Enhanced spectrum of primordial perturbations, galaxy formation and small scale structure

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The standard structure formation scenario is successful on linear scales. Several apparent problems affect it however at galactic scales, such as the small scale problems at low redshift and more recent issues involving early massive galaxy and black hole formation. As these problems arise where complex baryonic physics becomes important, the associated unknowns are often assumed to be behind the problems. But the same scales are also those where the primordial spectrum is relatively unconstrained, and there are several ways in which it can be modified. We focus on that arising from effects possibly associated with the crossing of high energy cutoff scale by fluctuation modes during inflation. Elementary arguments show that adiabatic evolution cannot modify the near scale invariance, we thus discuss a simple model for the contrary extreme of sudden transition. Numerical calculations and simple arguments suggest that its predictions, for parameters considered here, are more generic than may be expected, with significant modifications requiring a rapid transition. We examine the implications of such a scenario, in this simplest form of sudden jump, on the matter power spectrum and halo mass function in light of the limitations imposed by particle production. We show that enhancement and oscillation in the power spectrum on currently nonlinear scales can potentially simultaneously alleviate both the apparent problem of early structure formation and, somewhat counterintuitively, problems at low redshift such as the apparent dearth of dwarf galaxies. We discuss consequences that can observationally constrain the scenario and its parameters, including an inflationary Hubble scale $\lesssim 10^{-8} M_{Pl}$, while touching on the possibility of simultaneous modification of power on the largest scales.

I. INTRODUCTION

Structure can condense from small density perturbations in a nearly homogeneous universe through gravitational instability. In the context of contemporary cosmology the density perturbations are seeded by quantum fluctuations in a primordial scalar field driving inflation, which later decays as the universe reheats and the standard model particles (and putative dark ones) arise (e.g., [1] [2]). The statistical properties of the primordial perturbations thus leave their mark on the cosmic microwave background and large scale structure of the galaxy distribution. On these scales, on which they can be inferred with precision, the properties of the perturbations are consistent with a nearly scale invariant primordial spectrum essentially determining their statistics [3].

At a phenomenological level, the simplest models of inflation do predict a near scale invariant primordial power spectrum if the inflaton potential is specified in such a way that the resulting Hubble parameter is nearly constant over a sufficient number of e-folds. Nevertheless, this prediction is not unique [4] [5]; indeed, little is known of the microscopic physics of inflation, or the wider particle physics model it may be a part of, and the coupling of the inflaton driving inflation to other fields may lead to changes in the potential that can ruin the predictions of standard slow roll models (e.g., [6] [7] and [10–12] for reviews). Stages of singular or rapid evolution of the potential or its derivatives, interrupting slow roll, leave imprints on the primordial spectrum of fluctuations [7] [13] [22]. Such effects may lead to a variety of ‘features’ and changes that break the scale free spectrum, and may have observable consequences associated with significant enhancement or suppression of power on large scales [23–29], as well as on smaller (galactic and sub-galactic) scales [30–36], including the formation of primordial black holes [37–38]. However the anomalous variation in the inflaton potential needs to be localized in order for single field slow roll inflation to proceed for a sufficient number of e-folds, and is also limited on large scales by observations suggesting a nearly scale invariant primordial power spectrum [3] [39].

In some models, such as DBI inflation, the breaking of scale invariance and associated features in the power spectrum are best represented in terms of sharp temporal variations in the sound speed of perturbations [40–46]. This is still accompanied by corresponding changes in the equation of state of the inflaton and therefore, as in the cases above, by anomalous background evolution. On the other hand, if inflation proceeds modestly longer than the minimal number of e-folds needed to solve problems such as the apparent causal connection in the CMB field (‘horizon problem’), primordial perturbations
are expected to arise from well within a high energy cutoff scale where new physics may transpire. This gives rise to what is often termed as the trans-Planckian problem. The minimal length scale that it entails can be introduced by modifying the commutation relations or introducing nonlinear dispersion relations for the propagation of fluctuations (early studies include [47–53], while a relatively recent review can be found in [54]).

As the scale of inflation $H$ is expected to be much smaller than the scale $k_c$, where new physics may appear, the background evolution is unmodified by possible new physics, but modifications to the scale invariant spectrum may arise solely from anomalous evolution of the fluctuation modes, rather than of the inflation potential. However, if some form of decoupling is assumed — e.g. motivated by the fact that the power spectrum is evaluated at horizon exit scales $H \ll k_c$ — then the corrections can be quite small; of order $(H/k_c)^q$, with $q \gtrsim 1$. In the context of local effective field theory $q = 2$ [55]; if the fluctuation modes are assumed to simply emerge from the quantum foam at conformal time $\eta$, depending on the comoving mode wavenumber $k$ — instead of the usual Bunch Davies initial conditions taken at $\eta \rightarrow -\infty$ — then $q = 1$ [50, 57]; if the modes emerge from the local adiabatic vacuum then $q = 3$ [58]. A covariant small scale cutoff, introduced by removing field configurations that are off shell by more than a Planck scale leads to $q = 1$ [59].

Much larger corrections are nevertheless possible in principle. This is the case, in particular, if the emergence of the inflating modes from the high energy cutoff scale is assumed to be preceded by non-adiabatic evolution arising from a nonlinear dispersion relation at energy scales $> k_c$ [48, 60, 62]. Modes then do not emerge from the foam in their lowest energy states. Excited states arise, providing anomalous initial conditions for further evolution and leading to enhancement and oscillations in the power spectrum. The most general parametrization of the effects of a non-adiabatic high energy scale exit would therefore appear to include both phenomena [51, 63].

Though physics at these scales is largely unknown, it can in principle be envisioned that the introduction of a cutoff scale in itself can modify the effective dynamics of the fluctuations. Indeed, the introduction of a minimal length scale and a large variety of phenomenological descriptions of ‘quantum spacetimes’, can be characterized by nonlinear dispersion relations [61, 63]. At the most intuitive level, a simple hydrodynamic analogy suggests such a modification to the propagation of fluctuations [60, 68]. This, much in the same way that an effective macroscopic description of wave propagation through a fluid or lattice may still be employed at wavelengths approaching the interparticle distances, provided this is phenomenologically taken into account through a modified dispersion relation. As, at scales smaller than the interparticle distances waves cannot propagate at all, it is in the transition between such a cutoff scale and the scale on which the standard effective macroscopic description applies that a nonlinear dispersion relation may describe the propagation of fluctuations. At this simplest intuitive level, one may expect the dynamics of a sound wave, initially moving in a medium where interparticle spacing is large and comparable with its wavelength, to keep memory of the anomalous evolution, even after it crosses and propagates in a medium with smaller interparticle spacing, where the effective theory is perfectly valid and ‘decoupling’ is guaranteed. The rough analogy here would be with an inflaton fluctuation mode inflating from wavenumbers above the high energy cutoff scale to ones below it, on its way to the horizon. Applications of more sophisticated ‘analogue’ models to the inflationary scenario show that modifications of the dispersion relation can indeed lead to significant changes of power spectrum of field correlations [69].

An important limitation on modifications of the power spectrum through the inclusion of excited states relates to the fact that these are necessarily associated with departures from a vacuum state. And too much excitation can lead to departures significant enough to prohibit inflation from starting and persisting in the first place [70–74]. However, as has been pointed out, the limitations that arise thus may not be very constraining [75–77]. In the present investigation we wish to examine whether, within the limits imposed by particle production, excitations of inflaton modes, stemming from the presence of high energy cutoff scale, can lead to significant and astrophysically interesting modifications of the primordial power spectrum.

At present, the non-observation of significant departures from scale invariance on large (linear) scales, where the primordial spectrum can be rather precisely inferred, seem to embody the main evidence against such modifications. Indeed, observations on scales on which the density perturbation is linear preclude even relatively small modifications [78–82]. Observations are however much less constraining at smaller scales, where they are limited by the Silk damping of the cosmic microwave background (CMB), and by nonlinear structure formation erasing the possibility of directly mapping the observed power spectrum to the primordial one. Precise CMB and large scale structure inference is therefore limited to scales $\gtrsim 10\text{ Mpc}$. For a horizon scale of $\sim 10\text{ Gpc}$ this spans three orders of magnitudes. More model dependent constraints are available from Lyman-α forest down to wavenumbers roughly corresponding to comoving spatial scales of order of Mpc. Beyond that, the spectrum is quite weakly constrained [83]. On the other hand, the smallest structures that form in the context of the standard cold dark matter scenario have earth mass and roughly solar system size $\sim 10^{-4}\text{pc}$. From such scales to the smallest scales at which the linear power spectrum can be directly recovered one counts 11 orders of magnitude — nine more than those separating the nonlinear scale to the horizon.

It is not inconceivable that the scale invariance of the primordial power spectrum does not hold in some parts
of the aforementioned range. On the contrary, despite the significant successes of the current model of structure formation on large scales [54], through the past couple of decades a variety of problems have arisen on galactic scales. There is a group of quite possibly related long-standing issues connected to the central densities of dark matter halos, and the abundance and dynamical properties of local dwarf galaxies [85-86]; and, in apparent contradiction, more recent issues related to an apparent preponderance of massive old galaxies and supermassive black holes at redshifts 3 ≲ z ≲ 9 that may pose a challenge the current ΛCDM-based structure formation paradigm [87-100].

Such problems appear in the highly nonlinear regime of structure formation; where small density perturbations, born of primordial ones in the presumed inflaton, have sufficiently grown under gravity to form gravitationally bound objects. Since it is also at such scales that complex baryonic physics becomes important, it was natural to suppose that the main determinant lies in complex baryonic physics of galaxy formation and evolution. For example, for the small scale problems at z = 0, processes as coupling of the baryons through dynamical friction [101-109] or through random potential fluctuations [110-116], were invoked. (In addition to suggestions modifying the dark matter particle physics models, as in warm dark matter [117-121], self interacting dark matter [122-127] and fuzzy dark matter [128-132].) Similar attempts are naturally ongoing in the case of the more recent early structure formation problem (some are discussed in Section V.C.2).

