Growing Multiplex Networks with Arbitrary Number of Layers

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This paper focuses on the problem of growing multiplex networks. Currently, the results on the inter-layer degree distribution of growing multiplex networks present in the literature pertain to the case of two layers, and are confined to the special case of homogeneous growth. In the present paper, we obtain closed-form solutions for the inter-layer degree distribution of heterogeneously growing multilayer networks with arbitrary number of layers in the steady state. Heterogeneous growth means that each incoming node establishes different numbers of links in different layers. We first briefly consider the problem for two layers: we consider the cases of preferential attachment and uniform attachment separately, and we obtain closed-form expressions for the inter-layer degree distribution and expected degree distributions in each layer. Then for both attachment schemes, we generalize the problem to \( M \) layers, and provide solutions for the joint degree distribution across all layers.

I. INTRODUCTION

The network framework is widely used for studying complex systems and their properties, through mapping units onto nodes and interactions onto links. The conventional conceptualization consists of a single graph, while many real-world systems exhibit different types of interactions between constituents. Such networked systems in which units have heterogeneous types of interactions can be mathematically modeled under the framework of multiplex networks. In these settings, units (nodes) are members of distinct networks simultaneously. For example, it would be an oversimplified depiction to talk about the social network of a given group of people, because the same pair of individuals can have distinct types of relations at the same time: they can be kins, friends, coworkers, they can be connected on the social media, etc. In other words, there can be multiple networks between the same set of people. This is the basic rationale behind the multiplex representation of networked systems.

The multiplex framework envisages different layers housing different types of links between the same set of nodes (so there is one set of nodes, multiples sets of links). For example, we can take a sample of individuals and constitute a social media layer, in which links represent interaction on social media, a kinship layer, a geographical proximity layer, and so on. Examples of studies that have conceptualized real networked systems under the multiplex framework include citation networks \[1\, 2\], online social media \[3\, 4\], interbank networks \[5\], airline networks \[6\], scientific collaboration networks \[11\, 7\], and web of connections and interactions in online games \[8\].

Theoretical tools for characterizing and quantifying the properties of multiplex networks are generalizations of the single-layer scenario to multiple layers \[9\, 11\]. For example, the adjacency matrix is generalized to the adjacency tensor, whose \( ijk \) element is the weight of the link from node \( i \) to node \( j \) in layer \( k \). Similarly, all nodal attributes which were scalars in the single-layer picture (such as various types of centralities, degree, and clustering) are generalized to vectors in the multiplex scenario. These new theoretical measures enable studying various phenomena on top of multiplex networks analytically. Examples include epidemics \[12\, 13\], pathogen-awareness interplay \[14\, 15\], percolation processes \[12\, 16\, 17\], random walks \[18\, 19\], evolution of cooperation \[20\, 23\], diffusion processes \[24\] and social contagion \[25\]. For thorough reviews, see \[23\, 26\, 27\].

Since many real networks are growing in size, growing multiplex networks have also attracted attention in the literature. The mean-field approach is a potent method for investigating the temporal evolution of the degrees of individual nodes, extracting its asymptotic behavior in order to find the asymptotic (tail behavior) degree distribution of each of the individual layers. This approach is undertaken in \[28\, 31\]. An alternative approach of tackling network growth problems is the rate equation approach, undertaken in \[28\, 29\, 32\]. The rate equation enables solving for the joint degree distribution of the system, so that we can obtain the fraction of nodes that have a given degree vector across layers.

Previous results on the inter-layer degree distribution of growing multiplex networks—attainable through the rate equation approach—are confined to the case of homogeneous two-layer growth, where the number of links established by each new incoming node is the same across layers \[28\, 29\]. Note that the possibility of heterogeneous growth is envisaged in \[28\] (in the Supplemental Material therein), and their implications for the mean-field scenario are correctly alluded to. In the present paper, we consider different rates of link growth across layers explicitly, and obtain the inter-layer degree distri-
butions. Moreover, we extend the problem to general $M$ layers. To our knowledge, no solution for the inter-layer degree distribution of growing multiplex networks with arbitrary number of layers exists in the literature.

The assumption of link growth heterogeneity is motivated by the empirical studies on the structure of multi-layered interacting systems, which report that different layers generally exhibit different connectivity patterns (consequently, different average degrees, and other structural properties). For example, in [5], the connectivity structure of the players of a massive online game is mapped onto a multiplex network of six layers, and the average degrees of the layers are different (ranging from 3 in the most sparse layer to 61 to the densest layer). In [6], the friendship ties of a group of students is mapped onto Facebook friends, picture friends and cohabitation, and these layers are shown to have different connectivity distributions, hence different average degrees. In [33], the Indian airline and railway transportation networks are mapped onto two layers, representing two distinct modes of transportation between geographic locations. The degree distributions of these layers are then depicted, and it is observed that they have different degree distributions (as well as different nearest-neighbor degree distributions). In [34], the international trade network is mapped onto 97 layers, each layer pertaining to one distinct commodity. The connectivity patterns are different across layers. Although in this example the layers represent weighted networks, the assertion that the connectivity patterns are heterogeneous still holds.

In the present paper, we consider heterogeneously-growing layers. First, we consider a simple two-layer system for expository purposes. We obtain $n_{k,\ell}$, the fraction of nodes with degree $k$ in the first layer and degree $\ell$ in the second layer. We also use this result to find $\bar{f}(k)$, the expected layer-2 degree of a node whose degree in layer 1 is $k$. We solve the problem for the cases of preferential and uniform growth, separately. We demonstrate that the expression for $\bar{f}(k)$ is identical under these two settings. For the special case of homogeneous growth, this result agrees with those of [29].

We then generalize both the preferential and the uniform setups to $M > 2$ layers. Each incoming node establishes $\beta_1, \beta_2, \ldots, \beta_M$ links in layers $1, 2, \ldots, M$, respectively. For both uniform and preferential growth we obtain $n_0(k)$, which is the fraction of nodes with vector degree $k$. That is, the fraction of nodes with degree $k_1$ in layer 1, degree $k_2$ in layer 2, and so on.

