HERMITE-HADAMARD’S INEQUALITIES FOR PREQUASIINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for prequasiinvex functions via fractional integrals.

1. Introduction and Preliminaries

Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \), then

\[
(1.1) \quad f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex mapping. For several recent results concerning the inequality (1.1) we refer the interested reader to [1, 2, 3, 6].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function \( f : [a, b] \to \mathbb{R} \) is said to be quasi-convex on \([a, b]\) if inequality

\[
f (tx + (1-t)y) \leq \max \{ f(x), f(y) \},
\]

holds for all \( x, y \in I \) and \( t \in [0, 1] \).

Clearly, any convex function is quasi-convex function. Furthermore there exist quasi-convex functions which are not convex (see [5]).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1. Let \( f \in L[a, b] \). The Riemann-Liouville integrals \( J_{a^+}^\alpha f \) and \( J_{b^-}^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a
\]

and

\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad x < b
\]

\[
\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt
\]

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Theorem 1. Let \( f : [a, b] \to \mathbb{R} \), be positive function with \( 0 \leq a < b \) and \( f \in L[a, b] \). If \( f \) is a quasi-convex function on \([a, b] \), then the following inequality for fractional integrals holds:

\[
\frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \leq \max \{f(a), f(b)\}
\]

with \( \alpha > 0 \).

Theorem 2. Let \( f : [a, b] \to \mathbb{R} \), be a differentiable mapping on \((a, b)\) with \( a < b \). If \( |f'| \) is quasi-convex on \([a, b] \), \( \alpha > 0 \), then the following inequality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \leq \frac{b-a}{\alpha + 1} \left(1 - \frac{1}{2^\alpha}\right) \max \{|f'(a)|, |f'(b)|\}.
\]

Theorem 3. Let \( f : [a, b] \to \mathbb{R} \), be a differentiable mapping on \((a, b)\) with \( a < b \) such that \( f' \in L[a, b] \). If \( |f'|^q \) is quasi-convex on \([a, b] \), and \( q > 1 \), then the following inequality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \leq \frac{b-a}{2(\alpha p + 1)^\frac{1}{q}} \left(\frac{\max \{|f'(a)|^q, |f'(b)|^q\}}{p}\right)^\frac{1}{q}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \alpha \in [0, 1] \).

Theorem 4. Let \( f : [a, b] \to \mathbb{R} \), be a differentiable mapping on \((a, b)\) with \( a < b \) such that \( f' \in L[a, b] \). If \( |f'|^q \) is quasi-convex on \([a, b] \), and \( q \geq 1 \), then the following inequality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \leq \frac{b-a}{(\alpha + 1)} \left(1 - \frac{1}{2^\alpha}\right) \left(\frac{\max \{|f'(a)|^q, |f'(b)|^q\}}{p}\right)^\frac{1}{q}
\]

with \( \alpha > 0 \).

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [10]. Weir and Mond [11] introduced the...
concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. Pini [12] introduced the concept of prequasiinvex function as a generalization of invex functions. Later, Mohan and Neogy [20] obtained some properties of generalized preinvex functions. Noor [13]-[15] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Yang et al. in [18] studied prequasiinvex functions, and semistrictly prequasiinvex functions and Barani et al. in [19] presented some generalizations of the right hand side of a Hermite-Hadamard type inequality for prequasiinvex functions.

In this paper we generalized the results in [9] for prequasiinvex functions. Now we recall some notions in invexity analysis which will be used throughout the paper (see [16], [17] and references therein).

Let \( f : A \to \mathbb{R} \) and \( \eta : A \times A \to \mathbb{R} \), where \( A \) is a nonempty set in \( \mathbb{R}^n \), be continuous functions.

**Definition 2.** The set \( A \subseteq \mathbb{R}^n \) is said to be invex with respect to \( \eta(\cdot, \cdot) \), if for every \( x, y \in A \) and \( t \in [0, 1] \),
\[
x + t\eta(y, x) \in A.
\]
The invex set \( A \) is also called a \( \eta \)-connected set.

It is obvious that every convex set is invex with respect to \( \eta(y, x) = y - x \), but there exist invex sets which are not convex [10].

**Definition 3.** The function \( f \) on the invex set \( A \) is said to be preinvex with respect to \( \eta \) if
\[
f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y), \ \forall x, y \in A, \ t \in [0, 1].
\]
The function \( f \) is said to be preconcave if and only if \( -f \) is preinvex.

