THE MALGRANGE-EHRENPREIS THEOREM FOR NONLOCAL SCHRÖDINGER OPERATORS WITH CERTAIN POTENTIALS

WOOCHEOL CHOI

Department of Mathematics Education, Incheon National University
Incheon 22012, Republic of Korea

YONG-CHEOL KIM *

Department of Mathematics Education, Korea University
Seoul 02841, Republic of Korea

(Communicated by Enrico Valdinoci)

Abstract. In this paper, we prove the Malgrange-Ehrenpreis theorem for nonlocal Schrödinger operators \( L_K + V \) with nonnegative potentials \( V \in L^q_{\text{loc}}(\mathbb{R}^n) \) for \( q > \frac{n}{2s} \) with \( 0 < s < 1 \) and \( n > 2s \); that is to say, we obtain the existence of a fundamental solution \( e^V \) for \( L_K + V \) satisfying
\[
(L_K + V)e^V = \delta_0 \quad \text{in } \mathbb{R}^n
\]
in the distribution sense, where \( \delta_0 \) denotes the Dirac delta mass at the origin. In addition, we obtain a decay of the fundamental solution \( e^V \).

1. Introduction. In 1954-1956, Malgrange and Ehrenpreis \([5, 6, 11]\) proved independently that any partial differential operator with constant coefficients which is not identically vanishing has a fundamental solution in the space \( D'(\mathbb{R}^n) \) of distributions. In this paper, we obtain an extension of their result to nonlocal Schrödinger operators with nonnegative potentials \( V \in L^q_{\text{loc}}(\mathbb{R}^n) \) for \( q > \frac{n}{2s} \) with \( 0 < s < 1 \) and \( n > 2s \).

We introduce integro-differential operators of the form
\[
L_K u(x) = \frac{1}{2} \text{p.v.} \int_{\mathbb{R}^n} \mu(u, x, y)K(y) \, dy, \quad x \in \mathbb{R}^n,
\]
where \( \mu(u, x, y) = 2u(x) - u(x + y) - u(x - y) \) and the kernel \( K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+ \) satisfy the property
\[
\frac{\lambda c_{n,s}}{|y|^{n+2s}} \leq K(y) = K(-y) \leq \frac{\Lambda c_{n,s}}{|y|^{n+2s}}, \quad s \in (0, 1), 0 < \lambda < \Lambda < \infty,
\]
where \( c_{n,s} \) is the normalization constant comparable to \( s(1-s) \) given by
\[
c_{n,s} \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2s}} \, d\xi = 1.
\]

2000 Mathematics Subject Classification. Primary: 47G20, 45K05, 35J60, 35B65, 35D30; Secondary: 60J75.

Key words and phrases. Fundamental solution, nonlocal Schrödinger operators, potential.

* Corresponding author: Yong-Cheol Kim.
Set $\mathcal{L} = \{ L_K : K \in \mathcal{K}\}$ where $\mathcal{K}$ denotes the family of all kernels $K$ satisfying (1.2). In particular, if $K(y) = c_{n,s}|y|^{-n-2s}$, $s \in (0,1)$, then $L_K = (-\Delta)^s$ is the fractional Laplacian and it is well-known that

$$\lim_{s \to 1^{-}} (-\Delta)^s = -\Delta u$$

for any function $u$ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

In what follows, we consider the nonlocal Schrödinger operators given by

$$L_V := L_K + V$$

(1.4)

where $K \in \mathcal{K}$ and $V \in L_{loc}^q(\mathbb{R}^n)$, $q > \frac{n}{2s} > 1$, $s \in (0,1)$, is a nonnegative potential. We are interested in the existence of a fundamental solution for the operator $L_V$. Let $\mathcal{D}'(\mathbb{R}^n)$ be the space of all distributions on $\mathbb{R}^n$. Given $f \in \mathcal{D}'(\mathbb{R}^n)$, we say that a real-valued Lebesgue measurable function $u$ on $\mathbb{R}^n$ satisfies the equation $L_V u = f$ in the sense of $\mathcal{D}'(\mathbb{R}^n)$, if the linear map $L_V u : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ given by

$$(L_V u, \varphi) = \int_{\mathbb{R}^n} u(y)L_V \varphi(y) \, dy$$

is well-defined and a distribution, and also $(L_V u, \varphi) = (f, \varphi)$ for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

For the distributional domain of the operator $L_V$, we consider a weighted $L^p$ space as follows; for $p \geq 1$ and $s \in (0,s)$, we define the space

$$L_{s}^p(\mathbb{R}^n) = \{ u \in L_{loc}^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} |u(x)|^p (1 + |x|)^{-(n+2s)} \, dx < \infty \}$$

equipped with the norm

$$\|u\|_{L_{s}^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \frac{|u(x)|^p}{(1 + |x|)^{n+2s}} \, dx \right)^{1/p}. $$

**Theorem 1.1.** There exists a fundamental solution $e_V \in L_{s}^q(\mathbb{R}^n)$ for the nonlocal Schrödinger operator $L_V$, i.e. it satisfies that

$$L_V e_V = \delta_0 \text{ in the sense of } \mathcal{D}'(\mathbb{R}^n),$$

where $\delta_0$ is the Dirac delta mass at the origin and $q'$ is the dual exponent of $q$. Moreover, there exists a universal constant $C > 0$ depending on $n, s, \lambda$ and $\Lambda$ such that

$$0 \leq e_V(x) \leq C \frac{1}{|x|^{n-2s}} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}. \quad (1.5)$$

**Remark.** (a) In case that $L_K = (-\Delta)^s$ and $V = 0$, its fundamental solution $\phi_0(x) = \frac{\Gamma(n/2-s)}{\pi^{n/2}n^{n/2-s}|x|^{n+2s}}$ was explicitly obtained for $n \neq 2s$ and moreover $\phi_0(x) = -\frac{1}{n} \ln |x|$ for $n = 2s$ in [1] (see also [2]). Also, the readers can refer to [10] and [15] for the Riesz kernel of $(-\Delta)^s$.

(b) In [7] and [1], $L_1^s(\mathbb{R}^n)$ was considered as the distributional domain of the operator $L_K$. But we need a smaller space $L_2^s(\mathbb{R}^n)$ than $L_1^s(\mathbb{R}^n)$ to control the potential $V$, which satisfies that

$$H^s(\mathbb{R}^n) \subset L_{s}^\infty(\mathbb{R}^n) \subset L_{s}^2(\mathbb{R}^n) \subset L_{s}^1(\mathbb{R}^n) \subset S_s'(\mathbb{R}^n) \subset S''(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n) \quad (1.6)$$

where $H^s(\mathbb{R}^n)$ is the homogeneous fractional Sobolev spaces, $S_s'(\mathbb{R}^n)$ is the dual space of the space $S_s(\mathbb{R}^n)$ consisting of all smooth functions $g \in C^\infty(\mathbb{R}^n)$ with the seminorms $|g|_{S_s(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} (1 + |x|)^{n+2s}|\partial^\alpha g(x)| < \infty$ for all multi-indices.
α = (α₁, · · · , αₙ) ∈ (ℕ ∪ {0})ⁿ, and S′(ℝⁿ) is the space of tempered distributions, i.e. the dual space of the Schwartz space S(ℝⁿ).

The paper is organized as follows. In Section 2, we define several function spaces and give the fractional Sobolev embedding theorem which was proved in [14, 4]. In Section 3, we define weak solutions of the nonlocal equation Lₓ = f in a bounded domain Ω ⊂ ℝⁿ with Lipschitz boundary and obtain a relation between weak solutions (weak subsolutions, weak supersolutions) and minimizers (subminimizers, superminimizers) of the energy functional for the operator Lₓ, respectively. In Section 4, we obtain a Rellich-Kondrachov compactness theorem, a weak maximum principle and a comparison principle for Lₓ. In Section 5, we obtain an extension of the Malgrange-Ehrenpreis theorem, a weak maximum principle and a comparison principle for Lₓ. Moreover we obtain a decay of the fundamental solution eₓ for Lₓ by using functional analysis stuffs. Therefore we obtain a decay of the fundamental solution eₓ.

