_a(D_1) = 11$. On the other hand, if we take a curve $D_2 \in |\mathcal{O}_X(\hat{D})(1)|$, $D_2$ also has the same degree and the arithmetic genus as $D_1$.

**Proposition 4.6** Assume that $P_a(D) = 3$ and $C.D = 6$. Then the following conditions are equivalent.

(a) $D$ is not an aCM curve.

(b) One of the following cases occurs.

   (b₁) $|2C - D| \neq \emptyset$

   (b₂) There exist an effective divisor $\hat{D}$ on $X$ with $P_a(\hat{D}) = 3$ and $C.\hat{D} = 4$, and two lines $\Gamma_1$ and $\Gamma_2$ on $X$ such that $\Gamma_1.\Gamma_2 = 0$ and $D + \Gamma_1 + \Gamma_2 \in |D|$.

   (b₃) There exist an effective divisor $\hat{D}$ on $X$ with $P_a(\hat{D}) = 3$ and $C.\hat{D} = 4$, and a divisor $\Delta$ on $X$ such that $\Delta^2 = -4$ and $\hat{D} + \Delta \in |D|$.

**Remark 4.2** Let $D$ be as in Proposition 4.6. Since $C.(D - C) = 1$ and $(D - C)^2 = -9$, by Lemma 3.1, $|D - C| = \emptyset$. Since $\chi(\mathcal{O}_X(2)(-\hat{D})) = 0$, by the Serre duality, $h^1(\mathcal{O}_X(2)(-\hat{D})) = 0$ if and only if $|2C - D| = \emptyset$.

On the other hand, since $C.(C - D) = -1$, we have $|C - D| = \emptyset$. By the Serre duality, we have $h^2(\mathcal{O}_X(D)) = 0$. Since $\chi(\mathcal{O}_X(D)) = 1$, by the Riemann-Roch theorem, $h^1(\mathcal{O}_X(D)) = 0$ if and only if $h^0(\mathcal{O}_X(D)) = 1$.

**Proof of Proposition 4.6.** (b) $\implies$ (a). Assume $|2C - D| \neq \emptyset$. Then, by Remark 4.2, $D$ is not aCM. Assume that $D$ satisfies the condition (b₂) or (b₃) as in (b). An effective divisor $\hat{D}$ on $X$ with $P_a(\hat{D}) = 3$ and $C.\hat{D} = 4$ is a plane quartic. Indeed, since $C.(C - D) = 1$ and $(C - D)^2 = -3$, by Lemma 3.1, we have $|C - \hat{D}| \neq \emptyset$. Since $|\hat{D}|$ is a pencil on $X$, $h^0(\mathcal{O}_X(D)) \geq 2$. By Remark 4.2, this implies that $h^1(\mathcal{O}_X(D)) \neq 0$, and hence, $D$ is not aCM.

(a) $\implies$ (b). Assume that $D$ does not satisfy the condition (b). Then we show that $D$ is aCM. First of all, we show that $h^1(\mathcal{O}_X(D)) = 0$. Assume that $h^1(\mathcal{O}_X(D)) \neq 0$. By Remark 4.2, we have $h^0(\mathcal{O}_X(D)) \geq 2$. Since $D^2 = -2$, $|D|$ has a fixed component $\Delta$. We set $\hat{D} = D - \Delta$. Since $|D - C| = \emptyset$, we have $|\hat{D} - C| = \emptyset$. By the ampleness of $C$, $C.\Delta \geq 1$ and hence, $C.\hat{D} \leq 5$. Since $h^0(\mathcal{O}_X(\hat{D})) \geq 2$, by Remark 3.1, we have $C.\hat{D} \geq 4$. Therefore, we have

$$(C.\hat{D}, C.\Delta) = (5, 1) \text{ or } (4, 2).$$
We consider the case where \((C, \tilde{D}, C, \Delta) = (5, 1)\). By Lemma 3.1, we have \(\Delta^2 = -3\). Since \(D^2 = -2\) and \(\tilde{D}\) is numerical effective, we have \((\tilde{D}^2, \tilde{D}, \Delta) = (1, 0)\). Since \(C, (C - \tilde{D}) = 0\) and \(\tilde{D}\) is not linearly equivalent to \(C\), we have \([C - \tilde{D}] = 0\). By the Riemann-Roch theorem, we have \(h^0(\mathcal{O}_X(\tilde{D})) \geq \chi(\mathcal{O}_X(\tilde{D})) = 3\). Since \(C, \tilde{D} = 5\), by Remark 3.1, we have \(h^0(\mathcal{O}_X(\tilde{D})) = 3\) and \(h^1(\mathcal{O}_X(\tilde{D})) = 0\). By Proposition 3.5, we can assume that \(\tilde{D}\) is smooth and irreducible. By the exact sequence

