Galois representations arising from twenty-seven lines on a cubic surface
and the arithmetic associated with Hessian polyhedra

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1. Introduction

In his celebrated lecture: Mathematical problems (see [H]), which was delivered before
the International Congress of Mathematicians at Paris in 1900, David Hilbert gave
his famous 23 problems. The 12th problem is the extension of Kronecker’s theorem
on abelian fields to any algebraic realm of rationality. Hilbert said: “The extension of
Kronecker’s theorem to the case that, in place of the realm of rational numbers or of
the imaginary quadratic field, any algebraic field whatever is laid down as realm of
rationality, seems to me of the greatest importance. I regard this problem as one of
the most profound and far-reaching in the theory of numbers and of functions. It will
be seen that in the problem the three fundamental branches of mathematics, number
theory, algebra and function theory, come into closest touch with one another, and I am
certain that the theory of analytical functions of several variables in particular would be
notably enriched if one should succeed in finding and discussing those functions which
play the part for any algebraic number field corresponding to that of the exponential
function in the field of rational numbers and of the elliptic modular functions in the
imaginary quadratic number field.”

In his paper [Wei], Weil said: “First of all, it will be necessary to extend the theory of
abelian functions to non-abelian extensions of fields of algebraic functions. As I hope to
show, such an extension is indeed possible. In any case, we face here a series of important
and difficult problems, whose solution will perhaps require the efforts of more than one
generation.” Later on, Weil [Wei3] pointed out: “We take up the non-commutative case.”

According to Tate ([Ta], see also [L1]), the biggest problem after Artin’s reciprocity law was to extend it in some way to non-abelian extensions, or, from an analytic point of view, to identify his non-abelian $L$-series. The breakthrough came out of Langlands’s discovery in the 1960’s of the conjectural principle of functoriality, which included a formulation of a nonabelian reciprocity law as a special case. It turns out that in the nonabelian case, one should describe the representations of $\text{Gal}(\overline{K}/K)$ rather than $\text{Gal}(K/K)$ itself. Thus, the global reciprocity law is formulated as a general conjectural correspondence between Galois representations and automorphic forms.

The Langlands program describes profound relations between motives (arithmetic data) and automorphic representations (analytic data). Wiles’s spectacular work on the Shimura-Taniyama-Weil conjecture, which established the proof of Fermat’s Last Theorem, can be regarded as confirmation of such a relation in the case of elliptic curves. In general, the arithmetic information wrapped up in motives comes from solutions of polynomial equations with rational coefficients. The analytic information from automorphic representations is backed up by the structure of Lie theory. Thus, the Langlands program translates large parts of algebraic geometry into the language of representation theory of Lie groups. The Langlands correspondence embodies a large part of number theory, arithmetic algebraic geometry and representation theory of Lie groups. The Artin conjecture on $L$-functions and the Ramanujan-Petersson conjecture would follow from the Langlands correspondence. Over number fields, the Langlands correspondence in its full generality seems still to be out of reach. Even its precise formulation is very involved.

The investigation of the relation between the arithmetic of Galois representations and the analytic behavior of corresponding $L$-series is one of the central topics in number theory and arithmetic geometry. One of the outstanding problems in this field is Artin’s conjecture which predicts the holomorphy of the Artin $L$-series of nontrivial irreducible complex representations of the absolute Galois group of number fields. The idea of describing extensions of $\mathbb{Q}$ via the action of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on certain groups and other algebraic objects is very fruitful. Many examples of constructions of abelian and non-abelian extensions of $\mathbb{Q}$ are based on this idea. A complete classification of all these extensions in terms of Galois representations and in terms of certain objects of analysis and algebraic geometry (automorphic forms and motives) is the central problem of the Langlands programme (see [MP]).

In fact, elliptic curves can be considered as a special kind of cubic hypersurfaces. A cubic hypersurface is a projective algebraic variety defined by a homogeneous equation $F_3(x_0, \cdots, x_n) = 0$ of degree three with coefficients in some ground field $k$. When $n = 2$, it gives a cubic curve, i.e., an elliptic curve. Now, we will study the case of $n = 3$, cubic surfaces.

Over an algebraically closed field $k$, every irreducible cubic surface is a rational
surface. The class of a hyperplane section $h$ of a surface $F$ is the canonical class $-K_F$.
Any smooth cubic surface can be obtained from the projective plane $\mathbb{P}^2$ by blowing up of 6 generic points. The appropriate birational mapping $\phi : \mathbb{P}^2 \to F$ is determined by the linear system of cubic curves passing through the 6 points. There are 27 straight lines on $F$, each of which is exceptional; they are the only exceptional curves on $F$. The configuration of these 27 lines is rich in symmetries. The automorphism group of the corresponding graph is isomorphic to the Weyl group of $E_6$. Cubic surfaces belong to the class of del Pezzo surfaces, i.e. two-dimensional Fano varieties. Over an algebraically non-closed field $k$, there are smooth cubic surfaces $F$ which are not birationally isomorphic to $\mathbb{P}^2$ over $k$. Among these surfaces one finds surfaces possessing $k$-points, and these are unirational over $k$. The group of birational automorphisms of a minimal surface has been determined and an arithmetic theory of cubic surfaces has been developed in [Ma] by the non-associative structures, such as quasi-groups and Moufang loops.

In his paper [Wei2], Weil showed that for nonsingular rational surfaces over an algebraic number field $K$, the Hasse-Weil $L$-function, except for a finite number of factors, is the same as a suitable Artin $L$-function for a certain extension of $K$. For instance, for a nonsingular cubic surface in the projective 3-space, one thus gets an $L$-function belonging to the extension of $K$ determined by the 27 straight lines on the surface. The Galois group for this is $W(E_6)$, a group of order $2^7 \cdot 3^4 \cdot 5$ and has a simple subgroup of index 2. Therefore, the Hasse-Weil $L$-function for a given cubic surface is essentially an Artin $L$-function of a definitely non-abelian type for one dimensional representations, i.e., characters. Here is a rather unexpected connection between number theory and algebraic geometry.

According to [Ma], let $k$ be a perfect field and let $V$ run through the smooth cubic surfaces over $k$. We have the realization problems.

1. Which subgroups of $W(E_6)$ are realized as an image of the Galois group by its representation on $\text{Pic}(V \otimes \bar{k})$ for some $V$?

2. Which extensions of the field $k$ correspond with kernels of such representations?

In his monograph [Ma], Manin said: “Of course, the answer must depend heavily on $k$. The derived group $W'(E_6)$ is simple, and the whole of it obviously can not be realized over a local base field which has only solvable extensions. On the other hand, $W(E_6)$ can be realized over some function field. The more interesting case of a number field is completely unknown. The search for a reasonable approach to problem (2) may involve a pattern for a non-commutative class field theory. Here, it appears that topological-analytic considerations and something like Hecke operators do not suffice.”

In the present paper, we will show that three apparently disjoint objects: Galois representations arising from twenty-seven lines on a cubic surface (number theory and arithmetic algebraic geometry), Picard modular forms (automorphic forms), rigid Calabi-Yau threefolds and their arithmetic (Diophantine geometry) are intimately related to Hessian polyhedra and their invariants. We investigate the Artin $L$-functions for six dimensional
representations associated to a kind of cubic surfaces. We study the linear representations of the Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) which constitutes a primary object of number theory and find the correspondence between some transcendental functions and representations. Moreover, there exists the duality which connects them with each other. Our linear representations come from the Picard groups of cubic surfaces. We construct a Galois representation whose image is a proper subgroup of \( W(E_6) \), the Weyl group of the exceptional Lie algebra \( E_6 \). We give a conjecture about the identification of two different kinds of \( L \)-functions which can be considered as a higher dimensional counterpart of the Langlands-Tunnell theorem. We obtain the structure of rational points in the rigid Calabi-Yau threefolds. More precisely, we will investigate the relations among invariant theory (algebra), Picard modular forms (analysis), Galois representations (arithmetic) and cubic surfaces (geometry and topology). Just as the applications of modular curves and elliptic modular functions to number theory, i.e., one uses \( GL(2) \) to study \( GL(1) \), we will apply Picard modular surfaces and Picard modular functions to number theory, i.e., we will use \( GL(3) \) (or \( U(3) \)) to study \( GL(1) \). As an application, we find an answer to Manin’s problems described as above. Moreover, we give a conjecture which involves the noncommutative class field theory. It should be pointed out that our Galois representations arise from twenty-seven lines on a cubic surface, while the previous Galois representations either arise from torsion points of an elliptic curve or an abelian variety (e.g. the Tate modules), or arise from Mordell-Weil lattices of an elliptic curve, i.e., both of them come from elliptic curves (cubic curves).

**Main Results.**

We have the following dictionaries.

1. \( GL(2) \)
   - elliptic curves (cubic curves)
   - elliptic modular forms
   - regular polyhedra
   - algebraic solutions of hypergeometric differential equations
   - \( A_4, S_4, A_5 \)
   - \( PSL(2, \mathcal{O}_K)/PSL(2, \mathcal{O}_K)(\sqrt{5}) = A_5 \)
   - Hilbert modular surfaces over \( K = \mathbb{Q}(\sqrt{5}) \) (Hirzebruch)
   - diagonal cubic surface of Clebsch and Klein, \( F_1, 27 \) real lines
   - two dimensional real surface
GL(3)
cubic surfaces
Picard modular forms
Hessian polyhedra
algebraic solutions of Appell hypergeometric partial differential equations
Hessian groups: $G_{648}, G_{1296}$
$U(2,1; \mathcal{O}_K)/U(2,1; \mathcal{O}_K)(1 - \omega) = S_4$ (see [Ho1], [Ho3])
Picard modular surfaces over $K = \mathbb{Q}(\sqrt{-3})$
our cubic surface, $F_2$, 15 real lines and 12 complex lines
two dimensional complex surface

Let us consider the following Galois representation:

$$
\rho : \text{Gal}(\overline{K}/K) \to \text{Aut}(X),
$$

where $X$ is some geometric object.

1. $X = V_n(k)$ is an $n$-dimensional linear space over a field $k$, such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{F}_p$, $\text{Aut}(V_n(k)) = GL(n,k)$ (Lie groups, $p$-adic Lie groups, finite groups)
2. $X = T_\ell(\mu)$ is the Tate module of the number field $K$, $\text{Aut}(T_\ell(\mu)) = GL(1, \mathbb{Z}_\ell)(\ell$-adic Lie groups)
3. $X = T_\ell(E)$ is the Tate module of the elliptic curve $E$, $\text{Aut}(T_\ell(E)) = GL(2, \mathbb{Z}_\ell)(\ell$-adic Lie groups)
4. $X = (E(k(C)), \langle , \rangle)$ is the Mordell-Weil lattice associated to a rational elliptic surface, where $k(C)$ is the function field of a smooth projective curve $C$ over an algebraically closed field $k$, $\text{Aut}(E(k(C)), \langle , \rangle) = \text{Aut}(E_r)$, $r = 6, 7, 8$ (finite groups)
5. $X = \mathcal{L}$ is the configuration of 27 lines on the cubic surface $S$ preserving the intersection behavior of the lines, $\text{Aut}(\mathcal{L}) = W(E_6)$ (finite groups)

From the point of view of finite simple groups, the significance of $W(E_6)$ comes from the following three pairs $(A_5, S_5)$, $(A_6, S_6)$, $(O_5(3), W(E_6))$ and Brauer’s theorem (see [Br]): If $G$ is a simple group of an order $g = 5 \cdot 3^a \cdot 2^b > 5$, then $G$ is isomorphic to one of the following three groups: the alternating group $A_5$ of order 60, the alternating group $A_6$ of order 360 or the orthogonal group $O_5(3)$ of order 25920. The first two pairs appeared in Klein’s celebrated book [Kl]. In this paper, we will study the third one.

One of the significance of the group $W(E_6)$ is that it possesses in terms of different Chevalley groups (see [Hu], [CC]):

$$
G_{25,920} \cong PU(3,1; \mathbb{F}_4) \cong PSp(4, \mathbb{Z}/3\mathbb{Z}),
$$
where $G_{25,920} \subset W(E_6)$ is the simple subgroup of index two consisting of all even elements. We have the action of the unitary reflection group of order 25, 920: $G_{25,920}$ on $\mathbb{P}^4 = \{(Y_0, Y_1, Y_2, Y_3, Y_4)\}$. The following theorems say that the invariants of $G_{25,920}$ (see section two for the details):

$$
\Phi, u, t, \Psi_1, C_6, C_9, \Phi_3, t_3, C_{12}, C_{18}
$$

can be expressed as the polynomials of the invariants $f_0$, $f_1$, $G$, $H$, $K$ and $C_6$, where

$$
\Phi = Y_0^4 + 8Y_0Y_1^3,
$$

$$
u = Y_0\psi + 6Y_1\varphi,
$$

$$
t = Y_0^6 - 20Y_0^3Y_1^3 - 8Y_1^6,
$$

$$
\Psi_1 = \psi(-Y_0^3 + 4Y_1^3) + 18\varphi Y_0^2Y_1,
$$

$$
C_6 = Y_2^6 + Y_3^6 + Y_4^6 - 10(Y_2^3Y_3^3 + Y_3^3Y_4^3 + Y_4^3Y_2^3),
$$

$$
C_9 = (Y_2^3 - Y_3^3)(Y_3^3 - Y_4^3)(Y_4^3 - Y_2^3),
$$

$$
\Phi_3 = -\psi^3Y_0 + 18\varphi\psi^2Y_1 + 108\varphi^3Y_0,
$$

$$
t_3 = \psi^3(Y_0^3 + 8Y_1^3) - 54\varphi\psi^2Y_0^2Y_1 + 324\varphi^2\psi Y_0Y_1^2 + 216\varphi^3(Y_0^3 - Y_1^3),
$$

$$
C_{12} = (Y_2^3 + Y_3^3 + Y_4^3)[(Y_2^3 + Y_3^3 + Y_4^3)^3 + 216Y_2^3Y_3^3Y_4^3],
$$

$$
C_{18} = (Y_2^3 + Y_3^3 + Y_4^3)^6 - 540Y_2^3Y_3^3Y_4^3(Y_2^3 + Y_3^3 + Y_4^3)^3 - 5832Y_2^6Y_3^6Y_4^6.
$$

The other three invariants are given by

$$
\Psi = Y_0^3Y_1 - Y_1^4,
$$

$$
\Psi_2 = -\psi^2Y_1^2 - 3\varphi\psi Y_0^2 + 18\varphi^2Y_0Y_1,
$$

$$
C_{12} = Y_2Y_3Y_4[27Y_2^3Y_3^3Y_4^3 - (Y_2^3 + Y_3^3 + Y_4^3)^3].
$$

Here, the invariants $f_0$, $f_1$, $G$, $H$, $K$ are given by

$$
f_0 = Y_0 + 2Y_1, \quad f_1 = Y_0 - Y_1.
$$

$$
G = (Y_2 - Y_3)(Y_3 - Y_4)(Y_4 - Y_2), \quad H = \psi + 6\varphi, \quad K = \psi - 3\varphi
$$

with

$$
\varphi = Y_2Y_3Y_4, \quad \psi = Y_2^3 + Y_3^3 + Y_4^3.
$$
Theorem 1.1 (see [Y3], Main Theorem 4). The invariants $G$, $H$ and $K$ satisfy the following algebraic equations, which are the form-theoretic resolvents (algebraic resolvents) of $G$, $H$, $K$:

\[
\begin{cases}
4G^3 + H^2G - C_6G - 4C_9 = 0, \\
H(H^3 + 8K^3) - 9C_{12} = 0, \\
K(K^3 - H^3) - 27C_{12} = 0.
\end{cases}
\]

Consequently,

\[
C_{18} = -\frac{1}{27}(H^6 - 20H^3K^3 - 8K^6).
\]

Theorem 1.2 (Main Theorem 1). The invariants $f_0$, $f_1$, $H$ and $K$ satisfy the following algebraic equations, which are the form-theoretic resolvents (algebraic resolvents) of $f_0$, $f_1$, $H$ and $K$:

\[
u = \frac{1}{3}(f_0H + 2f_1K).
\]

\[
\begin{cases}
\Phi = \frac{1}{9}f_0(f_0^3 + 8f_1^3), \\
\Psi = \frac{1}{9}f_1(f_0^3 - f_1^3), \\
t = -\frac{1}{27}(f_0^6 - 20f_0^3f_1^3 - 8f_1^6).
\end{cases}
\]

\[
\begin{cases}
\Psi_1 = \frac{1}{9}[(f_0^3 - 4f_1^3)H - 6f_0^2f_1K], \\
\Psi_2 = \frac{1}{9}(-f_0^2HK + 2f_0f_1K^2 - f_1^2H^2), \\
\Phi_3 = \frac{1}{9}f_0(H^3 - 4K^3) - 6f_1H^2K, \\
t_3 = \frac{1}{27}[f_0^3(H^3 + 8K^3) - 18f_0^2f_1H^2K + 36f_0f_1^2HK^2 + 8f_1^3(H^3 - K^3)] \\
= \frac{1}{27}[(f_0^3 + 8f_1^3)H^3 - 18f_0^2f_1H^2K + 36f_0f_1^2HK^2 + 8(f_0^3 - f_1^3)K^3].
\end{cases}
\]

We find the following nonlinear duality:

\[
f_0 \leftrightarrow H, \quad f_1 \leftrightarrow K.
\]
Here, $f_0$ and $f_1$ are linear, $H$ and $K$ are cubic. The functions $u$ and $t_3$ are invariant under the above nonlinear dual transformation. Under the nonlinear dual transformation (1.9),

\[(1.10)\quad C_{12} \longleftrightarrow \Phi, \quad C_{12} \longleftrightarrow -\frac{1}{3} \Psi, \quad C_{18} \longleftrightarrow t, \quad \Psi_1 \longleftrightarrow \Phi_3.\]

We will study the analogies between Hessian invariants and Picard modular forms, which can be considered as the higher dimensional counterpart of the analogies between regular polyhedral invariants and elliptic modular forms studied by Klein.

**Conjecture 1.3.**

\[\mathbb{C}[z_1, z_2, z_3]^\text{Hessian groups} = \mathbb{C}[C_6, C_9, C_{12}].\]

*The map*

\[\phi : \mathbb{C}[C_6, C_9, C_{12}] \rightarrow \mathbb{C}[G_2, G_3, G_4]\]

given by

\[\phi(h(C_6, C_9, C_{12})) = h(G_2, G_3, G_4)\]

is an algebra isomorphism.

Here,

\[
\begin{align*}
C_6(z_1, z_2, z_3) &= z_1^6 + z_2^6 + z_3^6 - 10(z_1^3z_2^3 + z_2^3z_3^3 + z_3^3z_1^3), \\
C_9(z_1, z_2, z_3) &= (z_1^3 - z_2^3)(z_2^3 - z_3^3)(z_3^3 - z_1^3), \\
C_{12}(z_1, z_2, z_3) &= (z_1^3 + z_2^3 + z_3^3)[(z_1^3 + z_2^3 + z_3^3)^3 + 216z_1^3z_2^3z_3^3].
\end{align*}
\]

$G_2$, $G_3$, $G_4$ are of weight 2, 3 and 4, respectively, which were introduced by Holzapfel in [Ho1].

Put

\[
X(z_1, z_2, z_3) = z_1^3, \quad Y(z_1, z_2, z_3) = z_2^3, \quad Z(z_1, z_2, z_3) = z_3^3.
\]

\[
\varphi = z_1z_2z_3, \quad Q_1(z_1, z_2, z_3) = z_1z_2^2 + z_2z_3^2 + z_3z_1^2, \quad Q_2(z_1, z_2, z_3) = z_1^2z_2 + z_2^2z_3 + z_3^2z_1.
\]

**Theorem 1.4 (Main Theorem 2).** Let

\[\begin{align*}
W_2 &= (X + Y + Z)^2 - 12(XY + YZ + ZX), \\
W_3 &= (X - Y)(Y - Z)(Z - X), \\
W_4 &= (X + Y + Z)[(X + Y + Z)^3 + 216XYZ], \\
W_6 &= (X + Y + Z)^6 - 540XYZ(X + Y + Z)^3 - 5832X^2Y^2Z^2, \\
W_8 &= Q_1^3 + Q_2^3 - (X + Y + Z + 6\varphi)(XY + YZ + ZX) - 6\varphi^2(X + Y + Z) - 9\varphi^3, \\
W_9 &= Q_1Q_2 - (XY + YZ + ZX) - \varphi(X + Y + Z) - 3\varphi^2, \\
W_3 &= Q_1^3 - Q_2^3 - (X - Y)(Y - Z)(Z - X).
\end{align*}\]
They satisfy the following four relations:

\[
\begin{align*}
\{ & W_2^3 - 3W_2W_4 + 2W_6 = 432W_3^2, \\
& 8U^3(W_6^2 - W_4^3 - 1728W_4^3) = 27W_3(W_4 - 9U^4)^3, \\
& W_2^3W_6 - 3W_2^2W_4W_3 + 2W_6W_4 = 432W_3^2, \\
& 8U^3(W_6^2 - W_4^3 - 1728W_4^3) = 27W_3(W_4 - 9U^4)^3,
\end{align*}
\]

where

\[
27U^8 - 18W_4U^4 - 8W_6U^2 - W_4^2 = 0,
\]

and the Jacobian:

\[
\frac{\partial(W_2, W_3, W_4, 2\mathfrak{M}_3)}{\partial(X, Y, Z, \varphi)} = 288W_2^2.
\]

We find the following correspondence:

\[
(G_2, G_3, G_4, z^2) \longleftrightarrow (W_2, W_3, W_4, 2\mathfrak{M}_4).
\]

Here,

\[
\Delta^2 = 16G_2^4G_4 - 128G_2^2G_4^2 - 4G_2G_4^3 + 144G_2G_4^2G_4 - 27G_3^4 + 256G_4^3, \quad z^6 = \Delta^2.
\]

\[
6912W_4^3 = W_2^6 + 9W_2^2W_4^2 + 432^2W_3^4 - 4W_4^4 - 6W_4^2W_4 - 864W_2^3W_3^2 + 2592W_2W_3W_4.
\]

**Theorem 1.5 (Main Theorem 3).** Let

\[
x = Q_1 - Q_2, \quad y = X + Y + Z + 6\varphi, \quad z = X + Y + Z - 3\varphi.
\]

Then \(x, y\) and \(z\) satisfy the following three equations:

\[
2^4 \cdot 3^8W_3^2(4\mathfrak{M}_3 - 4x^3 - xy^2 + W_2x + 4W_3)^3
\]

\[
= 2^{10} \cdot 3^{11}W_3^2\mathfrak{M}_2^2(\mathfrak{M}_3 + W_3) - (4\mathfrak{M}_3 - 4x^3 - xy^2 + W_2x + 4W_3)\mathfrak{M}_2^2 \times

\times \{2^8 \cdot 3^{11}W_3^2\mathfrak{M}_2^2 + [(9W_4 - y(y^3 + 8z^3))^2 - 2^6 \cdot 3^{10}W_2\mathfrak{M}_2^2] +

+ 2^5[27W_4 - z(z^3 - y^3)][9W_4 - y(y^3 + 8z^3)] + 2^8[27W_4 - z(z^3 - y^3)]^2\}.
\]

\[
[9W_4 - y(y^3 + 8z^3)]^4 + 2^{12}[9W_4 - y(y^3 + 8z^3)][3^{18}W_3^4 + [27W_4 - z(z^3 - y^3)]^3]
\]

\[
= 2^{10} \cdot 3^{12}W_3^4W_4.
\]
\[ [9W_4 - y(y^3 + 8z^3)]^6 - 2^{11} \cdot 5[9W_4 - y(y^3 + 8z^3)]^3 \{3^{18}W_3^4 + [27W_4 - z(z^3 - y^3)]^3 \} - 2^{21} \{3^{18}W_3^4 + [27W_4 - z(z^3 - y^3)]^3 \}^2 = 2^{18} \cdot 3^{30}W_6^6. \]

We have the following table which describes the relation between Hessian polyhedra, invariant theory and Picard modular forms.

| Hessian polyhedra | Picard modular functions |
|-------------------|--------------------------|
| \((C_6, C_9, C_{12})\) | \((G_2, G_3, G_4)\), weight 2, 3, 4 |
| \((G, H, K)\) | \((\xi_1, \xi_2, \xi_3)\), weight 1 |
| \((\varphi, X, Y, Z, Q_1, Q_2)\) | \((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)\), weight 1 |

It is known that the Hessian group of order 216 is generated by the five generators (see [Y3]):

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix},
\]

\[
E = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.
\]

The actions of \(A, B, C\) and \(E\) on \((X, Y, Z, \varphi, Q_1, Q_2)\) are given as follows:

\[
A(X, Y, Z, \varphi, Q_1, Q_2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ \varphi \\ Q_1 \\ Q_2 \end{pmatrix},
\]

\[(1.20)\]

\[
B(X, Y, Z, \varphi, Q_1, Q_2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ \varphi \\ Q_1 \\ Q_2 \end{pmatrix},
\]

\[(1.21)\]
\[ C(X, Y, Z, \varphi, Q_1, Q_2) \]

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & \omega \\
\omega & \omega & \omega
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\varphi \\
Q_1 \\
Q_2
\end{pmatrix},
\]

(1.22)

\[ (\sqrt{-3})^3 E(X, Y, Z, \varphi, Q_1, Q_2) \]

\[
\begin{pmatrix}
1 & 1 & 1 & 6 & 3 & 3 \\
1 & 1 & 1 & 6 & 3\omega & 3\omega \\
1 & 1 & 1 & 6 & 3\bar{\omega} & 3\bar{\omega} \\
1 & 1 & 1 & -3 & 0 & 0 \\
3 & 3\bar{\omega} & 3\omega & 0 & 0 & 0 \\
3 & 3\omega & 3\bar{\omega} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\varphi \\
Q_1 \\
Q_2
\end{pmatrix},
\]

(1.23)

**Proposition 1.6.** The space curves \( W_2 = \mathcal{W}_3 = \mathcal{U}_3 = \mathcal{U}_2 = 0, W_3 = \mathcal{W}_3 = \mathcal{U}_3 = \mathcal{U}_2 = 0, W_4 = \mathcal{W}_3 = \mathcal{U}_3 = \mathcal{U}_2 = 0, W_6 = \mathcal{W}_3 = \mathcal{U}_3 = \mathcal{U}_2 = 0 \) and \( \mathcal{W}_4 = \mathcal{W}_3 = \mathcal{U}_3 = \mathcal{U}_2 = 0 \) are invariant curves on the invariant surface \( \mathcal{W}_3 = \mathcal{U}_3 = \mathcal{U}_2 = 0 \) under the action of the subgroup of Hessian group generated by \( A, B, C \) and \( E \).

Put

\[ g_1 = E(x), \quad g_2 = EC(x), \quad g_3 = EC^2(x), \quad g_4 = x, \quad g_5 = C(x), \quad g_6 = C^2(x). \]

We find that

\[ S : \begin{cases} 
g_1 + g_2 + g_3 = 0, 
g_4 + g_5 + g_6 = 0, 
g_1^3 + g_2^3 + g_3^3 = g_4^3 + g_5^3 + g_6^3, \end{cases} \]

(1.25)

which is a cubic surface.

The study of the general cubic surface dates from 1849, in which year the 27 lines were found by Cayley and Salmon; the discovery of the Sylvester pentahedron followed two years later, and the theory attained a remarkable degree of elegance and completeness with the introduction of the plane representation, made independently by Clebsch and Cremona in 1866 (see [Seg]). Nevertheless, even today the study of what Cayley has called ‘the complicated and many-sided symmetry’ of the relations between the 27 lines can not be said to have been exhausted, especially Schl"afli’s well-known notation for the lines readily lend itself to the description and deeper investigation of that symmetry.

As far as the equation defining a cubic surface is concerned, we have the following result, claimed by Sylvester (1851) and Steiner, and proved by Clebsch (1861). It is the so-called pentahedral form of the cubic.
Theorem 1.7 (see [Hu]). A general cubic form \( F(x_0 : x_1 : x_2 : x_3) \) can be put, in a unique way, in the form

\[
a_1y_1^3 + a_2y_2^3 + a_3y_3^3 + a_4y_4^3 + a_5y_5^3 = 0,
\]

where the coordinates \( y_i \) are linear in \((x_0 : x_1 : x_2 : x_3)\) and satisfy

\[
y_1 + y_2 + y_3 + y_4 + y_5 = 0.
\]

A second special form of the equation, the hexahedral form, is given by the polar hexagons:

Theorem 1.8 (see [Hu]). A general cubic form \( F \) can be put in a four-dimensional family of forms:

\[
z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 = 0,
\]

where the \( z_i \) are linear in the \( x_i \).

In fact, the pentahedral form of the cubic is closely related to the Clebsch-Klein diagonal cubic surface. In 1873 Klein proved that the famous diagonal surface of Clebsch, which is the surface in \( \mathbb{P}^4(\mathbb{C}) \) with equations

\[
x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \quad x_0 + x_1 + x_2 + x_3 + x_4 = 0,
\]

can be obtained from \( \mathbb{P}^2(\mathbb{C}) \) by blowing up 6 points in \( \mathbb{P}^2(\mathbb{C}) \) in a special position, namely the 6 points in \( \mathbb{P}^2(\mathbb{R}) = S^2/\{\pm 1\} \) corresponding to the 12 vertices of an icosahedron inscribed in \( S^2 \). Hirzebruch blew up 10 more points, namely those corresponding to the 20 vertices of the dual dodecahedron. The resulting surface \( Y \) can also be obtained from the Clebsch diagonal surface by blowing up 10 Eckhardt points, that is points, where 3 of the 27 lines on the surface meet. Let \( \mathcal{O}_K \subset \mathbb{Q}(\sqrt{5}) \) be the ring of integers and \( \Gamma \subset SL(2, \mathcal{O}_K) \) the congruence subgroup mod 2. Hirzebruch proved that the icosahedral surface \( Y \) is the minimal resolution of \( \overline{\mathbb{H}^2}/\Gamma \).

The significance of our cubic surface is that it is the hypersurface of the hexahedral form of the cubic.

