A finite range pairing force for density functional theory in superfluid nuclei

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The problem of pairing in the $^1S_0$ channel of finite nuclei is revisited. In nuclear matter forces of separable form can be adjusted to the bare nuclear force, to any phenomenological pairing interaction such as the Gogny force or to exact solutions of the gap equation. In finite nuclei, because of translational invariance, such forces are no longer separable. Using well known techniques of Talmi and Moshinsky we expand the matrix elements in a series of separable terms, which converges quickly preserving translational invariance and finite range. In this way the complicated problem of a cut-off at large momenta or energies inherent in other separable or zero range pairing forces is avoided. Applications in the framework of the relativistic Hartee Fock approach show that the pairing properties are depicted on almost the same footing as by the original pairing interaction not only in nuclear matter, but also in finite nuclei. This simple separable force can be easily applied for the investigation of pairing properties in nuclei far from stability as well as for further investigations going beyond mean field theory.

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Along with improved techniques to investigate more precisely nuclear systems being considered as well known, the recent generation of radioactive beam facilities enables us to examine exotic systems with extreme isospin values. Therefore experimental and theoretical studies of nuclei far from the valley of $\beta$-stability are presently at the forefront of nuclear science. Experiments with radioactive nuclear beams have already in the past discovered a number of new structure phenomena in exotic nuclei with extreme isospin values, and the next radioactive-beam facilities in construction will present new exciting opportunities for the study of the nuclear many-body systems.

Nuclei far from stability have also an important influence on astrophysical processes. Therefore the study of such nuclei has wide ranging applications in modern nuclear astrophysics. Unfortunately many of the nuclei of interest in this context have such large neutron excess, that it will be impossible in near future and probably even excluded in far future to investigate them on earth by experiments in the laboratory. Therefore it is extremely important to provide a powerful theory for a reliable description of nuclei close to the limits of stability. It should be based on a consistent treatment of both ground and excited states and should allow for predictions of nuclear properties in areas, which are hard or impossible to access by future experiments. Ab initio calculations and multi-configuration mixing within the shell-model are definitely a goal, but, so far, they can only be applied in light nuclei. At present, for a universal description of nuclei all over the periodic table, Density Functional Theory (DFT) based on the mean-field concept provides a very reasonable concept. DFT has been introduced in the sixties in atomic and molecular physics and shortly after that in nuclear physics under the name ‘density dependent Hartree-Fock theory’. Today it is widely used for all kinds of quantum mechanical many-body systems. DFT can, in principle, provide an exact description of many-body dynamics, if the exact density functional is known, but for systems such as nuclei one is far from a microscopic derivation and the most successful applications determine the functional in a phenomenological way. Starting from basic symmetries the parameters are adjusted to characteristic experimental data in finite nuclei and nuclear matter. In addition nuclei are self bound systems. One usually considers densities in intrinsic frames and it is still under debate whether density functional theory can be exact under these circumstances. Nonetheless in practice DFT provides in many nuclei all over the periodic table an amazingly successful description of the complicated many-body system.

Conventional DFT with a functional $E[\rho]$ depending only on the single particle density can be applied in nuclear physics practically only in a few doubly closed shell nuclei. In all nuclei with open shells, and this is the vast majority of all nuclei in the periodic table, the inclusion of particle-particle (pp) correlations is essential for a correct description of structure phenomena. Although, in principle the effective pp-interaction is isospin dependent with a $T = 0$ and a $T = 1$ part, for the vast majority of pairing effects in nuclei only the $T = 1$ part is important. In fact, little is known on the effective $T = 0$ part. It is still an open question, whether the effective pp-interaction in the $T = 0$ channel is strong enough to produce a pairing condensate with $\rho > \kappa$. We therefore restrict ourselves in the following discussion to $T = 1$ pairing correlations between like particles.

