RATIONAL SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS INVOLVING NORMS

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Abstract. In this note we present some results concerning the unirationality of the algebraic variety \( S_f \) given by the equation
\[
N_{K/k}(X_1 + \alpha X_2 + \alpha^2 X_3) = f(t),
\]
where \( k \) is a number field, \( K = k(\alpha) \), \( \alpha \) is a root of an irreducible polynomial \( h(x) = x^3 + ax + b \in k[x] \) and \( f \in k[t] \). We are mainly interested in the case of pure cubic extensions, i.e. \( a = 0 \) and \( b \in k \setminus k^3 \). We prove that if \( \deg f = 4 \) and the variety \( S_f \) contains a \( k \)-rational point \((x_0, y_0, z_0, t_0)\) with \( f(t_0) \neq 0 \), then \( S_f \) is \( k \)-unirational. A similar result is proved for a broad family of quintic polynomials \( f \) satisfying some mild conditions (for example this family contains all irreducible polynomials). Moreover, the unirationality of \( S_f \) (with non-trivial \( k \)-rational point) is proved for any polynomial \( f \) of degree 6 with \( f \) not equivalent to the polynomial \( h \) satisfying the condition \( h(t) \neq h(\zeta_3 t) \), where \( \zeta_3 \) is the primitive third root of unity. We are able to prove the same result for an extension of degree 3 generated by the root of polynomial \( h(x) = x^3 + ax + b \in k[x] \), provided that \( f(t) = t^6 + a_4 t^4 + a_1 t + a_0 \in k[t] \) with \( a_1 a_4 \neq 0 \).

1. Introduction

Let \( k \) be a number field and \( K/k \) be an algebraic extension of degree \( n \). There is a lot of papers devoted to the study of \( k \)-rational solutions of Diophantine equations of the form
\[
N_{K/k}(X_1 \omega_1 + \ldots + X_n \omega_n) = f(t),
\]
where \( N_{K/k} \) is a full norm form for the extension \( K/k \), \( \{\omega_1, \ldots, \omega_n\} \) is a fixed basis of the extension and \( f \) is a polynomial over \( k \). The main problem here is the question whether the Hasse principle, or in other words local to global principle, holds for the smooth proper model of a hypersurface given by the equation \( 1 \). For example, if \( f(t) \) is constant then the local to global principle holds for \( 1 \) (Hasse). If \( n = 2 \) and \( \deg f = 3 \) or 4 then the variety defined by \( 1 \) is called a Châtelet surface. The arithmetic of these surfaces is well understood. In particular, in \( 2, 3 \) it is proved that the Brauer-Manin obstruction for the Hasse principle and weak approximation is the only one. Moreover, the existence of a \( k \)-rational solution implies \( k \)-unirationality. These results are unconditional. However, the most general result in this area is obtained under Schinzel’s hypothesis (H) and says that if \( K \) is a cyclic extension of a number field \( k \), and \( f(t) \) is a separable polynomial of arbitrary degree, then the Brauer-Manin obstruction to the Hasse

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principle and weak approximation is the only one for the smooth and projective model $X$ of the variety given by the equation (1). Moreover, if there is no Brauer-Manin obstruction to the Hasse principle then the $k$-rational points are Zariski dense in $X$.

Most of the results in this area were proved using algebraic considerations (via the computation of the Brauer-Manin obstructions) or a combination of algebraic methods together with analytic techniques (see for example [5]). However, only few papers present constructions which allow to produce new solutions from a given $k$-rational solution of (1). As was mentioned in [5, p.162], usually this is a rather difficult problem.

