Life and Death in a Cage and at the Edge of a Cliff

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Abstract

The survival probabilities of a “prisoner” diffusing in an expanding cage and a “daredevil” diffusing at the edge of a receding cliff are investigated. When the diffuser reaches the boundary, he dies. For “marginal” boundary motion, i.e., the cage length grows as $\sqrt{At}$ or the cliff location recedes as $x_0(t) = -\sqrt{At}$ and the daredevil diffuses within the domain $x > x_0$, the survival probability of the diffuser exhibits non-universal power-law behavior, $S(t) \sim t^{-\beta}$, which depends on the relative rates of boundary and diffuser motion. Heuristic approaches are applied for the cases of “slow” and “fast” boundary motion which yield approximate expressions for $\beta$. An asymptotically exact analysis of these two problems is also performed and the approximate expressions for $\beta$ coincide with the exact results for nearly entire range of possible boundary motions.

1. Introduction

Consider an inebriated “prisoner” who diffuses within a one-dimensional cage defined by $(-L(t), L(t))$ and dies whenever he touches the walls (Fig. 1). We are interested in determining the probability for such a prisoner to survive until time $t$, $S(t)$. Mathematically, the time evolution of this process is governed by the diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = D \frac{\partial^2 c(x,t)}{\partial x^2},$$

within the domain $-L(t) \leq x \leq L(t)$, subject to the initial condition $c(x, t = 0) = \delta(x)$, and the absorbing boundary condition $c(x = \pm L(t), t) = 0$. Here $c(x, t)$ is the prisoner density at position $x$ and time $t$, and $D$ is the diffusion coefficient. The absorbing boundary condition imposes the death of the prisoner whenever he touches the walls of the cage.
Correspondingly, the survival probability $S(t)$ is given by the spatial integral of the prisoner density,

$$S(t) \equiv \int_{-L(t)}^{L(t)} c(x, t) \, dx.$$  \hspace{1cm} (2)

In cage of fixed size $2L$, the survival probability decays as $\exp(-\pi^2 Dt/4L^2)$ in the long-time limit [1]. More interesting behavior arises when we aid the prisoner by allowing the cage walls to recede. This obviously increases the prisoner lifetime and can also dramatically change the form of the prisoner survival probability. In the following, we consider power-law growth of the cage length, $L(t) \simeq (At)^\alpha$, as $t \to \infty$. Two domains of behavior arise which are determined by the relative magnitudes of the basic length scales of the system, namely, the cage length $2L(t)$ and the diffusion length $\sqrt{Dt}$. When $L(t) \ll \sqrt{Dt}$, which arises when $\alpha < 1/2$ or when $\alpha = 1/2$ and $A \ll D$, the prisoner diffuses faster than the cage walls recede and it is plausible to apply an approximation based on the assumption that the prisoner probability density is “close” to that in the case where the cage walls are static. This is the basis of the well-known adiabatic approximation [2]. Conversely, for a rapidly expanding cage, i.e., $\alpha > 1/2$ or $\alpha = 1/2$ and $A \gg D$, the cage expands faster than the prisoner diffuses and approximations based on a “free” prisoner should be appropriate.

In the interesting situation where the diffusion length of the prisoner and the cage size are comparable, both the adiabatic and the “free prisoner” approximations predict that the prisoner survival probability decays as a power law in time, $S(t) \sim t^{-\beta}$, but with a non-universal exponent which depends on $A/D$. To ascertain the accuracy of these heuristic approaches, we analyze of the diffusion equation with the moving boundary condition and find that the prisoner probability density may be written in terms of parabolic cylinder functions. From this analysis, we determine the exponent $\beta$ for all $A/D$ and verify that the results of the above two heuristic approaches are asymptotically exact.

