Conditions for avoiding large cancellation in type-I seesaw mechanism with four-zero texture

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In this letter, we search for conditions that the mass matrix of light neutrinos \( m_\nu \) is not a result of large cancellations in the type-I seesaw formula with four-zero texture. For the Yukawa matrix of neutrinos \( Y_\nu \) and heavy Majorana mass matrix \( M_R \), these conditions have the form \( (Y_\nu)_{12} \propto (m_\nu)_{12} \Rightarrow (Y_\nu)_{12} \propto (M_R)_{12} \). We call them alignment conditions because they align the second rows or columns of the three neutrino mass matrices. If these conditions do not hold, the large mixing in \( m_\nu \) is a result of fine-tuning due to the cancellation of several large terms. Then they are also required from the viewpoint of naturalness.

They place rough restrictions on flavor structures of neutrinos. Under these conditions, \( Y_\nu \) must have a mild hierarchy. For \( M_R \), the 12 submatrix has the same mild hierarchy as \( Y_\nu \) and \( m_\nu \). However, the 23 submatrix has a strong hierarchy without some fine-tuning. Therefore, it is likely that \( Y_\nu \) and \( M_R \) have qualitatively different flavor structures. Furthermore, since they incorporate CP phases of the matrix elements, these non-cancellation conditions imply an existence of universal generalized CP symmetry in the neutrino sector.

I. INTRODUCTION

Researches on the peculiar flavor structure of the Standard Model may provide some hints to the flavor puzzle and the theoretical origin of the Higgs boson. On the other hand, treatments of the flavor strongly depend on what we consider the Higgs boson is. Therefore, studies of mass matrices in a model-independent way is one dominant approach.

Among them, the four-zero texture \((M_f)_{11} = (M_f)_{13,31} = 0\) is still one of viable texture \([2, 25]\). This system has a nice property called seesaw invariance in the type-I seesaw mechanism \([26, 28]\). By imposing the four-zero texture on the right-handed neutrino mass \( M_R \) and the Yukawa matrix \( Y_\nu \), the mass of light neutrino \( m_\nu \) also becomes the four-zero texture. However, its type-I seesaw relation has many terms and its physical meaning is unclear. In this letter, we explore conditions that no large cancellation occurs in this type-I seesaw relation, its properties, and

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consequences.

This letter is organized as follows. In the next section, we explore conditions under which the seesaw relation with the four-zero texture does not have large cancellations. Sec. 3 gives a survey of properties of such conditions and their restrictions on the flavor structure of neutrinos. In Sec. 4, we discuss the consequences of imposing these conditions on $m_\nu$ with the trimaximal mixing condition. The final section is devoted to a summary.

II. TYPE-I SEESAW MECHANISM AND FOUR-ZERO TEXTURE

In this section, we discuss cancellations between terms of the neutrino mass matrices in the type-I seesaw mechanism with four-zero texture. Suppose that the Dirac neutrino mass matrix $M_D$ and the Majorana mass matrix of right-handed neutrinos $M_R$ have the following four-zero texture;

$$M_D = \frac{v}{\sqrt{2}} Y_\nu = \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu^* & D_\nu & B_\nu \\ 0 & B_\nu^* & A_\nu \end{pmatrix}, \quad M_R = \begin{pmatrix} C_R & D_R & B_R \\ D_R^* & C_R & B_R \\ 0 & B_R & A_R \end{pmatrix}. \quad (1)$$

The Hermiticity of $M_D$ can be justified by the parity symmetry of the left-right symmetric model \[29–31\]. Since the phases of $B_\nu$ and $C_\nu$ can be removed by redefinition of fields, $M_D$ and $Y_\nu$ are set to be real matrices without loss of generality. Thus, let $A_\nu \sim D_\nu$ be real parameters and $A_R \sim D_R$ be complex ones.

In a model with the type-I seesaw mechanism, the mass matrix of light neutrinos $m_\nu$ also have the four-zero texture \[3, 5\]:

$$m_\nu = M_D M_R^{-1} M_D^T \quad (2)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{F_\nu^2}{A_R} & \frac{A_\nu F_\nu}{A_R} \\ 0 & \frac{A_\nu F_\nu}{A_R} & \frac{A_\nu^2}{A_R} \end{pmatrix} + \begin{pmatrix} 0 & \frac{C_\nu^2}{C_R} & \frac{2 C_\nu D_\nu}{C_R} & \frac{C_\nu D_\nu}{C_R} \\ \frac{2 C_\nu D_\nu}{C_R} & \frac{2 C_\nu^2 D_R}{C_R} & \frac{2 C_\nu D_\nu}{C_R} & \frac{B_\nu C_\nu}{C_R} \\ \frac{2 C_\nu D_\nu}{C_R} & \frac{B_\nu C_\nu}{C_R} & \frac{B_\nu^2 C_\nu}{C_R} & \frac{B_\nu C_\nu}{C_R} \end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ c & d & b \\ 0 & b & a \end{pmatrix}. \quad (3)$$

