On the number of real eigenvalues of products of random matrices and an application to quantum entanglement

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Abstract
The probability that there are $k$ real eigenvalues for an $n$-dimensional real random matrix is known. Here, we study this for the case of products of independent random matrices. Relating the problem of the probability that the product of two real two-dimensional random matrices has real eigenvalues to an issue of optimal quantum entanglement, this is fully analytically solved. It is shown that in $\pi/4$ fraction of such products the eigenvalues are real. Being greater than the corresponding known probability ($1/\sqrt{2}$) for a single matrix, it is shown numerically that the probability that all eigenvalues of a product of random matrices are real tends to unity as the number of matrices in the product increases indefinitely. Some other numerical explorations, including the expected number of real eigenvalues, are also presented, where an exponential approach of the expected number to the dimension of the matrix seems to hold.

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(Some figures may appear in colour only in the online journal)

While the study of the spectra of random matrices has been extensive and applications have been too numerous and varied to state briefly [1], that of products of random matrices is relatively less extensive, even though it is well motivated [2–4]. For example, products of random matrices could describe Jacobian matrices of chaotic systems and the rate of exponential increase of the largest eigenvalue gives the Lyapunov exponent. A similar consideration arises for problems involving disordered systems, with the matrices being transfer operators and the Lyapunov exponent being localization lengths. Recent studies of the spectral properties of products of random matrices include [5, 6].

The Ginibre ensemble of random matrices is the simplest to construct, as these are $n \times n$ matrices with all the $n^2$ entries being i.i.d. random variables drawn from a normal distribution such as $N(0, 1)$ with zero mean and unit variance [7]. If the entries are complex, then the real and imaginary parts are independent random variables. For the purposes of this work,
attention is restricted to the real ensemble. It is known that the eigenvalues of such matrices can have a significant fraction of eigenvalues that are themselves real. Explicit expressions for \( p_{n,k} \), the probability that \( k \) eigenvalues are real for a random \( n \times n \) real matrix, have been found. Although these are not simple, there are elegant formulae for \( E_n \), the expected number of real eigenvalues as well as the probability that there are exactly \( n \) real eigenvalues \([8–10]\). For example, it is known that \( \lim_{n \to \infty} E_n / \sqrt{n} = \sqrt{2}/\pi \) and \( p_{n,n} = 2^{-m(n-1)/4} \) \([8]\).

Therefore, it is interesting to study the number of real eigenvalues of products of random matrices. If there are \( K \) matrices in the product, then let the probability that it has \( k \) real eigenvalues be denoted by \( p_{n,k}^{(K)} \). It is shown below that \( p_{2,2}^{(K)} = \pi/4 \), and therefore it is larger than the probability that a two-dimensional random matrix has real eigenvalues, which is \( p_{2,2} = 1/\sqrt{2} \). Numerical results indicate that \( p_{2,2}^{(K)} \) monotonically increases to 1 as \( K \) increases to \( \infty \); thus, the probability that there are real eigenvalues increases with the number of matrices in the product. Numerical results also indicate identical conclusions for matrices of dimensions larger than 2, namely that \( p_{n,n}^{(K)} \) is a monotonically increasing function of \( K \) and seems to tend to unity. The distribution of the matrix elements for \( K > 1 \) is naturally not independent, but that the correlations lead to this is a somewhat surprising result. Readers not interested in quantum entanglement may go directly to the paragraph following equation (8).

One direct application of the result for \( p_{2,2}^{(K)} \) to a problem in quantum entanglement \([11]\) is to find the fraction of real ‘optimal’ states \([12–14]\) of rank 2. A set of pure states of two qubits \(|\phi_i\rangle, i = 1, \ldots, k \rangle \) are C-optimal if for any \( \{p_i, i = 1, \ldots, k, \sum_i p_i = 1, \ p_i > 0 \} \) one has

\[
C \left( \rho = \sum_{i=1}^{k} p_i |\phi_i\rangle \langle \phi_i| \right) = \sum_{i=1}^{k} p_i C (|\phi_i\rangle \langle \phi_i|),
\]

with \( C(\cdot) \) being the concurrence function \([12, 15]\), a measure of entanglement between the two qubits. In general, the rhs is larger than the lhs, the concurrence being a convex function, and in this sense the set of states leads to optimally entangled mixtures if the equality is satisfied.