However, it is also precisely at the nonlinear scales, where baryonic physics becomes important, that the primordial power spectrum is relatively unconstrained. That modifications thereof can be relevant to small scale problems associated with galaxy formation has long been realized [30], but not as extensively investigated as the 'baryonic solutions' discussed above.

In this study we investigate the effect on the power spectrum from field excitations, stemming from nonadiabatic transition through a high energy cutoff regime corresponding to currently nonlinear scales, and within the limits imposed by particle production. We attempt to do this in generic terms, starting from well defined initial conditions, with linear dispersion relation (but with sound speed different from unity) and examining the effect of the transition. As this solely affects the fluctuation modes, the equation of state of the inflaton, and thus the background evolution, remains unmodified (unlike in cases such as DBI inflation mentioned above), this helps isolate the effect of excitations on the spectrum. We then look for associated effects on the matter power spectrum and dark matter halo mass function.

In the next section, after illustrating in simplest terms how the power spectrum is essentially an adiabatic invariant of the dynamics of inflaton fluctuations, we present and discuss a simple model representing the other extreme of a sudden transition (in the Appendix, we show results that suggest it is generic for the parameters considered here). In Section III we discuss what this model entails in more formal terms, evaluating the limits on power spectrum modification in terms of particle production. In Section IV we study, within these limits, the possible modifications on the matter power spectrum and halo mass function. We discuss possible astrophysical consequences and constraints, before presenting our conclusions in Section V.

II. ADIABATICITY, SCALE INVARIANCE AND THE SUDDEN EXTREME

A. The evolution of fluctuations

The general quadratic action for inflationary perturbations with sound speed $c_s$ can be expressed in terms of the Mukhanov-Sasaki (MS) variable $v$ as [133-135]:

$$S^{(2)} = \frac{1}{2} \int d^4x \left( \dot{v}^2 - c_s^2 (\nabla v)^2 + \frac{z''}{z} v^2 \right),$$

(1)

where $z = a^2 \phi_0^2 / H c_s$, $\phi_0$ is the background inflaton field, $H = a' / a$, and the primes denote derivative with respect to conformal time $\eta$. The evolution of each Fourier mode $v_k(\eta)$ is governed by the MS equation

$$v''_k + \left( c_s^2 k^2 - \frac{z''}{z} \right) v_k = 0.$$  

(2)

The MS variable is related to the curvature perturbations by $v = z R$. This is a quantity of fundamental interest, as it relates primordial quantum fluctuations to the observables, such as CMB anisotropies; the power spectrum of the large scale galaxy distribution; and, ultimately, (more indirectly) the formation of smaller scale structures, such as the dark matter halos hosting galaxies. The dimensionless power spectrum of such perturbations is given by

$$\Delta_k^2(k) = \frac{k^3}{2\pi^2} |R_k|^2,$$  

(3)

where the right-hand side is evaluated at the horizon ($c_s k = a H$), as in the absence of isocurvature perturbations, the comoving curvature perturbations $R$ are conserved on super-horizon scales [136]. Scale-invariant perturbations correspond to $\Delta_k^2(k) = \text{const}$. Departures from this can arise if $c_s$, or the inflationary Hubble scale $H$, depend on time.

In the standard inflationary scenario, a massless field, and quasi de Sitter evolution is assumed (and so $H$ is nearly constant throughout the inflationary stage). The associated slow roll parameters, defined as

$$\epsilon \equiv - \frac{\dot{H}}{H^2} \quad \eta \equiv \frac{\dot{\epsilon}}{H \epsilon} \quad \text{and} \quad \kappa \equiv \frac{\dot{c}_s}{H c_s},$$

(4)
are always much smaller than unity. Any departure from scale invariance is small, and is usually quantified by the spectral tilt parameter $n_s$:

$$n_s - 1 = \frac{d \ln \Delta^2_k}{d \ln k} = -2 \epsilon - \dot{\eta} - \kappa. \quad (5)$$

The slow roll parameters being small implies that $n_s \approx 1$.

To first order in the slow roll parameters, and assuming canonical kinetic terms, one can also show that

$$\frac{z''}{z} = \frac{1}{\eta^2} \left( 2 + 3 \epsilon + \frac{3}{2} \eta \right). \quad (6)$$

In this case, to a good approximation, one can rewrite (2) as

$$v''_k + \left( c_s^2 k^2 - \frac{2}{\eta^2} \right) v_k = 0. \quad (7)$$

If no non-standard dispersion relation is invoked then $c_s = 1$, and the standard scenario may be fully recovered.

### B. Adiabaticity, adiabatic invariants and primordial power spectrum

#### 1. General context

In the context of equation (2), setting $c_s = 1$ can be interpreted as the result of assuming a massless field with linear and dispersion relation between the physical frequency $\omega_{\text{phys}}$ and the physical wavenumber $k_{\text{phys}} = \frac{k}{a}$, $\omega_{\text{phys}} = k_{\text{phys}}$. However, as discussed in the introduction, this is not a necessity; a modification of the equation of state of the inflaton (e.g. such as in DBI inflation), or modification of the dispersion relation due to modes probing a high energy cutoff scale, beyond which new physics may arise, can change the situation.

In the latter case, beyond a cutoff scale $k_c$, one can introduce the relevant modification by replacing the square of the comoving wavenumber $k^2$ in (2) with

$$k^2 \rightarrow k_{\text{eff}}^2(k, \eta) \equiv a^2(\eta) \omega_{\text{phys}}^2 \left[ \frac{k}{a(\eta)} \right], \quad (8)$$

the main requirement being that the new dispersion relation recovers the linear one for scales $k \ll k_c$. This dispersion relation is thus necessarily time dependent, as it must transit between two regimes. It can be used to parametrize and reflect the effect of a varying sound speed in equations (2) and (7), the latter applying when the background dynamics is well approximated by standard slow roll. Indeed, in this context, Eq. (2) can be rewritten as

$$v''_k + \left[ k_{\text{eff}}^2(k, \eta) - \frac{z''}{z} \right] v_k = 0. \quad (9)$$

(A more rigorous derivation, based on a variational principle, can be found here [138]).

How does the extra time dependence, that thus arises, affect the power spectrum derived from the above equation? As noticed in several studies, mere time dependence in itself is not sufficient to alter the nearly scale invariant nature of the primordial spectrum of fluctuations. The adiabaticity condition — that is, $|\frac{d \omega}{d \eta}|/\omega^2 < 1$ — must be violated. A well known example where this condition is indeed violated invokes the Corley-Jacobson dispersion relation

$$k_{\text{eff}}^2(k, \eta) = k^2 - k^2 |b_m| \left[ \frac{k}{k_{\text{phys}}(\eta)} \right]^{2m} \quad (10)$$

(where $b_m$ is a constant and $m$ an integer). The studies [138] [137] [139], indeed indicated that a modification of the power spectrum, in the form of a change in the spectral index and superimposed oscillations, was possible. However, several criticisms were raised, including the possibility of complex frequencies arising at early times, rendering the quantum field theory ill-defined, and problems related to setting the initial conditions in non-adiabatic regime. To circumvent such issues, a new dispersion relation [138] was proposed, which exhibits linear behaviour in the small and large wavenumbers, but has intermediate concave region where the adiabaticity is violated locally.

Here we will be considering a simpler scenario, which assumes standard Bunch Davies type initial conditions, with modified sound speed but still linear dispersion relation. The effective sound speed transits to the standard relation $\omega = k$ as the boundary around $k_c$ is crossed. In this simple controlled context, we wish to estimate the rapidity and steepness of the transition required in order to produce palpable change in the power spectrum. It turns out that such a transition must be quite rapid, as we now discuss.

#### 2. The power spectrum as an adiabatic invariant

We now wish to show, in explicit simple terms, that the primordial power spectrum is in fact an adiabatic invariant of the evolution of inflationary perturbation, and thus cannot be significantly modified by any changes in the dispersion relation that keeps the dynamics of the perturbations sufficiently adiabatic.

We will be interested in the case when the dispersion relation is modified due to the fluctuation modes probing scales beyond a high energy cutoff, before they inflate into lower energy scales on their way to horizon exit. Only the effective speed of mode propagation is modified and slow roll inflation of a massless field is assumed to hold in all stages. So, Eq. (7) holds to a good approximation. However, because of the non-standard dispersion relation assumed, $c_s$ in that equation will not be necessarily unity in all stages. In fact its variation would incorporate changes parametrized by $k_{\text{eff}}$ in Eq. (9) above.

In principle $c_s$ in (7) can be either larger or smaller than unity. Perhaps a scenario in which modes do not
propagate at all for $k_{\text{phys}} \gg k_c$, and then do so at increasing $c_s \to 1$ as they emerge from the 'quantum foam' at $k_{\text{phys}} \sim k_c$, is appealing; it qualitatively connects, for example, to waves propagating in a lattice, which are scattered and dispersed to smaller speeds as one approaches the interparticle spacing, before ceasing to propagate. However 'analogue' models with superluminal speeds $c_s > 1$, beyond the cutoff scale, have also been proposed \textsuperscript{[65]}. As we discuss in Section \textbf{III} for our purposes both situations lead to similar results.

