On weak solutions to the 2D Savage–Hutter model of the motion of a gravity-driven avalanche

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\textbf{ABSTRACT}
We consider the Savage–Hutter system consisting of two-dimensional depth-integrated shallow water equations for the incompressible fluid with the Coulomb-type friction term. Using the method of convex integration we show that the associated initial value problem possesses infinitely many weak solutions for any finite energy initial data. On the other hand, the problem enjoys the weak-strong uniqueness property provided the system of equations is supplemented with the energy inequality.

\textbf{ARTICLE HISTORY}
Received 23 February 2015
Accepted 28 November 2015

\textbf{KEYWORDS}
Avalanche flow; convex integration; Savage–Hutter system; weak solution

\textbf{MATHEMATICS SUBJECT CLASSIFICATION}
Primary: 35Q86; 35L60; Secondary: 76T25

\section{Introduction}
The theory for gravity-driven avalanche flows is qualitatively similar to that of compressible fluid dynamics. We consider a relatively simple Savage–Hutter model based on the assumption that the material is incompressible, with an isotropic pressure distribution through its depth, and a Coulomb sliding friction, (Gray et al. [15] and [18] for a comprehensive overview). The time evolution of the flow height $h = h(t,x)$ and depth-averaged velocity $u = u(t,x)$ are described through a system of balance laws—the Savage–Hutter system:

\begin{align}
\partial_t h + \text{div}_x(hu) &= 0, \\
\partial_t (hu) + \text{div}_x(hu \otimes u) + \nabla_x(ah^2) &= h\left(-\gamma \frac{u}{|u|} + f\right),
\end{align}

where $|\cdot|$ is an Euclidean metric, $a \geq 0$, $\gamma \geq 0$, and $f$ are given (smooth) functions of spatial coordinate, $x \in \Omega \subset \mathbb{R}^2$ (Gray and Cui [14], Zahibo et al. [24]). For the sake of simplicity, we restrict ourselves to the periodic boundary conditions supposing accordingly that $\Omega$ is the “flat” torus

$$\Omega = ([0, 1][0, 1])^2.$$  

The system (1.1), (1.2) is supplemented with the initial conditions

$$h(0, \cdot) = h_0, \quad u(0, \cdot) = u_0.$$  

The term $\frac{u}{|u|}$ has to be understood as a multivalued mapping, which for nonzero velocities takes the value $\frac{u}{|u|}$, whereas for $u = 0$ takes the values in the whole closed unit ball. For the
physical justification for such formulation with a simple argument on one-dimensional steady solutions, we refer to Gwiazda [16].

In the absence of the driving force on the right-hand side of (1.2), the Savage–Hutter system coincides with barotropic Euler system describing the motion of a compressible inviscid fluid. As observed by Gray and Cui [14], the solutions of the Savage–Hutter system develop shock waves and other singularities characteristic for hyperbolic system of conservation laws. Accordingly, any mathematical theory based on the classical concept of (smooth) solutions fails as soon as we are interested in global-in-time solutions to the system (1.1), (1.2), and/or in solutions emanating from singular initial data. Hence in analog to the development of the theory for Euler equations, the problem of existence of weak solutions has so far remained open. The issue of measure-valued solutions for the two-dimensional model considered here was studied in Gwiazda [17]. Again, similar as for Euler flows, the author follows the concept of generalization by DiPerna and Majda [9] to capture both oscillations as well as concentration effects. The existence of entropy weak solution to the corresponding problem in one-dimensional setting was shown in Gwiazda [16]. Various modifications of the model to more complex topographies were considered in [1, 2, 13] and [19, 20] for computational results.

In this paper, we consider the weak solutions to the problem (1.1–1.4) determined by means of a family of integral identities:

$$\int_0^T \int_\Omega (h \partial_t \varphi + h \mathbf{u} \cdot \nabla_x \varphi) \, dx \, dt = - \int_0^T \int_\Omega h_0 \varphi(0, \cdot) \, dx \, dt$$

(1.5)

for any $$\varphi \in C^1([0, T) \times \Omega)$$;

$$\int_0^T \int_\Omega (h \mathbf{u} \cdot \partial_t \Phi + h \partial_t \Phi + h \mathbf{u} \otimes \mathbf{u} : \nabla x \Phi + ah^2 \text{div}_x \Phi) \, dx \, dt$$

$$= \int_0^T \int_\Omega h (\gamma \mathbf{B} - \mathbf{f}) \cdot \Phi \, dx \, dt$$

$$- \int_\Omega h_0 \mathbf{u}_0 \cdot \Phi(0, \cdot) \, dx$$

(1.6)

for any $$\Phi \in C^1([0, T) \times \Omega; \mathbb{R}^2)$$, where

$$\mathbf{B} = \mathbf{B}_\mathbf{u}(t, x) = \left\{ \begin{array}{ll}
\frac{\mathbf{u}(t, x)}{|\mathbf{u}(t, x)|} & \text{if } \mathbf{u}(t, x) \neq 0, \\
\in \overline{B_1(0)} & \text{if } \mathbf{u}(t, x) = 0,
\end{array} \right.$$  

(1.7)

where $$B_1(0)$$ is the unit ball in $$\mathbb{R}^2$$ with respect to the Euclidean metric. Hence speaking about weak solutions we in fact mean a triplet $$[h, \mathbf{u}, \mathbf{B}_\mathbf{u}]$$, whereas $$\mathbf{B}_\mathbf{u}$$ is the selection from the multivalued graph.

Using the method of convex integration, recently adapted to the incompressible Euler system by De Lellis and Székelyhidi [7], and Chiodaroli [3], we show that the Savage–Hutter system is always solvable but not well posed in the class of weak solutions. More specifically, we show that the problem (1.1–1.4) admits infinitely many weak solutions on a given time interval $$(0, T)$$ and for any sufficiently smooth initial data, (Section 2). The method of convex integration is used to construct large sets of weak solutions. Starting from the seminal results of Scheffer [21] and Shnirelman [22] on existence of compactly supported 2D weak solutions to the Euler equations and the previously mentioned result of De Lellis and Székelyhidi [7], the theory was extensively developed for constructing weak solutions with bounded energy.
by Wiedemann [23], in a nontrivial way extended for compressible Euler in the so-called \textit{variable coefficient} variant of the approach [3, 5], and recently to more complex systems such as equations of a compressible heat conducting gas [4], Euler-Korteweg-Poisson system [10].

