Partial algebraization and a q-deformed harmonic oscillator

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Abstract

From the algebraic treatment of the quasi-solvable systems, and a q-deformation of the associated $su(2)$ algebra, we obtain exact solutions for the q-deformed Schrödinger equation with a 3-dimensional q-deformed harmonic oscillator potential.

1 Introduction

In spite of the fact that the Schrödinger equation was introduced many years ago, the development of techniques, and specially, algebraic techniques, elaborated with the purpose of solving this equation, is still a living subject. Between the most remarkable and general works done to this date we can mention the following ones: 1. - The SO(2,2) treatment for the hypergeometric Natanzon potentials [1], studied by Alhassid et al [2]. 2. - The SUSYQM method for the shape-invariant potentials by Gendeshtein, Cooper, Dutt [3]. 3. - The SO(2,1) treatment for the hypergeometric-confluent Natanzon potentials studied in [4]. 4. - The partial algebraization method, which allows to obtain part of the spectrum for the so-called quasi-solvable systems, as we can see in [5]. We should also mention the works done by Kostelecký, Nieto and Man’ko, about algebraic descriptions including spin [6].

The following question arises: what can be done in the context of deformations? In other words, Is it possible to extend some of the previous approaches to include quantum groups?

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At this point we mention the works done by: Dayi and Duru (Coulomb and Morse potentials) [7], Cooper (Morse) [8], Daskaloyannis et al (Morse) [9], De Freitas and Salamó (Pöschl-Teller I and II) [10]. The references [7], [8] and [9] are related to the algebraic treatments of the “classical” systems.

In particular, it would be interesting to obtain algebraic solutions for the q-Schrödinger equation. This is, a Schrödinger equation in which the usual commutation relation between the position and momentum operators, 
\[ [X, P] = i, \]
is substituted by
\[ XP − qPX = i, \]
which can be done by associating the momentum operator to the deformed derivative, instead to the usual one, as we will see later on.

In this paper we work out such an algebraic method for the particular case in which the potential is a 3-dimensional q-harmonic oscillator. By q-harmonic oscillator we mean a q-dependent potential that reduces to the usual harmonic oscillator in the proper limit of \( q \to 1 \). The method is strongly based on the ideas of partial algebraization. It relies on the use of a particular deformation of the \( su(2) \) algebra (a quadratic one), which we call \( D_q^n(su(2)) \). In the same spirit of the partial algebraization method, we express the Hamiltonian of the system in terms of the generators defining this \( D_q^n(su(2)) \) q-algebra. This allows us to obtain exact solutions for part of the spectrum as well as the corresponding eigenfunctions.

This paper is organized as follows: In section 2 we make a review of the partial algebraization method. This is required to understand what follows in the paper, though it is a somewhat long section, so in order to avoid confusion, the reader should keep in mind that our goal is the algebraization of the q-Schrödinger equation with a q-harmonic oscillator. This is the central point of section 3 as well as the main part of this paper. In the mentioned section 3 we define and construct the \( D_q^n(su(2)) \) algebra. After rotating the q-Schrödinger equation, in analogy with what is done in the partial algebraization method, we continue the analogy, relating this rotated equation with a quadratic plus a linear expression of the generators defining \( D_q^n(su(2)) \). In section 4 we use a particular representation of \( D_q^n(su(2)) \) to show how we can get the so desired exact solutions of our q-Schrödinger equation. Next, some particular examples are shown. Finally, in section 5 we discuss the obtained results.
2 Partial Algebraization

The idea of the method of partial algebraization [5] consists in to express the Schrödinger equation
\[
\left( -\frac{1}{2} \frac{d^2}{dx^2} + (V(x) - E) \right) \Psi(x) = 0
\] (1)
in terms of the generators of SU(2). This can be done by first doing a non-unitary transformation on (1) in the following way
\[
\Psi(x) = \bar{\Psi}(x) \exp(-a(x))
\] (2)
thus (1) is now given by
\[
H_G \bar{\Psi}(x) = E \bar{\Psi}(x)
\] (3)
with
\[
H_G = -\frac{1}{2} \frac{d^2}{dx^2} + A(x) \frac{d}{dx} + \Delta V
\] (4)
where
\[
\Delta V = V(x) + \frac{1}{2} \frac{dA(x)}{dx} - \frac{1}{2} A^2(x)
\] (5)
and \(A(x)\) is defined as
\[
A(x) = \frac{da(x)}{dx}
\] (6)
The realization for the SU(2) algebra is taken to be
\[
J_+ = 2ju - u^2 \frac{d}{du}, \quad J_0 = -j + \frac{d}{du}, \quad J_- = \frac{d}{du}
\] (7)
with the usual commutation relations: \([J_+, J_-] = 2J_0, [J_0, J_+] = J_+, [J_0, J_-] = -J_-\). The index \(j\) labels the representations of dimensions \(2j+1\). The carrier space is given by monomials of degree \(n\) on the variable \(u\), i.e., \(u^n\). By other hand, since the eigenvalues of \(J_0\) lies between \(-j\) and \(j\), then \(0 \leq n \leq 2j\), thus the representation space \(R^j\) is given by
\[
R^j = \{1, u, u^2, ... u^{2j}\}
\] (8)
The algebraization of the Schrödinger equation is made assuming that the Hamiltonian \(H_G\) given in (4), may be written as
\[ H_G = \sum_{a,b=0,\pm,a \geq b} C_{ab} J_a J_b + \sum_{a=b=0,\pm} C_a J_a \] (9)

