LOWER BOUNDS ON THE CANONICAL HEIGHT OF NON-TORSION POINTS ON MORDELL CURVES

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Abstract. Le Boudec establishes a lower bound on the minimal canonical height of the non-torsion rational points of a natural density 1 subset of the family of quadratic twists of the congruent number elliptic curve. We establish similar results for the sextic twist family of Mordell curves given by $y^2 = x^3 + d$.

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1. Introduction

1.1. Summary. In Le Boudec’s work [LB16], a lower bound on the minimal non-zero canonical height of rational points of curves in a general quadratic twist family $E(d)(A, B) := dy^2 = x^3 + Ax + B$ [LB16, Theorem 1] is established. Stronger bounds are proven upon specialization to an analysis on the minimal non-zero canonical height of rational points of curves in the family of quadratic twists of the congruent number curve, given by $dy^2 = x^3 - x$ for $d \in \mathbb{Z}_{>1}$ square-free. To formally state these results, we make the following definition:

Definition 1.1. Let $\log \eta_d(A, B) := \min \{\hat{h}_{E(d)}(P), P \in E(d)(Q) \setminus E(d)(Q)_{\text{tors}}\}$. We use the convention that if $E(d)(Q) \setminus E(d)(Q)_{\text{tors}}$ is empty, then $\eta_d(A, B) = +\infty$.

Above, $\hat{h}_E$ represents the canonical, or Néron-Tate, height on $E$. Before stating the main Theorems of [LB16, LB18], let us set a target. By the analogy between number fields and elliptic curves, we have the following conjecture [LB16, Conjecture A]:

Date: January 2021.
2010 Mathematics Subject Classification. Primary: 11G05, Secondary: 14G05, 14G25.
I would like to thank Professor Chao Li of Columbia University for his mentorship during this project, which has been invaluable. This paper has been made possible by a grant funded by the Rabi Scholars Program at Columbia University, to which I am grateful. Both Li’s and the Program’s cooperation with me during the COVID-19 pandemic has been exceptional and much excellent work has been done on their part to enhance my research experience. Finally, the Rabi Scholars Program has done an excellent job to keep summer research for undergraduates within the program during the pandemic running as smooth as can be.
Conjecture 1.2. Define $S(X)$ to be the set of positive square-free integers up to $X$. Let $\epsilon > 0$ be fixed and let $A, B \in \mathbb{Z}$ such that $4A^3 + 27B^2 \neq 0$. Then, the set of square-free $d \in \mathbb{Z}_{\geq 1}$ such that
\[ \eta_d(A, B) > e^{d^{1/2-\epsilon}} \] (1.1)
has natural density 1 in $S(X)$.

The progress towards this conjecture is shown by the following three bounds:

Theorem 1.3. Let $\epsilon > 0$ be fixed and let $A, B \in \mathbb{Z}$ such that $4A^3 + 27B^2 \neq 0$. For almost every square-free $d \in \mathbb{Z}_{\geq 1}$, we have
\begin{itemize}
  \item (LB16 Theorem 1]) $\eta_d(A, B) > d^{1/4-\epsilon}$. \\
  \item (LB16 Theorem 2]) $\eta_d(-1, 0) > d^{5/8-\epsilon}$. \\
  \item (LB18 Theorem 1]) A positive proportion of square-free integers $d \in \mathbb{Z}_{\geq 1}$ are congruent numbers satisfying the lower bound
    \[ \eta_d(-1, 0) > d^{0.845}. \] (1.2)
\end{itemize}

With regard to concrete results and closeness to the aforementioned conjecture, these are the best ones in the literature. We seek to expand these results to a different family of elliptic curves. In particular, we choose to do so in thinking about a larger twist family: the sextic twist family of the Mordell curve, given by $E_d : y^2 = x^3 + d$, where $d \in \mathbb{Z}_{\neq 0}$ is now sixth-power free.

1.2. Conjectural Bound for $\zeta_d$. In analogy to [LB16], we define the quantity:

Definition 1.4. Let $\log \zeta_d := \min(\hat{h}_E(P), P \in E_d(\mathbb{Q}) \setminus E_d(\mathbb{Q})_{\text{tors}})$. We use the convention that if $E_d(\mathbb{Q}) \setminus E_d(\mathbb{Q})_{\text{tors}}$ is empty, then $\zeta_d = +\infty$.

