Bounded solutions of finite lifetime to differential equations in Banach spaces

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Abstract

Consider a smooth vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) and a maximal solution \( \gamma : [a,b] \to \mathbb{R}^n \) to the ordinary differential equation \( x' = f(x) \). It is a well-known fact that, if \( \gamma \) is bounded, then \( \gamma \) is a global solution, i.e., \( ]a,b[ = \mathbb{R} \). We show by example that this conclusion becomes invalid if \( \mathbb{R}^n \) is replaced with an infinite-dimensional Banach space.

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Introduction and statement of result

The starting point for our journey is a well-known result in the theory of ordinary differential equations: If the image \( \gamma([a,b]) \) of a maximal solution \( \gamma : [a,b] \to U \) to a differential equation

\[
x' = f(x)
\]

with locally Lipschitz right-hand side \( f : U \to \mathbb{R}^n \) on an open subset \( U \subseteq \mathbb{R}^n \) is relatively compact in \( U \), then \( \gamma \) is globally defined, i.e., \( ]a,b[ = \mathbb{R} \) (cf. [7, Chapter I, Theorem 2.1], [9, Korollar in 4.2.III], [11, Corollary 2 in §2.4]). In the special case \( U = \mathbb{R}^n \), this entails that bounded maximal solutions \( \gamma : ]a,b[ \to \mathbb{R}^n \) are always globally defined, exploiting that bounded sets and relatively compact subsets in \( \mathbb{R}^n \) coincide by the Theorem of Bolzano-Weierstrass (see, e.g., Corollaire 1 in [1, Chapter IV, §1, no. 5], or [12, Lemma 2.4] for this fact).

The first criterion applies equally well if \( \mathbb{R}^n \) is replaced with a Banach space (cf. [10, Chapter IV, Corollary 1.8]). However, bounded maximal solutions to ordinary differential equations in infinite-dimensional Banach spaces need not
be globally defined. Non-autonomous examples with locally Lipschitz right hand sides were given in [5] (in the Banach space $c_0$) and for Banach spaces admitting a Schauder basis in [3], [4]. By now, it is known that the pathology occurs for suitable autonomous systems on every infinite-dimensional Banach space, with locally Lipschitz right-hand side [8].

In the current note, we describe an easy, instructive example of a non-global, bounded solution to a vector field on a separable Hilbert space. In contrast to all of the cited literature, the vector field we construct is not only locally Lipschitz, but smooth (i.e., $C^\infty$).

**Theorem.** There exists a smooth vector field $f: \mathcal{H} \to \mathcal{H}$ on the real Hilbert space $\mathcal{H} := \ell^2(\mathbb{Z})$ of square summable real sequences $(a_n)_{n \in \mathbb{Z}}$, such that the ordinary differential equation $x' = f(x)$ has a bounded maximal solution $\gamma$ which is not globally defined.

Our strategy is to describe, in a first step, a smooth curve $\gamma: ]-1,1[ \to \mathcal{H}$ whose restrictions to $]-1,0]$ and $[0,1[$ have infinite arc length (Section 1). In a second step, we construct a smooth vector field $f: \mathcal{H} \to \mathcal{H}$ such that

$$\gamma'(t) = f(\gamma(t)) \quad \text{for all} \ t \in ]-1,1[,$$

ensuring that $\gamma$ is a solution to $x' = f(x)$ (Section 2). Thus $\gamma$ is not globally defined and it has to be a maximal solution because otherwise the arc length on one of the subintervals would be finite (contradiction).

1 The long and winding road

We fix a function $h: \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $h$ is smooth ($C^\infty$) with compact support inside $[-2,1]$;

(ii) $h(-2) = h(-1) = h(1) = 0$;

(iii) $h(0) = 1$;

(iv) $h'(t) > 0$ for all $t \in [-1,0[$, $h'(t) < 0$ for $t \in ]0,1[$ (whence $h(t) > 0$ there) and $h'(0) = 0$. 

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The existence of such a function is shown in Section 3. Using this function, we can define a smooth curve with values in the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$ via

$$\eta: \mathbb{R} \to \mathcal{H}, \quad t \mapsto \sum_{k \in \mathbb{Z}} h(t-k)e_k,$$

where $(e_k)_{k \in \mathbb{Z}}$ denotes the standard orthonormal basis of $\mathcal{H}$. Note that this sum is locally finite since $h$ has compact support; hence $\eta$ is smooth.

