Would one rather store squeezing or entanglement in continuous variable quantum memories?

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Abstract

Given two quantum memories for continuous variables (e.g., the collective pseudo-spin of two atomic ensembles) and the possibility to perform passive optical operations (typically beam-splitters) on the optical modes before or after the storage, two possible scenarios arise resulting in generally different degrees of final entanglement. Namely, one could either store an entangled state and retrieve it directly from the memory, or rather store two separate single-mode squeezed states and then combine them with a beam-splitter to generate the final entangled state. In this paper, we address the question of which of these two options yields the higher entanglement. By adopting a well established description of QND feedback memories, and a simple but realistic noise model, we analytically determine the optimal choice for several regions of noise parameters and quantify the advantage it entails, not only in terms of final entanglement but also in terms of the capability of the final state to act as a shared resource for quantum teleportation. We will see that, for ‘ideal’ or ‘nearly ideal’ memories, the more efficient of the two options is the one that better protects the quadrature subject to the largest noise in the memory (by increasing it and making it more robust).

Quantum memories are set to be a crucial component of future quantum computers. Even in the short and medium term, the development of effective quantum memories would pave the way for the implementation of a variety of quantum information protocols. For instance, in quantum communication – where distributing quantum correlations over long distances is a primary, and difficult, requirement – quantum repeaters assisted by good enough quantum memories could solve the problem of entanglement distribution in a not so distant future. Hence, considerable efforts are currently being devoted to improving the performance of quantum memories, in terms of both reliability and storage times.

Central among the controllable degrees of freedom that are benefiting from such technological developments are the so-called quantum continuous variables, as one customarily refers to the degrees of freedom described by pairs of canonical operators, like second quantized electromagnetic fields or motional degrees of freedom of massive particles. Continuous variables, mostly in their optical form, are very well suited for quantum communication and key distribution, due to the ease with which they can be distributed and to the comparative reluctance to interact with their environment, and hence to undergo decoherence. However, the very same reluctance that makes them so resilient in the face of environmental noise also implies that establishing quantum correlations between optical continuous variables, as necessary to exploit the advantages offered by quantum communication, is very difficult, and that only a very limited amount of entanglement can be generated in practice. A standard scheme to obtain continuous variable entanglement consists in mixing two single-mode squeezed states – with properly oriented optical phases – into a 50:50 beam-splitter. More generally, continuous variable entanglement always requires some degree of squeezing: the statistical variance of some (composite) degree of freedom of the system has to be below one unit of vacuum noise for entanglement to exist. Continuous variable squeezing and entanglement are therefore closely related resources in the sense that, given the former, the latter can be generated by “passive” optical operations, like beam-splitting, which are much less demanding than squeezing to realise in the laboratory.

Besides the production of quantum entanglement, the other, already mentioned, basic ingredient for the implementation of quantum communication is the capacity to store states reliably in quantum memories. Over the last ten years, a number of candidate strategies to store flying optical continuous variables into static degrees of freedom, typically utilising atomic ensembles, has been put forward and developed. Notwithstanding the potential advantages offered by other schemes – like, notably, EIT-based approaches – the most promising, and currently most effective, among these strategies are based on quantum non demolition (QND) feedback. In such schemes the light degrees of freedom are mapped into, and then successively retrieved from, the collective pseudo-spin of an atomic cloud. This approach has allowed for the demonstration of the proper, ‘quantum’ storage of coherent states and for the teleportation of quantum information between light and matter.

Very recently, the successful quantum storage of squeezing and entanglement in these memories has been reported as well. The possibility of storing entanglement and squeezing poses a relevant theoretical question: given some
precious single-mode squeezing, which has to be generated in costly and delicate parametric down conversion processes, is it better to store it as it is in a quantum memory, and then use it to produce entanglement after retrieval from the memory, or would one rather first use it to create entanglement and then map and retrieve the global entangled state? In other words, is one better off by storing squeezing or entanglement in a continuous variable quantum memory? This inquiry is dedicated to such a question. We will specialise to QND feedback memories, and carry out a comparative study between two cases, labeled with a and b respectively, where the memory will act after or before an entangling 50:50 beam-splitter (see Fig. 1). The figure of merit we will adopt is the final entanglement obtained in the two cases, i.e. the entanglement that, in such scenarios, would be available “on demand” after the memory’s operation. We will also extend our study to a directly operational figure of merit: the capability of the final state to act as a shared channel for the quantum teleportation of coherent states.