\[
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(\tilde{D}) \longrightarrow \mathcal{O}_{\tilde{D}}(\tilde{D}) \longrightarrow 0,
\]
we have \(h^0(\mathcal{O}_{\tilde{D}}(\tilde{D})) = 2\). Since \(\tilde{D}^2 = 1\) and \(P_a(\tilde{D}) > 0\), this is a contradiction.

We consider the case where \((C, \tilde{D}, C, \Delta) = (4, 2)\). Since \(h^0(\mathcal{O}_X(\tilde{D})) \geq 2\), by Remark 3.1, we have \(h^0(\mathcal{O}_X(\tilde{D})) = 2\). Since \(\tilde{D}^2 \geq 0\), we have \(\chi(\mathcal{O}_X(\tilde{D})) \geq 3\). Hence, by the Riemann-Roch theorem, we have \([C - \tilde{D}] \neq \emptyset\). Therefore, \(\tilde{D}\) is a plane quartic and hence, we have \(P_a(\tilde{D}) = 3\) and \(C, \tilde{D} = 4\). Since \(C, \Delta = 2\), \(\Delta^2\) is even, and by Remark 3.2, we have \(\Delta^2 \leq -4\). By Lemma 3.2, we have \(\Delta^2 = -4, -6, \text{ or } -12\), and if \(\Delta^2 = -12\), then \(\Delta\) is a double line, that is, there exists a line \(\Gamma\) on \(X\) such that \(\Delta \in [2\Gamma]\). Since \(\tilde{D}^2 = 0\) and \(D^2 = -2\), we have the contradiction \(2\tilde{D} \cdot \Gamma = 5\). If \(\Delta^2 = -4\), then \(\Delta\) is a plane conic, and if \(\Delta^2 = -6\), then \(\Delta\) is a divisor consisting of two skew lines on \(X\). Hence, \(D\) satisfies the condition \((b_2)\) or \((b_3)\) as in \((b)\). This contradicts the first hypothesis. Therefore, we have \(h^1(\mathcal{O}_X(D)) = 0\). Since \([D - C] = [\emptyset]\), by Corollary 3.1, we have \(h^1(\mathcal{O}_X(-D)) = 0\). Moreover, since \([2C - D] = [\emptyset]\), by Remark 4.2, we have \(h^1(\mathcal{O}_X(2)(-D)) = 0\). Since \(C, D + 5 < 15\), by Proposition 4.1, it is sufficient to show that \(h^1(\mathcal{O}_X(3)(-D)) = 0\). First of all, since \([2C - D] = [\emptyset]\), by the exact sequence

\[
0 \longrightarrow \mathcal{O}_X(1)(-D) \longrightarrow \mathcal{O}_X(2)(-D) \longrightarrow \mathcal{O}_C(2)(-D) \longrightarrow 0,
\]
we have \(h^0(\mathcal{O}_C(2)(-D)) = 0\). On the other hand, \(h^0(\mathcal{O}_C(D)(-1)) = 0\). In fact, since \(C, (D - C) = 1\), if \(h^0(\mathcal{O}_C(D)(-1)) > 0\), then there exists a point \(P \in C\) such that \(\mathcal{O}_C(D) \cong \mathcal{O}_C(1)(P)\). This means that \(\mathcal{O}_C(2)(-D) \cong \mathcal{O}_C(1)(-P)\). However, this is a contradiction. Since \(h^1(\mathcal{O}_X(D)(-1)) = h^1(\mathcal{O}_X(2)(-D)) = 0\), by the exact sequence

\[
0 \longrightarrow \mathcal{O}_X(D)(-2) \longrightarrow \mathcal{O}_X(D)(-1) \longrightarrow \mathcal{O}_C(D)(-1) \longrightarrow 0,
\]
we have \(h^1(\mathcal{O}_X(3)(-D)) = h^1(\mathcal{O}_X(D)(-2)) = 0\). Therefore, \(D\) is a\text{CM}.

