PERIODIC SOLUTIONS OF SOME CLASSES OF CONTINUOUS SECOND–ORDER DIFFERENTIAL EQUATIONS

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Abstract. We study the periodic solutions of the second–order differential equations of the form $\ddot{x} \pm x^n = \mu f(t)$, or $\ddot{x} \pm |x|^n = \mu f(t)$, where $n = 4, 5, \ldots$, $f(t)$ is a continuous $T$–periodic function such that $\int_0^T f(t) dt \neq 0$, and $\mu$ is a positive small parameter. Note that the differential equations $\ddot{x} \pm x^n = \mu f(t)$ are only continuous in $t$ and smooth in $x$, and that the differential equations $\ddot{x} \pm |x|^n = \mu f(t)$ are only continuous in $t$ and locally–Lipschitz in $x$.

1. Introduction. The periodic solutions of the second–order differential equations
$$\ddot{x} + x^3 = f(t),$$
where $f(t)$ is a $T$–periodic function have been studied by several authors. Thus, Morris [9] proves that if $f(t)$ is $C^1$ and its average is zero (i.e. $\int_0^T f(t) dt = 0$), then the differential equation (1) has periodic solutions of period $kT$ for all positive integer $k$. Ding and Zanolin [6] proved the same result without the assumption that the average of $f(t)$ be zero. Almost there is no results on the stability of these periodic solutions, but Ortega [10] proved that the differential equation (1) has finitely many stable periodic solutions of a fixed period.

On the other hand other authors have studied more general problems as the following one: when an equilibrium or a limit cycle of an autonomous differential system can be continued as a periodic solution when the autonomous system is periodically perturbed. This question of persistence is very classical, but for dimension two and for an equilibrium Bucică and Ortega [3] found a complete characterization of the persistence of a such periodic solution. These authors use more general results on the persistence of periodic solutions of autonomous systems under periodic

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perturbations obtained by Capietto, Mawhin and Zanolin in [4]. The main tool used by these authors for obtaining their results is the Brouwer degree theory. See also the interesting bibliography on these kind of problems given in [4].

Our goal is to extend the mentioned results on the periodic solutions of the second–order differential equation (1) to the second–order differential equations of the form

\[ \ddot{x} \pm x^n = \mu f(t), \]

and

\[ \ddot{x} \pm |x|^n = \mu f(t), \]

where \( n = 4, 5, \ldots \), \( f(t) \) is a continuous \( T \)-periodic function such that \( \int_0^T f(t) dt \neq 0 \), and \( \mu > 0 \) is a small parameter.

Note that the differential equations (2) are only continuous in \( t \) and smooth in \( x \), and that the differential equations (3) are only continuous in \( t \) and locally–Lipschitz in \( x \). As far as we know these kind of differential equations have not been studied up to know.

We note that the results of Morris [9] for the differential equation (1) are more general than our results for the equation (2) with \( n = 3 \), because he does not need that periodic perturbation of the autonomous equation be small. Moreover the techniques of the proof of Morris are completely different from our techniques which are based in the averaging theory.

We also study the linear stability or instability of the periodic solutions that we find of second–order differential equations (2) and (3). We recall that this kind of stability is defined using the local Poincaré map around the periodic solution, thus the linear stability of the periodic solution is the linear stability of the fixed point of the Poincaré map associated to the periodic solution. Consequently, if the real part of some eigenvalue of the fixed point has modulus larger than one then the periodic solution is unstable. If all the real parts of the eigenvalues of the fixed point are smaller than one then the periodic orbit is asymptotically stable. For more information about this kind of stability and the differences with the Liapunov stability see for instance the book [7].

1.1. Statement of the main results. Our main results are the following two theorems.

**Theorem 1.1.** Consider the second–order differential equations

\[ \ddot{x} \pm x^n = \mu f(t), \]

where \( n = 4, 5, \ldots \), \( f(t) \) is continuous, \( T \)-periodic function such that \( \int_0^T f(t) dt \neq 0 \), and \( \mu > 0 \) is a small parameter. Then, for \( n \geq 4 \) and \( \mu > 0 \) sufficiently small there exist two periodic solutions \( x_{\pm}(t, \mu) \) of period \( T \) of the differential equation (4) such that

\[ x_{\pm}(0, \mu) = \pm \mu^{1/n} \left| \frac{1}{T} \int_0^T f(t) dt \right|^{1/n} + O(\mu^{(n-1)/(2n)}), \]

where \( \pm \frac{1}{T} \int_0^T f(t) dt > 0 \) when \( n \) is even. Moreover the periodic solution \( x_-(t, \mu) \) is unstable for the equation \( \ddot{x} + x^n = \mu f(t) \) if \( n \) is even, and for the equations \( \ddot{x} \pm x^n = \mu f(t) \) if \( n \) is odd.
Theorem 1.1 is proved in section 2.

