EIGHT TYPES OF QUTRIT DYNAMICS GENERATED BY UNITARY EVOLUTION COMBINED WITH 2+1 PROJECTIVE MEASUREMENT

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ABSTRACT. We classify the Markov chains that can be generated on the set of quantum states by a unitarily evolving 3-dim quantum system (qutrit) that is repeatedly measured with a projective measurement (PVM) consisting of one rank-2 projection and one rank-1 projection. The dynamics of such a system can be visualized as taking place on the union of a Bloch ball and a single point, which correspond to the respective projections. The classification is given in terms of the eigenvalues of the $2 \times 2$ matrix that describes the dynamics arising on the Bloch ball, i.e., on the 2-dim subsystem. We also express this classification as the partition of the numerical range of the unitary operator that governs the evolution of the system. As a result, one can easily determine which of the eight possible chain types can be generated with the help of any given unitary.

KEYWORDS. quantum trajectories, Markov chains, projections, unitary matrices

MSC2020. Primary: 81-10 Secondary: 60J20, 37N20

1. INTRODUCTION

Consider performing successive (isochronous) measurements on a $d$-dimensional ($d \geq 2$) quantum mechanical system that between each two consecutive measurements undergoes deterministic time evolution governed by a unitary operator (Fig. 1.1). We model the joint evolution of such a system with a Partial Iterated Function System (PIFS), which is a notion that slightly (yet significantly) generalizes that of an Iterated Function System (IFS) with place-dependent probabilities. The Markov chain that is generated on the set of quantum states corresponds to the so-called discrete quantum trajectories, see, e.g., [2, 9, 37, 38, 41]. The sequences of measurement outcomes that are emitted by the system need not, however, be Markovian (this was first noted in [8]), but can be described by a hidden Markov model. The presence of long-term correlations between the outcomes can be interpreted as the system encoding in its current state some information about the outcomes of previous measurements.

Fig. 1.1. Repeatedly measured quantum system that between each two consecutive measurements evolves in accordance to a unitary operator $U$. State dynamics $(\tilde{\rho}_0, \tilde{\rho}_1, \tilde{\rho}_2, \ldots)$ is Markovian.
Suppose a unitarily evolving system is repeatedly measured with a rank-1 POVM, i.e., with a measurement that consists of (suitably rescaled) one-dimensional orthogonal projections. Then to each possible outcome there corresponds a single post-measurement state. (We assume the standard Lüders instrument is in use.) It follows that the Markov chain generated by this system can be easily recovered from the sequences of emitted outcomes. In consequence, symbolic dynamics is Markovian as well, and so long-term correlations cannot form in the sequences of outcomes. Therefore, for a unitarily evolving and repeatedly measured quantum system to have potential for information storage it is necessary that the measurement contain operators of ranks higher than one. Then the sequences of outcomes can be non-Markovian and the system gains the ability to exhibit long-term correlations between outcomes. Crutchfield & Wiesner dedicated a series of papers [20, 43, 55, 56, 57] to the phenomenon of intrinsic quantum computation, i.e., the way in which quantum systems store and manipulate information. Among other things, they investigated a specific example of a unitarily evolving three-dimensional quantum system (qutrit) which is repeatedly measured with a measurement consisting of two projectors, one of which is of rank two (and so the other necessarily of rank one).

The class of quantum measurements that consist exclusively of projections (PVMs) is distinguished by the fact that to each measurement outcome there corresponds a distinct subsystem whose dimension is equal to the rank of the projection corresponding to this outcome. In consequence, the dynamics of the system can be thought of as taking place on the union of pairwise disjoint Bloch bodies. In particular, the above-mentioned qutrit system can be visualized as the union of a Bloch ball (corresponding to the rank-2 projection) and a point (corresponding to the rank-1 projection), and so we refer to it as the ball & point system, see Fig. 1.2.

The aim of this paper is to classify the Markov chains that the ball & point system can generate on the set of quantum states. We distinguish eight types of chains, and each type is constituted by chains that show qualitatively the same behaviour with respect to the underlying Bloch ball and, in consequence, share the same limiting properties. The resulting classification can serve as a stepping stone to deriving an explicit formula for quantum dynamical entropy of the ball & point system via Blackwell integral entropy formula [11], see also [49, Thm. 5.5]. The main step in obtaining this classification is to examine the types of dynamics that arises on the two-dimensional subsystem, i.e., on the Bloch ball, when the deterministic time evolution of the system gets intertwined with the process of measurement.

![Fig. 1.2. Qutrit measured with a PVM consisting of one rank-2 projection and one rank-1 projection: the ball & point system.](image)
Also, we show how this classification can be transferred onto the numerical range of the unitary operator that governs the time evolution of the ball & point system. As a by-product, it follows that to determine which of the eight possible types of Markov chains can actually be generated by the ball & point systems evolving in accordance to a given unitary, it is sufficient to have a look at the numerical range of this unitary. In summary, we obtain the following classifying theorems:

I. classification of the non-unitary ball dynamics: of its fixed points (Obs. 4.12) and generic trajectories (Obs. 4.13);

II. classification of the Markov chains in terms of the eigenvalues of the matrix that describes the ball dynamics (Thm. 4.20);

III. classification of the Markov chains as a partition of the numerical range of the unitary operator that governs the time evolution of the ball & point system (Thm. 4.33).

This paper is organized as follows. The preliminary Section 2 lays down the basic framework, recalling the mathematical description of quantum theory and providing the definition of Partial Iterated Function Systems. In Section 3 we define the PIFS that models a unitarily evolving and repeatedly measured quantum system. The final Section 4 is dedicated to a thorough study of the ball & point qutrit system. In Subsection 4.1 we classify the types of Markov chains that such a system can generate in terms of the eigenvalues of the matrix describing the dynamics induced by the system on the Bloch ball. In Subsection 4.2 we investigate how this classification is reflected in the numerical range of the unitary operator. This numerical range is, generically, a triangle spanned by the eigenvalues of the unitary in question and its subset corresponding to the so-called elliptic chains turns out to be contained in a cubic curve. We examine this curve more closely in Subsection 4.3 and identify it as the Musselman cubic (Remark 4.39). Lastly, in Subsection 4.4 we discuss the specific unitary operator investigated by Crutchfield & Wiesner and expand on their results by identifying all types of Markov chains that the ball & point system governed by this unitary can generate.

2. Preliminaries

2.1. Quantum states & measurements. Fix \( d \geq 2 \). The set of \( d \)-dimensional quantum states is defined as \( \mathcal{S}(\mathbb{C}^d) := \{ \rho \in \mathcal{L}(\mathbb{C}^d) : \rho \geq 0, \, \operatorname{tr} \rho = 1 \} \), where \( \mathcal{L}(\mathbb{C}^d) \) denotes the space of (bounded) linear maps on \( \mathbb{C}^d \). The extreme points of \( \mathcal{S}(\mathbb{C}^d) \) form the set \( \mathcal{P}(\mathbb{C}^d) \) of pure states. It follows that \( \mathcal{P}(\mathbb{C}^d) \) is the set of one-dimensional orthogonal projections on \( \mathbb{C}^d \) and \( \mathcal{S}(\mathbb{C}^d) = \text{conv} \, \mathcal{P}(\mathbb{C}^d) \). For \( w \in \mathbb{C}^d \setminus \{0\} \) we let \( \rho_w \) denote the pure state corresponding to \( w \), i.e., the orthogonal projection on \( \text{span}\{w\} \). Clearly, \( \mathcal{P}(\mathbb{C}^d) \) can be put in one-to-one correspondence with rays in \( \mathbb{C}^d \), i.e., with the complex projective Hilbert space \( \mathbb{C} \mathbb{P}^{d-1} \).

Let \( \mathcal{U}(\mathbb{C}^d) \) denote the set of unitary operators on \( \mathbb{C}^d \). Deterministic time evolution of a quantum system is said to be governed by \( U \in \mathcal{U}(\mathbb{C}^d) \) if it is given by the unitary channel

\[
\Lambda^U : \mathcal{S}(\mathbb{C}^d) \ni \rho \mapsto U \rho U^* \in \mathcal{S}(\mathbb{C}^d).
\]

Unitary channels are examples of state automorphisms, which, following Kadison’s approach [36], we define as affine bijections on \( \mathcal{S}(\mathbb{C}^d) \), see also [23, Sec. 5.3]. These transformations represent symmetries in quantum formalism, i.e., describe freedom in choosing a particular mathematical representation of physical objects.
Another class of maps that give rise to state automorphisms is that of antiunitaries. Recall that $W: \mathbb{C}^d \to \mathbb{C}^d$ is called an antiunitary operator if $W(u + \gamma v) = W u + \overline{\gamma} W v$ and $\langle W u | W v \rangle = \langle v | u \rangle$ for all $u, v \in \mathbb{C}^d$, $\gamma \in \mathbb{C}$. We let $\overline{\mathcal{U}}(\mathbb{C}^d)$ stand for the set of antiunitary operators on $\mathbb{C}^d$. For $W \in \overline{\mathcal{U}}(\mathbb{C}^d)$ we define its adjoint $W^*$ via $\langle u | W^* v \rangle = \langle W u | v \rangle$, where $u, v \in \mathbb{C}^d$. One can show that $W^* \in \overline{\mathcal{U}}(\mathbb{C}^d)$ as well as $WW^* = I_d$. It follows that

$$\Lambda^W: S(\mathbb{C}^d) \ni \rho \mapsto W \rho W^* \in S(\mathbb{C}^d)$$

is indeed a state automorphism.\(^1\) It is a well-known result by Kadison that there are no other state automorphisms but those induced by unitary or antiunitary operators, see, e.g., [23, p. 101] or [30, Thm. 2.63].

Let $k \in \mathbb{N}$ and put $I_k := \{1, \ldots, k\}$. The measurement of a $d$-dimensional quantum system with $k$ possible outcomes is given by a positive operator valued measure (POVM), i.e., a set of positive semi-definite (non-zero) Hermitian operators $\Pi_1, \ldots, \Pi_k$ on $\mathbb{C}^d$ that sum to the identity operator, i.e.,

$$\sum_{i=1}^k \Pi_i = I_d. \tag{2.2}$$

We say that $\Pi$ is a projection valued measure (PVM) or a Lüders–von Neumann measurement [40] if $\Pi_i$ is a projection for every $i \in I_k$. We then have $k \leq d$ and the projections constituting $\Pi$ are necessarily orthogonal as self-adjoint projections on a Hilbert space. Moreover, they are mutually orthogonal, i.e., $\Pi_i \Pi_j = 0$ for $i, j \in I_k, i \neq j$ [29, p. 46].

If the state of the system before the measurement is $\rho \in S(\mathbb{C}^d)$, then the Born rule dictates that the probability of obtaining the $i$-th outcome is equal to $\text{tr}(\Pi_i \rho)$, where $i \in I_k$ [16]. Generically, the measurement process alters the state of the system, but the POVM alone is not sufficient to determine the post-measurement state. This can be done by defining a measurement instrument (in the sense of Davies and Lewis [21]) compatible with $\Pi$, see also [18], [17, Ch. 10], [30, Ch. 5]. Among infinitely many instruments generating the same measurement statistics we only consider here the so-called generalised Lüders instruments, disturbing the initial state in the minimal way, where the state transformation reads

$$\rho \mapsto \frac{\sqrt{\Pi_i } \rho \sqrt{\Pi_i } }{\text{tr}(\Pi_i \rho)}, \tag{2.3}$$

provided that the measurement has yielded the result $i \in I_k$ [22, p. 404], see also [5, 6].

2.2. Partial Iterated Function Systems. Recall that $I_k := \{1, \ldots, k\}$. By $S_k$ we denote the set of all permutation on $I_k$. Let $X$ be an arbitrary set.

**Definition 2.1.** [49, p. 59] The triple $(X, \mathcal{F}, \{p_i\}_{i \in I_k})$ is called a Partial Iterated Function System (PIFS) on $X$ if $p_i: X \to [0, 1]$, $\sum_{j \in I_k} p_j = 1$, and $F_i: \{x \in X: p_i(x) > 0\} \to X$, where $i \in I_k$. We call $F_1, \ldots, F_k$ evolution maps.

The action of a PIFS transforms a given initial state $x \in X$ into a new state $F_i(x)$ with (place-dependent) probability $p_i(x)$, and the symbol $i$ corresponding to this evolution is then emitted ($i \in I_k$). Thus, the repeated action of a PIFS generates a Markov chain

\(^1\)In contrast to unitary channels, state automorphisms induced by antiunitary operators are not completely positive. Hence, they describe symmetries that are physically unobservable, e.g., time reversal.
on $X$ and yields sequences of symbols from $I_k$, which can be modelled by a hidden Markov chain. The probabilities and evolution maps corresponding to the strings of symbols are defined inductively in the following natural way. Let $n \in \mathbb{N}$, $i := (i_1, \ldots, i_n) \in I_k^n$ and $j \in I_k$. For $n = 1$ both $p_i$ and $F_i$ are given. The probability of the system outputting $i_j := (i_1, \ldots, i_n, j) \in I_k^{n+1}$ is defined as

$$p_{ij}(x) := \begin{cases} p_j(F_i(x))p_i(x) & \text{if } p_i(x) > 0 \\ 0 & \text{if } p_i(x) = 0 \end{cases}$$ (2.4)

and the corresponding evolution map is defined as $F_{ij}(x) := F_j(F_i(x))$ if $p_i(x) > 0$.

For PIFSs acting on the set of quantum states we have a natural notion of conjugacy defined with the help of state automorphisms:

**Definition 2.2.** Let $F = (S(\mathbb{C}^d), F_i, p_i)_{i \in I_k}$ and $\tilde{F} = (S(\mathbb{C}^d), \tilde{F}_i, \tilde{p}_i)_{i \in I_k}$ be PIFSs. Also, let $V \in U(\mathbb{C}^d) \cup \overline{U(\mathbb{C}^d)}$. We say that $\tilde{F}$ is $V$-conjugate to $F$ if there exists $\sigma \in S_k$ such that

$$\tilde{F}_i = \Lambda^V \circ F_{\sigma(i)} \circ \Lambda^V \quad \text{and} \quad \tilde{p}_i = p_{\sigma(i)} \circ \Lambda^V$$

for every $i \in I_k$, where $\Lambda^V$ is the state automorphism induced by $V$, i.e., $\Lambda^V(\rho) = V\rho V^*$ for $\rho \in S(\mathbb{C}^d)$. When there is no need to specify the conjugating map, we simply say that $F$ and $\tilde{F}$ are conjugate and denote this by $F \sim \tilde{F}$.

The notion of a PIFS generalizes, slightly but significantly, that of an Iterated Function System (IFS) with place-dependent probabilities (see, e.g., [3, 47]) by allowing each evolution map to remain undefined on the states that have zero probability of being subject to the action of this map. Such a generalization is necessary in considering quantum measurements because the state transformation associated with a given measurement outcome cannot be defined on the states with zero probability of producing this outcome, see (2.3). IFSs acting on the set of pure quantum states have been examined in the framework of Event Enhanced Quantum Theory (EEQT) [12, 13, 14]; in particular, IFSs acting on the Bloch sphere were investigated in [31, 35], see also [32, 33, 34]. A generalization to systems that act in the space of all quantum states was proposed in [39].

### 3. Evolution & Measurement Quantum PIFSs

Fix a POVM $\Pi = \{\Pi_1, \ldots, \Pi_k\}$ and $U \in U(\mathbb{C}^d)$. In what follows we define the PIFS corresponding to a quantum system that evolves in accordance to $U$ and is repeatedly measured with $\Pi$. Let $i \in I_k$. Taking into account the Born rule and the unitary evolution prior to the measurement, for $\rho \in S(\mathbb{C}^d)$ we define the probability of obtaining the outcome $i$ as

$$p_i(\rho) := \text{tr}(\Pi_i U\rho U^*),$$

and the evolution map $F_i$ is defined as the composition of the unitary channel (2.1) with the state transformation due to $\Pi$ described in (2.3), i.e.,

$$F_i(\rho) := \frac{\sqrt{\Pi_i} U\rho U^* \sqrt{\Pi_i}}{\text{tr}(\Pi_i U\rho U^*)},$$

provided that $p_i(\rho) > 0$. Clearly, $(S(\mathbb{C}^d), F_i, p_i)_{i \in I_k}$ is a PIFS. We shall denote it by $F_{U, \Pi}$ and refer to it as the PIFS generated by $U$ and $\Pi$. 