Apparently, the approach of convex integration allows to avoid the difficulties arising from the multivalued formulation of the friction term. In case of measure-valued solutions, which were generated by the approximate sequences of corresponding viscous problems, one could not exclude the case of $\mathbf{u} = 0$, cf. Gwiazda [17] and also the same difficulty in 1D case [16]. In the current framework, we choose $h$ and the energy $E$ and the consequent steps lead to finding the momentum. Although the friction term was nonlinear in terms of $\mathbf{u}$, the situation turns out to be significantly better as we express the friction term as a product of a scalar function only of $h$ and $E$ and a function of $hu$, which is linear. The linearity then obviously provides weak continuity. And what is the most essential, postulating $h > 0$ and the energy being sufficiently large yields that the velocity $\mathbf{u}$ is nonzero, and hence we are away from the set where the friction term is multivalued.

Next, in Section 3, we augment the weak formulation (1.5–1.7) by the energy inequality. Moreover, using the relative energy method introduced by Dafermos [6] and developed in [11], we show the weak–strong uniqueness principle: A weak solution satisfying the energy inequality necessarily coincides with the strong solution emanating from the same initial data as long as the latter exists. The paper is concluded by some remarks concerning the implications of the convex integration technique on the well-posedness problem considered in the class of finite energy weak solutions (Section 4).

We conclude the introduction with a remark that the current studies are a negative result for numerical simulation conducted for this system as the analysis shows that solutions with a sufficiently large energy exist but are nonunique. Nevertheless, this does not exclude there might exist solutions with a small energy, which are unique.

2. \textbf{Infinitely many weak solutions}

We start with a brief description of the procedure of constructing the solutions. An essential tool is the Baire category method arising from the theory of differential inclusions. Note, however, that instead of the abstract arguments one could provide the constructive, however essentially longer proof by adding oscillatory perturbations. Similarly, we will start with regular initial data [4]

$$h_0 \in C^2(\Omega), \quad \mathbf{u}_0 \in C^2(\Omega; \mathbb{R}^2), \quad h_0 > 0 \quad \text{in} \ \Omega. \tag{2.1}$$

and reformulate the problem in different variables to obtain the system corresponding to an incompressible Euler system. The regularity of data appears to be important in the oscillatory lemma, which the same as in case of compressible Euler flow is formulated in the variable coefficient forms with the coefficients generated by the data. For the new system we will construct a family of subsolutions. Indeed, we rewrite it as a linear system coupled with a nonlinear constraint. First, completing the space of subsolutions in appropriate topology we consider a family of functionals, which turn out to be lower semicontinuous. In the next step we will conclude that as the limit object is a pointwise limit of continuous functions, namely it is \textit{Baire-1 function}, then the set of continuity points is infinite. To show that the points where the functional vanishes are solutions to the system we will use an oscillatory lemma, namely Lemma 2.1.
In the first step, using the standard Helmholtz decomposition, we may write
\[ h_0 u_0 = v_0 + V_0 + \nabla_x \Psi_0, \]
where \( \text{div}_x v_0 = 0, \int_\Omega \Psi_0 \, dx = 0, \int_\Omega v_0 \, dx = 0, V_0 \in \mathbb{R}^2. \) (2.2)

The main result proved in the section reads:

**Theorem 2.1.** Let \( T > 0 \) and the initial data \( h_0, u_0 \) satisfying (2.1) be given. Suppose that \( a, \gamma \in C^2(\Omega), a, \gamma \geq 0, f \in C^1([0, T] \times \Omega; \mathbb{R}^3). \)

Then the problem (1.1–1.4) admits infinitely many weak solutions in \((0, T) \times \Omega.\) The weak solutions belong to the class
\[
h, \partial_t h, \nabla_x h \in C^1([0, T] \times \Omega),
\]
\[
u \in C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^3)) \cap L^\infty((0, T) \times \Omega; \mathbb{R}^3), \quad \text{div}_x u \in C([0, T] \times \Omega),
\]
\[
B \in L^\infty((0, T) \times \Omega; \mathbb{R}^2).
\]

Above, by \( C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^3)), \) we mean the space of functions continuous in time with respect to the weak topology of \( L^2(\Omega; \mathbb{R}^3).\) The remaining part of this section is devoted to the proof of Theorem 2.1.

### 2.1. Convex integration ansatz

Similarly to the decomposition (2.2), we look for solutions in the form
\[ h u = v + V + \nabla_x \Psi, \quad \text{where div}_x v = 0, \int_\Omega \Psi(t, \cdot) \, dx = 0, \int_\Omega v(t, \cdot) \, dx = 0, V = V(t) \in \mathbb{R}^2. \]

Thus the continuity equation (1.1) reads
\[ \partial_t h + \Delta_x \Psi = 0 \quad \text{in } (0, T) \times \Omega, \quad h(0, \cdot) = h_0, \quad \Psi(0, \cdot) = \Psi_0. \] (2.3)

Now, we can choose \( h = h(t, x) \in C^2([0, T] \times \Omega) \) in such a way that
\[ h(0, \cdot) = h_0, \quad \partial_t h(0, \cdot) = -\Delta_x \Psi_0, \quad h(t, \cdot) > 0, \quad \int_\Omega h(t, \cdot) \, dx = \int_\Omega h_0 \, dx \quad \text{for all } t \in [0, T], \]
and compute
\[ -\Delta_x \Psi(t, \cdot) = \partial_t h(t, \cdot), \quad \int_\Omega \Psi(t, \cdot) \, dx = 0. \]

