IMPROVED BOHR’S INEQUALITY FOR SIMPLY CONNECTED DOMAINS

STAVROS EVDORIDIS, SAMINATHAN PONNUSAMY, AND ANTTI RASILA

Abstract. In this paper, we study the Bohr phenomenon for functions that are defined on a general simply connected domain of the complex plane. We improve known results of R. Fournier and St. Ruscheweyh for a class of analytic functions. Furthermore, we examine the case where a harmonic mapping is defined in a disk containing \( \mathbb{D} \) and obtain a Bohr type inequality.

1. Introduction and Main Results

Let \( \mathbb{D}(a; r) = \{ z : |z - a| < r \} \), and let \( \mathbb{D} := \mathbb{D}(0; 1) \), the open unit disk in the complex plane \( \mathbb{C} \). For a given simply connected domain \( \Omega \) containing \( \mathbb{D} \), let \( \mathcal{H}(\Omega) \) denote the class of analytic functions on \( \Omega \), and let \( \mathcal{B}(\Omega) \) be the class of functions \( f \in \mathcal{H}(\Omega) \) such that \( f(\Omega) \subseteq \mathbb{D} \). The Bohr radius for the family \( \mathcal{B}(\Omega) \) is defined to be the positive real number \( B = B_\Omega \in (0, 1) \) given by (see [10])

\[
B = \sup \{ r \in (0, 1) : M_f(r) \leq 1 \text{ for all } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}(\Omega), \ z \in \mathbb{D} \},
\]

where \( M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n \) is the majorant series associated with \( f \in \mathcal{B}(\Omega) \) in \( \mathbb{D} \). If \( \Omega = \mathbb{D} \), then it is well-known that \( B_\mathbb{D} = 1/3 \), and it is described precisely as follows:

Theorem A. (The Classical Bohr (radius 1/3) Theorem) If \( f \in \mathcal{B}(\mathbb{D}) \), then \( M_f(r) \leq 1 \) for \( 0 \leq r \leq 1/3 \). The number 1/3 is best possible.

The inequality \( M_f(r) \leq 1 \), for \( f \in \mathcal{B}(\mathbb{D}) \), fails to hold for any \( r > 1/3 \). This can be seen by considering the function \( \varphi_a(z) = (a - z)/(1 - az) \) and by taking \( a \in (0, 1) \) such that \( a \) sufficiently close to 1.

Theorem A was originally obtained by H. Bohr in 1914 [5] for \( 0 \leq r \leq 1/6 \). The optimal value 1/3, which is called the Bohr radius for the disk case, was later established independently by M. Riesz, I. Schur, and F.W. Wiener. Proofs have also been given by Sidon [26] and Tomić [27]. Over the past two decades there has been significant interest in Bohr-type inequalities. See [2, 3, 6–11, 13–16, 19–21, 23, 24] and the references therein. The paper [4] carried Bohr’s theorem to prominence for the case of several complex variables. A series of papers by several authors followed this article extending and generalizing this
phenomenon to many different situations. Readers are referred to [1] and [11, Chapter 8] for more information about Bohr’s inequality and related investigations.

For $0 \leq \gamma < 1$, we consider the disk $\Omega_\gamma$ defined by

$$\Omega_\gamma = \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1 - \gamma} \right| < \frac{1}{1 - \gamma} \right\}.$$

It is clear that the unit disk $\mathbb{D}$ is always a subset of $\Omega_\gamma$. In 2010, Fournier and Ruscheweyh [10] extended Bohr’s inequality in the following form.

**Theorem B.** ([10, Theorem 1]) For $0 \leq \gamma < 1$, let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then,

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq \rho_\gamma := \frac{1 + \gamma}{3 + \gamma},$$

Moreover, $\sum_{n=0}^{\infty} |a_n| \rho_\gamma^n = 1$ holds for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{B}(\Omega_\gamma)$ if and only if $f(z) = c$ with $|c| = 1$.

We are now in a position to state an improved version of this result.

**Theorem 1.** For $0 \leq \gamma < 1$, let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then, we have

$$\sum_{n=0}^{\infty} |a_n| r^n + \frac{8}{9} \left( \frac{S_r(1-\gamma)}{\pi} \right) \leq 1 \text{ for } r \leq \frac{1 + \gamma}{3 + \gamma},$$

where $S_r$ denotes the area of the image of the disk $\mathbb{D}(0; r)$ under the mapping $f$. Moreover, the inequality is strict unless $f$ is a constant function. The bound $8/9$ and the number $(1 + \gamma)/(3 + \gamma)$ cannot be replaced by a larger quantity.