We are working with a field $k$ of characteristic 0 and an algebraic extension $K/k$ of degree $n$. We take $\omega_i = \alpha^i$ for $i = 1, \ldots, n$, where $\alpha \in K$ is chosen in such a way that $K = k(\alpha)$. We thus are interested in the equation

$$N_{K/k}(X_1, \ldots, X_n) = f(t),$$

where in order to shorten the notation we put

$$N_{K/k}(X_1, \ldots, X_n) := N_{K/k}(X_1 + \alpha X_2 \ldots + \alpha^{n-1} X_n),$$

i.e. $N_{K/k}$ will denote a norm form, and $N_{K/k}$ denotes the corresponding field norm. In the sequel by a non-trivial solution of (2) we mean a solution $(X_1, \ldots, X_n, t)$ which satisfies $f(t) \neq 0$. The present paper is a contribution to the subject in that we show that in some cases the existence of one $k$-rational solution of (2) implies the existence of infinitely many $k$-rational solutions. This is obtained mainly by constructing parametric solution of the corresponding equation, or, in a more geometric language, by constructing of a $k$-rational curve lying on the corresponding algebraic variety. Of course, we are interested only in the existence of $k$-rational curves which are not contained in the fiber of the map $\Phi : S_f \ni (X_1, \ldots, X_n, t) \mapsto t \in \mathbb{P}^1(k)$. Our argument is based on a similar approach proposed by Mestre in a series of papers [6, 7, 8] devoted to the study of the existence of rational points on (generalized) Châtelet surfaces, i.e. surfaces defined by (2) with $n = 2$ and $\deg f \geq 5$.

Let us describe the content of the paper in some details. In Section 2, we prove that if $K/k$ is a pure cubic extension generated by the root of the polynomial $h(x) = x^3 + b \in k[x]$, $f \in k[t]$ is of degree 4, and the variety $S_f$ defined by the equation (2) contains a non-trivial $k$-rational point, then $S_f$ is unirational over $k$. In particular, in this case the set of $k$-rational points on $S_f$ is Zariski dense. We prove a similar result for $f \in k[t]$ of degree 5, provided that $f$ satisfies some mild conditions. In particular, if $f$ is an irreducible polynomial, then $S_f$ is $k$-unirational. We also prove that if $f \in k[t]$ is a monic polynomial of degree 6, $S_f$ contains non-trivial $k$ rational point and the polynomial $f$ is not equivalent to a polynomial $h \in k[t]$ satisfying the condition $h(t) \neq h(\zeta_3 t)$, then the variety $S_f$ given by equation (2) is $k$-unirational. This result is particularly interesting in the light of a recent work of Várilly-Alvarado and Viray [9]. Indeed, in the case under consideration the variety $S_f$ is a so called Châtelet threefold (in the terminology of [9]). The authors of the cited paper asked whether the existence of a $k$-rational point on $S_f$ implies $k$-unirationality [9, Problem 6.2]. Our result shows that $S_f$ is $k$-unirational for a broad class of polynomials. Moreover, if $k$ is a number field with a real embedding, we prove that for each polynomial $f(t) = a_0 t^6 + \sum_{i=0}^{4} a_{6-i} t^i \in k[t]$ and any given $\epsilon > 0$ there exists a polynomial $g(t) = c_0 t^6 + \sum_{i=0}^{4} c_{6-i} t^i \in k[t]$ which is close to $f$, 

$$g(t) = f(t) - \epsilon,$$
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i.e. \( |a_i - c_i| < \epsilon \) for \( i = 0, 2, \ldots , 6 \), and such that for any \( b \in k \setminus k^3 \) and a pure cubic extension \( K/k \) generated by the root of the polynomial \( h(x) = x^3 + b \), the variety \( \mathcal{S}_g \) is unirational over \( k \).

In Section 3 we consider the variety \( \mathcal{S}_f \) defined by the equation (\ref{equation2}) involving a norm form of an extension \( K/k \) generated by the root of an irreducible polynomial \( h(x) = x^3 + ax + b \in k[x] \). We prove that if \( f(t) = t^6 + a_4t^4 + a_1t + a_0 \in k[t] \), \( a_1a_4 \neq 0 \) then the variety \( \mathcal{S}_f \) is unirational over \( k \). Moreover, we give a remark concerning the unirationality of slightly more general varieties defined by equations of the form \( F(x, y, z) = f(t) \), where \( F \) is a homogenous form of degree 3 and \( f \) is a polynomial.

