TOEPLITZ OPERATORS ACTING ON LARGE WEIGHTED BERGMAN SPACES

HICHAM ARROUSI, INYOUNG PARK, AND JORDI PAU

ABSTRACT. We completely characterize the boundedness of the Toeplitz operator $T_\mu : A^p_\omega \to A^q_\omega$, $1 \leq p, q < \infty$ on large weighted Bergman spaces. A full description of the membership in the Schatten ideal $S_p(A^2_\omega)$ for $0 < p < \infty$ is also obtained.

1. Introduction

Let $H(D)$ denote the space of all analytic functions on $D$, where $D$ is the open unit disk in the complex plane $\mathbb{C}$. A weight is a positive function $\omega \in L^1(D, dA)$, with $dA(z) = \frac{dx dy}{\pi}$ being the normalized area measure on $D$. For $0 < p < \infty$, the weighted Bergman space $A^p_\omega$ is the space of all functions $f \in H(D)$ such that

$$
\|f\|_{A^p_\omega} = \left( \int_D \omega(z)^{1/2} |f(z)|^p \, dA(z) \right)^{1/p} < \infty.
$$

We are going to study Toeplitz operators acting on these weighted Bergman spaces, for a certain class $E$ of weights. The prototype is the exponential type weight

$$
\omega_\alpha(z) = \exp \left( \frac{-1}{(1 - |z|^2)^\alpha} \right), \quad \alpha > 0,
$$

but the class $E$ also contains non-radial weights. For the weights $\omega$ in the class $E$, the point evaluations $L_z$ are bounded linear functionals on $A^p_\omega$ for each $z \in D$. In particular, the space $A^2_\omega$ is a reproducing kernel Hilbert space: for each $z \in D$, there are functions $K_z \in A^2_\omega$ with $\|L_z\| = \|K_z\|_{A^2_\omega}$ such that

$$
L_z f = f(z) = \langle f, K_z \rangle_\omega,
$$

where

$$
\langle f, g \rangle_\omega = \int_D f(z) \overline{g(z)} \omega(z) \, dA(z)
$$

is the natural inner product in $L^2(D, \omega dA)$. The function $K_z$ is called the reproducing kernel for the Bergman space $A^2_\omega$ and has the property that $K_z(\xi) = \overline{K_z(z)}$. The study of the properties of the Bergman spaces with exponential type weights has attracted a lot of attention in recent years [1, 4, 6, 7, 8, 12, 13], and new techniques different from the ones used on standard Bergman spaces are required. Notice that, in [11], the space $A^p_\omega$ is denoted by $A^p(\omega^{p/2})$.

Let $\omega \in E$. The Toeplitz operator $T_\mu^\omega$ with symbol $\mu$ is given by

$$
T_\mu f(z) = T_\mu^\omega f(z) := \int_D f(\xi) \overline{K_z(\xi)} \omega(\xi) \, d\mu(\xi), \quad f \in H(D).
$$
We suppose that $\mu$ is a finite positive Borel measure on $\mathbb{D}$ that satisfies the condition
\[(1.2) \quad \int_{\mathbb{D}} |K_z(\xi)|^2 \omega(\xi) \, d\mu(\xi) < \infty.\]

Then, the Toeplitz operator $T_\mu$ is well-defined on a dense subset of $A^p_\omega$, $1 \leq p < \infty$. In fact, by [1, Corollary 6.4], the set $E$ of finite linear combinations of reproducing kernels is dense in $A^p_\omega$, and it follows from the condition (1.2) and the Cauchy-Schwarz inequality that $T_\mu(f)$ is well defined for any $f \in E$.

**Theorem 1.1.** Let $\omega \in \mathcal{E}$ and $1 \leq p \leq q < \infty$. Then $T_\mu : A^p_\omega \to A^q_\omega$ is bounded if and only if for each $\delta > 0$ small enough, one has
\[(1.3) \quad C_\mu := \sup_{z \in \mathbb{D}} \tau(z)^{2(\frac{1}{q} - \frac{1}{p})} \hat{\mu}_\delta(z) < \infty.\]

Moreover,
\[\|T_\mu\|_{A^p_\omega \to A^q_\omega} \asymp C_\mu.\]

Here $\hat{\mu}_\delta$ is the averaging function defined as
\[\hat{\mu}_\delta(z) = \frac{\mu(D(\delta \tau(z)))}{\tau(z)^2}, \quad z \in \mathbb{D}.\]

We refer to Section 2 for the definition of the function $\tau(z)$ associated to the weight $\omega$, and $D(\delta \tau(z))$ denotes an euclidian disk centered at $z$ and radius $\delta \tau(z)$.

**Theorem 1.2.** Let $\omega \in \mathcal{E}$, $1 \leq q < p < \infty$ and let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. The following conditions are equivalent:

(i) The Toeplitz operator $T_\mu : A^p_\omega \to A^q_\omega$ is bounded.

(ii) For each sufficiently small $\delta > 0$, the function $\hat{\mu}_\delta$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{D}, dA)$.

Moreover, we have
\[\|T_\mu\|_{A^p_\omega \to A^q_\omega} \asymp \|\hat{\mu}_\delta\|_{L^{\frac{pq}{p-q}}(\mathbb{D})}.\]

The corresponding description on compactness is also obtained. Also (see Theorem 6.6) we prove that, for $0 < p < \infty$, the Toeplitz operator is in Schatten class $S^p_\mu(A^2_\omega)$ if and only if the averaging function $\hat{\mu}_\delta$ is in $L^p(\mathbb{D}, d\lambda_\tau)$, where $d\lambda_\tau = \tau(z)^{-2}dA(z)$. This completes the characterization obtained by Lin and Rochberg in [9], where the necessity of the previous condition when $0 < p < 1$ was left open.

Throughout this work, the letter $C$ will denote an absolute constant whose value may change at different occurrences. We also use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \leq Cb$, and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$.

2. Preliminaries and Basic Properties

A positive function $\tau$ on $\mathbb{D}$ is said to be in the class $\mathcal{L}$ if satisfies the following two properties:

(A) There is a constant $c_1$ such that $\tau(z) \leq c_1 (1 - |z|)$ for all $z \in \mathbb{D}$;

(B) There is a constant $c_2$ such that $|\tau(z) - \tau(\zeta)| \leq c_2 |z - \zeta|$ for all $z, \zeta \in \mathbb{D}$. 

We also use the notation
\[ m_\tau := \min\left\{ \frac{1}{2}, c_1^{-1}, c_2^{-1} \right\}, \]
where \( c_1 \) and \( c_2 \) are the constants appearing in the previous definition. For \( a \in \mathbb{D} \) and \( \delta > 0 \), we use 
\( D(\delta \tau(a)) \) to denote the euclidian disc centered at \( a \) and radius \( \delta \tau(a) \). It is easy to see from conditions (A) and (B) (see [12, Lemma 2.1]) that if \( \tau \in \mathcal{L} \) and \( z \in D(\delta \tau(a)) \), then
\[ \frac{1}{2} \tau(a) \leq \tau(z) \leq 2 \tau(a), \]
for sufficiently small \( \delta > 0 \), that is, for \( \delta \in (0, m_\tau) \). This fact will be used many times in this work.

**Definition 2.1.** We say that a weight \( \omega \) is in the class \( \mathcal{L}^* \) if it is of the form \( \omega = e^{-2\varphi} \), where \( \varphi \in C^2(\mathbb{D}) \) with \( \Delta \varphi > 0 \), and \( (\Delta \varphi(z))^{-1/2} < \tau(z) \), with \( \tau(z) \) being a function in the class \( \mathcal{L} \). Here \( \Delta \) denotes the classical Laplace operator.

The following result is from [12, Lemma 2.2] and gives the boundedness of the point evaluation functionals on \( A^p_\omega \).

**Lemma A.** Let \( \omega \in \mathcal{L}^* \), \( 0 < p < \infty \), and let \( z \in \mathbb{D} \). If \( \beta \in \mathbb{R} \) there exists \( M \geq 1 \) such that
\[ |f(z)|^p \omega(z)^\beta \leq \frac{M}{\delta^2 \tau(z)\beta} \int_{D(\delta \tau(z))} |f(\xi)|^p \omega(\xi)^\beta \, dA(\xi), \]
for all \( f \in H(\mathbb{D}) \) and all \( \delta > 0 \) sufficiently small.

A consequence of the above result is that the Bergman space \( A^p_\omega \) is a Banach space when \( 1 \leq p < \infty \) and a complete metric space when \( 0 < p < 1 \). The following lemma on coverings is due to Oleinik, see [11].

**Lemma B.** Let \( \tau \) be a positive function in \( \mathbb{D} \) in the class \( \mathcal{L} \), and let \( \delta \in (0, m_\tau) \). Then there exists a sequence of points \( \{z_j\} \subset \mathbb{D} \), such that the following conditions are satisfied:

(i) \( z_j \notin D(\delta \tau(z_k)), \ j \neq k. \)

(ii) \( \bigcup_j D(\delta \tau(z_j)) = \mathbb{D}. \)

(iii) \( \bar{D}(\delta \tau(z_j)) \subset D(3\delta \tau(z_j)), \) where \( \bar{D}(\delta \tau(z_j)) = \bigcup_{z \in D(\delta \tau(z_j))} D(\delta \tau(z)), \ j = 1, 2, \ldots \)

(iv) \( \{D(3\delta \tau(z_j))\} \) is a covering of \( \mathbb{D} \) of finite multiplicity \( N. \)

The multiplicity \( N \) in the previous Lemma is independent of \( \delta \), and it is easy to see that one can take, for example, \( N = 256. \) Any sequence satisfying the conditions in Lemma [B] will be called a \( (\delta, \tau) \)-lattice.

**Definition 2.2.** A weight \( \omega \) is in the class \( \mathcal{E} \) if \( \omega \in \mathcal{L}^* \) and its associated function \( \tau \) satisfies the condition

(E) For each \( m \geq 1 \), there are constants \( b_m > 0 \) and \( 0 < t_m < 1/m \) such that
\[ \tau(z) \leq \tau(\xi) + t_m |z - \xi|, \ \text{for} \ |z - \xi| > b_m \tau(\xi). \]

The class \( \mathcal{E} \) contains the exponential type weights given by (1.1), but also includes non-radial weights (see [1] for an example).
2.1. Reproducing kernels estimates. The next result (see [3, 8, 12] for (a), and [1] for part (b)) provides useful estimates involving reproducing kernels.

