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Abstract. In this note we shall show that a lattice \( Z\omega_1 + Z\omega_2 \) in \( \mathbb{C} \) has \( \mathbb{Q} \)-linearly dependent quasi-periods if and only if \( \omega_2/\omega_1 \) is equivalent to a zero of the Eisenstein series \( E_2 \) under the action of \( SL_2(\mathbb{Z}) \) on the upper half plane of \( \mathbb{C} \).

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1. Introduction

Let \( \mathcal{L} = Z\omega_1 + Z\omega_2 \) be a lattice in \( \mathbb{C} \) with \( \omega_2/\omega_1 \in \mathbb{H} \), the upper half plane of \( \mathbb{C} \). Let \( \sigma(z; \omega_1, \omega_2) \) and \( \zeta(z; \omega_1, \omega_2) \) respectively be the Weierstrass sigma and zeta functions associated to \( \mathcal{L} \). Let \( g_2 \) and \( g_3 \) be the invariants of \( \mathcal{L} \). The numbers \( \eta_1(\mathcal{L}) = \eta(\omega_1) = 2\zeta(\omega_1/2; \omega_1, \omega_2) \), \( \eta_2(\mathcal{L}) = \eta(\omega_2) = 2\zeta(\omega_2/2; \omega_1, \omega_2) \) are called the quasi-periods associated to \( \mathcal{L} \). When \( \mathcal{L} \) is clear from the context, we simply write \( \eta_1, \eta_2 \) instead of \( \eta_1(\mathcal{L}) \) and \( \eta_2(\mathcal{L}) \) respectively. One of the long standing open problem in transcendental number theory is to find the dimension of the vector space \( V_{\mathcal{L}} \) generated by

\[ 1, \omega_1, \omega_2, \eta_1, \eta_2, \pi \]

over \( \overline{\mathbb{Q}} \), the algebraic closure of \( \mathbb{Q} \). Starting from the work of Siegel [10], Schneider [9], Baker [1], Coates [3,4] and finally by Masser [8], it is now known that for a lattice \( \mathcal{L} \) with algebraic invariants \( g_2, g_3 \), the vector space \( V_{\mathcal{L}} \) has dimension 4 in the CM case and 6 in the non-CM case. This is because in the CM case, there are two linear relations among the numbers in (1). The first one is

\[ \tau \omega_1 - \omega_2 = 0 \]

where \( \tau = \omega_2/\omega_1 \in \overline{\mathbb{Q}} \) and the other one is given by

\[ C\eta_1 - \tau \eta_2 - \kappa \omega_2 = 0, \]

(2)

where \( C \) is the constant term of the minimal polynomial of \( \tau \) over \( \mathbb{Q} \) and \( \kappa \in \mathbb{Q}(\tau, g_2, g_3) \) (see [8, Lemma 3.1] or [2, Theorem 8] for more details). Masser also proved that the number \( \kappa \) in (2)
vanishes if and only if \( \tau \) is congruent to \( i = \sqrt{-1} \) or \( \rho = e^{2\pi i/3} \) under \( \text{SL}_2(\mathbb{Z}) \); and in that case, \( \eta_1 \) and \( \eta_2 \) are linearly dependent over \( \mathbb{Q}(\tau) \).

Apart from lattices with algebraic invariants, there are two more cases for which we know the dimension of \( V_L \). For example, if \( \omega_1 = 1 \) and \( \omega_2 = i \) then by Siegel [10] at least one of the \( g_2, g_3 \) is not algebraic. And by (2), the quotient \( \eta_2/\eta_1 = -i \) in this case. (Note that we used (2) to find the ratio \( \eta_2/\eta_1 \); because, as we shall see later that, \( \eta_2/\eta_1 \) depends only on \( \omega_2/\omega_1 \) and not on \( g_2, g_3 \); this ratio can also be obtained from (4) and (9) below by choosing an appropriate \( \gamma \). Hence by the Legendre’s relation [7, p. 241] the vector space \( V_L \) has dimension two. Similarly, if \( \omega_1 = 1 \) and \( \omega_2 = \rho \) then in this case also at least one of the \( g_2, g_3 \) is not algebraic and by (2) we have \( \eta_2/\eta_1 = \rho^{-1} \). Hence in this case also the vector space \( V_L \) has dimension two. Except for these cases the author is not aware of any other lattices \( L \).

The following corollary is immediate.

**Corollary 1.** Let \( L = Z\omega_1 + Z\omega_2 \) be a lattice in \( \mathbb{C} \) with \( \tau = \frac{\omega_2}{\omega_1} \in \mathbb{H} \). Then \( \eta_1 \) and \( \eta_2 \) are \( \mathbb{Q} \)-linearly dependent if and only if \( \tau \) is congruent to a zero of \( E_2(z) \) under \( \text{SL}_2(\mathbb{Z}) \).