The following properties of $\mathcal{F}_{\tilde{U},\tilde{\Pi}}$ will come in handy later on. Let $i \in I_k$. First, we note that $\mathcal{F}_{\tilde{U},\tilde{\Pi}}$ preserves $\mathcal{P}(\mathbb{C}^d)$. Indeed, let $w$ be a unit vector in $\mathbb{C}^d$ and consider $\rho_w$. It follows that $p_i(\rho_w) = ||\sqrt{\Pi_i}Uw||^2$ and, provided that $\sqrt{\Pi_i}Uw \neq 0$, we obtain

$$F_i(\rho_w) = \rho\sqrt{\Pi_i}Uw.$$ 

(3.1)

Secondly, for $\rho \in \mathcal{S}(\mathbb{C}^d)$ we put

$$\Lambda_i^{U,\Pi}(\rho) := \sqrt{\Pi_i}U\rho U^*\sqrt{\Pi_i}$$

and observe that $p_i(\rho)$ and $F_i(\rho)$ can be recovered from the value of $\Lambda_i^{U,\Pi}(\rho)$ as, respectively, $\text{tr}(\Lambda_i^{U,\Pi}(\rho))$ and $\Lambda_i^{U,\Pi}(\rho)/\text{tr}(\Lambda_i^{U,\Pi}(\rho))$. Hence, it comes as no surprise that PIFS conjugacy can be expressed as follows:

**Observation 3.1.** Let $U, \tilde{U} \in \mathcal{U}(\mathbb{C}^d)$ and let $\Pi, \tilde{\Pi}$ be POVMs. Then $\mathcal{F}_{\tilde{U},\tilde{\Pi}}$ is $V$-conjugate to $\mathcal{F}_{U,\Pi}$ if and only if there exists $\sigma \in S_k$ such that

$$\Lambda_i^{U,\Pi} = \Lambda^V \circ \Lambda_i^{U,\Pi} \circ \Lambda^V$$

for every $i \in I_k$, where $V \in \mathcal{U}(\mathbb{C}^d) \cup \mathcal{U}(\mathbb{C}^d)$.

Since physical systems are invariant under (anti)unitary change of coordinates and under phase transformation, one expects both these operations to lead to PIFSs conjugate to the initial one.

**Proposition 3.2.** We have

(i) $\mathcal{F}_{e^{i\gamma}U,\Pi} \sim \mathcal{F}_{U,\Pi}$ for $\gamma \in \mathbb{R}$;

(ii) $\mathcal{F}_{VVV^*, \Pi VV^*} \sim \mathcal{F}_{U,\Pi}$ for $V \in \mathcal{U}(\mathbb{C}^d) \cup \mathcal{U}(\mathbb{C}^d)$.

**Proof.** (i) Let $\gamma \in \mathbb{R}$ and note that $\Lambda_j^{e^{i\gamma}U,\Pi} = \Lambda_j^{U,\Pi}$ for every $j \in I_k$, which, via Obs. 3.1, gives $\mathcal{F}_{e^{i\gamma}U,\Pi} \sim \mathcal{F}_{U,\Pi}$, as desired.

(ii) Let $V \in \mathcal{U}(\mathbb{C}^d) \cup \mathcal{U}(\mathbb{C}^d)$. Put $\tilde{U} := VVV^*$ and $\tilde{\Pi} := VV^*$, i.e., $\tilde{\Pi}_j = VV^*$ for $j \in I_k$. Since $\sqrt{\Pi_j}V^* = V\sqrt{\Pi_j}V^*$, for every $j \in I_k$ we obtain

$$\Lambda_j^{U,\Pi}(\rho) = \sqrt{\Pi_j}U\rho U^*\sqrt{\Pi_j}$$

$$= (V\sqrt{\Pi_j}V^*)(VV^*)(V\sqrt{\Pi_j}V^*)$$

$$= V\sqrt{\Pi_j}UV^*\rho U^*\sqrt{\Pi_j}V^*$$

$$= \Lambda^V(\Lambda_j^{U,\Pi}(\rho)).$$

It follows from Obs. 3.1 that $\mathcal{F}_{U,\Pi}$ is $V^*$-conjugate to $\mathcal{F}_{U,\Pi}$, which concludes the proof. \hfill \Box

From now on we fix the maximally mixed state $\rho_\ast := \mathbb{I}_d/d$ as the initial state of our system. The \textit{Markov chain generated by $\mathcal{F}_{U,\Pi}$} has the state space

$$S_\ast := \bigcup_{n \in \mathbb{N}} \bigcup_{i \in I_k} \{ F_i(\rho_\ast) : p_i(\rho_\ast) > 0 \},$$
its initial distribution \( \pi_\ast(\rho) : \rho \in S_\ast \) is given by

\[
\pi_\ast(\rho) := \begin{cases} p_i(\rho) & \text{if } \rho = F_i(\rho_\ast) \text{ for some } i \in I_k, \\ 0 & \text{otherwise,} \end{cases}
\]

and the transition matrix \( P_\ast \in \mathbb{R}^{S_\ast \times S_\ast} \) reads

\[
P_\ast(\rho, q) := \sum_{i \in I_k} p_i(\rho) \quad \text{for } \rho, q \in S_\ast.
\]

In what follows we discuss in more detail the Markov chain generated by \( F_{U, \Pi} \) when \( \Pi = \{ \Pi_1, \ldots, \Pi_k \} \) is a PVM. Clearly, the state space \( S_\ast \) is contained in the union of images of the evolution maps. We therefore let \( j \in I_k \) and discuss the domain and image of \( F_j \). We adopt the following notation. For \( \Theta \) being a non-trivial \( r \)-dimensional subspace of \( \mathbb{C}^d \) we put \( S(\Theta) := \{ \rho \in S(\mathbb{C}^d) : \text{im } \rho \subset \Theta \} \) and \( P(\Theta) := S(\Theta) \cap P(\mathbb{C}^d) \). Note that \( S(\Theta) = \text{conv } P(\Theta) \) and \( P(\Theta) \sim \mathbb{P}_\Theta \). Moreover, we can identify \( \rho \in S(\Theta) \) with \( \rho|_\Theta : \Theta \to \Theta \), so \( S(\Theta) \sim S(\mathbb{C}^r) \) as well as \( P(\Theta) \sim P(\mathbb{C}^r) \sim \mathbb{CP}^{r-1} \).

**Proposition 3.3.** We have \( \text{dom } F_j = S(\mathbb{C}^d) \setminus S(U^* \ker \Pi_j) \).

**Proof.** Let \( \rho \in S(\mathbb{C}^d) \). By spectral decomposition, there exist \( \mu_i \in [0, 1] \) and unit vectors \( w_i \in \mathbb{C}^d \), where \( i \in I_d \), such that \( \sum_{i=1}^d \mu_i = 1 \) and \( \rho = \sum_{i=1}^d \mu_i w_i w_i^* \). It follows that

\[
p_j(\rho) = \sum_{i=1}^d \mu_i \text{tr}(\Pi_j U w_i U^* w_i^*) = \sum_{i=1}^d \mu_i ||\Pi_j U w_i||^2.
\]

(3.2)

In consequence, \( p_j(\rho) = 0 \) iff for each \( i \in I_d \) we have \( w_i \in \ker(\Pi_j U) \) or \( \mu_i = 0 \). Since \( \ker(\Pi_j U) = U^* \ker \Pi_j \), we conclude that \( \rho \notin \text{dom } F_j \) iff \( \rho|_\Theta \in P(U^* \ker \Pi_j) \) for each \( i \in I_d \) such that \( \mu_i \neq 0 \), which is equivalent to \( \rho \in \text{conv } P(U^* \ker \Pi_j) = S(U^* \ker \Pi_j) \), as desired. \( \square \)

**Proposition 3.4.** We have \( \text{im } F_j = S(\text{im } \Pi_j) \) and \( \text{im } (F_j|_{P(\mathbb{C}^d)}) = P(\text{im } \Pi_j) \).

**Proof.** Since \( \text{im } (F_j|_{P(\mathbb{C}^d)}) \subset P(\text{im } \Pi_j) \), see (3.1), it follows easily (via spectral decomposition) that \( \text{im } F_j \subset S(\text{im } \Pi_j) \). To prove that the converse inclusions also hold, we first show that \( F_j(\rho) = U\rho U^* \) if \( \rho \in S(U^* \text{im } \Pi_j) \), i.e., the action of \( F_j \) on \( S(U^* \text{im } \Pi_j) \) coincides with that of the unitary channel \( \Lambda^U \). Let \( \rho \in S(U^* \text{im } \Pi_j) \). Clearly, \( \rho \in \text{dom } F_j \). Put \( r_j := \text{rank } \Pi_j \). By spectral decomposition, \( \rho = \sum_{i=1}^{r_j} \mu_i w_i w_i^* \) with unit vectors \( w_i \in U^* \text{im } \Pi_j \) and \( \mu_i \in [0, 1] \) such that \( \sum_{i=1}^{r_j} \mu_i = 1 \). For each \( i \in I_j \) we have \( U w_i \in \text{im } \Pi_j \), so \( \Pi_j U w_i U^* = U w_i \). Hence, \( p_j(\rho) = \sum_{i=1}^{r_j} \mu_i ||U w_i||^2 = 1 \), see (3.2), and so \( F_j(\rho) = \sum_{i=1}^{r_j} \mu_i U w_i U^* = U \rho U^* \), as desired. It remains to observe that \( S(U^* \text{im } \Pi_j) \ni \rho \mapsto F_j(\rho) = U \rho U^* \in S(\text{im } \Pi_j) \) is bijective and both it and its inverse preserve pure states. Thus, \( S(\text{im } \Pi_j) = S(F_j|_{S(U^* \text{im } \Pi_j)}) \subset \text{im } F_j \) and \( P(\text{im } \Pi_j) = P(F_j|_{P(U^* \text{im } \Pi_j)}) \subset P(F_j|_{P(\mathbb{C}^d)}) \), which concludes the proof. \( \square \)

Hence, as \( \text{im } \Pi_j \sim \mathbb{C}^r \), we have \( \text{im } F_j \sim S(\mathbb{C}^r) \), where \( r_j := \text{rank } \Pi_j \) and \( j \in I_k \). Moreover, \( \text{im } \Pi_1, \ldots, \text{im } \Pi_k \) are pairwise orthogonal, so \( \text{im } F_1, \ldots, \text{im } F_k \) are pairwise disjoint. Therefore, \( S_\ast \) is a countable subset of the union of pairwise disjoint Bloch bodies whose dimensions are determined by the ranks of the projections constituting \( \Pi \). For instance, a three-dimensional quantum system (qutrit) measured with a PVM consisting of one rank-2
projection and one rank-1 projection can be visualized as the union of a Bloch ball and a point.

As for the initial distribution $\pi_*$ of this chain, in the first step of the evolution we obtain

$$p_j(\rho_*) = \text{tr}(\Pi_j \rho_*) = r_j/d \quad \text{and} \quad F_j(\rho_*) = \Pi_j/r_j \sim I_{r_j}/r_j,$$

where again $r_j := \text{rank } \Pi_j$ for $j \in I_k$. It follows that $\pi_*(\rho) = r_i/d$ if $\rho = \Pi_i/r_i$ for some $i \in I_k$, and $\pi_*(\rho) = 0$ otherwise. That is, a projective measurement performed on the maximally mixed state yields the locally maximally mixed state of the subsystem corresponding to the measurement result with probability proportional to the dimension of this subsystem.

4. BALL & POINT QUTRIT SYSTEM

4.1. Classification of Markov chains. Fix $U \in \mathcal{U}(\mathbb{C}^3)$ and a PVM $\Pi = \{\Pi_1, \Pi_2\}$ such that $\text{rank } \Pi_1 = 2$, $\text{rank } \Pi_2 = 1$. In this section, we classify the Markov chains that $\mathcal{F}_{U, \Pi}$ can generate on $\mathcal{S}(\mathbb{C}^3)$.

Let $z$ stand for a unit vector from $\mathbb{C}^3$ such that $\text{im } \Pi_2 = \text{span } \{z\}$, i.e., $\text{im } F_2 = \{\rho_2\}$. Clearly, we have $\Pi_1 = I_3 - \rho_z$, so $\Pi = \Pi(z)$ is fully determined by $[z] \in \mathbb{C}P^2$. Putting $\Theta := \text{span } \{z\}$, we see that $\Theta = \text{im } \Pi_1 = \ker \Pi_2$ and $\text{span } \{z\} = \text{im } \Pi_2 = \ker \Pi_1$; hence, Propositions 3.3 & 3.4 give $\text{dom } F_1 = \mathcal{S}(\mathbb{C}^3) \setminus \{\rho_{U^*z}\}$ and $\text{dom } F_2 = \mathcal{S}(\mathbb{C}^3) \setminus \text{S}(U^* \Theta)$, as well as $\text{im } F_1 = \mathcal{S}(\Theta) \sim \mathcal{S}(\mathbb{C}^2)$ and $\text{im } F_2 = \{\rho_2\}$. Thus, the dynamics of the system takes place on the union of the Bloch ball and the point representing $\text{im } F_1$ and $\text{im } F_2$, respectively.

We put $\rho_m$ for $\frac{1}{2} \Pi_1 \sim \frac{1}{2} I_2$, i.e., $\rho_m$ is the maximally mixed state of $\mathcal{S}(\Theta)$, and so it occupies the centre of the Bloch ball representing $\mathcal{S}(\Theta)$. In the first step of the evolution of this system we obtain $p_1(\rho_*) = \frac{2}{3}$ and $p_2(\rho_*) = \frac{1}{3}$, while the post-measurement states read $F_1(\rho_*) = \rho_m$ and $F_2(\rho_*) = \rho_2$, see (3.3) and Fig. 4.1. Consequently, the initial distribution of the chain generated by $\mathcal{F}_{U, \Pi}$ reads $\pi_* = \frac{2}{3} \delta_{\rho_m} + \frac{1}{3} \delta_{\rho_2}$, where $\delta_{\rho}$ is the Dirac-delta measure at $\rho \in \mathcal{S}(\mathbb{C}^d)$.

![Fig. 4.1. First step of the evolution of the ball & point system](image)

In what follows, we show that the state space of this chain has a fairly simple structure; namely, it consists of the states generated by iterating $F_1$ on $\rho_2$ and on $\rho_m$. Indeed, we obviously have $\{F_1(\rho_*), F_2(\rho_*)\} = \{\rho_m, \rho_2\}$; let us therefore consider $\iota = (i_1, \ldots, i_n) \in I_2^n$ such that $p_1(\rho_*) > 0$, where $n \geq 2$. There are three cases to be considered, depending on the presence and position of ‘2’ in $\iota$. Firstly, if $i_n = 2$, then $F_1(\rho_*) = F_2 \circ F(\iota_{1, \ldots, i_{n-1}})(\rho_*) = \rho_2$. Secondly, if there exists $j \in I_{n-1}$ such that $i_j = 2$ and $i_l = 1$ for every $l \in \{j + 1, \ldots, n\}$, i.e.,
we have \( t = (i_1, \ldots, i_{j-1}, 2, 1, \ldots, 1) \), then

\[
F_i(\rho_s) = F_{(1,\ldots,1)} \circ F_2 \circ F_{(i_1,\ldots,i_{j-1})}(\rho_s) = F_{1^{n-j}}(\rho_2),
\]

where \( 1^r \) denotes the string consisting of \( r \) consecutive 1’s, i.e., \( 1^r := (1, \ldots, 1) \in I_2^r, r \in \mathbb{N} \).

The last case is that of \( \iota = 1^n \), which gives \( F_i(\rho_s) = F_{1^{n-1}}(\rho_m) \). It follows that

\[
S_s = \{ \rho_2, \rho_m \} \cup \{ F_1^r(\rho_2) : p_1^r(\rho_2) > 0, r \in \mathbb{N} \} \cup \{ F_1^r(\rho_m) : p_1^r(\rho_m) > 0, r \in \mathbb{N} \}.
\]

Let \( n(\rho) \) denote the maximum number of iterations of \( F_1 \) that can be applied to \( \rho \in \mathcal{S}(\mathbb{C}^d) \). If there exists \( s \in \mathbb{N} \) such that \( p_1^s(\rho) = 0 \), then \( p_1^q(\rho) = 0 \) for every \( q > s \). In this case we set \( n(\rho) := \max\{ r \in \mathbb{N} : p_1^r(\rho) > 0 \} \); otherwise, we set \( n(\rho) := \infty \). We obtain

\[
S_s = \{ F_1^n(\rho_2) : n = 0, \ldots, n(\rho_2) \} \cup \{ F_1^n(\rho_m) : n = 0, \ldots, n(\rho_m) \}.
\] (4.1)

To further investigate \( S_s \), it is convenient to consider separately the case of \( n(\rho_2) = 0 \), i.e., when \( p_1(\rho_2) = 0 \), and so the Bloch ball cannot be accessed from \( \rho_2 \). Note that if the system occupies \( \rho_2 \), then it goes to the ball with probability \( p_1(\rho_2) = ||\Pi_1 Uz||^2 \) or stays in \( \rho_2 \) with complementary probability \( p_2(\rho_2) = ||\Pi_2 Uz||^2 = |z|Uz\|^2 \). We put \( \omega := (z|Uz) \). By \( \sigma(A) \) we denote the set of eigenvalues of a linear map \( A \).

