Lucas, Fibonacci, and Chebyshev polynomials from matrices

Jerzy Kocik
Department of Mathematics, Southern Illinois University, Carbondale, IL62901
jkocik@siu.edu

Abstract
A simple matrix formulation of the Fibonacci, Lucas, Chebyshev, and Dixon polynomials is presented. It utilizes the powers and the symmetric tensor powers of a certain matrix.

Keywords: Matrices, Chebyshev polynomials, Fibonacci polynomials, Dixon polynomials, symmetric tensor powers.

MSC: 11B39

1 Introduction
Chebyshev polynomials play an important role in presentation of many basic mathematical concepts. They have a number of definitions. Here we present a novel definitions of the Chebyshev polynomials $T_n(x)$ and $U(x)$ in terms of the traces of certain powers of matrices:

$$T_n(x) = \frac{1}{2} \text{Tr} \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix}^n,$$

$$U_n(x) = \frac{1}{2} \text{Tr} \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix} \odot^n,$$

where “$\odot^n$” denotes the $n$-th symmetric tensor power. Similarly, Lucas and Fibonacci polynomials $L_n(x)$ and $F_n(x)$ may be interpreted as follows:

$$L_n(x) = \text{Tr} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n,$$

$$F_n(x) = \text{Tr} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \odot^n.$$

All these polynomials are special cases of the regular Dixon polynomials, which may also be formulated in terms of matrices:

$$\tilde{D}_n(x,y) = \text{Tr} \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix}^n,$$

$$\tilde{D}_n(x,y) = \text{Tr} \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix} \odot^n.$$

We review basic definitions in the next section, and then derive the above formulas.
2 Polynomials: Lucas, Fibonacci, Chebyshev, Dixon

Chebyshev polynomials emerge from the trigonometric identities, like

\[
\cos(2\varphi) = 2\cos^2(\varphi) - 1, \quad \cos(3\varphi) = 4\cos^3(\varphi) - 3\cos^2\varphi
\]

The cosine of a multiple angle is expressible in terms of polynomials in \(\cos(\varphi)\) due to the recurrence property:

\[
\cos((n + 1)\varphi) = \cos(n\varphi)\cos\varphi - \sin(n\varphi)\sin\varphi = 2\cos(n\varphi)\cos\varphi - \cos((n - 1)\varphi)
\]

Following this induction rule, and a similar one for the function of sine, one defines the two types of Chebyshev polynomials presented in Table 1.

**Definition 2.1.** Chebyshev polynomials of the first and the second kind, \(T(x)\) and \(U(x)\), respectively, are polynomials in variable \(x\) defined in a recursive way as follows:

\[
\begin{align*}
T_0 &= 1, & T_1 &= x, \quad T_{n+1} &= 2x T_n - T_{n-1} \\
U_0 &= 1, & U_1 &= 2x, \quad U_{n+1} &= 2x U_n - U_{n-1}
\end{align*}
\]

(2.1)

To eliminate the powers of 2, we may adjust the definitions of the standard Chebyshev polynomials to their “reduced versions”, defined and exemplified in the middle part of Table 1. It is a small price to pay, as the references to trigonometric identities are still valid:

\[
\cos(n\varphi) = \frac{1}{2}T_n(2\cos\varphi) \quad \sin(n\varphi) = \frac{U_{n-1}(2\cos\varphi)}{\sin(\varphi)}
\]

Another adjustment concerns the sign in the recursive rules, from minus to plus. This leads to polynomials that are known as Lucas and Fibonacci, presented in Table 1. Their evaluations at \(x = 1\) produce the Lucas and Fibonacci sequences, respectively [6, 3].

These four types of polynomials have an obvious generalization introduced in 1897 by Leonard Dickson [1]. We shall distinguish between the standard and the regular versions, the first being the original Dixon’s version. The differ by the sign convention.

