A note on the critical barrier for the survival of $\alpha$–stable branching random walk with absorption

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Abstract. We consider a branching random walk with an absorbing barrier, where the step of the associated one-dimensional random walk is in the domain of attraction of an $\alpha$-stable law with $1 < \alpha < 2$. We shall prove that there is a barrier $an^{1/\alpha}$ and a critical value $a_\alpha$ such that if $a < a_\alpha$, then the process dies; if $a > a_\alpha$, then the process survives. The results generalize previous results in literature for the case $\alpha = 2$.

Keywords. branching random walk, $\alpha$–stable spine, absorption, critical barrier.

1 Introduction

We consider a discrete-time one-dimensional branching random walk. It starts with an initial ancestor particle located at the origin. At time 1, the particle dies, producing a certain number of new particles. These new particles are positioned according to the distribution of the point process $\Theta$. At time 2, the above particles die, producing new particles positioned (with respect to the birth place) according to $\Theta$, and the process goes on with the same mechanism. We assume the particles produce new particles independently of each other at the same generation and of everything up to that generation. This system can be seen as a branching tree $T$ with the origin as the root.

For each vertex $x$ on $T$, we denote its position by $V(x)$. The family of the random variables $(V(x))$ is usually referred to as a branching random walk (Biggins [2]).

Throughout the paper, we assume:

$$\mathbb{E}\left(\sum_{|x|=1} 1\right) > 1, \quad \mathbb{E}\left(\sum_{|x|=1} e^{-V(x)}\right) = 1, \quad \mathbb{E}\left(\sum_{|x|=1} V(x)e^{-V(x)}\right) = 0,$$

where $|x|$ denotes the generation of $x$. This assumption is referred to in the literature as the boundary case; see for example Biggins and Kyprianou [4]. Every branching random walk satisfying certain mild integrability assumptions can be reduced to this case by some renormalization; see Jaffuel [11] for more details. Note that (1.1) implies $T$ is a super-critical Galton-Watson tree.

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Denote $\mathbb{N} = \{0, 1, 2, \cdots \}$ and $\mathbb{N}^* = \{1, 2, \cdots \}$. We define a “barrier” by a function $\varphi : \mathbb{N} \to \mathbb{R}$ and consider the branching random walk with absorption: On $\mathcal{T}$, all the individuals $x$ such that $V(x) > \varphi(|x|)$, i.e. born above the barrier, are immediately removed and do not reproduce.

A natural question is whether the process survives or not. Kesten [12], Derrida and Simon [6, 7], Harris J. and Harris S. [9] have studied the continuous analog of this process, the branching Brownian motion with absorption. Biggins et al [5] solved the corresponding question on the linear barriers. Under certain conditions (see (1.1)–(1.4)), Jaffuel [11] refined above result by considering a more general barrier. He found a barrier $a n^{\frac{1}{3}}$ and a critical value $\hat{a}$: the process dies when $a < \hat{a}$ and survives when $a > \hat{a}$.

Before stating the results in literatures, we introduce some notation. We denote by $u_i$ the ancestor of $u$ in generation $i$ and $\mathcal{T}_n := \{u \in \mathcal{T} : |u| = n\}$ the population at time $n$. And we say $x < y$ iff individual $x$ is an ancestor of individual $y$. Define an infinite path $u$ through $\mathcal{T}$ as a sequence of individuals $u = (u_i)_{i \in \mathbb{N}}$ such that $\forall i \in \mathbb{N}, |u_i| = i, u_i < u_{i+1}$, and denote their collection by $\mathcal{T}_\infty$. For $A \subset \mathcal{T}$, $\#A$ denotes the number of individuals in $A$.

**Theorem 1.1. (Biggins et. al. [5]).** Under condition (1.1), we have

$$\mathbb{P}\left( \exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq i\varepsilon \right) = \begin{cases} 0, & \text{if } \varepsilon \leq 0, \\ > 0, & \text{if } \varepsilon > 0. \end{cases}$$

To present the result in Jaffuel [11], we need more conditions.

1. $\exists \delta > 0, \mathbb{E}(\#\mathcal{T}_1^{1+\delta}) < +\infty$,
2. $\exists \varrho > 0, \mathbb{E}\left(\sum_{|u|=1} e^{-(1+\varrho)V(u)}\right) < +\infty$,
3. $\sigma^2 := \mathbb{E}\left(\sum_{|u|=1} V(u)^2 e^{-V(u)}\right) < +\infty$.

Jaffuel [11] refined Theorem 1.1 by replacing the linear barrier $i\varepsilon$ with a barrier $\varphi(i) := ai^{1/3}$.

**Theorem 1.2. (Jaffuel [11]).** Let $\hat{a} = \frac{3}{2}(3\pi^2\sigma^2)^{1/3}$. Assuming (1.1)–(1.4), we have

$$\mathbb{P}\left( \exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq ai^{1/3} \right) \begin{cases} > 0, & \text{if } a > \hat{a}, \\ = 0, & \text{if } a < \hat{a}. \end{cases}$$

The aim of the present paper is to replace condition (1.4) by

$$\mathbb{E}\left(\sum_{|x|=1} 1_{\{|V(x)| \geq y\}} e^{-V(x)}\right) \sim \frac{c}{y^a}, \quad y \to +\infty,$$
where $c \in (0, \infty)$ and $\alpha \in (1, 2]$. Actually, (1.5) turns out to be

$$
E \left( \sum_{|x|=1} 1_{\{V(x) \geq y\}} e^{-V(x)} \right) \sim \frac{c}{y^{\alpha}}, \quad y \to +\infty
$$

(1.6)

under (1.3). Now, if we define a random variable $X$ by

$$
P(X \leq x) = E \left( \sum_{|u|=1} 1_{\{V(u) \leq x\}} e^{-V(u)} \right), \quad x \in \mathbb{R},
$$

(1.7)

then we shall see from Lemma 2.1 and (1.5) that

$$
P(X > x) \sim cx^{1-\alpha}, \quad P(X < -x) = o(x^{1-\alpha}), \quad x \to +\infty.
$$

which means that $X$ is in the domain of attraction of a strictly $\alpha$-stable random variable $Y$ with characteristic function of the form

$$
G_{\alpha}(t) = \exp \{-c_{0}|t|^\alpha(1 - it|t|\tan \frac{\pi \alpha}{2})\}, \quad c_{0} > 0, \quad \alpha \in (1, 2].
$$

(1.8)

Denote by $(Y_t, t \in [0, 1])$ the strictly $\alpha$-stable Lévy process such that $Y_1$ has the same law as $Y$. Under (1.1) and (1.5), we call $(V(x))$ a stable branching random walk. We mention that the convergence of derivative martingale and additive martingale for stable branching random walk are studied in a recent paper [10]; The asymptotic behavior of the position of $N$-branching random walk with $\alpha$-stable spine has been studied in Mallein [13].

Theorem 1.3 and Proposition 1.5 are our main results.

