Trunk of Satellite and Companion Knots

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Abstract

We study the knot invariant called trunk, as defined by Ozawa, and the relation of the trunk of a satellite knot with the trunk of its companion knot. Our first result is $\text{trunk}(K) \geq n \cdot \text{trunk}(J)$ where trunk$(\cdot)$ denotes the trunk of a knot, $K$ is a satellite knot with companion $J$, and $n$ is the winding number of $K$. To upgrade winding number to wrapping number, which we denote by $m$, we must include an extra factor of $\frac{1}{2}$ in our second result $\text{trunk}(K) > \frac{1}{2} m \cdot \text{trunk}(J)$ since $m \geq n$. We also discuss generalizations of the second result.

1 Introduction

Knots and links are core objects in the study of three manifolds. The most important tools to study them are their numerical and homological invariants. There is an important family of invariants for knots (and links) which come from Morse theory. Among them are bridge number, width and trunk. For these Morse-type invariants, an important question is how they behave under the operations of connected sum and taking satellites. Those operations are interesting because they are the most important ways to construct more complicated knots out of the simple ones, allowing us to understand more complex knots better. Understanding the behaviors of the invariants under those operations would then contribute to the study of the properties of knots and links.

Bridge number was first introduced by Schubert [12] in the 1950s, and it has broad connections and applications in many aspects of knot theory. Its behavior
has been understood completely by the work of Schubert [12] and Schultens [13]:

\[ b(K_1 \sharp K_2) = b(K_1) + b(K_2) - 1, \]

\[ b(K) \geq m \cdot b(J). \]

Here \( b(\cdot) \) is the bridge number of a knot and \( \sharp \) means the connected sum, defined in [10]. \( K_1 \) and \( K_2 \) are knot classes. In the second inequality, \( K \) is a satellite knot with companion \( J \) and wrapping number \( m \).

Width was first defined by Gabai [3] in his proof of the Property R conjecture. It is also closely related to the study of meridional surfaces in the knot complements and was an essential part of the proof of the knot complement conjecture by Gordon and Luecke [4]. Its behavior under the connected sum was understood by Blair and Tomova [1], Rieck and Sedgwick [9] and Scharlemann and Schultens [11]:

\[ \max\{\omega(K_1), \omega(K_2)\} \leq \omega(K_1 \sharp K_2) \leq \omega(K_1) + \omega(K_2) - 2. \]

However, the behavior of width under taking satellites still remains a mystery. A partial result was proved by Guo and Li in [5]:

\[ \omega(K) \geq n^2 \cdot \omega(J), \]

where \( K \) is a satellite knot with companion \( J \) and \( n \) is the winding number of \( K \). This is not fully satisfactory as there are many important examples including Whitehead doubles which all have winding number zero, so inequality [11] will not yield anything nontrivial. On the other hand, the wrapping number is always non-zero so we expect to replace the winding number \( n \) by wrapping number \( m \) in the inequality [11] and this leads to the following conjecture:

**Conjecture 1.** Suppose \( K \) is a satellite knot with companion \( J \) and wrapping number \( m \), then we have

\[ \omega(K) \geq m^2 \cdot \omega(J). \]

The special case where \( K \) is the Whitehead double was proved by Guo and Li [7] but the general case is still open.

In this paper we present our results on trunk, which can be regarded as a
simplified version of width. The study of trunk would possibly shed some light on
the width case. The first thing we do is adapt the main result in \cite{5} to knot trunk
and prove the following theorem:

**Theorem 1.** Suppose $K$ is a satellite knot with companion $J$ and winding
number $n$, then we have

$$\text{trunk}(K) \geq n \cdot \text{trunk}(J).$$

(2)

We also study the case for wrapping number and obtain a lower bound of
trunk($K$) in terms of trunk($J$) and the wrapping number $m$. By definition, the
wrapping number is the least geometric intersection number of $K$ with any merid-
ian disk of the tubular neighborhood of $J$ and is always non-zero (by the definition
of taking satellites). However, we cannot get a result as strong as inequality (2)
when we use the wrapping number as we have a factor of a half in our bound:

**Theorem 2.** Suppose $K$ is a satellite knot with companion $J$ and wrapping
number $m$, then we have

$$\text{trunk}(K) > \frac{1}{2} \cdot m \cdot \text{trunk}(J).$$

We still make the following conjecture:

**Conjecture 2.** Suppose $K$ is a satellite knot with companion $J$ and wrapping
number $m$, then we have

$$\text{trunk}(K) \geq m \cdot \text{trunk}(J).$$

To bound the trunk of $K$, we need to study the intersection of a particular
knot $k$ with the regular level $h^{-1}(r)$ of the standard Morse function $h$ on $S^3$
and a regular value $r \in \mathbb{R}$. Since our knot $k$ is contained in a tubular neighborhood $V$
of the companion knot, we can first study the intersection $V \cap h^{-1}(r)$. By definition,
if a connected component $P \subset V \cap h^{-1}(r)$ is a meridian disk, then it intersects $K$
at least $m$ times. Hence the proof of Theorem 2 relies on the following key lemma:

**Key lemma.** Among all the relevant components (defined more precisely in
Section 4) of $V \cap h^{-1}(r)$, at least half of them are meridian disks.
There are some topological requirements for the intersection $V \cap h^{-1}(r)$. These requirements tell us how the components of $V \cap h^{-1}(r)$ are arranged on the regular level $h^{-1}(r)$ which is a 2-sphere. Then we translate this problem into a purely combinatorial one about arranging pieces on a 2-sphere and prove the key lemma in that setting.

The paper is organized as follows: In Section 2 we introduce some basic definitions about knot invariants and satellite knots. In Section 3 we summarize the result in [5] and prove $\text{trunk}(K) \geq n \cdot \text{trunk}(J)$. In Section 4 we explain how to translate the problem into combinatorics and prove the Key lemma. In Section 5 we discuss the wrapping number further and make some slight generalizations of Theorem 2.