**Notation and preliminaries**

Since all the states involved in this problem are, with very good approximation, Gaussian states, we can take advantage of the Gaussian formalism \[\text{[11]}\] in our analysis, whereby quantum states will be entirely characterised by their covariance matrices, whose entries represent all the second statistical moments of the canonical operators (first moments, being independent from the correlations, are irrelevant to our discussion and will henceforth be neglected).

If the two-mode quantum system in question is described by the vector of canonical operators \(\hat{\mathbf{r}} = [\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2]\), the covariance matrix (CM) \(\sigma\) of the state \(\rho\) is defined as \(\sigma_{jk} = \text{Tr}[\hat{q}_j \hat{p}_k \rho] - 2\text{Tr}[\hat{q}_j \rho] \text{Tr}[\hat{p}_k]\). In all that follows, we shall adopt natural units \((\hbar = 1)\) and report all the variances in units of vacuum noise (such that the vacuum state has CM equal to the identity matrix).

In this notation, the action of the memory can be described as a Gaussian quantum channel, acting on the covariance matrix \(\sigma\) as \[\text{[12]}\]

\[
\sigma \rightarrow X\sigma X^T + Y,
\]

with \(X = \text{diag}(\xi_1, \xi_1, \xi_2, \xi_2)\) and \(Y = \text{diag}(y_{q1}, y_{p1}, y_{q2}, y_{p2})\) (that is, the original state is recovered up to some multiplicative losses and additive noise on the system’s canonical operators). The relationships \(\xi_i^2 \geq (1 - \sqrt{y_{q_i}y_{p_i}})\) for \(i = 1, 2\) guarantee that the channel is a physical map \[\text{[12]}\]. Notice that this description accounts for all the noise observed in experiments, including losses at the input and output of the memory cells, uncertainties in the initial atomic collective pseudo-spin along the direction which is not addressed by the feedback loop, and additional noise of spurious nature (atomic decoherence, unsuppressed noise due to imperfections in the feedback loop, further technical noise, see \[\text{[10]}\]). Later on, we will see how the matrix \(Y\) relates to practical parameters, and consider realistic values of such parameters.

For the moment being, let us just mention that in an ‘ideal’ implementation of the memory, where the only residual noise is due to the initial uncertainty in the atomic pseudo-spin, one would have \(\xi_1 = \xi_2 = 1\) and \(y_{q1} = y_{p1} = 0\) \[\text{[7]}\]. This operating regime will be referred to as ‘ideal memories’ in the following. Since each atomic cloud will be generally different from another, \(\xi_1, y_{q1}\) and \(y_{p1}\) will generally differ from \(\xi_2, y_{q2}\) and \(y_{p2}\).

The two possibilities we are about to compare can be described by two different final quantum states, with different covariance matrices: \(\sigma_{a}\) with CM \(\sigma_{sa}\) where the noisy channel describing the memory is applied after the beam-splitter, corresponding to storing entanglement [Fig. 1(a)], and \(\sigma_{b}\) with CM \(\sigma_{s\theta}\) where the noisy channel is applied before the beam-splitter, corresponding to the storage of squeezing [Fig. 1(b)]. One has

\[
\sigma_{a} = X\sigma_{0}R^T X^T + Y, \quad (1)
\]

\[
\sigma_{b} = RX\sigma_{0}R^T \sigma_{0}R + RYR^T, \quad (2)
\]

where \(R = e^{i\pi/4}\) describes the action of the 50:50 beamsplitter in phase space, in terms of the \(4 \times 4\) anti-symmetric generator \(J\) with entries \(J_{jk} = \delta_{j+k,2} - \delta_{j+k,0}\) for \(1 \leq j, k \leq 4\). Note that if the noises in the two memories were perfectly identical then \([R, X] = [R, Y] = 0\) and \(\sigma_{a} = \sigma_{b}\), so that there would be no difference between the two cases.

