Kardar-Parisi-Zhang model for the fractal structure of cumulus cloud fields

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Abstract

We model the ascent of warm, moist air in the Earth’s atmosphere by turbulent convection and expansion with the KPZ equation, familiar in the physics literature on surface growth. Clouds form in domains where the interface between the rising air and its surrounding air achieves an elevation higher than that necessary for condensation. The model predictions are consistent with the perimeter fractal dimension and the cumulative frequency-size distribution of cumulus cloud fields observed from space.
In a pioneering study, Lovejoy [1] computed the fractal dimension of the perimeter of rain and cloud areas from scales of 1 to 1000 kilometers to be $1.35 \pm 0.05$. Rys and Waldvogel [2] carried out the same analysis to characterize the shape of hail clouds. For scales above 3 km they obtained a fractal dimension consistent with Lovejoy’s result. At scales below 3 km, the authors found that severely convective hail storms have perimeters with the usual Euclidean dimension of 1. Cahalan and Joseph [3] and Zhu et al. [4] extended their methodology, including the calculation of cumulative frequency-size distributions of cumulus cloud fields. They found cumulative frequency-size distributions, the number of clouds greater than or equal to an area $A$, to be a power-law function of area with an exponent close to $-1$ for some cumulus cloud scenes up to spatial scales of 10 km.

Two models have been proposed to explain aspects of the fractal structure of cumulus cloud fields. Hentschel and Procaccia [5] have considered the turbulent mixing of an initially compact cloud using a theory of turbulent diffusion to explain Lovejoy’s result. Their model does not appear to favor any particular cloud size distribution. Nagel and Raschke [6] have proposed a cellular automaton model of the atmosphere as a lattice of particles subject to a buoyant uplift upon the initiation of condensation and a nearest neighbor interaction to model entrainment of fluid by a nearby updraft. They were able to match Lovejoy’s result, but only for a particular percentage of cloud cover. Both papers model cloud dynamics only after the onset of condensation. It may be essential to model the dynamics of the ascending warm, moist air (and the descending air which replaces it) prior to condensation to explain the scaling of cumulus cloud fields up to scales of 1000 km as observed. Studies solving the equations of fluid motion have been applied to the problem of cumulus cloud formation [7] but are of too limited a spatial bandwidth to address the observed scale-invariance.

In this paper we apply a nonlinear stochastic differential equation known in the physics literature on surface growth as the Kardar-Parisi-Zhang (KPZ) equation [8] to model the interface between the ascending air and its surrounding air. The model incorporates the expansion (contraction) of ascending (descending) air, its random turbulent convection, and the entrainment of fluid by a nearby updraft. Clouds form in those domains where the
interface lies above some threshold elevation. We use the results of Kondev and Henley on the perimeter fractal dimension and size distribution of contour loops of random Gaussian surfaces to relate the KPZ Hausdorff measure to the observables of cumulus cloud fields.

We will apply the KPZ equation to the evolution of a thin fluid layer, originally horizontal, in the atmosphere. Two principal processes act on a warm, moist air mass in the atmosphere: 1) as the air is heated from below with long-wavelength outgoing radiation, convective instabilities transport the air vertically and 2) expansion occurs as ascending air enters regions of lower pressure higher in the atmosphere. Since it is impossible to determine where the convective instabilities will develop, a stochastic model for this transport is appropriate. We will model the force of convective instabilities on the fluid layer as a Gaussian white noise. A Gaussian white noise force, combined with the drag induced on the fluid layer by the air it displaces, is consistent with the observed Gaussian, uncorrelated (above the Lagrangian timescale: on the order of minutes in the atmosphere) velocity fluctuations in a stably or neutrally stratified atmosphere. The viscosity of air results in a shear force between an updraft and nearby air which results in an effective surface tension of the fluid layer. The force of convection and effective surface tension will result in a vertical velocity of the fluid layer (since the forces are balanced by the drag exerted by the adjacent fluid) based on these parameterizations as

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(x, y, t)$$

where $h$ is the elevation of the layer and $\eta(x, y, t)$ is Gaussian white noise.

In addition to the convective transport, the pressure gradient with height causes ascending (descending) air to expand (contract). The simplest model of this expansion and contraction is a constant growth of the interface directed everywhere perpendicular to the interface with a non-zero upward component for the layer as a whole denoted by $r$. This model corresponds to a constant pressure difference between the ascending air and the air above it. The local vertical component of growth is equal to $r(1 + (\nabla h)^2)^{\frac{1}{2}}$. If we assume that the gradients of the interface are small, or if we compare our model to only large-scale
structure, we can approximate this expression as \( r + \frac{r}{2}(\nabla h)^2 \). This Taylor expansion procedure is the same formulation employed by Kardar, Parisi, and Zhang \[8\] to motivate the nonlinear term \((\nabla h)^2\) to model lateral growth on atomic surfaces. The resulting differential equation for the height of the interface is

\[
\frac{\partial h}{\partial t} = \nu \nabla^2 h + r + \frac{r}{2}(\nabla h)^2 + \eta(x, y, t) \tag{2}
\]

This is the KPZ equation.

