TAMENESS FOR SET THEORY I

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Abstract. The paper is a first of two and aims to show that (assuming large cardinals) set theory is a tractable (and we dare to say tame) first order theory when formalized in a first order signature with natural predicate symbols for the basic definable concepts of second and third order arithmetic, and appealing to the model-theoretic notions of model completeness and model companionship.

Specifically we develop a general framework linking generic absoluteness results to model companionship and show that (with the required care in details) a $\Pi_2$-property formalized in an appropriate language for second or third order number theory is forcible from some $T \supseteq \text{ZFC} + \text{large cardinals}$ if and only if it is consistent with the universal fragment of $T$ if and only if it is realized in the model companion of $T$.

The paper is accessible to any person who has a fair acquaintance with set theory and first order logic at the level of an undergraduate course in both topics; however bizarre this may appear (given the results we aim to prove) no knowledge of forcing or large cardinals is required to get the proofs of its main results (if one accepts as black-boxes the relevant generic absoluteness results). On the other hand familiarity with the notions of model completeness and model companionship is essential. All the necessary model-theoretic background will be given in full detail.

The present work expands and systematize previous results obtained with Venturi.

The key model-theoretic result of this paper is that the definable (and conservative) extension of any $T \supseteq \text{ZFC}$ introducing predicates for the $\Delta_0$-definable (class) relations, function symbols for the $\Delta_0$-definable (class) functions, and predicates for the lightface definable projective subsets of $\mathcal{P}(\kappa)$ has as model companion the $T$-provable fragment of the theory of $H_{\kappa^+}$ in this signature (cfr. Thm. 3).

We also give evidence that any existence proof of the model companion of some $T$ extending $\text{ZFC} + \text{large cardinals}$ comes in pairs with generic absoluteness results for $T$.

Specifically we use Thm. 3 (and variations of it) to show that these results couple perfectly with Woodin’s generic absoluteness for second order number theory (cfr Thm. 1), the theory of $H_{\kappa_3}$ assuming Woodin’s axiom $\ast$ (cfr Thm. 2, Thm. 5, Thm. 7), and the author’s generic absoluteness results for the theory of $H_{\kappa_2}$ (cfr Thm. 4).

We proceed stating our main results.

Notation 1. Let $T$ be a $\tau$-theory. $T_\psi$ is the family of $\Pi_1$-sentences $\psi$ for $\tau$ which are provable from $T$. Accordingly we define $T_\exists$, $T_\forall_\exists$, etc.

Let $\tau_{ST}$ be a signature containing predicate symbols $R_\psi$ of arity $m$ for all bounded $\varepsilon$-formulae $\psi(x_1,\ldots,x_m)$, function symbols $f_\theta$ of arity $k$ for all bounded $\varepsilon$-formulae $\theta(y,x_1,\ldots,x_k)$, constant symbols $\omega$ and $\emptyset$. $\text{ZFC}_{\tau_{ST}} \supseteq \text{ZFC}$ is the $\tau_{ST}$-theory obtained adding axioms which force in each of its $\tau_{ST}$-models $\emptyset$ to be interpreted by the empty set,
Let $\omega$ to be interpreted by the first infinite ordinal, each $R_{\omega}$ as the class of $k$-tuples defined by the bounded formula $\psi(x_1, \ldots, x_k)$, each $f_{\theta}$ as the $l$-ary class function whose graph is the extension of the bounded formula $\theta(x_1, \ldots, x_l, y)$ (whenever $\theta$ defines a functional relation). Essentially ZFC$_{\text{ST}}$ is set theory axiomatized in a language admitting predicate symbols for $\Delta_0$-predicates, $\Delta_0$-definable functions, and a constant for the first infinite cardinal (see Notation 2 and Fact 1 below for details).

Let $\sigma_{\text{ST}}$ be a signature containing predicate symbols $S_\phi$ of arity $n$ for all $\tau_{\text{ST}}$-formulæ $\phi(x_1, \ldots, x_n)$; let $\sigma_\omega = \sigma_{\text{ST}} \cup \tau_{\text{ST}}$. ZFC$_{\omega}^*$ is the $\sigma_\omega$-theory obtained adding axioms which force in each of its $\sigma_\omega$-models each predicate symbol $S_\phi$ of arity $n$ to be interpreted as the subset of $P(\omega^n)$ defined by $\tau_{\text{ST}}$-formula $\phi^P(\omega^n)(x_1, \ldots, x_n)$. Essentially ZFC$_{\omega}^*$ extends ZFC$_{\text{ST}}$ adding predicate symbols for the lightface definable projective sets$^3$ (again see Notation 2 and Fact 1 below for details).

**Theorem 1.** Let $T$ be a $\sigma_\omega$-theory such that$^6$

$$T \supseteq \text{ZFC}_{\omega}^* \land \text{there are class many Woodin cardinals.}$$

Then $T$ has a model companion $T^*$. Moreover TFAE for any $\Pi_2$-sentence $\psi$ for $\sigma_\omega$:

1. For all universal $\sigma_\omega$-sentences $\theta$ such that $T + \theta$ is consistent, so is $T_\psi + \theta + \psi$;
2. $T$ proves that some forcing notion $P$ forces $\psi^H_{\omega_1}$;
3. $T \models \psi^H_{\omega_1}$;
4. $\psi \in T^*$.

**Theorem 2.** Let $\sigma_{\omega,\text{NS}_{\omega_1}}$ be the extension of $\sigma_\omega$ with a unary predicate symbol $\text{NS}_{\omega_1}$ and a constant symbol $\omega_1$. Consider the $\{\in, \omega_1, \text{NS}_{\omega_1}\}$-sentences:

1. $\theta_{\omega_1} \equiv \omega_1$ is the first uncountable cardinal,
2. $\theta_{\text{Stat}} \equiv \forall x (x \subseteq \omega_1 \text{ is non-stationary } \Leftrightarrow \text{NS}_{\omega_1}(x))$.

Let $ZFC_{\omega,\text{NS}_{\omega_1}}^*$ be the theory $ZFC_{\omega}^* + \theta_{\text{Stat}} + \theta_{\omega_1}$.

Let also $\theta_{SC}$ be the $\in$-sentence:

$$ZFC_{\omega}^* + \theta_{\text{Stat}} + \theta_{\omega_1}.$$ 

TFAE for any

$$T \supseteq \text{ZFC}_{\omega,\text{NS}_{\omega_1}}^* + \theta_{SC}$$

and for any $\Pi_2$-sentence $\psi$ for $\sigma_{\omega,\text{NS}_{\omega_1}}$:

1. For all universal $\sigma_{\omega,\text{NS}_{\omega_1}}$-sentences $\theta$ such that $T + \theta$ is consistent, so is $T_\psi + \theta + \psi$;
2. $T$ proves that some forcing notion $P$ forces $\psi^H_{\omega_2}$;
3. $T_\psi + ZFC_{\omega,\text{NS}_{\omega_1}}^* + \theta_{SC} + (\ast)\text{-UB} \models \psi^H_{\omega_2}$.

See Remark 1(6) for some information on $(\ast)$-UB.

In this article we will give a self-contained proof of Thm. 1 and of a weaker variation of Thm. 2 (cfr. Thm. 4). Thm. 2 is an easy corollary of results which we will formulate in this paper at a later stage (since they need more terminology than what has been introduced so far to be properly stated), and which will be proved in a sequel of this paper.

$^3$For a set or definable class $Z$ and a $\tau_{\text{ST}}$-formula $\psi, \psi^Z$ denotes the $\tau_{\text{ST}}$-formula obtained from $\psi$ requiring all its quantifiers to range over $Z$.

$^5$We decide to use $P(\omega^\omega)$ rather than $P(\omega)$ (or any other uncountable Polish space) to simplify slightly the coding devices we are going to implement to prove Thm. 1. Similar considerations brings us to focus on $P(\omega^\omega)$ rather than $P(\omega)$ in the formulation of Thm. 4 and on $P(\kappa^\omega)$ rather than $P(\kappa)$ in the formulation of Thm. 3. At the price of complicating slightly the relevant proofs one can choose to replace $\alpha^\omega$ by $\alpha$ all over for $\alpha$ any among $\omega, \omega_1, \kappa$.

$^6$It is not relevant for this paper to define Woodin cardinals. A definition is given in [13, Def. 1.5.1], for example.
(since their proof is considerably more involved, and its inclusion here would make the length of this paper grow exponentially).

Some of the following remarks are technical and require a strong background in set theory. The reader can safely skip them without compromising the comprehension of the remainder of this paper.

**Remark 1.**

1. The theories $T$ considered in all the above theorems are definable and conservative extensions of their $\in$-fragment; more precisely: for any of the above signatures $\tau$ there is a recursive list of axioms $T_\tau \subseteq T$ such that any $\in$-structure admits a unique extension to a $\tau$-structure which models $T_\tau$ (see Fact 1 below).

The key but trivial observation is that in the new signatures one can express the same concepts one can express in the signature $\in$, but using for many of these concepts formulae of much lower complexity according to the prenex normal form stratification. For example:

- In $\sigma_\omega$ projective determinacy is expressible by means of a family of contably many atomic sentences (see item 5 below).
- There is an uncountable cardinal is expressible by the $\Sigma_2$-sentence for $\tau_{ST}$ ($\text{and } \sigma_\omega$).

$$\exists x [(x \text{ is an ordinal } \land \omega \in x) \land \forall f [(f \text{ is a function } \land \text{dom}(f) \in x) \to \text{ran}(f) \neq x]$$

(and this concept cannot be expressed in this signature by a formula of lower complexity, even in $\sigma_\omega$).

- On the other hand the above sentence is $\text{ZFC}_{\omega, \text{NS}_{\omega_1}}$-equivalent to the universal $\tau_{ST} \cup \{\omega_1\}$-sentence:

$$[(\omega_1 \text{ is an ordinal } \land \omega \in \omega_1) \land \forall f [(f \text{ is a function } \land \text{dom}(f) \in \omega_1) \to \text{ran}(f) \neq \omega_1].$$

Our focus will be to understand which concepts are expressible by universal sentences and which are expressible by $\Pi_2$-sentences in the appropriate signatures.

One of the basic intuition leading to the above theorems is that the axiomatization of set theory in the signature $\{\in\}$ make unnecessarily complicated the formalization of many basic set theoretic properties; however if one adds the “right” predicates and constant symbols to denote certain basic properties (i.e. the $\Delta_0$-properties) and certain more complicated ones of which we have however a clear grasp (i.e. the projective sets and the non-stationary ideal), the logical complexity of set-theoretic concepts lines up with our understanding of them. Once this operation is performed, the two theorems above show (assuming large cardinals) that for $\Pi_2$-properties consistency with the universal fragment of $T$ overlaps with forcibility over models of $T$ and with provability with respect to the right extension of $T$.

2. Theorems 1 and 2 are special instantiation of a method which pairs the notion of model companionship with generic absoluteness results. Roughly the equivalence between (1) and (3) of the two theorems follow from the existence of a model companion for $T$ in the appropriate signature, while the equivalence of (2) and (3) follows from generic absoluteness results.

3. The reader may wonder why Thm. 1 does not conflict with Gödel's incompleteness theorem. Let $T_0$ be the theory

$$\text{ZFC}_\omega + \text{ there are class many Woodin cardinals}.$$  

The Gödel $\in$-sentence $\text{Con}(T_0)$ and its negation become atomic $\tau_{ST}$-sentences (since all their quantifiers range over $\omega$), hence a part of the universal (or of the $\Pi_2$) theory of any complete extension of $T_0$. However there are complete extensions of $T_0$ containing $\text{Con}(T_0)$ and others containing its negation, therefore the equivalences set forth in Thm. 1 are not violated letting the $\psi$ of the theorem be $\text{Con}(T_0)$. 


Note on the other hand that the content of Thm. 1 is that “almost” any question of second order arithmetic (see the next item) is decided by large cardinal axioms: apart from Gödel sentences, it is clearly open whether there are more interesting arithmetic (or even projective) statements (such as Goldblatt’s conjecture or Schanuel’s conjecture or Riemann’s hypothesis) which are independent of these axioms. Similar considerations apply to Theorem 2.

(4) Every lightface projective set (i.e. any definable subset without parameters of the structure (P(N), ∈, ⊆)) is the extension of a quantifier free formula in σω. Letting φn(x, y) by a τST-formula defining a universal set for Σn+1 sets, it is not hard to see that projective determinacy (according to the notation of [10, Section 20.A]) is given by an axiom scheme of τST-sentences in which quantifiers range just over subsets of P(ω<ω). In particular projective determinacy is expressed by a family of atomic sentences for σω in ZFC∗.

(5) The negation of the Continuum hypothesis CH is expressible in the signature τST∪ {ω1} ⊆ σω,NSω1 as the Π2-sentence ψ−CH:

(ω1 is the first uncountable cardinal)∧
∧∀f[(f is a function ∧ dom(f) = ω1) → ∃r(r ⊆ ω ∧ r ∉ ran(f))].

Most of third order number theory is expressible in this signature by a Π2-sentence, for example this is the case for Suslin’s hypothesis, every Aronszjain tree is special, and a variety of other statements.

(6) It is out of the scopes of the present paper to define (⋆)-UB; it will be essentially used only in the sequel of this work; 3 of Thm. 2 is the unique place of this paper where this statement will ever be mentioned. For the convenience of the interested reader we include its definition in Section 5. Let us just briefly say that (⋆)-UB is the strong form of Woodin’s axiom (⋆) asserting that NSω1 is saturated together with the existence of an L(UB)-generic filter for Woodin’s Pmax-forcing7 (where L(UB) is the smallest transitive model of ZF containing all the universally Baire sets).

Our ambition is to make the remainder of this paper self-contained and accessible to any person who has a fair acquaintance with set theory and first order logic. From now on, with large cardinal axioms, forcing axioms is needed or assumed on the reader, all it is required is just to accept as meaningful the statement of these theorems.

The following piece of notation will be used.

Notation 2.

• τST is the extension of the first order signature {∈} for set theory which is obtained by adjoining predicate symbols Rφ of arity n for any ∆0-formula φ(x1,...,xn), function symbols of arity k for any ∆0-formula θ(y,x1,...,xk) and constant symbols for ω and ∅.

• σST is the signature containing a predicate symbol Sφ of arity n for any τST-formula φ with n-many free variables.

• σκ = σST∪ τST∪ {κ} with κ a constant symbol.

• ZFC− is the -theory given by the axioms of ZFC minus the power-set axiom.

• TST is the τST-theory given by the axioms

7See [12] for details on Pmax.
(∀x∃y \mathcal{R}_φ(y, x)) → (\forall x \mathcal{R}_φ(f(x), x))

for all $\Delta_0$-formulae $\phi(x)$, together with the $\Delta_0$-sentences

$$\forall x \in 0 \neg (x = x),$$

$\omega$ is the first infinite ordinal

(the former is an atomic $\tau_{ST}$-sentence, the latter is expressible as the $\Pi_1$-sentence for $\tau_{ST}$ stating that $\omega$ is a non-empty limit ordinal contained in any other non-empty limit ordinal).

- $T_\kappa$ is the $\sigma_{ST} \cup \{\kappa\}$-theory given by the axioms

\[
\forall x_1 \ldots x_n [\mathcal{S}_\psi(x_1, \ldots, x_n) \leftrightarrow (\bigwedge_{i=1}^n x_i \subseteq \kappa^{<\omega} \land \psi^{\mathcal{P}(\kappa^{<\omega})}(x_1, \ldots, x_n))]
\]  

as $\psi$ ranges over the $\in$-formulae.

- $ZFC_{ST}^-$ is the $\tau_{ST}$-theory

$$ZFC^- \cup T_{ST}$$

- $ZFC^-_\kappa$ is the $\tau_{ST} \cup \{\kappa\}$-theory

$$ZFC^-_{ST} \cup \{\kappa \text{ is an infinite cardinal}\};$$

- $ZFC^*_\kappa$ is the $\sigma_\kappa$-theory

$$ZFC^-_\kappa \cup T_\kappa;$$

- $ZFC^*_\omega$ is

$$ZFC^*_\kappa \cup \{\kappa \text{ is the first infinite cardinal}\};$$

- Accordingly we define $ZFC_{ST}^*, ZFC^-_\kappa, ZFC^*_\kappa, ZFC^*_\kappa, ZFC^*_{ST}.$

\textbf{Fact 1.} Every $\sigma_\kappa$-formula is $T_\kappa \cup T_{ST}$-equivalent to an $\{\in, \kappa\}$-formula.

Moreover assume $\kappa$ is a definable cardinal (i.e. $\kappa = \omega$ or $\kappa = \omega_1$); more precisely assume there is an $\in$-formula $\psi_\kappa(x)$ such that

$$ZFC^- \vdash \exists x [\psi_\kappa(x) \land (x \text{ is a cardinal})].$$

Then every $\sigma_\kappa$-formula is $ZFC^*_\kappa + \psi_\kappa(\kappa)$-equivalent to an $\in$-formula.

\textbf{Proof.} The axioms of $T_{ST}$ and $T_\kappa$ are cooked up exactly so that one can prove the result by a straightforward induction on the $\sigma_\kappa$-formulae (see also the proof of Prop. 3.21). □

Theorem 1 is an immediate corollary of Woodin’s generic results for second order number theory (cfr. [18]) coupled with the following theorem:

\textbf{Theorem 3.} Assume $T \supseteq ZFC^*_\kappa$ is a $\sigma_\kappa$-theory. Then $T$ has a model companion $T^*$. Moreover for any $\Pi_2$-sentence $\psi$ for $\sigma_\kappa$, TFAE:

1. $\psi \in T^*$;
1. $T \vdash \psi^{H_{\kappa^+}}$;
3. For all universal $\sigma_\kappa$-sentences $\theta$, $T_\psi + \theta$ is consistent if and only if so is $T_\psi + \theta + \psi$.

We note that approximations to Thm. 3 for the case $\kappa = \omega$, and to Thm. 1 already appears in [15].

The present paper give a self-contained proof of Theorems 1 and 3. We defer to a second paper the proof of Theorem 2 (which reposes on the recent breakthrough by Asperò and Schindler establishing that Woodin’s axiom $(*)$ follows from $MM^{++}$ [1]); here we will prove a weaker version of it (cfr. Thm. 4) at the end of Section 2.

We prove rightaway Thm. 1 assuming Thm. 3:

\textbf{Proof.} Woodin’s generic absoluteness results for second order number theory give that 1(3) and 1(2) are equivalent (we give here a self-contained proof of this particular instance of Woodin’s results in Theorem 4.7). Theorem 3 gives the equivalence of 1(1) and 1(3). □
The proof and statement of Thm. 3 require familiarity with set theory at the level of an undergraduate book (for example [8] coupled with [11, Chapters III, IV] is far more than sufficient) as well as familiarity with the notion of model companionship.

To complete this introductory section it is convenient to sort out how the definable extensions $\text{ZFC}_T^*$, $\text{ZFC}_N$, $\text{ZFC}_N^*$, $\text{NS}_1^{-}$, $\text{NS}_2^{-}$ behave with respect to forcing. A central role is played by large cardinal axioms. The reader can safely skip this remark without compromising the reading of the sequel of this paper.

Remark 2. We outline here the invariance under forcing of the $\Pi_1$-theory of $V$ in certain natural signatures; since the universal fragment of a theory $T$ determines completely its model companion, the fact that in certain signatures $\tau$ forcing cannot change the $\Pi_1$-theory of $V$ (in combination with Levy’s absoluteness theorem) is the key to understand why set theory can have a model companion in some of these signatures, and why the properties of the model companion theory are paired with generic absoluteness results.

