Estimating the Ratio of Means in a Zero-inflated Poisson Mixture Model*

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Abstract
The problem of estimating the ratio of the means of a two-component Poisson mixture model is considered, when each component is subject to zero-inflation, i.e., excess zero counts. The resulting zero-inflated Poisson mixture (ZIPM) model can be treated as a three-component Poisson mixture model with one degenerate component. The EM algorithm is applied to obtain frequentist estimators and their standard errors, the latter determined via an explicit expression for the observed information matrix. Bayes and empirical Bayes estimators also are obtained by means of conjugate priors and their data-based variants. Lastly, the ZIPM distribution and the ZTP (zero-truncated Poisson) distribution are compared.

*Key words: Zero-inflated Poisson mixture, ratio of means, maximum likelihood estimator, EM algorithm, information matrix, standard error, Bayes estimator, conjugate prior, empirical Bayes estimator, zero-truncated Poisson distribution.
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1. Introduction.

Consider an ecological study aimed at determining the relative reproductive rate of a newly discovered invasive subspecies A of ant compared to that of the native subspecies B. The available data is indirect, consisting only of counts of nests in several standardized sites, rather than direct observations of individuals. Furthermore, the nests of the two subspecies are indistinguishable, (possibly) differing only in their relative numbers per site. If the expected numbers of nests per site for A and B are denoted by $\mu$ and $\nu$ respectively, it is desired to estimate their ratio $\theta \equiv \mu/\nu$, where $0 < \theta < \infty$.

Because little is known about the characteristics of A, no further constraint can be imposed on $\theta$, which renders the problem unidentifiable as stated, i.e. $(\mu, \nu)$ is indistinguishable from $(\nu, \mu)$. However, it is reasonable to assume that the newly discovered subspecies A is less prevalent than the established subspecies B, at least initially. This assumption will be incorporated into the mixture model introduced below, rendering it identifiable.

Furthermore, it is typical of such field studies that data is lost due to uncontrollable factors such as rain, resulting in excessive numbers of zero counts. As is commonly done, we shall adopt the zero-inflated Poisson (ZIP) distribution to represent this feature (cf. Lambert (1992)).

Let $N_{ij}$ denote the number of ant nests observed on day $i$ at site $j$. Let $I \equiv \{1, \ldots, I\}$ and $J \equiv \{1, \ldots, J\}$ be the corresponding index sets, and set $K = I \times J$, $K = |K| = IJ$. For $(i,j) \in K$, consider random variables (rvs)

\begin{align*}
Y_j & \sim \text{Bernoulli}(\pi), \\
M_{ij} \mid Y_j & \sim \text{Poisson}\{t_i[0^Y_j\mu + (1 - 0^Y_j)\nu]\}; \\
N_{ij} & = Z_{ij}M_{ij}, \\
Z_{ij} & \sim \text{Bernoulli}(\epsilon);
\end{align*}

where $0^0 = 1$, $\{Y_j\}$ and $\{Z_{ij}\}$ are mutually independent, and $\{M_{ij}\}$ and $\{Z_{ij}\}$ are conditionally mutually independent given $\{Y_j\}$. Thus $M_{ij}$ is a $\pi$-mixture of Poisson$(t_i\mu)$ and Poisson$(t_i\nu)$ rvs, where each $t_i > 0$ is known, reflecting a daily feature common to all sites, such as temperature, and $\mu, \nu \in (0, \infty)$ are unknown. Here $N_{ij}$ is a zero-inflated Poisson mixture (ZIPM) rv with zero-inflation parameter $1 - \epsilon \in (0, 1)$.

The main goal of this paper is the problem of estimating the ratio $\theta \equiv \mu/\nu$ based solely on the observed data $\{N_{ij}\}$, with $\{Y_j\}$, $\{M_{ij}\}$, and $\{Z_{ij}\}$...
unobserved. As noted above, for identifiability of \((\mu, \nu)\), and therefore of \(\theta\), a restriction must be imposed: we assume that \(0 < \pi \leq 1/2\), corresponding to the assumption that subspecies A occurs less frequently than subspecies B. Here \(\theta, \lambda \in (0, \infty)\), where \(\lambda \equiv \nu\) is viewed as a nuisance parameter. In terms of \((\theta, \lambda)\), (2) can be rewritten as

\[
M_{ij} | Y_j \sim \text{Poisson}(t_i \theta^Y_j \lambda).
\]

Both frequentist and Bayesian analyses will be presented.

Two well-known preliminary problems will serve as guideposts for the main problem. Section 2 reviews the case where \(\{Y_j\}\) are observed; here inference about \(\theta\) is based solely on the Poisson rvs \(\{M_{ij}\}\), with \(\{Z_{ij}\}\) and \(\{N_{ij}\}\) irrelevant. The maximum likelihood estimators (MLEs) \(\hat{\mu}, \hat{\nu}, \hat{\theta}\) and associated confidence intervals are straightforward. For Bayesian analysis (cf. Laurent and Lagrand [2012]), the integrated likelihood function (cf. (10))

\[
f_\delta(m | y; \theta) = \int_0^\infty f(m | y; \theta, \lambda) \gamma_\delta(\lambda) d\lambda
\]

w.r.t. a gamma prior probability density function (pdf) \(\gamma_\delta(\lambda)\) is obtained. A family of conjugate prior pdfs \(\phi_{\alpha,\beta}(\theta)\) is easily obtained (17) from \(f_\delta(m | y; \theta)\), leading to explicit posterior pdfs, Bayes estimators, and Bayesian credible intervals. Alternatively, the maximum integrated likelihood estimator (MILE), obtained by maximizing \(f_\delta(m | y; \theta)\) w.r.t. \(\theta\), is readily determined.

The case where \(\{Y_j\}\) are unobserved but \(\{M_{ij}\}\) are observed is reviewed in Section 3. This can be viewed as a two-component Poisson mixture model for the \(\{M_{ij}\}\); again \(\{Z_{ij}\}\) and \(\{N_{ij}\}\) are irrelevant. A standard application of the EM algorithm yields the MLEs \(\hat{\pi}, \hat{\mu}, \hat{\nu}, \hat{\theta}\), and hence \(\hat{\theta}\), then their standard errors are approximated via the observed information matrix \(I_m\), obtained explicitly in (50).

For Bayesian analysis in Section 3, the integrated likelihood function

\[
f_{\delta,\delta}(y, m | \theta) = \int_0^{1/2} \int_0^\infty f(y, m | \pi, \lambda, \theta) \vartheta(\pi) \gamma_\delta(\lambda) d\pi d\lambda
\]

w.r.t. \(\gamma_\delta(\lambda)\) and any proper prior pdf \(\vartheta(\pi)\) for \(\pi \in (0, \frac{1}{2})\) is obtained (cf. (55)). From this the integrated likelihood \(f_{\delta,\delta}(m | \theta)\) of \(M\) itself can be found explicitly (cf. (57)). No conjugate prior family is available, but for any prior
pdf \( \phi(\theta) \) the posterior pdf \( f_{\theta,\delta}(\theta \mid \mathbf{m}) \propto f_{\theta,\delta}(\mathbf{m} \mid \theta)\phi(\theta) \) can be simulated via MCMC methods, yielding Bayes estimators and credible intervals.

Alternatively, the conjugate prior \( \phi_{\alpha,\beta} \) in (17) can be replaced by a data-based version that depends on the unobserved \( \{Y_j\} \), whose values are then imputed by the EM algorithm, thereby yielding empirical Bayes posterior pdfs, estimators, and credible intervals.

The main problem, where only the ZIPM rvs \( \{N_{ij}\} \) are observed, is treated in Section 4. This can be viewed as a three-component Poisson mixture model where one of the components is degenerate at 0.\(^1\) Now the EM algorithm yields the MLEs \( \hat{\pi}, \epsilon, \hat{\mu}, \hat{\nu} \), and hence \( \hat{\theta} \), then their standard errors are approximated via the observed information matrix \( I_n \), obtained explicitly with some effort in (75)-(87), a main contribution of this study.

For Bayesian analysis in Section 4, the integrated likelihood function

\[
f_{\theta,\eta,\kappa,\delta}(\mathbf{y}, \mathbf{z}, \mathbf{n} \mid \theta) = \int_{0}^{1/2} \int_{0}^{1} \int_{0}^{\infty} f(\mathbf{y}, \mathbf{z}, \mathbf{n} \mid \pi, \epsilon, \theta, \lambda)\psi(\pi)\xi(\epsilon)\gamma(\lambda)d\pi d\epsilon d\lambda
\]

w.r.to a gamma prior pdf \( \gamma(\lambda) \), a beta prior \( \xi(\epsilon) \) for \( \epsilon \in (0, 1) \), and any proper prior pdf \( \psi(\pi) \) for \( \pi \in (0, 1/2] \) is obtained (cf. (92)). From this the integrated likelihoods \( f_{\theta,\eta,\kappa,\delta}(\mathbf{z}, \mathbf{n} \mid \theta) \) and \( f_{\theta,\eta,\kappa,\delta}(\mathbf{n} \mid \theta) \) can be obtained explicitly, cf. (93) and (94), although the latter is computationally challenging. Again no conjugate prior family is available, but for any prior pdf \( \phi(\theta) \) the posterior pdf \( f_{\theta,\eta,\kappa,\delta}(\theta \mid \mathbf{n}) \propto f_{\theta,\eta,\kappa,\delta}(\mathbf{n} \mid \theta)\phi(\theta) \) can be simulated via MCMC methods to obtain Bayes estimators and credible intervals.

Alternatively, the conjugate prior \( \phi_{\alpha,\beta} \) in (17) can be replaced by a data-based version that depends on the unobserved \( \{Y_j\} \) and \( \{Z_{ij}\} \), whose values are imputed by the EM algorithm, again yielding empirical Bayes posterior pdfs, estimators, and credible intervals.

The paper concludes with a comparison of the conditional ZIPM distribution and the ZTP (zero-truncated Poisson) distribution in Section 5.

\(^1\)A three-component mixture model with two degenerate components, one non-degenerate Poisson component, and i.i.d. observations was considered by Arora and Chaganty (2021).
**Notation:** Column vectors and arrays denoted by Roman letters appear in bold type, their components in plain type; caps denote rvs:

\[
\begin{align*}
t & \equiv (t_1, \ldots, t_I)' \in \mathbb{R}^I, \\
y & \equiv (y_1, \ldots, y_J)' \in \{0, 1\}^J, \\
z & \equiv (z_{ij}) \in \{0, 1\}^K, \\
m & = (m_{ij}) \in \mathbb{Z}_+^K, \\
n & = (n_{ij}) \in \mathbb{Z}_+^K, \\
Y & \equiv (Y_1, \ldots, Y_J)' \in \{0, 1\}^J, \\
Z & \equiv (Z_{ij}) \in \{0, 1\}^K,
\end{align*}
\]

where \( \mathbb{Z}_+ \) is the set of nonnegative integers. Sums and products will range over the index sets \( I \) and \( J \) unless otherwise specified, e.g.,

\[
\sum_i = \sum_{i=1}^I, \quad \prod_j = \prod_{j=1}^J, \quad \sum_{i,j} = \sum_{i=1}^I \sum_{j=1}^J.
\]

etc. Summation over one or neither of the indices \( i, j \) involving \( m_{ij}, n_{ij}, z_{ij}, \) or their random (capitalized) versions will be indicated by simply dropping the indices that are summed over, e.g.,

\[
\begin{align*}
m_i & = \sum_j m_{ij}, \\
m & = \sum_{i,j} m_{ij}, \\
N_j & = \sum_i N_{ij}, \\
n & = \sum_{i,j} n_{ij}.
\end{align*}
\]

2. **First preliminary problem: \( Y \) and \( M \) observed.**

Because \( Y \) is observed, the sets

\[
\begin{align*}
S & \equiv S(Y) : = \{ j \in J \mid Y_j = 1 \}, \\
T & \equiv T(Y) : = \{ j \in J \mid Y_j = 0 \},
\end{align*}
\]

are known, where \( S \cup T = J \). Therefore the ratio

\[
r : = r(Y) = |S|/J
\]

also is known, with \( 1 - r = |T|/J \). The conditional probability mass function
(pmf) of \( M \equiv \{M_{ij}\} \) given \( Y \) is

\[
f_{\mu,\nu}(m \mid y) = \prod_{i,j} e^{-t_i \theta_j^\lambda (t_i \theta_j^\lambda)^{m_{ij}}} / m_{ij}! \\
= \left( \prod_{i} \prod_{j \in S} e^{-t_i \mu_j^{m_{ij}}} \right) \left( \prod_{i} \prod_{j \in T} e^{-t_i \nu_j^{m_{ij}}} \right) \cdot \Xi_t(m) \\
= e^{-K \bar{r} \mu_j^{m_{S}}} \cdot e^{-K \bar{r} \nu_j^{m_{T}}} \cdot \Xi_t(m) \\
(7) \\
= e^{-K \bar{r} \mu_j^{m_{S}}} \cdot e^{-K \bar{r} \nu_j^{m_{T}}} \cdot \Xi_t(m),
\]

where \( \bar{t} = \frac{1}{T} \sum_{i} t_i \),

\[
m_S = \sum_{j \in S} m_j,
\]

\[
m_T = \sum_{j \in T} m_j,
\]

\[
\Xi_t(m) = (\prod_{i} t_i^{m_i}) \left( \prod_{i,j} m_{ij}! \right)^{-1}
\]

Thus, conditional on \( Y \), \((M_S, M_T)\) is a sufficient statistic for \((\mu, \nu)\), where

\[
M_S \equiv \sum_{j \in S} M_j \sim \text{Poisson}(K \bar{r} \mu),
\]

\[
M_T \equiv \sum_{j \in T} M_j \sim \text{Poisson}(K \bar{r}(1-r) \nu),
\]

with \( M_S \) and \( M_T \) independent.

**2.1. Frequentist analysis.** From (7), the MLEs of \( \mu, \nu, \) and \( \theta \equiv \mu / \nu \) are

\[
\hat{\mu} = \frac{M_S}{K \bar{r}},
\]

\[
\hat{\nu} = \frac{M_T}{K \bar{r}(1-r)},
\]

\[
\hat{\theta} = \frac{(1-r)M_S}{r M_T}.
\]

Based on \( \hat{\theta} \), approximate \((1 - \alpha)\) confidence intervals for \( \theta \) can be developed in several ways,\(^2\) two of which are presented here.