2. Preliminaries. Let ℱⁿ be the family of all real-valued Lebesgue measurable functions on ℝⁿ. Let Ω be a bounded domain in ℝⁿ with Lipschitz boundary and let K ∈ K. For s ∈ (0, 1), let Xˢ(Ω) be the function space of all f ∈ ℱⁿ on ℝⁿ such that u|₀ ∈ L²(Ω) and

$$\iint \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}}
\, dx \, dy < \infty$$

where ℝ²ⁿ := (ℝⁿ × ℝⁿ) \ (Dˢ × Dˢ) for a set D ⊂ ℝⁿ. We also denote by

$$X₀ⁿ(Ω) = \{ v \in Xⁿ(Ω) : v = 0 \text{ a.e. in } ℝⁿ \cup Ω \}. \quad (2.1)$$

Note that Xⁿ(Ω) and X₀ⁿ(Ω) are not empty, because C₀ⁿ(Ω) ⊂ X₀ⁿ(Ω). Then we see that (Xⁿ(Ω), || · ||ₓⁿ(Ω)) is a normed space, where the norm || · ||ₓⁿ(Ω) is defined by

$$||u||ₓⁿ(Ω) := ||u||ₓⁿ(Ω) + \left( \iint \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}} \, dx \, dy \right)^{1/2} < \infty. \quad (2.2)$$

For p ≥ 1 and s ∈ (0, 1), let Wˢ,p(Ω) be the usual fractional Sobolev spaces with the norm

$$||u||Wˢ,p(Ω) := ||u||ₘₚ(Ω) + [u]Wˢ,p(Ω) < \infty \quad (2.3)$$

where the seminorm ||[ · ]||Wˢ,p(Ω) is defined by

$$[u]Wˢ,p(Ω) = \left( \iint \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dx \, dy \right)^{1/p}. \quad \text{In what follows, we write } Hˢ(Ω) = Wⁿ,²(Ω). \text{ When } Ω = ℝⁿ \text{ in (2.3), we can similarly define the spaces } Wⁿ,ᵖ(ℝⁿ) \text{ and } Hⁿ(ℝⁿ) = Wⁿ,²(ℝⁿ) \text{ for } s \in (0, 1).

By [14], there exists a constant c > 1 depending only on n, s and Ω such that

$$||u||X₀ⁿ(Ω) ≤ ||u||ₓⁿ(Ω) \leq c ||u||X₀ⁿ(Ω) \quad \text{for any } u \in X₀ⁿ(Ω), \quad (2.4)$$

where

$$||u||X₀ⁿ(Ω) := \left( \iint \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}} \, dx \, dy \right)^{1/2}. \quad (2.5)$$
Thus \( \| \cdot \|_{X^s_0(\Omega)} \) is a norm on \( X^s_0(\Omega) \) which is equivalent to (2.2). Moreover it is known [14] that \( (X^s_0(\Omega), \| \cdot \|_{X^s_0(\Omega)}) \) is a Hilbert space with inner product

\[
\langle u, v \rangle_{X^s_0(\Omega)} := \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy. \tag{2.6}
\]

Let \( X^s_0(\Omega)^* \) be the dual space of \( X^s_0(\Omega) \); that is, the family of all bounded linear functionals on \( X^s_0(\Omega) \). Then we know that \( (X^s_0(\Omega)^*, \| \cdot \|_{X^s_0(\Omega)^*}) \) is a Hilbert space, where the norm \( \| \cdot \|_{X^s_0(\Omega)^*} \) is given by

\[
\| u \|_{X^s_0(\Omega)^*} := \sup \{ u(v) : v \in X^s_0(\Omega), \| v \|_{X^s_0(\Omega)} \leq 1 \}, \quad u \in X^s_0(\Omega)^*.
\]

For \( s > 0 \), we consider the class \( H^s(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : |\xi|^s |\hat{u}(\xi)| \text{ is in } L^2(\mathbb{R}^n) \} \) whose norm is defined by

\[
\| u \|_{H^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^s)^2 |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2},
\]

where the Fourier transform of \( u \) is defined by

\[
\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i(x, \xi)} u(x) \, dx.
\]

Then, by the Plancherel theorem, it is easy to check that \( H^s(\mathbb{R}^n) \) is \( H^s(\mathbb{R}^n) \) and they are norm-equivalent.

We also define the homogeneous fractional Sobolev spaces \( \dot{H}^s(\mathbb{R}^n) \) by the closure of \( S(\mathbb{R}^n) \) with respect to the norm

\[
\| u \|_{\dot{H}^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}.
\]

From a direct calculation [4], it turns out that

\[
\| u \|_{\dot{H}^s(\mathbb{R}^n)} = 2 c_{n,s}^{-1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \tag{2.7}
\]

where \( c_{n,s} \) is the universal constant given in (1.3).

**Lemma 2.1.** (a) If \( f \in X^s_0(\Omega) \), then \( f \in H^s(\mathbb{R}^n) \) and moreover

\[
\frac{c_{n,s}}{2} \| f \|_{H^s(\mathbb{R}^n)} = \| f \|_{X^s_0(\Omega)} \leq \| f \|_{H^s(\mathbb{R}^n)} = \| f \|_{X^s_0(\Omega)} \leq c \| f \|_{X^s_0(\Omega)}
\]

where \( c > 1 \) is the constant given in (2.4).

(b) If \( f \in H^s(\mathbb{R}^n) \), then \( \| f \|_{X^s_0(\Omega)} \leq \| f \|_{H^s(\mathbb{R}^n)} \leq \| f \|_{H^s(\mathbb{R}^n)} \).

**Proof.** (a) It easily follows from (2.7) and Lemma 5 in [14]. (b) It is also very straightforward. \( \square \)

Next we state the fractional Sobolev embedding theorem, which was proved in [14, 4].

**Proposition 2.2.** If \( 0 \leq s < \frac{n}{2} \), then the space \( \dot{H}^s(\mathbb{R}^n) \) is continuously embedded in \( L^{\frac{2n}{n-2s}}(\mathbb{R}^n) \), i.e. there is a constant \( C > 0 \) depending only on \( n, s \) such that

\[
\| u \|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leq C \| u \|_{\dot{H}^s(\mathbb{R}^n)}.
\]
3. Weak solutions and minimizers for $L_K + V$. In this section, we define weak solutions, weak subsolutions and supersolutions of the nonlocal equation $L_K u + V u = 0$ on $\Omega$. To comprehend them well, we obtain a relation between weak solutions (weak subsolutions and weak supersolutions) and minimizers (sub-minimizers and superminimizers) of the energy functional for the nonlocal operator $L_K + V$, respectively.

From now on, we always assume that $V$ is a nonnegative potential in $L_{\text{loc}}^q(\mathbb{R}^n)$ for $q > \frac{n}{2s}$ with $s \in (0, 1)$ and $n > 2s$. We denote by $L^2_q(\Omega)$ the weighted $L^2$ class of all real-valued measurable functions $g$ on $\mathbb{R}^n$ satisfying

$$
\|g\|^2_{L^2_q(\Omega)} := \int_{\Omega} |g(y)|^2 V(y) \, dy < \infty.
$$

We consider a bilinear form defined by

$$
\langle u, v \rangle_K = \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) K(x - y) \, dx \, dy \quad \text{for } u, v \in X^s(\Omega).
$$

From (b) of Corollary 5.2 below, we see in advance that

$$
\langle u, v \rangle_K = \langle u, v \rangle_{X^s_0(\Omega)} = \int_{\mathbb{R}^n} L^{1/2}_K u(y) L^{1/2}_K v(y) \, dy
$$

(3.1)

for $u, v \in X^s_0(\Omega)$.

**Definition 3.1.** Let $V \in L_{\text{loc}}^q(\mathbb{R}^n)$ for $q > \frac{n}{2s}$ with $s \in (0, 1)$ and $n > 2s$. Then we say that a function $u \in X^s_0(\Omega)$ is a weak solution of the nonlocal equation

$$
L_V u := L_K u + V u = f \quad \text{in } \Omega
$$

(3.2)

where $f \in X^s_0(\Omega)^*$, if it satisfies the weak formulation

$$
\langle u, v \rangle_K + \int_{\mathbb{R}^n} V(y) u(y) \varphi(y) \, dy = \langle f, \varphi \rangle
$$

(3.3)

for any $\varphi \in X^s_0(\Omega)$. Here $\langle \cdot, \cdot \rangle$ denotes the pair between $X^s_0(\Omega)^*$ and $X^s_0(\Omega)$.

In fact, it turns out that the weak solution of the equation (3.2) is the minimizer of the energy functional

$$
\mathcal{E}_V(v) = \|v\|^2_{X^s_0(\Omega)} + \|v\|^2_{L^q_2(\Omega)} - 2 \langle f, v \rangle, \quad v \in Y^s_0(\Omega) := X^s_0(\Omega) \cap L^2_q(\Omega),
$$

(3.4)

where $Y^s_0(\Omega)$ be a Hilbert subspace of $X^s_0(\Omega)$ which is endowed with the norm

$$
\|u\|_{Y^s_0(\Omega)} := \sqrt{\|u\|^2_{X^s_0(\Omega)} + \|u\|^2_{L^q_2(\Omega)}} < \infty, \quad u \in Y^s_0(\Omega).
$$

We consider function spaces $Y^s_0(\Omega)^+$ and $Y^s_0(\Omega)^-$ defined by

$$
Y^s_0(\Omega)^\pm = \{ v \in X^s(\Omega) : v^\pm \in X^s_0(\Omega) \}
$$

where $X^s(\Omega) := X^s(\Omega) \cap L^2_q(\Omega)$ be the subspace of $X^s(\Omega)$ which is endowed with the norm

$$
\|u\|_{X^s(\Omega)} := \sqrt{\|u\|^2_{X^s(\Omega)} + \|u\|^2_{L^q_2(\Omega)}} < \infty, \quad u \in X^s(\Omega).
$$

Then we see that $X^s_0(\Omega) = Y^s_0(\Omega)^+ \cap Y^s_0(\Omega)^-$. We now define weak subsolutions and supersolutions of the nonlocal equation (3.2) as follows.
Definition 3.2. We say a function \( u \in Y_0^s(\Omega)^- (Y_0^s(\Omega)^+) \) is a weak sub-solution (weak supersolution) of the nonlocal equation (3.2), if it satisfies that
\[
\langle u, v \rangle_K + \int_{\mathbb{R}^n} V(y)u(y)\varphi(y) \, dy \leq (\geq) \langle f, \varphi \rangle
\]
for every nonnegative \( \varphi \in X_0^s(\Omega) \). Also we say that a function \( u \) is a weak solution of the nonlocal equation (3.2), if it is both a weak sub-solution and a weak supersolution. So any weak solution \( u \) of the equation (3.2) must be in \( X_0^s(\Omega) \) and satisfies (3.3).

