Remarks on the Cayley-Hamilton theorem

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Abstract

We revisit the classical theorem by Cayley and Hamilton, “each endomorphism is a root of its own characteristic polynomial”, from the point of view of Hasse–Schmidt derivations on an exterior algebra.

1 Formulation of the Main Result

Let $\mathbb{A}$ be a commutative ring with unit, $\mathbb{M}$ a free $\mathbb{A}$-module of rank $r$ and $\bigwedge \mathbb{M} = \bigoplus_{i=0}^{r} \bigwedge^i \mathbb{M}$ its exterior algebra.

To any endomorphism $f$ of $\mathbb{M}$ we associate (see Section 3) the unique $\mathbb{A}$-module homomorphism

$$f(t) : \bigwedge \mathbb{M} \longrightarrow \bigwedge \mathbb{M}[[t]], \quad f(t) = \sum_{j \geq 0} f_j t^j, \quad f_j \in \text{End}_\mathbb{A}(\bigwedge \mathbb{M}),$$

such that

- $f(t)(\alpha \wedge \beta) = f(t)\alpha \wedge f(t)\beta, \quad \forall \alpha, \beta \in \bigwedge \mathbb{M};$
- $f(t)|_M = 1_M + \sum_{j \geq 1} f^j t^j,$

where $1_M$ is the identity endomorphism of $\mathbb{M}.$

Let

$$\det(1_M t - f) = t^r - e_1 t^{r-1} + \ldots + (-1)^r e_r \quad (1)$$

be the characteristic polynomial of $f.$ We prove, in Section 3, the following generalization of the Cayley-Hamilton theorem.

1.1 Theorem. For each $j = 1, \ldots, r,$ the sequence $\{f_k\}_{k \geq 0}$ satisfies the following linear recurrent relation of order $j,$

$$f_{i+j} - e_1 f_{i+j-1} - 1_M + (-1)^j e_j f_{i-j} + (-1)^j e_j f_{i-j+3} = 0,$$

for all $i \geq 0.$
Let $E_r(t) := \det (I_M - ft)$. Using (1), we get explicitly

$$E_r(t) = 1 - e_1 t + \ldots + (-1)^r e_r t^r. \quad (2)$$

For each $i \geq 0$, define

$$U_i(f) = f_i - e_1 f_{i-1} + \ldots + (-1)^r e_r f_{i-r} \in \text{End}_A(\bigwedge M), \quad (3)$$

with the convention that $f_k = 0$ if $k < 0$.

1.2 Corollary. We have

$$f(t) = \frac{U_0(f) + U_1(f)t + \ldots + U_{r-1}(f)t^{r-1}}{E_r(t)}. \quad (4)$$

Moreover, for each $j = 1, \ldots, r$ and $\forall \alpha \in \bigwedge^{r-j+1} M$,

$$f(t)(\alpha) = \frac{U_0(f)(\alpha) + U_1(f)(\alpha)t + \ldots + U_{j-1}(f)(\alpha)t^{j-1}}{E_r(t)}. \quad (5)$$

1.3 Remark. In the case $j = r$, Theorem 1.1 gives

$$f_{i+j}|_M - e_1 f_{i+j-1}|_M + \ldots + (-1)^j e_j f_i|_M = 0, \quad \forall i \geq 0.$$ 

Thus the sequence $\{f^k\}_{k \geq 0}$, where $f^0 := I_M$, satisfies the linear recurrence relation of order $r$,

$$f^{i+r} - e_1 f^{i+r-1} + \ldots + (-1)^r e_r f^i = 0, \quad \forall i \geq 0.$$

This is the classic Cayley-Hamilton theorem.

1.4 If $A$ contains the rational numbers, then the formal Laplace transform $L$ sending series $\sum_{n \geq 0} a_n t^n$ to $\sum_{n \geq 0} a_n n! t^n$, is invertible, [2]. In this case we can make some additional observations.