**Theorem A.** Let $K_z$ be the reproducing kernel of $A^2_\omega$, where $\omega$ is a weight in the class $E$. Then

(a) For each $z \in \mathbb{D}$, one has

$$\|K_z\|^2_{{A^2_\omega}} \omega(z) \asymp \frac{1}{\tau(z)^2}, \quad z \in \mathbb{D}. \quad (2.2)$$

(b) For each $M \geq 1$, there exists a constant $C > 0$ (depending on $M$) such that for each $z, \xi \in \mathbb{D}$ one has

$$|K_z(\xi)| \leq C \frac{1}{\tau(z)} \frac{1}{\tau(\xi)} \omega(z)^{-1/2} \omega(\xi)^{-1/2} \left(\frac{\min(\tau(z), \tau(\xi))}{|z - \xi|}\right)^M. \quad (2.3)$$

For weights in the class $E$, and points close to the diagonal, one has the following well-known estimate (see [9, Lemma 3.6] for example)

$$|K_z(\xi)| \asymp \|K_z\|_{A^2_\omega} \cdot \|K_\xi\|_{A^2_\omega}, \quad \xi \in D(\delta \tau(z))$$

for all $\delta \in (0, m_\tau)$ sufficiently small.

We also need the following result appearing in [2, Lemma 2.2].

**Lemma C.** Let $\omega \in E$, and $K_z$ be the reproducing kernel for $A^2(\omega)$. For $0 < p < \infty$ and $\alpha \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |K_z(\xi)|^p \omega(\xi)^{p/2} \tau(\xi)^{\alpha} dA(\xi) \leq C \omega(z)^{-p/2} \tau(z)^{\alpha - 2(p - 1)}.$$

As a direct application of Lemma C and (2.3), one gets the following estimate for the $p$-norm of the reproducing kernels. Let $\omega \in E$ and $0 < p < \infty$. Then, for each $z \in \mathbb{D}$, one has

$$\|K_z\|_{A^p_\omega} \asymp \omega(z)^{-1/2} \tau(z)^{-2(p - 1)/p}. \quad (2.4)$$

3. CARLESON TYPE MEASURES

Let $0 < p, q < \infty$. We say that $\mu$ is a $q$-Carleson measure for $A^p_\omega$ if there exists a finite positive constant $C$ such that

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leq C \|f\|^q_{A^p_\omega},$$

for all $f \in A^p_\omega$. Thus, $\mu$ is $q$-Carleson for $A^p_\omega$ when the inclusion $I_\mu : A^p_\omega \to L^q(\mathbb{D}, \mu)$ is bounded. Next results were essentially proved in [12]. Since the conditions on the weights are slightly different, we give a sketch of the proofs.

**Theorem 3.1.** Let $\omega \in E$ and $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Let $0 < p \leq q < \infty$. Then $I_\mu : A^p_\omega \to L^q(\mathbb{D}, \mu)$ is bounded if and only if for each sufficiently small $\delta > 0$ we have

$$K_{\mu, \omega} := \sup_{a \in \mathbb{D}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} \omega(\xi)^{-q/2} d\mu(\xi) < \infty. \quad (3.1)$$

Moreover, in that case, $K_{\mu, \omega} \asymp \|I_\mu\|^q_{A^p_\omega \to L^q(\mathbb{D}, \mu)}$. 


Proof. Suppose first that \( I_\mu : A_\omega^p \to L^q(\mathbb{D}, d\mu) \) is bounded. If \( \delta \in (0, m_\tau) \) is sufficiently small then, due to (2.3), part (a) in Theorem [A] and the fact that \( \tau(z) \asymp \tau(a) \) for \( z \in D(\delta \tau(a)) \), we get

\[
|K_a(z)| \asymp \|K_a\|_{A_\omega^p} \cdot \left| |K_a|_{A_\omega^q} \asymp \tau(a)^{-2} \omega(a)^{-1/2} \omega(z)^{-1/2}, \quad z \in D(\delta \tau(a)).
\]

This gives

\[
K_{\mu, \omega}(a) := \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} \omega(z)^{-q/2} d\mu(z) \asymp \frac{\tau(a)^{2q} \omega(a)^{q/2}}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} |K_a(z)|^q \, d\mu(z).
\]

Thus, taking into account the estimate for the norm in \( A_\omega^p \) of the reproducing kernel \( K_a \) given in (2.4) together with the boundedness of \( I_\mu \), we obtain

\[
K_{\mu, \omega}(a) \lesssim \|K_a\|_{A_\omega^p}^{-q} \int_{\mathbb{D}} |K_a(z)|^q \, d\mu(z) \lesssim \|K_a\|_{A_\omega^p}^{-q} \cdot \|I_\mu K_a\|_{L^q(\mathbb{D}, \mu)} \lesssim \|I_\mu\|_{A_\omega^p \to L^q(\mathbb{D}, \mu)}.
\]

It follows that

\[
K_{\mu, \omega} = \sup_{a \in \mathbb{D}} K_{\mu, \omega}(a) \lesssim \|I_\mu\|_{A_\omega^p \to L^q(\mathbb{D}, \mu)}.
\]

Conversely, suppose that (3.1) holds, and let \( f \in A_\omega^p \). Let \( \{z_j\} \) be a \((\delta, \tau)\)-lattice. Then, by the properties of the lattice given in Lemma [B] and Lemma [A], we have

\[
\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq \sum_j \int_{D(\delta \tau(z_j))} |f(z)|^q \, d\mu(z)
\]

\[
\lesssim \sum_j \int_{D(\delta \tau(z_j))} \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(\xi)|^p \omega(\xi)^{p/2} dA(\xi) \right)^{q/p} \omega(z)^{-q/2} \, d\mu(z)
\]

\[
\lesssim \sum_j \left( \int_{D(3\delta \tau(z_j))} |f(\xi)|^p \omega(\xi)^{p/2} dA(\xi) \right)^{q/p} K_{\mu, \omega}(z_j).
\]

Since \( q/p \geq 1 \), using the finite multiplicity of the covering, we obtain

\[
\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \lesssim K_{\mu, \omega} \cdot \|f\|_{A_\omega^p}^q,
\]

which completes the proof. \( \square \)

For each \( a \in \mathbb{D} \), we use the notation \( k_{p,a} \) for the normalized reproducing kernels in \( A_\omega^p \), that is,

\[
k_{p,a} = K_{p,a} / \|K_{p,a}\|_{A_\omega^p}.
\]

**Lemma 3.2.** Let \( \omega \in \mathcal{E} \) and \( 0 < p < \infty \). Then \( k_{p,a} \) converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \(|a| \to 1^-\).
Proof. Let \( r \in (0, 1), 0 < p < \infty \) and \( M \) be specified later. For \( |z| \leq r \) and \( |a| \geq \frac{1 + r}{2} \), by applying Theorem A we have
\[
|K_a(z)|^p \lesssim \frac{\omega(z)^{-p/2} \omega(a)^{-p/2}}{\tau(z)^p \tau(a)^p} \left( \min \left( \frac{\tau(z)}{|z - a|}, \frac{\tau(a)}{|a - z|} \right) \right)^{Mp}.
\]
Since \( |a - z| \geq \frac{1 + r}{2} \) and \( \|K_a\|^{-p}_{A^p_\omega} \asymp \omega(a)^{p/2} \tau(a)^{2(p-1)} \), we obtain
\[
|k_{p,a}(z)|^p \omega(z)^{p/2} \tau(z)^p = |K_a(z)|^p \|K_a\|^{-p}_{A^p_\omega} \omega(a)^{p/2} \tau(a)^p \lesssim \frac{\tau(a)^{(M+1)p-2}}{|z - a|^{Mp}} \lesssim \frac{\tau(a)^{(M+1)p-2}}{(1 - r)^{Mp}}.
\]
Because \( \tau(a) \to 0 \), taking \( M \) big enough (\( M > \frac{2}{p} - 1 \)) yields that \( k_{p,a} \) converges uniformly on compact subsets of \( \mathbb{D} \) to zero when \( |a| \to 1^- \). \( \square \)

**Theorem 3.3.** Let \( \omega \in \mathcal{E} \) and \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \). Let \( 0 < p \leq q < \infty \). Then \( I_\mu : A^q_\omega \to L^q(\mathbb{D}, \mu) \) is compact if and only if for each sufficiently small \( \delta > 0 \) we have
\[
(3.2) \quad \lim_{r \to 1^-} \sup_{|z| > r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta \tau(a))} \omega(\xi)^{-q/2} d\mu(\xi) = 0.
\]
Proof. One only needs to follow the proof given in \([12]\) with the help of Lemma 3.2 and minor modifications. The details are left to the interested reader. \( \square \)

For the case \( 0 < q < p < \infty \), we need the following lemma.

**Lemma 3.4.** Let \( \omega \in \mathcal{E}, 0 < p < \infty \) and, for \( \delta \in (0, m_r) \), let \( \{z_k\} \) be a \( (\delta, \tau) \)-lattice on \( \mathbb{D} \). The function given by
\[
F(z) := \sum_k \lambda_k \omega(z_k)^{1/2} \tau(z_k)^{2(p-1)} K_{z_k}(z)
\]
belongs to \( A^p_\omega \) for every sequence \( \lambda = \{\lambda_k\} \in l^p \). Moreover, \( \|F\|_{A^p_\omega} \lesssim \|\lambda\|_{l^p} \).

Proof. We left as an exercise for the reader to check that the partial sums defining \( F \) converges uniformly on compact subsets of \( \mathbb{D} \) showing that \( F \) defines an analytic function on \( \mathbb{D} \). For \( 0 < p \leq 1 \), using Lemma C we get
\[
\|F\|^p_{A^p_\omega} \lesssim \sum_k |\lambda_k|^p \omega(z_k)^{p/2} \tau(z_k)^{2(p-1)} \|K_{z_k}\|^p_{A^p_\omega} \lesssim \sum_k |\lambda_k|^p.
\]
For the case \( p > 1 \), let
\[
M(z) := \sum_{k=0}^{\infty} \tau(z_k)^2 \omega(z_k)^{1/2} |K_{z_k}(z)|.
\]
By Hölder’s inequality we have
\[
\|F\|^p_{A^p_\omega} \leq \int_{\mathbb{D}} \left( \sum_k |\lambda_k| \omega(z_k)^{1/2} \tau(z_k)^{2(p-1)} |K_{z_k}(z)| \right)^p \omega(z)^{p/2} dA(z) \lesssim \int_{\mathbb{D}} \left( \sum_k |\lambda_k|^p \omega(z_k)^{1/2} |K_{z_k}(z)| \right) M(z)^{p-1} \omega(z)^{p/2} dA(z).
\]
On the other hand, using Lemma A, the lattice properties and Lemma C we have

$$M(z) \lesssim \sum_{k=0}^{\infty} \int_{D(\delta \tau(z_k))} |K_z(\xi)| \omega(\xi)^{1/2} dA(\xi) \lesssim \int_{D} |K_z(\xi)| \omega(\xi)^{1/2} dA(\xi) \lesssim \omega(z)^{-1/2}. $$

Therefore, applying Lemma C again we obtain

$$\|F\|_{A_p^\omega}^p \lesssim \int_{\mathbb{D}} \left( \sum_k |\lambda_k|^p \omega(z_k)^{1/2} |K_{z_k}(z)| \right) \omega(z)^{1/2} dA(z) \lesssim \sum_k |\lambda_k|^p \omega(z_k)^{1/2} \int_{D} |K_{z_k}(z)| \omega(z)^{1/2} dA(z) \lesssim \|\lambda\|_{\ell^p}. $$

**Theorem 3.5.** Let $\omega \in \mathcal{E}$ and let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Let $0 < q < p < \infty$. The following conditions are equivalent:

(a) $I_\mu : A_p^\omega \to L^q(\mathbb{D}, \mu)$ is compact.