- The standard absoluteness results of Kunen’s book [11, Ch. IV] show that if $G$ is $V$ generic for some forcing notion $P \in V$, $V \subseteq V[G]$ for $\tau_{ST}$.
- Shoenfield’s absoluteness Lemma entails that if $G$ is $V$ generic for some forcing notion $P \in V$, $V \prec V[G]$ for $\tau_{ST}$.

This holds since $H_{\omega_1} \prec V$ and $H_{\omega_1}[G] \prec V[G]$ (cfr. Lemma 4.1), and $H_{\omega_1} \prec H_{\omega_1}[G]$ (see for example [17, Lemma 1.2]) for the signature $\tau_{ST}$.

- Major results of the Cabal seminar bring that assuming the existence of class many Woodin cardinals in $V$, if $G$ is $V$ generic for some forcing notion $P \in V$, $V \subseteq V[G]$ for $\sigma_\omega$ (roughly because $H_{\omega_1}^V \prec H_{\omega_1}[G]$ by Thm. 4.7, while $H_{\omega_1}^V \prec V$ and $H_{\omega_1}[G] \prec V[G]$ by Lemma 4.1) for the signature $\sigma_\omega$. More generally the same large cardinal assumptions and argument yield that $V \subseteq V[G]$ also for the signature extending $\tau_{ST} \cup \text{UB}$ with predicate symbols for all universally Baire sets of $V$ (instead of considering just the lightface projective sets as done by $\sigma_\omega$).

- Assume $G$ is $V$ generic for some forcing notion $P \in V$, $V \subseteq V[G]$ for $\tau_{ST} \cup \{\omega_1, \text{NS}_{\omega_1}\}$ if and only if $P$ is stationary set preserving: for the atomic predicates $\text{NS}_{\omega_1}$ the formula $\neg \text{NS}_{\omega_1}(S)$ is preserved between $V$ and $V[G]$ for all $S \subseteq \omega_1$ in $V$ only in this case. The sentence $\omega_1$ is the first uncountable cardinal is preserved only if $P$ does not collapse $\omega_1$.

- Assuming the existence of class many Woodin cardinals in $V$ for any forcing $P \in V$ (i.e. also if $P$ is not stationary set preserving or collapses $\omega_1$), for any $G$ $V$-generic for $P$, $V[G]$ and $V$ satisfy the same $\Pi_1$-sentences for $\sigma_\omega, \text{NS}_{\omega_1}$ (Thm. 6).

- On the other hand the signature $\sigma_\kappa$ with $\kappa \geq \omega_1$ behaves badly with respect to forcing; one has to put severe limitation on the type of forcings $P$ considered in order to maintain that $V \subseteq V[G]$ or just that $V$ and $V[G]$ satisfy the same universal $\sigma_\kappa$-sentences (see Remark 2.6 to appreciate the difficulties). However we will prove an interesting variation of Thm. 2 for $\sigma_\kappa$ in case $\kappa$ is interpreted by $\omega_1$ (cfr. Thm. 4).

These results combined together give the following argument for the proof of (2) implies (1) of Thm. 2 (mutatis mutandis for the proof of (2) implies (1) of Thm. 1): let $\psi$ be a $\Pi_2$-sentence for $\sigma_\omega, \text{NS}_{\omega_1}$ satisfying (2). Given some $\Pi_1$-sentence $\theta$ for $\sigma_\omega, \text{NS}_{\omega_1}$ consistent with $T$, find $M$ model of $T + \theta$. By (2) some forcing $P \in M$ forces $\psi^H_{\omega_2}$. By Thm. 6 and Levy’s absoluteness Lemma 4.1, the theory $T_V + \theta + \psi$ holds in $H_{\omega_2}^V$ whenever $N$ is a generic extension of $M$ by $P$.

The paper is organized as follows:
Section 2 proves Thm. 3 (WARNING: familiarity with the notion of model companionship is required). We also include in its last part a proof of a weaker variation of Thm. 2 (cfr. Thm. 4).

Section 3 gives a detailed account of model completeness and model companionship.

Section 4 gives a self-contained proof of the form of Levy absoluteness and of the particular form of Woodin’s generic absoluteness results we employ in this paper. The reader can safely skip it without compromising the comprehension of the remainder of the paper (WARNING: familiarity with the notion of model companionship is required).

Section 5 collects the main results we will prove in a sequel of this paper.

The paper contains (overly?) detailed proofs of every non-trivial result (many of which can be also found elsewhere i.e. most—if not all— of those appearing in sections 3 and 4), this has been made at the expenses of its brevity. Our hope is that this approach makes the paper accessible to all scholars with a basic knowledge of set theory and model theory.

The reader unfamiliar with the notion of model companionship and its main implications should start with Section 3, rather than with Sections 2 or 1.

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The opportunity to present preliminary versions of these results in the set theory seminar of the Équipe has also given me the possibility to improve them substantially. I thank all the people attending it for their many useful comments, in particular Alessandro Vignati.

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1. Some comments

Correct signatures for set theory. A first basic idea is that bounded formulae express “simple” properties of sets. The Levy stratification of set-theoretic properties consider those expressed by bounded formulae the simplest; then the complexity increases as unbounded quantifiers lines up in the prenex normal form of a formula. In particular the Levy stratification matches exactly with the stratification of $\tau_{\text{ST}}$-formulae according to the number of alternating quantifiers in their $\text{ZFC}_{\text{ST}}$-equivalent prenex form.

Assume instead we measure the complexity of a set theoretic property $P$ according to the number of alternating quantifiers of the prenex normal form of its $\in$-formalization. Then many basic properties already have high complexity: the formula $z = \{x, y\}$ is expressed by a $\Pi_1$ formula for $\in$; the $\in$-formula expressing $f$ is a function by means of Kuratowski pairs to define relations has already so many quantifiers that one cannot estimate their numbers at first glance, etc. If we resort to the axiomatization of set theory given by $\text{ZFC}_{\text{ST}}$, this

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8Our ambition is that this section could serve as a compact self-contained account of the key properties of model companion theories.

9We included these results here, because the versions of these results we found in the literature were not exactly fitting to our set up. Again our purpose for this section is to simplify the reader’s task, as well to give minor improvements of known results.
problem is overruled, and these two properties are expressed by atomic \(\tau_{ST}\)-formulae\(^{10}\). In particular reformulating \(\text{ZFC}\) using the signature \(\tau_{ST}\) recalibrates the complexity of formulae letting arbitrarily complex \(\in\)-formulae become atomic, while not changing the set of \(\text{ZFC}\)-provable theorems, and stratifies set theoretic properties in complete accordance with the Levy hierarchy\(^{11}\).

**Levy absoluteness and model companionship.** Mostowski collapsing theorem and the axiom of choice allow to code a set by a well founded relation on its hereditary cardinality, and in this way translate in an “absolute manner” questions about sets in \(H_{\kappa^+}\) to questions about \(\mathcal{P}(\kappa)\). The content of Theorem 3 is that we can give a very precise model-theoretic meaning to the term “absolute manner”: any \(\sigma\)-formula is \(\text{ZFC}_\kappa^- + \{\text{all sets have size } \kappa\}\)-equivalent to a universal \(\sigma\)-formula and to an existential \(\sigma\)-formula, i.e. it is a provably \(\Delta_1\)-property in this theory. What happens is that we encode complicated questions about the power-set of \(\kappa\) by means of atomic predicates, since the axioms listed in 1 amount to a method to eliminate quantifiers ranging over \(\mathcal{P}(\kappa)\). So Theorem 3 is another way to reformulate that the first order theory of \(H_{\kappa^+}\) reduces to the first order theory of \(\mathcal{P}(\kappa)\) in an absolute manner.

Remark also that for all models \((V, \in)\) of \(\text{ZFC}\) and all cardinals \(\kappa \in V\) and all signatures

\[
\tau_\kappa \subseteq \tau \subseteq \tau_\kappa \cup \mathcal{P}(\mathcal{P}(\kappa))
\]

\((H_{\kappa^+}^V, \tau^V) \prec_1 (V, \tau^V)\) is the unique transitive substructure of \(V\) containing \(\mathcal{P}(\kappa)\) which models \(\text{ZFC}_\kappa^-\) and the \(\Pi_2\)-sentence for \(\tau_\kappa\)

\[
\forall X \exists f (f : \kappa \to X \text{ is surjective}).
\]

In particular if a model companion of the \(\tau\)-theory of \(V\) exists, this can only be the \(\tau\)-theory of \(H_{\kappa^+}\).

**Generic invariance of the \(\Pi_1\)-theory of \(V\) in a given signature.** We say that a signature \(\sigma\) is generically tame for a \(\sigma\)-theory \(T\) extending \(\text{ZFC}_\kappa\) if the \(\Pi_1\)-consequences of \(T\) must be preserved through forcing extensions of models of \(T\) (which brings the implication \((2) \to (1)\) of Theorem 1 — as well as the corresponding implications of Theorems 2, 5, 7 — by the argument sketched in Remark 2).

Theorem 6 shows that this generic invariance holds for all\(^{12}\) \(\sigma \subseteq \tau_{\text{NS}_{\omega_1}} \cup \text{UB}^V\) where \(\text{UB}^V\) denotes the family of universally Baire sets of some \((V, \in)\) which models \(\text{ZFC}+\text{large cardinals}\).

Theorem 6 is close to optimal: a (for me surprising) fact remarked by Boban Veličkovič is that Thm. 6 cannot possibly hold for any \(\sigma \supseteq \tau_{\Sigma^1_0} \cup \{\omega_1, \omega_2\}\), where \(\omega_2\) is a constant which names the second uncountable cardinal:

\[
\square_{\omega_2} \text{ is a } \Sigma_1\text{-statement for } \tau_{\omega_2} = \tau_{\Sigma^1_0} \cup \{\omega_1\} \cup \{\omega_2\}:
\]

\[
\exists \{C_\alpha : \alpha < \omega_2\}:
\]

\[
\forall \alpha \in \omega_2 (C_\alpha \text{ is a club subset of } \alpha) \land
\]

\[
\land \forall \alpha \in \beta \in \omega_2 (\alpha \in \text{lim}(C_\beta) \to C_\alpha = C_\beta \cap \alpha) \land
\]

\[
\land \forall \alpha (\text{otp}(C_\alpha) \leq \omega_1)
\].

\(^{10}\)There are atomic \(\tau_{ST}\)-formulae whose \(\text{ZFC}_{ST}\)-equivalent prenex \(\in\)-formula of least complexity has an arbitrarily large number of alternating quantifiers.

\(^{11}\)Nonetheless there are \(\in\)-sentences whose least complexity \(\text{ZFC}_{ST}\)-equivalent \(\tau_{ST}\)-sentence in prenex normal form has an arbitrary finite number of alternating quantifiers, examples are given by lightface definable universal sets for \(\Sigma^1_1\)-properties (cfr. [15, Thm. 4.6]).

\(^{12}\)See Notation 3 for the definition of \(\tau_{\text{NS}_{\omega_1}}\).
\( \square_{\omega_2} \) is forcible by very nice forcings (countably directed and \(< \omega_2\)-strategically closed), and its negation is forcible by Coll(\(\omega_1, < \delta\)) whenever \(\delta\) is supercompact.

In particular the \(\Pi_1\)-theory for \(\tau_{\omega_2}\) of any forcing extension \(V[G]\) of \(V\) can be destroyed in a further forcing extension \(V[G][H]\), hence is not invariant across forcing extensions of \(V\) in any possible sense, assuming large cardinals in \(V\).

Theorems 1, 2, 5, 7 show that the strong form of consistency given by \((1)\) of Theorem 1 can characterize forcibility (at least for \(\Pi_2\)-sentences in the appropriate signature), if large cardinals enter the picture.

**Model companionship and generic absoluteness.** The first order theory of \(P(\kappa)\) for any infinite \(\kappa\) is very sensitive to forcing; but this depends on two parameters: whether or not we assume large cardinals, and what is the signature in which we look at the first order theory of \(P(\kappa)\).

Theorem 6 shows that we can “tune” the signature \(\sigma\) so that for any \(\sigma\)-theory \(T\) extending ZFC+large cardinals:

- the signature is expressive (i.e. many questions of second or third order arithmetic can be encoded by simple sentences, i.e. \(\Pi_2\)-sentences for \(\sigma\));
- the signature is not too expressive (i.e. the questions of second or third order arithmetic whose truth value can be changed by means of forcing cannot be encoded by \(\Pi_1\)-sentences for \(\tau\); in particular the \(\Pi_1\)-fragment of ZFC in the new signature is invariant across the generic multiverse, cfr. Thm. 6).

These two conditions entail that \((1)(2)\) implies \((1)(1)\) (respectively \((2)(2)\) implies \((2)(1)\)). Generic absoluteness results give that \((1)(2)\) is equivalent to \((1)(3)\) (respectively \((2)(2)\) is equivalent \((2)(3)\)).

Model completeness of the relevant theories gives the missing implication from \((1)\) to \((3)\) of Theorems 1, 2, 5, 7.

**Model companionship and generic absoluteness for second order number theory.** The standard argument used in set theory to assert that \(\Delta_0\)-properties are simple, is their invariance between transitive models, which in turns imply that their truth values cannot be changed by means of forcing.

Now consider second order number theory i.e.: the theory of the structure \((P(\omega), \in)\); modulo the by-interpretation which identifies a hereditarily countable set with the graph of the transitive closure of its singleton (see Section 2), the theory of \((P(\omega), \in)\) has the same set of theorems as the first order theory of the structure \((H_{\omega_1}, \tau_{ST})\), which in turns (by Fact 1) has the same set of theorems as the structure \((H_{\omega_1}, \sigma_\omega)\). The first order theory of \(H_{\omega_1}\) in any of these signatures can vary (by means of forcing) if one denies the existence of large cardinals (for example there can be lightface definable projective well-orders, or not): on the other hand a major result of Woodin is that assuming large cardinals, the first order theory of \((H_{\omega_1}, \in)\) is invariant with respect to forcing. The equivalence of \((2)\) and \((1)\) in Theorem 1 says that this theory is fixed by any reasonable method to produce its models, not just forcing.

Now we combine these results with the clear picture given by projective determinacy of the theory of projective sets: much in the same way we accept bounded formulae as “simple” predicates and make them equivalent to atomic formulae by means of ZFC_{ST}, if we accept as true large cardinal axioms, we are forced to consider projective sets of reals as “simple” predicates; ZFC_{\omega} includes them among the atomic predicates. Once we do so the first order theory of \(H_{\omega_1}\) is “tame” i.e. model complete, hence it realizes all \(\Pi_2\)-sentences which are consistent with its universal fragment (cfr. Fact 3.11); moreover large cardinals make provably true many of these \(\Pi_2\)-sentences, for example projective determinacy.
Model companionship and generic absoluteness for the theory of $\mathcal{P}(\omega_1)$. Theorem 2 extends the above considerations to the signature $\sigma_{\omega,\text{NS}_{\omega_1}}$. In this case a theory $T$ extending $\text{ZFC} + \text{large cardinals}$ is just able to say that:

- A $\Pi_2$-sentence for $\sigma_{\omega,\text{NS}_{\omega_1}}$ is consistent with the universal fragment of $T$ if it is $T$-provably forcible (cfr. 2 implies 1 of Thm. 2, see Remark 2 for a proof).
- The theory $T^*$ given by all $\Pi_2$-sentences $\psi$ for $\sigma_{\omega,\text{NS}_{\omega_1}}$ such that $\psi^{H_{\omega_2}}$ is provably forcible is consistent (cfr. 2 implies 3 of Thm. 2, one of the main results of Woodin on $\mathbb{P}_{\max}$ [12, Thm. 7.3]).
- Recently Asperò and Schindler proved that $\text{MM}^{++}$ implies ($\ast$)-$\text{UB}$ [1]. An immediate corollary of their result is that $T^*$ holds in the $H_{\omega_2}$ of models of $\text{MM}^{++}$. This result allow to prove the missing implications in Thm. 2 (or in Thm. 5, 7).

We will prove the assertions in the above items using Thm. 2, 5, 7 and Asperò and Schindler’s result in a sequel of this paper.

Model completeness and bounded forcing axioms. Let us now spend some more words relating model completeness to bounded forcing axioms and ($\ast$)-$\text{UB}$. Model companionship and model completeness capture in a model theoretic property the notion of “generic” structure for the models of a theory; this notion is recurrent in various domains (not only restricted to model theory), we mention two occurring in model theory: in many cases the Fraïssé limit of a given family $\mathcal{F}$ of finite(ly generated) structures for a signature $\tau$ is generic for the structures in $\mathcal{F}$; the algebraically closed field are generic with respect to the class of fields. Generic structures of a universal theory $T$ realize as many $\Pi_2$-properties as it is consistently possible while remaining a model of $T$. The standard examples of generic structures for a first order theory $T$ are given by $T$-existentially closed model, i.e. models which are $\Sigma_1$-substructures of any superstructure which realizes (the universal fragment of) $T$. We will make this rigorous in Section 3.

Compare these observations with the formulation of bounded forcing axioms as principles of generic absoluteness (as done by Bagaria in [5]) stating that $H_{\omega_2}^{\Sigma_1}$ is a $\Sigma_1$-substructure of any generic extension of $V$ obtained by forcings in the appropriate class.

In essence Theorem 2 and Thm. 5 outline that forcing axioms provide means to produce models of $H_{\omega_2}$ which are existentially closed for their universal theory and realize as many $\Pi_2$-sentences as the iteration theorems producing them makes possible.

Why $\text{CH}$ is false. Summing up on the above considerations, we believe we can give a strong argument against $\text{CH}$:

Assume we adopt the stance that:

- Large cardinal axioms are true.
- We consider set theory as formalized by a definable extension $T$ of $\text{ZFC} + \text{large cardinals}$ in a signature $\sigma$ where $\text{CH}$ can be corrected formalized, i.e. $T$ is a definable extension of $\text{ZFC}_{\omega_1}$ in the signature $\sigma \supseteq \tau_{ST} \cup \{\omega_1\}$ including a constant symbol for the first uncountable cardinal, so that:
  - $\neg \text{CH}$ is formalized by a $\Pi_2$-sentence for $\tau_{ST} \cup \{\omega_1\}$, (cfr. Remark 1(5)).
  - The $\Pi_1$-fragment of $T$ is invariant across forcing extensions (so that the basic facts about $\mathcal{P}(\omega_1)$ — i.e those expressible by $\Pi_1$-sentences for $\sigma$ — are not changed by means of forcing (cfr. (2) implies (1) of Thm. 2 holds for $T$).

Furthermore to select which among all possible $T$ in the signature $\sigma$ gives the true “axiomatization” of set theory, we adopt the following criteria:

- $T$ should maximize the family of $\Pi_2$-sentences for $\sigma$ which are consistent with its $\Pi_1$-consequences (cfr. Thm. 2(1));
• there should be a simple and manageable axiom system for $T$ (cfr. Thm. 2(3) or even $T$ has a model companion $T^*$).

With these premises, we conclude that Theorem 2 (also with Thm. 5, 7) implies that CH is false (since $\neg$CH is provably forcible from $T$).

