A SUPERSTRING THEORY
IN FOUR CURVED SPACE-TIME DIMENSIONS

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ABSTRACT

Neveu-Schwarz-Ramond type heterotic and type-II superstrings in four dimensional curved space-time are constructed as exact $N = 1$ superconformal theories. The tachyon is eliminated with a GSO projection. The theory is based on the $N=1$ superconformal gauged WZW model for the anti-de Sitter coset $SO(3, 2)/SO(3, 1)$ with integer central extension $k = 5$. The model has dynamical duality properties in its space-time metric that are similar to the large-small ($R \rightarrow 1/R$) duality of tori. To first order in a $1/k$ expansion we give expressions for the metric, the dilaton, the Ricci tensor and their dual generalizations. The curvature scalar has several singularities at various locations in the 4-dimensional manifold. This provides a new singular solution to Einstein’s equations in the presence of matter in four dimensions. A non-trivial path integral measure which we conjectured in previous work for gauged WZW models is verified.

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Sometime ago exact conformal theories based on the anti-de Sitter (ADS) cosets $SO(d - 1, 2)_- - k / SO(d - 1, 1)_- - k$ were introduced as models for strings propagating in curved space-time in $d = 2, 3, \cdots, 26$ dimensions [1]. The $N = 1$ superconformal models that generalize these were also expressed as Kazama-Suzuki cosets of the form $SO(d - 1, 2)_- - k \times SO(d - 1, 1)_1 / SO(d - 1, 1)_- - k + 1$ for $d = 2, 3, \cdots, 15$. The $d = 2$ bosonic model was interpreted as a string propagating on the manifold of a two dimensional black hole [2]. Similarly, in three dimensions the bosonic model describes the ADS string propagating in a singular and more complicated manifold which is a new solution to Einstein’s equations with matter [3] [4] [5]. These manifolds have the interesting property of duality which signals a shortest distance in string theory. The string geodesics, which correspond to the classical solutions of the gauged WZW model, have been obtained in [4] for any dimension, including $d = 4$.

In this paper we will investigate the most interesting case. Namely, the supersymmetric ADS string in four dimensions and $k = 5$ which in a certain sense is unique. We construct heterotic and type-II string theories based on this coset. The action for our model, in the conformal gauge, has four parts $S = S_0 + S_1 + S_2 + S_3$ with

$$S_0(g) = \frac{k}{8 \pi} \int_M d^2 \sigma \, Tr(g^{-1} \partial_+ g \, g^{-1} \partial_- g) - \frac{k}{24 \pi} \int_B Tr(g^{-1} dg \, g^{-1} dg \, g^{-1} dg)$$

$$S_1(g, A) = -\frac{k}{4 \pi} \int_M d^2 \sigma \, Tr(A_- \partial_+ g g^{-1} - \tilde{A}_+ g^{-1} \partial_- g + A_- g \tilde{A}_+ g^{-1} - A_- A_+)$$

$$S_2(\psi_+, A_-) = -\frac{k}{4 \pi} \int_M d^2 \sigma \, \psi_+^\mu \, (i D_- \psi_+)^\nu \eta_{\mu \nu}, \quad S_3(\chi_-) = \frac{k}{4 \pi} \int_M d^2 \sigma \, \sum_{a=1}^{22} \chi_-^a \, i \partial_+ \chi_-^a$$

(1)

In addition, there are ghost actions $S_4(b_L, c_L, \beta_L, \gamma_L)$ for left movers and $S_5(b_R, c_R)$ for right movers that are added due to the superconformal or conformal gauge fixing respectively. This action has $(1, 0)$ superconformal symmetry (see below) and is appropriate for the heterotic string. The type-II string requires $(1, 1)$ superconformal symmetry. Its action follows if $\chi_-^a$ is removed and replaced by $\psi_-^\mu$ that appears with a gauge covariant kinetic term just like $\psi_+^\mu$. Then $S_3$, $S_5$ are replaced by $S_3(\psi_-, A_+)$ and $S_5(b_R, c_R, \beta_R, \gamma_R)$.