There are 27 lines on the cubic surface \( S \):

\[
l_1 : \quad g_1 = g_2 + g_3 = g_4 = g_5 + g_6 = 0,
\]

\[
l_2 : \quad g_1 = g_2 + g_3 = g_5 = g_4 + g_6 = 0,
\]

\[
l_3 : \quad g_1 = g_2 + g_3 = g_6 = g_4 + g_5 = 0,
\]

\[
l_4 : \quad g_2 = g_1 + g_3 = g_4 = g_5 + g_6 = 0,
\]

\[
l_5 : \quad g_2 = g_1 + g_3 = g_5 = g_4 + g_6 = 0,
\]
\[ l_6 : \quad g_2 = g_1 + g_3 = g_6 = g_4 + g_5 = 0, \]
\[ l_7 : \quad g_3 = g_1 + g_2 = g_4 = g_5 + g_6 = 0, \]
\[ l_8 : \quad g_3 = g_1 + g_2 = g_5 = g_4 + g_6 = 0, \]
\[ l_9 : \quad g_3 = g_1 + g_2 = g_6 = g_4 + g_5 = 0, \]

and
\[ l_{j,1} : \quad g_1 = \omega^j g_4, \quad g_2 = \omega^j g_5, \quad g_3 = \omega^j g_6, \quad g_1 + g_2 + g_3 = 0, \]
\[ l_{j,2} : \quad g_1 = \omega^j g_4, \quad g_2 = \omega^j g_6, \quad g_3 = \omega^j g_5, \quad g_1 + g_2 + g_3 = 0, \]
\[ l_{j,3} : \quad g_1 = \omega^j g_5, \quad g_2 = \omega^j g_4, \quad g_3 = \omega^j g_6, \quad g_1 + g_2 + g_3 = 0, \]
\[ l_{j,4} : \quad g_1 = \omega^j g_5, \quad g_2 = \omega^j g_6, \quad g_3 = \omega^j g_4, \quad g_1 + g_2 + g_3 = 0, \]
\[ l_{j,5} : \quad g_1 = \omega^j g_6, \quad g_2 = \omega^j g_4, \quad g_3 = \omega^j g_5, \quad g_1 + g_2 + g_3 = 0, \]
\[ l_{j,6} : \quad g_1 = \omega^j g_6, \quad g_2 = \omega^j g_5, \quad g_3 = \omega^j g_4, \quad g_1 + g_2 + g_3 = 0, \]

where \( j \equiv 0, 1, 2(\mod 3) \).

Let us consider the configuration of the 27 lines (forgetting the cubic surface). That means we consider simply the set of 27 lines together with the incidence relations they satisfy. There is a group of automorphisms of the configuration meaning the permutations of the set of 27 lines, preserving the incidence relations.

There are three difficulties: 1. The realization of 27 lines on the cubic surface by algebraic equations. 2. Under the action of the subgroup of \( W(E_6) \), 27 lines on the cubic surface may not lie on the same cubic surface. 3. Under the action of the subgroup of \( W(E_6) \), the incidence relations of 27 lines may not preserve.

The double six is given by

\[
N = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6
\end{pmatrix} = \begin{pmatrix}
l_{1,1} & l_{2,2} & l_{2,3} & l_{1,4} & l_{1,5} & l_{2,6} \\
l_{2,1} & l_{1,2} & l_{1,3} & l_{2,4} & l_{2,5} & l_{1,6}
\end{pmatrix}.
\]

We find that

\[
c_{12} = l_1, \quad c_{13} = l_9, \quad c_{14} = l_{0,5}, \quad c_{15} = l_{0,4}, \quad c_{16} = l_5.
\]

\[
c_{23} = l_{0,6}, \quad c_{24} = l_6, \quad c_{25} = l_8, \quad c_{26} = l_{0,3}.
\]

\[
c_{34} = l_2, \quad c_{35} = l_4, \quad c_{36} = l_{0,2}.
\]

\[
c_{45} = l_{0,1}, \quad c_{46} = l_7, \quad c_{56} = l_3.
\]
We find that the action of the group $H$ generated by $A$, $B$, $C$ and $E$ on the 27 lines of our cubic surface $S$ gives the permutations of the lines which preserve the intersection behavior of the lines.

**Theorem 1.9 (Main Theorem 4).** The group $H$ generated by $A$, $B$, $C$ and $E$ is a subgroup of the group of automorphisms of the configuration of 27 lines on the cubic surface: $\text{Aut}(\mathcal{L}) = W(E_6)$.

This gives the Galois representation:

$$(1.31) \quad \rho : G_Q = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\mathcal{L}) = W(E_6).$$

Here,

$$(1.32) \quad \text{Im}(\rho) = H = \langle A, B, C, E \rangle, \quad \text{ker}(\rho) = \text{Gal}(\overline{\mathbb{Q}}/\mathcal{K}).$$

We have

$$(1.33) \quad \text{Im}(\rho) \cong \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\ker(\rho) = \text{Gal}(\overline{\mathbb{Q}}/\mathcal{Q}) \cong \text{Gal}(\mathcal{K}/\mathbb{Q}).$$

Hence,

$$(1.34) \quad \text{Gal}(\mathcal{K}/\mathbb{Q}) \cong H.$$

**Conjecture 1.10 (Main Conjecture).** Let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W(E_6)$ be a Galois representation. When the image of $\rho$: $\text{Im}(\rho) \cong H \leq W(E_6)$, if $\rho$ is odd, i.e., $\det(\rho(\sigma_c)) = -1$, where $\sigma_c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the complex conjugate, then there exists a Picard modular form $f$ of weight one such that

$$\text{L}(\rho, s) = \text{L}(f, s)$$

up to finitely many Euler factors, where $\text{L}(\rho, s)$ is the Artin $L$-function associated to the Galois representation $\rho$ and $\text{L}(f, s)$ is the automorphic $L$-function associated to the Picard modular form $f$. If $\rho$ is even, i.e., $\det(\rho(\sigma_c)) = 1$, then there exists an automorphic form $f$ associated to $U(2, 1): \Delta f = s(s - 2)f$ with $s = 1$ and (see [Y1])

$$\Delta = (z_1 + \overline{z}_1 - z_2\overline{z}_2) \left[ (z_1 + \overline{z}_1) \frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + (z_2 + \overline{z}_2) \frac{\partial^2}{\partial z_2 \partial \overline{z}_2} + (z_1 + \overline{z}_1) \frac{\partial^2}{\partial z_1 \partial \overline{z}_2} + (z_2 + \overline{z}_2) \frac{\partial^2}{\partial z_2 \partial \overline{z}_1} \right],$$

$$(z_1, z_2) \in \mathcal{S}_2 = \{ (z_1, z_2) \in \mathbb{C}^2 : z_1 + \overline{z}_1 - z_2\overline{z}_2 > 0 \},$$

such that

$$\text{L}(\rho, s) = \text{L}(f, s)$$

up to finitely many Euler factors, where $\text{L}(\rho, s)$ is the Artin $L$-function associated to the Galois representation $\rho$ and $\text{L}(f, s)$ is the automorphic $L$-function associated to the automorphic form $f$. 
Note that the group $W(E_6)$ is combinatorially defined, while the subgroup $H$ is defined on algebraic variety. Thus we give a connection between combinatoric and algebraic geometry. Furthermore, the Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H$ comes from number theory, the Picard modular forms come from analysis and representation theory of Lie groups. Here, Galois symmetry and geometric symmetry meet together. Therefore, our theorem and conjecture gives a connection which involves combinatoric, algebraic geometry, number theory, analysis and representation theory. More precisely, we find that Hessian polyhedra and their invariants play a role of bridge which connects Galois representations arising from 27 lines on a cubic surface (arithmetic) with Picard modular forms (analysis). In fact, there are two aspects about Hessian polyhedra. One is Hessian group which is related to analysis, the other is the system of invariants which is related to algebra. Thus, Hessian polyhedra give applications to number theory and algebraic geometry, especially arithmetic algebraic geometry and noncommutative class field theory.

**Theorem 1.11 (Main Theorem 5).** The variety associated with Hessian polyhedra (1.31)

$$E : \quad y^2 = x^3 - 27C_{12}x + 54C_{18}$$

can be expressed as the fiber product of two isogenous, semi-stable, rational elliptic modular surfaces

$$E_{1,t} : \quad (t - 1)(z_1^3 + z_2^3 + z_3^3) - 3(t + 2)z_1z_2z_3 = 0$$

and

$$E_{2,t} : \quad y^2 = x^3 - 3t(t^3 + 8)x - 2(t^6 - 20t^3 - 8).$$

Hence, $E$ is a rigid Calabi-Yau threefold. There are only finitely many rational points in $E_{1,t}$ and $E_{2,t}$. However, there exist infinitely many nontrivial rational points in $E$. More precisely, $P = (3C_6, 108C_9) \in E$ is of infinite order.

The present paper consists of five sections. In section two, we study Hessian polyhedra and cubic forms associated to the finite simple group $G_{25,920}$. In section three, we study Hessian polyhedra and Picard modular forms. In section four, we investigate Hessian polyhedra and Galois representations arising from 27 lines on a cubic surface. In the last section, we study Hessian polyhedra and the arithmetic of rigid Calabi-Yau threefolds.

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### 2. Hessian polyhedra and cubic forms associated to $G_{25,920}$

Let us recall some basic facts about the unitary reflection groups of order 25,920 (see [Hu]). We have actions of $G_{25,920}$ on $\mathbb{P}^3$ and on $\mathbb{P}^4$, both of which in fact are generated by unitary reflections.
Burkhardt (see [Bu1] and [Bu2]) determined the invariants of $G_{25,920}$ acting on $\mathbb{P}^4$. Let the coordinates be given as

$$Y_0 = Y_{00}, Y_1 = Y_{10}, Y_2 = Y_{01}, Y_3 = Y_{11}, Y_4 = Y_{12}$$

where the $Y_{\alpha\beta}$ are the theta functions of

$$\mathbb{P}^4 = \{ Y_{\alpha\beta} = \frac{1}{2}(X_{\alpha\beta} + X_{-\alpha-\beta}) \}.$$

Here

$$X_{\alpha\beta} = \Theta \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{3} & \frac{\beta}{3} \end{bmatrix} (\tau, z), \quad \tau \in \mathbb{H}_2, z \in \mathbb{C}^2, \alpha, \beta \in \mathbb{Z}/3\mathbb{Z}.$$

Using the isomorphism $G_{25,920} \cong PSp(4,\mathbb{Z}/3\mathbb{Z})$, it suffices to use generators of the symplectic group $PSp(4,\mathbb{Z}/3\mathbb{Z})$ to get generators of the action of $G_{25,920}$ on $\mathbb{P}^4$. However, Burkhardt used instead the corresponding hyperelliptic curves, and a certain Weierstrass form for them to describe the level 3 structure. The generators of the group $G_{25,920}$ are transformations $B$, $C$, $D$ and $S_2$, which act on $\mathbb{P}^4 = \{(Y_0, Y_1, Y_2, Y_3, Y_4)\}$ as follows:

$$B(Y_0, Y_1, Y_2, Y_3, Y_4) = \frac{1}{\sqrt{-3}}(Y_0 + 2Y_1, Y_0 - Y_1, Y_2 + Y_3 + Y_4, Y_2 + \omega Y_3 + \omega^2 Y_4, Y_2 + \omega^2 Y_3 + \omega Y_4),$$

$$C(Y_0, Y_1, Y_2, Y_3, Y_4) = (Y_0, Y_1, Y_4, Y_2, Y_3),$$

$$D(Y_0, Y_1, Y_2, Y_3, Y_4) = (-Y_0, -Y_2, -Y_1, -Y_3, -Y_4),$$

$$S_2(Y_0, Y_1, Y_2, Y_3, Y_4) = (Y_0, \omega^2 Y_1, Y_2, \omega^2 Y_3, \omega^2 Y_4).$$

In the form of matrices, we have

$$B = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 1 \\ 1 & \omega \\ 1 & \omega^2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & \omega^2 \\ \omega^2 & 1 \end{pmatrix}.$$ 

Note that

$$B^4 = C^3 = D^2 = S_2^3 = I.$$
and

\[
B^2 = - \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

In fact, \(B, C, D, S_2 \in SL(5, \mathbb{C})\).

Consider the hyperplane \(S\) given by \(S = \{Y_0 = 0\}\) and the plane \(J = \{Y_0 = Y_1 = 0\}\). The stabilizer of each is a subgroup of order 648, but these two subgroups of order 648 are not conjugate to each other; indeed,

\[
N(S) = \langle C, D, S_2 \rangle, \quad N(J) = \langle B, C, S_2 \rangle.
\]

Hence, \(G_{25,920}\) is generated by the subgroup of order 648 acting on \(J\), generated by \(B, C\) and \(S_2\), and by the centralizer of \(J\). Burkhardt shows this to be a homogenous tetrahedral group, and its invariants are:

\[
\Phi = Y_0^4 + 8Y_0Y_1^3, \\
\Psi = Y_0^3Y_1 - Y_1^4, \\
t = Y_0^6 - 20Y_0^3Y_1^3 - 8Y_1^6,
\]

which satisfy the relation:

\[
\Phi^3 - 64\Psi^3 = t^2.
\]

Furthermore, set

\[
\varphi = Y_2Y_3Y_4, \\
\psi = Y_2^3 + Y_3^3 + Y_4^3, \\
u = Y_0\psi + 6Y_1\varphi.
\]

All invariants of \(G_{25,920}\) can be written as linear combinations of the following:

\[
\Phi, u, t, \Psi_1, C_6, C_9, \Phi_3, t_3, C_{12}, C_{18},
\]

where

\[
\Psi_1 = \psi(-Y_0^3 + 4Y_1^3) + 18\varphi Y_0^2Y_1, \\
\Psi_2 = -\psi^2Y_1^2 - 3\varphi\psi Y_0^2 + 18\varphi^2Y_0Y_1, \\
\Phi_3 = -\psi^3Y_0 + 18\varphi\psi^2Y_1 + 108\varphi^3Y_0, \\
t_3 = \psi^3(Y_0^3 + 8Y_1^3) - 54\varphi\psi^2Y_0^2Y_1 + 324\varphi^2\psi Y_0Y_1^2 + 216\varphi^3(Y_0^3 - Y_1^3),
\]
and

$$\begin{aligned}
C_6 &= Y_2^6 + Y_3^6 + Y_4^6 - 10(Y_2^3 Y_3^3 + Y_3^3 Y_4^3 + Y_4^3 Y_2^3), \\
C_9 &= (Y_2^3 - Y_3^3)(Y_3^3 - Y_4^3)(Y_4^3 - Y_2^3), \\
C_{12} &= (Y_2^3 + Y_3^3 + Y_4^3)[(Y_2^3 + Y_3^3 + Y_4^3)^3 + 216Y_2^3 Y_3^3 Y_4^3], \\
C_{18} &= (Y_2^3 + Y_3^3 + Y_4^3)^6 - 540Y_2^3 Y_3^3 Y_4^3(Y_2^3 + Y_3^3 + Y_4^3)^3 - 5832Y_2^6 Y_3^6 Y_4^6.
\end{aligned}$$

They satisfy the following relations:

$$\begin{aligned}
\Psi_1^2 &= 16\Psi_2 + u^2\Phi, \\
\Psi_1^3 &= 2\Phi^2\Phi_3 - tt_3 - 3u^2\Phi\Psi_1 - u^3t,
\end{aligned}$$

and

$$\begin{aligned}
432C_9^2 &= C_9^3 - 3C_6 C_{12} + 2C_{18}, \\
1728C_{12}^3 &= C_{18}^3 - C_{12}^3.
\end{aligned}$$

Burkhardt calculates the following expressions (see [Bu1], [Bu2], [Hu]). There are invariants of degrees 4, 6, 10, 12, 18 and 45. The invariant of degree 45 is just the product of the 45 reflection hyperplanes defining the arrangement in $\mathbb{P}^4$. The others are given by:

$$\begin{aligned}
J_4 &= \Phi + 8u, \\
J_6 &= t + 20\Psi_1 - 8C_6, \\
J_{10} &= \frac{1}{24}(\Phi\Psi_1 + ut + 2\Phi C_6 + 2u\Psi_1 - 2\Phi_3 - 2uC_6), \\
J_{12} &= \frac{1}{24}(3t\Psi_1 + 3u\Phi^2 + 19\Psi_1^2 - 9u^2\Phi - 10C_6 t - 11t_3 + 9u^3 - 2C_6\Psi_1 - 4C_{12} + 4C_6^2), \\
J_{18} &= \frac{1}{864}(72t\Psi_2 + 9u\Phi^2\Psi_1 + 9u^2\Phi t + 288C_6\Psi_3^3 + \\
+ 4\Phi^2\Phi_3 - 18tt_3 - 42u^2\Phi\Psi_1 - 20u^3t - 18C_6 t \Psi_1 + \\
- 18C_6 \Phi^2 u + 84C_{12} t - 72u\Phi\Phi_3 + 162u^3\Psi_1 + \\
- 240C_6 \Psi_2 + 12C_6^2 u\Phi - 6C_6^2 t + 24\Psi_1 C_{12} - 36u^2\Phi_3 + \\
- 6C_6 t_3 - 18C_6 u^3 - 12C_6^2 \Psi_1 - 4C_{18} + 6C_6 C_{12} - 2C_6^3).
\end{aligned}$$

There is a relation between these, which takes the form $J_{45}^2 = \text{rational expression in the other invariants}$. There is also an invariant of degree 40: the product of the 40 Steiner primes. Denote this by $J_{40}$, the following relation holds:

$$3^{33}F_{40} = [J_4^2(2^9 J_{12} - J_4^3)] - 3 \cdot 2^{18}J_{10}^2 - 2^{19}[J_4 J_6 - 3 \cdot 2^8 J_{10}][J_4^3 J_{18} - 2^{11} J_{10}^3].$$
Now, we put

\[(2.8)\quad f_0 = Y_0 + 2Y_1, \quad f_1 = Y_0 - Y_1.\]

Then

\[Y_0 = \frac{1}{3}(f_0 + 2f_1), \quad Y_1 = \frac{1}{3}(f_0 - f_1).\]

Recall that (see [Y3])

\[(2.9)\quad H = \psi + 6\varphi, \quad K = \psi - 3\varphi.\]

Thus,

\[
\psi = \frac{1}{3}(H + 2K), \quad \varphi = \frac{1}{9}(H - K).
\]

We have the following theorem which says that the above invariants of \(G_{25,920}\) can be expressed as the polynomials of the invariants \(f_0, f_1, H\) and \(K\).

**Theorem 2.1** (Main Theorem 1). The invariants \(f_0, f_1, H\) and \(K\) satisfy the following algebraic equations, which are the form-theoretic resolvents (algebraic resolvents) of \(f_0, f_1, H\) and \(K\):

\[(2.10)\quad u = \frac{1}{3}(f_0H + 2f_1K).
\]

\[(2.11)\quad \begin{cases}
\Phi = \frac{1}{9}f_0(f_0^3 + 8f_1^3), \\
\Psi = \frac{1}{9}f_1(f_0^3 - f_1^3), \\
t = -\frac{1}{27}(f_0^6 - 20f_0^3f_1^3 - 8f_1^6),
\end{cases}
\]

\[(2.12)\quad \begin{cases}
\Psi_1 = \frac{1}{9}[(f_0^3 - 4f_1^3)H - 6f_0^2f_1K], \\
\Psi_2 = \frac{1}{9}(-f_0^2HK + 2f_0f_1K^2 - f_1^2H^2), \\
\Phi_3 = \frac{1}{9}[f_0(H^3 - 4K^3) - 6f_1H^2K], \\
t_3 = \frac{1}{27}[(f_0^3 + 8f_1^3)H^3 - 18f_0^2f_1H^2K + 36f_0f_1^2HK^2 + 8f_1^3(H^3 - K^3)] \\
= \frac{1}{27}[(f_0^3 + 8f_1^3)H^3 - 18f_0^2f_1H^2K + 36f_0f_1^2HK^2 + 8(f_0^3 - f_1^3)K^3].
\end{cases}
\]
Proof. We have
\[ u = \frac{1}{9}[(f_0 + 2f_1)(H + 2K) + 2(f_0 - f_1)(H - K)] \]
\[ = \frac{1}{3}(f_0H + 2f_1K). \]
\[ \Phi = \frac{1}{81}(f_0 + 2f_1)[(f_0 + 2f_1)^3 + 8(f_0 - f_1)^3] \]
\[ = \frac{1}{9}(f_0 + 2f_1)f_0(f_0^2 - 2f_0f_1 + 4f_1^2) \]
\[ = \frac{1}{9}f_0(f_3^3 + 8f_1^3). \]
\[ \Psi = \frac{1}{81}(f_0 - f_1)[(f_0 + 2f_1)^3 - (f_0 - f_1)^3] \]
\[ = \frac{1}{9}(f_0 - f_1)f_1(f_0^2 + f_0f_1 + f_1^2) \]
\[ = \frac{1}{9}f_1(f_3^3 - f_1^3). \]
\[ t = \frac{1}{729}[(f_0 + 2f_1)^6 - 20(f_0 + 2f_1)^3(f_0 - f_1)^3 - 8(f_0 - f_1)^6] \]
\[ = -\frac{1}{27}(f_0^6 - 20f_0^3f_1^3 - 8f_1^6). \]
\[ \Psi_1 = \frac{1}{9}\psi(f_0^3 - 6f_0^2f_1 - f_1^3) + \frac{2}{3}\varphi(f_0^3 + 3f_0^2f_1 - 4f_1^3) \]
\[ = \frac{1}{9}f_0^3(\psi + 6\varphi) - \frac{2}{3}f_0^2f_1(\psi - 3\varphi) - \frac{4}{9}f_1^3(\psi + 6\varphi) \]
\[ = \frac{1}{9}(f_0^3H - 6f_0^2f_1K - 4f_1^3H). \]
\[ \Psi_2 = \frac{1}{9}[-\psi^2(f_0^2 - 2f_0f_1 + f_1^2) - 3\varphi(\psi^2 + 4f_0f_1 + 4f_1^2) + 18\varphi^2(f_0^2 + f_0f_1 - 2f_1^2)] \]
\[ = -\frac{1}{9}f_0^2(\psi + 6\varphi)(\psi - 3\varphi) + \frac{2}{9}f_0f_1(\psi - 3\varphi)^2 - \frac{1}{9}f_1^2(\psi + 6\varphi)^2 \]
\[ = \frac{1}{9}(-f_0^2HK + 2f_0f_1K^2 - f_1^2H^2). \]
\[ \Phi_3 = \frac{1}{3}f_0(-\psi^3 + 18\varphi\psi^2 + 108\varphi^3) + \frac{2}{3}f_1(-\psi^3 - 9\psi\psi^2 + 108\varphi^3). \]

Here,
\[ -\psi^3 + 18\varphi\psi^2 + 108\varphi^3 = \frac{1}{3}(H^3 - 4K^3), \]
\[-\psi^3 - 9\varphi\psi^2 + 108\varphi^3 = -H^2K.\]

Hence,

\[
\Phi_3 = \frac{1}{9} [f_0(H^3 - 4K^3) - 6f_1H^2K].
\]

\[
t_3 = \frac{1}{3} \psi^3 (f_0^3 - 2f_0^2f_1 + 4f_0f_1^2) - 2\varphi\psi^2(f_0^3 + 3f_0^2f_1 - 4f_1^3) + 12\varphi^2\psi(f_0^3 - 3f_0f_1^2 + 2f_1^3) + 72\varphi^3(f_0^2f_1 + f_0f_1^2 + f_1^3)
\]
\[
= \frac{1}{3} f_0^3(\psi^3 - 6\varphi\psi^2 + 36\varphi^2\psi) - \frac{2}{3} f_0^2 f_1(\psi^3 + 9\varphi\psi^2 - 108\varphi^3) + \frac{4}{3} f_0 f_1^2(\psi^3 - 27\varphi^2\psi + 54\varphi^3) + 8 f_1^3(\varphi\psi^2 + 3\varphi^2\psi + 9\varphi^3).
\]

Here,

\[
\psi^3 - 6\varphi\psi^2 + 36\varphi^2\psi = \frac{1}{9}(H^3 + 8K^3),
\]
\[
\psi^3 + 9\varphi\psi^2 - 108\varphi^3 = H^2K,
\]
\[
\psi^3 - 27\varphi^2\psi + 54\varphi^3 = HK^2,
\]
\[
\varphi\psi^2 + 3\varphi^2\psi + 9\varphi^3 = \frac{1}{27}(H^3 - K^3).
\]

Hence,

\[
t_3 = \frac{1}{27} [f_0^3(H^3 + 8K^3) - 18 f_0^2 f_1 H^2 K + 36 f_0 f_1^2 H K^2 + 8 f_1^3(H^3 - K^3)].
\]

Note that (see [Y3])

\[
\begin{align*}
C_{12} &= \frac{1}{9} H(H^3 + 8K^3), \\
C_{12} &= \frac{1}{27} K(K^3 - H^3), \\
C_{18} &= -\frac{1}{27}(H^6 - 20H^3K^3 - 8K^6),
\end{align*}
\]

we find the following nonlinear duality:

\[
(2.14) \quad f_0 \longleftrightarrow H, \quad f_1 \longleftrightarrow K.
\]

Here, \(f_0\) and \(f_1\) are linear, \(H\) and \(K\) are cubic. The functions \(u\) and \(t_3\) are invariant under the above dual transformation. Under the dual transformation (2.14),

\[
(2.15) \quad C_{12} \longleftrightarrow \Phi, \quad C_{12} \longleftrightarrow -\frac{1}{3} \Psi, \quad C_{18} \longleftrightarrow t, \quad \Psi_1 \longleftrightarrow \Phi_3.
\]
Recall that (see [Y3])
\[ Z_1 = 432 \frac{C_6^2}{C_6^3}, \quad Z_2 = 3 \frac{C_12}{C_6^2} \]

Now, we put
\[ (2.16) \quad Z_3 = \frac{\Phi^3}{64\Psi^3}, \quad Z_4 = \frac{\Psi_3^2}{16\Psi_2}, \quad Z_5 = \frac{2\Phi_3}{3u^2\Phi_1}, \quad Z_6 = \frac{\Psi_4^2}{3u^2\Phi}, \quad Z_7 = \frac{ut}{3\Phi_1}. \]

Then
\[ (2.17) \quad \frac{t^2}{64\Psi^3} = Z_3 - 1, \quad \frac{u^2\Phi}{16\Psi_2} = Z_4 - 1, \quad \frac{tt_3}{3u^2\Phi_1} = Z_5 - Z_6 - Z_7 - 1. \]

Recall that (see [Y3])
\[ r_1 = \frac{G^2}{C_6}, \quad r_2 = \frac{H^2}{C_6}, \quad r_3 = \frac{K^2}{C_6}, \]

where \( G = (Y_2 - Y_3)(Y_3 - Y_4)(Y_4 - Y_2) \). Put
\[ (2.18) \quad r_4 = \frac{f_1}{f_0}. \]

By Theorem 2.1, We find that
\[ (2.19) \quad Z_3 = \frac{1}{64} \frac{(1 + 8r_4^3)^3}{r_4^3(1 - r_4^3)^3}. \]
\[ (2.20) \quad Z_4 = \frac{1}{16} \frac{[(1 - 4r_4^3)\sqrt{r_2} - 6r_4\sqrt{r_3}]^2}{r_4(1 - r_4^3)(\sqrt{r_2}r_3 - 2r_4r_3 + r_4^2r_2)}. \]
\[ (2.21) \quad Z_5 = \frac{2}{3} \frac{(1 + 8r_4^3)(r_2\sqrt{r_2} - 4r_3\sqrt{r_3} - 6r_4r_2\sqrt{r_3})}{(\sqrt{r_2} + 2r_4\sqrt{r_3})^2[(1 - 4r_4^3)\sqrt{r_2} - 6r_4\sqrt{r_3}].} \]
\[ (2.22) \quad Z_6 = \frac{1}{3} \frac{[(1 - 4r_4^3)\sqrt{r_2} - 6r_4\sqrt{r_3}]^2}{(\sqrt{r_2} + 2r_4\sqrt{r_3})^2(1 + 8r_4^3).} \]
\[ (2.23) \quad Z_7 = -\frac{1}{3} \frac{[(\sqrt{r_2} + 2r_4\sqrt{r_3})(1 - 20r_4^3 - 8r_4^6)]}{(1 + 8r_4^3)[(1 - 4r_4^3)\sqrt{r_2} - 6r_4\sqrt{r_3}].} \]
We have

\[
\begin{align*}
\sqrt{-3}f_0(B(Y_0, Y_1, Y_2, Y_3, Y_4)) &= f_0 + 2f_1, \\
\sqrt{-3}f_1(B(Y_0, Y_1, Y_2, Y_3, Y_4)) &= f_0 - f_1.
\end{align*}
\]

\[(2.24)\]

\[
\begin{align*}
f_0(C(Y_0, Y_1, Y_2, Y_3, Y_4)) &= f_0, \\
f_1(C(Y_0, Y_1, Y_2, Y_3, Y_4)) &= f_1.
\end{align*}
\]

\[(2.25)\]

Note that

\[
CD = \begin{pmatrix}
-1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix}, \quad DC = \begin{pmatrix}
-1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix}.
\]

\[
C^2D = \begin{pmatrix}
-1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix}, \quad DC^2 = \begin{pmatrix}
-1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix}.
\]

\[
(CD)^2 = (DC^2)^2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

\[
(DC)^2 = (C^2D)^2 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]

The above matrices are the permutation matrices.

On the other hand, Maschke determined the invariants of \(G_{25,920}\) acting on \(\mathbb{P}^3 = \{(z_0, z_1, z_2, z_3)\} \). According to Maschke [Mas], \(G_{25920}\) is generated by the following elements: \(M, N, P, Q\) and \(R\), where

\[M = FAFC^2EFA^2C, \quad N = AEA^2, \]

\[P = FAFECA^2CJ, \quad Q = BEJ^3, \quad R = F.\]
Here,

\[ A(z_0, z_1, z_2, z_3) = (z_0, z_2, z_3, z_1), \]
\[ B(z_0, z_1, z_2, z_3) = (-z_0, z_1, z_3, z_2), \]
\[ C(z_0, z_1, z_2, z_3) = (z_0, z_1, \omega z_2, \omega^2 z_3), \]
\[ D(z_0, z_1, z_2, z_3) = (\omega z_0, z_1, \omega z_2, \omega z_3), \]
\[ E(z_0, z_1, z_2, z_3) = (z_0, \frac{1}{\sqrt{-3}} (z_1 + z_2 + z_3), \frac{1}{\sqrt{-3}} (z_1 + \omega z_2 + \omega^2 z_3), \frac{1}{\sqrt{-3}} (z_1 + \omega^2 z_2 + \omega z_3)), \]
\[ F(z_0, z_1, z_2, z_3) = (-z_2, z_1, -z_0, -z_3), \]
\[ J(z_0, z_1, z_2, z_3) = (iz_0, iz_1, iz_2, iz_3). \]

In the form of matrices, we have

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \]
\[ C = \begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix}, \quad D = \begin{pmatrix} \omega & 1 \\ 1 & \omega \end{pmatrix}, \]
\[ E = \frac{1}{\sqrt{-3}} \begin{pmatrix} \sqrt{-3} & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad F = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \]
\[ J = iI. \]

Note that \( A, B, C, D, E, F, J \in SU(4). \)

We find that

\[ B^2 = F^2 = (AB)^2 = (BA)^2 = (BFB)^2 = I. \]
\[ (ABF)^2 = (BFA)^2 = (ABF \cdot BFA)^2 = -I. \]
\[ BFB = FBF, \quad ABF = -FAB. \]

Here,

\[ AB = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]
Moreover, the Jacobian of 

\[ ABF = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad BFA = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \]

\[ BFB = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad ABF \cdot BFA = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]

They play the similar role of the permutation matrices.

