Special geometry and symplectic transformations†

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ABSTRACT
Special Kähler manifolds are defined by coupling of vector multiplets to $N = 2$ supergravity. The coupling in rigid supersymmetry exhibits similar features. These models contain $n$ vectors in rigid supersymmetry and $n + 1$ in supergravity, and $n$ complex scalars. Apart from exceptional cases they are defined by a holomorphic function of the scalars. For supergravity this function is homogeneous of second degree in an $(n + 1)$-dimensional projective space. Another formulation exists which does not start from this function, but from a symplectic $(2n)$- or $(2n + 2)$-dimensional complex space. Symplectic transformations lead either to isometries on the manifold or to symplectic reparametrizations. Finally we touch on the connection with special quaternionic and very special real manifolds, and the classification of homogeneous special manifolds.

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1. Introduction

In nonlinear sigma models, the spinless fields define a map from the $d$-dimensional Minkowskian space-time to some ‘target space’, whose metric is given by the kinetic terms of these scalars. Supersymmetry severely restricts the possible target-space geometries. The type of target space which one can obtain depends on $d$ and on $N$, the latter indicating the number of independent supersymmetry transformations. The number of supersymmetry generators ('supercharges') is thus equal to $N$ times the dimension of the (smallest) spinor representation. For realistic supergravity this number of supercharge components cannot exceed 32. As 32 is the number of components of a Lorentz spinor in $d = 11$ space-time dimensions, it follows that realistic supergravity theories can only exist for $d \leq 11$. For the physical $d = 4$ dimensional space-time, one can have supergravity theories with $1 \leq N \leq 8$.

As clearly exhibited in table \[1\], the more supercharge components one has, the more restrictions one finds. When the number of supercharge components exceeds 8, the target spaces are restricted to symmetric spaces. For $\kappa = 16$ components, they are specified by an integer $n$, which specifies the number of vector multiplets. This row continues to $N = 1$, $d = 10$. Beyond 16 supercharge components there is no freedom left. The row with 32 supercharge components continues to $N = 1$, $d = 11$. Here we treat the case of 8 supercharge components. This is the highest value of $N$ where the target space is not yet restricted to be a symmetric space, although supersymmetry has already fixed a lot of its structure. We will mostly be concerned with $N = 2$ in $d = 4$ dimensions. The target space factorizes into a quaternionic and a Kähler manifold of a particular type \[i\], called \textit{special} \[3\]. The former contains the scalars of the hypermultiplets (multiplets without vectors). The latter contain the scalars in vector multiplets. Recently the special Kähler structure received a lot of attention, because it plays an important role in string compactifications. Also quaternionic manifolds appear in this context, and also here it is a restricted class of special quaternionic manifolds that is relevant. In lowest order of the string coupling constant these manifolds are even ‘very special’ Kähler and quaternionic, a notion that we will define below.

In the next section we describe the actions of $N = 2$ vector multiplets. First we consider rigid supersymmetry. We explain the fields in the multiplets, their description in superspace and how this leads to a holomorphic prepotential. Then we exhibit how the structure becomes more complicated in supergravity, where the space of physical scalars is embedded in a projective space. This became apparent by starting from the superconformal tensor calculus.

In section \[4\] we discuss the symplectic transformations, which play an important role in the recent developments of weak–strong coupling dualities. First we repeat the general idea (and elucidate it for $S$ and $T$ dualities), and then show what

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Table 1
Restrictions on target-space manifolds according to the type of supergravity theory. The rows are arranged such that the number $\kappa$ of supercharge components is constant. $\mathcal{M}$ refers to a general Riemannian manifold, $SK$ to ‘special Kähler’, $VSR$ to ‘very special real’ and $Q$ to quaternionic manifolds.

| $\kappa$ | $d = 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ |
|----------|---------|---------|---------|---------|---------|
| 2        | $\mathcal{M}$ | $N = 1$ | $N = 2$ | $N = 1$ | $N = 1$ |
| 4        | Kähler  | Kähler  | Kähler  | $N = 4$ | $N = 2$ | $N = 1$ |
| 8        | $Q$     | $Q$     | $SK \oplus Q$ | $VSR \oplus Q$ | $O \oplus Q$ |
| 16       | ...     | ...     | $SO(6,n)$ | $SU(1,1)$ | ... |
| 32       | ...     | ...     | $E_7$ | $SU(8)$ | $d = 10$ |

is the extra structure in $N = 2$ theories. There are two kind of applications, either as isometries of the manifolds (symmetries of the theory), or as equivalence relations of prepotentials (pseudosymmetries). We illustrate both with explicit examples. These will also exhibit formulations without a prepotential, showing the need for a formulation that does not rely on the existence of a prepotential. This formulation is given at the end of the section. Some further results will be mentioned in section 4.

In all of this we confine ourselves to special geometry from a supersymmetry/supergravity perspective. The connection with the geometry of the moduli of Calabi-Yau spaces is treated in the lectures of Pietro Fré.

