Median Optimal Treatment Regimes

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Abstract

Optimal treatment regimes are personalized policies for making a treatment decision based on subject characteristics, with the policy chosen to maximize some value. It is common to aim to maximize the mean outcome in the population, via a regime assigning treatment only to those whose mean outcome is higher under treatment versus control. However, the mean can be an unstable measure of centrality, resulting in imprecise statistical procedures, as well as unrobust decisions that can be overly influenced by a small fraction of subjects. In this work, we propose a new median optimal treatment regime that instead treats individuals whose conditional median is higher under treatment. This ensures that optimal decisions for individuals from the same group are not overly influenced either by (i) a small fraction of the group (unlike the mean criterion), or (ii) unrelated subjects from different groups (unlike marginal median/quantile criteria). We introduce a new measure of value, the Average Conditional Median Effect (ACME), which summarizes across-group median treatment outcomes of a policy, and which the median optimal treatment regime maximizes. After developing key motivating examples that distinguish median optimal treatment regimes from mean and marginal median optimal treatment regimes, we give a nonparametric efficiency bound for estimating the ACME of a policy, and propose a new doubly robust-style estimator that achieves the efficiency bound under weak conditions. To construct the median optimal treatment regime, we introduce a new doubly robust-style estimator for the conditional median treatment effect. Finite-sample properties are explored via numerical simulations and the proposed algorithm is illustrated using data from a randomized clinical trial in patients with HIV.

Keywords: double robustness, optimal treatment regime, conditional treatment effect, precision medicine, quantile.

1 Introduction

Optimal treatment regime methods aim to learn policies that map subject covariates to a decision, typically with the goal of maximizing a population criterion. The task of assigning treatment decisions based on individual characteristics is common and crucially important in many disciplines, with applications ranging from personalized medicine to loan approval to educational program assignment. By now there is a large literature on theory, methods, and applications of optimal treatment regimes [Murphy, 2003, Robins, 2004, Laber et al., 2014, Chakraborty, 2013, Schulte et al., 2014, van der Laan and Luedtke, 2014, Kosorok and Laber, 2019]; we mostly refer to this and related work for more general background and details.
As more individualized policies are deployed in practice, it is of particular importance to examine the
target value for which the treatment regimes are optimized. Traditionally, the mean outcome in the
population has been used as the criterion. An important reference in this stream for our purposes
is van der Laan and Luedtke [2014], which studies doubly robust estimation of the mean outcome
under both a known policy and the unknown optimal policy, showing how nonparametric efficiency
bounds can be achieved in both cases. This is part of a larger literature on semiparametric efficiency
theory and doubly robust estimation (also known as one-step or debiased or double machine learning
estimation) [Newey, 1990, Bickel et al., 1993, van der Vaart, 2002, van der Laan and Robins, 2003,
Tsiatis, 2007, Kennedy, 2016, Chernozhukov et al., 2018].

However, compared to the median, means can be a poor and/or sensitive measure of the centrality of
a distribution, for example in the presence of skew or contamination [Huber et al., 1967, Casella and
Berger, 2002, Huber, 2004]. As a result, a recent literature has evolved studying alternative target
values for learning policies. Linn et al. [2017], Wang et al. [2018], Luedtke et al. [2020] propose the
marginal quantile to be the criterion in settings where the outcome distribution is skewed or the tail
of the outcome distribution is of interest. For simplicity we refer to these approaches as marginal
median-based, since the issues we detail with marginal median objectives are the same as those for
generic quantiles. In presenting the marginal quantile optimal treatment regime, Linn et al. [2017]
also considers a marginal cumulative distribution function related target value where the goal is to
find a policy that maximizes the probability of the outcome being above a given threshold. Qi et al.
[2019] uses conditional value-at-risk [Rockafellar and Uryasev, 2002] on the outcome distribution
as the criterion to ensure the policy to be risk-averse. Unfortunately, in addition to not having a
closed-form optimal policy, the marginal median approaches yield optimal treatment decisions for
subjects with covariates $X = x$ that depend on outcomes of other subjects with different covariates
$X \neq x$. We view this as allowing a form of unfairness, which will be discussed in more detail
shortly.

Motivated by these concerns, in this paper we propose a treatment regime which assigns treatment
to those subjects who have a higher conditional median outcome under treatment versus control,
and which is optimal with respect to a new objective we call the Average Conditional Median Effect
(ACME). Crucially, the optimal policy for the ACME has two key properties, which in general do
not both hold for either mean or marginal median optimal treatment regimes:

- “Within-group robustness”: Within a group, i.e., for subjects with the same observed covariates,
the treatment decision is based on the median of the conditional outcome distribution, and
thus the policy is robust to outliers (e.g., when a small fraction of the group has extreme
outcomes), unlike the mean optimal treatment regime.

- “Across-group fairness”: Across groups, i.e., for subjects with different observed covariates,
the treatment decision for a given group cannot depend on outcomes of a different group,
unlike the marginal median optimal treatment regime.

Remark 1. Here we use the term “fair” rather loosely; in contrast there has been lots of recent work
defining fairness more formally [Dwork et al., 2012, Hardt et al., 2016, Chouldechova and Roth,
2020, Mehrabi et al., 2019]. We also acknowledge that other definitions of fairness could be plausible
in our setup, and really only use the above for a convenient shorthand. We describe in much more
detail why we refer to the second property as fairness-related in an example in Section 4.3.

Our main contributions are as follows. We reflect on existing standard target values and their optimal
treatment regimes and propose the median optimal treatment regime\(^1\), that assigns treatment to an individual based on their conditional median treatment effect (Section 3). A new measure of policy value, namely the average conditional median effect, is defined. The proposed regime promotes within-group robustness and across-group fairness, in comparison to the mean optimal and marginal median optimal policies (Section 4). In Section 5, we establish the local asymptotic minimax bound for estimating the ACME, and construct a doubly robust-style estimator along with a simple algorithm that achieves the bound under mild conditions (Section 6). To learn the median optimal treatment regime, we propose a new doubly robust-style estimator for the Conditional Median Treatment Effect (CMTE) in Section 7. Finally, we use numerical simulations to show finite-sample properties of the estimator and illustrate the algorithm using a dataset from a randomized clinical trial on HIV patients (Section 8).

2 Preliminaries

We are given independent and identically distributed (iid) samples \(Z_i = (X_i, A_i, Y_i)\) drawn from a distribution \(P\) where \(X_i \in \mathcal{X} \subseteq \mathbb{R}^d\) are covariates, \(A_i \in \{0, 1\}\) is a binary treatment assignment and \(Y_i \in \mathcal{Y}\) is a continuous real-valued outcome of interest. We use \(Y_a\) to denote the potential outcome under treatment \(A = a\) [Rubin, 1974]. Throughout we let \(m(v | w) = \inf\{v \in \mathbb{R} : P(V \leq v | W = w) \geq 1/2\}\) denote the median of a generic random variable \(V\) given \(W = w\). To simplify the presentation, we introduce the following notation for components of the distribution \(P\):

\[
\begin{align*}
\pi_a(x) &= P(A = a | X = x), \\
F_a(y | x) &= P(Y \leq y | X = x, A = a), \\
f_a(y | x) &= \frac{d}{dy}F_a(y | x), \\
m_a(x) &= m(Y | X = x, A = a), \\
F_{a,m}(x) &= F_a(m_a(x) | X = x), \\
f_{a,m}(x) &= f_a(m_a(x) | X = x), \\
\sigma_a(x) &= \sqrt{\text{Var}(Y | X = x, A = a)}.
\end{align*}
\]

We note that \(\pi_1(x)\) is called the propensity score, i.e., chance of receiving treatment given covariates, and \(\pi_0(x) = 1 - \pi_1(x)\). We assume \(F_a(y | x)\) to be absolutely continuous and hence \(f_a(y | x)\) exists. Throughout the paper, we refer to \(F_a(y | x)\) as the conditional cumulative distribution function, \(f_a(y | x)\) as the conditional density, \(m_a(x)\) as the conditional median, \(f_{a,m}(x)\) as the conditional density at the conditional median, and \(\sigma_a(x)\) as the conditional standard deviation. We assume the following standard conditions for identifying causal effects:

**Assumption 1.** The following causal assumptions hold

\[
\begin{align*}
(\text{Consistency}) & \quad Y = AY^1 + (1 - A)Y^0, \\
(\text{Positivity}) & \quad P(\epsilon \leq \pi_1(X) \leq 1 - \epsilon) = 1 \text{ for some } \epsilon > 0, \\
(\text{Exchangeability}) & \quad A \perp Y^a | X \text{ for } a = 0, 1.
\end{align*}
\]

\(^1\)We use marginal median optimal treatment regimes to refer to policies that maximize marginal median values.
Additional Notation  We use $P\hat{f} := \int \hat{f}(z) dP(z)$ to denote the expectation conditioned upon the randomness of $\hat{f}$ and $P_n f := P_n f(Z) = \frac{1}{n} \sum_{i=1}^{n} f(Z_i)$. We denote the Euclidean norm for a real-valued vector $x \in \mathbb{R}^d$ to be $\|x\|_2$ and the $L_2(P)$ norm of a function $f$ to be $\|f\| := (\int f(z)^2 dP(z))^{1/2}$. Finally, $a \lesssim b$ is equivalent to $a \leq C b$ for some positive constant $C$.

3  Proposed Target Policy & Value

In this section, we first review common target values and their optimal policies in the current literature (Section 3.1). In Section 3.2, we introduce a new policy and a new measure of a policy’s value, based on conditional medians. We show that under Assumption 1 these policies and values are identified.

3.1  Standard Policies & Values

Let $d : \mathcal{X} \to \{0, 1\}$ denote a deterministic policy that assigns a treatment based on input covariates and $D$ be the set of all such measurable functions. Traditionally, most optimal treatment regime research [Murphy, 2003, Zhang et al., 2012] has focused on finding the policy that maximizes the mean outcome

$$E[Y^d] := E[d(X)Y^1 + (1 - d(X))Y^0],$$

which under Assumption 1 is equivalent to $E[Y^d] = E_X [\mu_d(X)]$, where $\mu_d(X) := d(X)\mu_1(X) + (1 - d(X))\mu_0(X)$ and $\mu_a(X) := E[Y | X, A = a]$. When the goal is to maximize the overall population mean $E[Y^d]$, it is well-known that the optimal policy is given by

$$d^\star_{MEAN}(X) := 1\{\mu_1(X) > \mu_0(X)\}.$$ 

This policy simply treats only those subjects whose conditional mean treatment effect is positive.

Remark 2. Often, the mean optimal policy $d^\star_{MEAN}$ is motivated by the fact that it maximizes $E[Y^d]$: however, we view its primary motivation as coming from the fact that it maximizes the conditional value $E[Y^d | X = x]$ for all $x \in \mathcal{X}$, i.e., it is mean optimal for subjects of any type. From this perspective, the marginal value $E[Y^d]$ is just a one-number summary of the overall performance of a policy, across a heterogeneous population of subjects with different covariates. As we discuss further in subsequent sections, we view the maximization of the conditional value as more fundamental and fair in the across-group sense introduced in Section 1 and detailed further in Section 4.3.

While the mean is a valuable measure of centrality in many cases, it is sensitive to outliers and so can be an inappropriate target value when some have extreme responses to treatment. This has led to recent work maximizing the marginal median

$$m(Y^d) := m \left( d(X)Y^1 + (1 - d(X))Y^0 \right).$$

In the general case, marginal quantile-based values are proposed where the goal is to find the optimal policy with respect to a quantile of the outcome [Linn et al., 2017, Wang et al., 2018, Kallus et al., 2019]. Under Assumption 1, we obtain that

$$m(Y^d) = \inf \left\{ m : P(Y^d \leq m) \geq 1/2 \right\}$$

$$= \inf \left\{ m : \int P(Y^d \leq m | X = x) dP(x) \geq 1/2 \right\}$$

4
\[
= \inf \left\{ m : \int \mathbb{P}(Y \leq m \mid A = d(x), X = x)d\mathbb{P}(x) \geq 1/2 \right\}.
\]

From here on we use \(m(Y^d)\) to refer to the identified quantity above, with the understanding that it corresponds the counterfactual marginal median only under Assumption 1.

Unfortunately, as illustrated in Proposition 2, the marginal median objective allows specific subjects’ optimal treatment assignments to be determined by different subjects, with different covariate values. We see this as a violation of across-group fairness. For example, if in some population one group suddenly started responding more to treatment, then under the marginal median objective, the optimal treatment for other groups could change – even if their response to treatment did not.

Unlike the mean optimal policy \(d^\ast_{\text{MEAN}}(x)\), in general, the optimal policy under the marginal median objective (which we denote as \(d^\ast_{\text{MME}}(x)\)) cannot be viewed as maximizing a parameter defined upon the conditional outcome distributions \([Y \mid X = x, A = a]\) for \(a \in \{0, 1\}\), and does not have a closed form that depends only on the conditional outcome distributions.

### 3.2 Median Optimal Treatment Regimes

As mentioned in Remark 2, we believe conditional values are generally more fundamental and useful than marginal values. Therefore here we propose a policy \(d^\ast_{\text{ACME}}\) that maximizes the conditional median \(m(Y^d \mid X = x)\) for all \(x \in \mathcal{X}\). This implies that

\[
d^\ast_{\text{ACME}}(X) := \mathbb{1}\{m_1(X) > m_0(X)\}.
\]

Using the conditional median instead of the conditional mean to measure centrality of the conditional outcome distributions, compared to the mean optimal policy \(d^\ast_{\text{MEAN}}\), \(d^\ast_{\text{ACME}}\) is more robust to outliers. Unlike the marginal median optimal policy \(d^\ast_{\text{MME}}\), for all \(x \in \mathcal{X}\), our proposed policy \(d^\ast_{\text{ACME}}(x)\) is more individualized, in the sense that optimal treatment assignments for subjects with \(X = x\) do not depend on outcomes of different subjects with \(X \neq x\). We summarize the overall performance of our proposed conditional median optimal policy (also denoted as median optimal treatment regime and median optimal policy) \(d^\ast_{\text{ACME}}\) across groups of subjects with different covariates using the average conditional median effect, which we introduce below.

**Definition 1.** Given a policy \(d \in \mathcal{D}\), we define the average conditional median effect (ACME) as

\[
\mathbb{E}_X[m(Y^d \mid X)] := \mathbb{E}_X \left[ m(d(X)Y^1 + (1 - d(X))Y^0 \mid X) \right].
\]

**Remark 3.** Traditionally a treatment “effect” is a contrast between (distributions of) potential outcomes; for simplicity we call the ACME parameter an effect even though it involves counterfactuals under only one treatment policy, and so is not a contrast.

**Identification** Under Assumption 1, we have that

\[
\mathbb{E}_X[m(Y^d \mid X)] = \mathbb{E}_X[d(X)m(Y^1 \mid A = 1, X) + (1 - d(X))m(Y^0 \mid A = 0, X)]
= \mathbb{E}_X[d(X)m(Y \mid A = 1, X) + (1 - d(X))m(Y \mid A = 0, X)]
= \mathbb{E}_X[m_d(X)],
\]

where \(m_d(X) := d(X)m_1(X) + (1 - d(X))m_0(X)\). For the rest of the paper, without further specification, we assume Assumption 1 to hold.
As illustrated at the beginning of Section 3.2, the ACME is used to summarize the performance of the conditional median optimal policy \( d_{\text{ACME}}^* \). We show here that, just as \( d_{\text{MEAN}}^* \) optimizes \( \mathbb{E}[\mathbb{E}[Y^{d^*} | X]] \), \( d_{\text{ACME}}^* \) optimizes the ACME \( \mathbb{E}[m(Y^{d^*} | X)] \), i.e., for all \( d \in \mathcal{D} \),

\[
\mathbb{E}_X[m(Y^{d^*} | X)] = \mathbb{E}_X[d(X)m_1(X) + (1 - d(X))m_0(X)]
\]

\[
= \mathbb{E}_X[m_0(X) + (m_1(X) - m_0(X))d(X)]
\]

\[
\leq \mathbb{E}_X[m_0(X) + (m_1(X) - m_0(X))d_{\text{ACME}}^*(X)].
\]

Remark 4. We note that in the case where the policies can only be measurable functions of a subset of covariates \( V \subseteq X \), the optimal treatment regime for ACME is not as straightforward as the one for the mean outcome, due to the nonlinearity of the median. Developing the \( V \)-specific optimal treatment regime for ACME is an avenue for future work.

