Upper bounds for the $MD$-numbers and characterization of extremal graphs\textsuperscript{1}

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Abstract

For an edge-colored graph $G$, we call an edge-cut $M$ of $G$ monochromatic if the edges of $M$ are colored with the same color. The graph $G$ is called monochromatic disconnected if any two distinct vertices of $G$ are separated by a monochromatic edge-cut. For a connected graph $G$, the monochromatic disconnection number (or $MD$-number for short) of $G$, denoted by $md(G)$, is the maximum number of colors that are allowed in order to make $G$ monochromatic disconnected. For graphs with diameter one, they are complete graphs and so their $MD$-numbers are 1. For graphs with diameter at least 3, we can construct 2-connected graphs such that their $MD$-numbers can be arbitrarily large; whereas for graphs $G$ with diameter two, we show that if $G$ is a 2-connected graph then $md(G) \leq 2$, and if $G$ has a cut-vertex then $md(G)$ is equal to the number of blocks of $G$. So, we will focus on studying 2-connected graphs with diameter two, and give two upper bounds of their $MD$-numbers depending on their connectivity and independent numbers, respectively. We also characterize the $\lfloor \frac{n}{2} \rfloor$-connected graphs (with large connectivity) whose $MD$-numbers are 2 and the 2-connected graphs (with small connectivity) whose $MD$-numbers archive the upper bound $\lfloor \frac{n}{2} \rfloor$. For graphs with connectivity less than $\frac{n}{2}$, we show that if the connectivity of a graph is in linear with its order $n$, then its $MD$-number is upper bounded by a constant, and this suggests us to leave a conjecture that for a $k$-connected graph $G$, $md(G) \leq \lfloor \frac{n}{k} \rfloor$.

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1 Introduction

Let $G$ be a graph and let $V(G)$, $E(G)$ denote the vertex-set and the edge-set of $G$, respectively. We use $|G|$ and $|\big|G\big|$ to denote the number of vertices and the number of edges of $G$, respectively, and call them the order and the size of $G$. If there is no confusion, we also use $n$ and $m$ to denote $|G|$ and $|\big|G\big|$, respectively, throughout this paper. Let $S$ and $F$ be a vertex subset and an edge subset of $G$, respectively. Then $G - S$ is the graph obtained from $G$ by deleting the vertices of $S$ together with the edges incident with vertices of $S$, and $G - F$ is the graph whose vertex-set is $V(G)$ and edge-set is $E(G) - F$. Let $G[S]$ and $G[F]$ be the subgraphs of $G$ induced, respectively, by $S$ and $F$. Let $G[S]$ and $G[F]$ be the subgraphs of $G$ induced, respectively, by $S$ and $F$. We use $\mathbb{[}r\mathbb{]}$ to denote the set $\{1, 2, \cdots, r\}$ of positive integers. If $r = 0$, then set $\mathbb{[}r\mathbb{]} = \emptyset$. For all other terminology and notation not defined here we follow Bondy and Murty [4].

For a graph $G$, let $\Gamma : E(G) \rightarrow \mathbb{[}r\mathbb{]}$ be an edge-coloring of $G$ that allows a same color to be assigned to adjacent edges. For an edge $e$ of $G$, we use $\Gamma(e)$ to denote the color of $e$. If $H$ is a subgraph of $G$, we also use $\Gamma(H)$ to denote the set of colors on the edges of $H$ and use $|\Gamma(H)|$ to denote the number of colors in $\Gamma(H)$. For an edge-colored graph $G$ and a vertex $v$ of $G$, the color-degree of $v$, denoted by $d_c(v)$, is the number of colors appearing on the edges incident with $v$.

The three main colored connection colorings: rainbow connection coloring [8], proper connection coloring [5] and proper-walk connection coloring [3], monochromatic connection coloring [6], have been well-studied in recent years. As a counterpart concept of the rainbow connection coloring, rainbow disconnection coloring was introduced in [7] by Chartrand et al. in 2018. Subsequently, the concepts of monochromatic disconnection coloring and proper disconnection coloring were also introduced in [12] and [1, 9]. We refer to [2] for the philosophy of studying these so-called global graph colorings. More details on the monochromatic disconnection coloring can be found in [13]. We will further study this coloring in this paper and get some deeper and stronger results.

For an edge-colored graph $G$, we call an edge-cut $M$ a monochromatic edge-cut if the edges of $M$ are colored with the same color. If there is a monochromatic $uv$-cut with color $i$, then we say that color $i$ separates $u$ and $v$. We use $C_{\Gamma}(u, v)$ to denote the set of colors in $\Gamma(G)$ that separate $u$ and $v$, and let $c_{\Gamma}(u, v) = |C_{\Gamma}(u, v)|$.

An edge-coloring of a graph is called a monochromatic disconnection coloring (or MD-coloring for short) if each pair of distinct vertices of the graph has a monochromatic edge-cut separating them, and the graph is called monochromatic disconnected. For a connected graph $G$, the monochromatic disconnection number (or MD-number for short) of $G$, denoted by $md(G)$, is defined as the maximum number of colors that are allowed in order to make $G$ monochromatic disconnected. An extremal MD-coloring of $G$ is an MD-coloring that uses $md(G)$ colors. If $H$ is a subgraph of $G$ and $\Gamma$ is an edge-coloring of $G$, we call $\Gamma$ an edge-coloring restricted on $H$.
Theorem 1.3. \[13\] If two graphs. The union of $G$ and $H$ is the graph $G \cup H$ with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The intersect of $G$ and $H$ is the graph $G \cap H$ with vertex-set $V(G) \cap V(H)$ and edge-set $E(G) \cap E(H)$. The Cartesian product of $G$ and $H$ is the graph $G \square H$ with $V(G \square H) = \{(u, v) : u \in V(G), v \in V(H)\}$, $(u, v)$ and $(x, y)$ are adjacent in $G \square H$ if either $ux$ is an edge of $G$ and $v = y$, or $vy$ is an edge of $H$ and $u = x$. If $G$ and $H$ are vertex-disjoint, then let $G \vee H$ denote the join of $G$ and $H$ which is obtained from $G$ and $H$ by adding an edge between every vertex of $G$ and every vertex of $H$.

For a graph $G$, a pendant vertex of $G$ is a vertex with degree one. The ends of $G$ is the set of pendant vertices, and the internal vertex set of $G$ is the set of vertices with degree at least two. We use $\text{end}(G)$ and $\text{I}(G)$ to denote the ends of $G$ and the internal vertex set of $G$, respectively. The independent number of $G$, denoted by $\alpha(G)$, is the order of a maximum independent set of $G$. For two vertices $u, v$ of $G$, we use $N(u)$ to denote the neighborhood of $u$ in $G$, and $N(u, v)$ to denote the set of common neighbors of $u$ and $v$ in $G$. The distance between $u$ and $v$ in $G$ is denoted by $d(u, v)$, and the diameter of $G$ is denoted by $\text{diam}(G)$. We call a cycle $C$ (path $P$) a $t$-cycle ($t$-path) if $|C| = t$ ($|P| = t$). If $t$ is even (odd), then we call the path an even (odd) path and the cycle an even (odd) cycle. A 3-cycle is also called a triangle. A matching-cut of $G$ is an edge-cut of $G$, which also forms a matching in $G$.

In \cite{12,13} we got the following results, which are restated for our later use.

Lemma 1.1. \cite{12}

1. If a connected graph $G$ has $r$ blocks $B_1, \ldots, B_r$, then $\text{md}(G) = \sum_{i \in [r]} \text{md}(B_i)$ and $\text{md}(G) = n - 1$ if and only if $G$ is a tree.

2. $\text{md}(G) = \left\lfloor \frac{|G|}{2} \right\rfloor$ if $G$ is a cycle, and $\text{md}(G) = 1$ if $G$ is a complete graph with order at least two.

3. If $H$ is a connected spanning subgraph of $G$, then $\text{md}(H) \geq \text{md}(G)$. Thus, $\text{md}(G) \leq n - 1$.

4. If $G$ is connected, then $\text{md}(\text{v} \vee G) = 1$.

5. If $v$ is neither a cut-vertex nor a pendant vertex of $G$ and $\Gamma$ is an extremal MD-coloring of $G$, then $\Gamma(G) \subseteq \Gamma(G - v)$, and thus, $\text{md}(G) \leq \text{md}(G - v)$.

Theorem 1.2. \cite{12} If $G$ is a 2-connected graph, then $\text{md}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 1.3. \cite{13} If $G_1$ and $G_2$ are connected graphs, then $\text{md}(G_1 \square G_2) = \text{md}(G_1) + \text{md}(G_2)$.

Lemma 1.4. \cite{13} If $G$ has a matching-cut, then $\text{md}(G) \geq 2$. 

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We will list some easy observations in the following, which will be used many times throughout this paper. Suppose $\Gamma$ is an $MD$-coloring of $G$. If $H$ is a subgraph of $G$, then $\Gamma$ is an $MD$-coloring restricted on $H$. Every triangle of $G$ is monochromatic. If $G$ is a 4-cycle, then its opposite edges have the same color. If $G$ is a 5-cycle, then there are two adjacent edges having the same color.

Let $V$ be a set of vertices and let $E \subseteq 2^V$. Then a hypergraph $H = (V, E)$ is a linear hypergraph if $|E_i| \geq 2$ and $|E_i \cap E_j| \leq 1$ for any $E_i, E_j \in E$. The size of $H$ is the number of hyperedges in $H$. A hyperedge-coloring of $H$ assigns each hyperedge a positive integer. A linear hypergraph $H$ (say the size of $H$ is $k$) is a linear hypercycle if there is a sequence of hyperedges of $H$, say $E_1, \ldots, E_k$, and there exist $k$ distinct vertices $v_1, \ldots, v_k$ of $H$, such that $E_1 \cap E_k = \{v_k\}$ and $E_i \cap E_{i+1} = \{v_i\}$ for $i \in [k-1]$. If we delete a hyperedge from a linear hypercycle and then delete the vertices only in this hyperedge, then we call the resulting hypergraph a linear hyperpath. A linear hypercycle (linear hyperpath) is called a linear hyper $k$-cycle (linear hyper $k$-path) if the size of this linear hypercycle (linear hyperpath) is $k$.

## 2 Preliminaries

We need some more preparations before proceeding to our main results.

**Lemma 2.1.** For two connected graphs $G_1$ and $G_2$, if $md(G_1 \cap G_2) = 1$ then $md(G_1 \cup G_2) = md(G_1) + md(G_2) - 1$.

