Interference Alignment with Quantized Grassmannian Feedback in the K-user Constant MIMO Interference Channel

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Abstract

A simple channel state information (CSI) feedback scheme is proposed for interference alignment on the K-user Constant Multiple-Input-Multiple-Output Interference Channel (MIMO-IC). The scaling of the number of feedback bits with the transmit power required to preserve the multiplexing gain that can be achieved under perfect CSI is derived. This result is obtained through a reformulation of the interference alignment problem in order to exploit the theory of quantization on the Grassmann manifold. We show that, for practical choices of the channel dimensions, the proposed scheme outperforms the scheme consisting in quantizing the normalized channel matrices, in the sense that it yields a better sum rate performance for the same number of feedback bits.

Index Terms

K-user MIMO interference channel, interference alignment, CSI quantization, Grassmann manifold, limited CSI feedback.

I. INTRODUCTION

Multiple-antenna transceivers are known to improve the performance of a wireless communication link compared to single-antenna systems. The increasing demand for high throughput and reliable transmission necessitates efficient use of Multiple-Input-Multiple-Output (MIMO) systems. In particular in multi-user networks where interference is a major concern, the availability of channel state information (CSI) at the transmitter is crucial in order to fully exploit the performance improvement of MIMO systems. In scenarios where the channel is not reciprocal (such as frequency-division duplex systems), the CSI has to be quantized and fed back to the transmitter. The mismatch between the true channel and the quantized channel results in a degradation in performance. This degradation is

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more severe when it comes to multi-user systems because the channel mismatch not only reduces the effective channel gain but also causes interference for the other users.

Interference alignment (IA) is a precoding method that achieves the optimal multiplexing gain (also called the degrees of freedom, DoF) over the $K$-user interference channel when perfect CSI is available at the transmitters \[1\]. This method designs the precoders such that the total interference at each receiver lies in a space with minimum dimensions so that the rest of the dimensions can be used for interference-free decoding. IA has been introduced for the $K$-user MIMO IC in \[2\].

Extensive research has been made on limited feedback schemes for single-user MIMO systems \[3\] and references therein. In \[4\], codebook design is investigated when the receiver selects the best unitary precoder from a finite codebook and feeds back the index of the selected precoder to the transmitter. \[4\] shows that the optimal design for such a codebook is equivalent to the Grassmannian subspace packing problem. Some useful quantization bounds on the Grassmann manifold are derived in \[5\], \[6\]. In \[7\], quantization of the precoding matrix using random vector quantization (RVQ) codebooks is investigated, providing insights on the asymptotic optimality of RVQ.

For the interference channel, the CSI feedback problem is explored in the context of interference alignment over frequency selective channels for single-antenna users in \[8\] and for multiple-antenna users in \[9\]. Both references provide DoF-achieving quantization schemes and study the required scaling of the number of feedback bits. From another point of view, \[10\] provides an analysis of the effect of imperfect CSI on the mutual information of the interference alignment scheme. The authors in \[11\] proposed a method to reduce the quantization error with respect to (w.r.t.) the classical scheme; the method involves a computationally heavy iterative algorithm which must be ran for each codeword and for each channel realization.

In this paper we present a feedback scheme based on a Grassmannian representation of CSI and we characterize the proper scaling of the number of feedback bits with the transmit power in order to preserve the multiplexing gain achievable under perfect CSI. Using the structure of the interference alignment equations, we remove redundant information in the channel quantization procedure. We also exploit the fact that the receive filters need not be evaluated at the transmitter side. This reduces the information that is required to design precoders at the transmitter side. Simulations are also provided to compare our proposed method to existing feedback methods.

The remainder of the paper is organized as follows. In Section \[II\] the system model is described. A reformulation of the CSI representation for the interference alignment problem is provided in Section \[III\]. The limited feedback method is presented in Section \[IV\] while the achievable rates and DoF are analyzed in Section \[V\]. Simulation results are presented in Section \[VI\] and conclusions are drawn in Section \[VII\].

**Notation:** Non-bold letters represent scalar quantities, boldface lowercase and uppercase letters indicate vectors and matrices, respectively. $I_N$ is the $N \times N$ identity matrix, while $0$ denotes an all-zeros matrix. The trace, conjugate, transpose, Hermitian transpose of a matrix or vector are denoted by $\text{tr}(\cdot), (\cdot)^*, (\cdot)^T, (\cdot)^H$ respectively.
The expectation operator is represented by $E(\cdot)$. The determinant of a matrix (or absolute value of a scalar) is represented by $| \cdot |$. $G_{n,d}$ denotes the complex Grassmann manifold of dimensions $(n, d)$, i.e. the set of all $d$-dimensional vector subspaces of an $n$-dimensional vector space over $\mathbb{C}$. The Frobenius norm of a matrix is denoted by $\| \cdot \|_F$ while the two-norm (spectral norm) of a matrix is represented by $\| \cdot \|_2$. A block diagonal matrix is denoted by $\text{Bdiag}(\cdot)$ with the argument blocks on its diagonal.

**II. System Model**

A MIMO interference channel is considered in which $K$ transmitters communicate with their respective receivers over a shared medium. For the sake of notational simplicity, we consider the symmetric case where each transmitter has $M$ antennas while each receiver is equipped with $N$ antennas, although the method applies to more general non-symmetric settings as well.

Transmitter $j$ employs a precoding matrix $V_j$ to transmit $d$ data streams to its respective receiver. The vector at receiver $i$ reads

$$y_i = H_{ii}V_ix_i + \sum_{1 \leq j \leq K, j \neq i} H_{ij}V_jx_j + n_i \tag{1}$$

in which $H_{ij} \in \mathbb{C}^{N \times M}$ is the channel matrix between transmitter $j$ and receiver $i$, $V_j \in \mathbb{C}^{M \times d}$ is a truncated unitary matrix ($V_j^HV_j = I_d$), and $x_j \in \mathbb{C}^d$ is the data vector of transmitter $j$. Furthermore, $n_i \in \mathbb{C}^{N \times 1}$ is the additive noise at receiver $i$ whose elements are distributed independently as $\mathcal{CN}(0, 1)$. Assuming Gaussian i.i.d. signaling with $E[x_jx_j^H] = \frac{P}{d}I_d$, $j = 1, \ldots, K$ and using truncated unitary precoders, the transmit signal-to-noise ratio (SNR) is equal to $P$. Following [12], we assume that the channel coefficients are generic; in particular, this condition is fulfilled by any channel model where the coefficients are drawn independently from a continuous distribution.

