General Regular Variation, Popa Groups and Quantifier Weakening

by

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To Harry I. Miller (22 Feb 1939 - 15/16 December 2018), man and mathematician, who died with his boots on

‘The soldiers’ music and the rites of war
Speak loudly for him.’ (Shakespeare, Hamlet Act V.2)

Abstract. We introduce general regular variation, a theory of regular variation containing the existing Karamata, Bojanic-Karamata/de Haan and Beurling theories as special cases. The unifying theme is the Popa groups of our title viewed as locally compact abelian ordered topological groups, together with their Haar measure and Fourier theory. The power of this unified approach is shown by the simplification it brings to the whole area of quantifier weakening, so important in this field.

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1 Introduction

We recall the definition of Beurling slowly varying functions $\varphi$ (see e.g. [BinGT § 2.11], [BinO7]): these are positive, measurable or Baire (i.e. have the Baire property, BP), are $o(x)$ at infinity (or $O(x)$, depending on context), and, with

$$x \circ_\varphi t := x + t\varphi(x)$$

the Popa (or circle) operation (§ 2 below), satisfy

$$\log \varphi(x \circ_\varphi t) - \log \varphi(x) \to 0 : \quad \varphi(x \circ_\varphi t)/\varphi(x) \to 1.$$

(B)
Such \( \phi \) will play the role of auxiliary functions below. For a suitable auxiliary function \( h \) and limit function \( K \), called the \textit{kernel}, consider also the limit relationship

\[
[f(x \circ \phi_t) - f(x)]/h(x) \to K(t), \quad (G)
\]

where \( f \) here is the function of primary interest (‘G for Goldie, G for general’: see e.g. [BinO6,10,11], [Ost2]). Specialising to \( \phi \equiv 1, h \equiv 1 \) gives

\[
f(x + t) - f(x) \to K(t) \quad (K)
\]

(‘K for Karamata’). This is the defining relationship for \textit{Karamata regular variation} written additively (see e.g. [BinGT Ch. 1-3]: one needs to be able to pass between the additive notation above, and the original multiplicative notation, using the familiar exp-log isomorphism between the additive group of reals (Haar measure Lebesgue measure) and the multiplicative group of positive reals (Haar measure \( dx/x \)). Specialising instead to \( \phi \equiv 1, h \) slowly varying (in Karamata’s sense: [BinGT, Ch. 1]) gives

\[
[f(x + t) - f(x)]/h(x) \to K(t), \quad (BKdH)
\]

the defining relationship for \textit{Bojanic-Karamata/de Haan regular variation} [BinGT, Ch. 3], while specialising to \( f = \log \phi, h = 1, K = 0 \) gives Beurling slow variation as above. We call the limit relationship \((G)\) above \textit{general regular variation}, as it contains the other three. Below we give a unified treatment, using the algebraicization provided by the \textit{Popa groups} of §2 below.

As usual (see e.g. [BinO1,9]), we pass between the measurable and Baire cases (in any form of regular variation) ‘bitopologically’ – by passing between the Euclidean and density topologies. The same will be true in the Popa groups below, which are isomorphic to the reals algebraically and bitopologically; we thus extend the terms Euclidean and density topologies to these Popa isomorphs also.

## 2 Popa groups

Above we have used the Popa operation as a simplifying notational device for the regular variation above (general or otherwise), involving limits as \( x \to \infty \). But its usefulness is far greater, and is not confined to limits, as emerged in [BinO7], [Ost1]. Here one allows other auxiliary functions \( h \), with corresponding circle operations \( \circ_h \). This is most useful when the circle
operation is associative, and this requires \( h \) to satisfy the \textit{Gołąb-Schintzel functional equation}:

\[
h(s \circ_h t) = h(s + h(s)t) = h(s)h(t) \quad (GS)
\]

(cf. [Jav]). Thus \((GS)\) expresses homomorphy in this context, which will occur in the regular variation context after the passage to the limit \( x \to \infty \). Indeed, such an \( h \) generates group structures on subsets of \( \mathbb{R} \), that are in fact isomorphic to the group \((\mathbb{R}_+, \times)\). It is to these \textit{Popa groups} [Pop] that we now turn.

Write \( GS \) for the set of \textit{positive} solutions \( h \) of \((GS)\). It emerges that, being thus bounded below, they are continuous and of the form

\[
\eta(t) = \eta_\rho(t) := \eta(t) = 1 + \rho t
\]

for \( t > -1/\rho \), with the parameter \( \rho \geq 0 \); for a proof see [Brz2] and [BrzM], or the more direct [Ost3, §5] – see also [AczD] and the surveys [Brz1] and [Jab5]; cf. [Jab2], [Ost1]. For \( \eta \in GS \), put

\[
\mathcal{G}_\eta^* := \{x \in \mathbb{R} : \eta(x) \neq 0\}.
\]

Equipped with \( \circ_\eta \), this is a group. When \( \eta = \eta_\rho \) this operation is given explicitly by

\[
x \circ_\rho y = x + y(1 + \rho x),
\]

so that \( \mathcal{G}_\rho^* = \{x \in \mathbb{R} : x \neq \rho^*\} \), where \( \rho^* = -1/\rho \), the \textit{Popa centre}. We interpret this to mean \( \rho^* = -\infty \) for \( \rho = 0 \) and to mean \( \rho^* = 0 \) for \( \rho = +\infty \).

The operation \( \circ_\rho \) may also be rendered by reference to the equation \((GS)\) in the current context:

\[
\eta_\rho(x \circ_\rho y) = \eta_\rho(x)\eta_\rho(y) \quad (x, y \in \mathcal{G}_\rho),
\]

and thereby to the underlying role of the multiplicative positive reals \( \mathbb{R}_+ \):

\[
x \circ_\rho y = \eta_\rho^{-1}(\eta_\rho(x)\eta_\rho(y)) = [(1 + \rho x)(1 + \rho y) - 1]/\rho
\]

(which gives for \( \rho = 1 \) the \textit{circle operation} of ring theory: cf. [Ost2, §2.1]). It emerges from here that (except for the case \( \rho = 0 \) where \( \rho^* = -\infty \) so that \( \mathcal{G}_0^* = \mathbb{R} \)) the following subgroups of \( \mathcal{G}_\rho^* \) are of greater significance:

\[
\mathcal{G}_\rho := \{x \in \mathbb{R} : 1 + \rho x > 0\} : \quad \mathcal{G}_\eta := \{x \in \mathbb{R} : \eta(x) > 0\},
\]

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by virtue of being isomorphic with \((\mathbb{R}_+, \times)\) when \(\rho > 0\). (Likewise, the groups \(G^\star_\rho\) are all isomorphic with \((\mathbb{R}^\star, \times)\), with \(\mathbb{R}^\star := \mathbb{R}\setminus\{0\}\).)

As \(\rho^* = 0\) for \(\rho = +\infty\), the group \((\mathbb{R}_+, \times)\) may itself be viewed conveniently as \(G_\infty\), or perhaps more accurately as the rescaled limit of \(G_\rho\) as \(\rho \to \infty\), as follows:

\[
(x \circ_\rho y)/\rho = [(1 + \rho x)(1 + \rho y) - 1]/\rho^2 = x/\rho + y/\rho + xy \to xy, \text{ as } \rho \to \infty.
\]

We note that one has \(1_G = 0\) for \(G = G_\rho\) except for \(G = G_\infty\), when \(1_G = 1\). The inverse of \(t\) in \(G_\eta\) will be denoted by \(t^{-1}_\eta\) (or \(t^{-1}_\rho\), if more convenient); here

\[
t^{-1}_\eta = -t/\eta(t).
\]

We will also need to designate location to either side of \(1_G = 0\), using the notation

\[
G^+_\rho := \{x \in G_\rho : x > 0 & 1 + \rho x > 0\}
\]

and \(G^-_\rho := \{x \in G_\rho : x < 0 & 1 + \rho x > 0\}\).

Viewing the Popa operation as a conjugacy via the isomorphism \(\eta_\rho\),

\[
x \circ_\rho y = [(1 + \rho x)(1 + \rho y) - 1]/\rho,
\]

(denotes that \(\circ_\rho\) may be expressed in terms of the ring operations of \(\mathbb{R}\), and so permits other features of \(\mathbb{R}\) to be imported into \(G_\rho\). There are several possibilities here. The Popa groups may inherit either of the two canonical topological structures of their isomorphs, again enabling bitopological passage between them (as in §1). Thus they inherit a Euclidean topology, from which they derive their own metric structures; this is generated by (open) intervals, and makes \(G_\rho\) a locally compact abelian topological group. In turn this allows reference to Haar measure, and so to the second possibility: the Haar-density topology of \(G_\rho\), which which agrees with the topology induced on \(G_\rho\) by the (Lebesgue) density topology on \(\mathbb{R}\) (corresponding to Lebesgue measure \(\lambda\)) and with the Haar-density topology of \(\mathbb{R}_+\). In particular, the two topologies make available as a tool the interior-point theorem of Steinhaus-Weil from measure theory [Ste] [Wei], and the Piccard-Pettis category analogue [Pic] [Pet] (cf. [BinO13]). Before identifying the (normalized) Haar measure of \(G = G_\eta\), written \(\eta_G\), we observe below that \(G\) has a natural order which coincides with the usual order on \(\mathbb{R}\). We also identify the associated canonical invariant metrics on \(G\), below.
We recall that by the Birkhoff-Kakutani Theorem ([Bir], [Kak1]; cf. [DieS, §3.3, §8]) we may equip any metrizable group $G$ with a (left-)invariant metric $d^L_G$, equivalently with a (group) norm $||g|| := d^L_G(g, 1_G)$, as in [BinO2] (‘pre-norm’ in [ArhT]) that generates its topology. Its defining features are:

(i) $||g|| = 0$ iff $g = 1_G$;
(ii) $||gh|| \leq ||g|| + ||h||$;
(iii) $||g^{-1}|| = ||g||$.

The group norm on $\mathbb{R}_+$ is also a limit of $||t||_\rho$ for $\rho \to \infty$, as we will see below.

