Partially integrable dynamics of hierarchical populations of coupled oscillators

Arkady Pikovsky and Michael Rosenblum

Department of Physics and Astronomy, Potsdam University,
Karl-Liebknecht-Str 24, D-14476 Potsdam-Golm, Germany
(Dated: November 2, 2018)

We consider oscillator ensembles consisting of subpopulations of identical units, with a general heterogeneous coupling between subpopulations. Using the Watanabe-Strogatz ansatz we reduce the dynamics of the ensemble to a relatively small number of dynamical variables plus constants of motion. This reduction is independent of the sizes of subpopulations and remains valid in the thermodynamic limits. The theory is applied to the standard Kuramoto model and to the description of two interacting subpopulations, where we report a novel, quasiperiodic chimera state.

PACS numbers: 05.45.Xt

Large populations of coupled oscillators occur in a variety of applications and models of natural phenomena, ranging from collective dynamics of multimode lasers and Josephson junction arrays to the pedestrian synchrony [1]; the analysis of the dynamics of these systems is a topic of high interest. Even in the context of the simplest, paradigmatic case of globally (all-to-all) connected, sine-coupled phase oscillators (the famous Kuramoto model and its generalizations), many problems remain yet unsolved, especially those related to a heterogeneous coupling and nontrivial collective dynamics. In this Letter we treat an important case of a hierarchically organized population. It can be viewed as a (finite or infinite) collection of interacting subpopulations, each consisting of a (finite or infinite) number of identical units; sizes of the subpopulations and couplings between them are generally different (cf. [2, 3] and references therein). Using the seminal approach of Watanabe and Strogatz (WS) [4], we demonstrate that each subpopulation can be described by only three dynamical variables plus constants of motion, determined by initial conditions. This partial integrability allows us to separate the full dynamics into a relatively small number of generally dissipative modes (their number is proportional to the number of subpopulations) with possibly nontrivial behavior, and the integrals of motion. In particular, our theory allows us to extend some recent results [3, 5] and to determine the conditions of their validity.

Our basic model is a generalization of the Kuramoto model [1], cf. [2, 3]:

$$\frac{d\phi_k^a}{dt} = \omega_a + \frac{1}{N} \sum_{b=1}^{M} \sum_{j=1}^{N_b} \varepsilon_{a,b} \sin(\phi_j^b - \phi_k^a - \alpha_{a,b}) .$$

Here we denote the subpopulations by indices $a,b = 1,\ldots,M$. Variable $\phi_k^a(t)$ is the phase of oscillator $k$ in subpopulation $a$; $k = 1,\ldots,N_a$, where $N_a$ is the size of the subpopulation, and $\omega_a$ is the natural frequency of its oscillators (we remind that all oscillators in a subpopulation are identical). The total number of oscillators is $N = \sum_{a} N_a$ and two constants $\varepsilon, \alpha$ describe the coupling with an arbitrary phase shift, cf. [2]. The system can be re-written as

$$\frac{d\phi_k^a}{dt} = \omega_a + \text{Im}(Z_a e^{-i\phi_k^a}) ,$$

$$Z_a = \sum_b n_b \varepsilon_{a,b} e^{-i\alpha_{a,b}} e^{i\Theta_b} ,$$

where $Z_a$ is the effective force acting on the oscillators of subpopulation $a$. Here we have introduced the relative population sizes $n_a = N_a/N$ and the complex mean fields for each subpopulation

$$X_a + iY_a = r_a e^{i\Theta_a} = N_a^{-1} \sum_{k=1}^{N_a} e^{i\phi_k^a} .$$

Note that all oscillators in a subpopulation obey the same equation, though generally they have different initial conditions $\phi_k^a(0)$. Thus, we can apply to each subpopulation the WS ansatz [4] that reduces the dynamics of the subpopulation to that of three variables $\rho_a(t), \Psi_a(0), \Phi_a(t)$, via the transformation [3]

$$\tan \left[ \frac{\phi_k^a - \Phi_a}{2} \right] = \frac{1 - \rho_a}{1 + \rho_a} \tan \left[ \frac{\psi_k^a - \Psi_a}{2} \right]$$

containing $N_a$ constants $\psi_k^a$, which are directly determined from the initial state $\phi_k^a(0)$ and additionally satisfy

