SHARP VANISHING ORDER OF SOLUTIONS TO STATIONARY SCHröDINGER EQUATIONS ON CARNOT GROUPS OF ARBITRARY STEP

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ABSTRACT. Based on a variant of the frequency function approach of Almgren ([Al]), under appropriate assumptions we establish an optimal upper bound on the vanishing order of solutions to stationary Schrödinger equations associated to sub-Laplacian on Carnot groups of arbitrary step. Such a bound provides a quantitative form of strong unique continuation and can be thought of as a subelliptic analogue of the recent results obtained by Bakri ([Bk]) and Zhu ([Zhu]) for the standard Laplacian.

1. Introduction

We say that the vanishing order of a function $u$ is $\ell$ at $x_0$, if $\ell$ is the largest integer such that $D^\alpha u = 0$ for all $|\alpha| \leq \ell$, where $\alpha$ is a multi-index. In the papers [DF1], [DF2], Donnelly and Fefferman showed that if $u$ is an eigenfunction with eigenvalue $\lambda$ on a smooth, compact and connected $n$-dimensional Riemannian manifold $M$, then the maximal vanishing order of $u$ is less than $C \sqrt{\lambda}$ where $C$ only depends on the manifold $M$. Using this estimate, they showed that $H^{n-1}(x : u_\lambda(x) = 0) \leq C \sqrt{\lambda}$ where $u_\lambda$ is the eigenfunction corresponding to $\lambda$ and therefore gave a complete answer to a famous conjecture of Yau ([Yau]). We note that the zero set of $u_\lambda$ is referred to as the nodal set. This order of vanishing is sharp. If, in fact, we consider $M = S^n \subset \mathbb{R}^{n+1}$, and we take the spherical harmonic $Y_\kappa$ given by the restriction to $S^n$ of the function $f(x_1, \ldots, x_n, x_{n+1}) = \Re(x_1 + ix_2)^\kappa$, then one has $\Delta_{S^n} Y_\kappa = -\lambda_\kappa Y_\kappa$, with $\lambda_\kappa = \kappa(\kappa+n-2)$, and the order of vanishing of $Y_\kappa$ at the North pole $(0, \ldots, 0, 1)$ is precisely $\kappa = C \sqrt{\lambda_\kappa}$.

In his work [Ku1] Kukavica considered the more general problem

$$\Delta u = V(x)u,$$

where $V \in W^{1,\infty}$, and showed that the maximal vanishing order of $u$ is bounded above by $C(1 + ||V||_{W^{1,\infty}})$. He also conjectured that the rate of vanishing order of $u$ is less than or equal to $C(1 + ||V||_{L^\infty}^{1/2})$, which agrees with the Donnelly-Fefferman result when $V = -\lambda$. Employing Carleman estimates, Kenig in [K] showed that the rate of vanishing order of $u$ is less than $C(1 + ||V||_{L^\infty}^{2/3})$, and that furthermore the exponent $2/3$ is sharp for complex potentials $V$ based on a counterexample of Meshov. (see [Me]).

Recently, the rate of vanishing order of $u$ has been shown to be less than $C(1 + ||V||_{W^{1,\infty}}^{1/2})$ independently by Bakri in [Bk] and Zhu in [Zhu]. Bakri’s approach is based on an extension of the Carleman method in [DF1]. In this connection, we also quote the recent interesting paper by Rüland [Ru], where Carleman estimates are used to obtain related quantitative unique continuation results for nonlocal Schrödinger operators such as $(-\Delta)^{s/2} + V$. On the other hand, Zhu’s approach is based on a variant of the frequency function approach employed by Garofalo and Lin in [GL1], [GL2], in the context of strong unique continuation problems. Such variant consists in studying the growth properties of the following average of the Almgren’s height function

$$H(r) = \int_{B_r(x_0)} u^2 (r^2 - |x - x_0|^2)^\alpha dx, \quad \alpha > -1,$$
first introduced by Kukavica in [Ku] to study quantitative unique continuation and vortex degree estimates for solutions of the Ginzburg-Landau equation. In [BK] and [Zhu] it was assumed that $u$ be a solution in $B_{10}$ to

$$\Delta u = Vu,$$

with $||V||_{W^{1,\infty}} \leq M$ and $||u||_{L^{\infty}} \leq C_0$, and that furthermore $\sup_{B_1} |u| \geq 1$. Then, it was proved that $u$ satisfies the sharp growth estimate

$$||u||_{L^{\infty}(B_r)} \geq Br^{C(1+\sqrt{M})},$$

where $B, C$ depend only on $n$ and $C_0$. Such an estimate has been recently extended to stationary Schrödinger equations associated to generalized Baouendi Grushin operators in [BG1] and also for elliptic equations with Lipschitz principle part at the boundary of Dini domains in [BG2]. Over here, we would like to refer to [Ba], [Gr1] and [Gr2] for a detailed account on Baouendi-Grushin operators and corresponding hypoellipticity results.

Therefore given the current interest in quantitative forms of strong unique continuation and the crucial role played by them in the past to get Hausdorff measure estimates on the nodal sets as in [DF1] and [DF2] has provided us with a natural motivation to study quantitative uniqueness for elliptic equations on Carnot groups. More precisely, we analyze equations of the form

$$\Delta_H u = Vu,$$

where $\Delta_H$ is the sub-Laplacian on a Carnot group $G$ (see (2.9) below) and the discrepancy $E_u$ of the solution $u$ (see (2.22) for the definition) at the identity $e$ satisfies the growth assumption (2.23). The growth assumption (2.23) can be thought of as the measure of a certain symmetry type property of $u$ and we have kept a brief discussion on this aspect in Section 2.

The assumptions on the potential function $V$ are specified in (2.19) in the next section. They represent the counterpart on $G$ with respect to certain non-isotropic dilations of the following Euclidean requirements

$$|V(x)| \leq M, \quad |<x,DV(x)>| \leq M,$$

for the classical Schrödinger equation $\Delta u = Vu$ in $\mathbb{R}^n$. Such non-isotropic dilations will be described in Section 2.

