Gravity, Two Times, Tractors, Weyl Invariance and Six Dimensional Quantum Mechanics

R. Bonezzi, E. Latini and A. Waldron

Dipartimento di Fisica, Università di Bologna, via Irnerio 46, I-40126 Bologna, Italy
and INFN sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy
bonezzi@bo.infn.it

INFN, Laboratori Nazionali di Frascati, CP 13, I-00044 Frascati, Italy
latini@lnf.infn.it

Department of Mathematics, University of California, Davis CA 95616, USA
emanuele, wally@math.ucdavis.edu

Abstract

Fefferman and Graham showed some time ago that four dimensional conformal geometries could be analyzed in terms of six dimensional, ambient, Riemannian geometries admitting a closed homothety. Recently it was shown how conformal geometry provides a description of physics manifestly invariant under local choices of unit systems. Strikingly, Einstein’s equations are then equivalent to the existence of a parallel scale tractor (a six component vector subject to a certain first order covariant constancy condition at every point in four dimensional spacetime). These results suggest a six dimensional description of four dimensional physics, a viewpoint promulgated by the two times physics program of Bars. The Fefferman–Graham construction relies on a triplet of operators corresponding, respectively to a curved six dimensional light cone, the dilation generator and the Laplacian. These form an $\mathfrak{sp}(2)$ algebra which Bars employs as a first class algebra of constraints in a six-dimensional gauge theory. In this article four dimensional gravity is recast in terms of six dimensional quantum mechanics by melding the two times and tractor approaches. This “parent” formulation of gravity is built from an infinite set of six dimensional fields. Successively integrating out these fields yields various novel descriptions of gravity including a new four dimensional one built from a scalar doublet, a tractor vector multiplet and a conformal class of metrics.
1 Introduction

Theories with extra dimensions have been heavily scrutinized since the time of Kaluza and Klein [1]. The terminus of this train of thought is String Theory which attempts to encode the couplings of four dimensional theories in the geometry of hidden higher dimensions. A simpler and more generic rationale for further dimensions, however, might follow a line of reasoning similar to Einstein’s original identification of time as an additional coordinate, along with a gauge principle—general coordinate invariance—guiding the construction of physical theories in terms of Riemannian geometry.

In this article, we focus on two fairly recent suggestions that physics is inherently six dimensional. Firstly, motivated by duality and holographic arguments, Bars observed that many seemingly different four dimensional particle models could be regarded as gauge fixed versions of a single underlying six dimensional model. In fact the idea of using six dimensions to describe four dimensional physics dates back to Dirac [2]. What is notable about Bars’ “two times physics” [3] (see [4] for an overview) is that it aims ultimately to describe any physical system, whereas Dirac’s work pertained only to models with conformal symmetry\(^1\).

\(^1\)In fact there is a extensive literature on the handling of four dimensional conformal theories using six dimensional methods. Pertinent contributions include Boulanger’s conformal tensor calculus [5], the conformal space method of [6], the conformal higher
The second approach relies on replacing Riemannian geometry with conformal geometry so that physics is described by conformal classes of metrics and all equations are manifestly locally Weyl invariant. This is achieved by utilizing the simple physical principle that no physical quantity can depend on local choices of unit system which implies there must exist a way to write any physical system in a Weyl invariant way \cite{10, 11}. Weyl invariance is intimately related to conformal symmetry, and for reasons very similar to those first observed by Dirac, manifest Weyl invariance can be achieved by grouping existing four dimensional physical quantities in six dimensional multiplets known as “tractors”. This approach relies heavily on tractor calculus \cite{12, 13, 14}, a mathematical machinery designed for efficiently handling conformal geometries. Not only does the tractor approach identify a simple gauge principle—local unit invariance—for constructing models, it also identifies the additional timelike coordinate in two times physics as the choice of scale.

In this article we map out the relationship between the two times and tractor approaches, since they are in fact highly complementary, and in doing so present seven different formulations of four dimensional Einstein gravity\footnote{\textsuperscript{2}}, several of which are novel; they are summarized by the action principles \cite{11, 3, 27, 28, 31, 34, 37}. Of these, the action \cite{27} can be viewed as a parent action depending on infinitely many fields living in a six dimensional spacetime while all other theories are gauge fixed versions of this parent action. This starting point was first proposed by Bars as part of his two times description of physics although not precisely as a four dimensional theory of gravity \cite{15}. This action comes from a BRST quantization of the worldline conformal group gauge symmetries of a two times particle model\footnote{Our results are valid for any spacetime dimensionality, and all formulæ will be presented as functions of $d$, the spacetime dimension. We will, however often use the shorthand “four” to stand for $d$-dimensional and “six” to stand for $(d+2)$-dimensional.}. The operators generating local worldline conformal transformations form the gravity multiplet of the model. Bars’ action couples this gravity multiplet to a scalar multiplet which can be viewed as a dilaton. This fits extremely well with the tractor description of gravity in terms of a conformal class of metrics coupled to a scale field—the gauge field for local changes of unit systems.

\footnote{Massless four dimensional spinning particles were obtained earlier from six dimensions by Siegel in \cite{16} and further studied in \cite{17}.}
There is an alternative proposal for a two times description of four dimensional gravity due to Bars [18]. It has the advantage that at least part of the equations for the generators of worldline conformal transformations follow from an action principle. On the other hand, unlike the action (27), it does not make the worldline conformal group \( \mathfrak{sp}(2) \) symmetry—a central component of the two times set-up—manifest. It turns out that the two approaches are in fact equivalent, a fact that follows rapidly using tractor technology.

The tractor approach takes standard four dimensional physical quantities and groups them in Weyl-multiplets labeled by \( SO(d, 2) \) representations known as tractors. These tractors are functions of four dimensional space-time. In particular, from the scale field \( \sigma \) (the spacetime dependent Planck’s constant), one builds a tractor vector \( I^M \) known as the scale tractor. Like any tractor, under Weyl transformations it undergoes a tractor gauge transformation which in turn defines a covariant derivative known as the tractor connection\(^5\). The beauty of this approach is that the Einstein condition amounts to the scale tractor being parallel with respect to this connection. The length of the scale tractor is therefore parallel for physical geometries and in fact measures the cosmological constant. Upon coupling to matter, it also provides a massive coupling constant. Remarkably, even though the small size of the cosmological constant might seem to make the length of the scale tractor inappropriate for setting particle physics mass scales, including backreaction immediately solves this “cosmological constant hierarchy problem” [20]. In fact, parallel scale tractors form the first part of a link between the tractor and two times descriptions of gravity.

The link between two times physics and tractors is completed by the ambient formulation of tractor calculus developed by [21, 14, 22]. The main idea underlying ambient tractors relies on the Fefferman–Graham description of four dimensional conformal geometries in terms of six dimensional Ricci flat geometries admitting a closed homothety [23]. The latter condition implies that the six dimensional ambient geometry enjoys a curved null cone with a dilation-like vector field. This allows four dimensional conformal geometries

\(^4\)For example, for a relativistic particle, from the four-velocity \( v_\mu \), the component of the four-acceleration \( a^\mu \) and the vanishing function, one can build a tractor “six-velocity” \( V_M = (\frac{1}{v \cdot v}, a_\mu v^\mu, 0) \) transforming as a multiplet under Weyl transformations according to [5].

\(^5\)In fact, the tractor connection also appears in the Yang–Mills-like construction of conformal supergravity [19].
to be realized as rays in this ambient lightcone. Bars’ \( \mathfrak{sp}(2) \) triplet of worldline conformal group Noether charges can be viewed, respectively, as the defining function for the ambient null cone, dilation generator and the harmonic condition obeyed by the Weyl tensor for a Ricci flat geometry. Essentially taking the old Fefferman–Graham ambient metric construction, alongside with the idea of describing unit invariant four dimensional physics with conformal geometry leads one directly to Bars’ two times physics program. Needless to say, this confluence of mathematical and physical technologies is likely to lead to major advances in both fields.

