On the maximum number of odd cycles in graphs without smaller odd cycles

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Abstract
We prove that for each odd integer \( k \geq 7 \), every graph on \( n \) vertices without odd cycles of length less than \( k \) contains at most \( (n/k)^k \) cycles of length \( k \). This extends the previous results on the maximum number of pentagons in triangle-free graphs, conjectured by Erdős in 1984, and asymptotically determines the generalized Turán number \( \text{ex}(n, C_k, C_{k-2}) \) for odd \( k \). In contrary to the previous results on the pentagon case, our proof is not computer-assisted.

KEYWORDS
cycle-free, cycles, extremal problems

1 | INTRODUCTION

In 1984, Erdős [4] conjectured that every triangle-free graph on \( n \) vertices contains at most \( (n/5)^5 \) cycles of length 5 and the maximum is attained at the balanced blow-up of a \( C_5 \). Győri [11] proved an upper bound within a factor 1.03 of the optimal. Using flag algebras method, Grzesik [10] and, independently, Hatami, Hladký, Král’, Norine, and Razborov [13] proved that any triangle-free graph on \( n \) vertices has at most \( (n/5)^5 \) copies of \( C_5 \), which is a tight bound for \( n \) divisible by 5. Michael [17] presented a sporadic counterexample to the characterization of the extremal cases by presenting a graph on eight vertices showing that not only a balanced blow-up of a \( C_5 \) can achieve the maximum. Recently, Lidický and Pfender [16], also using flag algebras, completely determined the extremal graphs for every \( n \) by showing that the graph pointed out by Michael is the only extremal graph which is not a balanced blow-up of a pentagon.

Here, we prove the extension of the above results by showing the following theorem.

**Theorem 1.** For each odd integer \( k \geq 7 \), any graph on \( n \) vertices without odd cycles of length smaller than \( k \) contains at most \( (n/k)^k \) cycles of length \( k \). Moreover, the balanced blow-up of a \( k \)-cycle is the only graph attaining this maximum.
It is worth mentioning that, in contrary to the previous results on the pentagon case, our proof is not using flag algebras and is not computer-assisted.

Estimating the maximum number of edges in an $H$-free graph on $n$ vertices, called the Turán number of $H$ and denoted by $\text{ex}(n, H)$, is one of the most well-studied problems in graph theory. The original Turán Theorem [20] solves it for cliques and the classical Erdős–Stone–Simonovits Theorem [5] determines the asymptotic behavior of $\text{ex}(n, H)$ for any other nonbipartite graph $H$. The remaining bipartite case contains many interesting and long-standing open problems, as well as important results, see, for example, surveys by Füredi and Simonovits [7], Sidorenko [19] or, in the case of cycles, the survey by Verstraëte [22].

Generalization of the Turán number, calculating the maximum possible number of copies of a graph $T$ in any $H$-free graph on $n$ vertices, denoted by $\text{ex}(n, T, H)$, is attracting recently a lot of attention. Some specific cases, including the above-mentioned case of $\text{ex}(n, C_5, C_3)$, were considered earlier, but systematic studies of this problem were initiated by Alon and Shikhelman [1]. Especially in the case of cycles many results lately appeared. In particular, Bollobás and Győri [3] proved that $\text{ex}(n, C_3, C_5) = \Theta(n^{3/2})$, Győri and Li [12] extended this result to obtain bounds for $\text{ex}(n, C_3, C_{2k+1})$, which were later improved by Alon and Shikhelman [1] and by Füredi and Özkahya [6]. Recently, Gishboliner and Shapira [9] proved a correct order of magnitude of $\text{ex}(n, C_k, C_{k-2})$ for all even cycles, together with the tight asymptotic value of $\text{ex}(n, C_k, C_{2k})$. Theorem 1 implies the tight asymptotic value of $\text{ex}(n, C_k, C_{k-2})$ for all odd $k$, unknown before.

**Corollary 2.** For any odd integer $k \geq 7$, $\text{ex}(n, C_k, C_{k-2}) = (n/k)^k + o(n^k)$.

The proof of the corollary is a standard application of the Graph Removal Lemma. If a large graph $G$ is $C_{k-2}$-free, then, by the Regularity Lemma, it has at most $o(n^k)$ copies of $C_{\ell}$ for any odd $\ell$ smaller than $k$. By the Graph Removal Lemma, we can eliminate all the copies by removing $o(n^2)$ edges, thus the number of copies of $C_k$ in a graph $G$ would change by at most $o(n^k)$.

