Aspects of Non-Abelian Toda Theories

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Abstract

We present a definition of the non-abelian generalisations of affine Toda theory related from the outset to vertex operator constructions of the corresponding Kac-Moody algebra $\hat{g}$. Results concerning conjugacy classes of the Weyl group of the finite Lie algebra $g$ to embeddings of $A_1$ in $g$ are used both to present the theories, and to elucidate their soliton spectrum. We confirm the conjecture of [1] for the soliton specialisation of the Leznov-Saveliev solution. The energy-momentum tensor of such theories is shown to split into a total derivative part and a part dependent only on the free fields which appear in the general solution, and vanish for the soliton solutions. Analogues are provided of the results known for the classical solitons of abelian Toda theories.
1 Introduction

The abelian Toda field theories associated with an affine Kac-Moody algebra are now well established in the literature as examples of integrable models. They exhibit a variety of interesting structures, and their zero-curvature formulation (see e.g. [2]) allows the use of powerful techniques from the theory of Lie algebras.

The theories can also be presented from a Lagrangian point of view, the Lagrangian being the usual bosonic kinetic term minus a potential, exponential in the field. This is the natural form if we want to calculate the canonical energy-momentum tensor, which has been shown [3] to have some interesting properties. We can also perform the standard perturbative canonical quantisation of the theory by treating terms higher than quadratic in the potential as an interaction between the so-called Toda particles. These are the quanta of the modes of the ‘free’ Lagrangian which remains. The masses of these particles have an elegant algebraic origin (see e.g. [4]). Their couplings also permit a Lie algebraic description, given by Dorey’s fusing rule [5], [4]. Because the theory is integrable and two-dimensional the S-matrix for the particles has to satisfy a very restrictive set of constraints. A solution to this problem has been conjectured and discussed by a number of authors [6], [7], [8], [9].

Using the grading structure of the underlying Lie algebra the equations of motion can be completely integrated to find a general solution [10], [2] for the matrix elements of the dynamical variable in some particular representations. This solution is given in terms of a free field, belonging to the same group or algebra as the dynamical field. Using this solution it can be shown [3] that the energy-momentum tensor is a sum of two parts; a total derivative, which can be integrated to yield a surface term, and a piece dependent only on the free fields.

Because the theories possess an infinite set of degenerate vacua we expect to find soliton solutions interpolating them. This possibility was first discussed by Hollowood [11] for the theories based on $\hat{A}_r$. He used the Hirota method, a technique later used by [12], [13] to develop the solutions for all of the theories. This technique has the disadvantage that the solutions are not given in terms of a general formula, rather they are iteratively defined. This tends to make calculation of the mass spectrum rather tricky, as well as the evaluation of the so-called topological charge which is the change in the solution between $x = -\infty$ and $x = \infty$.

A parallel approach was developed in [3], where a specialisation of the general solution which leads to the N-soliton solution had been given. This allowed discussion of the spectra of conserved charges without having to find the explicit solution for a particular theory. It also revealed a surprising connection with the principal vertex operator construction of an affine algebra. (A restricted solution of the $\hat{A}_1$ theory which contains the soliton solutions was discussed by [4]. These authors used an intuitively attractive ‘dressing transformation’ method and also noted a connection with vertex operators.) The mass formula obtained by the methods of [3] reveals that the coupling constant must be imaginary for them to be positive. With this understanding the masses are found
to be proportional to the masses of the particles in the Toda theory associated with the dual Lie algebra. This bears a resemblance to duality conjectures in the theory of self-dual Yang-Mills monopoles, where technical difficulties appear to have prevented their resolution.

The present work arises from a conjecture in [1] for the solitonic specialisation of the general solution of the generalised or non-abelian Toda theories defined in [10]. Little progress has been made in the understanding of these theories, principally on account of the complexity of their original presentation. Here we show that there is a description which allows us to make further progress with them.

The structure of this paper is as follows. In section 2 we review the Leznov-Saveliev presentation and solution of a Toda-type system, which depends on the definition of some particular elements of a graded Lie algebra. In section 3 we describe the original definition of the non-abelian affine Toda theories based on the work of [10]. By analogy with the abelian Toda case [3] we give an alternative characterisation of these theories, and discuss the means by which it can be related to the original one. Section 4 is devoted to discussion of the abelianisation procedure for deducing the conserved charges of these theories, and we produce an explicit formula for the lowest one. The purpose of section 5 is to demonstrate the splitting of the energy-momentum tensor, one part being equal to the conserved charge calculated in the previous section, and the other a topological ‘improvement’ term. Section 6 uses this expression to calculate the energy and momentum of the N-soliton solution. Finally we indicate directions for future progress in section 7.

2 Leznov-Saveliev Solution

In this section we present the most general definition of a Toda theory as a zero-curvature condition, and its solution due to Leznov and Saveliev [10]. Our treatment and notation will be very much along the lines of this reference. For a detailed discussion of many applications of group-theory to integrable systems we refer to the monograph [2].

2.1 Zero-Curvature Presentation of Toda Systems

Let us establish some notation. To define a Toda theory we need a Lie group, \( \Gamma \), whose Lie algebra \( \gamma \) (assumed simple, semisimple algebras are a trivial generalisation) has a gradation over the integers. We label the graded subspaces

\[
\gamma = \bigoplus_{p \in \mathbb{Z}} \gamma_p. \tag{2.1}
\]

It is occasionally helpful to require that \( I(\gamma_p) = \gamma_{-p} \), where \( I \) is the Chevalley involution with respect to a Cartan subalgebra contained in \( \gamma_0 \). The Toda field, \( h = u^2 \), lies in \( \Gamma_0 \), the exponential of the subalgebra \( \gamma_0 \). Define also two elements \( X_\pm \in \gamma_{\pm 1} \). Working in light-cone coordinates \( x^\pm = (t \pm x)/\sqrt{2} \), the

\[ x_1^\pm = (t^\pm x)/\sqrt{2} \]

\[ x_2^\pm = (t^\pm x)/\sqrt{2} \]
connection with components
\[ A_+ = u^{-1} \partial_+ u + uX_+ u^{-1} \]
\[ A_- = -\partial_- uu^{-1} + u^{-1} X_- u \]  \hspace{0.5cm} (2.2)
yields as its zero-curvature condition
\[ \tilde{d} \tilde{A} + \tilde{A} \wedge \tilde{A} = 0 \]  \hspace{0.5cm} (2.3)
the Toda equation
\[ \partial_- \left( h^{-1} \partial_+ h \right) - \left[ X_+, h^{-1} X_- h^{-1} \right] = 0. \]  \hspace{0.5cm} (2.4)
Of course this connection is not unique, any gauge-transformed version will also yield (2.4) as its zero-curvature condition. Often it is useful to transform with either \( u \) or \( u^{-1} \) yielding
\[ A_+ = hX_+ h^{-1} \]
\[ A_- = -\partial_- hh^{-1} + X_- \]  \hspace{0.5cm} (2.5)
or
\[ A_+ = h^{-1} \partial_+ h + X_+ \]
\[ A_- = h^{-1} X_- h. \]  \hspace{0.5cm} (2.6)

This presentation of Toda systems as a zero-curvature condition is very useful from the point of view of extracting the general solution, and thus demonstrating the integrability (at least in that sense of the word). This demonstration is quite complicated, but in the end yields very simple and elegant results.

2.2 General Solution
First let us define \( T \in \Gamma \) such that
\[ \tilde{A} = T^{-1} \tilde{d} T. \]  \hspace{0.5cm} (2.7)
Thus \( T \) is just the inverse of the classical monodromy associated with the linear problem defined by a curvature-free connection. We define the subalgebras \( \gamma_\pm \) to be the direct sums of the graded subspaces with respectively positive and negative grades. (When \( \Gamma \) is a simple finite-dimensional Lie algebra these subalgebras will be nilpotent.) From these definitions we get the subgroups \( \Gamma_\pm \) by exponentiation. We introduce [10] the modified Gaussian decompositions of \( T \)
\[ T = M_+ N_- g_+ = M_- N_+ g_-, \]  \hspace{0.5cm} (2.8)
where \( M_\pm \in \Gamma_\pm \), likewise for \( N_\pm \), and \( g_\pm \in \Gamma_0 \). The existence of this decomposition is not proven in [10], but it seems likely that it will be valid except on the union of some submanifolds of \( \Gamma \) of dimensions less than that of \( \Gamma \).
Now suppose that we impose
\[ \partial_{\pm} M_{\pm}(x^{\pm}) = M_{\pm}(x^{\pm}) L_{\pm}(x^{\pm}), \quad (2.9) \]
where \( L_{\pm} \) are some arbitrary elements of \( \Gamma_{\pm 1} \). The bulk of the derivation of the general solution consists of showing that this choice is actually sufficient to reproduce the most general form for the connection \( \tilde{A} \), given the definitions 2.2, 2.3, 2.4 above. The method of proof begins by deriving some relations which the various components in the decomposition 2.8 must satisfy on account of the two equivalent possibilities. Re-write this equivalence in the form
\[ \begin{align*}
R &= M_{+}^{-1} M_{-} \equiv N_{-} g N_{+}^{-1}, \quad (2.10) \\
\end{align*} \]
where \( g = g_{+} g_{-}^{-1} \). The group element \( R \) thus defined satisfies the equations of motion
\[ \begin{align*}
\partial_{+} R &= -L_{+} R, \quad (2.11) \\
\partial_{-} R &= R L_{-}, \quad (2.12) \\
\end{align*} \]
using the definition of \( R \) purely in terms of \( M_{\pm} \). Now substitute in the equivalent definition in terms of \( N_{\pm} \) and \( g \). From 2.11 we obtain
\[ N_{-} \partial_{+} g N_{+}^{-1} + \partial_{+} N_{-} g N_{+}^{-1} + N_{-} g \partial_{+} N_{+}^{-1} = -L_{+} N_{-} g N_{+}^{-1} \quad (2.13) \]
which can be re-written in the form
\[ g^{-1} \partial_{+} g + \partial_{+} N_{+}^{-1} N_{+} g^{-1} N_{-} g = -g^{-1} N_{-}^{-1} L_{+} N_{-} g. \quad (2.14) \]
We can now use the grading structure in \( \gamma \) to decompose this expression into two separate equations. The rhs of 2.14 contains only grades +1 and below. We can separate the components of strictly positive grade (i.e. +1) on both sides to obtain
\[ \partial_{+} N_{+}^{-1} N_{+} = -g^{-1} L_{+} g. \quad (2.15) \]
Subtracting this off 2.14 yields
\[ \partial_{+} g^{-1} + N_{+}^{-1} \partial_{+} N_{-} = -N_{+}^{-1} L_{+} N_{-} + L_{+}. \quad (2.16) \]
An exactly analogous procedure for equation 2.12 gives
\[ g L_{-} g^{-1} = N_{-} \partial_{-} N_{-} \quad (2.17) \]
and
\[ g^{-1} \partial_{-} g + \partial_{-} N_{+}^{-1} N_{+} = N_{+}^{-1} L_{-} N_{+} - L_{-} \quad (2.18) \]
Substitution of 2.8 into 2.7 results in an expression for \( \tilde{A} \) which we can be reduced using 2.15, 2.18 to give the following simple expressions for \( A_{\pm} \):
\[ \begin{align*}
A_{+} &= g^{-1} \partial_{+} g_{-} + g_{+}^{-1} L_{+} g_{+}, \quad (2.19) \\
A_{-} &= g_{-}^{-1} L_{-} g_{-} + g_{+}^{-1} \partial_{-} g_{+} \quad (2.20) \\
\end{align*} \]
This is most of the work that we need to deduce the general solution. Comparing this result with an expression for the connection in terms of the dynamical variable $h$, for example 2.6, we find that the choices

