RELATIVE-ERROR STABILITY OF NUMERICAL ALGORITHMS
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Abstract. We formalize the definition of a stable algorithm that is (i) adapted to the use of multiple and variable precision arithmetic, (ii) sufficiently close to the actual practice of computing to be useful, and (iii) sufficiently robust from a mathematical point of view as to allow for the rigorous proof of theorems. This allows us to state some widely satisfied hypotheses, depending only on two functions \( f \) and \( g \), under which the composition of a stable algorithm for \( f \) and a stable algorithm for \( g \) is a stable algorithm for the composition \( f \circ g \).

Key words. condition number, numerical stability, numerical algorithm, composition theorem, amenability, relative error

AMS subject classifications. 68Q17, 65G50, 65G99, 68Q99

1. Introduction. This paper stems from a misleadingly simple question: what does it mean that an algorithm for computing a mathematical function is stable?

Let us start with a concise—but necessarily incomplete—overview of the modern history of this question. Stored-program computers were developed conceptually from the 1930s [35] and physically from the 1940s. As computers became increasingly common tools for engineers, mathematicians, and scientists, the reliability of their approximate computations has been of great interest. Especially in numerical linear algebra, the two related concepts of stability of an algorithm and condition of a problem have been simultaneously used to investigate this topic. Both ideas arguably originated in the 1940s with the pioneering work of von Neumann and Goldstine [37] and then Turing [36] on linear systems of equations; Turing is credited for introducing the term “condition number.” The study of stability was further developed in the 1950-1960s by Wilkinson, in the form of backward error analysis; his books [38, 39] contain many of his findings. Later decades saw early statements of a formal definition of stability: in this sense, de Jong [15] and Bunch [11] paved the way for our study. In the 1990s, Higham published the first edition of his monograph [21], which is still regarded as a cornerstone reference on the stability of algorithms in linear algebra. Trefethen and Bau also address the topic in several chapters of their equally celebrated book [34]. A classic reference for the theory of condition is the 1966 paper by Rice [30], where a general mathematical framework was built up. A more geometric approach to condition numbers is summarized in the monograph [12] by Bürgisser and Cucker. This viewpoint arose in the context of the intricate link between complexity and computation. For earlier developments in this area, we refer to Blum, Cucker, Shub, and Smale [8]. Finally, we note that stochastic approaches to stability and condition have also been proposed: Stewart’s stochastic perturbation theory [32], smoothed analysis [31], smoothed condition numbers [2], or mixed stochastic analysis [24].
To illustrate some of these ideas in more detail, consider the problem studied by Turing [36]: given a square, nonsingular matrix \( A \) and a vector \( b \), we want to solve \( Ax = b \) for \( x \). Suppose that the coefficients of \( A \) come from, say, experimental measurements so that their precision is limited. Assuming exactness of subsequent computations, we can compare the solution of \( Ax = b \) obtained using the “measured” value of \( A \) to the ideal solution associated with the “true” value of \( A \). How does the accuracy of the solution \( x \) depend on the errors made in estimating \( A \)? The question soon evolved into two distinct, yet related, concepts, discussed next.

**Condition:** The intrinsic hardness of problems. Here, one seeks to measure the first-order sensitivity of the solution with respect to variations in the input parameters. This is captured by the condition number of the problem. For example, the relative condition number of matrix inversion is given by Turing’s famous expression \( \kappa(A) = \|A\|\|A^{-1}\| \). It answers the question posed above by quantifying by how much a relative error in the coefficient matrix \( A \) can amplify the relative error in the solution \( x \). If the solution to \( Ax = b \) is computed with fixed precision, the above condition number must be compared with the precision. The larger their product, the fewer digits of the approximate solution one can trust, irrespective of the computational method. Although it often features in analyses of approximated computations, we stress that the condition number is a property of a problem that is independent of the choice or even the existence of an algorithm.

**Stability:** The reliability of an algorithm. Here, one looks for a measure of how much an algorithm can be trusted when executed in finite-precision arithmetic: even if some problem instance is well-behaved in the sense that its condition number is small, it can happen that an algorithm does not yield a satisfactory answer. This is captured by the more qualitative concept of stability. An algorithm is unstable if it can produce answers whose error is substantially in excess of what one expects from the problem’s condition number; on the other hand, it is stable if this is guaranteed not to happen. A classic example of an unstable algorithm for the overdetermined least-squares problem \( Cx = d \), where \( C \) is a left-invertible rectangular matrix, consists of transforming it into the algebraically equivalent linear system of normal equations \( (C^TC)x = C^Td \) and solving it by Gaussian elimination with full pivoting [7]. While both of these algorithms are stable, their composition nevertheless yields a disappointingly unstable algorithm for solving the original problem. The culprit is the condition number of the Gram matrix \( C^TC \) which is the square of \( \kappa(C) \) [21]. Fortunately, stable algorithms to solve \( Cx = d \) exist, e.g., one can compute a reduced singular value decomposition of \( C = USV^T \), solve the diagonal system \( Sy = U^Td \), and recover \( x = Vy \).

While the formal definition of the condition number is well established [12, 30], the notion of numerical stability is subtler and contextually flexible. Modern-classic references such as [21, 34] consider three different definitions of increasing strength:

1. A forward stable algorithm (also called weakly stable in [11]) gives an answer that is close to the actual exact answer for the given input. In the above example, a forward stable algorithm for \( Ax = b \) finds \( \tilde{x} \approx x \), where \( x \) is the exact solution. The approximate equality \( \approx \) is allowed to hide a linear dependence on the condition number \( \kappa(A) \).

2. A mixed forward-backward stable algorithm (sometimes called numerically stable [15, 21]) gives an almost exact answer to a nearby problem. In the \( Ax = b \) example above, a mixed forward-backward stable algorithm would find \( \tilde{x} \approx y \), where \( y \) is the exact solution to \( \tilde{A}y = b \) for some \( \tilde{A} \approx A \). Now the approximate equality \( \approx \) is allowed to depend polynomially on the size of the
matrix $A$ but, unlike forward stability, it is independent of $\kappa(A)$.

3. A backward stable algorithm (also called strongly stable in [11]) provides the exact answer to a nearby problem. In the $Ax = b$ example above, a backward stable algorithm would find $\tilde{x}$ such that $A\tilde{x} = b$ for some $\tilde{A} \approx A$. Here, the approximate equality $\approx$ is independent of $\kappa(A)$.

These definitions supported the analysis of numerical algorithms for decades. Yet, when they were introduced, mathematical software was typically implemented in one fixed precision floating-point arithmetic, such as either double or single precision. This paradigm is now changing [1,4,6,13,17,18,22,23] as we currently witness an expanding use of both high-precision (for enhanced accuracy) and low-precision (for enhanced speed) arithmetic. Some recent algorithms even combine multiple precisions: the idea is to work with a lower precision for steps where rounding errors are not a concern and with a higher precision for steps where more care is necessary; see [1] for a survey of numerical algorithms in this setting. Others are implemented in variable precision: this flexibility is key to be able to adapt both to an accuracy required by the user and to the condition number of the problem instance.

The aforementioned evolution in algorithm design calls for an update of the concept of stability. Moreover, the flexibility in existing definitions of stability prevents us from rigorously proving results that are useful in practice. The main goal of this paper is filling these gaps. That is, we will

1. formalize numerical stability of a numerical algorithm to approximate a mathematical function (treated in Section 2 to 4);
2. exploit this formalization to prove a composition theorem that gives a sufficient condition under which the composition of two stable algorithms yields another stable numerical algorithm (treated in Section 5); and
3. illustrate how these definitions and techniques can be exploited to prove stability of elementary functions, prove instability of fast matrix multiplication, and indicate the limits of the composition theorem (treated in Section 6 to 8).

Our approach builds on the classic notions of condition number and stability. The strongest notions of stability are too rigid for our goals. For example, as we discuss later, the composition of two backward stable algorithms is not always backward stable. We therefore adopt the weakest of the classic notions of stability, that is, forward stability. Although forward stability is sometimes overlooked as too weak, we argue that there are several reasons to give it more credit:

1. It allows us to produce formal results concerning the stability of compositions of algorithms. We view this as a fundamental cornerstone for a mathematical theory of stability to be sufficiently powerful.
2. The final users of numerical algorithms are often not numerical analysts, but rather scientists and engineers. Being fully based on forward rather than backward errors, forward stability is arguably easier to interpret for such non-experts. Forward stability analysis additionally exposes the hardness of the problem as measured by the condition number, while a backward error analysis may inadvertently give a non-expert the impression that the numerically computed outputs can be trusted regardless of the problem’s condition.
3. As mentioned above, increasingly often algorithms are being implemented in either a multiple or variable precision setting [4,6,13,17,18,22,23]. Many of the variable precision algorithms precisely follow the philosophy of guaranteeing a user-specified upper bound on the forward error. A forward stable algorithm is useful for this goal even if it is not backward stable.
4. When the goal is to prove instability, forward stability is actually the strongest
concept, by contraposition.

The outline and main results of this paper are as follows. Section 2 recalls Rice’s condition number (Definition 2.1) and investigates it in Pryce’s coordinatewise relative error metric (Definition 2.3). The main result of this section is the characterization of the corresponding condition number in Proposition 2.5. The key insight of this paper is that properties involving this condition number can establish the stability of compositions of stable algorithms. For such a statement to be precise, however, we need to carefully consider what we mean by “an algorithm for a problem”. Therefore, Section 3 clarifies the model of numerical computations that we adopted. It is based on Cucker, Malajovich, and Shub’s approaches to approximate computations in the Blum–Shub–Smale (BSS) model, which we recall in Appendix B. Thereafter, in Section 4, the classic definitions of backward, mixed, and forward stability are formalized in this model in Definition 4.1 to 4.3. The main innovations of this paper are presented in Section 5. First, we introduce the concept of amenable problems in Definition 5.1. This captures a wide class of problems that are well-behaved in their domain of definition and the growth of their condition number. Next, a compatibility condition is introduced in Definition 5.4, which ensures that a computational problem is decomposed into subproblems in such a way that it can lead to numerical stability. The main result, Theorem 5.6, on the forward stability of a composition of forward-stable algorithms is proven under the assumption of amenable and compatible problems. Note that we use problem-specific properties (condition) to establish algorithm-specific properties (stability). The helpful consequence is that every composition of forward-stable algorithms is forward stable if the amenability and compatibility of the corresponding problems holds. Finally, we investigate the string of implications from backward to mixed to forward stability, under the hypothesis of amenability in Proposition 5.7 and Theorem 5.8. All of the main results are employed in Section 6 in demonstrating several elementary amenable problems and forward stable algorithms, in the coordinatewise relative error metric, to solve them. Conversely, Section 7 demonstrates that Strassen’s algorithm for $2 \times 2$ matrix multiplication is not forward stable in the coordinatewise relative error metric, even though it satisfies Brent’s well-known forward error bound. The surprising example of the sine function, a non-amenable function on $\mathbb{R}$ that cannot be realized as a composition of well-behaved (i.e., compatible and amenable) functions, is featured in Section 8. The conclusions and outlook of our study are presented in Section 9.

2. The condition number in coordinatewise relative error. In this paper, we restrict ourselves to computational problems that can be modeled by a function $f$ from a subset of one real vector space to another. We start by recalling Rice’s definition of condition number [30] in this context, assuming to have fixed (possibly different) distance functions on the domain and codomain of $f$.

Recall that given a subset $D \subseteq \mathbb{R}^n$, a map $\text{dist}_D : D \times D \to [0, \infty]$ is a distance function if (i) it is symmetric, (ii) it satisfies the triangle inequality $\text{dist}_D(x, z) \leq \text{dist}_D(x, y) + \text{dist}_D(y, z)$ for $x, y, z \in D$, and (iii) $\text{dist}_D(x, y) = 0 \iff x = y$. If there is no ambiguity on the nature of the set $D$, we omit the subscript and simply write dist.

**Definition 2.1 (Condition number).** Given a function $f : S \to T$, where $S \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^n$ and two distance functions dist$_S$ and dist$_T$ defined on $S$ and $T$ respectively, the condition number $\kappa(f, x)$ of $f$ at $x \in S$ is

$$
\kappa(f, x) = \lim_{\epsilon \to 0^+} \sup_{y \in S, \text{dist}_S(x, y) < \epsilon} \frac{\text{dist}_T(f(x), f(y))}{\text{dist}_S(x, y)} \in [0, \infty].
$$
If \( x \) is an isolated point for the topology induced by the metric on \( S \), then we set by convention \( \kappa(f, x) = 0 \). We also define

\[ \tilde{\kappa} = 1 + \kappa. \]

The limsup in Definition 2.1 is always defined since we accept the value \( \infty \).

The condition number is a geometric invariant that depends on the function \( f \) but not on the algorithm we use to compute it. There is also an explicit dependence of the condition number on the choice of the distance functions \( \text{dist}_S \) and \( \text{dist}_T \). In addition, condition numbers satisfy the following sensible composition inequality.

