LOCAL-GLOBAL INVARIANTS
OF FINITE AND INFINITE GROUPS:
AROUND BURNSIDE FROM ANOTHER SIDE

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ABSTRACT. This expository essay is focused on the Shafarevich–Tate set of
a group G. Since its introduction for a finite group by Burnside, it has been
rediscovered and redefined more than once. We discuss its various incarnations
and properties as well as relationships (some of them conjectural) with other
local-global invariants of groups.

PREFACE

These notes focus on a local-global invariant \( \text{III}(G) \) of a group \( G \), its various
incarnations, applications and possible generalizations. The cyrillic III refers to
one of the names of this invariant, the Shafarevich–Tate set. At a glance, this
text is a bulk of definitions, vague questions and conjectures, mostly compiled from
numerous sources. However, some striking parallels with other invariants of \( G \) give
a hope that something more sensible is extractable from this eclectic material.

For a finite group \( G \), the invariant we are talking about was introduced by Burn-
side (naturally, under a different name) as early as in 1911, in the second edition of
his famous book [Bur1]. Soon enough, in the paper [Bur2], which was published in
1913, he constructed the first example of \( G \) with nontrivial \( \text{III}(G) \), so the present
article can be viewed as a modest commemoration of the 100th anniversary of this
event. Since then, \( \text{III}(G) \) has been rediscovered more than once, each time revealing
some new features. The main goal of this paper is to try to put some order in
its numerous avatars and attract the attention of experts in both finite and infi-
te groups, as well as in geometric group theory (and maybe of those whose main
interests lie outside group theory), to intriguing interrelations and applications. To-
wards this end, an attempt was made to make the bibliography as comprehensive
as possible, which resulted in disproportionately unbalanced structure of the text,
with a modestly sized body followed by a long tail. However, one can hope that
the reader will find here something beyond a mere list of references.

As a precaution, it must be said that one should not be deceived by a misleading
hint in the title: the paper has nothing in common with the well-known monograph
by Kostrikin, where Burnside’s name is identified (by default, as a sort of common
practice in certain circles) with a notoriously hard group-theoretic problem (which,
by the way, also carries a distinctive local-global flavour). We are indeed looking
from another side, focusing on the development of a far less advertised contribution
of Burnside’s.

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1. Main object

**Definition 1.1.** [On1] Let $G$ be a group acting on itself by conjugation, $(g, x) \mapsto gxg^{-1}$, and let $H^1(G, G)$ denote the first cohomology pointed set. The set of cohomology classes becoming trivial after restriction to every cyclic subgroup of $G$ is denoted $\text{III}(G)$ and called the Shafarevich–Tate set of $G$.

**Definition 1.2.** For the lack of a better term, we say that a group $G$ with one-element Shafarevich–Tate set is $X$-rigid. This term will be explained later, after clarifying relationships with some rigidity phenomena.

**Observation 1.3.** $X$-rigidity is often a crucial step to establishing important properties of $G$, or of a whole class of groups. On the other hand, groups with nontrivial $\text{III}(G)$ often provide interesting examples (or even allow one to refute long-standing conjectures). Some instances will be given below.

**Remark 1.4.** The terminology of Definition 1.1 is originated in the prototype of $\text{III}(G)$, dating back to the 1950's when it appeared in the context of a high-brow approach to diophantine equations and has been remaining since then one of the favourite objects of arithmetic geometers: given an algebraic group $A$ defined over a number field $k$, $\text{III}(A)$ is defined as the set of cohomology classes $H^1(\Gamma, A(\bar{k}))$ (where the absolute Galois group $\Gamma = \text{Gal}(\bar{k}/k)$ acts naturally on $\bar{k}$-points of $A$) that become trivial after restriction to every $\Gamma_v = \text{Gal}(\bar{k}_v/k_v)$, where $v$ runs over all places of $k$. In the purely group-theoretic setting as above, a much more down-to-earth description is available.

The following important remark is due to M. Mazur (see [On3]).

**Observation 1.5.** A map $f : G \to G$ is a cocycle if and only if the map $g : G \to G$ defined by $g(x) = f(x)x$ is an endomorphism. Furthermore, $f$ is a coboundary if and only if $g$ is an inner automorphism, and the restriction of $f$ to the cyclic subgroup generated by $x \in G$ is a coboundary if and only if $g(x)$ is conjugate to $x$. Denote by $\text{End}_c(G)$ (resp. $\text{Aut}_c(G)$) the set of endomorphisms (resp. automorphisms) $g$ of $G$ such that $g(x)$ is conjugate to $x$ for all $x \in G$.