$$\sum_{k=1}^{N_a} \cos \psi_k^a = \sum_{k=1}^{N_a} \sin \psi_k^a = 0 .$$

Due to an arbitrary shift of constants $\psi_k$ with respect to $\Psi$, only $N_a - 3$ of constants $\psi_k^a$ are independent. The WS method is valid generally, provided the number of oscillators in a subpopulation is larger than three, and the initial state does not have too many clusters, see [4] for a detailed discussion of these conditions and of how $\rho_a(0), \Psi_a(0), \Phi_a(0)$, and $\psi_k^a$ can be computed from $\phi_k^a(0)$. With account of Eq. (4), we write the WS equations for
ally depends not only on the global variables $\zeta$ and $\gamma$, but also on the constants of motion $\psi_k^0$. Equations \ref{eq:10} \ref{eq:11}, as well as equivalent equations \ref{eq:7} \ref{eq:9}, together with the definitions \ref{eq:8} \ref{eq:12} are exact and complete; they show that the dynamics of a hierarchical ensemble of oscillators can be reduced to $3M$ ODEs plus $N - 3M$ constants of motion. Before proceeding with the analysis of these equations and examples, let us discuss how a thermodynamic limit $N \to \infty$ can be introduced in this picture. There are two main ways of performing this.

(i) Suppose that the number of subpopulations $M$ remains finite, but their sizes grow $N, N_a \to \infty$ in a way that $n_a = \text{const.}$. In this case only Eq. \ref{eq:12} is affected and should be now written as an integral

$$
\gamma_a(z_a, \zeta_a) = \int_{-\pi}^{\pi} \frac{1}{1 + z_a^* e^{i(\zeta_a + \psi)}} \sigma_a(\psi) \, d\psi .
$$

Here $\sigma_a(\psi)$ is the distribution of the constants of motion $\psi$ in the subpopulation $a$, additionally it satisfies (cf. \ref{eq:6})

$$
\int_{-\pi}^{\pi} \sigma_a(\psi) e^{i\psi} \, d\psi = 0 .
$$

In this limit the ensemble is described by a set of $3M$ ODEs, where the right hand sides depend on the variables via integrals \ref{eq:13}. The integrals of motion are now the functions $\sigma_a(\psi)$.

(ii) In another limiting case, we keep the size of each subpopulation $N_a$ finite but let the number of subpopulations grow $M \to \infty$. Considering indices $a, b$ as continuous variables, we write instead of Eq. \ref{eq:3}

$$
Z(a) = \int db \, n(b)[\varepsilon(a, b) + \eta(a, b)]\gamma(b)z(b) .
$$

Now Eqs. \ref{eq:10} \ref{eq:11} \ref{eq:12} \ref{eq:13} become a system of integral equations; still it is simpler than the original equation \ref{eq:11} as at each value of the continuous parameter $a$ we have only three real time-dependent variables.

Certainly, one can also perform both thermodynamic limits simultaneously. Then the ensemble is described by the system \ref{eq:10} \ref{eq:11} \ref{eq:12} \ref{eq:13}.

Next we study an important case when Eqs. \ref{eq:10} \ref{eq:11} decouple. To this end we represent the fraction in Eqs. \ref{eq:12} \ref{eq:13} as a series

$$
\gamma_a = 1 + (1 - |z_a|^2) \sum_{l=2}^\infty C_l^n (-z_a^e e^{i\zeta_a})^l ,
$$

where complex constants $C_l^n$ depend only on the distribution of the constants of motion

$$
C_l^n = \frac{1}{N_a} \sum_{k=1}^{N_a} e^{i\psi_k} \quad \text{or} \quad C_l^n = \int_{-\pi}^{\pi} \sigma_a(\psi) e^{i\psi} \, d\psi ,
$$

and we used that $C_l^n = 0$ due to Eqs. \ref{eq:13} \ref{eq:13}. Obviously, the governing equations simplify, if $C_l^n = 0$ for $l \geq 2$ and all $a$, and, hence, $\gamma = 1$. Then the force $Z$ does not depend on the phase variable $\zeta$ and Eq. \ref{eq:10} decouples
from Eq. (11). It is easy to see from Eqs. (17) that $C_l^a$, which are in fact Fourier coefficients of the distribution of the constants of motion $\psi$, vanish in the thermodynamic limit of type (i), if $\sigma(\psi) = 1/2\pi$. However, if the number of oscillators in a subpopulation $N_a$ is finite, then even for a uniform spreading of $\psi_k$, the discrete sum in (17) yields $|C_l^a| = 1$, $\arg(C_l) = \psi_l$ for $l = N_a, 2N_a, \ldots$, and we get

$$
\gamma_a = 1 + \frac{1 - |z_a|^2}{1 - \left[ -z_a e^{i(\zeta_a + \psi_l)} \right]^{N_a}} . \tag{18}
$$

Thus, the deviation of $\gamma_a$ from unity is exponentially small in the size of the subpopulation and, therefore, can be neglected for large $N_a$. This is exactly the case, where the complex bunch amplitude $\rho$ is equal to the mean field amplitude $\rho$, because in (12) $\gamma = 1$.