where \( C_{ab} \) and \( C_a \) are constants. We obtain

\[ H_G = -\frac{1}{2} P_4(u) \frac{d^2}{du^2} + P_3(u) \frac{d}{du} + P_2(u) \] (10)

with \( P_4(u), P_3(u) \) and \( P_2(u) \) given by

\[
\begin{align*}
P_4(u) &= -2C_{++} u^4 + 2C_{+-} u^3 + 2(C_{+-} - C_{00})u^2 - 2C_{0-} u - 2C_{--} \\
P_3(u) &= 2C_{++}(1 - 2j)u^3 + ((3j - 1)C_{++} - C_+^2)u^2 \\
&\quad + (2jC_{+-} + (1 - 2j)C_{00} + C_0)u - jC_{0-} + C_- \\
P_2(u) &= 2j(2j - 1)C_{++} u^2 + 2j(C_+ - jC_{+0})u + j^2C_{00} - jC_0
\end{align*}
\] (11)

A more compact form can be achieved if we define a new set of parameters as follows

\[
\begin{align*}
a_4 &= -2C_{++}, \quad a_3 = 2C_{+0}, \quad a_2 = 2(C_{+-} - C_{00}), \quad a_1 = -2C_{0-} \\
a_0 &= -2C_{--}, \quad b_2 = C_+ - jC_{+0}, \quad b_1 = 2jC_{+-} + (1 - 2j)C_{00} + C_0 \\
b_0 &= -jC_{0-} + C_-, \quad c_0 = j^2C_{00} - jC_0
\end{align*}
\] (12)

therefore (11) is now given as

\[
\begin{align*}
P_4(u) &= a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0 \\
P_3(u) &= (2j - 1)a_4 u^3 + ((j - \frac{1}{2})a_3 - b_2)u^2 + b_1 u + b_0 \\
P_2(u) &= j(1 - 2j)a_4 u^2 + 2jb_2 u + c_0
\end{align*}
\] (13)

We now proceed to identify the expressions for \( H_G \) given in (4) and (10). Since the variables \( x \) and \( u \) are involved, a change of variables must be done, as follows

\[ x = \int P_4^{-\frac{1}{2}}(u) \frac{d}{du} = \phi(u) \] (14)

which also allows to eliminate the term multiplying the second derivative in (10). In this way we get
\[ H_G = -\frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{4P_3(u) + P_4'(u)}{4P_4^2(u)}\right) \frac{d}{dx} + P_2(u) \]  

(15)

The substitution \( u = \phi^{-1}(x) \) has to be done in the polynomials occurring in (10). Furthermore, \( P_4'(u) \) is the derivative of \( P_4(u) \) respect to \( u \) with the subsequent substitution of \( u \) as a function of \( x \).

We can see now more clearly the reason for transforming the Schrödinger equation with the non-unitary operator \( e^{-a(x)} \), since as a consequence of that, we get a term with first derivative which makes possible the identification of (4) and (10). From this identification we get

\[ A(x) = \frac{4P_3(u) + P_4'(u)}{4P_4^2(u)} \]  

(16)

and

\[ \Delta V = V(x) + \frac{1}{2} A'(x) - \frac{1}{2} A^2(x) = P_2(x) \]  

(17)

Substituting (16) in (17) we get the general expression for the potentials which may be algebraized by \( su(2) \)

\[ V(x) = P_2(u) + \frac{1}{8P_4(u)}(P_4'(u) + 4P_3(u))^2 - \frac{1}{8} P_4''(u) - \frac{1}{2} P_3'(u) \]

\[ + (P_4'(u) + 4P_3(u)) \frac{P_4'(u)}{16P_4(u)} \]  

(18)

The set of parameters characterizing this potential are: \( a_4, a_3, a_2, a_1, a_0, b_2, b_1, b_0 \) and \( c_0 \).