**Proof.** Let $G = G_1 \cup G_2$ and $\Gamma$ be an extremal $MD$-coloring of $G$. Then $|\Gamma(G_1 \cap G_2)| = 1$ and $\Gamma$ is an $MD$-coloring restricted on $G_1$ (and also $G_2$). So, $md(G_1 \cup G_2) = |\Gamma(G_1)| + |\Gamma(G_2)| - |\Gamma(G_1 \cap G_2)| \leq md(G_1) + md(G_2) - 1$. On the other hand, since $E(G_1 \cap G_2)$ is monochromatic under any $MD$-coloring of $G_1 \cup G_2$, let $\Gamma_i$ be an $MD$-coloring of $G_i$ for $i \in [2]$ such that $\Gamma_1(G_1 \cap G_2) = \Gamma_2(G_1 \cap G_2) = \Gamma(G_1) \cap \Gamma(G_2)$. Let $\Gamma'$ be an edge-coloring of $G$ such that $\Gamma'(e) = \Gamma_i(e)$ if $e \in E(G_i)$, and let $w$ be a vertex of $G_1 \cap G_2$. Then for any two vertices $u, v$ of $G_1 \cup G_2$, if $u, v \in V(G_i)$, then $C_{\Gamma_i}(u, v) \subseteq C_{\Gamma_i}(u, v)$; if $u \in V(G_1) - V(G_2)$ and $v \in V(G_2) - V(G_1)$, then $(C_{\Gamma_1}(u, w) \cup C_{\Gamma_2}(v, w)) \subseteq C_{\Gamma'}(u, v)$. So, $\Gamma'$ is an $MD$-coloring of $G$, i.e., $md(G_1 \cup G_2) \geq |\Gamma(G_1 \cap G_2)| = md(G_1) + md(G_2) - 1$. Therefore, $md(G_1 \cup G_2) = md(G_1) + md(G_2) - 1$. $\blacksquare$

**Lemma 2.2.** Let $G$ be a connected graph and let $G'$ be a graph obtained from $G$ by replacing an edge $e = ab$ with a path $P$. Then $md(G') \geq md(G) + \left\lceil \frac{|P|-1}{2} \right\rceil$.

**Proof.** Let $\Gamma$ be an extremal $MD$-coloring of $G$. Let $|P| = t$ and let $P = ae_1 c_1 \cdots e_t b$. Let $\Gamma'$ be an edge-coloring of $G'$ such that $\Gamma'(f) = \Gamma'(f)$ when $f \in E(G) - e$, $\Gamma'(e_i) = \Gamma'(e_{i-1} \cdots e_1) = |\Gamma'(G)| + i$ for $i \in \left[\frac{|P|-1}{2}\right]$, $\Gamma'(e_c) = \Gamma'(e_{c+1})$ when $t$ is odd, and $\Gamma'(e_c) = \Gamma'(e_{c+1})$ when $t$ is even. It is easy to verify that $\Gamma'$ is an $MD$-coloring of $G'$. Thus, $md(G') \geq md(G) + \left\lceil \frac{|P|-1}{2} \right\rceil$. $\blacksquare$
Lemma 2.3. Suppose \( u, v \) are nonadjacent vertices of \( G \) and \( \Gamma \) is an extremal \( MD \)-coloring of \( G \). Let \( C_\Gamma(u, v) = \{t\} \) and \( e \) an extra edge, and let \( \Gamma' \) be an edge-coloring of \( G \cup e \) that is obtained from \( \Gamma \) by coloring the added edge \( e \) with color \( t \). Then \( \Gamma' \) is an \( MD \)-coloring of \( G \cup e \) and \( md(G) = md(G \cup e) \).

Proof. Let \( H_t \) be the graph obtained from \( G \) by deleting all the edges with color \( i \). Let \( G' = G \cup e \). If \( \Gamma' \) is not an \( MD \)-coloring of \( G' \), then there are two vertices \( x, y \) of \( G' \) such that \( C_{\Gamma'}(x, y) = \emptyset \). If \( t \in C_\Gamma(x, y) \), since \( x, y \) are in different components of \( H_t \), we have \( t \in C_{\Gamma'}(x, y) \), a contradiction. If \( t \notin C_\Gamma(x, y) \), then let \( j \in C_\Gamma(x, y) \). Then there are two components \( D_1, D_2 \) of \( H_j \) such that \( x \in V(D_1) \) and \( y \in V(D_2) \). Since \( j \) does not separate \( x, y \) in \( G' \), the edge \( e \) connects \( D_1 \) and \( D_2 \), say \( u \in V(D_1) \) and \( v \in V(D_2) \). Thus, the color \( j \) separates \( u, v \) in \( G \), which contradicts that \( C_\Gamma(u, v) = \{t\} \). Therefore, \( \Gamma' \) is an \( MD \)-coloring of \( G' \). Since \( |\Gamma'(G')| = |\Gamma(G)| \) and \( \Gamma \) is an extremal \( MD \)-coloring of \( G \), we have \( md(G') \geq md(G) \). Since \( G \) is a connected spanning subgraph of \( G' \), by Lemma 2.3, we have \( md(G) \geq md(G') \). So, \( md(G) = md(G') \). 

Suppose \( \Gamma \) is an \( MD \)-coloring of \( G \) and \( G_i \) is the subgraph of \( G \) induced by the set of edges with color \( i \), which, in what follows, is called the \textit{color} \( i \text{-induced subgraph} \) of \( G \). Then for any component \( D_1 \) of \( G_i \) and any component \( D_2 \) of \( G_j \), we have \( |V(D_1) \cap V(D_2)| \leq 1 \); otherwise, suppose \( u, v \in V(D_1) \cap V(D_2) \). Then \( C_\Gamma(u, v) = \emptyset \), a contradiction. We use \( \mathcal{H}_\Gamma \) to denote a hyperedge-colored hypergraph with vertex-set \( V(G) \) and hyperedge-set \( \{V(D) \mid D \text{ is a component of some } G_i\} \), and the hyperedge \( F \) has color \( i \) if \( F \) corresponds to a component of \( G_i \). Let \( H_\Gamma \) be a graph with \( V(H_\Gamma) = V(G) \) and \( E(H_\Gamma) = \{uv \mid u, v \text{ are in the same component of some } G_i\} \).

Then each hyperedge of \( \mathcal{H}_\Gamma \) corresponds to a clique of \( H_\Gamma \), and any two hyperedges of \( \mathcal{H}_\Gamma \) (any two cliques of \( H_\Gamma \)) share at most one vertex. Thus, \( \mathcal{H}_\Gamma \) is a linear hypergraph. If \( F \) is a hyperedge of \( \mathcal{H}_\Gamma \) and \( u, v \in F \), then \( c_\Gamma(u, v) = 1 \). According to Lemma 2.3, we have the following result.

Lemma 2.4. If \( \Gamma \) is an extremal \( MD \)-coloring of \( G \), then \( md(G) = md(H_\Gamma) \).

Suppose \( \Gamma \) is an \( MD \)-coloring of \( G \) and \( C \) is a hyper \( k \)-cycle of \( \mathcal{H}_\Gamma \). Then there is a \( k \)-cycle \( C \) of \( H_\Gamma \) such that any adjacent edges of \( C \) have different colors. Thus, \( k \) \( \neq \) 3, 5. Moreover, if \( k = 4 \), then the opposite hyperedges of \( C \) have the same color.

3 Graphs with diameter two

In this section, we show that \( md(G) \leq 2 \) for a 2-connected graph \( G \) if \( diam(G) \leq 2 \). However, for any integer \( d \geq 3 \), we can construct a 2-connected graph \( G \) such that
diam(G) = d and md(G) can be arbitrarily large. Thus, it makes sense to focus on studying the graphs with diameter two, since graphs with diameter 1 are complete graphs and their MD-numbers are 1.

**Theorem 3.1.** Suppose G is a graph with diam(G) = 2. Then

1. if G has a cut-vertex, then md(G) is equal to the number of blocks of G;
2. if G is a 2-connected graph, then md(G) ≤ 2;
3. if any two nonadjacent vertices of G has at least two common neighbors, then md(G) ≤ 2, and the equality holds if and only if G = K_s □ K_t, where s, t ≥ 2.

**Proof.** The proof of statement (1) goes as follows. If v is a cut-vertex of G and diam(G) = 2, then v connects every vertex of V(G − v). Thus, for each block D of G, D − v is connected and D = (D − v) ∨ v, i.e., md(D) = 1. Therefore, md(G) is equal to the number of blocks of G.

Next, for the proof of statement (2) suppose Γ is an MD-coloring of G with |Γ(G)| ≥ 3. Then each hypercycle (hyperpath) of the above mentioned hypergraph HΓ is a hypercycle (linear hyperedge). We now prove that there is a rainbow hyper 3-path (the colors of the three hyperedges are pairwise differently) in HΓ. Since HΓ does not have hyper 3-cycle, the union of three consecutive hyperedges forms a hyper 3-path. If every vertex z of G has dΓ(z) ≤ 2, then there is a rainbow hyper 3-path in HΓ. If there is a vertex x of G with dΓ(x) ≥ 3, then there are three hyperedges, say D1, D2 and D3, such that x is the common vertex of them. Then the colors of D1, D2 and D3 are pairwise differently. Since G is a 2-connected graph, there is a vertex w of V(D1) − {x} with dΓ(w) ≥ 2 (otherwise, x is a cut-vertex of G; a contradiction). Then there is a hyperedge F of G, such that w is a common vertex of F and D1. Thus, either F ∪ D1 ∪ D2 or F ∪ D1 ∪ D3 is a rainbow hyper 3-path.

Let P be a rainbow hyper 3-path of H and let V(Dᵢ) ∩ V(Dᵢ₊₁) = {uᵢ} for i ∈ [2]. Let u ∈ V(D₁) − {u₁} and v ∈ V(D₃) − {u₂}. We use P_u,v to denote a minimum hyperpath connecting u and v. Since diam(G) = 2, the size of P_u,v is either one or two. Let C = P_u,v ∪ P. If P_u,v is a hyperedge, then C is a hyper 4-cycle. Since D₁ and D₃ are opposite hyperedges of C and they have different colors, a contradiction. If P_u,v is a hyper 2-path, then let F₁, F₂ be hyperedges of P_u,v, and let V(F₁) ∩ V(F₂) = {u₃}. If u₃ ∉ {u₁, u₂}, then C is a hyper 5-cycle, a contradiction. If u₃ ∈ {u₁, u₂}, then C contains a hyper 3-cycle, a contradiction.

