Dynamics of fermions coupling to a $U(1)$ gauge field in the limit

$e^2 \rightarrow \infty$

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Abstract

We study in this paper the properties of a gas of fermions coupling to a $U(1)$ gauge field at wavevectors $q < \Lambda << k_F$ at dimensions larger than one, where $\Lambda << k_F$ is a high momentum cutoff and $k_F$ is the fermi wave vector. In particular, we shall consider the $e^2 \rightarrow \infty$ limit where charge and current fluctuations at wave vectors $q < \Lambda$ are forbidden. Within a bosonization approximation, effective actions describing the low energy physics of the system are constructed, where we show that the system can be described as a fermion liquid formed by chargeless quasi-particles which has vanishing wavefunction overlap with the bare fermions in the system.

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I. INTRODUCTION

In the last few years, there has been enormous interests in the study of $U(1)$ gauge theories of fermionic systems in dimensions higher than one, as a result of appearance of effective gauge theories in the $t-J$ type models [1] and also in the studies of the $\nu = \frac{1}{2}$ Fractional Quantum Hall state [2,3]. In particular, the behaviour of the systems in the strong-coupling regime where charge and current fluctuations are forbidden (confined) are of special interests. In a previous paper we studied the effects of a longitudinal gauge field (Coulomb interaction) on a system of spinless fermions in the limit $e^2 \to \infty$. Within a bosonization approximation, we found that the ground state and low energy excitations of the system are described by a fermi liquid with chargeless quasi-particles [4]. In this paper we shall generalize our approach [4] to consider the $e^2 \to \infty$ limit of a system of spin $S = 1/2$, charge $e$ fermions minimally coupled to a $U(1)$ gauge field $(A_0, \vec{A})$ in dimensions higher than one. The gauge field dynamics is described by an effective long distance action $L_{\text{gauge}} = F^2_{\mu\nu}$ with high momentum cutoff $\Lambda << k_F$, where $k_F$ is the fermi wavevector. In this limit any non-uniform charge and current density fluctuations $<\rho(\vec{q})> \neq 0$ and $<\vec{j}(\vec{q})> \neq 0$ with wave vector $q < \Lambda$ in the system cost infinite energy and are forbidden. As a result any physical state $|\psi>$ that survives in this limit satisfies the constraints $\rho(\vec{q})|\psi> = 0$ and $\vec{j}(\vec{q})|\psi> = 0$, where $\rho(\vec{q})$ and $\vec{j}(\vec{q})$ are the charge and current density operators, respectively.

In this paper we shall show that by generalizing our previous approach [4], the projection to the physical states $|\psi>$ where density and current fluctuations are forbidden can be carried out in the $e^2 \to \infty$ limit in a bosonization approximation [3-8], and physical (chargeless) single-particle operators that commute with both charge and current operators of the system can be constructed. The method we use is similar in spirit to the field theory scheme proposed by Shanjar and Murthy [9] for the Fractional Quantum Hall effect, though the details are different. Within the approximation effective low energy Lagrangians that describe the chargeless particle-hole as well as single-particle excitations in the system are obtained. The physical sector of the system is described by a liquid of chargeless fermions.
and corresponds to a kind of ‘marginal’ fermi liquid in the original fermion description. The organization of our paper is as follows: in section II we shall outline our mathematical formulation of the bosonization procedure where an approximate bosonized action of the system describing both spin and charge particle-hole excitations will be derived. In section III we shall study in detail the eigenstates and eigen-energies of the particle-hole excitation spectrum in the bosonization approximation for arbitrary values of coupling constant $e^2$. In particular we shall show how the projection to the physical Hilbert space where charge and current fluctuations are forbidden is obtained in the $e^2 \rightarrow \infty$ limit. In section IV we shall consider the single particle excitations where $S = 1/2$, chargeless single fermion operators that commute with density and current operators are constructed and the equation of motion for these fermion operators is derived. An effective action in terms of these chargeless fermion operators is constructed where we shall show that the system forms a marginal fermi liquid. Our results will be summarized in section V where some further comments will be given.

II. MATHEMATICAL FORMULATION

We shall work in the Coulomb gauge $\nabla \cdot \vec{A} = 0$ where the $A_0$ component of the gauge field is integrated out, resulting in an instantaneous Coulomb interaction $v(q) = 4\pi e^2/q^2$ at $q < \Lambda$ between fermions. The resulting Hamiltonian of the system is,

$$
H = \sum_{i,\sigma} \frac{1}{2m} \left[ \vec{p}_{i\sigma} - e \frac{\vec{A}(\vec{r}_{i\sigma})}{c} \right]^2 + \frac{1}{2L^d} \sum_{\vec{q} \neq 0, q < \Lambda} \frac{1}{2} v(q) \rho(\vec{q})\rho(-\vec{q}) + H_{\text{photon}},
$$

where $\vec{p}_{i\sigma}$ and $\vec{r}_{i\sigma}$ are the momentum and position operators of the $i^{th}$ fermion with spin $\sigma$. $\rho(\vec{q}) = \sum_{\vec{k}\sigma} f_{\vec{k}+\vec{q}/2\sigma}^+ f_{\vec{k}-\vec{q}/2\sigma}$ is the density operator for the fermions. $f(f^+)$'s are fermion annihilation (creation) operators. $L^d$ is the volume of the system and $c$ is the velocity of light.

$$
H_{\text{photon}} = \sum_{\vec{q} \neq 0, q < \Lambda, \lambda} \omega_q \left( a^+_\lambda(\vec{q})a_\lambda(\vec{q}) + \frac{1}{2} \right),
$$

where $\omega_q = \frac{\epsilon}{c} \sqrt{\frac{2m}{\hbar^2}}$ is the photon frequency.
where $a(a^+)_{\vec{q}\lambda}$’s are photon annihilation (creation) operators with wavevector $\vec{q}$ and polarization $\lambda$. $\omega_q = cq$. We have set $\hbar = 1$ to simplify notation. Notice that the Hartree ($\vec{q} = 0$) interaction energy does not appear in the Hamiltonian as in usual Coulomb gas problem where overall charge neutrality is maintained by presence of a uniform background of opposite sign charges. After Fourier transforming and in second quantized form, we obtain

$$\sum_{\vec{q}\lambda} \frac{1}{2m} \left[ \vec{p}_{\vec{q}\sigma} - e \vec{A}(\vec{r}_{\vec{q}\sigma}) \right]^2 \rightarrow \sum_{\vec{k}\sigma} \left( \frac{k^2}{2m} \right) f^+_k f_k - \frac{e}{L_{d/2}} \sum_{\vec{q}\lambda} \vec{j}_p(\vec{q}).\vec{A}(\vec{q}, \lambda)$$

$$+ \sum_{\vec{q}\lambda} \left( \frac{e^2 n_0}{2m} \right) \vec{A}(\vec{q}, \lambda).\vec{A}(-\vec{q}, \lambda),$$

where $\vec{j}_p(\vec{q}) = \sum_{\vec{k}\sigma} (\vec{k}/m) f^+_k f_k - \vec{A}(\vec{q}, \lambda).\vec{A}(\vec{q}, \lambda)$ is the paramagnetic current operator, and

$$\vec{A}(\vec{q}, \lambda) = \left( \frac{2\pi}{\omega_q} \right)^{\frac{3}{2}} \vec{\xi}(\vec{q}, \lambda) (a_\lambda(\vec{q}) + a^+_\lambda(-\vec{q})),$$

where $\vec{\xi}(\vec{q}, \lambda)$ is the polarization vector of photons with momentum $\vec{q}$ and polarization mode $\lambda$. Note that all sums over $\vec{q}$’s are restricted to $\vec{q} \neq 0, |\vec{q}| < \Lambda$ in Eq. (3) and in all the following equations. We have also replaced the fermion density operator $\rho(\vec{r})$ by its expectation value $n_0$ in the diamagnetic term in Eq. (3a). This approximation can be justified in the $e^2 \rightarrow \infty$ limit where density fluctuations of fermions are forbidden. We shall discuss this more carefully in section IV.