Let us see now how this algebraization lead us to the energy spectrum and the corresponding eigenfunctions of the potential obtained in (18). Since the carrier space of \( su(2) \) is given by (8), then a basis for the eigenfunctions of \( H_G \) is

\[ \{ \tilde{\Psi} \} = \{ 1, u, u^2, \ldots u^{2j} \} \cup \{ \tilde{\Psi}_{2j+2}, \tilde{\Psi}_{2j+3}, \ldots \} \]  

(19)

where \( \{ \tilde{\Psi}_{2j+2}, \tilde{\Psi}_{2j+3}, \ldots \} \) is orthogonal to \( R^j \). So, the matrix representation of \( H_G \) is given by

\[ H_G = \begin{pmatrix} H_{G1} & 0 \\ 0 & H_{G2} \end{pmatrix} \]  

(20)

where \( H_{G1} \) is the matrix associated to \( R^j \) and \( H_{G2} \) is an infinite matrix associated to the second subspace, which is unknown. Therefore, since \( H_{G1} \)
is a hermitic matrix of dimension \((2j + 1) \times (2j + 1)\), it is possible to diagonalize it, obtaining a part of the energy spectrum and their corresponding eigenfunctions. This is the reason why the potentials treated in this way are known as quasi-solvables, since only a part of the spectra may be obtained. Nevertheless, it is possible to find the complete spectra with this technique for some potentials, as we will see later.

The eigenfunctions of \(H_{G1}\) will be linear combinations of elements of \(R^j\), therefore if \(H_{G1}\tilde{\Psi} = E\tilde{\Psi}\), then \(\tilde{\Psi}(u) = P_{2j}(u)\), where \(P_{2j}(u)\) is a polynomial on \(u\) of degree \(2j\). Using (2,6,16), the eigenfunctions of \(H_{G1}\) turns out to be of the form

\[
\Psi(u) = \tilde{\Psi}(x)e^{-a(x)} = P_{2j}(u(x))e^{-\int \frac{4P_3 + P'_4}{4u^2} dx}
\]

and the corresponding eigenvalues are obtained after the diagonalization of \(H_{G1}\).

Let us see one example. Let us take: \(a_4 = a_3 = a_1 = a_0 = c_0 = 0\). Thus \(P_4(u) = a_2u^2\), from (14) we get

\[
u = e^{kx}
\]

with \(k = \sqrt{a_2}\). The potential for this set of parameters is obtained from (18) and is given by

\[
V(x) = \frac{b_0^2}{2k^2}e^{2kx} + \frac{b_2}{k^2}(2jk^2 - b_1)e^{kx} + \frac{b_0}{k^2}(k^2 + b_1)e^{-kx} + \frac{b_2}{2k^2}e^{-2kx} + \frac{1}{8k^2} \left( (k^2 + 2b_1)^2 - 8b_0b_2 \right)
\]

(22)

\[
\phi(x) = \frac{1}{2k}(k^2 + 2b_1) + \frac{b_0}{k}e^{-kx} - \frac{b_2}{k}e^{kx}
\]

hence, after using (6)

\[
a(x) = \frac{1}{2k}(k^2 + 2b_1) x - \frac{b_0}{k^2}e^{-kx} - \frac{b_2}{k^2}e^{kx}
\]

(23)

\(H_{G}\) is then obtained from (15),

\[
H_{G} = -\frac{1}{2} a_2u^2 \frac{d^2}{du^2} + (-u^2b_2 + b_1u + b_0) \frac{d}{du} + 2jb_2u
\]
In the case $j = 0$, the representation space will be simply $R^0 = \{1\}$, and we will find only one energy eigenvalue: $E_0 = 0$ with $\Psi_0 = e^{-a(x)}$ its corresponding eigenstate. If $j = \frac{1}{2}$ we have then $R^{1/2} = \{1, u\}$. Representing each element of this space (i.e., 1 and $u$) by $|0\rangle$ and $|1\rangle$ respectively, we have then the following matrix elements: $\langle 0|H_G|0\rangle = 0$, $\langle 0|H_G|1\rangle = b_0$, $\langle 1|H_G|0\rangle = b_2$, and $\langle 1|H_G|1\rangle = b_1$. So, the expression for $H_{G1}$ in a matrix representation is given by

$$H_{G1} = \begin{pmatrix} 0 & b_0 \\ b_2 & b_1 \end{pmatrix}$$

whose eigenvalues are

$$E_{\pm} = \frac{1}{2}b_1 \pm \frac{1}{2}\sqrt{b_1^2 + 4b_2b_0}$$

with the corresponding eigenstates

$$\Psi_{\pm}(x) = N \left( 1 \pm \frac{E_{\pm}}{b_0} e^{kx} \right) e^{-a(x)}$$

where $a(x)$ is given in (23) and $N$ is a normalization constant. Notice that if we set $b_2 = 0$ in (22) we obtain the Morse potential which is exactly solvable. We expect then that with a proper choice of parameters one could obtain certain potentials that can be solved in a closed form, this is the point that we would like to discuss now.