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2 Preliminaries

We will start with some necessary definitions.

Definition 2.1. A knot is a smooth embedding

$$k : S^1 \hookrightarrow S^3$$

where $S^1$ is the unit circle in $\mathbb{R}^2$ and $S^3$ is the unit sphere in $\mathbb{R}^4$.

Definition 2.2. A knot class is a set of knots that are all isotopic to each other. See [10] for the definition of an isotopy.

We shall fix a Morse function throughout the paper. We consider $h : S^3 \to \mathbb{R}$ to be the standard height function $h(x, y, z, w) = w$ restricted to the unit sphere $S^3 \subset \mathbb{R}^4$. The pre-images of $\pm 1$ are denoted by $\pm \infty$. 
Definition 2.3. With the above notation, a knot $k$ is called Morse if the composition

$$h \circ k : S^1 \to \mathbb{R}$$

is a Morse function (see [8] for the definition of a Morse function). A critical point of $k$ means a point $p \in S^1$ so that

$$\nabla (h \circ k)(p) = 0.$$

The value $h \circ k(p) \in \mathbb{R}$ for a critical point $p \in S^1$ is called a critical value. Non-critical values are called regular values. For a number $a \in (-1, 1)$, the pre-image $h^{-1}(a) \subset \mathbb{R}^3$ is called a level. It is either critical or regular depending on $a$.

Convention 2.4. The following conventions will be used throughout the paper:

1. We will only consider knots that are Morse and whose critical points are all at different levels.
2. Knots are denoted by a lowercase letter like $k$, while knot classes are generally denoted by a capital letter like $K$.
3. By a knot we can either mean the embedding $S^1 \to S^3$ or the image of the embedding. We do not distinguish between them.

Notation 2.5. Let $k$ be a knot in $S^3$. Denote the critical levels of $k$ by $c_i$, and pick regular levels $r_i$ between two consecutive critical levels $c_i$ and $c_{i+1}$, so that:

$$c_1 < r_1 < c_2 < r_2 < \ldots < c_{s-1} < r_{s-1} < c_s.$$

For each regular level $r_i$, we define $w_i = |h^{-1}(r_i) \cap k|$, that is, $w_i$ is the number of intersections of this regular level with $k$.

2.1 Trunk and Width of Knots

Now we will define two invariants known as trunk and width for knots and knot classes.

Definition 2.6. The trunk number of a knot $k$ is given by the formula

$$\text{trunk}(k) = \max_{1 \leq i \leq s-1} w_i(k).$$
**Definition 2.7.** The *width number* of a knot $k$ is given by the formula

$$\omega(k) = \sum_{i=1}^{s-1} w_i.$$ 

**Definition 2.8.** We can extend the definitions of the trunk and width numbers of a knot to also apply to knot classes:

- The *trunk number of a knot class* $K$ is given by $\text{trunk}(K) = \min_{k \in K} \text{trunk}(k)$.
- The *width number of a knot class* $K$ is given by $\omega(K) = \min_{k \in K} \omega(k)$.

**Example 2.9.** Suppose $K$ is the trefoil knot. Let $k$ be the particular embedding as depicted in Figure 1. There are three regular levels (dotted lines) in the figure. They intersect the knot 2, 4, 2 times, counting from bottom to top, and the maximum is 4, which by definition is the trunk of this knot. The width of the knot is $2 + 4 + 2 = 8$. These also happen to be the width and trunk of the trefoil knot class.

![Figure 1: Width and trunk of a trefoil knot.](image)

**Remark 2.10.** In general, a knot $k$ such that $\text{trunk}(k) = \text{trunk}(K)$ may not satisfy $\omega(k) = \omega(K)$. There exists a deformation to increase the width of any nontrivial knot without increasing its trunk. For example, the trefoil in Example 2.9 has trunk 4 and minimum width 8. It can be deformed (noting that it still remains
in trefoil class as specified in Definition \(2.2\) so that its regular levels intersect it 2, 4, 2, 4 and 2 times. Then, the trunk would still be minimized at 4 but the width would be \(2 + 4 + 2 + 4 + 2 = 14\), which is not minimal. It is conjectured by Ozawa that any knot \(k\) where \(\omega(k) = \omega(K)\) also satisfies \(\text{trunk}(k) = \text{trunk}(K)\), although in [2], Zupan and Davies produce probable counterexamples to Ozawa’s conjecture.

**Definition 2.11.** Let \(T^2 = S^1 \times S^1\) be a two dimensional torus and \(D\) be a smoothly embedded disk on \(T^2\). A curve \(\alpha \subseteq T^2\) is *inessential* if \(\alpha = \partial D\) for some \(D \subseteq T^2\) and is *essential* otherwise.

### 2.2 Satellite and Companion Knots

Here, we will define the process of forming a *satellite knot*, which is one of the main ways to construct complicated knots from simple ones. We also define two invariants pertaining to satellite knots inside a solid torus.

**Definition 2.12.** Let \(\hat{V}\) be the standard solid torus defined as

\[
\hat{V} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \left( R - \sqrt{x_1^2 + x_2^2} \right)^2 + x_3^2 \leq r^2 \}
\]

where \(r, R\) are fixed so that \(0 < r < R\).

**Definition 2.13.** The *inner core* \(\hat{j}\) of a solid torus, or the circle in its center, is given by:

\[
\hat{j} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = R^3, x_3 = 0 \}.
\]

**Definition 2.14.** A *meridian disk* of \(\hat{V}\) is a properly embedded disk \(D\) whose boundary \(\partial D \subset \partial \hat{V}\) is essential on \(\partial \hat{V}\).