Our paper is organized as follows: In section 2 we review how Einstein gravity can be recovered in the tractor framework as a parallel condition on the scale tractor, and we fix conventions and notations. In particular we define the tractor connection and we introduce the main tractor operators. In section 3 we set out the ambient description of tractors and introduce the triplet of \( \mathfrak{sp}(2) \) operators underlying the two times approach. We discuss the latter in detail in section 4 where we introduce the most general deformation of the flat \( \mathfrak{sp}(2) \) algebra which contains an infinite tower of background fields. In section 5 we give the main new results based on a detailed analysis of Bars’ BRST parent field theory action. By careful gauge choices and identification of the dilaton field we produce the slew of new descriptions of four dimensional gravity mentioned above as well as establishing the link between tractor and two times approaches. In appendix A we give a succinct tractor analysis of Bars’ alternate proposal for a two times gravity theory. In our conclusions (section 6) we discuss the six dimensional quantum mechanical origin of four dimensional gravity, a candidate master theory generating the \( \mathfrak{sp}(2) \) and dilaton dynamics, a frame-like formulation of two times physics and the relation between the towers of auxiliary fields of the two times approach and an unfolding of the full (non-linear) four dimensional Einstein’s equations.

2 Gravity and Parallel Scale Tractors

It is well known that the Einstein–Hilbert gravitational action can be viewed as the gauge fixed version of a conformally improved scalar field theory \[24. \]

\[
S[\varphi, g] = -\frac{4(d-1)}{(d-2)} \int d^d x \sqrt{-g} \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{8} \frac{d-2}{d-1} R \varphi^2 \right],
\]

(1)
which is invariant under local Weyl rescalings $\Omega(x)$, transforming $\varphi \mapsto \Omega^{2-d} \varphi$
and
$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}.$$  

(2)

On the one hand this seems a rather trivial observation because choosing the gauge in which $\varphi$ is constant and equal to $\kappa^{-1}$, one recovers the usual gravity action $S(g, \kappa^{-1}) = -\frac{1}{2\kappa^2} \int d^d x \sqrt{-g} R$. To see that this is in fact a statement of fundamental importance, first note that the Weyl transformation (2) defines the equivalence class relation $g_{\mu\nu} \sim \Omega^2 g_{\mu\nu}$ of a conformal class of metrics $[g_{\mu\nu}]$, so that physics can be cast in terms of conformal, rather than Riemannian geometry. Secondly, note that the Weyl transformation (2) amounts to making local redefinitions of unit systems, which along with general coordinate invariance, is a symmetry that any formulation of physics must enjoy.

So far there is no hint of any six dimensional quantities. To see these, we attempt to write the Weyl invariant formulation (1) of Einstein–Hilbert gravity as the square of a single vector $I^M$

$$S[g, \sigma] = \frac{d(d-1)}{2} \int d^d x \sqrt{-g} \sigma I^M I_M.$$  

(3)

The six component vector

$$I^M = \begin{pmatrix} \sigma \\ \nabla^m \sigma \\ -\frac{1}{d} \left[ \Delta + P \right] \sigma \end{pmatrix},$$

(4)

is called the “scale tractor” and is distinguished by its transformation properties under Weyl transformations. Here the scalar $\sigma = \varphi^{2-d}$ is simply a relabeling of the dilaton $\varphi$ so that it has unit Weyl weight

$$\sigma \mapsto \Omega \sigma.$$  

The field $\sigma$ is often called the “scale” since it measures the relative choice of unit system from point to point in spacetime. Also, it is often convenient to work with the Schouten tensor $P_{\mu\nu}$ which is the trace adjusted Ricci-type tensor, defined by

$$P_{\mu\nu} = \frac{1}{d-2} \left( R_{\mu\nu} - \frac{1}{2(d-1)} g_{\mu\nu} R \right).$$
and its trace is denoted $P = P_\mu^\mu$.

The main features of the action (3) are

- It depends on conformal classes of metrics, embedded in the double equivalence class $[g_{\mu\nu}, \sigma] \sim [\Omega^2 g_{\mu\nu}, \Omega \sigma]$. This allows for manifest Weyl invariance while still specifying a canonical metric $g^0_{\mu\nu}$ in the conformal class satisfying $[g_{\mu\nu}, \sigma] \sim [g^0_{\mu\nu}, \kappa \pi^2]$.

- The measure $\sqrt{-g} \sigma^{-d}$ is separately Weyl invariant, as is also the square of the scale tractor $I^2$. This holds because the scale tractor $I^M$ transforms under particular local $SO(d, 2)$ transformations known as tractor gauge transformations.

- Einstein’s equations amount to the scale tractor being parallel with respect to the tractor connection, exactly the covariant derivative implied by tractor gauge transformations.

- The “length” of the scale tractor measures the cosmological constant. Hence Ricci flatness implies a lightlike scale tractor.

Let us explain these points and the key ingredients of tractor calculus in more detail.

From the four dimensional viewpoint, a six-component multiplet $(V^+, V^m, V^-)$ with $m = 0, \ldots, d - 1$, forms a weight $w$ tractor vector $V^M, M = +, m, -$, if under Weyl transformations it obeys the tractor gauge transformation:

$$V^M \mapsto \Omega^w U^M_N V^N,$$

$$U^M_N = \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon^m & \delta^m_n & 0 \\ -\frac{\Upsilon^2}{2t} & -\frac{\Upsilon}{t} & \frac{1}{\Omega} \end{pmatrix},$$

(5)

where $\Upsilon_\mu = e_\mu^m \Upsilon_m = \Omega^{-1} \partial_\mu \Omega$. In section 3 we will see that tractors naturally live as six-vectors in a six dimensional, signature $(4, 2)$ spacetime endowed with a curved light-cone structure. The reduction to four dimensions induces a tractor-covariant connection:

$$\mathcal{D}_\mu = \begin{pmatrix} \partial_\mu & -e_{\mu n} & 0 \\ P^m_\mu \nabla^m_{\mu n} & e^m_\mu \\ 0 & -P_{\mu n} & \partial_\mu \end{pmatrix},$$

(6)
such that
\[ D_\mu V^M \mapsto \Omega^w U^M_N \left[ D_\mu + w \Upsilon_\mu \right] V^N. \]

By means of the tractor connection one can construct a weight $-1$ tractor-vector operator, the so called “Thomas $D$-operator”, which acting on weight $w$ tractors reads:

\[ D^M = \begin{pmatrix} w(d + 2w - 2) \\ (d + 2w - 2)D^m \\ -(D_\mu D^\mu + w P) \end{pmatrix}. \]  
(7)

Acting with the Thomas $D$-operator on the scale $\sigma$, we obtain a weight 0 tractor-vector, the scale tractor

\[ I^M = \frac{1}{d} D^M \sigma, \]

which has components exactly given by (1).