In the framework of DFT pairing correlations are taken into account in the form of Hartree-Bogoliubov theory, where the energy functional $E[\rho, \kappa]$ depends not only on the normal density $\rho = \langle a^+ a \rangle$ but also on the pairing density $\kappa = \langle a^+ a^+ \rangle$. Both densities can be combined to the so-called Valatin density $R$ with the property $R = \rho + \kappa$. Because of this property, the pairing force in DFT is restricted to the $T = 1$ channel.
$\mathcal{R}^2 = \mathcal{R}$. This shows clearly that the Hartree-Bogoliubov version of DFT is a generalized mean field theory. In Ref. \cite{15, 16, 17} relativistic Hartree-Bogoliubov (RHB) theory has been introduced for the treatment of pairing correlations in relativistic DFT. It turns out that pairing itself is a non-relativistic effect influencing only the vicinity of the Fermi surface. The effect of the pairing field on the small components can be neglected to a very good approximation \cite{18}.

For nuclei close to the $\beta$-stability line, pairing has been included in non-relativistic DFT \cite{19} or in the relativistic DFT \cite{20} in the form of the simple constant gap approximation, where the pairing gap $\Delta$ is obtained from odd-even mass differences. The occupation numbers $v_k^2$ are then determined by the BCS-ansatz. The pairing tensor is diagonal with the elements $\kappa_k = u_k v_k$ where $u_k^2 + v_k^2 = 1$ and the pairing part of the density functional is given by the trace of $\kappa$: $E_{\text{pair}}[\kappa] = -\Delta \sum_k \kappa_k$. Of course, this sum diverges and therefore one has to restrict the sum over $k$ to a pairing window. This introduces an additional parameter, which is not well determined by experiment. Several prescriptions can be found in the literature to treat the pairing window \cite{21, 22}.

This way to include pairing correlations corresponds to the seniority model \cite{23}, which is also sometimes called monopole pairing, where only pairs coupled to angular momentum $J = 0$ feel the effective pp-interaction. Of course, this is a very simplified description of pairing correlations in nuclei and therefore this force is often replaced by a zero range force

$$V_{pp}^{\text{eff}} = V_0 \delta(r_1 - r_2) \alpha (1 - P^r)$$

(1)

which is sometimes chosen to be density dependent \cite{24}. For zero range forces as in Eq. (1) the factor $\frac{1}{2}(1 - P^r)$ projects on the $^1S_0$ channel. This force is simple to handle in $r$-space and it is therefore often used in non-relativistic HF-BCS-calculations with Skyrme forces \cite{25} and in relativistic density functionals based on point-coupling models \cite{26, 27, 28}. Unfortunately the corresponding pairing energy diverges in this case too. As in the case of seniority pairing one needs a pairing window. Sharp pairing windows have the tendency to lead to flip-flop solutions in self-consistent theories and therefore one uses in most of the present applications soft pairing windows \cite{29} with rather arbitrary cut off parameters.

For nuclei far from stability the BCS approximation presents only a poor approximation. In particular, in drip-line nuclei the Fermi level is found close to the particle continuum. The lowest particle-hole ($ph$) or particle-particle ($pp$) modes are often embedded in the continuum, and the coupling between bound and continuum states has to be taken into account explicitly. In these cases the BCS model does not provide a correct description of the scattering of nucleonic pairs from bound states to the positive energy continuum because several levels in the continuum become partially occupied leading to a gas of nucleons surrounding the nucleus. Including the system in a box of finite size leads to unreliable predictions for nuclear radii depending on the size of this box. In the non-relativistic case, it has been shown that the Hartree-Fock-Bogoliubov (HFB) theory in the continuum provides a very elegant solution to this problem \cite{30, 31, 32} and this method has been applied also to investigations of drip line and halo nuclei within covariant DFT \cite{33, 34}.

The HFB theory presents a unified description of $ph$- and $pp$-correlations \cite{35} on a mean-field level by using two average potentials: the self-consistent Hartree-Fock field $\hat{F}$ which encloses all the long range $ph$-correlations, and a pairing field $\Delta$ which sums up the $pp$-correlations.

