Geometry and integrability of the M-LXIX equation

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1 Introduction

The deep interrelation between many nonlinear differential equations of the classical differential geometry of surfaces and modern soliton equations is well established now [1-10].

In this paper, we will show that the (1+1)-dimensional spin system called the Myrzakulov LXIX (M-LXIX) equation is integrable. We do it by the establishing the equivalence between this and the Gauss-Codazzi (GC) equation. Integrability of the GC equation was established by V.E.Zakharov in [11].

2 The M-LXIX equation

Consider the M-LXIX equation [2]

\[ S_t = \frac{1}{\sqrt{S_x^2}}(-\sqrt{S_x^2} - u^2 S_x + u S \wedge S_x) \]  

\[ u_x = v \sqrt{S_t^2 - u^2} \]  

\[ v_t = -S \cdot (S_t \wedge S_x) \]  

where \( S = (S_1, S_2, S_3) \) is the spin vector, \( S^2 = \beta = \pm 1 \), \( u, v \) are scalar functions. In this paper we study integrability of the M-LXIX equation and its geometry.

3 L-equivalent of the M-LXIX equation

Let us introduce the orthogonal basis defined by the vectors

\[ e_1 = S, \quad e_2 = \frac{1}{k} S_x, \quad e_3 = \frac{1}{k} S \wedge S_x \]

where

\[ k^2 = S_x^2. \]

Hence and from (1) we get

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}_x = A \begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}, \quad \begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}_t = B \begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}
\]

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where
\[
A = \begin{pmatrix} 0 & k & 0 \\ -\beta k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix}.
\]  
(5)

In our case we put \(\omega_1 = 0\), \(\beta = \pm 1\). The compatibility condition of these equations gives
\[
k_t = \omega_3 x + \tau \omega_2 
\]  
(6a)
\[
\tau_t = -k \omega_2 
\]  
(6b)
\[
\omega_{2x} = \tau \omega_3. 
\]  
(6c)

This system is the L-equivalent (Lakshmanan equivalent [1]) counterpart of the M-LXIX equation (1).

4 The M-LXIX equation as the particular case of the M-0 equation

Consider the (1+1)-dimensional Myrzakulov 0 (M-0) equation [2]. This equation can be written in the different forms. Here we work with the following form of the (1+1)-dimensional M-0 equation
\[
e_{1t} = \omega_3 e_2 - \omega_2 e_3
\]  
(7a)
\[
\tau_t - \omega_{1x} = e_1 \cdot (e_{1x} \wedge e_{1t}).
\]  
(7b)

Let
\[
\tau = f_x + v, \quad \omega_1 = f_t
\]  
(8)
where \(f\) is some function. Let \(f = constant\). Then from (7) we obtain the equation (1). This means that the M-LXIX equation (1) is the particular case of the M-0 equation (7).

5 Geometry of the M-LXIX equation

We consider a surface immersed into the three-dimensional Euclidean space \(R^3\) generated by a position vector \(r = r(x, t)\) which is a function of two parameters - local coordinates - \(x\) and \(t\). The first and second fundamental forms are given by
\[
I = dx^2 + dy^2 + dz^2 = g_{ij} dx^i dx^j = Edu^2 + 2Fdu dv + Gdv^2
\]  
(9)

and
\[
II = g_{ij} dx^i dx^j = Ldu^2 + 2Mdu dv + Ndv^2.
\]  
(10)
The Gaussian curvature $K$ and mean curvature $H$ are of the form

$$K = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{LN - M^2}{EG - F^2},$$

and

$$H = \frac{1}{2}(k_1 + k_2) = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$  

(11)

(12)

As well known that everywhere outside umbilic points the first and second fundamental forms $I$ and $II$ can be diagonalized simultaneously (see e.g. [10]). Thus choosing the curvature lines as the coordinate lines, one, generally, has

$$I = g_{11}dt^2 + g_{22}dx^2, \quad II = d_{11}dt^2 + d_{22}dx^2.$$  

(13)