**Definition 2.15.** Let \(\hat{k}\) be a knot inside \(\hat{V}\) such that \(\hat{k}\) intersects any meridian disk in \(\hat{V}\). Let \(f\) be a smooth embedding from \(\hat{V}\) to \(S^3\) and let \(f(\hat{j}) = j\) and \(f(\hat{k}) = k\).

The knot \(k\) is a *satellite knot* with companion \(j\).
Definition 2.16. On an oriented surface, we can define the sign of the intersection of two oriented curves. A negative intersection is represented by $-1$ and a positive intersection is represented by $+1$. See Figure 3.

Definition 2.17. The winding number $n(\hat{k})$ is the absolute value of the sum of all intersections with signs of any fixed meridian disk with $\hat{k}$. Note that we can also define the winding number analogously as $n(k)$, since $n(k) = n(\hat{k})$ by definition. The signs are defined as in Definition 2.16. The winding number is independent of which meridian disk is chosen due to homology theory (see [6] for the full argument), and it is the number of times the satellite knot $k$ travels along $j$. For brevity, we will denote winding number by $n$ since we only consider one satellite knot.
Definition 2.18. The \textit{wrapping number} $m(\hat{k})$ is the minimal geometric intersection number of $k$ with any meridian disk. It will similarly be denoted by $m$ for brevity.

![Figure 4: Satellite knot with winding number 0 and wrapping number 2.](image)

3 Bounding Trunk with Winding Number

In this section we will prove Theorem 3.1. Along the proof, we will also review the main ideas in [5].

Theorem 3.1. Suppose $K$ is a satellite knot with a non-trivial companion $J$ and the winding number is $n$. Then we have: \[ \text{trunk}(K) \geq n \cdot \text{trunk}(J). \]

Proof. We pick a knot $k \in K$ such that trunk($k$) = trunk($K$). There will be a corresponding companion $j$ and a solid torus $V$ containing $j$ and $k$ as in Definition 2.12. As in [4], we can assume that $h|_{\partial V}$ is Morse and all critical points of $h|_{\partial V}$ are in distinct levels and assume that $V$ does not contain the two critical points ±∞ of $S^3$. Let $c_1, ..., c_s$ be all critical values of $h|_{\partial V}$. We define

$$M = V \setminus \bigcup_{i=1}^{n} h^{-1}(c_i).$$

We construct a graph out of this where vertices correspond to components of $M$ and two vertices are connected by an edge if the corresponding two components of $M$ are separated by a critical level. We call this graph $\Gamma_R(V)$.

Remark 3.2. The graph of this type was first introduced in the paper [11] by Scharlemann and Schutens. Later Guo and Li made a similar construction in [5].
Here we use the same construction as in Guo and Li’s paper, where this graph is called a Reeb graph.

**Proposition 3.3** (Guo, Li [5]). The graph $\Gamma_R(V)$ has the following properties:

1. There is a unique loop $l \subset \Gamma_R(V)$. We can embed $l$ into $V$.
2. The loop $l$ represents a generator in $H_1(V) \cong \mathbb{Z}$.
3. The loop $l \subset V$ can be also considered as a knot $l \subset S^3$ and its knot class $L$ is a connected sum of the companion $J$ with another knot $J'$:

$$L = J \# J'.$$

**Theorem 3.4** (Davies, Zupan [2]). For two knots $K_1, K_2$ we have $\text{trunk}(K_1 \# K_2) = \max\{\text{trunk}(K_1), \text{trunk}(K_2)\}$.

From Proposition 3.3 and Theorem 3.4 we have $\text{trunk}(L) = \text{trunk}(J \# J') \geq \text{trunk}(J)$. Therefore, to prove Theorem 3.1 we simply need to show that $\text{trunk}(K) \geq n \cdot \text{trunk}(L)$. We will need the following two lemmas:

**Lemma 3.5** (Guo, Li [5]). We can isotope $l$ into such a position $l'$, so that for any regular value $r \in \mathbb{R}$, we have the following property: suppose all components of intersection $h^{-1}(r) \cap V$ are

$$h^{-1}(r) \cap V = P_1 \cup P_2 \cup ... P_t,$$

then each component $P_i$ intersects $l$ at most once.

**Lemma 3.6** (Guo, Li [5]). Let $l'$ be given as in the above lemma. Given a planar surface $P$ where $|P \cap l'| = 1$, we have $|P \cap k| \geq n$.

We can choose a regular level $r$ such that $|h^{-1}(r) \cap l| = \text{trunk}(l)$. The above two lemmas apply here to conclude:

$$|h^{-1}(r) \cap k| = n \cdot \text{trunk}(l).$$

\[\square\]
4 Wrapping Number Theorem

We have found a lower bound for \( \text{trunk}(K) \) using the winding number, but we still would like to find a stronger bound using the wrapping number. One reason for this is that if the winding number \( n = 0 \), then Theorem 3.1 does not give anything nontrivial; however, the wrapping number \( m \) is always positive, so any bound using it will be nontrivial. The whole proof of Theorem 3.1 works well with the wrapping number except Lemma 3.6. This occurs because the proof of Lemma 3.6 uses the homology interpretation of winding number, and there is no analogous interpretation of the wrapping number. However, we can prove the following key lemma in place of Lemma 3.6 and conclude our main theorem.

4.1 Definitions and Lemmas

Lemma 4.1. Suppose \( r \) is a regular level of \( h|_{\partial V} \), and all components of \( h^{-1}(r) \cap V \) are

\[
h^{-1}(r) \cap V = P_1 \cup \ldots \cup P_t.
\]

Then among those components which have non-trivial intersection with \( l' \), more than \( \frac{1}{2} \) of them have exactly one essential (see Definition 2.11) boundary component on \( \partial V \).