In order to avoid the complicated problems of a pairing cut-off Gogny derived his energy functional \cite{4} from a finite range force of Brink-Booker type. The finite range guarantees that the force decreases as a function of the momentum transfer and the gap equation converges without any problems. Thus a pairing cut off is not necessary. The parameters of this force have been adjusted very carefully in a semi-phenomenological way by Gogny and his collaborators \cite{36, 37} to characteristic properties of the microscopic effective interactions and to experimental data. Over the years this method has turned out to be a very successful way to describe pairing correlations in nuclei and it is often used as benchmark for more microscopic investigations \cite{38, 39}.

Of course, mean field calculations with finite range forces turn out to require a substantial numerical effort, in particular in three-dimensional applications to triaxial nuclei \cite{40} and to rotating systems \cite{41, 42, 43} or in applications going beyond mean field such as the Generator Coordinate Method (GCM) \cite{44} connected with projection to good angular momentum \cite{27, 45} and particle number \cite{28, 46}. Since one needs nowadays systematic investigations over a wide range of nuclei \cite{47} it is highly desirable to find an effective interaction in the pairing channel which is numerically simpler without loosing the nice properties of this force. In this investigations we propose a new realistic pairing force, which is carefully adjusted in a separable form in momentum space to nuclear matter properties of the Gogny force. It turns out that this force is simple enough, that its matrix elements in finite nuclei can be expressed as a rather limited sum of separable terms.

We start our investigations in symmetric nuclear matter. The effective pairing force in the $pp$-channel can be represented by a sum of all diagrams irreducible in $pp$-direction. In lowest order it corresponds to the bare $NN$ interaction. Recently Duguet \cite{48} proposed a microscopic effective interaction to treat pairing correlations in nuclear matter in the $^1S_0$ channel. Starting from an ansatz separable in momentum space, he derived after several approximations an effective pairing force with zero range for practical applications in the context of Skyrme calculations in $r$-space for finite nuclei. We start from a similar ansatz in nuclear matter, however, instead of reducing it to zero range, we transform the force obtained in this way from momentum space to $r$-space. Then calculations in finite nuclei are carried out in terms of an
expansion in the eigenfunctions of a harmonic oscillator. This is particularly simple for a Gaussian ansatz leading to analytical expressions for the matrix elements. In Refs. 49, 50, a similar technique was used based on spherical Bessel functions.

The wave functions in infinite nuclear matter are characterized by the momentum \( \mathbf{k} \) and the spin \( s \)

\[
\varphi_{ks}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} / \sqrt{\lambda_s}. \tag{2}
\]

The antisymmetric matrix element of the pairing interaction \( V_{\text{sep}}^{1S_0} \) in the plane-wave basis has the form

\[
\langle k_1s_1,k_2s_2|1/2(1-P_\sigma) V_{\text{sep}}^{1S_0}|k'_1s'_1,k'_2s'_2\rangle_a = \frac{1}{2} \langle k_1,k_2|V_{\text{sep}}^{1S_0}|k'_1,k'_2\rangle_a \delta_{s_1s'_1} \delta_{s_2s'_2} - \delta_{s_1s'_2} \delta_{s_2s'_1}, \tag{3}
\]

where

\[
\langle k_1,k_2|V_{\text{sep}}^{1S_0}|k'_1,k'_2\rangle_a = \langle k|V_{\text{sep}}^{1S_0}|k'\rangle (2\pi)^3 \delta(K - K'). \tag{5}
\]

\( K = k_1 + k_2 \) and \( k = 1/2(k_1 - k_2) \) are the total and relative momentum of a particle pair, respectively. It has been pointed that the bare interaction in the \( 1S_0 \) channel is to a good approximation separable and nonlocal at low energy. The center of mass part of the matrix element is therefore approximated by a separable form,

\[
\langle k|V_{\text{sep}}^{1S_0}|k'\rangle = -Gp(k)p(k'). \tag{6}
\]

The isospin quantum number is not specified in this expression, since the form of the matrix elements in the \((T = 1)\) channel is trivial. \( G \) is the strength parameter of this pairing interaction.