Consider the series:

$$u_{-j}(t) := L^{-1} \left( \frac{t^j}{E_r(t)} \right) \in A[[t]], \quad 0 \leq j \leq r - 1,$$

which, as it was shown in [4], form an $A$-basis of solutions to the linear ODE

$$\begin{cases} y^{(r)}(t) - e_1 y^{(r-1)}(t) + \ldots + (-1)^r e_r y(t) = 0, \\ y(t) \in A[[t]]. \end{cases} \quad (5)$$
Equality (4) then implies
\[ \sum_{n \geq 0} f_n \frac{t^n}{n!} = \sum_{j=0}^{r-1} U_j(f) u_{-j}(t). \] (6)

1.5 Corollary. If \( A \) is a \( \mathbb{Q} \)-algebra, then \( \sum_{n \geq 0} f_n \frac{t^n}{n!} \) solves (5) in \( \text{End}_A(\wedge M)[[t]] \).

By restricting (6) to \( M \) and recalling that \( \sum_{n \geq 0} f_n \frac{t^n}{n!} = \exp(ft) \), one obtains the refinement by Leonard [5, 1996] and Liz [6, 1998] of Putzer’s method [7, 1966] to compute the exponential of a complex valued square matrix.

1.6 Corollary. If \( A \) is a \( \mathbb{Q} \)-algebra and \( f \in \text{End}_A(M) \) has the characteristic polynomial (1), then
\[ \exp(ft) = v_0(t) 1_M + v_1(t)f + \cdots + v_{r-1}(t)f^{r-1}, \]
where \( v_j(t) \) is the unique solution to (5) in \( A[[t]] \) with the initial condition \( v_j^{(i)}(t) = \delta_{ij}, \ 0 \leq i, j \leq r - 1 \).

2 Derivations on Exterior Algebra

2.1 Preliminaries. (See [1, 3]) Fix an \( A \)-basis \( b_0, \ldots, b_{r-1} \) of the ring \( M \). Consider \( \wedge M = \bigoplus_{j=0}^{r-1} \wedge^j M \), where \( \wedge^0 M = A \), and \( \wedge^j M, 1 \leq j \leq r \), is the \( A \)-module generated by \( \{b_{i_1} \wedge \ldots \wedge b_{i_j}\} \) with the relation:
\[ b_{i_{\sigma(1)}} \wedge \ldots \wedge b_{i_{\sigma(j)}} = \text{sgn}(\sigma)b_{i_1} \wedge \ldots \wedge b_{i_j}, \quad \sigma \in S_j, \]
where \( S_j \) denotes the permutation group on \( j \) elements. In particular, \( \wedge^1 M = M \). The exterior algebra structure on \( \wedge M \) is given by juxtaposition
\[ \wedge : \wedge^h M \times \wedge^k M \longrightarrow \wedge^{h+k} M, \quad (b_H, b_K) \mapsto b_H \wedge b_K, \]
Hence of \( \left( \bigwedge^h M \right) \) and \( \left( \bigwedge^k M \right) \) respectively.

Let \( t \) be an indeterminate over \( \bigwedge M \) and denote by \( \bigwedge M[[t]] \) and \( \text{End}_A(\bigwedge M)[[t]] \) the corresponding rings of formal power series with coefficients in \( \bigwedge M \) and \( \text{End}_A(\bigwedge M) \) respectively. If \( D(t) = \sum_{i \geq 0} D_i t^i \), \( \tilde{D}(t) = \sum_{j \geq 0} \tilde{D}_j t^j \in \text{End}_A(\bigwedge M)[[t]] \), their product is defined as:

\[
D(t) \tilde{D}(t) \alpha = D(t) \sum_{j \geq 0} \tilde{D}_j \alpha \cdot t^j = \sum_{j \geq 0} (D(t) \tilde{D}_j \alpha) \cdot t^j, \quad \forall \alpha \in \bigwedge M.
\]

Given \( D(t) \in \text{End}_A(\bigwedge M)[[t]] \), we use the same notation for the induced \( A \)-homomorphism \( D(t) : \bigwedge M \to \bigwedge M[[t]] \) mapping \( \alpha \in \bigwedge M \) to \( D(t) \alpha = \sum_{i \geq 0} D_i \alpha \cdot t^i \in \bigwedge M[[t]] \).

