Reflected backward stochastic differential equations with two optional barriers

Tomasz Klimsiak, Mauryce Rzymowski and Leszek Słomiński

Abstract

We consider reflected backward stochastic differential equations with two general optional barriers. The solutions to these equations have the so-called regulated trajectories, i.e. trajectories with left and right finite limits. We prove the existence and uniqueness of $L^p$ solutions, $p \geq 1$, and show that the solutions may be approximated by a modified penalization method.

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1 Introduction

In the present paper we study the existence, uniqueness and approximations of $L^p$, $p \geq 1$, solutions of reflected backward stochastic differential equations (RBSDEs) with monotone generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and two optional barriers $L, U$ satisfying the so-called generalized Mokobodzki condition.

The notion of RBSDE with one reflecting continuous barrier was introduced by El Karoui, Kupoudjian, Pardoux, Peng and Quenez [9], who proved the existence and uniqueness of solutions of equations with Lipschitz continuous generator and square-integrable data. RBSDEs with two continuous barriers were for the first time considered by Cvitanić and Karatzas [5] under the same assumptions on the generator and the data. In [5], a solution is a triple $(Y, Z, R)$ of $\mathbb{F}$-progressively measurable processes such that $Y$ is continuous and

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) \, dr + \int_t^T dR_r - \int_t^T Z_r \, dB_r, \quad t \in [0, T], \quad (1.1)$$

where $B$ is a standard $d$-dimensional Brownian motion and $\mathbb{F}$ is the standard augmentation of the natural filtration generated by $B$. Moreover, it is required that

$$L_t \leq Y_t \leq U_t, \quad t \in [0, T], \quad (1.2)$$

and $R$ is a continuous process of finite variation such that $R_0 = 0$ and the following minimality condition is satisfied:

$$\int_0^T (Y_r - L_r) \, dR_r^+ + \int_0^T (U_r - Y_r) \, dR_r^- = 0. \quad (1.3)$$
Here $R^+, R^-$ stand for the positive and negative part of the Jordan decomposition of the measure $dR$. In [5] the existence and uniqueness of a solution is proved. Note also that in [3, 9] important connections between solutions of RBSDEs and suitably defined optimal stopping problems were established.

Since the pioneering works [5, 9] reflected BSDEs have been intensively studied by many authors. The results of [5, 9] were generalized to equations with $L^p$ data and càdlàg barriers (see, e.g., [4, 12, 13, 14, 15, 19, 20, 23]). The assumption that the barriers are càdlàg implies that the first component $Y$ of a solutions is also a càdlàg process. Therefore this assumption is sometimes too strong when one think on applications of RBSDEs to optimal stopping problems, because it is known that in general solutions of such problems have merely regulated trajectories (see [8]). It is worth noting here that in [16, 22] RBSDEs with non-càdlàg (progressively measurable) barriers and càdlàg solutions are considered. However, in the definition of a solutions adopted in [16, 22] its first component $Y$ need not satisfy (1.2), but satisfies an essentially weaker condition saying that $L_t \leq Y_t \leq U_t$ for a.e. $t \in [0, T]$.

To our knowledge, RBSDEs with barriers which are not càdlàg and whose solution satisfies (1.2) are treated only in the papers [1, 10, 11, 17]. Among them, only [11] deals with equations with two barriers. In the present paper we generalize the existence and uniqueness results (1.2) are treated only in the papers [1, 10, 11, 17]. Among them, only [11] deals with equations with two barriers. In the present paper we generalize the existence and uniqueness results from [11] in several directions. We consider the case of $L^p$-data with $p \geq 1$ (in [11] only the case of $p = 2$ is considered). As for the generator, we assume that it is Lipschitz continuous with respect to $z$ and only continuous and monotone with respect to $y$ (in [11] it is assumed that $f$ is Lipschitz continuous with respect to $y$ and $z$). Moreover, we assume that the generator and the barriers satisfy the so-called generalized Mokobodzki condition which says that there exists a semimartingale $X \in \mathcal{M}_{loc} + \mathcal{V}^p$ such that $L_t \leq X_t \leq U_t$, $t \in [0, T]$, and

$$E\left(\int_0^T |f(r, X_r, 0)| dr\right)^p + |X| < \infty,$$

where $|X| := (E \sup_{t \leq T} |X|^p|^{1/p}$ for $p > 1$ and $|X| := \sup_{t \in \Gamma} E |X_t|$ (here $\mathcal{M}_{loc}$ is the space of local martingales and $\mathcal{V}^p$ is the space of finite variation processes with $p$-integrable variation, and $\Gamma$ denotes the set of all $\mathbb{F}$-stopping times). In [11] the standard Mokobodzki condition is assumed. It says that $L \leq X \leq U$ for some semimartingale $X \in \mathcal{M}_{loc} + \mathcal{V}^2$ such that $|X|^2 < \infty$. This condition automatically implies (1.4) with $p = 2$ in case $f$ is Lipschitz continuous.

The assumptions on $\xi$ and $f$ adopted in the present paper are the same as in our previous paper [17] devoted to equations with one lower barrier, and our definition of a solution is a counterpart to the definition introduced in [17]. For a process $\eta$, let $\Delta^+ \eta_t = \eta_{t+} - \eta_t$, $\Delta^- \eta_t = \eta_t - \eta_{t-}$, i.e. $\Delta^+ \eta_t$, $\Delta^- \eta_t$ denote the right and left jump of $\eta$ at $t$. Our definition says that a triple $(Y, Z, R)$ of $\mathbb{F}$-progressively measurable processes is a solution of RBSDE on the interval $[0, T]$ with terminal time $\xi$, right-hand side $f$ and optional barriers $L, U$ (RBSDE($\xi, f, L, U$) for short) if $Y, R$ are regulated processes, $R$ is a finite variation process with $R_0 = 0$, (1.1) and (1.2) hold true, and the following minimality condition is satisfied:

$$\int_0^T (Y_{t^-} - \liminf_{s \uparrow t} L_s) dR^+_t + \sum_{t < T} (Y_t - L_t) \Delta^+ R^+_t + \int_0^T (\liminf_{s \uparrow t} U_s - Y_{t^-}) dR^-_t + \sum_{t < T} (U_t - Y_t) \Delta^+ R^-_t = 0,$$

where $R^+_t, R^-_t$ are càdlàg parts of the processes $R^+, R^-$. We show that if the barriers $L, U$ are regulated, then $\Delta^- R^+_t = (Y_t - L_t^-)^-$, $\Delta^- R^-_t = (Y_t - U_t^-)^+$, and $\Delta^+ R^+_t = (Y_t^+ - L_t)^-$, and...
\[ \Delta^+ R_t^- = (Y_{t+} - U_t)^+ \]. If the barriers are càdlàg (resp. càglàd) then \( \Delta^+ R = 0 \) (resp. \( \Delta^- R = 0 \)). Consequently, if \( L, U \) are continuous, then condition (1.5) reduces to (1.3). Moreover, if the barriers are càdlàg, then condition (1.3) reduces to the minimality condition considered in [12]. In the present paper, we generalize the existence, uniqueness and approximation results proved in [17]. It is worth pointing out, however, that the proofs are essentially more complicated and in many points different from those in [17].

Our main results are proved in Sections 3 and 4. In Section 3, we consider equations with general two optional barriers (they need not be regulated). We show that there exists a unique solution \((Y, Z, R)\) to RBSDE\((\xi, f, L, U)\) such that \(|Y|_p + E|R_T^+|^p + E|R_T^-|^p < \infty\) and \(E\int_0^T |Z_r|^p dr)^{p/2} < \infty\) in case \(p > 1\), and \(E\int_0^T |Z_r|^q dr)^{q/2} < \infty\) for \(q \in (0, 1)\) in case \(p = 1\). In case \(p = 1\), we assume additionally that \(f\) satisfies condition (Z) introduced in the paper [1] devoted to usual (nonreflected) BSDEs. The proof of the existence part is divided into two steps. In the first step, we assume that \(f\) does not depend on \(z\) and we solve the following decoupling system

\[
\begin{align*}
Y_t^1 &= \text{ess sup}_{t \leq \tau \leq T} E(Y_{\tau}^2 + \int_t^T f(r, Y_r^1 - Y_r^2) dr + L_{\tau} 1_{\tau < T} + \xi 1_{\tau=T} |F_t}), \\
Y_t^2 &= \text{ess sup}_{t \leq \tau \leq T} E(Y_{\tau}^1 1_{\tau < T} - U_{\tau} 1_{\tau < T} |F_t)).
\end{align*}
\]

This system may be equivalently formulated as a system of RBSDEs with lower optional barriers (see [1], [10], [17]). Putting \( Y = Y^1 - Y^2 \), we obtain a solution of RBSDE\((\xi, f, L, U)\). Note that in the linear case, i.e. when \(f\) does not depend on \(y\) as well, this method was considered in the context of Dynkin games problem by Bismut [2], [3] (see also [18], [21]). Next, to solve the nonlinear problem, we apply a fixed point argument in case \(p > 1\), and Picard iteration procedure in case \(p = 1\).

In Section 4, under the additional assumption that the barriers \(L, U\) are regulated, we propose another approach to the existence problem. We consider two penalization schemes based on BSDEs with penalty term and RBSDEs with one barrier and penalty term. In the first one, we show that there exists a unique solution \((Y^n, Z^n)\) of generalized BSDE of the form

\[
Y_t^n = \xi + \int_t^T f(r, Y_r^n, Z_r^n) dr - \int_t^T Z_r^n dB_r + n \int_t^T (Y_r^L - L_r)^- dr + \sum_{t \leq \tau_{n,i} < T} (Y_{\sigma_{n,i}^+} - L_{\sigma_{n,i}})^- - n \int_t^T (Y_r^U - U_r)^+ dr - \sum_{t \leq \tau_{n,i} < T} (Y_{\tau_{n,i}^+} - U_{\tau_{n,i}})^+, \quad (1.6)
\]

where \(\{\{\sigma_{n,i}\}\} \) (resp. \(\{\{\tau_{n,i}\}\}\)) is a suitably defined array of stopping times exhausting the right jumps of \(L\) (resp. \(U\)). We prove that

\[
Y_t^n \to Y_t, \quad t \in [0, T]. \quad (1.7)
\]

Moreover, for every \(\gamma \in (0, 2)\),

\[
E\left(\int_0^T |Z_r^n - Z_r|^\gamma dr\right)^{p/2} \to 0 \quad (1.8)
\]

if \(p > 1\), and

\[
E\left(\int_0^T |Z_r^n - Z_r|^\gamma dr\right)^{q/2} \to 0, \quad q \in (0, 1), \quad (1.9)
\]

if \(p = 1\). We also prove that if \(\Delta^- R = 0\), then \(|Y^n - Y|_p \to 0\), and (1.8) holds true with \(\gamma = 2\). To prove (1.7)–(1.9) we first show the convergence of penalization schemes based on
RBSDEs. In this scheme, \((\bar{Y}^n, \bar{Z}^n, \bar{K}^n)\) (resp. \((Y^n, Z^n, A^n)\)) is a solution to reflected BSDE with upper barrier \(U\) (resp. lower barrier \(L\)) and the generator being a sum of \(f\) and an additional penalty term (depending on \(n\)) involving \(L\) (resp. \(U\)) and the right-side jumps of \(L\) (resp. \(U\)). We prove that \((\bar{Y}^n, \bar{Z}^n, \bar{K}^n), (Y^n, Z^n, A^n)\) converge to \((Y, Z, R)\) in the sense of (1.7) and

\[
\bar{Y}_t^n \leq Y_t^n \leq \sum_n^n, \quad t \in [0, T].
\]

The advantage of these approximations is that \(\{\bar{Y}^n\}\) is nondecreasing and \(\{Y^n\}\) is nonincreasing.

2 Preliminaries

Let \(B\) be a standard Wiener process defined on some probability space \((\Omega, \mathcal{F}, P)\) and let \(\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}\) be the standard augmentation of the filtration generated by \(B\). Recall that a function \(y : [0, T] \to \mathbb{R}^d\) is called regulated if for every \(t \in [0, T]\) the limit \(y_{t+} = \lim_{u \downarrow t} y_u\) exists, and for every \(s, t \in [0, T]\) the limit \(y_{s-} = \lim_{u \uparrow s} y_u\) exists. For any regulated function \(y\) on \([0, T]\), we set \(\Delta^+ y_t = y_{t+} - y_t\) if \(0 \leq t < T\), and \(\Delta^- y_s = y_s - y_{s-}\) if \(0 < s \leq T\). We also set \(\Delta^+ y_T = \Delta^- y_0 = 0\), \(\Delta y_t = \Delta^+ y_t + \Delta^- y_t\), \(t \in [0, T]\) and \(y_{t\pm} = y_t\) if \(t < T\), and \(y_{T\pm} = y_T\).

Note that \(y^0\) is a càdlàg function such that \(y_{t\pm} = \Delta^+ y_t + y_t, t \in [0, T]\). It is known that each regulated function is bounded and has at most countably many discontinuities (see, e.g., [2, Chapter 2, Corollary 2.2]).

For \(x \in \mathbb{R}^d, z \in \mathbb{R}^{d \times n}\), we set \(|x|^2 = \sum_{i=1}^d |x_i|^2\), \(|z|^2 = \text{trace}(z^* z)\). \((\cdot, \cdot)\) denotes the usual scalar product in \(\mathbb{R}^d\) and \(\text{sgn}(x) = 1_{\{x \neq 0\}} x/|x|\).

For a fixed stopping time \(\tau\), we denote by \(\Gamma_\tau\) the set of all \(\mathbb{F}\)-stopping times taking values in \([\tau, T]\). We put \(\Gamma := \Gamma_0\). We denote by \(\mathbb{L}^p, p > 0\), the space of random variables \(X\) such that \(\|X\|_p \equiv E(|X|^p)^{1/p} < \infty\). We denote by \(\mathcal{S}\) the set of all \(\mathbb{F}\)-adapted regulated processes, and by \(\mathcal{S}^p, p > 0\), the subset of \(Y \in \mathcal{S}\) such that \(E \sup_{0 \leq t \leq T} |Y_t|^p < \infty\). Given a regulated \(\mathbb{F}\)-adapted process \(X\), we set,

\[
|X|_p = \begin{cases} (E \sup_{t \leq T} |X_t|^p)^{1/(1/p)}, & \text{for } p \neq 1, \\ \sup_{\tau \in \Gamma} E |X_\tau|, & \text{for } p = 1. \end{cases}
\]

\(\mathcal{H}\) is the set of \(\mathbb{F}\)-progressively measurable processes \(X\) such that \(P(\int_0^T |X_t|^2 dt < \infty) = 1\), and \(\mathcal{H}^p, p > 0\), is the set of all \(X \in \mathcal{H}\) such that \(\|X\|_{\mathcal{H}^p} \equiv \left(\int_0^T |X_s|^2 ds\right)^{1/2}\|p < +\infty\).

We say that an \(\mathbb{F}\)-progressively measurable process \(X\) is of class (D) if the family \(\{X_\tau, \tau \in \Gamma\}\) is uniformly integrable. We equip the space of processes of class (D) with the norm \(|| \cdot ||_1\).

For \(\tau \in \Gamma\), we denote by \([\tau]\) the set \(\{(\omega, t) : \tau(\omega) = t\}\). An increasing sequence \(\{\tau_k\} \subset \Gamma\) is called a chain if \(\forall \omega \in \Omega \exists n \in \mathbb{N} \forall k \geq n \tau_k(\omega) = T\).

\(\mathcal{M}\) (resp. \(\mathcal{M}_{(\omega)}\)) is the set of all \(\mathbb{F}\)-martingales (resp. local martingales). \(\mathcal{M}^p, p \geq 1\), denotes the space of all \(M \in \mathcal{M}\) such that \(E([M]_T)^{p/2} < \infty\), where \([M]\) stands for the quadratic variation of \(M\).

\(\mathcal{V}\) (resp. \(\mathcal{V}^+\)) denotes the space of \(\mathbb{F}\)-progressively measurable process of finite variation (resp. increasing) such that \(V_0 = 0\), and \(\mathcal{V}^p\) (resp. \(\mathcal{V}^{p+}\)), \(p \geq 1\), is the set of processes \(V \in \mathcal{V}\) (resp. \(V \in \mathcal{V}^+\)) such that \(E[V_T]^p < \infty\), where \([V]_T\) denotes the total variation of \(V\) on \([0, T]\). For \(V \in \mathcal{V}\), we denote by \(V^*\) the càdlàg part of the process \(V\), and by \(V^d\) its purely jumping part consisting of right jumps, i.e.

\[
V_t^d = \sum_{s < t} \Delta^+ V_s, \quad V_t^* = V_t - V_t^d, \quad t \in [0, T].
\]
Let \( V^1, V^2 \in \mathcal{V} \). We write \( dV^1 \leq dV^2 \) if \( dV^1, dV^2 \) and \( \Delta^+ V^1 \leq \Delta^+ V^2 \) on \([0, T]\).

In the whole paper all relations between random variables hold \( P\text{-a.s.} \). For process \( X \) and \( Y \), we write \( X \leq Y \) if \( X_t \leq Y_t, t \in [0, T] \).

We assume that \( V \in \mathcal{V} \), the barriers \( L, U \) are \( \mathbb{F}\)-adapted optional processes, \( L_T \leq \xi \leq U_T \), and the generator is a map

\[
\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \ni (\omega, t, y, z) \mapsto f(\omega, t, y, z) \in \mathbb{R},
\]

which is \( \mathbb{F}\)-adapted for fixed \( y, z \). We will need the following assumptions.

(H1) There is \( \lambda \geq 0 \) such that \( |f(t, y, z) - f(t, y, z')| \leq \lambda|z - z'| \) for \( t \in [0, T], y \in \mathbb{R}, z, z' \in \mathbb{R}^d \).

(H2) There is \( \mu \in \mathbb{R} \) such that \( (y - y')(f(t, y, z) - f(t, y', z)) \leq \mu(y - y')^2 \) for \( t \in [0, T], y, y' \in \mathbb{R}, z \in \mathbb{R}^d \).

(H3) \( \xi, \int_0^T |f(r, 0, 0)| \, dr, |V|_T \in \mathbb{L}^p \).

(H4) For every \( (t, z) \in [0, T] \times \mathbb{R}^d \) the mapping \( \mathbb{R} \ni y \to f(t, y, z) \) is continuous.

(H5) \( [0, T] \ni t \mapsto f(t, y, 0) \in L^1(0, T) \) for every \( y \in \mathbb{R} \).

(H6) There exists a process \( X \in \mathcal{M}_{loc} + \mathcal{V}^p \) such that \( X \in \mathcal{S}^p, L \leq X \leq U \) and \( \int_0^T |f(r, X_r, 0)| \, dr \in \mathbb{L}^1 \).

(H6*) There exists a process \( X \in \mathcal{M}_{loc} + \mathcal{V}^1 \) such that \( X \) is of class \((D)\), \( L \leq X \leq U \) and \( \int_0^T |f(r, X_r, 0)| \, dr \in \mathbb{L}^1 \).

(Z) There exists a progressively measurable process \( g \) and \( \gamma \geq 0, \alpha \in [0, 1) \) such that

\[
|f(t, y, z) - f(t, y, 0)| \leq \gamma(g_t + |y| + |z|)^\alpha, \quad t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d.
\]

**Remark 2.1.** If \( X \in \mathcal{S} \) and \( X \) is of class \((D)\), then \( X \in \mathcal{S}^q \) for \( q \in (0, 1) \). To see this, we let \( \sigma_a = \inf \{ t \geq 0 : |X_t| > a \} \land T \). Then for \( q \in (0, 1) \) and \( b > 0 \),

\[
E \sup_{t \leq T} |X_t|^q = E \sup_{t \leq T} |X_t^\oplus|^q \leq b + \int_b^\infty P(\sup_{t \leq T} |X_t^\oplus|^q > a) \, da \\
\leq b + \int_b^\infty \frac{E|X_t^\oplus|}{a^{1/q}} \, da \leq b + |X_1| \int_b^\infty \frac{1}{a^{1/q}} \, da.
\]

Taking infimum over \( b > 0 \), we get

\[
E \sup_{t \leq T} |X_t|^q \leq \frac{1}{1 - q} |X_1|^q.
\]

**Definition 2.2.** We say that a pair \((Y, Z)\) of \( \mathbb{F}\)-progressively measurable processes is a solution of BSDE with right-hand side \( f + dV \) and terminal value \( \xi \) (BSDE(\( \xi, f + dV \)) in abbreviation) if

(a) \( Y \) is a regulated process and \( Z \in \mathcal{H} \),

(b) \( \int_0^T |f(r, Y_r, Z_r)| \, dr < \infty \),

(c) \( Y_t = \xi + \int_t^T f(r, Y_r, Z_r) \, dr + \int_t^T dV_r - \int_t^T Z_r \, dB_r, t \in [0, T] \).
The following definition of a solution of reflected BSDE with one optional barrier was introduced in [17].

