A NOTE ON THE RATIONAL HOMOLOGICAL DIMENSION OF
LATTICES IN POSITIVE CHARACTERISTIC

SAM HUGHES

ABSTRACT. We show via $\ell^2$-homology that the rational homological dimension of a
lattice in a product of simple simply connected Chevalley groups over global function
fields is equal to the rational cohomological dimension and to the dimension of the
associated Bruhat–Tits building.

1. INTRODUCTION

Let $k$ be the function field of an irreducible projective smooth curve $C$ defined over a
finite field $F_q$. Let $S$ be a finite non-empty set of (closed) points of $C$. Let $\mathcal{O}_S$ be the ring
of rational functions whose poles lie in $S$. For each $p \in S$ there is a discrete valuation $\nu_p$ of $k$ such that $\nu_p(f)$ is the order of vanishing of $f$ at $p$. The valuation ring $\mathcal{O}_p$ is the ring
of functions that do not have a pole at $p$, that is

$$\mathcal{O}_S = \bigcap_{p \notin S} \mathcal{O}_p.$$ 

Let $\bar{k}$ denote the algebraic closure of $k$. Let $G$ be an affine group scheme defined over $\bar{k}$ such that $G(\bar{k})$ is almost simple. For each $p \in S$ there is a completion $k_p$ of $k$ and the group $G(k_p)$ acts on the Bruhat–Tits building $X_p$. Thus, we may embed $G(\mathcal{O}_S)$
diagonally into the product $\prod_{p \in S} G(k_p)$ as an arithmetic lattice.

The rational cohomological dimension of a group $\Gamma$ is defined to be

$$\text{cd}_Q(\Gamma) := \sup \{ n : H^n(\Gamma; M) \neq 0, \ M \text{ a } \mathbb{Q}\Gamma\text{-module} \},$$

the rational homological dimension is defined completely analogously as

$$\text{hd}_Q(\Gamma) := \sup \{ n : H_n(\Gamma; M) \neq 0, \ M \text{ a } \mathbb{Q}\Gamma\text{-module} \}.$$ 

In [Gan12] it is shown that $\text{cd}_Q(G(\mathcal{O}_S)) = \prod_{p \in S} \dim X_p$. In light of this Ian Leary
asked the author what is $\text{hd}_Q(G(\mathcal{O}_S))$?

Mathematical Institute, Andrew Wiles Building, University of Oxford, Oxford OX2 6GG, UK

E-mail address: sam.hughes@maths.ox.ac.uk.

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Theorem A. Let $G$ be a simple simply connected Chevalley group. Let $k$ and $O_S$ be as above, then

$$\text{hd}_Q(G(O_S)) = \text{cd}_Q(G) = \prod_{p \in S} \dim X_p.$$  

More generally, we obtain the following.

Corollary B. Let $\Gamma$ be a lattice in a product of simple simply connected Chevalley groups over global function fields with associated Bruhat–Tits building $X$, then $\text{hd}_Q(\Gamma) = \text{cd}_Q(\Gamma) = \dim X$.

The author expects these results are well-known, however, they do not appear in the literature so we take the opportunity to record them here.

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2. $\ell^2$–HOMOLOGY AND MEASURE EQUIVALENCE

Let $\Gamma$ be a group. Both $\Gamma$ and the complex group algebra $\mathbb{C}\Gamma$ act by left multiplication on the Hilbert space $\ell^2\Gamma$ of square-summable sequences. The group von Neumann algebra $\mathcal{N}\Gamma$ is the ring of $\Gamma$-equivariant bounded operators on $\ell^2 G$. The non-zero divisors of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}\Gamma$ can be identified with the ring of affiliated operators $U\Gamma$.

There are inclusions $Q\Gamma \subseteq \mathcal{N}\Gamma \subseteq \ell^2\Gamma \subseteq U\Gamma$ and it is also known that $U\Gamma$ is a self-injective ring which is flat over $\mathcal{N}\Gamma$. For more details concerning these constructions we refer the reader to [Lüc02] and especially to Theorem 8.22 of Section 8.2.3 therein. The von Neumann dimension and the basic properties we need can be found in [Lüc02, Section 8.3].

