On restricted completions of chordal and trivially perfect graphs

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Abstract

Let $G$ be a graph having a vertex $v$ such that $H = G - v$ is a trivially perfect graph. We give a polynomial-time algorithm for the problem of deciding whether it is possible to add at most $k$ edges to $G$ to obtain a trivially perfect graph. This is a slight variation of the well-studied Edge Completion, also known as Minimum Fill-In, problem. We also show that if $H$ is a chordal graph, then the problem of deciding whether it is possible to add at most $k$ edges to $G$ to obtain a chordal graph is NP-complete.

1 Introduction

We consider finite, simple, and undirected graphs. A chordal graph is a graph with no induced cycles of length at least four. A graph $G$ is trivially perfect if for every induced subgraph $G'$ of $G$, the cardinality of a maximum independent set of $G'$ is equal to the number of maximal cliques of $G'$. A split graph is a graph whose vertex set can be partitioned into an independent set and a clique. Given as input a graph $G$ and an integer $k$, the problem of deciding whether it is possible to add at most $k$ edges to $G$ to obtain a graph belonging to a graph class $\Pi$ can be formalized as follows.

\textbf{Π completion}

\textbf{Input:} A graph $G$ and an integer $k$.

\textbf{Question:} Is there a graph $H \in \Pi$ such that $V(H) = V(G)$, $E(G) \subseteq E(H)$ and $|E(H)| = |E(G)| - k$?

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The Chordal completion problem was proved to be NP-complete in [16]. There are polynomial-time approximation algorithms [12], exponential-time exact algorithms [6, 7], and parameterized algorithms [1, 2, 4, 10] for this problem. The Trivially perfect completion problem is also known to be NP-complete [13] and a polynomial kernel for this problem was presented in [5]. We consider the following slight variation of this problem.

Π p-completion
Input: A graph $G$ having a set $T \subseteq V(G)$ such that $|T| \leq p$ and $G - T \in \Pi$ and an integer $k$.
Question: Is there a graph $H \in \Pi$ such that $V(H) = V(G)$, $E(G) \subseteq E(H)$ and $|E(H)| - |E(G)| \leq k$?

Given graphs $G$ and $H$ and a graph class $\Pi$, we say that $H$ is a $\Pi$ completion of $G$ if $G$ is a spanning subgraph of $H$ and $H \in \Pi$. Every edge of $E(H) \setminus E(G)$ is called a fill edge and we denote $\text{fill}(H) = |E(H)| - |E(G)|$. Note that for any positive integer $p$, the $\Pi p$-completion problem is a restriction of the $\Pi$ completion problem in the sense that an input for $\Pi p$-completion is also an input for $\Pi$ completion.

We remark that the input of the $\Pi p$-completion problem is formed only by a graph $G$ and an integer $k$, i.e., to know more than one set $T \subset V(G)$ such that $|T| \leq p$ and $G - T \in \Pi$ can help in the search for a minimum $\Pi$ completion of $G$, since each such set has a different structure that makes the graph $G$ not to belong to $\Pi$, however, this does not change the answer because the question of the problem asks only about $G$ and $k$. We also remark that the case $p = 1$ was proposed in [11] with another formulation and this is the problem we investigate in this work. In fact, the definition given in [11] has as input a graph $G \in \text{COGRAPH}$ and asks for a minimum COGRAPH completion of the graph obtained by adding one vertex $v$ to $G$ (jointly with some edges incident to $v$), which, as observed above, is equivalent to the COGRAPH 1-completion problem.

The text is organized as follows. In Section 2, we show that Chordal 1-completion is NP-complete. We also prove that given a graph $G$ having a vertex $v$ such that $G - v$ is a split graph and an integer $k$, it is NP-complete to decide whether $G$ has chordal completion with at most $k$ edges. In Section 3, we present an algorithm for solving the Trivially perfect 1-completion problem in polynomial time.

We conclude this section introducing some notations. For a natural number $k$, we denote $\{1, \ldots, k\}$ by $[k]$. We use $H \subset G$ to say that $H$ is a proper subgraph of $G$. The neighborhood of $v \in V(G)$ is denoted by $N_G(v)$.

2 Chordal and split graphs

Consider a graph $G$. A set of vertices $S$ of $G$ is said to be a vertex separator if there exist two vertices $v$ and $w$ of $G$ such that $v$ and $w$ are in distinct connected component of $G - S$. The text is organized as follows. In Section 2, we show that Chordal 1-completion is NP-complete. We also prove that given a graph $G$ having a vertex $v$ such that $G - v$ is a split graph and an integer $k$, it is NP-complete to decide whether $G$ has chordal completion with at most $k$ edges. In Section 3, we present an algorithm for solving the Trivially perfect 1-completion problem in polynomial time.

We conclude this section introducing some notations. For a natural number $k$, we denote $\{1, \ldots, k\}$ by $[k]$. We use $H \subset G$ to say that $H$ is a proper subgraph of $G$. The neighborhood of $v \in V(G)$ is denoted by $N_G(v)$.
A minimal separator set is called minimal if it is minimal under inclusion. We will denote by $S(G)$ the set of all minimal vertex separators of $G$.

**Theorem 1.** Let $G$ be a graph. The graph $G$ is chordal if and only if every minimal vertex separator set is a clique.

**Remark 2.** Let $G$ be a graph. If $H$ is a II completion of $G$ and $S$ is a vertex separator of $H$, then $S$ is a vertex separator of $G$.

**Lemma 3.** Let $G$ be a graph and let $H$ be a minimum chordal completion of $G$. Then, for every clique $C$ of $H$ properly contained in $N_H(v)$ for $v \in V(G)$, there exists a vertex $x \in V(H) \setminus C$ such that $vx \in E(G)$.

**Proof.** Suppose, by contradiction, that for every vertex $x \in N_G(v) \setminus C$, $vx$ is a fill edge. Let us call $F'$ to the set of fill edges $vx$ with $x \in N_G(v) \setminus C$. Consider the graph $H' = H - F'$. Notice that $v$ is a simplicial vertex of $H'$ and $F' \neq \emptyset$. In addition, $H' - v = H - v$ and thus $H'$ is a chordal graph. Consequently, $H'$ is a chordal completion of $G$ with less fill edges than $H$, contradicting that $H$ is a minimum chordal completion of $G$. The contradiction arises from supposing that for every vertex $x \in N_G(v) \setminus C$, $vx$ is a fill edge. Therefore, $N_G(v) \cap (V(H) \setminus C) \neq \emptyset$. □

**Lemma 4.** Let $G$ be a graph having a vertex $v$ such that $G - v$ is a non-complete connected chordal graph, and let $H$ be a minimum chordal completion of $G$. If no clique separator $S \in S(G)$ is contained in $N_H(v)$, then $N_G(v) \in S(H)$.

**Proof.** Since $N_H(v)$ is a vertex separator of $H$, $H$ is a chordal graph, and $N_H(v)$ contains no minimal clique separator in $S(G)$, Theorem 1 implies that $N_H(v)$ is a minimal clique separator of $H$. So, by Lemma 3 $N_H(v) = N_v(G)$, which means that $N_v(G) \in S(H)$. □

**Theorem 5.** Given a co-bipartite graph $G$ and an integer $k$, it is NP-complete to decide whether there exists a chordal completion of $G$ with at most $k$ fill edges.

Given a co-bipartite graph $F$ with $(A, B)$ a bipartition of $F$ and an integer $k$, set $K = k + \frac{1}{2}|A|(|A| - 1)$ and define $F^*$ as the graph with vertex set $V(F) \cup \{c_1, \ldots, c_{K+1}\}$ and $E(F^*) = (E(F) \setminus E(F[A])) \cup \{c_i u : i \in [K+1] \text{ and } u \in V(F) \setminus \{c_i\}\}$. In other words, $F^*$ is the graph obtained from $F$ by deleting all edges with both endpoints in $A$ and adding $K$ universal vertices to the resulting graph. Notice that $F^*$ is a split graph and thus a chordal graph. Denote by $G$ the graph with vertex set $V(G) = V(F) \cup \{v\}$ where $v$ is a new vertex and edge set $E(G) = E(F^*) \cup \{va : a \in A\}$. Write $C = \{c_1, \ldots, c_{K+1}\}$.

**Lemma 6.** Let $F$ be a co-bipartite graph with $(A, B)$ a bipartition of $F$, let $k$ be an integer and let $G$ be constructed from $F$ as above. Then, $F$ has a minimum chordal completion with at most $k$ fill edges if and only if $G$ has a minimum chordal completion with at most $K$ edges.

**Proof.** Suppose that $F$ has a minimum chordal completion with at most $k$ fill edges. Let us call $M$ to that set of fill edges. Consequently, a chordal completion $H$ of $G$ with $K$ edges can be obtained by adding all the edges with both endpoints in $A$ plus the edges of the set $M$. In order
to show that $H$ is indeed a chordal completion of $G$, it suffices to observe that for any vertex $u \in V(H) \setminus V(F)$, $N_H(u)$ is a clique or $u$ is a universal vertex of $H$, i.e., $u$ does not belong to an induced $C_p$ for $p \geq 4$.

Conversely, suppose that $G$ has a minimum chordal completion $H$ with at most $K$ edges. Let us call $M$ to that set of fill edges. Notice that every minimal clique separator of $G - v$ contains the clique $C$.

We claim that for any $a, a' \in A$, it holds that $aa' \in E(H)$. Then, suppose the contrary and let $a, a' \in A$ such that $aa' \notin E(H)$. Since $|M| \leq K$, there is $c_i \in C$ such that $vc_i \notin E(H)$, which implies that $vac_a$ is an induced $C_4$ of the chordal graph $H$, a contradiction. Hence, for any $a, a' \in A$, it holds that $aa' \in E(H)$.

Since $H$ has at most $K$ fill edges, $N_H(v)$ contains no minimal clique separator of $S(G)$. Therefore, by Lemma 4, $N_G(v)$ is a minimal separator of $H$ and thus, by Theorem 3, $N_G(v)$ is a clique in $H$. By Lemma 2, $N_H(v) = N_G(v)$. So, there is no fill edge having an endpoint in $C \cup \{v\}$. Since every edge with both endpoints in $A$ is a fill edge, the amount of edges with both endpoints in $A$ is $\frac{1}{2} |A|(|A| - 1)$ and $C$ is a clique formed by universal vertices of $H - v$, it follows that the remaining $k' \leq k$ edges have one endpoint in $A$ and the other in $B$. Therefore, adding these $k'$ edges to $F$ gives a chordal completion of $F$ with at most $k$ edges.

By combining Theorem 5 and Lemma 6, we obtain the following result.

**Theorem 7.** The Chordal 1-completion problem is NP-complete. In addition, given a graph $G$ having a vertex $v$ such that $G - v$ is a split graph and an integer $k$, it is NP-complete to decide whether $G$ has a chordal completion with at most $k$ edges.

### 3 Trivially perfect graphs

Trivially perfect graphs were introduced by Wolk [14] as comparability graphs of trees. Golumbic [8] called them trivially perfect graphs because it is easy to show that they are perfect from a characterization of perfect graphs, and showed that these graphs are precisely those graphs that are $\{P_4, C_4\}$-free. This characterization implies that the class of trivially perfect graphs is the subclass of cographs which are also chordal graphs or interval graphs. In this section, we present a polynomial-time algorithm for finding a minimum trivially perfect completion of a graph $G$ having a vertex $v$ such that $G - v$ is trivially perfect.