where $F_\nu = (B_\nu C_R - B_R C_\nu)/C_R$. In general, $a \sim d$ are also complex parameters. Note that the first matrix has determinant zero and its rank is equal to one. If $|A_\nu| \gtrsim |F_\nu|$ holds, the contributions of the first matrix in Eq. (3) can be neglected except for the 33 components, because the parameters $a, b, d$ are expected to be similar magnitude from the bi-maximal $\theta_{23}$. By assuming $Y_\nu$ and $M_D$ have the following hierarchy,

$$|A_\nu| \gg |B_\nu|, |D_\nu|, |C_\nu|, \quad (4)$$
this requirement is rewritten as

\[ |A_\nu| \gtrsim |F_\nu| = |B_\nu - C_\nu \frac{B_R}{C_R}| \iff \left| \frac{A_\nu}{C_\nu} \right| \gtrsim \left| \frac{B_R}{C_R} \right| . \tag{5} \]

The condition (\(5\)) compares degrees of hierarchies between \(M_R\) and \(M_D\). Under Eq. (\(5\)), the form of \(m_\nu\) becomes rather concise;

\[ m_\nu \simeq \begin{pmatrix} 0 & \frac{C^2_\nu}{C_R} \frac{C_\nu D_\nu}{C_R} + \frac{C_\nu D_\nu}{C_R} (D_\nu - C_\nu \frac{B_R}{C_R}) \frac{B_\nu C_\nu}{C_R} \frac{A^2_\nu}{A_R} \end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ c & d & b \\ 0 & b & a \end{pmatrix}. \tag{6} \]

In this case, we obtain

\[ \frac{b}{c} \simeq \frac{B_\nu}{C_\nu}, \tag{7} \]

a relation of ratios between elements of \(m_\nu\) and \(Y_\nu\).

On the other hand, when Eqs. (\(3\)) and (\(\overline{3}\)) do not hold and the hierarchy of \(M_R\) is so strong

\[ |A_\nu/C_\nu| \ll |B_R/C_R|, \]

the terms containing \(F_\nu\) in Eq. (\(\overline{3}\)) are quite larger than the other matrix elements \(m_\nu \sim C^2_\nu/C_R \sim A^2_\nu/A_R\), and large cancellations are required among the terms. In this situation, the large mixings of neutrinos are a result of fine-tuning between terms with large values. Therefore, the condition (\(\overline{7}\)) is also required from the viewpoint of naturalness.

A similar relation holds for the 22 components of \(m_\nu\). Since \(|b| \sim |d|\) holds in Eq. (\(\overline{3}\)), a condition for a large cancellation is

\[ \frac{2C_\nu D_\nu}{C_R} \simeq \frac{C^2_\nu D_R}{C^2_R}, \quad \frac{|C_\nu D_\nu|}{C_R} \gg \frac{C_\nu B_\nu}{C_R}. \tag{8} \]

This can be rephrased as the following,

\[ \left| D_\nu - C_\nu \frac{D_R}{C_R} \right| \simeq |D_\nu| \gg |B_\nu|. \tag{9} \]

On the contrary, a condition for such cancellation not to occur is

\[ \left| D_\nu - C_\nu \frac{D_R}{C_R} \right| \lesssim |B_\nu|. \tag{10} \]

In this case \(d \simeq C_\nu D_\nu/C_R\) holds and we obtain

\[ \frac{d}{c} \simeq \frac{D_\nu}{C_\nu}. \tag{11} \]

Note that these relations include phases of \(b, c\) and \(d\). Since \(B_\nu, C_\nu,\) and \(D_\nu\) are taken as real parameters, relative phases of \(b, c\) and \(d\) must be almost zero (or \(\pi\)) in this basis. For example,
such a situation is realized by the diagonal reflection symmetries (DRS) \[22–24\]:

\[
R M^*_{\nu,D,R} R = M_{\nu,D,R}, \quad R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{\nu,D,R} = \begin{pmatrix} F_{\nu,D,R} & iC_{\nu,D,R} & iE_{\nu,D,R} \\ iC_{\nu,D,R} & D_{\nu,D,R} & B_{\nu,D,R} \\ iE_{\nu,D,R} & B_{\nu,D,R} & A_{\nu,D,R} \end{pmatrix}.
\] (12)

By redefining phases of \( l_L \) and \( \nu_R \), all the mass matrices \( m_\nu \) and \( M_{D,R} \) in the neutrino sector have only real parameters, and indeed the conditions (7) and (11) are real. More generally, this situation can be achieved by an imposition of the same generalized CP symmetry (GCP) \[32–35\] on the neutrino sector. For this purpose, it is sufficient that the GCPs of \( M_D \) and \( M_R \) are the same in the seesaw mechanism,

\[
X M^*_{D,R} X^\dagger = M_{D,R}, \quad \Rightarrow \quad X m^*_\nu X^T = m_\nu,
\] (13)

with a unitary matrix \( X \).

