Quantum-Enhanced Doppler Radar/Lidar

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We propose a quantum-enhanced protocol to estimate the radial velocity $v$ of a moving target, using a frequency-entangled squeezed state composed of a signal and an idler beam as a probe state. The signal beam illuminates the moving object and it is reflected with its frequency shifted due to the Doppler effect. Then, a joint measurement between signal and idler beams is performed to estimate the velocity of the object. We aim at benchmarking this protocol against the classical one, which comprises a coherent state with the same energy illuminating the object. Indeed, employing squeezing and frequency entanglement as quantum resources provides a precision enhancement in the estimation of the velocity of the object. We identify three distinct parameter regimes. First, the frequency entanglement-dominant regime, where the advantage is proportional to the degree of frequency entanglement and mostly insensitive to the photon number. Second, the squeezing-dominant regime, with a quantum advantage that is higher than the standard quantum limit. Third, the mixed regime, where both squeezing and frequency entanglement are comparable and the proposed quantum protocol attains the Heisenberg limit. We show that an optimal measurement to achieve these results is frequency-resolved photon-number counting. Losses in the signal beam are considered for the high frequency entanglement regime. The protocol shows resilience, outperforming the classical protocol for all channel transmissivities given a large enough frequency entanglement.

I. INTRODUCTION

Quantum metrology exploits quantum mechanical resources, such as entanglement and squeezing, to measure a physical parameter with higher resolution than any strategy with classical resources. Many quantum metrology protocols in the photonic regime [1] have been proposed such as quantum illumination (QI) [2–5], quantum enhanced position and velocity estimation [6–12], quantum phase estimation [13–14], transmission parameter estimation [15–19], noise estimation [20], and estimation of separation between objects [21–22], among others. In these protocols, information about an object is retrieved by interrogating it with a signal beam. In the most general strategy, this signal is correlated or entangled with an idler beam, which is retained in the lab to perform a joint measurement at the end of the protocol. Indeed, the scheme can be seen as an interferometer setup, in which a channel depending on the parameter of interest is only applied to the signal mode.

Of particular interest for remote sensing applications is the QI protocol, where the aim is to detect the presence of a weakly reflecting target with an error probability smaller than using the best classical strategy. Here, a quantum advantage in the error probability exponent can be achieved by using a global measurement (up to 6 dB) [23–27], or by using local measurements (up to 3 dB) [28–30]. This advantage is achieved in a very noisy environment, such as the case of room-temperature microwave band, by a large bandwidth two-mode squeezed-vacuum state [3]. This requires a signal with a very low photon number per mode, which in the microwave regime is challenging to transmit open-air. Since amplifying the signal has been shown to break the quantum advantage [31–32], QI as originally thought remains an elusive achievement so far.

Once the presence of a target is established, properties like its location and velocity are also of interest. These can be estimated via signal arrival time and frequency measurement making use of the Doppler effect. Giovanetti et al. showed in [6], that the so-called GLM states defined in the frequency domain can attain the Heisenberg limit (HL), which is a $1/N$ scaling of the estimation error of the arrival time, where $N$ is the total number of photons. Equivalently, GLM states defined in the time domain reach the HL for the estimation error of frequency. This constitutes a quadratic improvement compared with the standard quantum limit (SQL) achieved by the classical protocol. In [11], the simultaneous estimation of location and radial velocity was considered using two GLM states in the frequency and time domain, respectively, that are transformed into two entangled signal and idler beams via a beam splitter. It was shown that the velocity and the location can simultaneously be estimated achieving the Heisenberg limit. This proves...
that frequency entanglement lifts the Arthurs-Kelly relation \[33\], which states that the location and velocity of an object can not be estimated with arbitrary precision using unentangled light. The work \[12\] further extended this by addressing the simultaneous estimation of relative location and velocity of two targets by means of two-photon entangled states. The main drawbacks of these previous works are the use of two-photon states, which does not allow for a photon-number-dependent analysis, and the use of the multiphoton GLM states, which are non-normalizable and thus not physical. In \[34\], a normalized version of the GLM state was introduced for range detection. Here, it was shown that the Heisenberg scaling persists for the normalized version. However, these GLM-type states are fragile in lossy channels. The loss of a single photon renders the state useless for retrieving information about the parameter. Although the robustness against losses of these GLM-type states may be improved by reducing their entanglement, this comes at the cost of decreasing the enhancement in the scaling of the error estimation. Furthermore, it is challenging to produce GLM states in the laboratory for photon numbers \( N > 2 \) \[8\].

In this article, we propose a protocol for a quantum Doppler radar/lidar, which estimates the radial velocity of a perfectly reflecting object. As a probe state, frequency-entangled twin-beams are used. The signal beam is sent against the moving object, which causes a frequency shift due to the Doppler effect. Finally, a joint measurement with the returned signal is performed. We propose as a probe state for the protocol a multimode generalization of the two-mode squeezed vacuum state, which is one of the most commonly produced states in quantum optics. The state is composed of photon pairs that share frequency entanglement. This photon-pair structure is resilient against losses, since the loss of a single photon only affects its partner, but not the other photon pairs. This is a crucial difference with GLM states, where the loss of a single photon means the loss of all the information about the parameter of interest due to the global entanglement. The quantum protocol is benchmarked against a classical protocol shining the object with the same amount of energy to make the comparison fair. Quantum Fisher information (QFI) is the main figure of merit for this comparison, although it gives the maximal amount of extractable information about the parameter of interest. Calculating the QFI for this multimode state is challenging, but by using properties of Gaussian states and introducing Schmidt modes, which effectively discretizes the frequency-continuous problem, we derive an analytical expression for the QFI. Two quantum resources can be identified in our resource quantum state, namely, squeezing and frequency entanglement. The performance of the quantum protocol is studied as a function of the photon number in three different parameter regimes, called high-frequency entanglement, high-squeezing, and mixed regime. The latter, for which a remarkable Heisenberg scaling can be attained, is called in this manner because neither squeezing nor frequency entanglement are dominant. We propose a measurement setup that saturates the quantum Cramér-Rao bound, consequently achieving the highest estimation accuracy of the velocity. It is noteworthy that the measurement setup can be performed separately in the signal and the idler, facilitating the experimental requirements.

The paper is structured as follows. In Section \[II\], the fundamentals of quantum estimation theory are introduced. In Section \[III\] we model the moving target as a perfectly-reflective mirror boosted at a relative constant velocity. Afterwards in Sections \[IV\] and \[V\] we introduce the probe states employed in both the quantum and classical protocols. As a figure of merit to benchmark their performance, we make use of the QFI. Then, in Section \[VI\] we discuss the different parameter regimes obtained and study when quantum advantage exists and how it behaves as a function of the signal photon number. In section \[VII\] an optimal measurement attaining the ultimate precision set by the quantum Cramér Rao bound is provided. Furthermore, the protocol is studied in the presence of losses in the signal beam and a lower bound for the performance of the quantum protocol compared to the classical one is given.

II. QUANTUM ESTIMATION THEORY

The objective of quantum estimation theory is to find the ultimate precision limit for the estimation of a parameter \( \lambda \) that is encoded in a quantum system. In our scenario, the probe state \( \rho \) that is emitted by the radar/lidar acquires information about \( \lambda \) during the reflection off the moving target, which transforms the state as \( \rho \rightarrow \rho_\lambda \). The classical Fisher information (FI) \( F(\lambda) \) is a measure of the information about the parameter \( \lambda \) that can be extracted by a given measurement corresponding to the positive operator-valued measure (POVM) \( \{ \Pi_x \} \) with \( \int dx \Pi_x = I \). The FI is given by

\[
F(\lambda) = \int dx \frac{1}{p_\lambda(x)} (\partial_\lambda p_\lambda(x))^2
\]

where \( p_\lambda(x) = \text{Tr}(\Pi_x \rho_\lambda) \) is the probability of measuring \( x \) given the parameter \( \lambda \). The FI is related to the Cramér-Rao bound which sets an ultimate precision limit for the estimation of \( \lambda \) that can be achieved performing an optimal measurement. The Cramér-Rao bound is given by \[35\]

\[
\text{Var}(\lambda) \geq \frac{1}{MF(\lambda)}
\]

where \( \lambda \) is an unbiased estimator that maps the measurement data of the \( M \) experiment repetitions to an estimate of the parameter \( \lambda \). The bound can be saturated using the maximum likelihood estimator in the limit of large \( M \) \[56\]. Maximizing the FI over all POVMs \( \{ \Pi_x \} \) yields the quantum Fisher information \( J(\lambda) \geq F(\lambda) \). Eq. \[2\]
for the QFI is called the quantum Cramér-Rao bound which sets the absolute precision limit for the estimation of $\lambda$. In the case of a pure-state manifold, i.e. when $\rho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$ for any $\lambda$, the QFI is given by \[ J(\lambda) = 4 \left( (\partial_\lambda \psi_\lambda | \partial_\lambda \psi_\lambda) - |\langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle|^2 \right). \] To prove a quantum advantage, we calculate the QFIs $J_q$ and $J_c$ of both the quantum and classical strategy. A quantum advantage is achieved, if the ratio is $J_q/J_c > 1$ assuming both strategies illuminate the object with the same energy and an optimal measurement is performed. An observable corresponding to the optimal measurement is given by $O_\lambda = \lambda + L_\lambda/J(\lambda)$, where $L_\lambda$ is the symmetric logarithmic derivative (SLD), which satisfies $L_\lambda \rho_\lambda + \rho_\lambda L_\lambda = 2\partial_\lambda \rho_\lambda$. As the optimal observable generally depends on the parameter itself, a prior guess about the parameter is required to construct the measurement. The measurement can then be adaptively optimised.

III. MODEL OF THE MOVING TARGET

We model the object of which we wish to estimate its constant velocity $v$ relative to emitter as a perfect mirror in a $1+1$ dimensional spacetime. For now, we assume an absence of noise and loss. As can be seen in Fig. 1, the quantum Doppler radar/lidar emits a signal beam towards the moving object, while also emitting an idler beam which is retained in the laboratory, such that a simultaneous measurement can be performed with the returned signal. The electromagnetic field of the signal beam obeys the Klein-Gordon equation $(\partial^2_t - \partial^2_x)\phi(t, x) = 0$ with the boundary condition $\phi(x_m) = 0$, where $x_m = vt$ is the location of the mirror, effectively splitting the spacetime into two parts. We assume the laboratory to be to the right of the mirror. The general solution of the Klein-Gordon equation satisfying the boundary condition is given by

$$\phi(x, t) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi \omega}} \left( e^{-i\omega(t+x)} - e^{-i\omega(t-x)} \right) a(\omega) + h.c.,$$

where $\lambda = (1 - v)/(1 + v)$ and the speed of light is set to 1. We choose to estimate the parameter $\lambda$ instead of $v$, as it naturally arises in the Doppler effect. The estimation error of $\lambda$ is related to the one of $\lambda$ via the error propagation formula for the QFI $J(v) = (\partial_\lambda \lambda(v))^2 J(\lambda(v))$. The Fourier coefficients $a(\omega)$ and their complex conjugates get promoted by canonical quantization to annihilation and creation operators which satisfy the relations $[a(\omega), a(\tilde{\omega})] = [a(\omega), a^\dagger(\tilde{\omega})] = 0$ and $[a(\omega), a^\dagger(\tilde{\omega})] = \delta(\omega - \tilde{\omega})$. Even though the notation $\hat{a}(\omega)$ is commonly used to distinguish the operators from the coefficients, we will employ just $a(\omega)$ for the operators from this point on, since only operators are used. The idler frequency mode is referred to as $b(\omega)$ and satisfies the same commutation relations. It commutes with the signal mode as both beams are spatially separated. The first term in Eq. 4 in brackets represents the incoming wave, while the second term is the outgoing wave which is Doppler shifted $\omega \to \omega/\lambda$. Now, let us derive the Bogolioubov transformation $U_\omega$ which maps the incoming modes $a(\omega)$ to the Doppler reflected outgoing modes, denoted as $a(-\omega)$. For this, a change of integration variables is performed in the second term in Eq. 4, leading to

$$\phi(x, t) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi \omega}} \left( e^{-i\omega(t+x)} a(\omega) + e^{-i\omega(t-x)} a(-\omega) + h.c. \right),$$

with the operator $a(-\omega) = -\lambda^{1/2} a(\lambda\omega)$. Thus, the process of reflection is described by the unitary transformation $U_\omega a(\omega) U_\omega^\dagger = -\lambda^{1/2} a(\lambda\omega)$. The prefactor $\lambda^{1/2}$ ensures a proper normalization and the change of sign is the $\pi$ phase shift that radiation experiences when reflected. The vacuum state $|0\rangle$, which satisfies $a(\omega)|0\rangle = b(\omega)|0\rangle = 0$, remains unchanged after Doppler reflection, that is $U_\lambda |0\rangle = |0\rangle$. In the most general framework, the outgoing mode also picks up a phase factor depending on the velocity and location of the object. Therefore, this phase can also be used to estimate the velocity, but generally at the cost of an additional knowledge about the location. Here, we will only be interested in the information about the velocity that is encoded in the spectrum of the light beams. The QFI $J_q$ derived here is a lower bound of the QFI in which phases are also taken into account.

