A Note an Admissible Mannheim Curves in Galilean Space $G_3$

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Abstract

The aim of this paper is to study the Mannheim partner curves in three dimensional Galilean space $G_3$. Some well known theorems are obtained related to Mannheim curves.

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1 Introduction

In 1850 J. Bertrand defined Bertrand curves, with the help of this definition Liu and Wang called the Mannheim pair which the principal normal vector of the first curve coincides with the binormal vector of the second curve and they obtained the necessary and sufficient conditions between the curvature and the torsions for a curve to be the Mannheim partner curves, [1]. Moreover, Orbay and Kasap examined on Mannheim curves in $E^3$, [2]. The geometry of the Galilean space $G_3$ has been treated in detail in Röschel's habilitation, [3]. Futhermore, Kamenarovic and Sipus studied about Galilean space, [4, 5]. The properties of the curves in the Galilean space are studied in [6, 7, 8]. In this paper, we gave some theorems and relations about the curvatures and torsions of admissible Mannheim curves in 3-dimensional Galilean space $G_3$.

2 Preliminaries

The Galilean space $G_3$ is a Cayley-Klein space equipped with the projective metric of signature (0,0,+,+), as in [9]. The absolute figure of the Galilean Geometry consist of an ordered triple $\{w, f, I\}$, where $w$ is the ideal (absolute) plane, $f$ is the line (absolute line) in $w$ and $I$ is the fixed elliptic involution of
points of \( f \), [4]. In the non-homogeneous coordinates the similarity group \( H_8 \) has the form
\[
\begin{align*}
\mathbf{x} &= a_{11} + a_{12} x \\
\mathbf{y} &= a_{21} + a_{22} x + a_{23} y \cos \varphi + a_{23} z \sin \varphi \\
\mathbf{z} &= a_{31} + a_{32} x - a_{23} y \sin \varphi + a_{23} z \cos \varphi
\end{align*}
\] (2.1)
where \( a_{ij} \) and \( \varphi \) are real numbers, [6].

In what follows the coefficients \( a_{12} \) and \( a_{23} \) will play the special role. In particular, for \( a_{12} = a_{23} = 1 \), (2.1) defines the group \( B_6 \subset H_8 \) of isometries of Galilean space \( G_3 \).

In \( G_3 \) there are four classes of lines:

i) (proper) non-isotropic lines- they don’t meet the absolute line \( f \).

ii) (proper) isotropic lines- lines that don’t belong to the plane \( w \) but meet the absolute line \( f \).

iii) unproper non-isotropic lines- all lines of \( w \) but \( f \).

iv) the absolute line \( f \).

Planes \( x = \text{constant} \) are Euclidean and so is the plane \( w \). Other planes are isotropic, [5].

Galilean scalar product can be written as
\[
< u_1, u_2 > = \begin{cases} 
  x_1 x_2 , & \text{if } x_1 \neq 0 \lor x_2 \neq 0 \\
  y_1 y_2 + z_1 z_2 , & \text{if } x_1 = 0 \land x_2 = 0
\end{cases}
\]
where \( u_1 = (x_1, y_1, z_1) \) and \( u_2 = (x_2, y_2, z_2) \). It leaves invariant the Galilean norm of the vector \( u = (x, y, z) \) defined by
\[
\| u \| = \begin{cases} 
  x , & x \neq 0 \\
  \sqrt{y^2 + z^2} , & x = 0.
\end{cases}
\]

Let \( \alpha \) be a curve given in the coordinate form
\[
\alpha : I \to G_3, \quad I \subset \mathbb{R} \\
t \to \alpha(t) = (x(t), y(t), z(t))
\] (2.2)
where \( x(t), y(t), z(t) \in C^3 \) and \( t \) is a real interval. If \( x'(t) \neq 0 \), then the curve \( \alpha \) is called admissible curve.