2. $N = 2$ actions

Table 2
Physical fields in $N = 2$, $d = 4$ actions

| spin | pure SG | $n$ vector m. | $s$ hyperm. |
|------|---------|---------------|-------------|
| 2    | 1       | 1             |             |
| 3/2  | 2       |               |             |
| 1    | 1       | $n$           |             |
| 1/2  | 2       | $2n$          | $2s$        |
| 0    | 2       | $2n$          | $4s$        |

We briefly introduce special Kähler manifolds in the context of $N = 2$, $d = 4$ supergravity. As exhibited in table 3, the physical multiplets of supersymmetry are vector and hypermultiplets, which can be coupled to supergravity. In this section we will not consider the hypermultiplets. The scalar sector of the $N = 2$ supergravity-Yang-Mills theory in four space-time dimensions defines the ‘special Kähler manifolds’. Without supergravity we have $N = 2$ supersymmetric Yang-Mills theory, which we will treat first. The spinless fields parametrize then a similar type of Kähler manifolds. The vector potentials, which describe the spin-1 particles, are accompanied by complex scalar fields and doublets of spinor fields, all taking values in the Lie algebra associated with the group that can be gauged by the vectors. In the second subsection we will see what the consequences are of mixing the vectors in the vector multiplets with the one in the supergravity multiplet.

2.1. Rigid supersymmetry

The superspace contains the anticommuting coordinates $\theta_\alpha^i$ and $\bar{\theta}_\dot{\alpha}i$ where $i = 1, 2$ and $\alpha, \dot{\alpha}$ are the spinor indices. The simplest superfields are, as in $N = 1$, the chiral superfields. They are defined by a constraint $\bar{D}^\dot{\alpha}i \Phi = 0$, where $\bar{D}^\dot{\alpha}$ is a covariant chiral superspace derivative, and $\Phi$ is a complex superfield. This constraint determines
its structure:

\[ \Phi = X + \theta^A, \lambda^A = \epsilon_{i j} \theta^A \theta^B F^+_{i j} + \epsilon_{A B} \vartheta_{\alpha \beta} Y_{i j} + \ldots, \]

where ... stands for terms cubic or higher in \( \theta \). New component fields can appear up to \( \theta^4 \), leading to \( 8 + 8 \) complex field components. All these fields do not form an irreducible representation of supersymmetry, but can be split into two sets of \( 8 + 8 \) real fields transforming irreducibly. We restrict ourselves to the set containing the fields already exhibited in (1), which leads to the vector multiplet (the others form a ‘linear multiplet’). The reduction is accomplished by the additional constraint

\[ D^A_i D^{\bar{i}}_j \epsilon_{A \bar{B}} = \epsilon_{i k} \epsilon_{j l} D^A(k) D^{\bar{j}}(l) \Phi \epsilon_{A \bar{B}}, \]

which for instance implies that the symmetric tensor \( Y_{i j} \) satisfies a reality constraint: \( Y_{i j} = \epsilon_{i k} \epsilon_{j l} Y^{k l} \), so that it consists of only 3 real scalar fields. But more importantly, we also obtain a constraint on the antisymmetric tensor: \( \partial^\mu (F^+_{\mu \nu} - F^-_{\mu \nu}) = 0 \), which is the Bianchi identity, which implies that \( F \) is the field strength of a vector potential. All the terms ... in (1) are determined in terms of the fields written down. Therefore the independent components of the vector multiplet are: \( X^A, \lambda^A, F^A_{\mu \nu}, Y^{i j} \) (where \( A = 1, \ldots, n \) denotes the possibility to include several multiplets). \( X^A \) and \( \lambda^A \) will describe the physical scalars and spinors, \( F^A \) are the field strengths of the vectors and \( Y^{i j} \) will be auxiliary scalars in the actions which we will construct.

As we have a chiral superfield, an action can be obtained by integrating an arbitrary holomorphic function \( F(\Phi) \) over chiral superspace. The action

\[ \int d^4 x \int d^4 \theta \ F(\Phi) + c.c. \]  

leads to the Lagrangian

\[ \mathcal{L} = g_{A B} \partial_\mu X^A \partial_\mu \bar{X}^B + g_{A B} \bar{\lambda}^i \vartheta_{\alpha \beta} Y_{i j} + \text{Im} (F^A_{\mu \nu} F^{A}_{\mu \nu}) + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{\text{4–fermi}} \]  

where the latter two terms are the couplings of the vector fields to the spinors and the terms quartic in fermions, which we do not write explicitly here. The metric in target space is Kählerian: \[ g_{A B}(X, \bar{X}) = \partial_A \partial_{\bar{B}} K(X, \bar{X}) \]

\[ K(X, \bar{X}) = i (\bar{F}_A(X) X^A - F_A(X) \bar{X}^A) \]

\[ F_A(X) = \partial_A F(X) ; \quad \bar{F}_A(X) = \partial_A \bar{F}(\bar{X}) \]

