Spin polarization independence of hard polarized fermion string scattering amplitudes

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Abstract

We calculate a class of polarized fermion string scattering amplitudes (PFSSA) at arbitrary mass levels. We discover that, in the hard scattering limit, the functional forms of the non-vanishing PFSSA at each fixed mass level are independent of the choices of spin polarizations. This result justifies and extends Gross conjecture on high energy string scattering amplitudes to the fermionic sector. In addition, this peculiar property of hard PFSSA is to be compared with the usual spin polarization dependence of the hard polarized fermion field theory scatterings.

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I. INTRODUCTION

One important characteristic of string scattering amplitudes (SSA) is its very soft exponential fall-off behavior in the hard scattering limit. This behavior is closely related to the existence of infinite linear relations among hard SSA of different string states at each fixed mass level. Moreover, these linear relations are so powerful that they can be used to solve all hard SSA and express them in terms of one amplitude. This means that there is only one hard SSA \( f(E, \theta) \) at each fixed mass level which is very different from the usual spin dependence of hard fermion field theory scatterings. This important high energy symmetry of string theory was first conjectured by Gross \cite{Gross1, Gross2, Gross3} and later corrected and proved by using the decoupling of zero norm states \cite{Gross4} in \cite{Gross5, Gross6, Gross7, Gross8, Gross9, Gross10}. For more details, see the recent review \cite{Gross11}.

However, all calculations that have been done so far are only for boson SSA of either the bosonic string theory \cite{Gross5, Gross6, Gross7, Gross8, Gross9, Gross10} or the NS sector (both GSO even and odd) of the fermionic string theory \cite{Gross10}. So it will be important and of interest to see whether one can extend Gross conjecture to the R sector of the fermionic string theory.

Since it is a nontrivial task to construct the general massive fermion string vertex operators, as the first step in this letter, we choose to calculate polarized fermion string scattering amplitudes (PFSSA) at arbitrary mass levels which involve the leading Regge trajectory fermion string state of the R sector \((\alpha' \equiv \frac{1}{2})\) \cite{Gross12}.

\[
\chi^{a}_{\alpha_{1}...\alpha_{n-1}m_{1}} \cdots \partial X^{m_{1}} \cdots i \partial X^{m_{n-1}} \left( i \partial X^{m_{n}} \right) \delta^{\beta}_{\alpha} - \frac{1}{8} \gamma^{\mu}_{\alpha \beta} k_{\mu} \psi^{m_{n}} \gamma_{\nu}^{\beta} \psi_{\nu} S_{\gamma} e^{- \frac{1}{2\phi^{2}} e^{ikX}}, \quad (1.1)
\]
in which the tensor-spinor wavefunction $\chi^{\alpha(m_1...m_n)}$ satisfies the on-shell conditions

$$k^m \chi^{\alpha(m_1...m_n)} = \eta^{mj} \chi^{\alpha(m_1...m_n)} = \chi^{\alpha(m_1...m_n)} \gamma^{m}_{\alpha\beta} = 0, M^2 = 2n,$$

(1.2)

which include a $\gamma$ traceless condition. One of the reasons for choosing this leading Regge trajectory state is that the corresponding vertex operator has been constructed in the literature [12]. The construction was mainly based on the complete construction of the first massive level states for both NS and R sectors [13].

On the other hand, since Gross conjecture was shown to be valid for both GSO even and odd states in the NS sector [10], for simplicity in this paper, we are going to ignore the GSO projection, and the other three string states in the SSA will be chosen to be one massless fermion and two tachyon states (GSO odd).

The state in Eq.(1.1) is a combination of $(\alpha^i_{-1})^{n}|\alpha\rangle_R$ and $(\alpha^i_{-1})^{n-1}(d^j_{-1})|\bar{\alpha}\rangle_R$ (in the light-cone gauge language). For the case of $n = 1$ [13], for example, the vector-spinor $\chi_\mu^\alpha$ is a 10D Majorana spinor that forms an irreducible massive representation of the Lorentz group. In the corresponding four dimensional case, the vector-spinor $\chi_\mu^\alpha$ transforms as the product of a four-vector and a Dirac spinor, and satisfies the Rarita-Schwinger equations

$$(\gamma \cdot \partial + M)\chi_\mu = 0,$$

(1.3)

$$\gamma^\mu \chi_\mu = 0,$$

(1.4)

which is the case of spin $s = \frac{3}{2}$ field equation of the more general Bargmann-Wigner equation with spin $s \geq \frac{1}{2}$. Note that Eq.(1.4) is similar to the $\gamma$ traceless condition in Eq.(1.2).