**Proposition 1.** (a) A group-norm on $G_\rho$ for $\rho \geq 0$ is given by

$$||t||_\rho := |\log(1 + \rho t)|(1 + \rho)/\rho.$$ 

(b) In particular, $||1||_\rho = \log(1 + \rho)/\rho$, and $||t||_\rho \to |t|$ as $\rho \to 0$ (for $t \neq 0$).

(c) A group-norm on $G_\infty = \mathbb{R}_+$ is given by

$$||t||_\infty := |\log t|.$$

**Proof.** (a) Here (i) is clear; as for (ii), we have

$$||s\circ t||_\rho = |\log(1 + \rho(s + t + \rho st))|(1 + \rho)/\rho = |\log(1 + \rho s)(1 + \rho t)|(1 + \rho)/\rho \leq ||s||_\rho + ||t||_\rho.$$

Then (iii) follows, since $\eta_\rho(t^{-1}) = \eta_\rho(t)^{-1}$ i.e. with $s = t_{\rho}^{-1}$

$$(1 + \rho s) = 1/(1 + \rho t).$$

(Or, from (conj) above, with $t$ for $y$ and its inverse $s = t_{\rho}^{-1}$ for $x$,

$$1_{\rho} = 0 = s \circ t = [(1 + \rho s)(1 + \rho t) - 1]/\rho : (1 + \rho s)(1 + \rho t) = 1.)$$

(b) The second assertion follows by L’Hospital’s rule (or as $\log(1 + \rho t) \sim \rho t$ for $\rho \sim 0$).

(c) The final assertion is similar to but simpler than in (a). □

**Remarks.** The inclusion of the scaling factor $(1 + \rho)$ is dictated by Haar-measure normalization concerns, below.
Proposition 2. For $\rho \geq 0$, the set $\mathbb{G}_\rho^+ = [0, \infty)$ is a sub-semigroup of $\mathbb{G}_\rho$; the induced order, $y \leq_\rho x$ iff $x \circ_\rho y^{-1} \in [0, \infty)$, coincides with $y \leq x$. Furthermore, for $c > 0$ and $a < b$,

$$a \circ_\rho c \leq b \circ_\rho c;$$

in particular, for the interval $(a, b)$,

$$(a, b) \circ_\rho c = (a \circ_\rho c, b \circ_\rho c):$$

the Euclidean topology on $\mathbb{G}_\rho$ is invariant under (positive) translation by $\circ_\rho$. Likewise, for $\rho > 0$, if $a < b$ and $c < d$, with $a, b, c, d \in \mathbb{G}_\rho$,

$$a \circ_\rho c \leq b \circ_\rho d.$$

and

$$s \leq t \text{ iff } s^{-1} \geq t^{-1} \text{ (s, t} \in \mathbb{G}_\rho).$$

Proof. For the first assertion observe that

$$0 \leq x - (1+\rho x)y/(1+\rho y) \text{ iff } 0 \leq x(1+\rho y) - (1+\rho x)y = x - y, \text{ as } 1+\rho y > 0.$$

From here, as $a \leq b$ and $c \leq d$,

$$a \circ_\rho c \leq b \circ_\rho c \text{ and } c \circ_\rho b < d \circ_\rho c: \quad a \circ_\rho c \leq b \circ_\rho d.$$

Finally, $s \leq t$ iff

$$-1/(1+\rho t) \geq -1/(1+\rho s): \quad 1-1/(1+\rho t) \geq 1-1/(1+\rho s): \quad -\rho t^{-1} \geq -\rho s^{-1}. \quad \square$$

Theorem 1 (Haar measure). Normalized Haar measure on the Popa group $\mathbb{G} = \mathbb{G}_\rho$, with the Euclidean topology giving the interval $(0, 1)$ measure $||1||_\rho$ for $\rho \geq 0$, has Radon-Nikodym derivative $(1+\rho)/\eta_\rho(g)$ w.r.t. $dg$, the Lebesgue measure on the additive reals, that is

$$d\eta_\rho(t) = (1+\rho) \, dt/\eta(t) = (1+\rho) \, dt/\eta_\rho(t)$$

$$= (1+\rho) \, dt/(1+\rho t), \text{ for } \eta = \eta_\rho.$$

In particular, as $1_\rho = 0$, the group norm satisfies

$$||x||_\rho = \eta_\rho((1_\rho, x)) = \int_0^x (1+\rho) \, dt/(1+\rho t) = \frac{1+\rho}{\rho} |\log(1+\rho x)|.$$
Proof. Since Haar measure is unique up to proportionality, begin by letting \( \tilde{\eta}_G \) be an arbitrary Haar measure for the group. As \( \tilde{\eta}_G \) and \( \lambda \) are absolutely continuous measures w.r.t. each other (both give (non-degenerate) intervals positive measure), the Radon-Nikodym derivative, which we write below as \( \delta(g) := \frac{d\tilde{\eta}_G}{d\lambda(g)} \), is well defined. To find the Radon-Nikodym derivative at \( g \), we compare the Lebesgue measure of an interval around \( g \) with its \( \tilde{\eta}_G \)-measure. Taking \( (a, b) \) an arbitrary interval around \( 0 = 1_\rho \), so that \( g \circ \eta(a, b) \) is a neighbourhood of \( g \),

\[
\begin{align*}
g \circ \eta(a, b) &= g + \eta_\rho (a, b) : \lambda(g \circ \eta(a, b)) = \eta_\rho (g) \lambda(a, b).
\end{align*}
\]

Now, taking limits below as \( a \uparrow 0, b \downarrow 0 \), and setting \( t = g \circ s = g + \eta(g)s \)

\[
\begin{align*}
\delta(1_\rho) &= \lim \frac{\tilde{\eta}_G((a, b))}{\lambda(a, b)} = \lim \frac{\tilde{\eta}_G(g \circ_\rho (a, b))}{\lambda(a, b)} \quad \text{(invariance)} \\
&= \lim \frac{\int_{g \circ_\rho (a, b)} \delta(t) \, dt}{\lambda(a, b)} = \eta_\rho (g) \lim \frac{\int_{(a, b)} \delta(g + \eta(g)s) \, ds}{\lambda(a, b)} = \eta_\rho (g) \delta(g) \quad \text{a.e.}
\end{align*}
\]

by the Lebesgue differentiation theorem [Sak, IV § 5], [Rud2, Th. 8.6]. So

\[
d\tilde{\eta}_G(t)/dt = \delta(g) = \delta(1_\rho)/\eta_\rho (g).
\]

So for the normalized measure \( \eta_G \) of the theorem, the Radon-Nikodym derivative at \( g \) is proportional to \( 1/\eta_\rho (g) \). The proportionality constant \( (1 + \rho) \) allows for the two extreme \( \rho \) values, to yield Lebesgue measure \( dt \) on the additive reals for \( \rho = 0 \), and Haar measure \( dt/t \) on the multiplicative reals \( \mathbb{R}_+ \) as \( \rho \to \infty \):

\[
d\eta_G(t) = \frac{1 + \rho}{1 + \rho t} \, dt \to dt \quad \text{as } \rho \to 0, \quad d\eta_G(t) = \frac{1 + \rho}{1 + \rho t} \, dt \to \frac{dt}{t} \quad \text{as } \rho \to \infty. \quad \square
\]

Remark. For \( \eta = \eta_\rho \) and \( \rho = 0 \), we interpret \( \rho^* = -1/\rho \) to mean \( -\infty \) (the Popa centre recedes to \( -\infty \)); then, since \( s \circ_0 t = s + t \), we recover the additive reals under ordinary Lebesgue measure, so \( \mathbb{G}_0 = \mathbb{R} \), by Prop. 1. Here, computing distance relative to \( 1_\rho = 0 \),

\[
||x||_\rho = \eta_\rho(0, x) = \int_0^x \frac{1 + \rho}{1 + \rho t} \, dt = \frac{1 + \rho}{\rho} |\log(1 + \rho x)| \to |x| \quad \text{as } \rho \to 0
\]
(the modulus signs are needed iff \( x < 0 \), when \( (x, 0) \) replaces \( (0, x) \) above).

As before, \( \log(1 + \rho x) \sim \rho x \) for \( \rho \sim 0 \).

In the limit as \( \rho \to \infty \) we interpret \( \rho^* = -1/\rho \) to mean 0 (the Popa centre approaches 0, from the left). Since \( 1_{\rho=\infty} = 1 \) is the unit in the multiplicative reals \( \mathbb{R}_+ := (0, \infty) \), computing distance now relative to 1, we retrieve

\[
\eta_\rho(1, x) = \int_1^x \frac{1 + \rho}{1 + \rho t} \, dt = \frac{1 + \rho}{\rho} \log \left| \frac{1 + \rho x}{1 + \rho} \right| \to |\log x| \text{ as } \rho \to \infty:
\]

\[
||x||_{\rho=\infty} = |\log x|.
\]

Recalling that

\[
x \circ_\rho y = x + y(1 + \rho x) = x + y + \rho xy = [(1 + \rho x)(1 + \rho y) - 1]/\rho,
\]

the corresponding conjugacy yields

\[
(x \circ_\rho y)/\rho = [(1 + \rho x)(1 + \rho y) - 1]/\rho^2 = x/\rho + y/\rho + xy \to xy, \text{ as } \rho \to \infty,
\]

so that \( G_\infty \) has domain \( \mathbb{R}_+ \) with \( \circ = \times \) (‘the multiplicative reals’).

This means that, up to scaling, there are just three Popa operations/groups, corresponding to \( \rho = 0, 1, \infty \), namely \( +, \circ, \times \) with \( \circ \) the circle operation of ring theory as above.