The scale tractor’s main importance is twofold: first, in tractor theories it controls the coupling of matter to scale in a Weyl-covariant way [10], parametrizing the breaking of local scale invariance in the $\sigma = constant$ physical gauge. On the other hand, $I^M$ is closely related to gravity itself: remarkably, the gravity-dilaton action (1), can be written entirely in terms of the scale tractor as in (3) where tractor indices are raised and lowered with the $SO(d,2)$ invariant metric

\[ \eta_{MN} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta_{mm} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]

To see that a tractor-parallel scale tractor, i.e. $D_\mu I^M = 0$, amounts to Einstein’s equations we explicitly compute the tractor derivative of $I^M$ that, once evaluated at the choice of constant scale $\sigma = \sigma_0$, reads

\[ D_\mu I^M \big|_{\sigma = \sigma_0} = \sigma_0 \begin{pmatrix} 0 \\ P_\mu^m - \frac{1}{d} e_\mu^m P \\ -\frac{1}{d} \partial_\mu P \end{pmatrix}. \]  
(8)

Setting this to zero says $R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R$ and $R = constant$, so that $g_{\mu\nu}$ is precisely an Einstein manifold. This happens at the choice of scale $\sigma = \sigma_0,
so we can say that the scale tractor is parallel when the metric is conformally Einstein:

$$\mathcal{D}_\mu I^M = 0 \iff g_{\mu\nu} = \Omega^2 g^0_{\mu\nu} \quad \text{with} \quad R_{\mu\nu}(g^0) \propto g^0_{\mu\nu}. $$

Moreover, if the scale tractor is parallel then its length squared $I^2 \equiv I^M I_M$ is constant, and proportional to the cosmological constant.

Geometrically the scale tractor can be viewed as coming from a vector perpendicular to a hypersurface in six dimensions. The intersection of that hypersurface with a (curved) lightcone defines a conformal class of metrics on the four dimensional intersection. This picture relies on a six dimensional ambient description of tractors which we describe in the next section. Given the significance of the scale tractor $I^M$, it would be extremely interesting to formulate four dimensional gravity in terms of an independent six component vector field. That result is obtained by combining ambient tractors with Bars’ two times physics proposal and is given in section 5.

3 Ambient Tractors

The importance of six-dimensional spacetimes for describing conformally invariant four-dimensional theories has been clear since the work of Dirac [2]. (Perhaps the simplest motivation for this is that the Minkowski space conformal group $SO(4,2)$ acts naturally on the flat Lorentzian space $\mathbb{R}^{4,2}$. ) Weyl invariance ensures rigid conformal symmetry whenever the metric enjoys conformal isometries; this suggests that four-dimensional conformal geometries can be studied in terms of six-dimensional Riemannian geometries. This was shown to be the case by Fefferman and Graham [23] who formulated the problem of constructing conformal invariants in terms of a six-dimensional ambient metric. This idea was extended to the tractor calculus description of conformal geometry in the series of articles [21 [14] (see also [22]).

Based on duality and holographic arguments, the two times approach of Bars advocates that four dimensional physics (irrespective of whether it enjoys rigid conformal symmetry or not) can be described using a six dimensional spacetime. The tractor approach of Gover et al uses the simple principle of invariance under local choices of unit system to argue that four dimensional physics should be formulated in terms of conformal geometry. Since the latter, in turn, enjoys an ambient six dimensional formulation, local unit invariance and tractors also support a formulation of four dimensional
physics using a six dimensional spacetime. In this section we give the main ingredients of the six dimensional ambient description of tractor calculus.

A four dimensional conformal manifold equipped with an equivalence class of metrics \([g_{\mu\nu}]\), with equivalence defined by local Weyl transformations

\[ g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}, \]

can be viewed as the space of rays in a five dimensional null hypersurface embedded in a six dimensional Riemannian ambient space with metric \(G_{MN}\). Specializing to the conformally flat case, consider the ambient space \(\mathbb{R}^{4,2}\) with the standard flat Lorentzian metric \(dX^M \eta_{MN} dX^N\), which enjoys a closed (and therefore hypersurface orthogonal) homothety given by the dilation/Euler operator \(X^M \frac{\partial}{\partial X^M}\). The zero locus of the homothetic potential \(X^M X_M \equiv X^2\) defines a five dimensional null cone so the space of null rays \(\xi^M\) subject to the equivalence relation \(\xi^M \sim \Omega \xi^M\) (where \(\Omega \in \mathbb{R}^+\)) is four dimensional and determines a (conformally flat) four dimensional conformal structure. The conformal class of metrics follows by letting \(\xi^M(x)\) be a section of the null cone. The ambient metric then pulls back to a four dimensional metric \(ds^2 = d\xi^M d\xi_M\). Choosing a different section \(\xi^M(x)\) results in a conformally related metric. For example, in the conformally flat setting, de Sitter, Minkowski and anti de Sitter space all inhabit the same conformal class. In this case the tractor connection of (6) is the pullback of the Cartan–Maurer form of \(SO(4,2)\) to the conformally flat four dimensional space time described as a coset \(SO(4,2)/P\) where \(P\) stabilizes a lightlike ray.

The above flat model of conformal geometry, as the space of lightlike rays in a six dimension ambient space, extends to curved spaces and general conformal structures as follows: A four dimensional conformal structure determines a Fefferman–Graham ambient metric which admits a hypersurface orthogonal homothety. In the flat case this homothety is generated by the Euler vector field whose components coincide with the standard Cartesian coordinates. In the curved ambient construction, the corresponding homothetic vector field will still be denoted by \(X^M\) (which are \(not\) generally coordinates for which we reserve the notation \(Y^M\)). The key identity is then the equation

\[ G_{MN} = \nabla_M X_N, \quad (9) \]

where \(G_{MN}\) is the ambient metric and \(\nabla\) is its Levi-Civita covariant derivative. This condition already suffices to uniquely determine a four dimensional
conformal structure. The symmetric part of (9) implies the homothetic conformal Killing equation while its antisymmetric part says that the one-form dual to $X^M$ is closed. Indeed this one form is exact

$$X_M = \frac{1}{2} \nabla_M X^2.$$ 

Clearly, the ambient metric is the double gradient of the homothetic potential $G_{MN} = \frac{1}{2} \nabla_M \partial_N X^2$. The zero locus of the potential $X^2$ defines a curved cone, a quotient of which recovers the four dimensional conformal manifold. Observe that the above identities for the ambient metric imply

$$X^M R_{MNRS} = 0 = (X^T \nabla_T + 2) R_{MN} R^R_S.$$ 

To ensure uniqueness of the ambient metric for a given four dimensional structure, Fefferman and Graham require that the ambient metric is formally Ricci flat in any odd dimension (to all orders), and Ricci flat to finite order in the defining function $X^2$ in even dimensions greater than or equaling four. For our purposes, uniqueness of the underlying four dimensional conformal structure is all we need, so we will typically work with six dimensional ambient metrics subject to (9) but need not impose six dimensional Ricci flatness.

The Rosetta Stone between six dimensional ambient space operators and the Thomas $D$-tractor operator (7) on a four dimensional conformal manifold was first given in [14] and simply reads

$$D_M \equiv \nabla_M (d + 2 X^N \nabla_N - 2) - X_M \Delta.$$ 

(10)

The canonical tractor of [13] corresponds to the vector field $X^M$ while tractor weights are eigenvalues of the operator $X^M \nabla_M$. (In [22], it was realized that these operators are related to a momentum space representation of the ambient space conformal group.) Tractor tensors $T^{M_1 \ldots M_s}(x)$ (sections of weighted tractor tensor bundles over four dimensional spacetime) can then be viewed as equivalence classes of six dimensional ambient space tensors

$$T^{M_1 \ldots M_s}(Y) \sim T^{M_1 \ldots M_s}(Y) + X^2 U^{M_1 \ldots M_s}(Y),$$ 

subject to a weight constraint

$$X^M \nabla_M T^{M_1 \ldots M_s} = w T^{M_1 \ldots M_s}.$$ 

(12)
The equivalence relation can also be handled by working with weight \( w - 2 \) ambient space tensors of the form

\[ \delta(X^2) \ T^{M_1 \ldots M_s}, \]

subject to the constraint \( X^2 = 0 \). It is not difficult to check that the ambient operator \((\ref{ambientoperator})\) is well defined on equivalence classes defined by the cone condition \((\ref{conecondition})\).

