On Triangle Counting Parameterized by Twin-Width

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Abstract

In this report we present an algorithm solving triangle counting in time \(O(d^3n + m)\), where \(n\) and \(m\), respectively, denote the number of vertices and edges of a graph \(G\) and \(d\) denotes its twin-width, a recently introduced graph parameter. We assume that a compact representation of a \(d\)-contraction sequence of \(G\) is given.

1 Introduction

Recently, the parameter twin-width was introduced by Bonnet et al. [5] over a series of papers. The twin-width of a graph \(G\), denoted by \(\text{tww}(G)\), roughly measures the distance of a graph from being a cograph, and has demonstrated to have various advantages that make it stand out as an important graph parameter. One of the main motivations to study twin-width comes from the fact that the class of bounded twin-width contains several interesting and diverse graphs as bounded boolean-width, bounded rank-width, bounded clique-width, unit interval, proper-minor closed or also \(K_t\)-minor free graphs [4]. On the algorithmic side, FO model checking is FPT on classes with bounded twin-width [5]. Moreover, various intractable problems like \textsc{independent set}, \textsc{dominating set}, and \textsc{clique} can be solved in time \(2^{O(n)}n\)-time [4]. It is also shown there that graphs of twin-width \(d\) admit an interval biclique partition of size \(O(d n)\). Using such an edge-partition, they show how to solve \textsc{all-pairs shortest paths} in time \(O(n^2 \log n)\).

In this report, we show how to solve the triangle counting problem on graphs with \(n\) vertices and \(m\) edges in time \(O(d^3n + m)\), with \(d\) denoting the twin-width of the graph. A graph \(G\) has twin-width at most \(d\) if there is a so-called \(d\)-contraction sequence of \(G\), which is defined in Section 2. Deciding if the twin-width of a graph is at most 4 is NP-hard [2]. Thus, we assume that a \(d\)-contraction sequence is given together with the graph \(G\).

The currently fastest (unparameterized) algorithm for triangle counting is due to Alon, Yuster, and Zwick [1], that have showed that triangle counting can be solved in time \(O(n^{3+\varepsilon})\) using fast matrix multiplication where \(2 \leq \omega < 2.372863\). For sparse graphs one can solve triangle counting in time \(O(m^{1.41}) = O(m^{1.41})\). It is conjectured that there is no \(O(n^{3-\varepsilon})\) time combinatorial algorithm. Within a parameterized framework, Courcelle, Ducoffe, and Popa gave an algorithm that runs in time \(O(cw^2(n + m))\) where \(cw\) denotes the clique-width of the input graph [8] whereas Kratsch and Nelles obtained an \(O(mw^{\omega-1} n + m)\) where \(mw\) is the modular-width of the input graph [8].

2 Preliminaries

All graphs considered are finite, undirected, and simple. We refer to [7] for the basic concepts and notions of graph theory. In particular, given a graph \(G = (V, E)\), we denote by \(V(G)\) its vertex set and by \(E(G)\) its edge set. For a graph \(G = (V, E)\) and a subset \(X \subseteq V\), we define the induced subgraph over the vertex set \(X\) as \(G[X] = (X, E')\) where \(E' = \{\{u, v\} \in V \mid u, v \in X\}\). We refer by \(N(v)\) the set of neighbors of a vertex \(v \in V(G)\), i.e., \(N(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}\). Given a vertex \(v \in V(G)\), the degree \(d(v)\) of \(v\) is the number of neighbors of \(v\), i.e., \(d(v) = |N(v)|\). Furthermore, we say that two vertices \(u, v \in V(G)\) are twins if \(N(v) \setminus \{u, v\} = N(u) \setminus \{u, v\}\). For any integers \(j, k \in \mathbb{N}\), we denote \([j, k] = \{j, j + 1, \ldots, k\}\) and in particular \([k] = [1, k] = \{1, 2, \ldots, k\}\).
**Twin-width.** A trigraph \( G \) is a triple \( G = (V, E, R) \) where \( E \) and \( R \) are two disjoint sets of edges. We refer to an edge in \( E \) as a black edge and to an edge in \( R \) as a red edge. By setting \( R = \emptyset \), one can interpret any graph \((V,E)\) as a trigraph \((V,E,\emptyset)\). A trigraph \( G = (V,E,R) \) with maximum red degree \( d \), i.e., maximum degree in the graph \((V,R)\), is called a \( d \)-trigraph. Furthermore, for any trigraph \( G = (V,E,R) \) and any vertex \( v \in V \), we denote by \( N_R(v) \) the set of red neighbors of \( v \), i.e., \( N_R(v) = \{u \in V \mid \{u,v\} \in R \} \).

