ON FUJITA INVARIANTS OF SUBVARIETIES OF A UNIRULED VARIETY

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Abstract. We show that if \(X\) is a smooth uniruled projective variety and \(L\) a big and semiample \(\mathbb{Q}\)-divisor on \(X\), then there exists a proper closed subset \(W \subset X\) such that every subvariety \(Y\) satisfying \(a(Y, L) > a(X, L)\) is contained in \(W\).

1. Introduction

If \(X\) is a smooth projective variety and \(L\) is a big \(\mathbb{Q}\)-divisor on \(X\), then the Fujita invariant, or \(a\)-constant is defined as follows
\[
a(X, L) = \inf\{t > 0 \mid K_X + tL \text{ is big}\}.
\]
Note that \(a(X, L) \in \mathbb{R}_{\geq 0}\) is well defined since \(K_X + tL\) is big for all \(t > 0\) sufficiently large, and that \(a(X, L) > 0\) if and only if \(K_X\) is not pseudo-effective. It is easy to see that the \(a\)-constant is a birational invariant in the sense that if \(\nu : X' \to X\) is a birational morphism of smooth varieties and \(L' = \nu^*L\), then \(a(X, L) = a(X', L')\). Therefore we may also define the \(a\)-constant for a big \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(L\) on an arbitrary normal projective variety \(X\) by letting
\[
a(X, L) := a(X', L')
\]
where \(\nu : X' \to X\) is a resolution of singularities and \(L' = \nu^*L\). Note that if \(X\) is smooth, then the \(a\)-constant is the usual pseudo-effective threshold, however if \(X\) is singular, these numbers may be different.

In [8], motivated by a conjecture of Batyrev and Manin that relates arithmetic properties of varieties with ample anticanonical class to geometric invariants, \(a\)-constants were intensively studied by Lehmann, Tanimoto and Tschinkel. They show that ([8, Theorem 1.1]), if \(X\) is a smooth uniruled projective variety and \(L\) an ample \(\mathbb{Q}\)-divisor on \(X\), then there exists a countable union of proper closed subsets \(W \subset X\) such that every subvariety \(Y\) satisfying \(a(Y, L) > a(X, L)\) is contained in \(W\). For the purpose of applications, it is expected that one may choose \(W\) to be a proper closed subset of \(X\). The purpose of this note is to prove that this is indeed the case:

Theorem 1.1. Let \(X\) be a smooth uniruled projective variety and \(L\) a big and semiample \(\mathbb{Q}\)-divisor on \(X\). Then there exists a proper closed subset \(W \subset X\) such that every subvariety \(Y\) satisfying \(a(Y, L) > a(X, L)\) is contained in \(W\).

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Note that this result is proven in [8, Theorem 1.2] assuming that a weak version of the BAB conjecture holds in dimension $n-1 = \dim X - 1$. We expect that Theorem 1.1 holds also if we just assume that $L$ is nef and big (rather than big and semiample).

Our idea is to replace the WBAB conjecture assumed in [8, Theorem 1.2] by constructing non-klt centers (see Proposition 2.8) and applying finiteness of $a$-constants (see Corollary 2.15). This is an application of a recent result of Di Cerbo [3] based on a boundedness result proved by Birkar [2].

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2. Preliminaries

In this paper we work over the field of complex numbers $\mathbb{C}$.

2.1. Facts on $a$-constants. In this subsection, for the convenience of the reader, we collect several facts about $a$-constants that were proven in [8].

Proposition 2.1 ([8, Proposition 4.1]). Let $X$ be a smooth projective variety and $L$ a big and nef $\mathbb{Q}$-divisor. Let $\mathcal{U} \to W$ be a family of subvarieties of $X$ such that $\mathcal{U} \to X$ is dominant. Then a general member $Y$ of the family $\mathcal{U}$ satisfies $a(Y, L) \leq a(X, L)$.

Theorem 2.2 ([8, Theorem 4.2]). Let $X$ be a smooth projective variety and $L$ a big and nef $\mathbb{Q}$-divisor. Let $\pi : \mathcal{U} \to W$ be a family of subvarieties of $X$. There exists a proper closed subset $V \subset X$ such that if a member $Y$ of the family $\mathcal{U}$ satisfies $a(Y, L) > a(X, L)$ then $Y \subset V$.

