On the computation of asymptotic critical values of polynomial maps and applications

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Abstract

Let \( f = (f_1, \ldots, f_p) \) be a polynomial tuple in \( \mathbb{Q}[z_1, \ldots, z_n] \) and let \( d = \max_{1 \leq i \leq p} \deg f_i \). We consider the problem of computing the set of asymptotic critical values of the polynomial mapping, with the assumption that this mapping is dominant, \( f : z \in \mathbb{K}^n \to (f_1(z), \ldots, f_p(z)) \in \mathbb{K}^p \) where \( \mathbb{K} \) is either \( \mathbb{R} \) or \( \mathbb{C} \). This is the set of values \( c \) in the target space of \( f \) such that there exists a sequence of points \( (x_i)_{i \in \mathbb{N}} \) for which \( f(x_i) \) tends to \( c \) and \( \|x_i\|_\kappa(df(x_i)) \) tends to 0 when \( i \) tends to infinity where \( df \) is the differential of \( f \) and \( \kappa \) is a function measuring the distance of a linear operator to the set of singular linear operators from \( \mathbb{K}^n \) to \( \mathbb{K}^p \).

Computing the union of the classical and asymptotic critical values allows one to put into practice generalisations of Ehresmann’s fibration theorem. This leads to natural and efficient applications in polynomial optimisation and computational real algebraic geometry.

Going back to previous works by Kurdyka, Orro and Simon, we design new algorithms to compute asymptotic critical values. Through randomisation, we introduce new geometric characterisations of asymptotic critical values. This allows us to dramatically reduce the complexity of computing such values to a cost that is essentially \( O(d^{2n(p+1)}) \) arithmetic operations in \( \mathbb{Q} \). We also obtain tighter degree bounds on a hypersurface containing the asymptotic critical values, showing that the degree is at most \( p^{n-p+1}(d-1)^n-p(d+1)^p \).

Next, we show how to apply these algorithms to unconstrained polynomial optimisation problems and the problem of computing sample points per connected component of a semi-algebraic set defined by a single inequality/inequation.

We report on the practical capabilities of our implementation of this algorithm. It shows how the practical efficiency surpasses the current state-of-the-art algorithms for computing asymptotic critical values by tackling examples that were previously out of reach.

Keywords: Asymptotic critical values, Polynomial optimisation, Gröbner bases
1. Introduction

Basic definitions and problem statement. Let \( z_1, \ldots, z_n \). Let \( f = (f_1, \ldots, f_p) \) be a polynomial tuple in \( \mathbb{K}[z] \) where \( \mathbb{K} \) is either \( \mathbb{R} \) or \( \mathbb{C} \) and let \( d = \max_{1 \leq i \leq p} \deg f_i \).

By a slight abuse of notation, we will also denote by \( f \), the polynomial mapping

\[
\mathbf{x} = (x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_p(x))
\]

and we assume this mapping is dominant. Furthermore, denote by \( d \) be a polynomial tuple in \( \mathbb{K}^{n \times p} \).

Then, the set of \( \nu \) introduced by Rabier in [1], given in [21, Definition 2.1]. Denote by \( L(\mathbb{K}^n, \mathbb{K}^p) \) the

Example 1. Let \( f = z_1^4 + (z_1z_2 - 1)^2 \). Since \( p = 1 \), \( \ker \text{Jac}(f^{[1]}) = \mathbb{K}^2 \). Thus, \( w_1(z) \) is simply the gradient of \( f \):

\[
w_1(z) = (4z_1^3 + 2z_2(z_1z_2 - 1), 2z_1(z_1z_2 - 1)).
\]

Let \( g = (z_1z_2, z_1z_3) \). Then, the Jacobian matrix associated to \( g \) is:

\[
\text{Jac}(g) = \begin{bmatrix}
z_2 & z_1 & 0 \\
z_3 & 0 & z_1
\end{bmatrix}.
\]

Therefore, \( \text{Jac}(g^{[1]}) = [z_3, 0, z_1] \) and so we restrict the linear mapping

\[
dg_1(z) : \gamma = (\gamma_1, \gamma_2, \gamma_3) \mapsto \gamma_1z_2 + \gamma_2z_1
\]

to the space \( \{ \gamma \in \mathbb{K}^3 \mid \gamma_1z_3 + \gamma_3z_1 = 0 \} \), the result being \( w_1(z) \). The construction of \( w_2(z) \) follows similarly.

Following [21, Definition 2.2], for \( z \in \mathbb{K}^n \), we consider the so-called Kuo distance

\[
\kappa(\text{df}(z)) = \min_{1 \leq j \leq p} \|w_j(z)\|.
\]

Then, the set of asymptotic critical values of the mapping \( f \) is defined to be the set:

\[
K_\infty(f) = \{ c \in \mathbb{C}^p \mid \exists (x_i)_{i \in \mathbb{N}} \subset \mathbb{C}^n \text{ s.t. } \|x_i\| \to \infty, f(x_i) \to c \text{ and } \|x_i\| \kappa(\text{df}(x_i)) \to 0 \}.
\]

By [21, Theorem 4.1], \( K_\infty(f) \) is an algebraic set of \( \mathbb{C}^p \). We also define the function \( \nu \) introduced by Rabier in [24], given in [21, Definition 2.1].
space of linear mappings from $\mathbb{K}^n$ to $\mathbb{K}^p$ and by $\Sigma$ the singular set of $L(\mathbb{K}^n, \mathbb{K}^p)$. The distance function to $\Sigma$ is given by

$$\text{dist}(A, \Sigma) = \inf_{B \in \Sigma} \|A - B\|.$$  

For $A \in L(\mathbb{K}^n, \mathbb{K}^p)$, $A^*$ denotes the adjoint operator of $A$. For example, in the setting of complex numbers, the adjoint operator is the complex conjugate transpose. Then,

$$\nu(A) = \inf_{\|\phi\|=1} \|A^*\phi\|.$$  

We give the following two properties of $\nu$ given in [21] that will be used in the proof of correctness of our algorithms. By [21, Proposition 2.2],

$$\nu(A) = \text{dist}(A, \Sigma).$$  

Furthermore, by [21, Corollary 2.1], the functions $\nu$ and $\kappa$ are equivalent in the following sense: For $A \in L(\mathbb{K}^n, \mathbb{K}^p)$,

$$\nu(A) \leq \kappa(A) \leq \sqrt{\nu(A)}.$$  

Therefore, as in the definition given by Rabier in [24], we can also define the asymptotic critical values in terms of the function $\nu$.

$$K_\infty(f) = \{c \in \mathbb{C}^p \mid \exists (x_t)_{t \in \mathbb{N}} \subset \mathbb{C}^n \text{ s.t. } \|x_t\| \to \infty, f(x_t) \to c \text{ and } \|x_t\|\nu(df(x_t)) \to 0\}. $$  

However, we will predominantly use the prior definition of the set of asymptotic critical values.

**Example 2.** Consider the polynomial $f = z_1^4 + (z_1z_2 - 1)^2$. This polynomial has a critical point at $(0, 0)$ that corresponds to the critical value 1. However, it is easy to see that $f$ takes values less than 1. Take the path parameterised by $t$: $z_1(t) = \frac{1}{t^2}, z_2(t) = t$. Then, as $t \to \infty$ we see $f(t) \to 0$. From Example 1, we know that:

$$w_1(z) = (4z_1^3 + 2z_2(z_1z_2 - 1), 2z_1(z_1z_2 - 1)).$$  

Then, along the path parameterised by $t$:

$$(z_1(t)^2 + z_2(t)^2)w_1(z(t)) = \left(\frac{4}{t^2} + \frac{4}{t}, 0\right) \to (0, 0) \text{ as } t \to \infty.$$  

Therefore, 0 is an asymptotic critical value of $f$.

The goal of this paper is to provide efficient algorithms which, on input $f$, compute finitely many polynomials whose simultaneous vanishing set contains $K_\infty(f)$.

**Motivations and prior works.** Denote by $K_0(f)$, the set of critical values of $f$

$$K_0(f) = \{c \in \mathbb{C}^p \mid \exists x \in \mathbb{C}^n \text{ s.t. } f(x) = c \text{ and } \text{rank}(df(x)) < p\}.$$  

The set of generalised critical values is defined as the union $K_0(f) \cup K_\infty(f)$; it will be denoted by $K(f)$. One strong property of $K(f)$ is the following. The mapping $f$ restricted to $\mathbb{K}^n \setminus f^{-1}(K(f))$ is a locally trivial fibration. Thus, for all connected open sets $U \subset \mathbb{K}^p \setminus K(f)$, for all $y \in U$ there exists a diffeomorphism $\varphi$ such that, with $\pi$ as the canonical projection map, the following diagram commutes [21, Theorem 3.1].
In other words, this definition of generalised critical values allows one to generalise Ehresmann’s fibration theorem to non-proper settings. This goes back to the problem of defining sets that contain the so-called bifurcation set of $f$ as introduced in \cite{24}. From that perspective, one crucial feature of $K_{\infty}(f)$, is that there is a generalised Sard’s theorem for this set, i.e. the codimension of $K_{\infty}(f)$ is greater than or equal to one \cite[Theorem 3.1]{21}.

These properties make the effective use of generalised critical values quite appealing in computational real algebraic geometry. In particular, in \cite{14, 27}, algorithms for

- computing an exact representation of the minimum of a given polynomial (i.e. the minimal polynomial of this minimum and an isolating interval),
- computing sample points for each connected component of a semi-algebraic set defined by a single inequality,

have been designed, relying on the computation of generalised critical values when $p = 1$. In this context, $f$ is assumed to lie in $\mathbb{Q}[z]$ and, in the works we refer to below, the chosen complexity model is the arithmetic one, i.e. one counts arithmetic operations in the base field $\mathbb{Q}$ without taking into account the growth of the bit sizes of the coefficients. We use the classical big-O notation $O(\phi(x))$ (where $\phi$ is a real valued function that is strictly positive for all large enough values of $x$) to denote the class of non-negative functions which up to a multiplicative constant, are bounded from above by $\phi(x)$ at infinity.

As far as we know, the first work that leads directly to an algorithm for computing $K_{\infty}(f)$ is given in \cite{21}. It is based on a geometric characterisation of $K_{\infty}(f)$ that allows one to apply algebraic elimination algorithms (e.g. algorithms for computing elimination ideals in polynomial rings) to get an algebraic description for $K_{\infty}(f)$. The tool of choice for performing algebraic elimination in \cite{17, 21} is Gröbner bases. In Section 2, we recall the geometry involved in this algorithm. At this stage, let us say that it builds equations defining locally closed sets in $\mathbb{C}^{n+p(n+2)}$.

It then considers the intersection of these sets with some linear subspaces chosen in such a way that the union of the projections of those intersections on the target space of $f$ contains $K_{\infty}(f)$.

Several attempts to improve this algorithmic pattern have been made. When $p = 1$, we mention \cite{27} which makes the connection between generalised critical values and properties of polar varieties. This yields a probabilistic algorithm whose runtime is in the order of $q^{O(n)}$ arithmetic operations in $\mathbb{Q}$. However, as noticed in \cite{19}, the argumentation in \cite{27} is incomplete. Still, this connection is exploited through the use of finite dimensional spaces of rational arcs to compute generalised critical values in \cite{18, 19}. Nevertheless, the complexity estimates in \cite{27} do not apply to \cite{18, 19}.

It should also be noted that the result in \cite{21} was generalised in \cite{17}. This allowed the design of an algorithm to compute the generalised critical values of a polynomial mapping restricted to an algebraic set, a setting not covered in this paper.
Main results. We build upon the geometric characterisations of $K_\infty(f)$ [21] to obtain new ones that allow us to design more efficient algorithms.

We introduce an element of randomisation to avoid some combinatorial steps in the algorithm designed in [17]. Next, we introduce another element of randomisation that reduces the computation of $K_\infty(f)$ to intersecting the Zariski closure of some locally closed subset of $\mathbb{C}^{n+p+1}$ with a linear affine subspace of codimension 2 such that the projection onto the target space of $f$ of this intersection contains $K_\infty(f)$.

The sets involved in the intersection process described above to geometrically characterise $K_\infty(f)$ can be related to some incidence varieties. We use this relation to design another algorithm that takes advantage of the determinantal setting. This allows us to obtain a faster algorithm in practice, though the theoretical complexity of both algorithms is essentially the same. We can now state our first main result.

**Theorem 3.** Let $f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p$ be a dominant polynomial mapping and let $d = \max_{1 \leq i \leq p} \deg f_i$. Then, the asymptotic critical values of $f$ are contained in a hypersurface of degree at most $p^{n-p+1}(d-1)^{n-p}(d+1)^p$.

We study the complexity of resulting algorithms by substituting Gröbner bases computations with the use of the geometric resolution algorithm designed in [13]. We first introduce some notation.

Given a polynomial sequence $g$ in $\mathbb{F}[z_1, \ldots, z_m]$, where $\mathbb{F}$ is a field and $\mathbb{F}$ is an algebraic closure of $\mathbb{F}$, we denote by $V(g) \subset \mathbb{F}^m$ the algebraic set defined by the simultaneous vanishing of the entries of $g$. We also recall the “soft-Oh” notation: $f(n) \in \tilde{O}(g(n))$ means that $f(n) \leq g(n) \log^O(1)(3 + g(n))$, see also [12, Chapter 25, Section 7]. Additionally, denote by $c$ the indeterminates $c_1, \ldots, c_p$.

**Theorem 4.** Let $f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p$ be a dominant polynomial mapping and let $d = \max_{1 \leq i \leq p} \deg f_i$.

There exists an algorithm which, on input $f$, computes a non-zero polynomial $g$ in $\mathbb{K}[c]$ such that $K_\infty(f) \subset V(g)$ using at most

$$O^*(p(p(d-1))^{2(p+1)(n-p)}(d+1)^{2p(p+1)})$$

arithmetic operations in $\mathbb{K}$.

Following the algorithmic scheme of [27], we show how to apply these new algorithms for computing asymptotic critical values, to the problem of computing sample points per connected component of a semi-algebraic set defined by a single inequality. Furthermore, we show how to use the generalised critical values to tackle the problem of computing an exact representation of the global infimum of a given polynomial. Note that this algorithm can decide if this infimum exists in $\mathbb{R}$ or if the given polynomial is unbounded from below.

We implemented all the aforementioned algorithms for computing asymptotic critical values in the MAPLE computer algebra system where we substitute the geometric resolution algorithm for Gröbner bases. For Gröbner bases computations, we rely on a combination of MSolve [5] and the FGrB library [10]. We used an extensive set of benchmark examples to illustrate the computational capabilities of these algorithms.