$$ L_\pm = r_\pm^1(x^\pm)X_\pm r_\pm^1(x^\pm), $$ (2.21)

$$ r_+ = g_+, $$ (2.22)

$$ h = r_-g_-, $$ (2.23)

reproduce 2.6 when substituted into 2.19, 2.20. Here $r_\pm(x^\pm)$ are arbitrary chiral fields. This establishes the correctness of the choice 2.9. From 2.23, 2.22 and 2.10 we deduce the solution for $h$ in the form

$$ h = r_-g_+^1r_+ = r_-N_+^{-1}M_-^{-1}M_+N_-r_+. $$ (2.24)

Of course this insufficient to explicitly evaluate a solution for $h$ as the fields $N_\pm$ are still undetermined. Leznov and Saveliev have found an elegant solution to this problem using the representation theory of $\Gamma$ and $\gamma$. There will be representations of $\gamma$ with special vectors which are annihilated by all the elements of $\gamma_+$, and in their duals there will be vectors which are annihilated by $\gamma_-$ on the right. Let us denote the sets of these vectors by $I_L$ and $I_R$ respectively. Taking matrix elements of $h^{-1}$ we find

$$ \langle \chi | h^{-1} | \psi \rangle = \langle \chi | r_-^{-1}M_+^{-1}M_-r_-^{-1} | \psi \rangle, $$ (2.25)

where $\langle \chi \rangle \in I_R$ and $| \psi \rangle \in I_L$. This expression is only useful when $\langle \chi \rangle$ and $| \psi \rangle$ belong to the same representation, of course. By this means we have eliminated the unknown $N_\pm$ factors at the expense of only being able to determine these matrix elements of $h^{-1}$. By taking all possible $\langle \chi \rangle | \psi \rangle$ we should, in principle, be able to reconstruct $h^{-1}$.

In the simplest examples, the abelian Toda theories which are usually studied, the gradation of $\gamma$ is just by the height of the roots. The highest-weight state of any representation will be annihilated by all of the step-operators corresponding to positive roots. Since in this case $\gamma_0$ is just the Cartan subalgebra, $h^{-1}$ lies in the maximal torus of $\Gamma$ and so we can reconstruct it by taking the diagonal matrix elements between highest-weight states of only the fundamental representations of $\gamma$. This result was used extensively in [8] and [13] to discuss the soliton solutions.

The purpose of establishing the above results is that the general solution can often be used to simplify expressions which we might wish to calculate. These properties will be used extensively in the rest of this paper.

3 Defining the Non-abelian Affine Theories

3.1 Gradations and Embeddings

The last section shows that, in order to define an integrable theory of Toda type, we need an integer-graded Lie algebra and some specified elements $X_\pm \in g_{\pm 1}$. Then the equation of motion 2.4 with the field $h$ lying in $\Gamma_0 = \exp(\gamma_0)$ will
always be integrable. The question then arises as to what further conditions we should impose on $X_\pm$ in order that (2.4) be of physical interest. There are a variety of approaches to this problem. In the original work of [10] both affine and finite non-abelian Toda theories were discussed. The essence of their approach is the following: we consider the inequivalent embeddings of $A_1$ subalgebras (with Chevalley basis $H, T_\pm$) in the finite algebra $g$. These were classified by Dynkin [16]. Dynkin proves that the expansion of $H$ over the generators $\{h_i\}$ of a Cartan subalgebra of $g$ is sufficient to uniquely determine the embedding, up to equivalence. The adjoint action of the embedded subalgebra can then be used to split $g$ into a set of submodules, each labelled by the spin $l$. Clearly the adjoint action of $\frac{1}{2}H$ provides a half-integer grading of the algebra, thus
\[
\left[ \frac{1}{2}H, g_M \right] = M g_M
\] (3.1)
and
\[
g = \bigoplus_{M=-L}^L g_M
\] (3.2)
where $L$ denotes the greatest value of $l$.

When all the $l$ are integers, the embedding is said to be integral, giving a gradation of $g$ over the integers. The embedding also singles out a natural choice of the elements $X_\pm$, namely $T_\pm \in g_{\pm 1}$. In this case (2.4) is often called conformal Toda, of either abelian or non-abelian type. Note that the field $h$ will generally not lie in an abelian group.

The much-studied abelian Toda models correspond to one particular embedding, called the principal embedding. In this case $H$ has the property that $\alpha_i(H) = 2$ for all the simple roots $\alpha_i$. The principal gradation given by the adjoint action of $\frac{1}{2}H$ is the usual gradation into step-operator subspaces of equal root height. In particular $g_0$ is just the Cartan subalgebra. The spectrum of $l$ in this special case runs over a set of integers called the exponents of the Lie algebra and $L$ is one less than the Coxeter number [17]. These abelian systems are well-represented in the literature for their wealth of structure, in particular for providing concrete example of W-algebras. (For a review of this vast subject see [18]. W-algebra structure of the more general non-abelian theories is discussed at the classical level in [14].)

All these types of theory based on finite $g$ can also be obtained by gauging the WZNW model (see [20]), which provides another way to obtain the Leznov-Saveliev solution 2.25.

In what remains of this paper we shall be interested in the affine Toda theories. In [10] the authors proceed to the affine non-abelian Toda theory via the above considerations. Having defined a gradation of $g$ using an $A_1$ embedding they consider $g$ as a subalgebra of the corresponding loop algebra in the usual way, i.e. $g \rightarrow g \otimes C[t, t^{-1}]$. They define a gradation in the loop algebra using the adjoint action of $\frac{1}{2}H + (L+1)d (d \equiv d/dt)$, and define elements

\footnote{From now on we will use $g$ when $\gamma$ is a finite algebra and $\hat{g}$ when it is a Kac-Moody algebra.}
\[ \hat{X}_\pm = X_\pm + t^{\pm 1}X_{\mp M}, \] where \( X_{\mp M} \) are some arbitrary elements of \( g_{\mp L} \); the \( \hat{X}_\pm \) are then used to define the theory via (2.4). This presentation has the advantage that each affine theory appears as a deformation of the corresponding finite theory. Unfortunately it has the disadvantage that the definition is too vague to allow much progress to be made. I try to explain how a related definition will have physical as well as calculational advantages.

### 3.2 Abelian Toda as a Model Toda Theory

The key to this alternative definition is the observation in [15] that, in the special case of the abelian theory, the elements \( \hat{X}_\pm \) can be related to the lowest elements of a graded Heisenberg subalgebra of the Kac-Moody algebra. From the above definitions we would obtain expressions

\[ \hat{X}_\pm = \sum_{i=1}^{r} \sqrt{k_i} X_{\pm \alpha_i} + t^{\pm 1}X_{\mp \psi} \]  

where \( k_i = \sum_j K_{ij}^{-1} \). The point is that in this case the coefficients of the step-operators can be changed by rescaling the coordinates and shifting the origin of the fields, without affecting the integrability of the theory. In [3] the choice

\[ \hat{X}_\pm = \sum_{i=0}^{r} \sqrt{m_i} \hat{X}_{\pm \alpha_i} \]  

was made. The advantage of this choice is that the commutation relation

\[ [\hat{X}_+, \hat{X}_-] = K \]  

is satisfied. Working in the unextended loop algebra (or equivalently a realisation of the Kac-Moody algebra in which \( K = 0 \)) this means that \( \hat{X}_+ \) and \( \hat{X}_- \) commute, and so by comparing with (2.4) we see that the theory admits solutions with \( h \) constant and commuting with \( \hat{X}_- \). In this abelian case this means that \( h \) has to lie in the centre of \( G \). The existence of well-defined degenerate vacuum solutions means that we would expect to find soliton solutions as the solutions of minimal energy which interpolate them. It is also the origin of the so-called affine Toda particles, which are the quanta of the modes obtained by diagonalising the quadratic term in the potential about this minimum, as usual in a perturbative quantum field theory. These particles have been extensively studied in the literature; their masses have an interesting algebraic origin, and an exact S-matrix has been postulated. (For an up-to-date account see e.g. [4], [6], [7]).

### 3.3 Kac-Peterson Construction

The problem, then, is to generalise this procedure to the rest of the affine Toda theories. Recall that we are searching for a definition of \( \hat{X}_\pm \in \hat{g}_{\pm L} \) such that

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\[ ^2 \text{Actually to make the equation of motion look more like those generally studied in elementary particle physics we generally introduce some extra factors in the definition of } \hat{X}_\pm \text{ - a mass scale and a coupling constant. These do not affect the integrability, of course.} \]
the commutation relation 3.5 will still be valid. To do this we describe an alternative presentation of the Kac-Moody algebra due to Kac-Peterson [21].