### 3. Hessian polyhedra and Picard modular forms

In his celebrated lecture [Kl], Klein set

\[ A_0 = z_1z_2, \quad A_1 = z_1^2, \quad A_2 = -z_2^2 \]

and

\[ A = A_0^2 + A_1A_2. \]

The invariants associated to the icosahedron: \( f, H \) and \( T \) (see [Kl] or [Y3]) can be expressed as the functions of \( A_0, A_1 \) and \( A_2 \):

\[ B = 8A_0^4A_1A_2 - 2A_0^2A_1^2A_2^2 + A_1^3A_2^3 - A_0(A_1^5 + A_2^5), \]

\[ C = 320A_0^6A_1^2A_2^2 - 160A_0^4A_1^3A_2^3 + 20A_0^2A_1^4A_2^4 + 6A_1^5A_2^5 + 4A_0(A_1 + A_2)(32A_0^3 - 20A_0^2A_1A_2 + 5A_1^2A_2^2)A_1^{10} + A_2^{10}, \]

\[ D = (A_1^5 - A_2^5)[-1024A_0^{10} + 3840A_0^8A_1A_2 - 3840A_0^6A_1^2A_2^2 + 1200A_0^4A_1^3A_2^3 - 100A_0^2A_1^4A_2^4 + A_1^{10} + A_2^{10} + 2A_1^5A_2^5 + A_0(A_1^5 + A_2^5)(352A_0^4 - 160A_0^2A_1A_2 + 10A_1^2A_2^2)]. \]

They satisfy the identity

\[ D^2 = -1728B^5 + C^3 + 720ACB^3 - 80A^2C^2B + 64A^3(5B^2 - AC)^2. \]

Moreover, the Jacobian of \( A, B \) and \( C \) satisfies the relation (see [F]):

\[
\frac{\partial(A, B, C)}{\partial(A_0, A_1, A_2)} = \begin{vmatrix}
\frac{\partial A}{\partial A_0} & \frac{\partial A}{\partial A_1} & \frac{\partial A}{\partial A_2} \\
\frac{\partial A}{\partial B} & \frac{\partial A}{\partial B} & \frac{\partial A}{\partial B} \\
\frac{\partial A}{\partial C} & \frac{\partial A}{\partial C} & \frac{\partial A}{\partial C}
\end{vmatrix} = -10D.
\]
The general Jacobian equation of the sixth degree is the following:

$$(z - A)^6 - 4A(z - A)^5 + 10B(z - A)^3 - C(z - A) + (5B^2 - AC) = 0.$$ 

Let $z_0, z_1, z_2, z_3, z_4, z_\infty$ be the six roots, put

$$y = (z_\infty - z_0)(z_1 - z_4)(z_2 - z_3).$$

Brioschi remarked that the square root of this expression is already rational in the $A$'s, and gives rise to an equation of the fifth degree. Let $x$ be this square root, Brioschi found for the five values of which $x$ is susceptible the following formula:

$$x_\nu = -\epsilon^\nu A_1(4A_0^2 - A_1A_2) + \epsilon^{2\nu}(2A_0A_1^2 - A_2^3) + \epsilon^{3\nu}(-2A_0A_2^2 + A_1^3) + \epsilon^{4\nu}A_2(4A_0^2 - A_1A_2),$$

with $\epsilon = \exp(2\pi i/5)$, while for the corresponding equation of the fifth degree he found this:

$$x^5 + 10Bx^3 + 5(9B^2 - AC)x - D = 0.$$ 

In fact, the Jacobian equation of the sixth degree with $A = 0$ is none other than that simplest resolvent of the sixth degree which Klein established in the case of the icosahedron: put $A_0 = z_1z_2$, $A_1 = z_1^2$, $A_2 = -z_2^2$ and correspondingly:

$$B = -f, \quad C = -H, \quad D = T.$$ 

The resolvent of the fifth degree is transformed into the following:

$$x^5 + 10Bx^3 + 45B^2x - D = 0.$$ 

The actions of the icosahedral group on $A_0$, $A_1$ and $A_2$ are given by

$$S : \quad A'_0 = A_0, \quad A'_1 = \epsilon^4A_1, \quad A'_2 = \epsilon A_2;$$

$$T : \quad \begin{cases} \sqrt{5}A'_0 = A_0 + A_1 + A_2, \\ \sqrt{5}A'_1 = 2A_0 + (\epsilon^2 + \epsilon^3)A_1 + (\epsilon + \epsilon^4)A_2, \\ \sqrt{5}A'_2 = 2A_0 + (\epsilon + \epsilon^4)A_1 + (\epsilon^2 + \epsilon^3)A_2; \end{cases}$$

$$U : \quad A'_0 = -A_0, \quad A'_1 = -A_2, \quad A'_2 = -A_1.$$ 

This leads to the investigation of the binary quadratic forms:

$$A_1z_1^2 + 2A_0z_1z_2 - A_2z_2^2$$
and Hilbert modular surfaces (see [Hi2]). Klein (see [Kl]) set
\[ \delta_\nu = (\epsilon^{4\nu} A_1 - \epsilon^\nu A_2)[(1 + \sqrt{5}) A_0 + \epsilon^{4\nu} A_1 + \epsilon^\nu A_2][(1 - \sqrt{5}) A_0 + \epsilon^{4\nu} A_1 + \epsilon^\nu A_2], \]
where \( \epsilon = \exp(2\pi i/5), \nu = 0, 1, 2, 3, 4 \) and found that
\[ \begin{aligned}
\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 &= 0, \\
\delta_0^3 + \delta_1^3 + \delta_2^3 + \delta_3^3 + \delta_4^3 &= 0.
\end{aligned} \]
Moreover,
\[ \begin{aligned}
S(\delta_0) &= \delta_1, S(\delta_1) = \delta_2, S(\delta_2) = \delta_3, S(\delta_3) = \delta_4, S(\delta_4) = \delta_0, \\
U(\delta_0) &= \delta_0, U(\delta_1) = \delta_4, U(\delta_2) = \delta_3, U(\delta_3) = \delta_2, U(\delta_4) = \delta_1.
\end{aligned} \]

\( T(\delta_0) = \delta_0 \). However, there does not exist the analogous transformation formulas for \( T(\delta_i) \) \( (i = 1, 2, 3, 4) \).

It is well-known that the group \( A_5 \) is isomorphic to the finite group \( I \) of those elements of \( SO(3) \) which carry a given icosahedron centered at the origin of the standard Euclidean space \( \mathbb{R}^3 \) to itself. The group \( I \) operates linearly on \( \mathbb{R}^3 \) (standard coordinates \( x_0, x_1, x_2 \)) and thus also on \( \mathbb{R}P^2 \) and \( \mathbb{C}P^2 \). We are concerned with the action on \( \mathbb{C}P^2 \) (see [Hi2] and [Hi3]). A curve in \( \mathbb{C}P^2 \) which is mapped to itself by all elements of \( I \) is given by a homogeneous polynomial in \( x_0, x_1, x_2 \) which is \( I \)-invariant up to constant factors and hence \( I \)-invariant, because \( I \) is a simple group. The graded ring of all \( I \)-invariant polynomials in \( x_0, x_1, x_2 \) is generated by homogeneous polynomials \( A, B, C, D \) of degrees 2, 6, 10, 15 with \( A = x_0^2 + x_1^2 + x_2^2 \). The action of \( I \) on \( \mathbb{C}P^2 \) has exactly one minimal orbit where “minimal” means that the number of points in the orbit is minimal. This orbit has six points, they are called poles. These are the points of \( \mathbb{R}P^2 \subset \mathbb{C}P^2 \) which are represented by the six axes through the vertices of the icosahedron. Klein uses coordinates
\[ A_0 = x_0, \quad A_1 = x_1 + ix_2, \quad A_2 = x_1 - ix_2 \]
and puts the icosahedron in such a position that the six poles are given by
\[ (A_0, A_1, A_2) = \left( \frac{\sqrt{5}}{2}, 0, 0 \right), \quad \left( \frac{1}{2}, \epsilon^\nu, \epsilon^{-\nu} \right) \]
with \( \epsilon = \exp(2\pi i/5) \) and \( 0 \leq \nu \leq 4 \).

The invariant curve \( A = 0 \) does not pass through the poles. There is exactly one invariant curve \( B = 0 \) of degree 6 which passes through the poles, exactly one invariant curve \( C = 0 \) of degree 10 which has higher multiplicity than the curve \( B = 0 \) in the poles and exactly one invariant curve \( D = 0 \) of degree 15. In fact, \( B = 0 \) has an ordinary double point (multiplicity 2) in each pole, \( C = 0 \) has a double cusp (multiplicity 4) in
each pole and \( D = 0 \) is the union of the 15 lines connecting the six poles. Klein gives formulas for the homogeneous polynomials \( A, B, C, D \) (determined up to constant factors). They generate the ring of all \( I \)-invariant polynomials. According to Klein the ring of \( I \)-invariant polynomials is given as follows:

\[
\mathbb{C}[A_0, A_1, A_2] = \mathbb{C}[A, B, C, D]/(R(A, B, C, D) = 0).
\]

The relation \( R(A, B, C, D) = 0 \) is of degree 30.

The equations for \( B \) and \( C \) show that the two tangents of \( B = 0 \) in the pole \((\sqrt{5}/2, 0, 0)\) are given by \( A_1 = 0, A_2 = 0 \). They coincide with the tangents of \( C = 0 \) in that pole. Therefore the curves \( B = 0 \) and \( C = 0 \) have in each pole the intersection multiplicity 10. Thus they intersect only in the poles.

When we restrict the action of \( I \) to the conic \( A = 0 \), we get the well-known action of \( I \) on \( \mathbb{CP}^1 \). The curves \( B = 0, C = 0, D = 0 \) intersect \( A = 0 \) transversally in 12, 20, 30 points respectively.

Put

\[
\Gamma(\sqrt{5}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathcal{O}_K) : \alpha \equiv \delta \equiv 1(\text{mod } \sqrt{5}), \beta \equiv \gamma \equiv 0(\text{mod } \sqrt{5}) \right\},
\]

where \( K = \mathbb{Q}(\sqrt{5}) \).

**Theorem 3.1** (Hirzebruch) (see [Hi2] and [Eb]). The ring of symmetric Hilbert modular forms for \( \Gamma(\sqrt{5}) \) is equal to \( \mathbb{C}[A_0, A_1, A_2] \) where \( A_0, A_1, A_2 \) have weight 1.

**Corollary 3.2** (see [Hi2] and [Eb]). The ring of symmetric Hilbert modular forms for \( SL(2, \mathcal{O}_K) \) of even weight is equal to \( \mathbb{C}[A, B, C] \).

Let

\[
\Gamma(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathcal{O}_K) : \alpha \equiv \delta \equiv 1(\text{mod } 2), \beta \equiv \gamma \equiv 0(\text{mod } 2) \right\},
\]

where \( K = \mathbb{Q}(\sqrt{5}) \).

**Theorem 3.3** (see [Hi1]). The graded ring of Hilbert modular forms of even weight relative to the group \( \Gamma(2) \) is the ring of polynomials in the five Eisenstein series \( E_0, \cdots, E_4 \), with the relations

\[
\sigma_2(E_0, \cdots, E_4) = 0, \quad \sigma_4(E_0, \cdots, E_4) = 0
\]

and their consequences, but no other relations. Here, \( \sigma_j \) is the \( j \)-th elementary symmetric function of \( E_0, \cdots, E_4 \).

**Corollary 3.4** (see [Hi1]). The ring \( M(SL(2, \mathcal{O}_K)) \) is generated by the four modular forms:

\[
\sigma_1 = \sum_{0 \leq i \leq 4} E_i, \quad \sigma_3 = \sum_{0 \leq i < j < k \leq 4} E_iE_jE_k, \quad \sigma_5 = \prod_{0 \leq i \leq 4} E_i, \quad \Delta = \prod_{0 \leq i < j \leq 4} (E_i - E_j)
\]
of weight 2, 6, 10, and 20, respectively, subject to the single relation:

\[ \Delta^2 = 3125\sigma_5^3 + 2000\sigma_5^2\sigma_3^2 + 256\sigma_5^2\sigma_1^2 + \\
-900\sigma_5^3\sigma_1 - 128\sigma_5\sigma_3^2\sigma_1^4 + 16\sigma_3^4\sigma_1^3 + 108\sigma_3^5. \]

Moreover, Hirzebruch found the connection between lattices, theta functions, coding theory and Hilbert modular surface associated with \( \mathbb{Q}(\sqrt{5}) \) (see [Hi], pp. 796-798). Let \( p \) be an odd prime and \( \zeta = e^{2\pi i/p} \). The field \( K = \mathbb{Q}(\zeta) \) contains the totally real field \( k = \mathbb{Q}(\zeta + \zeta^{-1}) \). For the rings of integers \( \mathcal{O}_K \) and \( \mathcal{O}_k \) we have

\[ \mathcal{O}_K = \mathbb{Z}\zeta + \cdots + \mathbb{Z}\zeta^{p-1}, \]
\[ \mathcal{O}_k = \mathbb{Z}(\zeta + \zeta^{-1}) + \cdots + \mathbb{Z}(\zeta^{p-1} + \zeta^{p-1}). \]

The Galois group \( G \) of \( k \) over \( \mathbb{Q} \) is cyclic of order \( p^2 \). Let \( \text{Tr} : k \to \mathbb{Q} \) be the trace, i.e.

\[ \text{Tr}(y) = \sum_{\sigma \in G} \sigma(y), \text{ for } y \in k. \]

We consider the following quadratic form \( Q \) over the \( \mathbb{Z} \)-lattice \( \mathcal{O}_K \)

\[ Q : \mathcal{O}_K \to \mathbb{Q}, \quad Q(x) = 2\text{Tr} \left( \frac{px}{p} \right). \]

It has discriminant \( \frac{1}{p^2} \).

The ideal \( (1 - \zeta) = p \) in \( \mathcal{O}_K \) has norm \( p \). Let \( \varrho : \mathcal{O}_K \to \mathbb{F}_p = GF(p) \) be the corresponding homomorphism. We also consider \( \varrho : \mathcal{O}_k \to \mathbb{F}_p^n \). For a self-dual code \( C \subset \mathbb{F}^n \) (\( n \) even) we define the lattice \( \Lambda = \varrho^{-1}(C) \). If we restrict \( Q \) to \( \Lambda \), then we obtain an integral even unimodular form on the lattice \( \Lambda \). It is well-known that \( n(p - 1) \equiv 0 \) (mod 8).

The theta functions in the Hilbert modular sense for the totally real field \( k \) depend on \( \frac{p-1}{2} \) variables \( z_\sigma \in \mathbb{H} \) (upper half-plane) indexed by the elements of the Galois group \( G \) of \( k \) over \( \mathbb{Q} \). For \( y \in k \)

\[ \text{Tr}(zy) \triangleq \sum_{\sigma \in G} z_\sigma y_\sigma \quad (y_\sigma = \sigma(y)). \]

The fundamental theta functions are

\[ \theta_j(z) = \sum_{x \in \mathbb{Z} + j} e^{2\pi i \text{Tr}(\frac{zx}{p})} \quad \text{for } j = 0, 1, \cdots, \frac{p-1}{2}. \]
The $\theta_j$ are Hilbert modular forms of weight 1 for the group

$$SL_2(O_k, \tilde{p}) = \{ A | A \in SL_2(O_k) \text{ and } A \equiv 1 \text{ mod } \tilde{p} \}$$

where $\tilde{p}$ is the prime ideal in $k$ lying over $p$

$$\tilde{p} = ((\zeta - \overline{\zeta})^2), \quad \tilde{p}^{\frac{p-1}{2}} = (p).$$

In fact, the group $SL_2(\mathbb{F}_p)$ acts on the $\frac{p+1}{2}$-dimensional vector space over $\mathbb{C}$ generated by the $\theta_j$. For $p \equiv 1 \text{ mod } 4$ this is an action of $PSL_2(\mathbb{F}_p)$.

The lattice $\Lambda$ gives a theta function

$$\theta_C(z) = \sum_{x \in \Lambda} e^{2\pi i \text{Tr}(z \frac{x^2}{p})}.$$

Consider the Lee weight enumerator $L_C(X_0, \cdots, X_{\frac{p-1}{2}})$.

**Theorem 3.5** (see [Hi], p.797).

$$\theta_C = L_C(\theta_0, \cdots, \theta_{\frac{p-1}{2}}).$$

$\theta_C$ is a Hilbert modular form of weight $n$ for the group $SL_2(O_k)$. The polynomial $L_C(X_0, \cdots, X_{\frac{p-1}{2}})$ is an invariant polynomial for the above mentioned representation of dimension $\frac{p+1}{2}$ of the group $SL_2(\mathbb{F}_p)$. All Hilbert modular forms $\theta_C$, $\theta_j$ occurring are symmetric, i.e. invariant under the action of the Galois group $G$ on the variables $z_\sigma$.

In his paper on $\mathbb{Q}(\sqrt{5})$, which is $k$ for $p = 5$, Hirzebruch proved that the ring of symmetric Hilbert modular forms for $SL_2(O_k, \tilde{p})$, $\tilde{p} = (\sqrt{5})$, equals $\mathbb{C}[A_0, A_1, A_2]$ where $A_0, A_1, A_2$ have weight 1. He proved that the ring of symmetric Hilbert modular forms for $SL_2(O_k)$ of even weight equals $\mathbb{C}[A, B, C]$ where $A, B, C$ are the Klein invariants of degrees 2, 6, 10. Moreover, he found that

$$A_0 = \theta_0, \quad A_1 = 2\theta_1, \quad A_2 = 2\theta_2.$$

Together with Theorem 3.5, this explains the above mentioned connection.

For $p = 3$ (where $k = \mathbb{Q}$). In this case $C$ is a ternary code. According to [Eb], the Lee weight enumerator of the code $C$ satisfies the following identity:

$$W_C(\theta_0, \theta_1) = \theta_C.$$

We consider the functions $\theta_0$ and $\theta_1$. The lattice $\Lambda$ is isomorphic to $A_2$. The lattice $A_2$ is the $\mathbb{Z}$-module $\mathbb{Z}^2$ with the quadratic form

$$(x, y)^2 = 2x^2 - 2xy + 2y^2.$$
Therefore,

\[(3.1) \quad \theta_0(z) = \sum_{(x,y) \in \mathbb{Z}^2} q^{x^2-xy+y^2}, \quad \theta_1(z) = q^{1\over 2} \sum_{(x,y) \in \mathbb{Z}^2} q^{x^2-xy+y^2+x-y}\]

with \(q = e^{2\pi iz}\). The functions \(\theta_0\) and \(\theta_1\) are modular forms of weight 1 for the group

\[\Gamma(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{3}, b \equiv c \equiv 0 \pmod{3} \right\}.\]

We have

\[\theta_0 \left( -{1\over z} \right) = z \over \sqrt{3} (\theta_0(z) + 2\theta_1(z)), \quad \theta_1 \left( -{1\over z} \right) = z \over \sqrt{3} (\theta_0(z) - \theta_1(z)).\]

\[\theta_0(z + 1) = \theta_0(z), \quad \theta_1(z + 1) = \omega \theta_1(z), \quad \omega = e^{2\pi i 2}.\]

The Eisenstein series \(E_4\) is the theta function of the unimodular \(E_8\)-lattice and

\[(3.2) \quad E_4 = \theta_0^4 + 8\theta_0 \theta_1^3, \quad E_6 = \theta_0^6 - 20\theta_0^3 \theta_1^3 - 8\theta_1^6.\]

**Theorem 3.6** (see [Eb]). *The algebra of all modular forms for the group \(\Gamma(3)\) is isomorphic to the polynomial algebra \(\mathbb{C}[\theta_0, \theta_1]\).*

The group \(SL_2(\mathbb{F}_3)\) acts on the polynomial algebra \(\mathbb{C}[\theta_0, \theta_1]\). The ring of invariant polynomials under this action is denoted by \(\mathbb{C}[\theta_0, \theta_1]^{SL_2(\mathbb{F}_3)}\).

**Theorem 3.7** (see [Eb]).

\[\mathbb{C}[\theta_0, \theta_1]^{SL_2(\mathbb{F}_3)} = \mathbb{C}[E_4, E_6].\]

The Hamming weight enumerator \(L_C = H_C = H_C(\theta_0, \theta_1)\) is a polynomial in the modular forms \(E_4\) and \(E_6^2\). For the ternary Golay code of length 12,

\[H_C = {5\over 8} E_4^3 + {3\over 8} E_6^2.\]

According to [CS], the Hamming weight enumerator classifies codewords according to the number of nonzero coordinates. More detailed information is supplied by the complete weight enumerator (c.w.e.), which gives the number of codewords of each composition. For example the c.w.e. of a ternary code \(C\) is

\[c.w.e. (x, y, z) = \sum_{u \in C} x^{n_0(u)} y^{n_1(u)} z^{n-1(u)},\]
where \( n_i(u) \) is the number of times \( i \in \mathbb{F}_3 \) occurs in \( u \). There is also a version of the MacWilliams identity for c.w.e.'s.

**Theorem 3.8 (Gleason) (see [CS]).** If \( C \) is a Type III code then \( W_C(x, y) \) is invariant under a group \( G_3 \) of order 48, and belongs to \( \mathbb{C}[\psi_4, \xi_{12}] \), where

\[
\psi_4 = x^4 + 8xy^3
\]

is the weight enumerator of \( C_4 \) and

\[
\xi_{12} = y^3(x^3 - y^3)^3.
\]

Equivalently, \( W_C(x, y) \) can be written as a polynomial in the weight enumerators of \( C_4 \) and \( C_{12} \).

To describe the complete weight enumerator, we first define some further polynomials. Let

\[
a = x^3 + y^3 + z^3, \quad p = 3xyz,
b = x^3y^3 + y^3z^3 + z^3x^3,
\beta_6 = a^2 - 12b = x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3),
\pi_9 = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3),
\alpha_{12} = a(a^3 + 8p^3).
\]

**Theorem 3.9 (see [CS]).** If \( C \) is a Type III code which contains the all-ones vector then the complete weight enumerator is invariant under a group \( G_4 \) of order 2592, and belongs to

\[
R \oplus \beta_6 \pi_9^2 R,
\]

where

\[
R = \mathbb{C}[\beta_6^2, \alpha_{12}, \pi_9^4].
\]

In other words the complete weight enumerator can be written uniquely as a polynomial in \( \beta_6^2 \), \( \alpha_{12} \) and \( \pi_9^4 \), plus \( \beta_6 \pi_9^2 \) times another such polynomial.

Here, \( G_3 \) is generated by \( \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \) and is the reflection group \( 3[6]2 \). \( G_4 \) is generated by all permutations, \( \text{diag}(1, 1, \omega) \) and

\[
\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega \\ 1 & \omega & \omega \end{pmatrix},
\]

has a center \( Z(G_4) \) of order 12, and \( G_4/Z(G_4) \) is the Hessian group of order 216.
It is well-known that a complete system of invariants for the Hessian groups (the corresponding geometric objects are Hessian polyhedra) has degrees 6, 9, 12, 12 and 18 and can be given explicitly by the following forms:

\[
\begin{align*}
C_6(z_1, z_2, z_3) &= z_1^6 + z_2^6 + z_3^6 - 10(z_1^3 z_2^3 + z_2^3 z_3^3 + z_3^3 z_1^3), \\
C_9(z_1, z_2, z_3) &= (z_1^3 - z_2^3)(z_2^3 - z_3^3)(z_3^3 - z_1^3), \\
C_{12}(z_1, z_2, z_3) &= (z_1^3 + z_2^3 + z_3^3)[(z_1^3 + z_2^3 + z_3^3)^3 + 216z_1^3 z_2^3 z_3^3], \\
\mathcal{C}_{12}(z_1, z_2, z_3) &= z_1 z_2 z_3[27z_1^3 z_2^3 z_3^3 - (z_1^3 + z_2^3 + z_3^3)^3], \\
C_{18}(z_1, z_2, z_3) &= (z_1^3 + z_2^3 + z_3^3)^6 - 540z_1^3 z_2^3 z_3^3(z_1^3 + z_2^3 + z_3^3)^3 - 5832z_1^6 z_2^6 z_3^6.
\end{align*}
\]

They satisfy the following relations:

\[
\begin{align*}
432C_9^2 &= C_6^3 - 3C_6 C_{12} + 2C_{18}, \\
1728\mathcal{C}_{12}^3 &= C_{18}^2 - C_{12}^3.
\end{align*}
\]

For the Hessian group 3[3][3][3], of order 6 \times 9 \times 12, the three forms are \(C_6, C_9, C_{12}\). For the closely related Hessian group 2[4][3][3], of order 6 \times 12 \times 18, they are \(C_6, C_{12}, C_9^2\) (see [Co]).

Note that

\[
\beta_6 = C_6, \quad \pi_9 = C_9, \quad \alpha_{12} = C_{12}, \quad R = \mathbb{C}[C_6^2, C_{12}, C_9^4]
\]

under the identification

\((x, y, z) = (z_1, z_2, z_3)\).

On the other hand,

\[
\begin{align*}
\psi_4 &= 9C_{12}, \\
\xi_{12} &= -27^3\mathcal{C}_{12}^3
\end{align*}
\]

under the identification

\((x, y) = (H, K)\).

Thus, we find a connection between Hessian invariants (Hessian groups) and coding theory.

Two theories apparently far apart, that of weight polynomials of codes over the field \(\mathbb{F}_2\) and that of theta functions of lattices over \(\mathbb{Z}\), are too similar to be strangers to each other. The most striking similarity is no doubt the following: The weight polynomials of certain codes \cdots generate the graded algebra of polynomials in two variables fixed by the operation of a subgroup \(H\) of \(GL(2, \mathbb{C})\); the theta functions of even unimodular lattices are modular forms which generate an algebra visibly isomorphic to the preceding
(see [BrE]). Since the times of Klein the analogies between invariant theory and modular forms have emerged.

Denote $E_2(z)$, $E_4(z)$, $E_6(z)$ the Eisenstein series of order 2, 4, 6 and the cusp form $\Delta(z)$ of weight 12 as

$$E_2(z) = 1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)q^m,$$
$$E_4(z) = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^m,$$
$$E_6(z) = 1 - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^m,$$
$$\Delta(z) = \frac{1}{1728} (E_4^3 - E_6^2) = q \prod_{m=1}^{\infty} (1 - q^m)^{24},$$

where $q = \exp(2\pi i z)$, and the sum of $r$th power of divisor function is $\sigma_r(m) := \sum_{d|m} d^r$. As is well known $E_4, E_6$ are modular forms of weight 4 and 6 but $E_2$ is not. The following result is due to Hecke.

**Theorem 3.10** (see [CMS]). The algebra of modular forms of weight multiple of 4 is $\mathbb{C}[E_4, \Delta]$. The algebra of modular forms of even weight is $\mathbb{C}[E_4, E_6]$.

Let $H_2 = \langle M_2, N_2 \rangle$, $M_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $N_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. $H_2$ is of order 192. There is a subgroup $G_2 \leq H_2$ of index 2 defined as the kernel of the following linear character defined on the generators of $H_2$ as

$$\chi(M_2) = -1, \quad \chi(N_2) = 1.$$

Define the following invariants for the group $H_2$ of degree 8 and 24, respectively:

$$\psi_8 = x^8 + 14x^4y^4 + y^8,$$
$$\nu_{24} = x^4y^4(x^4 - y^4)^4.$$

Then

$$\mathbb{C}[x, y]^{H_2} = \mathbb{C}[\psi_8, \nu_{24}],$$
$$\mathbb{C}[x, y]^{G_2} = \mathbb{C}[\psi_8, k_{12}],$$

where

$$k_{12} = x^{12} - 33(x^8y^4 + x^4y^8) + y^{12}.$$

**Theorem 3.11** (see [CMS]). The map

$$\phi_1 : \mathbb{C}[\psi_8, \nu_{24}] \to \mathbb{C}[E_4, \Delta]$$
defined by
\[ \phi_1(h(\psi_8, \nu_{24})) = h(E_4, \Delta) \]
is an algebra isomorphism. The map
\[ \phi_2 : \mathbb{C}[\psi_8, k_{12}] \to \mathbb{C}[E_4, E_6] \]
defined by
\[ \phi_2(h(\psi_8, k_{12})) = h(E_4, E_6) \]
is an algebra isomorphism.

Let \( G_3 = \langle M_3, N_3 \rangle \), \( M_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \), \( N_3 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \), \( \omega = \exp(2\pi i/3) \). Then \( G_3 \cong SL_2(\mathbb{F}_3) \). Primary invariants for \( G_3 \) are
\[ \psi_4 = x^4 + 8xy^3, \]
\[ k_6 = x^6 - 20x^3y^3 - 8y^6. \]
\[ \mathbb{C}[x, y]^{G_3} = \mathbb{C}[\psi_4, k_6]. \]

**Theorem 3.12** (see [CMS]). The map
\[ \phi_3 : \mathbb{C}[\psi_4, k_6] \to \mathbb{C}[E_4, E_6] \]
given by
\[ \phi_3(h(\psi_4, k_6)) = h(E_4, E_6) \]
is an algebra isomorphism.

