Quantum Coupled Nonlinear Schrödinger System with Different Masses

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Abstract.
In this letter, I have considered one-dimensional quantum system with different masses $m$ and $M$, which does not appear integrable in general. However I have found an exact two-body wave function and due to the extension of the integrable system to more general system, it was concluded that the rapidity or quasi-momentum in the integrable system should be regarded as a modification of velocity rather than that of momentum. I have also considered the three-body wave function and argued its integrable condition.

Recently solitary waves in a classical particle chain consisting of different masses were discovered [1,2]. In the one-dimensional classical particle chain with alternate different masses $m$ and $M$ whose ratio is a certain value given by the cyclotomic polynomial, there is a solitary wave for an appropriate initial condition.

In this letter, I will consider a kind of quantum version of such system.

Here, I will investigate the non-relativistic one-dimensional quantum system,

$$\left( -\sum_{i=1}^{N} \partial_{x_{i}}^{2} - \gamma \sum_{j=1}^{n} \partial_{y_{j}}^{2} + c \sum_{j=1}^{n} \sum_{i=1}^{N} \delta(x_{i} - y_{j}) \right) \Psi(x, y) = E \Psi(x, y), \tag{1}$$

where $1/\gamma$ is a normalized mass, $\gamma \geq 1$. By an appropriate scale transformation, this system can be interpreted as quantizing particles of masses $m$ and $M$ with the relation $M/m = \gamma$. This system is a generalization of the integrable hard core boson system with the same mass ($\gamma = 1$) [3-6]. Even though the system given by (1) does not appear integrable for general mass ratio $\gamma$, I expect that by investigating such exotic system, physical meaning of the integrable system might be revealed.

In this letter, I will study the system (1) parallelling to the argument of Thacker for the integrable system ($\gamma = 1$) in [5-6]. For later convenience, I will express this equation (1) by the second quantized field,

$$H = \int dx \left( \partial_{x} \Phi^{\ast} \partial_{x} \Phi + \gamma \partial_{x} \phi^{\ast} \partial_{x} \phi + c \phi^{\ast} \phi \Phi^{\ast} \Phi \right), \tag{2}$$

where $\phi$ and $\Phi$ are non-relativistic bosonic fields with canonical equal time commutation relations

$$[\Phi(x, t), \Phi^{\ast}(y, t)] = \delta(x - y), \quad [\phi(x, t), \phi^{\ast}(y, t)] = \delta(x - y), \tag{3}$$

and the other relations which are commutative. For $\gamma = 1$ case, this is essentially the same as the nonlinear Schrödinger system [3-6].

Hamiltonian commutes with the particle number operators $\hat{N} := \int \Phi^{\ast} \Phi dx$ and $\hat{n} := \int \phi^{\ast} \phi dx$. Thus I can precisely deal with $n$ and $N$ particles state, where $N$ and $n$ are the eigenvalues of the operators $\hat{N}$ and $\hat{n}$ respectively. Let $k$ be a momentum eigenstate of $\Phi$ particle and $k/\sqrt{\gamma}$ be that of $\phi$. I will introduce the creation operators,

$$A_{k}^{\dagger} = \int dx e^{ikx} \Phi^{\ast}(x). \tag{4.a}$$
and
\[ a_i^\dagger = \int dy e^{i k y} \phi^*(y). \] (4.b)

The non-perturbative momentum state will be denoted as
\[ \left| k_1, \cdots, k_N, k_{N+1}/\sqrt{\gamma}, \cdots, k_{n+N}/\sqrt{\gamma} \right> := \prod_{i=1}^{N} A_{k_i}^\dagger \prod_{j=N+1}^{n+N} a_{k_j/\sqrt{\gamma}}^\dagger |0\>, \] (5)

where \(|0\>\) is the vacuum state of this system. The total energy \( \omega_k \) and the total momentum \( P_k \) of the state are expressed as respectively,
\[ \omega_k = \sum_{i=1}^{N} k_i^2 + \sum_{j=1}^{n} k_j^2, \] (6.a)
and
\[ P_k = \sum_{i=1}^{N} k_i + \sum_{j=1}^{n} k_j/\sqrt{\gamma}. \] (6.b)

Following Thacker’s argument of the ordinary quantum nonlinear Schrödinger equation [6,7], I will compute the proper states of this system using the perturbation theory. By means of an old fashion perturbative method, an exact state is expressed by [6,7]
\[ |\Psi(k_1, \cdots, k_N, k_{N+1}/\sqrt{\gamma}, \cdots, k_{n+N}/\sqrt{\gamma})> = \sum_{i=0}^{\infty} (G_0(\omega_k)V)^i |k_1, \cdots, k_N, k_{N+1}/\sqrt{\gamma}, \cdots, k_{n+N}/\sqrt{\gamma}>, \] (7)

where \( G_0 \) is the Green function operator for free particles,
\[ G_0(\omega) = \frac{1}{\omega - H_0 + i\epsilon} \] (8)

and \( H_0 \) and \( V \) are the free terms and last term in (2) respectively.