Given a finite simple Lie algebra \( g \), let us fix a Cartan subalgebra \( H_0 \), and a set of simple roots \( \alpha_i \) with respect to it. Consider an element \( w \) belonging to some particular conjugacy class of the Weyl group. It is well known [22] that \( w \) may be lifted to an inner automorphism \( \sigma_w \), implemented by conjugation with \( S_w \), of \( g \) which acts as \( w \) on \( H_0 \), which is canonically isomorphic to the root space \( H_0^* \). This automorphism is of not unique since any redefinition of \( S_w \) of the form \( S_w \to T_1 S_w T_2 \), where \( T_1, T_2 \) are elements of the maximal torus of \( g \), i.e. they commute with of all of \( H_0 \), will also have the right action on \( H_0 \).

Let us denote the order of \( w \) by \( m \). Then \( S_w \) splits \( g \) into a direct sum of eigenspaces of eigenvalues \( \kappa = e^{2\pi ik} \), where \( k \in \mathbb{Z} \). (Note that \( (S_w)^m \) is not necessarily the identity, this occurs for much the same reason that the spinor representation of \( SO(3) \) is double-valued.) When all of the \( k \) lie in \( \mathbb{Z}/m \) we say that the class is integral. We denote the eigenspace of eigen value \( \kappa \) by \( g_\kappa \).

\( S_w \) can be written in the form

\[
S_w = \exp (2\pi ix).
\] (3.6)

Imposing the additional requirements that \((x, H_0) = 0\), and \([x, g_0] = 0\) we are now in a position to define a related basis for the loop algebra \( \tilde{g} = g \otimes \mathbb{C}[t, t^{-1}] \). We write

\[
a'(k) = t^k t^{-x} a_\kappa t^x
\] (3.7)

where \( a_\kappa \in g_\kappa \) is the projection of \( a \in g \) onto the subspace \( g_\kappa \). Note that this is an element of \( \tilde{g} \), as can be checked by taking \( t \to e^{2\pi i t} \). \( a'(k) \) can in turn be lifted to the full Kac-Moody algebra \( g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}d \oplus \mathbb{C}K \) using the formula

\[
a(k) = a'(k) - \delta_{k,0}(x, a)K
\] (3.8)

The algebra is \( \mathbb{Z}/2 \)-graded using the adjoint action of \( d' = m(d + x) \), i.e.

\[
[d', a(k)] = mka(k);
\] (3.9)

as we shall later show, this \( d' \) is not necessarily the same as the one defined earlier for some appropriate choice of embedding. Note that 3.8 is equivalent to the condition that \((d', a(k)) = 0\). From the definition 3.8 we get the following commutation relation, which is valid provided that either \( a \) or \( b \) is a pure eigenvector of the conjugation by \( S_w \):

\[
[a(k), b(l)] = [a, b](k + l) + K \delta_{k+l,0}(a, b)k.
\] (3.10)

The last term in this expression is most easily deduced by taking the inner product of both sides of the equation with \( d' \), and using the invariance of the Killing form. Recalling that \( H_0 \) is invariant under \( \sigma_w \) we can decompose it into a direct sum of graded eigenspaces, and choose a basis \( h_\kappa^r \in g_\kappa \) such that

\[
(h_{\kappa_1}^{r_1}, h_{\kappa_2}^{r_2}) = m \delta_{r_1, r_2} \delta_{1, \kappa_1 \kappa_2}.
\] (3.11)
The index $\tau$ allows for possible degeneracy of the spectrum. With this convention and the definition $\hat{E}_{m\kappa}^\tau \equiv h_{\kappa}^\tau(k)$ \[3.11\] yields

$$\left[ \hat{E}_I^\tau, \hat{E}_J^\nu \right] = \delta_{IJ} \delta_{\tau \nu} IK. \tag{3.12}$$

This is a graded Heisenberg subalgebra of $\hat{g}$. At the end of this section we will also detail a suitable basis for the rest of the algebra, although this is not necessary to define the theory we will need it later.

We repeat the fact that the grade $\pm 1$ elements $\hat{X}_+$ and $\hat{X}_-$ commute, up to an element of the centre, was crucial for the existence of a non-trivial vacuum structure, and hence soliton solutions, of abelian affine Toda theory. We now explain a means to achieve this structure in the more general case.

### 3.4 Conjugacy Classes and $A_1$ Embeddings

The question we have to answer is this: under what conditions will the graded Heisenberg algebras which are automatically produced by the Kac-Peterson procedure contain elements with grades $\pm 1$. The possible conjugacy classes of the Weyl groups were discussed by Carter \[23\]. Unfortunately the picture that emerges is clouded by a number of exceptional cases, but the vast majority of the classes are easy to describe.

Firstly, we define a regular semisimple subalgebra of $g$ to be a semisimple subalgebra which is a $\mathbb{C}$-span of some subset of the step operators. Such subalgebras were classified by Dynkin \[16\], as a step in the classification of non-conjugate embeddings of $A_1$ into $g$. Aside from a few exceptions, the $A_1$ embeddings are principal in some regular semisimple subalgebra (RSS). Notice that, as a consequence of the definition, the root system of this subalgebra is just a subsystem of that of $g$. Carter’s result is that the majority of classes of the Weyl group of $g$ contain elements which are in the Coxeter class of some RSS. In the case of the algebras of types $A_r$, $B_r$, $C_r$ and $G_2$ all of the classes are of this form.

Before turning to the exceptional cases we discuss the implications for the construction of affine Toda theories of this result. Clearly, to have a grade 1 element\[9\] we must have an eigenvalue of the action of $w$ on the root space which is $e^{2\pi i/m}$. We investigate under what conditions this will be the case.

Let us fix the Cartan subalgebra $H_0$ of $g$. Fix also a regular semisimple subalgebra $g_R$ of $g$, and thus the element $w$, and the automorphism $\text{Ad}S_w$. $g_R$ has a Cartan subalgebra $H_R \subset H_0 \[10\]$. We can orthogonally decompose with respect to the Killing form, $H_0 = H_R \oplus H_\perp$, where $(H_R, H_\perp) = 0$. Because $H_\perp$ is thus orthogonal to all of the simple roots of $g_R$ it commutes with the whole of it. This fact is of interest in the discussion of some the physical structure of the non-abelian theories (appendix \[8\]).

Now, the action of the Coxeter element is well known (originally from the work of \[11\]). In particular the root $s$ splits into eigenvectors of eigenvalues $\kappa = e^{2\pi ik}$ where $mk$ runs over the exponents of the algebra, and 1 is always

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3The non-degeneracy of the Killing form on the Heisenberg subalgebra means that the existence of a grade 1 element guarantees the existence of a grade -1 element.
a member of this set. When \( w \) belongs to the Coxeter class of some \( \mathfrak{g}_R \subset \mathfrak{g} \) the position is not that different; \( H_R \) will decompose into subspaces according to the exponents of \( \mathfrak{g}_R \). Because \( H_\perp \) commutes with all of \( \mathfrak{g}_R \) it commutes with \( S_w \) and so its eigenvalue under conjugation will simply be unity. When \( \mathfrak{g}_R \) is actually simple this means that there is always a unique eigenvalue \( e^{2\pi i/m} \), and so there is a natural and unique choice of \( \hat{X}_+ \) as the corresponding grade 1 element of the Heisenberg subalgebra. Now when \( g_R \) is not simple we know that it decomposes into a direct sum of simple ideals. The order of the Coxeter element of \( g_R \) is the least common multiple of the orders of the Coxeter elements of these ideals, which are just their Coxeter numbers. The only circumstance in which \( w \) can have the appropriate eigenvalue, and thus allow us to define \( \hat{X}_+ \), is when this least common multiple is actually equal to the greatest of these Coxeter numbers (of which all the others must be factors). If there is more than one ideal with this Coxeter number then the eigenvalue will be similarly degenerate. Corresponding to such \( g_R \) will be a whole class of non-abelian Toda theories.

The relationship of this definition to that already discussed is now clearer. Instead of adding \( t^{\pm 1}X_\pm M \) to \( X_\pm \) to produce elements \( \hat{X}_\pm \) of appropriate grade, we define a different gradation of the Kac-Moody algebra, using \( d' = m(d + x) \) in place of \( \frac{1}{2}H + (L + 1)d \), and append \( t^{\pm 1}X_\mp M_R \), where \( M_R \) denotes the maximal root of the embedded semi-simple subalgebra \( \mathfrak{g}_R \), as opposed to the maximal root(s) of the whole algebra. Only when \( M_R \) is one of these will the new definition of the theories be consistent with the previous one.

The next task is to deal with the remaining, exceptional classes of the Weyl group. These are classified in \cite{23}, to which we refer for further details as no particular picture emerges. \cite{23} also contains a table of the characteristic polynomials of such classes. Some of them do indeed have roots \( e^{2\pi i/m} \) and so would seem to be suitable for defining exceptional Toda theories.

This completes our discussion of the possibilities for defining Toda theories. From now on we assume that we have a chosen one of the suitable \( w \) and specified the elements \( \hat{X}_\pm \). We have already remarked that the admissibility of soliton solutions is guaranteed with the new approach, and we shall make that more concrete later on.

### 3.5 \( \hat{F}(\alpha, z) \) and Vertex Operators

So far we have only made use of the Kac-Peterson basis for the purpose of defining the theory, and to do this we only needed the affinisations of the elements of the Cartan subalgebra of \( \mathfrak{g} \). To actually use the Leznov-Saveliev method of solution we will need a basis for the rest of \( \hat{\mathfrak{g}} \) which we now discuss.

A logical basis to try would be that spanned by elements of the form \( X_\alpha(k) \), i.e. by affinising the eigenvector components of the step-operators of \( \mathfrak{g} \). Let us denote the number of roots on the \( w \)-orbit of \( \alpha \) by \( m_\alpha \). We can use the formula

\[
a_\kappa = \frac{1}{2m_\alpha} \sum_{p=0}^{2m_\alpha-1} \kappa^{-p} \sigma^p_w(a),
\]

(3.13)
where $\kappa^{2m_\alpha} = 1$ to explicitly decompose an element $a \in \mathfrak{g}$. When applied to $X_\alpha$ we see that each of the graded components must be a linear combination of the step-operators $X_\beta$, where $\beta$ lies on the same $w$-orbit of the root system as $\alpha$. From the expression 3.13 we can see that

$$ (X_{\sigma^q(\alpha)})^\kappa = \kappa^q (X_\alpha)^\kappa. \quad (3.14) $$

This means that to get a proper basis we only need to use the graded components of one representative of each of the orbits. Furthermore, since there are $m_\alpha$ linearly independent step operators on a $\sigma_w$-orbit, we conclude that there are precisely $m_\alpha$ non-zero $(X_\alpha)^\kappa$. There are just two possibilities for what the eigenvalues $\kappa$ can be, depending on the properties of $\sigma_w$. In the first case $\sigma_m (X_\alpha) = X_\alpha$. Then we can see by inspection of 3.13 that only the $(X_\alpha)^\kappa$ for which $\kappa = \kappa^m (X_\alpha)$ can be non-zero. In the second case, $\sigma_m (X_\alpha) = -X_\alpha$; this time $\kappa^m = -1$ is the appropriate condition. We stress here that a class may be integral, and yet still have orbits of order $m_\alpha$ upon whose step operators $\sigma_m = -1$.

