\[ \Delta N \text{ and the stochastic conveyor belt of Ultra Slow-Roll} \]

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We analyse field fluctuations during an Ultra Slow-Roll phase in the stochastic picture of inflation and the resulting non-Gaussian curvature perturbation, fully including the gravitational backreaction of the field’s velocity. By working to leading order in a gradient expansion, we first demonstrate that consistency with the momentum constraint of General Relativity prevents the field velocity from having a stochastic source, reflecting the existence of a single scalar dynamical degree of freedom on long wavelengths. We then focus on a completely level potential surface, \( V = V_0 \), extending from a specified exit point \( \phi_e \), where slow roll resumes or inflation ends, to \( \phi \to +\infty \). We compute the probability distribution in the number of e-folds \( N \) required to reach \( \phi_e \) which allows for the computation of the curvature perturbation. We find that, if the field’s initial velocity is high enough, all points eventually exit through \( \phi_e \) and a finite curvature perturbation is generated. On the contrary, if the initial velocity is low, some points enter an eternally inflating regime despite the existence of \( \phi_e \). In that case the probability distribution for \( N \), although normalizable, does not possess finite moments, leading to a divergent curvature perturbation.

**I. INTRODUCTION**

The \( \Delta N \) formalism is a very convenient way to compute the curvature perturbation generated during inflation. Its basic tenet is that quantum fluctuations stretched to superhorizon scales introduce randomness in the total number of e-folds at different points in the universe, with this number of e-folds counted from a given initial spatially flat time-slice to a given final uniform \( \phi \) hypersurface where \( \phi_e = \) a constant field \( \phi_e \) and defined by a constant field hypersurface where \( \phi = \phi_e \), such as inflation ending or the commencement of another distinct phase.

The curvature perturbation generated during a phase of Ultra Slow Roll (USR) has attracted considerable attention recently [22-29] due to the possibility that it leads to an enhanced curvature perturbation and a related enhanced primordial black hole production - see [30] for a recent review on cosmological implications of primordial black holes. In this work we revisit the problem, taking the scalar sector of gravity fully into account. We place the computation within the framework of a long wavelength (leading gradient) approximation to the equations of General Relativity for an inhomogeneous universe: we retain full non-linearities but drop terms that are second order in spatial gradients and properly take into account the field’s velocity and the corresponding gravitational backreaction. Quantum fluctuations are then consistently included as a random forcing of the dynamical equation of the scalar field, a well established approximation for IR quantum fields in inflationary spacetimes [10, 32, 37]. We find that imposing the 0th Einstein equation, the GR momentum constraint, leads to the field \( \phi \)...
being the only dynamical stochastic variable. The field’s velocity is constrained and does not obey an independent equation involving different stochastic kicks at different spatial points, unlike what a naive “separate universe” argument would imply. This is a non-linear generalization of the linear perturbation theory result for $k/aH \to 0$ \cite{4, 31}.

We apply the formalism to a simple problem: the curvature fluctuation generated in an extreme version of USR where the field is injected at some point $\phi_{in}$ with velocity $\Pi_{in}$ on a totally flat potential $V = V_0$. We find two separate regimes, depending on the distance from $\phi_{in}$ to the exit point $\phi_e$ and the initial velocity. If this distance is larger than the length of the classical trajectory, corresponding to a low injection velocity $\Pi_{in}$, the field experiences what we call the stochastic conveyor belt model for USR: in some parts of the universe the initial velocity is forgotten and the field experiences the infinite semi-line $\phi \to \infty$, never fully reaching $\phi_e$. The resulting probability distribution for the total number of e-folds $N$ is normalizable but does not have finite moments, leading to the infinite inflation observed in \cite{33, 39}. However, if the distance is smaller than the length of the classical trajectory, corresponding to a high injection velocity $\Pi_{in}$, graceful exit does occur and inflation eventually terminates in all points of the Universe. The curvature perturbation is then finite and is described by a highly non-Gaussian probability distribution that we compute.

Obviously, the infinite inflation regime will not be reached in realistic single field models where USR takes place only on a finite portion of the potential. This paper then serves an expository function for the developed techniques, involving mainly the use of the Hamilton-Jacobi equation for inflationary evolution, the consistent inclusion of stochastic fluctuations, and the description of the stochastic conveyor belt mechanism. These techniques will be used to analyse more realistic USR potentials in a forthcoming publication \cite{40}.

II. LONG WAVELENGTH SCALAR PERTURBATIONS

We start by recalling the long wavelength approach of Ref. \cite{3}, see also \cite{11}, which will take us to the starting point of our stochastic analysis. Considering the metric in its ADM parametrization,

$$g_{00} = -N^2 + \gamma^{ij} N_i N_j, \quad g_{0i} = N_i, \quad g_{ij} = \gamma_{ij},$$

(1)

where $N$ and $N_i$ are the lapse function and shift vector respectively, the Einstein equations for gravity plus a single scalar field $\phi$ give the GR energy and momentum constraints (00 and 0i Einstein equations)

$$K_{ij} \dot{K}^{ij} - \frac{2}{3} K^2 - (3) R + 16 \pi G \varepsilon = 0,$$

(2)

$$\dot{K}^{ij}_{il} - \frac{2}{3} K_{il} + 8 \pi G \Pi \phi_i = 0,$$

(3)

the dynamical equations for the extrinsic curvature tensor of the 3-slices $K_{ij} = \dot{K}_{ij} + \frac{1}{3} K \gamma_{ij}$,

$$\frac{\partial K}{\partial t} - N^i K_{ij} = -N^i \gamma_{ij}$$

+ $N \left[ \frac{3}{4} K_{ij} \dot{K}^{ij} + \frac{1}{2} K^2 + \frac{3}{4} (3) R + 4 \pi G S \right],$$

(4)

$$\frac{\partial \dot{K}^i_k}{\partial t} + N^i \dot{K}^j_k - N^j_k \dot{K}^i_l - N^i \dot{K}^j_k = -N^i \gamma_{kl}$$

+ $\frac{1}{3} N^j_l \gamma^i_k + N \left[ K \dot{K}^i_k + (3) \Pi \dot{K}^i_k - 8 \pi G S \right],$$

(5)

(stemming from the ij Einstein equation) and the equation of motion for the scalar field

$$\frac{1}{N} \left( \frac{\partial \Pi}{\partial t} - N^i \Pi_{ij} \right) - K \Pi - \frac{1}{N} N_i \phi^i - \phi^i \phi^i + dV = 0.$$  

(6)

In the above, the field momentum $\Pi$ is defined as

$$\Pi = \frac{1}{N} \left( \frac{\partial \phi}{\partial t} - N^i \phi^i \right),$$

(7)

the extrinsic curvature 3-tensor is

$$K_{ij} = -\frac{1}{2N} \left( \frac{\partial \phi^i}{\partial t} - N_{ij} - N_{ji} \right),$$

(8)

and the scalar’s energy density and stress tensor on the 3-slices read

$$\varepsilon = \frac{1}{2} \left( \Pi^2 + \phi^i \phi^i \right) + V(\phi),$$

(9)

and

$$S_{ij} = \phi_i \phi_j + \gamma_{ij} \left( \frac{1}{2} \Pi^2 - \frac{1}{2} \phi^i \phi^i \right) - V(\phi).$$

(10)

A vertical bar denotes a covariant derivative $\nabla_t$ the 3-metric $\gamma_{ij}$ which is also used to raise or lower spatial indices.

The approximation we use to study the non-linear long wavelength configurations relevant for inflation is to only keep terms containing the leading order in spatial derivatives. This is underpinned by the expectation that on scales $aL > H^{-1}$ the dynamics is dominated by time derivatives such that for any quantity $Q$ the inequality $|e^{-\alpha} \nabla Q| < |\partial_t Q|$ will be true, where $\partial_t$ denotes the local expansion rate (see Eqs. \cite{15} and \cite{17} below), a statement that in inflation is expected to eventually hold for all scales of interest. Furthermore, to simplify the equations we choose to consider coordinate systems constructed such that $N_i = 0$. This gauge choice fixes three gauge degrees of freedom, leaving one gauge function un-fixed; its elimination can be achieved e.g. by further choosing a specific form for the lapse function $N$. Under these assumptions, we get from \cite{3} that the traceless part of the extrinsic curvature evolves according to

$$\frac{\partial \dot{K}^i_k}{\partial t} = NK \dot{K}^i_k.$$  

(11)
The 3-metric can be further decomposed as
\[ \gamma_{ij} = e^{2\alpha} h_{ij} \]  
(12)
where $\text{Det} h_{ij} = 1$ and therefore $h^{ij} \frac{\partial}{\partial x^i} h_{ij} = 0$. We then have
\[ K = -3\dot{\alpha} \equiv -3 \frac{1}{N} \frac{\partial \alpha}{\partial t}, \]  
(13)
from where we directly obtain
\[ \tilde{K}^i_j = C^i_j(x)e^{-3\alpha}, \quad \text{Tr}[C^i_j(x)] = 0. \]  
(14)
Since during inflation $\alpha(t, x)$ represents the local generalization of the number of e-folds, it grows approximately linearly in time, and we can take it as a proxy for time in inflation. Therefore, Eq. (14) tells us that the anisotropic expansion rate $\tilde{K}^i_j$ – which is the non-linear generalization of the canonical momentum associated with gravitational waves – declines extremely rapidly (exponentially fast) during inflation. We are thus dynamically led to $\tilde{K}^i_j = 0$ and the most general 3-metric on long wavelengths can be written as
\[ \gamma_{ij}(t, x) = e^{2\alpha(t, x)} h_{ij}(x) \]  
(15)
with the long wavelength spacetime metric taking the form
\[ ds^2 = -N^2(t, x) dt^2 + e^{2\alpha(t, x)} h_{ij}(x) dx^i dx^j. \]  
(16)
and the 3-tensor $h_{ij}$ is not dynamical in this approximation, at least classically. Furthermore, we restrict the lapse function $N(t, x)$ to vary slowly enough in space such that its spatial gradients can be neglected. Later on we will consider scalar quantum fluctuations and the accompanying tensor fluctuations would provide $h_{ij}$ with a stochastic source from subhorizon tensor modes entering the long wavelength sector and with an amplitude set by the uncertainty principle. In this work we focus on the dynamics of the scalar sector of gravity, leaving that of the stochastic evolution of the tensor sector for future study.