**Conjecture 1.** We conjecture that the constant $8/9$ can be replaced by a decreasing function $t(\gamma)$ from $[0, 1)$ onto $[8/9, 16/9]$.

For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk, we write

$$\|f_0\|_r = \sum_{n=1}^{\infty} |a_n|^2 r^{2n},$$

where $f_0(z) = f(z) - f(0)$. Very recently, the authors have shown in [23] an improvement of Theorem A: if $f \in \mathcal{B}(\mathbb{D})$, then for every $r \leq 1/3$

$$\sum_{n=0}^{\infty} |a_n| r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|_r \leq 1.$$

In the next result we extend this improved version to $\mathcal{B}(\Omega_\gamma)$ and thus, we have the following refinement of Theorem B.

**Theorem 2.** For $0 \leq \gamma < 1$, let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then, we have

$$\sum_{n=0}^{\infty} |a_n| r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|_r \leq 1 \text{ for } r \leq r_0 := \frac{1 + \gamma}{3 + \gamma},$$

and the number $r_0$ cannot be improved.
Theorem 3. Let
\[ \lambda = \lambda(\Omega) = \sup_{f \in \mathcal{B}(\Omega)} \left\{ \frac{|a_n|}{1 - |a_0|^2} : a_0 \neq f(z) = \sum_{n=0}^{\infty} a_n z^n, \ z \in \mathbb{D} \right\}. \]

Theorem 4. Let \( \Omega \supset \mathbb{D} \) be a simply connected domain and \( f \in \mathcal{B}(\Omega) \), with \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( z \in \mathbb{D} \). Then, we have
\[ B_1(r) := \sum_{n=0}^{\infty} |a_n| r^n + 2 \left( \frac{1 + \lambda}{1 + 2\lambda} \right)^2 \frac{S_r}{\pi} \leq 1 \] for \( r \leq \frac{1}{1 + 2\lambda} \),
where \( S_r \) denotes the area of the image of the disk \( \mathbb{D}(0, r) \) under the mapping \( f \).

Recall that a complex-valued function \( f = u + iv \) in a simply connected domain \( \Omega \) is called harmonic in \( \Omega \) if it satisfies the Laplace equation \( \Delta f = 4f_{xx} = 0 \), i.e., \( u \) and \( v \) are real harmonic in \( \Omega \). It follows that every harmonic mapping \( f \) admits a representation of the form \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \Omega \). This representation is unique up to an additive constant. The Jacobian \( J_f \) of \( f \) is given by \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 \).

We say that a locally univalent function \( f \) is sense-preserving if \( J_f(z) > 0 \) in \( \Omega \). Consequently, a harmonic mapping \( f \) is locally univalent and sense-preserving in \( \Omega \) if and only if \( J_f(z) > 0 \) in \( \Omega \); or equivalently if \( h' \neq 0 \) in \( \Omega \) and the dilatation \( \omega_f := g'/h' \) of \( f \) has the property that \( |\omega_f| < 1 \) in \( \Omega \) [18].

If a locally univalent and sense-preserving harmonic mapping \( f = h + \overline{g} \) on \( \Omega \) satisfies the condition \( |\omega_f(z)| \leq k < 1 \) for \( \Omega \), then \( f \) is called \( K \)-quasiregular harmonic mapping on \( \Omega \), where \( K = (1 + k)/(1 - k) \geq 1 \) (cf. [12, 22]). Obviously \( k \to 1 \) corresponds to the limiting case \( K \to \infty \). Harmonic extensions of the classical Bohr theorem have been established in [9, 15, 17, 20, 21].

In the following, we consider a harmonic mapping in \( \Omega_{\gamma} \) and obtain Bohr’s inequality for its restriction to the unit disk.

Theorem 1. Let \( f = h + \overline{g} \) be a harmonic mapping in \( \Omega_{\gamma} \), with \( |h(z)| \leq 1 \) on \( \Omega_{\gamma} \). If \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=1}^{\infty} b_n z^n \) in \( \mathbb{D} \) and \( |g'(z)| \leq k|h'(z)| \) for some \( k \in [0, 1] \), then
\[ \sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq 1 \] for \( r \leq r_0 := \frac{1 + \gamma}{3 + 2k + \gamma} \).

The radius \( r_0 \) is the best possible.

