The Dirichlet problem for CMC surfaces in Heisenberg space

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We study constant mean curvature graphs in the Riemannian 3-dimensional Heisenberg spaces $\mathcal{H} = \mathcal{H}(\tau)$. Each such $\mathcal{H}$ is the total space of a Riemannian submersion onto the Euclidean plane $\mathbb{R}^2$ with geodesic fibers the orbits of a Killing field. We prove the existence and uniqueness of CMC graphs in $\mathcal{H}$ with respect to the Riemannian submersion over certain domains $\Omega \subset \mathbb{R}^2$ taking on prescribed boundary values.

1 Introduction

In recent years, there has been much research on minimal and constant mean curvature surfaces (CMC) in the simply connected homogeneous 3-manifolds, other than space forms. Figueroa, Mercuri and Pedrosa [5] gave many interesting such surfaces in $\mathcal{H}$, each invariant by Killing vector fields of the ambient space. Daniel [4] and Abresch-Rosenberg [1], [2] have also obtained some interesting results on these surfaces. For example, the latter authors proved that the only immersed $H$-surfaces in $\mathcal{H}$ which are homeomorphic to the 2-sphere are precisely the rotational $H$-spheres. We mention that the classical Alexandrov Theorem is not yet known in $\mathcal{H}$: “Is a compact embedded $H$-surface a rotational sphere”.

It is natural (and we believe important) to solve the Dirichlet problem in $\mathcal{H}$; we do this here.

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2 Preliminaries

2.1 The Heisenberg space

Let \( \mathcal{H} \) denote the three-dimensional Heisenberg Lie group endowed with a left invariant metric. In fact, we have a one-parameter family of metrics indexed by bundle curvature by a real parameter \( \tau \neq 0 \). The spaces are simply connected homogeneous Riemannian manifolds carrying a 4-dimensional isometry group. In global exponential coordinates they are \( \mathbb{R}^3 \) endowed in standard coordinates with the metrics

\[
ds^2 = dx^2 + dy^2 + \left( \tau (ydx - xdy) + dz \right)^2.
\]

A global orthonormal tangent frame is given by

\[
E_1 = \partial_x - \tau y \partial_z, \quad E_2 = \partial_y + \tau x \partial_z, \quad E_3 = \partial_z.
\]

The corresponding Riemannian connection is \( \bar{\nabla}_{E_j} E_j = 0 \), \( 1 \leq j \leq 3 \), and

\[
\bar{\nabla}_{E_1} E_3 = \bar{\nabla}_{E_3} E_1 = -\tau E_2, \quad \bar{\nabla}_{E_2} E_3 = \bar{\nabla}_{E_3} E_2 = \tau E_1
\]

\[
\bar{\nabla}_{E_1} E_2 = -\bar{\nabla}_{E_2} E_1 = \tau E_3.
\]

In particular,

\[
[E_1, E_2] = 2\tau E_3 \quad \text{and} \quad [E_1, E_3] = 0 = [E_2, E_3].
\]

The Heisenberg space is a Riemannian submersion \( \pi: \mathcal{H} \to \mathbb{R}^2 \) over the standard flat Euclidean plane \( \mathbb{R}^2 \) whose fibers are the vertical lines. Thus the fibers are the trajectories of a unit Killing vector field and hence geodesics. The horizontal vector fields \( E_1, E_2 \) are basic since they are the horizontal lifts of the vector fields of the orthonormal coordinate base of \( \mathbb{R}^2 \), namely, \( \pi_*(E_1) = \partial_x \) and \( \pi_*(E_2) = \partial_y \).