II. MODEL 1: PREFERENTIAL ATTACHMENT IN TWO LAYERS

The network is constructed by a set of nodes and two sets of links. There are two layers, each housing one set of links. This means that node $x$ can have a set of neighbors in layer 1 and a different set of neighbors in layer 2. Similarly, the degree of node $x$ in layer 1 can differ from its degree in layer 2. We denote the degree of node $x$ in layer 1 by $k_x$ and in layer 2 by $\ell_x$. We denote by $N_{k,\ell}(t)$ the number of nodes that have degree $k$ in the layer 1 and degree $\ell$ in layer 2. The fraction of such nodes is denoted by $n_{k,\ell}(t)$.

At the outset, the network comprises $N(0)$ nodes, with $L_1(0)$ links in layer 1 and $L_2(0)$ links in layer 2. The network grows by the successive addition of new nodes—one at each timestep. Each new node, upon birth, establishes $\beta_1$ links to the existing nodes in layer 1 and $\beta_2$ links in layer 2.

In this model, the probability that node $x$ receives a link in layer 1 from an incoming node is proportional to $k_x$. Similarly, in layer 2, the probability that node $x$ receives a link from the newcomer is proportional to $\ell_x$. At time $t$, when a new node is introduced the values of $N_{k,\ell}$ can consequently change. If a node with degree $k-1$ in layer 1 and degree $\ell$ in layer 2 receives a link in the first layer, its degree in layer 1 increments and turns into $k$, and $N_{k,\ell}$ increments consequently. If a node with degree $k$ in layer 1 and degree $\ell - 1$ in layer 2 receives a link in layer 2, its degree in layer 2 increments and consequently, $N_{k,\ell}$ increments. Moreover, if a node that has degree $k$ in layer 1 and degree $\ell$ in layer 2 receives a link in either of the layers, $N_{k,\ell}$ decrements consequently. Finally, each incoming node has degree $\beta_1$ in layer 1 and degree $\beta_2$ in layer 2 upon birth, so each new incoming node increments $N_{1,2}$ by one. The following rate equation captures the evolution of the expected value of $N_{k,\ell}$ by addressing the said events with their respective probabilities of occurrence:

$$N_{k,\ell}(t + 1) = N_{k,\ell}(t) + \beta_1 \frac{(k-1)N_{k-1,\ell}(t) - kN_{k,\ell}(t)}{L_1(0) + 2\beta_1 t} + \beta_2 \frac{(\ell-1)N_{k,\ell-1}(t) - \ell N_{k,\ell}(t)}{L_2(0) + 2\beta_2 t} + \delta_{k,\beta_1}\delta_{\ell,\beta_2}.$$ \hspace{1cm} (1)

We can use this to write the rate equation for $n_{k,\ell}$.

Using the substitution $N_{k,\ell} = (N(0) + t)n_{k,\ell}$, we obtain

$$[N(0) + t][n_{k,\ell}(t+1) - n_{k,\ell}(t)] = n_{t+1}(k, \ell) = \beta_1 \frac{(k-1)N_{k-1,\ell}(t) - kN_{k,\ell}(t)}{L_1(0) + 2\beta_1 t} + \beta_2 \frac{(\ell-1)N_{k,\ell-1}(t) - \ell N_{k,\ell}(t)}{L_2(0) + 2\beta_2 t} + \delta_{k,\beta_1}\delta_{\ell,\beta_2}.$$ \hspace{1cm} (2)

Now we focus on the steady state, that is, the limit as $t \to \infty$. In this time regime, the time variations of $n_{k,\ell}$ vanish, and we also have the following simplifications

$$\lim_{t \to \infty} \beta_1 \frac{N(0) + t}{L_1(0) + 2\beta_1 t} = \frac{1}{2}, \hspace{1cm} (3)$$

$$\lim_{t \to \infty} \beta_2 \frac{N(0) + t}{L_2(0) + 2\beta_2 t} = \frac{1}{2}. \hspace{1cm} (4)$$
So (2) transforms into the following equation in the steady state:

\[ n_{k\ell} = \frac{(k-1)n_{k-1,\ell} - kn_{k\ell}}{2} + \frac{(\ell-1)n_{k,\ell-1} - \ell n_{k\ell}}{2} + \delta_{k,1} \delta_{\ell,2}, \quad (4) \]

This can be equivalently expressed as follows

\[ n_{k\ell} = \frac{k-1}{k+\ell+2} n_{k-1,\ell} + \frac{\ell-1}{k+\ell+2} n_{k,\ell-1} + \frac{2\delta_{k,1} \delta_{\ell,2}}{2 + \beta_1 + \beta_2}. \quad (5) \]

This difference equation is solved in Appendix A. The answer is

\[ n_{k,\ell} = \frac{2\beta_1(\beta_1+1)\beta_2(\beta_2+1)}{(2+\beta_1+\beta_2)k(k+1)(\ell+1)} \left( \frac{k^{\beta_1+1}}{k^{\beta_1+2}} - \frac{\ell^{\beta_2+1}}{\ell^{\beta_2+2}} \right), \quad \beta_1 \neq 0, \beta_2 \neq 0, \quad (6) \]

Figure 1 is a depiction of the joint degree distribution for the symmetric case of \( \beta_1 = \beta_2 = 3 \). Figure 2 pertains to the asymmetric case of \( \beta_1 = 10 \) and \( \beta_2 = 10 \).

![Figure 1](image1.png)

**FIG. 1:** (Color online) The inter-layer joint degree distribution for preferential growth with \( \beta_1 = 3 \) and \( \beta_2 = 3 \). Theoretical result is presented in (6). Since the values decay fast in \( k \) and \( \ell \), we have depicted the logarithm of the inverse of this function, for a smoother output and better visibility. The joint distribution attains its maximum at \( k = \beta_1 + \ell = \beta_2 \). The contours are symmetric with respect to the bisector because \( \beta_1 > \beta_2 \).

