Gravity solutions for the D1-D5 system with angular momentum

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We construct a large family of supergravity solutions that describe BPS excitations on $AdS_3 \times S^3$ with angular momentum on $S^3$. These solutions take into account the full backreaction on the metric. We find that as we increase the energy of the excitation, the energy gap to the next non-BPS excitation decreases. These solutions can be viewed as Kaluza-Klein monopole “supertubes” which are completely non-singular geometries. We also make some remarks on supertubes in general.

1. Introduction

Perhaps one of the most distinctive aspects of gravity is that time slows down near heavy objects due to gravitational redshift.

We now have many cases where we have dual descriptions of gravitational theories in terms of ordinary quantum field theories via the AdS/CFT correspondence. It is interesting then to find situations where this effect is under some degree of control, so that we can understand it from the field theory point of view. Black holes are extreme examples where
this redshift factor goes to zero. In this paper we consider configurations where this redshift factor is important but does not go to zero.

We focus on \( AdS_3 \times S^3 \) compactifications, and we consider states with angular momentum on \( S^3 \) that are BPS. These states are also called “chiral primary” states. When these states carry large amounts of angular momentum, their back-reaction on the metric cannot be ignored. In this paper we construct exact gravity solutions which take this backreaction into account. We indeed find that there is an important redshift effect that implies, among other things, that the energy gap to the next non-BPS excitation decreases as we increase the angular momentum. This gap goes to zero for certain states that are on the verge of forming black holes.

These solutions can be found by noticing that the D1/D5 system with angular momentum blows up into a Kaluza-Klein monopole supertube, U-dual to the one described in \([1]\). Since the Kaluza-Klein monopole is non-singular, these geometries are non-singular. The configuration with maximal angular momentum, which corresponds to a supertube with circular shape, has a near horizon geometry equal to \( AdS_3 \times S^3 \) in global coordinates. Supertubes with non-circular shapes correspond to chiral primary excitations on the \( AdS_3 \times S^3 \) vacuum.

The solutions are also interesting since they provide non-singular gravity solutions for configurations that are 1/4 BPS in toroidally compactified string theory. Different gravity solutions are related to different microscopic states.

Previous work on the subject focused on gravity solutions with conical singularities. We show that these conical singularities are not a good description of the long distance properties of generic chiral primaries, i.e. the non-singular solutions are different, even at long distances. Some very special chiral primaries can give conical metrics with opening angles of the form \( 2\pi/N \). Conical metrics with non-integer angles are not a good approximation to any of the non-singular metrics. Singular geometries more closely related to chiral primaries can be found in \([2]\). We will show that our solutions look like the solutions in \([2]\) at long distances.

In this paper we also analyze some aspects of the geometry of supertubes in other dimensions and in various limits.

This paper is organized as follows. In section two we describe the construction of the gravity solutions. In section three we discuss the relation of these gravity solutions to the problem of chiral primaries in \( AdS_3 \times S^3 \). In section four we describe general non-singular solutions with plane wave asymptotic boundary conditions, which can be thought of as
arising from particles propagating on plane wave backgrounds. In section five we discuss some aspects of the gravitational geometry of supertubes in various dimensions. This section is a bit disconnected from the previous part of the paper.

2. Gravity solutions for the D1-D5 system with angular momentum

In this section we consider ten dimensional supergravity compactified on $S^1 \times T^4$, and consider a system of $Q_1$ D1 branes wrapped on $S^1$ and $Q_5$ D5 branes wrapped on all the compact directions. We are interested in constructing solutions which carry angular momentum and are 1/4 BPS. In other words, they are as BPS as the D1 and D5 branes with no angular momentum. Since there are four non-compact transverse directions, the angular momenta live in $SO(4) \sim SU(2)_L \times SU(2)_R$. The angular momentum is bounded by $J_L, J_R \leq \frac{k}{4} \equiv Q_1 Q_5$. For large values of $Q$ the angular momentum can be macroscopic and can have an important effect on the geometry of the configuration. This was initially explored in [4][5] who found that the geometry with maximal angular momentum was non-singular. In the meantime, studies of other 1/4 BPS configurations with angular momentum have given rather interesting results. The best known example is the so called “supertube” which is a configuration carrying D0 and fundamental string charges with angular momentum, which is described in terms of a tubular D2 brane with electric and magnetic fields on its worldvolume [1]. The configuration with maximal angular momentum consists of a tubular D2 brane with a radius square proportional to the product of the two charges. The configuration does not carry any net D2 brane charge. Tubes with arbitrary cross sections are also possible, but they carry less angular momentum [1].

The D0-F1 system is U-dual to D1-D5. Under this U-duality the above D2 brane goes over to a Kaluza Klein monopole which is wrapped on $T^4$ and a circle in the four non-compact dimensions. The special circle of the KK monopole is the $S^1$ common to the D1 and D5 branes. The gravity solution for a circular KK monopole was found in [4,5] (based on the general solutions in [6]) though it was not given this description (which is not that obvious by just looking at the metric). This solution is non-singular because the KK monopole has a non-singular geometry.

Now we construct similar solutions with arbitrary shapes which are also non-singular. The technique we use to find the solution is based on the observation that this system is U-dual to fundamental strings with momentum along the string. Microscopic configurations

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1 In appendix B we explain how to obtain the solutions for the $K3$ case.
of the system are given by strings carrying traveling waves along them, in other words, strings with only left (or only right) moving excitations. For this case there are gravity solutions that closely correspond to given microscopic states. Namely these solutions describe an oscillating string with an arbitrary profile $F(v)$, where $v$ is a lightcone coordinate along the string. By a chain of dualities these can be mapped to the D1-D5 system so that we find the solution (see appendix B)

$$ds^2 = f_1^{-1/2} f_5^{-1/2} [-(dt - A_idx^i)^2 + (dy + B_idx^i)^2] + f_1^{1/2} f_5^{1/2} dx \cdot dx$$

$$+ f_1^{1/2} f_5^{-1/2} dz \cdot dz$$

$$e^{2\phi} = f_1 f_5^{-1},$$

$$C^{(2)}_{ti} = \frac{B_i}{f_1}, \quad C^{(2)}_{ty} = f_1^{-1} - 1,$$

$$C^{(2)}_{iy} = - \frac{A_i}{f_1}, \quad C^{(2)}_{ij} = C_{ij} + f_1^{-1} (A_i B_j - A_j B_i)$$

The functions $f_1, f_5$ and $A_i$ appearing in this solution are related to the profile $F(v)$

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x - F|^2}, \quad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{\dot{F}^2 dv}{|x - F|^2}, \quad A_i = - \frac{Q_5}{L} \int_0^L \frac{\dot{F}_i dv}{|x - F|^2}$$

and the forms $B_i$ and $C_{ij}$ are defined by the duality relations

$$dC = -*_4 df_5, \quad dB = -*_4 dA.$$  

where the $*_4$ is defined in the four non-compact spatial dimensions. The one brane charge is given by

$$Q_1 = Q_5 \langle |\dot{F}|^2 \rangle = Q_5 \frac{1}{L} \int_0^L |\dot{F}|^2 dv$$

The length $L$ that appears in these formulas is

$$L = \frac{2\pi n_5}{R} = 2\pi n_5 R'$$

where $n_5$ is the number of fivebranes and $R$ is the radius of the $y$ circle, while $R'$ is the radius in the original fundamental string description. $n_5$ is the original number of strings which becomes the number of fivebranes. We see that we are taking a configuration where

\[\text{For simplicity we have set } g = \alpha' = V_4 = 1 \text{ in the above formulas. In that case } Q_1 = n_1 \text{ and } Q_5 = n_5, \text{ otherwise } Q_i \text{ have dimensions of length square and denote the contribution of the onebranes and fivebranes to the gravitational radius of the configuration, while } n_{1,5} \text{ are integers.}\]
the string is multiply wound. This will be important for later considerations. Configurations where the string consists of independent pieces can be obtained by adding the corresponding contributions in the harmonic functions (2.2). The solutions are parameterized by the profile $F(v)$ which describes a trajectory in the four non-compact dimensions. Note that the final solution (2.1) is time independent. We will see that the $v$ dependence of $F$ translates into a dependence of the solution on the non-compact dimensions. The angular momentum of the solution (2.1) is given by

$$J_{ij} = \frac{Q_5 R}{L} \int_0^L (F_i \dot{F}_j - F_j \dot{F}_i) dv$$

(2.6)

It can be checked that the angular momentum is always smaller than $n_1 n_5$. We will later concentrate on the two $U(1)$ components $J_{j_1} = J_{12}, J_{j_2} = J_{34}$ and define $2J_{L,R} = J_{j_1} \pm J_{j_2}$.

Note that all these solutions correspond to different ground states of the D1/D5 system. This system has a large degeneracy, of order $e^{2\pi \sqrt{2Q_1 Q_5}}$.

2.1. An argument showing the solution is non-singular

Looking at the metric (2.1), one might think that it is singular if $x = F(v_0)$ for some value of $v_0$, since the harmonic functions (2.2) diverge there. However, it was shown in [4,5] that the maximally rotating solution is non-singular. The maximally rotating solution corresponds to a circular profile

$$F_1 + iF_2 = ae^{i\omega v} \quad F_3 = F_4 = 0, \quad \text{with} \quad \omega = \frac{2\pi}{L} = \frac{R}{n_5}$$

(2.7)

From the expression for the charges (2.4) we get that the radius is

$$a = \frac{\sqrt{Q_1 Q_5}}{R}$$

(2.8)

On the other hand if we have a circular profile with a frequency $\omega' = n\omega$ (and $a' = a/n$), we would get a geometry which has a conical singularity of opening angle $2\pi/n$.

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3 More precisely, these solutions are particular combinations of states of the theory. Classically there is an infinite number of solutions since they are parameterized by continuous parameters. In the quantum theory we should quantize the moduli space of solutions and that will give us a finite number. This quantization is expected to give us the same as quantizing the left movers on a string, though we did not verify it explicitly.
Let us now look at the geometry corresponding to a more general profile $F(v)$. We will analyze the metric near the potential singularity $x = F(v_0)$ and show that for a generic profile $F(v)$ the solution is completely regular. By generic we mean a profile satisfying two conditions:

(i) the profile does not have self–intersections (if $v_1 \neq v_2$, then $F(v_1) \neq F(v_2)$);

(ii) the derivative $\dot{F}(v)$ never vanishes.

Looking at the vicinity of the singularity for such profile, we find

$$f_5 \approx \frac{Q_5}{L} \int_{-L/2}^{L/2} \frac{dv}{|x - F|^2} \approx \frac{Q_5}{L} \int_{-L/2}^{L/2} \frac{dv}{x_\perp^2 + (\dot{F})^2 v^2} = \frac{Q_5}{L} \frac{\pi}{|\dot{F}| x_\perp},$$

$$f_1 \approx \frac{Q_5}{L} \frac{\pi |\dot{F}|}{x_\perp}, \quad A_i \approx -\frac{Q_5}{L} \frac{\pi \dot{F}_i}{|\dot{F}| x_\perp}$$

We have split the coordinates of the transverse space around the point $F(v_0)$ into a longitudinal piece, $x_l$, along $\dot{F}(v_0)$ and a transverse piece $x_\perp$.

The asymptotics (2.9) can be used to show that there are no singularities in the longitudinal piece of the metric

$$ds_{l}^2 \equiv |\dot{F}| \left[ f_5 dx_l^2 - f_1^{-1} |A_i|^2 dx_l^2 \right],$$

but they are not good enough for finding the finite contribution to $ds_l$. We will refer to the appendix F of [9] where more careful analysis was done, and give the result

$$ds_{l}^2 = |\dot{F}| C dx_l^2$$

where $C$ is a positive numerical coefficient whose value depends on global properties of the profile

$$C(v_0) = \frac{1}{|\dot{F}(v_0)|^2} \left\{ \frac{Q_5}{L} \int_0^L \frac{dv (\dot{F}(v) - \dot{F}(v_0))^2}{(F(v) - F(v_0))^2} + (1 + |\dot{F}(v_0)|^2) \right\}$$

Let us now analyze the metric in the space transverse to the singularity

$$ds_{\perp}^2 \equiv |\dot{F}| \left[ f_5(dx_\perp^2 + x_\perp^2 d\Omega_2^2) + f_1^{-1}(B_i dx^i)^2 \right]$$

In order to compute the leading order terms in the metric it is important to compute $B_i$ which is dual to $A_i$. We only need to compute this to leading order in $x_\perp$ so that we find

$$B_\psi \sim -(\cos \theta - 1) \frac{\pi Q_5}{L}$$
where the metric in the flat transverse space is parameterized as
\[ ds^2_0 = dx^2 + dx^2_\perp + x^2_\perp (d\theta^2 + \sin^2 \theta d\psi^2) \] (2.15)

Note that the range of \( \theta \) is \( 0 \leq \theta < \pi \). Then the transverse metric (2.13) becomes
\[ ds^2 = \frac{4Q_5 \pi}{L} \left[ (d\sqrt{x_\perp})^2 + x_\perp \left\{ \left( \frac{d\theta}{2} \right)^2 + \sin^2 \frac{\theta}{2} d\psi^2 \right\} \right] \] (2.16)

Let us now look at the complete metric
\[ ds^2 = |\hat{F}| C dx^2_\perp + ds^2_\perp + \frac{L x_\perp}{\pi Q_5} \left\{ dy^2 + 2B_i dx^i dy - dt^2 + 2A_i dx^i dt \right\}, \] (2.17)

Near the singularity we get
\[ ds^2 = \frac{4Q_5 \pi}{L} \left\{ (d\sqrt{x_\perp})^2 + x_\perp \left[ \left( \frac{d\theta}{2} \right)^2 + \sin^2 \frac{\theta}{2} (dy + \frac{dy}{R})^2 + \cos^2 \frac{\theta}{2} \frac{dy^2}{R^2} \right] \right\} \] (2.18)

where we used (2.3). Let us introduce new coordinates
\[ \chi = \frac{y}{R}, \quad \tilde{\psi} = \psi + \chi, \quad \tilde{\theta} = \frac{\theta}{2}, \quad \rho = \sqrt{x_\perp} \] (2.19)

We see that \( \chi = \frac{y}{R} \) has periodicity \( 2\pi \), and thus the change of coordinates from \( \psi \) to \( \tilde{\psi} \) is well defined. We also note that \( \tilde{\theta} \) has a range \( 0 \leq \tilde{\theta} < \pi/2 \) and the metric (2.18) becomes:
\[ ds^2 = \frac{4Q_5 \pi}{L} \left\{ d\rho^2 + \rho^2 \left[ d\tilde{\theta}^2 + \sin^2 \frac{\tilde{\theta}}{2} d\tilde{\psi}^2 + \cos^2 \frac{\tilde{\theta}}{2} d\chi^2 \right] \right\} \] (2.20)

The first line in the above expression gives the metric of a flat four dimensional space from which we conclude that the geometry is regular near the string profile. Note also that the Killing vector \( \partial_t \) becomes light like at \( \rho = 0 \).