In the \( 1S_0 \) channel we find the following gap equation in the plane wave basis, the usual BCS equation

\[
\Delta(k) = -\int_0^\infty \frac{k'^2 dk'}{2\pi^2} \sqrt{\epsilon(k') - \mu} \frac{\Delta(k')}{2E(k')}, \tag{7}
\]

with the quasi-particle energy

\[
E(k) = \sqrt{(\epsilon(k) - \mu)^2 + \Delta^2(k)} \tag{8}
\]

and the in medium on-shell single particle energies \( \epsilon(k) \) associated with the state \( \varphi_k \). \( \mu \) is the chemical potential determined by the density and \( v(k,k') = -Gp(k)p(k') \). For a given pairing interaction, one can solve the BCS gap equation and calculate the corresponding gaps as a function of the density, i.e. as a function of the Fermi momentum in nuclear matter. This relationship between the gap at the Fermi surface and the Fermi momentum determines the properties of the pairing correlations.

Inserting the separable interaction \( Gp(k)p(k') \) into Eq. (7), the solution of the gap equation is trivial \( \Delta(k) = \Delta_0 p(k) \), where \( \Delta_0 \) is the gap at zero momentum satisfying the equation

\[
1 = \int_0^\infty \frac{k'^2 dk'}{4\pi^2} \frac{Gp^2(k)}{\sqrt{\epsilon(k) - \mu} + \Delta_0 p^2(k)} \tag{9}
\]

and therefore depending on the Fermi momentum \( k_F \).

The gap at the Fermi surface is obtained through

\[
\Delta(k_F) = \Delta_0(k_F)p(k_F) \tag{10}
\]

which shows that the bell shaped curve \( \Delta(k_F) \) determines the form of the functional dependence of the function \( p(k) \) in Eq. (6).

For RHB calculations in finite nuclei it is common to use the Gogny force [54] or a zero-range \( \delta \)-interaction [26]. We apply this method to derive a separable forces, i.e. the function \( p(k) \) in Eq. (10) by mimicking the non-relativistic Gogny force in the \( 1S_0 \) channel of nuclear matter. Of course this procedure depends on the self energies \( \epsilon(k) \). When we deal with the Gogny force these self energies Eqs. (7) and (9) are no longer free single particle energies. Medium corrections have to be included, for instance the Brueckner-Hartree-Fock approximation. For simplicity, however, we approximate the self energy in this investigation in a phenomenological way by the single particle energies of the form

\[
\epsilon(k) = V(\mu) + \sqrt{k^2 + M^*(\mu)}, \tag{11}
\]

where the effective mass \( M^* = M + S = M + g_\sigma \sigma \) and the effective vector field \( V = g_\omega \omega \) are given by the effective

![FIG. 1: Comparison of \( 1S_0 \) pairing gaps at the Fermi surface \( \Delta(k_F) \) as a function of the density for the Gogny forces D1 and D1S (solid curves) and the corresponding separable forces (dashed curves), where the function \( p(k) \) in Eq. (6) is represented in panel (a) by one Gaussian in Eq. (15) and in panel 3 by a sum of three Gaussians in Eq. (13).](image-url)
...with the parameter set NL3. The nonlinear Klein Gordon equations of the RMF model are determined by the solution of the effective interaction for the area of low energies. It is also found in Refs. [48, 57]. The Gogny force is an effective interaction for the nuclear medium. In the case of bare nucleon-nucleon forces with a strong short range repulsion this would be no longer the case. It is known that, by integrating out the high momentum components of the bare force, one obtains from most of the realistic NN-potentials the same interaction a low momenta $V_{\text{low } k}$. This low momentum NN interaction $V_{\text{low } k}$ can describe the two-nucleon system at low energy very well. Although the matrix elements of various realistic bare NN interactions are scattered, their low momentum part $V_{\text{low } k}$ has the same shape. It has been found that the matrix element of the separable form of AV18 is very close to that of $V_{\text{low } k}$ obtained from AV18 and very different from that of the potential AV18 itself. Therefore we plot in Fig. 2 also the matrix elements of $V_{\text{low } k}$. This illustrates the physical content of the separable force as an effective interaction for the area of low energies. It is also shown in Fig. 2 that the matrix elements of the Gogny force D1S have a very similar behavior to that of $V_{\text{low } k}$. This indicates that the Gogny force has a clear link to the bare NN force, especially for the parameterization D1.