The isometries of the space are the translations generated by the Killing vector fields

\[
F_1 = \partial_x + \tau y \partial_z, \quad F_2 = \partial_y - \tau x \partial_z, \quad F_3 = \partial_z,
\]

and the rotations about the \( z \)-axis corresponding to

\[
F_4 = -y \partial_x + x \partial_y.
\]

The translations corresponding to \( F_1 \) and \( F_2 \) are, respectively,

\[
(x, y, z) \mapsto (x + t, y, z + \tau ty)
\]

and

\[
(x, y, z) \mapsto (x, y + t, z - \tau tx)
\]

where \( t \in \mathbb{R} \). Thus, by the group of isometries vertical planes go to vertical planes, and Euclidean lines go to Euclidean lines. For additional information, we refer to [4].
2.2 Graphs

We denote by $S_0 \subset \mathcal{H}$ the surface whose points satisfy $z = 0$. Given a domain $\Omega \subset \mathbb{R}^2$ throughout the paper we also denote by $\Omega$ its lift to $S_0$. We define the graph $\Sigma(u)$ of $u \in C^0(\bar{\Omega})$ on $\Omega$ as

$$\Sigma(u) = \{(x, y, u(x, y)) \in \mathcal{H} : (x, y) \in \Omega\}.$$ 

Consider the smooth function $u^* : \mathcal{H} \to \mathbb{R}$ defined as $u^*(x, y, z) = u(x, y)$ and set $F(x, y, z) = z - u^*(x, y, z)$. Then $\Sigma(u) = F^{-1}(0)$, and therefore

$$2H = \text{div} \left( \frac{\nabla F}{|\nabla F|} \right).$$

Here $\text{div}$ and $\nabla$ denote the divergence and gradient in $\mathcal{H}$ and the mean curvature function $H$ of the graph is with respect to the downward pointing normal vector.

We have

$$\nabla F = -(\tau y + u_x)E_1 + (\tau x - u_y)E_2 + E_3.$$ 

Since $E_1, E_2$ are basic, using the Riemannian submersion one shows that the $H$-graph equation is

$$\text{div}_{\mathbb{R}^2} \left( \frac{\alpha}{W} \partial_x + \frac{\beta}{W} \partial_y \right) + 2H = 0 \quad (1)$$

where

$$\alpha = \tau y + u_x, \quad \beta = -\tau x + u_y$$

and

$$W^2 = 1 + \alpha^2 + \beta^2.$$ 

It follows easily that $\Sigma(u)$ has mean curvature function $H$ if and only if $u$ is a solution of the following PDE

$$Q_H(u) := \frac{1}{W^3} \left( (1 + \beta^2)u_{xx} + (1 + \alpha^2)u_{yy} - 2\alpha \beta u_{xy} \right) + 2H = 0 \quad (2)$$

for $\alpha, \beta$ and $W$ as above. We remark that this is the Euclidean mean curvature equation for $\tau = 0$.

2.3 Cylinders and cones

Let $\gamma : I \to S_0 \subset \mathcal{H}$ be a smooth curve parametrized on an interval $I \subset \mathbb{R}$ where $S_0$ is as above. We assume that $\gamma = \gamma(s)$ is parametrized so that $\gamma = \pi \circ \gamma$ carries a parametrization by arc length. Thus $\gamma(s) = (x(s), y(s), 0)$ satisfies $(x')^2 + (y')^2 = 1$. 

3
The vertical cylinder $C_{\gamma} \subset H$ over $\gamma$ is the surface generated by taking through each point of $\gamma$ the vertical geodesic fiber. Thus $C_{\gamma}$ is parametrized by $\varphi: I \times \mathbb{R} \to H$ given by

$$\varphi(s, t) = (x(s), y(s), t).$$

Then, the mean curvature $H_{C}$ (taken to be non-negative) of $C_{\gamma}$ is

$$H_{C}(s, t) = H_{C}(s) = \frac{k(s)}{2}$$

where $k(s)$ is the geodesic curvature function of $\bar{\gamma}$ with respect to the Euclidean metric. Notice that $H_{C}$ is independent of the parameter $\tau$. To see that (3) holds, first observe that the horizontal lift $T$ of $\bar{\gamma}' = d\bar{\gamma}/ds$ to each point of $C_{\gamma}$ forms a horizontal unit tangent vector field. Since $C_{\gamma}$ is ruled by vertical geodesics, it follows that the mean curvature of $C_{\gamma}$ is $2H_{C} = \langle \bar{\nabla}_{T}T, N \rangle$, where $N$ is the Gauss map of the cylinder $C_{\gamma}$ chosen so that $H_{C}$ is non-negative. But $N$ is the horizontal lift of a unit normal vector field $\eta$ to $\bar{\gamma}$ in $\mathbb{R}^{2}$, and hence $\langle \bar{\nabla}_{T}T, N \rangle = \langle D_{\gamma}\bar{\gamma}', \eta \rangle = k$, where $D$ denotes the Euclidean connection.

