Non-vanishing of derivatives of $L$-functions of Hilbert modular forms in the critical strip

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Abstract

In this paper, we show that, on average, the derivatives of $L$-functions of cuspidal Hilbert modular forms with sufficiently large parallel weight $k$ do not vanish on the line segments $\Im(s) = t_0, \Re(s) \in (\frac{k-1}{2}, \frac{k}{2} - \epsilon) \cup (\frac{k}{2} + \epsilon, \frac{k+1}{2})$. This is analogous to the case of classical modular forms.

Keywords: Hilbert modular forms, Derivatives of $L$-functions, Non-vanishing of $L$-functions

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1 Introduction

In [3], Kohnen proved that given any real number $t_0$ and $\epsilon > 0$, then for $k$ large enough, the average of the normalized $L$-functions $L^*(f, s)$ with $f$ varying over a basis of Hecke eigenforms of weight $k$ on $\text{SL}_2(\mathbb{Z})$ does not vanish on the line segment $\Im(s) = t_0, (k-1)/2 < \Re(s) < k/2 - \epsilon, k/2 + \epsilon < \Re(s) < (k+1)/2$.

Recently in [4], Kohnen, Sengupta and Weigel extended their method and showed a non-vanishing result for the derivatives of $L$-functions associated to modular forms of integer weight on the full group. In particular, they show that for $k$ large enough,

$$\sum_{j=1}^{d} \frac{1}{[k_{j},k_{j}]} \frac{d^n}{ds^n} [L^*(f_{k,j}, s)]$$

does not vanish on the line segment $\Im(s) = t_0, (k-1)/2 < \Re(s) < k/2 - \epsilon, k/2 + \epsilon < \Re(s) < (k+1)/2$.

In [2], we generalized the result of [3] to the context of cuspidal Hilbert modular forms of parallel weight $k$. In order to describe our work, we introduce the following notation. Let $F$ be a totally real number field of degree $n$ over $\mathbb{Q}$. Let $\mathcal{O}_F$ be the ring of integers of $F$, and assume its narrow class number is equal to 1. For $k \in 2\mathbb{N}$, we denote by $\mathcal{S}_k(\text{SL}_2(\mathcal{O}_F))$ the space of Hilbert cusp forms of parallel weight $k$ for $\text{SL}_2(\mathcal{O}_F)$. Our main result in [2] is the following theorem.

**Theorem 1** Let $k \in 2\mathbb{N}$, and let $B_k(\mathcal{O}_F)$ be a basis of normalized Hecke eigenforms of $\mathcal{S}_k(\text{SL}_2(\mathcal{O}_F))$. Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$. Then there exists a constant $C$ depending only on $t_0$, $\epsilon$...
and $F$ such that for $k > C$ the average
\[ \sum_{f \in B_k(O_F)} \frac{\Lambda(f,s)}{[f,f]} \]
is non-vanishing for any $s = \sigma + it_0$ with $\sigma \in \left(\frac{k-1}{2}, \frac{k}{2} - \epsilon\right) \cup \left(\frac{k}{2} + \epsilon, \frac{k+1}{2}\right)$.

Here, we extend the result in [2] by showing that the derivatives of $L$-functions of cuspidal Hilbert modular forms with sufficiently large parallel weight $k$ do not vanish on the line segments
\[ \Im(s) = t_0, \quad \Re(s) \in \left(\frac{k-1}{2}, \frac{k}{2} - \epsilon\right) \cup \left(\frac{k}{2} + \epsilon, \frac{k+1}{2}\right). \]

More precisely, we prove the following theorem.

**Theorem 2** Let $B_k(O_F)$ be a basis of normalized Hecke eigenforms of $S_k(SL_2(O_F))$. Let $t_0 \in \mathbb{R}$, $\epsilon > 0$ and $\ell \in \mathbb{N}$. Then there exists a constant $C$ depending only on $t_0$, $\epsilon$ and $F$ such that for $k > C$ the average
\[ \frac{1}{[f,f]} \frac{d^\ell}{ds^\ell} \left( \Lambda(f,s) \right) \]
is non-vanishing for any $s = \sigma + it_0$ with $\sigma \in \left(\frac{k-1}{2}, \frac{k}{2} - \epsilon\right) \cup \left(\frac{k}{2} + \epsilon, \frac{k+1}{2}\right)$.

We obtain the following corollary as a direct consequence.