**Lemma 2.2.** If \( f = g \circ h \) is a composition of functions, then

\[
\kappa(f, x) \leq \kappa(g, h(x)) \kappa(h, x).
\]

**Proof.** See Appendix A.1

The choice of the distances in Definition 2.1 depends on the context. As real arithmetic operations are substituted in a numerical algorithm by floating-point operations in the standard model of floating-point arithmetic [21], this results in small relative forward errors. Therefore, we focus on a metric that measures relative errors.

Let first us clarify what we mean by relative error. Usually, in numerical analysis, the relative error of an approximation \( \tilde{x} \in \mathbb{R} \) to an exact value \( x \in \mathbb{R} \) is defined as \( \frac{|x - \tilde{x}|}{|x|} \). Unfortunately, this does not define a distance on \( \mathbb{R} \) because symmetry and the triangle inequality can fail. Instead, we adopt the coordinatewise relative error metric that was already introduced by Pryce [29] in 1984, building on Olver’s Rp error analysis [27, 28]. Herein, the distance in \( \mathbb{R}^d \) between \( x \) and \( y \) is defined as the length of the shortest curve joining \( x \) and \( y \), where the length is ultimately determined by the choice of a Riemannian metric on \( \mathbb{R} \) that captures the concept of relative errors. This coordinatewise relative error metric and the surprising topology it induces on \( \mathbb{R}^d \) is stated next.

**Definition 2.3** (Coordinatewise relative error metric [29]). The topology associated with the coordinatewise relative error in \( \mathbb{R}^d \) is defined by \( 3^d \) connected components \( U_j \) in \( \mathbb{R}^d \), corresponding to the sign pattern (an element of \( \{-1, 0, 1\}^d \) of each vector. The connected component \( U_0 = \{0\} \) consists of just one point. On each of the other components \( U_1, \ldots, U_{3^d-1} \), we define a Riemannian metric as follows. Let \( v, w \) be a pair of tangent vectors lying in the subspace spanned by the elements of \( U_j \) and based at the point \( x \in U_j \). We associated with them the inner product

\[ \langle v, w \rangle_x = \sum_{i \in \chi(x)} \frac{v_i w_i}{|x_i|^2}, \]

where \( \chi(x) = \{i \mid x_i \neq 0\} \) contains the nonzero indices of \( x \), which defines the tangent space to \( U_j \) at \( x \). The induced norm of a tangent vector \( v \) is \( \|v\|_x = \langle v, v \rangle_x^{1/2} \). The length of a \( C^1 \) curve \( \gamma : [a, b] \to U_j \) is then defined by

\[
\text{Length}(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)} \, dt = \int_a^b \left( \sum_{i \in \chi(\gamma(t))} \frac{v_i^2(t)}{|\gamma_i(t)|^2} \right)^{1/2} \, dt,
\]

where \( \gamma'(t) = (v_1(t), \ldots, v_d(t)) \) is the derivative of \( \gamma \) at \( t \), and \( \gamma_i(t) \) is the \( i \)th component function of \( \gamma \). The distance between any pair of points \( x, y \in \mathbb{R}^d \) is 0 if \( x = y \), \( \infty \) if \( x \in U_j \) and \( y \in U_k \) with \( j \neq k \) lie on different connected components, and

\[
\text{dist}(x, y) = \inf_{\gamma} \text{Length}(\gamma)
\]
otherwise. The infimum is taken over all \( C^1 \) curves contained in the connected component \( U_i \) with extremes \( x \) and \( y \). This distance function satisfies the axioms of distance, including the triangle inequality \( \text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z) \) for all \( x, y, z \in \mathbb{R}^d \).

From now on whenever we refer to \( \mathbb{R}^d \) we assume that it is endowed with the topology and the metric structure of Definition 2.3.

The infimum of (2.2) is actually a minimum; that is, there exists a curve with extremes \( x, y \) such that its length is equal to \( \text{dist}(x, y) \). This curve is called a minimizing \textit{geodesic} between these two points. The existence follows from the Hopf–Rinow Theorem, see for example [19, Theorem 1.10]. In \( \mathbb{R}^n \), the relative error distance between two points \( x \) and \( y \) with the same sign is either \( \text{dist}(0,0) = 0 \) or

\[
\text{dist}(x, y) = \int_0^1 \frac{\|x - y\|_{\infty + (1-t)y}}{\|x + (1-t)y\|} \, dt = \int_0^1 \frac{|x - y|}{|x + (1-t)y|} \, dt = \left| \log \frac{x}{y} \right|.
\]

If \( x \) and \( y \) do not have the same sign, then \( \text{dist}(x, y) = \infty \). The coordinatewise relative error metric in \( \mathbb{R}^d \) is the product metric of \( \mathbb{R} \) when the latter is endowed with the relative error metric. Hence, the squared distance between two points \( v = (v_1, \ldots, v_d) \) and \( w = (w_1, \ldots, w_d) \) is \( \text{dist}(v, w)^2 = \text{dist}(v_1, w_1)^2 + \cdots + \text{dist}(v_d, w_d)^2 \). In particular, for \( u \in (0, \frac{1}{2}) \) we get after some manipulations that if \( w_i = v_i(1+\delta_i) \) for some \( \delta_i \in (-u, u) \), then \( \text{dist}(v, w) < 2\sqrt{d}u \). The following fact will also be helpful.

**Lemma 2.4.** Let \( f : S \to \mathbb{R}^n \) be as in Definition 2.1. Let \( x, y \in S \). If there is a minimizing geodesic \( \gamma(t) \) joining \( x \) and \( y \) and such that \( \kappa(f, z) \leq C \) for all \( z \in \gamma \), then

\[
\text{dist}(f(x), f(y)) \leq C \text{dist}(x, y).
\]

**Proof.** See Appendix A.2. \( \square \)

With the metric structure clarified, we further characterize the condition number from Definition 2.1. Recall that for \( A \in \mathbb{R}^{m \times n} \) with singular value decomposition \( A = U \Sigma V^T \), the Moore–Penrose pseudoinverse of \( A \) can be defined as \( A^+ = V \Sigma^T U^T \in \mathbb{R}^{n \times m} \) where \( \Sigma^T \) is obtained by transposing \( \Sigma \) and changing the non-zero elements \( \sigma_i \) of \( \Sigma \) to \( \sigma_i^{-1} \). The pseudoinverse of the zero matrix \( 0 \in \mathbb{R}^{m \times m} \) is \( 0 \in \mathbb{R}^{m \times n} \).

**Proposition 2.5 (Condition number in coordinatewise relative error).** Let \( f : S \to \mathbb{R}^n \) be a differentiable map with \( S \subset \mathbb{R}^m \) open, and let the connected components \( U_i \subset \mathbb{R}^m \) be as in Definition 2.3. Let \( \kappa(f, x) \) be the condition number from Definition 2.1 with respect to the coordinatewise relative error metric on both the domain and codomain. Then, the following holds:

(i) if \( x \in S \) and there exists an open neighborhood \( N \) of \( x \) such that \( f(N) \) is contained in one connected component of \( \mathbb{R}^n \), then

\[
\kappa(f, x) = \| \text{diag}(f(x))^{-1} D_x f \text{diag}(x) \|_2 ,
\]

where \( \| \cdot \|_2 \) is the spectral 2-norm, \( D_x f \) is the derivative of \( f \) at \( x \), and \( \text{diag}(z) \) is the diagonal matrix whose entries are the elements of a vector \( z \in \mathbb{R}^d \). In particular, if \( f \) takes an open neighborhood of \( x \) to a constant, then \( \kappa(f, x) = 0 \).

(ii) in all other cases \( \kappa(f, x) = \infty \).

**Proof.** See Appendix A.3. \( \square \)

**Remark 2.1.** Item (ii) in Proposition 2.5 corresponds to \( x \neq 0 \), \( f_i(x) = 0 \) for some \( i \) and \( f_j(y) \neq 0 \) for some \( y \) arbitrarily close to \( x \). In all other cases, recalling also that \( 0 \) is an isolated point in our chosen topology, item (i) applies, and then the
presence of the Moore–Penrose pseudoinverse in (2.3) implies that only the non-zero components of $x$ and $f(x)$ contribute to $\kappa(f, x)$.

Two important special cases of equation (2.3) arise for $m = 1$ and $n = 1$, simplifying to respectively

$$\kappa(f, x) = |x| \sqrt{\sum_{i \in \chi(f(x))} \left( \frac{f_i(x)}{|f_i(x)|} \right)^2} \quad \text{and} \quad \kappa(f, x) = \frac{1}{|f(x)|} \sqrt{\sum_{i \in \chi(x)} \left( x_i \frac{\partial f}{\partial x_i} \right)^2}.$$  

For univariate scalar-valued functions, i.e., $m = n = 1$, both reduce to the familiar expression of the relative condition number

$$\kappa(f, x) = \frac{|x| \cdot |f'(x)|}{|f(x)|}.$$  

Remark 2.2. It follows from Proposition 2.5 that, if $f_i(x) = 0$ for some $i$, then either $\kappa(f, x) = \infty$ or $f_i(x) = 0$ in a neighborhood of $x$. In particular, this clarifies the special values in the renowned formula (2.4):

- If $x = 0$ then $\kappa(f, x) = 0$, regardless of the value of $f'(x)$ and $f(x)$.
- If $x \neq 0$ and $f'(x) = 0$ in an open neighborhood of $x$, then $\kappa(f, x) = 0$.
- If $x \neq 0$, $f(x) = 0$ but $f'(x)$ is not constantly 0 in an open neighborhood of $x$, then $\kappa(f, x) = \infty$ even if $f'(x) = 0$.

3. A formal model of numerical algorithms. Informally, an algorithm is a sequence of instructions that can be programmed in any programming language, like C, Julia, Matlab, or Python. The formal modern–classic definition of an algorithm that we adopt in this paper is the BSS model, proposed by Blum, Shub, and Smale [9] and developed by the same authors and Cucker in [8]; it is recalled in Appendix B.

3.1. Scalable functions. The BSS model takes into account that many algorithms, especially those in numerical linear algebra, are scalable: they can be applied to problems of arbitrary dimension. In practice, mathematical algorithms like computing matrix factorizations, e.g., QR, LU, and SVD, of $n \times n$ matrices are usually implemented for all $n \geq 1$, rather than having separate implementations for each $n \in \mathbb{N}$.

We thus reconsider the concept of a function by letting it have input and output of varying dimensions: we call this a scalable function. By $\cup$ we denote the disjoint union of sets.

Definition 3.1 (A scalable function). A scalable function is of the form

$$f : \bigcup_k S_k \to \bigcup_k \mathbb{R}^{n_k},$$

where $S_k \subseteq \mathbb{R}^{m_k}$ and $m_k, n_k \in \mathbb{N}$ are sequences, either both infinite or both finite and with the same length.

When applied to $x \in S_k$ the function can be denoted by $f_k : S_k \to \mathbb{R}^{n_k}$, but for brevity we simply write $f(x)$. From now on we denote $M_k = \max(m_k, n_k)$, that is the maximum of the dimension of the input and the output for every $k$, since in practice an element of $S_k$ is represented by a vector in $\mathbb{R}^{m_k}$.

3.2. Standard floating-point arithmetic. Next, we recall the formal properties usually assumed in floating-point arithmetic [21]. A floating-point number system $\mathbb{F} \subseteq \mathbb{R}$ with base 2 is a subset of the real numbers of the form

$$\pm m \cdot 2^{-t},$$

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where \( t \in \mathbb{Z}, t > 2 \), is the precision and \( e \in \mathbb{Z} \). \( e_{\text{min}} \leq e \leq e_{\text{max}} \) is bounded. Moreover, the mantissa \( m \in \mathbb{Z} \) is either 0 or satisfies \( 2^{t-1} \leq m \leq 2^t - 1 \), ensuring a unique representation. In other words, \( x \in \mathbb{R} \) if \( x = 0 \) or \( x \) is of the form \( x = \pm 2^t \cdot [0.a_1 \ldots a_t]_2 \) for some \( e \) in the range and \( a_1, \ldots, a_t \in \{0, 1\} \), \( a_1 \neq 0 \). The unit roundoff is \( u = 2^{-t} \).

To achieve an axiomatization that is tame enough to prove theorems, the first mathematical concession that we must make is to relax the boundedness of \( e \). In other words, for the theory below we assume that \( e_{\text{min}} = -\infty \) and \( e_{\text{max}} = \infty \). With this assumption the floating-point system is defined by either the unit roundoff \( u \) or the precision \( t \). Therefore we will denote \( \mathbb{F} \) by \( \mathbb{F}_u \).

We denote by \( \mathbb{F}_u : \mathbb{R} \to \mathbb{F}_u \) the roundoff map that takes \( x \in \mathbb{R} \) to the number in \( \mathbb{F}_u \) that is closest to \( x \) in absolute value. Ties can be broken by the usual schemes, such as rounding to odd, to even, or away from 0. Our results do not depend on this choice. From now on we assume that maps \( \mathbb{F}_u \) exist for all \( 0 < u < \frac{1}{2} \).