Now in the case of Carnot groups, unlike the Euclidean case, the reader should notice that although we have an additional assumption (2.23) on the discrepancy $E_u$ of $u$, it is still not very restrictive in the sense that strong unique continuation property is in general not true for solutions to (1.4). This follows from some interesting work of Bahouri (see [Bah]) where the author showed that unique continuation is not true for even smooth and compactly supported perturbations of the sub-Laplacian. Therefore, one cannot expect any quantitative estimates to hold either without further assumptions. Once we introduce the appropriate notion in Section 2, the reader will also clearly see that the discrepancy $E_u$ is identically zero in the Euclidean case, i.e. when we view $G = \mathbb{R}^n$ as a Carnot group of step 1. On the other hand, it turns out that so far, only with this growth assumption on $E_u$ that we have in (2.23), strong unique continuation property (supc) for (1.4) is known. This follows from the interesting work of Garofalo and Lanconelli (see [GLa]) in the case when $G = \mathbb{H}^n$ (Heisenberg group which is a Carnot group of step 2). Such a result has been recently generalized to Carnot groups of arbitrary step by Garofalo and Rotz in [GR]. It is to be noted that the results in [GLa] and [GR] follow the circle of ideas in the fundamental works [GL1] and [GL2].

The purpose of our work is to therefore derive sharp quantitative estimates for equations (1.4) in the setup of [GR] where supc is known so far, i.e. with the growth assumption on the discrepancy term $E_u$ as in (2.23). Our main result Theorem 2.1 should be seen as a subelliptic
generalization of the above mentioned Euclidean results in [BK] and [Zhu]. As the reader will realize, such a generalization relies on the deep link existing between the growth properties of a certain generalized Almgren frequency and the sub-elliptic structure of $G$. It turns out that in the end, they beautifully combine.

In this paper, similar to [Zhu], and [BG1], we work with an appropriate weighted version of the Almgren's Frequency which is somewhat different from the one introduced in [GR]. Having said that, we do follow [GR] closely in parts. Since we are interested in the question of sharp vanishing order estimates, it is worth emphasizing that as opposed to Theorem 7.3 in [GR], we require some kind of monotonicity of the generalized frequency that is introduced in Section 3 and not just the boundedness of the frequency (see Theorem 3.1). Moreover, in order to recover the sharp vanishing order in our subelliptic situation, we also need to keep track of how the several constants that appear in our computations depend on the subelliptic $C^1$ norm of $V$ as in (2.19) and this entails some novel work. As the reader will notice in Section 3, it turns out that we have to substantially modify an argument used in the proof of Theorem 7.3 in [GR]. This constitutes one of the delicate aspects of our work and makes our proof quite different from that of the Laplacian as in [Zhu] and also from that of the Baouendi Grushin operators as in [BG1].

The paper is organized as follows. In Section 2, we introduce some basic notations, gather the relevant preliminary results from [DG], [GR], [GLa] and [GV1] and state our main result. In Section 3, we establish a monotonicity theorem for the generalized weighted Almgren type frequency that we introduce and we then subsequently prove our main result.

Acknowledgment: The author would like to thank his former PhD. advisor Prof. Nicola Garofalo for introducing him to the very interesting subject of unique continuation and whose fundamental work on this subject has been his constant inspiration. The author would also like to thank him for clarifying several results obtained in [GR].

2. Preliminaries and Statement of main result

2.1. Preliminaries. In this section, we state some preliminary results that is relevant to our work and is similar to the one as in Section 2 in [GR]. Henceforth in this paper we follow the notations adopted in [GR] with a few exceptions. For most of the discussion in this section, one can find a detailed account in the book [BLU]. We recall that a Carnot group of step $h$ is a simply connected Lie group $G$ whose lie algebra $g$ admits a stratification $g = V_1 \oplus \ldots \oplus V_h$ which is $h$ nilpotent., i.e., $[V_1, V_j] = V_{j+1}$ for $j = 1, \ldots, h-1$ and $[V_j, V_h] = 0$ for $j = 1, \ldots, h$. A trivial example is when $G = \mathbb{R}^n$ and in which case $g = V_1 = \mathbb{R}^n$. The simplest non-Abelian example of a Carnot group of step 2 is the Heisenberg group $\mathbb{H}^n$, i.e. in $\mathbb{R}^{2n+1}$, we let $(x, y, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ and the group operation is as follows

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x' y - x y'))$$

In such a case, we have that $V_1$ is spanned by

$$X_i = \partial_{x_i} + 2y_i \partial_t, \quad i = 1, \ldots, n$$

$$Y_j = \partial_{y_j} - 2x_j \partial_t, \quad j = 1, \ldots, n$$

and $V_2$ is spanned by $\partial_t$. We note that the following holds,

$$[X_i, Y_j] = -4\delta_{ij} \partial_t$$

and therefore $V_1$ generates the whole lie algebra. We would like to mention that over here, we identify the lie algebra $g$ with the left invariant vector fields.

Now in a Carnot group $G$, by the above assumptions on the Lie algebra, we see that any basis of the horizontal layer $V_1$ generates the whole $g$. We will respectively denote by

$$L_g(g') = gg', \quad R_g(g') = g'g$$

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the left and right translation by an element \( g \in \mathbb{G} \).

The exponential mapping \( \exp : \mathfrak{g} \to \mathbb{G} \) defines an analytic diffeomorphism onto \( \mathbb{G} \). We recall the Baker-Campbell-Hausdorff formula, see for instance section 2.15 in [V],

\[
\exp(c_1)\exp(c_2) = \exp(c_1 + c_2 + \frac{1}{2}[c_1, c_2] + \frac{1}{12}\{[c_1, [c_1, c_2]] - [c_2, [c_1, c_2]]\} + ..)
\]

where the dots indicate commutators of order four and higher. Each element of the layer \( V_j \) is assigned a formal degree \( j \). Accordingly, one defines dilations on \( \mathfrak{g} \) by the rule

\[
\Delta_\lambda c = \lambda c_1 + \ldots \lambda^h c_h
\]

The anisotropic dilations \( \delta_\lambda \) on \( \mathbb{G} \) are then defined as

\[
\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1} g
\]

Throughout the paper, we will indicate by \( dg \) the bi-invariant Haar measure on \( \mathbb{G} \) obtained by lifting via the exponential map \( \exp \) the Lebesgue measure on \( \mathfrak{g} \). Let \( m_j = \text{dim} V_j \). One can check that

\[
(d \circ \delta_\lambda)(g) = \lambda^Q dg
\]

where \( Q = \sum_{j=1}^{h} j m_j \). \( Q \) is referred to as the homogeneous dimension of \( \mathbb{G} \) and is in general different from the topological dimension of \( \mathbb{G} \) which is \( \sum_{j=1}^{h} m_j \).