The difference between our model and usual Coulomb gas problem has to be emphasized here. In usual Coulomb gas problem the high momentum cutoff $\Lambda$ is taken to be infinity, or satisfies $\Lambda >> k_F$. In this limit a Wigner crystal where electrons position are fixed is expected to be formed (in 3D) when $e^2$ becomes large because the potential energy term becomes dominating at length scale $\sim k_F^{-1}$. In our problem where $\Lambda << k_F$, the interaction is effective only at length scale $>>$ inter-particle spacing and formation of Wigner crystal is not warranted. In particular we found in Ref.[4] that the ground state of the system has large degeneracies in the absence of the kinetic energy term and the system may remain in a liquid state when kinetic energy term is switched on. We note that bosonization method is a
natural tool to study the liquid state of the problem in this limit \[7, 8, 10\]. The bosonization procedure can be formulated most easily by introducing the Wigner function operators

\[ \rho_{k\sigma}(\vec{q}) = f_{k+\vec{q}/2\sigma}^+ f_{\vec{k}-\vec{q}/2\sigma}. \]

In the path-integral formulation the Wigner operators are introduced in the Imaginary-time action of the system through Lagrange multiplier fields,

\[ S = \int_0^\beta d\tau \left[ \sum_{k\sigma} f_{k\sigma}^+ (\tau) \left( \frac{\partial}{\partial \tau} + \frac{k^2}{2m} - \mu \right) f_{k\sigma}(\tau) - \sum_{k\sigma, \vec{q}} i\lambda_{k\sigma}(\vec{q}, \tau) \left( \rho_{k\sigma}(\vec{q}, \tau) - f_{k+\vec{q}/2\sigma}^+ (\tau) f_{\vec{k}-\vec{q}/2\sigma}(\tau) \right) \right] + \frac{1}{L^d} \sum_{\vec{q}, \vec{k}, \vec{k}'} v(\vec{q}) \rho_{\vec{k}}^\dagger (\vec{q}, \tau) \rho_{\vec{k}'} (-\vec{q}, \tau) - \frac{\sqrt{2e}}{L^d/2} \sum_{\vec{k}, \vec{q}, \lambda} \left( \begin{pmatrix} \vec{k}, \vec{A}(\vec{q}, \lambda, \tau) \end{pmatrix} - \vec{k}, \vec{A}(\vec{q}, \lambda, \tau) \right) \rho_{\vec{k}}^\dagger (\vec{q}, \tau) \right] + S'_{\text{photon}}, \]

where \( \mu \) is the chemical potential,

\[ S'_{\text{photon}} = S_{\text{photon}} + \sum_{\vec{q}, \lambda} \left( \frac{e^2 n_0}{2m} \right) \vec{A}(\vec{q}, \lambda, \tau) \cdot \vec{A}(-\vec{q}, \lambda, \tau), \]

where \( S_{\text{photon}} \) is the pure photon action, \( \rho_{\vec{k}}(\vec{q}) = \frac{1}{\sqrt{2}} \sum_{\sigma} \rho_{\vec{k}\sigma}(\vec{q}) \) are density Wigner function operators, and \( \lambda_{k\sigma}(\vec{q})'s \) are Lagrangian multiplier fields introduced to enforce the constraint that the Wigner operators are given by \( \rho_{k\sigma}(\vec{q}) = f_{k+\vec{q}/2\sigma}^+ f_{\vec{k}-\vec{q}/2\sigma}. \) In particular, the original Hamiltonian \( (1) \) is recovered once the \( \lambda_{k\sigma}(\vec{q}) \) field is integrated out. For later convenience we shall also introduce spin Wigner operators \( \sigma_{\vec{k}}(\vec{q}) = \frac{1}{\sqrt{2}} \sum_{\sigma} \sigma \rho_{\vec{k}\sigma}(\vec{q}). \)

The photon action \( S'_{\text{photon}} \) can be diagonalized easily to obtain

\[ S'_{\text{photon}} = \sum_{\vec{q}, \lambda, i\omega_n} (-i\omega_n + \Omega_q) b_{\lambda}^\dagger (\vec{q}, i\omega_n) b_{\lambda} (\vec{q}, i\omega_n), \quad (5a) \]

where \( b(b^\dagger)'s \) are new photon eigenmodes with eigenfrequencies \( \Omega_q = \sqrt{\omega^2 + \omega_p^2} \), where \( \omega_p = \sqrt{4\pi n_0 e^2 / m} \) is the fermion plasma frequency. The vector field \( \vec{A}(\vec{q}, \lambda) \) can be written in terms of these new eigenmodes as,

\[ \vec{A}(\vec{q}, \lambda) = \left( \frac{2\pi}{\Omega_q} \right) \sqrt{\frac{e}{2m}} \hat{\xi}(\vec{q}, \lambda) (b_{\lambda}(\vec{q}) + b_{\lambda}^\dagger (-\vec{q})). \quad (5b) \]

Integrating out the fermion fields \( f(f^\dagger)'s \) we obtain an action in terms of \( \rho_{\vec{k}\sigma}(\vec{q}) \) and \( \lambda_{k\sigma}(\vec{q}) \) fields,

\[ S = \frac{1}{\beta} F_0 - Tr \ln \left[ \hat{1} - \hat{G}_0 \hat{\lambda} \right] - \sum_{\vec{k}, \vec{q}, i\omega_n, \sigma} i\lambda_{k\sigma}(\vec{q}, -i\omega_n) \rho_{k\sigma}(\vec{q}, i\omega_n) + \frac{1}{L^d} \sum_{\vec{q}, i\omega_n, \vec{k}, \vec{k}'} v(\vec{q}) \rho_{\vec{k}, i\omega_n}(\vec{q}) \rho_{\vec{k}', -\vec{q}, -i\omega_n} + S'_{\text{photon}}, \quad (6a) \]

\[ -\frac{\sqrt{2e}}{L^d/2} \sum_{\vec{k}, \vec{q}, \lambda, i\omega_n} \left( \begin{pmatrix} \vec{k}, \vec{A}(\vec{q}, \lambda, i\omega_n) \end{pmatrix} \right) \rho_{\vec{k}}^\dagger (\vec{q}, -i\omega_n) + S'_{\text{photon}}, \]
where $F_0$ is the free energy for an non-interacting Fermi gas. $\hat{G}_0$ and $\hat{\lambda}$ are infinite matrices in wave vector and frequency space, with matrix elements given by

\[
[\hat{G}_0]_{k\sigma,k'\sigma'} = \delta_{\sigma\sigma'} \delta_{k,k'} g_0(k), \quad g_0(k) = \frac{1}{i\omega_n - \xi_k}, \tag{6b}
\]

and

\[
[\hat{\lambda}]_{k\sigma,k'\sigma'} = \frac{i}{\sqrt{\beta}} \lambda \frac{e^{i\bar{q}.\bar{k}}}{2\pi} (\bar{k} - \bar{k}', i\omega_n - i\omega_{n'}) = \delta_{\sigma\sigma'} \frac{i}{\sqrt{\beta}} \lambda \frac{e^{i\bar{q}.\bar{k}}}{2\pi} (k - k'), \tag{6c}
\]

where $k = (\bar{k}, i\omega_n)$ and $\xi_k = k^2 / 2m - \mu$. The $\text{Tr} \ln [1 - \hat{G}_0 \hat{\lambda}]$ term can be expanded in a power series of $i\lambda \bar{k}_\sigma(q)$ field,

$$
\text{Tr} \ln [1 - \hat{G}_0 \hat{\lambda}] = -\text{Tr} [\hat{G}_0 \hat{\lambda}] - \frac{1}{2} \text{Tr} [\hat{G}_0 \hat{\lambda}]^2 - \frac{1}{3} \text{Tr} [\hat{G}_0 \hat{\lambda}]^3 + O(\lambda^4).
$$

Keeping terms to second order in $\lambda$ (Gaussian approximation), we obtain

$$
\text{Tr} \ln [1 - \hat{G}_0 \hat{\lambda}] \sim \frac{1}{2\beta} \sum_{k,q,\sigma} g_0(k + \frac{q}{2}) g_0(k - \frac{q}{2}) \lambda \bar{k}_\sigma(q) \lambda \bar{k}_\sigma(-q). \tag{7}
$$

Notice that the first order term in $\lambda$ gives the usual Hartree self-energy and is excluded in our Hamiltonian. The $i\lambda \bar{k}_\sigma(q)$ fields in Action (6a) can be integrated out in Gaussian approximation, resulting in an quadratic action $S$ in terms of $\rho_{\vec{r}}(q), \sigma_{\vec{r}}(q)$ and photon fields only. We obtain $S = S_{\rho-p} + S_{\sigma}$, where

\[
S_{\rho-p} = \frac{1}{2} \sum_{\bar{k},\bar{k}',\bar{q},i\omega_n} \left[ \frac{1}{\chi_{0\bar{k}}(\bar{q}, i\omega_n)} \left( \delta_{\bar{k},\bar{k}'} \right) + \frac{2\nu(q)}{L^d} \right] \rho_{\bar{k}}(\bar{q}, i\omega_n) \rho_{\bar{k}'}(-\bar{q}, -i\omega_n) \tag{8a}
\]

\[
-\sqrt{2e} \sum_{\bar{k},\bar{q},i\omega_n} \frac{\bar{k}.\bar{A}(\bar{q}, \lambda, i\omega_n)}{m} \rho_{\bar{k}}(\bar{q}, -i\omega_n) + S'_{\text{photon}}
\]

and

\[
S_{\sigma} = -\frac{1}{2} \sum_{\bar{k},\bar{q},i\omega_n} \frac{1}{\chi_{0\bar{k}}(\bar{q}, i\omega_n)} \sigma_{\bar{k}}(\bar{q}, i\omega_n) \sigma_{\bar{k}}(-\bar{q}, -i\omega_n), \tag{8b}
\]

where

$$
\chi_{0\bar{k}}(\bar{q}, i\omega_n) = \frac{1}{\beta} \sum_{\Omega} g_0(\bar{k} + \bar{q}/2, i\omega_n + i\Omega_n) g_0(\bar{k} - \bar{q}/2, i\Omega_n) = \frac{n_{\bar{k}+\bar{q}/2} - n_{\bar{k}+\bar{q}/2}}{i\omega_n - \frac{k\bar{q}}{m}}, \tag{8c}
$$