In the above, $S_0$ is the global WZW model [6] with $g(\sigma^+, \sigma^-) \in SO(3, 2)$. By itself this piece has $SO(3, 2)_L \times SO(3, 2)_R$ symmetry. Since $SO(3, 2)$ has a non-Abelian compact subgroup $SO(3)$ the quantum path integral could be defined uniquely only for $k = integer$
Indeed, we take $k = 5$ which is the value required by the total Virasoro central charge for the supersymmetric left movers

$$c_L = \frac{3kd}{2(k - d + 1)} = 15 \quad \text{for} \quad d = 4, \quad k = 5.$$  \hfill (2)

$c_L$ is cancelled by the super ghost system of $S_4$. It is seen that $d = 4$ is unique in the sense that, for $d < 10$, it is the only case with integer $k$ solution. The other acceptable integer $k$ solution occurs for $d = 10$ ($k = \infty$) which is the original Neveu-Schwarz-Ramond flat string. For type-II the central charge of the supersymmetric right-movers is also $c_R = 15$. However, for the heterotic string the bosonic part $SO(3, 2)_{-k}/SO(3, 1)_{-k}$ gives

$$c_R(\text{bose}) = \frac{10k}{k - 3} - \frac{6k}{k - 2} = 15$$  \hfill (3)

for $k = 5$ (already fixed in the action). Since the ghosts in $S_5(b_R, c_R)$ contribute $-26$ we require a $c_R(\chi) = 11$ contribution from the free fermions $\chi^\alpha$. Therefore the action $S_3$ contains 22 free fermions. This action could be viewed as giving rise to $SO(22)_1$ current algebra theory for right movers. There are many other ways of obtaining $c_R = 11$ as exact conformal theories based on current algebras. Perhaps the most interesting one is $E_7 \times SU(3) \times SU(2) \times U(1)$ (all at level $k = 1$) since it contains just the gauge group of the Standard Model and a “hidden” $E_7$.

The second piece in the action $S_1$ gauges the Lorentz subgroup $H = SO(3, 1)$ which is embedded in $SO(3, 2)_L \times SO(3, 2)_R$ with a deformation. As explained in the action of the gauge group could be deformed on the left or the right of the group element $g$. If the matrix representation of the gauged Lorentz algebra on the left is $t_a$ and the one on the

1. The easiest way to see this point is to write $g$ in parametric form $g = abc$ with $a \in SO(3), b \in SO(2)$ and $c \in SO(3, 2)/SO(3) \times SO(2)$ and apply the Polyakov-Wiegman formula. Then $S_0(g)$ decomposes into several pieces one of which is $S_0(a)$ that can be defined only for integer $k$ since $SO(3)$ is compact. The remaining pieces do not present a problem.

2. There are also negative integer $k$ solutions for $d = 11, 12, 13, 15, 19, 20, 25, 28, 40, 55, 100, \infty$. To insure a single time coordinate one must take now the coset $SO(4, 1)/SO(3, 1)$ since $k$ is negative. However, one expects problem with ghosts etc. for superstring theories with $d > 10$. Of course, $k$ can also be taken integer for other $d < 10$ only if one admits direct products of conformal theories.

3. Some examples are $[(E_8)_1 \times SU(4)_1], [(E_7)_1 \times SU(5)_1], [(E_7)_1 \times SU(3)_1 \times SU(2)_1 \times U(1)], [(E_6)_1 \times SO(10)_1], [(E_6)_1 \times SU(4)_2], [SO(10)_2 \times SU(3)_1]$, etc.
right is \( \tilde{t}_a \) then gauge invariance is satisfied by \( \tilde{t}_a = g_0^{-1}t_ao \) or \( \tilde{t}_a = g_0^{-1}(-t_a)^Tg_0 \), where \( g_0 \) is any constant group element in complexified \( SO(3, 2) \) (including \( g_0 \)'s not continuously connected to the identity) and \( t^T \) is the transpose of the matrix. In this notation the action \( S_1 \) is expressed in terms of \( A_\pm = A_\pm t_a \) and \( \tilde{A}_\pm = A_\pm \tilde{t}_a \) with the same \( SO(3, 1) \) gauge potential \( A_\pm (\sigma^+, \sigma^-) \). The simplest case of \( \tilde{t}_a = t_a \) corresponds to the standard vector subgroup. The remaining cases generalize the vector/axial gauging options that were first noticed for the 2d black hole \([10] [11] [12] [13]\) and thus provide a generalization of the concept of duality. Examples are given in \([4]\).