2. SOLUTIONS OF THE EQUATION \( N_{K/k}(X_1, X_2, X_3) = f(t) \) WITH PURE CUBIC EXTENSION \( K/k \) AND \( f \) OF DEGREE \( \leq 6 \)

Let \( k \) be a field of characteristic 0 and \( K/k \) be an extension of degree 3 generated by the root, say \( \alpha \), of the irreducible polynomial \( h(x) = x^3 + ax + b \) defined over \( k \). We are interested in the rational points lying on the variety defined by the equation

\[
\mathcal{S}_f : N_{K/k}(X_1, X_2, X_3) = f(t),
\]

where \( f \in k[t] \). In this section we consider the case of \( f \) of degree \( \leq 6 \). Since we are interested in \( k \)-unirationality of \( \mathcal{S}_f \), we make the assumption that the set of \( k \)-rational points on \( \mathcal{S}_f \) is non empty. To be more precise, we assume that there is a nontrivial \( k \)-rational point lying on \( \mathcal{S}_f \), i.e. there is a \( P = (x_0, y_0, z_0, t_0) \in \mathcal{S}_f(k) \) such that \( f(t_0) \neq 0 \). In particular the point \( P \) is a smooth point on \( \mathcal{S}_f \). In this section we consider the case of a pure cubic extension \( K/k \), i.e. \( K \) is generated by the root of a polynomial \( h \) with \( a = 0 \). Let us recall that in this case

\[
N_{K/k}(X_1, X_2, X_3) = X_1^3 + bX_2^3 + b^2X_3^3 + 3bX_1X_2X_3.
\]

Before we state our results let us note that \( \mathcal{S}_f \) is isomorphic with \( \mathcal{S}_g \), where \( g(t) = \sum_{i=1}^6 c_i t^i + 1 \). Indeed, making a change of variables \( t \mapsto t + t_0 \) we can assume that \( f(0) = c_0 = N_{K/k}(u, v, w) \neq 0 \) for some \( u, v, w \in k \). Multiplying this equation by \( c_0^{-1} = N_{K/k}(u', v', w') \), with \( u', v', w' \) chosen in such a way that \( N_{K/k}(u, v, w)N_{K/k}(u', v', w') = 1 \), and using the multiplicative property of a norm form, we get the desired form of our equation. It is clear that \( \mathcal{S}_f \) is \( k \)-unirational if and only if \( \mathcal{S}_g \) is \( k \)-unirational.

We are ready to prove the following result.

\textbf{Theorem 2.1.} Let \( k \) be a field of characteristic 0 and let \( K = k(\alpha) \), where \( \alpha^3 + b = 0 \) with \( b \in k \setminus k^3 \). Put \( g(t) = 1 + \sum_{i=1}^6 c_i t^i \in k[t] \) and let us suppose that

\[
(c_2, c_4, c_6) \neq \left( \frac{5c_1^2}{12}, -\frac{1}{144}c_1(5c_1^3 - 72c_3), -\frac{1}{144}c_1^2(c_1^3 - 12c_3) \right).
\]

Then the variety \( \mathcal{S}_g \) is \( k \)-unirational.

\textbf{Proof.} Let \( G = G(X_1, X_2, X_3, t) \) be a polynomial defining the variety \( \mathcal{S}_g \). We note that \( \mathcal{S}_g \) contains the \( k \)-rational point \((1, 0, 0, 0) \). We use it in order to construct a \( k \)-rational curve lying on \( \mathcal{S}_g \). More precisely, we are looking for a rational curve, say \( \mathcal{L} \), lying on \( \mathcal{S}_g \). We assume that \( \mathcal{L} \) can be parameterized by rational functions with parameter \( u \) in the following way:

\[
\mathcal{L} : X_1 = pt^2 + qt + 1, \quad X_2 = rt^2, \quad X_3 = st^2 + ut, \quad t = T,
\]