Consequently, the original problem (1.1–1.4) reduces to finding the functions \( v, V \) satisfying (weakly)
\[
\partial_t v + \partial_t V + \text{div}_x \left( \frac{(v + V + \nabla_x \Psi)}{h} \otimes (v + V + \nabla_x \Psi) \right) + (a h^2 + \partial_t \Psi) I
\]
\[ = h \left( -\gamma \frac{v + V + \nabla_x \Psi}{|v + V + \nabla_x \Psi|} + f \right), \] (2.4)
\[
\text{div}_x v = 0, \quad \int_\Omega v(t, \cdot) \, dx = 0, \] (2.5)
\[
v(0, \cdot) = v_0, \quad V(0) = V_0. \] (2.6)
2.2. Kinetic energy

We denote

\[ E = \frac{1}{2} \frac{1}{h} |v + V + \nabla_x \Psi|^2 \]  

(2.7)

the kinetic energy associated with the Savage–Hutter system. Analogously, we rewrite (2.4) in the form

\[ \partial_t v + \partial_t V + \text{div}_x \left( \frac{(v + V + \nabla_x \Psi) \otimes (v + V + \nabla_x \Psi)}{h} - \frac{1}{2} \frac{|v + V + \nabla_x \Psi|^2}{h} \right) + \nabla_x (E - \Lambda + ah^2 + \partial_t \Psi) = -\gamma \left( \frac{h}{2E} \right)^{1/2} (v + V + \nabla_x \Psi) + hf, \]

(2.8)

where \( \Lambda = \Lambda(t) \) is a spatially homogeneous function to be determined below. Finally, for

\[ E = \Lambda - ah^2 - \partial_t \Psi, \]

(2.9)

Equation (2.8) reduces to

\[ \partial_t v + \partial_t V + \text{div}_x \left( \frac{(v + V + \nabla_x \Psi) \otimes (v + V + \nabla_x \Psi)}{h} - \frac{1}{2} \frac{|v + V + \nabla_x \Psi|^2}{h} \right) = -\gamma \left( \frac{h}{2E} \right)^{1/2} (v + V + \nabla_x \Psi) + hf. \]

(2.10)

2.3. Determining \( V \)

The spatially homogeneous function \( V \) is determined as the unique solution of the ordinary differential equation

\[ \partial_t V - \left[ \frac{1}{|\Omega|} \int_{\Omega} \gamma \left( \frac{h}{2E} \right)^{1/2} \right] V = \frac{1}{|\Omega|} \int_{\Omega} \left[ \gamma \left( \frac{h}{2E} \right)^{1/2} (v + \nabla_x \Psi) + hf \right] dx, \ V(0) = V_0. \]

(2.11)

Note that \( V = V[v] \) depends on \( v \) and also on the function \( \Lambda \) in (2.9).

With such a choice of \( V \), Equation (2.10) reads

\[ \partial_t v + \text{div}_x \left( \frac{(v + V[v] + \nabla_x \Psi) \otimes (v + V[v] + \nabla_x \Psi)}{h} - \frac{1}{2} \frac{|v + V[v] + \nabla_x \Psi|^2}{h} \right) \]

\[ = -\gamma \left( \frac{h}{2E} \right)^{1/2} (v + V[v] + \nabla_x \Psi) + \frac{1}{|\Omega|} \int_{\Omega} \gamma \left( \frac{h}{2E} \right)^{1/2} (v + V[v] + \nabla_x \Psi) dx \]

\[ + hf - \frac{1}{|\Omega|} \int_{\Omega} hf \ dx. \]

(2.12)
Finally, we find a tensor \( \mathbb{M} = \mathbb{M}[v] \) such that \( \mathbb{M}(t,x) \in \mathbb{R}^{2\times2}_{\text{sym},0} \), where \( \mathbb{R}^{2\times2}_{\text{sym},0} \) is the set of \( 2 \times 2 \) symmetric traceless matrices for any \( t,x \), and

\[
\text{div}_x \mathbb{M} = -\gamma \left( \frac{h}{2E} \right)^{1/2} (v + V[v] + \nabla_x \Psi) + \frac{1}{|\Omega|} \int_{\Omega} \gamma \left( \frac{h}{2E} \right)^{1/2} (v + V[v] + \nabla_x \Psi) \, dx \\
+ hf - \frac{1}{|\Omega|} \int_{\Omega} hf \, dx, \\
\int_{\Omega} \mathbb{M}(t,\cdot) \, dx = 0 \quad \text{for any } t \in [0,T].
\]

We can take

\[
\mathbb{M} = \nabla_x m + \nabla_x^t m - \text{div}_x m^\|,
\]

where \( m \) is the (unique) solution of the elliptic equation

\[
\text{div}_x \left( \nabla_x m + \nabla_x^t m - \text{div}_x m^\| \right) = -\gamma \left( \frac{h}{2E} \right)^{1/2} (v + V[v] + \nabla_x \Psi) + \frac{1}{|\Omega|} \int_{\Omega} \gamma \left( \frac{h}{2E} \right)^{1/2} (v + V[v] + \nabla_x \Psi) \, dx \\
+ hf - \frac{1}{|\Omega|} \int_{\Omega} hf \, dx, \\
\int_{\Omega} m \, dx = 0.
\]

Indeed Desvilletes and Villani showed a variant of Korn’s inequality [8, Section IVI, Proposition 11]

\[
\| \nabla_x m + \nabla_x^t m - \text{div}_x m^\| \|_{L^2(\Omega;\mathbb{R}^{2\times2})} \geq \frac{1}{2} \| \nabla_x m \|_{L^2(\Omega;\mathbb{R}^{2\times2})},
\]

in particular, the problem (2.14), (2.15) admits a unique solution.

Summarizing, we write Equation (2.12) in a concise form

\[
\partial_t v + \text{div}_x \left( \frac{(v + V[v] + \nabla_x \Psi) \otimes (v + V[v] + \nabla_x \Psi)}{h} - \frac{1}{2} \frac{|v + V[v] + \nabla_x \Psi|^2}{h} - \mathbb{M}[v] \right) = 0,
\]

where \( V \) and \( \mathbb{M} \) are determined by means of (2.11) and (2.13), respectively.

