The second cohomology of $sl(m|1)$ with coefficients in its enveloping algebra is trivial

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Abstract

Using techniques developed in a recent article by the authors, it is proved that the 2–cohomology of the Lie superalgebra $sl(m|1); m \geq 2$, with coefficients in its enveloping algebra is trivial. The obstacles in solving the analogous problem for $sl(3|2)$ are also discussed.
1 Introduction

The present work is a direct sequel to a recent article by the authors [1] dealing with the cohomology of Lie superalgebras (for a list of pertinent references, see Ref. [1]). One of the main goals of these papers is to prove or disprove, for as many basic classical simple Lie superalgebras $L$ as possible, that

$$H^2(L, U(L)) = \{0\} ,$$

where $U(L)$ is the enveloping algebra of $L$, endowed with the adjoint action of $L$. As explained in Ref. [1], Eq. (1.1) implies that the associative superalgebra $U(L)$ does not admit of any non–trivial formal deformations in the sense of Gerstenhaber [2]. In Ref. [1], we have shown that Eq. (1.1) holds for the $osp(1|2n)$ algebras and for $sl(2|1)$. Here, we are going to prove that it is true for $sl(m|1)$ with $m \geq 2$.

The setup of our paper is the following. In Sec. 2 we introduce our notation and specialize some of the results of Ref. [1] to the case of present interest, the Lie superalgebras $sl(m|1)$ with $m \geq 2$. Sec. 3 contains the proof of Eq. (1.1) for these algebras. A short discussion follows in Sec. 4.

The paper is closed by an appendix, in which we consider the analogous problem for the algebra $sl(3|2)$. Unfortunately, up to now we have not been able to prove or disprove Eq. (1.1) in this case. However, we think it may be worthwhile to show the obstacles which one has to overcome if one wants to proceed along the lines described in Ref. [1].

We close this introduction by specifying some of our conventions. Throughout the present work the base field $\mathbb{K}$ is assumed to be algebraically closed and to have characteristic zero. The multiplication in a Lie superalgebra will be denoted by a pointed bracket $\langle , \rangle$. All modules over a Lie superalgebra are assumed to be graded.

2 Reminder of some previous results

Let us first explain our notation and conventions. For later use (in the appendix) we describe them for an arbitrary special linear Lie superalgebra $sl(m|n)$, with $m, n \geq 1$ and $m \neq n$. Quite generally, we follow Ref. [1], in particular, see Example 4 in Sec. 3. Thus we use the generators

$$X_{ij} = E_{ij} - \frac{1}{m-n} \sigma_i \delta_{ij} I ,$$

(with $i, j \in \{1, 2, \ldots, m+n\}$) of $sl(m|n)$, where the $E_{ij}$ are the standard basic $(m+n) \times (m+n)$ matrices, where $I$ is the $(m+n) \times (m+n)$ unit matrix, and where

$$\sigma_i = \begin{cases} 1 & \text{if } 1 \leq i \leq m \\ -1 & \text{if } m+1 \leq i \leq m+n . \end{cases}$$


Consequently, we have
\[ \sum_{i=1}^{m+n} X_{ii} = 0, \]  
(2.3)
and the Cartan subalgebra \( h \) of \( sl(m|n) \), consisting of the diagonal matrices in \( sl(m|n) \), is spanned by the elements \( X_{ii} \). The usual basic linear forms on the Cartan subalgebra of \( gl(m|n) \), associating to a diagonal matrix its \( i \)th diagonal element, yield by restriction to \( h \) the linear forms \( \varepsilon_i \in h^* \) given by
\[ \varepsilon_i(X_{jj}) = \delta_{ij} - \frac{1}{m-n} \sigma_j, \]  
(2.4)
which satisfy the equation
\[ \sum_{i=1}^{m+n} \sigma_i \varepsilon_i = 0. \]  
(2.5)
In terms of the \( \varepsilon_i \), every linear form \( \Lambda \) on \( h \) (in particular, any weight of an \( sl(m|n) \)–module) can be uniquely written in the form
\[ \Lambda = \sum_{i=1}^{m+n} L_i \varepsilon_i, \]  
(2.6)
with scalars \( L_i \) such that
\[ \sum_{i=1}^{m+n} L_i = 0. \]  
(2.7)
In fact, one has
\[ L_i = \Lambda(X_{ii}). \]  
(2.8)
With a slight abuse of notation, we shall write
\[ \Lambda = (L_1, L_2, \ldots, L_m|L_{m+1}, L_{m+2}, \ldots, L_{m+n}). \]  
(2.9)
Let us also mention that we are going to use the so–called distinguished system of simple roots.