For short, we will use “TP completion” standing for trivially perfect completion and write “$(G, F, v) \in 1TP$” if $G$ is connected a graph, $v \in V(G)$ and $F = G - v$ is a trivially perfect graph. We denote the number of connected components of a graph $G$ by $\omega(G)$. A rooted forest is a set of rooted trees. It is known [15] that there is a bijection $\alpha$, up to isomorphism, among the trivially perfect graphs and the rooted forests. In the remaining of the text, given a trivially perfect graph $G$, we will reserve the corresponding calligraphic letter to represent $\alpha(G)$, i.e., $G = \alpha(G)$ and $G = \alpha^{-1}(G)$. We adopt the notation $|G| = |E(G)|$ and $|G| = |V(G)|$. If $G$ is connected, then we also denote the root of $G$ by $g$. Note that $g$ is a universal vertex of $G$ and since $G$ can have other universal vertices, $g$ can have descendants that are also universal
vertices of $G$. We denote the trees of a rooted forest using the same letter with an underlined superscript indicating its position in a non-increasing ordering of their number of vertices, i.e., given a rooted forest $G$, one of the rooted trees of $G$ with maximum number of vertices is chosen arbitrarily to be $G^1$, and for rooted trees $G^i$ and $G^j$ such that $|G^i| \geq |G^j|$ it holds if $i < j$. The root of $G^i$ is denoted by $g^i$.

Let $T$ be a rooted tree and let $u, w \in V(T)$. We remark that the neighborhood of $u$ in $T$ is formed by the descendant and the ancestral vertices of $u$ in $T$. We denote $T_u$ for the subtree of $T$ rooted at $u$, $\pi(u)$ is the parent of $u$ in $T$ and we can also use $T_u$ meaning $\alpha^{-1}(T_u)$. If $u$ is ancestral of $w$, we denote the set of vertices that are descendants of $u$ and ancestral of $w$ in $T$ by $[u, w]_T$. We also use $(u, w)_T = [u, w]_T \setminus \{u\}$, $[u, w]_T = [u, w]_T \setminus \{w\}$ and $(u, w)_T = [u, w]_T \setminus \{u, w\}$. Every vertex of $(t, w)_T$ is a proper ancestral of $w$ in $T$. The level of $u$ in $V(T)$ is 1 plus the distance of $u$ to the root of $T$ and is denoted by $\ell_T(u)$. Observe that $|T| = \sum_{T_i \in T} (|T_i| - 1)\).

Every rooted tree $T$ has also a special vertex called the base of $T$, for which we reserve the letter $\overline{t}$. When not explicitly designated, the base of a tree is equal to its root. The base appears in the following definitions.

- For every $u, w \in V(T)$ such that $u$ is ancestral of $w$ and $w$ is ancestral of $\overline{t}$. If $w \neq \overline{t}$, then the subtree $T_{u, w}$ is defined as $T_u - T_x$ such that $x$ is the child of $w$ that is ancestral of $\overline{t}$. The base of $T_{u, w}$ is $w$. If $w = \overline{t}$, then we define $T_{u, w} = T_u$.
- Given another rooted tree $\overline{G}$, denote by $\langle \overline{G}, T \rangle = \langle \overline{G}, T, \overline{t} \rangle$ the rooted tree obtained by adding the edge $\overline{gt}$ with root $g$ and base $\overline{t}$.

For the following two definitions, let $k = \ell_T(\overline{t})$ and for every $j \in [k]$, denote by $u_j$ the ancestral of $\overline{t}$ in $T$ such that $\ell_T(u_j) = j$.

- For $1 \leq i \leq j \leq k$, define the average of $T_{u_i, u_j}$ as $a(T_{u_i, u_j}) = \frac{\sum_{\overline{t} \leq i < j} |T_{u_i, u_j}|}{j - i + 1}$. We will use $a(T) = a(T, \overline{t})$.
- We say that $T_{u_i, u_j}$ is the leading subtree of $T$ if $i \in [k]$ is the minimum number such that $a(T_{u_i, u_j}) \geq a(T_{u_i, u_j})$ for every $j \in [k]$.

Given a rooted tree $\overline{G}$, note that $G_{g, \overline{G}} = \overline{G}$ and that if $\overline{g} = g$, then $G_{g, g} = \overline{G}$. Note also that if $u$ and $w$ are twins in $G$ and $u$ is ancestral of $w$ that is a proper ancestral of $\overline{t}$ in $\overline{G}$, then $V(G_{u, w}) = \{u\}$. For $k \geq 3$ and rooted trees $T^1, \ldots, T^k$, we define $\langle T^1, \ldots, T^k \rangle = \langle T^1, \ldots, T^{k-1} \rangle, T^k \rangle$. Note that every rooted tree $T$ can be decomposed into $\langle T_{u, \pi(u_1)}, T_{u_1, \pi(u_2)}, \ldots, T_{u_k, \overline{t}} \rangle$ for any $k \leq \ell_T(\overline{t})$ where $t$ is ancestral of $u_1$, $u_i$ is ancestral of $u_{i+1}$ for $i \in [k - 1]$ and $u_k$ is ancestral of $\overline{t}$. Hence, given a possibly disconnected trivially perfect graph $F$ and a connected TP completion $H$ of $F$, there is a TP completion $G$ of $F$ with $\omega(G) = \omega(F)$ such that $H = \{X_{x^1, x_1}^1, \ldots, X_{x^k, x_k}^1\}$ for some $k \geq \omega(F)$ where $X^i$ is a tree of $G - \{X_{x^1, x_1}^1, \ldots, X_{x^i-1, x_{i-1}}^{i-1}\}$ for every $i \in [k]$ and $x_i$ is ancestral of $\overline{t}$ in $X^i$. We say that $H$ is a merge of $G$; and if for $i \in [k]$, $X_{x^i, x_i}^i$ is the leading
subtree with maximum average among the trees of $G - \{X^{2}_{x_{1},\ldots,x_{i-1},x_{i+1}}\}$, then we say that $H$ is a leading merge of $G$.

**Lemma 8.** Let $T^{1}$ and $T^{2}$ be rooted trees. For $j \in \{1, 2\}$, let $T^{j}$ be a rooted tree. If $a(T^{1}) \geq a(T^{2})$, then $|G_{1}| \leq |G_{2}|$ where $G_{1} = \langle T^{1}, T^{2} \rangle$ and $G_{2} = \langle T^{2}, T^{1} \rangle$.

**Proof.** Write $\ell_{1} = \ell_{T^{1}}(t)$ and $\ell_{2} = \ell_{T^{2}}(t)$. By definition, $a(T^{1}) = \frac{|T^{1}|}{\ell_{1}} \geq \frac{|T^{2}|}{\ell_{2}} = a(T^{2})$. Then, $\ell_{2}|T^{1}| \geq \ell_{1}|T^{2}|$. It remains to note that $|G_{1}| = \ell_{1}|T^{2}| + |T^{1}| + |T^{2}|$ and that $|G_{2}| = \ell_{2}|T^{1}| + |T^{1}| + |T^{2}|$. \[\square\]

**Lemma 9.** Let $F$ be a family of rooted trees and let $H$ be a merge of $F$. If $H$ is not a leading merge of $F$, then there are vertices $u_{1}, u_{2}, u_{3} \in V(F)$ such that $u_{1} \in [h, u_{2})_{H}$, $u_{2} \in (u_{1}, u_{3})_{H}$, $u_{3} \in (u_{2}, h]_{H}$ and $|H'| < |H|$ where $H' = (H_{h,\pi(u_{1})}, H_{u_{2}, u_{3}}, H_{u_{1},\pi(u_{2})}, H - H_{h,u_{3}})$.

**Proof.** Suppose by contradiction that the result does not hold. We can assume that $F$ is a family with minimum number of vertices for which the result does not hold. Denote the leading subtree of $F$ with maximum average as $T^{1}$. Then, $\overline{t^{1}}$ has an ancestral in $H$ not belonging to $T^{1}$. Denote the set of ancestrals of $t^{1}$ in $H$ by $\{w_{1}, \ldots, w_{p}\}$ and denote by $G^{1}, \ldots, G^{q}$ the partition of $H_{h,\overline{t^{1}}}$ into subtrees such that for every $i \in [q]$, $G^{i} = H_{w_{i}, u_{i}}$ is such that

- either $\{w_{j}, \ldots, w_{k}\} \subseteq V(T^{1})$, $j = 1$ or $w_{j-1} \notin V(T^{1})$, and $k = p$ or $w_{k+1} \notin V(T^{1})$;
- or $\{w_{j}, \ldots, w_{k}\} \cap V(T^{1}) = \emptyset$, $j = 1$ or $w_{j-1} \in V(T^{1})$, and $k = p$ or $w_{k+1} \in V(T^{1})$.

On the one hand, $G^{i}$ is a subtree of $T^{1}$ if and only if $i$ is even. By Lemma 8, $a(G^{i}) \geq a(G^{i+1})$ for every $i \in [k - 1]$. Therefore, $a(H_{h,\overline{t^{1}}} - T^{1}) > a(T^{1})$, which contradicts the fact that the maximum leading subtree of $F$ is $T^{1}$.

On the other hand, $G^{i}$ is a subtree of $T^{1}$ if and only if $i$ is odd. Using Lemma 8 again, we have that $a(H_{h,\overline{t^{1}}} - T^{1}) > a(T^{1})$. However, the fact that the maximum leading subtree of $F$ is $T^{1}$ implies that $a(G^{1}) < a(T^{1})$, which means that $a(H_{h,\overline{t^{1}}} - T^{1}) > a(T^{1})$, a contradiction with the fact that the maximum leading subtree of $F$ is $T^{1}$. \[\square\]

Let $(G, F, v) \in 1\text{TP}$. Note that if we add to $G$ the necessary edges to make $v$ a universal vertex, then the obtained graph $H$ is trivially perfect. We call this completion the **universal TP completion** of $(G, v)$, or simply $U(G, v)$. Let $R$ be a subtree of a tree of $F$. We denote by $\hat{R}$ the subtree of $R$ induced by $r$ and the vertices of the subtrees $R_{u}$ such that $u$ is a child of $r$ with $V(R_{u}) \cap N_{G}(v) = \emptyset$.

### 3.1 Slim TP completion

Given a trivially perfect graph $F$ and $S \subseteq V(F)$, we say that a graph $H$ is a **slim TP completion** of $(F, S)$ if $H$ is a connected trivially perfect completion of $(F, S)$, $S$ is a clique of $H$ and $\ell_{H}(s) \leq |S|$ for every $s \in S$. A slim TP completion is minimum if there is no other with less edges than it. The task of computing a minimum slim TP completion of $(F, S)$ takes part of the algorithm for solving **TRIVIALLY PERFECT 1-COMPLETION** in polynomial time (Algorithm 2).
In such algorithm, the base of \( \mathcal{H} \) will be the vertex \( s \in S \) such that \( \ell_{\mathcal{H}}(s) = |S| \). We show in the sequel how to find a minimum slim TP completion in polynomial time.

**Lemma 10.** Let \( F \) be a trivially perfect graph, let \( S \subseteq V(F) \) be such that \( V(F') \cap S \neq \emptyset \) for every connected component \( F' \) of \( F \) and let \( u \in V(F) \) such that every proper ancestral of \( u \) in \( F \) belongs to \( S \). If \( H \) is a minimum slim TP completion of \( (F, S) \), then \( H[V(F_u)] \) is a minimum slim TP completion of \( (F_u, S) \).

**Proof.** Suppose by contradiction that \( H^1 = H[V(F_u)] \) is not a minimum slim TP completion of \( (F_u, S) \). We can assume that \( u \) has been chosen maximizing its level in \( F \).