For a trigraph \( G = (V,E,R) \) and two vertices \( u,v \in V \), we define \( G/u,v = (V', E', R') \) as the trigraph obtained from \( G \) by contracting \( u \) and \( v \) into a new vertex \( w \) and after updating the edge sets in the following way: A vertex \( x \) is linked to the new vertex \( w \) in \( G/u,v \) by a black edge if and only if \( x \) is linked to \( u \) and to \( v \) in \( G \) by a black edge. Moreover, \( x \) is not adjacent to \( w \) if \( x \) is neither adjacent to \( u \) nor to \( v \) in \( G \). In all other cases \( x \) is linked to \( w \) by a red edge. Formally, \( V' = (V \setminus \{u,v\} \cup \{w\}) \) with \( \{w,x\} \in E' \) if and only if \( \{u,x\} \in E \) and \( \{v,x\} \in E \); \( \{w,x\} \notin E' \cup R' \) if and only if \( \{u,x\} \notin E \cup R \) and \( \{v,x\} \notin E \cup R \) and \( \{w,x\} \in R' \) otherwise. All edges that are not incident to \( u \) nor to \( v \) remain unchanged in \( G/u,v \). Notice that \( u \) and \( v \) do not need to be adjacent. For any integer \( d \geq 0 \), if both \( G \) and \( G/u,v \) are \( d \)-trigraphs, \( G/u,v \) is called a \( d \)-contraction. A trigraph is \( d \)-collapsible if there exists a sequence of \( d \)-contractions which contracts \( G \) to a single vertex. The minimum integer \( d \geq 0 \) such that \( G \) is \( d \)-collapsible is called the twin-width of \( G \), denoted by \( \text{tww}(G) \). In other words, for any graph \( G \) with \( \text{tww}(G) = d \), there exists a sequence of trigraphs \( G_n, G_{n-1}, \ldots, G_1 \) with \( G_n = G \), \( G_1 = K_1 \) (the clique of size 1) and \( G_k \) is a \( d \)-contraction of \( G_{k+1} \) for \( k \in [n-1] \). To represent such a contraction sequence efficiently, it is sufficient to only specify the vertices that get contracted:

**Definition 1** (Compact representation of a \( d \)-sequence). Let \( G = G_n, G_{n-1}, \ldots, G_1 = K_1 \) be a \( d \)-contraction sequence of an \( n \)-vertex graph \( G = (V,E) \) with \( V = \{v_1, v_2, \ldots, v_n\} \). Then, we call \( (v_{k+1}, v_k) \) a compact representation of a \( d \)-contraction sequence. The graph \( G_{k+1} \) results from \( G_k \) by contracting the two vertices \( v_{k+1} \) and \( v_k \) into a new vertex \( v_{2n-k+1} \) for \( k \in [2,n] \).

Finally, for a vertex \( v \in V(G_k) \), we denote by \( v(G) \) the subset of vertices in \( G \) eventually contracted into \( v \) in \( G_k \) and we denote \( G' = G[v(G)] \).