In our case, we have the following:
**Theorem 3.13.**
\[ \mathbb{C}[H, K]^{G_3} = \mathbb{C}[C_{12}, C_{18}]. \]

The map
\[ \varphi : \mathbb{C}[C_{12}, C_{18}] \to \mathbb{C}[E_4, E_6] \]
given by
\[ \varphi(h(C_{12}, C_{18})) = h(E_4, E_6) \]
is an algebra isomorphism.

Consequently,
\[ \varphi(h(C_{12}, -C_{12}^3)) = h(E_4, \Delta). \]

Now, we will study the analogies between Hessian invariants and Picard modular forms, which can be considered as the higher dimensional counterpart of the analogies between regular polyhedral invariants and elliptic modular forms studied by Klein.
Let us recall some basic facts about Picard modular forms. In his monograph [Ho1], Holzapfel proved the following results:

**Theorem 3.14 (see [Ho1]).**

1. The monodromy group of the hypergeometric differential equation system considered by Picard is the principal congruence subgroup $U(2, 1; \mathbb{Z}[\omega])(1 - \omega)$ of the full Eisenstein lattice of the ball $U(2, 1; \mathbb{Z}[\omega])$ with respect to the principal ideal $\mathbb{Z}[\omega] \cdot (1 - \omega)$, $\omega = \exp(2\pi i/3)$.

2. The rings of automorphic forms of $U(2, 1; \mathbb{Z}[\omega])(1 - \omega)$ and $U(2, 1; \mathbb{Z}[\omega])$ (of Nebentypus $\chi$) are polynomial rings

\[
\bigoplus_{m=0}^{\infty} [U(2, 1; \mathbb{Z}[\omega])(1 - \omega), m]_{\chi} = \mathbb{C}[\xi_1, \xi_2, \xi_3],
\]

\[
\bigoplus_{m=0}^{\infty} [U(2, 1; \mathbb{Z}[\omega]), m]_{\chi} = \mathbb{C}[G_2, G_3, G_4],
\]

with $\xi_1, \xi_2, \xi_3$ of weight 1, $G_k$ of weight $k$, $k = 2, 3, 4$.

A key result is the following one:

\[
\bigoplus_{m=0}^{\infty} [SU(2, 1; \mathbb{Z}[\omega])(1 - \omega), m] = \mathbb{C}[\xi_1, \xi_2, \xi_3, \zeta],
\]

with $\xi_k$ as above for $k = 1, 2, 3$, $\zeta$ of weight 2,

\[
\zeta^3 = (\xi_3^2 - \xi_2^2)(\xi_3^2 - \xi_1^2)(\xi_2^2 - \xi_1^2).
\]

3. $\mathbb{B}^2 / U(2, 1; \mathbb{Z}[\omega])(1 - \omega) \cong \mathbb{P}^2 - \{\text{four points in general position}\}$.

4. If $F_1$, $F_2$, $F_3$ are three fundamental solutions of Picard’s system of differential equations, then there is a basis $\xi_1', \xi_2', \xi_3'$ of the space of automorphic forms of weight 1 with respect to a group isomorphic to $U(2, 1; \mathbb{Z}[\omega])(1 - \omega)$, such that $(\xi_1' : \xi_2' : \xi_3')$ is the inverse of the multi-valued map $(F_1 : F_2 : F_3)$:

\[
\begin{array}{ccc}
\widetilde{\mathbb{B}}^2 & \xrightarrow{=} & \widetilde{\mathbb{B}}^2 \\
\uparrow (F_1 : F_2 : F_3) & & \downarrow (\xi_1' : \xi_2' : \xi_3') \\
\mathbb{P}^2 - \{\text{four points}\} & \xrightarrow{=} & \mathbb{P}^2 - \{\text{four points}\}
\end{array}
\]

where $\widetilde{\mathbb{B}}^2$ is projectively equivalent to $\mathbb{B}^2$ in $\mathbb{P}^2$. 
Let $\hat{B}_2/U(2, 1; \mathbb{Z}[\omega]) = \mathbb{B}^2/U(2, 1; \mathbb{Z}[\omega])$ be the Baily-Borel compactification, here $\mathbb{B}^2 = B_2 \cup \partial U(2, 1; \mathbb{Z}[\omega])$ the set of $U(2, 1; \mathbb{Z}[\omega])$-cusps. Then we have the following commutative diagram:

$$
P \xrightarrow{\sim} C(P) : Y^3 = X^4 + G_2(P)X^2 + G_3(P)X + G_4(P)$$

$\mathbb{B}^2 \xrightarrow{\text{quotient map}} \{\text{Picard curves}\}/\text{projective equivalence}$

$$
\mathbb{B}^2/\mathbb{U}(2, 1; \mathbb{Z}[\omega]) \xrightarrow{\text{quotient map}} \mathbb{B}^2/\mathbb{U}(2, 1; \mathbb{Z}[\omega])$$

($Y^3 = X^4$ is excluded from the set of Picard curves. On $\mathbb{B}^2 - U(2, 1; \mathbb{Z}[\omega]) \cdot D \subset \mathbb{B}^2$, $D = \{(z, 0) : |z| < 1\}$ a subdisc of $\mathbb{B}^2$, projective equivalence can be replaced by isomorphism.) The image of $\mathbb{B}^2 - U(2, 1; \mathbb{Z}[\omega]) \cdot D$ is the set of isomorphism classes of all smooth Picard curves.

(6) The images of the $U(2, 1; \mathbb{Z}[\omega])$-fixed point set of $\mathbb{B}^2$ is the set of classes of Picard curves with larger automorphism groups (larger than $\mathbb{Z}/3\mathbb{Z}$).

(7) The Lagrange resolvent map, which makes correspond to each polynomial of degree four a polynomial of degree three, admits by a change of fibres procedure over the moduli space of Picard curves (change Picard curves to elliptic curves) a geometric interpretation as a morphism $\mathbb{B}^2/U(2, 1; \mathbb{Z}[\omega]) \to H/SL(2, \mathbb{Z})$. There is a lift along the automorphic form (quotient) morphisms $\mathbb{B}^2 \to \mathbb{B}^2/U(2, 1; \mathbb{Z}[\omega])$, $H \to H/SL(2, \mathbb{Z})$.

Moreover, Picard modular forms can be considered as theta constants (see [Ho2]). Theta functions $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$ with characteristics $a, b \in \mathbb{Q}^g$ are holomorphic functions on $\mathbb{C}^g \times \mathbb{H}_g$, $\mathbb{H}_g$ the generalized Siegel upper half plane uniformizing the moduli space of principally polarized abelian varieties of dimension $g$. Explicitly the theta functions

$$
\vartheta \begin{bmatrix} a \\ b \end{bmatrix} : \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}
$$

are defined by

$$
\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp \left[ \pi i^t(n + a)\Omega(n + a) + 2\pi i^t(n + a)(z + b) \right].
$$

The restrictions $\vartheta|0 \times \mathbb{H}_g$,

$$
\theta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (0, \Omega)
$$
are called theta constants (with characteristics $a, b$).

**Theorem 3.15** (Feustel, Shiga) (see [Ho2]). Let $\theta_i(\Omega) = \vartheta_i(0, \Omega)$, $i = 0, 1, 2$, be the theta constants on $\mathbb{H}_3$ restricting the theta functions

$$\vartheta_{k}(z, \Omega) = \vartheta_0\left[\begin{array}{ccc} 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 \end{array}\right](z, \Omega), \quad k = 0, 1, 2, \quad z \in \mathbb{C}^3.$$

Set

$$Th_1 = \theta_0^3 + \theta_1^3 + \theta_2^3, \quad Th_2 = -3\theta_0^3 + \theta_1^3 + \theta_2^3,$$
$$Th_3 = \theta_0^3 - 3\theta_1^3 + \theta_2^3, \quad Th_4 = \theta_0^3 + \theta_1^3 - 3\theta_2^3,$$

and

$$th_i(\tau) = Th_i(\tau), \quad i = 1, 2, 3, 4, \quad \tau \in \mathbb{B}^2,$$

where the embedding $\ast : \mathbb{B}^2 \hookrightarrow \mathbb{H}_3$. Then the functions $th_i(\tau)$ are the normalized Picard modular forms satisfying

$$t_1 + t_2 + t_3 + t_4 = 0, \quad t_1, t_2, t_3 \text{ are linearly independent},$$

and the functional equations

$$\gamma^*(t_i) = (\det \gamma)^2 \cdot \text{sgn}(\gamma) \cdot j_\gamma \cdot t_{\tau(i)} \quad \text{for } i = 1, 2, 3, 4.$$

The analogies between elliptic modular forms and Picard modular forms are given as follows:

1. $\bigoplus_{m=0}^{\infty} [SL(2, \mathbb{Z}), m] = \mathbb{C}[g_2, g_3],$
2. $\bigoplus_{m=0}^{\infty} [SL(2, \mathbb{Z})(2), m] = \mathbb{C}[\epsilon_1, \epsilon_2],$

where $g_i$ is of weight $i$, $i = 2, 3$, $\epsilon_1, \epsilon_2$ are of weight 1.

2. $\bigoplus_{m=0}^{\infty} [U(2, 1; \mathcal{O}_K), m]_\chi = \mathbb{C}[G_2, G_3, G_4],$
3. $\bigoplus_{m=0}^{\infty} [U(2, 1; \mathcal{O}_K(1 - \omega)), m]_\chi = \mathbb{C}[\xi_1, \xi_2, \xi_3],$

where $G_k$ is of weight $k$, $k = 2, 3, 4$, $\xi_1, \xi_2, \xi_3$ are of weight 1.
This leads us to give the following conjecture, which is the higher dimensional counterpart of Theorem 3.11 and Theorem 3.12.

**Conjecture 3.16.**

\[ (3.7) \quad \mathbb{C}[z_1, z_2, z_3]^{\text{Hessian groups}} = \mathbb{C}[C_6, C_9, C_{12}] . \]

The map

\[ (3.8) \quad \phi : \mathbb{C}[C_6, C_9, C_{12}] \to \mathbb{C}[G_2, G_3, G_4] \]

given by

\[ (3.9) \quad \phi(h(C_6, C_9, C_{12})) = h(G_2, G_3, G_4) \]

is an algebra isomorphism.

We have the following correspondences

\[ (3.10) \quad (C_6, C_9, C_{12}) \longleftrightarrow (G_2, G_3, G_4), \]

\[ (3.11) \quad (G, H, K) \longleftrightarrow (\xi_1, \xi_2, \xi_3). \]

\[ (3.12) \quad (C_{12}, C_{18}) \longleftrightarrow (E_4, E_6) = (g_2, g_3), \]

\[ (3.13) \quad (H, K) \longleftrightarrow (\varepsilon_1, \varepsilon_2). \]

Note that

\[ C_{12} = C_{12}(z_1, z_2, z_3) \longleftrightarrow G_4, \quad \text{three-variable function.} \]

\[ C_{12} = C_{12}(H, K) \longleftrightarrow g_2, \quad \text{two-variable function.} \]

1. **GL(2):** \( \varepsilon_1, \varepsilon_2: \) theta functions of weight one, elliptic modular functions
2. **GL(3):** \( \xi_1, \xi_2, \xi_3: \) theta functions of weight one, Picard modular functions

\[ GL(2) : \quad (x, y) \longleftrightarrow (\theta_0(z), \theta_1(z)), z \in \mathbb{H}. \]

\[ E_4(z) = \theta_0(z)^4 + 8\theta_0(z)\theta_1(z)^3, \quad E_6(z) = \theta_0(z)^6 - 20\theta_0(z)^3\theta_1(z)^3 - 8\theta_1(z)^6. \]

\( \theta_0(z), \theta_1(z) \) are of weight 1.

\[ (3.14) \quad U(2, 1) : \quad (z_1, z_2, z_3) \longleftrightarrow (\theta_0(w_1, w_2), \theta_1(w_1, w_2), \theta_2(w_1, w_2)), (w_1, w_2) \in \mathcal{S}_2, \]
where $\mathcal{S}_2 = \{ (z_1, z_2) \in \mathbb{C}^2 : z_1 + \overline{z_1} > z_2 \overline{z_2} \}$. \( \vartheta_0(w_1, w_2) \), \( \vartheta_1(w_1, w_2) \) and \( \vartheta_2(w_1, w_2) \) are of weight \( \frac{1}{3} \). By

\[
9C_{12} = H(H^3 + 8K^3), \quad -27C_{18} = H^6 - 20H^3K^3 - 8K^6,
\]

we have

\[
\begin{align*}
(H, K) & \leftrightarrow (\theta_0(z), \theta_1(z)), \\
(9C_{12}, -27C_{18}) & \leftrightarrow (E_4(z), E_6(z)).
\end{align*}
\]

Note that

\[
\theta(z) = \vartheta_0(w_1, w_2)^3 + \vartheta_1(w_1, w_2)^3 + \vartheta_2(w_1, w_2)^3 + 6\vartheta_0(w_1, w_2)\vartheta_1(w_1, w_2)\vartheta_2(w_1, w_2),
\]

\[
\theta_1(z) = \vartheta_0(w_1, w_2)^3 + \vartheta_1(w_1, w_2)^3 + \vartheta_2(w_1, w_2)^3 - 3\vartheta_0(w_1, w_2)\vartheta_1(w_1, w_2)\vartheta_2(w_1, w_2).
\]

Hence,

\[
\vartheta_0(w_1, w_2)^3 + \vartheta_1(w_1, w_2)^3 + \vartheta_2(w_1, w_2)^3 = \frac{1}{3}[\theta_0(z) + 2\theta_1(z)],
\]

\[
\vartheta_0(w_1, w_2)\vartheta_1(w_1, w_2)\vartheta_2(w_1, w_2) = \frac{1}{9}[\theta_0(z) - \theta_1(z)].
\]

The group \( PSL(2, \mathbb{Z})/\Gamma(3) \) is the tetrahedral group. The theta functions \( \theta_0(z), \theta_1(z) \) define a mapping

\[
\frac{\theta_0}{\theta_1} : \overline{\mathbb{H}}/\Gamma(3) \to \mathbb{P}^1(\mathbb{C}).
\]

This mapping is a bijection. The group \( PSL(2, \mathbb{Z})/\Gamma(3) \) acts on \( \overline{\mathbb{H}}/\Gamma(3) \). Under the above mapping this action corresponds to the action of the tetrahedral group on a tetrahedron lying inside the Riemann sphere \( \mathbb{P}^1(\mathbb{C}) \) (see [Eb]).

In our case, the theta functions \( \vartheta_0(w_1, w_2) \), \( \vartheta_1(w_1, w_2) \) and \( \vartheta_2(w_1, w_2) \) define a mapping

\[
\left( \frac{\vartheta_1}{\vartheta_0}, \frac{\vartheta_2}{\vartheta_0} \right) : \mathcal{S}_2/U(2, 1; \mathcal{O}_K)(1 - \omega) \to \mathbb{P}^2(\mathbb{C}),
\]

where \( K = \mathbb{Q}(\sqrt{-3}) \) and \( \mathcal{O}_K = \mathbb{Z}[\omega] \).

Let \( \mathcal{S}_2 \) be the Siegel domain

\[
\mathcal{S}_2 = \{ (z_1, z_2) \in \mathbb{C}^2 : z_1 + \overline{z_1} > |z_2|^2 \}.
\]
The imaginary quadratic field $K = \mathbb{Q}(\sqrt{-3})$ is called the field of Eisenstein numbers. Its ring of integers
\[ \mathcal{O} = \mathcal{O}_K = \{ a/2 + b\sqrt{-3}/2 : a, b \in \mathbb{Z}, a \equiv b (\text{mod} 2) \} = \mathbb{Z} + \mathbb{Z}[\omega] \]
with $\omega = \frac{-1 + \sqrt{-3}}{2}$ is called the ring of Eisenstein integers. We define the Cayley transform $C: \mathbb{B}^2 := \{ (w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 < 1 \} \to \mathcal{S}_2$, where
\[ C = \begin{pmatrix} -\omega & -\omega \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \in GL(3, \mathcal{O}_K). \]
It is known that
\[ U(2, 1) = \{ g \in GL(3, \mathbb{C}) : gI_{2,1}g^* = I_{2,1} \}, \]
where $I_{2,1} = \text{diag}\{1, 1, -1\}$.
Let $G = CU(2, 1)C^{-1}$, $K = C(U(2) \times U(1))C^{-1}$ and $G(\mathcal{O}) = CU(2, 1; \mathcal{O}_K)C^{-1}$, then $G(\mathcal{O})$ is an arithmetic subgroup of $G$ and $S := G(\mathcal{O}) \backslash G / K$ is called a Picard modular surface. Denote $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then $CI_{2,1}C^* = J$. It follows that
\[ G = \{ g \in GL(3, \mathbb{C}) : gJg^* = J \}. \]
This is the other realization of $U(2, 1)$. For simplicity, from now on, we use the same symbol $U(2, 1)$ to denote as $G$. In fact, $G = \{ g \in GL(3, \mathbb{C}) : g^*Jg = J \}$.

The principal congruence subgroup $U(2, 1; \mathcal{O}_K)(1 - \omega)$ of $U(2, 1; \mathcal{O}_K)$ is the monodromy group of the Appell hypergeometric partial differential equations with parameters $(a, b, b', c) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$. Moreover, $U(2, 1; \mathcal{O}_K)(1 - \omega)$ is the modular group of the algebraic family of projective plane curves $C$ of equation type
\[ C: \quad y^3 = p_4(x), \quad \deg p_4(x) = 4 \]
(Picard curves) endowed with a fixed order of the ramification points of the 3-sheeted Galois covering $C \to \mathbb{P}^1$ induced by the projection $(x, y) \mapsto x$. More precisely, the noncompact surface $\mathcal{S}_2/U(2, 1; \mathcal{O}_K)(1 - \omega)$ classifies the isomorphy classes of these curves endowed with four ordered points. $\mathcal{S}_2/U(2, 1; \mathcal{O}_K)(1 - \omega)$ is isomorphic to $\mathbb{P}^2 - \{4 \text{ points}\}$. The four points come from four cusps lying on $\partial \mathcal{S}_2$. $SU(2, 1; \mathcal{O}_K)(1 - \omega)$ has up to $SU(2, 1; \mathcal{O}_K)(1 - \omega)$-equivalence exactly four cusps and three elliptic points $S_0, S_1, S_2 \in \mathcal{S}_2$ with stationary groups of order 3. For the smooth compactification $\bar{\mathcal{S}}_2/SU(2, 1; \mathcal{O}_K)(1 - \omega)$ of $\mathcal{S}_2/SU(2, 1; \mathcal{O}_K)(1 - \omega)$ one needs four elliptic curves $E_0, E_1, E_2, E_3$. So $\bar{\mathcal{S}}_2/SU(2, 1; \mathcal{O}_K)(1 - \omega)$ or $(\bar{\mathcal{S}}_2/SU(2, 1; \mathcal{O}_K)(1 - \omega), E_0 + E_1 + E_2 + E_3)$ is an elliptic bounded surface (see [Ho4]).
According to Picard (see [Pi] or [Ho1]), the five generators of the principal congruence subgroup \( U(2, 1; \mathcal{O}_K)(1 - \omega) \) are given as follows:

\[
g_1 = \begin{pmatrix} 1 & \omega - \omega & 1 - \omega \\ 0 & \omega & 1 - \omega \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & \omega - 1 & 1 - \omega \\ 0 & \omega & 1 - \omega \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
g_4 = \begin{pmatrix} \omega & 0 & 1 - \omega \\ 0 & 1 & 0 \\ 1 - \omega & 0 & -2\omega \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 0 & 0 \\ \omega - \omega & \omega & 0 \\ 1 - \omega & 1 - \omega & 1 \end{pmatrix}.
\]

In fact (see [Y2]),

\[
\begin{align*}
g_1 &= U_1^{-4}T_1^{-1}T_2^{-2}, \\
g_2 &= U_1^{-4}T_1^{-2}T_2^{-1}, \\
g_3 &= U_1^4, \\
g_4 &= U_2^3S^3[T_1, T_2]S^3(S^4U_2)^{-1}[T_1, T_2], \\
g_5 &= S^3U_1^{-4}T_1^{-1}T_2S^3.
\end{align*}
\]

Here,

\[
T_1 = \begin{pmatrix} 1 & 1 & -\omega \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & \omega & -\omega \\ 1 & 1 & \omega \\ 1 & 1 & 1 \end{pmatrix}, \quad S = -\omega J = \begin{pmatrix} \omega & -\omega \\ -\omega & \omega \end{pmatrix},
\]

\[
U_1 = \begin{pmatrix} 1 & -\omega \\ -\omega & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -1 & -\omega \\ -\omega & -1 \end{pmatrix}.
\]

\[
[T_1, T_2] := T_1T_2T_1^{-1}T_2^{-1} = \begin{pmatrix} 1 & 0 & \omega - \omega \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\]

Note that \( g_3^3 = g_4^3 = I \) and \( g_3g_4 = g_4g_3 \). Hence,

\[
\langle g_3, g_4 \rangle \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.
\]

In [Y3], we set

\[
\varphi = z_1z_2z_3, \quad \psi = z_1^3 + z_2^3 + z_3^3, \quad \chi = z_1^3z_2^3 + z_2^3z_3^3 + z_3^3z_1^3.
\]

Put

\[
(3.17) \quad X(z_1, z_2, z_3) = z_1^3, \quad Y(z_1, z_2, z_3) = z_2^3, \quad Z(z_1, z_2, z_3) = z_3^3.
\]
(3.18) \( Q_1(z_1, z_2, z_3) = z_1z_2^2 + z_2z_3^2 + z_3z_1^2, \quad Q_2(z_1, z_2, z_3) = z_1^2z_2 + z_2^2z_3 + z_3^2z_1. \)

The functions \( \varphi, \psi, \chi, X, Y, Z \) and \( Q_1, Q_2 \) satisfy the relations:

\[
\begin{align*}
\varphi^3 &= XYZ, \\
\psi &= X + Y + Z, \\
\chi &= XY + YZ + ZX, \\
\chi + 3\varphi^2 + \varphi\psi &= Q_1Q_2, \\
\psi\chi + 6\varphi\chi + 6\varphi^2\psi + 9\varphi^3 &= Q_1^3 + Q_2^3, \\
(X - Y)(Y - Z)(Z - X) &= Q_1^3 - Q_2^3.
\end{align*}
\]

Let

(3.20)

\[
\begin{align*}
W_2 &= (X + Y + Z)^2 - 12(XY + YZ + ZX), \\
W_3 &= (X - Y)(Y - Z)(Z - X), \\
\mathfrak{W}_3 &= XYZ - \varphi^3, \\
W_4 &= (X + Y + Z)[(X + Y + Z)^3 + 216XYZ], \\
\mathfrak{W}_4 &= \varphi[27XYZ - (X + Y + Z)^3], \\
W_6 &= (X + Y + Z)^6 - 540XYZ(X + Y + Z)^3 - 5832X^2Y^2Z^2, \\
\mathfrak{W}_3 &= Q_1^3 + Q_2^3 - (X + Y + Z + 6\varphi)(XY + YZ + ZX) - 6\varphi^2(X + Y + Z) - 9\varphi^3, \\
\mathfrak{W}_2 &= Q_1Q_2 - (XY + YZ + ZX) - \varphi(X + Y + Z) - 3\varphi^2, \\
\mathfrak{W}_3 &= Q_1^3 - Q_2^3 - (X - Y)(Y - Z)(Z - X).
\end{align*}
\]

**Theorem 3.17.** The invariants \( W_2, W_3, \mathfrak{W}_3, W_4, \mathfrak{W}_4 \) and \( W_6 \) satisfy the following algebraic relations:

(3.21)

\[
\begin{align*}
W_2^3 - 3W_2W_4 + 2W_6 &= 432W_3^2, \\
8U^3(W_6^2 - W_4^3 - 1728\mathfrak{W}_4^3) &= 27\mathfrak{W}_3(W_4 - 9U^4)^3,
\end{align*}
\]

where

(3.22)

\[ 27U^8 - 18W_4U^4 - 8W_6U^2 - W_4^2 = 0. \]

**Proof.** By the definition of \( W_2, W_3, W_4 \) and \( W_6 \), we have

\[ W_2^3 - 3W_2W_4 + 2W_6 = 432W_3^2. \]
We find that
\[(3.23)\quad W_6^2 - W_4^3 - 1728W_4^3 = 1728W_3^3[27XYZ - (X + Y + Z)^3]^3.\]

Put
\[(3.24)\quad U = X + Y + Z, \quad V = XYZ.\]

Then
\[(3.25)\quad \begin{cases} &U(U^3 + 216V) = W_4, \\ &U^6 - 540U^3V - 5832V^2 = W_6. \end{cases}\]

By the first equation in (3.25), we have
\[(3.26)\quad V = \frac{W_4 - U^4}{216U}.\]

Substituting the above expression (3.26) in the second equation in (3.25), we obtain that
\[27U^8 - 18W_4U^4 - 8W_6U^2 - W_4^2 = 0.\]

Set \(t = U^2\). Let
\[(3.27)\quad t^4 - \frac{2}{3}W_4t^2 - \frac{8}{27}W_6t - \frac{1}{27}W_4^2 = (t^2 + kt + l)(t^2 - kt + m).\]

Then
\[(3.28)\quad \begin{cases} &l + m - k^2 = -\frac{2}{3}W_4, \\ &k(m - l) = -\frac{8}{27}W_6, \\ &lm = -\frac{1}{27}W_4^2. \end{cases}\]

By the first two equations in (3.28), we have
\[(3.29)\quad \begin{cases} &l = \frac{1}{2k}(k^3 - \frac{2}{3}W_4k + \frac{8}{27}W_6), \\ &m = \frac{1}{2k}(k^3 - \frac{2}{3}W_4k - \frac{8}{27}W_6). \end{cases}\]

Substituting these two expressions in the third equation in (3.28), we have
\[(3.30)\quad k^6 - \frac{4}{3}W_4k^4 + \frac{16}{27}W_4^2k^2 - \frac{64}{729}W_6^2 = 0,\]
i.e.,

$$(k^2 - \frac{4}{9}W_4)^3 = \frac{64}{729}(W_6^2 - W_4^3).$$

Hence,

$$k = \pm \frac{2}{3}\sqrt{W_4 + \sqrt{W_6^2 - W_4^3}}.$$  \hspace{1cm} (3.31)

Consequently,

$$l = -\frac{1}{9}W_4 + \frac{2}{9}\sqrt{W_6^2 - W_4^3} \pm \frac{2}{9}\frac{W_6}{\sqrt{W_4 + \sqrt{W_6^2 - W_4^3}}}.$$  \hspace{1cm} (3.32)

$$m = -\frac{1}{9}W_4 + \frac{2}{9}\sqrt{W_6^2 - W_4^3} \pm \frac{2}{9}\frac{W_6}{\sqrt{W_4 + \sqrt{W_6^2 - W_4^3}}}.$$  \hspace{1cm} (3.33)

Thus,

$$t = \frac{1}{2}(-k \pm \sqrt{k^2 - 4l}), \text{ or } t = \frac{1}{2}(k \pm \sqrt{k^2 - 4m}).$$  \hspace{1cm} (3.34)

We have

$$W_6^2 - W_4^3 - 1728\mathfrak{m}_3^3 = 1728\mathfrak{m}_3(27V - U^3)^3 = \frac{27}{8}\mathfrak{m}_3(W_4 - 9U^4)^3.$$  \hspace{1cm} (3.35)

Therefore,

$$8U^3(W_6^2 - W_4^3 - 1728\mathfrak{m}_3^3) = 27\mathfrak{m}_3(W_4 - 9U^4)^3.$$  \hspace{1cm}

**Theorem 3.18.** The invariants $W_2, W_3, \mathfrak{m}_3, W_4, \mathfrak{m}_4, \mathfrak{u}_3, \mathfrak{v}_2$ and $\mathfrak{v}_3$ satisfy the following differential relations:

$$\frac{\partial(W_2, W_3, W_4, \mathfrak{m}_3)}{\partial(X, Y, Z, \varphi)} = 288\mathfrak{m}_4^2.$$  \hspace{1cm} (3.35)

$$\frac{\partial(\mathfrak{m}_3, \mathfrak{m}_2)}{\partial(Q_1, Q_2)} = 3(\mathfrak{u}_3 + W_3).$$  \hspace{1cm} (3.36)
Proof. We have
\[
\begin{align*}
\frac{\partial W_2}{\partial X} &= 2X - 10Y - 10Z, \\
\frac{\partial W_2}{\partial Y} &= 2Y - 10Z - 10X, \\
\frac{\partial W_2}{\partial Z} &= 2Z - 10X - 10Y, \\
\frac{\partial W_2}{\partial \varphi} &= 0.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial W_3}{\partial X} &= (Y - Z)(Y + Z - 2X), \\
\frac{\partial W_3}{\partial Y} &= (Z - X)(Z + X - 2Y), \\
\frac{\partial W_3}{\partial Z} &= (X - Y)(X + Y - 2Z), \\
\frac{\partial W_3}{\partial \varphi} &= 0.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial W_4}{\partial X} &= 4(X + Y + Z)^3 + 216XYZ + 216(X + Y + Z)YZ, \\
\frac{\partial W_4}{\partial Y} &= 4(X + Y + Z)^3 + 216XYZ + 216(X + Y + Z)ZX, \\
\frac{\partial W_4}{\partial Z} &= 4(X + Y + Z)^3 + 216XYZ + 216(X + Y + Z)XY, \\
\frac{\partial W_4}{\partial \varphi} &= 0.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial W_3}{\partial X} &= YZ, \\
\frac{\partial W_3}{\partial Y} &= ZX, \\
\frac{\partial W_3}{\partial Z} &= XY, \\
\frac{\partial W_3}{\partial \varphi} &= -3\varphi^2.
\end{align*}
\]

Hence,
\[
\frac{\partial(W_2, W_3, W_4, W_3)}{\partial(X, Y, Z, \varphi)} = -24\varphi^2\Delta,
\]
where
\[ \Delta = [(X + Y + Z)^3 + 54YZ(2X + Y + Z)] \times \\
\times [- (X + Y + Z)^3 - 9Y(X + Y + Z)^2 + 27X^2(Y + Z) + 27X(Y^2 + Z^2)] + \\
+ [(X + Y + Z)^3 + 54ZX(2Y + Z + X)] \times \\
\times [- (X + Y + Z)^3 - 9Y(X + Y + Z)^2 + 27Y^2(Z + X) + 27Y(Z^2 + X^2)] + \\
+ [(X + Y + Z)^3 + 54XY(2Z + X + Y)] \times \\
\times [- (X + Y + Z)^3 - 9Z(X + Y + Z)^2 + 27Z^2(X + Y) + 27Z(X^2 + Y^2)]. \]

By the proof of Proposition 4.3 in [Y3], we have
\[ \Delta = -12[27XYZ - (X + Y + Z)^3]^2. \]

Thus,
\[ \frac{\partial(W_2, W_3, W_4, W_5)}{\partial(X, Y, Z, \varphi)} = 288\varphi^2[27XYZ - (X + Y + Z)^3]^2 = 288\varphi^2. \]

On the other hand, we find that
\[ \frac{\partial(Q_3, Q_2)}{\partial(Q_1, Q_2)} = 3(Q_1^3 - Q_2^3) = 3(U_3 + W_3). \]

Combine Theorem 3.17 with Theorem 3.18, we get the proof of Theorem 1.4 (Main Theorem 2).

