On deformations of $A_\infty$-algebras

A A Sharapov$^{1,4,6,*}$ and E D Skvortsov$^{2,3,5}$

1 Physics Faculty, Tomsk State University, Lenin ave. 36, Tomsk 634050, Russia
2 Albert Einstein Institute, Am Mühlenberg 1, D-14476, Potsdam-Golm, Germany
3 Lebedev Institute of Physics, Leninsky ave. 53, 119991 Moscow, Russia

E-mail: sharapov@phys.tsu.ru

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Abstract
A simple method for constructing deformations of $A_\infty$-algebras is proposed. It combines the standard resolution technique with a ‘master equation’ approach. As an illustration we consider algebras of polynomials on quantum superspaces. Specifically, by introducing suitable resolutions, we construct explicit deformations of these algebras in the category of minimal $A_\infty$-algebras. The application of these deformations to higher spin gravities is briefly discussed.

Keywords: strong homotopy algebras, deformation theory, quantum polynomial algebra, higher spin gravities

1. Introduction

One of the main concerns of modern algebra is the weakening of various algebraic structures in a coherent way. Thus to each type of ‘classical’ algebras, like associative, Lie, Gerstenhaber etc, one can associate its strong homotopy analog called, respectively, $A_\infty$-, $L_\infty$-, $G_\infty$-, … algebras. While the classical algebras are defined in terms of binary multiplication operations obeying certain relations, the strong homotopy algebras involve the whole family of $n$-ary operations $\{m_n\}$, from 1 to \(\infty\), hence the name$^7$. The operations obey infinite sets of defining relations in such a way that the binary maps $m_2$ satisfy the ‘classical’ relations up to homotopy determined by $m_1$ and $m_3$. As usual, the need for such generalizations of classical algebraic structures stems from various problems of physics and mathematics [1].

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$^6$Author to whom any correspondence should be addressed.
$^7$There are also versions involving $m_0$, they are called non-flat.
In physics, for instance, strong homotopy algebras typically control the structure of classical equations of motion. This is best illustrated by an example of string field theory \[2–4\], where the classical dynamics are governed by the generalized Maurer–Cartan (MC) equation

$$m_1(\Phi) + m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + \cdots = 0. \quad (1.1)$$

Here \(\Phi\) is the string field taking values in the corresponding algebra. The type of the algebra constituted by \(m\)'s depends on string's topology: it is the \(A_\infty\) for open strings and \(L_\infty\) for closed \[5–7\]. In this particular situation the role of the differential \(m_1\) is played by the BRST operator, while the higher structure maps correspond to the tree-level string amplitudes. The defining conditions of strong homotopy algebra reincarnate then in the form of the gauge symmetry transformations

$$\delta_A \Phi = m_1(\Lambda) + m_2(\Phi, \Lambda) + \cdots,$$

\(\Lambda\) being an infinitesimal gauge parameter. Given the relation between the low energy string and field theories it is little wonder that the \(L_\infty\)- and \(A_\infty\)-algebras show up in the structure of conventional field theories as well \[8–10\].

Our interest to strong homotopy algebras is mostly inspired by their applications to higher spin gravities. Like string field theory, higher spin gravities involve infinite collections of fields of all spins, whose interaction is governed by higher spin symmetries. At the level of formal consistency the problem of introducing interactions \[11\] (see \[12–14\] for a review) is known to be equivalent to constructing an appropriate \(L_\infty\)- or \(A_\infty\)-algebra. We call such theories formal higher spin gravities \[15\]. The first structure map \(m_1\) is given now by the exterior differential \(d\) on forms and the truncated system of maps \(\{m_n\}_{n=2}^\infty\) defines a strong homotopy algebra by itself. The homotopy algebras of the form \(\{m_n\}_{n=2}^\infty\) are called minimal. These constitute an important category in the world of homotopy algebras as it is known that each \(L_\infty\)- or \(A_\infty\)-algebra is quasi-isomorphic to a minimal one \[16, 17\]. For minimal algebras the second structure map \(m_2\) satisfies the ‘classical’ relations, so that the equations of motion for massless higher spin fields admit a consistent truncation

$$d\Phi = -m_2(\Phi, \Phi). \quad (1.2)$$

This is nothing but the usual MC equation associated to a differential graded algebra, the higher spin algebra. From the physical viewpoint, equation (1.2) describes the dynamics of free higher spin fields, even though the right hand side contains the fields non-linearly. The genuine interaction vertices come from the higher structure maps \(m_n, n \geq 3\). By construction, these deform the free gauge symmetry of equation (1.2) in a consistent way, so that the full nonlinear system possesses the same number of physical degrees of freedom\(^8\). Thus, given the form of free field equation (1.2), the problem of switching on formally consistent interactions appears to be equivalent to the deformation of the underlying higher spin algebra in the category of minimal algebras.

Abstract considerations of the deformation problem for strong homotopy algebras can be found in \[18, 19\]. Here we are concerned with developing practical methods for constructing deformations of \(A_\infty\)-algebras, particularly minimal deformations of graded associative algebras. Let us outline our approach to the problem. Given an \(A_\infty\)-algebra, we first construct its resolution. This is defined by a pair of compatible \(A_\infty\)-structures forming a double complex,

\(^8\)It is important to stress that \(L_\infty\)- or \(A_\infty\)-algebras solve only the problem of formal consistency and further (physical) restrictions are necessary in order to have well-defined equations. Still the existence of higher spin gravities is well justified on the basis of CFT dual descriptions, see e.g. \[20\].
see section 3. The choice of a resolution is highly ambiguous. This ambiguity, however, provides some flexibility when dealing with particular algebras. As with any double complex, we can define the total $A_\infty$-structure, which, by construction, is quasi-isomorphic to the original one. If the resolution is ‘good enough’, the total $A_\infty$-structure admits a plenty of linear deformations, i.e. formal deformations that terminate at the first order. In many interesting cases such deformations are easy to construct and classify, especially, if one restricts to the class of non-flat deformations. The use of non-flat linear deformations is the key point of our approach. The desired deformation of the original $A_\infty$-algebra is then induced by a linear deformation of the total $A_\infty$-structure. In order to carry out the final step we propose a set of ‘master equations’, whose solutions provides us with explicit recurrent formulas for deformations, as explained in section 4.

In the last section 4, we illustrate our method by constructing minimal deformations of polynomial algebras on quantum superspaces. This class of algebras, being of some interest on its own, is closely related to higher spin algebras in various dimensions. In particular, we compute the cohomology relevant to the minimal deformations of these algebras and make comments on physical interpretation of some other cocycles.

2. $A_\infty$-algebras and their deformations

Throughout this paper, $k$ is an arbitrary field of characteristic zero and all unadorned tensor products $\otimes$ and Homs are taken over $k$. We start with reminding some basic definitions and constructions concerning $A_\infty$-algebras.

Let $V = \bigoplus V^l$ be a $\mathbb{Z}$-graded vector space and let $T(V) = \bigoplus_{n \geq 0} V^\otimes n$ denote its tensor algebra with the convention that $T^0(V) = k$. The spaces $T(V)$ and $\text{Hom}(T(V),V)$ naturally inherit the grading of $V$. Furthermore, the $\mathbb{Z}$-graded vector space $\text{Hom}(T(V),V) = \bigoplus \text{Hom}^l(T(V), V)$ is known to carry the structure of a graded Lie algebra with respect to the Gerstenhaber bracket [21]. This is defined as follows. Given a pair of homogeneous homomorphisms $f \in \text{Hom}(T^n(V), V)$ and $g \in \text{Hom}(T^m(V), V)$, we set

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f,$$

(2.1)

where

$$(f \circ g)(v_1 \otimes v_2 \otimes \cdots \otimes v_{m+n-1}) =$$

$$= \sum_{i=0}^{n-1} (-1)^{|g|} \sum_{|\eta|=i} |\eta| f(v_1 \otimes \cdots \otimes v_i \otimes g(v_{i+1} \otimes \cdots \otimes v_{i+n}) \otimes \cdots \otimes v_{m+n-1})$$

and $|g|$ stands for the degree of $g$ as a linear map between graded vector spaces\(^9\).