**Proposition 4.1.** The following conditions are equivalent:

(i) \( n(\rho_2) = 0 \);

(ii) \( |\omega| = 1 \);

(iii) \( z \) is an eigenvector of \( U \);

(iv) \( \Pi_1 U|_{\Theta} \) is unitary;

(v) \( \omega \in \sigma(U) \).

**Proof.**

(i) \( \Leftrightarrow \) (ii) \( n(\rho_2) = 0 \) iff \( p_1(\rho_2) = 0 \) iff \( p_2(\rho_2) = |\omega|^2 = 1 \).

(i) \( \Leftrightarrow \) (iii) \( n(\rho_2) = 0 \) iff \( p_1(\rho_2) = 0 \) iff \( \rho_2 \notin \text{dom} F_1 = \mathcal{S}(\mathbb{C}^3) \setminus \{ \rho U^*z \} \) iff \( \rho_2 = \rho U^*z \) iff \( U^*z \in \text{span}\{z\} \) iff \( z \) is an eigenvector of \( U \).

(iii) \( \Leftrightarrow \) (iv) Since \( U \) has an orthonormal eigenbasis and \( \Theta = \text{im} \Pi_1 \) is the orthogonal complement of \( \text{span}\{z\} \), it follows that \( z \) is an eigenvector of \( U \) iff \( \Theta \) is spanned by two eigenvectors of \( U \) iff \( \Theta \) is \( U \)-invariant, which in turn equivalent to \( \Pi_1 U|_{\Theta} = U|_{\Theta} \). This implies the unitarity of \( \Pi_1 U|_{\Theta} \). We now show that the converse implication also holds. Assume that \( \Pi_1 U|_{\Theta} \) is unitary and let \( w \in \Theta \). It follows that \( ||\Pi_1 Uw|| = ||w|| = ||Uw|| \), thus also \( ||\Pi_2 Uw|| = 0 \). Hence, \( Uw \in \text{ker} \Pi_2 = \Theta \). Since \( \Pi_1 \) acts on \( \Theta \) as the identity, we have \( \Pi_1 Uw = Uw \). Therefore, \( \Pi_1 U|_{\Theta} = U|_{\Theta} \), as desired.

(iii) \( \Rightarrow \) (v) Let \( \mu \in \mathbb{C} \) be an eigenvalue of \( U \) associated with \( z \). Clearly, \( Uz = \mu z \) implies that \( \omega = (z|Uz) = \mu \in \sigma(U) \), as desired.

(v) \( \Rightarrow \) (ii) Every eigenvalue of \( U \) is of unit modulus. \( \square \)
• **Case of** \( n(\rho_2) = 0 \). It follows from Prop. 4.1 that both \( \text{span}\{z\} \) and its orthogonal complement \( \Theta \) are invariant subspaces of \( U \). Thus, intuitively, there is no interaction between these two parts of the system as they are subject to separate unitary dynamics (one of which is trivial) induced by the suitable restrictions of \( U \). In consequence, the first measurement causes the system to get trapped into one of these subsystems. Furthermore, unitary dynamics on \( S(\Theta) \) corresponds to a rotation of the Bloch ball, see [19, Ch. 3, Sec. 5], [30, Example 2.51]. Obviously, every rotation fixes the centre of the ball. Hence, if the first measurement sends the system to the ball, it arrives at a fixed point \( \rho_m \) and loops there infinitely, see Fig. 4.2.

Formally, in addition to \( p_2(\rho_2) = 1 \) and \( n(\rho_2) = 0 \), we have \( p_1(\rho_m) = 1 \) along with \( \Pi_1(\rho_m) = \rho_m \) and \( n(\rho_m) = \infty \), because \( \Pi_1 U|_\Theta = U|_\Theta \) gives \( A_1^U \Pi_1(\rho_m) = \Pi_1 U \rho_m U^* \Pi_1 = U \rho_m U^* = \rho_m \). We conclude that \( S_* = \{ \rho_2, \rho_m \} \) and \( P_* = \mathbb{I}_{S_*} \).

**Fig. 4.2.** Dynamics of the ball & point system in the case of \( n(\rho_2) = 0 \)

• **Case of** \( n(\rho_2) \geq 1 \). From Prop. 4.1 we have \( Uz \notin \text{span}\{z\} \), and so \( \Pi_1 Uz \neq 0 \). Putting \( v := \Pi_1 Uz \), we see that \( \rho_v = F_1(\rho_2) \) is the pure state in \( S(\Theta) \) at which the system arrives from \( \rho_2 \) with probability \( p_1(\rho_2) = ||v||^2 > 0 \). Thus, (4.1) takes the form

\[
S_* = \{ \rho_2 \} \cup \{ F_1^n(\rho_v) : n = 0, \ldots, n(\rho_v) \} \cup \{ F_1^n(\rho_m) : n = 0, \ldots, n(\rho_m) \},
\]

see also Fig. 4.3. Clearly, if the result of the first measurement is ‘1’, then the system embarks on the trajectory of \( \rho_m \) under \( F_1 \); otherwise, it goes to \( \rho_2 \), from which the trajectory of \( \rho_v \) can be accessed. Generically, in each step the system can either follow the trajectory in the Bloch ball or go to \( \rho_2 \). (In Prop. 4.21 we show that there exists exactly one state in \( S(\Theta) \) from which the system cannot go to \( \rho_2 \) and this state is pure.) Note that after the first visit in \( \rho_2 \) the system can explore the ball only along the trajectory of \( \rho_v \); in particular, the trajectory of \( \rho_m \), once left, cannot be re-entered.

In order to establish \( S_* \), we need to determine the trajectories of \( \rho_m \) and \( \rho_v \) under \( F_1 \). To this end, we need more insight into how \( F_1 \) acts on \( S(\Theta) \). Recall that

\[
F_1 \mid_{S(\Theta)} : \quad S(\Theta) \setminus \{ \rho_Uz \} \ni \rho \mapsto \frac{A \rho A^*}{\text{tr}(A \rho A^*)} \in S(\Theta),
\]

where \( A := \Pi_1 U|_\Theta \). Note that \( F_1 \) is defined on the whole of \( S(\Theta) \) iff \( \rho_Uz \notin S(\Theta) \) iff \( U^*z \notin \Theta \) iff \( \omega \neq 0 \). Moreover, \( F_1 \mid_{S(\Theta)} \) is invertible iff \( A \) is invertible, and its inverse then reads

\[
\rho \mapsto \frac{A^{-1} \rho (A^{-1})^*}{\text{tr}(A^{-1} \rho (A^{-1})^*)}. \quad \text{We have already pointed out that} \quad F_1 \text{ preserves pure states, see (3.1), and now we can easily see that so does its inverse.}
\]
Proposition 4.2. $A$ is invertible iff $\omega \neq 0$

Proof. We show that $0 \in \sigma(A)$ iff $\omega = 0$. Assume $0 \in \sigma(A)$ and let $v \in \Theta$ be an eigenvector of $A$ associated with null eigenvalue. It follows that $Uv \in \ker \Pi_1 = \text{span}\{z\}$, and so $U^*z \in \Theta$. We get $\omega = \langle U^*z|z \rangle = 0$, as desired. Conversely, $\omega = 0$ gives $U^*z \in \Theta$; hence, $\Pi_1 U(U^*z) = 0$, and so $A$ has a null eigenvalue, which concludes the proof. \hfill $\Box$

Corollary 4.3. $F_1|_{S(\Theta)}$ and $F_1|_{P(\Theta)}$ are bijections iff $\omega \neq 0$.

Corollary 4.4. If $\omega \neq 0$, then $n(\rho_v) = n(\rho_m) = \infty$.

The following three steps lead to the characterization of the trajectories of $\rho_v$ and $\rho_m$ under $F_1$, and then to the classification of the Markov chains that can be generated by $F_{U,\Pi}$ in the case of $A$ being non-unitary. First, we establish some further properties of $A$. Next, we discuss the types of dynamics that $F_1$ can induce on the Bloch ball, and for each of these types we describe its generic trajectories and singular points. Finally, for each type we decide whether $\rho_v$ or $\rho_m$ is a generic or singular point of $F_1$.

*Step I. Properties of $A$.* The following considerations include the case of $A$ being unitary. Recall that $\omega := \langle z|Uz \rangle$ and $v := \Pi_1 Uz$. Let $u_1, u_2$ be any two vectors that span $\Theta$. Fixing $\{u_1, u_2, z\}$ as the basis of $\mathbb{C}^3$, we obtain the matrix representation

$$U \sim \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ * & * & \omega \end{bmatrix},$$

(4.3)
where $a_{ij}, v_j \in \mathbb{C}$ with $i, j \in \{1, 2\}$. Observe that $[a_{11} \ a_{12}]$ and $[v_1, v_2]^T$ represent $A$ and $v$, respectively, in the basis $\{u_1, u_2\}$ of $\Theta$.

Obviously, the eigenvalues of $A$ depend only on $tr A$ and $det A$. Moreover, we have

$$tr A = tr U - \omega \quad \text{and} \quad det A = \varpi det U. \quad (4.4)$$

Indeed, the trace formula is obvious by (4.3), while that for the determinant can be deduced from the following

**Theorem.** [58, Thm. 2.3] Let $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$ be a complex block matrix, where $V_1, V_4$ are square matrices. If $V$ is invertible and $V^{-1} = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$ is partitioned conformally to $V$, then $det V_1 = det V_4 det V$.

If $V$ in the above theorem is unitary, then $V^{-1} = V^*$, so $det V_1 = \overline{det V_4} det V$. This particular result is due to Muir [44, p. 339], see also [48]. In our situation the submatrix $V_4$ is of size $1 \times 1$, in which case the proof simplifies further, see [58, p. 172].

The next theorem allows us to determine the set $s(A)$ of singular values of $A$.

**Theorem.** [58, Thm. 6.3] Let $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$, where $V_1 \in \mathbb{C}^{n \times n}$ and $V_4 \in \mathbb{C}^{m \times m}$, be a unitary block matrix. If $m = n$, then $V_1$ and $V_4$ have the same singular values. If $n > m$ and the singular values of $V_4$ read $s_1, \ldots, s_m$, then the singular values of $V_1$ read $s_1, \ldots, s_m, 1, \ldots, 1$.

Applying the above theorem to $U$ with $n = 2$ and $m = 1$, we obtain $s(A) = \{||\omega||, 1\}$.

**Remark 4.6.** Note that $\sigma(A)$ and $s(A)$ depend on $z \in \mathbb{C}^3$ only via $\omega = (z|Uz) \in \mathbb{C}$. We further investigate this fact in Subsection 4.2.

Clearly, $\max\{||\lambda_1||, ||\lambda_2||\} \leq ||A|| = \max s(A) = 1$. Further, from the determinant formula in (4.4) we obtain $||\lambda_1\lambda_2|| = ||\omega||$, and so $||\omega|| = 1$ iff $||\lambda_1|| = ||\lambda_2|| = 1$. Thus, via Prop. 4.1 we get

**Proposition 4.7.** $A$ is unitary iff $||\lambda_1|| = ||\lambda_2|| = 1$.

In what follows we consider a Schur form of $A$. (Recall that a Schur form of $V \in \mathbb{L}(\mathbb{C}^d)$ is an upper triangular matrix which represents $V$ in an orthonormal basis comprising an eigenvector of $V$ and which has the eigenvalues of $V$ as diagonal entries.) Let us put $e_1$ for a normalised eigenvector of $A$ associated with $\lambda_1$ and let $w \in \Theta$ be a unit vector orthogonal to $e_1$. With respect to the orthonormal basis $\{e_1, w\}$ of $\Theta$ we have

$$A \sim \begin{bmatrix} \lambda_1 & x \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad A^* \sim \begin{bmatrix} \overline{\lambda_1} & 0 \\ w & \overline{\lambda_2} \end{bmatrix}, \quad (4.5)$$

where $x := \langle e_1|Aw \rangle$. Obviously, $w$ is an eigenvector of $A$ iff $x = 0$. From (4.5) we obtain

$$tr(A^*A) = ||\lambda_1||^2 + ||\lambda_2||^2 + |x|^2. \quad (4.6)$$

On the other hand, the singular values of $A$ are the square roots of the eigenvalues of $A^*A$ and $s(A) = \{||\lambda_1\lambda_2||, 1\}$. Thus,

$$tr(A^*A) = ||\lambda_1\lambda_2||^2 + 1. \quad (4.7)$$

Combining (4.6) & (4.7), we get $||\lambda_1||^2 + ||\lambda_2||^2 + |x|^2 = 1 + ||\lambda_1\lambda_2||^2$. It follows that

$$|x|^2 = (1 - ||\lambda_1||^2)(1 - ||\lambda_2||^2), \quad (4.8)$$

and so $x = 0$ iff $\max\{||\lambda_1||, ||\lambda_2||\} = 1$. As a result, we obtain
**Proposition 4.8.** A is normal (i.e., $A$ has an orthonormal eigenbasis) iff $\max\{|\lambda_1|, |\lambda_2|\} = 1$.

Next, we briefly consider the case of $A$ having a double eigenvalue. Namely, if $\lambda_1 = \lambda_2$ and $A$ is not defective (i.e., it has two linearly independent eigenvectors), then $A = \lambda_1 I_\Theta$ and every orthonormal basis of $\Theta$ constitutes an orthonormal eigenbasis of $A$. With the help of Prop. 4.7 & Prop. 4.8 we obtain

**Proposition 4.9.** If $\sigma(A) = \{\lambda_1\}$, then $A$ is defective iff $|\lambda_1| < 1$. Otherwise, $A$ is unitary.

**Step II. Ball dynamics.** Let $[v]$ stand for the equivalence class of $v \in \mathbb{C}^d \setminus \{0\}$ in $\mathbb{CP}^{d-1}$. Clearly, spectral decomposition theorem and the linearity of $\Lambda_{\Sigma, \Pi}$ allow us to deduce the dynamics generated by $F_1$ on $S(\Theta)$, see (4.2), from the action of $F_1$ on $P(\Theta)$. Using (3.1), we can translate $F_1|_{P(\Theta)}$ into the following (partial) map

$$A : P\Theta \ni [w] \mapsto [Aw] \in P\Theta \quad \text{if} \quad Aw \neq 0.$$  

In what follows we first briefly discuss the dynamics induced by $A$ on $P\Theta \sim \mathbb{CP}^1$, and then analyse how it extends to $S(\Theta)$.

Recall that the bijection $\mathbb{CP}^{d-1} \ni [z] \mapsto [Vz] \in \mathbb{CP}^{d-1}$ induced by a non-singular map $V \in \mathcal{L}(\mathbb{C}^d)$ is called a homography. Hence, if $0 \notin \sigma(A)$, then $A$ is a homography on $P\Theta \sim \mathbb{CP}^1$. Homographies on $\mathbb{CP}^1$ are isomorphic to the group of Möbius maps, which are orientation preserving conformal automorphisms of the Riemann sphere: the homography $[z] \mapsto [Wz]$ on $\mathbb{CP}^1$ induced by a non-singular matrix $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to the Möbius transformation $\hat{C} \ni z \mapsto \frac{az + b}{cz + d} \in \hat{C}$, see [1, 4, 7, 33, 46]. Based on the number and type of fixed points, non-identity Möbius maps are classified into three types (see also Fig. 4.4):

- **loxodromic**: two fixed points, one of which is attractive and the other repulsive;
- **elliptic**: two fixed points and both are neutral (i.e., neither attractive nor repulsive);
- **parabolic**: one fixed point, which is attractive (and unstable).

Now we examine how the dynamics induced by $A$ on $\mathbb{CP}^1$ depends on the eigenvalues of $A$. Let us adopt the following notation. If $A$ is not defective, then we let $e_j$ stand for a normalised eigenvector of $A$ associated with $\lambda_j$ ($j = 1, 2$). Here and henceforth, with no loss of generality, we assume that $|\lambda_2| \leq |\lambda_1|$ and put $\psi := \text{Arg} \frac{\lambda_2}{\lambda_1}$, provided that $0 \notin \sigma(A)$.

![Fig. 4.4. Möbius maps on the Riemann sphere](image-url)
Clearly, the eigenbasis \( \{e_1, e_2\} \) of \( A \) is unique (up to phases) and with respect to it we have \( A \sim \text{diag}(\lambda_1, \lambda_2) \). If \( A \) is defective, then \( e_1 \) will stand for a normalised eigenvector of \( A \) and \( g_1 \) for a unit vector orthogonal to \( e_1 \). It follows that \( g_1 \) is a generalised eigenvector of \( A \) since \( \ker(A - \lambda_1 I_2)^2 = \mathbb{C}^2 \). The orthonormal generalised eigenbasis \( \{e_1, g_1\} \) of \( A \) is unique (up to phases) and with respect to it we have \( A \sim \begin{bmatrix} \lambda_1 & x \\ 0 & \lambda_2 \end{bmatrix} \), where \( x = \langle e_1 | A g_1 \rangle \) satisfies \( |x| = 1 - |\lambda_1|^2 \in (0, 1) \), see (4.8) and Prop. 4.9.

