Method of sections in analytical calculations of pneumatic tires

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Abstract. Analytical calculations in the pneumatic tire theory are more preferable in comparison with experimental methods. The method of section of a pneumatic tire shell allows to obtain equations of intensities of internal forces in carcass elements and bead rings. Analytical dependencies of intensity of distributed forces have been obtained in tire equator points, on side walls (poles) and pneumatic tire bead rings. Along with planes in the capacity of secant surfaces cylindrical surfaces are used for the first time together with secant planes. The tire capacity equation has been obtained using the method of section, by means of which a contact body is cut off from the tire carcass along the contact perimeter by the surface which is normal to the bearing surface. It has been established that the Laplace equation for the solution of tasks of this class of pneumatic tires contains two unknown values that requires the generation of additional equations. The developed computational schemes of pneumatic tire sections and new equations allow to accelerate the pneumatic tire structure improvement process during engineering.

1. Introduction

Pneumatic-tire wheels have been widely adopted as movers and bearing elements of land transport vehicles. A pneumatic-tire wheel consists of a pneumatic tire, performed in the form of a torus, mounted on a metal rim. The internal airproof cavity of a tire is filled with compressed air. The structure and manufacturing technology of pneumatic tires characterize the development level of the modern engineering. However, the theory of pneumatic tires doesn’t reflect to the fullest extent the pneumatic and physical essence of pneumatic tires and is need of further development.

The relationship of environmental factors with processes inside a tire is established in mechanics by means of the method of sections. The theory of pneumatic tires is based on works of V.L. Biderman [1] during the 60s of the past century, in which the methods of sections were not properly adopted in calculations of pneumatic tires. The usage of the Laplace equation is a positive start, where non-traditional concentrated forces are used, but intensities of distributed forces per unit length of the tire section.

Insufficient attention is given in works of V.L. Biderman [1], B.L. Bukhin [2], V.I. Knoroz [3] to the research of the interaction of the tire contract area with bearing surface from the point of view of its influence on capacity, tire wear, tire drag, road stability and other parameters.
In the work of A. E. Belkin and others [4] the mechanics of tires is divided into external and internal one. The external mechanics bears relationship to the car dynamics and characteristics, and the internal one – to processes occurred inside a tire.

From the moment of the computer appearance a great number of researchers use the finite element method for the research of internal processes inside a tire.

In work [5] S.V. Sheshenin notes that models taking into account huge tire deflections, viscoelastic properties of rubber are very complicated. That’s why the matrix of resulting linear equation systems is turned out to be ill-conditioned.

In work [6] M.A. Salaani investigates the problems of the tire contact area with the bearing surface using Hermitian cubic functions and Hooke’s law. The distribution model of normal and tangential forces of tire traction gives reliable results.

In work [7] J. Wallaschek and B. Wies notes that the modeling of tires has reached a sufficiently high level, allowing to take into account the deformation rates of elastic elements, friction force of a tire tread with road and many other factors.

In work [8] P.E. Stelzig considers a pneumatic tire like an object subjected to huge deformations within specified severe limitations. He considers the operation of cords and strips, forming the supporting carcass of a tire. A complex process modeling unit is used, taking into account pre-distortions for ensuring the approximation of special functions of the limitation of variations, maintaining natural kinematic functions.

The fulfilled review shows that the natural complication of research methods has occurred. The method of sections is insufficiently used, allowing to establish the relation of external factors of tires with internal process in tire elements.

2. Task definition

Computational schemes of tire carcass sections were developed and equations were derived for the determination of the tire capacity, as well as equations of internal force intensities in the most dangerous points and elements of the carcass: in tire equator points, on bead rings etc. The obtained results are based on equilibrium equations using linear intensities of distributed forces along the section perimeters.

3. Theory

In the work of V.N. Tarasov [9] for the first time it is suggested to use the method of section for the research of pneumatic tire parameters. The cutoff of the contact body from the tire carcass was performed along the contact perimeter by the closed secant surface which is normal to the bearing surface.

As a result of the application of the method of sections a pneumatic-tire wheel is considered as a mechanism, where the cutoff contact body interacts with the carcass, at the same time the compressed air inside the tire is the main physical factor, determining the pneumatic-tire wheel capacity and operability.

Thanks to the method of sections in the theory of pneumatic tire the internal forces transfer into the category of external forces, which ensure the carcass equilibrium without the cutoff contact body and equilibrium of the free cutoff contact body. The tire is considered as a thin-wall shell, loaded with the internal pressure. From the equilibrium condition the intensity of forces in the carcass walls is determined.