6
\( n_{\vec{k}} = \theta(-\xi_{\vec{k}}) \) at zero temperature is the free fermion occupation number. \( S_{\rho-p} \) and \( S_{\sigma} \) can be expressed in terms of canonical boson fields by introducing

\[
\rho_{\vec{k}}(\vec{q}, i\omega_n) = \sqrt{|\Delta_{\vec{k}}(\vec{q})|} \left( \theta(\Delta_{\vec{k}}(\vec{q})) b^+_{\vec{k}}(\vec{q}, i\omega_n) + \theta(-\Delta_{\vec{k}}(\vec{q})) b_{\vec{k}}(-\vec{q}, -i\omega_n) \right), \tag{9a}
\]

\[
\sigma_{\vec{k}}(\vec{q}, i\omega_n) = \sqrt{|\Delta_{\vec{k}}(\vec{q})|} \left( \theta(\Delta_{\vec{k}}(\vec{q})) s^+_{\vec{k}}(\vec{q}, i\omega_n) + \theta(-\Delta_{\vec{k}}(\vec{q})) s_{\vec{k}}(-\vec{q}, -i\omega_n) \right),
\]

where \( \Delta_{\vec{k}}(\vec{q}) = n_{\vec{k} - \vec{q}/2} - n_{\vec{k} + \vec{q}/2} \). Correspondingly, we also have

\[
\rho_{\vec{k}}(-\vec{q}, -i\omega_n) = \sqrt{|\Delta_{\vec{k}}(\vec{q})|} \left( \theta(\Delta_{\vec{k}}(\vec{q})) b^+_{\vec{k}}(\vec{q}, i\omega_n) + \theta(-\Delta_{\vec{k}}(\vec{q})) b_{\vec{k}}(-\vec{q}, -i\omega_n) \right), \tag{9b}
\]

\[
\sigma_{\vec{k}}(-\vec{q}, -i\omega_n) = \sqrt{|\Delta_{\vec{k}}(\vec{q})|} \left( \theta(\Delta_{\vec{k}}(\vec{q})) s^+_{\vec{k}}(\vec{q}, i\omega_n) + \theta(-\Delta_{\vec{k}}(\vec{q})) s_{\vec{k}}(-\vec{q}, -i\omega_n) \right).
\]

Putting eqs. (9) back into \( S_{\rho-p} \), we obtain after some straightforward manipulations,

\[
S_{\rho-p} = \frac{1}{2} \sum_{\vec{k}, \vec{q}, i\omega_n} (-i\omega_n + \frac{\vec{k} \cdot \vec{q}}{m}) b^+_{\vec{k}}(\vec{q}, i\omega_n) b_{\vec{k}}(\vec{q}, i\omega_n) + \frac{1}{2} \sum_{\vec{q}, i\omega_n} (-i\omega_n + \Omega_q) b^+_{\vec{q}}(\vec{q}, i\omega_n) b_{\vec{q}}(\vec{q}, i\omega_n) \tag{10a}
\]

\[
+ \frac{1}{L^d} \sum_{\vec{k}, \vec{q}, i\omega_n} v(q) \sqrt{|\Delta_{\vec{k}}(\vec{q})|\Delta_{\vec{k}'}(\vec{q})} \theta(\Delta_{\vec{k}}(\vec{q})) \theta(\Delta_{\vec{k}'}(\vec{q})) \times \left( b^+_{\vec{k}}(\vec{q}, i\omega_n) b_{\vec{k}'}(\vec{q}, i\omega_n) + b_{\vec{k}}(-\vec{q}, -i\omega_n) b^+_{\vec{k}}(\vec{q}, i\omega_n) + b_{\vec{k}}(-\vec{q}, -i\omega_n) b^+_{\vec{k}}(-\vec{q}, -i\omega_n) \right)
\]

\[
- \frac{1}{L^d/2} \sum_{\vec{k}, \vec{q}, \lambda, i\omega_n} m_{\vec{k}}(\vec{q}, \lambda) \sqrt{|\Delta_{\vec{k}}(\vec{q})|\theta(\Delta_{\vec{k}}(\vec{q}))(b^+_{\vec{k}}(\vec{q}, i\omega_n) - b_{\vec{k}}(-\vec{q}, -i\omega_n))
\]

\[
\times (b_{\vec{k}}(\vec{q}, i\omega_n) + b^+_{\vec{k}}(-\vec{q}, -i\omega_n)),
\]

and

\[
S_{\sigma} = \frac{1}{2} \sum_{\vec{k}, \vec{q}, i\omega_n} (-i\omega_n + \frac{\vec{k} \cdot \vec{q}}{m}) s^+_{\vec{k}}(\vec{q}, i\omega_n) s_{\vec{k}}(\vec{q}, i\omega_n), \tag{10b}
\]

where

\[
m_{\vec{k}}(\vec{q}, \lambda) = \frac{2e}{m} \left( \frac{\pi}{\Omega_q} \right)^{\frac{d}{2}} 2^{\frac{d}{2}} \xi_{\vec{k}}(\vec{q}, \lambda).
\]

\( S_{\rho-p} \) describes a coupled system of (fermion) density particle-hole pair excitations and photons, whereas \( S_{\sigma} \) describes a system of free (fermion) spin particle-hole excitations. The density- and spin- particle-hole pair excitations are described by boson fields \( b(b^+)(\vec{q}) \) and \( s(s^+)(\vec{q}) \), respectively, satisfying usual boson commutation rules \([b(s)(\vec{k}), b(s)^+(\vec{k}')] = \)
\[\delta_{\vec{k}\vec{k}'} \delta_{\vec{q}\vec{q}'} \quad \text{and} \quad [b(s)_{\alpha}, b(s)_{\beta}] = [b(s)_{\alpha}^+, b(s)_{\beta}^+] = 0, \text{etc.} \] In this form the dynamics of the original fermion system is described completely in terms of charge- and spin- boson fields (bosonized).

Notice that we have restricted ourselves to the Gaussian approximation in deriving \(S_{\rho-p}\) and \(S_{\sigma}\). Higher order interaction terms between bosons will appear in a cumulant expansion of the \(\lambda_{\vec{k}}(q)\) fields \([10]\). The convergence of the cumulant expansion is formally controlled in our model by the small parameter \(\epsilon \sim (\Lambda/k_F)\), where a term of order \([b^m b^{\dagger(n-m)}]\) in the cumulant expansion has to leading order a term \(\sim \epsilon^{2(d-1)(n-1)}\) in each increasing order of cumulant expansion \([11,4]\). Notice however that the smallness of \(\epsilon\) does not guarantee that all calculated physical quantities will converge uniformly in the cumulant expansion. In particular, we shall see that infra-red divergence appears in higher-order expansion of the transverse gauge field \([12,13]\). We shall discuss in more detail the consequences of the infra-red divergences in section V. Notice also that in the \(\vec{q} \to 0\) limit, \(\Delta_{\vec{k}}(\vec{q}) \to -\delta(\epsilon_\vec{k} - \mu)(\frac{\vec{k} \cdot \vec{q}}{m})\) and the usual "tomographic" bosonization procedure based on subdivision of Fermi surface into disjoint patches at small \(\vec{q}\) is recovered \([6-8]\). Our bosonization procedure can be viewed as a generalization of the tomographic bosonization method for small wave vector \(\vec{q}\) to arbitrary values of \(|\vec{q}| < \Lambda\).

It is straightforward to show that the Gaussian approximation is essentially the same as usual RPA theory for interacting fermions \([1,6,8]\), except the additional assumption that particle-hole excitations can be treated as independent bosons in bosonization theory \([4]\). Note that particle-hole pairs are not all independent in a fermion system because of the Pauli exclusion principle. The assumption of independent bosons is a major approximation in the Gaussian theory in two- and three- dimensions. This is in contrast to what happens in one dimension where the entire particle-hole excitation spectrum can be represented rigorously by bosons \([8]\) when the fermion spectrum is linearized near the Fermi surface.