**Corollary 3** Let $t_0$, $\epsilon$, $\ell$ and $C$ be as in Theorem 2. Then for $k > C$ and any $s = \sigma + it_0$ with $\sigma \in \left(\frac{k-1}{2}, \frac{k}{2} - \epsilon\right) \cup \left(\frac{k}{2} + \epsilon, \frac{k+1}{2}\right)$, there exists a Hecke eigenform $f \in S_k(SL_2(O_F))$ such that $\frac{d^\ell}{ds^\ell} \left( \Lambda(f,s) \right) \neq 0$.

## 2 Setting and preliminaries

In this note, we work over a totally real number field $F$ of degree $n$ over $\mathbb{Q}$ with ring of integers $O_F$. The group of units in $O_F$ is denoted by $O_F^\times$. For simplicity of exposition, we assume that the narrow class number of $F$ is 1.

The absolute norm of an ideal $a \subset O_F$ is given by $N(a) = [O_F : a]$. The trace and the norm over $\mathbb{Q}$ of an element $x \in F$ are denoted by $\text{Tr}(x)$ and $N(x)$, respectively. We denote by $\mathcal{D}_F$ the different ideal of $F$ and by $d_F$ its discriminant over $\mathbb{Q}$. We have the relation $\mathcal{D}_F = (d_F)$ and $N(\mathcal{D}_F) = |d_F|$.

The real embeddings of $F$ are denoted by $\sigma_j : x \mapsto x_j := \sigma_j(x)$ for $j = 1, \ldots, n$. We say $x \in F$ is totally positive and write $x \gg 0$ if $x_j > 0$ for all $j$. Moreover, we use $X^+$ to denote the set of all totally positive elements in a subset $X$ of $F$.

To simplify exposition, we will often make use of the following notation. For $c, d \in F$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $s \in \mathbb{C}$, we set
\[ N(cz + d)^s = \prod_{j=1}^n (cz_j + d_j)^s. \]
Moreover, we set
\[ \text{Tr}(cz) = \sum_{j=1}^n c_jz_j. \]
Let us now recall the following results which are crucial for establishing Theorem 2 (see the proof of Lemma 8 below).

**Lemma 4** (Trotabas [7, Lemma 2.1]) There exist constants $C_1$ and $C_2$ depending only on $F$ such that

$$\forall \xi \in F, \exists \epsilon \in \mathcal{O}_F^{\times +}, \forall j \in \{1, \ldots, n\} : \quad C_1 |N(\xi)|^{1/n} \leq |(\epsilon \xi_j)| \leq C_2 |N(\xi)|^{1/n}.$$  

**Lemma 5** (Luo [5]) For $\lambda > 0$, we have

$$\sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{j < 1} \eta_j^\lambda < \infty. \quad (2)$$

### 3 Fourier expansion of the Kernel function for Hilbert modular forms

Let us now define the kernel function for Hilbert modular forms over $F$. Let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{H}^n$, $k \in \mathbb{Z}$ and let $s \in \mathbb{C}$ with $1 < \Re(s) < k - 1$. We have

$$R_s(z) = \gamma_k(s) \sum_{(a \ b) \in T} N(cz + d)^{-k} N(az + b)^{-s} N(cz + d)^{-s'}. \quad (3)$$

where $T = \left\{ \left( \begin{array}{cc} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{array} \right) : \epsilon \in \mathcal{O}_F^{\times +} \right\}$ and $\gamma_k(s) = \frac{(i^s \Gamma(s) \Gamma(k-s))^n}{|\mathcal{O}_F^{\times} : \mathcal{O}_F^{\times +}|}$.

We have shown earlier in [2] that $R_s(z)$ is a cusp form of parallel weight $k$ for the modular group $\text{SL}_2(\mathcal{O}_F)$, and its Fourier expansion at the infinite cusp is given as follows.

**Proposition 6** The function $R_s(z)$ has the following Fourier expansion:

$$R_s(z) = \sum_{v \in \mathcal{O}_F^{\times^{-1}}} \sum_{v > 0} r_{s,k}(v) \exp\left(2\pi i \text{Tr}(vz)\right),$$

where

$$r_{s,k}(v) = \frac{(2\pi)^n \Gamma^n(k-s)}{\sqrt{|d_F|}} N(v)^{s-1} + (-1)^{nk} \frac{(2\pi)^n (k-s) \Gamma^n(s)}{\sqrt{|d_F|}} N(v)^{k-s-1}$$

$$+ \gamma_k(s) \left( \frac{2\pi i}{\Gamma^n(k)} N(v)^{k-1} \sum_{(a,c) \in \mathcal{O}_F \times \mathcal{O}_F} \prod_{(a,c) \neq 0} \frac{N(c)^{s-k} N(a)^{k-s}}{\Gamma^n(c) \sqrt{|d_F|}} \text{exp}\left(2\pi i \text{Tr}\left(\frac{v_0}{c}\right)\right) \right)$$

$$\times \prod_{j=1}^{n} F_1 \left( s, k, -\frac{2\pi i v_j}{a_j c_j} \right).$$

To clarify the notation in the above equation, we note that summation $\sum^\times$ indicates that $(a, c)$ runs over all non-associated pairs only (in our setting, the pairs $(a_1, c_1)$ and $(a_2, c_2)$ are called associated if $(a_1, c_1) = u(a_2, c_2)$ for some totally positive unit $u \in \mathcal{O}_F^{\times +}$.

