We give an extension of Casimir $\mathcal{W} A_n$ algebras including a vertex operator which depends on non-simple roots of $A_{n-1}$.
1. Introduction

Conformal symmetry plays an important role in string theory and also in statistical physics. Its underlying symmetry algebra is Virasoro algebra. Casimir $W A_N$ algebras are the higher spin extensions of this algebra. The idea to extend the Casimir $W A_N$ algebras with the introduction of a vertex operator also seems to be relevant in two-dimensional field theories. This construction was presented first by V.A. Fateev and A.B. Zamolodchikov for $A_2$ and there are several works dealing with these constructions in the literature. The purpose of this work is to establish a method in this direction. Rather than the potential fields, we tried to extend these studies for primary fields by calculating explicitly all OPEs between primary fields and a vertex operator. This calculation was performed in Mathematica. Rather than the bosonic vertex operator definition in Ref.[12] . We then give a general definition of vertex operator, which depends on non-simple roots of $A_{N-1}$. The starting point in this work based on a bosonic vertex operator definition in Ref.[12].

This paper is organized as follows : In Sec. 2, we recapitulate primary basis for Casimir $W_N$ algebra by utilizing known Miura transformation with Feigin-Fuchs type of free massless scalar fields. In Sec.3, we constructed a vertex operator extension of the Casimir $W_N$ algebras by calculating explicitly all nontrivial OPEs between primary fields and a vertex operator. Finally, all these used techniques work only for primary fields. This calculation was performed in Mathematica. [13,14].

2. The Casimir $W_N$ Algebras Basis

In this section we recapitulate primary basis for the Casimir $W_N$ algebras from the Feigin-Fuchs type of free massless scalar field realization point of view. [7-10].

The Casimir $W_N$ algebra is an associative algebra generated by a set of chiral currents $U_i (z)$, of conformal dimension $k (k = 1, \cdots , N)$. The following Miura transformation gives the construction of Casimir $W_N$ algebra

\[ R_N (z) = - \sum_{k=0}^{N} U_k (z) (\alpha_0 \partial)^{N-k} =: \prod_{j=1}^{N} (\alpha_0 \partial - h_j (z)) : , \]

where symbol $: :$ shows normal ordering. Here $\alpha_i$ is a free parameter. $\varphi(z)$ has $N-1$ component which are Feigin Fuchs-type of free massless scalar fields. This transformation determines completely the fields $\{ U_i (z) \}$ with

\[ h_j (z) = i \mu_j \partial \varphi(z) \]

Here, $\mu_i$'s, ($i = 1, \cdots , N$) are the weights of the fundamental (vector) representation of $A_{N-1}$, satisfying $\sum_{i=1}^{N} \mu_i = 0$ and $\mu_i, \mu_j = \delta_{ij} - \frac{1}{N}$. The simple roots of $A_{N-1}$ are given by $\alpha_i = \mu_i - \mu_{i+1}$, ($i = 1, \cdots , N-1$). The Weyl vector of $A_{N-1}$ is denoted as $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha^+$ where $\alpha^+$ are the positive roots of $A_{N-1}$. A free scalar field $\varphi(z)$ is a single-valued function on the complex plane and its mode expansion is given by

\[ i \partial \varphi(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} . \]

Canonical quantization gives the commutator relations

\[ [a_m, a_n] = m \delta_{m+n,0} , \]

and these commutator relations are equivalent to the contraction

\[ \partial \varphi(z) \partial \varphi(w) = \frac{1}{(z-w)^2} . \]

By using single contraction $\partial \varphi(z) \partial \varphi(w)$, a contraction of $h_j (z)$ with itself is given by

\[ h_j (z) h_j (w) = \delta_{j} - \frac{1}{(z-w)^2} . \]
The fields \( \{ U_k(z) \} \) can be obtained by expanding \( R_{\alpha}(z) \). We present a first few one as in the following

\[
U_0(z) = -1, \quad U_1(z) = \sum_i h_i(z) = 0, \quad U_2(z) = - \sum_{i<j} (h_i h_j)(z) + \alpha_o \sum_i (i-1) \partial h_i(z)
\]  

One can see that \( U_2(z) = T(z) \) has spin-2, which is called the stress-energy tensor, \( U_k(z) \) has spin-k. The standard OPE of \( T(z) \) with itself is

\[
T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots
\]

where the central charge, for \( A_{N-1} \), is given by

\[
c = (N-1)(1-N(N+1)\alpha_o^2).
\]

A primary field \( \phi_n(z) \) with conformal spin-h must provide the following OPE with \( T(z) \)

\[
T(z) \phi_n(w) = \frac{h \phi_n(w)}{(z-w)^2} + \frac{\partial \phi_n(w)}{(z-w)} + \cdots
\]