The Gerstenhaber bracket is graded skew-symmetric,

$$[f, g] = - (-1)^{|f||g|}[g, f],$$

and satisfies the graded Jacobi identity

$$[[f, g], h] = [f, [g, h]] - (-1)^{|f||g|}[g, [f, h]].$$

\(^9\)Here we follow [6] in defining the degree of multi-linear maps. The conventional $\mathbb{Z}$-grading [21] on $\text{Hom}(T(V), V)$ is related to ours by suspension: $V \rightarrow V[-1], V[-1]^l := V^{l+1}$. Of course, this results in additional minus signs in the definition of the Gerstenhaber bracket.
We denote this graded Lie algebra by $L$.

By definition, an $A_\infty$-structure on $V$ is given by an element $m \in \text{Hom}^1(T(V), V)$ of degree 1 satisfying the MC equation

$$[m, m] = 0.$$  \tag{2.2}

The pair $(V, m)$ is called the $A_\infty$-algebra. Expanding the MC element $m$ into the sum $m = m_0 + m_1 + m_2 + \cdots$ of homogeneous multi-linear maps $m_n \in \text{Hom}(T^n(V), V)$ and substituting it back into (2.2), we get an infinite sequence of homogeneous relations on $m$’s, known as Stasheff’s identities [22]. An $A_\infty$-algebra is called flat if $m_0 = 0$. (By definition, the zero structure map $m_0 = m_0(1)$ is just an element of $V^1$.) In the flat case, the first structure map $m_1 : V^1 \to V^2$ squares to zero, $[m_1, m_1] = 2m_1^2 = 0$, making $V$ into a complex of vector spaces. A flat $A_\infty$-algebra is called minimal if $m_1 = 0$. For minimal algebras the second structure map $m_2 : V \otimes V \to V$ makes the space $V[-1]$ into a graded associative algebra with respect to the product

$$uv = (-1)^{|u|m_2|u \otimes v|}, \quad (2.3)$$

The associativity condition is encoded by the Stasheff identity $[m_2, m_2] = 0$. From this perspective, a graded associative algebra is just an $A_\infty$-algebra with $m = m_2$. More generally, an $A_\infty$-algebra with $m = m_1 + m_2$ is equivalent to a differential graded algebra $(V[-1], d)$ with the dot product (2.3) and the differential $d = m_1$. Again, the Leibniz rule

$$d(uv) = (du)v + (-1)^{|u|-1} uv d$$

is equivalent to the Stasheff identity $[m_1, m_2] = 0$.

In this paper, we are interested in formal deformations of $A_\infty$-algebras. Let

$$L = L \otimes k[[t]] = \bigoplus_{n=0}^{\infty} \text{Hom}(T^n(V), V) \otimes k[[t]]$$

(2.4)

denote the completed tensor product of $L$ and $k[[t]]$, with $t$ being a formal deformation parameter. The $\mathbb{Z}$-grading and the Gerstenhaber bracket on $L$ extend naturally to the space $\mathcal{L} = \bigoplus L^n$ making the latter into a graded Lie algebra over $k[[t]]$. By definition,

$$[f \otimes \alpha, g \otimes \beta] = [f, g] \otimes \alpha \beta, \quad \alpha(g \otimes \beta) = g \otimes \alpha \beta$$

for all $f, g \in L$ and $\alpha, \beta \in k[[t]]$. The natural augmentation $\epsilon : k[[t]] \to k$ induces the $k$-homomorphism $\pi : \mathcal{L} \to L$ of graded Lie algebras that sends the deformation parameter to zero. The MC elements of the algebra $\mathcal{L}$ are naturally identified with $A_\infty$-structures on the $k[[t]]$-vector space $\mathcal{V} = V \otimes k[[t]]$. By definition,

$$(f \otimes \alpha_0)(v_1 \otimes \alpha_1, \ldots, v_n \otimes \alpha_n) = f(v_1, \ldots, v_n) \otimes \alpha_0 \alpha_1 \cdots \alpha_n$$

for all $f \in \text{Hom}(T^n(V), V)$, $v_i \in V$ and $\alpha_j \in k[[t]]$.

We say that an $A_\infty$-structure $m'$ on $\mathcal{V}$ is a deformation of an $A_\infty$-structure $m$ on $V$ if $\pi(m') = m$. In other words, the deformed $A_\infty$-structure has the form

$$m' = m + m^{(1)}t + m^{(2)}t^2 + \cdots, \quad m^{(n)} \in L.$$  \tag{2.5}

A deformations $m'$ is said to be trivial if there exists an element $f \in \mathcal{L}^0$ such that

$$m' = e^f m e^{-f} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\text{ad}_f)^n m.$$
We are interested in explicit constructions of nontrivial deformations.

3. Deformations by means of resolutions

We say that two $A_{\infty}$-structures $m'$ and $m''$ on the same vector space $V$ are compatible, if

$$[m', m''] = 0. \quad (3.1)$$

In this case, the sum $m = m' + m''$ is again an $A_{\infty}$-structure (actually, any linear combination of the two gives an $A_{\infty}$-structure).

Suppose now that the $\mathbb{Z}$-grading on $V$ comes from a bi-grading, that is, $V = \bigoplus V^{p,q}$ and $V' = \bigoplus_{p+q=1} V^{p,q}$. We will denote the bi-degree of a homogeneous element $a \in V$ by $\deg a = (p,q)$ and refer to $|a| = p + q$ as the total degree of $a$. The double gradation of $V$ allows one to consider $A_{\infty}$-structures that are homogeneous with respect to the first and second degrees. Let us consider a pair of compatible $A_{\infty}$-structures $m'$ and $m''$ of bi-degrees

$$\deg m' = (1,0), \quad \deg m'' = (0,1). \quad (3.3)$$

In the special case that $m' = m'_1$ and $m'' = m''_1$ the double $A_{\infty}$-algebra $(V, m', m'')$ degenerates to a double complex of vector spaces. In the following we will mostly interested in the case where the second $A_{\infty}$-structure is given simply by a differential $d = m''_1$, while the first one is arbitrary. Then the compatibility condition (3.1) takes the form

$$dm'_1(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = - \sum_{i=1}^k (-1)^{|v_1|+\cdots+|v_{i-1}|} m'_1(v_1 \otimes \cdots \otimes dv_i \otimes \cdots \otimes v_k) \quad \text{for all } k = 0, 1, \ldots, \text{and } v_i \in V. \quad (3.3)$$

By construction, the complex $(V,d)$ splits into the direct sum of subcomplexes $(V^{p,q},d)$ labeled by the first degree. Let us further assume that $V^{p,q} = 0$ for all $q < 0$, and that the differential $d : V^{p,q} \to V^{p,q+1}$ is acyclic in positive $q$ degrees, that is, $H^q(V,d) = H^q(V,d) = 0$. Due to the compatibility and degree conditions the first $A_{\infty}$-structure $m'$ on $V$ can be consistently restricted onto the subspace $W = H^0(V,d)$ making the latter into an $A_{\infty}$-algebra. Let us denote this restriction by $m'_W = m'|_W$. In this situation we say that the triple $(V, m', d)$ is a resolution of the $A_{\infty}$-algebra $(W, m_W)$ and refer to the second degree as the resolution degree.

The main idea behind our approach is to deform the $A_{\infty}$-algebra $(W, m_W)$ by deforming its suitable resolution. The construction goes as follows.

Summing up the compatible $A_{\infty}$-structures, we endow $V$ with the `total’ $A_{\infty}$-structure $m = m' + d$. Then, following the general philosophy of deformation theory [24], we introduce the differential graded Lie algebra (DGLA) $\mathcal{L} = \bigoplus \mathcal{L}^n$, where

$$\mathcal{L}^n = \bigoplus_{p+q=n} \mathcal{L}^{p,q}, \quad \mathcal{L}^{p,q} = \text{Hom}^{p,q}(T(V), V) \otimes k[[t]].$$

Notice that we include into $\mathcal{L}$ only homomorphisms of non-negative resolution degree. The Lie bracket in $\mathcal{L}$ is given by the Gerstenhaber bracket, while the adjoint action of $m$ endows $\mathcal{L}$ with the differential

$$\partial = [m,-], \quad \partial : \mathcal{L}^n \to \mathcal{L}^{n+1}. \quad (3.3)$$
Let $\mathcal{L} = t\mathcal{L}$ denote the differential ideal of $\mathcal{L}$ that governs the formal deformations of the $A_\infty$-algebra $(V, m)$.