In the next, we furnish the definition of subminimizer and superminimizer of the functional (3.4) to get better understanding of weak subsolutions and supersolutions of the nonlocal equation (3.2).

Definition 3.3. (a) We say that a function \( u \in Y_0^s(\Omega)^- \) is a subminimizer of the functional (3.4) over \( Y_0^s(\Omega)^- \), if it satisfies that
\[
\mathcal{E}_V(u) \leq \mathcal{E}_V(u + \varphi)
\]
for all nonpositive \( \varphi \in X_0^s(\Omega) \). Also we say that a function \( u \in Y_0^s(\Omega)^+ \) is a superminimizer of the functional (3.4) over \( Y_0^s(\Omega)^+ \), if it satisfies (3.6) for all nonnegative \( \varphi \in X_0^s(\Omega) \).

(b) We say that a function \( u \) is a minimizer of the functional (3.4) over \( X_0^s(\Omega) \), if it is both a subminimizer and a superminimizer. So any minimizer \( u \) must be in \( X_0^s(\Omega) \) and satisfies (3.6) for all \( \varphi \in X_0^s(\Omega) \).

Lemma 3.4. If \( s \in (0, 1) \), then there is a unique minimizer of the functional (3.4). Moreover, a function \( u \in Y_0^s(\Omega)^-(Y_0^s(\Omega)^+) \) is a subminimizer (superminimizer) of the functional (3.4) over \( Y_0^s(\Omega)^-(Y_0^s(\Omega)^+) \) if and only if it is a weak sub-solution (weak supersolution) of the nonlocal equation (3.2). In particular, a function \( u \in X_0^s(\Omega) \) is a minimizer of the functional (3.4) if and only if it is a weak solution of the equation (3.2).

Proof. Using standard method of calculus of variation, we proceed with our proof. We now take any minimizing sequence \( \{u_k\} \subset X_0^s(\Omega) \). By applying Theorem 4.1 below, we can take a subsequence \( \{u_{k_j}\} \subset X_0^s(\Omega) \) converging strongly to \( u \in L^2(\Omega) \). So there exists a subsequence \( \{u_{k_j}\} \) of \( \{u_k\} \) which converges a.e. in \( \Omega \) to \( u \in X_0^s(\Omega) \). Thus by applying Fatou’s lemma we can show that the energy functional \( \mathcal{E}_V \) is weakly semicontinuous in \( X_0^s(\Omega) \). This implies that \( u \) is a minimizer of (3.4). Its uniqueness also follows from the strict convexity of the functional (3.4).

Next, we show the equivalency only for the weak supersolution case, because the other case can be done in a similar way. First, if \( u \in Y_0^s(\Omega)^+ \), then we observe that
\[
\mathcal{E}_V(u + \varphi) - \mathcal{E}_V(u) = 2(u, \varphi)_K + \int_{\mathbb{R}^n} V(y)u(y)\varphi(y) \, dy - 2(f, \varphi)
+ \|\varphi\|_{X_0^s(\Omega)}^2 + \|\varphi\|_{L^2(\Omega)}^2
\]
for all nonnegative \( \varphi \in X_0^s(\Omega)^- \). This implies that a weak supersolution \( u \in Y_0^s(\Omega)^+ \) of the equation (3.2) is a superminimizer of the functional (3.4) over \( Y_0^s(\Omega)^+ \).

On the other hand, we suppose that \( u \in Y_0^s(\Omega)^+ \) is a superminimizer of the functional (3.4). Then it follows from (3.7) that
\[
2(u, \varphi)_K + 2 \int_{\mathbb{R}^n} V(y)u(y)\varphi(y) \, dy - 2(f, \varphi) + \|\varphi\|_{X_0^s(\Omega)}^2 + \|\varphi\|_{L^2(\Omega)}^2 \geq 0
\]
for all nonnegative $\varphi \in X_0^s(\Omega)$. Since $\varepsilon \varphi \in X_0^s(\Omega)$ and it is nonnegative for any $\varepsilon > 0$ and $\varphi \in X_0^s(\Omega)$, we obtain that

$$2\langle u, \varphi \rangle_K + 2\int_{\mathbb{R}^n} V(y)u(y)\varphi(y)\,dy - 2\langle f, \varphi \rangle + \varepsilon\|\varphi\|^2_{X_0^s(\Omega)} + \varepsilon\|\varphi\|^2_{L^2(\Omega)} \geq 0$$

for any $\varepsilon > 0$. Taking $\varepsilon \to \infty$, we can conclude that

$$\langle u, \varphi \rangle_K + \int_{\mathbb{R}^n} V(y)u(y)\varphi(y)\,dy - 2\langle f, \varphi \rangle \geq 0$$

for any nonnegative $\varphi \in X_0^s(\Omega)$. Hence $u$ is a weak supersolution of the equation (3.2). Therefore we are done. \qed

4. Rellich-Kondrachov compactness theorem, a weak maximum principle and a comparison principle for $l_k + v$. In this section, we obtain the Rellich-Kondrachov compactness theorem, a weak maximum principle and a comparison principle for $L_K + V$ which will play a crucial role in proving the existence of a fundamental solution for the nonlocal Schrödinger operators in the next section.

First we get a compactness result $Y_0^s(\Omega) \hookrightarrow L^2(\Omega)$ by using the fact that $X_0^s(\Omega)$ and $Y_0^s(\Omega)$ are norm-equivalent and the precompactness of $Y_0^s(\Omega)$ in $L^2(\Omega)$.

**Theorem 4.1.** Let $n \geq 1$, $s \in (0,1)$ and $2s < n$. If $u \in Y_0^s(\Omega)$, then there is a universal constant $C > 0$ depending on $n, s$ and $\Omega$ such that

$$\|u\|_{L^2(\Omega)} \leq C \|u\|_{Y_0^s(\Omega)}.$$ 

Moreover, any bounded sequence in $Y_0^s(\Omega)$ is precompact in $L^2(\Omega)$.

**Proof.** We observe that $Y_0^s(\Omega) = X_0^s(\Omega)$ and they are norm-equivalent, because it follows from Lemma 2.1 and the fractional Sobolev inequality [4] that

$$\|u\|_{L^2(\Omega)} \leq C \|V\|^{1/2}_{L^2(\Omega)} \|u\|_{X_0^s(\Omega)}$$

with a universal constant $C > 0$ depending on $n, s$ and $\Omega$, and so

$$\|u\|^2_{X_0^s(\Omega)} \leq \|u\|^2_{Y_0^s(\Omega)} \leq (1 + C^2 \|V\|_{L^2(\Omega)}) \|u\|^2_{X_0^s(\Omega)} \tag{4.1}$$

for any $u \in X_0^s(\Omega)$. Applying Lemma 2.1 and the fractional Sobolev inequality again, we conclude that

$$\|u\|_{L^2(\Omega)} \leq C \|u\|_{H^s(\Omega)} \leq C \|u\|_{X_0^s(\Omega)} \leq C \|u\|_{Y_0^s(\Omega)}.$$ 

For the proof of the second part, take any bounded sequence $\{u_k\}$ in $Y_0^s(\Omega)$. Then it is also a bounded sequence in $X_0^s(\Omega)$. Thus by Lemma 8 [14] there is a subsequence $\{u_{kj}\}$ of the sequence and $u \in L^2(\Omega)$ such that $u_{kj} \to u$ in $L^2(\Omega)$ as $j \to \infty$. Hence we complete the proof. \qed

Next, we give a weak maximum principle and a comparison principle for the nonlocal Schrödinger operators $L_V$ as follows.

**Theorem 4.2.** If $u$ is a weak supersolution of the nonlocal equation $L_V u = 0$ in $\Omega$ such that and $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$, then $u \geq 0$ in $\Omega$.