Despite the approximations in the Gaussian theory, the bosonized form of the action has the advantage that within the approximation the full excitation spectrum as well as the wavefunctions of the system can be obtained easily. This allows us to study the properties of the system in great detail, as we shall see in the following.
III. DENSITY PARTICLE-HOLE EXCITATION SPECTRUM

The eigenstates and eigenvalue spectrum described by the action $S_{ρ−p}$ can be obtained by diagonalizing the bosonized action (10a) using a generalized Bogoliubov transformation. We introduce for each wave vector $\vec{q}$ the Bogoliubov transformation [14]

\[
b_k(\vec{q}) = \sum_{k'} \left[ \alpha_{kk'}^{>\gamma k'}(\vec{q}) + \beta_{kk'}^{<\gamma k'}(\vec{q}) \right], \\
b_{-k}(−\vec{q}) = \sum_{k'} \left[ \alpha_{kk'}^{<\gamma k'}(−\vec{q}) + \beta_{kk'}^{>\gamma k'}(−\vec{q}) \right],
\]

where $k = \vec{k}, \lambda(k' = \vec{k}', \lambda')$, and with correspondingly,

\[
\gamma_k(\vec{q}) = \sum_{k'} \left[ \alpha_{kk'}^{>\gamma k'}(\vec{q}) - \beta_{kk'}^{<\gamma k'}(−\vec{q}) \right], \\
\gamma_{-k}(−\vec{q}) = \sum_{k'} \left[ \alpha_{kk'}^{<\gamma k'}(−\vec{q}) - \beta_{kk'}^{>\gamma k'}(\vec{q}) \right],
\]

where we require that the $\gamma(\gamma^+)_k(\vec{q})$ operators diagonalize the Hamiltonian, i.e.

\[
H_{ρ−p} = \sum_k E_k(\vec{q}) \gamma_k^+(\vec{q}) \gamma_k(\vec{q}) + E_G,
\]

where $E_k(\vec{q})$ are the eigen-energies and $E_G$ is the ground-state energy of the system. The eigenstates may represent dressed particle-hole excitations ($k = \vec{k}$) or dressed photons ($k = \lambda$). Notice that additional collective modes may appear in the system and is also included in the sum $\sum_k$. The matrix elements $\alpha$ and $\beta$ satisfies the orthonormality condition

\[
\sum_{k''} \left[ \alpha_{kk''}^{>\gamma} \alpha_{k''k''}^{>\gamma^*} - \beta_{kk''}^{<\gamma} \beta_{k''k''}^{<\gamma^*} \right] = \delta_{kk'}, \\
\sum_{k''} \left[ \alpha_{kk''}^{<\gamma} \beta_{k''k''}^{>\gamma^*} - \beta_{kk''}^{<\gamma} \alpha_{k''k''}^{>\gamma^*} \right] = 0.
\]

Writing down the equation of motions for $b_k(\vec{q})$’s in terms of $\gamma(\gamma^+)_k(\vec{q})$’s [14], we obtain the Bogoliubov equations

\[
(E_{k'}(\vec{q}) - \frac{|\vec{k}, \vec{q}|}{m}) \alpha_{kk'}^{>\gamma} = -\theta(\Delta_{\check{k}}(\vec{q})) \sqrt{|\Delta_{\check{k}}(\vec{q})|} \left[ \frac{+(-)1}{L^{d/2}} \sum_\lambda m_{\check{k}}(\vec{q}, \lambda) (\alpha_{\check{k}k'}^{>\gamma} + \beta_{\check{k}k'}^{<\gamma^*}) \right] \left[ \frac{2v(\vec{q})}{L^{d}} \sum_{\vec{k''}} \theta(\Delta_{\check{k''}}(\vec{q})) \sqrt{|\Delta_{\check{k''}}(\vec{q})|} (\alpha_{\check{k''k'}^{>\gamma}} + \beta_{\check{k''k'}^{<\gamma^*}}) \right],
\]
\[ (E_{k'}(q) + \frac{\vec{k}\cdot\vec{q}}{m})\beta_{kk'}^{<(<)} = -\theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)} \left[ \frac{(-1)^{d/2}}{L^d/2} \sum_{\lambda} m_{\lambda k'}(q, \lambda)(\alpha_{\lambda k'}^{>(<)} + \beta_{\lambda k'}^{>(<)}) \right] + \frac{2\nu(q)}{L^d} \sum_{k'} \theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)}(\alpha_{k'k'}^{>(<)} + \beta_{k'k'}^{>(<)}) \] ,

and

\[ (E_{k'}(q) - \Omega_q)\alpha_{\lambda k'}^{>(<)} = \frac{(-1)^{d/2}}{L^d/2} \sum_{k'} m_{\lambda k'}(q, \lambda)\theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)}(\alpha_{k'k'}^{>(<)} - \beta_{k'k'}^{>(<)}) , \]

\[ (E_{k'}(q) + \Omega_q)\beta_{\lambda k'}^{>(<)} = \frac{(-1)^{d/2}}{L^d/2} \sum_{k'} m_{\lambda k'}(q, \lambda)\theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)}(\alpha_{k'k'}^{>(<)} - \beta_{k'k'}^{>(<)}) . \]

In general we find that there exists two kinds of solutions to these equations: (i) particle-hole continuum, with \( E_{k'}(q) = |\vec{k}\cdot\vec{q}|/m \), and (ii) collective modes, including (renormalized) photons with energy \( E_\lambda(q) \) determined by the eigenvalue equation \( E_\lambda(q)^2 - \Omega_q^2 - \frac{4\pi^2}{m^2}\chi_\lambda(q, E_\lambda(q)) = 0 \), and plasmons, with energy \( E_0(q) \) given by the eigenvalue equation \( 1 - \nu(q)\chi_0(q, E_0(q)) = 0 \), where

\[ \chi_\lambda(q, \omega) = \frac{2}{L^d(d-1)} \sum_{k} (\vec{k},\vec{k}_t)\chi_{0\vec{k}}(q, \omega) , \]

is the paramagnetic transverse current susceptibility, \( \vec{k}_t = \vec{k} - \vec{q}(\vec{k},\vec{q}) \) and

\[ \chi_0(q, \omega) = \frac{2}{L^d} \sum_{k} \chi_{0\vec{k}}(q, \omega) , \]

is the Lindhard function. The energies of these collective modes are outside the particle-hole continuum. We now consider the solutions in more detail. First we consider the particle-hole excitations. After some lengthy algebras, we obtain:

\[ \alpha_{kk'}^{>(<)} = \delta_{kk'} + \frac{\theta(\Delta_{k'}(q))\theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)\Delta_{k'}(q)}}{L^d/2} \times \left( 2\nu_{eff}(q, |\vec{k}'\cdot\vec{q}|/m) - \vec{k}_t,\vec{k}_{eff}(q, \vec{k}') \right) \]

\[ \beta_{kk'}^{>(<)} = \frac{\theta(\Delta_{k'}(q))\theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)\Delta_{k'}(q)}}{L^d/2} \times \left( 2\nu_{eff}(q, -|\vec{k}'\cdot\vec{q}|/m) + \vec{k}_t,\vec{k}_{eff}(q, \vec{k}') \right) \]

\[ \alpha_{\lambda k'}^{>(<)} = -\frac{\omega_{k'}(q, \Omega_q)\omega_{k'}(q, \Omega_q)}{L^d/2} \times \left( \frac{1}{2\theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)}} \langle \bar{\xi}_\lambda(q)\bar{\xi}_\lambda(q) \rangle \right) \]

\[ \beta_{\lambda k'}^{>(<)} = \frac{\omega_{k'}(q, \Omega_q)\omega_{k'}(q, \Omega_q)}{L^d/2} \times \left( \frac{1}{2\theta(\Delta_{k'}(q))\sqrt{\Delta_{k'}(q)}} \langle \bar{\xi}_{\lambda}(q)\bar{\xi}_{\lambda}(q) \rangle \right) \]
for the particle-hole continuum spectrum $\tilde{k}'$, where $v_{\text{eff}}(q, \omega) = v(q)/(1 - v(q)\chi_0(\tilde{q}, \omega)$ is the RPA effective interaction, and $\tilde{A}_{\text{eff}}(\tilde{q}, \tilde{k}') = (\frac{8\pi e^2}{m^2}k_f^2)/((\tilde{k}', \tilde{q})/m)^2 - \Omega_q - \frac{4\pi e^2}{m^2}\chi_1(\tilde{q}, |\tilde{k}', \tilde{q}|/m))$.

We also obtain $\alpha(\beta)_{\tilde{k}'}^\text{L} = -\alpha(\beta)_\lambda^\text{L}$, and $\alpha(\beta)_{\tilde{k}'}^\text{L} = \alpha(\beta)_\lambda^\text{L}$.