First, consider that \( u \notin S \). If \( S \cap V(F_u) = \emptyset \), then observe that \( V(F_u) \subseteq V(\mathcal{H} u', u) \), where \( u' = \pi_F(u) \), which means that no edges have been added joining vertices of \( F_u \). Then, we can assume that \( S_u := S \cap V(F_u) \neq \emptyset \). By the definition of slim TP completion, every vertex of \( S_u \) is ancestral of \( u \) in \( \mathcal{H} \), which means that \( H^1 \) is a slim TP completion of \( (F_u, S) \). Recall that \( V(\mathcal{H}_{s, u}) = \{s\} \) if \( s \in S_u \) is not the base of \( \mathcal{H} \). Furthermore, the only vertices of \( V(F_u) \) that are ancestors of \( u \) in \( \mathcal{H} \) are the vertices of \( S_u \). Let \( H^2 \) be a minimum slim TP completion of \( (F_u, S) \). We know that \( |H^2| < |H^1| \). Now, observe that if we replace \( H^1 \) by \( H^2 \) in \( H \), the resulting graph is a slim TP completion of \( (F, S) \) with less fill edges than \( H \), which is a contradiction.

Consider now that \( u \in S \). Denote by \( u_1, \ldots, u_k \) the children of \( u \) in \( F \) such that \( F_{u_i} \) has vertices of \( S \) for \( i \in [k] \). Then, by the choice of \( u \), it holds that \( F_i = H[V(F_{u_i})] \) is a minimum slim TP completion of \( (F_{u_i}, S) \) for every \( i \in [k] \). Write \( F' = \{F_1, \ldots, F_k\} \). Since \( H^1 \) is a slim TP completion of \( (F_u, S) \), we conclude that \( \mathcal{H}^2 = \mathcal{H}^1 - \mathcal{H}^1_{h_1, h_1} \) is a merge of \( F' \). Since \( H^1 \) is not minimum, by Lemma 9, \( \mathcal{H}^2 \) is not a leading merge of \( F' \). Lemma 9 also implies that there are vertices \( u_1, u_2, u_3 \in V(H^2) \) such that \( u_1 \in [h^2, u_2) \cap \mathcal{H}^2, u_2 \in (u_1, u_3) \cap \mathcal{H}^2, u_3 \in (u_2, \overline{h^2}) \cap \mathcal{H}^2 \) and \( |H^1| < |H^2| \) where \( H' = \langle \mathcal{H}^2_{h_1, \pi(u_1)}, \mathcal{H}^2_{u_2, u_3}, \mathcal{H}^2_{u_1, \pi(u_2)}, \mathcal{H}^1 - \mathcal{H}^1_{h_2, u_3} \rangle \). On the one hand, \( \mathcal{H}^2_{u_1, u_3} = \mathcal{H}^2_{u_2, u_3} \). Then, Lemma 9 implies that replacing \( \mathcal{H}^2 \) by \( H' \) in \( \mathcal{H} \) yields a slim TP completion of \( (F, S) \) with less edges than \( H \).

On the other hand, \( \mathcal{H}^2_{u_1, u_3} \neq \mathcal{H}^2_{u_1, u_3} \). Then, write \( [u_1, u_3]_\mathcal{H} = \{w_1, \ldots, w_p\} \) and denote by \( G^1, \ldots, G^q \) the partition of \( \mathcal{H}_{u_1, u_3} \) into subtrees such that for every \( i \in [q] \), \( G^i = \mathcal{H}_{w_j, w_k} \) is such that

- either \( \{w_j, \ldots, w_k\} \subseteq V(\mathcal{H}_{u_1, u_3}^2), j = 1 \) or \( w_{j-1} \notin V(\mathcal{H}_{u_1, u_3}^2), \) and \( k = p \) or \( w_{k+1} \notin V(\mathcal{H}_{u_1, u_3}^2), \)
- or \( \{w_j, \ldots, w_k\} \cap V(\mathcal{H}_{u_1, u_3}^2) = \emptyset, j = 1 \) or \( w_{j-1} \in V(\mathcal{H}_{u_1, u_3}^2), \) and \( k = p \) or \( w_{k+1} \in V(\mathcal{H}_{u_1, u_3}^2). \)

Since \( a(\mathcal{H}_{u_2, u_3}^2) > a(\mathcal{H}_{u_1, \pi(u_2)}^2) \), we conclude that there is \( i \in [k - 1] \) such that \( a(G^i) > a(G^{i+1}) \). Then, apply Lemma 8 in these two trees, a slim TP completion of \( (F, S) \) with less edges than \( H \) is constructed, which is a contradiction. \( \square \)

**Lemma 11.** The following hold for a connected trivially perfect graph \( F \) and \( S \subseteq V(F) \).
(i) If \( f \not\in S \), then \( H \) is a minimum slim TP completion of \((F, S)\) if and only if \( H \) is the graph obtained from \( F \) by adding the necessary edges to make every vertex of \( S \) a universal vertex.

(ii) If \( f \in S \), then \( \langle F_{f, f}, \mathcal{H}' \rangle \), where \( \mathcal{H}' \) is a minimum slim TP completion of \((F - V(\tilde{F}), S \setminus \{f\})\), is a minimum slim TP completion of \((F, S)\). Furthermore, \( \mathcal{H}' \) is a leading merge of a minimum slim TP completions of the trees of \( F - \tilde{F} \).

**Proof.** \(\square\) Let \( H' \) be a slim TP completion of \((F, S)\). Since \( f \not\in S \), every vertex \( s \in S \) is ancestral of \( f \) in \( \mathcal{H}' \). Note that \( s \in S \) is ancestral of \( f \) in \( H' \) if and only if \( s \) is a universal vertex of \( H' \).

Now, the result follows from the fact that the addition of the necessary edges to make every vertex of \( S \) a universal vertex produces a slim TP completion of \((F, S)\).

\(\square\) Since \( f \) is a universal vertex of \( F \), \( f \) is a universal vertex of every TP completion of \((F, S)\), i.e., \( f \) can be chosen as the root of any minimum slim TP completion of \((F, S)\). Let \( \mathcal{H} \) be a minimum slim TP completion of \((F, S)\) with root \( f \). Then, we can write \( \mathcal{H} = \langle F_{f, f}, \mathcal{H}' \rangle \). Note that \( H' \) is a slim TP completion of \((F - V(F_{f, f}), S \setminus \{f\})\). Now, the minimality of \( \mathcal{H} \) implies that \( \mathcal{H}' \) is also minimum. Furthermore, since \( f \in S \), Lemma \(\text{[10]}\) implies every \( H[F_u] \) is a minimum slim TP completion of \((F_u, S)\) for every child \( u \) of \( f \) in \( \mathcal{F} \). Then, Lemma \(\text{[9]}\) implies that \( \mathcal{H}' \) is a leading merging of minimum slim TP completions of the trees of \( \mathcal{F} - \tilde{\mathcal{F}} \).

**Lemma 12.** Let \( F \) be a trivially perfect graph of order \( n \) and let \( S \subseteq V(F) \) be such that \( V(F') \cap S \neq \emptyset \) for every connected component \( F' \) of \( F \). A minimum slim TP completion of \((F, S)\) can be computed in \( O(n^2) \) steps.

**Proof.** By Lemmas \(\text{[9]}\) and \(\text{[10]}\) a minimum slim TP completion of \((F, S)\) can be constructed by by computing minimum slim TP completion of \((F', S \cap V(F'))\) for every connected component of \( F \) and then doing a leading merge of the obtained trees.

According to Lemma \(\text{[11]}\) for every connected component \( F' \) of \( F \), if \( f' \not\in S \), then a minimum slim TP completion of \((F', S \cap V(F'))\) can be obtained by adding the necessary edges to make all vertices of \( V(F') \cap S \) adjacent to all vertices of \( F' \). If \( f' \in S \), then, applying recursion, a minimum slim TP completion of \((F', S \cap V(F'))\) is \( \langle F', H' \rangle \), where \( H' \) is a minimum slim TP completion of \((F' - V(\tilde{F}'), S \setminus \{f'\})\).

Since a leading merge can be computed in \( O(n) \), to add edges to make that some vertices become universal can also be done in \( O(n) \) steps (considering the tree representation of a trivially perfect graph), and the sum of the inputs of the recursive calls is smaller than \( n \), we conclude that the total time complexity \( T(n) \) of this algorithm can be expressed as \( T(n) = T(n-1) + O(n) \), which gives \( T(n) = O(n^2) \).

\(\square\)

### 3.2 Properties of a minimum TP completion

Now, we present some properties of minimum TP completions that are used in the polynomial-time algorithm for solving the TRIVIALLY PERFECT 1-COMPLETION problem.
Lemma 13. Let $G$ be a connected graph and let $(G, F, v) \in 1TP$. Then, every minimum TP completion $H$ of $G$ different of $U(G, v)$ having $v$ as the base satisfies the following properties where $S = N_G(v)$ and $\omega = \omega(F)$.

(i) $h \in S \cup \{f^1, \ldots, f^{\omega}\}$.

(ii) For every $u \in [h, v)_H$, it holds $V(H_{u,w}) \subseteq V(F^\omega)$ for some $i \in [\omega]$.

(iii) If $h \in V(F^\omega)$, then $h$ has a child in $H$ belonging to $\{v\} \cup V(F^\omega)$.

(iv) If $h \in V(F^\omega)$, then the number of fill edges that are not incident to $h$ is less than $|F^\omega|$.

(v) If $\ell_H(v) \geq 2$ and $\omega \geq 2$, then $\text{fill}(H) \geq |F^\omega|$.

(vi) If $\ell_H(v) \geq 3$, then $h \in V(F^\omega) \cup V(F^\omega)$.

(vii) If $\ell_H(v) \geq 3$, then $|F^\omega| > |F| - |F^\omega| - |F^\omega|$.

(viii) If $\ell_H(v) \geq 3$, then $|F^\omega| > \frac{|F|^2}{2}$.

(ix) For $u \in [h, v]_H$, $\mathcal{H}_u$ is a minimum TP completion of $G[V(H_u)]$.

(x) If $|V(H_{h,w})| = 1$, then there is $h^* \in (h, v)_H$ with $|V(H_{h^*,w})| \geq 2$ and $i \in [\omega]$ such that $[h, h^*] \subset V(F^\omega)$.

(xi) If $h \notin \{f^1, \ldots, f^{\omega}\}$, then $vf^\omega \notin E(H)$ and $\mathcal{H} = \langle \mathcal{P}^i, Y \rangle$ where $i \in [\omega]$ is such that $h \in V(F^i)$ and $\mathcal{P}^i$ is a minimum slim TP completion of $(F^i, S \cap V(F^\omega))$.

Proof. Suppose the contrary. Therefore, $h$ is a non-universal vertex of $V(F^\omega)$ for some $i \in [\omega]$. Observe that $h$ and $f^\omega$ are not twins in $F^\omega$ neither in $H$, but they are twins in $H[V(F^\omega)]$.