A. Reasons for conditions

Here, we discuss why such conditions are necessary. When the mass matrices (1) are expressed as \( M_D = (y_1, y_2, y_3) \) and \( M_R = (M_1, M_2, M_3) \) by 3 dimensional vectors \( y_i \) and \( M_i \), the inverse matrix of \( M_R \) can be written as

\[
M^{-1}_R = \frac{1}{\text{Det} M_R} \begin{pmatrix} A_R D_R - B_R^2 & -A_R C_R & B_R C_R \\ -A_R C_R & 0 & 0 \\ B_R C_R & 0 & -C_R^2 \end{pmatrix} = \frac{1}{\text{Det} M_R} \begin{pmatrix} (M_2 \times M_3)^T \\ (M_3 \times M_1)^T \\ (M_1 \times M_2)^T \end{pmatrix}. \] (14)

The rank one matrix in \( m_\nu \) \[3\] is generated from a part of \( M_R^{-1} \):

\[
m_\nu \supset \frac{1}{\text{Det} M_R} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} \begin{pmatrix} -B_R^2 & 0 & B_R C_R \\ 0 & 0 & 0 \\ B_R C_R & 0 & -C_R^2 \end{pmatrix} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{F^2_{\nu}}{A_R} & \frac{A_\nu F_\nu}{A_R} \\ 0 & \frac{A_\nu F_\nu}{A_R} & \frac{A_\nu^2}{A_R} \end{pmatrix}. \] (15)

Then the restriction (3) on \( F_\nu \) is rewritten as a condition on the third row of \( M_R \) as follows

\[
|F_\nu| \lesssim |A_\nu| \quad \Rightarrow \quad |(M_1 \times M_2)^T \cdot y_2| \lesssim |(M_1 \times M_2)^T \cdot y_3|. \] (16)
Here, the inner product $\cdot$ does not involve Hermitian conjugation and is in general a complex number. The rest of $m_\nu$ comes from

$$m_\nu \geq \frac{1}{\text{Det} M_R} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} \begin{pmatrix} A_R D_R & -A_R C_R & 0 \\ -A_R C_R & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & A_\nu \end{pmatrix} = \begin{pmatrix} 0 & \frac{C_\nu^2}{C_R} & 0 \\ \frac{C_\nu^2}{C_R} & 2C_\nu D_\nu - \frac{C_\nu^2 D_R}{C_R} & B_c C_\nu \\ 0 & B_c C_\nu & 0 \end{pmatrix}.$$  

(17)

In a similar way, the condition (10) containing $D_R$ can be rewritten as follows under the assumption that Eq. (16) holds:

$$|D_\nu - C_\nu \frac{D_R}{C_R}| \lesssim |B_\nu| \Rightarrow |(M_2 \times M_3)^T \cdot y_2| \lesssim |(M_2 \times M_3)^T \cdot y_3|.$$  

(18)

Since the inner products between the first and third rows of $M_R^{-1}$ and $y_2$ are small, these conditions are rewritten as

$$(M_2 \times M_3)^T \cdot y_2 \simeq (M_1 \times M_2)^T \cdot y_2 \simeq 0 \Leftrightarrow y_2 \simeq \alpha M_2 \simeq \beta m_2.$$  

(19)

Here $\alpha, \beta$ are some complex numbers and $m_\nu = (m_1, m_2, m_3)$. In an equivalent notation, we arrive at the following result,

$$(M_D)_{i2} \simeq \alpha (M_R)_{i2} \simeq \beta (m_\nu)_{i2}.$$  

(20)

These conditions align the second rows (or columns) of the mass matrices. Therefore, we call Eqs. (7) and (11), or Eq. (20) as the alignment conditions. The discussion here is similar to the sequential dominance [36–40]. However, it is different in that $M_R$ is not diagonalized. The result obtained on the trimaximal condition later is also different from the constrained sequential dominance [41–44].

### III. PROPERTIES OF ALIGNMENT CONDITIONS

In this section, we will examine the properties of the alignment conditions. First of all, let us explore parameter region where these conditions (7) and (11) hold. The Majorana mass matrix of $\nu_R$ is represented by $M_D$ and $m_\nu$ as

$$M_R = M_D^T m_\nu M_D$$  

(21)

and

$$= \begin{pmatrix} 0 & \frac{F_R}{a} & \frac{A_R F_R}{a} \\ \frac{F_R}{a} & A_\nu F_R & \frac{A_\nu^2 F_R}{a} \end{pmatrix} + \begin{pmatrix} \frac{C_\nu^2}{c} & C_\nu D_\nu & B_c C_\nu \\ C_\nu D_\nu & \frac{C_\nu^2 (D_\nu - \frac{4}{c} C_\nu)}{c} & B_c C_\nu \\ 0 & B_c C_\nu & 0 \end{pmatrix} = \begin{pmatrix} 0 & C_R & 0 \\ C_R & D_R & B_R \\ 0 & B_R & A_R \end{pmatrix}.$$  

