Chaos in Static Axisymmetric Spacetimes I : Vacuum Case

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Abstract

We study the motion of test particle in static axisymmetric vacuum spacetimes and discuss two criteria for strong chaos to occur: (1) a local instability measured by the Weyl curvature, and (2) a tangle of a homoclinic orbit, which is closely related to an unstable periodic orbit in general relativity. We analyze several static axisymmetric spacetimes and find that the first criterion is a sufficient condition for chaos, at least qualitatively. Although some test particles which do not satisfy the first criterion show chaotic behavior in some spacetimes, these can be accounted for the second criterion.

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1 Introduction

Chaos has become one of the most important ideas used to understand various non-linear phenomena in nature. We know many features of chaos in the Newtonian dynamics. However, we do not know, so far, so much about those in general relativity (GR). If gravity is strong, we have to use the Einstein’s theory of gravitation. We may find new types of chaotic behavior in strong gravitational fields, which do not appear in Newtonian dynamics. Although a chaotic behavior of the non-linear gravitational field itself may be much more interesting to study, there may be a fundamental difficulty to discuss its chaotic dynamics even in the well-studied Bianchi IX model, due to a gauge invariance, in particular a choice of time coordinate, in GR\(^1\). Before we devote ourselves to studying such a subject, we need to analyze problems with a clear setting to better understand chaos in GR. One such approach is to study test particle motion in a fixed curved background \(^2\)–\(^8\). We have an appropriate time coordinate, the proper time of a test particle, which is a natural invariant time\(^6\) or an infinite observer’s time, which has an invariant meaning because of the existence of a time-like Killing vector\(^7\).

In the present paper, we study test particle motion in static axisymmetric vacuum spacetimes and search for a generic criterion for chaos to occur. In Newtonian dynamics, a connection between chaos and a local instability of a test particle in a given potential \(V\) has been intensively discussed in two ways. One way is to study local instability measured by eigenvalues of the tidal-force matrix, which is derived from linearization of the equations of motion, i.e.,

\[
\frac{d^2n^i}{dt^2} = -V^i_j n^j, \tag{1.1}
\]

where \(n^i\) is a separation vector between two close orbits and \(V^i_j \equiv \partial^i \partial_j V\) is the tidal-force matrix\(^9\). When the eigenvalues of \(V^i_j\) are all negative at some point, the orbit becomes locally unstable in any direction. Then there is a possibility that such a local instability determines a global chaotic behavior of a test particle. The other way to analyze a local instability is to use the curvature of a fictitious space\(^10\). In Newtonian dynamics, the orbit of a test particle in a given potential \(V\) does not follow a geodesic except for a free particle, which is not interesting in this context. The equations of motion is

\[
\frac{d^2x^i}{dt^2} = -\partial^i V. \tag{1.2}
\]

If the potential \(V\) does not directly depend on the time parameter \(t\), the energy of particle \(E\) is conserved and the particle moves inside the classically-allowed region \(D = \{x|V(x) \leq E\}\). Then we make a conformal transformation of a three dimensional metric,

\[
\tilde{f}_{ij} = 2[E - V(x)]f_{ij}, \tag{1.3}
\]

where \(f_{ij}\) is the metric of physical flat space. Using this conformal transformation and transforming the time parameter from \(t\) to \(\tilde{t}\) by

\[
d\tilde{t} = 2[E - V(x)]dt, \tag{1.4}
\]

\(^*\)The Greek and Latin indices are used as those of 4-dimensional Lorentzian coordinates and of the 3-dimensional spatial coordinates, respectively.
we obtain an affine parameterized geodesic equation in a fictitious space with metric $\tilde{f}_{ij}$. The equation for a deviation vector $\tilde{n}$ orthogonal to the velocity $\tilde{v}$ in the fictitious space is

$$\frac{d^2\tilde{n}}{dt^2} = -\nabla_{\tilde{n}} \tilde{U},$$

(1.5)

where a potential $\tilde{U}$ is given by $\tilde{v}$ and $\tilde{n}$, such as

$$\tilde{U} = \frac{1}{2} \tilde{K} |\tilde{v}|^2 |\tilde{n}|^2.$$

(1.6)

$\tilde{K}$ is a sectional curvature spanned by $\tilde{v}$ and $\tilde{n}$ in the fictitious space and is defined as

$$\tilde{K}(\tilde{v}, \tilde{n}) = \tilde{R}_{ijkl} \tilde{v}^i \tilde{n}^j \tilde{v}^k \tilde{n}^l / |\tilde{v}|^2 |\tilde{n}|^2,$$

(1.7)

where $\tilde{R}_{ijkl}$ is the Riemann tensor given by the metric $\tilde{f}_{ij}$ in the fictitious space. Since $\tilde{n}=0$ is an unstable point of the potential $\tilde{U}$ if $\tilde{K} < 0$, a local instability is determined by a sign of the sectional curvature $\tilde{K}(\tilde{v}, \tilde{n})$.

It has been shown that the first way to measure a local instability by $V_j^i$ does not always predict the chaotic motion of a test particle. Some counter-examples are known[11]–[13], where it was pointed out that this non-correlation is caused by an adiabatic property in Eq.(1.1)[12, 13]. In fact, the tidal-matrix does not have any information about a particle’s velocity, that is, how the orbit passes through each point in a phase space. Hence, it seems to be insufficient to determine the chaotic behavior of each orbit only from the tidal-force matrix. However, see the discussion in §3.3. for a relativistic case.

On the other hand, for the second criterion of local instability, as was shown in the above, the local instability is determined by a sectional curvature of a 2-dimensional plane spanned by the velocity and deviation vector of the orbit. In this case, the local instability is determined not only by the position of a particle in configuration space but also by its velocity. It seems to be a better criterion to determine chaotic behavior. In fact, it is known that in the case that sectional curvatures of the particle orbit in a compact region are all negative, the geodesic becomes chaotic[10]. If a configuration space is two dimensional, because the sectional curvature is always equal to the scalar curvature, a negative scalar curvature always leads to chaos.

Such a correlation between the scalar curvature and chaos has been well-examined in Newtonian dynamics. For example, in the two dimensional Kepler problem, all bounded orbits pass through only the positive curvature region[14], which is consistent with the integrability of the system.

However, it turns out that the second method as well as the first one does not always predict an occurrence of chaos even in two dimensional case. For example, in the Henon-Heiles system, which is a model of a star moving in a galaxy, we cannot predict the change of the particle behavior from regular to chaotic by the sign of scalar curvature because it is always positive although chaos occurs in some situations[15]. Furthermore there are some examples in which stable periodic orbits exist even in the region of negative scalar curvature[13, 14]. This failure seems to stem from the following two main obstacles:
(1) The scalar curvature diverges on the boundary of the allowed region of a particle in Riemannian case\(^*\).

This singularity is inevitable as long as we use a conformal transformation (1.3). In pseudo-Riemannian case, although a orbit before transformation can cross the boundary of \(D\), the curvature after the transformation again diverges on the boundary. So we cannot follow the orbit near the boundary in the fictitious space for Riemannian and pseudo-Riemannian cases. Such a problem has been pointed out by several authors\([13, 16]\).

(2) The conformal transformation (1.3) is insufficient to get an affine-parameterized geodesic equation. We also need the transformation of time parameter (1.4). However, the time parameter is important when judging the chaotic behavior of a test particle as was pointed out in the vacuum Bianchi IX problem \([1]\). It may change a criterion depending on time parameter we choose.