First, consider that $f^\omega v \in E(H)$. On the one hand, $f^\omega$ is ancestral of $v$ in $H$. Since $f^\omega$ is not a universal vertex of $H$, $\omega \geq 2$ and there is $w \in V(F^\omega)$ for $j \neq i$ that is ancestral of $f^\omega$ in $H$ such that $|V(H_{w,w})| \geq 2$. We can assume that $w$ has been chosen with minimum level. Note that every vertex of $F^\omega$ for $k \neq i$ has a fill edge to $h$ and every vertex of $V(F^\omega)$ for $\ell \neq j$ has a fill edge to $w$, which implies that $\text{fill}(H) \geq |V(F)| > \text{fill}(U(G, v))$, which is not possible. On the other hand, $v$ is ancestral of $f^\omega$ in $H$. In this case, every vertex of $V(F^\omega) \setminus S$ has a fill edge to $v$. Since $v \neq h$, $\omega \geq 2$ and then every vertex of $F^\omega$ for $j \neq i$ has a fill edge to $h$. Therefore, we have that $\text{fill}(H) \geq \text{fill}(U(G, v))$.

Now, consider that $f^\omega w \notin E(H)$. Let $u$ be the vertex of $F^\omega$ that is ancestral of $v$ in $H$ with maximum level. Such vertex there exists because $h \in V(F^\omega)$. Observe that $f^\omega \in V(H_{u,w})$ and that for every $u' \in V(F^\omega)$ that is a proper ancestral of $u$ in $H$, it holds $H_{u',w} = \{u'\}$, i.e., $H_{h,h} = \{h\}$. Now, construct a tree $H^*$ by deleting $h$ from $H$ and reinserting $h$ as the parent of all children of $u$ except the one that is ancestral of $v$. Now, it remains to observe that $H^*$ is also a TP completion of $G$ having less fill edges than $H$ since $E(H^*) \subset E(H)$, which is a contradiction.
Suppose by contradiction that there is \( u \in [h, v)_H \) such that \( V(H_{u,a}) \) contains vertices of \( V(F^2) \) and of \( V(F^2) \) for different \( i, j \in [\omega] \). Without loss of generality, we can assume that \( u \in V(F^2) \). Let \( s \in S \cap V(F^2) \) and \( w \in V(F^2) \cap V(H_{u,a}) \). We can assume that \( w \) has been chosen minimizing its level in \( H \). Since \( su \in E(H) \), we have that \( s \not\in V(H_{u,a}) \) and that \( uw \not\in E(H) \). Since both \( w \) and \( s \) are adjacent to \( f_\omega \) in \( H \), we conclude that \( f_\omega \in [h, u)_H \). Let \( w' \) be the vertex of \( [f_\omega, u)_H \) with maximum level such that \( w' \in V(F^2) \) and \( W \) be the induced subgraph of \( H \) containing \( w \) and the descendants of \( w \) in \( H \) that belong to \( V(F^2) \).

Now, let \( H' \) be the tree obtained from \( H \) by deleting \( V(W) \) and adding the rooted tree \( W \) as a subtree of \( w' \). Since \( |H'| < |H| \), we have a contradiction.

Suppose by contradiction that every vertex of \( \{v\} \cup V(F^2) \) has level at least 3 in \( H \). Therefore, \( \text{fill}(H) \geq |F^2| + 2 \left( \sum_{j \in [\omega] \setminus \{1\}} |F_j^2| \right) \) for some \( i \in [\omega] \setminus \{1, 2\} \). Since \( |F^2| \geq |F^2_\omega| \), it holds that \( \text{fill}(H) \geq |F^2_\omega| + |F^2_i| + 2 \left( \sum_{j \in [\omega] \setminus \{1, 2\}} |F_j^2| \right) \geq |V(F)| > \text{fill}(U(G, v)) \), which is a contradiction.

For every vertex \( w \in V(F - F^2) \), the fill edge \( hw \) exists in \( H \), which means that there are at least \( |F - F^2| \) fill edges in \( H \) with one extreme in \( h \). Then, if the number of fill edges that are not incident to \( h \) is at least \( |F^2| \), it holds that \( \text{fill}(H) \geq |V(G)| > \text{fill}(U(G, v)) \), which is a contradiction.

If \( h \in V(F^2) \), then every vertex of \( F^2 \) has a fill edge to \( h \), and if \( h \in V(F^2) \) for \( i \geq 2 \), then every vertex of \( F^2 \) has a fill edge to \( h \). Since \( |F^2| \geq |F^2_\omega| \), the result does follow.

Suppose by contradiction that \( \ell_H(v) \geq 3 \) and \( h \in V(F^2) \) for some \( i \geq 3 \). Denote by \( v_2 \) the ancestral of \( v \) that is child of \( h \) in \( H \). If \( v_2 \in V(F^2) \), then \( wh \) and \( wv_2 \) are fill edges in \( H \) for every \( w \in V(F^2) \) for \( j \neq i \), which means that \( \text{fill}(H) \geq 2 \left( \sum_{j \in [\omega] \setminus \{i\}} |F_j^2| \right) \geq |V(G)| > \text{fill}(U(G, v)) \) because \( |F^2| \leq |F^2_\omega| \), which is not possible. Therefore, \( v_2 \in V(F^2) \) for some \( k \in [\omega] \setminus \{i\} \).

Now, we know that \( v_2 \) is ancestral of every vertex of \( V(F^2) \) for any \( j \not\in \{i, k\} \). Therefore, \( wh \) and \( wv_2 \) are fill edges in \( H \) for every \( w \in V(F^2) \) with \( j \not\in \{i, k\} \). Furthermore, every vertex \( w \in V(F^2) \) has a fill edge to \( h \), which implies that \( \text{fill}(H) \geq |F^2_\omega| + 2 \left( \sum_{j \in [\omega] \setminus \{i,k\}} |F_j^2| \right) \geq |V(G)| > \text{fill}(U(G, v)) \) because \( |F^2| \geq |F^2_\omega| \), which is a contradiction.

Suppose by contradiction that \( |F^2| \leq |F| - |F^2_\omega| - |F^2_\omega| \). If there is \( j \in [\omega] \) such that \( V(H_{v_2,l_2}) \subseteq V(F^2_\omega) \), where \( v_2 \) is the ancestral of \( v \) that is child of \( h \) in \( H \), then for every vertex \( u \) of \( F - F^2_\omega \), the fill edges \( uh \) and \( uv_2 \) exist in \( H \), which means that

\[
\text{fill}(H) \geq 2 \left( \sum_{i \in [\omega] \setminus \{j\}} |F_i^2| \right) \geq 2 \left( \sum_{i \in [\omega] \setminus \{1\}} |F_i^2| \right).
\]
Since \( \sum_{i \in [\omega] \setminus \{1,2\}} |F^i| \geq |F^1|, \) \( \text{fill}(H) \geq |V(G)| > \text{fill}(U(G,v)) \), which is not possible. Then, without loss of generality, we can say that \( V(H_{h,h}) \subseteq V(F^2) \) and \( V(H_{v_2,v_2}) \subseteq V(F^2) \) for different \( i, j \in [\omega] \). Therefore,

\[
\text{fill}(H) \geq \min\{|F^2|, |F^2|\} + 2 \left( \sum_{k \in [\omega] \setminus \{i,j\}} |F^k| \right) \geq |F^2| + \left( \sum_{k \in [\omega] \setminus \{i,j\}} |F^k| \right),
\]

because \( \sum_{k \in [\omega] \setminus \{i,j\}} |F^k| \geq |F^1| \). Since \( |F^2| + \min\{|F^2|, |F^2|\} \geq |F^2| + |F^2| \), it holds that \( \text{fill}(H) \geq |V(G)| > \text{fill}(U(G,v)) \), which is a contradiction.

By (12), we know that \( |F^2| > \ldots |F^2| \). Since \( |F^2| \geq |F^2| \), it holds that \( |F^2| > |F^2| \).

Suppose that there is a minimum TP completion \( R \) of \( G[V(H_u)] \) having less fill edges than \( H_u \). Now, observe that \( \langle H_{h,\pi(H_\omega)}, R \rangle \) is a TP completion of \( G \) with less fill edges than \( H \), which is a contradiction.

Let \( i \in [\omega] \) such that \( h \in V(F^2) \) and let \( x \in [h,v]_H \) such that \( x \notin V(F^2) \) but every proper ancestral of \( x \) in \( H \) belongs to \( V(F^2) \). Suppose by contradiction that \( |V(H_{h',h'})| = 1 \) for every proper ancestral of \( x \) in \( H \). Note that there is a fill edge joining every vertex not in \( F^2 \) to \( h \); and that there is a fill edge joining every vertex in \( F^2 \) to \( x \). Therefore, \( \text{fill}(H) \geq |F| > \text{fill}(U(G,v)) \), which is a contradiction.

By (1), we know that \( h \in S \). Let \( i \in [\omega] \) such that \( h \in V(F^i) \).

If \( |S \cap V(F^2)| = 1 \), then we claim that \( V(H) = V(F^2) \), which proof the result for this case. Suppose the contrary and let \( w \) be a vertex of \( V(F^2) \setminus V(H) \) with minimum level in \( H \). Denote by \( W \) the subgraph of \( F^i \) by the descendant vertices of \( w \) in \( H \). Now, let \( H' \) be the tree obtained from \( H \) by deleting \( V(W) \) and adding the rooted tree \( W \) as a subtree of \( h \). Since \( |H'| < |H| \), we have a contradiction.

Then, assume that \( |S \cap V(F^2)| \geq 2 \). Therefore, \( |V(H)| = 1 \). By (12), there is \( h^* \in (h,v)_H \) with \( |V(H_{h^*,h^*})| \geq 2 \) such that \( [h,h^*] \subseteq V(F^2) \). Choosing \( h^* \) with minimum level in \( H \), we conclude that \( V(H_{h^*,h^*}) = \{h' \} \) for every \( h' \in [h,h^*]_H \). Furthermore, \( f^2 \notin [h,h^*]_H \), because otherwise, \( f^2 \) could be chosen as the root of \( H \). Since \( f^2 \) is a universal vertex of \( F^i \), \( |V(H_{h^*,h^*})| \geq 2 \) implies that \( f^2 \in V(H_{h^*,h^*}) \setminus \{h^*\} \), which means that \( f^2 \notin E(H) \). Therefore, every vertex of \( S \cap V(F^2) \) is ancestral of \( v \) in \( H \) and \( \ell_H(s) \leq |S \cap V(F^2)| \) for every \( s \in S \cap V(F^2) \), which means that \( H_{h^*,h^*} \) is a slim TP completion of \( (F^2, S \cap V(F^2)) \) and \( H = \langle H_{h^*,h^*}, \gamma \rangle \). Finally, the fact that \( H \) is a minimum TP completion of \( G \) implies that \( H_{h^*,h^*} \) is a minimum slim TP completion of \( (F^2, S \cap V(F^2)) \). \( \square \)
3.3 Computing a minimum TP completion

We begin by presenting the general idea of the proposed algorithm. Let \((G, F, v) \in 1\text{TP}\) and let \(H\) be a minimum TP completion of \(G\). Because of items \((1), (2)\) and \((3)\) of Lemma 13 either \(\ell_H(v) \leq 2\) or there are at most 4 possibilities to \(h\), namely, \(h \in \{f^1, f^2, s_1, s_2\}\), where \(s_i\) is the root of a minimum slim TP completion \(P_i\) of \(F^i\) for \(i \in [2]\). We will see that to construct all trees \(T\) such that \(T\) is a TP completion of \(G\) and \(\ell_T(v) \leq 2\) can be done in polynomial time. To the other cases, we can write \(H = (\hat{H}, H^*), \) where either \(\hat{H} = F^i\) for \(i \in [2]\) or \(\hat{H} = P^i\) for \(i \in [2]\) and \(H^*\) is a minimum TP completion of \(G - \hat{H}\). These ideas lead to an algorithm for computing a minimum TP completion of \(G\). However, because of the recursion, we cannot guarantee a polynomial-time complexity. Therefore, in Algorithm 2, we exploit these and other properties in order to replace this big recursive call by few recursive calls with small inputs in order to obtain a polynomial-time number of steps.