(22)
where \( F_R = (c B_\nu - b C_\nu)/c \). Imposing the inequality (\ref{eq:inequality}) on the 12 and 23 components of \( M_R \), we obtain

\[
\frac{|B_R|}{|C_R|} = \left| \frac{A_\nu (c B_\nu - b C_\nu) + a B_\nu C_\nu}{a C_\nu^2} \right| \lesssim \frac{|A_\nu|}{|C_\nu|}. \tag{23}
\]

Since the term \( a B_\nu C_\nu \) can be neglected from the hierarchy (\ref{eq:hierarchy}), the deviation of \( B_\nu \) from Eq. (\ref{eq:range}) can be expressed in terms of a complex parameter \( \epsilon \);

\[
B_\nu \equiv \frac{b}{c} C_\nu + \epsilon, \quad |\epsilon| = \left| B_\nu - \frac{b}{c} C_\nu \right| \lesssim \frac{a C_\nu}{c}. \tag{24}
\]

Thus, this \( \epsilon \) is restricted to a region with a width of about \( C_\nu \). For example, by assuming \( Y_\nu \simeq Y_e \), the size of this limit is about \( C_e \simeq \sqrt{m_\mu m_\tau} \sim 7 \, \text{MeV} \) with \( B_e \simeq m_\mu \sim 100 \, \text{MeV} \). The range is sufficiently large to allow renormalization and threshold corrections. From \( |a| \sim |b| \), the magnitude of \( B_\nu \) for this range of \( |\epsilon| \) becomes \( 0 \lesssim |B_\nu| \lesssim 2 b C_\nu/c \).

Similarly, we define

\[
D_\nu \equiv \frac{d}{c} C_\nu + \delta, \tag{25}
\]

for the 22 component. Eq. (\ref{eq:range}) and (\ref{eq:range2}) leads to

\[
\left| D_\nu - C_\nu \frac{D_R}{C_R} \right| = \left| -(D_\nu - \frac{d}{c} C_\nu) \right| \lesssim |\delta| \lesssim |B_\nu| \lesssim \frac{b C_\nu}{c}. \tag{26}
\]

From \( |a| \sim |b| \sim |d| \), the allowed ranges of \( \epsilon \) and \( \delta \) are comparable.

### A. Flavor structure and alignment conditions

The alignment conditions place rough constraints on the hierarchy of \( Y_\nu \) and \( M_R \). For \( Y_\nu \), if the conditions (\ref{eq:hierarchy}) hold, then the hierarchy of \( Y_\nu \) becomes mild and cascade-like. Here, waterfall and cascade texture are matrices of the following form [15].

\[
M_{\text{waterfall}} \sim \begin{pmatrix}
\delta^2 & \lambda \delta & \delta \\
\lambda \delta & \lambda^2 & \lambda \\
\delta & \lambda & 1
\end{pmatrix}, \quad M_{\text{cascade}} \sim \begin{pmatrix}
\delta & \delta & \delta \\
\delta & \lambda & \lambda \\
\delta & \lambda & 1
\end{pmatrix}. \tag{27}
\]

Although the original paper assumed \( 1 \gg \lambda \gg \delta \), this letter includes a situation with \( \lambda \gtrsim \delta \). If the four-zero texture arises from sequential breaking of flavor symmetry such as \( SU(3)_F \) (or its subgroup), the cascade type is desirable for flavor structures of \( Y_\nu \) and \( M_D \). The observation of the CKM matrix also suggests a cascade type for Hermitian \( Y_d \).
Furthermore, the bi-maximal mixing of $\theta_{23}$ implies $|a| \sim |b| \sim |d|$ in $m_\nu$ (3). These conditions place constraints on the sizes of matrix elements. First, the condition $|a| \sim |b|$ leads to

$$\frac{B_\nu C_\nu}{C_R} \sim \frac{A^2_\nu}{A_R} \iff \frac{A_R A_\nu}{B_\nu C_\nu}.$$  

(28)

Thus, the magnitude of $M_R$ for the third generation is enhanced by $A_\nu / B_\nu$ and much larger than that of $Y_\nu$. If there is also no significant cancellation between the 22 components, $d \sim b$ also leads to

$$\frac{D_R}{C_R} \sim \frac{D_\nu}{C_\nu} \sim \frac{B_\nu}{C_\nu} \simeq \frac{b}{c}.$$  

(29)

This suggests a relatively mild hierarchical structure for light generations of $M_R$, similar to that of $Y_\nu$ and $m_\nu$.