4 Policy Comparisons

In this section, we compare the three policies \( d_{\text{MEAN}}^* \), \( d_{\text{MME}}^* \) and \( d_{\text{ACME}}^* \). In Section 4.1, we give a simple condition under which the mean and median optimal treatment regimes are equivalent, and point out that the mean optimal policy is not necessarily robust within groups (defined in Section 1). In Section 4.2, we illustrate that the marginal median optimal treatment regime does not ensure across-group fairness (defined in Section 1). In Section 4.3, we use a motivating example to summarize the differences among the three policies.

4.1 \( d_{\text{MEAN}}^* \) and \( d_{\text{ACME}}^* \)

To characterize the difference between the mean optimal treatment regime \( d_{\text{MEAN}}^* \) and our proposed median optimal treatment regime \( d_{\text{ACME}}^* \), we begin by stating a simple condition that ensures the two to be equivalent: \( d_{\text{MEAN}}^*(x) = d_{\text{ACME}}^*(x) \) if and only if the conditional mean and median treatment effects are always the same sign. One sufficient condition for the two policies to be equivalent is thus given below.

**Proposition 1.** If the conditional mean and median treatment effects are separated from zero relative to the conditional outcome standard deviation, in the sense that

\[
|\mu_1(x) - \mu_0(x)| > \sigma_1(x) + \sigma_0(x) \quad \text{and} \quad |m_1(x) - m_0(x)| > \sigma_1(x) + \sigma_0(x),
\]

then \( d_{\text{MEAN}}^*(x) = d_{\text{ACME}}^*(x) \).

An important note about the mean optimal policy \( d_{\text{MEAN}}^* \) is that it does not necessarily satisfy the within-group robustness mentioned in Section 1, in that it is sensitive to outliers. For example, as illustrated in Section 4.3, its decision can be very affected by a small subgroup that benefits a lot from the treatment, even if the majority of the subgroup is harmed by the decision. In contrast, the median remains unchanged as long as the amount of mass below and above it remains unchanged, while the mean is determined by how the masses are distributed, e.g., the mean can be arbitrarily high (or low) by moving a small fraction of the mass above (or below) the median to extreme values [Huber et al., 1967, Casella and Berger, 2002, Huber, 2004]. Further comparisons between the mean optimal and our proposed median optimal treatment regime are given in Section 4.3.
4.2 $d^*_\text{MME}$ and $d^*_\text{ACME}$

To illustrate ideas, here we consider a simple Gaussian model to compare the marginal median optimal treatment regime $d^*_\text{MME}$ and our proposed median optimal treatment regime $d^*_\text{ACME}$. This provides a counterexample showing that $d^*_\text{MME}(x)$ is determined not just by the conditional outcome distribution $[Y \mid X = x, A = a]$ for $a \in \{0, 1\}$ but also by the conditional outcome distribution at other (different) covariate values. That is, even if the conditional distribution $[Y \mid X = x, A = a]$ for $a \in \{0, 1\}$ is unchanged, the marginal median optimal policy $d^*_\text{MME}(x)$ can be different depending on $[Y \mid X = x', A = a]$ for $x' \neq x$. On the other hand, our median optimal policy $d^*_\text{ACME}(x)$ is fair across groups, in that the optimal decision remains the same so long as the conditional distribution for $X = x$ is unchanged.

Specifically consider a data-generating process where

\[ X \sim \text{Bernoulli}(1/2) \]
\[ Y \mid X = x, A = a \sim N(\mu_a(x), \sigma_a^2(x)) \]

for some unspecified (for now) mean $\mu_a(x)$ and standard deviation $\sigma_a(x)$. For any policy $d \in \mathcal{D}$, the ACME is $\mathbb{E}[m_d(X)] = (\mu_d(0) + \mu_d(1)) / 2$, and the marginal median value $m(Y^d)$ satisfies

\[ \Phi \left( \frac{m(Y^d) - \mu_d(0)}{\sigma_d(0)} \right) + \Phi \left( \frac{m(Y^d) - \mu_d(1)}{\sigma_d(1)} \right) = 1, \]

which implies that

\[ m(Y^d) = \frac{\sigma_d(1)\mu_d(0) + \sigma_d(0)\mu_d(1)}{\sigma_d(0) + \sigma_d(1)}, \]

i.e., it is a standard deviation-weighted average of outcome regressions (where each regression is weighted by the standard deviation of the other group). In the next proposition we give a counterexample showing how, somewhat surprisingly, the marginal median optimal policy for subjects with $X = 1$, for example, can depend on the means and variances for different subjects with $X = 0$.

**Proposition 2.** Suppose outcomes are higher and more variable under treatment for $X = 1$, i.e.,

\[ \mu_1(1) > \mu_0(1) \quad \text{and} \quad \sigma_1(1) > \sigma_0(1), \]

and the variances differ more than the means in the sense that $\frac{\mu_1(1)}{\mu_0(1)} < \frac{\sigma_1(1)}{\sigma_0(1)}$. Then there exist conditional distributions $Q_a$ and $\overline{Q}_a$ for $[Y \mid X = 0, A = a]$, $a \in \{0, 1\}$ such that

\[ \mathbbm{1}\{\mu_1(1) > \mu_0(1)\} = d^*_\text{MME}(1; Q_0, Q_1) \neq d^*_\text{MME}(1; \overline{Q}_0, \overline{Q}_1) = \mathbbm{1}\{\mu_1(1) \leq \mu_0(1)\}, \]

where $d^*_\text{MME}(x; Q_0, Q_1)$ is the marginal median optimal policy when $[Y \mid X = x, A = a]$ follows distribution $Q_a$.

**Remark 5.** When $\sigma_1(0) = \sigma_0(0) = \sigma_1(1) = \sigma_0(1)$, it follows that $m(Y^d) = \mathbb{E}[m_d(X)]$ for all $d \in \mathcal{D}$ and $d^*_\text{ACME}(x) = \mathbbm{1}\{\mu_1(x) > \mu_0(x)\}$ is an optimal policy with respect to $m(Y^d)$.

**Remark 6** (Distribution shift). Importantly, we also note that among the three optimal policies, only the marginal median optimal policy $d^*_\text{MME}$ varies under different covariate distribution $\mathbb{P}(X)$. This is crucial when distribution shift is a concern, which is common in many current setups [Quionero-Candela et al., 2009, Mo et al., 2020]. Therefore, under distribution shift between training and testing times, the marginal median optimal policy $d^*_\text{MME}$ could be suboptimal at testing time even if optimal at training. For example, if during test time $\mathbb{P}(X = 0) = 0$ and $\mathbb{P}(X = 1) = 1$, then
then in the case when $d^\ast_{\text{MME}}(1) = 1\{\mu_1(1) \leq \mu_0(1)\}$ during training and $\mu_1(1) \neq \mu_0(1)$, $d^\ast_{\text{MME}}$ is not optimal (with respect to the marginal median) during test time since $m(Y^d) = \mu_d(1)$. Unlike $d^\ast_{\text{MME}}$, our proposed median optimal treatment regime $d^\ast_{\text{ACME}}$ remains the same under covariate shifts.

As illustrated in Proposition 2, the optimal decision for a specific subject of covariate $x$ with respect to the marginal median depends on not just its own conditional distribution $[Y \mid X = x, A = a]$ for $a \in \{0, 1\}$ but also the conditional distributions at other covariate values. On the other hand, ACME allows the optimal decision for $x$ to be only influenced by its own conditional outcome distribution at that $x$. In Section 4.3, we give an illustration that summarizes the differences among the three policies.

![Figure 1: The density of $[Y \mid X = G_2, A = a]$ in Case I and Case II.](image)

### 4.3 Simple Illustration Comparing $d^\ast_{\text{MEAN}}$, $d^\ast_{\text{MME}}$, and $d^\ast_{\text{ACME}}$

Here we give a simple illustration that showcases the differences among the three policies, and among the corresponding values they maximize. We will see how the ACME optimal policy $d^\ast_{\text{ACME}}$ exhibit both within-group robustness and across-group fairness, while the more standard policies $d^\ast_{\text{MEAN}}$ and $d^\ast_{\text{MME}}$ may not.

Consider a population of two groups: one with college degrees ($G_1$) and the other without ($G_2$). The binary treatment $A$ indicates whether a job training program is assigned to the individual and the outcome $Y$ is the subjects’ improvement in income. In Case I, suppose the groups and outcomes follow the distribution:

\[
X \sim \text{Bernoulli}(1/2) \\
Y \mid X = G_1, A = a \sim \mathcal{N}(9, 1) \\
Y \mid X = G_2, A = a \sim (1 - a)\mathcal{N}(0, 1) + a(\mathcal{N}(22, 1) + .8 \mathcal{N}(-5, 1))
\]

In other words, there is no treatment effect for subjects with a college degree, whose outcome distribution is always the same Gaussian. However, under control subjects without a college degree have outcomes that follow a Gaussian centered at 0, while under treatment the outcomes follow a Gaussian mixture. The outcome distribution for $G_2$ is given in Figure 1. When examining the optimal treatment under each value, we find that both the mean optimal policy $d^\ast_{\text{MEAN}}$ and marginal median optimal policy $d^\ast_{\text{MME}}$ treat all subjects without a college degree (Table 1), while $d^\ast_{\text{ACME}}$ does not treat them. Compared to $d^\ast_{\text{MEAN}}$, our proposed median optimal policy $d^\ast_{\text{ACME}}$ is robust against outliers, hence its decisions promotes within-group robustness in the sense that the decisions are less prone to only benefiting a small subgroup, e.g., as in our $A = 1$ case for the group without a college degree.
degree. On the other hand, although intuitively the marginal median optimal policy $d_{\text{MME}}^*$ also gives robust decisions, it overlooks characteristics of specific subgroups. In particular, treatment decisions for those with $X = x$ can depend on other subjects’ conditional outcome distribution with $X \neq x$, violating across-group fairness. This is illustrated by comparing the optimal marginal median decision for subjects without a college degree under Case I and Case II where in Case II the conditional outcome distribution for subjects with a college degree under treatment and control is changed to

$$Y \mid X = G_1, A = a \sim N(-4, 2).$$

Surprisingly, as in Proposition 2, the marginal median optimal policy $d_{\text{MME}}^*$ under Case II is different for individuals without a college degree compared to the $d_{\text{MME}}^*$ under Case I, even though it was only the outcome distribution for subjects with a college degree that changed. This illustrates across-group unfairness of the marginal median value. In contrast, the median optimal decision for individuals without a college degree has remained unchanged.

|                  | Case I | Case II |
|------------------|--------|---------|
| Policy           | Treat $G_1$ | Treat $G_2$ | Treat $G_1$ | Treat $G_2$ |
| $d_{\text{MEAN}}^*$ | — | ✓ | — | ✓ |
| $d_{\text{MME}}^*$ | — | ✓ | — | — |
| $d_{\text{ACME}}^*$ | — | — | — | — |

Table 1: The table shows the treatment assignments under the mean optimal policy $d_{\text{MEAN}}^*$, the marginal median optimal policy $d_{\text{MME}}^*$, and the median optimal policy $d_{\text{ACME}}^*$ in Case I and Case II. Since in both cases, the conditional outcome distributions for $G_1$ are the same under treatment and control, “—” is used to indicate that the optimal decision for $G_1$ can be either $a = 0$ or $a = 1$. Though the conditional outcome distributions for $G_2$ remain unchanged in both cases, the optimal marginal median decision for $G_2$ has changed.

## 5 Efficiency Bound

In this section, using semiparametric theory [Newey, 1990, Bickel et al., 1993, van der Vaart, 2002, van der Laan and Robins, 2003, Tsiatis, 2007, Kennedy, 2016, Díaz, 2017, Chernozhukov et al., 2018], we study the nonparametric efficiency bound for estimating the ACME $\psi_d := E[m_d(X)]$ of a given policy $d$, given iid samples from distribution $P$. One of the key tools we will use throughout the paper is the efficient influence function $\phi_d(Z)$ (Corollary 2) which serves as a first-order derivative in a von Mises-type expansion of the target parameter (Lemma 3).

There are a number of reasons why characterizing efficient influence functions is important. The variance of the efficient influence function gives a minimax lower bound for estimating $\psi_d$ (Theorem 3) and so provides a benchmark to compare against when constructing estimators. In particular, as used in Section 6, efficient influence functions suggest a doubly robust-style bias-corrected estimator. Doubly robust estimators attain fast parametric rates in nonparametric settings where nuisance functions (e.g., $m_a, \pi_a, f_{a,m}$ in our context) are estimated at slower rates. We begin with presenting the von Mises-type expansion (a distributional Taylor expansion) for ACME, which is crucial for the efficiency bound (Theorem 3) and the estimation guarantees for our proposed doubly robust-style estimator (Theorem 5).
Lemma 3. Given a policy $d \in D$, for the average conditional median effect $\psi_d$ defined in (2), the following decomposition holds for all distributions $\mathbb{P}$ and $\mathbb{P}$:

$$\psi_d(\mathbb{P}) - \psi_d(\mathbb{P}) = \int \phi_d(Z; \mathbb{P})d(\mathbb{P} - \mathbb{P}) + R_d(\mathbb{P}, \mathbb{P}),$$

where $\phi_d(Z; \mathbb{P}) = \xi_d(Z; \mathbb{P}) - \psi_d(\mathbb{P})$,

$$\xi_d(Z; \mathbb{P}) = \frac{Ad(X)1/2 - 1\{Y \leq m_1(X)\}}{\pi_1(X)f_{1,m}(X)} + \frac{(1-A)(1-d(X))1/2 - 1\{Y \leq m_0(X)\}}{\pi_0(X)f_{0,m}(X)} + m_d(X),$$

and

$$R_d(\mathbb{P}, \mathbb{P}) = \int_X d(x) \left( \frac{\pi_1(x)F_{1,m}(x) - F_{1,m}(x)}{\pi_1(x)f_{1,m}(x)} \right)$$

$$+ (1 - d(x)) \left( \frac{\pi_0(x)F_{0,m}(x) - F_{0,m}(x)}{\pi_0(x)f_{0,m}(x)} \right) d\mathbb{P}(x).$$

($\pi_1, m_a$, and $f_{a,m}$ are the propensity score, the conditional median and the conditional density at the conditional median defined under $\mathbb{P}$ and $F_{a,m}(x) = F_a(m_a(x) | x)$.

Proofs of the results in this section can be found in Appendix A.2. Lemma 3 has several important implications. It implies that the efficient influence function of ACME is $\phi_d(Z)$ as shown in Corollary 2. The efficient influence function plays a crucial role in the bias term for the plug-in estimator $\psi_d(\mathbb{P})$, which we will correct to obtain the doubly robust-style estimator given later on in (8). The decomposition result in (4) (along with an alternate expression in (5)) is then used to show the convergence rate of the proposed estimator.

Throughout the rest of the paper, we rely on the following model assumption, which ensures that certain boundedness and mild smoothness conditions hold.

Assumption 2. The true distribution $\mathbb{P}$ lies in $\mathcal{P}$ where $\mathcal{P}$ is the statistical model given by

$$\mathcal{P} := \left\{ \mathbb{P} \Big| \forall \epsilon \in (0, 1), \mathbb{P}(\epsilon < \pi_1(X) \leq 1 - \epsilon) = 1, \right.$$  

$$\forall a \in \{0, 1\}, \exists 0 < M_0 \leq M_1, \mathbb{P}(M_0 \leq f_{a,m}(X) \leq M_1) = 1, \text{ and}$$  

$$\forall a \in \{0, 1\}, y \in \mathcal{Y}, f_a(y | X) \text{ is differentiable and L-Lipschitz continuous in } y \text{ almost surely} \right\}.$$

The conditions of Assumption 2 only require some boundedness of the propensity score and conditional outcome density, as well as some weak smoothness of the conditional density in $y$. Importantly, Assumption 2 does not require any smoothness of the propensity score or outcome density in $X$.