For the collective modes, we obtain for the longitudinal (plasmon) mode,

$$\alpha_{\tilde{k}0}^> = \frac{1}{L^{d/2}} \frac{2\theta(\Delta_\lambda(\tilde{q})\sqrt{|\Delta_\lambda(\tilde{q})|}}{(E_\lambda(\tilde{q}) - |k_{\tilde{q}}| m)} \alpha_\lambda^\text{L},$$

(15b)

$$\beta_{\tilde{k}0}^< = \frac{1}{L^{d/2}} \frac{2\theta(\Delta_\lambda(\tilde{q})\sqrt{|\Delta_\lambda(\tilde{q})|}}{(E_\lambda(\tilde{q}) + |k_{\tilde{q}}| m)} \beta_\lambda^\text{L},$$

and $\alpha(\beta)_{\tilde{k}0}^\text{L} = \alpha(\beta)_{\tilde{k}0}^\text{L}$, $\alpha(\beta)_{\tilde{k}0}^\text{L} = 0$. Notice that in bosonization theory, plasmon arises from non-perturbative interaction effect and cannot be obtained from analytic continuation of the non-interacting boson modes. Similarly, we also obtain for the (renormalized) photon modes,

$$\alpha_{\tilde{k}\lambda}^> = -\frac{\theta(\Delta_\lambda(\tilde{q})\sqrt{|\Delta_\lambda(\tilde{q})|}}{E_\lambda(\tilde{q}) - |k_{\tilde{q}}| m} \frac{(k.\xi_\lambda(\tilde{q})C(q))}{L^{d/2}} \alpha_\lambda^\text{L},$$

(15c)

$$\beta_{\tilde{k}\lambda}^< = -\frac{\theta(\Delta_\lambda(\tilde{q})\sqrt{|\Delta_\lambda(\tilde{q})|}}{E_\lambda(\tilde{q}) + |k_{\tilde{q}}| m} \frac{(k.\xi_\lambda(\tilde{q})C(q))}{L^{d/2}} \beta_\lambda^\text{L},$$

$$\alpha_{\tilde{k}\lambda'}^< = \frac{\delta_{\lambda\lambda'}}{E_\lambda(\tilde{q}) - \Omega_q m} \frac{e(\pi)}{\Omega_q} \frac{1}{2} \chi_1(\tilde{q}, E_\lambda(\tilde{q}))C(q),$$

$$\beta_{\tilde{k}\lambda'}^> = -\frac{\delta_{\lambda\lambda'}}{E_\lambda(\tilde{q}) + \Omega_q m} \frac{e(\pi)}{\Omega_q} \frac{1}{2} \chi_1(\tilde{q}, E_\lambda(\tilde{q}))C(q),$$

where

$$C(q) = \left( \frac{(\frac{4\pi e^2}{m^2})}{2E_\lambda(\tilde{q}) - (\frac{4\pi e^2}{m^2})[\partial\chi_1(\tilde{q}, \omega)]_{\omega = E_\lambda(\tilde{q})}} \right)^{\frac{1}{2}},$$

and with $\alpha(\beta)_{\tilde{k}\lambda}^\text{L} = \alpha(\beta)_{\lambda\lambda'}^\text{L}$ and $\alpha(\beta)_{\tilde{k}\lambda}^\text{L} = -\alpha(\beta)_{\tilde{k}\lambda'}^\text{L}$.

Next we examine the solutions of the bosonized Hamiltonian in the $e^2 \to \infty$ limit. First we consider the collective modes. Using the result that $\chi_0(\tilde{q}, \omega) \to \frac{n_0q^2}{m^2}$, and $\chi_1(\tilde{q}, \omega) \to \frac{2n_0q^2}{m^2} \epsilon$ in the limit $\omega \gg k_F q/m$, where $n_0$ is the fermion density. $\frac{n_0q^2}{m^2}$ and $\epsilon$ is the
average kinetic energy per fermions in the free fermion ground state, it is easy to see that in the limit \( e^2 \rightarrow \infty \), the collective mode frequencies \( E_0(q), E_\lambda(q) \) all approaches the plasma frequency \( \omega_p = (\frac{4\pi n_0 e^2}{m})^{\frac{1}{2}} \). Notice that \( \omega_p \rightarrow \infty \) as \( e^2 \rightarrow \infty \), indicating that the collective modes are outside the physical spectrum in this limit.

Despite the vanishing of collective excitations in the physical spectrum, the particle-hole excitation spectrum with excitation energies \( |\vec{k},\vec{q}|/m \) survives in bosonization theory in the limit \( e^2 \rightarrow \infty \). In this limit \( v_{\text{eff}}(q,|\vec{k}',\vec{q}|/m) \rightarrow -1/\chi_0(q,|\vec{k}',\vec{q}|/m) \) and \( \vec{k},\vec{A}_{\text{eff}}(\vec{q},\vec{k}') \rightarrow -2\vec{k},(\vec{k}'/m) \) and \( e^2 \rightarrow \infty \). The coefficients \( \alpha_{\vec{k}k} \)'s and \( \beta_{\vec{k}k} \)'s remain regular, indicating that the particle-hole excitation spectrum survives under effect of confinement. It is instructive to examine the charge and (transverse ) current fluctuations carried by the (eigen)-particle-hole excitations by examining the commutators \([\rho(\vec{q}),\gamma_{\vec{k}}(\vec{q}')]\) and \([\vec{j}_t(\vec{q}),\gamma_{\vec{k}}(\vec{q}')]\), where

\[
\vec{j}_t(\vec{q}) = \sum_{\vec{k},\sigma} e\vec{k}_t \rho_{\vec{k}\sigma}(\vec{q}) - (L^d/2) \sum_{\lambda} e^2 n_0 \vec{A}_t(-\vec{q},\lambda),
\]

is the transverse current operator. In particular, we expect that these commutators should vanish in the \( e^2 \rightarrow \infty \) limit, where charge and current fluctuations are forbidden. Using Eqs. (14), (15a) and usual boson commutation rules, it is straightforward to show that

\[
[e\rho(\vec{q}),\gamma_{\vec{k}}(\vec{q}')] = \delta_{\vec{q}\vec{q}'} \theta(\Delta_{\vec{k}}(\vec{q})) \sqrt{|\Delta_{\vec{k}}(\vec{q})|} \times \frac{-e}{1 - v(q)\chi_0(q,|\vec{k},\vec{q}|/m)}, \tag{16}
\]

and

\[
[\vec{j}_t(\vec{q}),\gamma_{\vec{k}}(\vec{q}')] = \delta_{\vec{q}\vec{q}'} \frac{-\sqrt{2}e}{m} \theta(\Delta_{\vec{k}}(\vec{q})) \sqrt{|\Delta_{\vec{k}}(\vec{q})|} \times \left( \frac{|\vec{k},\vec{q}| - (cq)^2}{|\vec{k},\vec{q}| - \Omega_q^2 - 4\pi e^2/m^2 \chi_t(q,|\vec{k},\vec{q}|/m)} \right), \tag{17}
\]

both vanishes in the limit \( e^2 \rightarrow \infty \).

Let us summarize our results obtained so far from bosonization theory. Within the Gaussian approximation, we obtain a RPA-like charge excitation spectrum with both collective modes and particle-hole excitations, and a free-fermion type spin particle-hole excitation spectrum. As the coupling constant \( e^2 \) increases, the energies of the charge collective modes rise continuously to infinity whereas both the charge and spin particle-hole excitation spectrum remains intact. The charge and current fluctuations carried by the charge particle-hole
excitations are projected out gradually as $e^2$ increases, resulting in chargeless particle-hole excitations in the confinement limit $e^2 \to \infty$. Notice that within the Gaussian approximation, the confinement state analytically continues to the usual Fermi liquid state and there is no phase transition in between. The theory thus suggests that the confinement state of a gas of fermions is a fermion liquid state formed by spin $S = 1/2$ and chargeless-quasi-particles. It also suggests that this is a rather unusual fermion liquid state, since bare fermions in the system carries spin $1/2$ and charge $e$, and the quasi-particles must have vanishing overlap with bare fermions if they carry zero charge. The nature of the liquid of chargeless fermions will be examined in more details in next section, where we shall construct explicitly "chargeless" fermion operators that describe the dynamics of the system in the limit $e^2 \to \infty$. The limitations of Gaussian theory will be uncovered in the process.

IV. SINGLE-PARTICLE PROPERTIES AND LOW-ENERGY EFFECTIVE LAGRANGIAN

In bosonization theory for one-dimensional systems, the single particle properties of the system can be determined once a rigorous representation of the single-particle operator in terms of density operators $\rho_L(q)$ and $\rho_R(q)$ are obtained \[6\]. In higher dimensions, this procedure becomes inadequate for two reasons: (1)The corresponding procedure requires that the bosonized representation of single-particle operator $\psi_{\sigma}(\vec{r})$ satisfies the commutation relations

$$[\psi_{\sigma}(\vec{r}), \rho_{\vec{k}}(\vec{q})] = e^{-i(\vec{k}+\vec{q}/2)\cdot\vec{r}} \int d^dr'e^{i(\vec{k}-\vec{q}/2)\cdot\vec{r}'} \psi_{\sigma}(\vec{r}')$$

for all possible momenta $\vec{k}$ and $\vec{q}$. We have not been able to find a representation which satisfies this criteria \[2,3,4,4\], and even if we can find such an representation, the theory would still be approximate because in dimensions higher than one, the boson representation using Wigner operators is not exact, (2)more importantly, unlike in one dimension where the elementary excitations are collective density- and spin-wave modes, bosonization theory
suggests that in dimensions higher than one the particle-hole excitation spectrum is fermi-liquid like, implying that fermionic quasi-particles exist in dimensions higher than one. It is important to construct directly quasi-particle operators in this case.