Equation (7) refers to a simple harmonic oscillator with variable frequency. If $c_s$ is constant, the variation solely comes from the second term in the bracket. To separate this effect from that connected to possible variation in $c_s$ at a high energy cutoff transition, we exploit the fact that $k_c \gg H$. This enables one neglect the second term in the brackets of equation (7), at scales ($\sim k_c$) around the high energy cutoff transition; as, when modes transit from beyond the cutoff scale $k_c$ to below it, the conformal time $\eta_c = -\frac{1}{a_cH} = -\frac{k}{Hk}$. The term in the brackets in the aforementioned equation is then $c_s^2 k^2 \left( 1 - \frac{2}{H^2} \frac{H^2}{k^2} \right)$. The second term inside this latter bracket is small compared to unity when

$$c_s^2 > 2 \left( \frac{H}{k_c} \right)^2. \quad (11)$$

Since we already assume that $k_c \gg H$, this is always the case when $c_s > 1$. We will also assume that this condition is satisfied when considering the case of $c_s < 1$ \textsuperscript{[140]}

This leaves us with an equation of a harmonic oscillator with frequency $\omega = c_s k$. The adiabatic invariant for a standard harmonic oscillator with specific energy $E$ and frequency $\omega$ is $J = \frac{E}{\omega}$. Taking the modulus of the amplitude and the velocity $v_k (\eta) = i \omega_k v_k (\eta)$, the energy of the oscillator is $E = \omega^2 |v_k|^2$. Whatever the evolution at scales above $k_c$, as long as it is adiabatic $J$ is conserved. Moreover, at scales $< k_c$ one must recover the standard linear dispersion relation, and so $\omega = k$. At such scales, relevant to eventual horizon crossing, one then has

$$J = k |v_k|^2. \quad (12)$$

Comparing this with the standard slow roll inflationary power spectrum

$$\Delta_x^2 (k) = \frac{kH^2}{2\pi^2} |v_k|^2, \quad (13)$$

one finds that they are equivalent up to a constant factor $\frac{H^2}{2\pi^2}$.

With sufficiently adiabatic adiabatic evolution through the transition at $k_c$ no significant change to the power spectrum can occur. Any appreciable effect could then result solely from variations in $H$ (if present), or from second term in bracket of Eq. (7), which also turns out to be quite modest, as may be expected given the quadratic correction at any physical scale ($k_{\text{phys}}/H)^{-2}$ For this implies again that the second terms in brackets of Eq. (7)

is smaller than the first until modes are close to existing the horizon. In Appendix \textbf{A} we show numerical calculations that corroborate this contention, in the context of the simple model described below.

C. A toy model of the sudden extreme

As we have seen, any adiabatic frequency change, due to nonstandard evolution of modes beyond a high energy cutoff scale, will not alter the nearly scale free form of the resulting power spectrum. We thus consider the opposite extreme; that of a sudden change in the sound speed at $k_c$, while employing the same approximation of neglecting the second term in the bracket of Eq. (7). The procedure again separates changes in the power spectrum due to variations in $c_s$ at around $k_c \gg H$ from any time dependence connected to the second term in the above equation at much smaller physical wavenumbers.

The change in sound speed across the transition is equivalent to a sudden change in frequency. To illustrate such a situation in simplest terms, we consider the effect of such a change on a simple harmonic oscillator, with initial amplitude $A$, frequency $\omega_{\text{in}}$ and phase $\phi$. Its evolution is given by

$$X_{\text{in}} (t) = A \cos (\omega_{\text{in}} t + \phi) \quad (14)$$
$$X'_{\text{in}} (t) = -A \omega_{\text{in}} \sin (\omega_{\text{in}} t + \phi), \quad (15)$$

with

$$A = \sqrt{X_{\text{in}}^2 (t) + \frac{V_{\text{in}}^2 (t)}{\omega_{\text{in}}^2}} \quad (16)$$
$$\phi = \arccos \left( \frac{X_{\text{in}} (t)}{A} \right) - \omega_{\text{in}} t. \quad (17)$$

Suppose that at some moment $t = t_s$ the spring constant is suddenly altered, and the corresponding frequency of the oscillator changes to $\omega_{\text{out}}$. Then, for $t \geq t_s$,

$$X_{\text{out}} (t) = B \cos (\omega_{\text{out}} (t - t_s)) + C \sin (\omega_{\text{out}} (t - t_s)) \quad (18)$$
$$X'_{\text{out}} (t) = -B \omega_{\text{out}} \sin (\omega_{\text{out}} (t - t_s)) + C \omega_{\text{out}} \cos (\omega_{\text{out}} (t - t_s)). \quad (19)$$

Matching the initial and final states at $t_s$ one obtains

$$B = X_{\text{in}} (t_s) \quad (20)$$
$$C = \frac{X_{\text{in}} (t_s)}{\omega_{\text{out}}}. \quad (21)$$

So the evolution after the jump can be expressed in terms of the initial state at the jump as

$$X_{\text{out}} (t) = X_{\text{in}} (t_s) \cos (\omega_{\text{out}} (t - t_s)) \quad (22)$$
with the amplitude and phase changing after the jump.

We now apply this toy model to attempt to mimic the evolution of fluctuations due to a sudden in frequency (or again, effectively sound speed) of propagation of inflaton fluctuations. In our approximation the evolution is effectively governed by two independent harmonic oscillators, due to the complexity of the mode functions in Eq. (7). Thus, for the real part,

\[ X_{\text{in}}(\eta) = A_r \cos(\omega_{\text{in}} \eta + \phi_r) \quad (23) \]

\[ \dot{X}_{\text{in}}(\eta) = -A_r \omega_{\text{in}} \sin(\omega_{\text{in}} \eta + \phi_r) , \quad (24) \]

and for the imaginary part we have

\[ X_{\text{in}}(\eta) = A_i \cos(\omega_{\text{in}} \eta + \phi_i) \quad (25) \]

\[ X'_{\text{in}}(\eta) = -A_i \omega_{\text{in}} \sin(\omega_{\text{in}} \eta + \phi_i) . \quad (26) \]

The sudden step will here correspond to conformal time \( \eta_c \), when an inflating mode crosses a physical the wavenumber \( k_c \), associated with the energy scale \( k_c \), where new physics may arise. Applying the step condition as previously, for the real part we thus find

\[ X_{\text{out}} = X_{\text{in}}(\eta_c) \cos[\omega(\eta - \eta_c)] + X'_{\text{in}}(\eta_c) \sin[\omega(\eta - \eta_c)] . \quad (27) \]

Similarly, for the imaginary part

\[ X_{\text{out}} = X_{\text{in}}(\eta_c) \cos[\omega(\eta - \eta_c)] + X'_{\text{in}}(\eta_c) \sin[\omega(\eta - \eta_c)] . \quad (28) \]

The complete solution then is

\[ v(\eta) = X_{\text{out}}(\eta) + iX_{\text{out}}(\eta) . \quad (29) \]

This can be evaluated at each \( k \), with \( \omega_{\text{out}} = k \), given \( \omega_{\text{in}} = c_s \cdot H_k \). As mentioned above (and checked in Appendix A1) usage of Eq. (29), in order to evaluate the effect on the primordial power spectrum of a sudden step in \( c_s \) and \( \omega \) at \( k_c \), returns a good approximation. The results of Appendix A2 also suggest that the sudden jump scenario itself turns out to be much more generic to any appreciable change in the power spectrum than may seem a priori. We now discuss how the power spectrum is evaluated and the modifications to the standard near scale invariant form that arise.

1. The power spectrum of primordial fluctuations

A mode corresponding to comoving wavenumber \( k \) crosses the high energy cutoff scale at \( \eta_c = -k_c / H_k \). At \( \eta \ll \eta_c \) we assume Bunch Davies type initial conditions but with \( \omega_k = c_s k \), with \( c_s \neq 1 \). Thus, before the transition in sound speed (and frequency),

\[ \nu_k(\eta) \rightarrow -1 / \sqrt{\omega_{\text{in}}} \eta \cdot \quad (30) \]

For the initial amplitudes one then has

\[ A_r = A_i = 1 / \sqrt{\omega_{\text{in}}} \cdot \phi_r = 0 \cdot \phi_i = \pi / 2 , \quad (31) \]

Modes with physical wavenumbers larger than \( k_c \) at the start of inflation undergo a frequency change such that \( \omega_{\text{in}} / \omega_{\text{out}} = c_s \) (where \( c_s \) refers to the value, different from unity, before the crossing). Modes with smaller wavenumbers do not cross the high energy cutoff scale and their frequency remains unmodified (we discuss how these are connected to current comoving scales in Section IV).

All modes eventually cross the horizon. Using equations (29) and (3), one can evaluate the power spectrum in the context of our simplified model when \( H \) is given. This is done at horizon crossing when \( \eta = \eta_H \). Alternatively, one can also use equation (7) to calculate the power spectrum numerically at superhorizon scales, as done in the Appendix for purpose of comparison and evaluating relevance of model.

In de Sitter inflation \( H \) is exactly constant, and all modes are assumed to exit the horizon at time \( \eta_H = -1 / k \). Since again a standard dispersion relation must reign beyond the high energy cutoff scale \( k_c \), one expects \( \omega_{\text{out}} = k \). All modes then leave the horizon at the same phase and oscillations implied by equations (27) and (28) do not appear in the power spectrum; only enhancement is found at scales undergoing the jumps. Numerically, equation (7) can be used to obtain similar results (cf. Appendix A1).

The Hubble parameter in more realistic models of inflation must vary slowly with time. The variations imply that modes do not leave the horizon at the same phase, and oscillations as well as enhancement appear in the primordial power spectrum. As a simple generic example, we will adopt power-law inflation [141][142] where (in proper time), \( a(t) \approx t^p \), with \( p > 1 \). This corresponds to an inflation potential of the form

\[ V(\phi) = V_0 \exp \left( -1 / M_P \sqrt{2 / p} (\phi - \phi_i) \right) , \quad (32) \]

with slow-roll parameters given by

\[ \epsilon_v = M_P^2 2 \left( V_0 / V \right) ^2 = 1 / p , \quad \eta_v = M_P^2 V_{\phi \phi} / V = 2 / p \cdot \quad (33) \]

where \( V_\phi = dV / d\phi \). The scale factor and the Hubble parameter become

\[ a(\eta) = \left( \eta / \eta_H \right) ^{2 / p} \cdot \quad H(\eta) = -p / (p - 1) \left( \eta / \eta_H \right) ^{p / (p - 1) - 1 / p} \cdot \quad (34) \]

with \( \eta_H = \eta_0 / \sqrt{p - 1} \cdot \) With these forms for the evolution of the scale factor and Hubble parameter one can again use Eq. (29) in conjunction with (3) to evaluate the power spectrum, or numerically integrate Eq. (2), which now
takes the following form

\[ v''_k(\eta) + \left( c_s^2 k^2 - \frac{2\eta^2 - p}{(1 - p)^2} \right) v_k(\eta) = 0. \] (35)

We will generally use \( H = 10^{-4} M_{Pl} \), \( p = 55 \), and \( \eta_c = -10^5 M_{Pl}^{-1} \), in order to get the correct normalization and tilt of the power spectrum on linear scales using power law inflation. The results, when rescaled accordingly, are valid for other values of \( H \), since the relative enhancement of the power spectrum depends only on the in and out frequency ratio of oscillations, as we have seen. The connection between the cutoff scale \( k_c \), where change in the power spectrum occur, and the corresponding comoving scale depends on \( H/k_c \) rather than the absolute values (Section \[ IV \ A \ 2 \]).