Moreover, for every co-prime pair $(a, c)$, we fix a choice of $(b_0, d_0) \in \mathcal{O}_F \times \mathcal{O}_F$ such that $ad_0 - bc_0 = 1$.

It was also shown in [2] that $R_s(z)$ can be expressed as:
Recall that \( \Lambda(s, f, s) \) satisfies the functional equation (see [6, page 654])

\[
\Lambda(k - s) = (-1)^{\nu} \Lambda(s).
\]

### 4 Proof of Theorem 2

In this section, we prove the main theorem of this paper following the recent work of Kohnen et al. [4]. In view of the functional equation (5), it suffices to consider the left hand side of the critical strip. Hence, we take \( s = \frac{k}{2} - \delta - it_0 \) where \( \epsilon < \delta < \frac{1}{2} \) and \( t_0 \in \mathbb{R} \).

Taking the first Fourier coefficients on both sides of (4) and using Proposition 6, we get

\[
(2\pi)^{|s|} \Gamma^n(k - s) + (-1)^{\nu} \left( (2\pi)^n \Gamma^{n(k - s)} \right)
\]

\[
\times \sum_{\substack{(a, c) \in \mathcal{O}_F \times \mathcal{O}_F \\
\gcd(a, c) = 1}} (N(a)^{|s|}) \exp \left( \frac{\pi}{2} i \text{Tr} \left( \frac{d_0}{c} \right) \right) \prod_{j=1}^n \frac{\Gamma(k - s)}{\Gamma(k)} \frac{\Gamma(s, k)}{\Gamma(s)}
\]

which is a basis of normalized Hecke eigenforms of \( \mathcal{S}_k(\text{SL}_2(\mathcal{O}_F)) \), and

\[
\Lambda(f, s) = |d_F|^{|s|} \Gamma^n \Gamma^{|s|} L(f, s).
\]

Taking the \( \ell \)-th derivative of both sides with respect to \( s \) gives

\[
\frac{1}{\sqrt{|d_F|}} \frac{d^\ell}{ds^\ell} \left( (2\pi)^{|s|} \Gamma^n(k - s) \right) + (-1)^{\nu} \frac{d^\ell}{ds^\ell} \left( (2\pi)^n \Gamma^{n(k - s)} \right)
\]

\[
\times \sum_{\substack{(a, c) \in \mathcal{O}_F \times \mathcal{O}_F \\
\gcd(a, c) = 1}} (N(a)^{|s|}) \exp \left( \frac{\pi}{2} i \text{Tr} \left( \frac{d_0}{c} \right) \right) \prod_{j=1}^n \frac{\Gamma(k - s)}{\Gamma(k)} \frac{\Gamma(s, k)}{\Gamma(s)}
\]

\[
= (-1)^{\nu} \pi^2 \Gamma^n(k - 1) \sum_{f \in \mathcal{B}_k(O_F)} \frac{\Lambda(f, s)}{|f|},
\]

where

\[
\frac{\Gamma(k - s)}{\Gamma(k)} \frac{\Gamma(s, k)}{\Gamma(s)} = \Gamma(k - s) \Gamma(s) \frac{\Gamma(s, k)}{\Gamma(k)} = 1F_1 \left( s, k - \frac{2\pi i}{a_j c_j} \right).
\]

Let us consider first the expression

\[
I_1 = \frac{1}{\sqrt{|d_F|}} \frac{d^\ell}{ds^\ell} \left( (2\pi)^{|s|} \Gamma^n(k - s) \right).
\]
We have

\[ I_1 = \frac{1}{\sqrt{|d_F|}} \sum_{j=0}^{\ell-1} \binom{\ell}{j} \frac{d^j}{d\psi^j} \left[ (2\pi)^{\psi} \right] \frac{d^{\ell-j}}{d\psi^{\ell-j}} \left[ \Gamma^n(k - s) \right] \]