First of all, we construct a non-aCM curve \(D\) satisfying the condition \((b_1)\) as in Proposition 4.6 \((b)\), by using a curve lying on a smooth quadric in \(\mathbb{P}^3\).

**Example 4.2** Let \(Z\) be a smooth quadric in \(\mathbb{P}^3\). Note that since \(Z \cong \mathbb{P}^1 \times \mathbb{P}^1\), if we let \(L_1\) and \(L_2\) be the classes of two skew lines on \(Z\), then \(\text{Pic}(Z) = \mathbb{Z}L_1 + \mathbb{Z}L_2\). Let \(H_Z\) be the hyperplane class of \(Z\). Then \(H_Z = L_1 + L_2\), and \(K_Z = -2H_Z\). Since \(|H_Z + 2L_2|\) is base point free and big, by the theorem of Bertini, we can take
a smooth irreducible curve $\tilde{D} \in |H_Z + 2L_2|$. Then $P_a(\tilde{D}) = 0$ and $H_Z.\tilde{D} = 4$. Moreover, $\tilde{D}$ has regularity $\leq 5$. Indeed, since $|5H_Z - 2L_2|$ is base point free and big, we have $h^1(O_Z(4)(-\tilde{D})) = h^1(K_Z(5)(-2L_2)) = 0$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow \mathcal{I}_{\tilde{D}}(4) \longrightarrow \mathcal{O}_Z(4)(-\tilde{D}) \longrightarrow 0,$$

we have $h^1(\mathcal{I}_{\tilde{D}}(4)) = 0$. Since $h^2(\mathcal{O}_Z(3)(-\tilde{D})) = h^3(\mathcal{O}_Z(5)(\tilde{D})) = 0$, by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{I}_{\tilde{D}}(3) \longrightarrow \mathcal{O}_Z(3)(-\tilde{D}) \longrightarrow 0,$$

we have $h^2(\mathcal{I}_{\tilde{D}}(3)) = 0$. Moreover, since $h^3(\mathcal{O}_{\mathbb{P}^3}) = 0$, we have $h^3(\mathcal{I}_{\tilde{D}}(2)) = 0$. Hence, by Theorem 4.2, there exists a smooth quintic hypersurface $X$ in $\mathbb{P}^3$ containing $\tilde{D}$. If we let $D$ be a non-zero effective divisor on $X$ which is linked to $\tilde{D}$ by the complete intersection $X \cap Z$, we have $P_a(D) = 3$ and $C.D = 6$.

On the other hand, we can give an example of a non-$a$CM curve $D$ satisfying the condition (b2) or (b3) as in Proposition 4.6 (b), by using two skew lines on the Fermat quintic hypersurface in $\mathbb{P}^3$.

**Example 4.3** Let $X$ be the quintic hypersurface in $\mathbb{P}^3$ defined by the equation $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$. Let $L_1$ and $L_2$ be lines on $X$ defined by the equations $x_0 + x_1 = x_2 + x_3 = 0$ and $x_0 + x_2 = x_1 + \xi x_3 = 0$, respectively. Here, $\xi$ is a primitive fifth root of unity. Note that $L_1$ and $L_2$ do not intersect (see Example 3.1). Moreover, let $\tilde{C}$ be the hyperplane section of $X$ defined by $x_1 + x_2 = 0$, and let $\Gamma$ be the line defined by the equation $x_1 + x_2 = x_0 + \xi^4 x_3 = 0$. Since $L_i.\Gamma = 0$ and $\tilde{C}.L_i = 1$ ($i = 1, 2$), $\tilde{C} - \Gamma$ is a plane quartic curve such that $(\tilde{C} - \Gamma).L_i = 1$ ($i = 1, 2$). Hence, if we set $\tilde{D} = \tilde{C} - \Gamma$ and $D = \tilde{D} + L_1 + L_2$, then $D$ is a non-$a$CM curve corresponding to the case (b2) as in Proposition 4.6 (b).