### III. BROKEN INVARIANCE AND PARTICLE PRODUCTION

#### A. General solution in terms of Bogoliubov coefficients

We now consider what the simplified sudden step model actually implies in terms of quantum fluctuations in an inflaton. For this purpose we translate it to the language Bogoliubov expansion and coefficients, recalling some familiar results in the process. In this context, the high energy cutoff transition will be seen to lead to be due to excitations of the field and particle production. In our model, when thus translated, the excitations will take place due to time-dependent transitions between time independent in and out states, and are therefore well defined. The excitations invariably lead to enhancement in the power spectrum.

#### B. Generic enhancement in power spectrum

Using (7), and again invoking the approximation of neglecting the \( \frac{\omega^2}{\eta} \) term due to \( k_c \gg H \), we get the mode function differential equation of massless scalar field in Minkowski spacetime, with \( \omega_k = k \). Assuming the field to be initially in the vacuum state \( |0\rangle \), the amplitude of the vacuum fluctuations (the square root of the power spectrum) are given in terms of the vacuum mode function corresponding to the Bunch Davies initial conditions (30). Then, non-adiabatic evolution (whether sudden or not), can transform this initial vacuum state \( |0\rangle \) to one with excitations, with respect to the old annihilation operator \( \hat{a}_k \).

To find the effect of such excitations on the power spectrum after the transition is complete, one can proceed as follows. First by writing the mode expansion of the field operator in terms of the annihilation operator \( \hat{b}_k \) of \(|0\rangle \) and its complex conjugate

\[ \hat{\chi} = \frac{1}{\sqrt{2}} \int (e^{i k \cdot x} \hat{\mu}_k \hat{b}_k^* + e^{-i k \cdot x} \mu_k \hat{b}_k^+) \frac{d^3 k}{(2\pi)^{3/2}}, \] (36)

and then computing the two point correlation function in the state \( |0\rangle \) using this operator. One can then define the amplitude of the quantum fluctuations in terms of the new mode function \( \mu_k(\eta) \) as

\[ \Delta_\mu(\eta) = \frac{1}{2\pi} k^{3/2} |\mu_k(\eta)|. \] (37)

This new normalized mode function is a linear combination of the old one and its complex conjugate. Using Bogoliubov coefficients, it can be written as

\[ \mu_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta). \] (38)

Thus we have

\[ \Delta_\mu(\eta) = \frac{1}{2\pi} k^{3/2} \left[ |\alpha_k|^2 + |\beta_k|^2 + 2 \Re(\alpha_k \beta_k^* e^{2i\omega_k \eta}) \right]^{1/2}. \] (39)

The second coefficient \( \beta \) refers to excitations away from the vacuum state; as we will see below it directly counts particle production. The ratio of the primordial power spectrum after and before the sudden change can be expressed as

\[ \frac{\Delta_\mu^2(\eta)}{\Delta^2(\eta)} = 1 + 2 |\beta_k|^2 + 2 \Re(\alpha_k \beta_k^* e^{2i\omega_k \eta}). \] (40)

Averaging over a period larger the periodic time of the system (or in generic inflation models, over horizon exists of the different modes with different \( H \)), eliminates the oscillating term. The main result is that excitations away from the vacuum state lead to typically larger RMS fluctuations and power spectrum.

An important point to note here is that the generic enhancement in the power spectrum will occur whether the 'jump' in frequency is upward — that is whether \( \omega_{in} < \omega_{out} \) — or the downward, with \( \omega_{in} > \omega_{out} \). Or, assuming a dispersion relation \( w = c_s k \) to govern the propagation of fluctuations before and after the jump, the power spectrum will be enhanced whether \( c_s \) is larger before the jump, or whether it is larger afterwards. This is seen explicitly below.

#### C. Relations between the coefficients and the frequencies

The above does not necessarily assume instantaneous transition jumps between the well defined in to out states, just that a time dependent transition occurred. In our simplified model we have two regions connected by a sudden jump, which enables us to calculate the Bogoliubov coefficients explicitly in terms of the in and out frequencies.
We label the initial vacuum as $|0_{\text{in}}\rangle$, and the final vacuum $|0_{\text{out}}\rangle$. Before the jump, we assume the scalar field is in the initial vacuum state, so we have the mode function

$$v_k^{(\text{in})}(\eta) = \frac{1}{\sqrt{\omega_{\text{in}}}} e^{i\omega_{\text{in}} \eta},$$

for $\eta < \eta_c$. Before the jump, the frequency $\omega_{\text{in}} = c_s k$ with $c_s \neq 1$. In order to connect with standard inflationary scenario, the frequency after the jump is $\omega_{\text{out}} = k$. The final frequency $\omega_{\text{out}}$ is therefore necessarily different from the initial one $\omega_{\text{in}}$. This causes excitations in the field, which modify the power spectrum.

After the jump, the mode function $v_k^{(\text{in})}(\eta)$ evolves into the superposition of $v_k^{(\text{out})}(\eta)$ and its complex conjugate:

$$v_k^{(\text{in})}(\eta) = \frac{1}{\sqrt{\omega_{\text{out}}}} \left[ \alpha_k e^{-i\omega_{\text{out}} (\eta - \eta_c)} - \beta_k e^{-i\omega_{\text{out}} (\eta - \eta_c)} \right].$$

The Bogoliubov coefficients $\alpha_k, \beta_k$ are determined by the requirement that the solution and its first derivative must be continuous at the jump, that is at $\eta = \eta_c$. The result is

$$\alpha_k = \frac{e^{-i\omega_{\text{in}} \eta_c}}{2} \left( \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} + \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \right),$$

$$\beta_k = \frac{e^{-i\omega_{\text{in}} \eta_c}}{2} \left( \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} - \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \right).$$

This explicitly shows that the time dependence introduced by assuming a sufficiently rapid transition from in to out states can lead to significant excitation in the inflaton field for large enough frequency ratio. As is clear, the absolute values of $\beta_k$ and $\alpha_k$ derived above do not depend on whether $\omega_{\text{in}} > \omega_{\text{out}}$ or the reverse. This again shows that generic enhancement in the power spectrum is expected, independent of the direction of the jump.

D. Limits from particle production

As we have seen, excitations of the inflaton generically lead to enhanced power spectrum. In section [IV] below, we will suggest that these may have important consequences at galactic scales, at both high and low redshifts, pertaining to such apparent problems as the dearth of dwarf galaxies, ‘too big to fail’ and early galaxy formation, while maintaining a standard spectrum at scales where it is highly constrained. But how much excitations of the field can one have without ruining the inflationary scenario itself? Indeed, the exponential expansion during inflation hinges on a dark energy equation of state, too much excitation and particle production can turn it instead into a radiation field, with deceleration replacing the exponential expansion.

The radiation energy density associated with the relativistic particles, which can be assumed to be produced through excitations of the field, is [76]

$$\langle \rho \rangle = \int_{k_{\text{phys}}=H}^{k_{\text{phys}}=H} d^3k_{\text{phys}} \omega_{\text{phys}}(k_{\text{phys}}) n_{k_{\text{phys}}},$$

where $k_{\text{phys}}$ and $\omega_{\text{phys}}$ are the physical wavenumbers and frequencies, $k_c = k_c$ is the cutoff scale for new physics and $H \approx \text{const.}$ is the Hubble scale of inflation. The occupation number of excited states can be expressed in terms of the second Bogoliubov coefficient as $n(k) = |\beta_k|^2$. In the relevant integration range the relation between the wavenumbers and frequencies is linear, and the integral is dominated by larger values of $k_{\text{phys}}$. In this case, $\langle \rho \rangle \approx \beta^2 k^4$, where $\beta$ corresponds to $\beta_k$ at larger values of $k$ dominating the integral.

In the context of the sudden step scenario $\beta_k$ is some non-zero constant for modes affected by the jump (and zero otherwise), and the above estimate is rigorously justified. In order for inflation to start and proceed then, $\beta^2 k_c^4$ must be smaller than the energy density scale of inflation $H^2 M_{\text{Pl}}^2$, where $M_{\text{Pl}}$ is the Planck mass. This leads to the condition

$$|\beta| < \frac{M_{\text{Pl}} H}{k_c^2}.$$  

If $k_c = M_{\text{Pl}}$ this is small for $H \ll k_c$. However much smaller cutoff scales may in principle be allowed (claimed to reach even the TeV scale e.g. [143, 144]; e.g. [145] for review). Even for large field inflation with $H \lesssim 3 \times 10^{-5} M_{\text{Pl}}$ and relatively conventional high scale $k_c \lesssim 10^{-3} M_{\text{Pl}}$, one finds $|\beta| \lesssim 30$ as an upper limit on $|\beta|$. In general, one only needs $k_c / M_{\text{Pl}} \approx H / k_c$ to get a Bogoliubov coefficient of order one.