Defining the local expansion rate as
\[ H(t, x) \equiv \frac{1}{N} \frac{\partial \alpha}{\partial t}, \]  
(17)
we are led to a set of long wavelength equations for the spatially dependent field $\phi(t, x)$ and expansion rate $H(t, x)$ comprising of the two GR constraints: the energy [2] and momentum [3] constraint (from now on we set $8\pi G = 1$)
\[ H^2 = \frac{1}{3} \left( \frac{1}{2} \Pi^2 + V(\phi) \right), \]  
(18)
and
\[ \partial_t H = -\frac{1}{2} \Pi \partial_t \phi, \]  
(19)
the evolution of the expansion rate [4],
\[ \frac{1}{N} \frac{\partial H}{\partial t} = -\frac{1}{2} \Pi^2, \]  
(20)
as well as the dynamical equations for the scalar field [6–7],
\[ \Pi = \frac{1}{N} \frac{\partial \phi}{\partial t}, \]  
(21)
\[ \frac{1}{N} \frac{\partial \Pi}{\partial t} + 3H\Pi + \frac{dV}{d\phi} = 0. \]  
(22)

Equations (18), (20), (21) and (22) are formally the same as those of homogeneous cosmology but are valid at each spatial point with a priori different values of the initial conditions for $\phi$ and $\Pi$. This is what is sometimes referred to as the “separate universe evolution”. However, not all spatially inhomogeneous initial conditions are allowed on long wavelengths as they must also satisfy the momentum constraint (19). As we will see, this restricts the possibility of assigning $\Pi$ independently of the initial value of $\phi$. Indeed, for the constraint (19) to be respected the local expansion rate $H$ must depend on the spatial position only through its dependence on the spatially varying $\phi$
\[ H(t, x) = H(\phi(t, x), t) \]  
(23)
and the field momentum must be given by
\[ \Pi = -2 \frac{dH}{d\phi}. \]  
(24)
Taking the time derivative of (23) and comparing with [20] we immediately find that
\[ \left( \frac{\partial H}{\partial \phi} \right)_\phi = 0, \]  
(25)
showing that the total spatio-temporal dependence of the expansion rate is solely determined through its dependence on $\phi(t, x)$
\[ H = H(\phi(t, x)) , \]  
(26)
and the field momentum is therefore given by
\[ \Pi = -2 \frac{dH}{d\phi} . \]  
(27)

Hence $\Pi$ is evidently also a function of $\phi$ alone, $\Pi = \Pi(\phi(t, x))$, with no explicit temporal or spatial dependence.
When (27) is inserted into the local energy constraint (19) it gives the Hamilton-Jacobi equation for the function $H(\phi)$

$$\left( \frac{dH}{d\phi} \right)^2 = \frac{3}{2} H^2 - \frac{1}{2} V(\phi). \quad (28)$$

Since (28) is a first order differential equation it admits a family of solutions of the form $H(\phi, C)$, with different solutions parametrized by an arbitrary constant $C$. A solution $H = H(\phi, C)$ to (28) along with

$$\frac{d\phi}{dt} = -2N\frac{dH}{d\phi} \quad (29)$$

provide a complete description and the inhomogeneous long wavelength scalar field configuration $\phi(x)$. The corresponding long wavelength metric (16) can then determined via

$$\frac{1}{N} \frac{\partial}{\partial t} \left( \frac{1}{N} \frac{\partial}{\partial x} \right) + 3H \frac{1}{N} \frac{\partial}{\partial t} + \frac{dV}{d\phi} = 0 \quad (30)$$

as expected. Although equations (28) - (30) are identical to those of homogeneous cosmology, including (31) which is implied by them, the momentum constraint (19) imposes that there is no freedom to choose initial values for $\Pi$ independently at each spatial point. Let us see why: If (28) is considered without reference to (19), a naive separate universe picture would imply that $C = C(x)$, i.e. for every point $x$ on the initial hypersurface there would be a separate integration constant $C(x)$. This simply encodes the freedom to choose the initial field momentum at that point independently of $\phi$. However, a spatially inhomogeneous $C(x)$ further leads to

$$\nabla H = (\partial C H) \nabla C + (\partial_\phi H) \nabla \phi \neq -\frac{1}{2} \Pi \nabla \phi. \quad (32)$$

Therefore, as long as $\partial C H \neq 0$ it follows that $\nabla C = 0$, otherwise the momentum constraint (19) would be violated. This restricts $C$ to be a global constant, meaning that the momentum $\Pi$ cannot be chosen arbitrarily at each point but, instead, all spatial points must be placed along one and the same integral curve of (28).

The restriction $\nabla C = 0$ is a direct consequence of the momentum constraint. However, it would not apply when integral curves of $H(\phi, C)$ exist for which $\partial C H = 0$. In this case setting $\nabla C \neq 0$ would not violate the momentum constraint since a spatially varying value of $C$ does not change the value of $H$ at different spatial points. Such integral curves are attractors of the long wavelength system as one can see by taking a derivative of (28) w.r.t. $C$ which leads to

$$\frac{\partial H}{\partial C} \propto a^{-3}. \quad (33)$$

This is equivalent to the decay of the second solution to (31) and, as was already pointed out in [9], if this decaying mode is neglected the momentum constraint places no further restriction on the long wavelength configuration; this is the case for example in Slow Roll inflation. In the case of motion on a constant potential examined in this work, the decaying mode (33) expresses the approach to a static field and de Sitter spacetime - see (38) where the constant $\phi_0$ in that formula plays the role of the general constant $C$ discussed here. Including this decaying mode is therefore crucial in studying the slide and slowing down of the field on the constant potential. More generally, the Hamilton-Jacobi treatment adopted here allows the $\Delta N$ formalism to be made fully and a priori consistent with the momentum constraint and include the decaying mode which is essential in studying Ultra Slow Roll.

We can conclude that this study goes beyond slow roll and is completely general, relying only on the long wavelength approximation. Note that slow-roll trivially satisfies the momentum constraint and is an attractor in which any dependence of $H$ on $C$ is altogether suppressed exponentially. The absence of a second dynamical inhomogeneous mode is unique to single scalar inflationary models. For a recent study on how the inflaton canonical momentum can be excited during inflation through its coupling to a light spectator scalar field, see Ref. [15].

Before closing this section we note that equations (28), (29) and (30) and the corresponding metric (16) are valid for any choice of time hyper-surfaces (any choice of $N$) as long as the corresponding spatial coordinate worldlines are constructed orthogonal to the time slices, keeping $N_i = 0$, and if all terms that are second order in spatial gradients are dropped, see [4]. For the reader’s convenience, we recall the demonstration of this fact in

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2 Equation (28), often referred to as the Hamilton-Jacobi equation in the literature, has been used in [42, 43] to study an alternative parameterization of possible homogeneous inflationary cosmologies, including USR [12, 14]. We stress that in the present, long wavelength context it describes an inhomogeneous universe as well, as originally demonstrated in [4].
III. HAMILTON-JACOBI SOLUTION IN ULTRA SLOW-ROLL

We now apply the above analysis to our extremal USR scenario where the field moves along a very flat and level part of the potential where \( V(\phi) \simeq V_0 \). Equation (36) then reads
\[
\left( \frac{dH}{d\phi} \right)^2 = \frac{3}{2} H^2 - \frac{1}{2} V_0 .
\]
Taking a derivative with respect to \( \phi \) gives,
\[
\frac{dH}{d\phi} \left( \frac{d^2 H}{d\phi^2} - \frac{3}{2} H \right) = 0 ,
\]
and the general solution can be written as
\[
H(\phi) = H_0 = \sqrt{\frac{V_0}{3}} \quad \text{or} \quad H(\phi) = A e^{-\sqrt{2} \phi} + B e^{\sqrt{2} \phi} \quad (36)
\]
where \( AB = \frac{V_0}{12} \). The constraints on the integration constants \( H_0 \) and \( AB \) are obtained by inserting the general solutions of (35) into the original Hamilton Jacobi equation (34). It is convenient to redefine \( A \) and \( B \) as
\[
A = \frac{H_0}{2} e^{\sqrt{2} \phi_0}, \quad B = \frac{H_0}{2} e^{-\sqrt{2} \phi_0}, \quad (37)
\]
where \( C \) and \( \phi_0 \) are global constants. With these definitions in mind, the second solution in (36) becomes,
\[
H(\phi) = H_0 \left( Ce^{-\sqrt{2} \phi} + C^{-1} e^{\sqrt{2} \phi} \right) = H_0 \cosh \left( \sqrt{\frac{3}{2}} (\phi - \phi_0) \right) , \quad (38)
\]
where we replaced \( C \) by \( \phi_0 = \sqrt{\frac{3}{2} \ln(C)} \). The corresponding momentum (velocity) is then from (24),
\[
\Pi = 0 , \quad \text{or} \quad \Pi = -\sqrt{6} H_0 \sinh \left( \sqrt{\frac{3}{2}} (\phi - \phi_0) \right) . \quad (39)
\]
The number of e-foldings \( \alpha \), from (17) and using the solution for \( H(\phi) \) (38) becomes
\[
\alpha = \int HN dt = -\frac{1}{6} \ln \left[ \sinh^2 \left( \sqrt{\frac{3}{2}} (\phi - \phi_0) \right) \right] . \quad (40)
\]
The above formulae completely describe the classical field evolution along a flat potential \( V = V_0 \), including the gravitational backreaction of a non-zero field velocity. However, they require some clarification with regards to the exact dynamics they describe. We see that if the field starts off with a finite velocity, it evolves asymptotically towards \( \phi_0 \), which it reaches only after an infinite amount of e-folds. The precise value of \( \phi_0 \) depends on the initial velocity imparted on \( \phi \) as well as its sign: If \( \Pi(\phi_0) > 0 \) then \( \phi_0 > \phi(t) \) and \( \phi \) moves asymptotically to the right towards \( \phi_0 \). If \( \Pi(\phi_0) < 0 \) then \( \phi_0 < \phi(t) \) and \( \phi \) moves asymptotically to the left towards \( \phi_0 \). Of course, if \( \Pi(\phi_0) = 0 \) then the field remains static and these are degenerate “trajectories” where \( \Pi = 0 \) and \( H = H_0 \) always. Note that they represent distinct solutions of (28) and the field does not transition from a \( \Pi \neq 0 \) to a \( \Pi = 0 \) state during its classical evolution, nor does its velocity change sign. The constant \( \phi_0 \) is the asymptotic end-point of different classical trajectories and it parametrizes different integral curves of the HJ equation (28); it is identified with the constant \( C \) of the general discussion in section 2. Note that \( \partial_{\phi_0} H|_{\phi = \phi_0} = 0 \) and therefore a static field