These facts can also be seen from the reconstruction of $M_R$. Substituting Eqs. (24) and (25) into Eq. (22), $M_R$ for given $Y_\nu$ and the upper limit of its absolute value are

$$M_R = \begin{pmatrix} 0 & \frac{C^2_\nu}{c} + \frac{d C^2_\nu}{c^2} + \frac{2 \delta C_\nu}{c} \frac{\epsilon A_\nu}{a} + \frac{b C^2_\nu}{c^2} + \frac{\epsilon C_\nu}{c} \frac{A^2_\nu}{a} & 0 \\ \frac{\epsilon A_\nu}{a} + \frac{b C^2_\nu}{c^2} + \frac{\epsilon C_\nu}{c} & \frac{A^2_\nu}{a} & 0 \\ 0 & \frac{A_\nu B_\nu}{a} & \frac{A^2_\nu}{a} \end{pmatrix} \lesssim \begin{pmatrix} C_\nu & 0 & 0 \\ 0 & C_\nu & 4 D_\nu A_\nu \\ 0 & A_\nu & \frac{c A^2_\nu}{a C_\nu} \end{pmatrix}. \quad (30)$$

In this case, $M_R$ has relatively mild hierarchy and satisfies Eqs. (28) and (29). On the other hand, in a situation like $|B_\nu|, |D_\nu| \gg |b C_\nu/c|$, Eq. (28) and Eq. (29) do not hold and $M_R$ will be a waterfall type matrix with strong hierarchy [20]:

$$M_R \simeq \begin{pmatrix} 0 & \frac{C^2_\nu}{c} & 0 \\ \frac{B^2_\nu}{a} + \frac{2 D_\nu C_\nu}{c} & \frac{A_\nu B_\nu}{a} & \frac{A^2_\nu}{a} \\ 0 & \frac{A_\nu B_\nu}{a} & \frac{A^2_\nu}{a} \end{pmatrix}. \quad (31)$$

In particular, if equalities hold for the conditions (6) and (11) with $\epsilon = \delta = 0$, the conditions also hold for $M_R$ (31),

$$\frac{D_\nu}{C_\nu} = \frac{d}{c} \iff \frac{D_R}{C_R} = \frac{D_\nu}{C_\nu}. \quad (32)$$

$$\frac{B_\nu}{C_\nu} = \frac{b}{c} \iff \frac{B_R}{C_R} = \frac{B_\nu}{C_\nu}. \quad (33)$$

Equivalently, the structure of $M_R$ is

$$M_R = \frac{C_\nu}{c} \begin{pmatrix} 0 & C_\nu & 0 \\ C_\nu & D_\nu & B_\nu \\ 0 & B_\nu & \frac{c A^2_\nu}{a C_\nu} \end{pmatrix} = \begin{pmatrix} 0 & C_R & 0 \\ C_R & D_R & B_R \\ 0 & B_R & A_R \end{pmatrix}. \quad (34)$$
In this case, a partially universal texture emerges in the three mass matrices of the neutrino sector \( m_\nu, M_D \) and \( M_R \). This suggests the universality of underlying flavor physics and is very attractive from the viewpoint of model building.

B. The condition for \( M_R \)

Let us consider the range where these conditions (32) and (33) hold for \( D_R \) and \( B_R \). First, by taking a ratio of the 12 and 22 components of Eq. (30),

\[
\frac{(M_R)_{22}}{(M_R)_{12}} = \frac{d}{c} + \frac{2 \delta}{C_\nu} + \frac{\epsilon^2 c}{a C_\nu^2}.
\]

From Eqs. (24) - (26), \(|\delta|, |\epsilon| \lesssim |b C_\nu/c|\) holds. The upper limit of the absolute value of Eq. (35) in this range will be about \(4d/c\), which is comparable to the allowed range of \( D_\nu/C_\nu \) of \(0 \lesssim |D_\nu/C_\nu| \lesssim 2d/c\).

However, the condition for \( B_R \) holds in a very narrow region. If we take the ratio of matrix elements of \( M_R \) (30) in the same way in Eq. (35),

\[
\frac{(M_R)_{23}}{(M_R)_{12}} = \frac{b}{c} + \frac{\epsilon}{C_\nu} + \frac{\epsilon c A_\nu}{a C_\nu^2}.
\]

From this, the range of \( \epsilon \) for which Eq. (33) is approximately valid is

\[
|\epsilon c A_\nu| \lesssim \frac{b}{c} \ \Rightarrow \ |\epsilon| \lesssim \frac{ab C_\nu^2}{c^2 A_\nu}.
\]

It means that the magnitude of \( \epsilon \) is constrained to about \(C_\nu^2/A_\nu\). For example, if we consider \(Y_\nu \simeq Y_e\), the range becomes very narrow, \(C_\nu^2/A_e \simeq m_e m_\mu/m_\tau \simeq 0.03 \text{ MeV}\). Such a strict restriction is not realistic for renormalization and or threshold corrections. On the contrary, if we consider the upper limit of \( \epsilon \) as Eq. (24), the third term of Eq. (36) for \( \epsilon \sim C_\nu \) is as large as \(A_\nu/C_\nu\). Therefore, the condition (33) for \( B_R \) seems to be less accurate than that of \( Y_\nu \).