We can further reinforce our case by remarking that:

• The same assumptions on $T$ and $T^*$ entail $2^{\aleph_0} = \aleph_2$ holds in any $\tau$-model $N$ of $T_\forall + \text{ZFC}$ in which $T^*$ holds in $H^{\aleph_0}_{\aleph_2}$; $2^{\aleph_0} = \aleph_2$ is not a $\Pi_2$-sentence for $\tau_{\omega_1}$, but it is a consequence of $\Pi_2$-sentences for $\tau_{\aleph_1} \cup \\{\omega_1\}$ which hold assuming BPFA. One such sentence is given by Caicedo and Velickovic in [6]:

$$\forall C \text{ ladder system on } \omega_1 \forall r \subseteq \omega \exists \alpha \exists f \left[ (f : \omega_1 \to \alpha \text{ is surjective}) \land \psi(C, r, \alpha) \right]$$

where $\psi(x, y, z)$ is a $\Sigma_1$-formula for $\tau_{\aleph_1} \cup \\{\omega_1\}$ which can be used to define for each ladder system $C$ an injective map $P(\omega) \to \omega_2$ with assignment $r \mapsto \alpha$ of the real $r$ to the ordinal $\alpha$ least such that $\psi(C, r, \alpha)$.

• The signature $\tau_{\text{NS}_{\omega_1}} \cup \text{UB}$ makes the $\Pi_1$-theory of $V$ invariant across the generic multiverse (cfr. Thm. 6); hence we can use forcing to detect which $\Pi_2$-sentences should belong to the model companion of set theory in any signature $\tau \subseteq \tau_{\text{NS}_{\omega_1}} \cup \text{UB}$ (if such a model companion exists); this is exactly the argument we used to argue for $\neg$CH.

• ($\ast$)-UB with a weak form of sharp for universally Baire sets can be equivalently formulated as the assertion that the $\tau_{\text{NS}_{\omega_1}} \cup \text{UB}$-theory of $V$ has as model companion the $\tau_{\text{NS}_{\omega_1}} \cup \text{UB}$-theory of $H_{\omega_2}$ (cfr. Thm. 5). This brings to light the complete accordance between the philosophy driving $P_{\text{max}}$ and bounded forcing axioms (that of maximizing the $\Pi_2$-sentences true in $H_{\omega_2}$) with the notion of model companionship.

Model companionship in set theory. Model companionship is a tameness notion which must be handled with care (see Section 3.5). We believe that the present paper presents a reasonable test to gauge the tameness of this notion: in set theory we are focusing mostly in two types of structures: generic extensions $V[G]$ of the universe of sets $V$ produced by (certain types of) forcings $P$, and the theory of $H^{|V[G]}$ of these generic extensions for suitably chosen (and definable) cardinals $\lambda$. We often study these structures working in signatures $\tau$ maintaining that $V \subseteq V[G]$ and $H^{|V[G]} \preceq_1 V[G]$ also for $\tau$ (in particular here and in a huge number of works one consider the case of $\tau$ being $\tau_{\text{NS}_{\omega_1}}$, $P$ being a stationary set preserving forcing, $\lambda$ being $\omega_2$, or the case $\tau$ being $\sigma_{\omega_1}$, $P$ being any forcing, $\lambda$ being $\omega_1$). The results of the present paper (and of its sequel) show that the axiomatization of set theory + large cardinals in these signatures is well behaved: first of all the models of its $\Pi_1$-fragment include all the structures of interest, i.e. all generic extensions of $V$ (eventually obtained by forcing of a certain kind), and all the initial segments of these generic extensions containing a large enough chunk of the universe. Moreover this theory admits a model companion and this model companion is uniquely determined by the family of $\Pi_2$-sentences which we can provably force to hold in the appropriate $H^{|V[G]}$ (with $\lambda = \omega_1$ or $\lambda = \omega_2$ decided by the signature). It has also to be noted that even the substructure relation is not that much affected by forcing; for example any $G$ $V$-generic for a stationary set preserving forcing $P$ maintains that $V \subseteq V[G]$ also for the signature $\sigma_{\omega_1, \text{NS}_{\omega_1}}$. (May be surprisingly) Thm. 6 shows that if $P$ is not stationary set preserving $V \subseteq V[G]$ fails for $\sigma_{\omega_1, \text{NS}_{\omega_1}}$, nonetheless $V$ and $V[G]$ will satisfy exactly the same $\Pi_1$-sentences for $\sigma_{\omega_1, \text{NS}_{\omega_1}}$.

It is in our eyes surprising the perfect matching existing between generic absoluteness results and the notion of model companionship which the present paper reveals.
2. The theory of $H_{\kappa^+}$ is the model companion of set theory

**Notation 2.1.** Given an $\in$-structure $(M, E)$ and $\tau$ a signature among $\tau^M, \sigma_\kappa, \ldots$, from now we let $(M, \tau^M)$ be the unique extension of $(M, E)$ defined in accordance with Notation 2 and Fact 1. In particular $(M, \tau^M)$ is a shorthand for $(M, S^M : S \in \tau)$. If $(N, E)$ is a substructure of $(M, E)$ we also write $(N, \tau^M)$ as a shorthand for $(N, S^M \upharpoonright N : S \in \tau)$.

**2.1. By-interpretability of the first order theory of $H_{\kappa^+}$ with the first order theory of $\mathcal{P}(\kappa)$.** Let’s compare the first order theory of the structure

$$(\mathcal{P}(\kappa), S^\mathcal{P}_\kappa : \phi \in \tau_{ST}^\mathcal{P})$$

with that of the $\tau_{ST}$-theory of $H_{\kappa^+}$ in models of ZFC$_{ST}$. We will show that they are ZFC$_{ST}$-provably by-interpretable with a by-interpretation translating $H_{\kappa^+}$ in a $\Pi_1$-definable subset of $\mathcal{P}(\kappa^2)$ and atomic predicates into $\Sigma_1$-relations over this set. This result is the key to the proof of Thm. 3 and it is just outlining the model theoretic consequences of the well-known fact that sets can be coded by well-founded extensional graphs.

**Definition 2.2.** Given $a \in H_{\kappa^+}$, $R \in \mathcal{P}(\kappa^2)$ codes $a$, if $R$ codes a well-founded extensional relation on some $\alpha \leq \kappa$ with top element 0 so that the transitive collapse mapping of $(\alpha, R)$ maps 0 to $a$.

- $\text{WFE}_\kappa$ is the set of $R \in \mathcal{P}(\kappa)$ which are a well founded extensional relation with domain $\alpha \leq \kappa$ and top element 0.
- $\text{Cod}_\kappa : \text{WFE}_\kappa \to H_{\kappa^+}$ is the map assigning $a$ to $R$ if and only if $R$ codes $a$.

The following theorem shows that the structure $(H_{\kappa^+}, \in)$ is interpreted by means of “imaginaries” in the structure $(\mathcal{P}(\kappa), \tau_{ST}^\mathcal{P})$ by means of:

- a universal $\tau_{ST} \cup \{\kappa\}$-formula (with quantifiers ranging over subsets of $\kappa^{<\omega}$) defining a set $\text{WFE}_\kappa \subseteq \mathcal{P}(\kappa^2)$.
- an equivalence relation $\equiv_\kappa$ on $\text{WFE}_\kappa$ defined by an existential $\tau_{ST} \cup \{\kappa\}$-formula (with quantifiers ranging over subsets of $\kappa^{<\omega}$)
- A binary relation $E_\kappa$ on $\text{WFE}_\kappa$ invariant under $\equiv_\kappa$ representing the $\in$-relation as the extension of an existential $\tau_{ST} \cup \{\kappa\}$-formula (with quantifiers ranging over subsets of $\kappa^{<\omega}$).

**Theorem 2.3.** Assume ZFC$_\kappa^-$. The following holds\(^{14}\):

1. The map $\text{Cod}_\kappa$ and $\text{WFE}_\kappa$ are defined by ZFC$_\kappa^-$-provably $\Delta_1$-properties in parameter $\kappa$. Moreover $\text{Cod}_\kappa : \text{WFE}_\kappa \to H_{\kappa^+}$ is surjective (provably in ZFC$_\kappa^-$), and $\text{WFE}_\kappa$ is defined by a universal $\tau_{ST} \cup \{\kappa\}$-formula with quantifiers ranging over subsets of $\kappa^{<\omega}$.

2. There are existential $\tau_{ST} \cup \{\kappa\}$-formulae (with quantifiers ranging over subsets of $\kappa^{<\omega}$), $\phi_\in, \phi_\equiv$ such that for all $R, S \in \text{WFE}_\kappa$, $\phi_\in(S, R)$ if and only if $\text{Cod}_\kappa(R) = \text{Cod}_\kappa(S)$ and $\phi_\equiv(S, R)$ if and only if $\text{Cod}_\kappa(R) \in \text{Cod}_\kappa(S)$. In particular letting

$$E_\kappa = \{ (R, S) \in \text{WFE}_\kappa : \phi_\in(S, R) \} ,$$

$$\equiv_\kappa = \{ (R, S) \in \text{WFE}_\kappa : \phi_\equiv(R, S) \} ,$$

$\equiv_\kappa$ is a ZFC$_\kappa^-$-provably definable equivalence relation, $E_\kappa$ respects it, and

\[^{14}\text{See [9, Section 25] for proofs of the case } \kappa = \omega; \text{ in particular the statement and proof of Lemma 25.25 and the proof of [9, Thm. 13.28] contain all ideas on which one can elaborate to draw the conclusions of Thm. 2.3.}

\[^{15}\text{Many transitive supersets of } H_{\kappa^+} \text{ are } \tau_{ST} \cup \{\kappa\}-\text{model of ZFC$_\kappa^-$ for } \kappa \text{ an infinite cardinal (see [11, Section IV.6]). To simplify notation we assume to have fixed a transitive } \tau_{ST} \cup \{\kappa\}-\text{model } N \text{ of ZFC$_\kappa^-$ with domain } N \supseteq H_{\kappa^+}. \text{ The reader can easily realize that all these statements holds for an arbitrary model } N \text{ of ZFC$_\kappa^-$ replacing } H_{\kappa^+} \text{ with its version according to } N.}
is isomorphic to \((H_{\kappa^+}, \in)\) via the map \([R] \mapsto \text{Cod}_\kappa(R)\).

Proof. A detailed proof requires a careful examination of the syntactic properties of \(\Delta_0\)-formulae, in line with the one carried in Kunen’s [11, Chapter IV]. We outline the main ideas, following Kunen’s book terminology for certain set theoretic operations on sets, functions and relations (such as \(\text{dom}(f), \text{ran}(f), \text{Ext}(R), \text{etc.}\)). To simplify the notation, we prove the results for a transitive model \((N, \in)\) which is then extended to a structure \((N, \tau^N_\kappa, \kappa^N)\) which models ZFC\(_\kappa^+\), and whose domain contains \(H_{\kappa^+}\). The reader can verify by itself that the argument is modular and works for any other model of ZFC\(_\kappa^+\) (transitive or ill-founded, containing the “true” \(H_{\kappa^+}\) or not).

(1) This is proved in details in [11, Chapter IV]. To define \(\text{WFE}_\kappa\) by a universal property over subsets of \(\kappa\) and \(\text{Cod}_\kappa\) by a \(\Delta_1\)-property over \(H_{\kappa^+}\), we proceed as follows:

- \(R\) is an extensional relation with domain contained in \(\kappa\) and top element \(0\) is defined by the \(\tau_{ST} \cup \{\kappa\}\)-atomic formula \(\psi_{\text{EXT}}(R)\) \(\text{ZFC}_{\kappa^+}\)-provably equivalent to the \(\Delta_0(\kappa)\)-formula:

\[
(R \subseteq \kappa^2) \land \\
(\text{Ext}(R) \in \kappa \lor \text{Ext}(R) = \kappa) \land \\
\forall \alpha, \beta \in \text{Ext}(R) [\forall u \in \text{Ext}(R) (u \ R \alpha \leftrightarrow u \ R \beta) \rightarrow (\alpha = \beta)] \land \\
\forall \alpha \in \text{Ext}(R) \neg (0 \ R \alpha).
\]

- \(\text{WFE}_\kappa\) is defined by the universal \(\tau_{ST} \cup \{\kappa\}\)-formula \(\phi_{\text{WFE}_\kappa}(R)\) (quantifying only over subsets of \(\kappa^{<\omega}\))

\[
\psi_{\text{EXT}}(R) \land \\
\forall f \subseteq \kappa^2 (f \text{ is a function } \rightarrow \exists n \in \omega \neg ([f(n+1), f(n)] \in R)).
\]

Its interpretation is the subset of \(\mathcal{P}(\kappa^{<\omega})\) of the \(\sigma_\kappa\)-symbol \(S_{\phi_{\text{WFE}_\kappa}}\).

- To define \(\text{Cod}_\kappa\), consider the \(\tau_{ST} \cup \{\kappa\}\)-atomic formula \(\psi_{\text{Cod}}(G, R)\) provably equivalent to the \(\tau_{ST} \cup \{\kappa\}\)-formula:

\[
\psi_{\text{EXT}}(R) \land \\
(\text{G is a function}) \land \\
(\text{dom}(G) = \text{Ext}(R)) \land \\
\forall \alpha, \beta \in \text{Ext}(R) [\alpha \ R \beta \leftrightarrow G(\alpha) \in G(\beta)].
\]

Then \(\text{Cod}_\kappa(R) = a\) can be defined either by the existential \(\tau_{ST} \cup \{\kappa\}\)-formula\(^{15}\)

\[
\exists G (\psi_{\text{Cod}}(G, R) \land G(0) = a)
\]

or by the universal \(\tau_{ST} \cup \{\kappa\}\)-formula

\[
\forall G (\psi_{\text{Cod}}(G, R) \rightarrow G(0) = a).
\]

(2) The equality relation in \(H_{\kappa^+}\) is transferred to the isomorphism relation between elements of \(\text{WFE}_\kappa\): if \(R, S\) are well-founded extensional on \(\kappa\) with a top-element, the Mostowski collapsing theorem entails that \(\text{Cod}_\kappa(R) = \text{Cod}_\kappa(S)\) if and only

\(^{15}\)Given an \(R\) such that \(\psi_{\text{EXT}}(R)\) holds, \(R\) is a well founded relation holds in a model of ZFC\(_\kappa^+\) if and only if \(\text{Cod}_\kappa\) is defined on \(R\). In the theory ZFC\(_\kappa^+\), \(\text{WFE}_\kappa\) can be defined using a universal property by a \(\tau_{ST} \cup \{\kappa\}\)-formula quantifying only over subsets of \(\kappa\). On the other hand if we allow arbitrary quantification over elements of \(H_{\kappa^+}\), we can express the well-foundedness of \(R\) also using the existential formula \(\exists G \psi_{\text{Cod}}(G, R)\). This is why \(\text{WFE}_\kappa\) is defined by a universal \(\tau_{ST} \cup \{\kappa\}\)-property in the structure \((\mathcal{P}(\kappa), \tau^\kappa_{S_{\text{ST}}}, \kappa)\), while the graph of \(\text{Cod}_\kappa\) can be defined by a \(\Delta_1\)-property for \(\tau_{ST} \cup \{\kappa\}\) in the structure \((H_{\kappa^+}, \tau^\kappa_{S_{\text{ST}}}, \kappa^\kappa)\).
if \((\text{Ext}(R), R) \cong (\text{Ext}(S), S)\). Isomorphism of the two structures \((\text{Ext}(R), R) \cong (\text{Ext}(S), S)\) is expressed by the \(\Sigma_1\)-formula for \(\tau_\kappa\):

\[
\phi_\kappa(R, S) \equiv \exists f \ (f \text { is a bijection of } \kappa \text { onto } \kappa \text { and } \alpha R \beta \text { if and only if } f(\alpha) S f(\beta)).
\]

In particular we get that \(S_{\phi_\kappa}(R, S)\) holds in \(H_\kappa^+\) for \(R, S \in WFE_\kappa\) if and only if \(\text{Cod}_\kappa(R) = \text{Cod}_\kappa(S)\).

Similarly one can express \(\text{Cod}_\kappa(R) \in \text{Cod}_\kappa(S)\) by the \(\Sigma_1\)-property \(\phi_\Sigma\) in \(\kappa\) stating that \((\text{Ext}(R), R)\) is isomorphic to \((\text{pred}_\Sigma(\alpha), S)\) for some \(\alpha \in \kappa\) with \(\alpha S 0\), where \(\text{pred}_\Sigma(\alpha)\) is given by the elements of \(\text{Ext}(S)\) which are connected by a finite path to \(\alpha\).

Moreover letting \(\cong_\kappa \subseteq WFE^2_\kappa\) denote the isomorphism relation between elements of \(WFE_\kappa\) and \(E_\kappa \subseteq WFE^2_\kappa\) denote the relation which translates into the \(\in\)-relation via \(\text{Cod}_\kappa\), it is clear that \(\cong_\kappa\) is a congruence relation over \(E_\kappa\), i.e.: if \(R_0 \cong_\kappa R_1\) and \(S_0 \cong_\kappa S_1\), \(R_0 E_\kappa S_0\) if and only if \(R_1 E_\kappa S_1\).

This gives that the structure \((WFE_\kappa/\cong_\kappa, E_\kappa/\cong_\kappa)\) is isomorphic to \((H_{\kappa^+}, \in)\) via the map \([R] \mapsto \text{Cod}_\kappa(R)\) (where \(WFE_\kappa/\cong_\kappa\) is the set of equivalence classes of \(\cong_\kappa\) and the quotient relation \([R] E_\kappa[S]\) holds if and only if \(R E_\kappa S\)).

This isomorphism is defined via the map \(\text{Cod}_\kappa\), which is by itself defined by a \(\text{ZFC}^-\_\kappa\)-provably \(\Delta_1\)-property for \(\tau_{ST} \cup \{\kappa\}\).

The very definition of \(WFE_\kappa, \cong_\kappa, E_\kappa\) show that

\[
WFE_\kappa = S^N_{\phi_{WFE_\kappa}},
\]

\[
\cong_\kappa = S^N_{\phi_{WFE_\kappa}(x) \land \phi_{WFE_\kappa}(y) \land \phi_\in(x,y)},
\]

\[
E_\kappa = S^N_{\phi_{WFE_\kappa}(x) \land \phi_{WFE_\kappa}(y) \land \phi_\in(x,y)}.
\]

\[
\square
\]

2.2. Model completeness for the theory of \(H_{\kappa^+}\).

**Theorem 2.4.** Any \(\sigma_\kappa\)-theory \(T\) extending

\[
\text{ZFC}^-_\kappa \cup \{\text{all sets have size } \kappa\}
\]

is model complete.

**Proof.** To simplify notation, we conform to the assumption of the previous theorem, i.e. we assume that the model \((N, \in)\) which is uniquely extended to a model of \(\text{ZFC}^-_\kappa\) every set has size \(\kappa\) on which we work is a transitive superstructure of \(H_{\kappa^+}\).

The statement every set has size \(\kappa\) is satisfied by a \(\text{ZFC}^-_\kappa\)-model \((N, \tau^-_{ST}, \kappa)\) with \(N \supseteq H_{\kappa^+}\) if and only if \(N = H_{\kappa^+}\). From now on we proceed assuming this equality.

By Robinson’s test 3.14 it suffices to show that for all \(\in\)-formulae \(\phi(x)\)

\[
\text{ZFC}^-_\kappa + \text{every set has size } \kappa \vdash \forall \bar{x} (\phi(\bar{x}) \leftrightarrow \psi_\phi(\bar{x})),
\]

for some universal \(\sigma_\kappa\)-formula \(\psi_\phi\).

We will first define a recursive map \(\phi \to \theta_\phi\) which maps \(\Sigma_n\)-formulae \(\phi\) for \(\{\in, \kappa\}\) quantifying over all elements of \(H_{\kappa^+}\) to \(\Sigma_{n+1}\)-formulae \(\theta_\phi\) for \(\tau_{ST} \cup \{\kappa\}\) whose quantifier range just over subsets of \(\kappa^{<\omega}\).