\(^2\)See Li, Tang, and Wong (2014) and references therein.
First, because \( \log \hat{\theta} = \log \hat{\mu} - \log \hat{\nu} \), the familiar normal approximation and propagation of error method shows that for large \( K \),
\[
\sqrt{K}(\hat{\mu} - \mu) \approx N(0, \frac{\mu}{r \nu (1-r)}),
\]
\[
\sqrt{K}(\hat{\nu} - \nu) \approx N(0, \frac{\nu}{r \mu (1-r)}),
\]
\[
\sqrt{K}(\log \hat{\mu} - \log \mu) \approx N(0, \frac{1}{r \nu (1-r)}),
\]
\[
\sqrt{K}(\log \hat{\nu} - \log \nu) \approx N(0, \frac{1}{r \mu (1-r)}),
\]
\[
\sqrt{K}(\log \hat{\theta} - \log \theta) \approx N(0, Q(\bar{t}; r; \hat{\mu}, \hat{\nu}));
\]
\[
Q(\bar{t}; r; \hat{\mu}, \hat{\nu}) := \frac{1}{r \mu} + \frac{1}{r \nu (1-r)}.
\]
Furthermore, \( \hat{\mu} \) and \( \hat{\nu} \) are consistent estimators of \( \mu \) and \( \nu \) respectively, so for sufficiently large \( K \),
\[
\sqrt{K}(\log \hat{\theta} - \log \theta) \approx N(0, Q(\bar{t}; r; \hat{\mu}, \hat{\nu})).
\]
This yields approximate \((1 - \alpha)\) confidence intervals for \( \log \theta \) and \( \theta \) given by
\[
\log \hat{\theta} \pm \sqrt{Q(\bar{t}; r; \hat{\mu}, \hat{\nu})} z_{\alpha/2},
\]
\[
\hat{\theta} e \pm \sqrt{Q(\bar{t}; r; \hat{\mu}, \hat{\nu})} z_{\alpha/2},
\]
respectively, provided that \( r \) is not close to 0 or 1.

A second way to obtain an approximate \((1 - \alpha)\) confidence interval for \( \theta \) is to consider the conditional distribution\(^4\) of \( M_S \) given \( M_S + M_T \) (and \( Y \)):
\[
[M_S \mid M_S + M_T = m] \sim \text{Binomial}(m, \eta),
\]
\[
\eta := \frac{r \mu}{r \mu + (1-r) \nu} = \frac{\theta}{\theta + (1-r)}
\]
so \( \eta \) is a strictly increasing function of \( \theta \). The conditional MLE of \( \eta \) is \( \hat{\eta} = \frac{M_S}{m} \), so if \( m \) is large\(^5\) then
\[
\sqrt{m}(\hat{\eta} - \eta) \approx N(0, \eta(1-\eta)),
\]
\[
\sqrt{m}[2 \sin^{-1}(\sqrt{\eta}) - 2 \sin^{-1}(\sqrt{\eta})] \approx N(0, 1),
\]
where \( \sin^{-1}(\sqrt{\eta}) \) is the well-known arcsine variance-stabilizing transformation\(^6\) for the binomial distribution. This yields approximate (conditional)

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\(^3\)\( z_{\alpha/2} \) is the upper \((1 - \frac{\alpha}{2})\)-quantile of the standard normal distribution \( N(0, 1) \).

\(^4\)Note that this conditional distribution does not depend on \( t_1, \ldots, t_I \).

\(^5\)Note that \( E(m) = E(M_S + M_T) = K\bar{t}r\theta + (1-r)\lambda \).

\(^6\)Refinements of this variance-stabilizing transformation are given by Guan (2009).
A nonzero scale parameter can be reduced to 1 simply by re-scaling $t_1, \ldots, t_J$. 

However, Laurent and Lagrand (2012, p.376) state that these conditional confidence intervals may be too conservative.
It follows from (10) that after integrating over \(\lambda\), \((M_S, M_T)\) is a sufficient statistic for \(\theta\) (still requiring that \(Y\) is known). For fixed \(\theta\) and \(\lambda\),

\[
\begin{align*}
[M_S | Y; \theta, \lambda] & \sim \text{Poisson}(K\tilde{r}\theta\lambda), \\
[M_T | Y; \theta, \lambda] & \sim \text{Poisson}(K\tilde{t}(1-r)\lambda),
\end{align*}
\]

with \(M_S\) and \(M_T\) independent, so the joint pmf of \((M_S, M_T)\) given \(Y = y\) is

\[
f(m_S, m_T | y; \theta, \lambda) = e^{-K\tilde{r}\theta\lambda}(K\tilde{r}\theta\lambda)^{m_S}/m_S! \cdot e^{-K\tilde{t}(1-r)\lambda}(K\tilde{t}(1-r)\lambda)^{m_T}/m_T! \\
= e^{-K\tilde{r}[r\theta+(1-r)]\lambda} \cdot (K\tilde{r})^{m_S}(1-r)^{m_T}/m_S!m_T!
\]

for \((m_S, m_T) \in \mathbb{Z}_+^2\). Thus the integrated pmf of \((M_S, M_T)\) given \(Y = y\) is

\[
f_\delta(m_S, m_T | y; \theta) = \int_0^\infty f(m_S, m_T | y; \theta, \lambda)\gamma_\delta(\lambda) d\lambda \\
= \frac{\Gamma(m_S + m_T + \delta)}{\Gamma(\delta)m_S!m_T!} \{K\tilde{r}[r\theta+(1-r)] + 1\}^{m_S+m_T+\delta}.
\]

similar to (10).

It may interest some to note that (13) can be expressed as a generalized bivariate negative binomial pmf:

\[
f_\delta(m_S, m_T | y; \theta) = \frac{\Gamma(m_S + m_T + \delta)}{\Gamma(\delta)m_S!m_T!} a(\theta)^{m_S} b(\theta)^{m_T} (1 - a(\theta) - b(\theta))^\delta,
\]

\[
a(\theta) : = \frac{K\tilde{r}\theta}{K\tilde{r}\theta + K\tilde{t}(1-r) + 1},
\]

\[
b(\theta) : = \frac{K\tilde{t}(1-r)}{K\tilde{r}\theta + K\tilde{t}(1-r) + 1}.
\]

From here there are two paths for inference about \(\theta\):

(i) Continue on the Bayesian path and impose a prior distribution\(^9\) on \(\theta\), from that obtain its posterior distribution; or

(ii) Carry out frequentist inference about \(\theta\) based on (10) (or (13)).

\(^9\)Under this approach, \(\theta\) is independent of \(\lambda\) a priori, which conforms to the assumption that \(\lambda\) is a nuisance parameter. Alternatively, some have proposed to treat \(\mu\) and \(\nu\) as independent a priori and impose separate priors on each. Under the latter approach, however, \(\theta = \mu/\nu = \mu/\lambda\) so \(\theta\) depends on \(\lambda\), hence \(\lambda\) is no longer a nuisance parameter.
We shall follow both paths in turn.

(i) A conjugate family of prior pdfs for $\theta$ is apparent from (10) and (13):

\[ \phi_{\alpha,\beta; K\bar{t},r}(\theta) = c(\alpha, \beta; K\bar{t}, r) \cdot \frac{\theta^{\alpha-1}}{(K\bar{t}[r\theta + (1-r)] + 1)^{\alpha+\beta}}, \quad 0 < \theta < \infty; \]

\[ (17) \]

\[ c(\alpha, \beta; K\bar{t}, r)) = \frac{\Gamma(\alpha + \beta)(Kt\bar{r})^\alpha[Kt(1-r)]^\beta}{\Gamma(\alpha)\Gamma(\beta)}. \]

where $\alpha, \beta > 0$. These are essentially $F$-densities and require that $r$ is known, which holds in the present case that $Y$ is observed. The prior mean is finite if $\beta > 1$ and is given by

\[ E_{\alpha,\beta;r}[\theta] = \frac{c(\alpha, \beta; K\bar{t}, r)}{c(\alpha + 1, \beta - 1; K\bar{t}, r)} = \frac{(1-r)\alpha}{r(\beta - 1)}. \]

\[ (19) \]

From (10) and (17), the posterior density of $\theta$ given $M = m$ is

\[ f_{\delta,\alpha,\beta}(\theta | y, m) \propto f_{\delta}(m | y; \theta)\phi_{\alpha,\beta; K\bar{t},r}(\theta) \]

\[ (20) \]

\[ \propto \frac{(m_S + \alpha)(m_T + \beta + \delta)^{m_S + m_T + \alpha + \beta + \delta}}{(K\bar{t}[r\theta + (1-r)] + 1)^{m_S + m_T + \alpha + \beta + \delta}}, \]

hence

\[ f_{\delta,\alpha,\beta}(\theta | y, m) = \phi_{m_S + \alpha, m_T + \beta + \delta; K\bar{t},r}(\theta). \]

\[ (21) \]

In particular, the Bayes estimator of $\theta$ is given by the posterior mean

\[ E_{m_S + \alpha, m_T + \beta + \delta;r}[\theta | y, m] = \frac{(1-r)(m_S + \alpha)}{r(m_T + \beta + \delta - 1)}. \]

\[ (23) \]

if $m_T + \beta + \delta > 1$. Bayesian posterior confidence intervals for $\theta$ can be obtained from (22).

(ii) From (10), after integrating w.r.to $\gamma_{\delta}(\lambda)$, the conditional log likelihood function of $M$ given $Y$ has the form

\[ \log f_{\delta}(m | y; \theta) = m_S \log \theta - (m_S + m_T + \delta) \log \{K\bar{t}[r\theta + (1-r)] + 1\} + h(m), \]
so the conditional maximum integrated likelihood estimator (MILE) of $\theta$ is

$$
\hat{\theta}_\delta(m \mid y) = \frac{[K\bar{t}(1-r) + 1]m_S}{Ktr(m_T + \delta)},
$$

which resembles (23). Theorem 2 of Fahrmeir (1987) applies to show that if $I_{M,\delta}(\theta \mid y)$ is large\(^{10}\) then

$$
\sqrt{K}[\hat{\theta}_\delta(m \mid y) - \theta] \approx N[0, KI_{M,\delta}^{-1}(\theta \mid y)],
$$

where $I_{M,\delta}(\theta \mid y)$ is the expected conditional information number

$$
I_{M,\delta}(\theta \mid y) = -E_\delta \left[ \frac{\partial^2}{\partial \theta^2} \log f_\delta(M \mid y; \theta) \mid y; \theta \right]
= E_\delta \left\{ \frac{M_S}{\theta^2} - \frac{(K\bar{t}r)^2(M_S + M_T + \delta)}{Ktr[\theta + (1-r)] + 1} \right\}
= \delta K\bar{t}r \left\{ \frac{1}{\theta} - \frac{K\bar{t}r}{Ktr[\theta + (1-r)] + 1} \right\}
= \frac{\delta K\bar{t}r[\bar{t}(1-r) + 1]}{\theta[Ktr[\theta + (1-r)] + 1]},
$$

$$
I_{M,\delta}^{-1}(\theta \mid y) = \frac{\theta}{\delta} \left[ \frac{\theta}{K\bar{t}(1-r) + 1} + \frac{1}{K\bar{t}r} \right].
$$

Here we used the facts that

$$
E_\delta(\lambda) = \delta; \quad E_\delta[M_S \mid y; \theta] = E_\delta \{E[M_S \mid y; \theta, \lambda] \mid y; \theta \}
= E_\delta \left\{ \sum_i \sum_{j \in S} m_{ij} \mid y; \theta, \lambda \right\}
= E_\delta \left\{ \sum_i \sum_{j \in S} t_i \theta \lambda \mid y; \theta \right\}
= K\bar{t}r\theta\delta;
$$

$$
E_\delta[M_T \mid y; \theta] = K\bar{t}(1-r)\delta.
$$

We note from (26) that $I_{M,\delta}(\theta \mid y)$ will be large if $K$ is large and $r$ is bounded away from 0 and 1. In this case (25) and (27) yield an approximate

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\(^{10}\)See condition (D) of Fahrmeir (1987), p.89.
(1 − α)-confidence interval for θ:

\[ \hat{\theta}_\delta \pm \frac{\hat{\theta}_\delta}{\delta} \left[ \frac{1}{Kt(1 - r) + 1} + \frac{1}{Ktr} \right] z_{\alpha/2}. \]

3. Second preliminary problem: Y unobserved, M observed.

Because Y is unobserved, i.e. missing, S, T, r, (1 − r), MS, and MT are unknown. Here \( M_{ij} \) is a \( \pi \)-mixture of \( \text{Poisson}(t_i\mu) \) and \( \text{Poisson}(t_i\nu) \) rvs, where \( \pi \) is the unknown mixing probability, cf. (2). Thus the (unconditional) pmf of the observed data array \( M \equiv (M_{ij}) \) is

\[
f_{\pi,\mu,\nu}(m) = \prod_{i,j} \left[ \pi e^{-t_i\mu} (t_i\mu)^{m_{ij}} + (1 - \pi) e^{-t_i\nu} (t_i\nu)^{m_{ij}} \right]/m_{ij}!
\]

\[
= \prod_{i,j} \left[ \pi e^{-t_i\mu} \mu^{m_{ij}} + (1 - \pi) e^{-t_i\nu} \nu^{m_{ij}} \right] \cdot \Xi_t(m).
\]

Note that the \( K \equiv IJ \) rvs \( M_{ij} \) are independent but non-identically distributed (ind) if \( t_1, \ldots, t_I \) are non-identical. The joint pmf of the complete (unobserved and observed) data \( (Y, M) \) is given by

\[
f_{\pi,\mu,\nu}(y, m) = f_\pi(y)f_{\mu,\nu}(m \mid y)
\]

\[
= \prod_j \pi^{y_j}(1 - \pi)^{1 - y_j} \cdot \prod_{i,j} \left[ e^{-t_i\mu} \mu^{m_{ij}} \right]^{y_j} \left[ e^{-t_i\nu} \nu^{m_{ij}} \right]^{1 - y_j} \cdot \Xi_t(m)
\]

\[
= \left[ \pi^{\bar{y}}(1 - \pi)^{1 - \bar{y}} \right]^J \cdot \prod_j \left[ \left( e^{-t_i\mu} \mu^{m_{ij}} \right)^{y_j} \left( e^{-t_i\nu} \nu^{m_{ij}} \right)^{1 - y_j} \right] \cdot \Xi_t(m),
\]

where \( \bar{y} = \frac{1}{J} \sum_j y_j \),

\[
\bar{m}_{\bar{y}} = \frac{1}{K} \sum_j m_{ij} y_j,
\]

\[
\bar{m}(1 - \bar{y}) = \frac{1}{K} \sum_j m_{ij}(1 - y_j) = \bar{m} - \bar{m}_{\bar{y}},
\]

\[
\bar{m} = \frac{1}{K} \sum_{i,j} m_{ij} = \frac{m}{K}.
\]
Thus $f_{\pi,\mu,\nu}(y, m)$ determines an exponential family with sufficient statistic $(\bar{Y}, \bar{M}Y, M(1-Y))$, where these are defined similarly to $(\bar{y}, \bar{my}, m(1-y))$.

**Identifiability:** In the mixture model determined by $f_{\pi,\mu,\nu}$ in (29), the parameters $\pi, \mu, \nu$ are not fully identifiable, since $f_{\pi,\mu,\nu} = f_{1-\pi,\nu,\mu}$. Thus, without further specification it is impossible to distinguish between $\theta$ and $1/\theta$. Equivalently, $|\log \theta|$ can be estimated but not $\log \theta$.

To deal with this, a restriction on the parametrization must be imposed. Often it is assumed that $\mu$ and $\nu$ are ordered, e.g. $\mu < \nu$ which is equivalent to $\theta < 1$, but this is inappropriate here. Instead we impose the restriction $\pi \leq 1/2$, which corresponds to the assumption that $\theta_j = \theta$ occurs less frequently than $\theta_j = 1$.