**III. Proposed Grassmannian Feedback Scheme for Interference Alignment**

Let us consider the interference alignment problem of [2], and assume that the channel state information is fed back from the receivers to the transmitters. The transmitters then compute the precoding matrices that satisfy the alignment conditions based on the received feedback. Specifically, assume that the $i$th receiver estimates the channel matrices $H_{ij}, \forall j \neq i$ and feeds back the necessary information to all the transmitters so that every transmitter is capable of solving the alignment problem. Alternatively, one could consider a single IA computation unit (to which all the CSI would be forwarded) where the precoders are computed and subsequently distributed to the transmitters. This distinction is immaterial, as the results presented here apply to both cases.

In this section we consider perfect CSI feedback in order to to highlight the intuition behind our limited feedback scheme. Assuming that global CSI is available at a given location, the precoders $V_i, i = 1 \ldots K$ must be designed
Eq. (6) shows that the rank-precoders computed from channel coefficients, precoders $C_i$Note that since $\text{rank}(C_i) = d$ almost surely (a.s.) the case for generic channel coefficients. We now show that the IA transmit precoders fulfills a similar equation to (3), i.e. for all $i \in \{1, \ldots, K\}$, i.e. for all $i \in \{1, \ldots, K\}$, it is sufficient that each receiver feeds back a point on the Grassmann manifold $G_{(K-1)M,N}$ representing the row space of $H_i$.

Proof: Let us consider perfect feedback of the row space of $H_i$. This form of feedback can be considered to take the form of the availability at the IA computation unit of a matrix $F_i$ of dimensions $(K-1)M \times N$ whose columns span the same subspace as the (transposed) rows of $H_i$ (here we assume that $H_i$ has full row rank, i.e. $N$, which is almost surely (a.s.) the case for generic channel coefficients). We now show that the IA transmit precoders computed by assuming $F_i^H$ as channel coefficients also align the interference on the true channel.

Let us consider an IA solution based on $F_i$, i.e. assume that there exist full-rank matrices $U_i$ and a full-rank block-diagonal matrix $V_{-i}$ that fulfill an equation similar to (3), i.e. for all $i \in \{1, \ldots, K\}$,  

$$U_i^H F_i^H V_{-i} = 0.$$  

Eq. (6) shows that the rank-$d$ matrix $C_i^{-1} U_i$ is able to cancel all interference at each receiver, i.e. the transmit precoders $V_1, \ldots, V_K$ also align interference over the true channels. Furthermore, under the assumption of generic channel coefficients, $\left(C_i^{-1} U_i\right)^H H_i V_i$ is full rank ($= d$) a.s. since $C_i^{-1} U_i$ is full column rank a.s. Therefore, the precoders computed from $F_i$ constitute a valid IA solution as per (2).
The required CSI feedback, here taking the form of the column space of $F_i$, is analogous to feeding back a point on the Grassmann manifold $G_{(K-1)M,N}$ for each one of the $K$ users.

Note that the scheme outlined above for the IC is in fact directly applicable to many other channel models where IA has been proposed, such as interfering multiple-access channels [15], [16] and broadcast channels [17], [18], as well as partially connected interference networks [19], [20].

IV. QUANTIZED CSI FEEDBACK

In this section we consider the case where the alignment equations are solved based on the (error-free) feedback of a quantized version of the CSI. In the first subsection we analyze our quantization scheme based on the Grassmannian representation outlined in the previous section. For comparison, in Section IV-B we analyze a simpler feedback method which was introduced in [9] and used as a baseline in [11]. We then discuss the relative merits of the two approaches.

A. Quantized feedback for the proposed scheme

Let us assume that receiver $i$ knows perfectly the state of its channels from all interfering transmitters, i.e. the coefficients of $H_i$, and performs the economy-size QR decomposition $H_i = F_i C_i$, where $F_i$ is a truncated unitary matrix, and $C_i$ is $N \times N$ and a.s. invertible, under the usual channel assumptions. Note that the use of the QR decomposition is a particular case of the decomposition used in the proof of Lemma 1: it ensures that $H_i$ and $F_i$ have the same column space, and adds the requirement that the columns of $F_i$ are orthonormal, which will simplify the subsequent analysis. Receiver $i$ quantizes the subspace spanned by the columns of $F_i$ using $N_f$ bits and feeds the index of the quantized codeword back to the unit in charge of computing the $V_i$'s. We further assume that the receivers and the computation unit share a predefined codebook $\mathcal{S} = \{S_1, ..., S_{2^{N_f}}\}$ which is composed of $2^{N_f}$ truncated unitary matrices of size $(K-1)M \times N$ and is designed using Grassmannian subspace packing. The quantized codeword at receiver $i$ is the point in $\mathcal{S}$ closest to $F_i$, i.e.

$$\hat{F}_i = \arg \min_{S \in \mathcal{S}} d_c(S, F_i) \tag{7}$$

in which $d_c(X, Y) = \frac{1}{\sqrt{2}} \|XX^H - YY^H\|_F$ is the chordal distance between $X$ and $Y$ in $G_{(K-1)M,N}$ [21].

Let us consider the scheme where the interference alignment problem is solved at the IA computation unit based on $\{\hat{F}_i^H\}^{K}_{i=1}$ to find $\{V_i\}^{K}_{i=1}, \{\hat{U}_i^H\}^{K}_{i=1}$ fulfilling

$$\hat{U}_i^H \hat{F}_i^H V_{-i} = 0, \ \forall i \in \{1, ..., K\}. \tag{8}$$

2For notational simplicity we drop the dependency of $\mathcal{S}$ on $i$, however the proposed analysis generalizes trivially to cases where $\mathcal{S}$ and $N_f$ are different across the receivers as will be seen in Section V-C.
At receiver $i$, inspired by the perfect feedback situation, we consider the receive filter $G^H_i = (C_i^{-1}F^H_i\hat{F}_i\tilde{U}_i)^H$. We define the interference term $e_i$ remaining at receiver $i$ after applying the filter $G^H_i$ to the signal $\mathbf{1}$ (we refer to $e_i$ as leakage) as

$$e_i = \sum_{1 \leq j \leq K, j \neq i} G^H_i H_{ij} V_j x_j,$$

and let $L_i = \text{tr} \left( E(\mathbf{e}_i\mathbf{e}_i^H) \right)$ denote its power. We now establish in Theorem 1 and Corollary 1 the growth rate of the number of feedback bits with the SNR which guarantees that $L_i$ remains bounded by a constant regardless of $P$ when $P \to \infty$. In section V, we will show that it also guarantees that the total multiplexing gain of the channel is achieved by the proposed scheme.