3 Algorithm

In the **triangle counting** problem, we are given a graph \( G = (V,E) \) and we are asked to count the number of triangles in \( G \), that is, the number of elements in the set \( T(G) = \{\{x,y,z\} \subseteq (V)^3 \mid \{x,y\}, \{y,z\}, \{z,x\} \subseteq E \} \). We will prove the following theorem:

**Theorem 1.** Let \( G = (V,E) \) be a graph with \( \text{tww}(G) = d \), and let a compact representation of a \( d \)-contraction sequence as defined in Definition 1 be given. Then, one can solve **triangle counting** in time \( O(d^2 n + m) \).

Using the compact representation of the \( d \)-contraction sequence, we gradually construct the graphs \( G = G_n, G_{n-1}, \ldots, G_1 = K_1 \). Consider a trigraph \( G_k = (V_k, E_k, R_k) \) of the contraction sequence of \( G \) for \( k \in [n] \) and a fixed triangle \( \{a,b,c\} \in G \) with \( a,b,c \in V(G) \). The vertices of the triangle can be in subgraphs corresponding to one, two, or three vertices of \( V_k \). More formally, we observe the following:

**Observation 1.** Let \( G = G_n, \ldots, G_1 = K_1 \) be a contraction sequence of a graph \( G \), let \( G_k = (V_k, E_k, R_k) \) a trigraph of the contraction sequence, and let \( \{a,b,c\} \) be a triangle in \( G \). Then, exactly one of the following statements is true (after possibly reordering \( a, b, \) and \( c \)):

- (i) \( a \in x(G), b \in y(G), c \in z(G) \) with \( \{x,y\}, \{y,z\}, \{z,x\} \subseteq E_k \)
- (ii) \( a \in x(G), b \in y(G), c \in z(G) \) with \( \{x,y\}, \{y,z\} \subseteq E_k, \{z,x\} \subseteq R_k \)
- (iii) \( a \in x(G), b \in y(G), c \in z(G) \) with \( \{x,y\} \subseteq E_k, \{y,z\}, \{z,x\} \subseteq R_k \)
- (iv) \( a \in x(G), b \in y(G), c \in z(G) \) with \( \{x,y\}, \{y,z\} \subseteq E_k \)
- (v) \( a \in x(G) \) and \( b, c \in y(G) \) with \( \{x,y\} \subseteq E_k \)
- (vi) \( a \in x(G) \) and \( b, c \in y(G) \) with \( \{x,y\} \subseteq R_k \)
- (vii) \( a, b, c \in x(G) \) for \( x \in V(G_k) \)

Let \( G_{k-1} = G_k/u,v \). A triangle \( T \) of \( G \) might transition from a case in \( G_{k-1} \) to a different case in \( G_k \), if \( T \) consists of vertices in \( u(G) \) or \( v(G) \). If \( T \) consists of vertices in both \( u(G) \) and \( v(G) \), one vertex less is needed in \( G_{k-1} \) to specify \( T \). If \( T \) does only admit a non-empty cut with one of the
Cases that are stored in the variable $t$ in the invariant are marked by a star. Selfloops are omitted. Transitions from an unmarked case to a marked case are represented by thick arrows.

![Diagram of case transitions.](image)

![Illustration of the different cases.](image)

Figure 1: Possible case transitions of a triangle in $G$ from $G_k$ to $G_{k-1}$ as described in Observation III. For every other vertex $y$ in $G_k$, the number of edges of the subgraphs of $G$ that get contracted into a vertex $x$ in $G_k$, i.e., the values $n_x = |V(G_x^*)|$ and $m_x = |E(G_x^*)|$ for each vertex $x \in V_k$. Also, we store for each red edge $\{x, y\} \in R_k$, the number of edges between $G^*$ and $G^*$, i.e., $\mu_{x,y} = |\{(a, b) \in E \mid a \in x(G), b \in y(G)\}|$.