**Definition 2.3.** We say that a triple \((Y, Z, K)\) of \(\mathbb{F}\)-progressively measurable processes is a solution of the reflected backward stochastic differential equation with right-hand side \(f + dV\), terminal value \(\xi\) and lower barrier \(L\) (RBSDE\((\xi, f + dV, L)\) in abbreviation) if

(a) \(Y\) is a regulated process and \(Z \in \mathcal{H}\),

(b) \(K \in \mathcal{V}^+, L_t \leq Y_t, t \in [0, T]\), and

\[
\int_0^T (Y_{r-} - \limsup_{s \uparrow r} L_s) dK^+_r + \sum_{r < T} (Y_r - L_r) \Delta^+ K_r = 0,
\]

(c) \(\int_0^T |f(r, Y_r, Z_r)| dr < \infty\),

(d) \(Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr + \int_t^T dK_r + \int_t^T dV_r - \int_t^T Z_r dB_r, \quad t \in [0, T]\).

**Definition 2.4.** We say that a triple \((Y, Z, K)\) of \(\mathbb{F}\)-progressively measurable processes is a solution of the reflected backward stochastic differential equation with right-hand side \(f + dV\), terminal value \(\xi\) and upper barrier \(U\) (RBSDE\((\xi, f + dV, U)\) in abbreviation) if \((-Y, -Z, K)\) is a solution of RBSDE\((-\xi, -\tilde{f} - dV, -U)\) with

\[
\tilde{f}(t, y, z) = f(t, -y, -z).
\]

The following theorem and lemma, which are easy modifications of [17, Theorem 2.10] and [17, Lemma 2.8], respectively, will be used in Section 4. We omit their proofs because are the same as the proofs of the corresponding results from [17].

**Theorem 2.5.** Assume that (H1), (H2), (H4), (H5) are satisfied, \((Y^n, Z^n) \in \mathcal{S} \otimes \mathcal{H}, D^n \in \mathcal{V}, K^n \in \mathcal{V}^+, t \mapsto f(t, Y^n_t, Z^n_t) \in L^1(0, T)\) and

\[
Y^n_t = Y^n_0 - \int_0^t f(s, Y^n_s, Z^n_s) ds - \int_0^t dK^n_s + \int_0^t dD^n_s + \int_0^t Z^n_s dB_s
\]

for \(t \in [0, T]\). Moreover, assume that

(a) \(dD^n \leq dD^{n+1}, n \in \mathbb{N}, \sup_{n \geq 0} E|D^n|_T < \infty\),

(b) \(\liminf_{n \to \infty} \left(\int_\sigma^\tau (Y^n_s - Y^n_t) d(K^n_s - D^n_s)^+ + \sum_{\sigma \leq s < \tau} (Y^n_s - Y^n_t) \Delta^+ (K^n_s - D^n_s)\right) \geq 0\) for any \(\sigma, \tau \in \mathcal{T}\) such that \(\sigma \leq \tau\),

(c) there exists a process \(C \in \mathcal{V}^{+1}\) such that \(\Delta^- K^n_t \leq \Delta^- C_t, t \in [0, T]\),

(d) there exist processes \(\underline{y}, \overline{y} \in \mathcal{V}^{+1} + \mathcal{M}_{loc}\) of class (D) such that

\[
E \int_0^T f^+(s, \underline{y}_s, 0) ds + E \int_0^T f^-(s, \overline{y}_s, 0) ds < \infty, \quad \underline{y}_t \leq Y^n_t \leq \overline{y}_t, \quad t \in [0, T],
\]

(e) \(E \int_0^T |f(s, 0, 0)| ds < \infty\),

(f) \(Y^n_t \nearrow Y_t, t \in [0, T]\).
Then $Y$ is regulated, $D \in \mathcal{V}$, where $D_t = \lim_{n \to \infty} D^n_t$, $t \in [0, T]$, and there exist $K \in \mathcal{V}^+$, $Z \in \mathcal{H}$ such that

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds - \int_0^t dK_s + \int_0^t dD_s + \int_0^t Z_s \, dB_s \quad t \in [0, T],$$

and

$$Z^n \to Z \quad dt \otimes \mathbb{P}\text{-a.e.}, \quad \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds \to 0 \quad \text{in probability} \quad \mathbb{P}.$$

Moreover, there exists a chain $\{\tau_k\} \subset \Gamma$ such that for every $p \in (0, 2)$,

$$E \int_0^{\tau_k} |Z^n_s - Z_s|^p \, ds \to 0. \quad (2.1)$$

If $|\Delta^- K_t| = 0$, $t \in [0, T]$, then (2.1) also holds for $p = 2$.

**Lemma 2.6.** Assume that (H1)–(H4) are satisfied. Let $D^n, D \in \mathcal{V}$ and $(Y^n, Z^n), (Y, Z) \in \mathcal{S} \otimes \mathcal{H}$ be such that $t \mapsto f(t, Y^n_t, Z^n_t), t \mapsto f(t, Y_t, Z_t) \in L^1(0, T)$ and

$$Y^n_t = Y^n_0 - \int_0^t f(s, Y^n_s, Z^n_s) \, ds - \int_0^t dD^n_s + \int_0^t Z^n_s \, dB_s, \quad t \in [0, T],$$

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds - \int_0^t dD_s + \int_0^t Z_s \, dB_s, \quad t \in [0, T].$$

If

(a) there exists a chain $\{\tau_k\}$, such that $\sup_{n \geq 0} E((D^n)_{\tau_k})^2 < \infty$,

(b) $\liminf_{n \to \infty}(\int_\sigma^T (Y_n - Y^n_n) \, dD^n_{s+} + \sum_{\sigma \leq s < \tau} (Y_n - Y^n_n) \Delta^+ D^n_s) \geq 0$ for all $\sigma, \tau \in \Gamma$ such that $\sigma \leq \tau$,

(c) there exists $C \in \mathcal{V}^{+,1}$ such that $|\Delta^-(Y_t - Y^n_t)| \leq |\Delta^- C_t|, t \in [0, T]$,

(d) there exist processes $y, \overline{y} \in \mathcal{V}^{+,1} + \mathcal{M}_{loc}$ of class (D) such that

$$\overline{y}_t \leq Y^n_t \leq y_t, \quad t \in [0, T], \quad E \int_0^T f^+(s, \overline{y}_s, 0) \, ds + E \int_0^T f^-(s, y_s, 0) \, ds < \infty,$$

(e) $Y^n_t \to Y_t, t \in [0, T]$,

then

$$Z^n \to Z \quad dt \otimes \mathbb{P}\text{-a.e.}, \quad \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)| \, ds \to 0 \quad \text{in probability} \quad \mathbb{P}$$

and there exists a chain $\{\tau_k\} \subset \Gamma$ such that for all $k \in \mathbb{N}$ and $p \in (0, 2)$,

$$E \int_0^{\tau_k} |Z^n_s - Z_s|^p \, ds \to 0. \quad (2.2)$$

If $\Delta^- C_t = 0$, $t \in [0, T]$, then (2.2) also holds for $p = 2$. 

7
3 Existence and uniqueness of solutions

3.1 Definition of a solution and comparison results

**Definition 3.1.** We say that a triple \((Y, Z, R)\) of \(\mathbb{F}\)-progressively measurable processes is a solution of the reflected backward stochastic differential equation with right-hand side \(f + dV\), terminal value \(\xi\), lower barrier \(L\) and upper barrier \(U\) (RBSDE(\(\xi, f + dV, L, U\))) if

(a) \(Y\) is a regulated process and \(Z \in \mathcal{H}\),

(b) \(R \in \mathcal{V}, L_t \leq Y_t \leq U_t, t \in [0, T]\), and

\[
\int_0^T (Y_{r-} - \limsup_{s \uparrow r} L_s) \, dR_r^+ + \sum_{r < T} (Y_r - L_r) \Delta^+ R_r^+ + \int_0^T (\liminf_{s \uparrow r} U_s - Y_{r-}) \, dR_r^- + \sum_{r < T} (U_r - Y_r) \Delta^+ R_r^- = 0,
\]

where \(R = R^+ - R^-\) is the Jordan decomposition of \(R\),

(c) \(\int_0^T |f(r, Y_r, Z_r)| \, dr < \infty\),

(d) \(Y_t = \xi + \int_0^T f(r, Y_r, Z_r) \, dr + \int_0^T dV_r + \int_0^T dR_r - \int_0^T Z_r \, dB_r, \ t \in [0, T]\).

Note that if \(L, U\) are regulated processes and \((Y, Z, R)\) is a solution to RBSDE(\(\xi, f, L, U\)) then

\[
\Delta^- R_t^+ = (Y_t - L_{t-} + \Delta^- V_t)^-, \quad \Delta^- R_t^- = (Y_t - U_{t-} + \Delta^- V_t)^+,
\]

and

\[
\Delta^+ R_t^+ = (Y_{t+} - L_t + \Delta^+ V_t)^-, \quad \Delta^+ R_t^- = (Y_{t+} - U_t + \Delta^+ V_t)^+.
\]

To check the first equality (the proofs of the other ones are similar) assume first that \(\Delta^- R_t^+ > 0\) and observe that by Definition 3.1(d),

\[
\Delta^- R_t^+ = -\Delta^- Y_t - \Delta^- V_t.
\]

Since by Definition 3.1(b), \(Y_{t-} = L_{t-}\), the desired equality holds true. Now assume that \(\Delta^- R_t^+ = 0\). Since \(\Delta^- R_t^- \geq 0\) and \(Y_{t-} \geq L_{t-}\),

\[
Y_t - L_{t-} + \Delta^- V_t = \Delta^- Y_t + Y_{t-} - L_{t-} + \Delta^- V_t = -\Delta R_t^+ + \Delta^- R_t^- + Y_{t-} - L_{t-} \geq 0,
\]

which completes the proof.

From the above equalities it follows in particular that if the barriers and \(V\) are càdlàg (resp. càglàd), then \(Y\) is càdlàg (resp. càglàd).

**Proposition 3.2.** Let \((Y^i, Z^i, R^i)\) be a solution of RBSDE(\(\xi^i, f^i + dV^i, L^i, U^i\)), \(i = 1, 2\). Assume that \(f^1\) satisfies (H1), (H2) and \(\xi^1 \leq \xi^2, f^1(\cdot, Y^2, Z^2) \leq f^2(\cdot, Y^2, Z^2)\ dt \otimes dP\)-a.s., \(dV^1 \leq dV^2, L^1 \leq L^2, U^1 \leq U^2\). If \((Y^1 - Y^2)^+ \in S^p\) for some \(p > 1\), then \(Y^1 \leq Y^2\).

**Proof.** Without loss of generality we may assume that \(\mu = -\frac{4\lambda^2}{p-1}\) (see [17] Remark 3.2]). By (H1), (H2) and the fact that \(f^1(\cdot, Y^2, Z^2) \leq f^2(\cdot, Y^2, Z^2)\ dt \otimes dP\)-a.s., we have

\[
((Y^1 - Y^2)^+)^{p-1}(f^1(r, Y^1_r, Z^1_r) - f^2(r, Y^2_r, Z^2_r)) \\
\leq ((Y^1 - Y^2)^+)^{p-1}(f^1(r, Y^1_r, Z^1_r) - f^1(r, Y^1_r, Z^2_r)) \\
\leq -\frac{4\lambda^2}{p-1}((Y^1_r - Y^2_r)^+)^p + \lambda((Y^1_r - Y^2_r)^+)^{p-1}|Z^1_r - Z^2_r|.
\]
Note that, by the minimality condition for $R^1, R^2$ and the assumption that $L^1 \leq L^2$ and $U^1 \leq U^2$, \begin{equation}
 1_{\{Y_{\cdot}^1 > Y_{\cdot}^2\}} d(R_r^1 - R_r^2)^+ \leq 1_{\{Y_{\cdot}^1 > Y_{\cdot}^2\}} dR_r^{1,+} + 1_{\{Y_{\cdot}^1 > Y_{\cdot}^2\}} dR_r^{2,+,*} = 0, \tag{3.2}
\end{equation}
and
\begin{equation}
1_{\{Y_{\cdot}^1 > Y_{\cdot}^2\}} \Delta^+(R_r^1 - R_r^2)^- \leq 1_{\{Y_{\cdot}^1 > Y_{\cdot}^2\}} \Delta^+ R_r^{1,+} + 1_{\{Y_{\cdot}^1 > Y_{\cdot}^2\}} \Delta^+ R_r^{1,-} = 0. \tag{3.3}
\end{equation}
By [17] Corollary A.5, for $\tau, \sigma \in \Gamma$ such that $\sigma \leq \tau$ we have \begin{align*}
((Y_\tau^1 - Y_\tau^2)^+) &+ \frac{p(p-1)}{2} \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-2} 1_{\{Y_r^1 > Y_r^2\}} |Z_r^1 - Z_r^2|^2 \, dr \\
& \leq ((Y_\tau^1 - Y_\tau^2)^+) + \frac{p(p-1)}{2} \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-1} (f^1(r, Y_r^1, Z_r^1) - f^2(r, Y_r^2, Z_r^2)) \, dr \\
& + p \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-1} d(V_r^1 - V_r^2)^+ + p \sum_{\sigma \leq \tau < \tau} ((Y_r^1 - Y_r^2)^+) p^{-1} \Delta^+(V_r^1 - V_r^2)^+ \\
& + p \sum_{\sigma \leq \tau < \tau} ((Y_r^1 - Y_r^2)^+) p^{-1} 1_{\{Y_r^1 > Y_r^2\}} d(R_r^1 - R_r^2)^+ \\
& + p \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-1} 1_{\{Y_r^1 > Y_r^2\}} \Delta^+(R_r^1 - R_r^2)^+ \\
& - p \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-1} (Z_r^1 - Z_r^2) dB_r.
\end{align*}
By the above inequality, \((3.1)\)–\((3.3)\) and the assumption that $dV^1 \leq dV^2$, we get
\begin{align*}
((Y_\sigma^1 - Y_\sigma^2)^+) &+ \frac{p(p-1)}{2} \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-2} 1_{\{Y_r^1 > Y_r^2\}} |Z_r^1 - Z_r^2|^2 \, dr \\
& \leq ((Y_\tau^1 - Y_\tau^2)^+) + \frac{4p\lambda^2}{p-1} \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) \, dr + p\lambda \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-1} |Z_r^1 - Z_r^2|^2 \, dr \\
& - p \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-1} (Z_r^1 - Z_r^2) dB_r. \tag{3.4}
\end{align*}
Note that
\begin{align*}
p\lambda((Y_\tau^1 - Y_\tau^2)^+) &p^{-1} |Z_r^1 - Z_r^2| \\
& = p((Y_r^1 - Y_r^2)^+) p^{-2} 1_{\{Y_r^1 > Y_r^2\}} \lambda(Y_r^1 - Y_r^2)^+ |Z_r^1 - Z_r^2| \\
& \leq p((Y_r^1 - Y_r^2)^+) p^{-2} 1_{\{Y_r^1 > Y_r^2\}} \left( \frac{4\lambda^2}{p-1} ((Y_r^1 - Y_r^2)^+) + \frac{p-1}{4} |Z_r^1 - Z_r^2|^2 \right) \\
& \leq \frac{4p\lambda^2}{p-1} ((Y_r^1 - Y_r^2)^+) + \frac{p(p-1)}{4} ((Y_r^1 - Y_r^2)^+) p^{-2} 1_{\{Y_r^1 > Y_r^2\}} |Z_r^1 - Z_r^2|^2.
\end{align*}
From this and \((3.4)\) it follows that
\begin{align*}
((Y_\sigma^1 - Y_\sigma^2)^+) &+ \frac{p(p-1)}{4} \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-2} 1_{\{Y_r^1 > Y_r^2\}} |Z_r^1 - Z_r^2|^2 \, dr \\
& \leq ((Y_\tau^1 - Y_\tau^2)^+) - p \int_{\sigma}^{\tau} ((Y_r^1 - Y_r^2)^+) p^{-1} (Z_r^1 - Z_r^2) dB_r. \tag{3.5}
\end{align*}
Let \( \{\tau_k\} \subset \Gamma \) be a localizing sequence for the local martingale \( \int_\sigma^\tau ((Y^1_r - Y^2_r)^+)^{p-1}(Z^1_r - Z^2_r) dB_r \). By (3.5) with \( \tau \) replaced by \( \tau_k \geq \sigma \), we have
\[
((Y^1_\sigma - Y^2_\sigma)^+) \leq ((Y^1_{\tau_k} - Y^2_{\tau_k})^+) - p \int_\sigma^{\tau_k} ((Y^1_r - Y^2_r)^+)^{p-1}(Z^1_r - Z^2_r) dB_r, \quad k \in \mathbb{N}.
\]
Taking the expectation and then letting \( k \to \infty \), we get \( E((Y^1_\sigma - Y^2_\sigma)^+) = E((\xi^1 - \xi^2)^+) = 0 \). Hence, by the Section Theorem (see, e.g., [6], Chapter IV, Theorem 86)), \( (Y^1_\sigma - Y^2_\sigma)^+ = 0 \), \( t \in [0, T] \).

**Lemma 3.3.** Let \( (Y^i, Z^i, R^i) \) be a solution of RBSDE(\( \xi^i, f^i + dV^i, L^i, U^i \)), \( i = 1, 2 \). Assume that \( f^1 \) satisfies (H2),(Z), \( Y^1, Y^2 \) are of class (D) and \( Z^1, Z^2 \in L^q((0, T) \otimes \Omega) \) for some \( q \in (\alpha, 1] \). Assume also that \( \xi^1 \leq \xi^2 \), \( f^1(\cdot, Y^2, Z^2) \leq f^2(\cdot, Y^2, Z^2) \) \( dt \otimes dP \)-a.s., \( dV^1 \leq dV^2 \), \( L^1 \leq L^2 \), \( U^1 \leq U^2 \). Then \( (Y^1 - Y^2)^+ \in S^p \) for \( p = \frac{q}{\alpha} \).

**Proof.** By [17] Corollary A.5, the assumptions, (H2), (3.2) and (3.3), for all \( \sigma, \tau \in \Gamma \), such that \( \sigma \leq \tau \) we have
\[
(Y^1_\sigma - Y^2_\sigma)^+ \leq (Y^1_\tau - Y^2_\tau)^+ + \int_\sigma^\tau 1_{\{Y^1_r > Y^2_r\}}(f^1(r, Y^1_r, Z^1_r) - f^2(r, Y^2_r, Z^2_r)) dr \\
+ \int_\sigma^\tau 1_{\{Y^1_r > Y^2_r\}} d(V^1_r - V^2_r)^+ + \sum_{\sigma \leq \tau < \tau} 1_{\{\tau^1_r > Y^2_r\}} \Delta^+(V^1_r - V^2_r) \\
+ \int_\sigma^\tau 1_{\{Y^1_r > Y^2_r\}} d(R^1_r - R^2_r)^+ + \sum_{\sigma \leq \tau < \tau} 1_{\{\tau^1_r > Y^2_r\}} \Delta^+(R^1_r - R^2_r) \\
- \int_\sigma^\tau 1_{\{\tau^1_r > Y^2_r\}} (Z^1_r - Z^2_r) dB_r \\
\leq \int_\sigma^\tau 1_{\{\tau^1_r > Y^2_r\}}(f^1(r, Y^2_r, Z^1_r) - f^1(r, Y^2_r, Z^2_r)) dr \\
- \int_\sigma^\tau 1_{\{\tau^1_r > Y^2_r\}} (Z^1_r - Z^2_r) dB_r. \quad (3.6)
\]
By (Z),
\[
|f^1(r, Y^2_r, Z^1_r) - f^1(r, Y^2_r, Z^2_r)| \leq |f^1(r, Y^2_r, Z^1_r) - f^1(r, Y^2_r, 0)| \\
+ |f^1(r, Y^2_r, 0) - f^1(r, Y^2_r, Z^2_r)| \\
\leq 2\gamma (g_r + |Y^1_r| + |Y^2_r| + |Z^1_r| + |Z^2_r|)^\alpha.
\]
Let \( \{\tau_k\} \) be a localizing sequence for the local martingale \( \int_\sigma^\tau 1_{\{\tau^1_r > Y^2_r\}} (Z^1_r - Z^2_r) dB_r \). From the above inequality and (3.6) we get
\[
(Y^1_\sigma - Y^2_\sigma)^+ \leq E((Y^1_\tau - Y^2_\tau)^+) + 2\gamma \int_0^T (g_r + |Y^1_r| + |Y^2_r| + |Z^1_r| + |Z^2_r|)^\alpha dr |F_\sigma).
\]
Since \( Y^1, Y^2 \) are of class (D), \( \{\tau_k\} \) is a chain and \( \xi^1 \leq \xi^2 \), letting \( k \to \infty \) in the above inequality we get
\[
(Y^1_\sigma - Y^2_\sigma)^+ \leq 2\gamma E(\int_0^T (g_r + |Y^1_r| + |Y^2_r| + |Z^1_r| + |Z^2_r|)^\alpha dr |F_\sigma).
\]
Let $p = q/\alpha$. By Doob’s inequality,

$$E \sup_{t \leq T} (Y_t^1 - Y_t^2)^+ \leq C_p E \left( \int_0^T (g_r + |Y_r^1| + |Y_r^2| + |Z_r^1| + |Z_r^2|)^q \, dr \right).$$

Hence $(Y^1 - Y^2)^+ \in S^p$.