The limits of integration in Eq. (45) assume that the whole interval between $k_c$ and $H$ is filled with excited states corresponding to modes that have already crossed $k_c$. In this case the time derivative $\frac{d(\rho)}{dt} \sim \beta^2 H^3 \dot{H}$ is much smaller in absolute value than the change in energy density of the inflaton $\sim M_{\text{Pl}}^2 H \dot{H}$ for values of $\beta^2$ of interest. However, at earlier times, when the modes are still crossing $k_c$, and the integration interval is variable, the time derivative of the backreaction energy $\frac{d(\rho)}{dt} \sim \beta^2 k^3_{\text{phys}}(k_c) k_{\text{phys}}(k_c) \approx \beta^2 H^3 k^4_{\text{phys}}(k_c)$ can be much larger (here $k_{\text{phys}}(k_c)$ refers to the physical wavenumber of the first scale that crosses $k_c$, and which decreases as it inflates towards the horizon, when $k_{\text{phys}}(k_c) = H$). This leads to the constraint $\beta^2 \lesssim \left( \frac{H^2 M_{\text{Pl}}^2}{k_c^4} \right)^4$, where $\epsilon = -\dot{H} / H^2$. A more detailed treatment gives a similar constraint, $\beta^2 \lesssim (6\pi)^2 \epsilon H^2 M_{\text{Pl}}^2 \left( \frac{k_c}{\nu_{\text{phys}}} \right)^4$. The effect of increasing energy density while the excited states are filling up can be quite complicated, as it would require evaluation of the modified evolution, taking into account the rescaling of the energy density (which itself can act as vacuum energy [79]). Here we just point out that, simply
assuming the usual relation $\epsilon = \frac{H^2}{8\pi G}$ to hold when the effect is small enough, leads to the condition

$$|\beta| \lesssim \left( \frac{2 \times 10^{-9}}{P_0} \right)^{1/2} \times 6.7 \times 10^4 \left( \frac{H}{k_{\text{phys}}(k_c)} \right)^2,$$  

(47)

with $P_0 \approx 2 \times 10^{-9}$ the standard characteristic value of the standard primordial power spectrum of scalar fluctuations. This rough estimate suggests that $|\beta|$ can be of order one, without affecting the power spectrum on larger scales exiting the horizon, if these scales exit when the spatial physical scale that first crosses the high energy threshold has inflated enough to be about 0.004 times the size of the horizon (so that the corresponding wavenumber $k_{\text{phys}}(k_c)$ is 1/0.004 times larger than the horizon). Note that this does not depend on the absolute value of $H$, or $H/k_c$. We further discuss the possible interpretation of this constraint in Section VA2.

As the ratio of the power spectrum modified by excitations to the vacuum power spectrum scales as $1 + 2\beta^2$, considerable modifications may be allowed in principle, if $\beta$ is of order one or larger. In the following we consider possible consequences of, and constraints on, such modification on currently nonlinear scales, where existing constraints are relatively weak and apparent problems with galaxy formation at low and high redshift arise.

IV. MATTER POWER SPECTRUM AND HALO MASS FUNCTION

In this section we examine some possible astrophysical implication of the sudden change of frequency at a high energy cutoff scale, discussed above. For this purpose we compute the linear matter power spectrum and the dark matter halo mass function. As on linear scales the power spectrum is strongly constrained, it is therefore the modified halo mass function, derived from it on nonlinear scales, that will be of interest, particularly in terms of its possible observable consequences on the galaxy mass function.

For the actual calculations we assume a $\Lambda$CDM universe with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.7$, and RMS dispersion in the density field at 8 $h^{-1}$Mpc at $z = 0$, $\sigma_8 = 0.8$.

A. The matter power spectrum

1. Evaluation procedure

The power spectrum of perturbation in the CDM is evaluated from

$$P(k, a) = \frac{4}{9} \frac{k^4 P_v(k)}{\Omega_m^2 H_0^2} T^2(k) D^2(a),$$  

(48)

where $P_v(k)$ is the primordial power spectrum, $D(a)$ the linear growth factor, and $H_0$ is the present value of the Hubble parameter.

As we will be primarily interested in generic consequences, rather than detailed comparison with data, for our purposes it will suffice to use the BBKS fitting form

$$T \left( x = \frac{k}{k_{\text{eq}}} \right) = \frac{\ln (1 + 0.171 x)}{0.171 x} [F(x)]^{-1/4},$$  

(49)

with $k_{\text{eq}} = 0.073 \Omega_m h^2$Mpc$^{-1}$ and

$$F(x) = 1 + 0.284 x + (1.18 x)^2 + (0.399 x)^3 + (0.490 x)^4.$$  

(50)

For the growth factor, we used the following form

$$D(z) = \frac{D^+(z)}{D^+(z = 0)},$$  

(51)

where

$$D^+(z) = \frac{5 \Omega_m}{2} H(z) \int_0^\infty \frac{(1 + z') dz'}{[H(z')/H_0]^3},$$  

(52)

with

$$H(z) = H_0 \sqrt{\Omega_m(1+z) + (1 - \Omega_m)}.$$  

(53)

2. Choice of jump scale

CMB and large scale structure observations place quite tight constraints on the amplitude of the primordial power spectrum on scales $\sim 10$Mpc or larger. We now show how smaller scales can be affected by a modified power spectrum, while larger scales remain unaffected, if inflation proceeds for approximately the number of e-folds needed to solve the horizon problem.

Observe that inflation takes place after the comoving spatial scale $k^{-1}(H_0) \sim H_0^{-1}$ exits the horizon; it is characterized by the minimum number of e-folds needed to solve the horizon problem: $N[k(H_0)] = \ln (a_{\text{end}}/a_{k(H_0)})$, where the subscripts denote the end of inflation and the epoch of horizon exit of the scale $k^{-1}(H_0)$. The condition can be written as

$$\frac{a_{k(H_0)}}{a_0} = \frac{H_0}{H},$$  

(54)

where, $a_0$ is the current scale factor. $H$ is the Hubble parameter at the horizon exit of the scale $k^{-1}(H_0)$ during inflation.

Given a comoving scale $k$, one may ask when it was equal to a given physical scale $k_c$ during inflation. This gives the following condition

$$k = \frac{a_c}{a_{k(H_0)} H}.$$  

(55)

As an example, we set $k(k_c) = 1$ Mpc$^{-1}$, $H = 10^{-4} M_{Pl}$, and $k_c = 1/M_{Pl}$. We then find that $a_c \sim a_{k(H_0)}$; that is, at the time the current horizon scale $H_0^{-1}$ exits the
and corresponding power law inflation model with $p$ context of the simplest scenario with constant $\beta$, can be affected if one insists on $|\beta| \approx 1$. In the de Sitter case due to all modes leaving the horizon at the same phase. The frequency ratio corresponds to a Boguliubov coefficient $|\beta| = 4.95$ (Eq. [44]). Note that, for power law inflation, there is net enhancement despite the strong oscillations, which appear symmetric around the uperurbed spectrum on the logscale. This will result in similar mass dispersions in the de Sitter and power law models, where the smoothing also leads to gradual enhancement despite the sudden jump at the cutoff scale (Fig. [2]).

horizon during inflation, the comoving spatial scale $\sim 1\text{Mpc}$ is of the order of the Planck length.

This general picture is reproduced even if the cutoff scale $k_c$ is not the Planck scale. All one needs is $H/k_c \approx 10^{-4}$. If inflation proceeds for a number of e-folds larger than the number $N_{\min} = \min(N[k(H_0)])$ required to solve the horizon problem, then the ‘jump scale’ can still correspond to $k(k_c) = 1\text{Mpc}^{-1}$ if $H/k_c < 10^{-4}$. In general, the number of e-folds allowed, with $k(k_c)$ corresponding to the smallest comoving spatial scale affected by the high energy cutoff transition, is

$$N = N_{\min} + \ln \left( \frac{k_c}{H} \left( \frac{k(H_0)}{k(k_c)} \right) \right), \quad (56)$$

The scale $k(H_0) \approx 10^{-4} \text{ Mpc}^{-1}$ is fixed by the present size of the horizon, while $k(k_c) = 1 \text{ Mpc}^{-1}$ happens to roughly correspond to the largest scale on which significant modification of the power spectrum would not affect its inference from galaxy cluster counts and lensing surveys (but, depending on the exact scale, not necessarily Lyman-\( \alpha \) bounds, as discussed in Section [IV C3]). Larger values of $k(k_c)$ are in principle possible, and in this case smaller structures are affected. However, if one takes into account our crude estimate of the time variation of the backreaction, this may be constrained. For, as mentioned in relation to Eq. (17), to maintain $|\beta|$ of order 1, one may need $H/k_{\text{phys}}(k_c) \gtrsim 0.004$. If $k(k_c) \approx 1 \text{ Mpc}^{-1}$, the comoving scale exiting the horizon when this is satisfied is $\approx 0.004\text{Mpc}^{-1}$. Larger scales, with smaller wavenumbers, can be affected if one insists on $|\beta| \gtrsim 1$. In the context of the simplest scenario with constant $\beta$, beyond the cutoff regime, the power spectrum on such scales may be thus modified on such scales. This may be allowed on comoving scales $k < 0.004 \text{ Mpc}^{-1}$, and may even be relevant to supposed anomalies of the CMB on large scales, but not on smaller spatial scales, where modifications are tightly constrained. That changes in the power spectrum on the largest scales may be connected with backreaction associated with initial evolution has already been noted (e.g. [149]), and may be of interest in the present context, but its proper examination is beyond our present scope.

Here we will be mainly interested in the enhancement of the power spectrum on large nonlinear scales, corresponding to $k \approx \text{Mpc}^{-1}$, because of the particularly interesting consequences for galaxy formation we discuss in Section [IV C]. Fig. [1] shows the resulting dimensionless matter power spectrum for a jump corresponding to a ratio of sound speeds (or in and out frequencies) of 100 on such scales. The ratio of frequency and sound speed corresponds to 100, associated with a Boguliubov coefficient $|\beta|$ of about 5. This large value is chosen as to clearly delineate phenomena associated with significant excitation on nonlinear structure formation. We will also explore the effect of smaller values and spatial cutoff scales below.

