Additive entropy underlying the general composable entropy prescribed by thermodynamic meta-equilibrium

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Abstract

We consider the meta-equilibrium state of a composite system made up of independent subsystems satisfying the additive form of external constraints, as recently discussed by Abe [Phys. Rev. E 63, 061105 (2001)]. We derive the additive entropy $S$ underlying a composable entropy $\tilde{S}$ by identifying the common intensive variable. The simplest form of composable entropy satisfies Tsallis-type nonadditivity and the most general composable form is interpreted as a monotonically increasing function $H$ of this simplest form. This is consistent with the observation that the meta-equilibrium can be equivalently described by the maximum of either $H[\tilde{S}]$ or $\tilde{S}$ and the intensive variable is same in both cases.

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Generalized entropic functionals which may yield consistent formulations of thermodynamics [1] and statistical mechanics [2] has attracted attention in recent years. Generalization of the standard entropy can also be a useful tool to describe effects of finiteness in physical systems [3]. The standard thermodynamic formalism is based on Boltzmann-Gibbs-Shannon entropy, which is additive. A particular generalization of the standard formalism that has been much studied recently is popularly called Tsallis statistics [4]. This formalism seems to provide an effective description of the meta-equilibrium states of certain complex systems. It is based on a generalized entropy that is nonadditive even when the subsystems forming the composite system are statistically independent. Tsallis-type nonadditivity may be considered as the simplest case of the most general nonadditivity rule for composable entropy, consistent with the existence of thermodynamic equilibrium [1, 5]. This so-called equilibrium within the Tsallis approach is actually a meta-equilibrium, which is described by a generalized zeroth law.

On the other hand, it has been suggested that the thermodynamic framework based on Tsallis entropy can be mapped to the framework based on additive (extensive) entropy [6]-[8]. Essentially, it means that a combination of an additive entropy and an appropriate intensive variable preserves the standard thermodynamic relations as well as statistical fluctuations.

In this letter, we derive the additive entropy underlying a composable entropy $\tilde{S}$ by identifying the intensive variable common to subsystems in meta-equilibrium. The case of the most general form for composable entropy
as treated in [1] is interpreted as a monotonically increasing function $H$ of the simplest composable entropy. This is consistent with the fact that the intensive variable should be same in the meta-equilibrium state corresponding to the maximum of either $H[\tilde{S}]$ or $\tilde{S}$.

Tsallis entropy is defined as

$$S_q^{(T)} = \frac{\sum_{i=1}^{W} p_i^q - 1}{1 - q},$$

(1)

where $p_i$ is the probability distribution set characterising the discrete microstates of the system and labelled $i = 1, ..., W$. The transformation relating additive and nonadditive entropies is given by

$$S_q^{(R)} = \frac{1}{(1 - q)\ln[1 + (1 - q)S_q^{(T)}]}.$$  

(2)

The additive entropy so obtained is well known as Renyi entropy [9].

For better understanding of this transformation, we first derive it directly by applying the generalized zeroth law for meta-equilibrium within Tsallis’ framework. In line with earlier works [1], we impose additive mean values as constraints. Thus we restrict to a special class of nonextensive models in which the entropy is non-additive, but the energy and other external constraints are additive. Furthermore, it is understood that the zeroth law in the standard Boltzmann-Gibbs formalism identifies an intensive variable common to the systems in mutual thermodynamic equilibrium. The generalized zeroth law serves a similar purpose within Tsallis approach.

Consider the meta-equilibrium between two systems $A$ and $B$, such that the maximum of the nonadditive entropy $\tilde{S}_q$ holds for the composite system
\[(A + B)\]

\[
\tilde{S}_q(A + B) = \tilde{S}_q(A) + \tilde{S}_q(B) + (1 - q)\tilde{S}_q(A)\tilde{S}_q(B),
\]

under the constraints which are fixed values of \(n\) number of additive quantities

\[
E_k(A + B) = E_k(A) + E_k(B), \quad k = 1, ..., n.
\]

As in standard thermodynamics, we define the meta-equilibrium entropy for the nonadditive case to be \(k_B\tilde{S}_q(\{E_k|k = 1, ..., n\})\), where \(k_B\) is the Boltzmann’s constant. Apparently, \(\tilde{S}_q\) is the explicit entropy function in terms of the set of given additive constraints \(\{E_k|k = 1, ..., n\}\) and is obtained from the optimum distribution \(\{p_i|i = 1, ..., W\}\) following from the maximisation of \(\tilde{S}_q(\{p_i\})\) [10].