As the next step, we evaluate the cohomology of the DGLA $(\mathcal{L}, \partial)$. To this end, we split the differential into the sum $\partial = \partial' + \partial''$ of the vertical and horizontal differentials

$$\partial' = [m', -], \quad \partial'' = [d, -]$$

and, using the bicomplex structure, endow $\mathcal{L}$ with a decreasing filtration associated to the first degree:

$$F^p\mathcal{L} = \bigoplus_{s \geq p} \bar{c}^s \mathcal{L}, \quad F^p\mathcal{L} \supset F^{p+1}\mathcal{L}, \quad \bigcup_{p \in \mathbb{Z}} F^p\mathcal{L} = \mathcal{L}.$$  

Since the resolution degree is bounded below, the filtration is regular and yields a spectral sequence

**Proposition 3.1.** The cohomology of the complex $(\mathcal{L}, \partial'')$ is centered in resolution degree zero, so that $H^q_{\partial''}(\mathcal{L}) = 0$ for $q > 0$.

**Proof.** To compute the groups $H^q_{\partial''}(\mathcal{L})$, we split the complex $(V, d)$ by introducing a contracting homotopy $h: V^{p,q} \to V^{p+1,q}$ together with the inclusion $i: W \to V$ and the projection $p: V \to W$ mappings associated to the subspace $W = H^0(V, d) \subset V$. Without loss in generality we may assume that

$$hd + dh = 1_V - ip, \quad h^2 = 0, \quad pi = 1_W, \quad ph = 0, \quad hi = 0.$$  

Then the operator $h$ induces a contracting homotopy $\tilde{h}: \mathcal{L}^{\bullet,q} \to \mathcal{L}^{\bullet,q-1}$ defined by

$$(\tilde{h}f)(v_1 \otimes \cdots \otimes v_n) = hf(v_1 \otimes \cdots \otimes v_n). \quad (3.4)$$

It is clear that

$$\tilde{h}\partial + \partial\tilde{h} = 1_{\mathcal{L}} - \tilde{i}p, \quad \tilde{h}^2 = 0,$$

where the operators $\tilde{i}$ and $\tilde{p}$ are defined similar to (3.4). Since $\ker(1 - ip) = W$, this means that any nontrivial cocycle (homomorphisms) of $(\mathcal{L}, \partial'')$ is cohomologous to one taking values in the subspace $W$. On the other hand, any homomorphism of $\mathcal{L}$ with values in $W$ has resolution degree zero and is automatically a nontrivial $\partial''$-cocycle. Thus,

$$H^q_{\partial''}(\mathcal{L}) \simeq H^q_{\tilde{p}}(\mathcal{L}) = \ker \partial'' \cap \mathcal{L}^{\bullet,0}. \quad (3.5)$$

We see that all the nonzero groups $E^1_{1,q}$, and hence $E^2_{q,q}$, are nested on the base $(q = 0)$. As a result the spectral sequence collapses at the second term giving the isomorphism

$$H^q(\mathcal{L}, \partial) \simeq E^2_{1,0} = H^q_{\tilde{p}}H^0_{\partial''}(\mathcal{L}), \quad (3.6)$$

see e.g. [23, theorem 4.4.1].

Rel. (3.5) identifies the group $H^q_{\partial''}(\mathcal{L})$ with the subspace $N = \ker \partial'' \cap \mathcal{L}^{\bullet,0} \subset \mathcal{L}$. Actually, $N$ is not just a subspace but a differential subalgebra of $(\mathcal{L}, \partial)$ as one can easily see. Notice also that $m' \in \mathcal{N} = \ker \partial \cap \mathcal{L}^{\bullet,0}$ and $\partial|_{\mathcal{N}} = \partial'$. This allows us to interpret Rel. (3.6) as an isomorphism of the cohomology groups:

$$H^q(\mathcal{L}, \partial) \simeq H^q(\mathcal{N}, \partial').$$

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In other words, the natural inclusion $\alpha : \bar{N} \to \bar{L}$ is a quasi-isomorphism of DGLA’s, so that the deformation problems for the $A_\infty$-structures $m$ and $m'$ are equivalent.

Finally, it is well known that each quasi-isomorphism of two DGLA’s admits a quasi-inverse homomorphism, inducing an isomorphism in cohomology. Let

$$\beta : \bar{L} \to \bar{N}$$

be quasi-inverse to $\alpha : \bar{N} \to \bar{L}$. With the help of $\beta$ we can map the MC elements of $(\bar{L}, \partial)$ to those of $(\bar{N}, \partial')$: If $\gamma$ an MC element of $\bar{L}$, then

$$\mu = \beta(\gamma) = \sum_{k=1}^{\infty} \frac{1}{k!} \beta_k(\gamma, \ldots, \gamma)$$

(3.8)

is an MC element of $\bar{N}$, with $\beta_k : \bar{L} \otimes k \to \bar{N}$ being some multi-linear maps. By construction,

$$[d, \mu] = 0$$

and each MC element (3.8) admits a consistent restriction to the subspace $W \subset V$, in the sense that $T(W) \subset T(V)$ and $\mu : T(W) \to W$. Let us denote this restriction by $\mu|_W$. Combining the quasi-isomorphism (3.7) with the restriction map, we can deform the original $A_\infty$-structure $m_W = m'|_W$ on $W$ by the formula

$$m'_W = m_W + \beta(\gamma)|_W$$

(3.9)

for any MC element $\gamma \in \bar{L}$.

In this paper, we focus upon a special class of deformations of $(V, m)$ that are represented by straight lines in the MC space of $(\bar{L}, \partial)$. Any such deformation is defined by an $A_\infty$-structure $\lambda$ which is compatible with $m$, i.e.

$$[\lambda, \lambda] = 0, \quad [m, \lambda] = 0.$$  

(3.10)

This ensures that the formal line

$$m' = m + t\lambda$$

(3.11)

defines a family of $A_\infty$-structures on $V$. Formula (3.9) yields then a formal deformation of the $A_\infty$-structure on $W$:

$$m'_W = m_W + \beta(t\lambda)|_W.$$  

(3.12)

As is seen from (3.8), the resulting $A_\infty$-structure $m'_W$ may contain higher orders in $t$, defining a formal curve rather than a line in the MC space.

A simple observation concerning the linear deformations (3.11) is that we can always satisfy the quadratic relation (3.10) by choosing $\lambda \in \text{Hom}(T^0(V), V)$. Having no arguments, the ‘homomorphism’ $\lambda$ automatically satisfies the first equation in (3.10) and we are left with the linear condition. The latter can easily be analyzed in many practical cases. For example, let $A = (V, m')$ be a graded associative algebra with $m' = m'_2$ and let $\lambda \in V$. Then the second equation in (3.10) tells us that

$$d\lambda = 0, \quad [m'_2, \lambda] = 0.$$  

In other words, $\lambda$ is just a $d$-cocycle belonging to the center of the associative algebra $A$. By construction, any such cocycle gives rise to a deformation of the associative algebra $A_W = (W, m_W)$ in the category of $A_\infty$-algebras. It is worth noting that the deformed $A_\infty$-structure (3.12) may well be flat, while its preimage (3.11) is not.
4. The master equations

In this section, we apply the method of our recent work \[25\] to construct the MC element 
\[\mu = \beta(t\lambda) \in \mathcal{L}^{1,0}\] determining the formal deformation (3.12). The main ingredient of our
approach is the operator 
\[\partial_{\mu} = \partial + [\mu, -]\]
together with the pair of auxiliary elements \(\Gamma \in \mathcal{L}^0\) and \(\Lambda \in \mathcal{L}^1\) of total degrees 0 and 1. The
unknowns \(\Gamma, \Lambda\) and \(\mu\) are supposed to satisfy the pair of master equations
\[\dot{\mu} = \Lambda - \partial_{\mu}\Gamma, \quad \dot{\Lambda} = [\Gamma, \Lambda]\] (4.1)
supplemented by the initial condition
\[\Lambda|_{t=0} = \lambda.\] (4.2)
Here the overdot stands for the derivative w.r.t. the formal parameter \(t\). The name and relevance of these equations to our problem are explained by the next statement.