Corollary 1. Let \( f = h + \overline{g} \) be a harmonic mapping in \( \Omega_{\gamma} \), with \( |h(z)| \leq 1 \) on \( \Omega_{\gamma} \). If \( h(z) = \sum_{n=0}^{\infty} a_n z^n \), \( g(z) = \sum_{n=1}^{\infty} b_n z^n \) in \( \mathbb{D} \) and \( f = h + \overline{g} \) is sense-preserving in \( \mathbb{D} \), then
\[ \sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq 1 \] for \( r \leq r_0 := \frac{1 + \gamma}{5 + \gamma} \).

The radius \( r_0 \) is the best possible.
2. Proofs of the main results

2.1. Necessary Lemma. Pick’s conformally invariant form of Schwarz’s lemma states that if \( f \in \mathcal{B}(\mathbb{D}) \), then

\[
|f(z)| \leq \frac{r + |f(0)|}{1 + |f(0)|r} \quad \text{and} \quad |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.
\]

Furthermore, it is well-known that the Taylor coefficients of \( f \in \mathcal{B}(\mathbb{D}) \) satisfy the inequalities:

\[
\frac{|f^{(n)}(0)|}{n!} \leq 1 - |f(0)|^2 \quad \text{for any } n \geq 1.
\]

More generally, we have the following sharp estimate for higher order derivatives.

Lemma C. (Ruscheweyh [25]) For \( f \in \mathcal{B}(\mathbb{D}) \), we have

\[
\frac{|f^{(n)}(\alpha)|}{n!} \leq \frac{1 - |f(\alpha)|^2}{(1 - |\alpha|)^{n-1}(1 - |\alpha|^2)}
\]

for each \( n \geq 1 \) and \( \alpha \in \mathbb{D} \). Moreover, for each fixed \( n \geq 1 \) and \( \alpha \in \mathbb{D} \),

\[
\sup_f (1 - |\alpha|)^{n-1} \frac{|f^{(n)}(\alpha)|}{n!} \frac{1 - |\alpha|^2}{1 - |f(\alpha)|^2} = 1,
\]

where the supremum is taken over all nonconstant analytic functions \( f \in \mathcal{B}(\mathbb{D}) \).

Lemma 1. Let \( g : \mathbb{D} \to \overline{\mathbb{D}} \) be an analytic function, and let \( \gamma \in \mathbb{D} \) be such that \( g(z) = \sum_{n=0}^{\infty} \alpha_n (z - \gamma)^n \) for \( |z - \gamma| < 1 - |\gamma| \). Then

\[
\sum_{n=0}^{\infty} |\alpha_n| r^n + \frac{8}{9} \left( \frac{S^{\gamma}}{\pi} \right) \leq 1 \quad \text{for} \quad r \leq r_0 := \frac{1 - |\gamma|^2}{3 + |\gamma|^2},
\]

where \( S^{\gamma} \) denotes the area of the image of the disk \( \mathbb{D}(\gamma; r(1 - |\gamma|)) \) under the mapping \( g \).

Proof. Without loss of generality, we may assume that \( \gamma \in [0, 1) \). Also, we note that \( z \in D_\gamma := \mathbb{D}(\gamma; 1 - \gamma) \) if and only if \( w = (z - \gamma)/(1 - \gamma) \) lies inside \( \mathbb{D} \).

For \( z \in D_\gamma \), define \( \phi : D_\gamma \to \mathbb{D} \) by \( w = \phi(z) = \frac{z - \gamma}{1 - \gamma} \). Then we have

\[
g(z) = \sum_{n=0}^{\infty} \alpha_n (1 - \gamma)^n \phi(z)^n = \sum_{n=0}^{\infty} b_n \phi(z)^n =: G(\phi(z)),
\]

in \( z \in D_\gamma \), where \( G(w) \) is an analytic function in \( \mathbb{D} \) with

\[
G(w) = \sum_{n=0}^{\infty} b_n w^n \quad \text{for } w \in \mathbb{D},
\]

so that \( g = G \circ \phi \) in \( D_\gamma \).

As in [14], for the area term of the function \( G \), we have the upper bound

\[
\frac{S^{\gamma}}{\pi} = \frac{1}{\pi} \text{Area } [G(\mathbb{D}(0; r))] \leq (1 - |b_0|^2)^2 \frac{r^2}{(1 - r^2)^2} = (1 - |\alpha_0|^2)^2 \frac{r^2}{(1 - r^2)^2}.
\]
Moreover,
\[
\text{Area}\left[G(\mathbb{D}(0; r))\right] = \text{Area}\left[g(\phi^{-1}(\mathbb{D}(0; r)))\right] = \text{Area}\left[g(\mathbb{D}(\gamma; r(1 - \gamma)))\right].
\]

