Odd graceful labeling on the *Ilalang graph* ($S_n, 3$) it’s variation

Z Amri$^{1,*}$, I Irvan$^1$, I Maryanti$^2$ and H Sumardi$^3$

$^1$ Program Studi Magister Pendidikan Matematika, PPs Universitas Muhammadiyah Sumatera Utara, Jl. Denai No. 217, Kota Medan 20226, Indonesia
$^2$ Program Studi Pendidikan Matematika, FKIP, Universitas Muhammadiyah Sumatera Utara, Jl. Kapt. Muktar Basri No. 3, Kota Medan 20226, Indonesia
$^3$ Program Studi Magister Pendidikan Matematika, Universitas Bengkulu, Jl. W. R. Supratman, Kandang Limun, Muara Bangka Hulu, Bengkulu 38371, Indonesia

*zulfiamri@umsu.ac.id

Abstract. A graph called odd graceful graph if there is an injective mapping $f: V(G) \rightarrow \{0,1,2,\ldots,2q - 1\}$ such that for each edge $xy \in E(G)$ labeled by $|f(u) - f(v)|$, and the results will form a label on the edge $\{1,3,5,\ldots,2q - 1\}$ which is bijective. Variabel $q$ is the number of edges. This paper purpose is to construct the formula of odd graceful labelling on the *ilalang graph* ($S_n, 3$) and it’s variation.

1. Introduction
A graph is a pair $G = (V, E)$ where $V$ is a nonempty set whose elements are called vertices and $E$ is a set of two-sets of vertices, whose elements are called edges (sometime links or lines). Generally, graph is a model that represents a trip, a computer connection networks, a television channel network in accordance with the definition of problem that act as vertices (objects) and edges (paths). Labeling on graph $G$ is the assignment of labels represented by integers to edges and/or vertices of a graph by specifics rules [1]. If the original set of labeling is just a set of vertices on the graph $G$, it is called the vertices labeling. If the labeling is just a mapping of the edge on the graph $G$ to a positive integer, then it is called an edge labeling. And when labeling is a mapping of a set of edges and a set of vertices to a set of positive integers it is called total labeling [2]. Galian [1] re-noted various researches about labeling which was given the title *A Dynamic Survey of Graph Labeling* has a reference of 2,264 titles which in 2010 were still 1,198 titles so that it can be said that in the past 6 years it had increased by 1,066 titles. This proves the development of labeling is quite rapid recently. Labeling recorded in *A Dynamic Survey of Graph Labeling* includes anti-magical labeling and its derivative variations, harmonious labeling and its derivative labeling of quantities and modification as well as labeling of graceful and its modification.

Graceful labeling was first put forward by Alex Rosa in 1966 which was originally called $\alpha$-labeling or $\alpha$-voluation. Some studies on labeling of graceful among them disclosed by Amri, et al namely for *ilalang graph* ($S_n, r$), $r \geq 3$ [3], centipede graph [4], H-star and A-star graphs [5] 8-Star graphs [6]. The graphs are in the Star Family graph, which is a graph induced from the $S_n$ star graph.
Modifications of graceful labeling include the labeling of graceful schemes, labeling $\hat{\rho}$ and graceful labeling odd.

A graph is said to have an odd graceful labeling if there is an injection mapping $f$ from $V(G)$ to \{0,1,2,3,...,2q-1\} so that each $xy$ edge is labeled $|f(x)-f(y)|$, the labeling results after ordering from the set \{1,3,5,...,2q-1\} \cite{7}. Each graph that satisfies the odd graceful labeling is called the odd graceful graph.

Gnanajothi proved that the odd graceful labeling graph class is between the $\alpha$-labeling graph class and the bipartite graph class and shows that any $\alpha$-labeling graph can be an odd graceful graph and that any odd or odd circle graph is not an odd graceful graph. He also showed the following graphs which were odd graceful graphs including the path graph $P_n$, the even graph $C_n$, the complete bipartite graph $K_{m,n}$, $P_n \odot K_1$ graph, book graph, $C_n \odot K_1$ graph, $C_4$ combined graph, $C_n \times K_1$ graph, cartepilar graph, product graph $P_n \times P_2$ \cite{7}.