\[ = \frac{1}{\sqrt{|d_F|}} \sum_{j=0}^{\ell-1} \binom{\ell}{j} (n \log (2\pi))^j (2\pi)^{\psi} \sum_{v_1+v_2+\ldots+v_n=\ell-j} \frac{(\ell-j)!}{v_1!v_2!\ldots v_n!} \prod_{1 \leq t \leq n} (-1)^{v_t} \Gamma^{(v_t)}(k - s) \]

\[ = \frac{(2\pi)^{\psi}}{\sqrt{|d_F|}} \sum_{j=0}^{\ell-1} (-1)^{\ell-j} \binom{\ell}{j} (n \log (2\pi))^j \sum_{v_1+v_2+\ldots+v_n=\ell-j} \frac{(\ell-j)!}{v_1!v_2!\ldots v_n!} \prod_{1 \leq t \leq n} \Gamma^{(v_t)}(k - s) \]

\[ + \frac{(2\pi)^{\psi}}{\sqrt{|d_F|}} \sum_{j=0}^{\ell-1} (-1)^{\ell-j} \binom{\ell}{j} (n \log (2\pi))^j \sum_{v_1+v_2+\ldots+v_n=\ell-j} \frac{(\ell-j)!}{v_1!v_2!\ldots v_n!} \prod_{1 \leq t \leq n} \Gamma^{(v_t)}(k - s). \]

Hence,

\[ \frac{\sqrt{|d_F|} I_1}{(2\pi)^{\psi} \Gamma^n(k - s)} = (n \log (2\pi))^\ell \]

\[ + \sum_{j=0}^{\ell-1} (-1)^{\ell-j} \binom{\ell}{j} (n \log (2\pi))^j \sum_{v_1+v_2+\ldots+v_n=\ell-j} \frac{(\ell-j)!}{v_1!v_2!\ldots v_n!} \prod_{1 \leq t \leq n} \frac{\Gamma^{(v_t)}(k - s)}{\Gamma(k - s)}. \]

(8)

In [4], it is observed that \( \frac{\Gamma^{(m)}(z)}{\Gamma(z)} \) can be expressed as a polynomial \( P \in \mathbb{Z}[\psi, \psi^{(1)}, \ldots, \psi^{(m)}] \), where \( \psi = \frac{1}{z} \) is the digamma function. It is important to note that the highest power of \( \psi \) appearing in the polynomial \( P \) is \( \psi^m \).

We also recall the following important estimates of the digamma function:

\[ \psi(z) \sim \log(z) - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k z^{2k}} \]

\[ \psi^{(m)}(z) \sim (-1)^{m-1} \left( \frac{(m-1)!}{2z^{m+1}} + \sum_{k=1}^{m} \frac{B_{2k} (2k + m - 1)!}{(2k)! 2k z^{2k+m}} \right), \]

for \( z \to \infty \) in \( |\arg(z)| < \pi \), where \( B_n \) is the \( n \)th Bernoulli number (see [1, 6.3.18 & 6.4.11]).

For \( s = \frac{k}{2} + \epsilon i 0 \) with \( \epsilon < \delta < \frac{1}{2} \), see that \( \frac{\Gamma^{(v_t)}(k-s)}{\Gamma(k-s)} = P_t (\log(k-s)) + o(1) \) where \( P_t \) is a polynomial with integer coefficients and degree \( v_t \). It follows that

\[ \prod_{1 \leq t \leq n} \frac{\Gamma^{(v_t)}(k - s)}{\Gamma(k - s)} = \prod_{1 \leq t \leq n} P_t (\log(k-s)) + o(1). \]

\[ \frac{\sqrt{|d_F|} I_1}{(2\pi)^{\psi} \Gamma^n(k - s)} = (n \log (2\pi))^\ell \]

\[ + \sum_{j=0}^{\ell-1} (-1)^{\ell-j} \binom{\ell}{j} (n \log (2\pi))^j \]
\[
\sum_{v_1 + v_2 + \cdots + v_n = \ell - j} \frac{(\ell - j)!}{v_1! v_2! \cdots v_n!} \prod_{1 \leq t \leq n} P_t (\log(k - s)) + o(1)
\]

\[
= \tilde{P} (\log(k - s)) + o(1),
\]

where \( \tilde{P} \) is a polynomial of degree \( \ell \) with integer coefficients and leading coefficient \((-1)^{\ell} \sum_{v_1 + \cdots + v_n = \ell - j} \frac{\ell!}{v_1! \cdots v_n!} \).