**Definition 2.2.** The standard Dixon polynomials \(P(x, y)\) are polynomials in two variables \(x\) and \(y\), defined in a recursive way as follows:

\[
P_0 = c, \quad P_1 = x, \quad P_{n+1} = x P_n - y P_{n-1}
\]

(2.2)

where \(c\) is a constant. If \(c = 2\) or \(c = 1\), the polynomials will be called of the first kind and the second kind, respectively. We shall use notation \(\hat{P}\) to indicate the initial constant value \(c\) in (2.2). (The minus sign is for historical consistency.)
Chebyshev first kind

\[ T_0 = 1 \]
\[ T_1 = x \]
\[ T_2 = 2x^2 - 1 \]
\[ T_3 = 4x^3 - 3x \]
\[ T_4 = 8x^4 - 8x^2 + 1 \]
\[ T_5 = 16x^5 - 20x^3 + 5x \]
\[ T_{n+1} = 2xT_n - T_{n-1} \]

\[ \cos(n\varphi) = T_n(\cos \varphi) \]

Reduced Chebyshev first kind

\[ \dot{T}_0 = 2 \]
\[ \dot{T}_1 = x \]
\[ \dot{T}_2 = x^2 - 2 \]
\[ \dot{T}_3 = x^3 - 3x \]
\[ \dot{T}_4 = x^4 - 4x^2 + 2 \]
\[ \dot{T}_5 = x^5 - 5x^3 + 5x \]
\[ \dot{T}_{n+1} = x\dot{T}_n - \dot{T}_{n-1} \]

\[ T_n(x) = \frac{1}{2}T(2x) \]

Lucas

\[ L_0 = 2 \]
\[ L_1 = x \]
\[ L_2 = x^2 + 2 \]
\[ L_3 = x^3 + 3x \]
\[ L_4 = x^4 + 4x^2 + 2 \]
\[ L_5 = x^5 + 5x^3 + 5x \]
\[ L_{n+1} = xL_n + L_{n-1} \]

Chebyshev second kind

\[ U_0 = 1 \]
\[ U_1 = 2x \]
\[ U_2 = 4x^2 - 1 \]
\[ U_3 = 8x^3 - 4x \]
\[ U_4 = 16x^4 - 12x^2 + 1 \]
\[ U_5 = 32x^5 - 32x^3 + 6x \]
\[ U_{n+1} = 2xU_n - U_{n-1} \]

\[ \sin(n\varphi) = \frac{U_{n-1}(\cos \varphi)}{\sin \varphi} \]

Reduced Chebyshev second kind

\[ \dot{U}_0 = 1 \]
\[ \dot{U}_1 = x \]
\[ \dot{U}_2 = x^2 - 1 \]
\[ \dot{U}_3 = x^3 - 2x \]
\[ \dot{U}_4 = x^4 - 3x^2 + 1 \]
\[ \dot{U}_5 = x^5 - 4x^3 + 3x \]
\[ \dot{U}_{n+1} = x\dot{U}_n - \dot{U}_{n-1} \]

\[ U_n(x) = \dot{U}(2x) \]

Fibonacci

\[ F_0 = 1 \]
\[ F_1 = x \]
\[ F_2 = x^2 + 1 \]
\[ F_3 = x^3 + 2x \]
\[ F_4 = x^4 + 3x^2 + 1 \]
\[ F_5 = x^5 + 4x^3 + 3x \]
\[ F_{n+1} = xF_n + F_{n-1} \]

Table 1: basic polynomials and their examples
Here are the first few standard Dixon polynomials given explicitly:

\[
\begin{align*}
\hat{P}_0 &= 2 \\
\hat{P}_1 &= x \\
\hat{P}_2 &= x^2 - 2y \\
\hat{P}_3 &= x^3 - 3xy \\
\hat{P}_4 &= x^4 - 4x^2y + 2y^2 \\
\hat{P}_5 &= x^5 - 5x^3y + 3xy^2
\end{align*}
\]

\[
\begin{align*}
\hat{^1P}_0 &= 1 \\
\hat{^1P}_1 &= x \\
\hat{^1P}_2 &= x^2 - y \\
\hat{^1P}_3 &= x^3 - 2xy \\
\hat{^1P}_4 &= x^4 - 3x^2y + y^2 \\
\hat{^1P}_5 &= x^5 - 4x^3y + 2xy^2
\end{align*}
\]

This is a generalization of the known polynomials and sequences:

\[
\begin{array}{c|cc}
\hat{P}(x,y) & c = 2 & c = 1 \\
\hline
y = 1 \rightarrow & \text{Chebyshev polynomials of the first kind} & \text{Chebyshev polynomials of the second kind} \\
\hline
y = -1 \rightarrow & \text{Lucas polynomial} & \text{Fibonacci polynomials} \\
\hline
y = -1, x = 1 \rightarrow & \text{Lucas numbers} & \text{Fibonacci numbers}
\end{array}
\]

One may find more appealing the altered definition of Dixon polynomials, namely with difference in (2.2) replaced by the sum:

**Definition 2.3.** The **regular Dixon polynomials** are defined by the following recurrence

\[
D_0 = c, \quad D_1 = x, \quad D_{n+1} = xD_n + yD_{n-1}
\]

(2.3)

where \(c\) is a constant (cf. Definition 2.2).