The equivalence relation \((\ref{conecondition})\) and weight constraint \((\ref{weightconstraint})\) do not define a unique extension of a four dimensional tractor to the six dimensional ambient space. For that, one needs to “fix a gauge” for the equivalence relation. A convenient choice is to require that six dimensional quantities are harmonic. The first example of this is the Ricci flat condition of Fefferman–Graham (because the remaining Weyl part of the ambient Riemann curvature is then harmonic). In fact, it is easily verified that the triplet of operators

\[ \{X^2, X^M \nabla_M + \frac{d+2}{2}, \Delta\}, \tag{13} \]

obey an \( \mathfrak{sp}(2) \) Lie algebra. This algebraic fact underlies Bars’ two times approach described in the next section.

## 4 Two Times Physics

A simple starting point for understanding two times physics, is the Howe dual pair \([26]\)

\[ \mathfrak{sp}(2(d+2)) \supset \mathfrak{sp}(2) \oplus \mathfrak{so}(d,2). \tag{14} \]

This Lie algebra statement—namely that \( \mathfrak{sp}(2) \) and \( \mathfrak{so}(d,2) \) are maximal cocommutants in \( \mathfrak{sp}(2(d+2)) \)—says that imposing as constraints an \( \mathfrak{sp}(2) \) subalgebra of the natural \( \mathfrak{sp}(2(d+2)) \) algebra acting on a \( d + 2 \) dimensional phase space, leaves a residual \( \mathfrak{so}(d,2) \) global symmetry algebra. This latter algebra generates the conformal isometries of \( d \)-dimensional Minkowski (or more generally conformally flat) spacetime.

Consider, for example, Bars’ approach to the relativistic particle \([27, 29]\). Instead of requiring worldline reparameterization invariance and therefore a four dimensional Hamiltonian constraint, Bars requires local worldline conformal invariance under \( \mathfrak{so}(2,1) \cong \mathfrak{sp}(2) \) which imposes a triplet of first class constraints. In four dimensions a three dimensional constraint algebra would be too constraining, but as is clear from the Fefferman–Graham
ambient space construction described above, if this constraint algebra acts in six dimensions as in \( (\text{13}) \), the null cone and weight constraints perform the reduction to four dimensions leaving a single Hamiltonian constraint just as in the standard approach. By making different gauge choices for the local \( \mathfrak{sp}(2) \) symmetry, one can obtain a plethora of four dimensional models—“holographic shadows”—all encompassed by a single six dimensional one \([\text{28}]\).

The above discussion pertains to single particle models propagating in fixed backgrounds. Our chief interest is a description of four dimensional field theories and in particular four dimensional gravity. For that, two main ingredients are required. Firstly we must quantize the underlying particle model so that, in turn, quantum mechanical wave functions can be reinterpreted as quantum fields. Secondly we need to write equations of motion for the background fields. Both steps can be achieved in a unified way by working with quantum mechanical operators. (An alternative approach employed heavily by Bars \([\text{29, 30}]\) is to employ phase space quantization technology \([\text{31}]\), but we find working directly with quantum mechanical operators to be more direct.)

Our model, described in detail in the next section, will be built from two multiplets, the first “gravity multiplet” will describe ambiently a conformal class of metrics along with an additional vector field intimately related to the scale tractor of section \([2]\). The second “dilaton multiplet” describes the dilaton or scale field (or in other words a spacetime-varying Planck’s constant). Equations of motion for the gravity multiplet have already been proposed by Bars \([\text{32}]\). Classically they amount to a triplet of Hamiltonians \( Q_{ij} = Q_{ji} \) \((i, j = 1, 2)\) on a \( 2(d+2) \) dimensional phase space subject to an \( \mathfrak{sp}(2) \) algebra under Poisson brackets

\[
\{ Q_{ij}, Q_{kl} \} = \varepsilon_{kj} Q_{il} + \varepsilon_{ki} Q_{jl} + \varepsilon_{lj} Q_{ik} + \varepsilon_{il} Q_{jk} .
\]  

(15)

Here one must solve for the \( Q_{ij} \) modulo gauge transformations corresponding to canonical transformations

\[
Q_{ij} \mapsto Q_{ij} + \{ \epsilon, Q_{ij} \} .
\]  

(16)

An elegant solution has been found by Bars \([\text{32}]\) by choosing Darboux coordinates \( \{ P_M, Y^N \} = \delta^N_M \), expanding in powers of the momentum \( P_M \) shifted by some vector field \( A_M(Y) \), and then partially fixing the gauge invariance \([13]\).
\[
Q = \left( X^M G_{MN}(Y) X^N \begin{array}{cc} X^M \tilde{P}_M & X^M \tilde{P}_M \\ X^M \tilde{P}_M & \Sigma(Y) + \tilde{P}_M G^{MN}(Y) \tilde{P}_N + H(\tilde{P}, Y) \end{array} \right), \tag{17}
\]

where
\[
\tilde{P}_M \equiv P_M + A_M(Y), \\
H(\tilde{P}, Y) \equiv \sum_{k=2}^{\infty} H^{M_1...M_k}(Y) \tilde{P}_{M_1} \cdots \tilde{P}_{M_k}.
\]

In addition, this result is intimately connected to ambient tractors, because the algebra (15) requires the metric \( G_{MN} \) appearing in (17) to obey the closed homothety condition (9). Moreover the vector field \( A_M \) appearing in \( \tilde{P}^M \) obeys
\[
X^M F_{MN} \equiv (\mathcal{L}_X + 1) A_N - \nabla_N (X^M A_M) = 0, \tag{18}
\]
and the scalar \( \Sigma \) and totally symmetric tensors \( H^{M_1...M_k} \) are subject to weight conditions
\[
(\mathcal{L}_X + 2) \Sigma \equiv (X^M \nabla_M + 2) \Sigma = 0, \\
(\mathcal{L}_X + 2) H^{M_1...M_k} \equiv (X^M \nabla_M + 2 - k) H^{M_1...M_k} = 0. \tag{19}
\]
Classically the tensors \( H^{M_1...M_k} \) must also be transverse to the homothetic vector field \( X^M \). The above solution still enjoys residual gauge symmetries of the form (16). The beauty of Bars’ solution is that these residual transformations amount to diffeomorphisms of the tensors \( X^M, G_{MN}, A_M, \Sigma \) and \( H^{M_1...M_k} \), abelian Maxwell gauge transformations of \( A_M \), as well as a certain class of higher rank symmetries of the symmetric tensors \( H^{M_1...M_k} \) which we will discuss in detail later.