On the other hand, let $L_3$ be the line on $X$ defined by the equation $x_0 + x_1 = x_2 + \xi^4 x_3 = 0$. Since $L_3.\Gamma = 1$, we have $L_3.\tilde{D} = 0$. The divisor $L_1 + L_3$ is a plane conic which is contained in the hyperplane section of $X$ defined by the equation $x_0 + x_1 = 0$. Since $(L_1 + L_3).\tilde{D} = 1$, if we set $D = \tilde{D} + L_1 + L_3$, $D$ is a non-$a$CM curve corresponding to the case (b3) as in Proposition 4.6 (b).

**Corollary 4.2** If $P_a(D) = C.D = 9$, then the following conditions are equivalent.

(a) $D$ is not an $a$CM curve.

(b) One of the following cases occurs.

(b1) $|D - C| \neq \emptyset$

(b2) There exist an effective divisor $\tilde{D}$ on $X$ with $P_a(\tilde{D}) = 3$ and $C.\tilde{D} = 4$, and two lines $\Gamma_1$ and $\Gamma_2$ on $X$ such that $\Gamma_1.\Gamma_2 = 0$ and $\tilde{D} + \Gamma_1 + \Gamma_2 \in |3C - D|$.

(b3) There exist an effective divisor $\tilde{D}$ on $X$ with $P_a(\tilde{D}) = 3$ and $C.\tilde{D} = 4$, and a divisor $\Delta$ on $X$ such that $\Delta^2 = -4$ and $\tilde{D} + \Delta \in |3C - D|$.
Proposition 4.7 Assume that Corollary 4.2, by Example 4.2, Example 4.3, and the proof of Corollary 4.2. Note that we can construct non-aCM curves satisfying the condition (b) as in Proposition 4.6, we have the assertion. □

Note that we can construct non-aCM curves satisfying the condition (b) as in Corollary 4.2, by Example 4.2, Example 4.3, and the proof of Corollary 4.2.

Proposition 4.7 Assume that \( P_a(D) = 5 \) and \( C.D = 7 \). Then the following conditions are equivalent.

(a) \( D \) is not an aCM curve.

(b) There exist effective divisors \( \Gamma_1 \) and \( \Gamma_2 \) on \( X \) with \( C.\Gamma_i = i \) \( (i = 1, 2) \), \( \Gamma_1^2 = -3 \), \( \Gamma_2^2 = -4 \), \( \Gamma_1 + \Gamma_2 = 0 \), and \( \Gamma_1 + \Gamma_2 \in |2C - D| \).

Proof. (b) \( \implies \) (a). Since the member of \( |2C - D| \) is not 1-connected, \( D \) is not aCM.

(a) \( \implies \) (b). Since \( (D - C)^2 = -8 \) and \( C.(D - C) = 2 \), by Lemma 3.2, we have \( |D - C| = \emptyset \). Assume that \( D \) does not satisfy the condition (b). Then we show that \( D \) is aCM. First of all, we show that \( h^1(O_X(D)) = 0 \). Since \( C.(C - D) = -2 \), we have \( |C - D| = \emptyset \). Assume that \( h^1(O_X(D)) \neq 0 \). Then, by the Riemann-Roch theorem, we have \( h^0(O_X(D)) \geq 3 \). Let \( \Delta \) be the fixed component of \( |D| \), and assume that it is not empty. We set \( \tilde{D} = D - \Delta \). Since \( C.\Delta \geq 1 \), we have \( C.\tilde{D} \leq 6 \). By Remark 3.1, we have \( C.\tilde{D} - 2 \geq h^0(O_X(\tilde{D})) \geq 3 \). Hence, \( C.\tilde{D} = 5 \) or \( 6 \) (resp. \( C.\Delta = 2 \) or \( 1 \)). Since \( C.(C - \tilde{D}) \leq 0 \) and \( \tilde{D} \) is not linearly equivalent to \( C \), we have \( |C - \tilde{D}| = \emptyset \). Therefore, since \( C.\tilde{D} - 2 \geq \chi(O_X(\tilde{D})) \), we have

\[
3C.\tilde{D} - 14 \geq \tilde{D}^2.
\]

We consider the case where \( C.\tilde{D} = 6 \). Since \( C.\Delta = 1 \), by Lemma 3.1, we have \( \Delta^2 = -3 \). Moreover, \( \tilde{D}^2 \) is even and, by the inequality (4.2), we have \( 0 \leq \tilde{D}^2 \leq 4 \). Assume that \( \tilde{D}^2 = 4 \). Since \( C.(\tilde{D} - C) = 1 \) and \( (\tilde{D} - C)^2 = -3 \), by Lemma 3.1, \( |\tilde{D} - C| \neq \emptyset \). This contradicts the fact that \( |D - C| = \emptyset \).