Now making the variations \(\delta\tilde{S}_q(A + B)\) and \(\delta E_k(A + B)\) for \(k = 1, ..., n\) vanish for the meta-equilibrium state, we obtain

\[
[1 + (1 - q)\tilde{S}_q(B)]\frac{\partial\tilde{S}_q(A)}{\partial E_k(A)} = [1 + (1 - q)\tilde{S}_q(A)]\frac{\partial\tilde{S}_q(B)}{\partial E_k(B)}. \quad (5)
\]

On rearranging (5), we can achieve separation of variables as

\[
\frac{1}{[1 + (1 - q)\tilde{S}_q(A)]}\frac{\partial\tilde{S}_q(A)}{\partial E_k(A)} = \frac{1}{[1 + (1 - q)\tilde{S}_q(B)]}\frac{\partial\tilde{S}_q(B)}{\partial E_k(B)} = \eta_k. \quad (6)
\]

The parameter \(\eta_k\), by definition, is common to both the subsystems \(A\) and \(B\), and is identified as the intensive variable. Dropping the index \(A\) or \(B\), for each subsystem the following equation holds for all values of \(k\)

\[
\frac{1}{[1 + (1 - q)\tilde{S}_q]}\frac{\partial}{\partial E_k} \tilde{S}_q(E_1, ..., E_k, ..., E_n) = \eta_k. \quad (7)
\]
Note that \( \frac{\partial S_q}{\partial E_k} \) defines the Lagrange multiplier associated with the maximum entropy method and so according to the above equation, intensive variables and the corresponding Lagrange multipliers are not identical, except when the entropy is additive.

The relation with an additive entropy can be established if we assume that the intensive variable \( \eta_k \) occurring above is the same as defined for an additive entropy \( S(E_1, ..., E_k, ..., E_n) \) and is given by

\[
\eta_k = \frac{\partial S}{\partial E_k}.
\]  

(8)

We remark here that the choice of the form of additive entropy is arbitrary here; one is free to use either BGS entropy or Renyi entropy. From (7) and (8), this implies that the nonadditive and the additive entropies are related in the following way

\[
\frac{1}{1-q} \ln \left[ 1 + (1-q) \tilde{S}_q \right] = S + c,
\]  

(9)

where the constant \( c \) has to be independent of \( E_k \). Thus the transformation of a nonadditive entropy to an additive one is recovered for the particular choice of \( c = 0 \).

Next, we explore this relation between nonadditive and additive entropies from the viewpoint of composable entropies as prescribed by the existence of meta-equilibrium. To proceed further, we recall the notion of composability of entropy [4, 12]. An arbitrary entropic form \( \tilde{S} \) is defined to be composable, if the total entropy \( \tilde{S}(A, B) \) for the composite system can be written as
\( \tilde{S}(A, B) = f[\tilde{S}(A), \tilde{S}(B)] \), where \( f[\cdot] \) is a certain bivariate function of the \( C^2 \) class and is symmetric in its arguments, \( f[\tilde{S}(A), \tilde{S}(B)] = f[\tilde{S}(B), \tilde{S}(A)] \).

Suppose the maximum of the general entropy determines a kind of meta-equilibrium between subsystems \( A \) and \( B \), under the given additive constraints \( \text{(1)} \). By equating the variations of the total entropy and total value of the constraint quantity to zero, we get

\[
\frac{\partial \tilde{S}(A, B)}{\partial \tilde{S}(A)} \frac{\partial \tilde{S}(A)}{\partial E_k(A)} = \frac{\partial \tilde{S}(A, B)}{\partial \tilde{S}(B)} \frac{\partial \tilde{S}(B)}{\partial E_k(B)}. \tag{10}
\]

To establish an intensive variable common to the two subsystems, a general case can be written in the following form \( \text{(11)} \)

\[
\frac{\partial \tilde{S}(A, B)}{\partial \tilde{S}(A)} = \frac{1}{\lambda} G[\tilde{S}(A, B)] \frac{dh[\tilde{S}(A)]}{d\tilde{S}(A)} h[\tilde{S}(B)], \tag{11}
\]

\[
\frac{\partial \tilde{S}(A, B)}{\partial \tilde{S}(B)} = \frac{1}{\lambda} G[\tilde{S}(A, B)] h[\tilde{S}(A)] \frac{dh[\tilde{S}(B)]}{d\tilde{S}(B)}, \tag{12}
\]

where \( h[\cdot] \) is some differentiable function; \( G[\cdot] \) is also an arbitrary function and \( \lambda \) is a constant. Therefore, using \( \text{(11)} \) and \( \text{(12)} \) in \( \text{(10)} \) and rearranging, an intensive variable common to systems \( A \) and \( B \) can be defined as

\[
\frac{1}{\lambda h[\tilde{S}(A)]} \frac{dh[\tilde{S}(A)]}{d\tilde{S}(A)} \frac{\partial \tilde{S}(A)}{\partial E_k(A)} = \frac{1}{\lambda h[\tilde{S}(B)]} \frac{dh[\tilde{S}(B)]}{d\tilde{S}(B)} \frac{\partial \tilde{S}(B)}{\partial E_k(B)} = \eta_k. \tag{13}
\]

Again, \( \frac{\partial \tilde{S}}{\partial E_k} = \tilde{\eta}_k \) is the Lagrange multiplier associated with system \( A \) or \( B \). Thus \( \text{(13)} \) defines the relation between the Lagrange multiplier and the intensive variable for a general composable entropy.

