THE KRENGEL’S THEOREM FOR COMPACT OPERATORS BETWEEN LOCALLY SOLID VECTOR LATTICES

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Abstract. Suppose $X$ is a locally solid vector lattice. It is known that there are several non-equivalent notions for compact operators on $X$. Furthermore, notion of the $AM$-property in $X$ as an extension for the $AM$-spaces in Banach lattices has been considered, recently. In this paper, we establish a variant of the known Krengel’s theorem for different types of compact operators between locally solid vector lattices.

1. MOTIVATION AND INTRODUCTION

Let us start with some motivation. Let $E$ be a Banach lattice. $E$ is called an $AM$-space provided that for each positive $x, y \in E_+$, we have $\|x \vee y\| = \|x\| \vee \|y\|$. The remarkable Kakutani’s theorem states that every $AM$-space is a closed sublattice of $C(K)$ for some compact Hausdorff space $K$. Now, suppose $E$ is a Banach lattice and $F$ is an $AM$-space. The Krengel’s theorem states that every compact operator $T : E \to F$ has a modulus which is defined by the Riesz-Kantorovich formulae; that is $|T|(x) = \sup \{|Ty| : |y| \leq x\}$ for each $x \in E_+$. So, we conclude that $AM$-spaces have many interesting properties among the category of all Banach lattices. Therefore, it is fascinating and significant to consider the $AM$-spaces and numerous applications in the operator theory to the locally solid vector lattices and operators between them. The first step has been done in [3]; namely, the $AM$-property which is the right extension for the $AM$-spaces. Moreover, Observe that there are several different ways to define bounded and compact operators between locally solid vector lattices. Some applications of the $AM$-property in these classes of operators have been obtained in [3,5]. In this paper, we are going to generalize the known Krengel’s theorem [2, Theorem 5.7] for different types of compact operators between locally solid vector lattices.

For undefined terminology and related notions, see [1,2]. All locally solid vector lattices in this note are assumed to be Hausdorff.

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2. MAIN RESULT

First, we recall the notion of the AM-property; for more details, see [5]. Suppose \( X \) is a locally solid vector lattice. We say that \( X \) has the AM-property provided that for every bounded set \( B \subseteq X \), \( B\alpha \) is also bounded with the same scalars; namely, given a zero neighborhood \( V \) and any positive scalar \( \alpha \) with \( B \subseteq \alpha V \), we have \( B\alpha \subseteq \alpha V \). Note that by \( B\alpha \), we mean the set of all finite suprema of elements of \( B \).

In this part, we recall the following useful fact; for more details, see [6, Lemma 3].

**Lemma 1.** Suppose \( X \) is a locally solid vector lattice with the AM-property and \( U \) is an arbitrary solid zero neighborhood in \( X \). Then, for each \( m \in \mathbb{N} \), \( U \vee \ldots \vee U = U \), in which \( U \) is appeared \( m \)-times.

Moreover, we have the following useful inequality in Archimedean vector lattices.

**Lemma 2.** Suppose \( E \) is an Archimedean vector lattice. Then for \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) in \( E \), the following inequality holds.

\[
x_1 \vee \ldots \vee x_n - y_1 \vee \ldots \vee y_n \leq (x_1 - y_1) \vee \ldots \vee (x_n - y_n).
\]

**Proof.** We proceed the proof by induction. For \( n = 2 \), we have

\[
x_1 \wedge x_2 - y_1 \wedge y_2 = (x_1 - (y_1 \wedge y_2)) \vee (x_2 - (y_1 \wedge y_2)) = (x_1 + ((-y_1) \wedge (-y_2))) \vee (x_2 + ((-y_1) \wedge (-y_2))) =
\]

\[
((x_1 - y_1) \wedge (x_1 - y_2)) \vee ((x_2 - y_1) \wedge (x_2 - y_2)) \leq (x_1 - y_1) \vee (x_2 - y_2).
\]

Now, suppose for \( n = k \), the statement is valid. We need prove it for \( n = k + 1 \). By using validness of the result for \( n = 2 \) and \( n = k \), we have

\[
x_1 \vee \ldots \vee x_k \vee x_{k+1} - y_1 \vee \ldots \vee y_k \vee y_{k+1} \leq ((x_1 \vee \ldots \vee x_k) - (y_1 \vee \ldots \vee y_k)) \vee (x_{k+1} - y_{k+1})
\]

\[
\leq (x_1 - y_1) \vee \ldots \vee (x_k - y_k) \vee (x_{k+1} - y_{k+1}).
\]

\[\Box\]

Recall that a subset \( B \) of a topological vector space \( X \) is said to be totally bounded if for each arbitrary zero neighborhood \( V \subseteq X \) there is a finite set \( F \) such that \( B \subseteq F + V \); for more information, see [2].

**Lemma 3.** Suppose \( X \) is a locally solid vector lattice with the AM-property. If \( B \subseteq X \) is totally bounded, then so is \( B\alpha \). In particular, sup \( B \) exists in \( X \) and sup \( B \subseteq B\alpha \).