Before we begin, let us set up a target analogous to Conjecture [1.2].

Conjecture 1.5. Let $\epsilon > 0$ be fixed and define $S_6(X) = \{d \in \mathbb{Z} : |d| \leq X, d \text{ sixth-power free}\}$. Then, the set of $d \in S_6(X) - \{0\}$ such that
\[ \zeta_d > e^{|d|^{1/6-\epsilon}} \] (1.3)
has natural density 1 in $S_6(X)$.

The exponent was chosen so that specializing $d$ to $d^3$ reproduces Conjecture [1.2].

Now, Goldfeld’s conjecture (see [Gol79]) allows us to simplify Conjecture [1.2] into the following question: “Is it true that for almost all square-free $d$ such that $\text{rank}(E_d(\mathbb{Q})) = 1$ we have the inequality in Conjecture [1.2]”? A similar simplification is conjectured for the family of Mordell curves given by $y^2 = x^3 + d$, as Goldfeld’s conjecture, with the quadratic twist family replaced by $E_d$, is also expected to hold.

Furthermore, the progress made toward Goldfeld’s conjecture in [Smi16 Theorem 1.5] is the key ingredient of the proof of [LB18 Theorem 1]. Because similar progress on Goldfeld’s conjecture as formulated in Conjecture [1.5] has been made (see [KL19 Theorem 1.8], and for its contextualization see [Li18 Theorem 5.4] and the sources therein), analogous improvement towards Conjecture [1.5] is expected in the case of the sextic twist family of Mordell curves.

Our target now is to make progress towards this conjecture via the following Theorem.

Theorem 1.6. Let $d \in \mathbb{Z}_{\neq 0}$ sixth-power free. Then, the set of $d \in S_6(X) - \{0\}$ such that
\[ \zeta_d > |d|^{2/9-\epsilon} \] (1.4)
where $\epsilon > 0$ has natural density 1 in $S_6(X)$.
The proof of this Theorem will combine the techniques used in the proof of the first and second bound of Theorem [1.3] The first bound does not require much heavy technology and only uses elementary parameterization data of the projective twist \(dy^2z = x^3 - xz^2\), which we provide in Lemma [2.5] The second bound requires a more technical geometry of numbers result ([LB16, Lemma 4]), which for us will be due to Heath-Brown ([HB02, Theorem 3]).

**Remark.** The proof of the second bound of Theorem [1.3] also requires 2-descent, which is not required for Theorem [1.6] It would be interesting to see if 3-descent may be used in our case to strengthen Theorem [1.6]. In this direction, it would be interesting to pursue stronger bounds in the quantitative arithmetic of projective varieties (see [Bro09] for an exposition) and further improve our results. Indeed, it is this field of study which allows for the greater simplicity of the proof of Theorem 1.6. In this direction, it would be interesting to see if 3-descent may be used in our case to improve our results.

2. Initial Estimate

Let us set the quantities

\[
N_\alpha(X) := \#\{d \in S_\delta(X) : \zeta_d < |d|^\alpha\} \tag{2.1}
\]

\[
N'_\alpha(X) := \sum_{d \in S_\delta(X)} \#\{P \in E_d(\mathbb{Q}) \setminus E_d(\mathbb{Q})_{\text{tors}} : e^{\hat{h}_{E_d}(P)} < |d|^\alpha\} \tag{2.2}
\]

for the remainder of the paper. We seek to maximize \(\alpha\) with respect to the condition that \(N_\alpha(X) \ll X\). However, each \(d\) counted by \(N_\alpha(X)\) is in correspondence with at least one point \(P \in E_d(\mathbb{Q})\) such that \(e^{\hat{h}_{E_d}(P)} < |d|^\alpha\). Hence, \(N_\alpha(X) \leq N'_\alpha(X)\).

2.1. Reducing Canonical Height to Polynomial Height. It is well-known that the definition of \(\hat{h}\) as cited earlier is not very effective in computing the canonical height. So, we consider the following asymptotic for a pseudo-computation of the canonical height (see [Sil09, §VIII.9, Theorem 9.3(e)] for the proof):

**Lemma 2.1.** Let \(f \in K(E)\) be an even function. Then,

\[
(deg f)\hat{h}_E(P) = h_f(P) + O(1), \tag{2.3}
\]

where the \(O(1)\) depends on \(E\) and \(f\).