It appears that our new algorithms outperform the state-of-the-art and can tackle examples which were previously out of reach.
Structure of the paper. In Section 2, we revisit the geometric characterisation of asymptotic critical values given in [21] and describe our first instance of randomisation. One can derive from this result, an algorithm that is similar to the deterministic one derived from [21], save for a combinatorial factor. Next, in Section 3, we go further and design two new algorithms, more efficient than the state-of-the-art, on which Theorem 3 and Theorem 4 rely. These theorems are then proved in Section 4. Section 5 returns to the applications given in [14, 27] in the context of the newly designed algorithms. We conclude with Section 6, which gives the results of our experiments with our algorithms for several families of polynomials, as well as for polynomials found in practice.

2. Preliminaries

Let \( f \in \mathbb{K}[x]^p \) be a dominant polynomial mapping. Firstly, let us recall that by [21, Theorem 3.1], \( K_\infty(f) \) has codimension at least 1 in \( \mathbb{C}^p \). Then, using the geometric description of \( K_\infty(f) \) provided in the proof of [21, Theorem 4.1], one can compute it. For the sake of completeness, we follow the proof given in [21, Theorem 4.1] with slight modifications to [21, Lemma 4.1] that are useful for the algorithm we describe in the next section.

**Notation 5.** Firstly, we introduce a few objects. We shall make use of the following change of coordinates to handle the asymptotic behaviour, sending \( z_s = 0 \) to \( \infty \):

\[
\tau_s(z) = \left( \frac{z_1}{z_s}, \ldots, \frac{z_{s-1}}{z_s}, \frac{1}{z_s}, \frac{z_{s+1}}{z_s}, \ldots, \frac{z_n}{z_s} \right).
\]

For each choice of \( s = 1, \ldots, n, j = 1, \ldots, p \) and point \( x \in \mathbb{K}^n \), let \( W_j^s(x) \) be the graph of \( x, w_j(x) \), a point in the Grassmannian of linear subspaces of \( \mathbb{C}^n \times \mathbb{C} \) that are of dimension \( n - p + 1 \), denoted by \( G_{n-p+1}(\mathbb{C}^n \times \mathbb{C}) \). Then, we consider the rational mappings

\[
M_j^s(f) : \mathbb{K}^n \setminus \{ z_s = 0 \} \to \mathbb{C}^p \times G_{n-p+1}(\mathbb{C}^n \times \mathbb{C}),
\]

\[
z \mapsto (f(\tau_s(z)), W_j^s(\tau_s(z))).
\]

The point \( W_j^s(z) \) is well-defined for \( z \) such that the kernel of \( \text{jac}(f^{[j]}) \) has dimension \( n - p + 1 \). Since we assume \( f \) is dominant, \( M_j^s(f) \) is well-defined outside of a proper Zariski closed subset of \( \mathbb{K}^n \).

Let \( \Lambda = G_{n-p+1}(\mathbb{C}^n \times 0) \). This is the set of \((n-p+1)\)-dimensional graphs of linear maps from \( \mathbb{C}^n \) to \( \mathbb{C} \) that are identically the zero map. We then intersect the Zariski closure of the graph of \( M_j^s(f) \) in the following way:

\[
L_j^s(f) = \text{graph} M_j^s(f) \cap \{ \{ z \in \mathbb{C}^n | z_s = 0 \} \times \mathbb{C}^p \times \Lambda \}.
\]

Define \( \pi : \mathbb{C}^n \times \mathbb{C}^p \times G_{n-k+1}(\mathbb{C}^n \times \mathbb{C}) \to \mathbb{C}^p \) to be the projection map and take

\[
K_j^s(f) = \pi(L_j^s(f)).
\]

By [21, Lemma 4.1],

\[
K_\infty(f) = \bigcup_{(s,j)=(1,1)}^{(n,p)} K_j^s(f).
\]
Example 6. Let \( f = z_1^4 + (z_1z_2 - 1)^2 \). As in Example 1, we have

\[
  w_1(z) = (4z_1^3 + 2z_2(z_1z_2 - 1), 2z_1(z_1z_2 - 1)).
\]

Then,

\[
  \tau_1(z) = \left( \frac{1}{z_1}, \frac{z_2}{z_1} \right), \quad \tau_2(z) = \left( \frac{z_1}{z_2}, \frac{1}{z_2} \right).
\]

We now illustrate the construction of \( K_1^1(f) \), taking care to exclude the variety \( V(z_1) \) where the mapping is not defined.

Firstly, we derive equations for \( M_1^1(f) \) by evaluating \( f \) and the linear map \( w_1 \) at \( \tau_1(z) \),

\[
  M_1^1(f) = \left( \frac{1}{z_1^2} + \left( \frac{z_2}{z_1^2} - 1 \right)^2, \frac{4}{z_1^2} + \frac{2z_2}{z_1^3}, \frac{2z_2}{z_1^3}, \frac{2z_2}{z_1^3} - \frac{2}{z_1^2} \right).
\]

Then, we consider the graph of these functions and so we introduce the variables \( c, u_1, \) and \( u_2 \). To describe this set algebraically, we consider the numerators of these rational functions with their respective value variables and remove the variety \( V(z_1) \) where \( M_1^1(f) \) is not defined. This leads to the following description of graph \( M_1^1(f) \)

\[
  \overline{V(cz_1^4 - z_1^2 + 2z_1^2z_2 - z_2^2 - 1, u_1z_1^2 + 2z_1^2z_2 - 2z_2^2 - 4, u_2z_1^2 + 2z_1^2 - 2z_2)} \setminus V(z_1).
\]

One can compute a finite list of polynomials whose simultaneous vanishing set is the above variety by a range of methods including Gröbner bases. Such methods are discussed in Subsection 3.4. Then, by setting \( z_1 = u_1 = u_2 = 0 \), one retrieves an algebraic description of the set \( L_1^1(f) \). In this case, we do not find any asymptotic critical values for \( s = 1 \) as the set \( L_1^1(f) \), and therefore, the set \( K_1^1(f) \), is empty.

Nonetheless, we consider \( K_2^1(f) \). As before, we introduce the variables \( c, u_1, \) and \( u_2 \) and consider the Zariski closure graph \( M_2^1(f) \) expressed by:

\[
  \overline{V(z_2^2 + z_1^2 - 2z_1z_2 + z_2^2 - 2z_1^2 - 2z_2^2 + 2z_1 - u_1z_1^2, 2z_1^2 - 2z_1z_2^2 - u_2z_2^2)} \setminus V(z_2).
\]

We perform a Gröbner basis computation and intersect with the variety \( V(z_2, u_1, u_2) \) to get the algebraic description of \( L_2^1(f) \):

\[
  L_2^1(f) = V(z_1, z_2, u_1, u_2, c).
\]

We arrive at the Zariski closure of \( K_2^1(f) \) by excluding all polynomials that involve any variables other than \( c \). In this case, we see that \( K_2^1(f) = V(c) \) and therefore 0 is the only possible asymptotic critical value of \( f \). In combination with Example 2, we conclude that \( K_∞(f) = \{0\} \).

Let \( g = (z_1z_2, z_1z_3) \). We investigate the case \( s = 1, j = 1 \), the other cases follow similarly. From Example 1, we have that

\[
  \text{Jac}(g) = \begin{bmatrix}
    z_2 & z_1 & 0 \\
    z_3 & 0 & z_1
  \end{bmatrix}.
\]

Therefore, \( \text{Jac}(g)[1] = [z_3, 0, z_1] \). We consider the points \( x \in \mathbb{K}^3 \) such that the matrix \( \text{Jac}(g)[1](x) \) has maximal rank. Thus, excluding evaluations at points in the variety \( V(z_1z_3) \), the kernel of the linear mapping has dimension 2.
It is easy to see that for such a fixed $x$, the kernel of $\text{Jac}(g^n)(x)$ is spanned by

$$B = \begin{pmatrix} -x_1/x_3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

We can then describe $w_1(z)$ as the product of $dg_j$ with this basis evaluated in $z$:

$$w_1(z) = dg_1(z)B(z) = \begin{pmatrix} -z_2z_1/z_3, z_1 \end{pmatrix}.$$ 

Thus, introducing the independent variables $c_1, c_2, u_1$ and $u_2$ and applying the transformation $\tau_1$, we consider the Zariski closure:

$$\text{graph } M_1(g) = \mathbb{V}(c_1z_1^2 - z_2, c_2z_1^2 - z_3, u_1z_1^2z_3 - z_2, u_2z_1^2 - 1) \setminus \mathbb{V}(z_1z_3).$$

After performing a Gröbner basis computation to compute a finite list of polynomials describing this set, we intersect with the variety $\mathbb{V}(z_1, u_1, u_2)$. We arrive at an algebraic description of $L_1(g)$:

$$L_1(g) = \mathbb{V}(1) = 0.$$ 

Thus, we find that $\overline{K_1(g)}$ is empty. Through similar analysis of the cases $s = 2, 3$ for $j = 1, 2$, one finds that $\overline{K_2(g)} = \mathbb{V}(c_1, c_2)$. Therefore, $(0, 0)$ is the only possible asymptotic critical value of $g$.

We extend this algebraic description of the asymptotic critical values with the following lemma. This lemma derives from [21, Lemma 4.1] with one major difference. We introduce a non-empty Zariski open subset $O_{GL}$ of $\text{GL}_n(\mathbb{K})$, the group of $n \times n$ invertible matrices with entries in $\mathbb{K}$. Essentially, we choose a random linear change of variables $A$, so that $A$ almost surely lies in $O_{GL}$, and consider the polynomial $f^A$ given by $f^A(z) = f(Az)$. By [26, Lemma 2.4], $K_\infty(f^A) = K_\infty(f)$, and so we can compute the asymptotic critical values of $f$ by computing those of $f^A$. We exploit this result in our algorithm by showing that choosing $A \in O_{GL}$ implies that for $1 \leq s \leq n$, whenever $z_s$ goes to $\infty$ in a path towards an asymptotic critical value, then so does $(Az)_1$. Thus, this element of randomisation removes the necessity of choosing $s$.

**Lemma 7.** Let $f \in \mathbb{K}[z]^p$ be a dominant polynomial mapping. Let $K_\infty(f)$ be the set of asymptotic critical values of $f$ and $K_2(f^A)$ be defined as in Notation 5. There exists a non-empty Zariski open subset $O_{GL}$ of $\text{GL}_n(\mathbb{K})$ such that for $A \in O_{GL}$ the following equality holds:

$$K_\infty(f) \subseteq \bigcup_{j=1}^p K_2(f^A).$$

**Proof of Lemma 7.** Suppose $c \in K_\infty(f)$. Then, there exists some sequence $(x_t)_{t \in \mathbb{N}}$ in $(\mathbb{C}^n)^\mathbb{N}$ such that as $t \to \infty$,

$$\|x_t\| \to \infty, f(x_t) \to c$$

and $\|x_t\| \kappa(\partial f(x_t)) \to 0$.

Then, by considering the real and complex parts of the latter two limits, one defines a finite number of polynomials with real coefficients that give constraints defining a
disc centred at $c$ and 0 respectively, a semi algebraic set in $\mathbb{R}^{2n}$ by the isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. Therefore, by the curve selection lemma at infinity [21, Lemma 3.3], which is obtained from a semialgebraic compactification of $\mathbb{R}^{2n}$ and the classical curve selection lemma [6, Theorem 2.5.5], and by [6, Proposition 8.1.12] there exists a Nash curve $\gamma : (0, 1) \rightarrow \mathbb{R}^{2n}$ such that

$$f(\gamma(t)) \rightarrow c, \|\gamma(t)\| \rightarrow \infty \text{ and } \|\gamma(t)\| \kappa(df(\gamma(t))) \rightarrow 0 \text{ as } t \rightarrow 0.$$ 

In the case $\mathbb{K} = \mathbb{C}$, one may then consider the Nash curve, a semialgebraic curve in the class $C^{\infty}$ defined from $(0, 1)$ to $\mathbb{C}^n$. In either case, since $\gamma$ is a Nash mapping, we can express each component of $\gamma$ as a Puiseux series in $t$ by a Taylor expansion at 0.

We denote this expansion $z(t)$. In this way, each component of the expansion of $\gamma$ has finitely many terms with negative exponents. In particular, the order of each component, the least value $r$ such that the coefficient of the term $t^r$ is non-zero, corresponds to the dominant term in the limit $t \rightarrow 0$.

Let $\lambda_i$ denote the coefficient of the term of $z_i(t)$ with exponent $\text{ord } z_i(t)$. Denote by $\mathcal{I}$ the index set of all combinations of the elements of the set $\{1, \ldots, n\}$. Then, we consider the finite set of linear equations of the $\lambda_i$:

$$\mathcal{Y} := \left\{ \sum_{j=1}^{\lfloor \phi \rfloor} a_{\phi,j} \lambda_{\phi,j} : a \in \mathbb{K}^n, \phi \in \mathcal{I} \right\}.$$ 

Consider the group of $n \times n$ invertible matrices $\text{GL}_n(\mathbb{K})$ with entries in $\mathbb{K}$. Then, the set $\mathcal{Y}$ gives the set of equations such that if any row of a matrix $T \in \text{GL}_n(\mathbb{K})$ satisfies any of these equations, there is a cancellation in the term of highest degree in the product $Tz(t)$. The zero set of each equation in $\mathcal{Y}$ defines a proper Zariski closed subset of $\text{GL}_n(\mathbb{K})$. Clearly, for any equation, these Zariski closed subsets are not dense. Therefore, since there are finitely many equations in $\mathcal{Y}$, the union of the zero sets of all equations in $\mathcal{Y}$ defines a proper Zariski closed subset of $\text{GL}_n(\mathbb{K})$. Thus, there exists a non-empty Zariski open subset $\mathcal{O}_{\text{GL}}^{-1}$ of $\text{GL}_n(\mathbb{K})$ such that no cancellation occurs in the highest degree so that all components of the product $Tz(t)$ grow at the same speed with $t$. Since $\mathcal{O}_{\text{GL}}^{-1} \subset \text{GL}_n(\mathbb{K})$, we may consider the proper Zariski closed subset of $\text{GL}_n(\mathbb{K})$, $\mathcal{O}_{\text{GL}}$, defined by

$$A \in \mathcal{O}_{\text{GL}} \iff A^{-1} \in \mathcal{O}_{\text{GL}}^{-1}.$$ 

Thus, choose $A \in \mathcal{O}_{\text{GL}}$. Consider the polynomial $f^A = f(Az)$ and the path $y(t) = A^{-1}z(t)$. Then, as $t \rightarrow \infty$ we have $\|y(t)\| \rightarrow \infty$ and $f^A(y(t)) \rightarrow c$. In particular, since there is no cancellation by the choice of $A$, we have that $y_1(t) \rightarrow \infty$. Furthermore, with the Rabi distance $\nu$ measuring the distance to the space of singular operators we have that $\|z(t)\|\nu(df(z(t))) \rightarrow 0$. By the genericity of $A$, we have $\|y(t)\|\nu(df^A(y(t))) \rightarrow 0$. Therefore, by [21, Corollary 2.1], we have

$$\|y(t)\|\kappa(df^A(y(t))) \rightarrow 0.$$ 

Now, choose $j$ such that $\kappa(df^A(y(t))) = \|w_j(y(t))\|$ where $w_j$ is the restriction of $df^A$ to the kernel of $\text{jac}(f^A^j)(z)$.
Since $G_{n-k+1}(\mathbb{K}^n \times \mathbb{K})$ is compact, by [23, Lemma 5.1], there is a limit $W_j'$ of graphs $y_i(t)w_j(y(t))$. Thus, by $\|y(t)\|\|w_j(z(t))\| \to 0$, $W_j' \in \Lambda$ so that $(0, c, W_j') \in E_j^3(f)$ and so $c \in K_j^3(f^A)$. Therefore,

$$K_\infty(f) = K_\infty(f^A) \subseteq \bigcup_{(s,j)=(1,1)}^{(n,p)} K_s^j(f^A) = \bigcup_{j=1}^p K_j^1(f^A).$$

In summary, the construction given in Notation 5 begets an algorithm to compute the asymptotic critical values of a dominant polynomial mapping. Moreover, Lemma 7 allows us to reduce the number of sets we must compute from $np$ to just $p$. We make further improvements and describe the following algorithms in Section 3.