**Proof.** From the assumption, we see that $u^- = 0$ in $\mathbb{R}^n \setminus \Omega$ and $u^+ \in X_0^s(\Omega)$, and thus $u^-, u \in X_0^s(\Omega)$. Since $u^+ u^- = 0$ in $\mathbb{R}^n$ and $u^+(x) u^-(y) \geq 0$ for all $x, y \in \mathbb{R}^n$,
we have that

\[ 0 \leq \langle u, u^- \rangle_{X_0^s(\Omega)} + \int_{\Omega} V(y)u(y)u^-(y) \, dy \]

\[ = \langle u^+ - u^-, u^- \rangle_{X_0^s(\Omega)} - \int_{\Omega} V(y)[u^-(y)]^2 \, dy \]

\[ \leq -\|u^-\|^2_{X_0^s(\Omega)} + \int_{\mathbb{R}^n} (u^+(x) - u^+(y))(u^-(x) - u^-(y))K(x - y) \, dx \, dy \]

\[ = -\|u^-\|^2_{X_0^s(\Omega)} + \int_{\mathbb{R}^n} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x - y) \, dx \, dy \]

\[ \leq -\|u^-\|^2_{X_0^s(\Omega)}. \]

This implies that \( u^- = 0 \) in \( \mathbb{R}^n \), and hence \( u \geq 0 \) in \( \Omega \). \qed

**Corollary 4.3.** If \( u \) is a weak subsolution of the nonlocal equation \( L_Vu = 0 \) in \( \Omega \) such that and \( u \leq 0 \) in \( \mathbb{R}^n \setminus \Omega \), then \( u \leq 0 \) in \( \Omega \).

**Corollary 4.4.** If \( u \) is a weak subsolution and \( v \) is a weak supersolution of the nonlocal equation (3.2) such that \( u \leq v \) in \( \mathbb{R}^n \setminus \Omega \), then \( u \leq v \) in \( \Omega \).

**Proof.** It immediately follows from Theorem 4.2. \qed

5. **Proof of the Malgrange-Ehrenpreis theorem for \( L_K + V \).** In this section, we study the existence of a fundamental solution \( \epsilon_V \) for the nonlocal Schrödinger operators \( L_V \), where \( V \) is a nonnegative potential with \( V \in L^q_{\text{loc}}(\mathbb{R}^n) \) for \( q > \frac{2n}{2s} \) and \( s \in (0,1) \) and \( n > 2s \).

Let \( T \) be an unbounded densely defined linear operator with domain \( \mathcal{D}(T) \) in a Hilbert space \( H \) with the inner product \( \langle \cdot, \cdot \rangle_H \). We denote by \( \mathcal{D}(T^*) \) the class of \( \eta \in H \) for which there exists a \( \nu \in H \) such that

\[ \langle T(\nu), \eta \rangle_H = \langle v, \nu \rangle_H \quad \text{for all } v \in \mathcal{D}(T). \]

For each \( \eta \in \mathcal{D}(T^*) \), we define \( T^*(\eta) = \nu \). Then we call \( T^* \) the *adjoint* of \( T \).

Let \( \Gamma(T) \) be the graph of such an operator \( T \); that is, it is the linear subspace

\[ \Gamma(T) = \{(u, v) \in H \times H : u \in \mathcal{D}(T) \text{ and } v = T(u)\}. \]

The operator \( T \) is said to be *closed*, if \( \Gamma(T) \) is closed in \( H \times H \). Also the operator \( T \) is said to be *closable*, if there is a closed extension \( T_0 \) of \( T \); that is, there is a closed operator \( T_0 \) with \( T \subset T_0 \) (i.e. \( \Gamma(T) \subset \Gamma(T_0) \)). We call \( \overline{T} \) the *closure* of \( T \), i.e. the smallest closed extension of \( T \). Then the following two facts are well-known [12]:

(a) If \( T \) is closable, then \( \Gamma(\overline{T}) = \overline{\Gamma(T)} \) and \( \overline{T} = \overline{T}^{**} \).

(b) If \( T_1, T_2 \) are densely defined operators with \( T_1 \subset T_2 \), then \( T_2^* \subset T_1^* \).

Let us denote by \( X_0^s := X_0^s(\mathbb{R}^n) \) the class of all \( u \in \mathcal{F}^n \) such that \( u \in X_0^s(\Omega) \) for some bounded domain \( \Omega \) in \( \mathbb{R}^n \). Similarly, we can define \( Y_0^s := Y_0^s(\mathbb{R}^n) \).

**Lemma 5.1.** If \( K \in \mathcal{K} \) for \( s \in (0,1) \), then the operator \( L_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is a densely defined operator with domain \( \mathcal{D}(L_K) = C_c^\infty(\mathbb{R}^n) \) and it is positive and symmetric. Moreover, there exists a unique closure \( L_K \) of \( L_K \) which is self-adjoint and \( \mathcal{D}(L_K) = H^{2s}(\mathbb{R}^n) \).
Proof. Note that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n) \subset X_0^s(\mathbb{R}^n)$. By Theorem 4.1, it is easy to check that $L_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a densely defined operator with domain $\mathcal{D}(L_K) = C_c^\infty(\mathbb{R}^n)$. We see that $L_K$ is nonnegative and symmetric, because $(L_Ku, u)_{L^2(\mathbb{R}^n)} = \|u\|_{K_0^2(\mathbb{R}^n)}^2 \geq 0$ and $(L_Ku, v)_{L^2(\mathbb{R}^n)} = (u, L_Kv)_{L^2(\mathbb{R}^n)}$ for any $u, v \in C_c^\infty(\mathbb{R}^n)$.

To prove the existence of the closure $\overline{L_K}$ which is self-adjoint, it suffices to show that $[L_K \pm i](C_c^\infty(\mathbb{R}^n))$ is dense in $L^2(\mathbb{R}^n)$ by verifying that its orthogonal complement is $[L_K \pm i](C_c^\infty(\mathbb{R}^n))^\perp = 0$ (refer to [12]). Indeed, let us take any $\varphi \in L^2(\mathbb{R}^n)$ satisfying

$$
\langle \varphi, [L_K \pm i]\varphi \rangle_{L^2(\mathbb{R}^n)} = 0 \quad \text{for all } \varphi \in \mathcal{D}(L_K) = C_c^\infty(\mathbb{R}^n).
$$

(5.2)

Since the Fourier transform $\mathfrak{F}(f) = \widehat{f}$ is a homeomorphism from $L^2(\mathbb{R}^n)$ onto itself, we have that

$$
\begin{align*}
0 &= \langle \varphi, [L_K \pm i]\varphi \rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{\varphi}, \mathfrak{F}([L_K \pm i]\varphi) \rangle_{L^2(\mathbb{R}^n)} \\
&= \langle \widehat{\varphi}, (m(\xi) \pm i)\widehat{\varphi} \rangle_{L^2(\mathbb{R}^n)} = \langle (m(\xi) \mp i)\widehat{\varphi}, \widehat{\varphi} \rangle_{L^2(\mathbb{R}^n)}
\end{align*}
$$

(5.3)

for all $\varphi \in \mathcal{D}(L_K) = C_c^\infty(\mathbb{R}^n)$, where $m(\xi)$ is the nonnegative function given by

$$
m(\xi) = \int_{\mathbb{R}^n} (1 - \cos(y, \xi))K(y) \, dy.
$$

(5.4)

Since $\mathfrak{F}(C_c^\infty(\mathbb{R}^n))$ is dense in $L^2(\mathbb{R}^n)$, it follows from the Plancherel theorem and (5.3) that $(m(\xi) \mp i)\widehat{\varphi} = 0$, and thus $\varphi = 0$.

Next, we show the uniqueness of the closure $\overline{L_K}$ which is self-adjoint. Indeed, if $S$ is another self-adjoint closed extension of $L_K$, then we see that $L_K \subset S$. Conversely, it follows from (5.1) that $[L_K]^* \subset S$, and so $S = S^* \subset [L_K]^* = L_K$.

Finally, we consider the multiplication operator $M_0 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with $\mathcal{D}(M_0) = \mathfrak{F}(C_c^\infty(\mathbb{R}^n))$ defined by $M_0(\widehat{\varphi}) = m\widehat{\varphi}$, where $m$ is the function in (5.4). Then, by the Plancherel theorem, we see that $L_K$ is unitarily equivalent to $M_0$. As in (5.3), we can also prove that there is a unique closure $\overline{M_0}$ of $M_0$ which is self-adjoint and $\mathcal{D}(\overline{M_0}) = \{ \varphi \in L^2(\mathbb{R}^n) : m\varphi \in L^2(\mathbb{R}^n) \}$. Thus we have that $\mathcal{D}(\overline{M_0}) = H^{2s}(\mathbb{R}^n)$, because we easily obtain that

$$\lambda|\xi|^{2s} \leq m(\xi) \leq \Lambda|\xi|^{2s}
$$

by (1.2) and (1.3). Since the Fourier transform $\mathcal{F}$ is a unitary isomorphism from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, the closure $\overline{L_K}$ of $L_K$ is unitarily equivalent to $\overline{M_0}$. Therefore we conclude that $\mathcal{D}(\overline{L_K}) = H^{2s}(\mathbb{R}^n)$.

In what follows, we write $L_K = \overline{L_K}$ for simplicity. As by-products of Lemma 5.1, we get a very useful nonlocal version of integration by parts and the norm equivalence between the space $X_0^s(\Omega)$ and the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ on a dense subspace $C_c^\infty(\Omega)$ of $X_0^s(\Omega)$.