**Remark 3.4.** Observe that if $f^1$, $f^2$ do not depend on $z$, then in Proposition 3.2, it is enough to assume that $(Y^1 - Y^2)^+$ is of class (D).

**Proposition 3.5.** Let $(Y, Z, R)$ be a solution of $\text{RBSDE}(\xi, f + dV, L, U)$. Assume that $p > 1$, $(H1)$, $(H2)$, $(H3)$ are satisfied, $Y \in S^p$, $Z \in H^p$, $R \in V^p$ or $p = 1$, $(H1)$, $(H2)$, $(H3)$, $(Z)$ are satisfied, $Y$ is of class (D), $Z \in H^q$, $q \in (0, 1)$ and $R \in V^1$. Then,

$$E \left( \int_0^T |f(r, Y_r, Z_r)| \, dr \right)^p < \infty. \quad (3.7)$$

**Proof.** We may assume that $\mu = 0$. By [17, Corollary A.5], for all $\sigma, \tau \in \Gamma$ such that $\sigma \leq \tau$, we have

$$|Y_\sigma| \leq |Y_\tau| + \int_{\sigma}^{\tau} \sgn(Y_r) f(r, Y_r, Z_r) \, dr + \int_{\sigma}^{\tau} \sgn(Y_r-) \, dV_r + \int_{\sigma}^{\tau} \sgn(Y_r-) \, dR_r^r + \sum_{\sigma \leq r < \tau} \sgn(Y_r) \Delta^+ R_r - \int_{\sigma}^{\tau} \sgn(Y_r) Z_r \, dB_r \quad (3.8)$$

By $(Z)$ and $(H2),

$$\sgn(Y_\tau) f(r, Y_r, Z_r) \leq -|f(r, Y_r, Z_r)| + 2|g_r + |Z_r|^{-1}| + 2|f(r, 0, 0)|. \quad (3.9)$$

whereas by $(H1)$ and $(H2),

$$\sgn(Y_\tau) f(r, Y_r, Z_r) \leq -|f(r, Y_r, Z_r)| + 2|Z_r| + 2|f(r, 0, 0)|. \quad (3.10)$$

From (3.8)–(3.10) and the assumptions we get the desired result. \hfill \Box

### 3.2 Existence of solutions for $f$ independent of $z$

**Theorem 3.6.** Assume that $f$ is independent of $z$. If $p > 1$ and $(H1)$–(H6) (resp. $p = 1$ and $(H1)$–(H5), (H6*)) are satisfied, then there exists a unique solution $(Y, Z, R)$ of $\text{RBSDE}(\xi, f + dV, L, U)$ such that $Y \in S^p$, $Z \in H^p$ and $R \in V^p$ (resp. $Y$ is of class (D), $Z \in H^q$, $q \in (0, 1)$, and $R \in V^1$).

**Proof.** Without loss of generality we may assume that $\mu = 0$. Let $(Y^{1,0}, Z^{1,0})$ be a solution of BSDE$(\xi, f + dV)$ such that if $p > 1$, $Y^{1,0} \in S^p$, $Z^{1,0} \in H^p$ and if $p = 1$, $Y^{1,0}$ is of class (D), $Z^{1,0} \in H^q$, $q \in (0, 1)$. Set $(Y^{2,0}, Z^{2,0}) = (0, 0)$. Moreover, for each $n \geq 1$ let $(Y^{1,n}, Z^{1,n}, K^{1,n})$ be a solution of $\text{RBSDE}(\xi, f_n + dV, L + Y^{1,n-1})$ with

$$f_n(r, y) = f(r, y - Y_r^{2,n-1}),$$

and let $(Y^{2,n}, Z^{2,n}, K^{2,n})$ be a solution of $\text{RBSDE}(0,0,Y^{1,n-1} - U)$ such that if $p > 1$, $Y^{1,n}, Y^{2,n} \in S^p$, $Z^{1,n}, Z^{2,n} \in H^p$, $K^{1,n}, K^{2,n} \in V^+ p$, and if $p = 1$, then $Y^{1,n}, Y^{2,n}$ are of class (D), $Z^{1,n}, Z^{2,n} \in H^q$, $q \in (0, 1)$, $K^{1,n}, K^{2,n} \in V^+ 1$. For each $n \geq 0$ the existence of the
above solutions follows from [17, Theorem 3.20]. In both cases \((p > 1, p = 1)\), by Proposition 3.5 we have
\[
E\left(\int_0^T |f(r, Y_r^{1,n} - Y_r^{2,n-1})| \, dr\right)^p < \infty.
\] (3.11)

The rest of the proof we divide into 4 steps.

**Step 1.** We show that the sequences \((Y^{1,n})_{n \geq 0}, (Y^{2,n})_{n \geq 0}\) are increasing. We proceed by induction. Clearly \(Y^{1,1} \geq Y^{1,0}\) and \(Y^{2,1} \geq Y^{2,0}\). Suppose that \(Y^{1,n} \geq Y^{1,n-1}\) and \(Y^{2,n} \geq Y^{2,n-1}\). Using (H2) we show that \(f_{n+1} \geq f_n\) and \(L + Y^{2,n} \geq L + Y^{2,n-1}\). Hence, by Proposition 3.2 and Remark 3.3, \(Y^{1,n+1} \geq Y^{1,n}\). By a similar argument, \(Y^{2,n+1} \geq Y^{2,n}\), so \((Y^{1,n})_{n \geq 0}, (Y^{2,n})_{n \geq 0}\) are increasing.

**Step 2.** Let \(Y^1 := \sup_{n \geq 1} Y^{1,n}, Y^2 := \sup_{n \geq 1} Y^{2,n}\). We show that \(Y^1, Y^2 \in \mathcal{S}^p\) if \(p > 1\), and if \(p = 1\), then \(Y^1, Y^2\) are of class (D). Let \(p > 1\). By (H6), there exists a process \(X \in (\mathcal{M}_{\text{loc}} + \mathcal{V}^p) \cap \mathcal{S}^p\) such that \(X \geq L\) and \(\int_0^T f^-(r, X_r, 0) \, dr \in \mathbb{L}^p\). If \(p = 1\), then by (H6*), there exists \(X\) of class (D) such that \(X \in \mathcal{M}_{\text{loc}} + \mathcal{V}^1, X \geq L\) and \(\int_0^T f^-(r, X_r, 0) \, dr \in \mathbb{L}^1\). Since the Brownian filtration has the representation property, there exist processes \(H \in \mathcal{M}_{\text{loc}}\) and \(C \in \mathcal{V}^p\) such that
\[
X_t = X_T - \int_t^T dC_r - \int_t^T H_r \, dB_r, \quad t \in [0, T].
\]

This equality can be written in the form
\[
\tilde{X}_t = \xi + \int_t^T f(r, \tilde{X}_r) \, dr + \int_t^T dV_r + \int_t^T dC'_r - \int_t^T H_r \, dB_r,
\]
where \(C'\) is some process in \(\mathcal{V}^p\), \(\tilde{X}_t = X_t, t \in [0, T]\), \(\tilde{X}_T = \xi\). Let \((\tilde{X}^1, \tilde{H}^1)\) be a solution of the following BSDE
\[
\tilde{X}^1_t = \xi + \int_t^T f(r, \tilde{X}_r) \, dr + \int_t^T dV_r + \int_t^T dC'_r - \int_t^T \tilde{H}_r \, dB_r, \quad t \in [0, T],
\]
and \((\tilde{X}^2, \tilde{H}^2)\) be a solution of the BSDE
\[
\tilde{X}^2_t = \int_t^T dC'_r - \int_t^T \tilde{H}_r \, dB_r, \quad t \in [0, T],
\]

such that if \(p > 1\), then \(\tilde{X}^1, \tilde{X}^2 \in \mathcal{S}^p, \tilde{H}^1, \tilde{H}^2 \in \mathbb{H}^p\), and if \(p = 1\), then \(\tilde{X}^1, \tilde{X}^2\) are of class (D), \(\tilde{H}^1, \tilde{H}^2 \in \mathbb{H}^q, \, q \in (0, 1)\). The existence of such solutions follows from [17, Theorem 3.20]. Let us note that \(\tilde{X} = \tilde{X}^1 - \tilde{X}^2\). It is easy to see that \((\tilde{X}^1, \tilde{H}^1, 0)\) is a solution of RBSDE(\(\xi, f_d + dV + dC^{++}, L + \tilde{X}^2\)) with \(\tilde{f}(r, x) = f(r, x - \tilde{X}^2)\) and \((\tilde{X}^2, \tilde{H}^2, 0)\) is a solution of RBSDE(0, \(dC^{--}, \tilde{X}^1 - U\)). Proceeding by induction we will show that for each \(n \in \mathbb{N}\), \(\tilde{X}^1 \geq Y^{1,n}\) and \(\tilde{X}^2 \geq Y^{2,n}\). For \(n = 0\), since \(\tilde{X}^2 \geq 0\), using (H2) we get \(\tilde{f} \geq f\). Hence, by Proposition 3.2 and Remark 3.3, \(\tilde{X}^1 \geq Y^{1,0}\). It is clear that \(\tilde{X}^2 \geq Y^{2,0}\) since \(Y^{2,0} = 0\). Suppose that \(\tilde{X}^1 \geq Y^{1,n}\) and \(\tilde{X}^2 \geq Y^{2,n}\). Using (H2) we show that \(f \geq f_{n+1}\), \(L + \tilde{X}^2 \geq L + Y^{2,n}, Y^{1,n} - U \leq \tilde{X} - U\). Hence by Proposition 3.2 and Remark 3.4, \(\tilde{X}^2 \geq Y^{1,n+1}, \tilde{X}^2 \geq Y^{2,n+1}\), so for each \(n \in \mathbb{N}\), \(\tilde{X}^1 \geq Y^{1,n}\) and \(\tilde{X}^2 \geq Y^{2,n}\). We have
\[
Y^{1,0} \leq Y^{1,n} \leq \tilde{X}^1, \quad Y^{2,0} \leq Y^{2,n} \leq \tilde{X}^2.
\] (3.12)

Therefore \(\tilde{Y}^1, \tilde{Y}^2 \in \mathcal{S}^p\) for \(p > 1\), and if \(p = 1\), then \(Y^1, Y^2\) are of class (D).
Step 3. We will show that there exist \( Z^1, Z^2 \in \mathcal{H}^p, K^1, K^2 \in \mathcal{V}^p \) if \( p > 1 \), and \( Z^1, Z^2 \in \mathcal{H}^q, q \in (0,1), K^1, K^2 \in \mathcal{V}^1 \) if \( p = 1 \) such that \((Y^1,Z^1,K^1)\) is a solution of \( \text{RBSDE}(\xi,f + dV, L + Y^2) \) with \( f(r,y) = f(r,y - Y^2) \) and \((Y^2,Z^2,K^2) \) is a solution of \( \text{RBSDE}(0,Y^1-U) \).

By (H4), \( f(r,Y^{1,n} - Y^{2,n-1}) \to f(r,Y^{1,0} - Y^2) \) as \( n \to \infty \). Furthermore, by (H2) and (3.12),
\[
 f(r,X^1_r) \leq f(r,Y^{1,n} - Y^{2,n-1}) \leq f(r,Y^{1,0} - X^2_r),
\]
Hence, by (H2), (H5) and the Lebesgue dominated convergence theorem,
\[
\int_0^T |f(r,Y^{1,n} - Y^{2,n-1})| dr - f(r,Y^{1,0} - Y^2) | dr \to 0.
\]
Observe that
\[
S^n_t = Y^{1,n}_t + \int_0^t f(r,Y^{1,n}_r - Y^{2,n-1}_r) dr + V_t \\
\geq L_t + Y^{2,n-1}_t + \int_0^t f(r,Y^{1,n}_r - Y^{2,n-1}_r) dr + V_t =: \bar{L}_t, \quad t \in [0,T],
\]
and that (3.11) implies that \( S^n \) is a supermartingale of class (D) on \([0,T] \). Letting \( n \to \infty \) in the above inequality we get
\[
S_t := Y^1_t + \int_0^t f(r,Y^1_r - Y^2_r) dr + V_t \geq L_t + Y^2_t + \int_0^t f(r,Y^1_r - Y^2_r) dr + V_t =: \bar{L}_t, \quad t \in [0,T].
\]
Set \( \tau_k = \inf \{ t \geq 0; \int_0^t |f(r,Y^1_r - Y^2_r)| dr \leq k \} \wedge T, \quad k \in \mathbb{N} \). By (H5), \( \{ \tau_k \} \) is a chain. From the definition \( \{ \tau_k \} \), and (3.12), (3.13) it follows that \( S \) is a supermartingale of class (D) on \([0,\tau_k]\), \( k \geq 0 \). It is clear that \( S \) majorizes \( \bar{L} \), so for \( \sigma \in \Gamma \) we have
\[
S_{\sigma \wedge \tau_k} \geq \text{ess sup}_{\tau \in \Gamma_{\sigma \wedge \tau_k}} E(\bar{L}_{\tau \wedge \tau_k} | \mathcal{F}_{\sigma \wedge \tau_k}).
\]
Hence
\[
Y^1_{\sigma \wedge \tau_k} \geq \text{ess sup}_{\tau \in \Gamma_{\sigma \wedge \tau_k}} E \left( \int_{\sigma \wedge \tau_k}^{\tau \wedge \tau_k} f(r,Y^1_r - Y^2_r) dr + \int_{\sigma \wedge \tau_k}^{\tau \wedge \tau_k} dV_r + (L_{\tau} + Y^2_{\tau}) 1_{\{\tau < \tau_k\}} + Y^1_{\tau_k} 1_{\{\tau = \tau_k\}} \right) | \mathcal{F}_{\sigma \wedge \tau_k}.
\]
(3.15)
To show the opposite inequality, we first note that the triple \((Y^{1,n},Z^{1,n},K^{1,n})\) is a solution of \( \text{RBSDE}(Y^{1,n}_\tau,f_n + dV, L + Y^{2,n-1}_\tau) \) on \([0,\tau_k]\), so by [17 Proposition 3.13], for \( \sigma \in \Gamma \) we have
\[
Y^{1,n}_{\sigma \wedge \tau_k} = \text{ess sup}_{\tau \in \Gamma_{\sigma \wedge \tau_k}} E \left( \int_{\sigma \wedge \tau_k}^{\tau \wedge \tau_k} f(r,Y^{1,n}_r - Y^{2,n-1}_r) dr + \int_{\sigma \wedge \tau_k}^{\tau \wedge \tau_k} dV_r \\
+ (L_{\tau} + Y^{2,n-1}_\tau) 1_{\{\tau < \tau_k\}} + Y^1_{\tau_k} 1_{\{\tau = \tau_k\}} | \mathcal{F}_{\sigma \wedge \tau_k} \right).
\]
(3.16)
By the definition of \( \tau_k \) and (3.13),
\[
E \int_0^{\tau_k} |f(r, Y_r^{1,n} - Y_r^{2,n-1}) - f(r, Y_r^1 - Y_r^2)| \, dr \to 0.
\]

By (3.16), (3.17) and [17] Lemma 3.19,
\[
Y_{\sigma \land \tau_k}^1 \leq \text{ess sup}_{\tau \in \Gamma_{\sigma \land \tau_k}} \left( \int_0^{\tau \land \tau_k} f(r, Y_r^1 - Y_r^2) \, dr + \int_{\sigma \land \tau_k}^{\tau \land \tau_k} dV_r + (L_\tau + Y_\tau^2)1_{\{\tau < \tau_k\}} + Y_{\tau_k}^1 1_{\{\tau = \tau_k\}} |\mathcal{F}_{\sigma \land \tau_k} \right).
\]

By the above inequality and (3.15),
\[
Y_{\sigma \land \tau_k}^1 \leq \text{ess sup}_{\tau \in \Gamma_{\sigma \land \tau_k}} \left( \int_0^{\tau \land \tau_k} f(r, Y_r^1 - Y_r^2) \, dr + \int_{\sigma \land \tau_k}^{\tau \land \tau_k} dV_r + (L_\tau + Y_\tau^2)1_{\{\tau < \tau_k\}} + Y_{\tau_k}^1 1_{\{\tau = \tau_k\}} |\mathcal{F}_{\sigma \land \tau_k} \right).
\]

On the other hand, by [17] Proposition 3.13, for every \( \sigma \in \Gamma \),
\[
Y_{\sigma}^{2,n} = \text{ess sup}_{\tau \in \Gamma_{\sigma}} \left( (Y_r^{1,n-1} - U_\tau)1_{\{\tau < T\}} |\mathcal{F}_{\sigma} \right).
\]

Since \( \{Y^{1,n}\}, \{Y^{2,n}\} \) are nondecreasing, letting \( n \to \infty \) and using standard properties of the Snell envelope we obtain
\[
Y_\sigma^2 = \text{ess sup}_{\tau \in \Gamma_{\sigma}} \left( (Y_r^1 - U_\tau)1_{\{\tau < T\}} |\mathcal{F}_{\sigma} \right).
\]

We have showed that \( S \) is a supermartingale on \([0, \tau_k]\), so by the Mertens decomposition there exist \( K^{1,k} \in Y^{1,+}, Z^{1,k} \in \mathcal{H} \) such that
\[
Y_t^1 = Y_{\tau_k}^1 + \int_t^{\tau_k} f(r, Y_r^1 - Y_r^2) \, dr + \int_t^{\tau_k} dV_r + \int_t^{\tau_k} dK_r^{1,k} - \int_t^{\tau_k} Z_r^{1,k} dB_r, \quad t \in [0, \tau_k].
\]

By [17] Corollary 3.11,
\[
\int_0^{\tau_k} (Y_r^1 - \lim_{s \uparrow r} \sup(L_s + Y_s^2)) \, dK_r^{1,k} + \sum_{r < \tau_k} (Y_r^1 - (L_r + Y_r^2)) \Delta^+ K_r^{1,k} = 0.
\]

Therefore \( (Y^1, Z^{1,k}, K^{1,k}) \) is a solution of RBSDE\( (Y_{\tau_k}^1, f + dV, L + Y^2) \) on \([0, \tau_k]\). By uniqueness, \( K_t^{1,k} = K_t^{1,k+1} \) and \( Z_t^{1,k} = Z_t^{1,k+1} \) for \( t \in [0, \tau_k] \), so using the fact that \( \{\tau_k\} \) is a chain we can define processes \( Z^1 \) and \( K^1 \) on \([0, T]\) by putting \( Z_t^1 = Z_t^{1,k}, K_t^1 = K_t^{1,k}, t \in [0, \tau_k] \).