3. Algorithms

In this section, we give a geometric result that allows us to introduce an additional element of randomisation. By next translating the geometric objects defined in Section 2 into an algebraic setting, we give an algebraic proof that the set of asymptotic critical values has codimension at least 1. Furthermore, we define an algorithm of algebraic elimination for computing a finite list of polynomials whose zero set contains the asymptotic critical values of the input dominant polynomial mapping. With the geometric result, this algorithm only needs to introduce $p + 1$ new indeterminates, rather than introducing $n + 1$ indeterminates as in the algorithm one derives from Lemma 7.

In addition to this, by making the relation to some incidence varieties, we reduce the number of introduced variables to just $p$, one for each value of our polynomial mapping. These reductions undeniably come with great complexity improvements and more efficient algorithms, particularly in the special case where $p = 1$. This second algorithm also provides an avenue to tighten the degree bound on the set of asymptotic critical values, which is investigated in Section 4.1. The complexity of the two algorithms described in the section is analysed in Section 4.2.

3.1. Geometric result

We start with a geometric proposition involving the following objects:

Let $\Gamma$ be a subspace of $\mathbb{K}^N$ of dimension $\theta$, for some $\theta < N$. Then, define the set $E \subset G_{N-\theta+1}(\mathbb{K}^N)$ to be the subset of the Grassmannian of subspaces of dimension $N - \theta + 1$ such that the projection of every subspace $E \in E$ onto $\Gamma$ has dimension 1.

**Proposition 8.** Let $W$ be a set of dimension $\alpha \geq \theta$ so that the Zariski closure of $W$, $V = \overline{W}$, is equidimensional in an ambient space of dimension $N$ and let $\Gamma$ and $E$ be defined as above. Suppose there exists a hypersurface $Z$ such that $Z \cap W = \emptyset$ and that $V \backslash Z = W$. Suppose the projection of $V$ onto $\Gamma$ has dimension $\theta$. Then, there exists a proper non-empty Zariski open subset $\mathcal{O}_E \subset E$ so that for $E \in \mathcal{O}_E$, $\overline{W \cap E} = V \cap E$.

**Proof of Proposition 8.** For any subspace $E$, $V \cap E$ is an algebraic set containing $W \cap E$. Therefore, we have that $\overline{W \cap E} \subset V \cap E$ and so it remains to show that $V \cap E \subset \overline{W \cap E}$.

By definition, the projection of $V$ onto the subspace $\Gamma$ is dominant. Then, by Thom’s transversality theorem [15, page 67] and by the definition of the set $E$, a generic element $E$ of $E$ intersects $V$ transversally. Thus, there exists a proper non-empty Zariski open
subset $O_1 \subset E$ so that for all $E \in O_1$, the intersection of $V$ and $E$ is transverse. Since $V$ is an equidimensional variety, by the genericity of $E$, $V \cap E$ is also equidimensional. By [32, Theorem 1.24], the intersection $V \cap E$ has dimension $\alpha - \theta + 1$ which is at least 1. Let $F = Z \cap V$, then $F$ has codimension 1 in $V$. By another application of Thom’s transversality theorem [15, page 67], there exists a proper non-empty Zariski open subset $O_2 \subset E$ so that for all $E \in O_2$, the intersection of $E$ and $F$ is transverse. Let $O_\mathcal{E} = O_1 \cap O_2$, a proper non-empty Zariski open subset of $\mathcal{E}$. Therefore, for all $E \in O_\mathcal{E}$, $V \cap E$ has dimension $\alpha - \theta$. Now, let $E \in O_\mathcal{E}$. Then, for all of the finitely many irreducible components $U$ of $V \cap E$ we want to show that $U \subset \overline{W \cap E}$. Let $U$ be one such irreducible component. Note that the dimension of $U$ is $\alpha - \theta + 1$ since $V \cap E$ is equidimensional. Furthermore, we have $\dim(U \cap F) < \alpha - \theta + 1$. This implies that $U \setminus F = U$. Combining this with

$$U \setminus F \subset (V \cap E) \setminus F = W \cap E,$$

we find by taking the Zariski closure that $U \subset \overline{W \cap E}$. Since this holds for all irreducible components of $V \cap E$, we conclude that $V \cap E \subset \overline{W \cap E}$. \hfill \Box

3.2. Algebraic description of asymptotic critical values

In this subsection we translate the objects $M^i_j(f)$, $L^i_j(f)$ and $K^i_j(f)$, defined as in Notation 5, to an algebraic setting that will allow the use of algebraic elimination algorithms. To this end, we must derive polynomials from which we can give varieties that are equal to these sets.

Let $f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p$ be a dominant polynomial mapping. Let $O_{\text{GL}} \subset \text{GL}_n(\mathbb{K})$ be a non-empty Zariski open subset such that any $A \in O_{\text{GL}}$ satisfies the genericity requirements of Lemma 7.

For each choice of $j$ we aim to compute a representation of $K^i_j(f^A)$. Let $\Lambda = \mathbb{G}_{n-p+1}(\mathbb{C}^n \times 0)$ and recall the definition of $K^i_j(f^A)$,

$$K^i_j(f^A) = \pi \left( L^i_j(f^A) \right) = \pi \left( \overline{\text{graph} M^i_j(f^A) \cap \{ \{z \in \mathbb{C}^n|z_1 = 0\} \times \mathbb{C}^p \times \Lambda \} } \right).$$

Consequently, the first step must be to compute a representation of the graph of $M^i_j(f^A)$. Let $W^i_j(z)$ denote the graph of the map defined by $z \mapsto z_1 w_j(z)$, then $M^i_j(f^A)$ is defined by

$$M^i_j(f^A) = (f^A(\tau_1(z)), W^i_j(\tau_1(z))).$$

Denote by $\mathcal{E}$ the set of $(n + p + 1)$-dimensional subspaces that are defined by the set of equations $u_1 - r_1 e = \cdots = u_{n-p+1} - r_{n-p+1} e = 0$, where $e$ is an indeterminate, for $r_1, \ldots, r_{n-p+1} \in \mathbb{K}$. Additionally, denote by $r$ the numbers $r_1, \ldots, r_{n-p+1}$.

**Corollary 9.** Let $f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p$ be a dominant polynomial mapping. Let $M^i_j(f^A)$ be defined as in Notation 5 with $A$ chosen to satisfy the genericity assumption of Lemma 7. Let $r \in \mathbb{K}$ be chosen so that the subspace $E \in \mathcal{E}$ they define satisfies the genericity condition of Proposition 8. Then, there exists a polynomial tuple $(g_1, \ldots, g_{n+1})$ and polynomial $h$ with entries in the polynomial ring $\mathbb{K}[z, c, e]$ such that,

$$M^i_j(f^A) = \overline{V(g_1, \ldots, g_{n+1}) \setminus V(h)},$$

$$L^i_j(f^A) = \overline{V(g_1, \ldots, g_{n+1}) \setminus V(h)} \cap V(z_1, e).$$

Furthermore, the dimension of $V(g_1, \ldots, g_{n+1})$ is $p$. 

11
Proof of Corollary 9. Since $f$ is a dominant polynomial mapping, it is clear that $f^A$ is also dominant. Thus, $M^1_j(f^A)$ is well-defined outside of a nowhere dense algebraic set. We must derive equations defining this set in the algorithm. Firstly, we must remove the set $V(z_1)$. Furthermore, we require that $W^1_1(\tau_1(z))$ be of dimension $n - p + 1$ to be an element of the Grassmannian $G_{n-p+1}(\mathbb{C}^n \times \mathbb{C})$. Recall that $w_j(z)$ is the restriction of $df^A_j$ to the kernel of the Jacobian matrix of $f$ with the $j$th row removed. Then, excluding $z_1 = 0$, $M^1_j(f^A)$ is well-defined so long as the determinant of the submatrix given by the first $p - 1$ columns of the Jacobian matrix $\text{jac}((f^A)_j)$ is not 0. We denote this determinant $\delta(z)$.

Next, we compute the Jacobian matrix $\text{jac}((f^A)_j)$ and a basis $B$ for its null space. We can accomplish this by evaluation interpolation techniques using a Kronecker substitution to reduce the problem to the univariate case. The details of this, along with the complexity analysis, is given in Section 4.2.

Since $\text{jac}((f^A)_j)$ has rank $p - 1$, the basis $B$ consists of $n - p + 1$ vectors, each with $n$ rational functions as entries. The denominators of these functions describe an algebraic set where the function $M^1_j(f^A)$ is not defined. By [1], $\delta(z)$ is the common denominator of these functions. Define $v_1(z), \ldots, v_{n-p+1}(z)$ to be such that $v_j(z)$ is the product of the gradient of $f^A_j$ with the $j$th element of the basis $B$. Let $\lambda(t) \in \mathbb{K}^n$ be a path such that

$$\lambda_1(t)v_1(\lambda(t)) = \cdots = \lambda_1(t)v_{n-p+1}(\lambda(t)) \to 0 \text{ as } t \to \infty$$

then, we have that $W^1_j(\lambda(t)) \to W$, for some $W \in \Lambda$. Then, apply the transformation $\tau_1$ to obtain the rational mappings $(\tau_1(z))v_1(\tau_1(z)), \ldots, (\tau_1(z))v_{n-p+1}(\tau_1(z))$. As discussed above, this is well-defined for $z \in \mathbb{K}^n \setminus V(z_1 \text{ numer}(\delta(\tau_1(z))))$. Consider the graph, $G$, of these rational mappings with value variables $u_1, \ldots, u_{n-p+1}$. We see that the intersection with the variety $V(u_1, \ldots, u_{n-p+1})$ captures the points where the linear map $w_j(z)$ is identically the zero map. Therefore, this gives an algebraic version of the intersection with the space $\Lambda$.

We can now define the polynomial tuple that gives our representation of the graph of $M^1_j(f^A)$. To do so, we introduce independent variables $c$ and $u_1, \ldots, u_{n-p+1}$. Define the polynomial tuple,

$$N = (f^1_A - c_1, \ldots, f^p_A - c_p, z_1v_1 - u_1, \ldots, z_1v_{n-p+1} - u_{n-p+1}).$$

Then, applying the transformation $\tau_1$ and taking the numerators of the resulting functions we have,

$$\text{graph } M^1_j(f^A) = V(\text{numer}(N(\tau_1(z)))) \setminus V(z_1 \text{ numer}(\delta(\tau_1(z))))).$$

From the above, it is now clear that the intersection with $\Lambda$ is accomplished by intersecting with the variety $V(u_1, \ldots, u_{n-p+1})$. Therefore,

$$L^1_j(f^A) = V(\text{numer}(N(\tau_1(z)))) \setminus V(z_1 \text{ numer}(\delta(\tau_1(z)))) \cap V(z_1, u_1, \ldots, u_{n-p+1}).$$

It is clear that for all $E$ where $r$ are all non-zero we have

$$V(u_1, \ldots, u_{n-p+1}) = E \cap V(c).$$
Furthermore, the intersection \( K \) of dimension 2 is contained in a set of codimension at least one. To do this, we will show that the sets we have been described, we may give an algebraic proof that the set of asymptotic critical values is not identically zero at any irreducible component of the graph of \( \tilde{M}_1(f^*) \) is an equidimensional variety of dimension \( n \) in an ambient space of dimension \( 2n + 1 \) and is such that
\[
\text{graph } M_1^j(f^*) \cap E = \text{graph } M_1^j(f^*) \cap E.
\]
Furthermore, the intersection graph \( M_1^j(f^*) \cap E \) has dimension \( p \). By [8, Theorem 4.3.4], since the polynomials
\[
u_1 - r_1 e, \ldots, u_{n-p+1} - r_{n-p+1} e
\]
are unaffected by the transformation \( \tau_1 \), for \( 1 \leq k \leq n - p + 1 \) we may replace \( u_k \) by \( r_k e \). Thus, we arrive at the tuple
\[
\tilde{N} = (f_1^* - c_1, \ldots, f_p^* - c_p, z_1 v_1(z) - r_1 e, \ldots, z_1 v_{n-p+1} - r_{n-p+1} e),
\]
so that
\[
\text{graph } M_1^j(f^*) = \text{graph } (\text{num}(\tilde{N}(\tau_1(z)))) \setminus \text{graph } (z_1 \text{ num}(\delta(\tau_1(z))))).
\]
We find that the polynomial tuple \( \tilde{N} \) and polynomial \( z_1 \text{ num}(\delta(\tau_1(z))) \) satisfy the statement. \( \square \)

Now that polynomials whose zero set contains the asymptotic critical values have been described, we may give an algebraic proof that the set of asymptotic critical values is contained in a set of codimension at least one. To do this, we will show that the sets we aim to derive in the algorithms in this section have codimension at least one. A critical fact meaning that the algorithms output will be a finite list of polynomials.