**Lemma 4.1.** The element \(\mu \in \mathcal{L}^{1,0}\) defined by equations (4.1) and (4.2) satisfies the MC
equation
\[\partial_{\mu} = -\frac{1}{2}[\mu, \mu]\]
whenever
\[\partial_{\lambda} = 0, \quad [\lambda, \lambda] = 0.\]

**Proof.** Let us denote
\[R = \partial_{\mu} + \frac{1}{2}[\mu, \mu], \quad T = \partial_{\mu}\Lambda, \quad S = [\Lambda, \Lambda].\]
Applying the operator \(\partial_{\mu}\) to both sides of the master equation (4.1), we find
\[\dot{R} = T + [\Gamma, R], \quad \dot{T} = S + [\Gamma, T],\] (4.3)
provided that \(\Lambda, \Gamma,\) and \(\mu\) obey (4.1). Acting by \(N\) on \(S\) and using the master equations once again, we get one more relation
\[\dot{S} = [\Gamma, S].\] (4.4)
Taken together, equations (4.3) and (4.4) constitute a closed system of linear ODEs
\[\dot{R} = T + [\Gamma, R], \quad \dot{T} = S + [\Gamma, T], \quad \dot{S} = [\Gamma, S].\]
Clearly, the equations have a unique solution \(R = 0, T = 0,\) and \(S = 0\) subject to the initial conditions
\[R(0) = 0, \quad T(0) = \partial_{\lambda} = 0, \quad S(0) = [\lambda, \lambda] = 0.\]

It remains to show that the master equation (4.1) do have a solution.
Lemma 4.2. Equations (4.1) and (4.2) have a unique solution satisfying the additional conditions
\[ \tilde{h} \Gamma = 0, \quad \tilde{p} \Gamma = 0, \]
where the operators $\tilde{h}$ and $\tilde{p}$ are defined by equation (3.4).

Proof. Let us expand $\Gamma$ and $\Lambda$ in homogeneous components:
\[ \Gamma = \sum_{n=0}^{\infty} \Gamma_n, \quad \Lambda = \sum_{n=0}^{\infty} \Lambda_n, \]
where
\[ \deg \Gamma_n = (-n, n), \quad \deg \Lambda_n = (1 - n, n). \]
On substitution of these expansions into the master equations (4.1), we obtain the system of homogeneous equations
\[ \dot{\mu} = \Lambda_0 - \partial'_\mu \Gamma_0, \]
\[ \partial'' \Gamma_n + \partial'_\mu \Gamma_{n+1} = \Lambda_{n+1}, \quad n = 0, 1, 2, \ldots, \]
\[ \dot{\Lambda}_n = \sum_{m=0}^{n} [\tilde{h}(\partial'_\mu \tilde{h})^m \Lambda_{n-k+1}], \quad n = 0, 1, 2, \ldots. \]
Here we introduced the shorthand notation $\partial'_\mu = \partial' + [\mu, -]$. Applying the contracting homotopy operator (3.4) to equation (4.8) and using conditions (4.5), we can formally solve (4.8) for $\Gamma$ as
\[ \Gamma_n = \sum_{k=0}^{\infty} (-\tilde{h}\partial'_\mu)^k \tilde{h} \Lambda_{n+k+1}. \]
Substituting this expression into the remaining equations (4.7) and (4.9), we get the system of ODEs
\[ \dot{\mu} = \sum_{m=0}^{\infty} (-\partial'_\mu \tilde{h})^m \Lambda_m, \quad \dot{\Lambda}_n = \sum_{k=0}^{n} \sum_{m=0}^{\infty} [\tilde{h}(\partial'_\mu \tilde{h})^m \Lambda_{k+m+1}, \Lambda_{n-k}], \]
where the overdot stands for the derivative in $t$. These last equations can be solved by iterations giving a unique solution subject to the initial conditions $\mu(0) = 0$ and $\Lambda(0) = \lambda$. In particular, if
\[ \lambda = \sum_{r=0}^{M} \lambda_r \]
is the expansion of $\lambda$ with respect to the resolution degree, then the first-order deformation is determined by

$$\dot{\mu}(0) = \sum_{r=0}^{M} (-\partial' \bar{h})^r \lambda_r.$$  \hfill (4.13)

The expression for the second-order deformation is more cumbersome. Differentiating the first equation in (4.11) and setting $t = 0$, we find

$$\ddot{\mu}(0) = - \sum_{m=0}^{M} \sum_{k=0}^{m-1} (-\partial' \bar{h})^k [\dot{\mu}(0), \bar{h}(-\partial' \bar{h})^{m-k-1} \lambda_m] + \sum_{m=0}^{M} (-\partial' \bar{h})^m \dot{\Lambda}_m(0),$$  \hfill (4.14)

where

$$\dot{\Lambda}_n(0) = \sum_{k=0}^{n} \sum_{m=0}^{M-k-1} [\bar{h}(-\partial' \bar{h})^m \lambda_{k+m+1}, \lambda_{n-k}]$$  \hfill (4.15)

and $\dot{\mu}(0)$ is given by (4.13). As is seen all the sums are finite and this property holds true in higher orders.

**Remark 4.3.** In case $\lambda \in \text{Hom}(T^0(V), V)$, equation (4.15) can be written as

$$\dot{\Lambda}_n(0) = \bar{h} \sum_{k=0}^{n} \sum_{m=0}^{M-k-1} (-\partial' \bar{h})^m \lambda_{k+m+1} \circ \lambda_{n-k}.$$  \hfill (4.16)

If in addition $\lambda_0 = 0$, then the second sum in (4.14) vanishes.

**Remark 4.4.** In the above proof, the convergence of the series (4.10) followed a posteriori, after solving the differential equations. In many interesting cases, however, it can be ensured a priori. Suppose, for example,

$$V = \bigoplus_{n \geq 0} V^{p,q},$$

that is, the first degree of homogeneous vectors is non-positive and bounded below by $-m$. Then both expansions in (4.6) are finite and the series (4.10) contains only finite number of terms. Such a situation occurs in the case of bimodules ($m = 1$) as discussed in section 5.

The space $\mathcal{L}$, being defined by (2.4), admits a natural decreasing filtration $F^p \mathcal{L} \supset F^{p+1} \mathcal{L}$, where

$$F^p \mathcal{L} = \prod_{n+2m \geq p} \text{Hom}(T^n(V), V) \otimes t^m[[t]].$$  \hfill (4.16)

If we now assume that the $A_\infty$-structure $m'$ is minimal, then the operator $\partial'_\mu = \partial' + [\mu, -]$ strictly increases the filtration degree and the series (4.10) is well defined as an element of the filtered space $\mathcal{L}$. 

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A A Sharapov and E D Skvortsov
5. Examples of deformations

In this section, we illustrate the above machinery of deformations by applying it to some bimodules over polynomial and Weyl algebras and to quantum polynomial superalgebras. Our interest to this class of examples is not purely algebraic. As indicated in example 5.2 below, these algebras and their deformations are of primary importance for higher spin gravities.