We see that \( (Y^1, Z^1, K^1) \) is a solution of RBSDE\( (\xi, f + dV, L + Y^2) \) on \([0, T]\). By [8], \( Y^2 \) is a supermartingale, so by the Mertens decomposition, there exist \( K^2 \in V^{1,+}, Z^2 \in \mathcal{H} \) such that
\[
Y_t^2 = \int_t^T dK_r^2 - \int_t^T Z_r^2 dB_r, \quad t \in [0, T],
\]
and by [17] Corollary 3.11,
\[
\int_0^T (Y_r^2 - \lim_{s \uparrow r} \sup(Y_s^1 - U_s)) \, dK_r^{2,s} + \sum_{r < T} (Y_r^2 - (Y_s^1 - U_s)) \Delta^+ K_r^2 = 0.
\]
Therefore \((Y^2, Z^2, K^2)\) is a solution of \(\text{RBSDE}(0, 0, Y^1 - U)\) on \([0, T]\). By [17, Theorem 3.20] and Remark 3.4, \(Z^1, Z^2 \in \mathcal{H}^p, K^1, K^2 \in \mathcal{S}^p\) if \(p \geq 1\), and \(Z^1, Z^2 \in \mathcal{H}^q, q \in (0, 1)\) and \(K^1, K^2 \in \mathcal{V}^{1, +}\) if \(p = 1\).

**Step 4.** Write \(Y := Y^1 - Y^2, Z := Z^1 - Z^2, R := K^1 - K^2\). We will show that \((Y, Z, R)\) is a solution of \(\text{RBSDE}(\xi, f + dV, L, U)\). We have

\[
Y_t = \xi + \int_t^T f(r, Y_r) \, dr + \int_t^T dV_r + \int_t^T dR_r - \int_t^T Z_r \, dB_r, \quad t \in [0, T].
\]

Obviously \(L \leq Y \leq U\). The process \(R\) satisfies the minimality condition because

\[
\int_0^T (Y_{r-} - \limsup_{s \uparrow r} L_s) \, dR_r^+ + \sum_{r < T} (Y_r - L_r) \Delta^+ R_r^+ \\
\leq \int_0^T (Y_{r-}^1 - \limsup_{s \uparrow r} (L_s + Y^2_s)) \, dK_r^1 + \sum_{r < T} (Y_r^1 - (L_r + Y^2_r)) \Delta^+ K_r^1 = 0
\]

and

\[
\int_0^T (\liminf_{s \uparrow r} U_s - Y_{r-}) \, dR_r^- + \sum_{r < T} (U_r - Y_r) \Delta^+ R_r^- \\
\leq \int_0^T (Y_{r-}^2 - \limsup_{s \uparrow r} (Y_s^1 - U_s)) \, dK_r^2 + \sum_{r < T} (Y_r^2 - (Y^1_r - U_r)) \Delta^+ K_r^2 = 0.
\]

The desired integrability of \(Y, Z\) and \(R\) follows from Step 2 and Step 3. Furthermore,

\[
E\left(\int_0^T |f(r, Y_r)| \, dr\right)^p < \infty, \quad p \geq 1.
\]

Indeed, by Proposition 3.2 and Remark 3.4, \(Y \leq Y \leq \bar{Y}\), where \(
\bar{Y}\) is the first component of a solution of \(\text{RBSDE}(\xi, f + dV, U)\) and \(Y\) is the first component of a solution of \(\text{RBSDE}(\xi, f + dV, L)\). By this and (H2), \(f(r, Y_r) \leq f(r, \bar{Y}_r) \leq f(r, Y_r)\). Since by [17, Theorem 4.1],

\[
E\left(\int_0^T |f(r, \bar{Y}_r)| \, dr\right)^p < \infty, \quad p \geq 1,
\]

the proof is complete.

We close this subsection with estimates for the difference of solutions of RBSDEs with generators not depending on \(z\). We will use them in the next subsection to study the existence of solutions of general RBSDEs.

**Proposition 3.7.** Let \((Y^i, Z^i, R^i)\) be solutions of \(\text{RBSDE}(\xi, f^i + dV, L, U)\), \(i = 1, 2\), where \(f^1, f^2\) do not depend on \(z\). Assume that \(f^1\) satisfies (H2). If \((Y^1 - Y^2) \in \mathcal{S}^p\) and \(\int_0^T |f^1(r, Y^2_r) - f^2(r, Y^2_r)| \, dr \in \mathcal{L}^p\) for some \(p > 1\), then \((Z^1 - Z^2) \in \mathcal{H}^p\) and there exists a constant \(C_p\) depending only on \(p\) such that

\[
E\left\{\sup_{t \leq T} |Y_t^1 - Y_t^2|^p + \int_0^T |Z_t^1 - Z_t^2| \, dr \right\}^{p/2} \leq C_p E\left(\int_0^T |f^1(r, Y^2_r) - f^2(r, Y^2_r)| \, dr\right)^p.
\]
Proof. Without loss of generality we may assume that $\mu \leq 0$. We know that
\[
Y_t^1 - Y_t^2 = \int_t^T \hat{f}(r, Y_r^1 - Y_r^2) \, dr + \int_t^T d(R_r^1 - R_r^2) - \int_t^T (Z_r^1 - Z_r^2) \, dB_r, \quad t \in [0, T],
\]
where $\hat{f}(r, y) = f^1(r, y + Y_r^2) - f^2(r, Y_r^2)$. Define $\tau_k = \inf \left\{ t \in [0, T] : \int_0^t |Z_r^1 - Z_r^2|^2 \, dr \geq k \right\} \wedge T, \ k \in \mathbb{N}$. By [17, Corollary A.5]
\[
|Y_t^1 - Y_t^2|^2 + \int_t^{\tau_k} |Z_r^1 - Z_r^2|^2 \, dr \leq |Y_{\tau_k}^1 - Y_{\tau_k}^2|^2 + 2 \int_t^{\tau_k} (Y_r^1 - Y_r^2)\hat{f}(r, Y_r^1 - Y_r^2) \, dr
+ 2 \int_t^{\tau_k} (Y_r^1 - Y_r^2)(R_r^1 - R_r^2)^* + 2 \sum_{t \leq r < \tau_k} (Y_r^1 - Y_r^2)\Delta^+(R_r^1 - R_r^2)
- 2 \int_t^{\tau_k} (Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r.
\] (3.19)

By the minimality condition and the fact that $U \geq Y^1 \geq L$ and $U \geq Y^2 \geq L$,
\[
\int_t^{\tau_k} (Y_r^1 - Y_r^2)(R_r^1 - R_r^2)^* + \sum_{t \leq r < \tau_k} (Y_r^1 - Y_r^2)\Delta^+(R_r^1 - R_r^2)
\leq \int_t^{\tau_k} (Y_r^1 - \limsup_{s \uparrow r} L_s) dR_r^{1,*+} + \int_t^{\tau_k} (Y_r^2 - \limsup_{s \uparrow r} L_s) dR_r^{2,*+}
+ \sum_{t \leq r < \tau_k} (Y_r^1 - L_r)\Delta^+ R_r^{1,+} + \sum_{t \leq r < \tau_k} (Y_r^2 - L_r)\Delta^+ R_r^{2,+}
+ \int_t^{\tau_k} (\liminf_{s \uparrow r} U_s - Y_r^1) dR_r^{1,*-} + \int_t^{\tau_k} (\liminf_{s \uparrow r} U_s - Y_r^2) dR_r^{2,*-}
+ \sum_{t \leq r < \tau_k} (U_r - Y_r^1)\Delta^+ R_r^{-1} + \sum_{t \leq r < \tau_k} (U_r - Y_r^2)\Delta^+ R_r^{-2} \leq 0.
\] (3.20)

Since $\mu \leq 0$, using (H2) we get
\[
2y\hat{f}(r, y) \leq 2\mu|y|^2 + 2|y|\|\hat{f}(r, 0)\| \leq 2|y||\hat{f}(r, 0)| = 2|y||f^1(r, Y_r^2) - f^2(r, Y_r^2)|.
\] (3.21)

By (3.19)–(3.21),
\[
\int_0^{\tau_k} |Z_r^1 - Z_r^2|^2 \, dr \leq \sup_{t \leq T} |Y_t^1 - Y_t^2|^2 + 2 \int_0^T |Y_r^1 - Y_r^2||f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr
+ 2 \left| \int_0^{\tau_k} (Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r \right|.
\]

Since
\[
2 \int_0^T |Y_r^1 - Y_r^2||f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr
\leq 2 \sup_{t \leq T} |Y_t^1 - Y_t^2| \int_0^T |f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr
\leq \sup_{t \leq T} |Y_t^1 - Y_t^2|^2 + \left( \int_0^T |f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr \right)^2,
\]

16
it follows from the above that
\[
\left( \int_0^{t_k} |Z_r^1 - Z_r^2|^2 \, dr \right)^{p/2} \leq C_p \left( \sup_{t \leq T} |Y_t^1 - Y_t^2|^p + \left( \int_0^T |f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr \right)^p \right.
\]
\[
\left. + \left| \int_0^{t_k} (Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r \right|^{p/2} \right). \tag{3.22}
\]
By the Burkholder-Davis-Gundy inequality,
\[
E \left| \int_0^{t_k} (Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r \right|^{p/2} \leq D_p E \left( \int_0^{t_k} |Y_r^1 - Y_r^2|^2 |Z_r^1 - Z_r^2|^2 \, dr \right)^{p/4}
\]
\[
\leq D_p E \left[ \sup_{t \leq T} |Y_t^1 - Y_t^2|^p \right. \left. \left( \int_0^{t_k} |Z_r^1 - Z_r^2|^2 \, dr \right)^{p/2} \right].
\]
Hence
\[
E \left| \int_0^{t_k} (Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r \right|^{p/2} \leq \frac{D_p^2}{2} E \sup_{t \leq T} |Y_t^1 - Y_t^2|^p + \frac{1}{2} E \left( \int_0^{t_k} |Z_r^1 - Z_r^2|^2 \, dr \right)^{p/2}.
\]
From the above and (3.22), for every \( k \geq 1 \),
\[
E \left( \int_0^T |Z_r^1 - Z_r^2|^2 \, dr \right)^{p/2} \leq C_p E \left( \sup_{t \leq T} |Y_t^1 - Y_t^2|^p \right. \left. + \left( \int_0^T |f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr \right)^p \right), \tag{3.23}
\]
Letting \( k \to \infty \) and applying Fatou’s lemma yields
\[
E \left( \int_0^T |Z_r^1 - Z_r^2|^2 \, dr \right)^{p/2} \leq C_p E \left( \sup_{t \leq T} |Y_t^1 - Y_t^2|^p \right. \left. + \left( \int_0^T |f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr \right)^p \right), \tag{3.23}
\]
so \( (Z^1 - Z^2) \in H^p \). To show the desired estimate for \( Y^1 - Y^2 \), we use once again [17, Corollary A.5]. We have
\[
|Y_t^1 - Y_t^2| \leq \int_t^T \text{sgn}(Y_r^1 - Y_r^2) \check{f}(r, Y_r^1 - Y_r^2) \, dr + \int_t^T \text{sgn}(Y_r^1 - Y_r^2) \, d(R_r^1 - R_r^2)^* + \sum_{t \leq r < T} \text{sgn}(Y_r^1 - Y_r^2) \Delta^+(R_r^1 - R_r^2) - \int_t^T \text{sgn}(Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r. \tag{3.24}
\]
Since \( f^1 \) satisfies (H2) and \( \mu \leq 0 \),
\[
\text{sgn}(y) \check{f}(y) \leq \lambda \mu |y| + |\check{f}(r, 0)| \leq |\check{f}(r, 0)| = |f^1(r, Y_r^2) - f^2(r, Y_r^2)|. \tag{3.25}
\]
Since \( \text{sgn}(Y_r^1 - Y_r^2) = |Y_r^1 - Y_r^2|^{-1}(Y_r^1 - Y_r^2) \mathbf{1}_{\{Y_r^1 - Y_r^2 \neq 0\}} \), it follows from (3.20), (3.24) and (3.25) that
\[
|Y_t^1 - Y_t^2| \leq \int_0^T |f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr - \int_0^T \text{sgn}(Y_r^1 - Y_r^2)(Z_r^1 - Z_r^2) \, dB_r.
\]
Hence
\[
|Y_t^1 - Y_t^2| \leq E \left( \int_0^T |f^1(r, Y_r^2) - f^2(r, Y_r^2)| \, dr \, \mathcal{F}_t \right).
\]
since \( \int_0^\infty \text{sgn}(Y^1_r - Y^2_r)(Z^1_r - Z^2_r) dB_r \) is a uniformly integrable martingale. This implies that

\[
E \sup_{t \leq T} |Y^1_t - Y^2_t|^p \leq E \left( \sup_{t \leq T} E \left( \int_0^T |f^1(r, Y^2_r) - f^2(r, Y^2_r)| dr |\mathcal{F}_t \right)^p \right) \\
\leq C_p E \left( \int_0^T |f^1(r, Y^2_r) - f^2(r, Y^2_r)| dr \right)^p.
\]

Combining the above inequality with (3.23) completes the proof. \( \square \)

3.3 Existence of solutions for general \( f \)

**Theorem 3.8.** Assume that \( p = 1 \) and (H1)-(H5), (H6*), (Z) are satisfied. Then there exists a unique solution \( (Y, Z, R) \) of RBSDE(\( \xi, f + dV, L, U \)) such that \( Y \) is of class (D), \( Z \in \mathcal{H}^q, q \in (0, 1) \) and \( R \in \mathcal{V}^1 \).

**Proof.** Without loss of generality we may assume that \( \mu = 0 \). We will use Picard’s iteration method. Set \( (Y^0, Z^0, R^0) = (0, 0, 0) \) and for \( n \geq 0 \) define

\[
Y^{n+1}_t = \xi + \int_t^T f(r, Y^{n+1}_r, Z^n_r) dr + \int_t^T dV_r + \int_t^T dR^{n+1}_r - \int_t^T Z^{n+1}_r dB_r, \quad t \in [0, T].
\]

Let \( f_n(r, y) = f(r, y, Z^{n-1}_r) \). For each \( n \geq 0 \), \( (Y^n, Z^n, R^n) \) is a solution of RBSDE(\( \xi, f_n + dV, L, U \)) such that \( Y^n \) is of class (D), \( Z \in \mathcal{H}^q, q \in (0, 1) \), and \( R \in \mathcal{V}^1 \). The existence of this solution follows from Theorem 2.7, since by induction, \( E \left[ \int_0^T |f_n(r, 0)| + |f_n(r, X_r)| dr \right] < \infty \), \( n \geq 1 \). By [17] Corollary A.5 and the minimality conditions for \( R^{n+1} \) and \( R^n \), for any \( \sigma, \tau \in \Gamma \), such that \( \sigma \leq \tau \) we have

\[
|Y^{n+1}_\sigma - Y^n_\sigma| \leq \int_\sigma^\tau \text{sgn}(Y^{n+1}_r - Y^n_r)(f(r, Y^{n+1}_r, Z^n_r) - f(r, Y^n_r, Z^{n-1}_r)) dr \\
- \int_\sigma^\tau \text{sgn}(Y^{n+1}_r - Y^n_r)(Z^{n+1}_r - Z^n_r) dB_r, \quad t \in [0, T]. \tag{3.26}
\]

By (H2) and (Z),

\[
\text{sgn}(Y^{n+1}_r - Y^n_r)(f(r, Y^{n+1}_r, Z^n_r) - f(r, Y^n_r, Z^{n-1}_r)) \leq \text{sgn}(Y^{n+1}_r - Y^n_r)(f(r, Y^n_r, Z^n_r) - f(r, Y^n_r, Z^{n-1}_r)) \leq 2\gamma (g_r + |Y^n_r| + |Z^n_r| + |Z^{n-1}_r|) \alpha.
\]

Let \( \{\tau_k\} \) be a localizing sequence for the local martingale \( \int_\sigma^\tau \text{sgn}(Y^{n+1}_r - Y^n_r)(Z^{n+1}_r - Z^n_r) dB_r \).

By the above inequality and (3.26), for \( n \geq 1 \) we have

\[
|Y^{n+1}_\sigma - Y^n_\sigma| \leq E \left( |Y^{n+1}_{\tau_k} - Y^n_{\tau_k}| + 2\gamma \int_0^T (g_r + |Y^n_r| + |Z^n_r| + |Z^{n-1}_r|) \alpha dr \right)_{\mathcal{F}_\sigma},
\]

Letting \( k \to \infty \) and using the fact that \( Y^{n+1}, Y^n \) are of class (D) we get

\[
|Y^{n+1}_\sigma - Y^n_\sigma| \leq 2\gamma E \left( \int_0^T (g_r + |Y^n_r| + |Z^n_r| + |Z^{n-1}_r|) \alpha dr \right)_{\mathcal{F}_\sigma}. \tag{3.27}
\]

Note that \( Z^n, Z^{n-1} \in \mathcal{H}^q, q \in (0, 1), Y^n \) is of class (D) and \( \{g_t\}_{t \in [0, T]} \) is integrable. Therefore the random variable \( I_n := \int_0^T (g_r + |Y^n_r| + |Z^n_r| + |Z^{n-1}_r|) \alpha dr \) belongs to \( L^q \) supposing that
\( \alpha \cdot q < 1 \). Fix \( \bar{q} \in (1, 2) \) such that \( \alpha \cdot \bar{q} < 1 \). Then, by Doob’s inequality and (3.27), \( Y^{n+1} - Y^n \in S^{q} \) for \( n \geq 1 \). Note that

\[
Y_t^{n+1} - Y_t^n = \int_t^T \hat{f}_n(r, Y_r^{n+1} - Y_r^n) \, dr + \int_t^T d(R_r^{n+1} - R_r^n) - \int_t^T (Z_r^{n+1} - Z_r^n) \, dB_r, \quad t \in [0, T],
\]

where \( \hat{f}_n(r, y) = f(r, y + Y^n_r, Z^n_r) - f(r, Y^n_r, Z^n_r^{-1}) \). By (Z) we have

\[
\int_0^T |f(r, Y^n_r, Z^n_r) - f(r, Y^n_r, Z^n_r^{-1})| \, dr \leq C \int_0^T (g_r + |Y^n_r| + |Z^n_r| + |Z^n_r^{-1}|)^\alpha.
\]

Since \( I_n \in L^\bar{q} \), it follows from Proposition 3.7 that \( Z^{n+1} - Z^n \in H^\bar{q} \) and there exists a constant \( C_q \) such that for all \( n \geq 1 \),

\[
\|(Y^{n+1} - Y^n, Z^{n+1} - Z^n)\|^{\bar{q}} \leq C_q E\left( \left( \int_0^T |f(r, Y^n_r, Z^n_r) - f(r, Y^n_r, Z^n_r^{-1})| \, dr \right)^{\bar{q}} \right),
\]

where

\[
\|(Y, Z)\| = \left( E\left( \sup_{t \leq T} |Y_t|^{\bar{q}} + \left( \int_0^T |Z_t|^2 \, dr \right)^{\bar{q}/2} \right) \right)^{1/\bar{q}}.
\]

Since \( f \) satisfies (H1), using Hölder’s inequality we get

\[
\|(Y^{n+1} - Y^n, Z^{n+1} - Z^n)\|^{\bar{q}} \leq C \left( \int_0^T |Z_r^n - Z_r^{n-1}|^2 \, dr \right)^{\bar{q}/2}
\]

for \( n \geq 2 \), where \( C = C_q \lambda^\bar{q} T^{\bar{q}/2} \). Therefore, for \( n \geq 2 \),

\[
\|(Y^{n+1} - Y^n, Z^{n+1} - Z^n)\|^{\bar{q}} \leq C^{n-1} \|(Y^2 - Y^1, Z^2 - Z^1)\|^{\bar{q}}.
\]

If \( C = C_q \lambda^\bar{q} T^{\bar{q}/2} < 1 \), then using the above inequality one can deduce that \( \{(Y^n - Y^1, Z^n - Z^1)\} \) is a Cauchy sequence, so \( (Y^n - Y^1, Z^n - Z^1) \) converges to some \((U, V)\) in \( S^\beta \times H^\beta \). Since \( (Y^1, Z^1) \in S^\beta \times H^\beta, \beta \in (0, 1) \), it follows that \( (Y^n, Z^n) \) converges to \( (Y, Z) := (U + Y^1, V + Z^1) \) in \( S^\beta \times H^\beta, \beta \in (0, 1) \). Moreover, \( Y^1 \) is of class \( (D) \), so \( Y^n \to Y \) in the norm \( \| \cdot \|_1 \). In the general case, we divide \([0, T]\) into a finite number of small intervals and use the standard argument.

