THE FORGETFUL MAP IN RATIONAL $K$-THEORY

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Abstract. Let $G$ be a connected reductive algebraic group acting on a scheme $X$. Let $R(G)$ denote the representation ring of $G$, and $I \subset R(G)$ the ideal of virtual representations of rank 0. Let $G(X)$ (resp. $G(G,X)$) denote the Grothendieck group of coherent sheaves (resp. $G$-equivariant coherent sheaves) on $X$. Merkurjev proved that if $\pi_1(G)$ is torsion-free, then the forgetful map $G(G,X) \to G(X)$ induces an isomorphism $G(G,X)/IG(G,X) \to G(X)$. Although this map need not be an isomorphism if $\pi_1(G)$ has torsion, we prove that without the assumption on $\pi_1(G)$, the map $G(G,X)/IG(G,X) \otimes \mathbb{Q} \to G(X) \otimes \mathbb{Q}$ is an isomorphism.

1. Introduction

Let $G$ be a connected reductive algebraic group acting on a scheme $X$. The $G$-equivariant coherent sheaves on $X$ are central to the study of $X$. These sheaves often have computable invariants, since the group action allows the use of tools such as localization theorems. Also, equivariant sheaves are an important source of sheaves on quotients by group actions, since if a quotient $X \to Y$ exists, then the sheaf of invariant sections of an equivariant sheaf on $X$ is a coherent sheaf on $Y$. It is natural to ask which coherent sheaves on $X$ admit $G$-actions. One positive result is due to Mumford, who proved that if $G$ is connected and $X$ is normal, and $L$ is any invertible sheaf on $X$, then some power of $L$ is $G$-linearizable [MFK, Corollary 1.6]. On the other hand, it is easy to find examples of coherent sheaves which do not admit $G$-actions. For example, PGL(2) acts on $\mathbb{P}^1$ but the sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$ does not admit an action of PGL(2) (see MFK, p. 33).

Merkurjev proved that from the point of view of $K$-theory, there is no obstruction to equivariance, as long as the fundamental group of $G$ is torsion-free (see Mer). Let $G(X)$ (resp. $G(G,X)$) denote the Grothendieck group of coherent sheaves (resp. $G$-equivariant coherent sheaves) on $X$. There is a forgetful map $G(G,X) \to G(X)$. Let $R = R(G)$ denote the representation ring of $G$, and $I \subset R$ the augmentation ideal, that is, the ideal of virtual representations of rank 0. The

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Grothendieck group $G(G, X)$ is an $R$-module. Merkurjev showed that if $\pi_1(G)$ is torsion-free, then the forgetful map induces an isomorphism $G(G, X)/IG(G, X) \to G(X)$.

If $\pi_1(G)$ is not torsion-free, this map can fail to be an isomorphism. For example, the fundamental group of $\text{PGL}(2)$ is $\mathbb{Z}/2\mathbb{Z}$, and the class $v = [\mathcal{O}_{\mathbb{P}^1}(1)] \in G(\mathbb{P}^1)$ is not in the image of $G(\text{PGL}(2), \mathbb{P}^1)$. However, if we tensor with $\mathbb{Q}$, this class is in the image. Indeed, $G(\mathbb{P}^1) = \mathbb{Z}[v]/(v^2 - 1)$, so after tensoring with $\mathbb{Q}$, we have $v = \frac{1}{2}(v^2 + 1)$. This element is in the image of the forgetful map since $v^2$ is the class of $\mathcal{O}_{\mathbb{P}^1}(2)$, which has a $G$-action.

This phenomenon holds more generally:

**Theorem 1.1.** Let $G$ be a connected reductive algebraic group acting on a scheme $X$. The forgetful map $G(G, X) \to G(X)$ induces an isomorphism $G(G, X)/IG(G, X) \otimes \mathbb{Q} \to G(X) \otimes \mathbb{Q}$. Hence the map $G(G, X) \otimes \mathbb{Q} \to G(X) \otimes \mathbb{Q}$ is surjective.

Merkurjev proves his theorem by using a spectral sequence relating equivariant and ordinary $K$-theory. The approach taken in this paper is different, and makes use of Brion’s analogue of Theorem 1.1 for Chow groups, along with the equivariant Riemann-Roch theorem proved by Edidin and the author. This use of Riemann-Roch explains the rational coefficients in the statement of our theorem.