As written, the reminder term $R_d(\mathbb{P}, \mathbb{P})$ in Lemma 3 does not appear to be second-order, i.e., involving products of differences of $\mathbb{P}$ and $\mathbb{P}$. However, in the next corollary we show that it is in fact second-order, under the model assumption (Assumption 2).
Corollary 1. For all distributions $\mathbb{P} \in \mathcal{P}$, given any distribution $\mathbb{F}$, the remainder term $R_d(\mathbb{F}, \mathbb{P})$ defined in (4) is equivalent to

$$
R_d(\mathbb{F}, \mathbb{P}) = \mathbb{P} \left\{ \frac{d(\bar{m}_1 - m_1)}{\bar{m}_1 f_{1,m}} \left( (\bar{\pi}_1 - \pi_1) \bar{f}_{1,m} + (\bar{f}_{1,m} - f_{1,m}) \pi_1 - (\bar{m}_1 - m_1) f'_{1,c,\pi_1} \right) + \frac{(1 - d)(\bar{m}_0 - m_0)}{\bar{m}_0 f_{0,m}} \left( (\pi_0 - \pi_0) \bar{f}_{0,m} + (\bar{f}_{0,m} - f_{0,m}) \pi_0 - (\bar{m}_0 - m_0) f'_{0,c,\pi_0} \right) \right\},
$$

where $f'_{a,c} \leq L$ is the derivative of $f_a(y | x)$ at $y = a(x)$ for some value $a(x)$ between $m_a(x)$ and $\bar{m}_a(x)$.

The fact that the remainder term is second-order implies that $\phi_d$ from Lemma 3 is the efficient influence function, as stated in the next corollary.

Corollary 2. For a given policy $d \in \mathcal{D}$, the efficient influence function for the average conditional median effect $\psi_d$ is $\phi_d(Z) = \xi_d(Z) - \psi_d$ where $\xi_d$ is defined in (3).

Since the variance of $\phi_d$ serves as a nonparametric efficiency bound, in the local minimax sense [van der Vaart, 2002], in the next result we give the exact form of this variance. This shows what factors drive the statistical difficulty in ACME estimation, and demonstrates the difference in efficiency for ACME versus other value measures.

Theorem 3. For a given policy $d \in \mathcal{D}$, the nonparametric efficiency bound for estimating $\psi_d$ is given by the variance $\sigma_d^2 := \text{Var}\{\phi_d(Z)\}$ where

$$
\sigma_d^2 = \mathbb{E} \left[ \frac{d(X)}{4\pi_1(X)f_{1,m}^2(X)} + \frac{1 - d(X)}{4\pi_0(X)f_{0,m}^2(X)} \right] + \text{Var}(m_d(X)).
$$

Theorem 3 shows how the efficiency bound for the ACME is driven by three main factors:

- the inverse propensity score $1/\pi_1(X)$,
- the inverse conditional density at the conditional median $1/f_{a,m}(X)$, and
- the heterogeneity of the conditional median $m_d(X)$, i.e., $\text{Var}(m_d(X))$.

In contrast, recall the nonparametric efficiency bound for the mean value $\mathbb{E}[Y^d]$ [Hahn, 1998, Theorem 1]:

$$
\sigma_{d,\text{MEAN}}^2 := \mathbb{E} \left[ \frac{d(X)\sigma_d^2(X)}{\pi_1(X)} + \frac{(1 - d(X))\sigma_d^2(X)}{\pi_0(X)} \right] + \text{Var}(\mu_d(X)),
$$

instead depends on the heterogeneity of the conditional outcomes $\sigma_a^2(X)$ and the heterogeneity of the conditional mean $\text{Var}(\mu_d(X))$ (along with the inverse propensity score).

Importantly, when the conditional distribution $[Y | X = x, A = a]$ is heavy-tailed, e.g., the outcome variance $\sigma_a^2(X)$ is very large, then in general the mean value efficiency bound $\sigma_{d,\text{MEAN}}^2$ would be severely affected, while the ACME analog $\sigma_d^2$ could still be quite small.

Next we formalize the local minimax lower bound property of the variance $\sigma_d^2$.

---

2Since $\mathbb{P} \in \mathcal{P}$, $f'_{a,c}$ exists and is bounded by $L$ almost surely.
Corollary 4. [van der Vaart, 2002, Corollary 2.6] For a given policy \( d \in D \) and any estimator \( \hat{\psi}_d \) learned using \( n \) iid samples from \( \mathbb{P} \), it follows that

\[
\inf_{\delta > 0} \lim_{n \to \infty} \inf_{\mathbb{P}, \mathbb{P}' \in \mathbb{P}} \sup TV(\mathbb{P}, \mathbb{P}') < \delta \quad \mathbb{E}_{\mathbb{P}} \left[ \left( \hat{\psi}_d - \psi_d(\mathbb{P}) \right)^2 \right] \geq \sigma^2_d,
\]

where \( TV(\mathbb{P}, \mathbb{P}') \) is the total variation distance between \( \mathbb{P} \) and \( \mathbb{P}' \) and \( \sigma^2_d \) is defined in (6).

Corollary 4 directly follows from van der Vaart [2002, Corollary 2.6]. It shows that without further assumptions, the asymptotic local minimax mean squared error of any estimator scaled by \( n \) can be no smaller than \( \sigma^2_d \). In Section 6, under mild conditions, we construct doubly robust-style estimators that achieve the local minimax lower bound shown in Corollary 4.

6 Estimation of the ACME

In this section, we propose efficient estimators of the ACME that utilize the efficient influence function given in Corollary 2. We start by presenting a simple plug-in estimator, which is a sample analogue of \( \psi_d \) for a given policy \( d \in D \). In Section 6.1, we present the doubly robust-style estimator for evaluating the ACME of a fixed given policy, as well as the optimal policy, and which improves on the simple plug-in. Specifically, under mild conditions, we show that in both cases the doubly robust-style estimator achieves the local minimax lower bound shown in Corollary 4. Finally, we present an algorithm for constructing the doubly robust-style estimator (Section 6.2).

Let \( Z^n = (Z_1, \ldots, Z_n) \) and \( Z^{n,0} = (Z_1^0, \ldots, Z_n^0) \) denote two independent samples. For a given policy \( d \in D \), the most natural estimator for estimating \( \psi_d \) is the plug-in estimator, i.e.,

\[
\hat{\psi}_{d,\text{pli}} = \mathbb{P}_n \{ d(X)\hat{m}_1(X) + (1 - d(X))\hat{m}_0(X) \},
\]

where \( \hat{m}_1 \) and \( \hat{m}_0 \) are learned using a separate sample \( Z^{n,0} \) from the sample \( Z^n \) used for taking the sample average \( \mathbb{P}_n \). While the plug-in estimator is intuitive, in general it inherits the slow convergence rate from the nuisance functions [Bickel et al., 1993, van der Vaart, 2002]. As suggested by the von Mises-type decomposition (Lemma 3), a natural estimator that improves upon the plug-in estimator is given in (8).

Remark 7. For a given set of iid data, we can obtain \( Z^n \) and \( Z^{n,0} \) by randomly splitting the data in half. In general, to obtain full sample size efficiency, one can split the data in folds and perform cross-fitting [Bickel and Ritov, 1988, Robins et al., 2008, Zheng and van der Laan, 2010, Chernozhukov et al., 2018], i.e., repeat the learning procedure for each split and average the results. Throughout this section, to ease notation, we present the analyses and results for the learning procedure with a single sample split, which can be easily extended to averages of multiple independent splits. If one wants to avoid sample splitting, our results would still hold under appropriate empirical process conditions (e.g., the nuisance functions taking values in Donsker classes).

6.1 Doubly Robust-Style Estimator of the ACME

To improve on the potential deficiencies of the simple plug-in above, we propose the doubly robust-style estimator \( \hat{\psi}_{d,\text{dr}} \):

\[
\hat{\psi}_{d,\text{dr}} = \mathbb{P}_n \left\{ \frac{1\{A = d(X)\}}{A\hat{m}_1(X) + (1 - A)\hat{m}_0(X)} \frac{1}{f_{A,\hat{m}}(X)} + \hat{m}_d(X) \right\},
\]

where \( f_{A,\hat{m}}(X) \) is the influence function of the doubly robust estimator.
where the nuisance functions $\hat{\pi}_1, \hat{m}_a$ and $\hat{f}_{a,\hat{m}}$ are learned using the separate sample $Z_{n,0}$, and $\hat{\pi}_0(X) = 1 - \hat{\pi}_1(X)$. Importantly, this estimator uses the efficient influence function (Corollary 2) to correct the bias of the plug-in.

**Remark 8.** As will be shown in Theorem 5, our estimator is not strictly doubly robust in the usual sense (i.e., its consistency requires consistent estimation of the conditional median). Nonetheless, we still call it doubly robust-style because it does have what is arguably the most crucial property of doubly robust estimators: its error involves second-order products, and so is “doubly small”.

In Section 6.1.1, we study the asymptotic convergence rate of $\hat{\psi}_{d,dr}$ for a given fixed policy $d$, showing it can converge at parametric rates even when nuisance functions are estimated nonparametrically, and follows a straightforward asymptotic normal distribution. Then in Section 6.1.2, we show that under mild conditions, our estimator $\hat{\psi}_{d,dr}$ still exhibits $\sqrt{n}$-convergence and asymptotic normality for the optimal ACME value, even after plugging in the learned policy $\mathbb{1}\{\hat{m}_1(X) > \hat{m}_0(X)\}$, under a margin condition.

### 6.1.1 ACME of a Fixed Policy

We begin with the case when a fixed policy $d$ is given and we aim to estimate the ACME of it. This can be useful if we have a specified class of policies, which we aim to optimize over; this includes the case where we use an independent sample or split to learn a policy, and then condition on that sample when estimating the value using another. In such cases, under mild assumptions, we show in Theorem 5 that the error of $\hat{\psi}_{d,dr}$ is the sum of a centered sample average term which is asymptotically normal and a term containing products of nuisance estimation errors.

**Theorem 5.** Assume

1. $\mathbb{P}(\epsilon \leq \hat{\pi}_1(X) \leq 1 - \epsilon) = 1$ for some $\epsilon \in (0, 1)$,
2. $\mathbb{P}(M_0 \leq \hat{f}_{a,\hat{m}}(X) \leq M_1) = 1$ for $a \in \{0, 1\}$ and $0 < M_0 \leq M_1$, and
3. $\|\hat{\xi}_d - \xi_d\| = o_P(1)$.

Then,

$$\hat{\psi}_{d,dr} - \psi_d = (\mathbb{P}_n - \mathbb{P})\hat{\xi}_d(\mathbb{P}) + O_P\left(\sum_{a=0}^1 \|\hat{m}_a - m_a\| \left(\|\hat{\pi}_a \hat{f}_{a,\hat{m}} - \hat{\pi}_a f_{a,m}\| + \|\hat{m}_a - m_a\|\right) + o_P(1/\sqrt{n})\right).$$

Importantly, Theorem 5 implies that $\hat{\psi}_{d,dr}$ attains a faster convergence rate than its nuisance estimators and can be asymptotically normal even when these nuisance functions only satisfy nonparametric sparsity, smoothness or other assumptions. We detail this further in the next corollary.

**Corollary 6.** Given $d \in \mathcal{D}$, under assumptions in Theorem 5, and the following two conditions:

1. $\|\hat{m}_a - m_a\| = o_P(n^{-1/4})$, and
2. For $a \in \{0, 1\}$, $\|\hat{\pi}_a \hat{f}_{a,\hat{m}} - \hat{\pi}_a f_{a,m}\| = O_P(n^{-1/4}),$

we have that $\hat{\psi}_{d,dr}$ is root-$n$ consistent and asymptotically normal with:

$$\sqrt{n}(\hat{\psi}_{d,dr} - \psi_d) \sim \mathcal{N}\left(0, \sigma_d^2\right).$$
Remark 9. Note that one sufficient condition for $\|\hat{\pi}_a f_{a,m} - \pi_a f_{a,m}\| = O_P(n^{-1/4})$ is that $\|\hat{\pi}_a - \pi_a\| = O_P(n^{-1/4})$ and $\|f_{a,m} - f_{a,m}\| = O_P(n^{-1/4})$ separately.

Corollary 6 shows that $\hat{\psi}_{d,dr}$ can achieve the nonparametric efficiency bound given in Corollary 4. For example, when $\pi_a$, $m_a$ and $f_{a,m}$ are $d$-dimensional functions in a Hölder smooth class with smoothness parameter being $\alpha$, $\beta$ and $\gamma$ respectively (i.e., their partial derivatives up to order $\alpha, \beta, \gamma$ exist and are Lipschitz) and are estimated with squared error $n^{-2\alpha/(2\alpha+d)}$, $n^{-2\beta/(2\beta+d)}$ and $n^{-2\gamma/(2\gamma+d)}$, then the conditions in Corollary 6 will be satisfied when $\alpha, \beta, \gamma \geq d/2$, i.e., when the smoothness is greater than half the dimension.

6.1.2 ACME of the Optimal Policy

Instead of evaluating a fixed policy $d$, one may want to evaluate the ACME of the truly optimal (but unknown) policy $d^*_{\text{ACME}} = 1\{\gamma(X) > 0\}$ where $\gamma(X) := m_1(X) - m_0(X)$ is the conditional median treatment effect (CMTE). This would give a benchmark for the best possible value that could be achieved by any policy: any improvements would have to come by way of changing the population or changing the treatment. A natural estimator for the optimal policy would simply plug in an estimate of $\gamma$: $\hat{d}^*_{\text{ACME}}(X) = 1\{\hat{\gamma}(X) > 0\}$. Here, the policy $\hat{d}^*_{\text{ACME}}$ is learned through the same sample $Z^{n,0}$ as the one used for estimating the nuisance functions (8). For estimating the CMTE $\gamma$, in addition to the plug-in $\hat{m}_1 - \hat{m}_0$, a doubly robust-style estimator will be discussed in Section 7. An immediate question to ask is whether the estimator $\hat{\psi}_{\hat{d}^*_{\text{ACME}},dr}$ for the learned policy $\hat{d}^*_{\text{ACME}}$ can still exhibit similar asymptotic convergence guarantees as $\hat{\psi}_{d,dr}$ for a fixed policy $d$. Our goal in this subsection is to answer this question. To simplify notation, we use $\psi_{d,dr} := \psi_{d^*_{\text{ACME}},dr}$ and $\hat{\psi}_{\hat{d}^*_{\text{ACME}},dr} := \hat{\psi}_{\hat{d}^*_{\text{ACME}},dr}$.

Importantly, the ACME of the optimal policy is a non-smooth functional, because of its dependence on the indicator function $1\{\gamma(X) > 0\}$. This challenge also arises in the mean value setting [Chakraborty et al., 2010, Laber and Murphy, 2011, Hirano and Porter, 2012, van der Laan and Luedtke, 2014, Laber et al., 2014, Luedtke and van der Laan, 2016]. One solution is to incorporate a margin condition [Tsybakov et al., 2004, Luedtke and van der Laan, 2016], as follows.

Assumption 3 (Margin Condition). For some $\alpha > 0$ and all $t > 0$, we have

$$\Pr\left(|\gamma(X)| \leq t\right) \leq (ct)^\alpha,$$

for some constant $c > 0$ such that $ct \leq 1$.