To summarize, what is happening is that the circle \( y \) is shrinking to zero size as we approach the string source but the presence of the field \( B_i \) implies that it is non-trivially fibered over the \( S^2 \) that is transverse to the line described by \( F \) in \( R^4 \) and we see that
the $y$ circle combines with the two sphere and $x_\perp$ to give a non-singular space, precisely as it happens for the Kaluza Klein monopole. Note also that a non-trivial condition on the harmonic functions characterizing the solution arises from demanding that the $B_i$ field leads to a well defined fibration of the $y$ circle on the four dimensional space away from the sources. This quantization condition on $B_i$ translates into a quantization condition for the field $A_i$, which is obeyed with a unit coefficient if we take and $A_i$ field as in (2.9) with $2\pi Q_i/L = R_y$.

In fact we can view these metrics as “supertubes” analogous to the ones described in [1] where the D1-D5 system blows up to a KK monopole that wraps the four directions of $T^4$ and a curve of shape given by the profile $F(v)$ in the non-compact directions. The circle of the KK monopole is the $y$ circle where the initial D1 and D5 are wrapped. The final geometry is non-singular if the system blows up into a single KK monopole. This is ensured if the function $F$ does not self intersect.

One would like to know what the topology of these solutions is. We can ignore the $T^4$ for this question. Before we put the D1-D5 system, the topology of the six dimensional space is $R \times R^4 \times S^1$. The topology of a fixed radius surface far away is $R \times S^3 \times S^1$. When we go in the radial direction the $S^3$ is filled in so that we have $R^4$. Let us start with the non-singular maximally rotating circular solution. The topology of a fixed radius surface far away is the same as above but when we go in we fill in the $S^1$ so that the final topology is $R \times R^2 \times S^3$. This is shown explicitly in appendix A. The topology for geometries that can be obtained as continuous deformations of the circle is still going to be the same. The actual metric and geometry of the solution will of course depend on many parameters. For example in the regime that $a^2 \gg Q_1, Q_5$ we find that the gravity solution has the shape of a ring whose gravitational thickness (of order $\sqrt{Q_1}$, or $\sqrt{Q_5}$) is much smaller than its radius.

An important lesson is that these configurations with D-brane charge can change the topology of the spacetime where they live. This situation is common in many examples of AdS/CFT. It is an example of a so called “geometric transition”.

2.2. Geometries with $A_N$ singularities

There are some singular geometries that are believed to be allowed in string theory. A particular example arises when we multiply each non-constant piece in the harmonic functions (2.2) by $N$. In that case we get a $Z_N$ singularity on the ring. In other words, instead of a single KK monopole we have $N$ coincident ones giving rise to an $A_{N-1}$ singularity.
The resulting geometry is a \( Z_N \) quotient of the geometries we discussed in the previous subsection. It is easy to see that we can find non-singular deformations of these geometries by separating the \( N \) copies of the harmonic functions in the transverse directions. When this separation is small we get that the \( A_{N-1} \) singularity becomes locally a smooth ALE space. This situation was explored in detail in [10] and we refer the reader to it for the details. Notice that if \( n_1 \) and \( n_5 \) are coprime and we are at a generic point in moduli space [11], then it is not possible to deform the \( A_{N-1} \) singularity by displacing the “center of mass” of the rings in the non-compact directions, but it is possible to deform it by combining the whole system into a single string. When we start with \( N \) coincident rings as in (2.7) we get a geometry that is a \( Z_N \) quotient of \( AdS_3 \times S^3 \). This has fixed points along an equator of \( S^3 \) and the origin of \( AdS_3 \). The metric can be written as the conical metric in Appendix C, with a coefficient \( \gamma^{-1} = N \). Conical metrics with arbitrary non-integer \( \gamma^{-1} \) have been considered in the literature. These metrics are suspicious since they would correspond to KK monopoles which do not obey the proper quantization condition. Indeed, if one looks at those metrics one finds that there are extra conical singularities as compared to the \( A_N \) case. These are easy to understand in the space parameterized by the flat coordinates \( x^i \). In this case we have “Dirac-strings” coming out of the KK monopole extended along the disk in the 12 plane with boundary on the ring. If the quantization condition is not obeyed, then the metric will be singular on this disk. Since we found a large family of non-singular metrics one might wonder if one could take a smooth metric which approximates these conical spaces with \( \gamma^{-1} \) not integer arbitrarily well. We argue in appendix C that this is not possible. In conclusion, conical metrics with \( \gamma^{-1} \) not integer are not a good approximation to the real solutions.

3. Geometries corresponding to chiral primaries.

We first need to take the decoupling limit of the solutions that we considered above (2.1). This amounts to dropping the ones in the harmonic functions \( f_1 \) and \( f_5 \) in (2.2). We can then see that the asymptotic geometry for large \( |x| \) is that of \( AdS_3 \times S^3 \). If we take the standard periodic conditions on the spinors along \( S^1 \) that preserve supersymmetry in the asymptotically flat context then we see that we are in the Ramond sector of the theory. Different solutions correspond to different Ramond vacua. These vacua can have various values of angular momenta ranging over \( -k/2 \leq J_{L,R} \leq k/2 \). The solution with a circular profile in the 12 plane corresponds to the Ramond vacuum with maximal angular
momentum $J_L = J_R = k/2$. Under spectral flow this state goes over to the NS vacuum. Spectral flow in the CFT is an operation that maps states in the R sector to states in the NS sector. It is a rather trivial operation involving only the overall U(1) R-charge of the states so we do not expect a significant change in the properties of the state when we perform it. In fact spectral flow amounts to a simple coordinate redefinition

$$\tilde{\phi} = \phi - \frac{t}{R}, \quad \tilde{\psi} = \psi + \frac{y}{R}$$

(3.1)

with these new variables the time independent configurations that we had in the Ramond sector can become time dependent if they depended on $\phi$. For example, the $\phi$ independent ring solution (2.7) becomes the time independent $AdS_3 \times S^3$ vacuum. Ramond sector solutions where the ring is deformed to other shapes, like an ellipse, for example, become time dependent when they are viewed as NS sector solutions. This is related to the fact that chiral primaries carry non-zero energy in the NS sector. Under spectral flow all RR vacua correspond to chiral primary states with

$$J_{L,R}^{NS} = J_{L,R}^R - \frac{k}{2}, \quad L_0^{NS} = |J_{L}^{NS}|, \quad \bar{L}_0^{NS} = |J_{R}^{NS}|.$$  

(3.2)

The physical properties of the solution in the interior of the space do not change when we do spectral flow, since it is just a redefinition of what we mean by energy and angular momentum. Note, in particular, that the statement that the $M = 0$ BTZ black hole is the Ramond ground state is imprecise. There are many Ramond ground states and they look quite different in the supergravity description depending on their angular momenta.

As discussed in [14][15][16] chiral primary states close to the NS ground state can be obtained by adding perturbative gravity modes on the NS ground state. They are particular gravity modes that are BPS. First we restrict to deformations of the ring into a more general shape in the 12 plane. There are two classes of deformations we can consider. One is a change in the shape of the ring and the other is small changes in the velocity with which we go around the ring. These two correspond to two towers of chiral primary states.

\[4\] Note, in particular, that spectral flow acting on the NS vacuum does not produce conical singularities as was asserted in [12][13]. A more detailed discussion about the action of spectral flow can be found in [3][4].
The supergravity chiral primary states are given as follows \([14][15][16]\). It is convenient to separate the three form field strengths in six dimensions into self dual and anti-self dual parts. The background fields in \(\text{AdS}_3 \times S^3\) are self dual. The chiral primary fields correspond to fluctuations in the anti-self dual part of the three form field strengths on the \(S^3\) (which also mix with fluctuations of scalar fields). These fields produce gravity modes with \((J_L, J_R) = 1/2, 1, 3/2, \cdots\). There is also one special tower of supergravity fields which starts at \((J_L, J_R) = 1, 3/2, \cdots\). These come from certain fluctuations in the metric of the three sphere.

The two classes of deformations of the ring that we discussed above correspond to two of these towers of chiral primaries. More precisely, changes in the velocity correspond to the tower associated to the anti-self dual component of the field strength whose self dual component is turned on in the background. The changes in shape correspond to the tower associated with deformations of the sphere. This can be seen by noticing that the lowest angular momentum deformation of the shape we can do is to deform the circle to an ellipse. This has angular momentum \(J_\phi = 2\) which corresponds to \(J_L = J_R = 1\). On the other hand we can change the velocity by \(v \rightarrow v + \epsilon \cos v\) which will introduce a mode with angular momentum \(J_\phi = 1\) which corresponds to \((J_L, J_R) = (1/2, 1/2)\). In summary, different Fourier modes in the expansion of \(F_{1,2}\) around the circular profile are in direct correspondence with chiral primary gravity modes with different values of angular momentum on \(S^3\) (i.e. different values of \(J_\phi\)).

There is another chiral primary tower with \((J_L, J_R) = (m + 1, m), \ m = 0, 1/2, \cdots\) and a similar one with \(J_L \leftrightarrow J_R\). These corresponds to oscillations of the ring into the 34 plane.

Finally we should consider many other chiral primaries that come from the anti-self dual components of other field strengths. These are easily described in the \(T^4\) theory as oscillations in the internal \(T^4\) directions of the initial fundamental string which we used to construct the solutions. We give the general solutions for those in appendix B. One nice aspect of those solutions is that we can easily find some solutions for which the metric is \(\phi\) independent. For example, choosing a simple profile where the string is also oscillating with frequency \(m\omega\) in the internal torus, we obtain in the near-horizon-limit the six dimensional metric (see appendix B)

\[
\frac{ds^2}{\sqrt{Q_1Q_5}} = (r^2 + \beta \cos^2\theta) \frac{1}{\sqrt{\alpha}} \left[ -(dt - \frac{\beta \sin^2\theta d\phi}{r^2 + \beta \cos^2\theta})^2 + (d\chi + \frac{\beta \cos^2\theta d\psi}{r^2 + \beta \cos^2\theta})^2 \right] + \sqrt{\alpha} dr^2 + r^2 + \beta \cos^2\theta (r^2 \cos^2\theta d\psi^2 + (r^2 + \beta) \sin^2\theta d\phi^2) \tag{3.3}
\]
where the function $\alpha$ is given by

$$\alpha = 1 - (1 - \beta) \left( \frac{\beta \sin^2 \theta}{r^2 + \beta} \right)^m$$  \hspace{1cm} (3.4)

and $\beta$ and $m$ are two parameters characterizing the solution. $m$ is the angular momentum of the single particle chiral primary we are exciting with $(J_L, J_R) = (\frac{m}{2}, \frac{m}{2})$. In other words we are considering a coherent state associated to this single particle chiral primary. The parameter $0 \leq \beta \leq 1$ measures the total angular momentum of the solution which is $J_{NS}^L = J_{NS}^R = n_1 n_5^2 (1 - \beta)$  \hspace{1cm} (3.5)

So that for $\beta = 1$ we get global $AdS_3 \times S^3$ and for $\beta = 0$ we get the singular geometry corresponding to the $M = 0$ BTZ black hole which from the NS point of view could be described as the extremal limit of a black hole that is rotating in the internal $S^3$. For $\beta > 0$ the geometry is non-singular. We can ask if the metric (3.3) goes over to the metric of a conical defect. Since the angular momentum is given by (3.3) we would expect that the opening angle of the corresponding conical defect should be $2\pi\beta$, or $\gamma = \beta$ in the notation of appendix C. These conical metrics were considered in [13, 4,5]. It turns out that the conical metric is not a good approximation to the long distance behavior of (3.3) since (3.3) contains terms that decay slowly at $r \to \infty$. In fact there is a massless field in $AdS$ with conformal weight two ($\Delta = \bar{\Delta} = 1$) which has a vev, this implies that the metric (3.3) differs from the $AdS$ metric by terms of order $1/r^2$. That is precisely the order of the difference between the conical metrics and the $AdS$ metric. This is discussed in more detail in appendix B.