\[
\frac{H(\phi)}{\sqrt{V_0}} \quad (34).
\]
can become inhomogeneous without violating the momentum constraint. Figure 3.1 summarises the different solutions. We therefore see that on long wavelengths and for a flat level potential:

- An evolving \((\Pi \neq 0)\) long wavelength field configuration always tends at asymptotically late times to the same field value at all spatial points, \(\phi(t, x) \rightarrow \phi_0\).
- An arbitrary inhomogeneous field configuration is only allowed for a static field, since \(\partial_{\phi_0} H|_{\phi=\phi_0} = 0\) in this case. This is a non-linear version of the growing mode of linearized perturbations.

As emphasized above, these represent two different solutions which classically do not evolve into each other. In the following section we discuss how this picture changes when quantum effects are included.

IV. STOCHASTIC EVOLUTION: THE CONVEYOR BELT OF ULTRA SLOW ROLL

Let us now incorporate a stochastic element in the evolution, modelling as usual quantum fluctuations stretched to long wavelengths. We do not need to specify the amplitude of the noise terms at this point so we keep them general for this discussion. Generically, any stochastic “add-on” to the dynamics, regardless of the microscopic origin of the extra noise terms, can be thought of as adding an extra stochastic “kick” to the classical drift determined by the dynamical equations. Suppose we introduce noise in both the field and its momentum: in a discretized form of the time evolution their values would be updated after a time step \(\Delta t\) as

\[
\Delta \phi = -2 \frac{\partial H}{\partial \phi} N \Delta t + \xi_\phi N \Delta t, \\
\langle \xi_\phi(t) \xi_\phi(t') \rangle = A \delta(t - t'), \\
\Delta \Pi = -3H \Pi N \Delta t + \xi_\Pi N \Delta t, \\
\langle \xi_\Pi(t) \xi_\Pi(t') \rangle = B \delta(t - t').
\]

(41) (42)

To be consistent with the constraints, \(H\) should be a solution to the Hamilton-Jacobi equation. For the case we are considering it is given by \([35]\) with either \(\phi > \phi_0\) or \(\phi < \phi_0\) depending on the sign of the velocity - see figure 3.1.

The energy and momentum constraints are not dynamical equations and therefore they are not to be accompanied by some form of an extra stochastic force. As is well known, the constraints are preserved by the classical dynamical evolution and any consistent stochastic extension of the dynamics should also preserve them while the stochastic kicks are incorporated - the stochastically updated field and momentum should respect them too. We can achieve this by elevating the constants appearing in the solution of the HJ equation, whose values parametrise different possible velocities for fixed field values, to stochastic variables. Let’s now see how this can be done in our case.

The constant \(\phi_0\) in \([39]\), characterizing different HJ solutions, can be linked to the velocity since we can write

\[
\phi_0 = \phi - \sqrt{2} \text{arcsinh} \left( - \frac{\Pi}{\sqrt{2} V_0^{3/2}} \right),
\]

and the choice of \(\Pi\) for fixed \(\phi\) which also defines the breadth of field values covered by motion on the flat potential gives the initial \(\Pi\) of the field. If many possible initial conditions for \(\Pi\) are contemplated, then \(\phi_0\) defines the different asymptotic resting points corresponding to different initial momenta \(\Pi_0\) for a given initial value of \(\phi\). A stochastic change in \(\Pi\) at fixed \(\phi\) would correspond to the field changing the HJ curve along which it evolves and this can be accommodated by promoting \(\phi_0\) to a stochastic variable which would change according to

\[
\Delta \phi_0 = \Delta \phi + \frac{\Delta \Pi}{3H} + \frac{1}{18H^3} \frac{\partial H}{\partial \phi} \Delta \Pi^2,
\]

(44)

leading to

\[
\Delta \phi_0 = \frac{NB}{18H^3} \frac{\partial H}{\partial \phi} N \Delta t + \left( \frac{1}{3H} \xi_\Pi + \xi_\phi \right) N \Delta t.
\]

(45)

At face value this provides a stochastic equation for \(\phi_0\) which would now take different values at different spatial points. However, as we stressed above, unless \(\frac{\partial H}{\partial \phi} = -\frac{\partial H}{\partial \phi} = \frac{\Pi}{2} = 0\), \(\phi_0\) should only take a global value if the momentum constraint is to be respected. This cannot be accommodated in \([45]\) since, for any choice of \(\xi_\phi\) and \(\xi_\Pi\), \(\phi_0\) necessarily develops inhomogeneities.

Hence, although one could a priory allow for stochastic changes in the velocity through stochastically jumping between different HJ trajectories on top of the stochastic \(\phi\) displacement, the momentum constraint prevents that if \(\Pi \neq 0\).

We are thus led to conclude that as long as \(\Pi \neq 0\) the whole long wavelength universe can only be located on different points of a single HJ trajectory with the following stochastic equation:

\[
\Delta \phi = -2 \frac{\partial H(\phi, \phi_0)}{\partial \phi} N \Delta t + \xi_\phi N \Delta t, \\
\langle \xi_\phi(t) \xi_\phi(t') \rangle = A \delta(t - t'),
\]

(46)

with either \(\phi > \phi_0\) or \(\phi < \phi_0\), depending on the fixed sign of the momentum. Once stochastic evolution takes the field past \(\phi_0\), memory of the initial velocity is lost.

\[^3\] Note that we are using Itô’s calculus here \([46, 47]\) and we therefore keep \(\Delta \Pi^2\) terms to follow changes to order \(\Delta t\). Other choices are possible along with corresponding calculi and the results are invariant since \(A\) and \(B\) are independent of \(\phi\).
and it simply diffuses by a free random walk on the flat potential surface \( V_0 \) obeying

\[
\Delta \phi = \xi_\phi(t, x) N \Delta t, \quad \langle \xi_\phi(t) \xi_\phi(t') \rangle = A \delta(t - t'). \quad (47)
\]

In this regime the field does jump between different HJ trajectories, i.e. different points on the \( H = H_0 \) surface. This is now allowed as these degenerate solutions are characterized by \( \partial_{\phi_0} H|_{\phi = \phi_0} = \Pi(\phi_0) = 0 \). Hence the momentum constraint is not violated by the universe occupying different solutions at different spatial points and stochastically jumping between them, becoming a collection of classically static field values that carry no extra energy.

Note that the quantum fluctuations have a remarkable effect. The classical phase-space, consisting of the set of trajectories that solve the HJ equation \([28]\) and which are shown in figure 3.1, is split into (a) regular (non-degenerate, HJ) trajectories, which are characterized by an initial momentum \( \Pi_{\text{in}} \), the corresponding field value, \( \phi_{\text{in}} = \phi(\Pi_{\text{in}}) \), and end at \( \phi_0 = \phi_0(\Pi_{\text{in}}) \) at which \( \Pi = 0 \); (b) degenerate trajectories characterized by \( \Pi_{\text{in}} = 0 \) and an arbitrary field value \( \phi_0 \). The quantum phase-space is very different however. A typical quantum/stochastic trajectory consists of a classical HJ branch \( \Pi_{\text{in}} \), which ends at \( \phi_0 = \phi(\Pi_{\text{in}}) \), supplemented by the set of all degenerate trajectories, \( \Pi = 0, \phi \in \mathbb{R} \). The point \( \phi_0 = \phi(\Pi_{\text{in}}) \) is a bifurcation point, at which the quantum trajectory splits into two branches: \( \phi > \phi_0 \) and \( \phi < \phi_0 \), see figure 3.1. This quantum phase space picture resembles a conveyor belt for the quantum field which starts at a point on one of the HJ branches, diffuses downwards towards the bifurcation point at \( \phi_0 \) and then continues diffusing along the set of points shown as the horizontal line \( \Pi = 0 \), \( H = H_0 \equiv \sqrt{V_0/3} \) in figure 3.1.