The exponent $\alpha$ in the margin condition characterizes the mass such that $m_1(X)$ and $m_0(X)$ are close. The lower $\alpha$ is, the weaker the margin condition is, i.e., the more mass the distribution of $\gamma(X)$ is allowed to have near zero. Similar margin conditions are used in classification literature [Tsybakov et al., 2004] and optimal treatment regimes [Luedtke and van der Laan, 2016]. Note that $\alpha = 0$ encodes no assumption, allowing $\gamma(X) = 0$ almost surely, while $\alpha = 1$ would hold as long as $\gamma(X)$ is continuously distributed with bounded density. The margin condition therefore provides a characterization on how hard the optimal decision problem is—intuitively, when $\gamma(X)$ is near zero, it is very hard to distinguish which subjects will benefit from treatment, while when $\gamma(X)$ is very different from zero, this is easy to distinguish. Under the margin condition, we show in Theorem 7 that the error of $\hat{\psi}_{\hat{d}^*_{\text{ACME}},dr}$ is similar to that of the error of $\hat{\psi}_{d^*_{\text{ACME}},dr}$ for the truly optimal policy $d^*$.

Theorem 7. Assume

1. $\Pr\left(\epsilon \leq \hat{\pi}_1(X) \leq 1 - \epsilon\right) = 1$ for some $\epsilon \in (0, 1)$,
2. \( \mathbb{P}(M_0 \leq \hat{f}_{a,\hat{m}}(X) \leq M_1) = 1 \) for \( a \in \{0,1\} \) and \( 0 < M_0 \leq M_1 \), and

3. \( ||\hat{\xi}_{d^*} - \xi_{d^*}|| = o_{\mathbb{P}}(1) \).

Then, under Assumption 3, we have that

\[
\hat{\psi}_{d^*,dr} - \psi_{d^*} = (\mathbb{P}_n - \mathbb{P}) \hat{\xi}_{d^*} + O_{\mathbb{P}} \left( \sum_{a=0}^{1} ||\hat{m}_a - m_a|| \left( ||\hat{\pi}_{a,\hat{m}} - \pi_{a,f_{a,m}}|| + ||\hat{m}_a - m_a|| \right) + ||\hat{\gamma} - \gamma||_\infty^{1+\alpha} + o_{\mathbb{P}}(1/\sqrt{n}) \right).
\]

Unlike the error of \( \hat{\psi}_{d,dr} \) for a fixed policy \( d \) presented in Theorem 5, the error of \( \hat{\psi}_{d^*,dr} \) depends on \( ||\hat{\gamma} - \gamma||_\infty^{1+\alpha} \). When \( \alpha \) gets higher, the error of \( \hat{\gamma} \) plays less of a role, as captured in Corollary 8.

Under the conditions in Corollary 8 we obtain asymptotic normality of the estimator and thus can construct asymptotic confidence intervals as illustrated in our experiments in Section 8.2.

**Corollary 8.** Under the assumptions of Theorem 7, and the following three conditions:

1. \( ||\hat{m}_a - m_a|| = o_{\mathbb{P}}(n^{-1/4}) \),

2. \( ||\hat{\gamma} - \gamma||_\infty = o_{\mathbb{P}} \left( n^{-1/(2(1+\alpha))} \right) \),

3. For \( a \in \{0,1\}, \ ||\hat{\pi}_{a,\hat{m}} - \pi_{a,f_{a,m}}|| = O_{\mathbb{P}}(n^{-1/4}) \),

we have that \( \hat{\psi}_{d^*,dr} \) is root-n consistent and asymptotically normal with:

\[
\sqrt{n}(\hat{\psi}_{d^*,dr} - \psi_{d^*}) \rightsquigarrow \mathcal{N}(0, \sigma_{d^*}^2).
\]

As shown in Corollary 8, the error of \( \hat{\psi}_{d^*,dr} \) shares the same convergence guarantee as \( \hat{\psi}_{d,dr} \) under the extra assumption that the margin condition holds and the estimation error of \( \hat{\gamma} \) is \( o_{\mathbb{P}} \left( n^{-1/(2(1+\alpha))} \right) \).

This suggests that when the CMTE \( \gamma \) can be estimated well, or when the margin condition holds in a strong sense, the doubly robust-style estimator when based on the estimated policy \( \hat{d}^* \) behaves as if the true optimal policy \( d^* \) was instead plugged in. In Section 6.2, we give an algorithm that describes the estimation procedure for our proposed estimator.

### 6.2 Construction of the Estimators

The construction of the estimator contains two main steps: nuisance training and policy evaluation (with cross-fitting an additional possible step). Let \( (D_1, D_2, D_3) \) denote three independent samples of \( n \) observations of \( (X_i, A_i, Y_i) \).

**Step 1 Nuisance training:**

(a) Use \( D_1 \) to construct propensity score estimates \( \hat{\pi}_1 \).

(b) Use \( D_1 \) to construct conditional median estimates \( \hat{m}_a \) for \( a \in \{0,1\} \), for example using quantile regression.

(c) Use \( D_2 \) to construct conditional density estimates at the estimated conditional median \( \hat{f}_{a,\hat{m}} \), for example by regressing \( \frac{1}{n} K \left( \frac{Y_1 - \hat{m}_a(X)}{\hat{m}_a(X)} \right) \) on \( X \) among those with \( A = a \), for \( K \) a standard kernel function.
Step 2 Evaluate the value of the given fixed policy \( d \) or the learned optimal policy \( \hat{d}^*(X) = 1\{\hat{\gamma}(X) > 0\} \) on \( D_3 \) through the estimator presented in (8).

Step 3 Cross-fitting (Optional): Repeat Step 1 and 2 two times by using the dataset in the order \((D_1, D_3, D_2)\), and \((D_2, D_3, D_1)\). Use the average of the resulting estimators as the final estimate of the ACME of the evaluated policy. We focus on three folds for simplicity, but alternatives using \( k > 3 \) folds are also possible.

In Section 8.2, we use an example to illustrate this estimation procedure. In the above we suggested quantile regression and a particular conditional density estimation procedure, but our convergence rate results show that any generic method could be used, as long as it satisfied the high-level \( L_2 \) error conditions listed there.

Remark 10. In the above we use sample splitting to avoid empirical process conditions. Specifically, we split \( D_1 \) from \( D_2 \) in order to avoid conditions when estimating the conditional density \( f_{a,m} \), and we split \( D_3 \) in order to avoid conditions in doing bias correction using the estimated influence function. This splitting could be omitted for if Donsker-type conditions are deemed acceptable.

7 Estimation of the CMTE & Policy Learning

In this section, we propose efficient estimators for the conditional median treatment effect \( \gamma(X) = m_1(X) - m_0(X) \), and use it to construct the median optimal treatment regime. Such an estimator is of independent interest, as it suggests ways to characterize the heterogeneity of the median treatment effect. After providing the doubly robust-style estimator \( \hat{\gamma}_{dr} \) in Section 7.1, we show how its estimation error is connected to the performance of the corresponding median optimal policy estimator \( \hat{d}_{dr}^*(X) = 1\{\hat{\gamma}_{dr}(X) > 0\} \) (Section 7.2). Unlike existing work on estimating the marginal and conditional quantile treatment effect [Chernozhukov and Hansen, 2005, Díaz, 2017, Firpo, 2007, Fortin et al., 2011, Frölich and Melly, 2013, Machado and Mata, 2005, Melly, 2005, Rothe, 2010], our proposed nonparametric estimator for the CMTE relies on pseudo-outcome regression.

7.1 Doubly Robust-Style Estimator of the CMTE

Following the conditional average treatment effect estimation procedure proposed in Kennedy [2020], we construct the doubly robust-style estimator \( \hat{\gamma}_{dr} \) as follows: Let \((D_1, D_2, D_3)\) denote three independent samples of \( n \) observations of \( Z_i = (X_i, A_i, Y_i) \).

Step 1 Nuisance training: Same as Step 1 in Section 6.2.

Step 2 Pseudo-outcome regression: Use \( D_3 \) to construct the pseudo-outcome

\[
\hat{g}(Z) = \hat{m}_1(X) - \hat{m}_0(X) + \frac{A - \hat{\pi}_1(X)}{\hat{\pi}_1(X)\hat{\pi}_0(X)} \cdot \frac{1}{\hat{f}_{A,\hat{m}}(X)} (Y - \hat{m}_A(X)),
\]

and regress it on covariates \( X \) to obtain \( \hat{\gamma}_{dr} \). The regression estimator \( \hat{\gamma}_{dr} \) is given by

\[
\hat{\gamma}_{dr}(x) = \hat{\gamma}_{dr} \{ \hat{g}(Z)|X = x \} = \sum_{i=1}^{n} w_i(x; X^n)\hat{g}(Z_i), \quad (9)
\]

where the weights \( w_i(x; X^n) \) are learned using \( X^n \) in sample \( D_3 \). Examples of such linear smoothers \( \hat{\gamma}_{dr} \) include kernel estimators, linear, ridge, local polynomial and RKHS regression, some random forests (e.g., Mondrian and kernel forests), as well as weighted combinations of aforementioned methods [Wasserman, 2006].
Step 3 Cross-fitting (Optional): Repeat Step 1 and 2 two times by using the dataset in the order 
\((D_1, D_3, D_2)\), and \((D_2, D_3, D_1)\). Use the average of the resulting estimators as the final \(\hat{\gamma}_\text{dr}\).

In a recent line of work [Nie and Wager, 2021, Kennedy, 2020], the conditional treatment effect
estimators are commonly compared with an oracle estimator \(\tilde{\gamma}\) that has access to the true nuisance
functions. We define the oracle estimator in our setting as follows.

**Definition 2 (Oracle).** Denote \(g(Z)\) to be the pseudo-outcome that depends on the true nuisance
functions:
\[
g(Z) = m_1(X) - m_0(X) + \frac{A - \pi_1(X)}{\pi_1(X)\pi_0(X)} \frac{1/2 - 1\{Y \leq m_A(X)\}}{f_{A,m}(X)}.
\]

Given independent and identically distributed samples \(\{X_i, O_i\}_{i=1}^n\) where \(O_i = \gamma(X_i) + \varepsilon_i\) and \(\varepsilon_i\) is
a mean-zero noise defined to be \(\varepsilon_i = g(Z_i) - \gamma(X_i)\), the oracle is given by
\[
\tilde{\gamma}(x) = \hat{E}_n\{O|X = x\}.
\]

In other words, the oracle \(\tilde{\gamma}\) regresses \(O\) on the covariates \(X\) using the same linear smoother \(\hat{E}_n\) as
the one used in \(\hat{\gamma}_\text{dr}\) (9). The performance of the oracle depends directly on the complexity (e.g.,
smoothness) of \(\gamma\) itself, since the outcome \(O\) is the sum of \(\gamma(X)\) and a mean-zero noise.

We illustrate in the following theorem that the mean squared error of \(\hat{\gamma}_\text{dr}\) can be upper bounded
by the oracle error incurred by \(\tilde{\gamma}\) and products of nuisance errors. This allows the CMTE to be
estimated at a faster rate even when the nuisance estimates are obtained at slower rates.

**Theorem 9.** Let \(\hat{\gamma}_\text{dr}\) and \(\tilde{\gamma}\) denote the conditional median treatment effect estimator and oracle
defined as above. Define the weighted norm \(\|\cdot\|_w\) and squared-weighted norm \(\|\cdot\|_{w^2}\) by
\[
\|v\|_w^2 = \|v(x)\|_w^2 = \sum_{i=1}^n \frac{|w_i(x; X^n)|}{\sum_j |w_j(x; X^n)|} \int |v(z)|^2 d\mathbb{P}(z|X_i),
\]
\[
\|v\|_{w^2}^2 = \|v(x)\|_{w^2}^2 = \sum_{i=1}^n \frac{|w_i(x; X^n)|^2}{\sum_j |w_j(x; X^n)|^2} \int |v(z)|^2 d\mathbb{P}(z|X_i).
\]

Assume
1. \(\mathbb{P}(\varepsilon \leq \hat{\pi}_1(X) \leq 1 - \varepsilon) = 1\) for some \(\varepsilon \in (0, 1)\),
2. \(\mathbb{P}(M_0 \leq \hat{f}_{a,\hat{m}}(X) \leq M_1) = 1\) for \(a \in \{0, 1\}\) and \(0 < M_0 \leq M_1\).
3. \(\text{Var}\{g(Z)|X = x\} \geq \sigma_{\text{min}}^2\) for all \(x \in \mathcal{X}\).

Then, for all \(x \in \mathcal{X}\), we have
\[
(\hat{\gamma}_\text{dr}(x) - \gamma(x))^2 \leq (\tilde{\gamma}(x) - \gamma(x))^2 + b(x) + O_{\mathbb{P}}(\|\tilde{g} - g\|_{w^2}^2 \mathbb{E}[(\tilde{\gamma}(x) - \gamma(x))^2]),
\]
where \(b(x) = (\sum_{i=1}^n |w_i(x; X^n)|^2) (\sum_{a=0}^1 \|\hat{m}_a - m_a\|_w^2 (\|\hat{\pi}_{a,\hat{m}} - \pi_{a,\hat{m}}\|_w + \|\hat{m}_a - m_a\|_w))^2\).

**Remark 11.** When the pseudo-outcome estimator \(\hat{g}\) is consistent in the squared-weighted norm, i.e.,
\(\|\hat{g} - g\|_{w^2} = o_{\mathbb{P}}(1)\), we have the third term in (10) to be \(o_{\mathbb{P}}(\mathbb{E}[(\tilde{\gamma}(x) - \gamma(x))^2])\).
The assumptions of Theorem 9 require $g(Z)$ to have variation for all $X = x$. Our upper bound on the squared error of $\hat{\gamma}_{dr}$ contains three terms: the first and last terms depend on the error of the oracle $\hat{\gamma}$, while the middle term relies on products of nuisance errors with respect to the weighted norm. When $\sum_j |w_j(x; X^n)| = 1$, the weighted norm $\| \cdot \|_w$ weighs the nuisance errors by $w_i(x; X^n)$ in a similar way to how the linear smoother $\hat{\gamma}_n$ weighs the pseudo-outcomes. As an example, when the linear smoother $\hat{\gamma}_n$ (e.g., nearest neighbor estimator) relies only on local information, these weighted norms ensure that the error at $x$ is also weighed by only its local information. In the more general case, the weights for the weighted norms are normalized so that they sum up to 1. As shown in Stone [1977], Györfi et al. [2002], $\sum_{i=1}^n |w_i(x; X^n)| = O_P(1)$ is a sufficient condition to ensure $\hat{\gamma}_n$ to be weakly universally consistent. In such cases, if $\| \hat{g} - g \|_w = o_P(1)$, then the squared error of $\hat{\gamma}_{dr}$ deviates from the error of the oracle $\hat{\gamma}$ by products of nuisance errors, suggesting that the CMTE can be estimated at a faster rate compared to the nuisance functions. For a more detailed discussion on sufficient and necessary conditions on $w_i(x; X^n)$ for ensuring the consistency of $\hat{\gamma}_n$, we refer the readers to [Stone, 1977].

7.2 Policy Learning

We briefly touch on the problem of learning the median optimal treatment regime. There are many approaches for policy learning. Among them, the two canonical ways are (1) empirical value maximization where the goal is to directly find the optimal policy through maximizing the empirical value of policies in a fixed policy class [Zhao et al., 2012, Zhang et al., 2013, Athey and Wager, 2021]; and (2) estimating the conditional treatment effect $\gamma(X)$ first and then plugging it into the closed form of the optimal treatment regime $d^*(X) = 1\{\gamma(X) > 0\}$ [Murphy, 2003, Robins, 2004, van der Laan and Luedtke, 2014]. The second approach is closely related to plug-in classifiers [Audibert and Tsybakov, 2007], since deciding on the optimal treatment can be viewed as a binary classification task. Another related line of work is robust policy learning [Xiao et al., 2019, Zhang et al., 2021]. Using a model-based approach, Xiao et al. [2019] learns conditional quantile treatment regimes through robust regression. On the other hand, Zhang et al. [2021] uses a heuristic two-stage nonparametric approach for robust policy learning. For a more comprehensive review on policy learning, we refer the readers to Athey and Wager [2021].