From now on we shall denote the quantities $X_\alpha (k)$ by $\hat{F}_{(km)}(\alpha)$. From 3.10 we deduce

$$ [\hat{E}^*_I, \hat{F}_J(\alpha)] = \alpha (h^*_\alpha) \hat{F}_{I+J}(\alpha), \quad (3.15) $$

with $\kappa = e^{2\pi i / M}$.

Kac-Peterson then show that, in the basic representation of the simply-laced affine algebras, the $\hat{F}_J(\alpha)$ can be realised as the modes of a vertex operator, constructed out of the $\hat{E}^*_I$. We shall only describe this construction briefly, extracting the features that are of importance in the rest of our argument. We define the quantities $\hat{F}(\alpha,z)$

$$ \hat{F}(\alpha,z) = \sum_k z^{-mk} \hat{F}_{mk}(\alpha). \quad (3.16) $$

Note that

$$ \hat{F}(w(\alpha),z) = \pm \hat{F}(\alpha,e^{2\pi i / m} z), \quad (3.17) $$

where the sign depends on how we choose the step operator basis for $\mathfrak{g}$. The commutation relation

$$ [\hat{E}^*_I, \hat{F}(\alpha,z)] = \alpha (h^*_\alpha) z^I \hat{F}(\alpha,z) \quad (3.18) $$

follows directly from the equation 3.13. From this it is straightforward to show that $U_\alpha(z)$, given by

$$ U_\alpha(z) = \exp \left( \sum_{k<0} \frac{1}{k} z^{-mk} h_\alpha(k) \right) \hat{F}(\alpha,z) \exp \left( \sum_{k>0} \frac{1}{k} z^{-mk} h_\alpha(k) \right), \quad (3.19) $$

has the property that it commutes with the Heisenberg subalgebra spanned by the $\hat{E}^*_I$. It can be shown that the various $U_\alpha(z)$ form an abelian group.

---

4 When the class of $w$ is integral we will only have integral powers of $z$ appearing in 3.10.
extended by the complex numbers. It turns that we have the vertex operator representation

\[ \hat{F}(\alpha, z) = \exp \left( - \sum_{k \neq 0} \frac{1}{k} z^{-mk} h_\alpha(k) \right) :U_\alpha(z) : \]  

(3.20)

where the colons denote normal ordering, familiar from quantum field theory. The space upon which \( \hat{F}(\alpha, z) \) acts is the tensor product of a certain representation of the extended group, and the Fock space built up by the \( E_i^- \), for which \( I < 0 \). \( U_\alpha(z) \) can be thought of as taking account of the zero modes, c.f. the homogeneous vertex operator construction discussed in [24], for example.

The only element of this construction that we will make use of concerns the expression for the normal-ordered product of two of the \( \hat{F}(\alpha, z) \). Using an argument essentially the same as that in [15] we find

\[ :\hat{F}(\alpha, z_1)\hat{F}(\beta, z_2): = S_{\alpha\beta}(z_1, z_2) \exp \left( - \sum_{k} \frac{1}{k} \left[ z_1^{-mk} h_\alpha(k) + z_2^{-mk} h_\alpha(k) \right] \right) :U_\alpha(z_1)U_\beta(z_2): . \]

(3.21)

The numerical ‘S-matrix’ factor \( S_{\alpha\beta} \) is given by

\[ S_{\alpha\beta}(z_1, z_2) = \prod_{n=0}^{m-1} \left( 1 - e^{-2\pi in/m} \frac{z_2}{z_1} \right) w^{n(\alpha)\beta} \]  

(3.22)

From this we can deduce the key fact that \((\hat{F}(\alpha, z))^2 = 0\). The reason this is important is that it means that \( \hat{F}(\alpha, z) \) is nilpotent within any unitary highest weight representation, and so the ‘group element’ \( \exp(\hat{F}(\alpha, z)) \) has a well-defined meaning.

In conclusion, it seems clear that this definition of the general affine Toda theory has at least one advantage over the traditional one, in that we now have a concrete presentation related from the outset to vertex operators.

## 4 Abelianisation

Having established a straightforward description of the Toda theories of affine type we now consider how to extract the simplest conserved currents using the abelianisation procedure (see [25] and references therein).

### 4.1 Review

We include a description of the abelianisation procedure in the spirit of [25], but substantially revised to suit the greater generality of the systems considered here. The essential principle is the following, suppose that we can find some gauge transformation of the connection \( 2.6 \) which will leave it lying in some subalgebra of \( \hat{g} \) commuting with a particular element \( X \), the centraliser \( Z(X) \) of \( X \). Such a gauge transformation will of course depend on the dynamical variables, just as the transformation linking \( 2.2 \) with \( 2.5 \) and \( 2.6 \) did. Suppose \( Y \) is some element of the centre of \( Z(X) \); taking the inner product of \( Y \) with
the zero-curvature condition $2.3$, the commutator term drops out and so we are left with
\[ \tilde{d} \left( \tilde{A}', Y \right) = 0. \] (4.1)

Note that we have denoted the transformed connection by $\tilde{A}'$. This can be rewritten as the conservation equation
\[ \delta(*) \tilde{A}', Y) = 0. \] (4.2)

From this conserved current we can of course extract a conserved charge in the usual fashion.

In the case of the affine Toda theories the grading structure gives us a means of actually implementing the abelianisation procedure. Starting with the connection $2.6$ we note that the only component with a positive grade is the $\hat{X}_+$. This raises the possibility that we might be able to use gauge transformations in $\hat{G}_-$, which would leave the positive-graded component invariant, since $(f \hat{X}_+ f^{-1}) \cap \hat{g}_+ = \hat{X}_+ \forall f \in \Gamma_-$. Now we use the following iterative procedure. Although such a gauge transformation must necessarily depend on an infinite number of parameters, it can in fact be described iteratively, using the following lemma:

**Lemma 1** Suppose that we have managed to perform a gauge transformation such that the only non-zero components of the connection which do not commute with $\hat{X}_+$ are in grades $n$ and below. Then there exists a gauge transformation by a $\hat{G}_-$ valued section of the form $\exp(\omega_{n-1})$, with $\omega_{n-1} \in \hat{g}_{n-1}$ which will cancel the component of $\tilde{A}$ in $\hat{g}_n$ which does not commute with $\hat{X}_+$.

The proof of this is quite straightforward, given the new definition of the non-abelian affine Toda theory. A transformation of this form will change the connection by adding $[\omega_{n-1}, \hat{X}_+] \in \hat{g}_n$, plus some terms of lower grade which are not of interest here. Thus all we have to prove is that $\hat{g}_n$ is the direct sum of its intersection with $Z(\hat{X}_+)$ and $\text{ad}(\hat{X}_+) \hat{g}_{n-1}$. This proof is straightforward if we use the results of the previous section. We established that there was a graded basis for the Kac-Moody algebra, such that $\hat{g}_n$ is spanned by the $\hat{E}_\tau$ and the orbit representatives $\hat{F}_n(\alpha)$. Using the commutation relation $3.17$ and the fact that $\hat{X}_+$ can be expanded over the $\hat{E}_\tau$, we see that the only elements of this basis for $\hat{g}_n$ which do not commute with $\hat{X}_+$ must indeed be the $\text{ad}\hat{X}_+$ images of elements of the basis of $\hat{g}_{n-1}$.

It is clear from this result that the desired transformation can be found iteratively by removing the non-commuting component of successively lower graded subspaces, starting with $\hat{g}_0$. By the use of an exactly analogous argument starting with the connection $2.5$ and proceeding in the opposite direction we will find another infinite set of conserved currents.

The gauge transformations are not unique, since any transformation using an element of the exponential of $Z(\hat{X}_+)$ will leave the connection in $Z(\hat{X}_+)$. We now exploit this freedom to find a particularly nice form for the lowest conserved currents, namely the chiral components of the energy-momentum tensor.
4.2 Explicit Expression for the Lowest Currents

From now on we restrict attention to the theories for which \( w \) belongs to an integral class (but see [26] for a partial discussion of the general case). In order to find explicit expressions for the currents we need to know the elements of the centre of \( Z(\hat{X}_+) \). A form for these is conjectured in appendix 11, but here we shall only need the canonical element, namely \( \hat{X}_+ \) itself. To calculate its inner product with the transformed connection we need only know the component of this connection lying within \( \hat{g}_{-1} \). This we can calculate using only the first two steps of the iterative procedure.

Starting from the form for the connection 2.6 we derive the condition for \( \omega_{-1} \), finding

\[
\left[ \hat{X}_+, \left[ \omega_{-1}, \hat{X}_+ \right] + h^{-1} \partial_+ h \right] = 0.
\] (4.3)

Performing the next transformation as well we find the component of \( A_+ \) with grade -1 to be

\[
\left[ \omega_{-2}, \hat{X}_+ \right] + \frac{1}{2} \left[ \omega_{-1}, \left[ \omega_{-1}, \hat{X}_+ \right] \right] + \left[ \omega_{-1}, h^{-1} \partial_+ h \right] - \partial \omega_{-1}.
\] (4.4)

Taking the inner product with \( \hat{X}_+ \) and using the invariance of the Killing form we get

\[
(\hat{X}_+, A_+) = \frac{1}{2} (h^{-1} \partial_+ h, [\hat{X}_+, \omega_{-1}]) - \partial_+ (d', [\hat{X}_+, \omega_{-1}])
\] (4.5)

The \( d' \) has been introduced purely for convenience. Now in general the solution of [13] for \( [\hat{X}_+, \omega_{-1}] \) will be \( h^{-1} \partial_+ h \) plus a term in the kernel of \( \text{ad} \hat{X}_+ \). Denote by \( P \) the orthogonal projection into this kernel, then \( [\hat{X}_+, \omega_{-1}] = (1 - P) h^{-1} \partial_+ h \). Substituting into (4.5) yields

\[
(\hat{X}_+, A_+) = \frac{1}{2} (h^{-1} \partial_+ h, h^{-1} \partial_+ h) - \partial_+ (d', h^{-1} \partial_+ h) + \ldots,
\] (4.6)

where the dots denote some terms which can be seen to be independent of \( x^- \) by applying the projection \( P \) to the equation of motion (or alternatively dispensed with by exploiting the remaining gauge freedom). These can be dropped (we discuss the field \( Ph^{-1} \partial_+ h \) in appendix [1]). We can easily show that the transformed \( (\hat{X}_+, A_-) \) vanishes due to the equation of motion, (aside from more terms similar to the above) so this means that we have a chiral conserved current given by \( J^- = (\hat{X}_+, A_+) \), \( J^+ = 0 \). An analogous expression for another chiral current can be derived by starting from 2.5 and performing gauge transformations with elements of \( \hat{G}_+ \).