We let \( Z \) to be the smooth vector field which corresponds to the infinitesimal generator of the non-isotropic dilations \( \delta_\lambda \). Note that \( Z \) is characterized by the following property

\[
\frac{d}{dr} u(\delta_r g) = \frac{1}{r} Z u(\delta_r g)
\]

Therefore if \( u \) is homogeneous of degree \( k \) with respect to \( \delta_\lambda \), i.e., \( u(\delta_r g) = r^k u(g) \), then we have that \( Zu = ku \).

Let \( \{e_1, \ldots, e_m\} \) be an orthonormal basis of the first layer \( V_1 \) of the Lie algebra. We define the corresponding left invariant smooth vector fields by the formula

\[
X_i(g) = dL_g(e_i), \quad i = 1, \ldots, m
\]

where \( dL_g \) denote the differential of \( L_g \). We assume that \( \mathbb{G} \) is endowed with a left-invariant Riemannian metric such that \( \{X_1, \ldots, X_m\} \) are orthonormal. We note that in this case, the bi-invariant Haar measure \( dg \) agrees with the Riemannian volume element (see for instance [BLU]). The corresponding subLaplacian is defined by the formula

\[
\Delta_H u = \sum_{i=1}^{m} X_i^2 u
\]

We note that by Hormander’s theorem, \( \Delta_H \) is hypoelliptic. We will indicate with \( e \) the identity element of \( \mathbb{G} \).

Let \( \Gamma(g, g') = \Gamma(g', g) \) be the positive fundamental solution of \( -\Delta_H \). It turns out that \( \Gamma \) is left invariant, i.e.,

\[
\Gamma(g, g') = \hat{\Gamma}(g^{-1} \circ g')
\]

For every \( r > 0 \), let

\[
B_r = \{g \in \mathbb{G} | \Gamma(g, e) > \frac{1}{r^{Q-2}}\}
It was proved by Folland [F1] that $\tilde{\Gamma}(g)$ is homogeneous of order $2 - Q$ with respect to the non-isotropic dilations (2.5). Therefore, if we define

$$\rho(g) = \Gamma(g)^{\frac{1}{Q-2}}$$

then $\rho$ is homogenous of degree 1. One can immediately see that $B_r$ can be equivalently characterized as

$$B_r = \{ g : \rho(g) < r \}$$

We let $S_r = \partial B_r$. We note that since $\Gamma$ is homogeneous of degree $2 - Q$, therefore

$$Z\Gamma = (2 - Q)\Gamma$$

Now by the strong maximum principle (Since $\Gamma(g,e)$ is harmonic for $g \neq e$), we have that $\Gamma(g,e) > 0$ for all $g \neq e$. Now since $Z\Gamma = <D\Gamma, Z>$, where $D\Gamma$ is the Riemannian gradient with respect to the left invariant metric, we conclude from (2.14) that $D\Gamma$ never vanishes. Therefore, by implicit function theorem, we conclude that the level sets $S_r$ are smooth hypersurfaces in $G$.

The position $d(g,g')$ defined by

$$d(g,g') = \rho(g^{-1} \circ g')$$

defines a pseudo-distance on $G$. In what follows, we denote by

$$\nabla_H u = \sum_{i=1}^{m} X_i u X_i$$

the horizontal gradient of $u$. We also let

$$|\nabla_H u|^2 = \sum_{i=1}^{m} (X_i u)^2$$

**2.2. Statement of the main result.** In order to state our main result, we first describe our framework. We will assume that $u$ is a solution to

$$\Delta_H u = Vu$$

in $B_1$. Since the regularity issues are not our main concern, we will assume apriori that $u, X_i u, X_i X_j u, Z u$ are in $L^2(B_1)$ with respect to the Haar measure $dg$. Concerning the potential $V$, we assume that it satisfies

$$|V| \leq K|\nabla_H \rho|^2 \quad |ZV| \leq K|\nabla_H \rho|^2$$

for some $K > 1$. An example of a smooth $V$ which satisfies (2.19) is given by $V(g) = \tilde{\nabla}(g)f(\rho|\nabla_H \rho|^2)$ where $\tilde{\nabla}$ is a smooth function defined on $G$ and $f : \mathbb{R} \to \mathbb{R}$ is a smooth compactly supported function which vanishes in a neighborhood of 0. The first condition in (2.19) is easy to see and for the second condition we note that

$$|ZV| \leq |(Z\tilde{\nabla})f(\rho|\nabla_H \rho|^2)| + |\tilde{\nabla}f'(\rho|\nabla_H \rho|^2)Z\rho|\nabla_H \rho|^2| + |\tilde{\nabla}f'(\rho|\nabla_H \rho|^2)\rho Z|\nabla_H \rho|^2|$$

Now since $\rho$ has homogeneity 1, $Z\rho = \rho$ and since $|\nabla_H \rho|^2$ has homogeneity 0 being the derivative of 1 homogenous function, we have that $Z(|\nabla_H \rho|^2) = 0$ for $g \neq e$. We note that the derivative in (2.20) is only computed for $g \neq e$ since $f$ vanishes in a neighborhood of 0. Therefore we obtain

$$|ZV| \leq |(Z\tilde{\nabla})f(\rho|\nabla_H \rho|^2)| + |\tilde{\nabla}f'(\rho|\nabla_H \rho|^2)\rho Z|\nabla_H \rho|^2|$$

From (2.21), it is easy to see that the second condition in (2.19) is satisfied for this choice of $V$.