Using the doubly robust-style estimator $\hat{\gamma}_{dr}$ presented in (9), we adopt the second approach and construct the median optimal treatment regime through $\hat{d}^*_{dr}(X) = 1\{\hat{\gamma}_{dr}(X) > 0\}$. We notice that the error of the policy $\hat{d}^*_{dr}$ is bounded by the error of $\hat{\gamma}_{dr}$. Under the assumption that for all $x \in X$, either $|\gamma(x)| > \delta$ for some $\delta > 0$ or $\gamma(x) = 0$, we obtain that

$$\mathbb{E}[I\{\hat{d}^*_{dr}(x) \neq d^*(x)\}] \leq \mathbb{P}(|\gamma(x)| \leq |\hat{\gamma}_{dr}(x) - \gamma(x)|) \leq \mathbb{E}[|\hat{\gamma}_{dr}(x) - \gamma(x)|] \leq \sqrt{\mathbb{E}[|\hat{\gamma}_{dr}(x) - \gamma(x)|^2]}.$$  

The first inequality follows from Lemma 4 and the second inequality holds since if $\gamma(x) = 0$, then $d^*(x)$ can be either 1 or 0, suggesting that $\hat{d}^*_{dr}(x)$ will always be optimal. The error of the learned policy $\hat{d}^*_{dr}$ can be upper bounded by the square root of the expected squared error given in Theorem 9. The assumption used here is stronger than the margin condition. It is of future interest to relax such assumption and study the performance of $\hat{d}^*_{dr}$ in more flexible settings.
8 Experiments

8.1 Numerical Simulation

To explore the finite-sample properties of the estimator, we simulate from the following data generating process:

\[
X \sim \mathcal{N}(0, I_5),
\]

\[
\text{logit}\{\pi_1(X)\} = X^\top \beta \text{ where } \beta = (.2, .2, .2, .2),
\]

\[
Y | X, A \sim \text{Lognormal}(X^\top \beta + A, .25).
\]

We note that there is heterogeneity of the conditional median treatment effects across covariates \(X\), i.e., \(m_1(X) - m_0(X) = \exp(X^\top \beta + 1) - \exp(X^\top \beta) = (e - 1)\exp(X^\top \beta)\). To inspect the rate of convergence of the estimator in terms of the nuisance estimation error, we constructed the nuisance estimators through the following procedure: \(\hat{\pi}_1(X) = \expit\{\logit(\pi_1(X)) + \epsilon_{1,n}\}\), \(\hat{m}_a(X) = m_a(X) + \epsilon_{2,n}\), and \(\hat{f}_{a,m}(X) = f_{a,m}(X) + \epsilon_{3,n}\) where \(\epsilon_{1,n}, \epsilon_{2,n}, \epsilon_{3,n}\) are independent samples drawn from \(\mathcal{N}(n^{-\alpha}, n^{-2\alpha})\). This construction ensures that the root mean square errors of \(\hat{\pi}_a, \hat{m}_a\) and \(\hat{f}_{a,m}\) are of order \(O(n^{-\alpha})\). The policy used for evaluation is \(d(X) = 1\{X_1 > 0\}\). Results shown in Figure 2 are averaged over 1000 rounds.

As we have seen in Figure 2, the doubly robust-style estimator converges faster compared to the plug-in estimator. When \(n = 5000\), we see the estimation error of the doubly robust-style estimator is close to 0 when \(\alpha = .25\), in line with what our theory suggests in Corollary 6.

8.2 Application: ACTG 175

We illustrate our proposed methods using the ACTG 175 dataset given in the R package speff2trial [Juraska et al., 2012]. The data are from a randomized clinical trial in a population of adults with HIV type I. The treatment is binary where \(A = 0\) stands for only using zidovudine as the therapy and \(A = 1\) represents combination therapies. The outcome of interest is CD4 T cell count after 96 ± 5 weeks. Covariates \(X\) include baseline CD4 and CD8 T cell count, age, weight, Karnofsky score, indicators for race, gender, hemophilia, homosexual activity, drug use, whether symptomatic, and previous zidovudine and antiretroviral use. Since the treatment is randomized, the propensity score is known to be \(\pi_1(x) = .75\) for all subjects. The number of observations is \(n = 1342\), after excluding subjects with missing outcomes.
Figure 3: 3a is the histogram of the outcomes for females under treatment and control. The difference in mean outcome under treatment versus control is 49, while the difference in median outcomes is only 1. 3b is the histogram of \( \hat{m}_1(X_i) - \hat{m}_0(X_i) \) where \( \hat{m}_a \) is estimated using quantile regression forests described in Section 8.2.

Figure 3a shows a histogram of outcomes under treatment versus control for females; the presence of skewed treatment responses suggest the median may be a more useful measure than the mean in this study. In fact, although the mean outcome in this group is quite different under treatment versus control (mean CD4 count is 341 for treated females, but only 292 for controls), the medians are similar (313 for treated, 312 for control).

Therefore we apply our proposed methods to estimate the value of the median optimal policy, and the value of a few competing policies. Specifically we use the estimator described in Section 6.2, splitting the sample into thirds (\( D_1, D_2 \) and \( D_3 \)) and using cross-fitting. In \( D_1 \) the conditional median estimate \( \hat{m}_a \) is obtained with quantile regression forests via the package quantregForest [Meinshausen, 2006] (recall the propensity score is known and so does not need to be estimated here). \( D_2 \) is then used to construct the density estimate \( \hat{f}_a, \hat{m}_a \), which we estimated by regressing a Gaussian kernel-weighted outcome centered at \( \hat{m}_a \) on \( X \) using the randomForest R package. The bandwidth \( h \) was chosen using Silverman’s rule [Silverman, 1986]. We considered estimating the ACME value of five policies using \( D_3 \): the observational policy \( d(X_i) = A_i \), a plug-in median optimal policy \( d(X_i) = 1\{\hat{m}_1(X_i) > \hat{m}_0(X_i)\} \), the treat-all policy \( d(X_i) = 1 \), the treat-none policy \( d(X_i) = 0 \), and a plug-in mean optimal policy \( d(X_i) = 1\{\hat{\mu}_1(X_i) > \hat{\mu}_0(X_i)\} \) where the regression functions \( \hat{\mu}_a \) are also estimated via random forests.

Figure 4 shows the estimated values with the proposed doubly robust-style estimator \( \hat{\psi}_{d,dr} \), as well as the plug-in \( \hat{\psi}_{d,p}\), for reference. 95% confidence intervals for \( \hat{\psi}_{d,dr} \) are obtained with the usual Wald interval based on the empirical variance \( \hat{\psi}_{d,dr} \pm 1.96\hat{\sigma}_d/\sqrt{n} \) where \( \hat{\sigma}_d \) is the sample standard deviation of the influence function estimates. The results show the median optimal policy gives the highest value (342.40 with 95% CI 8.20), with the mean optimal policy and treat-all policy close behind. The treat-none policy does substantially worse than even the observational (random assignment) policy. Figure 3b shows a histogram of the conditional median treatment effects \( \hat{m}_1(X_i) - \hat{m}_0(X_i) \), indicating a modest amount of effect heterogeneity.

9 Discussion

In this paper, we proposed a treatment policy based on conditional median treatment effects, and a new ACME value measure which is maximized for this policy. Importantly, our proposed approach...
Figure 4: The results for the ACTG 175 data analysis. Black triangles show the estimated ACME values of five different policies, computed with our proposed estimator (8), and the black lines show the estimated 95% confidence intervals. For comparison, the yellow circles are estimates from the plug-in estimator (7).

avoids both within-group lack of robustness issues with the mean, as well as across-group unfairness issues with the marginal median. We argue that optimal treatment policies should be defined in terms of the conditional distribution of outcomes, given covariates, rather than the marginal, in order to avoid across-group unfairness. We study nonparametric efficiency bounds and propose provably optimal doubly robust-style estimators, showing their finite-sample properties in simulations and in an illustration analyzing effects of combination therapy in treating HIV. We would generally argue that the mean is most useful as a measure of centrality when it is close to the median, and otherwise the median should be preferred; this suggests median effects should at least be used more widely than they are at present.

There are many opportunities for future related work. In addition to developing $V$-specific median optimal treatment regimes and analyzing $\hat{d}^*_0$, in more general settings, one could also consider more general (conditional) quantile optimal treatment regimes that replace conditional medians by conditional quantiles. All of this could also be adapted to numerous other settings, including continuous or time-varying treatments, or mediation problems, or settings where treatments may be confounded so that sensitivity analysis or instrumental variables may be used, etc.

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References

Susan Athey and Stefan Wager. Policy learning with observational data. *Econometrica*, 89(1): 133–161, 2021.
Jean-Yves Audibert and Alexandre B Tsybakov. Fast learning rates for plug-in classifiers. *The Annals of statistics*, 35(2):608–633, 2007.

Peter J Bickel and Ya’acov Ritov. Estimating integrated squared density derivatives: sharp best order of convergence estimates. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 381–393, 1988.

Peter J Bickel, Chris AJ Klaassen, Peter J Bickel, Ya’acov Ritov, J Klaassen, Jon A Wellner, and Ya’Acov Ritov. *Efficient and adaptive estimation for semiparametric models*, volume 4. Johns Hopkins University Press Baltimore, 1993.

George Casella and Roger L Berger. *Statistical inference*, volume 2. Duxbury Pacific Grove, CA, 2002.

Bibhas Chakraborty. *Statistical methods for dynamic treatment regimes*. Springer, 2013.

Bibhas Chakraborty, Susan Murphy, and Victor Strecher. Inference for non-regular parameters in optimal dynamic treatment regimes. *Statistical methods in medical research*, 19(3):317–343, 2010.

Victor Chernozhukov and Christian Hansen. An iv model of quantile treatment effects. *Econometrica*, 73(1):245–261, 2005.

Victor Chernozhukov, Denis Chetverikov, Mert Demirer, Esther Duflo, Christian Hansen, Whitney Newey, and James Robins. Double/debiased machine learning for treatment and structural parameters. *The Econometrics Journal*, 21(1):C1–C68, 2018.

Alexandra Chouldechova and Aaron Roth. A snapshot of the frontiers of fairness in machine learning. *Communications of the ACM*, 63(5):82–89, 2020.

Iván Díaz. Efficient estimation of quantiles in missing data models. *Journal of Statistical Planning and Inference*, 190:39–51, 2017.

Cynthia Dwork, Moritz Hardt, Toniann Pitassi, Omer Reingold, and Richard Zemel. Fairness through awareness. In *Proceedings of the 3rd innovations in theoretical computer science conference*, pages 214–226, 2012.

Sergio Firpo. Efficient semiparametric estimation of quantile treatment effects. *Econometrica*, 75(1):259–276, 2007.

Nicole Fortin, Thomas Lemieux, and Sergio Firpo. Decomposition methods in economics. In *Handbook of labor economics*, volume 4, pages 1–102. Elsevier, 2011.

Markus Frölich and Blaise Melly. Unconditional quantile treatment effects under endogeneity. *Journal of Business & Economic Statistics*, 31(3):346–357, 2013.

László Györfi, Michael Kohler, Adam Krzyzak, Harro Walk, et al. *A distribution-free theory of nonparametric regression*, volume 1. Springer, 2002.

Jinyong Hahn. On the role of the propensity score in efficient semiparametric estimation of average treatment effects. *Econometrica*, pages 315–331, 1998.

Moritz Hardt, Eric Price, and Nathan Srebro. Equality of opportunity in supervised learning. In *Proceedings of the 30th International Conference on Neural Information Processing Systems*, pages 3323–3331, 2016.
Keisuke Hirano and Jack R Porter. Impossibility results for nondifferentiable functionals. *Econometrica*, 80(4):1769–1790, 2012.

Peter J Huber. *Robust statistics*, volume 523. John Wiley & Sons, 2004.

Peter J Huber et al. The behavior of maximum likelihood estimates under nonstandard conditions. In *Proceedings of the fifth Berkeley symposium on mathematical statistics and probability*, volume 1, pages 221–233. University of California Press, 1967.

M Juraska, PB Gilbert, X Lu, M Zhang, M Davidian, and AA Tsiatis. speff2trial: Semiparametric efficient estimation for a two-sample treatment effect. *R package version*, 1(4), 2012.

Nathan Kallus, Xiaojie Mao, and Masatoshi Uehara. Localized debiased machine learning: Efficient estimation of quantile treatment effects, conditional value at risk, and beyond. *arXiv preprint arXiv:1912.12945*, 2019.

Edward H Kennedy. Semiparametric theory and empirical processes in causal inference. In *Statistical causal inferences and their applications in public health research*, pages 141–167. Springer, 2016.

Edward H Kennedy. Optimal doubly robust estimation of heterogeneous causal effects. *arXiv preprint arXiv:2004.14497*, 2020.

Edward H Kennedy, Sivaraman Balakrishnan, Max G’Sell, et al. Sharp instruments for classifying compliers and generalizing causal effects. *Annals of Statistics*, 48(4):2008–2030, 2020.

Michael R Kosorok and Eric B Laber. Precision medicine. *Annual review of statistics and its application*, 6:263–286, 2019.

Eric B Laber and Susan A Murphy. Adaptive confidence intervals for the test error in classification. *Journal of the American Statistical Association*, 106(495):904–913, 2011.

Eric B Laber, Daniel J Lizotte, Min Qian, William E Pelham, and Susan A Murphy. Dynamic treatment regimes: Technical challenges and applications. *Electronic Journal of Statistics*, 8(1):1225, 2014.

Kristin A Linn, Eric B Laber, and Leonard A Stefanski. Interactive q-learning for quantiles. *Journal of the American Statistical Association*, 112(518):638–649, 2017.

Alex Luedtke, Antoine Chambaz, et al. Performance guarantees for policy learning. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 56, pages 2162–2188. Institut Henri Poincaré, 2020.

Alexander R Luedtke and Mark J van der Laan. Statistical inference for the mean outcome under a possibly non-unique optimal treatment strategy. *Annals of statistics*, 44(2):713, 2016.

José AF Machado and José Mata. Counterfactual decomposition of changes in wage distributions using quantile regression. *Journal of applied Econometrics*, 20(4):445–465, 2005.

Ninareh Mehrabi, Fred Morstatter, Nripsuta Saxena, Kristina Lerman, and Aram Galstyan. A survey on bias and fairness in machine learning. *arXiv preprint arXiv:1908.09635*, 2019.

Nicolai Meinshausen. Quantile regression forests. *Journal of Machine Learning Research*, 7(Jun):983–999, 2006.

Blaise Melly. Decomposition of differences in distribution using quantile regression. *Labour economics*, 12(4):577–590, 2005.
Weibin Mo, Zhengling Qi, and Yufeng Liu. Learning optimal distributionally robust individualized treatment rules. *Journal of the American Statistical Association*, pages 1–16, 2020.

Susan A Murphy. Optimal dynamic treatment regimes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65(2):331–355, 2003.

Whitney K Newey. Semiparametric efficiency bounds. *Journal of Applied Econometrics*, 5(2):99–135, 1990.

Xinkun Nie and Stefan Wager. Quasi-oracle estimation of heterogeneous treatment effects. *Biometrika*, 108(2):299–319, 2021.

Zhengling Qi, Jong-Shi Pang, and Yufeng Liu. Estimating individualized decision rules with tail controls. *arXiv preprint arXiv:1903.04367*, 2019.

Joaquin Quiñonero-Candela, Masashi Sugiyama, Anton Schwaighofer, and Neil D Lawrence. *Dataset shift in machine learning*. The MIT Press, 2009.

James Robins, Lingling Li, Eric Tchetgen, Aad van der Vaart, et al. Higher order influence functions and minimax estimation of nonlinear functionals. In *Probability and statistics: essays in honor of David A. Freedman*, pages 335–421. Institute of Mathematical Statistics, 2008.

James M Robins. Optimal structural nested models for optimal sequential decisions. In *Proceedings of the second seattle Symposium in Biostatistics*, pages 189–326. Springer, 2004.

R Tyrrell Rockafellar and Stanislav Uryasev. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26(7):1443–1471, 2002.

Christoph Rothe. Nonparametric estimation of distributional policy effects. *Journal of Econometrics*, 155(1):56–70, 2010.

Donald B Rubin. Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of Educational Psychology*, 66(5):688, 1974.