Let \( \tilde{f}(r) = f(r, y, Z_r) \) and \((\tilde{Y}, \tilde{Z}, \tilde{R})\) be a solution of RBSDE\((\xi, \tilde{f} + dV, L, U)\) such that \( \tilde{Y} \) is of class \( (D) \), \( \tilde{Z} \in H^\beta, q \in (0, 1), \tilde{R} \in \mathcal{V}^1 \). The existence of the solution follows from Theorem 3.6. Repeating the reasoning following (3.27), but with \( Y^{n+1} \) replaced by \( \tilde{Y} \), we get

\[
\|(\tilde{Y} - Y^n, \tilde{Z} - Z^n)\|^{\bar{q}} \leq C_q E\left( \left( \int_0^T |f(r, Y^n_r, Z_r) - f(r, Y^n_r, Z^n_r^{-1})| \, dr \right)^{\bar{q}} \right).
\]

Since \( f \) satisfies (H1), using Hölder’s inequality we obtain

\[
\|(\tilde{Y} - Y^n, \tilde{Z} - Z^n)\|^{\bar{q}} \leq C E\left( \left( \int_0^T |Z_r - Z_r^{n-1}|^2 \, dr \right)^{\bar{q}/2} \right) \to 0, n \to \infty
\]

for \( n \geq 2 \), where \( C = C_q \lambda^\bar{q} T^{\bar{q}/2} \). Hence \((\tilde{Y}, \tilde{Z}) = (Y, Z)\). Therefore the triple \((Y, Z, \tilde{R})\) is a solution of RBSDE\((\xi, f + dV, L, U)\). This completes the proof. \( \square \)

**Theorem 3.9.** Assume that \( p > 1 \) and (H1)-(H6) are satisfied. Then there exists a unique solution \((Y, Z, R)\) of RBSDE\((\xi, f + dV, L, U)\) such that \( Y \in \mathcal{S}^p, Z \in \mathcal{H}^p \) and \( R \in \mathcal{V}^p \).
Proof. Consider the space $S^p \oplus \mathcal{H}^p$ equipped in the norm

$$
\|(Y, Z)\|_{S^p \oplus \mathcal{H}^p} = \left( E \left( \sup_{t \leq T} |Y_t|^p + \left( \int_0^T |Z_r|^2 \, dr \right)^{p/2} \right) \right)^{1/p}.
$$

Define $\phi : S^p \oplus \mathcal{H}^p \rightarrow S^p \oplus \mathcal{H}^p$ as $\phi((G, H)) = (Y, Z)$, where $(Y, Z, R)$ is the unique solution of RBSDE$(\xi, \tilde{f} + dV, L, U)$ with $\tilde{f}(t, y) = f(t, y, H_t)$ such that $Y \in S^p$, $Z \in \mathcal{H}^p$, $R \in \mathcal{V}^p$. The existence and uniqueness of such solution follows from Theorem 3.20. Let $(Y^1, Z^1), (Y^2, Z^2) \in S^p \oplus \mathcal{H}^p$ and $(G, H), (G', H') \in S^p \oplus \mathcal{H}^p$ be such that $(Y, Z) = \phi(G, H)$ and $(Y', Z') = \phi(G', H')$. By Proposition 3.7, there exists a constant $C_p$ such that

$$
\|(Y - Y', Z - Z')\|_{S^p \oplus \mathcal{H}^p} \leq C_p E \left( \left( \int_0^T |f(r, Y'_r, H'_r) - f(r, Y'_r, H'_r)| \, dr \right)^p \right).
$$

Since $f$ satisfies (H1), applying Hölder’s inequality yields

$$
\|(Y - Y', Z - Z')\|_{S^p \oplus \mathcal{H}^p} \leq C E \left( \left( \int_0^T |H_r - H'_r|^2 \, dr \right)^{p/2} \right)
$$

with $C = C_p \lambda^p T^{p/2}$. Hence

$$
\|(Y - Y', Z - Z')\|_{S^p \oplus \mathcal{H}^p} \leq C \|(G - G', H - H')\|_{S^p \oplus \mathcal{H}^p}.
$$

If $C = C_p \lambda^p T^{p/2} < 1$, then $\phi$ is a contraction, so by Banach’s fixed-point theorem, there exists $(Y, Z)$ such that $\phi(Y, Z) = (Y, Z)$. We set $R = R$. Then the triple $(Y, Z, R)$ is a solution of RBSDE$(\xi, f + dV, L, U)$. To get the existence of a solution in the general case we divide $[0, T]$ into a finite number of small intervals and use the standard argument. 

4 Penalization methods for RBSDEs with two regulated barriers

In this section we assume additionally that the barriers $L, U$ are $\mathbb{F}$-adapted regulated processes. We consider approximation of the solution of RBSDE$(\xi, f + dV, L, U)$ by modified penalization methods.

4.1 Monotone penalization method via RBSDEs

By [17] Theorem 3.20, for each $n \geq 1$ there exists a solution $(\tilde{Y}^n, \tilde{Z}^n, \tilde{A}^n)$ of RBSDE, with upper barrier $U$, of the form

$$
\tilde{Y}_t^n = \xi + \int_t^T f(r, \tilde{Y}_r^n, \tilde{Z}_r^n) \, dr + \int_t^T dV_r - \int_t^T d\tilde{A}_r^n - \int_t^T \tilde{Z}_r^n \, dB_r
$$

$$
+ n \int_t^T (\tilde{Y}_r^n - L_r)^- \, dr + \sum_{t \leq \sigma_{n,i} < T} (\tilde{Y}_{\sigma_{n,i}}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}}^-) \tag{4.1}
$$

such that if $p > 1$, then $\tilde{Y}^n \in S^p$, $\tilde{Z}^n \in \mathcal{H}^p$, $\tilde{A}^n \in \mathcal{V}^{p, q}$, and if $p = 1$, then $\tilde{Y}^n$ is of class (D), $\tilde{Z}^n \in \mathcal{H}^q$, $q \in (0, 1)$, $\tilde{A}^n \in \mathcal{V}^{+1}$. In (11), $\{\sigma_{n,i}\}$ is an array of stopping times exhausting the right-side jumps of $L$ and $V$. It is defined inductively as follows. We set $\sigma_{1,0} = 0$, and then

$$
\sigma_{1,i} = \inf \{ t > \sigma_{1,i-1}; \Delta^+ L_t < -1 \text{ or } \Delta^+ V_i < -1 \} \wedge T, \ i = 1, \ldots, k_1
$$

20
for some $k_1 \in \mathbb{N}$. Next, for $n \in \mathbb{N}$ and given array $\{\{\sigma_{n,i}\}\}$, we set $\tilde{\sigma}_{n+1,0} = 0$,

$$\tilde{\sigma}_{n+1,i} = \inf\{t > \tilde{\sigma}_{n+1,i-1}; \Delta^+L_t < -1/(n+1) \text{ or } \Delta^+V_t < -1/(n+1)\} \wedge T, \ i \geq 1.$$

Let $j_{n+1}$ be chosen so that $P(\tilde{\sigma}_{n+1,j_{n+1}} < T) \leq \frac{1}{n}$. We put

$$\sigma_{n+1,i} = \tilde{\sigma}_{n+1,i}, \quad i = 1, \ldots, j_{n+1}, \quad \sigma_{n+1,i+j_{n+1}} = \tilde{\sigma}_{n+1,j_{n+1}} \lor \sigma_{n,i}, \quad i = 1, \ldots, k_n,$$

$k_{n+1} = j_{n+1} + k_n + 1$. Finally, we put $\sigma_{n+1,k_{n+1}} = T$. Since $\Delta^+L_t < -1/n$ or $\Delta^+V_t < -1/n$ implies that $\Delta^+L_t < -1/(n+1)$ or $\Delta^+V_t < -1/(n+1)$, it follows from our construction that

$$\bigcup_{i=1}^{k_{n+1}} [\sigma_{n,i}] \subset \bigcup_{i=1}^{k_{n+1}} [\sigma_{n+1,i}] \ n \in \mathbb{N}. \quad (4.2)$$

Observe that, on each interval $(\sigma_{n,i-1}, \sigma_{n,i}), \ i = 1, \ldots, k_n + 1$, the triple $(\tilde{Y}^n, \tilde{Z}^n, \tilde{A}^n)$ is a solution of the classical RBSDE of the form

$$\tilde{Y}^n_t = L_{\sigma_{n,i}} \lor (\tilde{Y}^n_{\sigma_{n,i}} + \Delta^+V_{\sigma_{n,i}}) \wedge U_{\sigma_{n,i}} + \int_t^{\sigma_{n,i}} f(\tilde{Y}^n_r, \tilde{Z}^n_r) \, dr + \int_t^{\sigma_{n,i}} dV_r - \int_t^{\sigma_{n,i}} d\tilde{A}^n_r - \int_t^{\sigma_{n,i}} \tilde{Z}^n_r \, dB_r + n \int_t^{\sigma_{n,i}} (\tilde{Y}^n_r - L_r^-) \, dt, \ t \in (\sigma_{n,i-1}, \sigma_{n,i})$$

with upper barrier $U$ and $\tilde{Y}^n_0 = L_0 \lor (\tilde{Y}^n_{0+} + \Delta^+V_0) \wedge U_0, n \in \mathbb{N}$. Therefore, to solve equation (4.1), we divide $[0, T]$ into a finite number of intervals $[0, \sigma_{n,1}], \ldots, (\sigma_{n,k_n-1}, T]$ and we solve the equation on each interval $(\sigma_{n,i-1}, \sigma_{n,i})$ starting from $(\sigma_{n,k_n-1}, T]$.

Note that (4.1) can be written in the shorter form

$$\tilde{Y}^n_t = \xi + \int_t^T f(\tilde{Y}^n_r, \tilde{Z}^n_r) \, dr + \int_t^T dV_r + \int_t^T d\tilde{A}^n_r - \int_t^T \tilde{Z}^n_r \, dB_r + \int_t^T dK^n_r, \quad (4.3)$$

where

$$K^n_t = n \int_0^t (\tilde{Y}^n_r - L_r^-) \, dr + \sum_{0 \leq \sigma_{n,i} < t} (\tilde{Y}^n_{\sigma_{n,i}} + \Delta^+V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-$$

$$\equiv K^{n,s}_t + K^{n,d}_t, \quad t \in [0, T].$$

(4.4)

**Proposition 4.1.** Let $\{\sigma_k; k = 0, \ldots, m\}$ be an increasing sequence of stopping times such that $\sigma_0 = 0$ and $\sigma_m = T$. Let $(\tilde{Y}^i, \tilde{Z}^i, \tilde{A}^i), \ i = 1, 2,$ be a solution of the RBSDE

$$Y^i_t = \xi^i + \int_t^T f^i(s, Y^i_r, Z^i_r) \, ds + \int_t^T dV_r - \int_t^T d\tilde{A}^i_r - \int_t^T Z^i_r \, dB_r$$

$$+ \sum_{t \leq \sigma_k < T} (Y^i_{\sigma_k} + \Delta^+V_{\sigma_k} - L_{\sigma_k})^-, \ t \in [0, T],$$

(4.5)

with upper barrier $U$, such that if $p > 1$, then $Y^i \in S^p$, $Z^i \in H^p$, $A^i \in V^{+p}$, and if $p = 1$, then $Y^i$ is of class (D), $Z^i \in H^q$, $q \in (0, 1)$, $A^i \in V^{+, 1}$, $i = 1, 2$. Assume that $p > 1$ and (H1)-(H6) are satisfied or $p = 1$ and (H1)-(H5), (H6*), (Z) are satisfied. Let $\xi^1 \leq \xi^2$, $f^1 \leq f^2$. Then $dA^1 \leq dA^2$. 

21
Proof. For \(i = 1, 2\) we have
\[
Y_t^i = L_{\sigma_{k+1}} \vee (Y_{\sigma_{k+1}} + \Delta^+ V_{\sigma_{k+1}}) \wedge U_{\sigma_{k+1}} + \int_t^{\sigma_{k+1}} f(s, Y_s^i, Z_s^i) \, ds + \int_t^{\sigma_{k+1}} dV_r
- \int_t^{\sigma_{k+1}} dA_r^i - \int_t^{\sigma_{k+1}} Z_r^i \, dB_r, \quad t \in (\sigma_k, \sigma_{k+1}], \quad k = 0, \ldots, m - 1.
\]
By Proposition 3.2 and Lemma 3.3, \(Y^1 \leq Y^2\), so \(L_{\sigma_{k+1}} \vee (Y_{\sigma_{k+1}} + \Delta^+ V_{\sigma_{k+1}}) \wedge U_{\sigma_{k+1}} \leq L_{\sigma_{k+1}} \vee (Y_{\sigma_{k+1}} + \Delta^+ V_{\sigma_{k+1}}) \wedge U_{\sigma_{k+1}}\). Now on all intervals \((\sigma_k, \sigma_{k+1}]\) we consider introduced in [17] penalization schemes for the \(\overline{\text{RBSDE}}(L_{\sigma_{k+1}} \vee (Y_{\sigma_{k+1}} + \Delta^+ V_{\sigma_{k+1}}) \wedge U_{\sigma_{k+1}}, f^i + dV, U)\), \(i = 1, 2\). They have forms
\[
Y_{t}^{i,n} = L_{\sigma_{k+1}} \vee (Y_{\sigma_{k+1}} + \Delta^+ V_{\sigma_{k+1}}) \wedge U_{\sigma_{k+1}} + \int_t^{\sigma_{k+1}} f(s, Y_s^{i,n}, Z_s^{i,n}) \, ds + \int_t^{\sigma_{k+1}} dV_r
- n \int_t^{\sigma_{k+1}} (Y_s^{i,n} - U_r)^+ \, dr + \sum_{t \leq \tau_{n,j}^{k} < \sigma_{k+1}} (Y_{\tau_{n,j}^{k}}^{i,n} + \Delta^+ V_{\tau_{n,j}^{k}} - U_{\tau_{n,j}^{k}})^+ \]
\[
- \int_t^{\sigma_{k+1}} Z_r^{i,n} \, dB_r, \quad t \in (\sigma_k, \sigma_{k+1}], \quad k = 0, \ldots, m - 1, n \in \mathbb{N},
\]
where \(\{\{\tau_{n,j}^{k}\}\}\) is an array of stopping times exhausting the right-side jumps of \(U\) and \(V\) defined similarly to the array \(\{\tau_{n,i}\}\) for (4.1). Set
\[
\int_t^{\sigma_{k+1}} dA_r^{i,n} := n \int_t^{\sigma_{k+1}} (Y_s^{i,n} - U_r)^+ \, dr - \sum_{t \leq \tau_{n,j}^{k} < \sigma_{k+1}} (Y_{\tau_{n,j}^{k}}^{i,n} + \Delta^+ V_{\tau_{n,j}^{k}} - U_{\tau_{n,j}^{k}})^+.
\]
By Proposition 3.2 and Lemma 3.3, \(Y_1^{i,n} \leq Y_2^{i,n}\). By this and (4.6), \(dA_1^{i,n} \leq dA_2^{i,n}\). Furthermore, by [10] Lemma 4.1 and [17] Theorem 4.1, \(A_1^{i,n} \rightarrow A_1^{i}, A_2^{i,n} \rightarrow A_2^{i}\) weakly in \(L^1\) for every \(\tau \in \Gamma\). Therefore, by the Section Theorem, \(dA_1^{i} \leq dA_2^{i}\) on \((\sigma_k, \sigma_{k+1}]\). In order to complete the proof we have to show that \(\Delta^+ A_1^{i,k} \leq \Delta^+ A_2^{i,k}\), \(k = i, \ldots, m - 1\). If \(\Delta^+ A_1^{i,k} = 0\), then this inequality is obvious. Let \(\Delta^+ A_1^{i,k} > 0\). Note that
\[
\Delta^+ A_1^{i,k} = \Delta^+ Y_1^{i,k} + \Delta^+ V_{\sigma_k} + (Y_{\sigma_k}^{i,k} - \Delta^+ V_{\sigma_k} - L_{\sigma_k})^-.
\]
By the the minimality condition, \(Y_1^{i,k} = U_{\sigma_k}\). Therefore
\[
\Delta^+ Y_1^{i,k} = Y_{\sigma_k}^{1,k} - Y_{\sigma_k}^{2,k} \leq Y_{\sigma_k}^{1,k} - Y_{\sigma_k}^{2,k} = Y_{\sigma_k}^{2,k} - U_{\sigma_k} \leq Y_{\sigma_k}^{2,k} - Y_{\sigma_k} = \Delta^+ Y_{\sigma_k}^{2,k}.
\]
If \((Y_{\sigma_k}^{1,k} + \Delta^+ V_{\sigma_k} - L_{\sigma_k})^- = 0\), then by (4.7) and (4.8),
\[
\Delta^+ A_1^{i,k} \leq \Delta^+ Y_{\sigma_k}^{2,k} + \Delta^+ V_{\sigma_k} \leq \Delta^+ Y_{\sigma_k}^{2,k} + \Delta^+ V_{\sigma_k} + (Y_{\sigma_k}^{2,k} + \Delta^+ V_{\sigma_k} - L_{\sigma_k})^- = \Delta^+ A_2^{i,k}.
\]
If \((Y_{\sigma_k}^{1,k} + \Delta^+ V_{\sigma_k} - L_{\sigma_k})^- \neq 0\), then by (4.7) we have
\[
\Delta^+ A_1^{i,k} = -Y_{\sigma_k}^{1,k} + L_{\sigma_k} = -U_{\sigma_k} + L_{\sigma_k} \leq 0,
\]
which is a contradiction. Hence \(\Delta^+ A_1^{i,k} \leq \Delta^+ A_2^{i,k}\), which completes the proof. \(\square\)

**Theorem 4.2.** Let \((\hat{Y}^n, \hat{Z}^n, \hat{A}^n), n \in \mathbb{N},\) be defined by (1.1).
(i) Assume that \( p > 1 \) and \((H1)–(H6)\) are satisfied. Then \( \bar{Y}_t^n \nrightarrow Y_t \), \( t \in [0,T] \), and for every \( \gamma \in [1,2) \),

\[
E \left( \int_0^T |Z^n_r - Z_r|^\gamma \, dr \right)^{p/2} \rightarrow 0,
\]

(4.9)

where \((Y, Z, R)\) is the unique solution of \( \text{RBSDE}(\xi, f + dV, L, U) \) such that \( Y \in \mathcal{S}^p \), \( Z \in \mathcal{H}^p \), \( R \in \mathcal{S}^p \). Moreover, if \( \Delta^- R_t^+ = 0 \) for \( t \in (0,T] \), then the above convergence also holds with \( \gamma = 2 \), and moreover, \( |Y^n - Y|_p \rightarrow 0 \).

(ii) Assume that \( p = 1 \) and \((H1)–(H5), (H6^*)\) and \((Z)\) are satisfied. Then \( \bar{Y}_t^n \nrightarrow Y_t \), \( t \in [0,T] \), and for all \( \gamma \in [1,2) \) and \( r \in (0,1) \),

\[
E \left( \int_0^T |Z^n_r - Z_r|^\gamma \, dr \right)^{r/2} \rightarrow 0,
\]

(4.10)

where \((Y, Z, R)\) is the unique solution of \( \text{RBSDE}(\xi, f + dV, L, U) \) such that \( Y \) is of class \( (D), Z \in \mathcal{H}^q, R \in \mathcal{S}^q, q \in (0,1) \). Moreover, if \( \Delta^- R_t^+ = 0 \) for \( t \in (0,T] \), then the above convergence also hold with \( \gamma = 2 \) and \( |Y^n - Y|_1 \rightarrow 0 \).