**Corollary 10.** Let \( f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p \) be a dominant polynomial mapping. Let \( 1 \leq j \leq p \) and let \( K_1^j(f) \in \mathbb{C}^p \) be defined as above. Then, \( \text{dim}(K_1^j(f)) \leq p - 1 \). Furthermore, \( K_\infty(f) \) has codimension at least 1 in \( \mathbb{C}^p \).

**Proof of Corollary 10.** By Corollary 9, there exists a tuple of polynomials \((g_1, \ldots, g_{n+1}) \in \mathbb{K}[z, c]^{n+1} \) and polynomial \( h \in \mathbb{K}[z, c, e] \) such that for \( A \in \text{GL}_n(\mathbb{K}) \) satisfying the genericity condition of Lemma 7,
\[
\text{graph } M_1^j(f^*) = \text{graph } (g_1, \ldots, g_{n+1}) \setminus \text{graph } (h),
\]
\[
L_1^j(f^*) = \text{graph } (g_1, \ldots, g_{n+1}) \setminus \text{graph } (h) \cap \text{graph } (z_1, e),
\]
where \( \text{graph } (g_1, \ldots, g_{n+1}) \) has dimension \( p \). Furthermore, by the proof of Corollary 9, \( z_1 \) is a factor of \( h \). Therefore, \( z_1 \) is not identically zero at any irreducible component of
\[ V(g_1, \ldots, g_{n+1}) \setminus V(h). \] By [32, Theorem 1.24], the intersection with \( V(z_1) \) to derive \( L_1^j(f^A) \) necessarily reduces the dimension by 1. Hence, \( \dim(L_1^j(f^A)) \leq p - 1 \). Recall that \( K_1^j(f) = \pi(L_1^j(f)) \) where \( \pi \) is the projection map onto the \( c \)-space. Since the projection cannot increase the dimension [32, Theorem 1.25], \( \dim(K_1^j(f)) \leq p - 1 \).

Furthermore, by Lemma 7 for such a matrix \( A \in \text{GL}_n(K) \) we have,

\[ K_\infty(f) \subseteq \bigcup_{j=1}^p K_1^j(f^A). \]

Since the above holds for all \( j \), we have that the finite union \( \bigcup_{j=1}^p K_1^j(f^A) \) has dimension at most \( p - 1 \). Thus, \( K_\infty(f) \) has codimension at least 1 in \( \mathbb{C}^p \).

The algorithms described in this section use the above results and constructions to compute the asymptotic critical values of a dominant polynomial mapping using algebraic methods. To present the following algorithms we introduce some functions and subroutines that will feature in our algorithms.

### 3.3. Subroutines

We introduce 3 subroutines that will be used across all the algorithms featured in this article.

**Eliminate(\( P, v, w \)):**

**Input:** \( P \), a finite basis of an ideal, \( I \), of a polynomial ring (with base field \( K \) and two lists of indeterminates, \( v \) and \( w \)) which we denote \( K[v, w] \).

**Output:** \( E \), a finite basis of the ideal \( I \cap K[w] \).

**Intersect(\( P_1, \ldots, P_k \)):**

**Input:** \( P_1, \ldots, P_k \), finite bases of ideals, \( I_1, \ldots, I_k \), of a polynomial ring.

**Output:** \( P \), a finite basis of the ideal \( \bigcap_{i=1}^k I_i \).

**Saturate(\( P_1, P_2 \)):**

**Input:** \( P_1, P_2 \), finite bases of ideals, \( I_1, I_2 \), of a polynomial ring.

**Output:** \( S \), a finite basis of the ideal \( I_1 : I_2^\infty \).

**Remark 11.** We remark that algorithms for these subroutines exist, in particular all can be accomplished using Gröbner bases. We refer to [8, page 122], [4, Proposition 6.19] and [3, 9] for algorithms for computing a finite basis for respectively elimination ideals, intersection of ideals and the saturation of ideals.
Algorithm 1: acv

Input: $f : K^n \rightarrow K^p$ a dominant polynomial mapping with components in the ring $K[z]$, the list $z$.

Output: ACV, a finite list of polynomials whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $f$.

1. Generate a random change of variables $A \in K^{n \times n}$ and set $f^A \leftarrow f(Az)$.
2. For $j$ from 1 to $p$ do
   3. Generate random numbers $r \in K$.
   4. $B \leftarrow$ Basis of the kernel of $\text{jac}(f^A)[j]$.
   5. $(v_1(z), \ldots, v_{n-p+1}(z)) \leftarrow \text{df}_A^j B$.
   6. $\delta(z) \leftarrow$ the determinant of the first $p-1$ columns of $\text{jac}(f^A)[j]$.
   7. $N(z) \leftarrow \{f_1^A(z) - c_1, \ldots, f_p^A(z) - c_p, z_1 v_1(z) - r_1 e, \ldots, z_{n-p+1} v_{n-p+1}(z) - r_{n-p+1} e\}$.
   8. $G \leftarrow \text{num}(N(\tau_1(z)))$.
   9. $G_s \leftarrow \text{Saturate}(G, z_1 \text{num}(\delta(\tau_1(z))))$.
   10. $L \leftarrow G_s \cup \{z_1, e\}$.
   11. $V_j \leftarrow \text{Eliminate}(L, \{z, e\}, \{c\})$.
12. ACV $\leftarrow$ Intersect($V_1, \ldots, V_p$).
13. Return ACV.

3.4. First algorithm

We first define the objects that will be crucial in the proof of correctness and termination of Algorithm 1.

Theorem 12. Let $f = (f_1, \ldots, f_p) \in K[z]^p$ be a dominant polynomial mapping. Suppose that $A \in \text{GL}_n(K)$ satisfies the genericity condition of Lemma 7 and that $r$ and the corresponding subspace $E \in \mathcal{E}$ satisfies the genericity condition of Proposition 8. Then, Algorithm 1 terminates and returns as output a finite basis whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $f$.

Proof of Theorem 12. Firstly, Algorithm 1 uses linear algebra and, as in Remark 11, multivariate polynomial routines that are correct and terminate. Hence, Algorithm 1 terminates in finitely many steps.

By Lemma 7, for such a matrix $A \in \text{GL}_n(K)$,

$$K_{\infty}(f) \subseteq \bigcup_{j=1}^p K_1^j(f^A).$$

By Corollary 10, for each $1 \leq j \leq p$, $K_1^j(f^A)$ has codimension at least 1 in $\mathbb{C}^p$. Thus, the finite union $\bigcup_{j=1}^p K_1^j(f^A)$ has codimension at least 1 in $\mathbb{C}^p$. The goal is then to compute $p$ finite sets of polynomials such that their zero sets are $K_1^1(f^A), \ldots, K_1^p(f^A)$. The union of these zero sets would then contain the the asymptotic critical values of $f$. Hence, choose $1 \leq j \leq p$. Then, generate random numbers $r \in K$.
By Corollary 9, we derive polynomial tuple \((g_1, \ldots, g_{n+1})\) and polynomial \(h\) with entries in the polynomial ring \(K[z, c, e]\) such that,
\[
\text{graph } M_1^j(f^A) = \overline{V(g_1, \ldots, g_{n+1}) \setminus V(h)},
\]
\[
L_1^j(f^A) = \overline{V(g_1, \ldots, g_{n+1}) \setminus V(h) \cap V(z_1, e)}.
\]
As in the proof of this corollary, to compute the polynomial tuple \((g_1, \ldots, g_{n+1})\) and polynomial \(h\), we first compute a basis, \(B\), of the kernel of \(\text{jac}(f^A)[j]\). This is accomplished by an evaluation interpolation method detailed in Section 4.2. Then, it is easy to see that \(h = z_1 \text{numer}(\delta_1(z))\), where \(\delta_1\) is computed by step 6 of Algorithm 1 and the tuple \((g_1, \ldots, g_{n+1}) = G\), where \(G\) computed by step 8 of Algorithm 1.

The next stage is to compute the Zariski closure of the graph of \(M_1^j(f^A)\). Thus, we must compute a finite list of polynomials whose zero set is
\[
\overline{V(g_1, \ldots, g_{n+1}) \setminus V(h)}.
\]
By [8, Chapter 4, Section 4, Theorem 10], we may do this through saturations. Therefore, we apply the subroutine Saturate to \(G\) to saturate by the ideal \(\langle z_1 \text{numer}(\delta_1(z)) \rangle\). We denote by \(G_s\) the finite list of polynomials that is returned by the Saturate subroutine and conclude:
\[
\text{graph } M_1^j(f^A) = \overline{V(G_s)}.
\]
Thus, as in Corollary 9, we compute \(L_1^j(f^A)\) by intersecting with the variety \(V(z_1, e)\). By [8, Chapter 4, Section 3, Theorem 4], we add the polynomials \(z_1, e\) to the list \(G_s\) to define the finite list of polynomials \(L\) so that
\[
\overline{V(L)} = \overline{V(G_s) \cap V(z_1, e)}.
\]
It remains to project onto the \(c\)-space. By [8, Chapter 4, Section 4, Theorem 4], we apply the subroutine Eliminate to the list \(L\) in order to eliminate all variables except \(c_1, \ldots, c_p\). The result is a finite list of polynomials, \(V_j\), whose zero set is the set \(K_1^j(f^A)\) which contains the set \(K_1^j(f^A)\) by definition. Thus, by Corollary 10, the algebraic set \(\overline{V(V_j)}\) has codimension at least 1 in \(\mathbb{C}^p\).

We perform these steps for all \(j\) from 1 to \(p\) to obtain \(V_1, \ldots, V_p\). By [8, Chapter 4, Section 3, Theorem 15], the final step of computing their union can be performed by applying the subroutine Intersect to the lists \(V_1, \ldots, V_p\). The output is a finite list of polynomials which we denote ACV. We conclude that
\[
K_\infty(f) \subseteq \bigcup_{j=1}^{p} K_1^j(f^A) \subseteq V(ACV).
\]

Example 13. We give an example of how to use the computer algebra system Maple to implement Algorithm 1 with FGB [10], implemented in C, to perform the Gröbner basis computations. We shall report solely the inputs as the outputs are impractical to give here. The details and the results of these computations can instead be found on the webpage: [https://www-pol.sys.lip6.fr/~ferguson/globalacv.html](https://www-pol.sys.lip6.fr/~ferguson/globalacv.html)
Let $g = (z_1 z_2, z_1 z_3)$. We give the calls required to compute the set $K_1^j(g^A)$ for some invertible matrix $A$ satisfying the genericity condition in Theorem 1. The case $j = 2$ follows similarly.

We begin by generating a random seed so we can generate a random matrix $A$ and define the polynomial $g_A$:

```maple
> randomize();
> A := LinearAlgebra:-RandomMatrix(3);
> vars := [z1,z2,z3];
> Asubs := {seq(vars[i] = add(A[i,j]*vars[j], j=1..3), i=1..3)};
> gA := subs(Asubs, g);
> jacgA := VectorCalculus:-Jacobian(gA, vars);
> dgA := jacgA[1..1,1..3];
```

We first must compute the polynomial mapping $w_1(z) = (v_1(z), v_2(z))$. To do so, we compute a basis of the kernel of $\text{jac}(g^A[1])$. We can then define $\tau_1$ and $G$.

```maple
> B := LinearAlgebra:-NullSpace(jacgA[2..2,1..3]);
> v1 := add(B[1][i]*dgA[i]*vars[1], i=1..3);
> v2 := add(B[2][i]*dgA[i]*vars[1], i=1..3);
> tau := {z1 = 1/z1, z2 = z2/z1, z3 = z3/z1};
> G := numer(subs(tau, [gA[1] - c1, gA[2] - c2, rand() * e - v1, rand() * e - v2]));
```

Recall that $M_1^j(g^A)$ is only defined where $\text{jac}(g^A)$ is full rank. Therefore, we must remove the algebraic set defined by the minors of this matrix, as well as the variety $V(z_1)$. By the choice of a generic matrix $A$, it suffices to remove the algebraic set defined by the minor given by the first $p-1$ columns of $\text{jac}(g^A[1])$.

```maple
> delta := jacgA[2,1];
> Gs := FGb:-fgb_gbasis_elim([op(G), t*numer(subs(tau, delta)) * vars[1]-1], 0, [t], [op(vars), e, c1, c2]);
> V := FGb:-fgb_gbasis_elim([op(Gs), vars[1], e], 0, [op(vars), e], [c]);
```

After performing these computations, similar to what we saw in Example 6, $K_1^j(g^A) = V(c_1, c_2)$. A similar computation for $j = 2$ would reveal the same. Thus, $(0,0)$ is the only possible asymptotic critical value of $g^A$ and therefore of $g$.

3.5. Improved algorithm

We can implement the idea of Algorithm 1 in a different way. For fixed $1 \leq j \leq p$, consider a basis $B$ of the kernel of $\text{jac}((f^A)[j])$ and define $(v_1, \ldots, v_{n-p+1}) = df^A B$. By [1], we may assume that the $v_i(z)$ have common denominator $\delta(z)$, the determinant of the first $p-1$ columns of $\text{jac}((f^A)[j])$. Then, define the list of polynomials $G$ by $\text{numer}(f^A_1(\tau_1(z)) - c_1, \ldots, f^A_{p}(\tau_1(z)) - c_p, v_1(\tau_1(z)) - z_1 r_1 e, \ldots, v_{n-p+1}(\tau_1(z)) - z_1 r_{n-p+1} e)$, and denote $\text{Eliminate}(G, e, \{z, c\})$ by $G'$. We force the map $z \mapsto (v_1(\tau_1(z)), \ldots, v_{n-p+1}(\tau_1(z)))$. 17
to be parallel to a generic vector \( \mathbf{r} \in \mathbb{K}^{n-p+1} \). We did this before by introducing a variable \( e \). Instead, let \( M \) denote the ideal generated by the numerators of the minors of the following matrix evaluated at \( \tau_1(z) \)

\[
\begin{bmatrix}
v_1 & \cdots & v_{n-p+1} \\
r_1 & \cdots & r_{n-p+1}
\end{bmatrix}.
\]

If the minors of this matrix are set to 0, there is a rank deficiency. This means that the two rows are parallel. In this setting, we would therefore not need to introduce a variable \( e \) to consider the linear subspace \( E \) from Algorithm 1, but instead, we include the minors of this matrix. It is easy to see that the minors discussed are exactly what is obtained by eliminating the introduced variable \( e \) from the ideal given by the basis \( G \). This is the content of the following lemma.

