Light-cone gauge integrals: Prescriptionlessness at two loops

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The only calculations performed beyond one-loop level in the light-cone gauge make use of the Mandelstam-Leibbrandt (ML) prescription in order to circumvent the notorious gauge dependent poles. Recently we have shown that in the context of negative dimensional integration method (NDIM) such prescription can be altogether abandoned, at least in one-loop order calculations. We extend our approach, now studying two-loop integrals pertaining to two-point functions. While previous works on the subject present only divergent parts for the integrals, we show that our prescriptionless method gives the same results for them, besides finite parts for arbitrary exponents of propagators.

Keywords: Quantum field theory, negative dimensional integration, non-covariant gauges, light-cone gauge.

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I. INTRODUCTION

Perhaps the trickiest gauge in what someone has termed the “gauge-market” is the light-cone gauge. The problem of spurious poles arising in such a choice is well-known and the insights gained throughout the years, concerning the problems related to naive implementation of dimensional regularization or Cauchy principal value to treat them, are registered in the pertinent literature as well as the ultimate remedy for the pathologies, that so tenaciously has defied for many years our correct understanding of them: causal prescriptions.

Prescriptions may be of good help for a time and as long as they can solve immediate problems. If, on the other hand we could somehow avoid them and at the same time understand why we can dispense with them would be no doubt a better picture to envisage. Moreover, using such prescriptions turn the calculations very laborious — partial fractioning and integration over components are, in fact, part of the ML prescription. On the other hand, negative dimensional integration method (NDIM) can dismiss the referred tricks and more: integrals can be evaluated for arbitrary exponents of propagators and dimension. Always preserving causality and gauge invariance.

Our aim is to iron out NDIM, that is, to perform a two-loop test to our prescriptionless method for light-cone gauge integrals. The aftermath of this is the establishment of a growing confidence in the novel methodology, which thoroughly dispense with prescriptions as well as partial fractioning of two-loop light-cone singularities that arise at this level of perturbative calculations. We give results in terms of hypergeometric series for arbitrary exponents of propagators and dimension, which enable us to select particular values for them and especially those that are relevant to physical amplitudes, with not only the pole pieces of integrals, but their finite parts as well.

II. LIGHT-CONE GAUGE AT TWO-LOOPS: A TOY INTEGRAL IN NDIM

The principle of gauge invariance is foundational in modern quantum field theory. It assures that gauge fields are invariant under gauge transformations. In fact, we can go so far as to say that the gauge principle is the key to our present understanding of elementary particles and their interactions. A related but very often misunderstood concept is the question of gauge independent quantities. This has to do with the fact that relevant physical quantities of interest in any phenomenon must be gauge independent, i.e., whatever the gauge choice we make use of, the resulting measurable quantity must be independent of the choice made. In other words, calculations of physical quantities performed in any gauge must give the same result.

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In this trail, we can say that the gauge choice we make is more or less a matter of taste and preference, and of practical reasons related to the amount of hard work we have to make in order to achieve the end results. The lure of the light-cone gauge has its reason, since it is a physical gauge, meaning that ghost fields decouple from matter and gauge fields, unitarity is preserved without auxiliary scalar fields with the “wrong” fermionic statistics, and the gauge propagator has a deceivingly “simple” structure. Yet, attached to the same coin, the other side of it has subtleties peculiar to this gauge.

Until Mandelstam and Leibbrandt introduced their prescriptions — which are in fact equivalent — loop integrals in the physical light-cone gauge were not well understood since calculations of Feynman diagrams in such a choice produced ill-defined integrals proportional to \( \Gamma(0) \) and integrals with pathological results such as double poles in single loop diagrams. Later on, all these pathologies were understood to be related to violation of causality associated to naive use of dimensional regularization or the use of principal value prescription to deal with light-cone singularities.