Now we turn to the solution of the RHB equations in finite nuclei. As discussed earlier the separable force has simple Gaussian form and it is fitted to reproduce the density dependence of the gap at the Fermi surface in nuclear matter derived from the Gogny force. This force is definitely not identical to the full Gogny force in the $1S_0$-channel. Therefore, in order to apply these separable pairing force in calculations of finite nuclei, one has to study to what degree such a separable force can describe the pairing properties of these systems. For that purpose we use RHB theory in several spherical isotope chains, e.g. in Sn- and in Pb-isotopes.

First, we transform the force (6) from the momentum channel to that of the original Gogny forces. Similar results have been found in Refs. [48, 57]. The Gogny force is an effective force in the nuclear medium. In the case of bare nucleon-nucleon forces with a strong short range repulsion this would be no longer the case. It is known that, by integrating out the high momentum components of the bare force, one obtains from most of the realistic NN-potentials the same interaction a low momenta $V_{\text{low } k}$. This low momentum NN interaction $V_{\text{low } k}$ can describe the two-nucleon system at low energy very well. Although the matrix elements of various realistic bare NN interactions are scattered, their low momentum part $V_{\text{low } k}$ has the same shape. It has been found that the matrix element of the separable form of AV18 is very close to that of $V_{\text{low } k}$ obtained from AV18 and very different from that of the potential AV18 itself. Therefore we plot in Fig. 2 also the matrix elements of $V_{\text{low } k}$. This illustrates the physical content of the separable force as an effective interaction for the area of low energies. It is also shown in Fig. 2 that the matrix elements of the Gogny force D1S have a very similar behavior to that of $V_{\text{low } k}$. This indicates that the Gogny force has a clear link to the bare NN force, especially for the parameterization D1.
with bital angular momentum

\[ \hat{b} \]

represents the wave function in spin and angles coupled to

space to the coordinate space and obtain

\[ V(r_1, r_2, r'_1, r'_2) = -G \delta(R - R') P(r)P(r') \frac{1}{(1 - P^3)} \]

(14)

where \( R = \frac{1}{2}(r_1 + r_2) \) and \( r = r_1 - r_2 \) are center of mass and relative coordinates respectively, and \( P(r) \) is obtained from the Fourier transform of \( p(k) \). Using the Gaussian ansatz \( \langle 12 \rangle \) we find

\[ P(r) = \frac{1}{(4\pi a^2)^{3/2}} e^{-\frac{r^2}{4a^2}}. \]

(15)

Because of the \( \delta \)-term in Eq. (14) that insures translational invariance this force is not completely separable in coordinate space. However, the matrix elements of this force can be represented by a sum of a few separable terms in a basis of spherical harmonic oscillator functions.