For \( N = 1 \) the Kähler potential could have been arbitrary. The presence of two independent supersymmetries implies that this Kähler metric, and even the complete action, depends on a holomorphic prepotential \( F(X) \), where \( X \) denotes the complex scalar fields. Two different functions \( F(X) \) may correspond to equivalent equations of motion and to the same geometry. From the equation \( g_{A B} = 2 \text{Im} F_{A B} \), it follows that

\[ F \approx F + a + q_A X^A + c_{A B} X^A X^B \]

where \( a \) and \( q_A \) are complex numbers, and \( c_{A B} \) real. But more relations can be derived from the symplectic transformations that we discuss shortly.

The fact that the metric is Kählerian implies that only curvature components with two holomorphic and two anti–holomorphic indices can be non–zero. In this case, these are determined by the third derivative of \( F \):

\[ R_{A B C D} = - F_{A E C F} F^{E F}_{B D} . \]

2.2. Vector multiplets coupled to supergravity

The general action for vector multiplets coupled to \( N = 2 \) supergravity was first derived using superconformal tensor calculus \[ \text{[1]} \]. In that approach one starts from the \( N = 2 \) superconformal group, which is

\[ SU(2,2|N = 2) \supset SU(2,2) \otimes U(1) \otimes SU(2) . \]

The bosonic subgroup, which we exhibited, contains, apart from the conformal group in \( d = 4 \), also \( U(1) \) and \( SU(2) \) factors. The Kählerian nature of vector multiplet couplings and the quaternionic nature of hypermultiplet couplings is directly related to the presence of these two groups.
The superconformal group is, however, mainly a useful tool for constructing actions which have just super-Poincaré invariance (see the reviews [1]). To make that transition, the dilatations, special conformal transformations and $U(1) \otimes SU(2)$ are broken by an explicit gauge fixing. The same applies to some extra $S$–supersymmetry in the fermionic sector.

To describe theories as exhibited in table 2 the following multiplets are introduced: (other possibilities, leading to equivalent physical theories, also exist, see [11,10]). The Weyl multiplet contains the vierbein, the two gravitinos, and auxiliary fields. We introduce $n+1$ vector multiplets:

$$(X^I, \lambda^I, A^I_\mu) \quad \text{with} \quad I = 0, 1, ..., n. \quad (9)$$

The extra vector multiplet labelled by $I = 0$ contains the scalar fields which are gauge–fixed in order to break dilatations and the $U(1)$, the fermion to break the $S$–supersymmetry, and the vector which corresponds to the physical vector of the supergravity multiplet in table 2. Finally, there are $s+1$ hypermultiplets, one of these contains only auxiliary fields and fields used for the gauge fixing of $SU(2)$. For most of this paper we will not discuss hypermultiplets ($s = 0$).

Under dilatations the scalars $X^I$ transform with weight 1. On the other hand an action similar to (5) can only be constructed if $F(X)$ has Weyl weight 2. This leads to the important conclusion that for the coupling of vector multiplets to supergravity, one again starts from a holomorphic prepotential $F(X)$, this time of $n+1$ complex fields, but now it must be a homogeneous function of degree two [10].

In the resulting action appears $-\frac{1}{2} i(\bar{X}^IF_I - X^I\bar{F}_I)eR$, where $R$ is the space–time curvature. To have the canonical kinetic terms for the gravitons, it is therefore convenient to impose as gauge fixing for dilatations the condition

$$i(\bar{X}^IF_I - \bar{F}_IX) = 1. \quad (10)$$

Therefore, the physical scalar fields parametrize an $n$–dimensional complex hypersurface, defined by the condition (10), while the overall phase of the $X^I$ is irrelevant in view of a local (chiral) invariance. The embedding of this hypersurface can be described in terms of $n$ complex coordinates $z^A$ by letting $X^I$ be proportional to some holomorphic sections $Z^I(z)$ of the projective space $P\mathbb{C}^{n+1}$ [12]. The bosonic part of the resulting action is (without gauging)

$$e^{-1}L=\frac{1}{2}R + g_{\alpha \beta} \partial_{\mu} z^\alpha \partial^\mu \bar{z}^\beta - \text{Im} \left( \mathcal{N}_{IJ}(z, \bar{z}) \bar{F}_{\mu}^I F^{+J}_\mu \right). \quad (11)$$

The $n$–dimensional space parametrized by the $z^\alpha (\alpha = 1, \ldots, n)$ is a Kähler space; the Kähler metric $g_{\alpha \beta} = \partial_{\alpha} \partial_{\beta} K(z, \bar{z})$ follows from the Kähler potential

$$e^{-K(z, \bar{z})} = i \bar{Z}^I(\bar{z}) F_I(Z(z)) - i Z^I(z) \bar{F}_I(\bar{Z}(\bar{z})) \quad (12)$$