**Proof.** Without loss of generality we may assume that \( \mu = 0 \).

**Step 1.** We show that for every \( n \in \mathbb{N} \) the triple \((\bar{Y}_t^n, \bar{Z}_t^n, K^n, \bar{A}^n)\) is a solution of \( \text{RBSDE}(\xi, f + dV, L^n, U) \) with \( L^n = L - (\bar{Y}_n^n - L^n) = L \land \bar{Y}_n^n \). It is clear that \( \bar{Y}_t^n \geq L_t^n \), \( t \in [0,T] \). We also have

\[
\int_0^T (\bar{Y}_r^n - L_r^n) \, dK_r^{n*} = n \int_0^T (\bar{Y}_r^n - L_r^n)(\bar{Y}_r^n - L_r)^- \, dr = n \int_0^T (\bar{Y}_r^n - L_r)^+(\bar{Y}_r^n - L_r)^- \, dr = 0
\]

and

\[
\sum_{r<T}(\bar{Y}_r^n - L_r^n)\Delta^+ K_r^n = \sum_{\sigma_{n,i}<T}(\bar{Y}_{\sigma_{n,i}}^n - L_{\sigma_{n,i}}^n)(\bar{Y}_{\sigma_{n,i}}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-
\]

\[
= \sum_{\sigma_{n,i}<T}(\bar{Y}_{\sigma_{n,i}}^n - L_{\sigma_{n,i}})^+(\bar{Y}_{\sigma_{n,i}}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- = 0.
\]

We will justify the last equality. Striving for contradiction, suppose that

\[
\sum_{\sigma_{n,i}<T}(\bar{Y}_{\sigma_{n,i}}^n - L_{\sigma_{n,i}})^+(\bar{Y}_{\sigma_{n,i}}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- \neq 0.
\]

Then there exists \( i \in \{1, \ldots, k_n\} \) such that \( \bar{Y}_{\sigma_{n,i}}^n - L_{\sigma_{n,i}} > 0 \) and \( \bar{Y}_{\sigma_{n,i}}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}} < 0 \). By the last inequality and (4.3), \( \Delta^+ \bar{Y}_{\sigma_{n,i}}^n = \Delta^+ K_{\sigma_{n,i}}^n - \Delta^+ V_{\sigma_{n,i}} = (\bar{Y}_{\sigma_{n,i}}^n + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^- - \Delta^+ V_{\sigma_{n,i}} \). Hence \( \bar{Y}_{\sigma_{n,i}}^n = L_{\sigma_{n,i}} \), which is a contradiction.

**Step 2.** We will show that there exists a process \( Z' \in \mathcal{H} \) and a chain \( \{\tau_k\} \) such that

\[
E \left( \int_0^{\tau_k} |Z^n_s - Z'_s|^\gamma \, ds \right) \rightarrow 0, \quad \gamma \in [1,2).
\]

(4.11)

Moreover, we will show that if \( p > 1 \), then \( Z' \in \mathcal{H}^p \) and (4.9) holds with \( Z \) replaced by \( Z' \), and if \( p = 1 \), then \( Z' \in \mathcal{H}^q \), \( q \in (0,1) \), and (4.10) holds with \( Z \) replaced by \( Z' \). To show this we will use [16, Lemma 4.2]. If \( p > 1 \), then by \((H6)\) there exists \( X \in \mathcal{M}_{loc} + \mathcal{V}^p \), \( X \in \mathcal{S}^p \) such that \( X \geq L \) and \( \int_0^T f^-(s, X_s, 0) \, ds \in \mathbb{L}^p \). If \( p = 1 \), then by \((H6^*)\) there exists \( X \in \mathcal{M}_{loc} + \mathcal{V}^1 \)
of class (D) such that $X \geq L$ and $\int_0^T f^-(s, X_s, 0) \, ds \in L^1$. Since the Brownian filtration has the representation property, there exist processes $H \in \mathcal{M}_{loc}$ and $C \in \mathcal{V}^p$ such that

$$X_t = X_T - \int_t^T dC_s - \int_t^T H_s \, dB_s, \quad t \in [0, T].$$

Set $X'_t = X_t$, $t \in [0, T]$, $X'_T = \xi$. Then for some $A', K' \in \mathcal{V}^{+, p}$ we have that

$$X'_t = \xi + \int_t^T f(s, X'_s, H_s) \, ds + \int_t^T dV_s + \int_t^T dK'_s - \int_t^T dA'_s - \int_t^T H_s \, dB_s, \quad t \in [0, T].$$

Let $(\hat{X}^n, \hat{H}^n, \hat{A}^n)$ be a solution of the \textbf{RBSDE}

$$\hat{X}^n_t = \xi + \int_t^T f(s, \hat{X}^n_s, \hat{H}^n_s) \, ds + \int_t^T dV_s + \int_t^T dK'_s - \int_t^T d\hat{A}^n_s - \int_t^T \hat{H}^n_s \, dB_s$$

$$+ \sum_{t \leq \xi_{n,i} < T} (\hat{X}^n_{\xi_{n,i}+} + \Delta^+ V_{\xi_{n,i}} - L_{\xi_{n,i}})^-, \quad t \in [0, T],$$

(4.12)

with upper barrier $U$, such that if $p > 1$, then $\hat{X}^n \in \mathcal{S}^p$, $\hat{H}^n \in \mathcal{H}^p$, $\hat{A}^n \in \mathcal{V}^{+, p}$, and if $p = 1$, then $\hat{X}^n$ is of class (D), $\hat{H}^n \in \mathcal{H}^q$, $q \in (0, 1)$, $\hat{A}^n \in \mathcal{V}^{+, 1}$. The existence of such solution follows from [17] Theorem 3.20. Note that $(X', H, A')$ is a solution of $RBSDE(\xi, f + dV + dK', X')$. Since $X' \leq U$, by Proposition [3.2] and Lemma [3.3], $\hat{X}^n \geq X' \geq L$. Thanks to this, we may rewrite (4.12) in the form

$$\hat{X}^n_t = \xi + \int_t^T f(s, \hat{X}^n_s, \hat{H}^n_s) \, ds + \int_t^T dV_s + \int_t^T dK'_s - \int_t^T d\hat{A}^n_s + n \int_t^T (\hat{X}^n_s - L_s)^- \, ds$$

$$+ \sum_{t \leq \xi_{n,i} < T} (\hat{X}^n_{\xi_{n,i}+} + \Delta^+ V_{\xi_{n,i}} - L_{\xi_{n,i}})^- - \int_t^T \hat{H}^n_s \, dB_s, \quad t \in [0, T].$$

By Proposition [3.2] and Lemma [3.3] again,

$$\hat{X}^n \geq \hat{Y}^n,$$  

(4.13)

and by Proposition [4.1]

$$d\hat{A}^n \geq d\hat{A}^n, \quad n \geq 1.$$  

(4.14)

Moreover,

$$\left(\hat{X}^n_{\xi_{n,i}+} + \Delta^+ V_{\xi_{n,i}} - L_{\xi_{n,i}}\right)^- \leq \left(X'_{\xi_{n,i}+} + \Delta^+ V_{\xi_{n,i}} - L_{\xi_{n,i}}\right)^-$$

$$= (\Delta^+ X'_{\xi_{n,i}} + \Delta^+ V_{\xi_{n,i}} + X'_{\xi_{n,i}} - L_{\xi_{n,i}})^-$$

$$\leq (\Delta^+ X_{\xi_{n,i}} + \Delta^+ V_{\xi_{n,i}})^-$$

$$\leq \Delta^+ C_{\xi_{n,i}} + \Delta^+ V_{\xi_{n,i}}.$$  

(4.15)

Let $(\tilde{X}, \tilde{H}, \tilde{A})$ be a solution of the following $RBSDE$

$$\tilde{X}_t = \xi + \int_t^T f(s, \tilde{X}_s, \tilde{H}_s) \, ds + \int_t^T dV_s + \int_t^T dK'_s - \int_t^T d\tilde{A}_s + n \int_t^T (\tilde{X}_s - L_s)^- \, ds$$

$$+ \int_t^T d|C|_s + \int_t^T d|V|_s - \int_t^T \tilde{H}_s \, dB_s, \quad t \in [0, T],$$

24
with upper barrier \( U \), such that if \( p > 1 \), then \( \tilde{X} \in \mathcal{S}^p \), \( \tilde{H} \in \mathcal{H}^p \), \( \tilde{A} \in \mathcal{V}^{+p} \), and if \( p = 1 \), then \( \tilde{X} \) is of class (D), \( \tilde{H} \in \mathcal{H}^1 \), \( q \in (0, 1) \), \( \tilde{A} \in \mathcal{V}^{+1} \). The existence of the solution follows from [17 Theorem 3.20]. The triple \((\tilde{X}, \tilde{H}, \tilde{A})\) does not depend on \( n \), because by Proposition 3.2 and Lemma 3.3, \( \tilde{X} \geq X^n \), so the term involving \( n \) on the right-hand side of the above equation equals zero. By the last inequality and (4.13),

\[
\tilde{X} \geq \tilde{Y}^n. \tag{4.16}
\]

By Proposition 4.1, \( d\tilde{A}^n \leq d\tilde{A} \), which by (4.14) implies that

\[
d\tilde{A}^n \leq d\tilde{A}. \tag{4.17}
\]

By Proposition 3.2 and Lemma 3.3

\[
\tilde{Y}^n \leq \tilde{Y}^{n+1}. \tag{4.18}
\]

From this, (4.16), (4.17) and [16 Lemma 4.2], if \( p > 1 \), then

\[
E(K^n_T)^p + E\left( \int_0^T |\tilde{Z}^n_s|^2 \, ds \right)^{\frac{q}{2}} \leq CE \left( \sup_{t \leq T}(|Y^n_t|^p + |\tilde{X}_t|^p) + \left( \int_0^T d|V|_s \right)^p \right.
\]

\[
+ \left( \int_0^T |f^-(s, \tilde{X}_s, 0)| \, ds \right)^p + \left( \int_0^T \tilde{X}_s^+ \, ds \right)^p
\]

\[
+ \left( \int_0^T |f(s, 0, 0)| \, ds \right)^p + \left( \int_0^T d\tilde{A}_s \right)^p, \tag{4.19}
\]

and if \( p = 1 \), then for every \( q \in (0, 1) \),

\[
E\left( \int_0^T |\tilde{Z}^n_s|^2 \, ds \right)^{q/2} \leq CE \left( \sup_{t \leq T}(|Y^n_t|^q + |\tilde{X}_t|^q) + \left( \int_0^T |f(s, 0, 0)| \, ds \right)^q \right.
\]

\[
+ \left( \int_0^T |f^-(s, \tilde{X}_s, 0)| \, ds \right)^q + \left( \int_0^T \tilde{X}_s^+ \, ds \right)^q
\]

\[
+ \left( \int_0^T d|V|_s \right)^q + \left( \int_0^T d\tilde{A}_s \right)^q. \tag{4.20}
\]

Now we will apply Theorem 2.5 to (4.1). We know that \( \tilde{Y}^n \) is of class (D), \( \tilde{Z}^n \in \mathcal{H} \), \( K^n \in \mathcal{V}^+ \), \( \tilde{A}^n \in \mathcal{V}^+ \) and \( t \mapsto f(t, \tilde{Y}^n_t, \tilde{Z}^n_t) \in L^1(0, T) \) and \( V \) is a finite variation process. By Proposition 4.1 \( d\tilde{A}^n \leq d\tilde{A}^{n+1}, n \in \mathbb{N} \). Let \( Y^n_t = \sup_{n \geq 1} \tilde{Y}^n_t, A_t = \lim_{n \to \infty} \tilde{A}^n_t, t \in [0, T] \) and \( D^n := \tilde{A}^n - V \). We will check assumptions (a)–(f) of Theorem 2.5

(a) We have shown that \( d\tilde{A}^n \leq d\tilde{A}^{n+1} \) and \( d\tilde{A}^n \leq d\tilde{A}^n \), \( n \geq 1 \). Hence \( dD^n \leq dD^{n+1} \) and \( \sup_{n \geq 1} E|D^n|_T < \infty \).

(b) Let \( \tau, \sigma \in \Gamma \) be stopping times such that \( \sigma \leq \tau \). By (4.17),

\[
\liminf_{n \to \infty} \left( \int_{\sigma}^{\tau} (Y^\prime_s - \tilde{Y}^n_s) \, d(K^n_s - D^n_s) + \sum_{\sigma < s < \tau} (Y^\prime_s - \tilde{Y}^n_s) \Delta^+(K^n_s - D^n_s) \right)
\]

\[
\geq - \liminf_{n \to \infty} \left( \int_{\sigma}^{\tau} (Y^\prime_s - \tilde{Y}^n_s) \, d\tilde{A}_s - dV_s + \sum_{\sigma < s < \tau} (Y^\prime_s - \tilde{Y}^n_s) (\Delta^+ \tilde{A}^n_s - \Delta^+ V_s) \right).
\]
By the Lebesgue dominated convergence theorem,
\[
\lim_{n \to \infty} \left( \int_0^\tau (Y'_s - \bar{Y}^n_s) \, (d\bar{A}_s - dV_s) + \sum_{\sigma \leq s < \tau} (Y'_s - \bar{Y}^n_s)(\Delta^+ \bar{A}_s - \Delta^+ V_s) \right) = 0.
\]

Therefore \( \lim \inf_{n \to \infty} \int_0^\tau (Y'_s - \bar{Y}^n_s) \, d(K^n_s - D^n_s) \geq 0. \)

(c) It is easy to see that \( \Delta^- K^n_t = 0, \) \( n \in \mathbb{N}, \) \( t \in [0, T]. \)

(d) Let \( \bar{y} = Y^1 \) and \( y = \bar{X}. \) Then \( \bar{y}, y \in \mathcal{Y}^1 + \mathcal{M}_{loc}, \) \( \bar{y}, y \) are of class (D) and by (H6), (H6*) and Proposition 3.3,
\[
E \int_0^T f^+ (s, \bar{y}_s, 0) \, ds + E \int_0^T f^- (s, y_s, 0) \, ds < \infty.
\]

By (4.16), \( \bar{y}_t \leq \bar{Y}^n_t \leq y_t, \) \( t \in [0, T]. \)

(e) It follows from (H3).

(f) By the definition of \( Y', \bar{Y}_t \), \( t \in [0, T]. \)

By Theorem 2.5 \( Y' \) is regulated and there exist processes \( K \in \mathcal{V}^+ \) and \( Z' \in \mathcal{H} \) such that
\[
Y'_t = \xi + \int_0^T f(s, Y'_s, Z'_s) \, ds + \int_0^T dV_s + \int_0^T dK_s - \int_0^T dA_s - \int_0^T Z'_s \, dB_s, \quad t \in [0, T]. \tag{4.21}
\]

Moreover \( \bar{Z}^n \to Z' \) in the sense of (2.1). This when combined with (4.19) and (4.20) implies that if \( p > 1, \) then \( Z' \in \mathcal{H}^p \) and (4.9) is satisfied if if \( p = 1, \) then \( Z' \in \mathcal{H}^q, \) \( q \in (0, 1), \) (4.10) holds, and there exists a chain \( \{\tau_k\} \subset \Gamma \) such that (4.11) is satisfied.

**Step 3.** We will show that \( EK_T^p + EA_T^p < \infty. \) The desired integrability of \( A \) follows from the integrability of \( \bar{A} \) and (4.17). To prove that \( EK_T^p < \infty, \) we show that
\[
\sup_{n \geq 1} E \left( \int_0^T |f(s, Y^n_s, Z^n_s)| \, ds \right)^p + E \left( \int_0^T |f(s, Y'_s, Z'_s)| \, ds \right)^p < \infty. \tag{4.22}
\]

If \( p > 1, \) then by (H1), (H2), (4.16) and (4.18)
\[
E \left( \int_0^T |f(s, Y^n_s, Z^n_s)| \, ds \right)^p \leq C_p \left( E \left( \int_0^T |f(s, \bar{X}_s, 0)| \, ds \right)^p + E \left( \int_0^T |\bar{Z}^n_s|^2 \, ds \right)^{p/2} \right).
\]

If \( p = 1, \) then by (Z),
\[
E \int_0^T |f(s, \bar{Y}^n_s, \bar{Z}^n_s)| \, ds \leq \gamma E \int_0^T (g_s + |\bar{Y}^n_s| + |\bar{Z}^n_s|) \, ds + E \int_0^T |f(s, \bar{Y}^n_s, 0)| \, ds.
\]

By Hölder's inequality, (H2), (4.16) and (4.18),
\[
E \int_0^T (g_s + |\bar{Y}^n_s| + |\bar{Z}^n_s|) \, ds + E \int_0^T |f(s, \bar{Y}^n_s, 0)| \, ds
\]
\[
\leq C \left\{ E \left( \int_0^T |\bar{Z}^n_s|^2 \, ds \right)^{\alpha/2} + E \int_0^T (g_s + |\bar{X}_s| + |Y^1_s|)^\alpha \, ds \right. \nonumber \\
\left. + E \int_0^T |f(s, Y^1_s, 0)| + |f(s, \bar{X}_s, 0)| \, ds \right\}.
\]
Applying Fatou’s lemma and using (4.19) and (4.20) we get (4.22). The desired integrability of $K$ follows from (4.22) and the integrability of $Y', Z'$ and $A,V$.

**Step 4.** We show that the minimality condition for $A$ is satisfied, i.e.

$$\int_0^T (U_{t^-} - Y'_{t^-}) dA_t^* + \sum_{t< T} (U_t - Y'_t) \Delta^+ A_t = 0. \quad (4.23)$$

Since the triple $(\bar{Y}^n, \bar{Z}^n, \bar{A}^n)$ is a solution of (4.1), we have

$$\int_0^T (U_{t^-} - \bar{Y}'_{t^-}) dA_t^{n,*} + \sum_{t< T} (U_t - \bar{Y}'_t) \Delta^+ \bar{A}_t^n = 0. \quad (4.24)$$

By the Vitali–Hahn–Saks theorem, $d\bar{A}_t^n \nrightarrow dA$ in the variation norm, i.e.

$$\Delta^+ \bar{A}_t^n \nrightarrow \Delta^+ A_t, \quad \Delta^- \bar{A}_t^n \nrightarrow \Delta^- A_t, \quad \|dA^{n,*} - dA^{*,c}\|_{TV} \to 0. \quad (4.25)$$

Letting $n \to \infty$ in the second term of (4.24) and applying the Lebesgue dominated convergence theorem we obtain

$$\sum_{t< T} (U_t - Y'_t) \Delta^+ A_t = 0.$$

Since $|dA^{n,*} - dA^{*,c}|_{TV} \to 0$ and $0 \leq U_t - \bar{Y}_t^n \leq U_t - Y'_t$, using (4.24) and the Lebesgue dominated convergence theorem we get $\int_0^T (U_t - Y'_t) dA_t^{n,*} = 0$. If $\Delta^+ A_t^n = 0$, then $(U_t - Y'_t) \Delta^+ A_t^n = 0$. If $\Delta^- A_t^n > 0$, then by (4.25) there exists $N \in \mathbb{N}$ such that $\Delta^- A_t^{n,*} > 0$ for $n \geq N$. By this and (4.24), $\bar{Y}_t^n = U_t$ for $n \geq N$. By Proposition 3.2 and Lemma 3.3, $Y'_t \geq \bar{Y}_t^n = U_t$, so $Y'_t = U_t$. Therefore

$$\sum_{t< T} (U_t - Y'_t) \Delta^- A_t^n = 0.$$

**Step 5.** We will show that $Y' \geq L$. By (4.16), (4.17) and (4.22), $\sup_{n \geq 1} EK^n_T < \infty$, so $\left\{ n \int_0^T (\bar{Y}_s^n - L_s)^{-} ds \right\}$ is bounded in $L^1(\Omega)$. Therefore, passing to a subsequence if necessary, we may assume that there exists a dense countable subset $Q \subset [0, T]$ such that for $P,a.e.\omega \in \Omega$, $(\bar{Y}_t^n - L_t)^{-} \to 0$ for $t \in Q$. Consequently, $Y'_t \geq L_t$ for $t \in Q$. Hence $Y'_t \geq L_t$, $t \in [0, T]$. We will show that $Y'_t \geq L_t$ for every $t \in [0, T]$. Let $t \in [0, T]$. Assume that $\Delta^+(L_t + V_t) \leq 0$. If $\Delta^+ A_t > 0$, then $Y'_t = U_t$, so obviously $Y'_t \geq L_t$. In case $\Delta^+ A_t = 0$, we have $\Delta^+ Y'_t = -(\Delta^+ V_t - \Delta^+ K_t)$. Therefore

$$Y'_t + V_t = -(\Delta^+ V_t + \Delta^+ Y'_t) + Y'_t + V_t \geq L_t + V_t \geq L_t + V_t,$$

so $Y'_t \geq L_t$. Assume now that $\Delta^+(L_t + V_t) < 0$. If $\Delta^+ A_t > 0$, then $Y'_t = U_t$, so $Y'_t \geq L_t$. If $\Delta^+ A_t = 0$, then by (4.25), $\Delta^+ \bar{A}_t^n = 0$, $n \geq 1$. Since $\Delta^+(L_t + V_t) < 0$, $t \in \bigcup \{[\sigma_{n,i}]\}$ for sufficiently large $n$. Hence $\Delta^+ K_t^n = (\bar{Y}_{t+}^n + \Delta^+ V_t - L_t)^-$. By this and (4.1),

$$\Delta^+ \bar{Y}_t^n = -\Delta^+ V_t - (\bar{Y}_{t+}^n + \Delta^+ V_t - L_t)^-.$$