IV. CLASSICAL PROTOCOL

In the classical protocol we take a coherent signal as the probe state. For a continuum of frequency modes, a
coherent state is defined as $|\psi\rangle = \exp[\alpha \int d\omega f(\omega)(a(\omega) - a^\dagger(\omega))]|0\rangle$, where we take the displacement constant $\alpha$ to be a real number for the sake of simplicity. We choose the normalized spectral amplitude to be Gaussian $f(\omega) = 1/(\sqrt{\pi} \sigma_c) \exp[-(\omega - \omega_c)^2/(2\sigma_c^2)]$, where $\omega_c$ is the mean frequency and $\sigma_c/\sqrt{2}$ is the mode bandwidth, which corresponds to the standard deviation of the frequency. Furthermore, we make the narrowband approximation $\omega_c \gg \sigma_c$, which allows us to change the limits of integration to $(-\infty, \infty)$. The reflected state is given by $U\lambda|\psi\rangle = |\psi\rangle = \exp[\alpha \int d\omega \lambda^2 f(\omega)(a(\omega) - a^\dagger(\omega))]|0\rangle$, where we have used $U\lambda e^A U_\lambda^\dagger = e^{A\lambda}$, and $A$ is the exponent of the coherent state. Thus, the state is still a coherent state after the reflection but with an amplitude $f(\omega) \rightarrow -\lambda^2 f(\lambda \omega)$. The mean frequency and the frequency variance are shifted due to the Doppler effect and given by $\omega_c/\lambda$ and $\sigma_c^2/2\lambda^2$. Therefore, estimating the frequency and the variance provides information about the parameter $\lambda$. The calculation of the QFI is straightforward, but we need to compute $[\partial_\lambda \psi]\lambda$. As $[\partial_\lambda \lambda \lambda, \lambda_\lambda] = 0$, we can write $|\psi\rangle = e^{\lambda\lambda} \lambda \lambda |0\rangle$. As a consequence, it follows that $\langle \psi|\partial_\lambda \psi\rangle = 0$ and $\langle \psi|\partial_\lambda \psi\lambda\rangle = \{\partial_\lambda \lambda \lambda, \lambda_\lambda\}|0\rangle$. This leads to the expression for the QFI:

$$J_c(\lambda) = \frac{4\alpha^2}{\lambda^2} \int d\omega \left( \frac{1}{2} f(\omega) + \omega \partial_\omega f(\omega) \right)^2 = \frac{2\alpha^2}{\lambda^2} \left( \frac{\sigma_c^2}{\lambda^2} + 1 \right) \approx \frac{2N_c \omega_c^2}{\lambda^2} \frac{\omega^2}{\sigma_c^2}.$$  

where $N_c = \alpha^2$ is the average number of photons, so $J_c(\lambda)$ follows the SQL scaling expected for a classical strategy. The first term in Eq. (5) in the bracket is due to the Doppler frequency shift, whereas the second term is due to the mode bandwidth shift. Therefore, the second term is negligible due to the narrow-bandwidth approximation. For a pulsed titanium-sapphire laser operating in the red to infrared wavelengths often used in experiments, a typical value for the ratio of mean frequency and bandwidth is $\omega_c/\sigma_c \sim 100$. For the detailed calculation see Appendix C1.

V. QUANTUM PROTOCOL

For the quantum protocol, we use a twin-beam multimode squeezed vacuum state. This state can be produced in the laboratory by non-linear optical processes, such as spontaneous parametric down-conversion (SPDC). In this process of SPDC, a pump beam, which is considered to be classical, interacts with a $\chi(2)$ non-linear optical medium. Photons of the pump field decay into a signal and idler photon pairs. The use of a waveguide for SPDC allows for reducing the number of spatial modes to one for each beam given by $a(\omega)$ (signal) and $b(\bar{\omega})$ (idler). The effective Hamiltonian describing the process is given by

$$H_I = \xi \int d\omega \int d\bar{\omega} f(\omega, \bar{\omega})a(\omega)b(\bar{\omega}) - h.c.,$$

where the coupling constant $\xi$, referred to as the squeezing parameter, is chosen to be real for simplicity, and proportional to the intensity of the classical pump beam and the strength of the interaction. The normalized joint spectral amplitude $f(\omega, \bar{\omega})$ depends on the specifics of the non-linear process and on the pump beam. In the case of SPDC, the joint spectral amplitude can be in many cases approximated as a double Gaussian which also simplifies analytic calculations:

$$f(\omega, \bar{\omega}) = \sqrt{\frac{2}{\pi\sigma}} \exp\left(-\frac{(\omega + \bar{\omega} - \omega_0)^2}{2\sigma^2}\right) \times \exp\left(-\frac{(\omega - \bar{\omega})^2}{2\epsilon^2}\right).$$

The first function of $\omega + \bar{\omega}$ comprises energy conservation of the photon decay process and it is inherited by the frequency mode spectrum of the pump beam, which is assumed to be Gaussian with mean frequency $\omega_0$ and variance $\sigma^2/2$. The second function of $\omega - \bar{\omega}$ corresponds to the phase matching condition, i.e. momentum conservation, and depends on the spatial properties of the pump beam and the non-linear medium. Thus, by modulating the pump beam, both functions composing $f(\omega, \bar{\omega})$ can independently be tailored. We again assume the narrow-bandwidth approximation $\omega_0 \ll \sigma$ and $\omega_0 \ll \epsilon$. The double Gaussian can be decomposed into its Schmidt modes as $f(\omega, \bar{\omega}) = \sum_n r_n \psi_n(\omega - \omega_0/2)\bar{\psi_n}(\bar{\omega} - \omega_0/2)$, where $\{\psi_n(\omega)\}$ is an orthonormal set closely related to the Hermite functions (further details in Appendix A). The relative weight $r_n^2$ of each individual mode is given by $r_n^2 = \frac{\sigma/\sigma_c}{\sqrt{\pi\sigma_c}}(\frac{\omega_0}{\sigma_c})^n$ with $\sum_n r_n^2 = 1$. The number of active modes is given by the Schmidt number $K = (\sum_n r_n^4)^{-1} = \frac{\sigma^4}{\sigma_c^4}$, which we interpret as a measure of frequency entanglement between the signal and idler photon pair. For $K = 1$, only one pair of modes is necessary to describe the state and the double Gaussian factorizes, which implies no frequency entanglement. For $K > 1$, the state is frequency entangled and the degree of entanglement grows monotonically with $K$. In Ref. 33, a technique to obtain $K > 400$ was proposed, which corresponds to an extremely high-frequency entanglement of the photon pair. The Schmidt modes capture the spectral structure of $f(\omega, \bar{\omega})$ in a discrete manner, and thus it is natural to introduce discrete annihilation and creation operators $a_n = \int d\omega \psi_n(\omega - \omega_0/2)a(\omega)$ and $b_n = \int d\omega \psi_n(\omega - \omega_0/2)b(\bar{\omega})$ which are smeared out versions of $a(\omega)$ and $b(\bar{\omega})$. The modes satisfy the commutation relations $[a_n, a_m] = [b_n, b_m] = [a_n, b_m] = 0$ and $[a_n, a_m^\dagger] = [b_n, b_m^\dagger] = \delta_{nm}$ due to the orthonormality of $\psi_n(\omega)$. The discrete description of the problem substantially facilitates the calculation of the QFI. The Hamiltonian in Eq. (6) is given in the discrete description.
As the Hamiltonians for the individual modes commute $[H_n, H_m] = 0$, the total squeezing operator $S = e^H$ of the SPDC process can be written as a product of squeezing operators for each individual mode $S = \otimes_{n=0}^{\infty} S_n$ with $S_n = e^{\xi H_n}$. The squeezing parameter of the squeezer corresponding to the $n$th mode is given by $\xi r_n$. Finally, we are able to express the probe state of the quantum protocol using discrete creation operators. Using the normal ordered representation of squeezing operators\cite{Heinosaari}, we find (see Appendix B 1 for details)

$$S(0) = \sum_{n=0}^{\infty} \frac{1}{\cosh(\xi r_n)} \exp(-\tanh(\xi r_n) a_n^\dagger b_n^\dagger) |0\rangle.$$  

(9)

Thus, the twin-beam multimode squeezed vacuum state is just the product state of independent two-mode squeezed vacuum states. Now, the reflected state $|\psi_\lambda\rangle = U_\lambda S(0)$ is

$$|\psi_\lambda\rangle = \sqrt{C} \exp\left(-\sum_{n=0}^{\infty} \tanh(\xi r_n) a_n^\dagger b_n^\dagger\right) |0\rangle,$$  

(10)

where we have transformed the product in Eq. (9) into a sum in the exponent and we have introduced the normalization constant $C = \prod_n 1/\cosh(\xi r_n)$, which is independent of $\lambda$. The operator $a_n^\dagger$ transforms into

$$a_n^\dagger \rightarrow -\int d\omega \lambda^{1/2} \psi_n(\lambda\omega - \omega/2) a^\dagger(\omega),$$

which picks up a phase shift and a $\lambda$-dependence, whereas $\xi$, $r_n$, and the idler modes $b_n$ remain $\lambda$-independent. The mean frequency of the transformed mode is given by $\omega_0/2\lambda = \omega$, as one would expect from the Doppler effect. The bandwidth of each mode is proportional to $\sqrt{\sigma}$, and it transforms into $\sqrt{\sigma}/\lambda \equiv \sigma$ after the reflection. In the continuous formalism, the joint spectral amplitude converts into $f(\omega, \omega') \rightarrow -\lambda^{1/2} f(\omega, \omega')$. Now, in order to calculate the QFI, we need to first evaluate the derivative $|\partial_\lambda |\psi_\lambda\rangle\rangle$. The only component of the state that depends on $\lambda$ is $a_n^\dagger a_n^\dagger$. The derivative can be calculated using the properties of the Hermite functions and we find that $|\partial_\lambda a_n^\dagger\rangle$ is a linear combination of creation operators $a_n^\dagger$ ranging from modes $n-2$ to $n+2$. As the derivative of the exponent in Eq. (10) commutes with the exponent itself, we find $|\partial_\lambda |\psi_\lambda\rangle\rangle = -\sum_n \tanh(\xi r_n) \langle(\partial_\lambda a_n^\dagger) b_n^\dagger S(0)$, see Appendix C 2. By using the transformation rule $S^\dagger a_n^\dagger S = a_n^\dagger \cosh(\xi r_n) - b_n^\dagger \sinh(\xi r_n)$ and the analogous rule for the idler mode, whose derivation is discussed in Appendix B 1 a), we finally find the analytic expression for the QFI (see Appendix C 2 for the full derivation). This splits up into frequency and bandwidth contributions as

$$J_\lambda = \sum_{n=0}^{\infty} \sinh^2(\xi r_n) \left(n \cosh^2(\xi r_{n-1}) \right.$$  

$$+ \left(n+1\right) \cosh^2(\xi r_{n+1}) \right)$$

and the bandwidth term as

$$Z_{\sigma} = \sum_{n=0}^{\infty} \sinh^2(\xi r_n) \left[n(n-1) \cosh^2(\xi r_{n-2}) \right.$$  

$$+ \left(n+1\right) \left(n+2\right) \cosh^2(\xi r_{n+2}) \right).$$  

(12)

The bandwidth contribution is suppressed by the factor $\sigma/\omega_0^2$, which is small due to the narrow-bandwidth approximation. For a typical SPDC process in potassium dihydrogen phosphate crystal pumped by a frequency doubled titanium-sapphire laser, this factor is approximately $\sqrt{\sigma/\omega_0^2} \sim 0.01$\cite{Heinosaari} and usually gets smaller in an experimental setup as the frequency entanglement, and consequently $K$, increases.