### 3.1. Frequentist analysis: The EM algorithm.

To obtain the MLEs $\hat{\pi}, \hat{\mu}, \hat{\nu}$ and thus $\hat{\theta} = \hat{\mu}/\hat{\nu}$, it is straightforward to apply the EM algorithm (cf. McLachlan and Krishnan (2008)) as follows:

For $j = 1, \ldots, J$ define

$$M_j = \{M_{i,j} \mid i = 1, \ldots, I\},$$

$$m_j = \{m_{i,j} \mid i = 1, \ldots, I\}.$$  

Because (32) is an exponential family, for $l = 0, 1, \ldots$, the $(l+1)$-st E-step simply imputes $y_j$ to be

$$\begin{align*}
(\hat{y}_j)_{l+1} &= \mathbb{E}_{\hat{\pi}_l, \hat{\mu}_l, \hat{\nu}_l}[Y_j \mid M = m] \\
&= \mathbb{P}_{\hat{\pi}_l, \hat{\mu}_l, \hat{\nu}_l}[Y_j = 1 \mid M_j = m_j] \\
&= \frac{\hat{\pi}_l \prod_i e^{-t_i \hat{\mu}_l (t_i \hat{\mu}_l)^{m_{ij}}}}{\hat{\pi}_l \prod_i e^{-t_i \hat{\mu}_l (t_i \hat{\mu}_l)^{m_{ij}}} + (1 - \hat{\pi}_l) \prod_i e^{-t_i \hat{\nu}_l (t_i \hat{\nu}_l)^{m_{ij}}} + (1 - \hat{\pi}_l) \prod_i e^{-t_i \hat{\nu}_l (t_i \hat{\nu}_l)^{m_{ij}}} + (1 - \hat{\pi}_l) \prod_i e^{-t_i \hat{\nu}_l (t_i \hat{\nu}_l)^{m_{ij}}}} \\
&= \frac{\hat{\pi}_l}{\hat{\pi}_l + (1 - \hat{\pi}_l) e^{-H(\hat{\nu}_l - \hat{\mu}_l) (\hat{\nu}_l / \hat{\mu}_l)^{m_{ij}}}}.
\end{align*}$$

by Bayes formula. From (32), the complete-data MLEs are found to be

$$\hat{\pi} = \bar{y},$$

$$\hat{\mu} = \frac{\bar{my}}{\bar{y}},$$

$$\hat{\nu} = \frac{m(1-y)}{\bar{t}(1-y)}.$$
Thus the \((l+1)\)-st M-step yields the updated estimates

\[
\hat{\pi}_{l+1} = \frac{1}{J} \sum_j (\hat{y}_j)_{l+1},
\]

(37)

\[
\hat{\mu}_{l+1} = \frac{m(\hat{y})_{l+1}}{\hat{t} (\hat{y})_{l+1}},
\]

(38)

\[
\hat{\nu}_{l+1} = \frac{m(1-\hat{y})_{l+1}}{\hat{t} (1-\hat{y})_{l+1}} = \frac{\bar{m} - m(\hat{y})_{l+1}}{\hat{t} - \hat{t} (\hat{y})_{l+1}},
\]

(39)

where \((\hat{y})_{l+1} = ((\hat{y}_1)_{l+1}, \ldots, (\hat{y}_J)_{l+1})\). If at any stage \(\hat{\pi}_{l+1}\) exceeds 1/2, replace it by 1/2.

Various improvements to the EM algorithm have been suggested to increase its speed of convergence, etc. See McLachlan and Krishnan (2008) for a thorough survey.

Finally, from (38) and (39) we obtain the following updated estimator of \(\theta\) (which does not depend on \(\hat{t}\)):

\[
\hat{\theta}_{l+1} = \frac{\hat{\mu}_{l+1}}{\hat{\nu}_{l+1}} = \frac{m(\hat{y})_{l+1}}{\hat{t} (\hat{y})_{l+1}} \frac{1 - (\hat{y})_{l+1}}{\bar{m} - m(\hat{y})_{l+1}} = \frac{1}{m(\hat{y})_{l+1}} - 1.
\]

(40)

Starting value \(\hat{\pi}_0\) for the EM algorithm: Under the restriction \(\pi \leq 1/2\), a simple way to choose \(\hat{\pi}_0\) is as follows. Plot a histogram of the entire data set \(\{m_{ij}\}\) and attempt to discern two prevalent mixture components, either by eye or by density estimation (cf. Silverman (1986)), then determine their relative weights. Take \(\hat{\pi}_0\) to be the lesser of these weights.

Standard error for the MLE \(\hat{\theta}\): For simplicity of notation, set \(\omega = (\pi, \mu, \nu)\) and \(\hat{\omega}_l = (\hat{\pi}_l, \hat{\mu}_l, \hat{\nu}_l)\). Assume that the EM iterates \(\omega_l\) converge to \(\hat{\omega} \equiv (\hat{\pi}, \hat{\mu}, \hat{\nu})\), the actual MLEs based on the observed data \(M\). Then if \(K \equiv IJ\) is large, it follows from Theorem 2 of Hoadley (1971) that

\[
\sqrt{K}(\hat{\omega} - \omega) \approx N_3[0, K I_M^{-1}(\omega)],
\]

(41)

where, with \(f_\omega(m)\) given by (29),

\[
I_M(\omega) \equiv -E_\omega[\nabla^2 f_\omega(M)]
\]

(42)
is the total expected information matrix \((3 \times 3)\) for the sample \(M\).\(^{11}\)

However, as noted by Efron and Hinkley (1978) and Louis (1982), observed information \(I_m(\omega)\) usually yields a better normal approximation and often is more readily computed than expected information, so we replace (41) and (42) by

\[
\sqrt{K(\hat{\omega} - \omega)} \approx N_3[0, KI_m^{-1}(\omega)],
\]

\[
I_m(\omega) \equiv -\nabla^2_\omega \log f_\omega(m) = -E_\omega[\nabla^2_\omega \log f_\omega(m) | M = m]
\]

(43)

\[
I_m(\omega) \equiv -\nabla^2_\omega \log f_\omega(Y, m) | M = m = E_\omega[\nabla^2_\omega \log f_\omega(Y | m) | M = m].
\]

From (32),

\[
\log f_\omega(Y, m) = J[\bar{Y} \log \pi + (1 - \bar{Y}) \log(1 - \pi)] + K[mY \log \mu - \bar{Y} \mu + m(1 - \bar{Y}) \log \nu - \bar{Y}(1 - \bar{Y}) \nu] + h_t(m);
\]

\[
\nabla_\omega \log f_\omega(Y, m) = \begin{pmatrix}
\frac{J(Y - \pi)}{\pi(1 - \pi)} \\
K \left[ \frac{mY}{\mu} \right] - \bar{Y} \\
K \left[ \frac{m(1 - Y)}{\nu} \right] - \bar{Y}
\end{pmatrix};
\]

(44)

\[
-\nabla^2_\omega \log f_\omega(Y, m) = \begin{pmatrix}
\frac{J[(1 - 2\pi)Y + \pi^2]}{\pi^2(1 - \pi)^2} & 0 & 0 \\
0 & K \left[ \frac{mY}{\mu^2} \right] & 0 \\
0 & 0 & K \left[ \frac{m(1 - Y)}{\nu^2} \right]
\end{pmatrix};
\]

where \(\bar{mY}\) and \(\bar{m(1 - Y)}\) are defined similarly to (33) and \(h_t(m)\) does not depend on \(\omega\).

Furthermore by (30), for fixed \(m\),

\[
f_\omega(y | m) = f_\omega(y, m) / f_\omega(m) \propto \prod_j \left( \pi e^{-\mu \mu^m_j} \right)^{y_j} \left( 1 - \pi \right) e^{-\nu \nu^m_j} \right)^{1 - y_j},
\]

\(^{11}\)For any smooth function \(g \equiv g(\omega)\), the gradient \(\nabla_\omega g\) is the \(3 \times 1\) vector \((\frac{\partial g}{\partial \pi}, \frac{\partial g}{\partial \mu}, \frac{\partial g}{\partial \nu})^t\)
and the Hessian \(\nabla^2_\omega g\) is the \(3 \times 3\) matrix of mixed partial derivatives \(\frac{\partial^2 g}{\partial \pi \partial \pi}, \frac{\partial^2 g}{\partial \pi \partial \mu}, \ldots, \frac{\partial^2 g}{\partial \nu \partial \nu}\).
hence $Y_1, \ldots, Y_J$ are conditionally independent given $M = m$ with

\begin{equation}
[Y_j \mid M = m] \sim \text{Bernoulli}(p_j),
\end{equation}

\begin{equation}
p_j \equiv p_j(\omega; \bar{t}; \bar{m}_j) = \frac{\pi e^{-t\mu_m^j}}{\pi e^{-t\mu_m^j} + (1 - \pi)e^{-t\nu_m^j}},
\end{equation}

where $\bar{m}_j = \frac{1}{J}m_j$. From (46),

\begin{align*}
E_\omega[\bar{Y} \mid M = m] &= \bar{p}; \\
E_\omega[\bar{m}Y \mid M = m] &= \bar{mp}; \\
E_\omega[\bar{m}(1 - Y) \mid M = m] &= \bar{m}(1 - p);
\end{align*}

where $\bar{mp}$ and $\bar{m}(1 - p)$ are defined similarly to (33)-(34) and

\begin{align*}
\bar{p} &= \frac{1}{J} \sum_j p_j, \\
\bar{p}(1 - p) &= \frac{1}{J} \sum_j p_j(1 - p).
\end{align*}

Furthermore,

\begin{align*}
E_\omega[(1 - 2\pi)\bar{Y} + \pi^2 \mid M = m] &= (1 - 2\pi)\bar{p} + \pi^2.
\end{align*}

Thus from (44), the first term in (43) is given by

\begin{equation}
-E_\omega[\nabla_\omega^2 \log f_\omega(Y, m) \mid M = m]
\end{equation}

\begin{equation}
= \begin{pmatrix}
\frac{J[(1-2\pi)\bar{p} + \pi^2]}{\pi^2(1-\pi)^2} & 0 & 0 \\
0 & K\left[\frac{mp}{\mu^2}\right] & 0 \\
0 & 0 & K\left[\frac{m(1-p)}{\nu^2}\right]
\end{pmatrix} =: D(\omega; \bar{t}; m).
\end{equation}
The second term in (43) is obtained as follows: From (46),
\[
f_\omega(Y \mid m) = \prod_j p_j^{Y_j}(1 - p_j)^{1 - Y_j};
\]
\[
\log f_\omega(Y \mid m) = \sum_j Y_j \log p_j + (1 - Y_j) \log(1 - p_j);
\]
\[
\nabla_\omega \log f_\omega(Y \mid m) = \sum_j \frac{(Y_j - p_j)}{p_j(1 - p_j)} \nabla_\omega p_j;
\]
\[
\nabla_\omega^2 \log f_\omega(Y \mid m) = \sum_j \left[ \frac{(Y_j - p_j)\nabla^2 p_j - (\nabla_\omega p_j)(\nabla_\omega p_j)'}{p_j(1 - p_j)} - \frac{(Y_j - p_j)(\nabla_\omega p_j)(\nabla_\omega p_j(1 - p_j))'}{p_j'(1 - p_j)^2} \right];
\]
\[
E_\omega[\nabla_\omega^2 \log f_\omega(Y \mid m) \mid M = m] = -\sum_j \frac{(\nabla_\omega p_j)(\nabla_\omega p_j)'}{p_j(1 - p_j)} = \sum_j (\nabla_\omega \log p_j)(\nabla_\omega \log(1 - p_j))'.
\]

From (47),
\[
\log p_j = \log \pi - \tilde{t}_\mu + I\tilde{m}_\mu \log(1 - \mu) - \log \gamma_j,
\]
\[
\gamma_j \equiv \gamma_j(\omega; \tilde{t}; \tilde{m}_j) := \pi(e^{-t_\mu \mu \tilde{m}_j})' + (1 - \pi)(e^{-\tilde{t}_\nu \nu \tilde{m}_j})',
\]
from which it can be shown that
\[
\nabla_\omega \log p_j = \left(\frac{e^{-\tilde{t}_\nu \nu \tilde{m}_j}}{\gamma_j} \frac{1}{\pi}, (1 - \pi)\left(\frac{\tilde{m}_j}{\mu} - \tilde{t}\right)I, -(1 - \pi)\left(\frac{\tilde{m}_j}{\nu} - \tilde{t}\right)I\right)';
\]
\[
\nabla_\omega \log(1 - p_j) = -\frac{p_j}{1 - p_j} \nabla_\omega \log p_j
\]
\[
= -\frac{\pi}{1 - \pi} \left( e^{i(\nu - \mu)} \left(\frac{\mu}{\nu}\right)^{\tilde{m}_j} \right)' \nabla_\omega \log p_j
\]
\[
= -\frac{(e^{-t_\mu \mu \tilde{m}_j})'}{(1 - \pi)\pi} \left( \frac{1}{\mu}, \pi\left(\frac{\tilde{m}_j}{\mu} - \tilde{t}\right)I, -\pi\left(\frac{\tilde{m}_j}{\nu} - \tilde{t}\right)I\right)';
\]
\[
(\nabla_\omega \log p_j)[\nabla_\omega \log(1 - p_j)]' = -\frac{(e^{-\tilde{t}(\mu + \nu \nu \mu) \tilde{m}_j})^{\gamma_j}}{\gamma_j^2} \delta_j \delta_j',
\]
\[
\delta_j \equiv \delta_j(\omega; \tilde{t}; m_j) := \left( \frac{1}{\sqrt{\pi(1 - \pi)}}, I\sqrt{\pi(1 - \pi)}\left(\frac{\tilde{m}_j}{\mu} - \tilde{t}\right), -I\sqrt{\pi(1 - \pi)}\left(\frac{\tilde{m}_j}{\nu} - \tilde{t}\right)\right)'.
\]
Therefore
\[
E_{\omega}[\nabla_{\omega} \log f_\omega(Y \mid m) \mid M = m] = - \sum_j \frac{e^{-t(\mu+\nu)}(\mu\nu)^{\bar{m}_j}}{\gamma_j^2} \delta_j \delta_j'.
\]
Thus by (43), (48), and (49), the observed information matrix is
\[
I_m(\omega) = D(\omega; \bar{t}; \bar{m}) - e^{-t(\mu+\nu)} \Delta(\omega; \bar{t}; \bar{m})' \Delta(\omega; \bar{t}; \bar{m});
\]
\[
\Delta(\omega; \bar{t}; \bar{m}) := \left( \frac{(\mu\nu)^{\bar{m}_j/2}}{\gamma_1} \delta_1, \ldots, \frac{(\mu\nu)^{\bar{m}_j/2}}{\gamma_j} \delta_j \right) .
\]
Now estimate \( I_m(\omega) \) in the normal approximation
\[
\sqrt{K} (\hat{\omega} - \omega) \approx N_3[0, KI_m^{-1}(\omega)]
\]
by replacing \( \omega \) in \( I_m(\omega) \) by its MLE \( \hat{\omega} \equiv (\hat{\pi}, \hat{\mu}, \hat{\nu}) \) to obtain
\[
\sqrt{K} (\hat{\omega} - \omega) \approx N_3[0, KI_m^{-1}(\hat{\omega})].
\]
This requires replacing \( \pi, \mu, \nu \) by \( \hat{\pi}, \hat{\mu}, \hat{\nu} \) wherever the former three appear in the entries of \( I_m(\omega) \), including in \( p_j, \delta_j, \) and \( \gamma_j \). For large \( K \) the \( 3 \times 3 \) matrix \( I_m(\hat{\omega}) \) is positive definite, hence invertible.