In conclusion, the flavor structure of \( M_R \) with the four-zero texture and alignment conditions can be similar to \( Y_\nu \) for the 12 submatrix. By contrast, it is difficult to make the same structures for the 23 submatrix of \( M_R \) and \( Y_\nu \) without fine-tunings. As you can see in Eq. (30), \( Y_\nu \) and \( M_R \) are likely to have qualitatively different flavor structures. However, even if \( B_\nu \) in \( Y_\nu \) at low energy does not satisfy the strict limit (17), it is quite possible that the universality (33) \( B_R/C_R = B_\nu/C_\nu \) is valid in high energy regions such as the GUT scale.
IV. APPLICATION TO TRIMAXIMAL MIXING AND MAGIC SYMMETRY

Here we will analyze a neutrino mass matrix with four-zero texture that satisfies the alignment condition and the trimaximal mixing [46–52]. The four-zero texture with a trimaximal condition is given by [53]

\[
m_{\nu T} = \begin{pmatrix}
0 & c & 0 \\
c & c + a & c + a \\
0 & c + a & a
\end{pmatrix}, \\
m_{\nu T} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -c \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.
\] (38)

This matrix satisfies the following magic symmetry [49], which is equivalent to the eigenvector condition (38);

\[
S_2 m_{\nu T} S_2 = m_{\nu T}, \\
S_2 = \begin{pmatrix}
\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\
\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\
-\frac{2}{3} & \frac{2}{3} & \frac{1}{3}
\end{pmatrix}, \\
S_2^2 = 1_3.
\] (39)

However, even if we impose the trimaximal condition (38) on the general \( m_{\nu} \) [3], there are so many terms and physical meanings are unclear. Besides, if the trimaximal mixing is a consequence of the cancellation among terms with large values, it is difficult to derive relations for the matrix elements of \( Y_\nu \) and \( M_R \). Therefore, here we impose the alignment conditions Eqs. (7) and (11).

From \( b = \frac{c}{C_\nu}(B_\nu - \epsilon) \) and \( d = \frac{c}{C_\nu}(D_\nu - \delta) \), the mass matrix \( m_{\nu} \) [3] is represented by the deviations \( \epsilon \) and \( \delta \) as

\[
m_{\nu} = \begin{pmatrix}
0 & \frac{c^2}{C_R} & 0 \\
\frac{c^2}{C_R} & \frac{C_\nu}{C_R}(D_\nu - \delta) & \frac{C_\nu}{C_R}(B_\nu - \epsilon) \\
0 & \frac{C_\nu}{C_R}(B_\nu - \epsilon) & \frac{A^2}{A_R}
\end{pmatrix} = \begin{pmatrix} 0 & c & 0 \\ c & d & b \\ 0 & b & a \end{pmatrix}.
\] (40)

By identifying Eq. (III) with \( m_{\nu T} \) (38), two new relations arise;

\[
c + a = \frac{c^2}{C_R} + \frac{A^2}{A_R} = \frac{C_\nu}{C_R}(B_\nu - \epsilon) = \frac{C_\nu}{C_R}(D_\nu - \delta).
\] (41)

Or, equivalently,

\[
C_\nu \left( 1 + \frac{a}{c} \right) = D_\nu - \delta = B_\nu - \epsilon.
\] (42)

As a result, the first and second generations of \( Y_\nu \) must have roughly the same structure as \( m_{\nu} \). Without the alignment condition, it is difficult to extract such a relation from only the trimaximal condition (38).
Under the alignment condition, the form of $M_D$ is found to be

$$M_D = \begin{pmatrix} 0 & C_{\nu} & 0 \\ C_{\nu} & \frac{(a+c)C_{\nu}}{c} + \delta \left( \frac{(a+c)C_{\nu}}{c} + \epsilon \right) \\ 0 & \frac{(a+c)C_{\nu}}{c} + \epsilon & A_{\nu} \end{pmatrix}. \tag{43}$$

By further imposing DRS \([12]\), $m_{\nu T}$ \([38]\) reproduces the MNS matrix with an accuracy of $O(10^{-2})$ and predicts $a \simeq 2c$ \([24]\). In this case, these parameters are evaluated to $B_{\nu} \sim D_{\nu} \sim 3C_{\nu}$. A reconstructed $M_R$ \([34]\) from $m_{\nu}$ and $M_D$ becomes

$$M_R = \begin{pmatrix} 0 & \frac{C_{\nu}^2}{c} & \frac{C_{\nu}^2}{c} \\ \frac{C_{\nu}^2}{a} \frac{c^2}{a^2} + \frac{(a+c)C_{\nu}^2}{c^2} + \frac{2\delta C_{\nu}}{c} & \frac{cA_{\nu}}{a} + \frac{(a+c)C_{\nu}^2}{c^2} + \frac{\epsilon C_{\nu}}{c} & 0 \\ 0 & \frac{cA_{\nu}}{a} + \frac{(a+c)C_{\nu}^2}{c^2} + \frac{\epsilon C_{\nu}}{c} & \frac{A_{\nu}^2}{a} \end{pmatrix}. \tag{44}$$