In each iteration $k$ of the algorithm, the two vertices $v_i$ and $v_j$, given in the compact representation of the contraction sequence, are contracted to form the trigraph $G_{k-1}$. The algorithm updates the auxiliary values $n_x$, $m_x$, and $\mu_{x,y}$ that have changed, for all $x, y \in V_k$. Informally, the algorithm then counts a triangle of $G$ whenever it becomes a case marked by a star in Observation III for the first time. Whenever a triangle is of a marked case, it cannot transition back to an unmarked case and, eventually, all triangles will be of Case (vii). Note, that for $G = G_0$, every triangle is of Case (i).

More precisely, the transition from Case (ii) to Case (ii) is dealt by the main Algorithm III. In addition, the procedure TriCOUNTONENEIGHBOR focuses on the transitions from Case (ii) and Case (iii) to Case (iii) whereas the procedure TriCOUNTTWOEIGHORS handles the transitions from Case (ii) and Case (iii) to Case (iii) and from Case (iii) to Case (iii).

Consider a trigraph $G_k$ that will be contracted into $G_{k-1} = G_k / v_i, v_j$ according to the contraction sequence. For simplicity, we define $u := v_i$ and $v := v_j$ as the two vertices in $V(G_k)$ that get contracted into the new vertex $w := v_{2n-k+1}$ of $G_{k-1}$.

For the vertex $w \in V(G_{k-1})$, the number of vertices in $G^w$ is the sum of these numbers in $G^w$ and $G^w$. For the number of edges, we also need to add the number of edges between $G^w$ and $G^w$. Therefore, $n_w = n_u + n_v$ and

$$m_w = \begin{cases} m_u + m_v + n_u \cdot n_v & \text{if } \{u, v\} \in E_k, \\ m_u + m_v + \mu_{u,v} & \text{if } \{u, v\} \in R_k, \\ m_u + m_v & \text{otherwise.} \end{cases}$$

For every other vertex $x \in V(G_{k-1}), x \neq w$, these values remain unchanged. Finally, for any vertex $x \in V(G_{k-1})$, such that $\{w, x\} \in R(G_{k-1})$ the number of edges between $G^w$ and $G^w$ can be computed as follows:

$$\mu_{w,x} = \begin{cases} n_u \cdot n_x & \text{if } \{u, x\} \in E_k \text{ and } \{v, x\} \notin (E_k \cup R_k), \\ n_v \cdot n_x & \text{if } \{u, x\} \notin (E_k \cup R_k) \text{ and } \{v, x\} \in E_k, \\ \mu_{u,x} & \text{if } \{u, x\} \in R_k \text{ and } \{v, x\} \notin (E_k \cup R_k), \\ \mu_{v,x} & \text{if } \{u, x\} \notin (E_k \cup R_k) \text{ and } \{v, x\} \in R_k, \\ \mu_{u,x} + n_u \cdot n_x & \text{if } \{u, x\} \in R_k \text{ and } \{v, x\} \in E_k, \\ \mu_{v,x} + n_u \cdot n_x & \text{if } \{u, x\} \in E_k \text{ and } \{v, x\} \in R_k, \\ \mu_{u,x} + \mu_{v,x} & \text{if } \{u, v\} \in R_k \text{ and } \{v, x\} \in R_k. \end{cases}$$

For every other vertex $y \in V(G_{k-1}), y \neq w$, such that $\{x, y\} \in R(G_{k-1})$, the value $\mu_{x,y}$ remains unchanged.
We give a pseudocode of the algorithm below. For algorithmic purposes, we assume that we are given a graph \( G = (V, E) \) that will, over the course of the algorithm, be updated into the successive trigraphs defined by the contraction sequence. Similarly, the variable \( t \) in the algorithm represents the number of triangles in \( G \) computed so far. The procedure \text{UpdateAuxiliaryValues} is not given but is explained previously whereas the procedures \text{TriCountOneNeighbor} and \text{TriCountTwoNeighbors} will be described in the next paragraphs. Finally, \text{UpdateGraph} performs the actual contraction of the graph \( G \). The pseudocode of this procedure is omitted.