(b) $I_\mu : A_p^\omega \to L^q(\mathbb{D}, \mu)$ is bounded.

(c) For each sufficiently small $\delta > 0$, the function

$$F_{\mu, q}(z) = \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} \omega(\xi)^{-q/2} d\mu(\xi)$$

belongs to $L^{q/p - q}(\mathbb{D}, dA)$.

Moreover, one has

$$\|I_\mu\|_{A_p^\omega \to L^q(\mathbb{D}, \mu)}^q \asymp \|F_{\mu, q}\|_{L^{q/p - q}(\mathbb{D})}. $$

**Proof.** It is obvious that (a) implies (b). To prove (b) ⇒ (c), for an arbitrary sequence $\lambda = \{\lambda_k\} \in \ell^p$, and $t \in (0, 1)$, consider the function

$$G_t(z) = \sum_k \lambda_k r_k(t) \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} K_{z_k}(z),$$

where $r_k(t)$ is a sequence of Rademacher functions (see Appendix A of [5]) and $\{z_k\}$ is a $(\delta, \tau)$-lattice on $\mathbb{D}$. By Lemma 3.4 we obtain $\|G_t\|_{A_p^\omega} \lesssim \|\lambda\|_{\ell^p}$. Thus, the boundedness of $I_\mu : A_p^\omega \to L^q(\mathbb{D}, \mu)$ gives

$$\int_{\mathbb{D}} |G_t(z)|^q d\mu(z) \lesssim \|I_\mu G_t\|_{L^q(\mathbb{D}, \mu)}^q \lesssim \|I_\mu\|_{A_p^\omega \to L^q(\mathbb{D}, \mu)}^q \|\lambda\|_{\ell^p}^q.$$

From here the proof follows the same lines as in [12]. Finally, the proof of (c) ⇒ (a) can be done exactly as in [12].
4. BOUNDEDNESS

In this section we describe the boundedness of Toeplitz operators on large Bergman spaces. For \( \delta \in (0, m_\tau) \), we consider the averaging function \( \hat{\mu}_\delta \) defined on \( \mathbb{D} \) by

\[
\hat{\mu}_\delta(z) := \frac{\mu(D(\delta \tau(z)))}{\tau(z)^2}, \quad z \in \mathbb{D}.
\]

**Proof of Theorem 1.1** Assume first that \( T_\mu : A_p^\omega \to A_q^\omega \) is bounded. For fixed \( a \in \mathbb{D} \), one has

\[
T_\mu K_a(a) = \int_\mathbb{D} |K_a(z)|^2 \, \omega(z) \, d\mu(z).
\]

By (2.3), there is \( \delta \in (0, m_\tau) \) such that \( |K_a(z)| \lesssim \|K_z\|_{A_2^\omega} \cdot \|K_a\|_{A_2^\omega} \), for every \( z \in D(\delta \tau(a)) \). This together with the norm estimate given in (2.2) and the fact that \( \tau(z) \approx \tau(a) \) for \( z \in D(\delta \tau(a)) \) gives

\[
T_\mu K_a(a) \geq \int_{D(\delta \tau(a))} |K_a(z)|^2 \, \omega(z) \, d\mu(z)
\]

\begin{equation}
\gtrsim \int_{D(\delta \tau(a))} \|K_z\|_{A_2^\omega}^2 \|K_a\|_{A_2^\omega}^2 \, \omega(z) \, d\mu(z)
\end{equation}

\[
\lesssim \frac{\mu(D(\delta \tau(a)))}{\omega(a) \tau(a)^4} = \frac{\hat{\mu}_\delta(a)}{\omega(a) \tau(a)^2}.
\]

Therefore, by Lemma A and the estimate of the norm of the reproducing kernels given in (2.4), we obtain

\[
\tau(a)^{2(\frac{1}{p} - \frac{1}{q})} \hat{\mu}_\delta(a) \lesssim \frac{\omega(a)^{1/2}}{\tau(a)^{2(\frac{1}{p} - \frac{1}{q})}} \left( |T_\mu K_a(a)|^q \omega(a)^{q/2} \right)^{1/q}
\]

\begin{equation}
\leq \frac{\omega(a)^{1/2}}{\tau(a)^{2(\frac{1}{p} - \frac{1}{q})}} \|T_\mu K_a\|_{A_2^\omega}
\end{equation}

\[
\leq \frac{\omega(a)^{1/2}}{\tau(a)^{2(\frac{1}{p} - \frac{1}{q})}} \|T_\mu\|_{A_p^\omega \to A_q^\omega} \cdot \|K_a\|_{A_2^\omega} \lesssim \|T_\mu\|_{A_p^\omega \to A_q^\omega}.
\]

Conversely, suppose that (1.3) holds. We first prove that

\[
\int_{\mathbb{D}} |K_z(\xi)| \omega(\xi)^{1/2} \, d\mu(\xi) \lesssim C_\mu \tau(z)^{2(\frac{1}{p} - \frac{1}{q})} \omega(z)^{-1/2}.
\]

Indeed, by Lemma A we have

\[
|K_z(\xi)| \omega(\xi)^{1/2} \lesssim \frac{1}{\tau(\xi)^2} \int_{D(\frac{1}{2} \tau(\xi))} |K_z(s)| \omega(s)^{1/2} \, dA(s).
\]
Thus, Fubini’s theorem, the fact that $\tau(s) \asymp \tau(\xi)$ for $s \in D(\delta \tau(\xi))$ and Lemma[1] yield
\[
\int_{\mathbb{D}} |K_2(\xi)| \omega(\xi)^{1/2} \, d\mu(\xi) \lesssim \int_{\mathbb{D}} |K_2(s)| \omega(s)^{1/2} \, \widehat{\mu}(s) \, dA(s)
\]
\[
\lesssim C_\mu \int_{\mathbb{D}} |K_2(s)| \omega(s)^{1/2} \, \tau(s)^{(2(\frac{1}{p} - \frac{1}{q}))} \, dA(s)
\]
\[
\lesssim C_\mu \tau(z)^{2(\frac{1}{p} - \frac{1}{q})} \omega(z)^{-1/2}.
\]
If $q > 1$, by Hölder’s inequality, we obtain
\[
|T_\mu f(z)|^q \leq \left( \int_{\mathbb{D}} |f(\xi)| |K_2(\xi)| \omega(\xi) \, d\mu(\xi) \right)^q
\]
\[
\leq \left( \int_{\mathbb{D}} |f(\xi)|^q \omega(\xi)^{\frac{q+1}{q}} |K_2(\xi)| \, d\mu(\xi) \right) \left( \int_{\mathbb{D}} |K_2(\xi)| \omega(\xi)^{1/2} \, d\mu(\xi) \right)^{q-1}.
\]
Using (4.3), we have
\[
|T_\mu f(z)|^q \lesssim C_\mu^{-1} \left( \int_{\mathbb{D}} |f(\xi)|^q \omega(\xi)^{\frac{q+1}{q}} |K_2(\xi)| \, d\mu(\xi) \right) \tau(z)^{2(\frac{1}{p} - \frac{1}{q})(q-1)} \omega(z)^{-\frac{(q-1)}{2}}.
\]
Therefore, by Fubini’s theorem and Lemma[1] we obtain
\[
\|T_\mu f\|_{A^\mu_q}^q = \int_{\mathbb{D}} |T_\mu f(z)|^q \omega(z)^{q/2} \, dA(z)
\]
\[
\lesssim C_\mu^{-q} \left( \int_{\mathbb{D}} |f(\xi)|^q \omega(\xi)^{\frac{q+1}{q}} \tau(z)^{2(q-1)(\frac{1}{p} - \frac{1}{q})} \, dA(z) \right) \omega(z)^{-\frac{(q-1)}{2}} \, d\mu(\xi)
\]
\[
\lesssim C_\mu^{-q} \int_{\mathbb{D}} |f(\xi)|^q \omega(\xi)^{q/2} \tau(\xi)^{(2(q-1)(\frac{1}{p} - \frac{1}{q}))} \, d\mu(\xi).
\]
If $q = p = 1$, we arrive at this point directly after the use of Fubini’s theorem. Consider the measure $\nu$ given by
\[
d\nu(\xi) := \omega(\xi)^{q/2} \tau(\xi)^{(2(q-1)(\frac{1}{p} - \frac{1}{q}))} \, d\mu(\xi).
\]
Since (1.3) holds, it follows from Theorem[8.1] that the identity $I_\nu : A^\mu_p \rightarrow L^q(\mathbb{D}, \nu)$ is bounded with $\|I_\nu\| \lesssim C_\mu^{-1/q}$. This gives
\[
(4.4) \quad \|T_\mu f\|_{A^\mu_q}^q \lesssim C_\mu^{-q} \int_{\mathbb{D}} |f(z)|^q \, d\nu(z) \lesssim C_\mu^{-q} \cdot \|f\|_{A^\mu_p}^q.
\]
This finishes the proof. □

Before going to the proof of Theorem 1.2 we need the following result.

**Lemma 4.1.** Let $\omega \in \mathcal{E}$, and $1 < q < p < \infty$. If $\hat{\mu}_\delta \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA)$, then
\[
J_{\delta,q} := \int_{\mathbb{D}} \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(\xi)| \omega(\xi)^{1/2} d\mu(\xi) \right)^q \lesssim \|\hat{\mu}_\delta\|_{L^{\frac{pq}{p-q}}(\mathbb{D})}^q \|f\|_{A^p_\omega}^q,
\]
for any $f \in A^p_\omega$.