The formal power series \( D(t) = \sum_{i \geq 0} D_i t^i \) is invertible in \( \text{End}_A(\bigwedge M)[[t]] \) (i.e. there exists \( D^{-1}(t) \in \text{End}_A(\bigwedge M)[[t]] \) such that \( D(t) D^{-1}(t) = D(t) D(t) = 1_{\bigwedge M} \)), if and only if \( D_0 \) is an automorphism of \( \bigwedge M \). If \( D(t) \) is invertible, we shall write its inverse as \( D^{-1}(t) = \sum_{i \geq 0} (-1)^i \tilde{D}_i t^i \). With this convention, \( D(t) D^{-1}(t) D(t) = 1_{\bigwedge M} \) if and only if

\[
\tilde{D}_0 D_j - \tilde{D}_i D_{j-1} + \ldots + (-1)^i \tilde{D}_j D_0 = 0, \quad \forall j \geq 1. \quad (7)
\]

### 2.2 Proposition

**The following statements are equivalent:**

i) \( D(t)(\alpha \wedge \beta) = D(t)\alpha \wedge D(t)\beta, \quad \forall \alpha, \beta \in \bigwedge M; \)

ii) \( D_1(\alpha \wedge \beta) = \sum_{j=0}^i D_j \alpha \wedge D_{i-j} \beta, \quad \forall i \geq 0. \)

**Proof.** i) \( \Rightarrow \) ii). By definition of \( D(t) \), write i) as

\[
\sum_{i \geq 0} D_i (\alpha \wedge \beta) t^i = \sum_{j_1 \geq 0} D_{j_1} \alpha \cdot t^{i_1} \wedge \sum_{j_2 \geq 0} D_{j_2} \beta \cdot t^{i_2}. \quad (8)
\]

Hence \( D_i (\alpha \wedge \beta) \) is the coefficient of \( t^i \) on the right hand side of \( (8) \), which is \( \sum_{j_1+j_2=i} D_{j_1} \alpha \wedge D_{j_2} \beta = \sum_{j=0}^i D_j \alpha \wedge D_{i-j} \beta. \)

ii) \( \Rightarrow \) i) We have

\[
D(t)(\alpha \wedge \beta) = \sum_{i \geq 0} D_i (\alpha \wedge \beta) t^i = \sum_{i \geq 0} \left( \sum_{i_1+i_2=j} D_{i_1} \alpha \wedge D_{i_2} \beta \right) t^i \\
= \sum_{i_1 \geq 0} D_{i_1} \alpha \cdot t^{i_1} \wedge \sum_{i_2 \geq 0} D_{i_2} \beta \cdot t^{i_2} \\
= D(t)\alpha \wedge D(t)\beta. \quad \blacksquare \quad (9)
\]
2.3 Definition. (Cf. [1]) Let $D(t) \in \text{End}_A(\bigwedge M)[[t]]$. The induced map $D(t) : \bigwedge M \to \bigwedge M[[t]]$ is called a Hasse–Schmidt derivation on $\bigwedge M$ (HS-derivation for short), if it satisfies the (equivalent) conditions of Proposition 2.2.

We denote by $\text{HS}(\bigwedge M)$ the set of all HS-derivation on $\bigwedge M$.

2.4 Remark. If $D(t) \in \text{HS}(\bigwedge M)$, then $D_1$ is an $A$–derivation of $\bigwedge M$, i.e. the usual Leibniz’s rule $D_1(\alpha \wedge \beta) = D_1\alpha \wedge \beta + \alpha \wedge D_1\beta$. holds.

2.5 Proposition. (Cf. [1, 3]) The product of two HS-derivations is a HS-derivation. The inverse of a HS-derivation is a HS-derivation.