**Theorem 1.** The interference leakage power (due to imperfect CSI) at receiver $i$ can be bounded as

$$L_i \leq 2P\Delta^2 \left( 1 + o\left( 2^{-N_G} \right) \right)$$

where $N_G = 2N((K-1)M-N)$ is the real dimension of $G_{(K-1)M,N}$. $\Delta = \frac{2}{(c^2 N_f)^{N_G}}$, and the constant $c$ is the coefficient of the ball volume in the Grassmann manifold,

$$c = \frac{1}{N((K-1)M-N)!} \prod_{i=1}^{N} \frac{((K-1)M-i)!}{N-i}.$$

**Proof:** The power of the interference leakage at receiver $i$ reads

$$L_i = \text{tr} \left( \frac{P}{d} \sum_{j=1,j \neq i}^{K} G^H_i H_{ij} V_j V_j^H H_i G_i \right)$$

$$= \frac{P}{d} \text{tr} \left( G^H_i H_i V_{-i} V_{-i}^H H_i G_i \right)$$

$$= \frac{P}{d} ||G^H_i H_i V_{-i}||^2_F. \quad (12)$$

Substituting $G^H_i = (C_i^{-1}F^H_i\hat{F}_i\tilde{U}_i)$ and $H_i = C_i^{-1}F^H_i$ gives

$$L_i = \frac{P}{d} ||\hat{U}_i^H F_i C_i^{-1} H_i C_i^H F_i V_{-i}||^2_F \quad (13)$$

Using the alignment equation (8) and the fact that $\hat{F}_i \tilde{F}_i = I_N$ yields $\hat{U}_i^H \hat{F}_i \tilde{F}_i \hat{F}_i \tilde{F}_i V_{-i} = 0$, therefore (13) can be rewritten as

$$L_i = \frac{P}{d} ||\hat{U}_i^H F_i (F_i^H - \hat{F}_i \hat{F}_i^H) V_{-i}||^2_F. \quad (14)$$
Using the facts that $\|X\|_F \leq \sqrt{\text{rank}(X)} \|X\|_2$, $\|X\|_2 \leq \|X\|_F$ and $\|XY\|_2 \leq \|X\|_2 \|Y\|_2$, we have

$$L_i = \frac{P}{d} \|\tilde{U}_i^H \hat{F}_i^H (F_i F_i^H - \hat{F}_i \hat{F}_i^H) V_{-i}\|_F^2$$

$$\leq P \|\tilde{U}_i^H \hat{F}_i^H (F_i F_i^H - \hat{F}_i \hat{F}_i^H) V_{-i}\|_2^2$$

$$\leq P \|\tilde{U}_i^H\|_2^2 \|\hat{F}_i^H\|_2^2 \|\hat{F}_i - \hat{F}_i\|_2^2 \|V_{-i}\|_2^2$$

$$= P \|(F_i \hat{F}_i^H - \hat{F}_i \hat{F}_i^H)\|_F^2$$

$$\leq 2P d_c^2(\hat{F}_i, F_i).$$

(15)

The second equality holds because $\tilde{U}_i^H$, $\hat{F}_i^H$ and $V_{-i}$ are truncated unitary matrices, which implies that their two-norm is 1.

From [21, Theorem 5], if a codebook is generated using the sphere-packing procedure, the maximum value of the quantization error in terms of the chordal distance can be upper bounded as

$$\max_{F_i \in G_{(K-1)M,N}} d_c(\hat{F}_i, F_i) \leq \Delta(1 + o \left(2^{-\frac{N_f}{\log P}}\right)),$$

(16)

for $\Delta$ as previously defined. The constant $c$ in (11) is obtained from [5, Corollary 1]. From (15) and (16), the leakage power can be upper bounded as

$$L_i \leq 2P \Delta^2 (1 + o \left(2^{-\frac{N_f}{\log P}}\right)).$$

(17)

**Corollary 1.** Quantizing CSI with

$$N_f = N((K-1)M - N) \log P$$

(18)

bits is sufficient to keep the interference leakage $L_i$ bounded by a constant for arbitrarily large $P$.

**Proof:** From (10), since $o \left(2^{-\frac{N_f}{\log P}}\right) \to 0$ for large $P$, it is obvious that $L_i$ is bounded by a constant if $2^{\frac{2N_f}{\log P}}$ scales at least linearly with $P$, in particular if we have

$$N_f = \frac{N_f}{2} \log P = N((K-1)M - N) \log P.$$  

(19)

**B. Quantized feedback for the normalized-channel quantization method**

For comparison, let us now consider the CSI quantization method from [9], [11] whereby at receiver $i$, the matrices representing the channels from the interferers are vectorized and normalized independently, yielding $K-1$ unit-norm vectors $z_{ij} = \frac{\vec{H}_{ij}}{\|\vec{H}_{ij}\|_2}$, $j \neq i$, which are subsequently quantized jointly on the composite Grassmann manifold $G^{K-1}_{M,N,1}$. We denote this technique by normalized-channel quantization. We now establish the scaling of the number of feedback bits required to achieve bounded interference leakage under this quantization scheme.
Each receiver feeds back a point \( \hat{Z}_i \) on the composite Grassmann manifold \( \mathcal{G}^{K-1}_{MN,1} \) (\( \hat{Z}_i \) is equivalently represented by \( (K-1) \) unit norm vectors \( \hat{z}_{ij}, \ j = 1, ..., K, j \neq i \)). Let \( \hat{N}_f \) denote the number of feedback bits per user in this scheme, and consider a codebook \( T \) of size \( 2^N_f \). The quantization consists in selecting an element in the codebook according to

\[
\hat{Z}_i = \arg \min_{T \in T} D_c(T, Z_i),
\]