5.1. Minimal deformations of bimodules

Let us start with some general remarks. Given an associative algebra $A$ and an $A$-bimodule $M$, one can define a new associative algebra $\mathcal{A}$, called the trivial extension of $A$ by the bimodule $M$. As a vector space $\mathcal{A} = A \oplus M$ and multiplication is defined by the formula

$$(a,m)(a',m') = (aa', am' + ma') \quad \forall a, a' \in A, \quad \forall m, m' \in M.$$ 

If no extra structure is assumed, one may only deform the pair $(A, M)$ in the category of bimodules over associative algebras. This is the concern of classical deformation theory. Notice, however, that the algebra $\mathcal{A}$ admits a natural grading. This is obtained by prescribing the spaces $A$ and $M$ the degrees 0 and 1, respectively. When treated as a graded associative algebra, $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ may have nontrivial deformations in the category of $A_\infty$-algebras. We say that the deformation is minimal if the resulting $A_\infty$-algebra is minimal. As a particular case, this includes the deformation problem for the original bimodule structure.

In the following, we restrict our consideration to a rather special yet important class of bimodules that originate from polynomial algebras endowed with automorphisms. Let $V$ be an $n$-dimensional vector space over $k$ and let $\vartheta : V \to V$ be an automorphism of $V$. The action of $\vartheta$ on $V$ induces an automorphism of the dual space $V^*$, which then extends to an automorphism of the symmetric algebra $A = S(V^*)$. Let $^\vartheta a$ denote the result of the action of $\vartheta$ on $a \in A$. Given the automorphism $\vartheta$, we can view the $k$-vector space $A$ as an $A$-bimodule with respect to the following left and right actions:

$$a \circ m = am, \quad m \circ a = m^\vartheta a \quad \forall a, m \in A.$$ 

As is seen the right action of $A$ on itself is twisted by $\vartheta$. We denote this $A$-bimodule by $A^\vartheta$. In a similar way one can introduce a left-twisted bimodule $^\vartheta A$.

We are interested in constructing deformations of the bimodule $A^\vartheta$ in the category of minimal $A_\infty$-algebras. As explained in section 3, this can be done by means of a suitable resolution of the associated graded algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, where $\mathcal{A}^0 = \Lambda$ and $\mathcal{A}^1 = A^\vartheta$. Quite apparently, the choice of a resolution is highly ambiguous and different resolutions may generate different classes of deformations. Below, we consider only two simple constructions.

5.2. Polynomial and Weyl bimodules

Given a symmetric algebra $A = S(V^*)$, we introduce the algebra of endomorphisms $\text{Hom}(A, A)$ of the $k$-vector space $A$, the product being the composition of endomorphisms. Letting $\Lambda(V)$ denote the exterior algebra of $V$, we define the algebra $B = \text{Hom}(A, A) \otimes \Lambda(V)$. The standard grading on $\Lambda(V)$ makes $B$ into a graded associative algebra with $B^\ell = \text{Hom}(A, A) \otimes \Lambda^\ell(V)$.

Choosing linear coordinates $\{x^i\}$ on $V$ and $\{p_i\}$ on $V^*$, we can identify $S(V^*)$ with the algebra of polynomials $k[x^1, \ldots, x^n]$. Then the $k$-vector space $\text{Hom}(A, A)$ appears to be isomorphic to the space of formal power series in $p$’s with coefficients in polynomial functions.
in $x$’s. Upon this identification, the composition of two endomorphisms $a(x,p)$ and $b(x,p)$ is described by the Moyal-type product\(^{10}\)

$$a \bullet b = a \exp \left( \frac{\partial}{\partial p} \right) b,$$

(5.1)

and homogeneous elements of $B^j$ are represented by differential forms

$$\omega = \omega^{i_1 \cdots i_l}(x,p) dp_{i_1} \wedge \cdots \wedge dp_{i_l}.$$ (5.2)

By abuse of notation, we use the same symbol $\bullet$ to denote the multiplication in $B$.

The usual exterior differential $d : B^l \to B^{l+1}$ with respect to $p$’s,

$$d\omega = \frac{\partial \omega^{i_1 \cdots i_l}}{\partial p_j} dp_j \wedge dp_{i_1} \wedge \cdots \wedge dp_{i_l},$$ (5.3)

makes $B$ into a differential graded algebra. Using the standard contracting homotopy $h : B^l \to B^{l-1}$,

$$h\omega = \int_0^1 dt t^{l-1} \omega^{i_1 \cdots i_l}(x,tp) p_{i_1} dp_{i_1} \wedge \cdots \wedge dp_{i_l},$$ (5.4)

one can see that the differential (5.3) is acyclic in positive degrees and $H^0(B,d) \simeq A$. Thus, the triple $(B, \bullet, d)$ provides us with a resolution of the associative algebra $A = S(V^*)$. To make contact with the notation of the previous sections, one should shift the degree of $B$ by $-1$ in order for the $\bullet$-product to define the map $m' = m_2'$, see (2.3). We will write $B = B[1]$ for the desuspension of $B$.

By dimensional reason, all the $A_\infty$-structures on $B$ must belong to the subspace $\bigoplus_{k=0,1,2} \text{Hom}(T^k(B),B)$. In particular, the $A_\infty$-structures of the space $\text{Hom}(T^1(B),B)$ are given by 2-forms, being necessarily of resolution degree 2, are represented by 2-forms

$$\lambda = \lambda^0(x,p) dp_i \wedge dp_j.$$ (5.5)

These forms automatically satisfy the defining condition $[\lambda, \lambda] = 0$, while compatibility with the $\bullet$-product requires $\lambda$ to lie in the center of the algebra $B$. Since $Z(B) = k \otimes \Lambda(V) \simeq \Lambda(V)$, the compatible $A_\infty$-structures of $\text{Hom}(T^1(B),B)$ are given by 2-forms (5.5) with constant coefficients. Hence, $d\lambda = 0$. Using the contracting homotopy (5.4), one can readily see that the first-order deformation (4.13) of $A$ is given by the Poisson bracket

$$\mu^{(1)}(a, b) = \frac{1}{2} \lambda^0 \frac{\partial a}{\partial x^i} \frac{\partial b}{\partial x^j}, \quad \forall a, b \in k[x^1, \ldots, x^n].$$

The whole deformation, being constructed by formulas (4.11), reproduces the Moyal $\ast$-product

$$a \ast b = a \exp \left( \frac{1}{2} \lambda^0 \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) b.$$ (5.6)

In order to construct more interesting examples of minimal deformations, e.g. involving higher structure maps, we should extend the algebra $A$ by its bimodule $A^\ast$. A suitable resolution of the extended algebra $\mathcal{A} = A \otimes A^\ast$ is obtained as follows. The action of $\partial$ on $V$ induces the action on the dual space $V^*$ and then on the space $B$. Notice that the $\bullet$-product on $B$ is

\(^{10}\) One can think of $\rho(x,p)$ as a normal symbol of a differential operator on $A = k[x^1, \ldots, x^n]$ (perhaps of infinite order). Then the $\bullet$-product corresponds to the composition of differential operators.
\[ \vartheta \text{-invariant. This allows us to define the } \vartheta \text{-twisted bimodule } B^d \text{ over } B \text{ as well as the trivial extension } \mathcal{B} = B \oplus B^d, \text{ where the first and second summands have degrees 0 and 1, respectively. The product of two elements of } \mathcal{B} \text{ reads} \]

\[ (a, b) \bullet (a', b') := (a \bullet a', a \bullet b' + b \bullet \vartheta a'). \tag{5.7} \]

The action of the differential (5.3) extends to \( \mathcal{B} \) in the following way:

\[ d(a, b) = (da, db). \]

It is obvious that \( H(\mathcal{B}, d) \simeq A \). Hence, upon desuspension, the differential graded algebra \((\mathcal{B}, \bullet, d)\) provides a resolution of its cohomology algebra \( A = A \oplus A^d \).

Note that a constant 2-form \( \lambda = \lambda^i \text{d}p_i \wedge \text{d}p_j \in \mathcal{B} \) belongs to the center of \( B \) iff it is \( \vartheta \text{-invariant. Any such form defines an } A_\infty \text{-structure, which is compatible with the } \bullet \text{-product (5.7). Converse is also true: any compatible } A_\infty \text{-structure } \lambda \in A \subset B \text{ generating a minimal deformation of } \mathcal{A} \text{ is given by a } \vartheta \text{-invariant 2-form with constant coefficients. Applying now the general formulas of lemma 4.2 together with the contracting homotopy (5.4), one can easily see that the corresponding deformation of } \mathcal{A} \text{ is defined by the Moyal } \ast \text{-product (5.6). More precisely,} \]

\[ (a, b) * (a', b') := (a * a', a * b' + b * \vartheta a'). \tag{5.8} \]

for \( a, a', b, b' \in A = k[x_1, \ldots, x_n] \). Again, this deformation gives no higher structure maps.