Hence, for the uniform distribution of constants of motion $\psi$, ensemble (14) admits a simplified description via Eq. (10), supplemented by an equation for $Z_a$, either in a discrete or in a continuous (for the thermodynamic limit of type (ii)) form:

$$
Z_a = \sum_b n_b \varepsilon_{a,b} e^{-i\alpha_{a,b}} z_b , \tag{19}
$$

$$
Z(a) = \int db n(b) \varepsilon(a,b) e^{-i\alpha(a,b)} z(b) . \tag{20}
$$

A relation between the distribution of the original phases $\phi_k$ and the uniform distribution of constants of motion $\psi_k$ follows from Eq. (5): one can see that different distributions of the phases $\phi_k$, parameterized by different values of $\rho$, correspond to the uniformly distributed constants $\psi_k$.

As a first application of our framework, we apply Eqs. (10,11,12,15) to the classical Kuramoto problem (cf. 3). We set $\varepsilon(a,b) = \varepsilon = \text{const}, \alpha = 0$, use the frequency as the subpopulation index $\omega = \omega$, and perform the thermodynamic limit (ii). As a result, in the case when $\gamma = 1$ and the variable $\zeta$ (as well as the constants of motion) does not influence the dynamics, we obtain exactly Eqs. (10,20), derived recently by Ott and Antonsen (OA) [3] under an assumption of a certain parameterization of the phase distribution. Considering the Lorentzian distribution of natural frequencies $n(\omega) = [\pi(\omega^2 + 1)]^{-1}$ and using analytic properties of $z(\omega)$ as a function of complex frequency $\omega$, OA have calculated the integral in Eq. (20) by the residue of the pole at $\omega = i$ and have obtained $Z = \varepsilon z(i)$. Substituting $Z$ into Eq. (10) for $\omega = i$, OA derived a closed equation for $\dot{Z} = Z/\varepsilon$, i.e. for the usual Kuramoto mean field of the whole population:

$$
\dot{Z} = \left( -1 + \frac{\varepsilon}{2} \right) \dot{Z} - \frac{\varepsilon}{2} |Z|^2 \dot{Z} , \tag{21}
$$

solved it, and in this way obtained explicitly the evolution of the mean field. From our derivation of the equations of motion we conclude, that the particular ansatz used in [3] corresponds to the case of uniformly distributed constants of motion $\psi_k$, what is equivalent to vanishing Fourier coefficients $C_l$. Next we discuss, what changes if the distribution of constants $\psi_k$ is not uniform, i.e. $C_l \neq 0$. Let us treat the effect of non-vanishing coefficients $C_l$ perturbatively, assuming that in the first approximation the OA ansatz is valid. Considering for simplicity the effect of $C_2 \neq 0$ only, we obtain a correction to the mean field by substituting (10) into (15):

$$
\Delta Z \approx \varepsilon \int d\omega z^* (\omega)(|z(\omega)|^2 - 1)e^{i2\zeta(\omega)}C_2(\omega) . \tag{22}
$$

Calculation of this integral by the residue yields

$$
\Delta Z \approx \varepsilon \varepsilon^* (i)(|z(i)|^2 - 1)e^{i2\zeta(i)}C_2(i) . \tag{23}
$$

From Eq. (11) it follows that in the first approximation $\zeta(i) = \zeta_0 + it$. Therefore $\Delta Z \propto e^{-2t}$. We conclude that the contribution from a nonuniform distribution of constants $\psi_k$ results in an exponentially decaying correction to the mean field. The characteristic time scale of this decay is $1/2$, to be compared with the characteristic time scale of the evolution of the mean field, which, according to (21), is $(\varepsilon(2 - 1)^{-1}$. Thus, close to criticality $\varepsilon_c = 2$, the approximation of vanishing constants $C_1$ works well after short transients; this is not surprising as near a bifurcation point the dynamics is typically effectively low-dimensional, dominated by a few normal modes. Far from criticality the time scale separation is not valid and the dynamics is generally high-dimensional.

As a second example we extend recent results of Abrams et al [3]. They studied two coupled subpopulations of identical oscillators, i.e. model (14) with $a = 1, 2, \omega_1 = \omega_2, N_1 = N_2$ and heterogeneous coupling $\varepsilon_{1,1} = \varepsilon_{2,2} = 2\mu, \varepsilon_{1,2} = \varepsilon_{2,1} = 2\nu$, and $\alpha_{a,b} = \alpha$, where $\nu = 1 - \mu$. Using the OA ansatz [3] Abrams et al derived equations for the complex order parameters $z_{1,2}$ and analyzed the so-called chimera state, where, e.g., the first subpopulation is fully synchronized, $\rho_1 = 1$, whereas the other one is only partially synchronized, $\rho_2 < 1$; they have found both static and time-periodic solutions for $\rho_2$. With our approach we describe the system exactly, by writing six Eqs. (7,9) for both subpopulations. Since we are interested in the chimera state in the second subpopulation, the first, synchronous one, is described by its phase $\Phi_1$ only. In this case $Z_1 = \mu e^{i(\Phi_1 - \alpha)} + \nu \Delta e^{i(\Phi_2 + \beta - \alpha)}, Z_2 = \nu e^{i(\Phi_1 - \alpha)} + \mu \Delta e^{i(\Phi_2 + \beta - \alpha)}$, where

