Effective connections and fields
associated with shear-free null congruences

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Abstract
A special subclass of shear-free null congruences (SFC) is studied, with tangent vector field being a repeated principal null direction of the Weyl tensor. We demonstrate that this field is parallel with respect to an effective affine connection which contains the Weyl nonmetricity and the skew symmetric torsion. On the other hand, a Maxwell-like field can be directly associated with any special SFC, and the electric charge for bounded singularities of this field turns to be “self-quantized”. Two invariant differential operators are introduced which can be thought of as spinor analogues of the Beltrami operators and both nullify the principal spinor of any special SFC.

KEY WORDS: shear-free congruences; Weyl-Cartan connections; Maxwell equations; quantization of electric charge; invariant spinor operators.

1 Introduction
Among the congruences of null geodesics (of rectilinear light rays in flat Minkowski space) the congruences with zero shear (shear-free congruences, SFC) are certainly distinguished. Recall that expansion, twist and shear are three optical invariants [19] which can be associated with every geodesic congruence and describe the deformations of infinitesimal orthogonal sections along the rays. Particularly, light rays emitted by an arbitrary moving point source form a simplest SFC (with zero twist) [18]. Complexification of this construction leads to twisting SFC including the famous Kerr congruence.

In flat (precisely, in conformally flat) space, SFC are closely related to twistor geometry, and all analytical SFC can be obtained algebraically via the twistor construction and the Kerr theorem [19]. These congruences represent also one of the two classes of solutions to complex eikonal equation [9]. On the other hand, SFC naturally arise in the framework of non-commutative analysis (over the algebra of biquaternions), and their defining equation turns to be equivalent to the (nonlinear) generalization of the

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Cauchy-Riemann differentiability conditions \[6, 8\]. On these grounds, an
unified algebraic field theory (nonlinear, over-determined, non-Lagrangian)
has been developed in our works \[5, 7, 8, 9\] in which the approach has been
called \textit{algebrodynamics}.

Indeed, SFC in flat space manifest also numerous connections with equations
for massless fields (Penrose transform and Ward construction \[21\] for
self-dual fields among them) but the interlinks are not direct. The first and
the most known result which relates the SFC and the solutions to massless
field equations (to Maxwell equations in particular) is the Robinson’s theo-
rem \[16\]. It asserts that there always exists an affine parameter along the
congruence such that the self-dual 2-form \(F_{\mu\nu} := \varepsilon_{AB} \xi_A \xi_B\) defined via
the the principal spinor \(\xi_A\) of the SFC is \textit{exact} and, therefore, satisfies the
homogeneous Maxwell equations. However, electromagnetic field obtained
in this way is null, and there is no straightforward generalization of the
Robinson theorem to general (non-null) fields. We mention here only the
works \[1, 2\] where SFC were applied to construct the Hertz potential of an
electromagnetic field on Riemannian space-times.

On the other hand, in the framework of algebrodynamics it was shown
that every SFC induces an effective affine connection with a Weyl non-
metricity and a skew symmetric torsion \[5\]. Defining equations for SFC
become then just the conditions for principal spinor of SFC to be \textit{covari-
antly constant} (parallel) with respect to this Weyl-Cartan connection.

In analogy with the unified Weyl theory, for any SFC one can define,
therefore, a \textit{gauge field} with potentials represented by the Weyl 4-vector.
From the integrability conditions it follows then that correspondent field
strengths should satisfy homogeneous Maxwell equations. Further on we
refer to this field as the \textit{gauge field of the congruence} (GFC). Besides, the
potentials of an \(SL(2, \mathbb{C})\) matrix gauge field can be defined via the compo-
nents of the same Weyl vector, and by virtue of integrability conditions \textit{it
obeys the Yang-Mills type equations} \[5\]. It is known also \[12\] (see also \[8\])
that the component of projective spinor defining tangent vector field of a
SFC necessarily satisfies both linear wave and nonlinear eikonal equations.