Next, we deal with the sum

\[
I_2 = \frac{(-1)^{\ell/2} (2\pi)^{\ell/2}}{\sqrt{|d_F|}} d^\ell \left[ (2\pi)^n (k - s)^{\ell} \Gamma^n(s) \right].
\]

We have

\[
I_2 = \frac{(-1)^{\ell/2} (2\pi)^{\ell/2}}{\sqrt{|d_F|}} \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{d^j}{ds^j} \left[ (2\pi)^{n-j} \right] \frac{d^{\ell-j}}{d^{\ell-j}} \left[ \Gamma^n(s) \right]
\]

\[
= \frac{(-1)^{\ell/2} (2\pi)^{\ell/2}}{\sqrt{|d_F|}} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (\ln(2\pi))^j \sum_{v_1 + v_2 + \cdots + v_n = \ell - j} \frac{(\ell - j)!}{v_1! v_2! \cdots v_n!} \prod_{1 \leq t \leq n} \Gamma^{(v_t)}(s).
\]

It follows that

\[
\sqrt{|d_F|} I_2 = \frac{(-1)^{\ell/2} (2\pi)^{\ell/2}}{(2\pi)^n \Gamma^n(k - s)} \sum_{j=0}^{\ell} \binom{\ell}{j} \ln(2\pi)^j \sum_{v_1 + v_2 + \cdots + v_n = \ell - j} \frac{(\ell - j)!}{v_1! v_2! \cdots v_n!} \prod_{1 \leq t \leq n} \Gamma^{(v_t)}(s).
\]

If \( s = \frac{k}{2} - \delta + it_0 \) with \( \epsilon < \delta < \frac{1}{2} \), we have

\[
\sqrt{|d_F|} I_2 = \frac{(-1)^{\ell/2} (2\pi)^{\ell/2}}{(2\pi)^n \Gamma^n(k - s)} \left[ \sum_{j=0}^{\ell} \binom{\ell}{j} \ln(2\pi)^j \right] \sum_{v_1 + v_2 + \cdots + v_n = \ell - j} \frac{(\ell - j)!}{v_1! v_2! \cdots v_n!} \prod_{1 \leq t \leq n} \Gamma^{(v_t)}(s)
\]

\[
= (-1)^{\ell/2} (2\pi)^{\ell/2} \frac{\Gamma^n(s)}{(2\pi)^n \Gamma^n(k - s)} \left[ \sum_{j=0}^{\ell} \binom{\ell}{j} \ln(2\pi)^j \right] \sum_{v_1 + v_2 + \cdots + v_n = \ell - j} \frac{(\ell - j)!}{v_1! v_2! \cdots v_n!} \prod_{1 \leq t \leq n} \Gamma^{(v_t)}(s).
\]

where \( \tilde{Q} \) is a polynomial of degree \( \ell \) having integer coefficients and leading coefficient \((-1)^{\ell} \sum_{v_1 + \cdots + v_n = \ell - j} \frac{\ell!}{v_1! \cdots v_n!} \).

Observe that (see [1, 6.1.23 & 6.1.47])

\[
\left| \frac{\Gamma^n(s)}{\Gamma^n(k - s)} \right| = \left| \frac{k}{2} + it_0 \right|^{-2\Delta} \left| 1 + O \left( \frac{1}{\left| \frac{k}{2} + it_0 \right|^2} \right) \right|
\]

uniformly in \( \epsilon < \delta < \frac{1}{2} \). It follows that \( \frac{\sqrt{|d_F|} I_2}{(2\pi)^n \Gamma^n(k - s)} \) tends to 0 as \( k \to \infty \).
We still have to estimate the following sum:

\[ I_3 = \frac{(-1)^{\ell} (2\pi)^{nk}}{\sqrt{|d_F| |O_F^\times : O_F^\times|}} \times \sum_{(a, c) \in O_F \times O_F} \frac{d^\ell}{ds^\ell} \left[ \frac{N(c)^{s-k}}{N(a)^s} \exp \left( \frac{\pi i}{2} \text{ins} \right) \exp \left( 2\pi i \text{Tr} \left( \frac{d_0}{c} \right) \right) \prod_{t=1}^{n} \nu_t \left( s, k_t - \frac{2\pi i}{a_t c_t} \right) \right]. \]

We have

\[ I_3 = \frac{(-1)^{\ell} (2\pi)^{nk}}{\sqrt{|d_F| |O_F^\times : O_F^\times|}} \sum_{(a, c) \in O_F \times O_F} \frac{N(c)^{s-k}}{N(a)^s} \log \left( \frac{N(c)}{N(a)} \right) \times \frac{d^{\ell-j}}{ds^{\ell-j}} \left[ \exp \left( \frac{\pi i}{2} \text{ins} \right) \exp \left( 2\pi i \text{Tr} \left( \frac{d_0}{c} \right) \right) \prod_{t=1}^{n} \nu_t \left( s, k_t - \frac{2\pi i}{a_t c_t} \right) \right]. \]