The cone $C_{\gamma} \subset H$ with vertex $P \in H \setminus S_{0}$ and base curve $\gamma$ as above is just the Euclidean cone in $\mathbb{R}^{3}$ constituted of straight lines from $P$ through points of $\gamma$. Thus $C_{\gamma}$ is parametrized by

$$\psi(s, t) = (1 - t)P + t\gamma(s)$$

where $t \in (0, +\infty)$.

Vertical lines remain invariant under the isometries of $H$. Thus the same holds for vertical cylinders. Also Euclidean lines are sent to Euclidean lines by isometries of $H$, and vertical planes as well. Thus cones are also invariant by isometries. Hence, to analyze the behavior of the mean curvature of a cone we may assume that the vertex is $P = (0, 0, c)$ where $c \neq 0$. Then, either a computation using (2) or by a direct computation, we obtain that the mean curvature $H = H(s, t)$ of $C_{\gamma}$ pointing down is given by

$$H = \frac{ct^{2}(x^{2} + y^{2} + c^{2})y''x' - x''y'}{2(\tau^{2}t^{4}(x^{2} + y^{2})(x'y - y'x)^{2} + 2c\tau t^{3}(xx' + yy')(x'y - y'x) + t^{2}(c^{2} + (x'y - y'x)^{2})^{3/2}}.$$

Here the sign of $H$ is non-negative when $\gamma$ is a convex Jordan curve in $\mathbb{R}^{2}$. In particular,

$$H(s, 1) \to H_{C}(s) \quad \text{as} \quad c \to +\infty.$$

and

$$H(s_{0}, t) \to +\infty \quad \text{as} \quad t \to 0^{+}$$

if $y''x' - x''y' > 0$ at $\gamma(s_{0})$.

We also have fixing $t = t_{0}$ and allowing $c \to +\infty$ that

$$2H(s_{0}, t_{0}) \to (y''x' - x''y')(s_{0}),$$

and this is also a proof that the mean curvature of a cylinder is given by (3).
3 The main result

We now state and prove the Dirichlet theorem in Heisenberg space $\mathcal{H}$.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^3$ boundary $\Gamma = \partial \Omega$ whose curvature function with respect to the inner orientation is $k > 0$. Let $H$ be a constant satisfying $0 \leq 2H < k$ and let $\varphi \in C^0(\Gamma)$ be given. Then there exists a smooth function $u$ satisfying $u|_{\Gamma} = \varphi$ whose graph $\Sigma(u)$ in $\mathcal{H}$ has constant mean curvature $H$.

Moreover, if $M$ is a compact embedded connected surface inside the vertical cylinder $\mathbb{C}_\Gamma$ over $\Gamma$ with constant mean curvature $H$, $\partial M = \partial \Sigma(u)$ and the mean curvature vector of $M$ points down, then $M = \Sigma(u)$.

**Proof:** First suppose that $H = 0$. In this case we prove a more general existence result. In fact, we allow $k \geq 0$, and $\varphi$ to have a finite number of discontinuities $E \subset \Gamma$, and at each discontinuity, $\varphi$ has a left and right limit. The Nitsche graph (see [7]) $\gamma$ of $\varphi$ is the graph of $\varphi$ on $\Gamma \setminus E$ together with the vertical segments over each point of $E$, joining the left and right limits of $\varphi$ at this point. The Nitsche graph $\gamma$ is a Jordan curve on the vertical cylinder $\mathbb{C}_\Gamma$ and its vertical projection to $\Gamma$ is a monotone (constant on the vertical segments) map.