We see that $G$ is $X$-rigid if and only if it satisfies the following condition:

\[(1.1) \quad \text{End}_c(G) = \text{Inn}(G).\]

**Remark 1.6.** Endomorphisms (or automorphisms) $g$ with the property that $g(x)$ is conjugate to $x$ for all $x \in G$ appear in literature under different names: pointwise inner, conjugating, class-preserving, etc. In this text they will be called **locally inner**. Note that any locally inner endomorphism is injective.

Condition (1.1) is sometimes called Property E (see, e.g., [AKT3]).

Since $\text{Inn}(G) \subseteq \text{Aut}_c(G) \subseteq \text{End}_c(G)$, it is convenient to subdivide the property of being $X$-rigid into two weaker ones: 1) $\text{Inn}(G) = \text{Aut}_c(G)$; 2) $\text{Aut}_c(G) = \text{End}_c(G)$. The first property is sometimes referred to as property A (see, e.g., [Gros]). It can be written down as $\text{Out}_c(G) = 1$ meaning that $G$ has no locally inner outer automorphisms. In this text, a group $G$ satisfying 1) will be called A-rigid, and a group satisfying 2) will be called B-rigid. In these terms, any $X$-rigid group is both A-rigid and B-rigid, and **vice versa**.
2. Zoo of rigid groups

2.1. B-rigid groups.

**Observation 2.1.** The following groups are B-rigid:

(i) finite;
(ii) profinite;
(iii) solvable;
(iv) cohopfian.

(i) is obvious because every locally inner endomorphism is injective and hence, as $G$ is finite, surjective. (ii) is proved in [On3]. (iii) is proved in [AE]. (iv) is obvious (recall that the property of $G$ to be cohopfian means that $G$ contains no proper subgroups isomorphic to $G$, or, equivalently, that every injective endomorphism of $G$ is surjective). Note that this property is related to other rigidity properties: for example, it is satisfied by rigid hyperbolic groups [RS] (see [Sel, Theorem 4.4] for a generalization), and irreducible lattices in semisimple Lie groups, except for free groups (because of their Mostow rigidity) [Pr]. (Note that free groups are also B-rigid and, moreover, III-rigid, see Observation 2.4(ii) below.) The cohopfian property also holds for some Kleinian [DP], [OP], [WZ], 3-manifold [PV], [WaWu], and braid groups [BM], [Belli], as well as for some torsion-free nilpotent groups [Beleg].

2.2. A-rigid groups. Our first observation is obvious.

**Observation 2.2.** All complete groups are A-rigid.

Recall that a group $G$ is complete if it is centreless and all its automorphisms are inner. See [Rob] for a survey of finite complete groups. As to infinite groups, typical examples of complete groups arise as groups of automorphisms $G = \text{Aut}(F)$, where $F$ is a free group (or a group that is “not so far” from free). The cases where $F$ is a free group, free nilpotent group of class two, or the quotient of a free group by an appropriate characteristic subgroup were treated, respectively, in [DF1]–[DF3] (for groups of finite rank) and [To1]–[To3] (for groups of infinite rank). The cases $F = \text{GL}(n, \mathbb{Z})$ ($n$ odd), $F = \text{PGL}(2, \mathbb{Z})$ ($n \geq 2$), $F = \text{SL}(n, \mathbb{Z})$ ($n \geq 3$) were considered in [Dy]. The cases $F = B_n$ (Artin braid group), $n \geq 4$, and $F = \text{Aut}(B_3)$ were established in [DG]. All infinite symmetric groups are complete [DM].

**Observation 2.3.** The following groups are A-rigid:

I) Finite groups:

(i) symmetric groups [OW2];
(ii) simple groups [FS];
(iii) $p$-groups of order at most $p^4$ [KV1];
(iv) $p$-groups having a maximal cyclic subgroup [KV2];
(v) extraspecial [KV2] and almost extraspecial $p$-groups;
(vi) $p$-groups having a cyclic subgroup of index $p^2$ [KV3]; [FN];
(vii) groups such that the Sylow $p$-subgroups are cyclic for odd $p$, and either cyclic, or dihedral, or generalized quaternion for $p = 2$ [He1] (see [Su], [WalC] for a classification of such groups);
(viii) Blackburn groups [He4], [HL];
(ix) abelian-by-cyclic groups [HJ];
(x) primitive supersolvable groups [La];
(xi) unitriangular matrix groups over $\mathbb{F}_p$ and the quotients of their lower central series [BVY];
(xii) central products of A-rigid groups [KV2].