It was shown for the bosonic SSA that at each fixed mass level $M^2 = 2(N - 1)$ only tensor states of the following form [8, 9]

$$|N, 2m, q\rangle \equiv (\alpha^T_{-1})^{N-2m-2q}(\alpha^L_{-1})^{2m}(\alpha^L_{-2})^q|0, k\rangle$$

(1.5)

are of leading order in energy in the hard scattering limit. In Eq.(1.5), $e^P = \frac{1}{M}(E, k, 0) = \frac{k^2}{M^2}$ the momentum polarization, $e^L = \frac{1}{M}(k, E, 0)$ the longitudinal polarization and $e^T = (0, 0, 1)$ the transverse polarization are the three polarizations on the scattering plane [5, 6]. In the hard scattering limit, one can identify $e^P = e^L$ [5, 6]. It was remarkable to discover that all the hard bosonic SSA at each fixed mass level share the same functional forms with the following ratios [8, 9]

$$T^{(N,2m,q)}_{T^{(N,0,0)}} = \left(-\frac{1}{M}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!!.$$  

(1.6)
Thus there is only one hard SSA $T^{(N,0,0)}$ at each fixed mass level. For the leading Regge trajectory states we are considering in this paper, we set $q = 0$.

In this paper we will be mainly concerning with the spinor polarizations in the SSA calculation. So, for simplicity, we will be writing

$$
\chi_{(m_1...m_n)}^\alpha \sim \epsilon_{m_1}...\epsilon_{m_n} u^\alpha \tag{1.7}
$$

where $u^\alpha$ satisfies the 10D Dirac type equation in Eq. (1.3). For the leading hard SSA of the Regge trajectory states, one can choose to put all the tensor polarizations $\epsilon_{m_1} = \cdots = \epsilon_{m_n} = \epsilon_T$.

### II. POLARIZED FERMION STRING SCATTERING AMPLITUDES (PFSSA)

In this section, we will calculate the PFSSA with the following four vertex operators in the 10D open superstring theory: the massless spinor

$$
V = u^\alpha S_\alpha e^{-\phi/2} e^{ikX},
$$

the massive spinor ($\alpha' = \frac{1}{2}$)

$$
V = (i\varepsilon_1 \partial X) \cdots (i\varepsilon_n \partial X) u^\alpha S_\alpha e^{-\phi/2} e^{ikX} \tag{2.1}
$$

$$
- \frac{1}{8} (i\varepsilon_1 \partial X) \cdots (i\varepsilon_{n-1} \partial X) u^\alpha \gamma_\mu \gamma_\nu \psi \gamma_\rho \psi S_\gamma e^{-\phi/2} e^{ikX}
$$

and two tachyons

$$
V = e^{-\phi} e^{ikX} \quad (-1 \text{ ghost}),
$$

$$
V = (k \cdot \psi) e^{ikX} \quad (0 \text{ ghost}).
$$

In the above, we have chosen the total ghost charges sum up to $-2$. The correlators of the worldsheet boson $X^\mu$, worldsheet fermion $\psi^\mu$, spin field $S_A$, and ghost field $\phi$ are

$$
\langle X^\mu (z_1) X^{\nu} (z_2) \rangle = -\eta^{\mu\nu} \ln |z_{12}| \tag{2.5}
$$

$$
\langle \psi^\mu (z_1) S_A (z_2) S_B (z_3) \rangle = \frac{(\Gamma^\mu C)_{AB}}{\sqrt{2} z_{12}^{\frac{1}{2}} z_{13}^{\frac{1}{2}} z_{23}^{\frac{1}{2}}} \tag{2.6}
$$

$$
\langle \psi^\mu (z_1) \psi^{\nu} (z_2) \rangle = \frac{\eta^{\mu\nu}}{z_{12}} \tag{2.7}
$$

$$
\langle \phi (z_1) \phi (z_2) \rangle = - \ln |z_{12}| \tag{2.8}
$$
where \( z_{ij} = z_i - z_j \), \( \Gamma^\mu \) are 10D Dirac matrices calculated in Eq. (3.33) and \( C \) matrix calculated in Eq. (3.36).