Some odd graceful graphs on separate combinations of circle graph $C_n$ with path graph $P_n$, namely $C_4 \cup P_n$ graph, $C_6 \cup P_n$ graph, $C_8 \cup P_n$ graph, and $C_{10} \cup P_n$ graph which are then generalized into $C_m \cup P_n$ graph with $m$ being an even number and $n$ is greater than $m-4$ \cite{8}. Finally, Moussa and Badr in 2016 have constructed odd graceful labeling on the ladder graph with $m-\text{pendant} \odot mk_1$, namely $L_n \odot mk_1$, $S(l_n) \odot mk_1$, $\Delta_k - \text{snake}$, $S(\Delta_k - \text{snake}) \odot mk_1$ \cite{9}. In this article, we want to construct an odd graceful labeling on $(S_n,3)$ Ilalang graph $(S_n,3)$ \cite{3} and construct a new graph class of variations in the $S_n$ star graph.

2. Methods
The method used in this research is the method of studying literature, studying the available works. The results of the study were continued by constructing the odd graceful labeling, compiling the equations and showing the equations fulfilling the odd labeling of graceful in general.

3. Odd graceful labeling on family of star graph
This section will show the definitions and examples of graphs used in this article, and then provide a formula so that the graph meets odd graceful labeling. Amri et al, have defined $(S_n,3)$ ilalang graph as a graceful graph \cite{3}. Here are given definitions and examples as well as $(S_n,3)$ ilalang graph notation.

Definition 1. \cite{2}. The Star Graph family is graphs built from a combination of $S_n$ star graphs.

Definition 2. \cite{3}. The $(S_n,3)$ ilalang graph is a graph constructed from three $S_n$ star graphs and then given a node c that is connected to each $S_n$ star graph through an arc.

Following are examples and $(S_n,3)$ ilalang graph notation (Figure 1).

![Figure 1. Example and notation of $(S_n,3)$ Ilalang Graph.](image-url)
**Definition 3.** \((S_n, 3) − (n − 2)\) *ilalang graph*, \(n \geq 4\), is a graph that constructed from *ilalang graph* \((S_n, 3)\) by eliminating \(n − 2\) nodes in one of \(S_n\) star graph.

For more clearly given the example in Figure 2 as well as the notation.

![Figure 2](image-url)

**Figure 2.** Example and notation of \(((S_n, 3) − (n − 2))\) *Ilalang Graph*.

**Definition 4.** [7] A graph \((G)\) is called having odd graceful if there is injective mapping from \(f: V(G) \rightarrow \{0, 1, 2, ..., 2q − 1\}\) so that every edge \(xy \in E(G)\) is labeled as \(|f(x) − f(y)|\), the result of the labelling on its edge will maps \(f': E(G) \rightarrow \{1, 3, 5, ..., 2q − 1\}\) bijectively.

**Theorem 5.** \((S_n, 3)\) *Ilalang graph* has odd graceful labelling.

**Proof.** Let the notation of \((S_n, 3)\) *ilalang graph* is given in Figure 1 above. It is shown that vertices set \(V(S_n, 3) = \{c_0, c_1, c_2, c_3, v_1^1, v_2^1, ..., v_n^1, v_1^2, v_2^2, ..., v_n^2, v_1^3, v_2^3, ..., v_n^3\}\) and edges set \(E(S_n, 3) = \{c_0c_1, c_0c_2, c_0c_3, c_1v_1^1, c_1v_2^1, ..., c_1v_n^1, c_2v_1^2, c_2v_2^2, ..., c_2v_n^2, c_3v_1^3, c_3v_2^3, ..., c_3v_n^3\}\) So, the number of vertices and edges are \(3n + 4\)and \(3n + 3\), respectively. Below defined the \(f\) labeling:

\[
\begin{align*}
  f(c_0) &= 4n + 5 \\
  f(c_1) &= 0 \\
  f(c_2) &= 2 \\
  f(c_3) &= 2n + 4 \\
  f(v_1^i) &= 6n + 7 − 2i; i = 1, 2, ..., n \\
  f(v_2^i) &= 4n + 5 − 2i; i = 1, 2, ..., n \\
  f(v_3^i) &= 3 + 2i; i = 1, 2, ..., n
\end{align*}
\]

\(f\) labeling which defined from equation (1) to (7) are injective mapping from the set \(V\) into the integer set \(\{0, 1, 2, ..., 2q − 1\}\), so each edge \(xy \in E\) is labeled by \(f'\) edge labeling which are induced from \(f\) by \(f'(xy) = |f(x) − f(y)|\), is obtained edge labeling on \((S_n, 3)\) *ilalang graph* as follow:

\[
\begin{align*}
  f'(c_0c_1) &= |f(c_0) − f(c_1)| = 4n + 5 \\
  f'(c_0c_2) &= |f(c_0) − f(c_2)| = 4n + 3 \\
  f'(c_0c_3) &= |f(c_0) − f(c_3)| = 2n + 1 \\
  f'(c_1v_1^1) &= |f(c_1) − f(v_1^1)| = 6n + 7 − 2i; i = 1, 2, ..., n \\
  f'(c_2v_2^1) &= |f(c_2) − f(v_2^1)| = 4n + 3 − 2i; i = 1, 2, ..., n \\
  f'(c_3v_3^1) &= |f(c_3) − f(v_3^1)| = 2n + 1 − 2i; i = 1, 2, ..., n
\end{align*}
\]

From equations (8) to (13), is obtained label for the edge of \(\{1, 3, 5, ..., 2q − 1\}\) so that \((S_n, 3)\) *ilalang graph* is odd graceful graph.

**Theorema 6.** *Ilalang graph* \(((S_n, 3) − (n − 2))\), \(n \geq 4\) has odd graceful labelling.
Proof. Let the notation of \((S_n, 3) - (n - 2)\) Ilalang graph is given in Figure 2 above. It is shown that vertices set \(V((S_n, 3) - (n - 2)) = \{c_0, c_1, c_2, c_3, c_4, c_5, v^1_1, v^1_2, ..., v^1_n, v^2_1, v^2_2, ..., v^2_n\}\) and edges set \(E((S_n, 3) - (n - 2)) = \{c_0c_1, c_0c_2, c_0c_3, c_3c_4, c_5c_5, c_1v^1_1, c_1v^1_2, c_2v^1_1, c_2v^1_2, ..., c_1v^1_n, c_2v^2_1, c_2v^2_2, ..., c_2v^2_n\}\). So, the number of vertices and edges are \(2n + 5\) and \(2n + 6\), respectively. Below defined the \(f\) labeling:

\[
\begin{align*}
    f(c_0) &= 2n + 9 \quad (14) \\
    f(c_1) &= 0 \quad (15) \\
    f(c_2) &= 2 \quad (16) \\
    f(c_3) &= 4 \quad (17) \\
    f(c_4) &= 3 \quad (18) \\
    f(c_5) &= 1 \quad (19) \\
    f(v^1_i) &= 4n + 11 - 2i; i = 1, 2, ..., n \quad (20) \\
    f(v^2_i) &= 2n + 7 - 2i; i = 1, 2, ..., n \quad (21)
\end{align*}
\]