**Remarks.**

1. The alternative normalization is \( \delta(1_\rho) = 1 \), as

\[
x \circ_\rho y = x + y + \rho xy \sim x + y \quad (x, y \to 0).
\]

2. Note that \( G_\rho \) for \( \rho \in \mathbb{R}_+ \) has only one idempotent, \( c = 0 \) (replaced by \( c = 1 \) in the case \( \rho = \infty \)):

\[
c = c \circ_\rho c : \quad 0 = c + \rho c^2 = c(1 + \rho c),
\]

\[
c = c^2 : \quad c = 1 \text{ in } \mathbb{R}_+ = (0, \infty).
\]

3. The origin of the Haar measures \( dt \), \( dt/t \) for the cases \( \rho = 0, \infty \) above are clear: the arithmetic operations + and \( \times \). From the canonical \( dt/t \) case one may infer the general \( \rho \in (0, \infty) \) case by a change of origin to the Popa centre \( -1/\rho \). That of the intermediate values \( \rho \in (0, \infty) \) is exemplified by the case \( \rho = 1 \), giving \( dt/(1 + t) \). This arises via the role of the Beck sequences in the proof of Theorem 3 and the Remark below it, and is an instance of the ergodic theorem (see e.g. Billingsley [Bil, Ch. 1 §4] and Remark 4 below).
same measure arises in the Gauss-Kuzmin theorem on continued fractions, and for the same reason (again, see [Bil, Ch. 1 §4]).

4. As mentioned above: see [BinO7, Prop. 11 (iv)] for the sense in which the Beck sequence of iterates above grows arithmetically, which links their averages with the arithmetic means in the (Birkhoff-Khinchin) ergodic theorem.

5. The limiting behaviour of the moving average \([U(x \circ \varphi_t) - U(x)]/\varphi(x)\) of \(U\) and the Tauberian one-sided conditions studied by Bingham and Goldie [BinG] emerge in §3 below directly from the asymptotic operator \(K_h^\varphi(t, x)\) with the specialization \(h = \varphi\). The group norms exhibited in Th. 1 above thus coincide with the measures of occupation ‘times’ (on \([1, x]\)) of the associated limiting velocity flow \(\frac{d\text{w}(t)}{dt} = \eta(t)\) for \(\eta = \eta^\varphi\). Here Lebesgue measure \(dt\) measures time, and equates with \(w'(t) \cdot dt/\eta(t)\), i.e. the Haar integral of the flow rate.

We recall that the dual of a locally compact abelian group \(G\), denoted \(\hat{G}\), comprises the continuous homomorphisms from \(G\) to \(T\), the unit circle in the complex plane \(\mathbb{C}\). For \(\eta_G\) a Haar measure on \(G\), the Fourier transform is defined by

\[
\hat{f}(\gamma) := \int_G f(g)\gamma(-g) \, d\eta_G(g) \quad (\gamma \in \hat{G});
\]

for background see [Rud1], [Loo]. We specialize this in Theorem 2 below to the Popa group \((G_\rho, \circ_\rho)\) for \(0 < \rho < \infty\). It is helpful to first consider the extreme cases \(\rho = 0\) and \(\rho = \infty\), corresponding respectively to the familiar cases \(G = (\mathbb{R}, +)\) and \(G = (\mathbb{R}_+, \times)\). In the first case \(\hat{G} = G = (\mathbb{R}, +)\) [Loo, 35C], and we may write

\[
\gamma(w) = e^{i\gamma w} \quad (\gamma \in \mathbb{R}),
\]

so that, for \(f \in L^1(\mathbb{R})\),

\[
\hat{f}(\gamma) = \int_{\mathbb{R}} f(w)e^{-i\gamma w} \, dw \quad (\gamma \in \mathbb{R}).
\]

We pass to the second case using the isomorphism \(w = \log v\) which, for \(f \in L^1(\mathbb{R}_+)\), yields both the Fourier and Mellin transforms as

\[
\hat{f}(\gamma) = \int_0^\infty f(v)e^{-i\gamma \log v} \, dv/v \quad (\gamma \in \mathbb{R}), \quad \hat{f}(z) = \int_0^\infty f(t)t^{-z} \, dt/t \quad (z \in \mathbb{C}),
\]

with characters represented multiplicatively by \(\gamma(t) = t^z\).
We turn to the Fourier transform in the context of a locally compact abelian group ([Rud1], [Loo]), specialized to the Popa-group \( \mathbb{G}_\rho \) for \( \rho > 0 \). As we shall see, the Fourier-Popa transform of \( f : \mathbb{G}_\rho \to \mathbb{R} \) is in fact the ordinary Fourier transform of an affinely related function \( f_\rho : \mathbb{R}_+ \to \mathbb{R} \), defined as follows:

\[
f_\rho(t) = \frac{1 + \rho}{\rho} f(\eta_\rho^{-1}(t)) = \frac{1 + \rho}{\rho} f((t - 1)/\rho),
\]

so that

\[
f_\rho(1/u) = \frac{1 + \rho}{\rho} f((\eta_\rho^{-1}(u))^{-1}) = \frac{1 + \rho}{\rho} f((1 - u)/(\rho u)).
\]

As expected, for \( \rho \to \infty \) we recover \( f \) by rescaling: \( f_\rho(\rho t) \to f(t) \).

**Theorem 2 (Fourier transform).** For the Popa group \( G = (\mathbb{G}_\rho, \circ_\rho) \) with \( 0 < \rho < \infty \), the characters \( \gamma \in \hat{G} \) are

\[
\gamma(u) := e^{i \gamma \log(1 + \rho u)} \quad (\gamma \in \mathbb{R}).
\]

So, writing \( +_\rho \) and \( -_\rho \) for the operations of \( \circ_\rho \) and inversion here, the Fourier transform corresponding to the canonical Haar measure of Theorem 1 is

\[
\hat{f}(\gamma) = \int_{\mathbb{G}_\rho} f(u) \gamma(-_\rho u)(1 + \rho) \ du/(1 + \rho u) = \int_0^\infty f_\rho(t)e^{i \log t^{-\gamma}} dt/t,
\]

that is

\[
\hat{f}(\gamma) = \int_{\mathbb{G}_\rho} f(u) \gamma(-_\rho u)(1 + \rho) \ du/(1 + \rho u) = \int_0^\infty f((t - 1)/\rho)e^{i \log t^{-\gamma}} dt/t.
\]

The corresponding Mellin transform is thus

\[
\tilde{f}(z) = \int_0^\infty f_\rho(t)t^{-z} dt/t = \int_0^\infty f_\rho(1/u)u^z \ du/u = \int_0^\infty f((1 - u)/(\rho u))u^z \ du/u.
\]

**Proof.** Applying the isomorphisms \( \eta_\rho : (\mathbb{G}_\rho, \circ) \to (\mathbb{R}_+, \times) \) and \( \log : (\mathbb{R}_+, \times) \to (\mathbb{R}, +) \) and using \( u, v, w \) as corresponding generic elements with \( w = \log v \) and \( v = 1 + \rho u \), the character representation for \( (\mathbb{R}, +) \) recalled above gives the character representation for \( (\mathbb{G}_\rho, \circ) \) as asserted. By (inv) above

\[
1 + \rho(-_\rho u) = 1/(1 + \rho u),
\]
so substitution for $\gamma(-\rho u)$ gives the Fourier transform as

$$\hat{f}(\gamma) = \int_{G_{\rho}} f(u)\gamma(-\rho u)(1+\rho) \, du/(1+\rho u) = \int_{-1/\rho}^{\infty} f(u)e^{-i\gamma\log(1+\rho u)}(1+\rho) \, du/(1+\rho u).$$

Putting $t = \eta_\rho(u) = 1 + \rho u$ gives

$$\hat{f}(\gamma) = \int_{0}^{\infty} f_\rho(t)e^{i\log t^{-\gamma}} \, dt/t.$$  

This gives the first form of the Mellin transform above; for the second, take $u = 1/t$. □

3 Asymptotic actions and functional equations

We begin with the Karamata asymptotic operator $K$ acting on $f : \mathbb{R}_+ \to \mathbb{R}_+$, as in $(K)$ of § 1:

$$K(t, x)f := \frac{f(xt)}{f(x)}.$$

Suppose that $f$ is Karamata regularly varying, i.e. that, as $x \to \infty$,

$$K(t, x)f := \frac{f(xt)}{f(x)} \to K_f(t).$$

Here we adopt a relatively new point of view on the classical theory, by making explicit use of what has so far been mostly implicit: the cocycle structure underlying the operator $K(t, x)$, cf. [Ell] [EllE]. It is this that characterizes the limit function $K_f$, the Karamata kernel of $f$. Indeed,

$$\frac{f(xts)}{f(x)} = \frac{f(xts)}{f(xt)} \cdot \frac{f(xt)}{f(x)},$$

In the limit this yields the multiplicative Cauchy functional equation,

$$K_f(st) = K_f(s)K_f(t). \quad (CFE)$$

We will need the Popa operation $\circ_h$ above to be associative, and (see Th. O below) this requires $h$ to satisfy the Golab-Schintzel equation:

$$h(s \circ_h t) = h(s + h(s)t) = h(s)h(t). \quad (GS)$$
Thus \((GS)\) expresses homomorphy in this context, which will occur after the passage to the limit \(x \to \infty\). Before taking this limit, one has instead ‘asymptotic associativity’, or ‘almost associativity’. The Popa notation \(x \circ_\varphi t = x + t\varphi(x)\) describes a \(t\)-translation modified locally at \(x\), or ‘accelerated at \(x\)’ by reference to the ‘accelerator’ \(\varphi\) \((positive)\). We will need the rate of acceleration and its asymptotic value for the \(t\)-translation:

\[
\eta_x(t), \text{ or } \eta^x(t) := \frac{\varphi(x \circ_\varphi t)}{\varphi(x)} = \frac{\varphi(x + t\varphi(x))}{\varphi(x)} \to \eta(t), \text{ or } \eta^x(t)
\]

(assumed to exist), so that \(\eta(t) \geq 0\). As we learn from the Uniform Convergence Theorem (UCT) below, for \(\varphi\) above Baire or measurable, convergence is necessarily \emph{locally uniform}. The relevance of such convergence is witnessed by

\textbf{Theorem O} \cite[Th. 0]{Ost1}. If \(\varphi(x) = O(x)\) and \(\eta^x(t) \to \eta(t) = \eta^x(t),\) \emph{locally uniformly in} \(t\), then \(\eta\) satisfies the Gołąb-Schinzel functional equation

\[
\eta(s \circ_\eta t) = \eta(s)\eta(t).
\]

(\(GS\))

\textit{Notational conventions.} In Theorem O above \(\eta^x\) contains the \(x\) which tends to infinity. After this passage to the limit, attention focuses on the limit function \(\eta(t)\) which will depend on a parameter \(\rho\), below. We allow ourselves to denote this limit by \(\eta_\rho(t)\) and let context speak for itself here. Below we will take \(GS := \{\eta_\rho : \rho \geq 0\}\) to denote the family of continuous \((positive)\) solutions of the equation \((GS)\).