The PFSSA we want to calculate can be written as

\[
A = A_1 + A_2
\]  
(2.9)

where

\[
A_1 = \int dz_2 \left| z_{13}z_{14}z_{34} \right| \left( u_1^{a_1} S_{1a_1} e^{-\phi_1} e^{ik_1X_1} (i\epsilon_1 \partial X_2) \cdots (i\epsilon_n \partial X_2) u_2^{a_2} S_{2a_2} e^{-\phi_2} e^{ik_2X_2} \right),
\]

\[
A_2 = -\frac{1}{8} \int dz_2 \left| z_{13}z_{14}z_{34} \right| \left( u_1^{a_1} S_{1a_1} e^{-\phi_1} e^{ik_1X_1} \right) \left( (i\epsilon_1 \partial X_2) \cdots (i\epsilon_n \partial X_2) u_2^{a_2} S_{2a_2} e^{-\phi_2} e^{ik_2X_2} \right).
\]

Let’s calculate \( A_1 \) first, whose correlator can be written as

\[
u_1^{a_1} u_2^{a_2} k_{4\mu} \left( S_{1a_1} S_{2a_2} \psi^{\mu}_4 \right) \left( e^{-\phi_1} e^{-\phi_2} e^{-\phi_3} \right) \left( e^{ik_1X_1} (i\epsilon_1 \partial X_2) \cdots (i\epsilon_n \partial X_2) e^{ik_2X_2} e^{ik_3X_3} e^{ik_4X_4} \right).
\]

(2.12)

The first correlators in Eq. (2.12) was calculated in Eq. (2.11), and the other two can be calculated to be

\[
\left( e^{-\phi_1} e^{-\phi_2} e^{-\phi_3} \right) = \left| z_{12} \right|^{-1} \left| z_{13} \right|^{-1} \left| z_{23} \right|^{-1} \left| z_{14} \right|^{-1} \left| z_{24} \right|^{-1} \left| z_{34} \right|^{-1}
\]

(2.13)

and

\[
\left( e^{ik_1X_1} (i\epsilon_1 \partial X_2) \cdots (i\epsilon_n \partial X_2) e^{ik_2X_2} e^{ik_3X_3} e^{ik_4X_4} \right)
= \left( \frac{k_1 \cdot \epsilon_1}{z_{21}} + \frac{k_3 \cdot \epsilon_1}{z_{23}} + \frac{k_4 \cdot \epsilon_1}{z_{24}} \right) \cdots \left( \frac{k_1 \cdot \epsilon_n}{z_{21}} + \frac{k_3 \cdot \epsilon_n}{z_{23}} + \frac{k_4 \cdot \epsilon_n}{z_{24}} \right).
\]

(2.14)

For the \( s - t \) channel amplitude, we take \( z_1 = 0, z_3 = 1, z_4 \to \infty (0 \leq z_2 \leq 1) \) and, for simplicity, set all \( \epsilon_1 = \cdots = \epsilon_n = \epsilon \), we get

\[
A_1 = \frac{1}{\sqrt{2}} u_1^{a_1} (\Gamma^\mu C)_{\alpha_1 \alpha_2} u_2^{a_2} k_{4\mu} (-1)^{\frac{3}{2}} \int_0^1 dz_2 z_{12}^k z_{13}^j (1 - z_2)^{k_2 k_3 - \frac{4}{3}} \left( \frac{k_1 \cdot \epsilon}{z_2} - \frac{k_3 \cdot \epsilon}{1 - z_2} \right)^n.
\]

(2.15)
Finally the integration in \( A_1 \) can be performed and we obtain
\[
A_1 = \frac{1}{\sqrt{2}} u_1^{\alpha_1} (\Gamma^\mu C)_{\alpha_1 \alpha_2} u_2^{\alpha_2} k_{4 \mu} \times (-1)^{\frac{3}{2}} (-k_3 \cdot \epsilon)^n B \left( \frac{-s}{2} + n; \frac{-t}{2} \right) F_D^{(1)} \left( \frac{-t}{2}; n; \frac{s}{2} - n + 1; \frac{-k_1 \cdot \epsilon}{k_3 \cdot \epsilon} \right) \tag{2.16}
\]
where \( s = -(k_1 + k_2)^2 \) and \( t = -(k_2 + k_3)^2 \) are the Mandelstam variables, and \( F_D^{(1)} \) is the Lauricella function \( (K = 1) \). \[13\]

\[
F_D^{(K)} (\alpha; \beta_1, \ldots, \beta_K; \gamma; x_1, \ldots, x_K) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 dt \, t^{\alpha - 1} (1 - t)^{\gamma - \alpha - 1} \cdot (1 - x_1 t)^{-\beta_1} (1 - x_2 t)^{-\beta_2} \ldots (1 - x_K t)^{-\beta_K}. \tag{2.18}
\]