In engineering for the calculation of thin-wall shells the Laplace equation is used in the works of V.L. Biderman [1], S.P. Timoshenko [10]

\[
\frac{T_m}{\rho_m} + \frac{T_l}{\rho_l} = p_w, \tag{1}
\]
where $T_m$, $T_t$ – correspondingly the intensity of distributed meridional and circumferential forces per unit length of the tire section (Figure 1); $\rho_m$, $\rho_t$ – shell radiuses, tangential to meridional and circumferential directions; $p_w$ – tire air pressure.

![Figure 1. Computational scheme to the Laplace equation for the tire carcass infinitely small element.](image)

Equation (1) contains two unknown values $T_m$ and $T_t$. In these cases it’s necessary to write down an additional equilibrium equation the Laplace equation.

In work [11] V.N. Tarasov and I.V. Boyarkina proposed the method of section of the tire carcass by cylindrical surface with $r_o$ radius along the main parallels of a pneumatic tire. At the same time the tire carcass is sectioned by the vertical surface $a-a$ along the tire equator (Figure 2).

![Figure 2. Equilibrium of sectioned tire elements: a) equilibrium of one fourth part of the torus b) equilibrium of three fourth part of the torus.](image)

Figure 2.a shows the equilibrium of the cut off one fourth part of the tire torus, and Figure 2.b shows the carcass equilibrium without the cut off element of the tirep. Point $I$ on the tire equator and point 2 on the tire side walls (poles) are considered as the most important for the tire carcass strength, and they determine the tire section geometry on Figure 2.b.

On Figure 2.b we have the system of parallel forces relative to the axis $Oy$, being in equilibrium, and the system of radial concurrent forces to the axis $Oy$, which is also in equilibrium. That’s why the indicated systems of forces cannot rotate the considered and they are in equilibrium. Conditional vectors $T_{m1}$, $T_{m2}$ represent intensities of specific forces per unit length of the section perimeter ($N/m$).

For these forces it is possible to derive the equilibrium equation in the form of the equality of air pressure active forces and carcass reaction forces for relative lengths of the section perimeters $2\pi R$, $2\pi r_o$

$$\sum F_{ky} = 0; \quad -T_{m1} \cdot 2\pi R + p_w \pi \left( R^2 - r_o^2 \right) = 0;$$

(2)
\[ T_{m2} \cdot 2\pi r_o - p_w \cdot 2\pi r_o 0,5b = 0, \]  \quad (3)

where \( R \) – radius of the torus internal surface for the tire equator; \( r_o \) – average radius of the side walls of the torus internal surface; \( b \) – width of the internal surface of the tire profile.

From equations (2), (3) expressions for the determination of intensities of specific meridional forces \( T_{m1} \) and \( T_{m2} \) in the tire section perimeters were obtained

\[ T_{m1} = \frac{p_w(R^2 - r_o^2)}{2R}, \]  \quad (4)

\[ T_{m2} = p_w 0,5b, \]  \quad (5)

Formulas (4), (5) in the pneumatic tire theory allow to determine the values of the meridional specific forces in points 1 and 2 on the equator and poles of the pneumatic tire.

Figure 3 shows the computational scheme for the determination of the radial intensity of the distributed forces \( T_{m1} \).

![Figure 3](image_url)

**Figure 3.** Equilibrium of the tire sectioned elements by the plane \( a-a \).

The equilibrium equation for Figure 3 is as follows:

\[ T_{m1} \cdot 2\pi R = \pi \left( R^2 - (0,5d)^2 \right) p_w. \]  \quad (6)

From where

\[ T_{m1} = \frac{p_w \left( R^2 - (0,5d)^2 \right) p_w}{2R}. \]  \quad (7)

Formula (7) doesn’t coincide with formula (4). That’s why formula (7) is the correct one, which gives the greater absolute result and it’s recommended for the determination of the intensity \( T_{m1} \).

In work [1] formula (4) is known, but more exact formula (7) was derived for the first time in this work.

Figure 4 shows the section for the analytic definition of the intensity of carcass radial distributed forces for pole points on the tire side wall.

The equilibrium equation of intensities of carcass reaction forces and tire air pressure is as follows:

\[ 2T_{m2} \cdot 2\pi r_o - 2\pi r_o bp_w = 0. \]  \quad (8)

From where

\[ T_{m2} = 0,5bp_w. \]  \quad (9)

Equation (9) coincides with equation (2), which is absent in technical literature and represents a new result in the pneumatic tire theory.
Figure 4. Equilibrium of tire elements, cut off by cylindrical secant surface with $r_o$ radius.

Figure 5 represents the computational scheme in order to write down the analytical equation of forces in bead rings. Cylindrical secant surface with $r_{B,K}$ radius cuts off the seat rim from the tire. The ring radius $r_{B,K}$ is greater than the radius of the seat rim disk.

The secant surface

Figure 5. Equilibrium of tire elements, cut off by cylindrical secant surface with $r_{B,K}$ radius.