For \(\varphi\) Baire/measurable \(\eta^x\) is likewise Baire/measurable and so, as a solution of \((GS)\), \emph{continuous}, by a theorem of Popa \cite{Pop}. Furthermore, non-negative solutions of \((GS)\), being bounded below, are likewise continuous, as noted in §2. In any case, here we are interested only in \emph{positive} solutions of \((GS)\), and these take the form \(\eta(t) = \eta_\rho(t) := 1 + \rho t\), for \(t > \rho^* := -1/\rho\) with \(\rho \geq 0\) (and 0 to the left of \(\rho^*\), though here we work in \(\mathbb{R}_+\)), by a theorem of Gołąb and Schinzel – for the literature see \cite{Brz1}, \cite{Jab5}, and \cite{Ost1}. For a discussion of circumstances when local boundedness implies the continuity of solutions, for the family relevant here of functional equations related to \((GS)\), see \cite{Jab3}.

Below, we will encounter \emph{two} auxiliary functions, \(h\) and \(\varphi\), the second of which will give such an \(\eta\) asymptotically \((so \eta \text{ satisfies } (GS) \text{ and } \circ_\eta \text{ is associative})\).
For the purposes of combining an $s$- and a $t$-translation, it is convenient to expand the accelerator notation to one parametrized locally at $x$:

$$s \circ_{\varphi x} t := s + t\eta_x(s) = s + t\frac{\varphi(x + s\varphi(x))}{\varphi(x)}.$$ 

So in the limit one has for $\eta = \eta^\varphi = \eta^\rho$:

$$\circ_{\varphi x} \rightarrow \circ_{\eta} = \circ_{\rho}.$$ 

This justifies our earlier reference to ‘asymptotic associativity’. A second reason for the term comes from a very convenient expression for a related form of associativity, one which otherwise the notation keeps hidden:

$$(x \circ_{\varphi} b) \circ_{\varphi} a = x \circ_{\varphi} (b \circ_{\varphi x} a)$$

As an immediate application of this framework, we can rephrase the Beurling asymptotics, clarifying the underlying cocycle structure. These, as we will see, lead to functional equations, whose solutions are discussed in §5 below – see also the surveys [Brz1] and [Jab5]; cf. [Ost1].

**Proposition 3 (Beurling regular variation).** For the Beurling asymptotic operator $K^\varphi$ acting on $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

$$K^\varphi(t,x)f := \frac{f(x + t\varphi(x))}{f(x)} = \frac{f(x \circ_{\varphi} t)}{f(x)},$$ 

suppose that $f$ is Beurling regularly varying, i.e. that, as $x \rightarrow \infty$,

$$K^\varphi(t,x)f := \frac{f(x + t\varphi(x))}{f(x)} = \frac{f(x \circ_{\varphi} t)}{f(x)} \rightarrow K_f(t).$$ 

The corresponding cocycle structure is

$$K^\varphi(t \circ_{\varphi x} s, x) = K^\varphi(s, x \circ_{\varphi} t)K^\varphi(t, x),$$ 

leading in the limit to the Chudziak-Jabłońska equation

$$K_f(t \circ_{\eta} s) = K_f(s)K_f(t). \quad (CJ)$$

**Proof.** We have

$$\frac{f(x + (s + t)\varphi(x))}{f(x)} = \frac{f(x + t\varphi(x) + (s/\eta_x) \cdot \varphi(x + t\varphi(x)))}{f(x + t\varphi(x))} \cdot \frac{f(x + t\varphi(x))}{f(x)},$$
so that in the limit
\[
K(s + t, x) = K(s/\eta_x(t), x + t\varphi(x))K(t, x).
\]

Here replacing \( s \) by \( s\eta_x(t) \) yields
\[
K(t + s\eta_x(t), x) = K(s, x + t\varphi(x))K(t, x). \quad \square
\]

We turn now to the general regular variation of the title and §1 (cf. [BinO14]).

Following [Ost1], the auxiliary function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is self-equivarying, \( \varphi \in SE \), if \( \varphi(x) = O(x) \) and \( \eta_x^\varphi(t) \to \eta(t) = \eta^\varphi(t) \), locally uniformly in \( t \), as in Theorem O. The auxiliary function \( h \) will be Beurling regularly varying as in Prop. 1, i.e. \( \varphi \)-regularly varying, in the sense of [BinO5].

**Proposition 4 (General regular variation).** For the general asymptotics
\[
K^\varphi_h(t, x) := \frac{f(x + t\varphi(x)) - f(x)}{h(x)} \to K_f(t),
\]
with \( \varphi \in SE \), the corresponding cocycle structure is
\[
K^\varphi_h(t + s\eta_x(t), x) = K^\varphi_h(t \circ_{\varphi} s, x) = K^\varphi_h(s, x)K^\varphi(t, x) + K^\varphi_h(t, x),
\]
leading in the limit to
\[
K_f(t + s\eta(t)) = K_f(s)K_h(t) + K_f(t),
\]
or, equivalently, to the Beurling-Goldie equation satisfied by \( K_f : \mathbb{G}_\eta \to \mathbb{G}_\sigma : \)
\[
K_f(t \circ \eta s) = K_f(t) \circ_\sigma K_f(s), \quad \text{for } \sigma(z) = K_h(K_f^{-1}(z)). \quad \text{(BG)}
\]

**Proof.** Here the underlying cocycle structure mixes products with addition:
with \( y := x \circ_\varphi t \),
\[
K^\varphi_h(s + t, x) \quad = \quad \frac{f(x + (s + t)\varphi(x)) - f(x)}{h(x)}
\]
\[
= \quad \frac{f(x + t\varphi(x) + (s/\eta_x)\varphi(x \circ_\varphi t)) - f(x \circ_\varphi t) h(x \circ_\varphi t)}{h(x \circ_\varphi t)} \frac{h(x \circ_\varphi t)}{h(x)} + \frac{f(x \circ_\varphi t) - f(x)}{h(x)}
\]
\[
= \quad \frac{f(y + (s/\eta_x)\varphi(y)) - f(y) h(x \circ_\varphi t)}{h(y)} \frac{h(y)}{h(x)} + K^\varphi_h(t, x)
\]
\[
= \quad K^\varphi_h(s/\eta_x, y)K^\varphi(t, x) + K^\varphi_h(t, x).
\]
In the limit, since \( x + t \varphi(x) = x(1 + t \varphi(x)/x) \to \infty \) and \( \varphi(x) = O(x) \),

\[
K_f(s + t) = K_f(s/\eta, x)K_h(t) + K_f(t),
\]
giving \((BG)\) as above. \(\Box\)

**Remark.** A measurable \( \varphi : \mathbb{R} \to \mathbb{R}_+ \) is said to be *Beurling slowly varying* if, as above, but with \( \varphi(x) = o(x) \) and \( \eta^\varphi(t) \equiv 1 \) (that is, \( \rho = 0 \) in the above); it is *self-neglecting* if the convergence \( \eta_\varphi(t) \to 1 \) is locally uniformly in \( t \) – see [BinGT § 2.11], [BinO5].

### 4 Subadditivity in Popa groups

**Definition.** For \( \rho, \sigma \in [0, \infty] \), call \( S : \mathbb{G}_\rho \to \mathbb{G}_\sigma \) *subadditive* (resp. *additive*) if

\[
S(x \circ \rho y) \leq S(x) \circ_\sigma S(y) \quad \text{resp.} \quad S(x \circ \rho y) = S(x) \circ_\sigma S(y),
\]
or in the notation of Theorem 2

\[
S(x + \rho y) \leq S(x) +_\sigma S(y) \quad \text{resp.} \quad S(x + \rho y) = S(x) +_\sigma S(y),
\]

As \( \mathbb{G}_0 = \mathbb{R} \) (the additive reals), when \( \rho = \sigma = 0 \), this yields the usual notion of subadditivity, resp. additivity.

In particular the solutions \( K : \mathbb{G}_\rho \to \mathbb{G}_\sigma \) to the equation \((BG)\) are additive. For fixed \( \rho, \sigma \in \mathbb{R}_+ \) with \( \sigma > 0 \), the canonical form depends on a parameter \( \kappa \in \mathbb{R} \) (Theorem 3 below, [Ost2-Hom], [Chu1,2]), as follows:

\[
K_\kappa(t) = \eta_{-1}^{-1}(\eta_\varphi(t)^\kappa) = [(1 + pt)^\kappa - 1]/\sigma, \quad \text{if also } \rho > 0.
\]

Above one has \( \eta_\rho : \mathbb{G}_\rho \to \mathbb{R}_+ \), and \( \eta_{-1}^{-1} : \mathbb{R}_+ \to \mathbb{G}_\sigma \). The case \( \kappa = 0 \) corresponds to the trivial solution \( K \equiv 1_\sigma = 0 \).

**Example.** Recalling that \( \eta_\rho(x \circ \rho y) = \eta_\rho(x)\eta_\rho(y) \), so that \( \eta_{-1}^{-1}(uv) = \eta_\rho^{-1}(u) \circ_\rho \eta_\rho^{-1}(v) \) (on substituting \( u = \eta_\rho(x) \) etc.),

\[
K(x \circ \rho y) = \eta_{-1}^{-1}(\eta_\rho(x \circ \rho y)^\kappa) = \eta_{-1}^{-1}(\eta_\rho(x)^\kappa \eta_\rho(y)^\kappa)
= \eta_{-1}^{-1}(\eta_\rho(x)^\kappa) \circ_\sigma \eta_{-1}^{-1}(\eta_\rho(y)^\kappa)
= K(x) \circ_\sigma K(y).
\]
In fact, for fixed \( \rho, \sigma \in \mathbb{R}_+ \), the only additive functions bounded above are of this form, as below. Theorem 3 below is our reformulation here of [Ost2-Hom, Prop. A]; cf. [Chu1,2].