The proof of the previous theorem gave \(\tau_{ST} \cup \{\kappa\}\)-formulae \(\theta_{x=y}, \theta_{x \in y}\) such that

\[
S^H_{\theta_{x=y}} = \{ (R, S) \in (WFE_\kappa)^2 : \text{Cod}_\kappa(R) = \text{Cod}_\kappa(S) \},
\]

\[
S^H_{\theta_{x \in y}} = E_\kappa = \{ (R, S) \in (WFE_\kappa)^2 : \text{Cod}_\kappa(R) \in \text{Cod}_\kappa(S) \}.
\]

Specifically (following the notation of that proof)

\[
\theta_{x=y} = \phi_{WFE_\kappa}(x) \land \phi_{WFE_\kappa}(y) \land \phi_\in(x,y),
\]

\[
\theta_{x \in y} = \phi_{WFE_\kappa}(x) \land \phi_{WFE_\kappa}(y) \land \phi_\in(x,y).
\]
Now for any \( \{\in, \kappa\}\)-formula \( \psi(\vec{x}) \), we proceed to define the \( \tau_{\text{ST}} \cup \{\kappa\}\)-formula \( \theta_\psi(\vec{x}) \) letting:

- \( \theta_{\psi \land \psi}(\vec{x}) \) be \( \theta_\psi(\vec{x}) \land \theta_\psi(\vec{x}) \),
- \( \theta_{\neg \psi}(\vec{x}) \) be \( \neg \theta_\psi(\vec{x}) \),
- \( \theta_{\exists y \psi(y, \vec{x})}(\vec{x}) \) be \( \exists y \theta_\psi(y, \vec{x}) \land \phi_{\text{WFE}_\kappa}(y) \).

An easy induction on the complexity of the \( \tau_{\text{ST}} \cup \{\kappa\}\)-formulæ \( \psi(\vec{x}) \) gives that for any \( \{\in, \kappa\}\)-definable subset \( A \) of \( (H_{\kappa^+})^n \) which is the extension of some \( \{\in, \kappa\}\)-formula \( \phi(x_1, \ldots, x_n) \)

\[
\{(R_1, \ldots, R_n) \in (\text{WFE}_\kappa)^n : (\text{Cod}_\kappa(R_1), \ldots, \text{Cod}_\kappa(R_n)) \in A\} = S_{\theta_\phi}^{H_{\kappa^+}},
\]

with the further property that \( S_{\theta_\phi}^{H_{\kappa^+}} \subseteq (\text{WFE}_\kappa)^n \) respects the \( \equiv_\kappa \)-relation\(^{16}\).

Now every \( \sigma_\kappa \)-formula is \( \text{ZFC}_\kappa^+ \)-equivalent to a \( \{\in, \kappa\}\)-formula\(^{17}\).

Therefore we can extend \( \phi \mapsto \theta_\phi \) assigning to any \( \sigma_\kappa \)-formula \( \psi(\vec{x}) \) the formula \( \theta_\phi(\vec{x}) \) for some \( \{\in, \kappa\}\)-formula \( \psi(\vec{x}) \) which is \( \text{ZFC}_\kappa^+ \)-equivalent to \( \psi(\vec{x}) \).

Then for any \( \{\in, \kappa\}\)-formula \( \phi(x_1, \ldots, x_n) \) \( H_{\kappa^+} \models \phi(a_1, \ldots, a_n) \) if and only if

\[
(W_{\text{FEE}}/\equiv_\kappa, E_{\kappa}/\equiv_\kappa) \models \phi([R_1], \ldots, [R_n])
\]

with \( \text{Cod}_\kappa(R_i) = a_i \) for \( i = 1, \ldots, n \) if and only if

\[
H_{\kappa^+} \models \forall R_1, \ldots, R_n [(\bigwedge_{i=1}^n \text{Cod}_\kappa(R_i) = a_i) \rightarrow \theta_\phi(R_1, \ldots, R_n)]
\]

if and only if

\[
H_{\kappa^+} \models \forall R_1, \ldots, R_n [(\bigwedge_{i=1}^n \text{Cod}_\kappa(R_i) = a_i) \rightarrow S_{\theta_\phi}(R_1, \ldots, R_n)].
\]

Since this argument can be repeated verbatim for any model of \( \text{ZFC}_\kappa^+ \)-every set has size \( \kappa \), and any \( \sigma_\kappa \)-formula is \( \text{ZFC}_\kappa^+ \)-equivalent to a \( \{\in, \kappa\}\)-formula, we have proved the following:

**Claim 1.** For any \( \sigma_\kappa \)-formula \( \phi(x_1, \ldots, x_n) \), \( \text{ZFC}_\kappa^+ \)-every set has size \( \kappa \) proves that

\[
\forall x_1, \ldots, x_n [\phi(x_1, \ldots, x_n) \leftrightarrow \forall y_1, \ldots, y_n [(\bigwedge_{i=1}^n \text{Cod}_\kappa(y_i) = x_i) \rightarrow S_{\theta_\phi}(y_1, \ldots, y_n)]].
\]

But \( \text{Cod}_\kappa(y) = x \) is expressible by an existential \( \tau_{\text{ST}} \cup \{\kappa\}\)-formula provably in \( \text{ZFC}_\kappa^+ \subseteq \text{ZFC}_\kappa^- \), therefore

\[
\forall y_1, \ldots, y_n [(\bigwedge_{i=1}^n \text{Cod}_\kappa(y_i) = x_i) \rightarrow S_{\theta_\phi}(y_1, \ldots, y_n)]
\]

is a universal \( \sigma_\kappa \)-formula, and we are done. \( \square \)

\(^{16}\)It is also clear from our argument that the map \( \phi \mapsto \theta_\phi \) is recursive (and a careful inspection reveals that it maps a \( \Sigma_n \)-formula to a \( \Sigma_{n+1} \)-formula).

\(^{17}\)The map assigning to any \( \sigma_\kappa \)-formula a \( \text{ZFC}_\kappa^+ \)-equivalent \( \{\in, \kappa\}\)-formula can also be chosen to be recursive.
2.3. **Proof of Thm. 3.** We can immediately prove Thm. 3.

*Proof.* By Thm. 2.3, any theory extending

\[ \text{ZFC}^\kappa_\kappa^- + \text{every set has size } \kappa \]

is model complete. Therefore so is

\[ T^* = T_\forall \cup \text{ZFC}^\kappa_\kappa^- + \text{every set has size } \kappa. \]

We need to show that \( T^* \) is the model companion of \( T \), and that \( T^* = T_i^* \) for \( i = 0, 1 \) where

\[ T^*_0 = \{ \psi : \psi \text{ is a } \Pi_2\text{-sentence for } \sigma_\kappa \text{ and } T \vdash \psi^H_{\kappa^+} \}, \]

and \( T^*_1 \) is the set of \( \Pi_2\)-sentences \( \phi \) such that

For all \( \Pi_1\)-sentences \( \theta \) for \( \tau T_\forall + \phi \) is consistent if and only if so is \( T_\forall + \phi + \theta \).

\( T^* \) **is the model companion of** \( T \): By Lemma 3.19(1). It suffices to verify that for every model \( M \) of \( T \), \( H_\kappa^M \) is a \( \Sigma_1 \)-elementary substructure of \( M \) which models \( T^* \). But this holds true by Lemma 4.1. Therefore \( T^* \) is a model companion for \( T \).

\( T^*_1 = T^* : \) By Lemma 3.19(3) the model companion of \( T \) is axiomatized by \( T^*_1 \).

\( T^*_0 = T^* : \) First assume \( \psi \) is a \( \Pi_2\)-sentence in \( T^* \) and \( M \) models \( T \). We must show that \( H_\kappa^M \) models \( \psi \). But this is the case since \( H_\kappa^M \) models \( T^* \).

Conversely assume \( \psi \) is a \( \Pi_2\)-sentence for \( \sigma_\kappa \) which holds in any \( H_\kappa^M \) for \( M \) a model of \( T \). We must show that \( \psi \in T^* \). We show that \( \psi \in T^*_1 \): (using Lemma 3.19(2)) it suffices to show that \( S_\forall \cup \{ \psi \} \) is consistent for any consistent \( S \supseteq T \): fix \( M \) a model of \( S \); by assumption \( H_\kappa^M \) models \( \psi \); by Lemma 4.1 applied to \( M \), we get that \( H_\kappa^M \) models \( S_\forall \); we conclude that \( S_\forall \cup \{ \psi \} \) is consistent.

The proof is completed. \( \square \)

Remark 2.5. Thm. 3 can be proved for many other signatures other than \( \sigma_\kappa \). It suffices that the signature in question adds new predicates just for definable subsets of \( \mathcal{P}(\kappa)^n \), and also that it adds family of predicates which are closed under definability (i.e. projections, complementation, finite unions, permutations) and under the map \( \text{Cod}_\kappa \). Under these assumptions we can still use Lemma 4.1 and Lemma 3.19 to argue for the evident declination of Thm. 3 to this set up. However linking it to generic absoluteness results as we did in Theorem 1 requires much more care in the definition of the signature. We will pursue this matter in more details in the next section and in a follow-up of this paper.

2.4. **A weak version of Theorem 2.** Let \( \text{ZFC}^\omega_\omega \supseteq \text{ZFC}_{ST} \) be the \( \sigma_\omega = \sigma_\omega \cup \{ \kappa \} \)-theory obtained adding axioms which force in each of its \( \sigma_\omega \)-models \( \kappa \) to be interpreted by the first uncountable cardinal, and each predicate symbol \( S_\phi \) to be interpreted as the subset of \( \mathcal{P}(\omega^\omega_1)^n \) defined by \( \phi^{P_\omega(\omega^\omega_1)}(x_1, \ldots, x_n) \) (see again Notation 2 and Fact 1 for details).

**Theorem 4.** Let \( T \) be a \( \sigma_\omega \)-theory extending \( \text{ZFC}^\omega_\omega \) with the \( \in \)-sentence:

There are class many superhuge cardinals, and such that \( T + \text{MM}^{+++} \) is consistent.

\( \text{TFAE for any } \Pi_2\text{-sentence } \psi \text{ for } \sigma_\omega : \)

\( (1) \) For all universal \( \sigma_\omega \)-sentences \( \theta \) such that \( T + \theta \) is consistent, so is \( T_\forall + \theta + \psi \);

\( (2) \) \( T + \text{MM}^{+++} \) proves that some stationary set preserving forcing notion \( P \) forces \( \psi^H_{\kappa^2} + \text{MM}^{+++} \);

\( (3) \) \( T + \text{MM}^{+++} \vdash \psi^H_{\kappa^2} \).

See Remarks 2.6(4) for some information on \( \text{MM}^{+++} \), and 2.6(3) for informations on superhugeness.

The proof of Theorem 4 is a trivial variation of the proof of Theorem 1:
Proof. [16, Thm. 5.18] gives that 4(3) and 4(2) are equivalent. Theorem 3 establishes the equivalence of 4(3) and 4(1).

Remark 2.6.

(1) Note that $\text{ZFC}_{\omega_1}^*$ is more expressive than $\text{ZFC}_{\omega_1}^{*,\text{NS}_{\omega_1}}$. The former adds predicate symbols for all subsets of $\mathcal{P}(\omega_1^{<\omega})^k$ defined by $\phi^{\mathcal{P}(\omega_1^{<\omega})}(x_1,\ldots,x_k)$ as $\phi$ ranges over the $\varepsilon$-formulae. The latter adds predicate symbols for all subsets of $\mathcal{P}(\omega_1^{<\omega})^k$ defined by $\phi^{\mathcal{P}(\omega_1^{<\omega})}(x_1,\ldots,x_k)$ as $\phi$ ranges over the $\varepsilon$-formulae and a unique predicate symbol for the subset of $\mathcal{P}(\omega_1)$ given by the non-stationary ideal.

More precisely for any model $\mathcal{M} = (M,E)$ of ZFC, if $\mathcal{M}_0$ is the unique extension of $\mathcal{M}$ to a $\sigma_{\omega_1}$-model of $\text{ZFC}_{\omega_1}^*$, and $\mathcal{M}_1$ is the unique extension of $\mathcal{M}$ to a $\sigma_{\omega,\text{NS}_{\omega_1}}$-model of $\text{ZFC}_{\omega,\text{NS}_{\omega_1}}^*$, we get that $R_{\psi}^{\mathcal{M}_0} = R_{\psi}^{\mathcal{M}_1}$ and $f_{\psi}^{\mathcal{M}_0} = f_{\psi}^{\mathcal{M}_1}$ for all bounded formulae $\psi$, $\omega^{\mathcal{M}_0} = \omega^{\mathcal{M}_1}$, $\omega_1^{\mathcal{M}_0} = \omega_1^{\mathcal{M}_1}$, but for any $\tau_{\text{ST}}$-formula $\phi$, $S_{\phi}^{\mathcal{M}_1} = S_{\phi}^{\mathcal{M}_0}$.

(2) A key distinction between the signatures $\sigma_{\omega_1}$ and $\sigma_{\omega,\text{NS}_{\omega_1}}$ is that (assuming large cardinals) $\text{CH}$ cannot be $T$-equivalent to a $\Sigma_1$-sentence\footnote{By Thm. 6.} in $\sigma_{\omega,\text{NS}_{\omega_1}}$ for any $T$ as in the assumptions of Thm. 2, while it is $\text{ZFC}_{\omega_1}$-equivalent to an atomic $\sigma_{\omega_1}$-sentence\footnote{Following the notation to be introduced in Section 2, CH can be expressed as the $\tau_{\text{ST}} \cup \{\omega_1\}$-sentence quantifying just over subsets of $\mathcal{P}(\omega_1^{<\omega})$:

$\exists R \subseteq \mathcal{P}(\omega_1^{<\omega}) \ WFE_{\omega_1}(R) \land \forall S \subseteq \mathcal{P}(\omega_1^{<\omega}) [(WFE_{\omega_1}(S) \land \text{Ext}(S) = \omega) \leftrightarrow S \in E_{\omega_1} R]$. The latter is equivalent to a $\sigma_{\omega_1}$-sentence in $\text{ZFC}_{\omega_1}^*$.

$-\text{CH}$ is the simplest example of the type of $\Pi_2$-sentences which exemplifies why Thm. 4(2) must be weakened with respect to Thm. 2(2) and why Thm. 2 needs a different proof strategy than the one we use here to establish Theorems 1 and 4 (see for details 7 below). On the other hand the family of $\Pi_2$-sentences $\psi$ to which Theorem 4 applies is larger than the ones considered in Theorem 2 because the signature $\sigma_{\omega_1}$ is more expressive than $\sigma_{\omega,\text{NS}_{\omega_1}}$ (as shown by the case for $\text{CH}$).

(3) $\delta$ is superhuge if it supercompact and this can be witnessed by huge embeddings.

A superhuge cardinal is consistent relative to the existence of a 2-huge cardinal.

(4) For a definition of $\text{MM}^{++}$ see [16, Def. 5.19]. We just note that $\text{MM}^{++}$ is a natural strengthening of ($\ast$)-UB (by the recent breakthrough of Asperò and Schindler [1]) and of Martin’s maximum (for example any of the standard iterations to produce a model of Martin’s maximum produce a model of $\text{MM}^{++}$ if the iteration has length a superhuge cardinal [16, Thm. 5.29]).

(5) We can prove exactly the same results of Thm. 4 replacing (verbatim in its statement) $\text{MM}^{++}$ by any of the axioms $\text{RA}_{\omega_1}(\Gamma)$ introduced in [4] or the axioms $\text{CFA}(\Gamma)$ introduced in [3]; in item 4(2) stationary set preserving forcing notion $P$ must be replaced by $P \in \Gamma$.

(6) We consider Thm. 4 weaker than Thm. 2, because in Thm. 2 one can choose the theory $T$ to be inconsistent with $\text{MAX}(\text{UB}) + (\ast)$-UB without hampering its conclusion (for example $T$ could satisfy CH, a statement denied by ($\ast$)-UB), and because 2(2) holds for all forcing notions $P$. The key point separating these two results is that the signature $\sigma_{\omega_1}$ is too expressive and renders many statements incompatible with forcing axioms formalizable by existential (or even atomic) $\sigma_{\omega_1}$-sentences (for example such is the case for $\text{CH}$).

(7) We can also give a detailed explanation of why we cannot use Thm. 3 to prove Thm. 2 as we did for Theorems 1 and 4. The key point is that the model companion
$T^*$ of some $T \supseteq \text{ZFC}_{\omega_1,\text{NS}_{\omega_1}} + \text{there are class many Woodin}$ may not be axiomatized by the set $T^{**}$ of $\Pi_2$-sentences $\psi$ for $\sigma_{\omega,\text{NS}_{\omega_1}}$ such that $T \vdash \psi^{H_{\omega_2}}$, and this is what we used in the proofs of Theorems 1, 4.

For example this is the case for the theory $T = \text{ZFC}_{\omega_1,\text{NS}_{\omega_1}} + \text{CH} + \text{there are class many Woodin}$: By Remark 1(5) CH is expressible by the $\Sigma_2$-sentence in $\tau_{\omega_1} \cup \{\text{NS}_{\omega_1}\}$ $\psi_{\text{CH}}$, which shows that (in view of Levy Absoluteness) CH and $\text{CH}^{H_{\omega_2}}$ are $T$-equivalent. Now $\neg \text{CH}$ is in the Kaiser hull of $T$ (which is a subset of $T^*$) being a $\Pi_2$-sentence compatible with $S_\psi$ for any complete $S \supseteq T$ in view of Thm. 6 and Fact 3.12.

3. Existentially closed structures, model completeness, model companionship

We present this topic expanding on [14, Sections 3.1-3.2]. We decided to include detailed proofs since their presentation is (in some occasions) rather sketchy, and their focus is not exactly ours.

The first objective is to isolate necessary and sufficient conditions granting that some $\tau$-structure $M$ embeds into some model of some $\tau$-theory$^{20}$ $T$.

**Definition 3.1.** Given $\tau$-theories $T$, $S$, a $\tau$-sentence $\psi$ separates $T$ from $S$ if $T \vdash \psi$ and $S \vdash \neg \psi$.

$T$ is $\Pi_n$-separated from $S$ if some $\Pi_n$-sentence for $\tau$ separates $T$ from $S$.

**Lemma 3.2.** Assume $S, T$ are $\tau$-theories. TFAE:

1. $T$ is not $\Pi_1$-separated from $S$ (i.e. no universal sentence $\psi$ is such that $T \vdash \psi$ and $S \vdash \neg \psi$).
2. There is some $\tau$-model $M$ of $S$ which can be embedded in some $\tau$-model $N$ of $T$.

See also [14, Lemma 3.1.1, Lemma 3.1.2, Thm. 3.1.3]

**Proof.** We assume $T, S$ are closed under logical consequences.

(2) implies (1): By contraposition we prove $\neg(1) \rightarrow \neg(2)$.

Assume some universal sentence $\psi$ separates $T$ from $S$. Then for any model of $T$, all its substructures model $\psi$, therefore they cannot be models of $S$.

(1) implies (2): By contraposition we prove $\neg(2) \rightarrow \neg(1)$.

Assume that for any model $M$ of $S$ and $N$ of $T$ $M \not\subseteq N$. We must show that $T$ is $\Pi_1$-separated from $S$.

Given a $\tau$-structure $M$ which models $S$, let $\Delta_0(M)$ be the atomic diagram of $M$ in the signature $\tau \cup M$.

The theory $T \cup \Delta_0(M)$ is inconsistent, otherwise $M$ embeds into some model of $T$: let $Q$ be a $\tau \cup M$-model of $\Delta_0(M) \cup T$ and $Q$ be the $\tau$-structure obtained from $Q$ omitting the interpretation of the constants not in $\tau$. Clearly $Q$ models $T$. The interpretation of the constants in $\tau \cup M$ inside $Q$ defines a $\tau$-substructure of $Q$ isomorphic to $M$.