Theorem 4.2. Suppose \( K \) is the satellite knot with companion \( J \) and the wrapping number is \( m \). Then we have:

\[
\text{trunk}(K) > \frac{1}{2} m \cdot \text{trunk}(J)
\]

Note in this section we will use the lowercase letter \( j \) for indices, rather than for the companion knot.

Proof of Theorem 4.2 by Lemma 4.1. Let \( k \) be a knot where \( \text{trunk}(k) = \text{trunk}(K) \) and we have the corresponding companion knot and its tubular neighborhood \( V \). We construct a loop \( l \subset V \) just like in Theorem 3.1. We can also isotope \( l \) into \( l' \) as in Lemma 3.5. Pick a regular level \( r \) so that \( |h^{-1}(r) \cap l'| = \text{trunk}(l') \). We can also look at the components of \( h^{-1}(r) \cap V \). Note that each component intersects \( l \) at most once and by Lemma 4.1, more than \( \frac{1}{2} \) of the components which have non-trivial intersection with \( l' \) have exactly one essential boundary component.
From Definition 2.18, each of these pieces intersects the knot $k$ at least $m$ times (the inessential boundaries can be capped off by disks arbitrarily closed to $\partial V$ but $k \subset \text{int}(V)$ so those piece can be viewed as meridian disks when studying their intersection with $k$). So $\text{trunk}(K) > \frac{1}{2}m \text{trunk}(L)$. From $\text{trunk}(L) \geq \text{trunk}(J)$, we get Theorem 4.2.

The rest of the section will be focused on the proof of Lemma 4.1.

Suppose $V \subset S^3$ is a solid torus so that $h|_{\partial V}$ is Morse. Suppose $r \in \mathbb{R}$ is a regular level of $h|_{\partial V}$ and

$$h^{-1}(r) \cap V = P_1 \cup ... \cup P_t.$$ 

The pieces $P_i$ are contained on the regular level $h^{-1}(r)$ which is diffeomorphic to a 2-sphere. There are some restrictions on the pieces from topological side. With those restrictions, the question can be solved entirely using combinatorics.

**Lemma 4.3.** Every $P_i$ where $|P_i \cap l'| = 1$ has an odd number of essential boundary components.

**Proof.** For each $i$, let

$$\partial P_i = \alpha_{i,1} \cup ... \cup \alpha_{i,s_i} \cup \beta_i.$$ 

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where $\alpha_j$ are essential boundary components and $\beta$ is the collection of inessential circles.

We have a boundary map:

$$\partial : H_2(V, \partial V) \to H_1(\partial V)$$

and this map can be described explicitly as follows. $H_1(\partial V) \cong \mathbb{Z} \oplus \mathbb{Z}$, and the two generators are represented by meridians and longitudes (with respect to some framing). Note also $H_2(V, \partial V) \cong \mathbb{Z}$ so the map $\partial$ is actually:

$$\partial(1) = (1, 0).$$

Then $|P_i \cap l'| = 1$ means that $[P_i, \partial P_i] = \pm 1 H_2(V, \partial V)$. From the definition of boundary map, we have

$$(\pm 1, 0) = \partial(\pm 1) = \partial([P_i, \partial P_i]) = [\partial P_i] = \sum_{j=1}^{s_i} [\alpha_{i,j}]$$

Since $\alpha_{i,j} \cap \alpha_{i,j'} = \emptyset$, we know that all $\alpha_{i,j}$, if given the correct orientation, would represent the same class in $H_2(\partial V)$. So suppose for all $j$, $[\alpha_{i,j}] = \pm(x, y)$. Since some of the $[\alpha_{i,j}]$ cancel each other out because of opposite orientations, we get

$$[\partial P_i] = \sum_{j=1}^{s_i} [\alpha_{i,j}] = l \cdot (x, y) = (1, 0)$$

for $|l| \leq s_i$. This implies $lx = \pm 1$ so $l = \pm 1$. Since $l \equiv s_i \pmod{2}$, we have that $s_i$ is odd as desired. \hfill \Box

As in the above proof for each $i$ we have

$$\partial P_i = \alpha_{i,1} \cup \ldots \cup \alpha_{i,s_i} \cup \beta_i$$

where the $\alpha_{i,j}$ are all the components of the boundary of $P_i$ which are essential. Note we have $h^{-1}(r) \cong S^2$, so each $\alpha_{i,j}$ bounds a (unique) disk $D_{i,j}$ such that

$$D_{i,j} \cap \text{int}(P_i) = \emptyset.$$
Then we have:

**Lemma 4.4.** Suppose the solid torus, and hence the companion knot $J$, is knotted. Then for any disk $D_{i,j}$ there exists a piece $P_z \subset \text{int}(D_{i,j})$.

*Proof.* If $D_{i,j}$ does not contain any piece $P_z$ in its interior, then

$$D_{i,j} \cap (S^3 \setminus \hat{V}) = \partial D_{i,j} = \alpha_{i,j}.$$  

Since $\alpha_{i,j}$ is essential, this means that actually the complement of $V$ in $S^3$ is compressible and this is absurd since $V$ is knotted. $\square$

**Lemma 4.5.** Suppose the solid torus, and hence the companion knot $J$, is knotted. Suppose a disk $D_{i,j}$ does not contain any other disks $D_{i',j'}$ in its interior. Then there exists a piece $P_x \subset \text{int}(D_{i,j})$ so that $P_x$ has only one (essential) boundary components.

*Proof.* This follows from a standard innermost argument and Lemma 4.4. $\square$

We can describe the combinatorial setting now.

