A Unified Study of
Conforming and Discontinuous Galerkin Methods*

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Abstract

A unified study is presented in this paper for the design and analysis of different finite element methods (FEMs), including conforming and nonconforming FEMs, mixed FEMs, hybrid FEMs, discontinuous Galerkin (DG) methods, hybrid discontinuous Galerkin (HDG) methods and weak Galerkin (WG) methods. Both HDG and WG are shown to admit inf-sup conditions that hold uniformly with respect to both mesh and penalization parameters. In addition, by taking the limit of the stabilization parameters, a WG method is shown to converge to a mixed method whereas an HDG method is shown to converge to a primal method. Furthermore, a special class of DG methods, known as the mixed DG methods, is presented to fill a gap revealed in the unified framework.

Keywords. Finite element methods, DG-derivatives, Unified study

1 Introduction

In this paper, we propose a general framework to derive most of the existing finite element methods (FEMs), and discuss their relationships. We will illustrate the main idea by using the following elliptic boundary value problem

\[
\begin{aligned}
-\text{div}(\alpha \nabla u) &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^d \) (\( d \geq 1 \)) is a bounded domain and \( \alpha : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a bounded and symmetric positive definite matrix, with its inverse denoted by \( c = \alpha^{-1} \).

Setting \( p = -\alpha \nabla u \), the above problem can be formally written in two different forms. The first, known as the primal formulation, can be written as

\[
\begin{aligned}
cp + \nabla u &= 0 & \text{in } \Omega, \\
-\nabla^* p &= f & \text{in } \Omega,
\end{aligned}
\]

which requires that \( u \in H^1_0(\Omega) \) and \( p \in L^2(\Omega) \).

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The second, known as the mixed formulation, can be written as

\[
\begin{cases}
cp - \text{div}^* u = 0 & \text{in } \Omega, \\
\text{div} p = f & \text{in } \Omega,
\end{cases}
\] (1.3)

which requires that \( u \in L^2(\Omega) \) and \( p \in H(\text{div}; \Omega) \).

The design of FEMs then becomes an appropriate discretization of \( \nabla \) and \( \text{div} \), which amounts to imposing certain continuity conditions on \( u \) and \( p \cdot n \) in the following five different approaches:

1. strongly (\( H^1 \) or \( H(\text{div}) \) conforming elements),
2. weakly (nonconforming),
3. via Lagrangian multiplier (hybrid methods),
4. via Lagrangian multiplier and stabilization — stabilized hybrid primal methods and stabilized hybrid mixed methods, and
5. via penalization (DG).

The resulting different types of FEMs can then be fully described in a uniform framework by the notion of DG-gradient — \( \nabla_{dg} \), DG-divergence — \( \text{div}_{dg} \). Here, \( \nabla_{dg} \) (see (1.4)) is a generalization of the piecewise gradient which allows \( u_h \) to be discontinuous across element boundaries, but uses an additional Lagrangian multiplier space on the element boundaries. We note that \( \nabla_{dg} \) and \( \text{div}_{dg} \) in this context, as shown in Table 2.1 below, directly correspond to the weak derivatives introduced by Wang and Ye [109]. We denote by \( \{ T_h \} \) a family of triangulations of \( \Omega \) which satisfy the minimal angle condition. Let \( h_K = \text{diam}(K) \) and \( h = \max\{ h_K : K \in T_h \} \). For any \( K \in T_h \), denote \( n_K \) as the outward unit normal of \( K \). Denote by \( E_h \) the union of the boundaries of the elements \( K \) of \( T_h \). Let \( P_h : L^2(\Omega) \to V_h \) be the \( L^2 \) projection and \( f_h = P_h f \).

**Definition 1.1 (DG-derivatives)** Let \( \hat{u}_h = (u_h, \hat{u}_h) \in \hat{V}_h \), and \( \hat{p}_h = (p_h, \hat{p}_h) \in \hat{Q}_h \). Then the DG-gradient \( \nabla_{dg} : V_h \mapsto \hat{Q}_h \) and the DG-divergence \( \text{div}_{dg} : Q_h \mapsto \hat{V}_h^* \) are defined by

\[
\langle \nabla_{dg} u_h, \hat{q}_h \rangle = \langle \nabla h u_h, q_h \rangle_{\Omega} - \sum_{K \in T_h} \langle u_h, \hat{q}_h \cdot n_K \rangle_{\partial K} \quad \forall u_h \in V_h, \forall q_h \in \hat{Q}_h, \tag{1.4}
\]

\[
\langle \text{div}_{dg} p_h, \hat{v}_h \rangle = \langle \text{div} h p_h, v_h \rangle_{\Omega} - \sum_{K \in T_h} \langle p_h \cdot n_K, \hat{v}_h \rangle_{\partial K} \quad \forall v_h \in \hat{V}_h, \forall p_h \in Q_h. \tag{1.5}
\]