Note that Maxwell field (GFC) which arises in the above procedure has
a lot of remarkable properties which are not inherent to fields introduced
by Robinson and Penrose. Specifically, this field is really a gauge one
possessing a residual “weak” gauge group closely related to transformations
in the projective twistor space \[8\]. Moreover, the strengths and the singular
loci of the GFC admit explicit representation in twistor variables \[3, 9\] and
can be obtained in a completely algebraic way. Finally, effective \textit{electric
charge} of bounded singularities of the GFC turns to be \textit{self-quantized}, i.e.
integer multiple to a minimal “elementary” one \[5, 10, 11\].

It is also well known that an effective Riemannian metric of the Kerr-
Schild type can be canonically associated with every SFC in Minkowski
space. Under this particular deformation of space-time geometry the congruence preserves all its properties, i.e. remains null, geodetic and shear-free. Singularities of curvature of the deformed metrics are completely determined by the congruence and correspond to the locus of branching points of the latter. It is especially interesting that curvature singularities and strength singularities of the above discussed GFC completely coincide in space and time and define, therefore, a unique particle-like object with nontrivial time evolution. Note also that in many cases the induced metric can be choosed to satisfy vacuum or electrovacuum Einstein equations: for instance, the Kerr and the Kerr-Newman metrics can be obtained in such a way.

In general relativity significance of SFC is also justified by the Goldberg-Sachs theorem [4]: for any Einstein space the Weyl conformal tensor is algebraically special iff the manifold admits a SFC which in this case defines one of its repeated principal null direction (PND). Particularly, it follows that in any vacuum space-time there are at most 2 independent SFC (for the space of Petrov type D) or a single SFC in spaces of type II, III or N. The generalization of this theorem is known [18]. Generically, if a manifold is not conformally Einstein (and, therefore, not conformally flat) the number of distinct SFC can't exceed four because every SFC necessarily defines a PND of the Weyl tensor. Recall for comparison that in a conformally flat space there exists an infinity of distinct SFC described by the Kerr theorem.

In the paper, we restrict ourselves by a special subclass of SFC which define a repeated PND of the Weyl spinor and can, thus, exist only on an algebraically special Riemannian space-time. We prove that, as it takes place for the flat case, any such SFC defines a vector field parallel with respect to an effective Weyl-Cartan connection. Further on we consider the fields which can be associated with a special SFC and which are, in fact, inherited from those defined for rectilinear SFC on the flat background and described above. In particular, we show that the GFC correspondent to a special SFC on an asymptotically flat space preserves its Coulomb-like structure and the property of self-quantization of effective electric charge. As an example, the situation for radial (special) SFC on the Schwarzschild background is analyzed. Finally, we adapt the wave and the eikonal Beltrami operators to the 2-spinor fields and demonstrate that they both nullify the principal spinor of any special SFC. This property also can be looked at as a generalization of similar property of SFC on the Minkowski space-time.

The notation used in the paper corresponds to the standard one [19].

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2In fact, only an example of a space with three independent SFC was presented in [15].
2 Special shear-free null congruences and related Weyl-Cartan connections

Let \( l^\mu \) be a null vector field tangent to the rays of a null congruence, and \( \xi_A \) be a related principal spinor field of the congruence. In abstract index notation \([19]\) the correspondence is settled by \( l^\mu = \xi^A \xi^A \). Condition for the congruence to be shear-free (and, therefore, geodetic) is then

\[
\xi^A \xi^B \nabla_{AA'} \xi_B = 0, \tag{1}
\]

where \( \nabla_{AA'} \) stands for the spinor derivative with respect to the Levi-Civita connection. Making use of the spinor algebra, it is easy to check that SFC defining condition (1) is equivalent to one of the following two forms:

\[
\nabla_{A'}(A \xi_B) = \phi_{A'}(A \xi_B) \tag{2}
\]
or

\[
\xi^A \nabla_{AA'} \xi_B = \eta_{A'} \xi_B \tag{3}
\]

where an auxiliary complex vector field \( \phi_{A'} \) and a 2-spinor field \( \eta_{A'} \) are introduced. It follows then that covariant derivative of the principal spinor can be always represented in the form

\[
\nabla_{AA'} \xi_B = \phi_{BA'} \xi_A + \varepsilon_{AB} \eta_{A'} \tag{4}
\]

