Linear RNNs Provably Learn Linear Dynamic Systems

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Abstract—We study the learning ability of linear recurrent neural networks with Gradient Descent. We prove the first theoretical guarantee on linear RNNs to learn any stable linear dynamic system using any a large type of loss functions. For an arbitrary stable linear system with a parameter \( pC \) related to the transition matrix \( C \), we show that despite the non-convexity of the parameter optimization loss if the width of the RNN is large enough (and the required width in hidden layers does not rely on the length of the input sequence), a linear RNN can provably learn any stable linear dynamic system with the sample and time complexity polynomial in \( \frac{1}{pC} \). Our results provide the first theoretical guarantee to learn a linear RNN and demonstrate how can the recurrent structure help to learn a dynamic system.

I. INTRODUCTION

Recurrent neural network(RNN) is a very important structure in machine learning to deal with sequence data. It is believed that using the recurrent structure, RNNs can lean complicated transformations of data over extended periods. Non-linear RNN has been proved to be Turing-Complete[1], thus can simulate arbitrary procedures. However, training RNN requires optimizing highly non-convex functions which are very hard to solve. On the other hand, it is widely believed[2] that deep linear networks can capture some important aspects of optimization in deep learning and there are a series of recent papers trying to study the properties of deep linear networks [3, 4, 5]. Meanwhile, learning linear RNN itself is not only an important problem in System Identification but also useful for the language modeling in natural language processing [6]. In this paper, we study the non-convex optimization problem for learning linear RNNs.

Suppose there is a \( d_p \)-order and \( d \)-dimension linear system with the following form:

\[
h_t = Ch_{t-1} + D x_t, \\
y_t = G h_t(x),
\]

where \( C \in \mathbb{R}^{d_p \times d_p}, D \in \mathbb{R}^{d_p \times d} \) and \( G \in \mathbb{R}^{d \times d_p} \) are unknown system parameters.

At time \( t \), this system output \( y_t \). It is nature to consider the system identification problem to learn the unkonw system parameters from its outputs. We consider a new linear (student) RNN with the form:

\[
h'_t = Wh'_t-1 + Ax_t, \\
y'_t = Bh'_t(x),
\]

with \( W \in \mathbb{R}^{m \times m}, A \in \mathbb{R}^{m \times d}, B \in \mathbb{R}^{m \times d_y} \) and train the parameters \( W, A, B \) to fit the output \( y'_t \) of \( [1] \).

Just like what is commonly done in machine learning, one may consider to collect data \( \{x'_t, y'_t\} \) then optimize the empirical loss:

\[
L(W, A, B) = \frac{1}{n \cdot T} \sum_{i=1}^{n} \sum_{t=1}^{T} L(y'_t, y'_t'),
\]

\[
= \frac{1}{n \cdot T} \sum_{i=1}^{n} \sum_{t=1}^{T} L(Gh_t(x'), Bh'_t(x')),
\]

with Gradient Descent Algorithm, where \( L \) is a convex loss function.

Therefore the following questions arise naturally:

- Can gradient descent learn the target RNN in polynomial time and samples?
- What kinds of random initializations (for example, how large widths will \( W \) be?) do we need to learn the target RNN?

These problems look easy since it is very basic and important for the system identification problem. However, the loss \( \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} L(y'_t, y'_t') \) is non-convex and in fact even for SISO(single-input single-output, which means \( x_t, y_t \in \mathbb{R}^1 \)) systems, this question is far from being trivial. Only after the work in [7], the SISO case is solved. In fact, as pointed out in [8], although the widely used method in system identification is the EM algorithm [9], yet it is inherently non-convex and EM method will get stuck in bad local minima easily.

One naive method is to only optimize the loss \( L(y'_t, y'_t') \), which is a convex loss function. However, when \( y'_t \) is not accurately observed, for example we can only observe \( y_t = \hat{y}_t + n_t \) and \( L(x, y) = ||x-y||^2 \) where \( n_t \) is white noise with \( \sigma \) variance at time \( t \), the results by optimizing \( L(y'_t, y'_t') \) may not be optimal for the entire sequence loss \( \frac{1}{T} \sum_{t=1}^{T} L(y_t, y'_t) \). In fact a naive estimate from \( L(y'_t, y'_t') \) will output an estimation with error \( \sigma^2 \) but \( \frac{1}{T} \sum_{t=1}^{T} L(y_t, y'_t) \) may output an estimation with error \( O(\sigma^2/\sqrt{T}) \). It is shown in [7] that under some independent conditions of the inputs, SGD (stochastic gradient descent) converges to the global minimum of the maximum likelihood objective of an unknown linear time-invariant dynamical system from a sequence of noisy observations generated by the system and over-parameterization is helpful. However, their methods heavily rely on the SISO property \( (x, y) \in \mathbb{R}^1 \) of the system, and the condition \( x_i \) for different \( t \) are i.i.d. from Gaussian distribution. Their method can not be generalized to the systems with \( x \in \mathbb{R}^d \) and \( d > 1 \).
It is still open under which conditions can SGD be guaranteed to find the global minimum of the linear RNN loss.

In this paper, we propose a new NTK method inspired by the work \cite{10} and the authors’ previous work \cite{11} on non-linear RNN. And this is completely different method from that in \cite{7} so we avoid the defect that the method in \cite{7} can only be used in the SISO case.

We show that if the width $m$ of the linear RNN $\mathcal{P}$ is large enough (polynomial large), SGD can provably learn any stable linear system with the sample and time complexity only polynomial in $\frac{1}{1-\rho_C}$ and independent of the input length $T$, where $\rho_C$ is roughly the spectral radius (see Section III-A) of the transition matrix $C$. Learning linear RNN is a very important problem in System Identification. And since Gradient Descent with random initialization is the most commonly used method in machine learning, we are trying to understand this problem in a “machine-learning style”. We believe this can provide some insights for the recurrent structure in deep learning.

II. PROBLEM FORMULATION

We consider the target linear system with the form:

$$p_t^i = C p_{t-1}^i + D x_t^i,$$

$$y_t^i = G p_t^i,$$

which is a stable linear system with $\|C D\| \leq c_p \cdot \rho_C^k$ for all $k \in \mathbb{N}$ and $\rho_C < 1$, where $c_p > 0$, $\|G\|, \|D\| = \Theta(1)$. For a given convex loss function $L$, we set the loss function

$$L(C, D, G) = \mathbb{E}_{x, y \sim \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} L(y_t, \tilde{y}_t).$$

We define the global minimum $OPT_{\rho_C}$ as

$$OPT_{\rho_C} = \inf_{C, D, G} \mathbb{E}_{x, y \sim \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} L(y_t, \tilde{y}_t).$$

with $\|C D\| \leq c_p \rho_C^k$, $k \in \mathbb{N}$, $c_0 > 0$ is an absolute constant.

Let the sequences $\{(x_t^i = (x_t^i, x_{t+1}^i, ..., x_{T-1}^i))_{t=1}^{K}\}$, $\{(y_t^i = (y_t^i, y_{t+1}^i, ..., y_{T-1}^i))_{t=1}^{K}\}$ be $K$ samples i.i.d. drawn from

$$\mathcal{D} = \{(x_1, x_2, ..., x_T) \times (y_1, y_2, ..., y_T) \in \mathbb{R}^{d \times T} \times \mathbb{R}^{d \times T}\}.$$

We consider a “student” linear system (RNN) to learn the target one. Let $f_t(\mathbf{W}, A, x^i)$ be the t-time output of a linear RNN with input $x^i$ and parameters $\mathbf{W} \in \mathbb{R}^{n \times m}$, $A \in \mathbb{R}^{m \times d}$:

$$h_t(x) = \mathbf{W} h_{t-1} + A x_t,$$

$$f_t(\mathbf{W}, A, x) = \mathbf{B} h_t(x).$$

Our goal is to use $f_t(\mathbf{W}, A, x^i)$ and the $K$ samples to fit the empirical loss function and keeping the generalization error bound small.

III. OUR RESULT

The main result is formulated in the PAC-Learning setting as follow:

**Theorem 1:** (Informal) Under the conditions in the last section, suppose the entries of $\mathbf{W}$ in the student RNN are randomly initialized by i.i.d. generated from $\mathcal{N}(0, \frac{\epsilon}{m})$. We use SGD algorithm to optimize. $\mathbf{W}_k, A_k$ are the $k$-th step outputs of SGD algorithm.

For any $\epsilon, \delta > 0$, and $0 < \rho_C < 1$, there exist parameters $m^* = poly(\frac{1}{1-\rho_C}, \epsilon^{-1}, \delta^{-1}, c_p)$ and $K = poly(\frac{1}{1-\rho_C}, \epsilon^{-1}, \delta^{-1}, c_p)$ such that if $m > m^*$, with probability at least $1 - \delta$, SGD can reach

$$\mathbb{E}_{x, y \sim \mathcal{D}} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{T} \sum_{t=1}^{T} L(y_t, \tilde{f}_t(\mathbf{W}_k, A_k, x))$$

$$\leq OPT_{\rho_C} + \epsilon,$$

in $K$ steps.

**Remark 3.1:** Theorem II induces gradient descent with linear RNNs can provably learn any stable linear system with the iteration and sample complexity polynomial in $\frac{1}{1-\rho_C}$. This result is consistent with the previous Gradient Descent based method \cite{7} to learn SISO linear systems.

In our theorem, all the parameters do not rely on the length $T$. Note that suppose at different time, $x_t^i$ are i.i.d. drawn from a distribution $\mathcal{D}'$, $n_t$ is the white noise and $T$ is large enough,

$$\mathbb{E}_{n_t, x_t \sim \mathcal{D}'} \frac{1}{T} \sum_{t=1}^{T} ||\tilde{y}_t + n_t - \tilde{f}_t(\mathbf{W}, A, x)||^2$$

$$\leq \lim_{T \to \infty} \mathbb{E}_{n_t, x_t \sim \mathcal{D}'} \frac{1}{T} \sum_{t=1}^{T} ||\tilde{y}_t + n_t - \tilde{f}_t(\mathbf{W}, A, x)||^2 + \epsilon.$$

Thus optimizing the large $T$ loss is enough to predict the complete dynamic behaviors of the target system.