In GR, however, the motion of a free test particle is always described by geodesic, which is still interesting to study. We do not need any conformal transformation to get geodesic equations as we had to do for Newtonian dynamics, so we do not have the above problems in GR. Hence, the second criterion based on the curvature may work well in GR. Here we study a connection between the curvature and the chaotic behavior of a test particle in static axisymmetric spacetimes. We will show that the locally unstable region (LU region) given by the Weyl curvature tensor becomes an important tool to determine chaotic behaviors of a test particle.

Apart from the analysis of a local instability, the existence of an unstable periodic orbit (UPO) and homoclinic orbit around it are also known to be important causes of chaotic behavior in the Newtonian dynamics. The location of the UPO depends on the energy and angular momentum of the test particle. The existence of a UPO is determined not only by the background spacetime but also by the orbit elements of a particle. This indicates that a homoclinic tangle causes strong chaos independently of the spacetime curvature. Several authors have shown that strong chaos occurs in a perturbed Schwarzschild spacetime via the homoclinic tangle around the UPO\([4, 5]\). We will also study a homoclinic tangle of test particle motion in static axisymmetric spacetimes.

In section 2, we will quickly review a formalism for obtaining the eigenvalues of the Riemann or Weyl tensor in static axisymmetric vacuum spacetimes. In section 3, we numerically study a correlation between chaos and the distribution of positive eigenvalues, and show that a local instability measured by the Weyl curvature can be used as a sufficient condition for chaos, at least quantitatively. In section 4, we will show that the UPO and a homoclinic orbit also play key roles in causing strong chaos. Finally, we will give our conclusions and some remarks in section 5.

### 2 Local instability and eigenvalues of the Weyl tensor

In GR, the orbit of a test particle is described by a geodesic and it is plausible to examine first whether or not chaos can be predicted by the sign distribution of a background curvature.

\(^*\)Here Riemannian and pseudo-Riemannian denote the manifold whose signature is \((+,+,+,...)\) and \((-,+,+,...)\), respectively.
The equation of a geodesic deviation $n^\mu$ is given by the Riemann curvature tensor $R^\mu_{\nu\rho\sigma}$ as

$$\frac{D^2 n^\mu}{D\tau^2} = -R^\mu_{\nu\rho\sigma} u^\nu n^\rho u^\sigma,$$

(2.1)

where $u^\mu$ is the 4-velocity of a test particle and $\tau$ is its proper time. Here we use that $n^\mu$ which is perpendicular to $u^\mu$, i.e., $n_\mu u^\mu = 0$. This condition will be held at any time if we choose the $u$ and $n$ which satisfy the condition $n_\mu u^\mu = 0$ and $(Dn_\mu / D\tau) u^\mu = 0$ initially. From (2.1), we find the evolution equation for the norm of the deviation vector, $\|n\|$, as

$$\frac{d^2}{d\tau^2} \|n\| = K(u, n) \|n\| + \frac{1}{2\|n\|} \left\| n \times \frac{Dn}{D\tau} \right\|^2,$$

(2.2)

where we have used the notation $\|V\| \equiv \left( |V_\mu V^\mu| \right)^{1/2}$ for a 4-vector $V^\mu$. The sectional curvature of a two-surface spanned by $u^\mu$ and $n^\mu$ is \(\hat{K}(u, n)\) \(\equiv -R(u, n, u, n) / \|n\|^2\), \(^{†}\)

(2.3)

and $(n \times Dn/D\tau)_\mu = u^\alpha \epsilon_{\alpha\mu\rho\sigma} n^\rho Dn_\sigma / D\tau$.

\(^{†}\)From (2.2), we see that a local instability of geodesic is determined by the sign of the sectional curvature $K(u, n)$. If $K(u, n)$ is positive at some point in a configuration space, the geodesic deviation $n^\mu$ becomes exponentially unstable there. The “averaged” value of $K(u, n)$ coincides with the scalar curvature $R$\(^{[17]}\). This indicates that a positive scalar curvature may correspond to a locally unstable geodesic. However, if the dimension of the manifold is greater than 2, the condition $R > 0$ does not mean that all sectional curvatures are positive. Then we cannot conclude that such an orbit is unstable in any direction. Moreover, in the case of geodesic in 4-dimensional spacetime, the 4-velocity $u^\mu$ is timelike at each tangent space. Since we cannot choose any pair of 4-vectors $(u, n)$ at each tangent space, the average for all directions seems to be useless. We have to find an alternative way to judge the connection between chaos and curvature.

Here we utilize the eigenvalues of the Riemann tensor which was proposed by Szydlowski et al\(^{[17]}\). We define a bivector $S_A(A = 1, \ldots, 6)$, which corresponds to

$$S_{\mu\nu} \equiv u_{[\mu} n_{\nu]} = (u_\mu n_\nu - u_\nu n_\mu) / 2,$$

(2.4)

and describe the Riemann tensor in a bivector formalism as $R^A_B(A, B = 1, \ldots, 6)$\(^{[18]}\). For a given point in a configuration space, $K(S)$ has in general six critical values. That is,

$$\partial K(S) / \partial S^A = 0 \quad \text{for} \quad A = 1, \ldots, 6,$$

(2.5)

if and only if $S^A$ satisfies

$$R^A_B S^B = \kappa S^A,$$

(2.6)

There are six eigenvalues of $R^A_B$ and corresponding eigenvectors, $\kappa$ and $S^A$. The critical value of $K(S)$ is equal to one of the eigenvalues in the direction of its eigenvector. The sectional

\(^{†}\)Here we have used the notation $R(u, v, m, n) \equiv R_{\nu\rho\sigma\mu} u^\nu v^\rho m^\sigma n^\sigma$. The sign difference with the definition of $K$ from (1.7) (and (50) in \(^{[17]}\)) comes from the Lorentzian signature of spacetime-manifold in GR. In our previous papers in the Proceedings\(^[3]\), we made a mistake in its sign, but there is no change in our results.
curvature in any direction is composed of a of the eigenvalues. Therefore, the eigenvalues of
the Riemann tensor may determine a locally unstable geodesic.

In the vacuum spacetime, the Riemann tensor $R_{\mu\nu\rho\sigma}$ coincides with the Weyl tensor
$C_{\mu\nu\rho\sigma}$, which is decomposed into two $3 \times 3$ symmetric matrices, a diagonal ‘electric part’ $\mathcal{E}$
and an off-diagonal ‘magnetic part’ $\mathcal{H}$ as (see [18])

$$C = \begin{pmatrix} \mathcal{E} & \mathcal{H} \\ -\mathcal{H} & \mathcal{E} \end{pmatrix}. \quad (2.7)$$

From the form of (2.7), six eigenvalues of the Weyl tensor are composed of three independent
 eigenvalues and their complex conjugates. If the spacetime is static, the six eigenvalues
 of $\mathcal{C}$ are real and degenerate into three, because the magnetic part $\mathcal{H}$ vanishes and the matrix $\mathcal{C}$ is symmetric.