As in [GR], we define the discrepancy $E_u$ at $e$ by

$$E_u = <\nabla_H u, \nabla_H \rho > - \frac{Z u}{\rho}|\nabla_H \rho|^2$$
Like in [GR], we will also assume that
\[
|E_u| \leq \frac{f(\rho)}{\rho} |\nabla_H \rho|^2 |u|
\]
where \( f : (0, 1) \to (0, \infty) \) is a continuous increasing function which satisfies the Dini integrability condition
\[
\int_0^1 \frac{f(t)}{t} dt < K_0, \quad |f| \leq K_1
\]
We now list a few classes of examples from [GR] in which the assumption (2.23) holds.

In the case when \( G = \mathbb{R}^n \), we have that \( Z = \Sigma_{i=1}^n x_i \partial_i \) and \( \nabla_H \rho = \frac{\rho}{|x|} \). Therefore, we clearly see that \( E_u \equiv 0 \) in this case.

For a general Carnot group \( G \), when \( u \) is radial, i.e., if \( u(g) = f(\rho(g)) \), then it follows from a straightforward calculation (See Proposition 9.6 in [GR]) that \( E_u \equiv 0 \).

Also, if we specialize to the case when \( G = \mathbb{H}^n \) and assume \( u \) to be polyradial, i.e., with \( g = (w_1, \ldots, w_n, t) \) where \( w_i = (x_i, y_i) \), we have that \( u(g) = \phi(|w_1|, \ldots, |w_n|, t) \), then \( E_u \equiv 0 \). (see Proposition 9.11 in [GR]). It is however not true that for general groups of Heisenberg type (see Section 9.1 in [GR] for the precise definition of groups of Heisenberg type), polyradial functions have zero discrepancy. (see Section 9 in [GR] for a counterexample). Nevertheless, given these examples, we would like to think of the growth condition (2.23) on \( E_u \) as the measure of a certain symmetry type property of \( u \).

We now state our main result.

**Theorem 2.1.** Let \( u \) be a solution to (2.18) in \( B_1 \) such that \( |u| \leq C_0 \) and the discrepancy \( E_u \) of \( u \) at \( e \) satisfies (2.23). Let \( V \) satisfy (2.19). Then there exists a universal \( a \in (0, 1/3) \), and constants \( C_1, C_2 \) depending on \( Q, C_0 \) and \( K_0, K_1 \) in (2.24) and also \( \int_{B_{1/3}} u^2 |\nabla_H \rho|^2 dg \) such that for all \( 0 < r < a \), one has
\[
||u||_{L^\infty(B_r)} \geq C_1 r^{C_2 \sqrt{K}}
\]

**Remark 2.2.** We note that in the elliptic case as in [BG2], [Zhu], we have that \( |\nabla_H \rho|^2 \equiv 1 \) and in such a case, the constant \( K \) in (2.19) can be taken to be \( C(||V||_{W^{1, \infty}} + 1) \) for some universal \( C \). We thus see that in the elliptic case, (2.2.5) reduces to the Euclidean result as in [Bk] and [Zhu] since \( E_u \equiv 0 \) in the Euclidean case. Therefore our estimate (2.25) gives sharp bounds on the vanishing order of \( u \) at the identity \( e \) in terms of a certain "subelliptic" \( C^1 \) norm of \( V \) with bounds as in (2.19). Hence, our result can be thought of as a subelliptic analogue of the sharp quantitative uniqueness result in the Euclidean case.

**Remark 2.3.** It is worth mentioning the case when \( G = \mathbb{H}^n \) and \( E_u \equiv 0 \). Now from the definition of the sublaplacian and the explicit representation of the horizontal vector fields for \( \mathbb{H}^n \) as in (2.1), we have that \( u \) solves the following equation
\[
\Delta_z u + \frac{|z|^2}{4} \partial_{tt} u + \partial_{\theta} \theta u = V u
\]
where \( \theta_u = \Sigma (x_j \partial_{y_j} u - y_j \partial_{x_j} u) \) and \( E_u = \frac{\partial_{tt} \theta u}{\partial_{\theta} u} \) (see for instance Lemma 9.8 in [GR]). Now when \( E_u \equiv 0 \), we get that \( \theta_u \equiv 0 \) and hence \( u \) solves the stationary Schrödinger equation corresponding to the Baouendi-Grushin operator
\[
\Delta_z u + \frac{|z|^2}{4} \partial_{tt} u = V u
\]
for which the sharp quantitative estimate (2.25) follows from Theorem 1.1 in [BG1].
However if we only assume that $E_u$ satisfies \((2.23)\), then from \((2.23)\) and the fact that $E_u = \frac{4}{\rho} \theta_u$ we can only assert that $u$ solves \((2.26)\) where $\theta_u$ satisfies the following growth condition,
\[(2.28)\]
\[|\theta_u| \leq \frac{f(\rho)\rho^2|\nabla_H \rho|^2|u|}{4t}\]
and in this case, our result is not implied by [BG1]. Therefore our result is new even for $G = H^n$.

**Remark 2.4.** It remains to be seen when the potential $V$ only satisfies
\[(2.29)\]
\[|V| \leq K|\nabla_H \rho|^2\]
instead of \((2.19)\), then if it can be shown that the vanishing order of the solution $u$ is bounded from above by $CK^{2/3}$. This would constitute the subelliptic analogue of the result in [K] to which we would like to come back in a future study.