**Theorem 1.3.** Let $a_{\alpha} = (1 + \alpha^{-1})(\alpha(1 + \alpha)C_{\ast})^{\frac{1}{1+\alpha}}$, where

$$
C_{\ast} = C_{\ast}(\alpha) := -\lim_{t \to \infty} \frac{1}{t} \log P(|Y_s| \leq \frac{1}{2}, s \leq t) \in (0, +\infty).
$$

(1.9)

Assume (1.1)–(1.3) and (1.5). Then

$$
P(\exists u \in \mathcal{T}_{\infty}, \forall i \geq 1, V(u_i) \leq a_{\alpha}) \begin{cases} 
> 0, & \text{if } a > a_{\alpha}, \quad (\text{lower bound}) \\
= 0, & \text{if } a < a_{\alpha}, \quad (\text{upper bound})
\end{cases}
$$

(1.10)

**Remark 1.4.** We observe that if $\alpha = 2$, then condition (1.3) and (1.7) reduce to (1.3) and (1.4). In this case, $C_{\ast} = C_{\ast}(2) = \frac{\pi^2}{2}$, which coincides with Theorem 1.2.

By taking derivative and discussing monotonicity of the function

$$
f(x) = x + \frac{1+\alpha}{x^\alpha} C_{\ast}, \quad x > 0.
$$

we can see that

$$
f'(\frac{\alpha a_{\alpha}}{1+\alpha}) = 0,
$$

and $\min_{x} f(x) = a_{\alpha}$. If $a > a_{\alpha}$, then the equation $a = x + \frac{1+\alpha}{x^\alpha} C_{\ast}$ has two solutions in $x$. Let $r_a$ be the larger solution. Clearly, $r_a > \frac{\alpha a_{\alpha}}{1+\alpha}$. 
Proposition 1.5. For $a > a_\alpha$, $\varepsilon \in (0, r_a)$, $N \in \mathbb{N}^*$, define
\[ B_N = \{ \forall k \geq 1, \#\{ u \in T_Nk : \forall i \leq Nk, (a - r_a)i^{1+\alpha} \leq V(u_i) \leq ai^{1+\alpha} \} \geq \exp(\frac{1}{2}N^{1+\alpha}(r_a - \varepsilon)) \} \].

Then for sufficiently large $N$, $\mathbb{P}(B_N) > 0$.

The remainder of the paper is organized as follows. In section 2, we prove several lemmas on the one-dimensional random walk associated with $(V(x))$, which will be used in the proofs of Theorem 1.3 and Proposition 1.5. In section 3, we prove Proposition 1.5 and the lower bound for the survival probability in Theorem 1.3. The upper bound in Theorem 1.3 are discussed in section 4. Our technical routes and proofs are based on Mallein [13], Aïdékon and Jaffuel [1] and Jaffuel [11].

2 Small deviations estimate and variations

Let $(X_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of copies of $X$ defined by (1.7). Let $S_0 = 0$, and for any $n \geq 1$,
\[ S_n := \sum_{i=1}^{n} X_i. \]

$S$ is then a mean-zero heavy-tailed random walk starting from the origin.

Lemma 2.1. (many-to-one lemma, Biggins and Kyprianou [3]). For any $n \geq 1$ and any measurable function $F: \mathbb{R}^n \to [0, +\infty)$,
\[ \mathbb{E}\left( \sum_{|u|=n} e^{-V(u)} F(V(u_i), 1 \leq i \leq n) \right) = \mathbb{E}(F(S_i), 1 \leq i \leq n). \]

Our proof of the lower bound of Theorem 1.3 requires the following bivariate version of the many-to-one lemma.

Lemma 2.2. (Gantert, Hu and Shi [8]). Suppose $X$ is a random variable defined by (1.7). Let $\nu$ be a random variable taking values in $\mathbb{N}^*$ such that for any nonnegative measurable function $f$,
\[ \mathbb{E}(f(X, \nu)) = \mathbb{E}\left( \sum_{|u|=1} e^{-V(u)} f(V(u), \#\Gamma(u)) \right). \]

Let $n \geq 1$ and $(X_i, \nu_i)_{1 \leq i \leq n}$ be i.i.d. copies of $(X, \nu)$. Then for any measurable function $F: (\mathbb{R} \times \mathbb{N}^*)^n \to [0, +\infty)$, it holds that
\[ \mathbb{E}\left( \sum_{|u|=n} e^{-V(u)} F(V(u_i), \#\Gamma(u_{i-1}), 1 \leq i \leq n) \right) = \mathbb{E}(F(S_i, \nu_i, 1 \leq i \leq n)), \]

where $\Gamma(u) := \{ v \in T : |v| = |u| + 1, v > u \}$.

Let $C[0,1]$ (respectively, $C[0,1]$) be the set of functions (respectively, continuous functions) from $[0,1]$ to $\mathbb{R}$. For $z \in \mathbb{R}$, we denote by $\mathbb{P}_z$ the probability associated with the branching random walk $(V(x))$ starting from $z$, and $\mathbb{E}_z$ the corresponding expectation. We breviate $\mathbb{P}_0$ by $\mathbb{P}$.
In the following Theorem 2.3 and Lemma 2.4 let \((c_n)\) be a sequence of positive real numbers such that
\[
\lim_{n \to \infty} c_n = +\infty, \quad \lim_{n \to \infty} \frac{c_n}{n^{1/\alpha}} = 0.
\]

**Theorem 2.3.** (Mogul’skiĭ [14]) Let \(f, g \in C[0,1]\), with \(f < g\) and \(f(0) < 0 < g(0)\). We have
\[
\lim_{n \to \infty} \frac{c_n^\alpha}{n} \log P\left(\frac{S_i}{c_n} \in \left[f\left(\frac{j}{n}\right), g\left(\frac{j}{n}\right)\right], \ 0 \leq j \leq n\right) = -C_* \int_0^1 \frac{ds}{(g(s) - f(s))^\alpha},
\]
where \(C_*\) is defined by (1.9).

The following three lemmas are some more sophisticated versions of above theorem. For the proofs of them, we shall borrow some ideas from Aïdékon and Jaffuel [1], which discussed the case \(\alpha = 2\).

**Lemma 2.4.** Let \(f, g \in C[0,1]\), with \(f < g\) and \(f(0) < 0 < g(0)\). For any sequences \((f_n)\) and \((g_n)\) of \(F[0,1]\) such that \(\|f_n - f\|_\infty \to 0\) and \(\|g_n - g\|_\infty \to 0\) as \(n \to \infty\), we have
\[
\lim_{n \to \infty} \frac{c_n^\alpha}{n} \log P\left(\frac{S_i}{c_n} \in \left[f_n\left(\frac{j}{n}\right), g_n\left(\frac{j}{n}\right)\right], \ 0 \leq j \leq n\right) = -C_* \int_0^1 \frac{ds}{(g(s) - f(s))^\alpha},
\]
where \(C_*\) is defined by (1.9).