We can find a general class of potentials that are exactly solvable through this method if we take the potential (18) to be independent of $j$. After a tedious calculation one find the following conditions for the coefficients occurring in (18)

\begin{align*}
2(a_2b_2 + a_4b_0) + a_3b_1 - a_4a_1 &= 0, \\
2(2a_1b_2 + a_3b_0) - a_3a_1 - 4a_4a_0 &= 0 \\
a_0(a_3 - 2b_2) &= 0, \\
a_3b_2 + 2a_4b_1 &= 0 \\
a_4a_0 &= 0, \\
a_4a_1 &= 0, \\
a_3^2 - 4a_4a_2 &= 0
\end{align*}

This system has only one interesting solution and is given by

$$a_4 = a_3 = b_2 = 0$$

(25)

Substituting (25) in (18) we obtain for $V(x)$
\[ V(x) = \frac{(2(a_2 + 2b_1)u + a_1 + 4b_0)(2(3a_2 + 2b_1)u + 3a_1 + 4b_0)}{32(a_2 u^2 + a_1 u + a_0)} \]

\[ -\frac{1}{2}b_1 - \frac{1}{4}a_2 + c_0 \]  

(26)

with

\[ \frac{du}{dx} = \sqrt{a_2 u^2 + a_1 u + a_0} \]  

(27)

The expression for \( V(x) \) given in (26) is the most general one that can be completely algebrized using the su(2) algebra.

Under the conditions that we have imposed, the “rotated” Hamiltonian \( H_G \) (15) turns out to be

\[ H_G = -\frac{1}{2}(a_2 u^2 + a_1 u + a_0) \frac{d^2}{du^2} + (b_1 u + b_0) \frac{d}{du} + c_0 \]  

(28)

So, if \( |n\rangle \) represents the element \( u^n \) of the carrier space \( R_j \), with \( j \) arbitrary, the matrix elements of \( H_G \) are given by

\[ \langle m | H_G | n \rangle = (-\frac{1}{2}a_2 n(n - 1) + nb_1 + c_0)\delta_{n,m} \]  

(29)

\[ + (-\frac{1}{2}a_1 n(n - 1) + nb_0)\delta_{n-1,m} - \frac{1}{2}a_0 n(n - 1)\delta_{n-2,m} \]

Therefore, \( H_G \) is a triangular superior matrix, which means that their eigenvalues are those of the diagonal, i.e.,

\[ E_n = -\frac{1}{2}a_2 n(n - 1) + nb_1 + c_0 \]  

(30)

In the following table we exhibit the cases that can be solved through this technique, as well as the corresponding parameters.

| Potential | \( a_2 \) | \( a_1 \) | \( a_0 \) | \( b_1 \) | \( b_0 \) | \( c_0 \) |
|-----------|----------|----------|----------|--------|--------|--------|
| \( \frac{1}{2}f \omega^2 x^2 + \frac{l(l+1)}{2x^2} \) | 0 | 1 | 0 | 2\( \omega \) | \( \frac{l}{2} - \frac{1}{4} \) | \( \omega(\frac{1}{2} - l) \) |
| \( \frac{1}{2} A^2 + \frac{1}{2} B^2 e^{-\alpha x} - B(A + \alpha) e^{-\alpha x} \) | \( \alpha^2 \) | 0 | 0 | \( \alpha A \) | \( -\alpha B \) | \( -\frac{1}{2}(A + \frac{1}{A}) \) |
| \( \frac{1}{2} A^2 + \frac{1}{2} B^2 - A^2 - \alpha A \) \( \text{sech}(\alpha x) \) | \( \alpha^2 \) | 0 | 1 | \( \alpha A - \frac{1}{2} \alpha^2 \) | \( B \) | 0 |
| \( \frac{1}{2} A^2 - \frac{1}{2} B^2 - 2A + \alpha \) \( \text{tanh}(\alpha x) \) | \( \alpha^2 \) | 0 | 0 | \( \alpha A - \frac{1}{2} \alpha^2 \) | \( B \) | 0 |
| \( \frac{1}{2} A^2 + \frac{1}{2} (B^2 - A^2 + \alpha A) \text{csch}(\alpha x)^2 \) | \( \alpha^2 \) | 0 | -1 | \( \alpha A - \frac{1}{2} \alpha^2 \) | \( -B \) | 0 |
| \( -\frac{1}{2} B(2A + \alpha) \text{coth}(\alpha x) \text{csch}(\alpha x) \) | | | | | | |

8
Introducing these values for the parameters in (30) we can check that we get the correct expressions for the energy eigenvalues for each potential, see for example the table in [11].

