PRIME SPECTRA OF ABELIAN 2-CATEGORIES AND CATEGORIFICATIONS OF RICHARDSON VARIETIES

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Abstract. We describe a general framework for prime, completely prime, semiprime, and primitive ideals of an abelian 2-category. This provides a noncommutative version of Balmer’s prime spectrum of a tensor triangulated category. These notions are based on containment conditions in terms of thick subcategories of an abelian category and thick ideals of an abelian 2-category. We prove categorical analogs of the main properties of noncommutative prime spectra. Similar notions, starting with Serre subcategories of an abelian category and Serre ideals of an abelian 2-category, are developed. They are linked to Serre prime spectra of \( \mathbb{Z}_+ \)-rings. As an application, we construct a categorification of the quantized coordinate rings of open Richardson varieties for symmetric Kac–Moody groups, by constructing Serre completely prime ideals of monoidal categories of modules of the KLR algebras, and by taking Serre quotients with respect to them.

1. Introduction

1.1. Noncommutative categorical prime spectra. Balmer’s notion of a prime spectrum of a tensor triangulated category \([1,2]\) is a major tool in homological algebra, representation theory, algebraic topology and other areas. It is defined for triangulated categories with a symmetric monoidal structure. As noted in \([4]\), Balmer’s construction and results more generally apply to braided monoidal triangulated categories.

The notion of a prime spectrum of a braided monoidal triangulated category is a categorical version of the notion of a prime spectrum of a commutative ring. In the classical case of noncommutative rings, there are four different notions of primality \([11]\): prime, completely prime, semiprime and primitive spectra. In this paper we develop categorical notions of all of them, and prove analogs of many of their main properties. We do this in the abelian setting. However, instead of simply considering abelian monoidal categories, we work with the more general setting of abelian 2-categories. It is necessary to consider this more general setting, because many of the monoidal categorifications of noncommutative algebras that have been constructed so far are in the setting of 2-categories, rather than monoidal categories, see \([17,28,33]\). Categorifications via 2-categories are even needed for relatively small algebras such as the idempotent version of the quantized universal enveloping algebra of \( \mathfrak{sl}_2 \); we refer the reader to \([28]\) for a very informative review of this particular categorification.

1.2. Thick and prime ideals of abelian 2-categories. A 2-category is a category enriched over the category of 1-categories. In other words, a 2-category \( \mathcal{T} \), has the
property that for every two objects $A_1, A_2$ of it, the morphisms $\mathcal{T}(A_1, A_2)$ form a 1-category and satisfy natural identity conditions. A 2-category with one object is the same thing as a strict monoidal category. An abelian 2-category is such a category for which the 1-categories $\mathcal{T}(A_1, A_2)$ are abelian and the composition bifunctors are biexact. We work with small 2-categories, i.e., with 2-categories whose objects form a set and for which all 1-categories $\mathcal{T}(A_1, A_2)$ are small. We denote by $\mathcal{T}_1$ the isomorphism classes of 1-morphisms of $\mathcal{T}$. For two subsets $X, Y \subseteq \mathcal{T}_1$, denote by

$$X \circ Y$$

the set of isomorphism classes of 1-morphisms of $\mathcal{T}$ having representatives of the form $fg$ for $f$ and $g$ representing classes in $X$ and $Y$ such that $fg$ is defined.

The different versions of prime ideals of abelian 2-categories which we develop are based on the notion of a thick subcategory of an abelian category and its 2-incarnation, the notion of a thick ideal of an abelian 2-category. Recall that a thick (sometimes called wide) subcategory of an abelian category is a nonempty full subcategory which is closed under taking kernels, cokernels, and extensions.

A thick ideal $\mathcal{I}$ of an abelian 2-category $\mathcal{T}$ is a collection of subcategories $\mathcal{I}(A_1, A_2)$ of $\mathcal{T}(A_1, A_2)$ for all objects $A_1, A_2$ of $\mathcal{T}$ such that

1. $\mathcal{I}(A_1, A_2)$ are thick subcategories of the abelian categories $\mathcal{T}(A_1, A_2)$ and
2. the composition bifunctors of $\mathcal{T}$ restrict to bifunctors

$$\mathcal{T}(A_2, A_3) \times \mathcal{I}(A_1, A_2) \to \mathcal{I}(A_1, A_3)$$

and

$$\mathcal{I}(A_2, A_3) \times \mathcal{T}(A_1, A_2) \to \mathcal{I}(A_1, A_3)$$

for all objects $A_1, A_2, A_3$ of $\mathcal{T}$.

We call a proper thick ideal $\mathcal{P}$ of an abelian 2-category $\mathcal{T}$ prime if for all thick ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{T}$, $\mathcal{I} \circ \mathcal{J} \subseteq \mathcal{P}_1$ implies that either $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$.

A thick ideal $\mathcal{I}$ of an abelian 2-category $\mathcal{T}$ is prime if and only if two nearly identical properties hold:

1. $\mathcal{I}$ is a thick ideal of $\mathcal{T}$
2. $\mathcal{I}$ is a proper thick ideal of $\mathcal{T}$
3. $\mathcal{I}$ is such that for every 2-morphism $f : A_1 \to A_2$, $\mathcal{I}(f) := \mathcal{I}(A_1, A_2)$ is a prime ideal of $\mathcal{T}(A_1, A_2)$.

We obtain categorical versions of the main properties of prime, semiprime, and completely prime ideals of noncommutative rings. In Section 3 it is proved that the following are equivalent for a proper thick ideal $\mathcal{P}$ of an abelian 2-category $\mathcal{T}$:

1. $\mathcal{P}$ is a prime ideal;
2. If $f, g \in \mathcal{T}_1$ and $f \circ f \subseteq \mathcal{P}_1$, then either $f \in \mathcal{P}_1$ or $g \in \mathcal{P}_1$;
3. If $\mathcal{I}$ and $\mathcal{J}$ are any thick ideals properly containing $\mathcal{P}$, then $\mathcal{I} \circ \mathcal{J} \subseteq \mathcal{P}_1$;
4. If $\mathcal{I}$ and $\mathcal{J}$ are any left thick ideals of $\mathcal{T}$ such that $\mathcal{I} \circ \mathcal{J} \subseteq \mathcal{P}_1$, then either $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$.

We call the set of such thick ideals of $\mathcal{T}$ the prime spectrum of $\mathcal{T}$, to be denoted by $\text{Spec}(\mathcal{T})$, and define a Zariski type topology on it. For a multiplicative subset $\mathcal{M}$ of $\mathcal{T}_1$ (see Definition 3.13) and a proper thick ideal $\mathcal{I}$ of $\mathcal{T}$ such that $\mathcal{I} \cap \mathcal{M} = \emptyset$, we prove that every maximal element of the set

$$X(\mathcal{M}, \mathcal{I}) := \{ \mathcal{K} \text{ a thick ideal of } \mathcal{T} \mid \mathcal{K} \supseteq \mathcal{I} \text{ and } \mathcal{K} \cap \mathcal{M} = \emptyset \}$$

is a prime ideal of $\mathcal{T}$. This implies that $\text{Spec}(\mathcal{T})$ is non-empty for every abelian 2-category.

Categorical versions of simple, noetherian and weakly noetherian noncommutative rings are given in Section 4. There we prove that for every weakly noetherian abelian 2-category $\mathcal{T}$ and a proper thick ideal $\mathcal{I}$ of $\mathcal{T}$, there exist finitely many minimal Serre
prime ideals over $\mathcal{I}$ and there is a finite list of minimal prime ideals over $\mathcal{I}$ (possibly with repetition) $\mathcal{P}(1),...,\mathcal{P}(m)$ such that the product

$$\mathcal{P}_1^{(1)} \circ ... \circ \mathcal{P}_1^{(m)} \subseteq \mathcal{I}_1.$$ 

In Section 3, we prove a categorical version of the Levitzki–Nagata theorem for semiprime ideals, and furthermore show that the following are equivalent for a proper thick ideal $\mathcal{Q}$ of $\mathcal{T}$:

1. $\mathcal{Q}$ is semiprime;
2. If $f \in \mathcal{T}_1$ and $f \circ \mathcal{T}_1 \circ f \subseteq \mathcal{Q}_1$, then $f \in \mathcal{Q}_1$;
3. If $\mathcal{I}$ is any thick ideal of $\mathcal{T}$ such that $\mathcal{I}_1 \circ \mathcal{I}_1 \subseteq \mathcal{Q}_1$, then $\mathcal{I} \subseteq \mathcal{Q}$;
4. If $\mathcal{I}$ is any thick ideal properly containing $\mathcal{Q}$, then $\mathcal{I}_1 \circ \mathcal{I}_1 \not\subseteq \mathcal{Q}_1$;
5. If $\mathcal{I}$ is any left thick ideal of $\mathcal{T}$ such that $\mathcal{I}_1 \circ \mathcal{I}_1 \subseteq \mathcal{Q}_1$, then $\mathcal{I} \subseteq \mathcal{Q}$.

1.3. Serre prime ideals of 2-categories and ideals of $\mathbb{Z}_+$-rings. Serre subcategories of abelian categories are a particular type of thick subcategories. A thick ideal $\mathcal{I}$ of an abelian 2-category $\mathcal{T}$ will be called a Serre ideal if $\mathcal{I}(A_1, A_2)$ is a Serre subcategory of $\mathcal{T}(A_1, A_2)$ for all objects $A_1, A_2$ of $\mathcal{T}$. For those ideals one can consider the Serre quotient $\mathcal{T}/\mathcal{I}$ which is an abelian 2-category under a mild condition on $\mathcal{I}$.

We define a Serre prime (resp. semiprime, completely prime) ideal of an abelian 2-category $\mathcal{T}$ to be a prime (resp. semiprime, completely prime) ideal which is a Serre ideal. Section 6 treats in detail these ideals, and proves that they are characterized by similar to (p3)-(p4) and (sp3)-(sp5) properties as in §1.2, but with thick ideals replaced by Serre ideals. In other words, these kinds of ideals can be defined entirely based on the notion of Serre ideals of abelian 2-categories, just like the more general prime ideals are defined in terms of thick ideals.

The set of Serre prime ideals of $\mathcal{T}$, denoted by Serre-Spec($\mathcal{T}$), has an induced topology from Spec($\mathcal{T}$). This topology is shown to be intrinsically given in terms of Serre ideals of $\mathcal{T}$. If $\mathcal{C}$ is a strict abelian monoidal category, an alternative topology which more closely resembles the topology of Balmer in [1] can also be put on Serre-Spec($\mathcal{C}$). Under this topology, Serre-Spec($\mathcal{C}$) is a ringed space.

Denote $\mathbb{Z}_+ := \{0, 1, \ldots\}$. The Grothendieck ring $K_0(\mathcal{T})$ of an abelian 2-category $\mathcal{T}$ is a $\mathbb{Z}_+$-ring in the terminology of [7, Ch. 3], see Definitions 2.3 and 6.8. In §6.4 we define the notions of Serre ideals and Serre prime (semiprime and completely prime) ideals of a $\mathbb{Z}_+$-ring $\mathcal{R}$. The set of Serre prime ideals of $\mathcal{R}$, denoted by Serre-Spec($\mathcal{R}$), is equipped with a Zariski type topology.

It is proved in Section 6 that, for an abelian 2-category $\mathcal{T}$ with the property that every 1-morphism of $\mathcal{T}$ has finite length, the functor $K_0$ induces bijections between the sets of Serre ideals, Serre prime (semiprime and completely prime ideals) of the abelian 2-category $\mathcal{T}$ and the $\mathbb{Z}_+$-ring $K_0(\mathcal{T})$. Furthermore, the map

$$K_0 : \text{Serre-Spec}(\mathcal{T}) \rightarrow \text{Serre-Spec}(K_0(\mathcal{T}))$$

is shown to be a homeomorphism.

For a Serre prime (resp. semiprime, completely prime) ideal $\mathcal{I}$ of $\mathcal{T}$, the Serre quotient $\mathcal{T}/\mathcal{I}$ is a prime (resp. semiprime, domain) abelian 2-category. If every 1-morphism of $\mathcal{T}$ has finite length, then

$$K_0(\mathcal{T}/\mathcal{I}) \cong K_0(\mathcal{T})/K_0(\mathcal{I}).$$

The point now is that if we have a categorification of a $\mathbb{Z}_+$-ring $\mathcal{R}$ via an abelian 2-category $\mathcal{T}$ (i.e., $K_0(\mathcal{T}) \cong \mathcal{R}$) and $\mathcal{I}$ is a Serre ideal of $\mathcal{R}$, then there is a unique Serre ideal $\mathcal{I}$ of $\mathcal{T}$ such that $K_0(\mathcal{I}) = \mathcal{I}$. Furthermore, the Serre quotient $\mathcal{T}/\mathcal{I}$ categorifies the
\( \mathbb{Z}_+\text{-ring} R/I, \) i.e., \( K_0(T/I) \cong K_0(T)/K_0(I). \) If \( I \) is a Serre prime (resp. semiprime, completely prime) ideal of the \( \mathbb{Z}_+\text{-ring} R, \) then the Serre quotient \( T/I \) is a prime (resp. semiprime, domain) abelian 2-category. We view this construction as a general way of constructing monoidal categorifications of \( \mathbb{Z}_+\text{-rings} \) out of known ones by taking Serre quotients. This is illustrated in Section 9 in the case of the quantized coordinate rings of open Richardson varieties for symmetric Kac–Moody algebras.

We expect that, in addition, Serre prime ideals of abelian 2-categories and \( \mathbb{Z}_+\text{-rings} \) will provide a framework for finding intrinsic connections between prime ideals of noncommutative algebras and totally positive parts of algebraic varieties. In the case of the algebras of quantum matrices, such a connection was previously found by exhibiting related explicit generating sets for prime ideals of the noncommutative algebras and minors defining totally positive cells \([10]\).

**Primitive ideals** of abelian 2-categories \( T \) are introduced in Section 7 as the annihilation ideals of simple exact 2-representations in the setting of \([34],[35]\), where is proved that all such ideals are Serre prime ideals of \( T \).

### 1.4. Prime spectra of additive 2-categories

One can develop analogous (but much simpler) theory of different forms of prime ideals of an additive 2-category \( T \), which is a 2-category such that \( T(A_1,A_2) \) are additive categories for \( A_1,A_2 \in T \) and the compositions

\[
T(A_2,A_3) \times T(A_1,A_2) \to T(A_1,A_3)
\]

are additive bifunctors for \( A_1,A_2,A_3 \in T \).

This can be done by following exactly the same route as Sections 3–5 but based off the notion of a **thick ideal of an additive 2-category**. Call a full subcategory of an additive category **thick** if its closed under direct sums, direct summands and isomorphisms. A **thick ideal** \( I \) of an additive 2-category \( T \) is a collection of subcategories \( I(A_1,A_2) \) of \( T(A_1,A_2) \) for all objects \( A_1,A_2 \) of \( T \) such that

1. \( I(A_1,A_2) \) are thick subcategories of the additive categories \( T(A_1,A_2) \) and
2. \( T_1 \circ I_1 \subseteq I_1, I_1 \circ T_1 \subseteq I_1. \)

Using the same conditions on containments with respect to thick ideals and 1-morphisms as in Sections 3–5, one defines **prime, semiprime, and completely prime ideals of additive 2-categories** and proves analogs of the results in those sections (though in a simpler way than the abelian setting). One also analogously defines a Zariski topology on the set \( \text{Spec}(T) \) of prime ideals of \( T \) by using containments with respect to thick subcategories of \( T \). There are no analogs of the Serre type ideals in this setting.

If an additive 2-category \( T \) has the property that each of its 1-morphisms has a unique decomposition as a finite set of indecomposables (e.g., if all additive categories \( T(A_1,A_2) \) are Krull–Schmidt), then the split Grothendieck ring \( K_0^{sp}(T) \) is a \( \mathbb{Z}_+\text{-ring} \), see Remark 2.4. Similarly to \([6,5]\) for such additive 2-categories \( T \), one shows that the map \( K_0^{sp}(-) \) gives

- a bijection between the sets of thick, prime, semiprime and completely prime ideals of \( T \) and the sets of Serre ideals, Serre prime, semiprime and completely prime ideals of the \( \mathbb{Z}_+\text{-ring} K_0^{sp}(T), \) and
- a homeomorphism \( \text{Spec}(T) \to \text{Serre-Spec}(K_0^{sp}(T)) \).

Call a **primitive ideal of an additive 2-category** \( T \) to be the annihilation ideal of a simple 2-representation of \( T \) in the setting of \([34],[35]\). Similarly to Section 7 one shows that each such ideal is a prime ideal of \( T \).

In a forthcoming publication we obtain analogs of the results in the paper for (noncommutative) prime spectra of triangulated 2-categories.
1.5. Categorifications of Richardson varieties via prime Serre quotients. We finish with an important example of Serre completely prime ideals of abelian 2-categories that can be used to categorify the quantized coordinate rings of certain closures of open Richardson varieties. For a symmetrizable Kac–Moody group $G$, a pair of opposite Borel subgroups $B_{\pm}$ and Weyl group elements $u \leq w$, the corresponding open Richardson variety is defined as the intersections of opposite Schubert cells in the full flag variety of $G$,

$$R_{u,w} := (B_u B_+)/B_+ \cap (B_+ w B_+)/B_+ \subset G/B_+.$$  

They have been used in a wide range of settings in representation theory, Schubert calculus, total positivity, Poisson geometry, and mathematical physics. For symmetric Kac–Moody groups, Leclerc [29] constructed a cluster algebra inside the coordinate ring of each Richardson variety of the same dimension. In the quantum situation, Lenagan and the second named author constructed large families of toric frames for all quantized coordinate rings of Richardson varieties that generate those rings [30].

Recently, for each symmetrizable Kac–Moody algebra $g$, Kashiwara, Kim, Oh, and Park [23] constructed a monoidal categorification of the quantization of a closure of $R_{u,w}$ in terms of a monoidal subcategory of the category of graded, finite dimensional representations of the Khovanov–Lauda–Rouquier (KLR) algebras associated to $g$. Their construction uses Leclerc’s interpretation of the coordinate ring of a closure of $R_{u,w}$ in terms of a double invariant subalgebra.