A. Scale of $c_p$ and the Comparison with Previous Results

In our main theorem \cite{1} the parameters are polynomial in $c_p$. We should note that although we assume the target system is stable, this only means

$$\rho(C) = \lim_{k \to \infty} \|C^k\|^{1/k} < 1.$$

In fact, suppose $C \in \mathbb{R}^{N \times N}$, generally we have (see e.g. corollary 3.15 in \cite{12}):

$$\|C^k\| \leq \sqrt{N} \sum_{j=0}^{N-1} \left(\begin{array}{c} N-1 \\ j \end{array}\right) \left(\frac{k}{j}\right) \|C\|^j \rho(C)^{k-j}.$$

When we set $\rho_C = \rho(C)$, $c_p$ can be very large and generally we should set $\rho_C > \rho(C) + \epsilon$ to make $c_p$ be polynomial in $N$.

On the other hand, the scale of $c_p$ is closely related to the so-called “acquisitiveness” systems.

**Definition 1:** Let $z \in \mathbb{C}$. A SISO $N$-order linear system with the transfer function $\frac{\alpha}{\beta}^{\nu}$ is called $\alpha$-acquisitiveness if

$$\{p(z)/z^{\nu} : |z| = \alpha\} \subseteq S,$$

where

$$S = \{z : Re(z) \geq (1+\tau_0) \cdot Im(z)\} \cap \{z : \tau_1 \leq Re(z) \leq \tau_2\}.$$


And we have

**Lemma 2:** (Lemma 4.4 in [7]) Suppose the target linear system is SISO and \( \alpha \)-acquiescent. For any \( k \in \mathbb{N} \),

\[
\| C^k D \| \leq 2 \pi n \alpha^{-2n} / \gamma_1 \cdot \alpha^k.
\]

Thus in the SISO case, under the “acquiescent” conditions, our Theorem 1 reduces to the main result in [7].

**Corollary 3:** (Corresponding to Theorem 5.1 in [7]) Supposing a SISO linear system is \( \rho C \)-acquiescent, it is learnable in polynomial time and polynomial samples. And for MIMO systems, our condition \( \| C^k D \| \leq c \rho^k \gamma C, k \in \mathbb{N} \) is a good generalization for MIMO systems.

B. Our Techniques

Our proof technique is closely related to the recent works on deep linear network [3], non-linear network with neural tangent kernel [13], and non-linear RNN [10], [16]. Similar to [13], we carefully upper and lower bound eigenvalues of this Gram matrix throughout the optimization process, using some perturbation analysis. At the initialization point, we consider the spectral properties of Gaussian random matrices. Using the linearization method, we can show these properties hold throughout the trajectory of gradient descent. Then we only need to construct a solution near the random initialization. And the distance from the solution to the initialization can be bounded by the stability of the system.

**Notions.** For two matrices \( A, B \in \mathbb{R}^{m \times n} \), we define \( \langle A, B \rangle = \text{Tr}(A^T B) \). We define the asymptotic notations \( O(\cdot), \Omega(\cdot), \Theta(\cdot), \text{poly}(\cdot) \) as follows. \( a_n, b_n \) are two sequences. \( a_n = \Theta(b_n) \) if \( \lim sup_{n \to \infty} |a_n / b_n| < \infty \), \( a_n = \Omega(b_n) \) if \( \lim inf_{n \to \infty} |a_n / b_n| > 0 \), \( a_n = \Theta(b_n) \) if \( a_n = \Omega(b_n) \) and \( a_n = O((b_n)^k) \). \( O(\cdot), \Omega(\cdot), \Theta(\cdot), \text{poly}(\cdot) \) are notions which hide the logarithmic factors in \( O(\cdot), \Omega(\cdot), \Theta(\cdot), \text{poly}(\cdot) \). \( \| \cdot \| \) and \( \| \cdot \|_2 \) denote the 2-norm of matrices. \( \| \cdot \|_1 \) denotes the 1-norm. \( \| \cdot \|_F \) is the Frobenius-norm.

IV. RELATED WORKS

**Deep Linear Network.** The provable properties of the loss surface for deep linear networks were firstly shown in [5]. In [3], it is shown that if the width of the L-layer deep linear network is large enough (only depends on the output dimension, the rank \( r \) and the condition number \( \kappa \) of the input data), randomly initialized gradient descent will optimize deep linear networks in polynomial time in \( L, r, \kappa \). Moreover, in [4], the linear ResNet is studied and it is shown that Gradient Descent provably optimizes wide enough deep linear ResNets and the width does not rely on the number of layers.

**Over-Parameterization.** Non-linear networks with one hidden node are studied in [17] and [18]. In these works, it is shown that, for a single-hidden-node ReLU network, under a very mild assumption on the input distribution, the loss is one point convex in a very large area. However, for the networks with multi-hidden nodes, the authors in [19] pointed out that spurious local minima are common and indicated that an over-parameterization (the number of hidden nodes should be large) assumption is necessary. Similarly, [7] showed that over-parameterization can help in the training process of a linear dynamic system. Another import progress is the theory about neural tangent kernel (NTK). The techniques of NTK for finite width network are studied in [13], [13], [19], [13] and [16].

**Learning Linear System.** Prediction problems of time series for linear dynamical systems can be traced back to Kalman [20]. In the case that the system is unknown, the first polynomial guarantees of running time and sample complexity bounds for learning single-input single-output (SISO) systems with gradient descent are provided in [7]. For MIMO systems, it was shown in [21],[6] that the spectral filtering method can be provably learned with polynomial guarantees of running time and sample complexity.

V. PROBLEM SETUP AND MAIN RESULTS

In this section, we introduce the basic problem setup and our main results.

Consider sequences \( \{ x^i = (x^i_1, x^i_2, \ldots, x^i_T) \} \) and the label \( \{ y^i = (y^i_1, y^i_2, \ldots, y^i_T) \} \) in the data set. \( x^i \in \mathbb{R}^d, y^i \in \mathbb{R}^{d_y} \) and \( \| x^i \|, \| y^i \| \leq O(1) \). We assume \( d_y \leq d = O(1) \) and omit them in the asymptotic symbols. We study the linear RNN as:

\[
\begin{align*}
& h_0(x) = 0, h_t(x) = \tilde{W} h_{t-1} + A x_t, \\
& \tilde{f}_t(\tilde{W}, A, x) = B h_t(x) \in \mathbb{R}^{d_y}.
\end{align*}
\]

Assume \( L^*(x) = L(y_t, x_t) \) is convex and locally Lipschitz convex function: for any \( x, y \), when \( \| x \| \leq C, \| y \| \leq C \),

\[
\| \nabla_x L^*(x) \| \leq l_0(1 + C).
\]

Then we perform algorithm [1]

**Algorithm 1** Learning Stable Linear System with SGD

**Input:** Sequences of data \( \{ x, y \} \), learning rate \( \eta \), initialization parameter \( 0 < \rho < 1 \).

**Initialization:** The entries of \( \tilde{W}_0 \) and \( A_0 \) are i.i.d. generated from \( N(0, \frac{1}{m}) \) and \( N(0, \frac{1}{d_y}) \). The entries of \( B \) are i.i.d. generated from \( N(0, \frac{1}{d_y}) \).

for \( k = 0, 1, 2, 3, \ldots, K-1 \) do

\[
\tilde{W}_{k+1} = \tilde{W}_k - \frac{\eta}{\rho} \sum_t \nabla_{\tilde{W}_k} L(y^t, \tilde{f}_t(\tilde{W}_k, A_k, x^t))
\]

Randomly sample a sequence \( x^t \) and the label \( y^t \).

\[
A_{k+1} = A_k - \frac{\eta}{\rho} \sum_s \nabla_{A_k} L(y^t, \tilde{f}_t(\tilde{W}_k, A_k, x^t))
\]

end for

In fact, we have

**Theorem 4:** Assume there is \( \delta \in [0, e^{-1}) \). Let \( \rho_1 = \frac{1}{1+10 \frac{\pi}{\sqrt{\rho_0}}}, 0 < \rho_0 < 1 \). Set the initialization parameter \( \rho = \rho_1 \cdot \rho_0^2 \). Given an unknown distribution \( D \) of sequences of \( \{ x, y \} \), let \( \tilde{W}_k, A_k \) be the output of Algorithm [1].
For any small $\epsilon > 0$, there are parameters $^1$

\[
T_{\text{max}} = \Theta\left(\frac{1}{\log(\frac{1}{\rho_0})}\right) \cdot \left(2\log\left(\frac{c_\rho}{1 - \rho_0}\right) + \log(\frac{1}{\epsilon})\right) + \log\sqrt{\log(T_{\text{max}}/\delta) + \frac{1}{2}\log(m)}\right)\}
\]

\[
b = \Theta\left(\frac{T_{\text{max}}^4 b^4}{\nu c^2}\right),
\]

\[
K = \Theta\left(\frac{T_{\text{max}}^4 b^4}{\nu c^2}\right),
\]

\[
m^* = \Theta\left(\frac{c_\rho^2 K^4 (1 - \rho_0)^8 c^2}{b^6}\right) + \Theta\left(\frac{1}{\delta}\right),
\]

\[
\nu = \Theta\left(\frac{c_\rho^2 (1 - \rho_0)^{12}}{T_{\text{max}}^4 \cdot \frac{b^6}{(1 + 2b)^6}}\right),
\]

such that with probability at least $1 - \delta$, if $m > m^*$, for any $C, D, G$, the algorithm outputs satisfy:

\[
\frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{T} \sum_{t=1}^{T} \left\{ L(y_t^k, \tilde{f}_t(W_k, A_k), x_t^k) - L(y_t^k, y_t^k(C, D, G)) \right\} \leq O(\epsilon),
\]

where $\tilde{y}_t^k(C, D, G)$ is the output from the linear system:

\[
p_t^k = Cp_t^k - 1 + Dx_t^k,
\]

\[
y_t^k = Gp_t^k,
\]

with $||C^k D|| \leq c_\rho d^k$, for all $k \in \mathbb{N}$, $D \in \mathbb{R}^{d_x \times d}$, $p_t \in \mathbb{R}^{d}$, $C \in \mathbb{R}^{d_x \times d}$, $G \in \mathbb{R}^{d \times d_y}$.

