Inhomogeneous Topological Superfluidity in One-Dimensional Spin-Orbit-Coupled Fermi Gases

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We theoretically predict an exotic topological superfluid state with spatially modulated pairing gap in one-dimensional spin-orbit-coupled Fermi gases. The emergence of this inhomogeneous topological superfluidity is induced by appealing to the conspiration of a perpendicular Zeeman magnetic field and an equally weighted Rashba and Dresselhaus spin-orbit coupling in one-dimensional optical lattices. Based on the self-consistent Bogoliubov-de Gennes theory, we confirm that this novel topological phase is a unique condensation of Cooper pairs, which goes beyond the conventional phase separation phenomena in the artificial confinement. The properties of the emergent Majorana bound states are investigated in detail by examining the associated $\mathbb{Z}_2$ topological number, the eigenenergy spectrum, and the wave functions of the localized Majorana end modes.

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Introduction.—The recent tireless pursuit of Majorana fermions in nanomaterials, solid state systems, and atomic Fermi gases has generated a huge storm of research in both fields of condensed matter and cold atom physics [1–7]. A series of intriguing heterostructures comprising conventional s-wave superconductors, topological insulators, ordinary semiconductors, as well as other kinds of substances have been proposed to possess the capacity of harboring non-Abelian Majorana zero modes at the interfaces of the sample through a proximity effect [8–13]. Experimental realizations of these proposals have reported the detections of the zero-bias mid-gap states in InSb nanowires contacted to the normal metal and superconducting electrodes [14].

A parallel extensive search for the Majorana bound states has also been underway in the systems of ultracold fermionic superfluids [15–21]. Particularly, the breakthrough of realizing synthetic gauge fields in cold atomic condensates [22–24] greatly stimulates the research of topological superfluidity in the spin-orbit-coupled (SOC) Fermi gases subjected to a large Zeeman magnetic field. One peculiar advantage of deploying ultracold atoms to probe the topological properties of a quantum fluid lies in the fact that via the standard techniques of optical lattices and Feshbach resonances, now we can not only precisely tune the inter-particle interactions over a wide range of parameters, but also have the freedom to switch the dimensions and modify the geometries of a quantum gas to experimentally simulate various theoretical models in modern physics [25]. In comparison with the well-studied two-dimensional structures, one-dimensional (1D) nanowires and optical lattices have recently attracted growing attention in detecting and engineering Majorana zero modes due to their simple 1D confinement geometry and the resultant reduction of the decoherence effects [18–20, 26–28]. To some extent, fermionic cold atoms trapped in a tube or a chain provide another ideal platform to further explore the new topological states of matter.

In the present Letter, instead of discussing the sophisticated approaches of materializing and manipulating Majorana fermions [29–31], we try a new route to theoretically explore the possibility of synthesizing a topologically nontrivial Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) superfluid state [32–35] in the harmonically trapped 1D atomic Fermi gases [37]. As a long-sought exotic quantum state, FFLO superfluidity is well-known for its fragility in the disordered environments, which severely prohibits the conclusive observations of this esoteric inhomogeneous pairing state in experiments for the past half-century. While, on the other hand, Majorana fermions in topological superfluids are gapless chiral edge modes, which are predicted to be immune to the weak disorders and perturbations. In view of this unconventional property, it will be very promising to study the strategies of stabilizing novel quantum states, like the FFLO superfluidity, with the aid of topologically protected non-Abelian Majorana end states in a highly-controllable ultracold atomic laboratory.