Proof. For the product of HS-derivations $D(t)$ and $\tilde{D}(t)$, the statement i) of Proposition 2.2 holds:

$$D(t)\tilde{D}(t)(\alpha \wedge \beta) = D(t)(\sum_{j \geq 0} \sum_{j_1 + j_2 = j} \tilde{D}_{j_1} \alpha \wedge \tilde{D}_{j_2} \beta)t^j = \sum_{j \geq 0} \sum_{j_1 + j_2 = j} D(t)D_{j_1} \alpha \cdot t^{j_1} \wedge D(t)D_{j_2} \beta \cdot t^{j_2} = D(t)\tilde{D}(t)\alpha \wedge D(t)\tilde{D}(t)\beta.$$  

Similarly, for $\overline{D}(t)$, the inverse of the derivation $D(t)$, we have

$$\overline{D}(t)(\alpha \wedge \beta) = \overline{D}(t)(D(t)\overline{D}(t)\alpha \wedge D(t)\overline{D}(t)\beta) = (\overline{D}(t)D(t))D(t)\alpha \wedge D(t)\beta) = \overline{D}(t)\alpha \wedge D(t)\beta.$$ 

The following property of HS-derivations on the exterior algebra was called in [3] the integration by parts formula.

2.6 Proposition. If $\overline{D}(t)$ is the inverse of a derivation $D(t)$, then

$$D(t)\alpha \wedge \beta = D(t)\alpha \wedge D(t)\overline{D}(t)\beta = D(t)(\alpha \wedge \overline{D}(t)\beta).$$  

■

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2.7 Remark. Equating the coefficients of $t^j$ on the left and right hand sides of (10) gives an equivalent set of conditions: for any $j \geq 0$,

$$D_j \alpha \wedge \beta = D_j(\alpha \wedge \beta) - D_{j-1}(\alpha \wedge D_1 \beta) + \ldots + (-1)^j \alpha \wedge D_j \beta. \quad (11)$$

2.8 Proposition. For any $g(t) = \sum_{j \geq 0} g_j t^j : M \to \wedge M[[t]]$ there exists a unique HS-derivation $D(g; t)$ on $\wedge M$ such that $D(g; t)|_M = g(t)$.

Proof. For the chosen $A$-basis of the ring $M$ (see the beginning of Section 2.1) we necessarily have $D(g; t)(b_j) = g(t)b_j$, $0 \leq j \leq r-1$. Hence, by Proposition 2.2 i), the action of $D(g; t)$ is uniquely defined on all the basis vectors of $\wedge M$:

$$D(g; t)(b_{i_1} \wedge \ldots \wedge b_{i_j}) = g(t)b_{i_1} \wedge \ldots \wedge g(t)b_{i_j}, \quad 1 \leq j \leq r. \quad (12)$$

3 Proof of the Main Results

Let $f \in \text{End}_A(M)$. Denote by $\overline{f}(t) = \sum_{j \geq 0} (-1)^j \overline{f}_j t^j$ the unique HS-derivation of $\wedge M$ (see Proposition 2.8) extending

$$(1_M - ft) : M \to M[[t]] \subseteq \wedge M[[t]].$$

3.1 Proposition. For each $i = 1, \ldots, r$, the restriction of $\overline{f}_j$ to $\wedge^i M$ vanishes, for all $j > i$.

Proof. Induction on $i$. For $i = 1$ the statement holds, by definition of $\overline{f}(t)$. Assume the statement true for $1 \leq k \leq i-1$, and consider $\alpha \in \wedge^i M$. Due to $A$-linearity, its enough to take $\alpha = m_1 \wedge \ldots \wedge m_i$, where $m_1, \ldots, m_i \in M$. For $j > i$ we have, according to Proposition 2.2 ii),

$$\overline{f}_j(m_1 \wedge \ldots \wedge m_i) = \sum_{k=0}^{j} \overline{f}_k(m_1) \wedge \overline{f}_{j-k}(m_2 \wedge \ldots \wedge m_{i-1}).$$

But each summand on the right hand side vanishes. Indeed, if $k = 0, 1$, then $j-k > i-1$ and by the induction hypothesis $\overline{f}_{j-k}(m_2 \wedge \ldots \wedge m_{i-1}) = 0$. If $k \geq 2$, then $\overline{f}_k(m_1) = 0$. \hfill \blacksquare

In particular, $\overline{f}_j$ vanishes on the entire $\wedge M$, for $j > r$. 