We also consider the related problem of an inebriated “daredevil” who diffuses in a one-dimensional semi-infinite domain $x > x_0(t)$, and fall to his death whenever $x = x_0(t)$ is reached. When the cliff location is fixed, it is well-known that the survival probability of the daredevil is $S(t) \equiv \int_{x_0}^{\infty} c(x, t) \, dt \sim t^{-1/2}$ [1]. Thus the daredevil is sure to fall off the cliff (although his mean lifetime is infinite). This same decay law continues to hold if the cliff recedes slowly, i.e., $x_0 \sim (At)^\alpha$ with $\alpha < 1/2$ [3]. Conversely, if the cliff recedes
from the daredevil at a constant velocity $v$, there is finite probability for the daredevil to survive which rapidly approaches unity when $vl_0/D > 1$. Here $l_0$ is the initial distance from the daredevil to the cliff. However, if the cliff recedes at the same rate as the daredevil diffuses, $x_0(t) \sim -\sqrt{At}$ with $A$ of the order of $D$, marginal behavior again arises in which the daredevil survival probability exhibits a non-universal power-law decay in time.

It is worth mentioning that the first-passage probability of diffusion processes in the presence of moving, absorbing boundaries has been investigated previously by mathematicians and a substantial literature exists (see e.g., [3-5] and references therein). However, the methodology used in these papers is quite different from the simple-minded approach that we will present. In addition to differences in approach, an advantageous aspect of our study is that the prisoner and daredevil problems can be treated within the same framework.

2. Heuristic Approaches for Prisoner Survival in an Expanding Cage

For a fixed cage $(-L, L)$, the solution to the diffusion equation (1) may be written as an eigenfunction expansion in which each eigenmode decays exponentially in time, with a different characteristic decay time. In the long time limit, only the most slowly decaying eigenmode remains and the density approaches

$$c(x, t) \propto e^{-Dt/4L^2} \cos\left(\frac{\pi x}{2L}\right).$$

Thus the prisoner survival probability decays exponentially in time.

Now suppose that the cage expands slowly, $L(t) \ll \sqrt{Dt}$. In this case, the adiabatic approximation shows that the density profile approaches the same form as in the fixed-cage case, except that the parameters in this probability distribution acquire time dependence to satisfy the moving boundary condition. The corresponding probability density is

$$c(x, t) \propto f(t) \cos\left(\frac{\pi x}{2L(t)}\right).$$

with the amplitude $f(t)$ to be determined. Substituting Eq. (4) into Eq. (1) leads to

$$\dot{f} = -\left(\frac{D\pi^2}{4L^2}\right) f - \left(\frac{\pi x}{2L^2}\right) \tan\left(\frac{\pi x}{2L}\right) \dot{L} f.$$

3
When \( L(t) \) grows as \((At)^\alpha\) with \( \alpha < 1/2 \), the second term on the right-hand side may be neglected and the leading behavior of the amplitude \( f(t) \) is given by

\[
f(t) \to \exp \left[ -\frac{\pi^2 D}{4} \int_0^t dt' L^{-2}(t') \right]. \tag{6}
\]

The full asymptotic behavior of \( f(t) \) is expected to also contain a power-law prefactor in \( L \). However, this prefactor is not accessible within our naive approach. The leading behavior of \( f(t) \) now gives

\[
S(t) = \int_{-L(t)}^{L(t)} c(x, t) \, dx \simeq \frac{4}{\pi} f(t) L(t). \tag{7}
\]

Thus for \( \alpha < 1/2 \) the leading behavior of the survival probability decays as a stretched exponential in time

\[
S(t) \to \exp \left[ -\frac{\pi^2 D}{4(1-2\alpha)A^{2\alpha}} t^{1-2\alpha} \right]. \tag{8}
\]

For the marginal case of \( \alpha = 1/2 \), the second term in Eq. (5) is no longer negligible. Following the above procedure nevertheless, we find a non-universal power-law behavior,

\[
S(t) \sim t^{-\beta}, \quad \text{with} \quad \beta = \frac{\pi^2 D}{4A}. \tag{9}
\]

Given the nature of the approximation employed, this prediction is anticipated to be valid in the limit \( A \ll D \).