**Definition 4.** The function \( f \) on the invex set \( A \) is said to be prequasiinvex with respect to \( \eta \) if
\[
f(x + t\eta(y, x)) \leq \max\{f(x), f(y)\}, \ \forall x, y \in A, \ t \in [0, 1].
\]
Every quasi-convex function is a prequasiinvex with respect to \( \eta(y, x) = y - x \), but the converse does not holds (see example 1.1 in [18]).

We also need the following assumption regarding the function \( \eta \) which is due to Mohan and Neogy [20]:

**Condition C:** Let \( A \subseteq \mathbb{R}^n \) be an open invex subset with respect to \( \eta : A \times A \to \mathbb{R} \). For any \( x, y \in A \) and any \( t \in [0, 1] \),
\[
\eta(y, y + t\eta(x, y)) = -t\eta(x, y)
\]
and
\[
\eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y).
\]
(Note that for every \( x, y \in A \) and every \( t \in [0, 1] \) from condition C, we have
\[
\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).
\]

In [19] Barani et al. proved the Hermite-Hadamard type inequality for prequasiinvex as follows:
Theorem 5. Let \( A \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \rightarrow \mathbb{R} \). Suppose that \( f : A \rightarrow \mathbb{R} \) is a differentiable function. If \( |f'| \) is prequasiinvex on \( A \) then, for every \( a, b \in A \) the following inequalities holds

\[
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{|\eta(b,a)|}{4} \max \{|f'(a)|, |f'(b)|\}.
\]

(1.7)

Theorem 6. Let \( A \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \rightarrow \mathbb{R} \). Suppose that \( f : A \rightarrow \mathbb{R} \) is a differentiable function. Assume that \( p \in \mathbb{R} \) with \( p > 1 \). If \( |f'|^\frac{1}{p} \) is preinvex on \( A \) then, for every \( a, b \in A \) the following inequalities holds

\[
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{|\eta(b,a)|}{2(p+1)^\frac{1}{p}} \left( \max \{|f'(a)|^\frac{1}{p}, |f'(b)|^\frac{1}{p}\} \right)^{\frac{p}{p-1}}.
\]

(1.8)

In [21], Iscan proved the following Lemma and established some inequalities for preinvex functions via fractional integrals

**Lemma 1.** Let \( A \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \rightarrow \mathbb{R} \) and \( a, b \in A \) with \( a < a + \eta(b, a) \). Suppose that \( f : A \rightarrow \mathbb{R} \) is a differentiable function. If \( f' \) is preinvex function on \( A \) and \( f' \in L[a, a + \eta(b, a)] \) then, the following equality holds:

\[
\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(a+1)}{2\eta^a(b,a)} \left[ J_{a^+}^a f(a + \eta(b, a)) + J_{(a+\eta(b,a))}^a f(a) \right] = \frac{\eta(b, a)}{2} \int_0^1 [t^a - (1-t)^a] f'(a + t\eta(b,a)) \, dt
\]

(1.9)

In this paper, using lemma [1] we obtained new inequalities related to the right side of Hermite-Hadamard inequalities for prequasiinvex functions via fractional integrals.

2. Main Results

**Theorem 7.** Let \( A \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \rightarrow \mathbb{R} \) and \( a, b \in A \) with \( a < a + \eta(b, a) \). If \( f : [a, a + \eta(b, a)] \rightarrow (0, \infty) \) is a prequasiinvex function, \( f \in L[a, a + \eta(b, a)] \) and \( \eta \) satisfies condition C then, the following inequalities for fractional integrals holds:

\[
\frac{\Gamma(a+1)}{2\eta^a(b,a)} \left[ J_{a^+}^a f(a + \eta(b, a)) + J_{(a+\eta(b,a))}^a f(a) \right] \leq \max \{f(a), f(a + \eta(b,a))\} \leq \max \{|f(a)|, |f(b)|\}
\]

(2.1)