Next we add a few comments on gradations. As is well–known, the algebra \( sl(m|n) \) has a natural \( \mathbb{Z} \)–gradation which is consistent with its \( \mathbb{Z}_2 \)–gradation \(^3\). Consequently, \( U(sl(m|n)) \) is \( \mathbb{Z} \)–graded as well. These \( \mathbb{Z} \)–gradations are easily described by means of the element
\[ D = -\sum_{j=1}^{m} X_{jj} = \sum_{k=m+1}^{m+n} X_{kk} = \frac{1}{m-n} \text{diag}(n, n, \ldots, n; m, m, \ldots, m). \]  
(2.10)
In fact, if \( X \in sl(m|n) \) or \( X \in U(sl(m|n)) \), then \( X \) is homogeneous of \( \mathbb{Z} \)–degree \( r \) if and only if
\[ \langle D, X \rangle = rX. \]  
(2.11)
(Here and in the following \( \langle \, , \rangle \) denotes the super commutator.)
On the other hand, if $V$ is a graded $sl(m|n)$–module (in the $\mathbb{Z}_2$–graded or $\mathbb{Z}$–graded sense), we can shift the gradation and obtain another graded $sl(m|n)$–module (see Ref. [4] or also Ref. [1]). If $V$ is finite–dimensional and simple, its gradation can be fixed by specifying the degree of a highest weight vector. In the following, we shall mainly be interested in finite–dimensional simple modules $V$ for which the coefficients $L_i$ of the highest weight $\Lambda$ are integral. In this case, there is a preferred choice of the gradation: It is given by demanding that an element $x \in V$ be homogeneous of $\mathbb{Z}$–degree $r$ if and only if
\[
D \cdot x = r \, x
\] (2.12)
(where the dot denotes the action of $D$ on $x$ in $V$). Then, for any finite–dimensional simple graded subquotient $V$ of $U(sl(m|n))$ (endowed with the adjoint action), the $\mathbb{Z}$–gradation of $V$ induced from that of $U(sl(m|n))$ is exactly the $\mathbb{Z}$–gradation specified by Eq. (2.12). Thus in the following the gradation will normally be given by that equation. (The vector module $W$ is an exception: For this module we normally choose $W_0 = W_\bar{0}$ and $W_1 = W_\bar{1}$, with $\dim W_0 = m$ and $\dim W_1 = n$, however, in Sec. 3 we make a different choice.) Actually, for our purposes the precise specification of the gradations is, to some extent, not even necessary: A shift of the gradation of the module of coefficients simply results in a shift of the gradation of the cohomology groups [1].

In the following (up to the appendix), we consider the algebras $sl(m|1)$ with $m \geq 2$ and set
\[
L = sl(m|1). \tag{2.13}
\]
We note that in this case we have
\[
D = X_{m+1,m+1}. \tag{2.14}
\]
(The case of the algebra $sl(2|1) \simeq sl(1|2)$ has been considered in more detail in Ref. [1].)

Our goal is to prove Eq. (1.1) for $L = sl(m|1)$. We are going to proceed as in Ref. [1] and show that
\[
H^2(L, V) = \{0\} \tag{2.15}
\]
for all finite–dimensional simple (graded) subquotients $V$ of the $L$–module $U(L)$.

Let $V = V(\Lambda)$ be a finite–dimensional simple $L$–module with highest weight
\[
\Lambda = (L_1, L_2, \ldots, L_m|L_{m+1}), \tag{2.16}
\]
where the $L_i$ satisfy Eq. (2.7). According to Ref. [1], we have
\[
H^n(L, V) = \{0\} \quad \text{for all } n \tag{2.17}
\]
whenever $\Lambda$ does not belong to one of the following two families of weights:
\[
\begin{align*}
(0) & \quad (p, 1, \ldots, 1| - p - (m - 1)) \quad \text{with } p \geq 1 \text{ integral,} \tag{2.18} \\
(1) & \quad (0, \ldots, 0, -q|q) \quad \text{with } q \geq 0 \text{ integral.} \tag{2.19}
\end{align*}
\]
The families are labelled by the number \(k \in \{0,1\}\) appearing in their definition.

Recall that these are the highest weights of those finite–dimensional simple \(L\)–modules for which all Casimir operators without a constant term are equal to zero.

Quite generally, if \(V\) is a module of this type, then its dual is likewise. In the present case, the module of the family \((1)\) with \(q = 0\) is trivial and hence self–dual, and it can be shown that the modules of the families \((0)\) and \((1)\) with \(p = q \geq 1\) are dual to each other. In particular, if \(V\) is a simple module of the family \((1)\) with highest weight \((0, \ldots, 0, -p|p)\) (where \(p \geq 1\) is an integer), then the representative \(D_V\) of \(D\) in \(V\) has exactly the eigenvalues

\[
p, p + 1, \ldots, p + m - 1,
\]

and the corresponding eigenvalues in a simple module belonging to the family \((0)\) are the numbers opposite to those in (2.20).

