Analytical solutions for the quantum parametric oscillator from corresponding classical dynamics via a complex Riccati equation

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Abstract. The time-dependent Schrödinger equation for quadratic Hamiltonians has Gaussian wave packets as exact solutions. For the parametric oscillator with frequency \( \omega(t) \), the width of these wave packets must be time-dependent. This time-dependence can be determined via the solution of a complex nonlinear Riccati equation or an equivalent real nonlinear Ermakov equation. All quantum dynamical properties of the system can easily be constructed from these solutions, e.g., uncertainties of position and momentum, their correlations, tunnelling currents, ground state energies, etc. In addition, the link to the corresponding classical dynamics is supplied via linearization of the Riccati equation to a complex Newtonian equation, actually representing two equations of the same kind: one for the real and one for the imaginary part. If the solution of one part is known, the missing (linear independent) solution of the other can be obtained via a conservation law for the motion in the complex plane. Knowing these two solutions, the solution of the Ermakov equation can be determined immediately plus the explicit expressions for all the quantum dynamical properties mentioned above. Furthermore, these two solutions also provide the time-dependent Green function for the systems propagation for any given initial condition, not only for Gaussians. It is also possible to obtain the corresponding Wigner function and generalized creation and annihilation operators in this way. In searching for problems with analytical solutions, the crucial point is to identify systems that have either a wave packet solution with time-dependent width (in closed form) that obeys a Riccati equation, or a classical parametric oscillator with frequency \( \omega(t) \) that still enables an analytical solution of the Newtonian equation. Comparing the wave packets with the solution of the diffusion equation – also a Gaussian with time-dependent width – leads to a real Riccati equation. A first attempt at “complexifying” this equation turns out to be problematic. Nevertheless, it provides a clue to time-dependent frequencies inversely proportional to time that lead to Newtonian equations with analytical solutions. Following a detailed analysis of the corresponding quantum dynamics and energetics, possible extensions of the method are indicated.

1. Introduction
One, if not THE, major difference between classical physics and quantum physics is that classical theories are essentially based on real quantities whereas complex quantities are in principle necessary for quantum theory. Take mechanics: one might think it sufficient to double the number of degrees of freedom by replacing one real variable with a complex variable consisting of a real and an imaginary part; not quite, in quantum theory! This is because these two parts are in this case not independent of each other but coupled via some conservation law. Taking the
conventional view of quantum mechanics as a linear theory, this coupling is not at all obvious as real and imaginary parts of the complex variable fulfil the same (linear) equation and there seems to be no relation that couples these two components. A nonlinear perspective of quantum mechanics changes the situation completely; this is now possible via a formulation in terms of (quadratic) nonlinear (NL) complex Riccati equations. Squaring a complex number yields a new complex number where real and imaginary parts of the original one are uniquely interrelated. In the quantum mechanical context, this leads to a conservation law that enables the calculation of the missing part of the complex variable, providing one part (real or imaginary) is known.

Furthermore, the complex NL Riccati equation can be linearized to a complex Newtonian equation. Likewise, the conservation law mentioned previously makes it possible to calculate one part of the complex variable fulfilling the Newtonian equation, again, if the other part is known. This ultimately means it is sufficient to solve the classical Newtonian equation of motion for the classical system in order to obtain, via its complex version, the conservation law and corresponding NL complex Riccati equation, including all the information about the quantum dynamics of the system.

In Section 2 it is shown how the Gaussian wave packet (WP) solutions of the time-dependent (TD) Schrödinger equation (SE) for systems with exact analytic solutions (essentially Hamiltonians that are at most quadratic in position and momentum) can be traced back to the solution of the complex Riccati equation mentioned earlier. Different ways of solving this equation are discussed. From these solutions, all quantum dynamical properties – like uncertainties of position and momentum, their correlations, tunnelling currents, ground state energies, etc. – can be obtained. In addition, the TD Green function for the system’s propagation can be constructed for any given initial condition, not only for Gaussians. Also the corresponding Wigner function and generalized creation and annihilation operators follow from this procedure.

The parametric oscillator with TD frequency $\omega(t)$ and potential $V = \frac{m}{2} \omega^2(t)x^2$ (in one dimension) is studied thoroughly in Section 3 as an example of a Hamiltonian quadratic in position variable with analytic solutions. An extension to more than one dimension is usually straightforward. It has been shown [1, 2] that for $\omega = \omega(t)$ the width of the Gaussian WP must be TD. It is also well-known that a solution of the diffusion equation is a Gaussian function with width spreading proportional to time $t$. Therefore, a first attempt is made to link the TDSE with the diffusion equation via the Riccati equation describing the time-dependence of the width of the Gaussian solutions that exist in both cases. However, the major difference is that the Riccati equation corresponding to the diffusion equation is a real one, whereas, in the quantum mechanical case it is a complex one and, essentially, the squaring of the complex variable provides the information linking the real and imaginary parts in a unique way.

A first “naive” attempt at “complexifying” the real Riccati equation using the specific form of real and imaginary parts that fulfil the complex Riccati equation proves not quite successful, but not futile. It implies that the frequency $\omega(t)$ should be inversely proportional to time $t$, i.e., $\omega(t) \propto \frac{1}{t}$. This finding, together with the above-mentioned information about the complex Riccati equation and the resulting linearized complex Newtonian equation (or its equivalent formulation in terms of a real NL Ermakov equation), makes it possible to solve the classical as well as the quantum mechanical problem of some particular parametric oscillators in closed form. The dynamical and energetic aspects of the quantum system are examined and possible extensions of the method indicated.