## VI. QUANTUM ADVANTAGE

Let us now compare the performance of the quantum and the classical protocols and find out under which conditions quantum advantage is achieved. For that, we examine the ratio $J_q/J_c$, where we have omitted the dependence on $\lambda$ for the sake of readability. In the case $J_q/J_c > 1$, the quantum strategy outperforms the classical one assuming that an optimal measurement is performed. For a fair comparison, the energy of the signal beams in both strategies are set to be equal. The energies of the signal beams of the classical and quantum radar/lidar are given by $E_c = \omega_c \sigma^2 = \omega_c N_c$ and $E_q = \omega_0 \sum_n \sin^2(\xi r_n) = \omega_0 N_q$ respectively, and $N_c$ and $N_q$ are the corresponding photon numbers. Setting $E_q = E_c = E_S$ and $N_q \equiv N_S$, which allows us to replace the displacement constant with $\alpha^2 = \omega_0/2 \sigma N_S$. The ratio of QFIs is thus given by

$$\frac{J_q}{J_c} = \frac{\omega_0 \sigma_c^2 Z_{\sigma} + \frac{\sigma_c^2}{\omega_0^2} Z_{\sigma}}{\omega_c \sigma \frac{Z_{\sigma} + \frac{\sigma}{\omega_0^2} Z_{\sigma}}{N_S}} \equiv G \left(\frac{Z_{\sigma} + \frac{\sigma}{\omega_0^2} Z_{\sigma}}{N_S} \right),$$  

(13)

where we have defined $G \equiv \omega_0 \sigma_c^2/\omega_0 \sigma \alpha$. The ratio does not depend on $\lambda$, however, it depends on the physical parameters of the classical and quantum probe states. We will now fix these parameters to make the comparison fair between the protocols. We choose $\omega_0/2 = \omega_c$. Thus, the mean frequencies of the quantum and classical signal beams are equal and consequently, their photon numbers $N_c = N_S$ as well. For the bandwidths, we choose $\sigma_c^2 = \sigma_\alpha/2$. With this choice, the bandwidth of the classical mode is equal to the minimal bandwidth of the quantum modes, which is the one with $n = 0$, while the modes with $n > 0$ have a larger bandwidth. This disadvantages
the quantum protocol in the comparison, since a larger bandwidth generally decreases the sensitivity for velocity estimation. The spectral properties of both the classical and quantum modes are fairly similar, which makes the comparison reasonably fair. For this parameter choice, that we form now on will use, we have $G = 1$. Only the bandwidth contribution in Eq. (19) depends on the physical parameters and is heavily suppressed because of $\sigma \epsilon / \omega_0^2 \ll 1$.

A. No frequency entanglement

Let us first study the case in which no frequency entanglement is present between signal and idler beams. In this case, we have $\sigma = \epsilon$ and $K = 1$. The state reduces to the well-known two-mode vacuum state $|\psi_\lambda\rangle = \exp(\xi(a_0^\dagger b_0 - a_0^\dagger b_0^\dagger)|0\rangle)$ with signal photon number $N_S = \sinh^2(\xi)$. We find that

$$\frac{J_q}{J_c} = 1$$

for all values of the squeezing parameter $\xi$ (Appendix C3). Thus, no quantum advantage is achieved, which proves that frequency entanglement $K > 1$ is necessary for quantum advantage assuming the probe states given in Eq. (9) and in the absence of losses and thermal noise. Both protocols obey the SQL $J_q, J_c \sim N_S$.

B. High frequency-entanglement regime

Let us now consider the case in which the frequency entanglement is the dominant quantum resource. We specify this regime by the condition $\xi \ll K^{1/2}$, which allows us to approximate the hyperbolic functions as $\sinh^2(\xi r_n) \approx \xi^2 r_n^2$ and $\cosh^2(\xi r_n) \approx 1$. The number of photons can be approximated as $N_S \approx \xi^2$, where we have only taken the first term into account (Appendix B1b). For the ratio of QFIs, we find

$$\frac{J_q}{J_c} \approx K + \frac{\sigma \epsilon}{\omega_0} \left( K^2 + 1 + \frac{1}{K+1} \right).$$

The first term, $K$, is the frequency contribution, whereas the remaining terms, which are suppressed by the factor $\sigma \epsilon / \omega_0^2$, correspond to the bandwidth contribution. The latter becomes dominant for $\omega_0^2 / \sigma \epsilon < K$. The ratio $J_q / J_c$ does not depend on the photon number, which again implies SQL scaling for both strategies. For $K > 1$, there is quantum advantage, since the ratio is larger than 1. Furthermore, the advantage is unbounded, which makes this protocol experimentally feasible in realistic scenarios, since the advantage can survive in the presence of losses, as we will see in Section VII.

C. High-squeezing regime

Let us now consider a regime in which squeezing is the dominant quantum resource and the frequency entanglement is relatively weak. As the absence of frequency entanglement means that every photon is contained in the 0th mode, a weak frequency entanglement corresponds to the case in which most of the photons reside in the 0th mode, while a relatively few photons are in higher modes. The fraction of photons in the $n$th mode is given by $r_n^2 = \sinh^2(\xi r_n) / \sum_k \sinh^2(\xi r_k)$, where $\sinh^2(\xi r_n)$ is the photon number of the $n$th mode of the signal beam. By increasing $\xi$ for a fixed $K$, the relative contribution of higher modes $n > 0$ decreases. The majority of the photons are in the 0th mode, i.e. $r_0^2 \gg r_1^2$, when $\xi \gg K^{3/2}$ holds. Additionally, in this case the photon number is given by $N_S \approx e^{2\xi r_0} / 4$. We further require that the hyperbolic functions can be approximated as $\sinh(\xi r_n) \approx \cosh(\xi r_n) \approx e^{\xi r_n} / 2$ for $n = 0, 1$, which holds for $K > 1$ and $\xi \gg \sqrt{1/(K-1)}$. This second condition ensures that the photon number in mode $n = 1$ is sufficiently high to allow us to differentiate this state from the non-frequency-entangled state with $K = 1$. It also allows us to put the QFI in an analytic form as a function of $N_S$ as

$$\frac{J_q}{J_c} \approx \frac{4 \sqrt{\xi^2 + 1}}{2} N_S \sqrt{\frac{\xi^2 + 1}{K+1}},$$

FIG. 2. The normalized frequency (solid lines) and bandwidth (dashed lines) contributions $8Z_{\text{PI}} / (K N_S)^{1/2}$ are plotted against the squeezing parameter $\xi$ for Schmidt numbers $K = 10$ and $K = 20$. The normalised QFI approaches 1, which indicates a scaling above the SQL in the limit $K^{3/2} \ll \xi$. The normalized bandwidth contribution is much smaller and goes to 0 for $K^{3/2} \ll \xi$. It is even further suppressed by the factor $\sigma \epsilon / \omega_0^2$. 

$\sigma$ and $\epsilon$ are given in Eq. (9) and in the absence of losses and thermal noise. Both protocols obey the SQL $J_q, J_c \sim N_S$. 

$\frac{J_q}{J_c} = 1$ (14)

for all values of the squeezing parameter $\xi$ (Appendix C3). Thus, no quantum advantage is achieved, which proves that frequency entanglement $K > 1$ is necessary for quantum advantage assuming the probe states given in Eq. (9) and in the absence of losses and thermal noise. Both protocols obey the SQL $J_q, J_c \sim N_S$. 

$\frac{J_q}{J_c} \approx K + \frac{\sigma \epsilon}{\omega_0} \left( K^2 + 1 + \frac{1}{K+1} \right)$. (15)
where it was used that only terms containing both $\xi r_0$ and $\xi r_1$ in Eq. (11) contribute significantly. This is the reason why the bandwidth terms are negligible. The quantum advantage is proportional to the photon number of the second mode with $n = 1$. In Fig. 2 both the normalized frequency (solid lines) and bandwidth (dashed lines) terms are plotted against $\xi$ for Schmidt numbers $K = 10$ and $K = 20$, confirming our analytical results. As a conclusion, increasing the squeezing for a fixed $K$ also increases the quantum advantage, so squeezing can be seen as a sensitivity-enhancing resource of the protocol. The details about the calculations performed in this Subsection can be found in the Appendix C.5.

D. The mixed regime

Now, let us study the intermediate parameter regime $K^{1/2} \ll \xi < K^{3/2}$. Using these conditions, we can derive the analytic approximation of the QFI

$$J_q \approx \frac{1}{2} + \frac{\sigma \epsilon K^{3/2}}{\omega_0^2 2^{5/2} \xi} N_S,$$

(17)

where the first term is again the frequency contribution and the second term the bandwidth contribution, following both a Heisenberg scaling $J_q \sim N_S^2$. For details about the calculations in this Subsection, see Appendix D. The bandwidth contribution becomes dominant when $\omega_0^2/\sigma \epsilon < K^{3/2}/2^{5/2} \xi$. However, we have that $\omega_0^2/\sigma \epsilon > 1$ and $K^{3/2}/\xi \gg 1$, so we cannot decide which contribution is dominant. For instance, in the experimental setup referred in Section VII we had that $\sqrt{\sigma \epsilon/\omega_0} \approx 0.01$, so the bandwidth can be safely neglected in the mixed regime, at least for values of $\xi$ and $K$ up to 100. Therefore, we will neglect the bandwidth contribution from this point on. In Fig. 3 the ratio $2\lambda^2 \sigma \epsilon J_q/N_S^2 \approx \mathcal{Z}_{\sigma}/N_S^2$ is plotted for both $\xi$ and $K$ up to the values of 100. Three distinct regions corresponding to the three parameter regimes can be appreciated. In the white area, which corresponds to a value of the ratio 1, we observe a behavior of $J_q \approx N_S^2/2$ for the QFI. The red area is the high frequency entanglement regime and the blue area is the high-squeezing regime. The parameter conditions of the three regimes

| Regime 1 | Regime 2 | Regime 3 |
|----------|----------|----------|
| $\xi < K^{1/2}$ | $\xi \gg K^{3/2}$ | $K^{1/2} \ll \xi < K^{3/2}$ |
| $J_q \approx K$ | $J_q \approx N_S^2$ | $J_q \approx N_S$ |

and their corresponding quantum advantages are summarized in Table I.

| TABLE I. Listed are the quantum advantages for the different parameter regimes. Regime 1 and 2 correspond to the high-frequency entanglement, the high-squeezing and the mixed regime respectively. Here, the contributions due to the bandwidth shift are neglected. |
|-----------------|-----------------|-----------------|
| Regime 1 | Regime 2 | Regime 3 |
| $\xi < K^{1/2}$ | $\xi \gg K^{3/2}$ | $K^{1/2} \ll \xi < K^{3/2}$ |
| $J_q \approx K$ | $J_q \approx N_S^2$ | $J_q \approx N_S$ |

and their corresponding quantum advantages are summarized in Table I.

VII. OPTIMAL MEASUREMENT AND LOSS

As we are estimating only the velocity of the object, there always exists at least one optimal measurement saturating the QFI, but it is not necessarily unique. Quantum estimation theory provides techniques to construct some of these observables, in particular the one related to the symmetric logarithmic derivative (SLD) $\partial_{\lambda} = \lambda + L_{\lambda}/J(\lambda)$. However, its implementation in a realistic experimental setup is a highly non-trivial task. In the case of a pure-state manifold, the SLD $L_{\lambda}$ can be written as $L_{\lambda} = |\partial_{\lambda} \psi(\lambda)|^2 + |\psi(\lambda)|^2 \partial_{\lambda} |\psi(\lambda)|$. Thus, only $|\partial_{\lambda} \psi(\lambda)|$ needs to be calculated, which has been done for the calculation of the QFI and it can be found in Appendix C.2. However, a construction of this observable in a lab in an optical setup is far from trivial. Furthermore, it depends on the parameter $\lambda$ itself, and we would like to have a measurement working on the whole range of velocities if possible. Otherwise, an adaptive measurement strategy could be followed [37].

Let us analyze a measurement based on frequency-resolved photon-counting of signal and idler photons, which is discussed in detail in Appendix E. This measurement corresponds to POVMs $\{ |\omega, \tilde{\omega} \rangle \langle \omega, \tilde{\omega}| \}$, where $|\omega, \tilde{\omega} \rangle = \frac{1}{\sqrt{\lambda_{nm}}} a^\dagger(\omega_1) \cdots a^\dagger(\omega_n) b^\dagger(\tilde{\omega}_1) \cdots b^\dagger(\tilde{\omega}_m) |0 \rangle$. We calculate the FI, $\bar{F}_q$, corresponding to this measurement for a generalization $|\tilde{\psi}(\lambda)\rangle$ of the probe state $|\psi(\lambda)\rangle$ given in Eq. (10). This generalized probe state contains phase factors depending on the kinetic properties of the target and a complex squeezing parameter, which were...
previously omitted in our analysis. We can show that the measurement outcomes do not depend on these phases. Indeed, both states $|\psi_\lambda\rangle$ and $|\tilde{\psi}_\lambda\rangle$ give rise to the same probability distribution of measurement outcomes. Thus, the POVMs $\{|\omega, \tilde{\omega}\rangle\langle\omega, \tilde{\omega}|\}$ are actually phase insensitive. Finally, we prove that $\tilde{F}_q = J_q$, so this measurement also saturates the QFI in Section VI. Let us remark that this measurement does not depend on the parameter $\lambda$, so it can be used for saturating the QFI for any velocity, and that it could in principle be experimentally feasible [49][50]. Finally, notice that, by using phases-sensitive measurements, the precision could be further enhanced, but this may require further information about the target, such as its location.