Since $\mathbb{C}_\Gamma$ is mean convex with respect to the inside of $\mathbb{C}_\Gamma$, there is a least area embedded minimal disk $\Sigma$ inside $\mathbb{C}_\Gamma$ with $\partial \Sigma = \gamma$.

We claim that $\Sigma$ is a $z$-graph over $\Omega$ and solves the Dirichlet problem as desired. First observe that $\Sigma$ is nowhere vertical. To see this, suppose $p \in \text{int } \Sigma$ and the tangent plane to $\Sigma$ at $p$ is vertical. Let $\beta \in \mathbb{R}^2$ be a line such that the vertical plane $P = \pi^{-1}(\beta)$ equals the tangent plane to $\Sigma$ at $p$. Then $P \cap \Sigma$ near $p$ is an analytic curve topologically equivalent to $Re(z^k)$, $k \geq 2$, in a neighborhood of $z = 0$. Each branch of these curves leaving $p$ must go to $P \cap \partial \Sigma = P \cap \gamma$, by the maximum principle, i.e., a cycle in $(\text{int } \Sigma) \cap P$ would bound a disk in $\Sigma$ and we could touch this disk at an interior point with another vertical plane (which is also a minimal surface). Now $P \cap \gamma$ consists of two points of $\Gamma$, or one or two vertical segments of $\gamma$, by convexity of $\Gamma$. Hence, at least two of the branches of $P \cap \Sigma$ leaving $p$, go to the same point, or vertical segments of $\gamma$. This yields a compact cycle $C \subset P \cap \Sigma$. $\Sigma$ is simply connected so $C$ bounds a disk $D \subset \Sigma$. Using vertical planes in $\mathcal{H}$, we can touch $D$ at an interior point so $D$ would equal this vertical plane; a contradiction. Thus $\Sigma$ is nowhere vertical in its interior.

Now $\Sigma$ separates the vertical cylinder over $\Gamma$ into two components. So $\Sigma$ can be oriented with the unit normal pointing up in its interior. Then each vertical line over a point in the interior of $\Omega$, intersects $\Sigma$ in exactly one point, since at two successive points of intersection the normal to $\Sigma$ would point up and down. This proves $\Sigma$ is a graph over the interior of $\Omega$. 

Now assume that $H \neq 0$ and $\varphi$ is continuous. We have seen that $u$ must be a solution of the Dirichlet problem
\[
\begin{aligned}
Q_H(u) &= 0 \\
|\Gamma| &= \varphi
\end{aligned}
\]
(4)
where $Q_H$ was given in (2). To prove the existence part of the theorem, we use the continuity method. We show that the subset
\[
Z := \{ t \in [0, 1] : \exists u_t \in C^3(\Omega) \text{ such that } Q_{tH}(u_t) = 0 \text{ and } u_t|_{\Gamma} = t\varphi \}
\]
is nonempty, open and closed in $[0, 1]$. We have that $Z$ is not empty since $0 \in Z$; $S_0$ is a minimal surface in $H$. Standard arguments from the theory of quasilinear elliptic PDE’s presented in [6] give that $Z$ is open (a consequence of the implicit function theorem). Moreover, any solution of $Q_H(u) = 0$ is smooth in $\Omega$. Finally, that $Z$ is closed follows from the theory in [6] once we show that a priori height and gradient estimates exist.

We have from (2) that any Euclidean plane in $\mathbb{R}^3$ is a minimal surface in $H$. In particular, each leaf of the foliation of isometric surfaces $z = z_0 = \text{constant}$ is minimal and diffeomorphic to the base $\mathbb{R}^2$ by the projection of the Riemannian submersion. It follows using the maximal principle that any solution $u$ of (4) satisfies
\[
u \geq \min_{\partial \Omega} \varphi.
\]