We remark that Theorem 1.1 remains true even if $G$ is not reductive, provided that $G$ has a Levi factor $L$ (which is automatic in characteristic 0), since then the forgetful maps from $G$-equivariant $K$-theory and Chow groups to the corresponding $L$-equivariant groups are isomorphisms. Also, we expect that a topological version of Theorem 1.1 holds for equivariantly formal spaces (since for these spaces the map from equivariant cohomology to ordinary cohomology is surjective). Finally, the completion theorem of [EG3] should have implications in this setting.

**Conventions:** We work over an algebraically closed field $k$, and assume that the $G$-actions are locally linear—that is, the schemes on which $G$ acts can be covered by $G$-invariant quasi-projective open subsets. This assumption is automatically satisfied for normal schemes. (We work in this setting in order to apply Brion’s results, which are proved under these hypotheses. We remark that that Merkurjev’s results, suitably stated, remain valid when the ground field is not algebraically closed.) Also, to make use of functorial properties of Riemann-Roch (see [Ful, Theorem 18.3(4)]) we will assume that our schemes can be equivariantly embedded in smooth schemes.
2. Equivariant K-theory, Chow groups, and Riemann-Roch

In this section we recall some basic facts about K-theory, Chow groups, and Riemann-Roch, in the equivariant and non-equivariant settings. We prove a result comparing topologies on equivariant Chow groups, and also prove a compatibility result between Riemann-Roch and forgetful maps. Both of these results are used in the proof of the main theorem. Our main references for equivariant Chow groups and equivariant Riemann-Roch will be [EG1] and [EG2], where more details can be found. If \( M \) is an abelian group, we write \( M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q} \).

Because we want to index Chow groups by codimension, we will assume all schemes and algebraic spaces considered are equidimensional; our results are valid without this assumption, but we would have to index Chow groups by dimension.

We begin with some definitions. Let \( G \) be a linear algebraic group acting on an algebraic space \( X \). Let \( G(\mathbb{G}, X) \) (resp. \( G(X) \)) denote the Grothendieck group of \( G \)-equivariant coherent sheaves (resp. coherent sheaves). There is a forgetful map

\[
\text{For} : G(\mathbb{G}, X) \to G(X)
\]

which takes the class of a \( G \)-equivariant coherent sheaf to the class of the same sheaf, viewed nonequivariantly. If we need to keep track of the space involved, we will denote this by \( \text{For}_X \). Note that \( G(\mathbb{G}, X) \) is a module for the representation ring \( R = R(G) \) of \( G \). Let \( I \subset R \) denote the augmentation ideal (the ideal of virtual representations of rank 0). Let \( G(\mathbb{G}, X)_\mathbb{Q} \) denote the \( I \)-adic completion of \( G(\mathbb{G}, X)_\mathbb{Q} \) (not the tensor product with \( \mathbb{Q} \) of the \( I \)-adic completion of \( G(\mathbb{G}, X) \)).

Let \( CH^i(X) \) denote the codimension \( i \) Chow group of \( X \); if \( X \) has pure dimension \( d \), then \( CH^i(X) = A_{d-i}(X) \). Write \( CH^*(X) = \bigoplus_i CH^i(X) \). Similarly, let \( CH^G_i(X) = A^G_{d-i}(X) \) denote the “codimension \( i \)” equivariant Chow group of \( X \), and \( CH^G_i(X) = \bigoplus CH^G_i(X) \). By definition, if \( V \) is a representation of \( G \) and \( U \) an open subset of \( V \) on which \( G \) acts freely, then \( CH^G_i(X) = CH^i((X \times U)/G) \). This definition is independent of the choice of \( V \) and \( U \) (see [EG1]). We will denote the mixed space \((X \times U)/G\) by \( X \times^G U \) or \( X_G \). Now, \( X \) is embedded in \( X_G \) as a fiber of the map \( X_G \to U/G \), and pullback along this embedding gives a map

\[
\text{For} : CH^G_i(X) \to CH^i(X).
\]

Note that \( CH^G_0(X) \) is a module for the graded ring \( S = CH^G_0(\text{pt}) \). Let \( J \subset S \) be the ideal spanned by the homogeneous elements of \( S \) of positive degree.
The following proposition is similar to [EG2, Prop. 2.1], which dealt with the case where $G$ is a subgroup of the group of upper triangular matrices. The proof is a minor modification of that proof.