On the other hand, the eigenvalues of \(D_{U(L)}\) are the numbers \(0, \pm 1, \ldots, \pm m\). Thus, from the simple modules in the families \((0)\) and \((1)\), only the trivial module with highest weight \((0, \ldots, 0|0)\) and the two contragredient modules with the highest weights \((1, \ldots, 1|m)\) resp. \((0, \ldots, 0, -1|1)\) can be isomorphic to a finite–dimensional simple subquotient of \(U(L)\). Consequently, our claim will be proved if we can show that Eq. (2.15) holds for each of these three modules.

Actually, our task can be simplified even more. As is well–known, the mapping

\[
\tau : L \rightarrow L , \quad \tau(A) = -^{st}A
\]

(where \(^{st}A\) is the super transpose of \(A\)) is an automorphism of the Lie superalgebra \(L\). Moreover, if \(V\) is a finite–dimensional simple \(L\)–module and if \(\rho\) is the representation afforded by \(V\), then the representation \(\rho \circ \tau\) is equivalent to the representation contragredient to \(\rho\). But if \(V^\tau\) denotes the graded vector space \(V\), endowed with the representation \(\rho \circ \tau\), then we know that \(H^n(L, V)\) and \(H^n(L, V^\tau)\) are isomorphic (see Eq. (2.34) of Ref. [1]). The upshot is that we only have to consider the trivial module and the simple module with highest weight \(-\varepsilon_m + \varepsilon_{m+1}\).

### 3 Completion of the proof of our claim

According to the previous section we have to show that the 2–cohomology of \(L\) with coefficients in the trivial module \(V(0) = K\) and in \(V(-\varepsilon_m + \varepsilon_{m+1})\) is trivial.

The case of the trivial \(L\)–module \(K\) is easy (see also Ref. [3]). According to Prop. 2.1 of Ref. [3] every cohomology class in \(H^2(L, K)\) contains an \(L_0\)–invariant cocycle. A short look at the representations of \(L_0\) carried by \(L_0\) and \(L_{\pm 1}\) shows that there exists, up to the normalization, a unique non–zero super–skew–symmetric \(L_0\)–invariant bilinear form on \(L\). Obviously,

\[
(A, B) \rightarrow \text{Tr}(\langle A, B \rangle)
\]

(3.1)
is such a form, but this form is a coboundary. Thus we have shown that

$$H^2(L, \mathbb{K}) = \{0\}.$$  \hfill (3.2)

The case of the module

$$V = V(-\varepsilon_m + \varepsilon_{m+1})$$  \hfill (3.3)

is more difficult. First of all, we need a suitable realization of this module. Since we are going to use the Lie superalgebra $gl(m|1)$, we introduce the abbreviation

$$G = gl(m|1).$$  \hfill (3.4)

Let $W$ be the vector module of $G$, but endowed with shifted $\mathbb{Z}$– and $\mathbb{Z}_2$–gradations such that

$$W_{-1} = W_1 , \quad \dim W_1 = m$$  \hfill (3.5)

$$W_0 = W_0 , \quad \dim W_0 = 1$$  \hfill (3.6)

(the reason for this unusual choice will become obvious below, see Remark 3.1), moreover, let $S(W, \varepsilon)$ be the super–symmetric algebra constructed over $W$ (with $\varepsilon$ the standard commutation factor of supersymmetry; see Ref. [4]). We don’t write a product sign for the multiplication in $S(W, \varepsilon)$. It is well–known that there exists a natural representation $\varrho_0$ of $G$ in $S(W, \varepsilon)$, defined such that, for every $A \in G$, the representative $\varrho_0(A)$ is the unique super derivation of $S(W, \varepsilon)$ which extends $A$.

Let $\varrho$ denote the representation obtained from $\varrho_0$ by a certain twist:

$$\varrho(A) = \varrho_0(A) - \text{Str}(A)id$$  \hfill (3.7)

(where Str denotes the super trace). Obviously, the components $S_n(W, \varepsilon)$ are invariant under $\varrho_0$ and $\varrho$, and it is easy to see that $S_{m-1}(W, \varepsilon)$, endowed with the representation of $L$ induced by $\varrho$, is isomorphic to the $L$–module $V$. This is the realization of $V$ that we are going to use in the sequel.

More explicitly, let $(\theta_i)_{1 \leq i \leq m}$ be a basis of $W_1$, let $z$ span the one–dimensional space $W_0$, and let $\frac{\partial}{\partial \theta_i}$ and $\frac{\partial}{\partial z}$ be the corresponding super derivations of $S(W, \varepsilon)$. Then $\varrho$ is given by

$$\varrho(E_{i,j}) = \theta_j \frac{\partial}{\partial \theta_i} - \delta_{ij}$$ \hfill (3.8)

$$\varrho(E_{i,m+1}) = \theta_i \frac{\partial}{\partial z}$$ \hfill (3.9)

$$\varrho(E_{m+1,i}) = z \frac{\partial}{\partial \theta_i}$$ \hfill (3.10)

$$\varrho(E_{m+1,m+1}) = z \frac{\partial}{\partial z} + 1$$ \hfill (3.11)
where \( i, j \in \{1, 2, \ldots, m\} \).