We need some definitions. Let \(G, F, v\) be a connected graph and \(v\) a vertex of \(V(G)\) such that \((G, F, v) \in 1\text{TP}\), and let \(k = \min\{2, \omega(F)\}\). For \(i \in [\omega(F)]\), \(\Upsilon_i(G, v)\) is a family containing at most two members. One of the members of \(\Upsilon_i\) is \(\hat{F}^i\), and if \(|N_G(v) \cap V(F^2)| = 1\), then the second member exists and it is the minimum slim TP completion of \((F^2, N_G(v) \cap V(F^2))\). Denoting by \(J^1\) a minimum slim TP completion of \((F^2, N_G(v) \cap V(F^2))\), define \(\Lambda(G, v) = \{U(G, v), J^1, \Upsilon_1, \Upsilon_2\}\).

Let \(H\) be a TP completion of \(G\). We will write \(H = \langle X_1, \ldots, X_k \rangle_\Lambda\) if for every \(i \in [k]\), it holds that \(X_i \in \Lambda(G - (X_1 \cup \ldots X_{i-1}), v)\).

Using the above definitions, we can state a result that synthesizes and is stronger than the ideas discussed in the first paragraph of this section.

**Corollary 14.** Let \((G, F, v) \in 1\text{TP}\). If \(H\) is a minimum TP completion of \(G\), then there is \(X^1 \in \Lambda(G, v)\) such that \(H = \langle X^1, \Upsilon \rangle\) where \(\Upsilon\) is a minimum TP completion of \(G - X^1\). Furthermore, we can write \(H = \langle X^1, \ldots, X^k \rangle_\Lambda\).

**Proof.** It is a consequence of items \((1), (2), (3)\) and \((4)\) of Lemma 13. 

We discuss now some subroutines that are used in Algorithm 2. We call attention to the fact that the variables are passed to these subroutines by reference.

- **MINOF(Q):** \(Q\) is a family of graphs all having the same vertex set. The algorithm returns the graph of \(Q\) with minimum number of edges.
- **ADD(T, \Gamma):** \(T\) is a rooted tree and \(\Gamma\) is a family of rooted trees. The algorithm adds \(T\) to \(\Gamma\).
- **FINDTOP(G, v, R, \Upsilon, Q):** Algorithm 1
- **SLIMTPC(F, S):** \(F\) is a trivially perfect graph and \(S \subseteq V(F)\). The algorithm returns a minimum slim TP completion of \((F, S)\). 

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Algorithm 1: \texttt{FindTop}

\begin{algorithm}
\begin{itemize}
  \item \textbf{input}: \((G, F, v) \in \text{1TP}\) and a subtree \(R\) of \(\mathcal{F}_{\perp}\) (it also receives references to the family \(\Upsilon\) and the set of TP completions \(Q\) of \(G\))
  \item \textbf{output}: It updates \(Q\) and \(\Upsilon\)
  \item \(T := \langle R, U(G - R, v) \rangle\)
  \item \(k := \min\{2, \omega(G - v)\}\)
  \item \textbf{for} \(i \in [k]\) \textbf{do}
    \begin{itemize}
      \item \(\Upsilon_i := \Upsilon_i(G - R, v)\)
      \item \textbf{for} \(J \in \Upsilon_i\) \textbf{do}
        \begin{itemize}
          \item \(T' := \langle R, J, U(G - (R \cup J)) \rangle\)
          \item \textbf{if} \(|T'| < |T|\) \textbf{then}
            \begin{itemize}
              \item \(T := T'\)
            \end{itemize}
        \end{itemize}
    \end{itemize}
\end{itemize}
\text{ADD}(T, Q)
\end{algorithm}

Lemma 15. \textit{Given }\((G, F, v) \in \text{1TP}\) \textit{and a subtree }\(R\) \textit{of }\(\mathcal{F}_{\perp}\), \textit{Algorithm 1 computes }\(\Upsilon_1(G - R, v), \Upsilon_2(G - R, v)\) \textit{and a minimum TP completion of }\(G\) \textit{of the form }\langle R, T \rangle\textit{ with }\ell_T(v) \leq 2 \textit{ in }O(n)\textit{ steps where }n\textit{ is the order of }G.\)

\textit{Proof.} We can assume that we are using the tree representations of the graphs. Observe that we can rewrite the statement of the lemma as follows: \textit{given }\((Y, Z, v) \in \text{1TP}\), \textit{Algorithm 1 computes }\(\Upsilon_1(Y, v), \Upsilon_2(Y, v)\) \textit{and a minimum TP completion }\(T\) \textit{of }\(Y\) \textit{with }\ell_T(v) \leq 2 \textit{ in }O(n^2)\textit{ steps where }n\textit{ is the order of }Y.\)

Note that if \(\{u\} = N_Y(v) \cap V(Z_i)\) for some \(i \in [2]\), then the minimum slim TP completion of \((Z_i, N_Y(v) \cap V(Z_i))\) is the graph obtained by making \(u\) adjacent to all vertices of \(Z_i\). This can be clearly done in \(O(n)\) steps. Therefore, \(\Upsilon_i(G - R, v)\) can be computed in \(O(n)\) steps in line 4.

It is clear that a minimum TP completion \(T\) of \(Y\) such that \(\ell_T(v) = 1\) is \(U(Y, v)\). Such tree is computed in \(O(n)\) steps in line 1. Now, by Corollary 14 a minimum TP completion \(T\) of \(Y\) such that \(\ell_T(v) = 2\) is the minimum tree \(\langle X^1, U(Y - X^1, v) \rangle\) which can be formed choosing \(X^1\) among the members of \(\Upsilon_1(Y, v) \cup \Upsilon_2(Y, v)\). This is done in lines 5 to 8 of this algorithm spending \(O(n)\) steps since we are working with their tree representations.

Despite that the Trivially Perfect 1-completion has as input a graph \(G\), the input of Algorithm 2 is formed by a graph \(G\) and a vertex \(v \in V(G)\) to guarantee that the same vertex \(v\) be used in all recursive calls.

Theorem 16. \textit{Algorithm 2 is correct.}\n
\textit{Proof.} Let \(H\) be a minimum TP completion of \(G\) and set \(v\) as the base of \(H\). According to Corollary 14 there is \(X^1 \in \Lambda(G, v)\) such that \(\mathcal{H} = \langle X^1, \mathcal{Y} \rangle\) where \(\mathcal{Y}\) is a minimum TP completion of \(G - X^1\). The algorithm is divided into two cases. If \(|\mathcal{F}_{\perp}| < \frac{2|\mathcal{F}|}{3}\), then it finishes in line 28 otherwise, it finishes in line 69. In both lines, a call to MinOf is made where the family \(Q\) containing some TP completions of \(G\) is passed as parameter. Therefore, we have to show that \(H\) belongs to \(Q\) when one of these two lines is reached.
Algorithm 2: MinTPC

input: Graph $G$ and $v \in V(G)$ such that $G - v$ is a trivially perfect graph

output: A minimum TP completion of $G$

1. $F := G - v$
2. $S := N_G(v)$
3. $\Upsilon := \emptyset$
4. $\text{FindTop}(G, v, \emptyset, \Upsilon, Q)$
5. $B := F - \Upsilon$
6. if $|F_1| < \frac{2|F_1|}{3}$ then
   7. for $D^2 \in \Upsilon_2$ do
      8. $\text{Add}(\langle F_1, B_1, D^2 \rangle, \Gamma)$
      9. if $|S \cap V(B_1)| = 1$ then
         10. $\text{Add}(\langle F_1, \text{slimTPC}(B_1, S \cap V(B_1)), D^2 \rangle, \Gamma)$
   for $D^1 \in \Upsilon_1$ do
      12. $\mathcal{L} := \text{MinOf} (\{ \langle D^2, D^1 \rangle, \langle D^1, D^2 \rangle \})$
      13. $\text{Add}(\langle \mathcal{L}, U(G - L, v), Q \rangle)$
      14. if $|D^1| = |F_1|$ then
         15. $\text{Add}(\langle \mathcal{L}, \Gamma \rangle)$
      else if $|\mathcal{L}| \geq \frac{|F_1|}{2}$ then
         16. $\text{Add}(\langle \mathcal{L}, \text{MinTPC}(G - L, v), Q \rangle)$
   if $|F_1 \cap S| = 1$ then
      18. $\text{Add}(\text{slimTPC}(F_1, S), \Gamma)$
   else if $|F_1 \cap S| = 2$ then
      20. $\text{Add}(\langle F_1, \text{slimTPC}(B_1, S), \Gamma \rangle)$
   for $T \in \Gamma$ do
      22. if $|T| \geq \frac{|F_1|}{2}$ then
         23. $\text{Add}(\langle T, \text{MinTPC}(G - T, v), Q \rangle)$
   $\text{FindTop}(G, v, \Upsilon, \emptyset, \Upsilon, Q)$
   $\text{FindTop}(G, v, \Upsilon, \emptyset, \Upsilon, Q)$
   return $\text{MinOf}(Q)$

/* It continues */
Algorithm 2: continuation of MinTPC

29 \( \mathcal{R} := \emptyset \)
30 \( \mathcal{B} := \mathcal{F} \)
31 while \( |\mathcal{R}| < \frac{|\mathcal{F}|}{3} \) and \( |\mathcal{B}| \geq \frac{|\mathcal{B}|}{3} \) do
32 \( \text{FindTop}(G, v, \mathcal{R}, \Upsilon, Q) \)
33 for \( \Upsilon' \in \Upsilon_2 \) do
34 if \( |\langle \mathcal{R}, \Upsilon' \rangle| \geq \frac{|\mathcal{F}|}{3} \) and \( \left( \pi_{F_1}(l_{\mathcal{B}}) = \tau \text{ or } |\langle \mathcal{R}_{r', \tau}, \Upsilon' \rangle| < \frac{|\mathcal{F}|}{3} \right) \) then
35 \( \text{Add}(\langle \mathcal{R}, \Upsilon' \rangle, \Gamma^1) \)
36 if \( l_{\mathcal{B}} \notin S \) then
37 \( \mathcal{T} := \text{slimTPC}(B_{\mathcal{B}}, S \cap V(B_{\mathcal{B}})) \)
38 if \( |\mathcal{R}| + |\mathcal{T}| \geq \frac{|\mathcal{F}|}{3} \) then
39 \( \text{Add}(\langle \mathcal{R}, \mathcal{T} \rangle, \Gamma^2) \)
40 \( \mathcal{R} := \langle \mathcal{R}, \mathcal{B}_{\mathcal{B}} \rangle \)
41 \( \mathcal{B} := \mathcal{B} - \mathcal{B}_{\mathcal{B}} \)
/* For \( \mathcal{M}^i \in \Gamma^1 \), denote by \( \mathcal{B}^i \) the instance of \( \mathcal{B}_{\mathcal{B}} \) at the moment that \( \mathcal{M}^i \) is added to \( \Gamma^1 \). */
42 for \( \mathcal{M}^i \in \Gamma^1 \) do
43 if there are \( \mathcal{M}^i, \mathcal{M}^k \in \Gamma^1 \) with \( |\mathcal{B}^i| < |\mathcal{B}^j| < |\mathcal{B}^k| \) then
44 \( r' := \text{vertex such that } \pi_{F_1}(r') = m^i \)
45 for \( r'' \in [r', \pi_{F_1}(l_{\mathcal{B}})]_{\mathcal{R}} \) do
46 \( \text{Add}(\langle \mathcal{M}^i, \mathcal{R}_{r', r''}, U(G - (V(\mathcal{M}^i) \cup V(\mathcal{R}_{r', r''})), v) \rangle, \Gamma^3) \)
47 else
48 \( \text{Add}(\langle \mathcal{M}^i, \text{MinTPC}(G - \mathcal{M}^i, v) \rangle, \Gamma^3) \)
49 \( \Gamma^3 := \text{set formed by the trees of } \Gamma^2 \text{ with minimum number of vertices} \)
50 \( \Gamma^4 := \text{set formed by the trees of } \Gamma^2 \setminus \Gamma^3 \text{ with minimum number of vertices} \)
51 add to \( \Gamma \) one of the trees \( \Upsilon \) of \( \Gamma^3 \) such that \( |\Upsilon| \) is minimum
52 add to \( \Gamma \) one of the trees \( \Upsilon \) of \( \Gamma^4 \) such that \( |\Upsilon| \) is minimum
/* It continues */
Algorithm 2: continuation of MinTPC