By Lemma C, it follows that
\[
|\alpha_n| = |g^{(n)}(\gamma)| \leq \frac{(1 - |\alpha_0|^2)}{(1 + \gamma)^n} = \frac{1 - |\alpha_0|^2}{(1 + \gamma)(1 - \gamma)^n}
\]
for the Taylor coefficients of \(g\), and therefore, we have the inequality
\[
\sum_{n=1}^{\infty} |\alpha_n| r^n \leq \frac{1 - |\alpha_0|^2}{1 + \gamma} \sum_{n=1}^{\infty} \frac{r^n}{(1 - \gamma)^n} = \frac{1 - |\alpha_0|^2}{1 + \gamma} \left(\frac{r}{1 - \gamma - r}\right).
\]

Thus, by (5) and (6), we find that
\[
\sum_{n=0}^{\infty} |\alpha_n| r^n + K \left(\frac{S^\gamma}{\pi}\right) \leq 1 + \Psi(r)
\]
which is less than or equal to 1 provided that \(\Psi(r) \leq 0\), where
\[
\Psi(r) = |\alpha_0| + \frac{(1 - |\alpha_0|^2)r}{(1 + \gamma)(1 - \gamma - r)} + \frac{K(1 - |\alpha_0|^2)^2 r^2}{(1 - r^2)^2} - 1,
\]
and \(|\alpha_0| \leq 1\), which is always true by the assumption. Now, it is a simple exercise to see that \(\Psi\) is an increasing function of \(r\) for all \(r < 1 - \gamma\), which in turn implies that
\[
\Psi(r) \leq \Psi(r_0) \text{ for } r \leq r_0 := \frac{1 - \gamma^2}{3 + \gamma}.
\]

Thus, to prove that \(\Psi(r) \leq 0\), it suffices to show that \(\Psi(r_0) \leq 0\) for all \(|\alpha_0| \leq 1\). Elementary computation gives that
\[
\Psi(r_0) = \frac{1 - |\alpha_0|^2}{2} \left[1 + 2K(1 - |\alpha_0|^2)\frac{(3 + \gamma)^2(1 - \gamma^2)^2}{[(3 + \gamma)^2 - (1 - \gamma^2)^2]^2} - \frac{2}{1 + |\alpha_0|}\right]
\]
where
\[
F(x) = 1 + 2KA^2(\gamma)(1 - x^2) - \frac{2}{1 + x}, \quad x \in [0, 1],
\]
and
\[
A(\gamma) = \frac{(3 + \gamma)(1 - \gamma^2)}{(3 + \gamma)^2 - (1 - \gamma^2)^2}.
\]

It remains to show that \(F(x) \leq 0\) for all \(x \in [0, 1]\) and \(\gamma \in [0, 1]\). To do this, we first observe that \(A(\gamma) > 0\) for \(\gamma \in [0, 1]\),
\[
F(0) = 2KA^2(\gamma) - 1 \quad \text{and} \quad \lim_{x \to 1^-} F(x) = 0.
\]

Also, we may write
\[
A(\gamma) = (M \circ N)(\gamma), \quad M(r) = \frac{r}{1 - r^2}, \quad \text{and} \quad N(\gamma) = \frac{1 - \gamma^2}{3 + \gamma}.
\]
It follows that $A'(\gamma) = M'(N(\gamma))N'(\gamma)$, where

$$N'(\gamma) = -\left(\frac{\gamma^2 + 6\gamma + 1}{(3 + \gamma)^2}\right),$$

showing that $M(r)$ is an increasing function of $r$ in $(0, 1)$ and $N$ is a decreasing function of $\gamma$ in $[0, 1)$. Therefore, it follows that $A$ is a decreasing function of $\gamma$ in $[0, 1)$, with $A(0) = 3/8$ and $A(1) = 0$, and thus, $A^2(\gamma)$ is decreasing on $[0, 1]$. Hence, we have

$$A^2(\gamma) \leq A^2(0) = \frac{9}{64}.$$ 

Finally, we obtain that

$$F' = -4KA^2(\gamma)x + \frac{2}{1 + x^2}$$

$$\geq \frac{2}{1 + x^2} \left(1 - K\frac{9}{8}\right),$$

which is positive for all $x \in (0, 1)$ whenever $K \leq 8/9$. Thus, $F$ is increasing on $[0, 1]$ for $K \leq 8/9$. In particular, $F(x) \leq 0$ for all $x \in [0, 1]$ and $\gamma \in [0, 1)$. This completes the proof of the lemma. \hfill \square