Let us now briefly discuss the effect of losses on the performance of the protocol for the POVMs $\{|\omega, \tilde{\omega}\rangle\langle\omega, \tilde{\omega}|\}$. We assume no losses in the idler beam. Photon loss in the signal beam can occur on the way to and from the target and/or during the interaction with the object (which generalizes the protocol to non-perfectly reflecting objects). The probability of losing a signal-photon is assumed to be frequency independent and it is modeled by a beam splitter. In this framework, the beam splitter commutes with the reflection operation, so only one beam splitter with effective reflectivity $\eta$ is required for the whole path. We derive a lower bound, $\tilde{F}_{q,ib}(\eta)$, for the FI of the lossy case, $\tilde{F}_q(\eta)$, corresponding to the frequency-resolved photon-counting measurement using the state $|\psi_\lambda\rangle$. In the following, as phases do not play any relevant role due to the phase-insensitive measurement, the tilde will be omitted. The lower bound only takes into account measurement outcomes where no photon has been lost, so the rest are discarded. This provides a pessimistic lower bound for the FI of the lossy scenario, because the remaining measurement outcomes still contain information about the parameter $\lambda$ thanks to the photon-pair structure of the state. The QFI of the classical protocol is $J_q(\eta) = (1-\eta)J_c$, so it is just decreased by the factor $(1-\eta)$. To compare the performances of both protocols in the presence of losses, we again consider the ratio of FIs, which leads to

$$\frac{F_{q,ib}(\eta)}{J_q(\eta)} \approx \frac{C}{C_\eta} K$$

where $\sqrt{C_\eta} \approx \prod_n 1/\cosh(\sqrt{1-\eta} \xi r_n)$ and $C$ is defined in Eq. (10). Quantum advantage corresponds to the condition $F_{q,ib}/J_q(\eta) > 1$. On the left plot of Fig. 4 this ratio is depicted against the loss parameter $\eta$ for $K = 10$ and $\xi = 1, 2$. For both $\xi = 1$ and $\xi = 2$, quantum advantage’s lower bound starts at around 0 when $\eta = 0$ and then it decreases with $\eta$. The lower bound of the higher photon state with $\xi = 2$ decreases more rapidly, as the probability of losing at least one photon increases, and it falls below 1 at $\eta \approx 0.4$. For the state with $\xi = 1$, it falls below 1 for a higher photon loss parameter $\eta \approx 0.8$. On the right plot of Fig. 4 we depict the ratio in Eq. (18) against the Schmidt number $K$ for squeezing parameters $\xi = 1, 2$ with a fixed loss parameter $\eta = 0.7$. The ratio increases approximately linearly with $K$, which implies that $C/C_\eta$ is only weakly dependent on $K$ in the limit $\xi \ll K^{1/2}$. Thus, for a given $\xi$ and $\eta$, there exists a sufficiently large $K$ such that $F_{q,ib}(\eta)/J_q(\eta) > 1$. This shows that quantum advantage can always be achieved by increasing the frequency entanglement $K$. For the other two regimes, the lower bound for the ratio in Eq. (18) decays to 0, which does not mean that quantum advantage is lost, but that the the pessimistic lower bound is useless in these regimes. We also expect resilience in these regimes due to the photon-pair structure of the state. For the detailed discussion, see Appendix F.

VIII. FURTHER PERSPECTIVES

Lastly, we want to emphasize that the protocol can be easily adapted to different frequency and/or bandwidth estimating protocols. Also, the target’s trajectory can be generalized to an accelerating one via a Bogoliubov transformation [51][52], but with an additional complication due to the presence of Cassimir radiation. For stationary targets, the protocol can be adapted to estimate the location, which boils down to the estimation of arrival times of the signal beam. The probe state written in the time domain has exactly the same structure as in the frequency domain, where the bandwidths of the double Gaussian change as $\sigma \to 2/\sigma$ and $\epsilon \to 2/\epsilon$. The Schmidt number remains unaltered under this transformation. Thus, the estimation of time arrival of signal photons is the same as the estimation of mean frequency of the signal photons. The quantum advantage behaves as in Table 1. Analogously, an optimal measurement is the measurement of photon arrival times.
IX. CONCLUSION

We have proposed a protocol for a quantum Doppler radar/lidar that estimates the radial velocity of a reflecting moving target using a twin beam with frequency entanglement and squeezing as quantum resources. This quantum protocol was benchmarked against a classical one by calculating the QFI for both strategies. We have identified three different parameter regimes, achieving quantum advantage in all of them. In the high-entanglement regime, the quantum advantage is equal to the degree of frequency entanglement and scales according to the SQL. In the high-squeezing regime, where the frequency entanglement becomes less relevant compared to squeezing, the quantum protocol exceeds the SQL. In the mixed regime, where both quantum resources are comparable, the quantum protocol follows the HL. We have found that frequency-resolved photon counting of signal and idler beam saturates the quantum Cramér-Rao bound. Finally, the effect of losses on the performance of the protocols was studied by modeling them with a frequency-independent reflectivity beam splitter. A lower bound for the FI in this lossy scenario was derived, showing a robust linear scaling with the amount of frequency entanglement in the high-entanglement regime. Indeed, by increasing the frequency entanglement, quantum advantage can be achieved for all transmittivity values of the lossy channel in this parameter regime, indicating the resilience of this protocol in more realistic scenarios.

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Appendix A: Double-Gaussian distribution

A double Gaussian function

\[ f(\omega, \tilde{\omega}) = \sqrt{\frac{2}{\pi \sigma \epsilon}} \exp \left( -\frac{(\omega - \tilde{\omega} - \omega_0)^2}{2\sigma^2} \right) \exp \left( -\frac{(\omega - \tilde{\omega})^2}{2\epsilon} \right) \]  

(A1)

can be decomposed into its Schmidt modes \( f(\omega, \tilde{\omega}) = \sum_n r_n \psi_n(\omega - \omega_0/2) \psi_n(\tilde{\omega} - \omega_0/2) \). The relative weight \( r_n^2 \) of each mode is given by \( r_n = \frac{2\sqrt{\sigma \epsilon}}{\sigma + \epsilon} \left( \frac{\sigma - \epsilon}{\sigma + \epsilon} \right)^n \). The number of effective modes is given by the Schmidt number \( K = \frac{\sigma^2 + \epsilon^2}{2\sigma \epsilon} \).

We can write the coefficients completely in terms of the Schmidt number \( r_n = (\mp 1)^n \sqrt{\frac{2}{K + 1}} \sqrt{\frac{K - 1}{K + 1}} \), where the \( (\mp 1)^n \) is for the case \( \epsilon > \sigma \). However, for the quantities we calculate like the photon number and QFI, the cases \( \sigma > \epsilon \) and \( \sigma < \epsilon \) are equivalent as they only depend on \( r_n^2 \). The Schmidt modes are given by \( \psi_n(\omega - \omega_0/2) = \sqrt{s} \varphi_n(s(\omega - \omega_0/2)) \) with \( s = \sqrt{\frac{2}{\sigma \epsilon}} \) and the Hermite functions (harmonic oscillator wavefunction) \( \varphi_n(\omega) = (2^n n! \sqrt{\pi})^{-1/2} H_n(\omega) e^{-\omega^2/2} \), where \( H_n(\omega) \) are the Hermite polynomials. The Harmonic oscillator wave functions have the following properties [54]

\[ \varphi_n'(y) = -\sqrt{\frac{n + 1}{2}} \varphi_{n+1}(y) + \sqrt{\frac{n}{2}} \varphi_{n-1}(y) \]  

(A2)

\[ y \varphi_n(y) = \sqrt{\frac{n + 1}{2}} \varphi_{n+1}(y) + \sqrt{\frac{n}{2}} \varphi_{n-1}(y), \]  

(A3)

with \( \varphi_n'(y) = \frac{d}{dy} \varphi_n(y) \). The center frequency is independent of \( n \) and given by

\[ \int_{-\infty}^{\infty} d\omega \omega \psi_n^2(\omega - \omega_0/2) = \int_{-\infty}^{\infty} d\omega \omega s \varphi_n^2(s(\omega - \omega_0/2)) = \int_{-\infty}^{\infty} \frac{dy}{s} \left( \frac{y + \omega_0}{s} \right) s \varphi_n^2(y) \]  

(A4)

\[ = \int_{-\infty}^{\infty} \frac{dy}{s} \left( \frac{n + 1}{2} \varphi_{n+1}(y) \varphi_n(y) + \frac{1}{2} \sqrt{\frac{n}{2}} \varphi_{n-1}(y) \varphi_n(y) + \frac{\omega_0}{2} \varphi_n^2(y) \right) \]  

(A5)

\[ = \frac{\omega_0}{2}, \]  

(A6)
Thus, the frequency’s variance is \(\sigma^2 = \frac{\omega^2}{2} - \frac{\omega_0^2}{4}\) using the properties in Eq. \((A3)\). For the Doppler shifted mode \(\lambda^{1/2} \psi_n(\omega \lambda - \omega_0/2)\) we obtain a mean frequency of \(\omega_0/(2\lambda)\). The second moment is given by

\[
\int d\omega \omega^2 \psi_n^2(\omega - \omega_0/2) = \int d\omega \omega^2 s \varphi_n^2(\omega - \omega_0/2)
\]

\[
= \int \frac{dy}{s} \left( \frac{y + \omega_0}{2} \right)^2 s \varphi_n^2(y) = \int \frac{dy}{s^2} \left( \frac{y^2 + y\omega_0 + \omega_0^2}{4} \right) \varphi_n^2(y)
\]

\[
= \int \frac{dy}{s^2} \left( \sqrt{\frac{n+1}{2}} \varphi_{n+1}(y) + \sqrt{\frac{n}{2}} \varphi_{n-1}(y) \right)^2 + \frac{\omega_0^2}{4} = \frac{1}{s^2} \left( \frac{n+1}{2} + \frac{n}{2} \right) + \frac{\omega_0^2}{4}
\]

Thus, the frequency’s variance is \(\frac{\sigma^2}{\lambda^2}(n+1/2)\) and for the Doppler shifted mode \(\frac{\sigma^2}{2\lambda^2}(n+1/2)\).

**Appendix B: Properties of the probe states**

1. Quantum probe state

The twin-beam multimode squeezed vacuum state can be written as a tensor product of individual squeezing operators

\[
S = \bigotimes_{n=0}^\infty S_n = \bigotimes_{n=0}^\infty \exp(\xi (a_n b_n - a_n^\dagger b_n^\dagger)).
\]

Also the vacuum state, that is independent of \(\lambda\), can be written as a product of the individual vacua of each mode \(|0\rangle = \bigotimes_n |0\rangle_n\). The squeezing operator acting on the vacuum thus yields

\[
S|0\rangle = \bigotimes_n S_n \otimes |0\rangle_m = \bigotimes_n S_n |0\rangle_n = \bigotimes_n \frac{1}{\cosh(\xi r_n)} \exp \left( - \tanh(\xi r_n) a_n^\dagger b_n^\dagger \right) |0\rangle_n
\]

\[
= \sqrt{C} \exp \left( - \sum_n \tanh(\xi r_n) a_n^\dagger b_n \right) |0\rangle,
\]

where \(\sqrt{C} = \prod_n \frac{1}{\cosh(\xi r_n)}\).

a. Transformation rules

The transformation of the annihilation and creation operators due to squeezing reduces to the well-known transformation of the two-mode squeezed vacuum state due to the tensor product structure of the squeezing operator

\[
S^\dagger a_{\lambda\lambda} S = \bigotimes_r S^\dagger_r a_r \bigotimes_q S^\dagger_q = S^\dagger_{\lambda\lambda} S_{\lambda\lambda}.
\]

With this, we find the relations

\[
S^\dagger a_{\lambda\lambda} S = a_{\lambda\lambda} \cosh(\xi r_n) - b_n^\dagger \sinh(\xi r_n)
\]

\[
S^\dagger b_n S = b_n \cosh(\xi r_n) - a_n^\dagger \sinh(\xi r_n)
\]

\[
S^\dagger a_{\lambda\lambda}^\dagger S = a_{\lambda\lambda}^\dagger \cosh(\xi r_n) - b_n^\dagger \sinh(\xi r_n)
\]

\[
S^\dagger b_n^\dagger S = b_n^\dagger \cosh(\xi r_n) - a_n \sinh(\xi r_n).
\]

b. Photon number

Next, let us calculate the photon number for both beams. The photon-number operator is given as

\[
N = N_a + N_b = \int d\omega a^\dagger(\omega) a(\omega) + \int d\omega b^\dagger(\omega) b(\omega)
\]
in the continuous formalism, where \( N_a \) (\( N_b \)) is the photon-number operator of the signal (idler). Using \( a(\omega) = \sum_n \psi_n(\omega - \omega_0/2) a_n \) and \( b(\omega) = \sum_n \psi_n(\omega - \omega_0/2) b_n \), we can rewrite \( N \) in terms of discrete operators

\[
N = \int d\omega \sum_{n=0}^{\infty} \psi_n(\omega - \omega_0/2) a_n^\dagger a_n + \int d\omega \sum_{m=0}^{\infty} \psi_m(\omega - \omega_0/2) b_m^\dagger b_m = \sum_n a_n^\dagger a_n + \sum_n b_n^\dagger b_n, \tag{B9}
\]

where \( a_n^\dagger a_n \) (\( b_n^\dagger b_n \)) is the photon-number operator of the signal (idler) beam of the \( n \)th mode. With the transformation rules, the photon number of the twin-beam multimode squeezed vacuum can be calculated

\[
\langle 0 | S^\dagger U^\dagger_\Lambda NU_\Lambda S | 0 \rangle = \langle 0 | \sum_n a_{k\lambda}^\dagger \cosh(\xi r_k) - b_k \sinh(\xi r_k) \rangle \left( a_{k\lambda}^\dagger \cosh(\xi r_k) - b_k \sinh(\xi r_k) \right) | 0 \rangle
+ \langle 0 | \sum_n b_k^\dagger \cosh(\xi r_k) - a_{k\lambda} \sinh(\xi r_k) \rangle \left( b_k \cosh(\xi r_k) - a_{k\lambda} \sinh(\xi r_k) \right) | 0 \rangle
= \sum_n \sinh^2(\xi r_n) + \sum_n \sinh^2(\xi r_n) = 2 \sum_n \sinh^2(\xi r_n). \tag{B11}
\]