By compactness (since $\Delta_0(M)$ is closed under finite conjunctions) there is a quantifier-free $\tau$-formula $\psi_{M}(\bar{x})$ and $\bar{a} \in M^{<\omega}$ such that $T \vdash \psi_{M}(\bar{a})$ is inconsistent. This gives that $T \vdash \neg \psi_{M}(\bar{a})$. Since $\bar{a}$ is a family of constants never occurring in $T$, we get $T \vdash \forall \bar{x} \neg \psi_{M}(\bar{x})$ and $M \models \exists \bar{x} \psi_{M}(\bar{x})$.

The theory $S \cup \{\neg \exists \bar{x} \psi_{M}(\bar{x}) : M \models S\}$ is inconsistent, since $\neg \exists \bar{x} \psi_{M}(\bar{x})$ fails in any model $M$ of $S$.

---

$^{20}$In what follows we conform to Notation 2.1 and feel free to confuse a $\tau$-structure $M = (M, \tau^M)$ with its domain $M$ and an ordered tuple $\bar{a} \in M^{<\omega}$ with its set of elements. Moreover we often write $M \models \phi(\bar{a})$ rather than $M \models \phi(\bar{a})|_{\bar{a}}$ when $M$ is $\tau$-structure $\bar{a} \in M^{<\omega}$, $\phi$ is a $\tau$-formula.
By compactness there is a finite set of formulae \( \psi_{M_1} \ldots \psi_{M_k} \) such that
\[
S + \bigwedge \{ \neg \exists \bar{x}_i \psi_{M_i}(\bar{x}_i) : i = 1, \ldots, k \}
\]
is inconsistent. This gives that
\[
S \vdash \bigvee_{i=1}^{k} \exists \bar{x}_i \psi_{M_i}(\bar{x}_i).
\]
The \( \tau \)-sentence \( \psi := \bigvee_{i=1}^{k} \exists \bar{x}_i \psi_{M_i}(\bar{x}_i) \) holds in all models of \( S \) and its negation
\[
\bigwedge \{ \neg \exists \bar{x}_i \psi_{M_i}(\bar{x}_i) : i = 1, \ldots, k \}
\]
is a conjunction of universal sentences derivable from \( T \). Hence \( \neg \psi \) separates \( T \) from \( S \).

The following Lemma shows that models of \( T_\forall \) can always be extended to superstructures which model \( T \).

**Lemma 3.3.** Let \( T \) be a \( \tau \)-theory and \( M \) be a \( \tau \)-structure. TFAE:

1. \( M \) is a \( \tau \)-model of \( T_\forall \).
2. There exists \( N \supseteq M \) which models \( T \).

**Proof.** (2) implies (1) is trivial.

Conversely let \( \Delta_0(M) \) be the \( \tau \cup M \)-theory given by the atomic diagram of \( M \).

**Claim.** \( T \) is not \( \Pi_1 \)-separated from \( \Delta_0(M) \) (in the signature \( \tau \cup M \)).

**Proof.** If not there are \( \bar{a} \in M^{<\omega} \), and a quantifier free \( \tau \)-formula \( \phi(\bar{x}, \bar{z}) \) such that
\[
T \vdash \forall \bar{z} \phi(\bar{a}, \bar{z}),
\]
while
\[
\Delta_0(M) \vdash \neg \forall \bar{z} \phi(\bar{a}, \bar{z}).
\]
The latter yields that
\[
\Delta_0(M) \vdash \exists \bar{x} \exists \bar{z} \neg \phi(\bar{x}, \bar{z}),
\]
and therefore also that
\[
M \models \exists \bar{x} \exists \bar{z} \neg \phi(\bar{x}, \bar{z}).
\]
On the other hand, since the constants \( \bar{a} \) do not appear in any of the sentences in \( T \), we also get that
\[
T \vdash \forall \bar{x} \forall \bar{z} \phi(\bar{x}, \bar{z}).
\]
This is a contradiction since \( M \) models \( T_\forall \). \( \Box \)

By the Claim and Lemma 3.2 some \( \tau \cup M \)-model \( \bar{P} \) of \( \Delta_0(M) \) embeds into some \( \tau \cup M \)-model \( \bar{Q} \) of \( T \). Let \( Q \) be the \( \tau \)-structure obtained from \( \bar{Q} \) omitting the interpretation of the constants not in \( \tau \). Then \( Q \) models \( T \) and contains a substructure isomorphic to \( M \).

**Corollary 3.4 (Resurrection Lemma).** Assume \( M \prec_1 N \) are \( \tau \)-structures. Then there is \( Q \supseteq N \) which is an elementary extension of \( M \).

**Proof.** Let \( T \) be the elementary diagram \( \Delta_\omega(M) \) of \( M \) in the signature \( \tau \cup M \). It is easy to check that any model of \( T \) when restricted to the signature \( \tau \) is an elementay extension of \( M \). Since \( M \prec_1 N \), the natural extension of \( N \) to a \( \tau \cup M \)-structure realizes the \( \Pi_1 \)-fragment of \( T \) in the signature \( \tau \cup M \). Now apply the previous Lemma. \( \Box \)

The Resurrection Lemma motivates the resurrection axioms introduced by Hamkins and Johnstone in [7], and their iterated versions introduced by the author and Audrito in [4].
3.1. Existentially closed structures. The objective is now to isolate the “generic” models of some universal theory $T$ (i.e. all axioms of $T$ are universal sentences). These are described by the $T$-existentially closed models.

**Definition 3.5.** Given a first order signature $\tau$, let $T$ be any consistent $\tau$-theory. A $\tau$-structure $M$ is $T$-existentially closed ($T$-ec) if

1. $M$ can be embedded in a model of $T$.
2. $M \prec_{\Sigma_1} \mathcal{N}$ for all $\mathcal{N} \supseteq M$ which are models of $T$.

In general $T$-ec models need not be models of $T$, but only of their universal fragment. A standard diagonalization argument shows that for any theory $T$ there are $T$-ec models, see Lemma 3.8 below or [14, Lemma 3.2.11].

A trivial observation which will come handy in the sequel is the following:

**Fact 3.6.** Assume $M$ is a $T$-ec model and $S \supseteq T$ is such that some $\mathcal{N} \supseteq M$ models $S$. Then $M$ is $S$-ec.

**Proposition 3.7.** Assume a $\tau$-structure $M$ is $T$-ec. Then:

1. $M \models T_{\forall}$.
2. $M$ is also $T_{\forall}$-ec.
3. If $\mathcal{N} \prec_{\Sigma_1} M$, then $\mathcal{N}$ is also $T$-ec.
4. Let $\forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y}, \bar{a})$ be a $\Pi_2$-sentence with $\psi(\bar{x}, \bar{y}, \bar{z})$ quantifier free $\tau$-formula and parameters $\bar{a}$ in $M^{<\omega}$. Assume it holds in some $\mathcal{N} \supseteq M$ which models $T_{\forall}$, then it holds in $M$.
5. Let $S$ be the $\tau$-theory of $M$. For any $\Pi_2$-sentence $\psi$ in the signature $\tau$ $TFAE$:
   - $\psi$ holds in some model of $S_{\forall}$.
   - $\psi$ holds in $M$.

**Proof.**

1: There is at least one super-structure of $M$ which models $T$, and any $\psi \in T_{\forall}$ holds in this superstructure, hence in $M$.

2: Assume $M \subseteq \mathcal{P}$ for some model $\mathcal{P}$ of $T_{\forall}$. We must argue that $M \prec_{1} \mathcal{P}$.

   By Lemma 3.3, there is $\mathcal{Q} \supseteq \mathcal{P}$ which models $T$.

   Since $M$ and $\mathcal{Q}$ are both models of $T$ and $M$ is $T$-ec, we get the following diagram:

\[
\begin{array}{ccc}
M & \overset{\Sigma_1}{\longrightarrow} & \mathcal{Q} \\
\ll & \nearrow \& \searrow & \\
\subseteq & \mathcal{P} & \subseteq
\end{array}
\]

Then any $\Sigma_1$-formula $\psi(\bar{a})$ with $\bar{a} \in M^{<\omega}$ realized in $\mathcal{P}$ holds in $\mathcal{Q}$, and is therefore reflected to $M$. We are done by Tarski-Vaught’s criterion.

3: Assume $\mathcal{N} \subseteq \mathcal{P}$ for some model of $T_{\forall}$. Let $\Delta_0(\mathcal{P})$ be the atomic diagram of $\mathcal{P}$ in the signature $\tau \cup \mathcal{P} \cup M$ and $\Delta_0(M)$ be the atomic diagram of $M$ in the same signature\(^{22}\).

**Claim 3.** $T_{\forall} \cup \Delta_0(\mathcal{P}) \cup \Delta_0(M)$ is a consistent $\tau \cup M \cup \mathcal{P}$-theory.

\(^{21}\)For example let $T$ be the theory of commutative rings with no zero divisors which are not fields in the signature $(+, \cdot, 0, 1)$. Then the $T$-ec structures are exactly all the algebraically closed fields, and no $T$-ec model is a model of $T$. By Thm. 3 $(H_{\omega_1}, \sigma_{\omega})$ is $S$-ec for $S$ the $\sigma_{\omega}$-theory of $V$, but it is not a model of $S$: the $\Pi_2$-sentence asserting that every set has countable transitive closure is true in $(H_{\omega_1}, \sigma_{\omega})$ but denied by $S$.

\(^{22}\)We are considering $\mathcal{P} \cup M$ as the union of the domains of the structure $\mathcal{P}, M$ amalgamated over $\mathcal{N}$; in particular we add a new constant for each element of $\mathcal{P} \setminus \mathcal{N}$, a new constant for each element of $M \setminus \mathcal{N}$, a new constant for each element of $\mathcal{N}$. 
Proof Assume not. Find \( \vec{a} \in (\mathcal{P} \setminus \mathcal{N})^{<\omega} \), \( \vec{b} \in (\mathcal{M} \setminus \mathcal{N})^{<\omega} \), \( \vec{c} \in \mathcal{N}^{<\omega} \) and \( \tau \)-formulae \( \psi_0(\vec{x}, \vec{z}) \), \( \psi_1(\vec{y}, \vec{z}) \) such that:

- \( \psi_0(\vec{a}, \vec{c}) \in \Delta_0(\mathcal{P}) \),
- \( \psi_1(\vec{b}, \vec{c}) \in \Delta_0(\mathcal{M}) \),
- \( T \cup \left\{ \psi_0(\vec{a}, \vec{c}), \psi_1(\vec{b}, \vec{c}) \right\} \) is inconsistent.

Then

\[
T \vdash \neg \psi_0(\vec{a}, \vec{c}) \lor \neg \psi_1(\vec{b}, \vec{c}).
\]

Since the constants appearing in \( \vec{a}, \vec{b}, \vec{c} \) are never appearing in sentences of \( T \), we get that

\[
T \vdash \forall \vec{z} (\forall \vec{x} \neg \psi_0(\vec{x}, \vec{z})) \lor (\forall \vec{y} \neg \psi_1(\vec{y}, \vec{z})).
\]

Since \( \mathcal{P} \) models \( T_\psi \), and

\[
\mathcal{P} \models \psi_0(\vec{x}, \vec{z})|_{\vec{x}/\vec{a}, \vec{z}/\vec{c}},
\]

we get that

\[
\mathcal{P} \models \forall \vec{y} \neg \psi_1(\vec{y}, \vec{c}).
\]

Therefore

\[
\mathcal{N} \models \forall \vec{y} \neg \psi_1(\vec{y}, \vec{c})
\]

being a substructure of \( \mathcal{P} \), and so does \( \mathcal{M} \) since \( \mathcal{N} \preceq_1 \mathcal{M} \). This contradicts \( \psi_1(\vec{b}, \vec{c}) \in \Delta_0(\mathcal{M}) \).

If \( \mathcal{Q} \) is a model realizing \( T_\psi \cup \Delta_0(\mathcal{P}) \cup \Delta_0(\mathcal{M}) \), and \( \mathcal{Q} \) is the \( \tau \)-structure obtained forgetting the constant symbols not in \( \tau \), we get that:

- \( \mathcal{P} \) and \( \mathcal{M} \) are both substructures of \( \mathcal{Q} \) containing \( \mathcal{N} \) as a common substructure;
- \( \mathcal{N} \preceq_1 \mathcal{M} \preceq_1 \mathcal{Q} \), since \( \mathcal{Q} \) realizes \( T_\psi \) and \( \mathcal{M} \) is \( T_\psi \)-ec.

We can now conclude that if a \( \Sigma_1 \)-formula \( \psi(\vec{c}) \) for \( \tau \cup \mathcal{N} \) with parameters in \( \mathcal{N} \) holds in \( \mathcal{P} \), it holds in \( \mathcal{Q} \) as well (since \( \mathcal{Q} \supseteq \mathcal{P} \), and therefore also in \( \mathcal{N} \) (since \( \mathcal{N} \preceq_1 \mathcal{Q} \)).

(4): Observe that for all \( \vec{b} \in \mathcal{M}^{<\omega} \), \( \exists \vec{y} \psi(\vec{b}, \vec{y}, \vec{a}) \) holds in \( \mathcal{N} \), and therefore in \( \mathcal{M} \), since \( \mathcal{M} \) is \( T \)-ec; hence \( \mathcal{M} \models \forall \vec{x} \exists \vec{y} \psi(\vec{x}, \vec{y}, \vec{a}) \).

(5): First of all note that \( \mathcal{M} \) is \( S \)-ec since \( S \supseteq T \) (by Fact 3.6). By Lemma 3.3 (applied to \( S_\psi + \psi \) and \( \mathcal{M} \)) any \( \Pi_2 \)-sentence \( \psi \) for \( \tau \) which holds in some model of \( S_\psi \) holds in some model of \( S_\psi \) which is a superstructure of \( \mathcal{M} \). Now apply 4.

\( \square \)

In particular a structure is \( T \)-ec if and only if it is \( T_\psi \)-ec, and a \( T \)-ec structure realizes all \( \Pi_2 \)-sentences which are consistent with its \( \Pi_1 \)-theory.

We now show that any structure \( \mathcal{M} \) can always be extended to a \( T \)-ec structure for any \( T \) which is not separated from the \( \Pi_1 \)-theory of \( \mathcal{M} \).

Lemma 3.8. [14, Lemma 3.2.11] Given a first order \( \tau \)-theory \( T \), any model of \( T_\psi \) can be extended to a \( \tau \)-superstructure which is \( T \)-ec.

Proof Given a model \( \mathcal{M} \) of \( T \), we construct an ascending chain of \( T_\psi \)-models as follows. Enumerate all quantifier free \( \tau \)-formulae as \( \{ \phi_\alpha(y, \vec{x}_\alpha) : \alpha < |\tau| \} \). Let \( \mathcal{M}_0 = \mathcal{M} \) have size \( \kappa \geq |\tau| + \aleph_0 \). Fix also some enumeration

\[
\pi : \kappa \to |\tau| \times \kappa^2\]

\[
\alpha \mapsto (\pi_0(\alpha), \pi_1(\alpha), \pi_2(\alpha))
\]

such that \( \pi_2(\alpha) \leq \alpha \) for all \( \alpha < \kappa \) and for each \( \xi < |\tau| \), and \( \eta, \beta < \kappa \) there are unboundedly many \( \alpha < \kappa \) such that \( \pi(\alpha) = (\xi, \eta, \beta) \).
Let now $M_\eta$ with enumeration $\left\{\bar{m}_\eta^\xi : \xi < \kappa \right\}$ of $M_\eta^{\omega}$ be given for all $\eta \leq \beta$. If $M_\beta$ is $T$-ec, stop the construction. Else check whether $T_\gamma \cup \Delta_0(M_\beta) \cup \left\{\exists y \phi_{\pi_0(\alpha)}(y, \bar{m}_{\pi_2(\alpha)})\right\}$ is a consistent $\tau \cup M_\beta$-theory; if so let $M_{\beta+1}$ have size $\kappa$ and realize this theory. At limit stages $\gamma$, let $M_\gamma$ be the direct limit of the chain of $\tau$-structures $\{M_\beta : \beta < \gamma\}$. Then all $M_\xi$ are models of $T_\gamma$, and at some stage $\beta \leq \kappa$ $M_\beta$ is $T_\gamma$-ec (hence also $T$-ec), since all existential $\tau$-formulae with parameters in some $M_\eta$ will be considered along the construction, and realized along the way if this is possible, and all $M_\eta$ are always models of $T_\gamma$ (at limit stages the ascending chain of $T_\gamma$-models remains a $T_\gamma$-model).

Compare the above construction with the standard consistency proofs of bounded forcing axioms as given for example in [2, Section 2]. In the latter case to preserve $T_\gamma$ at limit stages we use iteration theorems\footnote{Assume $G$ is $V$-generic for a forcing which is a limit of an iteration of length $\omega$ of forcings $\{P_n : n < \omega\}$. In general $H^{V[G]}_{\omega_2}$ is not given by the union of $H^{V[G\upharpoonright P_n]}_{\omega_2}$, hence a subtler argument is needed to maintain that $H^{V[G]}_{\omega_2}$ preserves $T_\gamma$.}

3.2. The Kaiser hull of a first order theory. The Kaiser Hull of a theory $T$ describes the smallest elementary class containing all the “generic” structures for $T$. For most theories $T$ the models of the respective Kaiser hulls realize exactly all $\Pi_2$-sentences which are consistent with the universal fragment of any extension of $T$.

**Definition 3.9.** [14, Lemma 3.2.12, Lemma 3.2.13] Given a theory $T$ in a signature $\tau$, its Kaiser hull $KH(T)$ is given by the $\Pi_2$-sentences of $\tau$ which holds in all $T$-ec structures.

**Definition 3.10.** A $\tau$-theory $T$ is $\Pi_n$-complete, if it is consistent and for any $\Pi_n$-sentence either $\phi \in T$ or $\neg \phi \in T$.

By Proposition 3.7.5 we get:

**Fact 3.11.** Given a $\Pi_1$-complete first order $\tau$-theory $T$, its Kaiser Hull is a $\Pi_2$-complete $\tau$-theory defined by the request that for any $\Pi_2$-sentence $\psi$

$$\psi \in KH(T) \quad \text{if and only if} \quad \{\psi\} \cup T_\gamma \text{ is consistent.}$$

In particular any model of the Kaiser hull of a $\Pi_1$-complete $T$ realizes simultaneously all $\Pi_2$-sentences which are individually consistent with $T_\gamma$. For theories $T$ of interests to us their Kaiser hull can be described in the same terms, but the proof is much more delicate. We start with the following weaker property which holds for arbitrary theories:

**Fact 3.12.** Given a $\tau$-theory $T$, its Kaiser hull $KH(T)$ contains the set of $\Pi_2$-sentences $\psi$ for $\tau$ such that for all complete $S \supseteq T$, $S_\gamma \cup \{\psi\}$ is consistent.

*Proof.* Assume $\psi$ is a $\Pi_2$-sentence such that for all complete $S \supseteq T$, $S_\gamma \cup \{\psi\}$ is consistent. We must show that $\psi$ holds in all $T$-ec models.