If labeling which defined from equations (14) to (21) are injective mapping from the set \(V\) into the integer set \(\{0, 1, 2, ..., 2q - 1\}\), so each edge \(xy \in E\) is labeled by \(f\) edge labeling which are induced from \(f\) by \(f'(xy) = |f(x) - f(y)|\), is obtained edge labeling on \((S_n, 3)\) Ilalang graph as follow:

\[
\begin{align*}
    f'(c_0c_1) &= |f(c_0) - f(c_1)| = 2n + 9 \quad (22) \\
    f'(c_0c_2) &= |f(c_0) - f(c_2)| = 2n + 7 \quad (23) \\
    f'(c_0c_3) &= |f(c_0) - f(c_3)| = 2n + 5 \quad (24) \\
    f'(c_3c_4) &= |f(c_3) - f(c_4)| = 1 \quad (25) \\
    f'(c_3c_5) &= |f(c_3) - f(c_5)| = 3 \quad (26) \\
    f'(c_1v^1_i) &= |f(c_1) - f(v^1_i)| = 4n + 11 - 2i; i = 1, 2, ..., n \quad (27) \\
    f'(c_2v^2_i) &= |f(c_2) - f(v^2_i)| = 2n + 5 - 2i; i = 1, 2, ..., n \quad (28)
\end{align*}
\]

The equations from (22) until (28), the label of each edge is distinctive and is forming \(\{1, 3, 5, ..., 2q - 1\}\) so that \((S_n, 3) - (n - 2)\) Ilalang graph is odd graceful graph. □

Example of Odd Graceful Labeling:

![Figure 3. Example of odd graceful labeling at Ilalang Graph and its variation.](image-url)
4. Conclusion

$(S_n, 3) - (n - 2)$, $n \geq 4$ Ilalang graph can be defined as a graph induced by $(S_n, 3)$ Ilalang graph by eliminating $n - 2$ vertices in one of the $S_n$ star graph, and the graph is called odd graceful graph.

Acknowledgments

Thank you to UMSU and LP2M who have provided funds along with other facilities so that we can complete this research.

References

[1] Galian JA 2016 Dynamic survey of graph labeling. Electronic Journal of Combinatorics, 17#ds6
[2] Bača and Miller 2008 Super Edge-Antimagic Graphs: A Wealth of Problems and Some Solution USA: Brown Walker Press
[3] Amri Z, Ahmad M, Huda N, Supriadi, and Sugeng KA 2011 Pelabelan Graceful, Skolem Graceful dan Pelabelan $\rho$ pada Graf $(S_n, 3)$ Prosiding Seminar Nasional UNY, Yogyakarta p M 131- M 136.
[4] Amri Z and Sugeng KA 2011 Pelabelan Graceful, Skolem Graceful dan Pelabelan $\rho$ pada Graf Kelabang Jurnal Eurika Pendidikan Matematika FKIP UMSU, Medan 4(2) pp 109-115
[5] Nurul H and Amri Z 2012 Pelabelan Graceful, Skolem Graceful dan Pelabelan Pada Graf A-Bintang dan H-Bintang Jurnal Matematika Murni dan Terapan EPSILON. Banjarbaru 6(2) pp 30-37
[6] Amri Z and Harahap TH 2017 Pelabelan Graceful dan Pelabelan Rho Topi Pada 8-Bintang dengan $c_3$ untuk $n$ Genap Jurnal EduTech: Jurnal Ilmu Pendidikan dan Ilmu Sosial, FKIP-UMSU, Medan 3(2) pp 1-5
[7] Gnanajothi RB 1991 Topic In Graph Theory, Ph.D. Thesis Madurai kamaraj University
[8] Moussa MI 2010 An Algorithm for Odd Graceful Labeling of The Union of Paths and Cycles Journal on Applications of Graph Theory in Wireless Ad hoc Networks and Sensor Networks (J GRAPH-HOC) 2(1)
[9] Moussa MI and Badr EM 2016 Ladder and Subdivision of Ladder Graphs with Pendant Edges are Odd Graceful International Journal on Applications of Graph Theory in Wireless Ad hoc Networks and Sensor Networks(GRAPH-HOC) 8(1)