The following corollary is immediate.

**Main Theorem.** Let \( L = Z\omega_1 + Z\omega_2 \) be a lattice in \( \mathbb{C} \) with \( \tau = \frac{\omega_2}{\omega_1} \in \mathbb{H} \). Then \( \eta_1 \) and \( \eta_2 \) are \( \mathbb{Q} \)-linearly dependent if and only if \( \tau \) is congruent to a zero of \( E_2(z) \) under \( \text{SL}_2(\mathbb{Z}) \).

We shall prove the Main Theorem in the next section. The proof relies on the formula expressing the quasi-periods in-terms of \( G_2 \) (see Lemma 3) and the transformation formula of \( E_2 \) given by

\[
E_2(\gamma \tau) = (\gamma + d)^2 E_2(\tau) + \frac{6c}{\pi^2} (\gamma + d)
\]

where \( \gamma \in \mathbb{H} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \).

### 2. Quasi-periods and Laurent’s expansions

Let \( \sigma(z; \tau) = \sigma(z; 1, \tau) \) and \( \zeta(z; \tau) = \zeta(z; 1, \tau) \) respectively be the Weierstrass sigma and zeta functions associated to the lattice \( L' = Z + Z\tau \) with \( \tau \in \mathbb{H} \). These two functions are connected by the relation \( \zeta(z; \tau) = \frac{\sigma'(z; \tau)}{\sigma(z; \tau)} \).

For \( \omega \in L' \setminus \{0\} \), we write

\[
\frac{1}{z - \omega} = -\frac{1}{\omega} - \frac{z}{\omega^2} - \frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots
\]

for \( z \) near the origin. Thus, we have

\[
\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = -\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots.
\]
Now summing over all non-zero periods of \( \mathcal{L}_\tau \) and adding the term \( 1/z \), we obtain
\[
\zeta(z; \tau) = \frac{1}{z} - \sum_{k=2}^{\infty} G_{2k} z^{2k-1} \quad (6)
\]
where \( G_{2k} = G_{2k}(\tau) = \sum_{\omega \in \mathcal{L}_\tau \setminus \{0\}} \omega^{-2k} \) for \( k \geq 2 \) (the coefficients of even powers of \( z \) in (6) are zero, since \( \zeta(z; \tau) \) is an entire function).

The next lemma gives a connection between quasi-periods and the values of generalized Eisenstein series \( G_2 \).

**Lemma 2.** Let \( \eta_1 \) be the quasi-period associated to the period 1 of the lattice \( \mathcal{L}_\tau = \mathbb{Z} + \mathbb{Z} \tau \) with \( \tau \in \mathbb{H} \). Then \( \eta_1 = G_2(\tau) \).

**Proof.** We follow the strategy as given in [7, Chapter 18]. Accordingly, we express the Laurent’s expansion of \( \zeta(z; \tau) \) near the origin into two different ways and then comparing the corresponding coefficients we obtain the required representation for \( \eta_1 \). The first one is given by (6). For obtaining the second representation, let \( q_z = e^{2\pi i z} \). Consider the function
\[
\phi_1(z) = (2\pi i)^{-1} \left( q_z - 1 \right) \prod_{n=1}^{\infty} \frac{1 - q_{z+nt}}{(1 - q_{zt})^2} \quad (7)
\]
Since \( \tau \in \mathbb{H} \), we have \( |q_{nt}| < 1/2^n \) for large values of \( n \), and hence, for such values
\[
\left\| \frac{q_{nt}}{(1 - q_{nt})^2} \right\| < \frac{1}{(2^n - 1)^2}
\]
It follows that the series
\[
\sum_{n=1}^{\infty} \left( \frac{1 - q_{z+nt}}{(1 - q_{zt})^2} - 1 \right)
\]
converges absolutely and uniformly on compact subsets of \( \mathbb{C} \). Thus, the function \( \phi_1 \) is entire. Moreover, it satisfying the following transformation formulas (see [7, p. 247] for more details):
\[
\phi_1(z + 1) = \phi_1(z) \quad \text{and} \quad \phi_1(z + \tau) = -\frac{1}{q_z} \phi_1(z).
\]
On the other hand, the entire function
\[
\phi_2(z) = e^{-\frac{1}{2} \eta_1 z^2} q_z^{1/2} \sigma(z; \tau)
\]
also satisfies
\[
\phi_2(z + 1) = \phi_2(z) \quad \text{and} \quad \phi_2(z + \tau) = -\frac{1}{q_z} \phi_2(z).
\]
Therefore, the quotient \( \phi_1(z)/\phi_2(z) \) is elliptic. The product in (7) shows that both \( \phi_1 \) and \( \phi_2 \) have a simple zero at each point of \( \mathbb{Z} + \mathbb{Z} \tau \) and no other zeros. Hence \( \phi_1(z)/\phi_2(z) \) must be constant. Taking limit \( z \to 0 \) we see that the constant is 1, and therefore \( \phi_1(z) = \phi_2(z) \). We thus have
\[
\sigma(z; \tau) = (2\pi i)^{-1} e^{\frac{1}{2} \eta_1 z^2} \left( q_z^{1/2} - q_z^{-1/2} \right) \prod_{n=1}^{\infty} \frac{1 - q_{z+nt}}{(1 - q_{zt})^2}.
\]
Since the series in (8) converges absolutely and uniformly on compact subsets of \( \mathbb{C} \), taking logarithmic derivative term by term on the right side of the above equation we obtain
\[
\zeta(z; \tau) = \eta_1 z + \pi i \left( \frac{q_z + 1}{q_z - 1} \right) + 2\pi i \sum_{n=1}^{\infty} \frac{q_{nt} - z}{1 - q_{nt}^2} - \frac{q_{z+nt}}{1 - q_{z+nt}}.
\]
If we restrict the values of \( z \) such that \(|q_z| < |q_{z}^{-1}|\), then we have
\[
\sum_{n=1}^{\infty} \left( \frac{q_{n-z}}{1-q_{n-z}} - \frac{q_{n+z}}{1-q_{n+z}} \right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( q_{n+z}^m - q_{n-z}^m \right) = \sum_{m=1}^{\infty} q_{n+z}^m - q_{n-z}^m.
\]