**Proof.** Let \(0 < \varepsilon < \frac{1}{2} \min\{\min_{[0,1]} |g - f|, |f(0)|, |g(0)|\}\). We can choose \(N \geq 1\) s.t. for any \(n \geq N\), \(\max_{[0,1]} |f_n - f| + \max_{[0,1]} |g_n - g| < \varepsilon\). Then for such \(n\), we have
\[
\left\{f + \varepsilon < \frac{S_i}{c_n} < g - \varepsilon\right\} \subset \left\{f_n < \frac{S_i}{c_n} < g_n\right\} \subset \left\{f - \varepsilon < \frac{S_i}{c_n} < g + \varepsilon\right\}
\]
Applying Theorem 2.3 we get
\[
-C_* \int_0^1 \frac{ds}{(g - f - 2\varepsilon)^\alpha}
\leq \liminf_{n \to \infty} \frac{c_n^\alpha}{n} \log P\left(\frac{S_i}{c_n} \in \left[f_n\left(\frac{j}{n}\right), g_n\left(\frac{j}{n}\right)\right], \ 0 \leq j \leq n\right)
\leq \limsup_{n \to \infty} \frac{c_n^\alpha}{n} \log P\left(\frac{S_i}{c_n} \in \left[f_n\left(\frac{j}{n}\right), g_n\left(\frac{j}{n}\right)\right], \ 0 \leq j \leq n\right)
\leq -C_* \int_0^1 \frac{ds}{(g - f + 2\varepsilon)^\alpha}.
\]
Letting \(\varepsilon \to 0\), we complete the proof. \(\square\)

**Lemma 2.5.** Let \(f, g \in C[0,1]\), with \(f < g\) and \(f(0) < 0 < g(0)\). Let \((f_n)\) and \((g_n)\) be sequences of \(F[0,1]\) such that \(\|f_n - f\|_\infty \to 0\) and \(\|g_n - g\|_\infty \to 0\) as \(n \to \infty\). Let \(\beta^*\) and \(\gamma^*\) be positive real numbers such that \(0 < \beta^* < \gamma^* \leq 1\). Let \(u^*, v^* \in \mathbb{R}\) s.t. \(f(\beta^*) \leq u^* < v^* \leq g(\beta^*)\). Let \(\gamma(n) > \beta(n)\) be the sequences of positive integers, and \((\mu_n)_{n}, (\nu_n)_{n}\) be sequences of reals such that
\[ n^{-\frac{1}{1+\alpha}} \mu_n \to u^*, \quad n^{-\frac{1}{1+\alpha}} \nu_n \to v^*, \quad \frac{\beta(n)}{n} \to \beta^*, \quad \gamma(n) \to \gamma^*, \]

and for any \( n \geq 1 \),

\[ f_n(\beta(n)/n)n^{\frac{1}{1+\alpha}} \leq \mu_n \leq \nu_n \leq g_n(\beta(n)/n)n^{\frac{1}{1+\alpha}}, \quad 0 \leq \beta(n) < \gamma(n) \leq n. \]

We also assume that

\[ \exists M \in \mathbb{N}^*, \forall m \in \mathbb{N}^*, \#\{n : \gamma(n) - \beta(n) = m\} \leq M. \quad (2.1) \]

Then

\[
\lim_{n \to \infty} n^{-\frac{1}{1+\alpha}} \log \left( \sup_{\mu_n \leq z \leq \nu_n} \mathbb{P}_z \left( \frac{S_k - \beta(n)}{n^{\frac{1}{1+\alpha}}} \in \left[ f_n(\frac{k}{n}), g_n(\frac{k}{n}) \right], \beta(n) < k \leq \gamma(n) \right) \right)
\leq -C_n \int_{\beta^*}^{\gamma^*} \frac{ds}{(g(s) - u^* - f(s) + v^*)^\alpha}.
\]

**Proof.** Here we set \( c_n = n^{\frac{1}{1+\alpha}} \). Notice that for \( m \in A := \{ m \in \mathbb{N}^* : \exists n \in \mathbb{N}^*, \gamma(n) - \beta(n) = m \} \),

\[
\left\{ \forall z \in [\mu_n, \nu_n], \forall k \leq m, c_n f_n \left( \frac{\beta(n)+k}{n} \right) \leq z + S_k \leq c_n g_n \left( \frac{\beta(n)+k}{n} \right) \right\}
\subset \left\{ \forall k \leq m, c_n f_n \left( \frac{\beta(n)+k}{n} \right) - \nu_n \leq S_k \leq c_n g_n \left( \frac{\beta(n)+k}{n} \right) - \mu_n \right\}. \quad (2.2)
\]

By (2.1), we can define a surjection \( \varphi : \{1, 2, \cdots, M\} \times A \to \mathbb{N}^* \) such that for any \( 1 \leq l \leq M \) and \( m \in A \), it holds that \( m = \gamma(\varphi(l,m)) - \beta(\varphi(l,m)) \). For such \( l, m \), define

\[
\tilde{f}_m(t) = \frac{f_n((1-t)\beta(n)+t\gamma(n))c_n - \nu_n}{c_m}, \quad \tilde{g}_m(t) = \frac{g_n((1-t)\beta(n)+t\gamma(n))c_n - \mu_n}{c_m},
\]

where \( n = \varphi(l,m) \). It is not difficult to see that \( n \sim (\gamma^* - \beta^*)^{-1} m \), and \( c_n \sim c_m (\gamma^* - \beta^*)^{-1+\alpha} \). Consequently, as \( m \to \infty \),

\[
\tilde{f}_m(t) \to \tilde{f}(t) = (\gamma^* - \beta^*)^{-1+\alpha} (f((1-t)\beta^* + t\gamma^*) - v^*),
\]

\[
\tilde{g}_m(t) \to \tilde{g}(t) = (\gamma^* - \beta^*)^{-1+\alpha} (g((1-t)\beta^* + t\gamma^*) - u^*).
\]

For each \( 1 \leq l \leq M \), applying Lemma 2.1 (with \( f_n \) and \( g_n \) replaced by \( \tilde{f}_n \) and \( \tilde{g}_n \))
to the right hand side of (2.2), we obtain as \( m \to \infty \),

\[
\log \mathbb{P} \left( \forall z \in [\mu_n, \nu_n], \forall k \leq m, c_n f_n \left( \frac{\beta(n) + k}{n} \right) \leq z + S_k \leq c_n g_n \left( \frac{\beta(n) + k}{n} \right) \right) \\
\leq \log \mathbb{P} \left( \forall k \leq m, c_n f_n \left( \frac{\beta(n) + k}{n} \right) - \nu_n \leq S_k \leq c_n g_n \left( \frac{\beta(n) + k}{n} \right) - \mu_n \right) \\
\leq \log \mathbb{P} \left( \forall k \leq m, \tilde{f}_m \left( \frac{k}{m} \right) \leq \frac{S_k}{c_m} \leq \tilde{g}_m \left( \frac{k}{m} \right) \right) \\
= -m^{1/(1+\alpha)} C_* \int_0^1 \frac{ds}{(\tilde{g} - f)^\alpha} (1 + o(1)) \quad \text{(by Lemma 2.4)} \\
= -(1 + o(1)) m^{-1} C_* \int_{\beta^*}^{\gamma^*} \frac{ds}{(g - f - u^* + v^*)^\alpha}.
\]

This bound holds when \( n \) runs along the subsequence \( (\varphi(l, m))_m \) for each \( 1 \leq l \leq M \), which covers all the values \( n \in \mathbb{N}^* \), and then the proof is finished. \( \square \)