In these notations the GC equation takes the form [10]

$$\left(\frac{d_{11}}{\sqrt{g_{11}}}\right)_x = \frac{d_{22}}{g_{22}}(\sqrt{g_{11}})_x = 0$$  

(14a)

$$\left(\frac{d_{22}}{\sqrt{g_{22}}}\right)_t = \frac{d_{11}}{g_{11}}(\sqrt{g_{22}})_t = 0$$  

(14b)

$$\left(\frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\right)_t + \left(\frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}\right)_x + \frac{d_{11}d_{22}}{\sqrt{g_{11}g_{22}}} = 0.$$  

(14c)

Following [10], we denote

$$\psi_1 = \frac{d_{11}}{\sqrt{g_{11}}}, \quad \psi_2 = \frac{d_{22}}{\sqrt{g_{22}}}, \quad p = \frac{\sqrt{g_{11}}}{\sqrt{g_{22}}}, \quad q = \frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}.$$  

(15)

In such notations GC equation (14) takes the form

$$\psi_{1x} = p\psi_2$$  

(16a)

$$\psi_{2t} = q\psi_1$$  

(16b)

$$q_t + p_x + \psi_1\psi_2 = 0.$$  

(16c)

Also we have the following system [10]

$$\ddot{\psi}_1 = p\ddot{\psi}_2$$  

(17a)

$$\ddot{\psi}_2 = q\ddot{\psi}_1$$  

(17b)

where $\ddot{\psi}_1 = \sqrt{g_{11}}, \quad \ddot{\psi}_2 = \sqrt{g_{22}}$. In such notations the three fundamental forms of the surface look like

$$I = \ddot{\psi}_1^2dt^2 + \ddot{\psi}_2^2dx^2, \quad II = \ddot{\psi}_1\ddot{\psi}_1dt^2 + \ddot{\psi}_2\ddot{\psi}_2dx^2, \quad III = \psi_1^2dt^2 + \psi_2^2dx^2.$$  

(18)
6 Integrability of the M-LXIX equation

Let us we compare the equations (6) and (16). We see that these equations have the same form after the following transformations

\[ k = \frac{\tilde{\psi}_2 t}{\psi_1}, \quad \tau = \psi_1, \quad \omega_2 = -\psi_2, \quad \omega_3 = -\frac{\psi_{1x}}{\psi_2}. \]  

(19)

Now let us rewrite the equation (16) in the form

\[ \psi_{1x} = \frac{\tilde{\psi}_{1x}}{\tilde{\psi}_2} \psi_2 \]  

(20a)

\[ \psi_{2t} = \frac{\tilde{\psi}_{2t}}{\psi_1} \psi_1 \]  

(20b)

\[ \left( \frac{\tilde{\psi}_{2t}}{\psi_1} \right)_t + \left( \frac{\tilde{\psi}_{1x}}{\psi_2} \right)_x + \psi_1 \psi_2 = 0. \]  

(20c)

Quite recently it was shown that the GC equation (20) is integrable by the dressing method [11]. So as follows from these results the M-LXIX equation (1) is also integrable by the dressing method. This is the main result of this paper. Finally we would like to note that the equation (6) (and the equation (20)) has the following Lax representation

\[ \phi_x = U \phi, \quad \phi_t = V \phi \]  

(21)

where

\[ U = \frac{1}{2i} \left( \begin{array}{cc} \tau & k \\ k & -\tau \end{array} \right), \quad V = \frac{1}{2i} \left( \begin{array}{cc} \omega_1 & \omega_3 + i \omega_2 \\ \omega_3 - i \omega_2 & -\omega_1 \end{array} \right). \]  

(22)

7 Conclusion

Concluding, we have found the L-equivalent counterpart (6) of the M-LXIX equation (1). This L-equivalent equation (6) has the same form with the GC equation (20) for the same surface studied by Konopelchenko [10]. This means that we have identified to the M-LXIX equation (1) the same surface given by the GC equation (20). This is the first main result of his paper. Integrability this GC equation is proved by Zakharov in [11]. These results mean that the M-LXIX equation (11) is integrable by the dressing method that is our second main result.

8 Acknowledgments

This work was supported by INTAS, grant 99-1782.
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