**Theorem 3.** Take \( \psi : \mathbb{G}_\rho \to \mathbb{G}_\sigma \) additive with \( \rho, \sigma \in [0, \infty] \). Then the lifting \( \Psi : \mathbb{R} \to \mathbb{R} \) defined by the canonical isomorphisms \( \log, \exp, \{\eta_\rho : \rho > 0\} \) of \( \psi \) to \( \mathbb{R} \) is bounded above on \( \mathbb{G}_\rho \) iff \( \Psi \) is bounded above on \( \mathbb{R} \), in which case \( \Psi \) and \( \psi \) are continuous. Then for some \( \kappa \in \mathbb{R} \) one has:

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Popa parameter} & \sigma = 0 & \sigma \in (0, \infty) & \sigma = \infty \\
\hline
\rho = 0 & \kappa t & \eta_\sigma^{-1}(e^{\kappa t}) & e^{\kappa t} \\
\rho \in (0, \infty) & \log \eta_\rho(t)^\kappa & \eta_\sigma^{-1}(\eta_\rho(t)^\kappa) & \eta_\rho(t)^\kappa \\
\rho = \infty & \log t^\kappa & \eta_\sigma^{-1}(t^\kappa) & t^\kappa \\
\hline
\end{array}
\]

**Proof.** The canonical isomorphisms are order-preserving and continuous. For \( \rho, \sigma > 0 \) the lifting is given by

\[
\Psi = \log \eta_\sigma \psi \eta^{-1}_\rho \exp ,
\]

and this still holds for extreme values of \( \rho, \sigma \) with \( \exp, \log \) replacing \( \eta_0, \eta_\infty \). For \( \Psi(x) = \kappa x \), a routine calculation gives \( \psi \) as in the table above. \( \square \)

**Remark.** Notice that the passage from first to the third column is effected via \( \exp / \log \), while the middle column to the first column requires scaling of the domain via the coefficient \( \kappa \):

\[
\begin{align*}
\lim_{\sigma \to 0} \eta_\sigma^{-1}(e^{\kappa t}) &= \lim_{\sigma \to 0} \frac{e^{\kappa \sigma t} - 1}{\sigma} = \kappa t \text{ (equiv. } \kappa t \sim \log(1 + \kappa \sigma t)/\sigma) ; \\
\lim_{\sigma \to 0} \eta_\sigma^{-1}(\eta_\rho(t)^{\kappa \sigma}) &= \lim_{\sigma \to 0} \frac{[(1 + \rho t)^{\kappa \sigma} - 1]}{\sigma} = \log \eta_\rho(t)^{\kappa} ; \\
\lim_{\sigma \to 0} \eta_\sigma^{-1}(t^{\kappa \sigma}) &= \lim_{\sigma \to 0} \frac{e^{\kappa \sigma \log t} - 1}{\sigma} = \log t^\kappa .
\end{align*}
\]

Note also

\[
\kappa t \sim \rho \log(1 + \kappa t/\rho), \text{ as } \rho \to \infty; \quad \lim_{\rho \to 0} \frac{\log \eta_\rho(t)^{\kappa}}{\rho} = \lim_{\rho \to 0} \frac{\kappa \log(1 + \rho t)}{\rho} = \kappa t .
\]
Definition. Call $S : G_\rho \to G_\sigma$ additively bounded on $\Sigma$ if for some $\kappa$
\[ S(t) \leq K_\kappa(t) \quad (t \in \Sigma). \]
This lifts to the Popa context the notion of linear boundedness used in [Bi-nO10].

In the results below recall that $0 = 1_\rho = 1_\sigma$; $B_\delta(x)$ is the open ball about $x$ of radius $\delta$.

Proposition 5. For $S : G_\rho \to G_\sigma$ subadditive:
(i) if $S$ is bounded above on some interval, say by $M$ on $B_\delta(a)$, then for any $b \in G_\rho^+$
\[ S(b \circ a) \circ_\sigma M^{-1} \leq S(x) \leq S(b \circ a^{-1}) \circ_\sigma M \quad (x \in B_\delta(b)) \]
(with $M^{-1}$ etc. the inverses in the corresponding groups); in particular it is locally bounded.
(ii) If $S$ is locally bounded, then $\liminf_{t \to 0} S(t) \geq 0$, so $S(0+) = 0$ if $(HS(S))$ holds.

Below (as in §1), ‘G for Goldie, G for general’:

Theorem G1. For subadditive $S : G_\rho^+ \to G_\sigma^+ \cup \{ -\infty, +\infty \}$ with $S(0+) = S(0) = 0 : S$ is continuous at $0$ iff $S(z_n) \to 0$, for some sequence $z_n \uparrow 0$, and then $S$ is continuous everywhere, if finite-valued.

Proof of Theorem G1. This is as in [BinO10], mutatis mutandis, as the group order is the usual order on the line (Prop. 4), and with $-x$ etc. replaced by $x^{-1}$ (equivalently by $-\rho x$ as in Theorem 2). It is critical here that one works in $G_\rho^+$ and $G_\sigma^+$. □

Theorem G2 [BinO8, Th. 3]. If $S : G_\rho \to G_\sigma$ is subadditive with $S(0) = 0$ and there is a symmetric set $\Sigma$ containing $0$ with:
(i) $S$ is continuous at $0$ on $\Sigma$;
(ii) for all small enough $\delta > 0$, $\Sigma_0^\delta$ is locally Steinhaus-Weil
then $S$ is continuous at $0$ and so everywhere.

In particular, this conclusion holds if there is a symmetric set $\Sigma$, Baire/measurable and non-negligible in each $(0, \delta)$ for $\delta > 0$, on which

$S(u) = K_{\kappa_{\pm}}(u)$ for some $\kappa_{\pm} \in \mathbb{R}$ and all $u \in G_\rho^+ \cap \Sigma$, or all $u \in G_\rho^- \cap \Sigma$ resp.
Proof of Theorem G2. W.l.o.g. $\sigma > 0$, as the case $\sigma = 0$ is similar but simpler. Since $S|\Sigma$ is continuous at 0 it is bounded above on $\Sigma_\delta := \Sigma \cap (\delta^{-1}, \delta)$ for some $\delta > 0$; but $\Sigma_\delta \circ \Sigma_\delta$ contains an interval, so $S$ is bounded on an interval, and so locally bounded by Prop. 5(i). If $S$ is not continuous at 0, then by Prop. 5(ii) $\lambda_+ := \limsup_{t \to 0} S(t) > \liminf_{t \to 0} S(t) \geq 0$. Choose a null sequence $\{z_n\}$ with $S(z_n) \to \lambda_+ > 0$. Let $\varepsilon := \min\{\lambda_+ / 6, 1 / \sigma\}$. W.l.o.g. $S(z_n) > \lambda_+ - \varepsilon$ for all $n$. By continuity on $\Sigma$ at 0, there is $\delta > 0$ with $|S(t)| < \varepsilon$ for $t \in \Sigma_{\delta}$. As before and using symmetry, $\Sigma_{\delta} \circ \Sigma_{\delta} = \Sigma_{\delta} \circ (\Sigma_{\delta})^{-1}$ contains an interval $I$ around 0. For any $n$ with $z_n \in I$, there are $u_n, v_n \in \Sigma_{\delta}$ with $z_n = u_n \circ \rho v_n$; then, as $\varepsilon < 1 / \sigma$, $S(z_n) \leq S(u_n) \circ_\sigma S(v_n) = S(u_n) + S(v_n)(1 + \sigma S(u_n)) \leq \varepsilon(2 + \sigma \varepsilon) < 3 \varepsilon < \lambda_+ / 2$.

So $3\lambda_+ / 4 = \lambda_+ - \varepsilon < S(z_n) \leq S(u_n) \circ_\sigma S(v_n) \leq 3 \varepsilon < \lambda_+ / 2$, a contradiction. So $S$ is continuous at 0 and so continuous everywhere (as in Theorem G1):

$$-\sigma S(-\rho h) \leq S(x + \rho h) - \sigma S(x) \leq S(h).$$

The last part follows since $\Sigma \cap (0, \delta)$, being Baire/measurable and non-negligible, has the SW property for each $\delta > 0$. □

Theorem G3. Let $\Sigma \subseteq [0, \infty)$ be locally SW accumulating at 0. Suppose $S : \mathbb{R} \to \mathbb{R}$ is subadditive with $S(0) = 0$ and $S|\Sigma$ is additively bounded above by $G(x) := K_\kappa(x)$, i.e. $S(\sigma) \leq K_\kappa(\sigma)$ for some $\kappa$ and all $\sigma \in \Sigma$, so that in particular,

$$\limsup_{\sigma \to 0, \sigma \in \Sigma} S(\sigma) \leq 0.$$

Then $S(x) \leq K_\kappa(x)$ for all $x > 0$, so

$$\limsup_{x \to 0} S(x) \leq 0,$$

and so $S(0+) = 0$.

In particular, if furthermore there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ with $z_n \uparrow 0$ and $S(z_n) \to 0$, then $S$ is continuous at 0 and so everywhere.