### 2.4. Application of the method of convex integration

To recast our problem in terms of the method proposed by De Lellis and Székelyhidi [7], we need to determine

- a (topological) space of subsolutions \( X_0 \);
- a lower semicontinuous functional \( I : X_0 \to \mathbb{R} \) such that the points of continuity of \( I \) coincide with the solution set of our problem.
2.4.1. Subsolutions

Let $\lambda_{\text{max}}[A]$ denote the maximal eigenvalue of a symmetric matrix $A$. Motivated by De Lellis and Székelyhidi [7], we introduce the set

$$X_0 = \left\{ w \mid w \in L^\infty((0, T) \times \Omega, R^2) \cap C^1((0, T) \times \Omega, R^2) \cap C_{\text{weak}}([0, T]; L^2(\Omega; R^2)), \right.$$ 

$$w(0, \cdot) = v_0, \quad \text{div}_x w = 0 \text{ in } (0, T) \times \Omega,$$

$$\partial_t w + \text{div}_x F = 0 \text{ in } (0, T) \times \Omega \text{ for some } F \in C^1((0, T) \times \Omega; R^{2 \times 2}_\text{sym}),$$

$$\lambda_{\text{max}} \left[ (w + V[w] + \nabla_x \Psi) \otimes (w + V[w] + \nabla_x \Psi) \right] - F - M[w] < E - \delta$$

$$\text{in } (0, T) \times \Omega \text{ for some } \delta > 0 \right\},$$

where $E$ is the kinetic energy introduced in (2.9).

The first observation is that the set $X_0$ is nonempty provided

$$\Lambda(t) \geq \Lambda_0 > 0 \text{ in } [0, T] \quad (2.17)$$

in (2.9), and $\Lambda_0$ is large enough. Here “large enough” means in terms of the initial data $f$ and the time $T$. Indeed, taking $w = v_0, F = 0$, we have to find $\Lambda_0$ such that

$$\lambda_{\text{max}} \left[ \frac{(v_0 + V[v_0] + \nabla_x \Psi) \otimes (v_0 + V[v_0] + \nabla_x \Psi)}{h} - M[v_0] \right] < E - \delta = \Lambda - ah^2 - \partial_t \Psi - \delta. \quad (2.18)$$

Since $V$ is given by (2.11), it is easy to check that $V[v_0]$, together with $\partial_t V[v_0]$, remains bounded in terms of the data and uniformly for all $\Lambda \geq \Lambda_0$. Furthermore, applying the standard elliptic estimates to (2.13), we get

$$\|M(t, \cdot)\|_{W^{1, q}(\Omega; R^{2 \times 2})} \leq c(q, \text{data}) \text{ for any } 1 < q < \infty, \ t \in [0, T].$$

Consequently, we may fix $\Lambda_0, \Lambda$ satisfying (2.17), and finally, the kinetic energy $E$ in (2.8) in such a way that the set of subsolutions $X_0$ is nonempty.

2.4.2. Uniform bounds

As shown by De Lellis and Székelyhidi [7], we have

$$\frac{1}{2} \frac{|w + V[w] + \nabla_x \Psi|^2}{h} \leq \lambda_{\text{max}} \left[ \frac{(w + V[w] + \nabla_x \Psi) \otimes (w + V[w] + \nabla_x \Psi)}{h} - F - M[w] \right], \quad (2.19)$$

where the equality holds only if

$$F + M[w] = \frac{(w + V[w] + \nabla_x \Psi) \otimes (w + V[w] + \nabla_x \Psi)}{h} - \frac{|w + V[w] + \nabla_x \Psi|^2}{h} \mathbb{I}. \quad (2.20)$$
Since $E$ has been fixed, we may deduce from (2.19) that
\[ |w + V[w]| \leq c(E, T, \text{data}) \text{ in } (0, T) \times \Omega, \tag{2.21} \]
and, going back to (2.11), we may infer that
\[ |w| + |V[w]| + |\partial_t V[w]| \leq c(E, T, \text{data}) \text{ in } (0, T) \times \Omega. \tag{2.22} \]
Furthermore, (2.15) yields
\[ \|M[w](t, \cdot)\|_{W^{1,q}(\Omega; R^2)} \leq c(E, T, \text{data}) \text{ in } (0, T) \text{ for any } 1 \leq q < \infty. \tag{2.23} \]

The set $X_0$ is endowed with the topology of the space $C_{\text{weak}}([0, T]; L^2(\Omega; R^2))$. In view of (2.21), such a topology is metrizable on $X_0$ and we denote $X_0$ the completion of $X_0$—a topological metric space. In accordance with (2.22), (2.23), and the compact embedding $W^{1,q} \hookrightarrow C(\Omega)$, $q > 2$, we obtain
\[ V[w_n] \to V[w] \text{ in } C([0, T]) \]
\[ M[w_n] \to M[w] \text{ in } C((0, T) \times \Omega) \]
whenever $w_n \to w$ in $C_{\text{weak}}([0, T]; L^2(\Omega; R^2))$. \tag{2.24}

2.4.3. Functional $I$ and infinitely many solutions

Following De Lellis and Székelyhidi [7], we introduce the functional
\[ I[v] = \int_0^T \int_{\Omega} \left[ \frac{1}{2} \frac{|v + V[v] + \nabla_x \Psi|^2}{h} - E \right] \, dx \, dt : \overline{X}_0 \to R. \]

To proceed, we need the following variant of the oscillatory lemma (cf. De Lellis and Székelyhidi [7, Proposition 3], Chiodaroli [3, Section 6, formula (6.9)]) proved in [10, Lemma 3.1]:

**Lemma 2.1.** Let $U \subset R \times R^N$, $N = 2, 3$ be a bounded open set. Suppose that
\[ g \in C(U; R^N), \quad \mathcal{W} \in C(U; R_{sym,0}^N), \quad e, r \in C(U), \quad r > 0, \quad e \leq \bar{e} \text{ in } U \]
are given such that
\[ \frac{N}{2} \lambda_{\text{max}} \left[ \frac{g \otimes g}{r} - \mathcal{W} \right] < e \text{ in } U. \]

Then, there exist sequences
\[ w_n \in C_c^\infty(U; R^N), \quad \mathcal{G}_n \in C_c^\infty(U; R_{sym,0}^N), \quad n = 0, 1, \ldots \]
such that
\[ \partial_t w_n + \text{div}_x \mathcal{G}_n = 0, \quad \text{div}_x w_n = 0 \text{ in } R^N, \]
\[ \frac{N}{2} \lambda_{\text{max}} \left[ \frac{(g + w_n) \otimes (g + w_n)}{r} - (\mathcal{W} + \mathcal{G}_n) \right] < e \text{ in } U, \]
and
\[ w_n \to 0 \text{ weakly in } L^2(U; R^N), \quad \liminf_{n \to \infty} \int_U \frac{|w_n|^2}{r} \, dx \, dt \geq c(\bar{e}) \int_U \left( e - \frac{1}{2} \frac{|g|^2}{r} \right)^2 \, dx \, dt. \tag{2.25} \]
**Remark 2.1.** It is important to note that the constant $c(\tilde{\tau})$ in (2.25) is independent of the specific form of the quantities $g, \mathbb{W}, e,$ and $r.$

In view of (2.19), we have

$$I[w] < 0 \quad \text{for any } w \in X_0,$$

and as a consequence of (2.24), $I : \mathcal{X}_0 \to (-\infty, 0]$ is a lower semicontinuous functional with respect to the topology of the space $C_{weak}([0, T]; L^2(\Omega; R^2)).$ Consequently, by virtue of Baire’s category argument, the set of points of continuity of $I$ in $\mathcal{X}_0$ has infinite cardinality. Our ultimate goal will be to show that

$$I[v] = 0 \quad \text{whenever } v \in \mathcal{X}_0 \text{ is a point of continuity of } I \text{ in } \mathcal{X}_0. \quad (2.26)$$

In view of (2.9), (2.11), (2.14), (2.19), and (2.20), it is easy to check that $v$ represents a weak solution of the problem (2.4–2.6), which completes the proof of Theorem 2.1.

To see (2.26), arguing by contradiction, we assume that $v \in \mathcal{X}_0$ is a point of continuity of $I$ such that

$$I[v] < 0.$$ 

Since $I$ is continuous at $v$, there exists a sequence $\{v_m\}_{m=1}^\infty \subset X_0$ (and the associated fluxes $\mathbb{F}_m$) such that

$$v_m \to v \text{ in } C_{weak}([0, T]; L^2(\Omega; R^2)), \quad I[v_m] \to I[v] \text{ as } m \to \infty.$$ 

As $[v_m, \mathbb{F}_m]$ are subsolutions, we get

$$\lambda_{\text{max}} \left[ \frac{(v_m + V[v_m] + \nabla_x \Psi) \otimes (v_m + V[v_m] + \nabla_x \Psi)}{h} - \mathbb{F}_m - M[v_m] \right] < E - \delta_m \quad \text{for some } \delta_m \searrow 0.$$ 

Now, fixing $m$ for a while, we apply Lemma 2.1 with

$$N = 2, \quad U = (0, T) \times \Omega, \quad r = h, \quad g = v_m + V[v_m] + \nabla_x \Psi,$$

$$\mathbb{W} = \mathbb{F}_m + M[v_m], \quad \text{and} \quad e = E - \delta_m/2.$$ 

Denoting $\{[w_{m,n}, G_{m,n}]\}_{n=1}^\infty$, the quantities resulting from the conclusion of Lemma 2.1, we consider

$$v_{m,n} = v_m + w_{m,n}, \quad \mathbb{F}_{m,n} = \mathbb{F}_m + G_{m,n}.$$ 

Obviously,

$$\partial_t v_{m,n} + \text{div}_x \mathbb{F}_{m,n} = 0, \quad \text{div}_x v_{m,n} = 0, \quad v_{m,n}(0, \cdot) = v_0,$$

and in accordance with Lemma 2.1,

$$\lambda_{\text{max}} \left[ \frac{(v_{m,n} + V[v_m] + \nabla_x \Psi) \otimes (v_{m,n} + V[v_m] + \nabla_x \Psi)}{h} - \mathbb{F}_{m,n} - M[v_m] \right] < E - \delta_m/2.$$ 

Consequently, in view of the continuity properties of the operators $v \mapsto V[v], \quad v \mapsto M[v]$, we may conclude that for each $m$ there exists $n = n(m)$ such that

$$[v_{m,n(m)}, \mathbb{F}_{m,n(m)}] \in X_0, \quad m = 1, 2, \ldots$$
Moreover, in view of (2.25), we may suppose
\[ v_{m,n(m)} \to v \quad \text{in } C_{\text{weak}}([0, T]; L^2(\Omega; R^2)) \text{ as } m \to \infty, \]
in particular,
\[ I[v_{m,n}] \to I[v] \quad \text{as } m \to \infty. \tag{2.27} \]

Since for each \( m \)
\[
\lim_{n \to \infty} \int_0^T \int_\Omega \left( \frac{1}{2} \left| v_m + w_{m,n} + \nabla_x \Psi \right|^2 - E \right) \text{dx dt} = 0,
\]
thus using once more Lemma 2.1 combined with Jensen's inequality, we observe that the sequence \( v_{m,n(m)} \) can be taken in such a way that
\[
\liminf_{m \to \infty} I[v_{m,n(m)}] \\
= \liminf_{m \to \infty} \int_0^T \int_\Omega \left( \frac{1}{2} \left| v_m + w_{m,n(m)} + \nabla_x \Psi \right|^2 - E \right) \text{dx dt} \\
= \liminf_{m \to \infty} \int_0^T \int_\Omega \left( \frac{1}{2} \sum \left| w_{m,n(m)} \right|^2 \right) \text{dx dt} \\
\geq I[v] + C_1 \liminf_{m \to \infty} \int_0^T \int_\Omega \left( E - \delta_m - \frac{1}{2} \left| v_m + \nabla_x \Psi \right|^2 \right) \text{dx dt} \\
\geq I[v] + C_2(T, |\Omega|) \liminf_{m \to \infty} \left( \int_0^T \int_\Omega \left( E - \delta_m - \frac{1}{2} \left| v_m + \nabla_x \Psi \right|^2 \right) \text{dx dt} \right)^2 \\
= I[v] + C_2(T, |\Omega|) (I[v])^2, \quad C_2(T, |\Omega|) > 0,
\]
which is compatible with (2.27) only if \( I[v] = 0 \).