Denote by $\mathcal{F}_{u,w}$ the closure of $R_{u,w}$ in the Schubert cell $(B_+ w B_+)/B_+ \subset G/B_+$. We construct a monoidal categorification of the quantization $U_q^-[w]/I_q(w)$ of the coordinate ring of $\mathcal{F}_{u,w}$ used in the construction of toric frames in [30]. Here $U_q^-[w]$ are the quantum Schubert cell algebras [5, 32] and $I_q(w)$ are the homogeneous completely prime ideals of these algebras that arose in the classification of their prime spectra in [32]. This classification was based on the fundamental works of Anthony Joseph on the spectra of quantum groups [15, 16] from the early 90s. It was proved in [18, 25, 37] that certain monoidal subcategories $\mathcal{C}_w$ of the categories of graded, finite dimensional modules of the KLR algebras associated to $g$ categorify the dual integral form $U_\mathcal{A}^{-}[w]^{\vee}$ where $\mathcal{A} := \mathbb{Z}[q^\pm 1]$. We prove that for a symmetrizable Kac–Moody algebra $g$, the ideals $I_w(u) \cap U_\mathcal{A}^{-}[w]^{\vee}$ have bases that are subsets of the upper global/canonical basis of $U_\mathcal{A}^{-}[w]^{\vee}$. From this we deduce that for symmetric $g$, $I_w(u) \cap U_\mathcal{A}^{-}[w]^{\vee}$ are Serre completely prime ideals of the $\mathbb{Z}_+\text{-ring } U_\mathcal{A}^{-}[w]^{\vee}$. The bijection from [1, 3] implies that the monoidal category $\mathcal{C}_w$ has a Serre completely prime ideal $\mathcal{I}_w(w)$ such that $K_0(\mathcal{I}_w(w)) = I_w(u)$, and thus, the Serre quotient $\mathcal{C}_w/\mathcal{I}_w(w)$ categorifies $U_\mathcal{A}^{-}[w]^{\vee}/(I_w(u) \cap U_\mathcal{A}^{-}[w]^{\vee})$:

$$K_0(\mathcal{C}_w/\mathcal{I}_w(w)) \cong U_\mathcal{A}^{-}[w]^{\vee}/(I_w(u) \cap U_\mathcal{A}^{-}[w]^{\vee}).$$

It is an important problem to connect the categorification of Kashiwara, Kim, Oh, and Park [23] of open Richardson varieties (via subcategories of KLR modules) to ours (via Serre quotients of categories of KLR modules).

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2. ABELIAN 2-CATEGORIES AND CATEGORIFICATION

This section contains background material on (abelian) 2-categories and categorification of algebras.

2.1. 2-CATEGORIES. A category $\mathcal{T}$ is said to be enriched over a monoidal category $\mathcal{M}$ if the space of morphisms between any two objects of $\mathcal{T}$ is an object in $\mathcal{M}$ and $\mathcal{T}$ satisfies natural axioms which relate composition of morphisms in $\mathcal{T}$ and the identity morphisms of objects of $\mathcal{T}$ to the monoidal structure of $\mathcal{M}$. We refer the reader to [24] for details.

A 2-category is a category enriched over the category of 1-categories. This means that for a 2-category $\mathcal{T}$, given two objects $A_1, A_2$ of it, the morphisms $\mathcal{T}(A_1, A_2)$ form a 1-category. The objects of these categories are denoted by the same symbol $\mathcal{T}(A_1, A_2)$ – they are the 1-morphisms of $\mathcal{T}$. The morphisms of the categories $\mathcal{T}(A_1, A_2)$ are the 2-morphisms of $\mathcal{T}$. For a pair of 1-morphisms $f, g \in \mathcal{T}(A_1, A_2)$, we will denote by $\mathcal{T}(f, g)$ the 2-morphisms between $f$ and $g$, i.e., the morphisms between the objects $f$ and $g$ in the category $\mathcal{T}(A_1, A_2)$.

We have 2 types of compositions of 1- and 2-morphisms. We follow the notation of [28]:

1. For a pair of objects $A_1, A_2$ of $\mathcal{T}$, the composition of morphisms in the category $\mathcal{T}(A_1, A_2)$ is called vertical composition of 2-morphisms of $\mathcal{T}$. In the globular representation of $\mathcal{T}$, such a composition is given by the following diagram

$$
\begin{array}{c}
\bullet & \overset{h}{\longrightarrow} & \bullet \\
\downarrow{\beta} & & \downarrow{\alpha} \\
\bullet & \overset{g}{\longrightarrow} & \bullet \\
\downarrow{f} & & \downarrow{\text{id}} \\
\bullet & \overset{\text{id}}{\longrightarrow} & \bullet
\end{array}
$$

The vertical composition of the 2-morphisms $\alpha \in \mathcal{T}(f, g)$ and $\beta \in \mathcal{T}(g, h)$ will be denoted by $\beta\alpha \in \mathcal{T}(f, h)$, where $f, g, h$ are objects of $\mathcal{T}(A_1, A_2)$.

2. For each three objects $A_1, A_2, A_3$ of $\mathcal{T}$, we have a bifunctor of 1-categories

$$\mathcal{T}(A_2, A_3) \times \mathcal{T}(A_1, A_2) \to \mathcal{T}(A_1, A_3).$$

The resulting composition of 1- and 2-morphisms of $\mathcal{T}$ is called horizontal composition. In the globular representation of $\mathcal{T}$, these compositions are given by the diagram

$$
\begin{array}{c}
\bullet & \overset{f_1}{\longrightarrow} & \bullet \\
\downarrow{\alpha_1} & & \downarrow{\text{id}} \\
\bullet & \overset{g_1}{\longrightarrow} & \bullet \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
\bullet & \overset{\text{id}}{\longrightarrow} & \bullet
\end{array}
$$

In this notation, the horizontal composition of 2-morphisms will be denoted by $\alpha_2 * \alpha_1$. The horizontal composition of 1-morphisms will be denoted by $f_2 f_1$.

A 2-category $\mathcal{T}$ has identity 1-morphisms $1_A \in \mathcal{T}(A, A)$ (for its objects $A \in \mathcal{T}$). The compositions and identity morphisms satisfy natural associativity and identity axioms.
2-categories are generalizations of monoidal categories, in the sense that a strict monoidal category is the same thing as a 2-category with one object:

To a strict monoidal category \( \mathcal{M} \), one associates a 2-category \( \mathcal{T} \) with one object \( A \) by taking \( \mathcal{T}(A,A) := \mathcal{M} \). The tensor product in \( \mathcal{M} \) is used to define composition of 1-morphisms of \( \mathcal{T} \). For \( f, g \in \mathcal{M} = \mathcal{T}(A,A) \), one defines the 2-morphisms \( \mathcal{T}(f,g) := \mathcal{M}(f,g) \). All 2-categories with 1 object arise in this way.

Recall that a 1-category \( \mathcal{C} \) is called small if its objects form a set and \( \mathcal{C}(A_1, A_2) \) is a set for all pairs of objects \( A_1, A_2 \in \mathcal{C} \). Throughout the paper we work with small 2-categories \( \mathcal{T} \), which are 2-categories satisfying the conditions that the objects of \( \mathcal{T} \) form a set and \( \mathcal{T}(A_1, A_2) \) is a small 1-category for all pairs of objects \( A_1, A_2 \) of \( \mathcal{T} \).

The set of objects of such a 2-category \( \mathcal{T} \) will be denoted by the same symbol \( \mathcal{T} \). The set of 1-morphisms of \( \mathcal{T} \) will be denoted by \( \mathcal{T}_1 \).

2.2. Abelian 2-categories and categorification.

Definition 2.1. We will say that a 2-category \( \mathcal{T} \) is an abelian 2-category if \( \mathcal{T}(A_1, A_2) \) are abelian categories for all \( A_1, A_2 \in \mathcal{T} \) and the compositions

\[
\mathcal{T}(A_2, A_3) \times \mathcal{T}(A_1, A_2) \to \mathcal{T}(A_1, A_3)
\]

are exact bifunctors for all \( A_1, A_2, A_3 \in \mathcal{T} \).

More generally, for a ring \( k \), we will say that \( \mathcal{T} \) is a \( k \)-linear abelian 2-category if \( \mathcal{T}(A_1, A_2) \) are \( k \)-linear abelian categories for \( A_1, A_2 \in \mathcal{T} \).

A multiring category in the terminology of [7] Definition 4.2.3] is precisely a \( k \)-linear abelian 2-category with one object.

Remark 2.2. Let \( k \) be a field. Recall that a \( k \)-linear abelian category \( \mathcal{C} \) is called locally finite if it is Hom-finite (i.e., \( \dim_k \mathcal{C}(A_1, A_2) < \infty \) for all \( A_1, A_2 \in \mathcal{C} \)) and each object of \( \mathcal{C} \) has finite length; we refer the reader to [7 §1.8] for details. Let \( \mathcal{LFAb}_{ex} \) be the monoidal category of locally finite abelian categories equipped with the Deligne tensor product ([6] and [7 §1.11]) and morphisms given by exact functors.

In this terminology, a \( k \)-linear abelian 2-category \( \mathcal{T} \) with the property that the 1-categories \( \mathcal{T}(A_1, A_2) \) are locally finite for all \( A_1, A_2 \in \mathcal{T} \) is the same thing as a category which is enriched over the monoidal category \( \mathcal{LFAb}_{ex} \). This is easy to verify, the only key step being the universality property of the Deligne tensor product with respect to exact functors [7 Proposition 1.11.2(v)]

We will denote by \( K_0(\mathcal{C}) \) the Grothendieck group of an abelian category \( \mathcal{C} \). To each abelian 2-category \( \mathcal{T} \) one associates the pre-additive category \( K_0(\mathcal{T}) \) whose objects are the objects of \( \mathcal{T} \) and morphisms are

\[
K_0(\mathcal{T})(A_1, A_2) := K_0(\mathcal{T}(A_1, A_2)) \quad \text{for} \quad A_1, A_2 \in \mathcal{T}.
\]

Given a pre-additive category \( \mathcal{F} \), one says that the 2-category \( \mathcal{T} \) categorifies \( \mathcal{F} \) if \( K_0(\mathcal{T}) \cong \mathcal{F} \) as pre-additive categories.

To a pre-additive category \( \mathcal{F} \), one associates a ring with elements

\[
\oplus_{A_1, A_2 \in \mathcal{F}} \mathcal{F}(A_1, A_2).
\]

The product in the ring is the composition of morphisms when it makes sense and 0 otherwise. In particular, the identity morphisms \( 1_A \) are idempotents of the ring for all objects \( A \in \mathcal{F} \). By abuse of notation, this ring is denoted by the same symbol \( \mathcal{F} \) as the original category.
Definition 2.3. For an abelian 2-category $\mathcal{T}$, the ring $K_0(\mathcal{T})$ is called the Grothendieck ring of $\mathcal{T}$. We say that $\mathcal{T}$ categorifies an $S$-algebra $R$, for a commutative ring $S$, if $K_0(\mathcal{T}) \otimes_Z S \cong R$.

Often, it is not sufficient to consider multiring categories (abelian monoidal categories) to obtain categorifications of algebras, and one needs the more general setting of 2-categories.

Remark 2.4. An additive 2-category is a 2-category $\mathcal{T}$ such that $\mathcal{T}(A_1, A_2)$ are additive categories for all $A_1, A_2 \in \mathcal{T}$ and the compositions $\mathcal{T}(A_2, A_3) \times \mathcal{T}(A_1, A_2) \to \mathcal{T}(A_1, A_3)$ are additive bifunctors for all $A_1, A_2, A_3 \in \mathcal{T}$. For such a category $\mathcal{T}$, one defines the pre-additive category $K^{sp}_0(\mathcal{T})$ whose objects are the objects of $\mathcal{T}$ and morphisms are the split Grothendieck groups $K^{sp}_0(\mathcal{T}(A_1, A_2))$ of the additive categories $\mathcal{T}(A_1, A_2)$ for $A_1, A_2 \in \mathcal{T}$.

We say that an additive 2-category $\mathcal{T}$ categorifies an $S$-algebra $R$ if $K^{sp}_0(\mathcal{T}) \otimes_Z S \cong R$.

3. The prime spectrum

In this section we define the prime spectrum of an abelian 2-category and a Zariski type topology on it. We prove two equivalent characterizations of prime ideals, extending theorems from classical ring theory. We also prove that maximal elements of the sets of ideals not intersecting multiplicative sets of 1-morphisms of 2-categories are prime ideals.

3.1. Thick ideals of abelian 2-categories.

Definition 3.1. A weak subcategory $\mathcal{I}$ of a 2-category $\mathcal{T}$ is

(1) a subcollection $\mathcal{I}$ of objects of $\mathcal{T}$ and

(2) a collection of subcategories $\mathcal{I}(A_1, A_2)$ of $\mathcal{T}(A_1, A_2)$ for $A_1, A_2 \in \mathcal{I}$,

such that the composition bifunctors restrict to bifunctors $\mathcal{I}(A_2, A_3) \times \mathcal{I}(A_1, A_2) \to \mathcal{I}(A_1, A_3)$ for $A_1, A_2, A_3 \in \mathcal{I}$.

A weak subcategory $\mathcal{I}$ of a 2-category $\mathcal{T}$ is not necessarily a 2-category on its own because it might not contain the identity morphisms $1_A$ for its objects $A \in \mathcal{I}$. Apart from this, a weak subcategory of a 2-category satisfies the other axioms for 2-categories. The relationship of a weak subcategory to a 2-category is the same as the relationship of a subring to a unital ring $R$. In the latter case, the subring does not need to contain the unit of $R$.

Definition 3.2. (1) A thick subcategory of an abelian category is a nonempty full subcategory which is closed under taking kernels, cokernels, and extensions.

(2) A thick weak subcategory of an abelian 2-category $\mathcal{T}$ is a weak subcategory $\mathcal{I}$ of $\mathcal{T}$ having the same set of objects and such that for any pair of objects $A_1, A_2 \in \mathcal{T}$, $\mathcal{I}(A_1, A_2)$ is a thick subcategory of the abelian category $\mathcal{T}(A_1, A_2)$.

(3) A thick ideal of an abelian 2-category $\mathcal{T}$ is a thick weak subcategory $\mathcal{I}$ of $\mathcal{T}$ such that, for all 1-morphisms $f$ in $\mathcal{T}$ and $g$ in $\mathcal{I}$, the compositions $fg, gf$ are 1-morphisms of $\mathcal{I}$ whenever they are defined.
Sometimes the term wide subcategory is used instead of thick, see for instance [13].
Every thick subcategory of an abelian category is closed under isomorphisms and taking direct summands of its objects (because one can take the kernels of idempotent endomorphisms of its objects).

For a thick weak subcategory \( \mathcal{I} \) of an abelian 2-category \( \mathcal{T} \), \( \mathcal{I}(A_1, A_2) \) is an abelian category for every pair of objects \( A_1, A_2 \in \mathcal{I} \) with respect to the same kernels and cokernels as the ambient abelian category \( \mathcal{T}(A_1, A_2) \).

In part (3), the compositions of 1-morphisms that are used are the horizontal compositions discussed in §2.1. More explicitly, a thick subcategory \( \mathcal{I} \) of \( \mathcal{T} \) is a thick ideal if for all \( f_1 \in \mathcal{T}(A_1, A_2) \), \( g_2 \in \mathcal{I}(A_2, A_3) \) and \( f_3 \in \mathcal{T}(A_3, A_4) \), we have

\[
g_2 f_1 \in \mathcal{I}(A_1, A_3) \quad \text{and} \quad f_3 g_2 \in \mathcal{I}(A_2, A_4),
\]
where \( A_1, A_2, A_3, A_4 \in \mathcal{T} \).

**Remark 3.3.** Let \( \mathcal{I} \) and \( \mathcal{J} \) be a pair of thick weak subcategories of \( \mathcal{T} \). Then

\[
\mathcal{I} \subseteq \mathcal{J} \quad \text{if and only if} \quad \mathcal{I}_1 \subseteq \mathcal{J}_1.
\]

In particular,

\[
\mathcal{I} = \mathcal{J} \quad \text{if and only if} \quad \mathcal{I}_1 = \mathcal{J}_1.
\]

**Example 3.4.** There exists a unique thick ideal of every 2-category \( \mathcal{T} \) whose set of 1-morphisms consists of the 0 objects of the abelian categories \( \mathcal{T}(A_1, A_2) \) for all \( A_1, A_2 \in \mathcal{T} \). This thick ideal will be denoted by \( 0_{\mathcal{T}} \). Every other thick ideal of \( \mathcal{T} \) contains \( 0_{\mathcal{T}} \).

For two subsets \( X, Y \subseteq \mathcal{T}_1 \), denote by

\[
X \circ Y \text{ the set of isomorphism classes of 1-morphims of } \mathcal{T} \text{ having representatives of the form } fg \text{ for } f \text{ and } g \text{ representing classes in } X \text{ and } Y \text{ such that } fg \text{ is defined.}
\]

In general, \( X \circ Y \) can be empty. For \( f, g \in \mathcal{T}_1 \) the composition \( f \circ g \) is either empty or consists of one element.

In this notation, a thick weak subcategory \( \mathcal{I} \) of \( \mathcal{T} \) is a thick ideal if and only if

\[
\mathcal{T}_1 \circ \mathcal{I}_1 \subseteq \mathcal{I}_1 \quad \text{and} \quad \mathcal{I}_1 \circ \mathcal{T}_1 \subseteq \mathcal{I}_1.
\]

**Definition 3.5.** A thick left (respectively right) ideal of an abelian 2-category \( \mathcal{T} \) is a thick weak subcategory \( \mathcal{I} \) of \( \mathcal{T} \) such that

\[
\mathcal{T}_1 \circ \mathcal{I}_1 \subseteq \mathcal{I}_1 \quad \text{(respectively } \mathcal{I}_1 \circ \mathcal{T}_1 \subseteq \mathcal{I}_1).
\]

**Remark 3.6.** Let \( A_1, A_2, B_1, B_2 \) be four objects of an abelian 2-category \( \mathcal{T} \) such that

\[
A_i \cong B_i \quad \text{for} \quad i = 1, 2.
\]

Then for every thick ideal \( \mathcal{I} \) of \( \mathcal{T} \), we have (noncanonical) isomorphisms of abelian categories

\[
\mathcal{I}(A_1, A_2) \cong \mathcal{I}(B_1, B_2).
\]

Indeed, let \( f_i \in \mathcal{T}(A_i, B_i) \) and \( g_i \in \mathcal{T}(B_i, A_i) \) be such that

\[
f_i g_i \cong 1_{B_i} \quad \text{and} \quad g_i f_i \cong 1_{A_i}
\]
for \( i = 1, 2 \) (where the isomorphisms are in the categories \( \mathcal{T}(B_i, B_i) \) and \( \mathcal{T}(A_i, A_i) \)). The functor giving the equivalence \( (3.1) \) is defined by

\[
h \mapsto f_2 h g_1
on the level of objects \( h \in \mathcal{I}(A_1, A_1) \) and
\[
\alpha \mapsto 1_{f_2} * \alpha * 1_{g_1}
\]
on the level of morphisms.

### 3.2. Prime ideals of abelian 2-categories.

A thick ideal \( \mathcal{I} \) of \( \mathcal{T} \) will be called *proper* if \( \mathcal{I} \neq \mathcal{T} \); by Remark 3.3 this is the same as \( \mathcal{I}_1 \subseteq \mathcal{T}_1 \).