The equilibrium equation of the distributed intensities $T_{B,K}$ and air pressure forces is as follows:

$$2T_{B,K}2\pi r_{B,K} = 2\pi r_{B,K}b_{B,K}p_w.$$  \hspace{1cm} (10)

$$T_{B,K} = 0.5b_{B,K}p_w.$$ \hspace{1cm} (11)

Let’s consider the possibilities of Laplace equation (1) usage for the pneumatic tire calculation. For point $l$ on the tire equator (see Figure 2,b) the Laplace equation is as follows:

$$\frac{T_{ml}}{0.5H} + \frac{T_{ol}}{R} = p_w,$$ \hspace{1cm} (12)

where $H$ – tire profile height.
Laplace equation (12) contains two unknown values.

For point 2 of the tire (see Figure 2,a) the Laplace equation is as follows:

\[ \frac{T_{m2}}{0.5b} + \frac{T_{t1}}{\infty} = p_w. \] (13)

In point 2 the tire carcass has only one curvature, then from equation (13) we’ll find

\[ T_{m2} = 0.5b_{BK} p_w. \] (14)

Obtained formula (14) coincides with solution (11).

The Laplace equation for pneumatic tires allows to solve a limited class of tasks for objects with one wall curvature. Such objects include, for example, a thin-wall cylindrical pipe, the Laplace equation for which is as follows:

\[ \frac{T_m}{r} + \frac{T_i}{\infty} = p_w, \] (15)

where \( r \) – pipe section radius.

Then the intensity for the pipe wall is equal to \( T_m = rp_w \). If the force \( (T_m \cdot 1) \) effects on one running meter of the pipe wall, then the pipe wall stress will be equal to

\[ \sigma = \frac{T_m}{\delta} = \frac{rp_w}{\delta}, \] (16)

where \( \delta \) – pipe wall thickness.

Formula (15) coincides with the known formula for the determination of the thin-wall pipe strength.

Figure 6 shows the computational schemes for the analytical determination of the pneumatic tire capacity.

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**Figure 6.** Computational scheme of the pneumatic-tire wheel capacity: 
*a*) general scheme of loading and deflection; 
*b*) tire shell equilibrium without cutoff contact; 
*c*) cutoff contact body equilibrium.
Figure 6.a shows the general scheme of the tire loading and deflection; Figure 6.b shows the cutoff of the contact body from the tire carcass along the contact perimeter by the closed surface, which is normal to the bearing surface; Figure 6.c shows the cutoff contact body equilibrium.

For the shell equilibrium without the cutoff contact, according to the Newton’s third law, lifting force is applied to the shell, which is equal to the product of the tire air pressure \( p_w \) and actual contour area of the contact \( A_{k0} \), force \( Q \) on the wheel axle and vertical force \( \Delta Q \) of the carcass.

For the non-cutoff tire on Figure 6.a the equilibrium equation is true:

\[
\sum F_{\xi} = 0; \quad -Q + N = 0; \quad Q = N = \sigma_k A_{k0},
\]

where \( \sigma_k \) – average specific pressures in the contour area of the tire contact.

The contact body equilibrium on Figure 6.c is ensured by the wheel lifting force \( p_w A_{k0} \), acting downwards on the contact body according to the Newton’s third law, by the support reaction \( N \) and vertical force of the carcass \( \Delta Q \).

In closed surface, cutting off contact from the tire shell, normal stresses \( \sigma \) and elementary moments \( M_k \) appear, which have no the resultant force, as they are mutually balanced according to the symmetry condition. Tangential vertical specific forces have the resultant vertical force of the carcass:

\[
\Delta Q = \int \tau dS,
\]

where \( S \) – vertical surface area, cutting off the contact body from the tire shell; \( \tau \) – tangential stresses in the secant surface.

The equilibrium equation of the cutoff contact body shown on Figure 6.c is as follows:

\[
\sum F_{\xi} = 0; \quad N - \Delta Q - p_w A_{k0} = 0.
\]

From which, taking into account (17) we’ll obtain

\[
Q = p_w A_{k0} + \Delta Q,
\]

where \( Q \) – wheel capacity, vertical force on the wheel axle, which is normal to the bearing surface; \( p_w A_{k0} \) – pneumatic tire lifting force, equal to the product of the tire air pressure \( p_w \) and actual contour area of the contact \( A_{k0} \); \( \Delta Q \) – vertical force of the tire carcass.

4. Research results

The determination of the vertical force \( \Delta Q \) according to formula (18) is a complex task, that’s why in this article we’ll determine the force \( \Delta Q \) from equation (20) according to formula:

\[
\Delta Q = Q - p_w A_{k0}.
\]

For the solution of equation (21) the new definition is introduced – theoretical contour area of the area contact, determined according to the Pascal formula:

\[
A_k = Q/p_w,
\]

where \( Q \) – capacity, normal force on the axle; \( p_w \) – tire air pressure.