3 Algebraization of the q-Schrödinger equation through $D_q^{(n)}(su(2))$

We would like to use the ideas of the partial algebraization method presented in the preceding section to solve the q-deformed Schrödinger equation. A deformation is done after associating the momentum operator $P$ to a deformed derivative. Then the standard commutation relation between the momentum and the position operator $X$ is replaced by

$$XP - qPX = i$$

In this way we assume that a q-deformed Schrödinger equation is given by

$$\left( -\frac{1}{2}D_q^2 + (V_q(x) - E) \right) \Psi_q = 0$$

(31)

where the deformed derivative is given by

$$D_q\Phi(x) = \frac{\Phi(x) - \Phi(qx)}{(1 - q)x}$$

In this paper we will obtain exact solutions of (31), for the particular case in which the potential $V_q(x)$ is a q-deformed harmonic oscillator, using algebraic techniques. For this purpose we will use the $D_q^{(n)}(su(2))$ algebra to be defined later on. A realization of this algebra is obtained from the $su(2)$ algebra that was used in the preceding section (7) after a change of variables. As it was seen before, we had to change variables in order to equate the rotated Hamiltonian with the combinations of generators given in (9). The change of variables that we are going to use is a simple one, namely $u = x^n$. This is done for simplicity in order to avoid the complications that arise when one consider the general case. For this case, we have that (7) becomes
\[ J_+ = 2jx^n - \frac{1}{n} x^{n+1} \frac{d}{dx} \]
\[ J_0 = -j + \frac{1}{n} x \frac{d}{dx} \]
\[ J_- = \frac{1}{n} x^{1-n} \frac{d}{dx} \]

(32)

If we make a formal replacement of \( \frac{d}{dx} \rightarrow D_q \) in (32) we have then

\[ J_+ = 2jx^n - \frac{1}{n} x^{n+1} D_q \]
\[ J_0 = -j + \frac{1}{n} x D_q \]
\[ J_- = \frac{1}{n} x^{1-n} D_q \]

(33)

It is straightforward to prove that the new set of generators satisfy the following commutations relations

\[ [J_0, J_-] = -J_- g(J_0) \]
\[ [J_0, J_-] = g(J_0) J_+ \]
\[ [J_+, J_-] = f(J_0) \]

(34)

where

\[ g(J_0) = j(1 - q^{-n}) + \frac{|n|}{n} q^{-n} + (1 - q^{-n}) J_0 \]

(35)

\[ f(J_0) = j^2(2 - q^n - q^{-n}) - j \frac{n}{|n|} ([n] + [-n]) \]
\[ + (2j(1 - q^{-n}) + \frac{1}{n}([n] - [-n]) J_0 + (q^n - q^{-n}) J_0^2 \]

(36)

with

\[ [n] \equiv \frac{1 - q^n}{1 - q} \]

This relations define the algebra \( D_q^n(su(2)) \), which is a special case of the general ones studied in [12]. Then the Casimir operator is of the form

\[ C_q = J_- J_+ + h(J_0) \]

(37)
where \( h(J_0) \) is found to be

\[
h(J_0) = q^n J_0^n + \frac{|n|}{n} J_0
\]

(38)

After evaluating \( J_- J_+ \) one obtains for \( C_q \)

\[
C_q = j \left( \frac{|n|}{n} + q^n j \right)
\]

(39)

It is easy to verify that the Delbecq-Quesne [12] condition for the existence of the Casimir operator

\[
h(J_0) - h(J_0 - g(J_0)) = f(J_0)
\]

(40)

is satisfied.

Our next goal is the algebraization of (31) following the steps of the preceding section. We first perform a “rotation” of the form

\[
\Psi_q(x) = \tilde{\Psi}_q(x) F(-a(x))
\]

(41)

where \( F(-a(x)) \) is such that

\[
D_q F(-a(x)) = -D_q(a(x))F(-a(x))
\]

(42)

in analogy with the undeformed case where \( F \) was the exponential. In what follows we will use the following notation

\[
D_q f \equiv f', \quad f(qx) \equiv f^*
\]

(43)

Using the fact that \((fg)' = f'g + f^*g'\), equation (31) reads as follows

\[
-\frac{1}{2} \bar{\Psi}'' + \frac{a'}{2} (\bar{\Psi}^* + \bar{\Psi}'') + \frac{(a'' - a^* a')}{2} \bar{\Psi}^{**} + V \bar{\Psi} = E \bar{\Psi}
\]

(44)

where we have suppressed the index \( q \) and the variable \( x \) for simplicity. Taking into account that \( \bar{\Psi}' = (\bar{\Psi} - \bar{\Psi}^*)/(1 - q)x \) we get the following relations

\[
\begin{align*}
\bar{\Psi}^* &= \bar{\Psi} - (1 - q)x \bar{\Psi}' \\
\bar{\Psi}'^* &= \bar{\Psi}' - (1 - q)x \bar{\Psi}'' \\
\bar{\Psi}'^* &= q \bar{\Psi}' - (1 - q)x \bar{\Psi}'' \\
\bar{\Psi}^{**} &= \bar{\Psi} - 2(1 - q)x \bar{\Psi}' + (1 - q)^2 x^2 \bar{\Psi}''
\end{align*}
\]
Making these changes we obtain from (44)

\[
H_G \tilde{\Psi} = -\frac{1}{2}(1 + a'(1 - q^2) x + (a'^* a' - a'')(1 - q^2) x^2) \tilde{\Psi}'' \\
+ \frac{1}{2}(a'(1 + q) + 2(a'^* a' - a'')(1 - q) x) \tilde{\Psi}' \\
+ (V - \frac{1}{2}(a'^* a' - a'')) \tilde{\Psi} \\
= E \tilde{\Psi}
\]

This is the expression of the rotated Hamiltonian that we will attempt to equate to

\[
H_G = \sum_{a,b=0,\pm,a\geq b} C_{ab} J_a J_b + \sum_{a,b=0,\pm} C_a J_a
\]

(46)

Notice the difference with the non-deformed case, here the change of variables \( u \to x \) has been done from the beginning.