53 if $|R| \ge \frac{|F|}{3}$ then
54 \hspace{1em} Add($R$, $\Gamma$
55 else
56 \hspace{1em} for $C \in \mathcal{T}_2$ do
57 \hspace{2em} for $r^* \in [\pi_{F^L}(b^2), \overline{\tau}]_R$ do
58 \hspace{3em} $D := \langle R_{r^*}, C, \mathcal{R}_{r^*}\rangle$
59 \hspace{3em} $\text{ADD}(<D, U(G - D)>)$
60 \hspace{1em} end for $r^*$
61 end for $C$
62 $b := \text{root of the maximum tree of } F - Z$
63 \hspace{1em} for $C \in \mathcal{T}_2$ do
64 \hspace{2em} $D := U(G, v)$
65 \hspace{3em} for $z^* \in [\pi_{F^L}(b'), \overline{\tau}]_Z$ do
66 \hspace{4em} if $|E((Z_{z^*}(v), C, Z_{z^*}))| < |E(D)|$ then
67 \hspace{5em} $D := \langle R_{r^*}, C, Z_{z^*}\rangle$
68 \hspace{4em} Add($<D, \text{MinTPC}(G - D, v), Q>$)
69 \hspace{3em} end if
70 end for $z^*$
71 end for $C$
72 return MinOf($Q$)

By Lemma 15, the possibilities for $\ell_H(v) \leq 2$ are considered in line 54 because in the call to FindTop done in this line we have $V(R) = \emptyset$. From now on, we assume that $\ell_H(v) \geq 3$. Hence, Lemma 13 (27) implies that $h \in V(F^L) \cup V(F^S)$, more precisely, $X^1 \in \mathcal{Y}_1 \cup \mathcal{Y}_2$. We denote by $v_2$ the ancestral of $v$ at level 2. If $\ell_H(v) \geq 4$, denote by $v_3$ the ancestral of $v$ at level 3.

Case 1: $|F^L| < \frac{2|F|}{3}$ (lines 8 to 28).

In lines 8, 11, 15, 19, 20 and 22, some rooted trees are saved in the family $\Gamma$. In line 25, for each $T$ added to $\Gamma$ with at least $\frac{|F|}{3}$ vertices, a recursive call with input $G - T$ is done, which returns as result a tree $T^*$. Then, the TP completion $\langle T, T^*\rangle$ is saved in $Q$ as a candidate to be a minimum TP completion of $G$. Other TP completions of $G$ are also saved in $Q$ in lines 13 and 17 and in the calls to FindTop (lines 3, 26 and 27). In line 28, a TP completion contained in $Q$ with minimum number of edges is chosen as a minimum TP completion of $G$. Therefore, we have to prove that $H$ is added to $Q$ in one of these lines.

We need to show that a tree $T$ saved in $\Gamma$ cannot satisfies $H = \langle T, T^*\rangle$ if $|T| < \frac{|F|}{3}$. By Lemma 13 (27), each tree added in one of the lines 15, 19, 20 and 22 has at least $\frac{|F|}{3}$ vertices because it contains $V(F^1)$. Note that each tree $T$ added to $\Gamma$ in line 8 or 11 is formed from vertices of three trees $T^1, T^2$ and $T^3$ such that $T^1$ is a subtree of $F^T$ and $T^2$ and $T^3$ are subtrees of different trees of $F$. Thus, suppose by contradiction that $T = H_{h,v_3}$ was added to $\Gamma$ in line 8 or 11 having less than $\frac{|F|}{3}$ vertices. Since $h \in V(F^L)$, $H$ has at least $\frac{|F|}{3}$ fill edges with one extreme in $h$ because $|F^L| < \frac{2|F|}{3}$. Furthermore, $H$ has another $\frac{|F|}{3}$ fill edges, each with one extreme in $v_2$ or $v_3$ and the other extreme in $H - H_{h,v_3}$, because every vertex of $H - H_{h,v_3}$ has a non-neighbor in $G$ belonging to $\{v_2, v_3\}$ since $v_2$ and $v_3$ belong
to different tres of $F$ and $|H_{h,v_2}| < \frac{|F|}{3}$, which is a contradiction because this implies that $fill(H) \geq |V(F)| > fill(U(G,v))$.

First, consider that $h \in V(F^\perp)$. Since $\ell_H(v) \geq 3$, Lemma 13 implies that $v_2 \in V(F^\perp)$. Because of Lemma 13, the possible cases for $H$ are the members of $\mathcal{Y}_2$. Recall that $\mathcal{Y}_i$ for $i \in \{\omega(F)\}$ was constructed in line 4 and contains at most two members, one is $F^\perp$ and the other is the slim TP completion of $(F^\perp, N_G(v) \cap V(F^\perp))$ if $|S \cap V(F^\perp)| = 1$.

All possibilities for $V(H) \subseteq V(F^\perp)$ and $V(H_{v_2,v_3}) \subseteq V(F^\perp)$ are considered in lines 13, 15 and 17. In fact, note that line 17 is executed only if $|H_{h,v_2}| \geq \frac{|F|}{6}$, condition that is checked in line 10. Therefore, suppose by contradiction that $|H_{h,v_2}| < \frac{|F|}{6}$. Since the case $\ell_H(v) = 3$ is covered in line 13 we can assume that $\ell_H(v) \geq 4$. Since $h \in V(F^\perp)$ and $v_2 \in V(F^\perp)$, $H$ has at least $\frac{3|F|}{6}$ fill edges each of them with exactly one extreme in $\{h, v_2\}$ because each vertex of $H - H_{h,v_2}$ has a fill edge to $h$ or to $v_2$. Now, Lemma 13 implies that $|W^1| \geq \frac{|F|}{6}$ where $W^1 = V(F^\perp) \setminus V(H_{v_2,v_3})$. The assumptions that $|F^\perp| < \frac{2|F|}{3}$ and $|H_{h,v_2}| < \frac{|F|}{6}$ imply that $|W^2| \geq \frac{|F|}{6}$ where $W^2 = V(H_{v_2,v_3}) \setminus (V(F^\perp) \cup V(F^\perp))$. Since $\ell_H(v) \geq 4$, if $v_3 \in V(F^\perp)$, then $H$ has at least $\frac{|F|}{6}$ fill edges all of them having one extreme in $v_3$ and the other in $W^2$; and if $v_3 \notin V(F^\perp)$, then $H$ has at least $\frac{|F|}{6}$ fill edges all of them having one extreme in $v_3$ and the other in $W^1$. Therefore, $fill(H) \geq |V(G)| > fill(U(G,v))$, which is a contradiction.

Consider now that $h \in V(F^\perp)$. By Corollary 11, $\mathcal{X}^1 \in \{F^\perp, J\}$, where $J$ is a minimum slim TP completion of $(F^\perp, N_G(v) \cap V(F^\perp))$. First, assume that $\mathcal{X}^1 = J$. The case $|S \cap V(F^\perp)| = 1$ is considered in line 19. The case $|S \cap V(F^\perp)| = 2$ is considered in lines 20 and 22. At last, assume by contradiction that $|S \cap V(F^\perp)| \geq 3$. Therefore, $\{h, v_2, v_3\} \subset V(F^\perp)$ and every vertex of $F - F^\perp$ has three fill edges each one with an extreme in each vertex of $\{h, v_2, v_3\}$. Since $|F^\perp| < \frac{2|F|}{3}$, it holds that $|F - F^\perp| \geq \frac{|F|}{3}$ and then that $fill(H) \geq |V(F)| > fill(U(G,v))$, which is a contradiction.

Now, assume that $\mathcal{X}^1 = F^\perp$. The possible cases for $v_2 \in V(F^\perp)$ are considered in lines 13, 15 and 17. As noted above, line 17 is executed only if $|H_{h,v_2}| \geq \frac{|F|}{6}$, condition that is checked in line 10. The same proof used above does hold here to show that $|H_{h,v_2}| < \frac{|F|}{6}$ cannot happen. The cases for $v_2 \in \{v\}$ or $v_2 \in V(F^i)$ for $i \in \{3, \ldots, \omega(F)\}$ are considered in line 26. The case $V(H_{v_2,v_3}) = V(F^\perp) \setminus (V(F^\perp) \cup V(F^\perp))$ is considered in line 20. Then, we can assume that $H_{v_2,v_3} = F^\perp$ where $B = F^\perp - F^\perp$. The cases for $v_3 \in \{v\}$ or $v_3 \in V(F^i)$ for $i \in \{3, \ldots, \omega(F)\}$ are considered in line 27. Since we have already shown that $\{h, v_2, v_3\} \notin V(F^\perp)$, we can assume that $v_3 \in V(F^\perp)$. The possibilities for this case are considered in lines 8 and 10.

Case 2: $|F^\perp| \geq \frac{2|F|}{3}$ (lines 20 to 69).

From now on, when we refer to $R, B$ and $\Gamma^1$ without mentioning the line of the algorithm we are considering, assume that their values are the instances of the homonym variables of the algorithm when the while loop finishes. The conditions of the while loop (line 31) and $|F^\perp| \geq \frac{2|F|}{3}$ imply that $V(R) \subseteq V(F^\perp)$. We need some claims.

The first claim deals with the case where $\varphi$ is ancestral of $v$ in $H$ and has two ancestors in $H$ that are not ancestors of $\varphi$ in $F$. 





Claim 17. If there are \(w_1, w_2 \notin [f, \tau]_{\mathcal{F}}\) such that \(w_1 \in [h, w_2]_H\) and \(w_2 \in (w_1, \tau)_{H}\), then there are \(x_1, x_2 \in [\tau, \tau]_R\) such that \(H_{w_1} = (\mathcal{R}_{\tau, \pi(x_1)}, W_1, \mathcal{R}_{x_1, x_2}, W_2)\), \(W_1 \in \Upsilon_2(G - H_{h, \pi(x_1)}, \nu)\), \(W_2 \in \Upsilon_2(G - H_{h, x_2}, \nu)\) and \(|\langle \mathcal{R}_{\tau, \pi(x_1), \nu} \rangle| \geq \frac{|F|}{3}\).

Proof of Claim 17 Corollary 14 and the construction of \(R\) (lines 19 and 40) imply that \(H_{w_1} = (\mathcal{R}_{\tau, \pi(x_1)}, W_1, \mathcal{R}_{x_1, x_2}, W_2)\), \(W_1 \in \Upsilon_2(G - H_{h, \pi(x_1)}, \nu)\) and \(W_2 \in \Upsilon_2(G - H_{h, x_2}, \nu)\).