Despite field fluctuations being generated, the above picture does not carry with it a well-defined curvature perturbation. A corresponding curvature perturbation emerges only when an exit point \( \phi_e \) is specified, where either inflation ends or another inflationary era follows by exiting the region where \( V(\phi) = V_0 \). If \( \phi_e \) lies outside the HJ branch, we are faced with the stochastic conveyor belt and a double first passage-time problem: Firstly to transition from a \( \Pi \neq 0 \) solution onto the \( H = H_0 \) surface (non-stochastic evolution does not allow this) and secondly to exit the \( H = H_0 \) region by reaching \( \phi_e \). If \( \phi_e \) is reached within the HJ branch we have a standard first passage-time problem and the conveyor is not operational. We analyse these cases in sections \( V \) and \( VI \) respectively.

V. THE CASE \( \phi_e < \phi_0 \): USR WITHOUT GRACEFUL EXIT

As we demonstrated above, the gravitationally consistent inclusion of velocity to the problem of diffusion on a flat potential leads naturally to a two stage process when \( \phi_e < \phi_0 \): 1) All spatial points diffuse along a single branch of the HJ solution until \( \phi_0 \) is crossed and then 2) each point that has crossed \( \phi_0 \) diffuses independently along the level \( V_0 \) potential. We therefore need to construct a first passage time probability distribution for the first stage and for that we require the kernel (to which we shall also refer to as the propagator) with exit boundary conditions at \( \phi_0 \), achieved by setting \( P_{\text{HJ}}(\phi_0, \alpha) = 0 \) \([48]\). The probability current at that point then injects probability for the second stage of the diffusion - one can thus think of the HJ branch as a “conveyor belt” feeding the second diffusive process at a single point \( \phi_0 \).

The stochastic equation describing the IR field dynamics reads,

\[
d\phi = -2 \frac{\partial H}{\partial \phi} d\tau + \xi_\phi d\tau, \quad (48)
\]

where \( \xi_\phi \) is the noise generated by the flow of modes between the UV and IR sectors of the theory. When treated perturbatively, due to an effectively time-dependent cutoff, the leading order contribution occurs at the tree level \([16]\) and on super-Hubble scales the noise is, to a good approximation, of Markovian type,

\[
\langle \xi_\phi(\tau) \xi_\phi(\tau') \rangle = A \delta(\tau - \tau'), \quad (49)
\]

where \( \tau = \int^t N(t, x) dt' \) denotes a reparametrization invariant time. The coupling between the ultraviolet and long wavelength modes can then be approximated by its tree level expression,

\[
A = \frac{(\sigma \alpha H)^3}{2 \pi^2} (1 - \epsilon) H|_{|\phi(\tau, k)|^2 = \sigma \alpha H} \left[ 1 + \mathcal{O}(\kappa^2 H^2) \right],
\]

where \( \kappa^2 = 16\pi G \) is the loop counting parameter of quantum gravity and \( \sigma \) sets the highest (ultraviolet cutoff) energy scale of the long-wavelength theory. For example, when \( \sigma = 1 \), the highest scale (smallest wavelength) is the Hubble scale, when \( \sigma \ll 1 \), the highest scale is much smaller than the Hubble scale (or equivalently wavelength much longer than \( H^{-1} \)). Since there can be no secular enhancement in the loop corrections in \([50]\), the loop suppression factor, \( \kappa^2 H^2 \lesssim 10^{-12} \) represents a fair estimate of the accuracy of the stochastic approximation scheme developed in this work. In order to estimate the noise amplitude \([50]\), in what follows we work in the approximation \( \epsilon \approx 0 \) and set \( \sigma \ll 1 \), in which case \( |\phi(\tau, k)|^2 \approx H_0^2/(2k^3) \) and the noise amplitude \([50]\) simplifies to,

\[
A \approx \frac{H_0^3}{4\pi}, \quad (51)
\]

which is the approximation we use below. Rigorous proof that Starobinsky’s stochastic inflation \([16]\) reproduces the correct infrared dynamics on de Sitter can be found in Ref. \([32, 97]\) for interacting scalar field theories and in Ref. \([50]\) for quantum scalar electrodynamics. These works demonstrate that stochastic inflation not only reproduces the leading infrared logarithms at each order
in perturbation theory, but (when summed up) they also reveal what happens at late times in the deep nonperturbative regime when the large logarithms overwhelm small coupling constants, a point first made in [19]. To directly compute the curvature perturbation we use the number of e-folds \( \alpha \) as the time variable. This is in fact required for consistency with standard cosmological perturbation theory, see e.g. [21]. We therefore have the following branches of the evolution:

**HJ Branch:** The Langevin equation on the HJ branch is

\[
\frac{d\phi}{d\alpha} = -\frac{\delta \ln H(\phi, \phi_0)}{\delta \phi} + \frac{H(\phi, \phi_0)}{2\pi} \xi(\alpha) \tag{52}
\]

with

\[
\langle \xi(\alpha) \xi(\alpha') \rangle = \delta(\alpha - \alpha') \tag{53}
\]

and an absorbing boundary condition at \( \phi = \phi_0 \).

\( H(\phi, \phi_0) \) is the solution to the HJ equation with \( \phi_0 \) determined by the initial velocity of the field. When expressed as a Fokker-Planck equation this implies that the probability density \( P_{\text{HJ}}(\phi, \alpha) \) on the HJ branch obeys,

\[
\frac{\partial P_{\text{HJ}}}{\partial \alpha} = -\frac{\partial}{\partial \phi} \left( -\frac{\delta \ln H}{\delta \phi} P_{\text{HJ}} \right) + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} \left( \frac{H^2}{4\pi^2} P_{\text{HJ}} \right) \tag{54}
\]

\[
= -\frac{\partial J}{\partial \phi}, \tag{55}
\]

to be solved with the boundary condition \( P_{\text{HJ}}(\phi_0, \alpha) = 0 \) which implies that once a random walker \( \phi \) among the ensemble ventures to \( \phi = \phi_0 \), it is removed - see e.g. Ref. [28] for a detailed discussion of this boundary condition's use in exit problems.

**HJ (de Sitter) branch:** Once the stochastically evolving field at a spatial point reaches \( \phi_0 \), it is removed from the HJ branch and is injected into the degenerate \( V = V_0 \) (de Sitter) branch, where \( H = H_0 = \sqrt{V_0/3} \). It then diffuses along the semi-infinite branch \( \phi \in [\phi_e, \infty) \) of the flat potential \( V = V_0 \) according to the Langevin equation,

\[
\frac{d\phi}{d\alpha} = \frac{H_0}{2\pi} \xi(\alpha) \tag{56}
\]

again with

\[
\langle \xi(\alpha) \xi(\alpha') \rangle = \delta(\alpha - \alpha') , \tag{57}
\]

where now the influx from the HJ branch must also be accounted for. When the exit point \( \phi_e \) of the \( V = V_0 \) branch is reached inflation may end, for example by entering a non-slow roll region or by instant reheating, or the field may enter a subsequent slow roll phase. In either case, the quantity of interest is the number of e-folds until \( \phi_e \) is reached which is a stochastic quantity.

We can write the probability distribution for \( \phi \) on the \( H_0 \) branch as

\[
P_{\text{HJ}}(\phi) = P_D(\phi) + P_* \delta(\phi - \phi_0) \tag{58}
\]

where \( P_D \) is that part which has diffused along the \( V = V_0 \) surface while \( P_* \) denotes the probability at \( \phi_0 \) leaking in from the HJ branch. Its contribution to the Fokker-Planck equation on the \( V = V_0 \) branch can be computed as follows: in a time interval between \( \alpha \) and \( \alpha + d\alpha \) the amount of random walkers flowing in from the HJ branch is

\[
dP_\alpha = \int_{\phi_0}^{\phi} [P_{\text{HJ}}(\phi, \alpha + d\alpha) - P_{\text{HJ}}(\phi, \alpha)] d\phi \tag{59}
\]

Therefore,

\[
\frac{\partial P_*}{\partial \alpha} = \int_{\phi_0}^{\phi} \frac{\partial P_{\text{HJ}}}{\partial \alpha} d\phi = J(\phi_0, \alpha) . \tag{60}
\]

and the Fokker-Planck equation for \( P_{\phi_0} \) can then be written as

\[
\frac{\partial P_{\phi_0}}{\partial \alpha} = \frac{1}{2} \frac{\partial^2}{\partial \phi^2} \left( \frac{V_0}{12\pi^2} P_{\phi_0} \right) + J(\phi_0, \alpha) \delta(\phi - \phi_0) , \tag{61}
\]

where, recalling that \( P_{\text{HJ}}(\phi_0) = 0 \) and \( \partial_\phi H(\phi_0) = 0 \),

\[
J(\phi_0, \alpha) = \frac{V_0}{24\pi^2} \frac{\partial P_{\text{HJ}}}{\partial \phi} \bigg|_{\phi_0} . \tag{62}
\]

The probability distribution for the number of e-folds it takes for the field to reach \( \phi_e \) can be obtained from knowledge of \( P_D \) by noting that once the random walker has been injected into the \( V_0 \) branch and has started diffusing, the probability it hasn’t yet crossed \( \phi_e \) by the time of \( N \) e-folds is the same as that of inflation lasting longer than \( N \) e-folds:

\[
\text{Prob(Inflationary duration} > N) = \int_{\phi_e}^{\phi_0} \rho(\alpha) d\alpha = \int_{N}^{\phi_0} P_D(\phi, N) d\phi \tag{63}
\]

where we denoted the probability that inflation lasts (more precisely \( \phi_e \) is reached) between \( \alpha \) and \( \alpha + d\alpha \) e-folds by \( \rho(\alpha) \). Therefore

\[
\rho(N) = -\frac{\partial}{\partial N} \int_{N}^{\phi_0} P_D(\phi, N) d\phi . \tag{64}
\]