Note that the fixed points of \( A \) coincide with the eigenspaces of \( A \) that are associated with non-zero eigenvalues, and \( A \) is undefined on the eigenspace of \( A \) associated with null eigenvalue, provided that \( 0 \notin \sigma(A) \). We exclude the case of \( A \) generating trivial (identity) dynamics, which happens when \( A \sim \lambda_1 I_2 \) with \( \lambda_1 \neq 0 \). This is equivalent to \( \sigma(A) = \{\lambda_1\} \) and \( |\lambda_1| = 1 \), see Prop. 4.9, and \( A \) is then necessarily unitary. There are three cases to be considered:

(G) the generic case of \( A \) being non-trivial, non-singular, and non-defective, i.e., \( 0 \notin \sigma(A) \) and \( \lambda_1 \neq \lambda_2 \);

(D) the case of \( A \) being defective and non-singular, i.e., \( \sigma(A) = \{\lambda_1\} \) and \( 0 < |\lambda_1| < 1 \);

(S) the case of \( A \) being singular, i.e., \( 0 \in \sigma(A) \).

First, we examine Cases (G) and (D), while Case (S) will be investigated separately later on (see p. 16). Let \( n, m \in \mathbb{N}, m \neq n \).

(G) \( A \) has two fixed points \([e_1], [e_2]\). For \( w = c_1 e_1 + c_2 e_2 \), where \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \), we have

\[
A^nw = c_1 \lambda_1^n e_1 + c_2 \lambda_2^n e_2,
\]

which gives

\[
A^n[w] = c_1[e_1] + c_2 [\frac{\lambda_2}{\lambda_1}]^n \exp(i\pi/n)[e_2],
\]

so we obtain the following subcases:

(G1) if \( |\lambda_1| > |\lambda_2| \), then \( A^m[w] \neq A^n[w] \) and \( A^n[w] \xrightarrow{n \to \infty} [e_1] \); thus, \([e_1]\) is attractive. The dynamics generated by \( A \) is loxodromic;

(G2) if \( |\lambda_1| = |\lambda_2| \), then both fixed points \([e_1]\) and \([e_2]\) of \( A \) are neutral and the dynamics generated by \( A \) is elliptic. Every non-fixed point is almost periodic\(^2\) and it is periodic iff \( \psi \) is commensurable with \( \pi \). Hence, we distinguish two further subcases:

(G2i) if \( \psi \notin \mathbb{Q}\pi \), then \( A^m[w] \neq A^n[w] \) and \( \{A^n[w] : n \in \mathbb{N}\} \) is a dense subset of \( \{c_1[e_1] + c_2 e^{i\gamma} [e_2] : \gamma \in \mathbb{R}\} \);

(G2ii) if \( \psi \in \mathbb{Q}\pi \setminus \{0\} \), then \( A^m[w] = A^n[w] \) iff \( m = n \mod \kappa \), where \( \kappa \) stands for the smallest natural number \( k \geq 2 \) such that \( \exp(i\pi) \) is a \( k \)-th root of unity.

(That is, \( \exp(i\pi) \) is a primitive \( \kappa \)-th root of unity.) In the particular case of \( \kappa = 2 \), i.e., \( \psi = \pi \), the dynamics generated by \( A \) is called circular.

\(^2\)Recall that in the case of compact metric spaces, a point is almost periodic iff it is Birkhoff-recurrent iff the closure of its trajectory is a minimal set of the dynamical system under consideration, see [25, p. 38]. This result goes back to Birkhoff [10, pp. 198-199], see also [52, pp. 71, 129] or [53, pp. 169-170], as well as [28, p. 30] for historical background.
In the eigenbasis of $A$ we have

$$A^n w = (c_1 \lambda_1^n + c_2 n x \lambda_1^{n-1}) e_1 + c_2 \lambda_1^n g_1$$

for $w = c_1 e_1 + c_2 g_1$, where $c_1, c_2 \in \mathbb{C}$ and $c_2 \neq 0$. It follows that

$$A^n[w] = \left(\frac{1}{n} + \frac{c_2}{\lambda_1}\right)[e_1] + \frac{c_2}{n} [g_1],$$

so $A^m[w] \neq A^n[w]$ and $A^n[w] \xrightarrow{n \to \infty} [e_1]$.

Let us analyse how the action of $A$ extends to the whole of the Bloch ball. As one can expect, we shall see that upon extending loxodromic and parabolic Möbius maps we do not obtain any fixed points in the interior of the ball and the trajectories of all non-fixed points converge to the sole attractive fixed point on the Bloch sphere. An elliptic Möbius map extends to a dynamical system that has two neutral fixed points on the sphere, and the convex hull of these two states constitutes the set of fixed points of the system, all of which are neutral, while non-fixed points are almost periodic.

Let $\rho \in S(\Theta) \setminus P(\Theta)$. It is easy to show that there exists a unit vector $\hat{w} \in \Theta$ such that $\rho = \alpha \rho_{e_1} + (1 - \alpha) \rho_{\hat{w}}$ with some $\alpha \in (0, 1)$. Let $n, m \in \mathbb{N}$ be such that $m \neq n$. It follows that

$$\operatorname{tr}(A^n \rho A^m) = \alpha |\lambda_1|^{2n} + (1 - \alpha) |A^n \hat{w}|^{2n} > 0;$$

hence, putting $r_n := \|A^n \hat{w}\|^{2n}/|\lambda_1|^{2n}$, we obtain

$$F_1^n(\rho) = \frac{A^n \rho A^m}{\operatorname{tr}(A^n \rho A^m)} = \frac{\alpha}{\alpha + (1 - \alpha) r_n} \rho_{e_1} + \frac{(1 - \alpha) r_n}{\alpha + (1 - \alpha) r_n} \rho_{A^n \hat{w}}.$$

(G) In the eigenbasis of $A$ we have $\hat{w} = c_1 e_1 + c_2 e_2$ with some $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, and so

$$\|A^n \hat{w}\|^2 = |c_1|^2 |\lambda_1|^{2n} + |c_2|^2 |\lambda_2|^{2n} + 2 |\lambda_1 \lambda_2|^n \Re(\overline{c_1} c_2 \exp(iv/n) \langle e_1 | e_2 \rangle).$$

Observe that $\rho_{\hat{w}}$ is a fixed point of $F_1$ iff $c_1 = 0$, in which case we have $\rho_{\hat{w}} = \rho_{e_2}$.

(G1) If $|\lambda_1| > |\lambda_2|$, then $F_1^n(\rho) \xrightarrow{n \to \infty} \rho_{e_1}$ and $F_1^m(\rho) \neq F_1^n(\rho)$, because

- if $c_1 \neq 0$, then $\rho_{A^n \hat{w}} \xrightarrow{n \to \infty} \rho_{e_1}$ and $\rho_{A^n \hat{w}} \neq \rho_{A^n \hat{w}}$;
- if $c_1 = 0$, then $r_n = |c_2|^2 |\lambda_2/\lambda_1|^{2n}$, so $r_n \xrightarrow{n \to \infty} 0$ and $r_m \neq r_n$.

(G2) If $|\lambda_1| = |\lambda_2|$, then $r_n = |c_1|^2 + |c_2|^2 + 2 \Re(\overline{c_1} c_2 \exp(iv/n) \langle e_1 | e_2 \rangle)$. It follows that

- if $c_1 = 0$, then $\rho$ is a fixed point of $F_1$, because $\rho_{A^n \hat{w}} = \rho_{e_2}$ and $r_n = |c_2|^2 = 1$, thus also $F_1^n(\rho) = \rho$;
- if $c_1 \neq 0$, then $\rho$ is almost periodic (because so is $\rho_{\hat{w}}$) and its periodicity depends on the commensurability of $\psi$ with $\pi$:

  (G2i) if $\psi \notin \mathbb{Q} \pi$, then $\rho$ is almost periodic but not periodic. In particular, $F_1^n(\rho) \neq F_1^n(\rho)$ and $(F_1^n(\rho))_{n \in \mathbb{N}}$ is non-convergent;

  (G2ii) if $\psi \in \mathbb{Q} \pi \setminus \{0\}$ and $\psi$ is a primitive $\kappa$-th root of unity, then $\rho$ is periodic of period $\kappa$.

(D) We have $F_1^n(\rho) \neq F_1^n(\rho)$ and $F_1^n(\rho_{e_1}) \xrightarrow{n \to \infty} \rho_{e_1}$, because $\rho_{A^n \hat{w}} \neq \rho_{A^n \hat{w}}$ and $\rho_{A^n \hat{w}} \xrightarrow{n \to \infty} \rho_{e_1}$. 

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Observation 4.10. Let \( \rho \in S(\Theta) \setminus P(\Theta) \). In Cases (G) and (D), \( \rho \) is a fixed point of \( F_1 \) iff \( |\lambda_1| = |\lambda_2| \) and \( \rho = \alpha \rho_{e_1} + (1 - \alpha)\rho_{e_2} \) with some \( \alpha \in (0, 1) \), i.e., \( \rho \in \text{conv}\{\rho_{e_1}, \rho_{e_2}\} \). The matrix representation of \( \rho \) with respect to the eigenbasis \( \{e_1, e_2\} \) of \( A \) reads

\[
\begin{bmatrix}
\alpha & \alpha \langle e_1 | e_2 \rangle \\
(1 - \alpha) \langle e_2 | e_1 \rangle & 1 - \alpha
\end{bmatrix}
\]

Observation 4.11. Generic trajectories in Cases (G1) and (D) are qualitatively the same, i.e., infinite and convergent to the sole attractive point \( \rho_{e_1} \) of the system, even though the overall dynamics on the ball is different.

It remains to investigate the case of \( A \) being singular, i.e., Case (S). Recall that \( A \) is then undefined at exactly one point in \( P(\Theta) \) which can be identified with the eigenspace of \( A \) associated with null eigenvalue. There are two subcases, corresponding to the multiplicity of zero as the eigenvalue of \( A \):

(S1) if \( \lambda_1 \neq \lambda_2 = 0 \), i.e., \( A \sim \text{diag}(\lambda_1, 0) \), then for every \( w = c_1 e_1 + c_2 e_2 \) \( (c_1, c_2 \in \mathbb{C}) \) we obtain \( Aw = c_1 e_1 \). Consequently, \( A[e_1] = [e_1] \), \( A \) is undefined at \([e_2]\), and \( A[w] = [e_1] \) for every \([w] \neq [e_1], [e_2]\);

(S2) if \( \lambda_1 = \lambda_2 = 0 \), i.e., \( A \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), then \( Ae_1 = 0 \) and \( Ag_1 = e_1 \), so for every \( w = c_1 e_1 + c_2 g_1 \) \( (c_1, c_2 \in \mathbb{C}) \) we obtain \( Aw = c_2 e_1 \). It follows that \( A \) has no fixed points, \( A \) is undefined at \([e_1]\), and if \([w] \neq [e_1]\), then \( A[w] = [e_1] \).

Extending the action of \( A \) to \( S(\Theta) \), we easily obtain (see also Fig. 4.5):

(S1) if \( \lambda_1 \neq \lambda_2 = 0 \), then \( \text{dom} F_1|_{S(\Theta)} = S(\Theta) \setminus \{\rho_{e_2}\} \). For \( \rho \neq \rho_{e_2} \) we have \( F_1(\rho) = \rho_{e_1} \).

(S2) if \( \lambda_1 = \lambda_2 = 0 \), then \( \text{dom} F_1|_{S(\Theta)} = S(\Theta) \setminus \{\rho_{e_1}\} \). For \( \rho \neq \rho_{e_1} \) we have \( F_1(\rho) = \rho_{e_1} \); in particular, \( F_1^n \) is undefined on \( \rho \) for \( n \geq 2 \).

Fig. 4.5. The action of \( F_1 \) on \( S(\Theta) \) in the case of \( A \) being singular

(S1) \( S(\Theta) \setminus \{\rho_{e_2}\} \) gets collapsed to \( \rho_{e_1} \), which is a fixed point of \( F_1 \);

(S2) \( S(\Theta) \setminus \{\rho_{e_1}\} \) gets collapsed to \( \rho_{e_1} \), at which \( F_1 \) is not defined.
For the sake of convenience, we now reassemble the results that concern the fixed points and generic trajectories of $F_1|_{S(\Theta)}$ in the case of $A$ being non-unitary.

**Observation 4.12.** Fixed points of $F_1|_{S(\Theta)}$ if $A$ is not unitary.

1. If $0 < |\lambda_2| < |\lambda_1| \leq 1$, then $\rho$ has an infinite trajectory convergent to $\rho_{e_1}$.
2. If $0 < |\lambda_2| = |\lambda_1| < 1$ and $\psi \notin \pi Q$, then the trajectory of $\rho$ is almost periodic but not periodic, so infinite and non-convergent.
3. If $0 < |\lambda_2| = |\lambda_1| < 1$ and $\psi \in \pi Q \setminus \{0\}$ is a primitive $\kappa$-th root of unity, then $\rho$ has a periodic trajectory with period $\kappa$.
4. If $\lambda_1 = \lambda_2$ with $0 < |\lambda_1| < 1$, then $\rho$ has an infinite trajectory convergent to $\rho_{e_1}$ (cf. Obs. 4.11).
5. If $\lambda_2 = 0$ and $0 < |\lambda_1| \leq 1$, then $F_1(\rho) = \rho_{e_1} \in \text{fix} F_1|_{S(\Theta)}$.
6. If $\lambda_1 = \lambda_2 = 0$, then $F_1(\rho) = \rho_{e_1} \notin \text{dom} F_1|_{S(\Theta)}$.

**Step III. Singularity.** The last step in determining the trajectories of $\rho_\nu$ and $\rho_m$ under $F_1$ is to decide for each of these states whether it is a generic or singular point of $F_1$, i.e., whether or not it belongs to $\text{dom} F_1|_{S(\Theta)} \setminus \text{fix} F_1|_{S(\Theta)}$. Throughout this step we assume that $A$ is non-unitary.

**Proposition 4.14.** If $A$ is not unitary, then $\rho_m$ is a generic point of $F_1$.

**Proof.** First, recall that there is at most one state in $S(\Theta)$ at which $F_1$ is undefined and this state is pure, see (4.2), so $\rho_m \in \text{dom} F_1|_{S(\Theta)}$

Next, by contradiction, suppose that $\rho_m \in \text{fix} F_1|_{S(\Theta)}$. Recall that, in the case of $A$ being non-unitary, a state from $S(\Theta) \setminus P(\Theta)$ can be a fixed point of $F_1$ only if $0 < |\lambda_1| = |\lambda_2| < 1$ and $\psi \neq 0$, see Obs. 4.12. In addition, $\rho_m \sim \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with respect to every basis of $\Theta$, so if $\rho_m \in \text{fix} F_1|_{S(\Theta)}$, then Obs. 4.10 implies that $\langle e_1|e_2 \rangle = 0$. Hence, $A$ has an orthogonal eigenbasis, thus also $|\lambda_1| = 1$, see Prop. 4.8. In consequence, $|\lambda_1| = |\lambda_2| = 1$, which is a contradiction. Therefore, $\rho_m \notin \text{fix} F_1|_{S(\Theta)}$, as claimed. 

Next, recall from Obs. 4.5 that $\rho_\nu$ is a singular point of $F_1$ iff $v = \Pi_1Uz$ is an eigenvector of $A := \Pi_1 U|_e$. In the next theorem we express this condition in terms of the eigenvalues of $A$. (Recall that they are assumed to be ordered as $|\lambda_2| \leq |\lambda_1|$.) The following lemma is [50, Lemma 1] adapted to the current context. For completeness, we include this lemma along with its proof.
Lemma 4.15. If $|\lambda_1| = 1$, then $e_1$ is an eigenvector of $U$ associated with $\lambda_1$.

Proof. Clearly, we have $1 = ||Ue_1||^2 = ||\Pi_1Ue_1||^2 + ||\Pi_2Ue_1||^2 = |\lambda_1|^2 + ||\Pi_2Ue_1||^2$. Hence, if $|\lambda_1| = 1$, then $\Pi_2Ue_1 = 0$. Thus, $Ue_1 \in \Theta$ and so $Ue_1 = \Pi_1Ue_1 = \lambda_1e_1$, as desired. \hfill \Box

Proposition 4.16. If $A$ is not unitary, then $v$ is an eigenvector of $A$ iff $|\lambda_1| = 1$. In that case the eigenvalue corresponding to $v$ is equal to $\lambda_2$.

Proof. We consider two cases, corresponding to whether or not $A$ is defective.

- Assume that $A$ is not defective.