**Definition 3.7.** We call \( \mathcal{P} \) a *prime ideal* of \( \mathcal{T} \) if \( \mathcal{P} \) is a proper thick ideal of \( \mathcal{T} \) with the property that for every pair of thick ideals \( \mathcal{I} \) and \( \mathcal{J} \) of \( \mathcal{T} \),
\[
\mathcal{I}_1 \circ \mathcal{J}_1 \subseteq \mathcal{P}_1 \quad \Rightarrow \quad \mathcal{I} \subseteq \mathcal{P} \text{ or } \mathcal{J} \subseteq \mathcal{P}.
\]
The set of all prime ideals \( \mathcal{P} \) of an abelian 2-category \( \mathcal{T} \) will be called the *prime spectrum* of \( \mathcal{T} \) and will be denoted by \( \text{Spec}(\mathcal{T}) \).

By Remark 3.3, the property on the right side of the implication can be replaced with \( \mathcal{I}_1 \subseteq \mathcal{P}_1 \) or \( \mathcal{J}_1 \subseteq \mathcal{P}_1 \).

### 3.3. Two equivalent characterizations of prime ideals.

The following lemma is straightforward.

**Lemma 3.8.** The intersection of any family of thick ideals is a thick ideal.

If \( \mathcal{M} \) is a collection of 1-morphisms of \( \mathcal{T} \) (i.e., \( \mathcal{M} \subseteq \mathcal{T}_1 \)), let \( \langle \mathcal{M} \rangle \) denote the smallest thick ideal of \( \mathcal{T} \) containing \( \mathcal{M} \), which exists by the previous lemma.

**Lemma 3.9.** For every two collections \( \mathcal{M} \subseteq \mathcal{T}_1 \) and \( \mathcal{N} \subseteq \mathcal{T}_1 \) of 1-morphisms of an abelian 2-category \( \mathcal{T} \), we have
\[
\langle \mathcal{M} \rangle_1 \circ \langle \mathcal{N} \rangle_1 \subseteq \langle \mathcal{M} \circ \mathcal{T}_1 \circ \mathcal{N} \rangle_1.
\]

**Proof.** We will first show that
\[
\langle \mathcal{M} \rangle_1 \circ \mathcal{N} \subseteq \langle \mathcal{M} \circ \mathcal{T}_1 \circ \mathcal{N} \rangle_1.
\]
The 1-morphisms of \( \langle \mathcal{M} \rangle \) are obtained from the elements of \( \mathcal{M} \) by successive taking of kernels and cokernels (of 2-morphisms between these elements), and extensions (between these elements), as well as compositions on the left and the right by elements in \( \mathcal{T}_1 \). We need to show that those operations, composed on the right with the elements of \( \mathcal{N} \), yield elements of the right hand side.

(1) Suppose that \( \alpha : f \to g \) is a 2-morphism for \( f, g \in \mathcal{T}_1 \) with the property that
\[
fn, gn \in \langle \mathcal{M} \circ \mathcal{T} \circ \mathcal{N} \rangle_1 \quad \text{for all} \quad n \in \mathcal{T}_1 \circ \mathcal{N}.
\]
Note that, for example, every 1-morphism in \( \mathcal{M} \) has this property. Let \( \kappa : k \to f \) be the kernel of \( \alpha \). Since exact functors preserve kernels, \( \kappa * \text{id}_n : kn \to fn \) is the kernel of \( \alpha * \text{id}_n : fn \to gn \). The thickness property of \( \langle \mathcal{M} \circ \mathcal{T} \circ \mathcal{N} \rangle \) implies that \( kn \in \langle \mathcal{M} \circ \mathcal{T}_1 \circ \mathcal{N} \rangle_1 \) for all \( n \in \mathcal{T}_1 \circ \mathcal{N} \).

Symmetrically, one shows that if \( \gamma : g \to c \) is the cokernel of \( \alpha \), then \( cn \in \langle \mathcal{M} \circ \mathcal{T}_1 \circ \mathcal{N} \rangle_1 \) for all \( n \in \mathcal{T}_1 \circ \mathcal{N} \).

(2) Next, assume that
\[
0 \to f \to g \to h \to 0
\]
is an exact sequence in one of the abelian categories \( \mathcal{T}(A_1, A_2) \), where \( f, h \) have the property that \( fn, hn \in \langle \mathcal{M} \circ \mathcal{T}_1 \circ \mathcal{N} \rangle_1 \) for all \( n \in \mathcal{T}_1 \circ \mathcal{N} \). Since horizontal composition in \( \mathcal{T} \) is exact, for any \( n \in \mathcal{T}_1 \circ \mathcal{N} \), we get a short exact sequence
\[
0 \to fn \to gn \to hn \to 0.
\]
Since the first and last terms are in \( \langle \mathcal{M} \circ T_1 \circ \mathcal{N} \rangle_1 \), so is the middle term.

Combining (1)-(2) and the fact that \( \langle \mathcal{M} \circ T_1 \circ \mathcal{N} \rangle_1 \) is stable under left compositions with elements of \( T_1 \), yields \( \text{(3.3)} \). Analogously, we derive \( \text{(3.2)} \) from \( \text{(3.3)} \) by using \( \langle \mathcal{M} \rangle_1 \) in place of \( \mathcal{M} \).

**Theorem 3.10.** A proper thick ideal \( P \) of an abelian 2-category \( T \) is prime if and only if for all \( m, n \in T_1 \), \( m \circ T_1 \circ n \subseteq P_1 \) implies that either \( m \in P_1 \) or \( n \in P_1 \).

**Proof.** Suppose \( P \) is a prime ideal of \( T \), and that \( m \circ T \circ n \subseteq P \) for some \( m, n \in T_1 \). Then by the previous lemma,

\[
\langle m \rangle_1 \circ \langle n \rangle_1 \subseteq \langle m \circ T_1 \circ n \rangle_1 \subseteq P_1,
\]

and so by primeness of \( P \), \( \langle m \rangle \subseteq P_1 \) or \( \langle n \rangle \subseteq P_1 \). Therefore, \( m \) or \( n \) is in \( P_1 \).

Now suppose \( P \) is a proper thick ideal of \( T \) with the property that for all \( m, n \in T_1 \), \( m \circ T_1 \circ n \subseteq P_1 \) implies that either \( m \in P_1 \) or \( n \in P_1 \). Let \( \mathcal{I} \) and \( \mathcal{J} \) be a pair of thick ideals of \( T \) such that

\[
\mathcal{I}_1 \circ \mathcal{J}_1 \subseteq P_1 \quad \text{and} \quad \mathcal{J}_1 \not\subseteq P_1.
\]

Then there is some \( j \in \mathcal{J}_1 \) with \( j \notin P_1 \). However, \( i \circ T_1 \circ j \subseteq P_1 \) for any \( i \in \mathcal{I}_1 \) since \( i \circ T_1 \subseteq I_1, j \in \mathcal{J}_1 \), and \( I_1 \circ \mathcal{J}_1 \subseteq P_1 \). The assumed property of \( P \) implies that \( I_1 \subseteq P_1 \). Therefore \( \mathcal{I} \subseteq P \) by Remark \( \text{3.3} \). \( \square \)

It is easy to see that Theorem \( \text{3.10} \) implies the following:

**Proposition 3.11.** A proper thick ideal \( P \) of an abelian 2-category \( T \) is prime if and only if for every pair of right thick ideals \( \mathcal{I} \) and \( \mathcal{J} \) of \( T \)

\[
\mathcal{I}_1 \circ \mathcal{J}_1 \subseteq P_1 \quad \Rightarrow \quad \mathcal{I} \subseteq P \quad \text{or} \quad \mathcal{J} \subseteq P.
\]

A similar characterization holds using left thick ideals.

**Theorem 3.12.** A proper thick ideal \( P \) of an abelian 2-category \( T \) is prime if and only if for all thick ideals \( \mathcal{I}, \mathcal{J} \) of \( T \) properly containing \( P \), we have that \( \mathcal{I}_1 \circ \mathcal{J}_1 \not\subseteq P_1 \).

**Proof.** The implication \( \Rightarrow \) is clear. Suppose \( P \) is a proper thick ideal which is not prime. Then there exist some thick ideals \( \mathcal{I} \) and \( \mathcal{J} \) of \( T \) with \( \mathcal{I}_1 \circ \mathcal{J}_1 \subseteq P_1 \) and \( \mathcal{I}, \mathcal{J} \not\subseteq P \).

Set \( \mathcal{M} := P_1 \cup \mathcal{I}_1 \) and \( \mathcal{N} := P_1 \cup \mathcal{J}_1 \).

By Remark \( \text{3.3} \), \( P_1 \) is properly contained in both \( \langle \mathcal{M} \rangle_1 \) and \( \langle \mathcal{N} \rangle_1 \). Lemma \( \text{3.9} \) implies that

\[
\langle \mathcal{M} \rangle \circ \langle \mathcal{N} \rangle \subseteq \langle \mathcal{M} \circ T_1 \circ \mathcal{N} \rangle.
\]

Observe that

\[
\langle \mathcal{M} \rangle \circ \langle \mathcal{N} \rangle \subseteq \langle \mathcal{M} \circ T_1 \circ \mathcal{N} \rangle,
\]

by the following. Consider the composition \( itj \) for some \( i \in \mathcal{M}, t \in T_1, j \in \mathcal{N} \). So, \( i \in \mathcal{I} \) or \( P_1 \); likewise, \( j \in \mathcal{J} \) or \( P_1 \). If at least one of the two 1-morphism \( i, j \) is in \( P_1 \), we have \( itj \in P_1 \) since \( P \) is a thick ideal; if \( i \in \mathcal{I} \) and \( j \in \mathcal{J} \) then \( i \circ t \in \mathcal{I}_1 \), so \( itj \in \mathcal{I}_1 \circ \mathcal{J}_1 \subseteq P_1 \) by assumption.

Therefore \( \langle \mathcal{M} \rangle \) and \( \langle \mathcal{N} \rangle \) are thick ideals properly containing \( P \) and \( \langle \mathcal{M} \rangle_1 \circ \langle \mathcal{N} \rangle_1 \subseteq P_1 \) (the last inclusion follows from \( \text{(3.4)} \)–\( \text{(3.5)} \) and the minimality of the thick ideal \( \langle - \rangle \)). Hence, \( P \) does not have the stated property, which completes the proof of the theorem. \( \square \)
3.4. Relation to maximal ideals.

Definition 3.13. A nonempty set \( M \subseteq T_1 \) will be called multiplicative if \( M \) is a subset of non-zero equivalence classes of objects of \( T(A, A) \) for some object \( A \) of \( T \) and \( M \cdot M \subseteq M \).

The condition that \( M \subseteq T(A, A) \) means that all 1-morphism in \( M \) are composable. Let us explain the motivation for this condition. Let \( R \) be a ring and \( \{e_s\} \) be a collection of orthogonal idempotents. If \( M \) is a multiplicative subset such that \( M \subseteq \bigcup_{s,t} e_sRe_t \) then \( M \subseteq e_sRe_s \) for some \( s \), because otherwise \( M \) will contain the 0 element of \( R \).

Theorem 3.14. Assume that \( M \) is a multiplicative subset of \( T_1 \) for an abelian 2-category \( T \) and that \( I \) is a proper thick ideal of \( T \) such that \( I_1 \cap M = \emptyset \).

Let \( P \) be a maximal element of the collection of thick ideals of \( T \) containing \( I \) and intersecting \( M \) trivially, equipped with the inclusion relation, i.e., \( P \) is a maximal element of the set

\[
X(M, I) := \{ K \text{ a thick ideal of } T | K \supseteq I \text{ and } K_1 \cap M = \emptyset \}.
\]

Then \( P \) is prime.

Proof. Fix such an ideal \( P \). Suppose \( Q \) and \( R \) are thick ideals properly containing \( P \). By Theorem 3.12 it is enough to show that \( Q \circ R \not\subseteq P \). Since

\[
I \subseteq P \subseteq Q \quad \text{and} \quad I \subseteq P \subseteq R,
\]

both \( Q_1 \) and \( R_1 \) must intersect nontrivially with \( M \), by the maximality assumption on \( P \). Let \( q \in Q_1 \cap M \) and \( r \in R_1 \cap M \). If \( Q \circ R \not\subseteq P \), then we would obtain that \( qr \in P \), because by the definition of multiplicative subset of \( T_1 \), each two elements of \( M \) are composable. However, since \( qr \in M \), this contradicts with the assumption that \( P_1 \cap M = \emptyset \).

Remark 3.15. The set \( X(M, I) \) from Theorem 3.14 is nonempty because \( I \in X(M, I) \). The union of an ascending chain of thick ideals in the set \( X(M, I) \) is obviously a thick ideal of \( T \). By Zorn's lemma, the set \( X(M, I) \) from Theorem 3.14 always contains at least one maximal element.

Corollary 3.16. (1) For each proper thick ideal \( I \) of an abelian 2-category \( T \), there exists a prime ideal \( P \) of \( T \) that contains \( I \).

(2) Let \( M \) be a multiplicative set of an abelian 2-category \( T \). Every maximal element of the set of thick ideals \( K \) of \( T \) such that \( K_1 \cap M = \emptyset \) is a prime ideal. The set of such thick ideals contains at least one maximal element.

Proof. (1) Since the thick ideal \( I \) is proper, there exists an object \( A \in T \) such that \( 1_A \notin I_1 \). Indeed, otherwise

\[
T(B, A) = T(B, A) \circ 1_A = I(B, A)
\]

for all objects \( A, B \in T \). The statement of part (1) follows from Theorem 3.14 applied to \( M := \{1_A\} \) for an object \( A \in T \) such that \( 1_A \notin I \).

(2) For each multiplicative subset \( M \) of an abelian 2-category \( T \), the thick ideal \( 0_T \) from Example 3.4 intersects \( M \) trivially. This part follows from Theorem 3.14 applied to the thick ideal \( I := 0_T \).
The second part of the corollary, applied to the multiplicative subset $\mathcal{M} := \{1_A\}$ for an object $A \in \mathcal{T}$, implies the following:

**Corollary 3.17.** The prime spectrum of every abelian 2-category $\mathcal{T}$ is nonempty.

**Definition 3.18.** (1) An abelian 2-category $\mathcal{T}$ will be called **prime** if $0_\mathcal{T}$ is a prime ideal of $\mathcal{T}$.

(2) An abelian 2-category $\mathcal{T}$ will be called **simple** if the only proper thick ideal of $\mathcal{T}$ is $0_\mathcal{T}$.

Corollary 3.17 implies that every simple abelian 2-category $\mathcal{T}$ is prime.

### 3.5. The Zariski topology.

**Definition 3.19.** Define the family of closed sets $V(I) := \{P \in \text{Spec}(\mathcal{T}) \mid P \supseteq I\}$ of $\text{Spec}(\mathcal{T})$ for all thick ideals $I$.

**Remark 3.20.** This topology is different from the one considered by Balmer [1]. The main reason for which we consider it is to ensure good behavior under the $K_0$ map, see Theorem [6.12](4).

**Lemma 3.21.** For each abelian 2-category $\mathcal{T}$, the above family of closed sets turns $\text{Spec}(\mathcal{T})$ into a topological space. The corresponding topology will be called the **Zariski topology** of $\text{Spec}(\mathcal{T})$.

It is easy to verify for that for every pair of thick ideals $I, J$ of $\mathcal{T}$ and for every (possibly infinite) collection $\{I_s\}$ of thick ideals of $\mathcal{T}$, similarly to the classical situation, we have

$$V(I) \cup V(J) = V((I_i \circ J_i)) \quad \text{and}$$

$$\bigcap_i V(I_s) = V\left(\left\langle \bigcup_i (I_s)_1 \right\rangle\right).$$

Finally, we also have $V(\mathcal{T}) = \emptyset$ and $V(0_\mathcal{T}) = \text{Spec}(\mathcal{T})$.

### 3.6. An example. Let $\Gamma$ be a nonempty set and $k$ be an arbitrary field. Let $k\text{Vect}_k$ be the category of finite dimensional $k$-vector spaces considered as $(k,k)$-bimodules. Let $\{k_a \mid a \in \Gamma\}$ be a collection of fields isomorphic to $k$ and indexed by $\Gamma$.

There is a unique $k$-linear abelian 2-category $\mathcal{M}_\Gamma(k)$ whose set of objects is $\Gamma$ and such that

$$\mathcal{M}_\Gamma(k)(a,b) := k_b \text{Vect}_{k_a} \quad \text{for } a, b \in \Gamma.$$ 

Its composition bifunctors are given by

$$- \otimes_{k_a} : \mathcal{M}_\Gamma(k)(b,c) \times \mathcal{M}_\Gamma(k)(a,b) \to \mathcal{M}_\Gamma(k)(a,c).$$

Its Grothendieck group is $K_0(\mathcal{M}) \cong M_\Gamma(\mathbb{Z})$ — the ring of square matrices with finitely many nonzero integer entries whose rows and columns are indexed by $\Gamma$. In the terminology of [22], $M_\Gamma(k)$ is a categorification of the matrix ring $M_\Gamma(k')$ for any field $k'$.

Analogously to the classical situation, we show:

**Lemma 3.22.** The abelian 2-categories $\mathcal{M}_\Gamma(k)$ are simple (and thus prime).

**Proof.** Let $I$ be a thick ideal of $\mathcal{M}_\Gamma(k)$ that properly contains the 0-ideal $0_{\mathcal{M}_\Gamma(k)}$. Then for some $a, b \in \Gamma$,

$$I(a,b) \neq 0.$$
Since $\mathcal{I}$ is thick, $\mathcal{I}(a, b)$ is a nonzero subcategory of $k_0 \text{Vect}_{k_0}$ that is closed under taking direct summands. Hence $\mathcal{I}(a, b)$ contains the 1-dimensional vector space in $k_0 \text{Vect}_{k_0}$, and so,

$$\mathcal{I}(a, b) = k_0 \text{Vect}_{k_0} = \mathcal{M}_\Gamma(k)(a, b).$$

Since all objects in $\mathcal{M}_\Gamma(k)$ are isomorphic to each other, Remark 3.6 implies that

$$\mathcal{I}(a', b') = \mathcal{M}_\Gamma(k)(a', b')$$

for all $a', b' \in \Gamma$. Thus $\mathcal{I} = \mathcal{M}_\Gamma(k)$, which completes the proof. \qed

4. Minimal primes in noetherian abelian 2-categories

In this section we define noetherian abelian 2-categories $\mathcal{T}$, and prove that for all proper thick ideals $\mathcal{I}$ of $\mathcal{T}$, there exist finitely many minimal primes over $\mathcal{I}$ and the product of their 1-morphism sets (with repetitions) is contained in $\mathcal{I}_1$.

4.1. Noetherian abelian 2-categories.

Definition 4.1. (1) An abelian 2-category will be called left (resp. right) noetherian if it satisfies the ascending chain condition on thick left (resp. right) ideals.

(2) An abelian 2-category will be called noetherian if it is both left and right noetherian.

(3) An abelian 2-category will be called weakly noetherian if it satisfies the ascending chain condition on (two-sided) thick ideals.

More concretely, an abelian 2-category is noetherian if for every chain of thick left ideals

$$\mathcal{I} \subseteq \mathcal{I}_2 \subseteq \ldots$$

there exists an integer $k$ such that $\mathcal{I}_k = \mathcal{I}_{k+1} = \ldots$ and such a property is also satisfied for ascending chains of thick right ideals.