### 2.2. Proof of Theorem 1.

For $0 \leq \gamma < 1$, let

$$\Omega_{\gamma} = \left\{ z \in \mathbb{C} : \left|z + \frac{\gamma}{1 - \gamma}\right| < \frac{1}{1 - \gamma}\right\}$$

and consider $f : \Omega_{\gamma} \to \overline{\mathbb{D}}$ as in the statement. We consider the analytic function $\phi : \mathbb{D} \to \Omega_{\gamma}$ defined by $\phi(z) = (z - \gamma)/(1 - \gamma)$. Then the composition $g = f \circ \phi$ is analytic in $\mathbb{D}$ and

$$g(z) = (f \circ \phi)(z) = \sum_{n=0}^{\infty} \frac{a_n}{(1 - \gamma)^n} (z - \gamma)^n \quad \text{for} \quad |z - \gamma| < 1 - \gamma.$$ 

Applying Lemma 1 to the function $g$ gives

$$\sum_{n=0}^{\infty} \left|\frac{a_n}{(1 - \gamma)^n}\right| \rho^n + \frac{8}{9} \left(\frac{S_{\gamma}}{\pi}\right) \leq 1 \quad \text{for} \quad \rho \leq \frac{1 - \gamma^2}{3 + \gamma},$$

where $S_{\gamma}$ is defined as in Lemma 1 for $g$; or equivalently,

$$\sum_{n=0}^{\infty} |a_n| \left(\frac{\rho}{1 - \gamma}\right)^n + \frac{8}{9} \left(\frac{S_{\gamma}}{\pi}\right) \leq 1 \quad \text{for} \quad \rho \leq \frac{1 + \gamma}{3 + \gamma}.$$ 

The desired inequality follows by setting $\rho = r(1 - \gamma)$.

In order to prove the sharpness of the result, we consider the composition of the functions $G : \Omega_{\gamma} \to \mathbb{D}$ with $G(z) = (1 - \gamma)z + \gamma$ and $\psi : \mathbb{D} \to \mathbb{D}$ with

$$\psi(z) = \frac{a - z}{1 - az},$$

where $a \in \mathbb{D}$ is such that $|a| < 1$ and $|a(1 - \gamma)| < 1$. Then $\psi$ is analytic in $\mathbb{D}$ and $\psi(G(z)) = z$. Therefore, we have

$$F'(\psi(G(z))) = -4K(1 - \gamma)^4 x + \frac{2}{(1 + x^2)^2} \geq \frac{2}{(1 + x^2)^2} \left(1 - K\frac{9}{8}\right),$$

which is positive for all $x \in (0, 1)$ whenever $K \leq 8/9$. Thus, $F$ is increasing on $[0, 1]$ for $K \leq 8/9$. In particular, $F(x) \leq 0$ for all $x \in [0, 1]$ and $\gamma \in [0, 1)$. This completes the proof of the lemma. \hfill \square
for \( a \in (0, 1) \). Then \( g_0 = \psi \circ G \) maps \( \Omega_\gamma \) univalently onto \( D \). This gives
\[
g_0(z) = \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)z} = A_0 - \sum_{n=1}^{\infty} A_n z^n, \quad z \in D,
\]
where \( a \in (0, 1) \),
\[
A_0 = \frac{a - \gamma}{1 - a\gamma} \quad \text{and} \quad A_n = \frac{1 - a^2}{a(1 - a\gamma)} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^n.
\]

For a given \( \gamma \in [0, 1) \), let \( a > \gamma \) and find that
\[
\sum_{n=0}^{\infty} |a_n| r^n + \frac{8}{9} \left( \frac{S_r(1 - \gamma)}{\pi} \right) = A_0 + \sum_{n=1}^{\infty} A_n r^n + \frac{8}{9} \sum_{n=1}^{\infty} n A_n^2 r^{2n} = \frac{a - \gamma}{1 - a\gamma} + \frac{1 - a^2}{a(1 - a\gamma)} \frac{(1 - \gamma)r}{1 - a\gamma - ar(1 - \gamma)} + \frac{8}{9} \frac{r^2(1 - a^2)^2(1 - \gamma)^4}{[(1 - a\gamma)^2 - a^2 r^2(1 - \gamma)^4]^2} = 1 - (1 - a) \Phi(r),
\]
where
\[
\Phi(r) = \frac{1 + \gamma}{1 - a\gamma} - \frac{1 + a}{1 - a\gamma} \frac{r(1 - \gamma)}{1 - a\gamma - ar(1 - \gamma)} - \frac{8}{9} \frac{(1 - a)(1 + a)^2(1 - \gamma)^4 r^2}{[(1 - a\gamma)^2 - a^2 r^2(1 - \gamma)^4]^2}.
\]
Then, \( \Phi \) is a strictly decreasing function of \( r \) in \((0, 1)\) and thus, for \( r > r_0 = (1 + \gamma)(3 + \gamma) \), we have
\[
\Phi(r) < \Phi(r_0) = \frac{1 + \gamma}{1 - a\gamma} - \frac{1 + a}{1 - a\gamma} \frac{1 - \gamma^2}{(1 - a\gamma)(3 + \gamma) - a(1 - \gamma^2)} - \frac{8}{9} \frac{(1 + a)^2(1 - \gamma)^4(1 + \gamma)^2(3 + \gamma)^2}{[(1 - a\gamma)^2 - a^2(1 - \gamma)^4(1 + \gamma)^2]^2},
\]
which tends to 0 as \( a \to 1 \). Therefore, \( \Phi \) is negative for \( r > r_0 \) and hence, \( 1 - (1 - a) \Phi(r) > 1 \).