Restricting oneself to the set of states that are real in the standard basis, it was shown in \([14]\) that when \( k = 2 \), a large fraction \((\approx 0.285)\) of pairs of states were in fact C-optimal. The sampling of states is such that each of the real states is chosen from a uniform distribution on the unit 3-sphere \( S^3 \), which simply arises from the normalization of the four real components. Strong evidence was provided that the number 0.285 . . . , obtained initially from numerical simulations, was in fact \((\pi - 2)/4\). Below it is shown that this is in fact \( p_{2,2}^{(K)} - 1/2 \), whose evaluation then confirms the result. For completeness, we state that when \( k = 3 \) about 5.12\% of triples were C-optimal, while it was also shown that there was not even one quadruplet of real states that were so. Therefore, the set of complex states is necessary for there to be C-optimal states in general. For \( k > 2 \) though, there does not seem to be a direct connection to the problem of products of random matrices.

If \( |\phi_1\rangle \) and \( |\phi_2\rangle \) are an optimal pair satisfying equation (1), then we refer to them below as ‘co-optimal’. Such optimal pairs satisfy the following conditions \([14]\):

\[
r_{11}r_{22} > 0, \quad \text{and} \quad r_{11}r_{22} - r_{12}^2 < 0,
\]

where \( r_{ij} = \langle \phi_i | \sigma_j \otimes \sigma_i | \phi_j \rangle \).

\[
(2)
\]

Here \( \sigma_j \) is one of the Pauli matrices. If \( |\phi_i\rangle \) is a real state of two qubits, then the concurrence \( C (|\phi_i\rangle \langle \phi_i|) = |r_{11}| \). Let \( |\phi_i\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \) be a maximally entangled state, so that \( C (|\phi_i\rangle \langle \phi_i|) = 1 \). What characterizes states that are co-optimal with this? If

\[
|\phi_2\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle
\]

(3)
is such a state, then the conditions stated above lead to $r_{11} = -1$, $r_{22} = 2(bc - ad)$ and $r_{12} = (a + d)/\sqrt{2}$. This implies the following:

$$ad - bc > 0, \quad (a + d)^2 - 4(ad - bc) > 0,$$

which are naturally formulated as conditions on the matrix of coefficients

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

being equivalent to the requirement that $\det(M_1) > 0$ and $M_1$ has only real eigenvalues. Of course if $\det(M_1) < 0$, the eigenvalues are anyway real. Thus, a state is co-optimal with a maximally entangled state if its matrix of coefficients has a positive determinant, yet has real eigenvalues.

Formulated as above, the fraction of states that are co-optimal with a maximally entangled state is closely allied to the question of the fraction of $2 \times 2$ real matrices that have real eigenvalues. The matrix elements can be drawn from a normal i.i.d. random process, such as $N(0, 1)$. That this gives us the same answer as sampling uniformly from the normalization sphere is evident, as the question of reality of eigenvalues of a matrix remains independent of overall multiplication by scalars. Thus, we obtain the fraction $f_{a/4}$ of states that are co-optimal with a maximally entangled state to be

$$f_{a/4} = p_{a/2} - 1 = \frac{1}{\sqrt{2}} - \frac{1}{2}.$$

The $-1/2$ arises as the fraction $p_{a/2}$ will also include all instances when $\det(M_1) < 0$, which are to be subtracted, and $\det(M_1)$ is equally likely to be positive or negative. Thus, about 20.7% of real states are co-optimal with a maximally entangled one.