### 3. Proof of Theorem 2.1

#### 3.1. Monotonicity of a generalized frequency.
Following [Zhu] and [BG1], for $\alpha > 0$ to be decided later, we let
\[(3.1)\]
\[H(r) = \int_{B_r} u^2 |\nabla_H \rho|^2 (r^2 - \rho^2)^\alpha \, dg\]
For notational convenience, we will let $|\nabla_H \rho|^2 = \psi$. Therefore with this new notation, we have that
\[(3.2)\]
\[H(r) = \int_{B_r} u^2 (r^2 - \rho^2)^\alpha \psi\]
By differentiating with respect to $r$, we get that
\[(3.3)\]
\[H'(r) = 2\alpha r \int u^2 (r^2 - \rho^2)^{\alpha-1} \psi\]
Using the identity
\[(3.4)\]
\[(r^2 - \rho^2)^{\alpha-1} = \frac{1}{r^2} (r^2 - \rho^2)^\alpha + \frac{\rho^2}{r^2} (r^2 - \rho^2)^{\alpha-1}\]
the latter equation can be rewritten as
\[(3.5)\]
\[H'(r) = \frac{2\alpha}{r} H(r) + \frac{2\alpha}{r} \int u^2 (r^2 - \rho^2)^{\alpha-1} \rho^2 \psi\]
Now by using the fact that $Z \rho = \rho$, we see that $(r^2 - \rho^2)^{\alpha-1} \rho^2$ can be rewritten as
\[(3.6)\]
\[(r^2 - \rho^2)^{\alpha-1} \rho^2 = -\frac{1}{2\alpha} Z(r^2 - \rho^2)^\alpha\]
Therefore we get that
\[(3.7)\]
\[H'(r) = \frac{2\alpha}{r} - \frac{1}{r} \int u^2 Z(r^2 - \rho^2)^\alpha \psi\]
Now we note that the following two identity holds
\[(3.8)\]
\[Z(|\nabla_H \rho|^2) = 0, \ g \neq e\]
and
\[(3.9)\]
\[\text{div}_G Z = Q\]
For (3.9), for instance the reader can refer to [DG]. Note that over here, $\text{div}_G$ denotes the Riemannian divergence on $G$. Now by using the Divergence theorem on $G$ with respect to its Riemannian structure and also by using (3.8), (3.9), we get that

$$H'(r) = \frac{2\alpha + Q}{r} H(r) + \frac{2}{r} \int uZ u(r^2 - \rho^2)^\alpha \psi$$

(3.10)

Over here, we crucially use the fact that since $|\nabla_H \rho|^2$ has homogeneity 0, therefore it is bounded and hence the integration by parts can be justified by an approximation type argument. Now by using (2.23), we get that

$$H'(r) = \frac{2\alpha + Q}{r} H(r) + \frac{1}{(\alpha + 1)r} I(r) + K(r)$$

(3.11)

where

$$|K(r)| \leq \frac{f(r)}{r} H(r)$$

(3.12)

(3.11) can hence be rewritten as

$$H'(r) = \frac{2\alpha + Q}{r} H(r) + \frac{1}{(\alpha + 1)r} I(r) + K(r)$$

(3.13)

where

$$I(r) = 2(\alpha + 1) \int u < \nabla_H u, \nabla_H \rho > (r^2 - \rho^2)^\alpha \rho - \int u < \nabla_H u, \nabla_H (r^2 - \rho^2)^{\alpha + 1} >$$

(3.14)

Now we note that the following identity holds (see for instance [GV1])

$$\text{div}_G X_i = 0$$

(3.15)

Therefore, by applying integrating by parts to (3.14) and by using the equation (2.18) and the identity (3.15) we get that,

$$I(r) = \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha + 1} + Vu^2 (r^2 - \rho^2)^{\alpha + 1}$$

(3.16)

We now define the generalized frequency of $u$ as

$$N(r) = \frac{I(r)}{H(r)}$$

(3.17)

The central result of this section which implies our main estimate (2.25) in Theorem 2.1 is the following monotonicity result of $N(r)$.

**Theorem 3.1.** For $\alpha = \sqrt{K}$, we have that there exists universal $\overline{C}$ depending on $Q, K_0, K_1$ such that

$$r \to e^{\overline{C} \int_0^r f(t) \frac{dt}{t}} (N(r) + \overline{C} K(r^2 + \int_0^r \frac{f(t)}{t} dt))$$

is monotone increasing on $(0, 1)$.

**Proof.** The proof will be divided into several steps. We first calculate $I'(r)$. By differentiating the expression in (3.16) with respect to $r$, we get that

$$I'(r) = 2(\alpha + 1) \int |\nabla_H u|^2 (r^2 - \rho^2)^\alpha + 2(\alpha + 1) r \int Vu^2 (r^2 - \rho^2)^\alpha$$

(3.19)

This can be rewritten as

$$I'(r) = \frac{2(\alpha + 1)}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha + 1} + \frac{2(\alpha + 1)}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha} + 2(\alpha + 1) r \int Vu^2 (r^2 - \rho^2)^\alpha$$

(3.20)
Using the fact that $Z\rho = \rho$, the second term on the right hand side of above expression can be rewritten as

\begin{equation}
(3.21) \quad \frac{2(\alpha + 1)}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^\alpha \rho^2 = -\frac{1}{r} \int |\nabla_H u|^2 Z(r^2 - \rho^2)^{\alpha + 1}
\end{equation}

At this point, we need the following Rellich type identity which corresponds to Corollary 3.3 in [GV]. This can be thought of as the sub-elliptic analogue of Rellich type identity established in [PW]. For a $C^1$ vector field $F$, we have that

\begin{equation}
(3.22) \quad 2 \int_{\partial B_r} Fu < \nabla_H u, N_H > dH^{n-1} + \int_{B_r} div F|\nabla_H u|^2 dg \\
- 2 \int_{B_r} X_i u [X_i, F] udg - 2 \int_{B_r} Fu \Delta_H udg \\
= \int_{\partial B_r} |\nabla_H u|^2 < F, \nu > dH^{n-1}
\end{equation}

We now apply the identity (3.22) to the vector field $F = (r^2 - \rho^2)^{\alpha + 1} Z$. We note that the boundary terms don’t appear due to the presence of the weight $(r^2 - \rho^2)^{\alpha + 1}$. Therefore we get,

\begin{equation}
(3.23) \quad -\frac{1}{r} \int |\nabla_H u|^2 Z(r^2 - \rho^2)^{\alpha + 1} = -\frac{1}{r} \int |\nabla_H u|^2 div F + \frac{Q}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha + 1}
\end{equation}

where we used the fact that $div F = Q$. Now by applying (3.22), we get that

\begin{equation}
(3.24) \quad -\frac{1}{r} \int |\nabla_H u|^2 Z(r^2 - \rho^2)^{\alpha + 1} = -\frac{2}{r} \int X_i u [X_i, F] udg - \frac{2}{r} \int Fu \Delta_H udg + \frac{Q}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha + 1}
\end{equation}