Phillip J Schulte, Anastasios A Tsiatis, Eric B Laber, and Marie Davidian. Q-and a-learning methods for estimating optimal dynamic treatment regimes. *Statistical Science: a review Journal of the Institute of Mathematical Statistics*, 29(4):640, 2014.

Bernard W Silverman. *Density estimation for statistics and data analysis*, volume 26. CRC press, 1986.

Charles J Stone. Consistent nonparametric regression. *The annals of statistics*, pages 595–620, 1977.

Anastasios Tsiatis. *Semiparametric theory and missing data*. Springer Science & Business Media, 2007.

Alexander B Tsybakov et al. Optimal aggregation of classifiers in statistical learning. *The Annals of Statistics*, 32(1):135–166, 2004.

Mark J van der Laan and Alexander R Luedtke. Targeted learning of an optimal dynamic treatment, and statistical inference for its mean outcome. 2014.

Mark J van der Laan and James M Robins. *Unified methods for censored longitudinal data and causality*. Springer Science & Business Media, 2003.

Aad W van der Vaart. *Asymptotic statistics*, volume 3. Cambridge university press, 2000.
Aad W van der Vaart. Semiparametric statistics. *Lecture Notes in Math.*, (1781), 2002.

Lan Wang, Yu Zhou, Rui Song, and Ben Sherwood. Quantile-optimal treatment regimes. *Journal of the American Statistical Association*, 113(523):1243–1254, 2018.

Larry Wasserman. *All of nonparametric statistics*. Springer Science & Business Media, 2006.

Wei Xiao, Hao Helen Zhang, and Wenbin Lu. Robust regression for optimal individualized treatment rules. *Statistics in medicine*, 38(11):2059–2073, 2019.

Baqun Zhang, Anastasios A Tsiatis, Eric B Laber, and Marie Davidian. A robust method for estimating optimal treatment regimes. *Biometrics*, 68(4):1010–1018, 2012.

Baqun Zhang, Anastasios A Tsiatis, Eric B Laber, and Marie Davidian. Robust estimation of optimal dynamic treatment regimes for sequential treatment decisions. *Biometrika*, 100(3):681–694, 2013.

Jinchun Zhang, Andrea B Troxel, and Eva Petkova. Robust index of confidence weighted learning for optimal individualized treatment rule estimation. *Stat*, 10(1):e374, 2021.

Yingqi Zhao, Donglin Zeng, A John Rush, and Michael R Kosorok. Estimating individualized treatment rules using outcome weighted learning. *Journal of the American Statistical Association*, 107(499):1106–1118, 2012.

Wenjing Zheng and Mark J van der Laan. Asymptotic theory for cross-validated targeted maximum likelihood estimation. 2010.
A Proofs

A.1 Proofs in Section 4

Proof of Proposition 1. Pick \( x \in X' \). For all \( a \in \{0, 1\} \), we have \( |\mu_a(x) - m_a(x)| \leq \sigma_a(x) \). If \( \mu_1(x) > \mu_0(x) \), then \( m_1(x) \geq \mu_1(x) - \sigma_1(x) > \mu_0(x) + \sigma_0(x) \geq m_0(x) \), where the second inequality holds because of the assumption that \( |\mu_1(x) - \mu_0(x)| > \sigma_1(x) + \sigma_0(x) \). Similarly, \( m_1(x) > m_0(x) \) implies that \( \mu_1(x) > m_1(x) - \sigma_1(x) > m_0(x) + \sigma_0(x) \geq \mu_0(x) \). Thus, we have that \( \mu_1(x) > \mu_0(x) \Leftrightarrow m_1(x) > m_0(x) \), which completes the proof.

Proof of Proposition 2. By definition, the marginal median optimal policy \( d_{\text{MME}}^* \) satisfies that for all \( d \in D \), \( m(Y^{d_{\text{MME}}}) \geq m(Y^d) \). We use a tuple \( (a_1, a_0) \) to represent a policy \( d \) where \( a_i = d(i) \), which gives us the following table:

| \((a_0, a_1)\) | \(m(Y^d)\) |
|----------------|----------------|
| \((0, 1)\)    | \(\frac{\sigma_1(1)\mu_0(0) + \sigma_0(0)\mu_1(1)}{\sigma_0(0) + \sigma_1(1)}\) |
| \((1, 1)\)    | \(\frac{\sigma_1(1)\mu_1(0) + \sigma_1(0)\mu_1(1)}{\sigma_0(1) + \sigma_1(1)}\) |
| \((0, 0)\)    | \(\frac{\sigma_0(1)\mu_1(0) + \sigma_0(0)\mu_1(1)}{\sigma_0(0) + \sigma_0(1)}\) |
| \((1, 0)\)    | \(\frac{\sigma_0(1)\mu_1(0) + \sigma_1(0)\mu_0(1)}{\sigma_1(0) + \sigma_0(1)}\) |

Table 2: Marginal medians of all possible policies in the setting described in Section 4.2.

Given \( \sigma_1(1) > \sigma_0(1) \) and \( \mu_1(1) > \mu_0(1) \) and \( \frac{\mu_1(1)}{\mu_0(1)} < \frac{\sigma_1(1)}{\sigma_0(1)} \), when we pick \( \{\mu_a(0), \sigma_a(0)\}_{a=0}^1 \) such that for \( a = 0 \) and \( a = 1 \),

\[
(\sigma_1(1) - \sigma_0(1))\mu_a(0) + (\mu_1(1) - \mu_0(1))\sigma_a(0) > \sigma_1(1)\mu_0(1) - \sigma_0(1)\mu_1(1),
\]

the optimal policy \( d_{\text{MME}}^* \) will assign the decision for \( x = 1 \) according to \( d_{\text{MME}}^*(1) = \mathbb{1}\{\mu_1(1) > \mu_0(1)\} \) since the conditions have ensured that \( m(Y^{(1,1)}) > m(Y^{(1,0)}) \) and \( m(Y^{(0,1)}) > m(Y^{(0,0)}) \). On the other hand, if (11) does not hold for \( a = 0 \) and \( a = 1 \), then the marginal median optimal policy \( d_{\text{MME}}^*(1) = \mathbb{1}\{\mu_1(1) \leq \mu_0(1)\} \).

A.2 Proofs in Section 5

Derivation of the influence function under discrete covariates and continuous outcome. Let \( \text{IF}(\cdot) \) denote the operator that returns the influence function of an input functional. In this analysis, we assume \( X \) to be discrete and \( Y \) to be continuous. Let \( p(x) \) denote the probability mass function of \( X \). We start with the chain rule of \( \text{IF}(\cdot) \).

\[
\phi_d(Z) = \text{IF}(\psi_d(F)) = \text{IF} \left( \sum_{x \in X} p(x)m_d(x) \right)
\]
\[
= \sum_{x \in X} p(x) \left( \text{IF}(m_1(x))d(x) + \text{IF}(m_0(x))(1 - d(x)) \right) + \text{IF}(p(x))m_d(x),
\]

where \( \text{IF}(p(x)) = \mathbb{1}\{X = x\} - p(x) \). Thus, it suffices to find \( \text{IF}(m_a(x)) \) where \( a \in \{0, 1\} \).
Let $\delta_z$ denote the Dirac measure at $z$. We use the submodel $\mathbb{P}_\epsilon(z) = (1 - \epsilon)\mathbb{P}(z) + \delta_{z'}$ for some $z' = (x', a', y')$ to find the influence function, which gives that

$$f_\epsilon(y|x,a) = \frac{(1 - \epsilon)f(y|x,a)p(X=x,A=a) + \epsilon\delta_{y'}}{(1 - \epsilon)p(X=x,A=a) + \epsilon\mathbb{1}\{x=x',a=a'\}},$$

where $f_\epsilon$ and $f$ are densities with respect to $\mathbb{P}_\epsilon$ and $\mathbb{P}$ respectively. Let $m_{a,\epsilon}(x)$ denote the median of the distribution $\mathbb{P}_\epsilon(Y|X=x,A=a)$. By the definition of the median, we have that

$$F_\epsilon(m_{a,\epsilon}(x)|x,a) = \int_{y \leq m_{a,\epsilon}(x)} f_\epsilon(y|x,a)dy$$

$$= \int_{y \leq m_{a,\epsilon}(x)} \frac{(1 - \epsilon)f(y|x,a)p(X=x,A=a) + \epsilon\delta_{y'}}{(1 - \epsilon)p(X=x,A=a) + \epsilon\mathbb{1}\{x=x',a=a'\}}dy = \frac{1}{2}$$

When $x \neq x'$ or $a \neq a'$, we have that

$$\mathbb{P}(X=x,A=a) \int_{y \leq m_{a,\epsilon}(x)} f(y|x,a)dy = \frac{\mathbb{P}(X=x,A=a)}{2},$$

which suggests that $m_a(x) = m_{a,\epsilon}(x)$. When $x = x'$ and $a = a'$, we have that

$$\int_{y \leq m_{a,\epsilon}(x)} \frac{(1 - \epsilon)f(y|x,a)p(X=x,A=a) + \epsilon\delta_{y'}}{(1 - \epsilon)p(X=x,A=a) + \epsilon}dy = \frac{1}{2}.$$  \hspace{1cm} (12)

It has two cases

* When $y' > m_{a,\epsilon}(x)$, (12) can be simplified into

$$F(m_{a,\epsilon}(x)|x,a) = \int_{y \leq m_{a,\epsilon}(x)} f(y|x,a)dy = \frac{(1 - \epsilon)p(X=x,A=a) + \epsilon}{2(1 - \epsilon)p(X=x,A=a)}$$

and $m_{a,\epsilon}(x) = F^{-1}\left(\frac{(1 - \epsilon)p(X=x,A=a) + \epsilon}{2(1 - \epsilon)p(X=x,A=a)}\right)|x,a).$

* When $y' \leq m_{a,\epsilon}(x)$, (12) can be simplified into

$$F(m_{a,\epsilon}(x)|x,a) = \int_{y \leq m_{a,\epsilon}(x)} f(y|x,a)dy = \frac{(1 - \epsilon)p(X=x,A=a) - \epsilon}{2(1 - \epsilon)p(X=x,A=a)}$$

and $m_{a,\epsilon}(x) = F^{-1}\left(\frac{(1 - \epsilon)p(X=x,A=a) - \epsilon}{2(1 - \epsilon)p(X=x,A=a)}\right)|x,a).$

Putting it altogether, we have that when $x \neq x'$ or $a \neq a'$:

$$\text{IF}(m_a(x)) = \frac{d}{d\epsilon} m_{a,\epsilon}(x)\bigg|_{\epsilon=0} = \mathbb{1}\{x=x'\}\mathbb{1}\{a=a'\}. $$

When $x = x'$ and $a = a'$, we have that

$$\text{IF}(m_a(x)) = \mathbb{1}\{x=x'\}\mathbb{1}\{a=a'\} \frac{d}{d\epsilon} m_{a,\epsilon}(x)\bigg|_{\epsilon=0} = -\frac{\mathbb{1}\{x=x'\}\mathbb{1}\{a=a'\}}{\mathbb{P}(X=x)p(A=a|X=x)} - \frac{\mathbb{1}\{y' \leq m_a(x)\} - 1/2}{f(m_a(x)|x,a)},$$

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where the last equality holds since for all $m_{a,e}(x) = F^{-1}(g(e)|x,a)$,

\[ \frac{d}{de} m_{a,e}(x) \bigg|_{e=0} = \left( \frac{1}{f(m_{a,e}(x)|x,a)} \frac{d}{de} g(e) \right) \bigg|_{e=0}, \]

and for example when $g(e) = \frac{(1-e)\mathbb{P}(X=x,A=a) + e}{2(1-e)\mathbb{P}(X=x,A=a)}$, i.e., when $y' > F^{-1}\left( \frac{(1-e)\mathbb{P}(X=x,A=a) + e}{2(1-e)\mathbb{P}(X=x,A=a)} \bigg| x,a \right)$, gives that

\[ \frac{d}{de} g(e) = \frac{1}{2\mathbb{P}(X=x,A=a)} \cdot \frac{d}{de} \mathbb{E} = \frac{1}{2(1-e)^2\mathbb{P}(X=x,A=a)}. \]

Therefore, we have obtained that

\[ \phi_d(Z) = \sum_{x \in \mathcal{X}} p(x) \left\{ \mathbb{I} \{ m_1(x) \} d(x) + \mathbb{I} \{ m_0(x) \} (1 - d(x)) \right\} + \mathbb{I}(p(x))m_d(X) \]

\[ = \sum_{x \in \mathcal{X}} p(x) \left\{ \frac{\mathbb{I} \{ X = x \} \mathbb{I} \{ A = 0 \}}{\mathbb{P}(X = x)\mathbb{P}(A = 1|X = x)} \frac{1}{f(m_1(X)|x,1)} d(x) \right\} + \frac{\mathbb{I} \{ X = x \} \mathbb{I} \{ A = 0 \}}{\mathbb{P}(X = x)\mathbb{P}(A = 1|X = x)} \frac{1}{f(m_0(X)|x,0)} \right\} d(x) \]

\[ + \mathbb{I}(p(x))m_d(x) \]

\[ = \frac{Ad(X)\mathbb{I} \{ X \leq m_1(X) \}}{\pi_1(X)} + \frac{(1-A)(1-d(X))}{\pi_0(X)} \frac{\mathbb{I} \{ X \leq m_0(X) \}}{f_0,m(X)} + m_d(X) - \psi_d. \]

where the last equality follows since the indicator function $\mathbb{I} \{ X = x \}$ picks out the values $x \in \mathcal{X}$ such that it equals $X$ and $p(x) = \mathbb{P}(X = x)$.

**Proof of Lemma 3.** By definition, since $\int \phi_d(Z;\mathbb{P})d\mathbb{P} = 0$, we have that

\[ R_d(\mathbb{P}, \mathbb{P}) = \psi_d(\mathbb{P}) - \psi_d(\mathbb{P}) + \int \phi_d(Z;\mathbb{P})d\mathbb{P} \]

\[ = -\psi_d(\mathbb{P}) + \int_{\mathcal{X}} \frac{\pi_1(x)d(x)}{\pi_1(x)} \frac{1}{f_{1,m}(x)} + \frac{\pi_0(x)(1-d(x))}{\pi_0(x)} \frac{1}{f_{0,m}(x)} + m_d(x)d\mathbb{P}(x) \]

\[ = \int_{\mathcal{X}} d(x) \left( \frac{\pi_1(x) - m_1(x) - \pi_1(x)}{\pi_1(x)} \frac{F_{1,m}(x) - F_{1,m}(x)}{f_{1,m}(x)} \right) \]

\[ + (1 - d(x)) \left( \frac{\pi_0(x) - m_0(x) - \pi_0(x)}{\pi_0(x)} \frac{F_{0,m}(x) - F_{0,m}(x)}{f_{0,m}(x)} \right) d\mathbb{P}(x), \]

where the second equality follows from iterated expectation, i.e., $\mathbb{E}_\mathbb{P}[\mathbb{I} \{ Y \leq m_a(X) \}]|X = x, A = a = F_{a,m}(x)$, and the third equality follows from rearranging terms and the fact that $F_{a,m}(x) = 1/2$. \[ \square \]