Lastly, an approximate confidence interval for \( \theta \equiv \mu/\nu \equiv g(\omega) \) is obtained from (51) via propagation of error: for \( \hat{\theta} = \hat{\mu}/\hat{\nu} \),
\[
\sqrt{K}(\hat{\theta} - \theta) \approx N[0, L(\nabla_\omega g(\omega) | \omega)' I_m^{-1}(\hat{\omega}) \nabla_\omega g(\hat{\omega}) | \hat{\omega}]]
\]
\[
= N \left[ 0, K \left( \frac{\partial g}{\partial \pi} | \hat{\omega} , \frac{\partial g}{\partial \mu} | \hat{\omega} , \frac{\partial g}{\partial \nu} | \hat{\omega} \right) I_m^{-1}(\hat{\omega}) \left( \frac{\partial g}{\partial \pi} | \hat{\omega} , \frac{\partial g}{\partial \mu} | \hat{\omega} , \frac{\partial g}{\partial \nu} | \hat{\omega} \right)' \right]
\]
\[
= N \left[ 0, K \left( \frac{1}{\hat{\nu}}, \frac{-\hat{\mu}}{\hat{\nu}^2} \right) I_m^{-1}(\hat{\omega}) \left( 0, \frac{1}{\hat{\nu}}, \frac{-\hat{\mu}}{\hat{\nu}^2} \right)' \right]
\]
\[
= N \left[ 0, K \left( \frac{1}{\hat{\nu}}, \frac{-\hat{\mu}}{\hat{\nu}^2} \right) (I_{22} - I_{21}I_{11}^{-1}I_{12})^{-1} \left( \frac{1}{\hat{\nu}}, \frac{-\hat{\mu}}{\hat{\nu}^2} \right)' \right]
\]
\[
\equiv N(0, \hat{\sigma}^2).
\]
where \( I_m(\hat{\omega}) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \) with \( I_{11} : 1 \times 1 \) and \( I_{22} : 2 \times 2 \). Thus computation of \( \hat{\sigma}^2 \) only requires the inversion of a \( 2 \times 2 \) matrix. This yields the following approximate \( (1 - \alpha) \) confidence interval for \( \theta \):
\[
\hat{\theta} \pm \frac{\hat{\sigma}}{\sqrt{n}} z_{\alpha/2}.
\]
3.2. Bayesian analysis. Rewrite the joint pmf (32) of the complete (unobserved and observed) data \((Y, M)\) in terms of \(\pi, \theta, \lambda\) as follows:

\[
f(y, m | \pi, \lambda, \theta) = \left[ \pi^\gamma (1 - \pi)^{1 - \gamma} \right]^J e^{-\{Kti[\gamma + (1 - \gamma)]\} \lambda m \theta^{Kmy}} \cdot \Xi_t(m),
\]

since \(K[\gamma y + m(1 - y)] = K \tilde{m} = m\). If in addition to the gamma prior density \(\gamma_\delta(\lambda)\) for \(\lambda\) we assume any proper prior density \(\vartheta(\pi)\) for \(\pi \in (0, \frac{1}{2}]\), then the integrated joint pmf of \((Y, M)\) is

\[
f_{\theta, \delta}(y, m | \theta) = \int_0^{1/2} \int_0^\infty f(y, m | \pi, \lambda, \theta) \vartheta(\pi) \gamma_\delta(\lambda) d\pi d\lambda
\]

\[
= g_\theta(J\bar{y}) \cdot \frac{\Gamma(m + \delta) \{(\prod_i t_i^{m_i}) \theta^{Kmy}\}}{\Gamma(\delta) m! \{Kti[\gamma \theta + (1 - \bar{y})] + 1\}^{m+\delta}}.
\]

\[
\equiv f_\vartheta(y) \cdot f_\delta(m | y; \theta),
\]

where \(m = \sum_{i,j} m_{ij}\) and

\[
g_\theta(j) = \int_0^{1/2} \pi^j (1 - \pi)^{J - j} \vartheta(\pi) d\pi, \quad 0 \leq j \leq J.
\]

The integrated likelihood \(f_{\theta, \delta}(m | \theta)\) of \(M\) itself can be found explicitly:

\[
f_{\theta, \delta}(m | \theta) = \sum_{y \in T} f_{\theta, \delta}(y, m | \theta)
\]

\[
= \frac{\Gamma(m + \delta) \{(\prod_i t_i^{m_i}) \theta^{Kmy}\}}{\Gamma(\delta) m!} \sum_{j=0}^J \sum_{\{y | \bar{y} = j\}} \frac{g_\theta(J\bar{y}) \theta^{Kmy}}{\{Kti[\gamma \theta + (1 - \bar{y})] + 1\}^{m+\delta}}
\]

\[
= \frac{\Gamma(m + \delta) \{(\prod_i t_i^{m_i}) \theta^{Kmy}\}}{\Gamma(\delta) m!} \sum_{j=0}^J \frac{g_\theta(j)}{\{Kti[(\frac{j}{J}) \theta + (1 - \frac{j}{J})] + 1\}^{m+\delta}} \sum_{\{y | \sum_{j} m_{ij} = j\}} \theta^{\sum_{j} m_{ij} y_j}
\]

\[
\equiv \frac{\Gamma(m + \delta) \{(\prod_i t_i^{m_i}) \theta^{Kmy}\}}{\Gamma(\delta) m!} \sum_{j=0}^J \frac{g_\theta(j) s_j(m; \theta)}{\{Kti[(\frac{j}{J}) \theta + (1 - \frac{j}{J})] + 1\}^{m+\delta}},
\]

where \(s_j(m; \theta)\) is the \(j\)-th elementary symmetric function of \(\{\theta^{m_j} | j = 1, \ldots, J\}\):

\[
s_j(m; \theta) = \sum_{\sigma \subseteq J, \ j \in \sigma} \prod_{j \in \sigma} \theta^{m_j} = \sum_{\sigma \subseteq J, \ |\sigma| = j} \theta^{m_{\sigma}},
\]
\( m_\sigma = \sum_{j \in \sigma} m_j \), and \( s_0(\mathbf{m}; \theta) = 1 \). Thus \((M_1, \ldots, M_J)\) is a sufficient statistic for \( \theta \) but \( f_{\theta, \delta}(\mathbf{m} | \theta) \) is not an exponential family, so no conjugate prior is available. However, for any prior density \( \phi(\theta) \) the posterior pdf \( f_{\theta, \delta}(\theta | \mathbf{m}) \propto f_{\theta, \delta}(\mathbf{m} | \theta) \phi(\theta) \), which is available explicitly via (57). Thus MCMC methods (cf. Robert and Casella (2004)) can be used to obtain the corresponding Bayes estimator and posterior confidence intervals.

Alternatively, it follows from (55) and (56) that

\[
(59) \quad f_\delta(\mathbf{m} | \mathbf{y}; \theta) = \frac{\Gamma(m + \delta)(\prod_i t_i^{m_i})}{\Gamma(\delta) m!} \theta^{K \overline{m_y}}\{K\bar{y}\theta + (1 - \bar{y})\} + 1\}^{m+\delta}.
\]

Comparing (59) to (10) suggests an empirical Bayes approach where \( \phi_{\alpha, \beta; K\hat{t}, \hat{y}}(\theta) \) in (17) is used as a data-based prior density for \( \theta \). Here \( \bar{y} \) and \( \overline{m_y} \) are unobserved, but their values can be imputed via the EM algorithm described above, as follows.

The EM algorithm will output

\[
\hat{\bar{y}} := \lim_{l \to \infty} \frac{1}{J} \sum_j (\hat{y}_j)_{l+1}
\]

\[
\hat{\overline{m_y}} := \lim_{l \to \infty} \frac{1}{K} \sum_j m_j(\hat{y}_j)_{l+1},
\]

\[
\hat{m}(1 - \bar{y}) := \lim_{l \to \infty} \frac{1}{K} \sum_j m_j(1 - (\hat{y}_j)_{l+1}),
\]

(recall (33)-(34)), where \((\hat{y}_j)_{l+1}\) is given by (36)-(39). Then from (20)-(23), noting that \( m_S = K \overline{m_y} \) and \( m_T = K \overline{m(1 - y)} \), and replacing \( r \) by \( \hat{y}, 1 - r \) by \( 1 - \hat{y}, \overline{m_y} \) by \( \overline{\overline{m_y}} \), and \( \overline{m(1 - y)} \) by \( \overline{m(1 - y)} \), we obtain the empirical Bayes posterior density (compare to (22))

\[
(60) \quad f_{\delta, \alpha, \beta}(\theta | \hat{\bar{y}}, \mathbf{m}) := \phi_{K\overline{m_y} + \alpha, K\overline{m(1 - y)} + \beta + \delta; K\hat{y}, \hat{\bar{y}}}(\theta)
\]

and empirical Bayes estimator

\[
(61) \quad \hat{\theta}_{\delta, \alpha, \beta}^{EB} := \frac{(1 - \hat{\bar{y}})(K \overline{\overline{m_y}} + \alpha)}{\hat{\bar{y}} \left( K \overline{m(1 - y)} + \beta + \delta - 1 \right)}.
\]
provided that $K\bar{m}(1-y) + \beta + \delta > 1$. Empirical Bayes posterior confidence intervals for $\theta$ can be obtained from (60).

**Remark 3.1.** Taking $\alpha = \beta = 0$ leads to the prior density $\phi_{0,0,K\bar{y},\hat{y}}(\theta) = \theta^{-1}$. This is no longer data-based but is an improper prior, hence cannot reflect actual prior knowledge about $\theta$. Nonetheless, proceeding formally from (60) and (61), we obtain the posterior density

$$f_{\delta,0,0}(\theta | \hat{y}, \bar{m}) := \phi_{K\bar{m}y, K\bar{m}(1-y) + \delta; K\bar{t}, \hat{y}}(\theta),$$

which is a proper density if $K\bar{m}y > 0$, and from this the empirical Bayes estimator

$$\hat{\theta}_{\delta,0,0}^{\text{EB}} := \frac{(1 - \hat{y})(K\bar{m}y)}{\hat{y} \left( K\bar{m}(1-y) + \delta - 1 \right)},$$

valid if $K\bar{m}(1-y) + \delta > 1$; this may have desirable frequentist properties. □

**Remark 3.2.** Suppose that we wish to apply the EM algorithm directly to obtain the MILE of $\theta$ based on the integrated joint likelihood $f_{\theta,\delta}(y, m | \theta)$ in (55). Since this is not an exponential family, the E-step is nontrivial, requiring the evaluation of

$$G(\theta | \theta_0; m) := E_{\theta,\delta} \left\{ \log \left[ \frac{f_{\theta,\delta}(Y | m; \theta)}{f_{\theta,\delta}(Y | m; \theta_0)} \right] \bigg| m; \theta_0 \right\} = E_{\theta,\delta} \left\{ \log \left[ \frac{f_{\theta,\delta}(Y, m | m; \theta)}{f_{\theta,\delta}(Y, m | m; \theta_0)} \right] \bigg| m; \theta_0 \right\} - \log \left[ \frac{f_{\theta,\delta}(m | \theta)}{f_{\theta,\delta}(m | \theta_0)} \right]$$

$$= E_{\theta,\delta} \left\{ K\bar{m}Y \log \left( \frac{\theta}{\theta_0} \right) - (m + 1) \log \left\{ \frac{K\bar{t}(\bar{Y} + (1 - \bar{Y}))}{K\bar{t}(\bar{Y} + (1 - \bar{Y})) + 1} \right\} \bigg| m; \theta_0 \right\}$$

$$- \log \left[ \sum_{j=0}^{J} \frac{g_{\theta}(j)s_j(m; \theta_0)}{K\bar{t}[(\frac{1}{y})\theta + (1 - \frac{1}{y})] + 1} \bigg]^{m+1} - \sum_{j=0}^{J} \frac{g_{\theta}(j)s_j(m; \theta_0)}{K\bar{t}[(\frac{1}{y})\theta_0 + (1 - \frac{1}{y})] + 1} \bigg]^{m+1} \right\},$$

by (55) and (57), which is to be maximized over $\theta$ in the M-step. However, explicit evaluation of the conditional expectation is problematic. In such cases, approaches such as Monte Carlo simulation have been proposed; cf. McLachlan and Krishnan (2008), Debavelaere and Allassonnière (2021). □
4. Main problem: $Y$, $Z$, $M$ unobserved, $N$ observed.

Here $N_{ij}$ is a zero-inflated Poisson mixture (ZIPM) rv: $N_{ij}$ is an $\epsilon$-mixture of $M_{ij}$ and $O_{ij}$, where $O_{ij}$ is degenerate at 0, so $O_{ij} \sim \text{Poisson}(\lambda = 0)$; while $M_{ij}$ is a $\pi$-mixture of $\text{Poisson}(t_i\mu) \equiv \text{Poisson}(t_i\theta\lambda)$ and $\text{Poisson}(t_i\nu) \equiv \text{Poisson}(t_i\lambda)$ rvs. Thus this problem can be viewed as a three-component Poisson mixture model with one degenerate component and non-i.i.d observations. The three weights are $\pi \epsilon$, $(1 - \pi) \epsilon$, and $1 - \epsilon$, with the identifiability constraint $0 < \pi \leq 1/2$.

For notational simplicity, set $\omega = (\pi, \epsilon, \mu, \nu)$. Under this three-component mixture model, the unconditional pmf of the observed data $N \equiv \{N_{ij}\}$ is

$$f_\omega(n) = \prod_{i,j} \left[ \pi \epsilon e^{-t_i \mu}(t_i \mu)^{n_{ij}} + (1 - \pi) \epsilon e^{-t_i \nu}(t_i \nu)^{n_{ij}} + (1 - \epsilon) 0^{n_{ij}} \right] / n!,$$

where $0^0 = 1$ and $n! = \prod_{i,j} n_{ij}!$. Again the $K \equiv IJ$ rvs $N_{ij}$ are independent but non-identically distributed (iid) if $t_1, \ldots, t_I$ are non-identical.