2. Quantum dynamics of Gaussian wave packets

There are only few cases where the SE can be solved analytically. Particularly for potentials that are at most quadratic in position variable, the TDSE possesses solutions that have the form of Gaussian WPs. A general ansatz for such a function can be written as [3, 4]
\[
\Psi_{WP}(x,t) = N(t) \exp \left\{ i \left[ y(t) \tilde{x}^2 + \frac{1}{\hbar} \tilde{x} + K(t) \right] \right\}
\]  
(1)

with the purely TD normalization factor \( N(t) \) (and the TD function \( K(t) \) that will not be relevant for the following discussion). The variable \( \tilde{x} \) is defined as \( x - \langle x \rangle \) where \( \langle x \rangle \) is the mean value of position, calculated via \( \langle x \rangle = \int_{-\infty}^{\infty} dx x \Psi^* x \Psi = \eta(t) \); similarly, \( \langle p \rangle = m \dot{\eta} \) is the mean value of momentum. Important for the quantum mechanical aspect of the system’s dynamics is the coefficient of \( \tilde{x}^2 \) in the exponent, \( y(t) \). As the wave function \( \Psi \) is generally a complex function, the TD coefficient \( y(t) \) is also assumed to be complex, i.e., \( y(t) = y_R(t) + i y_I(t) \). The imaginary part is related to the position uncertainty via \( y_I = \frac{1}{2\sigma^2} \), with the mean square deviation of position \( \langle \tilde{x}^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \). The square root of \( \langle \tilde{x}^2 \rangle \) essentially determines the width of the Gaussian function. This function is completely determined by two parameters, the width and the maximum of the bell-shaped curve. In this quantum mechanical context, both parameters can be TD. The equations of motion that determine this time-dependence can be obtained by inserting the WP(1) into the TDSE,

\[
i \hbar \frac{\partial}{\partial t} \Psi_{WP}(x,t) = \left\{ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right\} \Psi_{WP}(x,t) ,
\]  
(2)

in the following for \( V(x,t) = \frac{m}{2} \omega^2 x^2 \) with the general case \( \omega = \omega(t) \) and the particular cases \( \omega = \omega_0 = \text{const.} \) (harmonic oscillator) and \( \omega = 0 \) (free motion). From the terms proportional to \( \tilde{x} = x - \eta \), one obtains

\[
i \dot{\eta} + \omega^2(t) \eta = 0 ,
\]  
(3)

i.e., the Newtonian equation of motion that determines the time-evolution of the maximum of the WP, located at \( x = \eta(t) \) and thus following the classical trajectory. Defining the complex variable \( \mathcal{C}(t) \) as \( \mathcal{C}(t) = \frac{2\hbar}{m} y(t) \), the terms proportional to \( \tilde{x}^2 \) provide the complex nonlinear (NL) Riccati equation

\[
\dot{\mathcal{C}} + \mathcal{C}^2 + \omega^2(t) = 0
\]  
(4)

which determines the time-evolution of the WP-width, as \( \mathcal{C}_1 = \frac{2\hbar}{m} y_1 = \frac{\hbar}{2m\langle x^2 \rangle} \). Therefore, in order to obtain the information about the typical quantum mechanical aspects of the system’s dynamics, this Riccati equation must be solved. One possible way of doing this \([5, 6]\) is to transform the inhomogeneous Riccati equation (4) into the (still NL and complex) homogeneous Bernoulli equation for \( \mathcal{V}(t) \),

\[
\dot{\mathcal{V}} + 2 \mathcal{C} \mathcal{V} + \mathcal{V}^2 = 0 ,
\]  
(5)

using, for the general solution of the Riccati equation, the ansatz

\[
\mathcal{C} = \hat{\mathcal{C}} + \mathcal{V}(t) ,
\]  
(6)

where \( \hat{\mathcal{C}} \) is a particular solution of the Riccati equation that can be found in some cases (particularly for constant \( \hat{\mathcal{C}} \)). The advantage of the Bernoulli equation is that it can be linearized to an inhomogeneous first order differential equation of the form

\[
\kappa \dot{\kappa} - 2 \mathcal{C} \kappa = 1
\]  
(7)

with \( \kappa = \frac{1}{\mathcal{V}(t)} \).

The general solution of the Riccati equation can then finally be written in the form
\[ C(t) = \tilde{C} + \frac{d}{dt} \ln [\mathcal{I}(t) + \kappa_0] \]  

(8)

where the (complex) parameter \( \kappa_0 \) is the initial value of \( \kappa(t) \) and \( \mathcal{I}(t) \) is an integral over exponential functions depending on the particular solution \( \tilde{C} \). For constant \( \tilde{C} \), the general solution can be written in the simple form

\[ C(t) = \tilde{C} + \frac{e^{-2\tilde{C}t}}{2\tilde{C}} \left( 1 - e^{-2\tilde{C}t} \right) + \kappa_0 \]  

(9)

where \( \tilde{C} \) can be real, imaginary or complex. A different way of treating the Riccati equation (4) can be taken by introducing a new variable \( \alpha(t) \) via

\[ C I(t) = \frac{1}{\alpha^2} \]  

(10)

From the imaginary part of Eq. (4) follows immediately

\[ C_R = \frac{\dot{\alpha}}{\alpha} \]  

(11)

As \( \alpha(t) \), via \( \alpha = \sqrt{\frac{2m}{\hbar} \langle \tilde{x}^2 \rangle} \), is directly proportional to the WP-width, only for constant width, i.e., \( \dot{\alpha} = 0 \), the real part \( C_R \) vanishes. In all other cases, the Riccati variable \( C(t) \) is complex (particularly also always for the parametric oscillator with \( \omega = \omega(t) \)). From the real part of Eq. (4), one finally obtains the equation for \( \alpha(t) \),

\[ \ddot{\alpha} + \omega^2(t)\alpha = \frac{1}{\alpha^3} \]  

(12)

known in the literature as the Ermakov equation [7], but also associated with the names Steen [8, 9, 10], Milne [11], Pinney [12], Lewis [13, 14] and Riesenfeld [14].

A special feature of Eq. (12) describing the evolution of the WP-width is quite interesting for the dynamics of the system. Eliminating \( \omega^2 \) using Eq. (3) that describes the evolution of the WP-maximum, leads to a dynamical invariant, the so-called Ermakov invariant

\[ I = \frac{1}{2} \left[ (\dot{\eta} \alpha - \eta \dot{\alpha})^2 + \left( \frac{\eta}{\alpha} \right)^2 \right] = \text{const} \]  

(13)

that is also a constant of motion for \( \omega = \omega(t) \), although in this case the Hamiltonian (and thus the energy) no longer has this property.