**Lemma 2.6.** Let \( f, g \in \mathcal{C}[0, 1] \), with \( f < g \) and \( f(0) < 0 < g(0) \). Let \((f_n)\) and \((g_n)\) be sequences of \( \mathcal{F}[0, 1] \) such that \( \|f_n - f\|_\infty \to 0 \) and \( \|g_n - g\|_\infty \to 0 \) as \( n \to \infty \). Let \( \beta^* \) and \( \gamma^* \) be positive real numbers such that \( 0 \leq \beta^* < \gamma^* \leq 1 \). Let \( 0 \leq \beta(n) < \gamma(n) \leq n \) be the sequences of positive integers such that

\[
\frac{\beta(n)}{n} \to \beta^*, \quad \frac{\gamma(n)}{n} \to \gamma^*,
\]

and assume (2.4). Then

\[
\limsup_{n \to \infty} n^{-\frac{1}{1+\alpha}} \log \left( \sup_z \mathbb{P} \left( \frac{S_{k-\beta(n)}}{n^{1+\alpha}} \in \left[ f_n \left( \frac{k}{n} \right), g_n \left( \frac{k}{n} \right) \right], \forall \beta(n) < k \leq \gamma(n) \right) \right) \leq -C_* \int_{\beta^*}^{\gamma^*} \frac{ds}{(g(s) - f(s))^\alpha},
\]

where \( \sup_z \) is taken over the set \( \{ z \in \mathbb{R} | n^{1/\alpha} f_n \left( \frac{\beta(n)}{n} \right) \leq z \leq n^{1/\alpha} g_n \left( \frac{\beta(n)}{n} \right) \} \).

**Proof.** Define

\[
p(z, n) := \mathbb{P} \left( \frac{S_{k-\beta(n)}}{n^{1+\alpha}} \in \left[ f_n \left( \frac{k}{n} \right), g_n \left( \frac{k}{n} \right) \right], \forall \beta(n) < k \leq \gamma(n) \right).
\]

Let \( \varepsilon > 0 \), and \( N \) be an integer such that \( N\varepsilon > g(\beta^*) - f(\beta^*) \). We define for \( j = 0, 1, \ldots, N \),

\[
\mu_n^j := n^{1+\alpha} f_n(\beta(n)/n)(N - j) + g_n(\beta(n)/n)j.
\]

Observe that

\[
\sup_{n^{1+\alpha} f_n \left( \frac{\beta(n)}{n} \right) \leq z \leq n^{1+\alpha} g_n \left( \frac{\beta(n)}{n} \right)} p(z, n) = \max_{0 \leq j \leq N-1} \sup_{\mu_n^j \leq z \leq \mu_n^{j+1}} p(z, n).
\]

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We apply Lemma 2.9 $N$ times, with $\mu_n = \mu^j_n$ and $\nu_n = u^{j+1}_n$, $j = 0, 1, \ldots, N - 1$ and get

$$
\lim_{n \to \infty} \sup_{n^{-\frac{1}{1+\alpha}}} \log \left( \sum_{n^{-\frac{1}{1+\alpha}}} p(z, n) \right) \leq -C_\star \int_{\beta^*} ds \frac{ds}{(g(s) - f(s) + \varepsilon)^\alpha}.
$$

By letting $\varepsilon \to 0$, we prove the lemma. □

Remark 2.7. Let $\varepsilon > 0$. Notice that the probability that $S_n$ stays between $f$ and $g$ is less than the probability that $S_n$ stays between $\tilde{f} := f - \varepsilon$ and $\tilde{g} := g + \varepsilon$. We can extend the upper bounds in Lemmas 2.4 and 2.6 to functions satisfying $f < g$, and $f(0) < 0 \leq g(0)$.

Theorem 2.8. (Prokhorov theorem [15]) If $\frac{S_n}{n^{1/\alpha}}$ converges in law to a strictly stable random variable $Y$, then the process $\{ \frac{S_{nt}}{n^{1/\alpha}} : t \in [0, 1] \}$ converges in law to an $\alpha$-stable Lévy process $\{Y_t, t \in [0, 1]\}$ in $D([0, 1])$ equipped with the Skorokhod topology such that $Y_1$ has the same law as $Y$.

Using an adjustment of the original proof of Mogul’skii, similarly to [13], we have two estimates for an enriched random walk. Recall that $(v_j)$ defined in Lemma 2.2 is a sequence of $N^*$-valued i.i.d. random variables.

Lemma 2.9. Let $f, g \in C[0, 1]$, with $f < g$ and $f(0) < 0 < g(0)$. We set $E_k^{(n)} = \{v_j \leq \exp\{n^{1/\beta}\}, j \leq k\}$ for some $\beta > 0$ and assume that

$$
\lim_{n \to \infty} n^{\alpha/(1+\alpha)} P \left( v_1 \geq \exp\{n^{1/\beta}\} \right) = 0.
$$

For any $f(0) < x < y < g(0)$, we have

$$
\lim_{n \to \infty} n^{-\frac{1}{1+\alpha}} \log \inf_{z \in [x, y]} P \left( \frac{S_j}{n^{1+\alpha}} \in \left[ f\left(\frac{j}{n}\right), g\left(\frac{j}{n}\right) \right], 0 \leq j \leq n, E_k^{(n)} \right) = -C_\star \int_0^1 ds \frac{ds}{(g(s) - f(s))^{\alpha}}.
$$

Moreover, for $b > 0$,

$$
\lim_{n \to \infty} n^{-\frac{1}{1+\alpha}} \log \inf_{z \in [x, g]} P \left( \frac{S_j}{n^{1+\alpha}} \in \left[ g(1) - b, g(1) \right], 0 \leq j \leq n, E_k^{(n)} \right) \geq -C_\star \int_0^1 ds \frac{ds}{(g(s) - f(s))^{\alpha}}.
$$

Proof. In the proof of [13] Lemma 2.6, replace $E_n$ by $E_k^{(n)}$ and let $c_n = n^{1+\alpha}$, $t_n = [A c_n]$ with $A > 0$. Then with the help of our Lemma 2.6 and Theorem 2.8 we can go along the line of [13] Lemma 2.6 to get the proof. The details are omitted. □

Replacing Lemma 2.4 by Lemma 2.9 in the proofs of Lemmas 2.5 and 2.6 we arrive at