The sample space of $(Y, (Z, N))$ is $\Upsilon \times \Omega$, where $\Upsilon = \{0, 1\}^J$ and

$$\Omega = \{ \{0, 1\}^K \times \{Z_+\}^K \} \cap \{ (z, n) \mid \forall i, j, z_{ij} = 0 \Rightarrow n_{ij} = 0 \} \cap \{ (z, n) \mid \forall i, j, n_{ij}(1 - z_{ij}) = 0 \} \cap \{ (z, n) \mid \prod_{i,j} 0^{n_{ij}(1 - z_{ij})} = 1 \},$$

with $Z_+$ the set of nonnegative integers. The joint pmf of the unobserved and observed data $(Y, Z, N)$ on $\Upsilon \times \Omega$ is

$$f_\omega(y, z, n) = f_\pi(y) f_\epsilon(z) f_{\mu, \nu}(n \mid y, z)$$

(65)

$$= \prod_{i,j} \pi^{y_{ij}}(1 - \pi)^{1 - y_{ij}} \prod_{i,j} \epsilon^{z_{ij}}(1 - \epsilon)^{1 - z_{ij}}$$

$$\cdot \prod_{i,j} \left[ e^{-t_i \mu(t_i \mu)^{n_{ij}}} e^{-t_i \nu(t_i \nu)^{n_{ij}}} (1 - y_{ij})^{z_{ij}} 0^{n_{ij}(1 - z_{ij})} \right] / n!$$

$$= \left[ \pi \epsilon (1 - \pi)^{1 - y} \right] J \left[ \epsilon (1 - \epsilon)^{1 - z} \right] K \left[ \frac{e^{-t_i \mu} \mu^{n_{ij}} e^{-t_i \nu} \nu^{n_{ij}}}{n!} \right]^J \prod_{i,j} \Xi_{ij} = \frac{n!}{n_{ij}(1 - z_{ij})}$$

(66)

$$= \left[ \pi \epsilon (1 - \pi)^{1 - y} \right] J \left[ \epsilon (1 - \epsilon)^{1 - z} \right] K \left[ \frac{e^{-t_i \mu} \mu^{n_{ij}} e^{-t_i \nu} \nu^{n_{ij}}}{n!} \right]^J \cdot \Xi_{ij}(z, n),$$

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where \( y = \{y_j\}, z = \{z_{ij}\}, n = \{n_{ij}\}, \)

\[
\bar{z} = \frac{1}{K} \sum_{i,j} z_{ij}, \\
\bar{t}z = \frac{1}{K} \sum_{i,j} t_i z_{ij} = \frac{1}{K} \sum_{i,j} t_i z_{ij}, \\
\bar{tyz} = \frac{1}{K} \sum_{i,j} t_i y_j z_{ij} = \frac{1}{K} \sum_{i,j} t_i y_j z_{ij},
\]

and similarly with \( y \) replaced by \( 1 - y \). To obtain (66) we have used the facts that for \( (y,z,n) \in \Upsilon \times \Omega, \)

\[
\bar{nyz} = \frac{1}{K} \sum_{i,j} n_{ij} y_j z_{ij}
= \frac{1}{K} \sum_{i,j} n_{ij} y_j \\
= \frac{1}{K} \sum_{j} n_{j} y_j \\
= \bar{ny},
\]

\[
\Xi_t(z, n) : = \prod_{i,j} t_i^{n_{ij} z_{ij}} 0^{n_{ij}(1-z_{ij})} \\
= \prod_{i,j} t_i^{n_{ij}} 0^{n_{ij}(1-z_{ij})} \\
= \prod_{i} t_i^{n_i} \cdot \prod_{i,j} 0^{n_{ij}(1-z_{ij})} \frac{n_i!}{n_i},
\]

and similarly with \( y \) replaced by \( 1 - y \). Thus \( f_\omega(y, z, n) \) determines an exponential family with support \( \Upsilon \times \Omega \) and sufficient statistic

\[
(\bar{Y}, \bar{Z}, \bar{t}Y \bar{Z}, \bar{t}(1-\bar{Y}) \bar{Z}, \bar{nY}, \bar{n(1-Y)}).
\]

### 4.1. Frequentist analysis: the EM algorithm.

To obtain the MLEs \( \hat{\epsilon}, \hat{\pi}, \hat{\mu}, \hat{\nu} \) and then \( \hat{\theta} = \hat{\mu}/\hat{\nu} \), it is again straightforward - albeit somewhat challenging, including notationally - to apply the EM algorithm, as follows:
For $i = 1, \ldots, I$ and $j = 1, \ldots, J$, define

\[
N_j = (N_{i,j} | i = 1, \ldots, I),
\]
\[
n_j = (n_{i,j} | i = 1, \ldots, I);
\]
\[
1^\#_{ij} \equiv 1^\#(n_{ij}) = 1 - 0^{n_{ij}},
\]
\[
1^\#_j \equiv 1^\#(n_j) = \sum_i 1^\#_{ij},
\]
\[
1^\# \equiv 1^\#(n) = \sum_j 1^\#_j,
\]
\[
t^\#_j \equiv t^\#(n_j) = \sum_i t_i 1^\#_{ij};
\]
\[
1^\approx_{ij} \equiv 1^\approx(n_j) = 0^{n_{ij}},
\]
\[
1^\approx_j \equiv 1^\approx(n_j) = \sum_i 1^\approx_{ij},
\]
\[
1^\approx \equiv 1^\approx(n) = \sum_j 1^\approx_j,
\]
\[
t^\approx_j \equiv t^\approx(n_j) = \sum_i t_i 1^\approx_{ij}.
\]

Here $1^\#_{ij}$ ($1^\approx_{ij}$) is the indicator function of the event $\{n_{ij} \neq 0\}$ ($\{n_{ij} = 0\}$), so $1^\#_j$ ($1^\approx_j$) is the number of nonzero (zero) $n_{ij}$ with $j$ fixed, etc. Because (65) is an exponential family, Bayes formula shows that for $l = 0, 1, \ldots$, the $(l+1)$-st E-step imputes $y_j$ as

\[
(\tilde{y}_j)_{l+1} = E_{\hat{\omega}_l}[Y_j | N = n]
\]
\[
= P_{\hat{\omega}_l}[Y_j = 1 | N_j = n_j]
\]
\[
= P_{\hat{\omega}_l}[Y_j = 1]P_{\hat{\omega}_l}[N_j = n_j | Y_j = 1]/P_{\hat{\omega}_l}[N_j = n_j]
\]
\[
= \hat{\pi}_l \prod_i [\hat{\epsilon}_l e^{-t_i \hat{\mu}_l} (t_i \hat{\mu}_l)^{n_{ij}} + (1 - \hat{\epsilon}_l)^0 n_{ij}]
\]
\[
= \hat{\pi}_l \prod_i [\hat{\epsilon}_l e^{-t_i \hat{\mu}_l} (t_i \hat{\mu}_l)^{n_{ij}} + (1 - \hat{\epsilon}_l)0^{n_{ij}}] + \hat{\pi}_l \prod_i [\hat{\epsilon}_l e^{-t_i \hat{\nu}_l} (t_i \hat{\nu}_l)^{n_{ij}} + (1 - \hat{\epsilon}_l)0^{n_{ij}}]
\]
\[
= \frac{A_{j,l}}{A_{j,l} + B_{j,l}};
\]

where $\hat{\omega}_l = (\hat{\epsilon}_l, \hat{\pi}_l, \hat{\mu}_l, \hat{\nu}_l)$ (cf. (70)-(73)) and

\[
A_{j,l} = \hat{\pi}_l \hat{\epsilon}_l^{1^\#} e^{-t_i^{1^\#} \hat{\mu}_l} (\prod_i t_i^{n_{ij}}) \hat{\mu}_l^{n_{ij}} \prod_i [\hat{\epsilon}_l e^{-t_i \hat{\mu}_l} + (1 - \hat{\epsilon}_l)]^{1^\approx_{ij}},
\]
\[
B_{j,l} = (1 - \hat{\pi}_l) \hat{\epsilon}_l^{1^\#} e^{-t_i^{1^\#} \hat{\nu}_l} (\prod_i t_i^{n_{ij}}) \hat{\nu}_l^{n_{ij}} \prod_i [\hat{\epsilon}_l e^{-t_i \hat{\nu}_l} + (1 - \hat{\epsilon}_l)]^{1^\approx_{ij}}.
\]
This simplifies to

\[(\hat{y}_j)_{t+1} = \frac{\hat{\pi}_l}{\hat{\pi}_l + (1 - \hat{\pi}_l)e^{-t_l^{(4)}(\hat{\nu}_l - \hat{\mu}_l)(\frac{\nu_l}{\mu_l})^{n_j}}} \prod_i \left[ \frac{\hat{\epsilon}_l e^{-t_l^{(4)}(1 - \hat{\epsilon}_l)}}{\hat{\epsilon}_l e^{-t_l^{(4)}(1 - \hat{\epsilon}_l)}} \right]^{1_{ij}} \],

which should be compared to (36).

Also at the \((l+1)\)-st E-step, \(z_{ij}, y_j z_{ij}\), and \((1 - y_j) z_{ij}\) are imputed as

\[E_{\omega_l}[Z_{ij} | \mathbf{N} = \mathbf{n}] = P_{\omega_l}[Z_{ij} = 1 | \mathbf{N}_{ij} = \mathbf{n}_{ij}] = P_{\omega_l}[Z_{ij} = 1]P_{\omega_l}[^{2}(N_{ij} = n_{ij} | Z_{ij} = 1)]/P_{\omega_l}[N_{ij} = n_{ij}]

= \frac{\hat{\epsilon}_l [\hat{\pi}_l e^{-t_l^{(4)}(t_l \hat{\mu}_l)^{n_{ij}}} + (1 - \hat{\pi}_l)e^{-t_l^{(4)}(t_l \hat{\nu}_l)^{n_{ij}}}] + (1 - \hat{\epsilon}_l)(1 - \hat{\pi}_l)}{\hat{\epsilon}_l [\hat{\pi}_l + (1 - \hat{\pi}_l)e^{-t_l^{(4)}(t_l \hat{\nu}_l - \hat{\mu}_l)(\frac{\nu_l}{\mu_l})^{n_{ij}}}] + (1 - \hat{\epsilon}_l)e^{-t_l^{(4)}(t_l \hat{\nu}_l - \hat{\mu}_l)(\frac{\nu_l}{\mu_l})^{n_{ij}}} 0^{n_{ij}}};

\[E_{\omega_l}[Y_j Z_{ij} | \mathbf{N} = \mathbf{n}] = P_{\omega_l}[Y_j = 1, Z_{ij} = 1 | \mathbf{N}_{j} = \mathbf{n}_{j}] = P_{\omega_l}[Y_j = 1, Z_{ij} = 1]P_{\omega_l}[N_{j} = n_{j} | Y_j = 1, Z_{ij} = 1]/P_{\omega_l}[N_{j} = n_{j}]

= \frac{\hat{\pi}_l \hat{\epsilon}_l \prod_i e^{-t_l^{(4)}(t_l \hat{\nu}_l)^{n_{ij}}}}{A_{j,l} + B_{j,l}}

= \frac{\hat{\pi}_l \hat{\epsilon}_l e^{-t_l^{(4)}\hat{\nu}_l} \prod_i \left[ \hat{\epsilon}_l e^{-t_l^{(4)}(1 - \hat{\epsilon}_l)} + (1 - \hat{\epsilon}_l) \right]^{-1_{ij}}}{\hat{\epsilon}_l^{1/4} \left\{ \hat{\pi}_l + (1 - \hat{\pi}_l)e^{-t_l^{(4)}(\hat{\nu}_l - \hat{\mu}_l)(\frac{\nu_l}{\mu_l})^{n_{ij}}} \prod_i \left[ \hat{\epsilon}_l e^{-t_l^{(4)}(1 - \hat{\epsilon}_l)} + (1 - \hat{\epsilon}_l) \right]^{-1_{ij}} \right\}^{-1_{ij}}};

\[(1 - y_j) z_{ij}]_{t+1} = (\hat{z}_{ij})_{t+1} - (y_j \hat{z}_{ij})_{t+1}.

From (66), the complete-data MLEs are found to be

\[\hat{\pi} = \bar{y},\]
\[\hat{\epsilon} = \bar{z},\]
\[\hat{\mu} = \frac{\sum_{ij} y_j z_{ij}}{\sum_{ij} y_j},\]
\[\hat{\nu} = \frac{\sum_{ij} n(1 - y_j) z_{ij}}{\sum_{ij} y_j(1 - y_j) z_{ij}}.\]
Thus the \((l+1)\)-st M-step yields the updated estimates

\[
\hat{\pi}_{l+1} = \frac{1}{J} \sum_j (\hat{y}_j)_{l+1},
\]

\[
\hat{\epsilon}_{l+1} = \frac{1}{K} \sum_{i,j} (\hat{z}_{ij})_{l+1},
\]

\[
\hat{\mu}_{l+1} = \frac{n(\hat{y})_{l+1}}{t(\hat{y}z)_{l+1}} = \frac{\sum_j n_j(\hat{y}_j)_{l+1}}{\sum_{i,j} t_i(y_j z_{ij})_{l+1}},
\]

\[
\hat{\nu}_{l+1} = \frac{n((1-y)z)_{l+1}}{t((1-y)z)_{l+1}} = \frac{\sum_{i,j} n_{ij}(1-\hat{y}_j)_{l+1}}{\sum_{i,j} t_i((1-\hat{y}_j)z_{ij})_{l+1}}.
\]

If, at any stage, either \(\hat{\pi}_{l+1}\) or \(\hat{\epsilon}_{l+1}\) exceeds \(1/2\), replace it by \(1/2\).

Finally, the updated estimator \(\hat{\theta}_{l+1} = (\hat{\pi}_{l+1}, \hat{\epsilon}_{l+1}, \hat{\mu}_{l+1}, \hat{\nu}_{l+1})\) is obtained from (72) and (73); here, unlike (40), it does depend on \(\{t_i\}\).

**Starting value \(\hat{\pi}_0\) for the EM algorithm:** Under the restriction \(\pi \leq 1/2\), a simple way to choose \(\hat{\pi}_0\) and \(\hat{\epsilon}_0\) is as follows. Plot a histogram of the entire data set \(\{n_{ij}\}\) and attempt to discern a spike at 0 and two prevalent mixture components above 0, either by eye or by density estimation, then determine their weights. Take \(1 - \hat{\epsilon}_0\) to be the weight of the spike at 0, then take \(\hat{\pi}_0\) to be the lesser of the relative weights of the two nonzero components.