To quantize the Hamiltonians \( Q_{ij} \), we look for operators acting on wavefunctions depending on coordinates \( Y^M \). We express these as expansions in the covariant derivatives \( \tilde{\nabla}_M = \nabla_M + A_M \). This amounts to a choice of quantum orderings for a basis of all operators acting on wavefunctions. More precisely, momenta \( P_M \) act on wavefunctions as derivatives \( \partial_M \), but we add subleading ordering terms to higher powers of momenta in order to maintain covariance. We then require that the quantum commutator of the \( Q_{ij} \)'s obeys the \( \mathfrak{sp}(2) \) algebra
\[
[Q_{ij}, Q_{kl}] = \varepsilon_{kij} Q_{il} + \varepsilon_{kij} Q_{ji} + \varepsilon_{lj} Q_{ik} + \varepsilon_{lb} Q_{jk}, \tag{20}
\]
14
modulo the quantum symmetry

\[ Q_{ij} \mapsto Q_{ij} + [\epsilon, Q_{ij}], \]  

(21)

whose parameter \( \epsilon \) is now itself an operator. This system of equations has been proposed by Bars in an equivalent phase space and star product quantization \[32\]. Quantization necessitates a slight modification of Bars’ classical solution to

\[ Q = \begin{pmatrix} X^2 & X^M \tilde{\nabla}_M + \frac{d+2}{2} \\ X^M \tilde{\nabla}_M + \frac{d+2}{2} & \Sigma + \tilde{\nabla}^2 + H(\tilde{\nabla}, Y) \end{pmatrix}, \]  

(22)

with

\[ H(\tilde{\nabla}, Y) \equiv \sum_{k=2}^{\infty} H^{M_1...M_k}(Y) \tilde{\nabla}_{M_1} \cdots \tilde{\nabla}_{M_k}. \]

Here the closed homothety, curvature and weight conditions are unaltered from their classical counterparts \[9\18\19\], but the transverse conditions on the symmetric tensors \( H^{M_1...M_k} \) are modified to read

\[ 2X_M H^{MM_2...M_k} + (k + 1)H^{M_M_2...M_k} = 0. \]  

(23)

From this we learn iteratively that the trace of \( H^{MN} \) vanishes, the trace of \( H^{MNR} \) is the part of \( H^{MN} \) parallel to \( X^M \) etcetera. More succinctly, the condition \(23\) just says

\[ [X^2, H(\tilde{\nabla}, Y)] = 0. \]

But now let us examine which gauge symmetries respect the quantum solution \(22\). Firstly, expanding the gauge parameter in powers of \( \tilde{\nabla}_M \)

\[ \epsilon(\tilde{\nabla}, Y) = -\alpha(Y) + \xi^M(Y) \tilde{\nabla}_M + \varepsilon(\tilde{\nabla}, Y), \]

where all terms of quadratic order and higher are stored in \( \varepsilon \), it is easy to verify that the zeroth and first order terms generate abelian gauge transformations

\[ A_M \mapsto A_M + \nabla_M \alpha, \]

and diffeomorphisms with parameter \( \xi^M \). These are desirable symmetries, so we do not want to gauge fix them at this juncture. We still have the higher order gauge freedoms in \( \varepsilon \), although these are not completely arbitrary:
Requiring $Q_{11} = X^2$ to be inert, the gauge parameter $\varepsilon$ obeys the same commutation relation with the homothetic potential as $H$

$$[X^2, \varepsilon] = 0.$$  (24)

Furthermore, invariance of $Q_{12}$ implies that

$$[X^M \tilde{\nabla}_M, \varepsilon] = 0.$$  

It follows that $\delta Q_{22} \equiv [\varepsilon, \Sigma + \tilde{\nabla}^2 + H]$ obeys the same conditions as $H$, namely

$$[X^2, \delta Q_{22}] = 0 = [X^M \tilde{\nabla}_M, Q_{22}] + 2Q_{22}.$$  

Now we define a vector

$$U_M \equiv \nabla_M \Sigma,$$

and note that

$$[\varepsilon, \Sigma] = \frac{1}{2} \varepsilon^{MN} \mathcal{L}_U G_{MN} + \varepsilon^{MN} U_M \tilde{\nabla}_N + \sum_{k=3}^{\infty} k \varepsilon^{M_1 \ldots M_k} (U_{M_1} \tilde{\nabla}_{M_2} \cdots \tilde{\nabla}_{M_k})_W,$$  (25)

where $(\bullet)_W$ denotes Weyl ordering in the symbols $(U, \tilde{\nabla})$.

We now make the assumption that the vector $U_M$ is non-vanishing. Certainly, the set of vanishing $U_M$ is measure zero (a situation similar to non-invertible metrics among the space of $4 \times 4$ matrices). Bars has suggested that models with vanishing $U_M$ might describe a novel “higher spin branch”, but we do not pursue this line of argument any further here. With $U_M$ non-vanishing the space of rank two and higher symmetric tensors $U_M \varepsilon^{M_1 \ldots M_k}$ appearing in the summation in formula (25) suffices to gauge away the operators $H(\tilde{\nabla}, Y)$. One might worry that this reintroduces new contributions to $Q_{22}$ at order zero and one in $\tilde{\nabla}$, but we have as yet not used the freedom to choose the first two terms in (25). Clearly, when $U_M \neq 0$, we can choose $\varepsilon^{MN} U_M$ to ensure that $Q_{22}$ has no term linear in $\tilde{\nabla}$. Finally, when $U_M$ is not a conformal Killing vector (notice that (24) implies that $\varepsilon^{MN}$ is trace-free) we can try to use the first term in (25) to remove $\Sigma$. A generic choice of metric $G_{MN}$ will not admit conformal Killing vectors so we may safely \footnote{It is possible that $\Sigma$ can still be gauged away even if the metric $G_{MN}$ admits conformal Killing vectors $U^M = \nabla^M \Sigma$. We have not analyzed this issue in detail, but it is interesting to note that the condition $\nabla_{(M} U_{N)} \propto G_{MN}$ along with the weight condition (19) for $\Sigma$ implies that $\Sigma$ is an eigenstate of the quadratic Casimir of the triplet of operators (13).} pick a gauge for which $\Sigma = 0$.  

\section*{6}
Thus, we arrive at our final solution for the quantum equations (20)

\[
Q(G_{MN}, A_M) = \begin{pmatrix}
X^2 & X^M \nabla_M + \frac{d+2}{2} \\
X^M \nabla_M + \frac{d+2}{2} & \nabla^2
\end{pmatrix}.
\]

(26)

It is parameterized, modulo diffeomorphisms and \(SO(1,1)\) gauge transformations by a metric \(G_{MN}\) and abelian gauge field \(A_M\) subject to the closed homothety and transverse curvature requirements in equations (9) and (18), respectively. This is the gravity multiplet of our model. It describes space-time geometry but does not describe gravitational dynamics. From the tractor viewpoint, that requires coupling to scale. Or in other words, a dilaton. Therefore, we now describe the coupling of the gravity multiplet to the dilaton multiplet.

5 Main Results: Gravity

In section 2 we saw that instead of formulating gravity in terms of an Einstein–Hilbert action functional depending on four-metrics, one could build from the square of the scale tractor \(I^M\) an equivalent action depending on the scale (or dilaton) \(\sigma\) and a conformal class of four dimensional metrics \([g_{\mu\nu}]\). The operator \(Q\) of the previous section depended on (i) a six dimensional metric \(G_{MN}\) with closed homothety and (ii) a six dimensional vector \(A_M\). Since the metric \(G_{MN}\) encodes a four dimensional conformal class of metrics \([g_{\mu\nu}]\) one can hope that the vector \(A_M\) is somehow related to the scale tractor and so that a theory built from the operator \(Q\) could amount to a tractor description of Einstein–Hilbert gravity. For this proposal to work, we still need to couple to a dilaton field, or in other words scalar matter. From a two times physics perspective this coupling should respect the gauge symmetry (21) as well as the \(\mathfrak{sp}(2)\) gauge symmetry generated by the operators \(Q\). A coupling to scalars with exactly these symmetries has been computed by Bars using first quantized BRST techniques [15] and reads

\[
S(Q, \Omega, \Theta, \Lambda, \Psi) = \frac{2(d-1)}{d-2} \int d^{d+2}Y \sqrt{G} \left[ \Omega Q_{22} + \Theta Q_{12} + \Lambda Q_{11} \right] \Psi.
\]

(27)

Our claim is that this action principle, along with the conditions (20) on the operator \(Q\) amounts to the tractor description of four dimensional Einstein–Hilbert gravity.
The action (27) depends (from a six dimensional viewpoint) on an infinite set of fields through the operator $Q$. However it also enjoys infinitely many local symmetries generated by an operator parameter $\epsilon$ as well as a local $\mathfrak{sp}(2)$ invariance with local parameters $(\lambda(Y), \theta(Y), \omega(Y))$

\[
Q \rightarrow Q + [\epsilon, Q], \\
\Psi \rightarrow \Psi + \epsilon \Psi, \\
\Omega \rightarrow \Omega - \epsilon^\dagger \Omega - Q_{11}^\dagger \theta + [Q_{12}^\dagger + 2] \omega, \\
\Theta \rightarrow \Theta - \epsilon^\dagger \Theta + Q_{11}^\dagger \lambda - Q_{22}^\dagger \omega - 4 \theta, \\
\Lambda \rightarrow \Lambda - \epsilon^\dagger \Lambda + Q_{22}^\dagger \theta - [Q_{12}^\dagger - 2] \lambda.
\]

Here the dagger operation is the standard adjoint with respect to the six dimensional measure appearing in (27). We are now ready to verify our claim that (27) is the theory of gravity.