Finally, we show statement (3). It is obvious that diam(G) ≤ 2, and G is a 2-connected graph when n ≥ 3. So, md(G) ≤ 2. Suppose G = Kᵢ □ Kᵢ and s, t ≥ 2. Then |N(u, v)| = 2 for any nonadjacent vertices u and v of G. By Lemma 1.1 (2) and Theorem 1.3 we have md(G) = md(Kₚ) + md(Kₜ) = 2.
Suppose $md(G) = 2$. Then $n \geq 3$ and $G$ is a 2-connected graph. Let $\Gamma$ be an extremal $MD$-coloring of $G$ and let $G_1, G_2$ be the colors $1, 2$ induced subgraphs of $G$, respectively. Since $md(G) = 2$, we have $d^c(v) \leq 2$ for each $v \in V(G)$. If $d^c(v) = 1$, by symmetry, suppose $v$ is in a component $D$ of $G_1$. Since $md(G) = 2$, we have $D \neq G$, i.e., there exists a vertex $u$ in $V(G) - V(D)$. Then $u, v$ are nonadjacent and $N(u, v) \subseteq D$. Let $\{a, b\} \subseteq N(u, v)$. Since $\Gamma(va) = \Gamma(vb) = 1$, we have $va \cup vb \cup ua \cup ub$ is a monochromatic 4-cycle, i.e., $u \in V(D)$, a contradiction. Thus, $d^c(v) = 2$ for each $v \in V(G)$. We use $D_u^1$ and $D_u^2$ to denote the components of $G_1$ and $G_2$, respectively, such that $V(D_u^1) \cup V(D_u^2) = u$.

Suppose there are $t$ components of $G_1$ and $s$ components of $G_2$. Since $G$ is a 2-connected graph, we have $s, t \geq 2$. Otherwise, if $s = 1$, then for each vertex $v$ of $G_1$, $v$ is a cut-vertex, a contradiction. We label the $t$ components of $G_1$ by the numbers in $[t]$ and label the $s$ components of $G_2$ by the numbers in $[s]$, respectively. We use $l_1(D)$ to denote the label of a component $D$ of $G_1$, and use $l_2(F)$ to denote the label of a component $F$ of $G_2$. For a vertex $u$ of $G$, since $d^c(u) = 2$, we use $(l_1(D_u^1), l_2(D_u^2))$ to denote $u$. For two vertices $u, v$ of $G$, let $u = (i, j)$ and let $v = (s, t)$. In order to show $G = K_i \sqcup K_j$, we need to show that $uv$ is an edge of $G$ when $i = s$ and $j \neq t$, or $i \neq s$ and $j = t$, and $u, v$ are nonadjacent vertices when $i \neq s$ and $j \neq t$. If $i \neq s$ and $j \neq t$, then $v \not\in V(D_u^1 \cup D_u^2)$. Since $N(u) \subseteq V(D_u^1 \cup D_u^2)$, $u, v$ are nonadjacent vertices of $G$. If, by symmetry, $i = s$ and $j \neq t$, then $D_u^1 = D_v^1$. Let $u' \in V(D_u^2) - \{u\}$. Then $u', v$ are nonadjacent. Since $N(v) \subseteq V(D_u^1 \cup D_v^2)$ and $N(u') \subseteq V(D_u^1 \cup D_v^2)$, we have

$$2 \leq |N(v, u')| \leq |V(D_u^1 \cup D_v^2) \cap V(D_u^1 \cup D_v^1)| = |D_u^1 \cap D_v^2| + |D_u^1 \cap D_v^2| \leq 2.$$ 

Thus, $D_u^1 \cap D_v^2 \subseteq N(v, u')$. Since $D_u^1 \cap D_v^2 = \{u\}$, we have $uv$ is an edge of $G$.

**Remark 1.** Suppose $L_1, \cdots, L_r$ are $r$ ($\geq 2$) internal disjoint odd paths with an order $2k_i + 2$ for each $i \in [r]$, and they have the same ends $\{u, v\}$. Let $L_i = uc_i x_{i1}^1 e_{2k_i} x_{i2}^1 \cdots x_{2k_i}^1 e_{2k_i+1} v$. Let $c_0 = 1$ and $c_i = \sum_{j=0}^{k_i} k_j$. If $k_i \geq 1$ for each $i \in [r]$, then let $\Gamma$ be an edge-coloring of $G$ such that $\Gamma(e_{ij}) = \Gamma(x_{ij}^1) = c_{i-1} + j$ and $\Gamma(e_{k_i+1}) = 1$ for each $i \in [r]$ and $j \in [k_i]$. Then $\Gamma$ is an $MD$-coloring of $G$ with $|\Gamma(G)| = |G| \cdot \frac{r}{2}$. Since $G$ is a 2-connected graph, we have $md(G) = \frac{|G|}{2}$.

**Theorem 3.2.** Suppose $G$ is a $\left\lceil \frac{n}{2} \right\rceil$-connected graph and $n \geq 4$. Then $md(G) \leq 2$ and
1. if \( n \) is even, then \( md(G) = 2 \) if and only if \( G = A_n \);

2. if \( n \) is odd, then \( md(G) = 2 \) if and only if \( G \in A_n \).

Proof. Since \( N(x) + N(y) \geq n - 1 \) for any two nonadjacent vertices \( x \) and \( y \), we have \( diam(G) \leq 2 \). So, \( md(G) \leq 2 \).

It is obvious that \( G \) is a \( \left\lceil \frac{n}{2} \right\rceil \)-connected graph if \( G = A_n \) or \( G \in A_n \). Moreover, by Lemma \( \text{L.3} \) and Theorem \( \text{3.1} \) we have \( md(G) = 2 \).

Now suppose \( G \) is a \( \left\lceil \frac{n}{2} \right\rceil \)-connected graph and \( md(G) = 2 \). Since \( n \geq 4 \), \( G \) is a 2-connected graph. We distinguish the following cases for our proof.

Case 1. \( n \) is even.

For any two nonadjacent vertices \( u, v \) of \( G \), \( |N(u) \cap N(v)| \geq 2 \). By Theorem \( \text{3.1} \) (3), \( G = K_s \Box K_t \), where \( s, t \geq 2 \). We need to prove that at least one of \( s, t \) equals two. Suppose \( H_1, H_2 \) are two cliques of order \( s, t \), respectively, and \( V(H_1) \cap V(H_2) = \{u\} \). Then \( N(u) \subseteq V(H_1 \cup H_2) \), i.e., \( s + t - 2 \geq \frac{n}{2} \). Since \( n = st \), we have \( t(s-2) \leq 2(s-2) \). Thus, either \( s = 2 \) or \( t = 2 \).

Case 2. \( n \) is odd.

Say \( n = 2k + 1 \) for some integer \( k \). Suppose \( \Gamma \) is an extremal \( MD \)-coloring of \( G \) and \( G_1, G_2 \) are the colors 1, 2 induced subgraphs, respectively.

Subcase 2.1. Every vertex \( v \) of \( G \) has \( d^\Gamma(v) = 2 \).

Suppose there are components \( D, F \) of \( G_1, G_2 \), respectively, such that \( V(G) \cap V(F) = \emptyset \). Then let \( u \in V(D) \) and \( v \in V(F) \). Since \( d^\Gamma(u) = d^\Gamma(v) = 2 \), there are components \( D' \) of \( G_1 \) and \( F' \) of \( G_2 \), such that \( V(D) \cap V(F') = \{u\} \) and \( V(F) \cap V(D') = \{v\} \). Since \( V(D) \cup V(F') - \{u\} \) and \( V(D') \cup V(F) - \{v\} \) are vertex-cuts of \( G \), we have \( |V(D) \cup V(F')| \geq k + 1 \) and \( |V(D') \cup V(F)| \geq k + 1 \). Since \( |V(D') \cap V(F')| \leq 1 \), we have \( n \geq |V(D) \cup V(F')| + |V(D') \cup V(F)| - |V(D') \cap V(F')| \geq 2k + 1 = n \), i.e., \( D \cup D' \cup F \cup F' = G \). Then \( u \) is a cut-vertex of \( G \), a contradiction. Therefore, for each component \( D \) of \( G_1 \) and each component \( F \) of \( G_2 \), we have \( |V(G) \cap V(F)| = 1 \). Then since \( d^\Gamma(v) = 2 \) for each \( v \in V(G) \), any two components of \( G_1 \) (and also \( G_2 \)) have the same order, say \( s \) (the order is \( s \)). Then \( s, t > 2 \); otherwise, suppose \( s = 2 \), i.e., \( G_1 \) is a matching. Since \( n \) is odd, we have \( V(G) - V(G_1) \neq \emptyset \). Thus, each vertex \( v \) of \( V(G) - V(G_1) \) has \( d^\Gamma(v) = 1 \), a contradiction. For a vertex \( x \) of \( G \), let \( D_1, D_2 \) be the components of \( G_1, G_2 \), respectively, containing \( x \). Then \( D_1 \cup D_2 - \{x\} \) is a vertex-cut of \( G \), i.e., \( s + t - 2 \geq k \). However, \( 2k + 1 = n = st \) and \( s, t > 3 \), a contradiction.

Subcase 2.2 There is a vertex \( v \) of \( G \) with \( d^\Gamma(v) = 1 \).

Suppose \( D \) is the component of \( G_1 \) containing \( v \). Then since \( D - \{v\} \) is a vertex cut of \( G \), we have \( |D| \geq k + 1 \). Since the set of vertices of \( D \) with color-degree two is a vertex-cut of \( G \), there are at least \( k \) vertices of \( D \), say \( v_1, \cdots, v_k \), such that \( d^\Gamma(v_i) = 2 \) for \( i \in [k] \). Let \( F_i \) be the component of \( G_2 \) containing \( v_i \) and let \( U = \bigcup_{i \in [k]} (V(F_i) - \{v_i\}) \). Then \( |U| \geq k \). Since \( n \geq |D| + |U| \geq 2k + 1 = n \), we have \( |D| = k + 1 \), \( |U| = k \),
and \(|F_i| = 2\) for \(i \in [k]\). Moreover, \(N(v) = \{v_1, \cdots, v_k\}\). Let \(V(F_i) - \{v_i\} = \{u_i\}\). For \(i, j \in [k]\), if \(u_iu_j\) is not an edge of \(G\), then \(U - \{u_i, u_j\} + v_j\) is a vertex-cut of \(G\) with order \(k - 1\), which contradicts that \(G\) is \(k\)-connected. For each \(v_i\), if there are two vertices \(v_j, v_l\) such that \(v_iv_j\) and \(v_iv_l\) are not edges of \(G\), then \(V(D) - \{v_i, v_j, v_l\} + u_i\) is a vertex-cut of \(G\) with order \(k - 1\), which contradicts that \(G\) is \(k\)-connected. Therefore, \(v_i\) connects all but at most one vertex of \(D - v\). So, \(G \in \mathcal{A}_n\).

4 Upper bounds

In this section, we give two upper bounds of the monochromatic disconnection number of a graph \(G\), one of which depends on the connectivity of \(G\), and the other depends on the independent number of \(G\). Note that for a \(k\)-connected graph \(G\), when \(k = 2\) (small) and \(k \geq \left[\frac{n}{3}\right]\) (large), from Theorems 1.2 and 3.2 we know that \(md(G) \leq \left[\frac{n}{k}\right]\). This suggests us to make the following conjecture.

Conjecture 4.1. Suppose \(G\) is a \(k\)-connected graph. Then \(md(G) \leq \left[\frac{n}{k}\right]\).