For \( 1 \leq j \leq \ell - 1 \), we have

\[
\frac{d^{\ell-j}}{ds^{\ell-j}} \left[ \exp \left( \frac{\pi i}{2} \text{ins} \right) \exp \left( 2\pi i \text{Tr} \left( \frac{d_0}{c} \right) \right) \prod_{t=1}^{n} \nu_t \left( s, k_t - \frac{2\pi i}{a_t c_t} \right) \right] \\
= \sum_{m=0}^{\ell-j} \binom{\ell-j}{m} \exp \left( 2\pi i \text{Tr} \left( \frac{d_0}{c} \right) \right) \left( \frac{\pi i}{2} \text{ins} \right)^m \exp \left( \frac{\pi i}{2} \text{ins} \right) \frac{d^{\ell-j-m}}{ds^{\ell-j-m}} \left[ \prod_{t=1}^{n} \nu_t \left( s, k_t - \frac{2\pi i}{a_t c_t} \right) \right] \\
= \sum_{m=0}^{\ell-j} \binom{\ell-j}{m} \exp \left( 2\pi i \text{Tr} \left( \frac{d_0}{c} \right) \right) \left( \frac{\pi i}{2} \text{ins} \right)^m \exp \left( \frac{\pi i}{2} \text{ins} \right) \\
\times \sum_{v_1 + v_2 + \cdots + v_n = \ell - j} (-1)^{v_1} \frac{d^{v_1}}{ds^{v_1}} \nu_t \left( s, k_t - \frac{2\pi i}{a_t c_t} \right). \\
\]

Therefore,

\[
\frac{\sqrt{|d_F| I_3}}{(2\pi)^m \Gamma^m (k-s)} = \frac{(-1)^{\ell} (2\pi)^{nk(k-s)}}{\Gamma^m (k-s) |O_F^\times : O_F^\times|} \sum_{(a, c) \in O_F \times O_F} \frac{N(c)^{s-k}}{N(a)^s} \exp \left( \frac{\pi i}{2} \text{ins} \right) \exp \left( 2\pi i \text{Tr} \left( \frac{d_0}{c} \right) \right) \\
\times \left[ \log \left( \frac{N(c)}{N(a)} \right) \prod_{t=1}^{n} \nu_t \left( s, k_t - \frac{2\pi i}{a_t c_t} \right) \\
+ \sum_{j=0}^{\ell-1} \log \left( \frac{N(c)}{N(a)} \right) \sum_{m=0}^{\ell-j} \binom{\ell-j}{m} \left( \frac{\pi i}{2} \text{ins} \right)^m \\
\times \sum_{v_1 + v_2 + \cdots + v_n = \ell - j} (-1)^{v_1} \frac{d^{v_1}}{ds^{v_1}} \nu_t \left( s, k_t - \frac{2\pi i}{a_t c_t} \right) \right].
\]
Let us now consider the sums

\[ E_{1,\ell}(s, k) = \sum_{(a, c) \in \mathcal{O}_F \times \mathcal{O}_F} \frac{N(c)^{s-k}}{N(a)^{s}} \exp \left( \frac{\pi i s}{2} \right) \exp \left( \frac{2\pi i Tr \left( d_0 \right)}{c} \right) \times \log \left( \frac{N(c)}{N(a)} \right) \prod_{t=1}^{n} \mathfrak{f}_1 \left( s, k_\ell - \frac{2\pi i}{a_1 c_\ell} \right) \]

and

\[ E_{2,\ell}(s, k) = \sum_{(a, c) \in \mathcal{O}_F \times \mathcal{O}_F} \frac{N(c)^{s-k}}{N(a)^{s}} \exp \left( \frac{\pi i s}{2} \right) \exp \left( \frac{2\pi i Tr \left( d_0 \right)}{c} \right) \times \log \left( \frac{N(c)}{N(a)} \right) \prod_{t=1}^{n} \mathfrak{f}_1 \left( s, k_\ell - \frac{2\pi i}{a_1 c_\ell} \right) \]

We have

\[ E_{1,\ell}(s, k) = \sum_{\eta \in \mathcal{O}_F^+} \sum_{c \in \mathcal{O}_F} \sum_{a \in \mathcal{O}_F} \frac{N(c)^{s-k}}{N(a)^{s}} \exp \left( \frac{\pi i s}{2} \right) \exp \left( \frac{2\pi i Tr \left( d_0 \right)}{c} \right) \times \log \left( \frac{N(c)}{N(a)} \right) \prod_{t=1}^{n} \mathfrak{f}_1 \left( s, k_\ell - \frac{2\pi i}{a_1 c_\ell} \right), \]  