Fix a point $(x_0, y_0, 0) \in \Omega$. Given $z_0 \in \mathbb{R}$, we consider the cone $C(z_0)$ with vertex $P = (x_0, y_0, z_0)$ constituted of straight lines from $P$ through points of the graph of $\varphi$ over $\Gamma$. Then, the piece $C_{\varphi}(z_0)$ of $C(z_0)$ from $P$ to the graph of $\varphi$ is contained inside the vertical cylinder over $\Gamma$. Notice that $C(z_0)$ is the cone $C_\Gamma(z_0)$ over $\hat{\Gamma} = C_\Gamma(z_0) \cap S_0$. Clearly, by choosing $z_0$ such that $|z_0|$ is large enough, the geodesic curvature of $\hat{\Gamma}$ with respect to the Euclidean metric is positive. In fact, the curve converges to $\Gamma$ as $|z_0| \to \infty$. Therefore, by our previous discussion on the mean curvature of vertical cylinders and cones we have that choosing $z_0$ large enough, say $z_0 = z_1$, and $z_0$ small enough, say $z_0 = z_2$, that $C(z_1)$ has mean curvature strictly larger than $H$ everywhere and $C(z_2)$ has negative mean curvature (this cone is going down). By the maximum principle, they are upper and lower barriers for the CMC $H$-graph equation on $\Omega$. Thus $C(z_1)$ and the above remark concerning planes below the graph of $\varphi$ provides an a priori height estimate for any solution of the Dirichlet problem (4) depending only on $\Omega$, $H$ and $\varphi$, that is,
\[
|u|_0 \leq C_0(\Omega, H, \varphi).
\]
Moreover, the cones also provide the following bound along $\Gamma$ for the norm of the Euclidean gradient of $u$
\[
|\nabla u| = \sqrt{u_x^2 + u_y^2} \leq C_1(\Omega, H, \varphi).
\]
The next result uses techniques developed in [3] to show that global estimates of the gradient reduces to the boundary estimates already obtained.

**Lemma 2.** Let \( u \in C^3(\Omega) \cap C^1(\bar{\Omega}) \) be a solution of (4). Assume that \( u \) is bounded in \( \Omega \) and that \( |Du| \) is bounded in \( \Gamma \). Then \( |Du| \) is bounded in \( \Omega \) by a constant that depends only on \( |u|_0 \) and \( \sup_{\Gamma} |Du| \).

**Proof:** To estimate \( |Du| = \sqrt{u_x^2 + u_y^2} \) in the interior of \( \Omega \) it suffices to obtain an estimate for \( \omega = \sqrt{\alpha^2 + \beta^2} \ e^{Au} \) for some positive constant \( A \) to be chosen later. If \( \omega \) achieves its maximum on \( \Gamma \) then we have the desired bound. Otherwise, \( \omega \) must reach its maximum at an interior point \( p_0 = (x_0, y_0) \) in \( \Omega \).

We may choose coordinates of the ambient space such that

\[
\beta(p_0) = -\tau x_0 + u_y(p_0) = 0.
\]

We denote

\[
v = \alpha(p_0) = \tau y_0 + u_x(p_0).
\]

The function \( \phi = \ln \omega = \ln \sqrt{\alpha^2 + \beta^2} \ e^{Au} \) also takes a maximum at \( p_0 \in \Omega \). That \( \phi_x(p_0) = 0 \) yields

\[
uu xx(p_0) = -Au u_x(p_0), \tag{5}
\]

and \( \phi_y(p_0) = 0 \) gives

\[
uu xy(p_0) = -\tau (Avx_0 + 1). \tag{6}
\]

Moreover, from \( \phi_{xx}(p_0) \leq 0 \) we obtain

\[
u u_{xxx}(p_0) \leq A^2 \nu^3 u_x(p_0) + A^2 \nu^2 u_x^2(p_0) - \tau^2 (Avx_0 + 2)^2, \tag{7}
\]

and \( \phi_{yy}(p_0) \leq 0 \) yields

\[
u u_{xxy}(p_0) \leq -Av^2 u_{yy}(p_0) + \tau^2 A^2 x_0^2 \nu^2 - u_{yy}^2(p_0). \tag{8}
\]