Thus, we have shown (2.26) and consequently, Theorem 2.1.

3. Dissipative solutions

The solutions “constructed” in the proof of Theorem 2.1 satisfy (2.9), more specifically,
\[
\frac{1}{2} h |u|^2 = E = \Lambda - ah^2 - \partial_t \Psi \quad \text{for a.a. } (t, x) \in (0, T) \times \Omega.
\]

In particular, as \( \Lambda \) has been chosen large, the total energy \( E_{\text{tot}} \) of the flow,
\[
E_{\text{tot}}(t) = \int_\Omega \left[ \frac{1}{2} h |u|^2 + ah^2 \right] (t, \cdot) \text{dx}
\]
may (and does in “most” cases) experience a jump at the initial time,
\[
\liminf_{t \to 0^+} E_{\text{tot}}(t) > \int_\Omega \left[ \frac{1}{2} h_0 |u_0|^2 + ah_0^2 \right] \text{dx}. \tag{3.1}
\]
Apparently, solutions satisfying (3.1) are “nonphysical” violating the first law of thermodynamics, at least if the forces $f$ are regular. This observation leads to a natural admissibility criterion based on the energy balance appended to the definition of weak solutions to eliminate the oscillatory solutions constructed in Theorem 2.1.

### 3.1. Energy inequality and dissipative solutions

For the sake of simplicity, suppose that $a > 0$ is a positive constant independent of $x$. Taking, formally, the scalar product of Equation (1.2) with $u$ and integrating the resulting expression over $\Omega \times (0, \tau)$, we obtain the energy inequality

$$E_{\text{tot}}(\tau) = \int_\Omega \left[ \frac{1}{2} h |u|^2 + ah^2 \right] (\tau, \cdot) \, dx + \int_0^\tau \int_\Omega h y B_u \cdot u \, dx \, dt \leq \int_\Omega \left[ \frac{1}{2} h_0 |u_0|^2 + ah_0^2 \right] \, dx$$

where the function $B$ was introduced in (1.7).

We say that $[h, u, B_u]$ is a dissipative weak solution to the Savage–Hutter system if, in addition to (1.5–1.7), the energy inequality (3.2) holds for a.a. $\tau \in (0, T)$.

### 3.2. Relative energy and weak–strong uniqueness

Our goal is to show that a dissipative and a strong solution emanating from the same initial data coincide as long as the latter exists. To this end, we revoke the method proposed by Dafermos [6] and later elaborated in [11], based on the concept of relative energy. We introduce the relative energy functional

$$\mathcal{E} \left( h, u \mid H, U \right) = \int_\Omega \left[ \frac{1}{2} h |u - U|^2 + P(h) - P'(H)(h - H) - P(H) \right] \, dx,$$

where

$$P(h) = ah^2.$$

Now, exactly as in Feireisl et al. [11, Section 3], we may derive the relative energy inequality

$$\mathcal{E} \left( h, u \mid H, U \right) (\tau) + \int_0^\tau \int_\Omega h y B_u \cdot (u - U) \, dx \, dt \leq \mathcal{E} \left( h_0, u_0 \mid H(0, \cdot), U(0, \cdot) \right)$$

$$+ \int_0^\tau \int_\Omega \left( \partial_t U + u \nabla_x U \right) \cdot (u - U) \, dx \, dt + \int_0^\tau \int_\Omega hf \cdot (u - U) \, dx \, dt$$

$$+ \int_0^\tau \int_\Omega \left( (H - h) \partial_t P'(H) + \nabla_x P'(H) \cdot (HU - hu) \right) \, dx \, dt$$

$$- \int_0^\tau \int_\Omega \nabla_x U \left( ah^2 - aH^2 \right) \, dx,$$

where $[h, u, B_u]$ is a dissipative weak solution of the Savage—Hutter system and $[H, U, B_U]$ are sufficiently smooth functions in $[0, T] \times \Omega$, $H > 0$.

We are ready to prove the following result:

**Theorem 3.1.** Let $[h, u, B_u]$ be a dissipative weak solution of the Savage–Hutter system in $(0, T) \times \Omega$ in the sense specified through (1.5–1.7), (3.2). Let $[H, U, B_U]$, $H > 0$ be a globally Lipschitz (strong) solution of the same problem, with

$$h_0 = H(0, \cdot), \ u_0 = U(0, \cdot).$$
Then

\[ h = H, \ u = U \quad \text{a.e. in} \ (0, T) \times \Omega. \]

As \( h, u \) are uniquely determined, then from the balance of momentum one can recover \( B_u \) such that \( B_u = B_U \) for almost all \((t, x) \in (0, T) \times \Omega\).