According to [Ho1], we denote the elementary symmetric functions of \( X_1, X_2, X_3, X_4 \) by
\[ \Sigma_1 = -(X_1 + X_2 + X_3 + X_4), \]
\[ \Sigma_2 = X_1X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4, \]
\[ \Sigma_3 = -(X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4), \]
\[ \Sigma_4 = X_1X_2X_3X_4, \]
and
\[ \Delta = \prod_{1 \leq i < j \leq 4} (X_i - X_j). \]

Then
\[ 27\Delta^2 = 4(\Sigma_2^3 - 3\Sigma_1\Sigma_3 + 12\Sigma_4)^3 - (27\Sigma_1^2\Sigma_4 + 27\Sigma_3^2 + 2\Sigma_2^3 - 72\Sigma_2\Sigma_4 + 9\Sigma_1\Sigma_2\Sigma_3)^2. \]

Their images in \( \mathbb{C}[x_1, x_2, x_3, x_4] = \mathbb{C}[X_1, X_2, X_3, X_4]/(X_1 + X_2 + X_3 + X_4) \) are denoted by \( G_1 = 0, G_2, G_3 \) and \( G_4 \), respectively. Hence,
\[ \Delta^2 = 16G_4^2G_4 - 128G_2G_4^2 - 4G_2^3G_3 + 144G_2^2G_3G_4 - 27G_3^4 + 256G_4^3. \]
Proposition 3.19 (see [Ho1]). There are two isomorphisms of graded rings:

\[ \bigoplus_{m=0}^{\infty} [U(2, 1; \mathcal{O}_K), m]_\chi \cong \mathbb{C}[G_2, G_3, G_4] \]

and

\[ \bigoplus_{m=0}^{\infty} [SU(2, 1; \mathcal{O}_K), m]_\chi \cong \mathbb{C}[G_2, G_3, G_4, z^2]. \]

The fundamental relation between the generators is \( z^6 = \Delta^2 \).

We find the following correspondence:

\( (G_2, G_3, G_4, z^2) \longleftrightarrow (W_2, W_3, W_4, \mathfrak{M}_4) \).

Here, \( W_2, W_3, W_4 \) and \( \mathfrak{M}_4 \) satisfy the single relation:

\[ 6912 \mathfrak{M}_4^3 = W_2^6 + 9W_2^2W_4^2 + 432W_3^4 - 4W_4^4 - 6W_2^4W_4 - 864W_2^2W_3^2 + 2592W_2W_3W_4. \]

The equations \( W_4 = 0 \) and \( \mathfrak{M}_4 = 0 \) define two surfaces of order 4. The intersection of the cubic surface \( \mathfrak{M}_3 = 0 \) with the quartic surface \( W_4 = 0 \) gives four elliptic curves

\[
\begin{aligned}
\varphi^3 + 6\varphi XY + (X + Y)XY &= 0, \\
\varphi^3 + 6\omega\varphi XY + (X + Y)XY &= 0, \\
\varphi^3 + 6\omega^2\varphi XY + (X + Y)XY &= 0, \\
\varphi^3 + (X + Y)XY &= 0.
\end{aligned}
\]

The intersection of the cubic surface \( \mathfrak{M}_3 = 0 \) with the quartic surface \( \mathfrak{M}_4 = 0 \) gives the other four elliptic curves

\[
\begin{aligned}
\varphi^3 - 3\varphi XY + (X + Y)XY &= 0, \\
\varphi^3 - 3\omega\varphi XY + (X + Y)XY &= 0, \\
\varphi^3 - 3\omega^2\varphi XY + (X + Y)XY &= 0, \\
\varphi^3 + (X + Y)XY &= 0.
\end{aligned}
\]

The equation \( W_2 = 0 \) defines a quadric surface, the equation \( W_3 = 0 \) is the union of three planes. The intersection of \( W_3 = 0 \) with \( \mathfrak{M}_3 = 0 \) gives three elliptic curves

\[ Y^2Z - \varphi^3 = 0, \quad Z^2X - \varphi^3 = 0, \quad X^2Y - \varphi^3 = 0. \]

The intersection of \( W_2 = 0 \) with \( \mathfrak{M}_3 = 0 \) gives an algebraic curve of degree six

\[ \varphi^6 - 10XY(X + Y)\varphi^3 + (X^4Y^2 - 10X^3Y^3 + X^2Y^4) = 0. \]
It is known that the Hessian group of order 216 is generated by the five generators (see [Y3]):

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix},
\]

\[
E = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.
\]

We have

\[
\begin{align*}
\varphi(A(z_1, z_2, z_3)) &= \varphi, \\
X(A(z_1, z_2, z_3)) &= Y, \\
Y(A(z_1, z_2, z_3)) &= Z, \\
Z(A(z_1, z_2, z_3)) &= X, \\
Q_1(A(z_1, z_2, z_3)) &= Q_1, \\
Q_2(A(z_1, z_2, z_3)) &= Q_2.
\end{align*}
\]

\[
\begin{align*}
\varphi(B(z_1, z_2, z_3)) &= \varphi, \\
X(B(z_1, z_2, z_3)) &= X, \\
Y(B(z_1, z_2, z_3)) &= Z, \\
Z(B(z_1, z_2, z_3)) &= Y, \\
Q_1(B(z_1, z_2, z_3)) &= Q_2, \\
Q_2(B(z_1, z_2, z_3)) &= Q_1.
\end{align*}
\]

\[
\begin{align*}
\varphi(C(z_1, z_2, z_3)) &= \varphi, \\
X(C(z_1, z_2, z_3)) &= X, \\
Y(C(z_1, z_2, z_3)) &= Y, \\
Z(C(z_1, z_2, z_3)) &= Z, \\
Q_1(C(z_1, z_2, z_3)) &= \omega Q_1, \\
Q_2(C(z_1, z_2, z_3)) &= \omega Q_2.
\end{align*}
\]
\[
\begin{aligned}
\varphi(D(z_1, z_2, z_3)) &= \overline{\omega} \varphi, \\
X(D(z_1, z_2, z_3)) &= X, \\
Y(D(z_1, z_2, z_3)) &= Y, \\
Z(D(z_1, z_2, z_3)) &= Z, \\
Q_1(D(z_1, z_2, z_3)) &= \omega z_1 z_2^2 + z_2 z_3^2 + \omega z_3 z_1^2, \\
Q_2(D(z_1, z_2, z_3)) &= \omega z_1^2 z_2 + z_2^2 z_3 + \overline{\omega} z_3^2 z_1. 
\end{aligned}
\]

\[
\begin{align*}
(\sqrt{-3})^3 \varphi(E(z_1, z_2, z_3)) &= X + Y + Z - 3\varphi, \\
(\sqrt{-3})^3 X(E(z_1, z_2, z_3)) &= X + Y + Z + 6\varphi + 3Q_1 + 3Q_2, \\
(\sqrt{-3})^3 Y(E(z_1, z_2, z_3)) &= X + Y + Z + 6\varphi + 3\overline{\omega}Q_1 + 3\omega Q_2, \\
(\sqrt{-3})^3 Z(E(z_1, z_2, z_3)) &= X + Y + Z + 6\varphi + 3\omega Q_1 + 3\overline{\omega} Q_2, \\
(\sqrt{-3})^3 Q_1(E(z_1, z_2, z_3)) &= 3X + 3\overline{\omega} Y + 3\omega Z, \\
(\sqrt{-3})^3 Q_2(E(z_1, z_2, z_3)) &= 3X + 3\omega Y + 3\overline{\omega} Z.
\end{align*}
\]

In the form of matrices, we have

\[
A(X, Y, Z, \varphi, Q_1, Q_2)
= \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\varphi \\
Q_1 \\
Q_2 \\
\end{pmatrix},
\]

\[
B(X, Y, Z, \varphi, Q_1, Q_2)
= \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\varphi \\
Q_1 \\
Q_2 \\
\end{pmatrix},
\]

\[
C(X, Y, Z, \varphi, Q_1, Q_2)
= \begin{pmatrix}
1 & 1 \\
1 & 1 \\
\omega & \overline{\omega} \\
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\varphi \\
Q_1 \\
Q_2 \\
\end{pmatrix},
\]
\[ (\sqrt{-3})^3 E(X, Y, Z, \varphi, Q_1, Q_2) \]

\[
\begin{pmatrix}
1 & 1 & 1 & 6 & 3 & 3 \\
1 & 1 & 1 & 6 & 3\bar{\omega} & 3\omega \\
1 & 1 & 1 & 6 & 3\omega & 3\bar{\omega} \\
1 & 1 & 1 & -3 & 0 & 0 \\
3 & 3\bar{\omega} & 3\omega & 0 & 0 & 0 \\
3 & 3\omega & 3\bar{\omega} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z \\
\varphi \\
Q_1 \\
Q_2
\end{pmatrix},
\]

(3.40)

We find that
\[
A^3 = B^2 = C^3 = E^4 = I.
\]

Theorem 3.19 (see [Y3], Main Theorem 4). The invariants \( G, H \) and \( K \) satisfy the following algebraic equations, which are the form-theoretic resolvents (algebraic resolvents) of \( G, H, K \):

\[
\begin{aligned}
4G^3 + H^2G - C_6G - 4C_9 &= 0, \\
H(H^3 + 8K^3) - 9C_{12} &= 0, \\
K(K^3 - H^3) - 27\mathcal{E}_{12} &= 0.
\end{aligned}
\]

Consequently,
\[ C_{18} = -\frac{1}{27}(H^6 - 20H^3K^3 - 8K^6). \]

Note that
\[ W_2 = C_6, W_3 = C_9, W_4 = C_{12}, \mathfrak{W}_4 = \mathcal{E}_{12}, W_6 = C_{18}. \]

Set \((x, y, z) = (G, H, K)\). Then
\[
x = Q_1 - Q_2, \quad y = X + Y + Z + 6\varphi, \quad z = X + Y + Z - 3\varphi.
\]

We have
\[
\frac{1}{9} y(y^3 + 8z^3) = (X + Y + Z)(X + Y + Z)^3 + 216\varphi^3 = W_4 - 216(X + Y + Z)\mathfrak{W}_3.
\]

(3.43)

\[
\frac{1}{27} z(z^3 - y^3) = \varphi[27\varphi^3 - (X + Y + Z)^3] = \mathfrak{W}_4 - 27\varphi\mathfrak{W}_3.
\]

(3.44)

Thus,
\[ X + Y + Z = \frac{9W_4 - y(y^3 + 8z^3)}{1944\mathfrak{W}_3}. \]
\[
\varphi = \frac{27\mathfrak{W}_4 - z(z^3 - y^3)}{729\mathfrak{W}_3}.
\]

\[
XYZ = \mathfrak{W}_3 + \varphi^3 = \mathfrak{W}_3 + \frac{[27\mathfrak{W}_4 - z(z^3 - y^3)]^3}{729\mathfrak{W}_3^3}.
\]

Note that \(U = X + Y + Z\) and \(V = XYZ\) satisfy the equations:

\[
U(U^3 + 216V) = W_4, \quad U^6 - 540U^3V - 5832V^2 = W_6.
\]

We obtain the following relations:

\[
[9W_4 - y(y^3 + 8z^3)]^4 + 2^{12}[9W_4 - y(y^3 + 8z^3)]\{3^{18}\mathfrak{W}_3^4 + [27\mathfrak{W}_4 - z(z^3 - y^3)]^3\}
= 2^{12} \cdot 3^{20} \mathfrak{W}_3^4 W_4.
\]

\[
[9W_4 - y(y^3 + 8z^3)]^6 - 2^{11} \cdot 5[9W_4 - y(y^3 + 8z^3)]\{3^{18}\mathfrak{W}_3^4 + [27\mathfrak{W}_4 - z(z^3 - y^3)]^3\}
- 2^{21} \{3^{18}\mathfrak{W}_3^4 + [27\mathfrak{W}_4 - z(z^3 - y^3)]^3\}^2 = 2^{18} \cdot 3^{30} \mathfrak{W}_3^4 W_6.
\]

Similarly, we have

\[
4x^3 + xy^2 - W_2x - 4W_3 = 4\mathfrak{U}_3 - 12(Q_1 - Q_2)\mathfrak{U}_2.
\]

Hence,

\[
Q_1 - Q_2 = \frac{4\mathfrak{U}_3 - 4x^3 - xy^2 + W_2x + 4W_3}{12\mathfrak{U}_2}.
\]

By the identity:

\[
(Q_1 - Q_2)^3 = Q_1^3 - Q_2^3 - 3Q_1Q_2(Q_1 - Q_2),
\]

where

\[
Q_1^3 - Q_2^3 = \mathfrak{U}_3 + W_3,
\]

\[
Q_1Q_2 = \mathfrak{U}_2 + (XY + YZ + ZX) + \varphi(X + Y + Z) + 3\varphi^2.
\]

Note that

\[
XY + YZ + ZX = \frac{1}{12} [(X + Y + Z)^2 - W_2]
= \frac{1}{2^8 \cdot 3^{11} \mathfrak{W}_3^3} [(9W_4 - y(y^3 + 8z^3))^2 - 2^6 \cdot 3^{10} W_2 \mathfrak{W}_3^2].
\]

We obtain the following relation:

\[
2^4 \cdot 3^8 \mathfrak{W}_3^2 (4\mathfrak{U}_3 - 4x^3 - xy^2 + W_2x + 4W_3)^3
= 2^{10} \cdot 3^{11} \mathfrak{W}_3^2 \mathfrak{U}_3 (4\mathfrak{U}_3 + W_3) - (4\mathfrak{U}_3 - 4x^3 - xy^2 + W_2x + 4W_3)\mathfrak{U}_2 \times
\]

\[
\times (2^8 \cdot 3^{11} \mathfrak{W}_3^2 \mathfrak{U}_2 + [(9W_4 - y(y^3 + 8z^3))^2 - 2^6 \cdot 3^{10} W_2 \mathfrak{W}_3^2] +
+ 2^5 [27\mathfrak{W}_4 - z(z^3 - y^3)][9W_4 - y(y^3 + 8z^3)] + 2^8 [27\mathfrak{W}_4 - z(z^3 - y^3)]^2}.\]
Therefore, we finish the proof of Theorem 1.5 (Main Theorem 3).

**Proposition 3.20.** The space curves $W_2 = \mathcal{W}_3 = \mathcal{V}_3 = \mathcal{U}_2 = 0$, $W_3 = \mathcal{W}_3 = \mathcal{V}_3 = \mathcal{U}_2 = 0$, $W_4 = \mathcal{W}_3 = \mathcal{V}_3 = \mathcal{U}_2 = 0$, $W_5 = \mathcal{W}_3 = \mathcal{V}_3 = \mathcal{U}_2 = 0$ and $W_6 = \mathcal{W}_3 = \mathcal{V}_3 = \mathcal{U}_2 = 0$ are invariant curves on the invariant surface $\mathcal{W}_3 = \mathcal{V}_3 = \mathcal{U}_2 = 0$ under the action of the subgroup of Hessian group generated by $A$, $B$, $C$ and $E$.

**Proof.** Note that on the surface $\mathcal{W}_3 = \mathcal{V}_3 = \mathcal{U}_2 = 0$, $XYZ = \varphi^3$,

\[ Q_1 Q_2 = XY + YZ + ZX + \varphi(X + Y + Z) + 3\varphi^2, \]
\[ Q_1^3 + Q_2^3 = (X + Y + Z + 6\varphi)(XY + YZ + ZX) + 6\varphi^2(X + Y + Z) + 9\varphi^3. \]

Hence,

\[
(\sqrt{-3})^9 \mathcal{V}_3(E(X, Y, Z, \varphi, Q_1, Q_2)) \\
= (X + Y + Z + 6\varphi)^3 + 27Q_1^3 + 27Q_2^3 - 27(X + Y + Z + 6\varphi)Q_1Q_2 + \\
- (X + Y + Z - 3\varphi)^3 \\
= 0.
\]

Similarly,

\[
(\sqrt{-3})^9 \mathcal{V}_3(E(X, Y, Z, \varphi, Q_1, Q_2)) \\
= 54(X^3 + Y^3 + Z^3 + 6XYZ) - 81(X^2Y + Y^2Z + Z^2X + XY^2 + YZ^2 + ZX^2) + \\
- 9(X + Y + Z)[3(X + Y + Z + 6\varphi)^2 - 27Q_1Q_2] + \\
- 18(X + Y + Z - 3\varphi)^2(X + Y + Z + 6\varphi) - 9(X + Y + Z - 3\varphi)^3 \\
= 243(X + Y + Z)[Q_1Q_2 - (X + Y + Z)\varphi - 3\varphi^2] - 729\varphi^3 + \\
- 243(X^2Y + Y^2Z + Z^2X + XY^2 + YZ^2 + ZX^2) \\
= 729(XYZ - \varphi^3) \\
= 0.
\]

\[
(\sqrt{-3})^6 \mathcal{V}_2(E(X, Y, Z, \varphi, Q_1, Q_2)) \\
= 9(X^2 + Y^2 + Z^2 - XY - YZ - ZX) - 3(X + Y + Z + 6\varphi)^2 + 27Q_1Q_2 + \\
- 3(X + Y + Z + 6\varphi)(X + Y + Z - 3\varphi) - 3(X + Y + Z - 3\varphi)^2 \\
= 27[Q_1Q_2 - (XY + YZ + ZX) - \varphi(X + Y + Z) - 3\varphi^2] \\
= 0.
\]

On the other hand, we have

\[
\begin{align*}
\mathcal{W}_3(A(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathcal{W}_3, \\
\mathcal{W}_3(B(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathcal{W}_3, \\
\mathcal{W}_3(C(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathcal{W}_3.
\end{align*}
\]
\[
\begin{align*}
\mathfrak{W}_3(A(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathfrak{W}_3, \\
\mathfrak{W}_3(B(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathfrak{W}_3, \\
\mathfrak{W}_3(C(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathfrak{W}_3, \\
\mathfrak{W}_2(A(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathfrak{W}_2, \\
\mathfrak{W}_2(B(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathfrak{W}_2, \\
\mathfrak{W}_2(C(X, Y, Z, \varphi, Q_1, Q_2)) &= \mathfrak{W}_2.
\end{align*}
\]

Hence, \( \mathfrak{W}_3 = \mathfrak{W}_3 = \mathfrak{W}_2 = 0 \) is an invariant surface. On this surface, we have
\[Q_1^3 - Q_2^3 = \pm (X - Y)(Y - Z)(Z - X).\]
Hence,
\[\sqrt{-3}^9 W_3(E(X, Y, Z, \varphi, Q_1, Q_2)) = 81 \sqrt{-3}(Q_1^3 - Q_2^3) = \pm 81 \sqrt{-3} W_3.\]
Furthermore,
\[
\begin{align*}
W_3(A(X, Y, Z, \varphi, Q_1, Q_2)) &= (Y - Z)(Z - X)(X - Y) = W_3, \\
W_3(B(X, Y, Z, \varphi, Q_1, Q_2)) &= (X - Z)(Z - Y)(Y - X) = -W_3, \\
W_3(C(X, Y, Z, \varphi, Q_1, Q_2)) &= (X - Y)(Y - Z)(Z - X) = W_3.
\end{align*}
\]
This implies that \( W_3 = \mathfrak{W}_3 = \mathfrak{W}_3 = \mathfrak{W}_2 = 0 \) is an invariant curve on the above surface. We have
\[\sqrt{-3}^6 W_2(E(X, Y, Z, \varphi, Q_1, Q_2)) = -27(X + Y + Z + 6\varphi)^2 + 324Q_1 Q_2,
\]
\[= -27[(X + Y + Z)^2 - 12(XY + YZ + ZX)]
\]
\[= -27W_2.\]
Moreover,
\[
\begin{align*}
W_2(A(X, Y, Z, \varphi, Q_1, Q_2)) &= W_2, \\
W_2(B(X, Y, Z, \varphi, Q_1, Q_2)) &= W_2, \\
W_2(C(X, Y, Z, \varphi, Q_1, Q_2)) &= W_2.
\end{align*}
\]
This implies that \( W_2 = \mathfrak{W}_3 = \mathfrak{W}_3 = \mathfrak{W}_2 = 0 \) is an invariant curve on the above surface. We have
\[
\sqrt{-3}^{12} W_4(E(X, Y, Z, \varphi, Q_1, Q_2))
\]
\[= 729(X + Y + Z + 6\varphi)[(X + Y + Z + 6\varphi)^3 + 24(Q_1^3 + Q_2^3) + 24(X + Y + Z + 6\varphi)Q_1 Q_2]
\]
\[= 729(X + Y + Z + 6\varphi)(X + Y + Z)[(X + Y + Z)^2 - 6(X + Y + Z)\varphi + 36\varphi^2]
\]
\[= 729(X + Y + Z)[(X + Y + Z)^3 + 216\varphi^3]
\]
\[= 729(X + Y + Z)[(X + Y + Z)^3 + 216XYZ]
\]
\[= 729W_4.\]
Moreover,
\[
\begin{align*}
W_4(A(X, Y, Z, \varphi, Q_1, Q_2)) &= W_4, \\
W_4(B(X, Y, Z, \varphi, Q_1, Q_2)) &= W_4, \\
W_4(C(X, Y, Z, \varphi, Q_1, Q_2)) &= W_4.
\end{align*}
\]

This implies that \( W_4 = \mathcal{W}_3 = \mathcal{Y}_3 = \mathcal{V}_2 = 0 \) is an invariant curve on the above surface.

On the above invariant surface one has
\[
W_6^3 - 3W_2W_4 + 2W_6 = 432W_3^2, \quad W_6^2 - W_4^3 = 1728W_3^3.
\]

Thus, \( W_6 = \mathcal{W}_3 = \mathcal{Y}_3 = \mathcal{V}_2 = 0 \) and \( \mathcal{W}_4 = \mathcal{W}_3 = \mathcal{Y}_3 = \mathcal{V}_2 = 0 \) are invariant curves on the above surface.

\[\square\]

4. Hessian polyhedra and Galois representations associated with cubic surfaces

Manin’s book [Ma] (see also [MT]) is devoted to the arithmetic of smooth cubic surfaces. We give an account of some of the results of this book that are characteristic of cubic surfaces only. A cubic surface \( X \) has a very specific property: every point \( x \in X \) generates a birational automorphism \( t_x \) of \( X \) by reflection in \( x \). This enables us to describe the group of birational automorphisms of \( X \). On the other hand, it allows us to introduce on \( X \) the structure of the ternary relation of collinearity of points. We recall that a symmetric quasigroup \( C \) is a set with a binary composition \( (x, y) \mapsto x \circ y \) such that the ternary relation \( x \circ y = z \) is symmetric under all permutations of \( x, y \) and \( z \), and such that \( x \circ y = y \circ x \) and \( x \circ (x \circ y) = y \) for any pair \( x, y \in C \). Suppose that an equivalence relation \( S \) is given on \( X(k) \), we say that \( S \) is admissible if collinearity of points induces the structure of a symmetric quasigroup on \( X(k)/S \). We say that a symmetric quasigroup \( C \) is abelian if for any element \( u \) the composition law \( xy = u \circ (x \circ y) \) turns \( C \) into an abelian group. If any three elements of \( C \) generate an abelian subquasigroup, we say that \( C \) is a CH-quasigroup. We fix an element \( u \) in \( C \) and set \( xy = u \circ (x \circ y) \), this composition law defines on \( C \) the structure of a commutative Moufang loop (CML), that is, it is commutative, has a unit element \( u \) and inverse, and moreover satisfies the three weak associativity properties: \( x(xy) = x^2y \), \( (xy)(xz) = x^2(yz) \) and \( x(y(xz)) = (x^2y)z \).

**Theorem 4.1** (see [Ma], [MT]). Let \( S \) be an admissible equivalence relation on a smooth cubic surface \( X \). Then \( X(k)/S \) is a CH-quasigroup, and in the corresponding CML the relation \( x^6 = 1 \) is satisfied identically; this CML is finite if and only if it is finitely generated, and in this case its order is \( 2^a \cdot 3^b \).

In our case, let \( H = \langle A, B, C, E \rangle \). Note that the order of \( H \) is \( 2^3 \cdot 3^2 \).

Let \( k \) be a perfect field, and let \( V \) run through the smooth cubic surfaces over \( k \). The classes of the lines on \( V \otimes \overline{k} \) generate the group \( N(V) = \text{Pic}(V \otimes \overline{k}) \) and the action of
the Galois group $G = \text{Gal}(\overline{k}/k)$ on $N(V)$ preserves symmetry and it implicitly contains an extremely large amount of information on the arithmetic and geometry of $V$.

Let $k$ be a global field. Let $\chi$ denote the character of the representation of $G$ on $N(V)$, and let $L(s, \chi, k)$ be the Artin $L$-function of the field $k$ corresponding to the character $\chi$.

**Theorem 4.2** (Weil) (see [Ma]). *The Hasse-Weil zeta function of the surface $V$ coincides (up to a finite number of Euler factors) with the product*

$$\zeta(s, k)\zeta(s - 2, k)\overline{L}(s - 1, \chi, k).$$

**Definition 4.3** (see [Ma]). Let $r \geq 1$ be an integer. We consider a composed object $\{N_r, \omega_r, (, )\}$, where

1. $N_r = \mathbb{Z}^{r+1} = \bigoplus_{i=0}^{r} \mathbb{Z}l_i$; $(l_i)$ a chosen basis.
2. $\omega_r = (-3, 1, \cdots, 1) \in N_r$.
3. $(, )$ is a bilinear form $N_r \times N_r \to \mathbb{Z}$ given by the formula

   $$(l_0, l_0) = 1, \quad (l_i, l_i) = -1 \quad \text{if} \quad i \geq 1, \quad (l_i, l_j) = 0 \quad \text{if} \quad i \neq j.$$

4. $R_r := \{l \in N_r : (l, \omega_r) = 0, \quad (l, l) = -2\}$.
5. $I_r := \{l \in N_r : (l, \omega_r) = (l, l) = -1\}$.

**Theorem 4.4** (see [Ma]). Let $V$ be a smooth cubic surface over a perfect field $k$.

1. The triple $\{N(V), \omega_V, \text{intersection number}\}$ is isomorphic to $\{N_6, \omega_6, (, )\}$ described in Definition 4.3.
2. The Galois group $G$ acts on $N(V)$ and preserves $\omega_V$ and the intersection number.
3. The set of classes of lines on $V \otimes \overline{k}$ goes over into $I_r$ under the isomorphism (1).

**Theorem 4.5** (see [Ma]). Let $3 \leq r \leq 8$.

1. The scalar product (with opposite sign) on $\mathbb{R} \otimes_{\mathbb{Z}} N_r \cong \mathbb{R}^{r+1}$ induces on the orthogonal complement of $\omega_r$ the structure of a Euclidean space. The set $R_r$ is a root system in it of type $A_1 \times A_2, A_4, D_5, E_6, E_7, E_8$, respectively (the sum of the indices is equal to $r$).
2. The following groups coincide:
   (a) the group of automorphisms of the lattice $N_r$ preserving $\omega_r$ and the scalar product;
   (b) the group of permutations of the vectors from $I_r$ preserving their pairwise scalar products;
   (c) the Weyl group $W(R_r)$ of the system $R_r$ generated by the reflections with respect to the roots.
In fact, Fano varieties in dimension two are called Del Pezzo surfaces. Over an algebraically closed field, these are: \( \mathbb{P}^2 \), \( \mathbb{P}^1 \times \mathbb{P}^1 \) and \( S_d \), where \( S_d \) is the blow up of \( \mathbb{P}^2 \) at \( 9 - d \) points, and the degree \( d = 1, \ldots, 8 \). However, starting with \( r = 9 \), the system \( R_r \) becomes infinite. Let \( V \) be a Del Pezzo surface of degree \( 1 \leq d \leq 7 \), and let \( f : V \to \mathbb{P}^2 \) be its representation in the form of a monoidal transformation of the plane with as centre the union of \( r = 9 - d \) points \( x_1, \ldots, x_r \). It is shown that (see [Ma]) the map \( D \mapsto (\text{class of } \mathcal{O}_V(D)) \in \text{Pic}(V) \) establishes a one-one onto correspondence between exceptional curves on \( V \) and exceptional classes in the Picard group. These classes generate the Picard group. When \( r = 6 \), the number of exceptional curves on \( V \) is 27.

The significance of cubic surfaces comes from the following: the higher the degree of a rational surface, the smaller the rank of its Picard module, and therefore the less freedom there is for the action of the Galois group. Del Pezzo surfaces of degree three are cubic surfaces in \( \mathbb{P}^3 \) and the Weyl group \( W(E_6) \) has enough large order. We have the following theorem (see [MT]): Any minimal rational surface \( X \) over \( k \) of degree \( d \leq 4 \) is birationally nontrivial. The triple \( \{N(V), \omega_V, \text{intersection number on } N(V)\} \) can be constructed for every surface \( V \). The action of the Galois group and the group of automorphisms of \( V \) on this triple yields essential invariants of the surface.

Note that when \( r = 6 \), we have \( |I_6| = 27 \), \( |R_6| = 72 \), \( |W(E_6)| = 2^7 \cdot 3^4 \cdot 5 \), \( |S_6| = 720 \), \( |W(E_6)/S_6| = 72 \). On the other hand, in our case, for \( H = \langle A, B, C, E \rangle \), we have \( |H| = 72 \).

According to [Hu], the modern point of view is to consider cubic surfaces as del Pezzo surfaces of degree 3. The combinatorics of the 27 lines is then encoded in the Picard group of the del Pezzo surface. In fact, the complement in \( \text{Pic}(S) \) to the hyperplane section, call it \( \text{Pic}^0(S) \), is isomorphic to the root lattice of \( E_6 \). The equation of the surface is given by the embedding of \( S \) by means of the linear system of elliptic curves through the six given points.