It is interesting to consider the limit of very large $m$ with $\beta$ fixed. In that limit we can set $\alpha = 1$ as long as we are at a distance of order $R_{AdS}/\sqrt{m}$ from the line at $r = 0$, $\theta = \pi/2$. The limiting metric is obtained from (3.3) by setting $\alpha = 1$. In this limit the solution has singularity along the circle $r = 0, \theta = \pi/2$ like the one present in the Aichelburg-Sexl metric [17]. The metric coincides with the solution in [18], which was expected to describe the metric of high momentum particles moving along a maximum circle of $S^3$.

Another interesting limit that we can take is a “plane wave” limit where we concentrate on distances which are small compared to the $AdS$ radius. In this limit, and in the region where $\alpha = 1$, the metric is

$$ds^2 = 2dx^+dx^- - (s^2 + u^2)(dx^+)^2 + ds^2 + du^2 + u^2d\tilde{\psi}^2 + s^2d\chi^2$$

$$+ (\beta - 1) \left[ 2dx^+dx^- - (s^2 + u^2)(dx^+)^2 - \frac{(dx^-)^2}{s^2 + u^2} \right]$$

$$+ \frac{\beta - 1}{s^2 + u^2} \left[ u^4d\tilde{\psi}^2 - 2s^2u^2d\chi d\tilde{\psi} + s^4d\chi^2 \right]$$  \hspace{1cm} (3.6)
where \( x^+ = t, \ x^- = \phi R^2_{AdS} \) and \( \tilde{\psi} = \psi + \chi \). This metric is singular at \( s = u = 0 \), but close to this point it is also necessary to take into account the full form for \( \alpha \) which we give in appendix B. The final metric is non-singular and is explicitly written in appendix B. We see that the behavior near \( r = s = 0 \) is of the form we expect for a metric which is carrying momentum density \( p_- \sim (\beta - 1) \) in the \( \phi \) or \( x^- \) direction.

An interesting aspect of (3.6) is that the metric does not asymptote to the plane wave metric at large \( u^2 + s^2 \). This is due to the fact that from the plane wave point of view we have constant \( p_- \) density and therefore infinite total \( p_- \). We show in section 4 that solutions with excitations localized in the \( x^- \) direction which carry finite total \( p_- \) are indeed asymptotic to the standard plane wave.

An interesting question we would like to understand is the behavior of the energy gap in these geometries. If we concentrate on excitations that are \( \phi \) and \( \psi \) independent then the energy gap can be computed easily in the case of \( m = 1 \) where we obtain (see appendix B)

\[
\omega_0 = 2 \sqrt{\beta} \tag{3.7}
\]

for the energy of the lowest energy excitation. The energy gap for large \( m \) is harder to estimate but we prove in appendix B that it is always lower than (3.7).

We see that as we increase the energy of the solution (by decreasing \( \beta \), see (3.5)) the redshift factor at the origin decreases, so that a clock runs more slowly there and also the energy gap to the next excitation is very low.

### 3.1. Remarks on the CFT description

A semi-quantitative explanation of this fact was given in [9,18]. The idea is that these chiral primaries will involve multiply wound strings. The energy gap for exciting such states becomes smaller as \( 1/w \) where \( w \) is the winding. This decrease of the energy gap can be seen even when the CFT is at its orbifold point, where the theory becomes a free CFT whose target space is \( \text{Sym}(T^4)^{n_1n_5} \) (for a full discussion of this system see [19]). The NS ground state is in the untwisted sector and can be interpreted as consisting of \( n_1n_5 \) singly wound strings. The energy gap to the next BPS excitation is of order one (we normalized the circle of the CFT to have radius one). High angular momentum single particle chiral primaries involve strings that are multiply wound. Non-BPS excitations on a string of winding number \( w \) go as \( 2/w \). In order to obtain a more precise match with the particular energy gaps we obtained above it seems that we need to go away from the
orbiﬁed point otherwise we can easily run into contradictions. For example, let us consider
the chiral primaries with \((J_L, J_R) = (1/2, 1/2)\) that come from the internal torus. These
would naively correspond, in the free orbifold picture, to states in the untwisted sector
(singly wound strings) where we have some excitations on some of the \(n_1 n_5\) singly wound
strings. More precisely, each single particle excitation corresponds to exciting one string
by adding a left and right moving fermion in the lowest state. Since all strings are singly
wound we get an energy gap of order one at the orbifold point independently of \(\beta\). On
the other hand, the gravity description of such states is given by (3.3) with \(m = 1\), for
which the energy gap is (3.7). Clearly we need to take into account that the supergravity
picture is valid only when we get away from the orbifold point in the CFT, which blurs the
distinction between singly wound and multiply wound strings. It would be highly desirable
to understand better this effect from the CFT point of view. In [9] some agreement was
found with this naive picture, since only very special geometries and chiral primaries were
used.

4. Solutions with plane wave asymptotics

From the general solutions in (2.1) it is also possible to obtain a general family of
solutions with plane wave asymptotic boundary conditions. The ﬁnal prescription is that
in order to obtain such solutions we should drop the ones in (2.2) and consider a proﬁle
\(F(v)\) which is a straight line in \(R^4\) with small wiggles in the various \(R^4\) and \(T^4\) coordinates.
This is rather analogous to our previous discussion where we took a proﬁle that was a circle
with some wiggles.

First we discuss how to obtain this as a limit of the general metric (2.1) corresponding
to a circular proﬁle with some small oscillations. It is a limit where we zoom in into a
small section of the circle where this section looks like a straight line with some oscillations.
More precisely, we rescale the coordinates and the proﬁle:

\[
    x_1 = a + \frac{x'_1}{R}, \quad x_2 = \frac{x'_2}{R}, \quad x_3 = \frac{x'_3}{R}, \quad x_4 = \frac{x'_4}{R}, \quad t = t'R, \quad y = \chi R, \quad R = \frac{\hat{R}}{\epsilon},
\]

\[
    F_1 = a \cos \omega v + \frac{\hat{F}_1(vR)}{R}, \quad F_2 = a \sin \omega v + \frac{\hat{F}_2(vR)}{R}, \quad F_3 = \frac{\hat{F}_3(vR)}{R}, \quad F_4 = \frac{\hat{F}_4(vR)}{R}
\]

(4.1)

and we deﬁne the new rescaled functions

\[
    \hat{f}_1 = \epsilon^2 f_1, \quad \hat{f}_5 = \epsilon^2 \hat{R}^4 f_5, \quad \hat{A}_i = \frac{1}{R^2} A_i, \quad \hat{B}_i = \frac{1}{R^2} B_i,
\]
Then we take a limit $\epsilon \to 0$ while the new coordinates and $\hat{R}$ remain fixed.

Defining the new parameter $\sigma = a\omega \hat{R}^2$ and dropping the primes in the new coordinates we find

\[
\hat{f}_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2},
\]
\[
\hat{f}_5 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2},
\]
\[
\hat{A}_2 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(\sigma + \hat{F}_2)dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2},
\]
\[
\hat{A}_\perp = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{F}_\perp dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2}.
\]

Here we introduced a new integration parameter $v' = vR$. Note that in terms of this parameter the argument of $\hat{F}_i$ is $\epsilon$–independent. In the derivation of (4.2) we also used the relation (2.5)

\[
\frac{Q}{L} = \frac{R}{2\pi}
\]

Note that the functions $\hat{F}_i(v')$ are not required to be periodic, they are arbitrary functions of $v'$. In terms of the functions (4.2) the metric becomes

\[
d s^2 = \frac{1}{\sqrt{\hat{f}_1 \hat{f}_5}} \left[ -(dt - \hat{A}_i dx^i)^2 + (dy + \hat{B}_i dx^i)^2 \right] + \sqrt{\hat{f}_1 \hat{f}_5} dx^i dx^i
\]

where $\hat{B}$ is defined by

\[
d \hat{B} = -ast d\hat{A}
\]

Let us examine the asymptotic behavior of this metric at large $x_\perp$. In this limit ($x_\perp \gg |\hat{F}|$) we get the following leading contributions to the harmonic functions\footnote{Here we assumed that $\langle \hat{F}_i \rangle = 0$. If this condition is not true, then string moves along the direction $\langle \hat{F}_i \rangle$ and we can account for this motion by redefining coordinate $x_2$.}

\[
\hat{f}_5 \approx \frac{1}{2\sigma x_\perp}, \quad \hat{f}_1 \approx \frac{1}{\beta} \frac{\sigma}{2x_\perp}, \quad \hat{A}_2 \approx -\frac{1}{2x_\perp}, \quad \hat{B}_\psi \approx -\frac{(\cos \theta - 1)}{2},
\]

all other components of the gauge fields $\hat{A}$ and $\hat{B}$ are subleading. In (4.4) we introduced a parameter $\beta \leq 1$:

\[
\beta \equiv \left( 1 + \frac{1}{\sigma^2} (\langle \hat{F}_2 \rangle^2 + |\hat{F}_\perp|^2) \right)^{-1}
\]
Substituting the expressions (4.4) in (4.3) and introducing the new coordinates:

\[ x^+ = t, \quad x^- = \frac{x_2}{\sqrt{\beta}}, \quad \tilde{\psi} = \psi + \chi \]

\[ u = \beta^{-1/4} \sqrt{2 x_\perp \sin \theta} / 2, \quad s = \beta^{-1/4} \sqrt{2 x_\perp \cos \theta / 2}, \]

we find

\[ ds^2 = -\beta \left[ 2 dx^+ dx^- + (u^2 + s^2)(dx^+)^2 \right] + (1 - \beta) \frac{(dx^-)^2}{u^2 + s^2} \]

\[ + du^2 + ds^2 + \frac{u^2 s^2}{u^2 + s^2} (d\tilde{\psi} - d\chi)^2 + \frac{\beta}{u^2 + s^2} (s^2 d\chi + u^2 d\tilde{\psi})^2 \]

which is indeed the same as (3.6) after some simple changes of signs. This is the general behavior of the metric for a configuration with uniform momentum density in the \( x^- \) direction. If the excitation is localized in the \( x^- \) direction, as we expect it to be for a finite \( p^- \) wavepacket, then the profile for the corresponding vibration will be such that \( |\hat{F}| \) differs significantly from zero only in a finite range of \( v' \). Then the averages entering the expression (4.5) vanish and \( \beta = 1 \). So in this case the metric asymptotically goes to the usual plane wave.\(^6\)

So far we have been looking at profiles which oscillate only in the noncompact directions. For the case of oscillations on the torus the six dimensional Einstein metric is still given by (4.6), but the function \( \hat{f}_1 \) should be replaced by \( \hat{f}_1 \)

\[ \hat{f}_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{(\sigma + \hat{F}_2)^2 + |\hat{F}_\perp|^2 + |\hat{F}|^2}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2} \right] dv' \]

\[ - \hat{f}_5^{-1} \hat{A}_a \hat{A}_a, \]

\[ \hat{A}_a = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{F}_a dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2} \]

The large \( x_\perp \) limit of the resulting solution still has the form (4.7), but \( \beta \) is now defined by

\[ \beta \equiv \left( 1 + \frac{1}{\sigma^2} ((\hat{F}_2)^2 + |\hat{F}_\perp|^2 + |\hat{F}|^2) \right)^{-1} \]

\(^6\) Note, in particular, that the coefficient of the term that goes as \((dx^-)^2/(s^2 + u^2)\) goes to zero, while in flat space it goes over to some function of \( x^- \). The difference is due to the rapid decay in the transverse coordinates of wavefunctions with fixed \( p^- \).