$X^I = e^{K/2} Z^I(z), \quad \bar{X}^I = e^{K/2} \bar{Z}^I(\bar{z}).$ The resulting geometry is known as special Kähler geometry [11]. The curvature tensor associated with this Kähler space satisfies the characteristic relation [13]

$$R_{\alpha \beta \gamma}^\delta = \delta^\delta_\beta \delta^\alpha_\gamma + \delta^\delta_\gamma \delta^\alpha_\beta - e^{2K} W_{\beta \gamma} \bar{W}^{\alpha \delta}, \quad (13)$$

where

$$W_{\alpha \beta \gamma} = i F_{IJK}(Z(z)) \frac{\partial Z^I}{\partial z^\alpha} \frac{\partial Z^J}{\partial z^\beta} \frac{\partial Z^K}{\partial z^\gamma}. \quad (14)$$

A convenient choice of inhomogeneous coordinates $z^\alpha$ are the special coordinates, defined by

$$z^A = X^A/X^0, \quad A = 1, \ldots, n, \quad (15)$$

or, equivalently,

$$Z^0(z) = 1, \quad Z^A(z) = z^A. \quad (16)$$

The kinetic terms of the spin-1 gauge fields in the action are proportional to the symmetric tensor

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \frac{\text{Im}(F_{IK}) \text{Im}(F_{JL}) X^K X^L}{\text{Im}(F_{KL}) X^K X^L}. \quad (17)$$

This tensor describes the field-dependent generalization of the inverse coupling constants and so-called $\theta$ parameters.

We give here some examples of functions $F(X)$ and their corresponding target spaces, which will be useful later on:

$$F = -i X^0 X^1 \frac{SU(1,1)}{U(1)}. \quad (18)$$
not all functions \( F \) symmetric tensor

The first three functions give rise to the manifold \( SU(1,1)/U(1) \). However, the first one is not equivalent to the other two as the manifolds have a different value of the curvature \( L \). The latter two are, however, equivalent by means of a symplectic transformation as we will show below. In the fourth example \( \eta \) is a constant non-degenerate real symmetric matrix. In order that the manifold has a non-empty positivity domain, the signature of this matrix should be \((+ - \cdots -)\). So not all functions \( F(X) \) allow a non-empty positivity domain. The last example, defined by a real symmetric tensor \( d_{ABC} \), defines a class of special Kähler manifolds, which we will denote as ‘very special’ Kähler manifolds. This class of manifolds is important in the applications discussed below.

3. Symplectic transformations

The symplectic transformations are a generalization of the electro-magnetic duality transformations. We first recall the general formalism for arbitrary actions with coupled spin-0 and spin-1 fields, and then come to the specific case of \( N = 2 \).

3.1. Pseudo-symmetries in general

We consider general actions of spin-1 fields with field strengths \( F_{\mu\nu} \) (now labelled by \( \Lambda = 1, \ldots, m \)) coupled to scalars. The general form of the kinetic terms of the spin-1 fields is

\[
L_1 = \frac{1}{4} (\text{Im} \ N_{\Lambda\Sigma} F_{\mu\nu} F_{\rho\sigma}^{\Lambda} ) \tag{23}
\]

We define

\[
G^{\mu\nu}_{++} = 2i \frac{\partial L}{\partial F_{\mu\nu}^{++}} = N_{\Lambda\Sigma} F^{+\Sigma\mu\nu} . \tag{24}
\]

The equations for the field strengths can then be written as

\[
\partial^\mu \text{Im} \ F_{\mu\nu}^{++} = 0 \quad \text{Bianchi identities}
\]

\[
\partial_\mu \text{Im} \ G_{\mu\nu}^{++} = 0 \quad \text{Equations of motion}
\]

This set of equations is invariant under \( GL(2m, \mathbb{R}) \) transformations:

\[
\left( \tilde{F}^+, \tilde{G}^+ \right) = S \left( F^+, G^+ \right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left( F^+, G^+ \right). \tag{25}
\]

However, the \( G_{\mu\nu} \) are related to the \( F_{\mu\nu} \) as in [24]. The previous transformation implies

\[
\tilde{G}^+ = (C + D\lambda) F^+ = (C + D\lambda)(A + B\lambda)^{-1} \tilde{F}^+ . \tag{26}
\]

Therefore the new tensor \( \tilde{N} \) is

\[
\tilde{N} = (C + D\lambda)(A + B\lambda)^{-1} . \tag{27}
\]

This tensor should be symmetric, as it is the second derivative of the action with respect to the field strength. This request leads to the equations which determine that \( S \in Sp(2m, \mathbb{R}) \), i.e.

\[
S^T \Omega S = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

or

\[
\begin{pmatrix} A^T C - C^T A = 0 \\ B^T D - D^T B = 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A^T D - C^T B = 1 \end{pmatrix} \tag{28}
\]