**Proof.** Since \( a, b \in A \) and \( A \) is an invex set with respect to \( \eta \), for every \( t \in [0,1] \), we have \( a + t\eta(b,a) \in A \). By prequasiinvexity of \( f \) and inequality (1.6) for every
$t \in [0, 1]$ we get
\[
\begin{align*}
  f (a + t \eta(b, a)) &= f (a + \eta(b, a) + (1 - t) \eta(a, a + \eta(b, a))) \\
  &\leq \max \{f(a), f(a + \eta(b, a))\}
\end{align*}
\]
and similarly
\[
\begin{align*}
  f (a + (1 - t) \eta(b, a)) &= f (a + \eta(b, a) + t \eta(a, a + \eta(b, a))) \\
  &\leq \max \{f(a), f(a + \eta(b, a))\}.
\end{align*}
\]
By adding these inequalities we have
\[
\begin{align*}
  f (a + t \eta(b, a)) + f (a + (1 - t) \eta(b, a)) &\leq 2 \max \{f(a), f(a + \eta(b, a))\}
\end{align*}
\]
Then multiplying both (2.3) by $t^{\alpha - 1}$ and integrating the resulting inequality with respect to $t$ over $[0, 1]$, we obtain
\[
\begin{align*}
  \int_0^1 t^{\alpha - 1} f (a + t \eta(b, a)) dt + \int_0^1 t^{\alpha - 1} f (a + (1 - t) \eta(b, a)) dt &\leq 2 \max \{f(a), f(a + \eta(b, a))\} \int_0^1 t^{\alpha - 1} dt.
\end{align*}
\]
i.e.
\[
\begin{align*}
  \frac{\Gamma(\alpha)}{\eta^{\alpha}(b, a)} \left[ J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b,a))^+}^\alpha - f(a) \right] &\leq \frac{2 \max \{f(a), f(a + \eta(b, a))\}}{\alpha}.
\end{align*}
\]
Using the mapping $\eta$ satisfies condition C the proof is completed. \hfill \Box

**Remark 1.** If in Theorem 7, we let $\eta(b, a) = b - a$, then inequality (2.1) become inequality (1.2) of Theorem 1.

**Theorem 8.** Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$ such that $f' \in L[a, a + \eta(b, a)]$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex function on $A$ then the following inequality for fractional integrals with $\alpha > 0$ holds:
\[
\begin{align*}
  \left| f(a) + f(a + \eta(b, a)) \right| &\leq \frac{\Gamma(\alpha + 1)}{2^\alpha \eta^\alpha(b, a)} \left[ J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b,a))^+}^\alpha - f(a) \right] \\
  &\leq \frac{\eta(b, a)}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \max \{|f'(a)|, |f'(b)|\}.
\end{align*}
\]
Proof. Using lemma \[1\] and the prequasiinvexity of \(|f'|\) we get

\[
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b, a)} \left[ J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b,a))^+}^\alpha f(a) \right] \right|
\leq \frac{\eta(b, a)}{2} \int_0^1 |t^\alpha - (1 - t)^\alpha| \max \{|f'(a)|, |f'(b)|\} dt
\]

\[
\leq \frac{\eta(b, a)}{2} \int_0^1 |t^\alpha - (1 - t)^\alpha| \max \{|f'(a)|, |f'(b)|\} dt
\]

\[
\leq \frac{\eta(b, a)}{2} \left\{ \int_0^1 |(1 - t)^\alpha - t^\alpha| \max \{|f'(a)|, |f'(b)|\} dt + \int_0^{1/2} |t^\alpha - (1 - t)^\alpha| \max \{|f'(a)|, |f'(b)|\} dt \right\}
\]

\[
= \eta(b, a) \max \{|f'(a)|, |f'(b)|\} \left( \int_0^{1/2} |(1 - t)^\alpha - t^\alpha| dt \right)
\]

\[
= \eta(b, a) \frac{1}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \max \{|f'(a)|, |f'(b)|\},
\]

which completes the proof. \[\square\]

Remark 2. a) If in Theorem 8 we let \(\eta(b, a) = b-a\), then inequality \[2.4\] become inequality \[1.3\] of Theorem 8.

b) If in Theorem 8 we let \(\alpha = 1\), then inequality \[2.3\] become inequality \[1.7\] of Theorem 8.

c) In Theorem 8 assume that \(\eta\) satisfies condition C. Using inequality \[2.2\] we get

\[
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b, a)} \left[ J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b,a))^+}^\alpha f(a) \right] \right|
\leq \frac{\eta(b, a)}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \max \{|f'(a)|, |f'(a + \eta(b, a))|\}
\]