(20)

where we have introduced \( Z_i = [z_{i1}, ..., z_{ii-1}, z_{ii+1}, ..., z_{iK}] \), \( T = [t_1, ..., t_{i-1}, t_{i+1}, ..., t_K] \) and \( D_c(T, Z_i) = \sqrt{\sum_{j=1, j \neq i}^{K} (1 - |t_j^H z_{ij}|^2)} \) is the chordal distance defined for the composite Grassmann manifold. The columns of \( \hat{Z}_i = [\hat{z}_{i1}, ..., \hat{z}_{ii-1}, \hat{z}_{ii+1}, ..., \hat{z}_{iK}] \) will be used to construct the quantized channel matrices \( \hat{H}_{ij} \) required at the IA computation unit to design precoders such that

\[
\hat{z}_{ij} = \mathrm{vec}(\hat{H}_{ij}) \quad \forall \ i \neq j.
\]

(21)

We assume that the interference alignment problem is then solved based on \( \hat{H}_{ij} \) to find \((\{V_i\}_{i=1}^K, \{\tilde{U}_i\}_{i=1}^K)\) fulfilling

\[
\tilde{U}_i^H \hat{H}_{ij} V_j = 0, \quad \forall i, j \in \{1, ..., K\}, \ j \neq i.
\]

(22)

Lemma 2. \( \hat{N}_f = (K-1)(MN-1) \log P \) bits at each receiver are sufficient to keep the interference leakage bounded using normalized-channel quantization.

Proof: See appendix A. \( \blacksquare \)

C. Comparisons of the two previous methods

Comparing the number of feedback bits required for our scheme with the case of normalized-channel quantization, we get

\[
\hat{N}_f - N_f = (N^2 - K + 1) \log P,
\]

(23)

i.e. the Grassmannian CSI quantization introduced in Section III outperforms the normalized-channel quantization method iff \( N^2 > K - 1 \). Note that this condition is independent from the number of transmit antennas.

In the particular case of a symmetric, square system \((M = N)\), we have the following result:

Lemma 3. If IA is feasible, \( \hat{N}_f > N_f \), i.e. the proposed scheme scheme always requires strictly less feedback than normalized-channel quantization.

Proof: A necessary condition for IA to be feasible is \([12]\)

\[
d \leq \frac{M + N}{K + 1}.
\]

(24)

With the assumption that \( M = N \) and using the fact that \( d \geq 1 \), \([24]\) gives

\[
K \leq \frac{2N}{d} - 1 < 2N.
\]

(25)
Another necessary condition for IA feasibility is \( N \geq 2d \), therefore \( N > 1 \) and consequently \( 2N < N^2 + 1 \). Combining with (25), we obtain \( K < N^2 + 1 \), which gives \( \tilde{N}_f > N_f \).

\[ \text{V. Achievable Rate Analysis} \]

In the previous section, we have used interference leakage as a proxy to evaluate how the quality of the available CSI influences alignment. Note however that having a bounded interference leakage is not sufficient in itself to ensure that the full DoF is achieved for asymptotically large \( P \) – in fact, the power of the signal of interest remaining after the receive filters could remain bounded too, or its rank could be reduced. In this section we show that this is not the case, and that the proposed CSI quantization scheme achieves the same DoF as IA under the perfect CSI assumption, provided that the proper scaling of \( N_f \) with \( P \) is respected.

In particular, we analyze the sum rate achieved by the proposed scheme, and we bound the rate loss (w.r.t. the interference-free case) due to CSI quantization in order to find a lower bound for the achievable sum rate. We prove that if the number of feedback bits is scaled according to (18), the rate loss is a constant value independent of \( P \), with the achievability of the total DoF as a corollary.

Note that the transmission scheme considered here is based on truncated unitary precoders \( V_j \), and therefore the transmitted signal is spatially white inside the \( d \)-dimensional subspace defined by the precoder. Clearly, this is suboptimal for finite values of the SNR, and spatial waterfilling in addition to IA would bring in performance improvement for \( d > 1 \). However, we remark that the performance gains of waterfilling vanish at asymptotically high SNR, provided that the channel is not rank deficient [22]. Therefore, the asymptotic analysis presented in this section holds regardless of whether spatial waterfilling is used in addition to IA or not.

\[ \text{A. Rate Loss Due to CSI Quantization} \]

We start with two lemmas that will be useful for the main proof. Let us consider the rate

\[ R_p^i = \log \left| I_d + \frac{P}{d} Q_S^i \right| \]

in which \( Q_S^i = G_i^H H_i V_i V_i^H H_i^H G_i \). This rate is achievable in the hypothetical case where the receive filter \( G_i^H \) defined in Section IV-A completely cancels all interference.

**Lemma 4.** Under the proposed CSI quantization scheme, one can make \( R_p^i \) grow asymptotically like \( d \log P \), provided that \( N_f \) is scaled according to (18).

**Proof:** Note that the proof is complicated by the fact that, in order to analyze \( R_p^i \) for increasingly large SNR, we need to consider quantization codebooks of increasing sizes; since \( Q_S^i = \tilde{U}_i^H \hat{F}_i^H F_i C_i^{-1} H_i V_i V_i^H H_i^H C_i^{-1} F_i^H \hat{F}_i \tilde{U}_i \), where \( \tilde{U}_i, \hat{F}_i \) and \( V_i \) all depend on the choice of the codebook, it is not clear whether \( Q_S^i \) admits a limit for
asymptotically large SNR \(^3\). Therefore, we resort to compactness arguments to show that there exists a series of codebooks of increasing size for which \(Q_S^g\) admits a limit.

We consider an infinite sequence of SNRs \(\mathcal{P} = \{P_n\}_{n \in \mathbb{N}}\) such that \(\lim_{n \to \infty} P_n = \infty\), as well as an infinite sequence of quantization codebooks \(\{S_n\}_{n \in \mathbb{N}}\), such that \(\|S_n\| = \frac{P_n^{N(K-1)M-N}}{N},\) following (18). For each SNR value \(P_n\), we let \(\hat{F}_{i,n}\) denote the point in \(S_n\) closest to \(F_i\), and \((V_{1,n}, \ldots, V_{K,n}, \hat{U}_{1,n}, \ldots, \hat{U}_{K,n}) \in \mathcal{G}_{M,d}^K \times \mathcal{G}_{N,d}^K\) a set of matrices constituting an IA solution based on \(\hat{F}_{i,n}\). In other words, we solve (7) and (8) for each \(n\), yielding an infinite series of solutions. Let us denote \(W_n = (\hat{F}_{1,n}, \ldots, \hat{F}_{K,n}, V_{1,n}, \ldots, V_{K,n}, \hat{U}_{1,n}, \ldots, \hat{U}_{K,n})\).