With these definitions and assumptions, the floating-point number system \( \mathbb{F}_u \) satisfies the well-known axioms:

- For all \( x \in \mathbb{R} \), \( \mathbb{F}_u(x) \in \mathbb{F}_u \).
- For all \( x \in \mathbb{R} \), \( \mathbb{F}_u(x) = x(1 + \delta) \) for some real \( |\delta| \leq u \).
- For all \( x \in \mathbb{F}_u \), \( \mathbb{F}_u(x) = x \).
- For all \( x \in \mathbb{F}_u \), \( \mathbb{F}_u(-x) = -\mathbb{F}_u(x) \).
- If \( u < u \) and \( x \in \mathbb{F}_u(x) \), then \( x = \mathbb{F}_u(x) \).
- If \( x, y \in \mathbb{R} \) and \( x \leq y \), then \( \mathbb{F}_u(x) \leq \mathbb{F}_u(y) \).
- For every operation \( \circ \in \{+, -, \times, /\} \), there is a corresponding floating-point operation \( \circ : \mathbb{F}_u \times \mathbb{F}_u \to \mathbb{F}_u \) such that for all \( x, y \in \mathbb{F}_u \) we have that

\[
    x \circ y = (x \circ y)(1 + \delta) \quad \text{for some real } |\delta| \leq u,
\]

with the unique exception of division by 0 that produces either NaN or \( \pm \infty \).

3.3. Numerical algorithms. Following [14, 25], we define a numerical algorithm as an algorithm in the BSS model where approximate computations of the algorithm are obtained by executing it in the foregoing model of floating-point arithmetic. We recall the formal definitions in Appendix B.

4. A formal model of numerical stability. In [14] and [25] the formal definition of numerical algorithm served to study complexity classes, but these works do not define numerical stability in this formal setting. In this section we fill this gap.

We denote a specific BSS machine that attempts to implement a function \( f \) by \( \hat{f} \). The output (if any) of the corresponding numerical algorithm on input \( x \) is denoted by \( \hat{f}^u(x) \). This emphasizes that the algorithm tries to approximate a function \( f \) and that the computations are carried out in \( \mathbb{F}_u \).

The concept of numerical stability was introduced to classify which of the numerical algorithms \( \hat{f}^u \) can be claimed to implement a given function \( f \), and to specify which ones are more accurate than others. The existing literature [21] gives flexible and informal definitions that involve “small errors” whose magnitude is deemed to depend on the context. In our context of numerical algorithms with arbitrary precision \( t \) and scalable functions, we believe that the magnitude of errors should be viewed relative to the size of the input and output ambient spaces \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively. Consequently, having defined \( M_k = \max\{m_k, n_k\} \), the definitions of numerical stability below allows errors to grow with \( M_k \), but at most polynomially.

Backward and mixed stability [21, Section 1.5] are defined independently of the condition number of the computational problem.
Remark 4.1. In the rest of the paper we will use “constants” depending polynomially on $M_k$. We tacitly assume without loss of generality that all these polynomials in $M_k$ are greater than 4 and monotonically non-decreasing. We did not optimize our proofs to yield the smallest possible polynomials, rather preferring clarity of exposition.

Definition 4.1 (Backward stability). Let $f : \mathcal{S}_k \to \mathbb{R}^{n_k}$ be a scalable function and $\hat{f}$ a BSS machine. Then, $\hat{f}$ is a backward stable algorithm for $f$ if there exist polynomials $a$ and $b$ such that for all $x \in S_k$,

$$0 < u \leq \frac{1}{a(M_k)} \Rightarrow \exists y : \hat{f}^u(x) = f(y) \text{ and } \text{dist}(x, y) \leq b(M_k)u.$$

In particular, an output must be produced for all such choices of $x$ and $u$.

Definition 4.2 (Mixed stability). Let $f : \mathcal{S}_k \to \mathbb{R}^{n_k}$ be a scalable function and $\hat{f}$ a BSS machine. Then, $\hat{f}$ is a mixed stable algorithm for $f$ if there exist polynomials $a$, $b$, and $c$ such that for all $x \in S_k$

$$0 < u \leq \frac{1}{a(M_k)} \Rightarrow \exists y : \text{dist}(\hat{f}^u(x), f(y)) \leq b(M_k)u \text{ and } \text{dist}(x, y) \leq c(M_k)u.$$

In particular, an output must be produced for all such choices of $x$ and $u$.

The final classic notion of stability is forward stability, which just says that an algorithm is stable if its output $\hat{f}^u(x)$ is close to the exact value $f(x)$, with the caveat that this distance may depend linearly on the condition number at $x$.

Definition 4.3 (Forward stability). Let $f : \mathcal{S}_k \to \mathbb{R}^{n_k}$ be a scalable function and $\hat{f}$ a BSS machine. Then, $\hat{f}$ is a forward stable algorithm implementing $f$ if there exists a stability polynomial $a$ such that either it holds for all $x \in S_k$

$$0 < u \leq \frac{1}{\kappa(f, x) a(M_k)} \Rightarrow \text{dist}(\hat{f}^u(x), f(x)) \leq a(M_k)\kappa(f, x)u,$$

or, equivalently, it holds for all $x \in S_k$ and all $\epsilon \in [0, 1]$ that

$$0 < u \leq \frac{\epsilon}{\kappa(f, x) a(M_k)} \Rightarrow \text{dist}(\hat{f}^u(x), f(x)) \leq \epsilon.$$

In particular, an output must be produced for all such choices of $x$ and $u$. If $\kappa(f, x) = \infty$, then the foregoing implications are vacuous and a forward stable numerical algorithm may either output any value, or not halt and output no value at all, for $u \neq 0$.

It is not hard to verify that the two conditions in Definition 4.3 are equivalent. The reason for giving two alternatives is twofold. First, each of them will be useful in our analyses. Second, they carry a different philosophy: the first criterion shows that given any sufficiently small unit roundoff $u$ a forward stable algorithm guarantees a certain accuracy, while the other shows that for every wanted accuracy $\epsilon$ one can find a unit roundoff such that a forward stable algorithms achieves the required accuracy.

The reason to use $\kappa$ instead of $\kappa$ in Definition 4.3 is that for some problems $\kappa$ may be equal to 0 or a very small number, and thus asking the error to depend linearly on $\kappa$ is just too much to be a realistic demand. We believe that most of the algorithms usually regarded as stable in numerical analysis satisfy Definition 4.3.
5. Stability of numerical algorithms for amenable problems. This section introduces the five fundamental innovations of this paper. First, we present the class of amenable computational problems. These are scalable functions that simultaneously admit a non-uniform bounded growth of the condition number and are well-behaved near the boundaries of their domains. Second, we propose a compatibility condition for amenable functions \( g \) and \( h \), under which we can prove amenability of the composition \( g \circ h \). Third, we prove the main theorem: composing two forward stable numerical algorithms \( \hat{g}^u \) and \( \hat{h}^u \) that respectively implement compatible amenable functions \( g \) and \( h \) results in a forward stable algorithm \( \hat{g}^u \circ \hat{h}^u \) implementing the amenable function \( g \circ h \). Fourth, under amenability a helpful chain of implications arises wherein backward implies mixed implies forward stability. Fifth, for a differentiable problem \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \), proving forward stability of \( \hat{f}^u \) in the relative error metric reduces to establishing the stability of all component functions \( \hat{f}_i^u \) for computing the \( f_i \)'s.

5.1. Amenable problems. The main goal of this article is to facilitate the composition of stable numerical algorithms, resulting in a new stable algorithm. Consider two scalable functions \( g \) and \( h \) that can be composed. We identified three obstacles that seem to prevent unbridled composition of forward stable numerical algorithms \( \hat{g}^u \) and \( \hat{h}^u \) implementing respectively \( g \) and \( h \):

1. \( \hat{g}^u \circ \hat{h}^u \) is no longer well-defined for some or even all \( u \);
2. the condition number of either \( g \) or \( h \) grows uncontrollably;
3. the maximum of the condition numbers \( \kappa(g, h(x)) \) and \( \kappa(h, x) \) is significantly larger than \( \kappa(g \circ h, x) \).

The issue with the first item is clear. The second obstacle may prevent the condition number from being useful since we want the condition number (a first order variation estimator) to provide reasonable bounds for moderately small values of \( u \). The third obstacle is behind the instability of the naive method to solve an overdetermined least-squares problem \( Cx = d \) discussed in the introduction, and was even exploited in [26] to prove forward instability of certain resultant-based methods to solve systems of polynomial equations, and in [3] to prove forward instability of pencil-based methods for computing tensor rank decompositions.

Observe that the second and third obstacles are formulated independently of the algorithms \( \hat{g}^u \) and \( \hat{h}^u \). Remarkably, the first obstacle can also be avoided by placing suitable restrictions only on the functions \( g \) and \( h \). The central idea of our notion of amenability is to prevent the occurrence of the first two obstacles.

**Definition 5.1 (Amenable function).** A scalable function \( f \) is amenable if there exists an amenability polynomial \( a \) such that:

\[(A.1) \text{ For all } x \in S_k, \text{ the ball } B_x = \left\{ y \in \mathbb{R}^m : \text{dist}(x, y) \leq \frac{1}{a(M_k)\tilde{\kappa}(f, x)} \right\} \text{ is contained in } S_k.\]

\[(A.2) \text{ For all } y \in B_x \text{ we have } \tilde{\kappa}(f, y) \leq a(M_k)\tilde{\kappa}(f, x).\]

**Remark 5.1.** Sometimes we want to deal with functions that are not scalable. In these cases we still talk about amenability but the polynomial \( a \) becomes just a constant. The same applies if a function is defined on \( \omega_k S_k \) where \( k \) runs over a finite set.

For all points \( x \in S_k \) where \( \tilde{\kappa}(f, x) = \infty \), the foregoing conditions are automatically satisfied. Thus, it suffices to verify amenability for all points \( x \) where \( \tilde{\kappa}(f, x) \) is finite.

The following lemma will be helpful to check if a given function is amenable.
Lemma 5.2. Given a scalable function \( f : \cup_k S_k \to \cup_k \mathbb{R}^{n_k} \), assume that there is a polynomial \( a \) such that for all \( k \in \mathbb{N} \) the following properties hold:

(i) Let \((x_0, x_1, x_2, \ldots) \subseteq S_k\) be a sequence such that
\[
\text{dist}(x_j, \partial S_k \cup I_k) \to 0,
\]
where \( \partial S_k \) is the boundary of \( S_k \) and \( I_k = \{ x \in S_k : \kappa(f, x) = \infty \} \) is the ill-posed locus. Then, \( \kappa(f, x_j) \to \infty \).

(ii) Let \( V_k = S_k \setminus I_k \) be the set where the condition number is finite and let \( \tilde{\kappa}_f : V_k \to \mathbb{R}, x \mapsto \tilde{\kappa}(f, x) \). The condition number of \( \tilde{\kappa}_f \) satisfies
\[
(5.1) \quad \kappa(\tilde{\kappa}_f, x) \leq \frac{a(M_k)}{4} \tilde{\kappa}(f, x), \quad \forall x \in S_k.
\]

Then, \( f \) is amenable with amenability polynomial \( a \).

Proof. See Appendix A.4. \( \square \)

Remark 5.2. If \( \kappa(f, x) \) is given by a smooth formula, then \( 5.1 \) is satisfied if
\[
(5.2) \quad \sqrt{\sum_i \left( x_i \frac{\partial \kappa}{\partial x_i} \right)^2} \leq q(M_k) \tilde{\kappa}(f, x)^2,
\]
for some polynomial \( q \).

It follows immediately in the notation of Lemma 5.2 that each \( V_k \) is an open subset of \( \mathbb{R}^{m_k} \) and that the restriction \( \kappa|_{V_k} \) is continuous for all \( k \in \mathbb{N} \).

Note that \( 5.2 \) essentially bounds the gradient of the condition number \( \kappa(f, x) \) as the square of the latter. Many condition numbers in linear algebra satisfy such a bound \([16,20]\).

As said in Section 2, once we have endowed \( \mathbb{R}^d \) with the coordinatewise relative metric, its induced topology has \( 3^d \) connected components. Fortunately, our definition of amenability takes care of that, as can be easily proved.

Lemma 5.3. Let \( \mathbb{R}^d \) be endowed with any metric and consider its metric topology. If \( f|_{S_\alpha} \) is amenable for a finite indexed collection of open sets \( S_\alpha \subseteq \mathbb{R}^d \), then \( f|_S \) is amenable, where \( S = \bigcup_\alpha S_\alpha \). Moreover, if all the \( f|_{S_\alpha} \) admit the same amenability polynomial, then so does \( f|_S \), even if \( \alpha \) runs through an infinite set.