Let \( \alpha \) be an admissible curve in \( G_3 \), is parameterized by arc length \( s \), is given by
\[
\alpha(s) = (s, y(s), z(s))
\]
where the curvature \( \kappa(s) \) and the torsion \( \tau(s) \) are
\[
\kappa(s) = \sqrt{y'^2(s) + z'^2(s)} \quad \text{and} \quad \tau(s) = \frac{\det [\alpha'(s), \alpha''(s), \alpha'''(s)]}{\kappa^2(s)},
\] (2.3)
respectively. The associated moving Frenet frame is
\[
\begin{align*}
T(s) &= \alpha'(s) = (1, y'(s), z'(s)) \\
N(s) &= \frac{1}{\kappa(s)} \alpha'(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)) \\
B(s) &= \frac{1}{\kappa(s)}(0, -z''(s), y''(s)).
\end{align*}
\] (2.4)
Here $T, N$ and $B$ are called the tangent vector, principal normal vector and binormal vector fields of the curve $\alpha$, respectively. Then for the curve $\alpha$, the following Frenet equations are given by

\[
\begin{align*}
T'(s) &= \kappa(s)N(s) \\
N'(s) &= \tau(s)B(s) \\
B'(s) &= -\tau(s)N(s)
\end{align*}
\]

(2.5)

where $T, N, B$ are mutually orthogonal vectors, [5].

3 Admissible Mannheim Curves in Galilean Space $G_3$

In this section, we defined the admissible Mannheim curve and gave some theorems related to these curves in $G_3$.

**Definition 3.1** Let $\alpha$ and $\alpha^*$ be an admissible curves with the Frenet frames along $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. The curvature and torsion of $\alpha$ and $\alpha^*$, respectively, $\kappa(s), \tau(s)$ and $\kappa^*(s), \tau^*(s)$ never vanish for all $s \in I$ in $G_3$. If the principal normal vector field $N$ of $\alpha$ coincidence with the binormal vector field $B^*$ of $\alpha^*$ at the corresponding points of the admissible curves $\alpha$ and $\alpha^*$. Then $\alpha$ is called an admissible Mannheim curve and $\alpha^*$ is an admissible Mannheim mate of $\alpha$. Thus, for all $s \in I$

\[
\alpha^*(s) = \alpha(s) + \lambda(s)N(s).
\]

(3.1)

The mate of an admissible Mannheim curve is denoted by $(\alpha, \alpha^*)$, see Figure 1.

![Figure 1. The admissible Mannheim partner curves](image)

**Theorem 3.1** Let $(\alpha, \alpha^*)$ be a mate of admissible Mannheim pair in $G_3$. Then function $\lambda$ is constant on $I$ and defined by equation (3.1).
Proof. Let $\alpha$ be an admissible Mannheim curve in $G_3$ and $\alpha^*$ be an admissible Mannheim mate of $\alpha$. Let the pair of $\alpha(s)$ and $\alpha^*(s)$ be of corresponding points of $\alpha$ and $\alpha^*$. Then the curve $\alpha^*(s)$ is given by (3.1). Differentiating (3.1) with respect to $s$ and using Frenet equations,

$$ T^* \frac{ds^*}{ds} = T + \lambda' N + \lambda \tau B \quad (3.2) $$

is obtained.

Here and here after prime denotes the derivative with respect to $s$. Since $N$ is coincident with $B^*$ in the same direction, we have

$$ \lambda'(s) = 0, \quad (3.3) $$

that is, $\lambda$ is constant. This theorem proves that the distance between the curve $\alpha$ and its Mannheim mate $\alpha^*$ is constant at the corresponding points of them. It is notable that if $\alpha^*$ is an admissible Mannheim mate of $\alpha$. Then $\alpha$ is also Mannheim mate of $\alpha^*$ because the relationship obtained in theorem between a curve and its Mannheim mate is reciprocal one.

Theorem 3.2 Let $\alpha$ be an admissible curve with arc length parameter $s$. $\alpha$ is an admissible Mannheim curve if and only if the torsion $\tau$ of $\alpha$ is constant.