**Corollary 5.2.** (a) If $K \in \mathcal{K}$ for $s \in (0, 1)$, then there is a unique positive self-adjoint square root operator $L_K^{1/2}$ of $L_K$, i.e. $L_K^{1/2} \circ L_K^{1/2} = L_K$, where $L_K^{1/2}u(\xi) = \sqrt{m(\xi)}u(\xi)$ for the multiplier $m(\xi)$ given in (5.4). Furthermore, it satisfies that $\mathcal{D}(L_K^{1/2}) = H^s(\mathbb{R}^n)$ and

$$
\langle L_Ku, v \rangle_{L^2(\mathbb{R}^n)} = \langle L_K^{1/2}u, L_K^{1/2}v \rangle_{L^2(\mathbb{R}^n)} = \langle u, v \rangle_K \quad \text{for any } u, v \in C_c^\infty(\mathbb{R}^n).
$$

(5.5)
Thus we have that $M_{\Omega}$ is positive and self-adjoint. Thus there is a positive self-adjoint square root operator as an operator $L$ on $\mathcal{D}(M_1) = \mathcal{F}(C_0^{\infty}(\mathbb{R}^n))$ defined by $M_1(\hat{\varphi}) = \sqrt{m} \hat{\varphi}$, it follows from Plancherel theorem that $L_{\Omega}^{1/2}$ is unitarily equivalent to $M_1$. Then as in (5.3) we obtain a unique closure $\overline{M_1}$ of $M_1$ which is self-adjoint and

$$\mathcal{D}(\overline{M_1}) = \{ \phi \in L^2(\mathbb{R}^n) : \sqrt{m} \phi \in L^2(\mathbb{R}^n) \}.$$ 

Since $\sqrt{\lambda} |\lambda| \leq \sqrt{\lambda} |\lambda| \leq \sqrt{\lambda} |\lambda|$, we have that $\mathcal{D}(L_{\Omega}^{1/2}) = H^s(\mathbb{R}^n)$ as in the proof of Lemma 5.1. Also the remaining parts can be achieved by straightforward calculation. (b) It easily follows from (a) and the density property of $C_0^{\infty}(\Omega)$ in $X_0^2(\Omega)$. (c) It easily follows from (5.5), (b) and Lemma 2.1. Therefore we complete the proof.

Let $H_\Omega = L^2(\Omega)$ be a Hilbert space with inner product $\langle u, v \rangle_{H_\Omega} = \langle Vu, Vv \rangle_{L^2(\Omega)}$.

**Lemma 5.3.** (a) The multiplication operator $M_\Omega : L^2(\Omega) \to L^2(\Omega)$ with domain $\mathcal{D}(M_\Omega) = H_\Omega$ given by $M_\Omega(u) = Vu$ is positive and self-adjoint.

(b) There is a unique positive self-adjoint square root operator $P$ of $M_\Omega$, i.e. $P \circ P = M_\Omega$.

**Proof.** (a) For any fixed $v \in L^2(\Omega)$, it is quite easy to check that the linear map $T_0 : H_\Omega \to \mathbb{R}$ defined by $T_0(u) = \langle M_\Omega u, v \rangle_{L^2(\Omega)}$ satisfies

$$|T_0(u)| \leq \|u\|_{H_\Omega} \|v\|_{L^2(\Omega)}.$$ 

By the Hahn-Banach theorem, we can extend $T_0$ to a continuous linear functional on $L^2(\Omega)$, and so by Riesz representation theorem there is a unique $M_\Omega v \in L^2(\Omega)$ such that

$$\langle u, \hat{v} \rangle_{L^2(\Omega)} = \langle M_\Omega(u), v \rangle_{L^2(\Omega)} = \langle u, M_\Omega^* v \rangle_{L^2(\Omega)} \text{ for all } u \in H_\Omega.$$ 

Thus we have that $M_\Omega^* v = Vv \in L^2(\Omega)$. This implies that $\mathcal{D}(M_\Omega^*) = H_\Omega = \mathcal{D}(M_\Omega)$.

Thus $M_\Omega$ is self-adjoint and it is also positive, because we have that $\langle M_\Omega(u), u \rangle_{L^2(\Omega)} = \|u\|_{L^2_\Omega}^2 \geq 0$.

(b) The existence and uniqueness of a positive self-adjoint square root operator $P$ of $M_\Omega$ can be obtained by Theorem 13.31 [13]. Hence we are done.

In what follows, by Lemma 5.1 and Lemma 5.3, for simplicity we may write

$$L_\Omega = L_{\Omega}^{1/2} + M_\Omega \text{ on } \Omega$$

as an operator $L_\Omega : L^2(\Omega) \to L^2(\Omega)$ with $\mathcal{D}(L_\Omega) = H^{2s}(\mathbb{R}^n) \cap L^2_{\Omega}(\Omega)$, which is positive and self-adjoint. Thus there is a positive self-adjoint square root operator $L_\Omega^{1/2}$ of $L_\Omega$. 


Lemma 5.4. We have the estimate
\[ \langle L_{V^\frac{1}{2}}, L_{V^\frac{1}{2}} \rangle_{L^2(\mathbb{R}^n)} = \langle u, v \rangle_{Y^0(\Omega)} + \langle Vu, v \rangle_{L^2(\Omega)} \]
for all \( u, v \in C_c^\infty(\Omega) \). Moreover, the map \( F : C_c^\infty(\Omega) \to L^2(\mathbb{R}^n) \) defined by \( F(\varphi) = L_{V^\frac{1}{2}} \varphi \) is injective and satisfies that
\[ \|F(\varphi)\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{Y^0(\Omega)} \quad \text{for any } \varphi \in C_c^\infty(\Omega). \] (5.6)

Proof. It easily follows from Corollary 5.2, Lemma 5.3 and Theorem 4.1.

Next, we obtain the existence and uniqueness of weak solution of the nonlocal equation \( L_V = h \) in \( \Omega \) for the forcing term \( h \in L^2(\Omega) \) and moreover for \( h \in Y_0^0(\Omega)^* \).

Lemma 5.5. For each \( h \in Y_0^0(\Omega)^* \), there is a unique weak solution \( u \in Y_0^0(\Omega) \) of the nonlocal equation \( L_V u = h \) in \( \Omega \) and \( \|u\|_{Y_0^0(\Omega)} \leq \|h\|_{Y_0^0(\Omega)^*} \). If \( h \in L^2(\Omega) \), then we have that \( \|u\|_{Y_0^0(\Omega)} \leq C \|h\|_{L^2(\Omega)} \).

Proof. We define a bilinear form \( a : Y_0^0(\Omega) \times Y_0^0(\Omega) \to \mathbb{R} \) by
\[ a(u, \phi) = \langle L_V u, \phi \rangle_{L^2(\Omega)} . \]
By Corollary 5.2, it is easy to check that
\[ a(u, u) = \|u\|_{Y_0^0(\Omega)}^2 \quad \text{and} \quad |a(u, \phi)| \leq \|u\|_{Y_0^0(\Omega)} \|\phi\|_{Y_0^0(\Omega)} \]
for any \( u, \phi \in Y_0^0(\Omega) \). Thus the existence result can be obtained by the Lax-Milgram theorem.

Since \( u \in Y_0^0(\Omega) \) is a weak solution of the equation \( L_V u = h \) in \( \Omega \), we have that
\[ \|u\|_{Y_0^0(\Omega)} = \langle L_V u, u \rangle_{L^2(\Omega)} = \langle h, u \rangle \leq \|h\|_{Y_0^0(\Omega)^*} \|u\|_{Y_0^0(\Omega)} , \]
and thus we have that \( \|u\|_{Y_0^0(\Omega)} \leq \|h\|_{Y_0^0(\Omega)^*} \). If \( h \in L^2(\Omega) \), then it follows from the dual form of Theorem 4.1 that
\[ \|u\|_{Y_0^0(\Omega)} \leq \|h\|_{Y_0^0(\Omega)^*} \leq C \|h\|_{L^2(\Omega)^*} = C \|h\|_{L^2(\Omega)} . \]
Hence we are done.

Let \( T : D(D) \to \mathbb{R} \) be a linear map. Then an equivalent condition of distribution is known as follows (see [13]): \( T \in D'(D) \) if and only if for each compact set \( Q \subset D \), there is an integer \( N = N(Q) > 0 \) and a constant \( C = C(Q) > 0 \) such that
\[ |T(\varphi)| \leq C \|\varphi\|_N \quad \text{for all } \varphi \in D_Q , \] (5.7)
where \( \|\varphi\|_N = \max\{\sup \{ |\partial^\alpha \varphi| : |\alpha| := \sum_{i=1}^n \alpha_i \leq N \} \} \) and \( D_Q \) denotes the class of all smooth functions in \( D \) which is supported in \( Q \). In what follows, we write \( \langle T, \varphi \rangle = T(\varphi) \).