In both signal and idler, the photon number is equal \( N_S = N_I \). Furthermore, the number of photons is independent of \( \lambda \), thus the photon number is conserved in the case of Doppler reflection for a mirror of constant velocity. By expanding the hyperbolic sine and using the geometric series, we find

\[
N_S = \sum_{n=0}^{\infty} \sinh(\xi r_n) = \xi^2 \left( 1 + \frac{\xi^2}{3K} + \frac{2 + 4\xi^4}{45 K^2} + \cdots \right). \tag{B12}
\]

The photon number of mode \( n \) is \( \sinh^2(\xi r_n) \).

c. Energy

Now, let us write the Hamiltonian in terms of discrete modes and calculate the energy of the twin-beam multimode squeezed vacuum. The Hamiltonian in the continuous formalism is

\[
H = H_a + H_b = \int d\omega \omega a^\dagger(\omega) a(\omega) + \int d\omega \omega b^\dagger(\omega) b(\omega). \tag{B13}
\]

For the Hamiltonian operator of the signal beam we find

\[
H_a = \sum_{n,m} \int d\omega \omega \lambda^{1/2} \psi_n(\omega \lambda - \omega_0/2) \lambda^{1/2} \psi_m(\omega \lambda - \omega_0/2) a_n^\dagger a_m \tag{B14}
= \lambda \sum_{n,m} \int d\omega \omega \varphi_n(s\lambda(\omega - \omega_0/2\lambda) \varphi_m(s\lambda(\omega - \omega_0/2\lambda)) a_n^\dagger a_m \tag{B15}
= \lambda \sum_{n,m} \int dy \left( \frac{y}{s\lambda} + \frac{\omega_0}{2\lambda} \right) \varphi_n(y) \varphi_m(y) a_n^\dagger a_m \tag{B16}
= \sum_{n,m} \int dy \left( \sqrt{\frac{n+1}{2}} \varphi_{n-1}(y) + \sqrt{\frac{n}{2}} \varphi_{n+1}(y) \right) \varphi_m(y) a_n^\dagger a_m + \frac{\omega_0}{2\lambda} \sum_n a_n^\dagger a_n \tag{B17}
= \sum_n \frac{\omega_0}{2\lambda} a_n^\dagger a_n + \frac{1}{s\lambda} \sqrt{\frac{n+1}{2}} a_{n-1}^\dagger a_n + \frac{1}{s\lambda} \sqrt{\frac{n}{2}} a_{n+1}^\dagger a_n + \frac{1}{s\lambda} \sqrt{\frac{n+1}{2}} a_{n-1}^\dagger a_{n+1}. \tag{B18}
\]

An analogous calculation for the idler mode yields

\[
H_b = \sum_n \frac{\omega_0}{2\lambda} b_n^\dagger b_n + \frac{1}{s\lambda} \sqrt{\frac{n+1}{2}} b_{n-1}^\dagger b_n + \frac{1}{s\lambda} \sqrt{\frac{n}{2}} b_{n+1}^\dagger b_n + \frac{1}{s\lambda} \sqrt{\frac{n+1}{2}} b_{n-1}^\dagger b_{n+1}. \tag{B19}
\]
Thus, in this basis the Hamiltonian operator is not diagonal. However, the expectation value of the non-diagonal terms is zero and we find for the energy

$$
\langle 0 | S^\dagger U^\dagger H U S | 0 \rangle = \frac{\omega_0}{2\lambda} N_a + \frac{\omega_0}{2} N_b = \frac{\omega_0}{2} \left( \frac{1}{\lambda} + 1 \right) \sum_n \sinh^2 (\xi r_n). \quad (B20)
$$

2. Coherent probe state

The coherent state $D|0\rangle = \exp(\alpha \int d\omega f(\omega) (a(\omega) - a^\dagger(\omega)))|0\rangle$ is an eigenstate of the continues annihilation operator $a(\omega)D|0\rangle = \alpha f(\omega)D|0\rangle$. With this, and well-known properties of Gaussian integrals, the photon number and the mean energy can be easily calculated

$$
\langle 0 | D^\dagger N_a D | 0 \rangle = \alpha^2 \int d\omega f^2(\omega) = \alpha^2 \quad (B21)
$$

and

$$
\langle 0 | D^\dagger H_a D | 0 \rangle = \alpha^2 \int d\omega \omega f^2(\omega) = \alpha^2 \omega_c. \quad (B22)
$$

For the reflected state, we find for the photon number $\alpha^2$ and the mean energy $\alpha^2 \omega_c / \lambda$, as expected.

Appendix C: Calculation of QFI

We note that the derivative with respect $\lambda$ of a state of the form $e^{A_\lambda}|0\rangle$ is given by

$$
\partial_\lambda e^{A_\lambda}|0\rangle = (\partial_\lambda A_\lambda)e^{A_\lambda}|0\rangle = e^{A_\lambda} \partial_\lambda A_\lambda|0\rangle, \quad \text{if } [A_\lambda, \partial_\lambda A_\lambda] = 0.
$$

1. QFI of classical protocol

In the case of the classical probe state the exponent is given by $A_\lambda = \alpha \lambda^{1/2} \int d\omega f(\omega)(a(\omega) - a^\dagger(\omega))$. The commutator is

$$
[A_\lambda, \partial_\lambda A_\lambda] = \int d\omega d\tilde{\omega} f(\omega \lambda) \partial_\lambda (\lambda^{1/2} f(\omega \lambda)) \left[ a(\omega) - a^\dagger(\omega), a(\tilde{\omega}) - a^\dagger(\tilde{\omega}) \right] = \int d\omega d\tilde{\omega} f(\omega \lambda) \partial_\lambda (\lambda^{1/2} f(\omega \lambda)) \left( - \frac{[a(\omega), a^\dagger(\tilde{\omega})]}{\delta(\omega - \tilde{\omega})} + \frac{[a(\tilde{\omega}), a^\dagger(\omega)]}{\delta(\omega - \tilde{\omega})} \right)
$$

$$
= 0 \quad (C3)
$$

Using $D^\dagger D = I$, we find for the inner product

$$
\langle \partial_\lambda \psi_\lambda | \partial_\lambda \psi_\lambda \rangle = \langle 0 | \partial_\lambda A_\lambda^\dagger D^\dagger \partial_\lambda A_\lambda | 0 \rangle = \langle 0 | \partial_\lambda A_\lambda^\dagger \partial_\lambda A_\lambda | 0 \rangle = -\langle 0 | \partial_\lambda A_\lambda \partial_\lambda A_\lambda | 0 \rangle \quad (C4)
$$

$$
= \alpha^2 \int d\omega \omega \left( \partial_\lambda (\lambda^{1/2} f(\omega \lambda)) \right)^2 = \alpha^2 \int d\omega \left( \frac{\lambda^{-1/2}}{2} f(\omega \lambda) + \lambda^{1/2} \omega f'(\omega \lambda) \right)^2 \quad (C5)
$$

$$
= \frac{\alpha^2}{\lambda^2} \int dy \left( \frac{1}{2} f(y) + y \partial_y f(y) \right)^2 = \frac{\alpha^2}{\lambda^2} \int dy f^2(y) \left( \frac{1}{2} - \frac{y^2 - y \omega_c}{\sigma_c^2} \right)^2 \quad (C6)
$$

$$
= \frac{\alpha^2}{\lambda^2} \int dy f^2(y) \left( \frac{1}{4} + \frac{\omega_c}{\sigma_c^2} + y^2 \left( \frac{\omega_c^2}{\sigma_c^2} - \frac{1}{\sigma_c^2} \right) - y^3 \frac{2 \omega_c}{\sigma_c^4} + y^4 \frac{1}{\sigma_c^4} \right) \quad (C7)
$$

$$
= \frac{\alpha^2}{2\lambda^2} \left( 1 + \frac{\omega_c^2}{\sigma_c^2} \right) \quad (C8)
$$
2. QFI of the quantum protocol

The only component of the probe state that depends on \( \lambda \) is the operator \( a_{n \lambda}^\dagger \). So let us calculate its derivative using the properties of the Hermite functions

\[
\partial_\lambda \left( \chi_{n \lambda} \right) = \partial_\lambda \left( \chi_{n \lambda-1 \lambda} \right)
\]

\[
= \frac{1}{2} \lambda \chi_{n \lambda-1 \lambda} + \lambda \chi_{n \lambda} \frac{\partial}{\partial \lambda} \lambda
\]

\[
= \frac{1}{2} \lambda \chi_{n \lambda-1 \lambda} + \lambda \chi_{n \lambda} \frac{\partial}{\partial \lambda} \lambda
\]

Let us set \( y = \lambda s \omega - s \omega_0 / 2 \), then \( s \omega = \frac{y}{\lambda} + \frac{n \omega_0}{2\lambda} \). We find for the second term

\[
\chi'_{n}(y) \cdot (y + s \omega_0 / 2) = \left( -\frac{n + 1}{2} \chi_{n+1}(y) + \sqrt{n} \chi_{n-1}(y) \right) (y + s \omega_0 / 2)
\]

\[
= \frac{n + 1}{2} \left( \sqrt{n} \chi_{n+2}(y) + \sqrt{n} \chi_{n}(y) \right) + \frac{n}{2} \left( \sqrt{n} \chi_{n}(x) + \sqrt{n} \chi_{n-2}(y) \right)
\]

\[
+ \left( \sqrt{n} \chi_{n+1}(y) \right) s \omega_0 / 2
\]

\[
= \left( \frac{n + 1}{2} + \frac{n}{2} \right) \chi_{n}(y) + \frac{n \omega_0}{2} \sqrt{n} \chi_{n-1}(y) - \frac{n \omega_0}{2} \sqrt{n} \chi_{n+1}(y)
\]

\[
+ \sqrt{n(n-1)} \chi_{n-2}(y) - \sqrt{n(n+1)2} \chi_{n+2}(y).
\]

With this

\[
\partial_\lambda \left( \chi_{n \lambda} \right) = \frac{1}{2} \lambda \chi_{n \lambda-1 \lambda} + \lambda \chi_{n \lambda} \frac{\partial}{\partial \lambda} \lambda
\]

\[
= \frac{1}{2} \lambda \chi_{n \lambda-1 \lambda} + \lambda \chi_{n \lambda} \frac{\partial}{\partial \lambda} \lambda
\]

We thus find for the derivative of the creation operator

\[
\partial_\lambda a_{n \lambda}^\dagger = \frac{1}{\lambda} \left( \frac{n \omega_0}{2} \sqrt{n} \chi_{n+1}(y) - \frac{n \omega_0}{2} \sqrt{n} \chi_{n-1}(y) + \sqrt{n(n-1)} \chi_{n-2}(y) - \sqrt{n(n+1)2} \chi_{n+2}(y) \right)
\]

\[
= \frac{1}{\lambda} \left( \alpha \chi_{n-1 \lambda} + \beta \chi_{n+1 \lambda} + \gamma \chi_{n-2 \lambda} + \delta \chi_{n+2 \lambda} \right)
\]

with \( \alpha = \frac{\omega_0}{\lambda} \sqrt{\frac{n+1}{2}}, \beta = -\frac{\omega_0}{\lambda} \sqrt{\frac{n+1}{2}}, \gamma = \frac{\sqrt{n(n-1)}}{4}, \) and \( \delta = -\frac{\sqrt{n(n+1)}}{4}. \)

Because the reflected state in Eq. 132 can be written as \(| \psi_\lambda \rangle = \sqrt{C} e^{B_\lambda} | \psi \rangle\) and the derivative of the creation operator \( \partial_\lambda a_{n \lambda}^\dagger \) is a linear combination of creation operators and thus \([\partial_\lambda B_\lambda, B_\lambda] = 0\), the derivative is given as \(| \partial_\lambda \psi_\lambda \rangle = \sqrt{C}(\partial_\lambda B_\lambda) e^{B_\lambda} | \psi \rangle\). The transformation rules imply \( \langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle = 0 \). Then, the QFI is given by \( 4(\partial_\lambda \psi | \partial_\lambda \psi) \).
Therefore, we only need to calculate the scalar product of the derivative of the state. We start by examining

\[
|\partial_\lambda \psi\rangle = \partial_\lambda S|0\rangle = \sqrt{C}\partial_\lambda \exp \left( -\sum_n \tanh(\xi_n) a_n^\dagger b_n^\dagger \right)|0\rangle
\]

\[
= \left( -\sum_n \tanh(\xi_n)\partial_\lambda a_n^\dagger b_n^\dagger \right) \sqrt{C} \exp \left( -\sum_m \tanh(\xi_m) a_m^\dagger b_m^\dagger \right)|0\rangle
\]

\[
= -\sum_n \tanh(\xi_n)\partial_\lambda a_n^\dagger b_n^\dagger S|0\rangle = -S \sum_n \tanh(\xi_n) S^\dagger \partial_\lambda a_n^\dagger S b_n^\dagger S|0\rangle
\]

\[
= -S \sum_n \tanh(\xi_n) S^\dagger \partial_\lambda a_n^\dagger S \left( b_n^\dagger \cosh(\xi_n) - \sinh(\xi_n) a_n \right) |0\rangle
\]

\[
= -S \sum_n \sinh(\xi_n) S^\dagger \partial_\lambda a_n^\dagger b_n^\dagger |0\rangle.
\]