Consider now minimal deformations that come from \( A_\infty \)-structures living in the space \( \text{Hom}(T^1(\mathcal{B}), \mathcal{B}) \). Each such structure defines and is defined by a differential \( D : B \to B \) that commutes with \( d \). Let us examine the differentials of the form

\[ D(a, b) = (b \bullet \gamma, 0), \tag{5.9} \]

where \( (a, b) \in B \) and \( \gamma = \gamma_i(x, p) \text{d}p_i \wedge \text{d}p_j \) is some 2-form of \( B \). It is clear that \( D^2 = 0 \).

Verification of the Leibniz identity for \( D \) and the \( \bullet \)-product (5.7) leads to the following conditions on \( \gamma \):

\[ \vartheta \gamma = \gamma, \quad \gamma \bullet a = \vartheta a \bullet \gamma, \quad \forall a \in B. \tag{5.10} \]

The second condition is enough to check only for the generators \( x^i \) and \( p_i \). Let us assume that the automorphism \( \vartheta : V \to V \) is diagonalizable, so that

\[ \vartheta p_i = q_i p_i, \quad \vartheta x^i = q_i^{-1} x^i \]

for some nonzero \( q_i \in k \). The direct check of (5.10) for the generators gives the differential equations

\[ (q_i^{-1} - 1)x^i \gamma = \frac{\partial \gamma}{\partial p_i}, \quad (q_i^{-1} - 1)p_i \gamma = \frac{\partial \gamma}{\partial x^i} \]

with the general solution

\[ \gamma = e^{\sum_{i=1}^n (q_i^{-1} - 1)x^i p_i} \lambda, \]

\( \lambda = \lambda^i \text{d}p_i \wedge \text{d}p_j \) being a 2-form with constant coefficients. Then the first condition in (5.10) requires the form \( \lambda \) to be \( \vartheta \)-invariant. Finally, the requirement \([D, d] = 0\) leads to the closedness condition

\[ d\gamma = 0 \iff \sum_{i=1}^n (q_i^{-1} - 1)x^i \text{d}p_i \wedge \lambda = 0. \]
To satisfy this last equation we have to assume that only two eigenvalues of $\vartheta$ are different from 1, say $q_1$ and $q_2$. Then we can take $\lambda = dp_1 \wedge dp_2$. It is clear that $\vartheta \lambda = \lambda$ iff $q_1$ and $q_2$ are mutually inverse to each other, so that

$$\gamma = e^{(q^{-1}-1)x^1p_1+(q^{-1}-1)x^2p_2}dp_1 \wedge dp_2$$

for some $q > 1$. Upon substitution to (5.9), this $\gamma$ generates a nontrivial deformation of the algebra $A$. Furthermore, the first-order deformation $\mu^{(1)}$ gives rise to the third structure map $m_3$. An explicit expression for $m_3$ is obtained by the general formula (4.13), where $\vartheta'$ is the Hochschild differential associated to the associative product (5.7), (2.3) and $\tilde{h}$ is determined by (5.4). After long but straightforward calculations one can find

$$m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = (-1)^{[\alpha_2]}h(D(\alpha_1) \bullet \alpha_2) \bullet \alpha_3$$

$$= (b_1 \Phi(a_2, a_3), b_1 \Phi(a_2, b_3) - b_1 \Phi(b_2, \vartheta a_3)). \quad (5.11)$$

Here $\alpha_i = (a_i, b_i) \in A$ and we introduced the notation

$$\phi(a, b) = -\varepsilon^{\alpha\beta} \int_{0<u<\alpha<1} \text{d}x \text{d}y \left( \frac{\partial a}{\partial x} \right) \left( 1 - w \right) x + w^\beta x \left( \frac{\partial b}{\partial y} \right) \left( 1 - u \right) x + u^\alpha x. \quad (5.12)$$

Thus, whenever $\text{rank}(\vartheta - 1) = 2$ and $\det \vartheta = 1$, there are two families of deformations of the algebra $A$: the first one is generated by the central 2-forms $\lambda \in Z(B)$, while the second is determined by the differentials $D \in \text{Der}(B)$ of the form (5.9). Since $\lambda \in B \subset B$, $D\lambda = 0$. This means that both the A$\infty$-structures on $B$ are compatible to each other and we may consider a 2-parameter family of deformations generated by $t\lambda + sD$. As we have seen, the $\lambda$-deformation just replaces the usual commutative multiplication of polynomials with the Moyal deformation (5.6). Actually, the Moyal deformation is not formal as for any given $a, b \in A$ the series (5.6) contains only finitely many terms. Hence, we can equate $t$ to any element of $k$, say 2. Suppose further that the form $\lambda$ is non-degenerate and $\omega = \lambda^{-1}$. Then $(V, \omega)$ is a symplectic vector space endowed with a symplectomorphism $\vartheta \in \text{Sp}(V)$. For $t = 2, s = 0$ the aforementioned family of deformations degenerates to a bimodule over the polynomial Weyl algebra, where the right action is twisted by $\vartheta$. Letting now $s$ to be a nonzero parameter, we get a formal deformation of the Weyl bimodule (5.8) in the category of A$\infty$-algebras. One could arrive at this deformation directly starting from a resolution of the Weyl bimodule. It turns out that an appropriate resolution is obtained from $(B, \bullet, d)$ by a mere replacement of the $\bullet$-product (5.1) with the following one:

$$a \bullet b = a \exp \left( \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} + \lambda \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} \right) b.$$ 

In [27], this resolution was called the Vasiliev resolution. The differential (5.9) is also modified. To satisfy equation (5.10) we should now take

$$\gamma = e^{(p, x - s) + \lambda(p, p)} \lambda_{\vartheta}(dp, dp), \quad \lambda_{\vartheta}(dp, dp) = \lambda(dp - \vartheta dp, dp - \vartheta dp).$$

Here the triangle brackets denote the natural pairing and $\lambda(u, v) = \lambda_{\vartheta}u_{ij}v_j$. Then the first-order deformation $\mu^{(1)} = m_3$ has a more complicated form

$$m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = (-1)^{[\alpha_2]}h(D(\alpha_1) \bullet \alpha_2) \bullet \alpha_3$$

$$= (b_1 \Phi(a_2, a_3), b_1 \Phi(a_2, b_3) - b_1 \Phi(b_2, \vartheta a_3)), \quad (5.11)$$

Here the triangle brackets denote the natural pairing and $\lambda(u, v) = \lambda_{\vartheta}u_{ij}v_j$. Then the first-order deformation $\mu^{(1)} = m_3$ has a more complicated form
Let \( \Phi(\vartheta) \) denote the generators of \( \Phi(\vartheta) \pi = 0 \) in such a way that
\( \partial_\vartheta \Phi(\vartheta) = 0 \).

Writing down the closedness condition for the dual vector space – Zumino complex. One can choose a basis
\( \{ \vartheta^1, \vartheta^2 \} \subset V^\ast \) used solely to define the Leibniz rule (5.15) for the differential in the corresponding Wess–Zumino complex. It is

It might be well to point out that the notion of parity for the generators of a quantum algebra has nothing to do with imposing on them commutation or anti-commutation relations as opposed to the case of superalgebras. It is

In this section, we will generalize the above example of deformation in two directions. For one
thing, we will consider more general extensions of polynomial algebras that involve several
automorphisms; for another, we will introduce more general class of resolutions to deform
these algebras.

5.3. Quantum polynomial superalgebras

In this section, we will generalize the above example of deformation in two directions. For one
thing, we will consider more general extensions of polynomial algebras that involve several
automorphisms; for another, we will introduce more general class of resolutions to deform
these algebras.