The only new case here is that of almost extraspecial groups (see, e.g., [CT] for the definition and classification). In particular, every such group is a central product of an extraspecial group and a cyclic group, so the result follows from (xii). See [Ya4] for a survey and some details.

II) Infinite groups:

(i) the absolute Galois group of $\mathbb{Q}$ [Ik];
(ii) the absolute Galois group of $\mathbb{Q}_p$ [Ik] (or, more generally, of any its finite extension [Ikeda (unpublished), JR]);
(iii) non-abelian free groups [Gros, Lemma 1]; goes back to [Ni];
(iv) non-abelian free profinite groups [Ja];
(v) so-called pseudo-$p$-free profinite groups [JR];
(vi) free nilpotent groups [En];
(vii) non-abelian free solvable groups [Rom];
(viii) nontrivial free products [Nes1];
(ix) one-relator groups of the form $\langle a, b | [a^m, b^n] = 1, m, n > 1 \rangle$ [TM];
(x) non-abelian free Burnside groups of large odd exponent [Ch], [At];
(xi) fundamental groups of compact orientable surfaces [Gros];
(xii) Artin braid groups $B_n$ and pure braid groups $P_n$ [DG], [Nes2];
(xiii) connected compact topological groups [McM];
(xiv) fundamental groups of closed surfaces with negative Euler characteristic [BKZ];
(xv) non-elementary subgroups $H$ of hyperbolic groups $G$ such that $H$ does not normalize any nontrivial finite subgroup of $G$ [MO, Corollary 5.4];
(xvi) some groups of automorphisms and birational automorphisms of the plane and space [De1]–[De4], [KS];
(xvii) unitriangular matrix groups over $\mathbb{Q}$ and the quotients of their lower central series, as well unitriangular matrix groups over $\mathbb{Z}$ [BVY];
(xviii) all finitely generated Coxeter groups [CMi].

2.3. III-rigid groups.

**Observation 2.4.** The following groups are III-rigid:

(i) groups appearing on the lists of both Observations 2.1 and 2.3;
(ii) free groups ([OW1] for finitely generated free groups and [AKT1] in general);
(iii) groups $SL(n, R)$, $PSL(n, R)$ and $GL(n, R)$ where $R$ is a euclidean domain [On4], [Wad1], [Wad2];
(iv) free products of at least two nontrivial groups [AKT3];
(v) amalgamated products $A \ast_H B$ where $H$ is a maximal cyclic subgroup of $A$ and $B$ [AKT3];
(vi) all Fuchsian groups $G(n, r, s)$ except, possibly, triangle groups $G(0, 0, 3)$ [AKT3];
(vii) almost all orientable Seifert groups, except possibly $G_1(0, 3)$ and $G_1(1, 1)$ [AKT4];
(viii) some “polygonal” [Ki] and “tree” products [KT];
The lists presented above look scattered, so some remarks and questions are in order.

**Remark 2.5.** In cases (i)–(v), (viii)–(x), (xii), (xiv), (xv) of Observation 2.3.II, the groups under consideration satisfy a stronger rigidity property: every normal automorphism (i.e., an automorphism preserving normal subgroups) is inner (in case (iii) this is proved in [Lub], [Lue], in case (xii) in [Nes2], and in case (xv) in [MO]; results of the latter paper were generalized in [MarMin]). Note that item (x) provides a “counter-example” to the statement made in the last paragraph of the preface. Groups in (i) are even more rigid: according to [Neu], every automorphism of \( \text{Gal}(\mathbb{Q}) \) is normal, and therefore it is inner.

**Remark 2.6.** The connectedness assumption in (xiii) is essential in light of the existence of finite groups that are not A-rigid.

**Remark 2.7.** For topological groups \( G \), another rigidity property is sometimes used. In [CDG] it is proved that if \( G \) is a connected linear real reductive Lie group, then every automorphism of \( G \) preserving unitary equivalence classes of unitary representations is inner. For compact groups this property is equivalent to be locally inner so one recovers A-rigidity, as in (xiii). Does A-rigidity hold in the noncompact case?

**Remark 2.8.** A-rigidity of a finitely generated group \( G \), together with its so-called conjugacy separability, implies that the group \( \text{Out}(G) \) is residually finite [Gros]. This observation provides an important source of residually finite groups, see [CMi] and references therein for details.