On the other hand, from (2) and (5) we have

\[
uu yy(p_0) = -2H (1 + \nu^2)^{1/2} + \frac{Av}{1 + \nu^2} u_x(p_0). \tag{9}
\]

Taking the derivative of (2) with respect to \( x \) and using (5) and (6) yields

\[
uu xx + (1 + \nu^2) \nuu xyy - 2Av^2 u_x u_{yy} - 2\tau^2 v (A^2 x_0^2 \nu^2 + 3Ax_0 v + 2) - 6AH \nu^2 (1 + \nu^2)^{1/2} u_x = 0
\]

at the point \( p_0 \). Multiplying the last equation by \( v \) and using (9) and inequalities (7) and (8) we obtain, after a long computation, that

\[
\frac{(v - \tau y_0)^2}{1 + \nu^2} + \tau^2 x_0^2 \leq \frac{1}{A^2} (AG_1(v) + G_2(v))
\]
where
\[ G_1(v) = \frac{2H\tau y_0(1 + v^2)^{1/2}}{v} + \frac{P(v)}{v^4}, \quad G_2(v) = -4H^2 + \frac{Q(v)}{v^4} \]
and \( \lim_{v \to \infty} P(v)/v^4 = \lim_{v \to \infty} Q(v)/v^4 \). Therefore,
\[ \lim_{v \to \infty} G_1(v) = 2H\tau y_0 \quad \text{and} \quad \lim_{v \to \infty} G_2(v) = -4H^2 < 0. \]
It follows that we can choose \( A > 0 \) such that
\[ \frac{(v - \tau y_0)^2}{1 + v^2} + \tau^2 x_0^2 \leq \frac{1}{2}. \]
This gives an upper bound for \( v^2 \), and hence for \( \omega = \sqrt{\alpha^2 + \beta^2} e^{Au} \). This concludes the proof of the Lemma.

Hence \( Z \) is closed, and this concludes the proof of the existence part of the Theorem for \( 0 \leq 2H < k \). Now we prove that the graph \( \Sigma = \Sigma(u) \) is unique. Suppose that \( M \) is an embedded \( H \)-surface inside the vertical cylinder \( C_\Gamma \) over \( \Gamma \) with \( \partial M = \partial \Sigma \). Then \( M \) separates \( C_\Gamma \) into two components and we assume the mean curvature vector of \( M \) points into the lower component. When the mean curvature vector points toward the upper component, our argument will show that \( M \) equals the graph of the function \( u \), equal to \( \varphi \) on \( \Gamma \), with mean curvature \( H \) and mean curvature vector pointing toward the upper component.

The mean curvature of the vertical cylinder over \( \Gamma \) is strictly larger than \( H \) and the mean curvature vector points inside the cylinder so the interior of \( M \) is disjoint from the cylinder by the comparison principle.

Denote by \( \Sigma(t) \) the surface \( \Sigma \) translated \( t \) by the flow of the Killing field \( \partial z \). Since \( \partial \Sigma \) is a \( z \)-graph, we have \( \partial \Sigma(t) \cap \Sigma(0) = \emptyset; \partial \Sigma(0) = \partial \Sigma \). Since \( M \) is compact there is a \( T > 0 \) such that \( \Sigma(T) \cap M = \emptyset \).

Now lower \( \Sigma(T) \) to \( \Sigma \) by the flow \( \partial z \), letting \( t \) go from \( T \) to 0. The mean curvature of each \( \Sigma(t) \) points down, so there can be no first contact of \( \Sigma(t) \) with \( M \) for \( t > 0 \), by the maximum principle. Thus \( M \) is below \( \Sigma \). Now choose \( T < 0 \) so that \( \Sigma(T) \cap M = \emptyset \). Move \( \Sigma(T) \) up to \( \Sigma \) by the flow \( \partial z \), letting \( t \) go from \( T \) to 0. There can be no first contact of \( \Sigma(t) \) with \( M \) for \( t \neq 0 \) by the maximum principle (the mean curvature vector of \( M \) points toward the downward component). Therefore \( M \) is above \( \Sigma \), and we obtain that \( M = \Sigma \). This concludes the proof of the Theorem.