We can see the entanglement between the two layers through the conditional average degree, which can be derived from (6). For the nodes who have degree \( k \) in layer 1, we can find their average degree in layer 2. Let us denote this quantity by \( \bar{\ell}(k) \). Analytically, we need to perform the following summation:

\[ \bar{\ell}(k) = \sum_{\ell} \frac{\ell n_{\ell|k}}{n_k} = \sum_{\ell} \ell \frac{n_{k,\ell}}{n_k} \]

\[ = \sum_{\ell} \frac{2\beta_1(\beta_1+1)\beta_2(\beta_2+1) \left( \frac{(\beta_1+\beta_2+2)}{\beta_1+2} \right) \left( k^{\beta_1+1} - \ell^{\beta_2+1} \right)}{(2+\beta_1+\beta_2)k(k+1)(\ell+1)} \]

\[ = \sum_{\ell} \frac{\beta_2(\beta_2+1) \left( \frac{(\beta_1+\beta_2+2)}{\beta_1+2} \right) \left( k^{\beta_1+1} - \ell^{\beta_2+1} \right)}{(2+\beta_1+\beta_2)(\ell+1)} \]

\[ = \sum_{\ell} \frac{\beta_2(\beta_2+1) \left( \frac{(\beta_1+\beta_2+2)}{\beta_1+2} \right) \left( k^{\beta_1+1} - \ell^{\beta_2+1} \right)}{(\ell+1)^{\beta_2+1}} \]

\[ \text{(7)} \]

In Appendix B we perform this summation. The answer is

\[ \bar{\ell}(k) = \frac{\beta_2}{\beta_1 + 1} (k + 2). \quad (8) \]

In the special case of \( \beta_1 = \beta_2 = m \), this reduces to \( \frac{m(k+2)}{1+m} \), which is consistent with the previous result in the literature [29].

We now focus on the distribution of total degree (i.e., the sum of degrees in the two layers). Let us denote \( k+\ell \) by \( q \). The joint distribution of \( q, k \) is simply \( n_{k,q-k} \). If we sum over all possible values of \( k \), we get the distribution of \( q \). Note that \( k \) is at least \( \beta_1 \), because every incoming node has an initial degree of \( \beta_1 \) in the first layer upon birth. Similarly, note that \( q - k_1 \) is at least \( \beta_2 \). Taking these two into account for the summation bounds, we
have:

\[ n(q) = \frac{2\beta_1(\beta_1 + 1)\beta_2(\beta_2 + 1) (\beta_1 + \beta_2 + 2)}{(2 + \beta_1 + \beta_2)} \times \sum_{k=\beta_1}^{q-\beta_2} \frac{(q-\beta_1 - \beta_2)}{k(k+1)(q-k)(q-k+1)(q+2)} \]

\[ = \frac{2\beta_1(\beta_1 + 1)\beta_2(\beta_2 + 1) (\beta_1 + \beta_2 + 2)}{(2 + \beta_1 + \beta_2)(q+2)(q+1)(q-1)} \sum_{k=\beta_1}^{q-\beta_2} \frac{(q-\beta_1 - \beta_2)}{(k+1)} . \]  

(9)

We use the following identity:

\[ \sum_{k=\beta_1}^{q-\beta_2} \frac{(q-\beta_1 - \beta_2)}{k-\beta_1} = \frac{(\beta_1 - 1)!((\beta_2 - 1)! (q-1))}{(\beta_2 + \beta_1 - 1)!} . \]  

(10)

This is proven in Appendix C. Plugging this result into (9), we arrive at

\[ n(q) = \frac{2(\beta_1 + \beta_2)(\beta_1 + \beta_2 + 1)}{q(q+1)(q+2)} u(q - \beta_1 - \beta_2) , \]  

(11)

where \( u(\cdot) \) is the Heaviside step function. This result means that the aggregated network is tantamount to one which grows under the preferential attachment mechanism where each incoming node establishes \( \beta_1 + \beta_2 \) links to existing nodes.

III. MODEL 2: UNIFORM ATTACHMENT IN TWO LAYERS

In this model, we assume that each incoming node establishes its link in both layers by selecting existing nodes uniformly at random. The rate equation (2) in the case of uniform attachment transforms into

\[ N_{k,\ell}(t+1) = N_{k,\ell}(t) + \beta_1 N_{k-1,\ell}(t) - N_{k,\ell}(t) \frac{N(0)}{t} + \delta_{k,\beta_1} \delta_{\ell,\beta_2} , \]  

(12)

Using the substitution \( n_{k,\ell}(t) = \frac{N_{k,\ell}(t)}{N(0)} \), this becomes

\[ \left[ N(0) + t \right] n_{k,\ell}(t+1) - n_{k,\ell}(t) + n_{t+1}(k,\ell) = \]

\[ + \beta_1 \frac{N_{k-1,\ell}(t) - N_{k,\ell}(t)}{N(0) + t} + \delta_{k,\beta_1} \delta_{\ell,\beta_2} , \]  

(13)

which simplifies to the following difference equation in the limit as \( t \rightarrow \infty \), that is, the steady state:

\[ n_{k,\ell} = \beta_1 \frac{n_{k-1,\ell} - n_{k,\ell}}{1} + \beta_2 \frac{n_{k,\ell-1} - n_{k,\ell}}{1} + \delta_{k,\beta_1} \delta_{\ell,\beta_2} . \]  

(14)

This can be simplified and equivalently expressed as follows

\[ n_{k,\ell} = \frac{\beta_1}{1 + \beta_1 + \beta_2} n_{k-1,\ell} + \frac{\beta_2}{1 + \beta_1 + \beta_2} n_{k,\ell-1} + \frac{\delta_{k,\beta_1} \delta_{\ell,\beta_2}}{1 + \beta_1 + \beta_2} . \]  

(15)

This difference equation is solved in Appendix C. The solution is

\[ n_{k,\ell} = \frac{\beta_1^{k-\beta_1} \beta_2^{\ell-\beta_2}}{(1 + \beta_1 + \beta_2)^k - \beta_1 + \ell - \beta_2 + 1} \]  

(16)

FIG. 3: (Color online) The inter-layer joint degree distribution for uniform growth with \( \beta_1 = 3 \) and \( \beta_2 = 3 \). Theoretical result is presented in [16]. Since the values decay fast in \( k \) and \( \ell \), we have depicted the logarithm of the inverse of this function, for a smoother output and better visibility. The joint distribution attains its maximum at \( k = \beta_1 \) and \( \ell = \beta_2 \). The contours are symmetric with respect to the bisector because \( \beta_1 \) and \( \beta_2 \) are equal.