Proof of Theorem G3. We are to show that $S(t) \leq K_\kappa(t)$ for all $t$. We may begin with the simplifying assumption that $K \equiv 1_\sigma = 0$, i.e. that $\kappa = 0,$
since $S'(t) := S(t) \circ \sigma(K,\tau(t))^{-1}_\sigma$ is linearly bounded above by $1_\sigma = 0$ on $\Sigma$, and $S'$ is subadditive, as $K$ is additive:

$$S'(u \circ v) := S(u \circ v) \circ \sigma K(u \circ v)^{-1}_\sigma \leq S(u) \circ \sigma S(v) \circ \sigma K(u)^{-1}_\sigma \circ \sigma K(v)^{-1}_\sigma.$$  

From now on the proof follows that of [BinO10, Th. 0‘], mutatis mutandis (interpreting $+$ as $+\rho$ and $-$ as $-\rho$ as in Theorem 2). □

5 Functional inequalities from asymptotic actions: the Goldie argument

We return to the Karamata asymptotic operator $K$ acting on $f : \mathbb{R}_+ \to \mathbb{R}_+$, as in (K) of § 3, but we now apply a natural alternative to the limits of § 3 when they cannot be assumed to exist. This is provided by the limsup operation, which in the Karamata setting is given by

$$K^*(t)f := \limsup K(t,x)f = \limsup \frac{f(xt)}{f(x)} := K^*_t(f).$$

This leads to an operator domain defined by

$$\mathbb{A}_f := \{ u : K_f(u) := \lim f(xt)/f(x) \text{ exists and is finite} \}.$$  

This is a subgroup of $\mathbb{R}_+$. For positive functions $f$, one has

$$\limsup \frac{f(xst)}{f(x)} \leq \limsup \frac{f(xst)}{f(xt)} \limsup \frac{f(xt)}{f(x)} : K^*(st)f \leq K^*(s)f \cdot K^*(t)f,$$

as $K(st,x) \leq K(s,xt)K(t,x)$. Here the limsup yields the multiplicative Cauchy functional inequality,

$$K^*_f(st) \leq K^*_f(s)K^*_f(t), \quad (CFI)$$

as well as a pair of equations restricted to $\mathbb{A}_f$:

$$\begin{cases}
K_f(st) = K_f(s)K_f(t) \\
K_f(t) = K^*_f(t)
\end{cases} \quad (s, t \in \mathbb{A}_f).$$

One seeks side-conditions on $f$ and imposes a density condition on $\mathbb{A}_f$ to deduce that $\mathbb{A}_f = \mathbb{R}_+$.  

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For the general asymptotics, with $\varphi \in SE$,

\[ K^*_{h\varphi}(t)f := \limsup f(x + t\varphi(x)) - f(x) \]

\[ \frac{h(x)}{h(x)}, \]

there is a corresponding operator domain defined by

\[ A_{hf} := \{ u : K_{hf}(u) := \lim[f(x + t\varphi(x)) - f(x)]/h(x) \text{ exists and is finite} \}, \]

(with $\varphi$ omitted when clear from context). As before, there is also a functional inequality:

\[ K^*_{hf}(t + s\eta(t)) \leq K^*_{hf}(s)K_h(t) + K^*_{hf}(t), \quad \text{with } K_h(t) := \lim h(x \circ \varphi t)/h(x), \]

where $K_h$ is assumed to exist for all $t$ (as in Prop. 4). The inequality may be reformulated in Popa-group language as the Beurling-Goldie inequality satisfied by $K^*_f : \mathbb{G}_\eta \to \mathbb{G}_\sigma$:

\[ K^*_{hf}(t \circ \eta s) \leq K^*_hf(t) \circ \sigma K^*_hf(s), \quad \text{for } \sigma(z) = K_h(K^*_{hf}^{-1}(z)). \quad (BGI) \]

However, there is no immediate justification for $A_{hf}$ being a subgroup, short of further hypotheses. Either an imposition of good behaviour of the limit, such as local uniformity in $t$, is needed, thus narrowing the domain, or a presumption of topologically good character of the domain itself, such as requiring $A_{hf}$ to contain a non-meagre subset. The latter may draw on additional axioms of set theory, for which see [BinO12]. For an extensive study of the uniformity assumptions, see [BinO7].

Henceforth we take for granted a domain $A$ that is a dense subgroup of an appropriate Popa group $G$, and a side-condition of right-sided continuity at $0 = 1_G$ imposed on $K^*_hf$ (so on $\mathbb{R}_+$).

Above we had the Beurling-Goldie equation ($BG$). Below, we restrict one or both of the arguments $u$ and $v$ to $A$, obtaining the ‘singly conditioned’ and ‘doubly conditioned’ Beurling-Goldie equations ($BG_A$) and ($BG_{AA}$). For the origins of the Goldie argument, see the Remark after Theorem 4 below.

We begin with an auxiliary result. (In the equation below $g(0)K(0) = 0$, so to avoid trivial (constant) solutions w.l.o.g. we assume both here and later that $g(0) = 1$.)

**Proposition 6** ([BojK, (2.2)], [BinGT, Lemma 3.2.1]; cf. [AczG]). Take $\eta \in GS$ and $g$ with $g(0) = 1$. If $K \neq 0$ satisfies

\[ K(u \circ \eta v) = g(v)K(u) + K(v) \quad (u, v \in A), \quad (BG_{AA}) \]

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with \( A \) a dense subgroup of \( \mathbb{G}_\eta \), then:

(i) the following is a subgroup of \( \mathbb{G}_\eta \) on which \( K \) is additive:

\[
A_g := \{ u \in A : g(u) = 1 \};
\]

(ii) if \( A_g \neq A \) and \( K \neq 0 \), there is a constant \( \kappa \neq 0 \) with

\[
K(t) \equiv \kappa(g(t) - 1) \quad (t \in A),
\]

and \( g \) satisfies

\[
g(u \circ_\eta v) = g(v)g(u) \quad (u, v \in A).
\]

(CJ)

(iii) So for \( A = \mathbb{G}_\eta^+ \) with \( \eta = \eta_\rho \) and \( g \) locally bounded at \( 0 \) with \( g \neq 1 \) except at \( 0 \):

\[
g(x) \equiv (1 + \rho t)^\gamma,
\]

for some constant \( \gamma \neq 0 \), and so \( K(t) \equiv cK_\gamma(t) \) for some constant \( c \), where

\[
K_\gamma(t) := [(1 + \rho t)^\gamma - 1].
\]

Proof. This is proved exactly as in [BinO6, Th. 1] with \( \circ_\eta \) or \( +_\rho \) replacing \( + \). One uses the Cauchy nucleus of \( K \) [Kuc, Lemma 18.5.1]. \( \Box \)

Example in the case \( \rho = 1 \). Below, put \( x = u + 1 \) and \( k(t) = g(t - 1) \):

\[
g((u + 1)(v + 1) - 1) = g(u + v + uv) = g(u)g(v) : g(xy - 1) = g(x - 1)g(y - 1);
\]

\[
k(xy) = k(x)k(y) : g(t) = k(t + 1) = (1 + t)\gamma;
\]

\[
K(t) = \kappa(g(t) - 1) = \kappa[(1 + t)^\gamma - 1].
\]

Theorem 4 (Generalized Goldie Theorem, cf. [BinO6, Th. 3]). If for \( \eta \in GS \) and \( A \) a dense subgroup of \( \mathbb{G}_\eta \),

(i) \( F^* : \mathbb{R} \to \mathbb{R} \) is positive and subadditive with \( F^*(0+) = 0 \);

(ii) \( F^* \) satisfies the singly-conditioned Beurling-Goldie equation

\[
F^*(u \circ_\eta v) = g(v)K(u) + F^*(v) \quad (u \in A)(v \in \mathbb{R}_+) \quad (BG_\kappa)
\]

for some non-zero \( K \) satisfying \( (BG_\kappa) \) with \( g \) continuous on \( \mathbb{R} \) and \( A_g = \{0\} \) (i.e. \( g(0) = 1 \) but otherwise \( g(v) \neq 1 \));
(iii) $F^*$ extends $K$ on $A$:

$$F^*(x) = K(x) \quad (x \in \mathbb{A}),$$

so that in particular $F^*$ satisfies $(BG_\mathbb{A})$, and indeed

$$F^*(u \circ \eta v) = g(v)F^*(u) + F^*(v) \quad (u \in \mathbb{A})(v \in \mathbb{G}_\eta^+):$$

- then for some $c > 0, \gamma \geq 0$

$$g(x) \equiv c(1 + \rho t)^{-\gamma} \text{ and } F^*(x) \equiv cK(x) = c[(1 + \rho t)^{-\gamma} - 1]/\rho \quad (x \in \mathbb{R}_+).$$

**Proof.** We write $\circ$ for $\circ \eta$, and $\mathbb{G}$ for $\mathbb{G}_\eta$. Put

$$G(x) := \int_0^x g(t) \, dt/\eta(t) : \quad G'(x) = g(x)/\eta(x).$$

By continuity of $g$ and Th. 1, $K$ is continuous on $\mathbb{A}$, so $K(u+) = K(u)$ for all $u \in \mathbb{A}$, and so in particular $K(0+) = 0$, which is also implied by (i) above. Also note that $F^*$ is right-continuous (and $F^*(u+) = K(u)$) on $\mathbb{A}$, and on $\mathbb{G}$ satisfies

$$\limsup_{v \downarrow 0} F^*(u \circ v) \leq g(0)F^*(u) + F^*(0+) = F^*(u).$$

We write $\delta^{i_0}$ for the $n$-fold product in $\mathbb{G}$ (inductively defined so that $\delta^{0_0} = 1_\mathbb{G} = 0$ and $\delta^{i_0} = \delta^{(n-1)_0} \circ \delta$).

Now we mimick the Goldie proof of [BinGT, §3.2.1] (extending [BinO6, Th. 3] to the current Popa context). For any $u, u_0$ with $u_0 \in \mathbb{A}$ and $u_0 > 0$, define $i = i(\delta) \in \mathbb{Z}$ for $\delta > 0$ so that $\delta^{(i-1)_0} \leq u < \delta^{i_0}$, and likewise for $u_0$ define $j = j_0(\delta)$. As $\mathbb{A}_g = \{0\}$, put $c_0 := K(u_0)/[g(u_0) - 1]$. For $m \in \mathbb{N}$

$$F^*(\delta^{i_0}) - F^*(\delta^{(m-1)_0}) = g(\delta^{(m-1)_0})K(\delta),$$

as $\delta^{i_0} \in \mathbb{A}$, so that on summing

$$F^*(\delta^{i_0}) = K(\delta) \sum_{m=1}^i g(\delta^{(m-1)_0}), \quad (**)$$

as $F^*(0) = 0$. Note that

$$\Delta_m := \delta^{i_0} - \delta^{(m-1)_0} = \delta \eta(\delta^{(m-1)_0}),$$

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so that $\Delta_m \to 0$ as $\delta \to 0$. But as $G'(t) = g(t)/\eta(t)$,

$$\sum_{m=1}^{i} g(\delta^{m_0})\Delta_m = \sum_{m=1}^{i} G'(\delta^{(m)0})\eta(\delta^{m_0})\delta \to \int_{0}^{u} G'(x) \, dx \quad (RI)$$

(for ‘Riemann Integral’). Without loss of generality $G(u_0) \neq 0$.