Equating the coefficients of \( \tilde{\Psi}'' \), \( \tilde{\Psi}' \) and \( \tilde{\Psi} \) of (45) with those that we get when acting with (46) on \( \tilde{\Psi} \), we get the following relations

\[
C_{++} x^{2n+2} q^{n+1} - C_{+0} x^{n+2} q + (C_{00} - C_{+-} q^{-n}) x^2 q \\
+ C_{0-} x^{2-n} q^{1-n} + C_{-} x^{2-2n} q^{1-n} \\
= -\frac{n^2}{2}(1 + a'(1 - q^2) x + (a'^* a' - a'')(1 - q^2) x^2)
\]

(47)

\[
C_{++}(\frac{n+1}{n^2} - 2j(1 + q^n)) x^{2n+1} + (C_{+0}(\frac{3j}{n} - \frac{1}{n^2}) - C_{+1}) x^{n+1} \\
+ (C_{+-}(\frac{2j}{n} - \frac{[1-n]}{n^2})) + C_{00}(\frac{1}{n^2} - \frac{2j}{n}) + C_{0} x \\
+ (C_{0-}(\frac{[1-n]}{n^2} - \frac{j}{n}) + C_{-} \frac{1}{n}) x^{1-n} + C_{-}(\frac{1-n}{n^2}) x^{1-2n} \\
= \frac{1}{2}(a'(1 + q) + 2(a'^* a' - a'')(1 - q) x)
\]

(48)

\[
C_{++} 2j(2j - \frac{n}{n}) x^{2n} + 2j(C_{+-} - C_{+0}) x^n + j(C_{00} - C_{0}) \\
= V(x) - \frac{1}{2}(a'^* a' - a'')
\]

(49)
From the above relations we see that \( a(x) \) can only be a combination of powers in \( x \), otherwise these expressions could not be satisfied.

Let us see the following case: \( n = 2 \), \( C_{++} = C_{--} = 0 \). Furthermore we let \( a' \) to be

\[
a' = \omega x - \frac{[l + 1]}{x}
\]  

(50)

therefore : \( a'' = \omega + \frac{[l+1]}{qx^2} \) and \( a^* = \omega qx - \frac{[l+1]}{q^2} \). Then we obtain from (47) the following conditions

\[
\begin{align*}
C_+ + 0 &= 2\omega^2 (1 - q^2) \\
C_{00}q^2 - C_{+-} &= 2\omega[l + 1](1 - q^4) \\
C_{0-} &= -2q - [l + 1]([l + 1] - 1 - q) (1 - q^2)
\end{align*}
\]

(51)

From (48) we have

\[
\begin{align*}
C_{+0}(3j - \frac{1}{2}) - C_+ &= 2\omega^2 q(1 - q) \\
C_{+-}(2j + \frac{1}{2q}) + C_{00}(\frac{1}{2} - 2j) + C_0 &= 2\omega[l + 1] \left( q + q^{-1} \right) (q - 1) + \omega (3q - 1) \\
C_{0-}(j + \frac{1}{2q}) - C_- &= [l + 1] (1 + q) + \frac{2}{q} [l + 1]([l + 1] - 1) (q - 1)
\end{align*}
\]

(52)

and finally from (49)

\[
(C_+ - C_{+0})2jx^2 + (C_{00} j - C_0) j = -\frac{1}{2}\omega^2 q x^2 + \frac{[l + 1][1 - [l + 1]]}{2q x^2}
\]

(53)

\[+ \frac{1}{2} \omega ([l + 1](q^{-1} + q) + 1) + V(x)\]

In this way (51) and (52) provide us with six equations relating \( C_{+0} \), \( C_{+} \), \( C_{00} \), \( C_{0} \), \( C_{+-} \), \( C_{0-} \) and \( C_{-} \) with the set \( \{ \omega, [l], q, j \} \). Since we have a set of six linear equations and seven variables, we will be free to choose one of those variables at will. Specifically, looking at the equations, we can see that \( C_{+0} \), \( C_{+} \), \( C_{0-} \) and \( C_{-} \) are fixed, and one of the variables \( C_{00} \), \( C_{0} \) and
$C_{+}$ will be free, and fixing one of them we fix the others two. This freedom on one of the variables corresponds in the classical (non-deformed) case, to the freedom that we have in choosing the ground state of the system.