The action \( S_2 \) contains the fermions \( \psi^\mu_+ \) with \( \mu = 0, 1, 2, 3 \) that belongs to the coset \( SO(3, 2)/SO(3, 1) \). The flat Minkowski metric \( \eta_{\mu\nu} = diag (1, -1, -1, -1) \) is used to contract the Lorentz indices. As shown in \([14]\) coset fermions lead to \( N = 1 \) superconformal symmetry. Indeed the super coset scheme \( SO(3, 2)_{-5} \times SO(3, 1)_1/\bar{SO}(3, 1)_{-4} \) for left movers requires that they appear with gauge covariant derivatives \( D_- \psi^\mu_+ = \partial_- \psi^\mu_+ - (A_-)^\mu_\nu \psi^\nu_+ \). The explicit supersymmetry transformations are written more conveniently in terms of the \( 5 \times 5 \) matrix \( \psi_+ = \begin{pmatrix} 0 & -\psi^\nu_+ \\ \psi^\mu_+ & 0 \end{pmatrix} \) that belongs to the \( G/H \) part of the Lie algebra \([15]\).

\[
\delta g = i\epsilon_- \psi_+ g, \quad \delta \psi_+ = \epsilon_- (gD_+ g^{-1})_{G/H}, \quad \delta \chi_a^- = 0, \quad \delta A_\pm = 0, \quad (4)
\]

with \( \partial_- \epsilon_-(\sigma^+) = 0 \). In a type-II theory \( \psi^\mu_- \) also mixes under supersymmetry with the group element \( g^{-1} \) with a transformation similar to the one above. The independent right-moving supersymmetry parameter in this case is \( \epsilon_+(\sigma^-) \).

This theory is supplemented with the original GSO projection \([16]\) adapted to four dimensions. Namely, we construct the operator \((-1)^F\) with the same prescriptions as \([16]\) and project onto the states \((-1)^F = 1\). Let us describe the effect on the ground states in the Neveu-Schwarz and Ramond sectors for left movers. In our coset scheme these are conveniently labelled by the scalar, vector and the two spinor representations of the fermionic \( SO(3, 1)_1 \). The GSO projection eliminates the scalar and one of the spinor representations so that the tachyon is eliminated from the theory. The remaining vector and Weyl spinor form the representations \((\tfrac{1}{2}, \tfrac{1}{2})\) and \((\tfrac{1}{2}, 0)\) of the Lorentz group in four dimensions. As is well known this is a covariant space-time supersymmetric vector multiplet and therefore signals the possibility of space-time supersymmetry in our heterotic model. The GSO projections for the type-II theory can be chosen such that the remaining ground state Weyl spinors for the left movers and right movers have either the opposite or
the same chirality. Accordingly the theory will be called type-IIA or type-IIB respectively. To see whether these theories are supersymmetric in curved space-time the target space supercharges have to be constructed explicitly by a curved space-time modification of the analysis of [17]. We will report on our efforts in this direction elsewhere.

In the remainder of this paper we will determine the sigma model geometry of the 4d super ADS string by finding the metric $G_{\alpha \beta}$, antisymmetric tensor $B_{\alpha \beta}$, dilaton $\Phi$ and curvature tensors in lowest order in an expansion in $1/k$. For either the heterotic or type-II cases the superconformal symmetry dictates the general form of the sigma model order by order in $1/k$. The leading forms are given in e.g. [18] [19]. The $1/k$ correction must be consistent with the vanishing of the beta functions. Since the computations of the beta functions to order $1/k$ are insensitive to the presence of fermions [19] one can simplify the calculation by concentrating on the purely bosonic theory $S_0 + S_1$.