**Proof.** For $z \in \mathbb{D}$ and $\xi \in D(\delta \tau(z))$, by Lemma A, Lemma B and (2.1), we obtain
\[
|f(\xi)| \omega(\xi)^{1/2} \lesssim \left( \frac{1}{\tau(\xi)^2} \int_{D(\delta \tau(\xi))} |f(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p} \lesssim \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p}.
\]
This gives
\[
\frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(\xi)| \omega(\xi)^{1/2} d\mu(\xi) \lesssim \hat{\mu}_\delta(z) \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(s)|^p \omega(s)^{p/2} dA(s) \right)^{1/p}.
\]
Therefore,
\[
J_{\delta,q} \lesssim \int_{\mathbb{D}} \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(s)|^p \omega(s)^{p/2} dA(s) \right)^{q/p} \hat{\mu}_\delta(z)^q dA(z).
\]
Applying Hölder’s inequality
\[
(4.5) \quad J_{\delta,q} \lesssim \left( \int_{\mathbb{D}} \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(s)|^p \omega(s)^{p/2} dA(s) dA(z) \right)^{q/p} \|\hat{\mu}_\delta\|_{L^{\frac{pq}{p-q}}(\mathbb{D})}^q.
\]
On the other hand, by Fubini’s theorem and $\tau(z) \asymp \tau(s)$, for $s \in D(\delta \tau(z))$, we have
\[
\int_{\mathbb{D}} \left( \frac{1}{\tau(z)^2} \int_{D(\delta \tau(z))} |f(s)|^p \omega(s)^{p/2} dA(s) \right) dA(z) \lesssim \|f\|_{A^p(\omega^{p/2})}^p.
\]
Combining this with (4.5), we get
\[
J_{\delta,q} \lesssim \|\hat{\mu}_\delta\|_{L^{\frac{pq}{p-q}}(\mathbb{D})}^q \|f\|_{A^p(\omega^{p/2})}^q.
\]
The proof is complete. □
Proof of Theorem 1.2 (i) ⇒ (ii) For an arbitrary sequence \( \lambda = \{ \lambda_k \} \in \ell^p \), consider the function

\[
G_t(z) = \sum_k \lambda_k r_k(t) \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} K_{z_k}(z), \quad 0 < t < 1,
\]

where \( r_k(t) \) is a sequence of Rademacher functions and \( \{ z_k \} \) is a \((\delta, \tau)\)-lattice on \( \mathbb{D} \). By Lemma 3.4, we have \( \| G_t \|_{A^q_\omega} \lesssim \| \lambda \|_{\ell^p} \). Thus, the boundedness of \( T_\mu : A^q_\omega \to A^q_\omega \) gives

\[
\| T_\mu G_t \|_{A^q_\omega} \lesssim \| T_\mu \|_{A^q_\omega \to A^q_\omega} \cdot \| \lambda \|_{\ell^p}.
\]

In other words, we have

\[
\int_\mathbb{D} \left| \sum_k \lambda_k r_k(t) \omega(z_k)^{1/2} \tau(z_k)^{2(\frac{p-1}{p})} T_\mu K_{z_k}(z) \right|^q \omega(z)^{q/2} dA(z) \lesssim \| T_\mu \|_{A^q_\omega \to A^q_\omega} \cdot \| \lambda \|_{\ell^p}.
\]

Integrating with respect to \( t \) from 0 to 1, applying Fubini’s theorem and invoking Khinchine’s inequality (see [10] for example), we obtain

\[
B := \int_\mathbb{D} \left( \sum_k |\lambda_k|^2 \omega(z_k) \tau(z_k)^{\frac{4(p-1)}{p}} |T_\mu K_{z_k}(z)|^2 \right)^{q/2} \omega(z)^{q/2} dA(z) \lesssim \| T_\mu \|_{A^q_\omega \to A^q_\omega} \cdot \| \lambda \|_{\ell^p}.
\]

Let \( \chi_k \) denote the characteristic function of the set \( D(3\delta \tau(z_k)) \). If \( 0 < q < 2 \), since the covering \( \{ D(3\delta \tau(z_k)) \} \) of \( \mathbb{D} \) has finite multiplicity \( N \), Hölder’s inequality gives

\[
\sum_k |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(\frac{p-1}{p})} |T_\mu K_{z_k}(z)|^q \chi_k(z)
\]

\[
\leq \left( \sum_k |\lambda_k|^2 \omega(z_k) \tau(z_k)^{\frac{4(p-1)}{p}} |T_\mu K_{z_k}(z)|^2 \right)^{q/2} \left( \sum_k \chi_k(z) \right)^{1-\frac{q}{2}}
\]

\[
\leq N^{1-\frac{q}{2}} \left( \sum_k |\lambda_k|^2 \omega(z_k) \tau(z_k)^{\frac{4(p-1)}{p}} |T_\mu K_{z_k}(z)|^2 \right)^{q/2}.
\]

For \( q \geq 2 \), we have the trivial inequality

\[
\sum_k |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(\frac{p-1}{p})} |T_\mu K_{z_k}(z)|^q
\]

\[
\leq \left( \sum_k |\lambda_k|^2 \omega(z_k) \tau(z_k)^{\frac{4(p-1)}{p}} |T_\mu K_{z_k}(z)|^2 \right)^{q/2}.
\]
All together, we obtain
\[
\sum_k |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(p-1)} \int_{D(3\delta\tau(z_k))} |T_{\mu_k}K_{z_k}(z)|^q \omega(z)^{q/2} dA(z)
\]
\[
= \int_{D} \sum_k |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(p-1)} |T_{\mu_k}K_{z_k}(z)|^q \chi(z) \omega(z)^{q/2} dA(z)
\]
\[
\leq \max\{1, N^{1-\frac{q}{2}}\} B.
\]
Hence, by Lemma \[A\], we arrive at the inequality
\[
(4.6) \quad \sum_k |\lambda_k|^q \omega(z_k)^{q/2} \tau(z_k)^{2q(p-1)+2} |T_{\mu_k}K_{z_k}(z_k)|^q \omega(z_k)^{q/2} \lesssim \|T_{\mu_k}\|_{A_p^q \rightarrow A_q^q}^q \|\lambda\|_{\ell^p}^q.
\]
As in (4.4), for \(\delta \in (0, m_\tau)\) small enough, we have
\[
|T_{\mu_k}K_{z_k}(z_k)| \gtrsim \frac{\omega(z_k)^{-1}\tilde{\mu}\bar{\delta}(z_k)}{\tau(z_k)^2}.
\]
That is,
\[
|T_{\mu_k}K_{z_k}(z_k)|^q \omega(z_k)^{q/2} \gtrsim \frac{\omega(z_k)^{-q}\tilde{\mu}\bar{\delta}(z_k)^q}{\tau(z_k)^{2q}}.
\]
Bearing in mind (4.6), this gives
\[
\sum_k |\lambda_k|^q \tau(z_k)^{2q(\frac{1}{p} - \frac{1}{q})} \tilde{\mu}\bar{\delta}(z_k)^q \lesssim \|T_{\mu_k}\|_{A_p^q \rightarrow A_q^q}^q \|\lambda\|_{\ell^p}^q.
\]
Then, by the duality between \(\ell^p/q\) and \(\ell^{\frac{p}{p-q}}\) we conclude that
\[
\sum_k \left( \tau(z_k)^{2q(\frac{1}{p} - \frac{1}{q})} \tilde{\mu}\bar{\delta}(z_k)^q \right)^{\frac{p}{p-q}} \lesssim \|T_{\mu_k}\|_{A_p^q \rightarrow A_q^q}^t,
\]
with \(t = pq/(p - q)\). Equivalently,
\[
\sum_k \tau(z_k)^{2} \tilde{\mu}\bar{\delta}(z_k)^{\frac{pq}{p-q}} \lesssim \|T_{\mu_k}\|_{A_p^q \rightarrow A_q^q}^t.
\]
This is the discrete version of our condition. To obtain the continuous version, simply note that
\[
\tilde{\mu}\bar{\delta}(z) \lesssim \tilde{\mu}\bar{\delta}(z_k), \quad z \in D(\delta\tau(z_k)).
\]
Then
\[
\int_{D} \tilde{\mu}\bar{\delta}(z)^{\frac{pq}{p-q}} dA(z) \lesssim \sum_k \int_{D(\delta\tau(z_k))} \tilde{\mu}\bar{\delta}(z)^{\frac{pq}{p-q}} dA(z) \lesssim \sum_k \tau(z_k)^{2} \tilde{\mu}\bar{\delta}(z_k)^{\frac{pq}{p-q}}.
\]
This finishes the proof of this implication.
(ii) ⇒ (i) First we begin with the easiest case \( q = 1 \). By Fubini’s theorem and Lemma C, we have

\[
\|T_\mu f\|_{A^1_\omega} \leq \int_D \left( \int_D |f(\xi)| |K_z(\xi)| \omega(\xi) \, d\mu(\xi) \right) \omega(z)^{1/2} \, dA(z)
\]

(4.7)

\[
= \int_D |f(\xi)| \left( \int_D |K_z(\xi)| \omega(z)^{1/2} \, dA(z) \right) \omega(\xi) \, d\mu(\xi)
\]

\[
\lesssim \int_D |f(\xi)| \omega(\xi)^{1/2} \, d\mu(\xi).
\]

Therefore, applying Theorem 3.5 with the measure \( d\nu(\xi) = \omega(\xi)^{1/2} \, d\mu(\xi) \), we obtain the desired result for this case.

To deal with the case \( 1 < q < \infty \), let \( \{z_j\} \) be a \((\delta, \tau)\)-lattice on \( D \). By Lemma A for \( \xi \in D(\delta\tau(z_j)) \), we have

\[
|K_z(\xi)| \omega(\xi)^{1/2} \lesssim \frac{1}{\tau(z_j)^2} \int_{D(3\delta\tau(z_j))} |K_z(s)| \omega(s)^{1/2} \, dA(s)
\]

Then

\[
|T_\mu f(z)| \leq \sum_j \int_{D(\delta\tau(z_j))} |f(\xi)| |K_z(\xi)| \omega(\xi) \, d\mu(\xi) \lesssim \sum_j I_\mu(f, z_j) K_\omega(z, z_j)
\]

with

\[ I_\mu(f, z_j) = \frac{1}{\tau(z_j)^2} \int_{D(\delta\tau(z_j))} |f(\xi)| \omega(\xi)^{1/2} \, dA(\xi), \]

and

\[ K_\omega(z, z_j) = \int_{D(3\delta\tau(z_j))} |K_z(s)| \omega(s)^{1/2} \, dA(s). \]

Since \( q > 1 \), we can use Hölder’s inequality to get

\[
|T_\mu f(z)|^q \lesssim \left( \sum_j I_\mu(f, z_j)^q K_\omega(z, z_j) \right) \left( \sum_j K_\omega(z, z_j) \right)^{q-1}.
\]

Since the covering has finite multiplicity, it follows that

\[
\sum_j K_\omega(z, z_j) \lesssim \|K_z\|_{A^1_\omega} \propto \omega(z)^{-1/2}.
\]

Therefore,
\[ \|T_\mu f\|_{A^q_\mu}^q \lesssim \sum_j I_\mu(f, z_j)^q \int_D K_\omega(z, z_j) \omega(z)^{1/2} dA(z). \]

By Fubini’s Theorem,

\[ \int_D K_\omega(z, z_j) \omega(z)^{1/2} dA(z) = \int_{D(3\delta(z_j))} \|K_s\|_{A^1_\omega} \omega(s)^{1/2} dA(s) \approx \tau(z_j)^2. \]

Thus, by Lemma 4.1 we finally obtain

\[ \|T_\mu f\|_{A^q_\mu}^q \lesssim \sum_j I_\mu(f, z_j)^q \tau(z_j)^2 \]

\[ \lesssim \int_D \left( \frac{1}{\tau(z)^2} \int_{D(2\delta(z))} |f(\xi)| \omega(\xi)^{1/2} d\mu(\xi) \right)^q dA(z) \]

\[ \lesssim \|\mu_{2\delta}\|_{L^{pq/(p-q)}(D)} \cdot \|f\|_{A^q_\mu}^q. \]

The proof is complete. \[\square\]

5. COMPACTNESS

We begin with a useful criteria for compactness of Toeplitz operators acting on large weighted Bergman spaces. The proof, usually omitted for these type of results, is standard and can be done following the lines of [15]. In order to offer no doubt on the validity of that, we offer the proof here. Before doing that we need a preliminary result.