The considered problem is closely related to the problem of finding the maximum number of induced cycles of a given length. Pippenger and Golumbic [18] conjectured in 1975 that for each $k \geq 5$, any graph on $n$ vertices contains at most $n^k/(k^k - k)$ induced $k$-cycles and the extremal graphs are iterated blow-ups of $C_k$. This conjecture was confirmed by Balogh, Hu, Lidický, and Pfender [2] for $k = 5$. In their original paper, Pippenger and Golumbic proved a general bound for each $k \geq 5$ within a multiplicative factor of $2e$. This was recently improved to $128e/81$ by Hefetz and Tyomkyn [14] and to 2 by Král’, Norin, and Volec [15]. Our main result is based on the method they developed.

## 2 | MAIN RESULT

Fix an odd integer $k \geq 7$ and let $G$ be any graph without $C_\ell$ for all odd $\ell$ between 3 and $k - 2$. Since there are no odd cycles smaller than $k$, each $k$-cycle in $G$ is induced.

We bound the number of $k$-cycles by bounding the probability that sampling vertices of $G$ one by one at random results in a fixed induced $k$-cycle. However, instead of sampling the vertices in the cycle order, we do it with a small shift and sample the fourth vertex before the third. This is to avoid the situation that a particular 3-vertex induced path in $G$ cannot be extended to a $k$-cycle, which happens, for example, when $G$ is a blow-up of a $k$-cycle.
For any $k$-cycle $v_0v_1...v_{k-1}$ contained in $G$, by a **good sequence** we denote a sequence $D = (z_i)_{i=0}^{k-1}$, where $z_i = v_i$ for $i \leq 1$ and $i \geq 4$, $z_2 = v_3$, and $z_3 = v_2$, that is, $v_2$ and $v_3$ are in the reversed order. Note that there are $2k$ different good sequences corresponding to a single induced $k$-cycle. For any vertices $v$ and $w$, by $d(v, w)$ we denote the minimum distance between the vertices $v$ and $w$ in $G$. Also, for $v \in V(G)$, write $N(v)$ for the neighborhood of a vertex $v$.

For a fixed good sequence $D$, we define the following sets:

- $A_0(D) = V(G)$,
- $A_1(D) = N(z_0)$,
- $A_2(D) = \{w \notin N(z_0) : d(z_1, w) = 2\}$,
- $A_3(D) = N(z_i) \cap N(z_2)$,
- $A_4(D) = \{w : z_0z_1z_2w$ is an induced path $\}$,
- $A_i(D) = \{w : z_0z_1z_2z_3...z_{i-1}w$ is an induced path $\}$ for $5 \leq i \leq k - 2$,
- $A_{k-1}(D) = \{w : z_0z_1z_2z_3...z_{k-2}w$ is an induced cycle $\}$.

We then define a **weight** $w(D)$ of a good sequence $D$ as

$$w(D) = \prod_{i=0}^{k-1} |A_i(D)|^{-1} = \frac{1}{n} \prod_{i=1}^{k-1} |A_i(D)|^{-1}.$$  

This quantity has the following probabilistic interpretation: suppose we want to sample $k$ vertices $w_0, ..., w_{k-1}$, so that $(w_i)_{i=0}^{k-1}$ is a good sequence. We start with choosing $w_0$ at random from all vertices of $G$. Next, we pick any neighbor of $w_0$ to be $w_1$. In general, $w_i$ is a random vertex from the set $A_i((w_i)_{i=0}^{k-1})$ (note that the definition of $A_i(D)$ depends only on first $i$ elements of a sequence $D$). Then, $w(D)$ is just the probability that the sequence $(w_i)_{i=0}^{k-1}$ obtained in this random process is equal to $D$.

In particular, the sum of weights of all good sequences is at most one, since it is the sum of probabilities of pairwise disjoint events.

Fix a $k$-cycle $v_0v_1...v_{k-1}$ in $G$, let $C = \{v_0, v_1, ..., v_{k-1}\}$ be the set of its vertices, and let $D_j = (v_j, v_{j+1}, v_{j+3}, v_{j+2}, v_{j+4}$, ..., $v_{j+k-1})$, for $0 \leq j \leq k - 1$, where the indices are considered modulo $k$, be all the good sequences with the same orientation corresponding to this cycle (half of the total number of good sequences corresponding to this cycle).