Some remarks are in order: First, these transformations act on the field strengths. They generally rotate electric into magnetic fields and vice versa. Such rotations, which are called duality transformations, because in four space-time dimensions electric and magnetic fields are dual to each other in the sense of Poincaré duality, cannot be implemented on the vector potentials, at least not in a local way. Therefore, the use of these symplectic transformations is only legitimate for zero gauge coupling constant. From now on, we deal exclusively with Abelian gauge groups. Second, the Lagrangian is not an invariant if \( C \) and \( B \) are not zero:

\[
\text{Im} \tilde{F}^+ G_{++} = \text{Im} \left( F^+ G_+ \right) + \text{Im} \left( 2F^+(C^T B)G_+ + F^+(C^T A)F^+ + G_+(D^T B)G_+ \right). \tag{29}
\]
If \( C \neq 0, B = 0 \) it is invariant up to a four–
divergence. Thirdly, the transformations can also
act on dyonic solutions of the field equations
and the vector \( \left( \eta^A_m, \eta_{eA} \right) \) of magnetic and electric
charges transforms also as a symplectic vector.
The Schwinger-Zwanziger quantization condition
restricts these charges to a lattice with minimal
surface area proportional to \( \hbar \). Invariance of this
lattice restricts the symplectic transformations to
a discrete subgroup:
\[
S \in Sp(2m, \mathbb{Z}). \tag{30}
\]
Finally, the transformations with \( B \neq 0 \) will be
non–perturbative. This can be seen from the fact
that they do not leave the purely electric charges
invariant, or from the fact that (27) shows that
these transformations invert \( N \) which plays the
role of the gauge coupling constant.

3.2. Pseudo–symmetries and proper symmetries

The transformations described above, change
the matrix \( N \), which are gauge coupling constants
of the spin-1 fields. This can be compared to dif-
fefomorphisms of the scalar manifold \( z \rightarrow \hat{z}(z) \)
which change the metric (which is the coupling
constant matrix for the kinetic energies of the
scalars) and \( N \):
\[
\hat{g}_{\alpha\beta}(\hat{z}(z)) \frac{\partial \hat{z}^\alpha}{\partial z^\gamma} \frac{\partial \hat{z}^\beta}{\partial z^\delta} = g_{\gamma\delta}(z) ; \hat{N}(\hat{z}(z)) = N(z). \tag{32}
\]
Both these diffeomorphisms and symplectic reparametrizations are ‘Pseudo–symmetries’:
\[
D_{\text{pseudo}} = Diff(\mathcal{M}) \times Sp(2m, \mathbb{R}). \tag{31}
\]
They leave the action form invariant, but change
the coupling constants and are thus not invar-
iances of the action.

If \( \hat{g}_{\alpha\beta}(z) = g_{\alpha\beta}(z) \) then the diffeomorphisms
become isometries of the manifold, and proper
symmetries of the scalar action. If these isome-
tries are combined with symplectic transforma-
tions such that
\[
\hat{N}(z) = N(z), \tag{33}
\]
then this is a proper symmetry. These are invar-
iances of the equations of motion (but not neces-
arily of the action as not all transformations can
be implemented locally on the gauge fields). To
extend the full group of isometries of the scalar
manifold to proper symmetries, one thus has to
embed this isometry group in \( Sp(2m; \mathbb{R}) \), and
arrives at the following situation:
\[
D_{\text{prop}} = Iso(\mathcal{M}) \subset Iso(\mathcal{M}) \times Iso(\mathcal{M}) \subset D_{\text{pseudo}}
\]

Let us illustrate how \( S \) and \( T \) dualities, treated
in Sen’s lectures \[7\], fit in this scheme as proper symmetries. The action he treats occurs in \( N = 4 \)
supergravity. The scalars are \( \lambda = \lambda_1 + i\lambda_2 \) and
a symmetric matrix \( M \), satisfying \( M\eta M = \eta^{-1} \)
where \( \eta = \eta^T \) is the metric of \( O(6, 22) \). Their
coupling to the spin-1 fields is encoded in the matrix
\( N = \lambda \eta + i\lambda_2 \eta M \eta \). \( \tag{33} \)
The transformations on the scalars should lead to\[7\] with (28). Let us first consider this for the \( T \)
dualities. These are transformations of \( O(6, 22) \):
\[
\tilde{F}^+ = A F^+ ; \quad \tilde{M} = A M A^T, \tag{34}
\]
(\( \lambda \) is invariant) where \( \eta = A^T \eta A \). This leads to
\( \tilde{N} = (A^T)^{-1} N A^{-1} \), which is of the form (27),
identifying \( D = (A^T)^{-1} \). The matrices \( C \) and \( B \)
are zero, which indicates that these symmetries are
realised perturbatively.