**Lemma 5:** For the parameters of the last theorem, when $m > m^*$ and $\epsilon$ is small enough, we have the following results:

1) $\frac{T_{\text{max}}^4 b^4}{K m^*} = \Theta(\epsilon)$

2) $l_0(1 + 2b) \cdot l_0^2(1 + 2b)^2 \cdot \frac{m \sqrt{m}}{(1 - \rho_0)^{1 + 2b}} \leq O(\epsilon)$

3) $K^2 \eta^2 \cdot \frac{m \sqrt{m}}{(1 - \rho_0)^{1 + 2b}} \cdot l_0^2(1 + 2b)^2 \cdot l_0(1 + 2b) \leq O(\epsilon)$

4) $\frac{m \sqrt{m}}{(1 - \rho_0)^{1 + 2b}} \leq O(\epsilon)$

5) $\frac{b^4 d^2 c \frac{T_{\text{max}}^4 b^4}{m^*}}{m^*} \leq O(\frac{1}{\sqrt{1 + 2b}})$

As for the population loss, we have:

**Theorem 6:** (Rademacher complexity for RNN) Under the condition in Theorem 4, with probability at least $1 - \delta$,

\[
\mathbb{E}_{x, y \sim D} \frac{1}{T} \sum_{t=1}^{T} \left\{ L(y_t, \tilde{f}_t(W_k, A_k, x)) - L(y_t, y_t) \right\} - \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{T} \sum_{t=1}^{T} \left\{ L(y_t^k, \tilde{f}_t(W_k, A_k), x^k) - L(y_t^k, y_t^k) \right\} \leq O(\epsilon).
\]

Since $\rho_1 \rightarrow 1$ as $m^* > \text{poly}(\frac{1}{\rho_0})$ large enough, we have $\rho_1 > \rho_0$ and $\rho > \rho_0^3$. Therefore from the above two theorems we have the following corollary:

**Corollary 7:** Let the initialization parameter be $\rho C < 1$. For any small $\epsilon > 0, \delta > 0$, there is a parameter $m^* = \text{poly}(\frac{1}{\epsilon}, \frac{1}{1 - \rho_0})$ such that with probability at least $1 - \delta$, if $m > m^*, K = \text{poly}(\frac{1}{1 - \rho_0}, 1, \frac{1}{\epsilon})$, the algorithm outputs satisfy:

\[
\mathbb{E}_{x, y \sim D} \frac{1}{T} \sum_{t=1}^{T} L(y_t, \tilde{f}_t(W_k, A_k, x)) \leq O(\rho C + O(\epsilon)).
\]

**Remark 5.1:** In this paper, our results only assume for some constant $C$:

\[
||\nabla_x L^*(x)|| \leq O(1 + C).
\]

This assumption is a very mild condition for the loss function $L(x)$, thus we, in fact, do not previously assume the form of the noise (for example, optimizing the square loss is to optimize the maximum likelihood objective of the Gaussian noise, and $l^1$ loss is to optimize that for Laplace noise) and our result can even apply to not only the the regression problems but also the classification problems. In this aspect, this result improves upon previous methods to learn linear dynamical systems in [71] and [8] which highly rely on the form of the square loss.

VI. PRELIMINARY PROPERTIES

Before proving Theorem 4 and 5, we need some properties of Gaussian random matrices and linear RNNs. The proof is in the Supplementary Materials.

To simplify symbols, in the latter part of the paper, we set $W = \frac{W_k}{\rho}$ and $^2$

\[
f_t(W, A) = \tilde{f}_t(W, A, x) = \sum_{t_0=0}^{t-1} \rho^0 B(\prod_{\tau=1}^{t_0} W) A X_{t_0-t_0}.
\]

Then

\[
W_{k+1} = W_k - \frac{\eta}{T \rho} \sum_{t=1}^{T} \nabla_{W_k} L(y_t^k, \tilde{f}_t(W_k, A_k, x^k)),
\]

\[
W_k = \frac{\eta}{T} \sum_{t=1}^{T} \nabla_{W_k} L(y_t^k, \tilde{f}_t(W_k, A_k, x^k)).
\]

Then we only need to consider $W_k$. The entries of $W_0$ are i.i.d. generated from $N(0, \frac{1}{m})$. Let $B(W_0, \omega) = \{W \mid ||W - W_0||_F \leq \omega\}$ and $B(A, \omega) = \{A \mid ||A - A_0||_F \leq \omega\}$.

A. Properties of Random Matrix

One of the key points in this paper is that with high probability, the spectral radius of matrix $W_0$ will be less than $\rho_1$ and when $m \rightarrow \infty$, $\rho_1 \rightarrow 1$. In fact, we have:

**Lemma 8:** With probability at least $1 - \text{exp}(\Omega(\text{log}^2 m))$ (it is larger than $1 - \delta$ when $m > m^*$), there exists $L = c_0 \cdot \sqrt{m} \log m \in \mathbb{N}$ such that for all $k \geq L$,

\[
||W_0^k|| \leq \rho_1^{-k},
\]

and for all $k < L$,

\[
||W_0^k|| \leq \rho_1^{-L},
\]

where $c_0 > 1$ is an absolutely constant. Meanwhile, for all $k \leq 2L$, with probability at least $1 - \text{exp}(\Omega(\text{log}^2 m))$,

\[
||W_0^k|| \leq 2 \sqrt{k}.
\]

$^1$We omit $x$ in $f_t(W, A, x)$ when it doesn’t lead to misunderstanding.
In the rest of this paper, all the probabilities are considered under the condition in Lemma 8.
As a corollary, let \( k \geq 2L, \omega_0 = \frac{1}{\rho_0} - 1 \) and when \( \| W - W_0 \| \leq \omega \leq \omega_0 \), we have
\[
\| (\rho_1 \cdot \rho_0^2 \cdot W)^{k} \| = \rho_1^k \sum_{i=0}^{k} C_i^k \cdot \| W_0^i \| \cdot \| W - W_0 \|^{k-i}
\]
\[
= \rho_1^k \sum_{i=0}^{k} C_i^k \cdot \| W_0^i \| \cdot (\rho_1 \cdot \rho_0^2)^{k-i} \cdot (1 - 1)\rho_1^{-i}
\]
\[
\leq \rho_1^k \sum_{i=0}^{k} C_i^k (\rho_1 \cdot \rho_0^2)^{2i} \cdot \| W_0^i \| \cdot \| W_0^i \| \cdot \| W - W_0 \|^{k-i}
\]
\[
\leq \rho_1^k k\rho_0^2 \sqrt{k}.
\]

**Lemma 11:** If \( m > \Omega(\log(\tau \cdot d)) \), with probability at least 1 - \( \delta \):
\[
\| B(\prod_{\tau=1}^{t} W)A \| \leq \sqrt{d\log(\tau \cdot d)/\delta}.
\]

**B. Properties of Linear RNN**

For any vector \( v_1, u_1 \in \mathbb{R}^d, v_2, u_2 \in \mathbb{R}^d \) with \( \| v_1 \| = || v_2 \| = \| u_2 \| = 1 \), if \( m^{1/2} > \Omega(\tau^d \cdot d) \), with probability at least 1 - \( \exp(-\Omega(m^2)) \), for any \( 0 \leq t, t' \leq \tau - 1, t \neq t' \):
\[
\| (u_1)^T B(\prod_{k=1}^{t} W_k) \cdot \{(t') W_0 \}^T B^T v_1 \| \leq 24\tau d^2 \log m,
\]
and
\[
\| (u_2)^T A_0^T (\prod_{k=1}^{t} W_k) \cdot \{(t') W_0 \}^T B^T v_1 \| \leq 24\tau d^2 \log m.
\]

**Lemma 12:** For any vector \( v_1, u_1 \in \mathbb{R}^d, v_2, u_2 \in \mathbb{R}^d \) with \( \| v_1 \| = || v_2 \| = \| u_2 \| = 1 \), if \( m^{1/2} > \Omega(\tau^d \cdot d) \), with probability at least 1 - \( \exp(-\Omega(m^2)) \), for any \( 0 \leq t, t' \leq \tau - 1, t \neq t' \):
\[
\| (u_1)^T B(\prod_{k=1}^{t} W_k) \cdot \{(t') W_0 \}^T B^T v_1 \| \leq 24\tau d^2 \log m,
\]
and
\[
\| (u_2)^T A_0^T (\prod_{k=1}^{t} W_k) \cdot \{(t') W_0 \}^T B^T v_1 \| \leq 24\tau d^2 \log m.
\]

Based on these results, we can only consider the properties of \( \rho_0^t B(\prod_{\tau=1}^{t} W)AX_{t-t_0} \) for bounded \( t_0 \). We have the following results:

**Lemma 10:** For any vector \( v_1 \in \mathbb{R}^d, v_2 \in \mathbb{R}^d \) with \( \| v_1 \| = \| v_2 \| = 1 \), if \( m > O(\tau^d \cdot d) \), with probability at least 1 - \( \exp(-\Omega(m/\tau^2)) \):
\[
0.9 \leq || \prod_{\tau=1}^{t} W_0 A_0 v_2 || \leq 1.1,
\]
\[
0.9 \leq \frac{1}{\sqrt{m}} || \prod_{\tau=1}^{t} W_0 B^T v_1 || \leq 1.1,
\]
for any \( 0 \leq t \leq \tau - 1 \) and \( \tau > 0 \).

**Lemma 14:** Under the condition in Theorem 4 with probability at least 1 - \( \delta \):
\[
\| f_t^\{lin\}(W, A) - f_t^\{lin,\tau\}(W, A) \| \leq \Theta(\sqrt{m^2} \rho_0^2 (1 - \rho_0)^3).
\]

Then since we set
\[
T_{max} > \Theta\left(\frac{1}{\log(\rho_0)}\right) \cdot \left\{3\log\left(\frac{1}{1 - \rho_0}\right) + \log b + \log\left(\frac{1}{\epsilon}\right) + \frac{1}{2} \log(m) + \log T_{max}\right\},
\]
We have
\[ ||f^t_{\text{lin}}(W, A) - f^t_{\text{lin}, T_{\text{max}}}(W, A)|| \leq \epsilon/b. \] (26)
Therefore \( f^t_{\text{lin}, T_{\text{max}}}(W, A) \) is a good approximation for \( f_t(W, A) \).

Note that from the above arguments and Lemma [11] we have
\[ ||f_t(W, A)|| \leq 2b, ||\nabla_f L|| \leq 2l_0 \cdot (1 + 2b), \]
and
\[ L(y_t, f_t(W, A)) \leq 4l_0 \cdot (b + 2b^2). \] (27)

VII. PROOF OF THEOREM [15]

Theorem [15] is a direct corollary of Theorem [13] and [16] below.