The physical light-cone gauge is defined through an external, constant, light-like four-vector \( n^\mu \), \( n^2 = 0 \), but it is now well-understood that this single vector is not sufficient to span the whole four-dimensional space-time, which in other words means that the gauge freedom is not totally removed. Residual gauge freedom remains to be removed by the introduction of the dual four-vector \( n^* \), \( n^2 = 0 \). This is done by hand in the ML-prescription whereas in our approach this is done naturally as a consequence of the general structure of the light-cone integral, defined over four-dimensional Minkowski space-time.

Our negative-dimensional approach can avoid the use of prescriptions [5] and provide physically acceptable results, i.e., causality preserving ones. The calculation we will present is the very first test beyond one-loop order without invoking the ML-prescription. We demonstrate that integration over components and partial fractioning tricks can be completely abandoned as well as parametric integrals. The important point to note [2] is that the dual light-like four-vector \( n^* \) is necessary in order to span the whole four-dimensional space [4], [5], when defining the gauge proper. Of course all these are, in the course of calculations dimensionally regularized into a \( D \)-dimensional space-time.

So far, we have tested our NDIM for integrals pertaining to one-loop class. Now we apply such technique to some massless two-loop integrals. Let us consider an integral studied by Leibbrandt and Nyeo [6]. The reason for this is two-fold: Firstly, they did not calculate it explicitly, but gave only the pole structure of it, showing that it does not contain any pathological features; and secondly because in our approach, a whole class of integrals is calculated simultaneously, their integral being a particular case of ours.

Let us define the following light-cone gauge integrals,

\[
C_3 = \int d^Dq d^Dk \frac{k^2}{q^2 (q-k)^2 (k-p)^2 (k \cdot n)(q \cdot n)},
\]

where in their calculation the ML-prescription

\[
\frac{1}{k \cdot n} = \lim_{\epsilon \to 0} \frac{k \cdot n^*}{(k \cdot n)(k \cdot n^*) + i\epsilon},
\]

must be understood. On the other hand, in the NDIM context the key point is to introduce the dual vector \( n^*_\mu \) in order to span the complete space [4], [5]. If we do not consider it, our result will violate causality, giving the Cauchy principal value of the integral in question, as we concluded in [5].

Our aim to is perform,

\[
\mathcal{N} = \int d^Dq d^Dk_1 (k_1^2)^i (q^2)^j (q-k_1)^2k (k-p)^2k_1 (k_1 \cdot n)^m (q \cdot n)^n (k_1 \cdot n^*)^r.
\]

The attentive reader will notice here that we have only put the factor \((k_1 \cdot n^*)^r\) and the other possible factor \((q \cdot n^*)^t\) is conspicuously absent. The latter could be taught as necessary on the grounds of equal footing ascribed to integrals in \( k_1 \) and \( q \). However, the NDIM technique requires only the former one. The reason for this is related to the fact that the integrals in \( k_1 \) and in \( q \) must generate factors of the form \( p \cdot n^* \) of the external momentum to ensure the complete spanning of the physical four-dimensional space-time [4]. From [5] we can readily see that only the \( k_1 \) integration contains the propagator \((k_1 - p)^2\) that can generate the needed \( p \cdot n^* \) in the external momentum, whereas the \( q \) integration is unable to do so, and the inclusion of \( q \cdot n^* \) is completely unnecessary. So here again we can see the power of NDIM. Although we treat each integration on the same footing, the propagators must be treated in a way to ensure the correct spanning of the physical four-dimensional space-time in the end result.

We will carry out this integral and then present results for special cases, including Leibbrandt and Nyeo’s \( C_3 \), where \( i = 1, r = 0 \) and the other exponents equal to minus one. Observe that the integral must be considered as a function of external momentum, exponents of propagators and dimension,

\[
\mathcal{N} = \mathcal{N}(i, j, k, l, m, r, s; P, D),
\]
where \( P \) represents \((p^2, p^+, p^-, \frac{1}{2}(n \cdot n^*))\), and we adopt the usual notation for the light-cone gauge.