To generalize the above, consider one state as $|\phi_1\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle$, $0 \leq \theta \leq \pi/4$, and the measure $f_0$ of states that are co-optimal with it, $\theta = \pi/4$ being what was just discussed. That any one state of the pair can be chosen such as this follows from the Schmidt decomposition. The uniform (Haar) distribution on the normalization sphere $S^3$ induces an invariant measure, say $\mu(\theta)$. Then, the fraction of pairs of states that are co-optimal is given by

$$\langle f \rangle = \int_0^{\pi/4} f_0 \mu(\theta) \, d\theta.$$  

The conditions of co-optimality of $|\phi_1\rangle$ and a general real two-qubit state $|\phi_2\rangle$, with

$$r_{11} = -\sin 2\theta, \quad r_{22} = 2(bc - ad), \quad r_{12} = a \cos \theta + b \sin \theta,$$

now translate to those on the product:

$$M_2 = \begin{pmatrix} \cos \theta & 0 \\ 0 & \sin \theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

as $\det(M_2) > 0$ and $M_2$ has real eigenvalues. Once again, in this problem it is equivalent to assuming that $(a, b, c, d)$ are i.i.d. $N(0, 1)$ numbers or uniformly distributed on the sphere $a^2 + b^2 + c^2 + d^2 = 1$.

Quite independent of the discussion above, but equivalently, one may start with a product of two random matrices, say $A_1A_2$, and perform a singular value decomposition of $A_1$ to obtain the product $O_1 A_1 O_2^T A_2 = O_1 A_1 O_2^T A_2 O_1^T = O_1 A_1 A_2 O_1^T$. Evidently, the spectrum of the original product is the same as that of $A_1 A_2$. Here, $O_1$ and $O_2$ are orthogonal matrices and $A_1$ is a diagonal matrix with positive elements, and $A_2 = O_2^T A_2 O_1$. Observe that if the elements of a matrix $A$ are i.i.d. $N(0, 1)$ distributed, those of the products $O A$ and $O A$, where $O$ is an arbitrary orthogonal matrix, are also identically distributed. Therefore, it follows that $A_2$ has elements that are i.i.d. $N(0, 1)$ distributed. Hence, one may well begin with the product in equation (8) without any loss of generality.
That the diagonal elements can be so taken, with $0 \leq \theta \leq \pi/4$, and distributed naturally according to the measure $\mu(\theta) = 2\cos 2\theta$, is now shown. The eigenvalues of $A_1A_1^T$ can be chosen as $\lambda > 0$ and $1 - \lambda \leq \lambda$, as division by an overall number, the $\text{tr}(A_1A_1^T)$, does not affect the nature of the reality of the eigenvalues of the product $A_1A_2$. With $A_1$ having elements drawn from an i.i.d. normal random process, namely the Ginibre ensemble, but with the trace restricted to unity, the distribution of the eigenvalues of $A_1A_1^T$ is known in general [16, 17]. For the special case of two-dimensional matrices, the distribution of the larger value can be read off as $P(\lambda) = (2\lambda - 1)/(\sqrt{\lambda(I - \lambda)}$. Hence, with the parametrization that $\lambda = \cos^2 \theta$ as above, the singular value being $\cos \theta$, the distribution $\mu(\theta)$ follows immediately.