Similar technique can be used to calculate \( A_2 \) whose correlator can be written as
\[
\begin{align*}
&\times (e^{i k_1 \cdot X_1} (i \epsilon_1 \partial X_2) \cdots (i \epsilon_{n-1} \partial X_2) e^{i k_2 X_2} e^{i k_3 X_3} e^{i k_4 X_4}). \tag{2.19}
\end{align*}
\]

The second and the third correlators in Eq.\((2.19)\) were calculated in Eq.\((2.13)\) and Eq.\((2.14)\) respectively. The first correlator can be written as
\[
\begin{align*}
\left\langle S_{1 \alpha_1} \gamma_\alpha^{\mu} k_{2 \mu} \psi_2 \gamma_\rho^{\beta} \psi_2^{\rho} S_{2 \gamma} \psi_4^{\nu} \right\rangle = \left\langle S_{1 \alpha_1} \gamma_\alpha^{\mu} k_{2 \mu} K^{\beta}_{\lambda} \psi_4^{\nu} \right\rangle \tag{2.20}
\end{align*}
\]
where the composite operators \( K^{\beta}_{\lambda} \) was defined to be
\[
K^{\beta}_{\lambda} = \psi_2 \gamma_\rho^{\beta} \psi_2^{\rho} S_{2 \gamma}. \tag{2.21}
\]

The correlation functions containing spin fields \( S_\alpha \) and the composite operators \( K^{\beta}_{\lambda} \) can be found in \[12\]. The computation of correlation functions with \( K^{\beta}_{\lambda} \) got simplified due to the \( \gamma \) traceless condition in Eq.\((1.2)\). The correlator in Eq.\((2.20)\) can then be calculated to be
\[
\begin{align*}
\left\langle S_{1 \alpha_1} \gamma_\alpha^{\mu} k_{2 \mu} K^{\beta}_{\lambda} \psi_4^{\nu} \right\rangle &= \frac{(10 - 2) \frac{1}{\sqrt{2 \gamma_{\alpha_2 \beta} \gamma_{\alpha_2 \beta}}} k_{2 \mu} \eta_{\gamma}^{\nu} C_{\alpha_1}}{\sqrt{2 \gamma_{\alpha_2 \beta} \gamma_{\alpha_2 \beta}}} \left( \frac{\gamma_{\alpha_2 \beta}}{\gamma_{\alpha_2 \beta}} \right) \tag{2.22}
\end{align*}
\]

By using Eq.\((2.13)\), Eq.\((2.14)\) and Eq.\((2.22)\), we get
\[
\begin{align*}
A_2 &= -\frac{1}{\sqrt{2}} u_1^{\alpha_1} u_2^{\alpha_2} \gamma^{\mu}_{\alpha_2 \beta} C_{\alpha_1}^{\mu} k_{2 \mu} (k_4 \cdot \epsilon_n) \\
\times \int dz_2 &\frac{z_2^{k_1 \cdot k_2 - \frac{1}{2}}}{\sqrt{\gamma_{12}^{z_2}} |z_{13}|^{z_1^{k_1 \cdot k_3 + \frac{1}{2}}} |z_{14}|^{z_1^{k_1 \cdot k_4 + \frac{1}{2}}} |z_{23}|^{z_2^{k_2 \cdot k_3 - \frac{1}{2}}} |z_{24}|^{z_2^{k_2 \cdot k_4 + \frac{1}{2}}} |z_{34}|^{z_3^{k_3 \cdot k_4 + 1}} \\
\times \left( \frac{k_1 \cdot \epsilon_1}{z_{21}} + \frac{k_3 \cdot \epsilon_1}{z_{23}} + \frac{k_4 \cdot \epsilon_1}{z_{24}} \right) \cdots \left( \frac{k_1 \cdot \epsilon_{n-1}}{z_{21}} + \frac{k_3 \cdot \epsilon_{n-1}}{z_{23}} + \frac{k_4 \cdot \epsilon_{n-1}}{z_{24}} \right). \tag{2.23}
\end{align*}
\]
For the $s-t$ channel amplitude, we take $z_1 = 0, z_3 = 1, z_4 \to \infty$ ($0 \leq z_2 \leq 1$) and, for simplicity, set all $\epsilon_1 = \cdots = \epsilon_n = \epsilon$ as before, we get