Now specialize to $u = c$ (This remains valid with $c = 0$ if $K(\delta) = 0$ for $\delta \in A \cap I$ for some interval $I = (0, \varepsilon)$, since then $F^*(u) = 0$ by right-continuity on $A$, as $F^*(\delta^{\varepsilon}) = 0$ for $\delta \in A \cap I$, by (**).)

We extend the domain of this equation from $A$ to the whole of $R$, using a key idea due to Goldie (see the Remark below).

Put $c := c_0[g(u_0) - 1]/G(u_0)$. As before, as $u_0 \in A$,

$$F^*(\delta^{\varepsilon}) \to G(u) \cdot F^*(u_0)/G(u_0).$$

Put $c_1 := c_0[g(u_0) - 1]/G(u_0)$.

Now specialize to $u \in A$, on which, by above, $F^*$ is right-continuous. Letting $\delta^{\varepsilon} \in A$ decrease to $u$, the inequality above becomes an equation:

$$K(u) = F^*(u) = c_1 G(u) \quad (u \in A).$$
For an arbitrary $u \in \mathbb{R}$, take $v \in A$ with $z := u - v > 0$, i.e. with $v < u$. Then

$$F^*(u) = F^*(v + z) = K(v)g(z) + F^*(z) \quad \text{(by (ii), as } v \in A \text{ and } z \in \mathbb{R}_+)$$

$$= c_1G(v)g(z) + F^*(z)$$

$$\to c_1G(u)g(0) + 0 = c_1G(u) \quad \text{(as } z \downarrow 0),$$

by continuity of $g$ and $G$, and $F^*(0+) = 0$. So

$$F^*(u) = c_1G(u) \quad (u \in \mathbb{G}).$$

Thus by (*) of Prop. 6, for some $\kappa$

$$c_1G(u) = F^*(u) = K(u) = \kappa[g(u) - 1] \quad (u \in A).$$

So, by density and continuity on $\mathbb{G}$ of $g$,

$$\kappa[g(u) - 1] = c_1G(u) \quad (u \in \mathbb{R}_+).$$

Thus $g$ is indeed differentiable; differentiation now yields

$$\kappa g'(u) = c_1g(u)/\eta(u) : \quad g'(u)/g(u) = (c_1/\kappa\eta(u)) \quad (u \in \mathbb{R}_+),$$

as $\kappa \neq 0$ (otherwise $K(u) \equiv 0$, contrary to assumption). So, as $g(0) = 1$, with $\gamma := -c_1/\kappa\rho$

$$\log g(u) = -\frac{c_1}{\kappa} \int_0^u \frac{dt}{1 + \rho t} = -\gamma \log(1 + \rho u) : \quad g(u) = (1 + \rho t)^{-\gamma}.$$

So

$$G(u) = \int_0^u g(t) \frac{dt}{\eta(t)} = \int_0^u (1 + \rho t)^{-\gamma-1}dt = [(1 + \rho u)^{-\gamma} - 1]/\rho.$$ 

So

$$G(u) = cK\gamma(u) : \quad F^*(u) = c_1G(u) = c_1[(1 + \rho u)^{-\gamma} - 1]/\rho \quad (u \in \mathbb{R}).$$

As $(1 + \rho u)^{-\gamma}$ is subadditive on $\mathbb{R}_+$ iff $\gamma \geq 0$ (cf. before Th. 1), $c_1 > 0$. □

**Remark.** Above, we have disaggregated the Goldie proof given in [BinGT, §3.2.1] into three steps. Firstly, we use the integral $G$ of the unknown auxiliary function $g$ (as in [BinO6, Th. 3], albeit here as a Haar integral), where
Goldie assumed \( g \) explicitly to be the exponential function \( e^{\gamma t} \). For Goldie this permits an explicit formula for the corresponding sums (for us the Riemann sums lead to a simple differential equation, which we can solve for \( g \), giving \( G \)). Secondly, we have partitioned the range of integration by use of a Beck sequence [Bec, Lemma 1.64] (iterating \( \circ \delta \)). Finally, the extension of the relation between \( F^* \) and \( G \) from \( A \to \mathbb{R}_+ \) makes explicit a remarkable achievement, due to Goldie (and left implicit in [BinGT, § 3.2.1]): establishment of left-sided continuity from the assumed right-sided continuity \( F^*(0+) = 0 \).

This overlooked feature was first made explicit in [BinO10] as Theorem 0 there (cf. Th. G1 above), yielding new results, and again put to further extensive use in [BinO11].

Armed with the results here we are now able to freely lift results from [BinO10] concerning when the solution \( K^*_{hf} : \mathbb{G}_\eta \to \mathbb{G}_\sigma \) of \((BGI)\) in fact solves \((BG)\) and so takes the form \( K^*_\kappa(u) \) for some \( \kappa \in \mathbb{R} \). We recall that in the interests of simplicity we assume that the domain of the asymptotic operator is a subgroup, leaving the reader to refer for results which guarantee this to [BinO7]. Below, we use linear to mean continuous and additive.

**Theorem 5 (Quantifier-Weakening Theorem, cf. [BinO10, Th. 6], [BinO7, Th. 6]).** With \( K^*_{hf} \) and \( \mathbb{A}_{hf} \) as above, suppose that

(i) \( \mathbb{A}_{hf} \) is a dense subgroup of \( \mathbb{G}_\eta \);
(ii) \( K^*_{hf} \) satisfies the one-sided Heiberg-Seneta boundedness condition

\[
\limsup_{u \downarrow 0} K^*_{hf}(u) \leq 0 \tag{HS}
\]

- then \( \mathbb{A}_{hf} = \mathbb{G}_\eta \) and \( K^*_{hf} \) is linear (continuous and additive):

\[
K^*_{hf}(u) = \lim_{x \to \infty} [f(u + x) - f(x)]/h(x) = K^*_\kappa(u)
\]

for some \( \kappa \in \mathbb{R} \), and all \( u \in \mathbb{G}_\eta \).

**Proof of Theorem 5.** As we assume here that \( \mathbb{A}_{hf} \) is a subgroup, referring to results in [BinO10, Props 3 and 6], \( K^*_f \) is a finite, subadditive, right-continuous extension of \( G \). So \( G \) is continuous on \( \mathbb{A}_f \), and so \( G(\sigma) = K^*_\kappa(\sigma) \), for all \( \sigma \in \mathbb{A}_f \). As \( \mathbb{A}_f \) is dense, by [BinO10, Prop. 7], \( K^*_f(u) = K^*_\kappa(u) \) for all \( u \). By [BinO10, Prop. 1], \( \mathbb{A}_f = \mathbb{G} \) and \( K^*_f(u) = G(u) \). □

We turn now to thinnings of the condition \((HS)\) of Theorem 5. For this we need some definitions from [BinO10].
Definitions. 1. Say that $\Sigma$ is *locally Steinhaus-Weil* (SW), or has the SW property locally, if for $x, y \in \Sigma$ and, for all $\delta > 0$ sufficiently small, the sets

$$
\Sigma^\delta := \Sigma \cap B_\delta(z),
$$

for $z = x, y$, have the *interior-point property*, that $\Sigma^\delta_x \pm \Sigma^\delta_y$ has $x \pm y$ in its interior. (Here $B_\delta(x)$ is the open ball about $x$ of radius $\delta$.) See [BinO3] for conditions under which this property is implied by the interior-point property of the sets $\Sigma^\delta_x - \Sigma^\delta_x$ (cf. [BarFN]); see also the rich list of examples below, which are used in [BinO8,10,11,13,14], [MilMO].

2. Say that $\Sigma \subseteq \mathbb{R}$ is *shift-compact* if for each *null sequence* $\{z_n\}$ (i.e. with $z_n \to 0$) there are $t \in \Sigma$ and an infinite $M \subseteq \mathbb{N}$ such that

$$
\{t + z_m : m \in M\} \subseteq \Sigma.
$$

See [BinO4], and for the group-action aspects, [MilO].

Examples of families of locally Steinhaus-Weil sets (see e.g. [BinO13]).

The sets listed below are typically, though not always, members of a topology on an underlying set.

(o) $\Sigma$ a usual (Euclidean) open set in $\mathbb{R}$ (and in $\mathbb{R}^n$) – this is the ‘trivial’ example;

(i) $\Sigma$ density-open subset of $\mathbb{R}$ (similarly in $\mathbb{R}^n$) (by Steinhaus’s Theorem – see e.g. [BinGT, Th. 1.1.1], [BinO13], [Oxt, Ch. 8]);

(ii) $\Sigma$ locally non-meagre at all points $x \in \Sigma$ (by the Piccard-Pettis Theorem – as in [BinGT, Th. 1.1.2], [BinO13], [Oxt, Ch. 8] – such sets can be ‘thinned out’, i.e. extracted as subsets of a second-category set, using separability or by reference to the Banach Category Theorem [Oxt, Ch.16]);

(iii) $\Sigma$ the Cantor ‘excluded middle-thirds’ subset of $[0,1]$ (since $\Sigma + \Sigma = [0,2]$);

(iv) $\Sigma$ universally measurable and open in the *ideal* topology ([LukMZ], [BinO9]) generated by omitting Haar null sets (by the Christensen-Solecki Interior-points Theorem of [Sol]);

(v) $\Sigma$ a Borel subset of a Polish abelian group and and open in the ideal topology generated by omitting *Haar meagre* sets in the sense of Darji [Dar] (by Jabłońska’s generalization of the Piccard Theorem, [Jab1, Th. 2], cf. [Jab3], and since the Haar-meagre sets form a $\sigma$-ideal [Dar, Th. 2.9]); for details see [BinO13].
If $\Sigma$ is Baire (has the Baire property) and is locally non-meagre, then it is co-meagre (since its quasi interior is everywhere dense).