Lemma 5.6. If \( u \in L^q(\mathbb{R}^n) \) and \( V \in L^q_{\text{loc}}(\mathbb{R}^n) \) is a nonnegative potential with \( q > \frac{n}{2s} > 1 \), \( s \in (0, 1) \), then the linear map \( L_V u : D(\mathbb{R}^n) \to \mathbb{R} \) defined by
\[ \langle L_V u, \varphi \rangle = \int_{\mathbb{R}^n} u(y) L_V \varphi(y) \, dy \] (5.8)
is a distribution, i.e., \( L_V u \in \mathbb{D}'(\mathbb{R}^n) \).

Proof. By [7], there is a constant \( c > 0 \) independent of \( \varphi \) such that
\[ |L_K \varphi(x)| \leq \frac{c \|\varphi\|_{2,s}}{(1 + |x|)^{n+2s}} \] (5.9)
for any \( \varphi \in \mathcal{D}(\mathbb{R}^n) \), where \( \| \varphi \|_{2,s} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^{n+2s} \sum_{|\alpha| \leq 2} |D^\alpha \varphi(x)|. \)

Take any compact set \( Q \subset \mathbb{R}^n \) and any \( \varphi \in \mathcal{D}_Q \). Since \( \varphi \in \mathcal{S}(\mathbb{R}^n) \), we have that

\[
|\varphi(x)| \leq \frac{\| \varphi \|_{(N_0,0)}}{(1 + |x|)^{n+2s}} \tag{5.10}
\]

where \( \| \varphi \|_{(N_0,0)} = \sup_{x \in Q} (1 + |x|)^{N_0} |\varphi(x)| \leq C(Q) \| \varphi \|_0 \) for \( N_0 = \lfloor \frac{n+2s}{q} \rfloor + 1 \). Thus it follows from (5.9), (5.10) and Hölder’s inequality that

\[
\langle L_V u, \varphi \rangle = C \| \varphi \|_{2,s} \int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}} \, dx
\]

\[
+ \left( \int_{\mathbb{R}^n} V^q(x) \, dx \right)^{1/q} \left( \int_{\mathbb{R}^n} |u(x)\varphi(x)|^q \, dx \right)^{1/q'}
\]

\[
\leq C(Q)(\|\varphi\|_2 + \|V\|_{L^q(Q)} \|\varphi\|_0)\|u\|_{L^q_v(\mathbb{R}^n)}
\]

\[
\leq C(Q)(1 + \|V\|_{L^q(Q)})\|u\|_{L^q_v(\mathbb{R}^n)}\|\varphi\|_2.
\]

Hence, by (5.7), we conclude that \( L_V u \in \mathcal{D}'(\mathbb{R}^n) \).

\[\square\]

**Lemma 5.7.** Let \( V \in L^q_{\text{loc}}(\mathbb{R}^n) \) be a nonnegative potential with \( q > \frac{n}{2s} > 1 \), \( s \in (0,1) \). If \( h \in L^{\frac{2s}{n-2s}}(\mathbb{R}^n) \) is nonnegative, then there is a function \( u \in \dot{H}^s(\mathbb{R}^n) \) such that

\[
L_V u = h \quad \text{in the sense of } \mathcal{D}'(\mathbb{R}^n).
\]

Moreover, there is an increasing sequence \( \{a_k\}_{k \in \mathbb{N}} \) with \( \lim_{k \to \infty} a_k = \infty \) such that \( u_{a_k} \in Y^s_0(B_{a_k}) \) satisfies the nonlocal equation \( L_V u_{a_k} = h \) in \( B_{a_k} \) in the weak sense and \( \lim_{k \to \infty} u_{a_k} = u \) a.e. in \( \mathbb{R}^n \).

**Proof.** We observe that

\[
h \in L^{\frac{2s}{n-2s}}(B_k) \subset L^2(B_k) \subset Y^s_0(B_k)^* \quad \text{for all } k \in \mathbb{N}.
\]

By Theorem 4.2 and Lemma 5.5, for each \( k \in \mathbb{N} \) there exists a nonnegative function \( u_k \in Y^s_0(B_k) \) which is a weak solution of the equation \( L_V u_k = h \) in \( B_k \), i.e.,

\[
\langle u_k, \varphi \rangle_{X^s_0(B_k)} + \langle Vu_k, \varphi \rangle_{L^2(B_k)} = \langle h, \varphi \rangle_{L^2(B_k)} \tag{5.13}
\]

for any \( \varphi \in Y^s_0(B_k) \). Taking \( \varphi = u_k \) in (5.13), it follows from Hölder’s inequality, Lemma 2.1 and the fractional Sobolev embedding on \( \dot{H}^s(\mathbb{R}^n) \) (Proposition 2.2) that

\[
\frac{c_{n,s}}{2} \|u_k\|^2_{\dot{H}^s(\mathbb{R}^n)} = \|u_k\|^2_{X^s_0(B_k)} \leq \|u_k\|^2_{Y^s_0(B_k)} \leq \|h\|_{L^{\frac{2s}{n-2s}}(\mathbb{R}^n)} \|u_k\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leq C \|h\|_{L^{\frac{2s}{n-2s}}(\mathbb{R}^n)} \|u_k\|_{\dot{H}^s(\mathbb{R}^n)} \tag{5.14}
\]

where \( C > 0 \) is a constant depending only on \( n, s \), but not on \( k \). This implies that

\[
\sup_{k \in \mathbb{N}} \|u_k\|_{L^{\frac{2s}{n-2s}}(\mathbb{R}^n)} \leq C \sup_{k \in \mathbb{N}} \|u_k\|^2_{\dot{H}^s(\mathbb{R}^n)} \leq C \|h\|_{L^{\frac{2s}{n-2s}}(\mathbb{R}^n)}.
\]

By weak compactness, there are a subsequence \( \{u_{k_i}\}_{i \in \mathbb{N}} \) and \( u \in \dot{H}^s(\mathbb{R}^n) \) such that

\[
u_{k_i} \rightharpoonup u \quad \text{in } L^{\frac{2s}{n-2s}}(\mathbb{R}^n) \quad \text{and} \quad u_{k_i} \to u \quad \text{in } \dot{H}^s(\mathbb{R}^n)
\]

(5.16)
Let us take any \( \varphi \in D(\mathbb{R}^n) \). Then there is some \( m \in \mathbb{N} \) such that \( \varphi \in D(B_m) \).

We note that \( L Vu_k = h \) in the sense of \( D'(B_m) \) for any \( k \geq m \). Thus, by (5.16), Corollary 5.2 and Lemma 5.6, we conclude that

\[
\langle L Vu, \varphi \rangle = \int_{\mathbb{R}^n} u(y) L V \varphi(y) \, dy = \lim_{i \to \infty} \int_{\mathbb{R}^n} u_{k_i}(y) L V \varphi(y) \, dy = \langle h, \varphi \rangle,
\]

since \( L V \varphi \in L^2(\mathbb{R}^n) \) by (5.9). This implies that \( L Vu = h \) in the sense of \( D'(\mathbb{R}^n) \).

Finally, we see from Lemma 2.1, (4.1) and Theorem 4.1 that the sequence \( \{u_{k_i}\} \) is precompact in \( L^2(B) \) for any ball \( B \) in \( \mathbb{R}^n \). This implies the required almost everywhere convergence.

Next, we shall prove our main theorem which is an extension of the Malgrange-Ehrenpreis theorem to nonlocal Schrödinger operators \( L V \).

For \( l \in \mathbb{N} \), let \( f_l(x) = l^n f(lx) \) where \( f \in C_c(B_1) \) is a nonnegative function with \( \|f\|_{L^1(\mathbb{R}^n)} = 1 \). By Lemma 5.7, there is a sequence \( \{u_l\}_{l \in \mathbb{N}} \subset \dot{H}^s(\mathbb{R}^n) \) such that

\[
L Vu_l = f_l \quad \text{in the sense of} \quad D'(\mathbb{R}^n). \tag{5.18}
\]

As a matter of fact, we see from a weak maximum principle (Theorem 4.2) and the proof of Lemma 5.7 that each \( u_l \) is nonnegative and, for each \( l \in \mathbb{N} \), there are a sequence \( \{a^l_k\}_{k \in \mathbb{N}} \) with \( \lim_{k \to \infty} a^l_k = \infty \) and a sequence \( \{u^l_k\}_{k \in \mathbb{N}} \) of nonnegative functions \( u^l_k \in Y_0(B_{a^l_k}) \) such that

\[
u_l = \lim_{k \to \infty} u^l_k \quad \text{a.e. in} \quad \mathbb{R}^n,
\]

\[
L Vu^l_k = f_l \quad \text{in} \quad B_{a^l_k} \quad \text{in the weak sense.} \tag{5.19}
\]

Then we obtain uniform estimates for the sequence \( \{u_l\}_{l \in \mathbb{N}} \) in the following lemma in order to prove the main theorem.