Using formulas (22) and (21) we can obtain the carcass vertical force formula:

\[
\Delta Q = (A_k - A_{k0}) p_w.
\]

From formula (23) it follows, that the force \( \Delta Q \) is an ambiguous value.

If \( A_k = A_{k0} \), then \( \Delta Q = 0 \) – it means that the load \( Q \) on the wheel axle is taken by compressed air in the tire.

If \( A_k > A_{k0} \) – it means that the carcass takes a part of the vertical force \( \Delta Q \), applied to the wheel axle.

If \( A_k < A_{k0} \), then \( \Delta Q < 0 \) – in this case the carcass vertical force \( \Delta Q \) possesses a negative value, the wheel lifting force appears to be an excessive one, i.e. it not only takes the force \( Q \), but also
creates an additional vertical tension force during the interaction of the contact body and shell carcass.

For the determination of the carcass vertical force $\Delta Q$ it’s necessary to determine the actual contour area of the contact $A_{\kappa\alpha}$ by the experimental or analytical method. Let’s consider the analytical solution of this task.

It’s established that the tire elements at the inlet to the contact radially deflect long before the approach to the bearing surface (see Figure 6,a). At the moment of the tread elements coming into contact, the tire possesses the radial initial deflection $\lambda_o$, which is connected with the normal tire deflection $\lambda$ by the expression:

$$\kappa = \frac{\lambda_o}{\lambda}$$

(24)

where $\kappa$ – coefficient of the tire initial deflection.

For truck pneumatic tires, using Figure 6,a, the angle $\alpha_1$ of the tire contact may be determined according to the formula from work [12]

$$\alpha_1 = \arccos \frac{\frac{0.5D}{\lambda} - 1}{\frac{0.5D}{\lambda} - \kappa}.$$  

(25)

The tire contact area length (see Figure 6,a) is determined according to the formula:

$$a_k = 2(0.5D - \kappa\lambda) \sin \alpha_1,$$

(26)

where $a_k$ – contact area length; $\kappa\lambda = \lambda_o$ – tire initial deflection.

In formulas (24)-(26) the coefficient $\kappa$ is the characteristic of the tire kinematic possibilities.

If $\kappa=0$, then the tire shell will possess perfect elasticity, in the result of which the tread elements come into contact without a preliminary deflection $\lambda_o=0$, and the contact area has the maximum length $a_k = a_{\max}$. Such imaginary tire has the greater tire drag in comparison with the tire which possesses the coefficient $\kappa<1$.

If conditionally set $\kappa=1$, then we obtain the tire, which $\lambda_o = \lambda$. and the tire contact length $a_k = 0$. In this limiting case the tire has no contact area, as $a_k = 0$, the load $Q$ on the wheel axle is taken by the tire carcass. Such tire has minimum rolling resistance.

It means that according to this indicator radial tires surpass cross-biased tires, because they have a tough breaker cylindrical strip under the tire tread.

For cross-biased tire 12.00-20 according to the research results the coefficient $\kappa$ has the value $\kappa=0.2 \pm 0.4$, for radial tires $\kappa \geq 0.5$.

The actual area of the tire contact is determined according to the formula:

$$A_{\kappa\alpha} = (a_k - b_k)b_k + \frac{\pi b_k^2}{4},$$

(27)

where $a_k$, $b_k$ – length and width of the tire contact area.

The tire contact area according to formula (27) is considered as the area of a simple oval.

5. Discussion of results

The application of the method of sections to pneumatic tires gives new impetus to the development of the theory of pneumatic tires. The tire contact contour area, tire carcass lifting force, tire carcass vertical load represent important components of the pneumatic tire theory.

New equations have been obtained for the determination of loads in tire elements, which represent the result of the developed computational schemes of the pneumatic tire sections by cylindrical surfaces with the usage of the intensity of distributed forces in the pneumatic tire sections during the generation of equations.

Each equation and formula proposed in this article is preceded by a new computational scheme of the tire section by planes or cylindrical surfaces, where intensities of distributed forces are used along the section perimeters. Formulas are proposed for the tire contact area with the bearing surface, which
are useful not only for the capacity calculation, but they also allow to determine the influence of the tire contact area form on the tire rolling resistance and wear.

6. Summary and conclusion
1. The application of the method of sections for pneumatic tires allows to deepen the theoretical knowledge relative to internal processes in pneumatic tires.
2. The contact body cutoff from the tire carcass has allowed to clarify the definition of the tire capacity, which depends on the contact contour area, geometric dimensions, air pressure and tire ply rating.
3. The developed computational schemes of the pneumatic tire sections and new equations will allow to accelerate the structure improvement process of pneumatic tires during their engineering.

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