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Lemma 2.10. Let \( f, g \in C[0,1] \), with \( f < g \) and \( f(0) < 0 < g(0) \). Let \( (f_n) \) and \( (g_n) \) be sequences of \( F[0,1] \) such that \( \|f_n - f\|_\infty \to 0 \) and \( \|g_n - g\|_\infty \to 0 \) as \( n \to \infty \). Let \( \beta^* \) and \( \gamma^* \) be positive real numbers such that \( 0 \leq \beta^* < \gamma^* \leq 1 \). Let \( 0 \leq \beta(n) < \gamma(n) \leq n \) be the sequences of positive integers such that:

\[
\frac{\beta(n)}{n} \to \beta^*, \quad \frac{\gamma(n)}{n} \to \gamma^*,
\]

and assume (2.1). Suppose that \( \{v_j\} \) is defined as in Lemma 2.11. We set \( E_k^{(n)} = \{v_j \leq \exp\{n^{1/\beta}\}, j \leq k \} \) for some \( \beta > 1 + \alpha \). Then for \( b > 0 \),

\[
\liminf_{n \to \infty} n^{-\frac{1}{1+\alpha}} \log \inf_z \mathbb{P}_z \left( \frac{S_{1+\alpha}(n)-\beta(n)}{n^{1+\alpha}} \in \left[ \frac{g(\gamma(n))}{n}, b, g(\gamma(n)) \right] \right),
\]

\[
\frac{S_{1+\alpha}(n)-\beta(n)}{n^{1+\alpha}} \in \left[ f\left(\frac{i}{n}\right), g\left(\frac{i}{n}\right) \right], \quad \beta(n) < j \leq \gamma(n), \quad E_k^{(n)} \geq -C_{\beta^*} \int_{\beta^*}^{\gamma^*} \frac{ds}{(g(s)-f(s))^{\alpha}},
\]

where the \( \inf_z \) is taken over the set \( \{z \in \mathbb{R} | f\left(\frac{\beta(n)}{n}\right)n^{1+\alpha} \leq z \leq g\left(\frac{\beta(n)}{n}\right)n^{1+\alpha} \} \).

The following lemma will be used to get the lower bound in Theorem 1.3.

Lemma 2.11. Let \( f, g \in C[0,1] \), with \( f < g \) and \( f(0) < 0 = g(0) \). Then there are \( M \geq 1 \) and \( \varepsilon_1 > 0 \) such that

\[
\lim_{\varepsilon_2 \to 0} \liminf_{n \to \infty} n^{-\frac{1}{1+\alpha}} \log P_n(M, \varepsilon_1, \varepsilon_2) = 0,
\]

where

\[
P_n(M, \varepsilon_1, \varepsilon_2) = \mathbb{P}\left( \exists u \in T_k, \forall i < k, \#\Gamma(u_i) \leq M, f\left(\frac{i}{n}\right) \leq \frac{V(u_i)}{n^{1+\alpha}} \leq g\left(\frac{i}{n}\right), -M\varepsilon_2 \leq \frac{V(u_k)}{n^{1+\alpha}} \leq -\varepsilon_1 \varepsilon_2 \right),
\]

with \( k := \lfloor \varepsilon_2 n^{1+\alpha} \rfloor \).

Proof. The proof is essentially similar to Jaffuel [11, Lemma 2.8] for the case \( \alpha = 2 \), so we omit it. \( \square \)

3 Lower bound for the survival probability

In this section we prove Proposition 1.3 and the lower bound for the survival probability in Theorem 1.3.

We consider the population surviving below the barrier \( i \mapsto ai^{1+\alpha} \): any individual born above the barrier would be removed and do not reproduce.

Suppose \( \lambda > 0 \) such that \( e^\lambda \in \mathbb{N} \). For any \( k \in \mathbb{N} \), we pick a particle \( z \) at position \( V(z) \) in generation \( e^{\lambda k} \), and denote by \( Y_k(z) \) the number of descendants it eventually has in generation \( e^{\lambda(k+1)} \). Instead of \( z \), we pick another particle \( \tilde{z} \) in the same generation \( e^{\lambda k} \) but positioned on the barrier at \( V(\tilde{z}) := a e^{\lambda k} \geq V(z) \), and suppose the number and displacements of the descendants of \( \tilde{z} \) are exactly the same as those of \( z \). Clearly, the
descendants of \( \tilde{z} \) are more likely to cross the barrier and be killed, hence, if we denote the number of its descendants by \( Y_k(\tilde{z}) \), then \( Y_k(\tilde{z}) \leq Y_k(z) \).

We hereby add a second absorbing barrier \( i \mapsto (a-b)i^{1+\alpha} \) for some \( 0 < b < a \) and kill any descendant of \( \tilde{z} \) born below it. We obtain that, almost surely,

\[
Z_k \leq Y_k(\tilde{z}) \leq Y_k(z),
\]

where

\[
Z_k := \#\{ \tilde{u} \in T_{e^{\lambda(k+1)}}, \tilde{u} > \tilde{z}, \forall e^{\lambda k} < i \leq e^{\lambda(k+1)}, (a-b)i^{1+\alpha} \leq V(\tilde{u}_i) \leq ai^{1+\alpha} \}.
\]

Clearly, \( Z_k \) is the number of descendants of \( \tilde{z} \) starting at time \( e^{\lambda k} \) at position \( ae^{\lambda k} \) over \( l_k := e^{\lambda(k+1)} - e^{\lambda k} \) generations. The individuals of \( \tilde{z} \) in generation \( i \) are killed if they are out of the interval: \( I_i := [(a-b)i^{1+\alpha}, ai^{1+\alpha}] \).

For \( u, v \in T \), let \( u_j := u \land v \in T \) be the lowest common ancestor of them. We split \( \mathbb{E}(Z_k^2) \) into the double sum over \( u, v \) according to the generation \( j \) as follows:

\[
\mathbb{E}(Z_k^2) = \mathbb{E}\left( \sum_{u > \tilde{z}, v > \tilde{z}, u_v = v} 1\{\forall e^{\lambda k} < i \leq e^{\lambda(k+1)}, V(u_i), V(v_i) \in I_i \} \right) = \sum_{j=0}^{l_k} D_{k,j},
\]

where \( D_{k,l_k} = \mathbb{E}(Z_k) \) and for \( j < l_k \),

\[
D_{k,j} := \mathbb{E}\left( \sum_{u > \tilde{z}, u_v = e^{\lambda(k+1)}} 1\{\forall e^{\lambda k} < i \leq e^{\lambda(k+1)}, V(u_i) \in I_i \} \sum_{v > u_{e^{\lambda k+j}}, v_v = e^{\lambda(k+1)}, v_{e^{\lambda k+j+1}} \neq u_{e^{\lambda k+j+1}}} 1\{\forall e^{\lambda k+j} < i \leq e^{\lambda(k+1)}, V(v_i) \in I_i \} \right).
\]

By Lemma 2.1 for \( x \in I_{e^{\lambda k+j}} \) we have

\[
F_{k,j}(x) := \mathbb{E}\left( \sum_{u > u_{e^{\lambda k+j}}, u_v = e^{\lambda(k+1)}} 1\{\forall e^{\lambda k+j} < i \leq e^{\lambda(k+1)}, V(v_i) \in I_i \} \mid V(u_{e^{\lambda k+j}}) = x \right)
= \mathbb{E}\left( e^{S_{k+j}} 1\{\forall 0 < i \leq l_k - j, x + S_i \in I_{e^{\lambda k+j+i}} \} \right)
\leq \exp\left\{ ae^{\lambda(k+1)} - a(e^{\lambda k} + j)^{1+\alpha} + b(e^{\lambda k} + j)^{1+\alpha} \right\} P\left( \forall 0 < i \leq l_k - j, x + S_i \in I_{e^{\lambda k+j+i}} \right) \tag{3.1}
\]

For some \( R_k > 0 \) (Its value need to be determined), we define a processes \( Z_k^{(k)} \) as follows. If an individual has a number of children greater than \( R_k \), then we remove all the descendants of it. We add a superscript \( (k) \) when dealing with this new process \( Z_k^{(k)} \).