**Standard error for the MLE \(\hat{\theta}\):** Recall that \(\omega = (\pi, \epsilon, \mu, \nu)\) and \(\hat{\omega}_t = (\hat{\pi}_t, \hat{\epsilon}_t, \hat{\mu}_t, \hat{\nu}_t)\) and assume as before that the EM iterates \(\omega_t\) converge to \(\hat{\omega} = (\hat{\pi}, \hat{\epsilon}, \hat{\mu}, \hat{\nu})\), the actual MLEs based on the observed data \(N\). Again we rely on the results of Hoadley and Efron/Hinkley to provide the normal approximation\(^{12}\)

\[
\sqrt{K}(\hat{\omega} - \omega) \approx N_4[0, K I_n^{-1}(\omega)],
\]

\[
I_n(\omega) \equiv -\nabla^2_{\omega} \log f_\omega(n)
\]

\[
= -E_\omega[\nabla^2_\omega \log f_\omega(n) \mid n]
\]

\[
= -E_\omega[\nabla^2_\omega \log f_\omega(Y, Z, n) \mid n] + E_\omega[\nabla^2_\omega \log f_\omega(Y, Z \mid n) \mid n],
\]

where \(I_n^{-1}\) is the \(4 \times 4\) observed information matrix. If \((Y, Z, n) \in Y \times \Omega\)

\(^{12}\)In this section the conditioning events \(Y = y\) and \(N = n\) are abbreviated as \(y\) and \(n\).
then by (66),

\[
\log f_\omega(Y, Z, n) = J[\bar{Y} \log \pi + (1 - \bar{Y}) \log (1 - \pi)] + K[\bar{Z} \log \epsilon + (1 - \bar{Z}) \log (1 - \epsilon)] + K[n_Y \log \mu - t_Y Z \mu + n(1 - Y) \log \nu - t(1 - Y) Z \nu] + \log \Xi(n, z);
\]

\[
\nabla_\omega \log f_\omega(Y, Z, n) = \begin{pmatrix}
J(\bar{Y} - \pi) \\ n_Y \epsilon(1 - \epsilon) \\ K - n_Y \\ K - \frac{n(1 - Y)}{\nu}
\end{pmatrix} / (1 - \epsilon) - tY Z
\]

(77)

\[
-\nabla_\omega^2 \log f_\omega(Y, Z, n) = \begin{pmatrix}
J(1 - 2\epsilon) \bar{Y} + \pi^2 \\ n_Y^2(1 - \pi)^2 \\ n_Y \epsilon^2(1 - \epsilon)^2 \\ n_Y \epsilon^2(1 - \epsilon)^2
\end{pmatrix} / (1 - \epsilon) - tY Z
\]

Furthermore by (65), for \((y, z, n) \in Y \times \Omega\) with \(n\) fixed,

\[
f_\omega(y, z | n) = f_\omega(y, z, n) / f_\omega(n) \propto \prod_j \pi^{y_j} (1 - \pi)^{1 - y_j}
\]

\[
\cdot \prod_{i,j} \left\{ \epsilon \left[ e^{-t_i \mu(n_{ij})} \right]^{y_j} \left[ e^{-t_i \nu(n_{ij})} \right]^{1 - y_j} \right\} \left[ (1 - \epsilon) 0^{n_{ij}} \right]^{1 - z_{ij}}.
\]

From this, \(\{Z_{ij}\}\) are conditionally independent given \(Y\) and \(N\), with

(78) \[Z_{ij} | y, n \sim \text{Bernoulli}(r_{ij}),\]

\[r_{ij} \equiv r(t_i, y_j, n_{ij}) \equiv r(\epsilon, \mu, \nu; t_i; y_j; n_{ij})\]

\[
= \frac{\epsilon \left[ e^{-t_i \mu(n_{ij})} \right]^{y_j} \left[ e^{-t_i \nu(n_{ij})} \right]^{1 - y_j} t_i^{n_{ij}}}{\epsilon \left[ e^{-t_i \mu(n_{ij})} \right]^{y_j} \left[ e^{-t_i \nu(n_{ij})} \right]^{1 - y_j} t_i^{n_{ij}} + (1 - \epsilon) 0^{n_{ij}}}
\]

(79) \[= 1 - 0^{n_{ij}} + \frac{0^{n_{ij}} \epsilon \left[ e^{-t_i \mu(n_{ij})} \right]^{y_j} \left[ e^{-t_i \nu(n_{ij})} \right]^{1 - y_j} t_i^{n_{ij}}}{\epsilon \left[ e^{-t_i \mu(n_{ij})} \right]^{y_j} \left[ e^{-t_i \nu(n_{ij})} \right]^{1 - y_j} t_i^{n_{ij}} + (1 - \epsilon)},\]
and

\[
f_\omega(y \mid n) \\
\propto \sum_z f_\omega(y, z \mid n) \\
\propto \prod_j \pi^{y_j}(1 - \pi)^{1 - y_j} \cdot \prod_{i,j} \{e^{-t_i \mu^n_{i,j}} \cdot e^{-t_i \nu^n_{i,j}} \cdot \frac{1 - \pi}{\pi} \cdot \prod_{i,j \mid n_{i,j} \neq 0} \{e^{-t_i \mu^n_{i,j}} \cdot e^{-t_i \nu^n_{i,j}} \cdot (1 - \pi) + (1 - \epsilon)\} \}
\]

\[
= \prod_j \pi^{y_j}(1 - \pi)^{1 - y_j} \\
\cdot \prod_{\{i,j \mid n_{i,j} \neq 0\}} \{e^{-t_i \mu^n_{i,j}} \cdot e^{-t_i \nu^n_{i,j}} \cdot \frac{1 - \pi}{\pi} \cdot \prod_{\{i,j \mid n_{i,j} = 0\}} \{e^{-t_i \mu^n_{i,j}} \cdot e^{-t_i \nu^n_{i,j}} \cdot (1 - \pi) + (1 - \epsilon)\} \}
\]

\[
\propto e^{1/\pi} \prod_j \left[\frac{e^{-t_j \mu^n_{j}}}{e^{-t_j \nu^n_{j}}} \cdot \left(1 - \pi\right)e^{-t_j \nu^n_{j}} \cdot \prod_{i,j \mid n_{i,j} = 0} \frac{1 - \pi}{\pi} \cdot e^{-t_j \nu^n_{j}} \cdot (1 - \pi) + (1 - \epsilon)\right]^{1/\pi}_{ij} \}
\]

\[
\propto \prod_j q_j^{y_j}(1 - q_j)^{1 - y_j}.
\]

where

\[
q_j \equiv q_j(n_j) \equiv q_j(\pi, \epsilon, \mu, \nu; t; n_j)
\]

\[
: = \frac{\pi e^{-t_j \mu^n_{j}} \cdot \prod_i \left[\frac{e^{-t_i \mu^n_{i,j}}}{e^{-t_i \nu^n_{i,j}}} \cdot \left(1 - \pi\right)e^{-t_i \nu^n_{i,j}} \cdot \prod_{i,j \mid n_{i,j} = 0} \frac{1 - \pi}{\pi} \cdot e^{-t_i \nu^n_{i,j}} \cdot (1 - \pi) + (1 - \epsilon)\right]^{1/\pi}_{ij} \}
\]

Thus \{Y_j\} are conditionally independent given \(N\), with

(80) \[Y_j \mid n \sim \text{Bernoulli}(q_j).\]

Therefore \(E_\omega[\hat{Y} \mid n] = \bar{q} \equiv \bar{q}(n)\), while

\[
E_\omega[\overline{nY} \mid n] = \frac{1}{K} \sum_{i,j} n_{i,j} E_\omega[Y_j \mid n]
\]

\[
= \frac{1}{K} \sum_{i,j} n_{i,j} q_j
\]

\[
= n\bar{q},
\]

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\[
E_\omega[n(1-Y) | \mathbf{n}] = \frac{1}{K} \sum_{i,j} n_{ij} (1 - q_j)
= n(1 - q).
\]

Furthermore,

\[
E_\omega[(1 - 2\pi)\bar{Y} + \pi^2 | \mathbf{n}] = (1 - 2\pi)\bar{q} + \pi^2.
\]

Next,

\[
E_\omega[\bar{Z} | \mathbf{n}] = E_\omega\{E_\omega[\bar{Z} | y, \mathbf{n}] | \mathbf{n}\};
= \frac{1}{K} \sum_{i,j} E_\omega\{r(t_i; Y_j, n_{ij}) | \mathbf{n}\};
\]

\[
= \frac{1}{K} \sum_{i,j} \{q_j r(t_i; 1, n_{ij}) + (1 - q_j) r(t_i; 0, n_{ij})\}
\equiv q r(1) + (1 - q) r(0).
\]

From (79), note that

\[
r(t_i; 1, n_{ij}) = 1 - 0^{n_{ij}} + \frac{0^{n_{ij}} \varepsilon e^{-t_i \mu} \mu^{n_{ij}} t_i^{n_{ij}}}{\varepsilon e^{-t_i \mu} \mu^{n_{ij}} t_i^{n_{ij}} + (1 - \varepsilon)};
\]

\[
r(t_i; 0, n_{ij}) = 1 - 0^{n_{ij}} + \frac{0^{n_{ij}} \varepsilon e^{-t_i \nu} \nu^{n_{ij}} t_i^{n_{ij}}}{\varepsilon e^{-t_i \nu} \nu^{n_{ij}} t_i^{n_{ij}} + (1 - \varepsilon)},
\]

and decompose \( \sum_{i,j} \) as \( \sum_{i,j | n_{ij} \neq 0} + \sum_{i,j | n_{ij} = 0} \); so (82) becomes

\[
E_\omega[\bar{Z} | \mathbf{n}] = \frac{1}{K} \sum_{i,j} \left[ 1 - 0^{n_{ij}} + \frac{q_j 0^{n_{ij}} \varepsilon e^{-t_i \mu} \mu^{n_{ij}} t_i^{n_{ij}}}{\varepsilon e^{-t_i \mu} \mu^{n_{ij}} t_i^{n_{ij}} + (1 - \varepsilon)} + \frac{(1 - q_j) 0^{n_{ij}} \varepsilon e^{-t_i \nu} \nu^{n_{ij}} t_i^{n_{ij}}}{\varepsilon e^{-t_i \nu} \nu^{n_{ij}} t_i^{n_{ij}} + (1 - \varepsilon)} \right]
\]

\[
= \frac{1}{K} + \frac{\varepsilon}{K} \sum_{i,j | n_{ij} = 0} \left[ \frac{q_j e^{-t_i \mu}}{e^{-t_i \mu} + (1 - \varepsilon)} + \frac{(1 - q_j) e^{-t_i \nu}}{e^{-t_i \nu} + (1 - \varepsilon)} \right]
\]

\[
= \frac{1}{K} + \frac{\varepsilon}{K} \sum_{i,j} \left[ \frac{q_j}{\varepsilon + (1 - \varepsilon) e^{t_i \mu}} + \frac{1 - q_j}{\varepsilon + (1 - \varepsilon) e^{t_i \nu}} \right]
\]

\[
: = \rho(\varepsilon, \mu, \nu; t; \mathbf{n}) \equiv \rho.
\]

Lastly,

\[
E_\omega[(1 - 2\varepsilon)\bar{Z} + \varepsilon^2 | \mathbf{n}] = (1 - 2\varepsilon)\bar{\rho} + \varepsilon^2.
\]
Therefore \(-E_\omega[\nabla_\omega^2 \log f_\omega(Y,Z,n) | n] | n\) is evaluated explicitly as follows (recall (76)-(77)):

\[
-\sum_{i,j} \left( \frac{J[1-2\pi]q+\pi^2}{\pi^2(1-\pi)^2} \right)
- \sum_{i,j} K \left[ \frac{2}{\epsilon^2(1-\epsilon)^2} \right] \log(1 - q_j) + (1 - Y_j) \log(1 - q_j)
- \sum_{i,j} K \left[ \frac{2}{\mu^2} \right] \log(1 - r_{ij}) + (1 - Z_{ij}) \log(1 - r_{ij})
- \sum_{i,j} K \left[ \frac{n(1-q)}{\mu^2} \right] \log(1 - r_{ij}) + (1 - Z_{ij}) \log(1 - r_{ij})

\]

For the second term in (76), it follows from (78) and (80) that

\[
f_\omega(Y,Z,n) = f_\omega(Y,n) f_\omega(Z,Y,n)
= \prod_j q_j (1-q_j)^{1-Y_j} \prod_i r_{ij} (1-r_{ij})^{1-Z_{ij}}
= \prod_j q_j (1-q_j)^{1-Y_j} \prod_{\{i,j\mid n_{ij}=0\}} r_{ij} (1-r_{ij})^{1-Z_{ij}},
\]

since \(n_{ij} \neq 0 \Rightarrow Z_{ij} = 1\) and \(r_{ij} = 1\) for \((Y,Z,n) \in \mathcal{Y} \times \Omega\). Thus

\[
\log f_\omega(Y,Z,n) = \sum_j [Y_j \log q_j + (1 - Y_j) \log(1 - q_j)]
+ \sum_{\{i,j\mid n_{ij}=0\}} [Z_{ij} \log r_{ij} + (1 - Z_{ij}) \log(1 - r_{ij})];
\]

\[
\nabla_\omega \log f_\omega(Y,Z,n) = \sum_j \left( \frac{(Y_j - q_j)}{q_j(1-q_j)} \nabla_\omega q_j + \sum_{\{i,j\mid n_{ij}=0\}} \frac{(Z_{ij} - r_{ij})}{r_{ij}(1-r_{ij})} \nabla_\omega r_{ij} \right);
\]

\[
\nabla_\omega^2 \log f_\omega(Y,Z,n) = \sum_j \left[ \frac{(Y_j - q_j) \nabla^2 q_j}{q_j(1-q_j)} (\nabla_\omega q_j)' - \frac{(Y_j - q_j) (\nabla_\omega q_j)(\nabla_\omega q_j)'}{q_j^2(1-q_j)^2} \right]
+ \sum_{\{i,j\mid n_{ij}=0\}} \left[ \frac{(Z_{ij} - r_{ij}) \nabla^2 r_{ij}}{r_{ij}(1-r_{ij})} (\nabla_\omega r_{ij})' - \frac{(Z_{ij} - r_{ij})(\nabla_\omega r_{ij})(\nabla_\omega r_{ij})'}{r_{ij}^2(1-r_{ij})^2} \right],
\]
where \( r_{ij} \equiv r(t_i; y_j, n_{ij}) \). Therefore

\[
E_\omega[\nabla_\omega^2 \log f_\omega(Y; Z \mid n) \mid n] = - \sum_j \frac{(\nabla_\omega q_j)(\nabla_\omega q_j)'}{q_j(1-q_j)} - \sum_{\{i,j|n_{ij}=0\}} E_\omega \left[ \frac{(\nabla_\omega r_{ij})(\nabla_\omega r_{ij})'}{r_{ij}(1-r_{ij})} \right] \mid n
\]

(86)

\[
= \sum_j (\nabla_\omega \log q_j)[\nabla_\omega \log(1-q_j)]'
\]

\[
+ \sum_{\{i,j|n_{ij}=0\}} E_\omega \left\{ (\nabla_\omega \log r(t_i; Y_j, n_{ij})) [\nabla_\omega \log(1-r(t_i; Y_j, n_{ij}))]' \right\} \mid n,
\]

where we used the facts that for any functions \( h(n) \) and \( h(y, n) \),

\[
E_\omega[(Y_j-q_j)h(n) \mid n] = h(n)E_\omega[(Y_j-q_j) \mid n] = 0,
\]

\[
E_\omega[(Z_{ij}-r_{ij})h(y, n) \mid n] = E_\omega \{ h(y, n)E_\omega[Z_{ij}-r_{ij} \mid y, n] \mid n \} = 0.
\]