Now, suppose by contradiction that \(|\langle \mathcal{R}_{\tau, \pi(x_1), \nu} \rangle| < \frac{|F|}{3}|\). If \(w_1 = h\), then \(w_1 \notin V(F)\) because \(W_1 \in \Upsilon_2(G - H_{h, \pi(x_1)}, \nu)\) and \(H_{h, \pi(x_1)} = \emptyset\), which means that \(H\) has at least \(\frac{2|F|}{3}\) fill edges each with one extreme in \(w_1\) because \(|F| \geq \frac{2|F|}{3}\). If \(w_1 \neq h\), then \(H\) has at least \(\frac{2|F|}{3}\) fill edges each with one extreme in \(w_1\) or \(h = f\) because \(|\langle \mathcal{R}_{\tau, \pi(x_1), \nu} \rangle| < \frac{|F|}{3}|\) and every vertex of \(H_{x_1}\) has in \(G\) one of these two vertices as a non-neighbor. Next, note that \(|F| \geq \frac{|F|}{3}|\). Since every vertex of \(F\) has a fill edge to \(w_2\), there are at least \(\frac{|F|}{3}|\) fill edges with both extremes in \(H_{w_2}\), which means that \(fill(H) \geq |F| > fill(U(G, \nu))\), a contradiction.

The second lemma says that for all but at most two trees added to \(\Gamma^1\) in line 55, a minimum TP completion is given by the universal TP completion.

Claim 18. Denote by \(\mathcal{R}_\ell, B_\ell\) and \(\Upsilon_\ell\) the instances of \(R, B, \Upsilon\), respectively, in line 63 of the \(\ell\)-th iteration of the while loop. If there are positive integers \(i < j < k\) such that \(|B^i| < |B^j| < |B^k|\), \(\mathcal{M}_\ell = (\mathcal{R}_\ell, \Upsilon_\ell) \in \Gamma^1\) for \(\ell \in \{i, j, k\}\) where \(\Upsilon_\ell \in \Upsilon_\ell\) and \(\mathcal{H} = (\mathcal{M}_i, \mathcal{P})\), then \(\mathcal{P} = (\mathcal{R}, \mathcal{M}_i, U(G - (V(\mathcal{M}_i) \cup V(\mathcal{R}_{\tau, \nu}))))\) where \(\pi_\mathcal{R}(r^\ell) = \pi_\mathcal{R}\) and \(r^\ell \in [\tau, \pi_\mathcal{F} \mathcal{F}(b^\ell)]\).

Proof of Claim 18 We begin remarking that the choice \(r''\) corresponds to the case where \(\mathcal{R}_\tau, r'' = \emptyset\). By the construction, we know that \(\mathcal{R}_i \subseteq \mathcal{R}_j \subseteq \mathcal{R}_k\) and \(V(\Upsilon_\ell) \subseteq V(B^j)\) for \(\ell \in \{i, j, k\}\), that \(\pi_\mathcal{F} \mathcal{F}(b^\ell) \in (\pi_\mathcal{R}, \pi_\mathcal{F} \mathcal{F}(b^\ell))\), that \(\pi_\mathcal{F} \mathcal{F}(b^\ell) \in (\pi_\mathcal{R}, \pi_\mathcal{F} \mathcal{F}(b^\ell))\), and that \(B^j\) and \(B^k\) have disjoint vertex subsets of \(V(\mathcal{F})\) both containing vertices of \(S\), see Figure 1.

Suppose by contradiction that the claim is false, i.e., \(\mathcal{P} \neq (\mathcal{R}_\tau, r'', U(G - (V(\mathcal{M}_i) \cup V(\mathcal{R}_{\tau, \nu}))))\) for any choice of \(r''\) in \([\pi_\mathcal{F} \mathcal{F}(b^\ell)]\). Applying Lemma 13 on \(H_{p}\), we conclude that the number of fill edges with both extremes in \(H_p = \mathcal{P}\) is less than \(|B^k|\). On the one hand, \(\mathcal{P} = (\mathcal{R}_\tau, \nu, S', \mathcal{K}'\prime)\), where \(S'\) is a minimum slim TP completion of \((F, N_G(v) \cap V(F))\) for some \(r^\ell \in [\nu]_R\) and \(\mathcal{K}'\prime\) is a minimum TP completion of \(G - (V(\mathcal{M}_i) \cup V(\mathcal{R}_{\tau, \nu})))\). We have a contradiction in the case because the construction of \(S'\) needs of at least \(|B^j|\) fill edges and \(|B^j| > |B^\ell|\). On the other hand, \(\mathcal{P} = (\mathcal{R}_\tau, \nu', \mathcal{K}''\prime)\) where \(\mathcal{K}''\prime\) is a minimum TP completion of \(G - (V(\mathcal{M}_i) \cup V(\mathcal{R}_{\tau, \nu'})))\). Lemma 13 leads to a contradiction in this case because the second greatest tree of \(G - (\mathcal{M}_i \cup \mathcal{R}_{\tau, \nu'})\) is \(\mathcal{B}^j\) and \(|B^j| \geq |B^\ell|\).

Claim 19. Only the two smallest sizes of trees \(\mathcal{M}\) added to family \(\Gamma^2\) in line 59 can satisfy \(H = (\mathcal{M}, \mathcal{K})\) where \(\mathcal{K}\) is a minimum TP completion of \(G - V(\mathcal{M})\).

Proof of Claim 19 Suppose that \(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in \Gamma^2\) when line 59 is reached with \(|\mathcal{M}_1| > |\mathcal{M}_2| > |\mathcal{M}_3|\). Note that \(\mathcal{M}_i\) was added to \(\Gamma^2\) before \(\mathcal{M}_i+1\) for \(i \in [2]\). For \(i \in [3]\), write \(\mathcal{M}_i = (\mathcal{R}_i, \mathcal{T}_i)\) where \(\mathcal{T}_i\) is the slim TP completion constructed in line 57 and \(\mathcal{H}_i = (\mathcal{M}_i, \mathcal{K}_i)\) where \(\mathcal{K}_i\) is a minimum TP completion of \(G - V(\mathcal{M}_i)\). Denote by \(W_i\) the subtree of \(\mathcal{F}\) induced by the
Recall that $T$ is the second greatest tree of $G - \bar{T}$. However, $F^2 \neq B^2_{(3)}$ because $F^2$ is not the second greatest tree of $G - F^1_{\bar{T},w_1}$. The number of iterations of the while loop is 6 which is the level of $\bar{T}$ in $F^1_{\bar{T},w_1}$.

vertices of $\bar{T}$. We know that $R^1 \subset R^2 \subset R^3$, that $T^3 \subset T^2 \subset T^1$ and that $P^1 \subset P^2 \subset P^3$ where $P^i = F^1_{\bar{T},w_1} - (R^i \cup T^i)$, see Figure 1.

Since $R^2 \cup T^2 \subset R^1 \cup T^1$ and $R^1 \subset R^2$, there exists $s_1 \in S \cap V(T_1 - (R_2 \cup T_2))$. Analogously, there exist $s_2 \in V(T_2 - (R_3 \cup T_3)) \cap S$ and $s_3 \in V(T_3) \cap S$. Note that $\{s_1, s_2, s_3\}$ is an independent set of $G$.

Let $w_a$ be the vertex of $W^1$ with minimum level such that $s_1 \in V(W^1_{w_a})$ and $s_2 \notin V(W^1_{w_a})$; and let $w_b$ be the vertex of $W^1$ with minimum level such that $s_2 \in V(W^1_{w_b})$ and $s_1 \notin V(W^1_{w_b})$. Recall that $T^1$ is a minimum slim TP completion of $(W^1, N_G(v) \cap V(W^1))$. Therefore, $s_1$ and $s_2$ are universal vertices in $T^1$. Since $w_a$ and $w_b$ are siblings and $V(W^2) \subseteq V(W^1_{w_a})$, it holds that there exist at least $|T^2|$ fill edges in $H$ each one joining $s_1$ to the vertices of $W^1_{w_a}$.

Let $w_c$ be the vertex of $W^2$ with minimum level such that $s_2 \in V(W^2_{w_c})$ and $s_3 \notin V(W^2_{w_c})$; and let $w_d$ be the vertex of $W^2$ with minimum level such that $s_3 \in V(W^2_{w_d})$ and $s_2 \notin V(W^2_{w_d})$. Recall that $T^2$ is a minimum slim TP completion of $(W^2, N_G(v) \cap V(W^2))$. Therefore, $s_2$ and $s_3$ are universal vertices in $T^2$. Since $w_c$ and $w_d$ are siblings and $V(W^3) \subseteq V(W^2_{w_c})$, it holds that there exist at least $|T^3|$ fill edges in $H$ each one joining $s_2$ to the vertices of $W^2_{w_d}$.

Note that every vertex $u \in V(P^3)$ has the three fill edges $u_{s_1}, u_{s_2}, u_{s_3}$ in $H^1$. Hence, we have that the number of fill edges joining vertices of $F^1_{\bar{T}}$ is at least $|T^2| + |T^3| + 3|P^2|$. By the
conditions of the while loop (line 31) and the fact that $|R| \geq \frac{2|F|}{3}$, we have that $|R| < \frac{|F|}{2}$, which means that $|T^i| + |P^i| \geq \frac{|F|}{2}$ for $i \in [3]$. Since $|P^3| > |P^2| > |P^1|$, we have that $|T^2| + |T^3| + 3|P^2| \geq |F|$. Since every vertex not in $F$ has a fill edge to $h$, it holds that $fill(H) > |V(F)| > fill(U(G, v))$, a contradiction.

As in the case where $|F| < \frac{2|F|}{3}$, some rooted trees are saved in the family $\Gamma$ (lines 39 and 42). Then, in line 61 for each $T$ added to $\Gamma$, a recursive call is done obtaining as result a tree $T'$. Then, the TP completion $(T, T')$ is saved in $Q$ as a candidate to be a minimum TP completion of $G$. Other TP completions of $G$ are saved in $Q$ in lines 39 and ?? and in the call to FINDTop (line 32). In line 60 the TP completion with minimum number of edges contained in $Q$ is chosen as a minimum TP completion of $G$. Therefore, we have to show that $H$ is added to $Q$ in one of these lines.

By Corollary 14 we can write $H = \langle \mathcal{X}^1, \ldots, \mathcal{X}^k \rangle_{\Lambda}$ for $k \geq 1$. Let $p \in [k]$ be such that $\mathcal{T} \in V(\mathcal{X}^p)$. We consider 3 subcases:

Case 2.1: $V(\mathcal{X}^i) \subseteq V(\mathcal{T})$ for every $i \in [p - 1]$.

First, consider that $\mathcal{X}^p = \mathcal{R}_T$. The case where $|R| \geq \frac{|F|}{3}$ is considered in line 64. If $|R| < \frac{|F|}{3}$, then $|\mathcal{T}| < \frac{|F|}{3}$, which means that if $H = \langle \mathcal{R}, K \rangle$ where $K$ is a minimum TP completion of $G - V(\mathcal{R})$, then Lemma 13 (iv) implies that $\ell_K(v) \leq 2$. This case is covered in line ??.