**Remark 2.9.** Regarding case (xvi) of Observation 2.3.II, let \( G \) be one of the following groups: \( \text{Aut}(\mathbb{A}^2_\mathbb{C}) \), the affine Cremona group of the polynomial automorphisms of the affine complex plane; \( \text{TAut}(\mathbb{A}^2_\mathbb{C}) \), the subgroup of the affine Cremona group consisting of the tame polynomial automorphisms (for \( n = 2 \) it coincides with the previous one); \( \text{Bir}(\mathbb{F}_2^n) \), the Cremona group of birational automorphisms of the complex projective plane. Then every automorphism of \( G \) is inner, up to composition with an automorphism of \( \mathbb{C} \) (see [De1], [KS], [De2], respectively). Thus the corresponding groups are all A-rigid. Moreover, in [De3] it was shown that every endomorphism of \( \text{Bir}(\mathbb{F}_2^n) \) is a composition of an inner automorphism with an endomorphism of \( \mathbb{C} \), and hence \( \text{Bir}(\mathbb{F}_2^n) \) is III-rigid (a nontrivial endomorphism of \( \mathbb{C} \) cannot be locally inner). See [De4] for a survey of other rigidity properties of the Cremona group.

**Remark 2.10.** Case (x) of Observation 2.4 is a consequence of the cohopfian property for freely indecomposable torsion-free hyperbolic groups (proved in [Sel]) and of [MO, Corollary 5.4]. If the hyperbolic group is freely decomposable, then one can use Observation 2.4(iv). The statement also follows from more general results of [BV]. (I thank the referee for this remark, as well as for pointing out case (xi) of Observation 2.4.)
Remark 2.11. Unitriangular matrix groups over prime fields and the quotients of their lower central series are all III-rigid, in view of Observations 2.1(iii), 2.3.I(xi) and 2.3.II(xvii). By [BVY], over a field that is not prime, none of these groups is A-rigid.

Question 2.12. Let $G$ be a split simple Chevalley group over a prime field $k$, $B$ be a Borel subgroup of $G$, $U$ be the unipotent radical of $B$. Let $G = U(k)$ be the group of $k$-rational points of $U$. Is it III-rigid? A similar question can be asked in the case where $k$ is replaced with $\mathbb{Z}$.

Remark 2.13. The case of symmetric groups $\text{Sym}(n)$ in Observation 2.3.I(i) is obvious because they all (except for $n = 2$ and $n = 6$) fall into the class of complete groups in view of a classical theorem of Hölder [Ho], and the exceptional cases are easily treated separately. We put this case as a separate item because of the following natural questions:

Question 2.14. (i) Are all Weyl groups III-rigid? (ii) What about other Coxeter groups? (iii) What about other reflection groups?

A partial answer to (ii) is provided by the paper [CMi], generalizing some earlier results [FH], [HRT], see Observation 2.3.II(xviii). It is applicable to finite Coxeter groups and thus answers (i) in the affirmative. (It would be instructive to get a more elementary independent proof of (i), using results of [Ba].) As to (iii), the case of finite complex reflection groups, where there is a classification due to Shephard and Todd (see, e.g., [Co]) and a nice description of automorphisms [MarMic], also looks tractable.

Remark 2.15. In view of Observations 2.3.I(ii), 2.3.II(xiii) and 2.4(iii), it is natural to ask what can happen if $G$ is the group of points of an arbitrary simple linear algebraic group (or some generalization of such). Note that the proofs of the results recorded above do not admit straightforward generalizations (some of them are purely computational, as in Observation 2.4(iii), and others use methods of functional analysis, as in Observation 2.3.II(xiii), in order to deduce the needed property from a stronger one; the proof of Observation 2.3.I(ii), relying on the classification of automorphisms, can apparently be generalized).

Here are several cases where one can expect III-rigidity and that seem tractable.

Conjecture 2.16. The following groups $G$ are III-rigid:

(i) $G = G(k)$, the group of $k$-points of a split simple Chevalley group $G$ defined over a sufficiently large field $k$;

(ii) the same as in (i), with $k$ replaced with some “good” ring;

(iii) the same as in (i), with any isotropic $k$-group $G$;

(iv) the same as in (i), with $G$ an anisotropic group splitting over a quadratic extension of $k$;

(v) $G \subset G(k)$ is a “big” subgroup possessing some rigidity properties in the sense of Mostow, Margulis, and others;

(vi) $G$ is a split Kac–Moody group over a sufficiently large field $k$.