4.2. Existence of minimal primes.

Lemma 4.2. In any abelian 2-category $\mathcal{T}$, for every thick ideal $\mathcal{I}$ and every prime ideal $\mathcal{P}$ containing $\mathcal{I}$, there is a minimal prime $\mathcal{P}'$ such that

$$\mathcal{I} \subseteq \mathcal{P}' \subseteq \mathcal{P}.$$

Proof. Let $\chi$ denote the set of primes which contain $\mathcal{I}$ and are contained in $\mathcal{P}$. We will use Zorn’s lemma to produce a minimal element of this set. We first show that any nonempty chain in $\chi$ has a lower bound in $\chi$. Take a nonempty chain of prime ideals in $\chi$, say

$$\mathcal{P}^{(1)} \supseteq \mathcal{P}^{(2)} \supseteq \ldots$$

Then define $\mathcal{Q} = \cap_{i=1}^{\infty} \mathcal{P}^{(i)}$. Since each $\mathcal{P}^{(i)}$ contains $\mathcal{I}$ and is contained in $\mathcal{P}$, $\mathcal{Q}$ is a thick ideal which has these properties. It remains to show that $\mathcal{Q}$ is a prime ideal. Take $f, g \in \mathcal{T}_1$ such that $f \circ \mathcal{T}_1 \circ g \subset \mathcal{Q}_1$, and $f \notin \mathcal{Q}_1$. Then $f$ is not in some $\mathcal{P}_1^{(i)}$. Therefore, $f \notin \mathcal{P}_1^{(j)}$ for $j \geq i$, and by the primeness of $\mathcal{P}^{(j)}$, $g \in \mathcal{P}_1^{(j)}$, for $j \geq i$ as well. Therefore, $g \in \mathcal{P}_1^{(k)}$ for all $k$, and thus, $g \in \mathcal{Q}_1$. This implies that $\mathcal{Q}$ is prime, and Zorn’s lemma completes the proof. \qed
4.3. Finiteness and product properties of minimal primes.

**Theorem 4.3.** In a weakly noetherian abelian 2-category $\mathcal{T}$, for every proper thick ideal $\mathcal{I}$, there exist finitely many minimal prime ideals over $\mathcal{I}$. Furthermore, there exists a finite list of minimal prime ideals over $\mathcal{I}$ (potentially with repetition) $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(m)}$ such that the product

$$\mathcal{P}^{(1)} \circ \ldots \circ \mathcal{P}^{(m)} \subseteq \mathcal{I}_1.$$

**Proof.** Denote the set

$$\chi := \{ \mathcal{I} \text{ a proper thick ideal of } \mathcal{T} \mid \# \text{ prime ideals } \mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(m)} \supseteq \mathcal{I} \text{ such that } \mathcal{P}^{(1)} \circ \ldots \circ \mathcal{P}^{(m)} \subseteq \mathcal{I}_1} \}.$$

Suppose that $\chi$ is nonempty. By the weakly noetherian property of $\mathcal{T}$, there exists a maximal element of $\chi$ (because every ascending chain in $\chi$ eventually stabilizes). Let $\mathcal{I}$ be a maximal element of $\chi$. The ideal $\mathcal{I}$ cannot be prime, since $\mathcal{I} \in \chi$. By Theorem 3.12 there exist proper thick ideals $\mathcal{J}$ and $\mathcal{K}$ such that $\mathcal{J} \circ \mathcal{K} \subseteq \mathcal{I}_1$, where $\mathcal{J}$ and $\mathcal{K}$ both properly contain $\mathcal{I}$. The latter property of $\mathcal{J}$ and $\mathcal{K}$, and the maximality of $\mathcal{I}$, imply that $\mathcal{J}, \mathcal{K} \notin \chi$. Hence, there exist two collection of prime ideals $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(m)} \supseteq \mathcal{J}$ and $\mathcal{Q}^{(1)}, \ldots, \mathcal{Q}^{(n)} \supseteq \mathcal{K}$ such that

$$\mathcal{P}^{(1)} \circ \ldots \circ \mathcal{P}^{(m)} \subseteq \mathcal{J} \quad \text{and} \quad \mathcal{Q}^{(1)} \circ \ldots \circ \mathcal{Q}^{(n)} \subseteq \mathcal{K}.$$

Then

$$\mathcal{P}^{(1)} \circ \ldots \circ \mathcal{P}^{(m)} \circ \mathcal{Q}^{(1)} \circ \ldots \circ \mathcal{Q}^{(n)} \subseteq \mathcal{I}_1,$$

giving a contradiction, since the ideals $\mathcal{P}^{(i)}$ and $\mathcal{Q}^{(j)}$ are prime and contain $\mathcal{I}$.

Hence, $\chi$ is empty. In other words, for every proper thick ideal $\mathcal{I}$ of $\mathcal{T}$ there exist prime ideals $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(m)} \supseteq \mathcal{I}$ such that

$$\mathcal{P}^{(1)} \circ \ldots \circ \mathcal{P}^{(m)} \subseteq \mathcal{I}_1.$$

Applying Lemma 4.2, we obtain that for each $\mathcal{P}^{(i)}$, there exists a minimal prime $\overline{\mathcal{P}}^{(i)}$ over $\mathcal{I}$ such that $\overline{\mathcal{P}}^{(i)} \subseteq \mathcal{P}^{(i)}$. Combining this with (4.1) gives

$$\overline{\mathcal{P}}^{(1)} \circ \ldots \circ \overline{\mathcal{P}}^{(m)} \subseteq \mathcal{I}_1$$

for the minimal primes $\overline{\mathcal{P}}^{(1)}, \ldots, \overline{\mathcal{P}}^{(m)}$ of $\mathcal{I}$.

Finally, we claim that every minimal prime ideal $\mathcal{P}$ over $\mathcal{I}$ is in the list $\overline{\mathcal{P}}^{(1)}, \ldots, \overline{\mathcal{P}}^{(m)}$. This implies that there are only finitely many primes of $\mathcal{T}$ that are minimal over $\mathcal{I}$. Indeed, we have

$$\overline{\mathcal{P}}^{(1)} \circ \ldots \circ \overline{\mathcal{P}}^{(m)} \subseteq \mathcal{P}_1,$$

and by the primeness of $\mathcal{P}$, we have

$$\mathcal{I} \subseteq \mathcal{P}^{(i)} \subseteq \mathcal{P}$$

for some $i$. Since $\mathcal{P}$ is minimal over $\mathcal{I}$, $\mathcal{P}^{(i)} = \mathcal{P}$. \hfill \square

The following corollary follows from applying Theorem 4.3 to $0_\mathcal{T}$.

**Corollary 4.4.** A weakly noetherian abelian 2-category has finitely many minimal prime ideals.

We also have the following corollary of Theorem 4.3.
Corollary 4.5. For a weakly noetherian abelian 2-category $\mathcal{T}$, all closed subsets of $\text{Spec}(\mathcal{T})$ (with respect to the Zariski topology) are finite intersections of finitely many sets of the form $V(P)$ for prime ideals $P$ of $\mathcal{T}$, recall §3.5.

5. The completely prime spectrum and the semiprime spectrum

In this section we define the notions of completely prime and semiprime ideals of abelian 2-categories, and give equivalent characterizations, one of which is an extension of the Levitzki–Nagata theorem from noncommutative ring theory.

5.1. Completely prime ideals.

Definition 5.1. A thick ideal $P$ of an abelian 2-category $\mathcal{T}$ will be called completely prime when it has the property that for all $f, g \in \mathcal{T}$:

$$f \circ g \subseteq P \quad \Rightarrow \quad f \in P \quad \text{or} \quad g \in P.$$

This is equivalent to saying that for all 1-morphisms $f$ and $g$ of $\mathcal{T}$, if $fg$ is not defined or $fg$ is a 1-morphism in $P$, then $f$ or $g$ is a 1-morphism in $P$. The stronger assumption, including the case of the condition when $fg$ is not defined, is needed to get the correct analog of a completely prime ideal of an algebra with a set of orthogonal idempotents.

Let $R$ be a ring and $\{e_s\}$ be a collection of orthogonal idempotents. If $I$ is a completely prime ideal of $R$ such that $I \subseteq \bigoplus_{s,t} e_s Re_t$, then for all $s, t \neq t', s'$,

either $e_s Re_t \subseteq I$ or $e_{t'} Re_{s'} \subseteq I$,

because $(e_s Re_t)(e_{t'} Re_{s'}) = 0$.

Theorem 3.10 implies the following:

Corollary 5.2. Every completely prime ideal of an abelian 2-category is prime.

Proof. Assume that $P$ is a completely prime ideal of $\mathcal{T}$. Let $f \in \mathcal{T}(A_3, A_4)$ and $g \in \mathcal{T}(A_1, A_2)$ be such that

$$f \circ \mathcal{T}_1 \circ g \subseteq P.$$ 

If $A_2 \neq A_3$, then $fg$ is not defined and the assumption on $P$ gives that either $f \in P$ or $g \in P$. If $A_2 = A_3$, then

$$fg = f1_{A_2}g \in f \circ \mathcal{T}_1 \circ g \subseteq P,$$

and, again by the assumption on $P$, we have that either $f \in P$ or $g \in P$. \qed

For every abelian 2-category $\mathcal{T}$, given an object $A$ of $\mathcal{T}$, consider the 2-subcategory $\mathcal{T}_A$ of $\mathcal{T}$ having one object $A$ and such that $\mathcal{T}_A(A, A) := \mathcal{T}(A, A)$. It is an abelian 2-category with one object (i.e., a multiring category). The next lemma shows that the completely prime ideals of an abelian 2-category $\mathcal{T}$ are classified in terms of the completely prime ideals of these multiring categories.

Lemma 5.3. Let $\mathcal{T}$ be an abelian 2-category.

(1) If $P$ is a completely prime ideal of $\mathcal{T}$, then there exists an object $A$ of $\mathcal{T}$ and a completely prime ideal $Q$ of the multiring category $\mathcal{T}_A$ such that

$$P(B, C) = \begin{cases} Q(A, A), & \text{if } B = C = A \\ \mathcal{T}(B, C), & \text{otherwise.} \end{cases}$$
(2) If $A$ is an object of $\mathcal{T}$ and $Q$ is a completely prime ideal of $\mathcal{T}_A$ such that

\[
\mathcal{T}(B, A) \circ \mathcal{T}(A, B) \subseteq Q(A, A)
\]

for every object $B$ of $\mathcal{T}$, then (5.1) defines a completely prime ideal $\mathcal{P}$ of $\mathcal{T}$.

**Proof.** (1) Since $\mathcal{P}$ is a proper thick ideal of $\mathcal{T}$, there exists an object $A$ of $\mathcal{T}$ such that

\[
1_A / \in \mathcal{P}.
\]

(Otherwise $\mathcal{P}$ will contain all 1-morphisms of $\mathcal{T}$ because

\[
\mathcal{T}(B, A) \circ 1_A = \mathcal{T}(B, A).
\]

This will contradict the properness of $\mathcal{P}$.)

Obviously $Q(A, A) := \mathcal{P}(A, A)$ defines a completely prime ideal of the multiring category $\mathcal{T}_A$. It remains to show that $\mathcal{P}$ is given by (5.1) in terms of $Q$.

(2) The condition (5.2) ensures that the weak thick subcategory $\mathcal{P}$ of $\mathcal{T}$ given by (5.1), is a thick ideal of $\mathcal{T}$. Its complete primeness is easy to show. □

**Definition 5.4.** A multiring category $\mathcal{T}$ will be called a **domain** if its zero ideal $0_\mathcal{T}$ is completely prime, i.e., if

\[
M \otimes N \cong 0 \Rightarrow M \cong 0 \text{ or } N \cong 0
\]

for all objects $M$ of $\mathcal{T}$.

An abelian 2-category $\mathcal{T}$ will be called **prime**, if its zero ideal $0_\mathcal{T}$ is prime.

**Example 5.5.** Let $H$ be a Hopf algebra over a field $k$. Denote by $H-\text{mod}$ the category of finite dimensional $H$-modules. It is a $k$-linear multiring category. This category is a domain: if $V, W \in H-\text{mod}$ are such that $V \otimes W \cong 0$, then

\[
\dim V \dim W = 0.
\]

Therefore, either $\dim V = 0$ or $\dim W = 0$. So, either $V \cong 0$ or $W \cong 0$. □

Let $\mathcal{T}$ be an abelian 2-category. The same proof shows that if

(1) $\eta : \mathcal{T}_1 \to R$ is a map such that $R$ is a domain and $\eta(fg) = \eta(f)\eta(g)$ for all $f, g \in \mathcal{T}$ for which the composition is defined, and

(2) $\mathcal{I}$ is a thick ideal of $\mathcal{T}$ such that $\mathcal{I}_1 = \eta^{-1}(0)$,

then $\mathcal{I}$ is a completely prime ideal of $\mathcal{T}$.

### 5.2. Semiprime ideals.

**Definition 5.6.** A thick ideal of an abelian 2-category will be called **semiprime** if it is an intersection of prime ideals. An abelian 2-category $\mathcal{T}$ will be called **semiprime**, if its zero ideal $0_\mathcal{T}$ is semiprime.

Theorem 4.3 implies that in a weak noetherian abelian 2-category every semiprime ideal is the intersection of the finitely many minimal primes over it.

The following theorem is a categorical version of the Levitzki–Nagata theorem.

**Theorem 5.7.** A thick ideal $\mathcal{Q}$ is semiprime if and only if for all $f \in \mathcal{T}_1$,

\[
f \circ \mathcal{T}_1 \circ f \subseteq \mathcal{Q}_1 \Rightarrow f \in \mathcal{Q}_1.
\]
Proof. First, suppose $Q = \bigcap_{s} P^{(s)}$ for some collection $\{P^{(s)}\}$ of primes of $T$. Suppose $f \in T_1$, and $f \circ T_1 \circ f \subseteq Q_1$. By primeness, $f \in P^{(s)}_{1}$ for all $s$. Therefore, $f \in Q_1$.

For the other direction, suppose that $Q$ is a thick ideal of $T$ having the property (5.4). Choose an element

$$g \in T_1, \ g \notin Q_1,$$

and set $g_0 := g$. It follows from (5.4) that $g_0 \circ T_1 \circ g_0 \not\subseteq Q_1$. Choose

$$g_1 \in g_0 \circ T_1 \circ g_0, \ g_1 \notin Q_1.$$ 

Again, since $g_1 \notin Q_1$, the condition (5.4) implies that $g_1 \circ T_1 \circ g_1 \not\subseteq Q_1$. Proceeding inductively in this manner, we construct a sequence of 1-morphisms $g_0, g_1, \ldots$ of $T$ such that

$$(5.5) \quad g_i \in g_{i-1} \circ T_1 \circ g_{i-1}, \ g_i \notin Q_1.$$ 

Since $g_i \in g_{i-1} \circ T \circ g_{i-1}$, we have $g_i \circ T \circ g_i \subseteq g_{i-1} \circ T \circ g_{i-1}$. Consider the set $S$ of thick ideals $I$ of $T$ such that

$$Q \subseteq I \quad \text{and} \quad g_i \notin I \quad \text{for all} \quad i = 0, 1, \ldots.$$ 

This set is nonempty because $Q \subseteq S$. Since the union of a chain of thick ideals is a thick ideal, we can apply Zorn’s lemma to get that $S$ contains a maximal element. Denote one such element by $P$. The proper thick ideal $P$ is prime. Indeed, if $J$ and $K$ are thick ideals that properly contain $P$, then by maximality of $P$, there are some $g_j \in J_1$ and $g_k \in K_1$. If $m$ is the max of $j$ and $k$, then $g_m$ is in both $J_1$ and $K_1$ by the first property in (5.5).

Hence,

$$g_{m+1} \in g_m \circ T_1 \circ g_m \subseteq J_1 \circ K_1 \quad \text{and} \quad g_{m+1} \notin P_1.$$ 

Therefore, $J_1 \circ K_1 \subseteq P_1$, and by Theorem 3.12 $P$ is prime. For every element $g \in T_1$ that is not in $Q_1$, we have produced a prime $P^{(g)}$ of $T$ such that

$$Q \subseteq P^{(g)} \quad \text{and} \quad g \notin P^{(g)}.$$ 

Therefore,

$$Q = \bigcap_{g \in T_1 \setminus Q_1} P^{(g)},$$

which completes the proof of the theorem. \qed

Theorem 5.8. Suppose $Q$ is a proper thick ideal in an abelian 2-category $T$. Then the following are equivalent:

1. $Q$ is semiprime;
2. If $I$ is any thick ideal of $T$ such that $I_1 \circ I_1 \subseteq Q_1$, then $I \subseteq Q$;
3. If $I$ is any thick ideal properly containing $Q$, then $I_1 \circ I_1 \not\subseteq Q_1$;
4. If $I$ is any right thick ideal of $T$ such that $I_1 \circ I_1 \subseteq Q_1$, then $I \subseteq Q$;
5. If $I$ is any left thick ideal of $T$ such that $I_1 \circ I_1 \subseteq Q_1$, then $I \subseteq Q$.

Proof. (1) $\Rightarrow$ (4): Suppose $Q$ is semiprime and $I$ is a right thick ideal with $I_1 \circ I_1 \subseteq Q_1$. Take any $i \in I_1$. Then $i \circ t \in I$ for all $t \in T_1$. Therefore, $i \circ T_1 \circ i \in Q_1$. Theorem 5.7 implies that $i \in Q$. Hence, $I_1 \subseteq Q_1$, and thus $I \subseteq Q$ by Remark 3.3.

(1) $\Rightarrow$ (5): This follows from a symmetric argument.

(4) $\Rightarrow$ (5) and (3): This is clear, since a thick ideal is also a right thick ideal, and a left thick ideal.

(3) $\Rightarrow$ (2): Suppose (3) holds, and $I$ is a thick ideal with $I_1 \circ I_1 \subseteq Q_1$. Then $(I_1 \cup Q_1)$ is a thick ideal containing $Q$. Since

$$(I_1 \cup Q_1) \circ (I_1 \cup Q_1) = (I_1 \circ I_1) \cup (Q_1 \circ I_1) \cup (I_1 \circ Q_1) \cup (Q_1 \circ Q_1) \subseteq Q_1,$$
applying Lemma 3.9, we obtain
\[
\langle I_1 \cup Q_1 \rangle_1 \circ \langle I_1 \cup Q_1 \rangle_1 \subseteq \langle (I_1 \cup Q_1) \circ (I_1 \cup Q_1) \rangle_1 = \langle (I_1 \cup Q_1) \circ (I_1 \cup Q_1) \rangle_1 \subseteq Q_1.
\]
From the assumption that the ideal \( Q \) has the property (3) and the fact that \( (I_1 \cup Q_1) \) is a thick ideal containing \( Q \), we get that \( \langle I_1 \cup Q_1 \rangle = Q \). Therefore, \( I_1 \subseteq Q_1 \), and thus \( I \subseteq Q \) by Remark 3.3.