Calculating the derivative $\eta'(t) = \sum_{k \in \mathbb{Z}} h'(t-k)e_k$, we see that $\eta'(t)$ is always non-zero. In fact, if $n \in \mathbb{Z}$ with $t \in [n, n+1]$, then $t - n - 1 \in [-1, 0]$ and thus $\langle e_{n+1}, \eta'(t) \rangle = h'(t - n - 1) \neq 0$. 

Figure 1: Graph of the function $h$

Figure 2: The curve $\eta$ on the interval $[0,1]$
By construction of $\eta$, we have $\eta(n) = e_n$ for each $n \in \mathbb{Z}$, which implies that $\eta$ has infinite arc length. Since the real-valued function $h$ is bounded, it follows that the curve $\eta$ is (norm-) bounded in the Hilbert space $H$.

Next, we fix a diffeomorphism $\varphi : ]-1,1[ \longrightarrow \mathbb{R}$ between the open interval $]-1,1[\text{ and the real line, e.g. } \varphi(t) = \tan(\frac{\pi}{2}t) \text{ or } \varphi(t) = \frac{1}{1-t^2}$. We now define

$$\gamma : ]-1,1[ \longrightarrow H, \quad t \mapsto \eta(\varphi(t)).$$

This curve is just a reparametrization of $\eta$ and hence shares some important properties with $\eta$, namely it is bounded in $H$, the derivative is always nonzero and it has infinite arc length. However, one important difference is that $\gamma$ is not globally defined, so if we are able to show that $\gamma$ is a maximal solution to a (time-independent) differential equation, then our theorem is established.

### 2 The surrounding landscape

Having constructed the curve $\gamma : ]-1,1[ \longrightarrow H$ in Section 1, we shall now define a smooth vector field $f : H \longrightarrow H$ such that $\gamma$ is a solution to the differential equation $x' = f(x)$. Since $\gamma$ (as well as its restriction to $]-1,0]$ and its restriction to $[0,1[$) has infinite arc length by construction, the solution is maximal, and our theorem follows.

Write $\langle x, y \rangle := \sum_{n \in \mathbb{Z}} x_n y_n$ for $x = (x_n)_{n \in \mathbb{Z}}$, $y = (y_n)_{n \in \mathbb{Z}}$ in $H$, and $\|x\| := \sqrt{\langle x, x \rangle}$. We shall use the following facts about distances (to be proven in Section 4):

1. The distance function

$$d_\gamma : H \rightarrow [0, \infty], \quad x \mapsto \inf \left\{ \|\gamma(t) - x\| : t \in ]-1,1[ \right\}$$

from the curve $\gamma$ is continuous on $H$. In particular, the set $U_r := \{x \in H : d_\gamma(x) < r\}$ is open and contains the image of $\gamma$, for each $r > 0$.

2. There is a number $\rho > 0$ such that for all $x \in U_\rho$ there exists a unique $\tau(x) \in ]-1,1[$ such that $\gamma(\tau(x))$ has minimum distance to $x$, that is $\|\gamma(\tau(x)) - x\| = d_\gamma(x)$.

3. The map $\tau : U_\rho \longrightarrow ]-1,1[$ is smooth.\(^{1}\)

\(^{1}\)See [2], [6] and [10] for differential calculus on Banach spaces.
The preceding properties entail that the squared distance function
\[ d^2_\gamma: U_\rho \to [0, \infty]: x \mapsto (d_\gamma(x))^2 = \|\gamma(\tau(x)) - x\|^2 \]
is smooth on the neighborhood \( U_\rho \) of \( \gamma \). This enables us to define the smooth vector field \( f \), using a suitable cut-off function \( \theta \):
\[ f: H \to H, \quad x \mapsto \begin{cases} \theta \left( d^2_\gamma(x) \right) \gamma'(\tau(x)) & \text{if } d_\gamma(x) < \rho; \\ 0 & \text{if } d_\gamma(x) > \rho/2. \end{cases} \]
Here, \( \theta: \mathbb{R} \to \mathbb{R} \) is a fixed smooth function with \( \theta(0) = 1 \) which vanishes outside of \( [-\rho^2/4, \rho^2/4] \). It is easily checked using the properties (a), (b) and (c) that the map \( f \) is well defined and smooth. The curve \( \gamma \) is a solution to the associated differential equation, since for all \( t \in ]-1, 1[ \):
\[ f(\gamma(t)) = \theta \left( d^2_\gamma(\gamma(t)) \right) \gamma'(\tau(\gamma(t))) \bigg|_{t=0}^{t=1} = \gamma'(t). \]
This shows that there is a smooth vector field \( f \) on \( H \) such that a maximal solution of the differential equation is bounded but has only finite lifetime.