Since \(\mathcal{G}_{M,d}^K \times \mathcal{G}_{N,d}^K\) is compact, as a Cartesian product of compact sets. Therefore, we can extract a convergent subsequence from \(\{W_n\}_{n \in \mathbb{N}}\). We let \(g(m) \in \mathbb{N}\) denote the index of the \(m\)-th element of the convergent subseries, where \(g\) is a monotonically increasing function. We also denote

\[
(F_1^*, \ldots, F_K^*, V_1^*, \ldots, V_K^*, \hat{U}_1^*, \ldots, \hat{U}_K^*) = \lim_{m \to \infty} W_{g(m)}. \tag{27}
\]

Letting \(Q_S^{i,n} = \hat{U}_{i,n}^H F_i^H F_i C_i H_i V_{i,n}^H V_i^H H_i^{-1} F_i^H F_i \hat{U}_{i,n}\), we can now write the limit \(\lim_{m \to \infty} Q_S^{i,g(m)} = Q_S^{i,*}\), where \(Q_S^{i,*} = \hat{U}_{i}^H F_i^* F_i^H F_i C_i^H H_i V_i^* V_i^H H_i^{-1} F_i^H F_i \hat{U}_{i}^*\). Therefore we have

\[
\lim_{m \to \infty} \frac{\log |I_d + \frac{P_{g(m)}}{d} Q_S^{i,g(m)}|}{\log P_{g(m)}} = \lim_{m \to \infty} \frac{\log |I_d + \frac{P_{g(m)}}{d} Q_S^{i,*}|}{\log P_{g(m)}} = \text{rank}\left(Q_S^{i,*}\right). \tag{28}
\]

Since \(F_i^*\) and \(F_i\) span the same subspace, \(F_i^* F_i\) is unitary. Therefore, considering the product of matrices in \(Q_S^{i,*}\), we note that \(\hat{U}_{i}^H F_i^* F_i C_i^H H_i V_i^* V_i^H H_i^{-1} F_i^H F_i \hat{U}_{i}\) has full row rank \(d\), \(V_i^*\) has full column rank \(d\), and both are independent of \(H_i\), from which we conclude that \(\text{rank}\left(Q_S^{i,*}\right) = d\) a.s., which proves the lemma.

We now turn to the rate \(R_q^i\) achievable by user \(i\) (after filtering by \(G_i^H\) at the receiver) with the proposed scheme under finite rate feedback; for this, let us consider the mutual information between \(x_i\) and \(y_i\),

\[
R_q^i = \log \left|I_d + \frac{P_d}{d} (Q_i^d + Q_i^t)\right| - \log \left|I_d + \frac{P_d}{d} Q_i^t\right| \tag{30}
\]

in which \(Q_i^d = G_i^H H_i V_i^H H_i^H G_i\).

**Lemma 5.** The rate achievable by the quantization scheme from Section IV-A can be lower bounded as

\[
R_q^i \geq R_p^i - d \log \left(1 + 2 \frac{P_d}{d} \Delta^2 \left(1 + o \left(2^{-\frac{N}{16d}}\right)\right)\right). \tag{31}
\]

\(^3\)Although it is clear that the subspace spanned by \(F_i\) admits a limit on the Grassmann manifold when \(N_f \to \infty\), the definition of \(\hat{U}_i\) and \(V_i\) as one (possibly among several) solution of (3) prevents the extension of the convergence result to those variables.

\(^4\)In order to obtain the same convergence properties for a point on \(G_{a,b}\) and for the corresponding unitary matrix representation \(F \in \mathbb{C}^{a,b}\), it is useful to make this representation unique, e.g. by requiring that the top square \(b \times b\) subblock of \(F\) is equal to \(I_b\). For the sake of notational simplicity, we omit these details.
\textit{Proof:} Consider the following quantity

\[
\Delta R_i = R^i_p - R^i_q
\]

(32)

\[
= \log \left| \mathbf{I}_d + \frac{P}{d} \mathbf{Q}_i \right| - A^i
\]

(33)

\[
\leq \log \left| \mathbf{I}_d + \frac{P}{d} \mathbf{Q}_i \right|,
\]

(34)

where \( A^i = \log \left| \mathbf{I}_d + \frac{P}{d} \mathbf{Q}_i \right| - \log \left| \mathbf{I}_d + \frac{P}{d} \mathbf{Q}_i \right| \) and the inequality follows from the fact that \( \mathbf{Q}_i \) is positive semi-definite and therefore \( A^i \geq 0 \). We can further bound \( \Delta R^i \) as

\[
\Delta R^i \leq d \log \left( \lambda_{\max} \left( \mathbf{I}_d + \frac{P}{d} \mathbf{Q}_i \right) \right)
\]

(35)

\[
= d \log \left( 1 + \frac{P}{d} \lambda_{\max} (\mathbf{Q}_i) \right)
\]

(36)

\[
= d \log \left( 1 + \frac{P}{d} \left\| \mathbf{G}_i^H \mathbf{H}_i \mathbf{V}_i \right\|^2_2 \right),
\]

(37)

where we used the fact that \( \left\| \mathbf{X} \right\|^2_2 = \lambda_{\max} (\mathbf{X} \mathbf{X}^H) \). From eqs. (12)–(15) we have \( \left\| \mathbf{G}_i^H \mathbf{H}_i \mathbf{V}_i \right\|^2_2 \leq 2d_c^2 (\mathbf{F}_i, \mathbf{F}_i) \), which gives

\[
\Delta R^i \leq d \log \left( 1 + 2 \frac{P}{d} d_c^2 (\mathbf{F}_i, \mathbf{F}_i) \right)
\]

(38)

\[
\leq d \log \left( 1 + 2 \frac{P}{d} \Delta^2 \left( 1 + o \left( 2^{-\frac{N_f}{N_G}} \right) \right) \right).
\]

(39)

Combining (39) with (32) yields (31).

\[ \Box \]

We can now state our main result:

**Theorem 2.** If IA with \( d \) DoF is feasible, the proposed CSI quantization scheme also achieves \( d \) DoF if \( N_f \) is scaled according to (18).

