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Abstract. We define linear and quadratic coherent states for the supersymmetric partners of the quantum infinite well through formal series expansions of the energy eigenfunctions of the systems and we study the appropriateness of this definitions as coherent states by means of their properties. In particular, we examine the localization in position and time evolution, minimum uncertainty relations and the behavior of the Wigner function.

1. Introduction

The concept of coherent state was introduced since the appearance of quantum mechanics on rather heuristic grounds, namely that a coherent state is a quantum state whose behavior is classical nonetheless [1]. A unique definition of such states remains elusive and a set of inequivalent definitions for a general system have been proposed in several works [2–6].

These definitions have lead to the construction of states that have shown to be most fruitful when studying quantum systems. A reason for this is the good localization in position that they tend to posses. The use of coherent states has included systems obtained through the supersymmetric technique [7–9]. This technique allows one to obtain a multi-parametric family of solvable Hamiltonians, departing from an initial one for whose stationary Schrödinger solution is known [10–19]. The thus obtained Hamiltonians are called supersymmetric (SUSY) partners of the initial one. Although the energy spectra of the SUSY partners are known through the technique, several other properties need to be understood once the Hamiltonian is obtained. This is where coherent states provide a better understanding of the dynamics of the system.

The SUSY partners are particularly interesting systems in that their spectrum is not connected in general by the action of the natural ladder operators given immediately by the technique [20–23]. Thus it is challenging to use those definitions of coherent states which rely on ladder operators. Several treatments have been developed to deal with this difficulty mainly focusing on the action of the various alternatives for constructing ladder operators of the system and in studying the formal series expansions of such states. As we will shortly see the SUSY partners of the infinite well [8, 9] (also known as the problem of a particle in a box) are paradigmatic yet simple examples of this behavior.
In this work we will define coherent states for the SUSY partners of the one-dimensional infinite well based on formal series expansions and the idea of good localization in position of these states.

This paper is organized as follows. In Sec. 2 we describe the supersymmetric partners of the one-dimensional infinite well. In Sec. 3 we give a brief enumeration of the alternatives for defining the coherent states of a quantum system. We thus concentrate our attention on two possible choices, i.e. the linear and quadratic coherent states for the SUSY infinite well and we examine their properties. In Sec. 4 we present our concluding remarks.

2. A family of SUSY partners of the 1D infinite well

It is well-known that the quantum infinite well may be described by the Hamiltonian

$$H = -\hbar^2 \frac{d^2}{dx^2} + V(x)$$

where the potential $V(x)$ is taken to be

$$V(x) = \begin{cases} 0, & 0 < x < \pi \\ \infty, & \text{otherwise} \end{cases}$$

(1)

The stationary eigenstates and the discrete energies of this system are

$$\psi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \quad E_n = \frac{\hbar^2 n^2}{2M}, \quad n = 1, 2, \ldots$$

(2)

In the following, we will use dimensionless units, setting $\hbar = 1$, $M = 1/2$.

SUSY partners of the infinite well have been constructed [7] starting from the usual intertwining relations

$$\tilde{H}Q = Q \hat{H}, \quad Q^\dagger \tilde{H} = \hat{H}Q^\dagger,$$

(3)

which involve the supercharges

$$Q = \frac{d^2}{dx^2} + \eta(x) \frac{d}{dx} + \epsilon + \frac{1}{2}(\eta^2(x) - \eta'(x))$$

(4)

and its adjoint $Q^\dagger$ where $\epsilon$ is an arbitrary constant. In the so-called confluent case [8], the technique produces a family of new potentials

$$\tilde{V}(x; k, \omega) = 2\eta'(x) = \begin{cases} \frac{32k^2 \sin(kx) \sin(2kx) + k(\pi \omega - x) \cos(2kx)}{\sin^2(2kx) + 2k(\pi \omega - x)^2}, & 0 < x < \pi \\ \infty, & \text{otherwise} \end{cases}$$

(5)

with two real parameters $k, \omega$. The first must satisfy the condition $\epsilon = k^2$ with $k = 1, 2, \ldots$ while $\omega$ is an arbitrary constant.