3 Details for Section 1

In Section 1 we used a function \( h: \mathbb{R} \to \mathbb{R} \) with certain properties (i)–(iv). We now prove the existence of \( h \). By the Fundamental Theorem,
\[ h: \mathbb{R} \to \mathbb{R}: x \mapsto \int_{-2}^{x} g(t) \, dt \]
with a suitable smooth function \( g: \mathbb{R} \to \mathbb{R} \). This reduces the problem of finding \( h \) to the problem of finding a function \( g \) with the following properties:
\[ (i)' \quad g \text{ is smooth with support inside } [-2, 1] \text{ and integral } \int_{-2}^{1} g(t) \, dt = 0; \]
\[ (ii)' \quad \int_{-2}^{-1} g(t) \, dt = 0; \]
\[ (iii)' \quad \int_{-1}^{0} g(t) \, dt = 1; \]
\[ (iv)' \quad g(t) > 0 \text{ for all } t \in [-1, 0[ , g(t) < 0 \text{ for all } t \in ]0, 1[, \text{ and } g(0) = 0. \]
It remains to construct such a function $g$. To this end, we start with a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ which is positive on $]-1, 1[$ and zero elsewhere, e.g.

$$
\psi : \mathbb{R} \to \mathbb{R} : t \mapsto \begin{cases} 
  e^{-\frac{1}{1-t^2}} & \text{if } |t| < 1 \\
  0 & \text{else.}
\end{cases}
$$

Using dilations and translations, we can create a function $\psi_{a,b}$ from the preceding one, which is positive on any given interval $]a, b[$:

$$
\psi_{a,b} : \mathbb{R} \to \mathbb{R} : t \mapsto \psi \left( -1 + 2 \frac{t-a}{b-a} \right).
$$

Now, we define the function $g$ as

$$
g := A \cdot \psi_{-2,-1} + B \cdot \psi_{-3/2,0} + C \cdot \psi_{0,1} \quad \text{(1)}
$$

with constants $A, B, C \in \mathbb{R}$ determined as follows:

Condition (iii)' requires that $B = (\int_{-1}^{0} \psi_{-3/2,0}(t) \, dt)^{-1}$. Thus $B > 0$.

Condition (ii)' requires that $A \int_{-2}^{-1} \psi_{-2,-1}(t) \, dt = -B \int_{-3/2}^{-1} \psi_{-3/2,0}(t) \, dt$ with $B$ as just determined. This equation can uniquely be solved for $A$ (with $A < 0$).

Condition (i)' requires that $C \int_{0}^{1} \psi_{0,1}(t) \, dt = - \int_{-2}^{0} g(t) \, dt = -1$ (where we used (iii)). This equation can be solved uniquely for $C$ (with $C < 0$).

Also (iv)' holds as $g(0) = 0$ by (1), $g(t) = B \psi_{-3/2,0}(t) > 0$ for $t \in [-1, 0[$ and $g(t) = C \psi_{0,1}(t) < 0$ for $t \in ]0, 1[$.

## 4 Details for Section 2

In this section, we prove the facts (a), (b) and (c) which were used in Section 2 to construct the vector field $f$.

(a) is easy to show: In fact, if a metric space $X$ is given and $A \subseteq X$ is a non-empty subset, then the distance function

$$
d_A : X \to [0, \infty] : x \mapsto \inf_{a \in A} d(x, a)
$$

is the infimum of a family of Lipschitz continuous functions on $X$ with Lipschitz constant 1. Hence $d_A$ is Lipschitz with constant 1 as well.
We now prove (b) and (c) using a so-called tubular neighborhood (a standard tool in differential geometry [10]). No familiarity with this method is presumed: All we need can be achieved directly, by elementary arguments.

We start with two easy lemmas concerning rotations:

**Lemma 4.1 (Rotation in \( \mathbb{R}^2 \))** Let \( v, w \in \mathbb{R}^2 \) be vectors in \( \mathbb{R}^2 \) with norm \( 1 \) and let \( R_{v,w} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the rotation around the origin mapping \( v \) to \( w \). Then \( R_{v,w}(w) = 2 \langle v, w \rangle w - v \).