The sum of two quadratic terms in (13) can also be re-written as the product of two linear terms (now expressed with \( C(t) \)) as

\[ I = \frac{1}{2} \alpha^2 \left[ (\dot{\eta} - C_R \eta)^2 + (C_I \eta)^2 \right] = \frac{1}{2} \alpha^2 \left[ (\dot{\eta} - C \eta) (\dot{\eta} - C^* \eta) \right] \]  

(14)

This is already formally similar to the harmonic oscillator Hamiltonian written in terms of creation and annihilation operators as

\[ H_{op} = \frac{1}{2m} p_{op}^2 + \frac{m}{2} \omega_0^2 x^2 = \hbar \omega_0 \left( a^+ a + \frac{1}{2} \right) \]  

(15)

with \( p_{op} = \frac{\hbar}{\sqrt{2m\omega}} \), the number operator \( a^+ a \) with the commutator \([a, a^+] = 1\) of the annihilation operator \( a \) and the creation operator \( a^+ \), defined as
\[ a = \sqrt{\frac{m}{2\hbar \omega_0}} \left( \frac{\text{pop}}{m} - i \omega_0 x \right), \]  
(16)

\[ a^+ = -i \sqrt{\frac{m}{2\hbar \omega_0}} \left( \frac{\text{pop}}{m} + i \omega_0 x \right), \]  
(17)

For the harmonic oscillator with constant frequency \( \omega_0 \) and constant width of the corresponding Gaussian function, \( i \omega_0 = i \frac{C_1}{\alpha} \) with constant \( \alpha \) is valid, i.e., \( i \omega_0 \) is simply the imaginary part of the complex Riccati variable. The real part \( C_r = \frac{\alpha}{\alpha} \) vanishes, as \( \dot{\alpha} = 0 \). For the general case with TD width of the Gaussian, a generalization of the creation and annihilation operators [15] can be obtained by replacing \( \eta \rightarrow x \), \( \dot{\eta} \rightarrow \text{pop} \) in the invariant (14) and taking into account the non-commutativity of \( x \) and \( \text{pop} \), leading to the corresponding operator

\[ I_{\text{op}} = \frac{\hbar}{m} \left[ a^+(t)a(t) + \frac{1}{2} \right] \]  
(18)

with the TD operators

\[ a(t) = i \sqrt{\frac{m}{2\hbar}} \alpha \left( \frac{\text{pop}}{m} - C x \right), \]  
(19)

\[ a^+(t) = -i \sqrt{\frac{m}{2\hbar}} \alpha \left( \frac{\text{pop}}{m} - C^* x \right). \]  
(20)

Finally, the Riccati equation (4) can also be linearized using the logarithmic derivative of a new complex variable \( \lambda(t) \) via \( C = \frac{\dot{\lambda}}{\lambda} \) to yield the complex Newtonian equation

\[ \ddot{\lambda} + \omega^2(t)\lambda = 0 \]  
(21)

with

\[ \lambda(t) = u + i \ z = \alpha e^{i \varphi}. \]  
(22)

Inserting the polar form into the above definition of \( C \) leads to

\[ C = \frac{\dot{\alpha}}{\alpha} + i \ \dot{\varphi}. \]  
(23)

Comparison with the definition as given in (10) and (11) shows that both definitions are consistent and the amplitude \( \alpha \) of \( \lambda \) is identical to the Ermakov variable \( \alpha(t) \), providing

\[ \dot{\varphi} = \frac{1}{\alpha^2} \]  
(24)

is valid. This can be proven easily by inserting the polar form of \( C \) into the imaginary part of the Riccati equation (4). Relation (24), written as a conserved quantity \( \dot{\varphi} \alpha^2 = 1 \), is at least formally similar to the conservation of angular momentum for the motion in a plane but here it is the motion of \( \lambda(t) \) in the complex plane (for further details, see [1, 2, 3, 4, 6]).

Written in Cartesian coordinates, the conservation law (24) takes the form

\[ \dot{z}u - \dot{u}z = 1 \]  
(25)

which is equivalent to the Wronskian determinant of two linear independent solutions \( u \) and \( z \) of the complex Newtonian equation (21).

Knowing one solution (e.g., \( z(t) \)), it is possible to calculate the other via
Knowing both solutions of the classical Newtonian equation immediately provides the solution for the dynamics of the WP width via

\[ \alpha^2 = u^2 + z^2. \]  

If \( \alpha \) is known, \( C_1 \) and \( C_\alpha \) can be determined and consequently all typical quantum mechanical properties like uncertainties and ground state energies:

\[ \langle x^2 \rangle(t) = \frac{\hbar}{2m} \alpha^2(t) = \frac{\hbar}{2m} \frac{1}{C_1(t)} \]  

\[ \langle p^2 \rangle(t) = \frac{m \hbar}{2} \left[ \dot{\alpha}^2(t) + \frac{1}{\alpha^2(t)} \right] = \frac{m \hbar C_\alpha^2(t) + C^2_1(t)}{2C_1(t)} \]  

\[ \langle [\dot{x}, \dot{p}] \rangle(t) = \langle \dot{x} \dot{p} + \dot{p} \dot{x} \rangle(t) = \hbar \alpha(t) \dot{\alpha}(t) = \hbar \frac{C_\alpha(t)}{C_1(t)} \]  

\[ U = \langle \dot{x}^2 \rangle \langle \dot{p}^2 \rangle = \frac{\hbar^2}{4} \left\{ 1 + (\alpha(t) \dot{\alpha}(t))^2 \right\} = \frac{\hbar^2}{4} \left\{ 1 + \left( \frac{C_\alpha(t)}{C_1(t)} \right)^2 \right\} \]  

\[ \bar{E} = \frac{\langle p^2 \rangle}{2m} + \frac{m}{2} \omega^2 \langle p^2 \rangle = \hbar \frac{1}{4} \left\{ \dot{\alpha}^2(t) + \frac{1}{\alpha^2(t)} + \omega^2 \alpha^2(t) \right\} = \hbar \frac{1}{4} \frac{1}{C_1} \left\{ C_\alpha^2(t) + C^2_1(t) + \omega^2 \right\}. \]  

So, with the solutions of the classical Newtonian equation it is possible to construct the solution of the Ermakov equation that describes the dynamics of the uncertainties and thus the quantum mechanical aspect. In other words, if the dynamics of the WP-maximum is known, the dynamics of the WP-width can be determined. This also works the other way round. Key point once more is the conservation law, this time not in the Cartesian form (25), but in the polar form (24). Again, as in (26), one integration is required to obtain the missing part, in this case \( \varphi = \int \frac{1}{\mu(t)} \, dt' \). Together with \( \alpha(t) \) this yields \( \lambda = e^{i\varphi} = u + i \, z = \alpha \cos \varphi + i \, \alpha \sin \varphi \), i.e., the two solutions of the Newtonian equation.