To study the four dimensional ADS bosonic string it is convenient to write the 10 parameter $SO(3,2)$ group element in parametric form as the product $g = ht$ where $h$ is in the left subgroup $h \in SO(3,1)_{L}$ and $t$ is in the coset $t \in SO(3,2)/SO(3,1)_{L}$. Furthermore $h$ and $t$ are given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & h_{\mu}^{\nu} \end{pmatrix}, \quad t = \begin{pmatrix} b & -bX^{\nu} \\ bX_{\mu} & (\eta_{\mu}^{\nu} - abX_{\mu}X^{\nu}) \end{pmatrix},$$

where the 6-parameter $SO(3,1)_{L}$ Lorentz group element $(\eta h^T \eta = h^{-1})$ can be written in the form $h_{\mu}^{\nu} = [(1 + \alpha)(1 - \alpha)^{-1}]_{\mu}^{\nu}$, with $\alpha_{\mu \nu} = -\alpha_{\nu \mu}$ when both indices are lowered. Furthermore, to insure that $t$ is a $SO(3,2)$ group element one can take $b = \epsilon(1+X^2)^{-\frac{1}{2}}$, $a = (1-b^{-1})/X^2$ with either $\epsilon = 1$ or $\epsilon = -1$ (when $\epsilon = -1$ the group element is far from the identity; this will play a role in duality). Let us first consider the standard undeformed theory ($t_{a} = \tilde{t}_{a}$) in which the Lorentz gauge transformation acts as $g' = \Lambda g \Lambda^{-1}$, which means $X'_{\mu} = \Lambda_{\mu}^{\nu}X_{\nu}$, $(\alpha')_{\mu}^{\nu} = (\Lambda \alpha \Lambda^{-1})_{\mu}^{\nu}$. Using the 6 parameter gauge freedom in $\Lambda$ one can “rotate” the group parameters to the form

$$\alpha_{02} = \alpha_{03} = \alpha_{12} = \alpha_{13} = 0$$

$$X_{0} = X_{2} = 0.$$  \hspace{1cm} (6)

This gauge fixing will introduce a Faddeev-Popov determinant that modifies the measure in the path integral. It will be evaluated below. The gauge fixed form for $X^{\mu}$ is appropriate for a space-like $X \cdot X < 0$. Since $X^2$ is a Lorentz invariant the cases of $X^2 > 0$, $X^2 < 0$, $X^2 = 0$ have to be gauge fixed separately. Fortunately it is possible to pass from one sign to
the other by analytic continuation (and renaming the basis $\mu = 0, 1, 2, 3$). Therefore one can concentrate on space-like $X^2 < 0$. (Similar remarks apply to the invariants $\alpha^{\mu\nu} \alpha_{\mu\nu}$, $\epsilon^{\mu\nu\lambda\sigma} \alpha_{\mu\nu} \alpha_{\lambda\sigma}$.) The remaining parameters can be written more conveniently by making a change of variables of the form

$$X_1 = -\tanh(2r) \cos \theta, \quad X_3 = -\tanh(2r) \sin \theta$$

$$\alpha_{01} = -\frac{\sinh(2t)}{\cosh(2t) + \epsilon'}, \quad \alpha_{23} = -\tan(\phi),$$

(7)

where $\epsilon' = \pm 1$ also plays a role in duality (when $\epsilon' = -1$ the subgroup element $h$ is far from the identity). Thus, $\alpha_{01} = \tanh(t)$ or $\coth(t)$. No such signs are necessary in the parametric form of $\alpha_{23}$ since the range of the compact angle $0 \leq \phi \leq 2\pi$ already accounts for $\cot(\phi) = \tan(\pi - \phi)$. Because of the change of variables in (7) the path integral measure is changed by a Jacobian. It will be taken into account below.

The gauge fields $(A_{\pm})_{\mu\nu}$ can be integrated out. The result is another determinant that modifies the measure. From prior experience in $d = 2, 3$ we have learned that the logarithm of this determinant is the dilaton field $\Phi = \log(\det M) + \text{const.}$ as will be seen below. Therefore, it is desirable to obtain this expression, which is gauge invariant prior to the gauge fixing described above. The quadratic piece in $S_1$ can be manipulated using the antisymmetry of $(A_{\pm})_{\mu\nu}$

$$Tr(A_- m A_+ m^T - A_- A_+) = Tr(A_- (m + 1) A_+ (m^T - 1)) = (A_-)_{\mu\nu} M^{\mu\nu,\lambda\sigma} (A_+)_{\lambda\sigma},$$

(8)