Remark 3.1. At this point it should be clear why we have chosen the gradation of \( W \) as given in Eqs. (3.5), (3.6). With this choice, the \( \theta_i \) are fermionic variables, and \( z \) is bosonic in the usual sense. In particular, \( z \) commutes with the \( \theta_i \). We could also work with the standard gradation of \( W \). Then \( S(W, \varepsilon) \) must be replaced by the super–Grassmann algebra constructed over \( W \), the \( \theta_i \) are still fermionic and \( z \) is bosonic, however, now \( z \) anticommutes with the \( \theta_i \). In principle, there is nothing wrong with this choice, but we wanted to avoid this unusual situation.

Regarded as an \( L_0 \)-module, (in fact, also as a \( G_0 \)-module,) \[ V = S_{m-1}(W, \varepsilon) \] (3.12) decomposes into \[ V = \bigoplus_{r=1}^{m} V_r , \] (3.13) where \[ V_r = z^{r-1} \bigwedge^{m-r} W_1 \] (3.14) is a simple \( L_0 \)-module with highest weight \[ -\varepsilon_{m-r+1} - \varepsilon_{m-r+2} - \ldots - \varepsilon_m + r \varepsilon_{m+1} \] (3.15) and highest weight vector \[ z^{r-1}\theta_1\theta_2\ldots\theta_{m-r}. \] (3.16) Note that according to Eq. (2.12) \( V_r \) is the \( \mathbb{Z} \)-homogeneous component of \( V \) of degree \( r \). The gradation inherited from \( W \) is obtained from this one by a shift.

We now are ready to determine \( H^2(L, V) \). Once again by Prop. 2.1 of Ref. [1], every cohomology class in \( H^2(L, V) \) contains an \( L_0 \)-invariant cocycle. To find these cocycles, we make a detour via \( G = gl(m|1) \). Let \[ f : L \times L \to V \] (3.17) be any bilinear mapping. Define the bilinear mapping \[ \tilde{f} : G \times G \to V \] (3.18) by setting \[ \tilde{f}(A, B) = f(A, B) \quad \text{for all } A, B \in L \] (3.19) \[ \tilde{f}(I, C) = -\tilde{f}(C, I) = 0 \quad \text{for all } C \in G \] (3.20) (recall that \( I \) denotes the \((m+1) \times (m+1)\) unit matrix). Then it is easy to see that \( f \) is an \( L_0 \)-invariant 2–cocycle if and only if \( \tilde{f} \) is a \( G_0 \)-invariant 2–cocycle (note that \( I \) acts on \( V \) by the zero operator). Consequently, it is sufficient to determine the
$G_0$–invariant 2–cocycles on $G$ (with values in $V$); the $L_0$–invariant 2–cocycles on $L$
are then simply obtained by restriction.

Let

$$g : G \times G \to V$$

(3.21)

be a super–skew–symmetric $G_0$–invariant bilinear mapping. According to our conventions, the invariance of $g$ under $D$ says that $g$ is homogeneous of degree 0 in the sense of $\mathbb{Z}$–gradations. Thus if $r$ and $s$ are two elements of $\{−1, 0, 1\}$, it follows that

$$g(G_r \times G_s) \subset V_{r+s}$$

(3.22)

and hence that

$$g(G_r \times G_s) = \{0\} \quad \text{if } r + s \leq 0. \quad (3.23)$$

Moreover, the restriction of $g$ to $G_1 \times G_1$ must be symmetric. But a look at the $\mathfrak{sl}(m)$–module structures of $S_2(G_1)$ (the symmetric tensor product of $G_1$ with itself) and $V_2$ shows that a non–zero symmetric $\mathfrak{sl}(m)$–invariant bilinear mapping of $G_1 \times G_1$ into $V_2$ does not exist. Thus we have

$$g(G_1 \times G_1) = \{0\}, \quad (3.24)$$

and all we have to do is to find the restriction of $g$ onto $G_0 \times G_1$, say.

To construct the $G_0$–invariant bilinear mappings $G_0 \times G_1 \to V_1$, we first define, for $i \in \{1, 2, \ldots, m\}$,

$$\eta_i = (-1)^{i-1}\theta_1 \ldots \hat{\theta}_i \ldots \theta_m = \frac{\partial}{\partial \theta_i}(\theta_1 \theta_2 \ldots \theta_m) \quad (3.25)$$

(as usual, the sign $\hat{\cdot}$ indicates that the element below it must be omitted). Obviously, the $\eta_i$ form a basis of $V_1$, moreover, we have

$$g(E_{ij})\eta_k = -\delta_{ik}\eta_j \quad (3.26)$$

for all $i, j, k \in \{1, 2, \ldots, m\}$. Combined with the known actions of $D$ and $I$, this shows explicitly that the $G_0$–modules $G_1$ and $V_1$ are isomorphic.