This is of course equivalent definition, the difference between the regular and standard Dixon polynomials is due to replacement \(y \rightarrow (-y)\), that is, \(P(x, y) = D(x, -y)\). The regular Dixon polynomials have all coefficient signs positive:

\[
\begin{align*}
\hat{D}_0 &= 2 \\
\hat{D}_1 &= x \\
\hat{D}_2 &= x^2 + 2y \\
\hat{D}_3 &= x^3 + 3xy \\
\hat{D}_4 &= x^4 + 4x^2y + 2y^2 \\
\hat{D}_5 &= x^5 + 5x^3y + 3xy^2
\end{align*}
\]

\[
\begin{align*}
\hat{^1D}_0 &= 1 \\
\hat{^1D}_1 &= x \\
\hat{^1D}_2 &= x^2 + y \\
\hat{^1D}_3 &= x^3 + 2xy \\
\hat{^1D}_4 &= x^4 + 3x^2y + y^2 \\
\hat{^1D}_5 &= x^5 + 4x^3y + 2xy^2
\end{align*}
\]
3 Polynomials via matrices

It is well-known that $2 \times 2$ determinant-1 matrices of the special linear group $SL(2)$ relate to Chebyshev polynomials, namely

$$\text{Tr } M^n = T_n(\text{Tr } M)$$

(Chebyshev polynomial of the first kind evaluated for $M$ with $\det M = 1$). A generalization of the above to matrices of arbitrary determinant was offered in [7]. We restate it below, using the language of symmetric powers and with adjusted terminology. (For details on the symmetric tensor powers see [4], Sec 7.2., and the Appendix below.)

**Proposition 3.1.** Let $M \in \text{Mat}(2, \mathbb{F})$ be any $2 \times 2$ matrix over some field $\mathbb{F}$. The trace of the $n$-th power and of the symmetric $n$-th power of $M$ coincide with the Dixon polynomial of the second and first kind, correspondingly, with the variables equal to $x = \text{Tr } M$ and $y = \det M$, i.e.,

$$(a) \quad \text{Tr } (M^n) = P_n(\text{Tr } M, \det M)$$

$$(b) \quad \text{Tr } (M^\otimes n) = \hat{P}_n(\text{Tr } M, \det M)$$

**Proof of the first statement:** Start with an observation that for any pair of $2 \times 2$ matrices $A, B \in \text{Mat}(2, \mathbb{F})$ this simple equation holds:

$$\text{Tr } (A^2B) = \text{Tr } A \cdot \text{Tr } AB - \det A \cdot \text{Tr } B$$

(show by direct inspection). Substitute $A = M$ and $B = M^n$. We get

$$\text{Tr } (M^{n+2}) = \text{Tr } M \cdot \text{Tr } M^n - \det A \cdot \text{Tr } M^n$$

(3.3)

Denote $\text{Tr } M = x$ and $\det M = y$. Then

$$\text{Tr } M^{n+2} = x\text{Tr } M^{n+1} - y\text{Tr } M^n$$

Since the traces satisfy the same rule as the polynomials and start with the same initial values: $M^0 = \text{Id}$ (identity matrix) and $\text{Tr } \text{Id} = 2$, as well as $M^1 = M$, thus denoting $x = \text{Tr } M$ we get the result.

**Proof of the second statement:** Note that the trace is invariant under similarity transformation of matrices, $M \rightarrow TMT^{-1}$ and every matrix may be brought to the lower-triangular form, we focus on matrix

$$M = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$$

We shall show that

$$\text{Tr } M^\otimes n = \sum_{i=0}^{n} a^{n-i}b^i = \sum a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + b^n$$

(3.4)

Indeed:

$$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ cx + by \end{bmatrix}$$
The k-th basis vector \( e_k \equiv x^{n-k}y^k \) is transformed to

\[
(ax)^{n-k}(cx + by)^k = (ax)^{n-k} \sum_i \binom{k}{i} a^i x^i b^{k-i}y^{k-i} \\
= \sum_i \binom{k}{i} a^{n-k} b^{k-i} c^i e_{k+i}
\]