Assume that \( \tilde{D}^2 = 2 \). Since \( |\tilde{D} - C| = \emptyset \), by the Riemann-Roch theorem, we have \( h^0(O_X(2)(-\tilde{D})) \geq \chi(O_X(2)(-\tilde{D})) = 2 \). Hence, we take \( D_0 \in |2C - \tilde{D}| \). Since \( |D_0 - C| = |C - \tilde{D}| = \emptyset \), by Remark 3.1, we have \( C.D_0 = 4 \) and \( h^0(O_C(D_0)) = 2 \). Hence, \( |O_C(D_0)| \) is a gonality pencil on \( C \). On the other hand, since \( D_0^2 < 0 \), \( |D_0| \) has a fixed component. However, since \( C \) is ample, this is a contradiction.

Assume that \( \tilde{D}^2 = 0 \). Since \( |\tilde{D}| \) is base point free, by Remark 3.4, there exist a smooth irreducible curve \( D_0 \) and a positive integer \( m \) such that \( mD_0 \in |\tilde{D}| \). Since \( C.D_0 = 2P_a(D_0) - 2 \equiv 0 \pmod{2} \), \( m = 1 \) or \( 3 \). If \( m = 1 \), then by the exact sequence

\[
0 \longrightarrow O_X \longrightarrow O_X(D_0) \longrightarrow O_{D_0}(D_0) \longrightarrow 0,
\]
we have \( h^0(\mathcal{O}_X(\hat{D})) = h^0(\mathcal{O}_X(D_0)) = 2 \). Since \( h^0(\mathcal{O}_X(D)) \geq 3 \), this is a contradiction. If \( m = 3 \), we have \( C.D_0 = 2 \). By the exact sequence (4.3), \( h^0(\mathcal{O}_X(D_0)) = 2 \). However, by Remark 3.1, this is a contradiction.

We consider the case where \( C.\hat{D} = 5 \). By the inequality (4.2), we have \( \hat{D}^2 \leq 1 \). Since \( \hat{D}^2 + 5 = 2P_a(\hat{D}) - 2 \), \( \hat{D}^2 \) is an odd number. Hence, we have \( \hat{D}^2 = 1 \). By Proposition 3.5, we can assume that \( \hat{D} \) is smooth and irreducible.

Since \( P_a(\hat{D}) > 0 \), we have \( h^0(\mathcal{O}_D(\hat{D})) = 1 \). Hence, by the exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(\hat{D}) \rightarrow \mathcal{O}_D(\hat{D}) \rightarrow 0,
\]

we have \( h^0(\mathcal{O}_X(\hat{D})) = 2 \). However, this contradicts the fact that \( h^0(\mathcal{O}_X(D)) \geq 3 \). By the above argument, we have \( \Delta = \emptyset \). Since \( D^2 = 1 \), by Proposition 3.5, we can assume that \( D \) is smooth and irreducible, and hence, by the similar reason as above, we have a contradiction. Therefore, we have \( h^1(\mathcal{O}_X(D)) = 0 \). Since \( |D - C| = \emptyset \), by Corollary 3.1, we have \( h^1(\mathcal{O}_X(-D)) = 0 \).

Next, we show that \( h^1(\mathcal{O}_X(2)(-D)) \neq 0 \). Assume that \( h^1(\mathcal{O}_X(2)(-D)) \neq 0 \). Since \( \chi(\mathcal{O}_X(2)(-D)) = 0 \) and \( |D - C| = \emptyset \), we have \( |2C - D| \neq \emptyset \). We take \( \hat{D} \in |2C - D| \). Since \( P_a(\hat{D}) = -1 \), there exists a non-trivial effective decomposition \( \hat{D} = \Gamma_1 + \Gamma_2 \) such that \( \Gamma_1, \Gamma_2 \leq 0 \). Since \( C.\hat{D} = 3 \), we may assume that \( C.\Gamma_i = i (i = 1, 2) \). Then we note that \( \Gamma_2^2 = -3 \), by Lemma 3.1. Hence, we have \( \Gamma_2^2 \geq \Gamma_2^2 + 2\Gamma_1.\Gamma_2 = -4 \). By Remark 3.2, we have \( \Gamma_2^2 = -4 \). This means that \( D \) satisfies the condition (b) and hence, we have \( h^1(\mathcal{O}_X(2)(-D)) = 0 \).