We thus find for the QFI

\[
J_q(\lambda) = 4(\partial_\lambda \psi \mid \partial_\lambda \psi) = \frac{4}{\lambda^2} \sum_n \sinh^2(\xi_n) \left( \alpha_n^2 \cosh^2(\xi_{n-1}) + \beta_n^2 \cosh^2(\xi_{n+1}) + \gamma_n^2 \cosh^2(\xi_{n-2}) + \delta_n^2 \cosh^2(\xi_{n+2}) \right)
\]

\[
= \frac{4}{\lambda^2} \sum_n \sinh^2(\xi_n) \left( \frac{\omega_n^2 s^2}{4} n + \frac{\omega_n^2 s^2}{4} n + 1 \right) - \frac{\omega_n^2 s^2}{4} \cosh^2(\xi_{n-1}) + \frac{n(n-1)}{4} \cosh^2(\xi_{n-2}) + \frac{(n+1)(n+2)}{4} \cosh^2(\xi_{n+2})
\]

The QFI splits into two parts, the part proportional to \( \omega_n^2 s^2 \) due to the frequency shift, and the remaining terms that are due to the bandwidth shift. This can be seen by writing the modes \( \psi_n \) as functions of the mean frequency \( \bar{\omega} = \omega_0/2\lambda \) and the variable related to the bandwidth \( \bar{\sigma} = (s\lambda)^{-1} = \sqrt{\sigma\epsilon/2\lambda^2} \). With this

\[
\partial_\lambda a_n^\dagger = \frac{\partial \bar{\omega}}{\partial \lambda} \partial_\sigma a_n^\dagger + \frac{\partial \bar{\sigma}}{\partial \lambda} \partial_\sigma a_n^\dagger,
\]

where

\[
\partial_\sigma a_n^\dagger = -\lambda \sqrt{\frac{2}{\sigma\epsilon}} \left( \sqrt{\frac{n}{2} a_{n-1\lambda}^\dagger} - \sqrt{\frac{n+1}{2} a_{n+1\lambda}^\dagger} \right)
\]

and

\[
\partial_\sigma a_n^\dagger = -\lambda \sqrt{\frac{2}{\sigma\epsilon}} \left( \sqrt{\frac{n(n-1)}{4} a_{n-2\lambda}^\dagger} - \sqrt{\frac{(n+1)(n+2)}{4} a_{n+2\lambda}^\dagger} \right).
\]

With this, the QFI splits up into a frequency and a bandwidth part

\[
J_q(\lambda) = \left( \frac{\partial \bar{\omega}}{\partial \lambda} \right)^2 J_q(\bar{\omega}) + \left( \frac{\partial \bar{\sigma}}{\partial \lambda} \right)^2 J_q(\bar{\sigma}).
\]
3. QFI for no frequency entanglement

In the case of no frequency entanglement $\sigma = \epsilon$, we have $K = 1$ and $r_n = 0$ for $n \neq 0$ and $r_0 = 1$. We thus find

$$J_q = 4(\partial_x \psi) = \frac{4}{\lambda^2} \sinh^2(\xi) \left( \frac{\omega_0^2}{4} + \frac{1}{2} \right) = \frac{1}{\lambda^2} \sinh^2(\xi) \left( \frac{\omega_0^2}{4\sigma^2} + \frac{1}{2} \right)$$

$$= \frac{\sinh^2(\xi)}{\lambda^2} \left( \frac{\omega_0^2}{\sigma^2} + 2 \right). \quad (C33)$$

As $\omega_0 \gg \sigma$, the bandwidth contribution can be neglected.

4. The high frequency-entanglement regime

In the case of $\xi \sqrt{\frac{2}{K+1}} \ll 1 \Leftrightarrow \xi \ll K^{1/2}$, we have $\sinh^2(\xi r_n) \approx \xi^2 \frac{2}{K+1} (\frac{\xi^2}{\sigma^2})^n$ and $\cos^2(\xi r_n) \approx 1$. With this, we find

$$J_q(\lambda) = 4(\partial_x \psi) = 4 \frac{\xi^2}{\lambda^2} \frac{2}{K+1} \sum_n \left( \frac{\sigma - \epsilon}{\sigma + \epsilon} \right)^{2n} \left( \frac{\omega_0^2}{4} \frac{n^2 + n + 1}{4} + \frac{n(n-1)}{4} + \frac{(n+1)(n+2)}{4} \right)$$

$$= 4 \frac{\xi^2}{\lambda^2} \frac{2}{K+1} \sum_n \left( \frac{\sigma - \epsilon}{\sigma + \epsilon} \right)^{2n} \left( \frac{\omega_0^2}{4} \frac{n^2 + n + 1}{2} \right). \quad (C35)$$

With $\sum_{n=0}^{\infty} \left( \frac{\xi^2}{\sigma^2} \right)^{2n} = \frac{K+1}{2}$, $\sum_{n=0}^{\infty} n \left( \frac{\xi^2}{\sigma^2} \right)^{2n} = \frac{K^2-1}{4}$, $\sum_{n=0}^{\infty} n^2 \left( \frac{\xi^2}{\sigma^2} \right)^{2n} = \frac{K^2-1}{4}$, we find

$$J_q(\lambda) = \frac{\xi^2}{\lambda^2} \frac{2}{K+1} \left( \frac{\omega_0^2}{\sigma} (K+1) + K^2 + K + 1 + \frac{2}{K} \right). \quad (C36)$$

In the case of $K \gg 1$, we obtain

$$J_q(\lambda) \approx \frac{\xi^2}{\lambda^2} \frac{2}{K+1} \left( \frac{\omega_0^2}{\sigma} (K+1) + K^2 + K + 1 + \frac{2}{K} \right). \quad (C37)$$

For $\frac{\omega_0^2}{\sigma^2} > K$, the frequency contribution is dominant. Conversely, for $\frac{\omega_0^2}{\sigma^2} < K$ the bandwidth contribution is dominant.

5. High-squeezing regime

We define the high-squeezing limit as the limit in which $\xi \gg 1$ and the 0th mode gives the dominant contribution, that is $r_1^2/r_0^2 \ll 1$, where $r_n^2 = \sinh^2(\xi r_n) / \sum_k \sinh^2(\xi r_k)$ is the relative weight of the $n$th mode. We find

$$\frac{r_1^2}{r_0^2} = \frac{\sinh^2 \left( \xi \sqrt{\frac{2}{K+1}} \frac{K}{K+1} \right)}{\sinh^2 \left( \xi \sqrt{\frac{2}{K+1}} \right)}. \quad (C38)$$

Let us assume, that both hyperbolic sines can be approximated as exponential functions $\sinh(x) \approx \exp(x)/2$, which is the case for $\xi \sqrt{\frac{2}{K+1}} \sqrt{\frac{K-1}{K+1}} \gg 1$. If $K \gg 1$, this condition is equivalent to $\xi / \sqrt{K-1} \gg 1$ and in the other limit $K \gg 1$ it is equivalent to $\xi \gg K^{1/2}$. The ratio of relative weights is then approximately given by

$$\frac{r_1^2}{r_0^2} \approx \frac{\exp \left( 2\xi \sqrt{\frac{2}{K+1}} \frac{K+1}{K+1} \right)}{\exp \left( 2\xi \sqrt{\frac{2}{K+1}} \right)} = \exp \left( 2\xi \sqrt{\frac{2}{K+1}} \left( \sqrt{\frac{K-1}{K+1}} \right) \right), \quad (C39)$$

which is small if the absolute value of the argument is small, that is $2\xi \sqrt{\frac{2}{K+1}} \left( 1 - \sqrt{\frac{K-1}{K+1}} \right) \gg 1$, which is equivalent to the condition $\xi \gg K^{3/2}$. In this case, the 0th mode mainly contributes and the signal photon number is thus
approximately given by

\[ N_S \approx \sum_n \sinh^2 (\xi r_n) \approx \frac{\exp \left( \frac{2\xi}{\sqrt{\frac{K-1}{K+1}}} \right)}{4}. \] (C41)

As the 0th mode yields the biggest contribution in terms of photon number, and the higher modes contain relatively few photons, we only consider the terms of the QFI due to the first few modes

\[ J_q(\lambda) = \frac{1}{\lambda^2} \left( \frac{\omega_0^2 s^2}{2} \left[ \sinh^2(\xi r_0) \cosh^2(\xi r_1) + \sinh^2(\xi r_1) \cosh^2(\xi r_0) \right] + 2 \sinh^2(\xi r_0) \cosh^2(\xi r_2) + 6 \sinh^2(\xi r_1) \cosh^2(\xi r_3) + \ldots \right). \] (C42)

The first two terms stem from the frequency contribution, the last two terms stem from the bandwidth contribution which is strongly suppressed because of \( \omega_0^2 s^2 \gg 1 \). Additionally, \( r_2^2 < r_1^2 \ll r_0^2 \), which justifies neglecting the bandwidth terms, as they do not contain summands that contain both \( r_0 \) and \( r_1 \) in the arguments of the hyperbolic functions. Only the first two frequency contributions contain \( r_0 \) and \( r_1 \) in the arguments of the hyperbolic functions, thus, these terms are the main ones contributing. We can rewrite the QFI in terms of the photon number using the fact that \( \cosh^2(\xi r_1) \) can also be approximated as an exponential function

\[ J_q(\lambda) \approx \frac{\omega_0^2}{8\lambda^2 \sigma^2} (4N_S)^{1+\frac{K-1}{K+1}}. \] (C43)

Appendix D: The mixed regime

In the mixed regime, we require \( \xi \gg K^{1/2} \) and \( \xi \ll K^{3/2} \), from which follows \( \xi \gg 1 \) and \( K \gg 1 \). To derive analytic expressions for the QFI and the photon number in this regime, we first consider sums of the form \( \sum_n \exp(xq^n) n^k \) with \( k = 0, 1 \). Let us start with \( k = 0 \).

\[
\sum_n \exp(xq^n) = \sum_n \sum_m \frac{(xq^n)^m}{m! n^n} = \sum_n \frac{x^n}{m!} \sum_m (q^n)^m n^n \quad (D1)
\]

\[
= \sum_m \frac{x^m}{m!} \frac{1}{1-q^m} \approx \sum_m \frac{x^m}{m! m(1-q)} \quad (D2)
\]

\[
= \frac{1}{x(1-q)} \sum_m \frac{x^{m+1} m!}{m!} \approx \frac{\exp(x)}{x(1-q)}.
\]

where we changed the summation order in Eq. \((D1)\), which is allowed because the summand is absolutely convergent. The approximation in Eq. \((D2)\) is valid if \( q \approx 1 \). For \( k = 1 \) we find using the same reasoning

\[
\sum_n \exp(xq^n) n = \sum_n \sum_m \frac{(xq^n)^m}{m! m^n} n = \sum_n \frac{x^n}{m!} \sum_m (q^n)^m n^n \quad (D4)
\]

\[
= \sum_m \frac{x^m}{m!} \frac{q^m}{(1-q)^2} \approx \sum_m \frac{x^m}{m! m^2(1-q)^2} \quad (D5)
\]

\[
= \frac{1}{x^2(1-q)^2} \sum_m \frac{x^{m+2} m^2!}{m!} \approx \frac{\exp(x)}{x^2(1-q)^2}. \quad (D6)
\]

For the bandwidth contribution we have a term with \( k = 2 \)

\[
\sum_n \exp(xq^n) n^2 = \sum_n \sum_m \frac{(xq^n)^m}{m! m^n} n^2 = \sum_m \frac{x^m}{m!} \sum_n (q^n)^m n^2 \quad (D7)
\]

\[
= -\sum_m \frac{x^m q^m (1+q^m)}{m! (-1+q^m)^3} - \sum_m \frac{x^m}{m! m^3 (q-1)^3} \quad (D8)
\]

\[
= -\frac{1}{x^3(q-1)^3} \sum_m \frac{x^{m+3} m^3!}{m!} \approx -\frac{2 \exp(x)}{x^3(q-1)^3}.
\]
Let us derive an analytic expression for the signal photon number $N_S = \sum_n \sinh^2(\xi \sqrt{\frac{2}{K+1}} \sqrt{\frac{K-1}{K+1}}^n)$. In this regime, we can approximate $\sinh^2(y) \approx \exp(2y)/4$ because of $\xi \gg K^{1/2}$. Furthermore, we identify $x = 2\xi \sqrt{\frac{2}{K+1}}$ and $q = \sqrt{\frac{K-1}{K+1}}$. We thus find

$$N_S \approx \frac{\exp \left(2\xi \sqrt{\frac{2}{K+1}}\right)}{4 \cdot 2\xi \sqrt{\frac{2}{K+1}} \left(1 - \sqrt{\frac{K-1}{K+1}}\right)}.$$  