Here are some comments. Let us start with (i). In this case, here is a sketch of a possible proof of A-rigidity: according to Steinberg, every automorphism is a composition of inner, graph, field and diagonal automorphisms (see [St], [Hum]);
consider graph, field and diagonal automorphisms separately; the first two types should move some semisimple conjugacy class, and the third moves some unipotent class. In case (ii), we have a Steinberg-like classification of automorphisms [Ab], [Bun], [Kli], and can proceed as in case (i). In cases (iii) and (iv), one can use [BT] and [We1], respectively. Groups appearing in (v) were discussed in [We2]. For groups in (vi), a classification of automorphisms is available [Ca], see also [CMu1], [CMu2]; the case of finite ground fields should be similar to [FS]. Perhaps one can also treat unitary forms of Kac–Moody groups over \( C \). As to B-rigidity, one can also use the approach of [BT] (apparently, to treat (iv)–(vi), it is to be generalized in an appropriate way).

Remark 2.17. Trying to extend A-, B-, or III-rigidity to other simple (or almost simple) groups, one should not be overoptimistic. The finitary symmetric group \( \text{FSym}(\Omega) \), where \( \Omega \) is an infinite countable set, is not B-rigid [AE]. Similar arguments can be used to show that the finitary alternating group \( \text{FAlt}(\Omega) \) (which is simple) is not B-rigid either. The group of automorphisms of each of these two groups is isomorphic to the infinite symmetric group \( \text{Sym}(\Omega) \) (see, e.g., [DM]), it contains locally inner outer automorphisms (conjugation by any element of \( \text{Sym}(\Omega) \) applied to a finitary permutation gives the same result as conjugation by some finitary permutation), hence none of these groups is A-rigid.

2.4. Genericity of rigid groups. After walking around the zoo of rigid groups in the previous sections, we would like to address the following question: are the species we described there rare or common? How probable is to meet any of them in wildlife? We will formulate an answer as a vague principle.

Principle 2.18. A reasonable property of a reasonable mathematical object lying inside a reasonable class of objects may not hold but it will hold at least for an object in general position (if not always), provided the class under consideration is enlarged or restricted, if necessary, in an appropriate way.

In loose terms, this means that the probability of the event that a randomly picked animal is a penguin, will be much higher if the samples allowed to test are restrictively placed within the Antarctic region.

Anyone can find lots of examples confirming this principle, looking at his/her favourite area of mathematics. Here is an example from number theory. Consider the following property of finite Galois extensions \( L/K \) of number fields: every element of \( K \) which is a norm everywhere locally is a norm globally (the Hasse principle). This property fails to hold for most Galois extensions. However, if we restrict our attention to cyclic extensions, it always holds (Hasse). In another direction, if we allow \( L/K \) to vary among all finite extensions (not necessarily normal), the Hasse principle holds in general position, i.e., for an extension \( L/K \) of degree \( n \) such that the Galois group \( \text{Gal}(M/K) \) of the normal closure \( M \) of \( L \) is the symmetric group \( \text{Sym}(n) \) [VK]. More generally, the Hasse principle for principal homogeneous spaces of algebraic tori holds for generic maximal tori in simple algebraic groups [VK], [Kl].

Some instances of Principle 2.18 in the context of III-rigidity will be given below. Given a class of groups \( G \), it is usually a challenging conceptual task to enlarge or restrict it in the spirit of this principle. Needless to add that to get a meaningful mathematical statement, one has to convert expressions such as “general position”, “random”, “generic”, “typical”, and similar euphemisms, into a precise definition.
Such a goal is far beyond the framework of the present article, and the interested reader is referred to relevant literature (see, e.g., [Grom1], [Grom2], [Ols], [Oll], [KS1], [KS2] and references therein) for different approaches to genericity.

Let us now go over to more concrete considerations.

**Examples 2.19.**

(i) Let $\mathcal{FP}$ denote the class of finite primitive permutation groups. Then the class of III-rigid groups is generic within $\mathcal{FP}$. Indeed, according to [LP], a random element of the symmetric group $\text{Sym}(n)$ is contained in a primitive group other than $\text{Sym}(n)$ or $\text{Alt}(n)$ with probability tending to zero as $n \to \infty$, and it remains to refer to Observations 2.3.1(i) and (ii). According to the same paper [LP], this statement remains true if one considers the class of finite transitive permutation groups.

(ii) Let $\mathcal{L}$ denote the class of linear groups over fields. Then the class of III-rigid groups is generic within $\mathcal{L}$. This follows from Observation 2.4(ii) because a random linear group is free [Ao].