The first step is use the gauge freedom $\epsilon$ to reach the gauge (26) for the operator $Q$. This yields a standard, generally covariant, six dimensional action depending only on finitely many fields $(G_{MN}, A_M, \Omega, \Theta, \Lambda)$

\[
S = \frac{2(d-1)}{d-2} \int d^{d+2}Y \sqrt{G} \left[ \Omega \nabla^2 + \Theta (X^M \nabla_M + \frac{d+2}{2}) + \Lambda X^2 \right] \Psi, \tag{28}
\]

with gauge invariance

\[
A_M \rightarrow A_M + \nabla_M \alpha, \\
\Psi \rightarrow \Psi - \alpha \Psi, \\
\Omega \rightarrow \Omega + \alpha \Omega - X^2 \theta - (X^M \nabla_M + \frac{d+2}{2} - 2) \omega, \\
\Theta \rightarrow \Theta + \alpha \Theta + X^2 \lambda - \nabla^2 \omega - 4 \theta, \\
\Lambda \rightarrow \Lambda + \alpha \Lambda + \nabla^2 \theta + (X^M \nabla_M + \frac{d+2}{2} + 2) \lambda. \tag{29}
\]

The action (28) is four dimensional gravity wearing a six dimensional disguise. To disrobe it further, we use the $SO(1,1)$ gauge symmetry $\alpha$ to choose a gauge

\[
X^M A_M = -w \quad \text{which implies} \quad X^N \nabla_N A_M = -A_M. \tag{30}
\]
Here $w$ is an arbitrary real number. We could equally well have chosen $w = 0$, but we prefer the above since it will imply the most general assignments of tractor weights to the scalar fields. In any case, $w$ will drop out at the end of our computation, and thereby serves as a check on our algebra. Notice that using (18), the potential $A_M$ now has weight $-1$ with respect to the weight operator $X^M \nabla_M$. Note that the vector $A_M$ still enjoys residual abelian gauge transformations with weight zero gauge parameter $X^M \nabla_M \alpha = 0$.

We now integrate out the Lagrange multipliers ($\Theta$, $\Lambda$) which imposes constraints

$$X^M \nabla_M \Psi = \left( w - \frac{d}{2} - 1 \right) \Psi, \quad X^2 \Psi = 0.$$  

Solving the latter constraint via

$$\Psi = \delta(X^2) \phi, \quad \phi \sim \phi + X^2 \chi,$$

and comparing with (11) and (12), we see that $\phi$ is a weight $w - \frac{d}{2} + 1$ tractor scalar.

There is still the freedom using the gauge parameter $\omega$ to gauge away $\Omega$ save for gauge transformations $\omega$ in the kernel of $X^M \nabla_M + w + \frac{d}{2} - 1$. Hence all that remains is the part of $\Omega$ of weight $-w - \frac{d}{2} + 1$. The remaining field content along with their weights are summarized in the following table

| Field $\quad$ | Weight $\quad$ |
|--------------|----------------|
| $\Omega$ $\quad$ | $-w - \frac{d}{2} + 1$ $\quad$ |
| $\phi$ $\quad$ | $w - \frac{d}{2} + 1$ $\quad$ |
| $A_M$ $\quad$ | $-1$ $\quad$ |

Integrating by parts to ensure no derivatives act on the delta function in $\Psi$, the action now takes the extremely simple form

$$S = \frac{2(d-1)}{d-2} \int d^{d+2} Y \sqrt{G} \delta(X^2) \ T, \quad (31)$$

where

$$T = \phi(\nabla^M - A^M)(\nabla_M - A_M) \Omega. \quad (32)$$

Since $T \sim T + X^2 U$, it is a tractor scalar with weight $-d$ (see the above table). We would like to express the action (31) as a four dimensional integral
over tractor-valued objects. To that end we need to express (32) in terms of ambient tractor operators: Using the ambient expression (10) for the Thomas $D$-operator, we easily derive the following ambient tractor identities

\[
\Delta \Omega - 2A^M \nabla_M \Omega = \frac{1}{w} A^M D_M \Omega, \\
\nabla^M A_M = \frac{1}{d-2} D_M A^M. 
\]

(There is no pole at $w = 0$ in the first identity, as can be easily verified by using the four dimensional component expression (7) for the Thomas $D$-operator.) Hence

\[
T = \phi \left( \frac{1}{w} A^M D_M - \frac{1}{d-2} (D_M A^M) + A^2 \right) \Omega. 
\]

The beauty of this expression is that $\delta(X^2)T$ now only depends on equivalence classes $A_M \sim A_M + X^2 B_M$, $\Omega \sim \Omega + X^2 \Xi$. Therefore all fields are now tractor valued. Hence we may replace the ambient space integral (31), with a four dimensional integral depending on tractors $(\phi, \Omega, A_M)$

\[
S = \frac{2(d-1)}{d-2} \int d^d x \sqrt{-g} \phi \left[ \frac{1}{w} A^M D_M - \frac{1}{d-2} (D_M A^M) + A^2 \right] \Omega. 
\]

Note that the integrand has weight $-d$ while the metric determinant has weight $d$ under Weyl transformations so this action principle is now manifestly Weyl invariant. Our claim is now that this tractor action is equivalent to the formulation of the Einstein–Hilbert action in terms of the square of the scale tractor (3).

To verify our final claim we must examine the remaining $SO(1,1)$ gauge symmetry

\[
A_M \mapsto A_M + \frac{1}{d-2} D_M \alpha, \\
\Omega \mapsto \Omega + \alpha \Omega, \\
\phi \mapsto \phi - \alpha \phi, 
\]

Bars handles delta-function valued ambient space integrals by developing a calculus for derivative of delta functions. The simple tractor analysis given here, obviates the need for such methods.
where the gauge parameter $\alpha$ is a weight zero tractor scalar. Notice that the gauge transformation of $A_M$ respects the condition $X^M A_M = -w$. Now observe that the action depends only algebraically on the $SO(1,1)$ gauge field $A_M$ and the pair of fields $(\phi, \Omega)$ form a doublet under this symmetry. Hence, we expect that upon integrating out $A_M$, only the gauge invariant combination $\phi \Omega$ should survive. This computation can be performed either using component expressions for the tractor quantities in (34) or directly using tractors. In components, one finds that the bottom slot $A^-$ of the gauge field decouples completely from the action and that integrating out the middle slot of $A_M$ sets it equal to the $SO(1,1)$ current $\frac{i}{2} \nabla_m \log (\Omega/\phi)$. This yields the four dimensional action for a conformally improved scalar field

$$S = \frac{2(d-1)}{d-2} \int d^d x \sqrt{-g} \varphi \left[ \Delta - \frac{d-2}{2} P \right] \varphi,$$

where $\varphi$ is the weight $1 - \frac{d}{2}$ scalar field defined by

$$\varphi^2 = \phi \Omega.$$

In other words it is the dilaton. Using the relationship between the dilaton and scale, $\varphi = \sigma^{1-\frac{d}{2}}$, we obtain as explained in section 2 the tractor version of the Einstein–Hilbert action in terms of the square of the scale tractor

$$S = \frac{d(d-1)}{2} \int d^d x \frac{\sqrt{-g}}{\sigma^d} I^M I_M.$$

This completes our demonstration that the $sp(2)$ invariant theory (27) amounts to a theory of four dimensional gravity. We now turn to implications of our results.