Suppose \(P\) is a \(k\)-path. Then \(md(K_r \square P) = md(K_r) + md(P) = k + 1\). Since \(n = |K_r \square P| = r(k + 1)\) and \(K_r \square P\) is an \(r\)-connected graph, the bound is sharp for \(k \geq 2\) if the conjecture is true.

The mean distance of a connected graph \(G\) is defined as \(\mu(G) = \left(\frac{n}{2}\right)^{-1}\Sigma_{u,v \in V(G)}d(u,v)\). Plesník in [14] posed the problem of finding sharp upper bounds on \(\mu(G)\) for \(k\)-connected graphs. Favaron et al. in [11] proved that if \(G\) is a \(k\)-connected graph of order \(n\), then

\[\mu(G) \leq \left[\frac{n + k - 1}{k}\right] \cdot \frac{n - 1 - \frac{k}{2} \left\lfloor \frac{n-1}{k}\right\rfloor}{n - 1},\]  

(1)

and the bound is sharp when \(n\) is even. If \(n\) is odd and \(k \geq 3\), then Dankelmann et al. in [10] proved that \(\mu(G) \leq \frac{n}{2k+1} + 30\) and this bound is, apart from an additive constant, best possible.

The following result gives a relationship between the monochromatic disconnection number and the connectivity of a graph, which means that if the connectivity of a graph is in linear of the order of the graph, then the monochromatic disconnection number of the graph is upper bounded by a constant.

Theorem 4.2. For any \(0 < \varepsilon < \frac{1}{2}\), there is a constant \(C = C(\varepsilon) < \frac{(1+\varepsilon)^2}{4\varepsilon^2(1-\varepsilon)}\), such that for any \(ε\)-\(n\)-connected graph \(G\), \(md(G) \leq C\).

Proof. Suppose \(\Gamma\) is an extremal \(MD\)-coloring of \(G\) and \(V(G) = \{v_1, \cdots, v_n\}\). We use \((i, j)\) to denote an unordered integer pair in this proof. For each color \(i\) of \(\Gamma(G)\), let

\[S_i = \{(j, l) : \text{the color } i \text{ separates } v_j \text{ and } v_l\}.

Then \(\Sigma_{i \in \Gamma}|S_i| = \Sigma_{j \neq l \in \Gamma}(v_j, v_l)\).
Claim 4.3. \(|S_i| \geq k(n - k)\) for each \(i \in \Gamma(G)\).

Proof. Let \(\varepsilon n = k\). The result holds obviously for \(k = 1\). Thus, let \(k \geq 2\). For each \(i \in \Gamma(G)\), let \(G_i\) be the color \(i\) induced subgraph of \(G\), and let \(H_i\) be the graph obtained from \(G\) by deleting all the edges with color \(i\). Then \(H_i\) is a disconnected graph. Suppose there is a component \(D\) of \(H_i\) with \(|D| > n - k\). Let \(U = \{v_j \mid v_j \in V(D) \cap V(G_i)\}\).

For a component \(B\) of \(G_{i},\) if \(V(B) \cap V(D) \neq \emptyset\), then \(|V(B) \cap V(D)| = 1\). Since \(B\) contains at least one vertex of \(V(G - D)\), we have \(|U| \leq |V(G - D)| < k\). Since \(|D| > n - k = n(1 - \varepsilon) > \varepsilon n = k\), \(U\) is a proper subset of \(V(D)\). So, \(U\) is a vertex-cut of \(G\). Since \(|U| < k\) and \(G\) is \(k\)-connected, this yields a contradiction. Thus, for each \(i \in \Gamma(G)\), there is no component of \(H_i\) with order greater than \(n - k\).

We partition the components of \(H_i\) into \(r\) parts such that \(r\) is minimum and the number of vertices in each part is at most \(n - k\). Suppose the \(r\) parts have \(n_1, \cdots, n_r\) vertices, respectively. Then \(\sum_{j \in [r]} n_j = n\). If \(r \geq 4\), then since \(r\) is minimum, \(n_l + n_j > n - k\) for each \(l, j \in [r]\). Thus,

\[
n(r - 1) = (r - 1) \sum_{i \in [r]} n_i = \sum_{i, j \in [r]} (n_l + n_j) > \binom{r}{2}(n - k),
\]

and then \(r(n - k) < 2n\). Since \(k < \frac{n}{2}\), this yields a contradiction. Therefore, \(r\) is equal to 2 or 3. If \(r = 2\), then \(|S_i| \geq n_1 \cdot n_2 \geq k(n - k)\). If \(r = 3\), then there is an \(n_l\) such that \(k \leq n_l \leq n - k\), say \(l = 1\). Otherwise, \(n_j < k\) for each \(j \in [3]\), then \(n = \sum_{j \in [3]} n_j < n\), a contradiction. Thus, \(|S_i| > n_1 \cdot (n_2 + n_3) \geq n(n - k)\).

By the inequality (1) above, we have

\[
\mu(G) \leq \left\lfloor \frac{n + k - 1}{k} \right\rfloor \cdot \frac{n - 1 - k}{n - 1} \left\lfloor \frac{n - 1}{k} \right\rfloor = \left\lfloor \frac{n + k - 1}{k} \right\rfloor \cdot \left(1 - \frac{k}{2(n - 1)} \left\lfloor \frac{n - 1}{k} \right\rfloor \right)
\]

\[
\leq \left(\frac{n + k - 1}{k} \left\lfloor \frac{n - 1}{k} \right\rfloor \right) \cdot \left(1 - \frac{k}{2(n - 1)} \left(\frac{n - 1}{k} - 1\right)\right)
\]

\[
= \frac{n + k - 1}{k} \cdot \frac{n + k - 1}{2(n - 1)} < \frac{(n + k)^2}{2k(n - 1)}.
\]

Since \(\sum_{i,j} d(v_i, v_j) = \mu(G) \cdot \left(\begin{array}{c} n \vspace{1pt} \\ 2 \end{array}\right)\), we have \(\sum_{i,j} d(v_i, v_j) < \frac{(n+k)^2n}{4k}\). It is obvious that \(d(v_i, v_j) > c_{\Gamma}(v_i, v_j)\) for any two vertices \(v_i, v_j\) of \(G\). Thus,

\[
md(G) \leq \frac{\sum_{i \in [r]} |S_i|}{k(n - k)} = \frac{\sum_{i,j} c_{\Gamma}(v_i, v_j)}{k(n - k)} \leq \frac{\sum_{i,j} d(u, v)}{k(n - k)} < \frac{(n + k)^2n}{4k^2(n - k)} = \frac{(1 + \varepsilon)^2}{4\varepsilon^2(1 - \varepsilon)}.
\]

The proof is thus complete.

Remark 2. Since \(\varepsilon < \frac{1}{3}\), we have \(\frac{(1 + \varepsilon)^2}{4\varepsilon^2(1 - \varepsilon)} < \frac{(\frac{1}{3})^2}{2\varepsilon^2} = \frac{2}{9\varepsilon^2}\). This means that when the connectivity of a graph increases, its MD-number could decrease, and the upper bound is 4 when \(\varepsilon\) is getting to \(\frac{1}{2}\).
The following result gives a relationship between the monochromatic disconnection number and the independent number of a graph.

**Theorem 4.4.** If $G$ is a 2-connected graph, then $\text{md}(G) \leq \alpha(G)$. The bound is sharp.

**Proof.** Let $P$ be a path and let $t \geq 2$ be an integer. Since $\alpha(K_t \square P) = |P| = \text{md}(K_t \square P)$, the bound is sharp if the result holds.

The proof proceeds by induction on the order $n$ of a graph $G$. If $n \leq 2\alpha(G)$, then since $G$ is a 2-connected graph, $\text{md}(G) \leq \alpha(G)$. If $G$ has a vertex $v$ such that $G - v$ is still 2-connected, then by Lemma 1.1 (5), we know $\alpha(G - v) \geq \text{md}(G)$. Since $\alpha(G - v) \leq \alpha(G)$, by induction, we have $\text{md}(G) \leq \text{md}(G - v) \leq \alpha(G - v) \leq \alpha(G)$. Thus, we only need to consider the graph $G$ with the property that $G - v$ is not a 2-connected graph for any vertex $v$ of $G$.

Let $u$ be a vertex of $G$ such that $G - u$ has a maximum component. Let $B = \{D_1, \cdots, D_s\}$ be the set of components of $G - u$ and let $D_r$ be a maximum component. Let $S$ be the set of cut-vertices of $G - u$. The block-tree of $G - u$, denoted by $T$, is a bipartite graph with bipartition $B$ and $S$, and a block $D_i$ has an edge with a cut-vertex $v$ in $T$ if and only if $D_i$ contains $v$. Then the leaves of $T$ are blocks, say $D_{k_1}, \cdots, D_{k_l}$. Since $G$ is 2-connected, there is a vertex $v_i$ of $D_{k_1} - S$ such that $u$ connects $v_i$ in $G$ for $i \in [l]$. We use $P_{i,j}$ to denote the subpath of $T$ from $D_{k_i}$ to $D_{k_j}$. We now prove that $T$ is a path and $D_i$ is an edge for $i \neq r$. If $T$ is not a path, then $l \geq 3$. There are two leaves of $T$, say $D_{k_1}$ and $D_{k_2}$, such that $D_r \in V(P_{1,2})$. Then $G - v_3$ has a component containing $V(D_r) \cup \{u\}$, which contradicts that $D_r$ is maximum. Thus, $T$ is a tree. Suppose $r \neq j$ and $D_j$ is not an edge, i.e., $D_j$ is a 2-connected graph. Since $T$ is a path, we have $W = V(D_j) - S - \{v_1, \cdots, v_l\} \neq \emptyset$. Let $u' \in W$. Then $G - u'$ has a component containing $V(D_r) \cup \{u\}$, which contradicts that $D_r$ is maximum. Thus, $D_i$ is an edge for $i \neq r$.

Without loss of generality, suppose $V(D_1) \cap V(D_{i+1}) = \{u_i\}$ for $i \in [s - 1]$. Then, $D_1, D_s$ are leaves of $T$, $D_i$ is an edge for $i \neq r$ and $S = \{u_1, \cdots, u_{s-1}\}$. Let $u_0 \in V(D_1 - S)$ and $u_s \in V(D_s - S)$ be two vertices adjacent to $u$.