**Theorem 15:** Under the condition in Theorem [4] for any \( A^*, W^* \) with \( ||A^* - A_0||_F, ||W^* - W_0||_F \leq \frac{R}{\sqrt{m}} \leq b \cdot T_{\text{max}}/\sqrt{m} \), let
\[ L^k(W_k, A_k) = \frac{1}{T} \sum_{t=1}^{T} L(y_k^t, f_t(W_k, A_k)), \]
\[ L_t^k(W_k, A_k) = L(y_k^t, f_t(W_k, A_k)). \]
with probability at least \( 1 - \delta \), the outputs of algorithm [1] satisfy:
\[ \frac{1}{K} \sum_{k=0}^{K-1} L_k^k(W_k, A_k) - L^k(W^*, A^*) \leq O(\epsilon). \] (28)

When the loss function is convex, one can easily see Theorem [15] follows. In our case, the proof of Theorem [15] is from the linearization Lemma [13] Lemma [13] says when \( ||A^* - A_0||_F, ||W^* - W_0||_F \) are small enough,
\[ f_t(W, A) = \sum_{t=0}^{t_0} 5^t B(t) \int_{\tau=1}^{t_0} W^\tau A x_{t-\tau}. \] (29)
will nearly be a linear function for \( W \) and \( A \). This is the main process to prove Theorem [15].

**Proof of the Theorem [15]**

From Lemma [9] for any \( t \),
\[ ||\nabla_W f_t(W, A)||_F, ||\nabla_A f_t(W, A)||_F \leq 32 \sqrt{m} \frac{\sqrt{m}}{(1 - \rho_0)^3} \]
when \( W \in B(W_0, \omega), A \in B(A_0, \omega) \). Meanwhile,
\[ W_{k+1} - W_k = \eta \nabla_W L^k(W_k, A_k), \]
\[ A_{k+1} - A_k = \eta \nabla_A L^k(W_k, A_k). \] (30)

Thus for any \( k \leq K \),
\[ ||W_{k} - W_0||_F \leq K \eta \cdot ||\nabla_W f||_F \cdot ||\nabla f||_L, \]
\[ ||A_{k} - A_0||_F \leq K \eta \cdot ||\nabla_A f||_F \cdot ||\nabla f||_L. \] (31)
In our case, from Eq [10] and [27], we have
\[ ||\nabla f||_L \leq l_0(1 + 2b), \]
\[ ||\nabla_W L||_F, ||\nabla_A L||_F \leq 32 \sqrt{m} \frac{\sqrt{m}}{(1 - \rho_0)^3} \cdot l_0(1 + 2b). \]
Thus \( ||W_{k} - W_0||_F, ||A_{k} - A_0||_F \leq K \eta \cdot \frac{32 \sqrt{m}}{(1 - \rho_0)^3} \cdot l_0(1 + 2b) \leq \omega. \)

From the convexity of \( L \) and Lemma [13], we have
\[ L^k(W_k, A_k) - L^k(W^*, A^*) \]
\[ \leq \frac{1}{T} \sum_{t=1}^{T} \nabla f_t L^k(W_k, A_k) \cdot [f_t(W_k, A_k) - f_t(W^*, A^*)], \]
\[ \leq \frac{1}{T} \sum_{t=1}^{T} \nabla f_t L^k(W_k, A_k) \cdot [\nabla f(W_k, A_k) \cdot [W_k - W^*] + \nabla_A f_t(W_k, A_k) \cdot [A_k - A^*] + \Theta(\frac{\sqrt{m} \omega^2}{(1 - \rho_0)^3}) \cdot l_0(1 + 2b), \]
\[ \leq \nabla_W L^k(W_k, A_k) \cdot [W_k - W^*] + \nabla_A L^k(W_k, A_k) \cdot [A_k - A^*] + \Theta(\frac{\sqrt{m} \omega^2}{(1 - \rho_0)^3}) \cdot l_0(1 + 2b). \]
Since
\[ W_{k+1} - W_k = -\eta \nabla W L^k(W_k, A_k), \]
\[ A_{k+1} - A_k = -\eta \nabla A L^k(W_k, A_k), \]
we have
\[ \frac{1}{K} \sum_{k=0}^{K-1} \{ L^k(W_k, A_k) - L^k(W^*, A^*) \}
\leq \frac{1}{K \eta} \sum_{k=0}^{K-1} \{ ||W_k - W_{k+1}||_F - ||W^* - W_k||_F \}
+ \Theta(\frac{\sqrt{m} \omega^2}{(1 - \rho_0)^3}) \cdot l_0(1 + 2b), \]
\[ \leq \frac{1}{K \eta} \sum_{k=0}^{K-1} \{ ||W_k - W^*||_F^2 - ||W_{k+1} - W^*||_F^2 \}
+ \eta [32 \sqrt{m} \frac{\sqrt{m}}{(1 - \rho_0)^3} \cdot l_0(1 + 2b)]^2 + \Theta(\frac{\sqrt{m} \omega^2}{(1 - \rho_0)^3}) \cdot l_0(1 + 2b), \]
\[ \leq O(\epsilon) + \Theta(\frac{\eta \sqrt{m} (1 + 2b)^2}{(1 - \rho_0)^3}) \]
\[ + \Theta([K \eta \cdot \frac{\sqrt{m} \omega^2}{(1 - \rho_0)^3} \cdot l_0(1 + 2b)]^2 \cdot \frac{\sqrt{m}}{(1 - \rho_0)^3}) \cdot l_0(1 + 2b), \]
\[ \leq O(\epsilon) + O(\epsilon) \]
\[ + \Theta((K \eta \cdot \frac{\sqrt{m} \omega^2}{(1 - \rho_0)^3} \cdot l_0(1 + 2b) \cdot \frac{\sqrt{m}}{(1 - \rho_0)^3}) \cdot l_0(1 + 2b)], \]
\[ \leq O(\epsilon). \]
Meanwhile \( \frac{n}{\sqrt{m}} \leq \omega \leq \omega_0 \leq \frac{1}{\rho_0} - 1 \). Thus
\[ \frac{1}{K} \sum_{k=1}^{K} L^k(W_k, A_k) - L^k(W^*, A^*) \leq O(\epsilon). \] (34)
Then Theorem 15 follows.

Based on Theorem 15 to prove the main result, we need to show there exits a \( \langle W^*, A^* \rangle \) with \( \|A^* - A_0\|_F, \|W^* - W_0\|_F \leq O(b \cdot T_{max}^2 / \sqrt{m}) \) and \( \|f_t(W^*, A^*) - \tilde{y}_t\|_F \) small so the target lies in the linearization domain. In fact we have:

**Theorem 16:** Under the condition in Theorem 4 with probability at least 1 – δ, there exist \( W^*, A^* \) with

\[
\|W^* - W_0\|_F, \|A^* - A_0\|_F \leq O(b \cdot T_{max}^2 / \sqrt{m}).
\]  

(35)

and

\[
\frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{T} \sum_{t=1}^{T} \left\{ L(y_t^k, f_t(W^*, A^*, x^k)) - L(y_t^k, \tilde{y}_t^k) \right\} \leq O(\epsilon).
\]

(36)

To prove Theorem 16 assume the target stable system is

\[
p_t^i = C p_t^{i-1} + D x_t^i,
\]

\[
\tilde{y}_t^i = G p_t^i,
\]

(36)

We construct

\[
W^* - W_0 = \sum_{t=1}^{T_{max}} (\rho^{-1})^{t-1} \sum_{t_1+t_2 = t} \{ (\prod_{\tau=1}^{t_1} W_0) P_{t_1}^1 (GC_{t_1}^{-1} D) \}
\]

\[
- (\rho^2)^{t-1}/(t-1) \cdot B \{ (\prod_{\tau=1}^{t} W_0) A_0 \} P_{t_2}^2 A_0 (\prod_{\tau=1}^{t} W_0),
\]

(37)

\[
A^* - A_0 = \{ (\prod_{\tau=1}^{t} W_0) \}^{T} B^{T} P_{t_{max}}^{1} [G D - BA_0],
\]

where

\[
P_{t_1}^1 = \{ B (\prod_{\tau=1}^{t_1} W_0) (\prod_{\tau=1}^{t_1} W_0)^T B^{T} \}^{-1},
\]

(37)

\[
P_{t_2}^2 = \{ A_0^T (\prod_{\tau=1}^{t_2} W_0)^T (\prod_{\tau=1}^{t_2} W_0) A_0 \}^{-1}.
\]

We can show our \( W^*, A^* \) satisfying

\[
\| f_t(W^*, A^*) - \sum_{t_0=0}^{t-1} G(I_{\tau=1} \prod_{\tau=1}^{t_0} C) D x_{t_0} \|_F \approx 0,
\]

using Lemma 12. To show \( \|W^* - W_0\|_F, \|A^* - A_0\|_F \leq O(b \cdot T_{max}^2 / \sqrt{m}) \), we need Lemma 11. The detailed proof is in Supplementary Materials.

Combining the above two theorems, we have

\[
\frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{T} \sum_{t=1}^{T} \left\{ L(y_t^k, f_t(W_k, A_k, x^k)) - L(y_t^k, \tilde{y}_t^k) \right\} \leq O(\epsilon).
\]

(38)

Therefore Theorem 15 follows.

**VIII. PROOF OF THEOREM 6**

In order to prove the bound for the population risk, we need the following results for Rademacher Complexity. These results can be found in Proposition A.12 of [23] and section 3.8 in [24].