**Proof.** A simple density arguments shows that \([H, U]\) may be used as test functions in the relative energy inequality (3.4). As the initial values coincide, the latter reads

\[
E(h, u \mid H, U)(\tau) + \int_0^\tau \int_\Omega h \gamma (B_u \cdot (u - U)) \, dx \, dt
\leq \int_0^\tau \int_\Omega h \left( \partial_t U + u \nabla_x U \right) \cdot (U - u) \, dx \, dt + \int_0^\tau \int_\Omega hf \cdot (u - U) \, dx \, dt
+ 2a \int_0^\tau \int_\Omega ((H - h) \partial_t H + \nabla_x H \cdot (HU - hu)) \, dx \, dt
- \int_0^\tau \int_\Omega \text{div}_x U \left( ah^2 - aH^2 \right) \, dx,
\]

where, furthermore,

\[
h \left( \partial_t U + u \cdot \nabla_x U \right) \cdot (U - u)
= h \left( \partial_t U + U \cdot \nabla_x U \right) \cdot (U - u) + h(u - U) \cdot \nabla_x U \cdot (U - u).
\]

As \( U \) is globally Lipschitz, the second term in (3.6) may be “absorbed” by the left-hand side of (3.5) via Gronwall’s argument. Furthermore,

\[
h \left( \partial_t U + U \cdot \nabla_x U \right) \cdot (U - u)
= -h \gamma B_U \cdot (U - u) - hf \cdot (u - U) - \frac{h}{H} \nabla_x (aH^2) \cdot (U - u).
\]

Consequently, (3.5) reduces to

\[
E(h, u \mid H, U)(\tau) + \int_0^\tau \int_\Omega h \gamma (B_u - B_U) \cdot (u - U) \, dx \, dt
\leq c \int_0^\tau E(h, u \mid H, U)(t) dt
+ 2a \int_0^\tau \int_\Omega ((H - h) \partial_t H + \nabla_x H \cdot (HU - hu)) \, dx \, dt
- \int_0^\tau \int_\Omega \text{div}_x U \left( ah^2 - aH^2 \right) \, dx.
\]

Finally, write

\[
\text{div}_x U \left( ah^2 - aH^2 \right) = \text{div}_x U \left( ah^2 - 2aH(h - H) - aH^2 \right) + 2a \text{div}_x U H(h - H),
\]
where, similarly to the above, the first term can be handled by Gronwall’s argument. Seeing the \([H, U]\) satisfies the equation of continuity (1.1) we conclude that
\[
\mathcal{E} \left( h, u \middle| H, U \right) (\tau) + \int_0^\tau \int_\Omega h \gamma (B_u - B_U) \cdot (u - U) \, dx \, dt \leq c \int_0^\tau \mathcal{E} \left( h, u \middle| H, U \right) (t) \, dt;
\]
whence, by Gronwall’s lemma, \(\mathcal{E} \left( h, u \middle| H, U \right) (\tau) = 0\) for a.a. \(\tau \in (0, T)\).

4. Concluding remarks

In view of Theorem 3.1, we are tempted to say that imposing the energy inequality (3.2) eliminates the nonphysical oscillatory solutions, the existence of which is claimed in Theorem 2.1. However, the method of convex integration may be used to obtain the following result:

**Theorem 4.1.** Under the hypotheses of Theorem 2.1, let \(T > 0\), and
\[
h_0 \in C^2(\Omega), \quad h_0 > 0,
\]
be given. Then there exists
\[
u_0 \in L^\infty(\Omega; \mathbb{R}^2)
\]
such that the Savage–Hutter system admits infinitely many dissipative weak solutions in \((0, T)\) starting from the initial data \([h_0; u_0]\).

**Sketch of the proof:**

Going back to (2.2), we make the following ansatz for the initial velocity \(v_0\):
\[
h_0 \tilde{u}_0 = v_0, \quad \text{with } \int_\Omega v_0 \, dx = 0, \quad \text{div}_x v_0 = 0.
\]

Accordingly, the potential \(\Psi_0\) vanishes identically and we may take in (2.3)
\[
h(t, \cdot) = h_0, \quad \Psi(t, \cdot) = \Psi_0 \quad \text{for all } t \in [0, T].
\]

For such a choice of the initial data, Theorem 2.1 yields the existence of (infinitely many) weak solutions \([h, u, B_u]\) to the system. Unfortunately, as observed earlier, these solutions typically violate the energy inequality at the initial time \(t = 0\). Up till now it was sufficient to require that \(\Lambda(t) \geq \Lambda_0\) for a sufficiently large \(\Lambda_0\) to provide in the construction that the set of subsolutions is nonempty. However, to ensure that solutions for \(t > 0\) are dissipative we need to specify the condition for \(\Lambda(t)\). For that reason, we rewrite the energy inequality (3.2) in terms of \(\Lambda\). We replace the term \(\int_0^\tau \int_\Omega h f \cdot u \, dx \, dt\) by the corresponding term that would exhibit maximal rate of dissipation, namely \(-\int_0^\tau \int_\Omega h |f| \cdot |u| \, dx \, dt\) and obtain

\[
\int_\Omega \left[ \Lambda(t) - \partial_t \Psi \right] (\tau, \cdot) \, dx + \int_0^\tau \int_\Omega \gamma \sqrt{h(\Lambda(t) - ah^2 - \partial_t \Psi)} \, dx \, dt \leq \int_\Omega \left[ \Lambda(0) \right] \, dx - \int_0^\tau \int_\Omega |f| \sqrt{h(\Lambda(t) - ah^2 - \partial_t \Psi)} \, dx \, dt.
\]

(4.1)

As the integral of \(\Psi\) vanishes, then the inequality simplifies as follows

\[
\Lambda(t) - \Lambda(0) \leq -\int_0^\tau \int_\Omega (\gamma + |f|) \sqrt{h(\Lambda(t) - ah^2 - \partial_t \Psi)} \, dx \, dt
\]

(4.2)
or equivalently to a differential inequality
\[
\frac{d}{dt} \Lambda(t) \leq - \int_{\Omega} (\gamma + |f|) \sqrt{h(\Lambda(t) - ah^2 - \partial_t \Psi)} \, dx
\] (4.3)
which we equip with the terminal condition \( \Lambda(T) = \Lambda_0 \).