Therefore the fields \( \{ U_k(z) \} \) are not primary because

\[
T(z) U_k(w) = \frac{1}{2} \sum_{s=1}^{k} \frac{(N-k+s)!}{(N-k)!} a_s^{-2} \left( (s-1)(N-1) + 2(k-1) \right) a_o^{-2} \frac{s-1}{N} \frac{U_{k-s}(w)}{(z-w)^{s+2}}
\]

\[
= kU_k(w) \frac{T(w)}{(z-w)^2} + \frac{\partial U_k(w)}{(z-w)} + (TU_k)(z) + \cdots
\]

Using above OPE, we defined \( \Omega \) (see also 6.8–10) some primary fields of the Casimir \( W_N \) algebra, whose relations are given by

\[
\mathcal{U}_3(z) = U_3(z) - \frac{(N-2)}{2} \alpha_o \partial T(z)
\]

\[
\mathcal{U}_4(z) = U_4(z) + \Omega_{uv_3} \partial U_3(z) + \Omega_{u^2} \partial^2 T(z) + \Omega_{TT} (TT)(z)
\]

where

\[
\Omega_{uv_3} = -\frac{(N-3)}{2} a_o
\]

\[
\Omega_{u^2} = \frac{(N-2)(N-3)}{4N(22+5c)} \left[ -3 + N(13 + 3N + 2c) a_o^2 \right]
\]

\[
\Omega_{TT} = \frac{(N-2)(N-3)}{2N(22+5c)} \left[ 5 - N(5N + 7) a_o^2 \right]
\]

and

\[
\mathcal{U}_5(z) = U_5(z) + \Omega_{uv_4} \partial U_4(z) + \Omega_{u^2v_3} \partial^2 U_3(z) + \Omega_{u^3} \partial^3 U_3(z) + \Omega_{u^2v_3} (U_2 U_3)(z) + \Omega_{u^2v_2} U_2 \partial U_2(z)
\]

where

\[
\Omega_{uv_4} = -\frac{(N-4)}{2} a_o
\]

\[
\Omega_{u^2v_3} = \frac{3}{4} \frac{(N-3)(N-4)}{N(114 + 7c)} \left[ -2 + N(20 + c + 2N) a_o^2 \right]
\]
\[ \Omega_{\alpha,\beta} = \frac{(N-2)(N-3)(N-4)\alpha}{12N(114+7c)} \left[ 9 - N(33 + c + 9N)\alpha^2 \right] \]

\[ \Omega_{\beta,\gamma} = \frac{(N-3)(N-4)}{N(114+7c)} \left[ 7 - N(13 + 7N)\alpha^2 \right] \]

\[ \Omega_{\gamma,\alpha} = \frac{(N-2)(N-3)(N-4)\alpha}{2N(114+7c)} \left[ -7 + N(13 + 7N)\alpha^2 \right] \]

(2.16)

As being in line with ref.15, we have calculated all OPEs of the Casimir \( W_4 \) algebra in a previous work.

3. Operator Product Expansions (OPEs) for Chiral Vertex Operators

A chiral vertex operators are defined by

\[ V_\beta(z) =: e^{i\beta \cdot \varphi(z)} : \] (3.1)

Here, a non-simple root \( \beta \), for \( A_{N-1} \), is given by

\[ \beta = \sum_{i=1}^{N-1} m_i \alpha_i \] (3.2)

and The Fubini-Veneziano field \( \varphi(z) \), which has conformal spin-0

\[ \varphi(z) = q - ip \ln z + i \sum_{n \neq 0} \frac{1}{n} a_n z^{-n} \] (3.3)

By using conformal spin-0 contraction \( \varphi(z) \varphi(w) = -\ln |z - w| \), The standard OPE of \( V_\beta(z) \) with \( V_\beta(w) \)

\[ V_\beta(z) V_\beta(w) = (z - w)^{\beta \bar{\beta}} : V_\beta(z) V_\beta(w) : \] (3.4)

We can tell that the operator \( V_\beta(z) \) carries a root \( \beta \).

\[ h_j(z) V_\beta(w) = -\frac{\theta_j}{z - w} V_\beta(w) + \cdots \] (3.5)

where

\[ \theta_j = \theta_j(\beta) \equiv (\beta, \mu_j) \] (3.6)

The OPE with the stress-energy tensor \( T(z) \) is

\[ T(z) V_\beta(w) = \frac{h(\beta)}{(z - w)^2} V_\beta(w) + \frac{\eta_1^{\beta} V_\beta(w)}{z - w} + \cdots \] (3.7)

where \( \eta_1^{\beta}(z) \), is given by

\[ \eta_1^{\beta}(z) = \sum_{i,j} (1 - \delta_{ij}) \theta_i h_j(z) \] (3.8)