If we prove that

$$\left( \sum_{j=0}^{k-1} w(D_j) \right)^{-1} \leq M$$

for some number $M$, then $\sum_{j=0}^{k-1} w(D_j) \geq M^{-1}$. Thus, by summing over all $k$-cycles (with both orientations) and using the fact that the sum of weights of all good sequences is at most one, we get that the total number of $k$-cycles is upper bounded by $M$.

Denote $n_{ij} = |A_i(D_j)|$. Since

$$\left( \sum_{j=0}^{k-1} w(D_j) \right)^{-1} = \left( \sum_{j=0}^{k-1} \prod_{i=0}^{k-1} n_{ij} \right)^{-1} = \left( \sum_{j=0}^{k-1} \left( \frac{n_{ij}}{2} \right)^{-1} \prod_{i=2}^{k-1} n_{ij}^{-1} \right)^{-1},$$
the maximum possible value of

\[ n \left( \sum_{j=0}^{k-1} \left( \frac{n_{i,j}}{2} - 1 \right) \prod_{i=2}^{k-1} n_{i,j} \right)^{-1} \]

is an upper bound on the number of \( k \)-cycles in \( G \).

Using the inequality between harmonic mean and geometric mean of \( k \) terms and the inequality between geometric mean and arithmetic mean of \( k(k-1) \) terms, we obtain

\[
\frac{n^{k-1}}{k} \left( \frac{1}{k} \sum_{j=0}^{k-1} \left( \frac{n_{i,j}}{2} - 1 \right) \prod_{i=2}^{k-1} n_{i,j} \right)^{\frac{1}{k-1}} \leq \frac{n}{k} \left( \prod_{j=0}^{k-1} n_{i,j} \right)^{\frac{1}{k-1}} = \frac{n}{k} \left( \frac{1}{k(k-1)} \sum_{j=0}^{k-1} \left( \frac{n_{i,j}}{2} + \sum_{i=2}^{k-1} n_{i,j} \right) \right)^{k-1}.
\]

Claim 3. The following inequality holds:

\[
\sum_{j=0}^{k-1} \left( \frac{n_{i,j}}{2} + \sum_{i=2}^{k-1} n_{i,j} \right) \leq n(k-1),
\]

with equality if and only if each vertex of \( G \) is connected to two vertices of \( C \) at distance two.

**Proof.** It is enough to prove that the contribution of any vertex \( w \in V(G) \) to the above sum is at most \( k - 1 \), and that such a contribution can only occur if \( w \) is connected to two vertices of \( C \) at distance two.

Notice that any vertex \( w \in V(G) \) has at most two neighbors in \( C \), since otherwise it creates a shorter odd cycle. For the same reason, each vertex \( w \) satisfies the following property:

(\( \star \)) There are at most three vertices in \( C \) at distance exactly 2 from \( w \), and any two such vertices are not adjacent.

If \( w \) has no neighbors in \( C \), then, for each \( j \), it can contribute only to \( n_{2,j} \). Moreover, if for some \( j \) we have \( d(w, v_{j-1}) = 2 \), then \( d(w, v_{j-1}) > 2 \) and \( d(w, v_{j+1}) > 2 \) by (\( \star \)), and so \( w \) does not contribute to \( n_{2,j} \) and \( n_{2,j-2} \). Therefore, such \( w \) contributes in total by at most \( k - 2 \).

Assume, then, that \( w \) has exactly one neighbor in \( C \)—from symmetry, let it be \( v_0 \). Because of having only one neighbor, for each \( j \), \( w \) does not contribute to \( n_{3,j} \) and \( n_{k-1,j} \).