**Proof.** Passing to a different coordinate system if necessary, we may assume that \( v = (1, 0) \) and \( w = (\cos \alpha, \sin \alpha) \). Then
\[
R_{v,w}(w) = (\cos(2\alpha), \sin(2\alpha)) = (\cos^2 \alpha - \sin^2 \alpha, 2 \cos \alpha \sin \alpha) = (2 \cos^2 \alpha - 1, 2 \cos \alpha \sin \alpha) = 2\langle e_1, (\cos \alpha, \sin \alpha) \rangle (\cos \alpha, \sin \alpha) - (1, 0),
\]
which indeed coincides with \( 2\langle v, w \rangle w - v \). \( \square \)

**Lemma 4.2 (Rotation in a real Hilbert space)** Let \( v, w \in \mathcal{H} \) be vectors of norm \( 1 \). We assume that \( v \neq -w \). Let \( R_{v,w} : \mathcal{H} \rightarrow \mathcal{H} \) be the rotation around \( 0 \) taking \( v \) to \( w \) and fixing every vector orthogonal to \( v \) and \( w \). Then the map \( R_{v,w} \) is given by the formula
\[
R_{v,w}(x) = x + \frac{2 \langle v, w \rangle + 1}{1 + \langle v, w \rangle} \langle x, v \rangle - \langle x, w \rangle w - \frac{\langle x, v + w \rangle}{1 + \langle v, w \rangle} v \quad \text{for all } x \in \mathcal{H}.
\]
In particular, the result depends smoothly on all parameters.

**Proof.** Since both sides of the equation are linear in \( x \), it is enough to check the following three special cases (where we used Lemma 4.1 for the second):

\[
R_{v,w}(v) = w; \quad R_{v,w}(w) = 2 \langle v, w \rangle w - v; \quad R_{v,w}(x) = x \quad \text{for all } x \in \{ v, w \}^\perp.
\]

All three cases are settled by straightforward calculations. \( \square \)

Note that the map \( R_{v,w} \) is not defined in the case that \( v = -w \) as the denominator \( 1 + \langle v, w \rangle \) becomes zero.

\[\text{As usual, for a subset } Y \subseteq \mathcal{H} \text{ we write } Y^\perp := \{ x \in \mathcal{H} : (\forall y \in Y) \langle x, y \rangle = 0 \}. \]
Definition 4.3 By the normal bundle of the curve \( \eta \), we mean the subset
\[
\mathcal{N} := \{(\eta(t), v) : t \in \mathbb{R}, v \in \mathcal{H} \text{ with } \langle \eta'(t), v \rangle = 0\}
\]
of \( \mathcal{H} \times \mathcal{H} \). It consists of all vectors with basepoint on the curve which are perpendicular to the curve.

Although the set \( \mathcal{N} \) carries the structure of a smooth vector bundle, we need not use the theory of vector bundles in what follows. Recall that \( \eta'(t) \neq 0 \) for all \( t \in \mathbb{R} \). It is useful to record further properties of \( \eta \). We shall use that
\[
\rho_0 := \min_{t \in [0,1]} \sqrt{h(t)^2 + h(t - 1)^2} > 0
\]
as \( h(t) > 0 \) for all \( t \in [-1,1[ \).

Lemma 4.4 (a) \( \eta : \mathbb{R} \to \mathcal{H} \) is injective.

(b) If \( n \in \mathbb{Z} \) and \( t \in [n, n+1[ \), then \( \eta(t) \) is a linear combination of \( e_n, e_{n+1} \) and \( e_{n+2} \).

(c) \( \|\eta(s) - \eta(t)\| \geq \rho_0 \) for all \( s, t \in \mathbb{R} \) such that \( |s - t| > 3 \).

(d) \( \frac{\eta'(t)}{\|\eta'(t)\|} \neq \frac{\eta'(0)}{\|\eta'(0)\|} \) for all \( t \in \mathbb{R} \).

Proof. (a) Let \( t \leq s \) in \( \mathbb{R} \) such that \( \eta(t) = \eta(s) \). There is \( n \in \mathbb{Z} \) such that \( t \in [n, n+1[ \). If \( s \geq n + 1 \) was true, then \( \langle e_n, \eta(s) \rangle = h(s - n) = 0 \) (as \( \text{supp}(h) \subseteq [-2,1] \)) while \( \langle e_n, \eta(t) \rangle = h(t - n) > 0 \) (since \( t - n \in [0,1[ \)). Thus we would get \( \eta(s) \neq \eta(t) \), a contradiction. As a consequence, \( s \in [n, n+1[ \) as well. Now \( h(s - n) = \langle e_n, \eta(s) \rangle = \langle e_n, \eta(t) \rangle = h(t - n) \) implies that \( s = t \), using that \( h|_{[0,1]} \) is strictly decreasing and hence injective.