In order to show this we start from the basis

\[ \langle n | m \rangle = \varphi_{nlm} / |r \rangle = R_{nl}(r, b)[Y_l(\hat{r}) \otimes \chi_{\frac{l}{2}}] |m \rangle \]

(16)

where \( R_{nl}(r, b) = \frac{b^{-\frac{3}{2}} R_{nl}(r/b)}{b} \) and \( [Y_l(\hat{r}) \otimes \chi_{\frac{l}{2}}] |m \rangle \) represents the wave function in spin and angles coupled to total angular momentum \( jm \). The radial wave function has the form

\[ R_{nl}(x) = \sqrt{\frac{2n!}{(n + l + \frac{3}{2})!}} x^l L_n^{l + \frac{1}{2}}(x^2)e^{-\frac{x^2}{2}} \]

(17)

with the radial quantum number \( n = 0, 1, \ldots \) and the orbital angular momentum \( l \). The quantity \( b = \sqrt{\hbar/(m\omega_0)} \) is the harmonic oscillator length. In the pairing channel we need only the two-particle wave functions coupled to angular momentum \( J = 0 \) and the projector \( \frac{1}{2}(1 - P^3) \) restricts us to the quantum numbers \( S = L = 0 \). Recoupling from the LS- to the jj-scheme therefore leads to the two-particle wave function

\[ |12\rangle_0 \equiv |\varphi_{n_1l_1j_1}(r_1), \varphi_{n_2l_2j_2}(r_2)\rangle J = 0 \]

(18)

\[ = \frac{j}{\sqrt{2l+1}} R_{n_1l_1}(r_1, b) R_{n_2l_2}(r_2, b)|\lambda = 0\rangle |S = 0\rangle \]

with \( j = \sqrt{2l+1} \) and \( s = \frac{1}{2} \). The functions \( |\lambda = 0\rangle = |Y_{l_1}(\hat{r}_1) \otimes Y_{l_2}(\hat{r}_2)\rangle_0 \) and \( |S = 0\rangle = |\chi_{\frac{l_1}{2}} \otimes \chi_{\frac{l_2}{2}}\rangle_0 \) are the angular and spin wave functions coupled to angular momentum \( \lambda = 0 \) and spin \( S = 0 \). All these wave functions are expressed in laboratory coordinates, while the separable pairing interaction in Eq. (14) is expressed in the center of mass frame by the center of mass coordinate \( R \) and the relative coordinates \( r \) of a pair. Therefore we transform to the center of mass frame using Talmi-Moshinsky brackets \( [59, 60, 61] \). We use the definition of Baranger \( [62] \)

\[ |n_1l_1, n_2l_2; \lambda \mu \rangle = \sum_{N L n} M_{n_1n_2l_1l_2}^{N L n l} |NL, n; \lambda \mu \rangle \]

(19)

\[ V_{121'}^{l_2l_2'} = \langle n_1l_1j_1, n_2l_2j_2|V|n_1l_1j_1, n_2l_2j_2\rangle_{J=0} \]

(23)

as a sum over the quantum numbers \( N, L, N', L', n, l, n', l' \) and \( J \) in Eq. (19). The integration over the center of mass
coordinates \( R \) and \( R' \) leads to \( N = N' \), \( L = L' \). Further restrictions occur through the fact that the sum contains integrals over the relative coordinates of the form

\[
\int R_{n l}(r, b_{r})Y_{l m}(\hat{r})P(r)d^{3}r. \tag{24}
\]

They vanish for \( l \neq 0 \) and this leads to \( L = l = 0 \). The quantum numbers \( n \) and \( n' \) are determined by the selection rule (21) and we are left with a single sum of separable terms

\[
V_{12}^{N} = -G \sum_{N} V_{12}^{N} V_{12}^{N} \tag{25}
\]

with the single particle matrix elements \( V_{12}^{N} \). For \( l_{1} = l_{2} = l \), \( j_{1} = j_{2} = j \) we find

\[
V_{12}^{N} = M_{n l n_{2} l_{2}}^{N \alpha = 0} \frac{j}{s_{l}} \int_{0}^{\infty} R_{n 0}(r, b_{r})P(r)r^{2}dr. \tag{26}
\]

For a Gaussian ansatz of \( P(r) \) in Eq. (15) this integral can be evaluated analytically

\[
V_{12}^{N} = \frac{1}{b^{3/2}} \frac{2^{1/4}}{\pi^{3/4}} \frac{1}{(1 + \alpha^{2})^{n+3/2}} \frac{j}{s_{l}} M_{n l n_{2} l_{2}}^{N \alpha = 0} \sqrt{(2n + 1)!} \frac{1}{2^{n+1+n!}}. \tag{27}
\]

where the parameter \( \alpha = a/b \) characterizes the width of the function \( p(r) \) in units of the oscillator length \( b \) and \( n \) is given by the selection rule (21) \( n = n_{1} + n_{2} + l - N \).