Let $P_1 = \bigcup_{i < r} D_i$ and let $P_2 = \bigcup_{i = r+1}^{s} D_i$. Then $P_1$ and $P_2$ are paths. There is an independent set $U_i$ of $P_i$ such that $U_i \cap V(D_r) = \emptyset$ and $|U_i| = \left\lceil \frac{|P_i| - 1}{2} \right\rceil$ for $i \in [2]$. Let $U$ be a maximum independent set of $D_r$. Then $U \cup U_1 \cup U_2$ is an independent set of $G - u$, i.e.,

\[
\alpha(G) \geq \alpha(G - v) \geq |U \cup U_1 \cup U_2| = \alpha(D_r) + \left\lceil \frac{|P_1| - 1}{2} \right\rceil + \left\lceil \frac{|P_2| - 1}{2} \right\rceil \geq \alpha(D_r) + \left\lceil \frac{|P_1| + |P_2| - 2}{2} \right\rceil = \alpha(D_r) + \left\lceil \frac{s - 1}{2} \right\rceil.
\]

Let $P = \{uw_0, uw_s\} \cup (\bigcup_{i \neq r} D_i)$ and let $G' = D_r \cup P$. Then $P$ is an $(s + 1)$-path and $G'$ is a 2-connected spanning subgraph of $G$. By Lemma 1.1 (3), we have
Let $\Gamma$ be an extremal $MD$-coloring of $G'$. Then $\Gamma$ is an $MD$-coloring restricted on $D_r$ and $P$. We call $D_r$ and each edge of $P$ the joints of $G'$. Let $C$ be the set of colors $c \in \Gamma(G')$ such that $c$ is in at least two joints of $G'$. For $c \in C$, we use $n_e$ to denote the number of joints of $G$ having edges colored with $c$. Then $md(G') = |\Gamma(G')| = |\Gamma(D_r)| = |P| - \Sigma_{e \in C}(n_e - 1)$. Since there is a color $c$ of $C$ such that $\Gamma(u_{r-1}, u_r)$ that separates $u_{r-1}$ and $u_r$, we have $c \in \Gamma(D_r) \cap \Gamma(P)$. By the same reason, for each $e \in E(P)$, either $\Gamma(e) = \Gamma(f)$ for an edge $f$ of $P - e$, or $\Gamma(e) \subseteq \Gamma(D_r)$. Thus, $\Sigma_{e \in C}(n_e - 1) \geq \left\lceil \frac{s - 2}{2} \right\rceil$. Therefore,

$$md(G) \leq md(G') = |\Gamma(D_r)| + |P| - \Sigma_{e \in C}(n_e - 1)$$

$$\leq \alpha(D_r) + s + 1 - \left\lfloor \frac{s + 2}{2} \right\rfloor = \alpha(D_r) + \left\lfloor \frac{s}{2} \right\rfloor$$

$$= \alpha(D_r) + \left\lfloor \frac{s - 1}{2} \right\rfloor \leq \alpha(G).$$

The proof is thus complete.

5 Characterization of extremal graphs

We knew that $md(G) = \left\lceil \frac{n}{2} \right\rceil$ if $G$ is a 2-connected graph. In this section, we characterize all the 2-connected graphs with $MD$-number $\left\lceil \frac{n}{2} \right\rceil$. We use $\mathcal{E} = (L_0; L_1, \cdots, L_t)$ to denote an ear-decomposition of $G$, where $L_0$ is a 2-connected subgraph of $G$ and $L_i$ is a path for $i \in [t]$. Let $Z_{\mathcal{E}} = \{L_i \mid i > 0 \text{ and } end(L_i) \subseteq V(L_0)\}$.

If $C$ is a cycle of $G$ and $v \in V(G) - V(C)$, then we use $\kappa(v, C)$ to denote the maximum number of $vv_i$-path $P_i$ of $G$, such that $V(P_i) \cap V(P_j) = \{v\}$ and $V(P_i) \cap V(C) = \{v_i\}$. We call $H = C \cup (\bigcup_{i=1}^{\kappa(v,C)} P_i)$ a $(v, C)$-umbrella of $G$ (or an umbrella for short) if $\kappa(v, C) \geq 3$. The vertices $v_1, \cdots, v_{\kappa(v,C)}$ divide $C$ into $\kappa(v,C)$ paths, say $P'_1, \cdots, P'_{\kappa(v,C)}$. We call $P_i$ a spoke of $H$ and call $P'_i$ a rim of $H$. If the size of each spoke is odd and the size of each rim is even, then we call the $(v, C)$-umbrella a uniform $(v, C)$-umbrella (or uniform umbrella for short).

A graph $G$ is called a $\theta$-graph if $G$ is the union of three internal disjoint paths $T_1, T_2$ and $T_3$ with $end(T_1) = end(T_2) = end(T_3)$. If each $T_i$ is an even path, then we call $G$ an even $\theta$-graph and call each $T_i$ a route.

Suppose $\mathcal{E} = (L_0; L_1, \cdots L_t)$ is an ear-decomposition of $G$. Then the concept normal ear-decomposition of $G$ is defined as follows.

- If $|G|$ is even, then $\mathcal{E}$ is a normal ear-decomposition of $G$ if $L_0$ is a cycle.
- If $|G|$ is odd and $G$ is not a bipartite graph, then $\mathcal{E}$ is a normal ear-decomposition of $G$ if $L_0$ is an odd cycle.
- If $|G|$ is odd and $G$ is a bipartite graph, then $\mathcal{E}$ is a normal ear-decomposition of $G$ if $L_0$ is either an umbrella or an even $\theta$-graph. Moreover, if $L_0$ is an even $\theta$-graph,
then for each $L_i \in \mathcal{E}$, $\text{end}(L_i)$ is contained in one route.

**Lemma 5.1.** If $G$ is a 2-connected graph, then $G$ has a normal ear-decomposition.

**Proof.** If $n$ is even or $G$ is a nonbipartite graph with $n$ odd, then $G$ has a normal ear-decomposition. If $G$ is a bipartite graph and $n$ is odd, then let $\mathcal{E} = \{L_0; L_1, \ldots, L_t\}$ be an ear-decomposition of $G$ with $L_0$ an even cycle. Since $n = |L_0| + \sum_{i=0}^{t} (|L_i| - 2)$ and $n$ is odd, there is an even path among the ears, say $L_i$. Since $H = \bigcup_{i=0}^{t-1} L_i$ is a 2-connected bipartite graph, there is an even cycle $C$ of $H$ containing $\text{end}(L_i)$. Moreover, $\text{end}(L_i)$ divides $C$ into two even paths. So, $L'_0 = C \cup L_i$ is an even $\theta$-graph, say the three routes are $T_1, T_2$ and $T_3$. Let $\mathcal{E}' = \{L'_0; L'_1, \ldots, L'_3\}$ be an ear-decomposition of $G$ and let $\text{end}(L'_j) = \{u_j, v_j\}$ for $j \in [8]$. If the ends of each $L'_j$ in $\mathcal{E}'$ are contained in one route, then $\mathcal{E}'$ is a normal ear-decomposition of $G$. Otherwise, suppose $L'_j \in \mathcal{E}'$, $u_j \in I(T_1)$ and $v_j \in I(T_2)$. Then $\kappa(u_j, T_2 \cup T_3) \geq 3$, i.e., there is a $(u_j, T_2 \cup T_3)$-umbrella, say $M$. Then there is a normal ear-decomposition of $G$ containing $M$. 

**Lemma 5.2.** Suppose $G$ is a 2-connected graph with $\text{md}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Let $\mathcal{E} = (L_0; L_1, \ldots, L_t)$ be an ear-decomposition of $G$ with $L_0$ a 2-connected subgraph of $G$ and $\text{end}(L_i) = \{a_i, b_i\}$ for $i \in [t]$. Then we have the following results.

1. If $H$ is a 2-connected subgraph of $G$, then each extremal MD-coloring of $G$ is an extremal MD-coloring restricted on $H$, and $\text{md}(H) = \left\lfloor \frac{\text{md}(G)}{2} \right\rfloor$.

2. If $n$ is even, then $G$ is a bipartite graph and $L_i$ is an odd path for $i \in [t]$.

3. If $n$ is odd, then when $|L_0|$ is even, exact one of $\{||L_1||, \ldots, ||L_t||\}$ is even; when $|L_0|$ is odd, $L_i$ is an odd path for $i \in [t]$.

**Proof.** Let $\Gamma$ be an extremal MD-coloring of $G$. Then for each $i \in [t]$, $\Gamma(L_i) \cap \Gamma(\bigcup_{l=0}^{i-1} L_l) \neq \emptyset$; otherwise, $C_\Gamma(a_i, b_i) = \emptyset$, a contradiction. Moreover, each color of $\Gamma(L_i) \cup \Gamma(\bigcup_{l=0}^{i-1} L_l)$ is used on at least two edges of $L_i$. Otherwise, suppose $p \in \Gamma(L_i) \cup \Gamma(\bigcup_{l=0}^{i-1} L_l)$ and color $p$ is only used on one edge $e = xy$ of $L_i$. Then since $\Gamma(\bigcup_{l=0}^{i-1} L_l) - e$ is connected, $C_\Gamma(x, y) = \emptyset$, a contradiction. Therefore,

$$\left\lfloor \frac{n}{2} \right\rfloor = \text{md}(G) = |\Gamma(L_0)| + \sum_{i=1}^{t} |\Gamma(L_i) - \Gamma(\bigcup_{l=0}^{i-1} L_l)|$$

$$\leq \text{md}(L_0) + \sum_{i=1}^{t} \left\lfloor \frac{|L_i| - 1}{2} \right\rfloor$$

$$\leq \left\lfloor \frac{|L_0|}{2} \right\rfloor + \sum_{i=1}^{t} \left\lfloor \frac{|L_i| - 1}{2} \right\rfloor$$

$$\leq \left\lfloor \frac{|L_0|}{2} \right\rfloor + \sum_{i \in [t]} \left\lfloor \frac{|L_i| - 1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor.$$
Then $|\Gamma(L_0)| = md(L_0) = \left\lfloor \frac{|L_0|}{2} \right\rfloor$ and $|\Gamma(L_i)| = \left\lfloor \frac{|L_i| - 1}{2} \right\rfloor$ for each $i \in [t]$. So, $\Gamma$ is an extremal MD-coloring restricted on $L_0$, and $md(L_0) = \left\lfloor \frac{|L_0|}{2} \right\rfloor$. Moreover, $|\Gamma(L_i) \cap \Gamma(\bigcup_{t=0}^{i-1} L_t)| = 1$ when $L_i$ is an odd path.

If $G$ is not a bipartite graph, $n$ is even and $L_0$ an odd cycle, then the above inequality does not hold. Thus, $G$ is a bipartite graph when $n$ is even. Moreover, $L_i$ is an odd path for each $i \in [t]$. If $n$ and $|L_0|$ are odd, then $L_i$ is an odd path for $i \in [t]$. If $n$ is odd and $|L_0|$ is even, then exact one of $\{|L_0|, \ldots, |L_t|\}$ is even.