Here, \( V_h \) and \( Q_h \) are the piecewise scalar and vector-valued discrete spaces on the triangulation \( T_h \), respectively, and \( \hat{V}_h \) and \( \hat{Q}_h \) are the piecewise scalar and vector-valued discrete spaces on \( E_h \), respectively. \( \hat{V}_h \) and \( \hat{Q}_h \) are defined as

\[
\hat{V}_h := V_h \times \hat{V}_h, \quad \hat{Q}_h := Q_h \times \hat{Q}_h.
\]

\( \hat{Q}_h \) and \( \hat{V}_h \) are the dual spaces of \( \hat{Q}_h \) and \( \hat{V}_h \), respectively.

As for the DG-derivatives defined in Definition 1.1, we will specify \( V_h, \hat{V}_h, Q_h \) and \( \hat{Q}_h \) at different concurrences. Let \( \hat{u}_h \) and \( \hat{p}_h \) be defined as single-valued functions on \( E_h \). They can be viewed as certain numerical traces of DG functions. We want to emphasize that the DG-derivatives on the discrete spaces are
globally defined. The dual operators of $\nabla_{dg}$ and $\text{div}_{dg}$ are denoted by $\nabla_{dg}^* : \tilde{Q}_h \mapsto V_h^*$ and $\text{div}_{dg}^* : \tilde{V}_h \mapsto Q_h^*$, respectively, such that

$$\langle \nabla_{dg}^* \tilde{p}_h, v_h \rangle = \langle \tilde{p}_h, \nabla_{dg} v_h \rangle \quad \forall v_h \in V_h, \quad \langle \text{div}_{dg}^* \tilde{u}_h, q_h \rangle = \langle \tilde{u}_h, \text{div}_{dg} q_h \rangle \quad \forall q_h \in Q_h.$$  

(1.6) (1.7)

We shall now use these DG-derivatives to formulate different types of Galerkin methods in §1.1–§1.3 below and summarize these different methods in Table 1.1. We further give some brief bibliographic comments on the development of these methods in §1.4.

### 1.1 Primal formulation

**Conforming and nonconforming methods** The first example of the FEMs using a primal formulation is the conforming or nonconforming finite element, which requires the continuity (or some weak continuity) of $u_h$ across the element boundaries.

$$\begin{cases}
  cp_h + \nabla_h u_h = 0 & \text{in } Q_h^*, \\
  -\nabla_h^* p_h = f_h & \text{in } V_h^*.
\end{cases} \quad (1.8)$$

Here, $\nabla_h$ is derived by taking the gradient piecewise on each element.

**Hybrid primal methods** The hybrid formulation enforces the (weak) continuity of the aforementioned primal method through a Lagrangian space on element boundaries, and it can be formally written as

$$\begin{cases}
  cp_h + \nabla_{dg} u_h = 0 & \text{in } \tilde{Q}_h^*, \\
  -\nabla_{dg}^* \tilde{p}_h = f_h & \text{in } \tilde{V}_h^*.
\end{cases} \quad (1.9)$$

**Stabilized hybrid primal methods** The hybrid primal methods are unstable for even-order elements. As a remedy, a stabilization term can be added to the first equation in (1.9) to obtain

$$\begin{cases}
  cp_h + S_{dg}^\eta \tilde{p}_h + \nabla_{dg} u_h = 0 & \text{in } \tilde{Q}_h^*, \\
  -\nabla_{dg}^* \tilde{p}_h = f_h & \text{in } \tilde{V}_h^*.
\end{cases} \quad (1.10)$$

where $\langle cp_h, \tilde{q}_h \rangle := (cp_h, q_h)$, and

$$\langle S_{dg}^\eta \tilde{p}_h, \tilde{q}_h \rangle := \sum_{K \in T_h} \langle \eta(p_h - \hat{p}_h) \cdot n_K, (q_h - \hat{q}_h) \cdot n_K \rangle_{\partial K}. \quad (1.11)$$

Here, $\eta > 0$ is a stabilization parameter. In most cases, the stabilized hybrid primal methods are also named weak Galerkin (WG) methods.