Thus we find, that the pairing matrix elements for the separable pairing interactions used in the RHB equation can be evaluated by the sum of separable terms in Eq. (25). In order to study the pairing properties in finite nuclei, we solve the RHB equation

\[
\left( \frac{\hbar_{D} - \mu}{\Delta} - h_{D} + \mu \right) \left( U \right) = E_{k} \left( \frac{U}{V} \right)_{k} \tag{28}
\]

self-consistently for the Dirac Hamiltonian \( h_{D} \) and the pairing field

\[
\Delta_{12} = G \sum_{N} P_{N} V_{12}^{N}, \tag{29}
\]

with the parameters

\[
P_{N} = \frac{1}{2} \sum_{12} V_{12}^{N} \kappa_{12} = \frac{1}{2} \text{Tr}(V_{N}^{T} \kappa) \tag{30}
\]

and the pairing tensor \( \kappa = UV^{T} \). The pairing energy in the nuclear ground state is given by

\[
E_{\text{pair}} = -G \sum_{N} P_{N}^{T} P_{N}. \tag{31}
\]

All the following calculations are carried out for the parameter set NL3 [55] by expanding the Dirac-Bogoliubov spinors in terms of 20 major oscillator shells [20].

In Fig. 4 we show the dependence of the pairing energy on the neutron number for the chain of the isotopes \(^{100}\text{Sn} \sim ^{160}\text{Sn} \) and \(^{164}\text{Pb} \sim ^{264}\text{Pb} \). As we see from the upper part of Fig. 4 good agreement is observed for the pairing energies calculated with the Gogny pairing force and its separable approximation. The largest discrepancy is less than 10%. From this comparison we can conclude that the separable pairing interaction can describe the pairing properties of finite nuclei on almost the same footing as its corresponding pairing interaction. Therefore, we can use the separable pairing interaction instead of its complicated original form. In fact, one could get even better agreement by using the ansatz (13) with three separable terms which produces in momentum space identical results as the Gogny force (see Fig. 1b).

As we see from the Eq. (25) the separable pairing interaction is not fully separable in the spherical harmonic oscillator basis. We have a sum over the quantum number \( N \) characterizing the major shells of the harmonic oscillator in the center of mass coordinate. In practical applications it turns out that this sum can be restricted to finite values \( N \leq N_{0} = 8 \). It is therefore enough to determine the \( N_{0} + 1 \) matrices \( V_{12}^{N} \) at the beginning of the iteration and to re-calculate the quantities \( P_{N} \) in Eq. (29) in each step of the iteration. As compared to calculations with the full Gogny force in the pairing channel this means a considerable reduction in memory and computer time.

In order to study the convergence with the number of separable terms \( N_{0} \) we show in Fig. 4 the pairing energies in the isotopic chains with \( Z = 50 \) (Sn) and \( Z = 82 \) (Pb).
for various values of \( N_0 \). We find that for the nuclei around the line of \( \beta \)-stability, \( N_0 = 5 \) is large enough to get the full pairing energy; but for the nuclei far from the \( \beta \)-stability line, we need at least \( N_0 = 8 \).

The left panels of Fig. 5 show the size of the individual matrix elements \( \Delta \) calculated with a value \( N_0 = 5 \) and \( N_0 = 8 \) as a function of their exact value (which is identical to the value at \( N_0 = 21 \)). We see that in particular the large matrix elements are very concentrated along the 45° line. Only for the very small pairing matrix elements we find small deviations. Similar results are obtained for the matrix elements of the pairing field \( \Delta_{12} \) of the nucleus \(^{244}\text{Pb}\). Here we find some deviations for \( N_0 = 5 \). However \( N_0 = 8 \) gives satisfactory agreement.