We shall now show that we can use the results needed for the derivation of the general solution to greatly simplify the expression 4.6.

4.3 Chiral Current in Terms of Free Fields

Using the expression for \( h \) given by 2.24 we can deduce the following formula for \( h^{-1} \partial_+ h \):

\[
h^{-1} \partial_+ h = r_+^{-1} \partial_+ r_+ - r_+^{-1} \partial_+ g g^{-1} r_+.
\] (4.7)
We find for the ‘kinetic term’ of the expression for the conserved charge
\[
\frac{1}{2}(h^{-1}\partial_+h, h^{-1}\partial_+h)
= \frac{1}{2}(r_+^{-1}\partial_+r_+, r_+^{-1}\partial_+r_+) - (\partial_+r_+^{-1}, \partial_+gg^{-1}) - (L_+, N_-L_+N_-^{-1})(4.8)
\]
To arrive at this formula we make extensive use of the grading properties of the inner product, and 2.16 from the derivation of the Leznov-Saveliev solution.

The second term is trickier to reduce. We get
\[
\partial_+(d', h^{-1}\partial_+h) = \partial_+ \left( d', \left( N_-^{-1}L_+N_-, N_-^{-1}\partial_+N_- \right) \right), \quad (4.9)
\]
where we have made use of 2.16 in particular. Expanding the derivative on the last term of this expression gives
\[
\left( d', \left[ N_-^{-1}L_+N_, N_-^{-1}\partial_+N_- + N_-^{-1}\partial_+r_+^{-1}N_-, N_-^{-1}L_+N_- \right] \right)
= \left( d', \left[ N_-^{-1}L_+N_-, N_-^{-1}\partial_+N_- - N_-^{-1}\partial_+r_+^{-1}N_- \right] \right). \quad (4.10)
\]
Now we examine the grading structure of this expression. Because this contains a commutator of a quantity of grades +1 and below with a quantity of grade zero and below, it follows that the only contribution to the inner product with \(d'\) must come from the grade 1 part of the first argument of the commutator, which is just \(L_+\). This gives us
\[
\left( L_+, N_-^{-1}\partial_+N_- - N_-^{-1}\partial_+r_+^{-1}N_- \right). \quad (4.11)
\]
This in turn leads to
\[
\left( L_+, -N_-L_+N_-^{-1} - N_-^{-1}\partial_+r_+^{-1}N_- \right), \quad (4.12)
\]
when we use 2.16. We recognise the first term as cancelling part of the ‘kinetic’ part of the conserved current 4.8. For the remainder of 4.12 we use 2.16 again to remove \(L_+\) yielding
\[
- \left( \partial_+gg^{-1}, \partial_+r_+^{-1} \right), \quad (4.13)
\]
which cancels with another term from the ‘kinetic’ part 4.8. Thus we are left with the remarkable result that
\[
(\hat{X}_+, A_+) = \frac{1}{2}(r_+^{-1}\partial_+r_+, r_+^{-1}\partial_+r_+) - \partial_+(d', r_+^{-1}\partial_+r_+). \quad (4.14)
\]
Put another way, the conserved current takes exactly the same form in terms of the free fields. Of course this makes it obvious that it must be chiral and conserved. The same result holds for the chiral current got from 2.5.

In 3 this result was proven for the abelian theories using 2.25 and representation theory. We see from the above that it is not actually necessary to take the matrix elements, indeed for the non-abelian theories it would probably prove too difficult anyway. 3 is still interesting because of the way these cancellations translate into representation theory, where they arise due orthogonality of different highest weight states in a tensor product decomposition.

We shall now discuss how the above result is of value in the study of the soliton solutions of the general theories.
5 Canonical Energy-Momentum Tensor

In this section I present the proof that the lowest conserved charges actually correspond to the chiral components of the improved energy-momentum tensor.

5.1 General Toda Action

In order to construct a canonical energy-momentum tensor we first need an action for the theory. This action $S$ takes the form

$$S = S_{\text{WZNW}} + 2\eta \int_{\partial B} \tilde{\omega} \left( (\hat{X}_+, h^{-1} \hat{X}_- h) - (\hat{X}_+, \hat{X}_-) \right),$$

(5.1)

where the Wess-Zumino-Novikov-Witten type action is defined to be

$$S_{\text{WZNW}} = \eta \left[ \frac{1}{2} \int_{\partial B} \tilde{\omega} g^{\mu\nu} \left( h^{-1} \partial_\mu h, h^{-1} \partial_\nu h \right) + \frac{1}{3} \int_{B} (h^{-1} \tilde{d}h \wedge h^{-1} \tilde{d}h \wedge h^{-1} \tilde{d}h) \right].$$

(5.2)

Here $\tilde{\omega}$ is the usual measure form on a metric manifold, and $B$ is a manifold whose boundary is space-time. See e.g. [24] or [27] for a review of the WZNW model. The equation of motion follows from the variation of this action in the usual way.

5.2 Splitting of Energy-Momentum Tensor

The canonical energy-momentum tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{|\det g|}} \frac{\delta S}{\delta g^{\mu\nu}}$$

(5.3)

Thus we get

$$T_{\mu\nu} = \eta \left[ \left( h^{-1} \partial_\mu h, h^{-1} \partial_\nu h \right) - \frac{g_{\mu\nu}}{2} \left( h^{-1} \partial_\alpha h, h^{-1} \partial_\alpha h \right) - 2g_{\mu\nu} \left( (\hat{X}_+, h^{-1} \hat{X}_- h) - (\hat{X}_+, \hat{X}_-) \right) \right].$$

(5.4)

Following much the same procedure to that of [3] we try to split this tensor into a traceless part plus a total derivative ‘improvement’. Define the total derivative term $C$ by

$$C_{\mu\nu} = 2\eta \left[ \left( d'_\nu, \partial_\mu \left( h^{-1} \partial_\nu h \right) - g_{\mu\nu} \partial_\alpha \left( h^{-1} \partial^\alpha h \right) \right) + g_{\mu\nu} \left( \hat{X}_+, \hat{X}_- \right) \right].$$

(5.5)

Note that this is symmetric; this can easily be shown by expanding the derivative and using the fact that $d'$ commutes with all of $\hat{g}_0$. Now we write

$$T_{\mu\nu} = C_{\mu\nu} + \Theta_{\mu\nu}.$$
Calculating the components of $\Theta$ in a light-cone basis (we need to use the equation of motion 2.4 to eliminate some terms) we find that it is diagonal, with

$$
\Theta_{++} = \eta \left[ (h^{-1} \partial_+ h, h^{-1} \partial_+ h) - 2 \left( d', \partial_+ \left( h^{-1} \partial_+ h \right) \right) \right],
$$

(5.7)

with an analogous expression for $\Theta_{--}$. It is no surprise that this should be just \ref{4.6} multiplied by a factor $\eta$.

Thus we find the same behaviour as in \ref{3}, the energy-momentum tensor splitting into two parts; one a traceless part dependent only on the free fields, and the other a total derivative. We can use this fact to calculate expressions for the energies and momenta of the soliton solutions of these theories.

6 Energy and Momentum of Soliton Solutions

In the previous section we obtained an expression for the energy and momentum of a solution of the affine Toda field theories constructed by the Leznov-Saveliev procedure. Of particular interest for physics are the soliton solutions, which, as we have already discussed, are expected to interpolate the degenerate vacua of the theory. In \ref{1} the specialisation of the general solution 2.25 which yields the soliton solutions was conjectured. We repeat this conjecture here for convenience.

6.1 Solitonic Specialisation

Let us discuss the physical significance of the splitting 5.6. $C_{\mu\nu}$ is a total derivative so it can be explicitly integrated to yield surface terms at infinity. We would expect the conserved charges of solitons, which are topological objects, to have this kind of behaviour. The other term, $\Theta_{\mu\nu}$, can be made to vanish simply by setting the free fields $r_\pm$ to zero. This rather drastic choice still leaves a large class of non-trivial solutions, as we shall show.

We can now solve the equations of motion 2.9 explicitly. Thus

$$
M_\pm = M_\pm(0) \exp \left( x_\pm \hat{X}_\pm \right),
$$

(6.1)

where $M_\pm(0)$ are some arbitrary group-valued constants. Inserting this into the solutions for matrix elements 2.25 gives

$$
\langle \chi | h^{-1} | \psi \rangle = \langle \chi | \exp \left( -x^+ \hat{X}_+ \right) \mathcal{G}_0 \exp \left( x^- \hat{X}_- \right) | \psi \rangle,
$$

(6.2)

The constant $\mathcal{G}_0$ is equal to $M_+(0)^{-1} M_-(0)$. In \ref{1} it is conjectured that the N-soliton solution is given by a choice of $\mathcal{G}_0$ of the form

$$
\mathcal{G}_0 = \exp \left( Q_1 \hat{F}(\gamma_1, z_1) \right) \cdots \exp \left( Q_i \hat{F}(\gamma_i, z_i) \right) \cdots \exp \left( Q_N \hat{F}(\gamma_N, z_N) \right).
$$

(6.3)

The parameters in this expression are:

\footnote{When the class of $w$ is integral we will only have integral powers of $z$ appearing in this expression.}
• \( z_i \) are some complex numbers which describe the rapidities of the component solitons, ordered so that \(|z_1| \geq |z_2| \geq \ldots \geq |z_i| \geq \ldots \geq |z_N|\).

• \( Q_i \) describe the positions of the solitons at some initial time.