The $4 \times 4$ matrix $m$ is given by

$$m = h(1 + abXX^T).$$

(9)

where $X_\mu$ forms a column matrix and $X^T$ is a row. Integrating out the $A_{\pm}$ gives $(\det(M))^{-1}$. This determinant may be written as products of the eigenvalues of the matrix $m$. To see this, go to a basis with diagonal $m = \text{diag} \ (m_0, m_1, m_2, m_3)$ and note that (8) takes the form $\sum_{\mu<\nu}(A_-)_{\mu\nu} (A_+)_{\mu\nu} (m_\mu m_\nu - 1)$. Thus,

$$\det(M) = \prod_{\mu<\nu=0}^3 (m_\mu m_\nu - 1)$$

$$= 1 + b^3 - \frac{1}{2} (1 + b)^2 ((Tr m)^2 - Tr(m^2))$$

$$+ b(1 + b) (Tr m Tr(m^{-1}) - 1) - b Tr(m^2) - b^2 Tr(m^2)^{-1}.$$
where the expression has been rewritten in terms of traces and determinants. This permits
restoring \( m \) to its general form in (9). Furthermore it is convenient to notice \( \det(h)\det(1 + abXX^T) = b \). Using (9) and calculating the traces one finds

\[
\det(M) = (1 - b^2)\{(1 - b)\left[1 - 2\left(\frac{X^T h X}{X^2}\right)^2\right] + \frac{1}{2} (Trh^2 - (Trh)^2) \}
\]

This is the gauge invariant form. The gauge fixed version is

\[
Ce^\Phi = det(M) = -2\epsilon \sinh^2(2r) [\cosh(2r) - \epsilon] \sin^2\theta \cos^2\theta \left[\cosh(2t) - \epsilon' \cos(2\phi)\right]^2.
\]

(12)

As indicated this leads to the identification of the dilaton field \( \Phi \) that solves the conformal
invariance conditions as will be seen below.

To obtain the effective action after the integration over \( A_{\pm} \) one solves the classical
equations for \( A_{\pm} \) and substitutes the solution in the action (1). Furthermore, one expo-
nentiates all the factors in the measure to produce the dilat on. After tedious algebra one
obtains the gauge fixed form of the bosonic effective action

\[
S_{eff} = \frac{k}{2\pi} \int d^2\sigma \sqrt{h} \left|h^{ij} G_{\alpha\beta} \partial_i \phi^\alpha \partial_j \phi^\beta - \frac{1}{4k} R^{(2)}(\Phi)\right], \quad \phi^\alpha = (t, r, \theta, \phi),
\]

(13)

where the target space metric \( G_{\alpha\beta} \) is given by the line element \( ds^2 = G_{\alpha\beta} d\phi^\alpha d\phi^\beta \) and the
antisymmetric tensor \( B_{\alpha\beta} \) is zero with our parametrization. The metric that emerges is

\[
\rho^2(r, \epsilon) = \frac{\cosh(2r) + \epsilon}{\cosh(2r) - \epsilon} = \tanh^2 r \text{ or } \coth^2 r, \quad \epsilon, \epsilon' = \pm 1 \text{ dual patches}
\]

(14)

The effects of \( \epsilon, \epsilon' \) can be reproduced by a dual theory defined by a deformation of the
gauge group with a constant \( g_0 \) as argued in [4]. The metric can be put in a diagonal form
by changing variables

\[
\begin{align*}
&z = \epsilon' \cos^2 \theta \cos(2\phi) + \sin^2 \theta \cosh(2t), \quad T = \cosh(2t), \quad w = \cos(2\phi) \\
&ds^2 = dr^2 + \frac{dz^2}{4\rho^2 (z - \epsilon' w)(T - z)} + \frac{\rho^2}{4} (T - \epsilon' w) \left[\frac{dw^2}{(z - \epsilon' w)(1 - w^2)} - \frac{dT^2}{(T - z)(T^2 - 1)}\right]
\end{align*}
\]

(15)
The ranges of the new parameters are

\[ z - \epsilon' w \geq 0, \quad T - z \geq 0, \quad T \geq 1, \quad |w| \leq 1. \]  

(16)

Next we list the components of the Ricci tensor in the new basis \( \phi^\alpha = (T, r, z, w) \). They will be needed to verify conformal invariance at one loop.