Our starting point is the generating functional for our negative-dimensional integrals,

\[
G_N = \int d^Dq\, d^Dk\, \exp \left[ -\alpha k^2 - \beta q^2 - \gamma (q-k)^2 - \theta (k-p)^2 - \phi (k \cdot n) - \omega (q \cdot n) - \eta (k \cdot n^*) \right],
\]

which after a little bit of algebra and integration over momenta yields,

\[
G_N = \left( \frac{\pi^2}{\lambda} \right)^{D/2} \exp \left\{ \frac{1}{\lambda} \left[ -g_1 p^2 - g_2 (p \cdot n) - g_3 (p \cdot n^*) + g_4 \left( \frac{1}{2} n \cdot n^* \right) \right] \right\},
\]

where

\[
g_1 = (\alpha \beta + \alpha \gamma + \beta \gamma) \theta, \quad g_2 = (\beta \phi + \gamma \omega + \gamma \phi) \theta, \quad g_3 = (\beta + \gamma) \eta \theta, \quad g_4 = 0
\]

and \( \lambda = \alpha \beta + \alpha \gamma + \beta \gamma + \beta \theta + \gamma \theta \).

Taylor expanding directly the exponentials we obtain,

\[
N = (-\pi)^{D/2} i!j!k!!m!!n!!r!!s!! \Gamma(1 - \sigma - D/2) \sum_{all=0}^\infty \frac{\delta}{X_1! \ldots X_{123}!} \frac{(p^2)^{X_{123}} (p^+)^{X_{456}} (p^-)^{X_{78}}}{Z_1! \ldots Z_{123}!} \left( \frac{n \cdot n^*}{2} \right)^{Y_{123}},
\]

where \( \sigma = i + j + k + l + m + r + s + D \), \( X_{123} = X_1 + X_2 + X_3 \), etc. and \( \delta \) represents the system of constraints \((8 \times 16)\) for the negative-dimensional integral. At the end of the day we have 12,870 possible solutions for such system. Most of them, 9,142, have no solution while 3,728 present solutions which can be written as hypergeometric series. Of course several of these will provide the same representation, and these solutions we call degenerate.

First of all, we present a result for the referred integral as a double hypergeometric series,

\[
N^A_2 = \pi^D f_2 A P_2 A \sum_{Z_j=0} \frac{(\sigma_n + D/2) Z_{45}(i + j + k + m + s + D) Z_{45}((D/2 + k) Z_n)}{(Z_1 Z_{123} Z_1 Z_{45}) (i + j + k + m + r + s + D) Z_{45}} \left( \frac{p^2 n \cdot n^*}{2 p^+ p^-} \right)^{Z_{45}},
\]

where

\[
f_2 A = (-m - s)(-i - j - k - D/2 - \sigma_n - D/2)(j + k + s + D)(i - s + r) \frac{-i i - j - k - D/2}{1 + r - s - D},
\]

are the coefficients in terms of Pochhammer symbols, \((x|y) \equiv (x)_y = \Gamma(x + y)/\Gamma(x)\) and

\[
P_2 A = (p^2)^{\sigma_n + i + j + k + D} (p^+)^{l + m + s - \sigma_n} (p^-)^{l + r - \sigma_n} \left( \frac{n \cdot n^*}{2} \right)^{\sigma_n - l},
\]

is the external kinematical configuration.

Observe that in the above series we must have \(|z| < 1\), where \(z = p^2 n \cdot n^*/2p^+ p^-\), in order to be convergent. Now we can consider the special case \((i = 1, j = k = l = m = s = -1, r = 0)\), studied in [7],

\[
C_3 = \pi^D \frac{\Gamma(5 - 2D) \Gamma(D - 1) \Gamma(D - 2) \Gamma(D - 2)}{\Gamma(1 - D/2) \Gamma(D - 3)} \left( \frac{p^2}{2} \right)^{2D - 5} (p^+)^{1 - D} (p^-)^{3 - D} \left( \frac{n \cdot n^*}{2} \right)^{D - 3} Z_{45},
\]

which clearly exhibits a double pole, as stated by Leibbrandt and Nyeo [7].