(D10)

Now, let us examine the frequency contribution of the QFI $J_q$. Both hyperbolic functions can be approximated as $\sinh^2 y \approx \cosh^2 y \approx \exp(2y)/4$ because of $\xi \gg K^{1/2}$. The condition $\xi \ll K^{3/2}$ implies $\exp(2\xi r_{n\pm 1}) \approx \exp(2\xi r_n)$. With this, we find that

$$Z_\sigma = \sum_n \sinh^2(\xi r_n) \left[(n-1) \cosh^2(\xi r_{n-2}) + (n+2) \cosh^2(\xi r_{n+2})\right]$$

$$\approx \frac{1}{16} \sum_n \exp \left(4\xi \sqrt{\frac{2}{K+1}} \sqrt{\frac{K-1}{K+1}}^n\right) \left[(2n+2) \cosh^2(\xi r_{n-2}) + (2n+2) \cosh^2(\xi r_{n+2})\right]$$

$$\approx \frac{1}{2} \left[\frac{\exp \left(4\xi \sqrt{\frac{2}{K+1}}\right)}{4 \sqrt{\frac{2}{K+1}} \left(1 - \sqrt{\frac{K-1}{K+1}}\right)^2} + \frac{\exp \left(4\xi \sqrt{\frac{2}{K+1}}\right)}{4 \sqrt{\frac{2}{K+1}} \left(1 - \sqrt{\frac{K-1}{K+1}}\right)^2}\right]$$

$$\approx \frac{1}{2} \left[\frac{\exp \left(4\xi \sqrt{\frac{2}{K+1}}\right)}{4 \sqrt{\frac{2}{K+1}} \left(1 - \sqrt{\frac{K-1}{K+1}}\right)^2}\right]$$

$$\approx \frac{N_S^2}{2}.$$  

(D14)

Let us now also look at the bandwidth contribution for which we have to approximate the sum

$$Z_\sigma = \sum_n \sinh^2(\xi r_n) \left[n(n-1) \cosh^2(\xi r_{n-2}) + (n+2)(n+1) \cosh^2(\xi r_{n+2})\right]$$

$$\approx \sum_n \frac{\exp(2\xi r_n)}{4} \left[(n^2 - n) \frac{\exp(2\xi r_{n-2})}{4} + (n^2 + 3n + 2) \frac{\exp(2\xi r_{n+2})}{4}\right]$$

$$\approx \frac{1}{16} \sum_n 2n^2 \exp(4\xi r_n)$$

$$\approx \frac{K^{3/2}}{25/2\xi} N_S^2.$$  

(D18)

In the mixed regime, the prefactor is big, that is $K^{3/2}/\sqrt{5/2\xi} \gg 1$. However, the bandwidth contribution is suppressed by the factor $\sigma\epsilon/\omega_0^2$, which is in most experimental setups is small, of the order of $10^{-4}$. A direct comparison between frequency and bandwidth contribution is made in Fig. 5. The bandwidth contribution can be safely neglected in the considered parameter range.

**Appendix E: Frequency-resolved photon counting**

We want to prove that a frequency-resolved photon counting attains the accuracy given by the QFI $J_q$ that was derived in the main text. Our proof starts as follows: We first show that $F(\lambda) = 4(\partial_\lambda \psi_\lambda | \partial_\lambda \psi_\lambda \rangle) = J_q(\lambda)$, if the POVM and the state satisfy the conditions

1) The POVM elements are one dimensional projectors, $\Pi_x = |x\rangle \langle x|.$

2) The measurement does not depend on the parameter $\lambda$, $\partial_\lambda |x\rangle = 0$.

3) The amplitudes are real, $(x|\psi_\lambda\rangle = \langle \psi_\lambda|x\rangle$. 

FIG. 5. The two terms $Z_\sigma$ and $Z_\omega$ that make up the QFI are compared by considering the contours of their ratio. The bandwidth term $Z_\sigma$ is up to 100 times bigger than its counterpart $Z_\omega$ in the mixed regime in the considered parameter range. Because the frequency contribution is further suppressed by the factor $\sigma/\omega_0^2 (\sim 10^{-4}$ typically), the bandwidth contribution can be neglected in most experimental scenarios.

Using these conditions, we obtain for the classical Fisher information

$$F(\lambda) = \int dx \frac{1}{\langle x | \psi_\lambda \rangle^2} (\partial_\lambda \langle x | \psi_\lambda \rangle)^2 = \int dx \frac{1}{\langle x | \psi_\lambda \rangle^2} (2\langle x | \psi_\lambda \rangle \langle x | \partial_\lambda \psi_\lambda \rangle)^2 = 4 \int dx \langle x | \partial_\lambda \psi_\lambda \rangle^2$$

which is indeed equal to the QFI.

Now, let us look at the frequency-resolved photon counting corresponding to the POVMs $|\nu, \tilde{\nu}\rangle\langle \nu, \tilde{\nu}|$ with

$$|\nu, \tilde{\nu}\rangle = \frac{1}{m!} a_1^\dagger (\nu_1) b_1^\dagger (\tilde{\nu}_1) \cdots a_m^\dagger (\nu_m) b_m^\dagger (\tilde{\nu}_m) |0\rangle.$$  

As we are in the lossless case, the number of idler and signal photons is equal. The measured frequencies are $\nu_1, \tilde{\nu}_1, \ldots, \nu_m, \tilde{\nu}_m$. Conditions 1) and 2) are clearly satisfied. Condition 3) is also satisfied if we take the state $|\psi_\lambda\rangle$ from the main text and thus, for this particular state the measurement is optimal. In the main text, we neglected the phase factors the state acquires due to its propagation. However, when a specific measurement is considered such as $|\nu, \tilde{\nu}\rangle\langle \nu, \tilde{\nu}|$, the phase factors could play a role. This is why we introduce them here, to keep the analysis as general as possible

$$|\tilde{\psi}_\lambda\rangle = \exp \left(-|\xi| e^{i\xi} \int d\omega \int d\tilde{\omega} \lambda^{1/2} f(\lambda, \tilde{\omega}) a(\omega) e^{i\omega \phi} b(\tilde{\omega}) e^{i\tilde{\omega} \theta} + h.c. \right) |0\rangle,$$

where we now also allow for a complex squeezing parameter $\xi = |\xi| e^{i\xi}$. The phase $\phi = \varphi(\lambda, L)$ of the signal mode generally depends on the parameter $\lambda$ and the distance $L$ between the moving object and the emitter. We also note that if multiple runs of the experiment are performed, the phase generally changes due to the movement of the object. With this more general state, it is not immediately clear that 3) is satisfied, most certainly it is not. So let us now study the performance of this state for the given measurement. First we introduce the discrete operators

$$\hat{a}_{n\lambda} = -\int d\omega \lambda^{1/2} a_n(\lambda \omega - \omega_0/2) e^{i\omega \phi} a(\omega)$$
$$\hat{b}_n = e^{i\xi} \int d\tilde{\omega} a_n(\tilde{\omega} - \omega_0/2) e^{i\tilde{\omega} \theta} b(\tilde{\omega}),$$
where we have absorbed the complex phase of the squeezing parameter into $\tilde{b}_n$. With this, the state can be written as

$$|\tilde{\psi}_\lambda\rangle = \sqrt{C} \exp\left(-\sum_n \tanh(|\xi| r_n) \tilde{a}_n^\dagger \tilde{b}_n^\dagger\right)|0\rangle$$  \hspace{1cm} (E7)

where $\sqrt{C} = \prod_n 1/\cosh(|\xi| r_n)$. Let us now write this state in terms of the continuous frequencies

$$|\tilde{\psi}_\lambda\rangle = \sqrt{C} \exp\left(-\int d\omega \sum_n \tanh(|\xi| r_n) \lambda^{1/2} \psi_n(\lambda \omega - \omega_0/2) \psi_n(\tilde{\omega} - \omega_0/2) e^{-i\omega \varphi} e^{-i\tilde{\omega} \theta} e^{-i\lambda^t} a^\dagger(\omega) b^\dagger(\tilde{\omega})\right)|0\rangle$$ \hspace{1cm} (E8)

$$= \sqrt{C} \exp\left(-\int d\omega t_\lambda(\omega, \tilde{\omega}) e^{-i\omega \varphi} e^{-i\tilde{\omega} \theta} e^{-i\lambda^t} a^\dagger(\omega) b^\dagger(\tilde{\omega})\right)|0\rangle$$ \hspace{1cm} (E9)

We have written the state in this form to easily find the $2m$-photon contributions. So let us project onto the $2m$-photon state $\langle \nu, \nu | = \frac{1}{m!}(0)a(\nu_1) b(\nu_1) \cdots a(\nu_m) b(\nu_m) \langle \nu, \nu |$

$$\langle \nu, \nu | \tilde{\psi}_\lambda\rangle = \frac{\sqrt{C}(-1)^m}{(m!)^2} \int d\omega_1 \int d\omega_1 \cdots \int d\omega_m \int d\omega m t_\lambda(\omega_1, \tilde{\omega}_1) \cdots t_\lambda(\omega_m, \tilde{\omega}_m) e^{-i\varphi(\omega_1 + \cdots \omega_m)} e^{-i\theta(\tilde{\omega}_1 + \cdots \tilde{\omega}_m)} e^{-im\zeta}$$

$$(0)a(\nu_1) b(\nu_1) \cdots a(\nu_m) b(\nu_m) a^\dagger(\omega_1) b^\dagger(\tilde{\omega}_1) \cdots a^\dagger(\omega_m) b^\dagger(\tilde{\omega}_m)|0\rangle$$ \hspace{1cm} (E10)

For the expectation value of the ladder operators we find

$$\langle 0 | a(\nu_1) b(\nu_1) \cdots a(\nu_m) b(\nu_m) a^\dagger(\omega_1) b^\dagger(\tilde{\omega}_1) \cdots a^\dagger(\omega_m) b^\dagger(\tilde{\omega}_m) |0\rangle = \sum_\gamma \delta(\nu_1 - \omega_\gamma_1) \cdots \delta(\nu_m - \omega_\gamma_m)$$

$$\times \sum_\gamma \delta(\tilde{\nu}_1 - \tilde{\omega}_\gamma_1) \cdots \delta(\tilde{\nu}_m - \tilde{\omega}_\gamma_m),$$ \hspace{1cm} (E11)

where $\sum_\gamma$ means that we sum over all permutations. There are no permutations between signal and idler frequencies $\nu_n \leftrightarrow \tilde{\nu}_m$, the permutations are only of the form $\nu_n \leftrightarrow \nu_m$ and $\tilde{\nu}_n \leftrightarrow \tilde{\nu}_m$. For these kind of permutations the phases $e^{-i\varphi(\omega_1 + \cdots \omega_m)} e^{-i\theta(\tilde{\omega}_1 + \cdots \tilde{\omega}_m)}$ are invariant and thus for every summand the same. Therefore, they can be factored out and we find

$$\langle \nu, \nu | \tilde{\psi}_\lambda\rangle \sim e^{-i\varphi(\omega_1 + \cdots + \omega_m)} e^{-i\theta(\tilde{\omega}_1 + \cdots + \tilde{\omega}_m)} e^{-im\zeta} \sum_\gamma t_\lambda(\omega_1, \tilde{\omega}_\gamma_1) \cdots t_\lambda(\omega_m, \tilde{\omega}_\gamma_m).$$ \hspace{1cm} (E12)

We thus find $|\langle \nu, \nu | \tilde{\psi}_\lambda\rangle|^2 = |\langle \nu, \nu | \tilde{\psi}_\lambda\rangle|^2$. The proposed measurement is phase insensitive. Thus, both states $\tilde{\psi}_\lambda$ and $\tilde{\psi}_\lambda$ give rise to the same probability distribution of measurement outcomes. With this, it immediately follows that $F_\lambda(\lambda) = 4 \langle L_1^\lambda \psi_\lambda | L_1^\lambda \psi_\lambda \rangle = J_q(\lambda)$, where $L_1^\lambda$ is the FI of the state $|\tilde{\psi}_\lambda\rangle$ and the measurement $|\tilde{\psi}_\lambda\rangle$. Thus, the proposed measurement attains the accuracy given by the QFI $J_q$, however, it does not attain the accuracy given by the QFI $J_q = 4 \langle \partial_\lambda \tilde{\psi}_\lambda | \partial_\lambda \tilde{\psi}_\lambda |^2 \rangle$ due to its phase insensitivity.