We use this Lemma to modify the condition \(e^{\hat{h}_{E_d}(P)} < |d|^\alpha\) in the definition of \(N'_\alpha(X)\). To start, we introduce the following definition:

**Definition 2.2.** Let \(g : C_1 \to C_2\) be a nonconstant map of smooth curves defined over \(K\), and let \(P \in C_1\). The **ramification index** of \(g\) at \(P\) is the quantity

\[
e_g(P) := ord_p(t_{g(P)} \circ g), \tag{2.4}
\]

where \(t_{g(P)} \in K(C_2)\) is a uniformizer at \(g(P)\), and \(g^* : K(C_2) \to K(C_1)\) is the usual injection of function fields induced by \(g\) and given by \(g^*(f) = f \circ g\).

The following Lemma will allow us to make the modification:

**Lemma 2.3.** For any \(P \in E_d(\mathbb{Q})\), we have the following asymptotic

\[
\hat{h}_{E_d}(P) = \frac{1}{6} h_f(P) + O(1), \tag{2.5}
\]

where \(f(P) = [x(P)^3/y(P)^2 : 1] \in K(E_d)\). Here, \(x\) and \(y\) are Weierstrass coordinates are the curve \(E_d\). Here, the \(O(1)\) constant may depend on \(E_1\) but neither on \(P\) nor \(d\).
Proof. We have an isomorphism \( \kappa : E_d \to E_1 \) with \( \kappa(x : y : z) = (d^{1/3}x : d^{1/2}y : z) \). Because the canonical height of an elliptic curve \( E \) is invariant under the algebraic representation of \( E \), we have that
\[
\hat{h}_{E_d}(P) = \hat{h}_{E_1}(\kappa(P)).
\]  
(2.6)
Since \( f \) is even in \( K(E_1) \), the assumption that \( \deg f = 6 \) yields
\[
\hat{h}_{E_1}(Q) = \frac{1}{6} h_f(Q) + O(1)
\]  
(2.7)
for any \( Q \in E_1(\overline{\mathbb{Q}}) \) by [Sil09, §VIII.9, Theorem 9.3(e)]. This also tells us that \( O(1) \) doesn’t depend on \( Q \). With this result we are done, since \( h_f(\kappa(P)) = h_f(P) \).

It remains to prove the assertion that \( \deg f = 6 \) where \( f = y^2/x^3 \). In light of [Sil09, §II.2, Proposition 2.6(a)], we may compute the degree by observing the preimage of the point \([1,0] \in \mathbb{P}^1 \), which are precisely the 2-torsion points. Let \( P_1, P_2, P_3 \) be the 2-torsion points of \( E_1 \) over \( \overline{\mathbb{Q}} \). The quantity \( e_f(P_i) \) is invariant under choice of \( i \), so we assume \( i = 1 \). This is also justified since \( x(P_i) \neq 0 \) for any \( i \).

From Definition 2.2, taking \( t_{[0,1]} = x \), we have
\[
e_f(P_1) = \text{ord}_{P_1}(t_{[0,1]} \circ f) = \text{ord}_{P_1}(y^2/x^3).
\]  
(2.8)
Since \( x(P_1) \neq 0 \), we may simplify the above to
\[
e_f(P_1) = \text{ord}_{P_1}(y^2).
\]  
(2.9)
We now prove that \( y \) is a uniformizer in \( \overline{\mathbb{Q}}(E_1) \) for \( P_1 \). This means showing \( m = (x - x(P_1), y) \subseteq \overline{\mathbb{Q}}[E_1] \). But \( x-x(P_1) = y^2(x-x(P_2))^{-1}(x-x(P_3))^{-1} \), and since \( (x-x(P_2))(x-x(P_3)) \neq 0 \) at \( P_1 \), it is in particular invertible, and so \( m = (x - x(P_1), y) = (y, y^2) = (y) \). Hence, \( \text{ord}_{P_1}(y) = 1 \), and so \( e_f(P_1) = 2 \). Hence our end result is that \( \deg f = 6 \).

By Lemma 2.3 we are thus content with replacing the condition \( e^{\hat{h}_{E_d}(P)} < |d|\alpha \) in the definition of \( N_{\alpha,\epsilon}(X) \) with \( e^{\hat{h}_{E_1}(P)} < |d|^{6\alpha} \).