$$
A(\rho_2, \Psi_2)e^{i(\rho_2, \Psi_2)} = \frac{1}{N_x} \sum_{k=1}^{N_x} \rho_2^2 e^{i\Psi_2} + e^{i\Psi_2^k/2} . \tag{24}
$$

Next, we note that the dynamics depends only on the phase difference $\delta = \Phi_1 - \Phi_2$ and, hence, write a closed
Following Abrams et al we want to analyze the full three-dimensional system \((25,27)\), which certainly can exhibit more complex solutions.

To verify our theoretical prediction, we have performed numerical simulations of the ensemble (1) for the same parameters, where Abrams et al obtained stationary and time-periodic solutions, but for different distributions of the constants \(\psi_k\). Namely, we took \(\psi_k\), uniformly distributed in the range \(-q\pi < \psi_k^{(2)} < q\pi\), where \(q \leq 1\) is a parameter. For \(q = 1\) we have reproduced the results of [3], while for \(q < 1\) the dynamics attains an additional time dependence and becomes periodic and quasiperiodic, respectively (see Fig. 1).

In conclusion, we have performed the exact reduction of the dynamics of hierarchically organized populations of coupled oscillators. Due to the partial integrability, only three dynamical variables remain relevant for each subpopulation, all other are constants of motion. This reduced description is independent of the subpopulation sizes and holds also in the thermodynamic limit. In an important particular case, when the distribution of the constants of motion is uniform, the governing equations further decouple and simplify. We have demonstrated that this case corresponds to the recently found particular ansatz of Ott and Antonsen [3]. The analysis of full equations has allowed us to estimate corrections to this particular solution due to non-uniformity of constants.

Application of our framework to the model by Abrams et al revealed existence of novel, quasiperiodically breathing chimera states.

In this Letter we restricted our attention to the simplest setups in order to demonstrate applicability of the theory. However, the method can be in a straightforward way extended to the cases of nonlinearly coupled populations [3], externally forced ensembles, etc. In these cases even a chaotic dynamics of the global variables can be expected. The main limitation of the theory is that the coupling in Eq. (1) has a sine form.

We acknowledge financial support from DFG (SFB 555).

[1] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence (Springer, Berlin, 1984); A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization. A Universal Concept in Nonlinear Sciences (CUP, Cambridge, 2001); S. H. Strogatz, Sync: The Emerging Science of Spontaneous Order (Hyperion, NY, 2003); E. Ott, Chaos in Dynamical Systems (CUP, 2nd edition, Cambridge, 2002); S. H. Strogatz, Physica D 143(1-4), 1 (2000); J. A. Acebron et al. Rev. Mod. Phys. 77(1), 137 (2005).

[2] E. Barreto et al. Phys. Rev. E 77, 036107 (2008).

[3] E. Ott and T. M. Antonsen, arXiv:0806.0004v1 (2008).

[4] S. Watanabe and H. Strogatz, Physica D 74, 197 (1994).

[5] D. M. Abrams et al. Phys. Rev. Lett. 101, 084103 (2008).

[6] H. Sakaguchi and Y. Kuramoto, Prog. Theor. Phys. 76(3), 576 (1986).

[7] M. Rosenblum and A. Pikovsky, Phys. Rev. Lett. 98, 064101 (2007).

[8] For convenience we use variables, different from those in [2].

---

FIG. 1: (Color online) Simulation of ensemble (1) for \(N_1 = N_2 = 64\), \(\alpha = \pi/2 - 0.1\), and different distributions of constants of motion \(\psi_k^{(2)}\). Mean fields \(X_2, Y_2\) are defined by Eq. (1). (a): \(\mu = 0.6\); uniform distribution of \(\psi_k^{(2)}\) results in the steady state (black plus), cf. [3], whereas nonuniform distributions with \(q = 0.9\) and \(q = 0.7\) yield limit cycle solutions (bold red and blue solid lines, respectively). (b)-(d): \(\mu = 0.65\); uniform distribution of \(\psi_k^{(2)}\) yields a limit cycle solution (b), cf. [3], whereas for \(q = 0.9\) (c) and for \(q = 0.7\) (d) we observe a new type of the chimera state with a quasiperiodic dynamics.