If we can justify that $\epsilon$ is as small as Eq. \((37)\), flavor structures such as Eqs. \((43)\) and \((44)\) are relatively easy to realize in the $SU(3)$ or $A_4$ model with a flavon with vacuum expectation value $\langle \varphi \rangle = (0, -1, 1)$. Thus, we can expect a universal flavor structure for $m_{\nu}, Y_{\nu}$ and $M_R$ with mild hierarchies. Furthermore, relations $A_{\nu,R} + C_{\nu,R} = B_{\nu,R}$ lead to the magic symmetry \([39]\) for $M_D$ and $M_R$, and matrix elements of $m_{\nu}, M_D$ and $M_R$ are concisely related as the previous study \([24]\).

Perhaps a solution $F_{\nu} = A_{\nu}$, or equivalently, $\epsilon C_{\nu}/C_R = a$ is similar to the sequential dominance \([36, 37]\) and seems to be natural because it separates the contributions of the two mass scales $a$ and $c$ in Eq. \((38)\). In this case, $B_{\nu} = C_{\nu}$ holds and further imposition of $B_{\nu} = D_{\nu}$ leads to $D_R = C_R$. This is also similar to the constrained sequential dominance (CSD) \([11]\).

V. SUMMARY

In this letter, we search for conditions that the mass matrix of light neutrinos $m_{\nu}$ is not a result of large cancellations in the type-I seesaw formula with four-zero texture. For the Yukawa matrix of neutrinos $Y_{\nu}$ and heavy Majorana mass matrix $M_R$, these conditions have the form $(Y_{\nu})_{i2} \propto (m_{\nu})_{i2} \Rightarrow (Y_{\nu})_{i2} \propto (M_R)_{i2}$. We call them alignment conditions because they align the second rows or columns of the three neutrino mass matrices. If these conditions do not hold, the large mixing in $m_{\nu}$ is a result of fine-tuning due to the cancellation of several large terms. Then they are also required from the viewpoint of naturalness.

They place rough restrictions on flavor structures of neutrinos. Under these conditions, $Y_{\nu}$ must have a mild hierarchy. For $M_R$, the 12 submatrix has the same mild hierarchy as $Y_{\nu}$ and $m_{\nu}$. However, the 23 submatrix has a strong hierarchy without some fine-tuning. Therefore, it is likely
that $Y_\nu$ and $M_R$ have qualitatively different flavor structures. Furthermore, since they incorporate CP phases of the matrix elements, these non-cancellation conditions imply an existence of universal generalized CP symmetry in the neutrino sector.

If the conditions are satisfied, information of $Y_\nu$ can be immediately extracted from a reconstructed $m_\nu$. By combining the alignment conditions with the trimaximal mixing condition, two such relations $(Y_\nu)_{22} \simeq (Y_\nu)_{23} \simeq (Y_\nu)_{12}[(m_\nu)_{12} + (m_\nu)_{33}] / (m_\nu)_{12}$ were obtained. Without the conditions, it is generally difficult to translate the trimaximal condition into such a relation for matrix elements in the type-I seesaw mechanism.