What is still missing is the information about which of the two Cartesian components is (up to a constant factor) identical to the classical trajectory and thus the maximum of the WP, \( u \) or \( z \). This can be ascertained using the TD Green function (also called Feynman kernel) of the problem. With the help of this function, the wave function \( \Psi(x, t) \) can be obtained starting from an initial wave function \( \Psi_\mathrm{WP}(x', t') \) according to

\[ \Psi_\mathrm{WP}(x, t) = \int dx' \, K(x, x', t, t' = 0) \, \Psi_\mathrm{WP}(x', t' = 0). \]  

Particularly for Gaussian wave functions it can be shown that the TD Green function is also a Gaussian function and can be expressed with the help of the TD parameters \( u(t) \) and \( z(t) \) according to [1, 2, 17]

\[ K(x, x', t, t' = 0) = \left( \frac{m}{2\pi \hbar \alpha_0 \sigma_0} \right)^{\frac{1}{2}} \exp \left\{ \frac{-im}{2\hbar} \left[ \frac{\dot{z}}{z} x^2 - 2x \left( \frac{x'}{\alpha_0} \right) + u \left( \frac{x'}{\alpha_0} \right)^2 \right] \right\}. \]
Inserting this kernel into (33) and choosing a Gaussian WP as initial state yields the TD Gaussian WP in the form

$$\Psi_{WP}(x,t) = \left(\frac{m}{\pi \hbar}\right)^{\frac{1}{4}} \left(\frac{1}{u + iz}\right)^{\frac{1}{4}} \exp\left\{\frac{i m}{2\hbar} \left[\frac{\dot{z}}{z} x^2 - \frac{(x - \frac{p_0 \alpha z}{m})^2}{z(u + iz)}\right]\right\}.$$

Comparison with the Gaussian WP as defined in (1) shows

$$z(t) = \frac{m}{\alpha_0 p_0} \eta(t)$$

and, using $\lambda = u + iz$ and the conservation law (25), the coefficient of the quadratic term in the exponent is $\frac{m}{2\hbar} \left(\frac{1}{\lambda}\right) = \frac{im}{2\hbar} \mathcal{C} = i y(t)$. So, $u$ and $z$ are identical to real and imaginary parts of the complex variable $\lambda(t)$ and the imaginary part $z(t)$ is, up to a constant factor, identical to the classical trajectory $\eta(t)$.

To conclude this section, the Wigner function of the problem is introduced. This quantum mechanical phase space distribution function can be obtained from the wave function in position space via the transformation

$$W(x,p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{+\infty} dq \, e^{ipq/\hbar} \Psi_{WP}^\ast(x + \frac{q}{2}) \Psi_{WP}(x - \frac{q}{2}).$$

Inserting our Gaussian WP into this Wigner transformation and using the relations between $C_t, C_R$ and the uncertainties as shown in (28)-(30), the Wigner function can be written as

$$W(x,p,t) = \frac{1}{\pi \hbar} \exp\left\{-\frac{2}{\hbar^2} \left[(p^2)x^2 - \langle [\tilde{x}, \hat{p}]_+, \tilde{x}\hat{p} + \langle x^2\rangle \hat{p}^2\rangle\right]\right\}.$$

Expressing the uncertainties in terms of $\alpha$ and $\dot{\alpha}$ and rearranging them in the exponent finally leads to

$$W(x,p,t) = \frac{1}{\pi \hbar} \exp\left\{-\frac{m}{\hbar} \left[\left(\frac{\dot{\alpha}}{\alpha} - \frac{\alpha}{m} \frac{\dot{p}}{m}\right)^2 + \left(\frac{\tilde{x}}{\alpha}\right)^2\right]\right\}.$$

This means the exponent of the Wigner function is essentially the Ermakov invariant (multiplied by $-\frac{2m}{\hbar^2}$), only $\eta$ is replaced by $\tilde{x} = x - \eta$ and $\dot{\eta}$ by $\frac{\dot{p}}{m} = \frac{p}{m} - \tilde{\eta}$. (For further details, see [16, 17].)

Figure 1 shows schematically the different treatments of the Riccati equation via the real Ermakov equation or the complex Newtonian equation and their corresponding interrelations.

3. Gaussian wave packets for the parametric oscillator

3.1. Gaussian functions with time-dependent width

It can be shown [1, 2] that a Gaussian WP solution of the TDSE with constant width exists for the harmonic oscillator with constant frequency $\omega = \omega_0$ only if this width is identical to the one of the corresponding ground state wave function. In all other cases, but especially for the parametric oscillator with $\omega = \omega(t)$, the WP width is TD. This time-dependence is obviously different for a different $\omega(t)$. For arbitrary $\omega(t)$, finding analytical solutions for the time-dependence of the WP width cannot be expected, nor for its maximum (obeying the corresponding classical Newtonian equation), as both quantities are not independent of each other, as demonstrated in the previous section. One might then ask if there are any cases of analytical solutions for the classical Newtonian and the corresponding Ermakov or Riccati equations. If there are, what are the corresponding frequencies $\omega(t)$?
\[ \mathcal{C} + \mathcal{C}^2 + \omega^2(t) = 0 \] complex Riccati

\[ \mathcal{C} = \frac{\lambda}{\dot{\lambda}} \]

\[ \ddot{\lambda} + \omega^2(t) \lambda = 0 \] complex Newton

\[ \lambda = u + iz = \alpha e^{i\varphi} \]

\[ z = \alpha \sin \varphi \]

\[ \frac{\ddot{\alpha}}{\dot{\alpha}} + \omega^2(t) \alpha = \frac{1}{\alpha^2} \] real Ermakov

\[ \varphi = \int \frac{1}{\alpha^2} dt' \]

\[ z = \alpha \sin \varphi \]

\[ \alpha = (z^2 + u^2)^{\frac{1}{2}} \]

\[ \alpha, \dot{\alpha} \]

\[ \langle \ddot{x}^2 \rangle, \langle \dot{p}^2 \rangle, \langle [\ddot{x}, \dot{p}]_+ \rangle \]

\[ u, E, \bar{v} = \frac{\mu}{\alpha} \]

\[ I = \frac{1}{2} \left[ (\dot{\eta} \alpha - \dot{\alpha} \eta)^2 + \left( \frac{\eta}{\alpha} \right)^2 \right] = \text{const} \] Ermakov invariant

\[ W(x, p, t) \]

\[ (a(t), a^+(t)) \rightarrow \text{CS} \]

\[ \text{Wigner Fct.} \]

\[ \text{generalized creat./annihil. ops.} \]