Suppose that $\bar{Y}_t^n < L_t$. Then

$$\bar{Y}_t^n - L_t + \Delta^+ V_t < \bar{Y}_t^n - \bar{Y}_t^n + \Delta^+ V_t = - (\bar{Y}_{t+}^n + \Delta^+ V_t - L_t)^-.$$
Consequently, $Y^n_t + \Delta^+ V_t - L_t < -(Y^n_t + \Delta^+ V_t - L_t)^-$, which is a contradiction. Thus $Y^n_t \geq L_t$, so $Y'_t \geq L_t$. Therefore
\[ Y'_t \geq L_t 1_{\{t < T\}} + \xi 1_{\{t = T\}} , \quad t \in [0, T] . \]

**Step 6.** We will show the minimality condition for $K$, i.e. we show that
\[ \int_0^T (Y'_r - L_r) \, dK^*_r + \sum_{r < T} (Y'_r - L_r) \Delta^+ K_r = 0 . \quad (4.26) \]

By (4.21), (4.22) and the integrability properties of $Y'$, $V$ and $A$, the process
\[ Y' + \int_0^T f(s, Y'_s, Z'_s) \, ds - V + A \]
is a supermartingale which majorizes the process $L 1_{\{t < T\}} + \xi 1_{\{t = T\}} + \int_0^T f(s, Y'_s, Z'_s) \, ds - V + A$. Hence
\[ Y'_t \geq \text{ess sup}_{\tau \in \Gamma_t} E \left( \int_0^\tau f(s, Y'_s, Z'_s) \, ds + \int_0^\tau dV_s - \int_0^\tau dA_s + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_\tau \right) . \quad (4.27) \]

Let $p > 1$. By [17, Proposition 3.13], Step 1 and the definition of $L^n$, for $t \in [0, T]$ we have
\[ Y^n_t = \text{ess sup}_{\tau \in \Gamma_t} E \left( \int_0^\tau f(s, Y^n_s, Z^n_s) \, ds + \int_0^\tau dV_s - \int_0^\tau dA_s + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_\tau \right) \]
\[ \leq \text{ess sup}_{\tau \in \Gamma_t} E \left( \int_0^\tau f(s, Y^n_s, Z^n_s) \, ds + \int_0^\tau dV_s \right. \]
\[ \left. - \int_0^\tau dA_s + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_\tau \right) . \quad (4.28) \]

Observe that by (4.9), (4.22) and the assumptions on $f$,
\[ E \int_0^T |f(r, Y^n_r, Z^n_r) - f(r, Y'_r, Z'_r)| \, dr \to 0 . \quad (4.29) \]

By (4.25), (4.28), (4.29) and [17, Lemma 3.19],
\[ Y'_t \leq \text{ess sup}_{\tau \in \Gamma_t} E \left( \int_0^\tau f(s, Y'_s, Z'_s) \, ds + \int_0^\tau dV_s - \int_0^\tau dA_s + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_\tau \right) . \]

This when combined with (4.27) gives
\[ Y'_t = \text{ess sup}_{\tau \in \Gamma_t} E \left( \int_0^\tau f(s, Y'_s, Z'_s) \, ds + \int_0^\tau dV_s - \int_0^\tau dA_s + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_\tau \right) . \]

Let $p = 1$. Since $Y^1 \leq \tilde{Y}^n \leq Y'$, $n \geq 1$, using (H2) we get
\[ f(t, Y'_t, 0) \leq f(t, \tilde{Y}^n_t, 0) \leq f(t, Y^1_t, 0) , \quad t \in [0, T] . \]

Define $\sigma_k = \inf \{ t \geq 0 : \int_0^t |f(r, Y^1_r, 0)| + |f(r, Y'_r, 0)| \, dr \geq k \} \wedge T$. It is clear that $\{ \sigma_k \}$ is a chain. We may assume that $\sigma_k = r_k$. Observe that by (4.10), the definition of $\sigma_k$ and the assumptions on $f$,
\[ E \int_0^{r_k} |f(r, \tilde{Y}^n_r, Z^n_r) - f(r, Y'_r, Z'_r)| \, dr \to 0 . \quad (4.30) \]
By \cite[Proposition 3.13]{17}, Step 1, (f) and the definition of $L^n$, for $t \in [0, \tau_k]$ we have

$$Y^n_t = \text{ess sup}_{\tau \in \Gamma_1, \tau_k \geq t} E \left( \int_{t}^{\tau} f(s, \bar{Y}^n_s, \bar{Z}^n_s) \, ds + \int_{t}^{\tau} dV_s - \int_{t}^{\tau} d\bar{A}^n_s + L^n_{\tau} \mathbf{1}_{\{\tau < \tau_k\}} + \bar{Y}^n_{\tau_k} \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t \right)$$

$$\leq \text{ess sup}_{\tau \in \Gamma_1, \tau_k \geq t} E \left( \int_{t}^{\tau} f(s, \bar{Y}^n_s, \bar{Z}^n_s) \, ds + \int_{t}^{\tau} dV_s - \int_{t}^{\tau} d\bar{A}^n_s + L_{\tau} \mathbf{1}_{\{\tau < \tau_k\}} + Y^n_{\tau_k} \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t \right).$$

By this, \cite[(4.27)]{17} gives

$$Y^n'_t \leq \text{ess sup}_{\tau \in \Gamma_1, \tau_k \geq t} E \left( \int_{t}^{\tau} f(s, Y'_s, Z'_s) \, ds + \int_{t}^{\tau} dV_s - \int_{t}^{\tau} dA_s + L_{\tau} \mathbf{1}_{\{\tau < \tau_k\}} + Y^n_{\tau_k} \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t \right).$$

This when combined with \cite[(4.27)]{17} gives

$$Y^n'_t = \text{ess sup}_{\tau \in \Gamma_1, \tau_k \geq t} E \left( \int_{t}^{\tau} f(s, Y'_s, Z'_s) \, ds + \int_{t}^{\tau} dV_s - \int_{t}^{\tau} dA_s + L_{\tau} \mathbf{1}_{\{\tau < \tau_k\}} + Y^n_{\tau_k} \mathbf{1}_{\{\tau = \tau_k\}} | \mathcal{F}_t \right).$$

By \cite[Corollary 3.11]{17} we have \cite[(4.25)]{17} satisfied on $[0, \tau_k]$, and since $\{\tau_k\}$ is a chain, we have it also on $[0, T]$.

**Step 7.** We will show that $(Y', Z', K - A) = (Y, Z, R)$. Put $R' = K - A$. Obviously $dR^+ \leq dK$ and $dR^- \leq dA$, so by \cite[(4.23)]{17} and \cite[(4.26)]{17},

$$
\int_0^T (Y_r - L_r) dR_r^{t+} + \sum_{r < T} (Y_r - L_r) \Delta^+ R_r^t = 0
$$

and

$$
\int_0^T (U_r - Y_r) dR_r^{t-} + \sum_{r < T} (U_r - Y_r) \Delta^- R_r^t = 0.
$$

Consequently, the triple $(Y', Z', R')$ is a solution of RBSDE$(\xi, f + dV, L, U)$ such that $Y' \in S^p$, $Z' \in \mathcal{H}^p$, $R' \in S^p$ in case $p > 1$, and in case $p = 1$ $Y'$ is of class (D), $Z' \in \mathcal{H}^q$, $q \in (0, 1)$, $R' \in \mathcal{V}^1$ in case $p = 1$. Hence, by Proposition \cite[3.2]{3.2} and Lemma \cite[3.3]{3.3} $(Y', Z', K - A) = (Y, Z, R)$.

**Step 8.** We will show that if $\Delta^- R^+ = 0$, then \cite[(4.3)]{4.3}, \cite[(4.10)]{4.10} hold with $\gamma = 2$ and $\bar{Y}^n - Y |_p \to 0$. Let $R^n = K^n - A^n$. By \cite[Corollary A.5]{17}, (H1) and (H2),

$$
\int_0^T |Z_r - \bar{Z}_r^n|^2 \, dr \leq 2\lambda \int_0^T |Y_r - \bar{Y}_r^n||Z_r - \bar{Z}_r^n| \, dr + 2 \int_0^T (Y_r - \bar{Y}_r^n) d(R - \bar{R}^n)_r^*
$$

$$+ 2 \sum_{0 \leq t < T} (Y_t - \bar{Y}_t^n) \Delta^+ (R_t - \bar{R}_t^n)^c + \sup_{0 \leq t \leq T} \left| \int_t^T (Y_r - \bar{Y}_r^n)(Z_r - \bar{Z}_r^n) \, dB_r \right| \leq \int_0^T (Y_r - \bar{Y}_r^n) d(R - \bar{R}^n)_r^*.
$$

By the the minimality condition and the assumption that $\Delta^- R^+ = 0$

$$
\sum_{0 \leq t < T} (Y_t - \bar{Y}_t^n) \Delta^+ (R_t - \bar{R}_t^n) \leq \sum_{0 \leq t < T} (Y_t - \bar{Y}_t^n) \Delta^+ R_t \leq \sum_{0 \leq t < T} (Y_t - \bar{Y}_t^n) \Delta^+ R_t^t
$$

and

$$
\int_0^T (Y_r - \bar{Y}_r^n) d(R - \bar{R}^n)_r^* \leq \int_0^T (Y_r - \bar{Y}_r^n) dR_r^{t+} = \int_0^T (Y_r - \bar{Y}_r^n) dR_r^{t+c}.
$$
By the above, \((4.31)\) and the Burkholder–Davis–Gundy inequality,

\[
E\left( \int_0^T |Z_r - Z^n_r|^2 \, dr \right)^{p/2} \leq C\left( E \sup_{0 \leq t \leq T} |Y_t - \bar{Y}^n_t|^p + (E \sup_{0 \leq t \leq T} |Y_t - \bar{Y}^n_t|^p)^{1/2}(E|R^n_{t_0}|^1)^{1/2}. \right) \tag{4.32}
\]

Furthermore, \(\Delta^{-}Y_t = \Delta^{-}R_t = -A^n_t\). Hence, if \(\Delta^{-}\bar{Y}^n_t = 0, n \geq 0\), then \(\Delta^{-}Y_t = 0\). Otherwise, i.e. if \(\Delta^{-}\bar{Y}^n_t = -A^n_t > 0\) for some \(n \geq 1\), then \(\Delta^{-}\bar{Y}^n_t > 0, n \geq N\) \((\{-A^n_t\} \) is nondecreasing), which implies that \(\bar{Y}^n_t = U_{t-}\) for \(n \geq N\). Since \(U_{t-} = \bar{Y}^n_t \leq Y_t \leq U_{t-}\), this implies that \(\bar{Y}^n_t = Y_{t-}, n \geq N\). Thus, in both cases, \(\Delta^{-}\bar{Y}^n_t \to \Delta^{-}Y_t, t \in [0, T]\). Moreover, by the construction of \(\bar{Y}^n_t, \Delta^{+}\bar{Y}^n_t \to \Delta^{+}Y_t, t \in [0, T]\). Consequently, by the generalized Dini theorem, \(\sup_{0 \leq t \leq T} |Y_t - \bar{Y}^n_t| \to 0\) as \(n \to \infty\). Therefore, by \((4.16)\) and the Lebesgue dominated convergence theorem, \(|Y - \bar{Y}^n|_p \to 0\). From this and \((4.32)\) we deduce that \((4.9)\) and \((4.10)\) hold with \(\gamma = 2\).

Analogously to \((4.1)\), we define \((Y^n, Z^n, A^n)\) as a solution of \(\text{RBSDE}\)

\[
Y^n = \xi + \int_0^T f(r, Y^n, Z^n) \, dr + \int_0^T \frac{dV_t}{r} + \int_0^T \frac{dK^n_r}{r} - \int_0^T Z^n r dB_r
\]

\[
- n \int_0^T (Y^n - U_r)^{-} \, dr - \sum_{t \leq \tau_{n,i} < T} (Y^n_{\tau_{n,i},+} + \Delta^{+}V_{\tau_{n,i},-} - U_{\tau_{n,i}})^+, \tag{4.33}
\]

with lower barrier \(L\), such that if \(p > 1\), then \(Y^n \in S^p, Z^n \in H^p, K^n \in V^{+,p}\), and if \(p = 1\), then \(Y^n \in S^q, Z^n \in H^q, q \in (0,1), K^n \in V^{+,1}\). Now \(\{\tau_{n,i}\}\) is defined as follows: we set \(\tau_{1,0} = 0\) and then

\[
\tau_{1,i} = \inf\{t > \tau_{1,i-1}; \Delta^{+}U_t > 1 \text{ or } \Delta^{+}V_t > 1\} \land T, i = 1, \ldots, k_1
\]

for some \(k_1 \in \mathbb{N}\). Next, for \(n \in \mathbb{N}\) and given array \(\{\tau_{n,i}\}\), we set \(\tilde{\tau}_{n+1,0} = 0, \tilde{\tau}_{n+1,i} = \inf\{t > \tilde{\tau}_{n+1,i-1}; \Delta^{+}U_t > 1/(n+1) \text{ or } \Delta^{+}V_t > 1/(n+1)\} \land T, i \geq 1\).

Let \(j_{n+1}\) be chosen so that \(P(\tilde{\tau}_{n+1,j_{n+1}} < T) \leq \frac{1}{n}\). We put

\[
\tau_{n+1,i} = \tilde{\tau}_{n+1,i}, i = 1, \ldots, j_{n+1}, \tau_{n+1,i} + j_{n+1} = \tilde{\tau}_{n+1,j_{n+1}} \land \tau_{n,i}, i = 1, \ldots, k_n, \]

\(k_{n+1} = j_{n+1} + k_n + 1\). Finally, we put \(\tau_{n+1,k_{n+1}} = T\).

**Theorem 4.3.** Let \((Y^n, Z^n, A^n), n \in \mathbb{N}\) be defined by \((4.33)\).

(i) Assume that \(p > 1\) and \((H1)-(H6)\) are satisfied. Then \(Y^n \prec Y, t \in [0, T]\), and for every \(\gamma \in [1, 2]\),

\[
E\left( \int_0^T |Z^n_r - Z_r|^\gamma \, dr \right)^{p/2} \to 0,
\]

where \((Y, Z, R)\) is the unique solution of \(\text{RBSDE}(\xi, f + dV, L, U)\) such that \(Y \in S^p, Z \in H^p, R \in S^p\). Moreover, if \(\Delta^{-}R_t = 0\) for \(t \in (0, T]\), then the above convergence also hold with \(\gamma = 2\) and \(|Y^n - Y|_p \to 0\).

(ii) Assume that \(p = 1\) and \((H1)-(H5), (H6^*)\), \((Z)\) are satisfied. Then \(Y^n \prec Y, t \in [0, T]\), and for all \(\gamma \in [1, 2]\) and \(r \in (0, 1)\),

\[
E\left( \int_0^T |Z^n_r - Z_r|^\gamma \, dr \right)^{r/2} \to 0,
\]
where \((Y, Z, R)\) is the unique solution of RBSDE\((\xi, f + dV, L, U)\) such that \(Y\) is of class \((D)\), \(Y \in \mathcal{S}^q, Z \in \mathcal{H}^q, R \in \mathcal{Y}^1, q \in (0,1)\). Moreover, if \(\Delta^{-} R_t = 0\) for \(t \in (0,T]\), then the above convergence also hold with \(\gamma = 2\) and \(|Y^n - Y| \to 0\).

**Proof.** By Definition 2.4, \((-\sum^n, -Z^n, K^n)\) is a solution of \(\overline{\text{RBSDE}}\) of the form

\[
-Y^n_t = -\xi - \int_t^T f(r, Y^n_r, Z^n_r) \, dr - \int_t^T dV_r - \int_t^T dK^n_r + \int_t^T Z^n_r \, dB_r
+ n \int_t^T (Y^n_r - U_r)^+ \, dr + \sum_{t \leq \sigma_{n,i} < T} (Y^n_{\sigma_{n,i}} + \Delta^+ V_{\sigma_{n,i}} - U_{\sigma_{n,i}})^+
\]

with upper barrier \(-L\). By Theorem 4.2, solutions of the above equation tend to the solution of \(\text{RBSDE}(\xi, f - dV, -U, -L)\) with \(\tilde{f}(t,y,z) = -f(t,-y,-z)\), from which the desired result follows. \(\blacksquare\)

### 4.2 Penalization method via BSDEs

In this section we consider approximation of solutions of RBSDE with two barriers by solutions of usual BSDEs. Let \((Y^n, Z^n)\) be a solution of BSDE of the form

\[
Y^n_t = \xi + \int_t^T f(r, Y^n_r, Z^n_r) \, dr + \int_t^T dV_r - \int_t^T Z^n_r \, dB_r + n \int_t^T (Y^n_r - L_r)^- \, dr
+ \sum_{t \leq \sigma_{n,i} < T} (Y^n_{\sigma_{n,i}} + \Delta^+ V_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-
- \sum_{t \leq \tau_{n,i} < T} (Y^n_{\tau_{n,i}} + \Delta^+ V_{\tau_{n,i}} - U_{\tau_{n,i}})^+
\]

such that if \(p > 1\), then \(Y^n \in \mathcal{S}^p, Z^n \in \mathcal{H}^p,\) and if \(p = 1\), then \(Y^n\) is of class \((D)\), \(Y^n \in \mathcal{S}^q, Z^n \in \mathcal{H}^q, q \in (0,1)\), where \(\{\sigma_{n,i}\}\) and \(\{\tau_{n,i}\}\) are defined by (4.1) and (4.33). One can find a solution of (4.35) inductively in the manner used to solve (4.1) and (4.33). More precisely, for fixed \(n \in \mathbb{N}\) let \(k_n\) be the number of stopping times \(\{\sigma_{n,i}\}\) and \(\{\tau_{n,i}\}\). We put \(m_n = 2k_n, \gamma_{n,0} = 0, \gamma_{n,1} = \sigma_{n,1} \wedge \tau_{n,1}\) and \(\gamma_{n,m} = \sigma_{n,m} \wedge \tau_{n,m}, m = 2, \ldots, m_n,\) where

\[
\tilde{\sigma}_{n,m} = \min\{\sigma_{n,i} : \sigma_{n,i} > \gamma_{n,m-1}, i = 1, \ldots, k_n\} \wedge T
\]

and

\[
\tilde{\tau}_{n,m} = \min\{\tau_{n,i} : \tau_{n,i} > \gamma_{n,m-1}, i = 1, \ldots, k_n\} \wedge T.
\]

Note that \(\{\gamma_{n,m}\}\) are stopping times such that \(\gamma_{n,m} = T\) and

\[
\bigcup_{i=1}^{k_n} [\sigma_{n,i}] \cup \bigcup_{i=1}^{k_n} [\tau_{n,i}] = \bigcup_{m=1}^{m_n} [\gamma_{n,m}].
\]

Moreover, for \(m = 1, \ldots, m_n\), on each interval \(\langle \gamma_{n,m-1}, \gamma_{n,m}\rangle\), the pair \((Y^n, Z^n)\) is a solution of the nonreflected BSDEs of the form

\[
Y^n_t = L_{\gamma_{n,m}} \vee (Y^n_{\gamma_{n,m}} + \Delta^+ V_{\gamma_{n,m}}) \wedge U_{\gamma_{n,m}} + \int_{\gamma_{n,m}}^{\gamma_{n,m}} f(r, Y^n_r, Z^n_r) \, dr + \int_{\gamma_{n,m}}^{\gamma_{n,m}} dV_r
- \int_t^{\gamma_{n,m}} Z^n_r \, dB_r + n \int_t^{\gamma_{n,m}} (Y^n_r - L_r)^- \, dr - n \int_t^{\gamma_{n,m}} (Y^n_r - U_r)^+ \, dr, t \in (\gamma_{n,m-1}, \gamma_{n,m}],
\]
with \( Y^n_t = L_0 \vee (\tilde{Y}_t^n + \Delta^+ V_0) \wedge U_0, n \in \mathbb{N} \). Therefore, to solve (4.3), we divide \([0, T]\) into the finite number of intervals \([0, \gamma_1], \ldots, (\gamma_{n,m-1}, T]\) and we solve this equation on these intervals \((\gamma_{n,m-1}, \gamma_{n,m}]\) inductively starting from the interval \((\gamma_{n,m-1}, T]\).