Proposition 2.3 ([8, Proposition 4.6]). Let $X$ be a smooth uniruled projective variety and $L$ a big and nef $\mathbb{Q}$-divisor. Then either

1. $X$ is covered by proper subvarieties $Y$ satisfying $a(Y, L) = a(X, L)$, or
2. $X$ is birational to a $\mathbb{Q}$-factorial terminal Fano variety $X'$ of Picard number 1.

Lemma 2.4 ([8, Lemma 4.7]). Let $X$ be a smooth projective variety and $L$ a big and nef $\mathbb{Q}$-divisor on $X$. Fix a constant $C$. Then the subset of Chow($X$) parametrizing subvarieties of $X$ that are not contained in $B_+(L)$ and are of $L$-degree at most $C$ is bounded.

2.2. Non-klt centers. We follow the standard notation and conventions of the minimal model program, see eg. [5].

Definition 2.5. Let $(X, \Delta)$ be a pair so that $X$ is a normal variety, $\Delta$ is an effective $\mathbb{Q}$-divisor, and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that a subvariety $V \subset X$ is a non-klt center of $(X, \Delta)$ if it is the image of a divisor of discrepancy at most $-1$. We will denote by Nklt($X, \Delta$) the union of all non-klt centers of $(X, \Delta)$. A non-klt place is a valuation corresponding to a divisor of discrepancy at most $-1$. A non-klt center is pure if $K_X + \Delta$ is log canonical at the generic point of $V$. If moreover there is a unique non-klt
place lying over the generic point of $V$, we will say that $V$ is an exceptional non-klt center.

The following is a weak form of Kawamata’s subadjunction theorem.

**Theorem 2.6** (Subadjunction, see [1] Proposition 5.1). Let $V \subset X$ be a non-klt center of a pair $(X, \Delta)$ which is lc at a general point of $V$. Let $\nu : V' \to V$ be the normalization. Then there is an effective $\mathbb{Q}$-divisor $\Delta_{V'}$ on $V'$ such that

$$\nu^*(K_X + \Delta)|_{V'} \sim_{\mathbb{Q}} K_{V'} + \Delta_{V'}.$$ 

We have the following connectedness lemma of Kollár and Shokurov for the non-klt locus (cf. Shokurov [9], Kollár [9] 17.4).

**Theorem 2.7** (Connectedness Lemma). Let $f : X \to Z$ be a proper morphism of normal varieties with connected fibers and $D$ a $\mathbb{Q}$-divisor such that $-(K_X + D)$ is $\mathbb{Q}$-Cartier, $f$-nef, and $f$-big. Write $D = D^+ + D^-$ where $D^+$ and $D^-$ are effective with no common components. If $D^-$ is $f$-exceptional (i.e. all of its components have image of codimension at least 2), then $\text{Nklt}(X, D) \cap f^{-1}(z)$ is connected for any $z \in Z$.

We can use the following proposition to construct non-klt centers.

**Proposition 2.8** (cf. [2] Lemma 3.2). Let $X$ be a $\mathbb{Q}$-factorial terminal Fano variety of dimension $n$. Assume $(-K_X)^n > (wn)^n$ for some positive rational number $w$. Then for every point $p \in X$ there is an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$ such that the unique minimal non-klt center $V_p \subset \text{Nklt}(X, \Delta_p)$ containing $p$ is exceptional.

**Proof.** Fix a point $p$. Fix a positive rational number $w'$ such that $(-K_X)^n > (w'n)^n > (wn)^n$. By [5] 6.7.1 Theorem, there is an effective $\mathbb{Q}$-divisor $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w'}K_X$ such that $(X, \Delta'_p)$ is not lc at $p$. Take $0 < t \leq 1$ the unique rational number such that $(X, t\Delta'_p)$ is log canonical but not klt at $p$. By [1] Proposition 3.2, Lemma 3.4, we can find an effective $\mathbb{Q}$-divisor $M_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$ and some rational number $a > 0$ such that for any rational number $0 < \epsilon \ll 1$, the pair $(X, (1-\epsilon)t\Delta'_p + eaM_p)$ has a unique minimal non-klt center $V_p$ passing through $p$ which is exceptional. Note that

$$(1-\epsilon)t\Delta'_p + eaM_p \sim_{\mathbb{Q}} -\frac{(1-\epsilon)t + ea}{w'}K_X$$

and $(1-\epsilon)t + ea < \frac{1}{w'}$ for $0 < \epsilon \ll 1$. Since $-K_X$ is ample, by adding a $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to a multiple of $-K_X$ to $\Delta'_p$, we conclude that there exists an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -\frac{1}{w}K_X$ and $(X, \Delta_p)$ has a unique minimal non-klt center $V_p$ passing through $p$ which is exceptional. \hfill \Box