**Lemma 14.** Let \( \mathbf{f} = (f_1, \ldots, f_p) \in \mathbb{K}[\mathbf{z}]^p \) be a dominant polynomial mapping. Let \( A \in \text{GL}_n(\mathbb{K}) \) and let \( \mathbf{r} \in \mathbb{K} \) so that the genericity assumptions of Theorem 12 hold. Let \( G, M \) and \( G' \) be defined as above. Then, the following equality holds:

\[
\langle \text{num}(f_1^A(\tau_1(z))) - c_1, \ldots, f_p^A(\tau_1(z)) - c_p \rangle + M = \langle G' \rangle.
\]

**Proof of Lemma 14.** We shall prove this by double inclusion.

Firstly, the numerators of the polynomials \( f_1^A - c_1, \ldots, f_p^A - c_p \) at \( \tau_1(z) \) are elements of \( G \) and \( \mathbb{K}[\mathbf{z}, \mathbf{c}] \), so it remains to show that the numerator of each minor is an element of \( \langle G' \rangle \). Let \( r_k v_1(\tau_1(z)) - r_k d_k(\tau_1(z)) \) be one such minor. Then, \( r_k (r_k z_1 e - d_k(\tau_1(z))) - r_k (r_1 z_1 e - v_1(\tau_1(z))) = r_k v_1(\tau_1(z)) - r_k d_k(\tau_1(z)). \) Taking the numerators of both sides we find that \( \text{num}(r_k v_1(\tau_1(z)) - r_k d_k(\tau_1(z))) \in \langle G' \rangle \).

Thus,

\[
\langle \text{num}(f_1^A(\tau_1(z))) - c_1, \ldots, f_p^A(\tau_1(z)) - c_p \rangle + M \subseteq \langle G' \rangle.
\]

On the other hand, let \( g \in \langle G' \rangle \). By the definition of \( G' \), \( g \in \langle G \rangle \) such that all \( e \)-terms are cancelled. That is,

\[
g = \text{num}(h_1(f_1^A(\tau_1(z))) - c_1) + \cdots + h_p(f_p^A(\tau_1(z)) - c_p) + \sum_{i=1}^p h_{i+1}(e z_i r_1 - v_i(\tau_1(z))) + \cdots + \sum_{i=n-p+1}^n h_{n+1}(e z_i r_{n-p+1} - v_{n-p+1}(\tau_1(z))),
\]

so that \( h_{p+1}, \ldots, h_{n+1} \in \mathbb{K}[\mathbf{z}, \mathbf{c}] \) are polynomials such that in the above sum, all terms involving \( e \) sum to 0. Consider the following monomial ordering, \( e > z_1 > \cdots > z_n > c_1 > \cdots > c_p \). Then, the leading term of each \( \text{num}(e z_i r_1 - v_i(\tau_1(z))) \) divides the leading term of the polynomial \( e z_i^d r(z) r_i \), for some \( d \) large enough. This leading term must cancel in the polynomial \( g \) as it involves \( e \). Therefore, \( (h_{p+1}, \ldots, h_{n+1}) \) is a syzygy on the leading terms of \( \text{num}(e z_i r_1 - v_i(\tau_1(z))) \). Recall that the \( S \)-polynomials generate the set of syzygies on the leading terms \( e z_i^d r_1 \) [8, page 111] and here the \( S \)-polynomials are simply the minors of the above matrix. It is therefore possible to rewrite \( h_{p+1}, \ldots, h_{n+1} \) as elements of \( M \). Thus,

\[
\langle G' \rangle \subseteq \langle \text{num}(f_1^A(\tau_1(z))) - c_1, \ldots, f_p^A(\tau_1(z)) - c_p \rangle + M.
\]

While this point of view allows us to drop the variable \( e \) and thus reduce the number of variables by 1, it makes us introduce many more equations, namely the \( \binom{n}{p} \) minors of the matrix. The following lemma actually ensures that only \( n - p \) of them are needed.
Lemma 15. Let \( f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n] \) and let \( r_1, \ldots, r_k \in \mathbb{K} \) with \( r_1 \neq 0 \). Consider the matrix
\[
\begin{bmatrix}
f_1 & \cdots & f_k \\
r_1 & \cdots & r_k
\end{bmatrix}
\]
Let \( I \in \mathbb{K}[x_1, \ldots, x_n] \) be the ideal generated by the minors of the matrix \( R \). Then, \( I = \langle r_2f_1 - r_1f_2, \ldots, r_kf_1 - r_1f_k \rangle \).

Proof of Lemma 15. Clearly, \( \langle r_2f_1 - r_1f_2, \ldots, r_kf_1 - r_1f_k \rangle \subset I \). Let \( M_{i,j} = r_jf_i - r_if_j \) be a minor of the matrix \( R \). Then, \( r_2M_{1,1} = r_1r_2f_1 - r_1r_2f_1 = r_1M_{1,1} \). Since \( r_1 \neq 0 \), we conclude that \( M_{i,j} \in \langle r_2f_1 - r_1f_2, \ldots, r_kf_1 - r_1f_k \rangle \). This holds for all \( 1 \leq i < j \leq k \) and so \( I \subset \langle r_2f_1 - r_1f_2, \ldots, r_kf_1 - r_1f_k \rangle \).

This approach with the minors leads us to the design of Algorithm 2.

**Algorithm 2: acv2**  
**Input:** \( f : \mathbb{K}^n \rightarrow \mathbb{K}^p \) a dominant polynomial mapping with components in the ring \( \mathbb{K}[z] \), the list \( z \).  
**Output:** ACV, a finite list of polynomials whose zero set has codimension at least 1 in \( \mathbb{C}^p \) and contains the set of asymptotic critical values of \( f \).

1. Generate a random change of variables \( A \in \mathbb{K}^{n \times n} \) and set \( f^A \leftarrow f(Az) \).
2. For \( j \) from 1 to \( p \) do
   3. Generate random numbers \( r \in \mathbb{K} \).
   4. \( B \leftarrow \text{Basis of the kernel of jac}((f^A)^{(j)}) \).
   5. \((v_1(z), \ldots, v_{n-p+1}(z)) \leftarrow df^A_jB \).
   6. \( \delta(z) \leftarrow \text{the determinant of the first } p - 1 \text{ columns of jac}((f^A)^{(j)}) \).
   7. \( N'(z) \leftarrow \{f^A_1 - c_1, \ldots, f^A_p - c_p, f_2v_1 - r_1v_2, \ldots, r_{n-p+1}v_1 - r_1v_{n-p+1}\} \).
   8. \( G'_z \leftarrow \text{num}r(N'(\tau_1(z))) \).
   9. \( G'_z \leftarrow \text{Saturate}(G'_z, z) \).
   10. \( L' \leftarrow G'_z \cup \{z_1\} \).
   11. \( V'_j \leftarrow \text{Eliminate}(L', \{z\}, \{c\}) \).
   12. ACV' \leftarrow \text{Intersect}(V'_1, \ldots, V'_p) \).
   13. Return ACV'.

Since Algorithms 1 and 2 are quite similar, we want to be able to reuse the proof of correctness of the former for the latter’s. The difference between these two algorithms is the stage of the algorithm where we eliminate the introduced variable \( e \). In Algorithm 1, this is in step 12; for Algorithm 2, as in Lemma 14, we consider an ideal equal to the one given if we eliminated \( e \) after step 9 in Algorithm 1. However, the steps in between involve a saturation with respect to an ideal that does not involve \( e \). This motivates the following lemma, which shall make use of the following notation.

For fixed \( 1 \leq j \leq p \), consider a basis \( B \) of the null space of \( \text{jac}((f^A)^{(j)}) \) and define \( v_1(z), \ldots, v_{n-p+1}(z) = df^A_jB \). By [1], we may assume that \( v_i(z) \) have common denominator \( \delta(z) \), the determinant of the first \( p - 1 \) columns of \( \text{jac}((f^A)^{(j)}) \). As in
Algorithm 1, define $G$ to be the list of polynomials
\[
\{\text{numer}(f_1(z) - c_1, \ldots, f_p(z) - c_p, z_1v_1(z) - r_1e, \ldots, z_{n-p+1}v_{n-p+1}(z) - r_{n-p+1}e)\},
\]
evaluated at $\tau(z)$. Define the following finite lists that are defined in Algorithm 1 or Algorithm 2:
\[
\begin{align*}
G_s &= \text{Saturate}(G, z_1 \text{ numer}(\delta(\tau_1(z)))) , \\
G' &= \text{Eliminate}(G, e, \{z, c\}) , \\
G'_s &= \text{Eliminate}(G_s, e, \{z, c\}) .
\end{align*}
\]
The relationship between these lists is investigated in the following lemma.

**Lemma 16.** Let $f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p$ be a dominant polynomial mapping. Let $A \in \text{GL}_n(\mathbb{K})$ and let $r \in \mathbb{K}$ so that the genericity assumptions of Theorem 12 hold. Let $G, G_s, G'$ and $G'_s$ be defined as above. Then, the following equality holds:
\[
\langle G'_s \rangle = \{\text{Saturate}(G', z_1 \text{ numer}(\delta(\tau_1(z))))\}.
\]

**Proof of Lemma 16.** By [8, Chapter 2, Section 7, Theorem 4], for a given ideal $I$ and term order $\geq$, there exists a unique reduced Gröbner basis of $I$ with respect to $\geq$. Note that given a Gröbner basis, which can be computed using Buchberger’s algorithm [8, Chapter 2, Section 7, Theorem 2], there exists an algorithm to compute a reduced Gröbner basis [4, Proposition 5.6]. Then, we shall prove this by considering algorithms which accomplish the subroutines Eliminate and Saturate by returning reduced Gröbner bases.

We will then show that the reduced Gröbner bases returned by these algorithms, in this case, are the same. The key is that the Saturate subroutine can be performed using Eliminate, and so we can perform both operations at the same time.

Thus, we first make explicit, the algorithms we shall use. The key is that both Eliminate and Saturate can be performed by computing elimination ideals. To compute elimination ideals, for an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$, we compute a Gröbner basis with respect to a lexicographic monomial ordering where $x_1 > \cdots > x_n$. By [8, Chapter 3, Section 1, Theorem 2], removing from this Gröbner basis all polynomials that involve the variables $x_1, \ldots, x_k$, for some $k < n$, gives a Gröbner basis of the ideal $I \cap \mathbb{K}[x_{k+1}, \ldots, x_n]$. For Saturate, by [8, Chapter 4, Section 4, Theorem 14], given an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ and a polynomial $g \in \mathbb{K}[x_1, \ldots, x_n]$, we can compute a Gröbner basis of $I : (g)\infty$ through the Eliminate subroutine. To do so, first compute a Gröbner basis $G$ of the ideal $I + (\ell g - 1)$, where $\ell$ is an independent variable, with respect to a lexicographic monomial ordering with $\ell > x_1 > \cdots > x_n$. Then, $G \cap \mathbb{K}[x_1, \ldots, x_n]$ is a Gröbner basis of the ideal $I : (g)\infty$ with respect to the ordering $x_1 > \cdots > x_n$. The last operation here is the elimination of the variable $\ell$. As in [8, Chapter 3, Section 1, Theorem 2], that is simply remove from $B$ all polynomials that involve $\ell$.

Therefore, we may perform both saturations and the elimination of $e$ with one Gröbner basis computation and an intersection. We simply add the polynomial $\ell z_1 \text{ numer}(\delta(\tau_1(z))) - 1$ to the list $G$ and consider a Gröbner basis with respect to a lexicographic monomial ordering with $e > \ell > z_1 > \cdots > z_n > c_1, \cdots, > c_p$. This is the output of Saturate($G', z_1 \text{ numer}(\delta(\tau_1(z)))$). We could alternatively eliminate $e$ last, which would make the result $G'_s$. As both ways involve intersecting with the same polynomial ring, $\mathbb{K}[z, c]$, the result is the same. \qed
We are now in a position to utilise Theorem 12 to give a proof of correctness for Algorithm 2.

**Theorem 17.** Let \( f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p \) be a dominant polynomial mapping. Suppose that \( A \in \text{GL}_n(\mathbb{K}) \) satisfies the genericity condition of Lemma 7 and that \( r \) and the corresponding subspace \( E \in \mathcal{E} \) satisfies the genericity condition of Proposition 8. Then, Algorithm 2 terminates and returns as output a finite basis whose zero set has codimension at least \( 1 \) in \( \mathbb{C}^p \) and contains the set of asymptotic critical values of \( f \).

**Proof of Theorem 17.** We will show that the only difference between Algorithm 1 and Algorithm 2 is when the independent variable \( e \) is eliminated (and subsequently, \( e \) is not added to the list \( G_s \) in step 11). We then show that this does not lose any asymptotic critical values. The proof that the result set has codimension at least \( 1 \) in \( \mathbb{C}^p \) is identical to the proof in Theorem 12.

The first seven steps of Algorithm 1 and Algorithm 2 are the same, where we choose \( r \) and \( A \) to satisfy the genericity assumptions in Theorem 12. Then, in step 8 of Algorithm 2, the list of polynomials and rational mappings

\[
N'(z) = \{ f_1^A(z) - c_1, \ldots, f_p^A(z) - c_p, r_2v_1(z) - r_1v_2(z), \ldots, r_{n-p+1}v_1(z) - r_1v_{n-p+1}(z) \}
\]

is defined. By Lemma 15, the ideal defined by this basis is equal to the following ideal

\[
\langle f_1^A(z) - c_1, \ldots, f_p^A(z) - c_p \rangle + M.
\]

Denote by \( G \) the following list of polynomials defined in step 8 of Algorithm 1

\[
\text{num}(f_1^A(\tau_1(z)) - c_1, \ldots, f_p^A(\tau_1(z)) - c_p, v_2(\tau_1(z)) - r_1v_2(z), \ldots, v_{n-p+1}(\tau_1(z)) - r_1v_{n-p+1}(z)).
\]

By Lemma 14, the ideal \( \text{num}(\langle N'(\tau_1(z)) \rangle) \) is generated by the basis returned by the subroutine \( \text{Eliminate}(G, e, \{ z, c \}) \). Therefore, we can conclude that the difference between Algorithm 1 and Algorithm 2 is the step when the variable \( e \) is eliminated. Then, Algorithm 2 terminates by Theorem 12. It remains to show that the variety generated by the output is of dimension at most \( p - 1 \) and contains the set of asymptotic critical values of \( f \).