Fix $M$ an existentially closed model for $T$ (it exists by Lemma 3.8); we must show that $M \models \psi$. Let $N \supseteq M$ be a model of $T$ and $S$ be the $\tau$-theory of $N$. Then $S$ is a complete theory and $M \models S_\gamma$ since $M \prec_1 N$ (being $T$-ec). Since $S \supseteq T$, $M$ is also $S$-ec (by Fact 3.6). Since $S_\gamma \cup \{\psi\}$ is consistent, and $S_\gamma$ is $\Pi_1$-complete, we obtain that $M$ models $\psi$, being an $S_\gamma$-ec model, and using Fact 3.11.\qed

We will show in Lemma 3.19 that the set of $\Pi_2$-sentences described in the Fact provides an equivalent characterization of the Kaiser hull for many theories admitting a model companion, among which those considered in the previous sections.
3.3. **Model completeness.** It is possible (depending on the choice of the theory \( T \)) that there are models of the Kaiser hull of \( T \) which are not \( T \)-ec\(^2\). Robinson has come up with two model theoretic properties (model completeness and model companionship) which describe the case in which the models of the Kaiser hull of \( T \) are exactly the class of \( T \)-ec models (even in case \( T \) is not a complete theory).

**Definition 3.13.** A \( \tau \)-theory \( T \) is **model complete** if for all \( \tau \)-models \( \mathcal{M} \) and \( \mathcal{N} \) of \( T \) we have that \( \mathcal{M} \sqsubseteq \mathcal{N} \) implies \( \mathcal{M} \prec \mathcal{N} \).

Remark that theories admitting quantifier elimination are automatically model complete. On the other hand model complete theories need not be complete\(^2\). However for theories \( T \) which are \( \Pi_1 \)-complete, model completeness entails completeness: any two models of a \( \Pi_1 \)-complete theory share the same \( \Pi_1 \)-theory, therefore if \( T_1 \supseteq T \) and \( T_2 \supseteq T \) with \( \mathcal{M}_1 \) a model of \( T_1 \), we can suppose (by Lemma 3.2) that \( \mathcal{M}_1 \sqsubseteq \mathcal{M}_2 \). Since they are both models of \( T \), model completeness entails that \( \mathcal{M}_1 \prec \mathcal{M}_2 \).

**Lemma 3.14.** [14, Lemma 3.2.7] (Robinson’s test) Let \( T \) be a \( \tau \)-theory. The following are equivalent:

(a) \( T \) is model complete.
(b) Any model of \( T \) is \( T \)-ec.
(c) Each existential \( \tau \)-formula \( \phi(\bar{x}) \) in free variables \( \bar{x} \) is \( T \)-equivalent to a universal \( \tau \)-formula \( \psi(\bar{x}) \) in the same free variables.
(d) Each \( \tau \)-formula \( \phi(\bar{x}) \) in free variables \( \bar{x} \) is \( T \)-equivalent to a universal \( \tau \)-formula \( \psi(\bar{x}) \) in the same free variables.

Remark that (d) (or (c)) shows that being a model complete \( \tau \)-theory \( T \) is expressible by a \( \Delta_0(\tau, T) \)-property in any model of \( \text{ZFC} \), hence it is absolute with respect to forcing.

**Proof.**

(a) implies (b): Immediate.

(b) implies (c): Fix an existential formula \( \phi(\bar{x}) \) in free variables \( x_1, \ldots, x_n \). Let \( \Gamma \) be the set of universal formulae \( \theta(\bar{x}) \) such that

\[
T \vdash \forall \bar{x} (\phi(\bar{x}) \rightarrow \theta(\bar{x})).
\]

Note that \( \Gamma \) is closed under finite conjunctions and disjunctions. Let \( \bar{c} = (c_1, \ldots, c_n) \) be a finite set of new constant symbols and \( \Gamma(\bar{c}) = \{ \theta(\bar{c}) : \theta(\bar{x}) \in \Gamma \} \).

It suffices to prove

\[
T \cup \Gamma(\bar{c}) \models \phi(\bar{c});
\]

if this is the case, by compactness, a finite subset \( \Gamma_0(\bar{c}) \) of \( \Gamma(\bar{c}) \) is such that

\[
T \cup \Gamma_0(\bar{c}) \models \phi(\bar{c});
\]

letting \( \bar{\theta}(\bar{x}) := \bigwedge \{ \psi(\bar{x}) : \psi(\bar{c}) \in \Gamma_0(\bar{c}) \} \), the latter gives that

\[
T \models \forall \bar{x} (\bar{\theta}(\bar{x}) \rightarrow \phi(\bar{x}))
\]

(since the constants \( \bar{c} \) do not appear in \( T \)).

\( \bar{\theta}(\bar{x}) \in \Gamma \) is a universal formula witnessing (c) for \( \phi(\bar{x}) \).

So we prove (2):

**Proof.** Let \( \mathcal{M} \) be a \( \tau \cup \{ c_1, \ldots, c_n \} \)-model of \( T \cup \Gamma(\bar{c}) \). We must show that \( \mathcal{M} \) models \( \phi(\bar{c}) \).

The key step is to prove the following:

\(^2\)This is the main issue we face in the proof of Thm. 2: we cannot prove that the theory \( T \) in its assumption has a model companion, we will only be able to compute that its Kaiser hull is described by 2(3).

\(^2\)For example the theory of algebraically closed fields is model complete, but algebraically closed fields of different characteristics are elementarily inequivalent.
Claim 4. \( T \cup \Delta_0(M) \cup \{ \phi(\vec{c}) \} \) is consistent (where \( \Delta_0(M) \) is the \( \tau \cup \{ c_1, \ldots, c_n \} \)-atomic diagram of \( M \)).

Assume the Claim holds and let \( \mathcal{N} \) realize the above theory. Then
\[
\mathcal{M} \subseteq \mathcal{N} \upharpoonright (\tau \cup \{ \vec{c} \}).
\]
Hence
\[
\mathcal{M} \upharpoonright \tau \subseteq \mathcal{N} \upharpoonright \tau.
\]
By (b)
\[
\mathcal{M} \upharpoonright \tau \triangleleft_1 \mathcal{N} \upharpoonright \tau.
\]
Now let \( b_1, \ldots, b_n \in \mathcal{M} \) be the interpretations of \( c_1, \ldots, c_n \). Then
\[
\mathcal{N} \upharpoonright \tau \models \phi(b_1, \ldots, b_n).
\]
Since \( \phi(\vec{x}) \) is \( \Sigma_1 \) for \( \tau \), we get that
\[
\mathcal{M} \upharpoonright \tau \models \phi(c_1, \ldots, c_n),
\]
and we are done.
So we are left with the proof of the Claim.

Proof. Let \( \psi(\vec{x}, \vec{y}) \) be a quantifier free \( \tau \)-formula such that \( \psi(\vec{c}, \vec{a}) \in \Delta_0(M) \) for some \( \vec{a} \in \mathcal{M} \).
Clearly \( \mathcal{M} \) models \( \exists \vec{y} \psi(\vec{c}, \vec{y}) \).
Then the universal formula \( \neg \exists \vec{y} \psi(\vec{c}, \vec{y}) \notin \Gamma(\vec{c}) \), since \( \mathcal{M} \) models its negation and \( \Gamma(\vec{c}) \) at the same time.
This gives that
\[
T \nvdash \forall \vec{x} (\phi(\vec{x}) \rightarrow \neg \exists \vec{y} \psi(\vec{x}, \vec{y})),
\]
i.e.
\[
T \cup \{ \exists \vec{x} [\phi(\vec{x}) \land \exists \vec{y} \psi(\vec{x}, \vec{y})] \}
\]
is consistent.
We conclude that
\[
T \cup \{ \phi(\vec{c}) \land \psi(\vec{c}, \vec{a}) \}
\]
is consistent for any tuple \( a_1, \ldots, a_k \in \mathcal{M} \) and formula \( \psi \) such that \( \mathcal{M} \) models \( \psi(\vec{c}, \vec{a}) \) (since \( \vec{c}, \vec{a} \) are constants never appearing in the formulae of \( T \)).
This shows that \( T \cup \Delta_0(M) \cup \{ \phi(\vec{c}) \} \) is consistent. \( \square \)

(2) is proved. \( \square \)

(c) implies (d): We prove by induction on \( n \) that \( \Pi_n \)-formulae and \( \Sigma_n \)-formulae are \( T \)-equivalent to a \( \Pi_1 \)-formula.
(c) gives the base case \( n = 1 \) of the induction for \( \Sigma_1 \)-formulae and (trivially) for \( \Pi_1 \)-formulae.
Assuming we have proved the implication for all \( \Sigma_n \) formulae for some fixed \( n > 0 \), we obtain it for \( \Pi_{n+1} \)-formulae \( \forall \vec{x} \psi(\vec{x}, \vec{y}) \) (with \( \psi(\vec{x}, \vec{y}) \) \( \Sigma_n \)) applying the inductive assumptions to \( \psi(\vec{x}, \vec{y}) \); next we observe that a \( \Sigma_{n+1} \)-formula is equivalent to the negation of a \( \Pi_{n+1} \)-formula, which is in turn equivalent to the negation of a universal formula (by what we already argued), which is equivalent to an existential formula, and thus equivalent to a universal formula (by (c)).
(d) implies (a): By (d) every formula is \( T \)-equivalent both to a universal formula and to an existential formula (since its negation is \( T \)-equivalent to a universal formula).
This gives that \( \mathcal{M} \prec \mathcal{N} \) whenever \( \mathcal{M} \subseteq \mathcal{N} \) are models of \( T \), since truth of universal formulae is inherited by substructures, while truth of existential formulae pass to superstructures.
We will also need the following:

**Fact 3.15.** Let \( \tau \) be a signature and \( T \) a model complete \( \tau \)-theory. Let \( \sigma \supseteq \tau \) be a signature and \( T^* \supseteq T \) a \( \sigma \)-theory such that every \( \sigma \)-formula is \( T^* \)-equivalent to a \( \tau \)-formula. Then \( T^* \) is model complete.

**Proof.** By the model completeness of \( T \) and the assumptions on \( T^* \) we get that every \( \sigma \)-formula is equivalent to a \( \Pi_1 \)-formula for \( \tau \subseteq \sigma \). We conclude by Robinson’s test.  

We will later show that model complete theories are the Kaiser hull of their universal fragment. This will be part of a broad family of tameness properties for first order theories which require a new concept in order to be properly formulated, that of model companionship.

### 3.4. Model companionship

Model completeness comes in pairs with another fundamental concept which generalizes to arbitrary first order theories the relation existing between algebraically closed fields and commutative rings without zero-divisors. As a matter of fact, the case described below occurs when \( T^* \) is the theory of algebraically closed fields and \( T \) is the theory of commutative rings with no zero divisors.

**Definition 3.16.** Given two theories \( T \) and \( T^* \) in the same language \( \tau \), \( T^* \) is the model companion of \( T \) if the following conditions holds:

1. Each model of \( T \) can be extended to a model of \( T^* \).
2. Each model of \( T^* \) can be extended to a model of \( T \).
3. \( T^* \) is model complete.

Different theories can have the same model companion, for example the theory of fields and the theory of commutative rings with no zero-divisors which are not fields both have the theory of algebraically closed fields as their model companion.

**Theorem 3.17.** [14, Thm. 3.2.14] Let \( T \) be a first order theory. If its model companion \( T^* \) exists, then

1. \( T^*_\forall = T^*_\forall^* \).
2. \( T^* \) is the theory of the existentially closed models of \( T^*_\forall \).

**Proof.**

1. By Lemma 3.3.
2. By Robinson’s test 3.14 \( T^* \) is the theory realized exactly by the \( T^*_\forall \)-ec models; by Proposition 3.7(2) \( M \) is \( T^*_\forall \)-ec if and only if it is \( T^*_\forall^* \)-ec; by (1) \( T^*_\forall^* = T^*_\forall \).

An immediate by-product of the above Theorem is that the model companion of a theory does not necessarily exist, but, if it does, it is unique and is its Kaiser hull.

**Theorem 3.18.** [14, Thm. 3.2.9] Assume \( T \) has a model companion \( T^* \). Then \( T^* \) is axiomatized by its \( \Pi_2 \)-consequences and is the Kaiser hull of \( T^*_\forall \).

Moreover \( T^* \) is the unique model companion of \( T \) and is characterized by the property of being the unique model complete theory \( S \) such that \( S^*_\forall = T^*_\forall \).

**Proof.** For quantifier free formulae \( \psi(\bar{x}, \bar{y}) \) and \( \phi(\bar{x}, \bar{y}) \) the assertion

\[
\forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y}) \leftrightarrow \forall \bar{y} \phi(\bar{x}, \bar{y})
\]

is a \( \Pi_2 \)-sentence.

Let \( T^{**} \) be the theory given by the \( \Pi_2 \)-consequences of \( T^* \).
Since $T^*$ is model complete, by Robinson’s test 3.14(c), for any $\Sigma_1$-formula $\exists \bar{y} \psi(\bar{x}, \bar{y})$ there is a universal formula $\forall \bar{y} \phi(\bar{x}, \bar{y})$ such that

$$\forall \bar{x} [\exists \bar{y} \psi(\bar{x}, \bar{y}) \iff \forall \bar{y} \phi(\bar{x}, \bar{y})]$$

is in $T^{**}$.

Again by Robinson’s test 3.14(c) $T^{**}$ is model complete.

Now assume $S$ is a model complete theory such that $S_\psi = T_\psi$. Clearly $T_\psi^{*} = T_\psi^{**} = T_\psi$.

By Robinson’s test 3.14(b) and Proposition 3.7(2), $S_\psi$ holds exactly in the $T_\psi$-ec models. Hence $T^* = T^{**}$ since $T^*_\psi = T^{**}_\psi$.

This shows that any model complete theory is axiomatized by its $\Pi_2$-consequences, that the model companion $T^*$ of $T$ is unique, that $T^*$ is also the Kaiser hull of $T$ (being axiomatized by the $\Pi_2$-sentences which hold in all $T$-ec-models), and is characterized by the property of being the unique model complete theory $S$ such that $T_\psi = S_\psi$. □

Thm. 3.18 provides an equivalent characterization of model companion theories (which is expressible by a $\Delta_0$-property in parameters $T$ and $T^*$, hence absolute for transitive models of $\text{ZFC}$).

We use the following criteria for model companionship in the proofs of Theorems 1, 4, 3.

**Lemma 3.19.** Let $T, T_0$ be $\tau$-theories with $T_0$ model complete. Assume that for every complete $\tau$-theory $S \supseteq T$ there is $M$ which models $T_0 + S_\psi$. Then:

1. $T^* = T_0 + T_\psi$ is the model companion of $T$.
2. $T^*$ is axiomatized by the the set of $\Pi_2$-sentences $\psi$ for $\tau$ such that $S_\psi \cup \{\psi\}$ is consistent for all complete $S \supseteq T$.
3. $T^*$ is axiomatized by the the set of $\Pi_2$-sentences $\psi$ for $\tau$ such that for all universal $\tau$-sentences $\theta T_\psi + \theta + \psi$ is consistent if and only if so is $T_\psi + \theta$.

**Proof.** By Fact 3.15 $T^*$ is model complete.

1. We need to show that any model of $T^*$ embeds into a model of $T$ and conversely. Assume $N$ models $T^*$. Then $N$ models $T_\psi$. By Lemma 3.3 there exists $M \supseteq N$ which models $T$. Conversely let $M$ model $T$ and $S$ be the $\tau$-theory of $M$. By assumption there is $N$ which models $T_0 + S_\psi$ (but this $N$ may not be a superstructure of $M$). Let $S^*$ be the $\tau$-theory of $N$. Then $S_\psi^* = S_\psi$, since $S_\psi$ and $S^*$ are $\Pi_1$-complete theories with $S_\psi^* \supseteq S_\psi$. Moreover $S^* \supseteq T^*$, since $S_\psi \supseteq T_\psi$.

**Claim 5.** The $\tau \cup M$-theory $S^* \cup \Delta_0(M)$ is consistent.

Assume the Claim holds, then $M$ is a $\tau$-substructure of a model of $S^* \supseteq T^*$ and we are done.

**Proof.** If not there is $\psi(\bar{a}) \in \Delta_0(M)$ such that $S^* \cup \{\psi(\bar{a})\}$ is inconsistent. This gives that

$$S^* \vdash \neg \psi(\bar{a}).$$

Since none of the constant in $\bar{a}$ occurs in $\tau$, we get that

$$S^* \vdash \forall \bar{x} \neg \psi(\bar{x}),$$

i.e. $\forall \bar{x} \neg \psi(\bar{x}) \in S_\psi^*$, $S_\psi$. But $M$ models $S_\psi$ and $\forall \bar{x} \neg \psi(\bar{x})$ fails in $M$; a contradiction. □

2. Assume $\psi \in T^*$ and $S$ is a complete extension of $T$, we must show that $S_\psi + \psi$ is consistent. By assumption there is $N$ which models $T^* + S_\psi$, and we are done.

3. Left to the reader.
Remark 3.20. We do not know whether the characterization of the model companion of $T$ given in Lemma 3.19(3) can be proved for all theories $T$ admitting a model companion: following the notation of the Lemma, it is conceivable that some $\tau$-theory $T$ has a model companion $T^*$ and there is some some univesal $\tau$-sentence $\theta$ such that for any model $M$ of $T\forall + \theta$ any superstructure of $M$ which models $T^*$ kills the truth of $\theta$. In this case no $\Pi_2$-sentence in the Kaiser hull of $T$ is consistent with the universal fragment of $T\forall + \theta$.

3.5. Is model companionship a tameness notion? Model completeness and model companionship are “tameness” notion for first order theories which must be handled with care.

Proposition 3.21. Given a signature $\tau$ consider the signature $\tau^*$ which adds an $n$-ary predicate symbol $R_\phi$ for any $\tau$-formula $\phi(x_1, \ldots, x_n)$ with displayed free variables.

Let $T_\tau$ be the following $\tau^*$-theory:

- $\forall \bar{x}(R_{\phi}(\bar{x}) \leftrightarrow R_\phi(\bar{x}))$ for all quantifier free $\tau$-formulae $\phi(\bar{x})$,
- $\forall \bar{x}[R_{\phi \land \psi}(\bar{x}) \leftrightarrow (R_\phi(\bar{x}) \land R_\psi(\bar{x}))]$ for all $\tau$-formulae $\phi(\bar{x}), \psi(\bar{x})$,
- $\forall \bar{x}[R_{\neg \phi}(\bar{x}) \leftrightarrow \neg R_\phi(\bar{x})]$ for all $\tau$-formulae $\phi(\bar{x})$,
- $\forall \bar{x}[\exists R_\phi(y, \bar{x}) \leftrightarrow R_{\exists \phi}(\bar{x})]$ for all $\tau$-formulae $\phi(\bar{x})$.

Then any $\tau$-structure $N$ admits a unique extension to a $\tau^*$-structure $N^*$ which models $T_\tau$. Moreover every $\tau^*$-formula is $T_\tau$-equivalent to an atomic $\tau^*$-formula. In particular for any $\tau$-model $N$, the algebras of its $\tau$-definable subsets and of the $\tau^*$-definable subsets of $N^*$ are the same.

Therefore for any consistent $\tau$-theory $T$, $T \cup T_\tau$ is consistent and admits quantifier elimination, hence is model complete.

Proof. By an easy induction one can prove that any $\tau$-formula $\phi(\bar{x})$ is $T_\tau$-equivalent to the atomic $\tau^*$-formula $R_\phi(\bar{x})$.

Another simple inductive argument brings that any $\tau^*$-formula $\phi(\bar{x})$ is $T_\tau$-equivalent to the $\tau$-formula obtained by replacing all symbols $R_\psi(\bar{x})$ occurring in $\phi$ by the $\tau$-formula $\psi(\bar{x})$. Combining these observations together we get that any $\tau^*$-formula is equivalent to an atomic $\tau^*$-formula.