Near the origin, we have
\[
i \left( \frac{q_{z} + 1}{q_{z} - 1} \right) = \cot \pi z = \sum_{k=0}^{\infty} (-1)^k \frac{2 \pi m z 2k+1}{(2k+1)!},
\]

and
\[
q_{z}^{-m} - q_{z}^{m} = -2i \sum_{k=0}^{\infty} (-1)^k \frac{(2 \pi m z 2k+1)}{(2k+1)!},
\]

where \( B_r \) is the \( r \)-th Bernoulli's number. Thus we have,
\[
\zeta(z; \tau) = \eta_{1} z + \pi \sum_{k=0}^{\infty} \frac{(-1)^k 2 2k B_{2k}(\pi z 2k+1)}{(2k)!} - 4 \pi \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q_{m \tau}}{1-q_{m \tau}} \frac{(2 \pi m z 2k+1)}{(2k+1)!}.
\]

Now comparing the coefficients of \( z \) on the above equation with that of (6) we get
\[
\eta_{1} = \frac{\pi^2 2^2 B_2}{2} - 8 \pi^2 \sum_{m=1}^{\infty} \frac{mq_{m \tau}}{1-q_{m \tau}} = \frac{\pi^2}{3} \left( 1 - 24 \sum_{m=1}^{\infty} \frac{mq_{m \tau}}{1-q_{m \tau}} \right) = \frac{\pi^2}{3} \left( 1 - 24 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m q_{m \tau}^n \right) = \frac{\pi^2}{3} \left( 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q_{n \tau}^n \right) = G_2(\tau),
\]

by (4). This completes the proof of the Lemma 2. □

There is a slight change in the notations used in the above lemma from that of [7, Chapter 18]. In [7], lattices in \( \mathbb{C} \) are written in the form \( \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) with the assumption \( \omega_1 / \omega_2 \in \mathbb{H} \). This implies that the quasi-period associated to the period 1 of the lattice \( \omega_2^{-1}(\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2) \) is denoted by \( \eta_2 \) in [7, Chapter 18]. Whereas, in our notation lattices in \( \mathbb{C} \) are written in the form \( \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) with the assumption \( \omega_2 / \omega_1 \in \mathbb{H} \). This implies that the quasi-period associated to the period 1 of the lattice \( \omega_1^{-1}(\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2) \) is denoted by \( \eta_1 \).

The following lemma is the homogeneous version of Lemma 2.