Using (58), (60) and (61) we obtain simply

\[
\rho(N) = \frac{V_0}{24\pi^2} \frac{\partial P_{\phi_0}(\phi, N)}{\partial \phi} \bigg|_{\phi_e} . \tag{65}
\]
A. Computing \( P_{HJ} \)

In order to obtain the current flowing into the \( V_0 \) branch from Eq. (62), we first need to compute \( P_{HJ} \), the probability distribution on the HJ branch. To obtain simple analytic expressions, we will make the approximation that \( \phi \) is close to \( \phi_0 \) on the HJ branch, corresponding to a small initial velocity. This is justified since

\[
\phi_{in} - \phi_0 = \arcsinh \left( -\frac{\Pi_{in}}{\sqrt{2}V_0^{1/2}} \right) \simeq -\frac{\Pi_{in}}{\sqrt{2}V_0^{1/2}} \ll 1 \quad (66)
\]

assuming that the field enters the USR regime from a previous slow roll phase. We will tackle the more general problem in an upcoming publication [?]. We therefore take the HJ branch stochastic dynamics to be (see \[38\])

\[
d\phi \simeq -3(\phi - \phi_0) + \frac{H_0}{2\pi} \xi(\alpha) \quad (67)
\]

with an exit boundary conditions at \( \phi_0 \). Setting \( \chi = \frac{\sqrt{2\pi}}{H_0} (\phi - \phi_0) \) the corresponding Fokker-Planck equation

\[
K_M(\chi, \alpha; \chi_{in}, \alpha_{in}) = \frac{1}{\sqrt{2\pi \sinh[3(\alpha-\alpha_{in})]}} \exp\left(-\frac{\coth[3(\alpha-\alpha_{in})](\chi^2 + \chi_{in}^2)}{2} + \frac{\chi \chi_{in}}{\sinh[3(\alpha-\alpha_{in})]}\right) \quad (71)
\]

which for small time intervals tends to

\[
\lim_{\alpha \to \alpha_{in}} K_M(\chi, \alpha; \chi_{in}, \alpha_{in}) = \delta(\chi - \chi_{in}) \cdot (72)
\]

Since a random walker is ‘removed’ upon reaching \( \chi = 0 \) (\( \phi = \phi_0 \)), for the problem at hand we do not require the free, but rather the absorptive kernel. Due to the symmetry of the effective potential in which the dynamics takes place, it can be obtained from the full kernel \[71\] by adding to it a free mirror kernel at \( -\chi \), giving

\[
\Psi(\chi, \alpha) = K_M(\chi, \alpha; \chi_{in}, \alpha_{in}) - K_M(-\chi, \alpha; \chi_{in}, \alpha_{in}) = \sqrt{\frac{2}{\pi}} \frac{1}{\sinh[3(\alpha-\alpha_{in})]} \exp\left(-\frac{\coth[3(\alpha-\alpha_{in})](\chi^2 + \chi_{in}^2)}{2}\right) \sinh\left(\frac{\chi \chi_{in}}{\sinh[3(\alpha-\alpha_{in})]}\right) \quad (73)
\]

which ensures the correct boundary condition is satisfied. The properly normalized \( P_{HJ} \) is then obtained from \[69\],

\[
P_{HJ}(\varphi, \alpha) = \frac{6\pi}{\sqrt{V_0}} \exp\left(\frac{3}{2}(\alpha-\alpha_{in}) - \frac{1}{2}(\chi^2 - \chi_{in}^2)\right) \Psi(\chi, \alpha) \quad (74)
\]

where \( C \) in Eq. \[69\] is chosen such that in the limit \( \alpha \to 0 \) for the probability density \( P_{HJ}(\chi, \alpha) \) reads,

\[
\frac{\partial P_{HJ}}{\partial \alpha} \simeq 3 \frac{\partial}{\partial \chi} (\chi P_{HJ}) + \frac{3}{2} \frac{\partial^2 P_{HJ}}{\partial \chi^2} \quad (68)
\]

Writing

\[
P_{HJ}(\chi, \alpha) = Ce^{\frac{3}{2}(\alpha - \frac{1}{2} \chi^2)} \Psi(\chi, \alpha) \quad (69)
\]

where \( C \) is a constant independent of \( \alpha \) and \( \chi \), but dependent on the choice of the initial state, \( \Psi(\chi, \alpha) \) obeys

\[
-\frac{1}{3} \frac{\partial \Psi}{\partial \alpha} = \frac{1}{2} \left( \frac{\partial^2}{\partial \chi^2} + \chi^2 \right) \Psi \quad (70)
\]

and the problem reduces to the quantum mechanical kernel for the simple harmonic oscillator (SHO), with a mass \( m \) and frequency \( \omega \) given by \( m\omega = \hbar H_0^2/(12\pi^2) \) in imaginary time \( \alpha = i\omega t/(3\hbar) \) (or \( t = -3\hbar\alpha/\omega \)), see e.g. \[51\]. The free propagator, also known in the literature on stochastic processes as the Mehler heat kernel \[52\], is given by

\[
P_{HJ}(\varphi, \alpha \to \alpha_{in}) = \frac{6\pi}{\sqrt{V_0}} \delta(\chi - \chi_{in}) = \delta(\phi - \phi_0) \quad (75)
\]

for \( \chi \geq 0, \chi_{in} > 0 \). A general probability distribution on the HJ branch can be obtained by convolving the absorptive kernel \[73\] with the initial probability distribution.

The injected current \[62\], \( J = \sqrt{V_0}/(4\pi) \partial_\chi P_{HJ}|_{\chi \to 0} \), into the flat \( V_0 \) branch is obtained by taking a derivative of \[74\],

\[
J(\alpha) = \frac{6\pi}{\sqrt{6\pi(3\Delta\alpha)^{3/2}}} \exp\left[\frac{3}{2} \Delta\alpha - \frac{1}{2} \left(\coth[3\Delta\alpha] - 1\right) \chi_{in}^2\right] \chi_{in} \quad (76)
\]

which rises at early times \( \Delta\alpha = \alpha - \alpha_{in} \ll 1 \) as,

\[
J(\alpha)|_{\Delta\alpha \ll 1} = \frac{\chi_{in}^2}{\sqrt{6\pi(3\Delta\alpha)^{3/2}}} e^{-\frac{\chi_{in}^2}{3\Delta\alpha}} \quad (77)
\]

5 Strictly speaking, in the limit \( \alpha \to \alpha_{in} \) the probability density \[74\] reduces to \( \delta(\chi - \chi_{in}) + \delta(\chi + \chi_{in}) \). Since the domain of validity of \[75\] is the HJ branch on which \( \chi \geq 0 \), the second delta function is discarded.
and we set, for simplicity, $\alpha$ exponentially, at the HJ branch, equation (61) must be supplemented

$$J(\alpha)|_{\Delta \alpha \gg 1} = \frac{6}{\sqrt{\pi}} \chi_{\text{in}} e^{-3\Delta \alpha} + \mathcal{O}(e^{-9\Delta \alpha}).$$  \hspace{1cm} (78)

The current $J(\alpha)$ for three different values of $\chi_{\text{in}}$ can be seen in figure V.A. The current increases from zero at $t = 0$ ($\alpha = \alpha_{\text{in}}$), peaks and then decays exponentially as $e^{-3(\alpha-\alpha_{\text{in}})}$, see (77) and (78).

### B. Computing $P_{V_0}$

Assuming that the initial field distribution lies entirely at the HJ branch, equation (61) must be supplemented by the initial condition $P(\chi, 0) = 0$ and the solution can therefore be written as

$$P_{V_0}(\phi, \alpha) = \int_{\alpha_{\text{in}}}^{\alpha} du G_e(\phi - \phi_0, \alpha - u) J(u),$$  \hspace{1cm} (79)

where $G_e(\phi - \phi', \alpha - \alpha')$ is the diffusive Green function with exit boundary condition at $\phi_e$, also known as the absorptive kernel. As above, it is straightforwardly constructed from the well known unrestricted diffusive kernel along an infinite interval

$$G(\phi, \phi', \Delta \alpha) = \sqrt{\frac{2\pi}{H_0^2 \Delta \alpha}} \exp\left(-\frac{2\pi^2}{H_0^2 \Delta \alpha} (\phi - \phi')^2\right)$$  \hspace{1cm} (80)

where $\Delta \alpha = \alpha - \alpha'$, by subtracting the same kernel but with $\phi'$ reflected on $\phi_e$: $\phi \to 2\phi_e - \phi$, giving

$$G_e(\phi, \phi', \Delta \alpha) = \sqrt{\frac{2\pi}{H_0^2 \Delta \alpha}} \left[ \exp\left(-\frac{2\pi^2}{H_0^2 \Delta \alpha} (\phi - \phi')^2\right) - \exp\left(-\frac{2\pi^2}{H_0^2 \Delta \alpha} (2\phi_e - \phi - \phi')^2\right) \right].$$  \hspace{1cm} (81)

This imposes the correct boundary conditions, $G_e(\phi_e, \phi', \Delta \alpha) = 0$ and $G_e(\phi, \phi', \Delta \alpha \to 0) = \delta(\phi - \phi')$ for $\phi, \phi' \geq \phi_e$. The limits of integration in (79) are determined by imposing that no current can be sourced before the beginning of inflation at $\alpha_{\text{in}}$ (lower limit) and that no current can be sourced in the future of $\alpha$ (upper limit).