**Theorem 17:** Let \( F_1, F_2, ..., F_d \) be \( d_y \) classes of functions \( \mathbb{R}^d \rightarrow \mathbb{R} \) and \( |L(\cdot)| \leq 4 \delta \cdot (b + 2b^2) \) be a bounded, \( l_0(1 + 2b) \)

Lipschitz function. For \( N \) samples \( x_i \) i.i.d. drawn from a distribution \( D \), with probability at least 1 – \( \delta \), we have

\[
\sup_{f_1 \in F_1, ..., f_d \in F_d} \mathbb{E}_{x \sim D} \left[ L(f_1(x), ..., f_d(x)) \right]
\]

\[
- \frac{1}{N} \sum_{i=1}^{N} L(f_1(x_i), ..., f_d(x_i))
\]

\[
\leq \Theta(l_0(1 + 2b)) \cdot \frac{d_y}{\sqrt{N}} + 4 \delta \cdot (b + 2b^2) \sqrt{\log(1/\delta)}.
\]

(39)

where \( \mathcal{R}(F) \) is defined as :

\[
\mathcal{R}(F) = \mathbb{E}_{\xi \sim \{\pm 1\}^N} \left[ \sup_{f \in F} \frac{1}{N} \sum_{i=1}^{N} \xi f(x_i) \right].
\]

Meanwhile, for linear functions, Rademacher Complexity is easy to be calculated:

**Theorem 18:** (Proposition A.12 in [22])

Let \( F \) be the function class \( \{ x \rightarrow f_0(x) + \langle w, x \rangle \in \mathbb{R} \} \) with \( f_0 \) a fixed function and \( \|w\| \leq B \), then

\[
\mathcal{R}(F) \leq \frac{B}{\sqrt{N}}.
\]

(40)

Now we will estimate the gap between the empirical and population loss:

\[
\sup_{f_1 \in F_1, ..., f_d \in F_d} \mathbb{E}_{x \sim D} \left[ L(f_1(x), ..., f_d(x)) \right]
\]

\[
- \frac{1}{N} \sum_{i=1}^{N} L(f_1(x_i), ..., f_d(x_i))
\]

\[
\leq \Theta(l_0(1 + 2b)) \cdot \frac{d_y}{\sqrt{N}} + 4 \delta \cdot (b + 2b^2) \sqrt{\log(1/\delta)}.
\]

(39)

Firstly, from Eq (10) and (26),

\[
|L(y_t, f_t(W_k, A_k)) - L(y_t, f_{t^0}^{lin,T_{max}}(W_k, A_k))| \leq O(\epsilon),
\]

and for all \( k \leq K \),

\[
\|W_k - W_0\|_F, \|A_k - A_0\|_F \leq \omega,
\]

where \( \omega = O(\frac{K \delta}{1 - \rho_0^3} l_0(1 + 2b)) \). Let \( e_j \) be the \( j \)-th orthogonal basis of \( \mathbb{R}^{d_y} \). We only need to consider the Rademacher complexity of the function class:

\[
\{ x \rightarrow e_j^T t f_{t}^{lin,T_{max}}(W_0 + \Delta W, A_0 + \Delta A, x) \}
\]

\[
\|\Delta W\|_F, \|\Delta A\|_F \leq \omega.
\]

(41)

And this is a class of linear functions. We can write it as \( x \rightarrow \langle F_0, x \rangle + \langle F, x \rangle \) with \( F_0 \) fixed. Apply Lemma 9 We have

\[
\|F\|_2 \leq 32 \frac{\sqrt{m}}{(1 - \rho_0)^3} \omega
\]

\[
\leq 32 \frac{\sqrt{m} \cdot [K \eta \cdot 32 \frac{\sqrt{m}}{(1 - \rho_0)^3} \cdot l_0(1 + 2b)].
\]

(42)
Then combining all the above results, Theorem [17] and Theorem [18] we have
\[
\|\mathbb{E}_{x,y \sim D} - \frac{1}{T} \sum_{t=1}^{T} \{L(y_t, f_t(W_k, A_k, x)) - L(y_t, \tilde{y}_t)\}
\]
\[
- \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{T} \sum_{t=1}^{T} \{L(y_t^k, f_t(W_k, A_k, x^k)) - L(y_t^k, \tilde{y}_t^k)\}
\]
\[
\leq \Theta \left( l_0 (1 + 2b) \left( \frac{\sqrt{m}}{(1 - \rho)^3} \right) \right)
\cdot \Theta \left( \frac{\eta m \sqrt{K}}{(1 - \rho)^3} \cdot l_0 (1 + 2b) \right) \sqrt{\frac{1}{K}}
\]
\[
+ O \left( \frac{l_0 \cdot (b + 2b^2 \sqrt{\log(1/\delta)} \sqrt{\frac{1}{K}}}{\sqrt{1 - \rho^2}} \right)
\]
\[
\leq O \left( l_0^2 (1 + 2b^2) \left( \frac{\eta m \sqrt{K}}{(1 - \rho)^6} \right) \right)
\]
\[
+ O \left( \frac{l_0 \cdot (b + 2b^2 \sqrt{\log(1/\delta)} \sqrt{K}}{\sqrt{1 - \rho^2}} \right)
\]
\[
\leq O(\epsilon).
\]
with probability at least 1 − δ. Our claim follows.

IX. CONCLUSION

We provided the first theoretical guarantee on learning linear RNNs with Gradient Descent. The required width in hidden layers does not rely on the length of the input sequence and only depends on the transition matrix parameter ρC. Under this condition, we showed that SGD can provably learn any stable linear system with transition matrix C satisfying \( \|C^k\| \leq O(\rho_k^k), k \in \mathbb{N} \), using \( \text{poly}(\frac{1}{1 - \rho C}, \epsilon^{-1}) \) many iterations and \( \text{poly}(\frac{1}{1 - \rho C}, \epsilon^{-1}) \) many samples. In this work we found a suitable random initialization which is available to optimize using gradient descent. This solves an open problem in System Identification and answers why SGD is available to optimize RNNs in practice. We hope this result can provide some insights for learning stable nonlinear dynamic systems using recurrent neural networks in deep learning.

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APPENDIX

LEMMA

The following concentration inequality is a standard result for subexponential distribution, which will be used many times.

**Lemma 19:** Standard Concentration inequality for Chi-square distribution:

Let $i = 1, 2, \ldots, m, v \in \mathbb{N}$ and $A_i \sim \chi^2(v)$ be $m$ i.i.d Chi-square distribution. Then with probability at least $1 - \exp(-\Omega(me^2))$,

$$\left| \sum_{i=1}^{m} \frac{1}{m} A_i - \mathbb{E} \chi^2(v) \right| \leq \epsilon.$$  

As a corollary, let $A \in \mathbb{R}^{m \times n}$ is a Gaussian random matrix, and the element $A_{i,j} \sim \mathcal{N}(0, \frac{1}{m})$. We can see for a fixed $v$ with $||v|| = 1$, with probability at least $1 - \exp(-\Omega(me^2))$,

$$||Av|| - 1 \leq \epsilon.$$  

**Proof of Lemma 19**

Firstly,

$$\left| \sum_{t=\tau}^{\infty} \rho^t \cdot B(t \prod_{\tau=1}^{t} W)QZ_t \right| \leq \sum_{t=\tau}^{\infty} \rho^t ||B(t \prod_{\tau=1}^{t} W)QZ_t||.$$  

(44)

Meanwhile, thanks to (18), for all $k$ in $\mathbb{N}$, $||\rho^k W^k|| \leq 2\sqrt{k} \rho_k^k$,

$$\rho^k ||B(t \prod_{\tau=1}^{t} W)|| \leq \frac{4\sqrt{m} \sqrt{k} \rho_k^k ||Q|| ||Z_t||}{(1 - \rho_0)^2}.$$  

Thus

$$\left| \sum_{t=\tau}^{\infty} \rho^t \cdot B(t \prod_{\tau=1}^{t} W)QZ_t \right| \leq \frac{4\tau \sqrt{m} \rho_0^\tau}{(1 - \rho_0)^2} ||Q||.$$  

(45)

The first part of Lemma 9 follows.

Now we consider

$$\left| \sum_{t_0=\tau}^{\infty} \sum_{t_1+t_2=t_0} \rho_0^{t_1} B(t_1-1 \prod_{\tau=1}^{t_1} W)Q(t_2-1 \prod_{\tau=1}^{t_2} W)AZ_{t_0} \right|.$$  

(46)

Note that

$$\left| \sum_{t_0=\tau}^{\infty} \sum_{t_1+t_2=t_0} \rho_0^{t_1} B(t_1-1 \prod_{\tau=1}^{t_1} W)Q(t_2-1 \prod_{\tau=1}^{t_2} W)AZ_{t_0} \right| \leq 8\sqrt{m} \rho_0^\tau ||Q||.$$  

Thus

$$\left| \sum_{t_0=\tau}^{\infty} \sum_{t_1+t_2=t_0} \rho_0^{t_1} B(t_1-1 \prod_{\tau=1}^{t_1} W)Q(t_2-1 \prod_{\tau=1}^{t_2} W)AZ_{t_0} \right| \leq 32 \frac{\sqrt{m} \rho_0^\tau \tau^2}{(1 - \rho_0)^3} ||Q||.$$  

(47)

Our results follow.

**Proof of Lemma 10**

For a fixed $v_2$, with probability at least $1 - \exp(-\Omega(me^2))$, $1 - \epsilon \leq ||A_0 v_2|| \leq 1 + \epsilon$.

Taking the $\epsilon$-net in $\mathbb{R}^d$, there is a set $\mathcal{N}$ with $|\mathcal{N}| \leq (3/\epsilon)^d$. For any vector $v$ in $\mathbb{R}^d$ with $||v|| = 1$, there is a vector $v'$ satisfying $||v - v'|| \leq \epsilon$. Thus with probability at least $1 - (3/\epsilon)^d \exp(-\Omega(me^2)) = 1 - \exp(-\Omega(me^2))$, for any vector $v_2$ in $\mathbb{R}^d$ with $||v_2|| = 1$,

$$1 - \epsilon - (1 + \epsilon) \leq ||A_0 v_2|| \leq (1 + \epsilon)^2.$$  

(49)

Thus

$$1 - 2\epsilon - \epsilon^2 \leq ||A_0 v_2|| \leq 1 + 2\epsilon + \epsilon^2.$$  

(50)

This is the same for $||W_0 A_0 v_2||$. With probability at least $1 - \exp(-\Omega(me^2))$,

$$1 - \epsilon \leq ||W_0 A_0 v_2|| \leq 1 + \epsilon.$$  

(51)

Let $h_t = \prod_{t_0=1}^{t} W_0 A_0 v_2$. When $t > 1$, we consider the Gram-Schmidt orthogonalization.

Let $U_t$ be the Gram-Schmidt orthogonalization matrix as:

$$U_t = GS(h_0, h_1, ..., h_t).$$  

(52)

We have

$$h_t = W_0 h_{t-1} = W_0 U_{t-2} W_{t-2}^T h_{t-1},$$

$$+ W_0 I - U_{t-2} U_{t-2}^T h_{t-1},$$

$$= \left[ W_0 U_{t-2}, \frac{W_0 (I - U_{t-2} U_{t-2}^T) h_{t-1}}{||I - U_{t-2} U_{t-2}^T h_{t-1}||} \right]$$

and $U_{t-2} = \left[ z_1, z_2 \right]$.  