(10)

and

\[ E_{2,\ell}(s, k) = \sum_{\eta \in \mathcal{O}_F^+} \sum_{c \in \mathcal{O}_F} \sum_{a \in \mathcal{O}_F} \frac{N(c)^{s-k}}{N(a)^{s}} \exp \left( \frac{\pi i s}{2} \right) \exp \left( \frac{2\pi i Tr \left( d_0 \right)}{c} \right) \times \log \left( \frac{N(c)}{N(a)} \right) \prod_{t=1}^{n} \mathfrak{f}_1 \left( s, k_\ell - \frac{2\pi i}{a_1 c_\ell} \right) \]

(11)

In (10) and (11), the notation \( \sum' \) indicates that the summation is restricted to non-associated elements only. In our setting, two non-zero elements \( x_1, x_2 \in \mathcal{O}_F \) are said to be associated if there exists \( u \in \mathcal{O}_F^\times \) such that \( x_1 = ux_2 \). To deal with \( E_{1,\ell}(s, k) \) and \( E_{2,\ell}(s, k) \), we need the following lemma.

**Lemma 7** Let \( s = \frac{k}{2} - \delta + it_0 \) where \( \epsilon < \delta < \frac{1}{2} \) and \( t_0 \in \mathbb{R} \). Let \( \ell \) be a non-negative integer. For all \( x \in \mathbb{R} \) and all sufficiently large \( k \), we have

\[ \frac{d^\ell}{ds^\ell} \left( \mathfrak{f}_1 (s, k, ix) \right) \ll_{\ell} \min \left\{ 1, |x|^{-1} (|s - 1| + |k - s - 1|) \right\}. \]

**Proof** By [1, 13.2.1], we have

\[ \mathfrak{f}_1 (s, k, ix) = \int_0^1 \exp(ixu)u^{k-s-1} (1-u)^{k-s-1} du. \]
It follows that \( |f_1(s, k, ix)| \leq 1 \) whenever \( \Re(s) > 1 \) and \( \Re(k - s) > 1 \). Moreover, by differentiating both sides with respect to \( s \), we get

\[
\frac{d^\ell}{ds^\ell} (f_1(s, k, ix)) \ll_{\ell} 1
\]

(12)

for any non-negative integer \( \ell \) (see [4, page 326]). On the other hand, integration by parts yields

\[
f_1(s, k, ix) = \int_0^1 \exp(ixu)u^{s-1}(1-u)^{k-s-1} \, du
\]

\[
= -\frac{s-1}{ix} \int_0^1 \exp(ixu)u^{-2}(1-u)^{k-s-1} \, du
\]

\[
+ \frac{k-s-1}{ix} \int_0^1 \exp(ixu)u^{s-1}(1-u)^{k-s-2} \, du.
\]

Taking the \( \ell \)-th derivative of both sides with respect to \( s \) yields

\[
\frac{d^\ell}{ds^\ell} (f_1(s, k, ix)) = -\frac{1}{ix} \int_0^1 \exp(ixu)u^{-2}(1-u)^{k-s-2} \, du
\]

\[
- \frac{s-1}{ix} \int_0^1 \exp(ixu)
\]

\[
\sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} (\log u)^j u^{-2}(\log(1-u))^{\ell-j}(1-u)^{k-s-1} \, du
\]

\[
+ \frac{k-s-1}{ix} \int_0^1 \exp(ixu)
\]

\[
\sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} (\log u)^j u^{s-1}(\log(1-u))^{\ell-j}(1-u)^{k-s-2} \, du.
\]

Hence,

\[
\frac{d^\ell}{ds^\ell} (f_1(s, k, ix)) \ll_{\ell} |x|^{-1} (|s-1| + |k-s-1|)
\]

(13)

whenever \( \Re(s) > 2 \) and \( \Re(k-s) > 2 \). The desired result follows from (12) and (13). □

**Lemma 8** Let \( s = \frac{k}{2} - \delta + it_0 \) where \( \epsilon < \delta < \frac{1}{2} \) and \( t_0 \in \mathbb{R} \). As \( k \to \infty \), we have \( E_{1,\ell}(s, k) = O(k^n) \) and \( E_{2,\ell}(s, k) = O(k^n) \) where the implied constant depends only on \( \ell, t_0, \delta \) and the field \( F \).