**Lemma 5.1.** Let \( \omega \in \mathcal{E}, 1 \leq p, q < \infty \) and \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \) satisfying (1.2). Let \( \{f_n\} \) be a bounded sequence in \( A^p_\omega \) converging to zero uniformly on compact subsets of \( \mathbb{D} \). Then \( T_\mu f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \).

**Proof.** Let \( \varepsilon > 0 \), and fix \( 0 < r < 1 \). Since \( \{f_n\} \) converges to zero uniformly on compact subsets, there exists a natural number \( n_0 \) such that \( \sup_{|\xi| \leq R} |f_n(\xi)| < \varepsilon \), for all \( n \geq n_0 \), with \( r + \frac{(1-r)}{2} \leq R < 1 \). Now we are going to prove that \( \sup_{|z| \leq r} |T_\mu f_n(z)| \to 0 \). By Theorem A for \( M \geq 1 \) and for all \( z, \xi \in \mathbb{D} \)

\[ |K_\xi(z)| \lesssim \|K_\xi\|_{A^2_\mu} \cdot \|K_\xi\|_{A^2_\mu} \left( \frac{\min(\tau(z), \tau(\xi))}{|z - \xi|} \right)^M \]

\[ \lesssim \|K_\xi\|_{A^2_\mu} \cdot \|K_\xi\|_{A^2_\mu} \frac{\tau(\xi)^M}{|z - \xi|^M}. \]

And by Lemma A

\[ |f_n(\xi)| \omega(\xi)^{1/2} \lesssim \|f_n\|_{A^p_\mu} \frac{\tau(\xi)^{2/p}}{\tau(\xi)^{2/p}}. \]
Applying this with our condition $(1.2)$ and the fact that $\|K_\xi\|_{A^p} \omega(\xi)^{1/2} \approx \frac{1}{\tau(\xi)}$, $\xi \in \mathbb{D}$, we have for $|z| \leq r$

$$|T_\mu f_n(z)| \leq \int_{|z| \leq R} |f_n(\xi)||K_z(\xi)| \omega(\xi) d\mu(\xi) + \int_{R < |\xi| < 1} |f_n(\xi)||K_z(\xi)| \omega(\xi) d\mu(\xi)$$

$$\lesssim \varepsilon + \int_{R < |\xi| < 1} \|f_n\|_{A^p} \|K_z\|_{A^p} \frac{\tau(\xi)^{M-(1+2/p)}}{|z - \xi|^M} d\mu(\xi)$$

$$\lesssim \varepsilon + \|K_z\|_{A^p} \int_{R < |\xi| < 1} \tau(\xi)^{M-(1+2/p)} \frac{d\mu(\xi)}{(1-r)^M}.$$

Taking $M > 1 + 2/p$ we can obtain the desired result. $\Box$

**Proposition 5.2.** Let $1 \leq p, q < \infty$ and $\omega \in \mathcal{E}$. Then $T_\mu : A^p_\omega \to A^q_\omega$ is compact, if and only if, for any bounded sequence $\{f_n\}$ in $A^p_\omega$ converging to zero uniformly on compact subsets of $\mathbb{D}$, one has

$$\lim_{n \to \infty} \|T_\mu f_n\|_{A^q_\omega} = 0.$$

**Proof.** Suppose first that $\lim_{n \to \infty} \|T_\mu f_n\|_{A^q_\omega} = 0$ for any bounded sequence $\{f_n\}$ in $A^p_\omega$ converging to zero uniformly on compact subsets of $\mathbb{D}$. We want to prove that $T_\mu$ is compact. Let $\{f_n\}$ be a bounded sequence in $A^p_\omega$. By Lemma A, $\{f_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Therefore, by Montel’s Theorem, there is a subsequence $f_{n_k}$ such that $f_{n_k} \to f$ uniformly on compact subsets of $\mathbb{D}$, for some $f \in H(\mathbb{D})$. Using Fatou’s lemma, it is easy to see that $f$ must be in $A^q_\omega$. By our assumption, $\lim_{n \to \infty} \|T_\mu f_{n_k} - T_\mu f\|_{A^q_\omega} = 0$, proving that $T_\mu$ is compact.

Conversely, assume that $T_\mu$ is compact and let $\{f_n\}$ be a bounded sequence in $A^p_\omega$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. If the conclusion is false, then there exists an $\varepsilon > 0$ and a subsequence $\{f_{n_j}\}$ such that

$$\|T_\mu f_{n_j}\|_{A^q_\omega} \geq \varepsilon, \quad \text{for all } j = 1, 2, 3, \ldots$$

Since $T_\mu$ is compact, we can find a further subsequence $\{f_{n_{j_k}}\}_k$ and $f \in A^q_\omega$ with $\|T_\mu f_{n_{j_k}} - f\|_{A^q_\omega} \to 0$, as $k \to \infty$. By Lemma A, for any $z \in \mathbb{D}$

$$|T_\mu f_{n_{j_k}} - f(z)| \lesssim \frac{\omega(z)^{-1/2}}{\tau(z)^{2/q}} \|T_\mu f_{n_{j_k}} - f\|_{A^q_\omega}.$$

Hence, $T_\mu f_{n_{j_k}} \to f \to 0$ uniformly on compact subsets of $\mathbb{D}$. Moreover, since $f_{n_{j_k}} \to 0$ uniformly on compact subsets of $\mathbb{D}$, from Lemma A we get $f \equiv 0$. Hence $\|T_\mu f_{n_{j_k}}\|_{A^q_\omega} \to 0$ as $k \to \infty$ which contradicts (5.1). The proof is complete. $\Box$

With the help of the previous proposition, we can characterize the compactness of the Toeplitz operator $T_\mu$, result that is stated below.

**Theorem 5.3.** Let $1 \leq p \leq q < \infty$ and $\omega \in \mathcal{E}$. Then $T_\mu : A^p_\omega \to A^q_\omega$ is compact if and only if for each $\delta \in (0, m_\tau)$ sufficiently small, one has

$$\lim_{r \to 1^-} \sup_{|z| > r} \tau(z)^{2(\frac{1}{q} - \frac{1}{p})} \tilde{\mu}_\delta(z) = 0.$$
By Lemma 3.2, \( k_{p,z} \) converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( |z| \to 1^- \). Thus, if \( T_\mu \) is compact, from Proposition 5.2, we obtain (5.2).

Conversely, suppose (5.2) holds, and let \( \{ f_n \} \) be a bounded sequence in \( A^p_\omega \) converging to zero uniformly on compact subsets of \( \mathbb{D} \). According to Proposition 5.2, we must show that \( \| T_\mu f_n \|_{A^p_\omega} \to 0 \). By (4.4) in the proof of Theorem 1.1, we have

\[
\| T_\mu f_n \|_{A^p_\omega} \lesssim \| f_n \|_{L^p(\mathbb{D}, d\nu)},
\]

with \( d\nu(\xi) := \omega(\xi)^{q/2} \tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})} d\mu(\xi) \). By Theorem 5.3, our assumption (5.2) implies that \( I_{\tilde{\omega}} : A^p_\omega \to L^q(\mathbb{D}, d\nu) \) is compact, which implies that \( \| f_n \|_{L^q(\mathbb{D}, d\nu)} \) tends to zero. Hence \( \| T_\mu f_n \|_{A^p_\omega} \to 0 \) proving that \( T_\mu \) is compact. The proof is complete. \( \square \)

**Theorem 5.4.** Let \( \omega \in \mathcal{E} \) and \( 1 \leq q < p < \infty \). Then \( T_\mu : A^p_\omega \to A^q_\omega \) is compact if and only if it is bounded, if and only if for each sufficiently small \( \delta > 0 \), the function \( \tilde{\mu}_\delta \) belongs to \( L^{\frac{m_\tau}{2\tau}}(\mathbb{D}, dA) \).

**Proof.** In view of Theorem 1.2, it is enough to show that \( T_\mu : A^p_\omega \to A^q_\omega \) is compact if \( \tilde{\mu}_\delta \) belongs to \( L^{\frac{m_\tau}{2\tau}}(\mathbb{D}, dA) \). Now, easy modifications in the proof of the sufficiency in Theorem 1.2 together with Proposition 5.2 gives the result. The details are left to the interested reader. \( \square \)

### 6. Membership in Schatten Classes

Let \( H \) be a separable Hilbert space, and \( 0 < p < \infty \). The Schatten class \( S_p = S_p(H) \) consists of those compact operators \( T \) on \( H \) for which its sequence of singular numbers \( \{ \lambda_n \} \) belongs to the sequence space \( \ell^p \) (the singular numbers are the square roots of the eigenvalues of the positive operator \( T^*T \), where \( T^* \) is the Hilbert adjoint of \( T \)). For \( p \geq 1 \), the class \( S_p \) is a Banach space with the norm \( \| T \|_p = (\sum_n |\lambda_n|^p)^{1/p} \), while for \( 0 < p < 1 \) one has the inequality \( \| S + T \|_p^p \leq \| S \|_p^p + \| T \|_p^p \). Also, one has \( T \in S_p \) if and only if \( T^*T \in S_{p/2} \). We refer to [16] Chapter 1 for a brief account on Schatten classes.

In this section, we are going to describe those positive Borel measures \( \mu \) for which the Toeplitz operator \( T_\mu \) belongs to the the Schatten ideal \( S_p(A^2_\omega) \), for \( \omega \in \mathcal{E} \). We recall the reader that we are still assuming that (1.2) holds. In order to obtain such a characterization, we need to introduce first some concepts.