where \( p, q, r, s, t \) are polynomials in \( u \). In this way we can construct a \( k \)-rational curve on \( \mathcal{S}_g \).
where \( p, q, r, s, T \) need to be determined. With \( X_i \) and \( t \) defined above, we get 
\[
G(X_1, X_2, X_3, t) = \sum_{i=1}^{6} C_i T^i,
\]
where 
\[
C_1 = 3q - c_1, \quad C_5 = b^2 u^3 + 3bru + 6pq + q^3 - c_3, \\
C_2 = 3p + 3q^2 - c_2, \quad C_4 = 3(b^2 u^2 + bqr u + brs + p^2 + pq^2) - c_4
\]
and \( C_5, C_6 \in k[p, q, r, s, u] \) depend on \( c_i \) for \( i = 1, \ldots, 5 \). The system \( C_1 = C_2 = C_3 = C_4 = 0 \) has exactly one solution with respect to \( p, q, r, s \) and it is given by:
\[
\begin{align*}
& p = \frac{1}{5} (3c_2 - c_1^2), \\
& q = \frac{1}{5} c_1, \\
& r = \frac{-27b^2 u^3 + 5c_1^3 - 18c_1 c_2 + 27c_3}{81bu}, \\
& s = \frac{u(27b^2 c_1 u^3 + 5c_1^3 + 27c_2 c_1^2 - 27c_3 c_2 - 27c_2^2 + 81c_4)}{3(54b^2 u^2 + 5c_1^2 - 18c_1 c_2 + 27c_3)}.
\end{align*}
\]
For \( p, q, r, s \) defined in this way we get \( C_i = \tilde{A}_i / D, i = 5, 6, \) and \( DG(X_1, X_2, X_3, T) = \tilde{A}_5 T^5 + \tilde{A}_6 T^6 \) for \( \tilde{A}_5, \tilde{A}_6 \in k[u] \) and \( D = 3^{12} b^2 u^3 (54b^2 u^3 + 5c_1^3 - 18c_1 c_2 + 27c_3)^3 \). We note that \( \deg_\tilde{u} \tilde{A}_6 = 18 \) and the leading coefficient of \( \tilde{A}_6 \) is \( 2^3 3^1 18b^{12} \). In particular \( A_6 \neq 0 \) as an element of \( k[u] \). We also have \( \deg_\tilde{u} \tilde{A}_5 = 15 \) and \( \tilde{A}_5 \neq 0 \) as an element of \( k[u] \) if and only if the condition \([1]\) is satisfied. In this case, we get a single non-zero solution of the equation \( T^5 (\tilde{A}_5 + \tilde{A}_6 T) = 0 \) with respect to \( T \). Indeed, we have
\[
T = \frac{\tilde{A}_5}{\tilde{A}_6} = \varphi(u) = \frac{2 \cdot 3^{19} b^{10} (5c_1^2 - 12c_2) u^{15} + \text{lower order terms in } \tilde{u}}{2^{3} 3^{18} b^{12} u^{18} + \text{lower order terms in } \tilde{u}}.
\]
Summing up, we see that the existence of a \( k \)-rational point \( P \) with \( f(t_0) \neq 0 \) implies that \( S_g \) contains a \( k \)-rational curve \( \mathcal{L} \), which is not contained in any hyperplane defined by the equation \( t = t_0 \) with \( t_0 \in k \). This allows us to define the base change \( t = \varphi(u) \) which gives the cubic surface \( S_{g_{\varphi \varphi}} \) defined over the field \( k(u) \) with a smooth \( k(u) \)-rational point. This immediately implies \( k(u) \)-unirationality of \( S_{g_{\varphi \varphi}} \) by \([1]\) Proposition 1.3 and thus \( k \)-unirationality of \( S_g \). Indeed, the map \( \Psi \) which guarantees unirationality of \( S_{g_{\varphi \varphi}} \) extends to a dominant rational map \( (\Psi, \varphi) \) which gives unirationality of \( S_g \) and thus \( S_f \). \( \square \)

**Corollary 2.2.** Let \( k \) be a field of characteristic zero and let \( K/k \) be a pure cubic extension. Consider the variety \( S_f \) with \( f \in k[t] \) of degree 4 and suppose that \( S_f \) contains a nontrivial \( k \)-rational point. Then \( S_f \) is \( k \)-unirational.