$(\Leftarrow)$ Assume that $|\lambda_1| = 1$. It follows from Lemma 4.15 that $e_1$ is an eigenvector of $U$ associated with $\lambda_1$. Let $w \in \Theta$ complete $\{e_1, z\}$ to an orthonormal basis of $\mathbb{C}^3$. We deduce from Prop. 4.8 that $w$ is an eigenvector of $A$ associated with $\lambda_2$. Clearly, span$\{w, z\}$ is invariant under $U$, so, in particular, $Uz \in$ span$\{w, z\}$. It follows that $\Pi_1Uz \in \Pi_1$(span$\{w, z\}$) = span$\{w\}$. Moreover, $\Pi_1Uz \neq 0$ since $||\Pi_1Uz||^2 = 1 - |\omega|^2$ and $|\omega| \neq 1$, because $A$ is not unitary, see Prop. 4.1. Hence, $v = \Pi_1Uz$ is an eigenvector of $A$ and it is associated with $\lambda_2$, as required.

$(\Rightarrow)$ Assume that $v$ is an eigenvector of $A$ associated with $\lambda_a, a \in \{1, 2\}$, i.e., we have $\Pi_1Uz = qe_a$ with some $q \in \mathbb{C} \setminus \{0\}$. Thus, $Uz = qe_a + \omega z$. Moreover, since $\Pi_1Ue_a = \lambda_a e_a$, we have $Ue_a = \lambda_a e_a + rz$ with some $r \in \mathbb{C}$. Therefore, span$\{e_a, z\}$ is $U$-invariant. Let $w \in \Theta$ be such that it completes $\{e_a, z\}$ to an orthonormal basis of $\mathbb{C}^3$. It follows that span$\{w\} \subset \Theta$ is $U$-invariant, and so $w$ is an eigenvector of both $U$ and $A$ associated with $\lambda_b$, where $b \in \{1, 2\}, b \neq a$. Hence, $|\lambda_b| = 1$ as an eigenvalue of $U$. Since $A$ is not unitary, we have $|\lambda_2| < 1$. We conclude that $b = 1$ and $a = 2$, i.e., $|\lambda_1| = 1$ and the eigenvalue of $A$ associated with $v$ is equal to $\lambda_2$, as claimed.

- It remains to show that if $A$ is defective, then $v$ is not an eigenvector of $A$, cf. Prop. 4.9. By contradiction, suppose that $v$ is an eigenvector of $A$, i.e., $\Pi_1Uz = qe_1$ with some $q \in \mathbb{C} \setminus \{0\}$. We deduce, arguing as above, that span$\{e_1, z\}$ is $U$-invariant, and so the ray in $\Theta$ that is orthogonal to $e_1$ is an eigenspace of $A$. This contradicts the assumption of $A$ being defective and concludes the proof. \hfill \Box

Corollary 4.17. From Thm. 4.16 it follows that $p_\nu$ is a singular point of $F_1$ iff $|\lambda_1| = 1$ and in that case we have $p_\nu = p_{e_2}$, thus also $p_1(p_\nu) = |\lambda_2|^2$. Hence,

- $\rho_\nu \in \text{fix} F_1|_{S(\Theta)}$ iff $|\lambda_1| = 1$ and $\lambda_2 \neq 0$;
- $\rho_\nu \notin \text{dom} F_1|_{S(\Theta)}$ iff $|\lambda_1| = 1$ and $\lambda_2 = 0$.

Remark 4.18. Using Obs. 4.13, Prop. 4.14, and Cor. 4.17, we can deduce $n(\rho_2)$ and $n(\rho_m)$ in the missing case of $0 \in \sigma(A)$, cf. Cor. 4.4. Namely, $n(\rho_2) = n(\rho_\nu) + 1$ and

- if $\lambda_1 \neq \lambda_2 = 0$ then $n(\rho_m) = \infty$ and (a) $n(\rho_\nu) = \infty$ if $|\lambda_1| < 1$;
  (b) $n(\rho_\nu) = 0$ if $|\lambda_1| = 1$;
- if $\lambda_1 = \lambda_2 = 0$ then $n(\rho_m) = 1$ and $n(\rho_\nu) = 1$.
Classification. In the following proposition we show that every pair of numbers from the unit disc in \( \mathbb{C} \) can constitute the spectrum of \( A \).

**Proposition 4.19.** Let \( \mu_1, \mu_2 \) belong to the unit disc in \( \mathbb{C} \). Then there exists a unitary matrix \( V \in \mathbb{C}^{3 \times 3} \) such that \( \mu_1, \mu_2 \) are the eigenvalues of the 2 \times 2 leading principal submatrix of \( V \).

**Proof.** Let \( \mu_1, \mu_2 \in \mathbb{C} \) be such that \( 0 \leq |\mu_1|, |\mu_2| \leq 1 \). Put

\[
\tilde{V} := \begin{bmatrix}
|\mu_1| & \sqrt{(1 - |\mu_1|^2)(1 - |\mu_2|^2)} & -|\mu_2|\sqrt{1 - |\mu_1|^2} \\
|\mu_2| & e^{i\arg(\mu_2) - i\arg(\mu_1)} & \sqrt{1 - |\mu_2|^2} e^{i\arg(\mu_2) - i\arg(\mu_1)} \\
\sqrt{1 - |\mu_1|^2} & -|\mu_1|\sqrt{1 - |\mu_2|^2} & |\mu_1\mu_2|
\end{bmatrix},
\]

where we adopt the convention that \( \arg 0 = 0 \). By direct calculation we verify that \( \tilde{V} \) is unitary. Thus, \( V := e^{i\arg(\mu_1)} \tilde{V} \) is also unitary. The 2 \times 2 leading principal submatrix of \( V \) reads \( \begin{bmatrix} \mu_1 & * \\ 0 & \mu_2 \end{bmatrix} \), so \( \mu_1 \) and \( \mu_2 \) are its eigenvalues, as desired. \( \square \)

Compiling all of the above results, we can finally classify the Markov chains that can be generated by \( F_\Pi \). We know from Prop. 4.14 that the trajectory of \( \rho_m \) is given by Obs. 4.13, provided that \( A \) is non-unitary. The trajectory of \( \rho_v \) depends on whether \( |\lambda_1| \) is of unit length, see Cor. 4.17, which splits Cases (G1) & (S1) into further subcases:

(a) if \( |\lambda_1| < 1 \), then \( \rho_v \) is generic and its trajectory is given by Obs. 4.13,
(b) if \( |\lambda_1| = 1 \), then \( \rho_v \) is singular and its trajectory can be deduced from Cor. 4.17.

In what follows, the states from \( S(\Theta) \setminus P(\Theta) \) are referred to as mixed states. Recall that if \( A \) is non-singular, then \( F_1 \) is a bijection on \( S(\Theta) \) and on \( P(\Theta) \), see Prop. 4.2 & Cor. 4.3.

Thus, the trajectory of a pure (resp. mixed) state under \( F_1 \) consists entirely of pure (resp. mixed) states.

**Theorem 4.20.** Classification of the types of chains that can be generated by \( F_\Pi \).

In brackets we indicate the case in Obs. 4.13 from which a given type originates, along with the subcase (a) or (b) corresponding to whether \( |\lambda_1| < 1 \) or \( |\lambda_1| = 1 \), respectively. See Fig. 4.6 for the transition diagrams.

**generic** [G1a+D]: If \( 0 < |\lambda_2| < |\lambda_1| < 1 \) or \( \lambda_1 = \lambda_2 \) with \( 0 < |\lambda_1| < 1 \), then \( \rho_v \) has an infinite trajectory over pure states and \( \rho_m \) has an infinite trajectory over mixed states; both these trajectories converge to \( \rho_{e_1} \).

**taupek** [G1b]: If \( 0 < |\lambda_2| < |\lambda_1| = 1 \), then \( \rho_v = \rho_{e_2} \) and the system loops there with probability \( |\lambda_2|^2 \), while \( \rho_m \) has an infinite trajectory convergent over mixed states to \( \rho_{e_1} \).

**infinity-elliptic** [G2i]: If \( 0 < |\lambda_2| = |\lambda_1| < 1 \) and \( \psi \notin \pi \mathbb{Q} \), then \( \rho_v \) has an infinite trajectory over pure states and \( \rho_m \) has an infinite trajectory over mixed states; both these trajectories are almost periodic.

**finite-elliptic** [G2ii]: If \( 0 < |\lambda_2| = |\lambda_1| < 1 \) and \( \psi \in \pi \mathbb{Q} \setminus \{0\} \), then \( \rho_v \) has a periodic trajectory over pure states and \( \rho_m \) has a periodic trajectory over mixed states. Their periods are equal to \( \kappa \) such that \( \psi \) is a primitive \( \kappa \)-th root of unity. If \( \kappa = 2 \), then this chain is called circular.
null [S]: If \( \lambda_2 = 0 \), then the system cannot loop over \( \rho_z \) since the probability of it going from \( \rho_z \) to the ball is equal to \( p_1(\rho_z) = 1 - |\omega|^2 = 1 \), see Prop. 4.2.

We distinguish three subtypes of null chains:

- **generic-null** [S1a]: if \( 0 < |\lambda_1| < 1 \), then \( F_1(\rho_m) = F_1(\rho_v) = \rho_{e_1} \), and the system loops over \( \rho_{e_1} \) with probability \( |\lambda_1|^2 \);

- **taupek-null** [S1b]: if \( |\lambda_1| = 1 \), then \( F_1(\rho_m) = \rho_{e_1} \), and the system loops over \( \rho_{e_1} \) with unit probability, while \( \rho_v = \rho_{e_2} \notin \text{dom} F_1|_S(\Theta) \), so the system returns from \( \rho_v \) to \( \rho_z \) with unit probability;

- **double-null** [S2]: if \( \lambda_1 = 0 \), then \( F_1(\rho_m) = F_1(\rho_v) = \rho_{e_1} \notin \text{dom} F_1|_S(\Theta) \), and the system goes from \( \rho_{e_1} \) to \( \rho_z \) with unit probability.

unitary: If \( |\lambda_2| = |\lambda_1| = 1 \), then \( \rho_z \) and \( \rho_m \) have trivial trajectories, see p. 10.

Clearly, Prop. 4.19 assures that all types of chains listed in Theorem 4.20 are realisable. Note that generic and \( \infty \)-elliptic chains have isomorphic transition diagrams, even though the trajectories of \( \rho_v \) and \( \rho_m \) have different limiting properties.

In the case of null chains, direct calculation gives \( p_2(\rho_v) = |\lambda_1|^2 \). Thus, in the transition diagram of the double-null chain the arrow from \( \rho_v \) to \( \rho_z \) is not present. Likewise, in the transition diagrams of generic and both kinds of elliptic chains it may happen that one arrow going to \( \rho_z \) from some state on the trajectory of \( \rho_v \) is actually non-existent, i.e., the corresponding probability is zero: in the following Prop. 4.21 we show that if \( A \) is not unitary, then there exists exactly one state in \( S(\Theta) \) which under the action of \( F_{\Pi\Pi} \) remains with unit probability in \( S(\Theta) \) and this state is pure. Clearly, the potential presence of such a state in the trajectory of \( \rho_v \) does not affect any limiting properties of the chain that are essential in calculating quantum dynamical entropy of \( F_{\Pi\Pi} \) via the Blackwell integral formula. It would, however, allow long-term correlations to appear in the sequence of measurement outcomes, which could be interpreted as the system exhibiting information storage.

**Proposition 4.21.** If \( A \) is not unitary, then there exists exactly one state \( \rho \in S(\Theta) \) such that \( p_1(\rho) = 1 \). Moreover, \( \rho \) is pure.

*Proof.* Since \( \Theta \) is a two-dimensional subspace of \( \mathbb{C}^3 \), we have \( \dim(\Theta \cap U^*\Theta) = 1, 2 \}. Clearly, \( \dim(\Theta \cap U^*\Theta) = 2 \) iff \( \Theta \) is \( U \)-invariant, which in turn is equivalent to \( A \) being unitary, a contradiction. Therefore, we have \( \dim(\Theta \cap U^*\Theta) = 1 \), i.e., there exists exactly one ray in \( \Theta \) with the property that its image under \( U \) is also contained in \( \Theta \). Putting \( u \) for a unit vector which spans this ray, we get \( U u = Au \), thus also \( ||Au|| = ||U u|| = 1 \). Hence, \( \rho_u \in \mathcal{P}(\Theta) \) satisfies \( p_1(\rho_u) = ||Au||^2 = 1 \).

Suppose now that there exists \( \rho \in S(\Theta) \setminus \mathcal{P}(\Theta) \) such that \( p_1(\rho) = 1 \). By spectral decomposition we have \( \rho = \gamma \rho_a + (1 - \gamma) \rho_b \), where \( \gamma \in (0, 1) \) and \( \rho_a, \rho_b \in \mathcal{P}(\Theta) \) are mutually orthogonal. It follows easily that \( 1 = p_1(\rho) = \gamma p_1(\rho_a) + (1 - \gamma) p_1(\rho_b) \) holds only if \( p_1(\rho_a) = p_1(\rho_b) = 1 \), which implies that \( \rho_a = \rho_b = \rho_u \). This contradicts the mutual orthogonality of \( \rho_a \) and \( \rho_b \), concluding the proof. \( \square \)
Fig. 4.6. Transition diagrams of the chains listed in Thm. 4.20
4.2. Numerical range. Let $U$ and $\Pi$ be as in the previous subsection, i.e., $U \in \mathcal{U}(\mathbb{C}^3)$ and a PVM $\Pi = \Pi(z) = \{\Pi_1, \Pi_2\}$ is such that $\Pi_2 = \rho_z$ and $\Pi_1 = \mathbb{I}_3 - \rho_z$ for some unit vector $z \in \mathbb{C}^3$. According to Theorem 4.20, the type of the chain generated by $\mathcal{F}_{U, \Pi}$ is determined by the eigenvalues of $\Pi_1 U \vert_\Theta$. In Remark 4.6 we observed that $\Pi$ enters the formula for the eigenvalues of $\Pi_1 U \vert_\Theta$ only through $\omega := (z \vert U z) \in \mathbb{C}$. In what follows, we explore how the type of the chain generated by $\mathcal{F}_{U, \Pi}$ depends on $U$ and $\omega$.

The numerical range of $T \in \mathcal{L}(\mathbb{C}^d)$ is defined as $\mathfrak{M}(T) := \{(z \vert T z) : z \in \mathbb{C}^d, \|z\| = 1\}$. It is well-known that the numerical range of a normal operator is the convex hull of its eigenvalues, so a polygon in $\mathbb{C}$. Therefore, the numerical range of $U \in \mathcal{U}(\mathbb{C}^3)$ is spanned by three points on the unit circle. Hence, generically, $\mathfrak{M}(U)$ corresponds to a triangle inscribed in the unit circle, unless some eigenvalues of $U$ coincide, in which case $\mathfrak{M}(U)$ reduces to a chord (if exactly two eigenvalues coincide) or to a point (if all three eigenvalues coincide). Obviously, the latter case is equivalent to $U$ being (up to an overall phase) the identity on $\mathbb{C}^3$.

Our first goal is to prove that every point in $\mathfrak{M}(U)$ corresponds to a family of conjugate PIFSSs, i.e., if for two unit vectors $z, \tilde{z} \in \mathbb{C}^3$ we have $(z \vert U z) = (\tilde{z} \vert U \tilde{z})$, then $\mathcal{F}_{U, \Pi(z)} \sim \mathcal{F}_{U, \Pi(\tilde{z})}$.

Throughout this (and the next) subsection the eigenvalues of $U$ are denoted by $e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}$, where $\phi_j \in [0, 2\pi)$, $j \in \{1, 2, 3\}$.

Lemma 4.22. Let $z, \tilde{z} \in \mathbb{C}^3$ be unit vectors. We have $(z \vert U z) = (\tilde{z} \vert U \tilde{z})$ iff there exists $V \in \mathcal{U}(\mathbb{C}^3)$ such that $UV = VU$ and $Vz = \tilde{z}$.

Proof. ($\Leftarrow$) We have $(\tilde{z} \vert U \tilde{z}) = (Vz \vert UVz) = (Vz \vert VUz) = (z \vert U z)$.

($\Rightarrow$) We consider the following three cases.

- Assume that $U$ has three different eigenvalues and fix an orthonormal eigenbasis $w_1, w_2, w_3$ of $U$ such that $U w_j = e^{i\phi_j} w_j$ for $j \in \{1, 2, 3\}$. We have $z = \sum_{j=1}^3 \alpha_j w_j$ with $\alpha_j \in \mathbb{C}$ and $\sum_{j=1}^3 |\alpha_j|^2 = 1$. Clearly, $(z \vert U z) = |\alpha_1|^2 e^{i\phi_1} + |\alpha_2|^2 e^{i\phi_2} + |\alpha_3|^2 e^{i\phi_3}$. Since $(\tilde{z} \vert U \tilde{z}) = (z \vert U z)$, the uniqueness (of normalised) barycentric coordinates in a triangle implies that there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that $\tilde{z} = \sum_{j=1}^3 \gamma_j w_j$. Let $V \in \mathcal{L}(\mathbb{C}^3)$ be such that $V w_j = e^{i\gamma_j} w_j$, $j \in \{1, 2, 3\}$. Clearly, $V$ is unitary and $Vz = \tilde{z}$. Since $U$ and $V$ have a common eigenbasis, we also have $UV = VU$.