Since \( C.D + 5 < 15 \), it is sufficient to show that \( h^1(\mathcal{O}_X(3)(-D)) = 0 \). Since \( h^1(\mathcal{O}_X(D)) = 0 \), by the Riemann-Roch theorem, we have \( h^0(\mathcal{O}_X(D)) = 2 \). Since \( |D - C| = \emptyset \) and \( h^1(\mathcal{O}_X(D)(-1)) = h^1(\mathcal{O}_X(2)(-D)) = 0 \), by the exact sequence

\[
0 \rightarrow \mathcal{O}_X(D)(-1) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_C(D) \rightarrow 0,
\]

we have \( h^0(\mathcal{O}_C(D)) = 2 \). Since \( h^0(\mathcal{O}_C(1)) = 3 \), we have \( h^0(\mathcal{O}_C(D)(-1)) = 0 \). Moreover, by the exact sequence

\[
0 \rightarrow \mathcal{O}_X(D)(-2) \rightarrow \mathcal{O}_X(D)(-1) \rightarrow \mathcal{O}_C(D)(-1) \rightarrow 0,
\]

we have \( h^1(\mathcal{O}_X(3)(-D)) = h^1(\mathcal{O}_X(D)(-2)) = 0 \). Hence, \( D \) is aCM.

A non-aCM curve satisfying the condition (b) as in Proposition 4.7 also can be constructed as a divisor on the Fermat quintic hypersurface in \( \mathbb{P}^3 \).

**Example 4.4** Let \( X \) be the quintic hypersurface in \( \mathbb{P}^3 \) defined by the equation \( x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0 \). Let \( C_1 \) and \( C_2 \) be the hyperplane sections defined by the equations \( x_0 + x_1 = 0 \) and \( x_0 + x_2 = 0 \), respectively. Moreover, let \( \Gamma_1 \) be the line on \( X \) defined by the equation \( x_0 + x_1 = x_2 + x_3 = 0 \), and let \( \Gamma_2 \) be the conic on \( X \) defined by the equation \( x_0 + x_2 = (x_1 + \xi x_3)(x_1 + \xi^2 x_3) = 0 \), where \( \xi \) is a primitive fifth root of unity. Since any irreducible component of \( \Gamma_2 \) does not
intersect $\Gamma_1$, we have $\Gamma_1, \Gamma_2 = 0$. Hence, if we set $D = C_1 + C_2 - \Gamma_1 - \Gamma_2$, then $D$ is a non-aCM curve corresponding to the case (b) as in Proposition 4.7.

**Corollary 4.3** Assume that $P_a(D) = 7$ and $C.D = 8$. Then the following conditions are equivalent.

(a) $D$ is not an aCM curve.

(b) There exist effective divisors $\Gamma_1$ and $\Gamma_2$ on $X$ with $C.\Gamma_i = i$ ($i = 1, 2$), $\Gamma_1^2 = -3, \Gamma_2^2 = -4$, $\Gamma_1, \Gamma_2 = 0$, and $\Gamma_1 + \Gamma_2 \in |D - C|$.

**Proof.** First of all, since $C.(D - 2C) = -2$, we have $|D - 2C| = \emptyset$. Since $\chi(O_X(3)(-D)) = 2$, we have $|3C - D| \neq \emptyset$. If we take $D \in |3C - D|$, since $P_a(D) = 5$ and $C.D = 7$, by Proposition 4.7, we have the assertion. □

If $D$ is the divisor on the Fermat quintic hypersurface $X$ in $\mathbb{P}^3$ as in Example 4.4, we can construct a non-aCM curve satisfying the condition (b) as in Corollary 4.3 as a curve which is linked to $D$ by the complete intersection of $X$ and a cubic hypersurface in $\mathbb{P}^3$ containing $D$.

**Proposition 4.8** Assume that $P_a(D) = 2$ and $C.D = 5$. Then the following conditions are equivalent.

(a) $D$ is not an aCM curve.