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4.1. The solution for a localized excitation

In order to understand the asymptotic behavior of the solution when the excitation is localized in $x^-$, we write down the explicit form for one such solution with a very simple profile. We pick a profile with a perturbation only in the torus direction.

\[
\hat{F}_1 = \begin{cases} 
0, & |v'| > v_0 > 0 \\
b(v_0 - |v'|), & |v'| < v_0
\end{cases}
\]

with all other components of $\hat{F}_a$ and $\hat{F}_i$ equal to zero. This profile gives following harmonic functions

\[
\begin{align*}
\hat{f}_5 &= \frac{1}{2\sigma x_\perp}, \\
\hat{A}_2 &= -\frac{1}{2x_\perp}, \\
\hat{B}_\psi &= -\frac{(\cos \theta - 1)}{2}, \\
\hat{A}_1 &= \frac{b}{2\pi \sigma x_\perp} \arctan \left\{ \frac{2(\sigma v_0)^2 x_\perp x^-}{[x_\perp^2 + (x^-)^2]^2 + (\sigma v_0)^2 (x_\perp^2 - (x^-)^2)} \right\}, \\
\hat{f}_1 &= \frac{\sigma}{2x_\perp} + \frac{b^2}{2\pi \sigma x_\perp} \arctan \left( \frac{2\sigma v_0 x_\perp}{x_\perp^2 + (x^-)^2 - (\sigma v_0)^2} \right) - \hat{f}_5^- \hat{A}_a \hat{A}_a,
\end{align*}
\]

In particular for $\sigma v_0 \ll x_\perp$ and arbitrary value of $x^-$ we get

\[
\hat{f}_1 = \frac{\sigma}{2x_\perp} + \frac{b^2 \sigma v_0}{2\pi \sigma x_\perp} \left\{ \frac{2x_\perp^2}{(x^-)^2 + x_\perp^2} + O\left( \frac{\sigma v_0}{x_\perp} \right) \right\}
\]

Thus in the leading order at large $x_\perp$ we get a usual plane wave \[20\], i.e. the metric (4.7) with $\beta = 1$.

5. The supertubes in different dimensions

The previous analysis of the D1-D5 system is special because the configuration blows up to a Kaluza-Klein monopole and leads to a non-singular situation. This will be the case also for all configurations with two charges that result from doing U-duality on the $T^4$. Of course, if we did T-duality on the $S^1$ where the D1 and D5 are wrapped we would get a singular metric since the KK monopole would become an NS 5 brane. The fact that the metric is non-singular is related to the fact that the theory has a non-zero energy gap for generic non-BPS excitations around the state with maximal angular momentum. This energy gap is also non-zero for other two charge systems in different number of dimensions. So we considered similar supergravity solutions in different dimensions but we found that they were all singular. In this section we summarize this discussion. For simplicity we
have concentrated on the solutions with maximal angular momentum in a given two plane of the non-compact transverse directions. Another question we consider is the following. We take the large radius limit of the ring and then look at the resulting geometry. It turns out that the following two limits do not commute, the near ring limit for fixed ring radius and the large radius first and then the small distance limit. The physical reason why they do not commute is that when one is approaching the ring one is exploring the IR region of the field theory living on the branes so that one is sensitive to the long distance geometry of the branes.

5.1. Solutions in different dimensions

In order to analyze such systems, we start with the F1-P1 system in $R^{1,d} \times S^1 \times T^{8-d}$ (with the appropriate powers in the harmonic functions), integrate the string sources along a ring (as explained in [21]), and then perform some dualities on it to get the desired system. We can write the solutions in different U-dual frames. We choose to work with the D0-F1 system blowing up to a D2 - the supertube of [1].

We start then with the 1/4 supersymmetric supergravity solution describing an oscillating string, wound around the $S^1$ and carrying right moving momentum [7], [8] in non-compact transverse dimensions

\[
d s^2 = -e^{2\Phi} du dv - (e^{2\Phi} - 1) \dot{F}^2 dv^2 + 2(e^{2\Phi} - 1) \dot{F} \cdot dx dv + dx^2_\alpha + dz^{2}_{8-d}
\]

\[
H(x, v)(-du dv + K(x, v) dv^2 + 2A_i(x, v) dx^i dv) + dx^2_a + d z^{2}_{8-d}
\]

\[
B_{uv} = \frac{1}{2}(e^{2\Phi} - 1) \quad ; \quad B_{vi} = -\dot{F}_i(e^{2\Phi} - 1) = H A_i
\]

\[
e^{-2\Phi} = 1 + \frac{Q}{|x - F(v)|^{d-2}}
\]

where the light cone coordinates are $u, v = t \pm y$ with $y \sim y + R_y$ and $x$ are $d$ noncompact directions and $z$ parameterize a $T^{8-d}$. $F(v)$ is a $d$-dimensional vector describing the location of the string.

Taking the ring profile: $(F_1^\alpha + i F_2^\alpha)(v) = a e^{i(\omega v + \alpha)}$, $F_3^\alpha = F_4^\alpha = 0$, and integrating the harmonic functions along $\alpha$, we get functions describing oscillating strings uniformly distributed along a ring:

\[
\langle H^{-1}(x) \rangle = 1 + \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\alpha}{|x - F^\alpha|^{d-2}} = 1 + \frac{Q}{\sigma^{d-2}} I_{1}^{(d-2)}(-\frac{2as}{\sigma^{2}})
\]

\[
\langle K(x) \rangle = \frac{Q}{2\pi} \int_0^{2\pi} d\alpha \frac{\partial_v F^\alpha \cdot d\alpha}{|x - F^\alpha|^{d-2}} = a^2 \omega^2 (\langle H^{-1} \rangle - 1) = \frac{a^2 \omega^2 Q}{\sigma^{d-2}} I_{1}^{(d-2)}(-\frac{2as}{\sigma^{2}}) \quad (5.2)
\]

\[
\langle A_\phi(x) \rangle = as \omega \frac{Q}{2\pi} \int_0^{2\pi} \frac{\partial_v F_\phi d\alpha}{|x - F^\alpha|^{d-2}} = \frac{as Q \omega s}{\sigma^{d-2}} I_{2}^{(d-2)}(-\frac{2as}{\sigma^{2}})
\]

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where \( s^2 \equiv x_1^2 + x_2^2 \) is the radial coordinate in the ring plane, \( w^2 \equiv x_3^2 + \cdots + x_d^2 \) is the perpendicular distance from the ring plane, \( \sigma^2 \equiv a^2 + s^2 + w^2 \), and where we defined the integrals:

\[
I_1^{(n)}(k) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\alpha}{(1 + k \cos \alpha)^{n/2}} \quad ; \quad I_2^{(n)}(k) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos \alpha d\alpha}{(1 + k \cos \alpha)^{n/2}}
\]

for integer \( n \)’s. The integrals above can be easily evaluated, and appear in appendix D. For odd \( n \) they involve elliptic functions.

Now, if we perform S-duality and then \( T_y \) duality on (5.1), we get the \( F^1D0 \rightarrow D2 \) supertube solution for any dimension \( d \), with \( H, K, A_\phi, I_1, I_2 \) as in (5.2), (5.3)

\[
ds^2 = \frac{1}{\sqrt{H^{-1}(1 + K)}} \left[ -dt^2 + 2A_\phi dt d\phi + H^{-1}dy^2 + s^2 \left( \frac{\alpha Q}{\sigma^2} \right)^2 \left( (I_1^{(d-2)})^2 - (I_2^{(d-2)})^2 \right) + \frac{1 + a^2 \omega^2}{a^2 \omega^2 Q} \sigma^{d-2} I_1 + \left( \frac{\sigma^{d-2}}{a \omega Q} \right)^2 d\phi^2 \right] + \sqrt{H^{-1}} \left( ds^2 + dW_{d-2}^2 + dz_{8-d}^2 \right)
\]

\[
e^{-2\Phi} = H^{3/2}(1 + K)
\]

\[
B_2 = -\frac{K}{1 + K} dt \wedge dy - \frac{A_\phi}{1 + K} d\phi \wedge dy
\]

\[
C_1 = -(H - 1) dt + HA_\phi d\phi \quad ; \quad C_3 = -\frac{A_\phi}{1 + K} dt \wedge d\phi \wedge dy
\]

where in the second line we wrote \( g_{\phi\phi} = \frac{1}{\sqrt{H^{-1}(1 + K)}} (s^2 H^{-1}(1 + K) - A_\phi^2) \) in terms of the integrals \( I_1, I_2 \). \( \Box \)

Analyzing the behavior of the different functions in the metric above, one finds that these solutions are everywhere regular, except maybe on a ring of radius \( a \) in the \( x \)-space \( (w = 0, s = a) \). In appendices C and D we show that these solutions are indeed singular on the ring in all dimensions except the for the D1-D5 system described above. We did not find any U-dual frame where the solutions were regular, except for \( d = 4 \) which is U-dual to the D1-D5 system. \( \Box \)

5.2. The large ring limit

In this subsection we take the limit where the radius of the ring becomes very large. This can be achieved by taking very large values of the charges. From the formula of the

\footnote{Note that \( I_1^{(d-2)}, I_2^{(d-2)} \) in (5.4) are evaluated at \( -\frac{2a^2}{\sigma^2} \).}

\footnote{Recently the \( d = 3 \) case was analyzed in \cite{22}.}
ring radius (2.8) we see that in this limit we expect to have a finite energy density per unit length along the ring. In this limit the ring becomes a straight line. We want to find the metric near the ring in this case. This metric has the form of the metric of a brane with some fluxes on it. In the D1-D5 it will be a KK monopole with some fluxes on it. These fluxes have a special value such that the supersymmetry that is preserved is independent of the orientation of the brane. Below we explain this in detail.

We can try take the limit where \( a \to \infty \) in (5.4), and see if the solutions we obtain really describe a flat \( D2 \) with fluxes.

From a worldvolume analysis [1] one finds that the radius of the supertube scales with the D0,F1 charges as \( a \sim \sqrt{Q_0 Q_s} \), where the two charges in our notations are \( Q, a^2 \omega^2 Q \). Keeping the ratio of the charges fixed, the scaling is

\[
a \sim Q \to \infty \quad ; \quad \delta \equiv a \omega \text{ fixed} \quad (5.5)
\]

Taking this limit for a fixed \( \rho \) in (5.4) and defining the coordinate \( x_\parallel \equiv a \phi \) gives the following metrics and fields for \( d \geq 4 \):

\[
ds^2 = \left[ 1 + \frac{\tilde{q}}{\rho^{d-3}} \right]^{1/2} \left\{ dy^2 + \frac{1}{1 + \frac{\tilde{q}}{\rho^{d-3}}} \left( 1 + \frac{\delta}{\rho^{d-3}} \right) dx^2 + \frac{2 \delta - \frac{\tilde{q}}{\rho^{d-3}}}{1 + \frac{\tilde{q}}{\rho^{d-3}}} dx_\parallel dt - \frac{dt^2}{1 + \frac{\tilde{q}}{\rho^{d-3}}} \right\} + \\
+ \left[ 1 + \frac{\tilde{q}}{\rho^{d-3}} \right]^{1/2} \left[ d\rho^2 + \rho^2 d\Omega_{d-2}^2 + dz_{18-d}^2 \right]
\]

\[
B_2 = -\frac{\delta \frac{\tilde{q}}{\rho^{d-3}}}{1 + \frac{\tilde{q}}{\rho^{d-3}}} (\delta dt \wedge dy + dx_\parallel \wedge dy)
\]

\[
C_1 = \frac{\frac{\tilde{q}}{\rho^{d-3}}}{1 + \frac{\tilde{q}}{\rho^{d-3}}} (dt + \delta dx_\parallel) \quad ; \quad C_3 = -\frac{\delta \frac{\tilde{q}}{\rho^{d-3}}}{1 + \frac{\tilde{q}}{\rho^{d-3}}} dt \wedge dx_\parallel \wedge dy
\]

\[
e^{-2\phi} = (1 + \frac{\tilde{q}}{\rho^{d-3}})^{-3/2} \left( 1 + \delta^2 \frac{\tilde{q}}{\rho^{d-3}} \right)
\]

where the effective charge \( \tilde{q} \) is given by

\[
\tilde{q} = \frac{Q}{a} \cdot \lim_{\rho \to 0} \left[ \left( \frac{Q}{a} \right)^{d-3} I_1^{(d-2)} \left( 1 + \frac{\rho^2}{2a^2} \right) \left( \frac{1}{1 + \frac{Q}{3\pi a} \sin \Theta + \frac{Q^2}{2a^2}} \right) \right]
\]

which for the different dimensions is:

\[
d = \begin{array}{cccccc}
4 & 5 & 6 & 7 & 8 \\
\tilde{q} = & \frac{Q}{2a} & \frac{Q}{\pi a} & \frac{Q}{4a} & \frac{2Q}{3\pi a} & \frac{3Q}{16a}
\end{array}
\]

(5.7)
For $d = 3$ we get a logarithmic singularity.

We would like to compare (5.6) with the metric and fields describing a D2-brane with F1 and D0 fluxes on a $T^{8-d}$. These can be generated by starting with the supergravity solution of a D2 in the $\tilde{t}, \tilde{y}, \tilde{x}_p$ directions \footnote{We choose a gauge where the Ramond-Ramond Gauge field vanishes at spatial infinity.}. Then T-dualizing in $\tilde{x}_p$ to obtain a D1 in the $\tilde{y}, \tilde{t}$ directions, smeared on the $\tilde{x}_p$ direction (with a harmonic function $f = 1 + \frac{q}{r^{d-3}}$). Then making a boost and a rotation with parameters $\alpha, \theta$ mixing $\tilde{t}, \tilde{y}, \tilde{x}_p$ to give $t, y, x_p$ \footnote{so that}

\begin{align*}
\tilde{t} &= \cosh \alpha t + \sinh \alpha (\cos \theta x_p + \sin \theta y) \\
\tilde{x}_p &= \cosh \alpha (\cos \theta x_p + \sin \theta y) + \sinh \alpha t \\
\tilde{y} &= (\cos \theta y - \sin \theta x_p)
\end{align*}

and finally making a T-duality in the $y$-direction \footnote{this procedure was explained for example in \cite{23}}.