(iii) In a similar spirit, in [KS2] it was shown that “random” quotients of the modular group $\text{PSL}(2, \mathbb{Z})$ are complete and cohopfian, hence III-rigid.

(iv) In view of overwhelming genericity of torsion-free hyperbolic groups (see, e.g., [Ols], [Oll, Chapter I]), Observation 2.4(i) shows that the class of III-rigid groups is generic within the class of all groups.

We finish this series of examples by a quote from [KS2]:

‘It seems likely that “endomorphism rigidity” is another general aspect of “randomness”: A random structure should not have any endomorphisms except those absolutely required by the nature of the structure.’

This quote can be rephrased in the context of rigidity properties considered above. Namely, the classes of A-, B-, and even III-rigid groups are much broader than most classically known classes of rigid groups: say, there are A-rigid groups that are not complete (and may be very far from complete, for example, free groups) and B-rigid groups that are not cohopfian. However, Examples 2.19 show that III-rigidity is another general aspect of “randomness”: A random group should not have any endomorphisms that behave locally as conjugations except those that behave in such a way globally. In other words, the group-theoretic “Hasse principle”, introduced in Definitions 1.1 and 1.2, should hold generically.

3. Properties of III($G$)

**Case 1. Finite groups**

Recall that in this case III($G$) is a group. It coincides with Out$_e(G)$, the group of locally inner outer automorphisms.

**Observation 3.1.**

(i) The group III($G$) is solvable [Sah].

(ii) The group III($G$) may be non-abelian [Sah].

(iii) If $G$ is solvable and each chief factor is complemented, then III($G$) is supersolvable [La].

(iv) If $G$ is supersolvable and its Frattini subgroup is trivial, then III($G$) is nilpotent [La].

(v) If $G$ is nilpotent of class $c$, then III($G$) is nilpotent of class at most $c - 1$ [Sah].

(vi) III($G$) does not depend of the isoclinism class of $G$ [Ya3].
Remark 3.2. Observation 3.1(ii) disproves an assertion made in [Bur1]. The smallest known counter-example is of order $2^{15}$ (in general, the construction gives, for each prime-power $q$, a group of order $q^{5m}$, $m \geq 3$).

Remark 3.3. (i) First examples of groups that are not III-rigid appeared among $p$-groups, they go back to Burnside [Bur2]. The smallest group that is not III-rigid is of order 32 (see [WalG]); a classification of groups of order $p^5$ with nontrivial III($G$) was obtained in [Ya3] (and made more precise in [Ya4]). More examples of non-III-rigid $p$-groups were found in [Ya1], [Ya4] among so-called Camina groups. See [Ya2] for some bounds for the order of III($G$).

(ii) Looking beyond $p$-groups, a classification of “minimal” groups $G$ with III($G$) $\neq 1$ in the class of solvable groups all of whose Sylow subgroups are abelian was obtained in [He1].

(iii) See [BVY] for some matrix counter-examples. Some general constructions of a similar spirit can be found in [Sah], [Sz].

Case 2. Infinite groups
Here much less is known.

The group Out$_c$(G) is finite if $G$ is hyperbolic [MS1], or, more generally, relatively hyperbolic [MS2].

The following observation attributed to Passman (see the introduction to [Sah]) implies that Out$_c$(G) may be an infinite simple group: this happens for $G = F\text{Sym}(\Omega)$, in which case Out$_c$(G) is isomorphic to the quotient Sym(\Omega)/F\text{Sym}(\Omega). (Note that this is in sharp contrast with Observation 3.1(i).)

In general, from [Mi1] it follows that Out$_c$(G) can be any countable group. Hence the set $\text{III}(G)$ can be as large as possible. In the examples of [Mi1], $G$ is finitely generated (in fact, 2-generated), satisfies Kazhdan’s property (T) and has exactly two conjugacy classes (including 1).

A possible conceptually interesting question is to understand the situation within some natural classes of groups where such pathologies do not arise. Perhaps, the first class to be considered is that of compact groups.

4. Shafarevich–Tate set vs. Bogomolov multiplier
Throughout this section, unless otherwise stated, $G$ is a finite group and $k$ is an algebraically closed field of characteristic zero. Recall that the Bogomolov multiplier $B_0(G)$ is defined as the subgroup of the Schur multiplier $H^2(G, \mathbb{Q}/\mathbb{Z})$ consisting of the cocycles becoming trivial after restricting to all abelian (or, equivalently, bicyclic) subgroup of $G$ [Bo]. This group coincides with the so-called unramified Brauer group of the quotient $V/G$, where $V$ is a vector $k$-space equipped with a faithful, linear, generically free action of $G$. The latter group is an important birational invariant of $V/G$, in particular, it equals zero whenever the variety $V/G$ is $k$-rational (or even retract $k$-rational). It was introduced by Saltman and used in constructing a counter-example to Noether’s problem [Sal]. The Bogomolov multiplier allows one to compute this group solely in terms of $G$.