Denote \( \text{Saturate}(G_s, z_1 \text{num}(\delta(\tau_1(z)))) \) by \( G_s' \) (the output of step 10 of Algorithm 1) and \( \text{Eliminate}(G_s', e, \{ z, c \}) \) by \( G_s'' \). By Lemma 16, \( G_s'' \) is equal to

\[
\text{Saturate}(\text{num}(N(\tau_1(z))), z_1 \text{ num}(\delta(\tau_1(z))))
\]

the output of step 10 of Algorithm 2. By [8, Chapter 3, Section 1, Theorem 2], the ideal generated by \( G_s' \) is contained in the ideal generated by \( G_s \). Thus, by [8, Chapter 1, Section 4, Proposition 8], \( \text{V}(G_s) \subseteq \text{V}(G_s') \) and so \( \text{V}(G_s \cup \{ z_1, c \}) \subseteq \text{V}(G_s' \cup \{ z_1 \}) \). Since these two varieties are the varieties generated by the outputs of step 11 of Algorithm 1 and Algorithm 2 respectively, and since the projection of \( \text{V}(G_s \cup \{ z_1, c \}) \) onto the subspace is the set \( K[I^d(f^A)] \) by Theorem 12, we can conclude that Algorithm 2 returns as output a basis whose zero set contains the set of asymptotic critical values of \( f \). Furthermore, as in the proof of Theorem 12, the variety \( \text{V}(G_s) \) has dimension at most \( p \) and therefore by [32, Theorem 1.24], \( \text{V}(G_s) \cap \text{V}(z_1) \) has dimension at most \( p - 1 \). The result then follows from [8, Chapter 9, Section 4, Theorem 8], for varieties \( X \) and \( Y \),

\[
\dim(X \cup Y) = \max(\dim(X), \dim(Y)).
\]
4. Degree bounds and complexity estimates

In this section we prove the main results stated in Section 1.

4.1. Proof of Theorem 3

In this subsection, we use Algorithm 2 to bound the degree of the set of asymptotic critical values of polynomial mappings. In both of the algorithms designed in this paper, for each choice of $1 \leq j \leq p$ assuming that $p > 1$, we must compute a basis of the kernel of $jac((f^A)^{|B|})$. Thus, we begin with a lemma bounding the degrees of the entries of such a basis.

**Lemma 18.** Let $f = (f_1, \ldots, f_p) \in K[z_1, \ldots, z_n]^p$ be a dominant polynomial mapping and let $A \in GL_n(K)$ satisfy the genericity condition of Lemma 7. Let $d = \max_{1 \leq i \leq p} \deg f_i$. Then, for all $1 \leq j \leq p$, there exists a basis $B$ of the kernel of $jac((f^A)^{|B|})$ such that the entries of $B$ are rational functions whose numerators and denominators have degree at most $(p - 1)(d - 1)$.

**Proof of Lemma 18.** First, fix some $1 \leq j \leq p$ and consider the matrix $jac((f^A)^{|B|})$. Since $f$ is dominant, by the genericity of $A$, $f^A$ is also dominant. Thus, the Jacobian matrix $jac((f^A)^{|B|})$ has rank $p - 1$ outside of a proper Zariski closed subset of $C^n$. By the generalised Cramer’s Rule for full rank matrices [1], we can find a basis of the null space of the Jacobian whose entries are rational functions of linear combinations of the minors of this matrix with a common denominator. This common denominator is itself the determinant of the submatrix comprised of the first $p - 1$ columns of $jac((f^A)^{|B|})$. Hence, the numerator and denominator of each entry has degree at most the degree of the maximal minors which is at most $(p - 1)(d - 1)$. \hfill \Box

We shall prove our main degree result by applying Bézout’s Theorem [16, Theorem 1] to the finite lists of polynomials defined in Algorithm 2. Therefore, we first give a lemma that will bound the degree of the lists $G$ and $G'$ that are defined in Algorithms 1 and 2 respectively. We recall the construction introduced in Corollary 9 and improved upon in Theorem 17. Let $A \in GL_n(K)$ and let $r \in K$ so that the genericity assumptions of Theorem 17 hold. For fixed $1 \leq j \leq p$ and a basis $B$ of the null space of $jac((f^A)^{|B|})$, define $v_1(z), \ldots, v_{n-p+1}(z) = \nabla f_j^A B$. Then,

$$G = \text{numer}(f_1^A(\tau_1(z)) - c_1, \ldots, f_p^A(\tau_1(z)) - c_p, \ r_1z\epsilon - v_1(\tau_1(z)), \ldots, r_{n-p+1}z\epsilon - v_{n-p+1}(\tau_1(z))),$$

$$G' = \text{numer}(f_1^A(\tau_1(z)) - c_1, \ldots, f_p^A(\tau_1(z)) - c_p, \ r_2v_1(\tau_1(z)) - r_1v_2(\tau_1(z)), \ldots, r_{n-p+1}v_1(\tau_1(z)) - r_1v_{n-p+1}(\tau_1(z))).$$

**Lemma 19.** Let $f = (f_1, \ldots, f_p) \in K[z_1, \ldots, z_n]^p$ be a dominant polynomial mapping. For fixed $1 \leq j \leq p$, let $G$ and $G'$ be finite lists of polynomials defined as above. Then, the degree of the highest dimension components of the algebraic sets $V(G)$ and $V(G')$ are at most $(p(d - 1) + 2)^{n-p+1}(d + 1)p$ and $(p(d - 1))^{n-p}(d + 1)p$ respectively.

**Proof.** By [8, Chapter 4, Section 6, Theorem 2], the variety $V(G)$ can be expressed in terms of its irreducible components $V(G) = W_1 \cup \cdots \cup W_k$. Since we are only concerned with the components of highest dimension, we may apply Bézout’s Theorem [16, Theorem 1] directly to bound this degree and similarly for the algebraic set $V(G')$. 22
Firstly, note that the degrees of the polynomials $f_1^A, \ldots, f_p^A$ are still bounded by $d$ since $A$ is a generic linear change of coordinates. Then for $1 \leq i \leq p$, the degrees of the numerator and denominator of $f_i^A(\tau_1(z))$ are bounded by $d$. Therefore, the degree of the numerator of $f_i^A(\tau_1(z)) - c_i$ is at most $d + 1$.

We must now bound the degrees of $v_1, \ldots, v_{n-p+1}$. To do this, first note that the degrees of the components of the gradient of $f_j^A$ are bounded by $d - 1$. Then, we invoke Lemma 18 to bound the degree of the entries of a basis $B$ of the null space of $\text{jac}((f^A)|_1)$. Thus, $v_1, \ldots, v_{n-p+1}$, the dot products of the basis vectors with the gradient of $f_j^A$, have degree at most $p(d-1)$.

Consider the list $G$. After applying the transformation $\tau_1(z)$ to the rational functions $v_1, \ldots, v_{n-p+1}$, we obtain a rational function whose numerator and denominator have degree $p(d-1)$. Thus, for $1 \leq i \leq n - p + 1$, the polynomial $\text{num}(r_i \tau_1(z) - v_i(\tau_1(z)))$ has degree $p(d-1) + 2$. Therefore, the list $G$ has $n-p+1$ polynomials of degree $p(d-1) + 2$ and $p$ polynomials of degree $d+1$.

Now consider the list $G'$. Recall that the entries of the basis $B$ have a common denominator. Therefore, the rational functions $v_1, \ldots, v_{n-p+1}$ also have a common denominator. Thus, when we now compute the minors of the matrix,

$$
\begin{bmatrix}
v_1(\tau_1(z)) & \cdots & v_{n-p+1}(\tau_1(z)) \\
r_1 & \cdots & r_{n-p+1}
\end{bmatrix},
$$

where $r_1, \ldots, r_{n-p+1}$ are generic elements of the base field, we again find a common denominator that does not increase the degree of the numerator. Hence, for $2 \leq i \leq n - p + 1$, the degree of $\text{num}(r_i \tau_1(z) - v_i(\tau_1(z)))$ is $p(d-1)$. Then, the list $G'$ has $n - p$ polynomials of degree $p(d-1)$ and $p$ polynomials of degree $d + 1$. One now applies Bézout’s Theorem [16, Theorem 1] to get the result.

Now that we have bounded the degrees of the objects considered in our algorithms, we may now prove our main degree result.

**Theorem 3.** Let $f = (f_1, \ldots, f_p) \in K[z_1, \ldots, z_n]^p$ be a dominant polynomial mapping. Let $d = \max_{1 \leq i \leq p} \deg f_i$. Then the asymptotic critical values of $f$ are contained in a hypersurface of degree at most $p^{n-p+1}(d-1)^{n-p}(d+1)^p$.

**Proof of Theorem 3.** Let $A \in \text{GL}_n(K)$ and let $r \in K$ so that the genericity assumptions of Theorem 17 hold. We consider the following finite lists of polynomials computed in Algorithm 2.

$$
G' = \text{num}(f_1^A(\tau_1(z)) - c_1, \ldots, f_p^A(\tau_1(z)) - c_p),
$$

$$
r_2 v_1(\tau_1(z)) - r_1 v_2(\tau_1(z)), \ldots, r_{n-p+1} v_1(\tau_1(z)) - r_1 v_{n-p+1}(\tau_1(z)),$$

$$
G'_s = \text{Saturate}(G', z_1 \text{ num}(\delta(\tau_1(z)))) ,
$$

$$
L' = G'_s \cup \{z_1\},
$$

$$
V'_j = \text{Eliminate}(L', \{z\}, \{c\}).
$$

By Lemma 7 and Theorem 17, $K(\infty(f)) \subset \bigcup_{j=1}^p V(V'_j)$. Thus, the degree of $K(\infty(f))$ is bounded by the degree of $\bigcup_{j=1}^p V(V'_j)$. We now aim to use the algebraic description of $V'_j$ as detailed above to give a bound on the degree of $V(V'_j)$. Then, the degree of $K(\infty(f))$ will be bounded by $p$ times that bound.
By [8, Chapter 4, Section 6, Theorem 2], the variety $V(G')$ can be expressed in terms of its irreducible components $V(G') = W_1 \cup \cdots \cup W_k$. By [8, Chapter 4, Section 4, Theorem 10], the saturation

$$(G'_s) = \langle G \rangle : \langle z_1 \text{num}(\delta(\tau_1(z))) \rangle^\infty$$

corresponds to the variety

$$V(G'_s) = V(G') \setminus V(z_1 \text{num}(\delta(\tau_1(z)))).$$

Thus, $V(G'_s)$ is the union of a subset of the irreducible components of $V(G')$. $V(G'_s) = W_{s_1} \cup \cdots \cup W_{s_\ell}$ such that $W_{s_j} \not\subseteq V(z_1 \text{num}(\delta(\tau_1(z))))$ for $1 \leq j \leq \ell$. Hence, we can bound the degree of $V(G'_s)$ by the so-called strong degree of $V(G')$, that is the sum of the degrees of the equidimensional components. Similarly, by [8, Chapter 4, Section 3, Theorem 4], adding polynomials to an ideal equates to intersecting the respective varieties. Therefore, by Bézout’s Theorem [16, Theorem 1] and since the polynomial, $z_1$, that we add has degree 1 we conclude that the degree of $V(L')$ is bounded by the degree of $V(G'_s)$. Furthermore, by [16, Lemma 2], since the projection is an affine map and since $V(L')$ is a constructible set in the Zariski topology, the degree of $V(V'_j)$ is bounded above by the degree of $V(L')$.

Note that we only need to consider the degree of the components of $V(G')$ of highest dimension since $V(G'_s)$ is contained in the union of these components. In summary, we can bound the degree of $K_1^1(f^A) \subseteq V(V'_j)$ by bounding the degree of $V(G')$. Thus, by Lemma 19, the degree of $K_1^1(f^A)$ is at most $(p(d-1))^{n-p}(d+1)^p$. Therefore, the degree of $K_\infty(f)$ is at most $p^{n-p+1}(d-1)^{n-p}(d+1)^p$. \hfill \square

4.2. Proof of Theorem 4

In this subsection, we assess the worst-case complexity of the three algorithms given in this paper. We apply the complexity results attained by the geometric resolution algorithm given in [13].

Let $M(n)$ be a cost function for multiplying two univariate polynomials of degree at most $n$ in terms of operations in the base field. For instance, $M(n) = O(n \log n \log \log n)$ using the Cantor–Kaltofen algorithm [7].

We also denote by $\omega$, $2 \leq \omega \leq 3$, the linear algebra complexity exponent. That is two matrices of size $n \times n$ over a field can be multiplied in $O(n^\omega)$ operations in the base field. At the time of writing, the best upper bound for $\omega$ is 2.3728639 due to Le Gall [22]. Denote by $\Omega = 1 + \omega$ the related constant of the complexity exponent of linear algebra over a ring. Finally, we denote the evaluation complexity of the polynomials of $G$ and $z_1 \text{num}(\delta(\tau_1(z)))$, defined in each Algorithm, by $L$.

Given a dominant polynomial mapping $(f_1, \ldots, f_p) \in \mathbb{K}[z]$ as input, for each choice of $1 \leq j \leq p$, the first steps of both the algorithms designed in this paper construct polynomials that will be the input of the algebraic elimination algorithms we use. In the case of our first algorithm, these polynomials are those described in Corollary 9. For the second algorithm, there are small differences described in Theorem 12. For both algorithms, the considered polynomials involve the computation of rational functions $v_1, \ldots, v_{n-p+1}$ and polynomial $\delta$. We recall the proof of Corollary 9 where these are defined. Let $B$ be a basis of the kernel of the Jacobian matrix $\text{jac}((f^A)^\delta))$. Then, define
Let \( f_1, \ldots, f_p \in \mathbb{K}[z] \) be a dominant polynomial mapping. Let the rational functions \( v_1, \ldots, v_{n-p+1} \) and polynomial \( \delta \) be defined as above. Then, computing \( v_1, \ldots, v_{n-p+1} \) and \( \delta \) requires at most \( O((p-1)^d n^{\omega}(d-1)^n) \) arithmetic operations in \( \mathbb{K} \).

Proof of Lemma 20. We consider a Jacobian matrix of size \((p-1) \times n\) whose entries have degree \( d-1 \) in \( n \) variables and we want to compute a basis of its null space. It consists of \( n-p+1 \) vectors of rational functions. By [1], the entries of these vectors have a common denominator, the polynomial \( \delta \), that is a maximal minor of \( \text{jac}((f^A)^{[i]}) \). Moreover, each numerator is a linear combination of maximal minors. Thus, these vectors are rational functions whose numerators and common denominator have degree \((p-1)(d-1)\) in \( n \) variables.