**Caveat.** 1. Care is needed in identifying locally SW sets: Matoůsková and Zelený [MatZ] show that in any non-locally compact abelian Polish group there are closed non-Haar null sets $A, B$ such that $A + B$ has empty interior. Recently, Jabłońska [Jab4] has shown that likewise in any non-locally compact abelian Polish group there are closed non-Haar meager sets $A, B$ such that $A + B$ has empty interior.

2. For an example on $\mathbb{R}$ of a compact subset $S$ such that $S - S$ does contains an interval, but $S + S$ has measure zero and so does not, see [CrnGH].

3. Here we are concerned with subsets $\Sigma \subseteq \mathbb{R}$ where such ‘anomalies’ are assumed not to occur.

We can now state some thinned variants of Th. 6.

**Theorem 6 (Thinned Quantifier Weakening Theorem; [BinO10, Th. 1′], cf. [BinO7, §6 Th. 5]).** Theorem 5 above holds with condition (ii) replaced by any one of the following:

(ii-a) $K_{hf}^*$ satisfies the Heiberg-Seneta boundedness condition thinned out to a symmetric set $\Sigma$ that is locally SW, i.e.

$$\limsup_{u \to 0, u \in \Sigma} K_{hf}^*(u) \leq 0;$$

(ii-b) $K_{hf}^*$ is linearly bounded above on a locally SW subset $\Sigma \subseteq \mathbb{R}_+ = (0, \infty)$ accumulating at 0, so that in particular

$$\limsup_{u \downarrow 0, u \in \Sigma} K_{hf}^*(u) \leq 0;$$

(ii-c) $K_{hf}^*$ is bounded above on a locally SW subset $\Sigma \subseteq \mathbb{A}_+$ accumulating at 0, that is, the following lim sup is finite:

$$\limsup_{u \downarrow 0, u \in \Sigma} K_{hf}^*(u) < \infty; \quad \text{(SW-HS($K_{hf}^*$))}$$

(ii-d) $S$ is bounded on a subset $\Sigma \subseteq A$ that is shift-compact (e.g. on a set that is locally SW, and so on an open set) and

$$A = A_{hf} := \{u : K_{hf}(u) := \lim_{x \to \infty} [f(u + x) - f(x)] / h(x) \text{ exists and is finite}\}.$$

**Proof.** This follows from the Popa variant of [BinO10, Theorem 1′], the proof of which follows from Theorems G2 and G3 of §4 above in place of [BinO10, Theorems 0′ and 0]. □
The classical *Quantifier Weakening Theorems* of regular variation ([BinGT, §1.4.3 and §3.2.5]) are re-stated below as Theorems K and BKdH. There, one needs as side-condition the Heiberg-Seneta condition $HS$ restated multiplicatively here as $(\limsup)$ (or a thinned version of it, as in Theorem 6). Recall from above the * notation (as in $g^*$) signifying that $\limsup$ replaces $\lim$.

**Theorem K** (cf. [BinGT, Th. 1.4.3]). Suppose that

$$\limsup_{\lambda \downarrow 1} K_1^*(\lambda) \leq 1.$$  \hspace{1cm} (lim sup-f)

Then the following are equivalent:

(i) there exists $\rho \in \mathbb{R}$ such that

$$f(\lambda x)/f(x) \to \lambda^\rho \quad (x \to \infty)(\forall \lambda > 0);$$

(ii) $g(\lambda) = \lim_{x \to \infty} f(\lambda x)/f(x)$ exists, finite for all $\lambda$ in a non-negligible set;

(iii) $g(\lambda)$ exists, finite, for all $\lambda$ in a dense subset of $(0, \infty)$;

(iv) $g(\lambda)$ exists, finite for $\lambda = \lambda_1, \lambda_2$ with $(\log \lambda_1)/\log \lambda_2$ irrational.

Theorem K is an immediate corollary of Theorem 5, as $(\limsup)$ iff $(\limsup-)$.

The final assertion follows from Kronecker’s theorem [HarW, Ch. 23].

**Theorem BKdH** (cf. [BinGT, Th. 3.2.5]). For $h$ with

$$\lim_{x \to \infty} h(\lambda x)/h(x) = \lambda^\rho \quad (\lambda > 0),$$

and

$$\limsup_{\lambda \downarrow 1} K_2^*(\lambda) \leq 0,$$  \hspace{1cm} (lim sup-h)

the following are equivalent:

(i) $K_{hf}(\lambda) := \lim_{x \to \infty} [f(\lambda x) - f(x)]/h(x)$ exists, finite for all $\lambda > 0$, and $K_{hf}(\lambda) = c\eta_\rho^{-1}(\lambda^\rho)$ for some $c$ and all $\lambda$ on a non-negligible set;

(ii) $K_{hf}(\lambda)$ exists, finite for all $\lambda$ in a non-negligible set;

(iii) $K_{hf}(\lambda)$ exists, finite, for all $\lambda$ in a dense subset of $(0, \infty)$;

(iv) $K_{hf}(\lambda)$ exists, finite for $\lambda = \lambda_1, \lambda_2$ with $(\log \lambda_1)/\log \lambda_2$ irrational.

Theorem BKdH is an immediate corollary of Theorem 4. As before the final assertion follows from Kronecker’s theorem.

The motivation for this paper was the treatment of Theorems K and BKdH above via Popa groups in [BinO7, §7] (specifically $(GFE)$ and $(GS)$ there and their equivalence), using the extra power of the extra generality here to provide a unified and simplified treatment.
6 Concluding Remarks

Beurling's Tauberian theorem. To extend the Wiener Tauberian Theorem (Theorem W, say) Beurling introduced (in unpublished lectures of 1957) his Tauberian Theorem (below), extending Theorem W from convolutions to ‘convolution-like’ operations. We need the Beurling convolution:

\[ F \ast_{\varphi} H(x) := \int F\left(\frac{x - u}{\varphi(x)}\right) H(u) \frac{du}{\varphi(x)} = \int F(-t) H(x \circ_{\varphi} t) \, dt. \]

This is an asymptotic version, involving the function \( \eta_x(.) \) of §3:

\[ \eta_x(t) := \varphi(x \circ_{\varphi} t)/\varphi(x), \]

of an ordinary convolution (below).

**Theorem B (Beurling’s Tauberian theorem).** For \( K \in L_1(\mathbb{R}) \) with Fourier transform \( \hat{K} \) non-zero on \( \mathbb{R} \), and \( \varphi \) Beurling slowly varying, that is

\[ \eta_x(t) \to 1 \quad (x \to \infty) \quad (t \geq 0) : \quad (BSV) \]

if \( H \) is bounded, and

\[ K \ast_{\varphi} H(x) \to c \int K(y)dy, \]

then for all \( F \in L_1(\mathbb{R}) \)

\[ F \ast_{\varphi} H(x) \to c \int F(y)dy \quad (x \to \infty). \]

This reduces to Theorem W on replacing \( \varphi \) by 1. For an elegant proof, see [Kor, IV.11].

In Theorem W, the argument in the integral above (with \( \varphi = 1 \)) is \( x - u \), and so is a convolution (written additively, or \( x/u \) multiplicatively). In Theorem B, the integral is merely ‘convolution-like’. Beurling was able to use his form of slow variation, (BSV), to reduce easily to convolution form, and so to Theorem W. His motivation was the Tauberian theorem for the Borel summability method, important in summability theory, complex analysis and probability [Kor, VI]. For applications in probability, see e.g. [Bin1,3].
Beurling convolution is an *asymptotic convolution*: to within a factor $\eta_x(t) \to 1$, it is the proper convolution
\[
(f \ast \varphi g)(x) := \int_{G_\rho} f(-t/\eta_x(t)) g(x \circ_{\varphi} t) \, d\eta_{G_\rho}(t) \quad (x \in G_\rho)
\]
For, given $x$ and $t$, solving for $s$ the equation
\[
x = (x \circ_{\varphi} t) \circ_{\varphi} s = x + t\varphi(x) + s\varphi(x + t\varphi(x))
\]
yields
\[
s = -t\varphi(x)/\varphi(x + t\varphi(x)) = -t/\eta_x(t)
\]
as the ‘inverse of $t$’ (relative to the binary operation $\circ_{\varphi}$ acting on the set $G_\rho$).

For $\varphi \in SE$, the corresponding asymptotic convolution is
\[
(f \ast \varphi g)(x) := \int f(-t/\eta_\rho(t)) g(x \circ_{\varphi} t) \, d\eta_{G_\rho}(t).
\]
For $\varphi(x) := \eta_\rho(x) \in GS$, since
\[
\eta_x(t) := \frac{\varphi(x + t\varphi(x))}{\varphi(x)} = \frac{\eta_\rho(x + t\eta_\rho(x))}{\eta_\rho(x)} = \eta_\rho(t),
\]
$(f \ast \varphi g)(x)$ becomes
\[
(f \ast \eta_\rho g)(x) := \int f(-t/\eta_\rho(t)) g(x \circ_{\eta_\rho} t) \, d\eta_{G_\rho}(t) = \int f(-t\rho) g(x + \rho t) \, d\eta_{G_\rho}(t),
\]
with the notation of Theorem 2. So in this case the asymptotic convolution becomes ordinary convolution for the Popa group $(G_\rho, \circ_{\varphi})$.

**Postscript.**

The whole area of regular variation stems from the pioneering work of Jovan Karamata (1902-1967) in 1930. The present paper stems from his work with Ranko Bojanic (1925-2017) of 1963 [BojK]. The first author offers here a reminiscence of his first meeting with Ranko Bojanic (in 1988, over dinner, at a conference at Ohio State University, Columbus OH). He asked Professor Bojanic why he and Karamata had stopped their work on regular variation in 1963. He replied unhesitatingly ‘Because we didn’t know what it was good for’. Analysts in general, and probabilists in particular, do now know what it is good for. Our aim here has been to demonstrate the power, and ongoing influence, of their work, with the benefit of 55 years worth of hindsight.
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