**Definition 4.6.** Suppose $A(s)$ is an embedding of $s$ many compact connected surfaces, or what we called *pieces*

$$P_1 \sqcup P_2 \ldots \sqcup P_s \hookrightarrow S^2,$$

so that all of the following hold:

1. For each $i$, let the boundary components of $P_i$ be

$$\partial P_i = \alpha_{i,1} \cup \ldots \cup \alpha_{i,s_i}.$$  

Then $s_i$ is either 1 or at least 3.

2. On the sphere $S^2$, each $\alpha_{i,j}$ bounds a disk $D_{i,j} \subset S^2$ so that $D_{i,j} \cap \text{int}(P_i) = \emptyset$. We have the following two requirements on $D_{i,j}$:
   
   (i). For each $D_{i,j}$, there exists a piece $P_x$ so that $P_x \subset \text{int}(D_{i,j})$.
   
   (ii). If a $D_{i,j}$ does not contain any other disks $D_{i',j'}$, then there exists a piece $P_k$ so that $P_k \subset D_{i,j}$ and $x_k = 1$.

We call such $A(s)$ an *arrangement* (of the surfaces on sphere).
Example 4.7. In Figure 6 we have two pictures. On the left we do not have an arrangement since one innermost piece is missing, violating requirement (2) in Definition 4.6. On the right, we have an arrangement where \( \lambda(A(5)) = 4 \).

![Not an arrangement](image1)

![Arrangement](image2)

Figure 6: Examples of piece configurations.

Definition 4.8. For an arrangement \( A(s) \), let \( \lambda(A(s)) \) be the number of pieces which have exactly one boundary component.

Now we are going to prove the following lemma:

Lemma 4.9. For any arrangement \( A(s) \), we have

\[
\lambda(A(s)) > \frac{s}{2}.
\]

Proof of Lemma 4.9 by Lemma 4.9. Suppose we have a regular value \( r \) and

\[
h^{-1}(r) \cap V = P_1 \cup ... \cup P_t.
\]

By Lemma 3.5 each component \( P_i \) intersects \( l' \) at most once. The components that do not intersect \( l' \) can be discarded and it is straightforward to check that the remaining components still satisfy Lemmas 4.3, 4.4 and 4.5. These correspond to the requirements (1) and (2) in Definition 4.6.

If there are no inessential boundary components for all \( P_i \), we are done. Now we consider what happens if any of the \( P_i \) have inessential boundaries.
Definition 4.10. Suppose $P_i$ is a piece and $\beta \subset \partial P_i$ is a boundary component of $P_i$ which is inessential. Then there is a (unique) disk $D \subset S^2$ so that $D \cap \text{int}(P_i) = \emptyset$. We call $\beta$ pseudo-essential if $D$ contains another pieces $P_j$ in its interior.

Let $\lambda_b(A(s))$ count pieces with exactly 1 essential boundary and $b$ many pseudo-essential boundaries. We claim that $\lambda_0(A(s)) + \lambda_1(A(s)) > \frac{1}{2}s$ and that this will prove Lemma 4.1.

Inessential circles which are not pseudo-essential can simply be ignored. If a piece $P_i$ has one essential boundary and at least two pseudo-essential boundary components, then this piece has already satisfied the conditions in Definition 4.6. Hence after removing all pieces with 1 essential boundary and 1 pseudo-essential boundary, it will result in a valid arrangement $A(s - \lambda_1(A(s)))$. We note $\lambda_0(A(s - \lambda_1(A(s)))) = \lambda_0(A(s))$.

We have by Lemma 4.9 that $\lambda_0(A(s - \lambda_1(A(s)))) > \frac{1}{2}(s - \lambda_1(A(s)))$, so clearly $\lambda_0(A(s)) + \lambda_1(A(s)) > \frac{1}{2}(s + \lambda_1(A(s))) > \frac{1}{2}s$ as desired. Hence we are done.

In the proof we will also need the following definition:

Definition 4.11. For an arrangement $A(s)$, let $\mu(A(s))$ be the maximum number of boundary components of a component among all components of $S^2 \setminus (P_1 \cup \ldots \cup P_s)$.

4.2 A Combinatorial Proof of Lemma 4.9

We use combinatorics and induction to prove Lemma 4.9.

Proof of Lemma 4.9. We proceed by induction and begin by only considering the cases where none of the $P_i$ have any inessential boundaries. We claim that

$$\lambda(A(2x)) \geq x + 1 \text{ and } \lambda(A(2x + 1)) \geq x + 2.$$ 

Further, we claim that $\mu(A(2x)) = 2$ if $\lambda(A(2x)) = x + 1$ and $\mu(A(2x + 1)) = 3$ if $\lambda(A(2x + 1)) = x + 2$. Additionally, we claim that if we have 3 adjacent circles in an arrangement $A(s)$ that minimizes $\lambda(A(s))$, then there is exactly one set of 3 adjacent circles rather than multiple sets and that all pieces have at least 3 boundaries or 1 boundary.


For our base cases, we let \( x = 1 \), and we note that we do have \( \lambda(A(2)) = 2 \) and \( \lambda(A(3)) = 3 \). We also have \( \mu(A(2)) = 2 \) and \( \mu(A(3)) = 3 \), so this is also consistent with our claim. There is only one unique arrangement for both \( A(2) \) and \( A(3) \). Further, in the arrangement \( A(3) \), there is exactly one set of 3 adjacent circles and all pieces have either 1 boundary or 3 boundaries. Thus our base cases satisfy our inductive hypothesis.

Now we assume that \( \lambda(A(2y)) \geq y + 1 \) and \( \lambda(A(2y + 1)) \geq y + 2 \). We also assume that \( \mu(A(2y)) = 2 \) and all pieces have either 1 boundary or 3 boundaries for any \( A(2y) \) such that \( \lambda(A(2y)) = y + 1 \). Similarly, we assume that \( \mu(A(2y + 1)) = 3 \) and all pieces have either 1 boundary or 3 boundaries for any \( A(2y + 1) \) such that \( \lambda(A(2y + 1)) = y + 2 \) and that if \( A(2y + 1) \) minimizes \( \lambda(A(2y + 1)) \), then it has exactly one set of 3 adjacent circles.