To find the conditional average degree, we first need the degree distribution of single layers. This is found previously for example in [29, 35]. The degree distribution in the first layer is

\[ n_k = \frac{1}{\beta_1} \left( \frac{\beta_1}{\beta_1 + 1} \right)^{k-\beta_1+1} \]  

(17)
The result is undertaken the steps similar to those in the previous sections) is also obtained in [29]—see Equations (S.40) expression under preferential and uniform attachment schemes) is also obtained in [29]—see Equations (S.40) and (S.45) therein.

This is identical to [8]. For the special case of $\beta_1 = \beta_2$, this result (that the expected degree has the same expression under preferential and uniform attachment schemes) is also obtained in [29]—see Equations (S.40) and (S.45) therein.

Finally, to obtain the distribution of total degree, we undertake the steps similar to those in the previous sec-

We need to compute

\[
\tilde{\ell}(k) = \sum_{\ell} \ell n_{k,\ell} = \sum_{\ell} \ell \frac{n_{k,\ell}}{n_k} \beta_1^{k-\beta_1} \beta_2^{\ell-\beta_2} \left( k-\beta_1 + \ell - \beta_2 \right)
\]

\[
= \sum_{\ell} \ell \left( \frac{1}{\beta_1 + \beta_2} \right)^{k-\beta_1 + \ell - \beta_2 + 1} \frac{1}{\beta_1 + \beta_2} \left( \frac{\beta_2}{\beta_1 + 1} \right)^{k-\beta_1 + \ell - \beta_2} \left( \frac{\beta_2}{\beta_1 + 1} \right)^{k-\beta_1 + \ell - \beta_2}
\]

\[
= \frac{(\beta_1 + 1)^{k-\beta_1 + 1}}{(\beta_1 + \beta_2 + 1)^{k-\beta_1 + 1}} \sum_{\ell} \frac{\beta_1^{\ell-\beta_2}}{(1 + \beta_1 + \beta_2)^{\ell}} \left( \frac{\beta_2}{\beta_1 + 1} \right)^{k-\beta_1 + \ell - \beta_2}
\]

We have performed this summation in Appendix E. The result is

\[
\tilde{\ell}(k) = \frac{\beta_2}{\beta_1 + 1} (k + 2).
\] (19)

This is similar to [17]. This means that the degree distribution of the aggregated network is identical to that of a uniformly growing network in which each newcomer forms $\beta_1 + \beta_2$ links to existing nodes.

IV. GENERALIZATION OF MODEL 1 TO $M$ LAYERS

Now let us consider there are $M$ layers, and $M$ corresponding types of link. Each incoming node establishes $\beta_m$ links in layer $m$ to existing nodes in that layer, where $m \in \{1, 2, \ldots, M\}$. The initial number of links at layer $m$ is denoted by $L_m(0)$. In this model, the degree of each node can be conveniently represented by a vector of length $M$. The degree of nodes $x$, denoted by $k_x$, stores the degree of node $x$ in layer $m$ as its $m$-th component. Let $N(k)$ denote the number of nodes whose vector of degrees is $k$. Let $n(k)$ denote the fraction of those nodes. If node $x$ receives a link in layer $m$, then its degree will change to $k_x + 1e_m$, where $e_m$ is the unit vector in $m$-th dimension, that is, it is a vector whose elements are all zero except its $m$-th element, which is unity. Let us denote the $m$th component of vector $k$ by $k_m$. The rate equation for $N(k)$ reads

\[
N_{t+1}(k) = N_t(k) + \sum_{m=1}^{M} \delta_{k_m, \beta_m}
\]

\[
+ \sum_{m=1}^{M} \beta_m (k_m - 1)N_t(k - e_m) - (k_m)N_t(k)
\]

\[
N_m(0) + 2\beta_m t
\]

(21)

Now we use $N_t(k) \approx \frac{n_t(k)}{N(0)}$, and the following limit:

\[
\lim_{t \to \infty} \beta_m \frac{N(0) + t}{L_m(0) + 2\beta_m t} = \frac{1}{2}.
\] (22)

With these substitutions, we can rewrite (21) in the steady state as follows
\[ n(k) = \frac{1}{2} \sum_{m=1}^{M} \left( (k_m - 1)n(k - e_m) - (k_m)n(k) \right) \]
\[ + \prod_{m=1}^{M} \delta_{k_m, \beta_m} \]  
(23)

This can be rearranged and expressed equivalently as follows

\[ n(k) = \sum_{m=1}^{M} \left[ \frac{k_m - 1}{2 + \sum_{m=1}^{M} k_m} n(k - e_m) \right] \]
\[ + \frac{2}{2 + \sum_{m=1}^{M} \beta_m} \prod_{m=1}^{M} \delta_{k_m, \beta_m} \]  
(24)

We first define the following auxiliary function:

\[ \phi(k) = \frac{(2 + \sum_{m=1}^{M} k_m)!}{\prod_{m=1}^{M} (k_m - 1)!} n(k) \]  
(25)

Inserting this into (24) yields the following simplified equation

\[ \phi(k) = \sum_{m=0}^{M} \phi(k - e_m) + 2 \left( \frac{2 + \sum_{m=1}^{M} k_m)!}{\prod_{m=1}^{M} (k_m - 1)!} \prod_{m=1}^{M} \delta_{k_m, \beta_m} \right) \]
\[ = \sum_{m=0}^{M} \phi(k - e_m) + 2 \left( \frac{2 + \sum_{m=1}^{M} \beta_m)!}{\prod_{m=1}^{M} (\beta_m - 1)!} \prod_{m=1}^{M} \delta_{k_m, \beta_m} \right) \]  
(26)

We define the \( M \)-dimensional Z-transform as follows:

\[ \Phi(z) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_M=0}^{\infty} \phi(z) z^{-k_1-k_2-\cdots-k_M} \]  
(27)

Plugging this into (26), we obtain

\[ \Phi(z) = 2 \left( \frac{2 + \sum_{m=1}^{M} \beta_m)!}{\prod_{m=1}^{M} (\beta_m - 1)!} \prod_{m=1}^{M} z^{-\beta_m} \right) \]  
(28)