For the \( S \) dualities, \( M \) is invariant. These
transformations are determined by the integers
\( s, r, q, p \) such that \( sp - qr = 1 \):
\[
\tilde{F}^+ = sF^+ + r\eta^{-1}NF^+; \quad \tilde{\lambda} = \frac{p\lambda + q}{r\lambda + s}.
\]
This leads to \( \tilde{N} = (pN + q\eta)(r\eta^{-1}N + s) \), which
is of the required form upon the identification
\[
S = \begin{pmatrix} sI & r\eta^{-1} \\ q\eta & pI \end{pmatrix}. \tag{35}
\]
Now, \( B \) and \( C \) are non-zero, which shows the non-
perturbative aspect of the \( S \)-duality.

3.3. Symplectic transformations in \( N = 2 \)

In \( N = 2 \) the tensor \( N \) is determined by the
function \( F \) as explained in section \[2\]. The defini-
tions of \( N \) in rigid and local supersymmetry can
be written in a clarifying way as follows:

\[
\begin{align*}
\text{rigid SUSY} & \quad \text{SUGRA} \\
\partial_C \tilde{F}_A &= N_{AB} \partial_C \tilde{X}^B & \partial_{\bar{C}} \tilde{F}_I &= N_{IJ} \partial_{\bar{C}} \tilde{X}^J \\
F_I &= N_{IJ} \tilde{X}^J 
\end{align*}
\]

(36)

From this definition it is easy to see that \( N \) transforms in the appropriate way if we define

\[
\begin{align*}
V &= \begin{pmatrix} X^A \\ F_A \end{pmatrix} & V &= \begin{pmatrix} X^I \\ F_I \end{pmatrix} \\
U_C &= \begin{pmatrix} \partial_C X^A \\ \partial_C F_A \end{pmatrix} & U_\alpha &= \begin{pmatrix} \partial_\alpha X^I \\ \partial_\alpha F_I \end{pmatrix}
\end{align*}
\]

(37)

(41)

and thus of the target-space manifold, in terms of the function \( \bar{F} \).

We have to distinguish two situations:

1. The function \( \bar{F}(X) \) is different from \( F(\tilde{X}) \), even taking into account (\ref{eq:37}). In that case the two functions describe equivalent classical field theories. We have a pseudo symmetry. These transformations are called symplectic reparametrizations (\ref{eq:37}). Hence we may find a variety of descriptions of the same theory in terms of different functions \( F \).

2. If a symplectic transformation leads to the same function \( F \) (again up to (\ref{eq:37})), then we are dealing with a proper symmetry. As explained above, this invariance reflects itself in an isometry of the target-space manifold. Henceforth these symmetries are called `duality symmetries', as they are generically accompanied by duality transformations on the field equations and the Bianchi identities. The question remains whether the duality symmetries comprise all the isometries of the target space, i.e. whether

\[
Iso(\mathcal{M}) \subset Sp(2(n+1), \Re) .
\]

We investigated this question in \( \text{\cite{7}} \) for the very special Kähler manifolds, and found that in that case one does obtain the complete set of isometries from the symplectic transformations. For generic special Kähler manifolds no isometries have been found that are not induced by symplectic transformations, but on the other hand there is no proof that these do not exist.

### 3.4. Examples (in supergravity)

We present here some examples of symplectic reparametrizations and duality symmetries in the context of \( N = 2 \) supergravity. First consider (\ref{eq:38}). If we apply the symplectic transformation

\[
S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{pmatrix}
\]

(41)

one arrives, using (\ref{eq:24}), at (\ref{eq:29}). So this is a symplectic reparametrization, and shows the equivalence of the two forms of \( F \) as announced above.
On the other hand consider
\[ S = \begin{pmatrix} 1 + 3\epsilon & \mu & 0 & 0 \\ \lambda & 1 + \epsilon & 0 & 2\mu/9 \\ 0 & 0 & 1 - 3\epsilon & -\lambda \\ 0 & -6\lambda & -\mu & 1 - \epsilon \end{pmatrix} \] (42)
for infinitesimal \( \epsilon, \mu, \lambda \). Then \( F \) is invariant. On the scalar field \( z = X^1/X^0 \), the transformations act as
\[ \delta z = \lambda - 2\epsilon z - \mu z^2/3 \, . \] (43)

They form an \( SU(1,1) \) isometry group of the scalar manifold. The domain were the metric is positive definite is \( \text{Im} \, z > 0 \). This shows the identification of the manifold as the coset space in \([19], [20]\).