Proof. Let $(\alpha, \alpha^*)$ be a mate of an admissible Mannheim curves, then there exists the relation

$$ T(s) = \cos \theta T^*(s) + \sin \theta N^*(s) $$
$$ B(s) = -\sin \theta T^*(s) + \cos \theta N^*(s) \quad (3.4) $$

and

$$ T^*(s) = \cos \theta T(s) - \sin \theta B(s) $$
$$ N^*(s) = \sin \theta T(s) + \cos \theta B(s) \quad (3.5) $$

where $\theta$ is the angle between $T$ and $T^*$ at the corresponding points of $\alpha(s)$ and $\alpha^*(s)$, (see Figure 1).

By differentiating (3.5) with respect to $s$, we get

$$ \tau B^* \frac{ds^*}{ds} = \frac{d(\cos \theta)}{ds} T + \sin \theta \kappa N + \cos \theta \tau N + \frac{d(\cos \theta)}{ds} B. \quad (3.6) $$

Since the principal normal vector field $N$ of the curve $\alpha$ and the binormal vector field $B^*$ of its Mannheim mate curve, then it can be seen that $\theta$ is a constant angle.

If we equations (3.2) and (3.5) is considered, then

$$ \lambda \tau \cot \theta = 1 \quad (3.7) $$

is obtained. According to Theorem 3.1 and constant angle $\theta$, $u = \lambda \cot \theta$ is constant. Then from equation (3.7), $\tau = \frac{1}{u}$ is constant, too.

Hence the proof is completed. □
Theorem 3.3 (Schell’s Theorem). Let \((\alpha, \alpha^*)\) be a mate of an admissible Mannheim curves with torsions \(\tau\) and \(\tau^*\), respectively. The product of torsions \(\tau\) and \(\tau^*\) is constant at the corresponding points \(\alpha(s)\) and \(\alpha^*(s)\).

Proof. Since \(\alpha\) is an admissible Mannheim mate of \(\alpha^*\), then
\[
(3.8) \quad \alpha = \alpha^* - \lambda B^*.
\]
By taking differentiation of last equation and using equation (3.4),
\[
(3.9) \quad \tau^* = \frac{1}{\lambda} \tan \theta
\]
can be given. By the helps of (3.7), the equation below is obtained easily;
\[
(3.10) \quad \tau\tau^* = \tan^2 \frac{\theta}{\lambda} = \text{constant}
\]
This completes the proof. □

Theorem 3.4 Let \((\alpha, \alpha^*)\) be an admissible Mannheim mate with curvatures \(\kappa, \kappa^*\) and torsions \(\tau, \tau^*\) of \(\alpha\) and \(\alpha^*\), respectively. Then their curvatures and torsions satisfy the following relations

i) \(\kappa^* = -\frac{d\theta}{ds^*}\)

ii) \(\kappa = \tau^* \frac{ds^*}{ds} \sin \theta\)

iii) \(\tau = -\tau^* \frac{ds^*}{ds} \cos \theta\).

Proof. i) Let us consideration equation (3.4), then we have
\[
(3.11) \quad < T, T^* > = \cos \theta.
\]
By differentiating last equation with respect to \(s^*\) and using the Frenet equations of \(\alpha\) and \(\alpha^*\), we reach
\[
(3.12) \quad < \kappa(s)N(s) \frac{ds}{ds^*}, T^*(s) > + < T(s), \kappa^*(s)N^*(s) > = -\sin \theta \frac{d\theta}{ds^*}.
\]
Since the principal normal \(N\) of \(\alpha\) and binormal \(B^*\) of \(\alpha^*\) are linearly dependent. By considering equations (3.4) and (3.12), we reach
\[
(3.13) \quad \kappa^*(s) = -\frac{d\theta}{ds^*}.
\]
If we take into consideration \(< T, B^* >, < B, B^* >\) scalar products and (2.5), (3.4), (3.5) equations, then we can easily prove ii) and iii) items of the theorem, respectively.

The relations given in ii) and iii) of the last theorem, we obtain
\[
\frac{\kappa}{\tau} = -\tan \theta = \text{constant}.
\]
□
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