For simplicity, and to fix ideas, we shall assume an initial CM given by: \(\sigma_{0} = \text{diag}(sN_1, sN_2/s, N_2/s, N_2s)\), with \(s \geq 1\), \(N_1 \geq 1\) and \(N_2 \geq 1\). For \(N_1 = N_2 = 1\), this CM describes two pure single-mode squeezed states with optical phases chosen so as to optimise the production of entanglement by a 50:50 beam-splitter \[\text{[4]}\]. Two different squeezing parameters could be chosen for the two modes but, in this context, it seems reasonable to endow the hypothetical experimenter with a fixed squeezing capability (different squeezing parameters can be treated with our approach, but would just

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**Figure 1:** The two cases being compared: in (a), the state is stored in the memory cells after the beam splitter has mixed and entangled the single-mode squeezed states; in (b), the state is stored before the entangling beam-splitting action. We assume, conceding something to idealisation, that all the noise is imputable to the storage and retrieval process. In practice, such a noise would certainly dominate over the mixing’s imperfections.
complicate the study without introducing any substantial conceptual difference). Finally, the parameters $N_1$ and $N_2$ describe the thermal broadening squeezed states are likely to undergo in practice.

The entanglement of the quantum states $\varrho_a$ and $\varrho_b$ can be characterised by evaluating their logarithmic negativities $E_N(\varrho_a)$ and $E_N(\varrho_b)$, an entanglement monotone which represents an upper bound – in “ebits” – to the distillable entanglement, i.e. to the maximal number of Bell pairs per copy of the entangled state which can be obtained by local operations and classical communications alone [13]. The logarithmic negativity is, for Gaussian states such as these, an increasing function of the smallest partially transposed symplectic eigenvalue $\tilde{\sigma}$, which is in turn a function of two partially transposed symplectic invariants $\text{Det } \sigma$ and $\Delta(a)$ [14]. If one expresses the covariance matrices in terms of $2 \times 2$ blocks, as in

$$\sigma_i = \begin{pmatrix} a_i & y_i \\ y_i & b_i \end{pmatrix} \quad \text{for } i = a, b,$$ (3)

the partially transposed symplectic invariants are $\text{Det } \sigma_i$ and $\Delta(a_i) = \text{Det } \sigma_i + \text{Det } \sigma_i - 2 \text{Det } y_i$ for $i = a, b$, and determine the smallest symplectic eigenvalue as follows

$$2\tilde{\sigma}_i^2 = \Delta(a_i) - \sqrt{\Delta^2(\sigma_i) - 4 \text{Det } \sigma_i} \quad \text{for } i = a, b.$$ (4)

Finally, the logarithmic negativity is given, in ebits, by

$$E_N(\varrho_i) = \max \left[ 0, - \log_2 \tilde{v}_i \right] \quad \text{for } i = a, b.$$ (5)

Note that for a state to be entangled one must have $\tilde{v} < 1$.

**Ideal memories**

In the case $\xi_1 = \xi_2$ and $y_{p1} = y_{q2} = 0$, we could derive a sharp analytical criterion to discriminate between the storage of squeezing and entanglement:

*Given the notation of the previous section and assuming $y_{p1} = y_{q2} = 0$, $\xi_1 = \xi_2$ and $1/s^2 \leq N_2/N_1 \leq s^2$, one has

$$y_{p2} \geq y_{q1} \iff E_N(\varrho_a) \geq E_N(\varrho_b).$$ (6)

That is, given this configuration of optical phases (fixed by the condition $s \geq 1$), storing entanglement is advantageous over storing single-mode squeezing if the noise acting on the second quadrature is larger than the noise acting on the first quadrature, and vice versa. The proof of the statement above can be found in appendix. Notice that the assumption (6) on the input state is very mild in that, even for a (very reasonable) squeezing parameter $s = 4$ (corresponding to 6 dB of squeezing), the assumption is violated if the input thermal noise affecting one mode is more than 16 times larger than the noise acting on the other mode. This condition will thus be met in most practical instances, which renders the criterion effectively independent from the parameters of the input state, as desirable.

Besides providing us with an analytical criterion valid in a relevant case, the finding above also sheds light on the physical origin of the advantage granted by one or the other strategy. *Given this configuration of optical phases, storing entanglement (i.e. applying the entangling beam-splitter before the retrieval from the memory) is advantageous if the noise acting on the second mode is larger than the noise acting on the first one: this is because, since $s > 1$, storing squeezing would mean that the larger noise ($y_{q2}$) would act on the (more delicate) squeezed quadrature $\varrho_2$. More generally, the optimal storage is the one whereby the variance of the noisiest quadrature is the larger before the storage takes place, and hence is the more robust in the face of the noise. Hence, our findings point to a way to improve the storage of continuous variable entanglement, provided that the additive noise that characterises each single-mode memory is known a priori. This would certainly be possible in practice, since each memory could be calibrated before use. However, it should be stressed that the relative advantage of storing two-mode entangled states or single-mode squeezing can only be determined if the noises associated to the two memory cells are known.*