As a second example, consider \([17]\). Using \([17]\) one obtains the matrix \( \mathcal{N} \) which determines (again with \( z = X^1/X^0 \))
\[ e^{-1} \mathcal{L}_I = -\frac{1}{2} \text{Re} \left[ z \left( F^0_\mu \right)^2 + z^{-1} \left( F^1_\mu \right)^2 \right] . \] (44)

This appears also in pure \( N = 4 \) supergravity in the so-called ‘\( SO(4) \) formulation’ \([18]\). Consider now the symplectic mapping \([19]\)
\[ S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \] (45)
leading to the transformations
\[ \tilde{X}^0 = X^0 \quad \tilde{X}^1 = -F_1 = iX^0 \] (46)
\[ \tilde{F}_0 = F_0 \quad \tilde{F}_1 = X^1 \, . \] (47)

This is an example where the transformation between \( \hat{X} \) and \( X \) is not invertible. Using \([39]\), we obtain \( \tilde{F} = 0 \). However, \( A + BN \) is invertible, and we can compute \( \tilde{N} \) using \([27]\), leading to
\[ e^{-1} \mathcal{L}_I = -\frac{1}{2} \text{Re} \left[ z \left( F^0_\mu \right)^2 + z \left( F^1_\mu \right)^2 \right] . \] (48)

(We performed here a symplectic transformation, but no diffeomorphism. We are still using the same variable \( z \)). This is the form familiar from the ‘\( SU(4) \) formulation’ of pure \( N = 4 \) supergravity \([19]\). This shows that there are formulations which can not be obtained directly from a superspace action.

In the final example, we will show that this particular formulation can be the most useful one. For that we consider the manifold
\[ SU(1,1) \otimes SU(r,2) \otimes SO(r) \otimes SO(2) . \] (49)

This is the only special Kähler manifold which is a product of two factors \([21]\). Therefore it appears in string theory where the first factor contains the dilaton-axion. The first formulation of this class of manifolds used a function \( F \) of the type \([22]\): \( F(X) = X^S X^r X^I \eta_{IJ} \), where \( \eta_{IJ} \) is the constant diagonal metric with signature \((+,-,\ldots,-)\) \([3]\). In this parametrization only an \( SO(r-1) \) subgroup of \( SO(r,2) \) is linearly realized (residing in \( A \) and \( D \) of \([23]\)). From a string compactification point of view one does not expect this. The full \( SO(r,2) \) should be a perturbative symmetry, as it is realized in the \( N = 4 \) theory described by Sen \([22,16]\). In the search for better parametrizations, by means of a symplectic reparametrization a function \( F \) of the square root type was discussed in \([22]\), which has \( SO(r) \) linearly realized. However, the solution was found in \([14]\), and was not based on a function \( F \) at all. The symplectic vector \( V \) contains then
\[ F_I = S \eta_{IJ} X^J , \] (50)
where \( S \) is one of the coordinates (representing the first factor of \([19]\)), and the \( X^J \) satisfy the constraint \( X^I \eta_{IJ} X^J = 0 \), where \( \eta_{IJ} \) is the \( SO(2,r) \) metric. For additional details on this example, see also \([24]\), where the perturbative corrections to the vector multiplet couplings are considered in the context of the \( N = 2 \) heterotic string vacua. This important example shows that under certain circumstances one needs a formulation that does not rely on the existence of a function \( F \).

### 3.5. Coordinate independent description

We want to be able to use more general coordinates than the special ones which appeared naturally in the superspace approach, and also to set up a formulation of the theory in which the symplectic structure is evident. First we will formulate this for the rigid case \([2]\).

We start by introducing the symplectic vector \( V \in \mathfrak{C}^m \), as in \([37]\), where now the \( F_A \) are
nates introduced before, corresponds to the special manifold. The choice of special coordinates introduced before, corresponds to \( X^\alpha(z) = z^\alpha \), \( F_A(z) = \frac{\partial F}{\partial X} (X(z)) \). By taking now derivatives with respect to \( z^\alpha \) one obtains \( U_\alpha \) analogous to the \( U_A \) in (37).

We define as metric on the special manifold

\[
g_{\alpha\beta} = i U_\alpha^T \Omega \bar{U}_\beta = i \langle U_\alpha, \bar{U}_\beta \rangle ,
\]

where we introduced a symplectic inner product \( \langle V, W \rangle \equiv V^T \Omega W \). The constraints which define the rigid special geometry can be formulated on the \( 2n \times 2n \) matrix

\[
\mathcal{V} \equiv \begin{pmatrix} U^T & 0 \\ 0 & U^T \end{pmatrix} = \begin{pmatrix} \partial_\alpha X^A & \partial_\gamma F_A \\ g^{\alpha\beta} \partial_\beta \bar{F}_A & g^{\gamma\delta} \partial_\delta \bar{F}_A \end{pmatrix} .
\]

This matrix should satisfy \( \mathcal{V} \Omega \mathcal{V}^T = -i \Omega \) and

\[
D_\alpha \mathcal{V} = A_\alpha \mathcal{V} \quad \text{with} \quad A_\alpha = \begin{pmatrix} 0 & C_{\alpha\beta\gamma} \\ 0 & 0 \end{pmatrix}
\]

for a symmetric \( C_{\alpha\beta\gamma} \) (being \( F_{ABC} \) in special coordinates); and \( D \) contains the Levi-Civita connection. The integrability condition of this constraint then implies the form of the curvature:

\[
R_{\alpha\beta\gamma\delta} = -C_{\alpha\gamma\delta} \bar{C}_{\beta\delta\gamma} g^{\epsilon\zeta} \quad \text{(compare this with (33)).}
\]