$$A_2 = -\frac{1}{\sqrt{2}} u_1^{\alpha_1} u_2^{\alpha_2} \gamma^\mu \sigma_{\alpha_2 \beta} C_{\alpha_1}^\beta k_{2\mu} (k_4 \cdot \epsilon) (-1)^{3} \int_0^1 dz_2 z_2^{\frac{n-2}{4} + n-2} (1 - z_2) \frac{k_3 \cdot \epsilon}{1 - z_2}^{n-1} \left( \frac{k_1 \cdot \epsilon}{z_2} \right)^{n-1}.$$  

Finally the integration in $A_2$ can be performed \cite{14} and we obtain

$$A_2 = -\frac{1}{\sqrt{2}} u_1^{\alpha_1} (\Gamma^\mu C)_{\alpha_1 \alpha_2} u_2^{\alpha_2} k_{2\mu} \times$$

$$(-1)^{\frac{3}{2}} (k_4 \cdot \epsilon) (-k_3 \cdot \epsilon)^{n-1} B \left(\frac{-s}{2} + n - 1; \frac{-t}{2} + 1\right) F_D^{(1)} \left(\frac{-t}{2} + 1; -n + 1; \frac{s}{2} - n + 2; \frac{-k_1 \cdot \epsilon}{k_3 \cdot \epsilon}\right).$$

This completes the calculation of the PFSSA.

### III. HARD SCATTERING LIMIT

In this section, we will calculate the hard scattering limit of the PFSSA we obtained in the previous section. We will concentrate on the spinor polarizations and ignore the parts of the tensor polarizations. To do so we need to solve $10D$ Dirac equation and calculate explicitly the two factors in Eq.\((2.16)\) and Eq.\((2.25)\)

$$u_1^{\alpha_1} (\Gamma^\mu C)_{\alpha_1 \alpha_2} u_2^{\alpha_2} k_{4\mu},$$

$$u_1^{\alpha_1} (\Gamma^\mu C)_{\alpha_1 \alpha_2} u_2^{\alpha_2} k_{2\mu}.$$  

We will follow the definition in \cite{16} to calculate the $10D$ Dirac matrices. The ground states of the R sector are degenerate and can be labeled by

$$s = (s_0, s_1, s_2, s_3, s_4)$$

where each of the $s_a$ is $\pm \frac{1}{2}$ in the $s$ basis. To simplify the notation, we will ignore the factor $\frac{1}{2}$ in the rest of the paper. There are $2^{25} = 32$ components of a $10D$ Dirac spinor. The $10D$ Dirac matrices can be calculated iteratively starting in $d = 2$, where

$$\Gamma^0 = i\sigma^2, \Gamma^1 = \sigma^1.$$  

Then in $d = 2k + 2$,

$$\Gamma^\mu = \gamma^\mu \otimes (-\sigma^3), \mu = 0, \cdots, d - 3,$$

$$\Gamma^{d-2} = I \otimes \sigma^1, \Gamma^{d-1} = I \otimes \sigma^2$$

7
where $\gamma^\mu$ is the $2^k \times 2^k$ Dirac matrices in $d-2$ dimensions and $I$ is the $2^k \times 2^k$ identity matrix. We list all the 10D Dirac matrices calculated in the following

\[
\begin{align*}
\Gamma^0 &= i\sigma^2 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\
\Gamma^1 &= \sigma^1 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\
\Gamma^2 &= -I_2 \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^3, \\
\Gamma^3 &= -I_2 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^3, \\
\Gamma^4 &= I_2 \otimes I_2 \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^3, \\
\Gamma^5 &= I_2 \otimes I_2 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^3, \\
\Gamma^6 &= -I_2 \otimes I_2 \otimes I_2 \otimes \sigma^1 \otimes \sigma^3, \\
\Gamma^7 &= -I_2 \otimes I_2 \otimes I_2 \otimes \sigma^2 \otimes \sigma^3, \\
\Gamma^8 &= I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma^1, \\
\Gamma^9 &= I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma^2.
\end{align*}
\] (3.33)