\section*{Algorithm 1}

\begin{itemize}
  \item \textbf{Input:} A graph \( G = (V, E) \) and a compact representation of a contraction sequence \((v_i, v_j)_{n\geq k\geq 2}\)
  \item \textbf{Output:} The number of triangles \( t \) in the graph \( G \)
\end{itemize}

1: \( t := 0 \) \hspace{1cm} \triangleright \text{number of triangles in } G \text{ of Case (i), (ii), (vi), and (vii)}
2: \( R = \emptyset \)
3: for every vertex \( x \in V \) do
4: \quad \( n_x := 1 \) \hspace{1cm} \triangleright \text{number of vertices in } G_x
5: \quad \( m_x := 0 \) \hspace{1cm} \triangleright \text{number of edges in } G_x
6: for \( n \geq k \geq 2 \) do
7: \quad \( u := v_{ik} \)
8: \quad \( v := v_{jk} \)
9: \quad \( w := v_{2n-k+1} \)
10: \text{UpdateAuxiliaryValues}(G, u, v, w)
11: \quad if \( \{u, v\} \in E \) then
12: \quad \quad \( t = t + n_u \cdot m_v + n_v \cdot m_u \) \hspace{1cm} \triangleright \text{Case (v) } \rightarrow \text{Case (vii)}
13: \quad \quad \text{for each } x \in N_R(w) \text{ do}
14: \quad \quad \quad \text{TriCountOneNeighbor}(G, u, v, w, x)
15: \quad \quad \text{TriCountTwoNeighbors}(G, u, v, w, x)
16: \quad \text{UpdateGraph}(G, u, v, w)
17: \text{return } t
\end{itemize}

In the procedure \text{TriCountOneNeighbor}, we focus on the red edge between the newly introduced vertex \( w \) and one of its red neighbors \( x \in N_R(w) \). We then consider the different edges between \( u, v, x \) to detect triangles in \( G \) that transition from Case (ii) and Case (vi) to Case (i). Finally, in the procedure \text{TriCountTwoNeighbors} we focus on the edges between the newly introduced vertex \( w \) and two of its neighbors \( x, y \in N_R(w) \). We then consider the different edges between \( u, v, x \) and \( y \) to detect triangles in \( G \) that transition from Case (i) and Case (ii) to Case (vi) and from Case (ii) to Case (vii).
1: procedure TriCountTwoNeighbors($G, u, v, w, x$)
2: for each $y \in V$ such that $\{x, y\} \in R$ and $\{w, y\} \in E$ do
3: if $\{u, x\} \in E$ then
4: $t = t + \mu_{x,y} \cdot n_u$
5: else if $\{v, x\} \in E$ then
6: $t = t + \mu_{x,y} \cdot n_v$
7: for each $y \in V$ such that $\{w, y\} \in R$ do
8: if $\{x, y\} \in E$ then
9: if $\{u, x\} \in E$ and $\{u, y\} \in E$ then
10: $t = t + n_u \cdot n_x \cdot n_y$
11: else if $\{v, x\} \in E$ and $\{v, y\} \in E$ then
12: $t = t + n_v \cdot n_x \cdot n_y$
13: if $\{u, x\} \in R$ and $\{u, y\} \in E$ then
14: $t = t + \mu_{u,x} \cdot n_y$
15: else if $\{v, x\} \in R$ and $\{v, y\} \in E$ then
16: $t = t + \mu_{v,x} \cdot n_y$
17: if $\{u, x\} \in E$ and $\{u, x\} \in R$ then
18: $t = t + \mu_{u,x} \cdot n_x$
19: else if $\{v, x\} \in E$ and $\{v, y\} \in R$ then
20: $t = t + \mu_{v,x} \cdot n_x$
21: else if $\{x, y\} \in R$ then
22: if $\{u, x\} \in E$ and $\{u, y\} \in E$ then
23: $t = t + \mu_{x,y} \cdot n_u$
24: else if $\{v, x\} \in E$ and $\{v, y\} \in E$ then
25: $t = t + \mu_{x,y} \cdot n_v$

We have now described the algorithm and can prove Theorem 1.