In static axisymmetry vacuum spacetime, we have two commuting Killing vectors, $\partial/\partial t$
and $\partial/\partial \phi$. Then the metric is described as

$$ds^2 = -e^{2U}dt^2 + e^{-2U}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (2.8)$$

where $U$ and $k$ are functions depending only on $\rho$ and $z$. By using this coordinate, we can
describe the $3 \times 3$ matrix $\mathcal{E}$ as the following form,

$$\mathcal{E} = \begin{pmatrix} \mathcal{C}_1^1 & \mathcal{C}_1^2 & O \\ \mathcal{C}_2^1 & \mathcal{C}_2^2 & O \\ O & O & \mathcal{C}_3^3 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_4^4 & \mathcal{C}_5^4 & O \\ \mathcal{C}_5^4 & \mathcal{C}_5^5 & O \\ O & O & \mathcal{C}_6^6 \end{pmatrix}, \quad (2.9)$$

where the suffices $1, \ldots, 6$ denote the tetrad components $(\hat{i}\hat{\rho}), (\hat{i}\hat{z}), (\hat{i}\hat{\phi}), (\hat{\rho}\hat{\phi})$, and $(\hat{\rho}\hat{z})$, respectively. The three eigenvalues of $\mathcal{E}_i^a$ and those eigenvectors, $\kappa_i$ ($i = 1, \ldots, 3$) and $S_{(i)}^a$, are defined by

$$\mathcal{E}_i^a S_{(i)}^b = \kappa_i S_{(i)}^a \quad \text{for} \quad i = 1, \ldots, 3 \quad (2.10)$$

From the form (2.9), the component $\mathcal{C}_3^3 = \mathcal{C}_6^6$ is one of the eigenvalues, $\kappa_{03}^c$. We also find
the remaining eigenvalues as

$$\kappa_{01}^c = \frac{1}{2} \left[ \mathcal{C}_1^1 + \mathcal{C}_2^2 + \sqrt{(C_1^1 - C_2^2)^2 + 4(C_2^2)^2} \right],$$

$$\kappa_{02}^c = \frac{1}{2} \left[ \mathcal{C}_1^1 + \mathcal{C}_2^2 - \sqrt{(C_1^1 - C_2^2)^2 + 4(C_2^2)^2} \right]. \quad (2.11)$$

Since the trace of the Weyl tensor (and then the trace of the electric part $\mathcal{E}$) vanishes, the
sum of the eigenvalues $\kappa_{0i}^c$ also vanishes, i.e.,

$$\kappa_{01}^c + \kappa_{02}^c + \kappa_{03}^c = 0. \quad (2.12)$$

In this case, we can divide the spacetime first into two regions by the signature of the
eigenvalues. One is a region where two eigenvalues are positive and the rest is negative,
and the other is a region with two negative and one positive eigenvalues. We shall further

\[ \text{We use units of } G = c = 1, \text{ but we explicitly write } G \text{ or } c \text{ when it may help our discussion.} \]
classify the former case into two types, i.e., one is the region where \( \kappa_{c01} > 0, \kappa_{c02} < 0 \) and \( \kappa_{c03} > 0 \) (\([+,−,+]\)-region) and the other is the region where \( \kappa_{c01} > \kappa_{c02} > 0 \) and \( \kappa_{c03} < 0 \) (\([+,+,−]\)-region). We shall call the \([+,+,−]\)-region a locally unstable (LU) region, which becomes important for chaos. The reason is as follows: First, since positive eigenvalues contribute to a local instability, the cases with two positive eigenvalues are likely to be more unstable locally than the case with just one positive eigenvalue. Although there are two regions with two positive eigenvalues (the \([+,−,+]\), \([+,+,−]\)-regions), the latter one (LU region), in which \( \rho-z \) plane orthogonal to the two Killing directions becomes unstable seems to be important because an instability in a Killing direction may not play any role in chaotic behavior of the orbit. Secondly, the LU region does not appear in the Petrov type D spacetime, in which a test particle motion is integrable[19] and does not show any chaotic behavior, but the other type of region exists even in the type D solutions. Hence we expect that the LU region seems to be more important for a local instability than any other region. It will be confirmed by numerical analysis in the next section. In the next section, we will numerically study a test particle motion in several background spacetimes and show a close relation between the above-defined LU region and chaotic behaviors of test particles.

3 Numerical analysis of local instability

In this section we study a correlation between the LU region and chaos of a test particle. We will show that the LU region given by the Weyl tensor plays a key role in causing the chaotic behavior of test particles. We analyze the chaotic behavior of a test particle by using the Poincaré map and the Lyapunov exponent. To define the Lyapunov exponent in GR, we use the proper time of each geodesic as a time-flow parameter. (see also [3, 7] for another choice.) We also define a measure distance \( \Delta \) in a phase space as

\[
\Delta^2 = g_{\mu\nu}n^\mu n^\nu + g_{\mu\nu} \frac{Dn^\mu}{D\tau} \frac{Dn^\nu}{D\tau}.
\]

The proper time, \( \tau \), should be normalized by a typical time scale in the system, e.g., \( GM/c^3 \), where \( M \) is a mass of a localized object. We can easily show that (3.1) is positive definite along each geodesic, because both \( n^\mu \) and \( Dn^\mu/D\tau \) are spacelike. So in our case we do not face the problem which Biesiada et.al. pointed out when defining a measure distance in a phase space with a Lorentzian signature [13]. By using this measure, we calculate the maximum Lyapunov exponent, which is defined by

\[
\lambda = \lim_{N \to \infty} \frac{1}{N\delta \tau} \sum_{n=1}^{N} \ln[\Delta(n\delta \tau)],
\]

where \( \Delta(n\delta \tau) \) is the value of \( \Delta \) after \( n\delta \tau \) time evolution, which is normalized to a unit length after each time interval \( \delta \tau \) [20]. The reciprocal of \( \lambda \) can be utilized as a standard time scale where the effect of chaos becomes conspicuous. We have integrated both the geodesic and the deviation equations using the predictor-corrector and the Burlisch-Store integrators [21]. We have not found any remarkable difference between the two methods.

We also map the LU region in static axisymmetric vacuum spacetimes and examine the chaotic behavior of test particles passing through such a region by the Poincaré map. The
allowed region where a test particle can move is obtained from the super-Hamiltonian

\[ H = \frac{1}{2} g_{\mu \nu} \left( \frac{dx^\mu}{d\tau} \right) \left( \frac{dx^\nu}{d\tau} \right). \]

Since \( H \) is conserved and fixed on the value \(-1.0\), we find that the square of meridian velocity

\[ v_\ast^2 \equiv \frac{1}{2} \left( g_{11} \left( \frac{dx^1}{d\tau} \right)^2 + g_{22} \left( \frac{dx^2}{d\tau} \right)^2 \right) \]

\[ = \frac{1}{2} \left( \left( \frac{d\rho}{d\tau} \right)^2 + \left( \frac{dz}{d\tau} \right)^2 \right) \exp (-2U + 2k) \]

for the particle with energy \( E \) and angular momentum \( L \) can be expressed as

\[ v_\ast^2(x, E, L) = \frac{(E^2 - V_{\text{eff}}^2(x, L))}{2\| \partial / \partial t \|^2}, \]

(3.4)

where \( x = (\rho, z) \) and \( \| \partial / \partial t \|^2 \equiv -g_{00} \). \( V_{\text{eff}}^2 \) is the effective potential for the particle with the angular momentum \( L \), which is defined by using the norms of two Killing vectors, \( \partial / \partial t \) and \( \partial / \partial \phi \) as follows.

\[ V_{\text{eff}}^2(x, L) \equiv \| \partial / \partial t \|^2 \left( 1 + \frac{L^2}{\| \partial / \partial \phi \|^2} \right), \]

(3.5)

where \( \| \partial / \partial \phi \|^2 \equiv g_{33} \). Since \( v_\ast^2 \) and \( \| \partial / \partial t \|^2 \) are positive in (3.4), the allowed region \( D_{\text{eff}} \) for the particle with \( E \) and \( L \) is given as

\[ D_{\text{eff}} \equiv \{ x | V_{\text{eff}}^2(x, L) \leq E^2 \}. \]

(3.6)

If \( D_{\text{eff}} \) is compact, i.e., if a test particle is trapped and never falls into a singularity, we call the particle geodesic a bound orbit. We discuss a chaotic behavior only for this bound orbit.