We begin with the calculation of $C$ matrix in Eq. (3.27) and Eq. (3.28), which is defined to be

\[ C = B_1 \Gamma^0 \] (3.34)

where

\[ B_1 = \Gamma^3 \Gamma^5 \Gamma^7 \Gamma^9 = -I_2 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1. \] (3.35)

So we have

\[ C = B_1 \Gamma^0 = -i\sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \] (3.36)

and

\[
\begin{align*}
\Gamma^0 C &= I_2 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1, \\
\Gamma^1 C &= \sigma^3 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1, \\
\Gamma^2 C &= \sigma^2 \otimes I_2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1.
\end{align*}
\] (3.37-3.39)

The next step is to solve 10D Dirac equation

\[ (ik \cdot \Gamma + M) u = 0, \] (3.40)
and calculate explicitly the spinors $u_1$ and $u_2$ in Eq. (3.27) and Eq. (3.28). In the CM frame, we have the kinematics

$$k_1 = \left( + \sqrt{p^2 + M_1^2}, -p, 0 \right),$$
$$k_2 = \left( + \sqrt{p^2 + M_2^2}, +p, 0 \right),$$
$$k_3 = \left( - \sqrt{q^2 + M_3^2}, -q \cos \theta, -q \sin \theta \right),$$
$$k_4 = \left( - \sqrt{q^2 + M_4^2}, +q \cos \theta, +q \sin \theta \right).$$

For our case here, $u_1$ is a massless spinor, so we have

$$k_1^\mu = (+p, -p, 0), \quad \mbox{(3.41)}$$

$$ik_1 \cdot \Gamma u_1 = 0. \quad \mbox{(3.42)}$$

The 10D Dirac equation can be calculated to be

$$\left[ i (-p) \Gamma^0 + i (-p) \Gamma^1 \right] u_1 = -ip \left[ i \sigma^2 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 + \sigma^1 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \right] u_1 = \left( i \sigma^2 + \sigma^1 \right) \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 u_1 = 0, \quad \mbox{(3.43)}$$

or

$$\left( \begin{array}{cc} 0 & 2 \\ 0 & 0 \end{array} \right) \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 u_1 = 0, \quad \mbox{(3.44)}$$

which can be solved to be

$$u_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes |\pm; \pm; \pm; \pm\rangle \quad \mbox{(3.45)}$$

where

$$|+\rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad |\rangle = \left( \begin{array}{c} 0 \\ 1 \end{array} \right). \quad \mbox{(3.46)}$$

For the massive $u_2$ we have

$$(ik_2 \mu \Gamma^\mu + M_2) u_2 = 0, \quad \mbox{(3.47)}$$

$$k_2^\mu = (+E_2, +p, 0) = \left( + \sqrt{p^2 + M_2^2}, +p, 0 \right). \quad \mbox{(3.48)}$$
The 10D Dirac equation can be calculated to be

\[
\left[ i (-E_2) \Gamma^0 + ip \Gamma^1 + M_2 \right] u_2
\]

\[
= \left[ i (-E_2) i \sigma^2 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 + ip \sigma^1 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 + M_2 \right] u_2
\]

\[
= \left[ (-i E_2 \sigma^2 + ip \sigma^1) \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 + M_2 I_2 \otimes I_2 \otimes I_2 \otimes I_2 \right] u_2 = 0. \quad (3.49)
\]

Let’s first assume

\[
u_2 = U_2 \otimes |\text{even # of ””}\rangle \quad (3.50)
\]

where \(U_2\) is a 2-spinor. If we put Eq.\((3.50)\) into Eq.\((3.49)\), we get

\[
\begin{pmatrix}
M_2 & -i E_2 + ip \\
-i E_2 + ip & M_2
\end{pmatrix} U_2 = 0,
\]

which can be solved to be

\[
U_2 = \begin{pmatrix} i \sqrt{\frac{E_2 - p}{2E_2}} \\ \sqrt{2E_2(E_2 - p)} \end{pmatrix}.
\]