Observation 4.1. (i) In many cases, the Bogomolov multiplier of III-rigid groups is zero. This is true at least for the groups listed in items (i)–(ix) of Observation 2.3.I. Moreover, for some of these groups the corresponding varieties $V/G$ are rational (or at least, retract rational). Indeed, in case (i)
rationality is classically known (E. Noether), it follows from the theorem on elementary symmetric functions. This is also known in cases (iii) [CK], (iv) [HuKa], (vi) [Ka1]. In case (ix) one knows retract rationality [Ka2].

The Bogomolov multiplier is zero in case (ii) [Ku]. Cases (v), (vii), (viii) were treated in [KK].

(i) III(G) is invariant under isoclinism, and so is $B_0(G)$: if $G_1$ and $G_2$ are isoclinic (i.e., have isomorphic quotients $G_i/Z(G_i)$ and derived subgroups $[G_i, G_i]$, and these isomorphisms are compatible), then we have $\text{III}(G_1) \cong \text{III}(G_2)$ and $B_0(G_1) \cong B_0(G_2)$. For III(G) we recall Observation 3.1(vi), and for $B_0(G)$ this was proved in [Mo2]; moreover, in [BB] it was proved that if $G_1$ and $G_2$ are isoclinic, then the corresponding linear generically free quotients $V/G_1$ and $V/G_2$ are stably birationally equivalent, thus answering questions posed in [HKK] in the affirmative.

(ii) There are examples of groups $G$ that are not III-rigid but $B_0(G) = 0$. Such examples can be found among groups of order $p^5$: for $p = 2$ one always has $B_0(G) = 0$ [CHKP] but there is a group with III(G) $\neq 1$ [WalG]; for every $p \geq 3$ there is an isoclinism family (denoted $\Phi_p$) for which III(G) $\neq 1$ [Ya4] but $B_0(G) = 0$ [HKK], [Mo1]. Note, however, that although $B_0(G) = 0$ for all groups $G$ of order 32, such a group can give rise to a homogeneous space $X = SL_3/G$ defined over a finite field $k$ so that the unramified Brauer group of $X$ is not zero [BDH]. Thus one may pursue in giving III(G) some birational flavour.

Question 4.2. (i) Do the groups listed in Observations 2.3.I(x),(xi) satisfy $B_0(G) = 0$?

(ii) Let $G = G_1 * G_2$ be a central product of groups such that $B_0(G_1) = B_0(G_2) = 0$. Is it true that $B_0(G) = 0$?

(iii) Do there exist III-rigid groups with nonzero Bogomolov multiplier?

It is a tempting task to find a conceptual explanation of the experimental data presented above. One can try to use a hint given in [GK], where the group III(G) was naturally embedded into the so-called second lazy cohomology group $H^2_{\ell}(O_k(G))$ of the Hopf algebra of $k$-valued functions on $G$. This object can be viewed as a far-reaching noncommutative generalization of the Schur multiplier of $G$ (and coincides with it in the case where $G$ is abelian). The construction of the embedding is far from obvious: first, the lazy cohomology is identified with the group of equivalence classes of invariant Drinfeld twists on $k[G]$, and then this latter group is mapped to a pointed set $B(G)$ consisting of the pairs $(A, b)$, where $A$ runs over normal abelian subgroups of $G$ and $b$ is a $k^*$-valued $G$-invariant non-degenerate alternating bilinear form on the Pontryagin dual $\hat{A}$. The fibre at the pointed element is then identified with III(G).

Of course, this construction cannot directly be used to reveal an eventual relationship between III(G) and $B_0(G)$ (say, because the lazy cohomology is not
invariant under isoclinism). However, a more thorough exploration does not seem completely hopeless.

Moreover, one can go beyond finite groups and try to embed an appropriate part of $\text{III}(G)$ (say, the group $\text{Out}_c(G)$ of locally inner outer automorphisms) into an appropriate cohomology group. In the case where $G$ is compact, one can use an approach presented in [NT].

5. Miscellaneous applications, ramifications, and generalizations

As mentioned in the preface, nonrigid groups often lead to interesting counter-examples. We list several instances of such a phenomenon. Throughout, unless otherwise stated, $G$ is a finite group.