### 3.1 Numerical results

The methods to find exact solutions in a static axisymmetric vacuum case are well-developed [22]–[25]. We can easily construct exact solutions with any mass distribution. In this paper, we use several known exact solutions to study a test particle motion.

First we shall analyze the Zipoy-Voorhees (ZV) solution [23]. The functions in a spacetime metric (2.8) are

\[ U = \frac{\delta}{2} \ln \left( \frac{r_1 + r_2 - 2m}{r_1 + r_2 + 2m} \right) \]

\[ k = \frac{\delta^2}{2} \ln \left( \frac{(r_1 + r_2 + 2m)(r_1 + r_2 - 2m)}{4r_1 r_2} \right), \]

(3.7)

(3.8)

where \( r_1 = [\rho^2 + (z - m)^2]^{1/2} \), \( r_2 = [\rho^2 + (z + m)^2]^{1/2} \) and \( m \) is a mass parameter. The singularity exists between \( \pm M / \delta \) on \( z \) axis. Here \( M \) is the gravitational mass of the singularity which is given as \( m \delta \) by using the parameters \( m \) and \( \delta \). The quadrupole moment \( Q \)
is also given as \( m^3 \delta^3 (1 - 1/\delta^2)/3 \). In the case of \( \delta = 1 \), \( Q \) vanishes and the spacetime just becomes the Schwarzschild solution. For the Schwarzschild spacetime, no LU region appears because two eigenvalues of \( \mathcal{E} \) are degenerate and negative. This comes from the fact that the Schwarzschild spacetime belongs to the Petrov type D. In the limit of \( \delta \to \infty \) fixing \( M = m \delta \), the singularity shrinks to the origin and the spacetime coincides with the Curzon spacetime \([23]\), with metric functions, \( U \) and \( k \), given by

\[
U = -G \frac{M}{r}, \quad k = -\frac{1}{2} G^2 \frac{M^2 \rho^2}{r^4},
\]

where \( r \equiv (\rho^2 + z^2)^{1/2} \). For the case of \( \delta \neq 1 \), the eigenvalues of \( \mathcal{E} \) are not degenerate, but both of the eigenvalues, \( \kappa_{01} \) and \( \kappa_{02} \) are not positive and no LU region appears (Fig.1). We have found that for any bound orbit both in the ZV spacetime and in the Curzon spacetime, the sectional curvature is always negative and no chaotic behavior of the orbit is seen at least from our analysis by use of the Lyapunov exponent and the Poincaré map.

Secondly we examine a system with \( N \)-point Curzon-type singularities on the symmetric axis \([24]\), whose metric functions, \( U \) and \( k \), in \((2.8)\) are

\[
U = -G \sum_{i=1}^{N} \frac{M_i}{r_i},
\]

\[
k = -\frac{1}{2} G^2 \sum_{i \neq j} M_i M_j \left[ \frac{r_i - r_j}{b_i - b_j} \right]^2 - \frac{1}{2} G^2 \sum_{i=1}^{N} \frac{M_i^2 \rho^2}{r_i^4},
\]

where \( M_i \) and \( b_i \) denote the mass and the position parameters of \( i \)-th singularity on z axis, respectively, and \( r_i \equiv [\rho^2 + (z - b_i)^2]^{1/2} \). We call this solution the \( N \)-Curzon spacetime because the \( N = 1 \) case corresponds to the Curzon solution.

We have first analyzed the case with a reflection symmetry on the equatorial plane. For the \( N \)-Curzon spacetime, LU regions exist and some bound orbits of test particles intersect within them. As we see from Fig.2, in 2-Curzon spacetime, the LU region appears on the equatorial plane. If the energy of the orbit gets larger than that at the UPOs (\( E_{\text{UPO}} \)), it fails to be bounded, so we just increase the energy up to \( E_{\text{UPO}} \) for a given angular momentum \( L \). When the energy approaches to \( E_{\text{UPO}} \) and it eventually exceeds some critical value \( E_{\text{cr}} < E_{\text{UPO}} \), tori in the phase space are broken (Fig.3), and strong chaos occurs in those orbits. On the other hand, the orbit which does not pass the LU region does not show any chaotic behavior, even if the orbit has the same energy as the chaotic one (Fig.4).

In 3-Curzon spacetime, where one of the point-mass singularities is at the origin and the other two are on the z axis at the same distance from the origin, two LU regions appear between those singularities. The result is similar to the case of 2-Curzon spacetime. Some orbit departs from the equatorial plane and eventually approaches to the LU region. Then, just after it crosses the LU region, the torus begins to break and the orbit becomes strongly chaotic.

\*In \([24]\), the last term in \((3.10)\) seems to be missed, otherwise the vacuum Einstein’s equations are not satisfied.
chaotic (Fig.5). On the other hand, no bound orbit not passing through the LU region shows any chaotic behavior (Fig.6).

Next we consider the spacetime with $N$ ZV-type singularities put on $z$ axis in order to see how the shape of singularity effects on the chaotic property of geodesics. It is known that a solution of $N$ ZV-type singularities located on the $z$ axis is derived by the inverse scattering method, where $N$ is an arbitrary natural number [25]. The most general form of the $N$-soliton solution with real-poles is given by

$$U = \frac{1}{2} \ln \left( \frac{i^N \zeta_1 \zeta_2 \cdots \zeta_N}{\rho^N} \right)^{\delta},$$  

$$k = \frac{1}{2} \ln \left[ \frac{\rho^{N/2} \prod_{n=1}^{N} (\zeta_n - \zeta_1)^2}{\prod_{n>l} (\rho^2 + \zeta_n^2) \prod_{l} \zeta_l^{N-2} C^{(N)}} \right].$$  

(3.11)

where

$$\zeta_n = w_n - z \pm \sqrt{(w_n - z)^2 + \rho^2}$$  

$$C^{(N)} = 2^{N/2} (N-2) \prod_{n>l} (w_{2n-1} - w_{2l-1})^2 (w_{2n} - w_{2l})^2.$$  

(3.12)

In the case $N=2$, if $w_1$ and $w_2$ are parameterized as $-m$ and $m$ and the signs in $\zeta_1$ and $\zeta_2$ are chosen to + and −, respectively, the solution (3.11) is asymptotically flat and coincides with the ZV solution. When $N=4$, setting $w_1 = z_1 - m_1$, $w_2 = z_1 + m_1$, $w_3 = z_2 - m_2$ and $w_4 = z_2 + m_2$, and choosing the signs in $\zeta_1 \sim \zeta_4$ as (+, −, +, −), the solution (3.11) is asymptotically flat and corresponds to the spacetime which contains two ZV type singularities with masses $m_1 \delta$ at $z_1$ and $m_2 \delta$ at $z_2$ on $z$ axis. We call this 2-ZV solution.

Here we examined 2-ZV solution with two singularities at $\pm 2GM/c^2$ on the $z$ axis in the case of $\delta = 1.0$. In this case, LU region appeared on equatorial plane in the same way as 2-Curzon spacetime and some of the bound orbits passing through the LU region become strongly chaotic (see Fig.7).

From the above analysis, the appearance of the LU region seems to be closely related to the chaotic behavior of geodesics. We may conclude that passing through the LU region is necessary to cause the chaotic motion of geodesic and the upper bound of $\lambda$ is determined by the value of positive eigenvalue $\kappa_1$ and $\kappa_2$ in an LU region.