Note that we are using in (5) and in the rest of the paper the following notation:

\[ x_+(0, \mu) = \mu^{1/n} \left( + \frac{1}{T} \int_0^T f(t) dt \right)^{1/n} + O(\mu^{(n-1)/(2n)}), \]

and

\[ x_-(0, \mu) = \mu^{1/n} \left( - \frac{1}{T} \int_0^T f(t) dt \right)^{1/n} + O(\mu^{(n-1)/(2n)}), \]

we only write (5).

**Theorem 1.2.** Consider the second–order differential equations

\[ \ddot{x} \pm |x|^n = \mu f(t), \]

where \( n = 4, 5, \ldots \), \( f(t) \) is continuous, \( T \)–periodic function such that \( \int_0^T f(t) dt \neq 0 \), and \( \mu > 0 \) is a small parameter. Then, for \( n \geq 4 \) and \( \mu \) sufficiently small there exist two periodic solutions \( x_\pm(t, \mu) \) of period \( T \) of the differential equation (8) such that

\[ x_\pm(0, \mu) = \pm \mu^{1/n} \left( \pm \frac{1}{T} \int_0^T f(t) dt \right)^{1/n} + O(\mu^{(n-1)/(2n)}), \]

where \( \pm \int_0^T f(t) dt > 0 \) when \( n \) is even. Moreover, the periodic solutions \( x_\pm(t, \mu) \) for the equation \( \ddot{x} - |x|^n = \mu f(t) \) are unstable.

Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be the 2–periodic function defined by

\[ g(t) = \begin{cases} t & \text{if } t \in [0, 1], \\ 2 - t & \text{if } t \in [1, 2]. \end{cases} \]

The following two corollaries follow easily from the previous two theorems.

**Corollary 1.** For \( \mu > 0 \) sufficiently small the equations \( \ddot{x} \pm x^4 = \mu g(t) \) have two periodic solutions \( x_\pm(t, \mu) \) such that \( x(0, \mu) = \pm \sqrt[4]{\mu/2} + O(\mu^{3/8}) \).

**Corollary 2.** For \( \mu \) sufficiently small then equations \( \ddot{x} + |x|^4 = \mu \sin^2 t \) have two periodic solutions \( x_\pm(t, \mu) \) such that \( x_\pm(0, \mu) = \pm \sqrt[4]{\mu/2} + O(\mu^{3/8}) \).

2. **Proof of the results.** In this section we shall prove Theorems 1.1 and 1.2 and Corollaries 1 and 2.

**Proof of Theorem 1.1.** Under the assumptions of Theorem 1.1 we write the second–order differential equation as the differential system of first order

\[ \dot{x} = y, \]
\[ \dot{y} = \mp x^n + \mu f(t). \]

Doing the change of variables

\[ x = \varepsilon^{2/(n-1)} X, \quad y = \varepsilon^{(n+1)/(n-1)} Y, \quad \mu = \varepsilon^{(2n)/(n-1)}, \]

with \( \varepsilon > 0 \), the differential system becomes

\[ \dot{X} = \varepsilon Y, \]
\[ \dot{Y} = \varepsilon (\mp X^n + f(t)). \]
We note that the change of variables \((11)\) is well defined because \(n > 1\). Now we apply the averaging theory of first order of the appendix. Using the notation of Theorem \(2.1\) of the appendix system \((12)\) can be written as system \((15)\) with \(x = (X, Y), H = (Y, \mp X^n + f(t)), R = (0, 0)\). The average function \(h(z)\) given in \((16)\) for system \((12)\) becomes
\[
h(X, Y) = (Y, \mp X^n + \frac{1}{T} \int_0^T f(t)dt).
\]
If \(n\) is even then the function \(h(X, Y)\) has two unique zeros
\[
(X^*_\pm, Y^*_\pm) = (\pm (\pm 1 \int_0^T f(t)dt)^{1/n}, 0).
\]
when \(\pm \frac{1}{T} \int_0^T f(t)dt > 0\) for the equation \(\ddot{x} \pm x^n = \mu f(t)\); note that only one of these two differential equations has two periodic solutions. If \(n\) is odd then the function \(h(X, Y)\) has two zeros,
\[
(X^*_\pm, Y^*_\pm) = ((\pm 1 \int_0^T f(t)dt)^{1/n}, 0),
\]
when \(\int_0^T f(t)dt \neq 0\) for both equations \(\ddot{x} \pm x^n = \mu f(t)\).

