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Volume 358, issue 11-12 (2020), p. 1207-1211.

<https://doi.org/10.5802/crmath.142>

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Harmonic Analysis / Analyse harmonique

Fourier Quasicrystals with Unit Masses

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Abstract. The sum of $\delta$-measures sitting at the points of a discrete set $\Lambda \subset \mathbb{R}$ forms a Fourier quasicrystal if and only if $\Lambda$ is the zero set of an exponential polynomial with imaginary frequencies.

Manuscript received 19th October 2020, accepted 30th October 2020.

1. Introduction

By a Fourier quasicrystal one usually means a complex measure with discrete support and spectrum. This concept goes back to works of Yves Meyer in the 1970-ies and it reappeared later in connection with an unexpected phenomenon in crystallography discovered by Dan Shechtman in the 1980-ies, see [5].

More precisely, following [6] we call a measure $\mu$ on $\mathbb{R}$ a crystalline measure, if it is an atomic measure which is a tempered distribution, its distributional Fourier transform $\hat{\mu}$ is an atomic measure and both the support $\Lambda$ and the spectrum $S$ of $\mu$ are locally finite sets. If in addition the measures $|\mu|$ and $|\hat{\mu}|$ are also tempered, then $\mu$ is called a Fourier quasicrystal (FQ).

The classical example of an FQ is the Dirac comb (the crystal)

$$\mu = \sum_{k \in \mathbb{Z}} \delta_k,$$

where $\delta_x$ is the unit mass at point $x$. Then the Poisson summation formula reads $\hat{\mu} = \mu$.

Examples of aperiodic quasicrystals were presented in [3] and then in [1, 6, 7]. Recently a new progress was achieved by P. Kurasov and P. Sarnak [2] who discovered examples of FQs with unit masses

$$\mu = \sum_{\lambda \in \Lambda} \delta_{\lambda},$$

(1)

where $\Lambda \subset \mathbb{R}$ is a uniformly discrete aperiodic set. An alternative construction of such measures was suggested by Y. Meyer [8].
Below we present one more construction and prove that it characterizes all FQs of form (1). A preliminary publication of our results was given in arXiv [9, 10].

The Theorem 1 below reveals a fundamental connection between FQs with unit masses and the zero sets \( Z(p) \) of exponential polynomials \( p \) with imaginary frequencies.

**Theorem 1.**

(i) Let \( p \) be an exponential polynomial

\[
 p(t) = \sum_{1 \leq j \leq N} c_j e^{2\pi i \gamma_j t}, \quad N \in \mathbb{N}, c_j \in \mathbb{C}, \gamma_j \in \mathbb{R},
\]

which has only simple real zeros. Then the measure \( \mu \) defined in (1) with \( \Lambda = Z(p) \) is an FQ.

(ii) Conversely, let \( \mu \) be an FQ of form (1). Then there is an exponential polynomial \( p \) of form (2) with real simple zeros such that \( \Lambda = Z(p) \).

We will sketch the proof of part (ii), see [10] for the proof of part (i). Using Theorem 1 (i) one may construct simple examples of aperiodic FQs.

**Lemma 2.** Fix a real number \( \epsilon \) satisfying \( 0 < |\epsilon| \leq 1/2 \) and set

\[
 p_\epsilon(t) := \sin(\pi t) + \epsilon \sin t.
\]

Then \( p_\epsilon \) has only simple real zeros and

\[
 Z(p_\epsilon) = \{ k + \epsilon_k : k \in \mathbb{Z} \}, \quad \epsilon_k \in [-1/6, 1/6].
\]

For a proof see [10].

Theorem 1 and Lemma 2 show that the sum of \( \delta \)-measures sitting at the points of \( Z(p_\epsilon) \) is an FQ.

Let \( p_\epsilon \) be given in (3). One may check that the numbers \( \epsilon_k \) in Lemma 2 satisfy \( \max_k |\epsilon_k| \to 0 \) as \( \epsilon \to 0 \). Therefore, the set \( Z(p_\epsilon) \) "approaches" the set of integers \( \mathbb{Z} \):

**Corollary 3.** For every \( \epsilon > 0 \) there is an aperiodic set

\[
 \Lambda = \{ k + \epsilon_k : k \in \mathbb{Z} \}, \quad 0 < |\epsilon_k| < \epsilon, k \in \mathbb{Z},
\]

such that the corresponding measure in (1) is an FQ.