### 3.2 Lyapunov exponent and Curvature

So far we have shown that the LU region plays a crucial role in causing strong chaos in a static axisymmetric spacetime, qualitatively. However, in order to apply chaos to real astrophysical phenomena, it is necessary to extract some quantitative information about the strength of chaos from the LU region. One way is to estimate a Lyapunov exponent $\lambda$. We show our results in Fig.8 for 2-Curzon spacetime and Table 1 for 2-ZV spacetime. In Fig. 8, we choose the same situation and initial condition for the orbit as in Fig.2, 3, and find the result that the inverses of Lyapunov exponents ($\lambda^{-1}$) are about $33\tau_P$ and $4.7\tau_P$ for (b).
and (c), respectively, where \( \tau_p \) is the average period around the symmetric axis for each particle orbit. For the 2-ZV spacetime, we studied two additional values \( \delta \) \((\delta = 0.65, 10.0)\) as well as the case of \( \delta = 1.0 \) depicted in Fig. 7, and found that as \( \delta \) increases, the value of \( \lambda \) becomes smaller and smaller (see Table 1). In this case, for each \( \delta \), \( \lambda^{-1} \) becomes (a)10.8\( \tau_p \), (b)9.4\( \tau_p \) and (c)8.0\( \tau_p \), respectively. These results indicate that the timescale on which chaos becomes conspicuous is about several Keplerian rotations. So the chaos we found may be strong enough to be potentially observed around compact objects in which the effects of GR are important.

As is shown in the above, a local instability is closely related to the occurrence of chaos, giving us the prospect of evaluating the Lyapunov exponent \( \lambda \) in terms of the curvature. Here we will examine it from a quantitative viewpoint. Since the local instability is determined by the sectional curvature \( K \) of \( \mathcal{K}(x, u, n) \) as we can see from (2.2), \( \lambda \) for chaos around LU region may be determined by some average of \( \mathcal{K}(x, u, n) \) as

\[
\lambda \approx \sqrt{\langle K \rangle},
\]

where \( \langle K \rangle \) is the averaged value of \( \mathcal{K}(x, u, n) \) inside the bound region \( D_{\text{eff}} \) which overlaps an LU region. In vacuum spacetime, the sectional curvature \( \mathcal{K}(u, n) \) at each point \( x \) for given four velocity \( u = u^{(a)}e_{(a)} \) and deviation vector \( n = n^{(a)}e_{(a)} \), can be expressed as a linear combination of \( \kappa_{0i} \), as follows

\[
\mathcal{K}(x, u, n) = \sum_{i=1}^{3} A_i(u, n)\kappa_{0i}(x),
\]

where

\[
A_i(u, n) \equiv \frac{[(u^{(0)}n^{(i)} - u^{(i)}n^{(0)})^2 - (u^{(j)}n^{(k)} - u^{(k)}n^{(j)})^2] / \|n\|^2}{\|n\|^2},
\]

\([i, j, k] \) is a permutation of \((1, 2, 3)\). Since we can eliminate \( \kappa_{03} \) by the condition \((2.12)\), the value of \( \langle K \rangle \) is determined by some average of \( \kappa_1 \) and \( \kappa_2 \). Here we utilize the geometrical mean, defined as

\[
\langle \kappa \rangle \equiv \sqrt{\kappa_1\kappa_2(x)},
\]

since \( \langle \kappa \rangle \) vanishes at the boundary of the LU region and takes the maximum value in the middle of the region (see Fig.6). Estimating \( \lambda \) from \( \langle \kappa \rangle \) as defined in \((3.14)\) seems to be a reasonable rough estimate, since other averages, such as the arithmetic mean of \( \kappa_1 \) and \( \kappa_2 \) give the same order of magnitude as \( \langle \kappa \rangle \) at least in the vicinity of LU region. We estimate the average values of \( \langle \kappa \rangle \) in the intersection of the LU region and \( D_{\text{eff}} \) as \( \sqrt{\langle \kappa \rangle_{\text{max}}} \), where \( \langle \kappa \rangle_{\text{max}} \) is the maximum value of \( \langle \kappa \rangle \) in the intersection of the two regions. In Table 1, we compare \( \sqrt{\langle \kappa \rangle_{\text{max}}} \) with the actual value of \( \lambda \) calculated for several chaotic orbits passing through the LU region. Unfortunately, the value of \( \lambda \) differs by approximately one from an averaged value \( \sqrt{\langle \kappa \rangle_{\text{max}}} \) for each \( \delta \) in Table 1. This deficit comes from the fact that the chaotic orbit not only moves on the meridian plane, but also rotates around the \( z \) axis. This rotation restricts the movement of the geodesic on the meridian plane according to the condition, \( u_z^2 \geq 0 \) in \((3.4)\). In fact, we do not see any chaotic motion for a geodesic passing through an LU region, if we choose the \( E \) and \( L \) so as to make the geodesic a stable periodic orbit. We also showed in Fig.3 that for the fixed value of \( L \), the chaos becomes stronger and stronger around an LU region as \( E \) increases and approaches the value for the
UPO. From \((3.14)\), these results mean that chaos becomes more and more conspicuous as the meridian velocity \(v_s\) increases. So it is natural to expect that \(\lambda\) can also be affected by \(v_s\). Since each component \(A_i(u, n)\) in \((3.14)\) is proportional to the combination of two components of \(u\), we utilize \(v_s^2\) and compare the product of its averaged value and \(<\kappa>\) with the numerical value of \(\lambda\). Here we take the maximum value of \(v_s\), i.e., \(v_{s,\text{max}}\), as the average value of \(v_s\) inside the part of the LU region that overlaps \(D_{\text{eff}}\).

As shown from Table 1, our estimate agrees well with the real Lyapunov exponent \(\lambda\) for the most chaotic orbit for each value of \(\delta\). As far as asymptotically flat vacuum spacetimes are concerned, the bound condition \((3.6)\) restricts the value of \(v_{s,\text{max}}\) to that less than the velocity of light. It reduces the estimate of \(\lambda\) for each geodesic to almost one tenth of \(\sqrt{<\kappa>_{\text{max}}}\), which is determined just from the curvature itself. It seems that \(\sqrt{<\kappa>_{\text{max}}}\) gives at most the upper limit of the Lyapunov exponent for the relativistic orbit passing through an LU region. The \(v_s/c\) dependence in \(\lambda\) is also important when characterizing the chaotic motion of geodesic in GR, in contrast to free particle motion in Newtonian mechanics, because the velocity dependence of local instability in GR completely disappears in the Newtonian limit, as we will show in the next subsection. So although our quantitative estimate of \(\lambda\) is rather empirical, the \(v_s\) dependence of sectional curvature \(K(u, n)\) seems to be essential, in order to make the criterion of local instability work well in GR.

### 3.3 Newtonian Dynamics vs. General Relativity

The results in §3.1 show that the LU region plays a crucial role in causing strong chaos at least in the vacuum spacetime, but using similar criteria to assess the presence of chaos by analyzing local instability in Newtonian dynamics is not always successful, as mentioned in §1. Here we shall compare our criterion in GR and those in Newtonian theory and show the reason why our criterion does work.

There are two criteria for local instability in Newtonian theory as discussed in §1: One is by the eigenvalues of the tidal matrix, and the other is the curvature of a fictitious space obtained by a conformal transformation. In §1, we pointed out several obstacles in the latter approach. Although our approach is technically quite similar to this, we do not face such obstacles, because we have a geodesic equation without conformal transformation and study test particle motion in a physical spacetime. The former approach in Newtonian theory is physically much closer to ours in GR, as we will discuss below.