**Proof** Upon taking absolute values, we get

\[
|E_{2,\ell}(s, k)| \leq \sum_{\eta \in \mathcal{O}_F^*} \sum_{\eta \in \mathcal{O}_F} \sum_{c \neq \mathcal{O}_F \gcd(a, c) = 1} |N(c)|^{-\frac{1}{2} - \delta} |N(a)|^\delta \sum_{j=0}^{\ell-1} \left| \log \left( \frac{|N(c)|}{|N(a)|} \right)^j \right|
\]

\[
\times \sum_{m=0}^{\ell-j} \binom{\ell-j}{m} (\frac{\pi}{2})^m \exp \left( \frac{\pi}{2} t_0 \right)
\]

\[
\times \prod_{v_1 + v_2 + \ldots + v_n = \ell-j-m} \frac{(\ell-j-m)!}{v_1!v_2! \ldots v_n!} \prod_{i=1}^n \left| \frac{d^i v_i}{ds^i} f_1(s, k, -\frac{2\pi t}{a_i\eta_i\ell}) \right|.
\]
By Lemma 7, we know that
\[
\frac{d^{n/2}}{ds^{n/2}} (f_1(s, k, ix)) \ll_{n/2} (|s - 1| + |k - s - 1|)^n \left| \frac{2\pi j}{\eta_j a_j c_j} \right|^{-\omega_j},
\]
where \(\omega_j\) is either 0 or 1 depending on whether \(\eta_j \geq 1\) or \(\eta_j < 1\) respectively. Hence, we get
\[
\prod_{j=1}^n \left| \frac{d^{n/2}}{ds^{n/2}} (f_1(s, k, ix)) \right| \ll_{n/2} (|s - 1| + |k - s - 1|)^n \prod_{\eta_j < 1} \left| \frac{2\pi}{\eta_j a_j c_j} \right|^{-1}.
\]
It follows that \(E_{2,\ell}(s, k)\) is
\[
\ll (|s - 1| + |k - s - 1|)^n \sum_{\eta \in \mathcal{O}_F^+} \sum_{c \in \mathcal{O}_F} \sum_{\eta_j < 1} \prod_{\eta_j < 1} \left| \frac{2\pi}{\eta_j a_j c_j} \right|^{-1} \left| \frac{N(c)}{n} \right|^{1/2 - \delta - \varepsilon} |N(a)| \left| \frac{k}{2} + \delta - it_0 \right|.
\]
By Lemma 4, we may choose the elements \(a, c\) in (14) such that
\[
N(a)^{1/n} \ll_{\ell} a_j \ll N(a)^{1/n} \quad \text{and} \quad |\mathcal{N}(c)|^{1/n} \ll_{\ell} c_j \ll |\mathcal{N}(c)|^{1/n},
\]
for all \(j \in \{1, \ldots, n\}\) with implicit constants depending only on \(F\). Therefore, we have
\[
E_{2,\ell}(s, k) \ll k^n \sum_{\eta \in \mathcal{O}_F^+} \sum_{\eta_j < 1} \prod_{\eta_j < 1} |\mathcal{N}(c)|^{-1/2 + 1/2 - \delta - \varepsilon} |N(a)|^{-1/2 + 1 + \delta - \varepsilon}.
\]
Lemma 5 allows us to factor out the sum over all \(\eta \in \mathcal{O}_F^+\) since it is convergent and depends only on \(F\). Thus, we get \(E_{2,\ell}(s, k) = O(k^n)\) for sufficiently large \(k\) as desired. We also emphasize that the implied constant in this estimate depends only on \(\delta, t_0, \ell\) and the field \(F\). The sum \(E_{1,\ell}(s, k)\) is treated similarly, and so we will not include the details here.

\(\square\)

It follows from Lemma 8 that
\[
\frac{\sqrt{|d_F|}}{(2\pi)^n} \Gamma^n(k - s) \ll \left| \frac{k^n(2\pi)^n k}{\Gamma^n(k - s)} \right|,
\]
which tends to 0 as \(k \to \infty\). Moreover, we have already established that as \(k \to \infty\) we have
\[
\frac{\sqrt{|d_F|}}{(2\pi)^n} \Gamma^n(k - s) \to 0,
\]
where as
\[
\frac{\sqrt{|d_F|}}{(2\pi)^n} \sim \hat{P} \left( \log \left( \frac{k}{2} + \delta - it_0 \right) \right),
\]
for some polynomial \(\hat{P}\) of degree \(\ell\). Applying these estimates to (7), we conclude that
\[
\sum_{f \in T_s(O_F)} \frac{1}{|f|} \frac{d^\ell}{ds^\ell} \left( \Lambda(f, s) \right)
\]
is non-vanishing for any \(s = \frac{k}{2} - \delta + it_0\) with \(\varepsilon < \delta < \frac{1}{2}\) and \(t_0 \in \mathbb{R}\).

Authors’ contributions
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