We first consider the denominator, \( \delta \), and proceed by Kronecker substitution [12, Chapter 8.4]. Since \( z_1 \) appears with degree at most \((p-1)(d-1)\), we set

\[
  z_2 = z_1^{(p-1)(d-1)+1}.
\]

Likewise, \( z_2 \) appears with degree at most \((p-1)(d-1)\) so we can set

\[
  z_3 = z_2^{(p-1)(d-1)+1} = z_1^{(p-1)(d-1)+1}.
\]

and so on until

\[
  z_n = z_1^{(p-1)(d-1)+1}.
\]

The result is a univariate polynomial whose highest degree monomial comes from

\[
  z_n^{(p-1)(d-1)} = z_1^{((p-1)(d-1)+1)^{n-1}(p-1)(d-1)},
\]

so of degree \( O((p-1)^n(d-1)^n) \). By the same Kronecker substitution, the entries of the minor corresponding to the denominator can be seen as univariate polynomials whose highest degree monomials come from

\[
  z_n^{d-1} = z_1^{((p-1)(d-1)+1)^{n-1}(d-1)}.
\]

Therefore, they are of degree \( O((p-1)^n(d-1)^n) \).

By fast multi-point evaluation techniques, we evaluate these \((p-1)^2\) entries of the minor in \( O((p-1)^n(d-1)^n) \) points in \( O^\omega((p-1)^{n+2}(d-1)^n) \) operations in the base field [12, Chapter 10.1].

We now perform a Gaussian elimination of size \((p-1) \times (p-1)\) for each of these evaluations in \( O((p-1)^n(d-1)^n) \). By the previous operations \([12, \text{Chapter 12.1}],\) it remains to interpolate the determinant as a univariate polynomial in \( O^\omega((p-1)^n(d-1)^n) \) operations [12, Chapter 10.2].

In total, the most expensive step is performing the Gaussian eliminations. Therefore, the overall complexity of computing \( \delta \) is \( O((p-1)^{n+\omega}(d-1)^n) \).
We then follow the same steps to compute each of the numerators. Essentially, this involves computing the determinants of all \( \binom{n}{p-1} \) maximal minors of \( \text{jac}((f^A)^{[1]}) \). To retrieve the numerators, we must simply compute linear combinations of these determinants and so this step is negligible to the complexity. Hence, approximating \( \binom{n}{p-1} \) by \( n^{p-1} \) as \( n \to \infty \), repeating this evaluation–interpolation method for each minor requires \( O((p-1)^n n^{p-1}(d-1)^n) \) operations.

Now that the basis \( B \) is computed, it remains to find its product with the gradient \( df_j \). Thus, by the same evaluation–interpolation techniques as before, we may compute each \( v_i \) in \( O^\ast((p-1)^n d^{-1}n^{p-1}) \) operations. This is negligible compared to the complexity of the Gaussian elimination step as before and so the overall complexity of computing \( v_1, \ldots, v_{n-p+1} \) and \( \delta \) is \( O((p-1)^{n+\omega}n^{p-1}(d-1)^n) \).

Once the initialising polynomials have been computed, the remaining steps of both algorithms designed in this paper rely on algebraic elimination algorithms. In particular we shall use the geometric resolution algorithm given in [13] in combination with the lifting algorithm of [30] to compute a parametric system whose solution set contains the set of asymptotic critical values of a given dominant polynomial mapping. Since both of the algorithms we analyse will use this framework, and in the end have essentially the same theoretical complexity, we give a full complexity analysis of Algorithm 2 and then show how the result generalises to Algorithm 1.

First, we recall the representation that is the output of geometric resolution algorithm of [13]. Consider polynomials \( g_1, \ldots, g_m, h \) in the polynomial ring \( \mathbb{K}[x_1, \ldots, x_m] \) and the zero-dimensional algebraic set \( S \) defined by \( g_1 = \cdots = g_m = 0 \), \( h \neq 0 \). Let \( D \) be the degree of this set and let \( T \) be a linear form of the input variables \( x_1, \ldots, x_m \). Then, the output of the geometric resolution algorithm is a representation

\[
\begin{align*}
q(T) &= 0 \\
q'(T)x_1 &= v_1(T) \\
&\quad \vdots \\
q'(T)x_m &= v_m(T),
\end{align*}
\]

where \( q \in \mathbb{Q}[T] \) is a univariate polynomials of degree at most \( D \) and \( v_1, \ldots, v_m \in \mathbb{Q}[T] \) are univariate polynomials of degree strictly less than \( D \). This is a representation of the set \( S \) outside of Zariski closed set \( V(q') \). We now give our main complexity result.

**Theorem 4.** Let \( f = (f_1, \ldots, f_p) \in \mathbb{K}[x]^p \) be a dominant polynomial mapping and let \( d = \max_{1 \leq i \leq p} \deg f_i \).

There exists an algorithm which, on input \( f \), computes a non-zero polynomial \( g \) in \( \mathbb{K}[c] \) such that \( K_\infty(f) \subset V(g) \) using at most

\[
O^\ast\left(p(p(d-1))^{2(p+1)(n-p)}(d+1)^{2p(p+1)}\right)
\]

arithmetic operations in \( \mathbb{K} \).

**Proof of Theorem 4.** We shall use Algorithm 2 which by Theorem 17 terminates, and returns a finite basis whose zero set has codimension at least 1 in \( \mathbb{C}^p \) and contains the set of asymptotic critical values of \( f \). First, fix some \( 1 \leq j \leq p \). Then, the first 8 steps of
Algorithm 2 are to compute the finite list of polynomials $G'$, which we denote $h_1, \ldots, h_n$, and the polynomial $\delta$. This has been analysed in Lemma 20.

We note that $G'$ consists of $n$ polynomials in $n + p$ variables and, by Theorem 17, defines a set of dimension $p$. Let $a_1, \ldots, a_p \in \mathbb{K}$ be generic elements of $\mathbb{K}$. Then, substituting $c_i$ for $a_i$ in the polynomials of $G'$ defines a zero-dimensional constructible set.

Using the geometric resolution algorithm of [13], we compute a representation of the system $h_1 = \cdots = h_n = 0$, $\text{numer}(\delta(T_1)) \neq 0$, $c_1 = a_1, \ldots, c_p = a_p$. By Lemma 19, this system has degree at most $(p(d - 1))^{n-p}(d+1)^p$. For ease of notation, we denote this degree $D$. Then, we have the representation

\[
\begin{align*}
q(T) & = 0 \\
q'(T)z_1 & = v_1(T) \\
& \vdots \\
q'(T)z_n & = v_n(T),
\end{align*}
\]

where the polynomials $q, v_1, \ldots, v_n \in \mathbb{K}[T]$ have degree at most $(p(d - 1))^{n-p}(d+1)^p$.

Now, using the lifting algorithm of [30] we obtain a parametric representation

\[
\begin{align*}
Q & = 0 \\
\frac{\partial Q}{\partial T}z_1 & = V_1 \\
& \vdots \\
\frac{\partial Q}{\partial T}z_n & = V_n,
\end{align*}
\]

where $Q, V_1, \ldots, V_n \in \mathbb{K}(c)[T]$ are polynomials in the linear form $T$ with coefficients in $\mathbb{K}(c)$. By [30, Theorem 1], the numerators and common denominators of the polynomials $Q, V_1, \ldots, V_n$ have degree at most $D$ in $c$.

Thus, with $G'_s$ defined as in Algorithm 2, this parameterises the set $V(G'_s)$. To now compute $V_j$, we must intersect with $V(z_1)$ and project onto the $c$-space. First, however, we convert our representation into one described by polynomials in $\mathbb{K}(c, T)$. To do so, we simply multiply each polynomial $Q, \frac{\partial Q}{\partial T}, V_1, \ldots, V_n$ by their respective common denominators. We denote the resulting polynomials $\tilde{Q}, \frac{\partial \tilde{Q}}{\partial T}, \tilde{V}_1, \ldots, \tilde{V}_n$ respectively. By [30, Theorem 1], these polynomials have degree at most $D$ in $c$ and degree at most $D$ in $T$. We claim that the polynomial

\[ g_j = \text{Res}_T \left( \tilde{Q}, \frac{\partial \tilde{Q}}{\partial T}, \tilde{V}_1 \right) \]

defines an algebraic set that contains $V(V_j)$. To see this, first note that the zero set of the polynomial $\text{Res}_T(\tilde{Q}, \tilde{V}_1) \in \mathbb{K}[c]$ contains all the points in the projection of the intersection with $V(z_1)$. Thus, this zero set contains $V(V_j)$ whenever this parametrisation is defined. Then, the zero set of $\text{Res}_T(\tilde{Q}, \frac{\partial \tilde{Q}}{\partial T})$ contains all the points of $V(V_j)$ where this parametrisation is not defined. Thus, the zero set of the product of these polynomials contains $V(V_j)$. One computes such a polynomial for each choice of $j$ and returns the product $g = \prod_{j=1}^{p} g_j$. 27
We now analyse the complexity of the algorithm described above. Firstly, by Lemma 20, the initialisation step of the algorithm requires at most
\[ O((p - 1)^n + \omega n^{p-1}(d - 1)^n) \]
arithmetic operations in \( \mathbb{K} \) for each choice of \( 1 \leq j \leq p \). Substituting \( \tau_1(z) \), finding the numerator and specialising \( c \) is negligible, so we now consider the complexity of the geometric resolution algorithm.

By [13, Theorem 1], computing a geometric resolution of the set \( V(G'_s) \) requires at most
\[ O(n(nL + n^\Omega)M(D)^2) \]
arithmetic operations in \( \mathbb{K} \). Assuming that \( d \geq 2 \) is fixed, we may bound the evaluation complexity, \( L \), by \( n(n + D) = O(n^{d+1}) \). Thus, by excluding logarithmic factors, we arrive at a simplification of the class
\[ O(n^{d+2}D_{p+1}). \]

Furthermore, by [30, Theorem 2], the lifting step requires at most
\[ O^\sim((nL + n^4)M(D)M(D^p) + np^2DM(D)M(D^{p-1})) = O^\sim((nL + n^4 + np^2)D_{p+1}) \]
arithmetic operations in \( \mathbb{K} \). By again assuming that \( d \geq 2 \) is fixed, we apply the same simplification of the evaluation complexity. Furthermore, assuming that \( n \geq p \), we arrive at a simpler form:
\[ O^\sim(n^{d+2}D_{p+1}). \]

The final step of importance is to compute \( 2p \) resultants. Recall that \( \tilde{Q}, \tilde{V}_1, \) and \( \tilde{\partial Q}/\partial T \) have degree at most \( D \) in \( c \) and \( T \). Thus, each Sylvester matrix has at most \( 2D \) columns and has entries of degree at most \( D \). Hence, the determinant of these matrices, the resultants we wish to compute, have degree at most \( 2D^2 \). We return to a Kronecker substitution to reduce to the bivariate case, leaving the variables \( c_1 \) and \( T \). Since the variables \( c \) each occur with degree at most \( 2D^2 \), we can set
\[ c_2 = c_1^{2D^2+1}, \ldots, c_p = c_1^{(2D^2+1)p-1}. \]

Therefore, with this substitution, we can write the entries of each Sylvester matrix as univariate polynomials in \( c_1 \) with degree in the class \( O(2^pD_{2p}) \).

By [12, Corollary 11.21], we can compute each bivariate resultant within \( O^\sim(2^pD_{2p+2}) \) arithmetic operations in \( \mathbb{K} \). We compute \( 2p \) resultants and so the overall complexity of computing the resultants is in the class
\[ O^\sim(p2^p+1D_{2p+2}). \]

In summary, the overall complexity for computing a polynomial whose zero set contains the asymptotic critical values of \( f \) is in the class:
\[ O^\sim(2^pD_{2p+2}) \]

The complexity of the resultant computation is dominant. Hence, this simplifies to the class:
\[ O^\sim\left(p(p(d - 1))^{2(p+1)(n-p)}(d + 1)^{2p(p+1)}\right). \]
Corollary 21. Let \( f = (f_1, \ldots, f_p) \in \mathbb{K}[z]^p \) be a dominant polynomial mapping and let \( d = \max_{1 \leq i \leq p} \deg f_i \). Then, Algorithm 1 returns a polynomial \( g \in \mathbb{K}[c] \) such that \( K_\infty(f) \subset V(g) \) within \( O^-(p(p(d - 1))^{2(p + 1)(n-p)(d + 1)^2p(p+1)}) \) arithmetic operations in \( \mathbb{K} \).

Proof of Corollary 21. Note that the procedure described in proving Theorem 4 can be used for Algorithm 1 with very few adjustments required. Firstly, note that the initialization step is almost identical except that we consider \( n + 1 \) polynomials in \( n + 2 \) variables rather than \( n \) polynomials in \( n + 1 \) variables. Therefore, the additional variable \( e \) found in Algorithm 1 will not change the complexity class. Furthermore, by Lemma 19, the list of polynomials considered in Algorithm 1 defines an algebraic set of degree at most \( (p(d - 1) + 2)^{n-p+1}(d+1)^p \). Comparing this to \( D \), the degree of the algebraic set defined by \( G' \) in Algorithm 2, which is equal to \( (p(d - 1))^{n-p}(d + 1)^p \), we conclude that this difference also will not change the complexity class. The last main difference is the resultant step. Indeed, we must also consider the intersection with the variety \( V(e) \). This results in the computation of one more resultant per choice of \( j \), \( \text{Res}_T(\tilde{Q}, \tilde{V}_e) \) where \( \tilde{V}_e \) is the lifted parametrisation of \( e \). However, the difference between computing \( 2p \) resultants and \( 3p \) resultants does not change the complexity class. \( \square \)

5. Applications

5.1. Solving Polynomial Optimisation Problems

In this subsection we present how to use the algorithms detailed in this paper to solve global polynomial optimisation problems.

Firstly, we review the problem we wish to solve. Consider a polynomial \( f \in \mathbb{Q}[z] \). We aim to compute the global infimum of this polynomial \( \inf_{x \in \mathbb{R}^n} f(x) = f^* \in \mathbb{R} \cup \{-\infty\} \).

We can solve this problem exactly by computing the generalised critical values of \( f \).

There are three cases:

- \( f^* \) is reached. Then, \( f^* \) is a critical value of \( f \);
- \( f^* \) is reached only at infinity, meaning that there is no minimiser \( x \in \mathbb{R}^n \) but instead a path \( x_t \in \mathbb{R}^n \) that approaches the infimum as \( \|x_t\| \to \infty \). Then, \( f^* \) is an asymptotic critical value of \( f \);
- \( f^* = -\infty \).