5.2. The compatibility condition. The third obstacle for stably composing numerical algorithms is decomposing a well-conditioned computational problem \( f \) into a composition \( f = g \circ h \) where either \( g \) or \( h \) is poorly conditioned (compared to \( f \)). This can lead to a forward unstable algorithm \([3,26]\). Based on this observation, we propose the following compatibility condition that excludes this possibility.

Definition 5.4 (Compatibility condition). Let \( g, h \) be scalable functions such that the composition \( g \circ h \) is well defined. Graphically, for \( k \geq 0 \)
\[
S_k \subseteq \mathbb{R}^{m_k} \xrightarrow{h} T_k \subseteq \mathbb{R}^{n_k} \xrightarrow{g} \mathbb{R}^{p_k}.
\]

We say that \( g, h \) are compatible if there exist compatibility polynomials \( b, c \) such that:

1. For all \( k \), we have \( n_k \leq b(M_k) \), where \( M_k = \max(m_k, p_k) \). In other words, the intermediate dimension is polynomially bounded by the maximum of the input and the output dimensions of the composition.
2. For all \( k \) and \( \forall x \in S_k \) there holds

\[
(C) \quad \tilde{\kappa}(g, h(x)) \hat{\kappa}(h, x) \leq c(M_k) \hat{\kappa}(g \circ h, x).
\]

That is, (2.1) can be reverted up to a factor which is polynomial in \( M_k \).

The compatibility condition guarantees that the composition of two amenable functions is again amenable, as is shown next.

**Proposition 5.5.** If \( g \) and \( h \) are compatible amenable functions, then \( g \circ h \) is also amenable.

**Proof.** Let \( a_g \) and \( a_h \) be the amenability polynomials of respectively \( g \) and \( h \), and let \( b, c \) be the polynomials appearing in the definition of compatibility. We will prove that \( f = g \circ h \) is amenable with amenability polynomial \( a(t) = a_g(b(t))a_h(b(t))c(t) \). Without loss of generality, we can assume that \( b(t) \geq t \).

For any given \( k \), the input and output dimensions of a function \( \varphi \) are denoted by \( m_k^\varphi, n_k^\varphi \) respectively, and \( M_k^\varphi = \max\{m_k^\varphi, n_k^\varphi\} \). We thus have \( M_k^g, M_k^h \leq b(M_k^\varphi) \) from the compatibility, which implies

\[
a(M_k^g) \geq a_g(M_k^h)a_h(M_k^h)c(M_k^\varphi),
\]

having exploited that all polynomials are non-decreasing by assumption. It suffices to verify amenability for all \( x \in S_k \) such that \( \tilde{\kappa}(g \circ h, x) \) is finite. It follows from compatibility that both \( \tilde{\kappa}(h, x) \) and \( \tilde{\kappa}(g, h(x)) \) are finite.

Let \( x \in S_k \), let \( y \in \mathbb{R}^{m_k} \) be such that

\[
\text{dist}(x, y) \leq \frac{1}{a(M_k^g)\hat{\kappa}(f, x)} \leq \frac{c(M_k^\varphi)}{a(M_k^g)\hat{\kappa}(h, x)} \leq \frac{1}{a_h(M_k^h)\hat{\kappa}(g, h(x))}.
\]

Then, \( y \in S_k \) by using (A.1) for \( h \). Hence, \( y \) is in the domain of \( f \) and (A.1) for \( f \) holds. Moreover, \( \tilde{\kappa}(h, y) \leq a_h(M_k^h)\hat{\kappa}(h, x) \) by (A.2) for \( h \). Similarly, for all \( y \in B_x \) it follows from Lemma 2.4 that

\[
\text{dist}(h(x), h(y)) \leq \text{dist}(x, y) \cdot \max_{z \in B_x} \hat{\kappa}(h, z)
\]

\[
\leq \text{dist}(x, y)a_h(M_k^h)\tilde{\kappa}(h, x) \leq \frac{a_h(M_k^h)\tilde{\kappa}(h, x)}{a(M_k^g)\hat{\kappa}(f, x)} \leq \frac{1}{a_g(M_k^h)\tilde{\kappa}(g, h(x))},
\]

so that \( \tilde{\kappa}(g, h(y)) \leq a_g(M_k^h)\tilde{\kappa}(g, h(x)) \) by (A.2) for \( g \). Consequently,

\[
\tilde{\kappa}(f, y) \leq \tilde{\kappa}(h(y)) \leq \tilde{\kappa}(h(x)) \leq a_h(M_k^h)a_g(M_k^h)\tilde{\kappa}(h, x)\tilde{\kappa}(g, h(x)) \leq a(M_k^\varphi)\tilde{\kappa}(f, x).
\]

Thus (A.2) holds for \( f \) as well, concluding the proof. \( \square \)

**5.3. The fundamental theorem.** Now we can prove the main result of this article on the composition, or concatenation, of forward stable numerical algorithms in the setting of compatible amenable computational problems.

**Theorem 5.6.** Let \( g \) and \( h \) be compatible amenable functions. For all forward stable algorithms \( g^u \) and \( h^u \) implementing respectively \( g \) and \( h \), the composition \( g^u \circ h^u \) is a forward stable algorithm implementing \( f = g \circ h \).

**Proof.** Let \( h : \omega_k S_k \rightarrow \omega_k T_k \) and \( g : \omega_k T_k \rightarrow \omega_k R^{p_k} \) with \( S_k \subset \mathbb{R}^{m_k}, T_k \subset \mathbb{R}^{n_k} \). Denote \( M_k^h = \max\{m_k, n_k\} \), \( M_k^g = \max\{n_k, p_k\} \) and \( M_k^\varphi = \max\{m_k, p_k\} \). Let \( a_g(t) \)
and \( a_h(t) \) be the amenability polynomials from Definition 5.1 for \( g \) and \( h \), respectively. Let \( \hat{a}_g(t) \) and \( \hat{a}_h(t) \) be the stability polynomials from Definition 4.3 of \( \hat{g} \) and \( h \), respectively. Finally, let \( b(t), c(t) \) be the compatibility polynomials from Definition 5.4 for \( g \) and \( h \), which by compatibility implies \( M_k^g, M_k^h \leq b(M_k^t) \). We will show that the theorem holds with stability polynomial

\[
a(t) = c(t)a_g(b(t))(\hat{a}_g(b(t)) + \hat{a}_h(b(t))).
\]

When \( \tilde{\kappa}(g \circ h, x) = \infty \), there is nothing to check. Thus, in the remainder of the proof we can assume by compatibility that both \( \kappa(g, h(x)) \) and \( \kappa(h, x) \) are finite.

Since \( g \) and \( h \) are compatible and \( \tilde{h}^u \) is forward stable, we have

\[
\begin{align*}
(5.3) \quad u & \leq \frac{1}{a(M_k^t)} \leq \frac{\kappa(f, x)}{a(M_k^t)\kappa(h, x)} \quad \Rightarrow \\
\kappa(\tilde{h}^u(x), h(x)) & \leq \kappa(M_k^h)\kappa(h, x)u \leq \frac{\kappa(M_k^h)\kappa(h, x)}{a(M_k^t)}\kappa(g, h(x)) \leq \frac{1}{a(M_k^t)}\kappa(g, h(x)).
\end{align*}
\]

where in the second use of (C) the consequent \( u \) was first bounded by \( \frac{1}{a(M_k^t)\kappa(f, x)} \). We know that the ball \( B_{h(x)} \) with radius \( \frac{1}{a(M_k^t)\kappa(g, h(x))} \) centered at \( h(x) \) is contained in \( T_k \). Consequently, \( g(\tilde{h}^u(x)) \) is well defined and a minimizing geodesic \( \gamma_h : [0, 1] \to T_k \) from \( h(x) \) to \( \tilde{h}^u(x) \) exists. Moreover, from (A.2) for \( g \) it follows that

\[
(5.4) \quad \kappa(g, \gamma_h(t)) \leq a_g(M_k^g)\kappa(g, h(x)), \quad t \in [0, 1].
\]

From Lemma 2.4, we have

\[
\begin{align*}
\kappa(\tilde{g}(\tilde{h}^u(x)), g(h(x))) & \leq \kappa(\tilde{h}^u(x), h(x)) \cdot \max_{t \in [0, 1]} \kappa(g, \gamma_h(t)) \\
& \leq a_g(M_k^g)\kappa(g, h(x)) \kappa(\tilde{h}^u(x), h(x)) \\
& \leq a_g(M_k^g)\kappa(M_k^h)\kappa(g, h(x))\kappa(h, x)u \quad (5.3)
\end{align*}
\]

\[
(5.5) \quad \kappa(g, \gamma_h(x)) \leq a_g(M_k^g)\kappa(g, h(x)).
\]

From (5.4) above, \( \kappa(\tilde{g}(\tilde{h}^u(x)), g(h(x))) \leq a_g(M_k^g)\kappa(g, \tilde{h}^u(x))u \leq a_g(M_k^g)\kappa(M_k^g)\kappa(g, h(x))u. \)

\[
(5.6) \quad \kappa(\tilde{h}^u(x), f(x)) \leq \kappa(\tilde{g}(\tilde{h}^u(x)), g(h(x))) + \kappa(\tilde{h}^u(x), g(h(x))) \leq a_g(M_k^g)\kappa(M_k^g)\kappa(g, h(x))u + a_g(M_k^g)\kappa(M_k^g)\kappa(f, x)u \\
\leq a_g(M_k^g)\kappa(M_k^g)\kappa(M_k^f) + a_g(M_k^g)\kappa(M_k^g)\kappa(M_k^f)\kappa(f, x)u.
\]

The theorem follows using again that \( M_k^g, M_k^h \leq b(M_k^t) \).
Remark 5.3. One can extend Theorem 5.6 to the composition of three or more maps by applying it repeatedly. For example, if the composition \( f \circ g \circ h \) makes sense and the three functions are amenable, one must check that \( g \) and \( h \) are compatible, and that \( f \) and \( g \circ h \) are compatible. Then, given stable algorithms \( \hat{f}, \hat{g}, \hat{h} \), from Theorem 5.6 we have a stable algorithm \( g \circ h \) for \( g \circ h \) and again from Theorem 5.6 the algorithm \( \hat{f} \circ \hat{g} \circ \hat{h} \) is stable for \( f \circ g \circ h \).

5.4. Establishing forward stability in amenable problems. The previous subsection introduced a potent tool for checking the forward stability of numerical algorithms for amenable problems: if the algorithm can be decomposed as a composition of two forward stable numerical algorithms whose functions are amenable and compatible, then the original algorithm is forward stable. Now we derive a chain of implications between the notions of stability introduced in Section 4. This yields another powerful tool for proving forward stability of numerical algorithms for amenable problems. In particular, it will be shown that both backward and mixed stable algorithms are also forward stable under the hypothesis of amenability. Since the presented notions of backward and mixed stability are essentially the classic ones, we postulate that most known backward and mixed stable algorithms in the literature for amenable problems are forward stable in the formal sense of Definition 4.3.

We start by observing that backward stability implies mixed stability.

Proposition 5.7. Let \( f \) be scalable. If \( \hat{f}^u \) is a backward stable numerical algorithm implementing \( f \), then it is mixed stable.

The reverse implication does not hold. For example, consider the function that maps a matrix to the orthogonal matrix in the QR factorization. A standard implementation of this method can be proved to be mixed stable; however, due to the fact that the output cannot be guaranteed to be orthogonal for finite \( u \) as irrational numbers are not representable in \( \mathbb{F}_u \), it is not backward stable.

When the underlying function \( f \) is amenable, mixed implies forward stability.

Theorem 5.8. Let \( f \) be amenable. If \( \hat{f}^u \) is a mixed stable numerical algorithm implementing \( f \), then it is forward stable.

Proof. Let \( a(t), b(t) \) and \( c(t) \) be the polynomials from Definition 4.2 for \( \hat{f}^u \). Let \( d(t) \) be the amenability polynomial in Definition 5.1 for \( f \). We now show that \( f^u \) is forward stable with stability polynomial \( p(t) = d(t)c(t) + a(t) + b(t) \).

We can assume \( \tilde{\kappa}(f, x) < \infty \), for otherwise forward stability is vacuously satisfied. Let \( u \leq \frac{1}{p(M_k)\tilde{\kappa}(f, x)} \). By the triangle inequality, we have

\[
\text{dist}(\hat{f}^u(x), f(x)) \leq \text{dist}(\hat{f}^u(x), f(y)) + \text{dist}(f(y), f(x)),
\]

where \( y \) is as in Definition 4.2. The first term is bounded by \( b(M_k)u \) by definition of mixed stability. To estimate the second addend, we rely on Lemma 2.4 to get

\[
\text{dist}(f(y), f(x)) \leq \text{dist}(x, y) \cdot \max_{t \in [0, 1]} \kappa(f, \gamma(t)),
\]

where \( \gamma(t) \) is the minimizing geodesic from \( x \) to \( y \). It exists from (A.1) for \( f \), since

\[
\text{dist}(x, y) \leq c(M_k)u \leq \frac{c(M_k)}{p(M_k)\tilde{\kappa}(f, x)} \leq \frac{1}{d(M_k)\tilde{\kappa}(f, x)}.
\]

By (A.2) for \( f \), we obtain

\[
\text{dist}(f(y), f(x)) \leq \text{dist}(x, y) \cdot \max_{t \in [0, 1]} \kappa(f, \gamma(t)) \leq d(M_k)c(M_k)\tilde{\kappa}(f, x)u.
\]
By putting everything together we obtain
\[ \text{dist}(\hat{f}^u(x), f(x)) \leq b(M_k)u + d(M_k)c(M_k)\tilde{\kappa}(f, x)u \leq p(M_k)\tilde{\kappa}(f, x)u, \]
which concludes the proof.