For the proof of Theorem 2, we modify the previous arguments slightly and prove the following:

**Lemma 2.** For \( \gamma \in [0, 1) \), let
\[
\Omega_\gamma := \left\{ z \in \mathbb{C} : |z + \frac{\gamma}{1 - \gamma}| < \frac{1}{1 - \gamma} \right\},
\]
and let \( f \) be an analytic function in \( \Omega_\gamma \), bounded by 1, with the series representation \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in the unit disk \( D \). Then,
\[
|a_n| \leq \frac{1 - |a_0|^2}{1 + \gamma} \quad \text{for} \quad n \geq 1.
\]
Proof. Clearly, \( \phi(z) = (z - \gamma)/(1 - \gamma) \in \Omega_I \) if and only if \( z \in \mathbb{D} \). Then \( g : \mathbb{D} \to \mathbb{D} \) defined by \( g(z) = (f \circ \phi)(z) \), is analytic in \( \mathbb{D} \); \( g(\gamma) = f(0) = a_0 \) and
\[
\frac{g^{(n)}(z)}{n!} = f^{(n)} \left( \frac{z - \gamma}{1 - \gamma} \right) \frac{1}{(1 - \gamma)^n}.
\]
In particular, at \( z = \gamma \), this gives
\[
(1 - \gamma)^n \frac{g^{(n)}(\gamma)}{(n!)^2} = \frac{f^{(n)}(0)}{n!} = a_n \quad \text{for } n \geq 1.
\]
By Lemma C we deduce that for \( n \geq 1 \),
\[
|a_n| = (1 - \gamma)^n \frac{|g^{(n)}(\gamma)|}{n!} \leq \frac{1 - |g(\gamma)|^2}{(1 + \gamma)} = \frac{1 - |a_0|^2}{1 + \gamma},
\]
and this completes the proof. \( \square \)

2.3. Proof of Theorem 2. Without loss of generality, we may assume that \( a_0 := a \in (0, 1) \). By applying Lemma 2 we obtain
\[
M_f(r) = \sum_{n=0}^{\infty} |a_n|r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|
\]
\[
\leq a + \frac{1 - a^2}{1 + \gamma} \sum_{n=1}^{\infty} r^n + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{1 + \gamma} \right)^2 \sum_{n=1}^{\infty} r^{2n}
\]
\[
= a + \frac{1 - a^2}{1 + \gamma} \frac{r}{1 - r} + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{1 + \gamma} \right)^2 \frac{r^2}{1 - r^2} = u(a) \quad \text{(say)}.
\]
Here \( u(a) = a + A(1 - a^2) + B(1 - a)(1 - a^2) + C(1 - a^2)^2 \) for \( a \in [0, 1] \), where
\[
A = \frac{1}{1 + \gamma} \frac{r}{1 - r}, \quad B = \frac{r^2}{(1 + \gamma)^2 (1 - r)(1 - r^2)} \quad \text{and} \quad C = \frac{1}{(1 + \gamma)^2 (1 - r)(1 - r^2)}.
\]
Now,
\[
u'(a) = 1 - 2Aa + B(3a^2 - 2a - 1) + 4C(a^3 - a)
\]
and
\[
u''(a) = -2A + 2B(3a - 1) + 4C(3a^2 - 1).
\]
Because \( B \) and \( C \) are non-negative, \( \nu'' \) is an increasing function of \( a \) in \([0, 1]\), it follows that
\[
u''(a) \leq \nu''(1) = -2A + 4B + 8C = \frac{2r}{(1 + \gamma)^2 (1 - r)(1 - r^2)} \Psi(r),
\]
where
\[
\Psi(r) = 4r^2 + 2r(1 - r) - (1 + \gamma)(1 - r^2) = (1 + r)(r(3 + \gamma) - (1 + \gamma))
\]
which is non-positive for \( r \leq r_0 = (1 + \gamma)/(3 + \gamma) \). Thus, we obtain that \( \nu''(a) \leq 0 \) for \( a \in [0, 1] \) and thus, \( \nu'(a) \) is decreasing on \([0, 1]\). Therefore, for \( r \leq \frac{1 + \gamma}{3 + \gamma} \), we have
\[
u'(a) \geq \nu'(1) = 1 - 2A = \frac{1 + \gamma - r(3 + \gamma)}{(1 + \gamma)(1 - r)} \geq 0 \quad \text{for all } a \in [0, 1].
from which it follows that $u(a) \leq u(1) = 1$ and this proves the stated inequality.