Proof of Theorem 1. Given the compact representation of the $d$-sequence $(v_{1k}, v_{2k})_{k \geq 2}$, the algorithm generates iteratively the contraction sequence $G = G_n, G_{n-1}, \ldots, G_1 = K_1$ with $G_{k-1} = G_k/v_{1k}, v_{2k}$ using the procedure UpdateGraph at the end of each iteration. The values $n_x, m_v$, and $\mu_{x,y}$ are updated by the procedure UpdateAuxiliaryValues in each iteration as described in the previous paragraph.

To prove that the final value of $t$ is equal to the number of triangles in $G$, we will prove that the following invariant is true at the beginning of each iteration, i.e., for each graph $G_k = (V_k, E_k, R_k)$ in the contraction sequence for $k \in [n]$:

\[ |T(G)| = t_k + \sum_{\{x, y\} \in E_k} n_x n_y + \sum_{\{x, y\} \in R_k} \mu_{x,y} n_y + \sum_{\{x, y\} \in E_k} (n_x m_y + m_x n_y) \]

We denote by $t_k$ the current value of $t$ at the start of iteration $k$ (and $t_1$ the final value after iteration $k = 2$). Recall that $|T(G)|$ denotes the number of triangles in $G$. For $k = n$, the value of $t_n$ is initialized to zero since $R_n = \emptyset$, $m_x = 0$, and $n_x = 1$ for all $x \in V_n$. Therefore, the invariant simplifies to the second summand only, which is indeed the desired number of all triangles in $G$. We will show that the value of the invariant will never change. Thus, for $i = 1, \ldots, n$, it then holds that $|T(G)| = t_i + 0 + 0 + 0$ and the correctness of Algorithm 1 follows.

As described in Observation 1, we distinguish seven cases of a possible occurrence of a triangle of $G$ in $G_k$. In the beginning, all triangles in $G$ are of Case 1 but some may change from a case to another one whenever $G_k$ gets contracted to $G_{k-1}$. For a fixed triangle, all possible case transitions are depicted in Figure 1. Notice that the triangles of $G$ of Case 1, 5, or 6, are counted directly by the corresponding sums in the invariant. We are left to show that the (current) value of $t_k$ is indeed the count of all triangles of $G$ that appear in $G_k$ of Case 2, 3, 4, 7, or 8. Notice that once a fixed triangle is of one of the latter cases, this triangle can never disappear back to an unmarked case.
By induction, we can assume that the invariant is true for $G_k$. To prove the invariant for $k-1$, we keep track of all triangles whose case changes from $G_k$ to $G_{k-1}$ regarding Observation [1]. Note that we only need to consider the triangles that are of a case that is not marked by a star in $G_k$, but in a case that is marked in $G_{k-1}$. Let $G_{k-1} = G_k / u, v$ and let $w$ be the new vertex of $G_{k-1}$.

Case (7) to Case (7b): Let $\{a, b, c\}$ be a triangle in $G$ that is of Case (7) in $G_k$, but of Case (7b) in $G_{k-1}$. This implies that there exist $x, y \in V_{k-1}$ with $a \in \omega(G)$, $b \in \chi(G)$, and $c \in \gamma(G)$ and $(w, x), (w, y) \in R_k$. Since $w$ is the contraction of $u$ and $v$, it holds that either $a \in \omega(G)$ with $(u, x), (u, y) \in R_k$ or $a \in \omega(G)$ with $(\{u, v\}, \{v, y\}) \in R_k$. In the former case, it is counted in the procedure TriCountTwoNeighbors, Line 10. In the latter case it is counted in Line 22.