**Proof.** Choose arbitrary solid zero neighborhood \( U \subseteq X \). By the assumption, there exists a finite set \( F \subseteq X \) such that \( B \subseteq F + U \). Assume that \( F = \{z_1, \ldots, z_m\} \). Put \( z_0 = z_1 \vee \ldots \vee z_m \). We claim that \( B\alpha \subseteq \{z_0\} + U \). Given any \( x_1, \ldots, x_n \in B \). There are
some $z_1, \ldots, z_n$ (possibly with the repetition), such that $x_i - z_i \in U$ for all $i = 1, \ldots, n$. Therefore, by using Lemma 2 and Lemma 1 we have

$$x_1 \vee \ldots \vee x_n - z_1 \vee \ldots \vee (z_n - x_n) \in U \vee \ldots \vee U = U.$$

Since $U$ is solid, similarly, we have

$$z_1 \vee \ldots \vee z_n - x_1 \vee \ldots \vee x_n \leq (z_1 - x_1) \vee \ldots \vee (z_n - x_n) \in U \vee \ldots \vee U = U.$$

This means that $(x_1 \vee \ldots \vee x_n) - z_0 \in U$.

Now, assume that $D$ is the set of all finite subsets of $B$ directed by the inclusion $\subseteq$. For each $\alpha \in D$, put $g_\alpha = \sup \alpha$. Observe that $\{g_\alpha\} \subseteq B^\vee$ satisfies $g_\alpha \uparrow$. By compactness of $B^\vee$, there exists a subnet of $(g_\alpha)$ that converges to some $g \in B^\vee$. Therefore, $\sup B = \sup B^\vee = \sup \{g_\alpha\} = g$. □

Now, we are able to consider a version of the Krengel’s theorem ([2, Theorem 5.7]) for each class of compact operators between locally solid vector lattices. First, we recall some preliminaries which are needed in the sequel.

Suppose $X$ and $Y$ are locally solid vector lattices and $T : X \to Y$ is a linear operator. $T$ is called nb-bounded if there is a zero neighborhood $U \subseteq X$ such that $T(U)$ is also bounded in $Y$; $T$ is said to be bb-bounded if it maps bounded sets into bounded sets.

Moreover, a linear operator $T : X \to Y$ is said to be nb-compact provided that there is a zero neighborhood $U \subseteq X$ such that $\overline{T(U)}$ is compact in $Y$; $T$ is bb-compact if for every bounded set $B \subseteq X$, $\overline{T(B)}$ is compact in $Y$. It is obvious that every nb-compact operator is nb-bounded and every bb-compact operator in bb-bounded. These classes of operators enjoy some topological and lattice structures; for a detailed exposition as well as related notions about bounded and compact operators see [3, 4, 5].

Krengel has proved that when the range of a compact operator $T$ between Banach lattices is an AM-space, then the modulus of $T$ exists and is also compact (see [2, Theorem 5.7]). In the following, we prove this remarkable result for nb-compact operators as well as for bb-compact operators when the range space has the AM-property.

**Theorem 4.** Suppose $X$ and $Y$ are locally solid vector lattices such that $Y$ possesses the AM-property and $T : X \to Y$ is an nb-compact operator. Then the modulus of $T$ exists and is also nb-compact.

**Proof.** There exists a zero neighborhood $U \subseteq X$ such that $T(U)$ is totally bounded in $Y$. Observe that for each $x \in U^+$, $T[-x, x]$ is totally bounded in $Y$ so that by Lemma 3 the supremum $|T|(x) = \sup \{ |Ty| : |y| \leq x \} = \sup T[-x, x]$ exists in $Y$. Thus, by [2, Theorem 1.14], the modulus of $T$ exists. According to Lemma 3, $\overline{T(U)^\vee}$ is also
compact and \(|T|(x) \in \overline{T(U)}^\vee\). Therefore, \(|T|(U_+) \subseteq \overline{T(U)}^\vee\). Since \(U \subseteq U_+ - U_+\), the proof would be complete.

\(\square\)

**Theorem 5.** Suppose \(X\) and \(Y\) are locally solid vector lattices such that \(Y\) possesses the AM-property and \(T : X \to Y\) is a \(bb\)-compact operator. Then the modulus of \(T\) exists and is also \(bb\)-compact.

**Proof.** The proof essentially has the same line. Fix a bounded set \(B \subseteq X\) such that \(T(B)\) is totally bounded in \(Y\); by replacing \(B\) with \(Sol(B)\), if necessary, we may assume that \(B\) is solid. Observe that for each \(x \in B_+, T[-x, x]\) is totally bounded in \(Y\) so that by Lemma 3, the supremum \(|T|(x) = \sup\{|Ty| : |y| \leq x\} = \sup T[-x, x]\) exists in \(Y\). Thus, by [2, Theorem 1.14], the modulus of \(T\) exists. According to Lemma 3, \(T(B)\) is also compact and \(|T|(x) \in \overline{T(B)}^\vee\). Therefore, \(|T|(B_+) \subseteq \overline{T(B)}^\vee\). Since \(B \subseteq B_+ - B_+\), we have the desired result. \(\square\)

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