(53)

Using this expression, the entries of $M_1 = W_0 U_{t-2}$ and $M_2 = W_0 (I - U_{t-2} U_{t-2}^T) h_{t-1}$ are i.i.d from $\mathcal{N}(0, \frac{1}{m})$. Therefore for fixed $z_1, z_2$, with probability at least $1 - \exp(-\Omega(me^2))$,

$$(1 - \epsilon) \sqrt{z_1^2 + z_2^2} \leq ||h_t|| \leq (1 + \epsilon) \sqrt{z_1^2 + z_2^2}.$$  

Take the $\epsilon$-net for $z_1, z_2$. The sizes of $\epsilon$-net for $z_1, z_2$ are $(3/\epsilon)^d (t+1)$ and $(3/\epsilon)^t$. Thus with probability at least $1 - (3/\epsilon)^d \exp(-\Omega(me^2)) = 1 - \exp(-\Omega(me^2))$,

$$(1 - \epsilon) \sqrt{z_1^2 + z_2^2} \leq ||h_t|| \leq (1 + \epsilon) \sqrt{z_1^2 + z_2^2}$$

for any $z_1, z_2$.

Therefore we have

$$(1 - \epsilon)^t \leq ||\prod_{t_0=1}^{t} W_0 A_0 v_2|| \leq (1 + \epsilon)^t.$$  

(54)

for any $0 \leq t \leq \tau$. Set $\epsilon = \frac{0.01}{\tau}$. The theorem for $||W_0 A_0 v_2||$ follows. For $||\sqrt{m} \left( \frac{W_0}{\sqrt{m}} \right)^t B^T v_1||$, the proof is the same.

This proves Lemma 10.
Proof of Lemma 8

Now we will prove Lemma 8. Similar to (54), for fixed \( v \in \mathbb{R}^m \), let \( L = \Theta(\sqrt{m/\log m}) \in \mathbb{N} \). With probability at least \( 1 - \text{Exp}(-\Omega(m/L^2)) \), for all \( 0 < t \leq L \), we have

\[
(1 - \frac{1}{L})^t ||v|| \leq || \sum_{t_0=1}^t W_0 v || \leq (1 + \frac{1}{L})^t ||v||.
\]  

Then for fixed normalized orthogonal basis \( v_i, i = 1, 2, \ldots, m \), with probability at least \( 1 - \text{Exp}(-\Omega(m/L^2)) \), for all \( 0 < t \leq L \), we have

\[
|| W_0 t v || \leq (1 + 1/L)^t ||v||.
\]  

Any \( v \in \mathbb{R}^m \) can be written as \( v = \sum_{i=1}^m a_i v_i \). When \( ||v|| = 1 \), we have

\[
\sum_i |a_i| \leq \sqrt{m}.
\]

Therefore with probability at least \( 1 - \text{Exp}(-\Omega(m/L^2)) \), for all \( 0 < t \leq L \), we have

\[
|| W_0 t v || \leq \sqrt{m}(1 + 1/L)^t ||v||.
\]  

Since \( L = \Theta(\sqrt{m/\log m}) \), \( L^2 > \sqrt{m} \), we have

\[
|| W_0 t || \leq L^2 (1 + 1/L)^t.
\]  

Note that \( 1/p_1 = 1 + 10 \cdot \frac{\log^2 m}{\sqrt{m}} > 1 + \log L \cdot \frac{\log m}{\sqrt{m}} \) so we can set \( L = \Theta(\sqrt{m/\log m}) \in \mathbb{N} \) larger than an absolute constant such that

\[
|| W_0 || \leq L^2 (1 + 1/L)^L \leq e^{\log L} \cdot e^{1 + 2 \log L} \leq (1 + 10 \cdot \frac{\log^2 m}{\sqrt{m}}) (\frac{\sqrt{m}}{\log^2 m}) \cdot 100 \cdot \log L \leq (1/p_1)^{100} \frac{\log m}{\sqrt{m}} \leq (1/p_1)^{L/2}.
\]  

And for \( k \leq L \)

\[
|| W_0 || \leq L^2 (1 + 1/L)^L \leq (1/p_1)^{L/2}.
\]  

If \( s > L \), we can write it as \( s = k \cdot L + r \) and \( k, r \in \mathbb{N} \), \( r \leq L \). Then

\[
|| W_0 || \leq || W_0 ||^k \cdot || W_0 || \leq \rho_1^{-Lk/2} \cdot \rho_1^{-L/2} \leq \rho_1^{-s}.
\]  

Thus we have

\[
|| W_0 || \leq \rho_1^{-s}.
\]

Proof of Lemma 10

Note that for a fixed \( v \in \mathbb{R}^m \),

\[
|| Bv ||
\]  

is a Gaussian random variable with variance \( ||v||^2 \). Meanwhile, let \( v_i, i = 1, \ldots, d \) be the orthogonal basis in \( \mathbb{R}^d \). We only need to bound \( \tau \cdot d \) many vectors \( B W_0 A v_i \).

Thus for any \( t \leq \tau \) and any \( v \), with probability at least \( 1 - (\tau \cdot d) \cdot \text{Exp}(-\Omega(m^2)) \), we have

\[
|| \sum_{t_0=1}^t W_0 A v_i || \leq (1 + \epsilon)^t.
\]  

Thus if \( m > \Omega(\log(\tau \cdot d/\delta)) \), with probability at least \( 1 - \delta \),

\[
|| B W_0 A || \leq \sqrt{d \log(\tau \cdot d/\delta)}.
\]

Lemma 11 follows.

Proof of Lemma 12

Consider

\[
|| u^T A \{(\sum_{\tau=1}^t W_0)^T \} \{(\sum_{\tau=1}^t W_0)\} A v ||,
\]  

with \( ||u|| = ||v|| = 1 \). Before we study the properties of the above equation, we need some very useful results in random matrices theory.

Lemma 20: (Lemma 5.36 in [24]) Consider a matrix \( B \) and \( \delta > 0 \). Let \( s_{\text{min}}(B), s_{\text{max}}(B) \) be the smallest and the largest singular values of \( B \) which satisfy

\[
1 - \delta \leq s_{\text{min}}(B) \leq s_{\text{max}}(B) \leq 1 + \delta.
\]  

Then

\[
|| B^T B - I || \leq 3 \max(\delta, \delta^2).
\]  

combining this lemma with (51), we have the following result.

Lemma 21: For any \( t \leq \tau \), let \( m^{1/2} > \tau \), \( F = W^T A \). With probability at least \( 1 - \text{Exp}(-\Omega(m^2)) \), we have

\[
|| F^T F - I || \leq \log m / \sqrt{m}.
\]  

Meanwhile, let \( x \in \mathbb{R}^d \), \( y = Ax \in \mathbb{R}^m \). We have:

\[
y^T W_0 y = w^T A x,
\]

where \( w' \in \mathbb{R}^m \) is a random vector and every entry of which is i.i.d drawn from \( N(0, ||y||^2) \). Thus with probability at least \( 1 - \text{Exp}(-\Omega(m^2)) \), we have

\[
|| y^T W_0 y || \leq \log m / \sqrt{m} ||y||^2.
\]  

Now let \( i, j \in [d] \) be the two indexes. Let \( e_1, e_2, \ldots, e_d \) be the orthonormal basis in \( \mathbb{R}^d \). We can write (66) into a linear combination of the below terms.

\[
|| e_i^T A \{(\sum_{\tau=1}^t W_0)^T \} \{(\sum_{\tau=1}^t W_0)\} A e_j ||,
\]

For any \( i \neq j \), we set

\[
z_{i,0} = A e_i, z_{j,0} = A e_j.
\]  

(73)
As shown in the last section, when $t \leq \tau$, with probability at least $1 - \tau \exp(-\Omega(m/\tau^2))$,
\[
\|z_{i,t}\| \leq (1 + 1/100\tau)^t. \tag{75}
\]

Now we consider Gram-Schmidt orthonormal matrix
\[
Z_{j,t} = GS(z_{i,0}, z_{j,0}, z_{i,1}, z_{j,1}, \ldots, z_{i,t}, z_{j,t}). \tag{76}
\]

It has at most $2(t+1)$ columns.

In order to prove the lemma, we consider $\|Z_{j,t}^T z_{i,t}\|$ using induction. When $t = 0$ and $t = 1$, from Lemma [21] and [71] we know with probability at least $1 - \exp(-\Omega(\log^2 m))$, $\|Z_{j,t}^T z_{i,t}\| \leq 2\log m/\sqrt{m}$.

For $t + 1$,
\[
Z_{j,t+1}^T z_{i,t+1} = Z_{j,t}^T z_{i,t} + Z_{j,t+1}^T W_0 z_{i,t} \tag{77}
\]

Note that thanks to the Gram-Schmidt orthogonalization, the elements in matrix $W_0(I - Z_{j,t}Z_{j,t}^T)$ are independent of $Z_{j,t+1}$.

Then with probability at least $1 - \exp(-\Omega(\log^2 m))$,
\[
\|Z_{j,t+1}^T W_0(I - Z_{j,t}Z_{j,t}^T)z_{i,t}\| \leq (1 + 1/100\tau)^t \|Z_{j,t+1}^T W_0 z_{i,t}\|. \tag{78}
\]

As for $Z_{j,t+1}^T W_0 z_{i,t}$, first note that the entries of matrix $W_0 Z_{j,t}$ are i.i.d from $N(0, 1/m)$. Then with probability at least $1 - \exp(-\Omega(m/\tau^2))$, $\|Z_{j,t+1}^T W_0 z_{i,t}\| \leq (1 + 1/50\tau) \|Z_{j,t+1}^T z_{i,t}\|$. \tag{79}

Meanwhile, there are at most $\exp(O(t + 1))$ fixed vectors forming an $\epsilon$-net for $Z_{j,t}^T z_{i,t}$.