• \( \gamma_i \) are some roots of \( g \). Roots belonging to the same orbit of \( w \) give the same species of soliton.

We will explain how the parameters can be given these identifications later on.

This form for \( G_0 \) has the feature that it allows \( 6.2 \) to be considerably simplified. From \( 3.15 \) it is obvious that the commutation relations

\[
[\hat{X}_\pm, \hat{F}(\alpha, z)] = q^\pm_\alpha z^{\pm 1} \hat{F}(\alpha, z) \tag{6.4}
\]

hold, with \( q^\pm_\alpha \) easily calculable from our knowledge of the definitions of \( \hat{X}_\pm \) (this is described in appendix \( 9 \)). Following \( 3 \) we shall use conventions so that \( q^-_\alpha = (q^+_\alpha)^* \). (The admissibility of this convention is a consequence of the existence of the Chevalley involution/definition of the adjoint of an element of the Lie algebra \( g \).) Using \( 6.4 \) we can re-write \( 2.25 \) as

\[
\langle \chi | h^{-1} | \psi \rangle = \langle \chi | \exp \left( Q_1 e^{-q^+_1 z_1 x^+ - q^-_1 z_1^{-1} x^-} \hat{F}(\gamma_1, z_1) \right) \ldots
\]

\[
\ldots \exp \left( Q_N e^{-q^+_N z_N x^+ - q^-_N z_N^{-1} x^-} \hat{F}(\gamma_N, z_N) \right) \exp \left( -x^+ x^- \frac{K(\hat{X}_+, \hat{X}_-)}{m} \right) \exp (K(\hat{X}_+, \hat{X}_-)) \tag{6.5}
\]

This explains why we identify \( \ln Q_i \) as describing the relative positions of the solitons. Given this form for the soliton solutions we now show how it is possible to explicitly calculate the energy and momentum of such a solution.

### 6.2 Calculating Surface Terms

Using the expression \( 5.5 \) for the improvement of the energy-momentum tensor we deduce that the light cone components of the energy and momentum of a soliton solution are of the form

\[
P^\pm = 2 \eta \int_{-\infty}^\infty dx \left( \pm \partial_x \left( \hat{d}' h^{-1} \partial_x h \right) + \frac{1}{\sqrt{2}} \left( \hat{X}_+, \hat{X}_- \right) \right). \tag{6.6}
\]

It is easy to show that the second, ‘classical renormalisation’, term will cancel with a similar contribution coming from the \( K \)-component of equation \( 6.5 \) substituted into the first term. This just leaves a portion which depends on which particular soliton solution we are considering. To evaluate this we need to introduce some elements of representation theory.

In Appendix \( 12 \) we introduce a particular highest weight representation \( V_{\tilde{\rho}} \) of the affine algebra with the following special properties:

• \( V_{\tilde{\rho}} \) is a highest weight representation with respect to the grading defined by \( d' \), i.e. all elements of positive grade annihilate the highest-weight state \( |\tilde{\rho} \rangle \).
• There is a Cartan subalgebra, $H_\Delta$, of $\hat{\mathfrak{g}}$ contained in $\hat{\mathfrak{g}}_0$. The orthogonal decomposition of $\hat{\mathfrak{g}}_0$ reads $\hat{\mathfrak{g}}_0 = H_\Delta \oplus \hat{\mathfrak{g}}_\Delta$. $|\hat{\rho}\rangle$ is annihilated by all of $\hat{\mathfrak{g}}_\Delta$.

• Given $h_\Delta \in H_\Delta$, $h_\Delta |\hat{\rho}\rangle = \hat{\rho}(h_\Delta)|\hat{\rho}\rangle$. Thus $|\hat{\rho}\rangle$ is a one-dimensional representation of both $\hat{\mathfrak{g}}_0$ and $\hat{\mathfrak{g}}_\Delta$.

• The following result holds for $s \in \hat{\mathfrak{g}}_0$:

$$ (d', s) = \langle \hat{\rho} | s | \hat{\rho} \rangle. \quad (6.7) $$

The utility of this representation is that we can replace the inner product with $d'$ occurring in 6.6 with a matrix element, which the Leznov-Saveliev solution allows us to calculate. We have

$$ P^\pm = \mp 2\eta \left[ (d', h^{-1}_\text{red} \partial \pm h\text{red}) \right]_{-\infty}^{\infty}, \quad (6.8) $$

where

$$ \langle \chi | h^{-1}_\text{red} | \psi \rangle = \langle \chi | \exp \left( Q_1 e^{-q_1^+ z_1^+ x^+ - q_1^- z_1^- x^-} \hat{F}(\gamma_1, z_1) \right) \cdots \exp \left( Q_N e^{-q_N^+ z_N^+ x^+ - q_N^- z_N^- x^-} \hat{F}(\gamma_N, z_N) \right) | \psi \rangle, \quad (6.9) $$

i.e. we have performed the classical renormalisation. Because $|\hat{\rho}\rangle$ is a one-dimensional representation we can re-write (6.8) to read

$$ P^\pm = \mp 2\eta \left[ \langle \hat{\rho} | h^{-1}_\text{red} | \hat{\rho} \rangle \langle \hat{\rho} | \partial \pm h\text{red} | \hat{\rho} \rangle \right]_{x = -\infty}^{\infty}. \quad (6.10) $$

The second matrix element can be expressed in terms of the first, which is given by the expression (6.9). Thus an explicit calculation is feasible. At this point it is appropriate to discuss the reality conditions that ought to be applied to both the definition of the theory and the solution (6.10) to produce physical soliton solutions.

6.3 Constraints on a Physical Toda Theory

As we remarked earlier it is customary in abelian Toda theories to introduce some extra constant parameters into the equation of motion (2.4). The reason for this is that the quadratic term in the potential can be diagonalised to yield a theory of bosons, whose interaction is governed by the higher terms. Of course this is the procedure used in quantum field theories since their inception. In this approximation, the abelian Toda equation is classically just a number of decoupled Klein-Gordon equations with real positive masses.

$$ \left( 2\partial_+ \partial_- + \mu_i^2 \right) \phi_i = 0. \quad (6.11) $$

This is done (e.g. [3]) by multiplying the definitions of $\hat{X}_\pm$ by $\pm \mu$, where $\mu$ is a real, positive mass parameter. We also customarily set $\eta = 1/\beta^2$, where $\beta$ is a coupling constant whose reality properties we shall decide on later.

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7 When we express $h$ as an exponential of some Lie algebra valued field.
Let us make these changes for the more general theories that we discuss here. We then get a revised version of equation 6.9:

\[ \langle \chi | h^{-1}_{\text{red}} | \psi \rangle = \langle \chi | \exp \left( Q_1 e^{\mu(q_1^{+} z^{+} - q_1^{-} z^{-})} \hat{F}(\gamma_1, z_1) \right) \ldots \langle \chi | \exp \left( Q_N e^{\mu(q_N^{+} z_N^{+} - q_N^{-} z_N^{-})} \hat{F}(\gamma_N, z_N) \right) | \psi \rangle. \]  

(6.12)

In particular the one-soliton solution is:

\[ \langle \chi | h^{-1}_{\text{red}} | \psi \rangle = \sum_{p}^{\mu} \langle \chi | (Q \hat{F}(\gamma, z))^{n} | \psi \rangle e^{n\mu(q^{+} z^{+} - q^{-} z^{-})}, \]  

(6.13)

where \( p \) is the greatest non-vanishing power of \( F(\gamma, z) \) within the representation.

Physically, a one-soliton solution has the property that it can be set to rest by a real Lorentz transformation. In two dimensions this just means changing the modulus of \( z \). The time dependence in the ansatz 6.13 enters through

\[ \frac{\mu}{\sqrt{2}} \left( q_1^{+} z - (q_1^{-})^{*} z^{-1} \right), \]  

(6.14)

and so we see that such a transformation can only be performed if

\[ q_1^{+} z = \epsilon |q_1| e^{\lambda}, \]  

(6.15)

with \( \lambda \) real, and \( \epsilon = \pm 1 \).

The conserved charges in equation 6.10 are found from the asymptotic behaviour of the solution 6.13. This is just a polynomial in

\[ \exp \left( \mu |q_1| \epsilon \left( e^{\lambda} + e^{-\lambda} \right) x/\sqrt{2} \right), \]  

(6.16)

as far as the space dependence is concerned, so its asymptotic behaviour depends on the sign of epsilon, being dominated by the terms of greatest or least degree in the sum. Finally we obtain

\[ P^{\pm} = -2 \frac{e^{\mp \lambda}}{\beta^2} |q_1| |p^{(\gamma)}_{\rho}|. \]  

(6.17)

Note that the factor epsilon cancels out. We can think of this as confirming that soliton and antisoliton possess the same energy. This expression means that the energy of a soliton with \( \beta \) real would be negative, and so we require that \( \beta \) be pure imaginary on physical grounds. This behaviour is familiar from the abelian theory [3]. For the invariant masses of the solitons we find

\[ M^2 = 2P^{+} P^{-} = \frac{8\mu^2}{\beta^2} |q_1|^{2} (p^{(\gamma)}_{\rho})^{2}. \]  

(6.18)

Having studied the one-soliton solution we are now in a position to examine the reality conditions appropriate to an N-soliton scattering solution, i.e. a solution with the group element of the form in 6.3 for which each one of the solitons can be set to rest by an appropriate Lorentz transformation. Using an obvious notation

\[ P^{\pm} = -2 \frac{e^{\mp \lambda}}{\beta^2} \sum_{i=1}^{N} |q_{\gamma i}| |p^{(i)}_{\rho}| e^{\pm \lambda_i}. \]  

(6.19)

20
This formula yields the physical interpretation of the rest of the parameters appearing in the expression 6.3.

In addition to the N-soliton scattering state we expect, by analogy with the abelian theories, that the spectrum will also contain breather solutions. A breather solution is a bound state of soliton and antisoliton whose individual energies and momenta are not real but whose sum is. This sum can be obtained from the two-soliton version of 6.19 by writing $\lambda_1 = \lambda + i\delta$, $\lambda_2 = \lambda - i\delta$ giving

$$P_{\text{breather}}^\pm = \frac{-4\mu |q| p}{\beta^2} e^{\pm\lambda} \cos \delta.$$  \hfill (6.20)

In fact it is possible to construct other N-soliton solutions for whom only the total energy-momentum is real. We assume that these will be ruled out by reality restrictions on the higher conserved charges. (See [15] for a discussion of this point.) Thus we expect the most general soliton solution which satisfies the physical constraints to be a scattering state of solitons and breathers which are individually physically allowed.