Let $p_0$ be the fraction of matrices $M_2$ that have real eigenvalues as $(a, b, c, d)$ are taken from $N(0, 1)$ and $\theta$ is fixed. This is realized each time the discriminant $\Delta_2 = (a \cos \theta + d \sin \theta)^2 - 4 \sin \theta \cos \theta (ad - bc) \geq 0$. This is rewritten as $\Delta_2 = (a \cos \theta - d \sin \theta)^2 + 2 \sin 2\theta \cos 2\theta \geq 0$, which is a condition on the sum of two statistically independent quantities. Using the fact that $x = (a \cos \theta - d \sin \theta)$ is distributed according to $N(0, 1)$ for all $\theta$ enables the following form:

$$p_0 = \int_0^\infty \Theta \left( \frac{\beta^2}{2} \right) \exp\left(\frac{\beta^2}{4} + \frac{\beta z}{2}\right) \frac{dz}{(2\pi)^{3/2}},$$

where $\beta = 1/\sin 2\theta$. Note that as $\theta \to 0$, $\beta \to \infty$, and $p_0 \to 1$. Taking the derivative with respect to $\beta$ converts the Heaviside step function into a Dirac delta function. Effecting a series of simplifications thereafter, including using polar coordinates for $y$ and $z$, results in the following remarkably simple equation:

$$\frac{dp_0}{d\beta} = \frac{1}{4\pi} \int_0^\pi \frac{\sin \phi}{\sin \phi + \beta} d\phi.$$

Integrating with respect to $\beta$ and incorporating the boundary condition at $\theta = 0$ gives

$$p_0 = 1 - \frac{1}{2\pi} \int_0^\pi \sqrt{\sin \phi} d\phi = 1 - \frac{1}{2\pi} \int_0^\pi \frac{(-1)^k \Gamma(k + \frac{1}{2}) \Gamma(\frac{k}{2} + \frac{3}{4})}{k!} \sin 2\theta^{k+\frac{3}{4}} d\phi.

The integral in this equation does not seem to acquire a simple form except when $\beta = 1$, which corresponds to $\theta = \pi/4$ and gives $p_{\pi/4} = 1/\sqrt{2}$, in agreement with the known result, stated previously as $p_{2, 2}$. It follows that $f_0 = p_0 - \frac{1}{2}$ is the fraction of states that are co-optimal with the state $\cos \theta |00\rangle + \sin \theta |11\rangle$. Note that as the concurrence in these states is $\sin 2\theta$, the fraction $f_0$ is simply a function of this. One can now use equation (7) to find the fraction of co-optimal pairs. The required invariant measure being $\mu(\theta) = 2\cos 2\theta$ is most well suited to express $\langle f \rangle$ as an infinite series as in equation (11), which may be identified with generalized hypergeometric functions.

Equivalently, one may use the integral in this equation to express $\int_{-\infty}^{\pi/4} p_0 \mu(\theta) d\theta$, the probability that the product of two random $2 \times 2$ matrices has real eigenvalues as

$$p_{2, 2}^{(2)} = \frac{1}{2\pi} \int_0^\pi \frac{\sinh^{-1}(\sin \phi)}{\sin \phi} d\phi.

This follows as the $\theta$ integral can be carried out in an elementary way, and also from the evaluation $\int_0^\pi \sqrt{(\sin x)/\sin x} dx = 2\pi$. The integral in equation (12) does not appear to be in standard tables, nor fully evaluated by mathematical packages, but as indicated from previous work, it is in fact simply $\pi^2/2$. Therefore, it seems interesting enough to warrant
Figure 1. The probability that all eigenvalues of a product of $K$ random $n$-dimensional matrices are real, based on 100,000 realizations.

Both the generalized hypergeometric functions appearing here are of the Saalschütz type, the sum of the top rows being 1 less than the sum of the bottom. Theorem 2.4.4 in [18] can be evoked for such functions, and it is remarkable that this is precisely the form of the rhs of the identity therein, which results in its evaluation as

$$\frac{1}{\sqrt{2\pi}} \Gamma^2\left(\frac{1}{4}\right) {}_3F_2\left[\begin{array}{c}
\frac{1}{4}, \\
\frac{1}{2}, \\
\frac{1}{2}, \\
\frac{5}{4}, \\
\end{array} \frac{1}{2} ; \\
\frac{5}{4} ; 1 \right] - \frac{1}{24\sqrt{2\pi}} \Gamma^2\left(-\frac{1}{4}\right) {}_3F_2\left[\begin{array}{c}
\frac{3}{4}, \\
\frac{3}{4}, \\
\frac{3}{4}, \\
\frac{3}{4}, \\
\end{array} \frac{3}{4} ; \\
\frac{3}{4} ; 1 \right].$$