Clearly, \( Z_k^{(k)} \leq Z_k \). Analogously to above discussion, we have

\[
\mathbb{E}\left( (Z_k^{(k)})^2 \right) = \sum_{j=0}^{l_k} D_{k,j}^{(k)} \tag{3.2}
\]
By [11 Page 1002-1003],

\[ D_{k,j}^{(k)} \leq (R_k - 1) \sup_{x \in l_{c\lambda_{k,j+1}}} F_{k,j+1}(x) \mathbb{E}(Z^{(k)}). \tag{3.3} \]

From the definition, it is not difficult to see that \( F_{k,j+1}(x) \leq F_{k,j+1}(x) \).

Define \( \beta(\rho, l) := [\rho l] + 1, \gamma(l) := l \) and write \( j = \beta(\rho, l_k) - 1 \) for any \( \rho \in (0, 1) \).

Lemma 2.2 yields that, uniformly in \( \rho \in (0, 1) \) and \( x \in T_{c\lambda_k + \beta(\rho, l_k)}, \)

\[
\limsup_{k \to \infty} l_k^{-\frac{1}{1+\alpha}} \log \mathbb{P}(\forall 0 < i \leq l_k - (j + 1), x + S_i \in I_{c\lambda_{k,j+1+i}}) \leq -C_* \int_\rho^1 \frac{1}{(g_2(t) - g_1(t))^\alpha} dt,
\]

where

\[
\begin{align*}
g_2(t) &:= a \left( t + \frac{1}{e^\lambda - 1} \right)^{\frac{1}{1+\alpha}} - \left( \frac{1}{e^\lambda - 1} \right)^{\frac{1}{1+\alpha}}, \tag{3.4} \\
g(t) &:= b \left( t + \frac{1}{e^\lambda - 1} \right)^{\frac{1}{1+\alpha}}, \quad g_1(t) = g_2(t) - g(t). \tag{3.5}
\end{align*}
\]

Combining with (3.3) and (3.4), we get that uniformly in \( \rho \in (0, 1), \)

\[
\limsup_{k \to \infty} l_k^{-\frac{1}{1+\alpha}} \log \frac{D_{k,\beta(\rho, l_k) - 1}^{(k)}}{\mathbb{E}(Z^{(k)})} \leq \limsup_{k \to \infty} l_k^{-\frac{1}{1+\alpha}} \log(R_k - 1) + g_2(1) - g_2(\rho) + g(\rho) - C_* \int_\rho^1 \frac{1}{(g_2 - g_1)^\alpha}. \tag{3.6}
\]

For any \( k \geq 1 \), we consider i.i.d. random variable \( X_i^{(k)}, 1 \leq i \leq l_k \) with the same distribution as \( X \) conditioned on \( v \leq R_k \) with \( (X, v) \) defined as in Lemma 2.2. Write \( S_j^{(k)} := \sum_{i=1}^{j} X_i^{(k)} \) for any \( 0 \leq j \leq l_k \). Let \( \delta > 0 \) and \( \varrho > 0 \) be the constants in (1.3).

For \( \varepsilon > 0 \), by Lemma 2.2 going along the line in [11 section 4.3], we have

\[
\mathbb{E}(Z^{(k)}_{i^*}) = \mathbb{E}\left( \sum_{u > \varepsilon, |u| = e^{\lambda R_k}} 1_{\{v e^{\lambda k_i} \leq e^{\lambda(k+1)}, V(u_i) \in I_i, \# \Gamma(u_i) \leq R_k\}} \right) \\
= \mathbb{E}\left( e^{S_i^{(k)}} 1_{\{v i \leq l_k, \alpha e^{\lambda k_i} \leq \lambda_{k+1}, S_i \in I_{c\lambda_{k+1}}, v_i \leq R_k\}} \right) \\
= \mathbb{E}\left( e^{S_i^{(k)}} 1_{\{v i \leq l_k, \alpha e^{\lambda k_i} \leq \lambda_{k+1}, S_i \in I_{c\lambda_{k+1}}\}} \bigg| V_i \leq R_k, \forall i \leq l_k \bigg) \mathbb{P}(v \leq R_k)^{l_k} \\
= \mathbb{P}(v \leq R_k)^{l_k} \mathbb{E}\left( e^{S_i^{(k)}} 1_{\{v i \leq l_k, \alpha e^{\lambda k_i} \leq \lambda_{k+1}, S_i \in I_{c\lambda_{k+1}}\}} \bigg| V_i \leq R_k \bigg) \\
\geq \mathbb{P}(v \leq R_k)^{l_k} \exp \left\{ \frac{1}{l_k^{1+\alpha}} (g_2(1) - \varepsilon) \right\} \\
\cdot \mathbb{P}(g_1(t) \leq \frac{S_{i^*}^{(k)}}{l_k^{1+\alpha}} \leq g_2(t), t \in [0, 1]; S_{i^*}^{(k)} \geq \frac{1}{l_k^{1+\alpha}} (g_2(1) - \varepsilon)), \tag{3.7}
\right.
\]
Let $\delta > 0$ and $\eta > 0$ be the constants in condition (1.3). By Hölder’s inequality,

$$
\mathbb{P}(\nu > R_k) \leq \left( \mathbb{E}(\# T_1^1 + \delta) \right)^{1/\eta} R_k^{-\delta/\eta} \left( \mathbb{E} \left[ \sum_{|u|=1} e^{-(1+\eta)V(u)} \right] \right)^{1/\eta}.
$$

We now choose $R_k := \lfloor e^{l_k/\epsilon} \rfloor$ for some $c > 1 + \alpha$. Therefore

$$
\lim_{k \to \infty} l_k^{-1/\alpha} \log (\mathbb{P}(\nu \leq R_k)^{l_k}) = 0.
$$

(3.8)

By the Markov property and using the notation of Lemma 2.11, there exist $M, \epsilon_1 > 0$ such that for sufficiently large $k$ and any small $\epsilon_2 > 0$,

$$
\mathbb{P}(g_1(t) \leq l_k^{1/\alpha} S_{\lfloor t l_k \rfloor}^{(k)} \leq g_2(t), t \in [0, 1]; S_k^{(k)} \geq l_k^{1/\alpha} (g_2(1) - \epsilon))
\geq P_k(M, \epsilon_1, \epsilon_2) \inf_{-M \leq z \leq -\epsilon_1 \epsilon_2 l_k^{1/\alpha}} H_k(z, \epsilon_2, g_1, g_2),
$$

(3.9)

where

$$
H_k(z, \epsilon_2, g_1, g_2) := \mathbb{P}_z \left( S_{\lfloor t l_k \rfloor}^{(k)} \geq l_k^{1/\alpha} (g_2(1) - \epsilon), \forall i \leq l_k - \lfloor \epsilon_2 l_k^{1/\alpha} \rfloor, g_1 \left( \frac{\lfloor \epsilon_2 l_k^{1/\alpha} \rfloor + i}{l_k} \right) \leq S_k^{(k)} \leq g_2 \left( \frac{\lfloor \epsilon_2 l_k^{1/\alpha} \rfloor + i}{l_k} \right) \right).
$$