Figure 1. SUSY potential as a function of $x$. 
Notice that the parameter \( k \) takes only positive integer values. For these potentials to be non singular, the admissible values of \( \omega \) are \( \omega \in \mathbb{R} \setminus \{0\} \cup \{1, \infty\} \). The parameter \( k \) controls the amount of oscillations and the potentials are invariant under the combined change \( \omega \to 1 - \omega \) and \( x \to \pi - x \). Two instances are illustrated on Fig. 1.

The normalized SUSY eigenstates \( \tilde{\psi}_n(x) \) are obtained from the intertwining relations (3) as:

\[
\tilde{\psi}_n(x; k, \omega) = \left(k^2 - n^2\right)^{-1/2} Q\psi_n(x), \quad (n \neq k).
\]

For \( n \neq k \), they are physical states, i.e. they are normalizable and such that \( \tilde{\psi}_n(0; k, \omega) = \tilde{\psi}_n(\pi; k, \omega) = 0 \), since \( \eta(0; k, \omega) = \eta(\pi; k, \omega) = 0 \). The corresponding energies are \( E_n = n^2 \) as in the non SUSY case.

For \( n = k \), we get \( Q\psi_k(x) = 0 \). The missing state is found \([8]\) to be

\[
\tilde{\psi}_k(x; k, \omega) = \sqrt{\frac{2}{\pi}} \sin(kx) \frac{2\pi k \sqrt{\omega(\omega - 1)}}{\sin(2kx) + 2k(\pi \omega - x)}.
\]

It is normalizable and such that \( \tilde{\psi}_k(0, k, \omega) = \tilde{\psi}_k(\pi, k, \omega) = 0 \). With this additional state the spectrum of \( \tilde{H} \) is complete.

\[\text{Figure 2. Potential of a SUSY infinite well with its first four energy eigenstates.}\]

Fig. 2 shows the spectrum of a SUSY partner of the infinite well where the second excited state \( k = 3 \) was used to produce the transformation. The base state and the first three excited states are shown too. For these states we have displaced the axis to their corresponding eigen-energies.

It is worth noticing that, since the infinite well possesses ladder operators \( A^+ \) and \( A^- \) \([24]\), then there are natural ladder operators for \( \tilde{H} \) given by

\[
L^\pm = QA^\pm Q^\dagger.
\]

3. Different sets of coherent states for the SUSY partners

For an arbitrary system, it is well-known there are distinct definitions of coherent states which in general yield to inequivalent states. Among these definitions we find the following ones.

- A Barut-Girardello coherent state \( \Phi_{Ge} \) is that which satisfies the following eigenvalue equation for an annihilation operator \( a \) of the system:

\[
a\Phi_{Ge}(z; x) = z\Phi_{Ge}(z; x), \quad z \in \mathbb{C}.
\]
A Gilmore-Perelomov coherent state is obtained through the action of the displacement operator \( D(z) = \exp(z a^\dagger - z^* a) \) on an extremal state \( \psi_0 \)

\[
D(z)\psi_0(x) = \exp(z a^\dagger - z^* a)\psi_0(x), \quad z \in \mathbb{C}.
\]

A coherent state \( \Phi_{\text{MUR}} \) is such that it minimizes the uncertainty relation

\[
\sigma_x\sigma_p = \frac{1}{2}, \quad \sigma_p^2 = \langle \Phi_{\text{MUR}} | O^2 | \Phi_{\text{MUR}} \rangle - \langle \Phi_{\text{MUR}} | O | \Phi_{\text{MUR}} \rangle^2.
\]

where \( x \) is the position and \( p \) is the momentum of the particle described by the system and \( \langle \Phi | O | \Phi \rangle \) is the mean value of the operator \( O \) when the system is in the state \( |\Phi\rangle \).

Notice that the first two definitions require that the system possesses ladder operators. For a solvable quantum system with Hamiltonian \( H_0 \) having discrete energy spectrum (finite or infinite), ladder operators may be defined such that they act on the energy eigenstates \( \psi_n(x) \) as

\[
a^- \psi_n(x) = \sqrt{f(n-1)} \psi_{n-1}(x), \quad a^+ \psi_n(x) = \sqrt{f(n)} \psi_{n+1}(x), \quad n = 1, 2, \ldots,
\]

which leads to a generalized Heisenberg algebra \((N\psi_n(x) \equiv n\psi_n(x))\):

\[
[a^-, N] = a^-, \quad [a^+, N] = -a^+, \quad [a, a^+] = f(N) - f(N-1),
\]

for a positive function \( f(n) \) such that \( f(0) = 0 \). Operatorial definitions of such ladder operators are well-known. For the case of the infinite well, see for example [25].