2. **Proof of Part (ii) of Theorem 1**

In what follows we consider the standard form of the Fourier transform

\[
 \hat{h}(u) := \int_{\mathbb{R}} e^{-2\pi i u t} h(t) \, dt, \quad h \in L^1(\mathbb{R}).
\]

Let us start with a result which may have intrinsic interest:

**Proposition 4.** Let \( \mu \) be a positive measure which is a tempered distribution, such that its distributional Fourier transform \( \hat{\mu} \) is a measure satisfying

\[
 |\hat{\mu}|(-R,R) = O(R^m), \quad R \to \infty, \quad \text{for some } m > 0,
\]

which means that \( |\hat{\mu}| \) is a tempered distribution. Then there exists \( C \) such that

\[
 \mu(a,b) \leq C(1 + b - a), \quad -\infty < a < b < \infty.
\]


Proof. It suffices to prove (5) for every interval \((a, b)\) satisfying \(b - a \geq 2\).

Fix any non-negative Schwartz function \(g(x)\) supported by \([-1/2, 1/2]\) and such that

\[
\int_{\mathbb{R}} g(x) \, dx = 1.
\]

Set

\[
f(x) := (g * \chi_{[-1/2, b+1/2]}) (x) \in S(\mathbb{R}).
\]

Clearly,

\[
|\hat{f}(t)| = |\hat{g}(t)\chi_{[-1/2, b+1/2]}(t)| \leq (1 + b - a) |\hat{g}(t)|.
\]

Using this inequality and (4), we get

\[
\int_{\mathbb{R}} f(x) \mu(dx) = \int_{\mathbb{R}} \hat{f}(t) \hat{\mu}(dt) \leq (1 + b - a) \int_{\mathbb{R}} |\hat{g}(t)| \, |\hat{\mu}|(dt) = C(1 + b - a).
\]

On the other hand, clearly,

\[
f(x) = g(x) * \chi_{[-1/2, b+1/2]}(x) = 1, \quad x \in (a, b).
\]

Hence,

\[
\int_{\mathbb{R}} f(t) \mu(dt) \geq \mu(a, b),
\]

which proves the Proposition 4.

Recall that a set \(\Lambda \subset \mathbb{R}\) is called uniformly discrete, if

\[
\inf_{\lambda', \lambda \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.
\]

A set \(\Lambda\) is called relatively uniformly discrete if it is a union of finite number of uniformly discrete sets.

Proposition 4 implies

Corollary 5. Let \(\mu\) be a measure of form (1) whose distributional Fourier transform is a measure satisfying (4). Then its support \(\Lambda\) is a relatively uniformly discrete set.

Assume \(\mu\) is an FQ of form (1). This means that a Poisson-type formula

\[
\sum_{\lambda \in \Lambda} f(\lambda) = \sum_{s \in S} a_s \hat{f}(s), \quad f \in S(\mathbb{R}),
\]

is true where \(S(\mathbb{R})\) denotes the Schwartz space, \(S\) is locally finite set and the coefficients \(a_s\) satisfy

\[
\sum_{s \in S, |s| < R} |a_s| \leq CR^m, \quad R > 1, \quad \text{for some } C, m > 0.
\]

To prove part (ii) of Theorem 1 we have to show that \(\Lambda = Z(p)\) for some exponential polynomial \(p\) of form (2). We will prove this under the additional restrictions that \(\Lambda\) is a symmetric set, \(-\Lambda = \Lambda\) and \(0 \notin \Lambda\). For the general case see [9].

Set

\[
\psi(z) := \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) = \prod_{\lambda \in \Lambda, \lambda > 0} \left(1 - \frac{z^2}{\lambda^2}\right), \quad z \in \mathbb{C}.
\]

The product converges (uniformly on compacts) due to Corollary 5.

Lemma 6. \(\psi\) is an entire function of order one and finite type, i.e. there exist \(C, \sigma > 0\) such that

\[
|\psi(z)| \leq Ce^{\sigma|z|}, \quad z \in \mathbb{C}.
\]

This lemma follows from Corollary 5 and the symmetry of \(\Lambda\) by standard estimates.

Lemma 7. The following representation is true:

\[
\frac{\psi'(z)}{\psi(z)} = -2\pi i \left(a_0/2 + \sum_{s \in S \cap (-\infty, 0)} a_s e^{-2\pi i s} z\right), \quad \text{Im } z > 0,
\]

where \(a_s\) are the coefficients in (6).
By (7), the series in (9) converges absolutely for every \( z, \Im z > 0 \).
Let us sketch a proof of Lemma 7. It follows from (8) that
\[
\frac{\psi'(z)}{\psi(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}, \quad z \in \mathbb{C}. \tag{10}
\]