Therefor $\|Z_{j,t+1}^T W_0 z_{i,t}\| \leq (1 + 1/50\tau) \|Z_{j,t+1}^T z_{i,t}\|$. \tag{80}

Therefore $\|Z_{j,t+1}^T W_0 z_{i,t}\| \leq (1 + 1/50\tau) \|Z_{j,t+1}^T z_{i,t}\|$. \tag{81}

When $m$ is large enough such that $\sqrt{2\log m} > 50\tau$, we have
\[
(1 + 1/50\tau) + (2\log m) \leq 3,
\]

Thus $\|Z_{j,t+1}^T z_{i,t+1}\| \leq (1 + 1/50\tau) \|Z_{j,t+1}^T z_{i,t}\|$. \tag{82}

As a corollary, this says suppose $\|u\| = ||v\| = 1$, $u^T v = 0$, for all $k \leq \tau$,
\[
\|u^T W^k v\| \leq 6 \log m/\sqrt{m}.
\]

Without loss of generality, we assume $t > t'$.

\[
\|e_i^T A^T \{\prod_{\tau=1}^{t} W_0^\tau\}\{\prod_{\tau=1}^{t'} W_0^\tau\} A e_i\|
\]

\[
= \left(\sum_{j=1}^{T_0} \left| \left| Z_{j,t}^T v_i \right| \right| + \left| \left| Z_{j,t}^T v_i \right| \right| \right) \leq 24\tau \frac{d^2 \log m}{\sqrt{m}}. \tag{83}
\]

Since $\|Z_{j,t}^T v_i \| \leq \frac{d \log m}{\sqrt{m}}$, \tag{84}

and $Z_{j,t} = GS(z_{i,0}, z_{j,0}, z_{i,1}, z_{j,1}, \ldots, z_{i,t}, z_{j,t})$, \tag{85}

we have $\|z_{i,t}^T v_{i,t} \| \leq \frac{d \log m}{\sqrt{m}}$. \tag{86}

Thus for any $u, v$,
\[
\|u^T A^T \{\prod_{\tau=1}^{t} W_0^\tau\}\{\prod_{\tau=1}^{t'} W_0^\tau\} A v\| \leq 24\tau \frac{d^2 \log m}{\sqrt{m}}. \tag{87}
\]

There is a similar argument for $B^T \{\prod_{\tau=1}^{t} W_0^\tau\} B$. \tag{88}

The theorem follows.

\section*{Proof of Lemma [13 and 14]}

Firstly, note that
\[
f_t(W, A) = BAX_t + \rho \cdot BWAX_{t-1} + \ldots + \rho^{t-1} \cdot B\{\prod_{\tau=1}^{t-1} W_0\} AX_1. \tag{89}
\]
We have
\[
\nabla_W f_t(W, A) : Z = BZX_{t-1} + \ldots \\
+ \rho^{t-1} \cdot \sum_{t_1 + t_2 = t - 1, t_1, t_2 > 0} \rho^{t-t_0} \cdot B(\prod_{r=1}^{t-1} W)Z(\prod_{r=1}^{t-1} W)AX_{t_0},
\]
\[
= \sum_{t_1 + t_2 + t_3 = t, t_1, t_2 > 0} \rho^{t-t_3} \cdot B(\prod_{r=1}^{t-1} W)Z(\prod_{r=1}^{t-1} W)AX_{t_3}.
\]

\[
\nabla_A f_t(W, A) : Z = BZX_t + \rho \cdot BWZX_{t-1} + \ldots \\
+ \rho^{t-1} \cdot B(\prod_{r=1}^{t-1} W)ZX_{t_0}.
\]

Note that from (8), \(|\rho^{t}W||| \leq 2\sqrt{t}\rho_{t0}^t \) for all \( t \in \mathbb{N} \). We have
\[
||R_{t0}(W', A') - R_{t0}(W, A)||_F \\
- \nabla_W R_{t0}(W, A) \cdot [W' - W], \\
- \nabla_A R_{t0}(W, A) \cdot [A' - A],
\]
\[
\leq \sum_{i+j = t-t_0-1} ||B|| \cdot \rho^{t-t_0} \cdot ||W^j|| \cdot ||W^j|| \\
\cdot ||W' - W|| \cdot ||A' - A|| \\
+ \sum_{i+j+k = t-t_0-2} ||B|| \cdot \rho^{t-t_0} \cdot ||W^i|| \cdot ||W^j|| \cdot ||W^k||
\]
\[
\leq 2\sqrt{m}t(t-t_0)^2\rho_{t0}^{t-t_0} \cdot (A(t-t_0) \cdot \omega^2 \\
+ 2\sqrt{m}t(t-t_0)^2\rho_{t0}^{t-t_0} \cdot (t-t_0)^2 \cdot \omega^2 \cdot 2 \\
\leq 32\sqrt{m}t(t-t_0)^3\omega^2 \rho_{t0}^{t-t_0}.
\]

Thus
\[
||f_t(W', A') - f_t(W, A) - \nabla_W f_t(W, A) \cdot [W' - W]||_F \\
+ \langle \nabla_A f_t(W, A), A' - A \rangle ||_F \\
\leq \sum_{t} 32\sqrt{m}t(t-t_0)^3\omega^2 \rho_{t0}^{t-t_0} \\
\leq 768 \sqrt{m}\omega^2 \frac{1}{(1 - \rho_0)^5}.
\]

Lemma 15 follows.

\[
f_t^{lin}(W, A) = f_t(W_0, A_0) + \nabla_W f_t(W_0, A_0) \cdot [W - W_0] \\
+ \nabla_A f_t(W_0, A_0) \cdot [A - A_0],
\]
\[
f_t^{lin, \tau}(W, A) = f_t^\tau(W_0, A_0) + \nabla_W f_t^\tau(W_0, A_0) \cdot [W - W_0] \\
+ \nabla_A f_t^\tau(W_0, A_0) \cdot [A - A_0],
\]
and
\[
\nabla_W f_t(W, A) : Z = BZX_{t-1} + \ldots \\
+ \rho^{t-1} \cdot \sum_{t_1 + t_2 = t - 1, t_1, t_2 > 0} \rho^{t-t_0} \cdot B(\prod_{r=1}^{t-1} W)Z(\prod_{r=1}^{t-1} W)AX_{t_0},
\]
\[
\sum_{t_1 + t_2 + t_3 = t, t_1, t_2 > 0} \rho^{t-t_3} \cdot B(\prod_{r=1}^{t-1} W)Z(\prod_{r=1}^{t-1} W)AX_{t_3}.
\]

Meanwhile
\[
\nabla_A f_t(W, A) : Z = BZX_t + \rho \cdot BWZX_{t-1} + \ldots \\
+ \rho^{t-1} \cdot B(\prod_{r=1}^{t-1} W)ZX_{t_0}.
\]

Using the same way as above, we have
\[
||f_t^{lin}(W, A) - f_t^{lin, \tau}(W, A)||_F \\
\leq \sum_{k=0} \sum_{j=k-1} ||B|| \cdot \rho^j \cdot ||W^j_0|| \cdot ||W^j|| \cdot ||W|| \cdot ||A_0|| \\
+ \sum_{k=0} ||B|| \cdot \rho^k \cdot ||W^k_0|| \cdot ||A||
\]
\[
\leq \tau \rho_0^\tau \cdot 8 \frac{\sqrt{m}}{(1 - \rho_0)^3}.
\]
from Lemma 9.

\[
\text{EXISTENCE: PROOF OF THEOREM 16}
\]
Firstly we briefly introduce the main steps of the proof.
1) From Lemma 13 for all \( t \in \mathbb{T} \) and \( W, W' \in B(W_0, \omega) \), \( A, A' \in B(A_0, \omega) \) with \( \omega \leq \omega_0 \),
\[
||f_t(W, A) - f_t(W_0, A_0) - \nabla_W f_t(W_0, A_0) \cdot [W - W_0] \\
- \nabla_A f_t(W_0, A_0) \cdot [A - A_0]|| \leq 768 \frac{\sqrt{m}\omega^2}{(1 - \rho_0)^5}.
\]

Therefore we consider the linearization function
\[
f_t^{lin}(W, A) = f_t(W_0, A_0) + \nabla_W f_t(W_0, A_0) \cdot [W - W_0] \\
+ \nabla_A f_t(W_0, A_0) \cdot [A - A_0].
\]
We have
\[
||f_t(W, A) - f_t^{lin}(W, A)|| \leq O(\frac{\sqrt{m}\omega^2}{(1 - \rho_0)^5}).
\]

2) From Lemma 14
\[
||f_t^{lin}(W, A) - f_t^{lin, T_{\text{max}}}(W, A)|| \leq O(\frac{\sqrt{m}\omega^2}{(1 - \rho_0)^5}).
\]
Therefore we only need to consider
\[
f_t^{lin, T_{\text{max}}}(W, A) = \sum_{t_0=0}^{T_{\text{max}}} \rho_{t0}^{t} B(\prod_{r=1}^{t_0} W_0)A_0 X_{t-t_0} \\
+ \sum_{t_0=0}^{T_{\text{max}}} \sum_{t_1+t_2=t_0} \rho_{t1}^{t} B(\prod_{r=1}^{t_1} W_0)[W - W_0(\prod_{r=1}^{t_1} W_0)A_0 X_{t-t_0} \\
+ \sum_{t_0=0}^{T_{\text{max}}} \rho_{t0}^{t} B(\prod_{r=1}^{t_0} W_0)[A - A_0] X_{t-t_0}.
\]
3) Finally we set

$$W^* - W_0 = \sum_{t_m=1}^{T_{max}} \frac{(\rho^{-1})^{-1} t_m^{-1} \sum_{t_1 + t_2 = t_m^{-1}} \{ \left(\prod_{\tau=1}^{t_1-1} W_0^{T} \right) B^T P_1^{t_1} (G C_{t_m}^{t_m^{-1}})^D}$$

Then from Lemma 10, 11 and 12,

$$A^* - A_0 = \left(\prod_{\tau=1}^{t_1-1} \frac{t_1}{W_0^{T}} \right) B^T P_1^{t_1-1} \left(G D - BA_0 \right),$$

where

$$P_1^t = \left\{ B \left(\prod_{\tau=1}^{t_1-1} W_0^{T} \right) \left(\prod_{\tau=1}^{t_1-1} \frac{t_1}{W_0^{T}} \right) \right\}^{-1},$$

$$P_2^t = \left\{ A_0^{T} \left(\prod_{\tau=1}^{t_1-1} W_0^{T} \right) \left(\prod_{\tau=1}^{t_1-1} \frac{t_1}{W_0^{T}} \right) \right\}^{-1}.$$

Then from Lemma 10, 11 and 12,

$$f_{t_{\text{lin}},T_{\text{max}}}^t(W^*, A^*) \approx \sum_{t_0=0}^{T_{max}} G \left(\prod_{\tau=1}^{t_0} C \right) DX_{t_{t_0}}.$$

With probability at least 1 - exp(-Ω(log^2 m)),

$$\| f_{t_{\text{lin}},T_{\text{max}}}^t(W^*, A^*) - \tilde{y}_t \| \leq O \left( b \cdot d \cdot \epsilon \cdot T_{\text{max}}^2 \log m \right),$$

and we can show \( \| W^* - W_0 \|_F, \| A^* - A_0 \|_F \leq O(b \cdot T_{\text{max}}^2/\sqrt{m}) \) using Lemma 11. The theorem follows.