Let us now briefly consider the quantitative advantage granted by the optimal choice by considering values taken from practical experiments. Under ideal conditions, one would have $y_{q1} = (1 - 1/Z_1^2)\Delta_{A1}$ and $y_{q2} = (1 - 1/Z_2^2)\Delta_{A2}$. Here, $Z_1$ and $Z_2$ are parameters which depend only on the optical detuning of the swap interaction and take the value $\sqrt{6.4}$ in our experiment of reference [10], whereas $\Delta_{A1}$ and $\Delta_{A2}$ are the initial variances of one of the quadratures of the collective atomic pseudo-spin in the two memory cells. The atomic clouds are initialised in spin-squeezed states in order to reduce $\Delta_{A1}$ and $\Delta_{A2}$. Although in principle the spin-squeezing of the atomic ensembles could be pushed further, in practical instances these quadratures still take values around 0.8 (in units of vacuum noise). It is reasonable to assume that the main difference between the noise added by the two memories would be due to differences between $\Delta_{A1}$ and $\Delta_{A2}$, which are far less controllable than $Z_1$ and $Z_2$. Hence, assuming $Z_1^2 = Z_2^2 = 6.4$, $s = 4$, $N_1 = N_2 = 1$, $\Delta_{A1} = 0.6$ and $\Delta_{A2} = 1$, one would have $E_N(\varrho_b) = 1.06$ ebits and $E_N(\varrho_a) = 0.94$ ebits: in this case, storing single-mode squeezing rather than an entangled state would protect 0.12 ebits of entanglement. As a rough rule of thumb, confirmed also for noisy input states (i.e., for $N_1$ and $N_2$ not equal 1), a difference of 0.25 vacuum units in the additive noises of the two cells is reflected by a difference of 0.1 ebits in the entanglement of $\varrho_a$ and $\varrho_b$.

**Noisy memories**

In practice, the functioning of the quantum memories we are dealing with is not only limited by the initial variance
Figure 2: Difference between the logarithmic negativities \( \delta E_N = E_N(\varrho_0) - E_N(\varrho_b) \) (a) and achievable teleportation fidelities \( \delta F = F_a - F_b \) (b) for \( N_1 = 1.4, \ N_2 = 1.2, \ s = 8, \ Z_1^2 = Z_2^2 = 6.4, \ G_1 = G_2 = 0.95, \ \Delta_{A1} = \Delta_{A2} = 0.1 \) and \( \Delta_{P1} = \Delta_{P2} = 0.3 \), as a function of \( \Delta_{M1} \) and \( \Delta_{M2} \). The planes \( \delta E_N = 0 \) and \( \delta F = 0 \) are also plotted, showing that \( \varrho_b \) is more entangled when \( \Delta_{M1} > \Delta_{M2} \).

of the atomic spin cloud, but also subject to input losses (as the light impinging on the memory cell is partially reflected) and to spurious additive noise, which has been estimated in recent experiments (see the Supplementary Material of Ref. [10]). For \( \varrho_{y1} \) and \( \varrho_{y2} \) non-vanishing and \( \xi_1 \neq \xi_2 \), no simple general criterion to determine which of \( \varrho_a \) or \( \varrho_b \) is more entangled could be derived. However, a systematic analytical comparison between the two cases was carried out. Such a comparison shows that \( \delta_y \geq 0 \) and \( \delta_p \geq 0 \) \( \Leftrightarrow \delta E_N \geq 0 \), where the new parameters are defined as \( \delta_y := y_{y2} - y_{y1}, \ \delta_p := y_{p2} - y_{p1} \) and \( \delta E_N := E_N(\varrho_b) - E_N(\varrho) \). If these assumptions on the additive noise are not satisfied, the optimal choice cannot be established in general, as the contribution of the noise acting on the \( \hat{q} \) quadratures counteracts the noise on the \( \hat{p} \) quadratures. The primary heuristic way to discriminate between the two cases is then simply to compare the differences in additive noises: thus one observes that \( \delta_y \geq \delta_0 \geq 0 \) or \( \delta_y \leq \delta_0 \leq 0 \) then \( \delta E_N \geq 0 \), for a wide range of parameters. The converse of this criterion, where \( \delta_y \geq \delta_0 \geq 0 \) and \( \delta E_N \leq 0 \), is however very often violated: regardless of the chosen optical phase (i.e., even for \( s < 1 \)), the optimality of the storage of squeezing turns out to be more stable with respect to oscillations in the additive noise. Accordingly, in all observed instances we found that, for \( y_{y1} = y_{y2} \) and \( y_{p2} = y_{p2} \), one has \( \delta E_N \geq 0 \): the storage of squeezing is more robust in the presence of ‘phase-insensitive’ noise in the memories.