For the sharpness of the radius, as in the case of Theorem 1, we consider the function

$$g_0(z) = \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)z} = A_0 - \sum_{n=1}^{\infty} A_n z^n, \ z \in \mathbb{D},$$

where $a \in (0, 1)$, and $A_n (n \geq 0)$ are given by (7). Now, for a given $\gamma \in [0, 1)$, we also let $a > \gamma$. Then

$$M_{g_0}(r) = A_0 + \sum_{n=0}^{\infty} A_n r^n + \left( \frac{1}{1 + A_0} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} A_n^2 r^{2n}$$

$$= \frac{a - \gamma}{1 - a\gamma} + \frac{1 - a^2}{a(1 - a\gamma)} \sum_{n=1}^{\infty} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^n r^n$$

$$+ \left( \frac{1 - a\gamma}{(1 + a)(1 - \gamma)} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{a(1 - a\gamma)} \right)^2 \sum_{n=1}^{\infty} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^{2n} r^{2n}$$

$$= 1 - \frac{1}{1 - a\gamma} \Phi(r),$$

where

$$\Phi(r) = 1 + \gamma - \frac{(1 + a)(1 - \gamma)r}{1 - a\gamma - a(1 - \gamma)r}$$

$$- \left( \frac{1 - a\gamma}{(1 + a)(1 - \gamma)} + \frac{r}{1 - r} \right) \frac{(1 + a)(1 - a^2)}{1 - a\gamma} \frac{(1 - \gamma)^2 r^2}{(1 - a\gamma)^2 - a^2(1 - \gamma)^2 r^2}. $$

The function $\Phi$ is strictly decreasing for $r$ in $(0, 1)$ and thus, if $r > r_0$, then $\Phi(r) < \Phi(r_0) \rightarrow 0$ as $a \rightarrow 1$. Therefore, $\Phi(r)$ is negative for $r > r_0$, as $a$ tends to 1, and hence $M_{g_0}(r) > 1$ for $r > r_0$. □

2.4. Proof of Theorem 3. By the definition of $\lambda$ given by (1), we have

$$|a_n| \leq \lambda(1 - |a_0|^2) \text{ for } n \geq 1. \tag{8}$$

For the term $S_r$ we have

$$\frac{S_r}{\pi} = \frac{1}{\pi} \iint_{|z| < r} |f'(z)|^2 \, dx \, dy \leq \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$$

$$\leq \lambda^2(1 - |a_0|^2)^2 \sum_{n=1}^{\infty} nr^{2n} = \lambda^2(1 - |a_0|^2)^2 \frac{r^2}{(1 - r^2)^2}. $$

Therefore, by using the expression for $B_1(r)$ given by (2) and the inequality (8), we obtain for $r \leq 1/(1 + 2\lambda)$ that

$$B_1(r) \leq |a_0| + \lambda(1 - |a_0|^2) \frac{r}{1-r} + 2 \left( \frac{1 + \lambda}{1 + 2\lambda} \right)^2 \lambda^2(1 - |a_0|^2)^2 \frac{r^2}{(1-r^2)^2} \leq |a_0| + \lambda(1 - |a_0|^2) \frac{1/(1 + 2\lambda)}{1 - 1/(1 + 2\lambda)}$$

$$+ 2 \left( \frac{1 + \lambda}{1 + 2\lambda} \right)^2 \lambda^2(1 - |a_0|^2)^2 \frac{1/(1 + 2\lambda)^2}{(1 - (1/(1 + 2\lambda)^2))^2}$$