**Theorem 22:** Consider a linear system:

$$p_t = C p_{t-1} + D X_t,$$

$$\tilde{y}_t = G p_t,$$

with \( \| C^k D \| < c_{k, \rho} \cdot \rho^k \) for any \( k \in \mathbb{N}, \rho < 1, D \in \mathbb{R}^{d_p \times d}, p_t \in \mathbb{R}^{d_p}, C \in \mathbb{R}^{d_p \times d}, G \in \mathbb{R}^{d \times d} \).

For given data \( \{x_t, y_t\} \), and any \( 0 < \rho < 1 \), if \( m > m^* \), with probability at least 1 - exp(-Ω(log^2 m)), there exist \( W^*, A^* \) satisfying that for all \( t \),

$$\| f_t(W^*, A^*) - \tilde{y}_t \|_F \leq O \left( \frac{\sqrt{m(\rho^3 T_{\text{max}})}}{1-\rho^3} \right)$$

$$+ O \left( \frac{b \cdot d^2 \epsilon \cdot T_{\text{max}} \log m}{m^{1/2}} \right),$$

and

$$\| W^* - W_0 \|_F, \| A^* - A_0 \|_F \leq 2c_{\rho, b} T_{\text{max}}^2/\sqrt{m}.$$

**Proof:**

Firstly, note that from Lemma 13,

$$\| f_t(W, A) - f_t(W_0, A_0) - \nabla_W f_t(W_0, A_0) \cdot [W - W_0]$$

$$- \nabla_A f_t(W_0, A_0) \cdot [A - A_0] \| \leq O \left( \frac{\sqrt{m} \omega^2}{1-\rho^3} \right).$$

Therefore let

$$f_{t_{\text{lin}}}^t(W, A) = f_t(W_0, A_0) + \nabla_W f_t(W_0, A_0) \cdot [W - W_0]$$

$$+ \nabla_A f_t(W_0, A_0) \cdot [A - A_0].$$

We have

$$\| f_t(W, A) - f_{t_{\text{lin}}}^t(W, A) \| \leq O \left( \frac{\sqrt{m} \omega^2}{1-\rho^3} \right).$$

Note that

$$\| W^* - W_0 \|_F, \| A^* - A_0 \|_F \leq O(b \cdot T_{\text{max}}^2/\sqrt{m}).$$

We can show

$$\| f_t(W^*, A^*) - f_{t_{\text{lin}}}^t(W^*, A^*) \| \leq O(\epsilon/b).$$

Now, from Lemma 14,

$$\| f_{t_{\text{lin}}}^t(W, A) - f_{t_{\text{lin}},T_{\text{max}}}^t(W, A) \| \leq O \left( \frac{\sqrt{m} \omega^2}{1-\rho^3} \right).$$

We only need to consider

$$f_{t_{\text{lin}},T_{\text{max}}}^t(W, A) = \sum_{t_0=0}^{T_{max}}$$

$$+ \sum_{t_0=0}^{T_{max}} \sum_{t_1 + t_2 = t} \rho^t B \left( \prod_{\tau=1}^{t_1-1} W_0 \right) \left(\prod_{\tau=1}^{t_2-1} W_0 \right) A_0 X_{t_{t_0}}$$

$$+ \sum_{t_0=0}^{T_{max}} \sum_{t_1 + t_2 = t} \rho^t B \left( \prod_{\tau=1}^{t_1-1} W_0 \right) \left(\prod_{\tau=1}^{t_2-1} W_0 \right) A_0 X_{t_{t_0}}.$$

Finally we set

$$W^* - W_0 = \sum_{t_m=1}^{T_{max}} \frac{(\rho^{-1})^{-1} t_m^{-1} \sum_{t_1 + t_2 = t_m^{-1}} \{ \left(\prod_{\tau=1}^{t_1-1} W_0^{T} \right) B^T P_1^{t_1} (G C_{t_m}^{t_m^{-1}})^D}$$

$$\cdot \{ G C_{t_m}^{t_m^{-1}} D \} = \frac{(\rho^{-1})^{-1} t_m^{-1} \sum_{t_1 + t_2 = t_M^{-1}} B \left( \prod_{\tau=1}^{t_1-1} W_0^{T} \right) A_0}$$

$$\cdot P_1^{t_1} \left(\prod_{\tau=1}^{t_1-1} W_0 \right) A_0 X_{t_{t_0}}$$

$$\cdot P_2^{t_2} \left(\prod_{\tau=1}^{t_2-1} W_0 \right) A_0 X_{t_{t_0}}$$

$$\cdot P_2^{t_2} \left(\prod_{\tau=1}^{t_2-1} W_0 \right) A_0 X_{t_{t_0}}.$$

$$A^* - A_0 = \left\{ \left(\prod_{\tau=1}^{t_1-1} W_0^{T} \right) B^T P_1^{t_1-1} \right\} \left(\prod_{\tau=1}^{t_1-1} W_0 \right) A_0 X_{t_{t_0}}.$$

where

$$P_1^{t_1} = \left\{ B \left( \prod_{\tau=1}^{t_1-1} W_0^{T} \right) B^T \right\}^{-1},$$

$$P_2^{t_2} = \left\{ A_0^{T} \left(\prod_{\tau=1}^{t_2-1} W_0^{T} \right) A_0 \right\}^{-1}.$$
When \( t_1, t_2 \leq T_{\text{max}} \), note that \( P_{t_1}^1 \) and \( P_{t_2}^2 \) are symmetric matrices. From Lemma [10]

\[
\|P_{t_1}^1\| \leq \frac{2}{m}, \quad (110)
\]

\[
\|P_{t_2}^2\| \leq 2.
\]

Moreover, from Lemma [11],

\[
\left|G(C)DX_{t-t_0}\right| \leq c_\rho \quad \text{since} \quad \rho(C) \leq \rho.
\]

Meanwhile, \((\rho^{-1})^{t_m-1}\|G(C)D\| \leq c_\rho \) since \( \rho(C) \leq \rho \).

Combining all the above results, Lemma [11] and

\[
|f_{t_{\text{lin}},T_{\text{max}}}(W^*, A^*) - \sum_{t_0=0}^{T_{\text{max}}-1} G(C)DX_{t-t_0}| 
\leq \mathcal{O}(T_{\text{max}}^2 \cdot 2b \cdot c_\rho \cdot \frac{d^2 \log m}{\sqrt{m}}),
\]

we have

\[
|f_{t_{\text{lin}},T_{\text{max}}}(W^*, A^*) - \sum_{t_0=0}^{T_{\text{max}}-1} G(C)DX_{t-t_0}| 
\leq \mathcal{O}(T_{\text{max}}^2 \cdot 2b \cdot c_\rho \cdot \frac{d^2 \log m}{\sqrt{m}}),
\]

\[
|f_{t_{\text{lin}},T_{\text{max}}}(W^*, A^*) - \tilde{y}_t| 
\leq \mathcal{O}(\frac{b \cdot d^2 c_\rho T_{\text{max}}^2 \log m}{m^{1/2}} + \frac{c_\rho T_{\text{max}}}{1-\rho}).
\]

The theorem follows.

Using Theorem [22] from the definition of \( T_{\text{max}} \),

\[
T_{\text{max}} > \Theta\left(\frac{1}{\log(\frac{1}{\rho})} \cdot \{\log(\frac{1}{1-\rho}) + \log(\frac{1}{\epsilon})\}\right),
\]

we have

\[
|f_t(W^*, A^*) - \tilde{y}_t| \leq \mathcal{O}(\frac{\epsilon}{L_0 \cdot (1+2b)}).
\]

Thus \( L(y_t, f_t(W^*, A^*)) - L(y_t, \tilde{y}_t) \leq \mathcal{O}(\epsilon) \). Theorem [16] follows.

This shows

\[
f_{t_{\text{lin}},T_{\text{max}}}(W^*, A^*)
\]

\[
= \sum_{t_0=0}^{T_{\text{max}}} \rho^{t_0} B(\prod_{\tau=1}^{t_0} W_0) A_0 X_{t-t_0}
\]

\[
+ \sum_{t_0=0}^{T_{\text{max}}} \sum_{t_1+t_2=t_0} \rho^{t_0} B(\prod_{\tau=1}^{t_0} W_0)
\]

\[
\cdot \sum_{t_m=1}^{T_{\text{max}}} (\rho^{-1})^{t_m-1} \sum_{t_1+t_2=t_m-1} \rho^{t_0} B(\prod_{\tau=1}^{t_0} W_0)
\]

\[
\cdot (\prod_{\tau=1}^{t_0} W_0)^{t_1} B^T P_{t_1}^{t_1} G(C)D^{t_m-1} D
\]

\[
- (\rho^{-1})^{t_m-1}/(t_m - 1) \cdot B(\prod_{\tau=1}^{t_0} W_0) A_0) P_{t_0}^{t_0} A_0^{T}(\prod_{\tau=1}^{t_0} W_0)
\]

\[
\cdot (\prod_{\tau=1}^{t_0} W_0) A_0 X_{t-t_0}
\]

\[
+ \sum_{t_0=0}^{T_{\text{max}}} \rho^{t_0} B(\prod_{\tau=1}^{t_0} W_0) (\prod_{\tau=1}^{t_0} W_0)^{t_0} B^T
\]

\[
P_{t_0}^{t_0} G(C)D - BA_0 X_{t-t_0}.
\]

Combining all the above results, Lemma[11] and

\[
\tilde{y}_t = \sum_{t_0=0}^{t_0} G(\prod_{\tau=1}^{t_0} C)DX_{t-t_0},
\]