Figure 1: Different treatments of the complex Riccati equation for the TDSE.

In a first attempt at answering these questions, it might be useful to start out with a comparison of other systems that are described by Gaussian functions with TD width. The diffusion equation, for instance, has solutions with this property. The diffusion process is a classical one which means the corresponding Gaussian function is real. In our quantum mechanical case, it is complex. For comparison, however, it is possible to use the Gaussian WP that describes the probability density \( \rho_S(x, t) = \Psi^\ast \Psi(x, t) \) where the subscript S indicates that \( \rho_S \) is related to the solution \( \Psi(x, t) \propto e^{i \frac{\bar{h}}{2m} C(t) \dddot{x}^2} \) of the SE.

This probability density fulfils the continuity equation

\[ \frac{\partial}{\partial t} \rho_S + \frac{\partial}{\partial x} (\rho_S v_-) = 0 \] (40)

and has (up to a normalization factor) the form

\[ \rho_S(x) \propto \exp \left\{ -\frac{m \dddot{x}^2}{\hbar} \right\} = \exp \left\{ -\frac{\dddot{x}^2}{2\langle \dddot{x}^2 \rangle_S} \right\}. \] (41)

The term \( \rho_S v_- \) is a convection current with velocity field \( v_- \) and, for Gaussian WPs, is written in the form

\[ v_- = \dot{\eta} + \mathcal{C}_R \dddot{x} = \dot{\eta} + \frac{\dot{\alpha}}{\alpha} \dddot{x}, \] (42)
i.e., the non-classical contribution, representing the tunnelling contribution to the current, is also completely determined by the solution of the complex Riccati or the equivalent Ermakov equation. In the diffusion equation, the convection current is replaced by the diffusion current $-D \frac{\partial}{\partial x}\rho_D$, leading to

$$\frac{\partial}{\partial t}\rho_D - D \frac{\partial^2}{\partial x^2}\rho_D = 0.$$  \hfill (43)

The corresponding diffusion distribution function $\rho_D$ is, again up to a normalization factor, given by

$$\rho_D(x) \propto \exp\left\{-Rx^2\right\} = \exp\left\{-\frac{x^2}{4Dt}\right\} = \exp\left\{-\frac{x^2}{2\langle\tilde{x}^2\rangle_D}\right\}.$$  \hfill (44)

The coefficient $R$ of the quadratic term is real and fulfils the Riccati/Bernoulli equation

$$\dot{R} + 4DR^2 = 0$$  \hfill (45)

with the solution $R = \frac{1}{\sqrt{4D}t}$.

Comparing the two mean-square deviations of position,

$$\langle \tilde{x}^2 \rangle_S = \frac{\hbar^2}{2m}\alpha^2$$  \hfill (46)

and

$$\langle \tilde{x}^2 \rangle_D = 2D t$$  \hfill (47)

one arrives at

$$\alpha^2 = 2t$$  \hfill (48)

if $D = \frac{\hbar}{2m}$ is assumed; an assumption that can also be found in the literature [18, 19], e.g., when considering the free motion TDSE as a diffusion equation with imaginary diffusion coefficient.

In a first “naive” attempt at “complexifying” the real Riccati equation (45), one could assume that from

$$C_t = \frac{1}{\alpha^2} = \frac{1}{2t}$$  \hfill (49)

also the missing real part $C_R$ could be obtained via

$$C_R = \frac{\dot{\alpha}}{\alpha} = \frac{1}{2t}.$$  \hfill (50)

Inserting (49) and (50) into the real part of the complex Riccati equation (4) would finally provide the corresponding frequency $\omega(t)$,

$$\dot{C}_R + C_R^2 - C_t^2 - \omega^2(t) = \dot{C}_R + \omega^2(t) = 0 \quad \rightarrow \quad \omega(t) = \frac{1}{\sqrt{2t}}.$$  \hfill (51)

From Sect. 2 it is known that the complex Riccati equation can be linearized to the Newtonian equation

$$\ddot{\lambda} + \omega^2(t)\lambda = 0$$  \hfill (52)
where real and imaginary parts of $\lambda$ must fulfil the classical equation of motion with the same frequency $\omega$ that also occurs in the Riccati equation.

An ansatz that fulfils Eq. (52) is $z = t^{\frac{3}{2}} \rightarrow \dot{z} = \frac{1}{2} t^{-\frac{1}{2}}$, leading to

$$\ddot{z} = -\frac{1}{4} t^{-\frac{3}{2}} = -\frac{1}{4} t^{\frac{3}{2}} z$$

$$\rightarrow \omega_{Ne}(t) = \frac{1}{2} t \neq \omega_{Ru}(t).$$

(53)

This obviously is in disagreement with (51).

Deeply related with this result is the question, if a Gaussian solution of the TDSE (or the corresponding continuity equation, respectively) for a parametric oscillator with specific $\omega(t)$ can be found that is also a solution of the diffusion equation. A Gaussian function fulfilling this requirement should obey the relation

$$\frac{\partial}{\partial x} (\varrho S v) = -D \frac{\partial^2}{\partial x^2} \varrho.$$  

(54)

The first problem is due to the drift motion of the Gaussian obeying the continuity equation, where the maximum moves according to $v_-$ with the classical velocity $\dot{\eta}(t)$, whereas the maximum of the Gaussian describing the diffusion process is usually fixed. But even if this problem could be circumvented by, e.g., changing to some moving coordinate frame or similar procedures, still the relation between the quantities $\alpha(t)$ that describe the spreading of the Gaussians have to be matched. This means, the relation

$$D \frac{2\hbar}{m} - \frac{4m^2}{\hbar^2} \frac{1}{\alpha^2} \dot{x}^2 = \dot{\alpha} \alpha - \frac{2m \dot{\alpha}}{\hbar} \dot{x}^2$$

(55)

has to be fulfilled.