The KPZ equation with a two-dimensional surface has been solved numerically by Amar and Family \[11\] and Moser, Wolf and Kertesz \[12\]. The solution is a surface with a Gaussian distribution of elevations and a variance which depends upon the linear size of the surface as \( V \propto L^{2H} \) where \( H \approx 0.4 \) is known as the Hausdorff measure or roughness exponent.

Kondev and Henley \[9\] have obtained the relationship between the fractal dimension of a contour loop of a Gaussian surface, \( D \), and its Hausdorff measure as \( D = 1.5 - \frac{H}{2} \). A contour loop is a connected subset of a surface with equal elevation. Since clouds form above a threshold elevation where condensation begins, their base perimeters, observable from satellite images as in Lovejoy’s work, may be associated with the contour loops of Kondev and Henley. Their relation, together with the Hausdorff measure \( H = 0.4 \), predicts a cloud perimeter fractal dimension of 1.3, consistent with the value 1.35 ± 0.05 observed by Lovejoy \[1\] and Rys and Waldvogel \[2\].

In addition, Kondev and Henley have given the size distribution of contour lengths (the probability that a randomly chosen contour loop has a length \( s \)) as \( N(s) \propto s^{-\tau} \) where \( \tau = 1 + \frac{2-H}{D} \). The cumulative distribution (the number of contours with length greater than \( s \)) is the integral of the noncumulative distribution, \( N(> s) \propto s^{-\frac{2-H}{D}} \). Since the length of a contour is related to the area it encloses by \( s \propto A^{\frac{D}{2}} \) (by definition), the cumulative distribution of areas enclosed by contours is \( N(> A) \propto A^{-\frac{2-H}{D}} \). For the KPZ Hausdorff measure of \( H = 0.4 \) this gives \( N(> A) \propto A^{-0.8} \).

In order to test the model predictions of the cumulative frequency-size distribution against cumulus cloud fields, we obtained global composite images from the GOES satellites.
prepared at the Space Science and Engineering Center at the University of Wisconsin, Madison for five days each in the months of October, 1995 and January, 1996. The days were each separated by at least three days to ensure that each scene was distinct. We analyzed cloud images only within 30 degrees latitude of the equator. Tropical clouds are ideal for study since they form in environments which are nearly uniform horizontally \[13\]. We divided each global scene into 60° x 60° scenes centered on South America, Africa, and the Western Pacific Ocean (regions of consistent large-scale cloud cover). To analyze smaller scales, we obtained images of the Earth photographed from the space shuttle. We analyzed 16 STS-67 images that satisfied the following criteria: 1) considerable cumulus cloud cover untainted by other types of clouds, 2) adequate contrast to define cloud shapes easily, 3) clouds were not conspicuously correlated with topography, and 4) clouds were photographed at a small look angle. Otherwise, the choice of the images was random. The resolution cell size of each image type was determined through calibration with respect to a recognizable geographic shape. The resolution cell sizes of the GOES composite and shuttle images were estimated to be 8100 km\(^2\) and 0.084 km\(^2\), respectively. We converted each image to a binary black and white image by making all pixels darker than a certain threshold black and all those lighter than the threshold white. All white areas are defined as clouds for our analysis. The cumulative frequency-size distribution of each image was computed and averaged with other images of its type at equal cloud numbers. Least-squares linear fits to the logarithms of the data averaged in logarithmically-spaced bins (so that the data was uniformly weighted in log space) yielded power-law exponents -0.72 and -0.82 for the GOES global composite and space shuttle images, respectively. These exponents are similar to the predicted exponent -0.8 based on the KPZ model. In order to compare the distributions of the two types of images, the cloud numbers for the space shuttle images were multiplied by a correction factor, discussed in Lovejoy \[1\], of the ratio of the GOES to the space shuttle resolution cell size to the 0.8 power. The average cumulative frequency-size distribution for the space shuttle images scaled in this way is plotted with the average distribution of the GOES images in Figure 2 along with a least-squares fit to the data with an exponent of -0.8.
We have shown that a simplified model describing the dynamics of ascending warm, moist air expanding and undergoing turbulent convection predicts a fractal structure for cumulus cloud fields consistent with observations of cumulus cloud fields based upon the perimeter fractal dimension and the cumulative frequency-size distribution.
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FIGURES

FIG. 1.

Average cumulative frequency-size distribution, the number of clouds greater than or equal to and area $A$, of GOES global composite and (appropriately scaled) space shuttle cloud images. The distributions are consistent with the KPZ model prediction $N(>A) \propto A^{-0.81}$. 