To construct the quasi-particle operators we start from the equation of motion of the bare fermion operator \( \psi_\sigma(\vec{r}) = \frac{1}{L^{d/2}} \sum_{\vec{k}} e^{i\vec{k}.\vec{r}} f_{\sigma} \) at imaginary time,

\[
\frac{\partial \psi_\sigma(\vec{r})}{\partial \tau} = [H, \psi_\sigma(\vec{r})]
\]

\[
= \frac{1}{2m} \nabla^2 \psi_\sigma(\vec{r}) - \frac{1}{L^d} \sum_{\vec{q}} v(q) \rho(\vec{q}) e^{-i\vec{q}.\vec{r}} \psi_\sigma(\vec{r}) - \frac{ie}{mL^{d/2}} \sum_{\vec{q}\lambda} e^{i\vec{q}.\vec{r}} \bar{A}(\vec{q}, \lambda).\nabla \psi_\sigma(\vec{r}).
\]

where we have replaced the fermion-density operator \( \rho(\vec{r}) \) by its expectation value \( n_0 \) in the diamagnetic term as we have done in sections II and III in deriving (18). We shall discuss the validity of this approximation at the end of this section. In the bosonization approximation, the operators \( \rho(\vec{q}) \) and \( A(\vec{q}, \lambda) \) can be decomposed as \( \rho(\vec{q}) = \rho_{ph}(\vec{q}) + \rho_c(\vec{q}) \), and \( A(\vec{q}, \lambda) = A_{ph}(\vec{q}, \lambda) + A_c(\vec{q}, \lambda) \), where \( v(q) \rho_{ph}(\vec{q}) \) and \( \frac{e\bar{A}_{ph}(\vec{q}, \lambda)}{c} \) describes the coupling of the fermion to particle-hole excitations through the scalar and vector field fluctuations, respectively, whereas \( v(q) \rho_c(\vec{q}) \) and \( \frac{e\bar{A}_c(\vec{q}, \lambda)}{c} \) describes coupling of the fermion to collective modes (plasmons and photons). It is straightforward to obtain

\[
v(q) \rho_{ph}(\vec{q}) = \sum_{\vec{k}} \sqrt{[\Delta_E(\vec{q})] \theta(\Delta_E(\vec{q})]} \left[ v_{eff}(q, \frac{[\vec{k}.\vec{q}]}{m}) \gamma^{+}_k(\vec{q}) + v_{eff}(q, \frac{[\vec{k}.\vec{q}]}{m}) \gamma^{-}_k(-\vec{q}) \right], \quad (19a)
\]

\[
v(q) \rho_c(\vec{q}) = L^{d/2} \left( -\frac{\partial \chi_{0}(q, \omega)}{\partial \omega} \right)^{-\frac{1}{2}} \omega = E_0(\vec{q}) \left[ \gamma^{+}_{0}(\vec{q}) + \gamma^{0}_{0}(-\vec{q}) \right],
\]

and

\[
\frac{e\bar{A}_{ph}(\vec{q}, \lambda)}{m} = \frac{-1}{\sqrt{2L^{d/2}}} \xi_{\lambda}(\vec{q}) \sum_{k} \theta(\Delta_E(\vec{q})) \sqrt{[\Delta_E(\vec{q})]} \left[ \bar{\xi}_{\lambda}(\vec{q}) A_{eff}(\vec{q}, \vec{k}) \gamma^{+}_{k}(\vec{q}) - \bar{\xi}_{\lambda}(\vec{q}) A^{*}_{eff}(\vec{q}, \vec{k}) \gamma^{+}_{-k}(-\vec{q}) \right] \quad (19b)
\]

\[
\frac{e\bar{A}_c(\vec{q}, \lambda)}{m} = \sqrt{2C(q)} \xi_{\lambda}(\vec{q}) \left( \gamma_{\lambda}(\vec{q}) + \gamma_{-\lambda}^{+}(-\vec{q}) \right).
\]

In particular, it is easy to see that in the \( e^2 \to \infty \) limit, the interaction between fermion and particle-hole excitations in both longitudinal \( v(q) \rho_{ph}(q) \) and transverse \( e\bar{A}_{ph}(\vec{q}, \lambda)/c \) channels remains regular and finite, and divergences in the \( e^2 \to \infty \) limit appear only through the interactions between fermion and collective excitations. Notice that infra-red divergences
in the interaction between fermion and particle-hole excitations also exist. However, they are not results of taking $e^2 \to \infty$ and are not considered here. We shall discuss the effects of infra-red divergences in the last section.

The divergence in interaction between fermions and collective modes suggests that we have to eliminate these interactions first to study dynamics of (physical) fermion operators in the limit $e^2 \to \infty$. To eliminate these interactions we look for a canonical transformation for the fermion operator [15],

$$\psi_{P_{\sigma}}(\vec{r}) = e^{\phi(\vec{r})} \psi_{\sigma}(\vec{r}).$$  \hspace{1cm} (20)

with corresponding equation of motion,

$$[H, \psi_{P_{\sigma}}(\vec{r})] = [H, e^{\phi(\vec{r})} \psi_{\sigma}(\vec{r})] + e^{\phi(\vec{r})} [H, \psi_{\sigma}(\vec{r})],$$ \hspace{1cm} (21)

where we shall choose $\phi(\vec{r})$ such that $[H, e^{\phi}]$ cancels the interaction term between fermion and collective modes, i.e.

$$[H, e^{\phi} \psi_{\sigma}(\vec{r})] = e^{\phi} \left[ \frac{1}{L^d/2} \sum_{\vec{q}} v(q) \rho_{\sigma}(\vec{q}) e^{-i\vec{q}.\vec{r}} \psi_{\sigma}(\vec{r}) + \frac{ie}{mL^d/2} \sum_{\vec{q} \lambda} e^{i\vec{q}.\vec{r}} \tilde{A}_{\sigma}(\vec{q}, \lambda) \nabla \psi_{\sigma}(\vec{r}) \right].$$

Furthermore, we shall assume that $\phi(\vec{r})$ depends linearly on the collective mode operators $\gamma_{0(\lambda)}(\vec{q})$ and $\gamma_{0(\lambda)}^+(\vec{q})$'s, and that $H$ can be replaced by the bosonized Hamiltonian [12] in evaluating the commutator $[H, e^{\phi}]$. Notice that the last assumption is justified in the $e^2 \to \infty$ or $\vec{q} \to 0$ limit, where both the longitudinal and transverse collective mode excitations become exact.

With these conditions, it is straightforward to show that $\phi(\vec{r}) = \phi_l(\vec{r}) + \tilde{W}(\vec{r}).\nabla$, where

$$\phi_l(\vec{r}) = \frac{1}{L^d/2} \sum_{\vec{q}} \frac{e^{-i\vec{q}.\vec{r}}}{E_0(\vec{q}) \left[ -\frac{\partial \chi_{0(\lambda)}(\omega)}{\partial \omega} \right]^2} (\gamma_0^+(\vec{q}) - \gamma_0(-\vec{q})),$$ \hspace{1cm} (22)

$$\tilde{W}(\vec{r}) = \sqrt{2i} \frac{L^d/2}{L^d/2} \sum_{\vec{q} \lambda} \frac{e^{i\vec{q}.\vec{r}} C(\vec{q})}{E_\lambda(\vec{q})} (\gamma^+_{\lambda}(-\vec{q}) - \gamma_{\lambda}(\vec{q})) \tilde{\xi}_{\lambda}(\vec{q}),$$

where $\phi_l$ and $\tilde{W}$ describes the dressing of fermion by (longitudinal) plasmon modes and (transverse) photon modes, respectively. Notice that $[\phi_l(\vec{r}), \tilde{W}(\vec{r}).\nabla] = 0$, because of decoupling between transverse and longitudinal fluctuations.
To show that $\psi_{P\sigma}(\vec{r})$ describes chargeless fermions in the $e^2 \to \infty$ limit, we examine the commutation relations between $\psi_{P\sigma}(\vec{r})$ and density and (transverse) current operators. It is straightforward to show that

$$\left[ e\rho(\vec{q}), \psi_{P\sigma}(\vec{r}) \right] = e \left( \frac{2\chi_0(q, E_0(\vec{q}))}{E_0(\vec{q})} \left[ -\frac{\partial\chi_0(q, \omega)}{\partial\omega} \right]_{\omega = E_0(\vec{q})} - 1 \right) e^{i\vec{q}.\vec{r}} \psi_{P\sigma}(\vec{r}), \quad (23a)$$

and

$$\left[ \vec{j}_t(\vec{q}), \psi_{P\sigma}(\vec{r}) \right] = \frac{ie}{m} \left( 1 - \frac{2C(q)^2}{E_\lambda(\vec{q})} \left[ \chi_t(\vec{q}, E_\lambda(\vec{q})) + n_0m \right] \right) e^{i\vec{q}.\vec{r}} \nabla_t \psi_{P\sigma}(\vec{r}). \quad (23b)$$

where $\nabla_t = \nabla - \hat{q}(\hat{q}.\nabla)$. It is easy to see that both commutators vanish in the limit $e^2 \to \infty$, when $E_{0(\lambda)}(\vec{q}) \to \omega_P \to \infty$. It is also easy to see that the $\psi_{P\sigma}$ operator has the same commutation relation with spin density operator $\sigma(\vec{q}) = \sum_{k} \sigma_k(\vec{q})$ as bare fermion operator $\psi_\sigma$. These results together imply that the dressed single particle operators $\psi_{P\sigma}(\vec{r})$'s represent spin $S = 1/2$, 'chargeless' fermions in the limit $e^2 \to \infty$.