**Lemma 2.9.** Keep the notation in Proposition 2.8. If $w > 2$, then $\dim V_p > 0$ for a general point $p$.

**Proof.** Assume to the contrary that there exist $p_1 \in X$ such that $V_{p_1} = \{p_1\}$ and $p_2 \in X \setminus \text{Supp}(\Delta_{p_1})$ such that $V_{p_2} = \{p_2\}$. Then $p_1$ and $p_2$ are contained in $\text{Nklt}(X, \Delta_{p_1} + \Delta_{p_2})$ and $p_2$ is isolated by construction. On the other hand,

$$-(K_X + \Delta_{p_1} + \Delta_{p_2}) \sim_{\mathbb{Q}} \left(1 - \frac{2}{w}\right)(-K_X)$$

and...
2.3. Finiteness of $a$-constants. We recall the main result of [3] in this subsection.

**Definition 2.10.** Let $X$ be a normal projective variety and $H$ a big $\mathbb{Q}$-divisor. We define the pseudo-effective threshold to be

$$\tau(X, H) := \inf \{ t \geq 0 \mid K_X + tH \text{ is big} \}.$$  

Note that if $X$ is smooth, $a$-constant and pseudo-effective threshold just coincide.

**Definition 2.11** (cf. [3, Definition 3.1]). Fix a positive integer $n$ and two positive real numbers $\epsilon$ and $\delta$. We define $D_n(\epsilon, \delta)$ to be the set of lc pairs $(X, \Delta)$ such that:

1. $X$ is a normal projective variety of dimension $n$,
2. $\Delta$ is a big $\mathbb{Q}$-divisor with coefficients $\geq \delta$, and
3. $(X, t\Delta)$ is $\epsilon$-lc and $K_X + t\Delta$ is pseudo-effective for some $0 \leq t \leq 1$.

**Definition 2.12** (cf. [3, Definition 3.2]). Fix a positive integer $n$ and two positive real numbers $\epsilon$ and $\delta$. We define the set $T_n(\epsilon, \delta) := \{ \tau(X, \Delta) \mid (X, \Delta) \in D_n(\epsilon, \delta) \}$.  

**Theorem 2.13** ([3, Corollary 3.6]). Fix a positive integer $n$ and three positive real numbers $\epsilon$, $\delta$ and $\eta$. Then the set $T_n(\epsilon, \delta) \cap [\eta, 1]$ is a finite set.

To apply this theorem in our situation, we have the following corollary.

**Definition 2.14.** Fix a positive integer $n$. We define $P_n$ to be the set of pairs $(Y, L)$ such that:

1. $Y$ is a normal projective variety of dimension $n$,
2. $L$ is a base point free big Cartier divisor.

**Corollary 2.15.** Fix a positive integer $n$ and a positive real number $\eta$. Then the set 

$$\{ a(Y, L) \mid (Y, L) \in P_n \} \cap [\eta, \infty)$$

is a finite set.

**Proof.** We may assume that $\eta \leq \frac{1}{4(n+1)}$.

Firstly, we show that the set 

$$\{ a(Y, L) \mid (Y, L) \in P_n \} \cap [\eta, \frac{1}{2}]$$

is a finite set. Take $(Y, L) \in P_n$ and assume that $a(Y, L) \in [\eta, \frac{1}{2}]$. Note that $a(Y, \frac{1}{2}L) = 2a(Y, L) \in [2\eta, 1]$. By taking a resolution, we may assume that $Y$ is smooth. In this case $a(Y, \frac{1}{2}L) = \tau(Y, \frac{1}{2}L)$. Replacing $L$ by a general element in $|L|$, we may assume that $L$ is irreducible and smooth. Moreover, $(Y, \frac{1}{2}L)$ is $\frac{1}{2}$-lc and $K_Y + \frac{1}{2}L$ is pseudo-effective, that is, $(Y, \frac{1}{4}L) \in D_n(\frac{1}{2}, \frac{1}{2})$. This implies that the set 

$$\left\{ a\left(Y, \frac{1}{2}L\right) \mid (Y, L) \in P_n \right\} \cap [2\eta, 1]$$
is finite by Theorem 2.13 and so is \( \{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap [\eta, \frac{1}{2}] \).