On the other hand, however, it can be shown that the initial data belonging to the trajectory \( \tau \mapsto [h(\tau, \cdot); u(\tau, \cdot)] \) give rise to subsolutions, in the sense specified in Section 2.4.1, that do satisfy the energy inequality up to the initial time \( \tau \). Indeed we may use the abstract result proved in [12, Theorem 6.1] to conclude that there exists a sequence of times \( \tau_n \to 0 \) such that the problem with the initial data
\[
[h(\tau_n, \cdot), u_0 = u(\tau_n, \cdot)]
\]
adopts a nonempty set of subsolutions satisfying the energy inequality including the initial time \( \tau_n \). In view of Theorem 2.1, these initial data give rise to infinitely many dissipative weak solutions. Moreover, as \( h \) was taken independent of \( t \), we have \( h(\tau_n, \cdot) = h_0 \).

Finally, we would like to mention that the same results hold in case of different choice of friction term. According to Hutter et al. [18], the Savage—Hutter model is well valid for sand avalanches, however for the case of snow avalanches there is often considered a second velocity-dependent contribution, e.g., \( h|u|u \). In a consequence, in terms of new variables, the friction term reads for some coefficients \( \gamma_1, \gamma_2 \geq 0 \) as follows
\[- \left( \gamma_1 \left( \frac{h}{2E} \right)^{1/2} + \gamma_2 \left( \frac{2E}{h} \right)^{1/2} \right) (v + V + \nabla_x \Psi).\]

**Funding**

The research of E.F. leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Program (FP7/2007-2013)/ ERC Grant Agreement 320078. P.G. is a coordinator of the International PhD Projects Program of the Foundation for Polish Science operated within the Innovative Economy Operational Program 2007–2013 (PhD Program: Mathematical Methods in Natural Sciences). The research of A.Ś.-G. has received funding from the National Science Centre, DEC-2012/05/E/ST1/02218.DEC-2012/05/E/ST1/02218

**References**

[1] Bouchut, F., Mangeney-Castelnau, A., Perthame, B., Vilotte, J.-P. (2003). A new model of Saint Venant and Savage–Hutter type for gravity driven shallow water flows. *C. R. Math. Acad. Sci. Paris* 336:531–536.
[2] Bouchut, F., Westdickenberg, M. (2004). Gravity driven shallow water models for arbitrary topography. *Commun. Math. Sci.* 2:359–389.
[3] Chiodaroli, E. (2014). A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Eqs.* 11:493–519.
[4] Chiodaroli, E., Feireisl, E., Kreml, O. (2015). On the weak solutions to the equations of a compressible heat conducting gas. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32:225–243.
[5] Chiodaroli, E., Kreml, O. (2014). On the energy dissipation rate of solutions to the compressible isentropic Euler system. *Arch. Rational Mech. Anal.* 214:1019–1049.
[6] Dafermos, C. M. (1979). The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.* 70:167–179.
[7] De Lellis, C., Székelyhidi, L., Jr. (2010). On admissibility criteria for weak solutions of the Euler equations. *Arch. Rational Mech. Anal.* 195:225–260.
[8] Desvillettes, L., Villani, C. (2005). On the trend to global equilibrium for spatially inhomogeneous kinetic systems: The Boltzmann equation. *Invent. Math.* 159:245–316.

[9] DiPerna, R. J., Majda, A. J. (1987). Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Commun. Math. Phys.* 108:667–689.

[10] Donatelli, D., Feireisl, E., Marcati, P. (2015). Well/ill posedness for the Euler-Korteweg-Poisson system and related problems. *Commun. Partial Differ. Equations* 40:1314–1335.

[11] Feireisl, E., Jin, B. J., Novotný, A. (2012). Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system. *J. Math. Fluid Mech.* 14:712–730.

[12] Feireisl, E. (2015). Weak solutions to problems involving inviscid fluids. IM Preprint No 2-2015.

[13] Fernández-Nieto, E. D., Bouchut, F., Bresch, D., Castro Daz, M. J., Mangeney, A. (2008). A new Savage–Hutter type model for submarine avalanches and generated tsunami. *J. Comput. Phys.* 227:7720–7754.

[14] Gray, J. M. N. T., Cui, X. (2007). Weak, strong and detached oblique shocks in gravity-driven granular free-surface flows. *J. Fluid Mech.* 579:113–136.

[15] Gray, J. M. N. T., Tai, Y.-C., Noelle, S. (2003). Shock waves, dead zones and particle-free regions in rapid granular free-surface flows. *J. Fluid Mech.* 491:161–181.

[16] Gwiazda, P. (2002). An existence result for a model of granular material with non-constant density. *Asymptot. Anal.* 30:43–60.

[17] Gwiazda, P. (2005). On measure-valued solutions to a two-dimensional gravity-driven avalanche flow model. *Math. Methods Appl. Sci.* 28:2201–2223.

[18] Hutter, K., Wang, Y., Pudasaini, S. P. (2005). The Savage–Hutter avalanche model: How far can it be pushed? *Philos. Trans. R. Soc. London Ser. A Math. Phys. Eng. Sci.* 363:1507–1528.

[19] Juez, C., Murillo, J., Garca-Navarro, P. (2013). 2D simulation of granular flow over irregular steep slopes using global and local coordinates. *J. Comput. Phys.* 255:166–204.

[20] Pelanti, M., Bouchut, F., Mangeney, A. (2008). A Roe-type scheme for two-phase shallow granular flows over variable topography. *M2AN Math. Modell. Numer. Anal.* 42:851–885.

[21] Scheffer, V. (1993). An inviscid flow with compact support in space-time. *J. Geom. Anal.* 3:343–401.

[22] Shnirelman, A. (1997). On the nonuniqueness of weak solution of the Euler equation. *Commun. Pure Appl. Math.* 50:1261–1286.

[23] Wiedemann, E. (2011). Existence of weak solutions for the incompressible Euler equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28:727–730.

[24] Zahibo, N., Pelinovsky, E., Talipova, T., Nikolkin, I. (2010). Savage–Hutter model for avalanche dynamics in inclined channels: Analytical solutions. *J. Geophys. Res.* 115:B3402, 1–18.