It is obvious that any states consisting of only \( \phi \) or \( \Phi \) are reduced to free ones. Thus I will be concerned only with a mixing state of \( \phi \) and \( \Phi \).

**Two-body collision.**

I will consider a non-trivial two-body wave function of \( \phi \) and \( \Phi \) here.

Before I start the quantum computation, I will mention elastic collision of two classical particles with mass \( \gamma \) and 1 for the initial momentums \( k_1 \) and \( k_2/\sqrt{\gamma} \). Due to the energy-momentum conservation law, the variation of these momentums at a collision, \( (k_1, k_2/\sqrt{\gamma}) \rightarrow (K_1, K_2/\sqrt{\gamma}) \), are given as [1,2]
\[ \left( \begin{array}{c} K_1 \\ K_2 \end{array} \right) = \Gamma_+ \left( \begin{array}{c} k_1 \\ k_2 \end{array} \right), \quad \Gamma_+ := \left( \begin{array}{cc} \beta & \sqrt{1-\beta^2} \\ \sqrt{1-\beta^2} & -\beta \end{array} \right), \] (9)

where \( \beta := (\gamma - 1)/(\gamma + 1) \) and then \( \gamma \equiv (1 + \beta)/(1 - \beta) \).

I will study the quantum analogue of above situation (9) given through the Hamiltonian (2). The old fashion perturbative calculation is performed by the diagram Fig.1 [5,6]. The first term which corresponds to the non-perturbative state is given by,
\[ \int dx dy e^{i(k_1 x + k_2 y/\sqrt{\gamma})} \phi^*(x) \Phi^*(y)|0\>. \] (10)
The second term in Fig.1 becomes

\[
\int \frac{dp_1}{2\pi} \frac{dp_2}{2\pi\sqrt{\gamma}} \left( \frac{1}{k_1^2 + k_2^2 - p_1^2 - p_2^2 + i\epsilon} \right) 2\pi e^i(k_1 + k_2/\sqrt{\gamma} - p_1 - p_2/\sqrt{\gamma}) A_{p_1}^\dagger A_{p_2}^\dagger |0\rangle = \frac{c}{k_1 - \sqrt{\gamma}k_2} \int dx dy \\
\left[ \theta(y-x) e^{i(k_1 x + k_2 y/\sqrt{\gamma})} + \theta(x-y) e^{i(K_1 x + K_2 y/\sqrt{\gamma})} \right] \Phi(x) \Phi(y) |0\rangle,
\]

where \( k_1 > k_2 \sqrt{\gamma} \). The higher order terms are given as the loop diagrams illustrated in Fig.1 and have the same structure, which are calculated as

\[
c \int \frac{dp_1}{2\pi} \frac{dp_2}{2\pi\sqrt{\gamma}} \left( \frac{\delta(k_1 + k_2/\sqrt{\gamma} - p_1 - p_2/\sqrt{\gamma})}{k_1^2 + k_2^2 - p_1^2 - p_2^2 + i\epsilon} \right) = \frac{ic}{2(\sqrt{\gamma}k_2 - k_1)}.
\]

Here the series of the diagrams in Fig.1 converges and the converged point is defined by \( \Delta(k_1, k_2) \),

\[
1 + 2 \left( \frac{ic}{2(\sqrt{\gamma}k_2 - k_1)} \right) + 2 \left( \frac{ic}{2(\sqrt{\gamma}k_2 - k_1)} \right)^2 + \cdots = \frac{k_1 - \sqrt{\gamma}k_2 - ic/2}{k_1 - \sqrt{\gamma}k_2 + ic/2} = e^{-i\Delta(k_1, \sqrt{\gamma}k_2)/2},
\]

where \( \Delta \) function has the property,

\[
\Delta(K_1, K_2) = -\Delta(k_1, k_2).
\]

Accordingly I obtain the full exact two-body wave function for \( k_1 < k_2 \sqrt{\gamma} \) and then \( K_2 \sqrt{\gamma} > K_1 \),

\[
|\Psi(k_1, k_2/\sqrt{\gamma})\rangle = \int dx dy \left[ \theta(x-y) \left( e^{i(k_1 x + k_2 y/\sqrt{\gamma})} + (e^{i(\Delta(k_1, K_2)/2) - 1}) e^{i(K_1 x + K_2 y/\sqrt{\gamma})} \right) + \theta(y-x) e^{i(k_1 x + k_2 y/\sqrt{\gamma} - \Delta(k_1, k_2)/2)} \right] \Phi(x) \Phi(y) |0\rangle.
\]

The constituent parts of this wave function are illustrated as Fig.2. In the region of \( x < y \), there are incident state and reflection state while the state in the region \( y < x \) is the transparent one. If \( \gamma = 1 \) and \( \phi = \Phi \), this wave function becomes the ordinary two-body state of quantum non-linear Schrödinger system [3-6].