To show that $\psi_{P\sigma}(\vec{r})$ and $\psi_{P\sigma}(\vec{r}')$ represent independent physical fermionic excitations in the system when $\vec{r} \neq \vec{r}'$ we examine the commutation relation between the dressed fermion operators. We obtain

$$\{ \psi_{P\sigma}(\vec{r}), \psi_{P\sigma}^+(\vec{r}') \} \sim \frac{\sqrt{2}}{n_0(\pi |\vec{r} - \vec{r}'|)^{d-1}} \hat{O}(\vec{r}, \vec{r}'),$$

in the limits $e^2 \to \infty$ and $|\vec{r} - \vec{r}'| \to \infty$, where

$$\hat{O} = \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \right) \left[ \psi_{\sigma^+}(\vec{r}')e^{\phi(\vec{r}')}(\nabla \psi_\sigma(\vec{r}'))e^{\phi^+(\vec{r}')} - (\nabla \psi_{\sigma'}^+(\vec{r}'))e^{\phi(\vec{r})}\psi_\sigma(\vec{r})e^{\phi^+(\vec{r})} \right].$$

The main contribution to the commutator comes from nonzero commutation relation between the plasmon cloud around one chargeless fermion ($\phi_t(\vec{r})$) and the other bare fermion ($\psi_{\sigma'}(\vec{r}')$). The non-vanishing commutator between chargeless fermions separated by large distance reflects the nonlocal nature of the chargeless fermion operator. Fortunately, the power law decay of the commutator between $\psi_{P\sigma}$ operators separated by large distances at dimensions larger than one indicates that they can be used as starting point to construct independent single fermion operators when describing the dynamics of the system at long
distances. Notice that in one dimension, such a construction is not possible because of distance independence of the commutation relation. In fact, the only fermionic operators that commute with density operator $\rho(\vec{q})$’s are the ladder operators \[ \hat{a}^\dagger, \hat{a} \] which raise or lower the number of particles in the system by one. There are only four independent ladder operators in the system, which change the number of left-going and right-going spin-$\sigma$ fermions and cannot be used to construct local quasi-particle excitations.

The relation between the fermion liquid formed by the chargeless and bare fermions can be seen by examining the relation between the two ground state expectation values $\langle \psi_{\sigma}^+(\vec{r})\psi_{\sigma}(\vec{r}') \rangle$ and $\langle \psi_{\sigma}^+(\vec{r})\psi_{\sigma}(\vec{r}') \rangle$. Assuming that the collective modes can be treated as independent excitations, we obtain to leading order in the projected Hilbert space,

$$\langle \psi_{\sigma}^+(\vec{r})\psi_{\sigma}(\vec{r}') \rangle \sim e^{-2\frac{\pi m e^2}{\hbar c n_0}} \int_{L-1}^{L} d^d q_1 d^d q_2 (1 - \cos(\vec{q} \cdot (\vec{r} - \vec{r}')) ) \langle \psi_{\sigma}^+(\vec{r})\psi_{\sigma}(\vec{r}') \rangle. \quad (24a)$$

The singular behaviour in the exponential factor comes from plasmon contributions and photons do not contribute to this order. Notice that the average density of bare fermions and dressed fermions are the same and the fermi surface volume of the dressed fermions is exactly the same as that of the bare fermions. It is also easy to see that

$$\langle \psi_{\sigma}^+(\vec{r})\psi_{\sigma}(\vec{r}') \rangle \sim e^{-\frac{\pi m e^2}{\hbar c n_0}} \int_{L-1}^{L} d^d q_1 d^d q_2 (1 - \cos(\vec{q} \cdot (\vec{r} - \vec{r}')) ) \langle \psi_{\sigma}^+(\vec{r})\psi_{\sigma}(\vec{r}') \rangle, \quad (24b)$$

which vanishes in the $e \to \infty$ limit, indicating that the bare and chargeless fermions have zero wavefunction overlap as expected. Using Eq. (24a), it is straightforward to show that the bare-fermion occupation number $n(\vec{k})$ has no discontinuity across fermi surface in the $e^2 \to \infty$ limit (3D), even if the chargeless fermions form a fermi liquid, in agreement with the marginal fermi liquid picture.

The bosonization result we obtained in previous section suggests that in the limit $e^2 \to \infty$ the low energy physics of our system may be described as a fermi liquid of the chargeless fermions. To see whether this is the case we consider the equation of motion for the chargeless fermion operators $\psi_{\sigma}$, assuming that they can be treated as canonical fermions. It is straightforward to show that
\[
\frac{\partial \psi_{P\sigma}(\vec{r})}{\partial \tau} = \frac{1}{2m} \left( \nabla^2 \psi_{P\sigma}(\vec{r}) - \nabla^2 \phi(\vec{r}) - : (\nabla \phi(\vec{r}))^2 : \right) \psi_{P\sigma}(\vec{r}) - 2\nabla \phi(\vec{r}).\nabla \psi_{P\sigma}(\vec{r}) - \frac{1}{L^d} \sum_{\vec{q}} \nabla \left( v(\vec{q}) \rho_{ph}(\vec{q}) e^{i\vec{q}\cdot\vec{r}} \psi_{P\sigma}(\vec{r}) \right) - \frac{i}{L^d/2} \sum_{\vec{q}\lambda} e^{i\vec{q}\cdot\vec{r}} \frac{e}{m} \bar{A}_{ph}(\vec{q}, \lambda).\nabla \psi_{P\sigma}(\vec{r}),
\]

where we have neglected constant energy terms coming from normal ordering of operators under deriving Eq. (23). Notice that direct couplings between fermions and collective modes are absent in the equation of motion of \( \psi_{P\sigma}(\vec{r}) \). However, interactions between chargeless fermions and particle-hole excitations remain in the equation of motion. Moreover, an indirect coupling to collective modes is also generated from the fermion kinetic energy term as is in the similar "small polaron" problem [13]. It is easy to see by direct power counting of \( e^2 \) in the \( \phi(\vec{r}) \) field that the coupling of the "dressed" fermion to collective excitations through the kinetic energy term is much weaker than the original fermion-plasmon and fermion-photon couplings. In particular, the self-energy correction of chargeless fermions from \( \phi(\vec{r}) \) fields remains finite in the \( e^2 \rightarrow \infty \) limit.

We shall now study the dynamics of the chargeless fermions in more detail. First we write down an effective action for the chargeless fermion operator \( \psi_{P\sigma}(\vec{r}) \), the effective action is constructed with the requirement that it reproduces the equation of motion (25) for \( \psi_{P\sigma}(\vec{r}) \). It is straightforward to show that the correct action is

\[
S_{eff}(\psi_{P}, \bar{\psi}_{P}) = \sum_{\sigma} \int_0^\beta d\tau d^d x \psi_{P\sigma}(\vec{x}, \tau) \left[ \frac{\partial}{\partial \tau} - \nabla^2 \frac{2m}{2} - \mu + i\bar{A}_{eff}(\vec{x}, \tau) . \nabla + \phi_{eff}(\vec{x}, \tau) \right] \psi_{P\sigma}(\vec{x}, \tau),
\]

where \( \phi_{eff}(\vec{q}) \sim v(\vec{q}) \rho_{ph}(\vec{q}) \) and \( \bar{A}_{eff}(\vec{q}) \sim e\bar{A}_{ph}(\vec{q})/m \), with dynamics given by

\[
S_{eff}(\phi) = -\frac{1}{2} \sum_{\vec{q}, \i\omega} \chi_{0}(\vec{q}, \i\omega) |\phi_{eff}(\vec{q}, \i\omega)|^2,
\]

\[
S_{eff}(\bar{A}) = -\frac{1}{2} \sum_{\vec{q}, \i\omega} \bar{\chi}_{t}(\vec{q}, \i\omega) \bar{A}_{eff}(\vec{q}, \i\omega) . \bar{A}_{eff}(-\vec{q}, -\i\omega)
\]

where \( \bar{\chi}_{t}(\vec{q}, \i\omega) = \chi_{t}(\vec{q}, \i\omega) + mn_0 \) is the total transverse current susceptibility of free fermions. We have neglected in \( S_{eff}(\psi_{P}, \bar{\psi}_{P}) \) the collective mode contributions to the dynamics of \( \psi_{P\sigma} \) through the kinetic energy term. The residue plasmon-fermion interaction induces a weak
short-ranged effective attractive interaction between chargeless fermions and may lead to superconductivity. We shall not discuss this possibility in this paper.