5.5. Stacking amenable differentiable functions. The previous subsection showed how existing results combined with amenability can be leveraged to prove stability. Now we show our tool to extend claims on univariate functions to vector, matrix and tensor–valued mappings.

**Proposition 5.9** (Stacking of amenable functions is amenable, and stacking of forward stable algorithms is forward stable). Let \( f : S_k \subseteq \mathbb{R}^{m_k} \to \mathbb{R}^{n_k} \) be a scalable function. Recall that we denote \( M_k = \max(m_k, n_k) \), and assume that there exists a polynomial \( a(t) \) such that:

1. Each coordinate function \( f_{k,i} : S_k \to \mathbb{R} \), \( 1 \leq i \leq n_k \), is, as a non-scalable, standalone function, amenable with amenability (constant) polynomial \( a(M_k) \).
2. There exist corresponding forward stable numerical algorithms \( \hat{f}_{k,i} \) for \( f_{k,i} \) with stability (constant) polynomial \( a(M_k) \).

Then, \( f \) itself is amenable as a scalable function with amenability polynomial \( A(t) = ta(t) \) and the “stacked” algorithm \( \hat{f}^u \) given by \( \hat{f}^u(x) = (\hat{f}_{k,1}^u(x), \ldots, \hat{f}_{k,n_k}^u(x)) \) is a forward stable algorithm for \( f \) with stability polynomial \( A(t) \).

**Proof.** The condition number of \( f_k = f|_{S_k} \) and its component functions \( f_{k,i} \) are related as follows:

\[ \max_{1 \leq i \leq n} \kappa(f_{k,i}, x) \leq \kappa(f_k, x) \leq \sqrt{\sum_{i=1}^{n_k} \kappa(f_{k,i}, x)^2} \leq \sqrt{n_k} \max_{1 \leq i \leq n} \kappa(f_{k,i}, x). \]

That is, the condition number of the stacked problem for any fixed \( k \) is proportional to the condition number of the worst function in the stacking for that fixed \( k \). This is indeed easy to check from the definition.

We check directly that \( f \) is amenable from Definition 5.1. Fix any \( k \geq 1 \) and \( x \in S_k \). Since all \( f_{k,i} \) are amenable with constant polynomial \( a(M_k) \), we have

\[ \text{dist}(x, y) \leq \frac{1}{a(M_k)\tilde{\kappa}(f_{k,i}, x)} \Rightarrow \begin{cases} y \in S_k \\ \tilde{\kappa}(f_{k,i}, y) \leq a(M_k)\tilde{\kappa}(f_{k,i}, x) \end{cases}. \]

Then, for all \( i = 1, \ldots, n_k \), with \( A(t) = ta(t) \) we have

\[ \text{dist}(x, y) \leq \frac{1}{A(M_k)\tilde{\kappa}(f_k, x)} \Rightarrow \text{dist}(x, y) \leq \frac{1}{a(M_k)\tilde{\kappa}(f_{k,i}, x)} \Rightarrow \tilde{\kappa}(f_{k,i}, y) \leq a(M_k)\tilde{\kappa}(f_{k,i}, x), \]

and hence the last item in Definition 5.1 is easily checked:

\[ \tilde{\kappa}(f_k, y) \leq \sqrt{\sum_{i=1}^{n_k} \tilde{\kappa}(f_{k,i}, y)^2} \leq a(M_k)\sqrt{\sum_{i=1}^{n_k} \tilde{\kappa}(f_{k,i}, x)^2} \leq \sqrt{n_k}a(M_k)\tilde{\kappa}(f, x) \leq A(M_k)\tilde{\kappa}(f, x). \]

We have thus proved that each \( f_k \) is amenable with constant \( A(M_k) \), so \( f \) is amenable with polynomial \( A(t) \).
Assume now that the functions $f_{k,i} : S_k \to \mathbb{R}$ have forward stable algorithms $\hat{f}_{k,i}^u$. Fix any $k > 1$. By Definition 4.3, and using the amenability polynomials $a$ and $A$, we have

$$\forall x \in S_k, \forall u < \frac{1}{a(M_k)\tilde{\kappa}(f_{k,i}, x)} : \text{dist}(\hat{f}_{k,i}^u(x), f_{k,i}(x)) \leq a(M_k)\tilde{\kappa}(f_{k,i}, x) u.$$ 

Hence,

$$\text{dist}(\hat{f}_{k}^u(x), f_k(x)) = \sqrt{\sum_{i=1}^n \text{dist}(\hat{f}_{k,i}^u(x), f_{k,i}(x))^2} \leq a(M_k) u \sqrt{\sum_{i=1}^n \kappa(f_{k,i}, x)^2} \leq A(M_k) \tilde{\kappa}(f_k, x) u,$$

proving that $\hat{f}_k^u$ is forward stable for $f_k$ with stability constant polynomial $A(M_k)$, that is, $\hat{f}_k^u$ is forward stable for $f$ with stability polynomial $A(t)$. \qed

Notable cases include replicating an input $n$ times and the identity map on $\mathbb{R}^n$.

6. Amenable problems and stable algorithms to solve them. We now describe a catalogue of elementary but fundamental problems that are amenable when we endow our input and output spaces with the coordinatewise relative distance. We also provide forward stable algorithms for solving these problems. These can then be exploited to investigate more complicated algorithms as in Section 7.

The next graph summarizes the results we establish in the next subsections:

This section can be skipped on a first reading, as its main goal is to establish amenability of the above problems, and prove the stability of basic algorithms for solving them.

6.1. Elementary functions. It is verified immediately that the identity map $\text{Id} : \mathbb{R} \to \mathbb{R}, x \mapsto x$, the constant map $\alpha : \mathbb{R} \to \mathbb{R}, x \mapsto \alpha$, and the inversion map $\cdot^{-1} : \mathbb{R}_0 \to \mathbb{R}_0, x \mapsto x^{-1}$ have constant condition numbers bounded by 1. Moreover, since they are defined everywhere (except possibly at 0), the conditions of Definition 5.1 are satisfied, so they are amenable. The algorithms implementing these functions as stated are forward stable because in all cases $\text{dist}(\hat{f}_k^u(x), f(x)) = \log \left| \frac{f(x)(1+\delta)}{f(x)} \right| \leq 2u$ where $|\delta| \leq u < 1/4$.

Let $\alpha \in \mathbb{R}$ be a constant. It can be verified that $x \mapsto x \circ \alpha$, where $\circ \in \{+, -, \cdot\}$, has a forward stable algorithm that consists of replacing $\circ$ by the floating-point implementation $\delta^\circ$ (use Lemma 5.2 for amenability). For division $\mapsto x/\alpha$ the same holds if $\alpha$ is nonzero and likewise for $x \mapsto \alpha/x$ with $x$ nonzero.

6.2. Multiplication and summation. The scalable function codifying the product, $\Pi : x = (x_1, \ldots, x_k) \mapsto x_1 \cdots x_k$ on the domain $D = \cup_k \mathbb{R}^k$, has condition number

$$\kappa(\Pi, x) = \left\| \begin{array}{cc} \frac{\partial}{\partial x_1} (x_1 \cdots x_k) \\ \frac{\partial}{\partial x_2} (x_1 \cdots x_k) \\ \vdots \\ \frac{\partial}{\partial x_k} (x_1 \cdots x_k) \end{array} \right\|_2 = \sqrt{k},$$

where $\cup_k \mathbb{R}^k$.
if all the \( x_i \) are different from 0, and otherwise \( \kappa(\Pi, \mathbf{x}) = 0 \) because \( \Pi \) is locally constant. As this is constant in the \( x_i \)'s in both cases, we obtain \( \kappa(\kappa_1, \mathbf{x}) = 0 \), so \( \Pi \) satisfies the hypotheses of Lemma 5.2. Hence, the function is amenable in all of the connected components of \( \mathbb{R}^k \) (with the topology of relative error), which from Lemma 5.3 implies that \( \Pi \) is amenable in \( \mathbb{R}^k \) and also in \( D \).

It is easy to see from the definition that the trivial algorithm for \( \Pi \) that generates the sequence \( x_1, x_1 x_2, \ldots, \Pi(\mathbf{x}) = x_1 \cdots x_k \) is backward stable: its output is \( \Pi(\mathbf{x}) \Pi_2^{k-1} (1 + \delta_i) \) where \( \delta_i \in (-u, u) \), that is the exact product of \( y_1, x_2, \ldots, x_k \) where the relative distance from \( y_1 \) to \( x_i \) is at most \( |\log(1-u)| \leq 2ku \) whenever \( u < 1/(4k) \). Hence, from Definition 4.1 with stability polynomials \( a(k) = b(k) = 4k \), the algorithm is backward stable and from Theorem 5.8 we deduce that it is forward stable.

The same strategy shows that, for any fixed \( k \in \mathbb{Z} \), the map \( x \to x^k \) is amenable (once we remove the obvious exceptions such as \( 0^{-1} \)) and the straightforward algorithm is forward stable.

A similar argument works with summation, which also defines a scalable function that we denote by \( \Sigma : \mathbb{R}_k \to \mathbb{R} \), \( (x_1, \ldots, x_k) \mapsto x_1 + \cdots + x_k \). The condition number can be computed from Proposition 2.5(i):

\[
\kappa(\Sigma, \mathbf{x}) = \left[ \frac{\partial}{\partial x_i} (x_1 + \cdots + x_k) \right]_{x_1 + \cdots + x_k = 0} = \frac{\sqrt{\sum_{i=1}^k x_i^2}}{|x_1 + \cdots + x_k|} = \|\mathbf{x}\|_2 \|\Sigma(\mathbf{x})\|_2,
\]

if \( \Sigma(\mathbf{x}) \neq 0 \). When \( \Sigma(\mathbf{x}) = 0 \) and \( \mathbf{x} \neq 0 \), Proposition 2.5(ii) entails that \( \kappa(\Sigma, \mathbf{x}) = \infty \), so the foregoing expression is valid for all \( \mathbf{x} \neq 0 \). It is a routine exercise to show that

\[
\kappa(\kappa_2, \mathbf{x}) \leq 2k\kappa(\Sigma, \mathbf{x}),
\]

showing again that \( \Sigma \) is amenable. As in the case of multiplication, it is easy to see that the naive algorithm for \( \Sigma \) is backward stable which again under amenability implies forward stable from Theorem 5.8.

### 6.3. Arithmetic operations on vectors, matrices and tensors.

Since we have seen that stacking amenable functions gives another amenable function, it follows immediately that the following maps with domain \( \mathbb{R}^{n_1 \times \cdots \times n_d} \) are amenable functions (recall Remark 5.1 on the role of the dimensions of the input and range):

1. **Multiplication** by a constant, i.e., \( X \mapsto \alpha X \) for \( \alpha \in \mathbb{R} \).
2. **Adding a constant**, i.e., \( [X_1, \ldots, X_d] \mapsto [X_1, \ldots, X_d + \alpha] \) where \( \alpha \in \mathbb{R} \).
3. **Adding a constant array**, i.e., \( X \mapsto X + A \) with \( A \in \mathbb{R}^{n_1 \times \cdots \times n_d} \).
4. **Adding \( n \) arrays**, i.e., \( (X_1, \ldots, X_n) \mapsto X_1 + \cdots + X_n \), where \( X_i \in \mathbb{R}^{n_1 \times \cdots \times n_d} \).
5. **Hadamard product** with a constant array \( A \in \mathbb{R}^{n_1 \times \cdots \times n_d} \), i.e., \( X \mapsto X \odot A = [X_{i_1 \cdots i_d} A_{i_1 \cdots i_d}] \).
6. **Hadamard product of \( n \) arrays**, i.e., \( (X_1, \ldots, X_n) \mapsto X_1 \odot \cdots \odot X_n \).
7. **Tensor product** with a constant array \( A \in \mathbb{R}^{m_1 \times \cdots \times m_e} \), i.e., \( X \mapsto X \otimes A = [X_{i_1 \cdots i_d} A_{j_1 \cdots j_e}] \), where the image is an element of \( \mathbb{R}^{n_1 \times \cdots \times n_d \times m_1 \times \cdots \times m_e} \).
8. **Tensor product of \( n \) arrays**, i.e., \( (X_1, \ldots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n \) (this time the domain is \( \mathbb{R}^{n_1 \times \cdots \times n_d} \)). As a special case, the **Kronecker product** of two matrices is amenable.