### 6.4 Particles and Solitons

In [3] a relationship between the masses of the Toda particles and solitons in abelian Toda was described. We now want to give arguments which show that this is probably also the case in the more general non-abelian theories. The difficulty here is that the equation of motion will not in general decompose into a set of decoupled Klein-Gordon equations for some $\hat{g}_0$ valued fields $ln h$. The non-abelian nature of $\hat{g}_0$ means that 2.4 will also contain some terms quadratic in the first derivatives of the fields. (In [28] the authors show that these theories can be thought of as integrable theories of particles moving in a black hole space-time.) In spite of this problem we still find that the soliton solution 6.13 is a polynomial in an exponential function, and the static solution of a set of decoupled Klein Gordon equations like 6.11 are just

$$\phi_i \sim e^{\pm\mu_i x}.$$ \hfill (6.21)

We might expect that as the soliton solutions asymptotically approach the vacuum the particle modes do decouple, and so $\langle h_{\text{red}} \rangle$ ought to be a polynomial in Klein-Gordon solutions. Using

$$M_{\text{particle}} \gamma = \mu \gamma h,$$ \hfill (6.22)

and comparing the dependences of 6.13 and 6.21 we find that

$$M_{\text{soliton}} \gamma = \frac{M_{\text{particle}} \gamma p_{\hat{p}}^{(\gamma)}}{|\beta|^2 \hbar}.$$ \hfill (6.23)

We stress again that this formula should not be taken too seriously until the uncertainty in the precise meaning of $M_{\text{particle}}$ is resolved.

We shall now describe an additional property of the soliton solutions.
6.5 Fusing Rule

In [15] a soliton analogue of Dorey's fusing rule for the scattering of Toda particles was discussed. This made use of the expressions 3.21, 3.22 for the normal-ordered product of two of the \( \hat{F}(\alpha, z) \) within a level one representation of a simply-laced \( \hat{g} \). By inspection of 3.22 we see that \( S_{\alpha\beta} \) will be singular at \( z_2 = e^{2\pi i n/m} z_1 \) whenever \( w^n \alpha \cdot \beta = -1 \), i.e. when \( w^n \alpha + \beta \) is a root of \( g \).

Suppose that we have a two-soliton solution (with no reality condition imposed), generated by

\[
G_0 = \exp \left( Q_1 \hat{F}(\alpha, z_1) \right) \exp \left( Q_2 \hat{F}(\beta, z_2) \right).
\]

(6.24)

Suppose that we simulataneously take the limits

\[
z_2 \rightarrow e^{2\pi i n/m} z_1, \\
Q_1, Q_2 \rightarrow 0,
\]

(6.25)

in such a way that the normal-ordered product of the two \( F \) remains finite.

By using the adjoint action of the \( \hat{E}_I^\tau \) we can show that the resulting \( \hat{G} \) is proportional to the exponential of a third \( F \), thus:

\[
G_0 = \exp \left( Q \hat{F}(w^n(\alpha) + \beta, z_2) \right).
\]

(6.26)

The constant \( Q \) is the constant of proportionality whose determination is irrelevant for our purposes.

What this means physically is the following, that a two-soliton solution can "fuse" to form a one soliton solution. The fusion is governed by the root system of \( g \). We know that the orbits of \( w \) correspond to the possible species of solution once we have removed the trivial solutions (see appendix [3]). The condition \( w^n \alpha \cdot \beta = -1 \) just means that \( w^n(\alpha) + \beta \) is a root, and so we have the following statement of the generalised fusing rule. Take a two soliton solution, with solitons of species \( \alpha \) and \( \beta \), such that there exist elements of the \( w \)-orbits of \( \alpha \) and \( \beta \) whose sum is also a root. The parameters of this solution can be chosen so that in the above limit the two solitons fuse to give a third whose species is \( w^n(\alpha) + \beta \). It is important to note that if the third soliton is physical then at least one of the originals must be unphysical. This is probably of significance in the quantum theory as it would seem to mean that one soliton cannot decay into two.

We now discuss the theory.

7 Discussion and Conclusions

We have seen that the non-abelian affine Toda theories can be defined in such a way that the classical soliton solutions can be extracted in simialr fashion to that used for the solitons of the abelian theory [3]. This was done by sticking
to group-valued Toda fields $h$ instead of the Lie algebra valued fields almost exclusively used in the literature. This sort of method is more in the spirit of the original solution (formula 2.25 of the theories, as we have discussed. This principle has allowed us to extend the result of [3] on the splitting of the energy-momentum tensor of the abelian theories to the whole class, which would seem computationally impossible using the means given there.

Using results of [23] and [16] on the relationship between conjugacy classes of the Weyl group of the Lie algebra $g$ and inequivalent embeddings of $A_1$ subalgebras we are able to give a presentation of the theories making contact with the elegant work of [21], which is essential if we are to study the soliton spectrum of the theories in a systematic fashion. We find that there are theories in addition to those originally discussed by Leznov and Saveliev in [11].

We have shown that many of the remarkable features of the soliton spectrum of the abelian theories can be extended to the general theories in a general fashion. In particular we have seen how the conserved quantities are topological and real, despite their densities being complex. Finally we note that we have not bothered to calculate any explicit soliton solutions. There is a very good reason for this. When the equation of motion 2.4 for the non-abelian theories are written out in terms of some parametrisation of the Toda field $h$ (usually in terms of some product of exponentials of some fields lying in $\hat{g}_0$, e.g. [11]) they look extremely messy. We want to stress that this is unnecessary, since we are able to deduce information about the theory in notation in which it appears natural without resorting to brute-force methods.

Let us now discuss possible extensions of the work. Of course the goal of any particle physicist is always to find a quantum theory, where we expect even greater richness. There have been several approaches to quantising the abelian theories which might be hoped to work in general. In [29], [30] the authors quantise the matrix elements of $h^{-1}$ which appear in the Leznov-Saveliev solution. They satisfy an exchange relation which makes contact with the theory of quantum algebras. It is possible that the present approach will also lend itself to this means of quantisation. An interesting alternative seems to be to try to quantise the free fields, zero-modes and all which underly the Leznov-Saveliev solution. This appears to be the course followed in [31], for the case of real, abelian, finite Toda theories. Such theories and their non-abelian generalisations have been of much interest recently on account of their having a W-algebra structure. At the classical level there is a discussion in [11]. The quantum theory as it currently stands is detailed in [11]. Such algebras for the affine Toda theories are discussed in e.g. [32] and other references in [1].

The above methods have the similarity that they quantise the whole of the theory. Perhaps of more relevance for the soliton solutions is the approach of Hollowood [33], where the quantum theory of the $A_r$ abelian Toda solitons is assumed to be encapsulated in their S-matrix, which can be conjectured and shown to satisfy a restrictive set of constraints. This is much the same thing as is done for the Toda particles. A further exciting possibility for soliton quantisation appears in the paper [34]. Here, the authors consider an N=2 supersymmetric field theory. They construct a combination of the spinor charges whose square is zero and can thus be used as a BRST-type operator for the quantum
theory. It turns out that the energy-momentum tensor is a variation under the symmetry generated by this charge, and so correlation functions become independent of position in space-time. Furthermore the particle excitations all become unphysical leaving only topological degrees of freedom. This seems very similar to our setting the free-field to zero and would thus be interesting to investigate further in the present context.

As well as the quantum theory, there is also the interesting possibility of further investigation at the classical level. We would like to rule out all multi-soliton solutions with real energies other than those given. It is probable that this happens if we restrict all of the rest of the conserved charges to be real, given some normalisation. These charges have been given for the $\hat{A}_r$ abelian theories in [35].

It is hoped that the non-abelian Toda theories will receive the attention that they appear to deserve, and that has already been focussed on the abelian theories.

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9 Structure Constants of the Kac-Peterson Basis

In this appendix we indicate how the coefficients $q^\pm_\alpha$ appearing in the commutation relation 6.4 and the energy-momentum formula 6.19 can be calculated.

First let us project down from the loop algebra into the Lie algebra $g$, by setting the formal parameter $t$ to unity, for example. The image of $\hat{X}_\pm$ under this projection will be elements $h_\pm \in H_R$, with eigenvalues $e^{\pm 2\pi i / m}$ under conjugation by $S_w$. The precise definition of these elements was discussed in section 3. The structure constants $q^\pm_\alpha$ are given by

$$q^\pm_\alpha = \alpha(h_\pm).$$

All we need to know, besides the root system of $g$, is an expansion of $h_\pm$ over the generators of $H_0$. In fact since we know the way that $H_R$ is embedded into $H_0$ we only need to know the expansion of $h_\pm$ over the generators of $H_R$. This problem has been solved in [4], where the authors use the properties of the Coxeter element discovered by Kostant [17] to expand the images of $h_\pm$ under the canonical isomorphism with $H_R^*$ over the simple roots of the algebra. Crucial to this procedure is the bicolouration property of finite type Dynkin diagrams; their points can be labelled ‘black’ or ‘white’ with no points of like colour adjacent. Defining $\delta_j B = 1$ if $i$ is black and zero otherwise yield the
following formula, valid for the simple roots of a simple algebra:

\[ q^+_{\alpha_j} = 2ie^{-i\delta_{j\mu}\theta} \sin \theta x_j(1), \]  

where \( \theta = \frac{\pi}{h} \), and \( x_j(1) \) are the components of the right Perron-Frobenius vector of the Cartan matrix of \( g \), this being the eigenvector of least eigenvalue. The formula in this form was lifted from [3].

The expression 9.2 can easily be generalised to the case where \( g_R \) is not semisimple, the constants \( q^\pm \) then being an appropriately normalised sum of such terms.