We define the \( \omega \)-Berezin transform \( B_{\omega,\mu} \) of the measure \( \mu \) as

\[
B_{\omega,\mu}(z) := \int_{\mathbb{D}} |k_z(\xi)|^2 \omega(\xi) d\mu(\xi), \quad z \in \mathbb{D},
\]

where \( k_z \) are the normalized reproducing kernels in \( A^2_\omega \). We also recall that, for \( \delta \in (0, m_\tau) \), the averaging function \( \tilde{\mu}_\delta \) is given by

\[
\tilde{\mu}_\delta(z) := \frac{\mu(D(\delta\tau(z))}{\tau(z)^2}, \quad z \in \mathbb{D}.
\]

We also consider the measure \( \lambda_\tau \) given by

\[
d\lambda_\tau(z) = \frac{dA(z)}{\tau(z)^2}, \quad z \in \mathbb{D}.
\]
Proposition 6.1. Let $0 < p < \infty$, and $\omega \in \mathcal{E}$. The following conditions are equivalent:

(a) The function $B_{\omega, \mu}$ is in $L^p(\mathbb{D}, d\lambda_{\tau})$.

(b) The function $\mu_{\delta}$ is in $L^p(\mathbb{D}, d\lambda_{\tau})$ for any $\delta \in (0, m_{\tau})$.

(c) The sequence $\{\mu_{\delta}(z_n)\}$ is in $\ell^p$ for any $(\delta, \tau)$-lattice $\{z_n\}$ with $\delta \in (0, \frac{m_{\tau}}{4})$.

Proof. (a) $\Rightarrow$ (b). This is essentially proved in the proof of Theorem 3.1, where the same computations give

$$B_{\omega, \mu}(z) = \int_{\mathbb{D}} |k_z(s)|^2 \omega(s) d\mu(s) \geq \mu_{\delta}(z).$$

Since $B_{\omega, \mu}$ is in $L^p(\mathbb{D}, d\lambda_{\tau})$, this gives (b).

(b) $\Rightarrow$ (c). Since $\mu_{\delta}(z_n) \lesssim \mu_{4\delta}(z)$, for $z \in D(\delta \tau(z_n))$, then

$$\sum_n \mu_{\delta}(z_n)^p \lesssim \sum_n \int_{D(\delta \tau(z_n))} \mu_{4\delta}(z)^p \frac{dA(z)}{\tau(z)^2} \lesssim \int_{\mathbb{D}} \mu_{4\delta}(z)^p d\lambda_{\tau}(z).$$

(c) $\Rightarrow$ (a). We have

$$B_{\omega, \mu}(z) \leq \|K_z\|_{A^2_{\omega}}^2 \sum_n \int_{D(\delta \tau(z_n))} |K_z(s)|^2 \omega(s) d\mu(s).$$

If $p > 1$, we use Lemma A in order to obtain

$$\int_{D(\delta \tau(z_n))} |K_z(s)|^2 \omega(s) d\mu(s) \lesssim \int_{D(\delta \tau(z_n))} \left( \frac{1}{\tau(s)^2} \int_{D(\delta \tau(s))} |K_z(\xi)|^2 \omega(\xi) dA(\xi) \right) d\mu(s)$$

$$\lesssim \left( \int_{D(3\delta \tau(z_n))} |K_z(\xi)|^2 \omega(\xi) dA(\xi) \right) \mu_{\delta}(z_n).$$

Thus, by Hölder’s inequality,

$$\left( \sum_n \int_{D(\delta \tau(z_n))} |K_z(s)|^2 \omega(s) d\mu(s) \right)^p$$

$$\lesssim \|K_z\|_{A^2_{\omega}}^{2(p-1)} \sum_n \left( \int_{D(3\delta \tau(z_n))} |K_z(\xi)|^2 \omega(\xi) dA(\xi) \right)^{p} \mu_{\delta}(z_n)^p.$$ 

This gives

$$\int_{\mathbb{D}} B_{\omega, \mu}(z)^p d\lambda_{\tau}(z) \lesssim \sum_n \mu_{\delta}(z_n)^p \int_{D(3\delta \tau(z_n))} \left( \int_{\mathbb{D}} |K_{\xi}(z)|^2 \|K_{\xi}\|_{A^2_{\omega}}^{-2} d\lambda_{\tau}(z) \right) \omega(\xi) dA(\xi).$$

Since $\|K_z\|_{A^2_{\omega}}^2 \asymp \tau(z)^{-2} \omega(z)^{-1}$, we have

$$\int_{\mathbb{D}} |K_{\xi}(z)|^2 \|K_{\xi}\|_{A^2_{\omega}}^{-2} d\lambda_{\tau}(z) \asymp \|K_{\xi}\|_{A^2_{\omega}}^2 \asymp \tau(\xi)^{-2} \omega(\xi)^{-1}.$$
Putting this estimate in the previous inequality, we finally get

$$\int_{\mathbb{D}} B_\omega \mu(z)^p d\lambda_\tau(z) \lesssim \sum_n \hat{\mu}_\delta(z_n)^p$$

finishing the proof for the case $p > 1$. Now, if $0 < p \leq 1$, proceeding as before using Lemma [A], we have

$$B_\omega \mu(z)^p \lesssim \|K_z\|_{A^2_\omega}^2 \sum_n \left( \int_{D(3\delta \tau(z_n))} |K_z(\xi)|^{2p} \omega(\xi)^p dA(\xi) \right)^{1/p} \hat{\mu}_\delta(z_n)^p \tau(z_n)^{2(p-1)/p}.$$ 

Since $0 < p \leq 1$, this gives

$$B_\omega \mu(z)^p \lesssim \|K_z\|_{A^2_\omega}^{-2p} \sum_n \left( \int_{D(3\delta \tau(z_n))} |K_z(\xi)|^{2p} \omega(\xi)^p dA(\xi) \right)^{1/p} \hat{\mu}_\delta(z_n)^p \tau(z_n)^{2(p-1)}.$$ 

Integrating this inequality against the measure $d\lambda_\tau(z)$, and using the norm estimate $\|K_z\|_{A^2_\omega} \omega(z) \asymp \tau(z)^{-2}$, we obtain that

$$\int_{\mathbb{D}} B_\omega \mu(z)^p d\lambda_\tau(z)$$

is less than constant times

$$\sum_n \hat{\mu}_\delta(z_n)^p \tau(z_n)^{2(p-1)} \int_{D(3\delta \tau(z_n))} \left( \int_{\mathbb{D}} |K_\xi(z)|^{2p} \omega(z)^p \tau(z)^{2(p-1)} dA(z) \right) \omega(\xi)^p dA(\xi).$$

By Lemma [C] we have

$$\int_{\mathbb{D}} |K_\xi(z)|^{2p} \omega(z)^p \tau(z)^{2(p-1)} dA(z) \lesssim \omega(\xi)^{-p} \tau(\xi)^{-2p}.$$ 

This, together with the previous estimate, yield

$$\int_{\mathbb{D}} B_\omega \mu(z)^p d\lambda_\tau(z) \lesssim \sum_n \hat{\mu}_\delta(z_n)^p.$$ 

The proof is complete. \(\square\)

Next Lemma is the analogue to our setting of a well known result for standard Bergman spaces.

**Lemma 6.2.** Let $\omega \in \mathcal{E}$, and $T$ be a positive operator on $A^2_\omega$. Let $\tilde{T}$ be the Berezin transform of the operator $T$ defined by

$$\tilde{T}(z) = \langle Tk_z, k_z \rangle_\omega, \quad z \in \mathbb{D}.$$ 

(a) Let $0 < p \leq 1$. If $\tilde{T} \in L^p(\mathbb{D}, d\lambda_\tau)$ then $T$ is in $S_p$.

(b) Let $p \geq 1$. If $T$ is in $S_p$ then $\tilde{T} \in L^p(\mathbb{D}, d\lambda_\tau)$. 
Proof. Let $p > 0$. The positive operator $T$ is in $S_p$ if and only if $T^p$ is in the trace class $S_1$. Fix an orthonormal basis $\{e_k\}$ of $A_2^\omega$. Since $T^p$ is positive, it belongs to the trace class if and only if $\sum_k \langle T^p e_k, e_k \rangle_\omega < \infty$. Let $S = \sqrt{T^p}$. Then

$$\sum_k \langle T^p e_k, e_k \rangle_\omega = \sum_k \|S e_k\|_{A_2^\omega}^2.$$ 

Now, Fubini’s theorem and Parseval’s identity, we have

$$\sum_k \|S e_k\|_{A_2}^2 = \sum_k \int_D |S e_k(z)|^2 \omega(z) dA(z) = \sum_k \int_D \langle S e_k, K_z \rangle_\omega^2 \omega(z) dA(z)$$

$$= \int_D \left( \sum_k |\langle e_k, S K_z \rangle_\omega|^2 \right) \omega(z) dA(z) = \int_D \|S K_z\|_{A_2}^2 \omega(z) dA(z)$$

$$= \int_D \langle T^p K_z, K_z \rangle_\omega \omega(z) dA(z) = \int_D \langle T^p k_z, k_z \rangle_\omega \|K_z\|_{A_2}^2 \langle K_z \rangle_\omega \omega(z) dA(z)$$

$$\simeq \int_D \langle T^p k_z, k_z \rangle_\omega d\lambda_\tau(z).$$

Hence, both (a) and (b) are consequences of the inequalities (see [16, Proposition 1.31])

$$\langle T^p k_z, k_z \rangle_\omega \leq \langle T k_z, k_z \rangle_\omega^p = [\langle T k_z, k_z \rangle_\omega^p], \quad 0 < p \leq 1$$

and

$$[\langle T\rangle_\omega^p] = \langle T k_z, k_z \rangle_\omega^p \leq \langle T^p k_z, k_z \rangle_\omega, \quad p \geq 1.$$ 

This finishes the proof of the lemma. \qed

**Proposition 6.3.** Let $\omega \in \mathcal{E}$. If $0 < p \leq 1$ and $B_{\omega, \mu}$ is in $L^p(\mathbb{D}, d\lambda_\tau)$, then $T_\mu$ belongs to $S_p(A_2^\omega)$. Conversely, if $p \geq 1$ and $T_\mu$ is in $S_p(A_2^\omega)$, then $B_{\omega, \mu} \in L^p(\mathbb{D}, d\lambda_\tau)$.

**Proof.** If $B_{\omega, \mu}$ is in $L^p(\mathbb{D}, d\lambda_\tau)$, then it is easy to see that $T_\mu$ is bounded on $A_2^\omega$ (just use the discrete version in Proposition 6.1 to see that the condition in Theorem 1.1 holds). Therefore, the result is a consequence of Lemma 6.2 since $\widetilde{T}_\mu(z) = B_{\omega, \mu}(z)$. \qed

Now we are almost ready for the characterization of Schatten class Toeplitz operators, but we need first some technical lemmas on properties of lattices. We use the notation

$$d_\tau(z, \zeta) = \frac{|z - \zeta|}{\min(\tau(z), \tau(\zeta))}, \quad z, \zeta \in \mathbb{D}.$$
Lemma 6.4. Let $\tau \in \mathcal{L}$, and $\{z_j\}$ be a $(\delta, \tau)$-lattice on $\mathbb{D}$. For each $\zeta \in \mathbb{D}$, the set

$$D_m(\zeta) = \{z \in \mathbb{D} : d_\tau(z, \zeta) < 2^m \delta\}$$

contains at most $K$ points of the lattice, where $K$ depends on the positive integer $m$ but not on the point $\zeta$.