**Proposition 2.1.** Let $G$ be a connected reductive algebraic group acting on an scheme $X$. Let $N = CH_G(X)_\mathbb{Q}$. The topologies on $N$ induced by the two filtrations $\{J^n N\}$ and $\{\bigoplus_{i \geq n} N^i\}$ coincide.

**Proof.** We must show two things. First, given any $n$, there exists an $r$ such that $J^r N \subset \bigoplus_{i \geq n} N^i$. For this we may take $n = r$, since $N$ is non-negatively graded and $J N^i \subseteq N^{i+1}$. Second, given any $n$, there exists an $r$ such that $\bigoplus_{i \geq r} N^i \subseteq J^n N$. Indeed, Brion proved that $N/JN \cong CH^*(X)$. Thus, $N/JN$ is 0 in degrees greater than $d = \dim X$, so $N^p = JN^{p-1}$ for $p > d$. Thus, for $p \geq n + d$, we have $N^p = J^n N^{p-n}$, so for $r = n + d$, we have $\bigoplus_{i \geq r} N^i \subseteq J^n N$, as desired. □

**Corollary 2.2.** Let $G$ be a connected reductive algebraic group acting on an scheme $X$. The $J$-adic completion of $CH_G(X)_\mathbb{Q}$ is isomorphic to the direct product $\prod_{i=0}^\infty CH^i_G(X)_\mathbb{Q}$.

**Proof.** Since the completion of $CH_G(X)_\mathbb{Q}$ with respect to the topology induced by the second filtration above is the direct product $\prod_{i=0}^\infty CH^i_G(X)_\mathbb{Q}$, this follows from the preceding proposition. □

In [EG2], the authors constructed an equivariant Riemann-Roch map

$$\tau_G^X: G(G, X) \to \prod_i CH^i_G(X)_\mathbb{Q},$$

with the same functorial properties as the non-equivariant Riemann-Roch map $\tau_X: G(X) \to CH^*(X)_\mathbb{Q}$ of [Ful]. The equivariant Riemann-Roch map induces an isomorphism

$$\hat{\tau}_X^G: \hat{R}(G, X)_\mathbb{Q} \to \prod_i CH^i_G(X)_\mathbb{Q}.$$ 

(In [EG2], $\hat{\tau}_X^G$ was denoted simply by $\tau_X^G$.) Also, there is an equivariant Chern character map $\text{ch}_G: R \to S$ which takes $I$ to $J$ and induces an isomorphism of the $I$-adic completion $\hat{R}$ of $R$ with the $J$-adic completion $\hat{S}$ of $S$. Using $\text{ch}_G$ to identify $\hat{R}$ with $\hat{S}$, the functorial properties of $\hat{\tau}_X^G$ (see [EG2, Theorem 3.1(c)]) imply that is an isomorphism of $\hat{R} = \hat{S}$-modules.

The forgetful maps in $K$-theory and Chow groups are compatible with the Riemann-Roch maps, by the following proposition. Let $\tau_X^{G,i}: G(G, X) \to CH^i_G(X)_\mathbb{Q}$ (resp. $\tau_X^i: G(X) \to CH^i(X)_\mathbb{Q}$) be the composition of the map $\tau_X^G$ (resp. $\tau_X$) with the projection to the component of degree $i$. 
Proposition 2.3. Let $G$ be a linear algebraic group acting on an algebraic space $X$. The following diagram commutes:

$$
\begin{array}{ccc}
G(G, X) & \xrightarrow{\tau^G,i} & \prod_i CH^i_G(X)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
G(X) & \xrightarrow{\tau^X} & CH^*(X)_{\mathbb{Q}}.
\end{array}
$$