In order to apply our study to a realistic experimental setting, let us now borrow the notation of reference [10], whereby \( \xi_1 = G_1, \ \xi_2 = G_2, \) and the entries of the \( Y \) matrix would be \( y_{y1} = (1 - 1/Z_1^2) \Delta_{A1} + (1 - 1/Z_2^2) + \Delta_{A1}, y_{y2} = (1 - 1/Z_1^2) \Delta_{A2} + (1 - 1/Z_2^2) + \Delta_{A2} \) and \( y_{p2} = (1 - 1/G_2^2) + \Delta_{P2} \), where \( G_1 \) and \( G_2 \) quantify the losses, \( \Delta_{A1}, \Delta_{A2}, \Delta_{P1} \) and \( \Delta_{P2} \) represent the additional noise, of various origin (accounting for the decoherence of the atoms during the storage and for other imperfections), while \( Z_1, Z_2, \Delta_{A1} \) and \( \Delta_{A2} \) were defined in the previous section. A typical comparison between case \( a \) and \( b \) in realistic conditions is shown in Fig. 2(a). For the set of parameters \( N_1 = N_2 = 1, \ s = 5, \ Z_1 = Z_2 = 6.4, \ G_1 = G_2 = 0.85, \ \Delta_{A1} = \Delta_{A2} = 0.2, \ \Delta_{P1} = \Delta_{P2} = 0.4, \ \Delta_{M1} = 0.9 \) and \( \Delta_{M2} = 0.6 \), which is well within current experimental reach, the state \( \varrho_0 \) is not entangled whereas the state \( \varrho_0 \) is entangled. In this instance, storing squeezing rather than entanglement into the memories would make the difference between an entangled or separable final state.

The effect of different loss factors in the two memory cells has also been investigated. If \( G_1 \neq G_2 \), one can actually discriminate on more general grounds between the storage of squeezing or entanglement: an extensive quantitative analysis indicates that if the other parameters are such that, for \( G_1 = G_2 = 1, E_N(\varrho_0) \geq E_N(\varrho) \), then the same inequality holds for different \( G_1 \) and \( G_2 \) as well (see Fig. 3(a)). Instead, as apparent in Fig. 3(b) if the other parameters are such that \( E_N(\varrho_0) \geq E_N(\varrho) \), this inequality may be reversed by varying the loss factors. In this sense, storing squeezing would offer an additional guarantee in practical instances where the loss factors may vary widely between different memory cells.

**Teleportation fidelities**

To put our results into a clear operational context, we intend now to adopt a different figure of merit in lieu of the logarithmic negativity: we will compare the storage of squeezing with the storage of entanglement in terms of how well the final states obtained allow one to perform the quantum teleportation of coherent states. The quality of the latter process can be estimated by the teleportation fidelities \( F_a \) and \( F_b \), defined as the overlap between the initial state and the teleported state when using, respectively, \( \varrho_a \) and \( \varrho_b \) as a shared entangled resource between sender and receiver [2]. Although they are dependent on the entanglement of \( \varrho_a \) and \( \varrho_b \), \( F_a \) and \( F_b \) cannot be expressed in terms of \( \xi_1 \) and \( \xi_2 \) for general Gaussian states (see later), so our previous analysis does not, strictly speaking, imply any precise re-
sult concerning teleportation fidelities. However, \( F_a \) and \( F_b \) can be treated analytically in our setting.