From (53) we get $V(x)$. We then need the coefficients $C_{+0}$ and $C_{+}$. The expression for $C_{+0}$ is given by (51) and then from (52) we obtain for $C_{+}$

$$C_{+} = \omega^{2}(1-q)(6j(1+q) - 3q - 1)$$

then from (52) $V(x)$ is finally given by

$$V(x) = \frac{1}{2q} \frac{(l + 1)(l + 1) - 1}{x^2} + V_{0}$$

where $\omega^{2}$ and $V_{0}$ are defined as

$$\omega_{q}^{2} = \omega^{2} \left( (1-q^{2})(8j^{2} - 2j) - 4jq(1-q) + \frac{q}{2} \right)$$

$$V_{0} = C_{00}j^{2} - C_{0}j - \frac{\omega^{2}}{2} + \frac{\omega l + 1}{2}(q + q^{-1})$$

We can say that $V(x)$ correspond to a q-deformed harmonic oscillator with a q-deformed angular frequency $\omega_{q}$ and with a q-deformed angular momentum. Notice that in the limit $q \to 1$ we obtain the classical potential.

4 Deformed spectrum and eigenfunctions

As we mentioned before, the carrier space of $SU(2)$, i.e. $R^l$, is given in the classical case by monomials on the variable $u$. This was the original variable defining the algebra, see (7) and (8). We can use this basis in the deformed case. Since we have chosen a change of variables given by $u = x^{2}$, the monomials will be of the form $x^{2k}$ ($k$ integer), these are represented as $|k\rangle$. Then from (33) we have

$$J_{+} |k\rangle = \left(2j - \frac{[2k]}{2} \right) |k + 1\rangle$$

$$J_{0} |k\rangle = \left(-j - \frac{[2k]}{2} \right) |k\rangle$$

$$J_{-} |k\rangle = \frac{[2k]}{2} |k - 1\rangle$$
If \( j = \frac{1}{4}[2m] \) for some integer \( m \), the representation will be finite-dimensional, otherwise will be infinite-dimensional and bounded below. Since we have expressed \( H_G \) in terms of the generators of the algebra, then we can represent it as a matrix acting on the corresponding basis. If the representation is infinite-dimensional it will be too hard to diagonalize \( H_G \) in order to obtain the spectrum, therefore, we will work with the finite case, so we have only to diagonalize a finite matrix as happened in the classical case.

According to (41), the eigenfunctions will be of the form,

\[
\Psi(x) = P(x^2)F(-a(x))
\]

where \( P(x^2) \) is a polynomial on \( x^2 \). Our next task is to evaluate the function \( F(-a(x)) \). Let us write \( F(-a(x)) = x^{l+1}f(x) \), then from equations (42) and (50) we get

\[
D_q(x^{l+1}f(x)) = \left(-\omega x + \frac{[l+1]}{x}\right)f(x)
\]

we obtain

\[
q^{l+1}x^{l+1}D_q(f(x)) = -\omega x^{l+2}f(x)
\] (58)

if we write \( f(x) \) as a power series \( f(x) = \sum_{n=0}^{\infty} \alpha_n x^n \), the coefficients \( \alpha_n \) are found to satisfy the following conditions

\[
\alpha_n = -\frac{\omega}{q^{l+1}\langle n \rangle} \alpha_{n-2}, \quad \text{for } n \text{ even}
\]

\[
\alpha_n = 0, \quad \text{for } n \text{ odd}
\]

Hence

\[
\alpha_{2n} = \alpha_0 \left( -\frac{\omega(1+q)}{q^{l+1}} \right)^n \frac{1}{\langle n \rangle_{q^2}!}
\]

where \( \alpha_0 \) is arbitrary and \( \langle n \rangle_{q^2} \) is defined in the same way as \( \langle n \rangle \), but with the replacement \( q \to q^2 \), and the factorial means

\[
\langle n \rangle_{q^2}! = \langle n \rangle_{q^2} [n-1]_{q^2} \cdots
\]

With these results we see that the function \( f(x) \) is

\[
f(x) = \sum_{n=0}^{\infty} \frac{1}{\langle n \rangle_{q^2}!} \left( -\frac{\omega(1+q)}{q^{l+1}} x^2 \right)^n
\]

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which is a deformed exponential with deformation parameter $q^2$ and with argument $-\frac{\omega(1+q)}{q^{l+1}}x^2$. So the function $F(-a(x))$ turns out to be

$$F(-a(x)) = x^{l+1} \exp_{q^2} \left( -\frac{\omega(1+q)}{q^{l+1}}x^2 \right)$$  \hspace{1cm} (59)$$

where we have set $\alpha_0 = 1$.