It should be noted that the phase factor of the wave function, which constitutes the S-matrix, is determined by the velocity \( k_1 \) and \( \sqrt{\gamma}k_2 \) rather than the momentum \( k_1 \) and \( k_2 / \sqrt{\gamma} \). Hence though the spectral parameters \( k_1 \) and \( k_2 \) which appear in \( \Delta(k_1, k_2) \) are sometimes called as "quasi-momentum" [3] for \( \gamma = 1 \) case, they should be interpreted as modification of the velocities.

**Three-body wave function.**

Next I will comment upon the computations of the three-body wave function in which two \( \Phi \) particles have the momentum \((k_1, k_2)\) and a \( \phi \) particle has \( k_3 / \sqrt{\gamma} \). Due to the classical properties of the one-dimensional chain [1,2], it will be clarified that wave function of these three particles can be calculated for a certain mass ratio even though this system is not integrable in general.

The classical elastic scattering of these particles are also expressed by this transformation matrices like (9). The collision of \((p_1, p_3 / \sqrt{\gamma})\) \(\rightarrow\) \((p'_1, p'_3 / \sqrt{\gamma})\) obeys the linear transformation,

\[
\begin{pmatrix}
p'_1 \\
p'_2 \\
p'_3
\end{pmatrix} = \Gamma_+ \begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}, \quad \Gamma_+ := \begin{pmatrix}
\beta & 0 & \sqrt{1 - \beta^2} \\
0 & 1 & 0 \\
\sqrt{1 - \beta^2} & 0 & -\beta
\end{pmatrix}.
\]

Similarly the collision \((p_2, p_3 / \sqrt{\gamma})\) \(\rightarrow\) \((p'_2, p'_3 / \sqrt{\gamma})\) is given by

\[
\begin{pmatrix}
p'_1 \\
p'_2 \\
p'_3
\end{pmatrix} = \Gamma_- \begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}, \quad \Gamma_- := \begin{pmatrix}
1 & 0 & 0 \\
0 & \beta & \sqrt{1 - \beta^2} \\
0 & \sqrt{1 - \beta^2} & -\beta
\end{pmatrix}.
\]
From ref. [1,2], the transition from the initial state \((k_1, 0, 0)\) to the final state \((0, k_1, 0)\) occurs which is expressed as

\[
\begin{pmatrix}
0 \\
k_1 \\
0
\end{pmatrix} = \Gamma_{-\Gamma_+ \Gamma_- \Gamma_+ \cdots \Gamma_{-\Gamma_+}} \begin{pmatrix}
k_1 \\
0 \\
0
\end{pmatrix} = (\Gamma_{-\Gamma_+})^n \begin{pmatrix}
k_1 \\
0 \\
0
\end{pmatrix},
\]

(18)

if the mass ratio is given by

\[
\gamma = -\frac{\cos\left(\frac{2\pi n}{2n+1}\right)}{\cos\left(\frac{2\pi n}{2n+1}\right) + 1},
\]

(19)

where \(n\) is a fixed integer parameter. It should be remarked that (19) contains \(\gamma = 1\) case as \(n = 1\). Thus in the classical particle chain system, (19) is a generalization of \(\gamma = 1\) case in which the initial momentum \(k_1\) of the first particle are completely transferred to that of the second particle and such transition propagates like a soliton [1,2]. In fact, recently a solitary wave was found for an initial condition in classical chain system with general \(n\) mass ratio.

Due to the time reversion symmetry of the newtonian mechanics, the relation also holds for (19),

\[
\begin{pmatrix}
k_2 \\
0 \\
0
\end{pmatrix} = (\Gamma_{+\Gamma_-})^n \begin{pmatrix}
0 \\
k_2 \\
0
\end{pmatrix}.
\]

(20)

One of purposes of this letter is to investigate a quantum system which inherits these novel properties of the classical system.

Noting above properties, I will consider the three-body wave function in this quantum system which is given by the diagram illustrated in Fig. 3. Here it should be also noted that while in the classical particle chain system in ref. [1,2] one should pay attentions upon the configuration space, in this quantum system (1) and (2), due to the quantum principle, I need not take care of the configuration space as long as the momentum representation.