Eq. (1) and therefore Eqs. (2), (3) are invariant under scalings of the principal spinor of the form

\[
\xi_A \mapsto \hat{\xi}_A = \alpha \xi_A \tag{5}
\]
\[
\phi_\mu \mapsto \hat{\phi}_\mu = \phi_\mu + \nabla_\mu \ln \alpha, \quad \eta_{A'} \mapsto \hat{\eta}_{A'} = \alpha (\eta_{A'} + \xi^A \nabla_{AA'} \ln \alpha) \tag{6}
\]

with \( \alpha(x) \) being an arbitrary (differentiable) complex function. In flat case it can be fixed so that the spinor \( \eta(x) \) identically vanishes and SFC condition (1) reduces to the form \([8]\)

\[
\xi^A \nabla_{AA'} \xi_B = 0 \quad \Leftrightarrow \quad \nabla_{AA'} \xi_B = \phi_{BA'} \xi_A \tag{7}
\]

Properties of the gauge field of the congruence (GFC) represented by 4-potentials \( \phi_{BA'} \) in Eq. (7) which in the flat case satisfies homogeneous Maxwell equations and for which the value of effective electric charge is necessarily self-quantized have been presented in the introduction. Further on we return to study its properties on a curved Riemannian background.

Consider now the reduced form of SFC condition (7). Differentiating it and commuting the spinor covariant derivatives in the l.h.s. we get

\[
\Psi_{ABC} \xi^B \xi^C \xi^D = 0, \tag{8}
\]
where $\Psi_{ABCD} = \Psi_{(ABCD)}$ is the Weyl spinor of conformal curvature. Thus, apart of the conformally flat case, reduction of the SFC equation to the form (7) can be made only for algebraically special spaces and only when the particular SFC defines a repeated principal null direction (RPND) of the Weyl conformal tensor. Recall by this that integrability condition for generic form of SFC equation (1) leads to a weaker condition $\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0$ which does not restrict the curvature implying only for the principal SFC spinor to represent one of the four null directions of the Weyl tensor (not necessarily repeated).

Conversely, let now an algebraically special space is equipped with a SFC defining a RPND of the Weyl tensor. Can its equation be reduced to the form (7), i.e. the spinor $\eta_{A'}$ be turned into zero? To ensure this, a spinor $\eta$ being given, the scaling parameter $\alpha(x)$ must be found which obeys the equation $\xi^A\nabla_{AA'}\ln \alpha = -\eta_{A'}$. Calculating the commutator of the derivatives and taking into account that the spinor $\eta_{A'}$ is related to $\xi_A$ by (3) we obtain the following necessary condition for the spinor $\tilde{\eta}_{A'}$ to vanish:

$$\xi^A\nabla_{AA'}\eta_{A'} = 0.$$  

(9)

For analytic congruences this is also a sufficient one (3). Using then Eq.(3) and taking the contraction $(\xi^A\nabla_{A}C')(\xi^C\nabla_{CC'})\xi_B$ in its l.h.s. we obtain that condition (2) is equivalent to the old one $\Psi_{ABCD}\xi^B\xi^C\xi^D = 0$ so that necessary condition (8) is also a sufficient one. Note that if the metric is conformally flat the IC are satisfied identically. So we have

**Lemma 1** The SFC equation (7) can be represented in the form (7) iff the SFC spinor is a RPND of the Weyl tensor or the space is conformally flat.

Further on SFC which admit a reduced representation (7) and define a RPND of the Weyl tensor will be referred to as special SFC (SSFC).