Let \( V \) be an \( n \)-dimensional vector space over \( k \) and let \( \Gamma \subset GL(V) \) be a finitely generated,
abelian subgroup acting semi-simply on \( V \). The group \( \Gamma \), being finitely generated and
abelian, is isomorphic to the direct product of cyclic groups \( \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_l} \). Let \( \{ \vartheta_1, \vartheta_2, \ldots, \vartheta_m \} \subset \Gamma \) denote the generators of \( \Gamma \). Since the action of \( \Gamma \) in \( V \) is semi-simple, one can choose a basis \( \{ p_i \} \subset V \) in such a way that
\( \partial_i p_i = q_{i a} p_i \), \( i = 1, \ldots, n \), \( a = 1, \ldots, m \),
for some nonzero \( q_{i a} \in k \). The action of \( \Gamma \) extends naturally to the symmetric algebra
\( S(V) \simeq k[p_1, \ldots, p_n] \). Geometrically, one can regard the generators \( p_i \) as coordinates on the
dual vector space \( V^\ast \).

Given the group \( \Gamma \), we extend the vector space \( V^\ast \) to a quantum superspace \( W \) by adding
\( m \) ‘odd coordinates’ \( \pi_a \). The coordinates are assumed to satisfy the commutation relations\(^{11}\)
\( p_i p_j - p_j p_i = 0, \quad \pi_a p_i - q_{i a} p_i \pi_a = 0, \quad \pi_a \pi_b - \pi_b \pi_a = 0, \quad (\pi_a)^2 = 0, \quad (5.14) \)
and we prescribe them the following degrees:
\[ |p_i| = 0, \quad |\pi_a| = -1. \]

\(^{11}\) It might be well to point out that the notion of parity for the generators of a quantum algebra has nothing to do with imposing on them commutation or anti-commutation relations as opposed to the case of superalgebras. It is

used solely to define the Leibniz rule (5.15) for the differential in the corresponding Wess–Zumino complex.
The Grassmann parity of the coordinates is induced by this $Z$-grading. The algebra generated by $p$’s and $\pi$’s satisfies the PBW property, so that any its element can be written as a $pr$-ordered polynomial $f(p, \pi)$. We will refer to this algebra as the algebra of quantum polynomials [28, 29].

The quantum superspace $W$ can be endowed with a differential calculus [30, 31]. By definition, the DG-algebra of differential forms $\Omega(W) = \bigoplus_{p \geq 0} \Omega^p(W)$ is generated by the coordinates $p_i$, $\pi_a$, and their differentials $dp_i$, $d\pi_a$ of degrees

$$|dp_i| = 1, \quad |d\pi_a| = 0.$$  

The exterior differential $d : \Omega^p(W) \to \Omega^{p+1}(W)$ is defined now as a degree 1 derivation of $\Omega(W)$ squaring to zero\(^\text{12}\):

$$d(\alpha \beta) = (d\alpha) \beta + (-1)^{|\alpha|} \alpha d\beta, \quad d^2 = 0. \quad (5.15)$$

The ideal generated by (5.14) in the free algebra on the generators $p$’s and $\pi$’s should now be extended to an ideal in the differential algebra freely generated by the coordinates and their differentials. Applying $d$ to Rel. (5.14) and assuming the differentials $dp_i$ and $d\pi_a$ to be linearly independent over $\Omega^0(W)$, we get

$$p_idp_j - dp_i p_j = 0, \quad \pi_adp_i + q_{ai}dp_i\pi_a = 0, \quad d\pi_adp_i - q_{ai}dp_i d\pi_a = 0,$$

$$\pi_ad\pi_a - d\pi_a\pi_a = 0, \quad \pi_ad\pi_b + d\pi_b\pi_a = 0, \quad a \neq b. \quad (5.16)$$

From the equation $d^2 = 0$ it then follows immediately that

$$dp_i dp_j + dp_j dp_i = 0, \quad d\pi_adp_i - q_{ai}dp_i d\pi_a = 0,$$

$$d\pi_ad\pi_b + d\pi_b d\pi_a = 0, \quad a \neq b. \quad (5.17)$$

Taken together Rel. (5.14)–(5.17) define the Wess–Zumino (WZ) complex associated to a quantum $R$-matrix obeying the additional condition $R^2 = 1$, see [30, 32].

It is known [33] that the cohomology of the WZ complex $(\Omega^*(W), d)$ is nested in degree zero,

$$H^*(\Omega, d) \simeq H^0(\Omega, d) \simeq k. \quad (5.18)$$

Moreover, it is not hard to write a contracting homotopy $h : \Omega^p(W) \to \Omega^{p-1}(W)$ leading to this conclusion, see [30].

The above WZ complex can further be extended to the so-called quantum Weyl superalgebra [30, 31, 34]. This is achieved by introducing the partial derivatives $\partial^i, \partial^a : \Omega(W) \to \Omega(W)$ through the relation

$$d = dp_i \partial^i + d\pi_a \partial^a. \quad (5.19)$$

It follows immediately that

$$\partial^i p_j = \delta^i_j, \quad \partial^a \pi_b = \delta^a_b, \quad \partial^i \pi_a = 0, \quad \partial^a p_i = 0.$$

Denoting $\partial^i = x^i$, $\partial^a = \theta^a$ and setting

$$|x^i| = 0, \quad |\theta^a| = 1,$$

we define $\mathcal{B}^*$ to be the DG-algebra generated by the elements

\(^{12}\)To simplify formulas, we do not write the wedge product.
\[ x^i, \theta^a, p_i, \pi_a, dp_i, d\pi_a \quad (5.20) \]

subject to Relns. (5.14), (5.16), (5.17), and

\[
x^i p_j - p_j x^i = \delta^i_j, \quad x^i \pi_a - q_{a\alpha} \pi_a x^i = 0, \\
\theta^a \pi_a + \pi_a \theta^a = 1, \quad \theta^a \pi_b - \pi_b \theta^a = 0, \quad p_i \theta^a - q_{a\alpha} \theta^a p_i = 0, \\
x^i dp_j - dp_j x^i = 0, \quad x^i d\pi_a - q_{a\alpha} d\pi_a x^i = 0, \quad dp_i \theta^a + q_{a\alpha} \theta^a dp_i = 0, \\
\theta^a d\pi_a - d\pi_a \theta^a = 0, \quad \theta^a d\pi_b + d\pi_b \theta^a = 0, \quad a \neq b, \\
x^i x^j - x^j x^i = 0, \quad \theta^a x^i - q_{a\alpha} x^i \theta^a = 0, \quad \theta^a \theta^b - \theta^b \theta^a = 0, \quad (\theta^a)^2 = 0 
\]

(no summation over repeated indices). The action of the differential \( d \) extends from \( \Omega(W) \) to \( B^1 \) by setting \( dx^i = d\theta^a = 0 \). The DG-algebra \( (B^1, d) \) enjoys the PBW property and we can represent its elements by ordered polynomials in the variables \( (5.20) \). The subalgebra generated by the elements \( (x^i, p_i, \theta^a, \pi_a) \) is called the quantum Weyl superalgebra \([4, 30]\); it contains the subalgebra \( A = \ker d \) generated by \( x^i \) and \( \theta^a \). The latter is clearly isomorphic to the algebra of quantum polynomials \( O^\theta(W) \). Furthermore, it follows from (5.18) that \( H(B^1, d) \simeq A \).

In order to make \( (B^1, d) \) into a resolution of the algebra \( A \) we prescribe the following bi-degrees to its generators:

\[ \deg x^i = (0, 0), \quad \deg p_i = (0, 0), \quad \deg \theta^a = (1, 0), \quad \deg \pi_a = (-1, 0), \]

\[ \deg dp_i = (0, 1), \quad \deg d\pi_a = (-1, 1). \]

Then \( [a] \) coincides with the total degree of the element \( a \in B^1 \). Although the pair \( (B^1, d) \) meets all the defining conditions of a resolution, it appears to be too small to generate nontrivial deformations of \( A \). For this reason we consider its completion, denoted by \( B \), with respect to the ideal generated by \( \{p_i\} \). The elements of \( B \) are formal power series in \( p_i \)'s with coefficients being polynomial functions in the other variables.