Departing from the action of the ladder operators on the energy eigenstates we can obtain the proposed coherent states for the system of interest as the formal series

\[
\Phi_{\text{Ge}}(z; x) = \sum_{n=1}^{d} c_n(z)\psi_n(x),
\]

where \( d \) is the dimension of the Hilbert space that may be finite or infinite.

The coefficients \( c_n(z) \) are different in the case of the Barut-Girardello definition from those in the Gilmore-Perelomov one and in both cases these coefficients depend on the function \( f(n) \).

When dealing with SUSY transformations it is common to find that the usual construction for the ladder operators of the resulting system is disconnected through the action of these operators. This yields the problem of having a definition of coherent states that takes into account only a proper subset of the basis of the Hilbert space. An alternative that has been proven [7] to work appropriately in this cases is a construction of the ladder operators with a linear function \( f(n) = n \). In such cases the coherent states are defined in each subset which is disconnected by the action of the ladder operators.

For the SUSY partners of the infinite well there are three subsets which are not connected by the action of the ladder operators \( L^\pm = QA^\pm Q^\dagger \). The first is composed by all \( \psi_n \) with \( n < k \), the second has one element, the eigenfunction \( \psi_k \), and the third is given by all \( \psi_n \) with \( n > k \). Nonetheless, let us assume that we can define formally a new set of ladder operators \( L^\pm \) for these systems such that they satisfy (12) and (13), then a generalized coherent state is formally given by

\[
\Phi_{\text{Ge}}(z; x) = \frac{1}{\sqrt{N(z)}} \sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{\rho(n-1)}} \psi_n(x),
\]

where \( 1/\sqrt{N(z)} \) is a normalization constant and \( \rho(n) \) is the generalized factorial of the function \( f(n) \) given by

\[
\rho(n) = \begin{cases} 
1, & \text{if } n = 0 \\
\prod_{i=1}^{n} f(n), & \text{if } n > 0.
\end{cases}
\]
Thus fixing the function \( f(n) \) will fix the coherent states given by (15). In what follows we will study the set of generalized coherent states given by two relevant choices for \( f(n) \), say the linear case \( f(n) = n \) and the quadratic case \( f(n) = n^2 \). Indeed, the first choice leads to the usual action (12) of the oscillator-like ladder operators. The second choice leads to the factorization of the SUSY Hamiltonian \( \tilde{H} = \mathcal{L}^{-} \mathcal{L}^{+} \).

### 3.1. Linear coherent states

When \( f(n) = n \), the linear coherent states of the SUSY partners of the infinite well are

\[
\tilde{\Phi}_L(x; k, \omega; z) = \frac{1}{\sqrt{|N_L|}} \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \tilde{\psi}_n(x; k, \omega),
\]

where \(|N_L| = \exp(|z|^2)|\).

This series can be associated to the standard series for the coherent states of the harmonic oscillator, although in this case the energy eigenfunctions \( \tilde{\psi}_n(x; k, \omega) \) are those of the SUSY partners of the infinite well.

### 3.2. Quadratic coherent states

When \( f(n) = n^2 \), the quadratic coherent states of the SUSY partners of the infinite well are

\[
\tilde{\Phi}_Q(x; k, \omega; z) = \frac{1}{\sqrt{|N_Q|}} \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \tilde{\psi}_n(x; k, \omega),
\]

where \(|N_Q| = I(0, 2|z|)\) and \(I(a, b)\) is the modified Bessel function of the first kind. Due to the commutation relations (13), these coherent states are related to SU(1,1)-coherent states.

### 3.3. Properties

Let us recall that the time evolution of an arbitrary state is given by the action of the evolution operator \( U_t = \exp(-i\tilde{H}t) \). We thus get for our coherent states

\[
U_t \left( \tilde{\Phi}_{Ge} \right) = \frac{1}{\sqrt{N(z)}} \sum_{n=1}^{\infty} \frac{z^{n-1} e^{-n^2t}}{\sqrt{n-1}} \tilde{\psi}_n(x; k, \omega),
\]

possibly up to a phase factor.