2.2. Reduction to \( d \) of Bounded Square Part. Let \( d = \text{sgn}(d)\sqrt{(d')^d} \) where \( d' \) is square-free. Fix \( \epsilon > 0 \), and define the following two quantities:
\[
N_{\alpha,\epsilon}(X) = \# \{ d \in S_6(X) : \Box \leq |d|^{\epsilon}, \zeta_d < |d|^{\alpha} \}
\]  
(2.10)
\[
N_{\alpha,\epsilon}^*(X) = \sum_{\Box \leq |d|^{\epsilon}} \# \{ P \in E_d(\mathbb{Q}) \setminus E_d(\mathbb{Q})_{\text{tors}} : e^{\hat{h}_{3,2}(P)} < |d|^{6\alpha} \}
\]  
(2.11)
With the exact reasoning as in the case of \( N_{\alpha}(X) \) and \( N_{\alpha}^*(X) \), we first observe that \( N_{\alpha,\epsilon}(X) \leq N_{\alpha,\epsilon}^*(X) \).

Lemma 2.4. \( N_{\alpha,\epsilon}(X) \ll X \) implies \( N_{\alpha}^*(X) \ll X \) (implies \( N_{\alpha}(X) \ll X \)).

Proof. Define \( 1 - \epsilon(X) \) to be the proportion of integers of \( S_6(X) \) present in \( \{ d \in S_6(X) : \Box(d) < X^\epsilon \} \). As \( X \to \infty \),
\[
\frac{\# \{ x \leq X : \Box(d) \geq |d|^{\epsilon} \cap ([-X, -X^{1/2}] \cup [X^{1/2}, X]) \}}{2(X - X^{1/2})} \to 0
\]  
(2.12)
as it is well-known the proportion integers with square part \( < X^{\epsilon/2} \) is \( \zeta(2)^{-1} \sum_{n=1}^{[X^{\epsilon/2}-1]} n^{-2} \to 1 \) as \( X \to \infty \). Replacing \( X^{1/2} \) by \( 0 \) in the above yields
\[
\frac{\# \{ x \leq X : \Box(d) \geq |d|^{\epsilon} \}}{2X} \to 0.
\]  
(2.13)
This is because this substitution introduces adds at most \(2[X^{1/2} - 1]\) to the numerator, which is negligible as the denominator prior to the substitution has an \(X\) term. Furthermore, because a positive proportion of integers are 6th power free, one obtains
\[
\frac{\#(d \in S_6(X) : \gcd(d) \geq |d|^\epsilon)}{\#S_6(X)} \to 0 \quad (2.14)
\]
Thus, \(\epsilon(X) \to 0\) as \(X \to \infty\) since \(|d| \leq X\), and so the above holds for \(|d|^\epsilon\) replaced by \(X^\epsilon\). Now suppose it was shown that \(N_{a,\epsilon}^*(X) = o((1 - \epsilon(X))X) = o(X)\). Then, there exists a function \(\delta(X)\) which tends to 0 as \(X \to \infty\) such that
\[
\limsup_{X \to \infty} \frac{N_{a,\epsilon}^*(X)}{X} \leq \limsup_{X \to \infty} \frac{\delta(X)(1 - \epsilon(X))X + \epsilon(X)X}{X} = \limsup_{X \to \infty} \epsilon(X) + \delta(X)(1 - \epsilon(X)) = 0. \quad (2.15)
\]
Since \(\liminf_{X \to \infty} \frac{N_{a,\epsilon}^*(X)}{X} \geq 0\), this implies that \(\limsup_{X \to \infty} \frac{N_{a,\epsilon}^*(X)}{X} = 0\). Now, since \(N_{a,\epsilon}^*(X) \leq N_{a,\epsilon}^*(X)\), it suffices to prove that \(N_{a,\epsilon}^*(X) = o(X)\) in order to show that \(N_{a,\epsilon}^*(X) = o(X)\).

This general-purpose Lemma will serve us well in proving both Lemma 2.6 and Theorem 1.6 as it allows us to dramatically cut down on the size of the variables in the parameterization data presented in Lemma 2.5 in favor of an asymptotic analysis.

### 2.3. Parameterization Data of \(E_d\)

We now introduce the parameterization Lemma crucial to the proof of both Lemma 2.6 and Theorem 1.6.