The inverse of this generating function is given by

\[ \phi(k) = 2 \left( \frac{2 + \sum_{m=1}^{M} \beta_m)!}{\prod_{m=1}^{M} (\beta_m - 1)!} \prod_{m=1}^{M} \right) \int \prod_{m=1}^{M} \frac{z_m^{-\beta_m-1}}{1 - \sum_{m=0}^{M} z_m^{-1}} (2\pi i)^M \]  
(29)

It is informative to look at the behavior of the integral for small \( M \), so we can speculate the answer for general \( M \). We consider the case of \( M = 4 \) for illustrative purposes. In this case, we need compute

\[ \frac{1}{(2\pi i)^4} \int \prod_{m=1}^{4} \frac{z_m^{k_m-\beta_m-1}}{1 - \sum_{m=0}^{4} z_m^{-1}} \]  
in order to find \( \phi(k) \). Let us denote this integral by \( I_4 \), and let us denote the \( M \)-dimensional integral appearing on the right hand side of (25) by \( I_M \). We perform the integration for \( I_4 \) by undertaking the following steps:
The pattern is clear and is readily generalizable for generic $M$. The value of the integral in $M$ dimensions is

$$I_M = \prod_{n=1}^{M} \left( \sum_{m=1}^{n} \frac{(k_m - \beta_m)}{k_n - \beta_n} \right).$$

(31)

Inserting this into (29), we arrive at

$$\phi(k) = 2 \frac{(2 + \sum_{m=1}^{M} \beta_m)!}{(\beta - 1)! (\beta + 2)!} \prod_{n=1}^{M} \left( \sum_{m=1}^{n} \frac{(k_m - \beta_m)}{k_n - \beta_n} \right).$$

(32)

Finally, from (25) we obtain

$$n(k) = \prod_{m=1}^{M} (k_m - 1)! \frac{(2 + \sum_{m=1}^{M} \beta_m)!}{(\beta - 1)! (\beta + 2)!} \prod_{n=1}^{M} \left( \sum_{m=1}^{n} \frac{(k_m - \beta_m)}{k_n - \beta_n} \right).$$

(33)

For the special case of $M = 2$ it is straightforward to verify that (33) reduces to (6). Also note that the last product on the right hand equals unity for the case of $M = 1$. In this case, we have

$$n(k) = \frac{(k - 1)! (\beta + 2)!}{(\beta - 1)! (k + 2)!} \frac{2 \beta (\beta + 1)}{k (k + 1) (k + 2)}. $$

(34)
This agrees with the degree distribution of the preferential attachment model, obtained for example in [36, 37].

V. GENERALIZATION OF MODEL 2 TO M LAYERS

Now we assume that there are $M$ layers, all of them growing under uniform attachment. The rate equation reads

$$N_{t+1}(k) = N_t(k) + \prod_{m=1}^{M} \delta_{k_m, \beta_m} + \sum_{m=1}^{M} \beta_m \frac{N_t(k - e_m) - N_t(k)}{N(0) + t}$$

(35)

For $n_t(k)$, this transforms into the following recurrence relation in the steady state

$$n(k) = \prod_{m=1}^{M} \delta_{k_m, \beta_m} + \sum_{m=1}^{M} \beta_m \left[ n_t(k - e_m) - n_t(k) \right]$$

(36)

This can be rearranged and recast as

$$n(k) = \frac{\prod_{m=1}^{M} \delta_{k_m, \beta_m}}{1 + \sum_{m=1}^{M} \beta_m} + \sum_{m=1}^{M} \beta_m \frac{\left[ n_t(k - e_m) \right]}{1 + \sum_{m=1}^{M} \beta_m}$$

(37)

Let us define

$$\begin{cases} B \equiv \sum_{m=1}^{M} \beta_m \\ q_m \equiv \frac{\beta_m}{B} \end{cases}$$

(38)

Taking the generating function of two sides of (37), we get

$$\psi(z) = \frac{1}{1 + B} \prod_{m=1}^{M} z_m^{-\beta_m} + \psi(z) \sum_{m=1}^{M} q_m z_m^{-1}.$$  

(39)

This can be equivalently expressed as follows

$$\psi(z) = \frac{1}{1 + B} \prod_{m=1}^{M} z_m^{-\beta_m} \cdot \frac{1}{1 - \sum_{m=1}^{M} q_m z_m^{-1}}.$$  

(40)

The inverse of this generating function yields the desired degree distribution:

$$n(k) = \frac{1}{1 + B} \left( \frac{1}{1 - (2\pi i)^M} \right)^{M} \frac{1}{1 - \sum_{m=1}^{M} q_m z_m^{-1}}.$$  

(41)

It is possible to pursue with generic $M$, but similar to Model 1, we take a more convenient approach. Let us invert this generating function for the example case of $M = 4$ for expository purposes, so that we can visibly speculate a pattern to generalize for generic $M$. We define $J_M$ as the integral that appears on the right hand side of (41). For $M = 4$, we compute $J_4$ by undertaking the following steps:
\[
J_4 \stackrel{\text{def}}{=} \frac{1}{(2\pi)^4} \int \int \int \frac{z_1^{k_1 - \beta_1 - 1} z_2^{k_2 - \beta_2 - 1} z_3^{k_3 - \beta_3 - 1} z_4^{k_4 - \beta_4 - 1}}{z_1^{k_1 - 1} - z_2^{k_2 - 1} - z_3^{k_3 - 1} - z_4^{k_4 - 1}} \, dz_1 dz_2 dz_3 dz_4 = \frac{1}{(2\pi)^4} \int \int \int \frac{z_1^{k_1 - \beta_1} z_2^{k_2 - \beta_2} z_3^{k_3 - \beta_3} z_4^{k_4 - \beta_4}}{z_1^{k_1 - 1} z_2^{k_2 - 1} z_3^{k_3 - 1} z_4^{k_4 - 1}} \, dz_1 dz_2 dz_3 dz_4
\]

One, we have (16) is straightforward to verify. For the case of a link in layer 2, the probability of receiving a link in layer 1 from the incoming node would be proportional to \(g_{11} k_x + g_{12} \ell_x\), and its probability of receiving a link in layer 2 would be proportional to \(g_{21} k_x + g_{22} \ell_x\).

VI. SUMMARY AND OPEN PROBLEMS

In this paper we focused on heterogeneous growth of multiplex networks. We considered both the preferential and uniform growth mechanisms. We analyzed the problem for \(M > 2\) layers and obtained the joint degree distributions.