This gives the following metric and gauge fields:

\begin{align*}
\mathbf{5.8}
\end{align*}

$$
\begin{align*}
&ds^2 = f^{-1/2}[-(1 - h^{-1} \frac{q \sinh^2 \alpha}{r^{d-3}})dt^2 + (1 + h^{-1} \frac{q \cosh^2 \alpha \cos^2 \theta}{r^{d-3}})dx_p^2 + 2h^{-1} \frac{q \sinh \alpha \cos \alpha \cos \theta}{r^{d-3}}dt dx_p + fh^{-1} dy^2] + f^{1/2}[dr^2 + r^2 d\Omega_2^2 + dz_{8-d}^2] \\
&B_2 = -h^{-1} \frac{q \sin \theta \cosh \alpha}{r^{d-3}}[\sinh \alpha dy \wedge dt + \cosh \alpha \cos \theta dy \wedge dx_p] \\
&C_1 = (f^{-1} - 1)[\cos \theta \cosh \alpha dt + \sinh \alpha dx_p] ; \quad C_3 = h^{-1} \frac{q \sin \theta \cosh \alpha}{r^{d-3}} dt \wedge dx_p \wedge dy \\
e^{2\phi} &= g^2 f^{3/2} h^{-1} \\
f &\equiv 1 + \frac{q}{r^{d-3}} \quad ; \quad h \equiv 1 + \frac{q \cosh^2 \alpha \sin^2 \theta}{r^{d-3}}
\end{align*}$$

Comparing (5.8) to (5.6) we find exact agreement if we choose $\sinh \alpha = \tan \theta = \delta$. All of the solutions (5.8) are 1/2 supersymmetric as they are dual to a D2. However only the subfamily of such solutions with $\sinh \alpha = \tan \theta$ would continue being supersymmetric (with 1/4 supersymmetry) if we start curving the brane, taking the direction $x_p$ and putting it on some closed curve, e.g. the ring. (The exact supersymmetries that this curvily shaped D2 with fluxes preserves can be found doing a worldvolume analysis, as done in \cite{1}, or as done for the D2D2 system in \cite{24}). Under a T-duality in the $S^1$ circle this system...
becomes a D1 brane that winds along the $S^1$, and moves along the $S^1$ as it stretches in the $x_p$ direction. The velocity is such that a brane that is stretched in the opposite direction along $x_p$ but with the same winding and velocity intersects the original brane at a point that moves with the speed of light \cite{25}. These configurations preserve 1/4 of the supersymmetries. These configurations are intimately related to the oscillating strings we started with. In fact strings carrying oscillations only in one direction will intersect with each other at points that move at the speed of light.

In the $d = 4$ case we can make a U-duality to the D1-D5 system so that the large ring radius (5.6) becomes a straight KK monopole carrying D1 and D5 fluxes

$$ds^2 = \left[1 + \frac{q_1}{\rho}\right]^{-1/2}\left[1 + \frac{q_5}{\rho}\right]^{-1/2}[-(dt - \frac{\sqrt{q_1 q_5}}{\rho}dx_\parallel)^2 + (dy - \sqrt{q_1 q_5}(1 - \cos \Theta)d\psi)^2]$$

$$+ \left[1 + \frac{q_1}{\rho}\right]^{1/2}\left[1 + \frac{q_5}{\rho}\right]^{1/2}[d\rho^2 + \rho^2 d\Theta^2 + \rho^2 \sin^2 \Theta d\psi^2 + dx_\parallel^2] + \sqrt{\frac{\rho + q_1}{\rho + q_5}}dz^{(4)}$$

$$e^{2\phi} = \frac{\rho + q_1}{\rho + q_5}$$

$$C_2 = -\frac{q_1}{\rho + q_1}(dt + \sqrt{\frac{q_5}{q_1}}dx_\parallel) \wedge (dy + \sqrt{\frac{q_5}{q_1}}\rho(1 - \cos \Theta)d\psi)$$

where we have defined the charge densities $q_i = Q_i/(2a)$ which are finite in the limit. This metric is non-singular if $R_y = 2\sqrt{q_1 q_5}$. This is a condition on the fluxes for a given radius $R_y$. If we U-dualize (5.8) we can get solutions which represent KK monopole with arbitrary values of the fluxes that are 1/2 BPS. What is special about the fluxes in (5.8) is that we can reverse the KK monopole charge, keeping the same values for $q_1, q_5$ so that the configuration with KK and anti-KK charges still preserve 1/4 of the supersymmetries. As shown in \cite{25} this configuration is U-dual to configurations with intersecting D-branes where the intersection point moves at the speed of light (see also \cite{20}).

Note that in the limit that we drop the 1 in the harmonic functions that appear in (5.9) we obtain a plane wave in six dimensions. We can get this as a limit where we scale the charges to infinity and the rest of the coordinates appropriately. The geometry (5.9) thus provides us with a spacetime which is asymptotically flat and that looks like a plane wave in a suitable near horizon limit.

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Appendix A. Topology of the solutions

If we have a single ring profile, such as the one in (2.7) then the harmonic functions (2.2) can be found explicitly and read

\[ f_5 - 1 = Q h^{-1}, \quad f_1 - 1 = a^2 \omega^2 Q h^{-1} \]
\[ h^2 = [(s + a)^2 + w^2][(s - a)^2 + w^2] \]
\[ A_\phi = 2a^2 \omega s^2 \left( \frac{1}{h + s^2 + a^2 + w^2} \right) \]

where \( s^2 = x_1^2 + x_2^2 \) and \( w^2 = x_3^2 + x_4^2 \).

In order to understand more clearly the topology of the metric it is convenient to write the metric in other coordinates such that the metric reads

\[
ds^2 = \frac{1}{\sqrt{f_1 f_5}} \left[ -(dt - \frac{a\sqrt{Q_1 Q_5}}{r^2 + a^2 \cos^2 \theta} \sin^2 \theta d\phi)^2 + (dy + \frac{a\sqrt{Q_1 Q_5}}{r^2 + a^2 \cos^2 \theta} \cos^2 \theta d\psi)^2 \right] + \\
+ \sqrt{f_1 f_5}[(r^2 + a^2 \cos^2 \theta)(\frac{dr^2}{r^2 + a^2} + d\theta^2) + r^2 \cos^2 \theta d\psi^2 + (r^2 + a^2) \sin^2 \theta d\phi^2] \\
+ \sqrt{f_1 f_5}dz_a dz_a \\
e^{2\phi} = f_1 f_5, \\
C_{(2)} = (1 - \frac{1}{f_5})(dt - \sqrt{\frac{Q_1}{Q_5}}a \sin^2 \theta d\phi) \wedge (dy - \sqrt{\frac{Q_1}{Q_5}}a \cos^2 \theta d\psi) \\
+ Q_1 \cos^2 \theta d\phi \wedge d\psi \\
f_{1,5} = 1 + \frac{Q_{1,5}}{r^2 + a^2 \cos^2 \theta} \]

(A.2)

This form of the metric arises naturally if we view the solution as a limit of the general five dimensional black hole solutions in [3]. The explicit coordinate change from the coordinates \( w, s \) in (A.1) to the ones in (A.2) is

\[ s^2 = (r^2 + a^2) \sin^2 \theta, \quad w = r \cos \theta \]

(A.3)

and \( \phi \) and \( \psi \) are again the phases of \( x_1 + ix_2 \) and \( x_3 + ix_4 \) respectively. Here there is a potential singularity when \( r = 0 \) and \( \theta = \pi/2 \). We can rewrite the metric (A.2) in a form
where its singularity structure is more transparent

\[
ds^2 = \sqrt{f_1 f_5} (r^2 + a^2 \cos^2 \theta) [d\theta^2 + h \cos^2 \theta (d\psi + \frac{a \sqrt{Q_1 Q_5}}{f_1 f_5 h (r^2 + a^2 \cos^2 \theta)^2} dy)^2 + \\
+ \tilde{h} \sin^2 \theta (d\phi + \frac{a \sqrt{Q_1 Q_5}}{f_1 f_5 h (r^2 + a^2 \cos^2 \theta)^2} dt)^2] + \\
+ \sqrt{f_1 f_5} (r^2 + a^2 \cos^2 \theta) \left[ \frac{r^2}{g} dy^2 + \frac{dr^2}{r^2 + a^2} \right] - \frac{1}{\sqrt{f_1 f_5}} (1 + \frac{\sin^2 \theta a^2 Q_1 Q_5}{f_1 f_5 h (r^2 + a^2 \cos^2 \theta)^3}) dt^2
\]

(A.4)

where the functions \( h, \tilde{h}, g \) are

\[
\begin{align*}
g &= Q_1 Q_5 + (Q_1 + Q_5) r^2 + (r^2 + a^2 \cos^2 \theta) r^2 \\
h &= \frac{g}{f_1 f_5 (r^2 + a^2 \cos^2 \theta)^2} \\
\tilde{h} &= \frac{Q_1 Q_5 + (Q_1 + Q_5) (r^2 + a^2) + (r^2 + a^2 \cos^2 \theta) (r^2 + a^2)}{f_1 f_5 (r^2 + a^2 \cos^2 \theta)^2}
\end{align*}
\]

(A.5)

The important properties of these functions are \( g(r = 0, \theta) = Q_1 Q_5, \ h(r, \theta = \pi/2) = 1, \ \tilde{h}(r, \theta = 0) = 1 \). These properties, together with (2.8), ensure that the solution is nonsingular. Note that after the coordinate redefinition \( \tilde{\psi} = \psi + y/R \) and \( \tilde{\phi} = \phi + t/R \) the metric near \( r \sim 0 \) looks like that of a deformed \( S^3 \).

In order to study the topology of the solution we notice that the time direction will just give a factor of \( R \), so we drop it from the discussion. The topology of a surface of large \( r \) is that of \( S^1 \times S^3 \). Near \( r = 0 \) we see that the \( S^1 \) circle shrinks to zero size while the sphere parameterized by \( \tilde{\psi}, \theta, \tilde{\phi} \) does not. Topologically this is basically the same as the sphere we had at infinity since the map \( (y, \psi, \theta, \phi) \rightarrow (y, \tilde{\psi}, \theta, \tilde{\phi}) \) can be continuously deformed to the identity. This means that the final topology of the spatial region \( r \leq r_0 \) is that of a \( D^2 \times S^3 \). So the \( S^3 \) is non-contractible.

It is interesting to understand what the deformed three sphere that we have at \( r = 0 \) looks like in the original “flat” coordinates \( s, w \). From (A.3) we see that \( r = 0 \) is the disk spanned by \( w = 0 \) and \( s < a \). On top of this we have the \( y \) circle. These together from a three sphere since the \( y \) circle shrinks to zero at the boundary of the disk. Note that in the decoupling limit, where \( Q_i \) become very large the functions in (A.3) become constant. Then the three sphere parameterized by the coordinates \( \tilde{\psi}, \theta, \tilde{\phi} \) is a round three sphere. Away from the decoupling limit it is not metrically a round three sphere.
Appendix B. Gravity duals of chiral primaries on the torus.

In the previous sections we discussed the geometries corresponding to chiral primaries associated with AdS$_3 \times$S$^3$. Such chiral primaries are universal and they do not depend on the structure of the internal manifold M in AdS$_3 \times$S$^3 \times$M. But there are also some chiral primaries associated with the internal manifold, and in this section we will discuss them for the simplest case where $M = T^4$. We comment on the K3 case at the end.

To construct the geometries corresponding to such chiral primaries, we will follow the steps outlined in section 2. Namely we will start from the vibrating string, perform the dualities to relate it to the D1-D5 system, and then perform spectral flow to go to the NS sector. The only difference is that now we will allow the string to vibrate not only in noncompact directions, but also on the torus. Since to go to the D1-D5 system we have to perform dualities in the torus directions, the geometry of the vibrating string should be translation invariant in these directions, and we can achieve this by “smearing” in the torus coordinates (just like we smeared the profile on the y direction by performing integration over $v$ in the string profiles (2.2)).