Using this information as well as the standard representation theory of $\mathfrak{sl}(m)$, we now can describe the super–skew–symmetric $G_0$–invariant bilinear mappings $G \times G \to V$, as follows. Define three bilinear maps $g_1, g_2, g_3$ of $G \times G$ into $V$ by

$$g_1(E_{i,j}, E_{m+1,k}) = -g_1(E_{m+1,k}, E_{i,j}) = \delta_{ik}\eta_j \quad (3.27)$$

$$g_2(E_{i,j}, E_{m+1,k}) = -g_2(E_{m+1,k}, E_{i,j}) = \delta_{ij}\eta_k \quad (3.28)$$

$$g_3(E_{m+1,m+1}, E_{m+1,k}) = -g_3(E_{m+1,k}, E_{m+1,m+1}) = \eta_k \quad (3.29)$$

where $i, j, k \in \{1, 2, \ldots, m\}$, and with the understanding that the values of $g_1, g_2, g_3$ on the remaining pairs of the standard basis elements of $G$ are equal to zero. Then
$g_1, g_2, g_3$ are super–skew–symmetric and $G_0$–invariant, and any bilinear map $g : G \times G \to V$ with these properties is a linear combination of them.

Now suppose that $g : G \times G \to V$ is a $G_0$–invariant 2–cocycle. Using the $G_0$–invariance of $g$ as well as the fact that $g$ vanishes on $G_0 \times G_0$, it is easy to see that

$$(\delta^2 g)(A, B, C) = g(\langle A, B \rangle, C)$$

(3.30)

for all $A, B \in G_0$ and $C \in G$. Hence the cocycle condition implies that $g$ vanishes on $sl(m) \times G$. Consequently, $g$ must be a linear combination of $g_2$ and $g_3$,

$$g = ag_2 + bg_3.$$  

(3.31)

A short calculation then shows that

$$(\delta^2 g)(E_{k,m+1}, E_{m+1,i}, E_{m+1,j}) = -(a + b)(\delta_{ki} \eta_j + \delta_{kj} \eta_i)$$

(3.32)

for all $i, j, k \in \{1, 2, \ldots, m\}$, which implies that

$$a + b = 0.$$  

(3.33)

Without loss of generality we now may assume that

$$a = 1,$$  

(3.34)

and then we have

$$g(A, E_{m+1,k}) = \text{Str}(A) \eta_k$$

(3.35)

for all $A \in G$ and all $k \in \{1, 2, \ldots, m\}$. Consequently, $g$ vanishes on $L \times L$, and according to our previous discussion, this implies that

$$H^2(L, V) = \{0\},$$

(3.36)

as claimed.

We close this section by the remark that $g$ as specified above is a 2–cocycle on $G$, and that $g$ is not a 2–coboundary. Thus we have

$$\dim H^2(G, V) = 1.$$  

(3.37)

4 Discussion

In the present paper we have shown that

$$H^2(L, U(L)) = \{0\}$$

(4.1)

for the Lie superalgebras $L = sl(m|1)$ with $m \geq 2$. Our method of proof was the following.
Because of the long exact cohomology sequence \([1]\), a sufficient (but not necessary) condition for (4.1) to hold is that

\[ H^2(L, V) = \{0\} \tag{4.2} \]

for all simple subquotients \(V\) of \(U(L)\) (these are automatically finite-dimensional).

Let \(\Lambda\) be the highest weight of \(V\). To prove that Eq. (4.2) holds for the modules \(V\) in question, we first used the results of Example 4 in Sec. 3 of Ref. \([1]\) (and hence Prop. 2.2 of that reference) to conclude that Eq. (4.2) is true if \(\Lambda\) does not belong to the families (0) and (1) defined by Eqs. (2.18), (2.19). By comparing the eigenvalues of \(D_{U(L)}\) and \(D_V\) we could then reduce the problem to the consideration of just three cases, finally, by using the automorphism (2.21) of \(L\), even to two cases. One of these cases was the trivial module, for which Eq. (4.2) could be proved immediately. In the other case, we found a nice realization of \(V\), which made the necessary calculations simple.

In view of our experience with \(sl(3|2)\) (see the appendix) it must be said that the case of the algebras \(sl(m|1)\) is particularly favourable. In more general cases (already for \(sl(3|2)\)) we certainly need more information on the adjoint \(L\)-module \(U(L)\) than just the eigenvalues of \(D_{U(L)}\). Also, most of the modules \(V\) one finally has to consider will not be well-known, and it may be very hard to find a suitable realization for them. (Recall that, at least for the \(sl(m|n)\) algebras with \(m \neq n\), these modules are maximally atypical \([1]\).)

All this seems to indicate that in our approach too many details are needed, and that more profound methods are necessary to solve our problem for more general algebras.