$$= 1 - (1 - |a_0|) \left[ 1 - \frac{1 + |a_0|}{2} - \frac{(1 + |a_0|)(1 - |a_0|^2)}{8} \right]$$

$$= 1 - \frac{(1 - |a_0|^2)}{8} F(|a_0|),$$

where

$$F(x) = \frac{8}{1+x} - 5 + x^2.$$  

We see that $F(0) = 3$, $F(1) = 0$, $F'(x) \leq 0$ in $[0,1]$ and thus, $F(x) \geq F(1) = 0$ for $x \in [0,1]$. This observation shows that $B_1(r) \leq 1$ for $r \leq 1/(1 + 2\lambda)$ and the proof of the theorem is completed.

□

2.5. **Proof of Theorem 4.** The function $h$ is analytic in $\Omega_\gamma$, with $|h(z)| \leq 1$, and hence Lemma 2 implies that

$$|a_n| \leq \frac{1 - |a_0|^2}{1 + \gamma}$$

for $n \geq 1$. By [17, Lemma 1], the condition $|g'(z)| \leq k|h'(z)|$ gives that

$$\sum_{n=0}^{\infty} |b_n|^2r^n \leq k^2 \sum_{n=0}^{\infty} |a_n|^2r^n$$

for $r < 1$, and thus

$$\sum_{n=1}^{\infty} |b_n|r^n \leq \sqrt{\sum_{n=0}^{\infty} |b_n|^2r^n} \sqrt{\sum_{n=1}^{\infty} r^n} \leq \frac{k(1 - |a_0|^2)}{1 + \gamma} \frac{r}{1-r}.$$  

Therefore, with $|a_0| = a \geq 0$,

$$N_f(r) = \sum_{n=0}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \leq a + (1+k) \frac{1-a^2}{1+\gamma} \frac{r}{1-r}$$

$$= 1 - \frac{1-a}{(1+\gamma)(1-r)} \{1+\gamma - r[1+\gamma + (1+a)(1+k)]\},$$

which is less than or equal to 1 whenever $r \leq r_0(a)$, where

$$r_0(a) = \frac{1+\gamma}{1+\gamma + (1+a)(1+k)}.$$
Bohr’s inequality for simply connected domains

This gives the condition

\[ r \leq \frac{1 + \gamma}{3 + 2k + \gamma} = r_0(1). \]

For the sharpness, we consider the function \( f_0 = h_0 + \overline{g_0} \), in \( \Omega_\gamma \), where

\[ h_0(z) = \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)z} = A_0 - \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{D}, \]

where \( a \in (0, 1) \), and \( A_n (n \geq 0) \) are given by (7), and

\[ f_0(z) = k\lambda [h_0(z) - A_0]. \]

Thus, we find that

\[
N_{f_0}(r) = A_0 + (1 + k\lambda) \sum_{n=1}^{\infty} A_n r^n \\
= \frac{a - \gamma}{1 - a\gamma} + (1 + k\lambda) \frac{1 - a^2}{a(1 - a\gamma)} \sum_{n=1}^{\infty} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^n r^n \\
= 1 - \frac{1 - a}{1 - a\gamma} \Phi(r),
\]

where

\[ \Phi(r) = 1 + \gamma - (1 + \lambda) \frac{(1 + a)(1 - \gamma) r}{1 - a\gamma - a(1 - \gamma) r}, \]

which is strictly decreasing for \( r \in [0, 1) \), and hence, for \( r > r_0(1) \)

\[ \Phi(r) < \Phi(r_0(1)) = 1 + \gamma - (1 + k\lambda) \frac{(1 + a)(1 - \gamma^2)}{(1 - a\gamma)(3 + 2k + \gamma) - a(1 - \gamma^2)}, \]

which approaches 0 as \( a \) and \( \lambda \) tend to 1. \( \square \)

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S. Evdoridis, AALTO UNIVERSITY, DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS, P. O. BOX 11100, FI-00076 AALTO, FINLAND.
E-mail address: stavros.evdoridis@aalto.fi

S. Ponnusamy, DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY MADRAS, CHENNAI-600 036, INDIA
E-mail address: samy@iitm.ac.in

A. Rasila, TECHNION – ISRAEL INSTITUTE OF TECHNOLOGY, GUANGDONG TECHNION, 241 DAXUE ROAD, SHANTOU 515063, GUANGDONG, PEOPLE'S REPUBLIC OF CHINA
E-mail address: antti.rasila@iki.fi; antti.rasila@gtiit.edu.cn