\textit{Proof:} From Lemma 5 we have

\[
R^i_q \geq R^i_p - d \log \left( 1 + 2 \frac{P}{d} \Delta^2 \left( 1 + o \left( 2^{-\frac{N_f}{N_G}} \right) \right) \right).
\]

(40)

Substituting \( P = 2^{\frac{2N_f}{N_G}} \) and \( \Delta = \frac{2}{(c_2^{N_f})^{\frac{2}{N_G}}} \) gives

\[
\lim_{P \to \infty} \frac{R^i_q}{\log P} \geq \lim_{P \to \infty} \frac{R^i_p - d \log \left( 1 + \frac{8}{c_2^{N_f} d} \left( 1 + o \left( 2^{-\frac{N_f}{N_G}} \right) \right) \right)}{\log P}
\]

(41)

\[
= \lim_{P \to \infty} \frac{R^i_p - d \log \left( 1 + \frac{8}{c_2^{N_f} d} \right)}{\log P}
\]

(42)

\[
\geq d,
\]

(43)
where we have used the fact that \( o \left( 2^{-\frac{N_f}{c}} \right) \to 0 \) since \( N_f \to \infty \) as \( P \to \infty \). (43) is obtained by noticing that when \( P \to \infty \), the argument of the logarithm in the numerator of (42) remains bounded by a constant, while \( R_p^i \) grows with \( d \log P \) according to Lemma 4.

Another observation is that when the number of feedback bits is scaled according to (18), the bound in (34) gets tighter as the SNR increases. This can be seen by noticing that
\[
\lim_{P \to \infty} A^i = \lim_{P \to \infty} \log \left| I_d + \frac{P}{d} Q_i^j \left( I_d + \frac{P}{d} Q_S^i \right)^{-1} \right| = \lim_{P \to \infty} \log \left| I_d + Q_i^j Q_S^i \right|^{-1}
\]
which goes to zero since \( Q_S^i \) is full rank almost surely and when feedback scales according to (18) we have \( ||Q_i^j||_2 \to 0 \) for \( P \to \infty \).

Note that Lemma 5 holds for any particular codebook and also for any realization of the channel matrices. We will now consider the average loss in the achievable rate when the quantization codebooks are random.

**B. Average Rate Loss under Random Vector Quantization**

If RVQ is used, the relevant performance metric is the average sum rate over all possible codebooks. Here we introduce the following corollary:

**Corollary 2.** The rate of user \( i \) can be lower bounded as
\[
E_S(R_q^i) \geq R_p^i - d \log \left( 1 + \frac{2P}{d} \frac{\Gamma\left(\frac{2}{N_m}\right)}{\Gamma\left(\frac{2}{2(2N_f)}\right)} \right),
\]
where \( \Gamma(\cdot) \) denotes the Gamma function.

**Proof:** Starting from (38), the average rate for user \( i \) reads
\[
E_S(R_q^i) \geq R_p^i - d E_S \left( \log \left( 1 + \frac{2P}{d} d^2_c(\hat{F}_i, F_i) \right) \right) \geq R_p^i - d \log \left( 1 + 2P E_S \left( d^2_c(\hat{F}_i, F_i) \right) \right)
\]
where the second inequality follows by application of Jensen’s inequality to the log function. The term \( E_S(d^2_c(\hat{F}_i, F_i)) \) represents the expected value of the distortion while using a random codebook, and can be further bounded using [21 Theorem 6], which can be summarized as follows: for asymptotically large codebook size, when using a random codebook for quantizing a matrix \( F \) arbitrarily distributed over an arbitrary manifold, the \( k \)-th moment of the chordal distance \( D^{(k)} = E_{S,F}(d^k_c(\hat{F}, F)) \) can be bounded as
\[
\frac{N_m}{(N_m + k)(c2^{N_f})^\frac{k}{c}} \leq D^{(k)} \leq \frac{\Gamma\left(\frac{k}{N_m}\right)}{\Gamma\left(\frac{k}{c2^{N_f}}\right)^\frac{k}{N_m}},
\]
where the codebooks have \( 2^{N_f} \) elements and \( N_m \) is the real dimension of the corresponding manifold. Using the upper bound in (47) for \( k = 2 \) over the Grassmann manifold, results in (45).
C. Per-User DoF for Asymmetric Feedback

An interesting consequence of the rate-loss analysis conducted previously can be observed when each receiver uses its own scaling of the CSI quantization codebook size with $P$. Formally, let $N^i_j$ denote the number of bits used by receiver $i$ to quantize $F^i$.

**Corollary 3.** If $N^i_j$ scales with $P$ such that

$$\alpha_i \triangleq \lim_{P \to \infty} \frac{N^i_j}{N((K-1)M-N) \log P}$$

exists and is finite, then the DoF achievable by user $i$ is

$$d^q_i \geq d^p_i \min (\alpha_i, 1),$$

where $d^p_i$ is the achievable DoF of this user with perfect CSI.

**Proof:** The proof follows simply from (31) by taking the limit of the lower bound when $P \to \infty$, and therefore holds both for RVQ and for codebooks generated using the sphere-packing procedure.

Practically, this means that the DoF achieved by a given user is independent of the quality of the feedback provided by the other users. This observation, obtained here for precoding based on the IA criterion, is consistent with the scaling obtained in [23] for centralized schemes using different precoding schemes such as zero-forcing.

VI. Simulation Results

This section presents simulations validating the results hitherto established. For relatively small codebook sizes (up to $2^{15}$), simulations were performed using RVQ codebooks – the results are presented in Section VI-A. Due to the lack of structured codebooks allowing a tractable implementation of the quantizer (7), the performance obtained for larger codebooks is extrapolated by using a perturbation method based on the analytical characterization of the distribution of the quantization error, the details of which being presented in Section VI-B.