**Proof of Corollary 1.** Recall that for (4), we have

\[ R_d(\mathbb{P}, \mathbb{P}) = \int_{\mathcal{X}} d(x) \left( \frac{1}{A_2} \frac{\pi_1(x)}{\pi_1(x)} \frac{F_{1,m}(x) - F_{1,m}(x)}{f_{1,m}(x)} \right) \]

\[ + \frac{\pi_0(x) - m_0(x) - \pi_0(x)}{\pi_0(x)} \frac{F_{0,m}(x) - F_{0,m}(x)}{f_{0,m}(x)} \right) d\mathbb{P}(x), \]

\[ + \frac{\pi_0(x) - m_0(x) - \pi_0(x)}{\pi_0(x)} \frac{F_{0,m}(x) - F_{0,m}(x)}{f_{0,m}(x)} \right) d\mathbb{P}(x), \]

\[ 28 \]
For $x \in \mathcal{X}$ such that $f_a(y|x)$ is differentiable and $L$-Lipschitz continuous in $y$, we can simplify $A_2$ as:

$$A_2 = \frac{F_1(x) - F_{1,m}(x)}{f_{1,m}(x)}$$

$$= \frac{1}{f_{1,m}(x)} \left( F_{1,m}(x) + f_{1,m}(x)(m_1(x) - m_1(x)) + \frac{f'(c_1(x)|x)}{2}(m_1(x) - m_1(x))^2 - F_{1,m}(x) \right)$$

$$= \frac{1}{f_{1,m}(x)} \left( f_{1,m}(x)(m_1(x) - m_1(x)) + \frac{f'_c(x)}{2}(m_1(x) - m_1(x))^2 \right),$$

where the second equality follows from Taylor’s theorem and $c_1(x)$ is a value in between $m_1(x)$ and $m_1(x)$. To simplify the notations, in the third equality (and the following usage of Taylor’s theorem), we denote the derivative of the conditional probability density function $f_a(-|X = x)$ at $c_a(x)$ to be $f'_a,c(x)$ where $c_a(x)$ is a value between $m_a(x)$ and $m_a(x)$ and satisfies $F_{a,m}(x) = F_{a,m}(x) + f_{a,m}(x)(m_a(x) - m_a(x)) + f'_a,c(x)(m_a(x) - m_a(x))^2/2$. This gives that

$$A_1 = m_1(x) - m_1(x) - \frac{\pi_1(x)}{\pi_1(x)} \frac{1}{f_{1,m}(x)} \left( f_{1,m}(x)(m_1(x) - m_1(x)) + \frac{f'_c(x)}{2}(m_1(x) - m_1(x))^2 \right)$$

$$= (m_1(x) - m_1(x)) \left( 1 - \frac{\pi_1(x)}{\pi_1(x)} \frac{f_{1,m}(x)}{f_{1,m}(x)} - \frac{f'_c(x)}{2}(m_1(x) - m_1(x))^2 \right).$$

Applying the same logic to $A_0$, we obtain that

$$R_d(\mathbb{P}, \mathbb{P}) = \int d(x) \left( (m_1(x) - m_1(x)) \left( 1 - \frac{\pi_1(x)}{\pi_1(x)} \frac{f_{1,m}(x)}{f_{1,m}(x)} - \frac{f'_c(x)}{2}(m_1(x) - m_1(x))^2 \right) ight)$$

$$+ (1 - d(x)) \left( (m_0(x) - m_0(x)) \left( 1 - \frac{\pi_0(x)}{\pi_0(x)} \frac{f_{0,m}(x)}{f_{0,m}(x)} - \frac{f'_c(x)}{2}(m_0(x) - m_0(x))^2 \right) \right)$$

$$= \mathbb{P} \left\{ \frac{d}{\pi_1 f_{1,m}} \left( \frac{m_1 - m_1}{\pi_1 f_{1,m} - \pi_1 f_{1,m}} - \frac{f'_c(x)}{2}(m_1 - m_1)^2 \right) \right\}$$

$$+ \frac{(1 - d)}{\pi_0 f_{0,m}} \left( \frac{m_0 - m_0}{\pi_0 f_{0,m} - \pi_0 f_{0,m}} - \frac{f'_c(x)}{2}(m_0 - m_0)^2 \right) \right\}$$

$$= \mathbb{P} \left\{ \frac{d}{\pi_1 f_{1,m}} \left( \frac{m_1 - m_1}{\pi_1 f_{1,m} + (f_{1,m} - f_{1,m})} \pi_1 - \frac{(m_1 - m_1)}{2} \right) \right\}$$

$$+ \frac{(1 - d)}{\pi_0 f_{0,m}} \left( \frac{(m_0 - m_0)}{\pi_0 f_{0,m} + (f_{0,m} - f_{0,m})} \pi_0 - \frac{(m_0 - m_0)}{2} \right) \right\}.$$

$$\Box$$

**Proof of Corollary 2.** Consider a parametric submodel $\mathbb{P}_\epsilon$, e.g., the probability density functions $p_\epsilon$ and $d$ of $\mathbb{P}_\epsilon$ and $\mathbb{P}$ satisfy that $p_\epsilon(z) = p(z)(1 + \epsilon h(z))$ where $\mathbb{E}[h] = 0$, $\|h\|_\infty < M$ and $\epsilon > 1/M$. To show that $\phi_d$ is an influence function, it suffices to show that the mean-zero $\phi_d$ satisfies path-wise differentiability, i.e.,

$$\frac{d\psi_d(\mathbb{P}_\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} = \int_Z \phi_d(Z; \mathbb{P}) \left( \frac{d}{d\epsilon} \log d\mathbb{P}_\epsilon \right) \bigg|_{\epsilon=0} d\mathbb{P}.$$
As shown in Lemma 3, we have that $\psi_d(P_\epsilon) = \psi_d(P) + \int_Z \phi_d(Z; P)d(P_\epsilon - P) - R_d(P, P_\epsilon)$, which gives that 

$$
\left. \frac{d\psi_d(P_\epsilon)}{d\epsilon} \right|_{\epsilon=0} = \frac{d}{d\epsilon} \int_Z \phi_d(Z; P)dP_\epsilon \bigg|_{\epsilon=0} - \frac{d}{d\epsilon} R_d(P, P_\epsilon) \bigg|_{\epsilon=0}
$$

$$
= \int_Z \phi_d(Z; P) \left( \frac{d}{d\epsilon} \log dP_\epsilon \right) \bigg|_{\epsilon=0} dP_\epsilon - \frac{d}{d\epsilon} R_d(P, P_\epsilon) \bigg|_{\epsilon=0}.
$$

Finally, as shown in Corollary 1, $R_d(P, P_\epsilon)$ only contains second-order products of errors between $P$ and $P_\epsilon$ and thus using the product rule on $\frac{d}{d\epsilon} R_d(P, P_\epsilon)$, one can get that 

$$
\left. \frac{d}{d\epsilon} R_d(P, P_\epsilon) \right|_{\epsilon=0} = 0.
$$

Since our model is nonparametric, there is only one influence function which is efficient [Bickel et al., 1993, Tsiatis, 2007]. Thus, $\phi_d$ is the efficient influence function of $\psi_d$.

**Proof of Theorem 3.** For a given policy $d \in D$, we have that

$$
\text{Var} \{ \phi_d(Z) \} = \mathbb{E} [\phi_d(Z)^2] = \mathbb{E}[(\xi_d(Z) - \psi_d)^2]
$$

$$
= \mathbb{E} \left\{ \left( \frac{\mathbb{E}Y|X,A \left[ (1/2 - \mathbb{I}\{Y \leq m_1(X)\})^2 \right]}{\pi_1(X)} \right)^2 \right\} + \mathbb{E} \left\{ \left( \frac{(1-A)(1-d(X))}{\pi_0(X)} \right)^2 \mathbb{I}\{Y \leq m_0(X)\} \right\} + \text{Var} \{m_d(X)\}
$$

$$
= \mathbb{E} \left\{ \frac{\mathbb{E}Y|X,A \left[ (1/2 - \mathbb{I}\{Y \leq m_1(X)\})^2 \right]}{\pi_1(X)f_{1,m}(X)} \right\} + \mathbb{E} \left\{ \frac{(1-A)(1-d(X))}{\pi_0(X)} \mathbb{I}\{Y \leq m_0(X)\} \right\} + \text{Var} \{m_d(X)\}
$$

$$
= \mathbb{E} \left\{ \frac{d(X)}{4\pi_1(X)f_{1,m}^2(X)} + \frac{(1-d(X))}{4\pi_0(X)f_{0,m}^2(X)} \right\} + \text{Var} \{m_d(X)\},
$$

where the first equality is true since $\mathbb{E}[\phi_d(Z)] = 0$, the forth equality uses the fact $A^2 = A$, the fifth equality holds as $\mathbb{E}Y|X,A=a[\mathbb{I}\{Y \leq m_a(X)\}] = 1/2$, and the last equality is true since $\text{Var}\{\mathbb{I}\{Y \leq m_1(X)\}|X, A = 1\} = \mathbb{P}(Y \leq m_1(X)|X, A = 1)$ $(1 - \mathbb{P}(Y \leq m_1(X)|X, A = 1)) = 1/4$.

**A.3 Proofs in Section 6**

**Proof of Theorem 5.** We start with the following decomposition

$$
\hat{\psi}_{d,dr} - \psi_d = \mathbb{P}_n \xi_d(\hat{P}) - \mathbb{P} \xi_d(\hat{P}) = \underbrace{\mathbb{P}_n - \mathbb{P}}_{T_1} \left( \xi_d(\hat{P}) - \xi_d(\hat{P}) \right) + \underbrace{\mathbb{P}_n - \mathbb{P}}_{T_2} \xi_d(\hat{P}) + \mathbb{P} \left( \xi_d(\hat{P}) - \xi_d(\hat{P}) \right) .
$$
When \( \hat{\xi}_d \) is learned from a separate sample from the empirical measure \( \mathbb{P}_n \), by Lemma 5, we obtain that

\[
T_1 = O_{\mathbb{P}} \left( \frac{\| \hat{\xi}_d - \xi_d \|}{\sqrt{n}} \right) = o_{\mathbb{P}}(1/\sqrt{n}),
\]

where the last equality follows from our assumption that \( \| \hat{\xi}_d - \xi_d \| = o_{\mathbb{P}}(1) \). On the other hand, when \( \xi_d \) and \( \hat{\xi}_d \) are contained in a Donsker class and \( \| \hat{\xi}_d - \xi_d \| = o_{\mathbb{P}}(1) \), by van der Vaart [2000, Lemma 19.24], we have that \( T_1 = o_{\mathbb{P}}(1/\sqrt{n}) \). Since \( \mathbb{P}_n \xi_d - \psi_d = 0 \), we have that

\[
T_3 = \mathbb{P} \left( \hat{\xi}_d - \xi_d \right) = \mathbb{P} \left\{ \frac{\pi_1 d F_{1,m} - F_{1,m}}{\pi_1} + \frac{\pi_0 (1-d) F_{0,m} - F_{0,m}}{\bar{f}_{0,m}} + \hat{m}_d - m_d \right\}
= \mathbb{P} \left\{ d \left( \hat{m}_1 - m_1 - \frac{\pi_1 F_{1,m} - F_{1,m}}{\pi_1} \right) + (1-d) \left( \hat{m}_0 - m_0 - \frac{\pi_0 F_{0,m} - F_{0,m}}{\bar{f}_{0,m}} \right) \right\}
= R_d \left( \mathbb{P}, \mathbb{P} \right),
\]

where the second equality holds since \( \mathbb{P}_n \xi_d = \psi_d = \mathbb{P} \{ m_d \} \) and \( \mathbb{E}_{Y|X,A=a}[\mathbb{P}\{ Y \leq \hat{m}_a(X) \}|X,A=a] = F_{a,\hat{m}}(X) \). Finally, using Lemma 1, we obtain that

\[
R_d(\mathbb{P}, \mathbb{P}) = \mathbb{P} \left\{ \frac{d (\hat{m}_1 - m_1)}{\pi_1 f_{1,m}} \left( (\pi_1 - \pi_1 f_{1,m} + (\bar{f}_{1,m} - f_{1,m}) \pi_1 - (\hat{m}_1 - m_1) \frac{f_{1,m}}{2} \right)
+ \frac{(1-d) (\hat{m}_0 - m_0)}{\bar{f}_{0,m}} \left( (\pi_0 - \pi_0 f_{0,m} + (\bar{f}_{0,m} - f_{0,m}) \pi_0 - (\hat{m}_0 - m_0) \frac{f_{0,m}}{2} \right) \right\}
= O_{\mathbb{P}} \left( \sum_{a=0}^1 ||\hat{m}_a - m_a|| \left( ||\hat{m}_a f_{a,m} - \pi_a f_{a,m}|| + ||\hat{m}_a - m_a|| \right) \right).
\]

\[\square\]

**Proof of Corollary 6.** Following from the proof of Theorem 5, we have that \( \hat{\psi}_{d,dr} - \psi_d = T_1 + T_2 + T_3 \) where \( T_1 = o_{\mathbb{P}}(1/\sqrt{n}) \). From Theorem 3, we have that \( \text{Var}\{ \xi_d \} = \text{Var}\{ \phi_d \} = \sigma^2_d < +\infty \) since \( \mathbb{P} \in \mathcal{P} \). By Central Limit Theorem, we have that \( T_2 = o_{\mathbb{P}}(1/\sqrt{n}) \) and

\[
\sqrt{n}T_2 = \sqrt{n}(\mathbb{P}_n - \mathbb{P})\xi_d(\mathbb{P}) = \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n \xi_d(Z_i) - \mathbb{E}[\xi_d(Z)] \right) \rightarrow \mathcal{N}(0, \text{Var}(\xi_d)).
\]

Using Slutsky’s theorem and the assumptions that ensure \( T_3 = o_{\mathbb{P}}(1/\sqrt{n}) \), we have

\[
\sqrt{n}(\hat{\psi}_{d,dr} - \psi_d) \rightarrow \mathcal{N}(0, \sigma^2_d).
\]

\[\square\]

**Proof of Theorem 7.** For any \( d \in \mathcal{D} \), by the fact that \( \mathbb{P}_n \xi_d = \psi_d \), we have that

\[
\hat{\psi}_{d,dr} - \psi_d = \mathbb{P}_n \hat{\xi}_{d^*} - \mathbb{P} \hat{\xi}_{d^*} + \mathbb{P} \xi_{d^*} - \mathbb{P} \xi_{d^*}
= \mathbb{P}_n \hat{\xi}_{d^*} - \mathbb{P} \xi_{d^*} + \mathbb{P} \gamma(d^* - d^*),
\]

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where the second equality uses that

\[
\mathbb{P}\xi_{\hat{d}^*} - \mathbb{P}\xi_{d^*} = \mathbb{P}\left\{ \frac{A(d^*(X) - d^*(X))}{\pi_1(X)} \frac{1}{2} - \mathbb{1}\{Y \leq m_1(X)\} \right\} \frac{\pi_0(X)}{f_{0,m}(X)} + \frac{(1 - A)(d^*(X) - d^*(X))}{\pi_0(X)} \frac{1}{2} - \mathbb{1}\{Y \leq m_0(X)\} + m_{\hat{d}^*}(X) - m_{d^*}(X) \right\} = \mathbb{P}\left\{ m_1(d^* - d^*) + m_0(d^* - d^*) \right\} = \mathbb{P}\gamma(d^* - d^*).
\]

The second equality above comes from the fact that \( E[|Y|] = 1\{Y \leq m_a(X)\} = 1/2 \). Further, by Lemma 4 and the fact that \( d^* = 1\{\gamma > 0\} \), we obtain that

\[
\mathbb{P}\gamma(d^* - d^*) \leq \mathbb{P}\gamma\mathbb{1}\{|\gamma| \leq |\hat{\gamma} - \gamma|\} \leq \mathbb{P}|\hat{\gamma} - \gamma|\mathbb{1}|\gamma| \leq |\hat{\gamma} - \gamma| \leq \|\hat{\gamma} - \gamma\|_\infty \lesssim |\hat{\gamma} - \gamma|^{1+\alpha},
\]

where the last inequality follows from the margin condition. On the other hand, similar to the proof of Theorem 5, we use the decomposition

\[
\mathbb{P}_n\xi_{\hat{d}^*} - \mathbb{P}\xi_{\hat{d}^*} = (\mathbb{P}_n - \mathbb{P})\left(\xi_{\hat{d}^*} - \xi_{d^*}\right) + (\mathbb{P}_n - \mathbb{P})\xi_{d^*} + \mathbb{P}\xi_{\hat{d}^*} - \xi_{\hat{d}^*}.
\]