Case (7b) to Case (7b): This implies that there exist $x, y \in V_{k-1}$ with $a \in \omega(G)$, $b \in \chi(G)$, and $c \in \gamma(G)$. Let us first assume that $(x, y) \in R_{k-1}$. Since $\{a, b, c\}$ is a triangle of Case (7b) in $G_k$, it holds that $\{u, x\}, \{u, y\} \in E_k$ or $\{v, x\}, \{v, y\} \in E_k$. In the former case, it is counted in the procedure TriCountTwoNeighbors, Line 14, and in the latter case, in Line 16. Now assume that $(x, y) \in E_{k-1}$, i.e., $\{w, x\}, \{w, y\} \in R_{k-1}$. Since $\{a, b, c\}$ is a triangle of Case (7b) in $G_k$, it now holds that either $(u, x) \in R_k$ and $(u, y) \in E_k$ (counted in the procedure TriCountTwoNeighbors, Line 24, or $(v, x) \in R_k$ and $(v, y) \in E_k$ (Line 19). Line 20. Note that since $(u, x), (u, y) \in R_{k-1}$, it cannot be that the first two or the last two cases occur simultaneously.

Case (7b) to Case (7b): Suppose there are $x, y \in V_{k-1}$ with $a \in \omega(G)$, $b \in \chi(G)$, and $c \in \gamma(G)$. Let us first assume that $(x, y) \in R_{k-1}$ and for the incident red edges of $G_k$ to $R_{k-1}$, it now holds that either $(u, x) \in R_k$ and $(u, y) \in E_k$ or $(v, x) \in R_k$ and $(v, y) \in E_k$. The former is counted in the procedure TriCountTwoNeighbors, Line 14, and the latter in Line 20.

Case (7b) to Case (7b) and Case (7) to Case (7b): Let $\{a, b, c\}$ be a triangle of Case (7b) in $G_{k-1}$, i.e., there exists $x \in V_{k-1}$ with $(x, w) \in R_{k-1}$ and such a triangle is counted in Algorithm 1, Line 12.

Thus, the number of all the triangles of $G$ that are of Case (7b), (7), (17), and (21) in $G_{k-1}$ is indeed computed and stored in the variable $t$ after the iteration $k$. Since the algorithm only computes $t$ whenever a triangle transitions from an unmarked case to a marked case after contraction, the value $t$ exactly is the desired value.

We store the graph in sorted adjacency lists, which can be initially realized in time $O(n^2 + m)$ using a linear-time sorting algorithm to sort the vertices $v_1, \ldots, v_n$. To contract the two vertices $u$ and $v$ in each iteration, we can scan the sorted adjacency lists of $u$ and $v$ to identify the red neighborhood and black neighborhood of $w$. Since $w$ has at most $d$ incident red edges and, for each black neighbor, we decrease the number of total edges by one, the total running time for every call of the procedure UpdateGraph, sums up to $O(dn + m)$. Since the auxiliary values only change for $w$ and for the incident red edges of $w$, they can be updated in time $O(d)$ per iteration. Eventually, it takes $O(dn)$ for every call of the procedure UpdateAuxiliaryValues. Finally, the procedures TriCountOneNeighbor and TriCountTwoNeighbors are called at most $d$ times per iteration, taking respectively $O(1)$ and $O(d)$ time. Thus, the overall running time of Algorithm 1 is $O(d^2 n^2 + m)$.

4 Conclusion

We have obtained an efficient parameterized algorithm for triangle counting parameterized by the twin-width tww of the input graph. As a matter of fact, the algorithm is adaptive as it runs in time $O(tww^2 n + m)$ whereas the best unparameterized combinatorial algorithms run in time $O(n^3)$. Our algorithm is based on dynamic programming and stores a few values that need to be updated at each contraction step.

Some future directions would be to extend this approach and design efficient algorithms to solve other tractable problems when parameterized by the twinwidth of the input graph. It would, furthermore, be highly interesting to find the most general parameter for which an adaptive algorithm for triangle counting exists. Finally, our algorithm is adaptive when compared to the best combinatorial algorithms, however, there exists a non-combinatorial algorithm that runs in time $O(n^{\omega})$ where $\omega < 2.372863$ [1]. An improvement would then be to obtain an $O(tww^{\omega-1} n + m)$-time algorithm.
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