Then we show that the set
\[
\{a(Y, L) \mid (Y, L) \in \mathcal{P}_n\} \cap \left[ \frac{1}{2}, \infty \right)
\]
is a finite set. Take \((Y, L) \in \mathcal{P}_n\) and assume that \(a(Y, L) \geq \frac{1}{2}\). By taking a resolution, we may assume that \(Y\) is smooth. By [8, Proposition 2.10], \(a(Y, L) \leq n + 1\). Now we consider \((Y, 2(n + 1)L) \in \mathcal{P}_n\). Note that \(a(Y, 2(n + 1)L) = \frac{1}{2(n + 1)}a(Y, L)\), hence \(a(Y, 2(n + 1)L) \in \left[ \frac{1}{4(n + 1)}, \frac{1}{2} \right]\). By the first step, \(a(Y, 2(n + 1)L)\) belongs to a finite set. Hence \(a(Y, L)\) belongs to a finite set. \(\square\)

3. Proof of Theorem 1.1

We prove the following proposition suggested by B. Lehmann.

**Proposition 3.1.** Fix a positive real number \(t\). Let \(X\) be a smooth projective variety and \(L\) a big and semiample \(\mathbb{Q}\)-divisor. Then there is a bounded family \(\mathcal{U}\) of subvarieties of \(X\) such that any subvariety \(Y\) not contained in \(B_+(L)\), with \(a(Y, L) > t\) is dominated by some members \(Z\) of \(\mathcal{U}\), such that \(a(Z, L) = a(Y, L)\).

**Proof.** Note that for a subvariety \(Y\) not contained in \(B_+(L)\), \(L|_Y\) is nef and big, and so \(a(Y, L)\) is well defined. Therefore we will only consider subvarieties not contained in \(B_+(L)\).

Replacing \(L\) by some multiple, we may assume that \(L\) is a base point free Cartier divisor.

We construct \(\mathcal{U}\) inductively by increasing induction on the dimension of \(Y\).

For a subvariety \(Y\) with \(a(Y, L) > t\) and \(\dim Y = 1\), it is easy to see that \(Y\) is a rational curve with
\[
\deg_Y(L) = Y : L = \frac{2}{a(Y, L)} < \frac{2}{t}.
\]
By Lemma 2.24 such \(Y\) form a bounded family \(\mathcal{U}_1\).

Suppose that we have constructed a bounded family \(\mathcal{U}_i\) of subvarieties such that every subvariety \(Y\) with \(a(Y, L) > t\) and \(\dim Y < i\) is dominated by some members \(Z\) of \(\mathcal{U}\) such that \(a(Z, L) = a(Y, L)\). We construct \(\mathcal{U}_{i+1}\) as follows. Suppose that \(Y\) is an \((i + 1)\)-dimensional subvariety satisfying \(a(Y, L) > t\). By taking a resolution, we may assume that \(Y\) is smooth. Proposition 2.3 shows that either

1. \(Y\) is covered by proper subvarieties \(Z\) with \(a(Z, L) = a(Y, L)\), or
2. \(Y\) is birational to a \(\mathbb{Q}\)-factorial terminal Fano variety \(Y'\) of Picard number 1.

In Case (1), by induction, \(Z\) is dominated by some members \(Z'\) of \(\mathcal{U}_i\) such that \(a(Z', L) = a(Z, L)\), and so is \(Y\).

In Case (2), by taking a resolution, we may assume \(\phi : Y' \rightarrow Y''\) is a morphism. By the proof of [8 Proposition 4.6], \(K_{Y''} + a(Y, L)\phi_*(L|_Y) \equiv 0\).

We define constant \(c_0 < 1\) and \(w \geq 2\) as follows: since \(L\) is base point free, we know that the set
\[
\{a(Z, L) \mid Z \text{ is a subvariety of } X\} \cap (t, \infty)
\]
is finite by Corollary 2.15. Hence we may take a rational number $c_0 < 1$ such that the set

$$\{a(Z, L) \mid Z \text{ is a subvariety of } X\} \cap [c_0a(Z', L), a(Z', L))$$

is empty for any subvariety $Z'$ with $a(Z', L) > t$. Take $w = \frac{1}{1-c_0}$. We may assume $w > 2$ by decreasing $c_0$.

If $(-K_{Y'})^{i+1} \leq (w(i + 1))^{i+1}$, then

$$(L|_Y)^{i+1} \leq (\phi_*(L|_Y))^{i+1} \leq \frac{(w(i + 1))^{i+1}}{a(Y, L)^{i+1}} < \frac{(w(i + 1))^{i+1}}{a(X, L)^{i+1}}.$$  

Then by Lemma 2.4 such $Y$ form a bounded family $\mathcal{U}_{l+1}$.