3.2 Corollary. \( \overline{f}(t) = 1_{\Lambda M} - f_1 t + \ldots + (-1)^r f_r t^r. \)

Let \( f(t) = \sum_{j \geq 0} f_j t^j \) be the inverse HS-derivation of \( \overline{f}(t). \)

3.3 Proposition. For each \( i \geq 1, \) \( f_i(m) = f^i(m), \forall m \in M. \)

Proof. Induction on \( i. \) Notice that \( f_1(m) = \overline{f}_1(m) = f(m). \) Now assume \( f_k(m) = f^k(m), \) for \( 1 \leq k \leq i - 1. \) According to (7), we have

\[
f_i(m) = \overline{f}_1 f_{i-1}(m) - \overline{f}_2 f_{i-2}(m) + \ldots + (-1)^{i-1} \overline{f}_i(m).
\]

By the induction hypothesis, \( f_k(m) \in M \) for \( 1 \leq k \leq i - 1, \) therefore Proposition 3.1 implies the vanishing of all terms on the right hand side, but the first one. Then

\[
f_i(m) = \overline{f}_1 f_{i-1}(m) = \overline{f}_1 (f^{i-1}(m)) = f(f^{i-1}(m)) = f^i(m). \]

Take the basis element \( \zeta := b_0 \wedge b_1 \wedge \ldots \wedge b_{r-1} \) of \( \Lambda^r M \) (see Section 2.1). It is unique up to the multiplication by an invertible in \( A. \) It is a common eigenvector of all of \( f_i \) and \( \overline{f}_i. \) Recall that \( e_1, \ldots, e_r \in A \) stay for the coefficients of the characteristic polynomial of \( f \) and of \( E_r(t), \) see (1), (2). It turns out that they are eigenvalues of \( \overline{f}_i|_{\Lambda^r M}. \)

3.4 Proposition. We have \( \overline{f}_i(\zeta) = e_i \zeta, \) \( 1 \leq i \leq r. \)

Proof. Recall that \( \overline{f}(t) \) is the HS-derivation extending \( 1_M - ft. \) Hence, like in the proof of Proposition 2.8,

\[
\overline{f}(t)(\zeta) = \overline{f}(t)(b_0 \wedge \ldots \wedge b_{r-1}) = (1_M - ft) b_0 \wedge \ldots \wedge (1_M - ft) b_{r-1} = \det (1_M - ft) (b_0 \wedge \ldots \wedge b_{r-1}) = \det (1_M - ft)(\zeta).
\]

Thus the eigenvalue of \( \overline{f}_i \) on \( \zeta \) is the coefficient of \( t^i \) in

\[
\det (1_M - ft) = E_r(t),
\]

see (2).

Define endomorphisms \( U_i(f), i \geq 0, \) via the equality

\[
E_r(t)f(t) = \sum_{i \geq 0} U_i(f)t^i. \tag{13}
\]
Comparing the coefficients of $t^i$ on the both sides gives the explicit formula (3).

### 3.5 Lemma

We have $U_j(f)\zeta = 0$ for all $j > 0$.

**Proof.** Denote by $h_j$ the eigenvalue of $f_j$ on $\zeta$, and write

$$f_i(\zeta) = h_i \zeta, \quad f(t)\zeta = H_r(t)\zeta, \quad H_r(t) = 1 + \sum_{j \geq 1} h_j t^j.$$

By construction, $E_r(t)H_r(t) = 1$. Equivalently, we get

$$h_j - e_1 h_{j-1} + \ldots + (-1)^j e_j = 0, \quad j \geq 1.$$

Therefore

$$U_j(f)\zeta = (f_j - \sum_{i=1}^r (-1)^i e_i f_{j-i})\zeta = (h_j - e_1 h_{j-1} + \ldots + (-1)^j e_j)\zeta = 0. \quad \blacksquare$$

### 3.6 Lemma

We have

$$U_i(f)\alpha \wedge \beta = \sum_{j=0}^{i} (-1)^j U_{i-j}(f)(\alpha \wedge f_j(\beta)). \quad (14)$$