Let us notice now that another evident form of representation of SSFC equation (7) is the condition for its principal spinor $\xi_A$ to be covariantly constant (parallel),

$$\tilde{\nabla}_{AA'}\xi_B = 0,$$

(10)

with respect to an effective spinor connection represented by

$$\tilde{\nabla}_{AA'}\xi_B \equiv \nabla_{AA'}\xi_B - \Gamma_{AA'B}^C\xi_C, \quad \Gamma_{AA'B}^C = \phi_{BA'}\varepsilon_{A'B}^C$$

(11)

. Thus Eq.(11) is a spinor image of the condition for tangent vector of the SSFC $l_\mu$ to be parallel with respect to affine connection of the form

$$\tilde{\nabla}_\mu l_\alpha = \nabla_\mu l_\alpha - 2(\delta_\mu^\beta a_\alpha + \delta_\alpha^\beta a_\mu - g_\mu a_\alpha \beta - \varepsilon_\mu^\alpha \beta \gamma b^\gamma)l_\beta$$

(12)

containing the Weyl nonmetricity 4-vector $a_\mu(x)$ and the skew symmetric torsion pseudo-trace 4-vector $b_\mu(x)$ represented respectively by the real
and imaginary parts of the unique complex gauge field (GFC) \( \phi_\mu(x) = a_\mu(x) + ib_\mu(x) \). Effective Weyl-Cartan connections of the type \( (12) \) were introduced (for the case of the flat metric background \( g_{\mu\nu} = \eta_{\mu\nu} \)) in \([5,8]\), in the framework of noncommutative analysis over the algebra of biquaternions. It has been also applied to formulate a geometrical version of the Weinberg-Salam electroweak theory \([14]\). Finally, it was dealt with in \([20]\) in connection with the theory of Einstein-Weyl spaces and the "heterotic geometry" arising in the SUSY nonlinear \(\sigma\)-models.

We are now in a position to formulate the main result of the section.

**Theorem 1**  The following three statements for null SFC on an algebraically special Riemannian manifold are equivalent:

1. SFC is a special one (SSFC), i.e. admits a defining equation of the reduced form \([7]\).
2. SFC defines a repeated principal null direction of the Weyl tensor.
3. Tangent vector field of a SFC is parallel with respect to effective Weyl-Cartan connection of the form \((12)\).

To conclude, we mention that in the space of Petrov types D (or N) both SFC (or one single SFC) if exist are of a special form (i.e. SSFC).

### 3 Gauge (electromagnetic) fields associated with shear-free null congruences

We are going now to study the properties of the GFC for a SSFC on a curved Riemannian background. By this it is noteworthy that the SSFC defining system \([7]\) possesses a residual "weak" gauge invariance of the form

\[
\xi_A \mapsto \alpha \xi_A, \quad \phi_\mu \mapsto \phi_\mu + \partial_\mu \ln \alpha \quad (13)
\]

where the gauge parameter \(\alpha(x)\) can not be an arbitrary function of coordinates. Indeed, it is easy to check that the gauge parameter should satisfy the equation

\[
\xi^A \nabla_{AA'} \alpha = 0 \quad (14)
\]

For analytic congruences there always exists two functionally independent solutions of this equation \([14]\) so that \(\alpha\) is constant on the two complex 2-surfaces canonically defined by SFC. In flat space it means that \(\alpha\) is a function of the twistor \(T_a = (\xi_A, \xi_A X^{AA'})\) corresponding to a ray of SSFC.

Thus, symmetry \([18,19]\) is a generalization of the "weak" gauge symmetry for the SSFC defining equation in the flat case \([17,18]\). In view of this gauge invariance we indeed may treat the GFC, i.e. the field represented by the potentials \(\phi_{AA'}\), as an electromagnetic-like field associated with (generated by) a SSFC.
However, it is necessary then to derive the dynamical equations of the GFC. For this, consider now the complete set of integrability conditions for the SSFC equation \((7)\) of general type. Rearranging the covariant derivatives and splitting the arising curvature spinor into canonical irreducible parts \(\Psi_{(ABCD)}\), \(\Phi_{(A'B')(CD)}\) and \(\Lambda\) we obtain