Theorem 9. Let \(A \subseteq \mathbb{R}\) be an open invex subset with respect to \(\eta : A \times A \to \mathbb{R}\) and \(a, b \in A\) with \(a < a + \eta(b, a)\) such that \(f' \in L[a, a + \eta(b, a)]\). Suppose that \(f : A \to \mathbb{R}\) is a differentiable function. If \(|f'|^q\) is prequasiinvex function on \(A\) for some fixed \(q > 1\) then the following inequality holds:

\[
(2.5) \quad \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b, a)} \left[ J_{a^+}^\alpha f(a + \eta(b, a)) + J_{(a+\eta(b,a))^+}^\alpha f(a) \right] \right|
\leq \frac{\eta(b, a)}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right) \left( \max \{|f'(a)|^q, |f'(b)|^q\} \right)^{1/q}
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\alpha > 0\).
Proof. From lemma 1 and using Hölder inequality with properties of modulus, we have
\[ \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b,a)} \left[ J^{\alpha}_a f(a + \eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))} f(a) \right] \right| \]
\[ \leq \frac{\eta(b,a)}{2} \int_0^1 |t^\alpha - (1-t)^\alpha| \, dt \leq \frac{\eta(b,a)}{2} \left( \int_0^1 |t^\alpha - (1-t)^\alpha| \, dt \right)^{\frac{1}{\alpha}} \]
\[ \leq \frac{\eta(b,a)}{2} \left( \int_0^1 |t^\alpha - (1-t)^\alpha| \, dt \right)^{\frac{1}{\alpha}} \left( \int_0^1 |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}}. \]
On the other hand, we have
\[ \int_0^1 |t^\alpha - (1-t)^\alpha| \, dt = \int_0^1 [(1-t)^\alpha - t^\alpha] \, dt + \int_0^1 [1 - (1-t)^\alpha] \, dt = \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2\alpha} \right). \]
Since \(|f'|^q\) is prequasiinvex function on \(A\), we obtain
\[ |f'(a + t\eta(b,a))|^q \leq \max \{ |f'(a)|^q, |f'(b)|^q \}, \quad t \in [0,1] \]
and
\[ \int_0^1 |t^\alpha - (1-t)^\alpha| \, dt \leq \int_0^1 |t^\alpha - (1-t)^\alpha| \, dt \leq \int_0^1 |t^\alpha - (1-t)^\alpha| \, dt \]
\[ = \max \{ |f'(a)|^q, |f'(b)|^q \} \left( \int_0^1 [(1-t)^\alpha - t^\alpha] \, dt + \int_0^1 [1 - (1-t)^\alpha] \right) \]
\[ = \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2\alpha} \right) \max \{ |f'(a)|^q, |f'(b)|^q \}. \]
From here we obtain inequality (2.5) which completes the proof. \(\square\)

**Remark 3.**  

a) If in Theorem 9, we let \(\eta(b,a) = b-a\) then inequality (2.5) become inequality (1.2), Theorem 4.

b) In Theorem 9, assume that \(\eta\) satisfies condition C. Using inequality (2.2) we get
\[ \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b,a)} \left[ J^{\alpha}_a f(a + \eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))} f(a) \right] \right| \]
\[ \leq \frac{\eta(b,a)}{(\alpha + 1)} \left( 1 - \frac{1}{2\alpha} \right) \left( \max \{ |f'(a)|^q, |f'(a + \eta(b,a))|^q \} \right)^{\frac{1}{q}}. \]

Theorem 10. Let \(A \subseteq \mathbb{R}\) be an open invex subset with respect to \(\eta : A \times A \rightarrow \mathbb{R}\) and \(a,b \in A\) with \(a < a + \eta(b,a)\) such that \(f' \in L[a, a + \eta(b,a)]\). Suppose that
Proof. From Lemma 1 and using Hölder inequality with properties of modulus, we have

\[
\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b, a)} \left[ J^\alpha_a f(a + \eta(b, a)) + J^\alpha_{a+\eta(b, a)} f(a) \right]
\]

\[
\leq \frac{\eta(b, a)}{2} \left( \max \{|f'(a)|^q, |f'(a + \eta(b, a))|^q \} \right)^{\frac{1}{q}}.
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\alpha \in [0, 1]\).

Remark 4. a) If in Theorem 10, we let \(\eta(b, a) = b - a\) then inequality (2.6) become inequality (1.4) of Theorem 3.

b) If in Theorem 10, we let \(\alpha = 1\) then inequality (2.6) become inequality (7.8) of Theorem 10.

c) In Theorem 10, assume that \(\eta\) satisfies condition C. Using inequality (2.2) we get

\[
\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b, a)} \left[ J^\alpha_a f(a + \eta(b, a)) + J^\alpha_{a+\eta(b, a)} f(a) \right]
\]

\[
\leq \frac{\eta(b, a)}{2(\alpha p + 1)^{\frac{1}{p}}} \left( \max \{|f'(a)|^q, |f'(a + \eta(b, a))|^q \} \right)^{\frac{1}{q}}.
\]
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