### 1.2 Mixed formulation

**Mixed methods** The first example of the FEMs using a mixed formulation is the mixed finite element method, which requires the continuity of $p_h$ across elements, i.e.,

$$\begin{cases}
  cp_h - \text{div}_h^* u_h = 0 & \text{in } Q_h^*, \\
  \text{div}_h p_h = f_h & \text{in } V_h^*.
\end{cases} \quad (1.12)$$

Here, $\text{div}_h$ is calculated by taking the gradient piecewise on each element.
Table 1.1: A unified framework of FEMs
Hybrid mixed methods To reduce the number of the globally-coupled degrees of freedom, hybrid mixed methods are also developed using the Lagrange multiplier technique. Similar to the hybrid primal methods (1.9), the mixed form (1.3) can be reformulated in terms of DG-divergence as

\[
\begin{align*}
\begin{cases}
  cp_h - \text{div}_{dg}^* \bar{u}_h = 0 & \text{in } Q_h^*, \\
  \text{div}_{dg} p_h = f_h & \text{in } \bar{V}_h^*.
\end{cases}
\end{align*}
\] (1.13)

Stabilized hybrid mixed methods Under some choices of the space \( Q_h \) and \( \bar{V}_h \), the hybrid mixed method is no longer stable. Again, as a natural device, a stabilization term is then added to the hybrid mixed methods to obtain

\[
\begin{align*}
\begin{cases}
  cp_h - \text{div}_{dg}^* \bar{u}_h = 0 & \text{in } Q_h^*, \\
  \text{div}_{dg} p_h + S^*_u \bar{u}_h = f_h & \text{in } \bar{V}_h^*.
\end{cases}
\end{align*}
\] (1.14)

where \( \langle cp_h, q_h \rangle := (cp_h, q_h) \), and

\[
\langle S^*_u \bar{u}_h, \bar{v}_h \rangle := \sum_{K \in T_h} \tau (\bar{u}_h - \hat{u}_h, \bar{v}_h - \hat{v}_h)_{\partial K}.
\] (1.15)

Here, \( \tau > 0 \) is a stabilization parameter. In most cases, the stabilized hybrid mixed methods are also named hybrid discontinuous Galerkin (HDG) methods.

Remark 1.2 If \( \alpha \) (or \( c \)) is piecewise constant, we can eliminate \( p_h \) in (1.14) to obtain the primal WG-FEM introduced in [83, 109]:

\[
(\alpha \text{div}_{dg}^* \bar{u}_h, \text{div}_{dg}^* \bar{v}_h)_{\Omega} + \sum_{K \in T_h} \eta (u_h - \hat{u}_h, v_h - \hat{v}_h)_{\partial K} = (f, v_h) \quad \forall \{v_h, \hat{v}_h\} \in \bar{V}_h.
\] (1.16)

1.3 Discontinuous Galerkin methods

Instead of using the Lagrange multiplier technique, a penalty term is added in the bilinear form of the discontinuous Galerkin (DG) method to force continuity. With the concepts of DG-gradient and DG-divergence defined as in Definition 1.1, most of the DG methods for approximating the elliptic problem can be written as

\[
\begin{align*}
\begin{cases}
  cp_h - \text{div}_{dg}^* \bar{u}_h = 0 & \text{in } Q_h^*, \\
  -\nabla_{dg}^* \hat{p}_h = f_h & \text{in } V_h^*, \\
  \hat{p}_h = \bar{p}(p_h, u_h) & \text{on } \mathcal{E}_h, \\
  \hat{u}_h = \bar{u}(p_h, u_h) & \text{on } \mathcal{E}_h.
\end{cases}
\end{align*}
\] (1.17a-d)

where \( \bar{p} \) and \( \bar{u} \) are the formulas for defining \( \hat{p}_h \) and \( \hat{u}_h \), respectively, in the terms \( p_h \) and \( u_h \), respectively.

If penalization is forced to impose the continuity of \( u_h \), we refer to the DG method as a primal DG method. In this paper, we also use penalization to impose the continuity of \( p_h \cdot n \), which is called mixed DG method.