The formulation can even be simplified in terms of a vielbein \( e^A_\alpha = \partial_\alpha X^A \) (being the unit matrix in special coordinates). Then the connection \( \hat{\Gamma}_{\alpha\beta}^\gamma = \partial_\beta e^A_\alpha \hat{e}^A_\alpha \) is flat, and there are holomorphic constraints

\[
\hat{\mathcal{V}} \equiv \begin{pmatrix} e^A_\alpha & \partial_\alpha F_A \\ 0 & e^A_\alpha \end{pmatrix} \quad \text{with} \quad \hat{A}_\alpha = \begin{pmatrix} \hat{\Gamma}_{\alpha\beta}^\gamma & -i C_{\alpha\beta\gamma} \\ 0 & \hat{\Gamma}_{\alpha\beta}^\gamma \end{pmatrix}.
\]

For Supergravity a similar definition of special geometry is possible. This formulation was first given in the context of a treatment of the moduli space of Calabi-Yau three-folds\(^7\). The particular way in which we present it here is explained in more detail in \([26]\). Now the symplectic vectors have \( 2(n + 1) \) components. We first impose the constraint \((33)\), which is written in a symplectic way as \( \langle \mathcal{V}, \mathcal{V} \rangle \equiv V^T \Omega V = -i \).

Then we define \( n \) holomorphic symplectic sections, parametrized by \( z^\alpha \), which are proportional to \( V \):

\[
V(z, \bar{z}) = e^{\frac{i}{2} K(z, \bar{z})} v(z) ,
\]

and the proportionality constant defines the Kähler potential. These equations are then invariant under ‘Kähler transformations’

\[
v(z) \rightarrow e^{f(z)} v(z) \quad \text{and} \quad K(z, \bar{z}) \rightarrow K(z, \bar{z}) - f(z) - f(\bar{z})
\]

\[
V \rightarrow e^{\frac{1}{2}(f(z) - f(\bar{z}))} V .
\]

Usually the \( F_I(z) \) (function which depend on \( X^I(z) \)). Then one has \( F_I = \partial_I F \), and the scaling symmetry implies that \( F \) is a holomorphic function homogeneous of 2nd degree in \( X^I \). But e.g. with \( F_1 \) this is not the case.

To make contact with the Picard-Fuchs equations in Calabi-Yau manifolds, a similar formulation as for the rigid case is useful. This is obtained by defining the \( (2n + 2) \times (2n + 2) \) matrix

\[
\mathcal{V} = \begin{pmatrix} V \\ \bar{V} \\ U_\alpha \end{pmatrix} ,
\]

which satisfies \( \mathcal{V} \Omega \mathcal{V}^T = i \Omega \). One then introduces a connection such that the constraints are \([27]\)

\[
D_\alpha \mathcal{V} = A_\alpha \mathcal{V} , \quad D_\alpha \mathcal{V} = A_\alpha \mathcal{V} .
\]

with e.g.

\[
A_\alpha = \begin{pmatrix} 0 & 0 & \delta_\alpha^\gamma \\ 0 & \delta_\alpha^\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} .
\]

The integrability conditions lead to the curvature tensor

\[
R_{\alpha\beta\gamma\delta} = g_{\alpha\beta} g_{\gamma\delta} + g_{\alpha\delta} g_{\beta\gamma} - C_{\alpha\beta\gamma} C_{\delta\epsilon} g^{\epsilon\delta} .
\]
4. Further results and conclusions

Special geometry is not confined to Kähler manifolds. There exist a c map, which can be obtained either from dimensional reduction of the field theory to 3 dimensions, or from superstring compactification mechanisms [4]. This maps special Kähler manifolds to a subclass of the quaternionic manifolds, which are then called special quaternionic. As already mentioned, a subclass of special manifolds are the ‘very special’ ones. These can be obtained from dimensional reduction of actions in 5 dimensions, characterised by a symmetric tensor $d_{ABC}$ [28]. This mapping is called the r map [29], and the manifolds in the 5-dimensional theory are called ‘very special real’ manifolds. These concepts were very useful in the classification of homogeneous [30] and symmetric [14] special manifolds. A study of the full set of isometries could be done systematically in these models. All this has been summarised in [26].

For string theory the implications of special geometry in the rigid theories for the moduli spaces of Riemann surfaces [25], and in the supergravity theories for Calabi-Yau spaces [2] [3] is extremely useful for obtaining non-perturbative results [31, 27, 28]. For these results we refer to [3] and to [22], where many more aspects of special manifolds in the context of topological theories, Landau-Ginzburg theories, etc. are discussed.

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