$T_\tau$ forces the $M^*$-interpretation of any relation symbol $R_\phi(\bar{x})$ in $\tau^* \setminus \tau$ to be the $M$-interpretation of the $\tau$-formula $\phi(\bar{x})$ to which it is $T_\tau$-equivalent. □

Observe that the expansion of the language from $\tau$ to $\tau^*$ behaves well with respect to several model theoretic notions of tameness distinct from model completeness: for example $T$ is a stable $\tau$-theory if and only if so is the $\tau^*$-theory $T \cup T_\tau$, the same holds for $\Pi_2$-theories, or for $\alpha$-minimal theories, or for $\kappa$-categorical theories.

The passage from $\tau$-structures to $\tau^*$-structures which model $T_\tau$ can have effects on the embeddability relation; for example assume $M \subseteq N$ is a non-elementary embedding of $\tau$-structures; then $M^* \nsubseteq N^*$: if the non-atomic $\tau$-formula $\phi(\bar{a})$ in parameter $\bar{a} \in M^{<\omega}$ holds in $M$ and does not hold in $N$, the atomic $\tau^*$-formula $R_\phi(\bar{a})$ holds in $M^*$ and does not hold in $N^*$.

However if $T$ is a model complete $\tau$-theory, then for $M \subseteq N$ $\tau$-models of $T$, we get that $M \prec N$: this entails that $M^* \nsubseteq N^*$, which (by the quantifier elimination of $T \cup T_\tau$) gives that $M^* \prec N^*$. In particular for a model complete $\tau$-theory $T$ and $M, N$ $\tau$-models of $T$, $M \subseteq N$ if and only if $M^* \subseteq N^*$.

Let us now investigate the case of model companionship. If $T$ is the model companion of $S$ with $S \neq T$ in the signature $\tau$, $T \cup T_\tau$ and $S \cup T_\tau$ are both model complete theories in the signature $\tau^*$. But $T \cup T_\tau$ cannot be the model companion of $S \cup T_\tau$, by uniqueness of the model companion, since each of these theories is the model companion of itself and they are distinct. Moreover if $T$ and $S$ are also complete, no $\tau^*$-model of $S \cup T_\tau$ can embed into a $\tau^*$-model of $T \cup T_\tau$: since $T$ is the model companion of $S$ and $S \neq T$, $T_\forall = S_\forall$.
and there is some \( \Pi_2 \)-sentence \( \psi \) \( \forall x \exists y \phi(x, y) \) with \( \phi \)-quantifier free in \( T \setminus S \). Therefore \( \forall x R_3 \psi(x) \in (T \cup T_\tau)_\psi \setminus (S \cup T_\tau)_\psi \); we conclude by Lemma 3.2, since \( T \cup T_\tau \) and \( S \cup T_\tau \) are complete, hence the above sentence separates \( (T \cup T_\tau)_\psi \) from \( (S \cup T_\tau)_\psi \).

3.6. **Summing up.** The results of this section gives that for any \( \tau \)-theory \( T \):

- The universal fragment of \( T \) describes the family of substructures of models of \( T \), and the \( \tau \)-ec models realize all \( \Pi_2 \)-sentences which are “absolutely” consistent with \( T_\psi \) (i.e. consistent with the universal fragment of any extension of \( T \)).
- Model companionship and model completeness describe (almost all) the cases in which the family of \( \Pi_2 \)-sentences which are “absolutely” consistent with \( T \) (as defined in the previous item) describes the elementary class given by the \( \tau \)-ec structures.
- One can always extend \( \tau \) to a signature \( \tau^* \) so that \( T \) has a conservative extension to a \( \tau^* \)-theory \( T^* \) which is model complete, but this process may be completely uninformative since it may completely destroy the substructure relation existing between \( \tau \)-models of \( T \) (unless \( T \) is already model complete).
- On the other hand for certain theories \( T \) (as the axiomatizations of set theory considered in the present paper), one can unfold their “tameness” by carefully extending \( \tau \) to a signature \( \tau^* \) in which only certain \( \tau \)-formulae are made equivalent to atomic \( \tau^* \)-formulae. In the new signature \( T \) can be extended to a conservative extension \( T^* \) which has a model companion \( T^*_\psi \), while this process has mild consequences on the \( \tau^* \)-substructure relation for models of \( T^*_{\psi} \) (i.e. for the pairs of interest of \( \tau \)-models \( M_0 \subseteq M_1 \) of a suitable fragment of \( T \), their unique extensions to \( \tau^* \)-models \( M^*_0 \) are still models of \( T^*_{\psi} \) and maintain that \( M^*_0 \subseteq M^*_1 \) also for \( \tau^* \)). This gives useful structural information on the web of relations existing between \( \tau^* \)-models of \( T^*_\psi \) (as outlined by Theorems 1, 4, 3).
- Our conclusion is that model completeness and model companionship are tameness properties of elementary classes \( \mathcal{E} \) defined by a theory \( T \) rather than of the theory \( T \) itself: these model-theoretic notions outline certain regularity patterns for the substructure relation on models of \( \mathcal{E} \), patterns which may be unfolded only when passing to a signature distinct from the one in which \( \mathcal{E} \) is first axiomatized (much the same way as it occurs for Birkhoff’s characterization of algebraic varieties in terms of universal theories).
- The results of the present paper shows that if we consider set theory together with large cardinal axioms as formalized in the signature \( \sigma_\omega, \sigma_{\omega_1}, \sigma_{\omega_1}^{\text{NS}_{\omega_1}}, \sigma_{\omega_1} \), we obtain (until now unexpected) tameness properties for this first order theory, properties which couple perfectly with well known (or at least published) generic absoluteness results. We do not have an abstract model theoretic justification for selecting these signatures out of the continuum many signatures which produce definable extensions of \( \text{ZFC} \). However the common practice of set theory (independently of our results) already motivate our choice, and our results validate it.

4. Auxiliary results

We collect here auxiliary results needed to prove Theorems 1 and 3. We prove all these results working in “standard” models of \( \text{ZFC} \), i.e. we assume the models are well-founded. This is a practice we already adopted in Section 2. We leave to the reader to remove this unnecessary assumption.

4.1. **Generalizations of Levy absoluteness.** We start with a natural generalization of Levy’s absoluteness we used in the proof of Thm. 3.

**Lemma 4.1.** Let \( \kappa \) be an infinite cardinal and \( A \) be any family of subsets of \( \bigcup_{n \in \omega} \mathcal{P}(\kappa)^n \). Let \( \tau_A = \tau_{ST} \cup A \).
Then:

\[(H_{\kappa^+}^V, \tau_A^V) \prec_{\Sigma_1} (V, \tau_A^V).\]

**Proof.** Assume for some \(\tau_A\)-formula \(\phi(x, y)\) without quantifiers\(^{26}\) and \(\bar{a} \in H_{\kappa^+}\)

\[(V, \tau_A^V) \models \exists y \phi(\bar{a}, y).\]

Let \(\alpha > \kappa\) be large enough so that for some \(b \in V_\alpha\)

\[(V, \tau_A^V) \models \phi(\bar{a}, b).\]

Then

\[(V_\alpha, \tau_A^V) \models \phi(\bar{a}, b).\]

Let \(A_1, \ldots, A_k\) be the subsets of \(\mathcal{P}(\kappa)^k\) which are the predicates mentioned in \(\phi\). By the downward Lowenheim-Skolem theorem, we can find \(X \subseteq V_\alpha\) which is the domain of a \(\tau_{A_1, \ldots, A_k}\)-elementary substructure of

\[(V_\alpha, \tau_{ST}, A_1, \ldots, A_k)\]

such that \(X\) is a set of size \(\kappa\) containing \(\kappa\) and such that \(A_1, \ldots, A_k, \kappa, b, \bar{a} \in X\). Since \(|X| = \kappa \subseteq X\), a standard argument shows that \(H_{\kappa^+} \cap X\) is a transitive set, and that \(\kappa^+\) is the least ordinal in \(X\) which is not contained in \(X\). Let \(M\) be the transitive collapse of \(X\) via the Mostowski collapsing map \(\pi_X\).

We have that the first ordinal moved by \(\pi_X\) is \(\kappa^+\) and \(\pi_X\) is the identity on \(H_{\kappa^+} \cap X\). Therefore \(\pi_X(a) = a\) for all \(a \in H_{\kappa^+} \cap X\). Moreover for \(A \subseteq \mathcal{P}(\kappa)^n\) in \(X\)

\[(3) \quad \pi_X(A) = A \cap M.\]

We prove equation (3):

**Proof.** Since \(X \cap V_{\kappa+1} \subseteq X \cap H_{\kappa^+}\), \(\pi_X\) is the identity on \(X \cap H_{\kappa^+}\), and \(A \subseteq V_{\kappa+1}\), we get that

\[\pi_X(A) = \pi_X[A \cap X] = \pi_X[A \cap X \cap V_{\kappa+1}] = A \cap M \cap V_{\kappa+1} = A \cap M.\]

\(\square\)

It suffices now to show that

\[(4) \quad (M, \tau_{ST}^V, \pi_X(A_1), \ldots, \pi_X(A_k)) \subseteq (H_{\kappa^+}, \tau_{ST}^V, A_1, \ldots, A_k).\]

Assume 4 holds; since \(\pi_X\) is an isomorphism and \(\pi_X(A_j) = \pi_X[A_j \cap X]\), we get that

\[(M, \tau_{ST}^V, \pi_X(A_1), \ldots, \pi_X(A_k)) \models \phi(\pi_X(b), \bar{a})\]

since

\[(X, \tau_{ST}^V, A_1 \cap X, \ldots, A_k \cap X) \models \phi(b, \bar{a}).\]

By (4) we get that

\[(H_{\kappa^+}, \tau_{ST}^V, \pi_X(A_1), \ldots, \pi_X(A_k)) \models \phi(\pi_X(b), \bar{a})\]

and we are done.

We prove (4): since \(M\) is transitive, any atomic \(\tau_{ST}\)-formula (i.e. any \(\Delta_0\)-property) holds true in \(M\) if and if it holds in \(H_{\kappa^+}\). It remains to argue that the same occurs for the \(\tau_{A_j}\)-formulae of type \(A_j(x)\), i.e. that \(A_j \cap M = \pi_X(A_j)\) for all \(j = 1, \ldots, n\); which is the case by (3). \(\square\)

**Remark 4.2.** Key to the proof is the fact that subsets of \(\kappa\) have bounded rank below \(\kappa^+\). If \(A \subseteq H_{\kappa^+}\) has elements of unbounded rank, the equality \(\pi_X(A) = A \cap M\) may fail: for example if \(A = H_{\kappa^+}, \pi_X(A) = H_{\kappa^+} \cap X\) while \(A \cap M = M\). This shows that 4 fails for this choice of \(A\).

\(^{26}\)A quantifier-free \(\tau_{A_1, \ldots, A_k}\)-formula is a boolean combination of atomic \(\tau_{ST}\)-formulae with formulae of type \(A_j(x)\). For example \(\exists x \in y A(y)\) is not a quantifier-free \(\tau_{ST}\)-formula, and is actually equivalent to the \(\Sigma_1\)-formula \(\exists x (x \in y) \land A(y)\).
4.2. Universally Baire sets and generic absoluteness for second order number theory. We collect here the properties of universally Baire sets and the generic absoluteness results for second order number theory we need to prove Thm. 1.

Notation 4.3. \( A \subseteq \bigcup_{n \in \omega} \mathcal{P}(\kappa)^n \) is projectively closed if it is closed under projections, finite unions, complementation, and permutations (if \( \sigma : n \to n \) is a permutation and \( A \subseteq \mathcal{P}(\kappa)^n \), \( \delta[A] = \{(a_\sigma(0)), \ldots, a_{\sigma(n-1)} : (a_0, \ldots, a_{n-1}) \in A\} \)).

Otherwise said, \( A \) is the class of lightface definable subsets of some signature on \( \mathcal{P}(\kappa) \).

4.3. Universally Baire sets. Assuming large cardinals there is a very large sample of projectively closed families of subsets of \( \mathcal{P}(\omega) \) which are are “simple”, hence it is natural to consider elements of these families as atomic predicates.

The exact definition of what is meant by a “simple” subset of \( 2^\omega \) is captured by the notion of universally Baire set.

Given a topological space \((X, \tau)\), \( A \subseteq X \) is nowhere dense if its closure has a dense complement, meager if it is the countable union of nowhere dense sets, with the Baire property if it has meager symmetric difference with an open set. Recall that \((X, \tau)\) is Polish if \( \tau \) is a completely metrizable, separable topology on \( X \).

Definition 4.4. (Feng, Magidor, Woodin) Given a Polish space \((X, \tau)\), \( A \subseteq X \) is universally Baire if for every compact Hausdorff space \((Y, \sigma)\) and every continuous \( f : Y \to X \) we have that \( f^{-1}[A] \) has the Baire property in \( Y \).

\( \text{UB} \) denotes the family of universally Baire subsets of \( X \) for some Polish space \( X \).

We adopt the convention that \( \text{UB} \) denotes the class of universally Baire sets and of all elements of \( \bigcup_{n \in \omega+1} (2^\omega)^n \) (since the singleton of such elements are universally Baire sets).

The theorem below outlines three simple examples of projectively closed families of universally Baire sets containing \( 2^\omega \).

Theorem 4.5. Let \( T_0 \) be the \( \tau_{ST} \)-theory \( \text{ZFC}^* \)-there are infinitely many Woodin cardinals and a measurable above and \( T_1 \) be the \( \tau_{ST} \)-theory \( \text{ZFC}^* \)-there are class many Woodin cardinals.

1. [13, Thm. 3.1.12, Thm. 3.1.19] Assume \( V \) models \( T_0 \). Then every projective subset of \( 2^\omega \) is universally Baire.

2. [13, Thm. 3.3.3, Thm. 3.3.5, Thm. 3.3.6, Thm. 3.3.8, Thm. 3.3.13, Thm. 3.3.14] Assume \( V \models T_1 \). Then \( \text{UB} \) is projectively closed.

To proceed further we now list the standard facts about universally Baire sets we will need:

1. [9, Thm. 32.22] \( A \subseteq 2^\omega \) is universally Baire if and only if for each forcing notion \( P \) there are trees \( T_A, S_A \) on \( \omega \times \delta \) for some \( \delta > |P| \) such that \( A = p[[T_A]] \) (where \( p : (2 \times \kappa)^\omega \to 2^\omega \) denotes the projection on the first component and \([T] \) denotes the body of the tree \( T \)), and

\[ P \Vdash T_A \text{ and } S_A \text{ project to complements,} \]

by this meaning that for all \( G \) \( V \)-generic for \( P \)

\[ V[G] \models (p[[T_A]] \cap p[[S_A]] = \emptyset) \land (p[[T_A]] \cup p[[S_A]] = (2^\omega)^V[G]) \]

2. Any two Polish spaces \( X, Y \) of the same cardinality are Borel isomorphic [10, Thm. 15.6].

3. Any Polish space is Borel isomorphic to a Borel subset of \([0,1]^{\omega} \) [10, Thm. 4.14], hence also to a Borel subset of \( 2^\omega \) (by the previous item).

4. Given \( \phi : \mathbb{N} \to \mathbb{N} \), \( \prod_{n \in \omega} 2^{\phi(n)} \) is Polish (it is actually homemomorphic to the union of \( 2^\omega \) with a countable Hausdorff space) [10, Thm. 6.4, Thm. 7.4].
Hence it is not restrictive to focus just on universally Baire subsets of $2^\omega$ and of its countable products, which is what we will do in the sequel.

**Notation 4.6.** Given $G$ a $V$-generic filter for some forcing $P \in V$, $A \in \mathcal{UB}^{V[G]}$ and $H$ $V[G]$-generic filter for some forcing $Q \in V[G],$

$$A^{V[G][H]} = \left\{ r \in (2^n)^{V[G][H]} : V[G][H] \models r \in p[[T_A]] \right\},$$

where $(T_A, S_A) \in V[G]$ is any pair of trees as given in item 1 above such that $p[[T_A]] = A$ holds in $V[G]$, and $(T_A, S_A)$ project to complements in $V[G][H]$.

4.4. **Generic absoluteness for second order number theory.** We decide to include a full proof of Woodin’s generic absoluteness results for second order number theory we used, it follows readily from [13, Thm. 3.1.2] and the assumptions that there exists class many Woodin limits of Woodin, we reduce these large cardinal assumptions to the existence of some of Woodin’s result. The theorem below is an improvement of [17, Thm. 3.1].

**Theorem 4.7.** Assume in $V$ there are class many Woodin cardinals. Let $\mathcal{A} \in V$ be a family of universally Baire sets of $V$ and $\tau_\mathcal{A} = \tau_S,T \cup \mathcal{A}$. Let $G$ be $V$-generic for some forcing notion $P \in V$.

Then

$$(H_{\omega_1}, \tau_\mathcal{A}^V) \prec (H_{\omega_1}^{V[G]}, \tau_S^{V[G]}), A^{V[G]} : A \in \mathcal{A}).$$

**Proof.** We proceed by induction on $n$ to prove the following stronger assertion

**Claim 6.** Whenever $G$ is $V$-generic for some forcing notion $P \in V$ and $H$ is $V[G]$-generic for some forcing notion $Q$ in $V[G]$,

$$(H_{\omega_1}^{V[G]}, \tau_S^{V[G]}, A^{V[G]} : A \in \mathcal{A}) \prec_n (H_{\omega_1}^{V[G][H]}, \tau_S^{V[G][H]}, A^{V[G][H]} : A \in \mathcal{A}).$$

**Proof.** It is not hard to check that for all $A \in \mathcal{A}$, $A^{V[G]} = A^{V[G][H]} \cap V[G]$ (choose in $V$ a pair of trees $(T, S)$ such that $A = p[[T]]$ and the pair $(T, S)$ projects to complements in $V[G][H]$, and therefore also in $V[G]$). Therefore $(H_{\omega_1}^{V[G]}, \tau_S^{V[G]}, A^{V[G]} : A \in \mathcal{A})$ is a $\tau_A$-substructure of $(H_{\omega_1}^{V[G][H]}, \tau_S^{V[G][H]}, A^{V[G][H]} : A \in \mathcal{A}).$

This proves the base case of the induction.

We prove the successor step.

Assume that for any $G$ $V$-generic for some forcing $P \in V$ and $H$ $V[G]$-generic for some forcing $Q \in V[G]$,

$$(H_{\omega_1}^{V[G]}, \tau_S^{V[G]}, A^{V[G]} : A \in \mathcal{A}) \prec_n (H_{\omega_1}^{V[G][H]}, \tau_S^{V[G][H]}, A^{V[G][H]} : A \in \mathcal{A}).$$

Fix $\hat{G}$ and $\hat{H}$ as in the assumptions of the Claim as witnessed by forcings $\hat{P} \in V$ and $\hat{Q} \in V[\hat{G}]$.

We want to show that

$$(H_{\omega_1}^{V[G]}, \tau_S^{V[G]}, A^{V[G]} : A \in \mathcal{A}) \prec_{n+1} (H_{\omega_1}^{V[G][\hat{P}]}, \tau_S^{V[G][\hat{P}]}, A^{V[G][\hat{P}]} : A \in \mathcal{A}).$$

Let $\gamma$ be a Woodin cardinal of $V$ such that $\hat{P} \ast \hat{Q} \in V_\gamma$ (where $\hat{Q} \in V^{\hat{P}}$ is chosen so that $\hat{Q}_G = \hat{Q}$).

Then $\gamma$ is also Woodin in $V[\hat{G}]$. Let $K$ be $V[\hat{G}]$-generic for $^{27}(T_\gamma^{\omega_1})^{V[\hat{G}]}$ with $\hat{H} \in V[K]$, so that $V[G][K] = V[G][\hat{H}][\hat{K}]$ for some $\hat{K} \in V[\hat{G}][K]$.  