The next step is to check that
\[
\sum_{\lambda \in \Lambda} \frac{1}{z - \lambda} = -2\pi i \left( a_0/2 + \sum_{s \in S \cap (-\infty, 0)} a_s e^{-2\pi i sz} \right), \quad \Im z > 0. \tag{11}
\]
This can be done as follows: For every fixed \( z, \Im z > 0 \), set
\[
e_z(u) = \begin{cases} 2\pi e^{-2\pi i zu} & u < 0 \\ 0 & u \geq 0 \end{cases}
\]
Then the inverse Fourier transform of \( e_z \) is the function \( i/(z - t) \). Fix any function \( h \in S(\mathbb{R}) \) such that \( h(0) = 1 \) and the Fourier transform \( \hat{H} := \hat{h} \) is even, non-negative and vanishes outside \((-1, 1)\).
Then use (6) with \( f(t) = h(e(t))/(z - t) \):
\[
\sum_{\lambda \in \Lambda} \frac{h(e(\lambda))}{z - \lambda} = -i \sum_{s \in S} a_s \left( e_z(u) * \frac{1}{e} H(u/e) \right)(s).
\]
Finally, to prove Lemma 7 one lets \( \varepsilon \to 0 \) and checks that the right and left hand-sides above converge to the corresponding sides of (11).

Now, it follows from (9) that there exists \( K \in \mathbb{C} \) such that
\[
\psi(z) = K \exp \left( -\pi i a_0 z + \sum_{s \in S \cap (-\infty, 0)} (a_s/s) e^{-2\pi i sz} \right), \quad \Im z > 0.
\]
Set
\[
p(z) := e^{\pi i a_0 z} \psi(z)/K = \exp \left( \sum_{s \in S \cap (-\infty, 0)} (a_s/s) e^{-2\pi i sz} \right), \quad \Im z > 0. \tag{12}
\]
Recall that \( S \) is a locally finite set. Therefore, by (7) the series above converges absolutely for every \( z, \Im z > 0 \).

Denote by \( S_k \) the sets
\[
S_1 := S \cap (-\infty, 0), \quad S_2 := S_1 + S_1, \quad S_3 := S_1 + S_1 + S_1, \ldots
\]
Denote by \( a_{s,k} \) the coefficients of the series
\[
\frac{1}{k!} \left( \sum_{s \in S \cap (-\infty, 0)} (a_s/s) e^{-2\pi i sz} \right)^k = \sum_{s \in S_k} a_{s,k} e^{-2\pi i sz}, \quad k \in \mathbb{N}, \Im z > 0.
\]
Then by (12) we get a representation
\[
p(z) = 1 + \sum_{k=1}^{\infty} \sum_{s \in S_k} a_{s,k} e^{-2\pi i sz},
\]
where the double series converges absolutely for every \( z, \Im z > 0 \). Set
\[
U := \{0\} \bigcup_{j=1}^{\infty} S_j \subset (-\infty, 0].
\]
One may check that \( U \) is a locally finite set and that \( p \) admits a representation
\[
p(z) = \sum_{u \in U} d_u e^{-2\pi i uz}, \quad \Im z > 0, \tag{13}
\]
where the series converges absolutely.

To prove part (ii) of Theorem 1 it remains to check that the series in the right hand-side of (13) contains only a finite number of terms. This can be done as follows: Since \( \psi \) is an entire
function of order one and finite type, the same is true for $p$. By (13), $p$ is bounded on every line $\text{Im} z = \text{const} > 0$. It follows that (see [4, Lecture 6, Theorem 2]) $p$ is an entire function of exponential type, i.e. it satisfies
$$ |p(x + iy)| \leq C e^{\sigma|y|}, \quad x, y \in \mathbb{R}, $$
with some $C, \sigma > 0$. Now, to check that in (13) we have $d_u = 0$ for every $u \in U$, $|u| > \sigma$, one simply integrates both sides against $e^{2\pi i uz}(\sin \epsilon z / \epsilon z)^2$, where $\epsilon > 0$ is so small that $|u| - \epsilon > \sigma$ and $U \cap (u - \epsilon, u + \epsilon) = \{u\}$.

We note that one can extend Theorem 1 to measures with integer masses,
$$ \mu = \sum_{\lambda \in \Lambda} c_{\lambda} \delta_{\lambda}, \quad c_{\lambda} \in \mathbb{N}, \lambda \in \Lambda. \quad (14) $$

Theorem 8.

(i) If a measure $\mu$ of form (14) is an FQ, then there is an exponential polynomial $p$ of form (2) with real zeros such that $\Lambda = Z(p)$ and $c(\lambda)$ is the multiplicity of zero $\lambda$.

(ii) Conversely, let $p$ be an exponential polynomial of form (2) with real zeros and let $c(\lambda)$ be the multiplicity of zero $\lambda$. Then the measure $\mu$ of form (14) where $\Lambda = Z(p)$ is an FQ.

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