In case \( m = 1 \), the quantum polynomial superalgebra \( A = A^0 \oplus A^1 \) is clearly isomorphic to the trivial extension of the polynomial algebra \( A = k[x^1, \ldots, x^m] \) by the bimodule \( A^0 \), where \( \theta \) is the automorphism associated to a single generator \( \theta \) of degree 1. This situation has been already considered in the previous subsection.

Let us now describe the \( A_{\infty} \)-structures from \( \text{Hom}(T^0(B), B) \simeq B \) that are compatible with the associative product and the differential \( d \) in \( B \). These are given by the \( d \)-cocycles belonging to the center of \( B \). First, we note that any nonzero element of the form \( \pi_i f^m \) cannot be a \( d \)-cocycle, while an element \( \theta^a g_a \) does not belong to the center \( Z(B) \) unless it is zero. So, we can restrict ourselves to \( \theta \)- and \( \pi \)-independent elements of \( B \). These constitute a differential subalgebra spanned by the forms

\[ f = g(x, p, dp)(d\pi_1)^{n_1}(d\pi_2)^{n_2} \cdots (d\pi_m)^{n_m}, \quad n_a = 0, 1, \ldots \]

Verifying the commutativity conditions

\[ x^i f - f x^i = 0, \quad p_i f - f p_i = 0, \quad (5.21) \]

we find

\[ f = e^{\sum_{i=1}^m (-1)^n_i \phi_{i,n_i} x^i} g(dp)(d\pi_1)^{n_1}(d\pi_2)^{n_2} \cdots (d\pi_m)^{n_m} \]

for some differential form \( g = g^{i_1 \cdots i_r} dp_{i_1} \cdots dp_{i_r} \) with constant coefficients. Renumbering the coordinates \( x^i \), if necessary, we may assume that
where $k$ depends on $n_a$. Then the closedness condition $df = 0$ restricts the form of basis cocycles to

$$f = e^{-\sum_{i=1}^{n} (1 - \prod_{a=1}^{n} q_a) \theta^i \partial \pi_1 \cdots \partial \pi_n \partial \alpha_i (\partial \pi_1)^{n_1} \cdots (\partial \pi_n)^{n_n}}$$

(5.23)

for some $\alpha_j > k$. Finally, the conditions

$$\theta^j f - (-1)^{|j|} f \theta^j = 0, \quad \pi_0 f - (-1)^{|f|} f \pi_0 = 0$$

(5.24)

impose the following set of restrictions on the numbers $n_a$ and $\alpha_j$:

$$q_{a_1}q_{a_2} \cdots q_{a_k}q_{a_\ell}(-1)^{\sum_{h=0}^{k-1} r_h} = 1, \quad \forall a = 1, 2, \ldots, m.$$  

(5.25)

Applying the differential to the second equations in (5.21) and (5.24) yields the other commutativity conditions

$$\partial f - (-1)^{|f|} f \partial = 0, \quad \partial \pi_0 f - f \partial \pi_0 = 0.$$  

Finally, note that the cocycles (5.23) are all non-trivial when viewed as elements of the sub-complex $(Z(B), d)$. In such a way we arrive at the next statement.

**Theorem 5.1.** The cohomology group $H(Z(B), d)$ is generated by the cocycles (5.23) with parameters obeying (5.22) and (5.25).

**Example 5.2.** Let $\Gamma = \mathbb{Z}_2$ act on $V = \mathbb{R}^2$ by the reflection $\theta x^i = -x^i, i = 1, 2$. Then the algebra $\mathcal{A} = \mathcal{A}_{\mathbb{Z}_2}$ is generated by the three elements $x^1, x^2$, and $\theta$ subject to the relations

$$x^i x^j - x^j x^i = 0, \quad x^i \theta + \theta x^i = 0, \quad \theta^2 = 0.$$  

(5.26)

Hence, all $q_i = -1$. The algebra $\mathcal{A} = \mathbb{A}^0 \oplus \mathbb{A}^1$ is isomorphic to the trivial extension of the polynomial algebra $\mathcal{A} = \mathbb{R}[x^1, x^2]$ by the right-twisted bimodule $\mathbb{A}^0$.

According to the above considerations, the space $H(B, d)$ is spanned by the cocycles

$$\partial \pi_1 \partial \pi_2 (\partial \pi)^{2m}, \quad e^{-2x^i \theta \partial \pi_1 \partial \pi_2 (\partial \pi)^{2m+1}, \quad (\partial \pi)^{2m}, \quad m = 0, 1, \ldots, \quad (5.27)$$

which define mutually compatible $A_\infty$-structures on $B = B_{\mathbb{Z}_2}$.

As an associative algebra $H(B, d)$ is generated by the four basis cocyles

$$\partial \pi_1 \partial \pi_2, \quad e^{-2x^i \theta \partial \pi_1 \partial \pi_2 (\partial \pi)^{2}}, \quad (\partial \pi)^{2}, \quad 1.$$  

(5.27)

Of these cocycles only the first two have total degree 1 when regarded as elements of the algebra $\mathcal{B}$. The first cocycle generates the usual Moyal’s deformation of the polynomial algebra, while the second leads to higher structure maps. In particular, the first-order deformation associated to the second cocycle gives a non-zero map $m_2$, which is similar in form to that considered in section 5.1.

The above result can easily be extended to the Klein group $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ acting on $V = \mathbb{R}^4$ by

$$\theta_1 x^\alpha = -x^\alpha, \quad \theta_1 y^\alpha = y^\alpha, \quad \theta_2 x^\alpha = x^\alpha, \quad \theta_2 y^\alpha = -y^\alpha,$$  

(5.28)
Let us summarise our results. In this paper, we proposed a simple method for the deformation of \( A_{\infty} \)-algebras. If the initial (undeformed) \( A_{\infty} \)-algebra is just an associative algebra \( A \), then the only input required by our method is a multiplicative resolution of \( A \). Although the proposed method is unable to solve the deformation problem in full generality, there are some interesting classes of associative algebras (e.g. symplectic reflection algebras [26]) where it does provide all non-trivial deformations in the category of minimal \( A_{\infty} \)-algebras [25, 27].
particular, most of the oscillator realizations of higher spin algebras fall into this class. As was explained in the Introduction, any deformation of a higher spin algebra to a minimal $A_\infty$-algebra gives rise to a formally consistent interaction for massless higher spin fields.

Another approach to formal higher spin gravities, and historically the first one, is Vasiliev’s equations. These equations is a field-theoretical realization of the idea of resolution for specific higher spin algebras and their realizations. In the case of the 4$d$ higher spin gravity, the relation of our approach to the earlier framework of Vasiliev [11] has been discussed in [14, 27], and [35]. In brief, the Vasiliev’s approach is aimed at constructing certain differential graded algebra (i.e. a flat $A_\infty$-algebra with $m_k = 0$, $\forall k \geq 3$), while our method results in a minimal $A_\infty$-algebra ($m_1 = 0$). The point now is that both the $A_\infty$-algebras appear to be quasi-isomorphic to each other defining thus physically equivalent theories. More precisely, these algebras provide, respectively, an anti-minimal and a minimal model of one and the same $A_\infty$-algebra. In practical terms, quasi-isomorphic algebras correspond to higher spin gravities that are related by addition/elimination of auxiliary fields and by field redefinition. The special convenience of our method is that it leads to field equations with much fewer auxiliary fields.

Within our approach the problem of constructing new examples of formal higher spin gravities is reduced to finding an appropriate resolution for a given higher spin algebra. We hope that the resolutions of the quantum polynomial algebras constructed in the present paper as well as the resolutions for the symplectic reflections algebras [25, 27] will lead to a large class of new higher spin gravities or to another type of integrable equations. It is pertinent to note in this connection that many equations of the form

$$m_1(\Phi) + m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + \cdots = 0$$

that originate from deformations of associative algebras appear to be integrable and their solutions and invariants can be written down with the help of a Lax pair [15].

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ORCID iDs

A A Sharapov © https://orcid.org/0000-0002-1519-865X

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