At this point, we note that the following identity holds (See for instance [DG])

\begin{equation}
(3.25) \quad [X_i, Z] = X_i
\end{equation}

Therefore by using (3.25), we have

\begin{equation}
(3.26) \quad [X_i, F] u = X_i (r^2 - \rho^2)^{\alpha + 1} Z + (r^2 - \rho^2)^{\alpha + 1} X_i = -2(\alpha + 1)\rho (r^2 - \rho^2)^{\alpha} X_i \rho Z + (r^2 - \rho^2)^{\alpha + 1} X_i
\end{equation}

By using (3.26) in (3.24) we get that,

\begin{equation}
(3.27) \quad -\frac{1}{r} \int |\nabla_H u|^2 Z(r^2 - \rho^2)^{\alpha + 1} = \frac{4(\alpha + 1)}{r} \int <\nabla_H u, \nabla_H \rho > \rho Z u (r^2 - \rho^2)^{\alpha} \\
- \frac{2}{r} \int V u Z u (r^2 - \rho^2)^{\alpha + 1} + \frac{Q - 2}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha + 1}
\end{equation}

Now by using the growth assumption (2.23) on the discrepancy $E_u$ we get that

\begin{equation}
(3.28) \quad -\frac{1}{r} \int |\nabla_H u|^2 Z(r^2 - \rho^2)^{\alpha + 1} = \frac{4(\alpha + 1)}{r} \int (Z u)^2 (r^2 - \rho^2)^{\alpha} \psi + \frac{Q - 2}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha + 1} \\
- \frac{2}{r} \int V u Z u (r^2 - \rho^2)^{\alpha + 1} + K_1(r)
\end{equation}

where

\begin{equation}
(3.29) \quad |K_1(r)| \leq \frac{4(\alpha + 1)f(r)}{r} \int (r^2 - \rho^2)^{\alpha} |u||Zu|\psi
\end{equation}
Therefore by substituting the above expression in (3.20) we get that,
\[(3.30)\]
\[I'(r) = \frac{2\alpha + Q}{r} \int \left| \nabla_H u \right|^2 (r^2 - \rho^2)^{\alpha + 1} + \frac{4(\alpha + 1)}{r} \int (Zu)^2 (r^2 - \rho^2)^{\alpha} \psi + 2(\alpha + 1)r \int Vu^2 (r^2 - \rho^2)^{\alpha} - \frac{2}{r} \int VuZu (r^2 - \rho^2)^{\alpha + 1} + K_1(r)\]
Recalling the definition of \(I(r)\), we can rewrite \(I'\) as
\[(3.31)\]
\[I'(r) = \frac{2\alpha + Q}{r} I(r) - \frac{2\alpha + Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha + 1} + \frac{4(\alpha + 1)}{r} \int (Zu)^2 (r^2 - \rho^2)^{\alpha} \psi + 2(\alpha + 1)r \int Vu^2 (r^2 - \rho^2)^{\alpha} - \frac{2}{r} \int VuZu (r^2 - \rho^2)^{\alpha + 1} + K_1(r)\]
Now by integrating by parts and again by using the fact that \(\text{div}_\mathbb{H} Z = Q\), we get that
\[(3.32)\]
\[-\frac{2}{r} \int VuZu (r^2 - \rho^2)^{\alpha + 1} = -\frac{1}{r} \int Zu^2 (V(r^2 - \rho^2)^{\alpha + 1} = \frac{1}{r} \int u^2 \text{div}_\mathbb{H} ((r^2 - \rho^2)^{\alpha + 1}) = -\frac{Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha + 1}\]
\[+ \frac{1}{r} \int u^2 ZV (r^2 - \rho^2)^{\alpha + 1} - \frac{2(\alpha + 1)}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha}\]
At this point, we note from (2.19) that the following estimate holds
\[(3.33)\]
\[\left| \frac{Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha + 1} \right| \leq CKrH(r)\]
and also
\[(3.34)\]
\[\left| \frac{1}{r} \int u^2 ZV (r^2 - \rho^2)^{\alpha + 1} \right| \leq CKrH(r)\]
for some universal \(C\). We now write the expression \(\frac{2\alpha + Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha + 1}\) as
\[(3.35)\]
\[\frac{2\alpha + Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha + 1} = (2\alpha + Q)r \int Vu^2 (r^2 - \rho^2)^{\alpha} - \frac{2\alpha + Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha} \rho^2\]
Therefore, by using (3.32), (3.33), (3.34) and (3.35) in (3.31) and also by using (2.19) we get that
\[(3.36)\]
\[I'(r) = \frac{2\alpha + Q}{r} I(r) + \frac{4(\alpha + 1)}{r} \int (Zu)^2 (r^2 - \rho^2)^{\alpha} \psi + O(1) KrH(r) + K_1(r)\]
Finally from the definition of \(N(r)\) as in (3.17) and from (3.10) and (3.36) we get that the following inequality holds
\[(3.37)\]
\[N'(r) = \frac{I'(r)}{H(r)} - \frac{H'(r)}{H(r)} N(r)\]
\[\geq -C_1 Kr \]
\[+ \frac{(4(\alpha + 1) \int (Zu)^2 (r^2 - \rho^2)^{\alpha} \psi - (4(\alpha + 1) \int (r^2 - \rho^2)^{\alpha} uZu \psi)(\int (r^2 - \rho^2)^{\alpha} u < \nabla_H \rho, \nabla_H u > \rho)}{r H^2(r)}\]
\[+ \frac{(4(\alpha + 1) f(r)(\int |u||Zu|(r^2 - \rho^2)^{\alpha} \psi)}{r H(r)}\]
where \(C_1\) is universal. Now by using (2.23), we get that
\[(3.38)\]
\[4(\alpha + 1) \int (r^2 - \rho^2)^{\alpha} u < \nabla_H \rho, \nabla_H u > \rho = 4(\alpha + 1) \int uZu (r^2 - \rho^2)^{\alpha} \psi + K_2(r)\]
where

\[ |K_2(r)| \leq 4(\alpha + 1)f(r)H(r) \]

Therefore by using (3.38) and (3.39) in (3.37) we get that

\[ N'(r) = \frac{I'(r)}{H(r)} - \frac{H'(r)}{H(r)}N(r) \geq -C_1Kr + \left( \frac{4(\alpha + 1)\int(Zu)^2(r^2 - \rho^2)^\alpha \psi}{rH(r)} \right) - \frac{4(\alpha + 1)(\int(r^2 - \rho^2)^\alpha uZu \psi)^2}{rH^2(r)} - \frac{8(\alpha + 1)f(r)(\int|u||Zu|(r^2 - \rho^2)^\alpha \psi)}{rH(r)} \]

At this point, we need a modified form of an argument used in the proof of Theorem 7.3 in [GR]. Before proceeding further, we make the following remark.