- Assume that $U$ has two different eigenvalues. With no loss of generality we assume that $e^{i\phi_1} \neq e^{i\phi_2} = e^{i\phi_3}$. There exists an orthonormal eigenbasis $\{w_1, w_2, w_3\}$ of $U$ such that $U w_j = e^{i\phi_j} w_j$ ($j \in \{1, 2, 3\}$) and $z = \alpha_1 w_1 + \alpha_2 w_2$ with $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfying $|\alpha_1|^2 + |\alpha_2|^2 = 1$. Since $(z \vert U z) = (z \vert U z) = |\alpha_1|^2 e^{i\phi_1} + |\alpha_2|^2 e^{i\phi_2}$ and the (normalised) barycentric coordinates in a segment are unique, there exist $\gamma_2, \gamma_3 \in \mathbb{R}$ such that $|\beta_2|^2 + |\beta_3|^2 = |\alpha_2|^2$ and $\tilde{z} = \alpha_1 e^{i\gamma_2} w_1 + \beta_2 w_2 + \beta_3 w_3$. Clearly, there exists $V \in \mathcal{L}(\mathbb{C}^3)$ such that $V w_1 = e^{i\gamma_2} w_1$ and $\alpha_2 V w_2 = \beta_2 w_2 + \beta_3 w_3$. We see that this $V$ satisfies $Vz = \tilde{z}$. Moreover, $U$ and $V$ share a one-dimensional eigenspace, namely that spanned by $w_1$, and every eigenvector of $V$ contained in $\text{span}\{w_2, w_3\}$ is also an eigenvector of $U$, because $\text{span}\{w_2, w_3\}$ is an eigenspace of $U$. Hence, $U$ and $V$ share an eigenbasis, and so they commute.

- Finally, if $U = e^{i\phi_1} \mathbb{I}_3$, then $U$ commutes with every $V \in \mathcal{L}(\mathbb{C}^3)$, and so it suffices to take any $V \in \mathcal{U}(\mathbb{C}^3)$ satisfying $Vz = \tilde{z}$.

\[\square\]
From the proof of Lemma 4.22 we can easily deduce how to construct the PVM $\Pi(z)$ corresponding to a given $\omega \in \mathfrak{M}(U)$, i.e., how to find a unit vector $z \in \mathbb{C}^3$ satisfying $\langle z|Uz \rangle = \omega$. First, we determine the barycentric coordinates of $\omega$ in $\mathfrak{M}(U)$, i.e., the coefficients $\alpha_0, \alpha_1, \alpha_2 \in [0,1]$ such that $\sum_{i=0}^3 \alpha_i = 1$ and $\sum_{i=0}^3 \alpha_i e^{i\phi_i} = \omega$. Putting $z := \sum_{i=1}^3 \sqrt{\alpha_i} w_j$, where $Uw_j = e^{i\phi_j} w_j$ with $j \in \{1,2,3\}$, we obtain $\langle z|Uz \rangle = \omega$, as desired.

In order to have the unit vector $z \in \mathbb{C}^3$ determining the measurement best exposed, we adjust notation for the projectors constituting $\Pi(z)$ by putting $\Pi_2 := \rho_2$ and $\Pi_3 := I_3 - \Pi_2$ as well as $\Theta_2 := \text{span}\{z\}^\perp$. Note that $\Pi(Vz) = V\Pi(z)V^*$ for $V \in \mathcal{U}(\mathbb{C}^3) \cup \overline{\mathcal{U}(\mathbb{C}^3)}$, i.e., $\Pi^{12} = V\Pi_2 V^*$ for $\Pi_2 \in \{1,2\}$.

**Theorem 4.25.** If $\langle z|Uz \rangle = \langle \tilde{z}|U\tilde{z} \rangle$, then $\mathcal{F}_{U,\Pi(z)} \sim \mathcal{F}_{U,\Pi(\tilde{z})}$, where $z, \tilde{z} \in \mathbb{C}^3$ are unit vectors.

**Proof.** Let $z, \tilde{z} \in \mathbb{C}^3$ be unit vectors satisfying $\langle z|Uz \rangle = \langle \tilde{z}|U\tilde{z} \rangle$. It follows from Lemma 4.22 that there exists $V \in \mathcal{U}(\mathbb{C}^3)$ such that $VU = VU$ and $Vz = \tilde{z}$; thus, we have $UV^* = V^*U$ and $\Pi(z) = V\Pi(\tilde{z})V^*$. For $\rho \in \Sigma(\mathbb{C}^3)$ and $i \in \{1,2\}$ we obtain

$$
\Lambda^U_{i,\Pi(z)}(\rho) = \Pi^2_i U \rho U^* \Pi^2_i = V \Pi^2_i V^* U \rho U^* \Pi^2_i V^* = V \Pi^2_i V^* \rho V U^* \Pi^2_i V^* = \Lambda^V \Lambda^U_{i,\Pi(z)}(\Lambda^V(\rho)).
$$

Observation 3.1 assures that $\mathcal{F}_{U,\Pi(z)}$ is $V^*$-conjugate to $\mathcal{F}_{U,\Pi(\tilde{z})}$, as desired. \hfill \Box

The above corollary allows us to write $\mathcal{F}_{U,\omega}$ for any $\mathcal{F}_{U,\Pi(z)}$ such that $\langle z|Uz \rangle = \omega$. The main goal of this subsection is to distinguish the subsets of $\mathfrak{M}(U)$ that correspond to particular types of chains, i.e., to describe the partition of $\mathfrak{M}(U)$ such that $\omega_1, \omega_2 \in \mathfrak{M}(U)$ are in the same subset if and only if the chains generated by $\mathcal{F}_{U,\omega_1}$ and $\mathcal{F}_{U,\omega_2}$ fall into the same case in Theorem 4.20. Actually, we have already seen in Proposition 4.1 that $\mathcal{F}_{U,\omega}$ generates the unitary chain iff $\omega \in \sigma(U)$, i.e., iff $\omega$ is a vertex of $\mathfrak{M}(U)$. In what follows we consider one by one the conditions on the eigenvalues of $A$ that in Theorem 4.20 characterise the remaining seven types of chains and express them in terms of $\mathfrak{M}(U)$. We adopt the following notation. Let $\omega \in \mathfrak{M}(U)$. We write $A_\omega$ for $\Pi_1 U|_{\Theta_2}$ if $z \in \mathbb{C}^3$ satisfies $\langle z|Uz \rangle = \omega$. By $\lambda_1(\omega), \lambda_2(\omega)$ we denote the eigenvalues of $A_\omega$, ordered as $|\lambda_1(\omega)| \geq |\lambda_2(\omega)|$, and put $\psi_\omega := \text{Arg} \frac{\lambda_2(\omega)}{\lambda_1(\omega)}$, provided that $0 \notin \sigma(A_\omega)$.

Let us start with null chains. Recall that $\mathcal{F}_{U,\omega}$ generates a null chain iff $\lambda_2(\omega) = 0$ and it depends on $|\lambda_1(\omega)|$ which of the three subtypes is actually generated. Also, recall that $0 \in \sigma(A_\omega)$ is equivalent to $\omega = 0$, see Prop. 4.2. Assuming $0 \in \mathfrak{M}(U)$, we deduce that $\lambda_2(0) = 0$ and $\lambda_1(0) = \text{tr} U$, see (4.4). If $\mathfrak{M}(U)$ is non-degenerate, then 0
is obviously its circumcentre, while \( \text{tr} \ U \) turns out to be its orthocentre. Indeed, putting 
\[ W_j := [\cos \phi_j, \sin \phi_j]^T \sim e^{i \phi_j}, \] where \( j \in \{1, 2, 3\} \), and 
\[ T := W_1 + W_2 + W_3 \sim \text{tr} \ U, \]
we obtain (see also Fig. 4.7)
\[ (T - W_1) \cdot (W_2 - W_3) = ||W_2||^2 - W_2 \cdot W_3 + W_3 \cdot W_2 - ||W_3||^3 = 0, \]
so \( T \) lies at the altitude dropped from \( W_1 \) to \( W_2W_3 \). We easily conclude that all altitudes of \( \mathfrak{W}(U) \) intersect at \( \text{tr} \ U \), as claimed.

![Fig. 4.7.](Image)

**Corollary 4.26.** \( F_{U, \omega} \) generates a null chain if and only if \( \omega = 0 \). The subtype of the null chain generated by \( F_{U, 0} \) is

(a) **generic-null** \( \Leftrightarrow 0 < |\text{tr} \ U| < 1 \) \( \Leftrightarrow \mathfrak{W}(U) \) is an acute-non-equilateral triangle;

(b) **double-null** \( \Leftrightarrow \text{tr} \ U = 0 \) \( \Leftrightarrow \mathfrak{W}(U) \) is the equilateral triangle;

(c) **taupek-null** \( \Leftrightarrow |\text{tr} \ U| = 1 \) \( \Leftrightarrow \mathfrak{W}(U) \) is a right-angled triangle or a diameter.
Next, we investigate the types of chains that can be generated at the boundary of $\mathcal{M}(U)$. As before, by $e_j$ we denote a normalised eigenvector of $A_\omega$ associated with $\lambda_j(\omega)$, $j \in \{1, 2\}$.

**Theorem 4.27.** Let $\omega \in \mathcal{M}(U)$. We have $\omega \in \partial \mathcal{M}(U)$ if and only if $|\lambda_1(\omega)| = 1$.

**Proof.** ($\Leftarrow$) Let $\omega \in \mathcal{M}(U)$ and assume that $|\lambda_1(\omega)| = 1$. From Lemma 4.15 we know that $\lambda_1(\omega) \in \sigma(U)$ and that $e_1$ is an eigenvector of $U$ associated with $\lambda_1(\omega)$. With no loss of generality we assume that $\lambda_1(\omega) = e^{i\phi_1}$. From (4.4) we obtain

$$e^{i\phi_2} + e^{i\phi_3} = \lambda_2(\omega) + \omega \quad \text{and} \quad e^{i\phi_2 + i\phi_3} = \lambda_2(\omega). \quad (4.9)$$

Let $a_1, a_2, a_3 \in [0, 1]$ stand for the normalised barycentric coordinates of $\omega$ in $\mathcal{M}(U)$, i.e.,

$$\omega = \sum_{j=1}^3 a_j e^{i\phi_j} \quad \text{and} \quad \sum_{j=1}^3 a_j = 1.$$  

From (4.9) it follows that $a_1(e^{i\phi_2} - e^{i\phi_1})(e^{i\phi_3} - e^{i\phi_1}) = 0$, which gives $a_1 = 0$ or $e^{i\phi_1} \in \{e^{i\phi_2}, e^{i\phi_3}\}$, and in either of these cases we conclude that $\omega \in \text{conv}\{e^{i\phi_2}, e^{i\phi_3}\} \subset \partial \mathcal{M}(U)$, as desired.

($\Rightarrow$) Let $\omega \in \partial \mathcal{M}(U)$. With no loss of generality we assume that $\omega = a e^{i\phi_1} + (1 - a) e^{i\phi_2}$, where $a \in [0, 1]$. Observe that for $\tilde{\omega} := (1 - a) e^{i\phi_1} + a e^{i\phi_2}$ we have $\text{tr} U = \omega + \tilde{\omega} + e^{i\phi_3}$ and $e^{i\phi_1 + i\phi_2} \tilde{\omega} = \tilde{\omega}$. Thus, again via (4.4), we obtain

$$\text{tr} A_\omega = \tilde{\omega} + e^{i\phi_3} \quad \text{and} \quad \det A_\omega = \tilde{\omega} e^{i\phi_3}.$$ 

It follows easily that $\lambda_1(\omega) = e^{i\phi_3}$ and $\lambda_2(\omega) = \tilde{\omega}$; in particular, we have $|\lambda_1(\omega)| = 1$, which concludes the proof. \qed

Clearly, if $|\lambda_1(\omega)| = 1$, then the determinant formula in (4.4) gives $|\lambda_2(\omega)| = |\omega|$. Therefore, for the boundary of $\mathcal{M}(U)$ we have the following classification of chain types.

**Corollary 4.28.** Let $\omega \in \partial \mathcal{M}(U)$. Then the Markov chain generated by $F_U, \omega$ is

(a) tautep-null $\iff \omega = 0 \iff \omega$ is the midpoint of the hypotenuse of right-angled $\mathcal{M}(U)$ or the midpoint of $\mathcal{M}(U)$ degenerate to a diameter (see also Cor. 4.26(c));

(b) unitary $\iff |\omega| = 1 \iff \omega$ is a vertex of $\mathcal{M}(U)$ (see also Prop. 4.1);

(c) tautep $\iff 0 < |\omega| < 1 \iff \omega$ is neither the midpoint of the hypotenuse of right-angled $\mathcal{M}(U)$ nor the midpoint of $\mathcal{M}(U)$ degenerate to a diameter nor a vertex of $\mathcal{M}(U)$.

**Remark 4.29.** The ($\Rightarrow$) part of the proof of Theorem 4.27 along with Proposition 4.8 imply that if $\omega = a e^{i\phi_1} + (1 - a) e^{i\phi_2}$ for some $a \in [0, 1]$, then $A_\omega$ is a normal operator with $\sigma(A_\omega) = \{e^{i\phi_3}, (1 - a) e^{i\phi_1} + a e^{i\phi_2}\}$. Collecting the results of Proposition 4.8, Lemma 4.15 and Theorem 4.27, we deduce that the following conditions are equivalent:

(i) $A_\omega$ is normal (unitarily diagonalisable);

(ii) an eigenvector of $U$ is orthogonal to $z$;

(iii) $\omega$ is a convex combination of at most two eigenvalues of $U$ (i.e., $\omega \in \partial \mathcal{M}(U)$),

(iv) $|\lambda_1(\omega)| = 1$.

One may want to compare these conditions with those in Propositions 4.1 & 4.7, which characterise the case of $A_\omega$ being unitary.
Remark 4.30. From [51, Thm. 14] and Remark 4.29 it follows that the probability measure induced by $\mathcal{F}_{U, \omega}$ on the space of sequences of measurement outcomes $\{1, 2\}^\mathbb{N}$ is non-ergodic (with respect to the shift operator) if and only if $\omega \in \partial \mathfrak{M}(U)$.

It remains to describe the subsets of $\operatorname{Int} \mathfrak{M}(U) \setminus \{0\}$ corresponding to finite- and $\infty$-elliptic chains. The first step is to characterise $\omega \in \mathfrak{M}(U) \setminus \{0\}$ such that $A_\omega$ induces elliptic dynamics on the Bloch sphere, which, as we recall, is equivalent to $|\lambda_1(\omega)| = |\lambda_2(\omega)| > 0$ with $\psi_\omega \neq 0$. Also, recall that elliptic dynamics is called circular if we additionally have $\psi_\omega = \pi$, which amounts to $\lambda_1(\omega) = -\lambda_2(\omega) \neq 0$.

**Proposition 4.31.** If $\omega \in \mathfrak{M}(U) \setminus \{0\}$, then $A_\omega$ generates elliptic dynamics iff

$$\frac{(\operatorname{tr} U - \omega)^2}{\omega \det U} \in [0, 4).$$

In particular, $A_\omega$ generates circular dynamics iff $\omega = \operatorname{tr} U$.

**Proof.** Let $\omega \in \mathfrak{M}(U) \setminus \{0\}$. Recall that $\omega \neq 0$ assures that $0 \notin \sigma(A_\omega)$ and $\det A_\omega \neq 0$. First, we discuss the case of $A_\omega$ generating circular dynamics. Clearly, $\lambda_1(\omega) = -\lambda_2(\omega)$ iff $\operatorname{tr} A_\omega = 0$ iff $\omega = \operatorname{tr} U$, as claimed. Let us now characterise elliptic-non-circular dynamics, i.e., with $\psi_\omega \neq \pi$. Denote by $\Delta_\omega$ the discriminant of the characteristic polynomial of $A_\omega$, i.e., $\Delta_\omega := (\operatorname{tr} A_\omega)^2 - 4 \det A_\omega$, and observe that

$$|\lambda_1(\omega)| = |\lambda_2(\omega)| \quad \text{and} \quad \psi_\omega \notin \{0, \pi\}$$

$$\iff \Delta_\omega \neq 0 \quad \text{and} \quad \operatorname{Arg}((\operatorname{tr} A_\omega)^2/\Delta_\omega) = \pi$$

$$\iff \operatorname{Arg}((\operatorname{tr} A_\omega)^2/\det A_\omega) = 0 \quad \text{and} \quad |(\operatorname{tr} A_\omega)^2| < 4|\det A_\omega|$$

$$\iff (\operatorname{tr} A_\omega)^2/\det A_\omega \in (0, 4),$$

and so the assertion of the proposition follows. \(\square\)

**Remark 4.32.** Consider $\omega \in \mathfrak{M}(U) \setminus \{0\}$. Analogously to the proof of Proposition 4.31, one can verify that the dynamics induced on the Bloch sphere by $A_\omega$ is parabolic iff $\frac{(\operatorname{tr} U - \omega)^2}{\omega \det U} = 4$ and $\omega \notin \sigma(U)$, and it is loxodromic iff $\frac{(\operatorname{tr} U - \omega)^2}{\omega \det U} \notin [0, 4]$. In fact, we recovered here a well-known criterion for deciding whether a Möbius transformation of $\tilde{\mathbb{C}}$ is elliptic, parabolic or loxodromic, see, e.g., [1, Prop. 2.16]. It is also straightforward to check that the remaining case of $\frac{(\operatorname{tr} U - \omega)^2}{\omega \det U} = 4$ and $\omega \in \sigma(U)$ corresponds to $A_\omega$ being proportional to the identity map, and so to the trivial dynamics on the Bloch sphere.