Using the definition of DG-derivatives, we summarize the relationships of different FEMs in Table 1.1. From the table, we observe that the primal and mixed methods are the discretization of primal and mixed forms, respectively. Under certain restrictions pertaining to finite element spaces, the (non)conforming...
primal and mixed FEMs are obtained. In addition, by introducing Lagrange multipliers, the hybrid primal
and hybrid mixed FEMs can be derived from primal and mixed FEMs, respectively. Generally speaking, these
hybrid methods are not stable, and specific finite element spaces need to be identified to make the schemes
work. By adding stabilization terms to the hybrid primal and hybrid mixed FEMs, one obtains the stabilized
hybrid primal (weak Galerkin) and stabilized hybrid mixed (hybrid discontinuous Galerkin) FEMs, which
are the approximation of the primal form and mixed form, respectively. Moreover, when the stabilization
terms dominate, the solution of the stabilized primal (mixed) FEMs approaches the approximation of the
mixed (primal) form under certain choices of discrete spaces, which completes the outer loop in Table 1.1.
We emphasize that only $\nabla_{dg}$ and $\nabla_{*dg}$ appear in the primal formulations, and only $\text{div}_{dg}$ and $\text{div}_{*dg}$ are found
in the mixed formulations.

Traditionally, the DG methods in our framework can be viewed as a compromise between the primal
and mixed formulations by introducing both $\nabla_{*dg}$ and $\text{div}_{*dg}$. Then different DG schemes can be obtained by
proper choices of numerical fluxes, see Table 6.1. With the help of this framework, we derive a new family
of mixed DG methods, which can be regarded as the dual form of primal DG methods.

1.4 Brief bibliographic comments

The idea of conforming FEMs can be traced back to the 1940s [75] and the Courant element [57]. After
a decade, many works, such as [77, 62, 65, 120, 121, 93, 36, 87], proposed more conforming elements and
presented serious mathematical proofs concerning error analysis and, hence, established the basic theory of
FEMs. The first nonconforming element, the Wilson element [115], was proposed for rectangle meshes. In
1972, Strang [101] developed the concept of variational crimes in a nonconforming finite element method.
The well-known Crouzeix-Raviart [58] finite element appeared in 1973. For more examples on nonconforming
elements, we refer the reader to [64, 63, 98, 89, 13] for second-order elliptic problems, and [112, 113, 76, 119,
118] for higher-order elliptic problems.

The mixed FEMs are tailored for approximating problems with several primary variables. These include
linear elasticity in a stress-displacement system, Stokes equations, and flow of a porous media within a
velocity-pressure system. The condition for the well-posedness of mixed formulations is known as inf-sup
or the Ladyzhenskaya-Babuška-Brezzi (LBB) condition [16]. Common simplicial mixed finite elements are
the Raviart-Thomas elements [95, 86] and the Brezzi-Douglas-Marini finite element [19, 17]. Another family
of mixed finite elements was proposed in [85] for tetrahedra, cubes, and prisms by Nédélec. Further, a
family of rectangular mixed finite elements in two- and three-space variables were designed in [18] by Brezzi,
Douglas, Fortin, and Marini. Mixed FEMs and their applications to various problems were summarized in
monographs by Brezzi, Fortin, and Boffi [21, 11].

The idea of hybrid methods was proposed in [65, 92, 93, 91, 68]. The discrete spaces used in the hybrid
primal FEMs are discontinuous [94, 116, 96], and Lagrange multipliers are adopted to force continuity
across the element boundary. For instance, based on a primal hybrid variational principle, Raviart and
Thomas [96] viewed the nonconforming finite elements as discontinuous spaces on which weak continuity
was imposed by multiplier space. Further work on this hybrid idea was done by Babuška, Oden and Lee
in [7], in which a flux variable was introduced in bilinear form. In 1985, hybridization was shown to be
more than an implementation trick by Arnold and Brezzi in [3]. More precisely, it was proven that the
new unknown introduced by hybridization could also be interpreted as the Lagrange multiplier associated
with a continuity condition on the approximate flux which contains additional information about the exact
solution. After yet nearly another decade, a new hybridization approach was put forward by Cockburn
and Gopalakrishnan [41], one in which the hybrid formulation not only simplifies the task of assembling
the stiffness matrix for the multiplier, but also can be used to establish links between apparently unrelated mixed methods. Their approach also allows new, variable degree versions of those methods to be devised and analyzed \cite{42, 43, 44, 45}. Recently, a hybrid mixed method for working with linear elasticity problem was presented in \cite{67}.