**Remark 3.2.** In the subsequent expressions, all the constants \( C_i \)'s, \( \tilde{C}_i \)'s that will appear are all universal and only depends on \( C_0, Q \) and \( K_0, K_1 \) as in (2.24).

Note that from the definition of \( E_u \) and the growth condition (2.22) that the following holds

\[ \int uZu(r^2 - \rho^2)^\alpha \psi = \frac{I(r)}{2(\alpha + 1)} + H_1(r) \]

where

\[ |H_1(r)| \leq f(r)H(r) \]

Now from the expression of \( I(r) \) as in (3.16) and also from the assumption on \( V \) as in (2.19), we have that

\[ I(r) + Kr^2H(r) \geq 0 \]

Since \( \alpha = \sqrt{K} \), by taking into account (3.43) we get that

\[ \frac{I(r)}{2(\alpha + 1)} + \sqrt{Kr^2H(r)} \geq 0 \]

By substituting (3.44) in (3.41), we obtain

\[ \int uZu(r^2 - \rho^2)^\alpha \psi - H_1(r) + \sqrt{Kr^2H(r)} \geq 0 \]

Now because of the bound in (3.42) we get from the above inequality that the following holds

\[ \int uZu(r^2 - \rho^2)^\alpha \psi + \sqrt{Kr^2H(r) + f(r)H(r)} \geq 0 \]

We now distinguish 2 cases. Either we have that

**Case 1:**

\[ \left( \int u^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} \leq \sqrt{2} \left( \int uZu(r^2 - \rho^2)^\alpha \psi + 8Kr^2H(r) + f(r)H(r) \right) \]

or

**Case 2:**

\[ \left( \int u^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} > \sqrt{2} \left( \int uZu(r^2 - \rho^2)^\alpha \psi + 8Kr^2H(r) + f(r)H(r) \right) \]
If Case 1 (i.e. (3.47)) occurs, then by applying Cauchy-Schwartz inequality to the expression

\[
\frac{4(\alpha + 1) \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi}{rH(r)} - \frac{4(\alpha + 1)(\int (r^2 - \rho^2)^\alpha uZu \psi)^2}{rH(r)^2}
\]

in (3.40), we see that the above expression is non-negative.

Now we estimate the term

\[
-8(\alpha + 1)f(r)(\int |u|Zu|(r^2 - \rho^2)^\alpha \psi)
\]

in (3.40) by using Cauchy-Schwartz and also by using the estimate (3.47) and consequently obtain

\[
(3.49)
\]

\[
\leq \frac{8(\alpha + 1)f(r)(\int uZu(r^2 - \rho^2)^\alpha \psi + 8\sqrt{K}rH(r) + f(r)H(r))}{rH(r)}
\]

Now from (3.41) we get that

\[
(3.50)
\]

Therefore by using (3.50) in (3.49) we have that the following holds

\[
(3.51)
\]

\[
N'(r) \geq -C_2 \frac{f(r)}{r} N(r) - C_3Kr
\]

\[
- \tilde{C}_1 K\frac{f^2(r)}{r} - \tilde{C}_2 K \frac{f(r)}{r}
\]

In (3.51), we crucially used the fact that \(\alpha = \sqrt{K} \leq K\). Now since \(|f| \leq K_1\) we obtain from (3.51) that the following holds

\[
(3.52)
\]

\[
N'(r) \geq -C_2 \frac{f(r)}{r} N(r) - C_4Kr - C_5 K \frac{f(r)}{r}
\]

If instead Case 2 (i.e. (3.48)) occurs, then there are 2 sub-cases. Either

**subcase 1**

\[
(3.53)
\]

\[
\int uZu(r^2 - \rho^2)^\alpha \psi \geq 0
\]

or

**subcase 2**

\[
(3.54)
\]

\[
\int uZu(r^2 - \rho^2)^\alpha \psi \leq 0
\]

If subcase 1 (i.e. (3.53)) occurs, then the argument is relatively simple. By using the inequality \((a + b)^2 \geq a^2\) when \(a, b \geq 0\) in (3.48) we get that

\[
(3.55)
\]

\[
(\int u^2(r^2 - \rho^2)^\alpha \psi)(\int (Zu)^2(r^2 - \rho^2)^\alpha \psi) \geq 2(\int uZu(r^2 - \rho^2)^\alpha \psi)^2
\]

From (3.55), it follows that

\[
(3.56)
\]

\[
\frac{4(\alpha + 1) \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi}{rH(r)} - \frac{4(\alpha + 1)(\int (r^2 - \rho^2)^\alpha uZu \psi)^2}{rH(r)^2}
\]

\[
\geq \frac{2(\alpha + 1) \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi}{rH(r)}
\]

(3.57)
By using the above inequality in (3.40), we get that
\[ N'(r) \geq -C_1 Kr \]
\[ + \frac{2(\alpha + 1) \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi}{rH(r)} - \frac{8(\alpha + 1)f(r)(\int |u||Zu|(r^2 - \rho^2)^\alpha \psi)}{rH(r)} \]