To compute the probability distribution for the field on the flat branch we use the convolution integral (79) with the absorptive kernel is (81) and the injected current (76) to obtain,

$$P_{V_0}(\phi, \alpha) = \frac{3 \chi_{\text{in}}}{H_0} \int_{0}^{\alpha} du \frac{\exp\left(-\frac{\chi^2}{6(\alpha-u)}\right) - \exp\left(-\frac{(\chi - 2\chi_e)^2}{6(\alpha-u)}\right)}{\sqrt{\alpha-u}} \exp\left[\frac{3}{2}u - \frac{1}{2} \left(\frac{\coth(3u) - 1}{\sinh(3u)}\right)^{3/2}\chi_{\text{in}}^2\right],$$  \hspace{1cm} (82)

where $\chi = \sqrt{12\pi} (\phi - \phi_0)/H_0$, $\chi_e = \sqrt{12\pi} (\phi_e - \phi_0)/H_0$ and we set, for simplicity, $\alpha_{\text{in}} = 0$.\footnote{One can always recover the dependence on $\alpha_{\text{in}}$ by noting that the integral (82) is a function of $\alpha - \alpha_{\text{in}}$.}

### C. Probability density for the e-fold number

From $P_{V_0}$ we can directly compute the e-fold probability density using (65)

$$\vartheta(N) = \frac{\sqrt{3}}{2\pi} (\chi_e \chi_{\text{in}}) \int_{0}^{N} du \exp\left[-\frac{\chi^2}{6(N-u)} + \frac{3}{2}u - \frac{1}{2} \left(\frac{\coth(3u) - 1}{\sinh(3u)}\right)^{3/2}\chi_{\text{in}}^2\right].$$  \hspace{1cm} (83)

where we note that $\chi_e < 0$ by definition. Although the above integrals cannot be evaluated analytically, an approximate evaluation of (82) can be performed by noting
that the dominant dependence on $u$ sits in the exponent and the integral can be well approximated by a steepest descent method presented in Appendix B. There is a very simple case, namely if the integral is dominated by $u \ll 1$ and if $\alpha \gg 1$, then it evaluates to,

$$P_{V_0}(\phi, \alpha) \approx \sqrt{\frac{12\pi}{H_0^2}} \frac{e^{-\chi_{\text{in}}^2/2}}{6\pi \Delta \alpha} \times \left[ \exp \left( -\frac{\chi^2}{6\Delta \alpha} \right) - \exp \left( -\frac{(\chi - 2\chi_e)^2}{6\Delta \alpha} \right) \right], \quad (84)$$

which is, up to the factor $e^{-\chi_{\text{in}}^2/2}$, equal to the absorptive kernel $G_e(\phi - \phi_0; \Delta \alpha) \left( \Delta \alpha = \alpha - \alpha_{\text{in}} \right)$ in Eq. (81). Therefore

$$\varrho(\mathcal{N}) \approx \sqrt{\frac{6}{\pi}} e^{-\frac{\chi_{\text{in}}^2}{2}} \frac{e^{-\frac{\chi^2}{N^3/2}}}{N^{3/2}} \quad (85)$$

We see that although $\varrho(\mathcal{N})$ is normalizable, the probability distribution does not decay fast enough as $\phi \rightarrow \infty$ and therefore all moments are infinite: $\langle \mathcal{N}^n \rangle \approx \infty$ for $n \geq 1$. A numerical evaluation of the probability density $\varrho(\mathcal{N})$ is plotted in figure 5.2. For large $\mathcal{N}$ the distribution tends to an $\propto N^{-3/2}$ decay, which is in agreement with our analytic estimate. This reflects the fact that if the precipice signified by $\phi_0$ is beyond $\phi_0$, the field settles into free diffusion along the half-line towards $\phi \rightarrow \infty$, a situation termed infinite inflation in [38] [39].

The endless diffusion towards $\phi \rightarrow \infty$ would of course not occur if the field was injected into the de Sitter branch from a prior slow-roll regime. We will deal with this in more detail in [?] where more complete models are studied. A simple way to regulate this infinite diffusion would be to erect a reflecting wall at, or close to $\phi_0$. In [23] this is shown to indeed lead to a distribution with finite moments and hence a finite curvature perturbation.

VI. THE CASE $\phi_e > \phi_0$: USR WITH GRACEFUL EXIT

We saw in the previous section that if the initial velocity of the field does not suffice to carry it beyond $\phi_e$, $\phi_e < \phi_0$, eternal inflation sets in on the semi-line $[\phi_e, \infty)$ and the curvature perturbation is infinite, as signified by the divergence of all moments of $\mathcal{N}$. We show in this section that this is not true when the exit point $\phi_e$ occurs on the HJ branch, i.e. before the asymptotic point $\phi = \phi_0$ at which the classical trajectory of $\phi$ would terminate. In other words we now assume that,

$$\phi_e \geq \phi_0, \quad (86)$$

and show that this model of inflation exhibits a graceful exit. The probability density $P_{HJ}(\phi, \alpha)$ is then of the form (74), but with $\Psi(\chi, \alpha)$ given by the absorptive kernel mirrored at $\chi_e$, see Eq. (73),

$$\Psi(\chi, \alpha) = K_M(\chi, \alpha; \chi_{\text{in}}, \alpha_{\text{in}}) - K_M(2\chi_e - \chi; \alpha; \chi_{\text{in}}, \alpha_{\text{in}})$$

$$= \frac{1}{\sqrt{2\pi \sinh[3(\alpha - \alpha_{\text{in}})]}} \left\{ \exp \left( -\frac{1}{2} \coth[3(\alpha - \alpha_{\text{in}})](\chi^2 + \chi_{\text{in}}^2) + \frac{\chi \chi_{\text{in}}}{\sinh[3(\alpha - \alpha_{\text{in}})]} \right) \right\} - \exp \left( -\frac{1}{2} \coth[3(\alpha - \alpha_{\text{in}})] \left\{ (2\chi_e - \chi)^2 + \chi_{\text{in}}^2 + \frac{(2\chi_e - \chi) \chi_{\text{in}}}{\sinh[3(\alpha - \alpha_{\text{in}})]} \right\} \right\}, \quad (87)$$

The probability $\varrho(\mathcal{N})$ in (64) that inflation ends in the interval $[\mathcal{N}, \mathcal{N} + d\mathcal{N}]$ of e-folds is then,

$$\varrho(\mathcal{N}) = -\frac{\partial}{\partial \mathcal{N}} \int_{\phi_e}^{\infty} d\phi P_{HJ}(\mathcal{N}, \phi)$$

$$= -\frac{\partial}{\partial \mathcal{N}} \left\{ \frac{1}{2} \text{erfc} \left[ \sqrt{1 + n} (\chi_e - e^{-3\Delta \mathcal{N}} \chi_{\text{in}}) \right] - \text{erf} \left[ \chi_{\text{in}}^2 - \chi_e^2 - (\chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e)^2 \right] \frac{1}{2} \text{erfc} \left[ \sqrt{n} \left( \chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e \right) \right] \right\}$$

$$= \frac{3\sqrt{\pi}}{\sqrt{n}} \left\{ 2(1+n) \chi_{\text{in}} - (3+2n) e^{-3\Delta \mathcal{N}} \chi_e \right\} \text{erf} \left[ -(1+n) (\chi_e - e^{-3\Delta \mathcal{N}} \chi_{\text{in}})^2 \right]$$

$$- 3e^{-3\Delta \mathcal{N}} \chi_e \left[ \chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e \right] e^{(\chi_{\text{in}}^2 - \chi_e^2) - (\chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e)^2} \text{erfc} \left[ \sqrt{n} \left( \chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e \right) \right], \quad (89)$$

$$= \frac{3\sqrt{\pi}}{\sqrt{n}} \left\{ 2(1+n) \chi_{\text{in}} - (3+2n) e^{-3\Delta \mathcal{N}} \chi_e \right\} \text{erf} \left[ -(1+n) (\chi_e - e^{-3\Delta \mathcal{N}} \chi_{\text{in}})^2 \right]$$

$$- 3e^{-3\Delta \mathcal{N}} \chi_e \left[ \chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e \right] e^{(\chi_{\text{in}}^2 - \chi_e^2) - (\chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e)^2} \text{erfc} \left[ \sqrt{n} \left( \chi_{\text{in}} - e^{-3\Delta \mathcal{N}} \chi_e \right) \right], \quad (90)$$
FIG. VI.1. The probability distribution in the USR model with graceful exit as a function of the number of e-folds $\Delta N = N - N_{\text{in}}$ with $N_{\text{in}} = 0$ and given in Eq. (90) for four choices of $(\chi_e, \chi_{in})$: (0, 1) (solid black); (0, 1.5) (solid orange); (0.3, 1) (long dashes) and (0.3, 2) (short dashes).

We see that for larger $\chi_e (\chi_{in})$ inflation gets shorter (longer), which is what one would expect.

where $n = 1/(e^{6\Delta N} - 1)$, $\Delta N = N - N_{\text{in}}$ and $\text{erf}(z) = 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt$ is the complementary error function. The distribution $\phi(N)$ in (88,90) is plotted in figure VI.1 for a few selected values of $\chi_e$ and $\chi_{in}$.

The distribution is again strongly non-Gaussian, however at large $N$ it falls-off exponentially as $\propto e^{-\Delta N}$, such that the moments of the curvature perturbation are all finite, implying that inflation terminates. The first term after the curly bracket in (89) is the standard result for the probability that the particle is located anywhere at $\chi > \chi_e$ and it approaches one when $\chi_e \to -\infty$, as it should, while the second term reduces the probability due to the absorbptive boundary condition at $\phi = \phi_0$, where inflation ends.