To show the relation between a criterion in the Newtonian theory and that in GR, we take the Newtonian limit of the deviation equation \((2.1)\). If the spatial components of the 4-velocity \(v^i \equiv u^i/u^0\) are much less than that of light \(c\), we can approximate the 4-velocity as \(u^i \approx (c, v^i)\). The deviation equation \((2.1)\) becomes

\[
\frac{D^2n^i}{D\tau^2} = -c^2R^i_{0j0n}n^j - cR^i_{0jk0}v^k - cR^i_{kj00}v^k - R^i_{ijkl}n^k v^j v^l.
\]

We also assume that the spacetime is approximately a flat Minkowski spacetime with small perturbations, i.e., the metric is expressed as \(g_{ab} = \eta_{ab} + h_{ab} (h_{ab} \ll 1)\), where \(\eta_{ab}\) denotes the Minkowski metric. The Riemann tensor \(R_{\alpha\beta\gamma\delta}\) is now

\[
R_{\alpha\beta\gamma\delta} = \frac{1}{2}(\partial_\beta\partial_\gamma h_{\alpha\delta} + \partial_\alpha\partial_\delta h_{\beta\gamma} - \partial_\alpha\partial_\gamma h_{\beta\delta} - \partial_\beta\partial_\delta h_{\alpha\gamma}).
\]
Introducing the Newtonian potential $V$ as $V \equiv -c^2h_{00}/2$, $R_{ij}^0$ becomes $\partial^i \partial_j V/c^2$. To the leading order of $h_{ab}$ and $v^i/c$, the terms with $v^i$ in (3.16) do not appear and Eq.(3.16) coincides with the tidal acceleration equation (1.1) in the Newtonian theory. To the same order, the sectional curvature (3.14) becomes

$$K(x, u, n) = -[(n^1)^2k_1 + (n^2)^2k_2 + (n^3)^2k_3]/c^2,$$

where $k_i (i = 1 \sim 3)$ are the eigenvalues of $V_{ij}$ and correspond to $-c^2k_{0i}^\perp$.

From these equations, we find that the deviation equation in the Newtonian limit becomes adiabatic and lose any information of the velocity of the orbit. For an axisymmetric system in the Newtonian theory, we can also define an LU region as we did in GR, i.e., a region where both $k_1$ and $k_2$ are negative. Then, for deviations perpendicular to the symmetric direction of $\partial/\partial\phi$, all orbits become locally unstable in an LU region regardless of those velocities because $n^3 = 0$ in (3.18) for such deviations. It seems that the same criterion for chaos applies in the Newtonian case.

However, this is not true. For example, we can find an LU region in the Newtonian model where two gravitational point sources are fixed on z-axis. Even if some bound orbit passes through the LU region, (in fact, we can find such an orbit), no chaotic behavior should be found because of the integrability of the system [26]. The reason why the LU region does not predict strong chaos is that its adiabatic nature cuts the connection between local instability described by an LU region and the global instability of a bound orbit. This is confirmed from our previous analysis in Sec.3.2, in which the Lyapunov exponent $\lambda$ depends on a particle velocity. Our result is also consistent with the failure of the use of the eigenvalues of the tidal matrix as a criterion for chaos in the Newtonian theory.

4 Homoclinic tangle around UPO and chaos in GR

In the previous section, we have shown that the LU region plays an important role in causing chaotic behavior of a test particle. However, in 2-Curzon spacetime, we have also found some orbits whose chaotic behavior cannot be explained by the passage through the LU region. In Fig.8, where two singularities with different mass ($M$ and 0.5$M$) are put at $z = -4M$ and 4$M$ on the z axis, we find chaotic behavior for test particles which do not pass through the LU region. In 2-ZV spacetime, we have found the similar examples. Hence, we seem to have another type of chaos, which is not understood by the criterion discussed in Sec.3.

Here we will explain such exceptional cases from another point of view, that is, the existence of UPO and a homoclinic tangle around it. In GR, a single point-mass system such as the Schwarzschild spacetime usually has a UPO. Around the UPO in the Schwarzschild spacetime, there exists an orbit of a test particle which gradually departs from the UPO point and eventually returns to it with infinite time interval, if the angular momentum is appropriately chosen. This orbit is called a homoclinic orbit and is known to play an important role in causing strong chaos. A heteroclinic orbit may also be defined as the orbit which gradually departs from a UPO point, but approaches another UPO point with infinite time interval.

†The sign difference comes from the fact that we treat the Euclidean space in the Newtonian theory, while the Lorentzian spacetime in GR.
For spacetimes with a reflection symmetry on the equatorial plane, we find a homoclinic orbit which departs from and returns to the UPO on the plane. This orbit is not chaotic. However, if some perturbations by other gravitational sources break such a reflection symmetry, the homoclinic (or heteroclinic) orbit no longer exists, and stable and unstable manifolds starting with the UPO point could be split and tangled in a complicated way. This splitting of the homoclinic orbit is called a homoclinic tangle and is known to cause strong chaos [27].

For example, Bombelli and Calzetta [4] showed by using the Melnikov method that when the Schwarzschild spacetime is perturbed by linear gravitational waves, strong chaos in a test particle motion appears through the tangle of a homoclinic orbit around a UPO point. Moeckel also showed by the similar method that strong chaos appears in the Schwarzschild spacetime with different perturbations, whose Newtonian limit reduces to the famous Hill’s problem [5].

In our case, if the background spacetime has a reflection symmetry as discussed in the previous section, there exists a homoclinic orbit on the equatorial plane. However, for the spacetime without a reflection symmetry, a homoclinic orbit does not exist and stable and unstable manifolds starting with the UPO point are tangled. Such a tangle is realized by breaking a reflection symmetry. In fact, as we have shown in Fig.8, the orbits become strongly chaotic even if they do not pass through any LU region.

We can easily distinguish this type of chaos from the previous one studied by a local instability. Choosing the equatorial plane as the Poincaré section, we find that a homoclinic orbit, if it exists, will appear on the boundary of a family of tori. Then we can easily check the occurrence of a homoclinic tangle by examining whether or not the boundary of tori is broken. For the case with a reflection symmetry, the boundary torus in the Poincaré map is not broken as seen in Fig.2, while in Fig.8, we see the boundary torus is broken, which means that a homoclinic tangle occurs in the case without a reflection symmetry.

In general, a homoclinic tangle requires periodic perturbations around a homoclinic orbit, because a stable manifold tangles with an unstable one infinitely through a periodicity of those manifolds. However our case does not need a periodicity of perturbations, because a homoclinic orbit itself is periodic around the z axis even before it is perturbed because of the axisymmetry. So it is enough to break the reflection symmetry on the z axis in order to cause an infinite tangle of stable manifold with unstable one, as we find in our numerical results.

A homoclinic tangle strongly depends on the properties of each geodesic such as the energy or angular momentum. So it is plausible that in this case the background curvature, which is independent of the properties of each geodesic, no longer determines the chaotic behaviors of a test particle. This may be the reason why we have seen strong chaos outside the LU region.

5 Concluding Remarks

We have examined two criteria for the chaotic motion of a test particle in static axisymmetric spacetime. We find that unlike Newtonian dynamics, there is a close relation in GR between chaos and the curvature of background spacetimes. We have used the eigenvalues of the Weyl tensor instead of the scalar curvature as the criterion for chaos, because it can be applied
even in the vacuum spacetime. In a point-singularity system, the LU region does not appear and no chaotic behavior is seen regardless of the “shape” of the singularity. On the other hand, in the plural-singularity systems such as $N$-Curzon spacetimes or 2-ZV spacetime, the LU regions appear between those singularities and we find that some orbits become strongly chaotic there, if the energy increases beyond some critical value.