Fermions interacting with gauge fields with effective interaction (26) have been studied in detail by a number of authors [2,3,12,13] and it is believed that interaction between fermions and transverse gauge field with effective dynamics (26) may lead to break down of fermi liquid theory and the formation of a marginal fermi liquid. If this is the case, the ground state formed by the chargeless fermions would itself be in a marginal fermi liquid state and our system can be viewed as a "double" marginal fermi liquid state of the original fermions. This is in contrast to (Gaussian) bosonization theory which predicts that the chargeless fermions should form a fermi liquid state. The failure of Gaussian theory in producing a marginal fermi liquid state shows the limitation of Gaussian theory. This point will be elaborated further in the last section.

It is interesting to make comparison between our approach and the usual treatment of the $U(1)$ gauge field in t-J model where the gauge field effectively imposes the constraints $\rho_s(q) + \rho_h(q) = 0$ and $\vec{j}_s(q) + \vec{j}_h(q) = 0$ [1], where $\rho_s(\vec{j}_s)$ and $\rho_h(\vec{j}_h)$ are the spin and hole densities (current densities) operators, respectively. In the usual treatment [1,16], the spins and holes were integrated out at Gaussian level, resulting in an effective action for gauge fields very similar to Eq. (26) except that the longitudinal and transverse response functions $\chi_0(q,\omega)$ and $\bar{\chi}_t(q,\omega)$ are replaced respectively by the sum of the longitudinal and transverse response functions from the spin- and hole- components, respectively. It is then assumed that the spinons and holons in the projected Hilbert space can be treated as free particles interacting with the gauge fields with dynamics given by $S_{\text{eff}}$ [1]. Although our analysis here includes only fermions and is not directly applicable to the t-J model, the form of our low energy effective Lagrangian is in agreement with this assumption, suggesting that the spinons in the usual treatment of t-J model are strongly related to our chargeless fermions. A detailed treatment of the t-J model using our approach will be reported in a different paper.

Finally we examine the validity of replacing the fermion density operator $\rho(\vec{r})$ by the
average fermion density $n_0$ in the diamagnetic term when we derive the equation of motions for the bare and chargeless fermions. We shall now show that this approximation is justified in the $e^2 \to \infty$ limit for the chargeless fermions $\psi_{P\sigma}$. We look at the correction term in the equation of motion for chargeless fermions, $[\Delta H, \psi_{P\sigma}(\vec{r})]$, where

$$\Delta H = \frac{e^2}{2m} \int d^dr \vec{A}(\vec{r}) \cdot \vec{A}(\vec{r})(\rho(\vec{r}) - n_0).$$

It is easy to show from Eqs. (19a), (23a) and (22) that $[\rho(\vec{q}), \psi_{P\sigma}(\vec{r})] \sim \frac{1}{e^2} \psi_{P\sigma}$, $e\vec{A}/m \sim e^{1/2}(\gamma_\lambda + \gamma_\lambda^\dagger)$ and $[e\vec{A}(\vec{r}), \psi_{P\sigma}(\vec{r})] \sim \nabla \psi_{P\sigma}$ in the $e^2 \to \infty$ limit. As a result we have

$$[\Delta H, \psi_{P\sigma}(\vec{r})] \sim (\vec{A}(\vec{r}'), \nabla \psi_{P\sigma}(\vec{r}))(\rho(\vec{r}) - n_0) + O(\frac{1}{e}).$$

The first term vanishes in the $e^2 \to \infty$ limit if the matrix elements $< n | (\rho(\vec{r}) - n_0) | m >$ vanish faster than $1/e^{1/2}$ in the physical Hilbert space of interests. This will be the case if the physical Hilbert space is spanned by the chargeless fermion operators we construct, because of the commutation rule between density operator and chargeless fermions (Eq. (23a)).

**V. SUMMARY**

Using a bosonization approach we studied in this paper a gas of spin $S = 1/2$ fermions interacting with a $U(1)$ gauge field with high momentum cutoff $q < \Lambda << k_F$. In particular we consider the $e^2 \to \infty$ limit where the gauge field becomes confining. We have analysed the problem in two steps. Within the Gaussian approximation we show in section III that a liquid state solution is obtained where the particle-hole excitation spectrum of the system is always fermi-liquid like, with the charge carried by the particle-hole excitation vanishing continuously in the $e^2 \to \infty$ limit. To verify the above picture we construct in section IV $S = 1/2$, chargeless fermionic operators that can be used as the starting point for constructing quasi-particles in the system. We find that the dynamics of these ”physical” single-particle operators are governed by a Lagrangian very similar to the effective Lagrangian obtained in conventional treatment of gauge field model \[1\]. The solution we obtained describes a
kind of 'double-marginal' fermi liquid state where (i) the chargeless fermions have vanishing wavefunction overlap with bare fermions in the system and (ii) they form a marginal fermi liquid state themselves.

It is useful to distinguish the effects of longitudinal and transverse gauge fields on our system, in particular how they contribute in the $e^2 \to \infty$ limit, to the construction of chargeless fermions and their effective dynamics. First we consider the longitudinal gauge field. The main effect of longitudinal gauge field appears through fermion-plasmon interaction which is singular in the limit $e^2 \to \infty$. In dimension $d = 2$, there exists an additional infra-red singularity associated with the response of plasmon field (orthogonality catastrophe effect) when charged particles are added to the system [17]. The effect exists for any finite values of $e^2 > 0$. The interaction between fermions and particle-hole excitations is finite and regular for any value of $e^2$. The singular fermion-plasmon interaction and the infra-red singularity at dimension $d = 2$ are eliminated together with our canonical transformation (20) [4], and the remaining interaction between the chargeless fermions and longitudinal gauge fields is regular and introduces no further singularities in the chargeless fermion system [4].

The situation is however, quite different with the transverse gauge field. There exists no infra-red singularity in the fermion-photon interaction at both two- and three-dimensions when $e^2$ is finite, and the singularity associated with the fermion-photon interaction in the limit $e^2 \to \infty$ is in fact weaker than the corresponding singularity associated with fermion-plasmon interaction, as can be seen from direct power counting of $e^2$ in $\phi_l$ and $\vec{W}$ fields. As a result, the canonical transformation (20) is dominated by effect of plasmons in the limit $e^2 \to \infty$. On the other hand, the interaction between bare fermions and particle-hole excitations through transverse gauge field is infra-red singular for any values of $e^2$ in both two- and three- dimensions [4], and the singularity carries over to the effective interaction (26) between dressed fermions and transverse gauge field. It is believed that the infra-red singularity in interaction between fermions and particle-hole excitations through transverse gauge field will lead to the formation of a marginal fermi liquid state [2,3,12,13].

The possibility of formation of an marginal fermi liquid state of the chargeless fermions
points out the limitation of bosonization theory in Gaussian approximation, where a fermi-liquid type particle-hole excitation spectrum is always obtained in the limit $e^2 \to \infty$. It is clear that higher-order terms in cumulant expansion of transverse gauge field has to be included in bosonization theory to obtain the correct marginal fermi liquid behavior \cite{2}, and a complete theory where particle-hole and single-particle excitations are treated self-consistently is still missing.

An immediate question that comes with this observation is that we have used the excitation spectrum in Gaussian approximation to construct chargeless fermion operators in our paper. If the particle-hole excitation spectrum is incorrect, our results based on chargeless fermion operators $\psi_{P\sigma}$ become questionable! Fortunately although the particle-hole excitation spectrum in Gaussian theory is questionable, the plasmon and photon excitations we obtained are exact in the $q \to 0$ or $e^2 \to \infty$ limits. In particular our canonical transformation (20) for the chargeless fermions $\psi_{P\sigma}$ involves only the collective mode operators and is independent of the Gaussian approximation! However, it has to be cautioned that in deriving the effective action $S_{eff}(\psi_p, \psi^+_p)$ which describes scattering of chargeless fermions with particle-hole excitations, it is assumed that the eigen-particle-hole excitations are described correctly by Gaussian theory. Strong modifications may occur if the spectrum of particle-hole excitation is modified strongly in the correct theory.

Another more fundamental question is how valid it to treat our chargeless fermions as canonical fermions, given that they obey a rather non-trival commutation relation. In particular, it is possible that the unusual commutation relation between fermions may lead to additional non-fermi liquid behaviour. At present we have no answer to these questions.

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