Thus the coefficient at \( e_k \) corresponds to \( i = 0 \) and therefore the k-th entry on the diagonal of \( M^\otimes n \) is \( a^n b^{n-k} \). The trace is therefore as in (3.4). Now, it is a simple algebraic manipulation to see that:

\[
da^n + a^{n-1}b + a^{n-2}b^2 + \ldots + b^n = (a + b) \cdot (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1}) \\
- ab \cdot (a^{n-2} + a^{n-3}b + a^{n-4}b^2 + \ldots + b^{n-2})
\]

This is equivalent to

\[
\text{Tr} M^\otimes n = \text{Tr} M \cdot \text{Tr} M^\otimes (n-1) - \det M \cdot \text{Tr} M^\otimes (n-2)
\]

The initial terms agree with (2.2) for \( c = 1 \); indeed, \( \text{Tr} M^\otimes 0 \equiv \text{Tr}[1] = 1 \) and \( \text{Tr} M^\otimes 1 \equiv \text{Tr} M \). \( \square \)

We are now ready justify the matrix representations of the Dixon polynomials:

**Proposition 3.2.** The following may be viewed as the definition of the standard Dixon polynomials

\[
\tilde{P}_n(x, y) = \text{Tr} \begin{bmatrix} x & -y \\ 1 & 0 \end{bmatrix}^{\otimes n} \quad \tilde{P}_n(x, y) = \text{Tr} \begin{bmatrix} x & -y \\ 1 & 0 \end{bmatrix} \otimes n
\]

**Proof.** This may be seen as a simple corollary to Proposition 3.1 by a smart choice of the matrix. Namely, observe that the following matrix has the trace and determinant matching with the variables \( x \) and \( y \):

\[
\text{Tr} \begin{bmatrix} x & -y \\ 1 & 0 \end{bmatrix} = x, \quad \det \begin{bmatrix} x & -y \\ 1 & 0 \end{bmatrix} = y
\]

Substitute this matrix to (3.1), and the result follows. \( \square \)
Corollary 3.3. In particular, the following are alternative definitions of Lucas, Fibonacci and Chebyshev, polynomials:

Dixon regular  
\[ \hat{D}_n(x) = \text{Tr} \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix}^{\otimes n} \]

Lucas and Fibonacci  
\[ L_n(x) = \text{Tr} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^{\otimes n} \quad F_n(x) = \text{Tr} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \]

Chebyshev reduced  
\[ \hat{T}_n(x) = \text{Tr} \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}^{\otimes n} \quad \hat{U}_n(x) = \text{Tr} \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix} \]

Chebyshev  
\[ T_n(x) = \text{Tr} \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix}^{\otimes n} \quad U_n(x) = \text{Tr} \begin{bmatrix} 2x & -1 \\ 1 & 0 \end{bmatrix} \]

Appendix

A.1 Symmetric powers

Here we present a quick way to calculate the symmetric tensor powers of a 2 \( \times \) 2 matrix. For this purpose, we use \( x \) and \( y \) as dummy variables (not to be confused with \( x \) and \( y \) of the Dixon polynomials). Let us define the following map from 2- to \((n+1)\)-dimensional standard vector space:

\[
\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \otimes^n \\ y \otimes^n \end{bmatrix} = [x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n]^T
\]

Denote \( v = [x, y]^T \). The \( n \)-th symmetric power of a 2\( \times \)2 matrix \( M \) is an \((n+1)\times(n+1)\) matrix \( M^{\otimes n} \) defined by

\[
(Mv)^{\otimes n} = M^{\otimes n}v^{\otimes n} \quad (3.5)
\]

Example: Consider the matrix

\[ M = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \]

The calculations for the second symmetric power are:

\[
Mv = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ x \end{bmatrix} \rightarrow \begin{bmatrix} (ax + by)^2 \\ (ax + by)x \\ x^2 \end{bmatrix} = \begin{bmatrix} a^2x^2 + 2abxy + b^2y^2 \\ ax^2 + bxy \\ x^2 \end{bmatrix} = \begin{bmatrix} a^2 & 2ab & b^2 \\ a & b & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}
\]

Thus the second power \( M^{\otimes 2} \) is the last square matrix, above. Other powers are calculated similarly from (3.5).

Remark: If the action of the matrices is defined on the dual space, i.e., the row vectors, the symmetric powers differ slightly, but the diagonals of both versions coincide.