For a normal ear-decomposition $\mathcal{E} = \{L_0; L_1, \ldots, L_t\}$ of a 2-connected graph $G$, if $L_0$ is an odd cycle and $L_i \in \mathbb{Z}_\mathcal{E}$, then $end(L_i)$ divides $L_0$ into an odd path and an even path, which are denoted by $f_o(\mathcal{E}, i)$ and $f_e(\mathcal{E}, i)$, respectively. If $L_0$ is an even cycle, $L_i \in \mathbb{Z}_\mathcal{E}$ and $e \in E(L_0)$, then we use $g(\mathcal{E}, i, e)$ to denote the subpath of $L_0$ with ends $end(L_i)$ and $g(\mathcal{E}, i, e)$ contains $e$. We define a function $f(\mathcal{E}, i, j)$ for $0 \leq i < j \leq t$ as follows.

$$f(\mathcal{E}, i, j) = \begin{cases} 
  f_o(\mathcal{E}, j) & i = 0, L_j \in \mathbb{Z}_\mathcal{E} \text{ and } L_0 \text{ is an odd cycle}; \\
  g(\mathcal{E}, i, e) & i = 0, L_j \in \mathbb{Z}_\mathcal{E} \text{ and } L_0 \text{ is an even cycle with } e \in E(L_0); \\
  a_jPb_j & i = 0, L_j \in \mathbb{Z}_\mathcal{E}, L_0 \text{ is an umbrella, } P \text{ is either a spoke or a rim of } L_0 \text{ such that } end(L_j) \subseteq V(P); \\
  a_jTb_j & i = 0, L_j \in \mathbb{Z}_\mathcal{E}, L_0 \text{ is an even } \theta\text{-graph, } T \text{ is one of the three routes such that } end(L_i) \subseteq V(T); \\
  a_jL_ib_j & i > 0 \text{ and } end(L_j) \subseteq V(L_i); \\
  K_4 & \text{otherwise.} 
\end{cases}$$

If $L_0$ is not an even cycle, then the function depends only on $\mathcal{E}, i$ and $j$. If $L_0$ is an even cycle and $i = 0$, then the function also depends on $e$. Thus, we need to fix an edge $e$ of $L_0$ in advance if $L_0$ is an even cycle.

**Lemma 5.3.** If $G$ is a uniform umbrella or an even $\theta$-graph other than $K_{2,3}$, then $|G|$ is odd and $md(G) = \left\lfloor \frac{|G|}{2} \right\rfloor$.

**Proof.** It is obvious that $|G|$ is odd. Fix an integer $k \geq 3$. Suppose $G'$ is either a minimum even $\theta$-graph other than $K_{2,3}$, or a minimum uniform umbrella with $k$ spokes.

If $G'$ is a minimum even $\theta$-graph other than $K_{2,3}$, then $G'$ and one of its extremal MD-colorings are depicted in Figure 1(1), which implies $md(G') = 3 = \left\lfloor \frac{|G'|}{2} \right\rfloor$.

If $G'$ is a minimum uniform umbrella with $k$ spokes, then each spoke is an edge and each rim is a 2-path. Suppose the $k$ spokes are $e_1 = vv_1, \cdots, e_k = vv_k$, and the $k$ rims are $P_1 = v_1f_1u_1f_2v_2, \cdots, P_k = v_kf_{2k-1}u_kv_{2k}$. We color each $e_i$ with $i$. The colors of the edges of $P_i$ obey the rule that opposite edges of any 4-cycle have the same color.
(see Figure 1). Since \( k \geq 3 \), we know that for \( v_1, \{e_1, f_2, f_{2k-1}\} \) is a monochromatic \( v_1v \)-cut (it is also a monochromatic \( v_1v \)-cut for \( i \neq 1 \), and a monochromatic \( v_1u_i \)-cut for \( i \neq 1, 2, k \}), \( \{e_2, f_1, f_{3}\} \) is a monochromatic \( v_1u_1 \)-cut and \( \{e_k, f_{2k}, f_{2k-3}\} \) is a monochromatic \( v_1u_k \)-cut. By symmetry, the edge-coloring is an \( MD \)-coloring of \( G' \) with \( k \) colors. Since \( G' \) is 2-connected and \( |G'| = 2k + 1 \), we have \( md(G') = k = \left\lceil \frac{|G'|}{2} \right\rceil \).

Suppose \( G \) is a uniform umbrella with \( k \) spokes (an even \( \theta \)-graph other than \( K_{2,3} \)). Then \( G \) is obtained from \( G' \) by replacing some edges with odd paths, respectively. W.l.o.g., suppose \( G \) is obtained from \( G' \) by replacing one edge with an odd path \( P \). Then by Lemma 2.2 we have \( md(G) \geq md(G') + \left\lceil \frac{|P| - 1}{2} \right\rceil = \left\lceil \frac{|G|}{2} \right\rceil \), i.e., \( md(G) = \left\lceil \frac{|G|}{2} \right\rceil \).

The proof is thus complete.

**Lemma 5.4.** If \( G \) is a bipartite graph of odd order and \( md(G) = \left\lfloor \frac{n}{2} \right\rfloor \), then each umbrella of \( G \) is a uniform umbrella.

**Proof.** Suppose \( G \) is a bipartite graph of odd order and \( md(G) = \left\lfloor \frac{n}{2} \right\rfloor \). Let \( H \) be a \((v, C)\)-umbrella of \( G \). We show that \( H \) is a uniform umbrella.

If \( \kappa(v, C) = 3 \), then let \( R_1, R_2 \) and \( R_3 \) be spokes of \( H \) and \( R_i \) be a \( vv_i \)-path. Then \( C \) is divided into three paths by vertices \( v_1, v_2 \) and \( v_3 \) (say, the three paths are \( W_1, W_2 \) and \( W_3 \), such that \( end(W_1) = \{v_1, v_2\} \), \( end(W_2) = \{v_2, v_3\} \) and \( end(W_3) = \{v_1, v_3\} \)). If each \( R_i \) is an odd path, then since \( G \) is a bipartite graph, each \( W_i \) is an even path, \( H \) be a uniform \((v, C)\)-umbrella of \( G \). If, by symmetry, \( R_1 \) is an even path and \( R_2, R_3 \) are odd paths, then \( W_1, W_3 \) are odd paths and \( W_2 \) is an even path. Then since \( (W_1 \cup W_3 \cup R_2 \cup R_3; R_1, W_2) \) is an ear-decomposition of \( H \) containing even paths \( R_1 \) and \( W_2 \), by Lemma 5.2 (1) and (3) this yields a contradiction. If, by symmetry, \( R_1 \) is an odd path and \( R_2, R_3 \) are even paths, then \( H \) is a uniform \((v_1, R_2 \cup R_3 \cup W_2)\)-umbrella. If each \( R_i \) is an even path, then \((C; R_1 \cup R_2, R_3) \) is an ear-decomposition of \( H \) containing two even paths, a contradiction.

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![Figure 1: Extremal MD-colorings of the minimum even θ-graph and the minimum uniform umbrella.](image-url)
If \( \kappa(v, C) \geq 4 \), then let \( W_1, W_2, W_3, W_4 \) be four spokes of \( H \) (let \( W_i \) be a \( vv_i \) path for \( i \in [4] \)). Then \( C \) is divided into two paths by \( v_2 \) and \( v_3 \) (say, the two paths are \( Y_1 \) and \( Y_2 \)). W.l.o.g., suppose \( W_1 \) is an even path. Then \( (Y_1 \cup W_2 \cup W_3; Y_2, W_4, W_1) \) is an ear-decomposition of \( H \). Since \( md(H) = \left\lfloor \frac{|H|}{2} \right\rfloor \) and \( W_1 \) is an even path, by Lemma 5.2, \( W_2 \). Then \( (C \cup W_3 \cup W_4; W_1, W_2) \) is an ear-decomposition of \( H \) containing two even paths, a contradiction. So, each spoke of \( H \) is an odd path. Since \( H \) is a bipartite graph, each rim of \( H \) is an even path. 

Lemma 5.5. If \( \mathcal{E} = (L_0; L_1, \cdots, L_t) \) is an ear-decomposition of \( G \). Then \( \mathcal{E} \) can have the following possible properties.

Q: If \( end(L_j) \cap I(L_i) \neq \emptyset \), then \( end(L_j) \subseteq V(L_i) \).

R: If \( end(L_j) \cap I(f(\mathcal{E}, k, i)) \neq \emptyset \), then \( f(\mathcal{E}, k, j) \) is a proper subpath of \( f(\mathcal{E}, k, i) \).

The concept standard ear-decomposition of \( G \) is defined as follows.

- If \( |G| \) is even, then \( \mathcal{E} \) is a standard ear-decomposition of \( G \) if \( L_0 \) is an even cycle.

- If \( |G| \) is odd and \( G \) is not a bipartite graph, then \( \mathcal{E} \) is a standard ear-decomposition of \( G \) if \( L_0 \) is an odd cycle and \( f_i(\mathcal{E}, i) \cap f_j(\mathcal{E}, j) \neq \emptyset \) for \( L_i, L_j \in Z_{\mathcal{E}} \).

- If \( |G| \) is odd and \( G \) is a bipartite graph, then \( \mathcal{E} \) is a standard ear-decomposition of \( G \) if \( L_0 \) is either a uniform umbrella or a even \( \theta \)-graph other than \( K_{2,3} \). Moreover, for each \( L_i \in Z_{\mathcal{E}} \), if \( L_0 \) is a uniform umbrella, then \( end(L_i) \) is contained in either a rim or a spoke; if \( L_0 \) is an even \( \theta \)-graph other than \( K_{2,3} \), then \( end(L_i) \) is contained in one route.

Therefore, a standard ear-decomposition of \( G \) is also a normal ear-decomposition of \( G \).

Lemma 5.5. If \( \mathcal{E} = (L_0; L_1, \cdots, L_t) \) is a standard ear-decomposition of \( G \) and \( \mathcal{E} \) has properties Q and R, then there exist integers \( 0 \leq k < r \leq t \) such that \( end(L_r) \subseteq V(L_k) \), and \( d(u) = 2 \) for each \( u \in I(f(\mathcal{E}, k, r)) \cup I(L_r) \).

Proof. For \( i \in [t] \), let \( end(L_i) = \{a_i, b_i\} \). We use \( m_r \) \( (n_r) \) to denote the minimum integer such that \( a_r \in V(L_{m_r}) \) \( (b_r \in V(L_{m_r})) \). Since \( I(L_0) = V(L_0) \), we have \( a_i \in I(L_{m_i}) \) and \( b_i \in I(L_{n_i}) \). Since \( \mathcal{E} \) has property Q, we know for each \( i \in [t] \), either \( end(L_i) \subseteq V(L_{m_i}) \), or \( end(L_i) \subseteq V(L_{n_i}) \). Let \( l_i \) be the minimum integer such that \( end(L_i) \subseteq V(L_{l_i}) \).

Let \( D \) be a digraph with vertex-set \( V(D) = \{s_0, s_1, \cdots, s_t\} \) and arc-set \( A(D) = \{(s_i, s_j) \mid f(\mathcal{E}, i, j) \neq K_4\} \). We use \( d_j \) to denote the length of a minimum directed path from \( s_0 \) to \( s_j \). If \( end(L_j) \cap I(L_i) \neq \emptyset \), then \( d_j = d_i + 1 \). Let \( U = \{j \mid d_j \text{ is maximum}\} \). If \( j \in U \), then \( d_G(u) = 2 \) for each \( u \in I(L_j) \).