B. Halo mass function

1. Evaluation procedure

On nonlinear scales, modifications of the primordial power spectrum are primarily encoded in the mass function of self gravitating dark matter objects, the halos hosting galaxies. We evaluate this mass function using
FIG. 2. RMS mass fluctuations corresponding to power spectra shown in Fig. 1. Note that the strong oscillations in the power spectrum, in the case of power law inflation, have little effect here, as they are smoothed over and integrated out as the dispersions are extracted from the power spectra. Note also that, despite the sharp jump in linear power spectra, the change in the RMS mass fluctuations beyond the cutoff scale is gradual.

the Press-Schecter formalism, which estimates the number of dark matter halos per unit mass and comoving volume in the universe, given the linear matter power spectrum via a spherical collapse model [150, 151]. This is given by

$$\frac{dn}{dm} = \frac{\rho_0}{M^2} f(\sigma) \left| \frac{d\ln \sigma}{d\ln M} \right|$$

where $\rho_0$ is the mean density at $z = 0$ and the fitting function $f(\sigma)$ is given

$$f(\sigma) = \frac{\sqrt{2}}{\pi} \frac{\delta_c}{2\sigma^2} \exp \left( -\frac{\delta_c^2}{2\sigma^2} \right),$$

where $\delta_c = 1.686$ is the critical overdensity for spherical collapse and $\sigma$ is the RMS variance of mass fluctuations within a sphere of radius $R$ that contains mass $M = \vartheta_f \rho_0 R^3$, where $\vartheta_f$ is some constant that depends on the filter function $W$. For Gaussian filter it is $\vartheta_f = (2\pi)^{3/2}$.

The filter function is characterized by its size $R$ or mass $M$. In the case of Gaussian filter we use here, the relation between them is

$$M = 4.37 \times 10^{12} \Omega_m h^{-1} \left( \frac{R}{h^{-1} \text{Mpc}} \right)^3 M_\odot$$

The mass variance is calculated through the integral,

$$\sigma^2(R) = \frac{1}{2\pi^2} \int_0^{\infty} k^2 P(k) W^2(kR) dk,$$

where $P(k)$ is the linear power spectrum and $W(kR)$ is the Fourier transform of the Gaussian filter function

$$W(kR) = \exp \left( -\frac{(kR)^2}{2} \right).$$

FIG. 3. Multiplicity function, describing the fraction of mass in dark halos of mass $M$, for the power spectra shown in Fig. 1 and dispersions of Fig. 2. As may already be expected from the latter figure, the results are similar in de Sitter and power law inflation (labelled DS and PL respectively), due to the smoothing and integration over the oscillations as the mass function is derived.

FIG. 2 shows the thus calculated dispersion for de Sitter and power low inflation models. As can be seen the strong oscillations in the power spectrum of the latter case are smoothed and integrated over, and the results are quite similar in the two cases. Also, despite the sharp jump in the corresponding power spectra, the change in the RMS mass fluctuations in the nonlinear regime beyond the cutoff scale is gradual.

Fig. 3 shows the resulting halo multiplicity function, which estimates the fraction of mass in halos of mass $M$, corresponding to the unperturbed and perturbed (with jump) matter power spectrum shown in Fig. 1. As may be expected given the mass dispersions shown in Fig. 2 the results are virtually identical in case of de Sitter and power law inflation, despite the strong oscillations in the
function with redshift, at fixed mass, being smaller in the case of enhanced spectrum than in the unperturbed case.

Finally, there is this somewhat counterintuitive effect, mentioned at the conclusion of the previous subsection, of a suppression of the multiplicity function contribution of halos at smaller masses, of \( \lesssim 10^{10} M_\odot \). This interesting result may be understood by recalling that enhancement in the power spectrum at comoving scales associated with masses of \( \gtrsim 10^{12} M_\odot \), implies that all smaller scales are also enhanced. And enhancement at smaller (length and mass) scales in turn implies that smaller halos form at higher redshift and that by the redshifts considered here they have already been typically subsumed in larger ones; that is, the typical mass scale, for a given fluctuation level at a given redshift, shifts up. This leads to a relative decrease in the number of halos with masses \( \lesssim 10^9 M_\odot \). As opposed to the case of the enhancement of the multiplicity of relatively high mass halos, the de-enhancement is here relatively larger at smaller \( z \), as the lower mass halos are now those that are rarer at such redshifts.

C. Interpretation and possible consequences

As, in the current model of structure formation, galaxies form in seeds provided by the potential wells of dark matter halos, the significant modifications to the halo mass function are expected to leave imprints on the associated galaxy stellar mass function. Pertinent questions here thus include whether those modifications have consequences for problems arising at small scales within the current standard scenario of structure formation, as outlined in the introduction; or, in contrast, whether such modifications can constrained also on nonlinear scales.

1. Small scale problems at low redshift and the dearth of dwarf galaxies

One straightforward consequence of the suppression of halo multiplicity at small scales pertains to the long-standing issue of the dearth of dwarf galaxies in the standard scenario: a galaxy like the Milky Way is expected, in the context of the ΛCDM with a standard primordial power spectrum, to have hundreds of satellites that are not observed, and some of the predicted hosting halos are too 'large' to have 'failed' to form galaxies. This is part of the so-called small scale problems of the standard scenario has given rise to various explanations, e.g in terms of baryonic physics, warm dark matter, fuzzy dark matter, as well as direct suppression of the small scale power spectrum.

As mentioned above, our somewhat counter-intuitive result is that an enhancement of power on small scales can also lead to a suppression in the number of small halos (as these 'overmerge' into larger entities). This suppression, of about an order of magnitude at \( z = 0 \) at the scales where issues such as the dearth of small galax-

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FIG. 4. Same as in Fig. 3 but at the indicated redshifts (using power law inflation model). Solid lines show the results with standard scale free primordial spectrum, which are compared to those obtained when the power spectrum is boosted at smaller scales, as a result of an imposed step (corresponding to a ratio of hundred fold) in mode frequency and propagation speed (cf. Fig. 1).
ies and too big to fail problems appear, can therefore be of relevance to apparent small scale crises arising in the context the standard model of structure formation.

2. The excess of early massive galaxies

The enhancement at higher mass scales may, on the other hand, have consequences for the more recently raised issues associated with early galaxy formation. These are extensions of longstanding phenomena related to what is referred to as ‘downsizing’ (e.g. [152]), required to account for preponderance of early massive galaxies, a phenomenon that does not appear entirely natural in a hierarchical structure formation scenario, where the smaller halos embodying the potential wells hosting the galaxies form first.

The problem of early galaxy formation has been termed ‘impossibly early’ in the context of the standard ΛCDM scenario of structure formation [91]. In that work, the authors attempt to derive the halo mass function at high-z, primarily from stellar mass functions derived using photometric spectral energy distribution templates and ultraviolet luminosity functions. The halo mass is then inferred by assuming a stellar to halo mass ratio of $M_*/M_h = 1/70$. If this local value of $M_*/M_h$ is used, then Fig. 1 of the aforementioned work suggests that the number density of massive galaxies can greatly exceed that of the halos they should inhabit for $z \gtrsim 4$ in the standard ΛCDM structure formation scenario. The discrepancy becomes more severe as one moves up in redshift and mass, reaching four orders of magnitude or more.

The above would seem to rule out the standard scenario of structure formation in the context of ΛCDM cosmology. However, a couple of caveats have been pointed out. First, regarding the assumption that $M_*/M_h$ does not vary with redshift. For, as can also be seen from Fig. 1 of [91], instead of moving the points inferred from the observed stellar number densities down orders of magnitude to fit the corresponding halo number densities, one can move the points horizontally to the left by an order of magnitude. This fitting procedure in effect invokes a $z$ (and $M_*$) dependent $M_*/M_h$, to replace the fiducial local value of $M_*/M_h = 1/70$ assumed by the authors. The procedure, requiring $M_*/M_h \sim 1/7$, is still in principle consistent with a universal baryon fraction of $1/6.3$, associated with the standard cosmological scenario, but only just [90] [153].

Another caveat that has been pointed out concerns the extraction of $M_*$ and associated number densities from the ultraviolet luminosity function at high $z$, which some of the data points of [91] relies on [154]. However, a multi-wavelength analysis of a sample of massive galaxies at $z > 3$ also leads to a cumulative mass function that can be consistent (within estimated errors) with $M_*/M_h$ approaching the universal baryon fraction at $z \sim 5.5$ and $M_* > 10^{11}M_\odot$ [155]. That work also shows (Fig. 14) that the number densities of massive galaxies are very difficult to reproduce in hydrodynamic numerical simulations — with significant underestimate for $z > 3$ — which may be expected, as their reproduction would seem to require that all available baryons reside inside galaxies, and their near total conversion to stars over a short time ($\sim$ Gyr). This would have as consequence the presence of a significantly ‘quenched’, quiescent population of massive galaxies already at high redshift. The presence of such a population, which is indeed observed, poses significant challenges. Synthesizing the stellar populations of one such object, observed at $z = 3.717$, for example, seems to again require prior evolution involving a $M_*/M_h$ reaching the universal baryon fraction [156] (see also [157]). There now appears to be a substantial population of such galaxies, observed at increasing redshift [158–164], and not easily reproduced by either hydrodynamic simulations [161] [162] or semi analytical models [159] [163].

Although questions as to the ultimate severity of these problems will only be settled with the next generation surveys (e.g. with the JWST), the situation warrants pointing out that they can in principle be alleviated by invoking small scale enhancement of the primordial power spectrum examined here.

Fig. 5 shows that significant enhancements can be achieved at mass scales $10^{12}M_\odot \lesssim M \lesssim 10^{13}M_\odot$, with a peak at a scale corresponding to highest dark matter to stellar mass ratio in standard modelling, at which the enhancements reach the ‘impossibly’ large levels claimed in [91]. Perhaps no less important is the slower evolution of the mass function for $z \gtrsim 4$, observed in Fig. 6 which is more consistent with the redshift evolution of the inferred stellar mass densities in [91] than the much faster evolution in the standard case (the slow evolution of the stellar mass function for $4 \lesssim z \lesssim 7$ was also observed for example by Song et. al. [165]). This would seem to
FIG. 6. Ratios of the mass functions at different redshifts, for the standard case (left) and that with modified power spectrum (right). Note the slower evolution (reflected in the smaller ratios) at higher \( z \) for most of the mass range in the modified case.

waive the apparent requirement of a \( M_*/M_h \) that is high dependent on redshift in order to fit the data.