Proof. Let $K$ be the number of points of the lattice contained in $D_m(\zeta)$. Due to the Lipschitz condition (B), we have

$$\tau(\zeta) \leq \tau(z_j) + c_2|\zeta - z_j| \leq (1 + c_2 2^m \delta) \tau(z_j) = C_m \tau(z_j).$$

Then

$$K \cdot \tau(\zeta)^2 \leq C_m^2 \sum_{z_j \in D_m(\zeta)} \tau(z_j)^2 \lesssim C_m^2 \cdot \text{Area} \left( \bigcup_{z_j \in D_m(\zeta)} D\left(\frac{\delta}{4} \tau(z_j)\right) \right).$$

As done before, we also have $\tau(z_j) \leq C_m \tau(\zeta)$, if $z_j \in D_m(\zeta)$. From this we easily see that

$$D\left(\frac{\delta}{4} \tau(z_j)\right) \subset D\left(c 2^m \delta \tau(\zeta)\right)$$

for some constant $c$. Since the sets $\{D(\frac{\delta}{4} \tau(z_j))\}$ are pairwise disjoints, we have

$$\bigcup_{z_j \in D_m(\zeta)} D\left(\frac{\delta}{4} \tau(z_j)\right) \subset D\left(c 2^m \delta \tau(\zeta)\right).$$

Therefore, we get

$$K \cdot \tau(\zeta)^2 \leq C_m^2 \cdot \text{Area} \left( D\left(c 2^m \delta \tau(\zeta)\right) \right) \lesssim C_m^2 2^{2m} \tau(\zeta)^2,$$

that implies $K \leq C 2^{4m}$. \hfill \Box

Next, we use the result just proved to decompose any $(\delta, \tau)$-lattice into a finite number of “big” separated subsequences.

Lemma 6.5. Let $\tau \in \mathcal{L}$ and $\delta \in (0, m_\tau)$. Let $m$ be a positive integer. Any $(\delta, \tau)$-lattice $\{z_j\}$ on $\mathbb{D}$ can be partitioned into $M$ subsequences such that, if $a_j$ and $a_k$ are different points in the same subsequence, then $d_\tau(a_j, a_k) \geq 2^m \delta$.

Proof. Let $K$ be the number given by Lemma 6.4. From the lattice $\{z_j\}$ extract a maximal $(2^m \delta)$-subsequence, that is, we select one point $\xi_1$ in our lattice, and then we continue selecting points $\xi_n$ of the lattice so that $d_\tau(\xi_n, \xi) \geq 2^m \delta$ for all previous selected point $\xi$. We stop once the subsequence is maximal, that is, when all the remaining points $x$ of the lattice satisfy $d_\tau(x, \xi) < 2^m \delta$ for some $\xi$ in the subsequence. With the remaining points of the lattice we extract another maximal $(2^m \delta)$-subsequence, and we repeat the process until we get $M = K + 1$ maximal $(2^m \delta)$-subsequences. If no point of the lattice is left, we are done. On the other hand, if a point $\zeta$ in the lattice is left, this means that there are $M = K + 1$ distinct points $x_\zeta$ (at least one for each subsequence) in the lattice with $d_\tau(\zeta, x_\zeta) < 2^m \delta$, in contradiction with the choice of $K$ from Lemma 6.4. The proof is complete. \hfill \Box
Now we are ready for the main result of this Section, that characterizes the membership in the Schatten ideals of the Toeplitz operator acting on $A^2_\omega$. For $p \geq 1$, the equivalence of $T_\mu$ being in $S_p$ and condition (c) was obtained in [9], where the sufficiency of (c) for $0 < p < 1$ was also established. They left open the necessity for $0 < p < 1$, a problem that is solved here.

**Theorem 6.6.** Let $\omega \in \mathcal{E}$ and $0 < p < \infty$. The following conditions are equivalent:

(a) The Toeplitz operator $T_\mu$ is in $S_p(A^2_\omega)$.

(b) The function $\hat{\mu}_\delta$ is in $L^p(\mathbb{D}, d\lambda_\tau)$ for $\delta \in (0, m_\tau)$.

(c) The sequence $\{\hat{\mu}_\delta(z_n)\}$ is in $\ell^p$ for any $(\delta, \tau)$-lattice $\{z_n\}$ with $\delta \in (0, \frac{m_\tau}{4})$.

(d) The function $B_{\omega, \mu}$ is in $L^p(\mathbb{D}, d\lambda_\tau)$.

**Proof.** By Proposition [6.1] the statements (b), (c) and (d) are equivalent. Also, according to Proposition [6.3] it remains to prove that (d) implies (a) for $p > 1$, and that (a) implies (c) when $0 < p < 1$.

Let $1 < p < \infty$, and assume that $\hat{\mu}_\delta \in L^p(\mathbb{D}, d\lambda_\tau)$ with $\delta \in (0, m_\tau)$. It is not difficult to see, using the equivalent discrete condition in (c) together with Theorem [5.3], that $T_\mu$ must be compact. For any orthonormal set $\{e_n\}$ of $A^2_\omega$, we have

$$
\sum_n \langle T_\mu e_n, e_n \rangle^p_{A^2_\omega} = \sum_n \left( \int_{\mathbb{D}} |e_n(z)|^2 \omega(z) d\mu(z) \right)^p.
$$

By Lemma [A] and Fubini’s theorem,

$$
\int_{\mathbb{D}} |e_n(z)|^2 \omega(z) d\mu(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{\tau(z)} \int_{D(\delta \tau(z))} |e_n(\zeta)|^2 \omega(\zeta) dA(\zeta) \right) d\mu(z)
$$

$$
\lesssim \int_{\mathbb{D}} |e_n(\zeta)|^2 \omega(\zeta) \hat{\mu}_\delta(\zeta) dA(\zeta).
$$

Since $p > 1$ and $\|e_n\|_{A^2_\omega} = 1$, we can apply Hölder’s inequality to get

$$
\left( \int_{\mathbb{D}} |e_n(z)|^2 \omega(z) d\mu(z) \right)^{p} \lesssim \int_{\mathbb{D}} |e_n(\zeta)|^2 \omega(\zeta) \hat{\mu}_\delta(\zeta)^p dA(\zeta).
$$

Putting this into (6.1) and taking into account that $\|K_\zeta\|^2_{A^2_\omega} \omega(\zeta) \asymp \tau(\zeta)^{-2}$, we see that

$$
\sum_n \langle T_\mu e_n, e_n \rangle^p_{A^2_\omega} \lesssim \int_{\mathbb{D}} \left( \sum_n |e_n(\zeta)|^2 \right) \omega(\zeta) \hat{\mu}_\delta(\zeta)^p dA(\zeta)
$$

$$
\lesssim \int_{\mathbb{D}} \|K_\zeta\|^2_{A^2_\omega} \omega(\zeta) \hat{\mu}_\delta(\zeta)^p dA(\zeta)
$$

$$
\asymp \int_{\mathbb{D}} \hat{\mu}_\delta(\zeta)^p d\lambda_\tau(\zeta).
$$

By [16] Theorem 1.27 this proves that $T_\mu$ is in $S_p$ with $\|T_\mu\|_{S_p} \lesssim \|\hat{\mu}_\delta\|_{L^p(\mathbb{D}, d\lambda_\tau)}$. 
Next, let \( 0 < p < 1 \), and suppose that \( T_\mu \in S_p(A^2_\omega) \). We will prove that (c) holds. The method for this proof has its roots in previous work of S. Semmes \([14]\) and D. Luecking \([10]\). Let \( \{z_n\} \) be a \((\delta, \tau)\)-lattice on \( \mathbb{D} \) with \( \delta \in (0, 1/2m_\tau) \). We want to show that \( \{\tilde{\mu}_\delta(z_n)\} \) is in \( \ell^p \). To this end, we fix a large positive integer \( m \geq 2 \) and apply Lemma 6.5 to partition the lattice \( \{z_n\} \) into \( M \) subsequences such that any two distinct points \( a_j \) and \( a_k \) in the same subsequence satisfy \( d_\nu(a_j, a_k) \geq 2^m \delta \). Let \( \{a_n\} \) be such a subsequence and consider the measure

\[
\nu = \sum_n \mu_{\chi_n},
\]

where \( \chi_n \) denotes the characteristic function of \( D(\delta \tau(a_n)) \). Since \( m \geq 2 \), the disks \( D(\delta \tau(a_n)) \) are pairwise disjoints. Since \( T_\mu \) is in \( S_p \) and \( 0 \leq \nu \leq \mu \), then \( 0 \leq T_\nu \leq T_\mu \) which implies that \( T_\nu \) is also in \( S_p \). Moreover, \( \|T_\nu\|_{S_p} \leq \|T_\mu\|_{S_p} \). Fix an orthonormal basis \( \{e_n\} \) for \( A^2_\omega \) and define an operator \( B \) on \( A^2_\omega \) by

\[
B \left( \sum_n \lambda_n e_n \right) = \sum_n \lambda_n k_{a_n},
\]

where \( k_{a_n} \) are the normalized reproducing kernels of \( A^2_\omega \). By Lemma 3.4, the operator \( B \) is bounded. Since \( T_\nu \in S_p \), the operator \( T = A^* T_\nu A \) is also in \( S_p \), with

\[
\|T\|_{S_p} \leq \|B\|^2 \cdot \|T_\nu\|_{S_p}.
\]

We split the operator \( T \) as \( T = D + E \), where \( D \) is the diagonal operator on \( A^2_\omega \) defined by

\[
Df = \sum_{n=1}^{\infty} \langle T e_n, e_n \rangle_\omega \langle f, e_n \rangle_\omega e_n, \quad f \in A^2_\omega,
\]

and \( E = T - D \). By the triangle inequality,

\[
(6.2) \quad \|T\|_{S_p}^p \geq \|D\|_{S_p}^p - \|E\|_{S_p}^p.
\]

Since \( D \) is positive diagonal operator, and \( B_\omega \nu(a_n) \geq \tilde{\nu}_\delta(a_n) = \tilde{\mu}_\delta(a_n) \), we have

\[
(6.3) \quad \|D\|_{S_p}^p = \sum_n \langle T e_n, e_n \rangle_\omega^p = \sum_n \langle T_\nu k_{a_n}, k_{a_n} \rangle_\omega^p \geq C_1 \sum_n \tilde{\mu}_\delta(a_n)^p.
\]