An immediate generalization of this work is to consider nonzero couplings in the preferential growth mechanism, so that the link reception probability of a node in each layer depends on its degrees in all layers, and then find the joint degree distribution via the rate equation approach (nonzero couplings are envisaged in [28][30], but the rate equation approach remains unsolved and no closed-form solution for the inter-layer degree distribution exists in the literature). For example, in the two-layer case, the connection kernel would depend on the degrees of existing nodes in both layers. Consider an existing node \(x\), who has degree \(k_x\) in layer 1 and degree \(\ell_x\) in layer 2 at time \(t\). Then its probability of receiving a link in layer 1 from the incoming node would be proportional to \(g_{11} k_x + g_{12} \ell_x\), and its probability of receiving a link in layer 2 would be proportional to \(g_{21} k_x + g_{22} \ell_x\).

Our solution presented in this paper pertains to the special case of \(g_{11} = g_{22} = 1\) and \(g_{12} = g_{21} = 0\). The rate equation corresponding to the general case (counterpart of Equation (1), with all four \(g_{ij}\)'s nonzero, would be as follows:
Finally, a related quantity which is ubiquitous in the studying of epidemics and various diffusion processes over networks is the nearest-neighbor degree distribution \([38-41]\). Let us denote this quantity by \(p_1(k,\ell|q, r)\). For a node with degree \(k\) in layer 1 and degree \(\ell\) in layer 2, if we select one of its neighbors in layer 1 randomly, then \(p_1(k,\ell|q, r)\) would be the probability that this neighbor has degree \(q\) in layer 1 and degree \(r\) in layer 2. Similarly, \(p_2(k,\ell|q, r)\) would be the probability that a randomly selected layer-2 neighbor of a node with degrees \(k,\ell\) will have degrees \(q,r\). This quantity is essential for studying dynamics on networks, and to obtain it for multiplex networks, one also needs exact expressions for the degree distributions—which the present paper focused on.

**Appendix A: Solving Difference Equation**

We need to solve

\[
n_{k\ell} = \frac{k - 1}{k + \ell} n_{k-1\ell} + \frac{\ell - 1}{k + \ell} n_{k\ell-1} + \frac{2\delta_{k\beta_1}\delta_{\ell\beta_2}}{2 + \beta_1 + \beta_2}. \tag{A1}
\]

We define the new sequence

\[
m_{k\ell} \overset{\text{def}}{=} \frac{(k + \ell + 2)!}{(k - 1)!(\ell - 1)!} n_{k\ell}. \tag{A2}
\]

The following holds

\[
\begin{align*}
\frac{k - 1}{k + \ell + 2} n_{k-1\ell} &= \frac{(k - 1)!(\ell - 1)!}{n} k_{k-1\ell} \\
\frac{\ell - 1}{k + \ell + 2} n_{k\ell-1} &= \frac{(k - 1)!(\ell - 1)!}{n} k_{k\ell-1}.
\end{align*} \tag{A3}
\]

Plugging these into (A1), we can recast it as

\[
m_{k\ell} = m_{k-1\ell} + m_{k\ell-1} + 2 \frac{(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)! (\beta_2 - 1)!} \delta_{k\beta_1}\delta_{\ell\beta_2}. \tag{A4}
\]

Now define the Z-transform of sequence \(m_{k\ell}\) as follows:

\[
\psi(z, y) \overset{\text{def}}{=} \sum_k \sum_{\ell} m_{k\ell} z^{-k} y^{-\ell}
\]

Taking the Z transform of every term in (A4), we arrive at

\[
\psi(z, y) = z^{-1} \psi(z, y) + y^{-1} \psi(z, y) + 2 \frac{(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)! (\beta_2 - 1)!} z^{-\beta_1} y^{-\beta_2}. \tag{A6}
\]
This can be rearranged and rewritten as follows

\[ \psi(z, y) = \frac{2}{1 - z^{-1} - y^{-1}} \frac{(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} z^{-\beta_1} y^{-\beta_2} \]  
\[ \text{(A7)} \]

The inverse transform is given by

\[ m_{k,\ell} = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} \oint \oint \frac{z^{k-\beta_1-1} y^{\ell-\beta_2-1} dzdy}{(-4\pi^2)(1 - z^{-1} - y^{-1})} \]
\[ = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} \oint \oint \frac{z^{k-\beta_1} y^{\ell-\beta_2} dzdy}{(-4\pi^2)(zy - z - y)} \]
\[ = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} \oint \oint \frac{z^{k-\beta_1} y^{\ell-\beta_2} dzdy}{(-4\pi^2)(y - 1)[z - \frac{y}{y-1}]} \]  
\[ \text{(A8)} \]

First we integrate over \( z \). We get

\[ m_{k,\ell} = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} \oint \frac{(y^{-1})^{k-\beta_1} y^{\ell-\beta_2} dy}{(2\pi i)(y - 1)} \]
\[ = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} \oint \frac{y^{k-\beta_1+\ell-\beta_2} dy}{(2\pi i)(y - 1)k-\beta_1+1} \]  
\[ \text{(A9)} \]

Now note that the residue of \( \frac{f(y)}{(y - 1)^n} \) for positive integer equals \( \frac{f^{(n-1)}(1)}{(n - 1)!} \), where the numerator denotes the \( (n - 1) \)th derivative of the function \( f(y) \), evaluated at \( y = 1 \). Also, note that the \( m \)th derivative of the function \( y^n \), for integer \( n \) and \( m \), equals \( \frac{m!}{(n - m)!} y^{n-m} \). Combining these two facts, we obtain

\[ m_{k,\ell} = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} \binom{k-\beta_1+\ell-\beta_2}{k-\beta_1} \]  
\[ \text{(A10)} \]

Using \( \text{(A2)} \), we arrive at

\[ n_{k,\ell} = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 - 1)!((\beta_2 - 1)!)} \frac{1}{k(k+1)\ell(\ell+1)} \binom{k-\beta_1+\ell-\beta_2}{k-\beta_1} \]  
\[ \text{(A11)} \]