**Proof.** According to definition (13), we write

$$\sum_{i \geq 0} (U_i(f)\alpha) t^i \wedge \beta = E_r(t)f(t)\alpha \wedge \beta, \quad (15)$$

then apply integration by parts formula (10), and again use (13),

$$= E_r(t)f(t)(\alpha \wedge \bar{f}(t)\beta)) = \sum_{i \geq 0} U_i(f)(\alpha \wedge \bar{f}(t)\beta)t^i \quad (16)$$

Comparing the coefficients of $t^i$ on the left hand side of (15) and the right hand side of (16) gives (14). \quad \blacksquare
3.7 Proof of Theorem 1.1. We shall show that \( U_k(f)\alpha = 0 \), for any \( k \geq j \) and any \( \alpha \in \bigwedge^{r-j+1} M \). For such \( \alpha \) and for any \( \beta \in \bigwedge^j M \), we have \( \alpha \wedge \beta \in \bigwedge^r M \). Write \( U_k(f)\alpha \wedge \beta \), according to Lemma 3.6, as
\[
U_k(f)(\alpha \wedge \beta) - U_{k-1}(f)(\alpha \wedge \mathcal{F}_1(\beta)) + \ldots + (-1)^k \alpha \wedge \mathcal{F}_k(\beta).
\]
We see that each term but the very last is zero, by Lemma 3.5. The very last term also is zero, by Proposition 3.1 as \( k > j - 1 \).

In particular \( U_j(f) \) vanishes on the entire exterior algebra \( \bigwedge M \) for all \( j \geq r \).

Now we restrict each \( U_k(f) \), \( 0 \leq k \leq r - 1 \), to \( M \) getting:
\[
U_k(f)|_M = p_k(f) = f^k - e_1 f^{k-1} + \ldots + (-1)^k e_k.
\]
Then (13) takes the form,
\[
\sum_{j \geq 0} f^j t^j = \frac{1_M + p_1(f)t + \ldots + (-1)^{r-1} p_{r-1}(f)t^{r-1}}{E_r(t)}. \quad (17)
\]

3.8 Proof of Corollary 1.6. Given a \( \mathbb{Q} \)-algebra \( A \), the \textit{formal Laplace transform} \( L : A[[t]] \to A[[t]] \) and its inverse \( L^{-1} \) act as follows, see [2],
\[
L \sum_{n \geq 0} a_n t^n = \sum_{n \geq 0} n! a_n t^n, \quad L^{-1} \sum_{n \geq 0} c_n t^n = \sum_{n \geq 0} c_n \frac{t^n}{n!}.
\]
Consider
\[
u_{-j} := L^{-1} \left( \frac{t^j}{E_r(t)} \right), \quad 0 \leq j \leq r - 1.
\]
Applying \( L^{-1} \) to (17), we get the following expression for \( \exp(ft) \),
\[
\exp(ft) = u_0 + p_1(f)u_{-1} + \ldots + p_{r-1}(f)u_{-r+1}.
\]
It will be convenient to re-write the series \( u_0, u_{-1}, \ldots, u_{-r+1} \) in terms of
\[
H_r(t) = \frac{1}{E_r(t)} = 1 + \sum_{j \geq 0} h_j t^j,
\]
introduced in the proof of Lemma 3.5. We obtain

\[ u_{-j} = L^{-1}(t^j H_r(t)) = \sum_{n \geq 0} h_{n-j} \frac{t^n}{n!}, \quad 0 \leq j \leq r - 1. \]

According to [4], these series form an \( A \)-basis of solutions to the ODE (5) in \( A[[t]] \). Hence \( \exp(ft) \) solves this ODE in \( A[[t]] \).

Take the standard \( A \)-basis of solutions, \( \{v_j(t)\}_{0 \leq j \leq r-1} \), where \( v_j(t) \) denotes the unique solution to (5) in \( A[[t]] \) satisfying \( v_j^{(i)}(t) = \delta_{ij}, \quad 0 \leq i, j \leq r - 1 \). In the standard basis, the coefficients are the initial conditions of the solution. In the case of \( \exp(ft) \), these are \( 1, f, \ldots, f^{r-1} \).

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