**Lemma 5.8.** (a) For each \( l \in \mathbb{N} \) and \( p \in [1, \frac{n}{n-2s}) \) with \( s \in (0, 1) \), there exists some constant \( C = C(p, \lambda, n, s) > 0 \) such that

\[
\|u_l\|_{L^p(B_r(x_0))} \leq C r^{\frac{\lambda}{n} - (n-2s)} \quad \text{for any} \quad x_0 \in \mathbb{R}^n \quad \text{and} \quad r > 0. \tag{5.20}
\]

(b) For each \( l \in \mathbb{N} \) and \( p \in [1, \frac{n}{n-2s}) \) and \( \gamma \in (0, s) \) with \( s \in (0, 1) \), there exists some constant \( C = C(p, n, s, \gamma) > 0 \) such that

\[
[u_l]_{W^{s, p}(B_r(x_0))} \leq C r^{\frac{\lambda}{n} - (n-2s + \gamma)} \quad \text{for any} \quad x_0 \in \mathbb{R}^n \quad \text{and} \quad r > 0. \tag{5.21}
\]

(c) If \( V \in L^q_{\text{loc}}(\mathbb{R}^n) \) is nonnegative for \( q > \frac{n}{n-2s} \) with \( s \in (0, 1) \), then for each \( l \in \mathbb{N} \) there is a constant \( C = C(p, \lambda, n, s) > 0 \) such that

\[
\|u_l\|_{L^q_{\text{loc}}(B_r(x_0))} \leq C \frac{\|V\|_{L^q(B_r(x_0))}}{r^{\frac{n-2s}{n}}} \quad \text{for any} \quad x_0 \in \mathbb{R}^n \quad \text{and} \quad r > 0, \tag{5.22}
\]

where \( p \in [1, \frac{n}{n-2s}) \) is the dual exponent of \( q > \frac{n}{n-2s} \).

(d) For each \( l \in \mathbb{N} \) and for any \( s \in (0, 1) \) and \( x \in \mathbb{R}^n \setminus \{0\} \) with \( |x| \geq 3/l \), we have that

\[
|u_l(x)| \leq C \frac{1}{|x|^{n-2s}} \tag{5.23}
\]

where \( C = C(n, s, \lambda, \Lambda) > 0 \) is a constant depending only on \( n, s, \lambda \) and \( \Lambda \).

**Proof.** (a) Motivated by [9], we consider a nonnegative function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) defined by

\[
\varphi = \beta^{1-\alpha} - (\beta + u^l_k)^{1-\alpha}
\]
where $\beta > 0$ and $\alpha \in (1,2)$ will be chosen later. Since each $u^k_i$ is supported in $B_{a^k_i}$ and $a(t) = (\beta + t)^{1-\alpha}$ is Lipschitz continuous on $(0,\infty)$, by Lemma 2.1 and the definition of $Y^\mu_0(B_{a^k_i})$ we see that $\varphi \in Y^\mu_0(B_{a^k_i}) \subset H^s(\mathbb{R}^n)$. Thus we can use $\varphi$ as a testing function. From the weak formulation of the nonlocal equation given in (5.19), we have that

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} (u^k_i(x) - u^k_i(y))(\varphi(x) - \varphi(y))K(x-y)\,dx\,dy \leq \beta^{1-\alpha}. \tag{5.24}$$

By the mean value theorem, we have that

$$\varphi(x) - \varphi(y) = (u^k_i(x) - u^k_i(y))\int_0^1 (\beta + tu^k_i(y) + (1-t)u^k_i(y))^\alpha \,dt. \tag{5.25}$$

Thus it follows from (5.24) and (5.25) that

$$\beta^{1-\alpha} \geq (\alpha - 1) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \left( \int_0^1 \frac{(u^k_i(x) - u^k_i(y))^2 \,dt}{(\beta + tu^k_i(x) + (1-t)u^k_i(y))^\alpha} \right) \,dx\,dy. \tag{5.26}$$

By Jensen’s inequality and the mean value theorem, we have that

$$\left( \frac{u^k_i(x)}{(\beta + u^k_i(x))^\alpha/2} - \frac{u^k_i(y)}{(\beta + u^k_i(y))^\alpha/2} \right)^2 = \left( \int_0^1 \frac{dt}{\beta + tu^k_i(x) + (1-t)u^k_i(y)} \right)^2 \leq 4 \int_0^1 \frac{|u^k_i(x) - u^k_i(y)|^2 \,dt}{(\beta + tu^k_i(x) + (1-t)u^k_i(y))^\alpha}. \tag{5.27}$$

Plugging (5.27) into (5.26) and using Proposition 2.2 (fractional Sobolev embedding theorem), we obtain that

$$\beta^{1-\alpha} \geq \frac{C(\alpha - 1)}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{u^k_i(x)}{(\beta + u^k_i(x))^\alpha/2} - \frac{u^k_i(y)}{(\beta + u^k_i(y))^\alpha/2} \right)^2 \,dx\,dy,$$

$$\geq \frac{C(\alpha - 1)}{4} \left( \int_{\mathbb{R}^n} \left( \frac{u^k_i(x)}{(\beta + u^k_i(x))^\alpha/2} \right)^{\frac{2n}{n-2\alpha}} \,dx \right)^{\frac{n-2\alpha}{n}}. \tag{5.28}$$

Take any $p \in [1, \frac{n}{n-2\alpha}]$ and choose some $\alpha \in (1,2)$ satisfying $\frac{2}{p} = 1 - \frac{(n-2\alpha)p}{2n}$. By Hölder’s inequality and (5.28), we have that

$$\|u^k_i\|_{L^p(B_r(x_0))}^p \leq C \left( \int_{\mathbb{R}^n} \left( \frac{u^k_i(x)}{(\beta + u^k_i(x))^\alpha/2} \right)^{\frac{2n}{n-2\alpha}} \,dx \right)^{\frac{(n-2\alpha)p}{2n}} \left( \int_{B_r(x_0)} (\beta + u^k_i(x))^\frac{2}{q^p} \,dx \right)^{\frac{1}{q^p}} \tag{5.29}$$

$$\leq C \beta^{(1-\alpha)p/2} \left( \int_{B_r(x_0)} (\beta + u^k_i(x))^{\frac{2}{q^p}} \,dx \right)^{\frac{1}{q^p}},$$

where $q'$ is the dual exponent of $q = \frac{2n}{(n-2\alpha)p}$, because $\frac{2}{p}q' = 1$. If we set

$$\beta = \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^k_i(x)^p \,dx \right)^{\frac{1}{p}} = \frac{|B_1(0)|^{-1/p}}{r^{n/p}} \left( \int_{B_r(x_0)} u^k_i(x)^p \,dx \right)^{\frac{1}{p}},$$

then

$$\|u^k_i\|_{L^p(B_r(x_0))} \leq C \left( \frac{u^k_i}{\beta} \right) \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^k_i(x)^p \,dx \right)^{\frac{1}{p}},$$

where

$$\left( \frac{u^k_i}{\beta} \right) \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^k_i(x)^p \,dx \right)^{\frac{1}{p}} \leq C \left( \frac{u^k_i}{\beta} \right) \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^k_i(x)^p \,dx \right)^{\frac{1}{p}},$$

which completes the proof.
This gives that, for each $l \in \mathbb{N}$,
\[
\|u^k_l\|_{L^p(B_r(x_0))} \leq C r^{\frac{\gamma}{2} - (n-2s)} \quad \text{for any } k \in \mathbb{N}.
\] (5.30)

Hence it follows from (5.19), (5.30) and Fatou’s lemma that
\[
\|u_l\|_{L^p(B_r(x_0))} \leq C r^{\frac{\gamma}{2} - (n-2s)} \quad \text{for each } l \in \mathbb{N}.
\]

(b) For $\gamma \in (0, s]$ with $s \in (0, 1)$, we set
\[
U_\gamma(x, y) = \frac{|u^k_l(x) - u^k_l(y)|}{|x-y|^{\gamma}}.
\]

For $p \in [1, 2)$, we write
\[
U^p_\gamma(x, y) = \left( \frac{U^2_\gamma(x, y)}{(\beta + u^k_l(x) + u^k_l(y))^\alpha} \right)^{\frac{p}{2}} ((\beta + u^k_l(x) + u^k_l(y))^\alpha |x-y|^{2(s-\gamma)})^{\frac{p}{2}}.
\]

Then it follows from Hölder’s inequality and (5.26) that
\[
\int \int \int \hspace{1cm} B^2_r(x_0) U^p_\gamma(x, y) \left[ \frac{|x-y|^n}{r^{n-\alpha}} \right] \leq \left( \int \int \int \hspace{1cm} B^2_r(x_0) \frac{U^2_\gamma(x, y)}{(\beta + u^k_l(x) + u^k_l(y))^\alpha} \right)^{\frac{p}{2}} \left( \int \int \int \hspace{1cm} B^2_r(x_0) \frac{(\beta + u^k_l(x) + u^k_l(y))^\alpha |x-y|^{2(s-\gamma)}}{|x-y|^{n-2(s-\gamma)p/(2-p)}} \right)^{\frac{2-p}{p}} (5.31)
\]

where $B^2_r(x_0) = B_r(x_0) \times B_r(x_0)$. If we choose $\beta$ satisfying
\[
\beta = \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^k_l(x)^{\frac{np}{2-p}} d\gamma \right)^{\frac{2-p}{p}},
\]
then (5.31) leads us to get that
\[
\int \int \int \hspace{1cm} B^2_r(x_0) U^p_\gamma(x, y) \left[ \frac{|x-y|^n}{r^{n-\alpha}} \right] \leq C r^{(s-\gamma)p} \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^k_l(x)^{\frac{np}{2-p}} d\gamma \right)^{\alpha} \left( \int_{B_r(x_0)} u^k_l(x)^{\frac{np}{2-p}} d\gamma \right)^{\frac{2-p}{p}} (5.32)
\]