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[1] H. Fritzsch and Z.-z. Xing, Phys. Lett. B353, 114 (1995), hep-ph/9502297.
[2] K. Kang and S. K. Kang, Phys. Rev. D56, 1511 (1997), hep-ph/9704253.
[3] A. Mondragon and E. Rodriguez-Jauregui, Phys. Rev. D59, 093009 (1999), hep-ph/9807214.
[4] J. L. Chkareuli and C. D. Froggatt, Phys. Lett. B 450, 158 (1999), hep-ph/9812499.
[5] H. Nishiura, K. Matsuda, and T. Fukuyama, Phys. Rev. D60, 013006 (1999), hep-ph/9902385.
[6] K. Matsuda, T. Fukuyama, and H. Nishiura, Phys. Rev. D61, 053001 (2000), hep-ph/9906433.
[7] H. Fritzsch and Z.-z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000), hep-ph/9912358.
[8] J. L. Chkareuli, C. D. Froggatt, and H. B. Nielsen, Nucl. Phys. B 626, 307 (2002), hep-ph/0109156.
[9] H. Fritzsch and Z.-z. Xing, Phys. Lett. B 555, 63 (2003), hep-ph/0212195.
[10] Z.-z. Xing and H. Zhang, Phys. Lett. B569, 30 (2003), hep-ph/0304234.
[11] Z.-z. Xing and H. Zhang, J. Phys. G 30, 129 (2004), hep-ph/0309112.
[12] M. Bando, S. Kaneko, M. Obara, and M. Tanimoto, Prog. Theor. Phys. 112, 533 (2004), hep-ph/0405071.
[13] K. Matsuda and H. Nishiura, Phys. Rev. D74, 033014 (2006), hep-ph/0606142.
[14] G. C. Branco, M. N. Rebelo, and J. I. Silva-Marcos, Phys. Rev. D 76, 033008 (2007), hep-ph/0612252.
[15] G. Ahuja, S. Kumar, M. Randhawa, M. Gupta, and S. Dev, Phys. Rev. D76, 013006 (2007), hep-ph/0703005.
[16] J. Barranco, F. Gonzalez Canales, and A. Mondragon, Phys. Rev. D82, 073010 (2010), 1004.3781.
[17] R. Verma, G. Ahuja, N. Mahajan, M. Gupta, and M. Randhawa, J. Phys. G 37, 075020 (2010), 1004.5452.
[18] M. Gupta and G. Ahuja, Int. J. Mod. Phys. A 26, 2973 (2011), 1206.3844.
[19] Z.-z. Xing and Z.-h. Zhao, Nucl. Phys. B 897, 302 (2015), 1501.06346.
[20] M. J. S. Yang, Phys. Rev. D 95, 055029 (2017), 1612.09049.
[21] M. J. S. Yang, Phys. Lett. B 806, 135483 (2020), 2002.09152.
[22] M. J. S. Yang, Chin. Phys. C 45, 043103 (2021), 2003.11701.
[23] M. J. S. Yang, Nucl. Phys. B 972, 115549 (2021), 2103.12289.
[24] M. J. S. Yang (2021), 2104.12063.
[25] A. Bagai, A. Vashisht, N. Awasthi, G. Ahuja, and M. Gupta (2021), 2110.05065.
[26] P. Minkowski, Phys. Lett. 67B, 421 (1977).
[27] M. Gell-Mann, P. Ramond, and R. Slansky, Conf. Proc. C790927, 315 (1979).
[28] T. Yanagida, Conf. Proc. C7902131, 95 (1979).
[29] J. C. Pati and A. Salam, Phys. Rev. D 10, 275 (1974), [Erratum: Phys.Rev.D 11, 703–703 (1975)].
[30] G. Senjanovic and R. N. Mohapatra, Phys. Rev. D 12, 1502 (1975).
[31] R. N. Mohapatra and J. C. Pati, Phys. Rev. D 11, 566 (1975).
[32] G. Ecker, W. Grimus, and H. Neufeld, Nucl. Phys. B 247, 70 (1984).
[33] M. Gronau and R. N. Mohapatra, Phys. Lett. B 168, 248 (1986).
[34] F. Feruglio, C. Hagedorn, and R. Ziegler, JHEP 07, 027 (2013), 1211.5560.
[35] M. Holthausen, M. Lindner, and M. A. Schmidt, JHEP 04, 122 (2013), 1211.6953.
[36] S. F. King, Phys. Lett. B 439, 350 (1998), hep-ph/9806440.
[37] S. F. King, Nucl. Phys. B 562, 57 (1999), hep-ph/9904210.
[38] S. F. King, Nucl. Phys. B 576, 85 (2000), hep-ph/9912492.
[39] S. F. King, JHEP 09, 011 (2002), hep-ph/0204360.
[40] S. Antusch and S. F. King, New J. Phys. 6, 110 (2004), hep-ph/0405272.
[41] S. F. King, JHEP 08, 105 (2005), hep-ph/0506297.
[42] S. Antusch, S. F. King, C. Luhn, and M. Spinrath, Nucl. Phys. B 856, 328 (2012), 1108.4278.
[43] S. Antusch, S. F. King, and M. Spinrath, Phys. Rev. D 87, 096018 (2013), 1301.6764.
[44] F. Björkeroth and S. F. King, J. Phys. G 42, 125002 (2015), 1412.6996.
[45] N. Haba, R. Takahashi, M. Tanimoto, and K. Yoshioka, Phys. Rev. D78, 113002 (2008), 0804.4055.
[46] P. F. Harrison and W. G. Scott, Phys. Lett. B535, 163 (2002), hep-ph/0203209.
[47] P. F. Harrison and W. G. Scott, Phys. Lett. B594, 324 (2004), hep-ph/0403278.
[48] R. Friedberg and T. D. Lee, HEPNP 30, 591 (2006), hep-ph/0606071.
[49] C. S. Lam, Phys. Lett. B640, 260 (2006), hep-ph/0606220.
[50] J. D. Bjorken, P. F. Harrison, and W. G. Scott, Phys. Rev. D74, 073012 (2006), hep-ph/0511201.
[51] X.-G. He and A. Zee, Phys. Lett. B645, 427 (2007), hep-ph/0607163.
[52] W. Grimus and L. Lavoura, JHEP 09, 106 (2008), 0809.0226.
[53] R. R. Gautam and S. Kumar, Phys. Rev. D94, 036004 (2016), 1607.08328.