This can be equivalently expressed as follows:

\[ n_{k,\ell} = \frac{2(\beta_1 + \beta_2 + 1)!}{(\beta_1 + \beta_2 + 2)k(k+1)\ell(\ell+1)} \binom{\beta_1+\beta_2+2}{\beta_1+1} \binom{k-\beta_1+\ell-\beta_2}{k-\beta_1} \]  
\[ \text{(A12)} \]

**Appendix B: Performing the Summation in** \( \text{(7)} \)

We need to calculate

\[ \tilde{\ell}(k) = \sum_{\ell} \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \frac{\binom{\beta_1+\beta_2+2}{\beta_1+1} \binom{k-\beta_1+\ell-\beta_2}{k-\beta_1}}{\binom{\ell+2}{\ell}} \]  
\[ \text{(B1)} \]

We use the following identity:

\[ \frac{1}{m(m+1)} = (n+1) \int_0^1 t^m(1-t)^{n-m} dt, \]  
\[ \text{(B2)} \]

To rewrite the binomial reciprocal of the coefficient as follows

\[ \frac{1}{\binom{k+\ell+2}{\ell}} = (k + \ell + 3) \int_0^1 t^\ell(1-t)^{k+2} dt. \]  
\[ \text{(B3)} \]

Also, from Taylor expansion, it is elementary to show that

\[ S_1(x, n) \overset{\text{def}}{=} \sum_{m} x^m \binom{m}{n} = \frac{x^n}{(1-x)^{n+1}}. \]  
\[ \text{(B4)} \]

This identity will be used in the steps below. Plugging \( \text{(B3)} \) into \( \text{(B5)} \), we have
\[
\bar{\ell}(k) = \sum_{\ell} \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \left( \frac{\beta_1 + \beta_2 + 2}{\beta_1 + 1} \right) \sum_{\ell} (k + \ell + 3) \left( \frac{k - \beta_1 + \ell - \beta_2}{k - \beta_1} \right) \int_0^1 t^{\ell}(1-t)^{k+\ell+2} dt \\
= \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \left( \frac{\beta_1 + \beta_2 + 2}{\beta_1 + 1} \right) \int_0^1 (1-t)^{k+\ell+2} t^{-k-2} \sum_{\ell} (k + \ell + 3) t^{k+\ell+2} \left( \frac{k - \beta_1 + \ell - \beta_2}{k - \beta_1} \right) dt \\
= \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \left( \frac{\beta_1 + \beta_2 + 2}{\beta_1 + 1} \right) \int_0^1 (1-t)^{k+\ell+2} t^{-k-2} d\frac{d}{dt} \left[ \sum_{\ell} t^{k+\ell+3} \left( \frac{k - \beta_1 + \ell - \beta_2}{k - \beta_1} \right) \right] dt \\
= \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \left( \frac{\beta_1 + \beta_2 + 2}{\beta_1 + 1} \right) \int_0^1 (1-t)^{k+\ell+2} t^{-k-2} d\frac{d}{dt} \left[ t^{k+\beta_1 + \beta_2} \left( \frac{t^{k-\beta_1}}{(1-t)^{k-\beta_1+1}} \right) \right] dt \\
= \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \left( \frac{\beta_1 + \beta_2 + 2}{\beta_1 + 1} \right) \int_0^1 (1-t)^{k+\ell+2} t^{-k-2} d\frac{d}{dt} \left[ \frac{t^{k+\beta_1 + \beta_2+3}}{(1-t)^{k-\beta_1+1}} \right] dt \\
= \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \left( \frac{\beta_1 + \beta_2 + 2}{\beta_1 + 1} \right) \int_0^1 (1-t)^{k+\ell+2} t^{-k-2} d\frac{d}{dt} \left[ \frac{t^{\beta_1+\beta_2+3}}{(1-t)^{k-\beta_1+1}} \right] dt \\
= \frac{\beta_2(\beta_2 + 1)}{(2 + \beta_1 + \beta_2)} \left( \frac{\beta_1 + \beta_2 + 2}{\beta_1 + 1} \right) \left[ (k + \beta_2 + 3) \int_0^1 t^{\beta_1+\beta_2} dt - (1 + \beta_1 + \beta_2) \int_0^1 (1-t)^{\beta_1+\beta_2+1} dt \right] \\
= \frac{\beta_2(\beta_2 + 1)(\beta_1 + \beta_2 + 2)}{(2 + \beta_1 + \beta_2)(1 + \beta_1 + \beta_2)!} \left( \frac{\beta_1+\beta_2+3}{\beta_1+\beta_2+1} \right) [k + \beta_2 + 3] (\beta_1+\beta_2+3) \\
= \frac{\beta_2(\beta_2 + 1)}{(\beta_1 + \beta_2+1)!} \left( \frac{\beta_1+\beta_2+3}{\beta_1+\beta_2+1} \right) [k + \beta_2 + 3] (\beta_1+\beta_2+3) \\
= \frac{\beta_2}{\beta_1 + 1} (k + 2) \\
\]

Appendix C: Proving the Identity Given in (10)

\[
\sum_{k} \frac{(q-\beta_1-\beta_2)}{(q-2)} \frac{(k-\beta_1-\beta_2)}{(k-1)} dt \\
= \sum_{k} \left( \frac{q - \beta_1 - \beta_2}{k - \beta_1} \right) (q-1) \int_0^1 (1-t)^{k-1} t^{q-k-1} dt \\
= (q-1) \int_0^1 t^{q-1-\beta_1-\beta_2-1} \sum_{k} \left( \frac{q - \beta_1 - \beta_2}{k - \beta_1} \right) \left( \frac{1-t}{t} \right)^{k-\beta_1} dt \\
= (q-1) \int_0^1 \sum_{k} t^{q-1-\beta_1-\beta_2} (1-t)^{\beta_1-1} \left( 1 + \frac{1-t}{t} \right)^{q-\beta_1-\beta_2} dt \\
= (q-1) \int_0^1 t^{\beta_2-1} (1-t)^{\beta_2-1} \left( \frac{q-\beta_1-\beta_2}{\beta_2 + (\beta_2 + 1)!} \right) \\
= \frac{(q-1)!}{(\beta_2 + (\beta_2 + 1)!)}. \\
\]

We use the property of the Gamma integral given in [B2] in order to deal with the binomial coefficient that is in the denominator of the summand. We have:
Appendix D: Solving Difference Equation