A. Performance results using RVQ

In this section, we evaluate the performance of the quantization scheme of Section IV-A with RVQ codebooks. The performance metric is the sum rate evaluated through Monte-Carlo simulations. The sum rate achievable over the MIMO IC using interference alignment precoders under the assumption that the input signals are Gaussian can be written as

$$R_{\text{sum}} = \sum_{i=1}^{K} \log \left| I_N + \frac{P}{d} \sum_{j=1}^{K} H_{ij} V_j V_j^H H_{ij}^H \right| - \sum_{i=1}^{K} \log \left| I_N + \frac{P}{d} \sum_{j=1, j \neq i}^{K} H_{ij} V_j V_j^H H_{ij}^H \right|. \quad (50)$$

A 3-user IC with $M = N = 2$ antennas per node and $d = 1$ data stream for each transmitter is considered. Entries of the channel matrices are generated according to $CN(0, 1)$ and the performance results are averaged over the channel realizations. Our proposed method is compared to the normalized-channel quantization method from Section IV-B where the interfering channel matrices toward a receiver are independently vectorized and quantized.
Fig. 1. Sum-rate comparison of quantization methods, for the 3-user MIMO IC, \( N = M = 2 \) using (50).

based on the idea of composite Grassmann manifold and finally the indices of the quantized vectors are fed back to the transmitters (denoted by “NCQ” in the figures).

For the proposed method, the codebook entries are independent \((K - 1)M \times N\) random truncated unitary matrices generated from the Haar distribution. For the normalized-channel quantization method, random unit norm vectors are used in the codebook construction. Figure 1 shows the achievable sum rate versus transmit SNR for \( N_f = 5 \) and 10 feedback bits when the precoders are designed based on the quantized feedback. Clearly the proposed scheme outperforms normalized-channel quantization for the same number of feedback bits. It can be also seen that for a fixed number of feedback bits, the sum-rate saturates at high SNR, while it grows unbounded (with the slope equal to the DoF) for the perfect CSI case.

The sum rate in (50) is achievable when optimum receivers (not including the projection filters \( G_i^H \)) are used at the receivers. Since the achievable scheme in Section IV is using the projection filters \( G_i^H \), we evaluated the performance achieved by this scheme, defined as

\[
R_{\text{sum}}' = \sum_{i=1}^{K} \log |G_i^H G_i| + \frac{P}{d} \sum_{j=1}^{K} G_i^H H_{ij} V_j V_j^H H_{ij}^H G_i - \sum_{i=1}^{K} \log |G_i^H G_i| + \frac{P}{d} \sum_{j=1, j \neq i}^{K} G_i^H H_{ij} V_j V_j^H H_{ij}^H G_i.
\]

(51)

Results are provided in Figure 2. The slope of the curves at high SNR gives an indication of the achieved DoF. It is clear from Figure 2 that the slope of the sum-rate curve with quantized feedback matches that of perfect CSI when the number of feedback bits is scaled according to (18) (here we have used \( N_f = [0, 7, 13, 20, 26] \) bits and the corresponding powers \( P = 2^{2N_f} \)). Conversely, when the codebook size is fixed, the performance always saturates at high SNR, with the achieved performance depending on the codebook size. Simulations were performed only up to 20 dB SNR due to the complexity associated to the growth of the codebook size with \( P \).
Fig. 2. Sum-rate according to (51) of the proposed method for different number of bits, for the 3-user MIMO IC, $N = M = 2$.

B. Perturbations on the Grassmann manifold

In order to validate the DoF results of Section V, an evaluation of the achieved sum-rate at high SNR is required. In order to deal with exponentially large codebooks, we propose to replace the quantization process with a perturbation which approximates the quantization error. As will be seen, this approach provides a good approximation of the sum rate, while avoiding the prohibitively complex simulation of RVQ schemes for very large codebooks.

Let us consider a point on $G_{n,p}$, represented by a $n \times p$ truncated unitary matrix $F$. Here, we assume that $n \geq 2p$, since it is otherwise more efficient to consider the null space of $F$ instead. Since the columns of $F$ are orthonormal, they can be completed to form an orthonormal basis of the $n$-dimensional space. In fact, according to [6], any other point on $G_{n,p}$ can be represented in the basis constituted by the columns of the unitary matrix $W = [F \ F^c]$ as

$$\tilde{F} = W \begin{bmatrix} C \\ S \\ 0_{n-2p} \end{bmatrix},$$  \hspace{1cm} (52)$$

for some $F^c$ in the null space of $F$ and

$$C = \begin{bmatrix} \cos \theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \cos \theta_p \end{bmatrix}, \quad S = \begin{bmatrix} \sin \theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sin \theta_p \end{bmatrix},$$  \hspace{1cm} (53)$$

where $\theta_1, \ldots, \theta_p$ are real angles. Clearly, for $\theta_1 = \ldots = \theta_p = 0$, we obtain $\tilde{F} = F$. More generally, the squared chordal distance between the two points on $G_{n,p}$ represented by $F$ and $\tilde{F}$ is

$$r = d^2_c(F, \tilde{F}) = \sum_{i=1}^{p} \sin^2 \theta_i.$$  \hspace{1cm} (54)$$
Therefore, in order to generate random perturbations of a certain chordal distance $\sqrt{r_0}$ from $F$, we propose to generate random values for the angles $\theta_1, \ldots, \theta_p$ such that $\sum_{i=1}^{p} \sin^2 \theta_i = r_0$, and to pick a random orthonormal basis $F^c$ of the null subspace of $F$. The perturbed matrix is then computed using (52).

The histogram (not shown) of the squared quantization error $d^2_c(F, \bar{F})$ obtained from an implementation of the quantizer (7) suggests that the Gaussian distribution is a good approximation for the probability density function of $r$. The parameters of this distribution can be obtained from the moments in (47). Note that (47) only provides bounds on $D^{(k)}$, however since those bounds are asymptotically tight when the codebook size increases, we arbitrarily choose to use the upper bound as an approximation of $D^{(k)}$, i.e.

$$\hat{r} \triangleq \frac{\Gamma\left(\frac{2}{N_G}\right)}{\frac{1}{2}c2^{N_f}} \approx D^{(2)}$$

is the average and

$$\sigma^2_r \triangleq \frac{\Gamma\left(\frac{4}{N_G}\right)}{\frac{1}{2}c2^{N_f}} - \hat{r}^2 \approx D^{(4)} - (D^{(2)})^2$$

is the variance. We propose generate the values for $r$ according to $\mathcal{N}(\hat{r}, \sigma^2_r)$ truncated to the interval $(0, \Delta^2)$. This process is summarized in Algorithm 1.

**Algorithm 1 Generating random perturbations around $F$**

- Draw a random realization of the squared chordal distance $r$ from $\mathcal{N}(\hat{r}, \sigma^2_r)$
- If $r < 0$ or $r > \Delta^2$ then generate a new sample
- Draw independent $s_1, \ldots, s_p$ uniformly from the interval $(0, 1)$
- Compute the angles $\theta_i = \sin^{-1}\left(\frac{s_i\sqrt{r}}{\sqrt{\sum_{i=1}^{p} s_i^2}}\right)$
- Generate a random orthonormal basis $F^c$ of the null space of $F$
- Compute $\bar{F}$ according to (52).