\[
R_{rr} = -4 \frac{\epsilon \cosh(2r) + 3}{\sinh^2(2r)}
\]
\[
R_{rz} = \frac{2 \epsilon}{\sinh(2r)} \left[ \frac{1}{T - z} - \frac{1}{z - \epsilon' w} \right]
\]
\[
R_{rT} = \frac{2 \epsilon}{\sinh(2r)} \frac{1}{T - z}
\]
\[
R_{rw} = \frac{2 \epsilon}{\sinh(2r)} \frac{-\epsilon'}{z - \epsilon' w}
\]
\[
R_{zz} = \frac{\epsilon}{\sinh^2(2r)} \cosh(2r) + 3 \sinh^2(2r)
\]
\[
R_{zT} = \frac{1}{2} \frac{1}{T - z} \left[ \frac{1}{z - \epsilon' w} - \frac{2}{T - z} \right]
\]
\[
R_{zw} = \frac{-\epsilon'}{2} \frac{1}{z - \epsilon' w} \left[ \frac{2}{z - \epsilon' w} - \frac{1}{T - z} \right]
\]
\[
R_{TT} = \frac{\epsilon \rho^3}{\sinh(2r)} \frac{T - \epsilon' w}{(T - z)(T^2 - 1)} + \frac{\rho^4}{2} \frac{(T - \epsilon' w)(z - \epsilon' w)}{(T - z)^2(T^2 - 1)} - \frac{1}{2} \frac{1}{(T - z)^2}
\]
\[
R_{Tw} = \frac{\epsilon'}{2} \frac{1}{(T - z)(z - \epsilon' w)}
\]
\[
R_{ww} = \frac{-\epsilon \rho^3}{\sinh(2r)} \frac{T - \epsilon' w}{(z - \epsilon' w)(1 - w^2)} - \frac{\rho^4}{2} \frac{(T - z)(T - \epsilon' w)}{(z - \epsilon' w)^2(1 - w^2)} - \frac{1}{2} \frac{1}{(z - \epsilon' w)^2}.
\]  

(17)

The scalar curvature deduced from the above is

\[
R = -\frac{16 \epsilon}{\cosh(2r) - \epsilon} - 4\rho^2 \left[ \frac{T - z}{z - \epsilon' w} + \frac{z - \epsilon' w}{T - z} \right] + \frac{4}{\rho^2} \frac{z^2 - 1}{(T - z)(z - \epsilon' w)}. \]  

(18)

It has singularities at

\[
r = 0, \quad T - z = 0 \ (\theta = \pm \frac{\pi}{2}), \quad z - \epsilon' w = 0 \ [(\theta = 0), \quad (t = 0 \text{ for } \epsilon' \cos 2\phi = 1)].
\]  

(19)

Next we turn to the question of conformal invariance. At one loop (which corresponds to the leading order in a \( 1/k \) expansion) the following equations must be satisfied
\[ R_{\alpha \beta} = D_{\alpha} D_{\beta} \Phi \]
\[ c_L = 6 + \frac{3}{4k} (D_{\alpha} \Phi D^{\alpha} \Phi + D^2 \Phi), \]

where \( 6 = 3d/2 \) includes the contributions of the left moving supersymmetric 4 bosons and 4 fermions. (Similarly, for the right movers one can write \( c_R \) by replacing the 6 by \( 4 + 11 \).)

These equations are in fact satisfied by the expressions given above for the metric, the dilaton and the Ricci tensor. In particular the dilaton contribution to the central charge reduces to just a constant \( c_L = 6 + 18/k \). This result coincides with the large \( k \) expansion of (2) \( c_L = 3d/2 + 3d(d - 1)/2(k - d + 1) \rightarrow 6 + 18/k \). Similarly it also coincides with the large \( k \) expansion of \( c_R \) given by (3) plus the required 11. This is another non-trivial check of our dilaton. The dilaton contribution to the central charge \( 18/k \) is indeed independent from the fermions that come along with the bosons either as left movers or right movers.