(b) There exist a line $\Gamma$ and an effective divisor $\tilde{D}$ on $X$ with $C.\tilde{D} = 4$, $P_a(\tilde{D}) = 3$, $\tilde{D}, \Gamma = 0$, and $\tilde{D} + \Gamma \in |D|$ or $|2C - D|$.

**Proof.** (b) $\implies$ (a). If $D$ satisfies the condition (b), the members of $|D|$ or $|2C - D|$ are not 1-connected. Hence, $D$ is not aCM.

(a) $\implies$ (b). Since $(C - D).C = 0$ and $P_a(D) = 2$, we have $|C - D| = |D - C| = \emptyset$. Assume that $D$ does not satisfy the condition (b). Then we show that $D$ is an aCM curve. First of all, we show that $h^1(O_X(D)) = 0$. Assume that $h^1(O_X(D)) \neq 0$. Since $\chi(O_X(D)) = 1$, we have $h^0(O_X(D)) \geq 2$. Since $D^2 = -3$, the fixed component of $|D|$ is not empty. Hence, let $\Gamma$ be the fixed component of $|D|$, and we take $\tilde{D} \in |D - \Gamma|$. Since $C.\tilde{D} \leq 4$, by Remark 3.1, we have $C.\tilde{D} = 4$ and $C.\Gamma = 1$. Hence, $\Gamma$ is a line on $X$. Since $\tilde{D}^2 + 2\tilde{D}.\Gamma = 0$, $\tilde{D}^2 \geq 0$, and $\tilde{D}.\Gamma \geq 0$, we have $\tilde{D}^2 = \tilde{D}.\Gamma = 0$. In this case, $D$ satisfies the condition (b). Hence, we have $h^1(O_X(D)) = 0$. Since $|D - C| = \emptyset$, by Corollary 3.1, we have $h^1(O_X(-D)) = 0$. Since $\chi(O_X(2)(-D)) = 1$, by the Serre duality and the Riemann-Roch theorem, we have $|2C - D| \neq \emptyset$. If we take a member $D_0 \in |2C - D|$, then $P_a(D_0) = 2$ and $C.D_0 = 5$. Hence, by symmetry, we have $h^1(O_X(2)(-D)) = 0$ and $h^1(O_X(3)(-D)) = h^1(O_X(D)(-2)) = 0$. Since $C.D + 5 < 15$, by Proposition 4.1, $D$ is aCM. □

**Example 4.5** Let $X$ be the quintic hypersurface in $\mathbb{P}^3$ defined by the equation $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$. Let $\Gamma$ and $\tilde{\Gamma}$ be lines on $X$ defined by the equations
$x_0 + x_1 = x_2 + x_3 = 0$ and $x_0 + x_2 = x_1 + x_3 = 0$, respectively. Note that $\Gamma$ and $\tilde{\Gamma}$ intersect at one point. If we let \( \tilde{C} \) be the hyperplane section of $X$ defined by the equation $x_0 + x_2 = 0$, then $\tilde{C}.\Gamma = 1$. Hence, we have $(\tilde{C} - \tilde{\Gamma}).\Gamma = 0$.

We set $\tilde{D} = \tilde{C} - \tilde{\Gamma}$ and $D = \tilde{D} + \Gamma$. Then $P_a(D) = 2$ and $\tilde{C}.D = 5$. Let $C$ be the hyperplane section defined by the equation $x_0 + x_1 = 0$, and we set $D_0 = C + \tilde{C} - D$. Then $D_0$ has the same arithmetic genus and degree as $D$.

The non-aCM curves satisfying the condition (b) as in Proposition 4.8 (b) can not be constructed on any smooth quartic hypersurface in $\mathbb{P}^3$. In fact, a curve $\tilde{D}$ with $P_a(\tilde{D}) = 3$ and $\deg \tilde{D} = 4$ on a smooth quartic hypersurface $Y$ in $\mathbb{P}^3$ is a hyperplane section of $Y$, and hence, any curve on $Y$ intersects $\tilde{D}$. Proposition 4.8 means that a curve of arithmetic genus 2 and degree 5 is not necessarily aCM if it does not lie on any smooth hypersurface of degree $d \leq 4$ in $\mathbb{P}^3$.

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