Thus we start with the metric of a vibrating string, smear it over the torus directions, and perform the following dualities

$$
\begin{pmatrix}
(P(5)) \\
(F1(5))
\end{pmatrix}
\rightarrow
\begin{pmatrix}
(P(5)) \\
(D1(5))
\end{pmatrix}
\rightarrow
\begin{pmatrix}
(P(5)) \\
(D5(56789))
\end{pmatrix}
\rightarrow
\begin{pmatrix}
(P(5)) \\
(NS5(56789))
\end{pmatrix}
\rightarrow
\begin{pmatrix}
(F1(5)) \\
(NS5(56789))
\end{pmatrix}
$$

(B.1)

This way we get an “F1-NS5” of type IIA theory$^{12}$

$$
ds^2 = \frac{1}{f_1} \left[ -(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + f_5 dx^i dx^i + dz^a dz^a
$$

$$
e^{2\Phi} = \frac{f_5}{f_1}, \quad B_{ty} = -1 + \frac{1}{f_1}, \quad B_{ti} = \frac{B_i}{f_1},
$$

$$
B_{yi} = \frac{A_i}{f_1}, \quad B_{ij} = C_{ij} - \frac{A_i B_j - A_j B_i}{f_1}, \quad C^{(1)}_a = f_5^{-1} A_a,
$$

$$
C^{(3)}_{abc} = f_5^{-1} \epsilon_{abcd} A_d, \quad C^{(3)}_{iya} = \frac{A_i A_a}{f_1 f_5}, \quad C^{(3)}_{ita} = \frac{B_i A_a}{f_1 f_5},
$$

$$
C^{(3)}_{ija} = \frac{(A_i B_j - A_j B_i) A_a}{f_1 f_5}, \quad C^{(3)}_{iya} = A_a f_5^{-1} \left( -2 + \frac{1}{f_1} \right),
$$

$$
C^{(5)}_{tyabc} = -\epsilon_{abcd} A_d f_5^{-1} \left[ 2 - \frac{1}{f_1 f_5} \right], \quad C^{(5)}_{iyabc} = -\epsilon_{abcd} \frac{A_i A_d}{f_1 f_5},
$$

$$
C^{(5)}_{tiabc} = -\epsilon_{abcd} \frac{B_i A_d}{f_1 f_5}, \quad C^{(5)}_{ijabc} = \epsilon_{abcd} \frac{(A_i B_j - A_j B_i) A_d}{f_1 f_5}
$$

$^{12}$ A simple further T-duality in one of the $T^4$ directions would give a solution in IIB
\[ \tilde{f}_1 \equiv f_1 - f_5^{-1} A_a A_a \]

One can now perform additional T duality along one of the torus directions followed by S duality, to get a D1-D5 system. We will do this step only with the metric. But in any case, if one wants to study the properties of six dimensional Einstein metric, then one gets the same results starting either from D1-D5 or F1-NS5. The functions in (B.2) are given by:

\[ f_5 = 1 + \frac{Q}{L} \int_0^L \frac{dz dv}{[(x - F)^2 + (z - F)^2]^2} = 1 + \frac{Q}{L} \int_0^L \frac{dv}{(x - F)^2}, \]
\[ f_1 = 1 + \frac{Q}{L} \int_0^L \frac{|\dot{G}|^2 dv}{(x - F)^2}, \quad A_i = -\frac{Q}{L} \int_0^L \frac{\dot{F}_i dv}{(x - F)^2}, \quad A_a = -\frac{Q}{L} \int_0^L \frac{\dot{F}_a dv}{(x - F)^2}. \] (B.3)

Here we introduced an eight dimensional vector \( G = (F_i, F_a) \).

Note that in (B.3) we have integrated over the position \( z \) of the string in the internal torus. This is done to obtain a solution that is independent of the internal coordinates. This implies that the dependence on \( F_a \) disappears from \( f_5 \) in (B.3), but does not disappear from \( f_1 \) and \( A_a \).

Let us analyze the metric (B.2) near the singularity. Near the singularity we get:

\[ f_5 = \frac{Q}{L} \frac{\pi}{|F|_{x_\perp}}, \quad f_1 = \frac{Q|\dot{G}|^2}{L} \frac{\pi}{|F|_{x_\perp}}, \quad f_1 - 1 - f_5^{-1} A_a A_a = \frac{Q \pi |\dot{F}|}{L x_\perp}, \]
\[ A_i = -\frac{Q}{L} \frac{\pi \dot{F}_i}{|F|_{x_\perp}}, \quad A_a = -\frac{Q}{L} \frac{\pi \dot{F}_a}{|F|_{x_\perp}}. \] (B.4)

The expressions for \( f_5 \), \( A_i \) and \( f_1 - 1 - f_5^{-1} A_a A_a \) do not depend on the profile in the internal directions \( F \), and thus the criteria for the absence of the singularity is the same as in the case with no vibrations on the torus, namely the profile should not self intersect in the \( x_1, \ldots, x_4 \) space and \( \dot{F} \) should never vanish.

In the case of type IIB string theory on \( AdS_3 \times S^3 \times K3 \) there are also chiral primaries that are associated to extra anti-self dual 3-form gauge fields in six dimensions that come

The simplest way to construct the harmonic functions is following. We can first decompactify torus directions and look at the vibrating string in eight noncompact directions. Then we can smear over positions of the string in \( z_1, \ldots, z_4 \) (which corresponds to integration over \( z \) in \( f_5 \)), and in the end compactify \( z_1, \ldots, z_4 \) on the torus.
from anti-self-dual two forms on $K3$. Using heterotic/IIA duality it is very simple to get these solutions too. We have to perform the chain of dualities

$$
\begin{align*}
\left( \frac{P(5)}{F1(5)} \right)_{\text{het/IIA}} & \rightarrow \left( \frac{NS5(56789)}{NS5(56789)} \right)_{T5} & \rightarrow 
\end{align*}
$$

so that in the end we get a solution of IIB on $K3$. In the heterotic theory the fundamental string can oscillate in the $T^4$ directions as well as in the 16 extra bosonic left moving directions on the heterotic worldsheet. Solutions of this type were discussed in [7][8]. It is in principle straightforward to perform the duality transformations, but we leave that for the interested reader.

**B.1. Example of the vibration on the torus.**

We consider the simplest example for the vibrations on the torus:

$$
F_1 = a \cos \omega v, \quad F_2 = a \sin \omega v, \quad F_1 = b \cos m \omega v, \quad F_2 = b \sin m \omega v,
$$

all other components are zero. The frequency $\omega$ is related to the radius $R$ of the $y$ direction by (2.7). As we already mentioned, the expressions for $f_5$ and $A_i$ remain the same as they were for $b = 0$, so to find the metric we only have to evaluate $\tilde{f}_1 = f_1 - f_5^{-1} A_a A_a$. Substituting the profile (B.6) in (B.2), we find:

$$
\tilde{f}_1 = f_1 - f_5^{-1} A_a A_a = 1 + \frac{Q \omega}{r^2 + a^2 \cos^2 \theta \left(a^2 + b^2 m^2\right)} - \frac{Q b^2 m^2}{Q + r^2 + a^2 \cos^2 \theta \left(a^2 \sin^2 \theta + \frac{r^2 + a^2}{r^2 + a^2}\right)}
$$

and the metric for the D1–D5 system becomes:

$$
\begin{align*}
 ds^2 &= \frac{1}{\sqrt{\tilde{f}_1 f_5}} \left[ -(dt - \frac{a^2 R}{r^2 + a^2 \cos^2 \theta} \sin^2 \theta d\phi)^2 + (dy + \frac{a^2 R}{r^2 + a^2 \cos^2 \theta} \cos^2 \theta d\psi)^2 \right] + \\
 &+ \sqrt{\tilde{f}_1 f_5} \left[ (r^2 + a^2 \cos^2 \theta) \left( - \frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + r^2 \cos^2 \theta d\psi^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \right] + \\
 &+ \sqrt{\tilde{f}_1 f_5} d\mathbf{z}^2
\end{align*}
$$

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Note that the total fivebrane charge is $Q_5 = Q$ and the onebrane charge is given by $Q_1 = Q\omega^2(a^2 + m^2b^2)$ where $\omega$ is as in (2.7). In particular we have:

$$R = \sqrt{\frac{Q_1Q_5}{a^2 + m^2b^2}}$$

In order to obtain (3.3) we need to drop the 1 in the harmonic functions in (B.3). This gives

$$\tilde{f}_1 - 1 = \frac{\alpha Q_1}{r^2 + a^2\cos^2 \theta}, \quad f_5 - 1 = \frac{Q_5}{r^2 + a^2\cos^2 \theta}, \quad A_\phi = \sqrt{\frac{a\sqrt{Q_1Q_5}\sin^2 \theta}{r^2 + a^2\cos^2 \theta}} \quad (B.11)$$

with $\alpha$ and $\beta$ defined by

$$\beta = \frac{a^2}{a^2 + m^2b^2}, \quad \alpha = 1 - (1 - \beta) \left( \frac{\beta \sin^2 \theta}{r^2 + \beta} \right)^m \quad (B.12)$$

Note that for large values of $m$ $\alpha$ is equal to one everywhere except for the small vicinity of the ring $r = 0, \theta = \frac{\pi}{2}$, and in the limit $m \to \infty$ the harmonic functions (B.11) reduce to the ones for the solution corresponding to a ring of rotating particles [18,2].

We define $\chi = y/R$ and rescale

$$t \to Rt, \quad r^2 \to r^2(a^2 + b^2m^2) = r^2\frac{Q_1Q_5}{R^2}, \quad (B.13)$$

then (B.7) becomes:

$$\frac{ds^2}{\sqrt{Q_1Q_5}} = (r^2 + \beta \cos^2 \theta) \frac{1}{\sqrt{\alpha}} \left[ -(dt - \frac{\beta \sin^2 \theta d\phi}{r^2 + \beta \cos^2 \theta})^2 + (d\chi + \frac{\beta \cos^2 \theta d\psi}{r^2 + \beta \cos^2 \theta})^2 + \frac{\sqrt{\alpha}dr^2}{r^2 + \beta} + \frac{\sqrt{\alpha}d\theta^2}{r^2 + \beta \cos^2 \theta} \right]$$

$$+ \frac{\sqrt{\alpha}d\phi^2}{r^2 + \beta \cos^2 \theta} \left( r^2 \cos^2 \theta d\psi^2 + (r^2 + \beta) \sin^2 \theta d\phi^2 \right) \quad (B.14)$$

Let us look at the limit $m \to \infty$ (which corresponds to $\alpha = 1$) and compare the above metric with the metric of the conical defect. To do this it is convenient to rewrite (B.14) for $\alpha = 1$ as

$$\frac{ds^2}{\sqrt{Q_1Q_5}} = - \left( r^2 + \frac{\beta - \beta^2}{2} \right) dt^2 + \left( r^2 + \frac{\beta - \beta^2}{2} \right) d\chi^2 + \frac{dr^2}{r^2 + \beta}$$

$$+ d\theta^2 + \cos^2 \theta (d\psi + \beta d\chi)^2 + \sin^2 \theta (d\phi + \beta dt)^2$$

$$+ \frac{(\beta - 1)\beta}{r^2 + \beta \cos^2 \theta} (\cos^4 \theta d\psi^2 - \sin^4 \theta d\phi^2) + \frac{\beta(1 - \beta)}{2} \cos 2\theta (-dt^2 + d\chi^2) \quad (B.15)$$
If we now introduce new coordinates:

\[ r' = r + \frac{\beta - \beta^2}{4r}(1 - \cos 2\theta), \quad \theta' = \theta - \frac{\beta(\beta - 1)}{4r^2}\sin 2\theta \]

then in the leading two orders at infinity the metric (B.15) becomes:

\[
\frac{ds^2}{\sqrt{Q_1Q_5}} = - \left( r'^2 + \beta^2 \right) dt^2 + r'^2 d\chi^2 + \frac{dr'^2}{r'^2 + \beta^2} \\
+ d\theta'^2 + \cos^2 \theta'(d\psi + \beta d\chi)^2 + \sin^2 \theta'(d\phi + \beta dt)^2 \\
+ (\beta - 1)\beta \cos 2\theta' \left[ \frac{1}{r'^2}(d\theta'^2 + \cos^2 \theta' d\psi^2 + \sin^2 \theta' d\phi^2) + (dt^2 - d\chi^2 + \frac{dr'^2}{r'^4}) \right]
\]

(B.16)

The first two lines give a metric of a conical defect, while the last line gives a perturbation, which corresponds to an \(AdS_3\) scalar with angular momentum \(l = 2\). This mode is a mixture of an overall rescaling of the sphere, AdS, and the three form field strengths. The fact that the correction to the \(AdS_3\) part of the metric in (B.16) is not just an overall factor is due to the fact that these scalar fluctuations also imply a change of the metric of the form \(\delta g_{\mu\nu} \sim \nabla_\mu \nabla_\nu \delta \phi\) where \(\delta \phi\) is the scalar fluctuation. More details and explicit formulas can be found in [16]. Note that the terms in the last line of (B.16) are of the same order as the terms of the \(AdS_3\) part of the metric in the first line. This implies that the conical defects are not a good approximation to these metrics.

### B.2. Plane wave limit of the solution

In this subsection we take the plane wave limit of the solution (B.14). Let us call \(\sqrt{Q_1Q_5} = \epsilon^{-2}\), then we define rescaled quantities by

\[
t = x^+, \quad \phi = \epsilon^2 x^- , \quad r = \epsilon \sqrt{\beta} s , \quad \frac{\pi}{2} - \theta = \epsilon u , \quad \tilde{m} = \frac{m}{\epsilon^2}
\]

(B.17)

In the \(\epsilon \rightarrow 0\) limit we get the metric

\[
ds^2 = \beta \alpha^{-1/2}[2dx^+dx^- - (s^2 + u^2)(dx^+)^2] + \alpha^{1/2}(ds^2 + du^2 + u^2 d\tilde{\psi}^2 + s^2 d\chi^2) \\
+ (\alpha^{1/2} - \beta \alpha^{-1/2}) \left[ \frac{(dx^-)^2}{s^2 + u^2} - \frac{u^4 d\tilde{\psi}^2 - 2s^2 u^2 d\chi d\tilde{\psi} + s^4 d\chi^2}{s^2 + u^2} \right]
\]

(B.18)

where now \(\alpha\) becomes

\[
\alpha = 1 - (1 - \beta)e^{-\tilde{m}(u^2 + s^2)}
\]

(B.19)
Note that $\beta$ remains fixed and $1 - \beta$ has the interpretation of momentum $p_-$ per unit length. This metric is non-singular. In the limit $\tilde{m} \to \infty$ it becomes the metric (3.6) which is singular. Of course for large $\tilde{m}$ the metric looks like (3.6) if $s^2 + u^2 > 1/\tilde{m}$ where we can approximate $\alpha \sim 1$.