We emphasize that in the present approach the intensive variable associated with a general composable entropy is \textit{independent of the function} \( G \).
which by definition, is not factorisable into contributions from systems $A$ and $B$.

For any subsystem ($A$ or $B$), we can rewrite (13) as

$$\frac{1}{\lambda h[S]} \frac{\partial h[\tilde{S}]}{\partial E_k} = \eta_k.$$  \hspace{1cm} (14)

Again invoking the relation (8), we note that the function $h$ is related to the additive entropy $S$ as $h[\tilde{S}] = \exp(\lambda \{S + c\})$, where $c$ has been identified before. Setting $\tilde{S} = 0$ for $S = 0$, we obtain

$$h[0] = \exp(\lambda c).$$  \hspace{1cm} (15)

Note that for $\lambda \to 0$, we have $h[0] \to 1$, a property assumed in [1]. Alternately, if the constant is set as $c = 0$, then $h[0]$ is unity for all values of $\lambda$. In general, we write

$$h[\tilde{S}] = h[0]\exp(\lambda S).$$  \hspace{1cm} (16)

This form of the function $h$ is solely determined by the requirement of the existence of an intensive variable associated with an additive entropy $S$. It is thus intrinsically independent of the form of the $G[::]$ function.

We remark here that Tsallis type nonadditivity of entropy is obtained as a special case of the above analysis, if we put $G$ as a constant equal to unity, and identify for each subsystem

$$h[\tilde{S}] = 1 + \lambda \tilde{S},$$  \hspace{1cm} (17)

which gives $h[0] = 1$. Then equation (13) is identical to (6) for $\lambda = (1 - q)$. 

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On combining (16) and (17), we have
\[ \tilde{S} = \exp(\lambda S) - 1, \] (18)
which satisfies
\[ \tilde{S}(A + B) = \tilde{S}(A) + \tilde{S}(B) + \lambda \tilde{S}(A) \tilde{S}(B). \] (19)

Next, we ascertain the relation between the additive entropy \( S \) and the composable entropy \( \tilde{S} \) for a general function \( G \). Following [1], it is convenient to set
\[ G[\tilde{S}(A, B)] = \left( \frac{dH[\tilde{S}(A, B)]}{d\tilde{S}(A, B)} \right)^{-1}. \] (20)
This allows to obtain the composability rule for entropy in terms of the function \( H \). The case of Tsallis type nonadditivity corresponds in this notation to \( H \) as an identity function. For concreteness, let us assume that the function \( G[\cdot] \) is positive, implying that \( H[\tilde{S}] \) is a monotonically increasing (differentiable) function of \( \tilde{S} \). We give an interpretation of this assumption later on.

So using (20), the relations (11) and (12) are rewritten as
\[ \frac{dH[\tilde{S}(A, B)]}{d\tilde{S}(A, B)} \frac{\partial \tilde{S}(A, B)}{\partial \tilde{S}(A)} = \frac{1}{\lambda} \frac{dH[\tilde{S}(A)]}{d\tilde{S}(A)} \frac{h[\tilde{S}(B)]}{h[\tilde{S}(A)]}, \] (21)
\[ \frac{dH[\tilde{S}(A, B)]}{d\tilde{S}(A, B)} \frac{\partial \tilde{S}(A, B)}{\partial \tilde{S}(B)} = \frac{1}{\lambda} h[\tilde{S}(A)] \frac{dh[\tilde{S}(B)]}{d\tilde{S}(B)}, \] (22)
which further imply
\[ \frac{\partial H[\tilde{S}(A, B)]}{\partial \tilde{S}(A)} = \frac{1}{\lambda} \frac{dH[\tilde{S}(A)]}{d\tilde{S}(A)} h[\tilde{S}(B)], \] (23)
\[ \frac{\partial H[\tilde{S}(A, B)]}{\partial \tilde{S}(B)} = \frac{1}{\lambda} h[\tilde{S}(A)] \frac{dh[\tilde{S}(B)]}{d\tilde{S}(B)}. \] (24)
On integrating (23) or (24), we obtain the composability rule for entropy as

$$H[\tilde{S}(A, B)] = \frac{1}{\lambda} h[\tilde{S}(A)] h[\tilde{S}(B)] + \text{constant.} \quad (25)$$