The Jacobian of the function \(h(X,Y)\) at these zeros is \(\pm nX^*_\pm (n-1)\). By Theorem \(2.1\) and Remark \(1\) we deduce that there are two periodic solutions \((X_\pm(t, \varepsilon), Y_\pm(t, \varepsilon))\) of system \((12)\) satisfying that
\[
(X_\pm(0, \varepsilon), Y_\pm(0, \varepsilon)) = (X^*_\pm, 0) + O(\varepsilon).
\]
From \((11)\) we have \(x = \mu^{1/n} X\). We conclude that for \(\mu > 0\) sufficiently small there exist two periodic solutions \(x_\pm(t, \mu)\) of period \(T\) of the differential equation \((1)\) such that
\[
x_\pm(0, \mu) = \mu^{1/n} X^*_\pm + O(\mu^{(n-1)/(2n)}).
\]
We note that for \(\mu > 0\) sufficiently small \(\mu^{1/n} \gg \mu^{(n-1)/(2n)}\) if and only if \(n > 3\), which holds by assumption.

The two eigenvalues of the corresponding Jacobian matrix of the average function \(h(X, Y)\) at the zero \((X^*, Y^*)\) are \(\pm \sqrt{n} X^* (n-1)\). If \(n\) is even and \(\pm \frac{1}{T} \int_0^T f(t)dt > 0\) the solution \((X_\pm(t, \varepsilon), Y_\pm(t, \varepsilon))\) of system \((12)\) provides an unstable periodic solution for the equation \(\ddot{x} + x^n = \mu f(t)\). If \(n\) is odd and \(\frac{1}{T} \int_0^T f(t)dt \neq 0\) the solution \((X_\pm(t, \varepsilon), Y_\pm(t, \varepsilon))\) of system \((12)\) provides an unstable periodic solution for the equation \(\ddot{x} \pm x^n = \mu f(t)\). Then from Theorem \(2.2\) of this appendix it follows the results on the instability of the periodic solutions stated in the theorem.

**Proof of Theorem 1.2** In the assumptions of Theorem \(1.2\) we write the second–order differential equation as the differential system of first order
\[
\dot{x} = y, \\
\dot{y} = \mp |x|^n + \mu f(t).
\]
Doing the change of variables (11), the differential system (13) becomes
\[ \begin{align*}
X &= \varepsilon Y, \\
Y &= \varepsilon (\mp |X|^n + f(t)).
\end{align*} \tag{14} \]

Note that we can apply the averaging theory of first order of the appendix because the function \(|X|^n\) is locally Lipschitz. Using the notation of Theorem 2.1 of the appendix system (12) can be written as system (15) with \(x = (X, Y)\), \(H = (Y, \mp |X|^n + f(t))\), \(R = (0, 0)\). The average function \(h(z)\) given in (16) for system (15) becomes
\[ h(X, Y) = \left( Y, \mp |X|^n + \frac{1}{T} \int_0^T f(t) dt \right). \]

The function \(h(X, Y)\) has the two zeros
\[ (X^*_\pm, Y^*_\pm) = \left( \pm \left( \frac{1}{T} \int_0^T f(t) dt \right)^{1/n}, 0 \right), \]
such zeros exist when \(\pm \int_0^T f(t) dt > 0\) and \(n\) is even, or when \(\int_0^T f(t) dt \neq 0\) and \(n\) is odd. The Jacobians of the function \(h(X, Y)\) at the zeros \((X^*_\pm, Y^*_\pm)\) are \(\pm n|X^*_\pm|^{n-1}\). By Theorem 2.1 and Remark 3 we deduce that there is two periodic solutions \((X_\pm(t, \varepsilon), Y_\pm(t, \varepsilon))\) of system (13) satisfying that
\[ (X_\pm(0, \varepsilon), Y_\pm(0, \varepsilon)) = (X^*_\pm, 0) + O(\varepsilon). \]