With better statistics, and firmer grip on observational systematics, it should be possible to distinguish between scenarios involving enhancement in the primordial power spectrum, such as the one presented here and reconciliation with data through improvement of the baryonic model; by invoking further ‘downsizing’ physics input, in terms of feedback, quenching and other ‘subgrid’ physics (assuming the data remain consistent with the strict upper bounds placed in the context of ΛCDM [96]). As the baryonic models become better constrained, there may be particular consequences that could also constrain (or confirm) the sort of scenario discussed here. We now discuss some of these.

3. Other observables and consequences

In the context of the enhanced spectrum scenario presented here, the clustering of halos, on mass scales and redshifts where numbers are predicted to be significantly enhanced, may be measurably different from the standard case. This is because the biasing with respect to the matter distribution would be expected to be different (since they would correspond to less rare density peaks). Combined clustering and abundance matching analysis in the context of a ‘halo model’ (e.g., [166]), particularly at higher redshifts [167], could thus in principle test, and place constraints on, scenarios involving primordial power spectrum enhancements. The galaxy-matter correlation function, entering into calculations of galaxy-galaxy weak lensing signals, should also be different in the present scenario from the standard case. The difference should again be especially significant at higher redshifts, where the abundance of high mass halos is strongly increased, making for a relatively clumpy matter distribution. The altered mass function could also have implication for reionization.

Another observable that can potentially place immediate constraints on the scenario discussed here is the Lyman-\( \alpha \) forest. Here, detailed comparison with data involves complex simulations that depend on assumptions regarding the state of the intergalactic medium, which become less robust at nonlinear scales [132, 168], where significant modification to the linear power spectrum may also only mildly affect the spectrum on nonlinear scales (as their imprint comes primarily from modifications in the mass function rather than spatial distribution of collapsed objects; and the modification in RMS fluctuations is gradual even with the jump in linear spectrum; cf. Fig. 2). Nevertheless, the modifications to the linear power spectrum considered above are large and fall within the region relevant to Lyman-\( \alpha \) observations. In Fig. 7 we show the relative change in the mass function for (about an order of magnitude) smaller perturbation in the power spectrum, as well as on scales where Lyman-\( \alpha \) bounds may be expected to be considerably weaker. As can be seen, in the former case, significant enhancement in the mass function can still occur at the right scale, so as to alleviate the apparent early galaxy formation problem. But the effect on reducing the number of small halos, relevant to the dearth of small galaxies and too big to fail problems at low \( z \), is small. In the case when the modification in the power spectrum is placed on a smaller scale, the decrease in number of small halos is again significant, but the enhancement at high redshifts happens at a smaller mass scale (the enhancements are also smaller because halos in the standard scenario are already more abundant of such scales).

Thus, potential resolution of either galactic scale issues or their simultaneous alleviation, through modification of the power spectrum, rather than (or in addition to) baryonic physics input, may be tested through distinguishing predictions. This is true in general and is not confined to our particular simple model of a sudden jump; such tests
will thus become more relevant if the small scale issues connected to the standard structure formation scenario are confirmed to persist with incoming observations.

In the context of the present scenario, such observations can potentially probe imprints (or lack thereof) of high energy cutoff physics on the relevant astrophysical scales, and place constraints on the duration of inflation, as the ratio of the Hubble scale of inflation to the high energy cutoff scale and the number of inflationary e-folds fix the scale at which the matter power spectrum and halo mass function is modified (cf. Section IV A 2). For the minimal number of e-folds required to solve the horizon problem for example, \( H/k_c \approx 10^{-4} \) is required to address the galactic scale issues discussed. Significant modifications also require \( k_c \lesssim H/k_c \) (Section III D). These tests can be stringent; as, given this scale, and the level of excitation determined by the Boguliubov \( \beta_k \), the predictions of the present simplest scenario of sudden jump through a high energy transition scale, are unique in terms of the expected effect on the power spectrum.

V. CONCLUSION

Slow roll inflation predicts a nearly scale invariant spectrum of primordial fluctuations, which is borne out by precise observations of the cosmic microwave background and large scale structure in the universe. Nevertheless, that prediction is not unique, a variety of effects invoking discontinuous or phased evolution between slow rolls, for example, can lead to anomalous 'features' in the spectrum. Excited states arising from the crossing of a high energy physics cutoff scale, can also lead to significant modifications to the scale free spectrum. Although these are ruled out at the scales where the aforementioned observations are effective, the primordial spectrum is relatively unconstrained on smaller, currently nonlin-

ear scales, where the matter distribution has collapsed into bound self gravitating objects, washing out the primordial signature by largely encoding it into the halo mass function.

On the contrary, at such scales — which span many more octaves of observable structure than the three that are probed in the linear regime — a variety of issues arise in the context of the standard model of structure formation; such as the 'small scale problems' at low redshift and the apparent problems involving early galaxy and supermassive black hole formation at higher \( z \) — extension of longstanding phenomena requiring 'downsizing' in galaxy formation. As these issues arise precisely at the scales where complex baryonic physics comes to play a central role in the standard scenario of structure formation, it was natural that extensive investigation of solutions in these terms have been pursued. However, as these are also the scales where the primordial spectrum of fluctuations is relatively weakly constrained, this aspect, with its effects and consequences, may also warrant further investigation.

Here we considered the effects of excited states arising from the transiting of fluctuation modes through a high energy cutoff scale. As the power spectrum of primordial fluctuations is effectively an adiabatic invariant of inflaton dynamics (Section II B 2), adiabatic evolution necessarily leaves the nearly scale free spectrum intact. We next considered a simple model of the opposite extreme; of a sudden jump across the transition. The initial conditions for the fluctuations before the jump are well defined, taking the Bunch Davies form, but with propagation speed \( c_s \neq 1 \). An intuitive, simple 'analogue' model approximated by such a transition corresponds to the case of a gas or lattice where sound waves do not propagate at all below the interparticle distance, then propagate at an anomalous speed in an effective macroscopic approximation, before finally propagating...
with the standard sound speed and dispersion relation as the wavelength become progressively larger than the interparticle distance.

In this context, the primordial spectrum is invariably enhanced rather than suppressed (whether the initial $c_s > 1$ or is $< 1$) for all scales undergoing the transition through a high energy cutoff (Section IV). This is accompanied by strong, tightly spaced oscillations in the power spectrum of generic (as opposed to de Sitter) models of inflation, where modes exit the horizon at different phases (due to the slowly varying Hubble parameter). Numerical calculations suggest that sufficiently non-adiabatic evolution, leading to significant modification of the power spectrum implies, for the parameters considered here, an effectively sudden transition (Appendix A2). The simple model of sudden jump, and its predictions, are in this context thus generic.

Given the excitation level induced in the inflaton field, and the current comoving scale corresponding to the jump across the high energy cutoff scale during inflation, the predictions of the simple sudden jump models are essentially unique (in terms of its effect on the matter power spectrum, mass variance and the dark matter halo mass function). The level of excitation can be quantified through the Bogoliubov coefficient $\beta_k \neq 0$ for scales that undergo the jump, and is easily evaluated in terms of the in and out frequency ratio (or equivalently $c_s$ ratio; Section III C). If assumed to be within a few orders of magnitude of the Planck scale, the jump scale corresponds to currently nonlinear scales if inflation proceeds for approximately the number of e-folds necessary to solve the horizon problem. This is also generally the case if $H/k_c \sim 10^{-4}$, with smaller ratios allowing for larger e-folds (Section IV A2).

Backreaction bounds on $|\beta|$ must be imposed, as ‘over-excitation’ of the inflaton would result in radiation domination rather than inflation; these may however still allow for major enhancements of the power spectrum $\sim 1 + 2\beta^2$ (and oscillations in the generic inflation case). As we discuss in Section III D this would be generally the case if $k_c \lesssim H/k_c M_{Pl}$. Such enhancements can have observable consequences, confirming or constraining the effect of excitations on structure formation on nonlinear scales. In order to impose modifications on such scales in particular, and still keep the excitations from overwhelming the inflaton vacuum state, one thus requires $H/k_c \sim 10^{-4}$ and $k_c \lesssim H/k_c M_{Pl}$. This implies $k_c \lesssim 10^{-4} M_{Pl}$ and $H \lesssim 10^{-8} M_{Pl}$. A rough estimate of the derivative of the backreaction suggests possible modification of the power spectrum on the largest scales, and may place tight constraints on the comoving scale at which enhancement of the small scale power spectrum can occur (to about a comoving Mpc; Section III D). That suppression on the largest scales can accompany the enhancement on small, nonlinear ones, is an interesting possibility that may be worth studying in detail.

To probe for possible characteristic signatures of the modifications on nonlinear scales, we evaluate (in Section IV) the dark halo multiplicity function, quantifying the fraction of mass in halos of mass $M$. The peak, resulting from power spectrum enhancement, is chosen to correspond to a few times $10^{12} M_\odot$. This is the mass scale where the highest mass to light ratio is inferred when associating galaxies with halos in the context of halo models derived within the standard scenario. It is also the scale where issues related to the apparent preponderance of early massive galaxies (particularly quiescent ones) appear (Section IV C2). For relatively small enhancements at small redshifts $z$, which can be adjusted for by changing the galaxy-halo occupation numbers, the enhancement at larger $z \sim 8$ is dramatic, as such massive halos are very rare at these redshifts in the standard scenario. The change in the number densities of massive galaxy-hosting halos with redshift is also much smaller than in the standard case. Combined, these effects may alleviate the apparent ‘impossibly’ early galaxy formation problem, even in the most extreme form claimed.

Perhaps more surprisingly, an enhancement of the spectrum at these intermediate nonlinear scales and below leads to suppression of small halos at low $z$, thus potentially alleviating longstanding issues related to the dearth of small galaxies, including those ‘too big to fail’, in the standard structure formation scenario. This is due to the enhanced spectrum leading to overmerging of small mass objects at high $z$, so as to lead to a suppression of such objects at low $z$.