For the choice $D = \frac{\hbar}{2m}$ (as assumed above) this leads to

$$1 - \frac{2m}{\hbar} \frac{1}{\alpha^2} \dot{x}^2 = \dot{\alpha} \alpha - \frac{2m}{\hbar} \frac{\dot{\alpha}}{\alpha} \dot{x}^2$$

(56)

or

$$\dot{\alpha} \alpha = 1.$$  

(57)

From the Gaussian solution of the diffusion equation follows

$$\alpha^2 = \frac{2m}{\hbar} \langle \dot{x}^2 \rangle = \frac{4m}{\hbar} D t = 2t,$$

(58)

what is in agreement with (49), (50) and (57).

On the other side, the width of the Gaussian fulfilling the continuity equation corresponding to the TDSE of a parametric oscillator with frequency $\omega(t)$ has to fulfil the Ermakov equation (12), where $\omega(t)$ is the same frequency that also occurs in the classical Newtonian equation for $\dot{\eta}(t)$.

From Eq. (57) follows that $\frac{\dot{\alpha}}{\alpha} = -\frac{1}{\alpha^2}$, so the Ermakov equation (12) would turn into

$$\omega^2(t) = 2 \left( \frac{1}{\alpha^4} \right) = 2 \left( \frac{1}{2} t \right)^2 = \left( \frac{1}{\sqrt{2} t} \right)^2.$$  

(59)

Therefore, the frequency of the Ermakov equation would be $\omega(t) = \frac{1}{\sqrt{2} t} = \omega_{Ru}(t)$, like in the Riccati equation (see Eq. (51)) and different from the frequency $\omega_{Ne}(t)$ that enters
the classical Newtonian equation of the problem. So, there is no Gaussian solution of the parametric oscillator that fulfils the corresponding continuity equation and at the same time also the diffusion equation.

According to these results, the approach via the WP-width $\alpha(t)$ and the Riccati equation seems problematic. However, with $z(t)$, one solution of the Newtonian equation is already known. As shown in Section 2, with a second linear independent solution $u(t)$, the Ermakov variable can be obtained via (27).

### 3.2. Approach via the Newtonian equation of motion

For $\omega(t) = \frac{1}{2t}$, a first solution of the Newtonian equation is given by $z = t^{\frac{1}{2}}$. From (26), it then follows that $u(t) = -t^{\frac{1}{2}} \ln t$ and, thus $^1$ from (27),

$$\alpha^2 = u^2 + z^2 = t \left[ 1 + \ln^2 t \right].$$  \hfill (60)

With this expression for $\alpha$, now $C_I$ and $C_R$ can be determined as

$$C_I = \frac{1}{\alpha^2} = \frac{1}{t \left[ 1 + \ln^2 t \right]}$$  \hfill (61)

$$C_R = \frac{\dot{\alpha}}{\alpha} = \frac{1}{2} \frac{\left( 1 + \ln t \right)^2}{t \left[ 1 + \ln^2 t \right]} = \frac{1}{2} \frac{1}{t} + \frac{\ln t}{t \left[ 1 + \ln^2 t \right]}$$  \hfill (62)

where $\frac{1}{2t}$ is actually also a particular solution $\tilde{C}$ of the Riccati equation. A tricky aspect is still the divergence for $t \to 0$; but this can be resolved.

In the following, the modified TD frequency

$$\omega(t) = \frac{1}{2} \left( t + b \right)$$  \hfill (63)

is considered with $b = \text{constant} \neq 0$ and dimension “time”.

Two linear-independent solutions of the (complex) Newtonian equation

$$\ddot{\lambda} + \left( \frac{1}{2 \left( t + b \right)} \right)^2 \lambda = 0$$  \hfill (64)

are

$$z = (t + b)^{\frac{1}{2}}$$  \hfill (65)

and

$$u = - (t + b)^{\frac{1}{2}} \ln(t + b),$$  \hfill (66)

leading to the WP width via

$$\alpha^2(t) = (t + b) \left[ 1 + \ln^2(t + b) \right]$$  \hfill (67)

with the well-behaved initial value $\alpha^2(0) = b \left[ 1 + \ln^2 b \right]$.

From (67), the imaginary and real parts of the Riccati variable can be obtained as

$^1$ In the following three equations it is assumed that $t \geq 0$, where the ln-terms diverge for $t = 0$
\[ C_t = \frac{1}{(t + b) \left[ 1 + \ln^2(t + b) \right]} \]  
\[ C_R = \frac{1}{2(t + b)} + \frac{\ln(t + b)}{(t + b) \left[ 1 + \ln^2(t + b) \right]} \]  

with initial values \( C_t(0) = \frac{1}{b \left[ 1 + \ln^2 b \right]} \) and \( C_R(0) = \frac{1}{2b} + \frac{\ln b}{b \left[ 1 + \ln^2 b \right]} \). The first term on the rhs of \( C_R \) is again a particular solution of the Riccati equation, \( \hat{C} = \frac{1}{2(t + b)} \). This (real) particular solution can also be utilized to prove the accuracy of (68) and (69) using the direct method of solving the Riccati equation outlined in Section 2; i.e., assuming that the general solution is given by \( C = \hat{C} + V(t) \). In this case, \( V(t) \) must fulfil the Bernoulli equation

\[ \dot{V} + \frac{1}{(t + b)} V + V^2 = 0 \]  

As \( \hat{C} \) is real, the imaginary part of the Bernoulli equation is identical to the imaginary part of the general solution, \( V_I = C_I \). With \( C_t \) as given in (68), it follows from \( V_R = -\frac{1}{2V_I} - \frac{1}{2(t + b)} \) that

\[ V_R = \frac{\ln(t + b)}{(t + b) \left[ 1 + \ln^2(t + b) \right]} \]  

which is exactly the missing second term on the rhs of (69), fulfilling the real part of the Bernoulli equation,

\[ \dot{V}_R + \frac{1}{(t + b)} V_R + V_R^2 = 0 \]  