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Appendix

A Problems with \(sl(3|2)\)

Once we had proved Eq. (1.1) for the algebras \(L = sl(m|1); m \geq 2\), we intended to investigate the algebras \(sl(m|n); m \neq n\), in general. In order to see what type of problems would arise, we first considered the algebra \(sl(3|2)\). Unfortunately, up to
now we have not been able to prove or disprove Eq. (1.1) in this case. Nevertheless, we think it may be worthwhile to present some of our intermediate results, in order to show the obstacles one has to overcome if one wants to proceed along the lines described in Ref. [1] and in the present paper.

In this appendix, we set
\[ L = sl(3|2) \]  
(A.1)
and use the notation introduced at the beginning of Sec. 2. We try to proceed as in Secs. 2 and 3. Let \( V(\Lambda) \) be a finite–dimensional simple \( L \)–module with highest weight
\[ \Lambda = (L_1, L_2, L_3|L_4, L_5), \]  
(A.2)
where the \( L_i \) satisfy Eq. (2.7). According to Ref. [1], we have
\[ H^n(L, V(\Lambda)) = \{0\} \quad \text{for all } n \]  
(A.3)
whenever \( \Lambda \) does not belong to one of the following three families of weights:
\begin{align*}
(0) & \quad (p, q, 2| - q - 1, -p - 1) \quad \text{with } p \geq q \geq 2 \\
(1) & \quad (p, 1, q| - q, -p - 1) \quad \text{with } p \geq 1 \geq q \\
(2) & \quad (0, p, q| - q, -p) \quad \text{with } 0 \geq p \geq q
\end{align*}
(A.4) (A.5) (A.6)
where, in all cases, \( p \) and \( q \) are integers.

We already know that the class of simple \( L \)–modules with these highest weights is closed under taking duals. The following list gives, on the same line, for each of the weights appearing in Eqs. (A.4) – (A.6), the highest weight of the corresponding dual module:
\begin{align*}
(p, q, 2| - q - 1, -p - 1), & \quad (0, 1 - q, 1 - p|p - 1, q - 1) \quad \text{with } p > q \geq 2 \\
(p, p, 2| - p - 1, -p - 1), & \quad (0, -p, -p|p, p) \quad \text{with } p \geq 2 \\
(p, 1, q| - q, -p - 1), & \quad (1 - q, 1 - p|p - 1, q - 2) \quad \text{with } p \geq 1 > q \\
(p, 1, 1| - 1, -p - 1), & \quad (0, 0, 1 - p|p - 1, 0) \quad \text{with } p \geq 2 \\
(1, 1, 1| - 1, -2), & \quad (0, -1, -1|1, 1) \quad \text{(A.7)} \\
(0, 0, 0|0, 0), & \quad (0, 0, 0|0, 0) \quad \text{(A.8)}
\end{align*}
(A.7) (A.8)
We note that, in this list, a weight appears twice if and only if the corresponding module is self–dual. Let us also mention that, apart from the trivial module, every module of the class (2) is dual to some of the modules in the classes (0) or (1). Similarly, every module of the class (0) is dual to some module of the class (2).

Next we note that the representative of \( D \) under the adjoint representation in \( U(L) \) has exactly the eigenvalues \( 0, \pm 1, \ldots, \pm 6 \). Consequently, for any subquotient \( V \) of \( U(L) \), the eigenvalues of the representative \( D_V \) must belong to this set. To apply this condition, we note that, for any weight \( \Lambda \) given by the Eqs. (2.6), (2.7), we have
\[ \Lambda(D) = -L_1 - L_2 - L_3 = L_4 + L_5. \]  
(A.13)
At this point, we meet a first complication. Whereas only a few modules of the types (0) and (2) pass the criterion above, there are infinitely many modules of type (1) which satisfy it. Thus we have to find some other conditions to narrow down the number of possibilities. Unfortunately, we don’t have any more detailed information on the structure of the $L$–module $U(L)$. Consequently, for the time being, we simply ignore the fact that the modules we are finally interested in are simple subquotients of $U(L)$. Instead, we consider all the simple $L$–modules $V(\Lambda)$, with $\Lambda$ a weight of the types (0), (1), (2) above, and we try to find a simple necessary criterion implying that $H^2(L, V(\Lambda)) = \{0\}$.

This can be achieved as follows. Once again because of Prop. 2.1 of Ref. [1], we are only interested in the $L_0$–invariant 2–cocycles with values in $V(\Lambda)$. In particular, such a cocycle can be identified with an $L_0$–module homomorphism of $L \wedge L$ (the super–exterior square of the adjoint $L$–module) into $V(\Lambda)$. But a non–zero homomorphism of this type exists if and only if there is at least one simple $L_0$–submodule of $L \wedge L$ which is isomorphic to an $L_0$–submodule of $V(\Lambda)$.

The $L_0$–module structure of $L \wedge L$ is easily determined: Suffice it to say that $L \wedge L$ decomposes into the direct sum of 27 simple $L_0$–submodules (not all non–isomorphic, of course).