The $\ell^2$-Betti numbers of a group $\Gamma$, denoted $b_i^{(2)}(\Gamma)$, are then defined to be the von-Neumann dimensions of the homology groups $H_i(\Gamma;U\Gamma)$. The following lemma is a triviality.

Lemma 2.1. Let $\Gamma$ be a discrete group and suppose that $b_i^{(2)}(\Gamma) > 0$, then the homology group $H_i(\Gamma;U\Gamma)$ is non-trivial.

Two countable groups $\Gamma$ and $\Lambda$ are said to be measure equivalent if there exist commuting, measure-preserving, free actions of $\Gamma$ and $\Lambda$ on some infinite Lebesgue measure space $(\Omega, m)$, such that the action of each of the groups $\Gamma$ and $\Lambda$ admits a finite measure.
fundamental domain. The key examples of measure equivalent groups are lattices in the same locally-compact group [Gro93]. The relevance of this for us is the following deep theorem of Gaboriau.

**Theorem 2.2** (Gaboriau’s Theorem [Gab02]). Suppose a discrete group $\Gamma$ is measure equivalent to a discrete group $\Lambda$, then $b_p(\Gamma) = 0$ if and only if $b_p(\Lambda) = 0$.

### 3. Proofs

**Proof of Theorem A.** We first note that the group $\Gamma := G(\mathcal{O}_S)$ is measure equivalent to the product $\Lambda := \prod_{p \in S} G(t_p)$ for some suitably chosen $t_p \in \mathcal{O}_p$. By [PST18, Theorem 1.6] (see also [Dym04; Dym06; Dav+07]) the group $G(t_p)$ has one non-vanishing $\ell^2$-Betti number in dimension $\dim X_p$. Hence, by the Künneth formula $\Lambda$ has one non-vanishing $\ell^2$-Betti number in dimension $d = \prod_{p \in S} \dim X_p$. Thus, by Gaboriau’s theorem, the group $\Gamma$ has exactly one non-vanishing $\ell^2$-Betti number in dimension $d$. It follows from Lemma 2.1 that $hd_\mathbb{Q}(\Gamma) \geq d$. The reverse inequality follows from the fact that $\Gamma$ acts properly on the $d$-dimensional space $\prod_{p \in S} \dim X_p$. $\square$

**Proof of Corollary B.** The proof of the corollary is entirely analogous. First, we split $G$ into a product of simple groups $\prod_{i=1}^n G_i$ corresponding to the decomposition of the Bruhat–Tits building $X = \prod_{i=1}^n X_i$. Let $\Lambda_i$ be a lattice in $G_i$ and let $\Lambda = \prod_{i=1}^n \Lambda_i$. Each $\Lambda_i$ has a non-vanishing $\ell^2$-Betti Number in dimension $\dim X_i$. In particular, $\Lambda$ has a non-vanishing $\ell^2$-Betti Number in dimension $\dim X = \prod_{i=1}^n \dim X_i$. By Gaboriau’s Theorem $\Gamma$ also has non-vanishing $\ell^2$-Betti Number in dimension $\dim X$. It follows from Lemma 2.1 that $hd_\mathbb{Q}(\Gamma) \geq d$. The reverse inequality follows from the fact that $\Gamma$ acts properly on the $d$-dimensional space $\prod_{p \in S} \dim X_p$. $\square$

**Remark 3.1.** A similar argument can be applied to lattices in products of simple simply-connected algebraic groups over locally compact $p$-adic fields. One obtains the analogous result for such a lattice $\Gamma$ that $cd_\mathbb{Q}(\Gamma) = hd_\mathbb{Q}(\Gamma) = \dim X$, where $X$ is the associated Bruhat–Tits building.

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