Proof. This can be proved using a change of groups argument along the lines of [EG2, Lemma 4.3]. Here we give a more direct proof. Let $V$ be a representation of $G$ and $U$ an open subset of $V$ on which $G$ acts freely, such that the codimension of $V - U$ is greater than $i$. By definition, $CH^i_G(X) = CH^i(X_G)$, where $X_G = X \times^G U$. Let $\pi : X \times U \to X$ denote the projection, and let $q : X \times U \to X_G$ denote the quotient map. If $\mathcal{F}$ is a coherent sheaf on $X_G$, then the pullback sheaf $q^*\mathcal{F}$ on $X \times U$ has a natural $G$-action. The assignment $\mathcal{F} \to q^*\mathcal{F}$ gives an equivalence of categories between the category of coherent sheaves on $X \times^G U$ and the category of $G$-equivariant coherent sheaves on $X \times U$ (this follows from Thomason’s work [Tho]; see [EG2] for a discussion). This equivalence yields an isomorphism $G(X_G) \to G(G, X \times U)$, which we denote by $q^*$.

Let $u \in U$ and let $v \in U/G$ be the image of $u$. Let $j : X \to X \times U$ take $X$ to $X \times \{u\}$, and let $k = q \circ j : X \to X_G$. Then $k$ is the inclusion of $X$ as the fiber of $X_G \to U/G$ over $v$. The normal bundle $N_k$ to $k$ is pulled back from the normal bundle to the inclusion of $v$ in $U/G$, so $N_k$ is trivial, and hence by [Ful, Theorem 18.3],

$$
(1) \quad \tau_X \circ k^! = k^* \circ \tau_{X_G}
$$

as maps $G(X_G) \to CH^*(X)_{\mathbb{Q}}$.

Let $\mathcal{V}$ denote the vector bundle $X \times^G (U \times V) \to X_G$. Define

$$
\rho_U : G(X_G) \to CH^*(X_G)
$$

by

$$
(2) \quad \rho_U(\beta) = \frac{\tau_{X_G}(\beta)}{Td(\mathcal{V})}.
$$

Let $\rho_U^i$ be the composition of $\rho_U$ with the projection onto the $i$-th component. Then by the definition of the equivariant Riemann-Roch
map (see [EG2]), τ\textsuperscript{G,i}_X is the top row of the following diagram:

\[
\begin{align*}
G(G, X) & \xrightarrow{\pi^!} G(G, X \times U) \xrightarrow{(q^*)^{-1}} G(X_G) \xrightarrow{k!} CH^i(X_G) = CH^i_G(X) \\
& \quad \downarrow k! \quad \downarrow k^* = \text{For} \\
G(X) & \xrightarrow{\tau_X^k} CH^*(X). 
\end{align*}
\]

Here \(\pi^!\) is the flat pullback in equivariant \(K\)-theory; if \(E\) is an equivariant coherent sheaf on \(X\) then \(\pi^! [E] = [\pi^* E]\), where \(\pi^* E\) is the pullback of the sheaf \(E\). Also, \(k^!\) and \(k^*\) are the Gysin morphisms associated to the regular embedding \(k\) (see [Ful]). The pullback along \(k\) of the vector bundle \(V\) is trivial, so \(k^* (Td(V)) = 1\). Hence (1) and (2) imply that the diagram commutes. To complete the proof of the proposition, it suffices to show that

\[
(3) \quad k^! \circ (q^*)^{-1} \circ \pi^! = \text{For}_X
\]
as maps \(G(G, X) \rightarrow G(X)\). Now, \(k^! = j^! q^!\), so the left hand side of (3) is

\[
(4) \quad j^! q^! (q^*)^{-1} \pi^!.
\]

By definition, \((q^*)^{-1}\) takes the class of an equivariant coherent sheaf \(F\) to the class of a nonequivariant sheaf \(E\) with \(q^* E = F\). On the other hand, \(q^! [E]\) is the class of \(q^* E\) (viewed as a nonequivariant coherent sheaf) in \(G(X \times U)\). Thus, the composition \(q^! (q^*)^{-1}\) is \(\text{For}_{X \times U} : G(G, X \times U) \rightarrow G(X \times U)\). Since the forgetful map commutes with flat pullback, (4) equals

\[
j^! \circ \text{For}_{X \times U} \circ \pi^! = j^! \pi^! \circ \text{For}_X = (\pi \circ j)^! \circ \text{For}_X = \text{For}_X,
\]
as desired. \(\square\)

### 3. Completions

The purpose of this section is to prove a simple result (Lemma 3.1) about completions. This lemma is certainly known (cf. [Bou], p. 247) for finitely generated modules, but because of a lack of a reference for non-finitely generated modules, a proof is included.