Observation 5.1. (i) Formanek showed [Fo] that if $\text{III}(G) \neq 1$ and $\varphi$ is an appropriately chosen locally inner outer automorphism of $G$, then the semidirect product $G_1 := G \rtimes \langle \varphi \rangle$ provides a counter-example to a conjecture by Roth [Rot], asserting that each irreducible representation of $G/Z(G)$ is a subrepresentation of the conjugation representation $G/Z(G)$ on $\mathbb{C}[G]$; see [KM] for minimal counter-examples in the class of $p$-groups. On the other hand, the Roth property turned out to be related to a recently developed “Lie theory” and “noncommutative differential geometry” on finite groups [LMR]. Does there exist any direct connection between $\text{III}(G)$ and these new theories?

(ii) The smallest non-$\text{III}$-rigid group of order 32 constructed in [WalG], as well as more complicated examples in [He1], played a crucial role in refuting some long-standing conjectures in the theory of integral group rings, including Higman’s isomorphism problem for integral group rings of finite groups (see [He2], [He4]). Some other interesting examples, based on [Sah], were obtained in [PW].

(iii) Examples of nilpotent Lie groups with $\text{III}(G) \neq 1$ were used in [GW] to construct compact Riemannian manifolds that are isospectral but not isometric.

(iv) Apart from $\text{III}(G)$, there are other local-global invariants of a similar flavour. One of such is related to the notion of Coleman automorphism [He3], [HeKi]: this is an automorphism of $G$ that becomes inner after restricting to each Sylow subgroup of $G$. An argument attributed to Ph. Gille (see [Pa]) shows that any group admitting a non-inner Coleman automorphism provides an example of a principal homogeneous space defined over a number field that has a rational zero-cycle of degree one (= has rational points in extensions of coprime degrees) but has no rational points. Similar arguments were used in [GGHZ] to construct non-isomorphic curves becoming isomorphic over extensions of coprime degrees. The existence of a principal homogeneous space of a connected linear algebraic group over an arbitrary field with the same property as above (with a rational zero-cycle of degree one and without rational points) is an open problem (known as Serre’s conjecture).

Remark 5.2. As shown in the papers [CMi], [BV] cited above, some classes of $\text{III}$-rigid groups $G$ are rigid in an even stronger sense: any endomorphism of $G$ which preserves the conjugacy classes of all elements of short length (in some suitable metric) must be inner. This observation gives rise to a number of testability
problems of the following flavour. We say that a subset $S$ of a III-rigid group $G$ is a test subset if the following holds: an endomorphism $\varphi$ of $G$ is inner if and only if $\varphi(s)$ is conjugate to $s$ for all $s \in S$. Then one can ask about the existence of “small” test subsets. Can one choose $S$ computable in some reasonable sense? finite? one-element? See [BV] for relevant discussions.

Finally, one can introduce a local-global invariant more general than $\text{III}(G)$.

**Definition 5.3.** Let $G$ be a group, let $\Gamma \leq G$ be a subgroup acting on $G$ by conjugation, and let $H^1(\Gamma, G)$ denote the first cohomology pointed set. Let $\mathcal{F}$ be a family of subgroups of $\Gamma$. We define

$$\text{III}_\mathcal{F}(\Gamma, G) := \ker[H^1(\Gamma, G) \rightarrow \prod_{\Gamma' \in \mathcal{F}} H^1(\Gamma', G)]$$

and call it the Shafarevich–Tate set with respect to $\mathcal{F}$.

In particular, if $\Gamma = G$, we abbreviate $\text{III}_\mathcal{F}(G, G)$ to $\text{III}(G)$.

With this terminology, if $G$ is finite then the solvable group $\text{III}(G)$ has several natural subgroups: $\text{III}_\mathcal{A}(G)$, $\text{III}_\mathcal{N}(G)$, $\text{III}_\mathcal{S}(G)$, where $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{S}$ stand for the family of all abelian, nilpotent and Sylow subgroups of $G$ respectively. The first one is nilpotent (of class at most 2) [Da3], the second one is abelian [Da3], and the third one is abelian too [HeKi] (earlier Dade [Da1] proved that it is solvable [Da1] and then that it is nilpotent [Da4]); in all known examples the first group is abelian too.

Using a set-up similar to Definition 5.3, D. Segal treated an arithmetic local-global problem of equivalence of binary forms [Seg], and T. Ono studied twists of hyperelliptic curves [On5], [On6].

The reader is welcome to provide more applications and interrelations.

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