For all aforementioned functions, the straightforward algorithms implementing the formulas are forward stable by Proposition 5.9.

### 6.4. Linear maps.

We consider the sequence of linear maps \( A_k : \mathbb{R}^k \to \mathbb{R}^{n_k} \) given in coordinates by a sequence of \( n_k \times k \) matrices \( A_k \). Consider first the case \( n_k = 1 \) for
all \( k \), so that the map is the scalable function \( f : \bigcup_k \mathbb{R}^k \to \mathbb{R} : x \mapsto A_k x = \sum_{i=1}^k a_i x_i \). As with summation, we note that if one \( x_i = 0 \), then \( f_k \) is equivalent to \( f_{k-1} \) where the argument \( x_i \) is dropped. As \( \kappa(f,0) = 0 \) by definition, it suffices to treat the case where all elements of \( x \) are nonzero. In this case, the condition number is verified to be \( \kappa(f_k, x) = \frac{1}{\|A_k x\|} \|\begin{pmatrix} a_1 x_1 & \cdots & a_k x_k \end{pmatrix}\|_2 \), a formula that holds even if \( A_k x = 0 \). We realize this map as the composition of \( g(x) = (a_1 x_1, \ldots, a_k x_k) \) and the addition map \( \Sigma_k \). Note that \( g \) is the Hadamard product of \( x \) with the vector of constants \( a = (a_1, \ldots, a_k) \), so it is amenable and the straightforward algorithm is stable. We have

\[
\kappa(g, x) = \sqrt{\lambda(x)} \leq \sqrt{k} \quad \text{and} \quad \kappa(\Sigma_k, x \odot a) = \frac{1}{\|a\|_2} \|x \odot a\|_2,
\]

hence showing that \( \kappa(f, x) = \kappa(\Sigma_k, x \odot a) \). Thus, Theorem 5.6 holds for compositions with the stable summation algorithm \( \Sigma^\alpha \) from subsection 6.2 with \( \alpha(t) = t + 4 \). This concludes the argument for \( n = 1 \). For general \( n > 1 \), \( f \) is amenable by stacking the \( n \) amenable component functions. The corresponding algorithm is forward stable.

Note that arbitrary linear combinations of a fixed set of vectors can be computed stably using the above algorithm.

**6.5. Inner product and Euclidean norm.** The final elementary example from linear algebra we consider is computing the inner product

\[
\langle \cdot, \cdot \rangle : \bigcup_k \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}, \quad (x, y) \mapsto x^T y,
\]

and its induced Euclidean norm \( \|x\|_2 = \sqrt{(x,x)} \). This inner product can be realized by composing \( \Sigma_k \) with the Hadamard product \( \odot_k \), both of which are amenable and have stable algorithms. Again, if either \( x_i = 0 \) or \( y_i = 0 \), then \( \langle \cdot, \cdot \rangle_k \) is equivalent to \( \langle \cdot, \cdot \rangle_{k-1} \) where both arguments \( x_i \) and \( y_i \) are dropped. It thus suffices to treat the case where all elements of \( x \) and \( y \) are nonzero. As we have \( \kappa(\odot_k, (x, y)) = \sqrt{2k} \) and, by Proposition 2.5(i),

\[
\kappa(\langle \cdot, \cdot \rangle, (x, y)) = \frac{1}{\|x^T y\|} \|\begin{pmatrix} x_1 y_1, \ldots, x_k y_k \end{pmatrix}\|_2 = \frac{\|x \odot y\|_2}{\|\Sigma(x \odot y)\|} = \kappa(\Sigma, x \odot y),
\]

(when \( x^T y = 0 \), the same formula holds), compatibility of \( \langle \cdot, \cdot \rangle = \odot \odot \) follows. Hence by the main theorem, the composition of stable algorithms for summation and Hadamard product is a stable algorithm for the standard inner product.

The induced norm can be computed via the three-part composition

\[
\|x\|_2 = \sqrt{\kappa(\langle \cdot, \cdot \rangle \circ (x \mapsto (x, x)))} = \sqrt{\kappa(\Sigma \odot \Sigma)} \|x\|_2^2.
\]

As before, taking the norm in which some elements are zero is the same as taking the norm of the nonzero part of the vector. Hence, we can assume that all \( x_i \neq 0 \). First we show that \( \langle \cdot, \cdot \rangle \circ (x \mapsto (x, x)) \) is a stable algorithm for \( x \mapsto \|x\|_2 \). We already know the maps that are amenable, so by Theorem 5.6 checking compatibility suffices. They are also compatible because \( \kappa(\| \cdot \|_2^2, x) = \frac{1}{\|x\|_2^2} \|x\|_2 = 2 \) and the copy map \( c_k : \mathbb{R}^k \to \mathbb{R}^{2k} \), \( x \mapsto (x, x) \) has \( \kappa(c, x) = \|I_k I_k\|_2 = \sqrt{2} \). The difficult one is

\[
\kappa(\langle \cdot, \cdot \rangle, (x, x)) = \frac{\|x \odot x\|_2^2}{\|x\|_2^2} \geq \frac{1}{\sqrt{k}} \frac{\|x \odot x\|_1}{\|x\|_2^2} = \frac{1}{\sqrt{k}}.
\]

\( ^{1} \)To be precise, we should rather write \( a_{k,i} \), since the coefficients of the matrix depend on \( k \) but we think there is no ambiguity and use the less cumbersome notation.

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Compatibility follows, so this is a stable algorithm for computing the squared norm.

Amenability of \( \sqrt{\cdot} \) on \( \mathbb{R}_{\geq 0} \) follows immediately from the fact that \( \kappa(\sqrt{\cdot}, x) = \frac{1}{2} \) is constant and the distance to zero is \( \infty \). It can be deduced from [40, p. 113] and classic roundoff error analysis that the square root of \( 4^{k-1} \leq g \leq 4^{k} \) can be computed mixed stably by the Babylonian method; that is, Newton’s method applied to \( F(x) = x^2 - g^{1/k} = 0 \) starting from \( \frac{1}{2} \), terminated as discussed on [40, p. 117], and multiplying the result with \( 2^k \). Forward stability follows by amenability and Theorem 5.8. Hence, for applying Theorem 5.6, we should check compatibility of \( \sqrt{\cdot} \) and \( \| \cdot \|_2^2 \). For their composition, \( \langle \cdot, \cdot \rangle \), we find, if \( x \neq 0 \),

\[
\kappa(\langle \cdot, x \rangle) = \frac{1}{\|x\|_2^2} \| (2x_1^2, \ldots, 2x_n^2) \|_2 = 2 \frac{\| x \otimes x \|_2}{\|x\|_2^2} \leq 2 \frac{\| x \otimes x \|_1}{\|x\|_2^2} = 2.
\]

Consequently, compatibility holds, proving that the composition of the straightforward algorithm for \( c \), a stable algorithm for \( \langle \cdot, \cdot \rangle \), and the Babylonian method for \( \sqrt{\cdot} \) is a forward stable numerical algorithm for computing the 2-norm of a nonzero vector. If the input is \( x = 0 \), the algorithm must output 0, which is clearly forward stable.

7. Strassen’s matrix multiplication is not forward stable. Recall that \( m \times n \times n \times p \) matrix multiplication \( C = A \cdot B \) can be written as a map \( \cdot : \mathbb{R}^{m \times n} \times \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{m \times p} \) that takes \( (A, B) \rightarrow [\{a_i, b_j\}]_{i,j} \) where \( a_i \) is the \( i \)th row of \( A \) and \( b_j \) is the \( j \)th column of \( B \). It follows from amenability and stability of inner products that the algorithm obtained by stacking forward stable algorithms for the inner product is a forward stable matrix multiplication algorithm.

While attempting to prove that the foregoing algorithm has an optimal bilinear complexity, Strassen made the fabulous discovery that it is, in fact, not optimal [33]. He presented instead an algorithm for \( 2 \times 2 \) matrix multiplication requiring only 7 multiplications rather than the 8 required by the above algorithm. His computation consists of composing the maps

\[
g : \mathbb{R}^7 \rightarrow \mathbb{R}^{2 \times 2}, \quad x \mapsto \begin{bmatrix} x_1 + x_4 - x_5 + x_7 \\ x_2 + x_4 \\ x_3 + x_5 \\ x_1 - x_2 + x_3 + x_6 \end{bmatrix}
\]

and

\[
h : \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^7, \quad (A, B) \mapsto \begin{bmatrix} (a_{11} + a_{22})(b_{11} + b_{22}) \\ (a_{21} + a_{22})b_{12} + a_{11}(b_{12} - b_{22}) \\ a_{22}(b_{21} - b_{11}) \\ (a_{11} + a_{12})b_{22} \\ (a_{21} - a_{11})(b_{11} + b_{12}) \\ (a_{12} - a_{22})(b_{21} + b_{22}) \end{bmatrix}
\]

so that \( \cdot = g \circ h \) for \( 2 \times 2 \) multiplication. These functions are stacked multivariate polynomials, each of which can be easily checked to be amenable using (5.2), so the stacked functions are amenable and the straightforward algorithms are stable. We now show that \( \cdot = g \circ h \) spectacularly violates the compatibility condition in the coordinatewise relative error metric. We follow the strategy from [3, 26] and show that the numerical excess factor, which for the composition of two functions \( g \circ h \) is

\[
\kappa(g, h(x))\kappa(h, x)
\]

becomes unbounded in the neighborhood of some specifically chosen \( A, B \in \mathbb{R}^{2 \times 2} \). In fact, it suffices to show this for a lower bound on the above expression, such as
\[ \kappa(g_{1,2}, h(A, B)) = \frac{1}{|a_{11}b_{12} + a_{12}b_{22}|} \| (a_{11}b_{12} - b_{22}), (a_{11} + a_{12})b_{22} \|_2, \]

while on the other hand \( \kappa(\cdot, (A, B)) \leq \sqrt{\sum_{i,j=1}^{2} \kappa(i,j, (A, B))}, \) where

\[ \kappa(i,j, (A, B)) = \frac{1}{|a_{1i}b_{1j} + a_{2i}b_{2j}|} \| (a_{1i}b_{1j}, a_{2i}b_{2j}) \|_2. \]

Observe that \( \kappa(g_{1,2}, h(A, B)) \) can be made large by letting \( a_{11}b_{22} \) dominate \( a_{11}b_{12}, a_{12}b_{22}, \) and their sum. As a concrete example, we can take a small \( 0 < \epsilon < 1 \) and define \( A_\epsilon = B_\epsilon = \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix}, \) so \( C_\epsilon = \begin{bmatrix} 1 + \epsilon^2 & 2\epsilon \\ 2\epsilon & 1 + \epsilon^2 \end{bmatrix} \). Then, we get

\[ \kappa(g_{1,2}, h(A_\epsilon, B_\epsilon)) = \frac{\sqrt{(1-\epsilon)^2 + (1+\epsilon)^2}}{2\epsilon} \text{ and } \kappa(i,j, (A_\epsilon, B_\epsilon)) = \begin{cases} \sqrt{(1+\epsilon^2)} & \text{if } i = j \\ \frac{1}{2\epsilon} \sqrt{2\epsilon} & \text{if } i \neq j \end{cases}. \]

Note that \( \kappa(g_{1,2}, h(A_\epsilon, B_\epsilon)) \geq \frac{1}{2\epsilon} \) while \( \kappa(i,j, (A_\epsilon, B_\epsilon)) \leq 1 \). Therefore, the ratio (7.1) is at least \( \frac{1}{4\epsilon^2} \). As \( \epsilon \) was arbitrary, there is no constant \( a \) that can control this quantity. Moreover, it is an exercise to verify that for all matrices \( A, B \in \mathbb{R}^{2 \times 2} \) in the relative error ball \( B_\delta(A_\epsilon) \times B_\delta(B_\epsilon) \) of fixed radii \( \delta \) centered at the \( 2 \times 2 \) identity matrix \( I_2 \) the compatibility condition will require \( a(k) > c_\delta \epsilon^{-1} \) for some constant \( c_\delta \) depending on the choice of \( \delta \). Taking the limit of all such balls (which are of constant radius in the relative error topology) as \( \epsilon \to 0 \) shows that compatibility fails for any finite choice of \( a \) at least on a set of positive Lebesgue measure.

Unless there is a fortuitous cancellation of rounding errors, a priori we should not expect that composing stable algorithms for \( g \) and \( h \) will result in a stable algorithm for \( 2 \times 2 \) matrix multiplication because the compatibility condition emphatically fails for these amenable problems at \( (A_\epsilon, B_\epsilon) = (I_2, I_2) \). Our theory strongly suggests that Strassen’s matrix multiplication is not a forward stable numerical algorithm in the relative error metric. We verify this numerically next. The experiments in this and the next section were conducted on a system consisting of an Intel Core i7-4770K CPU with 3.5GHz clock speed and 32GB main memory running Ubuntu 18.04.4. The Matlab/Octave code we used is included in the supplementary files.