In the complementary case of a rapidly growing cage, \( L(t) \gg \sqrt{Dt}, \) i.e., \( \alpha > 1/2 \) or \( \alpha = 1/2 \) and \( A \gg D \), it is plausible to assume that the prisoner density profile approaches a Gaussian but with a decaying overall integral. Thus we are led to hypothesize

\[
c(x, t) \simeq \frac{S(t)}{\sqrt{4\pi Dt}} \exp \left( -\frac{x^2}{4Dt} \right). \tag{10}
\]

Although this distribution does not satisfy the absorbing boundary condition, the inconsistency is expected to be negligible, since the density is exponentially small at the cage walls. The decay of the mass may be found by equating the flux to the cage walls, \( 2j = -2D \frac{\partial c}{\partial x} \), with the mass loss. An elementary computation shows that the survival probability approaches a constant for \( \alpha > 1/2 \). In the marginal case of \( \alpha = 1/2 \),

\[
\dot{S} = -\frac{S}{t} \sqrt{\frac{A}{4\pi D}} \exp \left( -\frac{A}{4D} \right), \tag{11}
\]
This again leads to power law behavior for the survival probability, $S \sim t^{-\beta}$ with

$$\beta = \sqrt{\frac{A}{4\pi D}} \exp\left(\frac{-A}{4D}\right). \quad (12)$$

The predictions of this “free prisoner” approximation should be accurate when $A \gg D$. As is intuitively expected, when $A/D$ decreases toward 0, $\beta$ diverges. This corresponds to the survival probability exhibiting faster than power-law decay, as predicted by the adiabatic approximation. Conversely, as $A/D \to \infty$ the prisoner diffuses more slowly than the cage expands and there is a finite chance to survive asymptotically, i.e., $\beta \to 0$.

3. Asymptotic Analysis for the Marginally Growing Cage

Let us now investigate more carefully the borderline case where the cage grows as $L(t) \simeq \sqrt{At}$. Within a scaling formulation, it is natural to hypothesize that the density can be written in terms of the dimensionless variables

$$\xi \equiv \frac{x}{L(t)}, \quad \sigma \equiv \frac{x}{\sqrt{Dt}},$$

as $t \to \infty$. Since both the cage length and the diffusion length grow at the same rate, it proves convenient to consider the basic variables to be $\xi$ and $\rho = \xi/\sigma = A/D$ and write the concentration as

$$c(x, t) \sim t^{-\beta-1/2} C_\rho(\xi). \quad (13)$$

The power law prefactor is chosen to ensure that the survival probability decays as $t^{-\beta}$, as defined previously.

Substituting Eq. (13) into Eq. (1), we find that the scaling function $C_\rho(\xi)$ satisfies the ordinary differential equation

$$\frac{D}{A} \frac{d^2 C}{d\xi^2} + \frac{1}{2} \xi \frac{dC}{d\xi} + \left(\beta + \frac{1}{2}\right) C = 0, \quad (14)$$

where the $\rho$ dependence has been dropped, for notational simplicity. Introducing $\xi = \eta \sqrt{2D/A}$ and $C(\xi) = e^{-\eta^2/4} D(\eta)$, transforms Eq. (14) into the canonical form for the parabolic cylinder equation [6]

$$\frac{d^2 D}{d\eta^2} + \left(2\beta + \frac{1}{2} - \frac{\eta^2}{4}\right) D = 0. \quad (15)$$
A spatially symmetric solution to Eq. (15) (appropriate for the prisoner starting in the middle of the cage) is

\[ D(\eta) = \frac{1}{2}(D_{2\beta}(\eta) + D_{2\beta}(-\eta)), \]  

(16)

with \( D_\nu(\eta) \) the parabolic cylinder function of order \( \nu \). Finally, the relation between the decay exponent \( \beta \) and \( A/D \) is determined by the absorbing boundary condition

\[ D_{2\beta}\left(\sqrt{\frac{A}{2D}}\right) + D_{2\beta}\left(-\sqrt{\frac{A}{2D}}\right) = 0. \]  

(17)