Now note that

\[
\log q_j = \log \pi - t^*_j \mu + n_j \log \mu + \sum_i 1_{ij}^n \log (e^{-t_{ij}\mu} -1) + 1] - \log \psi_j;
\]

\[
\psi_j := \pi e^{-\sum_j n_j} \prod_i [\epsilon (e^{-t_{ii}\mu} -1) + 1]^{1_{ij}^n} + (1-\pi) e^{-\sum_j n_j} \prod_i [\epsilon (e^{-t_{ij}\nu} -1) + 1]^{1_{ij}^n};
\]

\[
\frac{\partial \psi_j}{\partial \pi} = e^{-\sum_j n_j} \prod_i [\epsilon (e^{-t_{ii}\mu} -1) + 1]^{1_{ij}^n} - e^{-\sum_j n_j} \prod_i [\epsilon (e^{-t_{ij}\nu} -1) + 1]^{1_{ij}^n},
\]

\[
\frac{\partial \psi_j}{\partial \epsilon} = \pi e^{-\sum_j n_j} \sum_i \frac{1_{ij}^n (e^{-t_{ii}\mu} -1)}{\epsilon (e^{-t_{ii}\mu} -1) + 1} \prod_i [\epsilon (e^{-t_{ij}\nu} -1) + 1]^{1_{ij}^n},
\]

\[
+ (1-\pi) e^{-\sum_j n_j} \sum_i \frac{1_{ij}^n (e^{-t_{ij}\nu} -1)}{\epsilon (e^{-t_{ij}\nu} -1) + 1} \prod_i [\epsilon (e^{-t_{ij}\nu} -1) + 1]^{1_{ij}^n},
\]

\[
\frac{\partial \psi_j}{\partial \mu} = \pi e^{-\sum_j n_j} \prod_i [\epsilon (e^{-t_{ii}\mu} -1) + 1]^{1_{ij}^n} \left[ \frac{n_j}{\mu} - t^*_j - \epsilon \sum_i \frac{1_{ij}^n t_{ij} e^{-t_{ii}\mu}}{\epsilon (e^{-t_{ii}\mu} -1) + 1} \right],
\]

\[
\frac{\partial \psi_j}{\partial \nu} = (1-\pi) e^{-\sum_j n_j} \prod_i [\epsilon (e^{-t_{ij}\nu} -1) + 1]^{1_{ij}^n} \left[ \frac{n_j}{\nu} - t^*_j - \epsilon \sum_i \frac{1_{ij}^n t_{ij} e^{-t_{ij}\nu}}{\epsilon (e^{-t_{ij}\nu} -1) + 1} \right];
\]

\[
\]
from which it can be shown that

\[
\frac{\partial \log q_j}{\partial \pi} = \frac{e^{-t_j^\pi \nu_j n_j}}{\pi \psi_j} \prod_i \left[ e(e^{-t_i \mu_i} - 1) + 1 \right]^{\gamma_j}.
\]

\[
\frac{\partial \log q_j}{\partial \epsilon} = \frac{(1 - \pi) e^{-t_j^\pi \nu_j n_j}}{\psi_j} \prod_i \left[ e(e^{-t_i \mu_i} - 1) + 1 \right]^{\gamma_j} \sum_i \frac{1}{e(e^{-t_i \mu_i} - 1) + 1} \left[ e(e^{-t_i \mu_i} - 1) + 1 \right].
\]

\[
\frac{\partial \log q_j}{\partial \mu} = \frac{(1 - \pi) e^{-t_j^\pi \nu_j n_j}}{\psi_j} \prod_i \left[ e(e^{-t_i \mu_i} - 1) + 1 \right]^{\gamma_j} \left[ n_j \mu - t_j^\mu - \epsilon \sum_i \frac{1}{e(e^{-t_i \mu_i} - 1) + 1} \right].
\]

\[
\frac{\partial \log q_j}{\partial \nu} = -\frac{(1 - \pi) e^{-t_j^\pi \nu_j n_j}}{\psi_j} \prod_i \left[ e(e^{-t_i \mu_i} - 1) + 1 \right]^{\gamma_j} \left[ n_j \nu - t_j^\nu - \epsilon \sum_i \frac{1}{e(e^{-t_i \mu_i} - 1) + 1} \right].
\]

These four partial derivatives determine the $4 \times 1$ column vector $\nabla_\omega \log q_j$.

Furthermore,

\[
\nabla_\omega \log (1 - q_j) = -\frac{q_j}{1 - q_j} \nabla_\omega \log q_j
\]

\[
= -\frac{\pi}{1 - \pi} \frac{e^{-t_j^\pi \mu_j n_j}}{\psi_j} \prod_i \left[ e(e^{-t_i \mu_i} - 1) + 1 \right]^{\gamma_j} \nabla_\omega \log q_j,
\]

hence

\[
\left( \nabla_\omega \log q_j \right) \left( \nabla_\omega \log (1 - q_j) \right)'
\]

\[
= -\frac{\pi(1 - \pi) e^{-t_j^\pi (\mu + \nu)(\mu \nu)} n_j}{\psi_j} \prod_i \left[ e(e^{-t_i \mu_i} - 1) + 1 \right]^{\gamma_j} \frac{1}{\pi(1 - \pi)} \phi_j \phi_j',
\]

where

\[
\phi_j \equiv \phi_j(\omega; t; n_j) := \left( \begin{array}{c}
\frac{1}{\pi(1 - \pi)} \\
\sum_i \frac{1}{e(e^{-t_i \mu_i} - 1) + 1} + 1 \right]^{\gamma_j} \phi_j(\omega; t; n_j) := \left( \begin{array}{c}
\frac{1}{\pi(1 - \pi)} \\
\sum_i \frac{1}{e(e^{-t_i \mu_i} - 1) + 1} + 1 \right]^{\gamma_j}
\end{array} \right).
\]
Next, for $n_{ij} = 0$,

$$r(t_i; 1, 0) = \frac{e e^{-t_i \mu}}{e(e^{-t_i \mu} - 1) + 1},$$

$$r(t_i; 0, 0) = \frac{e e^{-t_i \nu}}{e(e^{-t_i \nu} - 1) + 1};$$

$$\log r(t_i; 1, 0) = \log \epsilon - t_i \mu - \log[e(e^{-t_i \mu} - 1) + 1],$$

$$\log(1 - r(t_i; 1, 0)) = \log(1 - \epsilon) - \log[e(e^{-t_i \mu} - 1) + 1];$$

$$\log r(t_i; 0, 0) = \log \epsilon - t_i \nu - \log[e(e^{-t_i \nu} - 1) + 1],$$

$$\log(1 - r(t_i; 0, 0)) = \log(1 - \epsilon) - \log[e(e^{-t_i \nu} - 1) + 1];$$

so with $\omega = (\pi, \epsilon, \mu, \nu)$,

$$\nabla_{\omega} \log r(t_i; 1, 0) = \begin{pmatrix} 0, \frac{1}{e(e^{-t_i \mu} - 1) + 1}, \frac{-t_i}{e(e^{-t_i \mu} - 1) + 1}, 0 \end{pmatrix},$$

$$\nabla_{\omega} \log(1 - r(t_i; 1, 0)) = \begin{pmatrix} 0, \frac{e^{-t_i \mu}}{(1 - \epsilon)[e(e^{-t_i \mu} - 1) + 1]}, \frac{e t_i e^{-t_i \mu}}{e(e^{-t_i \mu} - 1) + 1}, 0 \end{pmatrix},$$

$$\nabla_{\omega} \log r(t_i; 0, 0) = \begin{pmatrix} 0, \frac{1}{e(e^{-t_i \nu} - 1) + 1}, 0, \frac{-t_i}{e(e^{-t_i \nu} - 1) + 1} \end{pmatrix},$$

$$\nabla_{\omega} \log(1 - r(t_i; 0, 0)) = \begin{pmatrix} 0, \frac{-e^{-t_i \nu}}{(1 - \epsilon)[e(e^{-t_i \nu} - 1) + 1]}, 0, \frac{e t_i e^{-t_i \nu}}{e(e^{-t_i \nu} - 1) + 1} \end{pmatrix}.$$

Thus

$$E_{\omega}\left\{ (\nabla_{\omega} \log r(t_i; Y_j, 0)) [\nabla_{\omega} \log(1 - r(t_i; Y_j, 0))]' \right| n \right\}$$

$$= (\nabla_{\omega} \log r(t_i; 1, 0))[\nabla_{\omega} \log(1 - r(t_i; 1, 0))]' q_j$$

$$+ (\nabla_{\omega} \log r(t_i; 0, 0))[\nabla_{\omega} \log(1 - r(t_i; 0, 0))]'(1 - q_j)$$

$$= -\frac{e(1 - \epsilon)e^{-t_i \mu}}{[e(e^{-t_i \mu} - 1) + 1]^2} \chi_i(1) \chi_i(1)' q_j$$

$$- \frac{e(1 - \epsilon)e^{-t_i \nu}}{[e(e^{-t_i \nu} - 1) + 1]^2} \chi_i(0) \chi_i(0)' (1 - q_j),$$

where

$$\chi_i(1) \equiv \chi(\epsilon; t_i; 1) := \begin{pmatrix} 0, \frac{1}{e(1 - \epsilon)}, t_i, 0 \end{pmatrix},$$

$$\chi_i(0) \equiv \chi(\epsilon; t_i; 0) := \begin{pmatrix} 0, \frac{1}{e(1 - \epsilon)}, 0, t_i \end{pmatrix}.$$
Therefore from (86), the second term in (76) is given by

$$(87)$$

$$E_\omega[^{\nabla^2}_\omega \log f_\omega(Y, Z \mid n) \mid n]$$

$$= -\pi(1 - \pi) \sum_j \frac{e^{-t_j^2(\mu+\nu)}(\mu\nu)^{n_j}}{\psi_j^2} \prod_i \{[\epsilon(e^{-t_i\mu} - 1) + 1][\epsilon(e^{-t_i\nu} - 1) + 1]\}^{\psi_j} \phi_j \phi_j'$$

$$= \epsilon(1 - \epsilon) \sum_{i,j} \frac{e^{-t_i\mu}}{[\epsilon(e^{-t_i\mu} - 1) + 1]^2} \chi_i(1) \chi_i(1)' q_j + \frac{e^{-t_i\nu}}{[\epsilon(e^{-t_i\nu} - 1) + 1]^2} \chi_i(0) \chi_i(0)' (1 - q_j).$$

Together with (85), this explicitly determines the observed information matrix $I_n(\omega)$ in (74)-(76).

Now estimate $I_n(\omega)$ in the normal approximation

$$\sqrt{K}(\hat{\omega} - \omega) \approx N_4[0, K I_n^{-1}(\omega)]$$

by replacing $\omega$ in $I_n(\omega)$ by its MLE $\hat{\omega} \equiv (\hat{\pi}, \hat{\epsilon}, \hat{\mu}, \hat{\nu})$, obtained via the EM algorithm, to obtain

$$(88)$$

$$\sqrt{K}(\hat{\omega} - \omega) \approx N_4[0, K I_n^{-1}(\hat{\omega})],$$

where $K = IJ$. This requires replacing $\pi, \epsilon, \mu, \nu$ by $\hat{\pi}, \hat{\epsilon}, \hat{\mu}, \hat{\nu}$ wherever the former appear in the entries of $I_n(\omega)$, including in $q_j, \rho, \psi_j, \phi_j, \chi_i$. For large $K$ the $4 \times 4$ matrix $I_n(\omega)$ is positive definite, hence invertible.

Lastly, an approximate confidence interval for $\theta \equiv \mu/\nu \equiv g(\omega)$ is obtained from (88) by propagation of error. For $\hat{\theta} = \hat{\mu}/\hat{\nu}$,

$$\sqrt{K}(\hat{\theta} - \theta) \approx N[0, K((\nabla_\omega g(\omega) |_{\hat{\omega}})' I_n^{-1}(\hat{\omega}) \nabla_\omega g(\hat{\omega}) \mid_{\hat{\omega}})]$$

$$= N \left[ 0, K \left( \frac{\partial g}{\partial \pi} \right) \left( \frac{\partial g}{\partial \epsilon} \right) \left( \frac{\partial g}{\partial \mu} \right) \left( \frac{\partial g}{\partial \nu} \right) \right] \left( \frac{\partial g}{\partial \pi} \right) \left( \frac{\partial g}{\partial \epsilon} \right) \left( \frac{\partial g}{\partial \mu} \right) \left( \frac{\partial g}{\partial \nu} \right)$$

$$= N \left[ 0, K \left( 0, 0, \frac{1}{\hat{\nu}}, -\frac{\hat{\mu}}{\hat{\nu}^2} \right) \right] \left( 0, 0, 0, \frac{1}{\hat{\nu}}, -\frac{\hat{\mu}}{\hat{\nu}^2} \right)'$$

$$= N \left[ 0, K \left( \frac{1}{\hat{\nu}}, -\frac{\hat{\mu}}{\hat{\nu}^2} \right) \left( I_{22} - I_{21} I_{11}^{-1} I_{12} \right)^{-1} \left( \frac{1}{\hat{\nu}}, -\frac{\hat{\mu}}{\hat{\nu}^2} \right) \right]'$$

$$(89)$$

$$\equiv N(0, \tau^2),$$

34
where $I_n(\hat{o}) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$ is the partitioning of $I_n(\hat{o})$ into $2 \times 2$ blocks. Thus computation of $\hat{\tau}^2$ only requires the inversion of two $2 \times 2$ matrices. This yields the following approximate $(1 - \alpha)$ confidence interval for $\theta$:

$$
\hat{\theta} \pm \frac{\hat{\tau}}{\sqrt{K}} z_{\alpha/2}.
$$

(90)

4.2. Bayesian analysis. Rewrite the joint pmf (66) of the complete (unobserved and observed) data $(Y, Z, N)$ in terms of the parameters $\pi, \epsilon, \theta, \lambda$ as follows:

$$
f(Y, Z, N \mid \pi, \epsilon, \theta, \lambda) = \left[ \pi^y(1 - \pi)^{1 - y} \right]^J \left[ \epsilon^z(1 - \epsilon)^{1 - z} \right]^K \cdot \left[ e^{-tyz\theta \lambda} \theta \lambda e^{-t(1 - y)z} \lambda \lambda e^{-t(1 - y)z} \right] \cdot \Xi_t(z, n)
$$

(91)

where $n = \sum_{i,j} n_{ij}$. If we assume the gamma prior density $\gamma_{\delta}(\lambda)$ for $\lambda$, any proper prior density $\vartheta(\pi)$ for $\pi \in (0, 1)$, and the beta($\eta, \kappa$) prior density

$$
\xi_{\eta, \kappa}(\epsilon) := \frac{\Gamma(\eta + \kappa)}{\Gamma(\eta)\Gamma(\kappa)} e^{\theta - 1}(1 - \epsilon)^{\kappa - 1} \eta_{(0, 1)}(\epsilon)
$$

for $\epsilon$, where $\eta, \kappa > 0$, then from (91) the integrated joint pmf of $(Y, Z, N)$ on $\Upsilon \times \Omega$ is

$$
f_{\vartheta, \eta, \kappa, \delta}(Y, Z, N \mid \theta) = \int_0^{1/2} \int_0^1 \int_0^\infty f(Y, Z, N \mid \pi, \epsilon, \theta, \lambda) \vartheta(\pi) \xi_{\eta, \kappa}(\epsilon) \gamma_{\delta}(\lambda) d\pi d\epsilon d\lambda
$$

(92)

(compare to (55)), where for $0 \leq j \leq J$ and $0 \leq \ell \leq K$,

$$
g_{\theta}(j) = \int_0^{1/2} \pi^j(1 - \pi)^{J - j} \vartheta(\pi) d\pi,
$$

$$
h_{\eta, \kappa}(\ell) = \frac{\Gamma(\eta + \kappa)\Gamma(\eta + \ell)\Gamma(\kappa + K - \ell)}{\Gamma(\eta)\Gamma(\kappa)\Gamma(\eta + \kappa + K)}..
$$
For \( \sigma \subseteq J \) and \( \tau \subseteq K \) let \( 1_{\sigma} \) and \( 1_{\tau} \) denote their indicator functions. From (92), the integrated joint pmf \( f_{\theta,\eta,\kappa,\delta}(z, n | \theta) \) of \((Z, N)\) can be expressed explicitly as follows:

\[
f_{\theta,\eta,\kappa,\delta}(z, n | \theta) = \sum_{y \in \mathcal{T}} f_{\theta,\eta,\kappa,\delta}(y, z, n | \theta)
= \frac{\Gamma(n + \delta) \prod t_i^{n_i}}{\Gamma(\delta) n!} \sum_{j=0}^{J} \sum_{\{y : J \bar{y} = j\}} g_{\theta}(J \bar{y}) h_{\eta,\kappa}(K \bar{z}) \theta^{K \bar{y} \bar{z}} \prod_{i \in J} t_i^{n_i} \sum_{\sigma \subseteq J, |\sigma| = j} \left( \sum_{\ell = 0}^{K} \sum_{\{z : \bar{v} \bar{z} = \ell\}} g_{\theta}(J \bar{y}) h_{\eta,\kappa}(K \bar{z}) \theta^{K \bar{y} \bar{z}} \prod_{i \in J} t_i^{n_i} \sum_{\sigma \subseteq J, |\sigma| = j} \sum_{\tau \subseteq K, |\tau| = \ell} \theta^{n_{\tau}} \Delta_{\ell,\sigma}(n | \theta) \right) \Omega(z),
\]

(93)

where

\[
\begin{align*}
n_{\sigma} &= \sum_{j} n_{j} 1_{\sigma}(j), \\
(tz)_{\sigma} &= \sum_{i,j} t_{i} 1_{\sigma}(j) z_{ij}, \\
(tz) &= \sum_{i,j} t_{i} z_{ij} = K \bar{t} z,
\end{align*}
\]

and \( n_{\emptyset} = (tz)_{\emptyset} = 0 \).