The constant is determined by the requirement that $\tilde{S}(A, B) = 0$ for $\tilde{S}(A) = \tilde{S}(B) = 0$. Finally on using (16), we obtain

$$H[\tilde{S}(A, B)] = \frac{h^2[0]}{\lambda} \left\{ \exp\left[\lambda S(A) + S(B)\right] - 1 \right\} + H[0]. \quad (26)$$

Using the fact that for an ordered system, say $B$, $S(B) = 0$, which implies $\tilde{S}(A, B) = \tilde{S}(A)$, we obtain the transformation between an additive entropy and the general composable entropy for a subsystem ($A$ or $B$) as follows

$$S = \frac{1}{\lambda} \ln \left[ 1 + \frac{\lambda}{h^2[0]} \left( H[\tilde{S}] - H[0] \right) \right]. \quad (27)$$

This relation may be taken as the most general transformation connecting additive and composable nonadditive entropies compatible with the generalized zeroth law. Note that it is not implied that any composable entropy can be mapped to an additive entropy. As we argue below, the identification of an intensive variable which is independent of the function $G$ or $H$, seems to imply that the composable entropy is either $\tilde{S}$ which satisfies Tsallis-type nonadditivity, or it is a monotonically increasing function $H$ of $\tilde{S}$.

To illustrate this conclusion, suppose that the meta-equilibrium is described by the maximum of $H[\tilde{S}(A, B)]$ under the additive constraints (4). Then making the variation $\delta H$ vanish under $\delta E_k(A) = -\delta E_k(B)$, the condition of equilibrium implies

$$\frac{\partial H[\tilde{S}(A, B)]}{\partial \tilde{S}(A)} \frac{\partial \tilde{S}(A)}{\partial E_k(A)} = \frac{\partial H[\tilde{S}(A, B)]}{\partial \tilde{S}(B)} \frac{\partial \tilde{S}(B)}{\partial E_k(B)}. \quad (28)$$
This can be rewritten as
\[
\frac{dH[\tilde{S}(A, B)]}{d\tilde{S}(A, B)} \frac{\partial \tilde{S}(A)}{\partial E_k(A)} \frac{\partial \tilde{S}(A)}{\partial E_k(B)} = \frac{dH[\tilde{S}(A, B)]}{d\tilde{S}(A, B)} \frac{\partial \tilde{S}(A, B)}{\partial E_k(B)},
\]
(29)

Now assuming the conditions (11), (12) and on using (20) we obtain the same relation as (13). Thus the intensive variable corresponding to the maximisation of $\tilde{S}(A, B)$ is identical to that for the maximisation of $H[\tilde{S}(A, B)]$, which is in accordance with the fact that $H$ has been assumed to be a monotonic function of $\tilde{S}$ and thus the equilibrium condition obtained from each is identical under similar constraints. Again note that for $H$ to be identical function, the above equations go neatly to the case of the simplest composable entropy.

It may be pointed out that although the intensive variable is the same for the maximisation of either $\tilde{S}$ or $H[\tilde{S}]$, yet the relation between intensive variable $\eta_k$ and the corresponding Lagrange multiplier $\tilde{\eta}_k$ depends on the specific entropy chosen for maximisation. For the case of Tsallis entropy, the intensive variable is given by [6, 7]
\[
\eta_k = \frac{\tilde{\eta}_k}{\{1 + (1 - q)S_q^{(T)}\}}.
\]
(30)

To obtain the corresponding relation for the case of most general composable entropy, we can apply the partial derivative with respect to $E_k$ to Eq. (27) and obtain
\[
\eta_k = \frac{\tilde{\eta}_k^{(H)}}{\{h^2[0] + \lambda(H[\tilde{S}] - H[0])\}} \frac{dH[\tilde{S}]}{d\tilde{S}}.
\]
(31)

Summarising, it has been shown in literature that an additive entropy underlies the Tsallis entropy, which along with an intensive variable, preserves
the standard thermodynamic structure. In this letter, we have looked at
the relation between nonadditive and additive entropies from the viewpoint
of the generalised zeroth law and the notion of composable entropies. We
have shown the mapping between Tsallis-type nonadditive entropy (which
is the simplest composable form consistent with meta-equilibrium) and an
additive entropy by identifying the intensive variable common to two sys-
tems in meta-equilibrium. The abovementioned mapping becomes possible
if this intensive variable is also the one associated with an additive entropy.
Further, we have argued that if the meta-equilibrium is alternately described
by the maximum of the most general composable entropy $H$, then $H$ can be
interpreted as a monotonically increasing function of the Tsallis-type nonad-
ditive entropy. The fact that the meta-equilibrium state is expected to be
physically the same in either maximisation problem, is consistent with the
observation that the intensive variable is same in both the cases.

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