The Petrov type D spacetime is integrable and the LU region does not exist for such a spacetime. From our analysis, we can conclude that the existence of the LU region may be closely related to the non-integrability of the spacetime as well as chaotic behavior of a test particle.

However we know that only the passing through this region does not determine the fate of the orbit completely. As we showed in Fig.2, the behavior of an orbit crossing an LU region depends strongly on its energy $E$ and angular momentum $L$. If the energy $E$ is less than some critical value, the orbit does not become chaotic even in an LU region. This is because the orbit does not move on the meridian plane orthogonal to the two Killing directions, while the LU region causes an instability in the direction parallel to this plane. In fact, a stable circular orbit can exist in an LU region for appropriate values of $E$ and $L$. It is never chaotic because it is strictly restricted to move on the meridian plane. This can be explained by the fact that a meridian motion determined by $v_\star$ vanishes from (3.4) for a stable circular orbit. It seems that the LU region can well cause chaotic motion only when the orbit has a high enough meridian velocity, $v_\star$ to be able to move freely on meridian plane. Our numerical results in sec.3.2 give empirical support for the $v_\star$ dependence of $\lambda$.

As we showed in Sec.3.3, the $v_\star$ dependence of $\lambda$ is also important when comparing chaotic motion of geodesics in GR to free particle motion in Newtonian mechanics. In the Newtonian case, the $v_\star$ dependence on the sectional curvature disappears and local instability cannot be used to estimate chaos. So $v_\star$ seems to play a key role in making LU region function well as the criterion for chaos in GR.

Although our criterion is a sensitive test for the occurrence of chaos, it is no more than a sufficient condition. In fact, we have also shown that the existence of the UPO and the homoclinic tangle around the UPO is an important cause of strong chaos. In particular, when a homoclinic tangle appears through perturbations, strong chaos occurs around a UPO even if the orbit does not pass through the LU region. In order to make the criterion more reliable, it seems to be necessary to include information about each geodesic, such as $E$ and $L$. The work of Szydlowsky et.al. develops this idea. They devised a way to judge local instability by utilizing the eigenvalues of the Riemann tensor for the geodesic after the conformal transformation, including $E$ [28]. However, this kind of approach that uses a conformal transformation has difficulties, as we pointed out in introduction. In fact, U. Yurtsever showed that the curvature determined after the conformal transformation does not always accurately predict the presence of chaos in static spacetime. So the problem will have to be considered further in the future.

From the astrophysical point of view, it is likely that such a homoclinic tangle could occur around compact objects with UPOs through perturbations by gravitational waves or other gravitational sources such as stars or galaxies. However, our results indicate that we have to take into account the curvature effect, which is characterized by LU region in axisymmetric static spacetimes, independently as a source of strong chaos. The reciprocal of the maximum Lyapunov exponent we got in these kinds of chaos is about several rotations.
of the Keplerian orbit. So the chaos characterized by curvature may also be expected to be observed around the compact objects in which the effect of GR is important. The general relativistic chaos determined by both of the two causes could be necessary to explain some unknown relativistic phenomena in astrophysics in the future.

In this paper we restrict our analysis to the case of static axisymmetric vacuum spacetimes. In this case, the Riemann tensor coincides with the Weyl tensor and the six eigenvalues degenerate into three, which makes our analysis simple. It is natural to ask whether or not our results hold for more general cases. If spacetime is not empty, the Riemann tensor is not the same as the Weyl tensor by the effect of local matter fluid and six eigenvalues become independent. Although the curvature analysis become more complicated, the way to analyze it is straightforward. We will present our analysis for the static case with this matter effect and show that our present results are still valid in the next paper. For stationary spacetimes with rotation, however, it turns out to be much more difficult to analyze, because the curvature $\mathcal{R}$ is not symmetric and its eigenvalues become complex. We are also very interested in non-stationary system such as a coalescing binary system, where we cannot introduce an effective potential. In spite of those difficulties, we believe that a sectional curvature and a homoclinic tangle are closely related to strong chaos. Such an analysis will be undertaken in the future.

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Table 1:

| case | $\delta$ | $Q/M^3$ | $\lambda/M^{-1}$ | $v_{\ast,\text{max}}\sqrt{\kappa_{\text{max}}}/M^{-1}$ | $v_{\ast,\text{max}}(E, L)$ | $\sqrt{\kappa_{\text{max}}}/M^{-1}$ |
|------|---------|--------|----------------|------------------------------------------------|-----------------------------|---------------------------------|
| (a)  | 0.65    | -0.4556| 4.6537$x\times10^{-3}$ | 4.3574$x\times10^{-3}$ | 5.3412$x\times10^{-2}$ | 8.1580$x\times10^{-2}$ |
| (b)  | 1.0     | 0.0    | 4.2510$x\times10^{-3}$ | 3.9369$x\times10^{-3}$ | 5.6232$x\times10^{-2}$ | 7.0010$x\times10^{-2}$ |
| (c)  | 10.0    | 0.3333 | 3.7543$x\times10^{-3}$ | 2.7760$x\times10^{-3}$ | 4.2481$x\times10^{-2}$ | 6.5347$x\times10^{-2}$ |

Figure Captions

FIG 1: The $[+,-,+]$-region (shaded) of the Curzon spacetime with mass parameter, $M$, and the bound region (dotted) for a test particle with energy, $E^2 = 0.895635 \, (\mu c)^2$, and angular momentum $L = 3.54 \, G\mu M/c$, where $\mu$ denotes the rest mass of the test particle. Although the $[+,-,+]$-region surrounds the singularity (black dot), no LU region appears.

FIG 2: (a) The $[+,-,+]$-regions (shaded) and LU region (lightly shaded) of the 2-Curzon spacetime with two equal masses, $M$, located at $\pm 2GM/c^2$ on the $z$ axis (black dots) and the bound region (dotted) of a test particle with the angular momentum $L = 6.8 \, G\mu M/c$ and energy, $E^2 = 0.88957 \, (\mu c)^2$ corresponding to $E_{\text{UPO}}$. The LU region on the equatorial plane intersects with the bound region (dark shaded). (b) The Poincaré map of the bound orbits of the test particle in 2-Curzon spacetime with the same parameters in (a). We choose the equatorial plane as the Poincaré section. The initial momenta are all $p_0 = 0$ except for the outermost one, which is $p_{\rho}/p_{\rho} = 0.01$. The initial positions are $z_0 = 0, \rho_0 = 3.13, 3.0(\text{chaotic}), 3.25(\text{chaotic}), 3.4, 2.83 \, GM/c^2$ from the innermost. Almost all of Tori are broken strongly.
FIG 3: The Poincaré map of the bound orbits of the test particle in 2-Curzon spacetime with the same parameters as those in Fig.2, except for the energy. We choose the equatorial plane as the Poincaré section. We increase the energy little by little up to the value of $E_{UPO}(\approx 0.88957\mu c^2)$ as $(E/\mu c^2)^2 = 0.8892[(a)], 0.88935[(b)], 0.88957[(c)]$ in the Fig.2. The initial data for each tori in (a) are $p_0^0 = 0, z_0 = 0$ and $\rho_0 = 3.0, 3.1, 3.2, 3.3$ $GM/c^2$ from the innermost. The outermost torus corresponds to $p_0^0/p_0^0 = 0.01, z_0 = 0$ and $\rho_0 = 3.0$ $GM/c^2$. In (b), the two small tori correspond to the initial conditions $p_0^0 = \pm 0.01\mu c, z_0 = 0$ and $\rho_0 = 2.97$ $GM/c^2$. Starting from the inside, the initial data for the Poincaré maps are $p_0^0 = 0, z_0 = 0$ and $\rho_0 = 3.0$ $GM/c^2$. For (c), the initial momenta are all $p_0^0 = 0$ except for the outermost one, which is $p_0^0/p_0^0 = 0.01$. The initial positions are $z_0 = 0, \rho_0 = 3.13, 3.0(chaotic), 3.25(chaotic), 3.4, 2.83$ $GM/c^2$ from the innermost. Tori are broken more strongly as the energy increases.