**Lemma 2.5.** Let \(d \in \mathbb{Z}\) be a sixth-power free integer and let \((x, y, z) \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}^2\) satisfying \(\gcd(x, y, z) = 1\) and
\[
y^2z = x^3 + dz^3. \quad (2.16)
\]
Then, we may uniquely write \(d = b_0^2d_1\), \(x = b_0b_1x_1\), \(y = b_0y_1\), and \(z = b_1^3\), where \((b_0, b_1, d_1, x_1, y_1) \in \mathbb{Z}_{\geq 1}^2 \times \mathbb{Z}^3\) subject to the equation
\[
y_1^2 = b_0x_1^3 + d_1b_1^6. \quad (2.17)
\]

*Proof.* Let \(b_0 = \gcd(x, y)\) and write \(x = b_0x_0\) and \(y = b_0y_1\), such that \(\gcd(x_0, y_1) = 1\). Substituting yields
\[
\frac{b_0^2}{b_1}y_1^2z = z^3x_0^3 + dz^3. \quad (2.18)
\]
Now, since \(\gcd(x, y, z) = 1\), we have that \(\gcd(b_0, z) = 1\). Because the left-hand side of equation 2.18 is divisible by \(b_0^2\), it must be true that \(b_0^2 \mid d\). Write \(d = b_0^2d_1\). Next, define \(b_1 = \gcd(x_0, z)\). Since \(\gcd(b_1, y) = \gcd(b_1, b_0) = 1\) (again since \(\gcd(x, y, z) = 1\)), we must have that \(b_1^3 \mid z\). We write \(x_0 = b_1x_1\) and \(z = ub_1^3\), with \(\gcd(ub_1^2, x_1) = 1\). We now prove that \(u = 1\). To start, we substitute in our parameterizations thus far into Equation 2.18
\[
y_1^2u = b_0x_1^3 + d_1u^3b_1^6 \quad (2.19)
\]
Since \(\gcd(x_1, v) = 1\) and \(u \mid z\), we have that \(\gcd(u, b_0) = 1\). Hence, \(u \mid x_1^3\). We also have that \(\gcd(x_1, ub_1^2) = 1\), and so \(\gcd(u, x_1^3) = 1\). Thus, \(u = 1\) as wished. We now have the simplified equation:
\[
y_1^2 = b_0x_1^3 + d_1b_1^6. \quad (2.20)
\]
completing the proof.

*Remark.* The idea behind the parameterization of this lemma comes from Lemma 2.3, where we consider the logarithmic height with respect to a function in the function field \(K(E_1)\) depending on both \(x\) and \(y\). This motivates the variable \(b_0\) in the proof of the first Lemma.
We can now make use of the previous two Lemmas to establish the promised trivial bound.

**Lemma 2.6.** Let \( d \in \mathbb{Z}_{>0} \) be sixth-power free. The set of \( d \in S_6(X) - \{0\} \) such that
\[
\zeta_d > |d|^{1/6 - \epsilon}
\] (2.21)
where \( \epsilon > 0 \) has natural density 1 in \( S_6(X) \).

**Proof.** Taking Equation 2.17 of Lemma 2.5
\[
y_1^2 = b_0x_1^3 + d_1b_1^6
\] (2.22)
observe that
\[
|d_1b_1^6| \ll \max(|y_1^2|, |b_0x_1^3|) \leq \max(|y_1^2|, |b_0x_1^3|)
\] (2.23)
\[
\leq \max(|y_1^2|, |b_0b_1x_1^3|)
\] (2.24)
By the proof of Lemma 2.4, we can take \( b_0 < |d|^{\epsilon} \) for almost all \( d \). Then, \( |d_1b_1^6| \geq |d|^{1-\epsilon}|b_1^6 \geq |d|^{1-\epsilon} \).
Thus, \( |d_1b_1^6| \ll \max(|y_1^2|, |b_0b_1x_1^3|) \) implies
\[
|d|^{1-\epsilon} \ll \max(|y_1^2|, |b_1x_1^3|).
\] (2.25)
Now by Lemma 2.5, the polynomial height implicit in the quantity \( N_d(X) \) is the right-hand side of the above inequality. Due to the factor of 6 in front of \( \alpha \), this result implies
\[
\zeta_d \gg |d|^{1/6 - \epsilon}
\] (2.26)
for almost all \( d \). \( \square \)

The goal now is to improve this bound, which is the content of Theorem 1.6 and whose proof we now move towards.