We observe that for any plausible arrangement \( A(s) \) we have \( \mu(A(s)) \geq 2 \). Additionally, suppose that \( A(s) \) is an arrangement that minimizes \( \lambda(A(s)) \) and \( A(s + 1) \) is an arrangement that minimizes \( \lambda(A(s + 1)) \). From this, we have:

\[
\lambda(A(s)) \leq \lambda(A(s + 1)) \leq \lambda(A(s)) + 1.
\]

Now we claim that from the existence of an arrangement \( A(2y + 1) \) where \( \mu(A(2y + 1)) = 3 \) and \( \lambda(A(2y + 1)) = y + 2 \), \( \exists A(2y + 2) \) such that

\[
\lambda(A(2y + 2)) = \lambda(A(2y - 1)) = y + 2.
\]

Since \( \mu(A(2y - 1)) = 3 \), we know that we have exactly one set three adjacent circles. Arbitrarily call the three circles \( C_1, C_2, C_3 \). To form \( P_{2y+2} \), we circle \( C_1 \) and \( C_2 \) with a new circle \( C_4 \). We circle \( C_1, C_4 \) with a new circle \( C_5 \). By doing so, we have a new piece \( P_{2y} \) whose boundaries are \( C_1, C_4 \) and \( C_5 \), while we have not increased the number of innermost essential circles. Further, since there was only one set of 3 adjacent circles to begin with and now it is gone, we have found an arrangement \( A(2y + 2) \) such that

\[
\lambda(A(2y + 2)) = \lambda(A(2y + 1)) = y + 2 \text{ and } \mu(A(2y + 2)) = 2.
\]
Additionally, since \(\lambda(A(s + 1)) \geq \lambda(A(s))\), we have that \(\lambda(A(2y + 2)) \geq y + 2\) for any \(A(2y + 2)\).

From this construction, we claim that if \(\lambda(A(2y + 2)) = y + 2\), then \(\mu(A(2y + 2)) = 2\).

Case 1: \(\lambda(A(2y + 2)) = y + 2\).

We know from our work above that since \(\lambda(A(2y + 2)) = y + 2\), we must have \(\mu(A(2y + 2)) = 2\). Since we never have more than two adjacent circles, we know that we cannot add a new piece inside either one because that would violate the requirement that all pieces must have an odd number of boundaries. Therefore, the new piece must be outside any two adjacent circles, implying that it is a meridian disk. Thus, we have that \(\lambda(A(2y + 3)) = y + 3\).

Case 2: \(\lambda(A(2y + 2)) \geq y + 3\).

From the inequality \(\lambda(A(s)) \leq \lambda(A(s + 1))\), this implies \(\lambda(A(2y + 3)) \geq y + 3\).

Now we claim that \(\lambda(A(2y + 3)) = y + 3\) implies \(\mu(A(2y + 3)) = 3\). To see this, we observe that we may remove a piece from \(A(2y + 3)\) with 3 boundaries such that at least one of its inner boundaries contains only a meridian disk, and then
remove that meridian disk.

This gives us an $A(2y + 1)$ where $\lambda(A(2y + 1)) = y + 2$. Since we know that $\lambda(A(2y + 1)) \geq y + 2$, our $A(2y + 1)$ is a minimal arrangement, so it has exactly 1 set of 3 adjacent circles from our inductive hypothesis. When we add a meridian disk, we either get 4 adjacent circles or two sets of 3 adjacent circles. Either way, when we add $P_{2y+3}$ with 3 boundaries such that one of its boundaries is directly around the meridian disk we just added, we end up with 1 set of 3 adjacent circles as desired.

Since we have considered both cases, we conclude that we must have $\lambda(A(2y + 3)) \geq y + 3$. We also know that $\exists A(2y + 3)$ such that $\lambda(A(2y + 3)) = y + 3$ because we can simply add a meridian disk to an arrangement $A(2y + 2)$ such that $\lambda(A(2y + 2)) = y + 2$. Finally, this clearly yields $\mu(A(2y + 3)) = 3$ and there is exactly one set of 3 adjacent circles.

This inductive proof shows that $\lambda(A(2x)) \geq x + 1$ and $\lambda(A(2x + 1)) \geq x + 2$. From this, if $A(n)$ is a minimal arrangement, we have:

$$\lim_{x \to \infty} \frac{\lambda(A(x))}{x} = \frac{1}{2}.$$ 

However, for any specific value of $x$, we have that $\frac{\lambda(A(x))}{x} > \frac{1}{2}$. Therefore, the fraction of the horizontal pieces that are meridian disks and intersect $k$ at least $m$ times is always strictly greater than $\frac{1}{2}$ as desired. 

\section{5 $\lambda(a)$ and $\mu(a)$}

Theorem 4.2 establishes a lower bound for the trunk of a satellite knot in terms of the trunk of the companion knot and the wrapping number of the satellite knot. We know from Definition 2.18 that the wrapping number is defined to be the minimal geometric intersection number of a meridian disk with the satellite knot. Recall that a piece is a meridian disk if and only if it has exactly 1 essential boundary. In this section, we extend the previous results to obtain a lower bound on the trunk of a satellite knot in terms of the minimal number of geometric intersections of pieces with more than one essential boundary with the satellite
Definition 5.1. $S(a)$ is the set of all connected planar surfaces $S \subset V$ such that $\partial S \subset \partial V$, $S$ represents a generator of $H_2(V, \partial V)$ and $S$ has no more than $a$ essential boundaries.