**Proof.** We are working with \( S_g \) where \( g(t) = 1 + \sum_{i=1}^{4} c_i t^i \) with \( c_4 \neq 0 \). We have \( S_g \simeq S_f \). In order to get the result we need to check whether the condition \([1]\) is satisfied for all \( c_i \in k \) for \( i = 1, 2, 3, 4 \). We see that \([1]\) is not satisfied if and only if \((c_2, c_4, c_5) = (5c_1^2 / 12, c_1^3 / 144, 0)\). In particular \( c_1 \neq 0 \). Making the (invertible) substitution \( t \mapsto 6t / c_1 \) we are left with the problem of proving unirationality of \( S_h \) with \( h(t) = (3t^2 + 2t + 1)^2 \). We assume that \( \mathcal{L} \) can be parametrized by rational functions with parameter \( u \) in the following way
\[
\mathcal{L} : X_1 = T + 1, \quad X_2 = u T, \quad X_3 = p T, \quad t = q T,
\]
where parameters \( p, q, T \) still need to be determined. For \( X_1, X_2, X_3, t \) defined in this way we get \( F = \sum_{i=1}^{4} C_i T^i \), where
\[
C_1 = 3 - 6q, \quad C_2 = 3 + 3bpu - 15q^2, \quad C_3 = 1 + b^2 p^3 + 3br - bu^3 - 18q^3, \quad C_4 = -9q^4.
\]
We solve the system $C_1 = C_2 = 0$ with respect to $p, q$ and get $p = 1/4bu, q = 1/2$. This substitution allows us to find the expression for $T$ in the form

$$T = \frac{-64b^2a^6 - 32bu^3 + 1}{36bu^3}.$$ 

The expression for $T$ together with the expressions for $p, q$ give equations (6) defining the rational parametric curve $L$ lying on $S_h$. Using now the same reasoning as at the end of the proof of Theorem 2.1, we get the result. \hfill \square

**Remark 2.3.** We were trying to prove $k$-unirationality of $S_g$ in the case when the polynomial $g \in k[t]$ is of degree 5 and does not satisfy the condition (4). Among other things we were trying to replace the polynomial $g(t)$ by the polynomial $h(Y) = (1 + vY)^6g(Y/(1 + vY))$. In this way we got the variety $S_h$ via the substitution

$$X_i = Y_i/(1 + vY)^2 \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad t = Y/(1 + vY).$$

Unfortunately, one can check that if $g$ does not satisfy the condition (4), then $h(T)$ does not satisfy the condition (4), too. Because all our efforts failed, we decided to state the following:

**Question 2.4.** Let $k$ be a field of characteristic 0 and let $K = k(\alpha)$, where $\alpha^3 + b = 0$ with $b \in k \setminus k^3$. Put $g(t) = 1 + \sum_{i=1}^{5} c_it^i \in k[t]$ with $c_5 \neq 0$ and let us suppose that the condition (4) is not satisfied. Is the variety $S_g$ unirational over $k$?

Let us note that if the polynomial $g$ does not satisfy the condition (4), then $g$ is reducible, namely

$$g(t) = -\frac{1}{144}(c_1^2t^2 + 6c_1t + 12)(c_1^3 + 12c_3)t^3 - c_1^2t^2 - 6c_1t - 12).$$

In particular, Theorem 2.1 implies that if $g$ is irreducible of degree 5 then $S_g$ is $k$-unirational and thus the set of $k$-rational points on $S_g$ is Zariski dense. It is clear that the same is true for a polynomial $f$ corresponding to $g$.

In a recent paper Várilly-Alvarado and Viray [9] introduced the notion of a Châtelet threefold, which is a variety defined by the equation (2) with $n = 3$ and $f \in k[t]$ of degree 6. The authors of this paper asked, whether the existence of a $k$-rational point on $S_f$ implies $k$-unirationality of $S_f$ [9, Problem 6.2]. The statement of the Theorem 2.1 gives us a broad family of polynomials $f$ such the variety $S_f$ is $k$-unirational. In the next corollary we make this result more explicit.

Before we state our result, let us recall that two polynomials $f_1, f_2 \in k[t]$ are equivalent if $\deg f_1 = \deg f_2$ and there exist $\alpha, \beta \in k$ such that $f_2(t) = f_1(\alpha t + \beta)$.