4 A further result

It would be interesting to know if Theorem \( \Box \) holds when we allow \( 2H = k \). In this section we give the following partial answer.
**Theorem 3.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^3$ boundary $\Gamma = \partial \Omega$ whose curvature function with respect to the inner orientation is $k > 0$. Let $H$ be a constant satisfying $|\tau|/\sqrt{3} < H \leq k/2$. Then there exists a smooth function $u$ satisfying $u|_\Gamma = 0$ whose graph $\Sigma(u)$ in $\mathcal{H}$ has constant mean curvature $H$.

We need a supersolution $w$ defined in a neighborhood of $\Gamma$ (better than the cones in the preceding section); $w$ is constructed in the following result.

**Proposition 4.** Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $Q_H(u) = 0$ in $\Omega$, $u|_\Gamma = 0$ and $|u|_0 < M$. If $0 < 2H \leq k$ on $\Gamma$, then there is a constant $C = C(H, \Omega, M)$ such that

$$\sup_{\Gamma} |Du| \leq C.$$

**Proof:** Let $\gamma: [0, \ell] \to \Gamma$ be a parametrization by arc length and let $\nu$ stand for the unit normal vector to $\Gamma$ pointing to $\Omega$. We parametrize a neighborhood $U$ of $\Gamma$ in $\Omega$ by

$$P = P(s, t) = \gamma(s) + t\nu(s)$$

for $(s, t) \in [0, \ell] \times [0, \epsilon]$, where $0 < \epsilon < 1/k(s)$. We compute (11) on $U$ making use of the orthonormal frame

$$P_t = \nu, \quad \frac{1}{\phi} P_s = \gamma'$$

where $\phi(s, t) = 1 - tk(s) > 0$. Notice that (11) can be written as

$$Q_H(u) = \text{div}_{\mathbb{R}^2} \left( \frac{Z}{\sqrt{1 + |Z|^2}} \right) + 2H = 0$$

where $Z(p) = -\tau Jp + Du(p)$ and $J$ is the standard complex structure in $\mathbb{R}^2$. Then,

$$W^3Q_H(u) = W^3\text{div}_{\mathbb{R}^2} \left( \frac{1}{W} Z \right) + 2HW^3 = -\frac{1}{2} \langle DW^2, Z \rangle + W^2\text{div}_{\mathbb{R}^2} Z + 2HW^3, \quad (11)$$

where $W^2 = 1 + |Z|^2$.

We compute $W^3Q_H(w) = 0$ for $w = w(t)$ to be chosen. Then $Dw = w_t P_t$ and

$$W^2 = 1 + |Z|^2 = w_t^2 - 2\theta w_t + A \quad (12)$$

where $\theta = \tau\langle JP, P_t \rangle = \tau\langle \gamma, \gamma' \rangle$ and $A = 1 + \tau^2|\gamma + t\nu|^2$. Moreover,

$$\text{div}_{\mathbb{R}^2} Z = \Delta w = w_{tt} - k_t w_t$$

where

$$k_t(s) = \langle D_{P_s/\phi} P_s/\phi, P_t \rangle,$$
and hence, \( k_0(s) = k(s) \). Thus,
\[
W^2 \Delta w = w_t^2 w_{tt} - 2\theta w_t w_{tt} - k_t w_t^3 + 2\theta k_t w_t^2 + A w_{tt} - Ak_t w_t.
\] (13)

Moreover,
\[
DW^2 = (2w_t w_{tt} - 2\theta w_t w_{tt} + A_t)P_t + (-2\theta_s w_t + A_s)\phi^{-2}P_s.
\]