Landscape of the phase diagrams.—Here we conceive a minimal 1D lattice model, which simultaneously hosts Majorana fermions at the trap edges and an inhomogeneous FFLO phase in the bulk, to demonstrate the existence of a novel inhomogeneous topological superfluid in the fermionic condensates. We believe that this finding may serve as the first step toward creating a topologically protected FFLO state in the cold atomic and/or solid state systems. The Hamiltonian describing the 1D SOC Fermi gases can be written as [7, 12, 16, 19–21]:

\begin{align}
H &= H_K + H_R + H_D + H_\Delta, \\
H_K &= -t \sum_{i,j,\sigma} \psi_{i\sigma} \psi_{j\sigma} + \sum_{i,\sigma} [V(r_i) + \hbar \sigma \mu] \psi_{i\sigma}^\dagger \psi_{i\sigma}, \\
H_R &= -\lambda_x \sum_i (\psi_{i\uparrow}^\dagger \psi_{i+\hat{x}\downarrow} - \psi_{i\downarrow}^\dagger \psi_{i+\hat{x}\uparrow} + \text{H.c.}), \\
H_D &= -\lambda_y \sum_i (\psi_{i\uparrow}^\dagger \psi_{i+\hat{x}\uparrow} - \psi_{i\downarrow}^\dagger \psi_{i+\hat{x}\downarrow} + \text{H.c.}), \\
H_\Delta &= -\sum_i (\Delta_\uparrow \psi_{i\uparrow}^\dagger \psi_{i\downarrow} + \Delta_\downarrow \psi_{i\downarrow}^\dagger \psi_{i\uparrow}),
\end{align}

where $\psi_{i\sigma}^\dagger$ ($\psi_{i\sigma}$) denotes the creation (annihilation) field operator of fermionic atom with spin $\sigma \equiv (\uparrow, \downarrow)$ at site $r_i$. $H_K$ is the kinetic term with the nearest neighbor hopping integral $t$, the 1D harmonic trapping potential $V(r_i) = m\omega^2 r_i^2/2$, and a
perpendicular Zeeman magnetic field \( h \), as well as the chemical potential \( \mu \). In consideration of the current experimental status that only 1D equally weighted Rashba and Dresselhaus (ERD) spin-orbit (SO) coupling has been accessible to ultracold atomic fermions, we explicitly adopt two different kinds of SO interactions for a general purpose [12, 21]. \( H_R \) represents the spin-flip Rashba type SO coupling with the strength \( \lambda_z \), while \( H_D \) is a spin-conserving Dresselhaus (110) SO interaction with the strength \( \lambda_y \). \( H_\Delta \) denotes the spin-singlet s-wave contact attraction between atoms with the gap order parameter \( \Delta_\lambda \equiv V_\lambda \langle \psi_i \psi_j \rangle \), whence \( V_\lambda \) is the pairing strength.

To visualize the landscape of the candidate ground states, we first focus on understanding the bulk properties of the 1D SOC Fermi gas by ignoring the trapping potential and imposing the periodic boundary condition. The resulting Hamiltonian in the momentum space can be expressed via the Nambu spinor \( \Psi^\dagger = (\psi_{i\uparrow}^\dagger, \psi_{i\downarrow}^\dagger, \psi_{i\downarrow}, \psi_{i\uparrow}) \) as

\[
H = \sum_k \mathcal{H}(k) \Psi_k \quad \text{up to a constant, where the Bogoliubov-de Gennes (BdG) Hamiltonian reads:}
\]

\[
\mathcal{H}(k) = \frac{1}{2} \begin{pmatrix}
\xi^+_{\pm k} & \eta^+_{\pm k} & 0 & -\Delta_q \\
-\eta^+_{\pm k} & \xi^+_{\pm k} & \Delta_q & 0 \\
0 & \Delta_q & -\xi^+_{\pm k} & \eta^+_{\pm k} \\
-\Delta_q & 0 & -\eta^+_{\pm k} & -\xi^+_{\pm k}
\end{pmatrix}.
\]

In Eq. (2), we have assumed that the real-space superfluid order parameter is composed of the pair condensation at a definite center-of-mass momentum \( q \): \( \Delta_q = \Delta s e^{iqr} \) [34, 35], and the dispersions \( \xi^\pm_{\pm k} = -2t \cos \left( \frac{\pi}{2} \pm k \right) - \mu \pm [h + 2\lambda_y \sin \left( \frac{\pi}{2} \pm k \right)] \); \( \eta^\pm_{\pm k} = 2i\lambda_z \sin \left( \frac{\pi}{2} \pm k \right) \). Note that because of the presence of ERD SO coupling, \( \Delta_q \) is now a complex number, but we can still choose \( \Delta_q \) to be real without loss of generality. Typically, the eigenenergy of \( \mathcal{H}(k) \) cannot be analytically evaluated, so we need to solve the problem numerically and minimize the mean-field thermodynamic potential \( E_g = \langle H \rangle / N + \Delta^2_q / V_\lambda \) at zero temperature to self-consistently extract the genuine values of \( q \) and \( \Delta_q \) for the ground state (\( N \), even, is the total number of sites in the 1D chain).