Thus the vertex operator \( V_\beta(z) \) is a conformal field of spin \( h(\beta) \) which is algebraic in \( \beta \), is given by

\[ h(\beta) = -\sum_{i<j} \theta_i \theta_j + \alpha_0 \sum_i (i-1) \theta_i \equiv U_2(\beta) \] (3.9)

* if we take \( \beta = \alpha_i \), a simple root, then \( \eta_1^{\beta} V_\beta(z) = \partial V_\beta(z) \).
For \( k > 2 \), similar OPEs between the \( U_k(z) \) and \( V_\beta(w) \) could not be calculated explicitly in general root \( \beta \) for \( A_{N-1} \), except the highest order singular term, since the fields \( U_i(z) \)'s are not primary, but primary OPEs will be given later in general form. We have

\[
U_i(z) V_\beta(w) = \frac{U_i(\beta)}{(z-w)^d} V_\beta(w) + \text{less sing. term's} \cdots
\]

(3.10)

Explicitly \( U_i(\beta) \)'s are some polynomial functions in \( \beta \). These are found to be

\[
U_3(\beta) = - \sum_{i < j < k} \theta_i \theta_j \theta_k + 2\alpha_0 \sum_{i < j} (i - 1) \theta_i \theta_j
\]

\[
+ \alpha_0 \sum_{i < j} (j - i - 1) \theta_i \theta_j + \alpha_0^2 \sum_i (i - 1) \theta_i
\]

(3.11)

\[
U_4(\beta) = - \sum_{i < j < k < l} \theta_i \theta_j \theta_k \theta_l + \alpha_0 \sum_{i < j < k} (i + j + k + 10) \theta_i \theta_j \theta_k
\]

\[
+ \alpha_0^2 \sum_{i < j} (-11 + 6 i + 6 j - i^2 - j^2) \theta_i \theta_j
\]

\[
+ \alpha_0^3 \sum_i (i - 1)(i - 3) \theta_i
\]

(3.12)

and

\[
U_5(\beta) = - \sum_{i < j < k < l < m} \theta_i \theta_j \theta_k \theta_l \theta_m + \alpha_0 \sum_{i < j < k < l} (i + j + k + l - 10) \theta_i \theta_j \theta_k \theta_l
\]

\[
+ \alpha_0^2 \sum_{i < j} (-35 + 10 i + i^2 + 10 j - i j - j^2 + 10 k - i j - j k + k^2) \theta_i \theta_j \theta_k
\]

\[
+ \alpha_0^3 \sum_{i < j} (i + j - 5)(10 - 5 i + i^2 - 5 j + j^2) \theta_i \theta_j
\]

\[
+ \alpha_0^4 \sum_i (i - 1)(i - 2)(i - 3)(i - 4) \theta_i
\]

(3.13)

Having found the primary fields \( U_j(z) \) we are now ready to compute the primary OPEs. The first result is

\[
\overline{U}_3(z) V_\beta(w) = \frac{\overline{U}_3(\beta)}{(z-w)^3} V_\beta(w) + \frac{(\zeta_1 V_\beta)(w)}{(z-w)^2} + \frac{(\zeta_2 V_\beta)(w)}{z-w} + \cdots
\]

(3.14)

where

\[
\overline{U}_3(\beta) = - \sum_{i < j < k} \theta_i \theta_j \theta_k + \alpha_0 \sum_{i < j} (i + j - N - 1) \theta_i \theta_j
\]

\[
+ \alpha_0^2 \sum_i (N - i)(i - 1) \theta_i
\]

(3.15)

\[
\zeta_1(z) = - \frac{\alpha_6}{2} \sum_{i=1}^{N-1} (2i - N) \gamma_i \sigma_j(z) + \sum_{i,j,k=j+1}^N (1 - \delta_{ij} - \delta_{ik}) \theta_i \theta_j \theta_k h_i(z)
\]

(3.16)

and

\[
\zeta_2(z) = - \alpha_0 \sum_{i=1}^{N-1} \tau_i \partial_i h_i(z) + h_{i+1}(z) - \sum_{i,j,k=j+1}^N (1 - \delta_{ij} - \delta_{ik}) \theta_i h_j h_k(z)
\]
\[ + \sum_{i,j,k=j+1}^{N} (1 - \delta_{ij} - \delta_{ik}) \theta_j \theta_k \partial h_i(z) \]  

(3.17)

where

\[ \gamma_i = (\beta, \alpha_i) \equiv \theta_i - \theta_{i+1} \]  

(3.18)

\[ \sigma_j(z) = \sum_{i=1}^{j} h_j(z) \]  

(3.19)

and

\[ \tau_j = (\beta, \lambda_j) \equiv \sum_{i=1}^{j} \theta_i \]  

(3.20)