On the other hand, since \( 0 < p < 1 \), by \([16]\) Proposition 1.29] we have

\[
(6.4) \quad \|E\|_{S_p}^p \leq \sum_n \sum_{k} \langle E e_n, e_k \rangle_\omega^p = \sum_{n,k:k \neq n} \langle T_\nu k_{a_n}, k_{a_k} \rangle_\omega^p \leq \sum_{n,k:k \neq n} \left( \int_{\mathbb{D}} |k_{a_n}(\xi)| \omega(\xi) d\nu(\xi) \right)^p \sum_{j} \left( \int_{D(\delta \tau(a_j))} |k_{a_n}(\xi)| \omega(\xi) d\mu(\xi) \right)^p.
\]
If \( n \neq k \), then \( d_\tau(a_n, a_k) \geq 2^m \delta \). Thus, for \( \xi \in D(\delta \tau(a_j)) \), it is not difficult to see that either
\[
d_\tau(\xi, a_n) \geq 2^{m-2} \delta \quad \text{or} \quad d_\tau(\xi, a_k) \geq 2^{m-2} \delta.
\]
Indeed, since \( n \neq k \), then either \( d_\tau(a_n, a_j) \geq 2^m \delta \) or \( d_\tau(a_j, a_k) \geq 2^m \delta \). Suppose that \( d_\tau(a_n, a_j) \geq 2^m \delta \). If \( d_\tau(\xi, a_n) < 2^{m-2} \delta \), then
\[
|a_n - a_j| \leq |a_n - \xi| + |\xi - a_j| < 2^{m-2} \delta \min(\tau(a_n), \tau(\xi)) + \delta \tau(a_j)
\]
\[
\leq 2^{m-1} \delta \min(\tau(a_n), \tau(a_j)) + \delta \tau(a_j).
\]
This directly gives a contradiction if \( \min(\tau(a_n), \tau(a_j)) = \tau(a_n) \), using the Lipschitz condition (B) we get
\[
|a_n - a_j| < 2^{m-1} \delta \min(\tau(a_n), \tau(a_j)) + \delta \tau(a_n) + c_2 \delta |a_n - a_j|
\]
Since \( c_2 \delta \leq 1/4 \), and \( m \geq 2 \), we see that this implies
\[
d_\tau(a_n, a_j) < \frac{4}{3}(2^{m-1} + 1)\delta \leq 2^m \delta.
\]
Thus, without loss of generality, we assume that \( d_\tau(\xi, a_n) \geq 2^{m-2} \delta \). For any \( n \) and \( k \) we write
\[
I_{nk}(\mu) = \sum_j \int_{D(\delta \tau(a_j))} |k_{a_n}(\xi)| |k_{a_k}(\xi)| \omega(\xi) d\mu(\xi).
\]
With this notation and taking into account (6.4), we have
\[
(6.5) \quad \|E\|_{S_p}^p \leq \sum_{n,k:k \neq n} I_{nk}(\mu)^p.
\]
By Theorem A with \( m = 2 \), we have
\[
|K_{a_n}(\xi)| \lesssim \|K_{a_n}\|_{A_2^2} \cdot \|K_{\xi}\|_{A_2^2} \ d_\tau(\xi, a_n)^{-2}.
\]
Apply this inequality raised to the power \( 1/2 \), together with the fact that \( d_\tau(\xi, a_n) \geq 2^{m-2} \delta \) to get
\[
|k_{a_n}(\xi)| = \frac{|K_{a_n}(\xi)|^{1/2}}{\|K_{a_n}\|_{A_2^2}} \ |K_{a_n}(\xi)|^{1/2} \lesssim 2^{-m} |k_{a_n}(\xi)|^{1/2} \cdot \|K_{\xi}\|_{A_2^2}^{1/2}
\]
We also have
\[
(6.7) \quad |k_{a_k}(\xi)| \leq |k_{a_k}(\xi)|^{1/2} \cdot \|K_{\xi}\|_{A_2^2}^{1/2}
\]
Putting (6.6) and (6.7) into the definition of \( I_{nk}(\mu) \), and using the norm estimate
\[
\|K_{\xi}\|_{A_2^2} \approx \tau(\xi)^{-1} \omega(\xi)^{-1/2},
\]
we obtain
\[
I_{nk}(\mu) \lesssim 2^{-m} \sum_j \frac{1}{\tau(a_j)} \int_{D(\delta \tau(a_j))} |k_{a_n}(\xi)|^{1/2} |k_{a_k}(\xi)|^{1/2} \omega(\xi)^{1/2} d\mu(\xi).
\]
By Lemma A for \( \xi \in D(\delta \tau(a_j)) \), one has
\[ |k_{a_n}(\xi)|^{1/2} \omega(\xi)^{1/4} \lesssim \left( \frac{1}{\tau(\xi)^2} \int_{D(\delta\tau(\xi))} |k_{a_n}(z)|^{p/2} \omega(z)^{p/4} dA(z) \right)^{1/p} \]

\[ \lesssim \tau(a_j)^{-2/p} \, S_n(a_j)^{1/p} , \]

with

\[ S_n(x) = \int_{D(3\delta\tau(x))} |K_{a_n}(z)|^{p/2} \omega(z)^{p/4} dA(z) . \]

In the same manner we also have

\[ k_{a_k}(\xi)|^{1/2} \omega(\xi)^{1/4} \lesssim \tau(a_j)^{-2/p} \, S_k(a_j)^{1/p} . \]

Therefore, there is a positive constant \( C_2 \) such that

\[
I_{nk}(\mu) \leq C_2 \cdot 2^{-m} \sum_j \frac{\tau(a_j)^{-4/p}}{\tau(a_j)} \, S_n(a_j)^{1/p} \cdot S_k(a_j)^{1/p} \cdot \mu(D(\delta\tau(a_j)))
\]

\[ = C_2 \cdot 2^{-m} \sum_j \tau(a_j)^{-1-4/p} \cdot S_n(a_j)^{1/p} \cdot S_k(a_j)^{1/p} \cdot \tilde{\mu}_\delta(a_j) . \]

Since \( 0 < p < 1 \), we get

\[
I_{nk}(\mu)^p \leq C_2^p \cdot 2^{-mp} \sum_j \tau(a_j)^{p-4} \cdot S_n(a_j) \cdot S_k(a_j) \cdot \tilde{\mu}_\delta(a_j)^p .
\]

Bearing in mind (6.5), this gives

\[
\| E \|_{S_p}^p \leq C_2^p \cdot 2^{-mp} \sum_j \tau(a_j)^{p-4} \cdot \tilde{\mu}_\delta(a_j)^p \left( \sum_{n,k} S_n(a_j) \cdot S_k(a_j) \right)
\]

\[ = C_2^p \cdot 2^{-mp} \sum_j (\tau(a_j)^{p-4} \cdot \tilde{\mu}_\delta(a_j)^p) \left( \sum_{n} S_n(a_j) \right)^2 . \]

On the other hand, we have

\[
\sum_n S_n(a_j) = \sum_n \int_{D(3\delta\tau(a_j))} |k_{a_n}(z)|^{p/2} \omega(z)^{p/4} dA(z)
\]

\[ = \int_{D(3\delta\tau(a_j))} \left( \sum_n |K_z(a_n)|^{p/2} \| K_{a_n} \|_{A_\omega}^{-p/2} \right) \omega(z)^{p/4} dA(z) . \]

By the \( A_\omega^2 \)-norm estimate and the “submean property” in Lemma [A]
\[
\sum_n |K_z(a_n)|^{p/2} \|K_{a_n}\|_{A^2_S}^{-p/2} \lesssim \sum_n |K_z(a_n)|^{p/2} \omega(a_n)^{p/4} \tau(a_n)^{p/2}
\]
\[
\lesssim \sum_n \left( \frac{1}{\tau(a_n)^2} \int_{D(\delta \tau(a_n))} |K_z(\zeta)|^{p/2} \omega(\zeta)^{p/4} dA(\zeta) \right) \tau(a_n)^{p/2}.
\]
Since \(\tau(a_n) \asymp \tau(\zeta)\) for \(\zeta \in D(\delta \tau(a_n))\), the disjointness of the disks together with Lemma C yield
\[
\sum_n |K_z(a_n)|^{p/2} \|K_{a_n}\|_{A^2_S}^{-p/2} \lesssim \int_D |K_z(\zeta)|^{p/2} \omega(\zeta)^{p/4} \tau(\zeta)^{\frac{p}{2} - 2} dA(\zeta)
\]
\[
\lesssim \omega(z)^{-\frac{p}{4}} \tau(z)^{\frac{p}{2} - 2(\frac{p}{2} - 1)}.
\]
Bearing in mind (6.9), we conclude that there exists another positive constant \(C_3\) such that
\[
\sum_n S_n(a_j) \leq C_3 \cdot \tau(a_j)^{2 - \frac{p}{2}}.
\]
Putting this in to (6.9) we finally get
\[
\|E\|_{S^p}^p \leq C_2^p \cdot C_3^2 \cdot 2^{-mp} \sum_j \hat{\mu}_j(a_j)^p.
\]
Combining this with (6.2), (6.3) and choosing \(m\) large enough so that
\[
C_2^p \cdot C_3^2 \cdot 2^{-mp} \leq C_1/2,
\]
then we deduce that
\[
\sum_j \hat{\mu}_j(a_j)^p \leq \frac{C_1}{2} \|T\|_{S^p}^p \leq C_4 \|T_\mu\|_{S^p}^p.
\]
Since this holds for each one of the \(M\) subsequences of \(\{z_n\}\), we obtain
\[
(6.10) \quad \sum_n \hat{\mu}_j(z_n)^p \leq C_4 M \|T_\mu\|_{S^p}^p
\]
for all locally finite positive Borel measures \(\mu\) such that
\[
\sum_n \hat{\mu}(z_n)^p < \infty.
\]
Finally, an easy approximation argument then shows that (6.10) actually holds for all locally finite positive Borel measures \(\mu\). The proof is complete. \(\square\)
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HICHAM ARROUSSI, DEPARTAMENT DE MATEMÀTICA APPLICADA I ANALISI, UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08007 BARCELONA, SPAIN
E-mail address: arroussihicham@yahoo.fr

INYOUNG PARK, CENTER FOR GEOMETRY AND ITS APPLICATIONS, POHANG UNIVERSITY OF SCIENCE AND TECHNOLOGY, POHANG 790-784
E-mail address: iypark26@postech.ac.kr

JORDI PAU, DEPARTAMENT DE MATEMÀTICA APPLICADA I ANALISI, UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08007 BARCELONA, SPAIN
E-mail address: jordi.pau@ub.edu