By Lemma 2.10 we get that

$$
\lim_{k \to \infty} l_k^{-1/\alpha} \log \inf_{-M \leq z \leq -\epsilon_1 \epsilon_2 l_k^{1/\alpha}} H_k(z, \epsilon_2, g_1, g_2) \geq -C_1 \int_0^1 \frac{1}{(g_2 - g_1)^\alpha}.
$$

(3.10)

Then put (3.8)–(3.10) into (3.7). Letting $\epsilon \to 0$ and recalling Lemma 2.11 (and many to one lemma), we arrive at

$$
\lim \inf_{k \to \infty} l_k^{-1/\alpha} \log \mathbb{E}(Z_k^{(k)}) \geq g_2(1) - C_2 \int_0^1 \frac{1}{(g_2 - g_1)^\alpha}.
$$

(3.11)

Now we have

**Lemma 3.1.** Choose $\lambda$ sufficiently large such that $e^\lambda \in \mathbb{N}^\ast$. For fixed $\theta \in (0, 1)$, set $\nu_k = \theta \mathbb{E} Z_k^{(k)}$ and define

$$
T_k = \mathbb{P}(Z_k^{(k)} \geq \nu_k).
$$

If $a > a_\alpha$, then

$$
\sum_{k=0}^\infty e^{-\nu_k T_{k+1}} < +\infty.
$$
Proof. The proof is similar to that of [11, Lemma 4.1], which is for the finite variance case.

Combining (3.11) with (3.6) yields that, uniformly in \( \rho \in (0, 1) \),

\[
\limsup_{k \to \infty} l_k^{-\frac{1}{1+\alpha}} \log D_{k,\beta}(\rho, l_k) \leq -g_2(\rho) + g(\rho) + C_* \int_0^\rho \frac{1}{(g_2 - g_1)^\alpha}.
\]

Together with (3.2) and the Paley-Zygmund inequality

\[
T_k \geq (1 - \theta)^2 \frac{\langle \mathbb{E}(Z_k^{(k)}) \rangle}{\mathbb{E}(Z_k^{(k)})^2},
\]

we have that

\[
\liminf_{k \to \infty} l_k^{-\frac{1}{1+\alpha}} \log T_k \geq \min_{0 \leq \rho \leq 1} \left\{ g_2(\rho) - g(\rho) - C_* \int_0^\rho \frac{1}{(g_2 - g_1)^\alpha} \right\}.
\]

Define

\[
G_\lambda(\rho) := -g_2(\rho) + g(\rho) + C_* \int_0^\rho \frac{dt}{g(t)^\alpha} + e^{-\lambda/(1+\alpha)} \left( -g_2(1) + C_* \int_0^1 \frac{dt}{g(t)^\alpha} \right).
\]

Denote \( f(t) = (t + \frac{1}{e^\lambda - 1})^{1+\alpha} \) for \( t \in [0, 1] \). By (3.21) and (3.25) we have \( g_2 = af - af(0) \) and \( g = bf \). Then

\[
G_\lambda(\rho) = af(0) + (b - a) f(\rho) + \frac{C_*}{b^{\alpha}} \int_0^\alpha \frac{dt}{f(t)^\alpha} + e^{-\lambda/(1+\alpha)} \left( af(0) - af(1) + \frac{C_*}{b^{\alpha}} \int_0^1 \frac{dt}{f(t)^\alpha} \right).
\]

Noting that \( f(1) = e^{\lambda/(1+\alpha)} f(0) \) and \( f' = \frac{1}{1+\alpha} f^{-\alpha} \), we have

\[
G_\lambda(\rho) = \left( b + \frac{(1+\alpha) C_* - a}{b^{\alpha}} \right) f(\rho) + e^{-\lambda/(1+\alpha)} \left( af(0) - \frac{(1+\alpha) C_*}{b^{\alpha}} f(0) \right).
\]

Since \( a > a_\alpha \), we can choose \( b \) such that \( b + \frac{(1+\alpha) C_*}{b^{\alpha}} < a \). For this \( b \), noticing that \( f \) is increasing on \( [0, 1] \), we obtain

\[
\max_{0 \leq \rho \leq 1} G_\lambda(\rho) = G_\lambda(0) = f(0) \left[ \left( b + \frac{(1+\alpha) C_* - a}{b^{\alpha}} \right) + e^{-\lambda/(1+\alpha)} \left( a - \frac{(1+\alpha) C_*}{b^{\alpha}} \right) \right] < 0,
\]

for sufficiently large \( \lambda \). Meanwhile,

\[
g_2(1) - C_* \int_0^1 \frac{dt}{g(t)^\alpha} = (f(1) - f(0)) \left( a - \frac{(1+\alpha) C_*}{b^{\alpha}} \right) > 0.
\]
Then for sufficiently large $\lambda$,

\[
A := \min_{0 \leq p \leq 1} \left( g_2(\rho) - g(\rho) - C_s \int_0^\rho \frac{dt}{g(t)^{\alpha}} + g_2(1) - C_s \int_0^1 \frac{dt}{g(t)^{\alpha}} \right)
\]

\[
= \min_{0 \leq p \leq 1} \left( -G_{\lambda}(\rho) + (1 - e^{-\frac{\lambda}{1+\alpha}}) \right) \left( g_2(1) - C_s \int_0^1 \frac{dt}{g(t)^{\alpha}} \right)
\]  

> 0.

This together with (3.11)–(3.13), yields that for sufficiently large $k$ (noting that $l_{k+1} > l_k$),

\[
\nu_k T_{k+1} \geq \theta \exp\{A \frac{1}{1+\alpha}\}.
\]

The proof is concluded. \hfill \Box

**Proof of Proposition 1.5** Suppose that $Z^{(k)}_k$, $Z_k$ and $\nu_k$ are defined as before. For any $n \geq 1$, define

\[
P_n := \mathbb{P}\left( \forall 1 \leq k \leq n, \#\{u \in T_{\lambda k} : \forall i \leq e^{\lambda k}, (a - r_a) i^{1+\alpha} \leq V(u_i) \leq a i^{1+\alpha} \} \geq \nu_{k-1} \right).
\]

If $1 \leq n_0 \leq n$, by the Markov property and independence of individuals in generation $e^{\lambda k}$, we have

\[
P_{n+1} \geq P_n(1 - (1 - \mathbb{P}(Z_n \geq \nu_n))^{\nu_{n+1}}).
\]

Observe that $Z^{(k)}_k \leq Z_k$. We have $\mathbb{P}(Z^{(n)}_n \geq \nu_n) \leq \mathbb{P}(Z_n \geq \nu_n)$ and

\[
P_{n+1} \geq P_n(1 - (1 - T_n)^{\nu_{n+1}}).
\]

By induction, we obtain

\[
P_n \geq P_{n_0} \prod_{k=n_0}^{n-1} (1 - (1 - T_k)^{\nu_{k-1}}) \geq P_{n_0} \prod_{k=n_0}^{n-1} (1 - e^{-\nu_{k-1} T_k}), \quad n > n_0.
\]

\[
\log P_n \geq \log P_{n_0} + \sum_{k=n_0}^{n} \log(1 - e^{-\nu_{k-1} T_k}). \quad (3.14)
\]