For other applications of symmetric tensor powers, see [2, 4].
Below, we present the regular and the symmetric powers of matrix
\[ M = \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix} \].

The regular powers are:
\[
\begin{align*}
M^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
M^1 &= \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix} \\
M^2 &= \begin{bmatrix} x^2 + y & xy \\ x & y \end{bmatrix} \\
M^3 &= \begin{bmatrix} x^3 + 2xy & x^2y + y^2 \\ x^2 + y & xy \end{bmatrix} \\
M^4 &= \begin{bmatrix} x^4 + 3x^2y + y^2 & x^3y + 2xy^2 \\ x^3 + 2xy & x^2y + y^2 \end{bmatrix} \\
M^5 &= \begin{bmatrix} x^5 + 3x^3y + 3xy^2 & x^4y + 3x^3y^2 + y^3 \\ x^4 + 3x^2y^2 + y^2 & x^3y + y^2 \end{bmatrix}
\]

The symmetric tensor powers are:
\[
\begin{align*}
M^{\otimes 0} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
M^{\otimes 1} &= \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix} \\
M^{\otimes 2} &= \begin{bmatrix} x^2 & 2xy & y^2 \\ x & y & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
M^{\otimes 3} &= \begin{bmatrix} x^3 & 3x^2y & 3xy^2 & y^3 \\ x^2 & 2xy & y^2 & 0 \\ x & y & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
M^{\otimes 4} &= \begin{bmatrix} x^4 & 4x^3y & 6x^2y^2 & 4xy^3 & y^4 \\ x^3 & 3x^2y & 3xy^2 & y^3 & 0 \\ x^2 & 2xy & y^2 & 0 & 0 \\ x & y & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

One may easily verify that the traces of the powers correspond to the regular Dixon polynomials of the second and first kind, \( \text{Tr } M^n = D_n \) and \( \text{Tr } M^{\otimes n} = D_n \) (and to the standard version, under replacement \( y \rightarrow -y \)). Substituting \( y = 1, -1 \) yields the further specifications to the other polynomials discussed: Chebyshev, Lucas and Fibonacci. Note that under substitution \( x = y = 1 \), the symmetric powers become the upper parts of the Pascal triangle in a matrix form.
A.2 A handful of properties

The matrix formulation of the polynomials leads to other interesting properties and symmetries. Here we mention just a few:

(1) \[ M_{n+1} = \begin{bmatrix} D_{n+1} & yD_n \\ D_n & yD_{n-1} \end{bmatrix}. \]

for instance:

\[ M_3 = \begin{bmatrix} x^3 + 2xy & x^2y + 2y^2 \\ x^2 + y & xy \end{bmatrix} = \begin{bmatrix} D_3 & yD_2 \\ D_2 & yD_1 \end{bmatrix}. \]

(2) \[ D_n + yD_{n-2} = D_n. \]

(3) \[ \det M^\otimes_n = (\det M)^{\frac{n(n+1)}{2}}. \]

(4) Off-diagonal sums:

\[ M_{1,1+k}^\otimes + M_{1,2+k}^\otimes + ... + M_{n-k,n}^\otimes = y^k D_{n-k}. \]

These and other properties will be discussed elsewhere.

References

[1] Dickson, L. E. . "The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group I,II". Ann. of Math. 11 (1/6): 65–120, 161–183 (1897-1897).

[2] Philip Feinsilver and Jerzy Kocik, Krawtchouk matrices from classical and quantum walks, Contemporary Mathematics, 287 2002 , pp. 83-96.

[3] Verner E. Hoggatt, Jr. and Calvin T. Long, Divisibility properties of generalized Fibonacci polynomials, Fibonacci Quart., 12:113–120, (1974).

[4] Jerzy Kocik, Krawtchouk matrices from the Feynman path integral and from the split quaternions, Contemporary Mathematics, 668, pp. 131-164 (2016). [arXiv:1604.00109]

[5] Thomas Koshy, Fibonacci and Lucas Numbers with Applications, Wiley (2017), 2nd edition.

[6] Edouard Lucas, Theorie des Fonctions Numeriques Simplement Periodiques, Amer J. Math., 1 (2):184–196 (1878). Continued in 1 (3):197–240, and 1 (4):289–321 in the same year.

[7] Robert Owczarek, Remarks on Chebyshev Polynomials, Fibonacci Polynomials and Kauffman Bracket Skein Modules, Journal of Knot Theory and Its Ramifications 27, (07), 1841007 (2018)