Finally we have carried out a comparison of the new separable pairing force in Eq. (25) with the full Brink-Booker part of the Gogny force in the pairing channel (called full Gogny force in the following), and with various zero range forces. In the upper panels of Fig. 6 we show the size of the corresponding pairing matrix elements as a function of the matrix elements of the full Gogny force with the parameter set D1S \( \text{[50]} \) and the lower panels present matrix elements of the pairing field \( \Delta_{12} \) in the oscillator basis for the finite nucleus \(^{244}\text{Pb}\) as a function of those calculated with the full Gogny force D1S. In the two panels on the left side we have used the separable pairing force with \( N_0 = 21 \) which is identical to the full sum of Eq. (25). The rest of the panels in Fig. 6 show calculations with three zero range forces \( \text{[11]} \) of various strength parameters. They have been determined in calculation for the finite nucleus \(^{244}\text{Pb}\) shown in the lower two panels, where a cut-off energy \( E_c \) has been used for the zero range forces. It has been smoothed by a Fermi function

\[
f(E) = \frac{1}{1 + \exp((E - E_c)/D_c)}
\]

as in Ref. \( \text{[29]} \). The smoothing parameter has been kept constant \( D_c = 0.5 \) MeV in all cases and three different values have been chosen for the cut-off energy, \( E_c = 18 \) MeV in the lower middle panel and \( E_c = 9, 27 \) MeV in the lower right panel. For each cut-off energy the value of the strength parameter \( V_0 \) has been chosen in such a way that the resulting average pairing gap in the canonical basis

\[
\Delta_{av} = \frac{1}{N} \sum_{\mu} |\Delta_{\mu\bar{\mu}}v_{\mu}^2|
\]

is equal to \( \Delta_{av} = 1.76 \) MeV, the corresponding average gap calculated with the full Gogny force D1S used in the abscissa of the three lower curves. We obtain the values \( V_0 = 485, 326, \) and \( 280 \) MeV-fm\(^3\) for \( E_c = 9, 18, \) and 27 MeV respectively.

We find that although the \( \delta \)-force can give the same average gap as the Gogny force D1S if the size of the strength is adjusted properly, the individual matrix elements of the forces and the matrix elements \( \Delta \) of the pairing field are very different from each other. Apart from many rather small matrix elements the rest of the matrix elements of the \( \delta \)-force cluster around rather constant values. Therefore the \( \delta \)-force behaves very much like a constant pairing force with a plateau. As seen in the two right lower panels of Fig. 6 the value of this plateau depends on the cut-off energy. On the other side our separable force concentrates along the 45° line especially for the large matrix elements. There are only small differences observed in the region of small pairing matrix elements. For the pairing potential \( \Delta \) we obtain a similar results in the lower panel of Fig. 6. Here it is clearly seen that the separable approximation is very similar to the full Gogny force.

Summarizing, we discuss in this investigation a very simple effective pairing interaction in the \( ^1S_0 \)-channel, which is of finite range, translational invariant and separable. This simple force can be easily applied in realistic applications of modern relativistic and non-relativistic density functional theory, in particular also in complicated calculations, such as for nuclei with triaxial shapes, for nuclei in the rotating frame, for the fission process, for QRPA calculations and for all kinds of investigations beyond mean field theory using techniques of projection, generator coordinates, or particle vibrational coupling. Investigations in this direction are in progress.

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FIG. 6: (color online) Upper panels: The pairing matrix elements of our separable force and of various $\delta$-forces with different cut-off parameters $E_c$ are compared with those of the Gogny force D1S. Lower panels: the matrix elements of the pairing potential $\Delta$ for the nucleus $^{244}$Pb calculated with the forces shown in the upper panels are compared with those obtained with the Gogny force D1S. Further details are given in the text.

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