We believe that our metric is a new singular solution of Einstein’s equations in four dimensions in the presence of matter. The manifold comes in many patches. First, there are the patches that follow from analytic continuation of our variables (in order to cover the possibilities of space-like and time-like \( X^2 \) etc.). These analytic continuations have to be done consistently with maintaining an \( SO(3, 2) \) group element up to a renaming of the indices (see e.g. [4]). In any one of these analytic continuations there can be only one time coordinate. Furthermore, there are the patches that follow from duality transformations as already indicated in the expression of the metric in the form \( \epsilon, \epsilon' = \pm 1 \). One has to keep in mind a similar situation in the 2d black hole and 3d ADS string in order to appreciate these comments by analogy. It is desirable to find somewhat more global (or fully global) coordinates that give a better description of this manifold and consequently lead to a satisfactory interpretation of what it represents (as it was possible to do for the 2d black hole). We are currently working on such a project.

We now consider the path integral measure. Let us ignore the fermions. To begin, one has a measure \( (Dg)(DA)F(g) \) where \( (Dg) \) is the Haar measure for the group \( G = SO(3, 2) \) and \( (DA) \) is the measure for the gauge field which one may define as the naive measure \( DA_{+}DA_{-}. \) The additional factor \( F(g) \) which is gauge invariant under the gauge subgroup \( H = SO(3, 1) \) is a priori allowed since the measure is required to be invariant only under \( H \) and not under the global symmetry \( G_L \times G_R. \) In [4] we found that conformal invariance required a nontrivial factor that took the form \( F = \exp(-\Phi)/\sqrt{-G} \) in the bosonic \( d = 2, 3 \)

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4 The physical metric in the low energy theory is \( G_{\mu \nu}^{phys} = e^\Phi G_{\mu \nu}. \) The curvature scalar for this metric has singularities of higher degrees at the same locations as before.
Theories. We conjectured that this form might be true for all gauged WZW theories. We now verify that indeed $F$ takes this form in the 4 dimensional theory.

The Haar measure for $g = ht \in SO(3,2)$ is $(Dg) = (Dh)(D^4X)/(1 + X^2)^{5/2}$ where $(D^4X)$ is the naive Minkowski (or Euclidean) measure, and $(Dh)$ is the measure for the subgroup $SO(3,1)$ which will be computed in terms of the parameters $\alpha_{\mu\nu}$ introduced above. Consider the line element

$$Tr(h^{-1}dh)^2 = Tr(\frac{1}{1-\alpha^2}d\alpha \frac{1}{1-\alpha^2}d\alpha)$$

which defines a group metric for the parameters $\alpha_{\mu\nu}$. The Haar measure is the square root of the determinant of this metric. The problem of computing the determinant is similar to computing $det(M)$ above. Using similar methods one obtains the Haar measure

$$(Dg) = (D^6\alpha)(D^4X)\frac{f(\alpha, X)}{f(\alpha, X)}, \quad f(\alpha, X) = det(1 - \alpha^2)^{3/2} (1 + X^2)^{5/2}. \quad (22)$$

Next consider the Faddeev-Popov determinant for the 6 gauge conditions (6). Let these 6 functions be denoted by $f_i(g)$. By applying a 6 parameter infinitesimal gauge transformation $\Lambda_{\mu\nu} = \eta_{\mu\nu} + \omega_{\mu\nu}$ one obtains $\delta f_i = \Delta_i^{\mu\nu} \omega_{\mu\nu}$. The Faddeev-Popov determinant is the determinant of the $6 \times 6$ matrix $\Delta_i^{\mu\nu}$. Using the vector and tensor transformation properties $\delta \alpha_{\mu\nu} = \omega_{\mu}^{\nu} \alpha_{\lambda\nu} + \omega_{\nu}^{\lambda} \alpha_{\mu\lambda}$, $\delta X_\mu = \omega_{\mu}^{\lambda} X_\lambda$ one can easily figure out the transformation of the gauge functions $f_i$ and then evaluate $\Delta_i^{\mu\nu}$ on the gauge slice $f_i = 0$. The result is

$$\begin{pmatrix}
\delta\alpha_{02} \\
\delta\alpha_{03} \\
\delta\alpha_{12} \\
\delta\alpha_{13} \\
\delta X_0 \\
\delta X_2
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & \alpha_{23} & \alpha_{01} & 0 & 0 \\
0 & -\alpha_{23} & 0 & 0 & \alpha_{01} & 0 \\
0 & \alpha_{01} & 0 & 0 & \alpha_{23} & 0 \\
0 & 0 & \alpha_{01} & -\alpha_{23} & 0 & 0 \\
-X_1 & 0 & -X_3 & 0 & 0 & 0 \\
0 & 0 & 0 & X_1 & 0 & X_3
\end{pmatrix}
\begin{pmatrix}
\omega_{01} \\
\omega_{02} \\
\omega_{03} \\
\omega_{12} \\
\omega_{13} \\
\omega_{23}
\end{pmatrix} \quad (23)$$