6 Conclusions and Outlook

In this article we formulated the Einstein–Hilbert action as a trace

$$S = \text{tr} \ Q \ P$$

over quantum mechanical operators $Q$ (as in (26)) and

$$P = \begin{pmatrix} |\Psi\rangle \langle \Lambda| & \frac{1}{2} |\Psi\rangle \langle \Theta| \\ \frac{1}{2} |\Psi\rangle \langle \Theta| & |\Psi\rangle \langle \Omega| \end{pmatrix}.$$
In this formulation, second quantization amounts to integrating over the space of operators $Q$ and $P$ in the path integral. This leads one to wonder whether quantum field theory effects, such as Weyl anomalies, can be understood from this six-dimensional quantum mechanical picture. An advantage of this two times approach is that it formulates gravity in terms of a very limited field content: the three components of $Q$ viewed as functions of a twelve dimensional phase space. Weyl and diffeomorphism symmetries are neatly encoded in the algebra (20) and its gauge invariance (21). A pressing question therefore is to compute anomalies in the $sp(2)$ symmetry.

Another benefit of the two times starting point (27) is that it yields a new tractor formulation of the conformally Einstein condition (see the action (34)). At the very least, this should have implications for conformal geometry; the triplet of tractor fields ($\phi, \Omega, A_M$) underly the scale tractor $I^M$. This observation deserves further investigation.

Another interesting avenue for further research is whether there exists a framelike formulation of two times physics. This is based on the simple observation that the operator (26) can be factorized as

$$Q = \left[ \left( X^M \right) \left( \tilde{\nabla} M \right) \right]_W.$$

The operator $V^M_i = (X^M \tilde{\nabla}^M)$ can then be interpreted as a two times frame field, so one could try to impose the Howe dual pair (14) decomposition as equations of motion for fundamental fields $V^M_i$. This might be particularly interesting when one considers the interpretation of the infinite tower of six dimensional auxiliary fields appearing in the parent action (27). In particular, one wonders whether these fields solve the problem posed, and partially solved in [33], of finding an unfolding of the full nonlinear Einstein’s equations. The relation between these two approaches may be clearer in a framelike formulation, since (unlike unfolding constructions) two times models are typically constructed in a metric formulation.

Finally, a gravitational two times action principle that simultaneously incorporates the benefits of both actions (27) and (39)—namely producing the $sp(2)$ algebra as equations of motion while maintaining manifest $sp(2)$ symmetry—would be very desirable. In fact, once we understand that our work implies that the coupling of the gravity multiplet (built from $sp(2)$ generators) to scalars really amounts to a gravity-dilaton coupling, then we can identify yet another action principle proposed by Bars as a candidate.
model for cosmological four dimensional Einstein gravity. Bars’ proposal is to produce the equations of motion for the operator \( Q \) from a Chern–Simons action
\[ S_{\text{CS}} = \int [Q \star Q + Q \star Q \star Q], \]
(where the Moyal star product \( \star \) is employed to produce operator equations of motion from phase space valued fields). Hence the sum of this action plus the BRST action \( S_{\text{BRST}} \) in
\[ S = S_{\text{CS}} + \lambda S_{\text{BRST}}, \] (38)
deforms the \( \mathfrak{sp}(2) \) relations by dilaton dependent terms (see [30] for explicit formulæ). A simple conjecture, therefore, is that these produce the cosmological constant coupling missing from the action (27). In particular, the relative coefficient \( \lambda \) in the total action (38) could be identified with the cosmological constant.

Acknowledgements

It is a pleasure to thank Itzhak Bars, Nicolas Boulanger, David Cherney, Tom Curtright, Olindo Corradini, Claudia de Rham, Rod Gover, Maxim Grigoriev, Djordje Minic, Bruno Nachtergaele, Andy Port, Abrar Shaukat, Per Sundell, Misha Vasiliev and Steven Weinberg for discussions. R.B. thanks the U.C. Davis Department of Mathematics for warm hospitality and the INFN, Sezione di Bologna, and the Università di Bologna Marco Polo program for financial support.

A An Alternative Six Dimensional Formulation of Gravity

In [18] Bars proposed the following six dimensional field theory model for gravity coupled to scalar field
\[ S = -\frac{1}{2} \int d^{d+2} Y \sqrt{G} \left[ \delta(W) \left( R(G) \varphi^2 + \alpha (\nabla \varphi)^2 - \lambda \varphi^{2d} \right) \right. \]
\[ \left. - \delta'(W) \left( (\Delta W - 4) \varphi^2 - \nabla_M W \nabla^M \varphi^2 \right) \right], \] (39)
with $\alpha = \frac{4(d-1)}{d-2}$ and for some $\lambda$ playing the rôle of the cosmological constant.

A distinguishing feature of this action is that the homothetic condition and the weight condition on $\varphi$ follow from its equations of motion; they indeed arise from the field equations for $G_{MN}$ and $\varphi$ instead of requiring closure of the $\mathfrak{sp}(2)$ algebra. Partially solving those equations, one obtains the following set of relations

$$W = X^2, \quad G_{MN} = \nabla_M X_N, \quad X^M \nabla_M \varphi = \left(1 - \frac{d}{2}\right)\varphi.$$  

Plugging these back in (39) we get the following model

$$S = -\frac{1}{2} \int d^{d+2}Y \sqrt{G} \delta(X^2) \left[R(G)\varphi^2 - \alpha \varphi \Delta \varphi - \lambda \varphi \varphi^2 \right]. \quad (40)$$

Now note that, introducing the scale tractor $I^M$ constructed from $\sigma = \varphi^{\frac{2}{d-2}}$ in the usual way (see section 2), the action (40) becomes

$$S = -\frac{1}{2} \int d^{d+2}Y \sqrt{G} \delta(X^2) \left[R(G)\varphi^2 + \frac{\alpha}{\sigma} \varphi I^M D_M \varphi - \lambda \varphi \varphi^2 \right],$$

that in turn, by using the relation $I^M D_M \sigma^k = k(d + k - 1)\sigma^{k-1}I^2$, can be rewritten as

$$S = -\frac{1}{2} \int d^{d+2}Y \sqrt{G} \delta(X^2) \frac{1}{\sigma^d} \left[R(G)\sigma^2 - d(d-1)I^2 - \lambda \right]. \quad (41)$$

Let us observe at this point that, as was shown by Fefferman and Graham in [23], a conformal class of $d$-dimensional metrics $[g_{\mu\nu}]$ determines a Ricci flat ambient space if $d$ is odd, and a Ricci flat ambient space modulo $(X^2)^{\frac{d}{d-2}}$. Hence, since the action (41) depends only on the conformal class of metrics $[g_{\mu\nu}]$ and includes the delta function $\delta(X^2)$, we can set to zero the curvature term in (41). In fact, another way to see this, is that we could have chosen a gauge in section 4 where $\Sigma = R(G)$.