The procedure is as follows: We first compute an algebraic representation of the generalised critical values of \( f \). We do this by computing a polynomial whose roots contain the asymptotic critical values by using the algorithms described in this paper or in the papers [17–19]. Then, using the gradient ideal as in [11] we can similarly compute a polynomial whose roots contain the critical values of \( f \). There are algebraic elimination algorithms that compute such polynomials with rational coefficients, for example Gröbner bases [8, Chapter 2] or the geometric resolution algorithm designed in [13], since we assumed that \( f \in \mathbb{Q}[z] \). Thus, after finding a common denominator, we may assume
these polynomials have integer coefficients. Then, we may use a real root isolation algorithm such as in [25], based on Descartes’ rule of sign [2, Theorem 2.44], to compute isolating intervals with rational endpoints for all real roots of these polynomials.

Let $C = \{c_1, \ldots, c_k\} \subset \mathbb{R}$ be the finite set of real algebraic numbers that are the real roots of the above polynomials. Then, the set $C$ contains the generalised critical values of $f$. By [21, Theorem 3.1], the polynomial $f$ with restricted domain $f : \mathbb{R}^n \setminus f^{-1}(K(f)) \to \mathbb{R} \setminus K(f)$ is a fibration over each connected component of $\mathbb{R} \setminus K(f)$. Therefore, since $C$ is finite, the same fibration property applies to the restriction $f : \mathbb{R}^n \setminus f^{-1}(C) \to \mathbb{R} \setminus C$. Hence, to decide the emptiness of each connected component of $\mathbb{R} \setminus C$, it is sufficient to decide the emptiness of one fibre for each connected component.

After computing the isolating intervals for the elements of $C$, we may now choose rational numbers $r_1, \ldots, r_k$ so that

$$r_1 < c_1 < r_2 < \cdots < r_k < c_k.$$

We must assess the emptiness of the fibres of these values. We do so using the algorithm designed in [29]. We consider, for $0 \leq i \leq k$, the ideal $(f - r_i)$. This algorithm requires a radical ideal such that $V(f - r_i)$ is smooth and equidimensional. Since $r_i$ is outside of these isolating intervals, we have that $V(f - r_i)$ is smooth and equidimensional. Furthermore, since $V(\sqrt{(f - r_i)}) = V(f - r_i)$, we may instead consider the square-free part of $(f - r_i)$, $\sqrt{(f - r_i)}$, to decide the emptiness of $V_\mathbb{R}(f - r_i) = V(f - r_i) \cap \mathbb{R}^n$.

Firstly, if $V_\mathbb{R}(f - r_0)$ is non-empty then we must be in the third case and so $f^* = -\infty$. For the remaining two cases, let $i$ be the least index such that $V_\mathbb{R}(f - r_i)$ is non-empty, if such an index exists. If $r_i$ is greater than the least critical value, which one may decide from the isolating intervals, then the least critical value is the minimum of $f$. Else, $c_{i-1}$ corresponds to an asymptotic critical value and is the infimum of $f$. If such an index does not exist, then the least critical value of $f$ is the minimum and if $f$ does not have any critical values, then the infimum is $c_k$.

The complexity of the algorithm for polynomial optimisation described is as follows. For a polynomial $f \in \mathbb{Q}[z]$ of degree $d$, we first compute the polynomial representation of $K(f)$. By Theorem 3 and [27, Theorem 4.3], we can compute this within $O^*(n^7d^{4n})$ arithmetic operations in $\mathbb{Q}$. By [17, Corollary 4.4], $f$ has at most $d^n$ generalised critical values. Thus, with $\beta$ bounding the bit-size of the input polynomial, isolating the real roots with the algorithm designed in [25] requires $O(\beta d^{4n})$ operations. We must then choose at most $d^n + 1$ points in $\mathbb{Q}$, the $r_1, \ldots, r_{d^n}$ as above, and decide the emptiness of each $V_\mathbb{R}(f - r_i)$. This requires the use of the algorithm designed in [29] at most $d^n$ times with each computation requiring $O(n^7d^{3n})$ operations. Thus, one can compute an isolating interval for the infimum of a polynomial $f \in \mathbb{Q}[z]$ of degree $d$ in $O^*(n^7d^{4n})$ arithmetic operations in $\mathbb{Q}$.

Example 22. Consider the polynomial $f = z_1^2z_2^2 + 2z_1z_3^2 + z_4^2 + z_1^2 + 3z_1z_2 + 2z_2^2$. First, we compute the set of generalised critical values. Note that in this simple example it is possible to find exactly the real algebraic numbers that contain the generalised critical values because the degrees of the polynomials we compute in our algorithms are small. We find that $K_0(f) = \{0\}$ and using Algorithm 1 we find $K_\infty(f) \subset \{-\frac{1}{3}\}$. Now, to show that $f^* = -\frac{1}{3}$ one must first show that $f$ is bounded from below. To do so, decide the emptiness of the real variety $V_\mathbb{R}(f - r)$ for some real number $r < -\frac{1}{3}$. For example, we can choose $r = -1$ and find that this variety is indeed empty. Finally, one must show
that $-\frac{1}{2}$ truly is an asymptotic critical value as Algorithm 1 computes a superset of the asymptotic critical values. Thus, one shows that $f$ takes values less than $0$ by once again deciding the emptiness of a fibre. So, consider the variety $V_R(f + \frac{1}{2})$ and find that it is not empty. This shows that $f$ takes values less than $0$ and by the fibration property satisfied by the generalised critical values we conclude that the infimum of $f$ is $-\frac{1}{2}$.

**Example 23.** Consider the polynomial $f = z_1^3 + z_1^2 z_2 - 2z_1 z_2 + 1$. We find that $K_0(f) = \{1\}$ and $K_\infty(f) \subset \{0\}$. We first test the third case. Take a value less than $0$, for example $-1$, and decide the emptiness of $V_R(f + 1)$. We find that this fibre is not empty and so by the fibration property, we conclude that $f^* = -\infty$.

For more information on solving polynomial optimisation problems, we refer to [14, 28, 31].

5.2. Deciding the emptiness of semi-algebraic sets defined by a single inequality

In this subsection, we continue to explore the applications of algorithms computing generalised critical values. Let $f \in \mathbb{Q}[z]$ be a polynomial with degree $d$ and consider the semi-algebraic set $S$ defined by the single inequality $f > 0$. The goal is to test the emptiness of the set $S$ and in the case that $S$ is not empty to compute at least one point in each connected component. There exists $e \in \mathbb{Q}^+$ small enough such that the problem is reduced to computing at least one point in each connected component of the real algebraic set $V_R(f - e)$. Such an $e$ is small enough in this sense if it is less than the least positive generalised critical value of the map $z \in \mathbb{R}^n \rightarrow f(z) \in \mathbb{R}$, we refer to [27, Theorem 5.1]. To decide when this is the case, one computes isolating intervals for the generalised critical values by [2, Algorithm 10.63]. Once an appropriate $e$ has been chosen, it remains to compute at least one point in each connected component of $V_R(f - e)$. This may be accomplished using the algorithm designed in [29]. To apply this algorithm, we require that $(f - e)$ is radical and $V(f - e)$ is equidimensional and smooth. Since $e$ is away from any generalised critical values we have that $V(f - e)$ is equidimensional and smooth. Moreover, if $(f - e)$ is not radical, we may simply take the square-free part instead as $V(\sqrt{(f - e)}) = V(f - e)$.

As in the previous application, the complexity of computing isolating intervals for all real generalised critical values is in the class $O^*(n^7d^m)$. After choosing an appropriate rational number $e$, it remains to apply the algorithm designed in [29]. This requires $O(n^7d^m)$ operations. Therefore, the overall complexity of deciding the emptiness of the semi-algebraic set defined by $f > 0$ is in the class $O^*(n^7d^m)$. Moreover, in the case where this set is not empty, at least one point in each connect component is computed.

**Example 24.** Consider the polynomial $f = z_1^3(1 - z_2) - (z_1 z_2^2 - 1)^2$. Again, in this simple example we obtain polynomials of degree at most 2 from our algorithms and so we can give explicitly the set containing the generalised critical values. The polynomial giving the asymptotic critical values is $e$ while for the critical values it is $229c^3 - 202c - 27$. Hence, we find that $K(f) \subset \{0, 1, \frac{27}{229}\}$. We note that the value $1$ is a critical value, hence we may decide immediately that the semialgebraic set defined by $f > 0$ is nonempty. Now, to compute at least one sample point in each connected component of this set, we must choose a suitable fibre to investigate. Thus, we choose a rational value greater than $0$ and less than the least generalised critical value, such as $\frac{1}{2}$, and use the algorithm in [29] to compute sample points for each connected component of $V_R(f - \frac{1}{2})$. We may do so because $(f - \frac{1}{2})$ is a radical ideal.
6. Experiments

The algorithms discussed in this paper have initially been implemented in the MAPLE computer algebra system using a combination of FGb [10] and MSolve [5], both implemented in C, to perform the Gröbner basis computations as well as to compute the degree of various objects described in the tables below. In this section, we present the experimental results of these implementations with computations performed on a computing server with 1536 GB of memory and an Intel Xeon E7-4820 v4 2GHz processor. Exceptionally, we also present the timings of these implementations when given polynomials from practice as input with computations performed on a computing server with 754 GB of memory and an Intel Xeon Gold 6244 3.6GHz processor. The computations were performed under finite fields before reconstructing the rational polynomial whose roots are the asymptotic critical values. For our timings, the entry $\infty$ has been given in the cases when the algorithm has not terminated within 2 days. Additionally, if a computation could not be performed we give the entry N/A. We give the remainder of the entries correct up to two significant figures.

We compare our algorithms to the one derived from the work of [21, Section 4] combined with our first element of randomisation from Lemma 7. We denote the resulting algorithm $acv_0$.

Our algorithms outperform $acv_0$ in every tested circumstance as expected. Additionally, in general, Algorithm 2 is faster than Algorithm 1. However, the polynomial system returned by Algorithm 2 can be of higher degree than that returned by Algorithm 1. Concerning the particular case we study in this section, the case $p = 1$, we have that the polynomial returned by Algorithm 1 is a factor of the output of Algorithm 2.

We also test how Algorithm 1 behaves when we perform the saturation step with two different methods. While both methods use Gröbner bases, the first method given in [8, Theorem 4.4.14] introduces a new variable which acts as the inverse of the polynomial one wishes to saturate by. The second method, given by Bayer [3] and described in [9, Exercise 15.41], works for homogeneous ideals. Thus, we also introduce a new variable to first homogenise our ideal. Then, to perform the saturation by the ideal $\langle z_1 \rangle$, we factor out all powers of $z_1$ from the homogeneous ideal. Setting the introduced variable to 1 returns a basis for the saturated ideal. We find that for generic dense polynomials, the first method is the best. However, for the particular families of polynomials with asymptotic critical values that we test out algorithms with the Bayer method is noticeably faster.

We give three families of polynomials that have asymptotic critical values for the purpose of testing our algorithms. For $n \geq 2$, let

$$f_n = z_1^2 + \sum_{i=2}^{n} (z_1 z_i - 1)^2, \quad g_n = \sum_{i=1}^{n} \prod_{j=1}^{n} \frac{z_j^2}{z_i^2}, \quad m_n = \sum_{i=1}^{n} \prod_{j=1}^{i} z_j^{-i}.$$  

For $n \geq 2$, each of these polynomials has an asymptotic critical value at 0. For $n \geq 3$, $f_n$ also has an asymptotic critical value at $n$. Additionally, we compare our algorithms with random dense polynomials, which do not have asymptotic critical values. To do so, we introduce the following notation. For a random dense polynomial in $k$ variables and degree $s$ we write $d_{sk}$.
From Table 1, we see that Algorithm 1 and Algorithm 2 surpass acv0 in all instances by a large factor. Moreover, for our specific polynomial families, the Bayer method of saturation is faster. However, for generic dense polynomials, the Bayer method is slower in all cases except $d = 2$. Additionally, for all tested examples, Algorithm 2 is the quickest. In particular, for generic dense polynomials with degree 4 in 6 variables, Algorithm 2 finishes within a few seconds while neither of the other two algorithms terminated within 48 hours.

Next, in Table 2, we present the timings of Algorithm 2 with polynomials coming from practice. For these computations, we now use MSOLVE [5] to perform the saturation step and when applicable we complete the computation with FGb [10]. The polynomials $f_{1,2}, f_{1,3}, f_{2,2}$ and $f_{2,3}$ are given in [20] while the polynomials $s_1, s_2$ and $s_3$ can be found on the webpage https://www-polsys.lip6.fr/~ferguson/sauter.html.

Finally, in Table 3, we give our results on the degree of hypersurfaces containing the asymptotic critical values. We compare the bounds given for Algorithm 2 in Theorem 3 to the degree of the output of Algorithm 2 and to the degree of the ideal generated by the basis $G'$ as computed in Algorithm 2. We compute the latter degree by using the $fgb_{\text{hilbert}}$ function of the FGb library [10] to find the numerator of the Hilbert series of $\mathbb{K}[z]/\langle G' \rangle$ and evaluating this at 1. We note that for the case of random dense
polynomials, the degree of the ideal generated by the basis $G'$ equals the bound we give on the degree of the asymptotic critical values.

| Polynomial | Algo. 2 | $G$ | $K_{\infty}(f)$ |
|------------|---------|-----|----------------|
| $f_5$      | 405     | 4   | 3              |
| $f_{25}$   | 1412147682405 | 4   | 3              |
| $g_5$      | 21609   | 90  | 1              |
| $g_6$      | 93934323 | 138 | 1              |
| $m_4$      | 43904   | 124 | 1              |
| $m_5$      | 25920000 | 572 | 1              |
| $d_2n_{20}$| 3       | 3   | 0              |
| $d_2n_{100}$| 3      | 3   | 0              |
| $d_3n_5$   | 64      | 64  | 0              |
| $d_3n_{17}$| 256     | 256 | 0              |
| $d_4n_4$   | 135     | 135 | 0              |
| $d_4n_6$   | 1215    | 1215| 0              |

Table 3: Comparison of degree bounds and degree reached during the algorithm.

Acknowledgements. The authors are supported by the ANR grants ANR-18-CE33-0011 SESAME, ANR-19-CE40-0018 DE RERUM NATURA and ANR-19-CE48-0015 ECARP, the PGMO grant CAMISADO and the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement N. 813211 (POEMA).

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