**Lemma 3.** Let \( \mathcal{L} = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) be a lattice in \( \mathbb{C} \) with \( \tau = \frac{\omega_2}{\omega_1} \in \mathbb{H} \). We have
\[
\eta_1 = \frac{G_2(\tau)}{\omega_1} \quad \text{and} \quad \eta_2 = \frac{\tau G_2(\tau) - 2 \pi i}{\omega_1}.
\]

**Proof.** By the Legendre's relation
\[
\omega_2 \eta_1(\mathcal{L}) - \omega_1 \eta_2(\mathcal{L}) = 2 \pi i,
\]
hence it is sufficient to show that \( \eta_1(\mathcal{L}) = \frac{G_2(\tau)}{\omega_1} \). Since \( \eta_1(\mathcal{L}) \) is homogeneous of degree \(-1\), it is enough to prove this lemma when \( \mathcal{L} = \mathbb{Z} + \mathbb{Z} \tau \) with \( \tau \in \mathbb{H} \). We are thus reduced to show that for \( \mathcal{L} = \mathbb{Z} + \mathbb{Z} \tau \) with \( \tau \in \mathbb{H} \), we have \( \eta_1(\mathcal{L}) = G_2(\tau) \); but, this is a consequence of Lemma 2. This completes the proof. □
3. Proof of the Main Theorem

Let \( L = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) be a lattice in \( \mathbb{C} \) with \( \tau = \frac{\omega_2}{\omega_1} \in \mathbb{H} \). By Lemma 3, the quotient \( \eta_2(L)/\eta_1(L) \) is a function of \( \tau \) and we denote it by \( F(\tau) \) (this function was first introduced and studied by Heins [5]). Hence by (4) and (9) we have

\[
F(\tau) = \frac{\tau E_2(\tau) + 6i/\pi}{E_2(\tau)}. \tag{10}
\]

It follows from this identity that \( \eta_1(L) \) and \( \eta_2(L) \) are \( \mathbb{Q} \)-linearly dependent if and only if \( F(\tau) \) is a rational number (it is convenient here to assume \( \infty \) is a rational). Hence we are reduced to show that \( F(\tau) \) is a rational number if and only if there exists a zero \( \tau' \) of \( E_2(z) \) and a matrix \( \gamma \in SL_2(\mathbb{Z}) \) such that \( \tau = \gamma \tau' \).

If \( F(\tau) = \infty \), then we have \( E_2(\tau) = 0 \). If \( F(\tau) = 0 \), then we have \( \tau E_2(\tau) + 6i/\pi = 0 \); and hence \( E_2(\tau) = \tau E_2(\tau) + 6i/\pi = 0 \). Suppose that \( F(\tau) \) is a rational number which is neither 0 nor \( \infty \), say \( q/p \), with \( (p,q) = 1 \). Then, by (10) we have

\[
(-pr + q)E_2(\tau) = \frac{6p}{\pi i} \tag{11}
\]

Choose \( r, s \in \mathbb{Z} \) such that \( pr - qs = -1 \). Then the matrix

\[
\gamma = \begin{pmatrix} s & -r \\ -p & q \end{pmatrix} \in SL_2(\mathbb{Z}).
\]

We set \( \tau' = \gamma \tau \). Then by (5),

\[
E_2(\tau') = (-p \tau + q) \left((-p \tau + q)E_2(\tau) - \frac{6p}{\pi i}\right),
\]

which is equal to zero by (11).

Conversely, let \( \tau' \) be a zero of \( E_2(z) \), and let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an element of \( SL_2(\mathbb{Z}) \). We shall show that \( F(\gamma \tau') \) is a rational number. If \( c = 0 \), then \( \gamma = T^b \) where \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Thus \( E_2(\gamma \tau') = 0 \), and hence \( F(\gamma \tau') = \infty \). If \( a = 0 \), then \( \gamma = \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} \), and hence \( \gamma \tau' = \frac{1}{i \tau + d} \). It follows from (5) that \( F(\gamma \tau') = 0 \). Now let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an element of \( SL_2(\mathbb{Z}) \) such that \( ac \neq 0 \). Then, by (5) we have

\[
0 = E_2(\tau') = E_2(\gamma^{-1}(\gamma \tau')) = (-c(\gamma \tau') + a)^2 E_2(\gamma \tau') - \frac{6c}{\pi i} (-c(\gamma \tau') + a).
\]

Since \( \tau' \) is not a rational number we must have

\[
(\gamma \tau' - a/c) E_2(\gamma \tau') + \frac{6}{\pi i} = 0.
\]

Again by (5), we have \( E_2(\gamma \tau') \neq 0 \), from this we conclude that \( F(\gamma \tau') = a/c \) is a rational number, and this completes the proof of the Main Theorem.

4. Concluding remarks

It is expected that the zeros of \( E_2 \) are transcendental; but so far none of them is known to be transcendental. One may ask whether transcendence of \( \omega_2/\omega_1 \) is a necessary condition for \( \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \) to have \( \mathbb{Q} \)-linearly dependent quasi-periods? The answer is no. For example, the quasi-periods associated to \( \mathbb{Z} + \mathbb{Z} i \) are \( \mathbb{Q} \)-linearly dependent. It is interesting to classify all lattices with \( \mathbb{Q} \)-linearly dependent quasi-periods.

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