**Theorem 4.4.** Let \((Y^n, Z^n), n \in \mathbb{N}\) be defined by (4.35).

(i) Assume that \( p > 1 \) and (H1)–(H6) are satisfied. Then \( Y^n_t \to Y_t, t \in [0, T], \) and for every \( \gamma \in [1, 2), \)

\[
E\left( \int_0^T |Z^n_r - Z_r|^\gamma \, dr \right)^{p/2} \to 0, \tag{4.36}
\]

where \((Y, Z, R)\) is the unique solution of RBSDE\((\xi, f + dV, L, U)\) such that \( Y \in S^p, Z \in \mathcal{H}^p, R \in \mathcal{S}^p. \) Moreover, if \( \Delta^- R_t = 0 \) for \( t \in (0, T], \) then the above convergence also hold with \( \gamma = 2 \) and \(|Y^n - Y|_p \to 0.\)

(ii) Assume that \( p = 1 \) and (H1)–(H5), (H6*), (Z) are satisfied. Then \( Y^n_t \to Y_t, t \in [0, T], \) and for all \( \gamma \in [1, 2) \) and \( r \in (0, 1), \)

\[
E\left( \int_0^T |Z^n_r - Z_r|^\gamma \, dr \right)^{r/2} \to 0, \tag{4.37}
\]

where \((Y, Z, R)\) is the unique solution of RBSDE\((\xi, f + dV, L, U)\) such that \( Y \) is of class (D), \( Y \in S^1, Z \in \mathcal{H}^q, R \in \mathcal{V}^q, \) \( q \in (0, 1). \) Moreover, if \( \Delta^- R_t = 0 \) for \( t \in (0, T], \) then the above convergence also hold with \( \gamma = 2 \) and \(|Y^n - Y|_1 \to 0.\)

**Proof.** Notice that (4.35) one can written in the shorter form

\[
Y^n_t = \xi + \int_t^T f(r, Y^n_r, Z^n_r) \, dr + \int_t^T dV_r - \int_t^T Z^n_r \, dB_r + \int_t^T dK^n_r - \int_t^T dA^n_r, \tag{4.38}
\]

where

\[
K^n_t = n \int_0^t (\tilde{Y}_r^n - L_r)^- \, dr + \sum_{0 \leq r_m, i < t} (Y^n_{\tau_{n,i}, +} + \Delta^+ \nu_{\tau_{n,i}, +} - L_{\tau_{n,i}})^- \\
= : K^{n,*}_t + K^{n,d}_t, \quad t \in [0, T] \tag{4.39}
\]

and

\[
A^n_t = n \int_0^t (Y^n_r - U_r)^+ \, dr + \sum_{0 \leq \nu_{n,i} < t} (Y^n_{\nu_{n,i}, +} + \Delta^+ \nu_{\nu_{n,i}, +} - U_{\nu_{n,i}})^+ \\
= : A^{n,*}_t + A^{n,d}_t, \quad t \in [0, T] \tag{4.40}
\]

**Step 1.** We show the convergence of \( \{Y^n\}. \) By Step 1 of Theorem 4.2 and Theorem 4.3, we know that \( Y^n \) is the first component of a solution of RBSDE\((\xi, f + dV, L^n, U)\) with \( L^n = L \wedge Y^n, \) and \( \tilde{Y}^n \) is the first component of a solution of RBSDE\((\xi, f + dV, L, U^n)\) with \( U^n = U \vee Y^n. \) As in the Step 1 of the proof of the Theorem 4.2 one can show that the triple \((Y^n, Z^n, K^n - A^n)\) is a solution of RBSDE\((\xi, f + dV, L^n, U^n)\). By Proposition 3.2 and Lemma 3.3

\[
\tilde{Y}^n \leq Y^n \leq \bar{Y}^n, \quad n \geq 1. \tag{4.41}
\]

By Theorem 4.2 and Theorem 4.3

\[
Y^n_t \to Y_t, \quad t \in [0, T]. \tag{4.42}
\]
Let \( \hat{S}_X(D) \), \( X \) that the representation property, there exist processes \( H, D \). We are going to check assumptions the remaining assumptions (a)-(d). Let \( L \) with lower barrier \( \hat{S}_X = X \). Then \( \hat{S}_X \) from \([17, \text{Theorem 3.18}])\). Note that \( X \) to \( (4.38). \) Since we know that \( X \in M_{\text{loc}} + \mathcal{V}^p, X \in \mathcal{S}^p \) such that \( U \geq X \geq L \) and \( \int_0^T f^-(s, X_s, 0) \, ds \in \mathbb{L}^p \). If \( p = 1 \), then by \( (H6) \), there exists \( X \) of class (D), \( X \in M_{\text{loc}} + \mathcal{V}^1 \), \( X \geq L \) and \( \int_0^T f^-(s, X_s, 0) \, ds \in \mathbb{L}^1 \). Since the Brownian filtration has the representation property, there exist processes \( H \in M_{\text{loc}} \) and \( C \in \mathcal{V}^p \) such that

\[
X_t = X_T - \int_t^T dC_s - \int_t^T H_s \, dB_s, \quad t \in [0, T].
\]

Set \( X'_t = X_t, t \in [0, T], X'_T = \xi \). Then for some \( A', K' \in \mathcal{V}^{+, p} \) we have

\[
X'_t = \xi + \int_t^T f(r, X'_r, H_r) \, dr + \int_t^T dV_r + \int_t^T dK'_r - \int_t^T dA'_r - \int_t^T H_r \, dB_r, \quad t \in [0, T].
\]

Let \( (\hat{X}^n, \hat{H}^n, \hat{K}^n) \) be a solution of the RBSDE

\[
\begin{align*}
\hat{X}_t^n = & \xi + \int_t^T f(r, \hat{X}_r^n, \hat{H}_r^n) \, dr + \int_t^T dV_r + \int_t^T d\hat{K}_r^n - \int_t^T dA'_r - \int_t^T \hat{H}_r^n \, dB_r \\
& + \sum_{t \leq \tau_n, t \leq T} (\hat{X}_{\tau_n, i}^n + \Delta^+ V_{\tau_n, i} - U_{\tau_n, i})^+, \quad t \in [0, T],
\end{align*}
\]

with lower barrier \( L^n \), such that if \( p > 1 \), then \( \hat{X}^n \in \mathcal{S}^p, \hat{H}^n \in \mathcal{H}^p, \hat{K}^n \in \mathcal{V}^{+, p} \) and if \( p = 1 \), then \( \hat{X}^n \) is of class (D), \( \hat{H}^n \in \mathcal{H}^q, q \in (0, 1), \hat{A}^n \in \mathcal{V}^{+, 1} \). The existence of the solution follows from \([17, \text{Theorem 3.18}])\). Note that \( (X', H, K') \) is a solution of RBSDE(\( \xi, f + dV - dA', X' \)). Since \( L^n \leq X' \), by Proposition 3.2 and Lemma 3.3 \( \hat{X}^n \leq X' \leq U \). Thanks to this, we may rewrite (4.43) in the form

\[
\begin{align*}
\hat{X}_t^n = & \xi + \int_t^T f(r, \hat{X}_r^n, \hat{H}_r^n) \, dr + \int_t^T dV_r + \int_t^T d\hat{K}_r^n - \int_t^T dA'_r - n \int_t^T (\hat{X}_r^n - U_r)^+ \, dr \\
& - \sum_{t \leq \tau_n, t \leq T} (\hat{X}_{\tau_n, i}^n + \Delta^+ V_{\tau_n, i} - U_{\tau_n, i})^+ - \int_t^T \hat{H}_r^n \, dB_r, \quad t \in [0, T].
\end{align*}
\]

By Proposition 3.2 and Lemma 3.3 again,

\[
\hat{X}^n \leq \sum^n,
\]

and by Proposition 4.1

\[
d\hat{K}^n \geq dK^n, \quad n \geq 1.
\]

Moreover,

\[
(\hat{X}_{\tau_n, i}^n + \Delta^+ V_{\tau_n, i} - U_{\tau_n, i})^+ \leq (X'_{\tau_n, i} + \Delta^+ V_{\tau_n, i} - U_{\tau_n, i})^+ \\
= (\Delta^+ X'_{\tau_n, i} + \Delta^+ V_{\tau_n, i} + X'_{\tau_n, i} - U_{\tau_n, i})^+ \\
\leq (\Delta^+ X_{\tau_n, i} + \Delta^+ V_{\tau_n, i})^+ \\
\leq \Delta^+ |C|_{\tau_n, i} + \Delta^+ |V|_{\tau_n, i}.
\]
Let \((\tilde{X}^n, \tilde{H}^n, \tilde{K}^n)\) be a solution of the RBSDE

\[
\begin{align*}
\tilde{X}_t^n &= \xi + \int_t^T f(r, \tilde{X}^n_r, \tilde{H}^n_r) \, dr + \int_t^T dV_r + \int_t^T d\tilde{K}_r^n - \int_t^T dA_r^n - n \int_t^T (\tilde{X}_r^n - U_r)^+ \, dr \\
&\quad - \int_t^T dC|_r - \int_t^T d|V_r| - \int_t^T \tilde{H}_r^n \, dB_r, \quad t \in [0, T],
\end{align*}
\]

with lower barrier \(L^n\), such that if \(p > 1\), then \(\tilde{X}^n \in \mathcal{S}^p\), \(\tilde{H}^n \in \mathcal{H}^p\), \(\tilde{K}^n \in \mathcal{V}^{+p}\) and if \(p = 1\), then \(\tilde{X}^n\) is of class (D), \(\tilde{H}^n \in \mathcal{H}^q\), \(q \in (0, 1)\), \(\tilde{K}^n \in \mathcal{V}^{+1}\). The existence of the solution follows [17] Theorem 3.18. By Proposition 3.2 and Lemma 3.3, \(\tilde{X}^n \leq \tilde{X}^n\). By this and (4.44),

\[
\tilde{X}^n \leq \tilde{Y}^n. \tag{4.46}
\]

Moreover, by Proposition 5.1 \(d\tilde{K}^n \leq d\tilde{K}^n\), which by (4.45) implies that

\[
dK^n \leq d\tilde{K}^n. \tag{4.47}
\]

By [16] Lemma 4.8, there exists a chain \(\{\tau'_k\}\) such that

\[
E\left(\sup_{t \leq \tau'_k} (|Y^1_t|^2 + |Y^1_t|^2)\right) < \infty, \quad k \geq 1. \tag{4.48}
\]

Define \(\tau''_k = \inf\{t \geq 0; \int_t^f |f(r, 0, 0)| \, dr + \int_t^f d|V_r| + \int_t^f f^-(r, Y^1_r, 0) \, dr \geq k\} \wedge T, \; k \in \mathbb{N}\). Since \(Y^{n+1} \leq Y^n\) and \(Y^n \leq Y^{n+1}\) by (4.41) we have that

\[
Y^1 \leq Y^n \leq Y^1. \tag{4.49}
\]

Hence

\[
|Y^n| \leq |Y^1| + |Y^n|. \tag{4.50}
\]

Set \(\tau_k = \tau'_k \wedge \tau''_k\). By (4.50) and [16] Proposition 4.3,

\[
E((\tilde{K}^n_{\tau_k})^2) \leq CE \left(\sup_{t \leq \tau_k} (|Y^1_{\tau_k}|^2 + |Y^1_{\tau_k}|^2 + \left(\int_0^{\tau_k} |f(r, 0, 0)| \, dr\right)^2 \right. \\
\left. + \left(\int_0^{\tau_k} d|V_r|\right)^2 + \left(\int_0^{\tau_k} f^-(r, Y^1_r, 0) \, dr\right)^2\right) < \infty. \tag{4.51}
\]

That (a) is satisfied now follows from (4.47), (4.51) and [16] Lemma 4.8. Since \((Y^n, Z^n, K^n - A^n)\) is a solution to RBSDE\((\xi, f + dV, L^n, U^n)\), for \(\sigma, \tau \in \Gamma\) such that \(\sigma \leq \tau\) we have

\[
\int_\sigma^\tau (Y_r - Y^n_r) \, dD^n_r + \sum_{\sigma \leq t < \tau} (Y_t - Y^n_t)\Delta^+ D^n_t \geq \int_\sigma^\tau (Y_r - Y^n_r) \, dV^*_r + \sum_{\sigma \leq t < \tau} (Y_t - Y^n_t)\Delta^+ V_t.
\]

From this, (4.50) and the Lebesgue dominated convergence theorem we get (b). Assumption (c) follows from the inequality \(|\Delta^- (Y_r - Y^n_r)| = |\Delta^- Y_r| \leq |R|_t + \Delta^- |V|_t\). Assumption (d) follows from (4.49) and Proposition 3.3. Since (a)–(c) are satisfied, Lemma 2.6 yields

\[
Z^n \to Z, \quad dt \otimes P\text{-a.e.} \tag{4.52}
\]

34
Step 3. We will show (4.36) and (4.37). By (4.47), (4.50) and [16] Lemma 4.2, if $p > 1$, then
\[
E\left( \int_0^T |Z^n_t|^2 \, ds \right)^{p/2} \leq CE\left( \sup_{t \leq T} (|\bar{Y}^n_t|^p + |\bar{Y}_t^n|^p) + \left( \int_0^T d|V|^p \right) + \left( \int_0^T |f^{-}(s, Y^n_1, 0)| \, ds \right)^p + \left( \int_0^T f(s, 0, 0) \, ds \right)^p + \left( \int_0^T d\bar{K}_s^n \right)^p \right),
\]
and if $p = 1$, then for every $q \in (0, 1)$,
\[
E\left( \int_0^T |Z^n_t|^2 \, ds \right)^{q/2} \leq CE\left( \sup_{t \leq T} (|\bar{Y}^n_t|^q + |\bar{Y}_t^n|^q) + \left( \int_0^T |f(s, 0, 0)| \, ds \right)^q + \left( \int_0^T d|V|^q \right) + \left( \int_0^T d\bar{K}_s^n \right)^q \right).
\]

Let $(\bar{X}, \bar{H})$ be a solution of the BSDE
\[
\bar{X}_t = \xi + \int_t^T f(r, \bar{X}_r, \bar{H}_r) \, dr + \int_t^T dV_r - \int_t^T dA'_r, \\
- \int_t^T d|C|_r - \int_t^T d|V|_r - \int_t^T \bar{H}_r \, dB_r, \quad t \in [0, T],
\]
such that if $p > 1$, then $\bar{X} \in \mathcal{S}^p$, $\bar{H} \in \mathcal{H}^p$, $\bar{K} \in \mathcal{V}^{+p}$ and if $p = 1$, then $\bar{X}$ is of class (D), $\bar{H} \in \mathcal{H}^q$, $q \in (0, 1)$, $\bar{K} \in \mathcal{V}^{+1}$. The existence of the solution follows from [17] Theorem 3.18. By Proposition 3.2 and Lemma 3.8, $\bar{X} \leq \bar{X}^n$, so by (4.46),
\[
\tilde{X} \leq \bar{X}^n \leq Y^1.
\]

By (4.55) and [16] Lemma 4.2, if $p > 1$, then
\[
E(\bar{K}^n_T)^p \leq CE\left( \sup_{t \leq T} (|\bar{Y}^n_t|^p + |\bar{X}_t|^p) + \left( \int_0^T d|V|^p \right) + \left( \int_0^T |f^{-}(s, Y^n_1, 0)| \, ds \right)^p + \left( \int_0^T f(s, 0, 0) \, ds \right)^p + \left( \int_0^T d\bar{K}_s^n \right)^p \right),
\]
and if $p = 1$, then for every $q \in (0, 1)$,
\[
E(\bar{K}^n_T)^q \leq CE\left( \sup_{t \leq T} (|\bar{Y}^n_t|^q + |\bar{X}_t|^q) + \left( \int_0^T d|V|^q \right) + \left( \int_0^T |f^{-}(s, Y^n_1, 0)| \, ds \right)^q + \left( \int_0^T f(s, 0, 0) \, ds \right)^q + \left( \int_0^T d\bar{K}_s^n \right)^q \right).
\]

In case $p > 1$, combining (4.53) with (4.56) we get
\[
E\left( \int_0^T |Z^n_t|^2 \, dr \right)^{p/2} \leq CE\left( \sup_{t \leq T} (|\bar{Y}^n_t|^p + |\bar{Y}_t^n|^p + |\bar{X}_t|^p) + \left( \int_0^T d|V|^p \right) + \left( \int_0^T |f^{-}(r, Y^n_1, 0)| \, dr \right)^p + \left( \int_0^T f(r, 0, 0) \, dr \right)^p + \left( \int_0^T dA'_r \right)^p + \left( \int_0^T d|C|^p \right)^p \right),
\]
and if $p = 1$, then for every $q \in (0, 1)$,
In case $p = 1$, combining (4.54) with (4.57) we get
\[
E\left(\int_0^T |Z_r^n|^2 \, dr\right)^{q/2} \leq CE\left(\sup_{t \leq T}|Y_t^n|^q + |Y_t^n|^q + \left(\int_0^T |f(r,0,0)| \, dr\right)^q + \left(\int_0^T |f^-(r,Y_t^n,0)| \, dr\right)^q + \left(\int_0^T d|V|_r\right)^q + \left(\int_0^T d|C|_r\right)^q\right)
\]
for $q \in (0,1)$. From (4.52) and (4.58), (4.59) we easily get (4.36) and (4.37).

**Step 4.** We will show that if $\Delta^- R = 0$, then (4.36) and (4.37) hold with $\gamma = 2$ and $|Y^n - Y|_p \to 0$. To this end, we first note that by (4.41),
\[
\sup_{t \leq T}|Y_t^n - Y_t| \leq \sup_{t \leq T}|Y_t^n - Y_t|_{\infty} + |Y^n - Y|_p.
\]
By this, Theorem 4.2 and Theorem 4.3, $|Y^n - Y|_p \to 0$. Now set $R^n = K^n - A^n, n \in \mathbb{N}$, and observe that by [17, Corollary 5.5], hypotheses (H1) and (H2) and the assumption that $\Delta^- R = 0$,
\[
\int_0^T |Z_r - Z_r^n|^2 \, dr \leq 2\lambda \int_0^T |Y_r - Y_r^n||Z_r - Z_r^n| \, dr + 2 \int_0^T (Y_r - Y_r^n) d(R - R^n)_r + 2 \sum_{0 \leq t < T} (Y_t - Y_t^n) \Delta^+ (R_t - R_t^n)
\]
\[
+ \sup_{0 \leq t \leq T} \int_0^T (Y_r - Y_r^n)(Z_r - Z_r^n) dB_r
\]
\[
\leq 2\lambda \int_0^T |Y_r - Y_r^n||Z_r - Z_r^n| \, dr + 2 \int_0^T (Y_r - Y_r^n) dR^c_r + 2 \sum_{0 \leq t < T} (Y_t - Y_t^n) \Delta^+ R_t + \sup_{0 \leq t \leq T} \int_0^T (Y_r - Y_r^n)(Z_r - Z_r^n) dB_r.
\]
Applying the the Burkholder–Davis–Gundy inequality yields
\[
E\left(\int_0^T |Z_r - Z_r^n|^2 \, dr\right)^{p/2} \leq C \left(E \sup_{0 \leq t \leq T} |Y_t - Y_t^n|^p + (E \sup_{0 \leq t \leq T} |Y_t - Y_t^n|^p)^{1/2} (E|R_T^n|)^{1/2}\right).
\]
It is clear that the above inequality implies (4.36) and (4.37) with $\gamma = 2$, which completes the proof.

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