We consider Strassen’s algorithm for multiplying matrices in the relative error ball \( N_\epsilon = B_\delta(A_\epsilon) \times B_\delta(B_\epsilon) \) with \( \delta = \frac{1}{2} \). As the numerical excess factor is proportional to \( \epsilon^{-1} \), we anticipate that the observed forward errors in the coordinatewise relative error metric will reflect this. For each \( \epsilon \), we sample \( A'_\epsilon \) and \( B'_\epsilon \) randomly in \( N_\epsilon \) as follows: first, sample \( 2 \times 2 \) matrices \( P_A \) and \( P_B \) with i.i.d. standard normally distributed elements. Then, normalize them as \( P'_A = \frac{P_A}{\|P_A\|_F} \) and \( P'_B = \frac{P_B}{\|P_B\|_F} \), where \( \| \cdot \|_F \) is the Frobenius norm. Finally, set

\[ A'_\epsilon = A_\epsilon \circ \exp(P'_A) \text{ and } B'_\epsilon = B_\epsilon \circ \exp(P'_B), \]

where \( \exp(\cdot) \) applies the exponential function elementwise to the entries of its array argument; it is not the matrix exponential. In this way \( \text{dist}(A_\epsilon, A'_\epsilon) = \text{dist}(B_\epsilon, B'_\epsilon) = \frac{1}{2} \).

We sample 1000 matrices like this for each of the 1000 values of \( \epsilon = 10^{-12}, 10^{-12+\frac{1}{10}}, 10^{-12+2\frac{1}{10}}, \ldots, 10^{-1} \). For each of these matrices, we compute \( A'_\epsilon B'_\epsilon \) using Strassen’s algorithm \( \hat{g}^u \circ \hat{h}^u \) in double-precision floating-point arithmetic, so \( u = 2^{-52} \), in Octave
4.2.2. We also computed a reference $C'_\epsilon = A'_\epsilon B'_\epsilon$ with a numerically forward stable algorithm that is obtained by stacking stable algorithms for computing inner products. Then, we recorded the forward error in the coordinatewise relative error metric and in the absolute error metric ($\text{dist}_{\text{abs}}(A, B) = \|A - B\|_F$). Strassen’s algorithm satisfies a forward error bound $[5,10,21]$ relative to the latter, although our theory casts serious doubts on its stability relative to the former.

The results of this experiment are summarized in Figure 7.1. There, the loss of precision (lop) $\|C - \hat{C}_\epsilon^u\|_{\text{max}} \leq c\|A\|_{\text{max}}\|B\|_{\text{max}}u + o(u) = cu + o(u)$, where $\hat{C}_\epsilon^u$ is the approximation of $C_\epsilon$ computed by Strassen’s algorithm. This also implies an upper bound of $2cu + o(u)$ on the absolute error $\text{dist}_{\text{abs}}(A, B)$. It can be observed in Figure 7.1 from the constant blue area that this bound indeed holds.

Finally, we remark that while Strassen’s matrix multiplication satisfies Brent’s bound on the forward error, this does not mean that it is forward stable in the sense of Definition 4.3. In fact, neither traditional nor Strassen’s matrix multiplication are forward stable numerical algorithms for matrix multiplication in the absolute error metric. Indeed, in the coordinatewise absolute error metric $\kappa(\cdot,(x,y)) = \|(x,y)\|_2$ but the standard model for floating-point arithmetic is wholly based on attaining relative accuracy; we “only” have $x^{u\cdot y} = (1 + \delta)x \cdot y$ where $|\delta| \leq u$ so that $|x^{u\cdot y} - x \cdot y| = |xy|\delta$, which in general is not bounded by $u\sqrt{x^2 + y^2} = (|x| + |y|)u$ when both $x$ and $y$ are large in magnitude. Thus, floating-point multiplication is not a forward stable algorithm.

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2One can verify that $\tilde{\kappa}(g, h(A, B)) = \tilde{\kappa}(g_1, h(A, B))$ and $\tilde{\kappa}(h, (A, B)) = 2.$
in the absolute error metric and we should not expect that using it in compositions will yield forward stable algorithms even if amenability and compatibility hold.

8. A sinful function. The final example we discuss is that of the innocuously-looking sine function in the relative error metric. Let us consider first the finite domain \( \Omega = [k_1 \pi, k_2 \pi] \) where \( k_1, k_2 \in \mathbb{N} \) and verify amenability. The condition number is

\[
\kappa(\sin, x) = \left| \frac{\cos x}{\sin x} \right|
\]

when \( x \neq 0 \) and 0 by definition otherwise. We will apply Lemma 5.2, to prove the amenability of \( \sin \). Since \( \lim_{x \to k \pi} \kappa(\sin, x) = \infty \) for all \( k \in \mathbb{N} \setminus \{0\} \), the first condition of Lemma 5.2 holds. Recall that the distance to 0 is \( \infty \) in the relative error metric, so whatever happens to \( \kappa(\sin, x) \) near zero is irrelevant for amenability. For the second condition, in the variant (5.2), we compute that

\[
\left| \frac{\partial \kappa_\sin}{\partial x} \right| = \frac{|\cos x|}{\sin x} \left| \frac{x^2}{\sin^2 x} \right| \leq \left| \frac{\cos x}{\sin x} \right| + \frac{x^2}{\sin^2 x}.
\]

When \( x \) is an integer multiple of \( \pi \), both the foregoing and \( \tilde{\kappa}(\sin, x)^2 \) equal \( \infty \). As the singularities of \( |x \partial \kappa_\sin / \partial x| \) and \( \tilde{\kappa}(\sin, x)^2 \) occur only at the roots of \( \sin x \), we can multiply both by \( \sin^2 x \) and subtract \( |x \cos x \sin x| \) to find

\[
\sin^2 x \left| \frac{\partial \kappa_\sin}{\partial x} \right| - |x \cos x \sin x| \leq x^2.
\]

On the other hand,

\[
C \sin^2 x \tilde{\kappa}(\sin, x)^2 - |x \cos x \sin x| = C \sin^2 x + (2C - 1)|x \cos x \sin x| + C x^2 \cos^2 x.
\]

As \( \sin^2 x \geq \frac{1}{2} \) whenever \( \cos^2 x \leq \frac{1}{2} \), it follows that taking \( C = 2(\max(|k_1|, |k_2|) \pi)^2 \) ensures \( |x \partial \kappa_\sin / \partial x| \leq C \tilde{\kappa}(\sin, x)^2 \), so that Lemma 5.2 holds because of Remark 5.2. Thus \( \sin \) is amenable on any domain of the form \([k_1 \pi, k_2 \pi]\) with \( k_1, k_2 \in \mathbb{N} \).

By contrast, considering \( \sin \) on \( \mathbb{R} \) does not result in an amenable function. The reason is that the condition number \( \kappa(\sin, x) \) is rapidly oscillating between 0 and \( \infty \) at \( \frac{\pi}{2} + k \pi \) and \( \infty \) at \( k \pi \). If \( x \) is a point where \( \kappa(f, x) = 0 \), then the ball \( B_x \) from Definition 5.1 has constant radius \( \frac{1}{2} \) in the relative error metric. Since the set of points where \( \kappa = \infty \) includes \( X = \{k \pi \ | \ 0 \neq k \in \mathbb{Z}\} \) and \( \text{dist}(k \pi, (k + 1) \pi) = \log \frac{k + 1}{k} \) is decreasing monotonically to zero as \( k \to \infty \), we see that for sufficiently large \( x \) any constant-radius ball in the relative metric will contain one or more points with infinite condition number. Hence, (A.2) fails for \( \sin \) on \( \mathbb{R} \). This also shows that Lemma 5.3 cannot be extended to countable unions.

Since the sine function is not amenable on \( \mathbb{R} \), it cannot be realized as a composition of compatible, amenable functions \( g_i, i = 1, \ldots, \ell \). This is rather unsettling. First, failure of compatibility of amenable functions often results in an unstable algorithm. To see why, note that the numerical excess factor (7.1) cannot be bounded by a constant polynomial \( c \) in Definition 5.4. Second, failure of one of the functions \( g_i \) to be amenable typically signals a highly sensitive function so that the maximum condition number in a small neighborhood of a point \( x \) is greater than a constant times \( \kappa(g_i, x) \). This makes it difficult to stably compose them in numerical algorithms. In this light, it is not a surprise that the following Matlab code reveals the dramatic growth of relative forward errors displayed in Figure 8.1.
Figure 8.1: The loss of precision when computing the sine of $x = 1 + \pi 2^k$ for $k = 1, \ldots, 100$ in Matlab. The reference value was computed using variable precision arithmetic with 10000 digits.

```matlab
lop = zeros(100,1);
PI = vpa('pi',10000);
SIN1 = sin(vpa('1',10000));
for k = 1 : 100
    trueInput = PI*vpa('2',10000)^k + 1;
    numAlgSin1 = sin(double(trueInput));
    lop(k) = abs(log(numAlgSin1 / SIN1)) / eps;
end
```

9. Conclusions. This paper formalized classic notions of backward, mixed, and forward stability of numerical algorithms in the BSS model with strong approximate computations, suitable for fixed, multiple, and variable precision floating-point algorithms. With these definitions in place, we derived a composition theorem for forward stable numerical algorithms, under the sufficient condition of amenability and compatibility of the mathematical problems that these numerical algorithms approximately solve. We employed the introduced tools to prove the stability of well-known, elementary algorithms. It was demonstrated that Strassen’s $2 \times 2$ matrix multiplication is not forward stable in the absolute and relative coordinatewise error metrics. Finally, we illustrated the limitations of the composition theorem, showing that it cannot prove the existence of a forward stable algorithm for the sine function over the reals.

We hope to have convinced the reader that the composition theorem can be a valuable tool in an applied or computational mathematician’s toolbox. This theorem gives clear guidance on how problems should be decomposed into subproblems, so that applying forward stable algorithms to the subproblems automatically stably solves the original problem. In our experience, a failure of the composition theorem for amenable problems often signals potential sources of numerical instability that merit further analysis. The composition theorem enables us to fully leverage the extensive and hard-earned catalogue of (backward, mixed, and forward) stable numerical algorithms from the last seven decades in the design of new forward stable algorithms for tomorrow’s challenges.

Appendix A. Proofs of the technical results.

A.1. Proof of Lemma 2.2. Let $h : S \to \mathbb{R}^n$, $g : T \to \mathbb{R}^m$ with $h(S) \subseteq T \subseteq \mathbb{R}^n$. First, assume that $x \in S$ is such that $\kappa(h, x)$ and $\kappa(g, h(x))$ are finite. Then, for any given $\delta \in (0, 1)$, let $\epsilon', \epsilon > 0$ be such that

$$
\sup_{x \in T, 0 < \text{dist}(h(x), z) \leq \epsilon} \frac{\text{dist}(g(h(x)), g(z))}{\text{dist}(h(x), z)} \leq \kappa(g, h(x)) + \delta, \text{ and}
$$
Hence, for sufficiently small $\epsilon'$, $\epsilon > 0$ exist from the definition of $\kappa(h, x)$ and $\kappa(g, h(x))$, and note also that without loss of generality we can assume that $\epsilon' < \epsilon < \delta < 1$. Then, for $y \in S$, $0 < \text{dist}(x, y) \leq \frac{\epsilon'}{2\tilde{\kappa}(h, x)}$ we have

$$\text{dist}(h(x), h(y)) \leq (\kappa(h, x) + \epsilon) \text{dist}(x, y) \leq \frac{\kappa(h, x) + \epsilon'}{2\tilde{\kappa}(h, x)} < \epsilon.$$

We thus have

$$\text{dist}(g(h(x)), g(h(y))) \leq (\kappa(g, h(x)) + \delta) \text{dist}(h(x), h(y)) \leq (\kappa(g, h(x)) + \delta)(\kappa(h, x) + \epsilon) \text{dist}(x, y).$$

We have proved that

$$0 < \text{dist}(x, y) \leq \frac{\epsilon'}{2\tilde{\kappa}(h, x)} \Rightarrow \frac{\text{dist}(f(x), f(y))}{\text{dist}(x, y)} \leq (\kappa(g, h(x)) + \delta)(\kappa(h, x) + \epsilon),$$

which readily implies $\kappa(f, x) \leq (\kappa(g, h(x)) + \delta)(\kappa(h, x) + \epsilon)$. Since the choice of $\delta$ is arbitrary, the claimed bound (2.1) follows.

Finally, (2.1) is valid if $\tilde{\kappa}(h, x) = \infty$ or $\tilde{\kappa}(g, h(x)) = \infty$, as $\tilde{\kappa}(g, h(x)) \geq 1$ and $\tilde{\kappa}(h, x) \geq 1$. This concludes the proof.