Then the Faddeev-Popov determinant is $\Delta = -X_1X_3(\alpha_{23}^2 + \alpha_{01}^2)^2$.

By changing variables as in (7) one picks up a Jacobian $(DX_1)(DX_3)(D\alpha_{01})(D\alpha_{23}) = (Dt)(Dr)(D\theta)(D\phi) J(t, r, \theta, \phi)$. In terms of these parameters $\phi^\alpha = (t, r, \theta, \phi)$ we have

$$\Delta = -\tanh^2(2r) \sin\theta \cos\theta \left(\cosh(2t) - \epsilon' \cos(2\phi)\right)^2 \frac{\cosh(2t) - \epsilon' \cos(2\phi)}{\cosh^4(\phi) (\cosh(2t) + \epsilon')^2}$$

$$J = \frac{4\epsilon' \sinh(2r)}{\cosh^3(2r) \cos^2(\phi) (\cosh(2t) + \epsilon')} \quad (24)$$

$$f(\alpha, X) = \frac{8}{\cosh^5(2r) \cos^6(\phi) (\cosh(2t) + \epsilon')^3}.$$
The integration measure becomes \((D^4\phi) F J \Delta/(f \det M)\). It is interesting to observe that the known factors combine to give \((D^4\phi) \frac{J \Delta}{f \det M} = \sqrt{-G} F\)

\[
\frac{J \Delta}{f \det M} = \sqrt{-G} = \frac{\rho(r, \epsilon)}{|\cos \theta \sin \theta|},
\]

(25)

A similar relation holds for the two dimensional black hole and, in the less trivial case, for the three dimensional ADS string [4]. We are by now convinced that one can prove a theorem which states that the various measure factors (except \(F\)) combine to give \(\sqrt{-G}\) in general for any gauged WZW model. We now require that the contribution from all the measure factors to the effective action be just the correct dilaton which we have already identified. This fixes \(F\)

\[
F = \frac{f}{J \Delta} = \frac{e^{-\Phi}}{\sqrt{-G}}.
\]

(26)

As pointed out earlier this factor must be consistent with gauge invariance under \(SO(3, 1)\) transformations. This is indeed the case since \(e^{-\Phi}\) is written in a manifestly invariant form in (11) and one can also rewrite

\[
G^{-1} = \frac{[1 - \epsilon \sqrt{1 + X^2}]^2}{4X^2} [1 - \frac{(4X^T hX - Trh)^2}{2Trh^2 - (Trh)^2 + 8}].
\]

(27)

Thus, using the invariant forms (11) (27) one has the correct measure in any gauge. It is interesting to speculate that (26) which yields two expressions for \(F\) may be valid to all orders in \(1/k\) when the exact dilaton and metric forms are inserted. It is natural to expect this since the first expression is purely group theoretical and is independent of \(k\), while the second expression depends on the higher order corrections to the dilaton and metric. This idea can be tested for the 2d black hole for which the exact dilaton and metric have been conjectured [12]. Using the exact expressions in [12] we find that indeed the measure factor \(F\) is independent of \(k\) to all orders and satisfies the relation (26).

We have formulated heterotic and type-II superstrings in curved space-time in four dimensions, as exact superconformal field theories. The tachyon instability has been eliminated through a GSO projection. These models are special since they can be analysed using current algebra techniques. In the next stage the quantum theory should be investigated in more detail than [1], including a discussion of modular invariance which is guided by the Lagrangian formulation. It is expected that the deformation \(g_0\) that appears in the action [1] plays a role in combining left and right movers in modular invariant physical states. This process is expected to lead to a determination of the number of families and related quantities which may be of phenomenological significance. Our curved space-time theory may be applicable to the physics of the early Universe.
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