Definition 5.2. Let $\mu(a) = \min_{S \in S(a)} |S \cap k|$.

By definition, $\mu(a + 1) \leq \mu(a)$.

Definition 5.3. Let $\lambda(a)$ be the largest possible value such that for any satellite knot $k$ with companion $j$ we have $\text{trunk}(K) \geq \lambda(a) \cdot \mu(a) \cdot \text{trunk}(J)$.

Suppose:

$$\mu = \lim_{a \to \infty} \mu(a).$$

Remark 5.4. Note we always have $m = \mu(1) \geq \mu \geq n$, where $n$ is the winding number and $m$ is the wrapping number. There indeed exist cases when $\mu = m$ or $\mu = n$. For instance, when the satellite knot $K$ is the Whitehead double knot, by (Li, Guo), we have that $\mu(a) = 2$ for any $a$. Further, we have that $\text{trunk}(K) = 2 \cdot \text{trunk}(J)$, making $\lambda(a) = 1$ in this case. Since we can always form the Whitehead double of any knot, we know that if $\lambda(a) > 1$, then the Whitehead double would be a counterexample. Therefore, we have $\lambda(a) \leq 1$.

Proposition 5.5. $\text{trunk}(K) \geq \frac{1}{2} (m + \mu) \cdot \text{trunk}(J) \geq \frac{1}{2} (m + n) \cdot \text{trunk}(J), \mu \cdot \text{trunk}(J)$.

Note that Remark 5.4 and Proposition 5.5 imply Theorem 3.1.

Proof. Let $k$ be a knot such that $\text{trunk}(k) = \text{trunk}(K)$ and we have the corresponding companion knot and its tubular neighborhood $V$. We construct a loop $l \subset V$ just like in Theorem 3.1. We can also isotope $l$ into $l'$ as in Lemma 3.5. Pick a regular level $r$ for which $|h^{-1}(r) \cap l'| = \text{trunk}(l')$. We can also look at the components of $h^{-1}(r) \cap V$. The intersection of a regular level $h^{-1}(r)$ with the solid torus is a set of horizontal pieces, each with an odd number of essential boundaries. Let $b_a$ denote the proportion of total pieces with exactly $a$ boundaries. It is obvious that:
We note that by Lemma 4.1, \( b_1 > \frac{1}{2} \). Then we have:

\[
\sum_{a=1}^{\infty} b_a = 1.
\]

\[
\text{We note that by Lemma 4.1, } b_1 > \frac{1}{2}. \text{ Then we have:}
\]

\[
\text{trunk}(K) \geq \text{trunk}(l') \sum_{a=1}^{\infty} b_a \mu(a)
\]

\[
\geq \text{trunk}(l') \left( \frac{1}{2} m + \sum_{a=3}^{\infty} b_a \mu(a) \right)
\]

\[
\geq \text{trunk}(l') \left( \frac{1}{2} m + \mu \cdot \sum_{a=3}^{\infty} b_a \right).
\]

Recalling that \( m \geq \mu \geq n \) and \( \text{trunk}(l') \geq \text{trunk}(L) \geq \text{trunk}(J) \) we get:

\[
\text{trunk}(K) \geq \text{trunk}(l') \left( \frac{1}{2} m + \mu \cdot \sum_{a=3}^{\infty} b_a \right)
\]

\[
\geq \text{trunk}(J) \cdot \frac{\mu + m}{2}
\]

\[
\geq \frac{1}{2} (m + n) \text{trunk}(J), \mu \cdot \text{trunk}(J).
\]

To bound \( \lambda(a) \), we will slightly alter Definition 4.6 for arrangement:

**Definition 5.6.** An arrangement \( A(s) \) is an embedding of surfaces as in Definition 4.6, but with the additional requirement that each piece has an odd number of boundary components.

We can see from Section 4 that it is only possible to have an even number of boundary components when we count the inessential boundary components. However, we can actually ignore inessential boundary components as we did in Section 4.

**Theorem 5.7.** Suppose \( a \) is odd. Then we have \( \lambda(a) > \frac{a}{a+1} \).

To prove Theorem 5.7, we will need the following lemma:
Lemma 5.8. Any arrangement as defined in Definition 5.6 can be constructed from two meridian disks using a sequence of the following two types of “moves”.

Move 1: adding a new meridian disk to the arrangement.

Move 2: replacing a meridian disk with a piece with some odd number a boundary components, with at least one piece contained in each of the a − 1 new boundary components.

Note that each step is by definition reversible, and both performing the move and reversing the move in a valid arrangement results in another valid arrangement. Also, the initial state consisting of two meridian disks is a valid arrangement.

Proof of Lemma 5.8. Given any arrangement \( A(s) \) (pieces on a 2-sphere), we repeat two procedures until it is not possible to continue.

Procedure 1: perform the reverse of Move 1 until each boundary component contains at most one meridian disk.

Procedure 2: perform the reverse of Move 2 on pieces with more than 1 boundary component that do not themselves contain another piece with more than 1 boundary.

If there exists a piece with more than 1 boundary component, then by the innermost argument, there exists a piece with more than 1 boundary component such that each boundary component contains only meridian disks within its interior (see Definition 4.10 for interior). After performing the first procedure to the fullest extent, it is possible to perform the second procedure to remove the “innermost” piece with more than one boundary component. Thus, the only possible arrangement where these procedures cannot be performed does not have any piece with more than one boundary component. This means it only has meridian disks. After reaching such an arrangement, we then perform procedure 1 to get two meridian disks.