Now that the model is completely written in terms of tractor objects it may be directly written in four dimensional language as

$$S = \frac{d(d-1)}{2} \int d^d x \sqrt{-g} \sigma^d \left[I^M I_M + \frac{\lambda}{d(d-1)} \right]. \quad (42)$$

When $\lambda = 0$, this model coincides with (36) demonstrating the equivalence of these two models in that case. The formulation (39) has the advantage that
it includes a cosmological constant and partially imposes the relations (20) as equations of motion coming from a variational principle. Its disadvantage is that the manifest $\mathfrak{sp}(2)$ symmetry is lost. In our conclusions we speculated that a third model proposed by Bars incorporates the best features of both models (27) and (39).

References

[1] T. Kaluza. Sitz. Preuss. Akad. Wiss. (1921) 966; O. Klein. Zeitschrift für Physik 37 (1926) 895.

[2] P. A. M. Dirac, Ann. Math. 37 (1936) 429.

[3] I. Bars and C. Kounnas, Phys. Lett. B 402 (1997) 25 [arXiv:hep-th/9703060]; I. Bars, Class. Quant. Grav. 18, 3113 (2001) [arXiv:hep-th/0008164]; I. Bars, C. Deliduman and O. Andreev, Phys. Rev. D 58, 066004 (1998) [arXiv:hep-th/9803188]; I. Bars, Phys. Rev. D 58, 066006 (1998) [arXiv:hep-th/9804028]; S. Vongehr, “Examples of black holes in two-time physics,” arXiv:hep-th/9907077; I. Bars, “Two-time physics,” arXiv:hep-th/9809034; I. Bars, Phys. Rev. D 59, 045019 (1999) [arXiv:hep-th/9810025]; I. Bars and C. Deliduman, Phys. Rev. D 58, 106004 (1998) [arXiv:hep-th/9806085]; I. Bars, C. Deliduman and D. Minic, Phys. Rev. D 59, 125004 (1999) [arXiv:hep-th/9812161]; I. Bars, C. Deliduman and D. Minic, Phys. Lett. B 457, 275 (1999) [arXiv:hep-th/9904063]; I. Bars, Phys. Lett. B 483, 248 (2000) [arXiv:hep-th/0004090]; I. Bars, C. Deliduman and D. Minic, Phys. Lett. B 466, 135 (1999) [arXiv:hep-th/9906223]; I. Bars, Phys. Rev. D 62, 085015 (2000) [arXiv:hep-th/0002140]; I. Bars, Phys. Rev. D 62, 046007 (2000) [arXiv:hep-th/0003100]; I. Bars and S. J. Rey, Phys. Rev. D 64, 046005 (2001) [arXiv:hep-th/0104135]; I. Bars, “Two-time physics,” arXiv:hep-th/9809034; I. Bars, AIP Conf. Proc. 589 (2001) 18 [AIP Conf. Proc. 607 (2001) 17] [arXiv:hep-th/0106021];

[4] I. Bars, “Gauge Symmetry in Phase Space, Consequences for Physics and Spacetime,” arXiv:1004.0688 [hep-th].

[5] N. Boulanger, J. Math. Phys. 46, 053508 (2005) [arXiv:hep-th/0412314].
[6] C. R. Preitschopf and M. A. Vasiliev, Nucl. Phys. B 549 (1999) 450 [arXiv:hep-th/9812113].

[7] O. V. Shaynkman, I. Y. Tipunin and M. A. Vasiliev, Rev. Math. Phys. 18 (2006) 823 [arXiv:hep-th/0401086]; M. A. Vasiliev, Nucl. Phys. B 829 (2010) 176 [arXiv:0909.5226 [hep-th]]; Nucl. Phys. B 829 (2010) 176 [arXiv:0909.5226 [hep-th]].

[8] X. Bekaert and M. Grigoriev, SIGMA 6 (2010) 038 [arXiv:0907.3195 [hep-th]].

[9] S. Weinberg, “Six-dimensional Methods for Four-dimensional Conformal Field Theories”, arXiv:1006.3480 [hep-th].

[10] A. R. Gover, A. Shaukat and A. Waldron, Nucl. Phys. B 812 (2009) 424 [arXiv:0810.2867 [hep-th]]; Phys. Lett. B 675 (2009) 93 [arXiv:0812.3364 [hep-th]]; A. Shaukat and A. Waldron, Nucl. Phys. B 829 (2010) 28 [arXiv:0911.2477 [hep-th]].

[11] A. Shaukat, “Unit Invariance as a Unifying Principle of Physics,” U.C. Davis Ph.D. dissertation, arXiv:1003.0534 [math-ph].

[12] T. Y. Thomas, Proc. N.A.S., 12, 352 (1926); “The Differential Invariants of Generalized Spaces,” Cambridge University Press, Cambridge, 1934.

[13] A. R. Gover, Adv. Math. 163, 206 (2001); A. Čap and A. R. Gover, Ann. Glob. Anal. Geom. 24, 231 (2003); T. N. Bailey, M. G. Eastwood and A. R. Gover, Rocky Mtn. J. Math. 24, 1 (1994). Rend. Circ. Mat. Palermo (2) Suppl. No. 59 (1999), 25–47.

[14] A. R. Gover and L. J. Peterson, Commun. Math. Phys. 235, 339 (2003).

[15] I. Bars and Y. C. Kuo, Phys. Rev. D 74 (2006) 085020 [arXiv:hep-th/0605267].

[16] W. Siegel, Int. J. Mod. Phys. A 3 (1988) 2713.

[17] S. M. Kuzenko and Z. V. Yarevskaya, Mod. Phys. Lett. A 11 (1996) 1653 [arXiv:hep-th/9512115].

[18] I. Bars, Phys. Rev. D 77 (2008) 125027 [arXiv:0804.1585 [hep-th]].
[19] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. Lett. 39, 1109 (1977); Phys. Lett. B 69 (1977) 304; Phys. Rev. D 17 (1978) 3179.

[20] R. Bonezzi, O. Corradini and A. Waldron, “Local Unit Invariance, Back-Reacting Tractors and the Cosmological Constant Problem,” arXiv:1003.3855 [hep-th].

[21] A. Čap and A. R. Gover, Ann. Glob. Anal. Geom. 24, 231 (2003).

[22] A. R. Gover and A. Waldron, ATMP to appear, arXiv:0903.1394 [hep-th].

[23] C. Fefferman and C.R. Graham, Conformal invariants, Elie Cartan et les Mathematiques d’Aujourdhui (Asterique, 1985) 95.

[24] B. Zumino, “Effective Lagrangians and Broken Symmetries”, Lectures on Elementary Particles and Quantum Field Theory, Brandeis University Summer Institute, 2, 437 (1970).

[25] S. Deser, Ann. Phys. 59, 248 (1970).

[26] R. Howe, Trans. Am. Math. Soc. 313 (1989) 539 [Erratum-ibid. 318 (1990) 823].

[27] I. Bars, Class. Quant. Grav. 18 (2001) 3113 [arXiv:hep-th/0008164].

[28] I. Bars, Phys. Rev. D 62 (2000) 046007 [arXiv:hep-th/0003100].

[29] I. Bars, Phys. Rev. D 64 (2001) 126001 [arXiv:hep-th/0106013].

[30] I. Bars and S. J. Rey, Phys. Rev. D 64 (2001) 046005 [arXiv:hep-th/0104135].

[31] C. K. Zachos, D. B. Fairlie and T. L. Curtright, “Quantum mechanics in phase space”, World scientific series in 20th century physics, Vol 34.

[32] I. Bars and C. Deliduman, Phys. Rev. D 64 (2001) 045004 [arXiv:hep-th/0103042].

[33] M.A. Vasiliev, Class. Quant. Grav. 11 (1994) 649.