The $L_0$–module structure of the modules $V(\Lambda)$ is much more difficult to obtain. Information on these modules could be extracted from the conjectured character formula in Ref. [6] or from the character formula proved in Ref. [7]. However, we haven’t tried to do that but rather argue more directly, as follows.

Let $\bar{V}(\Lambda)$ be the Kac module with highest weight $\Lambda$ (see Ref. [8]). If there is a simple $L_0$–submodule of $L \wedge L$ which is isomorphic to a submodule of $V(\Lambda)$, this is even more the case with $V(\Lambda)$ replaced by $\bar{V}(\Lambda)$ (since $V(\Lambda)$ is a quotient of $\bar{V}(\Lambda)$). But the structure of the $L_0$–module $\bar{V}(\Lambda)$ can be determined by the standard representation theory of $L_0$, and then it is not difficult to check whether the condition above is satisfied. (All this is a straightforward but cumbersome task: Note that $\bar{V}(\Lambda)$ may be the direct sum of up to 64 simple $L_0$–submodules.)

The calculation sketched above yields a finite list of highest weights $\Lambda$. But there is still one more observation to be made. Obviously, the $L$–module $L \wedge L$ is self–dual. Thus if there exists a non–trivial $L_0$–module homomorphism of $L \wedge L$ into $V(\Lambda)$, then there is also one into the dual module $V(\Lambda)^*$. Since we have been working with the Kac modules $\bar{V}(\Lambda)$, there may be — and indeed are — weights $\Lambda$ in the list above for which the highest weight of the dual module $V(\Lambda)^*$ is not in the list. All these weights $\Lambda$ may be excluded as well.

The upshot of all this is that, for any finite–dimensional simple $L$–module $V(\Lambda)$ with highest weight $\Lambda$, the inequality $H^2(L, V(\Lambda)) \neq \{0\}$ implies that $\Lambda$ is one of the following weights:

\begin{align}
(0, 0, 0) & \quad \text{[A.14]} \\
(1, 1, 1) & \quad \text{[A.15]}
\end{align}

\begin{align}
& (0, -1, -1, 1, 1) \\
& (0, -1, -1, 1, 1)
\end{align}
\[(2,1,1|-1,-3), \ (0,0,-1|1,0) \quad \text{(A.16)}\]
\[(1,1,0|0,-2) \quad \text{(A.17)}\]
\[(3,1,1|-1,-4), \ (0,0,-2|2,0) \quad \text{(A.18)}\]
\[(2,1,0|0,-3), \ (1,1,-1|1,-2) \quad \text{(A.19)}\]

and, in addition,
\[(3,1,0|0,-4), \ (1,1,-2|2,-2) \quad \text{(A.20)}\]
\[(2,1,-1|1,-3). \quad \text{(A.21)}\]

As before, if two weights stand on the same line, the corresponding $L$–modules are dual to each other; if there is only one weight on a line, the corresponding $L$–module is self–dual. The reader can easily convince himself/herself that, for each of these weights, the representative $D_V(\Lambda)$ takes its eigenvalues in the set \{-6, -5, \ldots, 6\}. Hence the assumption that $V(\Lambda)$ is isomorphic to a subquotient of $U(L)$ doesn’t imply any further restrictions.

Our next task would be to calculate $H^2(L,V(\Lambda))$ for the weights $\Lambda$ given above. To do this we need much more information on the modules $V(\Lambda)$. In particular, we need the $L_0$–module structure of these modules. In a painstaking analysis of the corresponding Kac modules $\tilde{V}(\Lambda)$, we have determined how the modules $V(\Lambda)$ decompose when regarded as $L_0$–modules. As a by–product, we have also found the composition factors of the Kac modules themselves. It turns out that the $V(\Lambda)$ in question admit a unique decomposition into simple $L_0$–submodules, i.e., all the simple $L_0$–modules contained in $V(\Lambda)$ have multiplicity one.

The more detailed information thus obtained allows us to rule out the weights in Eqs. (A.20), (A.21). For these weights, $V(\Lambda)$ doesn’t contain a simple $L_0$–submodule which is isomorphic to an $L_0$–submodule of $L \wedge L$. Thus we are left with the weights in Eqs. (A.14) – (A.19). Using the automorphism (2.21) for $L = \mathfrak{sl}(3|2)$, we only have to consider one of the weights on each line. Thus, there are six cases to consider.

The first case is the trivial $L$–module $\mathbb{K}$ with the highest weight (A.14). It is known from Ref. [5] and easy to see by means of Prop. 2.1 in Ref. [1] that
\[H^2(L, \mathbb{K}) = \{0\}. \quad \text{(A.22)}\]

The weights in (A.13) correspond to the covector and vector modules of $L$. Somewhat unexpectedly, for these modules $V$ we have
\[\dim H^2(L, V) = 1. \quad \text{(A.23)}\]

This is bad news: It shows that in order to prove Eq. (1.1) for $L = \mathfrak{sl}(3|2)$ we need more detailed information on the structure of the $L$–module $U(L)$.