Let \( i \) be an integer in \( U \) such that \( |f(\mathcal{E}, l_i, i)| \) is minimum. If there is a vertex \( v \) of \( I(f(\mathcal{E}, l_i, i)) \) such that \( d_G(v) \geq 3 \), then there is a path \( L_k \) such that \( v \in end(L_k) \cap \)
$I(f(\mathcal{E}, l_i, i))$. Since $\mathcal{E}$ has property $\mathbf{R}$, $f(\mathcal{E}, l_i, k)$ is a proper subpath of $f(\mathcal{E}, l_i, i)$, i.e., $|f(\mathcal{E}, l_i, k)| < |f(\mathcal{E}, l_i, i)|$. Since $|f(\mathcal{E}, l_i, i)|$ is minimum, we have $k \notin U$. Then there is a path, say $L_p$, such that $\text{end}(L_p) \cap I(L_k) \neq \emptyset$. Thus, $d_p > d_k = d_i$, a contradiction. Hence, $d_G(u) = 2$ for each $u \in I(f(\mathcal{E}, l_i, i))$. 

**Theorem 5.6.** Suppose $G$ is a 2-connected graph and $\mathcal{E} = (L_0; L_1, \ldots L_t)$ is a normal ear-decomposition of $G$. Then $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if $\mathcal{E}$ is a standard ear-decomposition of $G$ that has properties $\mathbf{Q}$ and $\mathbf{R}$, $L_i$ is an odd path for each $i \in [t]$, and $f(\mathcal{E}, i, j)$ is an odd path if $f(\mathcal{E}, i, j) \neq K_4$.

**Proof.** For $i \in [t]$, let $\text{end}(L_i) = \{a_i, b_i\}$.

For the necessity, suppose $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$. If $n$ is even, then $L_0$ is an even cycle. By Lemma 5.2 (2), $G$ is a bipartite graph and $L_i$ is an odd path for $i \in [t]$. Since $f(\mathcal{E}, i, j) \cup L_j$ is an even cycle, $f(\mathcal{E}, i, j)$ is an odd path. If $n$ is odd, then since $\mathcal{E}$ is normal, $|L_0|$ is odd. By Lemma 5.2 (4), $L_i$ is an odd path for $i \in [t]$. Suppose there are integers $i, j$ such that $f(\mathcal{E}, i, j)$ is an even path. If $i = 0$ and $L_0$ is an odd cycle, then $f(\mathcal{E}, i, j) = f_o(i, j)$ is an odd path, a contradiction. If $i > 0$ and $L_0$ is an odd cycle, then $H = L_j \cup (\bigcup_{c=0}^{i-1} L_c)$ is a 2-connected subgraph of $G$ and $(L_0; L_1 \cdot \cdot \cdot , L_{i-1}, L_i \cup L_j - I(f(\mathcal{E}, i, j)), f(\mathcal{E}, i, j))$ is an ear-decomposition of $H$ with $L_0$ an odd cycle and $f(\mathcal{E}, i, j)$ an even path, and by Lemma 5.2 (1) and (3) this yields a contradiction. If $L_0$ is an umbrella or an even $\theta$-graph other than $K_{2,3}$, then $G$ is a bipartite graph. Since $f(\mathcal{E}, i, j) \cup L_j$ is an even cycle and $L_j$ is an odd path, $f(\mathcal{E}, i, j)$ is an odd path, a contradiction. Thus, $f(\mathcal{E}, i, j)$ is an odd path if $n$ is odd.

We need to prove that $\mathcal{E}$ is standard and $\mathcal{E}$ has properties $\mathbf{Q}$ and $\mathbf{R}$ below.

**Claim 5.7.** $\mathcal{E}$ is standard.

**Proof.** If $n$ is even, then since $G$ is a bipartite graph, $L_0$ is an even cycle. Thus, $\mathcal{E}$ is standard.

If $G$ is not a bipartite graph and $n$ is odd, then $L_0$ is an odd cycle. Suppose $\mathcal{E}$ is not a standard ear-decomposition of $G$. Then there are paths $L_i$ and $L_j$ of $Z_G$ such that $E(f_e(\mathcal{E}, i)) \cap E(f_e(\mathcal{E}, j)) = \emptyset$. Let $D = L_i \cup L_j \cup [L_0 - I(f_e(\mathcal{E}, i) \cup f_e(\mathcal{E}, j))]$. Then $D$ is 2-connected subgraph of $L_0 \cup L_j \cup L_i$. Since $(D; f_e(\mathcal{E}, i), f_e(\mathcal{E}, j))$ is an ear-decomposition of $L_0 \cup L_i \cup L_j$ and $f_e(\mathcal{E}, i), f_e(\mathcal{E}, j)$ are even paths, by Lemma 5.2 (1) and (3) this yields a contradiction. Thus, $\mathcal{E}$ is standard.

If $G$ is a bipartite graph, $n$ is odd and $L_0$ is an even $\theta$-graph, then $L_0 \neq K_{2,3}$. Otherwise $L_0$ is a 2-connected subgraph of $G$ with $md(L_0) = 1 < \left\lfloor \frac{|L_0|}{2} \right\rfloor$, and by Lemma 5.2 (1) this yields a contradiction. Thus, $\mathcal{E}$ is standard.

If $G$ is a bipartite graph, $n$ is odd and $L_0$ is an umbrella, then suppose the rims of $L_0$ are $W_1, \cdot \cdot \cdot , W_k$, where $k \geq 3$ and $W_i$ is a $v_i v_{i+1}$-path for $i \in [k - 1]$. Suppose the spokes are $R_1, \cdot \cdot \cdot , R_k$, where $R_i$ is a $v_i v_{i}$-path. Let $C = \bigcup_{k \in [k]} W_i$. Since $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$,
by Lemma 5.2, \( L_0 \) is a uniform umbrella, i.e., each \( W_i \) is an even path and each \( R_i \) is an odd path. Suppose there is a path \( L_i \) of \( Z_\varepsilon \) such that \( \text{end}(L_i) \) is neither contained in any spoke nor contained in any rim. If \( a_i \in I(R_j) \) and \( b_i \in V(L_0) - V(R_j) \), then \( a_i \) divides \( R_j \) into two subpaths \( R_j^1 = vL_ia_i \) and \( R_j^2 = a_iL_jv_j \). Since \( k \geq 3 \), w.l.o.g., let \( b_j \notin I(W_k) \). Then \( H_s = R_j^3 \cup L_j \cup (\bigcup_{i \neq k} W_i) \cup (\bigcup_{i \neq j} R_i) \) is a 2-connected graph for \( s \in [2] \). Since \( L_j \) is an odd path, one of \( R_j^1 \) and \( R_j^2 \) is an even path, say \( R_j^1 \). Since \( (H_2; W_k, R_j^1) \) is an ear-decomposition of \( L_0 \cup L_i \) and \( W_k, R_j^1 \) are even paths, by Lemma 5.2 (1) and (2) this yields a contradiction. If \( \text{end}(L_i) \subseteq V(C) \), then since \( G \) is a bipartite graph, \( L_i \) is an odd path and each \( W_j \) is an even path, we have \( |\text{end}(L_i) \cap \{v_1, \cdots, v_k\}| \leq 1 \). Therefore, there is a rim \( W_j \) such that \( a_i \) divides \( W_j \) into two odd paths \( W_j^1 = v_jW_ja_i \) and \( W_j^2 = a_iW_jv_{j+1} \). (w.l.o.g., suppose \( 1 \leq j < k \)). Since there is no rim containing \( \text{end}(L_i) \), we have \( b_i \notin V(W_j) \). Note that \( \text{end}(L_i) \) divides \( C \) into two subpaths \( C^1 \) and \( C^2 \) such that \( v_j \in V(C^1) \) and \( v_{j+1} \in V(C^2) \). Since \( k \geq 3 \), by symmetry, suppose \( |C^1 \cap \{v_1, \cdots, v_k\}| \geq 2 \). Then there is an integer \( l \in [k] - \{i+1\} \) such that \( C^1 \) contains \( v_l \) and \( v_l \). Then there is an ear-decomposition \( (C'; P'_1, P'_2, \cdots) \) of \( L_0 \cup L_i \) such that \( C' = C^1 \cup L_i, P'_1 = R_i \cup R_l \) and \( P'_2 = W_i^2 \cup R_{i+1} \). Since \( P'_1 \) and \( P'_2 \) are even paths, by Lemma 5.2 (3) this yields a contradiction. Thus \( \mathcal{E} \) is standard.

**Claim 5.8.** \( \mathcal{E} \) has property \( Q \).

**Proof.** Let \( m_i (n_i) \) be the minimum integer such that \( a_i \in V(L_{m_i}) \) \((b_i \in V(L_{n_i})) \). Since \( I(L_0) = V(L_0) \), we have \( a_i \in I(L_{m_i}) \) and \( b_i \in I(L_{n_i}) \). Let \( l_i \) be an integer such that \( \text{end}(L_i) \cap I(L_{l_i}) \neq \emptyset \).

Suppose \( \mathcal{E} \) does not have property \( Q \). Then there are integers \( 0 \leq j < r \leq t \) such that \( a_r \in I(L_j) \) and \( b_r \notin V(L_j) \). Since \( b_r \in I(L_{n_r}) \), by symmetry, suppose \( j > l_{n_r} \).

For convenience, let \( l_{n_r} = i \). Since \( L_j \) is an odd path, let \( a_jL_ja_r \) be an even path. Let \( l = \max\{m_j, n_j, n_r\} \) and \( H = L_j \cup L_r \cup (\bigcup_{h=0}^k L_h) \). Then \( H \) is a 2-connected graph with an ear-decomposition \( (L_0; L_1, \cdots, L_t, a_rL_jb_j \cup L_r, a_jL_ja_r) \). If \( L_0 \) is an odd cycle, or a uniform umbrella, or an even \( \theta \)-graph other than \( K_{2,3} \), then since \( |L_0| \) is odd and \( a_jL_ja_r \) is an even path, by Lemma 5.2 (1) and (3) this yields a contradiction. If \( L_0 \) is an even cycle, then by Lemma 5.2 (1) and (2) this yields a contradiction.