To build the original arrangement from these two remaining meridian disks, we reverse each move made in the deconstruction process. □

Proof of Theorem 5.7. We prove \( \lambda(a) > \frac{a}{a+1} \) for all odd \( a \) with an inductive argument. Let \( A(s_t) \) be the arrangement after \( t \) moves have been performed on an arrangement \( A(s) \). Let \( x_t \) be the number of pieces with at most \( a \) boundaries,
and let \( y_t \) be the total number of pieces in \( A(s_t) \). From Definition 5.3 and the same argument as in the proof of Theorem 4.2, \( \lambda(a) \) is greater than or equal to the minimum possible value of \( \frac{x_t}{y_t} \). Note that we start with \( A(s_0) = A(2) \), which is the arrangement with two meridian disks, and in this case, \( x_0 = 2 \) and \( y_0 = 2 \).

Performing Move 1 results in \( x_{t+1} = x_t + 1 \) and \( y_{t+1} = y_t + 1 \) so \( \frac{x_{t+1}}{y_{t+1}} \geq \frac{x_t}{y_t} \).

Performing Move 2 with a piece of \( c \) boundaries such that \( c \leq a \) results in \( x_{t+1} = x_t + (c - 1) \) and \( y_{t+1} = y_t + (c - 1) \) so \( \frac{x_{t+1}}{y_{t+1}} \geq \frac{x_t}{y_t} \).

Performing Move 2 with a piece of \( c \) boundaries such that \( c > a \) results in \( x_{t+1} = x_t + (c - 2) \) and \( y_{t+1} = y_t + (c - 1) \). Note that each pieces have an odd number of boundary components, so \( c \) is odd and \( c \geq a + 2 \). A smaller number of essential boundaries in the piece added always causes the biggest decrease in \( \frac{x_t}{y_t} \) since \( x_t \leq y_t \) by definition. Thus, the arrangement with minimal \( \frac{x_t}{y_t} \) is constructed by repeatedly performing Move 2, converting meridian disks to pieces with \( a + 2 \) boundaries:

\[
\lim_{t \to \infty} \frac{x_t}{y_t} = \lim_{t \to \infty} \frac{2 + t \cdot a}{2 + t \cdot (a + 1)} = \frac{a}{a + 1}
\]

Thus, \( \lambda(a) > \frac{a}{a + 1} \).

**Proposition 5.9** (Zupan, Davies [2]). \( \omega(K) \geq \frac{1}{2} \cdot \text{trunk}(K)^2 \) for any knot class \( K \).

**Corollary 5.10.** If \( \omega(J) = \frac{1}{2} \cdot \text{trunk}(J)^2 \), then \( \omega(K) > \left( \frac{a}{a + 1} \right)^2 \cdot \mu(a)^2 \cdot \omega(J) \). When \( a = 1 \), we get \( \omega(K) > \frac{1}{4} m^2 \omega(J) \). As \( a \) approaches infinity, we get \( \omega(K) \geq \mu^2 \omega(J) \).

**Proof.** Suppose by contradiction that \( \omega(K) \leq \left( \frac{a}{a + 1} \right)^2 \cdot \mu(a)^2 \cdot \omega(J) \). Then we have:

\[
\left( \frac{a}{a + 1} \right)^2 \cdot \mu(a)^2 \cdot \omega(J) \geq \omega(K) \geq \frac{1}{2} \cdot \text{trunk}(K)^2 > \frac{1}{2} \left( \frac{a}{a + 1} \right)^2 \mu(a)^2 \cdot \text{trunk}(J)^2.
\]

Simplifying this, we get that \( \omega(J) > \frac{1}{2} \cdot \text{trunk}(J)^2 \), which violates the restriction that \( \omega(J) = \frac{1}{2} \cdot \text{trunk}(J)^2 \).
6 Conclusion

6.1 Summary

With Theorem 3.1, we bounded the trunk of a satellite knot with the winding number to get the inequality \( \text{trunk}(K) \geq n \cdot \text{trunk}(J) \). Then, we used Lemma 4.1 to prove Theorem 4.2, which gave us a bound with the wrapping number but with a factor of \( \frac{1}{2} \). We extended this in Proposition 5.5 to include the winding number and get that \( \text{trunk}(K) \geq \frac{1}{2}(m + n) \text{trunk}(J) \). We also generalized the wrapping number invariant to a set of \( \mu(a) \), and proved Theorem 5.7, which states that \( \text{trunk}(K) > \frac{a}{a+1}\mu(a) \text{trunk}(J) \). In Corollary 5.10, we applied Theorem 5.7 and Proposition 5.9 to get a conditional bound on the width of a satellite knot.

6.2 Future Directions of Study

Currently, the strongest conjectures in this field are \( \text{trunk}(K) \geq m \cdot \text{trunk}(J) \) and \( \omega(K) \geq m^2 \omega(J) \). They would be consistent with a theorem that has already been proved: \( b(K) \geq m \cdot b(J) \) where \( b(\cdot) \) denotes the bridge number (half the number of critical points) of a knot. In Theorem 4.2 and Proposition 5.5, we offer partial results towards the conjecture regarding trunk. To make further progress, it would be interesting to try to incorporate the bridge number inequality. This inequality provides a relationship between the number of critical points of a satellite knot and its companion. Since width and trunk are defined by the number of intersections that regular levels have with a knot, which are directly affected by the knot’s critical points, the bridge number inequality may prove to be useful to prove stronger inequalities in the future.

Another future direction of work is to look at examples of specific satellite knots and try to obtain better bounds than \( \frac{1}{2}m \). The basic family of examples, the Whitehead doubles, has been fully studied by Guo and Li [7]. Thus, it makes sense to consider other examples of satellite knots with wrapping number 2 but winding number 0, for example, the pattern drawn in Figure 7.

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Figure 7: Potential pattern for future study where $m = 2$ and $n = 0$.

The pattern in Figure 7 is the simplest example for which the techniques from [7] fail to work. Understanding this example would be another interesting way to make further progress in bounding the trunk of a satellite knot in terms of its companion.

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