So the first class of solutions of \(u_2\) is

\[
u_2^{(1)} = \begin{pmatrix} i \sqrt{\frac{E_2 - p}{2E_2}} \\ \sqrt{2E_2(E_2 - p)} \end{pmatrix} \otimes |\text{even # of ””}\rangle. \quad (3.53)
\]

Alternatively, we can assume

\[
u_2 = U_2 \otimes |\text{odd # of ””}\rangle. \quad (3.54)
\]

For this case, Dirac equation reduces to

\[
\begin{pmatrix}
M_2 & i E_2 - ip \\
-i E_2 - ip & M_2
\end{pmatrix} U_2 = 0,
\]

and we get the second class of solutions

\[
u_2^{(2)} = \begin{pmatrix} i \sqrt{\frac{E_2 - p}{2E_2}} \\ \sqrt{2E_2(E_2 - p)} \end{pmatrix} \otimes |\text{odd # of ””}\rangle. \quad (3.56)
\]

We are now ready to calculate the vector components of \(u_1 \Gamma^\mu Cu_2\) in Eq.\((3.27)\) and Eq.\((3.28)\) which are to be contracted with \(k_1\) and \(k_2\). One needs only calculate the first three components of the vector.
On the other hand, it is crucial to note that the last three components of $\Gamma^0 C$, $\Gamma^1 C$ and $\Gamma^2 C$ in Eq.(3.37), Eq.(3.38) and Eq.(3.39) are all off-diagonal matrices. In order to get non-vanishing amplitudes, one is forced to choose different spin sign factors for each of the last three spin components of $u_1$ and $u_2$. We will see that the choice of $u_2^{(2)}$ in Eq.(3.56) give leading order amplitudes in the hard scattering limit, while the choice of $u_2^{(1)}$ in Eq.(3.53) give subleading order amplitudes.

For the first case, as an example, we choose $u_1$ as

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |+;+;+;+\rangle$$

and $u_2$ as

$$u_2^{(1)} = \left( \frac{i \sqrt{E_2 - p}}{2E_2} \otimes |\text{even # of } - \rangle \right) = \left( \frac{i \sqrt{E_2 - p}}{2E_2 M_2} \right) \otimes |-;--;--;-\rangle. \quad (3.58)$$

The first three component of $u_1 \Gamma^\mu Cu_2^{(1)}$ can be calculated to be

$$u_1 \Gamma^0 C u_2^{(1)} = -i \sqrt{\frac{E_2 - p}{2E_2}}, \quad (3.59)$$

$$u_1 \Gamma^1 C u_2^{(1)} = -i \sqrt{\frac{E_2 - p}{2E_2}}, \quad (3.60)$$

and

$$u_1 \Gamma^2 C u_2^{(1)} = 0. \quad (3.61)$$

So we have in Eq.(3.27)

$$u_1 \Gamma^\mu C u_2^{(1)} k_{4\mu} = -i \sqrt{\frac{E_2 - p}{2E_2}} (E_4 + q \cos \theta) \quad (3.62)$$

and in Eq.(3.28)

$$u_1 \Gamma^\mu C u_2^{(1)} k_{2\mu} = -i \sqrt{\frac{E_2 - p}{2E_2}} (-E_2 + p). \quad (3.63)$$

For the second case, as an example, we choose $u_1$ as in Eq.(3.57) and $u_2$ as

$$u_2 = \left( \frac{i \sqrt{E_2 - p}}{-2E_2 M_2} \otimes |\text{odd # of } - \rangle \right) = \left( \frac{i \sqrt{E_2 - p}}{-2E_2 M_2} \right) \otimes |+;--;--;-\rangle. \quad (3.64)$$
The first three component of \( u_1 \Gamma^\mu Cu_2^{(2)} \) can be calculated to be

\[
u_1 \Gamma^0 Cu_2^{(2)} = 0,
\]

(3.65)

\[
u_1 \Gamma^1 Cu_2^{(2)} = 0
\]

(3.66)

and

\[
u_1 \Gamma^2 Cu_2^{(2)} = \frac{M_2}{\sqrt{2E_2 (E_2 - p)}}.
\]

(3.67)

So we have in Eq.(3.27)

\[
u_1 \Gamma^\mu Cu_2^{(2)} k_{4\mu} = \frac{M_2}{\sqrt{2E_2 (E_2 - p)}} q \sin \theta
\]