Since \(\phi\) particle does not interact with itself, only the loop diagram appears consisting of only \(\phi\) and the first \(\Phi\) particles or \(\phi\) and the second \(\Phi\) particles. Since in the loop diagram, one does not need impose the energy-momentum conservation, the loop diagram does not generates new momentum ratio as I showed in (12). On the other hand, even in the virtual states, the momentum and the energy must be conserved in the intermediate states except the loop diagrams. This situation is essentially the same as that of the calculus of the Bethe ansatz for the case of \(\gamma = 1\). In the "collision" graph of Fig. 3, at which the thin line and thick line are incoming and outgoing, the momentum transfer occurs. For the initial condition \((k_1, k_2, k_3/\sqrt{\gamma})\), in the perturbative calculation, there appear infinite types of momentum

\[
(\Gamma_{-\Gamma_+})^m \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad \Gamma_{+}(\Gamma_{-\Gamma_+})^m \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix},
\]

(21)

where \(m\) is a non-negative integer. Thus in general this system cannot be exactly calculated by the perturbation scheme and thus neither appear integrable.

However when the mass ratio is given by (19), the kinds of momentums appearing in the perturbative calculation becomes, at most, \(4n + 1\). In other words, I obtain the relation

\[
(\Gamma_{-\Gamma_+})^{2n+1} = 1
\]

(22)
The relation (22) is proved as follows;

\[
(\Gamma_- \Gamma_+)^{2n+1} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = (\Gamma_- \Gamma_+)^{n+1} (\Gamma_- \Gamma_+)^n \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \Gamma_- (\Gamma_+ \Gamma_-)^n \Gamma_+ \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \Gamma_- \Gamma_+ \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = (\Gamma_- \Gamma_+)^n \Gamma_- \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = (\Gamma_- \Gamma_+)^n \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}
\]

\[
= \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}.
\]

where \(\cdot\)'s are unknown quantities. Here I used (18), (20) and the relations;

\[
\Gamma_+ \begin{pmatrix} k_1 \\ k_2 \\ \cdot \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ \cdot \end{pmatrix}, \quad \Gamma_- \begin{pmatrix} \cdot \\ k_2 \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ k_2 \\ \cdot \end{pmatrix}, \quad \Gamma_+^2 = \Gamma_-^2 = 1.
\]

By the energy-momentum conservation, the third component in (23) is determined as \(k_3\) and \((\Gamma_- \Gamma_+)^{2n+1}\) must be unit matrix.

As the transitions between the momentums of the first and second \(\Phi\) particles occur \(n\) times in the perturbative diagram mediated by the \(\phi\) particle, the momentum completely exchanges; \((k_1, \cdot, \cdot) \rightarrow (\cdot, k_1, \cdot)\) and similarly \((\cdot, k_2, \cdot) \rightarrow (k_2, \cdot, \cdot)\). After \((4n + 2)\) "collisions" in the calculus, the initial momentum \((k_1, k_2, k_3)\) recovers. Since such shuttles of the momentum repeat in the calculus of the perturbation scheme, there appear, at most, \(4n + 1\) kinds of momentums in the calculation. Accordingly if the mass ratio given by (19), three-body wave function is classified by, at most, \(4n + 1\) momentum states and can be calculated in principal like that in ref.[6]. Due to the relation (12), it is expected that the phase factor consists of the (pseudo-)velocities.

**Discussion.**

In (13), there appears the ratio \(k_1/(\sqrt{k_2})\) in the phase of collision. This ratio corresponds to the ratio of the velocity rather than the momentum of the classical particles. Hence I emphasis that the quasi-momentum in [3] should be interpreted as the modification of the velocity. I believe that this picture plays an important role if one goes beyond the integrable model to more realistic model.

Furthermore using the diagrams, I conjectured that the three-body wave function of three particles can be exactly calculated for the special mass ratio which was calculated in ref.[1,2]. Thus this quantum system partially succeeds the properties of solitary wave in the classical system of the hard core chain [1,2].

Further computations will be reported somewhere [7].

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**References**

1. S. Ishiwata, S. Matsutani and Y. Onishi, to appear in Phys. Lett. A.
2. S. Ishiwata and S. Matsutani, Advances in soliton theory and its applications/ The 30th anniversary of the Toda lattice, 1996 in Japan.
4. C. N. Yang, Phys. Rev. 168 (1968), 1920.
5. H. B. Thacker, Rev. Mod. Phys. 53 (1981), 253.
6. H. B. Thacker, Phys. Rev. D 11 (1975), 838.
7. S. Matsutani, in preparation.

**Figure Captions.**

Fig.1: Sum of graphs for the two-body wave function.
Fig.2: Resonance of two-body wave function.
Fig.3: Sum of graphs for the three-body wave function.