Outside the region \(|z| < 1\), there is a solution which in principle could be obtained by analytic continuation from the above result, provided the formulae for such analytic continuation were known. However, as far as we are aware of, this is not the case. NDIM on the other hand, gives several hypergeometric series representations for Feynman
loop integrals which are all connected by analytic continuation, and analytic continuation formulas can in principle be implied from them. We quote only the final result valid when \( |z| > 1 \),

\[
N_2^B = \pi^D f_2^B P_2^B \sum_{X_i=0}^{\infty} \frac{(D/2 + k|X_7)(D/2 + j + s|X_8)(-r|X_78)}{X_7!X_8!(1 + m - r + s|X_78)(1 - l - r - D/2|X_78)} \left( \frac{2p^+p^-}{p^2 \cdot n^*} \right)^{X_78},
\]

where

\[
f_2^B = (-m - s|r)(\sigma + D/2| - l - r - D/2)(-i - j - k - D/2| - l - r - D/2)(-l|j + l + s + D/2)
\times (j + k + s + D| - j - s - D/2| - j - k - D/2)(-k|l + r + D/2),
\]

and \( P_2^B = (p^2)^{i+j+k+l+r+D} (p^+)^{m-r-s} \). An important point to observe here is that, hypergeometric series do not allow negative integer denominator parameters, for they are not well defined in this case. Care must therefore be taken because of the factor \((1 + m - r + s|X_78)\) present in the denominator.

There are also results given as triple hypergeometric series,

\[
N_3^A = \pi^D P_1 \Gamma_1 F_1 \left( \frac{p^2 \cdot n^*}{2p^+p^-} \right) + \pi^D P_2 \Gamma_2 F_2 \left( \frac{p^2 \cdot n^*}{2p^+p^-} \right),
\]

where

\[
F_1 \left( \frac{p^2 \cdot n^*}{2p^+p^-} \right) = \sum_{Y_i=0}^{\infty} \frac{(D/2 + l|Y_{123})(-r|Y_{123})(-m|Y_{12})(D/2 + k|Y_1)(-s|Y_3)(D/2 + j + s|Y_2)}{Y_1!Y_2!Y_3!(1 - \sigma + l|Y_{123})(D + j + k + s|Y_{12})(1 + i + j + k + l + D|Y_{123})} \left( \frac{p^2 \cdot n^*}{2p^+p^-} \right)^{Y_{123}},
\]

and

\[
F_2 \left( \frac{p^2 \cdot n^*}{2p^+p^-} \right) = \sum_{X_i=0}^{\infty} \frac{(D/2 + k|X_7)(-i|X_72)(-\sigma + r|X_{123})(-i - j - k - l - r - D|X_{123})(-j - k - D/2|X_3)}{X_7!X_2!X_3!(1 - i - j - k - l - D|X_{123})} \left( \frac{p^2 \cdot n^*}{2p^+p^-} \right)^{X_{123}},
\]

The coefficient factors of external momenta are defined in the following table and the Pochhammer symbols are,

\[
\Gamma_1 = (-l|\sigma)(-i - j - k - D/2|i)(j + k + s + D| - k - D/2)(-j - i - k - l - D)(\sigma + D/2| - \sigma + k)
\times (-k|k + l + D/2),
\]

and

\[
\Gamma_2 = (\sigma + D/2|i + j + k + D/2)(j + k + s + D| - k - D/2)(-m - s|\sigma - r)(-j - k - D/2)
\times (-k - i - j - l - r - D)(-l|k + l + D/2)(1 + \sigma|\sigma - 2\sigma),
\]