Applying $\log(1 + x) \sim x(x \to 0^+)$, by Lemma 3.1 we have $\sum_{k=n_0}^{\infty} \log(1 - e^{-\nu_{k-1} T_k}) > -\infty$. Then there exists $p > 0$ such that for all sufficiently large $n$, we have $P_n \geq p > 0$. By 3.11 and recalling $l_k := e^{\lambda(k+1)} - e^{\lambda k}$, we have for sufficiently large $\lambda$ and $k$,

\[
\nu_k = \theta \mathbb{E}(Z_k^{(k)}) \geq \theta \exp \left\{ t^{1/(1+\alpha)} \left( g_2(1) - C_s \int_0^1 \frac{1}{(g_2 - g_1)^{\alpha}} \right) \right\}
\]

\[
\geq \theta \exp \left\{ t^{1/(1+\alpha)} (r_a - \epsilon) \right\}
\]

\[
\geq \theta \exp \left\{ (1 - e^{-\lambda}) \frac{1}{1+\alpha} \cdot e^{\lambda(k+1)/(1+\alpha)} (r_a - \epsilon) \right\}
\]

\[
\geq \exp \left\{ \frac{1}{2} N^{1+\alpha}(r_a - \epsilon) \right\}, \quad (3.15)
\]
by choosing large $\lambda$ such that $N = e^\lambda \in \mathbb{N}^*$ and $1 - e^{-\lambda} > 1/2$. Consequently,

\[
\begin{align*}
\mathbb{P}(B_N) &= \mathbb{P}\left( \forall k \geq 1, \# \{ u \in T_{N^k} : \forall i \leq N^k, (a - r_u)i^{1+\alpha} \leq V(u_i) \leq ai^{1+\alpha} \} \geq \exp \left\{ \frac{1}{2} N^{1+\alpha} (r_a - \varepsilon) \right\} \right) \\
&\geq \mathbb{P}\left( \forall k \geq 1, \# \{ u \in T_{N^k} : \forall i \leq N^k, (a - r_u)i^{1+\alpha} \leq V(u_i) \leq ai^{1+\alpha} \} \geq \nu_{k-1} \right) \\
&= \lim_{n} P_n > 0.
\end{align*}
\]

**Proof of the lower bound of Theorem 1.3.** The proof is immediate by Proposition 1.5. □

4 Upper bound for the survival probability

The idea and technical route of the upper bound are similar to [11, Section 3.4] (which is for the cases $\alpha = 2$). We only explain the sketch of the proofs and omit the details.

Fix $a > 0$. Clearly,

\[
\begin{align*}
\mathbb{P}\left( \exists u \in T_{\infty}, \forall i, V(u_i) \leq ai^{1+\alpha} \right) &= \lim_{n \to \infty} \mathbb{P}\left( \exists u \in T_n, \forall i \leq n, V(u_i) \leq ai^{1+\alpha} \right). \quad (4.1)
\end{align*}
\]

Let $h$ be some continuous function from $[0, 1]$ to $[0, +\infty)$. Lemma 4.1. For $a \in (0, a_\alpha)$, we have

\[
\limsup_{n \to \infty} n^{-\frac{1}{1+\alpha}} \log \mathbb{P}\left( \exists u \in T_n, \forall i \leq n, V(u_i) \leq ai^{1+\alpha} \right) \leq -K, \quad (4.2)
\]

where $K := \min(K_1, K_2)$, and

\[
K_1 := -a + C_s \int_0^1 \frac{dt}{h(t)^\alpha},
\]

\[
K_2 := \min_{0 \leq \rho \leq 1} \left\{ -a \rho^{\frac{1}{1+\alpha}} + h(\rho) + C_s \int_0^\rho \frac{dt}{h(t)^\alpha} \right\} \quad (4.3)
\]

for some non-negative continuous function $h$ on $[0, 1]$.

**Proof.** The proof can be obtained by the method of [11] sections 3.1-3.3], if we replace $aj^{1/3}$, $s_2$ and $s$ therein by $aj^{1+\alpha}$, $K_2$ and $K$, respectively, and apply our Lemma 2.4 and Lemma 2.6 (instead of Lemma 2.4 and Proposition 2.5 in [11]), with $g(t) = at^{1+\alpha}$ and $f(t) = at^{1+\alpha} - h(t)$. We omit the details here. □

Set $a \in (0, a_\alpha)$. With the help of Lemma 4.1 if we can find a function $h$ such that $K > 0$, then the proof of the upper bound of Theorem 1.3 is completed. In the following we do this work.

Add the constraint $h(1) = 0$ (but assume $\int_0^1 \frac{dt}{h(t)^\alpha} < \infty$). Taking $\rho = 1$, we see that $K_2 \leq K_1$. As a result, $K = K_2$. If we can choose $h$ in such a way that $h(0) > 0$ and
\[-a\rho^{1/\alpha} + h(\rho) + C_s \int_0^\rho \frac{dx}{h(x)} \text{ does not depend on } \rho, \text{ then by (4.3), } K = K_2 \equiv h(0). \text{ In this case, } h \text{ should be the solution of the equation:}
\[
\forall t \in [0,1], \quad -at^{\frac{1}{1+\alpha}} + h(t) + C_s \int_0^t \frac{dx}{h(x)^\alpha} = K, \tag{4.4}
\]
where \(K\) is some positive constant, the value of which is to be set later in such a way that \(h(1^-) = 0\). According to the discussion above, this value of \(K\) will give a bound for the rate of decay of the survival probability.

Equivalently, equation (4.4) may be written as \(h(0) = K\) and \(\forall t \in (0,1),\)
\[
h'(t) = \frac{a}{1+\alpha} t^{\frac{\alpha}{1+\alpha}} - \frac{C_s}{h(t)^\alpha}. \tag{4.5}
\]

By the Picard-Lindelöf theorem, this ordinary equation admits a unique maximal solution \(h\) defined on an interval \([0, t_{\text{max}})\). Actually, as [11, Proposition 3.6], we now have

**Proposition 4.2.** Let \(h\) be the unique maximal solution of equation (4.5) with initial condition \(h(0) = 1\). If \(a < a_\alpha\), then \(t_{\text{max}} < +\infty\) and \(h(t) \to 0\) as \(t \to t_{\text{max}}\).

**Proof of the upper bound of Theorem 1.3.** For \(a < a_\alpha\), suppose that \(h\) is the unique maximal solution of equation (4.5) with initial condition \(h(0) = 1\). By Proposition 4.2, \(t_{\text{max}} \in (0, \infty)\). Define \(\epsilon = 1/t_{\text{max}}\) and \(h_\epsilon(t) = e^{-1/(1+\alpha)}h(\epsilon t)\). Direct calculation yields that \(h_\epsilon\) is the solution of equation (4.5) on \([0,1)\) with initial condition \(h_\epsilon(0) = e^{-1/(1+\alpha)}\). Choosing \(K = h_\epsilon(0) = e^{-1/(1+\alpha)}\) and applying Lemma 4.1 in (4.1), we obtain the desired result.

□

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