One can easily determine the limiting behaviors of \( \beta = \beta(A/D) \) for \( A/D \to 0 \) and \( A/D \to \infty \) and thus check the validity of the heuristic predictions given in Eqs. (9) and (12), respectively. In the former case, the exponent \( \beta \) is large and hence, from Eq. (15), the profile approaches the cosine form. Thus Eq. (9) provides the correct asymptotics. In the latter case, \( A \gg D \), \( \beta \to 0 \) and Eq. (15) approaches the Schrödinger equation for the ground state of the harmonic oscillator, for which \( D_0(\eta) = \exp(-\eta^2/4) \). For small but finite \( \beta \) it is natural to seek a perturbative solution

\[ D(\eta) = \exp(-\eta^2/4) + \beta A(\eta) + \ldots. \]  

(18)

Substituting this expansion into Eq. (15) yields an inhomogeneous linear equation for the correction \( A(\eta) \):

\[ \frac{d^2A}{d\eta^2} + \left(\frac{1}{2} - \frac{\eta^2}{4}\right)A = -2\exp(-\eta^2/4). \]  

(19)

Introducing \( B(\eta) \) through \( A(\eta) = \exp(-\eta^2/4)B(\eta) \), we obtain \( B'' - \eta B' = -2 \). By solving this latter equation, the perturbative solution for \( D(\eta) \) is

\[ D(\eta) = e^{-\eta^2/4} - 2\beta e^{-\eta^2/4} \int_0^\eta d\eta_1 e^{\eta_1^2/2} \int_0^{\eta_1} d\eta_2 e^{-\eta_2^2/2} + O(\beta^2). \]  

(20)

Combining Eq. (20) and the absorbing boundary condition \( D(\sqrt{A/2D}) = 0 \) we reproduce (after a straightforward but lengthy computation) the asymptotics given by Eq. (12). Thus we have rigorously justified the previous heuristic predictions for the decay exponent \( \beta \).

It is also interesting to consider the mean lifetime of the prisoner, \( \langle t \rangle = \int_0^\infty dt S(t) \). This quantity is finite for \( \beta > 1 \) and infinite for \( \beta < 1 \). The borderline case of \( \beta = 1 \), corresponds to the second excited state of the wavefunction in the harmonic oscillator.
potential in Eq. (15), for which the solution for $D(\eta)$ is $D(\eta) = (1 - \eta^2)e^{-\eta^2/4}$. The boundary condition of Eq. (17) now gives $A = 2D$. Thus the borderline case between a finite and an infinite mean survival time corresponds to $A = 2D$. It is gratifying that for this case of $A = 2D$, the simple-minded adiabatic approach gives $\beta \approx 1.234$. This provides a sense for the accuracy of the adiabatic approach in the regime $A < 2D$.

4. Survival of a Daredevil at the Edge of a Cliff

Let us now turn to the case of an inebriated daredevil who survives if he remains within the semi-infinite domain $x_0(t) \leq x < \infty$. As mentioned in the introduction, the most interesting case is where $x_0(t) \simeq -\sqrt{At}$ as $t \to \infty$, corresponding to the cliff receding at the same rate at which diffusion tends to transport the daredevil to the cliff. This situation can be analyzed by methods similar to those applied for the prisoner in the marginally growing cage.

It is convenient to first change variables from $(x, t)$ to $(x' = x - x_0(t), t)$ to fix the absorbing boundary at the origin. Thus the initial diffusion equation is transformed to the convection-diffusion equation (where the prime is now dropped)

$$
\frac{\partial c}{\partial t} - \frac{x_0}{2t} \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2}, \quad \text{for } 0 \leq x < \infty,
$$

(21)

with the fixed absorbing boundary condition $c(x = 0, t) = 0$. As in the case of the prisoner in the marginally expanding cage, we apply the same scaling assumption, Eq. (13), for the probability density of the daredevil. Substituting this form into Eq. (21) gives

$$
\frac{D}{A} \frac{d^2 C}{d\xi^2} + \frac{1}{2} (\xi - 1) \frac{dC}{d\xi} + \left( \beta + \frac{1}{2} \right) C = 0.
$$

(22)

Transforming from $\xi = -\frac{x}{x_0}$ and $C(\xi)$ to $\eta$ and $D(\eta)$ in a slightly different way than previously,

$$
\xi - 1 = \sqrt{\frac{2D}{A}} \eta, \quad C(\xi) = \exp \left( -\frac{\eta^2}{4} \right) D(\eta),
$$