Furthermore, the integrated likelihood \( f_{\theta,\eta,\kappa,\delta}(n | \theta) \) of \( N \) itself can be obtained explicitly from (92) as follows. Setting \( \Omega_n = \{z : (z, n) \in \Omega\} \),

\[
f_{\theta,\eta,\kappa,\delta}(n | \theta) = \sum_{y \in \mathcal{T}} \sum_{z \in \Omega_n} f_{\theta,\eta,\kappa,\delta}(y, z, n | \theta)
= \frac{\Gamma(n + \delta) \prod t_i^{n_i}}{\Gamma(\delta) n!} \sum_{j=0}^{J} \sum_{\{y : J \bar{y} = j\}} \sum_{\{z : \bar{v} \bar{z} = 0\}} g_{\theta}(J \bar{y}) h_{\eta,\kappa}(K \bar{z}) \theta^{K \bar{y} \bar{z}} \prod_{i \in J} t_i^{n_i} \sum_{\sigma \subseteq J, |\sigma| = j} \left( \sum_{\ell = 0}^{K} \sum_{\{z : \bar{v} \bar{z} = \ell\}} g_{\theta}(J \bar{y}) h_{\eta,\kappa}(K \bar{z}) \theta^{K \bar{y} \bar{z}} \prod_{i \in J} t_i^{n_i} \sum_{\sigma \subseteq J, |\sigma| = j} \sum_{\tau \subseteq K, |\tau| = \ell} \theta^{n_{\tau}} \Delta_{\ell,\sigma}(n | \theta) \right) \Omega_n(z),
\]

(94)

\[
\begin{align*}
\Omega_n(z) &= \{z : (z, n) \in \Omega\} \\
&= \{z : (z, n) \in \Omega\},
\end{align*}
\]
where

\[ n_{\sigma} = \sum_{j} n_{j} 1_{\sigma}(j), \quad (95) \]

\[ \Delta_{\ell,\sigma}(n | \theta) = \sum_{\tau \subseteq K, |\tau| = \ell} \frac{1_{\Omega_{n}}(1_{\tau})}{\{t_{\sigma,\tau}(\theta - 1) + t_{\tau} + 1\}^{n+\delta}}, \quad (96) \]

\[ t_{\sigma,\tau} = \sum_{i,j} t_{i} 1_{\sigma}(j) 1_{\tau}(i,j), \]

\[ t_{\tau} = \sum_{i,j} t_{i} 1_{\tau}(i,j), \]

and \( t_{\emptyset,\tau} = t_{\sigma,\emptyset} = t_{\emptyset} = 0 \).

Because \( f_{\vartheta,\eta,\kappa,\delta}(n | \theta) \) is not an exponential family, no conjugate prior is available. However, for any prior density \( \phi(\theta) \) the posterior pdf

\[ f_{\vartheta,\eta,\kappa,\delta}(\theta | n) \propto f_{\vartheta,\eta,\kappa,\delta}(n | \theta) \phi(\theta), \]

which can be obtained explicitly\(^{13}\) via (94)-(96). Thus MCMC methods (Robert and Casella (2004)) can be used to simulate the posterior distribution of \( \theta \) and thereby obtain the corresponding Bayes estimator and posterior confidence intervals.

Alternatively, we can adopt an empirical Bayes approach as in Section 3.2. For the data-based prior pdf \( \phi_{\alpha,\beta;K_{t^2\bar{r}}}(\theta) \) (cf. (17)), where \( \bar{r} = \frac{t_{yz}}{t_{z}} \), it follows from (92) and (17) that the integrated posterior pdf of \( \theta \), given the complete data \((y, (z, n)) \in \Upsilon \times \Omega\), satisfies

\[ f_{\vartheta,\eta,\kappa,\delta}(\theta | y, z, n) \propto f_{\vartheta,\eta,\kappa,\delta}(y, z, n | \theta) \phi_{\alpha,\beta;K_{t^2\bar{r}}}(\theta) \]

\[ \propto \frac{\phi_{K_{n^{-y}} + \alpha, \bar{r} (1 - y) + \beta + \delta; K_{t^2\bar{r}}}(\theta)}{\{K_{n^{-y}} + \alpha + \beta + \delta; K_{t^2\bar{r}}(\theta)\}}, \quad (97) \]

since \( n = K_{n^{-y}} + K_{n(1 - y)} \); note that (97) does not depend on \( \vartheta, \eta, \kappa \). Here \( K_{n^{-y}}, K_{n(1 - y)}, K_{t^2\bar{r}}, K_{t^2\bar{r}z} \), and thus \( \bar{r} \), are unobserved, but we can impute their values via the above-discussed EM algorithm as follows:

\(^{13}\)In principle. There are a total of \( 2^{J} \) subsets \( \sigma \subseteq J \) and \( 2^{K} \) subsets \( \tau \subseteq K \) that appear in the summations in (94), where \( K = IJ \), so exact calculation of \( f_{\vartheta,\eta,\kappa,\delta}(n | \theta) \) is infeasible if \( K \) is large. Instead, Monte Carlo simulation over \((\sigma, \tau)\) can be used to approximate \( f_{\vartheta,\eta,\kappa,\delta}(n | \theta) \).
The EM algorithm will output

\[ K\hat{ny} = \lim_{l \to \infty} \sum_j n_j(\hat{y}_j)_{l+1}, \]

(98)

\[ Kn(1 - y) = n - K\hat{ny}, \]

(99)

\[ K\hat{tz} = \lim_{l \to \infty} \sum_{i,j} t_i(\hat{z}_{ij})_{l+1}, \]

(100)

\[ K\hat{tyz} = \lim_{l \to \infty} \sum_{i,j} t_i(\hat{y}_j z_{ij})_{l+1}, \]

(101)

\[ \hat{\bar{r}} = \frac{\hat{tyz}}{\hat{tz}}, \]

(102)

where \((\hat{y}_j)_{l+1}, (\hat{z}_{ij})_{l+1}, \) and \((\hat{y}_j z_{ij})_{l+1}\) appear in (67)-(69). Now refer to (20)-(23) and replace \(r\) by \(\hat{\bar{r}}\), \(m_S\) by \(K\hat{ny}\), and \(m_T\) by \(Kn(1 - y)\), thus we obtain the empirical Bayes integrated posterior density

\[ f_{\delta, \alpha, \beta}(\theta | \hat{y}, \hat{z}, n) := \phi_{K\hat{ny} + \alpha, K\hat{ny}(1 - y) + \beta + \delta, K\hat{tz}, \hat{\bar{r}}} (\theta) \]

and empirical Bayes estimator\(^{14}\)

\[ \hat{\theta}_{\delta, \alpha, \beta}^{EBZIP} := \frac{(1 - \hat{\bar{r}})(K\hat{ny} + \alpha)}{\hat{\bar{r}} \left( K\hat{ny}(1 - y) + \beta + \delta - 1 \right)}, \]

(103)

provided that \(K\hat{ny}(1 - y) + \beta + \delta > 1\). Empirical Bayes integrated posterior confidence intervals for \(\theta\) can be obtained from (103).

**Remark 4.1.** Taking \(\alpha = \beta = 0\) yields the prior density \(\phi_{0,0; K\hat{ny}, \hat{\bar{r}}} (\theta) = \theta^{-1}\). This is no longer data-based but is improper, hence cannot reflect actual prior knowledge about \(\theta\). However, proceeding formally from (103) and (104), we obtain the posterior density

\[ f_{\delta, 0, 0}(\theta | \hat{y}, \hat{z}, n) := \phi_{K\hat{ny}, K\hat{ny}(1 - y) + \delta, K\hat{ny}, \hat{\bar{r}}} (\theta), \]

(105)

\(^{14}\)Note that the imputed values of \(\{z_{ij}\}\) occur in \(\hat{\theta}_{\delta, \alpha, \beta}^{EBZIP}\) through \(K\hat{ny}\) and \(Kn(1 - y)\) as well as through \(\hat{\bar{r}}\); see (98)-(99), (67), and (70)-(73).
which is a proper density if $K \hat{m} \gamma > 0$, and from this the estimator

$$
\hat{\theta}_{EBZIP}^{\hat{\gamma},0,0} := \frac{(1 - \hat{r})(K \hat{m} \gamma)}{\hat{r} (K \hat{n}(1 - y) + \delta - 1)},
$$

valid if $K \hat{n}(1 - y) + \delta > 1$, and which may have desirable frequentist properties. \hfill \Box

**Remark 4.2.** Direct determination of the MILE of $\theta$ based on $f_{\vartheta,\eta,\kappa,\delta}(n \mid \theta)$ in (94) again appears problematic. As in Remark 3.2, one might attempt to obtain this MILE by applying the EM algorithm to $f_{\vartheta,\eta,\kappa,\delta}(y,z,n \mid \theta)$ in (92) or to $f_{\vartheta,\eta,\kappa,\delta}(z,n \mid \theta)$ in (93), but again the E-steps are challenging. \hfill \Box

**Remark 4.3.** Note that the term $1_{\Omega_n}(1_{\tau})$ in $\Delta_{\ell,\sigma}(n \mid \theta)$ (cf. (96)) depends on $n$ only through

$$
1 - 0^n := (1 - 0^{n_{ij}} \mid (i,j) \in \mathcal{K}) = \{1_{ij}^x \mid (i,j) \in \mathcal{K} \},
$$

i.e., the indicator function over $\mathcal{K} \equiv I \times J$ of the set of nonzero $n_{ij}$'s. Thus we obtain the following interesting fact from (94)-(96) and the Factorization Criterion: $(N_1, \ldots, N_J; 1 - 0^N)$ is a sufficient statistic\footnote{This holds for any choice of the prior pdf $\vartheta(\pi)$.} for $\theta$ based on the integrated likelihood $f_{\vartheta,\eta,\kappa,\delta}(n \mid \theta)$. If we recall that in the non-ZIP model of Part I, $(M_1, \ldots, M_J)$ is a sufficient statistic for $\theta$ based on the integrated likelihood $f_{\vartheta,\delta}(m \mid \theta)$ for $\theta$ given in (57), then this shows that in the Bayesian framework, after integrating over the parameters $\pi, \epsilon, \lambda$, the statistic $1 - 0^N$ is the only additional information needed for inference about $\theta$ when zero-inflation is present. This raises the interesting question of determining the joint distribution of $(N_1, \ldots, N_J; 1 - 0^N)$ based on the integrated likelihood $f_{\vartheta,\eta,\kappa,\delta}(n \mid \theta)$. \hfill \Box
5. Conditional ZIPM = ZTP?

Consider two subsets of the index set $\mathcal{K}$ and two subarrays of the data array $\mathbf{N} \equiv (N_{ij})$:

\[
\begin{align*}
\Omega_Z^\pm &= \{(i,j) \mid Z_{ij} = 1\}, \\
\Omega_N^\pm &= \{(i,j) \mid N_{ij} \neq 0\}, \\
N_Z^\pm &= (N_{ij} \mid Z_{ij} = 1) = (M_{ij} \mid Z_{ij} = 1), \\
N_N^\pm &= (N_{ij} \mid N_{ij} \neq 0) = (M_{ij} \mid M_{ij} \neq 0).
\end{align*}
\]

Both $\Omega_Z^\pm$ and $\Omega_N^\pm$ are random subsets, $\Omega_Z^\pm$ is unobserved, $\Omega_N^\pm$ is observed, and $\Omega_N^\pm \subseteq \Omega_Z^\pm$, so $N_N^\pm \subseteq N_Z^\pm$. Because $\mathbf{M}$ is independent of $\mathbf{Z}$, $N_Z^\pm$ is a random subarray of the i.n.i.d. array $(M_{ij})$, where membership in this subarray depends only on $\mathbf{Z}$. Thus $N_N^\pm$ is also a (smaller) random subarray of the i.n.i.d. array $(M_{ij})$, where membership depends on both $\mathbf{Z}$ and the events $\{M_{ij} \neq 0\}$.

The latter fact suggest a question: Is the conditional distribution of the two-component ZIPM r.v $N_{ij}$ given $N_{ij} \neq 0$ the same as the distribution of the mixture of the conditional distributions of the two Poisson components given that each is non-zero? The latter conditional distribution is the well-known zero-truncated Poisson (ZTP) distribution, also called positive Poisson, which has been thoroughly studied (cf. Johnson, Kemp, and Kotz (2005)). The ZTP distribution model also is an exponential family, with pmf given by

\[
g_\lambda(x) = \frac{\lambda^x}{(e^\lambda - 1)x!}, \quad x = 1, 2, \ldots.
\]

If the answer to the above question is yes, then estimation of $\pi, \mu, \nu$ and thus $\theta$ could be based on only the set of non-zero $N_{ij}$. That is, discard all 0’s and view the remaining $N_{ij}$ as $\pi$-mixtures of two ZTP components with parameters $t_i\mu$ and $t_i\nu$. Because this involves only two mixture components rather than three as above, both being exponential families, and neither is degenerate, estimation methods such as the EM algorithm would be easier to carry out.

Unfortunately the answer to the question is no. If we abbreviate $N_{ij}$ by $N$, $M_{ij}$ by $M$, and $Z_{ij}$ by $Z$, then the question can be expressed as follows:

\[
\begin{align*}
\text{Is } P[N = x \mid N \neq 0] &= \frac{\pi \mu^x}{(e^\mu - 1)x!} + \frac{(1 - \pi) \nu^x}{(e^\nu - 1)x!}, \quad x = 1, 2, \ldots?
\end{align*}
\]
However, for $x \geq 1$,

$$P[N = x \mid N \neq 0] = \frac{P[ZM = x, ZM \neq 0]}{P[ZM \neq 0]}$$

$$= \frac{P[M = x, Z = 1, M \neq 0]}{P[Z = 1, M \neq 0]}$$

$$= \frac{P[M = x, M \neq 0]}{P[M \neq 0]}$$

$$= \frac{\pi \mu^x e^{\mu x}}{1 - \pi e^{\mu x} - (1 - \pi) e^{\nu x}} \cdot \frac{1 + (1 - \pi) \nu^x}{e^{\nu x} - 1},$$

since $M$ and $Z$ are independent, so the question becomes:

Is

$$\frac{\pi \mu^x}{e^{\mu x}} \cdot \frac{1 + (1 - \pi) \nu^x}{e^{\nu x} - 1}, \quad x = 1, 2, \ldots?$$

After some algebra, this equation simplifies to

$$\left(\frac{\mu}{\nu}\right)^x = \left(\frac{e^\mu - 1}{e^\nu - 1}\right),$$

which cannot hold for all $x \geq 1$ unless $\mu = \nu$.

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