For the mathematical theory behind the above methods, we refer to the monographs by Ciarlet \cite{35} and by Brenner and Scott \cite{14}. For more detailed discussion on mixed and hybrid methods, we refer to the monographs \cite{21, 97, 11}.

The idea of using DG methods for elliptic equations can be traced back to the late 1960s \cite{79}, and the same approach was studied again in \cite{5}. Recently, DG methods have been applied to purely elliptic problems; examples include the interior penalty methods studied in \cite{6, 59, 114, 2}, and the local DG method for elliptic problem in \cite{54}. DG methods for diffusion and elliptic problems were considered in \cite{25, 26}. Additional problems utilizing DG methods can be found in \cite{15, 80, 81, 69}. A review of the development of DG methods up to 1999 can be found in \cite{48} by Cockburn, Karniadakis, and Shu. In \cite{4}, Arnold, Brezzi, Cockburn, and Marini unified the analysis of DG methods for elliptic problems. An abstract theory for using quasi-optimal nonconforming and DG methods for elliptic problems was very recently presented in \cite{104, 105}.

In 2004, a new hybridization approach was presented in \cite{41} by Cockburn and Gopalakrishnan. This new hybridization approach was also applied to a DG method in \cite{28}. Using the LDG method to define the local solvers, a super-convergent LDG-hybridizable Galerkin method for second-order elliptic problems was designed in \cite{38}. In 2009, a unified analysis for the hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems was presented in \cite{46} by Cockburn, Gopalakrishnan, and Lazarov. In 2010, a projection-based error analysis of HDG methods was presented in \cite{47} by Cockburn, Gopalakrishnan, and Sayas, in which a projection operator was tailored to obtain the $L^2$ error estimates for both potential and flux. A projection-based analysis of the HDG methods used for convection-diffusion equations for semi-matching nonconforming meshes was presented in \cite{32}. A connection between the staggered DG method and an HDG method was shown in \cite{34}. Applications of HDG methods to linear and nonlinear elasticity problems can be found in \cite{100, 66, 78}. More references to the recent developments of HDG methods can be found in \cite{37}, to other applications in \cite{31, 49, 53}, to super-convergence analysis in \cite{51, 52, 39, 40}, and to a posteriori error analysis in \cite{55, 56, 50}.

With the introduction of weak gradient and weak divergence, Wang and Ye \cite{109, 110} proposed and analyzed a WG method for a second order elliptic equation formulated as a system of two first-order linear equations. Wang and Wang \cite{108} gave a summary of the idea and applications of WG methods for various problems since the publication of \cite{109}. Wang and Wang \cite{107} also developed a primal-dual WG finite method for second-order elliptic equations in non-divergence form, and then they further extended their methods to Fokker-Planck type equations in \cite{106}. Applications of WG to other problems can be found. See \cite{111} for Stokes equations, \cite{30} for Darcy-Stokes flow, and \cite{84} for Maxwell equations.

In view of derivations, the stabilized hybrid mixed method aims to properly choose the numerical trace of the flux and can be viewed as a stabilization approach for the hybrid mixed method, while the stabilized hybrid primal method stems from the proper definitions of the weak gradient and weak divergence and can be viewed as a stabilization approach for the hybrid primal method. In the contexts of HDG and WG, their relationship has been discussed in \cite{37}.

The rest of the paper is organized as follows. In §2, we present some preliminary materials. In §3, using a second-order elliptic problem as an example, we show that most of the existing FEMs can be rewritten in compact form by using the concept of DG-derivatives and discuss their relationships. In §4, we present the well-posedness of WG methods under the specific, parameter-dependent norms and further the convergence
analysis of WG. We present a similar result for the HDG methods in §5. In §6, we discuss the DG and derive the mixed DG methods by making a dual choice of numerical traces of primal DG methods. In §7, we analyze the relationship between various FEMs and show that WG methods converge to mixed methods and that HDG methods converge to primal methods under the limitation of parameter. We give a brief summary in the last section.

Throughout this paper, we shall use letter $C$, which is independent of mesh-size and stabilization parameters, to denote a generic positive constant which may stand for different values at different occurrences. The notations $x \lesssim y$ and $x \gtrsim y$ mean $x \leq Cy$ and $x \geq Cy$, respectively.