Now by applying Cauchy-Schwartz inequality with \( \varepsilon \), i.e. the inequality
\[ 2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon} \]
to the term
\[ \frac{8(\alpha + 1)f(r)(\int |u||Zu|(r^2 - \rho^2)^\alpha \psi)}{rH(r)} \]
in (3.58) for small enough \( \varepsilon \), we get that
\[ N'(r) \geq -C_1 Kr - C_6 K \frac{f(r)}{r} \]

If instead subcase 2 (i.e. (3.54)) occurs, then we first note that (3.48) trivially implies that
\[ \int u^2(r^2 - \rho^2)^\alpha \psi \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi > \sqrt{2} \int uZu(r^2 - \rho^2)^\alpha \psi + \sqrt{K} r^2 H(r) + f(r)H(r) \]

Now by squaring the above inequality in (3.61) (where we taking into account that the right hand side in the above inequality is non-negative due to (3.46)) and then by using \((a + b)^2 \geq a^2 + 2ab\) for \( b \geq 0 \) with
\[ a = \int uZu(r^2 - \rho^2)^\alpha \psi \]
\[ b = \sqrt{K} r^2 H(r) + f(r)H(r) \]
we get that
\[ H(r)(\int (Zu)^2 (r^2 - \rho^2)^\alpha \psi) \geq 2\left( \int uZu(r^2 - \rho^2)^\alpha \psi \right)^2 \]
\[ + 4\sqrt{K} r^2 (\int uZu(r^2 - \rho^2)^\alpha \psi)H(r) + 4f(r)(\int uZu(r^2 - \rho^2)^\alpha \psi)H(r) \]

Now we note that (3.60) and (3.54) together imply that
\[ -\sqrt{K} r^2 H(r) - f(r)H(r) \leq \int uZu(r^2 - \rho^2)^\alpha \psi \leq 0 \]

Therefore by using (3.63) in (3.40) and then by subsequently using the estimate (3.64) we get that,
\[ N'(r) \geq -C_7 Kr \]
\[ + \frac{2(\alpha + 1) \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi}{rH(r)} - \frac{8(\alpha + 1)f(r)(\int |u||Zu|(r^2 - \rho^2)^\alpha \psi)}{rH(r)} \]
\[ - C_8 K \frac{f(r)}{r} - \frac{8(\alpha + 1)f(r)(\int |u||Zu|(r^2 - \rho^2)^\alpha \psi)}{rH(r)} \]

where we used the fact that \(|f| \leq K_1\) and \( \alpha = \sqrt{K} \). In order to get to (3.65), we also used the fact that \( (\alpha + 1) \leq 2\alpha \) since \( \alpha > 1 \).
Now if we consider the term $16K^{3/2}r^3$ in (3.65), we note that it appears with a negative sign on the right hand side and the exponent of $K$ in that term is $\frac{3}{2}$ which is more than 1. This would not let us conclude the desired monotonicity result in (3.18). Therefore, we have to get rid of this term in the final expression of $N'$. In order to do so, we first note that since (3.46) holds, therefore we get that the following inequality holds

$$\int uZu(r^2 - \rho^2) \frac{\alpha}{r} + 8\sqrt{Kr^2H(r) + f(r)H(r)} \geq 7\sqrt{Kr^2H(r)}$$

(3.66)

Now because we are in Case 2, (3.48) and (3.66) together imply that

$$\int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r} \geq 94Kr^4H(r)$$

(3.67)

By squaring the above inequality and by cancelling off $H(r)$ from both sides, we get that

$$\int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r} \geq 94Kr^4H(r)$$

By dividing both sides by $rH(r)$ we get

$$\frac{(\alpha + 1) \int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r}}{rH(r)} \geq 94Kr^3/2r^3$$

(3.69)

Therefore by writing

$$\frac{2(\alpha + 1) \int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r}}{rH(r)} = \frac{(\alpha + 1) \int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r}}{rH(r)} + \frac{(\alpha + 1) \int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r}}{rH(r)}$$

and by using (3.69) in (3.65), we get that

$$N'(r) \geq -C_7Kr - C_8Kf(r) + 94Kr^3/rH(r) + \frac{(\alpha + 1) \int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r}}{rH(r)} - 16Kr^3/2r^3 - \frac{8(\alpha + 1)f(r)(\int |u||Zu|(r^2 - \rho^2) \frac{\alpha}{r})}{rH(r)}$$

(3.70)

Therefore we see that the estimate (3.69) allows us to get rid of the undesirable term $16Kr^3/rH(r)$ and now from (3.70) one can easily infer that the following inequality holds

$$N'(r) \geq -C_7Kr - C_8Kf(r) + 94Kr^3/rH(r) + \frac{(\alpha + 1) \int (Zu)^2(r^2 - \rho^2) \frac{\alpha}{r}}{rH(r)} - 8(\alpha + 1)f(r)(\int |u||Zu|(r^2 - \rho^2) \frac{\alpha}{r})$$

(3.71)

Again by applying Cauchy-Schwartz inequality with $\varepsilon$ to the term

$$\frac{8(\alpha + 1)f(r)(\int |u||Zu|(r^2 - \rho^2) \frac{\alpha}{r})}{rH(r)}$$

with an appropriate choice of $\varepsilon$, we get that the following estimate holds

$$N'(r) \geq -C_7Kr - C_8Kf(r)$$
Therefore in conclusion, we have that in all the cases, either the estimate (3.52), (3.60) or (3.72) holds. Each of these estimates implies that for some universal constant \( \tilde{C}_0 \), the following inequality holds

\[
N'(r) \geq -\tilde{C}_0 \left( \frac{f(r)}{r} N(r) + Kr + Kf(r) \right)
\]

(3.73) now follows from (3.73) in a standard way. \( \square \)

3.2. Proof of estimate (2.25) in Theorem 2.1. We note that although (3.19) in the monotonicity Theorem 3.1 is different from its counterpart Theorem 3.1 in [BG1], nevertheless it still implies that the following inequality holds

\[
N(r) \leq \tilde{C}_1 (N(s) + \tilde{C}_2 K), \quad \text{for } 0 < r < s < 1.
\]

(3.74)

Using (3.74), we can argue in the same way as in Section 4 in [BG1] to conclude that our desired estimate (2.25) in Theorem 2.1 holds.

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