Simulations were performed in order to validate experimentally the perturbation method proposed above. The sum-rate performance achieved by IA for the CSI obtained from the perturbation method is plotted against the performance obtained for the actual quantization scheme in Figure 3. It is clear that the proposed perturbation method accurately approximates the Grassmannian quantization process, even for small codebooks.

### C. Validation of the DoF results

We now use the quantization error model introduced in the previous section to analyze the CSI feedback scheme from Section IV-A in the high SNR regime. Figure 4 depicts the sum rate performance using the perturbation method compared to perfect CSI and to the lower bound derived in (45). The slope of the sum rate at high SNR regime obtained for the quantizer with $N_f = \frac{N_c}{2} \log P$ bits is identical to that of perfect CSI, as is the case for the lower bound derived in (45).

---

5Experiments have shown no noticeable performance difference when using the lower bound instead.
Fig. 3. Comparison of the perturbation scheme from Section VI-B (solid) to the real quantizer (7) (dashed), for the 3-user MIMO IC, $N = M = 2$.

Fig. 4. Sum rate performance using the perturbation method compared to perfect CSI and the lower bound derived in (45), for the 3-user MIMO IC, $N = M = 2$.

VII. CONCLUSION

A new CSI feedback scheme for interference alignment on the K-user MIMO interference channel was proposed consisting in a parsimonious representation based on the Grassmann manifold. We characterized the scaling of the number of feedback bits with the SNR required in order to preserve the multiplexing gain achievable using perfect CSI. Simulations results confirm that our scheme provides a better sum rate performance compared to quantization of the normalized channel matrices for the same number of feedback bits. Furthermore, considering quantization on
the Grassmann manifold, we introduced a model for the chordal distance of the quantization error which facilitates the performance analysis of schemes requiring intractably large codebooks.

**APPENDIX A**

**PROOF OF LEMMA**

Similar to (12), the power of the interference leakage at receiver $i$ can be written as

$$
\bar{L}_i = \text{tr} \left( \frac{P}{d} \sum_{j=1,j \neq i}^{K} \tilde{U}_i^H H_{ij} V_j V_j^H H_{ij}^H \tilde{U}_i \right)
$$

(57)

$$
= \frac{P}{d} \sum_{j=1,j \neq i}^{K} \| \tilde{U}_i^H H_{ij} V_j \|_F^2
$$

(58)

$$
= \frac{P}{d} \sum_{j=1,j \neq i}^{K} \| \tilde{U}_i^H (H_{ij} - \alpha \hat{H}_{ij}) V_j \|_F^2
$$

(59)

$$
= \frac{P}{d} \sum_{j=1,j \neq i}^{K} \| \tilde{U}_i^H \|_F^2 \| (H_{ij} - \alpha \hat{H}_{ij}) \|_F^2 \| V_j \|_F^2
$$

(60)

$$
\leq P d \sum_{j=1,j \neq i}^{K} \| H_{ij} - \alpha \hat{H}_{ij} \|_F^2
$$

(61)

$$
= P d \sum_{j=1,j \neq i}^{K} \| \text{vec}(H_{ij}) - \alpha \text{vec}(\hat{H}_{ij}) \|_2^2
$$

(62)

$$
= P d \sum_{j=1,j \neq i}^{K} \| \text{vec}(H_{ij}) \|_2^2 \left\| z_{ij} - \alpha \frac{\tilde{z}_{ij}}{\| \text{vec}(H_{ij}) \|_2} \right\|_2^2
$$

(63)

for an arbitrary scalar $\alpha$. In particular, choosing $\alpha = \tilde{z}_{ij}^H z_{ij} / \| \text{vec}(H_{ij}) \|_2$ yields

$$
\bar{L}_i \leq P d \sum_{j=1,j \neq i}^{K} \| \text{vec}(H_{ij}) \|_2^2 \left\| z_{ij} z_{ij}^H z_{ij} - \tilde{z}_{ij} \tilde{z}_{ij}^H z_{ij} \right\|_2^2
$$

(64)

$$
\leq P d \sum_{j=1,j \neq i}^{K} \| \text{vec}(H_{ij}) \|_2^2 \left\| z_{ij} z_{ij}^H z_{ij} - \tilde{z}_{ij} \tilde{z}_{ij}^H z_{ij} \right\|_F^2 \| z_{ij} \|_2^2
$$

(65)

$$
= 2 P d \sum_{j=1,j \neq i}^{K} \| \text{vec}(H_{ij}) \|_2^2 (1 - |z_{ij}^H \tilde{z}_{ij}|^2)
$$

(66)

$$
\leq 2 P dB_{\text{max}} \sum_{j=1,j \neq i}^{K} (1 - |z_{ij}^H \tilde{z}_{ij}|^2)
$$

(67)

$$
= 2 P dB_{\text{max}} D_c^2 (Z_i, \tilde{Z}_i)
$$

(68)

where $B_{\text{max}} = \max_j \| \text{vec}(H_{ij}) \|_2^2$.

Employing the bound on the composite Grassmann manifold from [9 theorem II.1], for every codebook we have
\[ \max_{\mathbf{Z}_i \in \mathcal{G}^{K-1}_{MN,1}} D_c(\mathbf{Z}_i, \hat{\mathbf{Z}}_i) \leq \frac{2}{(\bar{c}^2 N_f)^{\bar{N}_G}} \triangleq \Delta \] which results in

\[ \bar{L}_i \leq 2PdB_{\max} \Delta^2 \leq \frac{8PdB_{\max}}{(\bar{c}^2 N_f)^{\bar{N}_G}} \] (69)

where \( \bar{N}_G = 2(K-1)(MN-1) \) is the real dimension of \( \mathcal{G}^{K-1}_{MN,1} \) and \( \bar{c} \) is a constant. It is clear from (69) that quantizing \( \mathbf{Z}_i \) with \( \bar{N}_f = \frac{\bar{N}_G}{2} \log P = (K-1)(MN-1) \log P \) bits at receiver \( i \) guarantees that the interference leakage remains bounded regardless of the SNR.

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