Eqs. (88,90) contain a complete information for the probability distribution of the number of e-folds in this simple model (where we assumed a small initial momentum, which allowed us to linearize $\phi - \phi_0$ in (67)). To get a better understanding of $\phi(N)$ in (88), we shall now calculate the first few moments of the number of e-folds,

$$\langle N^k \rangle = \int_{N_{\text{in}}}^{\infty} N^k \phi(N) dN, \quad (k = 0, 1, 2, \cdots).$$

Let us first look at the zeroth moment,

$$\langle 1 \rangle = \frac{1}{2} \left[ 1 + \text{erf}[\chi_e] + e^{-\chi_e^2} \right] \quad (\chi_e \geq 0).$$

When $0 \leq \chi_e \lesssim 1$ this is, as one would expect, of the order one. One can account for the fact that (92) is not exactly equal to one by dividing $\langle N^0 \rangle$ by (1). The moments of $N$ are considerably more difficult to calculate, and therefore in what follows for simplicity we consider the case, $\chi_e = 0$ (or $\phi_e = \phi_0$). Then the probability distribution (88,90) reduces to

$$\phi(N) dN = \frac{6}{\sqrt{\pi}} (1 + n) \chi_{in} e^{-n \chi_{in}^2} dN.$$  

It pays off to convert this into the probability per unit $dn = -6n(1 + n) dN$,

$$G(n) dn = \frac{\chi_{in}}{\sqrt{\pi n}} e^{-n \chi_{in}^2} dN,$$

such that the $k$-th moment in (91) gives,

$$\langle N^k \rangle = \frac{2}{\sqrt{\pi}} \int_0^\infty dy e^{-y^2} \left[ \frac{1}{6} \ln \left( 1 + \frac{\chi_{in}^2}{y^2} \right) \right]^k,$$

where $(n\choose k) = n!/[k!(n - k)!]$ is the binomial coefficient.

The first moment in (96) can be expressed in terms of a generalized hypergeometric function,

$$\langle N \rangle = \frac{\pi}{6} \text{erf}(i \chi_{in}) - \frac{\chi_{in}^2}{3} \times _2 F_2 \left( \frac{1}{2}, 1; \frac{3}{2}, 2; \chi_{in}^2 \right).$$

The higher moments are harder to evaluate analytically. Nevertheless, one can show that the following confluent hypergeometric function generates all the moments,

$$G(\alpha, \chi_{in}) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy e^{-y^2} \left( 1 + \frac{\chi_{in}^2}{y^2} \right)^\alpha = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} - \frac{\alpha}{6} \right) \times U \left( -\frac{\alpha}{6}, 1; \chi_{in}^2 \right) = 1 F_1 \left( \frac{1}{6}, 1; \chi_{in}^2 \right) \times \frac{\Gamma \left( \frac{1}{2} - \frac{\alpha}{6} \right)}{\Gamma \left( -\frac{\alpha}{6} \right)} (\chi_{in}^2)^{1/2} \times 1 F_1 \left( \frac{1}{2} - \frac{\alpha}{6}, \frac{3}{2}; \chi_{in}^2 \right)$$

in the sense that

$$\langle N^k \rangle = \left( \frac{\partial^k}{\partial \alpha^k} G(\alpha, \chi_{in}) \right)_{\alpha = 0} \quad (k = 1, 2, 3, \cdots),$$

where $U$ denotes the confluent hypergeometric function. In figure VI.2 we show the first few moments in Eq. (95) and their fluctuations around the mean $\langle N \rangle$ defined in Eq. (96). For simplicity we choose $\chi_e = 0$ and plot our results as a function of $\chi_{in}$. We see that the distribution is highly non-Gaussian, which is one of the main results of this work. Because we have calculated $G(\alpha)$ with the assumption of small $\phi - \phi_0$, in figure VI.2 we plot the results only for $\chi_{in} < 1$. The principal conclusion is that the non-Gaussianities produced in USR are quite large and grow with $\chi_{in}$, or the length of the USR supporting potential segment. On the other hand, from figure VI.1 we see that a larger $\chi_{in}$ implies a larger average number of e-folds of USR $\langle N \rangle$, from which we conclude that
a longer USR phase generates larger non-Gaussianities. This observation can be of crucial importance for the generation of primordial black holes. These results are in broad agreement with the findings of [23] and it would be interesting to make a more quantitative comparison, recalling that we have fully and consistently included the gravitational effects of the field’s velocity.

VII. CONCLUSION AND DISCUSSION

In this paper we established a consistent formalism for describing the quantum evolution of the large scale curvature perturbation, generated during inflation on very flat portions of the potential $V(\phi)$, fully taking into account the scalar gravitational back-reaction and the finite classical velocity for the field. This was achieved by combining a long wavelength approximation to the Einstein equations with the stochastic picture of inflationary quantum fluctuations. We found that the 0-t Einstein equation, usually neglected in the widely used “separate universe” approach, leads to a single stochastic equation for the scalar field but not its velocity, the latter being fully determined by the former even beyond slow roll through a unique solution to the Hamilton-Jacobi equation (28).

We then focused on a completely level potential $V = V_0$ where inflation occurs in an ultra slow roll (USR) regime. We assumed that $\phi \in [\phi_0, +\infty)$ and that the field is injected with some finite velocity $\Pi_{in}$ at $\phi_{in}$. We showed that on large (super-Hubble) scales USR is a phase space attractor, in the sense that gravitational constraints fully fix the field velocity in terms of the field $\phi$, up to a global constant $\phi_0$ determined by the initial velocity and marking the end point towards which the classical field evolution asymptotes. This is accurate up to small, exponentially decaying gradient corrections, which are highly suppressed, and thus completely irrelevant on very large scales. The value $\phi_e$ demarcated an exit point where inflation either ends or the field enters into another region of the potential, presumably one supporting slow roll. The stochastic number of e-folds required to reach $\phi_e$ directly gives the curvature perturbation.

The inflation dynamics depends crucially on the distance between the entry and exit points $|\phi_{in} - \phi_e|$ and on the initial field velocity $\Pi_{in}$. As we argue in section VI if $|\phi_{in} - \phi_0| > |\phi_{in} - \phi_e|$ the field performs a graceful exit, with $\phi_e$ being eventually reached at all spatial points. If, on the other hand, $|\phi_{in} - \phi_0| < |\phi_{in} - \phi_e|$, the quantum phase space becomes larger than the classical one such that USR proceeds in two distinct phases, discussed in detail in section VII. When the quantum particle reaches the end point $\phi_0$ of the classical trajectory, it will start diffusing along the set of classical trajectories marked by $\langle \Pi \rangle = 0$ and arbitrary $\phi$, implying that the point $\phi_0$ acts as a bifurcation point of the quantum phase space, at which the quantum trajectory splits into two branches, see figure 3.1. Consequently, a conveyor belt picture of the quantum particle phase space emerges and leads to a phase where some random walkers exit but most are trapped in an eternal de Sitter epoch as the field freely diffuses towards $\phi \rightarrow +\infty$. While in the former case a well defined probability distribution for the curvature perturbation emerges, in the latter, although normalizable, the distribution has no finite moments indicating an infinite curvature perturbation. This behavior of course depends on there not being a barrier in reaching $\phi \rightarrow +\infty$, a situation not valid in more complete inflationary models.

This is a preliminary study in many respects. In a realistic inflationary model, the flat potential portion will be finite and even when $|\phi_{in} - \phi_e|$ is large and the conveyor belt is operational, the field’s diffusion towards large values will be halted, although it may still lead to a greatly enhanced curvature perturbation. Furthermore, in passing to equation (67) we linearized in the field perturbation $\phi - \phi_0$, which is equivalent to assuming a small initial field velocity $\Pi_{in}$. This was done for simplicity and to obtain the semi-analytic results presented here but is not necessary. This paper was largely expository of the methods developed and we will return with a more general treatment and more realistic USR models in a forthcoming publication [?]. Finally, we dropped the tensor modes which are non-dynamical classically. This statement will no longer hold when their quantum fluctuations are taken into account. We reserve a more sophisticated non-linear treatment of the IR stochastic tensors for the future.

Acknowledgements:
In this appendix, following [4], we recall that under changes of the time hypersurfaces $t \to T(t,x)$ the long wavelength equations (17), (20), (21) and (22) remain invariant and the long wavelength spatial metric (15) retains its form. These statements are valid up to terms which are second order in spatial gradients and are therefore dropped within the long wavelength approximation.

Starting with coordinates $(t, x^i)$, consider a change in the choice of constant time hyper-surfaces (the spacetime time-slicing) defined by a new time coordinate $T(t,x^i)$. To keep $N^i = 0$ in the new coordinate system, new spatial coordinates $X^i$ must also be chosen which are orthogonal to the $T = \text{const}$ surfaces. We now examine how such the transformation between the old and new coordinates can be obtained.