As mentioned, the halo mass function, in itself, cannot place strong constraints on enhancements of the primordial power spectrum on currently nonlinear scales. This is because one can always vary the galaxy halo occupation number to fit the data (in the standard scenario, the early galaxy formation issues at high $M$ and $z$ arises because this seems to sometimes require very large stellar mass fraction, which the enhanced halo mass function here may resolve; Section IV C2). Major modifications in the spectrum of primordial fluctuation are also eventually encoded in more minor modifications to the nonlinear matter power spectrum, as these enter primarily through the modified halo mass function rather than the statistics of the spatial distribution. Nevertheless, the scenario of an enhanced primordial power spectrum at scales corresponding to currently nonlinear ones, can be tested through its signature on halo biasing. The fact that more massive halos would be less rare may be expected to particularly impact such observables as galaxy-mass correlations and clustering signals (especially at higher redshift where the effects of enhancement at higher mass scales are more prominent). To solve both the aforementioned problems — of high-$z$ massive galaxies and dearth of local dwarfs — simultaneously, in the context of the present simple scenario of excitation through sudden transitions, also entails significant modifications at scales probed by Lyman-$\alpha$ observations (the required modifications at such scales are much more modest if one only needs to alleviate either of them; Section IV C3). Such observations, coupled with simulations
with the type of modified spectrum inferred here, may therefore place constraints on the present scenario and variants thereof. In the context of the simple model considered here, this would include constraints on $H/k_\star$, $k_\star$, $\beta$, and the number of inflationary e-folds, as discussed above.

Thus, current and future observations, coupled with modelling and simulations, can be used to constrain the scenario discussed here — or any other that involves solving the aforementioned problems through enhancement of the power spectrum at currently nonlinear scales, including variants possibly involving phased or discontinuous stages of inflation with relatively localized peaks in the primordial spectrum. This will become perhaps more pressing if next generation surveys (e.g. employing the JWST) confirm problems related to early galaxy formation. On smaller (sub-galactic) scales still, primordial power spectrum enhancement may problems related to early supermassive black hole formation, and the formation of the first dark matter objects; it may be tested through such effects as CMB spectral distortions (although the associated constraints are currently relatively quite weak $[83]$).

To conclude, enhanced power of primordial fluctuations on currently nonlinear scales may help alleviate small scale issues associated with standard model of structure formation. As it makes testable predictions, this scenario can, in principle, be distinguished from other solutions, for example those invoking baryonic physics, which also becomes important at those same scales. In the process, constraints can be placed on inflationary models. In the particular, in the simplest scenario proposed here, the predictions regarding the effect on the matter power spectrum and quantities derivable thereof are essentially unique, given the field excitation levels and the comoving scale of the high energy transition leading to these. These are in turn related to the characteristic inflationary and cutoff scales, and the possibility of the inclusion of excited states. The observational constraints on the latter, even in the nonlinear regime, can be potentially tighter than those inferred simply from the requirement that excitations be limited so as to allow for inflation. An interesting possibility, not examined in detail here, is that modifications of power on the largest scales may in principle be associated with the small scale changes.

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**Appendix A: Comparison of simplified model with numerical solution, and the effect of relaxing the sudden jump condition**

In this appendix we test the approximation of the simplified model introduced in Section II C introduced to evaluate the effect of non-adiabatic transition at a high energy cutoff scale $k_\star$ on the primordial power spectrum. There are two approximations that were invoked; the sudden step and the neglect the term proportional to $-\frac{1}{\omega^2}$ in the Mukhanov-Sasaki equations $[2]$ and $[7]$. We start by examining the latter, then we discuss the former.

1. **Model versus numerical solution of Mukhanov-Sasaki equation with step**

We evolve the dynamics of fluctuation modes numerically, using the MS equation $[2]$ for de Sitter and power law inflationary backgrounds, while replacing the term $k^2$ with

$$k^2 \rightarrow k^2_{\text{eff}}(k, \eta) \equiv a^2(\eta)\omega^2_{\text{phys}}\left[\frac{k}{a(\eta)}\right].$$  \hspace{1cm} (A1)

where

$$\omega^2_{\text{phys}}\left[\frac{k}{a(\eta)}\right] = \left(\frac{k}{a(\eta)} + \frac{\delta k}{a(\eta)}\mathcal{H}\left[\frac{k}{k_c, a(\eta)} - 1\right]\right)^2,$$  \hspace{1cm} (A2)

and $\mathcal{H}$ is the Heaviside step function. The parameter $\delta$ quantifies the size of the step. Such that the sound speed past the step given by $c_s = 1 + \delta$; it can be positive or negative, corresponding to an upward and downward jump in sound speed respectively. As discussed in Section III, in the context of the sudden step scenario they are equivalent in terms of the effect on the power spectrum.

The primordial spectra are evaluated, for both the analytical model and numerical calculations), as described in Section II C. The results are shown and compared in figures $[5]$ and $[9]$ for the case of de Sitter and power law inflation respectively. As noted in Section II C, in the de Sitter case all modes exit the horizon at exactly the same phase, and any initial shift in phase, and change in effective frequency due to the second term in bracket of the MS equation, leads to corresponding constant difference in the final power spectrum between the numerical and analytical results. Nevertheless, the relative error in the ratio of the perturbed to unperturbed power spectrum is still of order 25% when the step frequency ratio is 10. It is an order of magnitude lower still when the change of frequency is hundred folds.

In the case of power law inflation the Hubble scale $H$ is not exactly constant. The modes exit the horizon at different phases, and this leads to the oscillations, which accompany the enhancement in Fig. $[9]$. The corresponding error is then primarily in phases, with the maxima and minima of the oscillations practically equal in the
simplified model and the numerical calculations. The change in phase is generally unimportant for calculating quantities with observable consequences; such as the mass fluctuations at a given spatial or mass scale, and halo mass multiplicity function. For these depend on integrals of the matter power spectrum, derived from the primordial one calculated here (this is discussed further in Section IV). The simple analytical model — with its simple interpretation in terms of well defined in and out states; Section III — thus turns out to be a good approximation.

2. Non-adiabaticity versus sudden jump condition

Oscillations can, in general, be considered adiabatic, if $\omega(\eta)$ changes only slightly over a characteristic time $\Delta \eta$ of order of one oscillation period. If the frequency $\omega$ changes from a value $\omega_1$ to another value $\omega_2$, on a characteristic timescale $\Delta \eta$, the change may thus be adiabatic if

$$|\omega_1 - \omega_2| < \omega_2^2 \Delta \eta. \quad (A3)$$

In the context of our sudden step model, $\omega_1$ and $\omega_2$ will correspond to $\omega_{in}$ and $\omega_{out}$, respectively. We take the ‘typical’ omega on the right hand side to correspond to the minimal frequency; supposing that the dynamics may be affected non-adiabatically if the change in frequency is larger than this. To examine to what extent that model may describe a more general situation, where change may be more gradual, we use the logistic function to parametrize the transition:

$$k_{\text{eff}}(\eta) = k + \frac{\delta k}{1 + \exp \left[ -\gamma \left( \frac{n}{n_c} - 1 \right) \right]}.$$

(A4)

Here the parameter $\gamma$ describes the stiffness of the transition, this being steep and steplike for $\gamma \gg 1$, and $\delta$ (which may be positive or negative) the scale of the step in the transition. Thus, in the high energy regime limit, the sound speed $c_s = 1 + \delta$, while $c_s = 1$ when the transition...
FIG. 10. The primordial power spectrum evaluated at different levels of violations of the adiabaticity condition (A3), when transition across the high energy cutoff scale is interpolated using a logistic function (Eq. A4), and numerically integrated. The numbers in the legend keys refer to the order of magnitude above the critical value of $\gamma$ required to violate the adiabaticity condition (Eq. A5). Left panel: interpolation between high energy sound speed $c_s = 100$ and standard regime ($c_s = 1$). Right panel: interpolation between high energy sound speed $c_s = 0.01$ and standard regime. The results are shown for de Sitter inflation, and wavenumbers on the horizontal axis are expressed in terms of the high energy cutoff scale $k_c$, with $k_c/H = 300$.

to the standard low-energy physics regime is complete. In these terms, the characteristic time over which $\omega$ changes between its initial and final value is $\eta_c$. The adiabaticity violation condition can then be written as

$$\gamma > \frac{\text{Min}(c_s^2)}{|c_s - 1|} \left( \frac{k_c}{H} \right), \quad (A5)$$

where we have used $\eta_c = \frac{k_c}{H}$, and $c_s \neq 1$ to the high energy limit sound speed. Since, as we have seen in Section II B 2, the power spectrum is essentially an adiabatic invariant of the dynamics of inflationary perturbations, it is necessary to satisfy this condition in order to modify the standard power spectrum.

Two cases are of particular interest in the context of the present study: $c_s \gg 1$, so that $\text{Min}(c_s) = 1$ and $c_c \ll 1$, when $\text{Min}(c_s) = c_s$. For sound speeds considered here, the minimal value that $gamma$ can reach is when $c_s = 0.01$, and this is of order one or more if $k_c/H \geq 10^4$, as required for keeping the significant changes in power spectrum in the nonlinear regime of structure formation (Section IV A 2). The transition is for such parameters necessarily stiff.

We now show that the transition is stiff even form much smaller $k_c/H = 300$, which is a minimal value, in the sense that with smaller values the effect of the second term in brackets of Eq. (7) becomes important at $\eta \sim \eta_c$ for $c_s = 0.01$ (see discussion relating to inequality 11). For it turns out that the adiabaticity condition needs to be quite strongly violated for sufficient change in the dynamics significantly affects the power spectrum (that significant changes occurs well beyond the adiabaticity breaking condition is common in dynamical systems [169]).

This can be seen from Fig. 10, where we show (left panel) that large changes only occur when $\gamma$ is orders of magnitudes above the value estimated from (A5). This is the case for both the $c_s = 100$ and $c_s = 0.01$, with the former being stiffer still as expected from (A5) [170]. The transition is stiffer still for smaller values of $c_s > 1$ and larger $c_s < 1$. Thus, for sound speed ratios considered here, our simple model of a sudden jump, and accompanying signature of a sudden break in the power spectrum, appears much more generic in describing the transition studied here than may seem a priori.

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