Let \(R\) be a Noetherian ring and \(I\) an ideal of \(R\). Let \(\hat{M}\) denote the \(I\)-adic completion of the \(R\)-module \(M\). We view \(\hat{M}\) as the set of coherent sequences \((m_1, m_2, \ldots)\); here \(m_k \in M/I^k M\), and coherent means that for all \(k\), the natural map \(M/I^{k+1} M \rightarrow M/I^k M\) takes \(m_{k+1}\) to \(m_k\). Since \(\hat{I} = I\hat{R}\) [AM Prop. 10.15], we have \(\hat{I}\hat{M} = I\hat{R}\hat{M} = \hat{M}\). The composition \(M \rightarrow \hat{M} \rightarrow M/I\hat{M}\) induces a map \(f : M/I\hat{M} \rightarrow M/I\hat{M}\).
Lemma 3.1. Let $R$ be a Noetherian ring and $I$ an ideal of $R$. For any $R$-module $M$, the map $f : M/IM \to \hat{M}/\hat{IM}$ is an isomorphism.

Proof. The exact sequence $0 \to IM \to M \xrightarrow{\pi} M/IM \to 0$ yields an exact sequence of completions

$$0 \to \hat{IM} \to \hat{M} \xrightarrow{\pi} \hat{M}/\hat{IM} = M/IM \to 0$$

(see [AM Cor. 10.3, 10.4]). The map $\hat{\pi} : \hat{M} \to M/IM$ takes the coherent sequence $\mu = (m_1, m_2, \ldots)$ to $m_1$. We claim that the subspaces $\hat{IM}$ and $\hat{M}$ of $\hat{M}$ are equal. This suffices, for then the map $p : \hat{M}/\hat{IM} \to M/IM$ (induced from $\hat{\pi}$) is an isomorphism. Indeed, the map $f : M/IM \to M/IM$ is induced from the map $M \to \hat{M}/\hat{IM}$ taking $m$ to $(m_1, m_2, \ldots)$, where we set $m_k = m \mod I^kM$. Since $p \circ f$ is the identity map of $M/IM$, the claim implies that $f$ is an isomorphism.

It remains to prove the claim. As noted above, ker $\hat{\pi} = \hat{IM}$. Clearly $IM \subseteq \ker \hat{\pi}$, so we must show the reverse inclusion.

Given an element $\mu = (m_1, m_2, \ldots) \in \hat{M}$, let $p_k(\mu) = m_k \in M/I^kM$. Let $a_1, \ldots, a_n$ generate $I$. Suppose that $\mu \in \ker \hat{\pi}$. We want to show that $\mu \in \hat{IM}$. Now, $p_1(\mu) = 0$, and $p_2(\mu) \in IM/\hat{I}^2M$. Let $\mu^1, \ldots, \mu^n$ be elements of $M$ such that $\sum a_i \mu^i \mod I^2M = p_2(\mu)$. Let $\hat{\mu}^i$ be the image of $\mu^i$ under $M \to \hat{M}$, and let

$$\mu(2) = \mu - \sum a_i \hat{\mu}^i.$$  

Then $p_1(\mu(2)) = p_2(\mu(2)) = 0$, so $p_3(\mu(2)) \in I^2M/\hat{I}^3M$. Let $\mu^{ij}$ be elements of $M$ such that $\sum a_ia_j \mu^{ij} \mod I^3M = p_3(\mu(2))$. Let $\hat{\mu}^{ij}$ be the image of $\mu^{ij}$ under $M \to \hat{M}$, and let

$$\mu(3) = \mu(2) - \sum a_ia_j \hat{\mu}^{ij}.$$  

Then $p_i(\mu(3)) = 0$ for $i \leq 3$. Proceeding inductively, suppose we have $\mu(k) \in \hat{M}$ with $p_i(\mu(k)) = 0$ for $i \leq k$. Then we can find elements $\mu^J \in M$, where $J$ runs over the collection of all $k$-element multisets with elements in $\{1, 2, \ldots, n\}$, such that if we define