It would be nice to understand why the asymptotic structure of (B.18) (where we can safely set $\alpha = 1$) is naively different from that of a usual plane wave. For example the transverse space is no longer $R^4$ in (3.6). This deserves further study.

B.3. Mass gap for the geometry (B.14).

Let us look at the spectrum of excitation on the background (B.14). For simplicity we will look at the minimally coupled scalar field, but our results will be true for more general excitations. Let us remind the reader that for a conical defect metric with opening angle $2\pi \gamma$ the mass gap is $E = 2\gamma$. Since we argued above that these conical metrics are not a good approximation to the metrics we consider we will perform an estimate of the mass gap in the metric (B.14).

For simplicity we will look at the modes of the scalar field which are constant in the $\phi, \psi, \chi$ directions. Then looking for the solution in the form $\Phi(r, \theta, t) = e^{-iEt}\Phi(r, \theta)$, we find the Klein–Gordon equation:

$$
\frac{1}{r} \partial_r (r(r^2 + \beta) \partial_r \Phi) + \frac{1}{\sin \theta \cos \theta} \partial_\theta (\sin \theta \cos \theta \partial_\theta \Phi) + \frac{E^2 \Phi}{(r^2 + \beta \cos^2 \theta)} \left[ \alpha - \frac{\beta^2 \sin^2 \theta}{r^2 + \beta} \right] = 0
$$

(B.20)

It is convenient to introduce new coordinates $x = r/\sqrt{\beta}$, $y = \cos \theta$. Then we find:

$$
\frac{1}{x} \partial_x (x(x^2 + 1) \partial_x \Phi) + \frac{1}{y} \partial_y (y(1 - y^2) \partial_y \Phi) + \frac{E^2 \Phi}{\beta(x^2 + y^2)} \left[ 1 - (1 - \beta) \left( \frac{1 - y^2}{1 + x^2} \right)^m \right] - \beta \frac{1 - y^2}{1 + x^2} = 0
$$

(B.21)

Note that for $m = 1$ the variables in this equation separate, and in particular we get spherically symmetric solutions which are normalizable near $x = 0$:

$$
\Phi = (x^2 + 1)^{-k} F(-k, 1 - k; 1; -x^2)
$$

where $E = 2k\sqrt{\beta}$. This function in normalizable near infinity if and only if $k$ is a positive integer. Thus for $m = 1$ we have a mass gap $E = 2\sqrt{\beta}$.  

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Let us now look at the more interesting case when $m > 1$. In this case the variables in the equation (B.21) do not separate, and we can’t find the exact spectrum. But we can get an upper bound on the mass gap using variational methods. First we rewrite (B.21) as a Schrödinger equation:

$$(H - E^2 V) \Phi = 0.$$ 

where $V$ is a positive potential ($E^2 V$ comes from the last term in (B.20)). This eigenvalue problem is the same as the one that arises when we have masses and springs, except that now the matrices are replaced by operators in a Hilbert space. Suppose this equation has a spectrum of eigenvalues $E_k$ with corresponding eigenfunctions $\Phi_k$ obeying $(H - E_k^2 V) \Phi_k = 0$. These will generically form a complete basis system in the space of normalizable functions. Then for any such function we get

$$\Phi = \sum a_m \Phi_m$$

We also note that

$$\langle \Phi_k | (H - E^2 V) | \Phi_l \rangle = (E_k^2 - E^2) \langle \Phi_k | V | \Phi_l \rangle = (E_l^2 - E^2) \langle \Phi_k | V | \Phi_l \rangle$$

From here we conclude that

$$\langle \Phi_k | V | \Phi_l \rangle = 0, \quad \text{if} \quad k \neq l$$

(as usual, if there is a degeneracy $E_k = E_l$, the above condition gives a choice of a basis). For a generic function $\Phi$ we get:

$$\langle \Phi | (H - E^2 V) | \Phi \rangle = \sum (E_k^2 - E^2) |a_k|^2 \langle \Phi_k | V | \Phi_k \rangle$$

Since $V$ is positive, $\langle \Phi_m | V | \Phi_m \rangle \geq 0$. To show that there is an eigenvalue $E_0 < E$ it is sufficient to find a normalizable state $|\Phi\rangle$ such as

$$\langle \Phi | (H - E^2 V) | \Phi \rangle < 0$$

Let us take a trial function

$$\Phi = \frac{1}{1 + x^2}.$$
Then taking an average of the left hand side of (B.21), we find

$$\langle \Phi | (H - E^2 V) | \Phi \rangle = \frac{1}{2} \int_0^\infty dx \partial_x (x(x^2 + 1) \partial_x \Phi)$$

$$- \int_0^\infty x dx \Phi^2 \frac{E^2}{\beta} \left[ \int_0^1 dy \frac{y}{(x^2 + y^2)} \left( 1 - (1 - \beta) \left( \frac{1 - y^2}{1 + x^2} \right)^m - \beta \frac{1 - y^2}{x^2 + 1} \right) \right]$$

$$= \int_0^\infty \frac{udu}{(1 + u)^3} - \frac{E^2}{\beta} \left\{ I_0 - (1 - \beta) I_m - \beta I_1 \right\}$$

(B.22)

Here we introduced the following integral

$$I_k = \int_0^\infty \frac{x dx}{(1 + x^2)^2} \int_0^1 dy \frac{(1 - y^2)^k}{(x^2 + y^2)} = \frac{1}{4} \int_0^\infty \frac{dx}{(x + 1)^{k+2}} \int_0^1 \frac{(1 - y)^k dy}{x + y}$$

$$= \frac{1}{4(k + 1)}$$

(B.23)

This gives

$$\langle \Phi | (H - E^2 V) | \Phi \rangle = \frac{1}{2} - \frac{E^2}{4\beta} \left\{ 1 - \frac{\beta}{2} - \frac{1 - \beta}{m + 1} \right\}$$

(B.24)

This expression becomes negative for $E > E_1$, where

$$E_1 = \sqrt{2\beta} \left\{ 1 - \frac{\beta}{2} - \frac{1 - \beta}{m + 1} \right\}^{-1/2}$$

(B.25)

so the mass gap is less than $E_1$. In particular, for all $m \geq 1$ we have $E_1 \leq 2\sqrt{\beta}$, so the mass gap is always less than this amount.

### Appendix C. No conical defects with arbitrary opening angles

One can easily write singular solutions with the same angular momentum as the solutions we have been considering. The simplest is a conical metrics of the form

$$\frac{ds^2}{R_{AdS}^2} = -(r^2 + \gamma^2) dt^2 + r^2 d\chi^2 + \frac{dr^2}{r^2 + \gamma^2} + d\theta^2 + \cos^2 \theta (d\psi + \gamma d\chi)^2 + \sin^2 \theta (d\phi + \gamma dt)^2$$

(C.1)

These metrics have a conical singularity at $r = (\pi/2 - \theta) = 0$. The singularity has a form which is rather similar to that of an $A_N$ singularity but with an opening angle which is $2\pi\gamma$ instead of $2\pi/N$. In addition, if $\gamma^{-1}$ is not an integer there are singularities at $r = 0$ and any $\theta$. 
When $\gamma^{-1}$ is an integer we can think of the metric (C.1) as arising from a “supertube” configuration with $N$ KK monopoles instead of just one KK monopole. Furthermore, it is possible to continuously deform the non-singular solutions that we had in this paper and get to these conical metrics. All we need to do is to take a profile $F(v)$ which wraps $N$ times around the origin. If it does not self intersect we will have a smooth metric and as we take the limit that $F$ is moving on the same circle $N$ times we get the conical defect metric (C.1) with $\gamma^{-1} = N$.

On the other hand the metrics (C.1) with $\gamma^{-1} \neq N$ should not be allowed from the KK monopole point of view since they would mean that we have fractional KK monopole charge. In fact this is the reason that the singularity for non-integer $\gamma^{-1}$ is more extended than for $\gamma^{-1}$ integer. In the former case there is a fractional “Dirac string” coming out of the fractional KK monopole which is responsible for this singularity. Despite this strange features one might ask the following question. Can we find a smooth metric that is arbitrarily close to the metric (C.1) with non integer $\gamma^{-1}$? When we say that a metric is very close to (C.1) we mean that the metric is equal to (C.1) up to very small corrections everywhere except very near the singularity. Namely, if we pick a $\gamma^{-1}$, say $3/2$, then we pick an $\epsilon$, say $\epsilon = 10^{-6}$, then we want to find a metric which only differs from (C.1) by terms of order $\epsilon$ once we are at $r > \epsilon$. We will now show that this is not possible\(^{14}\).

Without loss of generality we can assume that the angular momentum is in the direction $J_{12}$ and all other components vanish. In general the angular momentum of any configuration is characterized by two invariants $J^2_L$ and $J^2_R$ but for conical metrics of the asymptotic form (C.1) we have $J^2_L = J^2_R$ so that using a rotation we can always put the angular momentum in the 12 plane. So suppose we have a metric that is very close to the metric of the conical defect for distances larger than some tiny distance $\epsilon$. Then the harmonic functions will be very similar to the harmonic functions that give (C.1). The harmonic functions for (C.1) are given by (A.1) except that $\omega$ now obeys $\omega Q = \gamma R$. Since the harmonic functions are close to each other the source for the hypothetical non-singular metric should be close to the source of the harmonic functions in (A.1). In particular, $f_5$ implies that the source is distributed near a ring in the 12 plane. So in the expressions we will find below we will approximate $F_1^2 + F_2^2 - (F_3^2 + F_4^2) \sim F_1^2 + F_2^2$, but we do not make any assumptions about $\dot{F}_{3,4}^2$.

\(^{14}\) So, for example, it is futile to try to find the dual description of the conical defect metrics with arbitrary $\gamma$ \cite{5}, since these metrics are not a good approximations to anything. It is OK to consider the ones with integer $\gamma^{-1}$.
It is now instructive to consider the large $r$ behavior of the metric. Using (B.2) we can read off all the harmonic functions of the form (B.3). The leading behavior of such functions is

$$f_5 = \frac{Q_5}{x^2} + \frac{2Q_5\langle F_i \rangle x_i}{x^4} + \langle 4F_i F_j - F^2 \delta_{ij} \rangle \frac{x_i x_j}{x^6}$$

$$f_1 = \frac{Q_5\langle \dot{G}^2 \rangle}{x^2} + \frac{2Q_5\langle F_i \dot{G}^2 \rangle x_i}{x^4} + \langle (4F_i F_j - F^2 \delta_{ij}) \langle \dot{G}^2 \rangle \rangle x_i x_j \frac{x_i x_j}{x^6},$$

$$A_i = -2Q_5\langle \dot{F}_i F_j \rangle \frac{x_j}{x^4}, \quad B_i = -Q_5\epsilon_{ijkl}\langle \dot{F}_k F_l \rangle \frac{x_j}{x^4}, \quad A_a = -2Q_5\langle \dot{F}_a F_j \rangle \frac{x_j}{x^4},$$

(C.2)

First let us note that by shifting the origin we can always set $\langle F_i \rangle = 0$. Then the ten dimensional dilaton will be of the form

$$e^{2\Phi} = \frac{f_1}{f_5} = \frac{Q_1}{Q_5} (1 + 2\frac{x^i}{x^2} \frac{\langle F_i \dot{G}^2 \rangle}{\langle \dot{G}^2 \rangle} + \cdots)$$

(C.3)

Since this decays very slowly for large $x$ we set its coefficient to zero. Similarly, by considering the fields that are excited by the torus fluctuations we conclude that we also need to set to zero $\langle F_a F_i \rangle = 0$.