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$$\sqrt{2\pi} {}_2F_1\left[\begin{array}{c}
\frac{1}{4}, \\
\frac{1}{4} \\
\end{array} \frac{1}{4} ; 1 \right] = \frac{\pi^2}{2}.$$ 

Here an identity of Gauss is used for ${}_2F_1$ at arguments of unity [18] and leads to $p^{(2)}_{2,2} = \pi/4$, and hence finally $\langle f \rangle = (\pi - 2)/4$.

The generalizations, dealt with numerically below, are to products of more than two $2 \times 2$ matrices as well as to higher dimensional matrices and for a variable number of products. The behavior of $p^{(K)}_{2,2}$ for $K \geq 2$ is seen in figure 1 and shows this monotonically increasing with $K$. In the same figure, the corresponding probabilities that all the eigenvalues are real for such products of three- and four-dimensional matrices are also shown. This increase in the probability that all eigenvalues are real is also reflected in the expected number of real eigenvalues. This is shown in figure 2 where this number $E_n^{(K)} = \sum_{k=0}^{n} k p^{(K)}_{n,k}$ is plotted as a function of $n$ for fixed values of number of products $K$ in the top panel. In the bottom panel, the expected number is shown as a function of $K$ for two- and four-dimensional matrices. An exponentially fast approach to the dimension of the matrix is observed here. Thus, $n - E_n^{(K)} \sim \exp(-\gamma_n K)$ seems to hold with $\gamma_n$ decreasing with increasing $n$ ($\gamma_2$ and $\gamma_4$ are the slope values in figure 2). Numerical results also indicate that the probability that there are $k$ real eigenvalues for $k < n$, while not necessarily monotonic, does eventually vanish with the number of products, leaving the dominant case as the one with all eigenvalues real. This is illustrated in figure 3 where the quantity $p^{(K)}_{n,k}$ for $k = 0, 2, 4, 6, 8$ corresponding to the probability of finding $k$ real eigenvalues.
A similar trend is observed for all other numerically tested dimensionalities, and hence there is a strong case that this is true in general.

For a fixed dimensionality, as the number of products increases more eigenvalues ‘condense’ from the complex plane onto the real axis. The distribution of the eigenvalues hence changes significantly as well. For a single random matrix, the eigenvalues are asymptotically distributed according to the circular law [19, 20], while the real eigenvalues are asymptotically uniformly distributed [9]. In figure 4 the eigenvalues of products of ten-dimensional real matrices are shown. This is shown for four values of $K$, namely 1, 2, 5 and 10, and the distortion from an approximately circular law is evident with the formation of two lobes.
The eigenvalues are divided by the norm of the products of the matrices so that the values are not exponentially increasing and can be compared.

In conclusion and summary, the number of real eigenvalues for products of real random matrices has been studied. The case of products of two two-dimensional random matrices was fully analytically solved, and it was shown that in the fraction of $\pi/4$ cases, the matrices had real eigenvalues. This solved a problem of entanglement, where it was shown that the fraction of optimal pairs of two qubit states is therefore $(\pi - 4)/2$. Generalizations show that with increasing number of products, all the eigenvalues tend to be real with probability approaching unity. Moreover, this seems valid for all matrix dimensions. Needless to say, the numerical results pose interesting challenges, as the resulting matrices have highly correlated matrix elements. For one thing, why all the eigenvalues become real is a natural question, and while admittedly tentative, it might have to do with the rapid convergence of vectors under repeated multiplication by random matrices. Thus, it is also interesting to explore connections, if any, between the exponential rates found in this paper and Lyapunov exponents.

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