(b) Let \( m \in \mathbb{Z} \) with \( h(t-m) = \langle e_m, \eta(t) \rangle \neq 0 \). Since \( \text{supp}(h) \subseteq [-2,1] \), we deduce that \( t-m \in [-2,1[ \), whence \( m \in [t-1, t+2[ \) and thus \( m \in [n-1, n+3[ \), which entails \( m \in \{n, n+1, n+2\} \). The assertion follows.

(c) As \( \text{supp}(h) \subseteq [-2,1] \) and \( |s - t| > 3 \), we cannot have both \( h(t-k) \neq 0 \) and \( h(s-k) \neq 0 \) for any \( k \in \mathbb{Z} \). Hence \( \eta(t) \) and \( \eta(s) \) are orthogonal vectors and thus \( \|\eta(s) - \eta(t)\| = \sqrt{\|\eta(s)\|^2 + \|\eta(t)\|^2} \geq \|\eta(t)\| = \sqrt{\sum_{k \in \mathbb{Z}} (h(t-k))^2} \geq \sqrt{h(t-n-1)^2 + h(t-n)^2} \geq \rho_0 \), with \( n \) as in (b).

(d) Note that \( \eta'(0) = h'(-1)e_1 \) (as \( h'(-2) = h'(0) = h'(1) = 0 \)), where \( h'(-1) > 0 \). If \( \eta'(t) \) was a negative real multiple of \( \eta'(0) \) and hence of \( e_1 \), then \( h'(t - 1) = \langle e_1, \eta'(t) \rangle < 0 \), thus \( t - 1 \in [-2, -1[ \) or \( t - 1 \in ]0,1[ \) (as \( h'_{[-1,0]} \geq 0 \)). In the first case, \( \langle e_0, \eta'(t) \rangle = h'(t) > 0 \), contrary to \( \eta'(t) \in -\mathbb{R}e_1 \). In the second case, \( \langle e_2, \eta'(t) \rangle = h'(t - 2) > 0 \), contrary to \( \eta'(t) \in -\mathbb{R}e_1 \). \( \square \)
Lemma 4.5 (Global parametrization of the normal bundle of $\eta$)

Let $H_0 := \{\eta'(0)\}^\perp$ and $R_t := R \frac{\eta'(0)}{\|\eta'(0)\|} \frac{\eta'(t)}{\|\eta'(t)\|}$ be the rotation turning $\frac{\eta'(0)}{\|\eta'(0)\|}$ to $\frac{\eta'(t)}{\|\eta'(t)\|}$, as introduced in Lemma 4.2. Then the following map is a bijection:

$$
\Psi : \mathbb{R} \times H_0 \rightarrow N, \quad (t, x) \mapsto (\eta(t), R_t(x)).
$$

**Proof.** First of all, the map $R_t = R \frac{\eta'(0)}{\|\eta'(0)\|} \frac{\eta'(t)}{\|\eta'(t)\|}$ is injective on an open neighborhood $V$ of $x$. For any $N \ni x$, there exists an open neighborhood $V_x$ of $x$ such that $\eta(t)(V_x)$ is disjoint from the compact set $K$.

Injectivity: Assume $\Psi(t_1, x_1) = \Psi(t_2, x_2)$. Since the curve $\eta : \mathbb{R} \rightarrow \mathcal{H}$ is injective (see Lemma 4.4(a)), we get $t_1 = t_2$. Now, the rotation map is clearly bijective and hence $x_1 = x_2$ which shows injectivity of $\Psi$.

Surjectivity: Let $(\eta(t), v) \in N$ be given. Because $R_t$ is a bijective isometry taking $\eta'(0)$ to a non-zero multiple of $\eta'(t)$, we have $R_t(\{\eta'(0)\}^\perp) = \{\eta'(t)\}^\perp$. Thus $\Psi(t) \times H_0 = \{\eta(t)\} \times \{\eta'(t)\}^\perp$, entailing the surjectivity of $\Psi$. \qed

We will use the preceding parametrization of the normal bundle to construct a parametrization of a tubular neighborhood of $\eta$. Before, we recall a simple lemma from the theory of metric spaces:

Lemma 4.6 (Local injectivity around a compact set) Let $X$ be a metric space and let $f : X \rightarrow Y$ be a continuous map to some topological space $Y$. We assume that $f$ is locally injective, i.e. each $x \in X$ has an open neighborhood $V_x$ in $X$ on which $f$ is injective. Assume furthermore that $f$ is injective when restricted to a non-empty compact set $K \subseteq X$. Then $f$ is injective on an $\varepsilon$-neighborhood $B_\varepsilon(K) := \{x \in X : d_K(x) < \varepsilon\}$ of $K$.