To demonstrate the validity of the above result, we look at the special case of high frequency entanglement $K \gg 1$ and low squeezing $|\xi| \ll 1$, where we can explicitly calculate the FI. The state can be approximated as the superposition of the vacuum and a two-photon state

$$|\psi_\lambda\rangle \approx |0\rangle + |\xi| e^{-i\xi} \int d\omega \int d\tilde{\omega} \lambda^{1/2} f(\lambda \omega, \tilde{\omega}) a^\dagger(\omega) b^\dagger(\tilde{\omega}) e^{-i\lambda^t} |0\rangle.$$ \hspace{1cm} (E13)

Thus, only zero and two-photon events are relevant, and the corresponding POVM operators are $\{ |0\rangle |0\rangle, a^\dagger(\omega) b^\dagger(\tilde{\omega}) |0\rangle |0\rangle b(\tilde{\omega}) a(\omega) \}$. For the calculation of the FI, the vacuum term does not contribute, as it does not depend on the parameter. The measurement of the signal and idler frequency corresponds to the probability distribution, the joint spectrum

$$p_\lambda(\omega, \tilde{\omega}) = |\langle 0 | a(\omega) b(\tilde{\omega}) | \psi_\lambda \rangle|^2 = \lambda \xi^2 f^2(\lambda \omega, \tilde{\omega}).$$ \hspace{1cm} (E14)
Plugging this into the definition of the FI yields

$$F(\lambda) = \int d\omega \int d\tilde{\omega} \frac{1}{\lambda^2 f^2(\lambda\omega, \tilde{\omega})} \left( \partial_\lambda \left( \lambda^{1/2} f(\lambda\omega, \tilde{\omega}) \right) \right)^2$$

(E15)

$$= \xi^2 \int d\omega \int d\tilde{\omega} \frac{1}{\lambda f^2(\lambda\omega, \tilde{\omega})} \left( 2 \left( \lambda^{1/2} f(\lambda\omega, \tilde{\omega}) \right) \partial_\lambda \left( \lambda^{1/2} f(\lambda\omega, \tilde{\omega}) \right) \right)^2$$

(E16)

$$= 4\xi^2 \int d\omega \int d\tilde{\omega} \left( \partial_\lambda \left( \lambda^{1/2} f(\lambda\omega, \tilde{\omega}) \right) \right)^2$$

(E17)

$$= 4\xi^2 \int d\omega \int d\tilde{\omega} \left( \frac{1}{2\lambda^{1/2}} f(\lambda\omega, \tilde{\omega}) + \lambda^{1/2} \partial_\lambda f(\lambda\omega, \tilde{\omega}) \right)^2$$

(E18)

$$= 4\xi^2 \int d\omega \int d\tilde{\omega} f(\lambda\omega, \tilde{\omega}) \left( \frac{1}{2\lambda^{1/2}} - \lambda^{1/2} \frac{(\omega\lambda + \tilde{\omega} - \omega_0) + (\omega\lambda - \tilde{\omega})}{\sigma^2} \right)^2$$

(E19)

$$= 4\xi^2 \int d\omega \int d\tilde{\omega} f(\omega, \tilde{\omega}) \left( \frac{1}{4} - \omega \left( \frac{\omega + \tilde{\omega} - \omega_0}{\sigma^2} + \frac{\omega - \tilde{\omega}}{\epsilon^2} \right) + \omega^2 \left( \frac{\omega + \tilde{\omega} - \omega_0}{\sigma^2} + \frac{\omega - \tilde{\omega}}{\epsilon^2} \right)^2 \right).$$

(E20)

Using the substitution $u = \omega + \tilde{\omega}$, $v = \omega - \tilde{\omega}$, with $2\omega = u + v$, the Jacobi determinant $|J| = 1/2$ and the fact that $f^2(\omega(u,v), \tilde{\omega}(u,v)) = \frac{2}{\pi \sigma \epsilon} \exp\left(-\frac{(u - \omega_0)^2}{\sigma^2}\right) \exp\left(-\frac{v^2}{\epsilon^2}\right)$ we can compute the integral using well-known identities for Gaussian integrals

$$F(\lambda) = \frac{4\xi^2}{\lambda^2} \int du \int dv \frac{1}{\pi \sigma \epsilon} \exp\left(-\frac{(u - \omega_0)^2}{\sigma^2}\right) \exp\left(-\frac{v^2}{\epsilon^2}\right) \left( \frac{1}{4} - u + v \frac{(u - \omega_0) + v}{2 \cdot \frac{1}{2}} + \frac{(u + v)^2}{4} \left( \frac{u - \omega_0}{\sigma^2} + \frac{v}{\epsilon^2} \right)^2 \right)$$

(E21)

$$= \frac{4\xi^2}{\lambda^2} \left( \left( \frac{1}{4} \right) + \left( -\frac{1}{2} \right) + \left( \frac{5}{2} \cdot \frac{4K}{\sigma \epsilon} + \frac{K^2}{4} - \frac{1}{2} \cdot \frac{4}{4} \right) \right)$$

(E22)

$$= \frac{\xi^2}{\lambda^2} \left( \frac{\omega_0^2 K}{\sigma \epsilon} + K^2 + 1 \right)$$

(E23)

$$\approx \frac{\xi^2 K}{\lambda^2} \left( \frac{\omega_0^2}{\sigma \epsilon} + K \right).$$

(E24)

The FI indeed coincides with $J_\eta(\lambda)$, as expected.

**Appendix F: Photon loss**

Let us now briefly discuss photon loss for the state $|\tilde{\psi}\rangle$. We assume that the idler beam suffers no losses as it can be immediately measured after its creation due to $[U_\lambda, b_n] = 0$. The signal beam, however, can suffer losses on the way from the emitter to the target, at the target itself due to imperfect reflection, and on the way from the target to the receiver. The loss is modelled as a frequency independent beam splitter. Because the beam splitter operator commutes with the reflection operator $U_\lambda$, we can introduce an effective loss parameter $\eta$, which gives the probability of losing a photon on the whole trip. The signal mode thus transforms as

$$U_{BS} a_{n\lambda}^\dagger U_{BS}^\dagger = \sqrt{1 - \eta} a_{n\lambda}^\dagger + \sqrt{\eta} c_{n\lambda}^\dagger$$

(F1)

where $c_{n\lambda}^\dagger$ is the mode of the lost photon, which the experimenter has no access to. We omitted the tilde for the creation operators that contain phase information for readability. We denote the vacuum of the $c_{n\lambda}$ mode as $|0\rangle_c$. Applying the beam splitter transformation to the state, one obtains

$$U_{BS} |\tilde{\psi}_\lambda\rangle |0\rangle_c = \sqrt{C} \left[ 1 + \sqrt{1 - \eta} \sum_n t_n a_n^\dagger b_n^\dagger + \sqrt{\eta} \sum_n t_n b_n^\dagger c_n^\dagger + \sqrt{1 - \eta} \sum_n t_n t_m (1 - \eta) a_n^\dagger a_m^\dagger + \sqrt{\eta} \sqrt{1 - \eta} (a_m^\dagger c_n^\dagger + a_n^\dagger c_m^\dagger) + \eta c_n^\dagger e_m^\dagger \right] b_n^\dagger b_m^\dagger + \cdots |0\rangle |0\rangle_c.$$  

(F2)
where we have introduced the abbreviation \( t_n = \tanh(\xi r_n) \). Now, let us gather all the terms that did not lose photons and rewrite the previous state as

\[
U_{BS}|\tilde{\psi}_\lambda(\omega)|0\rangle_c = \sqrt{C} \exp \left( -\sqrt{1 - \eta} \sum_n t_n a_n^\dagger b_n^\dagger \right) |0\rangle_0 + \cdots = \sqrt{\frac{C}{C_\eta}} \sqrt{C_\eta} \exp \left( -\sum_n \tanh(g_n) a_n^\dagger b_n^\dagger \right) |0\rangle_0 + \cdots \tag{F3}
\]

\[
= \sqrt{\frac{C}{C_\eta}} \bigotimes_n S(g_n)|0\rangle_0 + \cdots = \sqrt{\frac{C}{C_\eta}} S_\eta|0\rangle_0 + \cdots \tag{F4}
\]

where we have introduced \( g_n = \arctanh(\sqrt{1 - \eta} \tan(\xi r_n)) \) and \( \sqrt{C_\eta} = \prod_n 1/\cosh(g_n) \). Thus, the states that have not suffered loss can be described as a product of squeezers \( \bigotimes_n S(g_n) = S_\eta \) with squeezing parameter \( g_n \). Let us define \( \rho_{\lambda,BS} = U_{BS}|\tilde{\psi}_\lambda(\omega)|0\rangle_c,|0\rangle_0 \langle \tilde{\psi}_\lambda(\omega)|U_{BS}^\dagger \). To obtain the state after loss, the inaccessible \( c_{n,\lambda} \) mode is traced out

\[
\rho_{\lambda,\eta} \equiv \text{Tr}_c(\rho_{\lambda,BS}) = \epsilon (|0\rangle_0 \rho_{\lambda,BS} |0\rangle_0 + \sum_r \epsilon (|0\rangle_0 \rho_{\lambda,BS}^r |0\rangle_0 + \cdots \tag{F5}
\]

\[
= U_{BS}|\tilde{\psi}_\lambda(\omega)|\tilde{\psi}_\lambda(\omega)|U_{BS}^\dagger + \cdots = \frac{C}{C_\eta} S_\eta|0\rangle_0 \langle S_\eta^I |0\rangle_0 + \cdots \tag{F6}
\]

Let us now derive a lower bound \( F_{\eta,lb} \) for the FI that corresponds to the measurement of frequency-resolved photon counting. It is a lower bound, because all measurements where at least one photon is lost are discarded. Thus, the only relevant contribution of the mixed state is \( \frac{C}{C_\eta} S_\eta|0\rangle_0 \langle S_\eta^I |0\rangle_0 \). By using the results from the previous Appendix, we find

\[
F_{\eta,lb}(\eta) = \frac{C}{C_\eta} \frac{1}{\lambda^2} \sum_{n=0}^\infty \sinh^2(g_n) \left[ \frac{\omega_0^2}{\sigma_\epsilon} \left( n \cosh^2(g_{n+1}) + (n + 1) \cosh^2(g_{n+1}) \right) + n(n-1) \cosh^2(g_{n-2}) + (n+1)(n+2) \cosh^2(g_{n+2}) \right]. \tag{F7}
\]

This lower bound of the FI is structurally similar to the QFI \( J_q \). The FI has an additional prefactor \( C/C_\eta \) and the coefficients have changed from \( \xi r_n \) to \( g_n \). In the limit of \( \xi \ll K^{1/2} \), we have \( g_n \approx (1 - \eta) \xi^2 r_n^2 \) and for the FI

\[
F_{\eta,lb}(\eta) \approx \frac{C}{C_\eta} \frac{\omega_0^2}{\lambda^2 \sigma_\epsilon} (1 - \eta) \xi^2 \left( K + \frac{\sigma_\epsilon}{\omega_0^2} K^2 \right). \tag{F8}
\]

The QFI of the classical protocol is simply given by \( J_c(\eta) = (1 - \eta) J_c \) as the photon loss channel only transforms the displacement constant by \( \alpha \to \sqrt{1 - \eta} \alpha \). Thus, in the limit we find for the ratio

\[
\frac{F_{\eta,lb}(\eta)}{J_c(\eta)} = \frac{C}{C_\eta} \left( K + \frac{\sigma_\epsilon}{\omega_0^2} K^2 \right). \tag{F9}
\]

By making various approximations, we find that \( C \) only weakly depends on \( K \) in the limit of \( \xi \ll K^{1/2} \), and for \( \xi \ll K \to \infty \) we find \( C \approx \exp(-\xi^2) \) and thus for the ratio

\[
\frac{F_{\eta,lb}(\eta)}{J_c(\eta)} \approx \exp(-\eta^2) \left( K + \frac{\sigma_\epsilon}{\omega_0^2} K^2 \right) \approx \frac{\sigma_\epsilon}{\omega_0^2} \exp(-\eta^2) K^2, \tag{F10}
\]

because \( C_\eta \approx \exp(-1 - \eta) \xi^2 \). In the this limit, the bandwidth term becomes dominant. By increasing \( K \), \( C \) remains approximately constant and the ratio \( F_{\eta,lb}(\eta)/J_c(\eta) \) grows quadratically with \( K \). The minimum degree of frequency entanglement needed to ensure a quantum advantage grows exponentially \( K_{\text{min}} \approx \frac{\omega_0^2}{\sqrt{\sigma_\epsilon}} \exp(\eta N_S/2) \) with the expected number of lost signal photons because of \( \eta \xi^2 \approx \eta N_S \). We note, that these results are only valid in the regime \( \xi \ll K^{1/2} \to \infty \).

For the high-squeezing and the mixed regime the prefactor \( C/C_\eta \) vanishes and we find for the ratio \( F_{\eta,lb}(\eta)/J_c(\eta) \to 0 \). That does not mean that there is no quantum advantage in these regimes, it only shows that the extremely pessimistic lower bound, which only takes measurements into account where no photon has
been lost, of the FI becomes useless in these regimes.
[53] E. Celeghini, M. Gadella, and M. A. del Olmo, Symmetry 13, 853 (2021).