FIG 4: The Poincaré map for the orbits of a test particle in the 2-Curzon spacetime with two equal mass parameter $M$ located at $\pm 2GM/c^2$ on z axis. The energy and angular momentum are fixed as $E^2 = 0.913 (\mu c^2)^2$ and $L = 6.94 G\mu$, respectively so as to satisfy the condition that only the orbit corresponding the most outer torus passes the LU region. The equatorial plane is chosen as the Poincaré section. The rests of the initial conditions are $p_0^0 = 0, z_0 = 0, \rho_0 = 15.0, 10.0, 4.0$ $GM/c^2$ from inside torus. The small torus around $(\rho, p^0) \sim (3 GM/c^2, 0)$ corresponds to the orbit with the initial position of $\rho_0 = 3.3$ $GM/c^2$. Only the torus which corresponds to the orbit passing through the LU region is broken.

FIG 5: Chaos in the 3-Curzon spacetime with three singularities of equal mass parameter, $M$, located at the origin and at $\pm 10GM/c^2$ on the z axis. Two LU regions appear between point singularities. The left figures [(a),(c),(e)] show the LU region and the time evolution of the locus of the bound orbit with $E^2 = 0.76 (\mu c^2)^2$ and $L = 5.5 G\mu M/c$. The initial conditions for all figures are $p_0^0/p_0^0 = 0.03, \rho_0 = 5.0$ $GM/c^2, z_0 = 0$. The right figures [(b),(d),(f)] are the Poincaré maps corresponding to the orbits on the left side. The orbit initially departs from the equatorial plane and eventually approaches the LU region. Then just after it crosses the LU region at the proper time $\tau \approx 100337.5$ $GM/c^3 (\sim 100 \times [the orbital period])$, the torus begins to break and the orbit becomes chaotic.
FIG 6: Chaos in the 3-Curzon spacetime with three singularities of equal mass parameter, $M$, located at the origin and at $\pm 10GM/c^2$ on the z axis (black dots). The LU-region is shown in (a) and (b) with the locus of the bound orbits with the same energy $E^2 = 0.81 (\mu c)^2$ and angular momentum $L = 6.0 G\mu M/c$ but with different initial conditions ($\rho_0 = 10.0 GM/c^2, z_0 = 0$, and $p_0^\rho = 0$ for (a) and $p_0^\rho/p_0^\rho = 0.6$ for (b)). The Poincaré maps corresponding to those orbits are shown in (c). The torus of the orbit which does not cross the LU region [(a)] is not broken, but it is broken for the orbit passing through the LU region [(b)].

FIG 7: Chaos in the 2-ZV spacetime with two singularities of equal mass parameter $M (m_1 = m_2 = M/\delta)$ located one the lines between $\pm GM/c^2$ and $\pm 3GM/c^2$ on z axis (black solid lines). The parameter $\delta$ is chosen to be 1.0, which means that each singularity corresponds to the Schwarzschild black hole, then $[+, -, +]$-regions do not appear around these singularities. The LU and the bound regions of the orbits are shown in (a), with $E^2 = 0.90913 (\mu c)^2$, $L = 6.9 G\mu M/c$. In (a), the bound region certainly intersects with the LU region. The Poincaré map of the chaotic orbit in the bound region is shown in (b). The rest of initial data for the orbit is $p_0^\rho = 0, \rho_0 = 2.6 GM/c^2, z_0 = 0$.

FIG 8: The Lyapunov exponents of the orbits with the same parameters as those in Figs.3(a) $\sim$ 3(c) and with the initial conditions $p_0^\rho = 0, \rho_0 = 3.0 GM/c^2$, and $z_0 = 0$. As we can see, the Lyapunov exponents for (b) and (c) converge to positive values such that the larger value corresponds to the larger energy of the particle.

FIG 9: The distribution of $<\kappa>$ on equatorial plane for 2-ZV spacetime with two singularities at $\pm 2GM/c^2$ on the z axis. We compared the $<\kappa>$ for the value of $\delta=0.65, 1.0, 10.0$. The region where $<\kappa> > 0$ coincides with an LU region. For each $\delta$, $<\kappa>$ has a peak in an LU region. As the value of $\delta$ increases, the peak becomes shorter and shorter.

FIG 10: Chaos in the 2-Curzon spacetime without a reflection symmetry. Two singularities with mass parameters $M$ and $0.5M$ are located at $4GM/c^2$ and $-4GM/c^2$ on the z axis (black dots), respectively. The LU and the bound regions of the orbit with $E^2 = 0.865 (\mu c)^2$, $L = 4.2 G\mu M/c$ are shown in (a), while (b) shows its Poincaré map. We chose the plane $z = 2GM/c^2$ as a Poincaré section. The initial conditions for the three large tori are $z_0 = 2.0 GM/c^2$ and $(\rho_0, p_0^\rho) = (8.0 GM/c^2, 0.075\mu c), (8.0 GM/c^2, 0.05\mu c), (7.5 GM/c^2, 0)$, and the three small tori correspond to the orbit with the initial data of $\rho_0 = 8.5 GM/c^2, z_0 = 2.0 GM/c^2$ and $p_0^\rho = -0.06\mu c$. The initial conditions for the chaotic orbits are $\rho_0 = 6.0 GM/c^2, z_0 = 2.0 GM/c^2$ and $p_0^\rho = 0$. Although the bound region does not intersect with the LU region, strong chaos appears, as seen in (b).
Table 1: Comparison of $v_{*,\text{max}}\sqrt{\left<\kappa\right>_{\text{max}}}$ with $\lambda$. The estimations are made for 2-ZV spacetime with two singularities at $\pm 2GM/c^2$ on the $z$ axis. $Q$ is a quadrupole moment and is given by $\delta$ as $\delta = (a)0.65, (b)1.0, (c)10.0$. In each case, $E$ and $L$ are determined so as to satisfy the condition, $E \sim E_{UPO}(L)$ and the bound region $D_{\text{eff}}$ overlaps an LU region, as $(E/(\mu c^2), L/(G\mu M/c)) = (a)(0.92713, 7.0), (b)(0.90913, 6.9), (c)(0.88959, 6.8)$. The rest of the initial condition are determined so that each geodesic becomes strongly chaotic as (a)$p_0^\rho = 0, \rho_0 = 2.1 \text{ GM}/c^2, z_0 = 0$, (b)$p_0^\rho = 0, \rho_0 = 2.6 \text{ GM}/c^2, z_0 = 0$, (c)$p_0^\rho = 0, \rho_0 = 2.85 \text{ GM}/c^2, z_0 = 0$. For each $\delta$, $v_{*,\text{max}}\sqrt{\left<\kappa\right>_{\text{max}}}$ is in good agreement with the Lyapunov exponent $\lambda$. As the value of $\delta$ increases, both of the values become consistently smaller and smaller.