### 3. A Direct Count Via Uniform Bounds

**3.1. Quantitative Arithmetic of Projective Varieties.** In recent decades much progress has been made on establishing bounds for the number of integer solutions in a given box of forms in 2 or 3 variables. We shall use the following Theorem of Heath-Brown [HB02, Theorem 3] to illustrate the power of this technique in our scenario.

**Theorem 3.1.** Let \( F \in \mathbb{Z}[x_1, x_2, x_3] \) be a form of degree \( d \) in three variables irreducible over \( \mathbb{Q} \), and suppose we insist \( |x_i| \leq B_i \) where \( B_i \geq 1 \). Let \( V = B_1B_2B_3 \) and let
\[
T = \max \prod_{i=1}^3 B_i^{f_i}
\] (3.1)
where the maximum is taken over all integer triples \((f_1, f_2, f_3)\) for which \( x_1^{f_1}x_2^{f_2}x_3^{f_3} \) occurs with non-zero coefficient. Let \( N(F; B_1, B_2, B_3) \) be the number of integers solutions of \( F \) subject to the aforementioned bounds on the \( x_i \). Then, for any \( \epsilon' > 0 \)
\[
N(F; B_1, B_2, B_3) \ll \epsilon' T^{-d}V^{d^{-1}+\epsilon'}.
\] (3.2)
If \( F \) is non-singular, one has
\[
N(F; B_1, B_2, B_3) \ll \epsilon' V^{2(3d)^{-1}+\epsilon'}
\] (3.3)

The remainder of this section focuses on applying this Theorem to a suitable polynomial, which we do in the following Lemma.
Lemma 3.2. \(N_{\alpha,\epsilon}(X) \ll X^{1/3+3\alpha+\epsilon}\).

Proof. Let us refer to the conditions on the variables of Lemma 2.5 as (*). Then,
\[
N_{\alpha,\epsilon}(X) \ll \#\{(b_0, b_1, d_1, x_1, y_1) \in \mathbb{Z}_{\geq 1}^2 \times \mathbb{Z}^3 : (\ast), |b_1 x_1| \ll X^{2\alpha}, |y_1| \ll X^{3\alpha}\}
\] (3.4)
where (\ast) includes the condition that \(y_1^2 = b_0 x_1^3 + d_1 b_1^6\). Now, consider this equation as one in the variables \(d_1\) and \(y_1\) with \(b_0, b_1, x_1\) fixed. First, we may take \(|b_0| < X^\epsilon\). Second, since \(|b_1 x_1| \ll X^{2\alpha}\), the number of possible pairs \((b_1, x_1)\) is bounded above by a constant multiple of
\[
\sum_{1 \leq a \leq X^{2\alpha}} \left\lfloor \frac{X^{2\alpha}}{a} \right\rfloor \ll X^{2\alpha} \log X
\] (3.5)
where the logarithm is natural. Thus, the maximal number of triples \((b_0, b_1, x_1)\) in consideration is \(\ll X^{2\alpha+\epsilon}\). Now, let us homogenize \(y_1^2 = b_0 x_1^3 + d_1 b_1^6\) with the variable \(v\):
\[
y_1^2 - v^2 b_0 x_1^3 - v d_1 b_1^6 = 0
\] (3.6)
which is a degree 2 form in the variables \(d_1, y_1,\) and \(v\). We seek integer solutions to this polynomial subject to the conditions
\[
|d_1| \ll X, |y_1| \ll X^{3\alpha}, |v| \leq 1.
\] (3.7)
Thus, in the notation of Theorem 3.1 one obtains \(V = X^{1+3\alpha}\) and \(d = 2\). Now, in Lemma 2.5 \(d_1 \neq 0\) since \(d \neq 0\), and \(b_1 \neq 0\) since it is defined as the positive GCD of two integers. Thus, Equation 3.6 is actually a non-singular form, and so applying the non-singular estimate of Theorem 3.1 yields \(\ll X^{1/3+\alpha}\) solutions. Combining this with the number of elements space of triples \((b_0, b_1, x_1)\) yields the result.

\[\square\]

3.2. **Proof of Theorem 1.6** By Lemma 2.4 it suffices to show that \(N_{\alpha,\epsilon} \ll X\). By Lemma 3.2 this means we choose \(\alpha\) such that \(\frac{1}{3} + 3\alpha + \epsilon < 1\). Since \(\epsilon\) can be chosen to be arbitrarily small, we can take \(\alpha < 2/9\).

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