$$\mu(k + 1) = \mu(k) - \sum_{|J|=k} a^J \hat{\mu}^J,$$  

then we have $p_j(\mu(k + 1)) = 0$ for $j \leq k + 1$. (Here $|J|$ is the number of elements in $J$, counted with multiplicity; $a^J = \prod_{j \in J} a_j$, where each $a_j$ occurs with its multiplicity in $J$; and $\hat{\mu}^J$ is the image of $\mu^J$ under $M \to \hat{M}$.) Then

$$\mu = \sum_k \sum_{|J|=k} a^J \mu^J;$$  

that is, the right hand side converges to the element $\mu \in \hat{M}$. Let $S_i$ be the collection of multisets whose smallest element is $i$. We can rewrite the preceding equation as

$$\mu = a_1 \sum_{J \in S_1} a^{J-(1)} \mu^J + a_2 \sum_{J \in S_2} a^{J-(2)} \mu^J + \cdots + a_n \sum_{J \in S_n} a^{J-(n)} \mu^J.$$ 

Each of the series $\sum_{J \in S_i} a^{J-(i)} \mu^J$ converges to an element of $\hat{M}$, so we conclude that $\mu \in I\hat{M}$, as desired. \qed

Remark 3.2. In the proof of the lemma, the claim that $\hat{M} = I\hat{M}$ admits a simpler proof if $M$ is finitely generated. Indeed, in this case the horizontal maps in the following commutative diagram are isomorphisms ([AM Prop. 10.13]):

$$\hat{R} \otimes_R IM \to \hat{M} \quad \Downarrow \quad \Downarrow \quad \hat{R} \otimes_R M \to \hat{M}.$$ 

The image in $M$ of $\hat{R} \otimes_R IM$ under the upper (resp. lower) composition is $I\hat{M}$ (resp. $\hat{M}$), so $\hat{M} = IM$ as desired.

4. Proof of Theorem

In this section we work with rational coefficients and tensor all Grothendieck groups and Chow groups with $\mathbb{Q}$. For simplicity we will omit this from the notation and simply write, for example, $G(G, X)$ for $G(G, X)_\mathbb{Q}$, or $R$ for $R_\mathbb{Q}$. If $M$ is an $R$-module we will write $M/I$ for $M/IM$, and if $N$ is an $S$-module we will write $N/J$ for $N/JN$.

Recall that by Corollary 2.2 we can identify the $J$-adic completion of $CH^*_G(X)$ with the direct product $\prod_{i=0}^{\infty} CH^i_G(X)$.

By Proposition 2.3 we have a commutative diagram

$$\begin{array}{ccc}
G(G, X) & \xrightarrow{\tau_X^G} & \prod_i CH^i_G(X) \\
\downarrow \text{For} & & \downarrow \text{For} \\
G(X) & \xrightarrow{\tau_X} & \prod_i CH^i(X).
\end{array}$$

Now, $\tau_X^G$ takes $IG(G, X)$ to $J\prod_i CH^i_G(X)$. Also, the forgetful maps factor as

$$G(G, X) \to G(G, X)/I \to G(X)$$

and

$$\prod_i CH^i_G(X) \to (\prod_i CH^i_G(X))/J \to CH^*(X).$$
Therefore, we obtain a commutative diagram

\[
\begin{array}{ccc}
G(G, X)/I & \xrightarrow{\overline{\tau}_X^G} & (\prod_i CH^i_G(X))/J \\
\downarrow & & \downarrow \\
G(X) & \xrightarrow{\tau_X} & \prod_i C^i_H(X),
\end{array}
\]

where \( \overline{\tau}_X^G \) is induced from \( \tau_X^G \). The map \( \tau_X \) is an isomorphism (see [Ful, Corollary 18.3.2]). We claim that \( \overline{\tau}_X^G \) is as well. Indeed, \( \overline{\tau}_X^G \) factors as

\[
G(G, X) \rightarrow \widehat{G}(G, X) \rightarrow \prod_i CH^i_G(X),
\]

and the map \( \overline{\tau}_X^G \) is an isomorphism. As observed in Section 2, if we use \( ch \) to identify \( \hat{R} \) with \( \hat{S} \), then \( \overline{\tau}_X^G \) is an isomorphism of \( \hat{R} = \hat{S} \)-modules. Hence \( \overline{\tau}_X^G \) induces an isomorphism

\[
\widehat{G}(G, X)/I \rightarrow \left( \prod_i CH^i_G(X) \right)/J.
\]