\[
\Psi_{ABCD}\xi^D = \xi_{(A}\varphi_{BC)} \quad (15)
\]

\[
6\Lambda\xi_A = \varphi_{AB}\xi^B - \frac{3}{2}\xi_A\Pi \quad (16)
\]

\[
\Phi_{(A'B')(CD)}\xi^D = \pi_{(A'B')(CD)}\xi^D + \frac{1}{2}\tilde{\varphi}_{A'B'}\xi_C \quad (17)
\]

where the spinors \(\varphi_{AB} := \nabla_{A'}(A\phi_{A'B})\) and \(\tilde{\varphi}_{A'B'} := \nabla_{A}(A'\phi_{B'})\) are respectively the antiself- and selfdual parts of the complex strength tensor of the GFC

\[
\mathcal{F}_{\mu\nu} := \nabla_{[\mu}\phi_{\nu]} = \varphi_{AB}\varepsilon_{A'B'} + \tilde{\varphi}_{A'B'}\varepsilon_{AB} \quad (18)
\]

and where also \(\Pi := \nabla_{\mu}\phi^\mu + 2\Phi\), \(\pi_{(A'B')} := \nabla_{(A'}\phi_{B')} - \phi_{(A'}\phi_{B')}\) and \(\Phi := \phi_{\mu\phi}^\mu = \frac{1}{2}\phi_{AA'}\phi_{AA'}\).

In the conformally flat space or in a space of the Petrov type N the left side of \((15)\) vanishes, \(\varphi_{AB} = 0\), and the GFC \(\mathcal{F}_{\mu\nu}\) is necessarily selfdual. Taking then into account the existence of 4-potential \(\phi_{\mu}\) (the exactness of the 2-form \(\mathcal{F}\)), we obtain that the complex GFC \(\mathcal{F}_{\mu\nu}\) (as well as its real part) satisfies homogeneous Maxwell equations.

In the case of space of the Petrov type III with the SSFC defining the triple PND one has \(\Psi_{ABCD}\xi^D = 0\). Then we get from Eq. \((15)\) \(\varphi_{AB}\xi^B = 0\) so that \(\varphi_{AB} = \lambda\xi_A\xi_B\) with \(\lambda(x)\) being a complex function. The last expression shows that in this case the antiselfdual part of the GFC strength has the familiar form of Robinson’s null field \([16]\). However, under considered conditions the Robinson’s theorem does not hold since the gauge freedom has been already used to vanish the spinor \(\eta_{A'}\), i.e. to bring the congruence to the reduced form.

Generally, making use of Eq. \((7)\) it is easy to check that for any SSFC the null Robinson-like field \(\psi_{AB} = \xi_A\xi_B\) satisfies the following equation:

\[
\nabla^{AA'}\psi_{AB} = -\phi^{AA'}\psi_{AB}, \quad (19)
\]

This equation links together the Robinson-like field \(\psi_{AB}\) and the considered GFC represented by the 4-potentials \(\phi_{AA'}\) and, moreover, is gauge invariant in the “weak” sense, i.e. invariant under the transformations of the type \([15, 14]\).

3Contrary to the GFC, the field \(\varphi_{AB}\) does not remain unchanged in these transformations being instead rescaled.
Let us pass now to general case of a SSFC in an algebraically special space and express the 4-potentials \( \phi_{AA'} \) through the spinor field \( \xi_A \). Contracting Eq.(7) with an arbitrary independent spinor \( \tau^A \) and introducing the normalized spinor \( \iota^A := (\xi_B \tau^B)^{-1} \tau^A \) such that \( \xi_A \iota^A = 1 \) we obtain for the potentials

\[
\phi_{AA'} = \iota^C \nabla_{CA'} \xi_A. \tag{20}
\]

We recall here that in the flat space the GFC \( \phi_{AA'} \) not only satisfies the homogeneous Maxwell equations but also possesses the charge quantization property. Specifically, the effective electric charge for every bounded singularity of this field (calculated via making use of the Gauss theorem) is self-quantized, i.e. discrete and integer multiple to a minimal charge (equal in dimensionless units to \( \pm 1/4 \)) which can be choosed as an analogue of the elementary one \([10, 11]\). The property is therein a consequence of the over-determined structure of SFC equations or, equivalently, of the topological restrictions resembling those responsible for quantization of the Dirac magnetic monopole \([3]\).