In terms of the \( 2 \times 2 \) submatrices of Eq. (1), the optimal teleportation fidelity that can be achieved for input coherent states is given by

\[
F_i = \frac{2}{\sqrt{\text{Det}(2I + \alpha_i + \beta_i - 2\sigma_i Y_i)}}, \quad \text{for } i = a, b, \tag{8}
\]

where \( I \) is the two-dimensional identity matrix and \( \sigma_z \) is the Pauli \( z \) matrix \([15]\). Some basic algebra leads to the following relationships for the fidelities: \( 1/F_a^2 = [1 + (y_{p1} + y_{q2})/2 + \Delta_N(\xi_1 + \xi_2)^2/4 + N_2s(\xi_1 - \xi_2)^2/4][1 + (y_{p1} + y_{q2})/2 + N_2/s(\xi_1 + \xi_2)^2/4 + N_1s(\xi_1 - \xi_2)^2/4] \) and \( 1/F_b^2 = [1 + y_{p1} + \Delta_N(\xi_1^2)[1 + y_{q2} + \Delta_N(\xi_2^2)] \). Upon immediate inspection, these equations allow one to retrieve the analogous of criterion \( 7 \) for the teleportation fidelities. For \( y_{p1} = y_{q2} = 0 \) (‘ideal’ memories) and \( \xi_1 = \xi_2 \), one has

\[
F_a \geq F_b \iff y_{q2} \geq y_{q1}. \tag{9}
\]

If none of the diagonal entries of \( Y \) vanish or if the loss factors \( \xi_1 \) and \( \xi_2 \) are different, the situation is slightly more involved, since the comparison between \( F_a \) and \( F_b \) depends in general also on the parameters of the initial state \( N_1, N_2 \) and \( s \).

However, as apparent when comparing Fig. 2(b) to Fig. 2(a) and Fig. 4 to Fig. 3, the qualitative behaviour of the teleportation fidelities \( F_a \) and \( F_b \) reflects very closely that of the entanglement measures \( E_N(\rho_b) \) and \( E_N(\rho_b) \), thus strengthening our previous analysis on operational grounds. Note that in Fig. 4(b) we report the difference between the quantities \( \max[F_i, 1/2] \) for \( i = 1, 2 \), because 1/2 is the threshold value that can be achieved by a ‘classical’ measure-and-prepare strategy, without the help of state \( g \): this quantity better quantifies the advantage provided by the final states.

Notice that the region of positive values in plot 4(b) is different from the region of positive values in plot 2(b), i.e., there are parameters’ sets such that \( F_b \geq F_a \) but \( E_N(\rho_b) \geq E_N(\rho_b) \). This fact leads to an interesting theoretical side remark: the teleportation fidelity of a two-mode Gaussian state cannot be related directly to the logarithmic negativity, not even for two-mode Gaussian states (at least without previous optimisation over local operations, see [17]). This counterexample only arose for \( \xi_1 \neq \xi_2 \). For \( \xi_1 = \xi_2 \) the difference in teleportation fidelities mirrors faithfully the difference in logarithmic negativities.

**Summary**

We have considered continuous variable quantum memories operated in atomic ensembles by QND feedback, and meaningfully compared the storage of single-mode squeezing with the storage of two-mode entanglement to find that:

- if the predominant additive noise in each memory acts on a single quadrature (ideal or near-ideal memories) and for comparable loss factors in the two memories, then the storage preserving the most entanglement is the one whereby the quadrature subject to the largest noise is made more robust (larger);

- if the differences \( \delta_{\hat{q}} \) and \( \delta_{\hat{p}} \) in the noises on the \( \hat{q} \) and \( \hat{p} \) quadratures influence the final state in opposite ways, the advantage offered by the storage of squeezing proves to be more stable than the one granted by the storage of entanglement (i.e., for \( s > 1 \), \( \delta_{\hat{q}} \geq \delta_{\hat{p}} \geq 0 \Rightarrow \delta E_N \geq 0 \) but \( \delta_{\hat{q}} \geq \delta_{\hat{p}} \geq 0 \Rightarrow \delta E_N \leq 0 \));

- in the case of phase-insensitive Gaussian noise \( (y_{p1} = y_{q2} \quad \text{for} \quad i = 1, 2) \), storing squeezing is advantageous;

- the optimality of the storage of squeezing is stable under asymmetric variations of the loss factors between the two memories (the same is not true for the storage of entanglement).

These findings were confirmed in terms of both entanglement – as quantified by the logarithmic negativity – and teleportation efficiency of the respective final states.

Notice that we did not consider the optimised production or protection of entanglement over any passive operation as in [14] and [17] (which would require an accurate knowledge of the input state and of the action of the memory), since we rather tried to identify conditions that would hold for generic inputs, and that would point to the best choice for a retrieval of the entanglement on demand.