Fix six points in \( \mathbb{P}^2 \), say \( x = (p_1, \cdots, p_6) \), such that the \( p_i \) are in general position, i.e., no three lie on a line, and not all six lie on a conic. Let \( \mathbb{P}^2_x \) denote the blow up of \( \mathbb{P}^2 \) at all six points, \( q_x : \mathbb{P}^2_x \to \mathbb{P}^2 \). Consider the following curves as classes in \( \text{Pic}(\mathbb{P}^2_x) \):

1. \( a_1, \cdots, a_6 \), the exceptional divisors over \( (p_1, \cdots, p_6) \);
2. \( b_1, \cdots, b_6, b_i \) the proper transform of the conic \( q_i \) passing through all points \( p_j \), \( j \neq i \);
3. \( c_{ik} \), the proper transform of the line \( \overline{p_i p_k} \).

If we consider the surface \( \mathbb{P}^2_x \), we have \( H^2(\mathbb{P}^2_x, \mathbb{Z}) = |l|\mathbb{Z} \oplus \mathbb{Z}a_i \). Let \( Q \) be the intersection form on \( H^2(\mathbb{P}^2_x, \mathbb{Z}) \), then the classes \( a_i, b_i, c_{ij} \) fulfill \( Q(a_i, a_i) = Q(b_i, b_i) = Q(c_{ij}, c_{ij}) = -1 \). In a well-known manner one takes a rank 6 subset, which is isomorphic to the root lattice of type \( E_6 \).

Consider the orthocomplement of the canonical class on \( \mathbb{P}^2_x \), and denote this by
Pic\(^0(\widetilde{\mathbb{P}}_x^2)\). Recall that the anti-canonical class is \(3l + \sum_{i=1}^{6} a_i\), and that the anti-canonical embedding of \(\widetilde{\mathbb{P}}_x^2\) is as a cubic surface. Consequently we may view Pic\(^0(\widetilde{\mathbb{P}}_x^2)\) as the orthocomplement of the hyperplane section class of Pic\((S_x)\), where \(S_x\) is the cubic surface which is the anti-canonical embedding. The following elements \(\lambda\) with \(Q(\lambda, \lambda) = -2\) form a basis of Pic\(^0(S_x)\):

\[
\begin{align*}
\alpha_0 &= l - a_1 - a_2 - a_3, \\
\alpha_1 &= a_1 - a_2, \\
\alpha_2 &= a_2 - a_3, \\
\alpha_3 &= a_3 - a_4, \\
\alpha_4 &= a_4 - a_5, \\
\alpha_5 &= a_5 - a_6.
\end{align*}
\]

These also form a base of a root system of type \(E_6\), by taking \(\alpha_1, \cdots, \alpha_5\) as the sub-root system of type \(A_5\). Since the classes \(a_i, b_i, c_{ij}\) are exceptional, they all represent elements of Pic\(^0(S_x)\).

According to [Ma], we have the realization problems.

(1) Which subgroups of \(W(E_6)\) are realized as an image of the Galois group by its representation on Pic\((V \otimes \overline{k})\) for some \(V\)?

(2) Which extensions of the field \(k\) correspond with kernels of such representations?

Shioda (see [Shi2] and [Shi3]) considered Galois representations and algebraic equations arising from Mordell-Weil lattices. Let \(E = E_\lambda\) be the elliptic curve, defined by one of the following equations, over \(K_0 = k_0(t)\) where \(k_0 = \mathbb{Q}(\lambda) = \mathbb{Q}(p_i, q_j).

\[(E_8)\quad y^2 = x^3 + x \left( \sum_{i=0}^{3} p_it^i \right) + \left( \sum_{i=0}^{3} q_it^i + t^5 \right)\]

\[\lambda = (p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3) \in \mathbb{A}^8\]

\[(E_7)\quad y^2 = x^3 + x(p_0 + p_1t + t^3) + \left( \sum_{i=0}^{4} q_it^i \right)\]

\[\lambda = (p_0, p_1, q_0, q_1, q_2, q_3, q_4) \in \mathbb{A}^7\]

\[(E_6)\quad y^2 = x^3 + x \left( \sum_{i=0}^{2} p_it^i \right) + \left( \sum_{i=0}^{2} q_it^i + t^4 \right)\]

\[\lambda = (p_0, p_1, p_2, q_0, q_1, q_2) \in \mathbb{A}^6\]

Let \(f : S_\lambda \to \mathbb{P}^1\) be the associated elliptic surface. The fibre \(f^{-1}(\infty)\) is an additive singular fibre of type II, III or IV according to the case \((E_r)\) for \(r = 8, 7\) or 6. Assume that \(\lambda\) satisfies the condition:

every fibre of \(f\) over \(t \neq \infty\) is irreducible.
Then the Mordell-Weil lattice $E(k(t))$ is isomorphic to $E_r^*$, the dual lattice of $E_r$, where $k$ is the algebraic closure of $k_0$, and we get the Galois representation

$$\varrho = \varrho_{\lambda} : \text{Gal}(k/k_0) \to \text{Aut}(E_r^*) = \text{Aut}(E_r).$$

Recall that $\text{Aut}(E_r) = W(E_r)$ for $r = 8$ or 7, and $W(E_r)\{\pm 1\}$ for $r = 6$, where $W(E_r)$ denotes the Weyl group of type $E_r$. In any case, we have $\text{Im}(\varrho_{\lambda}) \subset W(E_r)$.

We assume that $\lambda$ is generic, i.e. $p_i$, $q_j$ are algebraically independent over $\mathbb{Q}$. Then the above condition holds.

**Theorem 4.6 (Shioda).** Let $\lambda$ be generic over $\mathbb{Q}$. Then

(i) the image of the Galois representation $\varrho_{\lambda}$ is the full Weyl group $W(E_r)$: $\text{Im}(\varrho_{\lambda}) = W(E_r)$.

(ii) Let $K_{\lambda}/k_0$ be the Galois extension corresponding to $\text{Ker}(\varrho_{\lambda})$, and let $\{P_1, \cdots, P_r\}$ be a basis of $E(k(t)) \simeq E_r^*$ consisting of minimal vectors. Further let $u_i = sp_i^\infty(P_i) \in K_{\lambda} \subset k$. Then $u_1, \cdots, u_r$ are algebraically independent over $\mathbb{Q}$, and we have

$$K_{\lambda} = k_0(u_1, \cdots, u_r) = \mathbb{Q}(u_1, \cdots, u_r),$$

$$\text{Gal}(\mathbb{Q}(u_1, \cdots, u_r)/\mathbb{Q}(p_i, q_j)) = W(E_r).$$

(iii) $W(E_r)$ acts on the vector space $\mathbb{Q}u_1 \oplus \cdots \oplus \mathbb{Q}u_r$ and hence on the polynomial ring $\mathbb{Q}[u_1, \cdots, u_r]$, and the ring of the invariants is:

$$\mathbb{Q}[u_1, \cdots, u_r]^{W(E_r)} = \mathbb{Q}[p_i, q_j].$$

In particular, $p_i$ and $q_j$ form the fundamental invariants of the Weyl group $W(E_r)$ and we can explicitly write

$$p_i \text{ or } q_j = J_d(u_1, \cdots, u_r),$$

where $J_d$ denotes a $W(E_r)$-invariant of degree $d$, $d \in \{2, 5, 6, 8, 9, 12\}$, $\{2, 6, 8, 10, 12, 14, 18\}$ or $\{2, 8, 12, 14, 18, 20, 24, 30\}$ for $r = 6, 7$ or 8.

**Theorem 4.7 (Shioda).** Every Galois extension of $\mathbb{Q}$ with Galois group $W(E_r)$ is obtained as $K_{\lambda}$ for some $\lambda \in \Lambda$. In other words, every such extension arises from the Mordell-Weil lattice of the elliptic curve $E_{\lambda}/\mathbb{Q}(t)$ for some $\lambda \in \mathbb{Q}^r$.

In the equation $(E_6)$, let $y = y' \pm t^2$ and $(x : y' : t : 1) = (X : Y : Z : W)$. Then we have a cubic surface $V = V_{\lambda}^\pm$ in $\mathbb{P}^3$:

$$Y^2W \pm 2YZ^2 = X^3 + X(p_0W^2 + p_1ZW + p_2Z^2) + q_0W^3 + q_1ZW^2 + q_2Z^2W.$$

It is smooth if and only if the above condition about $\lambda$ holds. Under this assumption, the narrow Mordell-Weil lattice $E_{\lambda}(k(t))^0 \simeq E_6$ is isomorphic to the primitive part of $NS(V)$. The 27 minimal sections of the form $P = (at + b, t^2 + dt + e)$ in the Mordell-Weil lattice $E_{\lambda}(k(t))$ are transformed into the 27 lines on $V = V_{\lambda}^+$ defined by the equation

$$X = aZ + bW, \quad Y = dZ + eW.$$
(Similarly, the 27 sections \(-P\) are mapped to the 27 lines on \(V_\lambda^{-}\).)

Therefore, Shioda answered the problems raised by Manin when the image is surjective, i.e. \(\text{Im}(\varrho) = W(E_6)\).

The Galois representations arising from Mordell-Weil lattices

\[
\varrho : G = \text{Gal}(k/k_0) \to \text{Aut}(E(k(C)), \langle , \rangle) = \text{a finite group},
\]

where \(k(C)\) is the function field of a smooth projective curve \(C\) over an algebraically closed field \(k\), are quite different from those arising from the torsion points of an elliptic curve or an abelian variety (e.g. the Tate modules), because we are dealing with points of infinite order here.

In fact, Mordell-Weil lattices \(E_6, E_7\) and \(E_8\) come from Klein singularities \(E_6, E_7\) and \(E_8\) which are intimately related to tetrahedron, octahedron and icosahedron. In contrast to Shioda’s approach, we study Galois representations and the algebraic equations arising from Hessian polyhedra. It should be pointed out that Shioda used elliptic cubic surfaces, which are not the cubic surfaces of general type. However, we will use the general type cubic surfaces of type \(F_2\).

For \((z_1, z_2, z_3) \in \mathbb{P}^2, X, Y, Z, \varphi, Q_1\) and \(Q_2\) are cubic polynomials of \(z_1, z_2\) and \(z_3\). We will study the invariants \(x, y\) and \(z\) as functions of \(X, Y, Z, \varphi, Q_1\) and \(Q_2\).

We have

\[
A(x) = x, \quad B(x) = -x, \quad B^2(x) = x.
\]

\[
C(x) = \overline{\omega}Q_1 - \omega Q_2, \quad C^2(x) = \omega Q_1 - \overline{\omega}Q_2, \quad C^3(x) = x.
\]

\[
E(x) = Y - Z, \quad E^2(x) = x.
\]

\[
EC(x) = X - Y, \quad EC^2(x) = Z - X.
\]

\[
E^2C(x) = \omega Q_1 - \overline{\omega}Q_2, \quad E^2C^2(x) = \overline{\omega}Q_1 - \omega Q_2.
\]

\[
A(y) = y, \quad B(y) = y, \quad C(y) = y.
\]

\[
E(y) = \frac{\sqrt{-3}}{3} (y + 2z), \quad E^2(y) = -y, \quad E^3(y) = -\frac{\sqrt{-3}}{3} (y + 2z), \quad E^4(y) = y.
\]

\[
A(z) = z, \quad B(z) = z, \quad C(z) = z.
\]

\[
E(z) = \frac{\sqrt{-3}}{3} (y - z), \quad E^2(z) = -z, \quad E^3(z) = -\frac{\sqrt{-3}}{3} (y - z), \quad E^4(z) = z.
\]

Put

\[
(4.1) \quad g_1 = E(x), \quad g_2 = EC(x), \quad g_3 = EC^2(x), \quad g_4 = x, \quad g_5 = C(x), \quad g_6 = C^2(x).
\]

We find that

\[
(4.2) \quad g_1 + g_2 + g_3 = 0, \quad g_4 + g_5 + g_6 = 0.
\]
(4.3) \[ g_1^3 + g_2^3 + g_3^3 = 3W_3 = 3C_9, \quad g_4^3 + g_5^3 + g_6^3 = 3(Q_1^3 - Q_2^3) = 3C_9. \]

Hence, we obtain the following variety:

\[
S : \begin{cases} 
  g_1 + g_2 + g_3 = 0, \\ 
  g_4 + g_5 + g_6 = 0, \\ 
  g_1^3 + g_2^3 + g_3^3 = g_4^3 + g_5^3 + g_6^3, 
\end{cases}
\]

which is a cubic surface in \( \mathbb{P}^5 \). Moreover, we get the following map:

\[
\mathbb{P}^2 \to \mathbb{P}^5 \\
(z_1, z_2, z_3) \mapsto (g_1, g_2, g_3, g_4, g_5, g_6)
\]

by which the projective plane \( \mathbb{CP}^2 \) is mapped biholomorphically to the cubic surface \( S \subset \mathbb{CP}^5 \).

We find that

\[
(4.5) \begin{cases} 
  E(g_1) = g_4, \\ 
  E(g_2) = g_6, \\ 
  E(g_3) = g_5, 
\end{cases} \begin{cases} 
  E(g_4) = g_1, \\ 
  E(g_5) = g_2, \\ 
  E(g_6) = g_3. 
\end{cases}
\]

\[
(4.6) \begin{cases} 
  A(g_1) = g_3, \\ 
  A(g_2) = g_1, \\ 
  A(g_3) = g_2, 
\end{cases} \begin{cases} 
  A(g_4) = g_4, \\ 
  A(g_5) = g_5, \\ 
  A(g_6) = g_6. 
\end{cases}
\]

\[
(4.7) \begin{cases} 
  B(g_1) = -g_1, \\ 
  B(g_2) = -g_3, \\ 
  B(g_3) = -g_2, 
\end{cases} \begin{cases} 
  B(g_4) = -g_4, \\ 
  B(g_5) = -g_6, \\ 
  B(g_6) = -g_5. 
\end{cases}
\]

\[
(4.8) \begin{cases} 
  C(g_1) = g_1, \\ 
  C(g_2) = g_2, \\ 
  C(g_3) = g_3, 
\end{cases} \begin{cases} 
  C(g_4) = g_5, \\ 
  C(g_5) = g_6, \\ 
  C(g_6) = g_4. 
\end{cases}
\]

Hence, the cubic surface \( S \) is invariant under the action of the subgroup generated by \( A, B, C \) and \( E \).

There are 27 lines on the cubic surface \( S \):

\[
l_1 : \quad g_1 = g_2 + g_3 = g_4 = g_5 + g_6 = 0, 
\]
where \( j \equiv 0,1,2 \pmod{3} \).

We find that

\[
\begin{align*}
l_2 : & \quad g_1 = g_2 + g_3 = g_5 = g_4 + g_6 = 0, \\
l_3 : & \quad g_1 = g_2 + g_3 = g_6 = g_4 + g_5 = 0, \\
l_4 : & \quad g_2 = g_1 + g_3 = g_4 = g_5 + g_6 = 0, \\
l_5 : & \quad g_2 = g_1 + g_3 = g_5 = g_4 + g_6 = 0, \\
l_6 : & \quad g_2 = g_1 + g_3 = g_6 = g_4 + g_5 = 0, \\
l_7 : & \quad g_3 = g_1 + g_2 = g_4 = g_5 + g_6 = 0, \\
l_8 : & \quad g_3 = g_1 + g_2 = g_5 = g_4 + g_6 = 0, \\
l_9 : & \quad g_3 = g_1 + g_2 = g_6 = g_4 + g_5 = 0,
\end{align*}
\]

and

\[
\begin{align*}
l_{j,1} : & \quad g_1 = \omega^j g_4, \quad g_2 = \omega^j g_5, \quad g_3 = \omega^j g_6, \quad g_1 + g_2 + g_3 = 0, \\
l_{j,2} : & \quad g_1 = \omega^j g_4, \quad g_2 = \omega^j g_6, \quad g_3 = \omega^j g_5, \quad g_1 + g_2 + g_3 = 0, \\
l_{j,3} : & \quad g_1 = \omega^j g_5, \quad g_2 = \omega^j g_4, \quad g_3 = \omega^j g_6, \quad g_1 + g_2 + g_3 = 0, \\
l_{j,4} : & \quad g_1 = \omega^j g_5, \quad g_2 = \omega^j g_6, \quad g_3 = \omega^j g_4, \quad g_1 + g_2 + g_3 = 0, \\
l_{j,5} : & \quad g_1 = \omega^j g_6, \quad g_2 = \omega^j g_4, \quad g_3 = \omega^j g_5, \quad g_1 + g_2 + g_3 = 0, \\
l_{j,6} : & \quad g_1 = \omega^j g_6, \quad g_2 = \omega^j g_5, \quad g_3 = \omega^j g_4, \quad g_1 + g_2 + g_3 = 0,
\end{align*}
\]

where \( j \equiv 0,1,2 \pmod{3} \).

We find that

\[
\begin{align*}
E(l_1) = l_1, & \quad A(l_1) = l_7, \quad B(l_1) = l_1, \quad C(l_1) = l_2, \\
E(l_2) = l_4, & \quad A(l_2) = l_8, \quad B(l_2) = l_3, \quad C(l_2) = l_3, \\
E(l_3) = l_7, & \quad A(l_3) = l_9, \quad B(l_3) = l_2, \quad C(l_3) = l_3, \\
E(l_4) = l_3, & \quad A(l_4) = l_1, \quad B(l_4) = l_7, \quad C(l_4) = l_5, \\
E(l_5) = l_6, & \quad A(l_5) = l_2, \quad B(l_5) = l_9, \quad C(l_5) = l_6, \\
E(l_6) = l_9, & \quad A(l_6) = l_3, \quad B(l_6) = l_8, \quad C(l_6) = l_4, \\
E(l_7) = l_2, & \quad A(l_7) = l_4, \quad B(l_7) = l_4, \quad C(l_7) = l_8, \\
E(l_8) = l_5, & \quad A(l_8) = l_5, \quad B(l_8) = l_6, \quad C(l_8) = l_9, \\
E(l_9) = l_8, & \quad A(l_9) = l_6, \quad B(l_9) = l_5, \quad C(l_9) = l_7, \\
E(l_{j,1}) = l_{2j,2}, & \quad A(l_{j,1}) = l_{j,4}, \quad B(l_{j,1}) = l_{j,1}, \quad C(l_{j,1}) = l_{j,4}.
\end{align*}
\]
\[ E(l_{j,2}) = l_{2j,1}, \quad A(l_{j,2}) = l_{j,6}, \quad B(l_{j,2}) = l_{j,2}, \quad C(l_{j,2}) = l_{j,3}. \]
\[ E(l_{j,3}) = l_{2j,5}, \quad A(l_{j,3}) = l_{j,2}, \quad B(l_{j,3}) = l_{j,6}, \quad C(l_{j,3}) = l_{j,6}. \]
\[ E(l_{j,4}) = l_{2j,3}, \quad A(l_{j,4}) = l_{j,5}, \quad B(l_{j,4}) = l_{j,5}, \quad C(l_{j,4}) = l_{j,5}. \]
\[ E(l_{j,5}) = l_{2j,6}, \quad A(l_{j,5}) = l_{j,1}, \quad B(l_{j,5}) = l_{j,4}, \quad C(l_{j,5}) = l_{j,1}. \]
\[ E(l_{j,6}) = l_{2j,4}, \quad A(l_{j,6}) = l_{j,3}, \quad B(l_{j,6}) = l_{j,3}, \quad C(l_{j,6}) = l_{j,2}. \]

It is well-known that the Segre cubic threefold is given by (see [Hu])
\[
S_3 : \begin{cases} 
  x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0, \\
  x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.
\end{cases}
\]

The hyperplane sections \( \{ x_i = 0 \} \) of \( S_3 \) are Clebsch diagonal cubic surfaces with equation
\[
S_3 : \begin{cases} 
  x_0 + x_1 + x_2 + x_3 + x_4 = 0, \\
  x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.
\end{cases}
\]

The relation between \( S_3 \) and the icosahedral group was studied by Hirzebruch. It turns out that \( S_3 \) is \( A_5 \)-equivariantly birational to the Hilbert modular surface for \( \mathcal{O}_K \) of level \( \sqrt{5} \) where \( K = \mathbb{Q}(\sqrt{5}) \) (see [Hu]).

We find that the hyperplane sections \( \{ x_i + x_j + x_k = 0 \} \) of \( S_3 \) are our cubic surfaces with equation
\[
S : \begin{cases} 
  g_1 + g_2 + g_3 = 0, \\
  (-g_4) + (-g_5) + (-g_6) = 0, \\
  g_1^3 + g_2^3 + g_3^3 + (-g_4)^3 + (-g_5)^3 + (-g_6)^3 = 0.
\end{cases}
\]

Our cubic surface can be expressed as
\[
(4.9) \quad S : \quad x_1^2 x_2 + x_1 x_2^2 + x_3^2 x_4 + x_3 x_4^2 = 0.
\]

The Hessian variety of \( S \) (up to a constant) is given by
\[
(4.10) \quad (x_1^2 + x_1 x_2 + x_2^2)(x_3^2 + x_3 x_4 + x_4^2) = 0,
\]

which are four planes.

Let us consider the group of permutations (see [Hu]), \( \text{Aut} \) of the 27 lines, by which we mean the permutations of the lines preserving the intersection behavior of the lines. For this it is useful to consider the famous double sixes and the notation for the 27 lines introduced by Schlafli. A double six is an array
\[
N = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}
\]
of 12 of the 27 lines with the property that two of these 12 meet if and only if they are in different rows and columns. (This notation distinguishes this particular set of 12, although any such double six is equivalent to it under Aut(\(\mathcal{L}\)).) The other lines are given by the \(\binom{6}{2} = 15\) \(c_{ij} = a_i b_j \cap a_j b_i\), where \(a_i b_j\) denotes the tritangent spanned by those two lines. There are 36 double sixes, namely the \(N\) above, 15 \(N_{ij}\) and 20 \(N_{ijk}\):

\[
N_{ij} = \begin{bmatrix}
a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} \\
a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in}
\end{bmatrix},
\]

\[
N_{ijk} = \begin{bmatrix}
a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} \\
c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n
\end{bmatrix}.
\]

Since a double six describes the intersection behavior of the lines, we see that \(S_6\) (the symmetric group on six letters) acts by permutations on a double six and a \(\mathbb{Z}_2\) acts by exchanging rows. Since there are 36 double sixes, we have \(|\text{Aut}(\mathcal{L})| = |S_6| \cdot 2 \cdot 36 = 51840\). In fact \(\text{Aut}(\mathcal{L})\) is the Weyl group \(W(E_6)\), and the 36 double sixes correspond to the positive roots. The 27 lines correspond to the 27 fundamental weights of \(E_6\), and many other sets of objects (lines, tritangents, etc.) correspond to natural sets of objects (roots, weights, etc.) of \(E_6\).

According to [Seg], a nonsingular cubic surface can only be of one of the five types: \(F_1\), \(F_2\), \(F_3\), \(F_4\) and \(F_5\). Our cubic surface is of \(F_2\) which has 15 real lines and 12 complex lines of the 2nd kind.

The 15 real lines of a cubic surface \(F_2\) are of two different sorts: 6 of them, \(l_{0,1}, \ldots, l_{0,6}\), constituting 2 complementary triplets, are elliptic; the 9 others \(l_1, \ldots, l_9\) are hyperbolic of the 1st kind and constitute a Steiner set. The remaining lines \(l_{1,1}, \ldots, l_{1,6}, l_{2,1}, \ldots, l_{2,6}\) of \(F_2\) form a self-conjugate double-six, which we call ‘of the 9th kind’, in which each pair of corresponding lines is a pair of conjugate complex lines of the 2nd kind; hence the \(\sigma\)-transformation inherent to it changes each line of \(F_2\) in its conjugate.

\(F_2\) has 15 other self-conjugate double-sixes, which are the 15 double-sixes permutable with the one just considered. Each sextuplet of such a double-six is consequently self-conjugate, and consists of 4 real lines and 2 conjugate complex lines of the 2nd kind; but, while in 9 of these 15 double-sixes (which we call ‘of the 4th kind’) the 4 real lines of each sextuplet are 2 elliptic and 2 hyperbolic, in the 6 others (which we call ‘of the 5th kind’, and which constitute 2 permutable triads of associate double-sixes) the 4 real lines of each sextuplet are 1 elliptic and 3 hyperbolic. If \(\omega\) is any one of the 15 double-sixes considered, its 8 real lines are the only real lines of \(F_2\) skew to a well-determined real line of \(F_2\) which is hyperbolic or elliptic according as \(\omega\) is of the 4th or 5th kind.

In fact, the Clebsch diagonal cubic surface is of \(F_1\) which has its 27 lines all real: 12 of them are elliptic, and 15 are hyperbolic of the 1st kind.

Let \(\Gamma_i\) be the group of the lines of a real cubic surface \(F_i\) and \(\mathfrak{S}\) be the group of the 27 lines of a cubic surface.
Segre [Seg] proved that the group $\Gamma_1$ inherent to a cubic surface $F_1$ is icosahedral, and can be defined as the group of the 60 substitutions of $\mathfrak{S}$ which transform into themselves each of the 2 sextuplets of elliptic lines and also the system of 5 principal tritangent planes of $F_1$, inducing among these planes a substitution of even class.

The group $\Gamma_2$ inherent to a cubic surface $F_2$ is of order 36; it is simply isomorphic with the direct product of 2 symmetric groups of degree 3, its transformations being uniquely defined by the property of performing an arbitrary substitution among the 3 elliptic left-handed lines of $F_2$ and an arbitrary substitution among the 3 elliptic right-handed lines of $F_2$. The group operates transitively among the 9 pairs of complementary triplets of the 1st kind of $F_2$, any 2 such pairs being transformed one into the other by 4 transformations of $\Gamma_2$; in particular, the subgroup of $\Gamma_2$ transforming into itself one of these pairs is trirectangular, and coincides with the subgroup of $\Gamma_2$ which leaves unchanged each of the 2 elliptic lines of the pair.

In our case, we find that the set of 15 real lines, i.e., $\{l_1, \cdots, l_9\}$ and $\{l_{0,1}, \cdots, l_{0,6}\}$ are invariant under the actions of $A$, $B$, $C$ and $E$, respectively. For the complex lines, we have

\[
E(l_{1,1}) = l_{2,2}, E(l_{1,2}) = l_{2,1}, E(l_{1,3}) = l_{2,5}, E(l_{1,4}) = l_{2,3}, E(l_{1,5}) = l_{2,6}, E(l_{1,6}) = l_{2,4},
\]

\[
E(l_{2,1}) = l_{1,2}, E(l_{2,2}) = l_{1,1}, E(l_{2,3}) = l_{1,5}, E(l_{2,4}) = l_{1,3}, E(l_{2,5}) = l_{1,6}, E(l_{2,6}) = l_{1,4}.
\]

We find that

\[
\begin{align*}
 l_{1,1} \cap l_{1,2} &= \overline{l_{2,1} \cap l_{2,2}} = (-2\omega, \omega, \omega, -2, 1, 1), \\
 l_{1,1} \cap l_{1,3} &= \overline{l_{2,1} \cap l_{2,3}} = (\omega, \omega, -2\omega, 1, 1, -2), \\
 l_{1,1} \cap l_{1,4} &= \overline{l_{2,1} \cap l_{2,4}} = \emptyset, \\
 l_{1,1} \cap l_{1,5} &= \overline{l_{2,1} \cap l_{2,5}} = \emptyset, \\
 l_{1,1} \cap l_{1,6} &= \overline{l_{2,1} \cap l_{2,6}} = (\omega, -2\omega, \omega, 1, -2, 1), \\
 l_{1,2} \cap l_{1,3} &= \overline{l_{2,2} \cap l_{2,3}} = \emptyset, \\
 l_{1,2} \cap l_{1,4} &= \overline{l_{2,2} \cap l_{2,4}} = (\omega, -2\omega, \omega, 1, 1, -2), \\
 l_{1,2} \cap l_{1,5} &= \overline{l_{2,2} \cap l_{2,5}} = (\omega, \omega, -2\omega, 1, -2, 1), \\
 l_{1,2} \cap l_{1,6} &= \overline{l_{2,2} \cap l_{2,6}} = \emptyset, \\
 l_{1,3} \cap l_{1,4} &= \overline{l_{2,3} \cap l_{2,4}} = (-2\omega, \omega, \omega, 1, -2, 1), \\
 l_{1,3} \cap l_{1,5} &= \overline{l_{2,3} \cap l_{2,5}} = (\omega, -2\omega, \omega, -2, 1, 1), \\
 l_{1,3} \cap l_{1,6} &= \overline{l_{2,3} \cap l_{2,6}} = \emptyset, \\
 l_{1,4} \cap l_{1,5} &= \overline{l_{2,4} \cap l_{2,5}} = \emptyset, \\
 l_{1,4} \cap l_{1,6} &= \overline{l_{2,4} \cap l_{2,6}} = (\omega, \omega, -2\omega, -2, 1, 1).
\end{align*}
\]
\[ l_{1,5} \cap l_{1,6} = l_{2,5} \cap l_{2,6} = (-2, \omega, \omega, 1, 1, -2). \]
\[
\begin{align*}
&\begin{cases}
  l_{1,1} \cap l_{2,1} = 0, \\
  l_{1,1} \cap l_{2,2} = 0, \\
  l_{1,1} \cap l_{2,3} = 0, \\
  l_{1,1} \cap l_{2,4} = (1, \overline{\omega}, \omega, \overline{\omega}, \omega, 1), \\
  l_{1,1} \cap l_{2,5} = (\overline{\omega}, 1, \omega, \overline{\omega}, 1), \\
  l_{1,1} \cap l_{2,6} = 0.
\end{cases} \\
&\begin{cases}
  l_{1,2} \cap l_{2,1} = 0, \\
  l_{1,2} \cap l_{2,2} = 0, \\
  l_{1,2} \cap l_{2,3} = (1, \omega, \overline{\omega}, \overline{\omega}, \omega, 1), \\
  l_{1,2} \cap l_{2,4} = 0, \\
  l_{1,2} \cap l_{2,5} = 0, \\
  l_{1,2} \cap l_{2,6} = (\overline{\omega}, \omega, 1, \omega, \overline{\omega}, 1).
\end{cases} \\
&\begin{cases}
  l_{1,3} \cap l_{2,1} = 0, \\
  l_{1,3} \cap l_{2,2} = (1, \overline{\omega}, \omega, \omega, \overline{\omega}, 1), \\
  l_{1,3} \cap l_{2,3} = 0, \\
  l_{1,3} \cap l_{2,4} = 0, \\
  l_{1,3} \cap l_{2,5} = 0, \\
  l_{1,3} \cap l_{2,6} = (\overline{\omega}, 1, \omega, \overline{\omega}, \omega, 1).
\end{cases} \\
&\begin{cases}
  l_{1,4} \cap l_{2,1} = (1, \omega, \overline{\omega}, \omega, \overline{\omega}, 1), \\
  l_{1,4} \cap l_{2,2} = 0, \\
  l_{1,4} \cap l_{2,3} = 0, \\
  l_{1,4} \cap l_{2,4} = 0, \\
  l_{1,4} \cap l_{2,5} = (\overline{\omega}, \omega, 1, \overline{\omega}, \omega, 1), \\
  l_{1,4} \cap l_{2,6} = 0.
\end{cases} \\
&\begin{cases}
  l_{1,5} \cap l_{2,1} = (\omega, 1, \overline{\omega}, \omega, \overline{\omega}, 1), \\
  l_{1,5} \cap l_{2,2} = 0, \\
  l_{1,5} \cap l_{2,3} = 0, \\
  l_{1,5} \cap l_{2,4} = (\omega, \overline{\omega}, 1, \omega, \overline{\omega}, 1), \\
  l_{1,5} \cap l_{2,5} = 0, \\
  l_{1,5} \cap l_{2,6} = 0.
\end{cases}
\end{align*}
\]
The double six is given by

\[
N = \left( \begin{array}{cccccc}
    a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
    b_1 & b_2 & b_3 & b_4 & b_5 & b_6
\end{array} \right) = \left( \begin{array}{cccccc}
    l_{1,1} & l_{1,2} & l_{1,3} & l_{1,4} & l_{1,5} & l_{1,6} \\
    l_{2,1} & l_{2,2} & l_{2,3} & l_{2,4} & l_{2,5} & l_{2,6}
\end{array} \right).
\]

We find that

\[
\begin{align*}
l_{1,1} \cap l_{1,2} &= l_{2,1} \cap l_{2,2} = (-2\omega,\omega,\omega,-2,1,1), \\
l_1 \cap l_{1,1} &= l_1 \cap l_{2,1} = (0,-\omega,\omega,0,-1,1), \\
l_1 \cap l_{1,2} &= l_1 \cap l_{2,2} = (0,\omega,-\omega,0,-1,1).
\end{align*}
\]

Note that \( \overline{l_1} = l_1 \). Hence, \( l_1 \cap l_{1,1} \in l_1, l_1 \cap l_{1,2} \in l_1, l_1 \cap l_{1,3} \in l_1, l_1 \cap l_{1,4} \in l_1, l_1 \cap l_{1,5} \in l_1, l_1 \cap l_{1,6} \in l_1 \). Thus, \( c_{12} = l_1 \).