At this point, we have changed our strategy: Maybe $H^2(L,U(L))$ is different from \{0\}. In order to show this we recall that $U(L)$, regarded as an $L$–module,
is canonically isomorphic to the super–symmetric algebra $S(L, \varepsilon)$ (see Ref. [9]). It is well–known that $S(L, \varepsilon)$ decomposes into the direct sum of its $\mathbb{Z}$–homogeneous components $S_n(L, \varepsilon); n \geq 0,$ and that these are $L$–submodules of $S(L, \varepsilon)$, moreover, $S_n(L, \varepsilon)$ is canonically isomorphic to the submodule of super–symmetric tensors in $L^{\otimes n}$. In particular, the submodule $S_0(L, \varepsilon)$ is isomorphic to $\mathbb{K}$, and we already know that $H^2(L, \mathbb{K})$ is trivial. The submodule $S_1(L, \varepsilon)$ is isomorphic to the adjoint $L$–module, and its highest weight $(1, 0, 0|0, -1)$ is not of type $(0), (1)$, or $(2)$. Hence $H^2(L, L)$ is trivial as well.

Thus we consider $S_2(L, \varepsilon)$. A detailed analysis shows that the $L$–module $S_2(L, \varepsilon)$ (uniquely) decomposes like

$$S_2(L, \varepsilon) \simeq V(2, 0, 0|−1, −1) \oplus V(1, 0, 0|0, −1) \oplus W,$$  \hspace{1cm} (A.24)

with an indecomposable but non–simple $L$–module $W$. The module $W$ has a Jordan–Hölder series

$$W = W_0 \supset W_1 \supset W_2 \supset W_3 = \{0\}$$  \hspace{1cm} (A.25)

such that $W/W_1$ and $W_2$ are trivial one–dimensional $L$–modules and such that

$$W_1/W_2 \simeq V(1, 1, 0|0, −2).$$  \hspace{1cm} (A.26)

Note that the latter module has the highest weight given in (A.17).

The module $W_2$ consists of the $L$–invariant elements in $S_2(L, \varepsilon)$, it is spanned by the so–called (quadratic) tensor Casimir (split Casimir) element $C$. (It is known that any $L$–invariant element of $S_2(L, \varepsilon)$ is proportional to $C$: Otherwise, $L$ would have two linearly independent quadratic Casimir elements or, equivalently, two linearly independent super–symmetric $L$–invariant bilinear forms.)

On the other hand, let $G$ be an $L_0$–invariant element of $W$ which does not belong to $W_1$. Any other element $G'$ with these properties has the form

$$G' = aG + bC$$  \hspace{1cm} (A.27)

with a non–zero constant $a$ and an arbitrary constant $b$. According to the preceding characterization of $W_2$, the element $G$ is not $L$–invariant. Actually, $G$ generates the $L$–module $W$ (but, of course, it is not a highest weight vector). Let us also note that $W_1$ is the unique maximal and $W_2$ the unique minimal (i.e., simple) $L$–submodule of $W$.

Remark A.1. Obviously, the module $S_2(L, \varepsilon)$ is self–dual, and so are the modules $V(2, 0, 0|−1, −1)$ and $V(1, 0, 0|0, −1)$ (the latter is isomorphic to the adjoint $L$–module). A moment’s thought then shows that $W$ is self–dual as well. This explains part of the structure of $W$. Note that a similar but even more complicated structure also exists for $sl(1|2)$ (see Eq. (B.3) of Ref. [1]).

The weights $(2, 0, 0|−1, −1)$ and $(1, 0, 0|0, −1)$ do not belong to the families $(0), (1), (2)$. Thus we have

$$H^n(L, S_2(L, \varepsilon)) \simeq H^n(L, W) \quad \text{for all } n,$$  \hspace{1cm} (A.28)
and a rather tedious calculation shows that
\[ H^n(L, W) = \{0\} \quad \text{for } n = 1, 2. \quad (A.29) \]
As is easily guessed, a lot of detailed knowledge about the modules \( V(1, 1, 0|0, -2) \)
and \( W \) is necessary to prove this result.

Thus the situation is rather unpleasant: We know of simple \( L \)-modules (the vector and covector modules) for which the 2–cohomology is non–trivial, but since we don’t have sufficient information on the \( L \)-module \( U(L) \), we do not know what this fact implies for \( H^2(L, U(L)) \). On the other hand, one of the candidates for a non–trivial cohomology (namely \( V(1, 1, 0|0, -2) \)) really is isomorphic to a subquotient of \( U(L) \), but this does not imply that \( H^2(L, U(L)) \) is non–trivial. (We stress that we have not shown that the 2–cohomology of \( L \) with values in \( V(1, 1, 0|0, -2) \) is trivial.)

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