Moreover, by noticing that \( \mathcal{H}(k) \) respects the built-in particle-hole symmetry, we can thus introduce an associated 1D \( \mathbb{Z}_2 \) number to characterize the nontrivial topological structure of the Bloch bands in the presence of both SO interactions and the Zeeman magnetic field. The \( \mathbb{Z}_2 \) number is defined to be \((-1)^{\nu} \), where the Berry phase \( \nu \) equals \( \frac{1}{\pi} \sum_{E(k) < 0} \int_0^{\pi} \langle \phi(k) | \hat{\nabla} \phi(k) \rangle dk \). \( |\phi(k)\rangle \) here are the eigenvectors of \( \mathcal{H}(k) \) with the negative eigenvalues in the first Brillouin zone. When the \( \mathbb{Z}_2 \) number is \(-1 \) (+1), the bulk system will be topologically nontrivial (trivial). It is worth mentioning that to engender a topological superfluid with the unconventional Majorana end states, besides the nontrivial band structure, the system also needs to sustain the channel of an effective spinless \( p \)-wave pairing. In our model [Eq. (2)], a nonvanishing \( \Delta_q \) will just encode the designed \( p \)-wave symmetry for the intraband pairing once projected into the basis that diagonalizes the non-interacting part of the Hamiltonian.

Therefore, it becomes practical to employ the pair condensation and the 1D \( \mathbb{Z}_2 \) topological number as a composite order parameter indexed by \( |\Delta_q| (-1)^{\nu} \) to discriminate between the distinct ground states and accordingly map out the phase diagrams of the SOC Fermi gases.

In the self-consistent calculation, we set the hopping integral \( t = 1 \) as the energy unit and the lattice constant \( a = 1 \) as the length unit; the \( s \)-wave attraction magnitude \( V_\lambda \) is fixed to be \( 2.5t \). We also introduce a parameter \( \lambda = 0.7t \) for the SO interactions, which determines the strengths of Rashba and Dresselhaus SO couplings in the following manner: \( \lambda_z^2 + \lambda_y^2 = \lambda^2 \). For simplicity, we only present the results at zero temperature.

Figure 1 shows the typical phase diagrams for the 1D Fermi gases at different intensities of SO couplings on the \( h-\mu \) plane. In panel (a), we keep \( \lambda_z = \lambda \) and switch off the Dresselhaus SO interaction. The resulting ground states in the limit of strong Rashba SO coupling are thus dominated by the homogeneous superfluid state with only the Rashba SO coupling \( (\lambda_z = \lambda, \lambda_y = 0) \), while panel (b) illustrates the emergence of an inhomogeneous topological superfluid phase in the ERD SOC Fermi gases \( (\lambda_z = \lambda_y = \sqrt{2} \lambda) [\lambda = 0.7t] \). The phase boundaries are symmetric with respect to \( \mu = 0 \).
logical portion of the spectrum, hence triggering the emergence of a new type of \textit{topo}-FFLO superfluid state with finite $q$, nonzero $\Delta_q$, and nontrivial $\mathbb{Z}_2$ number “$-1$”. The physical underpinning of this inhomogeneous topological superfluidity is mainly stemming from the inversion \textit{asymmetry} of the Bloch bands, which means that with fixed ERD SO coupling, the increase of $\hbar$ will not only modify the topology of the band structure via spoiling the time reversal symmetry, but it also facilitates the effective $p$-wave superfluid pairing at a nonvanishing center-of-mass momentum $q$. It is even appealing to perceive that the inclusion of Dresselhaus SO coupling would also efficiently \textit{enlarge} the domain of FFLO state in the phase diagram. Finally, further depletion of the atom occupation at large $\hbar$ will result in a normal gas (NG) state with completely suppressed pair condensation.