(Here we are using the fact that \( \hat{I} = \hat{I} \hat{R} \), so \( \hat{G}(G, X)/\hat{I} = \hat{G}(G, X)/I \); similarly, \( (\prod_i C^i_H_G(X))/\hat{J} = (\prod_i C^i_H_G(X))/J \). We can write \( \overline{\tau}_X^G \) as the composition

\[
G(G, X)/I \rightarrow \hat{G}(G, X)/\hat{I} \rightarrow (\prod_i C^i_H_G(X))/J.
\]

Since the second map is an isomorphism, and by Lemma 3.1 the first map is an isomorphism as well, we conclude that \( \overline{\tau}_X^G \) is an isomorphism, proving the claim.

Now, we have a commutative diagram

\[
\begin{array}{ccc}
CH^*_G(X) & \longrightarrow & \prod_i CH^i_G(X) \\
\downarrow & & \downarrow \\
CH^*(X) & \longrightarrow & \prod_i C^i_H(X)
\end{array}
\]

(the bottom equality is because \( C^i_H(X) \) is zero for \( i < 0 \) or \( i > \dim X \)). From this we obtain a commutative diagram

\[
\begin{array}{ccc}
CH^*_G(X)/J & \longrightarrow & (\prod_i C^i_H_G(X))/J \\
\downarrow & & \downarrow \\
CH^*(X) & \longrightarrow & \prod_i C^i_H(X)
\end{array}
\]
The top map is an isomorphism by Corollary 2.2 and Lemma 3.1, and Brion proved that the left vertical map is an isomorphism. Hence, combining diagrams (4) and (6), we obtain a commutative diagram

\[
\begin{array}{ccc}
G(G, X)/I & \longrightarrow & CH^*_G(X)/J \\
\downarrow & & \downarrow \\
G(X) & \xrightarrow{\tau_X} & CH^*(X),
\end{array}
\]

Since the top, bottom, and right vertical maps are isomorphisms, we conclude that the left vertical map is an isomorphism as well. This completes the proof.

**Example 4.1.** We return to the example of \( G = \text{PGL}(2) \) acting on \( \mathbb{P}^1 \), considered in the introduction. Let \( B \) denote the stabilizer in \( G \) of the point \([1 : 0]\) and let \( T \) denote the maximal torus which is the image of the diagonal matrices in \( \text{GL}(2) \) under the quotient map \( \text{GL}(2) \to \text{PGL}(2) \). Then \( \mathbb{P}^1 \) can be identified with \( G/B \), and a standard change of groups argument (see e.g. [EG2, Prop. 3.2]) implies

\[
G(G, G/B) = G(B, \text{pt}) = R(B) = R(T).
\]

Since we are working with rational coefficients, \( R(T) \simeq \mathbb{Q}[u, u^{-1}] \) and this isomorphism can be chosen so that \( u \) corresponds to \([\mathcal{O}_{\mathbb{P}^1}(2)]\) in \( G(G, \mathbb{P}^1) \). We may view \( R(G) \) as the subring \( \mathbb{Q}[u + u^{-1}] \) of \( R(T) \); then the ideal \( I \subset R(G) \) is generated by \( u + u^{-1} - 2 \), so \( G(G, \mathbb{P}^1)/I = \mathbb{Q}[u, u^{-1}]/((u - 1)^2) \). Also, if \( v = [\mathcal{O}_{\mathbb{P}^1}(1)] \in G(\mathbb{P}^1) \), then \( G(\mathbb{P}^1) = \mathbb{Q}[v]/((v - 1)^2) \). The forgetful map \( G(G, \mathbb{P}^1) \to G(\mathbb{P}^1) \) takes \( u \) to \( v^2 \), and induces an isomorphism \( G(G, \mathbb{P}^1)/I = G(\mathbb{P}^1) \). However, if we were working with integer coefficients, the forgetful map would not be surjective, since in that case \( v \) is not in the image.

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