\( (2) \Rightarrow (1) \): Suppose (2) holds, and \( f \in T \) is a 1-morphism such that \( f \circ T_1 \circ f \subseteq Q_1 \).
Lemma 3.9 implies that
\[
\langle f \rangle_1 \circ \langle f \rangle_1 \subseteq \langle f \circ T_1 \circ f \rangle_1 \subseteq Q_1.
\]
Therefore, by (2), \( \langle f \rangle_1 \subseteq Q_1 \), and so, \( f \in Q_1 \). Hence, \( Q_1 \) is semiprime.

We have the following corollary from the characterizations (4) and (5) of semiprime ideals in the previous theorem. For a subset \( X \subseteq T_1 \), denote by \( X^{\circ n} := X \circ \cdots \circ X \) the \( n \)-fold composition power.

**Lemma 5.9.** If \( Q \) is a semiprime ideal of the abelian 2-category \( T \), and \( I \) is a right or left thick ideal with \( (I_1)^{\circ n} \subseteq Q_1 \), then \( I \subseteq Q \).

**Proof.** We prove the statement by induction on \( n \). For \( n \geq 2 \), we have
\[
((I_1)^{\circ (n-1)})^{\circ 2} = (I_1)^{\circ n} \circ (I_1)^{\circ n-2} \subseteq Q_1.
\]
Theorem 5.8 implies that \( (I_1)^{\circ (n-1)} \subseteq Q_1 \), and so by the inductive assumption, \( I \subseteq Q \). 

\[\square\]

6. **The Serre prime spectra of abelian 2-categories and \( \mathbb{Z}_+ \)-rings**

In this section we define and investigate the notions of Serre prime, semiprime, and completely prime ideals of abelian 2-categories and \( \mathbb{Z}_+ \)-rings. We establish that the corresponding topological spaces for abelian 2-categories and \( \mathbb{Z}_+ \)-rings are homeomorphic. We also describe the relations of the first set of notions to the notions of prime, completely prime and semiprime ideals of abelian 2-categories, and the second set of notions to the prime spectra of rings.

6.1. **Serre ideals of abelian 2-categories.** Recall that a Serre subcategory of an abelian 1-category is a subcategory which is closed under subobjects, quotients, and extensions. Every Serre subcategory \( I \) of an abelian category \( C \) is thick, and in particular, is closed under isomorphisms. For such a subcategory, one forms the Serre quotient \( C/I \) which has a canonical structure of abelian category [41, §10.3]. By [36, Theorem 5], for every Serre subcategory \( I \) of an abelian category \( C \), we have the exact sequence
\[
K_0(I) \to K_0(C) \to K_0(C/I) \to 0.
\]

**Definition 6.1.** (1) We call a thick ideal \( I \) of an abelian 2-category \( T \) a **Serre ideal** if for every two objects \( A_1, A_2 \in T \),
\[
I(A_1, A_2) \text{ is a Serre subcategory of } T(A_1, A_2).
\]
(2) A Serre prime (resp. semiprime, completely prime) ideal \( P \) of an abelian 2-category \( T \) is a prime (resp. semiprime, completely prime) ideal which is a Serre ideal.

In the terminology of Definition 3.1, a Serre ideal of an abelian 2-category \( T \) is a weak subcategory \( I \) with the same set of objects such that

(1) for any pair of objects \( A_1, A_2 \in T \), \( I(A_1, A_2) \) is a Serre subcategory of the abelian category \( T(A_1, A_2) \) and
objects, with the morphism 1-categories

for every Serre ideal 1. We will say that 1 is a left (resp. right) Serre ideal of 2 if condition (1) is satisfied and 1 ⊆ 1 (resp. 1 ⊆ 1).

Proposition 6.2. For every Serre ideal 1 of an abelian 2-category 2 such that 1A ⊈ 1(A, A) for all objects A ∈ 2, one can form the Serre quotient 2/1 with the same set of objects, with the morphism 1-categories

(2/1)(A1, A2) := 2(A1, A2)/1(A1, A2) for A1, A2 ∈ 2,

and with identity 1-morphisms given by the images of 1A. This quotient is an abelian 2-category.

The proof of the proposition is direct, using (6.1) and the following well known fact:

If, for i = 1, 2, 1i are abelian categories, 1i are Serre subcategories, and 1 : 1i → 12 is an exact functor such that 1(1i) ⊆ 12, then the induced functor 1 : 1i/1i → 12/12 is exact.

This follows from the commutativity of the square diagram consisting of the compositions of functors 1i → 1i/1i → 12/12 and 1i → 12/12, the exactness of the projection functors 1i → 1i/1i (see Exercise 10.3.2(4)), and the fact that every exact sequence in 1i/1i is isomorphic to one coming from an exact sequence in 1i.

It is easy to prove that, similarly to the ring theoretic case, we have the following:

Lemma 6.3. A proper Serre ideal 1 of a multiring category 2 is completely prime, if and only if the Serre quotient 2/1 is a domain in the sense of Definition 5.4.

Analogously to Lemmas 3.8 and 3.9 one proves the following result. We leave the details to the reader.

Lemma 6.4. Let 2 be an abelian 2-category.

(1) The intersection of any family of Serre ideals of 2 is a Serre ideal of 2. In particular, for any subset 1 ⊆ 2, there exists a unique minimal Serre ideal of 2 containing 1; it will be denoted by ⟨1⟩S.

(2) For 1, 2 ⊆ 2, we have

⟨1⟩S ∩ ⟨2⟩S ⊆ ⟨1 ∩ 2⟩S.

6.2. Serre prime ideals of abelian 2-categories. Similarly to the proofs of Theorems 3.10 3.12 3.14 and 5.8 using Lemma 6.4 one proves the following result:

Theorem 6.5. Let 2 be an abelian 2-category.

(1) The following are equivalent for a proper Serre ideal 1 of 2:

(a) 1 is a Serre prime ideal;
(b) If 1 and 2 are any Serre ideals of 2 such that 1 ∩ 2 ⊆ 1, then either 1 ⊆ 1 or 2 ⊆ 1;
(c) If 1 and 2 are any Serre ideals properly containing 1, then 1 ∩ 2 ⊆ 1;
(d) If 1 and 2 are any left Serre ideals of 2 such that 1 ∩ 2 ⊆ 1, then either 1 ⊆ 1 or 2 ⊆ 1.

(2) Let 1 be a nonempty multiplicative subset of 2 (cf. Definition 3.13) and 1 be a Serre ideal of 2 such that 1 ∩ 1 = 0. Let 1 be a maximal element of the collection of Serre ideals of 2 containing 1 and intersecting 1 trivially, equipped with the inclusion relation, i.e., 1 is a maximal element of the set

X(1, 1) := {K a Serre ideal of 2 | K ⊇ 1 and K ∩ 1 = 0}.

Then 1 is Serre prime ideal.
The following are equivalent for a proper Serre ideal \( Q \) of \( T \):

(a) \( Q \) is a Serre semiprime ideal;
(b) If \( I \) is any Serre ideal of \( R \) such that \( I_1 \circ I_1 \subseteq Q_1 \), then \( I \subseteq Q \);
(c) If \( I \) is any Serre ideal properly containing \( Q \), then \( I_1 \circ I_1 \not\subseteq Q_1 \);
(d) If \( I \) is any left Serre ideal of \( T \) such that \( I_1 \circ I_1 \subseteq Q_1 \), then \( I \subseteq Q \).

In the proof of part (1) of the theorem, the key step is to show that a proper Serre ideal \( I \) of \( T \) satisfying the property (b) is a Serre prime ideal of \( T \). This is proved by showing that property (b) implies that for all \( m, n \in T_1 \),

\[
m \circ T_1 \circ n \subseteq P_1 \Rightarrow m \in P_1 \text{ or } n \in P_1.
\]

This fact is verified by repeating the proof of Theorem 3.10, but using Lemma 6.4(2) in place of Lemma 3.9.

The set \( X(M, I) \) in part (3) of the theorem is nonempty because \( I \in X(M, I) \). The union of an ascending chain of Serre ideals in the set \( X(M, I) \) is obviously a Serre ideal of \( T \). By Zorn’s lemma, the set \( X(M, I) \) always contains at least one maximal element.

Similarly to the proof of Corollary 3.16, we obtain:

**Corollary 6.6.** For every proper Serre ideal \( I \) of an abelian 2-category \( T \), there exists a Serre prime ideal \( P \) of \( T \) that contains \( I \).

Analogously to the proof of Theorem 4.3 one proves the following:

**Proposition 6.7.** For every abelian 2-category \( T \) satisfying the ACC on (2-sided) Serre ideals, given a proper Serre ideal \( I \) of \( T \), there exist finitely many minimal Serre prime ideals over \( I \). Furthermore, there is a finite list of minimal Serre prime ideals over \( I \) (possibly with repetition) \( P^{(1)}, ..., P^{(m)} \) such that the product

\[
P^{(1)}_1 \circ ... \circ P^{(m)}_1 \subseteq I_1.
\]

Let Serre-Spec\((T)\) denote the set of Serre prime ideals of an abelian 2-category \( T \). Similarly to \( [3,5] \) one shows that it is a topological space with closed subsets given by

\[
V^S(I) = \{ P \in \text{Serre-Spec}(T) \mid P \supseteq I \}
\]

for the Serre ideals \( I \) of \( T \). We will refer to this as to the Zariski topology of Serre-Spec\((T)\).

Proposition 6.7 implies that, if \( T \) satisfies the ACC on Serre ideals, then every closed subset of Serre-Spec\((T)\) is a finite intersection of subsets of the form \( V^S(P) \) for some \( P \in \text{Serre-Spec}(T) \). In particular, this property holds for all weakly noetherian abelian 2-categories \( T \).

The set-theoretic inclusion

\[
(6.2) \quad \text{Serre-Spec}(T) \hookrightarrow \text{Spec}(T)
\]

realizes Serre-Spec\((T)\) as a topological subspace of Spec\((T)\). Indeed, Lemma 6.4(1) implies that for every thick ideal \( I \) of \( T \) we have

\[
V(I) \cap \text{Serre-Spec}(T) = V^S(I^S).
\]

**6.3. The Serre prime spectrum as a ringed space.** For the following subsection, assume that \( \mathcal{C} \) is an abelian monoidal category. In the case when it is strict this is the same as an abelian 2-category with one object. All constructions and results in the paper are valid for abelian monoidal categories without the strictness assumption. By Remark 3.20 and the embedding \( (6.2) \), the Zariski topology we have thus far endowed Serre-Spec\((\mathcal{C})\) with is different from the topology used by Balmer in \([1]\). The motivation for this consists of the applications to categorification, which we develop below. However,
if \( C \) is an abelian monoidal category, we can consider an analogue of Balmer’s topology on \( \text{Serre-Spec}(C) \), where we define the closed sets of \( \text{Serre-Spec}(C) \) to be
\[
V_S^C(X) = \{ P \in \text{Serre-Spec}(C) \mid X \cap P = \emptyset \}
\]
for any set of objects \( X \) in \( C \). Analogously to Section 2 of [1], one shows that this collection defines a topological space. It may be equipped with a sheaf of commutative rings in a similar manner to [1]. Let \( U = V^C \) be an open set of \( \text{Serre-Spec}(C) \), where \( V^C = \bigcap_{P \in U} P \). Note that \( C \) is a Serre ideal, since it is an intersection of Serre ideals. We define a presheaf of commutative rings in the following way:
\[
U \mapsto \text{End}_{C/C}(\mathbf{1}, \mathbf{1}),
\]
where \( \mathbf{1} \) is the image of 1 (the unit object of \( C \) with respect to the monoidal product) in the Serre quotient \( C/C \). Recalling Proposition 6.2, \( C/C \) has a canonical structure as an abelian monoidal category. By e.g. Proposition 2.2.10 in [7], \( \text{End}_{C/C}(\mathbf{1}, \mathbf{1}) \) is a commutative ring. The sheafification of this presheaf gives \( \text{Serre-Spec}(C) \) the structure of a ringed space. The question about the construction of a ringed space structure on the spectra of abelian monoidal categories was raised by Michael Wemyss.

6.4. \( \mathbb{Z}_+ \)-rings and their Serre prime ideals. Recall that \( \mathbb{Z}_+ := \{0,1,\ldots\} \).

We will use the following slightly weaker terminology for \( \mathbb{Z}_+ \)-rings compared to [7, Definition 3.1.1]:

**Definition 6.8.** We will call a ring \( R \) a \( \mathbb{Z}_+ \)-ring if it is a free abelian group and has a \( \mathbb{Z} \)-basis \( \{ b_\gamma \mid \gamma \in \Gamma \} \) such that for all \( \alpha, \beta \in \Gamma \),
\[
b_\alpha b_\beta = \sum_{\gamma \in \Gamma} n_{\alpha,\beta}^{\gamma} b_\gamma
\]
for some \( n_{\alpha,\beta}^{\gamma} \in \mathbb{Z}_+ \).

In addition, [7] Definition 3.1.1] requires that a \( \mathbb{Z}_+ \)-ring \( R \) be a unital ring and
\[
1 = \sum_{\gamma \in \Gamma} n_{\gamma} b_\gamma \quad \text{for some} \quad n_{\gamma} \in \mathbb{Z}_+.
\]

We do not require a \( \mathbb{Z}_+ \)-ring to be unital and to have the above additional property in order to apply the notion to the Grothendieck rings of abelian 2-categories with infinitely many objects.

For a \( \mathbb{Z}_+ \)-ring \( R \), denote
\[
R_+ := \bigoplus_{\gamma \in \Gamma} \mathbb{Z}_+ b_\gamma.
\]

For \( r, s \in R \), denote
\[
r \leq s \quad \text{if} \quad s - r \in R_+.
\]

**Definition 6.9.** Let \( R \) be a \( \mathbb{Z}_+ \)-ring.

1. A left (resp. right) ideal \( I \) of \( R \) will be called a a left (resp. right) Serre ideal if it has the properties that
\[
I = (I \cap R_+) - (I \cap R_+) \quad \text{and} \quad s \in R_+, r \in I \cap R_+, s \leq r \Rightarrow s \in I.
\]

2. A Serre ideal of \( R \) is a 2-sided ideal \( I \) of \( R \) which satisfies (6.4).

3. A Serre prime ideal of \( R \) is a proper Serre ideal \( P \) of \( R \) that has the property that
\[
IJ \subseteq P \quad \Rightarrow \quad I \subseteq P \quad \text{or} \quad J \subseteq P.
\]
for all Serre ideals $I, J$ of $R$.

(4) A Serre semiprime ideal of $R$ is an ideal which is the intersection of Serre prime ideals.

(5) A Serre completely prime ideal of $R$ is a proper Serre ideal $P$ that has the property that for all $r, s \in R_+$,

$$rs \in P \implies r \in P \text{ or } s \in P.$$ 

For a subgroup $I$ (under addition) of a $\mathbb{Z}_+$-ring $R$, the property (6.4) is equivalent to

$$I = \bigoplus_{\gamma \in \Gamma'} \mathbb{Z}b_\gamma \text{ for some subset } \Gamma' \subseteq \Gamma.$$ 

In particular, the right and 2-sided Serre ideals of $R$ satisfy (6.5). Using this fact one easily proves the following theorem, by following the strategy of the proofs of Proposition 3.1, Theorem 3.7 and Corollary 3.8 in [11].

**Theorem 6.10.** Let $R$ be a $\mathbb{Z}_+$-ring.

(1) The following are equivalent for a proper Serre ideal $P$ of $R$:

(a) $P$ is a Serre prime ideal;
(b) If $I$ and $J$ are two Serre ideals of $R$ properly containing $P$, then $IJ \not\subseteq P$;
(c) If $I$ and $J$ are two left Serre ideals of $R$ such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$;
(d) For all $\alpha, \beta \in \Gamma$,

$$b_\alpha R b_\beta \subseteq P \implies b_\alpha \in P \text{ or } b_\beta \in P.$$ 

(2) A proper Serre ideal $P$ of $R$ is a completely prime Serre ideal if and only if for all $\alpha, \beta \in \Gamma$,

$$b_\alpha b_\beta \subseteq P \implies b_\alpha \in P \text{ or } b_\beta \in P.$$ 

(3) The following are equivalent for a proper Serre ideal $Q$ of $R$:

(a) $Q$ is a Serre semiprime ideal;
(b) If $I$ is any Serre ideal of $R$ such that $I^2 \subseteq Q$, then $I \subseteq Q$;
(c) If $I$ is any Serre ideal of $R$ properly containing $Q$, then $I^2 \not\subseteq Q$;
(d) For all $r \in R_+$,

$$rRr \subseteq P \implies r \in P.$$ 

Denote by $\text{Serre-Spec}(R)$ the set of Serre prime ideals of a $\mathbb{Z}_+$-ring $R$. Similarly to §6.2 one proves that $\text{Serre-Spec}(R)$ is a topological space with closed subsets

$$V^S(I) = \{P \in \text{Serre-Spec}(T) \mid P \supseteq I\}$$

for the Serre ideals $I$ of $R$. We will call this the Zariski topology of $\text{Serre-Spec}(R)$.

### 6.5. $\mathbb{Z}_+$-rings and abelian 2-categories

For an abelian category $\mathcal{C}$ denote by $\mathcal{C}_s$ the equivalence classes of its simple objects.

**Lemma 6.11.** Assume that $\mathcal{C}$ is an abelian category in which every object has finite length. Then the following hold:

(1) Every two Jordan-Hölder series of an object of $\mathcal{C}$ contain the same collections of simple subquotients counted with multiplicities and, as a consequence,

$$K_0(\mathcal{C}) \cong \bigoplus_{A \in \mathcal{C}_s} \mathbb{Z}[A].$$
(2) The Serre subcategories of \( C \) are in bijection with the subsets of \( C_s \). The Serre subcategory corresponding to a subset \( X \subseteq C_s \) is the full subcategory \( S(X) \) of \( C \) whose objects have Jordan-Hölder series with simple subquotients isomorphic to objects in \( X \).

(3) For every Serre subcategory \( \mathcal{I} \) of \( C \),

\[
K_0(C/\mathcal{I}) \cong K_0(C)/K_0(\mathcal{I}).
\]

**Proof.** The first part of the lemma is [7, Theorem 1.5.4].

(2) Clearly, for every subset \( X \subseteq C_s \), the subcategory \( S(X) \) of \( C \) is Serre. Assume that \( \mathcal{I} \) is a Serre subcategory of \( C \). Denote by \( X \) the isomorphism classes of simple objects of \( C \) which belong to \( \mathcal{I} \). Since \( \mathcal{I} \) is closed under taking subquotients and isomorphisms, \( \mathcal{I} \subseteq S(X) \). Because \( \mathcal{I} \) is closed under extensions, \( S(X) \subseteq \mathcal{I} \). Thus, \( \mathcal{I} = S(X) \).