We have seen above that our metrics will generically have a particular operator of weight $(1, 1)$ with a non-vanishing expectation value. This will give rise to a deformation of the metric that can be sensed far away. If we are interested in having a metric which is very close to the metric of a conical defect then we want to make the coefficient of this operator as small as possible. The operator we discussed is a $n = 2$ spherical harmonic on $S^3$ so that its coefficients have the form of a quadrupole moment $Q_{ij}$. In particular we can look at the following combination:

$$\sqrt{\tilde{f}_1 f_5} - \frac{1}{\sqrt{\tilde{f}_1 f_5}} [A_i A_i - B_i B_i] = \frac{\sqrt{Q_1 Q_5}}{x^2} \frac{Q_{ij} x_i x_j}{x^6} + O(x^{-5})$$

then we notice that the quadrupole moment $Q_{ij}$ vanishes for a conical defect. For a general metric $Q_{ij}$ be computed by using (C.2) and performing a computation very similar to the one we did near (B.15), (B.16). We find

$$Q_{11} + Q_{22} \sim \left[ (F_1^2 + F_2^2) - (F_3^2 + F_4^2) \right] (1 + \frac{\langle \dot{G}^2 \rangle}{\langle G^2 \rangle}) - 8\langle \dot{F}_1 F_2 \rangle^2 \frac{\langle G^2 \rangle}{\langle G^2 \rangle}$$

(C.4)

where expectation values mean averages over $v$ and we used that the angular momentum is in the 12 plane. We want (C.4) to vanish in order to have a metric close to (C.1). As

15 Note that $2\langle F_1 \dot{F}_2 \rangle = \langle F_1 \dot{F}_2 - F_2 \dot{F}_1 \rangle \sim J_{12}$. 

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we argued above we can neglect \((F_3^2 + F_4^2)\) relative to \((F_1^2 + F_2^2)\) in (C.4). It is possible to show that the result we get after neglecting such a term is always positive and it only vanishes when the profile is precisely a ring and the motion has constant velocity. In order to show that let us multiply all terms in (C.4) by \(\langle|\dot{G}|^2\rangle\). Defining \(F_1 + iF_2 = re^{i\phi}\) we then find
\[
\left(\langle r^2 \dot{\phi}^2 \rangle - \langle r^2 \dot{\phi} \rangle^2 \right) + \left(\langle r^2 \dot{\phi}^2 \rangle - \langle r^2 \dot{\phi} \rangle^2 \right) + \text{other terms} \tag{C.5}
\]
where all other terms are non negative. Using the formula \(\langle ab \rangle^2 \leq \langle a^2 \rangle \langle b^2 \rangle\) (and the equal sign holds only if \(a/b = \text{constant}\)) we see that all terms are non negative so that if (C.3) vanishes then all terms should be zero. Setting the first term in (C.3) to zero we get that \(\dot{\phi} = \text{constant}\). Setting the second to zero we get \(r = \text{constant}\). Setting to zero all other terms in (C.3) we get that \(|\dot{F}|^2 = \dot{F}_3^2 + \dot{F}_4^2 = 0\).

What we have shown so far is that if the metric is close to the conical defect then the profile closely tracks a profile with constant \(r\) and \(\dot{\phi}\). Since \(\phi\) has to be single valued this implies that only integer values of \(\gamma^{-1}\) are allowed.

We also see that generic chiral primaries with \(J_{L,R}^{NS} < k/2\) do not produce conical metrics (C.1) but the metrics that we have discussed in our paper. Only very special chiral primaries produce metrics close to (C.1) with integer \(\gamma^{-1}\).

Appendix D. Evaluation and Expansion of the integrals \(I_{1}^{(n)}(k), I_{2}^{(n)}(k)\)

The integrals we defined in (5.3)
\[
I_{1}^{(n)}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\alpha}{(1 + k \cos \alpha)^{n/2}} \quad ; \quad I_{2}^{(n)}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\cos \alpha d\alpha}{(1 + k \cos \alpha)^{n/2}} \tag{D.1}
\]
are not hard to evaluate. For even \(n\) the integrals are
\[
I_{1}^{(2)}(k) = \frac{1}{\sqrt{1 - k^2}} , \quad I_{1}^{(4)}(k) = \frac{1}{(1 - k^2)^{3/2}} , \quad I_{1}^{(6)}(k) = \frac{(2 + k^2)}{2(1 - k^2)^{5/2}} , \quad I_{1}^{(8)}(k) = \frac{(2 + k^2)}{2(1 - k^2)^{7/2}}
\]
\[
I_{2}^{(2)}(k) = -\frac{(1 - \sqrt{1 - k^2})}{k\sqrt{1 - k^2}} , \quad I_{2}^{(4)}(k) = -\frac{k}{(1 - k^2)^{3/2}} , \quad I_{2}^{(6)}(k) = -\frac{3k}{2(1 - k^2)^{5/2}} \tag{D.2}
\]
For odd \( n \), the integrals involve elliptic functions

\[
2\pi I_1^{(1)}(k) = 4 \frac{1}{\sqrt{1+k}} K\left(\sqrt{\frac{2k}{1+k}}\right)
\]

\[
2\pi I_1^{(3)}(k) = 4 \frac{\sqrt{1+k}}{1-k^2} E\left(\sqrt{\frac{2k}{1+k}}\right)
\]

\[
2\pi I_1^{(5)}(k) = \frac{4\sqrt{1+k}}{3(1-k^2)^2} \left[-(1-k)K\left(\sqrt{\frac{2k}{1+k}}\right) + 4E\left(\sqrt{\frac{2k}{1+k}}\right)\right]
\]

\[
2\pi I_2^{(1)}(k) = \frac{4}{k\sqrt{1+k}} \left[(1+k)E\left(\sqrt{\frac{2k}{1+k}}\right) - K\left(\sqrt{\frac{2k}{1+k}}\right)\right]
\]

\[
2\pi I_2^{(3)}(k) = \frac{4\sqrt{1+k}}{k(1-k^2)} \left[E\left(\sqrt{\frac{2k}{1+k}}\right) - (1-k)K\left(\sqrt{\frac{2k}{1+k}}\right)\right]
\]

\[
2\pi I_2^{(5)}(k) = \frac{4\sqrt{1+k}}{3k(1-k^2)^2} \left[-(1+3k^2)E\left(\sqrt{\frac{2k}{1+k}}\right) + (1-k)K\left(\sqrt{\frac{2k}{1+k}}\right)\right]
\]

They also obey the relations:

\[
I_1^{(n)}(-k) = I_1^{(n)}(k), \quad I_2^{(n)}(k) = -I_2^{(n)}(-k), \quad I_2^{(n)}(k) = -\frac{2}{n-2} \partial_k I_1^{(n-2)}(k). \tag{D.4}
\]

We were interested in evaluating the integrals at \( k = -\frac{2\sigma}{\sigma^2} \) so that in the near ring limit \( k \to -1 \) where the functions (D.2) and (D.3) are singular. Let us find the leading contribution near \( k = -1 \). For \( n > 1 \) we find

\[
I_1^{(n)}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha}{(1+k-2k\sin^2(\alpha/2))^{n/2}} \approx \frac{\sqrt{2}}{2\pi} \int_{-\infty}^{\infty} \frac{d\beta}{(1+k+\beta^2)^{n/2}}
\]

\[
= \frac{\sqrt{2}}{2\pi} (1+k)^{-n/2+1/2} B\left(\frac{1}{2}, \frac{n-1}{2}\right), \tag{D.5}
\]

\[
I_2^{(n)}(k) \approx \frac{\sqrt{2}}{2\pi} (1+k)^{-n/2+1/2} B\left(\frac{1}{2}, \frac{n-1}{2}\right).
\]

For \( n = 1 \) one can extract the leading asymptotic from the elliptic function

\[
I_1^{(1)} = I_2^{(1)} = \frac{\sqrt{2}}{2\pi} \ln \frac{32}{1+k} \tag{D.6}
\]

We also need the expression for

\[
I_3^{(n)} \equiv (I_1^{(n)})^2 - (I_2^{(n)})^2 \tag{D.7}
\]

and the asymptotics (D.5), (D.6) is not enough to evaluate it. Nevertheless, we can rewrite the leading asymptotics of this expression as

\[
I_3^{(n)} \approx 2I_1^{(n)}(I_1^{(n)} - I_2^{(n)}) \tag{D.8}
\]
and the problem is reduced to evaluation of the leading behavior of

\[ I_4^{(n)} = I_1^{(n)} - I_2^{(n)} \approx \frac{1}{2\pi} \int_0^{2\pi} \frac{2\sin^2(\alpha/2)d\alpha}{(1 + k - 2k \sin^2(\alpha/2))^{n/2}} \]  

(D.9)

This integral can be written as

\[ I_4^{(n)} \approx \frac{2}{n-2} (k + 1)^{2-n/2} \frac{\partial}{\partial k} \left[ (k + 1)^{n/2-1} I_1^{(n-2)} \right] \]  

(D.10)

for \( n > 1 \), and the integrals for \( I_4^{(1)} \) and \( I_4^{(2)} \) can be evaluated explicitly. This gives the following asymptotics:

\[ I_4^{(1)} = \frac{2\sqrt{2}}{\pi}, \quad I_4^{(2)} = 1, \quad I_4^{(3)} = \frac{1}{\pi \sqrt{2}} \ln \frac{32}{1 + k}, \quad I_4^{(n)} = \frac{B\left(\frac{1}{2}, \frac{n-3}{2}\right)}{\sqrt{2\pi}} \frac{(1 + k)^{3-n}}{n-2} (n > 3) \]  

(D.11)

Using above expressions we can find the leading asymptotics of \( I_3^{(n)} \)

\[ I_3^{(1)} = \frac{8}{\pi^2} (\ln \frac{8a}{\rho}), \quad I_3^{(2)} = \frac{2a}{\rho}, \quad I_3^{(3)} = \frac{8a^2}{\pi^2 \rho^2} (\ln \frac{8a}{\rho}), \quad I_3^{(4)} = \left(\frac{a}{\rho}\right)^4, \quad I_3^{(5)} = \frac{64 a^6}{9 \pi^2 \rho^6}, \quad I_3^{(6)} = \frac{3}{4} \left(\frac{a}{\rho}\right)^8 \]  

(D.12)

where we have used

\[ 1 + k \approx \rho^2 \frac{2a^2}{\rho^2} \]

**Appendix E. The near ring solutions**

In this appendix we expand (5.4) around the ring to examine its behavior.

Expanding (5.4) for small \( \rho \) , where \( \rho \) is the distance from the ring, \( s = a + \rho \sin \Theta \), \( w = \rho \cos \Theta \), using the expansions of \( I_1, I_2 \) around \(-1\) which appear in Appendix D , we find the following near-ring metrics for the different dimensions\(^{16}\). However, as \( g_{t\phi} \) remains finite in the limit, the metrics we obtain are nondegenerate.

\[ ds^2 \sim \epsilon^3. \]  

For \( d \geq 5 \) the scaling is \( \rho, z^i \sim \epsilon^2, \phi, t \sim \epsilon, \epsilon \to 0 \). then \( ds^2 \sim \epsilon^3 \). In these limits, \( g_{tt} \) always scales to zero as \( g_{tt} \sim \left(\frac{\rho}{a}\right)^{3(d-3)/2} \sim \epsilon^{3(d-3)} \).

---

\(^{16}\) More rigorously, all of the limits above should be thought of as scaling limits where \( a, Q, \omega \) remain constant and the coordinates scale. For \( d = 4 \) this scaling is \( \rho, z^i \sim \epsilon^2, y, \phi, t \sim \epsilon, \epsilon \to 0 \). then \( ds^2 \sim \epsilon^3 \). For \( d \geq 5 \) the scaling is \( \rho, z^i \sim \epsilon^2, \phi \sim 1, y \sim \epsilon^{-(d-5)}, t \sim \epsilon^{-2(d-5)} \) and then the metric scales as \( ds^2 \sim \epsilon^{d-3} \). In these limits, \( g_{tt} \) always scales to zero as \( g_{tt} \sim \left(\frac{\rho}{a}\right)^{3(d-3)/2} \sim \epsilon^{3(d-3)} \). however, as \( g_{t\phi} \) remains finite in the limit, the metrics we obtain are nondegenerate.
This form of the metric is valid only for $\rho \ll a$ where the log is strictly positive. For larger values of $\rho$, one needs to retain more terms in the expansion of the elliptic functions. A U-dual system of this $d = 3$ solution was recently considered in [22], where it was lifted to an M-theory solution with zero gauge fields. That solution is singular, as can be verified by calculating its curvature invariants.
* $d = 8$:

\[
ds^2 \approx \frac{1}{a^2 \omega^2} \left[ \frac{16 a^6}{3 Q} \right]^{3/2} \left( \frac{\rho}{a} \right)^{5/2} \left[ \frac{3 \omega Q}{8 a^4} dt d\phi + \frac{3Q}{16a^6} dy^2 + \frac{3 \omega^2 Q^2}{256 a^8} \left( \frac{a}{\rho} \right)^3 d\phi^2 \right] + \\
+ \sqrt{\frac{3Q}{16a^6} \left( \frac{a}{\rho} \right)^{5/2} \left[ d\rho^2 + \rho^2 d\Omega_6^2 \right]}
\]

(E.6)

Looking at these metrics, one can see that only for $d = 4$, we obtain a $g_{\phi \phi}$ which scales with $\rho$ like the other metric components parallel to the brane. For the other dimensions $d > 4$, we find that $g_{\phi \phi}$ goes to zero much slower than the other parallel components, as we approach the brane. However, one must bare in mind that what we should obtain are supergravity solutions describing a brane with fluxes on a ring. The effects of the curvature evidently affect the metric near the brane for all $d > 4$. This is related to I.R. phenomena on the worldvolume theory on the brane. Whether a solution is singular or not might depend on the U-duality frame in which it is presented. We did not find any frame where the solutions we have above for $d \neq 4$ ($d = 4$ is U-dual to the D1-D5 system) are non-singular.
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