Given the new time surface $T(t,x^i)$, a set of spatial coordinates $X^i$ is chosen on a $T = T_0$ hypersurface and then orthogonally projected to thread all other $T = \text{const}$ hypersurfaces and labelling spatial coordinates in them too. Along constant $X^i$ curves the old coordinates $x^\mu$ will change as $dx^\mu = T^\mu_\nu ds$ where $s$ is an arbitrary parameter. Along such lines, $T$ will change as

$$dT = T_\mu dx^\mu = T^\mu_\nu T_\nu^\nu ds \quad (100)$$

which implies

$$\left( \frac{\partial x^\mu}{\partial T} \right)_X = \frac{T^\mu_\alpha T^\alpha_\nu}{T_\alpha^\alpha T_\nu^\nu} \quad (101)$$

which defines 4 of the 16 components of the transformation matrix between the old and new coordinates. To determine the 12 remaining components consider the transformation matrix

$$B^\mu_k = \left( \frac{\partial x^\mu}{\partial X^k} \right)_T \quad (102)$$

which should be chosen such that

$$T_{\mu \nu} B^\mu_k = 0 \quad (103)$$

in order to keep $g_{TX^i} = 0$. If condition (103) is satisfied on the $T = T_0$ hypersurface it will always be satisfied. This can be seen by taking the $T$ derivative of $B^\mu_k$ to find

$$\left( \frac{\partial B^\mu_k}{\partial T} \right)_X = \left( \frac{\partial}{\partial X^k} \left( \frac{\partial x^\mu}{\partial T} \right)_X \right)_T = \left( \frac{\partial}{\partial X^k} \left( \frac{T^\mu_\alpha T^\alpha_\nu}{T_\alpha^\alpha T_\nu^\nu} \right) \right)_T B^\mu_k \left( \frac{T^\mu_\alpha T^\alpha_\nu}{T_\alpha^\alpha T_\nu^\nu} \right) \quad (104)$$

In turn, this relation can be used to show that

$$\partial_T (T^\mu_\nu B^\mu_k) = 0 \quad (105)$$

and hence that $N_i$ is kept zero on all $T$ time-slices in the $(T,X^i)$ coordinates.

From (103) we have

$$B^0_ k = - \frac{B^l_ k T^l_0}{T_0} \quad (106)$$

which, when substituted in (104) gives

$$\left( \frac{\partial B^l_k}{\partial T} \right)_X = \left[ \left( \frac{T^l_0}{T_0^\alpha T_0^\alpha} \right)_m - \frac{T^m_0}{T_0} \left( \frac{T^l_0}{T_0^\alpha T_0^\alpha} \right)_0 \right] B^m_k \quad (107)$$

The r.h.s. is second order in spatial gradients and is dropped within our approximation scheme, implying that to this order in the gradient expansion $B^0_k$ is independent of $T$.

$$B^l_k \equiv B^l_k(X). \quad (108)$$

On the other hand, by integrating $x^j$ along a line of constant $X^i$, we obtain

$$x^j = f^j(X) + \int \frac{T^j}{T^0} dT \quad (109)$$

This is consistent with (108); a derivative of the second term w.r.t. $X^j$ involves two spatial gradients $\partial \partial x^j$ as can be seen by using (106). Furthermore, any function evaluated at $x^i$ will read

$$g(x^i) = g(f^j(X)) + \int \frac{T^j}{T^0} dT \quad (110)$$

Hence, within our approximations

$$g(x^i) = g(f^j(X)) = \tilde{g}(X^i) \quad (111)$$

and the 3-metric (15) in the $(T,x)$ coordinates reads

$$\gamma_{\nu \lambda} = e^{2\alpha(t,x)} B^l_k(X) B^k_l(X) h_{00}(x) = e^{2\alpha(t,x)} B^l_k(X) B^k_l(X) h_{00}(x) \quad (112)$$

i.e it again has the form of a locally defined conformal factor times a time independent 3-metric which is only a function of the 3 new spatial coordinates $X^i$. The simplest choice for the new spatial coordinates is of course $f^j(X) = X^j$. 

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Regarding the dynamical equations in the new time $T$, we note that for any time dependent quantity $Q$,\[\left(\frac{\partial Q}{\partial T}\right)_{X_j} = \frac{1}{T_0} \left(\frac{\partial Q}{\partial t}\right)_{x_j} + \frac{T^{-k}}{T_0 T^0} \left(\frac{\partial Q}{\partial x^k}\right)_t . \] (113)

Dropping the second term on the r.h.s. as second order in spatial gradients and noting that the two lapse functions, $N_t$ and $N_T$, associated with the time coordinates $t$ and $T$ respectively are related by $N_t = N_T \partial T / \partial t$, we have
\[\frac{1}{N_T} \left(\frac{\partial Q}{\partial T}\right)_{X_j} = \frac{1}{N_t} \left(\frac{\partial Q}{\partial t}\right)_{x_j} . \] (114)

up to second order in spatial gradients. This can be used to show the invariance of the long wavelength dynamical equations under changes of the time slicing.

**APPENDIX B: STEEPEST DESCENT FOR $P_{V_0}$**

The integral in $[32]$ is dominated by the dependence on $u$ in the exponent, which diverges in both limits of integration, and hence is dominated by some intermediate $u$, at which the function in the exponent minimizes. To study the integral in more detail, we write it in the form
\[P_{V_0} = \frac{3\chi_{in}}{H_0} \left[I(\chi, \alpha)-I(\chi-2\chi_e, \alpha)\right] , \] (115)
\[I(\chi, \alpha) = \int_0^\alpha e^{-S(\chi, u)} du \] (116)

where
\[S(\chi, u) = \frac{\chi^2}{6(\alpha-u)^2} + \frac{1}{2} \ln(\alpha-u) - 6u \]
\[-\frac{3}{2} \ln(2) + n(6u)\chi^2_{in} - \frac{3}{2} \ln[n(6u)], \] (117)

and where $n(x) = 1/(e^x-1)$ is the Bose-Einstein function of its argument. The integral [116] is then performed by expanding $S(\chi, u)$ in [117] around the local minimum $u_0$ (at which $S'_0 \equiv [\partial_u S(\chi, u)]_{u=u_0} = 0$) as,
\[S(u) \approx S(u_0) + \frac{1}{2} S''_0(u-u_0)^2 + O((u-u_0)^3) , \] (118)

where $S''_0 = [\partial^2_u S(\chi, u)]_{u=u_0}$. Upon dropping the higher orders $O((u-u_0)^3)$, the integral [116] becomes simple to evaluate,
\[I(\chi, \alpha) = \sqrt{\frac{\pi}{2S'_0}} \left[ \text{Erf} \left( \frac{\sqrt{S''_0}}{2} (\alpha-u_0) \right) + \text{Erf} \left( \frac{\sqrt{S''_0}}{2} u_0 \right) \right] \] (119)

where the result is meaningful if $S''_0 > 0$. To complete the evaluation, we need $u_0$ and $S''_0$, and hence we need the first and second derivative of [117].

\[S'_0 = \frac{\chi^2}{6(\alpha-u)^2} - \frac{1}{2(\alpha-u)} - 6(n+1)\chi^2_{in} + 3+9n \]
\[= 0 \] (120)
\[S''_0 = \frac{\chi^2}{3(\alpha-u)^3} - \frac{1}{2(\alpha-u)^2} + 36n(n+1)(n+2)\chi^2_{in} - 54n(n+1) , \] (121)

where we made use of $\partial_n n(6u) = -6(n+1), \partial_n^2 n(6u) = 36(n+1)(n+2), u_0$ is found by setting $S'_0 = 0$ in [120]. If $\alpha \gg u$, the problem of solving [120] reduces to finding the positive root of a quadratic equation in $n = n(6u)$, which is easily solved for $n_0 = n(6u_0)$, and hence also for $u_0$. 

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[1] S. W. Hawking, “The Development of Irregularities in a Single Bubble Inflationary Universe,” Phys. Lett. 115B (1982) 295.
[2] A. A. Starobinsky, “Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations,” Phys. Lett. 117B (1982) 175.
[3] A. H. Guth and S. Y. Pi, “The Quantum Mechanics of the Scalar Field in the New Inflationary Universe,” Phys. Rev. D 32 (1985) 1899.
[4] D. S. Salopek and J. R. Bond, “Nonlinear evolution of long wavelength metric fluctuations in inflationary models,” Phys. Rev. D 42 (1990) 3936.
[5] M. Sasaki and E. D. Stewart, “A General analytic formula for the spectral index of the density perturbations produced during inflation,” Prog. Theor. Phys. 95 (1996) 71 [astro-ph/9507001].
[6] M. Sasaki and T. Tanaka, “Superhorizon scale dynamics of multiscalar inflation,” Prog. Theor. Phys. 99 (1998), 763-782 [arXiv:gr-qc/9801017 [gr-qc]].
[7] D. H. Lyth, K. A. Malik and M. Sasaki, “A General proof of the conservation of the curvature perturbation,” JCAP 0505 (2005) 004 [astro-ph/0411220].
[8] D. H. Lyth and Y. Rodriguez, “The Inflationary prediction for primordial non-Gaussianity,” Phys. Rev. Lett. 95 (2005) 121302 [astro-ph/0504045].
[9] N. S. Sugiyama, E. Komatsu and T. Futamase, “$\delta N$ formalism,” Phys. Rev. D 87 (2013) no.2, 023530 [arXiv:1208.1073 [gr-qc]].
[10] J. Garriga, Y. Urakawa and F. Vernizzi, “$\delta N$ formalism from superpotential and holography,” JCAP 02 (2016), 036 [arXiv:1509.07339 [hep-th]].
[11] N. C. Tsamis and R. P. Woodard, Phys. Rev. D 69 (2004) 084005.
[12] W. H. Kinney, “Horizon crossing and inflation with large eta,” Phys. Rev. D 72 (2005) 023515.
[13] M. H. Namjoo, H. Firouzjahi and M. Sasaki, “ Violation of non-Gaussianity consistency relation in a single field inflationary model,” EPL 101 (2013) no.3, 39001.
[50] T. Prokopec, N. C. Tsamis and R. P. Woodard, “Stochastic Inflationary Scalar Electrodynamics,” Annals Phys. 323 (2008) 1324 [arXiv:0707.0847 [gr-qc]].

[51] J. J. Sakurai and J. Napolitano, “Modern quantum Mechanics,” 2nd edition, Cambridge University Press, 2017.

[52] Pauli, W., Wave Mechanics: Volume 5 of Pauli Lectures on Physics (Dover Books on Physics, 2000) ISBN 0486414620.