[^27]: $T_\gamma^{\omega_1}$ denotes here the countable tower of height $\gamma$ denoted as $Q_{<\gamma}$ in [13, Section 2.7].
Hence we have the following diagram:

\[
\begin{array}{c}
(H_{\omega_1}^{V[G]}, \tau^{V[G]}_{ST}, A^{V[G]} : A \in \mathcal{A}) \\
\xrightarrow{\Sigma_n} (H_{\omega_1}^{V[G][K]}, \tau^{V[G][K]}_{ST}, A^{V[G][K]} : A \in \mathcal{A}) \\
\xrightarrow{\Sigma_n} (H_{\omega_1}^{V[G][H]}, \tau^{V[G][H]}_{ST}, A^{V[G][H]} : A \in \mathcal{A})
\end{array}
\]

obtained by inductive hypothesis applied both on \(V[G]\), \(V[G][H]\) and on \(V[G][H][K]\), \(V[G][H][K]\), and using the fact that \((H_{\omega_1}^{V[G][K]}, \tau^{V[G][K]}_{UB})\) is a fully elementary superstructure of \((H_{\omega_1}^{V[G]}, \tau^{V[G]}_{UB})\) [13, Thm. 2.7.7, Thm. 2.7.8].

Let \(\phi \equiv \exists x \psi(x)\) be any \(\Sigma_{n+1}\) formula for \(\tau_{\mathcal{A}}\) with parameters in \(H_{\omega_1}^{V[G]}\). First suppose that \(\phi\) holds in \((H_{\omega_1}^{V[G]}, \tau^{V[G]}_{ST}, A^{V[G]} : A \in \mathcal{A})\), and fix \(\bar{a} \in V[G]\) such that \(\psi(\bar{a})\) holds in \((H_{\omega_1}^{V[G]}, \tau^{V[G]}_{ST}, A^{V[G]} : A \in \mathcal{A})\). Since

\[
(H_{\omega_1}^{V[G]}[\bar{a}], \tau^{V[G]}_{ST}, A^{V[G]}[\bar{a}] : A \in \mathcal{A}) \prec_n (H_{\omega_1}^{V[G][\bar{a}][H]}, \tau^{V[G][\bar{a}][H]}_{ST}, A^{V[G][\bar{a}][H]} : A \in \mathcal{A}),
\]

we conclude that \(\psi(\bar{a})\) holds in \((H_{\omega_1}^{V[G][\bar{a}][H]}, \tau^{V[G][\bar{a}][H]}_{ST}, A^{V[G][\bar{a}][H]} : A \in \mathcal{A})\), hence so does \(\phi\).

Now suppose that \(\phi\) holds in \((H_{\omega_1}^{V[G][\bar{a}][H]}, \tau^{V[G][\bar{a}][H]}_{ST}, A^{V[G][\bar{a}][H]} : A \in \mathcal{A})\) as witnessed by \(\bar{a} \in H_{\omega_1}^{V[G][\bar{a}][H]}\).

Since

\[
(H_{\omega_1}^{V[G][\bar{a}][H]}, \tau^{V[G][\bar{a}][H]}_{ST}, A^{V[G][\bar{a}][H]} : A \in \mathcal{A}) \prec_n (H_{\omega_1}^{V[G][\bar{a}][K]}, \tau^{V[G][\bar{a}][K]}_{ST}, A^{V[G][\bar{a}][K]} : A \in \mathcal{A}),
\]

it follows that \(\psi(\bar{a})\) holds in \((H_{\omega_1}^{V[G][\bar{a}][K]}, \tau^{V[G][\bar{a}][K]}_{ST}, A^{V[G][\bar{a}][K]} : A \in \mathcal{A})\), hence so does \(\phi\). Since

\[
(H_{\omega_1}^{V[G]}, \tau^{V[G]}_{ST}, A^{V[G]} : A \in \mathcal{A}) \prec (H_{\omega_1}^{V[G][\bar{a}][K]}, \tau^{V[G][\bar{a}][K]}_{ST}, A^{V[G][\bar{a}][K]} : A \in \mathcal{A}),
\]

the formula \(\phi\) holds also in \((H_{\omega_1}^{V[G][\bar{a}][K]}, \tau^{V[G][\bar{a}][K]}_{ST}, A^{V[G][\bar{a}][K]} : A \in \mathcal{A})\).

Since \(\phi\) is arbitrary, this shows that

\[
(H_{\omega_1}^{V[G][\bar{a}][K]}, \tau^{V[G][\bar{a}][K]}_{ST}, A^{V[G][\bar{a}][K]} : A \in \mathcal{A}) \prec_{n+1} (H_{\omega_1}^{V[G][\bar{a}][H][K]}, \tau^{V[G][\bar{a}][H][K]}_{ST}, A^{V[G][\bar{a}][H][K]} : A \in \mathcal{A}),
\]

concluding the proof of the inductive step for \(\bar{G}\) and \(\bar{H}\).

Since we have class many Woodin, this argument is modular in \(\bar{G}, \bar{H}\) as in the assumptions of the inductive step, because we can always find some Woodin cardinal \(\gamma\) of \(V\) which remains Woodin in \(V[G]\) and is of size larger than the poset in \(V[G]\) for which \(\bar{H}\) is \(V[G]\)-generic. The proof of the inductive step is completed. 

5. Further results

We introduce without a few comments the results whose proof is deferred to a second paper, together with the relevant terminology and definitions. The following supplements Notation 2.

**Notation 3.**

- \(\tau_{\mathcal{NS}_{\omega_1}}\) is the signature \(\tau_{ST} \cup \{\omega_1\} \cup \{\mathcal{NS}_{\omega_1}\}\) with \(\omega_1\) a constant symbol, \(\mathcal{NS}_{\omega_1}\) a unary predicate symbol.
- \(T_{\mathcal{NS}_{\omega_1}}\) is the \(\tau_{\mathcal{NS}_{\omega_1}}\)-theory given by \(T_{ST}\) together with the axioms

\[
\omega_1 \text{ is the first uncountable cardinal,}
\]

\[
\forall x \ [(x \subseteq \omega_1 \text{ is non-stationary}) \leftrightarrow \mathcal{NS}_{\omega_1}(x)].
\]
• \( \text{ZFC}_{\text{NS}_{\omega_1}}^- \) is the \( \tau_{\text{NS}_{\omega_1}} \)-theory
\[ \text{ZFC}_{\text{ST}}^- + T_{\text{NS}_{\omega_1}}. \]

• Accordingly we define \( \text{ZFC}_{\text{NS}_{\omega_1}}^- \).

**Theorem 5.** Let \( V = (V, \in) \) be a model of
\[ \text{ZFC} + \text{MAX}(\text{UB}) + \text{there is a supercompact cardinal and class many Woodin cardinals,} \]
and \( \text{UB} \) denote the family of universally Baire sets in \( V \).

**TFAE**
(1) \( (V, \in) \) models \( (\ast)\)-UB;
(2) \( \text{NS}_{\omega_1} \) is precipitous\(^{28} \) and the \( \tau_{\text{NS}_{\omega_1}} \cup \text{UB}\)-theory of \( V \) has as model companion the \( \tau_{\text{NS}_{\omega_1}} \cup \text{UB}\)-theory of \( H_{\omega_2} \).

Here is the definition of \( \text{MAX}(\text{UB}) \) and \( (\ast)\)-UB:

**Definition 1.** \( \text{MAX}(\text{UB}) \): There are class many Woodin cardinals in \( V \), and for all \( G \) \( V \)-generic for some forcing notion \( P \in V \):

(1) Any subset of \( (2^{\omega_1})^V[G] \) definable in \( (H_{\omega_1}^V[G] \cup \text{UB}^V[G], \in) \) is universally Baire in \( V[G] \).
(2) Let \( H \) be \( V[G]\)-generic for some forcing notion \( Q \in V[G] \). Then\(^ {29} \):
\[ (H_{\omega_1}^V[G] \cup \text{UB}^V[G], \in) \preceq (H_{\omega_1}^V[G]^H \cup \text{UB}^V[G]^H, \in). \]

\( \text{MAX}(\text{UB}) \) is a form of sharp for the universally Baire sets (a slight weakening of the conclusion of [13, Thm. 4.17]). It holds in any forcing extension of \( V \) where a supercompact of \( V \) becomes countable. We will comment in details on \( \text{MAX}(\text{UB}) \) in the sequel of this paper.

See [12] for a definition of \( P_{\text{max}} \) and [13, Section 1.6, pag. 39] for a definition of saturated ideal on \( \omega_1 \).

**Definition 2.** Let \( A \) be a family of dense subsets of \( P_{\text{max}} \).

• \( (\ast)\)-\( A \) holds if \( \text{NS}_{\omega_1} \) is saturated and there exists a filter \( G \) on \( P_{\text{max}} \) meeting all the dense sets in \( A \).
• \( (\ast)\)-\( \text{UB} \) holds if \( \text{NS}_{\omega_1} \) is saturated and there exists an \( L(\text{UB})\)-generic filter \( G \) on \( P_{\text{max}} \).

Woodin’s definition of \( (\ast) \) [12, Def. 7.5] is equivalent to \( (\ast)\)-\( A \) + there are class many Woodin cardinals for \( A \) the family of dense subsets of \( P_{\text{max}} \) existing in \( L(\mathbb{R}) \).

**Theorem 6.** Assume \( V \) models that there are class many Woodin cardinals and \( \text{UB} \) is the family of universally Baire sets in \( V \). Then the \( \Pi_1 \)-theory of \( V \) for the language \( \tau_{\text{NS}_{\omega_1}} \cup \text{UB} \) is invariant under set sized forcings.

**Notation 4.**

• Given a family \( A \) of predicate symbols:
\[ \sigma_A = \tau_{\text{ST}} \cup A, \]

\(^{28}\) See [13, Section 1.6, pag. 41] for a definition of precipitousness and a discussion of its properties. A key observation is that \( \text{NS}_{\omega_1} \) being precipitous is independent of \( \text{CH} \) (see for example [13, Thm. 1.6.24]), while \( (\ast)\)-\( \text{UB} \) entails \( 2^\kappa = \kappa^+ \) (for example by the results of [12, Section 6]). Another key point is that we stick to the formulation of \( P_{\text{max}} \) as in [12] so to be able in its proof to quote verbatim from [12] all the relevant results on \( P_{\text{max}}\)-preconditions we will use. It is however possible to develop \( P_{\text{max}} \) focusing on Woodin’s countable tower rather than on the precipitousness of \( \text{NS}_{\omega_1} \) to develop the notion of \( P_{\text{max}}\)-precondition. Following this approach in all its scopes, one should be able to reformulate Thm. 5(2) omitting the request that \( \text{NS}_{\omega_1} \) is precipitous. We do not explore this venue any further neither here nor in the sequel of this paper.

\(^{29}\) Elementarity is witnessed via the map defined by \( A \mapsto A^{V[G][H]} \) for \( A \in \text{UB}^V[G] \) and the identity on \( H_{\omega_1}^{V[G]} \) (See Notation 4.6 for the definition of \( A^{V[G][H]} \)).
\[ - \sigma_{A, \text{NS}_{\omega_1}} = \tau_{\text{NS}_{\omega_1}} \cup A, \]
\[ - \sigma_\omega \text{ is } \sigma_A \text{ for } A = \sigma_{ST}, \]
\[ - \sigma_{\omega, \text{NS}_{\omega_1}} \text{ is } \sigma_{A, \text{NS}_{\omega_1}} \text{ for } A = \sigma_{ST}. \]

- Let UB denote the family of universally Baire sets, and L(UB) denote the smallest transitive model of ZF which contains UB.

\[ T_{l\text{-UB}} \text{ is the } \sigma_{\omega, \text{NS}_{\omega_1}} \text{-theory given by the axioms} \]
\[ \forall x_1 \ldots x_n [S_\psi(x_1, \ldots, x_n) \leftrightarrow (\bigwedge_{i=1}^n x_i \subseteq \omega^{<\omega} \wedge \psi^{L(UB)}(x_1, \ldots, x_n))] \]

as \( \psi \) ranges over the \( \epsilon \)-formulae.

- \( ZFC_{l-UB}^- \) is the \( \sigma_\omega \)-theory

\[ \text{ZFC}_{ST}^- \cup T_{l\text{-UB}}; \]

- \( ZFC_{l-UB, \text{NS}_{\omega_1}}^- \) is the \( \sigma_{\omega, \text{NS}_{\omega_1}} \)-theory

\[ \text{ZFC}_{\text{NS}_{\omega_1}}^- \cup T_{l\text{-UB}}; \]

- Accordingly we define \( ZFC_{\text{NS}_{\omega_1}}, ZFC_{l\text{-UB}}, ZFC_{l\text{-UB, NS}_{\omega_1}} \).

**Theorem 7.** Let \( T \) be any \( \sigma_{\omega, \text{NS}_{\omega_1}} \)-theory extending

\[ ZFC_{l\text{-UB, NS}_{\omega_1}}^- + \text{MAX}(UB) \]

There is a supercompact cardinal and class many Woodin cardinals.

Then \( T \) has a model companion \( T^* \).

Moreover TFAE for any for any \( \Pi_2 \)-sentence \( \psi \) for \( \sigma_{\omega, \text{NS}_{\omega_1}} \):

- (A) \( T^* \vdash \psi \).
- (B) For any complete theory \( S \supseteq T \), \( S \vdash \text{consistent} \).
- (C) \( T \vdash \exists P (P \text{ is a partial order } \wedge \forall P \psi^H_{\omega^2}). \)
- (D) \( T_V + ZFC_{l\text{-UB, NS}_{\omega_1}}^- + \text{MAX}(UB) + (\ast)\text{-UB} \vdash \psi^H_{\omega^2}. \)
- (E) \( T \vdash (\mathbb{P}^{\text{max}} \vdash \psi^H_{\omega^2})^{L(UB)}. \)

We immediately obtain Thm. 2 as a corollary of Thm. 7 and Thm. 6:

**Proof.** Note that every lightface projective set is in \( L(UB) \) (since the quantifier defining the set range over \( P(\omega) \subseteq L(UB) \)); hence we can assume that \( ZFC_{\omega, \text{NS}_{\omega_1}}^* \) is a fragment of \( ZFC_{l\text{-UB, NS}_{\omega_1}}^- \) the interpretation of \( S_\theta \) according to \( ZFC_{\omega, \text{NS}_{\omega_1}}^* \) is the same of \( S_{\theta^{\omega, \omega}} \) according to \( ZFC_{\omega, \text{NS}_{\omega_1}}^* \) which has the same interpretation of \( S_{\theta^{\omega, \omega}} \) according to \( ZFC_{l\text{-UB, NS}_{\omega_1}}^* \). Therefore a \( \Pi_2 \)-sentence for \( \sigma_\omega \) in the theory \( ZFC_{\omega, \text{NS}_{\omega_1}}^* \) can be regarded as a \( \Pi_2 \)-sentence also for the theory \( ZFC_{l\text{-UB, NS}_{\omega_1}}^- \).

(3) implies (2): If \( P \) forces \( MM^{++} \), by Asper\ö and Schindler’s result, \( P \vdash (\ast)\text{-UB} \); hence \( P \vdash \psi^H_{\omega^2} \) by (3).

(2) implies (1): Given some complete \( S \supseteq T \), and a model \( M \) of \( S \), find \( N \) forcing extension of \( M \) which models \( \psi^H_{\omega^2} \). By Thm. 6 and Lemma 4.1, \( H^N_{\omega^2} \models S_V \) and we are done.

(1) implies (3): assume \( \mathcal{M} \) models

\[ T_V + ZFC_{\omega, \text{NS}_{\omega_1}}^* + \theta_{SC} + (\ast)\text{-UB}; \]

find \( N \) forcing extension of \( \mathcal{M} \) which models

\[ T_V + ZFC_{\omega, \text{NS}_{\omega_1}}^* + \text{MAX}(UB). \]
By Thm. 6 and (1), $\psi$ is consistent with the $\Pi_1$-theory of $\mathcal{N}$. By the equivalence of (A) with (D) of Thm. 3 applied to the $\Pi_1$-complete theory of $\mathcal{N}$, we get that $\mathcal{N}$ models $\psi^{\mathcal{H}_2}$ is forcible by $\mathbb{P}_{\text{max}}$ over $L(\mathcal{UB})$. Since all the universally Baire predicates appearing in $\psi$ are projective and lightface definable, $\mathcal{N}$ models $\psi^{\mathcal{H}_2}$ is forcible by $\mathbb{P}_{\text{max}}$ over $L(\mathbb{R})$. Since $L(\mathbb{R})^\mathcal{M}$ and $L(\mathbb{R})^\mathcal{N}$ are elementarily equivalent (without any need to appeal to $\text{MAX}(\mathcal{UB})$, but just to $\theta_{\text{SC}}$ and [13, Thm. 3.1.2]), we get that $\mathcal{M}$ models $\psi^{\mathcal{H}_2}$ is forcible by $\mathbb{P}_{\text{max}}$ over $L(\mathbb{R})$. Since $\mathcal{M} \models (\ast)\text{-}\mathcal{UB}$, we conclude that $\psi^{\mathcal{H}_2}$ holds in $\mathcal{M}$.

□

References

[1] D. Asperó and R. Schindler. $\text{MM}^{++}$ implies $(\ast)$. https://arxiv.org/abs/1906.10213, 2019.
[2] David Asperó and Joan Bagaria. Bounded forcing axioms and the continuum. *Ann. Pure Appl. Logic*, 109(3):179–203, 2001.
[3] David Asperó and Matteo Viale. Category forcings. In preparation, 2019.
[4] Giorgio Audrito and Matteo Viale. Absoluteness via resurrection. *J. Math. Log.*, 17(2):1750005, 36, 2017.
[5] J. Bagaria. Bounded forcing axioms as principles of generic absoluteness. *Arch. Math. Logic*, 39(6):393–401, 2000.
[6] A. E. Caicedo and B. Veličković. The bounded proper forcing axiom and well orderings of the reals. *Math. Res. Lett.*, 13(2-3):393–408, 2006.
[7] Joel David Hamkins and Thomas A. Johnstone. Resurrection axioms and uplifting cardinals. *Arch. Math. Logic*, 53(3-4):463–485, 2014.
[8] Karel Hrbacek and Thomas Jech. *Introduction to set theory*, volume 220 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, third edition, 1999.
[9] T. Jech. *Set theory*. Springer Monographs in Mathematics. Springer, Berlin, 2003. The third millennium edition, revised and expanded.
[10] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
[11] K. Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1980. An introduction to independence proofs.
[12] P. B. Larson. Forcing over models of determinacy. In *Handbook of set theory. Vols. 1, 2, 3*, pages 2121–2177, Springer, Dordrecht, 2010.
[13] Paul B. Larson. *The stationary tower*, volume 32 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004. Notes on a course by W. Hugh Woodin.
[14] K. Tent and M. Ziegler. *A course in model theory*. Cambridge University Press, 2012.
[15] G. Venturi and M. Viale. The model companions of set theory. https://arxiv.org/abs/1909.13372, 2019.
[16] Matteo Viale. Category forcings, $\text{MM}^{+++}$, and generic absoluteness for the theory of strong forcing axioms. *J. Amer. Math. Soc.*, 29(3):675–728, 2016.
[17] Matteo Viale. Martin’s maximum revisited. *Arch. Math. Logic*, 55(1-2):295–317, 2016.
[18] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *De Gruyter Series in Logic and its Applications*. Walter de Gruyter GmbH & Co. KG, Berlin, revised edition, 2010.