Since \(x = \varepsilon^{2/(n-1)} X\) and \(\mu = \varepsilon^{(2n)/(n-1)}\), we have \(x = \mu^{1/n} X\). So for \(\mu > 0\) sufficiently small there exists two periodic solutions \(x_\pm(t, \mu)\) of period \(T\) of the differential equation (14) such that
\[ x_\pm(0, \mu) = \mu^{1/n} X^*_\pm + O(\mu^{(n-1)/(2n)}). \]

The two eigenvalues of the corresponding Jacobian matrix of the average function \(h(X, Y)\) at the zeros \((X^*_\pm, 0)\) are \(\pm \sqrt{-n|X^*_\pm|^{n-1}}\) for the equation \(\ddot{x} + |x|^n = \mu f(t)\), and at the zeros \((X^*_\pm, 0)\) are \(\pm \sqrt{n|X^*_\pm|^{n-1}}\) for the equation \(\ddot{x} - |x|^n = \mu f(t)\). Again by Theorem 2.2 it follows that the periodic solutions \(x_\pm(t, \mu)\) are unstable for the equation \(\ddot{x} - |x|^n = \mu f(t)\). This completes the proof of the theorem.

Appendix: Averaging theory of first order. In this section we present the first order averaging method as it was extended in [1], where the differentiability of the vector field is not needed. The sufficient conditions for the existence of a simple isolated zero of the average function are given in terms of the Brouwer degree, see [5] for precise definitions.

**Theorem 2.1.** We consider the following differential system
\[ \dot{x}(t) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{15} \]
where \(H : \mathbb{R} \times D \to \mathbb{R}^n\), \(R : \mathbb{R} \times D \times (\mp 1, \varepsilon f) \to \mathbb{R}^n\) are continuous functions, \(T\)-periodic in \(t\), and \(D\) is an open subset of \(\mathbb{R}^n\). We define \(h : D \to \mathbb{R}^n\) as
\[ h(z) = \frac{1}{T} \int_0^T H(s, z) ds, \tag{16} \]
and assume that
(i) \(H\) and \(R\) are locally Lipschitz in \(x\);
(ii) for \( a \in D \) with \( h(a) = 0 \), there exists a neighborhood \( V \) of \( a \) such that \( h(z) \neq 0 \) for all \( z \in V \setminus \{a\} \) and \( d_B(h, V, a) \neq 0 \) (where \( d_B(h, V, a) \) denotes the Brouwer degree of \( h \) in the neighborhood \( V \) of \( a \)).

Then, for \( |\varepsilon| > 0 \) sufficiently small, there exists an isolated \( T \)-periodic solution \( x(t, \varepsilon) \) of system (15) such that \( x(0, \varepsilon) \to a \) as \( \varepsilon \to 0 \).

If the average function \( h(z) \) is differentiable in some neighborhood of a fixed isolated zero \( a \) of \( h(z) \), then we can use the following remark in order to verify the hypothesis (ii) of Theorem 2.1. For more details see again [8].

Remark 1. Let \( h : D \to \mathbb{R}^n \) be a \( C^1 \) function, with \( h(a) = 0 \), where \( D \) is an open subset of \( \mathbb{R}^n \) and \( a \in D \). Whenever \( a \) is a simple zero of \( h \) (\( \det(Dh(a)) \neq 0 \)), i.e the determinant of the Jacobian matrix of the function \( h \) at \( a \) is not zero), there exists a neighborhood \( V \) of \( a \) such that \( h(z) \neq 0 \) for all \( z \in V \setminus \{a\} \). Then \( d_B(h, a, V, 0) \in \{-1, 1\} \).

In [2] Theorem 2.1 is improved as follows.

Theorem 2.2. Under the assumptions of Theorem 2.1, for small \( \varepsilon \) the condition \( \det(Dh(a)) \neq 0 \) ensures the existence and uniqueness of a \( T \)-periodic solution \( x(t, \varepsilon) \) of system (15) such that \( x(0, \varepsilon) \to a \) as \( \varepsilon \to 0 \), and if all eigenvalues of the matrix \( Dh(a) \) have negative real parts, then the periodic solution \( x(t, \varepsilon) \) is stable. If some of the eigenvalue has positive real part the periodic solution \( x(t, \varepsilon) \) is unstable.

The averaging theory for studying periodic solutions is very useful see for instance [5].

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