Our results could have significant impact as operational guidelines for the storage and retrieval of continuous variable entanglement, which will be a ubiquitous prerequisite in the areas of quantum communication and information processing alike.

**Appendix – Derivation of analytical results**

Although the smallest symplectic eigenvalues \( \tilde{v}_a \) and \( \tilde{v}_b \) could in principle be evaluated analytically, their expres-
sions are extremely complex, and do not allow for direct manipulation. To treat our comparison analytically, we will instead first assume $\xi_1 = \xi_2 = 1$ and define a class of covariance matrices $\sigma(\delta)$: $\sigma_0 = R_\delta R^T + R_\delta R^T$, with $R_\delta = e^{3\delta}$, and $J$ with entries $\tilde{J}_{jk} = \delta_{j+k,2} - \delta_{j+k,2}$, such that $R_{N/4} = R$ and, hence, $\sigma_0 = \sigma_J$ and $\sigma_{N/4} = \sigma_J$.

The infinitesimal variations of the symplectic invariants with respect to $\delta$ can be obtained by applying the formula $\det(M + \delta N) = \det M + \det(N)\tr(MN^{-1})\delta + O(\delta^2)$, whence one gets:

\[
\frac{d\Delta(\sigma)}{d\delta} = [(a - c)\delta p + (b - d)\delta q - 4\delta \delta q \sin(2\delta)] \cos(2\delta),
\]

(10)

\[
\frac{d\det(\sigma)}{d\delta} = [A\delta p + B\delta q + \delta \delta q \sin(2\delta)] \cos(2\delta),
\]

(11)

with $A = (b - d)(a + y_{12})(c + y_{12}), B = (a - c)(b + y_{12})(d + y_{12}), C = (c - ab + (c - a)y_{12} + (d - b)y_{12})$ and $D = (a - c)(b - d), \delta p = y_{p2} - y_{p1}$ and $\delta q = y_{q2} - y_{q1}$ and $a = N_1 s, b = N_1 s, c = N_2 s, d = N_2 s$. Eq. (11) implies

\[
\sqrt{\Delta^2(\sigma) - 4\Delta(\sigma)d\sigma^2} = \frac{d\det(\sigma)}{d\delta} - \frac{d\Delta(\sigma)}{d\delta} d\sigma.
\]

(12)

For $\delta p = y_{p2} = 0$, which is the case for ideal memories, inserting (10) and (11) into (12) yields

\[
\frac{d\sigma^2}{d\delta} = \frac{[h(a - c)\delta q - \delta^2 p (b - d)\delta q]}{\sqrt{\Delta^2(\sigma) - 4\det(\sigma)}} \cos(2\delta).
\]

(13)

Now, the condition $\delta < 0$ implies $(a - c) = (N_1 s - N_2 s) \geq 0$ and $(b - d) = (N_1 s - N_2 s) \leq 0$, so that $\frac{d\sigma^2}{d\sigma} = k\delta q$, where $k$ is positive for $0 \leq \delta \leq \pi/4$. Therefore, $\tilde{\sigma} = \tilde{\sigma} \leq \tilde{\sigma}_{N/4} = \tilde{\sigma}$, if $\delta q \geq 0$ and $\tilde{\sigma} = \tilde{\sigma} \geq \tilde{\sigma}_{N/4} \geq \tilde{\sigma}$, if $\delta q \leq 0$, which leads to (2) because of Eq. (5).

If $\xi_1 = \xi_2 = 1$, i.e., if the losses of the two memory cells are equal, one has $[X, R] = 0$, such that Eq. (11) becomes $\sigma_0 = R_X \delta q X^T R_Y + Y$, and one can redefine the initial CM as $\sigma'_0 = X_0 \delta q X^T$, which takes the same form as $\sigma_0$ by redefining $N_i \rightarrow N_i \xi_i^2$ for $i = 1, 2$. Our previous proof of the criterion $y_{q2} \geq y_{q1} \Leftrightarrow E_X(q_2) \geq E_X(q_1)$ then applies under the same condition (6), which is unaffected by the transformation of the $N_i$’s since $\xi_1 = \xi_2$.

Notice that $\sigma'_0$ might not be a physical CM, in that it could violate Robertson–Schrödinger inequalities. This is however irrelevant for our proof.

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