Finally, let us consider $\omega \in \operatorname{Int} \mathfrak{M}(U) \setminus \{0\}$ such that $A_\omega$ generates elliptic dynamics. To distinguish between finite-elliptic and $\infty$-elliptic chains, we need to decide whether or not $\psi_\omega$ is commensurable with $\pi$. Putting $\Upsilon(\omega) := \min\{\psi_\omega, 2\pi - \psi_\omega\}$, by direct calculation we obtain

$$\cos \frac{\Upsilon(\omega)}{2} = \frac{1}{2} \left| \operatorname{tr} A_\omega/\sqrt{\det A_\omega} \right|,$$

and so the assertion of the proposition follows. \(\square\)
Theorem 4.33 (Thm. 4.20 revisited, see also Fig. 4.8). Let $\omega \in \mathcal{M}(U)$. The Markov chain generated by $\mathcal{F}_{U,\omega}$ is

- **unitary** if $\omega \in \sigma(U)$;
- **null** if $\omega = 0$ and the subtype of this chain is
  - **generic-null** if $\mathcal{M}(U)$ is an acute-non-equilateral triangle;
  - **double-null** if $\mathcal{M}(U)$ is an equilateral triangle;
  - **taupek-null** if $\mathcal{M}(U)$ is a right-angled triangle or a diameter;
- **finite-elliptic** if $\omega \in \partial \mathcal{M}(U) \setminus (\sigma(U) \cup \{0\})$;
- **$\infty$-elliptic** if $\omega \in \text{Int} \mathcal{M}(U) \setminus \{0\}$, $\frac{(\text{tr} U - \omega)^2}{\sqrt{\det U}} \in [0, 4)$ and $\text{arccos} \frac{|\text{tr} U - \omega|}{2\sqrt{|\omega|}} \in \pi \mathbb{Q}$;
- **generic** otherwise.

![Fig. 4.8. Type of the Markov chain generated by $\mathcal{F}_{U,\omega}$ for $\omega \in \mathcal{M}(U)$ in the case of $\mathcal{M}(U)$ being non-degenerate and (a) acute-non-equilateral; (b) equilateral; (c) right-angled (and isosceles). The lines corresponding to finite- or $\infty$-elliptic chains were found numerically.](image_url)

Theorem 4.33 allows us to determine the type of the Markov chain generated by $\mathcal{F}_{U,\omega}$ by inspecting $\mathcal{M}(U)$ and $\omega$ instead of the eigenvalues of $A_\omega$. Furthermore, taking a look at $\mathcal{M}(U)$, one can quickly tell which of the eight possible chain types can be generated with help of $U$ as one varies the ray in $\mathbb{C}P^2$ that specifies the measurement, which translates into varying $\omega$ over $\mathcal{M}(U)$. Clearly, the only non-trivial problem here is to decide whether elliptic chains can be generated. In the next subsection we shall prove that the subsets of $\mathcal{M}(U)$ where finite- and $\infty$-elliptic chains are generated are both infinite whenever $\mathcal{M}(U)$ is non-degenerate. Also, we shall see that these two subsets are contained in a cubic plane curve. Therefore, if $\text{Int} \mathcal{M}(U) \neq \emptyset$, then $\{\omega \in \mathcal{M}(U): \mathcal{F}_{U,\omega} \text{ generates a generic chain}\}$ is indeed generic in both topological and measure-theoretic sense.
4.3. Elliptic chains. Throughout this subsection we assume that $\text{Int} \mathfrak{M}(U) \neq \emptyset$. We shall now prove that the two sets $\{ \omega \in \mathfrak{M}(U) : \mathcal{F}_{U, \omega} \text{ generates a finite-elliptic chain} \}$ and $\{ \omega \in \mathfrak{M}(U) : \mathcal{F}_{U, \omega} \text{ generates an } \infty \text{-elliptic chain} \}$ are both infinite. Let us adopt the following notation. Put $\hat{f}(z) := \frac{\text{tr}(U_z)^2}{\det U}$, $z \in \mathbb{C} \setminus \{0\}$, and 

$$E := \{ \omega \in \text{Int} \mathfrak{M}(U) : A_\omega \text{ generates elliptic dynamics} \}$$

$$= \{ \omega \in \text{Int} \mathfrak{M}(U) : \hat{f}(\omega) \in [0, 4) \}.$$ 

Additionally, we consider $f(z) := \frac{\text{tr}(U_z - z)^2}{\det U}$, $z \in \mathbb{C}$, and 

$$C := \{ z \in \mathbb{C} : \Im \hat{f}(z) = 0 \}.$$ 

Clearly, $C$ is a cubic plane curve. Observe that $\hat{f}(z) = f(z)/|z|^2$, and so $\Im \hat{f}(z) = 0$ iff $\Im \hat{f}(z) = 0$ or $z = 0$. Hence, in particular, $E \subset C$. Further, note that $\{0, \text{tr} U\} \subset C$, i.e., $C$ contains both the circumcentre and the orthocentre of $\mathfrak{M}(U)$. It follows easily that the eigenvalues of $U$, i.e., the vertices of $\mathfrak{M}(U)$, also belong to $C$. Indeed, it suffices to note that

$$f(e^{i\phi_1}) = (e^{i\phi_2} + e^{i\phi_3})^2 e^{-i\phi_2 - i\phi_3} = 2 + 2 \cos(\phi_2 - \phi_3) \in [0, 4] \subset \mathbb{R}, \quad (4.12)$$

where, as before, $e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}$ with $\phi_1, \phi_2, \phi_3 \in [0, 2\pi)$ stand for the eigenvalues of $U$.

We now shall prove that the two sets

$$\begin{align*}
&\mathfrak{M}(U) \cap \text{Int} \mathfrak{M}(U) \\
&\mathfrak{M}(U) \cap \text{Int} \mathfrak{M}(U) \cap B(1, r) \neq \emptyset \quad \text{for every } r > 0.
\end{align*}$$

Proposition 4.34. Let $\mu \in \sigma(U)$. If the interior vertex angle of $\mathfrak{M}(U)$ at $\mu$ is not right, then $\mathfrak{C} \cap \text{Int} \mathfrak{M}(U) \cap B(1, r) \neq \emptyset$ for every $r > 0$.

Proof. By Prop. 3.2(i) we may, with no loss of generality, so adjust the overall phase that $U \sim \text{diag}(1, e^{ip_1}, e^{ip_2})$, $\varphi_1, \varphi_2 \in [0, 2\pi)$, and the vertex angle of $\mathfrak{M}(U)$ at 1 is not right. Fix $r > 0$. We show that $\mathfrak{C} \cap \text{Int} \mathfrak{M}(U) \cap B(1, r) \neq \emptyset$. For $\varepsilon \in (0, 1)$ we consider

$$\gamma_\varepsilon : [0, \varepsilon] \ni t \mapsto (1 - \varepsilon) + (\varepsilon - t) e^{ip_1} + t e^{ip_2} \in \mathfrak{M}(U).$$

It can easily be seen that $\gamma_\varepsilon([0, \varepsilon])$ corresponds to a segment in $\mathfrak{M}(U)$ which connects the side between 1 and $e^{ip_1}$ with that between 1 and $e^{ip_2}$, and which is parallel to the side between $e^{ip_1}$ and $e^{ip_2}$.

Clearly, there exists $\delta \in (0, 1)$ such that $\gamma_\varepsilon([0, \varepsilon]) \subset B(1, r)$ if $\varepsilon \in (0, \delta)$. Indeed, let $\varrho \in (0, r]$ satisfy $\varrho < \min\{|1 - e^{ip_1}|, |1 - e^{ip_2}|\}$, i.e., $\partial B(1, \varrho)$ intersects both the side of $\mathfrak{M}(U)$ connecting 1 and $e^{ip_1}$ as well as that connecting 1 and $e^{ip_2}$. For $j \in \{1, 2\}$ denote $\delta_j := \varrho/|1 - e^{ip_j}|$. Putting $\delta := \min\{\delta_1, \delta_2\}$ and letting $\varepsilon \in (0, \delta)$, we obtain

$$|1 - \gamma_\varepsilon(0)| = |1 - (1 - \varepsilon) - \varepsilon e^{ip_1}| = \varepsilon|1 - e^{ip_1}| = \varepsilon \varrho/\delta_1 < \varrho$$

so $\gamma_\varepsilon(0) \in B(1, \varrho)$. Likewise, $\gamma_\varepsilon(\varepsilon) \in B(1, \varrho)$. Hence, $\gamma_\varepsilon([0, \varepsilon]) \subset B(1, \varrho) \subset B(1, r)$, as claimed.

Moreover, for every $\varepsilon \in (0, 1)$ we have $\gamma_\varepsilon((0, \varepsilon)) \subset \text{Int} \mathfrak{M}(U)$. Therefore, to conclude that $\mathfrak{C} \cap \text{Int} \mathfrak{M}(U) \cap B(1, r) \neq \emptyset$, it suffices to show that $\mathfrak{C} \cap \gamma_\varepsilon((0, \varepsilon)) \neq \emptyset$ for some $\varepsilon \in (0, \delta)$.
where $\delta$ is as above. To this end, let us consider $(0, 1) \ni \varepsilon \mapsto \gamma(0) = (1 - \varepsilon) + \varepsilon e^{i\varphi_1}$, which, obviously, is a parametrisation of the side of $\mathfrak{M}(U)$ between 1 and $e^{i\varphi_1}$. By direct computation we obtain

$$\Im f(\gamma(0)) = 2\varepsilon (1 - \varepsilon)(1 - \cos \varphi_1) [\sin(\varphi_2 - \varphi_1) + \varepsilon (\sin(\varphi_1 - \varphi_2) + \sin \varphi_2)].$$

Note that $\sin(\varphi_2 - \varphi_1) \neq 0$, because $\varphi_1 \neq \varphi_2$ since $\text{Int } \mathfrak{M}(U) \neq \emptyset$, and $\varphi_1 \neq \varphi_2 \pm \pi$ since the vertex angle at 1 is not right. Consequently, there exists $\varepsilon_1 \in (0, \delta)$ such that for $\varepsilon \in (0, \varepsilon_1)$ we have $\text{sgn } \Im f(\gamma(0)) = \text{sgn } \sin(\varphi_2 - \varphi_1)$. Analogously, for $(0, 1) \ni \varepsilon \mapsto \gamma(\varepsilon) = (1 - \varepsilon) + \varepsilon e^{i\varphi_2}$ there exists $\varepsilon_2 \in (0, \delta)$ such that $\text{sgn } \Im f(\gamma(\varepsilon)) = \text{sgn } \sin(\varphi_1 - \varphi_2)$ for $\varepsilon \in (0, \varepsilon_2)$.

Finally, letting $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$, we have $\text{sgn } \Im f(\gamma(\varepsilon)) = -\text{sgn } \Im f(\gamma(\varepsilon))$. Since $\Im f \circ \gamma : [0, \varepsilon] \to \mathbb{R}$ is a continuous function which takes on values of different signs at the endpoints of the interval constituting its domain, Bolzano’s (intermediate value) theorem assures the existence of $t \in (0, \varepsilon)$ such that $\Im f(\gamma(t)) = 0$. Therefore, $\mathcal{C} \cap \gamma(\varepsilon((0, \varepsilon)) \neq \emptyset$, which concludes the proof.

**Corollary 4.35.** Since $\mathcal{C} \cap \text{Int } \mathfrak{M}(U)$ is a semi-algebraic set, Theorem 4.34 and the Curve Selection Lemma (see, e.g., [42, §3] or [15, Thm. 2.5.5]) imply that for each $\mu \in \sigma(U)$ there exists a non-trivial interval contained in $\mathfrak{M}(U)$.

Next, we show that $E$ coincides with $C$ in the vicinity of $\mu \in \sigma(U)$, provided that the (interior) vertex angle of $\mathfrak{M}(U)$ at $\mu$ is not right.

**Proposition 4.36.** Let $\mu \in \sigma(U)$. If the interior vertex angle of $\mathfrak{M}(U)$ at $\mu$ is not right, then there exists $\varepsilon > 0$ such that $\mathcal{C} \cap \text{Int } \mathfrak{M}(U) \cap B(\mu, \varepsilon) \subset E$.

**Proof.** Let the overall phase be so adjusted that $U \sim \text{diag}(1, e^{i\varphi_1}, e^{i\varphi_2})$, where $\varphi_1, \varphi_2 \in [0, 2\pi)$, and the vertex angle of $\mathfrak{M}(U)$ at 1 is not right. First, we show that $\tilde{f}(1) \in (0, 4)$. Indeed, from (4.12) we obtain $\tilde{f}(1) = f(1) = 2 + 2 \cos(\varphi_1 - \varphi_2)$. Again, since $\text{Int } \mathfrak{M}(U) \neq \emptyset$, we have $\varphi_1 \neq \varphi_2$, and from the assumption that the vertex angle of $\mathfrak{M}(U)$ at 1 is not the right angle we have $\varphi_1 \neq \varphi_2 \pm \pi$. Therefore, $\cos(\varphi_1 - \varphi_2) \in (-1, 1)$, and so $\tilde{f}(1) \in (0, 4)$, as claimed. Clearly, $\tilde{f}$ is continuous at 1, and thus so is $\mathbb{R} \circ \tilde{f}$; hence, there exists $\varepsilon > 0$ such that $\mathbb{R} \circ \tilde{f}(B(1, \varepsilon)) \subset (0, 4)$.

Let $\omega \in \mathcal{C} \cap \text{Int } \mathfrak{M}(U) \cap B(1, r)$. It follows that $\omega \neq 0$ and $\Re \tilde{f}(\omega) \in (0, 4)$ since $\omega \in B(1, r)$ with $r < 1$, while $\omega \in \mathcal{C}$ guarantees that $\Im \tilde{f}(\omega) = \Im f(\omega) = 0$. Therefore, $\omega \in \text{Int } \mathfrak{M}(U) \setminus \{0\}$ and $\tilde{f}(\omega) \in (0, 4)$, which implies that $\omega \in E$, as desired.

Finally, recall from (4.11) that if $\omega \in E$, then the (smaller) angle between the eigenvalues of $A_\omega$ is given by

$$\Upsilon(\omega) = 2 \arccos \frac{\text{tr } U - \omega}{2 \sqrt{|\omega|}}.$$

**Proposition 4.37.** There is a non-trivial interval contained in $\Upsilon(E)$.

**Proof.** Here we adjust the overall phase so that $\det U = 1$. Recall that $\tilde{f}(E) \subset [0, 4)$. Let $\omega \in E$ and observe that

$$\Upsilon(\omega) = 2 \arccos \left(\frac{1}{2} \sqrt{\tilde{f}(\omega)}\right).$$

29
Clearly, it suffices to show that a non-trivial interval is contained in $\tilde{f}(E)$. Recall that Cor. 4.35 & Prop. 4.36 assure that there exists $\mu \in \sigma(U)$ and a smooth arc $g: [0, 1] \to \mathcal{M}(U)$ such that $g(0) = \mu$ and $g(t) \in E$ for $t \in (0, 1)$. Since $\tilde{f}$ is continuous, it actually suffices to show that $\tilde{f} \circ g: (0, 1] \to [0, 4) \subset \mathbb{R}$ is not constant.