By the same method as above, we obtain that

\[
\begin{align*}
c_{12} &= l_1, & c_{13} &= l_9, & c_{14} &= l_{0,5}, & c_{15} &= l_{0,4}, & c_{16} &= l_5. \\
c_{23} &= l_{0,6}, & c_{24} &= l_6, & c_{25} &= l_8, & c_{26} &= l_{0,3}. \\
c_{34} &= l_2, & c_{35} &= l_4, & c_{36} &= l_{0,2}. \\
c_{45} &= l_{0,1}, & c_{46} &= l_7, & c_{56} &= l_3.
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
E(a_1) &= a_2, & A(a_1) &= a_4, & B(a_1) &= a_1, & C(a_1) &= a_4. \\
E(a_2) &= a_1, & A(a_2) &= a_6, & B(a_2) &= a_2, & C(a_2) &= a_3. \\
E(a_3) &= a_5, & A(a_3) &= a_2, & B(a_3) &= a_6, & C(a_3) &= a_6. \\
E(a_4) &= a_3, & A(a_4) &= a_5, & B(a_4) &= a_5, & C(a_4) &= a_5. \\
E(a_5) &= a_6, & A(a_5) &= a_1, & B(a_5) &= a_4, & C(a_5) &= a_1.
\end{align*}
\]
\[ E(a_6) = a_4, \quad A(a_6) = a_3, \quad B(a_6) = a_2. \]
\[ E(b_1) = b_2, \quad A(b_1) = b_4, \quad B(b_1) = b_1, \quad C(b_1) = b_4. \]
\[ E(b_2) = b_1, \quad A(b_2) = b_6, \quad B(b_2) = b_2, \quad C(b_2) = b_3. \]
\[ E(b_3) = b_5, \quad A(b_3) = b_2, \quad B(b_3) = b_6, \quad C(b_3) = b_6. \]
\[ E(b_4) = b_3, \quad A(b_4) = b_5, \quad B(b_4) = b_5, \quad C(b_4) = b_5. \]
\[ E(b_5) = b_6, \quad A(b_5) = b_1, \quad B(b_5) = b_4, \quad C(b_5) = b_1. \]
\[ E(b_6) = b_4, \quad A(b_6) = b_3, \quad B(b_6) = b_3, \quad C(b_6) = b_2. \]
\[ E(c_{12}) = c_{12}, \quad A(c_{12}) = c_{46}, \quad B(c_{12}) = c_{12}, \quad C(c_{12}) = c_{44}. \]
\[ E(c_{13}) = c_{25}, \quad A(c_{13}) = c_{24}, \quad B(c_{13}) = c_{16}, \quad C(c_{13}) = c_{46}. \]
\[ E(c_{14}) = c_{23}, \quad A(c_{14}) = c_{45}, \quad B(c_{14}) = c_{15}, \quad C(c_{14}) = c_{45}. \]
\[ E(c_{15}) = c_{26}, \quad A(c_{15}) = c_{14}, \quad B(c_{15}) = c_{14}, \quad C(c_{15}) = c_{14}. \]
\[ E(c_{16}) = c_{24}, \quad A(c_{16}) = c_{34}, \quad B(c_{16}) = c_{13}, \quad C(c_{16}) = c_{24}. \]
\[ E(c_{23}) = c_{15}, \quad A(c_{23}) = c_{26}, \quad B(c_{23}) = c_{26}, \quad C(c_{23}) = c_{36}. \]
\[ E(c_{24}) = c_{13}, \quad A(c_{24}) = c_{56}, \quad B(c_{24}) = c_{25}, \quad C(c_{24}) = c_{35}. \]
\[ E(c_{25}) = c_{16}, \quad A(c_{25}) = c_{16}, \quad B(c_{25}) = c_{24}, \quad C(c_{25}) = c_{13}. \]
\[ E(c_{26}) = c_{14}, \quad A(c_{26}) = c_{36}, \quad B(c_{26}) = c_{23}, \quad C(c_{26}) = c_{23}. \]
\[ E(c_{34}) = c_{35}, \quad A(c_{34}) = c_{25}, \quad B(c_{34}) = c_{56}, \quad C(c_{34}) = c_{56}. \]
\[ E(c_{35}) = c_{56}, \quad A(c_{35}) = c_{12}, \quad B(c_{35}) = c_{46}, \quad C(c_{35}) = c_{16}. \]
\[ E(c_{36}) = c_{45}, \quad A(c_{36}) = c_{23}, \quad B(c_{36}) = c_{36}, \quad C(c_{36}) = c_{26}. \]
\[ E(c_{45}) = c_{36}, \quad A(c_{45}) = c_{15}, \quad B(c_{45}) = c_{45}, \quad C(c_{45}) = c_{15}. \]
\[ E(c_{46}) = c_{34}, \quad A(c_{46}) = c_{35}, \quad B(c_{46}) = c_{35}, \quad C(c_{46}) = c_{25}. \]
\[ E(c_{56}) = c_{46}, \quad A(c_{56}) = c_{13}, \quad B(c_{56}) = c_{34}, \quad C(c_{56}) = c_{12}. \]

For
\[
N = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6
\end{pmatrix},
\]
we have
\[
E(N) = \begin{pmatrix}
a_2 & a_1 & a_5 & a_3 & a_6 & a_4 \\
b_2 & b_1 & b_5 & b_3 & b_6 & b_4
\end{pmatrix}.
\]
\[ A(N) = \begin{pmatrix} a_4 & a_6 & a_2 & a_5 & a_1 & a_3 \\ b_4 & b_6 & b_2 & b_5 & b_1 & b_3 \end{pmatrix} \]

\[ B(N) = \begin{pmatrix} a_1 & a_2 & a_6 & a_5 & a_4 & a_3 \\ b_1 & b_2 & b_6 & b_5 & b_4 & b_3 \end{pmatrix} \]

\[ C(N) = \begin{pmatrix} a_4 & a_3 & a_6 & a_5 & a_1 & a_2 \\ b_4 & b_3 & b_6 & b_5 & b_1 & b_2 \end{pmatrix} \]

For  
\[ N_{12} = \begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix} \]

we have  
\[ E(N_{12}) = \begin{pmatrix} a_2 & b_2 & c_{15} & c_{13} & c_{16} & c_{14} \\ a_1 & b_1 & c_{25} & c_{23} & c_{26} & c_{24} \end{pmatrix} \]

\[ A(N_{12}) = \begin{pmatrix} a_4 & b_4 & c_{26} & c_{56} & c_{16} & c_{36} \\ a_6 & b_6 & c_{24} & c_{54} & c_{14} & c_{34} \end{pmatrix} \]

\[ B(N_{12}) = \begin{pmatrix} a_1 & b_1 & c_{26} & c_{25} & c_{24} & c_{23} \\ a_2 & b_2 & c_{16} & c_{15} & c_{14} & c_{13} \end{pmatrix} \]

\[ C(N_{12}) = \begin{pmatrix} a_4 & b_4 & c_{36} & c_{35} & c_{31} & c_{32} \\ a_3 & b_3 & c_{46} & c_{45} & c_{41} & c_{42} \end{pmatrix} \]

For  
\[ N_{123} = \begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix} \]

we have  
\[ E(N_{123}) = \begin{pmatrix} a_2 & a_1 & a_5 & c_{46} & c_{34} & c_{36} \\ c_{15} & c_{25} & c_{12} & b_3 & b_6 & b_4 \end{pmatrix} \]

\[ A(N_{123}) = \begin{pmatrix} a_4 & a_6 & a_2 & c_{13} & c_{35} & c_{15} \\ c_{26} & c_{24} & c_{46} & b_5 & b_1 & b_3 \end{pmatrix} \]

\[ B(N_{123}) = \begin{pmatrix} a_1 & a_2 & a_6 & c_{34} & c_{35} & c_{45} \\ c_{26} & c_{16} & c_{12} & b_5 & b_4 & b_3 \end{pmatrix} \]

\[ C(N_{123}) = \begin{pmatrix} a_4 & a_3 & a_6 & c_{12} & c_{25} & c_{15} \\ c_{36} & c_{46} & c_{34} & b_5 & b_1 & b_2 \end{pmatrix} \]

The other \( N_{ij} \) and \( N_{ijk} \) can be calculated similarly. We find that the action of the group \( H \) generated by \( A, B, C \) and \( E \) on the 27 lines gives the permutations of the lines which preserve the intersection behavior of the lines. Hence, we conclude that the group \( H \) is a subgroup of \( \text{Aut}(\mathcal{L}) = W(E_6) \).
Theorem 4.8 (Main Theorem 4). The group $H$ generated by $A$, $B$, $C$ and $E$ is a subgroup of $\text{Aut}(L) = W(E_6)$.

For the complex conjugate $\sigma_c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have

$$\sigma_c(N) = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix}.$$  
$$\sigma_c(N_{12}) = \begin{pmatrix} b_1 & a_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ b_2 & a_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}.$$  
$$\sigma_c(N_{123}) = \begin{pmatrix} b_1 & b_2 & b_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & a_4 & a_5 & a_6 \end{pmatrix}.$$  

This leads us to study the representation:

(4.16) $\rho : G_{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(L)$,

i.e.,

(4.17) $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow W(E_6)$.

Here,

(4.18) $\text{Im}(\rho) = H = \langle A, B, C, E \rangle$, $\ker(\rho) = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{K})$.

We have

(4.19) $\text{Im}(\rho) \cong \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\ker(\rho) = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\text{Gal}(\overline{\mathbb{Q}}/\mathbb{K}) \cong \text{Gal}(\mathbb{K}/\mathbb{Q})$.

Hence,

(4.20) $\text{Gal}(\mathbb{K}/\mathbb{Q}) \cong H$.

The (geometrical, analytical and algebraical) properties of the Hessian group $G = \langle A, B, C, D, E \rangle$ have been studied in our previous paper [Y3]. Note that $A^*A = B^*B = C^*C = I$, but $E^*E \neq I$. We have $\text{Tr}(A) = 3$, $\text{det}(A) = 1$. $\text{Tr}(B) = 2$, $\text{det}(B) = 1$. $\text{Tr}(C) = 3$, $\text{det}(C) = 1$. $\text{Tr}(E) = 0$.

$$\text{det}(E) = \frac{1}{(\sqrt{-3})^{18}} \text{det} \begin{pmatrix} 1 & 1 & 1 & 9 & 3 & 3 \\ 1 & 1 & 1 & 9 & 3\omega & 3\omega \\ 1 & 1 & 1 & 9 & 3\overline{\omega} & 3\overline{\omega} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 3\omega & 3\omega & 0 & 0 & 0 \\ 3 & 3\overline{\omega} & 3\overline{\omega} & 0 & 0 & 0 \end{pmatrix} = -1.$$
Note that

\[
E^2 = \begin{pmatrix}
-1 & 0 & -1 \\
-1 & 0 & -1 \\
-1 & 0 & -1
\end{pmatrix}.
\]

In fact, the elements of \( H \) have integer determinant and trace.

Now let us recall the definition of Artin \( L \)-functions (see [Ar1], [G]). Suppose \( K \) is a number field, and \( E \) is a finite Galois extension of \( K \) with Galois group \( G = \text{Gal}(E/K) \). By a representation of \( G \) we understand a homomorphism \( \sigma \) of \( G \) into \( GL(V) \), the group of invertible linear transformations of a complex vector space of dimension \( n \). Given a finite place \( p \) of \( K \), we say a prime \( \mathfrak{P} \) of \( \mathcal{O}_E \) lies over \( p \) if \( \mathfrak{P} \) appears in the factorization of \( p \mathcal{O}_E \) into prime ideals of \( \mathcal{O}_E \). Given such a pair \( \mathfrak{P}/p \), we have:

1. The decomposition group \( D_{\mathfrak{P}} = \{ g \in G : g(\mathfrak{P}) = \mathfrak{P} \} \);
2. The inertia subgroup \( I_{\mathfrak{P}} = \{ g \in D_{\mathfrak{P}} : g(x) = x \pmod{\mathfrak{P}} \) for all \( x \in \mathcal{O}_E \} \);
3. The Frobenius automorphism \( \text{Fr}_{\mathfrak{P}} \) generating the cyclic group \( D_{\mathfrak{P}}/I_{\mathfrak{P}} \) which is isomorphic to \( \text{Gal}(\mathcal{O}_E/\mathfrak{P} : \mathcal{O}_K/p) \).

Let \( V_{\mathfrak{P}} \) denote the subspace of \( V \) formed by vectors invariant by \( \sigma(I_{\mathfrak{P}}) \). Then \( \sigma(\text{Fr}_{\mathfrak{P}}) \) is defined on \( V_{\mathfrak{P}} \), and the Euler factor

\[
L_p(\sigma, s) = \frac{1}{\det(I - \sigma(\text{Fr}_{\mathfrak{P}}) \cdot Np^{-s}|_{V_{\mathfrak{P}}})^{-1}
\]

depends only on \( p \), not \( \mathfrak{P} \). The Artin \( L \)-function attached to \( \sigma \) is given by

\[
L(\sigma, s) = L_{E/K}(\sigma, s) = \prod_p L_p(\sigma, s)
\]

which is convergent for \( \text{Re}(s) > 1 \).

In [Ar2], Artin studied representations with a finite image \( \text{Im}(\rho) \) (see also [MP]). In this case the representation \( \rho \) is always rational for some number field and is semisimple. Hence \( \rho \) is uniquely determined (up to equivalence) by its character \( \chi_\rho \). If \( \rho \) is an arbitrary rational representation then the function \( L(\rho, s) \) is uniquely defined by its character \( \chi_\rho \). In view of this fact one often uses the notation \( L(\chi_\rho, s) = L(\rho, s) \).

For arbitrary \( n \), there is the following remarkable Reciprocity Conjecture:

**Conjecture 4.9 (Langlands).** Suppose \( E \) is a finite Galois extension of \( F \) with Galois group \( G = \text{Gal}(E/F) \), and \( \sigma : G \to GL_n(\mathbb{C}) \) is an irreducible representation of \( G \). Then there exists an automorphic cuspidal representation \( \pi_\sigma \) on \( GL_n \) over \( F \) such that \( L(s, \pi_\sigma) = L(s, \sigma) \).

The truth of this conjecture implies the truth of Artin conjecture on the entirety of the Artin \( L \)-function \( L(s, \sigma) \).
Conjecture 4.10 (Main Conjecture). Let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W(E_6)$ be a Galois representation. When the image of $\rho$: $\text{Im}(\rho) \cong H \leq W(E_6)$, if $\rho$ is odd, i.e., $\det(\rho(c)) = -1$, where $c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the complex conjugate, then there exists a Picard modular form $f$ of weight one such that

$$L(\rho, s) = L(f, s)$$

up to finitely many Euler factors, where $L(\rho, s)$ is the Artin $L$-function associated to the Galois representation $\rho$ and $L(f, s)$ is the automorphic $L$-function associated to the Picard modular form $f$. If $\rho$ is even, i.e., $\det(\rho(c)) = 1$, then there exists an automorphic form $f$ associated to $U(2, 1)$: $\Delta f = s(s - 2)f$ with $s = 1$ and (see [Y1])

$$\Delta = (z_1 + \overline{z}_1 - z_2\overline{z}_2) \left[ (z_1 + \overline{z}_1) \frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2}{\partial z_2 \partial \overline{z}_2} + z_2 \frac{\partial^2}{\partial z_1 \partial z_2} + \overline{z}_2 \frac{\partial^2}{\partial z_1 \partial \overline{z}_2} \right],$$

$$(z_1, z_2) \in \mathcal{S}_2 = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 + \overline{z}_1 - z_2\overline{z}_2 > 0\},$$

such that

$$L(\rho, s) = L(f, s)$$

up to finitely many Euler factors, where $L(\rho, s)$ is the Artin $L$-function associated to the Galois representation $\rho$ and $L(f, s)$ is the automorphic $L$-function associated to the automorphic form $f$.

Here, we find that three topics: $L$-functions, Galois representations and automorphic forms or, equivalently, representations, which figure prominently in the modern higher arithmetic, appear naturally in our conjecture. The $L$-functions are attached to both the Galois representations and the automorphic representations and are the link that joins them. Our conjecture, like Langlands reciprocity law and Artin’s abelian reciprocity law, gives a correspondence independent of any particular geometric construction.

(1) tetrahedron, $A_4$ (Langlands) (see [L2]), elliptic modular forms of weight 1.
(2) octahedron, $S_4$ (Tunnell), elliptic modular forms of weight 1.
(3) icosahedron, $A_5$ (Artin conjecture), elliptic modular forms of weight 1.
(4) Hessian polyhedra, $H$ (our conjecture), Picard modular forms of weight 1.

Note that the group $W(E_6)$ is combinatorially defined, while the subgroup $H$ is defined on algebraic variety. Thus we give a connection between combinatoric and algebraic geometry. Furthermore, the Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H$ comes from number theory, the Picard modular forms come from analysis and representation theory of Lie groups. Here, Galois symmetry and geometric symmetry meet together. Therefore, our theorem and conjecture gives a connection which involves combinatoric, algebraic geometry, number theory, analysis and representation theory.

5. Hessian polyhedra and the arithmetic of rigid Calabi-Yau threefolds
A complete system of invariants for the Hessian groups (the corresponding geometric objects are Hessian polyhedra) has degrees 6, 9, 12, 12 and 18 and can be given explicitly by the following forms:

\[
\begin{align*}
C_6(z_1, z_2, z_3) &= z_1^6 + z_2^6 + z_3^6 - 10(z_1^3 z_2^3 + z_2^3 z_3^3 + z_3^3 z_1^3), \\
C_9(z_1, z_2, z_3) &= (z_1^3 - z_2^3)(z_2^3 - z_3^3)(z_3^3 - z_1^3), \\
C_{12}(z_1, z_2, z_3) &= (z_1^3 + z_2^3 + z_3^3)[(z_1^3 + z_2^3 + z_3^3)^3 + 216 z_1^3 z_2^3 z_3^3], \\
\mathcal{C}_{12}(z_1, z_2, z_3) &= z_1 z_2 z_3 [27 z_1^3 z_2^3 z_3^3 - (z_1^3 + z_2^3 + z_3^3)^3], \\
C_{18}(z_1, z_2, z_3) &= (z_1^3 + z_2^3 + z_3^3)^6 - 540 z_1^3 z_2^3 z_3^3 (z_1^3 + z_2^3 + z_3^3)^3 - 5832 z_1^6 z_2^6 z_3^6.
\end{align*}
\]

They satisfy the following relations:

\[
\begin{align*}
432 C_9^2 &= C_6^3 - 3 C_6 C_{12} + 2 C_{18}, \\
1728 \mathcal{C}_{12}^3 &= C_{18}^2 - C_{12}^3.
\end{align*}
\]

The first relation can be considered as an elliptic curve with two parameters $C_{12}$ and $C_{18}$:

\[
(5.1) \quad 432 C_9^2 = C_6^3 - 3 C_6 C_{12} + 2 C_{18}.
\]

The second one can be considered as an elliptic curve with a parameter $\mathcal{C}_{12}$:

\[
(5.2) \quad C_{18}^2 = C_{12}^3 + 1728 \mathcal{C}_{12}^3.
\]

When $C_9 = 0$, the equation (5.1) reduces to

\[
C_6^3 - 3 C_6 C_{12} + 2 C_{18} = 0.
\]

In fact, Wirtinger studied the algebraic function $z$ of two variables $x, y$ defined by the equation (see [BE])

\[
z^3 + 3xz + 2y = 0.
\]

In modern terms: the projection of the surface $X$ with this equation to the $(x, y)$-plane is the seminiversal unfolding of the 0-dimensional $A_2$-type singularity $z^3 = 0$. The discriminant curve $D \subset \mathbb{C}^2$ has the equation

\[
x^3 + y^2 = 0, \quad \text{i.e.,} \quad 1728 \mathcal{C}_{12}^3 = 0.
\]

The fundamental group of $\mathbb{C}^2 - D$ operates on the fibre over the base point by the monodromy representation. Wirtinger calculates $\pi_1(\mathbb{C}^2 - D)$ and finds a presentation with two generators and one relation $sts = tst$. In modern terms: $\pi_1$ is the braid
group on 3 strings. The monodromy representation is the canonical homomorphism of this group to the symmetric group $S_3$. In his computation of $\pi_1(\mathbb{C}^2 - D)$, Wirtinger used an idea of Heegaard. Heegaard reduced the complex geometry of an algebroid covering $(X, x) \to (\mathbb{C}^2, 0)$ with a singularity $(D, 0)$ of the discriminant to a situation of 3-dimensional topology. He considered a small 4-ball $B \subset \mathbb{C}^2$ centered at 0 with boundary $\partial B = S^3$, a 3-sphere. The intersection $L = D \cap S^3$ is a knot or link in $S^3$. In Wirtinger’s example it is the trefoil knot. This established a link between the geometry of singularities of complex surfaces and 3-dimensional topology.

Put $x = 3C_6, \ y = 108C_9$

in the elliptic curve (5.1), we have

(5.3) \quad E : \quad y^2 = x^3 - 27C_{12}x + 54C_{18}.

The $j$-invariant is given by

(5.4) \quad j = -\frac{C_{12}^3}{C_{12}^3}.

According to [Y3], put

$\varphi = z_1z_2z_3, \quad \psi = z_1^3 + z_2^3 + z_3^3, \quad \chi = z_1^3z_2^3 + z_2^3z_3^3 + z_3^3z_1^3.$

$G = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1).$

$H = \psi + 6\varphi, \quad K = \psi - 3\varphi.$

\[
\begin{align*}
C_6 &= \psi^2 - 12\chi, \\
C_{12} &= \psi(\psi^3 + 216\varphi^3), \\
C_{12} &= \varphi(27\varphi^3 - \psi^3), \\
C_{18} &= \psi^6 - 540\varphi^3\psi^3 - 5832\varphi^6.
\end{align*}
\]

**Theorem 5.1** (see [Y3], Main Theorem 4). The invariants $G$, $H$ and $K$ satisfy the following algebraic equations, which are the form-theoretic resolvents (algebraic resolvents) of $G$, $H$, $K$:

\[
\begin{align*}
4G^3 + H^2G - C_6G - 4C_9 &= 0, \\
H(H^3 + 8K^3) - 9C_{12} &= 0, \\
K(K^3 - H^3) - 27C_{12} &= 0.
\end{align*}
\]
We have
\[ C_{12} = \frac{1}{9} H(H^3 + 8K^3), \quad C_{12} = \frac{1}{27} K(K^3 - H^3). \]

Hence,
\[ j = -27 \frac{H^3(H^3 + 8K^3)^3}{K^3(K^3 - H^3)^3}. \]

Put
\[ t = \frac{H}{K}. \]

Then
\[ j = -27 \frac{t^3(t^3 + 8)^3}{(1 - t^3)^3}. \]

Note that
\[ \phi = \frac{1}{9} (H - K), \quad \psi = \frac{1}{3} (H + 2K). \]

Hence,
\[ C_{18} = -\frac{1}{27} (H^6 - 20H^3K^3 - 8K^6). \]

Now, we get
\[ E : \quad y^2 = x^3 - 3H(H^3 + 8K^3)x - 2(H^6 - 20H^3K^3 - 8K^6). \]

Put
\[ X = \frac{x}{K^2}, \quad Y = \frac{y}{K^3}. \]

We have
\[ E_{2,t} : \quad Y^2 = X^3 - 3t(t^3 + 8)X - 2(t^6 - 20t^3 - 8). \]

For the elliptic curve
\[ E_{2,t}/\mathbb{Q}(t) : \quad y^2 = x^3 - 3t(t^3 + 8)x - 2(t^6 - 20t^3 - 8), \]

we find that
\[ P_2 = (3t^2, 4(t^3 - 1)) \in E_{2,t}. \]
Moreover,

\[(5.11) \quad [2]P_2 = (3t^2, -4(t^3 - 1)), \quad -P_2 = (3t^2, -4(t^3 - 1)).\]

Hence, \([2]P_2 = -P_2\), i.e., \([3]P_2 = O\). So, \(P_2\) is a 3-division point in \(E_{2,t}\).

The discriminant of the elliptic curve \((5.9)\) is given by

\[(5.12) \quad \Delta = 2^{12} \cdot 3^3 \cdot (t - 1)^3(t^2 + 1)^3.\]

The bad fibers occur at

\[t = 1, \omega, \overline{\omega}, \infty.\]

They are of types \((I_3, I_3, I_3, I_3)\) by [Her], p.336. Correspondingly,

\[
\begin{aligned}
& t = 1, \text{i.e., } \varphi = 0, \text{i.e., } z_1z_2z_3 = 0, \\
& t = \omega, \text{i.e., } \psi - 3\overline{\omega}\varphi = 0, \text{i.e., } (z_1 + z_2 + \overline{\omega}z_3)(\omega z_1 + z_2 + \omega z_3)(\overline{\omega}z_1 + z_2 + z_3) = 0, \\
& t = \overline{\omega}, \text{i.e., } \psi - 3\omega\varphi = 0, \text{i.e., } (z_1 + \omega z_2 + z_3)(\omega z_1 + z_2 + \omega z_3)(\overline{\omega}z_1 + \overline{\omega}z_2 + z_3) = 0, \\
& t = \infty, \text{i.e., } \psi - 3\varphi = 0, \text{i.e., } (z_1 + z_2 + z_3)(z_1 + \omega z_2 + \overline{\omega}z_3)(z_1 + \overline{\omega}z_2 + \omega z_3) = 0,
\end{aligned}
\]

where \(\omega = \exp(2\pi i/3)\). They are the 12 lines of the Hessian arrangement on \(\mathbb{P}^2\).