By the fact that \( \mathbb{P}\phi_{\hat{d}^*} = \psi_{\hat{d}^*} + \mathbb{P}\xi_{\hat{d}^*} = 0 \), we have that

\[
\mathbb{P}\left(\xi_{\hat{d}^*} - \xi_{d^*}\right) = \mathbb{P}\left\{ \frac{\pi_1d^*}{\pi_1} F_{1,m} - F_{1,m} \hat{\pi}_1 + \frac{\pi_0(1 - d^*)}{\pi_0} F_{0,m} - F_{0,m} \hat{\pi}_0 + \hat{m}_{\hat{d}^*} - m_{\hat{d}^*} \right\} = \mathbb{P}\left\{ d^* \left( \hat{m}_1 - m_1 - \frac{\pi_1}{\pi_1} F_{1,m} - F_{1,m} \hat{\pi}_1 \right) + (1 - d^*) \left( \hat{m}_0 - m_0 - \frac{\pi_0}{\pi_0} F_{0,m} - F_{0,m} \hat{\pi}_0 \right) \right\} = R_{\hat{d}^*}(\mathbb{P},\mathbb{P}) = O_P \left( \sum_{a=0}^1 \|\hat{m}_a - m_a\| \left( \|\hat{\pi}_a f_{a,m} - \pi_a f_{a,m}\| + \|\hat{m}_a - m_a\| \right) \right).
\]

Finally, for the term \((\mathbb{P}_n - \mathbb{P})\left(\xi_{\hat{d}^*} - \xi_{d^*}\right)\): when \( \xi_{\hat{d}^*} \) is estimated from a separate sample \( Z^{n,0} \) from the empirical measure over \( Z^n \), by Lemma 5 we have that

\[
(\mathbb{P}_n - \mathbb{P})\left(\xi_{\hat{d}^*} - \xi_{d^*}\right) = O_P \left( \frac{\|\xi_{\hat{d}^*} - \xi_{d^*}\|}{\sqrt{n}} \right) = o_P(1/\sqrt{n}),
\]

where the last equality follows from our assumption that \( \|\xi_{\hat{d}^*} - \xi_{d^*}\| = o_P(1) \). When \( \xi_{d^*} \) and \( \xi_{\hat{d}^*} \) are contained in a Donsker class and \( \|\xi_{\hat{d}^*} - \xi_{d^*}\| = o_P(1) \), by van der Vaart [2000, Lemma 19.24], we have that \((\mathbb{P}_n - \mathbb{P})\left(\xi_{\hat{d}^*} - \xi_{d^*}\right) = o_P(1/\sqrt{n}) \).

\[\square\]

**Proof of Corollary 8.** From Theorem 7, we have that

\[
\hat{\psi}_{\hat{d}^*,d^*} - \psi_{d^*} = (\mathbb{P}_n - \mathbb{P})\xi_{d^*} + O_P\left(1/\sqrt{n}\right) + \]

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\[
\|\hat{\gamma} - \gamma\|_\infty^{1+\alpha} + \sum_{a=0}^1 \|\bar{m}_a - m_a\| \left( \|\bar{\pi}_a \hat{f}_{a, \bar{m}} - \pi_a f_{a,m}\| + \|\bar{m}_a - m_a\| \right)\].

The conditions in Corollary 8 ensure that \( R = o_{\overline{P}}(1/\sqrt{n}) \), which gives that
\[
\sqrt{n}(\psi_{d^*, dx} - \psi_{d^*}) \rightsquigarrow N\left(0, \sigma_{d^*}^2\right),
\]
since \( \text{Var}\{\xi_{d^*}\} = \text{Var}\{\phi_{d^*}\} = \sigma_{d^*}^2 \).

### A.4 Proofs in Section 7

**Proof of Theorem 9.** We begin with a decomposition:

\[
\begin{align*}
\hat{\gamma}_{dr}(x) - \gamma(x) & = \sum_{i=1}^n w_i(x; X^n)\hat{g}(Z_i) - \gamma(x) \\
& = \left( \sum_{i=1}^n w_i(x; X^n)g(Z_i) - \gamma(x) \right) + \sum_{i=1}^n w_i(x; X^n)h(X_i) + \sum_{i=1}^n w_i(x; X^n) (\hat{g}(Z_i) - g(Z_i) - h(X_i)) \\
& = (\hat{\gamma}(x) - \gamma(x)) + \sum_{i=1}^n w_i(x; X^n)h(X_i) + \sum_{i=1}^n w_i(x; X^n) (\hat{g}(Z_i) - g(Z_i) - h(X_i)),
\end{align*}
\]

where \( h(x) = \mathbb{E}[\hat{g}(Z) - g(Z)|D^n, X = x] \) and \( D^n = (D_1, D_2) \) is the training data for obtaining \( \hat{g} \).

To bound \( h(x) \), we obtain that
\[
\begin{align*}
\hat{\gamma}(x) - \gamma(x) & = \frac{\hat{m}_1(x) - m_1(x)}{\hat{\pi}_1(x)} - \frac{\pi_1(x) F_1, \hat{m}(x) - F_1, m(x)}{\hat{f}_1, \hat{m}(x)} \\
& = \left( \hat{m}_1(x) - m_1(x) \right) \left( 1 - \frac{\pi_1(x) F_1, m(x)}{\hat{\pi}_1(x) \hat{f}_1, \hat{m}(x)} \right) - \frac{f_1, e(x) \pi_1(x) (\hat{m}_1(x) - m_1(x))^2}{2\hat{\pi}_1(x) \hat{f}_1, \hat{m}(x)} \\
& \quad - \left( \hat{m}_0(x) - m_0(x) \right) \left( 1 - \frac{\pi_0(x) f_0, m(x)}{\hat{\pi}_0(x) \hat{f}_0, \hat{m}(x)} \right) - \frac{f_0, e(x) \pi_0(x) (\hat{m}_0(x) - m_0(x))^2}{2\hat{\pi}_0(x) \hat{f}_0, \hat{m}(x)} \\
\lesssim & \sum_{a=0}^1 |\hat{m}_a(x) - m_a(x)| \left( |\bar{\pi}_a(x) \bar{f}_{a, \bar{m}}(x) - \pi_a(x) f_{a,m}(x)| + |\bar{m}_a(x) - m_a(x)| \right),
\end{align*}
\]

where \( f_{a,c}(x) \leq L \) is the derivative of \( f_a(y | x) \) at \( y = c_a(x) \) for some value \( c_a(x) \) between \( m_a(x) \) and \( \bar{m}_a(x) \) and the third equality follows similarly to the proof of Corollary 1. This gives that
\[
\begin{align*}
& \sum_{i=1}^n w_i(x; X^n)h(X_i) \\
\lesssim & \sum_{i=1}^n \left( |\hat{m}_a(X_i) - m_a(X_i)| \left( |\bar{\pi}_a(X_i) \bar{f}_{a, \bar{m}}(X_i) - \pi_a(X_i) f_{a,m}(X_i)| + |\bar{m}_a(X_i) - m_a(X_i)| \right) \right) \\
= & \sum_{a=0}^1 \sum_{i=1}^n |w_i(x; X^n)||\hat{m}_a(X_i) - m_a(X_i)| \left( |\pi_a(X_i) f_{a,m}(X_i) - \bar{\pi}_a(X_i) \bar{f}_{a, \bar{m}}(X_i)| + |\bar{m}_a(X_i) - m_a(X_i)| \right).
\end{align*}
\]
\[
\begin{align*}
&= \frac{1}{n} \sum_{a=0}^{n} \left| w_i(x; X^n) |\hat{m}_a(X_i) - m_a(X_i)| \cdot |\pi_a(X_i) f_{a,m}(X_i) - \hat{\pi}_a(X_i) \hat{f}_{a,m}(X_i)| \right. \\
&\quad + \left. \sum_{a=0}^{1} \sum_{i=1}^{n} \left| w_i(x; X^n) \right| \cdot |\hat{m}_a(X_i) - m_a(X_i)| \cdot |\hat{m}_a(X_i) - m_a(X_i)| \right. \\
&\quad \leq \frac{1}{n} \sum_{a=0}^{n} \left| w_i(x; X^n) |\hat{m}_a(X_i) - m_a(X_i)|^{2} \cdot \sqrt{\sum_{i=1}^{n} \left| w_i(x; X^n) \right| |\pi_a(X_i) f_{a,m}(X_i) - \hat{\pi}_a(X_i) \hat{f}_{a,m}(X_i)|^{2} \right. \\
&\quad \left. + \sum_{a=0}^{1} \sum_{i=1}^{n} \left| w_i(x; X^n) \right| \cdot |\hat{m}_a(X_i) - m_a(X_i)|^{2} \cdot \sqrt{\sum_{i=1}^{n} \left| w_i(x; X^n) \right| |\hat{m}_a(X_i) - m_a(X_i)|^{2} \right. \\
&\quad = \left( \sum_{i=1}^{n} | w_i(x; X^n) | \right) \cdot \left( \sum_{a=0}^{1} \left| \hat{m}_a - m_a \right|_w \left( \left| \hat{\pi}_a \hat{f}_{a,m} - \pi_a f_{a,m} \right|_w + \left| \hat{m}_a - m_a \right|_w \right) \right). 
\end{align*}
\]

To bound \( G(x) := \sum_{i=1}^{n} w_i(x; X^n) (\hat{g}(Z_i) - g(Z_i) - h(X_i)) \), we observe that by definition,

\[ \mathbb{E}[\hat{g}(Z_i) - g(Z_i) - h(X_i)|D^n, X^n] = 0, \]

where \( X^n \) are the covariates in \( D_3 \). We also have

\[
\text{Var}\{G(x)|D^n, X^n\} = \text{Var}\left\{ \sum_{i=1}^{n} w_i(x; X^n) (\hat{g}(Z_i) - g(Z_i) - h(X_i)) |D^n, X^n \right\} \\
= \sum_{i=1}^{n} w_i(x; X^n)^2 \text{Var}\{\hat{g}(Z_i) - g(Z_i) |D^n, X^n\} \\
= \sum_{i=1}^{n} w_i(x; X^n)^2 \text{Var}\{\hat{g}(Z_i) - g(Z_i) |D^n, X^n\} \\
\leq \left\| \hat{g} - g \right\|_w^2 \sum_{i=1}^{n} w_i(x; X^n)^2, 
\]

where the third equality holds since \( \text{Var}\{\hat{g}(Z_i) - g(Z_i) |D^n, X^n\} = \mathbb{E}[\left(\hat{g}(Z_i) - g(Z_i)\right)^2|D^n, X^n] - \mathbb{E}[\hat{g}(Z_i) - g(Z_i)|D^n, X^n]^2 \leq \mathbb{E}[\left(\hat{g}(Z_i) - g(Z_i)\right)^2|D^n, X^n] \). We note that

\[
\mathbb{E}[\left(\hat{\gamma}(x) - \gamma(x)\right)^2] = \mathbb{E}\left[\left( \sum_{i=1}^{n} w_i(x; X^n) (g(Z_i) - \gamma(X_i)) + \sum_{i=1}^{n} w_i(x; X^n) \gamma(X_i) - \gamma(x) \right)^2 \right] \\
= \mathbb{E}\left[\left( \sum_{i=1}^{n} w_i(x; X^n) (g(Z_i) - \gamma(X_i)) \right)^2 \right] + \mathbb{E}\left[\left( \sum_{i=1}^{n} w_i(x; X^n) \gamma(X_i) - \gamma(x) \right)^2 \right] \\
= \mathbb{E}\left[ \sum_{i=1}^{n} w_i(x; X^n)^2 \text{Var}\{g(Z_i)|X_i\} \right] + \mathbb{E}\left[ \left( \sum_{i=1}^{n} w_i(x; X^n) \gamma(X_i) - \gamma(x) \right)^2 \right] \\
\geq \sigma_{\min}^2 \mathbb{E}\left[ \sum_{i=1}^{n} w_i(x; X^n)^2 \right],
\]

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where the second and third equality hold since $\mathbb{E}[g(Z_i)|X^n] = \gamma(X_i)$ and all samples are independent. Putting it altogether, using Markov’s inequality, we obtain

$$
\mathbb{P} \left( \frac{|G(x)|}{\|\hat{g} - g\|_{w^2} \sqrt{\mathbb{E}[(\gamma(x) - \gamma(x))^2]}} \geq t \right) = \mathbb{E} \left[ \mathbb{P} \left( \frac{|G(x)|}{\|\hat{g} - g\|_{w^2} \sqrt{\mathbb{E}[(\gamma(x) - \gamma(x))^2]}} \geq t \bigg| D^n, X^n \right) \right]
$$

$$
\leq \frac{1}{t^2} \cdot \mathbb{E} \left[ \text{Var} \{G(x)|D^n, X^n\} \right] \leq \frac{1}{t^2} \cdot \frac{1}{\sigma_{\text{min}}^2}.
$$

Thus, we have that $G(x) = O_P(\|\hat{g} - g\|_{w^2} \sqrt{\mathbb{E}[(\gamma(x) - \gamma(x))^2]})$ and $G(x)^2 = O_P(\|\hat{g} - g\|_{w^2}^2 \mathbb{E}[(\gamma(x) - \gamma(x))^2])$.

The proof completes by realizing that $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$. \hfill \qed

**B Auxiliary Lemmas**

Below are the auxiliary lemmas we have used to prove the results in this paper. For completeness, we include their proofs here as well.

**Lemma 4.** [Kennedy et al., 2020, Lemma 1] Let $\hat{f}, f$ be functions taking any real values. Then

$$
|\mathbb{1}\{\hat{f} > 0\} - \mathbb{1}\{f > 0\}| \leq \mathbb{1}\{|f| \leq |\hat{f} - f|\}.
$$

**Proof.** It follows that

$$
|\mathbb{1}\{\hat{f} > 0\} - \mathbb{1}\{f > 0\}| = \mathbb{1}\{f, \hat{f} \text{ have opposite signs}\} \leq \mathbb{1}\{|f| \leq |\hat{f} - f|\},
$$

since $|\hat{f} - f| = |\hat{f}| + |f|$ when $f, \hat{f}$ have opposite signs. \hfill \qed

**Lemma 5.** [Kennedy et al., 2020, Lemma 2] Let $P_n$ denote the empirical measure over $Z^n = (Z_1, \ldots, Z_n)$ and $\hat{\phi}$ be a function estimated from a sample $Z^{n,0} = (Z_1^0, \ldots, Z_n^0)$, which is independent of $Z^n$, then for any function $\phi$,

$$(P_n - P)(\hat{\phi} - \phi) = O_P \left( \frac{\|\hat{\phi} - \phi\|}{\sqrt{n}} \right).$$

**Proof.** First, we notice that

$$
\text{Var} \left\{ (P_n - P)(\hat{\phi} - \phi) \bigg| Z^{n,0} \right\} = \text{Var} \left\{ P_n(\hat{\phi} - \phi) \bigg| Z^{n,0} \right\} = \frac{1}{n} \text{Var} \left\{ \hat{\phi} - \phi \bigg| Z^{n,0} \right\} \leq \frac{\|\hat{\phi} - \phi\|_2^2}{n},
$$

where the first equality is true since conditioned on $Z^{n,0}$, $P(\hat{\phi} - \phi)$ is a constant. Then by applying the law of total expectation and Chebyshev’s inequality, we obtain that

$$
\mathbb{P} \left( \frac{|(P_n - P)(\hat{\phi} - \phi)|}{\|\hat{\phi} - \phi\|_2/\sqrt{n}} \geq t \right) = \mathbb{E} \left[ \mathbb{P} \left( \frac{|(P_n - P)(\hat{\phi} - \phi)|}{\|\hat{\phi} - \phi\|_2/\sqrt{n}} \geq t \bigg| Z^{n,0} \right) \right]
$$

$$
\leq \mathbb{E} \left\{ \text{Var} \left\{ (P_n - P)(\hat{\phi} - \phi) \bigg| Z^{n,0} \right\} \right\} \leq \frac{1}{t^2},
$$

where we have utilized the fact that $\mathbb{E} \left\{ (P_n - P)(\hat{\phi} - \phi) \bigg| Z^{n,0} \right\} = \mathbb{E} \{(\bar{\phi} - \phi)Z^{n,0}\} - P(\hat{\phi} - \phi) = 0$. \hfill \qed