(3) It easily follows from parts (1) and (2) that the first map in (6.1) is injective. The resulting short exact sequence from (6.1) implies the third part of the lemma. □

For an abelian 2-category \( T \), denote by \((T_1)_s\) the isomorphism classes of simple 1-morphisms of \( T \). Recall Definition 3.1. For a subset \( X \subseteq (T_1)_s \), denote by \( S(X) \) the (unique) weak subcategory of \( T \) such that

\[
S(X)(A_1, A_2) := S(X \cap \mathcal{T}(A_1, A_2))
\]

for all \( A_1, A_2 \in \mathcal{T} \).

**Theorem 6.12.** Assume that \( T \) is an abelian 2-category with the property that every 1-morphism of \( T \) has finite length. (In other words, every object of \( T(A_1, A_2) \) has finite length for all objects \( A_1, A_2 \in T \).) Then the following hold:

(1) The weak subcategories \( \mathcal{I} \) of \( T \) with the property that \( \mathcal{I}(A_1, A_2) \) is a Serre subcategory of \( T(A_1, A_2) \) for all \( A_1, A_2 \in T \) are parametrized by the subsets of \((T_1)_s\). For \( X \subseteq (T_1)_s \), the corresponding subcategory is \( S(X) \).

(2) The Grothendieck ring \( K_0(T) \) is a \( \mathbb{Z}_+ \)-ring and

\[
K_0(T) \cong \bigoplus_{f \in (T_1)_s} \mathbb{Z}[f].
\]

If, in addition, \( T \) has finitely many objects, then \( K_0(T) \) has the property (6.3) and, more precisely,

\[
1 = \sum_{A \in T} [1_A].
\]

(3) The map \( K_0 \) defines a bijection between the sets of left (resp. right, 2-sided) Serre ideals of \( T \) and of \( K_0(T) \).

(4) The map \( K_0 \) defines a homeomorphism

\[
K_0 : \text{Serre-Spec}(T) \xrightarrow{\cong} \text{Serre-Spec}(K_0(T)).
\]

It is a bijection between the subsets of completely prime (resp. semiprime) ideals of \( T \) and \( K_0(T) \).

**Proof.** Part (1) follows from Lemma 6.11(2).

(2) The fact that \( K_0(T) \) is a \( \mathbb{Z}_+ \)-ring follows from the fact that for every abelian category \( C \) and \( B \in C \),

\[
[B] \in \bigoplus_{A \in C_s} \mathbb{Z}_+[A].
\]

The second statement in part (2) is obvious.
(3) We consider the case of left Serre ideals, the other two cases being analogous. Let \( \mathcal{I} \) be a left Serre ideal of \( \mathcal{T} \). By part (1) of the theorem, \( \mathcal{I} = \mathcal{S}(X) \) for some \( X \subseteq (\mathcal{T}_1)_s \). Therefore, the subset

\[
K_0(\mathcal{I}) = \bigoplus_{f \in X} \mathbb{Z}[f] \subseteq K_0(\mathcal{T})
\]

has the property (6.3). Since \( \mathcal{T}_1 \circ \mathcal{I}_1 \subseteq \mathcal{I}_1 \), we have \( K_0(\mathcal{T})K_0(\mathcal{I}) \subseteq K_0(\mathcal{I}) \), and thus, \( K_0(\mathcal{I}) \) is a left Serre ideal of \( K_0(\mathcal{T}) \).

Next, let \( I \) be a left Serre ideal of \( K_0(\mathcal{T}) \). By (6.3),

\[
I = \bigoplus_{f \in X} \mathbb{Z}[f]
\]

for some \( X \subseteq (\mathcal{T}_1)_s \). Let \( \mathcal{I} \) be the weak subcategory \( \mathcal{S}(X) \) of \( \mathcal{T} \). Clearly, \( K_0(\mathcal{I}) = I \). To show that \( \mathcal{I} \) is a left Serre ideal of \( \mathcal{T} \), it remains to prove that \( \mathcal{T}_1 \circ \mathcal{I}_1 \subseteq \mathcal{I}_1 \), i.e., that

\[
g_2 f_1 \in \mathcal{I}(A_1, A_3) \quad \text{for all} \quad g_2 \in \mathcal{T}(A_2, A_3), f_1 \in \mathcal{I}(A_1, A_2)
\]

for all objects \( A_1, A_2, A_3 \) of \( \mathcal{T} \). Since \( I \) is a left Serre ideal,

\[
[g_2 f_1] = [g_2][f_1] \in I = \bigoplus_{f \in X} \mathbb{Z}[f],
\]

and thus, \( g_2 f_1 \in \mathcal{S}(X) = \mathcal{I} \).

It is straightforward to verify that the above two maps \( \mathcal{I} \mapsto K_0(\mathcal{I}) \) and \( I \mapsto \mathcal{I} \) are inverse bijections between the left Serre ideals of \( \mathcal{T} \) and \( K_0(\mathcal{T}) \).

(4) Similarly to part (3) one proves that the map \( K_0 \) defines a bijection between the prime (resp. completely prime, semiprime) ideals of the abelian 2-category \( \mathcal{T} \) and the \( \mathbb{Z}_+ \)-ring \( K_0(\mathcal{T}) \). In the first case one uses the characterization of Serre prime ideals of an abelian 2-category in Theorem 6.5(1)(b) vs. the definition of Serre prime ideals of a \( \mathbb{Z}_+ \)-ring. In the second case one uses the definitions of completely prime ideals in the two settings. In the third case one uses the characterizations of Serre semiprime ideals in the two settings given in Theorems 6.5(3)(b) and 6.10(3)(b).

The fact that the map

\[
K_0 : \operatorname{Serre-Spec}(\mathcal{T}) \to \operatorname{Serre-Spec}(K_0(\mathcal{T}))
\]

is a homeomorphism follows from the definitions of the collections of closed sets in the two cases in terms of Serre ideals and the bijection in part (3) of the theorem. \( \square \)

We have the following immediate corollary of part (3) of the theorem and Lemma 6.11

**Corollary 6.13.** Let \( \mathcal{T} \) be an abelian 2-category which is a categorification of the \( \mathbb{k} \)-algebra \( R \otimes_{\mathbb{Z}} \mathbb{k} \) for a \( \mathbb{Z}_+ \)-ring \( R \). If \( I \) is a Serre ideal of \( R \) and \( \mathcal{I} \) is the unique Serre ideal of \( \mathcal{T} \) with \( K_0(\mathcal{I}) = I \) as in Theorem 6.13(3), then \( \mathcal{T}/\mathcal{I} \) is a categorification of the \( \mathbb{k} \)-algebra \( (R/I) \otimes_{\mathbb{Z}} \mathbb{k} \).

6.6. **Serre prime ideals of \( \mathbb{Z}_+ \)-rings vs prime ideals.** Let \( R \) be a \( \mathbb{Z}_+ \)-ring. In general, \( \operatorname{Serre-Spec}(R) \) is not a subset of the prime spectrum \( \operatorname{Spec}(R) \) of \( R \). Similarly a Serre completely prime ideal of \( R \) is not necessarily a completely prime ideal of \( R \) (in the classical sense), and a Serre semiprime ideal of \( R \) is not necessarily a semiprime ideal of \( R \). The point in all three cases is that the notions of Serre type are formulated in terms of inclusion properties concerning elements of \( R_+ \), while the classical notions are formulated in terms of inclusion properties concerning elements of the full ring \( R \).

**Example 6.14.** Consider the commutative \( \mathbb{Z}_+ \)-ring \( R := \mathbb{Z}[x]/(x^2 - 1) \) with positive \( \mathbb{Z} \)-basis \( \{1, x\} \). The 0-ideal of \( R \) is Serre prime while it is not a prime ideal of \( R \).
Example 6.15. Consider the setting of Example 5.5 and assume that \( H \) is a finite dimensional Hopf algebra over the field \( k \). The 0 ideal of \( H - \text{mod} \) is Serre completely prime, and by Theorem 6.12(4), 0 is a Serre completely prime ideal of \( K_0(H - \text{mod}) \). However, 0 is not a completely prime ideal of the ring \( K_0(H - \text{mod}) \) because \( K_0(H - \text{mod}) \otimes Z \mathbb{Q} \) is a finite dimensional algebra over \( \mathbb{Q} \) and thus definitely has 0 divisors. Furthermore the 0 ideal of \( K_0(H - \text{mod}) \) is not even semiprime, except for the special case when the algebra \( K_0(H - \text{mod}) \otimes Z \mathbb{Q} \) is semisimple (because the radical of this finite dimensional algebra is nilpotent). \( \square \)

On the other hand, the following lemma provides a simple but important fact about getting Serre prime (resp. completely prime, semiprime) ideals of a \( \mathbb{Z}_+ \)-ring from particular types of prime (resp. completely prime, semiprime) ideals of a \( R \) in the classical sense.

**Lemma 6.16.** Assume that \( R \) is a \( \mathbb{Z}_+ \)-ring with a positive basis \( \{ b_\gamma \mid \gamma \in \Gamma \} \). If 

\[
I = \bigoplus_{\gamma \in \Gamma'} \mathbb{Z} b_\gamma \text{ for some subset } \Gamma' \subseteq \Gamma
\]

and \( I \) is a prime (resp. completely prime, semiprime) ideal of \( R \) in the classical sense, then \( I \) is a Serre prime (resp. completely prime, semiprime) ideals of \( R \).

**Proof.** The first property of \( I \) is equivalent to the one in (6.4). The assumption that \( I \) is a prime (resp. completely prime, semiprime) ideal of \( R \) implies that it satisfies the condition (b) in Theorem 6.10(1) in the first case, the condition in Theorem 6.10(2) in the second case, and the condition (d) in Theorem 6.10(3) in the third case. For example, if \( I \) is a semiprime ideal of \( R \) in the classical sense, it satisfies the condition (d) in Theorem 6.10(3) for all \( r \in R \). Now the lemma follows from Theorem 6.10. \( \square \)

**Remark 6.17.** Let \( R \) be a \( \mathbb{Z}_+ \)-ring categorified by an abelian 2-category \( T \). By Theorem 6.12(3) and Lemma 6.16 the prime ideals of \( R \) that are categorifiable are precisely the precisely the ones that are thick; that is the set 

\[
\text{Spec}(R) \cap \text{Serre-Spec}(R).
\]

7. The Primitive Spectrum

In this section we describe the relationship between the annihilation ideals of simple 2-representations of abelian 2-categories and the Serre prime ideals of these categories.

7.1. 2-representations. Following Mazorchuk–Miemietz [34], define a 2-representation of a 2-category \( T \) to be a strict 2-functor \( F \) from \( T \) to \text{Cat}, the 2-category of all small categories. That is, \( F \) sends objects of \( T \) to small categories, 1-morphisms of \( T \) to functors between categories, and 2-morphisms of \( T \) to natural transformations between functors.

Recall that the category of additive functors between two abelian categories has a canonical structure of an abelian category.

**Definition 7.1.** A 2-representation \( F \) of a 2-category \( T \) will be called exact, if

1. \( F(A) \) is an abelian category for every object \( A \) in \( T \);  
2. \( F(f) \) is an additive functor for all 1-morphisms \( f \) in \( T \);  
3. For any exact sequence of 1-morphisms in \( T \), 

\[
0 \to f \to g \to h \to 0,
\]

the sequence 

\[
0 \to F(f) \to F(g) \to F(h) \to 0
\]
is an exact sequence of 1-morphisms in $\text{Cat}$.

Following, Mazorchuk, Miemietz, and Zhang [35 Section 3.3], we call a 2-representation $\mathcal{F}$ simple if the collection of categories

$$\{\mathcal{F}(A) \mid A \in \mathcal{T}\}$$

has no nonzero proper $\mathcal{T}$-invariant ideals. Such an ideal $X$ is a subset of the disjoint union of the set of morphisms of the categories $\mathcal{F}(A)$ for $A \in \mathcal{T}$ with the following properties:

1. $ab$ and $ba$ are in $X$ for all $a \in X$ and all morphisms $b$ in $\mathcal{F}(A)$ such that the composition is well-defined;
2. $\mathcal{F}(f)(a) \in X$ for all $f \in T_1$ and $a \in X$;
3. There is some morphism $a \in X$ which is not a zero morphism.

7.2. Annihilation ideals of 2-representations.

Definition 7.2. Given an exact 2-representation $\mathcal{F}$ of the abelian 2-category $\mathcal{T}$, define its annihilation ideal $\text{Ann}(\mathcal{F})$ to be the weak subcategory of $\mathcal{T}$ having the same set of objects, set of 1-morphisms given by

$$\text{Ann}(\mathcal{F})_1 := \{f \in T_1 \mid \mathcal{F}(f) \text{ is a zero functor}\},$$

and set of 2 morphisms

$$\text{Ann}(\mathcal{F})(f, g) := \mathcal{F}(f, g) \text{ for all } f, g \in \text{Ann}(\mathcal{F})_1.$$

Lemma 7.3. The annihilation ideal $\text{Ann}(\mathcal{F})$ of every exact 2-representation $\mathcal{F}$ of an abelian 2-category $\mathcal{T}$ is a Serre ideal of $\mathcal{T}$.

Proof. The proof is a direct verification of the necessary properties.

To verify the ideal property of $\text{Ann}(\mathcal{F})$, chose $f \in \text{Ann}(\mathcal{F})_1$ and $g \in T_1$ such that the composition is defined. Then $\mathcal{F}(f)$ is a zero functor, and therefore, $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$ is also a zero functor. So, $fg \in \text{Ann}(\mathcal{F})_1$. Likewise, $T_1 \circ \text{Ann}(\mathcal{F})_1 \subseteq \text{Ann}(\mathcal{F})_1$.

To verify that $\text{Ann}(\mathcal{F})(A_1, A_2)$ is a Serre subcategory of the abelian category $\mathcal{T}(A_1, A_2)$ for all object $A_1$ and $A_2$ of $\mathcal{T}$, consider an exact sequence $0 \to f \to g \to h \to 0$ in $\mathcal{T}(A_1, A_2)$. By Definition 7.2(3), $0 \to \mathcal{F}(f) \to \mathcal{F}(g) \to \mathcal{F}(h) \to 0$ is an exact sequence in the abelian category of additive functors between the abelian categories $\mathcal{F}(A_1)$ and $\mathcal{F}(A_2)$.

If $f, h \in \text{Ann}(\mathcal{F})$, then $\mathcal{F}(f)$ and $\mathcal{F}(h)$ are both the zero functor and $\mathcal{F}(g)$ must also be the zero functor. Hence, $g \in \text{Ann}(\mathcal{F})_1$. Likewise, assuming instead that $g \in \text{Ann}(\mathcal{F})_1$, we get that $f, h \in \text{Ann}(\mathcal{F})_1$. Hence, $\text{Ann}(\mathcal{F})$ is a Serre ideal of $\mathcal{T}$. \[\square\]

Finally we have the following theorem, analogons to the relationship between prime ideals of rings and annihilators of simple representations, see e.g. [11, Proposition 3.12].

Theorem 7.4. Suppose $\mathcal{F}$ is a simple exact 2-representation of an abelian 2-category $\mathcal{T}$. Then $\text{Ann}(\mathcal{F})$ is a Serre prime ideal of $\mathcal{T}$.

Proof. We use the assumption of the simplicity of $\mathcal{F}$ to show that $\text{Ann}(\mathcal{F})$ satisfies the condition in Theorem 3.12 form which we obtain that $\text{Ann}(\mathcal{F})$ is a prime ideal of $\mathcal{T}$. The fact that $\text{Ann}(\mathcal{F})$ is a Serre ideal was established in Lemma 7.3.

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are thick ideals such that

$$\mathcal{I}_1 \circ \mathcal{J}_1 \subseteq \text{Ann}(\mathcal{F})_1,$$

and neither $\mathcal{I}$ nor $\mathcal{J}$ is contained in $\text{Ann}(\mathcal{F})$. Then we claim that the set

$$X := \{a(\mathcal{F}(j)(b))c \mid a, b, c \text{ morphisms such that the composition is defined}, j \in \mathcal{J}_1\}$$

is an ideal of $\text{Ann}(\mathcal{F})$. First, we show that $X$ is contained in $\text{Ann}(\mathcal{F})_1$. For this, we use that $\text{Ann}(\mathcal{F})_1$ is a Serre ideal of $\mathcal{T}$, and the fact that $\mathcal{I}_1 \circ \mathcal{J}_1 \subseteq \text{Ann}(\mathcal{F})_1$. Finally, we show that $X$ is an ideal of $\text{Ann}(\mathcal{F})$. For this, we use that $\text{Ann}(\mathcal{F})_1$ is a Serre ideal of $\mathcal{T}$, and the fact that $\mathcal{I}_1 \circ \mathcal{J}_1 \subseteq \text{Ann}(\mathcal{F})_1$. \[\square\]
forms a nonempty $\mathcal{T}$-invariant ideal, contradicting the simplicity of $\mathcal{F}$. It is clear that this set is an ideal, i.e., closed under composition on the left and right by any morphisms of $\text{Cat}$ with appropriate source and target. We must show that it is invariant under $\mathcal{T}$, that it is nonzero, and that it is a proper subset of all morphisms of the categories $\mathcal{F}(A)$ for all objects $A$ of $\mathcal{T}$.

First, assume that $g \in \mathcal{T}_1$. Then

$$\mathcal{F}(g)(a(\mathcal{F}(j)(b))c) = \mathcal{F}(g)(a)\mathcal{F}(g)(\mathcal{F}(j)(b))\mathcal{F}(g)(c) = \mathcal{F}(g)(a)\mathcal{F}(gj)(b)\mathcal{F}(g)(c),$$

which is clearly in $X$ whenever the composition is defined, since $gj \in \mathcal{J}_1$. Hence, $X$ is $\mathcal{T}$-invariant.

Next, we show that $X$ is a proper subset. For all $i \in \mathcal{I}_1$ and $a(\mathcal{F}(j)(b))c \in X$, we have

$$\mathcal{F}(i)(a(\mathcal{F}(j)(b))c) = \mathcal{F}(i)(a)\mathcal{F}(ij)(b)\mathcal{F}(i)(c) = \mathcal{F}(i)(a)0\mathcal{F}(i)(c) = 0$$

whenever the composition is defined. If $X$ equals the set of all morphisms of the collection of abelian categories \{\mathcal{F}(A) \mid A \in \mathcal{T}\}, then this would imply that $\mathcal{F}(i)$ is a zero functor for all $i \in \mathcal{I}_1$. Therefore, $\mathcal{I}_1 \subseteq \text{Ann}(\mathcal{F})_1$. Applying Remark 3.3 and the assumption that $\mathcal{I}$ is a thick ideal gives that $\mathcal{I}$ is contained in $\text{Ann}(\mathcal{F})$, which is a contradiction.

By a similar argument, one shows that $X$ contains nonzero morphisms. Since $\mathcal{J}$ is not contained in the annihilator of $\mathcal{F}$ by assumption, there is some $j \in \mathcal{J}_1$ such that $\mathcal{F}(j)$ is not the zero functor, and hence there is some morphism $b$ such that $\mathcal{F}(j)(b)$ is a nonzero morphism. Then by letting $a$ and $b$ be the appropriate identity morphisms, we see that $\mathcal{F}(j)(b)$ is a nonzero morphism in $X$.