\textit{Topo}-FFLO superfluid in a trap.—With an overall picture about the landscape of the phase diagrams, we now proceed to concentrate on studying in depth the detailed real-space configurations of the emergent Majorana bound states localized at the ends of the chain in the \textit{topo}-FFLO phase under the realistic harmonic trapping and ERD SO coupling. After performing the canonical transformations: $\psi_{i\uparrow} = \sum_n \left[ u_n(r_i) \gamma_{n\uparrow} - v^*_n(r_i) \gamma_{n\downarrow} \right]$ and $\psi_{i\downarrow} = \sum_n \left[ v_n(r_i) \gamma_{n\uparrow} - u^*_n(r_i) \gamma_{n\downarrow} \right]$, we can obtain the self-consistency BdG equations from Eq. (1) as follows: $[H, \gamma_{n\sigma}] = -E_n \gamma_{n\sigma}$ and $[H, \gamma_{n\sigma}^\dagger] = E_n \gamma_{n\sigma}$. Since the system enjoys the particle-hole symmetry $\{H, \Xi\} = 0$ ($\Xi \equiv \sigma_z K$, $K$ is complex conjugation), the Bogoliubov quasiparticle operators $\gamma$’s satisfy the relations of $\gamma_{E} = \gamma_{-E}$, which implies that the system will become topologically nontrivial and accommodate Majorana zero modes at the domain walls separating topological and non-topological regimes if $E = 0$ and $\gamma_0 = \gamma_0$. The extended real-space computation [Fig. 2] is conducted on a $501 \times 1$ optical lattice with an open boundary condition. The trapping potential $m \omega^2 / 2$ is of about $0.0001 \hbar$. We then set $\hbar = 0.6 \hbar$, $\lambda_z = \lambda_y = 0.5 \hbar$, and $\mu = -1.3 \hbar$.

Figure 2 summarizes the spatial profiles of the superfluid order parameter as well as the fermion density distributions in the confined 1D tube along $\hat{x}$-direction. Due to the presence of a perpendicular Zeeman field, the real and imaginary parts of $\Delta_i$ display rapid oscillations across the zero point in antisymmetric ways throughout the whole region of the quantum gas with a period of about 15 sites [see Figs. 2(a) and (b)] [36]. To gain a concrete understanding of this spatial variation, we expand $\Delta_i$ in terms of a spectrum of plane waves $\Delta_i = \sum_{q'} \Delta_{q'} e^{i q' r_i}$. It becomes clear from Fig. 2(d) that the imposition of a harmonic confinement induces the modulation of multiple Fourier modes $\Delta_{q'}$ in the interval ranging from $q' \sim 0.5$ to $1.0$. With such observations, we regard that at the given parameters, the bulk system has been entering an inhomogeneous FFLO state. For completeness, we have added the density distributions $n_{i\sigma}$ in Fig. 2(c) as well, from which the bimodal structure of the spin $\uparrow$ atomic distribution is detectable.

More remarkably, when mapping out the corresponding spectrum of the system, we find that interestingly, the 1D quantum gas inside the tube also possesses the nontrivial topological properties. As shown in Fig. 3(a), the quasiparticle states with energies $\pm E$ are distributed symmetrically to $E = 0$ as anticipated. In the bottom inset of panel (a), we further give out explicitly the eigenenergy of the lowest excitation modes. By switching on ERD SO interaction, the minimal value of $|E_n|$ becomes exponentially small ($\sim 10^{-11}$), which is an indication of the emergence of unpaired Majorana fermions at the confinement edges.

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**FIG. 2:** Spatial profiles of the superfluid order parameter $\Delta_i$ and the atomic densities $n_{i\sigma}$ for the 1D ERD SOC Fermi gases in the real-space confinement. The real part of $\Delta_i$ is plotted in panel (a); its imaginary part is in panel (b). The atomic density distributions are shown in panel (c) with $n_{i\uparrow}$ solid line and $n_{i\downarrow}$ dashed line. Fourier transforms of $\Delta_i$ at finite momenta $q'$ are depicted in the panel (d).