Let us repeat the equation we need to solve for easy reference

\[ n_{k,t} = \frac{\beta_1}{1 + \beta_1 + \beta_2} n_{k-1,t} + \frac{\beta_2}{1 + \beta_1 + \beta_2} n_{k,t-1} + \frac{\delta_{k,\beta_1} \delta_{t,\beta_2}}{1 + \beta_1 + \beta_2}. \]  

(D1)

Let us define the following quantities from brevity:

\[
\begin{align*}
q_1 &\overset{\text{def}}{=} \frac{\beta_1}{1 + \beta_1 + \beta_2} \\
q_2 &\overset{\text{def}}{=} \frac{\beta_2}{1 + \beta_1 + \beta_2}
\end{align*}
\]

(D2)

Taking the Z transform from both sides of (D1), we get

\[
\psi(z, y) = q_1 z^{-1} \psi(z, y) + q_2 y^{-1} \psi(z, y) + \frac{z^{-\beta_1} y^{-\beta_2}}{1 + \beta_1 + \beta_2}.
\]

(D3)

This can be rearranged and recast as

\[
\psi(z, y) = \frac{1}{1 - q_1 z^{-1} - q_2 y^{-1}} \frac{z^{-\beta_1} y^{-\beta_2}}{1 + \beta_1 + \beta_2}.
\]

(D4)

This can be inverted through the following steps

\[
\begin{align*}
n_{k,t} &= \frac{1}{(1 + \beta_1 + \beta_2)(2\pi i)^2} \int \int \psi(z, y) z^{-k-1} y^{-t-1} dz dy \\
&= \frac{1}{(1 + \beta_1 + \beta_2)(2\pi i)^2} \int \int \frac{z^{-k-1} y^{-t-1}}{1 - q_1 z^{-1} - q_2 y^{-1}} dz dy \\
&= \frac{1}{(1 + \beta_1 + \beta_2)(2\pi i)^2} \int \int \frac{z^{-k-1} y^{-t-1}}{z - \frac{q_1 y}{y - q_2}} \frac{1}{y - q_2} dz dy \\
&= \frac{1}{(1 + \beta_1 + \beta_2)(2\pi i)^2} \int \int \frac{y^{t+\beta_2} \left( \frac{y q_1}{y - q_2} \right)^{k+\beta_1}}{y - q_2} dz dy \\
&= \frac{1}{(1 + \beta_1 + \beta_2)(2\pi i)^2} \int \int \frac{y^{t+\beta_2} \left( \frac{y q_1}{y - q_2} \right)^{k+\beta_1}}{y^{k+\beta_1+\beta_2+1}} dz dy \\
&= \frac{1}{(1 + \beta_1 + \beta_2)} \frac{1}{(k+\beta_1+\beta_2)!} q_1^{k-\beta_1} q_2^{\ell-\beta_2} \\
&= \frac{1}{(1 + \beta_1 + \beta_2)} \frac{1}{(k+\beta_1+\beta_2)!} q_1^{k-\beta_1} q_2^{\ell-\beta_2}.
\end{align*}
\]

(D5)

After inserting the expressions for \( q_1, q_2 \) from [D2], this becomes

\[
n_{k,t} = \frac{\beta_1^{k-\beta_1} \beta_2^t \beta^{-\beta_2}}{(1 + \beta_1 + \beta_2)^{k+\beta_1+\beta_2+1}}.
\]

(D6)

Appendix E: Performing the Summation

We need to perform the following summation:

\[
\bar{\ell}(k) = \frac{(\beta_1 + 1)^{k-\beta_1+1}}{(1 + \beta_1 + \beta_2)^{k+\beta_1+\beta_2+1}} \sum_{\ell} \frac{\beta_1^{k-\beta_1+\ell} \beta_2^{\ell-\beta_2}}{(1 + \beta_1 + \beta_2)^\ell}.
\]

(E1)

Let us denote \( k - \beta_1 \) by \( k' \) and \( \ell - \beta_2 \) by \( \ell' \). Also let us denote \( \frac{\beta_2}{1 + \beta_1 + \beta_2} \) by \( x \). We need to evaluate the following sum: \( \sum_{\ell'} (\beta_2 + \ell') x^{k'} \). Let us use (B4) and define

\[
S_1(x, k') \overset{\text{def}}{=} \sum_m x^m \left( \frac{m}{n} \right) = \frac{x^n}{1 - x}.
\]

(E2)

We have:

\[
\sum_{\ell'} (\beta_2 + \ell') x^{k'} = \beta_2 x^{-k'} S_1(x, k') + x \sum_{\ell'} \ell' x^{k'-1} \left( \frac{k'}{k'} \right) = \beta_2 x^{-k'} S_1(x, k') + \frac{d}{dx} \left( x^{-k'} S_1(x, k') \right) = \beta_2 x^{k'} \left( \frac{x^{k'}}{1 - x} \right)' + \frac{d}{dx} \left( \frac{x^{k'}}{1 - x} \right)'
\]

(E3)

Replacing \( x \) with \( \frac{\beta_2}{1 + \beta_1 + \beta_2} \) and \( k' \) by \( k - \beta_1 \) and inserting this result into (E1), we get

\[
\frac{1}{1 - \left( \frac{\beta_2}{1 + \beta_1 + \beta_2} \right)^{k+\beta_1+\beta_2+2}} \left[ \beta_2 + \frac{\beta_2}{1 + \beta_1 + \beta_2} (k - \beta_1 + 1 - \beta_2) \right] = \frac{1 + \beta_1 + \beta_2}{(1 + \beta_1)^{k+\beta_1+\beta_2+2}} \left[ \beta_2 + \frac{\beta_2}{1 + \beta_1 + \beta_2} (k - \beta_1 + 1 - \beta_2) \right] = \frac{1 + \beta_1 + \beta_2}{(1 + \beta_1)^{k+\beta_1+\beta_2}} \left[ \beta_2 (k + 2) + \frac{\beta_2}{1 + \beta_1 + \beta_2} \right].
\]

(E4)

Plugging this into (E1), we get

\[
\bar{\ell}(k) = \frac{\beta_2 (k + 2)}{1 + \beta_1}.
\]

(E5)

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