One can repeat all the above computations for \( j = 4 \) and 5. The final results have also been calculated but we will not give the explicit results here because the expressions are quite long. For the purpose of illustration we give only the highest order singular term:

\[ \mathcal{U}_4(z) V_\beta(w) = \frac{\mathcal{U}_4(\beta)}{(z-w)^4} V_\beta(w) + \text{less sing. terms} \cdots \]  

(3.21)

where

\[ \mathcal{U}_4(\beta) = U_4(\beta) - 3 \Omega_{\alpha_3 \alpha_4} U_3(\beta) + 6 \Omega_{\alpha_2 \alpha_2} U_2(\beta) + \Omega_{\alpha \alpha} U_2(\beta) \left( U_2(\beta) + 2 \right) \]  

(3.22)

We also obtained the following composite OPE to use,

\[ (TT)(z) V_\beta(w) = \frac{h(\beta)}{(z-w)^4} V_\beta(w) + \left( \frac{2 h(\beta) + 1}{(z-w)^3} \right) (\eta_1^\beta V_\beta)(w) \]

\[ + \frac{2 h(\beta)(TV_\beta)(w)}{(z-w)^2} + \frac{(\eta_2^\beta V_\beta)(w)}{(z-w)^2} + \frac{2 h(\beta)(\partial TV_\beta)(w)}{z-w} + \frac{2(T \eta_1^\beta V_\beta)(w)}{z-w} + \cdots \]  

(3.23)

where \( \eta_1^\beta(z) \) is already defined in (3.8) and \( \eta_2^\beta(z) \) is given by

\[ \eta_2^\beta(z) = \left( \sum_{i,j}^{N} (1 - \delta_{ij}) \theta_i h_j(z) \right)^2 + \sum_{i,j}^{N} (1 - \delta_{ij}) \theta_i \partial h_j(z) \]  

(3.24)

and finally

\[ \mathcal{U}_5(z) V_\beta(w) = \frac{\mathcal{U}_5(\beta)}{(z-w)^5} V_\beta(w) + \text{less sing. terms} \cdots \]  

(3.25)

where

\[ \mathcal{U}_5(\beta) = U_5(\beta) - 4 \Omega_{\alpha \alpha_4} U_4(\beta) + 12 \Omega_{\alpha_2 \alpha_3} U_3(\beta) - 24 \Omega_{\alpha_3 \alpha_2} U_2(\beta) \]

\[ + \Omega_{\alpha \alpha_2} U_3(\beta) \left( U_2(\beta) + 3 \right) - 2 \Omega_{\alpha_2 \alpha \alpha} U_2(\beta) \left( U_2(\beta) + 3 \right) \]  

(3.26)

Here the highest order singular terms are exposed.

To summarize, we have given a systematic algorithm to compute all nontrivial OPEs between primary fields and a vertex operator in the Casimir \( W_N \) algebras basis.

* if we take \( \beta = \alpha_i \), a simple root, then \( (\eta_2^\beta V_\alpha)(z) = \partial^2 V_\alpha(z) \).
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References

1. M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, Vols. 1, 2 (Cambridge Univ. Press, Cambridge, 1987).
2. C. Itzykson, H. Saluer and J.B. Zuber, eds, Conformal Invariance and Applications to Statistical Mechanics (World Scientific, Singapore, 1988).
3. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.
4. J. Thierry-Mieg, Generalizations of the Sugawara construction, in: Nonperturbative Quantum Field Theory, eds G. ’t Hooft et al., Proc. Cargese School 1987 (Plenum Press, New York, 1988) p.567.
5. F. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. B304 (1988) 348; 371.
6. K. Hornfeck, Phys. Lett. B275, 355 (1992).
7. H.T.Özer, “On the construction of $W_N$-algebras in the form of $A_{N-1}$ Casimir algebras”, Mod. Phys. Lett. A11, 1139 (1996).
8. V.A. Fateev and A.B. Zamolodchikov, Nucl. Phys. B280 (1987) 644.
9. V.A. Fateev and S.L. Lukyanov, Sov. Sci. Rev. A Phys. Vol.15 Part.2 (1990) pp.47-77.
10. V.A. Fateev and S.L. Lukyanov, Int. J. Mod. Phys. A3 (1988) 507.
11. I.K. Kostov, Nucl. Phys. B33 (1988) 559.
12. B. Feigin, E. Frenkel, Integral of motion and quantum groups, [hep-th/9310022].
13. K. Thielemans, "A Mathematica\textsuperscript{TM} package for computing operator product expansions (OPEdefs 3.1)"\textsuperscript{,} Theoretical Phys. Group, Imperial College, London(UK).
14. S. Wolfram, Mathematica\textsuperscript{TM}, (Addison-Wesley,1990).
15. Blumenhagen et al., Nucl. Phys. B361. 255 (1991).