**Corollary 2.5.** Let $k$ be a field of characteristic 0 and let $K = k(\alpha)$, where $\alpha^3 + b = 0$ with $b \in k \setminus k^3$. Let $f \in k[t]$ be of degree 6 and suppose that $f$ is not equivalent to the polynomial $h \in k[t]$ satisfying the condition $h(t) = h(\zeta_3t)$, where $\zeta_3$ is the primitive third root of unity. Let us also suppose that $S_f$ contains a nontrivial $k$-rational point. Then the variety $S_f$ given by (3) is $k$-unirational.

**Proof.** First of all let us note that the existence of a non-trivial $k$-rational point on $S_f$ with $f$ of degree 6 and the fact that the norm form is multiplicative, implies that $S_f \simeq S_h$, where $h(t) = t^6 + \sum_{i=1}^{6} c_it^i$ for some $c_j \in k, j = 2, 3, \ldots, 6$. From our assumption on $f$ we know that at least one among the elements $c_2, c_4, c_5$ is non-zero. Making the change of variables $X_i = Y_i/T$ for $i = 1, 2, 3$ and $t = 1/T$ we get that $S_f \simeq S_g$ with $g(T) = 1 + \sum_{i=2}^{6} c_iT^i$. We can apply now Theorem 2.1 to the variety $S_g$. It is $k$-unirational provided that the condition (4) is satisfied. In
our case we have $c_1 = 0$ and thus (3) is not satisfied if and only if $c_2 = c_4 = c_5 = 0$ which is not the case.

Using the corollary above in the case of a number field $k$ with real embedding in $\mathbb{R}$, we deduce the following interesting result.

**Theorem 2.6.** Let $K/k$ be a number field with $k \subset \mathbb{R}$ and put $f(t) = a_6 t^6 + \sum_{i=0}^4 a_{6-i} t^i \in k[t]$ with $a_0 \neq 0$. Then, for each $\epsilon > 0$ there exists a polynomial $g(t) = c_0 t^6 + \sum_{i=0}^4 c_{6-i} t^i \in k[x]$ such that $|a_i - c_i| < \epsilon$ for $i = 0, 2, \ldots, 6$ and for each pure cubic extension $K/k$ of degree 3, the variety $S_g$ given by the equation $N_{K/k}(X_1, X_2, X_3) = g(t)$ is $k$-unirational.

**Proof.** We are working with $S_h \simeq S_f$, where $h(t) = t^6 f(1/t)$. We note that for any given $a_0 \in k^*$ we can find a triple $u, v, w \in k$ such that $|N_{K/k}(u, v, w) - a_0| < \epsilon$ and $N_{K/k}(u, v, w) \neq 0$, which is a consequence of the density of the image of the norm map $N_{K/k} : k^3 \to k$. Then we take $c_0 = N_{K/k}(u, v, w)$. If $h(t) \neq h(\zeta_3 t)$ we take $c_i = a_i$ for $i = 2, \ldots, 6$. If $h(t) = h(\zeta_3 t)$ then we take $c_i = a_i$ for $i = 3, 6$ and we take $c_2 = c_4 = c$ for any $c \in k$ which satisfies $|c| < \epsilon$. Then we put $g(t) = c_0 t^6 + \sum_{i=0}^4 c_{6-i} t^i$ and note that $S_g$ contains a $k$-rational point at infinity. Moreover, $S_g \simeq S_{h'}$, where $h'(t) = t^6 g(1/t)$. From the Corollary (2.5) we get the result.

The presented results motivate us to state the following:

**Conjecture 2.7.** Let $k$ be a number field and $K/k$ be a cyclic extension of degree 3. Let $f \in k[t]$ be a polynomial of degree 6 and let us suppose that there exists a non-trivial $k$-rational point on $S_f$. Then $S_f$ is $k$-unirational.

We finish this section with the following simple result.