With the expressions for \( \alpha^2 \), \( C_t \) and \( C_R \), the analytical form of the quantum uncertainties, ground state energy and tunnelling contribution to the probability current can be given explicitly:

\[ \langle \hat{x}^2 \rangle(t) = \frac{\hbar}{2m} \alpha^2(t) = \frac{\hbar}{2m} (t + b) \left[ 1 + \ln^2(t + b) \right] \]  
\[ \langle \hat{p}^2 \rangle(t) = \frac{m \hbar}{2} \left[ \dot{\alpha}^2(t) + \frac{1}{\alpha^2(t)} \right] = \frac{m \hbar}{2} \left( \frac{1}{t + b} \left[ \frac{1}{4} \left( 1 + \ln(t + b) \right)^2 + \frac{1}{2} \ln(t + b) + 1 \right] \right) \]  
\[ \langle [\hat{x}, \hat{p}]_+ \rangle(t) = \hbar \alpha(t) \dot{\alpha}(t) = \frac{\hbar}{2} \left[ 1 + \ln(t + b) \right]^2 \]  
\[ U(t) = \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle = \frac{\hbar^2}{4} \left\{ 1 + (\alpha \dot{\alpha})^2 \right\} = \frac{\hbar^2}{4} \left\{ 1 + \frac{1}{4} \left[ 1 + \ln(t + b) \right]^4 \right\} \]  
\[ \tilde{E}(t) = \frac{\langle \hat{p}^2 \rangle}{2m} + \frac{m}{2} \omega^2(t) \langle \hat{x}^2 \rangle = \frac{\hbar}{4} \left\{ \dot{\alpha}^2 + \frac{1}{\alpha^2} + \omega^2(t) \alpha^2 \right\} = \frac{\hbar}{4} \left\{ \frac{1}{(t + b)} \left[ \frac{1}{2} (1 + \ln(t + b))^2 + 1 \right] \right\} \]
Table 1: Particular values of the quantum uncertainties, ground state energy and tunnelling contribution to the probability current for $t = 0$ and $t \to \infty$.

$$\tilde{v}_-(t) = v - \langle v \rangle = \frac{\dot{\alpha}}{\alpha} \tilde{x} = \left\{ \frac{1}{2(t + b)} + \frac{\ln(t + b)}{(t + b)} \right\} \tilde{x}. \quad (78)$$

The particular values for $t = 0$ and $t \to \infty$ are given in Table 1.

The plots for $\langle \tilde{x}^2 \rangle(t) = \sigma_x^2(t)$, $\langle \tilde{p}^2 \rangle(t) = \sigma_p^2(t)$, $\frac{1}{2}([\tilde{x}, \tilde{p}]_+) = \sigma_{xp}$, $U(t) = \sigma_x^2 \sigma_p^2$ are given in Figures 2 and 3. The time-dependence of $\tilde{E}(t)$ and $\tilde{v}_-$ (essentially given by $C_R = \frac{\omega}{\alpha}$) is shown in Figure 4.

![Graph 2](image_url)

**Figure 2**: Graphs of $\sigma_x^2(t) = \langle \tilde{x}^2 \rangle(t)$ and $\sigma_p^2(t) = \langle \tilde{p}^2 \rangle(t)$ as function of time.

Obviously, the position uncertainty and thus the WP width are diverging for $t \to \infty$ whereas the momentum uncertainty is approaching zero. The product of both is going to infinity for $t \to \infty$ while the energy contribution $\tilde{E}$ is going to zero in this case, as in a (classical) dissipative system (without fluctuating force).

### 3.3. Classical dynamics and energetics for this parametric oscillator

According to (36), $z(t)$ is proportional to the classical trajectory $\eta(t)$. Therefore, the classical Newtonian equation for this parametric oscillator takes the form

$$\ddot{\eta} + \left( \frac{1}{2(t + b)} \right)^2 \eta = 0 \quad (79)$$

with the solution

$$\eta(t) = v_0 \tau^{\frac{1}{2}}(t + b)^{\frac{1}{2}} \quad (80)$$

with $v_0 = \frac{1}{2} b^{\frac{1}{2}}$ where the constant $\tau$ with dimension time has been introduced to get the correct dimension for $\eta(t)$. The corresponding velocity $\dot{\eta}$ is then given by

$$\dot{\eta}(t) = \frac{1}{2} v_0 \tau^{\frac{1}{2}}(t + b)^{-\frac{1}{2}} = v_0 \frac{\tau^{\frac{1}{2}}}{2(t + b)^{\frac{1}{2}}} \quad (81)$$
and $\tilde{E}(t)$ as function of time.

Figure 4: Graphs of $	ilde{E}(t) = \frac{1}{2m} \sigma_p^2(t) + \frac{m}{2} \left( \frac{1}{2} (t + b) \right)^2 \sigma_x^2(t)$ and $C_R = \frac{\dot{\alpha}}{\alpha}$ as function of time.

with $\dot{\eta}_0 = v_0 \frac{t}{2 b^2}$. From $\dot{\eta}_0 = v_0$ follows $b = \frac{t}{4}$.

The classical energy of the system then acquires the form

$$E = T + V = \frac{m}{2} \dot{\eta}^2 + \frac{m}{2} \left( \frac{1}{2(t+b)} \right)^2 \eta^2$$

$$= \frac{m}{2} v_0^2 \frac{\tau}{2(t+b)} \propto \frac{1}{t}$$

(82)
and decreases inversely proportional to time $t$. The initial energy $E_0$ is given by $E_0 = \frac{m}{2} v_0^2 \frac{\tau}{2\omega}$, the final energy for $t \to \infty$ approaches zero, $E_\infty \to 0$, as in a dissipative system.