**Claim 5.9.** \( \mathcal{E} \) has property \( R \).

**Proof.** If \( \mathcal{E} \) does not have property \( R \), then there are integers \( r, i, j \) such that \( \text{end}(L_j) \cap I(f(\mathcal{E}, r, i)) \neq \emptyset \) and \( f(\mathcal{E}, r, j) \) is not a subpath of \( f(\mathcal{E}, r, i) \). Since \( \mathcal{E} \) has property \( Q \), \( f(\mathcal{E}, r, j) \) is a subpath of \( L_r \). Then \( \text{end}(L_j) \) and \( \text{end}(L_j) \) appear alternately on \( L = f(\mathcal{E}, r, i) \cup f(\mathcal{E}, r, j) \), say \( a_i, a_j, b_i, b_j \) are consecutively on \( L \). Here, \( L \) is a subpath of the path \( L_r \) if \( r > 0 \); \( L \) is a subpath of either a rim or a spoke of \( L_r \) if \( r = 0 \) and \( L_0 \) is a uniform umbrella; \( L \) is a subpath of a route if \( r = 0 \) and \( L_0 \) is an even \( \theta \)-graph other than \( K_{2,3} \); \( L \) is a subpath of a cycle \( L_r \) if \( r = 0 \) and \( L_0 \) is a cycle. Let \( W^1 = a_iLa_j, W^2 = a_jLb_i \) and \( W^3 = b_iLb_j \). Since \( f(\mathcal{E}, r, i) \) and \( f(\mathcal{E}, r, j) \) are odd paths,
either $W^1, W^3$ are even paths and $W^2$ is an odd path, or $W^2$ is an even path and $W^1, W^3$ are odd paths. Let $H = \bigcup_{l=0}^{L} L_i \cup L_i \cup L_j$.

Suppose $W^1, W^3$ are even paths and $W^2$ is an odd path. Let $H'$ be a graph obtained from $H$ by removing $W^1$ and $W^3$. Then $H'$ is a 2-connected graph. Since $(H'; W^1, W^3)$ is an ear-decomposition of $H$ and $W^1, W^3$ are even paths, by Lemma 5.2 this yields a contradiction.

Suppose $W^2$ is an even path and $W^1, W^3$ are odd paths. Let $H_t$ be a graph obtained from $H$ by removing $W^3$ for $i \in [3]$. It is obvious that each $H_t$ is a 2-connected graph. If $L_0$ is an even cycle, then $(H_2; W^2)$ is an ear-decomposition of $G$, and by Lemma 5.2 (1) and (2) this yields a contradiction. If $r = 0$ and $L_0$ is an odd cycle, then $P = L_0 - I(L)$ is an even path and $C = H_2 - I(P)$ is an even cycle. Since $(C; P, W^2)$ is an ear-decomposition of $H$ and $P, W^2$ are even paths, by Lemma 5.2 (1) and (3) this yields a contradiction. If $r = 0$ and $L_0$ is an even $\theta$-graph, then suppose $T_1, T_2$ and $T_3$ are routes of $L_0$, and suppose $L$ is a subpath of $T_1$. Then $(H_2 - I(T_2); T_2, W^2)$ is an ear-decomposition of $H$ and $T_2, W^2$ are even paths, a contradiction. If $r = 0$ and $L_0$ is a uniform umbrella, then there is a rim $W$ of $L_0$ such that $L$ is not a subpath of $W$. Then $(H_2 - I(W); W, W^2)$ is an ear-decomposition of $H$ and $W, W^2$ are even paths, a contradiction. If $r > 0$ and $n$ is odd, then $(L_0; \cdots, W^2)$ is an ear-decomposition of $H$. Since $|L_0|$ is odd and $W^2$ is an even path, by Lemma 5.2 (1) and (3) this yields a contradiction.

Now for the sufficiency, suppose $E = (L_0; L_1, \cdots, L_t)$ satisfies all conditions of the theorem, i.e., $E$ is a standard ear-decomposition of $G$ that has properties $Q$ and $R$. $L_i$ is an odd path for $i \in [t]$, and $f(E, j, i)$ is an odd path when $f(E, j, i) \neq K_t$. Recall the definitions of digraph $D$, set $U$ and integer $l_i$ in Lemma 5.3. We choose an integer $r$ from $U$ such that $|f(E, l_r, r)|$ is minimum. For convenience, let $l = l_r$. Then for each vertex $u$ of $I(f(E, l_r, r)) \cup I(L_r)$, we have $d_G(u) = 2$. The proof proceeds by induction on $t$. By Lemmas 5.1 (2) and 5.3, the result holds for $t = 0$.

If $L_r$ is not an edge, then let $G'$ be a graph obtained from $G$ by replacing $f(E, l, r)$ with an edge $f = a_r b_r$, let $G'_1 = G' - I(L_r)$ and $G'_2 = L_r \cup f$. Let $L_0 = [L_r - I(f(E, l, r)) - E(f(E, l, r))] \cup f$. Let $E'$ be an ear-decomposition of $G'_1$ obtained from $E$ by removing $L_r$, and then replacing $L_r$ with $L$. If $l > 0$, then since $f(E, l, r)$ is an odd path, $L$ is an odd path and $E'$ satisfies all the conditions. If $l = 0$ and $L_0$ is a uniform umbrella (an odd cycle or an even cycle), then $L$ is also a uniform umbrella (an odd cycle, an even cycle), i.e., $E'$ satisfies all the conditions in this case. If $l = 0$ and $L_0$ is an even $\theta$-graph, then $E'$ satisfies all the conditions except for $L = K_2,3$. Thus, $E'$ satisfies all the conditions unless $L = K_2,3$.

If $L \neq K_2,3$, then $E'$ satisfies all the conditions. Since the number of paths in $E'$ is $t - 1$, by the induction hypothesis we have $md(G'_1) = \begin{bmatrix} |G'_1| \\ 2 \end{bmatrix}$. Since $G'_2$ is an even cycle, we have $md(G'_2) = \frac{|G'_2|}{2}$. Thus, by Lemma 2.1 $md(G') = md(G'_1) + md(G'_2) - 1 = \frac{|G'|}{2}$.  

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Since $G$ is a graph obtained from $G'$ by replacing $f$ with the odd path $f(\mathcal{E}, l, r)$, by Lemma 5.5, we have $md(G) \geq md(G') + \left\lfloor \frac{|f(\mathcal{E}, l, r)|}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

If $L = K_{2,3}$, then $l = 0$ and $r = 1$. Since $r \in U$, $d_r$ is maximum and $d_r = 1$ (the definition $d_r$ is in the proof of Lemma 5.5). Thus, $L_i \in Z_E$ for each $i \in [t]$. Let $T_1, T_2$ and $T_3$ be routes of $L_0$ with $|T_1| \leq |T_2| \leq |T_3|$. Then $T_1$ and $T_2$ are 2-paths and $f(\mathcal{E}, 0, r)$ is a subpath of $T_3$ with $|f(\mathcal{E}, 0, r)| = |T_3| - 1$. Since $L_0 \neq K_{2,3}$, we have $|f(\mathcal{E}, 0, r)| = |T_3| - 1 \geq 4$. For each $L_i$, if $\text{end}(L_i) \cap I(T_j) \neq \emptyset$ for $j \in [2]$, then $|f(\mathcal{E}, 0, i)| = 2 < |f(\mathcal{E}, l, r)|$, a contradiction; if $\text{end}(L_i) = \text{end}(T_3)$, then $f(\mathcal{E}, 0, i)$ is an even path, a contradiction. Thus, $f(\mathcal{E}, 0, i)$ is a proper subpath of $T_3$ and $|f(\mathcal{E}, 0, i)| = |f(\mathcal{E}, 0, r)|$ for each $i \in [t]$. If $\text{end}(L_i) \neq \text{end}(L_r)$ for $i, j \in [t]$, then $\text{end}(L_i) \cap I(f(\mathcal{E}, 0, r)) \neq \emptyset$ and $f(\mathcal{E}, 0, i)$ is not a proper subpath of $f(\mathcal{E}, 0, r)$, i.e., $\mathcal{E}$ does not have property $R$, a contradiction. Therefore, $\text{end}(L_i) = \text{end}(L_j)$ for each $i, j \in [t]$. Let $H = T_2 \cup T_3 \cup (\bigcup_{i \in [t]} L_i)$. Then $H$ is a graph constructed in Remark 1. Thus, $md(H) = \frac{|H|}{2}$. Suppose $\Gamma$ is an extremal MD-coloring of $H$ (see Remark 1).

Let $T_1 = u_1a_1e_2v_2$ and $T_2 = u_1f_1b_2v_2$. Since $G = H \cup T_1$, let $\Gamma'$ be an edge-coloring of $G$ such that $\Gamma(e) = \Gamma'(e)$ for each $e \in E(H)$, and $\Gamma(e_1) = \Gamma'(f_2)$ and $\Gamma(e_2) = \Gamma'(f_1)$. Then $\Gamma'$ is an MD-coloring of $G$ with $\left\lfloor \frac{n}{2} \right\rfloor$ colors, i.e., $md(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

If $L_i$ is an edge, then replace $L_i$ by $L_i \cup L_r - I(f(\mathcal{E}, l, r))$ and replace $L_r$ by $f(\mathcal{E}, l, r)$. Then the new ear-decomposition also satisfies all the conditions. Moreover, $d_r$ is maximum and $|f(\mathcal{E}, l, r)| = 2$ is minimum in the new ear-decomposition. Since $L_r$ is not an edge in the new ear-decomposition, this case has been discussed above.

Remark 3. Recalling the proof of Lemma 5.7, we can find a normal ear-decomposition for a given 2-connected graph in polynomial time. For a normal ear-decomposition $\mathcal{E}$ of $G$, deciding whether $\mathcal{E}$ satisfies all the conditions of Theorem 5.6 can be done in polynomial time. Thus, given a 2-connected graph $G$, deciding whether $md(G) = \left\lfloor \frac{|G|}{2} \right\rfloor$ is polynomially solvable.

Corollary 5.10. If $G$ is a 2-connected graph with $md(G) = \left\lfloor \frac{|G|}{2} \right\rfloor$, then $G$ is a planar graph.

Proof. By Theorem 5.6 there is a standard ear-decomposition $\mathcal{E} = \{L_0; L_1, \ldots, L_t\}$ of $G$ that has properties $Q$ and $R$. Since $G$ is a planar graph if $G$ is a cycle, an umbrella or a $\theta$-graph, the result holds for $t = 0$. Our proof proceeds by induction on $t$. Suppose $t > 0$. By Lemma 5.5, there are integers $k, i$ such that $f(\mathcal{E}, k, i)$ is a path of order at least two, and $d_G(u) = 2$ for each $u \in I(f(\mathcal{E}, k, i)) \cup I(L_i)$. Let $G'$ be a graph obtained from $G$ by removing $L_i$. By Lemma 5.2 (1), $md(G') = \left\lfloor \frac{|G'|}{2} \right\rfloor$. By the inductive hypothesis, $G'$ is a planar graph. Since $d_G(u) = 2$ for each $u \in I(f(\mathcal{E}, k, i))$, there is a face $F$ of $G'$ such that $f(\mathcal{E}, k, i)$ is a subpath of $F$. Therefore, $L_i$ can be embedded in $F$ and $G$ is a planar graph.

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