Now, consider that $\mathcal{X}^p \neq \mathcal{R}_T$, i.e., $\mathcal{X}_p$ is a minimum slim TP completion $T$ of $\mathcal{F}$ for some $r' \in [r, T] \setminus S$. This case is considered in lines 38 to 39. Because of the if condition of line 38 we have to prove that we only need to save $\langle \mathcal{R}, \mathcal{T} \rangle$ in line 39 if $\frac{|F|}{3} + |\mathcal{T}| \geq \frac{|F|}{3}$. Suppose by contradiction that $|\mathcal{T}| < \frac{|F|}{3} - |\mathcal{R}|$ and write $H = \langle \mathcal{R}, \mathcal{W} \rangle$. Observe that the vertices of $T$ form the biggest tree of $F - \mathcal{R}$, which implies, by items (iv) and (v) of Lemma 13, that $\ell_{\mathcal{W}}(v) \leq 2$. However, these cases have already been considered in line 39. Then, we can assume that $|\mathcal{T}| \geq \frac{|F|}{3} - |\mathcal{R}|$ or equivalently that $\frac{|R|}{3} + |\mathcal{T}| \geq \frac{|F|}{3}$.

According to Claim 13 we can discard the trees added to $\Gamma^2$ in line 39 having more vertices than the two smallest sizes. Now, consider two trees $\mathcal{T}$ and $\mathcal{T}'$ added to $\Gamma^2$ in line 39 such that $V(\mathcal{T}) = V(\mathcal{T}')$. Note that if $|T| > |T'|$, then $H$ cannot be $\langle \mathcal{T}, T'' \rangle$, because $\langle \mathcal{T}', T'' \rangle$ has less fill edges than $\langle \mathcal{T}, T'' \rangle$. The trees that cannot be part of a minimum solution by these observations are eliminated in lines 49 and 52.

Case 2.2: There is exactly one $i \in [p - 1]$ such that $\mathcal{X}^i \in \mathcal{Y}_2(G - \mathcal{H}_{h, \pi(x'^i)}$, $v)$.

We start this case proving that the proper ancestors in $Z$ of the vertex $b$ chosen in line 62 do not need to be considered in the for loop beginning in line 65. Let $Z$ and $C$ be chosen in lines 60, 62, and 63 of a same iteration of the for loop beginning in line 60 respectively. Therefore, it suffices to show that for any $z' \in [z, \pi_x(b)$, it holds $|E((Z_{z, \pi(z')}, P)) | \geq |E(U(G, v))|$ where $P$ is a minimum TP completion of $G - V(Z_{z, \pi(z')})$, $S \cap V(Z_{z, \pi(z')})$). Write $B = \mathcal{F} - Z$ and $\mathcal{D} = \mathcal{F} - Z_{z, z'}$. Note that $b = b_{\mathcal{D}}$ and that $|V(D_{\mathcal{D}})| < |V(D_{\mathcal{D}})|$. By Lemma 13 (iv), the number of fill edges of $\langle Z_{z, \pi(z')}, P \rangle$ that are not incident to $z'$ is smaller than $|V(D_{\mathcal{D}})|$. However, since there is exactly one $i \in [p - 1]$ such that $\mathcal{X}^i \in \mathcal{Y}_2(G - \mathcal{H}_{h, \pi(x'^i)}$, $v)$, there are $|V(B_{\mathcal{D}})|$ fill edges
in \(\langle \mathcal{Z}_{z, \pi(x')} : \mathcal{P} \rangle\) that are not incident in \(z'\), a contradiction. Hence, in line 65 we only need to consider the vertices \(z'\) belonging to \(\pi_{F}(b, \pi)^{2}\).

On the one hand, \(X_{p} = \mathcal{R}_{r}\). Then, we can write \(\mathcal{H}_{r} = \langle \mathcal{R}_{r}, \mathcal{K} \rangle\) where \(\mathcal{K}\) is a minimum TP completion of \(G - V(\mathcal{H}_{r, p})\). If \(|\mathcal{R}| < \frac{|\mathcal{J}|}{3}\), then \(|\mathcal{B}| < \frac{|\mathcal{I}|}{3}\). Hence, by Lemma 13, it holds that \(\ell_{\mathcal{K}}(v) \leq 2\). This case is covered in lines 50 to 59. The case where \(|\mathcal{R}| \geq \frac{|\mathcal{J}|}{3}\) is considered in lines 62 to 68 using the tree \(\mathcal{R}\) saved in line 64.

On the other hand, \(X_{p} \neq \mathcal{R}_{r}\), i.e., \(X_{p}\) is a slim TP completion \(\mathcal{T}\) of \(\mathcal{J}_{r}\) for some \(r' \in [r, \pi]_{\mathcal{R}} \setminus S\). This case is considered in lines 62 to 68 using the tree \(\langle \mathcal{R}, \mathcal{T} \rangle\) saved in \(\Gamma^{2}\) in line 39.

**Case 2.3:** There are at least two \(i \in [p - 1]\) such that \(X^{i} \in \Upsilon_{r}(G - \mathcal{H}_{r, \pi(x')}, v)\).

Therefore, there are \(y, y' \in [r, \pi]_{\mathcal{R}}\) and \(j, k \in [p - 1]\) such that \(\mathcal{H}_{r, x} = \langle x^{1}, \ldots, x^{\ell} \rangle = \langle \mathcal{R}_{r, \pi(y)}, X^{j}, \mathcal{R}_{r, y'}, X^{k} \rangle, X^{j} \in \Upsilon_{r}(G - \mathcal{R}_{r, \pi(y)}, v)\) and \(X^{k} \in \Upsilon_{r}(G - (\mathcal{R}_{r, y'} \cup X^{j}), v)\). By Claim 17, we know that \(|\langle \mathcal{R}_{r, \pi(y)}, X^{j} \rangle| \geq \frac{|\mathcal{J}|}{3}\). Furthermore, we claim that we can discard \(\langle \mathcal{R}_{r, \pi(y)}, X^{j} \rangle\) if \(\pi(x^{j}) \neq \pi(y)\) and \(|\langle \mathcal{R}_{r, \pi(x^{j})}, X^{j} \rangle| \geq \frac{|\mathcal{J}|}{3}\). Indeed, by the algorithm, we know that \(|\mathcal{R}_{r, \pi(y)}| < \frac{|\mathcal{J}|}{3}\).

Since \(|\langle \mathcal{R}_{r, \pi(y)}, X^{j} \rangle| \geq \frac{|\mathcal{J}|}{3}\), we have that \(|X^{j}| > |\mathcal{H}_{r}(y)|\), which implies that \(|\langle \mathcal{R}_{r, \pi(x^{j})}, X^{j}, \ldots, X^{k} \rangle| \geq \frac{|\mathcal{J}|}{3}\). These constraints are guaranteed by the if conditions in line 53.

Finally, every tree saved in \(\Gamma^{1}\) in line 35 is considered in the for loop beginning in line 42. If \(\mathcal{M}^{1}\) satisfies the if conditions of line 13 then, by Claim 18, the possibilities for \(\mathcal{M}^{1}\) are covered in for loop beginning in line 46. Otherwise, a recursive call for \(\mathcal{M}^{1}\) is made in line 48.

**Theorem 20.** Algorithm 2 runs in \(O(n^{7})\) steps where \(n\) is the order of the input graph.

**Proof.** The algorithm is divided into two cases, one is when \(|\mathcal{J}| < \frac{2|\mathcal{J}|}{3}\) and the other is when \(|\mathcal{J}| \geq \frac{2|\mathcal{J}|}{3}\). Then, we can express the number of steps of the algorithm by \(T(n) = T_{1}(n) + T_{2}(n)\), where \(T_{1}(n)\) is associated with the first case and \(T_{2}(n)\) with the second case. The algorithm has five lines with recursive calls, namely, lines 17 and 25 for \(|\mathcal{J}| < \frac{2|\mathcal{J}|}{3}\), and lines 48, 61, and 68 for \(|\mathcal{J}| \geq \frac{2|\mathcal{J}|}{3}\).

For the case \(|\mathcal{J}| < \frac{2|\mathcal{J}|}{3}\), we will show that \(T_{1}(n) \leq 2T_{1}(\frac{5n}{6}) + 6T_{1}(\frac{2n}{3})\). The input graph of each recursive call in line 17 has order at most \(\frac{5|\mathcal{J}|}{6}\) because it is checked in line 16 if \(|\mathcal{L}| \geq \frac{|\mathcal{J}|}{6}\).

We have to show that line 17 is executed at most twice. The tree \(\mathcal{L}\) considered in line 17 is formed by one member of \(\Upsilon_{1}\) and one member of \(\Upsilon_{2}\). We know that \(|\Upsilon_{1}| \leq 2\) and \(|\Upsilon_{2}| \leq 2\).

However, if \(|\Upsilon_{1}| = 2\), then there is a tree in \(\Upsilon_{1}\) with the same order as \(\mathcal{J}_{r}\). Then, for this tree, line 15 is executed instead of line 17 which means that line 17 is executed at most twice.

The input graph of each recursive call in line 25 has order at most \(\frac{2|\mathcal{J}|}{3}\) because it is checked in line 24 if \(|\mathcal{L}| \geq \frac{|\mathcal{J}|}{3}\). Line 25 is executed at most 6 times because \(|\mathcal{L}| \leq 6\) when this line is reached. This is true because at most two trees are added to it in line 8 because \(|\Upsilon_{2}| \leq 2\), at most two in line 13 because at most one member of \(\Upsilon_{1}\) has \(\mathcal{J}_{r}\) vertices, at most one in line 20 and at most one in line 22.

For the case \(|\mathcal{J}| \geq \frac{2|\mathcal{J}|}{3}\), we show that \(T_{2}(n) \leq 13T_{2}(\frac{2n}{3})\). The order of the input graph of each recursive call in line 48 is clearly at most \(\frac{2|\mathcal{J}|}{3}\) because it is checked in the if conditions of line 34 if \(|\langle \mathcal{R}, \mathcal{J} \rangle| \geq \frac{|\mathcal{J}|}{3}\). The order of the input graph of every recursive call in lines 61 and 68 is at most \(\frac{2|\mathcal{J}|}{3}\) because it is checked in the if conditions of lines 35 and 36 respectively, if the
tree has at least $\frac{|F|}{3}$ before it be added to $\Gamma$. It remains to show that the number of recursive calls for this case is at most 13.

Denoting by $B_{(j)}$ the instance of $\mathcal{B}$ in line 39 of the $j$-th iteration of the while loop, it holds that $|B_{(i_1)}| < |B_{(i_2)}|$ if $i_1 < i_2$. Therefore, the if condition of line 13 and the fact that $|Y_2| \leq 2$ for any instance of $\mathcal{Y}$ imply that the number of recursive calls in line 45 is at most 4.

At most two trees added to family $\Gamma^2$ in line 39 are saved in $\Gamma$ in lines 51 to 52. Since one more tree can be added to $\Gamma$ in line 54, the family $\Gamma$ has at most 3 elements when line 60 is reached. For each member of $\Gamma$, at most 3 recursive calls are done in lines 61 and 68, which result in at most 9 for this subcase. Then, the total number of recursive calls of this case is at most 13.

Using Lemma 11 it is easy to see that the remaining operations of the algorithm cost $O(n^5)$. Using the Master Theorem [3], we conclude that $T_2(n) = O(n^{\log_2 13}) = O(n^7)$. We use induction on $n$ to prove that $T_1(n) < n^7$ for $n$ sufficiently large. $T_1(n) \leq 6T_1(\frac{5n}{2}) + 2T_1(\frac{5n}{6}) + n^6 \\ \leq 6(\frac{5n}{2})^7 + 2(\frac{5n}{6})^7 + n^6 < \frac{9}{10n}n^7 + n^6 < n^7$. Therefore $T(n) = O(n^7)$.

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