Now we assume that $(-K_{Y'})^{i+1} > (w(i + 1))^{i+1}$. By Proposition 2.8, for a general point $p \in Y'$, there exists an effective $\mathbb{Q}$-divisor $\Delta'_p \sim_{\mathbb{Q}} -\frac{1}{w}K_{Y'}$ such that $V'_p \subset \text{Nkl}(Y', \Delta'_p)$ is the minimal exceptional non-klt center containing $p$. Note that by Lemma 2.9 and $w > 2$, $\dim V'_p > 0$. Let $\nu : \tilde{V}'_p \rightarrow V'_p$ be the normalization. For any $\mathbb{Q}$-Cartier divisor $G$ on $V'_p$, we denote $G_{|\tilde{V}'_p} = \nu^*G$.

By Theorem 2.0 there is an effective $\mathbb{Q}$-divisor $\Delta_{\tilde{V}'_p}$ such that

$$(K_{\tilde{V}'_p} + \Delta_{\tilde{V}'_p})_{|\tilde{V}'_p} \sim_{\mathbb{Q}} K_{\tilde{V}'_p} + \Delta_{\tilde{V}'_p}.$$  

Note that since $K_{Y'} + a(Y, L)\phi, L \equiv 0$, we have

$$K_{\tilde{V}'_p} + \Delta_{\tilde{V}'_p} + \left(1 - \frac{1}{w}\right)a(Y, L)\phi, L_{|\tilde{V}'_p} \sim_{\mathbb{Q}} 0.$$  

Let $V_p$ be the strict transform of $V'_p$ on $Y$. Let $\tilde{V}_p$ be a common resolution of $V'_p$ and $V_p$, $f : \tilde{V}_p \rightarrow V_p$, $g : \tilde{V}_p \rightarrow \tilde{V}'_p$. Then

$$K_{\tilde{V}_p} + \left(1 - \frac{1}{w}\right)a(Y, L)f^*(L|_{V_p}) = g^*(K_{\tilde{V}'_p} + \Delta_{\tilde{V}'_p} + \left(1 - \frac{1}{w}\right)a(Y, L)\phi, L_{|\tilde{V}'_p}) - g_*^{-1}\Delta_{\tilde{V}'_p} + E$$

$$\sim_{\mathbb{Q}} -g_*^{-1}\Delta_{\tilde{V}'_p} + E,$$

where $E$ is a $g$-exceptional $\mathbb{Q}$-divisor. Note that the $\mathbb{Q}$-divisor $-g_*^{-1}\Delta_{\tilde{V}'_p} + E$ is not big. Hence $K_{\tilde{V}_p} + \left(1 - \frac{1}{w}\right)a(Y, L)f^*(L|_{V_p})$ is not big and therefore

$$a(V_p, L) \geq \left(1 - \frac{1}{w}\right)a(Y, L) = c_0a(Y, L).$$

By the definition of $c_0$, this implies that $a(V_p, L) \geq a(Y, L)$. Since $p$ is a general point, $Y$ is dominated by such $V_p$. By induction, $V_p$ is dominated by some members $Z$ of $\mathcal{U}_t$ such that $a(Z, L) = a(V_p, L) \geq a(Y, L)$. Hence $Y$ is dominated by some members $Z$ of $\mathcal{U}_t$ such that $a(Z, L) \geq a(Y, L)$. By Proposition 2.1 by taking general members, $Y$ is dominated by some members $Z$ of $\mathcal{U}_t$ such that $a(Z, L) = a(Y, L)$.

Hence we may take $\mathcal{U}_{t+1} = \mathcal{U}_t \cup \mathcal{U}'_{t+1}$, and the proof is completed. \qed

**Proof of Theorem 14.** Take $t = a(X, L)$ in Proposition 3.1 there is a bounded family $\mathcal{U}$ of subvarieties of $X$ such that any subvariety $Y$ not contained in $\mathcal{B}_+(L)$, with $a(Y, L) > a(X, L)$ is dominated by some members $Z$ of $\mathcal{U}$, such that $a(Z, L) = a(Y, L) > a(X, L)$. By Theorem 2.2 there exists a proper
closed subset $W \subset X$ such that any member $Z$ of the family $U$ satisfying $a(Z, L) > a(X, L)$ is contained in $W$. Hence any subvariety $Y$ with $a(Y, L) > a(X, L)$ is contained in $W$. □

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