**Theorem 2.8.** Let $K$ be a field of characteristic 0 and let $K = k(\alpha)$, where $\alpha^3 + b = 0$ with $b \in k \setminus k^3$. Put $f(t) = t^3m + a_2 t^m + a_1 t + a_0 \in k[t]$ with $a_1 \neq 0$. Then the variety $S_f$ is $k$-unirational.

**Proof.** Let $F = F(X_1, X_2, X_3, t)$ be a polynomial defining the variety $S_f$. We put

$$X_1 = t^m, \quad X_2 = u, \quad X_3 = \frac{a_2}{3bu}.$$

For $X_i$ defined in this way the polynomial $F$ (in the variable $t$) is of degree 1 with the root

$$t = \varphi(u) = -\frac{27b^2 u^6 + 27ba_0 u^3 - a_0^3}{27ba_1 u^3},$$

which under the assumption $a_1 \neq 0$ is a non-constant element of $k(u)$. We thus see that the cubic surface $S_{f_\varphi}$ is $k(u)$-unirational and this implies the $k$-unirationality of $S_f$.

3. **Solutions of the equation $N_{K/k}(X_1, X_2, X_3) = f(t)$ for general cubic extension and $f$ of degree 6**

We consider now the variety $S_f$ given by the equation (3) for a general extension $K/k$ of degree 3 and a monic polynomial $f \in k[t]$ of degree 6. We thus assume that $K = k(\alpha)$, where $\alpha$ is a root of an irreducible polynomial $h(x) = x^3 + ax + b \in k[x]$ with $a \neq 0$. Unfortunately, in this case we were unable to prove the $k$-unirationality of $S_f$ for all polynomials $f$ which satisfy $f(t) \neq f(\zeta_3 t)$. However, we prove the following result.
Theorem 3.1. Let \( k \) be a field of characteristic 0 and put \( K = k(\alpha) \), where \( \alpha^3 + a\alpha + b = 0 \) and \( f(t) = t^6 + a_4 t^4 + a_1 t + a_0 \in k[t] \) with \( a_1 a_4 \neq 0 \). Then the variety \( S_f \) given by \( \mathbf{3} \) is unirational over \( k \).

Proof. In this case the norm form takes the form \( N_{K/k} = N_{K/k}(X_1, X_2, X_3) \), where
\[
N_{K/k} = X_1^3 - bX_2^3 + a^2 X_3^3 + (aX_2 + 3bX_3)X_1X_2 - (2aX_1^2 - a^2 X_1 X_3 - abX_2 X_3)X_3.
\]
Let \( G = G(X_1, X_2, X_3, t) \) be the polynomial defining the variety \( S_f \). We use exactly the same approach as in the proof of Theorem 2.1. This time we just take \( X_1 = t^2 + p \), where \( p \) needs to be determined. We thus get
\[
G(X_1, X_2, X_3, t) = \sum_{i=0}^{6} C_i t^i,
\]
where
\[
C_2 = a^2 X_3^2 - 4apX_3 + aX_2^2 + 3bX_2X_3 + 3p^2, \quad C_3 = 0, \quad C_4 = 3p - a_4 - 2aX_3.
\]
Eliminating \( p \) from the equation \( C_4 = 0 \) we are left with the equation \( C_2 = 0 \) defining a curve, say \( C \), in the plane \( (X_2, X_3) \). The equation for \( C \) can be rewritten in the form
\[
C : (2a^2 X_3 - 9bX_2)^2 = 4a^2 a_4^2 + 3(4a^3 + 27b^2)X_2^2.
\]
The curve \( C \) is of genus 0 and has a rational point \( (X_2, X_3) = (0, a_4/a) \) and thus can be parameterized by rational functions. A parametrization of \( C \) together with the expression for \( p \) is given by
\[
X_2 = \frac{4a_4 u}{3(4a^3 + 27b^2) - u^2}, \quad X_3 = \frac{a_4 (12a^3 + 81b^2 + 18bu + u^2)}{a(3(4a^3 + 27b^2) - u^2)}; \quad p = \frac{a_4 + 2aX_3}{3}.
\]
For \( X_2, X_3 \) and \( p \) chosen in this way we have the equality \( DG(X_1, X_2, X_3, t) = A_0 + A_1 t \), where \( \deg A_0 = 6 \) and \( D = A_1 = -27a^3 a_4 (12a^3 + 81b^2 - u^2)^3 \). From the assumption on \( a_1 \) we know that \( DA_1 \neq 0 \). Careful analysis of the coefficients of the polynomial \( A_0 \) shows that if the coefficients of \( f \) satisfy \( a_1 a_4 \neq 0 \) then the function \( t = \varphi(u) = -A_0/A_1 \) satisfies \( \varphi \in k(u) \setminus k \). Thus, we have found a rational curve lying on \( S_f \). Finally, the same argument as at the end of the proof of Theorem 2.1 gives \( k \)-unirationality of the variety \( S_f \). \( \square \)