Therefore, for the elliptic curve \((5.3)\), we have

\[(5.13) \quad P_1 = (3C_6, 108C_9) \in E,\]

\[(5.14) \quad P_2 = (3H^2, 4(H^3 - K^3)) = (3(\psi + 6\varphi)^2, 108\varphi(\psi^2 + 3\psi\varphi + 9\varphi^2)) \in E.\]

Note that

\[t = \frac{H}{K} = \frac{\psi + 6\varphi}{\psi - 3\varphi}.\]

This gives the following identity:

\[(5.15) \quad E_{1,t} : \quad (t - 1)(z_1^3 + z_2^3 + z_3^3) - 3(t + 2)z_1z_2z_3 = 0,\]

which is a family of elliptic curves over \(\mathbb{P}^1\). In the singular fiber: \(t = 1\), \(E_{1,t}\) degenerates into \(z_1z_2z_3 = 0\). In the singular fiber: \(t = \infty\), \(E_{1,t}\) degenerates into \((z_1 + z_2 + z_3)(z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0\). In the singular fiber: \(t = \omega\), \(E_{1,t}\) degenerates into \((z_1 + \omega z_2 + z_3)(\omega z_1 + z_2 + \omega z_3)(\overline{\omega}z_1 + \overline{\omega}z_2 + z_3) = 0\). In the singular fiber: \(t = \overline{\omega}\), \(E_{1,t}\) degenerates into \((z_1 + z_2 + \overline{\omega}z_3)(\omega z_1 + z_2 + \omega z_3)(\overline{\omega}z_1 + z_2 + \overline{\omega}z_3) = 0\). They are the 12 lines of the Hessian arrangement on \(\mathbb{P}^2\). Now, \(E\) decomposes into the pair \((E_{1,t}, E_{2,t})\).
By the linear transformation \( t = 9r + 1 \), \( E_{1,t} \) becomes one of the Beauville’s families of elliptic curves, which are modular elliptic surfaces for some conjugate of the given group (see [B] and [Ve2]). The equation in \( \mathbb{P}^2 \) is given by

\[
III: \quad z_1^3 + z_2^3 + z_3^3 = \left( \frac{1}{r} + 3 \right) z_1 z_2 z_3.
\]

The group \( \Gamma = \Gamma(3) \). The number of components of singular fibers is \((3, 3, 3, 3)\). The Picard-Fuchs equation for this family of elliptic curves is given by

\[
III: \quad r(27r^2 + 9r + 1)f'' + (9r + 1)^2 f' + 3(9r + 1)f = 0.
\]

The solution of this equation in terms of modular forms is

\[
III: \quad r = \eta(3\tau)^3 \eta\left(\frac{1}{3}\tau\right)^{-3}, \quad f = \eta\left(\frac{1}{3}\tau\right)^3 \eta(\tau)^{-1}.
\]

In our present case, the Picard-Fuchs equation becomes

\[
(5.16) \quad (t^3 - 1) \frac{d^2 f}{dt^2} + 3t^2 \frac{df}{dt} + tf = 0.
\]

In fact, for \( C = \mathbb{P}^1 \), \( C = \mathbb{P}^1 - \{1, \infty, \text{roots of } t^2 + t + 1 = 0\} \) and the family \( f: X \to C \) is the modular family for the group \( \Gamma_1(3) \). The Picard-Fuchs equation (for the holomorphic 1-forms on \( X_t \)) is (see [Pe])

\[
(t - 1)(t^2 + t + 1) \frac{d^2 f}{dt^2} + 3t^2 \frac{df}{dt} + tf = 0.
\]

The solutions are given by

\[
(5.17) \quad t = 9 \frac{\eta(3\tau)^3}{\eta\left(\frac{1}{3}\tau\right)^3} + 1, \quad f = \frac{\eta\left(\frac{1}{3}\tau\right)^3}{\eta(\tau)}.
\]

\( \Gamma(3) \) is a genus zero principal congruence subgroup of \( PSL(2, \mathbb{Z}) \), the index of \( \Gamma(3) \) in \( PSL(2, \mathbb{Z}) \) is \( \mu = 12 \). As a rational function of the Hauptmodul \( \rho \), the \( j \)-function can be expressed as (see [MS])

\[
j(\rho) = \frac{\rho^3(\rho + 6)^3(\rho^2 - 6\rho + 36)^3}{(\rho - 3)^3(\rho^2 + 3\rho + 9)^3},
\]

which is also the \( J \)-invariant of the semi-stable arithmetic elliptic surface over \( \mathbb{P}^1 \). In the elliptic curve \( E_{1,t} \), set

\[
(5.18) \quad \rho = \frac{z_1^3 + z_2^3 + z_3^3}{z_1 z_2 z_3} = \frac{3(t + 2)}{t - 1}.
\]
We find that

\[(5.19) \quad j(\rho) = 27 \frac{t^3(t^3 + 8)^3}{(t^3 - 1)^3},\]

which is the \(j\)-invariant of the elliptic curve \(E_{2,t}\).

Let us recall the following well-known results (see [Shi1]): let \(k\) be a field of characteristic \(\neq 3\) containing 3 cubic roots of unity. Consider the elliptic curve

\[E_3 : \quad x^3 + y^3 + z^3 - 3\mu xyz = 0\]

defined over \(K_3 = k(\mu)\), \(\mu\) being a variable over \(k\). Then \(E_3\) has exactly 9 \(K_3\)-rational points (i.e. base points of the pencil) and they are of order 3.

**Proposition 5.2**. (Deuring Normal Form) (see [Sil]). Let \(E/K\) be an elliptic curve over a field with \(\text{char}(K) \neq 3\). Then \(E\) has a Weierstrass equation over \(\overline{K}\) of the form

\[E_\alpha : \quad y^2 + \alpha xy + y = x^3, \quad \alpha \in \overline{K}, \alpha^3 \neq 27.\]

This equation has discriminant and \(j\)-invariant

\[\Delta = \alpha^3 - 27, \quad j = \alpha^3(\alpha^3 - 24)^3/(\alpha^3 - 27).\]

We will give some basic facts about families of elliptic curves with points of order 3: the Hessian family (see [Hus]).

**Definition 5.3.** The Hessian family of elliptic curves \(E_\alpha : y^2 + \alpha xy + y = x^3\) is defined for any field of characteristic different from 3 with \(j\)-invariant \(j(\alpha) = j(E_\alpha)\) of \(E_\alpha\) given by

\[j(\alpha) = \frac{\alpha^3(\alpha^3 - 24)^3}{\alpha^3 - 27}.\]

The curve \(E_\alpha\) is nonsingular for \(\alpha^3 \neq 27\), that is, \(\alpha\) is not in \(3\mu_3\), where \(\mu_3\) is the group of third roots of unity. Over the line \(k\) minus 3 points, \(k - 3\mu_3\), the family \(E_\alpha\) consists of elliptic curves with a constant section \((0, 0)\) of order 3 where \(2(0, 0) = (0, -1)\).

The Hessian family \(E_\alpha\) has three singular fibres which are nodal cubics at the points of \(3\mu_3 = \{3, 3\omega, 3\omega^2\}\), where \(\omega^2 + \omega + 1 = 0\).

There are other versions of the Hessian family which in homogeneous coordinates take the form

\[H_\mu : \quad u^3 + v^3 + w^3 = 3\mu uvw,\]

or in affine coordinates with \(w = -1\), it has the form

\[u^3 + v^3 = 1 - 3\mu uv.\]
If we set $y = -v^3$ and $x = -uv$, we obtain $x^3/y - y = 1 + 3\mu x$, or

$$E_{3\mu}: \quad y^2 + 3\mu xy + y = x^3.$$ 

This change of variable defines what is called a 3-isogeny of $H_\mu$ onto $E_{3\mu}$. There are nine cross-sections of the family $H_\mu$ given by

$$(0, -1, 1), \quad (0, -\omega, 1), \quad (0, -\omega^2, 1),$$

$$(1, 0, -1), \quad (\omega, 0, -\omega^2), \quad (\omega^2, 0, -\omega),$$

$$(-1, 1, 0), \quad (-1, \omega^2, 0), \quad (-1, \omega, 0).$$

The family $H_\mu$ is nonsingular over the line minus $\mu_3$, and choosing, for example, $0 = (-1, 1, 0)$, one can show that these nine points form the subgroup of 3-division points of the family $H_\mu$.

For $E_{3t}: y^2 + 3txy + y = x^3$, we have

$$j_1 = \frac{27t^3(9t^3 - 8)^3}{t^3 - 1}.$$ 

For $E_{2,t}: y^2 = x^3 - 3t(t^3 + 8)x - 2(t^6 - 20t^3 - 8)$, we have

$$j_2 = \frac{27t^3(t^3 + 8)^3}{(t^3 - 1)^3}.$$ 

Let $r = t^3 - 1$, then

$$\begin{align*}
   j_1 &= j_1(r) = \frac{27}{r}(r + 1)(9r + 1)^3, \\
   j_2 &= j_2(r) = \frac{27}{r^3}(r + 1)(r + 9)^3.
\end{align*}$$

Note that

$$j_1 \left(\frac{1}{r}\right) = j_2(r).$$

This gives a one-to-one correspondence between two families of elliptic curves $E_{3t}$ and $E_{2,t}$.

We will make use of certain elliptic modular surfaces associated to torsion free congruence subgroups of $SL(2, \mathbb{Z})$. Each of these has four singular fibers, all of which are of type $I_b$ for various $b$. This is actually a complete list of rational elliptic surfaces with exactly four semi-stable singular fibers (see [B] and [Her] for more details). Combine with the results in [B] and [Her], we find that two families of elliptic curves $E_{1,t}$ and $E_{2,t}$ are isogenous. According to [Shi1], Theorem 5.2, the points of $E_{1,t}$ and $E_{2,t}$ over $\mathbb{Q}(t)$ are of finite order because the group of sections of any elliptic modular surface is finite. In fact, $E_{1,t}$ is defined over $\mathbb{Q}(t)$, $E_{2,t}$ is defined over $\mathbb{Q}[t]$. 

Isogenous elliptic curves are related by the modular polynomials (see [La]). For elliptic curves $E$ and $F$ with $j$-invariants $j_E$ and $j_F$ respectively, there is an isogeny $\lambda : E \to F$ with kernel isomorphic to the cyclic group of order $n$ if and only if

$$\Phi_n(j_E, j_F) = 0,$$

where $\Phi_n(x, y)$ is the modular polynomial of order $n$. The roots of $\Phi_n(X, j(\tau)) = 0$ are $j(\alpha\tau)$ where $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $a, b, d \in \mathbb{Z}$ with $(a, b, d) = 1$, $ad = n$ and $0 \leq b < d$. Such $\alpha$ are called primitive.

According to [Shi2], suppose that the minimal Weierstrass equation of $E$ over $K = k(t)$ is given by

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t), \quad a_i(t) \in k[t].$$

Then the associated elliptic surface $f : S \to \mathbb{P}^1$ is a rational surface if and only if

$$\deg a_i(t) \leq i \quad (\text{all } i) \quad \text{and} \quad \text{Sing}(f) \neq \emptyset.$$ 

In case $\text{char}(k) \neq 2, 3$, we can take the equation in a more familiar form:

$$y^2 = x^3 + p(t)x + q(t), \quad p(t), q(t) \in k[t].$$

Then the above condition is equivalent to $\deg p(t) \leq 4$, $\deg q(t) \leq 6$ and $\Delta = 4p(t)^3 + 27q(t)^2$ is not a constant, i.e. $\Delta \notin k$.

For $E_{2,t}$, we have $p(t) = -3t(t^3 + 8)$, $q(t) = -2(t^6 - 20t^3 - 8)$ and $\Delta = -2^8 \cdot 3^3(t^3 - 1)^3$.

Hence, $E_{2,t}$ is a rational elliptic surface.

We need the following theorems due to Shioda (see [Shi1]).

**Proposition 5.4** (see [Shi1], Proposition 4.2). The elliptic modular surface $B_\Gamma$ has $t_1$ singular fibres of types $I_b$ $(b \geq 1)$, $t_2$ singular fibres of types $I_b^*$ $(b \geq 1)$ and $s$ singular fibres of type $IV^*$, where $t_1$, $t_2$ and $s$ are respectively the number of cusps of the first kind, the number of cusps of the second kind and the number of elliptic points for $\Gamma$.

**Theorem 5.5** (see [Shi1], Theorem 5.2). Let $B_\Gamma$ be the elliptic modular surface attached to $\Gamma$.

1. If $\Gamma$ has torsion (i.e. $s > 0$), then the group of sections $S(B_\Gamma)$ is either trivial or a cyclic group of order $3$.
2. If $\Gamma$ has a cusp of the second kind (i.e. $t_2 > 0$), then the group of sections $S(B_\Gamma)$ is either trivial or isomorphic to one of the groups

$$\mathbb{Z}/(2), \mathbb{Z}/(4) \quad \text{or} \quad \mathbb{Z}/(2) \times \mathbb{Z}/(2).$$

3. If $\Gamma$ is torsion-free and all cusps are of the first kind, then the group of sections $S(B_\Gamma)$ is isomorphic to a subgroup of $\mathbb{Z}/(m) \times \mathbb{Z}/(m)$, where $m$ denotes the least common multiple of $b_i$'s $(1 \leq i \leq t_1)$. Here we suppose that the singular fibres of $B_\Gamma$ are of types $I_{b_i}$ $(1 \leq i \leq t_1)$. 

Combine with the above results, we find that the Mordell-Weil groups for \(E_{1,t}\) and \(E_{2,t}\) are finite groups. More precisely, the Mordell-Weil groups of \(E_{1,t}\) and \(E_{2,t}\) over \(\mathbb{Q}(t)\) are isomorphic to the subgroups of \(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}\).

Let \(Y \to \mathbb{P}^1, Y' \to \mathbb{P}^1\) be two rational elliptic surfaces. Under some mild technical conditions on the singular fibers, the fibre product \(Y \times_{\mathbb{P}^1} Y' \to \mathbb{P}^1\) admits a nice resolution \(W\) which is a projective simply connected threefold with trivial canonical bundle. It can be showed that some of them are rigid, i.e., admit no nontrivial deformation (see [Sc]).

According to [Sc], let \(r : Y \to \mathbb{P}^1\) and \(r' : Y' \to \mathbb{P}^1\) denote two relatively minimal, rational, elliptic surfaces with sections. Write \(S\) (resp. \(S'\)) for the images of the singular fibers of \(Y\) (resp. \(Y'\)) in \(\mathbb{P}^1\). The fiber product \(p : W = Y \times_{\mathbb{P}^1} Y' \to \mathbb{P}^1\) is nonsingular except at points in the fibers over \(S'' = S \cap S'\). In order that the singularities of \(W\) be no worse than ordinary double points, we shall assume that the singular fibers of \(r\) and \(r'\) above the points in \(S''\) are either irreducible nodal rational curves or cycles of smooth rational curves. In the notation of Kodaira such fibers are of type \(I_b\), where \(b > 0\) denotes the number of irreducible components. Such fibers are also called semi-stable.

The dualizing sheaf of \(W\) is most readily computed by regarding \(W\) as the hypersurface in \(Y \times Y'\) obtained by pulling back the diagonal in \(\mathbb{P}^1 \times \mathbb{P}^1\) via the map \(r \times r'\). For a relatively minimal, regular elliptic surface, \(r : Y \to \mathbb{P}^1\), the canonical sheaf is given by

\[
\omega_Y \simeq r^* \mathcal{O}_{\mathbb{P}^1}(p_g(Y) - 1) \otimes \mathcal{O}_Y(\sum_i (m_i - 1)F_i),
\]

where the second term is the contribution of the multiple fibers. An easy computation using the adjunction formula reveals that the dualizing sheaf \(\omega_W\) is trivial exactly when \(p_g(Y) = p_g(Y') = 0\) and there are no multiple fibers. Of course these conditions are fulfilled when both \(Y\) and \(Y'\) are rational and have sections. However it follows from Castelnuovo’s rationality criterion and the vanishing of the Tate-Shafarevich group for rational elliptic surfaces with sections that this is the only case when the conditions for triviality of \(\omega_W\) are fulfilled. In this case any small resolution has trivial canonical bundle.

| Level | Congruence subgroup | Number of components in singular fibers |
|-------|---------------------|----------------------------------------|
| 3     | \(\Gamma(3)\)       | 3, 3, 3, 3                              |
| 4     | \(\Gamma_1(4) \cap \Gamma(2)\) | 4, 4, 2, 2                              |
| 5     | \(\Gamma_1(5)\)     | 5, 5, 1, 1                              |
| 6     | \(\Gamma_1(6)\)     | 6, 3, 2, 1                              |
| 8     | \(\Gamma_0(8) \cap \Gamma_1(4)\) | 8, 2, 1, 1                              |
| 9     | \(\Gamma_0(9) \cap \Gamma_1(3)\) | 9, 1, 1, 1                              |

**Proposition 5.6 (see [Sc]).** Let \(r : Y \to \mathbb{P}^1\) and \(r' : Y' \to \mathbb{P}^1\) denote relatively minimal rational elliptic surfaces with section. Suppose that the fibers of \(r\) and \(r'\) above
all points in $S''$ are semi-stable. Then the fiber product $Y \times_{P^1} Y'$ has only ordinary double point singularities. The projective variety, $\overline{W}$, obtained by blowing up the nodes has no infinitesimal deformations, exactly when all fibers of $r$ and $r'$ are semi-stable and one is in one of the following four cases, each of which actually occurs:

1. $Y$ and $Y'$ appear in Table 1 and are isogenous. In particular $S = S' = S''$ has 4 elements.
2. $Y$ and $Y'$ appear in Table 1, each has at least one $I_1$ fiber, and the map $r'$ has been modified by an automorphism of $P^1$ so that $\sharp(S'') = 3$. Furthermore the singular fibers of $r$ and $r'$ which do not lie above points of $S''$ are of type $I_1$.
3. $Y$ and $Y'$ are not isogenous and $S = S' = S''$ has 5 elements.
4. $Y$ appears in Table 1, $S = S''$, $\sharp(S') = 5$, and the singular fiber of $r'$ which does not map to $S''$ has type $I_1$.

Therefore, we find that the fiber product $E = E_{1,t} \times_{P^1} E_{2,t}$ is a rigid Calabi-Yau threefold.

The fiber product can be considered as a kind of topological surgery. We will study the structure of rational points under this topological surgery. From the point of view of Diophantine geometry, fiber products can be used to generate some new rational points in Calabi-Yau threefolds. For example, for the self-fiber product

$$E_{1,t} \times_{P^1} E_{1,t} : \quad (x^3 + y^3 + z^3)rst = (r^3 + s^3 + t^3)x y z,$$

there are some trivial new rational points, such as $x = r$, $y = s$, $z = t$ and so on. It will be interesting to find some nontrivial new rational points in the fiber products. We will prove that there are infinitely many nontrivial new rational points in the fiber product $E = E_{1,t} \times_{P^1} E_{2,t}$.

In actually dealing with specific elliptic curves over $\mathbb{Q}$ we find the following theorem due to Lutz and Nagell very useful and it gives a manageable method for determining $E_{\text{tor}}(\mathbb{Q})$.

**Theorem 5.7** (Lutz-Nagell) (see [Sil]). Let $E/\mathbb{Q}$ be an elliptic curve with Weierstrass equation

$$y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{Z}.$$

Suppose $P \in E(\mathbb{Q})$ is a non-zero torsion point. Then

1. $x(P), y(P) \in \mathbb{Z}$.
2. Either $[2]P = O$, or else $y(P)^2$ divides $4A^3 + 27B^2$.

For the elliptic curve (5.3), we know that $P = (3C_6, 108C_9) \in E$. Here, $A = -27C_{12}$ and $B = 54C_{18}$. Note that $A, B \in \mathbb{Z}$ when $(z_1, z_2, z_3) \in \mathbb{Z}^3$. In fact, $E$ can be defined over $\mathbb{Q}[z_1, z_2, z_3]$. Suppose that $P$ is a torsion point in $E$, then $P$ is a torsion point in
$E$ with $(z_1, z_2, z_3) \in \mathbb{Z}^3$. We find that

\[(5.20) \quad [2]P = \left(-6C_6 + \frac{(C_6^2 - C_{12})^2}{64C_9^2}, -108C_9 + \frac{9C_6(C_6^2 - C_{12})}{8C_9} - \frac{(C_6^2 - C_{12})^3}{512C_9^3}\right).\]

If $[2]P = O$, then $C_9 = 0$ for every $(z_1, z_2, z_3) \in \mathbb{Z}^3$, which is impossible! Thus, $[2]P \neq O$. By Lutz-Nagell theorem, we have that $108^2C_9^2$ divides $2^8 \cdot 3^{12}C_{12}$, i.e., $C_9^2$ divides $2^4 \cdot 3^6C_{12}^3$ for every $(z_1, z_2, z_3) \in \mathbb{Z}^3$, which is also impossible! Therefore, $P$ is of infinite order.

Now, we obtain the following theorem:

**Theorem 5.8 (Main Theorem 5).** The variety associated with Hessian polyhedra

\[(5.21) \quad E : \quad y^2 = x^3 - 27C_{12}x + 54C_{18}\]

can be expressed as the fiber product of two isogenous, semi-stable, rational elliptic modular surfaces

\[E_{1,t} : \quad (t - 1)(z_1^3 + z_2^3 + z_3^3) - 3(t + 2)z_1z_2z_3 = 0\]

and

\[E_{2,t} : \quad y^2 = x^3 - 3t(t^3 + 8)x - 2(t^6 - 20t^3 - 8).\]

Hence, $E$ is a rigid Calabi-Yau threefold. There are only finitely many rational points in $E_{1,t}$ and $E_{2,t}$. However, there exist infinitely many nontrivial rational points in $E$. More precisely, $P = (3C_6, 108C_9) \in E$ is of infinite order.

The developments in physics stimulated the interest of mathematicians in Calabi-Yau varieties. There are some important conjectures for these special varieties. One of these is the modularity conjecture for Calabi-Yau threefolds defined over $\mathbb{Q}$ (see [Me] and [Yu]).

Let us recall some basic facts about Calabi-Yau varieties (see [Yu]).

**Definition 5.9.** Let $X$ be a smooth complex projective variety of dimension $d$. $X$ is called a Calabi-Yau variety if

1. $H^i(X, \mathcal{O}_X) = 0$ for every $i$, $0 < i < d$.
2. The canonical bundle of $X$ is trivial.

We introduce the Hodge cohomology groups and the Hodge numbers

\[H^{i,j}(X) := H^j(X, \Omega^i_X), \quad h^{i,j}(X) := \dim H^{i,j}(X).\]

By the duality induced from complex conjugation, $h^{i,j}(X) = h^{j,i}(X)$, and the Serre duality on the Hodge cohomology groups asserts that $h^{i,j}(X) = h^{d-j,d-i}(X)$. Also there is the Hodge decomposition

\[H^k(X, \mathbb{C}) = \bigoplus_{i+j=k} H^{i,j}(X).\]
The \( k \)-th Betti number \( b_k(X) := \dim H^k(X, \mathbb{C}) \) and the Euler characteristic is

\[
\chi(X) = \sum_{k=0}^{2d} (-1)^i b_k(X).
\]

The condition (1) is that \( h^{i,0}(X) = 0 \) for every \( i, 0 < i < d \), and the condition (2) implies that the geometric genus \( p_g := h^{d,0}(X) = 1 \).

The dimension three objects of our interest here are Calabi-Yau threefolds. Note that Calabi-Yau threefolds are Kähler manifolds, so that \( h^{1,1} > 0 \). The Hodge diamond of a Calabi-Yau threefold is given by

\[
\begin{array}{cccc}
1 & & & \\
 & 0 & & 0 \\
0 & & h^{1,1} & 0 & \\
1 & h^{2,1} & h^{1,2} & 1 & \\
 & 0 & h^{2,2} & 0 & \\
 & & 0 & & \\
1 & & & \\
\end{array}
\]

The Betti numbers are

\[ b_0 = b_6 = 1, \quad b_1 = b_5 = 0, \quad b_2 = b_4 = h^{1,1}, \quad b_3 = 2(1 + h^{2,1}). \]

The Euler characteristic is \( \chi = 2(h^{1,1} - h^{2,1}) \).

**Definition 5.10.** A Calabi-Yau threefold \( X \) is said to be rigid if \( h^{2,1} = 0 \).

The name comes from the fact that this condition implies \( H^1(X, T_X) = 0 \), and hence that \( X \) has no infinitesimal complex deformations. In this case, the middle cohomology

\[ H^3(X) = H^{3,0}(X) \oplus H^{0,3}(X) \cong \mathbb{C}^2 \]

is two-dimensional.

Now we will formulate the modularity conjecture for rigid Calabi-Yau threefolds over \( \mathbb{Q} \) (see [Yu]). Let \( X \) be a rigid Calabi-Yau threefold over \( \mathbb{Q} \). We know that an integral model for \( X \) always exists. Let \( p \) be a good rational prime. Then the characteristic polynomial of the endomorphism \( \text{Frob}_p \) on \( H^3_{\text{ét}}(X, \mathbb{Q}_\ell) \) is

\[
P^p_3(T) := \det(1 - \text{Frob}^*_p T| H^3_{\text{ét}}(X, \mathbb{Q}_\ell)).
\]

Then it is given by

\[
P^p_3(T) = 1 - t_3(p)T + p^3 T^2 \in \mathbb{Z}[T]
\]

with \( t_3(p) \in \mathbb{Z}, |t_3(p)| \leq 2p^{3/2} \). Furthermore, by the Lefschetz fixed point formula, \( t_3(p) \) can be expressed in terms of the number of rational points on \( X \) over \( \mathbb{F}_p \). Let \( t_i(p) := \)
\text{tr}(\text{Frob}_p^*|H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)) \text{ denote the trace of the endomorphism } \text{Frob}_p^* \text{ on } H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell). \text{ Then the Lefschetz fixed point formula asserts that }

\#X(\mathbb{F}_p) = \sum_{i=0}^{6} (-1)^i t_i(p).

For a rigid Calabi-Yau threefold, this formula gives rise to the identity:

\[ t_3(p) = 1 + p^3 + (1 + p)t_2(p) - \#X(\mathbb{F}_p) \]

where \( t_2(p) \leq ph^{1,1} \) and the equality holds when all cycles generating \( H^{1,1}(X) \) are defined over \( \mathbb{Q} \). The \( L \)-series of \( X \) is given by putting together all the local data:

\[ L(X_{\mathbb{Q}}, s) := L(H^3_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell), s) = (\ast) \prod_{p: \text{good}} \frac{1}{1 - t_3(p)p^{-s} + p^{3-2s}} \]

where \( (\ast) \) is the factor corresponding to bad primes.

Now, we give the modularity conjecture for rigid Calabi-Yau threefolds over \( \mathbb{Q} \).

**Conjecture 5.11** (see [Yu]). Let \( X \) be a rigid Calabi-Yau threefold defined over \( \mathbb{Q} \). Then there is a cusp form \( f \) of weight \( 4 \) on some \( \Gamma_0(N) \) where \( N \) is divisible only by bad primes, such that the two 2-dimensional \( \ell \)-adic Galois representations \( \rho_{X,\ell} \) and \( \rho_{f,\ell} \) associated to \( X \) and \( f \), respectively, have the isomorphic semi-simplifications \( (\rho_{X,\ell} \sim \rho_{f,\ell}) \). Consequently,

\[ L(X, s) = L(f, s). \]

More precisely, if we write \( f(q) = \sum_{m=1}^{\infty} a_f(m)q^m \) with \( q = e^{2\pi i z} \), then

\[ t_3(p) = a_f(p) \]

for all primes not dividing \( N \).

The fiber products of rational elliptic surfaces give some examples of rigid Calabi-Yau threefolds and their modularity. It is known that (see [B], [Sc] and [Seb]) there are exactly six torsion free genus zero congruence subgroups of \( PSL(2, \mathbb{Z}) \), and the list is complete. They are \( \Gamma(3), \Gamma(4) \cap \Gamma(2), \Gamma(5), \Gamma(6), \Gamma_0(8) \cap \Gamma_1(4) \) and \( \Gamma_0(9) \cap \Gamma_1(3) \) in the Table 1. The modularity of the rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \) corresponding to these subgroups are given as follows:

**Theorem 5.12** (see [SY], [Ve1]). Let \( \Gamma \) be one of the six groups in the Table 1. Then the rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \) associated to \( \Gamma \) are all modular. More precisely, let \( \widetilde{S}_\Gamma := S_{\Gamma} \times_{\mathbb{C}^*} S_{\Gamma} \) be a rigid Calabi-Yau threefold defined over \( \mathbb{Q} \) associated to \( \Gamma \). Then there is a cusp form of weight 4 on \( \Gamma \) such that \( L(\widetilde{S}_\Gamma, s) = L(f, s) \).

In particular, when \( \Gamma = \Gamma(3) \), the singular fibers are of types \( (I_3, I_3, I_3, I_3) \). The Hodge number \( h^{1,1}(X) = 36 \). The cusp form of weight 4 is given by \( \eta(q^3)^8 \). Here \( \eta(q) \) is
Dedekind eta-function \( \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) with \( q = e^{2\pi iz}, z \in \mathbb{H} \). The equation for the rigid Calabi-Yau threefold is

\[(x^3 + y^3 + z^3)r st = (r^3 + s^3 + t^3)xyz.\]

Note that in the above construction, we only use the self-fiber products of semi-stable, rational elliptic modular surfaces. It will be interesting to study the fiber products of two isogenous, semi-stable, rational elliptic modular surfaces, such as our fiber product \( E_{1,t} \times_{\mathbb{P}^1} E_{2,t} \). A natural question is: which modular form corresponds to this fiber product?

According to Vafa (see [V]), sometimes the physics of the singularities are unconventional. For example when a 4-cycle (say a \( \mathbb{CP}^2 \)) shrinks in a Calabi-Yau threefold, it gives rise to very interesting unconventional new physical theories which were not anticipated! This is thus a great source of insight into new physics. In particular what types of singularities occur as well as what are the ways to resolve them will be of extreme importance for unravelling aspects of this new physics. It is tempting to speculate that these singularities may also lead to new invariants for four manifolds.

For the rigid Calabi-Yau threefold \( E \) in (5.21) where \( C_{12} = C_{12}(z_1, z_2, z_3) \) and \( C_{18} = C_{18}(z_1, z_2, z_3) \) with \((z_1, z_2, z_3) \in \mathbb{CP}^2\), we construct infinitely many subvarieties of \( E \):

\[(5.22)\]

\[Y_m = \{(z_1, z_2, z_3, x([m]P), y([m]P)) : (z_1, z_2, z_3) \in \mathbb{CP}^2, P = (3C_6, 108C_9) \in E, m \in \mathbb{Z}\}.

It will be interesting to study the singularity when \( Y_m \) shrinks in \( E \). This singularity maybe involves vanishing cycles and Picard-Lefschetz theory.

Note that the Calabi-Yau threefold \( E \) in (5.21) can be considered as an elliptic curve defined over \( \mathbb{Q}(T_1, T_2) \):

\[(5.23)\]

\[\frac{E}{\mathbb{Q}(T_1, T_2)} : \quad y^2 = x^3 - 27C_{12}x + 54C_{18},\]

where \( T_1 = z_1/z_3, T_2 = z_2/z_3 \) and

\[(5.24)\]

\[
\begin{align*}
C_{12} &= C_{12}(T_1, T_2) = (T_1^3 + T_2^3 + 1)[(T_1^3 + T_2^3 + 1)^3 + 216T_1^3T_2^3],
C_{18} &= C_{18}(T_1, T_2) = T_1T_2[27T_1^3T_2^3 - (T_1^3 + T_2^3 + 1)^3],
\end{align*}
\]

Let \( k \) be a number field and \( A \) an abelian variety over \( K = k(T_1, \ldots, T_n) \) where the \( T_i \) are indeterminates over \( k \). By a theorem of Néron (see [Se]), \( A(K) \) is a finitely generated group. In our case, \( E(\mathbb{Q}(T_1, T_2)) \) is finitely generated by Néron’s theorem. The \( j \)-invariant

\[(5.25)\]

\[j = j(T_1, T_2) = -\frac{C_{12}(T_1, T_2)^3}{C_{12}(T_1, T_2)^3} \notin \mathbb{Q}.
\]
In fact, $E$ is an elliptic threefold (see [Mi] for more details).

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