**FIG. 3:** (color online). Part of the quasiparticle spectrum $E_n$ of the trapped 1D ERD SOC Fermi gases. The bottom inset indicates the numeric value of the lowest eigenenergy of $|E_n|$ [panel (a)]. Panels (b) and (c) depict the amplitudes of the wave functions for the zero energy states. All the parameters are the same as in Fig. 2.
Compared to the previous theoretical investigations on the 1D topological superfluid with the Rashba SO coupling \cite{18-20}, we highlight here that there is only one pair of Majorana fermions survived in our system. The second lowest eigenenergy of $|E_n|$ equals 0.0034 $\neq$ 0. Since these gapless chiral edge states live only inside the domain walls or boundaries separating topologically distinct regimes, the appearance of this remaining pair of Majorana zero modes is signifying that besides the vacuum and the normal gas state, there should just exist one united quantum phase formed by the fermionic atoms in the trapped 1D optical lattice. This unique condensation of Cooper pairs is exactly the inhomogeneous topological superfluid state we discussed, which is beyond the conventional phase separation phenomena in confined atomic systems. Furthermore, it shall be emphasized that the lifting of the second lowest-lying states from zero energy is mainly driven by the real-space penetration between the topological and the FFLO superfluids. This mutual merging of these two quantum phases comes into a unified topological FFLO state.

In Fig. 3(b) and (c), we plot the amplitudes of the wave functions for the unpaired Majorana fermions, which are localized at the phase boundaries as advertised. One distinguishing feature of the amplitude distribution is again the presence of the spatial modulation, which serves as another compelling evidence for the phase unification.

Similar to the mechanism proposed for the semiconducting heterostructures \cite{7,12}, this revealed coexistence of topological order with FFLO superfluidity might be yielded by the conspiration of a spin-singlet–pairing mediated $p$-wave superfluid instability in the topologically nontrivial Bloch bands with the Zeeman field facilitated breaking of time reversal as well as inversion symmetries. The non-Abelian Majorana fermions are further emerging from the phase twist of the orbital motion accompanying both spin-flip Rashba and spin-conserving Dresselhaus interactions \cite{16,17}. In our point of view, the lattice model Eq. (1) demonstrates a new method to create the inhomogeneous topological superfluidity in the cold atomic Fermi systems, which might be realized via the current experimental facilities \cite{22-24,37}. It also uncloaks several interesting aspects concerning the interplay between different sorts of SO interactions with the perpendicular magnetic field.

**Comments and conclusions.**—Quantum fluctuation effects, which are ignored at the mean-field treatment, shall become pronounced and important in low-dimensional systems. Nevertheless, the exact bosonization and DMRG analyses on such 1D systems have shown that both the FFLO superfluidity and the Majorana bound states are robust and stable against the critical quantum fluctuation corrections \cite{38-43}, therefore, more or less, confirming and erecting our mean-field predictions on the inhomogeneous topological superfluidity.

The non-Abelian statistical properties of the Majorana edge modes \cite{13,17} in the background of an inhomogeneous attractive pairing might lead to more fascinating phenomena in the SOC quantum gases, which will be pursued further in the future work. Local perturbations induced by disorders can also be used to probe the topological and superfluid properties of this novel spatially modulated state \cite{44}. It would be interesting to thoroughly examine the “stiffness” of this topo-FFLO state in the disordered and fluctuating environments.

Summarizing, we theoretically study the phase diagram of an ERD SOC Fermi gas in the 1D optical lattice, and successfully identify an exotic topological FFLO superfluid state in the region of strong SO couplings and Zeeman magnetic field. Detailed structures of the order parameters and the Majorana end modes associated with this topo-FFLO phase are manifestly uncovered in real space through resorting to the BdG formulation. Our work might open a new direction to harness Majorana fermions in protecting the “delicate” quantum phases topologically.

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