Remark 3.2. It is natural to ask whether the method employed in order to get the \( k \)-unirationality of the varieties considered in this paper can be used in other situations. More precisely, one can ask the following.

Question 3.3. Let \( f \in k[t] \). How general an indecomposable form \( F \in k[X_1, X_2, X_3] \) of degree 3 can be such that a variety defined by the equation \( F(X_1, X_2, X_3) = f(t) \) is unirational over \( k \) for most choices of \( f \) of fixed degree?

For example, let us consider the case of a monic polynomial \( f \in k[t] \) of degree 6. It would be rather unexpected if taking the form
\[
F(X_1, X_2, X_3) = X_1^3 + aX_2^3 + bX_3^3 + (cX_1 + dX_2 + eX_3)X_2X_3,
\]
we could prove the \( k \)-unirationality of the hypersurface defined by the equation
\[
S : F(X_1, X_2, X_3) = f(t);
\]
where \( f(t) = t^6 + \sum_{i=0}^{6} a_i t^i \in k[t] \) and \( a, b, c, d, e \in k \) satisfy certain conditions. We note that for a generic choice of \( a, b, c, d, e \in k \) the form \( F \) is absolutely irreducible, i.e., is irreducible as a polynomial in \( \bar{k}[X_1, X_2, X_3] \). Let
$G(X_1, X_2, X_3, t) = F(X_1, X_2, X_3) - f(t)$ be the polynomial defining the hypersurface $S$. In order to verify the $k$-unirationality of $S$, it is enough to take

$$X_1 = t^2 + \frac{a_4}{3}, \quad X_2 = \frac{b_3 - bu^3}{cu}, \quad X_3 = ut + \frac{u(3beu^4 - 3a_3cu - a_1^2c + 3a_2c)}{3c(2bu^3 + a_3)}.$$

Indeed, for $X_1, X_2, X_3$ chosen in this way we note an equality $DG(X_1, X_2, X_3, t) = C_1t + C_0$, where $C_0, C_1 \in k[u]$ depend on the coefficients $a, b, c, d, e$ and $a_i$ for $i = 0, \ldots, 4$. Moreover, we have $D = 27c^3u^3(2bu^3 + a_3)^3$. If $C_0C_1 \neq 0$ as a polynomial in $k[u]$, we get a solution $t = \varphi(u) = -C_0/C_1$. We have $\deg C_1 = 17$ and $\deg C_0 = 18$. The expression for $t$ together with the expressions for $X_1, X_2, X_3$ given by $7$ yield a parametrization (in the parameter $u$) of a rational curve on $S$ with $f(\varphi(u)) \neq 0$. The existence of a rational curve lying on $S$ allows us to define a rational base change $t = \varphi(u)$. Then the (cubic) surface $S_u : F(X_1, X_2, X_3) = f(\varphi(u))$ (treated as a surface over the field $k(u)$) contains a smooth $k(u)$-rational point $P$ with coordinates given by $7$ and thus $S_u$ is $k$-unirational over $k(u)$. As an immediate consequence we get the $k$-unirationality of $S$ over $k$.

It is possible to write explicit conditions on the coefficients of the polynomial $f$ and the form $F$ which will guarantee that $\varphi \in k(u) \setminus k$. For example, if $abceac_3 \neq 0$ then $\varphi \in k(u) \setminus k$.

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