(23)

we find that $D(\eta)$ satisfies the same parabolic cylinder equation (Eq. (16)) as in the prisoner problem. However, slightly different boundary conditions apply. Death of the daredevil at the edge of the cliff implies

$$
D(-\sqrt{A/2D}) = 0.
$$

(24a)
On the other hand, \( S(t) \equiv \int_0^\infty dx \ c(x, t) \leq 1 \) implies the boundary condition at \( \eta = \infty \),

\[
\mathcal{D}(\eta = \infty) = 0. \tag{24b}
\]

Mathematically, the determination of \( \beta \) and \( \mathcal{D}(\eta) \) is equivalent to finding the ground state energy and wavefunction of a quantum particle in a potential composed of an infinite barrier at \( \eta = -\sqrt{A/2D} \) and the harmonic oscillator potential for \( \eta > -\sqrt{A/2D} \). Higher excited states do not contribute in the long time limit. A general solution to Eq. (22) satisfying \( \mathcal{D}(\infty) = 0 \) is

\[
\mathcal{D}(\eta) = \mathcal{D}_{2\beta}(\eta), \tag{25}
\]

and the absorbing boundary condition \( \mathcal{D}_{2\beta}(-\sqrt{A/2D}) = 0 \) determines the relation between the decay exponent \( \beta \) and \( A/D \).

Since Eq. (25) provides only an implicit relation \( \beta = \beta(A/D) \), it is useful to determine the limiting behaviors of \( \beta \) for small and large values of \( A/D \). These relations can be derived directly from Eq. (21) and elementary facts about the quantum mechanics of the harmonic oscillator, rather than relying on mathematical properties of the parabolic cylinder functions. In the limit of slowly moving boundary, \( A \ll D \), the wall is close to the origin in the \( \eta \)-variable. When the wall is exactly at the origin, the ground state of the truncated potential is obviously the first excited state for pure harmonic oscillator, namely, \( 2\beta = 1 \) and \( \mathcal{D}(\eta) = \eta \exp(-\eta^2/4) \). For \( A \ll D \), we therefore expect that \( 0 < 1 - 2\beta \ll 1 \). This again suggests the perturbative solution,

\[
\mathcal{D}(\eta) = \eta \exp(-\eta^2/4) + (1 - 2\beta) \mathcal{A}(\eta) + \ldots. \tag{26}
\]

Substituting this expansion into the parabolic cylinder equation yields for \( \mathcal{A}(\eta) \):

\[
\frac{d^2 \mathcal{A}}{d\eta^2} + \left( \frac{3}{2} - \frac{\eta^2}{4} \right) \mathcal{A} = \eta e^{-\eta^2/4}. \tag{27}
\]

Introducing \( \mathcal{B}(\eta) \) through \( \mathcal{A}(\eta) = \eta \exp(-\eta^2/4) \mathcal{B}(\eta) \) we find \( \mathcal{B}'' - (\eta - 2/\eta) \mathcal{B}' = 1 \). Solving this equation subject to the boundary condition Eq. (24b) gives \( \mathcal{B}(\eta) \), from which one ultimately obtains

\[
\mathcal{D}(\eta) = \eta e^{-\eta^2/4} + (1 - 2\beta) \eta e^{-\eta^2/4} \int_\eta^\infty d\eta_1 \eta_1^{-2} e^{\eta_1^2/2} \int_\eta^{\infty} d\eta_2 \eta_2^{-2} e^{-\eta_2^2/2} + \ldots. \tag{28}
\]
Applying the absorbing boundary condition Eq. (24a) gives

$$\beta \approx \frac{1}{2} - \sqrt{\frac{A}{4\pi D}}. \quad (29)$$

One can treat the opposite limit $A \gg D$ similarly. In terms of the coordinate $\eta$, the location of the wall goes to $-\infty$. Hence the unperturbed ground state for this system is just the ground state for the pure harmonic oscillator, namely, $\beta = 0$ and $D(\eta) = \exp(-\eta^2/4)$. Following the same perturbative approach as in the complementary case of $A \ll D$, we find

$$D(\eta) = e^{-\eta^2/4} + 2\beta e^{-\eta^2/4} \int_\eta^\infty d\eta_1 e^{\eta_1^2/2} \int_{\eta_1}^\infty d\eta_2 e^{-\eta_2^2/2} + O(\beta^2). \quad (30)$$

Combining Eq. (30) with the absorbing boundary condition one finds the same expression for $\beta$, Eq. (12), as was found for a finite cage.