9.1 Vanishing Structure Constants

When \( w \) is not a Coxeter element of the Weyl group of \( g \) there will generally be some \( w \)-orbits of roots \( \alpha \) which are orthogonal to \( h^\pm \). We can see from the form of the solution given in 6.9, for example, that the \( \exp(Q\hat{F}(\alpha,z)) \) for such roots lead to trivial solutions of the equations of motion. It seems reasonable to suppose that these group elements should be omitted altogether as not contributing anything to the soliton spectrum, merely shifting the field \( h \) by a constant group element, whose Adjoint action on \( \hat{X}^\pm \) is trivial. We get a trivial solution whatever the value of \( Q \). This leads us to the conclusion that the existence of such \( \hat{F}(\alpha,z) \) implies that the constant, vacuum, solutions of the theory are not a discrete set. This should not come as any surprise if we recall the remarks of section 4, about some irritating portions of the conserved currents. The projection operator \( P \) when applied to the equation of motion 2.4 gives

\[ \partial_- P(h^{-1}\partial_+ h) = 0. \]

The existence of this rather trivial portion of the field was first noted by Leznov and Saveliev in the original description [10]. It seems that this cannot simply be factored out of the theory since the motion of the rest of the fields will still depend on the solution of 9.3. It could well be that there will be a valid interpretation of these fields, but this awaits further study.

10 Vertex Operators in other Representations

In this appendix we indicate, along much the same lines as for the principal vertex operator construction in [15], how the existence of vertex operator representations for the \( \hat{F}(\alpha,z) \) in the level one representations of simply-laced affine algebras allows us to deduce properties of the \( \hat{F}(\alpha,z) \) in the other representations. These are necessary for the construction of the matrix elements in the Leznov-Saveliev solution 2.25. First let us describe how the results can be extended to the rest of the fundamental representations of the simply-laced algebras (and we know that the general highest weight representations can be constructed by taking symmetrisated tensor product of these, though these representations are not really necessary for our purposes).

We expect that all of the fundamental representations can be found in the decompositions of tensor products of the level one representations (known
henceforth as the minimal representations). In principle then, we could use
the vertex operator construction in such a tensor product, and then decompose
to obtain the fundamental representation. In practice this computation would
seem to be difficult, but it can be done (see the simple example of $D_4$ in [15]
for example). However, we do not really need to perform these calculations as the
explicit form of the solution is of no real interest to us (see the remarks in the
discussion). What is interesting from our point of view is the asymptotic be-
haviour of the solution, which is governed by the greatest non-vanishing power
of $\hat{F}(\alpha, z)$ within a representation. We have already seen that the square of
$\hat{F}(\alpha, z)$ must vanish within the minimal representations. How can we use this
to answer the above question?

Notice that a representation of level $x$ can only be found in the tensor
product of $x$ minimal representations.\footnote{Note that the construction by tensor products of a given representatio
n is not necessarily unique.} This is obvious from the fact that $x$ is
the eigenvalue of $K$. Within such a representation the maximum non-vanishing
power of $\hat{F}(\alpha, z)$ must be $\leq x$, since any greater power would have all of its
constituent terms with a quadratic or greater power acting on some fundamental
representation. I am unable to prove the equality, although it seems unlikely
that any power of $\hat{F}(\alpha, z)$ not actually required to vanish should do so. We
shall thus assume that the greatest non-vanishing power of $\hat{F}(\alpha, z)$ in a level $x$
representation of a simply-laced $\hat{g}$ is $x$.

In this paper we are particularly interested in the representation with highest
weight $\hat{\rho}$, for the purpose of evaluating the energy and momentum of the soliton
solutions. The level of this representation is $m$, since

$$x = \hat{\rho}(K) = (d', K) = m, \quad (10.1)$$

from the definition of $d'$.

Now we discuss the value of vertex operator constructions for the non-simply
laced algebras. In [15] it was noted that every non-simply laced algebra, in-
cluding the twisted algebras, can be obtained as a subalgebra of a simply-laced
algebra invariant under the canonical lift of one of its diagram automorphisms.
Such automorphisms are of course outer. The value of this observation was
that the highest weight representation theory of the simply-laced algebra, and
in particular the principal vertex operator method for constructing an arbitrary
such representation, can be used to describe the representation theory of the
$\hat{F}(\alpha, z)$. In particular it turns out that the $\hat{F}(\alpha, z)$ for the non-simply laced
algebra are just invariant linear combinations of those for the simply-laced al-
gebra. In the present, more general, case, given a Weyl group element $w$ (and
equivalently a regular semisimple subalgebra $g_R$) of a non-simply laced algebra
$g$ embedded in a simply-laced algebra $g^2$, we see that there will be a corre-
sponding $g^2_R$ as the embedded image of $g_R$, which is preserved by the diagram
automorphism. A little thought along the same lines as in [15] reveals that
we expect the $\hat{F}(\alpha, z)$ of the non-simply laced loop algebra to again be linear
combinations of the $\hat{F}(\alpha, z)$ of the simply-laced loop algebra, since we can in
fact be precise about the action of the automorphism on the $\hat{E}_I$. This means
that the simply-laced $\hat{F}(\alpha, z)$ provide a permutation representation of the diagram automorphisms (up to signs); to get the non-simply laced $\hat{F}(\alpha, z)$ we just identify the trivial one-dimensional representations. For more detail on all of these points again refer to [15].

Finally if the non-simply laced $\hat{F}(\alpha, z)$ is a linear combination of $y$ simply-laced $\hat{F}(\alpha, z)$, then its $xy + 1^{th}$ power must vanish in a representation of level $x$.

There are other approaches to the construction of representations of the non-simply-laced affine algebras. The extension of the Kac-Peterson construction to the twisted algebras was discussed in the conclusion of the original article [21]. This would presumably be useful to present and discuss twisted non-abelian affine Toda theories. An alternative approach to the construction of homogeneous vertex operator constructions was found in [36] where the authors found that the sum of $\hat{F}(\alpha, z)$ can actually be expressed as a single vertex operator if we introduce some fermionic Fock space as well as that of the homogeneous Heisenberg subalgebra. It is possible that this sort of approach could be used for the more general $\hat{F}(\alpha, z)$ discussed here.

11 On the Centre of $Z(\hat{X}_+)$

In the discussion of the abelianisation procedure in section 4, for example in equation 4.12, the local conserved charges which can be obtained by this method were shown to correspond to the elements of the centre of $Z(\hat{X}_+)$. There is a natural conjecture for the form of these elements, given the procedure of section 3.4 for constructing an appropriate $\hat{X}_+$. We proceed in exactly the same way as in the appendix of [25]. First consider the element $h_+ \in g$ of which $\hat{X}_+$ is the grade 1 affinisation. If an element of the loop algebra $\tilde{g}$ belongs to the centre of $Z(\hat{X}_+)$ then it follows that its natural projection into $g$, for example by setting the formal parameter $t$ to unity, belongs to the centre of $Z(h_+)$. Because $h_+$ is an eigenvector of the conjugation by $S_w$, with eigenvalue $e^{2\pi i/m}$, it follows that $Z(h_+)$ is a $C$-span of eigenvectors of the conjugation. This in turn means that its centre is also of this form. Let $C \in g \otimes g$ be the Casimir tensor of $g$, invariant under the adjoint action of $g$. It is easy to show that elements of $g$ of the form

$$h^{(n)} = \text{Tr}_L h^n C$$

must lie in the centre of $H(h_+)$. $\text{Tr}_L$ denote the trace in some representation of the left hand factor of the tensor product. Note that $h^{(n)}$ has eigenvalue $e^{2\pi in/m}$ under the conjugation. It seems reasonable to suppose that the non-zero $h^{(n)}$ actually span the whole of the centre. This conjecture could perhaps be verified using results in [17] and [16].

12 The Dual Weyl Representation

In this appendix we discuss the properties of a particular highest weight representation of the algebra $\hat{g}$, which we shall call the dual Weyl representation.
Focus on the element $x$ which defines the grading of $\hat{g}$. We note that $x \in g_R$ is semisimple (ad-diagonalisable), which means that there must be at least one Cartan subalgebra $H_\triangle$ of $g$ which contains it, and this Cartan subalgebra is clearly contained in $g_0$. We can make a particular choice for $H_\triangle$ by using the fact that there is a unique Cartan subalgebra of $g_R$ which commutes with $x$ (because $x$ is a regular element of $g_R$). Call this $H'_R$ and fix

$$H'_R \oplus H_\perp = H_\triangle \subset g_0 \subset \hat{g}. \quad (12.1)$$

We can guarantee (see e.g. [22]) that $x$ will lie in the dominant Weyl chamber of some system of simple roots (strictly the image of this chamber under the canonical isomorphism between $H_\triangle^*$ and $H_\triangle$). The advantage of this choice is that the step-operators $e_\alpha$ corresponding to the positive roots will have non-negative grades, and those corresponding to the negative roots will have non-positive grades. Implementing the Kac-Peterson procedure we get a correspondingly $d'_-$-graded basis $t'_e_\alpha$ of the loop algebra. This shows that we can legitimately define highest weight representations with respect to the $d'_-$-gradation, i.e. representations generated from a vector $|\Lambda\rangle$ which is annihilated by all of the step operators of positive grade, and which satisfies

$$h_\triangle |\Lambda\rangle = \Lambda(h_\triangle) |\Lambda\rangle \forall h_\triangle \in H_\triangle \subset \hat{g} \quad (12.2)$$

This picture of the Kac-Peterson basis as being the affinisation of a step-operator basis defined with respect to a different Cartan subalgebra of $g$ allows us to be more specific about the action of $\hat{g}_0$ on the highest weight representations defined in this way. Suppose that $h_\alpha \in H_\triangle$ is the Cartan element corresponding to each simple root $a$ of the Kac-Moody algebra $\hat{g}$. It is well known [37] that there exists a highest weight representation for which the $\Lambda(h_\alpha)$ are equal to any set of non-negative integers. In particular there must exist a representation with highest weight state $|\hat{\rho}\rangle$ with $(d', h_a) = \hat{\rho}(h_a)$ since

$$(d', h_a) = (d', [e_\alpha, e_{-\alpha}]) = ([d', e_\alpha], e_{-\alpha}) \quad (12.3)$$

are all non-negative integers.

We need to know the action of $\hat{g}_0$ on the highest weight state. Orthogonally decompose $\hat{g}_0 = H_\triangle \oplus \hat{g}_\triangle$, where $\hat{g}_\triangle$ is a direct sum of step operators with $d'$-grades zero: if it includes a step operator for a particular root $b$ it also includes that for $-b$. Because $(d', h_b) = \hat{\rho}(h_b) = 0$ we know that neither of $\hat{\rho} \pm b$ can be weights of the representation. Thus we conclude that $|\hat{\rho}\rangle$ is annihilated by $\hat{g}_\triangle$.

This representation is put to use in section 6.

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