Take any $p \in [1, \frac{n}{n-\alpha})$. We may select some $\alpha \in (1, 2)$ satisfying $\frac{np}{2-p} \in [1, \frac{n}{n-2s})$, because $h(t) = \frac{t}{2-2s} \geq t$ for any $t \in [1, 2)$ and $h(\frac{n}{n-\alpha}) = \frac{n}{n-2s}$. Applying (5.30) to (5.32), we obtain that
\[
\int \int \int \hspace{1cm} B^2_r(x_0) U^p_\gamma(x, y) \left[ \frac{|x-y|^n}{r^{n-\alpha}} \right] \leq C r^M
\]
where the power index $M$ of $r$ is given by

$$M = (s - \gamma)p - \frac{n(2 - p)(1 - \alpha)}{2\alpha} + \left[ \frac{n(2 - p)}{\alpha p} - (n - 2s) \right] \frac{\alpha p}{2 - p} \left[ \frac{(2 - p)(1 - \alpha)}{2\alpha} + \frac{2 - p}{2} \right]$$

Thus we have that, for each $l \in \mathbb{N}$,

$$|u_l^k|_{W^\gamma,r(B_0(x_0))} \leq C r^M \quad \text{for any } k \in \mathbb{N}. \quad (5.33)$$

Hence, (5.19), (5.33) and Fatou’s lemma imply the second part.

(c) By Hölder’s inequality, we have that

$$\| u_l \|_{L^1(B_r(x_0))} \leq \| u_l \|_{L^p(B_r(x_0))} \| V \|_{L^q(B_r(x_0))}$$

where $p \in [1, \frac{n}{n-2s}]$ is the dual exponent of $q > \frac{n}{2s}$. Therefore, by (5.20) and (5.34), we obtain that

$$\| u_l \|_{L^1(B_r(x_0))} \leq \frac{C \| V \|_{L^q(B_r(x_0))}}{r^{n-2s-n/p}}$$

with a universal constant $C = C(p, \lambda, n, s) > 0$. Thus we are done.

(d) Take any $x \in \mathbb{R}^n$ with $|x| \geq 3/4$. Then we note that $B_{2r}(x) \cap B_{1/4} = \emptyset$ where $r = \frac{1}{4} |x|$. By Lemma 4.5 [3], we have the estimate

$$\| u_l^k \|_{L^\infty(B_r(x))} \leq \frac{C}{r^{n/2}} \| u_l^k \|_{L^2(B_{2r}(x))} \quad (5.35)$$

with some universal constant $C > 0$. Using a standard argument in [8], for any $p \in [1, \infty)$ there is a universal constant $C = C(n, s, p) > 0$ such that

$$\| u_l^k \|_{L^\infty(B_r(x))} \leq \frac{C}{r^{n/p}} \| u_l^k \|_{L^p(B_{2r}(x))}. \quad (5.36)$$

Thus it follows from (5.30) and (5.36) with $p = 1$ that

$$\| u_l^k \|_{L^\infty(B_r(x))} \leq \frac{C}{r^n} \| u_l^k \|_{L^1(B_{2r}(x))} \leq \frac{C}{r^{n-2s}}.$$  

Therefore this and (5.19) imply (5.23). Hence we complete the proof. \hfill \Box

**Proof of Theorem 1.1.** We take any value $\gamma \in (0, s)$ and $p \in [1, \frac{n}{n-2s}]$. Then, from (5.19), Lemma 5.8 [(5.20), (5.21),(5.22)] and Theorem 7.1 [4], the standard diagonalization process yields that there exist a subsequence $\{u_{l_i}\}_{i \in \mathbb{N}}$ of $\{u_l\}_{l \in \mathbb{N}}$ and $\epsilon_V \in W_{\text{loc}}^{\gamma,p}(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ such that

\[
\begin{cases}
  u_{l_i} \to \epsilon_V & \text{in } W^{\gamma,p}(B_r(x_0)), \\
  u_{l_i} \to \epsilon_V & \text{in } L^p(B_r(x_0)), \\
  V u_{l_i} \to V \epsilon_V & \text{in } L^1(B_r(x_0)), \\
  u_{l_i} \to \epsilon_V & \text{a.e. in } B_r(x_0)
\end{cases}
\]  

(5.37)

for any $(x_0, r) \in \mathbb{R}^n \times (0, \infty)$. We write $\{u_l\}_{l \in \mathbb{N}}$ instead of $\{u_{l_i}\}_{i \in \mathbb{N}}$, for simplicity. The estimate (1.5) is now easily derived from (d) of Lemma 5.8.

Take any $\varphi \in C_c^\infty(\mathbb{R}^n)$. Since $q' < \frac{n}{n-2s}$ and $|u_l(x)| + |\epsilon_V(x)| \leq C' |x|^{n-2s}$, it is easy to check that $\epsilon_V \in L^{q'}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ by (1.6) and we may also apply
the Lebesgue’s dominated convergence theorem to yield that $u_l$ converges to $\epsilon_V$ in $L^1_+(\mathbb{R}^n)$. Thus we have that
\[
\lim_{l \to \infty} \left| \int_{\mathbb{R}^n} u_l(x)(L_K + V)\varphi(x)dx - \int_{\mathbb{R}^n} \epsilon_V(x)(L_K + V)\varphi(x)dx \right|
\leq \lim_{l \to \infty} \int_{\mathbb{R}^n} |u_l(x) - \epsilon_V(x)||L_K\varphi(x)| dx \\
+ \lim_{l \to \infty} \left| \int_{\mathbb{R}^n} (u_l(x) - \epsilon_V(x))V(x)\varphi(x) dx \right| = 0.
\]

Also we note that
\[
\lim_{l \to \infty} \int_{\mathbb{R}^n} u_l(x)(L_K + V)\varphi(x)dx = \lim_{l \to \infty} \int_{\mathbb{R}^n} f_l(x)\varphi(x)dx = \varphi(0).
\]

Using the above two limits, we arrive at the following
\[
\int_{\mathbb{R}^n} \epsilon_V(x)(L_K + V)\varphi(x)dx = \varphi(0).
\]

Finally, by Lemma 5.6, we see that $L_V\epsilon_V$ is the required distribution. Hence we complete the proof. \qed

**Acknowledgments.** We would like to thank the referee for helpful comments which lead to improve our original manuscript. Yong-Cheol Kim was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (MSIP-2017R1A2B1005433) and also was supported by School of Education, Korea University Grant in 2018. Woocheol Choi was supported by the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2017R1C1B5076348).

**REFERENCES**

[1] C. Bucur, Some observations on the Green function for the ball in the fractional Laplace framework, *Comm. Pure and Appl. Anal.*, 15 (2016), 657–699.

[2] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. P.D.E.*, 32 (2007), 1245–1260.

[3] W. Choi and Y.-C. Kim, $L^p$-mapping properties for nonlocal Schrödinger operators with certain potential, preprint, arxiv:math/0605406.

[4] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, 136 (2012), 521–573.

[5] L. Ehrenpreis, Solution of some problems of division. I. Division by a polynomial of derivation, *Amer. J. Math.*, 76 (1954), 883–903.

[6] L. Ehrenpreis, Solution of some problems of division. I. Division by a punctual distribution, *Amer. J. Math.*, 77 (1955), 286–292.

[7] M. Fall and T. Weth, Liouville theorems for a general class of nonlocal operators, *Potential. Anal.*, 45 (2016), 187–200.

[8] Q. Han and F. Lin, *Elliptic Partial Differential Equations*, Courant Lecture Notes in Mathematics, American Mathematical Society, 1997.

[9] T. Kuusi, G. Mingione and Y. Sire, Nonlocal equations with measure data, *Comm. Math. Phys.*, 337 (2015), 1317–1368.

[10] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, New York, 1972.

[11] B. Malgrange, Existence et approximation des solutions des équations aux d’érivées partielles et des équations de convolution, *Ann. Inst. Fourier*, 6 (1955/56), 271–355.

[12] M. Reed and B. Simon, *Functional Analysis I*, Methods of Modern Mathematical Physics. Academic Press.

[13] W. Rudin, *Functional Analysis*, 2nd edition, International Series in Pure and Applied Mathematics. McGraw-Hill, 1991.
[14] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.*, **33** (2013), 2105–2137.

[15] E. M. Stein, *Singular Integrals and Differentiability*, Princeton Univ. Press, 1970.

Received August 2017; revised January 2018.

*E-mail address:* choiwc@inu.ac.kr

*E-mail address:* ychkim@korea.ac.kr