### 5. Summary and Discussion

We have presented a heuristic and an asymptotically exact approach to determine the survival probability and the density distribution for (i) a prisoner in a growing cage, $(-L(t), L(t))$, and (ii) a daredevil in the domain $x > x_0(t)$ with a cliff at $x = x_0(t)$. We were primarily concerned with the “marginal” case where $L(t) \cong \sqrt{At}$ and $x_0 \cong -\sqrt{At}$ (with $A$ of the order of $D$), so that boundary of the system recedes at the same rate at which diffusion tends to bring the diffuser (prisoner or dared evil) toward his demise. In these marginal situations, the survival probability of the diffuser exhibits a non-universal power-law decay in time. The value of the decay exponent in the limiting cases of $A \gg D$ and $A \ll D$ can be obtained by simple arguments. These limiting behaviors are found to coincide with the results from an asymptotic analysis of the underlying equation of motion.

For the prisoner problem, our results can be straightforwardly extended to general spatial dimension. Following the same adiabatic and “free prisoner” approximations that were applied in one dimension, we find that the decay exponent $\beta$ becomes

$$\beta = \begin{cases} \frac{j_d D}{A} & \text{adiabatic;} \\ \frac{1}{1(\frac{d}{2})} \left( \frac{A}{4D} \right)^{d/2} \exp\left( -\frac{A}{4D} \right) & \text{free.} \end{cases} \quad (31)$$

Here $j_d$ is the first positive root of the spherical Bessel function $J_{d/2-1}(x)$. Similarly, in the case of marginal cage growth, the $d$-dimensional analog of the scaling ansatz, Eq. (13),
can be applied, leading to a generalization of Eq. (14). In three dimensions, in particular, the additional transformation $C(\xi) = F(\xi)/\xi$ leads to the same parabolic cylinder equation (14) for $F$, but with the parameter $\beta + \frac{1}{2}$ replaced by $\beta + 1$.

It is instructive to compare the behavior of the survival probabilities given here with those of the related problem where the absorbing boundaries themselves undergo diffusive motion. These are situations for which exact solutions have been given previously [7]. For example, consider the survival of a diffusing daredevil when the position of the cliff also diffuses with a diffusivity $A$. This situation is trivially isomorphic to the case of a static cliff and a daredevil with diffusivity $D + A$. Thus the survival probability decays universally as $t^{-1/2}$. As might be expected, a cliff which is systematically receding from a diffusing daredevil with $L(t) \propto \sqrt{t}$ leads to a larger survival probability compared to the case of a stochastically moving cliff.

On the other hand, the survival of a diffusing prisoner inside a cage where both walls diffuse (each with diffusivity $A$) is more interesting. A variety of exact solutions show that the prisoner survival probability decays non-universally, $S(t) \sim t^{-\beta}$, with decay exponent $\beta = \pi/2 \cos^{-1}(D/(D + A))$ [7]. For rapidly diffusing walls, $\beta \to 1$, while the corresponding limit for systematically receding walls is $\beta \to 0$. Clearly if cage walls are receding rapidly, the prisoner is more likely to survive compared to the case where the cage walls are diffusing rapidly. On the other hand, in the limit of slowly diffusing walls, $\beta \to \sqrt{\frac{\pi D}{8 A}}$. Strangely, this is almost the square-root of the corresponding expression for $\beta$ quoted in Eq. (9). Intriguingly, the prisoner is more likely to survive in a stochastically and slowly growing cage than in a cage which grows systematically at the same average rate.

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Figure Caption

Figure 1. (a) The prisoner in the expanding cage, and (b) the daredevil at the edge of a receding cliff.