(3.68)

and in Eq.(3.28)

\[
u_1 \Gamma^\mu Cu_2^{(2)} k_{2\mu} = 0.
\]

(3.69)

In the hard scattering limit, the energy order of Eq.(2.17) and Eq.(2.26) are the same. To calculate the leading order amplitudes in Eq.(2.9), we need the results calculated in Eq.(3.62), Eq.(3.63), Eq.(3.68) and Eq.(3.69). We note that

\[
p = \sqrt{E_2^2 - M_2^2} = E_2 \left(1 - \frac{M_2^2}{2E_2^2} + \cdots \right),
\]

(3.70)

so one gets in the hard scattering limit

\[
\frac{M_2}{\sqrt{2E_2 (E_2 - p)}} \to 1,
\]

(3.71)

\[
\sqrt{\frac{E_2 - p}{2E_2}} \to 0.
\]

(3.72)

Finally the only leading order amplitude in the hard scattering limit is

\[
u_1 \Gamma^\mu Cu_2^{(2)} k_{4\mu} = E \sin \theta.
\]

(3.73)

We conclude that for the choice of Eq.(3.56), \( A_1 \) in Eq.(2.9) gives the leading order amplitudes in the hard scattering limit. One can count the number of leading order amplitudes. There are \( 2^4 = 16 \) choices of spin polarizations for \( u_1 \). Once the polarization of \( u_1 \) is fixed, each of the last three spin signs of \( u_2 \) are fixed to be of the different sign with \( u_1 \), and the
second spin sign of $u_2$ is then fixed by the condition that the total number of ($-$) spin sign is odd.

In sum, among $2^4 \times 2^4 = 256$ PFSSA, only 16 of them are of leading order in energy in the hard scattering limit. More importantly, all the 16 leading order amplitudes share the same functional forms and are independent of the choices of spin polarizations. This result justifies and extends Gross conjecture [1–3] on high energy string scattering amplitudes to the fermionic sector.

IV. DISCUSSION

In contrast to the PFSSA considered in this paper, in the more familiar polarized fermion field scattering amplitude (PFFSA) calculation in quantum field theory, the leading order non-vanishing hard (ie. massless limit) PFFSA are in general NOT proportional to each other. We give two examples here. In QED, for the lowest order process of $e^- e^+ \rightarrow \mu^- \mu^+$, there are 4 non-vanishing among 16 hard PFFSA [17]:

\begin{equation}
M(e_R e_L^+ \rightarrow \mu_R \mu_L^+) = M(e_L e_R^+ \rightarrow \mu_L \mu_R^+) \sim (1 + \cos \theta) = 2 \cos^2 \frac{\theta}{2},
\end{equation}

\begin{equation}
M(e_R e_L^+ \rightarrow \mu_L \mu_R^+) = M(e_L e_R^+ \rightarrow \mu_R \mu_L^+) \sim (1 - \cos \theta) = 2 \sin^2 \frac{\theta}{2},
\end{equation}

and they are not all proportional to each other. Note that the usual unpolarized cross section obtained by summing over final spins and averaging over the initial spins in the hard scattering limit is

\begin{equation}
\frac{1}{4} \sum_{\text{spins}} |M|^2 \sim (1 + \cos^2 \theta).
\end{equation}

The second example is the lowest order process of the elastic scattering of a spin-one-half particle by a spin-zero particle such as $e^- \pi^+ \rightarrow e^- \pi^+$. The non-vanishing amplitudes were shown to be [18]:

\begin{equation}
M(e_R \pi^+ \rightarrow e_R \pi^+) = M(e_L \pi^+ \rightarrow e_L \pi^+) \sim \cos \frac{\theta}{2},
\end{equation}

\begin{equation}
M(e_R \pi^+ \rightarrow e_L \pi^+) = M(e_L \pi^+ \rightarrow e_R \pi^+) \sim \sin \frac{\theta}{2}.
\end{equation}

They are again not all proportional to each other.

This paper is the first attack by the present authors to probe high energy, higher spin fermion string scatterings. There are many interesting related issues which remained to be
studied. To name a few examples, are there linear relations among hard fermion SSA so that all the fermion SSA can be solved and expressed in terms of one amplitude? can these relations be extended to connect hard SSA of string states of NS sector and R sector? We will come back to these interesting topics in the near future.

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