**Proof.** The product space $X \times X$ becomes a metric space if we define the distance between $(x_1, x_2)$ and $(x'_1, x'_2)$ as the maximum of $d(x_1, x'_1)$ and $d(x_2, x'_2)$. For $x \in X$ and $(x_1, x_2) \in V_x \times V_x$, we have $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$. The set $C$ of all pairs $(x_1, x_2)$ on which $f$ fails to be injective can therefore be written as

$$
C := \left\{ (x_1, x_2) \in X \times X : f(x_1) = f(x_2) \text{ and } x_1 \neq x_2 \right\}
$$

$$
= \left\{ (x_1, x_2) \in (X \times X) \setminus \bigcup_{x \in X} (V_x \times V_x) : f(x_1) = f(x_2) \right\},
$$

showing that $C$ is a closed subset of the product $X \times X$. The set $C$ is disjoint to the compact set $K \times K$ since $f$ is injective on $K$. Let $\varepsilon$ be the distance between the sets $C$ and $K \times K$. It follows that $f$ is injective on $B_\varepsilon(K)$. \qed
Lemma 4.7 (Existence of a tubular neighborhood) Consider the map
\[ \Phi: \mathbb{R} \times \mathcal{H}_0 \to \mathcal{H}, \quad (t, x) \mapsto \eta(t) + R_t(x), \]
which is the composition of the parametrization map \( \Psi \) from Lemma 4.5 and the addition in the Hilbert space \( \mathcal{H} \).

Then there exists a constant \( \rho > 0 \) such that \( \Phi \) maps the open set
\[ \Omega_\rho := \{ (t, x) \in \mathbb{R} \times \mathcal{H}_0 : \| x \| < \rho \} \]
diffeomorphically onto the open set
\[ U_\rho := \{ x \in \mathcal{H} : d_{\eta(\mathbb{R})}(x) < \rho \} . \]
Moreover, for all \( (t, x) \in \Omega_\rho \), the unique point on \( \eta(\mathbb{R}) \) with minimum distance to \( \Phi(x, t) \) is \( \eta(t) \).

Proof. Observe first that \( \Phi \) is a smooth map as a composition of smooth maps. Next, we calculate the directional derivative of \( \Phi \) at a point \((t_0, 0)\) in a direction \((t, x)\):

\[
\lim_{s \to 0} \frac{\Phi((t_0, 0) + s(t, x)) - \Phi(t_0, 0)}{s} = \lim_{s \to 0} \frac{1}{s}(\Phi(t_0 + st, sx) - \Phi(t_0, 0)) = \lim_{s \to 0} \frac{1}{s}(\eta(t_0 + st + R_{t_0+st}(sx) - \eta(t_0) - R_{t_0}(0)) = \lim_{s \to 0} \frac{1}{s}(\eta(t_0 + st) - \eta(t_0)) + \lim_{s \to 0} R_{t_0+st}(x) = \eta'(t_0) \cdot t + R_{t_0}(x).
\]

Hence, the derivative of \( \Phi \) at \((t_0, 0)\) is the linear mapping \( \mathbb{R} \times \mathcal{H}_0 \to \mathcal{H} \), \((t, x) \mapsto \eta'(t_0) \cdot t + R_{t_0}(x) \) which is invertible.

By the Inverse Function Theorem, there is an open neighborhood \( \Omega_{t_0} \) of \((t_0, 0)\) in \( \mathbb{R} \times \mathcal{H}_0 \) such that \( \Phi|_{\Omega_{t_0}} \) is a diffeomorphism onto its open image \( \Phi(\Omega_{t_0}) \).

For the moment, let us restrict our attention to the compact set \( [0, 4] \times \{0\} \subseteq \mathbb{R} \times \mathcal{H}_0 \) on which \( \Phi \) is injective (as so is \( \eta \)). Then, by Lemma 4.6, there is \( \rho > 0 \) such that \( \Phi \) is injective on \( [0, 4] \times B^{\mathcal{H}_0}_{\rho}(0) \) (where \( B^{\mathcal{H}_0}_{\rho}(0) := \{ x \in \mathcal{H}_0 : \| x \| < \rho \} \)). Since \( [0, 4] \times \{0\} \) is covered by open sets on which \( \Phi \) is a
diffeomorphism, after shrinking \( \rho \) we may assume that \( \Phi \) takes \([0, 4[ \times B^H_\rho(0)\) diffeomorphically onto an open set. We may also assume that \( \rho < \frac{\rho_0}{2} \), for \( \rho_0 \) as in \((2)\). Then \( \Omega_\rho := \mathbb{R} \times B^H_\rho(0) \) has all the required properties:

Exploiting the self-similarity of \( \eta \), let us show that \( \Phi \) is injective on the set \([n, n + 4[ \times B^H_\rho(0) \) for each \( n \in \mathbb{Z} \) and that \( \Phi \) restricts to a diffeomorphism from \([n, n + 4[ \times B^H_\rho(0) \) onto an open subset of \( \mathcal{H} \). To this end, let

\[
S_n : \mathcal{H} \to \mathcal{H}
\]

be the bijective isometry determined by \( S_n(e_k) = e_{k+n} \) for all \( k \in \mathbb{Z} \). Then \( \eta(t+n) = S_n\eta(t) \) for all \( t \in \mathbb{R} \) and thus also \( \eta'(t+n) = S_n\eta'(t) \). Hence

\[
\Phi(t+n, x) = \eta(t+n) + R_{t+n}(x) = S_n\eta(t) + S_nR_ir^1S_{-1}^1R_{t+n}(x) = S_n\Phi(t, R^1_{t}S^1_{t-n}R_{t+n}(x)).
\]

The map \( \Theta_n : \mathbb{R} \times \mathcal{H}_0 \to \mathbb{R} \times \mathcal{H}_0 \), \( \Theta_n(t, x) := (t, R^1_{t}S^1_{t-n}R_{t+n}(x)) \)

is a bijection and smooth (using that the mapping

\[
\mathbb{R} \times \mathcal{H}_0 \to \mathcal{H}, \quad (t, x) \mapsto R^1_{t}(x) = R_{\frac{d'(t)}{\|d'(t)\|}}\left(0\right)
\]

is smooth). Also \( \Theta_n^{-1} \) is smooth, as \( \Theta_n^{-1}(t, x) = (t, R^{-1}_{t-n}S_nR_{t}(x)) \). Note that \( R^{-1}_{t}S^1_{t-n}R_{t+n} \) is a bijective isometry which fixes \( \eta'(0) \) and hence takes \( \mathcal{H}_0 = \{\eta'(0)\} \) onto itself. Hence \( \Theta_n \) is a diffeomorphism that maps \([0, 4[ \times B^H_\rho(0) \) (as well as \([0, 4[ \times B^H_\rho(0) \)) onto itself. Now \( \Phi(t, x) = S_n\Phi(\Theta_n(t-n, x)) \)

by the above, whence \( \Phi \) takes \([n, n + 4[ \times B^H_\rho(0) \) diffeomorphically onto an open set, and is injective on \([n, n + 4[ \times B^H_\rho(0) \) (as desired).

\( \Phi \) is injective on \( \Omega_\rho \): Let \((s, x), (t, y) \in \mathbb{R} \times B^H_\rho(0) \) with \( \Phi(s, x) = \Phi(t, y) \). If we had \(|s-t| > 3\), then \(|\Phi(s, x) - \Phi(t, y)| = \|\eta(s)-\eta(t) + R_s(x) - R_t(y)| \geq \|\eta(s)-\eta(t)\| - \|R_s(x)\| - \|R_t(y)\| \geq \rho_0 - 2\rho > 0 \) would follow, contradiction. Thus \(|s-t| \leq 3 \) and hence \( s, t \in [n, n+4] \) for some \( n \in \mathbb{Z} \). Thus \((s, x) = (t, y)\), by injectivity of \( \Phi \) on \([n, n + 4[ \times B^H_\rho(0) \).

\( \Phi(\Omega_\rho) \) is open and \( \Phi|_{\Omega_\rho} \) is a diffeomorphism onto its image: We just verified that \( \Phi|_{\Omega_\rho} \) is injective. Since \( \Omega_\rho = \bigcup_{n \in \mathbb{Z}} [n, n + 4[ \times B^H_\rho(0) \) and \( \Phi \) takes each of the sets \([n, n + 4[ \times B^H_\rho(0) \) diffeomorphically onto an open subset of \( \mathcal{H} \), the assertion follows.