Fix $a \in \mathbb{R}$. We will show that there are finitely many solutions to $\tilde{f}(z) = a$ in $\mathbb{C}$. Clearly, for $z \in \mathbb{C} \setminus \{0\}$ this equation can be equivalently written as $(\text{tr } U - z)^2 = a\pi$. Letting $x, y \in \mathbb{R}$ be such that $z = x + iy$, we obtain

$$
\begin{align*}
\begin{cases}
(\Re \text{tr } U - x)^2 - (\Im \text{tr } U - y)^2 = ax \\
2(\Re \text{tr } U - x)(\Im \text{tr } U - y) = -ay
\end{cases}
\end{align*}
$$

Each of these equations describes a hyperbola (possibly degenerated to two intersecting lines). The hyperbola given by the upper equation has asymptotes with slopes equal to $\pm 1$, while the other hyperbola has asymptotes parallel to the axes. Hence, these hyperbolas have at most four points in common, and so the equation $\tilde{f}(z) = a$ holds at finitely many points, as claimed. In consequence, $\tilde{f} \circ g$ cannot be constant on $(0, 1]$. It follows that $\{\tilde{f}(g(t)) : t \in (0, 1]\}$ contains a non-trivial interval, and thus so does $\Upsilon(E)$, as desired. \hfill \Box

Let $\omega \in E$ and recall that the chain generated by $F_{U, \omega}$ is finite-elliptic iff $\Upsilon(\omega) \in \pi \mathbb{Q}$; otherwise, this chain is $\infty$-elliptic, see Thm. 4.33. It follows from Prop. 4.37 that $\Upsilon(E) \cap (\pi \mathbb{Q})$ and $\Upsilon(E) \setminus (\pi \mathbb{Q})$ are both infinite, so we have the following

**Theorem 4.38.** If $\operatorname{Int} \mathcal{M}(U) \neq \emptyset$, then $\{\omega \in \mathcal{M}(U) : F_{U, \omega} \text{ generates a finite-elliptic chain}\}$ and $\{\omega \in \mathcal{M}(U) : F_{U, \omega} \text{ generates an } \infty \text{-elliptic chain}\}$ are both infinite.

**Remark 4.39.** Let us now take a closer look at the cubic curve $C$ in which $E$ is contained. It turns out that if the polynomial defining $C$ is irreducible, then $C$ is the Musselman (third) cubic, which was introduced in [45] and catalogued in [26] as K028. Indeed, let the overall phase be so adjusted that $\det U = 1$. Then $C$ is the zero-set of the following polynomial

$$
\mathcal{M}(x, y) = y^3 - 3x^2y + 2\beta x^2 + 4\alpha xy - 2\beta y^2 - 2\alpha\beta x + \beta^2 y - \alpha^2 y,
$$

(4.13)

where $\alpha := \Re \text{tr } U$, $\beta := \Im \text{tr } U$. Substituting in (4.13) the Cartesian coordinates $(x, y)$ with the barycentric coordinates $(a, b, c)$, $a + b + c = 1$, via the conversion formula

$$
x = a \cos \phi_1 + b \cos \phi_2 + c \cos \phi_3,
$$

$$
y = a \sin \phi_1 + b \sin \phi_2 + c \sin \phi_3,
$$

where $e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}$ denote the eigenvalues of $U$, we arrive at the barycentric equation of the Musselman cubic as given in [26].

The Musselman cubic is an equilateral cubic, i.e., it has three asymptotes that concur and the (smaller) angle of intersection of every two of these asymptotes is equal to $\pi/3$, see also Fig. 4.9(a)–(c). This can be easily deduced by examining the cubic terms in (4.13). Namely, the slopes of the asymptotes satisfy the equation $s^3 = 3s$ ($s \in \mathbb{R}$), which implies that these slopes read $s_1 = 0 = \tan 0, s_2 = \sqrt{3} = \tan \frac{\pi}{3}, s_3 = -\sqrt{3} = \tan \frac{2\pi}{3}$. The point of the concurrence of the asymptotes turns out to be the midpoint between the centroid and orthocentre of $\mathcal{M}(U)$ [26]. The Musselman cubic has a node (an ordinary double point, (an ordinary double point,
i.e., a point where exactly two branches intersect and they have distinct tangent lines) at the orthocentre of \( \mathfrak{W}(U) \), i.e., at \( \text{tr } U \), where two of its branches intersect orthogonally [24, p. 96].

Let us further inspect \( M(x, y) \) and discuss the potential degenerations of \( C \). Namely,

\[
\frac{\partial M}{\partial x}(x, y) = \frac{\partial M}{\partial y}(x, y) = 0 \iff (x, y) \in \{(\alpha, \beta), (\frac{\alpha}{3}, \frac{\beta}{3})\} \sim \{\text{tr } U, \frac{1}{3}\text{tr } U\}.
\]

We already know that \( \text{tr } U \in C \), i.e., \( M(\alpha, \beta) = 0 \). As for the other point, we get \( M(\frac{\alpha}{3}, \frac{\beta}{3}) = 0 \) iff \( \beta \in \{0, \alpha\sqrt{3}, -\alpha\sqrt{3}\} \), which turns out to be equivalent to the fact that \( \mathfrak{W}(U) \) is isosceles. Obviously, \( (\alpha, \beta) \) and \( (\frac{\alpha}{3}, \frac{\beta}{3}) \) coincide iff \( \alpha = \beta = 0 \) iff \( \mathfrak{W}(U) \) is equilateral.

Thus, if \( \mathfrak{W}(U) \) is isosceles but not equilateral, then \( C \) has two singular points. It follows that \( C \) degenerates into the union of a line and a hyperbola, see Fig. 4.9(d)-(e); indeed, it can be easily verified that if \( \alpha \neq 0 \) and \( \beta = 0 \), then

\[
M(x, y) = y(3x^2 - y^2 - 4\alpha x + \alpha^2),
\]

and so \( C \) consists of the \( x \)-axis and of the hyperbola given by the equation

\[
\frac{(x - 2\alpha/3)^2}{(\alpha/3)^2} - \frac{y^2}{(\alpha/\sqrt{3})^2} = 1.
\]

---

**Fig. 4.9.** \( C \) (in green) along with \( \partial \mathfrak{W}(U) \) (in blue) and \( \text{tr } U \) (the purple dot):

(a) \( \text{tr } U \in \text{Int } \mathfrak{W}(U) \)

(b) \( \text{tr } U \in \partial \mathfrak{W}(U) \)

(c) \( \text{tr } U \notin \mathfrak{W}(U) \)

(d) \( \alpha \neq 0, \beta = 0 \)

(e) \( \alpha \neq 0, \beta = \alpha\sqrt{3} \)

(f) \( \alpha = \beta = 0 \)
Observe that the transverse axis of this hyperbola coincides with the \( x \)-axis and its vertices are located at \( \alpha = \text{tr} \ U \) and \( \frac{2}{3} \alpha = \frac{1}{3} \text{tr} \ U \). Similarly, one can verify that if \( \alpha \neq 0 \) and \( \beta = \pm \alpha \sqrt{3} \), then \( C \) consists of the median of base of the triangle corresponding to \( W(U) \) and of a hyperbola whose transverse axis coincides with this median and which has vertices at \( \text{tr} \ U \) and \( \frac{1}{3} \text{tr} \ U \). Lastly, if the triangle is equilateral (\( \alpha = \beta = 0 \)), then
\[
M(x, y) = y(\sqrt{3}x - y)(\sqrt{3}x + y),
\]
and so \( C \) degenerates into the union of three triangle medians, see Fig. 4.9(f).

In consequence, \( M(x, y) \) is reducible whenever the triangle corresponding to \( W(U) \) is isosceles. We now argue that the converse also holds. If \( W(U) \) is not isosceles, then \( (\alpha, \beta) \) is the only singular point of \( C \) and this point is a node since \( \det H_M(\alpha, \beta) = -4(\alpha^2 + \beta^2) < 0 \), where \( H_M(\alpha, \beta) \) stands for the Hessian matrix of \( M \) at \( (\alpha, \beta) \). Clearly, a reducible cubic curve is the union of an irreducible conic and a line or of three lines, see also [54, p. 677] or [27, p. 70]. It follows that a reducible cubic has exactly one singular point which is a node iff this cubic degenerates into a parabola and a line parallel to the symmetry axis of this parabola, or into a hyperbola and a line parallel to an asymptote of this hyperbola. In what follows we exclude both these possibilities, which allows us to conclude that \( M(x, y) \) is irreducible if \( W(U) \) is not isosceles, as claimed.

Namely, assume that the Musselman cubic degenerates into a conic and a line, i.e., that
\[
M(x, y) = (Ax^2 + Bxy + Cy^2 + Dx + Ey + F)(Gx + Hy + J),
\]
where \( A, B, C, D, E, F, G, H, J \in \mathbb{R} \). Comparing the right-hand side coefficients with those of \( M \), see (4.13), we deduce that
\[
\begin{align*}
AG &= 0 \\
CH &= 1 \\
CG + BH &= 0 \\
BG + AH &= -3
\end{align*}
\]
In what follows we will repeatedly use the obvious fact that \( A = 0 \) or \( G = 0 \), as well as \( C \neq 0 \) and \( H \neq 0 \).

(1) Assume that the conic is a parabola, i.e., \( B^2 = 4AC \).

- If \( A = 0 \), then also \( B = 0 \), which contradicts \( BG + AH = -3 \).
- If \( G = 0 \), then \( BH = 0 \). It follows that \( B = 0 \), thus also \( AC = 0 \), and so \( A = 0 \). This again contradicts \( BG + AH = -3 \).

(2) Assume that the conic is a hyperbola, i.e., \( B^2 > 4AC \). Also, assume that the line \( Gx + Hy + J = 0 \) is parallel to an asymptote of this hyperbola, i.e., that \( -G/H \) is equal to the slope of an asymptote, which reads \( \frac{1}{2}(-B \pm \sqrt{B^2 - 4AC})/C \).

- If \( A = 0 \), then the slopes of the asymptotes are equal to 0 and \( -B/C \). If \( G/H = 0 \), then \( G = 0 \), which contradicts \( BG + AH = -3 \). If \( G/H = B/C \), then \( G = B = 0 \) since we also have \( CG + BH = 0 \). This contradicts \( B^2 > 4AC \).
- If \( G = 0 \), then also \( B = 0 \) since \( CG + BH = 0 \). The asymptotes have slopes \( \pm \sqrt{-A/C} \). If \( 0 = G/H = \pm \sqrt{-A/C} \), then \( A = 0 \), which contradicts \( B^2 > 4AC \).
4.4. The Crutchfield & Wiesner example. In this final subsection we examine the ball & point system whose evolution is governed by $U \in \mathcal{U}(\mathbb{C}^3)$ represented in the standard basis $e_1, e_2, e_3$ of $\mathbb{C}^3$ as

$$U \sim \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$ 

This unitary matrix has been discussed extensively by Crutchfield & Wiesner in a series of papers [20, 43, 55, 56, 57]. The measurements they have considered are the PVMs of the form $\Pi(z) = \{\rho_z, \mathbb{I}_3 - \rho_z\}$ with $z$ taken from the standard basis of $\mathbb{C}^3$. This has led to two different chains, namely those corresponding to $\omega = \langle z|Uz \rangle \in \{0, \frac{1}{\sqrt{2}}\}$, because $\langle e_1|Ue_1 \rangle = \frac{1}{\sqrt{2}}$ and $\langle e_2|Ue_2 \rangle = \langle e_3|Ue_3 \rangle = 0$.

Since $\sigma(U) = \{1, \exp(i\gamma), \exp(-i\gamma)\}$ with $\gamma$ such that $\cos \gamma = \frac{1}{4}(\sqrt{2} - 2)$, which gives $\gamma \approx 98.42^\circ$, it follows that $\mathfrak{M}(U)$ is an acute-non-equilateral isosceles triangle. Note that $\det U = 1$ and $\text{tr} U = \frac{1}{\sqrt{2}}$. Hence, $\mathcal{F}_{U,0}$ generates a generic-null chain and $\mathcal{F}_{U, \frac{1}{\sqrt{2}}}$ generates a circular chain. Let us determine the subset $\mathcal{E}$ of $\mathfrak{M}(U)$, where elliptic chains are generated. Let $\omega = x + iy \in \text{Int} \mathfrak{M}(U)$ with $x, y \in \mathbb{R}$. From (4.14) and (4.15) we know that $(\text{tr} U - \omega)^2/\omega \in \mathbb{R}$ holds on the $x$-axis (apart from the origin) and on the hyperbola given by

$$18(x - \frac{\sqrt{2}}{3})^2 - 6y^2 = 1,$$

which intersects the $x$-axis at $\left(\frac{\sqrt{2}}{3}, 0\right) \sim \text{tr} U$ and $\left(\frac{\sqrt{2}}{3}, 0\right) \sim \frac{1}{2} \text{tr} U$. First, we examine the $x$-axis. Let $\omega \in \mathfrak{M}(U) \cap \mathbb{R}$ and put $\Delta_\omega := \omega^2 - (4 + \sqrt{2})\omega + \frac{1}{2}$. It follows easily that $(\text{tr} U - \omega)^2/\omega \in [0, 4)$ iff $\Delta_\omega < 0$ iff $\omega \in (\omega_b, 1]$, where $\omega_b := \frac{1}{2}\sqrt{2} + 2 - \sqrt{4 + 2\sqrt{2}}$. We conclude that $\mathcal{F}_{U, \omega}$ generates an elliptic chain iff $\omega \in (\omega_p, 1]$. Next, we examine the hyperbola. Let $\omega = x + iy$ with $x, y \in \mathbb{R}$. Then $(\text{tr} U - \omega)^2/\omega \in (0, 4)$ is equivalent to

$$0 \leq x^3 - \sqrt{2}x^2 + \frac{1}{2}x + y^2(\sqrt{2} - 3x) < 4(x^2 + y^2)$$

which holds on the hyperbola (4.16) iff $x \in (\sqrt{2} - 2, \frac{1}{2} \text{tr} U] \cup \{\text{tr} U\}$. Restricting to $\mathfrak{M}(U)$, we get $x \in (\omega_b, \frac{1}{2} \text{tr} U] \cup \{\text{tr} U\}$, where $\omega_b := \cos \gamma \approx -0.146$. Hence, $\mathcal{E}$ consists of the segment $(\omega_p, 1]$ in the $x$-axis and of the part of a hyperbola branch contained between the vertices $\exp(\pm i\gamma)$. In Fig. 4.10 we show all chain types that can be generated by $\mathcal{F}_{U, \omega}$ for $\omega \in \mathfrak{M}(U)$.

Let us compute the eigenvalues $\lambda_1, \lambda_2$ of $A_\omega$ along the $x$-axis. Let $\omega \in \mathfrak{M}(U) \cap \mathbb{R} = [\omega_b, 1]$. The discriminant of the characteristic polynomial of $A_\omega$, which reads

$$p_A(\lambda) = \lambda^2 - (\text{tr} U - \omega)\lambda + \det U\omega$$

$$= \lambda^2 - \left(\frac{1}{\sqrt{2}} - \omega\right)\lambda + \omega,$$

is given by $\Delta_\omega$. It follows that

(i) if $\omega \in [\omega_b, \omega_p)$, then $\Delta_\omega > 0$ and $\lambda_{1,2} = \frac{1}{2\sqrt{2}} - \frac{1}{2}\omega \pm \sqrt{\Delta_\omega}$; in particular, $|\lambda_1| \neq |\lambda_2|$;

(ii) if $\omega = \omega_p$, then $\Delta_\omega = 0$ and $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}\sqrt{2 + \sqrt{2}} - 1$;

(iii) if $\omega \in (\omega_p, 1]$, then $\Delta_\omega < 0$ and $\lambda_{1,2} = \frac{1}{2\sqrt{2}} - \frac{1}{2}\omega \pm i\sqrt{\Delta_\omega}$; in particular, $\lambda_1 = \overline{\lambda_2}$, and so $|\lambda_1| = |\lambda_2|$;
The following table summarises how the properties of the system, in particular the type of dynamics induced on the Bloch sphere, depend on $\omega$ varying between $\omega_b$ and 1.

| value of $\omega$ | eigenvalues of $A_\omega$ | dynamics induced by $A_\omega$ | type of chain generated by $F_{U,\omega}$ |
|------------------|---------------------------|--------------------------------|----------------------------------|
| $\omega_b \approx -0.146$ | $\lambda_1 = 1$, $\lambda_2 = \omega_b$ | loxodromic | taupek |
| $(\omega_b, 0)$ | $\lambda_2 < 0 < \lambda_1$, $|\lambda_1| \neq |\lambda_2|$ | loxodromic | generic |
| 0 | $\lambda_1 = \text{tr} U$, $\lambda_2 = 0$ | singular | generic-null |
| $(0, \omega_p)$ | $0 < \lambda_2 < \lambda_1$ | loxodromic | generic |
| $\omega_p \approx 0.094$ | $\lambda_1 = \lambda_2$ | parabolic | generic |
| $(\omega_p, \text{tr} U)$ | $\lambda_1 = \overline{\lambda_2}$ | elliptic | finite- or $\infty$-elliptic |
| $\text{tr} U = \frac{1}{2} \sqrt{2}$ | $\lambda_1 = -\lambda_2 = i^{2-1/4}$ | circular | circular |
| $(\text{tr} U, 1)$ | $\lambda_1 = \overline{\lambda_2}$ | elliptic | finite- or $\infty$-elliptic |
| 1 | $\lambda_1 = \overline{\lambda_2} = e^{i\gamma}$ | elliptic | unitary |

Fig. 4.10. Numerical range of the unitary matrix introduced by Crutchfield & Wiesner along with the types of Markov chains that can be generated by the ball & point system whose evolution is governed by this unitary.
5. ACKNOWLEDGMENTS

The author is grateful to Wojciech Słomczyński for numerous comments that substantially improved this manuscript.

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