The analytic continued integral, valid for the other kinematical region is given by

\[
N_3^B = \pi^D P_3 \Gamma_3 F_3 \left( \frac{2p^+p^-}{p^2 \cdot n^*} \right) + \pi^D P_4 \Gamma_4 F_4 \left( \frac{2p^+p^-}{p^2 \cdot n^*} \right),
\]

where

\[
F_3 \left( \frac{2p^+p^-}{p^2 \cdot n^*} \right) = \sum_{Z_i=0}^{\infty} \frac{(-\sigma|Z_{123})(-i|Z_{12})(j + s + D/2|Z_2)(-j - k - D/2|Z_3)(D/2 + k|Z_1)(D/2 + l|Z_{123})}{Z_1!Z_2!Z_3!(1 + m + s + D/2|Z_{123})(l + r + D/2|Z_{123})} \left( \frac{2p^+p^-}{p^2 \cdot n^*} \right)^{Z_{123}},
\]

and
\[ F_4 \left( \frac{2p^+ p^-}{p^2 n \cdot n^*} \right) = \sum_{X_i=0}^{\infty} \frac{(D/2 + k|X_4|)(-s|X_3|)(i + j + k + r + D|X_{456}|)(-m|X_{46})}{X_4! X_3! X_6!(1 - l - m - s - D/2|X_{456})} \times \frac{(D/2 + j + s|X_6|)(-\sigma + r|X_{456})}{(D + j + k + s|X_{46})(1 - m + r - s|X_{456})} \left( \frac{2p^+ p^-}{p^2 n \cdot n^*} \right)^{X_{456}}, \]  

The coefficient factors of external momenta are defined in the table and the Pochhammer symbols are,

\[ \Gamma_3 = (-m - s - l - D/2)(j + k + s + D) - k - D/2)(-k|k + l + D/2)(-j|k - D/2)(-l|k + l + D/2)(1 + rl + D/2)(-1)^{-l-D/2}, \]

and

\[ \Gamma_4 = (-j - k - D/2 - i)(j + k + s + D) - k - D/2)(\sigma + D/2 - 2\sigma - D/2 + r)(-k|k + l + m + s + D/2) \times (-l|\sigma - m - s)(-j|j + k + D/2)(1 + rl - m - s)(-1)^{-2l - m - s}. \]

| Factors |
|----------|
| \( P_1 \) | \( (p^2)^{\sigma - r - s}(p^+)^{m + s}(p^-)^r \) |
| \( P_2 \) | \( \left( \frac{n - n^*}{l} \right)^{-\sigma + m + r + s}(p^+)^{\sigma - r}(p^-)^{\sigma - m - s} \) |
| \( P_3 \) | \( (p^2)^{D/2 + i + j + k}(p^+)^{D/2 + i + m + s}(p^-)^{D/2 + l + r}(p^2)^{l - D/2} \) |
| \( P_4 \) | \( \left( \frac{n - n^*}{l} \right)^{m + s}(p^+)^{\sigma - r}(p^-)^{m + r - s} \) |

Factors for hypergeometric series representing integral \( N_3 \), eq.(14) and eq.(19).

In this section we calculated two-loop light-cone gauge integrals. NDIM is a technique which can avoid invoking prescriptions, partial fractioning and integration over components. Moreover, several integrals can be calculated at once since it is as easy to deal with arbitrary exponents of propagators as to particular values for them.

### III. CONCLUSION

The light-cone gauge is known as the “trickiest” of the non-covariant gauges. This is because unphysical poles, generated by loop integrals, do appear and may violate causality if care is not taken as to the correct treatment of those poles. The cure for such problem came in the form of prescriptions: Mandelstam-Leibbrandt and causal Cauchy principal value ones. Recently, we proposed a third way to carry these integrals out, a method which can abandon prescriptions. NDIM is such a prescriptionless technique and in this work we tested it beyond the one-loop level. We studied two-loop integrals, which can also be handled easily, and give results which are compatible with the ones obtained using ML-prescription.

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