Therefore, $X$ is a nonzero, proper $\mathcal{T}$-invariant ideal, which contradicts our assumption that $\mathcal{F}$ is simple. This gives that $\text{Ann}(\mathcal{F})$ is a Serre prime ideal of $\mathcal{T}$. \hfill $\square$

**Definition 7.5.** The primitive spectrum of an abelian 2-category $\mathcal{T}$, denoted $\text{Prim}(\mathcal{T})$, is the subset of Serre-Spec($\mathcal{T}$) consisting of all primes $\mathcal{P}$ for which there exists a simple exact 2-representation $\mathcal{F}$ of $\mathcal{T}$ with $\mathcal{P} = \text{Ann}(\mathcal{F})$.

8. QUANTUM SCHUBERT CELL ALGEBRAS, CANONICAL BASES AND PRIME IDEALS

This section contains background material on quantum groups and quantum Schubert cell algebras, and their canonical bases defined by Kashiwara and Lusztig. We recall facts about the homogeneous completely prime ideals of the quantum Schubert cell algebras and their relations to quantizations of Richardson varieties.

8.1. Quantum groups, canonical bases and quantum Schubert cell algebras. Let $\mathfrak{g}$ be a (complex) symmetrizable Kac–Moody algebra with Cartan matrix $(a_{ij})_{i,j=1}^r$ and Cartan subalgebra $t \subset \mathfrak{g}$. Denote the Weyl group of $\mathfrak{g}$ by $W$. Let $\{\alpha_i \mid 1 \leq i \leq r\} \subset t^*$ and $\{s_i \mid 1 \leq i \leq r\}$ be the sets of simple roots of $\mathfrak{g}$ and simple reflections of $W$, respectively. Denote by $\{\alpha_{i^-} \mid 1 \leq i \leq r\} \subset t$ and $\{\varpi_i \mid 1 \leq i \leq r\} \subset t^*$ the sets of simple coroots and fundamental weights of $\mathfrak{g}$. Thus, $(\alpha_i^\vee, \alpha_j) = a_{ij}$. Let $(.,.)$ be the standard nondegenerate symmetric bilinear form on $t^*$ satisfying

$$\langle \alpha_i^\vee, \lambda \rangle = \frac{2\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha_i \rangle} \quad \text{for } \lambda \in t^* \quad \text{and} \quad \langle \alpha_i, \alpha_i \rangle = 2 \quad \text{for short roots } \alpha_i.$$

Then

$$d_i := \frac{\langle \alpha_i, \alpha_i \rangle}{2} \in \mathbb{Z}_+.$$

Let

$$Q, \ P, \ P_+ \subset t^*$$
be the root and weight lattices of \( g \), and the set of its dominant integral weights. Denote
\[
P^\vee := \{ h \in t \mid \langle h, P \rangle \subset \mathbb{Z} \} \subset t \quad \text{and} \quad Q_+ := \bigoplus \mathbb{Z}_+ \alpha_i \subset t^* .
\]

Let \( U_q(g) \) be the quantized universal enveloping algebra of \( g \) over \( \mathbb{Q}(q) \) with generators \( e_i, f_i, q^h \) for \( 1 \leq i \leq r, h \in P^\vee \) and relations as in [19]. We will use the Hopf algebra structure of \( U_q(g) \) with coproduct given by
\[
(8.1) \quad \Delta(e_i) = e_i \otimes 1 + q^{d_{ij}} \otimes e_j, \quad \Delta(f_i) = f_i \otimes q^{-d_{ij}} + 1 \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h
\]
for \( h \in P^\vee, 1 \leq i \leq r \). Let \( U_q^\pm(g) \) and \( U_q^0(g) \) be the unital subalgebras of \( U_q(g) \) generated by \( \{e_i \mid 1 \leq i \leq r\} \) (resp. \( \{f_i \mid 1 \leq i \leq r\} \) and \( \{q^h \mid h \in P^\vee\} \). Denote the (symmetric) \( q \)-integers and factorials
\[
q_i := q^{d_i}, \quad [n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \quad \text{and} \quad [n]_i! := [1]_i \cdots [k]_i .
\]
Denote by * and \( \varphi \) the \( \mathbb{Q}(q) \)-linear anti-automorphisms of \( U_q(g) \) defined by
\[
e_i^* := e_i, \quad f_i^* := f_i, \quad (q^h)^* := q^{-h}, \quad \text{and} \quad \varphi(e_i) := f_i, \quad \varphi(f_i) := e_i, \quad \varphi(q^h) := q^h
\]
for \( 1 \leq i \leq r, h \in P^\vee \). The composition \( \varphi^* := \varphi \circ * = * \circ \varphi \), which is a \( \mathbb{Q}(q) \)-linear automorphism of \( U_q(g) \), satisfies
\[
\varphi^*(e_i) = f_i, \quad \varphi^*(f_i) = e_i \quad \text{and} \quad \varphi^*(q^h) = q^{-h} .
\]
This composition is denoted by \( ^\vee \) in [21]; we will use the above notation to avoid interference with later notation.

The Hopf algebra \( U_q(g) \) is graded by the root lattice \( Q \) by setting
\[
(8.2) \quad \deg e_i = \alpha_i, \quad \deg f_i = -\alpha_i, \quad \deg q^h = 0 .
\]
The homogeneous components of a subspace \( Y \) of \( U_q(g) \) of degree \( \gamma \in Q \) will be denoted by \( Y_\gamma \). Denote by \( e_i^\gamma \) the \( \mathbb{Q}(q) \)-linear skew-derivations of \( U_q^-(g) \) such that
\[
e_i^\gamma(f_j) = \delta_{ij} \quad \text{and} \quad e_i^\gamma(xy) = e_i^\gamma(x)y + q^{-\langle \alpha_i, \gamma \rangle}xe_i^\gamma(y) \quad \text{for} \quad x \in U_q^-(g), y \in U_q^-(g) .
\]
Kashiwara’s (nondegenerate, symmetric) bilinear form \( \langle -, - \rangle_K : U_q^-(g) \times U_q^-(g) \to \mathbb{Q}(q) \) is defined by
\[
(1, 1)_K = 1 \quad \text{and} \quad \langle f_i, xy \rangle_K = \langle x, e_i^\gamma(y) \rangle_K
\]
for all \( 1 \leq i \leq r \) and \( x, y \in U_q^-(g) \).

**Remark 8.1.** This differs slightly from the conventional choice for Kashiwara’s form \( \langle -, - \rangle_K : U_q^-(g) \times U_q^-(g) \to \mathbb{Q}(q) \), which is defined by
\[
(1, 1)_K = 1 \quad \text{and} \quad \langle f_i, xy \rangle_K = \langle x, e_i^\gamma(y) \rangle_K
\]
for all \( 1 \leq i \leq r \) and \( x, y \in U_q^-(g) \) in terms of the \( \mathbb{Q}(q) \)-linear skew-derivations \( e_i^\gamma \) of \( U_q^-(g) \) given by
\[
e_i^\gamma(f_j) = \delta_{ij} \quad \text{and} \quad e_i^\gamma(xy) = e_i^\gamma(x)y + q^{\langle \alpha_i, \gamma \rangle}xe_i^\gamma(y) \quad \text{for} \quad x \in U_q^-(g), y \in U_q^-(g) .
\]
The two forms are related by
\[
(8.3) \quad \langle x, y \rangle_K = \overline{\langle x, y \rangle} \quad \text{for} \quad x, y \in U_q^-(g) ,
\]
where \( x \mapsto \overline{x} \) denotes the \( \mathbb{Q} \)-linear automorphism of \( \mathbb{Q}(q) \) given by \( \overline{q} = q^{-1} \) and the bar involution of \( U_q(g) \) (the skew-linear automorphism of \( U_q(g) \) given by \( \overline{f_i} = f_i \)). Using
one converts dualization results with respect to one form to such results for the other.

Let \( A := \mathbb{Z}[q^\pm] \) and \( U^\pm_q(\mathfrak{g}) \) be the (divided power) integral forms of \( U^\pm_q(\mathfrak{g}) \), which are the \( A \)-subalgebras of \( U^\pm_q(\mathfrak{g}) \) generated by \( e_i^{(k)} = e_i^k/[k]! \) and \( f_i^{(k)} = f_i^k/[k]! \) for \( 1 \leq i \leq r, k \in \mathbb{Z}_+ \), respectively. The dual integral form \( U^{-}_A(g) \) is the \( A \)-subalgebra given by

\[
U^{-}_A(g)^\vee = \{ x \in U^{-}_q(g) \mid (x, U^{-}_A(g))_K \subset A \}.
\]

Kashiwara [19] and Lusztig [31] defined the canonical/lower global basis of \( U^\pm_q(\mathfrak{g}) \) and the dual canonical/upper global basis of \( U^\pm_q(\mathfrak{g})^\vee \). These bases have a number of remarkable properties; for instance, they descend to bases of integrable highest weight modules by acting on highest weight vectors. We will denote by \( B^\text{low} \) the lower global basis of \( U^\pm_q(\mathfrak{g}) \) and by \( B^\text{up} \) the upper global basis of \( U^{-}_A(g) \).

The lower global basis \( B^\text{low} \) and the upper global basis \( B^\text{up} \) form a pair of dual bases of \( U^\pm_q(g) \) and \( U^{-}_A(g)^\vee \) with respect to the pairing \((-,-)_K\). For \( b \in B^\text{low} \), denote by \( b^\vee \in B^\text{up} \) the corresponding dual element, so

\[
(b, c^\vee)_K = \delta_{b,c} \quad \text{for} \quad b, c \in B^\text{low}.
\]

The lower global bases \( B^\text{low} \) satisfy the invariance properties

\[
(B^\text{low})^* = B^\text{low} \quad \text{and} \quad \varphi(B^\text{low}) = \varphi^*(B^\text{low}) = B^\text{low},
\]

see [20, Theorem 2.1.1], [21, Theorem 4.3.2] and [18, Theorem 8.3.4].

To each Weyl group element \( w \), one associates the quantum Schubert cell algebras \( U^{-}_q[w] \subseteq U^{-}_q(g) \). They can be defined in two ways. Starting from a reduced expression

\[
w = s_{i_1} \cdots s_{i_N}
\]
of \( w \), consider the roots

\[
\beta_1 := \alpha_{i_1}, \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})
\]

and the root vectors

\[
\{ f_{\beta_j} := T_{i_1} \cdots T_{i_{j-1}}(f_j) \mid 1 \leq j \leq N \}
\]

using Lusztig’s braid group action [32, 14] on \( U_q(\mathfrak{g}) \). De Concini, Kac, and Procesi [5], and Lusztig [32, §40.2] defined the algebra \( U^-_q[w] \) as the unital \( \mathbb{Q}(q) \)-subalgebra of \( U^-_q(g) \) with generating set \( (8.6) \), and proved that this is independent on the choice of reduced expression of \( w \). Berenstein and Greenstein [3] conjectured that

\[
U^-_q[w] = U^-_q(g) \cap T^e(w) (U^-_q(g)),
\]

and Kimura [27] and Tanisaki [39] proved this property. It can be used as a second definition of the algebras \( U^-_q[w] \). Kimura proved [26, Theorem 4.5] that

\[
B^\text{up}[w] := B^\text{up} \cap U^-_q[w]
\]
is an \( A \)-basis of the \( A \)-algebra

\[
U^-_A[w]^\vee := U^-_q[w] \cap U^-_A(g)^\vee.
\]

We will refer to this algebra as to the dual integral form of \( U^-_q[w] \). The set \( B^\text{up}[w] \) is called the upper global basis of \( U^-_A[w]^\vee \).
8.2. Homogeneous completely prime ideals of the algebras $U_q[w]$. Denote the Hopf subalgebras $U^{\geq 0} := U_q^+(g)U_q^0(g)$ and $U^{\leq 0} := U_q^-(g)U_q^0(g)$ of $U_q(g)$. The Rosso-
Tanisaki form

$$(\cdot, \cdot)_{RT} : U^{\leq 0} \times U^{\geq 0} \to \mathbb{Q}(q^{1/d})$$

(for an appropriate $d \in \mathbb{Z}_+$) is the Hopf algebra pairing satisfying

$$(y, xx')_{RT} = (\Delta(y), x' \otimes x)_{RT}, \quad (yy', x)_{RT} = (y \otimes y', \Delta(x))_{RT}$$

for $y, y' \in U^{\leq 0}$, $x, x' \in U^{\geq 0}$, and normalized by

$$(f_i, e_j)_{RT} = \delta_{ij}, \quad (q^h, q^{h'})_{RT} = q^{(h, h')}, \quad (f_i, q^{h'})_{RT} = (q^h, e_i)_{RT} = 0$$

for $1 \leq i, j \leq r$, $h, h' \in P^\vee$. This is a slightly different normalization than the usual one \cite[Eq. (6.12)(2)]{14} needed in order to match this form to Kashiwara’s one. The two normalizations for $(\cdot, \cdot)_{RT}$ are related to each other by a Hopf algebra automorphism of $U^{\leq 0}$ coming from the torus action associated to its $Q$-grading.

For $\gamma \in Q_+$ let

$$\{x_{\gamma,i}\} \quad \text{and} \quad \{y_{\gamma,i}\}$$

be a set of dual bases of $(U_q^-(g))_{-\gamma}$ and $(U_q^+(g))_{\gamma}$ with respect to $(\cdot, \cdot)_{RT}$. The quasi-
$R$-matrix of $U_q(g)$ is

$$R := \sum_{\gamma \in Q_+} \sum_{i} y_{\gamma,i} \otimes x_{\gamma,i} \in U_q^+(g) \otimes U_q^-(g)$$

where the completed tensor product is with respect to the descending filtration \cite[§4.1.1]{32}.

For $\lambda \in P_+$, we will denote by $V(\lambda)$ the irreducible $U_q(g)$-module of highest weight $\lambda$ and by $V(\lambda)^0$ its restricted dual

$$V(\lambda)^0 := \oplus_{\nu \in P}(V(\lambda)_\nu)^*$$

where

$$V(\lambda)_\nu := \{ v \in V(\lambda) \mid q^h v = q^{(\nu,h)} v, \quad \forall h \in P^\vee \} \quad \text{for} \quad \nu \in P$$

are the (finite dimensional) weight spaces of $V(\lambda)$. Let $v_\lambda$ be a fixed highest weight vector of $V(\lambda)$. Denote by $B(\lambda)^{\text{low}}$ the lower global basis of (the integral form $U^-_{\text{int}}(g)v_\lambda$) of $V(\lambda)$. It is an $A$-basis of $U_q^+(g)v_\lambda$ and a $Q(q)$-basis of $V(\lambda)$. For $w \in W$, let $v_{w,\lambda}$ be the unique element of $V(\lambda)_w\lambda$ which belongs to $B(\lambda)^{\text{low}}$. Let

$$V_{\pm}(\lambda) := U_q^+(g)v_{w,\lambda} \subseteq V(\lambda)$$

be the associated Demazure modules. For $v \in V(\lambda)$ and $\xi \in V(\lambda)^*$, denote the corresponding matrix coefficient of $V(\lambda)$ considered as a functional on $U_q(g)$:

$$c_{\xi, v} \in (U_q(g))^*$$

given by $c_{\xi, v}(x) = \xi(x \cdot v)$ for $x \in U_q(g)$.

A subspace $U$ of $U_q(g)$ will be called homogeneous if

$$U = \bigoplus_{\gamma \in Q} U_\gamma \quad \text{where} \quad U_\gamma := U \cap U_q(g)_\gamma.$$

**Theorem 8.2.** \cite[Theorem 3.1(a)]{42} Let $g$ be a symmetrizable Kac–Moody algebra and $w \in W$ be a Weyl group element. For all $u \in W$ such that $u \leq w$, the set

$$I_u(w) = \{ (c_{\xi, v_{w,\lambda}} \otimes \text{id}, R)^* \mid \xi \in V(\lambda)^0, \xi \perp V_{u}^-(\lambda), \lambda \in P_+ \}$$

is a homogeneous completely prime ideal of $U_q^+[w]$. 
The original proof of this theorem that was given in [42] extensively used the works of Joseph [15, 16] and Gorelik [12] which were written for the case when $\mathfrak{g}$ is finite dimensional (complex simple Lie algebra). In [44] Lemmas 3.4 and 3.5] a simplified, self-contained proof was given that established the validity of the theorem when $U_q(\mathfrak{g})$ is defined over a base field $k$ of arbitrary characteristic; the needed results of Joseph [15, 16] and Gorelik [12] were reproved in this setting. The proofs of these facts in [44] work in the full generality of symmetrizable Kac–Moody algebras.

The Rosso-Tanisaki form $(-,-)_{RT}$ satisfies

$$ (y^*, x^*)_{RT} = (y, x)_{RT} \quad \text{for} \quad y \in U_{\leq 0}^-, x \in U_{\geq 0}, $n

[44] Lemma 6.16]. Therefore,

$$ (8.9) \quad \mathcal{R}^* \otimes \mathcal{R} = \mathcal{R}, $$n

and thus, the ideals $I_w(u)$ are also given by

$$ I_w(u) = \{ (\langle \xi, \psi_{\lambda}, \phi, \phi \rangle \otimes \text{id}, \mathcal{R}) \mid \xi \in V(\lambda^0), \xi \perp V_{u^{-}}(\lambda), \lambda \in P_+ \}. $$p

We will need the following relation between the bilinear forms $(-,-)_K$ and $(-,-)_{RT}$, and the corresponding expression for $\mathcal{R}$ in terms of global bases.

**Proposition 8.3.** Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. For all $x_1, x_2 \in U_{q^-}(\mathfrak{g})$, we have

$$ (8.10) \quad (x_1, \varphi^*(x_2))_{RT} = (x_1, x_2)_K. $$p

The quasi-$R$-matrix of $U_q(\mathfrak{g})$ is given by

$$ (8.11) \quad \mathcal{R} = \sum_{b \in B_{low}} \varphi^*(b) \otimes b' \mathcal{R} = \sum_{b \in B_{low}} \varphi(b) \otimes (b')^*, $$p

recall (8.4).