![Figure 3](image)

**Figure 3.** Probability density of the linear and quadratic coherent states for the case \( \tilde{V}(x; k = 3), \omega = 2 \) at times \( t = 0, 0.03, 0.06 \) and \( t = 0.09 \) from left to right, respectively.

Fig. 3 gives the probability density \( P_z = |\tilde{\Phi}|^2 \) of both types of coherent states at different times. At \( t = 0 \), the linear coherent states show a better localization in the position than the
quadratic ones. As the states evolve in time the initial localization is lost, however, this effect is stronger in the case of the linear coherent states whose probability density rapidly spreads over the domain of the well.

Figure 4. Dispersion in position $\sigma_x$ (- - -) and momentum $\sigma_p$ (−−−) and their product $\sigma_x\sigma_p$ (_____ for a linear and a quadratic coherent state for $\hat{V}(x; k = 3, \omega = 2)$.

With respect to the third definition given at the beginning of the section, we compute the dispersion in position $\sigma_x$ and momentum $\sigma_p$, as well as their product, for the linear and quadratic coherent states. In order to do the calculations, we make an approximation of the states keeping 100 terms in the series. Fig. 4 shows that the product $\sigma_x\sigma_p$ obeys the uncertainty relation, however it is not minimal for any value of $|z|$ other than $z = 0$. As expected for $z = 0$ the dispersions for the linear and the quadratic coherent states are those of the ground state.

Figure 5. Mean value of $x$ (- - -) and $p$ (−−−) for $\tilde{\Phi}_L$ and $\tilde{\Phi}_Q$ for $\hat{V}(x; k = 3, \omega = 2)$ and $z = 1$.

On the other hand, the mean values of the observables corresponding to position and momentum evolve in time as shown in Fig. 5. We see that they evolve periodically with period equal to $2\pi$, similar to a bouncing between the walls of the well. Although the temporal stability is not fully attained because the eigen-energies are quadratic functions of $n$, periodicity is still expected since the time evolution is a phase shift on these states. The oscillations of these mean values behave similarly in both cases.

Finally, Fig. 6 shows the Wigner function of these coherent states, namely

$$W(x, p) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \tilde{\Phi}^* \left( x + \frac{y}{2}, t; k, \omega; z \right) \tilde{\Phi} \left( x - \frac{y}{2}, t; k, \omega; z \right) e^{-ipy} dy,$$

(20)

where we use fixed values of $t$, $k$, $\omega$ and $z$. Notice that in both cases the Wigner function has regions in its domain where it is negative. On the other hand, notice that these functions are not displaced versions of the Wigner function of the ground state $\tilde{\psi}_1$ which can be seen in Fig 7. These features are in contrast with the usual coherent states of the harmonic oscillator whose Wigner function is non-negative and is found to be that of its ground state translated in the phase space.
Figure 6. Wigner function of $\tilde{\Phi}_L$ and $\tilde{\Phi}_Q$ for the case $\tilde{V}(x; k = 3, \omega = 2)$, $z = 1$ and $t = 0$

Figure 7. Wigner function of $\tilde{\psi}_1$ for the case $\tilde{V}(x; k = 3, \omega = 2)$

4. Conclusion

The eigenfunctions of the supersymmetric partners of the quantum infinite well fail to be completely connected through the action of its natural ladder operators. This makes difficult obtaining coherent states through the usual definitions. Thus we introduced a formal definition of such states given by formal series that could show the expected behavior of a coherent state in terms of an arbitrary positive function. Two natural choices have been considered leading to linear and quadratic coherent states. The linear ones show a better localization in the position at time $t = 0$ but the probability density spreads quickly as time evolves. We also analyzed the time evolution of the mean position and momentum. These mean values oscillate in a similar manner in both states. Finally, the Wigner functions of the linear and quadratic coherent states were found to be negative in regions of their domains and they are not displaced versions of the Wigner function of the ground state. Both of this characteristics differ from those of the usual coherent states of the harmonic oscillator.

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