A.2. Proof of Lemma 2.4. Let $\gamma : [0, d] \to \mathbb{R}^n$ be the minimizing geodesic from $x$ to $y$, parameterized by arc-length in such a way that $\text{dist}(\gamma(a), \gamma(b)) = b - a$ for $0 \leq a \leq b \leq d$, and consider the univariate function $\psi(t) = \text{dist}(f(x), f(\gamma(t)))$, which is continuous since $\kappa(f, x) < \infty$ for $x$ in the curve. Given $\delta > 0$, consider the set

$$S = \{t : \psi(t) \leq (C + \delta)t \leq [0, d]$$

and let $\sigma = \sup S$; since $0 \in S$ and $S$ is bounded $\sigma$ exists. Moreover $\sigma \in S$ as $S$ is closed. Suppose $\sigma < d$. Then,

$$\psi(\sigma + \epsilon) \leq \psi(\sigma) + \text{dist}(f(\gamma(\sigma)), f(\gamma(\sigma + \epsilon))).$$

The first addend is bounded by $(C + \delta)\sigma$. To estimate the second addend observe that, by definition of $C$, the condition number, and lim sup, there exists $\eta$ such that whenever $\epsilon < \eta$ then

$$\text{dist}(f(\gamma(\sigma)), f(\gamma(\sigma + \epsilon))) \leq \left(\frac{C + \delta}{2}\right)(C + \delta)(\sigma + \epsilon) < (C + \delta)\epsilon.$$

Hence, for sufficiently small $\epsilon$, $\psi(\sigma + \epsilon) < (C + \delta)(\sigma + \epsilon)$ which implies $\sigma + \epsilon \in S$ and contradicts the definition of $\sigma$. We conclude $\sigma = d$, which proves the lemma.

A.3. Proof of Proposition 2.5. Let $\mathbb{R}^n = \bigcup_{i=0}^{m-1} U_i$ (respectively $\mathbb{R}^n = \bigcup_{j=0}^{m-1} V_j$) be the decomposition of the domain $\mathbb{R}^n$ (resp. the codomain $\mathbb{R}^n$) into its connected components induced by the coordinate-wise relative error metric. The linear space spanned by the elements of $U_i$ will be denoted by $\overline{U}_j$; it is the subspace of vectors whose $i$th component is zero if $i \notin \chi(x)$. 24
Let \( j > 0 \) be fixed and assume \( x \in U_j \). If there is no open neighborhood \( N_x \subseteq U_j \) of \( x \) such that \( f(N_x) \subseteq V_k \) for some \( k \), then it follows from Definition 2.1 and Definition 2.3 that \( \kappa(f, x) = \infty \). Otherwise,

\[
\kappa(f, x)^2 = \lim_{\epsilon \to 0} \sup_{y \in \partial U_j} \sum_{i \in \chi(f(x))} \left( \frac{\log \frac{|f(x+\epsilon y)|}{|f(x)|}}{|f(x)|} \right)^2 = \lim_{\epsilon \to 0} \sup_{y \in \partial U_j} \epsilon^2 \sum_{i \in \chi(f(x))} \left( \frac{(Df_i)(y)}{f_i(x)} \right)^2 + o(\epsilon^2),
\]

having used the definition of the derivative to expand \( f_i(x+\epsilon y) \) as \( f_i(x) + \epsilon Df_i(x) y + o(\epsilon) \).

On the other hand, the square of the right-hand term of (2.3) equals

\[
\kappa(f, x)^2 = \sup_{z \in \partial U_j} \sum_{i \in \chi(f(z))} \left( \frac{(Df_i)(\text{diag}(x) z)}{f_i(x)} \right)^2 = \sup_{y \in \partial U_j} \sum_{i \in \chi(f(x))} \left( \frac{(Df_i)(y)}{f_i(x)} \right)^2 + o(\epsilon^2),
\]

where we have changed coordinates by setting \( y = \text{diag}(x) z \). The proposition follows.

**A.4. Proof of Lemma 5.2.** Let \( x \in S_k \) be arbitrary, \( \tilde{\kappa}(f, x) < \infty \), and let \( y \in B_x \) be arbitrary. Denote the minimizing geodesic connecting \( x \) and \( y \) by \( \gamma(t) : [0, d] \to \mathbb{R}^n_u \), where \( d = \text{dist}(x, y) \leq \frac{1}{\alpha(M_k) \kappa(f, x)} \).

First we show that \( \gamma \subset V_k \) under the conditions of Lemma 5.2. Assume it is not true. Since \( \gamma(0) = x \in V_k \subseteq S_k \setminus \partial S_k \) (using (i)), there exists a supremum \( t \in [0, d] \) such that \( \gamma(s) \in V_k \) for all \( 0 \leq s < t \) but \( \gamma(t) \notin V_k \). From item (ii) in the assumptions and the fact that \( \tilde{\kappa}(f, \gamma(s)) < \infty \) for all \( s \in [0, t] \) because \( \gamma([0, t]) \subset V_k \), we have that \( s \mapsto \tilde{\kappa}(f, \gamma(s)) \) is a continuous function. Let

\[
C(s) = \sup_{\alpha \in [0, s]} \tilde{\kappa}(f, \gamma(\alpha)),
\]

which is a continuous function of \( 0 \leq s < t \). From Lemma 2.4 and (5.1) we have, for all \( s \in [0, t] \),

\[
\log \left( \frac{\tilde{\kappa}(f, \gamma(s))}{\tilde{\kappa}(f, x)} \right) = \text{dist}(\tilde{\kappa}(f, \gamma(s)), \tilde{\kappa}(f, x)) \leq \sup_{\alpha \in [0, s]} \kappa(\tilde{\kappa}, f, \gamma(\alpha)) s \leq \frac{a(M_k)}{4} \sup_{\alpha \in [0, s]} \tilde{\kappa}(f, \gamma(\alpha)) s \leq \frac{a(M_k)}{4} C(s) s.
\]

In particular, since \( 0 < t \leq d \leq \frac{1}{\alpha(M_k) \kappa(f, x)} \), it follows that

\[
\log \left( \frac{C(s)}{\tilde{\kappa}(f, x)} \right) \leq \frac{a(M_k)}{4} C(s) s \leq \frac{C(s)}{4 \tilde{\kappa}(f, x)}, \quad s \in [0, t).
\]

This means that the set \( R = \{ C(s) / \tilde{\kappa}(f, x) : s \in [0, t] \} \) is contained in the set \( \{ \alpha \in (0, \infty) : \log(\alpha) \leq a / 4 \} \), which is itself contained in \( (0, 2] \cup [8, \infty) \). Since \( R \) is connected (as \( C \) is continuous) and contains the point 1 (since \( C(0) = \tilde{\kappa}(f, x) \)), we must have \( R \subseteq (0, 2] \). That is, \( C(s) \leq 2 \tilde{\kappa}(f, x) \) for all \( s \in [0, t] \). Since the hypothesis includes that the condition number explodes to \( \infty \) when \( \gamma(s) \) approaches the boundary \( \partial S_k \) or the limit point has infinite condition number, this contradicts the existence of \( t \in [0, d] \) such that \( \gamma(t) \notin V_k \). We conclude that \( \gamma \subset V_k \), and hence \( B_x \), the geodesic ball of
radius $\frac{1}{\alpha_M}r(f,x)$, is contained in $V_k \subseteq S_k$ as well. This establishes the first item in Definition 5.1.

As the argument above applies for all $t \in [0,d]$ we can also conclude

$$\tilde{r}(f, y) \leq C(d) \leq 2\tilde{r}(f, x),$$

which establishes the second item in Definition 5.1. This concludes the proof.

Appendix B. The BSS model of complexity and computation. We recall in this appendix some known definitions to answer formal questions like “what is an algorithm?” and “what is an approximate computation?”

B.1. The Blum–Shub–Smale model of computation. The BSS computational model [8,9] is a formalization of the concept of algorithm, similar to the classic Turing machine but permitting exact computation between real numbers. Indeed, a BSS machine is sometimes called a real number Turing machine. Formally, a BSS machine is an algorithm? and “what is an approximate computation?”

We label the nodes as follows. Let $N$ be the number of non-terminating nodes in the computation graph $G$. The input node is $v_0 \in V$. The $N'$ output nodes are $v_{N'+1}, v_{N'+2}, \ldots, v_{N'+N'}$. All other nodes are numbered from 1 to $N \in \mathbb{N}$. The sequence of states $(0, v_0, s(0)), (1, v_1, s(1)), \ldots$ is called the exact computation sequence with input $i$ if $s(0) = i$ is the input and each $(j, v_i, s(j)) \in N \times V \times S$ such that $(v_0, v_i) \in E$, $(v_i, v_{i+1}) \in E$ for all $j$, and the correct branch is taken in branch nodes. The BSS machine terminates for input $i$ if its exact computation sequence is finite. At a branch node $j$, the two possible outgoing edges can be thought of as “go to line $k$” instructions, thus allowing loops and conditionals. The purpose of the fifth nodes is to access to arbitrary positions in the sequence. Since we think of $S$ as an internal random access memory and of elements $s$ as finite collections of variables with assigned values, fifth nodes allow to recall any such variable from memory.

A BSS machine can be described by the computation graph $G$ which describes the program flow in a graphical way or by any reasonable pseudocode, with the caveat
that exact computation of rational functions with real numbers is allowed. The use of other functions such as for example \( \sqrt{x} \), \( \log x \), and \( \sin x \) is not permitted in a pseudocode description of a BSS machine; these methods should be implemented by separate subroutines.

We cannot be sure that a given machine will finish on every input, nor can we guarantee that the running time will be bounded by some constant independent of the input. A good part of the theoretical efforts in the study of BSS machines has been devoted to understand deep questions related to these points, see [8, 12]. These subtle issues are not featured in this paper, but we must be aware that a machine is not a function defined on \( \overline{I} \), since it may produce no output (i.e., it may run forever) in some input. The set of points \( \Omega \subseteq \overline{I} \) where a machine produces some output is called the halting set of the machine, and thus we can capture a part of the machine by considering it as a function defined on \( \Omega \), but \( \Omega \) is unknown a priori.

Note also that the result of a computation node might not be well defined for some values of the input since division by 0 may occur. Thus, formally, before any computation node where a division is performed we include a branch node that checks whether the denominator equals 0; if so, the branch node repeats forever so that the machine will never finish on that input.

B.2. Approximate computations. Numerical analysis studies the behaviour of algorithms in the presence of roundoff errors introduced by floating-point arithmetic. In recent works [14, 25], the BSS model was used to formalize intuitive concepts commonly used in the study of algorithms with a focus on complexity theory for approximate computations. Both define a numerical algorithm as a BSS machine whose computations are made in floating-point arithmetic.\(^3\) We recall the definition based on strong approximate computations from [25].

**Definition B.1 (Strong approximate computation [25, Definition 7.1]).** Let \( M(I, O, S, G) \) be a BSS machine and let \( 0 < u < \frac{1}{4} \) be a unit roundoff. The strong \( u \)-computation for \( M \) on input \( i \in I \) is the exact computation sequence \((j, v_i, s(j))\) obtained by the BSS machine \( M^u(I, O, S, \tilde{G}) \), where \( \tilde{G} \) is constructed from \( G \) by replacing all

- real operations \( x \circ y \) with \( \circ \in \{+, -, \cdot, \div\} \) by their floating-point counterpart \( x \hat{\circ} y \),
- real constants \( c \in \mathbb{R} \) by \( \text{fl}(c) \), and
- real inputs or outputs \( x \in \mathbb{R} \) by \( \text{fl}_u(x) \).

For every precision \( t > 2 \) (or, equivalently, roundoff \( u < \frac{1}{t} \)) the machine \( M^u \) has access to \( u \) and \( t \) as internal constants, which formally means that its input is \((i, u, t)\). The halting set \( \Omega^u \) is the set of inputs for which the strong \( u \)-computation terminates. If \( i \in \Omega^u \) we write \( M^u(i) \) for the output (i.e., the last state) of the strong \( u \)-computation. With this notation, we can look at \( M^u \) as a function \( M^u : \Omega^u \to O \). We call \( M^u \) a numerical algorithm.

Two different BSS machines with the same output will in general produce distinct strong \( u \)-computation sequences with possibly other outputs for the same \( x \). Even the halting sets of two such computations can be different. This is because the roundoff errors made in each floating-point operation can accumulate in various ways; a classic example is the machine computing \( x \to x + 1 \) by adding 1 and another doing the same

\(^3\)Cucker [14, Definition 2] gives a definition that is similar to that of weak and strong approximate computations in [25, Section 7], although there are some subtle differences in their definitions of complexity classes. Our theory is quite robust and does not critically depend on these details.
by adding 0.1 ten times.

As pointed out in [14], since integer numbers can be represented exactly in a $b$-ary expansion, we can assume that a strong $u$-computation differs from the original exact computation of the BSS algorithm only in its treatment of reals and the real operations. Integers and the elementary integer operations $\sigma_Z \in \{+Z, -Z, \cdot Z, \div Z\}$ need not be replaced by floating-point operations.

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