Generally, in an algebraically special space the GFC \( \varphi_{AA'} \) is not necessarily selfdual and, generically, does not satisfies the homogeneous Maxwell equations. However, if the space is asymptotically flat, in view of Eq.(15) it satisfies the Maxwell equations with effective sources of geometrical origin defined by the (derivatives of) conformal curvature of the manifold. Correspondent corrections to electromagnetic field caused by the presence of these (extended) sources are proportional to \( 1/r^3 \) and considerable only at distances of the order of gravitational radius of field distribution. As to the effective electric charge which is calculated at the asymptotic, it remains self-quantized and, as before, multiple in value to the minimal elementary one.

As a simple example, let us consider one of the radial SSFC, say \( l = (dt - dr)/(1 - \cos \theta) \) of the Schwarzschild space-time. Calculating the (real part of) associated GFC we find that it is electric in nature, with a single nonzero radial component \( E_r = q(1/r^2 - 6Gm/(c^2r^3)) \) where the dimensionless charge \( q = 1/4 \) is equal to elementary \( q = e \) in physical units. The second term is proportional to \( 1/r^3 \), produced by the volume charge density \( \rho = 3eGm/(c^2r^4) = 3eR_{grav}/(2r^4) \) and comparable with the first Coulomb term at a distance \( r \approx R_{grav} \).

4 Invariant spinor differential operators

Let us derive now some additional fundamental constraints which hold for the principal spinor of a SSFC. In fact, for SFC in Minkowski space it was proved in \([12]\) (see also \([8]\)) that every component of SSFC spinor \( \xi_A \)
satisfies the eikonal equation flat space while their ratio satisfies the wave 
equation.

We are going now to generalize these equations to the case of a SSFC in 
on arbitrary algebraically special space. For this, taking into account the 
spinor (i.e., not scalar) nature of the field functions, we must firstly write 
correspondent equations in a manifestly invariant form. In result we come 
to the following statement.

**Theorem 2** For any SSFC the spinor $\xi_A$ satisfies the two sets of equations 
of the form

$$E_{(BC)}(\xi) \equiv \nabla_{AA'}\xi_B \nabla^{AA'}\xi_C = 0$$

$$D(\xi) \equiv \xi^C\nabla^{AA'}\nabla_{AA'}\xi_C = 0$$ (21, 22)

**Proof.** To check the first three equations it’s sufficient to use expressions 
for covariant derivatives of $\xi$ from the SSFC equation [17]. On the other 
hand, differentiating Eq. [17] and taking then the contraction with $\xi^B$ one 
gets $D(\xi) = \varphi_{AB}\xi^A\xi^B$, and the last term vanishes for every SSFC in con-
sequence of the integrability condition [16].

As the defining Eq. [17] of SSFC, so the Eqs. [21, 22] are invariant under 
“weak” rescalings [13, 14] of the spinor $\xi$. This means, in particular, that 
esential are only the restrictions they impose on the ratio of two com-
ponents of the spinor $\xi_A$, say, on the function $G = \xi_1/\xi_0$. In flat case 
from Eqs. [21, 22] one gets $D(\xi) = \eta^{\mu\nu}\partial_\mu G\partial_\nu G = 0$.

Thus, the above introduced invariant differential operators $E_{(BC)}(\xi)$ 
and $D(\xi)$ can, in fact, be regarded as spinor analogues of the two known 
Beltrami operators the latter acting on scalars. On the other hand, these 
operators both nullify the principal spinor of every SSFC.

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