**Proof.** The $Q$-grading of $U_{\leq 0}$ specializes to a $\mathbb{Z}_+$-grading via the group homomorphism $Q : \mathbb{Z} \to \mathbb{Z}$ given by $\alpha_i \mapsto -1$. The corresponding graded components will be denoted by $(U_{\leq 0})_l$. Set

$$ (U_{\leq 0})_{\geq l} := (U_{\leq 0})_l \oplus (U_{\leq 0})_{l+1} \oplus \ldots $$p

For $x := f_{i_1} \ldots f_{i_k}$ and $h := d_1 \alpha_i^\vee + \ldots + d_k \alpha_k^\vee$, we have

$$ \Delta(x) - x \otimes q^{-h} - \sum_{j=1}^k q^{\langle \alpha_j, \alpha_i^\vee \rangle} f_{i_1} \ldots f_{i_{j-1}} f_{i_{j+1}} \ldots f_{i_k} \otimes f_{i_j} q^{-h_d} \alpha_i^\vee \in U_{\leq 0} \otimes (U_{\leq 0})_{\geq 2}, $$p

i.e.,

$$ \Delta(x) - x \otimes q^{-h} - \sum_{i=1}^r c_i^\mu(x) \otimes f_i q^{-h_d} \alpha_i^\vee \in U_{\leq 0} \otimes (U_{\leq 0})_{\geq 2}. $$p

This property, and the two properties of the Rosso-Tanisaki form

$$ ((U_{\leq 0})_\gamma, (U_{\geq 0})_\nu)_{RT} = 0 \quad \text{for} \quad \gamma + \nu \neq 0, $$p

$$ (xq^h, yq^h')_{RT} = q^{-h_h'}(x, y)_{RT} \quad \text{for} \quad x, y \in U_{q^-}(\mathfrak{g}), h, h' \in P^\vee $$p

(see [44] Eqs. 6.13(1)-(2)]) imply that the bilinear form $(-,-)$ on $U_q^-$ given by

$$ \langle x, y \rangle := (x, \varphi^*(y))_{RT} $$p

satisfies $\langle x, f_i y \rangle = \langle e_i^\mu(x), y \rangle$ for $x, y \in U_{q^-}(\mathfrak{g})$ and $1 \leq i \leq r$. The uniqueness property of the Kashiwara form implies that this form equals $(-,-)_K$, which proves (8.10).
The invariance property \((8.5)\), the relation \((8.10)\) between the bilinear forms \((-,-)_{RT}\) and \((-,-)_{K}\), and the orthogonality property \((8.4)\) imply that
\[
\{ \varphi^*(b) \mid b \in B_{low}^\prime \} \quad \text{and} \quad \{ b^\prime \mid b \in B_{low} \}
\]
are a pair of dual bases of \(U_q^+(g)\) and \(U_q^-(g)\) with respect to the pairing \((-,-)_{RT}\). This gives the first equality in \((8.11)\). The second equality follows from \((8.9)\) and the first invariance property in \((8.5)\).

\[\square\]

8.3. Quantizations of Richardson varieties. Let \(G\) be the Kac–Moody group (over \(\mathbb{C}\)) corresponding to \(g\). Let \(B_\pm\) be opposite Borel subgroups of \(G\). For \(u, w \in W\), the open Richardson variety associated to the pair \((u, w)\) is the locally closed subset of the flag variety \(G/B^+\) defined by
\[
R_{u,w} := (B^- w B^+)/B^+ \cap (B_+ w B_+)/B_+.
\]
It is nonempty if and only if \(u \leq w\) in which case it has dimension \(\ell(w) - \ell(u)\) (in terms of the standard length function \(\ell : W \rightarrow \mathbb{Z}_+\)). We have the stratifications of \(G/B^+\) into unions of Schubert cells
\[
G/B^+ = \bigsqcup_{w \in W} (B_+ w B_+)/B^+ = \bigsqcup_{u \in W} (B^- u B^+)/B^+.
\]
and open Richardson varieties
\[
G/B^+ = \bigsqcup_{u \leq w} \bigsqcup_{u, w \in W} R_{u,w}.
\]
Denote the closure
\[
\overline{R}_{u,w} := \text{Cl}_{(B_+ w B_+)/B^+}(R_{u,w})
\]
of \(R_{u,w}\) in the Schubert cell \((B_+ w B_+)/B_+\).
For \(w \in W\), define the dual extremal vectors
\[
\xi_{w,\lambda} \in V(\lambda)^*_{w,\lambda} \quad \text{by} \quad \xi_{w,\lambda}(v_{w,\lambda}) = 1
\]
(keeping in mind that \(\dim V(\lambda)_{w,\lambda} = 1\)). Denote the image of the corresponding extremal matrix coefficient in \(U_q^-[w]\):
\[
\Delta_{\lambda, w, \lambda} := \langle \xi_{w,\lambda} \otimes \text{id}, \mathcal{R} \rangle^* \in U_q^-[w] \quad \text{for} \quad \lambda \in P_+.
\]

**Proposition 8.4.** For all symmetrizable Kac–Moody algebras \(g\) and \(u \leq w \in W\), the factor ring \(U_q^-[w]/I_w(u)\) is a quantization of the coordinate ring \(\mathbb{C}[\overline{R}_{u,w}]\) of the closure of the open Richardson variety \(R_{u,w}\) in the Schubert cell \((B_+ w B_+)/B_+\). The localization
\[
(U^+[-w]/I_w(u))[\Delta_{\omega_i, w, \omega_i}^- 1 \leq i \leq r]
\]
of this ring is a quantization of the coordinate ring \(\mathbb{C}[R_{u,w}]\).

These facts were stated in \([43, \text{pp. 274-275}]\) for finite dimensional complex simple Lie algebras \(g\), but the proofs given there carry over to the symmetrizable Kac–Moody case directly.
9. Categorifying Richardson Varieties

In this section, we prove that the ideals \( I_w(u) \cap U^-_A[u] \) of \( U^-_A[u] \) are Serre completely prime ideals for all symmetric Kac–Moody algebras \( g \) and \( u \leq w \in W \). We then use Theorem 6.12(4) to construct a (domain) multiring category which categorifies the quantization of the coordinate ring of the closure of the open Richardson variety \( R_{u,w} \) in the Schubert cell \( (B_+wB_+)/B_+ \). This category is obtained as a factor of a multiring category consisting of graded, finite dimensional representations of the corresponding KLR algebras.

9.1. The categorifications of \( U^-_A(g) \) and \( U^-_A[w] \), and relations to dual canonical bases.

For each symmetrizable Kac–Moody algebra \( g \), Khovanov, Lauda [25] and Rouquier [37] defined a family of (graded) quiver Hecke algebras over a base field \( k \), which we will call KLR algebras. They proved that the category \( C \) which is the direct sum of the categories of finite dimensional graded modules of the KLR algebras associated to \( g \) has the following properties:

**Theorem 9.1.** (Khovanov–Lauda [25] and Rouquier [37]) For each symmetrizable Kac–Moody algebra \( g \) and base field \( k \), \( C \) is a \( k \)-linear multiring category such that

\[
K_0(C) \cong U^-_A(g)^\vee.
\]

The action of \( q \) on the right hand side and the shift of grading autoequivalence of \( C \) are related via

\[
[M(k)] = q^k[M] \quad \text{for all objects } M \text{ of } C.
\]

The theorem implies that for every symmetrizable Kac–Moody algebra \( g \), \( U^-_A(g)^\vee \) is a \( \mathbb{Z}_+ \)-ring with positive \( \mathbb{Z}_+ \)-basis \( \{[M] \} \) where \( M \) runs over the isomorphism classes of the simple objects of \( C \). (Here we disregard the structure of \( U^-_A(g)^\vee \) as an \( A \)-module and view it just as a ring.) For symmetric Kac–Moody algebras \( g \), the relation between this basis and the upper global basis of \( U^-_A(g)^\vee \) is given by the next theorem. For it we recall that the dual of each graded finite dimensional representation of a KLR algebra has a canonical structure of a KLR module which is also graded, finite dimensional. This gives a canonical duality endofunctor of \( C \).

**Theorem 9.2.** (Varagnolo–Vasserot [40] and Rouquier [38]) For each symmetric Kac–Moody algebra \( g \) and base field \( k \) of characteristic 0, under the isomorphism (9.1), the upper global basis corresponds to the set of isomorphism classes of the self-dual simple modules in the category \( C \).

The theorem implies that in these cases \( U^-_A(g)^\vee \) is a \( \mathbb{Z}_+ \)-ring with positive \( \mathbb{Z}_+ \)-basis

\[
q^\mathbb{Z}_B^{up}.
\]

For each symmetric Kac–Moody algebra \( g \) and \( w \in W \), in [18, §11.2] Kang, Kashiwara, Kim and Oh constructed a monoidal subcategory \( C_w \) of \( C \) as the smallest monoidal Serre subcategory closed under shifts, containing a certain set of simple self-dual modules of the KLR algebras (18, Definition 11.2.1) and, using [9], they proved:

**Theorem 9.3.** (Kang–Kashiwara–Kim–Oh [18]) For each symmetric Kac–Moody algebra \( g \),

\[
K_0(C_w) \cong U^-_A[w]^\vee.
\]
Combining Theorems 9.2 and 9.3 gives that, under the isomorphism (9.2), the elements of the upper global basis \( B_{\text{up}}^+[w] \) of \( U^-_\mathcal{A}[w] \) (recall (8.7)) correspond to the isomorphism classes of the simple self-dual objects \( \mathcal{C}_w \). In particular, \( U^-_\mathcal{A}[w] \) is a \( \mathbb{Z}_+ \)-ring with a positive \( \mathbb{Z} \)-basis

\[
q^\mathbb{Z} B_{\text{up}}^+[w].
\]

9.2. Serre completely prime ideals of the \( \mathbb{Z}_+ \)-rings \( U^-_\mathcal{A}[w] \) and the multiring categories \( \mathcal{C}_w \). For \( u \leq w \), denote the ideals

\[
I_w(u)_\mathcal{A} := I_w(u) \cap U^-_\mathcal{A}[w]^\vee
\]

of \( U^-_\mathcal{A}[w] \). Theorem 8.2 implies that for every symmetrizable Kac–Moody algebra \( \mathfrak{g} \), \( I_w(u)_\mathcal{A} \) are completely prime ideals of \( U^-_\mathcal{A}[w] \) in the classical sense. The following is the main result of this section.

**Theorem 9.4.** (1) Let \( \mathfrak{g} \) be a symmetrizable Kac–Moody algebra and \( u \leq w \in W \). The ideal \( I_w(u)_\mathcal{A} \) has an \( \mathcal{A} \)-basis given by

\[
B_{\text{up}}^-[w] \cap I_w(u)_\mathcal{A}.
\]

Furthermore, it is a Serre completely prime ideal of the \( \mathbb{Z}_+ \)-ring \( U^-_\mathcal{A}[w] \).

Denote by \( X_w(u) \) the set of isomorphism classes of self-dual simple objects \( M \) of \( \mathcal{C}_w \) such that \( [M] \in I_w(u)_\mathcal{A} \). Let

\[
\mathcal{I}_w(u) := S(X_w(u)[k], k \in \mathbb{Z})
\]

be the full subcategory of \( \mathcal{C}_w \) whose objects have Jordan-Hölder series with simple subquotients isomorphic to shifts of objects in \( X_w(u) \) as in Lemma 6.11(2).

(2) Let \( \mathfrak{g} \) be a symmetric Kac–Moody algebra and \( u \leq w \in W \). For all base fields \( k \) of characteristic 0, \( \mathcal{I}_w(u) \) are Serre completely prime ideals of the \( k \)-linear multiring category \( \mathcal{C}_w \). For the corresponding Serre quotient \( \mathcal{C}_w/\mathcal{I}_w(u) \), we have

\[
K_0(\mathcal{C}_w/\mathcal{I}_w(u)) \cong U^-_\mathcal{A}[w]^\vee/I_w(u)_\mathcal{A}.
\]

By the first part of Theorem 9.4(1),

\[
(U^-_\mathcal{A}[w]^\vee/I_w(u)_\mathcal{A}) \otimes_\mathcal{A} \mathbb{Q}(q) \cong U^-[w]/I_w(u)
\]

and by Proposition 8.3 \( U^-[w]/I_w(u) \) is a quantization of the coordinate ring \( \mathbb{C}[\mathcal{R}_{u,w}] \) of the closure of the Richardson variety \( R_{u,w} \) in the Schubert cell \( (B_+wB_+)/B_+ \). This fact and Theorem 9.4(2) imply that the Serre quotient \( \mathcal{C}_w/\mathcal{I}_w(u) \), which is a domain in the sense of Definition 5.4, is a monoidal categorification of the quantization of \( \mathbb{C}[\mathcal{R}_{u,w}] \).

9.3. Proof of Theorem 9.4. Recall that \( B(\lambda)^{\text{low}} \) denoted the lower global basis of the irreducible module \( V(\lambda) \) for \( \lambda \in P_+ \). We will need two facts about the lower global bases of Demazure modules proved by Kashiwara:

**Theorem 9.5.** (Kashiwara [20]) For every symmetrizable Kac-Moody algebra \( \mathfrak{g} \) and dominant integral weight \( \lambda \in P_+ \), the intersection

\[
B_w^\pm(\lambda)^{\text{low}} := B(\lambda)^{\text{low}} \cap V_w^\pm(\lambda)
\]

is a \( \mathbb{Q}(q) \)-basis of the Demazure module \( V_w^\pm(\lambda) \).

The sets \( B_w^\pm(\lambda)^{\text{low}} \) are called the lower global bases of the Demazure modules \( V_w^\pm(\lambda) \). The plus case was proved in [20, Proposition 3.2.3(i)] and the minus in [20, Proposition 4.1]. The following theorem describes the relationship between the canonical/low global bases \( B_w^\pm(\lambda) \) of the Demazure modules and the action of the canonical/low global bases of \( U^-_\mathcal{A}(\mathfrak{g}) \) acting on the corresponding extremal weight vectors.
**Theorem 9.6.** (Kashiwara [21,22]) Let \( g \) be a symmetrizable Kac-Moody algebra \( g \) and \( \lambda \in P_+ \) be a dominant integral weight. Denote the subset

\[
\mathfrak{B}_w^+(\lambda)^\text{low} := \{ b \in B^\text{low}_+ \mid b \cdot v_{w,\lambda} \neq 0 \}
\]

of the lower global basis of \( U^+_\lambda(g) \). Then there is a bijection between this set and the lower global basis of the Demazure module \( V^+_w(\lambda) \) given by

\[
\eta_w : \mathfrak{B}_w^+(\lambda)^\text{low} \overset{\cong}{\rightarrow} B^+_w(\lambda)^\text{low} \text{ given by } \eta_w(b) := b \cdot v_{w,\lambda}.
\]

The corresponding fact to this theorem for the negative Demazure modules (where everywhere plus is replaced by minus) was proved in [20, Proposition 4.1].

The following proposition is a stronger form of the statement of the first part of Theorem 9.6 (1).

**Proposition 9.7.** For all symmetrizable Kac–Moody algebras \( g \), and \( u \leq w \in W \), the ideal \( I_w(u) \) of the quantum Schubert cell algebra \( U^{-}_q[w] \) has a \( \mathbb{Q}(q) \)-basis given by

\[
\bigcup_{\lambda \in P_+} \{ b^\vee \mid b \in \varphi^{-1}\eta^{-1}_w(B^+_w(\lambda)\bigcap B^-_u(\lambda)) \}.
\]

**Proof.** For \( \lambda \in P_+ \), consider the basis of \( V(\lambda)^\circ \) (cf. (8.8)) which is dual to the lower global basis \( B(\lambda)^\text{low} \) of \( V(\lambda) \). Given \( v \in B(\lambda)^\text{low} \), denote by \( v^\vee \) the corresponding dual element, so

\[
v_1^\vee(v_2) = \delta_{v_1,v_2} \text{ for } v_1, v_2 \in B(\lambda)^\text{low}.
\]

Theorem 9.5 implies that

\[
\{ \xi \in V(\lambda)^\circ \mid \xi \perp V^-_w(\lambda) \} = \text{Span}_{\mathbb{Q}(q)} \{ B^+_w(\lambda)^\text{low} \bigcap B^-_u(\lambda)^\text{low} \}
\]

\[
\oplus \text{Span}_{\mathbb{Q}(q)} \{ B(\lambda)^\text{low} \bigcap (B^+_w(\lambda)^\text{low} \bigcup B^-_u(\lambda)^\text{low}) \}.
\]

For \( v \in B(\lambda)^\text{low} \bigcap (B^+_w(\lambda)^\text{low} \bigcup B^-_u(\lambda)^\text{low}) \), we have \( v^\vee \perp V_w^+(\lambda) \), and thus,

\[
\langle c^v_{v^\vee,w,\lambda} \otimes \text{id}, \mathcal{R} \rangle = 0.
\]

Therefore, the subspace

\[
\{ \langle c_{\xi^\vee,w,\lambda} \otimes \text{id}, \mathcal{R} \rangle^* \mid \xi \in V(\lambda)^\circ \mid \xi \perp V^-_w(\lambda) \} \subset I_w(u)
\]

is spanned by

\[
\{ \langle c_{v^\vee,w,\lambda} \otimes \text{id}, \mathcal{R} \rangle^* \mid v \in B^+_w(\lambda)^\text{low} \bigcap B^-_u(\lambda)^\text{low} \}.
\]

The proposition now follows from the identity

\[
(9.3) \quad \langle c_{v^\vee,w,\lambda} \otimes \text{id}, \mathcal{R} \rangle^* = (\varphi^{-1}\eta^{-1}_w(v))^\vee \text{ for } v \in B^+_w(\lambda)^\text{low}.
\]

To show this, first note that Theorem 9.6 implies that for \( v \in B^+_w(\lambda)^\text{low} \) and \( b \in B^+_w(\lambda)^\text{low} \),

\[
\langle v^\vee, b \cdot v_{w,\lambda} \rangle = \begin{cases} 1, & \text{if } b = \eta^{-1}_w(v) \\ 0, & \text{otherwise.} \end{cases}
\]

Using this and the second part of Proposition 8.3, for \( v \in B^+_w(\lambda)^\text{low} \), we obtain

\[
\langle c_{v^\vee,w,\lambda} \otimes \text{id}, \mathcal{R} \rangle^* = \sum_{b \in B^+_w(\lambda)^\text{low}} \langle v^\vee, \varphi(b) \cdot v_{w,\lambda} \rangle b^\vee = (\varphi^{-1}\eta^{-1}_w(v))^\vee,
\]

which shows (9.3) and completes the proof of the proposition. \( \square \)
Proof of Theorem 9.4. (1) The first statement in part (1) follows from Proposition 9.7. By Theorem 8.2, $I_w(u)$ is a completely prime ideal of $U_q[w]$, and therefore, the contraction $I_w(u) \cap U_A[w]$ is a completely prime ideal of $U_A[w]$. The ideal $I_w(u) \cap U_A[w]$ has a $\mathbb{Z}$-basis consisting of elements that belong to $q^\mathbb{Z}_B[w]$, which, by Theorem 9.2, is precisely the positive basis of the $\mathbb{Z}_+$-ring $U_A[w]$. Now we apply Lemma 6.16, which gives that $I_w(u) \cap U_A[w]$ is a Serre completely prime ideal of $U_A[w]$.

Part (2) follows from part (1), Theorem 6.12(4) (applied to the category $C_w$) and the isomorphism $K_0(C_w) \cong U_A[w]^\vee$ from Theorem 9.3. □

It is possible that Theorem 9.4(2) holds for symmetrizable Kac–Moody algebras $\mathfrak{g}$ by arguments that avoid the use of Theorem 9.2.

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