To contribute to \( n_{ij} \) for \( i \notin \{2, 3, k-1\} \), \( w \) needs to be connected to \( v_{i+j-1} \), and so it can contribute only to \( n_{i,0} \) and \( n_{i,k-1-i+1} \) for \( 4 \leq i \leq k-2 \). Finally, \( w \) can contribute to \( n_{2,j} \) only if \( d(w, v_{j+1}) = 2 \) and \( w \notin N(v_j) \). By (\( \star \)), there are at most three vertices in \( C \) at distance 2 from \( w \), but one of them is \( v_1 \) and \( w \in N(v_0) \), so \( w \) contributes to \( \sum_{j=0}^{k-1} n_{2,j} \) by at most 2. It follows that in this case \( w \) contributes to the considered sum in total by at most \( k - 3 + \frac{1}{2} \).
Finally, assume that $w$ has exactly two neighbors in $C$. These neighbors have to be at distance 2 in $C$, as otherwise it creates an odd cycle of length shorter than $k$. From symmetry, let $v_{k-1}$ and $v_1$ be the neighbors of $w$. Then, $d(w, v_i) = 2$ for $i = k - 2, 0, 2$, and there are no more $i$ with this property by $(\star)$. Therefore, $w$ contributes only to $n_{1,k-1}, n_{1,2}, n_{2,k-3}, n_{3,k-2}$, and $n_{i,k-1}$ for $4 \leq i \leq k - 1$, hence $w$ contributes to the considered sum in total by $k - 1$. □

Using the above claim, we immediately get the wanted bound $(n/k)^k$ for (1). It follows that the total number of $k$-cycles in $G$ is at most $(n/k)^k$, as desired.

If a graph $G$ is achieving this bound, then $n$ needs to be divisible by $k$ and we need to have equalities in all the inequalities we considered. In particular, for each $k$-cycle, all the other vertices of $G$ need to be connected with exactly two vertices of the cycle, which are at distance 2 (as in the blow-up of a $k$-cycle). Since we used the arithmetic and geometric mean (AM-GM) inequality, all the blobs need to have the same size. Thus, one can easily deduce that the only graph attaining the maximum is the balanced blow-up of a $k$-cycle.

In the case of $n$ not divisible by $k$, to prove this way an exact bound on the number of $k$-cycles, one cannot use the AM-GM inequalities, but bound (1) using Claim 3 in a bit more sophisticated way. Trying just to maximize (1) over all such choices of $n_{ij} \in \mathbb{N}$ that the inequality from Claim 3 holds, one can get a higher value than for the numbers $n_{ij}$ corresponding to blow-ups of a $k$-cycle, but such values may not be realizable by any graph. Therefore, using this approach, one would have to take into consideration also other relations between the numbers $n_{ij}$. Still, if $n$ is big enough in relation to $k$, then the maximum of (1) needs to be achieved when there is an equality in Claim 3. Thus, for $n$ big enough, the only graph achieving the maximum number of $k$-cycles is a balanced blow-up of a $k$-cycle.

3 | CONCLUDING REMARKS AND OPEN PROBLEMS

In our proof, basically the only place where we are using that $k$ is an odd number is to say that if a $k$-cycle is not induced (or, more generally, there is a short path in the graph between distant vertices of this cycle), then the graph contains a smaller odd cycle. This is not the case if $k$ is an even number. Moreover, we do not have an analogue of Theorem 1 for even $k$, as forbidding any even cycle prevents from having big blow-ups of a single edge. Nevertheless, one can carefully analyze the proof to obtain the following result on induced even cycles.

Observation 4. For each even integer $k \geq 8$, any graph on $n$ vertices without induced cycles $C_\ell$ for $\ell = 3$ and $5 \leq \ell \leq k - 1$ and without induced $C_6$ with one or two main diagonals contains at most $(n/k)^k$ induced cycles of length $k$.

It seems possible that the same construction (balanced blow-up of a $k$-cycle) gives the best possible number of induced $k$-cycles also if we only forbid triangles.

Conjecture 1. For each integer $k \geq 5$, any triangle-free graph on $n$ vertices contains at most $(n/k)^k$ induced cycles of length $k$.

Looking from the other side, if we forbid $C_\ell$ for some odd $\ell$ and try to maximize the number of $C_k$ for some larger odd $k$, it seems that, asymptotically, the best is always to take a balanced blow-up of an $(\ell + 2)$-cycle.
Conjecture 2. For any odd integers \( k > \ell \geq 3 \), it holds \( \text{ex}(n, C_k, C_{\ell}) = \left(\frac{k}{k - (\ell + 2)}\right) + \left(\frac{k}{k - 3(\ell + 2)}\right) + \cdots + \left(\frac{n^{\ell + 2}}{(\ell + 2)^2}\right) + o(n^k) \).

Using publicly available software Flagmatic [21], one can numerically verify that Conjecture 1 holds for \( k \leq 8 \) and Conjecture 2 holds for \( k \leq 7 \).

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