We now show that \( \Phi(\Omega_\rho) = U_\rho \). Since \(|\eta(t) - \Phi(t, x)| = \|R_t(x)\| \leq \rho \) if \((t, x) \in \Omega_\rho \), we have \( \Phi(\Omega_\rho) \subseteq U_\rho \). For the converse inclusion, let \( p \in U_\rho \).
To see that \( p \in \Phi(\Omega_p) \), we first show that the distance \( d_{\eta(\mathbb{R})}(p) \) is attained, i.e., there is \( s \in \mathbb{R} \) such that \( d_{\eta(\mathbb{R})}(p) = \|\eta(s) - p\| \). If this was wrong, we could choose a sequence \( s_k \in \mathbb{R} \) such that \( \|\eta(s_k) - p\| \to d_{\eta(\mathbb{R})}(p) \). Then \( |s_k| \to \infty \) (otherwise, \( s_k \) had a bounded subsequence inside \([-R, R]\) for some \( R > 0 \), and then the minimum of the continuous function \( s \mapsto \|\eta(s) - p\| \) on this compact interval would coincide with \( d_{\eta(\mathbb{R})}(p) \), a contradiction). For each \( k \in \mathbb{N} \), there is \( n_k \in \mathbb{Z} \) such that \( s_k \in [n_k, n_k + 1] \). Then \( |n_k| \to \infty \) as well. After passing to a subsequence, we may assume that \( s_k - n_k \in [0, 1] \) converges to some \( \Delta \in [0, 1] \). Writing \( p_m := \langle e_m, p \rangle \) for \( m \in \mathbb{Z} \), we have \( \|p\|^2 = \sum_{m \in \mathbb{Z}} p_m^2 \) and \((p_m)_{m \in \mathbb{Z}} \in \ell^2(\mathbb{Z})\). Lemma \[4.4\](b) now shows that

\[
\|\eta(s_k) - p\|^2 = |h(s_k - n_k) - p_{n_k}|^2 + |h(s_k - n_k - 1) - p_{n_k+1}|^2 + |h(s_k - n_k - 2) - p_{n_k+2}|^2 + \sum_{m \notin \{n_k, n_k+1, n_k+2\}} p_m^2.
\]

Letting \( k \to \infty \) (and using that \( p_m \to 0 \) as \( |m| \to \infty \)), we deduce that

\[
d_{\eta(\mathbb{R})}(p)^2 = h(\Delta)^2 + h(\Delta - 1)^2 + h(\Delta - 2)^2 + \|p\|^2 \geq \rho_0^2
\]

and thus \( d_{\eta(\mathbb{R})}(p) \geq \rho_0 \). But \( d_{\eta(\mathbb{R})}(p) < \rho \leq \rho_0 \), contradiction. Hence, there exists \( s \in \mathbb{R} \) such that \( d_{\eta(\mathbb{R})}(p) = \|\eta(s) - p\| \).

By the preceding, the distance between the points \( \eta(r) \) and \( p \) (as a function on \( r \)) is minimized for \( r = s \). Since \( \frac{d}{dr}\|\eta(r) - p\|^2 = 2\langle \eta'(r), \eta(r) - p \rangle \), we deduce that the derivative \( \eta'(s) \) has to be orthogonal to \( \eta(s) - p \). Thus \( y := R_s^{-1}(p - \eta(s)) \in \mathcal{H}_o \) and \( p = \Phi(s, y) \). Since \( \|y\| = \|p - \eta(s)\| = d_{\eta(\mathbb{R})}(p) < \rho \), we have \((s, y) \in \Omega_p \) and hence \( p = \Phi(s, y) \in \Phi(\Omega_p) \). Thus \( U_\rho = \Phi(\Omega_p) \).

If also \( p = \Phi(t, x) \) for some \((t, x) \in \Omega_p \), then \((t, x) = (s, y)\) by injectivity of \( \Phi \) and thus \( s = t \). Hence \( \eta(t) \) is the unique point in \( \eta(\mathbb{R}) \) which minimizes \( \|\eta(t) - \Phi(t, x)\| \).

We are now in the position to prove the facts (b) and (c) stated in Section \[2\].

To prove (b), we use the number \( \rho > 0 \) constructed in Lemma \[4.7\] for the curve \( \eta \). Since the curves \( \gamma \) and \( \eta \) differ only by a re-parametrization, the existence of a unique nearest point remains true.

To obtain (c), we may write the function

\[
\tau : U_\rho \longrightarrow ]-1, 1[\]
which assigns to each point $x \in U_\rho$ the index $t \in ]-1, 1[$ such that $\gamma(t)$ has minimum distance to $x$ as follows:

$$\tau = \varphi^{-1} \circ \pi_R \circ \Phi^{-1}$$

where $\Phi: \Omega_\rho \rightarrow U_\rho$ is the diffeomorphism from Lemma 4.7, the mapping $\pi_R: \Omega_\rho \rightarrow \mathbb{R}$, $(t, x) \mapsto t$ denotes the projection onto the first component and $\varphi: ]-1, 1[ \rightarrow \mathbb{R}$ is the diffeomorphism used to define the curve $\gamma$. As a composition of smooth maps, the map $\tau$ is smooth. The proof is complete. □

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