The decrease in energy is given by

$$\dot{E} = - \frac{m}{2} v_0^2 \frac{\tau}{2} \left( \frac{1}{t+b} \right)^2 = -2\omega(t) E = -4 \omega(t) T < 0 .$$

(83)

Comparison with a dissipative system with a friction force linear proportional to velocity, i.e., the equation of motion

$$\ddot{\eta} + \gamma \dot{\eta} + \omega^2 \eta = 0 ,$$

(84)

shows that, in this case, the decrease in energy is

$$\dot{E} = -2\gamma T = -4 \frac{\gamma}{2} T .$$

(85)

So, the time-dependent frequency would correspond to half of the friction parameter $\gamma$,

$$\omega(t) \doteq \frac{\gamma}{2} .$$

(86)

The plots of $\eta(t), \dot{\eta}(t), E(t)$ and $\dot{E}(t)$ are shown in Figures 5 and 6.
4. Conclusions
The TDSE for potentials that are at most quadratic in position variable possesses exact solutions that have the form of Gaussian WPs. These functions are uniquely determined by two parameters, the maximum and the width. In this case, both parameters are TD. The time-dependence of the maximum obeys the classical Newtonian equation of motion for the given potential; the time-evolution of the width is determined by a complex NL Riccati equation or, equivalently, by a real NL Ermakov equation. For the free motion \( \omega = 0 \) and the harmonic oscillator \( \omega = \omega_0 = \text{const.} \), both equations can be solved in closed form. For TD frequency \( \omega = \omega(t) \), it is the explicit time-dependence of \( \omega(t) \) that determines whether analytic solutions exist or not.

Independent of the particular \( \omega(t) \), the solutions of the Riccati or Ermakov equations provide the complete information about the quantum dynamics of the system. These solutions also provide the necessary information to describe the dynamics in terms of TD Green functions or Feynman kernels, generalized creation/annihilation operators or Wigner functions.

Furthermore, the complex Riccati equation can also be linearized to a complex Newtonian equation where real and imaginary parts of the complex variable \( \lambda(t) \) fulfilling this equation are two linear-independent solutions of the second-order Newtonian differential equation. However, these two solutions are not independent of each other but coupled via a kind of conservation law that is related to an invariant that can be obtained from the coupled pair of differential equations provided by the Newtonian and the Ermakov (or Riccati) equation of the problem. Essentially, the solution of the classical Newtonian equation makes it possible to obtain a second linear-independent solution with both providing the solution of the Ermakov equation and thus all information about the quantum dynamics of the system.

On the other hand, the interrelations between the classical and quantum aspects of the system’s dynamics also lead to the solution of the classical dynamics, once the solution of the quantum dynamics describing the time-dependence of the WP width is known.

In looking for a TD frequency \( \omega(t) \) that allows analytic solutions for the WP maximum and width, a comparison with the time-dependence of the Gaussian function that is a solution of the diffusion equation suggests a frequency that is inversely proportional to time, \( \omega(t) \propto \frac{1}{t} \).

One problem is that the width of the Gaussian function describing the classical diffusion process obeys a real Riccati equation, whereas the quantum dynamics is determined by a complex Riccati equation. Finding the missing imaginary part of the Riccati equation when only the real part is known is not trivial. A more promising attempt is to solve the corresponding classical Newtonian equation of motion for the given \( \omega(t) \), construct the second linear-independent solution via the above-mentioned conservation law which finally leads to the solution of the real Ermakov or complex Riccati equation.

This procedure proved quite successful for \( \omega(t) = \frac{1}{t^2} \) except for the problematic aspect of divergencies for \( t \to 0 \) and \( t < 0 \) (already on the classical level; but could be avoided by replacing \( t \) with \(|t|\)). The divergence for \( t \to 0 \) can be circumvented by changing \( \omega(t) \) to \( \omega(t) = \frac{1}{|a(t+b)|} \).

In particular, the case for \( a = 2 \) was considered in this paper. The dynamics and energetics of the parametric oscillator with frequency \( \omega(t) = \frac{1}{2(t+b)} \) for the cases \( b = 0 \) and \( b \neq 0 \) can be compared with that of a damped harmonic oscillator with linear friction.

For \( b = 0 \), time-reversal seems not to affect the equation of motion at first sight, i.e., Eq. (79) with \( b = 0 \), as in this case \( \omega^2(t) = \omega^2(−t) \). However, Eq. (80) for the classical trajectory and Eq. (81) for the classical velocity would, for \( t < 0 \), lead to purely imaginary values. This seemingly unphysical behaviour could be resolved by replacing \( t \) with its absolute value \(|t|\) for \( t < 0 \). Despite this apparent reversibility of the motion, the energy still decreases inversely-proportional to time and approaches zero, like in the dissipative, frictionally-damped system, i.e., the system displays dissipation without irreversibility.

For \( b \neq 0 \), the situation changes concerning the equation of motion as, in this case,
\( \omega(t) \neq \omega(-t) \). So now, also the equation of motion is no longer invariant under time-reversal since the addition of the constant \( b \) breaks the time-reversal symmetry! Still, \( \dot{E} < 0 \), i.e., the energy decreases similarly to the situation in a dissipative system. Regarding the trajectory and velocity, the same essentially applies as in the case \( b = 0 \), i.e., replacing \( t \) with \( |t| \) prevents the problem of imaginary values.

Comparison of the frequency \( \omega(t) \) with \( \frac{\gamma}{2} \) in (86) is analogous to the aperiodic limit of the damped harmonic oscillator, i.e., the case \( \omega_0 = \frac{\gamma}{2} \). Indeed, it can be shown that there are analytical solutions for the parametric oscillator with the more general frequency

\[
\omega(t) = \frac{1}{a(t+b)}
\]

for arbitrary constant \( b \). However, the systems display qualitatively different behaviour for different values of \( a \). The case of \( a = 2 \) has been discussed in detail in this work and is formally similar to the aperiodic limit of the damped harmonic oscillator, i.e., \( \omega_0 = \frac{\gamma}{2} \).

The case \( a < 2 \) resembles the harmonic oscillator with undercritical damping, \( \omega_0 < \frac{\gamma}{2} \) and exhibits oscillations with damped amplitudes of the velocity or momentum (contrary to the position!).

Finally, the case \( a > 2 \) behaves like an overcritically damped harmonic oscillator without oscillatory motion and \( \omega_0 > \frac{\gamma}{2} \). The latter two cases will be discussed in detail in a forthcoming paper.

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