In order to evaluate the energy spectrum we need the matrix elements of the deformed Hamiltonian. These elements are easy to compute from the expressions (46) together with (55), (56) and (57), we get

$$\langle n | H_G | k \rangle = \left( 2j - \frac{[2k]}{2} \right) \left( C_+ + C_{+0} \left( -j + \frac{[2k]}{2} \right) \right) \delta_{n,k+1}$$

$$+ \left( C_{+-} \left( \frac{[2k]}{2} \right) \left( 2j - \frac{[2k]}{2} \right) \right) + C_{00} \left( -j + \frac{[2k]}{2} \right) \delta_{n,k}$$

$$+ \left( \frac{[2k]}{2} \right) \left( C_{0-} \left( -j + \frac{[2k]}{2} \right) + C_- \right) \delta_{n,k-1}$$  \hspace{1cm} (60)$$

To illustrate the method, let’s see some examples. The simplest possible cases are $j = 0$ and $j = \frac{1}{4}[2]$. In the case $j = 0$ we have obviously one energy eigenvalue, which is $E = 0$ and its corresponding eigenfunction is then

$$\Psi_0(x) = x^{l+1} \exp_{q^2} \left( -\frac{\omega(1+q)}{q^{l+1}}x^2 \right)$$

For $j = \frac{1}{4}[2] = \frac{1}{4}(1+q)$, the representation space will be of dimension 2 and will be given by $R^2 = \{1, x^2\}$. As it was said before, we still have a free parameter ($C_{+-}$, $C_{00}$ or $C_0$). For convenience, we choose the fixing condition given by

$$C_{00}j - C_0 = 0$$

In this way, the matrix representation of $H_{G1}$ is nicely given by

$$H_{G1} = \begin{pmatrix} 0 & 2j(C_- - jC_{0-}) \\ 2j(C_+ - jC_{+0}) & 2j(C_0 + 2jC_{++}) \end{pmatrix}$$  \hspace{1cm} (61)$$

So obtaining the values for $C_{+0}$, $C_{+-}$, $C_{0-}$, $C_+$, $C_0$ and $C_-$ from (51) and (52), substituting them in the above matrix, and diagonalizing it, we can get two energy eigenvalues for the system in question, as well as the corresponding eigenfunctions. The two eigenvalues are the following ones:
\[ E_{q\pm} = \left( p_1[l + 1] + p_2 \pm 2\sqrt{p_3[l + 1]^2 + p_4[l + 1] + p_5} \right) \frac{(1 + q)}{8q} \omega \]

where \( p_i, i = 1 \ldots 5 \), are polynomials on \( q \) given by

\[
\begin{align*}
    p_1 &= -2(q^2 + 1)(q - 1)^2 \\
    p_2 &= 2q(3q - 1) \\
    p_3 &= (1 + q)^2(q - 1)^6 \\
    p_4 &= 2q(-1 + 2q)(1 + q)^2(q - 1)^3 \\
    p_5 &= q^2(1 + q)(4q^2 - 3q + 1)
\end{align*}
\]

We can see that in the limit \( q \to 1 \) we get the expected eigenvalues for the resulting potential:

\[ V(x) \to \frac{1}{2} \omega^2 x^2 + \frac{l(l + 1)}{2x^2} + \left( \frac{3}{2} - l \right) \omega \]

i.e.

\[
\begin{align*}
    E_{q^-} &\to 0 \\
    E_{q^+} &\to 2\omega
\end{align*}
\]

It is possible also to get the eigenfunctions associated to each eigenvalue. They will be of the form

\[ \Psi_{\pm}(x) = \left( A_{\pm} + B_{\pm}x^2 \right) x^{l+1} \exp_{q^2} \left( -\frac{\omega(1 + q)}{q^{l+1}} x^2 \right) \]

where \( A_{\pm} \) and \( B_{\pm} \) are some complicated functions of \( \omega \) and \( [l + 1] \), which we will not show here.

5 Final Comments

We have seen how using a q-algebra of the kind studied in [12] we can get exact solutions for part of the energy eigenvalues and eigenfunctions of the q-Schrödinger equation in the case in which the potential is a 3-dimensional q-deformed harmonic oscillator. The deformation of the potential is a very specific one, i.e. with a very specific q-dependence, as shown in (54). We
could also want to construct any q-dependent potential such that in the limit $q \to 1$ we get the usual classical potential. Our solution is only valid for the specific case we was considering, so a general solution for any possible choice of the q-dependence remains as an open and very hard question. Even harder is to consider potentials others than a harmonic oscillator, like the Morse potential or so. This is because of the somehow complicated properties of the deformed derivative.

A remarkable aspect of our solution is the fact that we can get only part of the spectrum, the rest remaining unknown. This shouldn’t be too astonishing, after all our method is based on the ideas of partial algebraization, which in most of the cases allows us to get only part of the spectrum, the only exceptions being the ones studied at the end of section 2. Nevertheless, considering that one of those exceptions was precisely the harmonic oscillator, it is curious that we cannot get the whole spectrum in the deformed case, but only a part of it.

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