Using \( JP_t = -\phi^{-1}P_s = -\gamma' \) and \( \phi^{-1}JP_s = P_t = \nu \), it is easy to see that
\[
\frac{1}{2} \langle DW^2, Z \rangle = w_t^2 w_{tt} - 2\theta w_t w_{tt} + \theta^2 w_{tt} + B w_t + C
\] (14)
where the functions \( B \) and \( C \) are bounded on \( U \) and do not depend on \( w \) or any of its derivatives. It follows from (11), (12), (13) and (14) that
\[
W^3 Q_H(w) = 2H(w_t^2 - 2\theta w_t + A)^{3/2} - k_t w_t^3 + 2\theta k_t w_t^2 + (A - \theta^2) w_{tt} - (Ak_t + B) w_t - C.
\]

For positive constants \( L \) and \( K \) choose
\[
w(t) = L \ln(1 + K^2 t).
\]
Then \( w(0) = 0 \) and \( w_{tt} = -w_t^2 / L \). Given \( M > 0 \) choose \( L = M / \ln(1 + K) \). Thus,
\[
w(t) = \frac{M}{\ln(1 + K)} \ln(1 + K^2 t).
\]

Hence,
\[
w(1/K) = M
\]
and
\[
w_t(0) = \frac{MK^2}{\ln(1 + K)}.
\]

We claim that we can choose \( K > 1/\epsilon \) large enough such that \( Q_H(w) < 0 \) for all \((s, t) \in [0, \ell] \times [0, 1/K] \). This fact, together with \( w(1/K) = M \) (recall that \(|u|_0 < M\)) allows us to use \( w \) as a barrier from above for \( Q_H \) and conclude the proof.

It suffices to show that \( Q_H(w) < 0 \) at \( t = 0 \) for \( K \) large enough. Since \( w_t(0) \to +\infty \) as \( K \to +\infty \), the claim is clear at points of \( \Gamma \) where \( 2H < k \). If \( 2H = k \) first observe that at \( t = 0 \)
\[
\lim_{K \to +\infty} (w_t^2 - 2\theta w_t + A)^{3/2} - w_t^3 + 2\theta w_t^2
\]
\[
\frac{w_t^2}{w_t^2 + 2\theta w_t^2} = -\theta.
\]

Then, we have that
\[
(A - \theta^2)w_{tt}(0) = -\frac{1}{L}(1 + \tau^2(|\gamma|^2 - \langle \gamma, \gamma' \rangle^2))w_t^2(0) < 0,
\]
and the claim follows from the fact that \( L \to 0^+ \) as \( K \to +\infty \).
Proof of Theorem: Let $\Omega(n)$ be the domain with boundary

$$P(s, 1/n) = \gamma(s) + \frac{1}{n}\nu(s)$$

for large $n$, so $\partial\Omega(n)$ is smooth. By Theorem 1 there exists an $H$-graph $\Sigma(n)$ with $\partial\Sigma(n) = \partial\Omega(n)$, since the curvature of $\partial\Sigma(n)$ is strictly greater than $2H$. Let $u_n$ be the function with graph $\Sigma(n)$

The curvature tensor of $\mathcal{H}$ is given for any $X,Y,Z \in \mathcal{T}\mathcal{H}$ by

$$R(X,Y)Z = -3\tau^2(X \wedge Y)Z + 4\tau^2R_1(\partial_z; X,Y)Z$$

where

$$R_1(\partial_z; X,Y)Z = \langle Y, Z \rangle \langle X, \partial_z \rangle \partial_z + \langle Y, \partial_z \rangle \langle Z, \partial_z \rangle X - \langle X, Z \rangle \langle Y, \partial_z \rangle \partial_z - \langle X, \partial_z \rangle \langle Z, \partial_z \rangle Y.$$

Thus the (not normalized) scalar curvature of $\mathcal{H}$ is $S = -\tau^2$.

By Theorem 1 of [8], there is a positive constant $L$ such that $|u_n|_0 \leq L$ for each $n$. By the maximum principle, $u_{n+1} > u_n$ on the domain of $u_n$. Since the $u_n$ are uniformly bounded by $L$, the function

$$u(x) = \lim_{n \to \infty} u_n(x),$$

is well defined for $x \in \Omega$ and is an $H$-graph in $\Omega$. Moreover, the upper barrier $w$ constructed in Proposition 4 shows that $u$ takes the value zero on the boundary.

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