## 2 Preliminaries

In this section, we shall describe some basic notation and properties of DG-derivatives.

### 2.1 DG notation

Given a bounded domain $D \subset \mathbb{R}^d$ and a positive integer $m$, $H^m(D)$ is the Sobolev space with the corresponding usual norm and semi-norm, which are denoted by $\| \cdot \|_{m,D}$ and $| \cdot |_{m,D}$, respectively. We abbreviate these by $\| \cdot \|_m$ and $| \cdot |_m$, respectively, when $D$ is chosen as $\Omega$. The $L^2$-inner product on $D$ and $\partial D$ are denoted by $\langle \cdot, \cdot \rangle_D$ and $\langle \cdot, \cdot \rangle_{\partial D}$, respectively. $\| \cdot \|_{0,D}$ and $\| \cdot \|_{0,\partial D}$ are the norms of Lebesgue spaces $L^2(D)$ and $L^2(\partial D)$, respectively, and $\| \cdot \| = \| \cdot \|_{0,\Omega}$.

Let $\mathcal{E}^i_h = \mathcal{E}_h \setminus \partial \Omega$ be the set of interior edges and $\mathcal{E}^\partial_h = \mathcal{E}_h \setminus \mathcal{E}^i_h$ be the set of boundary edges. Further, let $h_e = \text{diam}(e)$. For $e \in \mathcal{E}^i_h$, we select a fixed normal unit direction, denoted by $n_e$. For $e \in \mathcal{E}^\partial_h$, we specify the outward unit normal as $n_e$. Let $e$ be the common edge of two elements $K^+$ and $K^-$, and let $n^i = n|_{\partial K^i}$ be the unit outward normal vector on $\partial K^i$ with $i = +, -$. For any scalar-valued function $v$ and vector-valued function $q$, let $v^\pm = v|_{\partial K^\pm}$, $q^\pm = q|_{\partial K^\pm}$. Then, we define averages $\{ \cdot \}, \{ \cdot \}$ and jumps $[ \cdot ], [ \cdot ]$ as follows:

\[
\{ v \} = \frac{1}{2}(v^+ + v^-), \quad \{ q \} = \frac{1}{2}(q^+ + q^-), \quad \{ q \} = \frac{1}{2}(q^+ + q^-) \cdot n_e \quad \text{on } e \in \mathcal{E}^i_h,
\]

\[
[v] = v^+ n^+ + v^- n^-, \quad [v] = [v] \cdot n_e, \quad [q] = q^+ n^+ + q^- n^- \quad \text{on } e \in \mathcal{E}^i_h,
\]

\[
[q] = q \cdot n, \quad \{ q \} = q \cdot n \quad \text{on } e \in \mathcal{E}^\partial_h.
\]

Here, we specify $n$ as the outward unit normal direction on $\partial \Omega$.

We define some inner products as follows:

\[
\langle \cdot, \cdot \rangle_{T_h} = \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_K, \quad \langle \cdot, \cdot \rangle_{E_h} = \sum_{e \in \mathcal{E}_h} \langle \cdot, \cdot \rangle_e, \quad \langle \cdot, \cdot \rangle_{\mathcal{E}^i_h} = \sum_{e \in \mathcal{E}^i_h} \langle \cdot, \cdot \rangle_e, \quad \langle \cdot, \cdot \rangle_{\mathcal{E}^\partial_h} = \sum_{K \in \mathcal{T}_h} \langle \cdot, \cdot \rangle_{\partial K}. \tag{2.1}
\]

We now give more details about the last inner product. For any scalar-valued function $v$ and vector-valued function $q$,

\[
\langle v, q \cdot n \rangle_{\partial T_h} = \sum_{K \in \mathcal{T}_h} \langle v, q \cdot n \rangle_{\partial K} = \sum_{K \in \mathcal{T}_h} \langle v, q \cdot n_K \rangle_{\partial K}.
\]

Here, we specify the outward unit normal direction $n$ corresponding to the element $K$, namely $n_K$. In addition, let $\nabla_h$ and $\text{div}_h$ be defined by the relation

\[
\nabla_h v|_K = \nabla v|_K, \quad \text{div}_h q|_K = \text{div} q|_K \quad \forall K \in \mathcal{T}_h.
\]
With the definition of averages and jumps, we have the following identity:

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\mathbf{q} \cdot \mathbf{n}_K) v \, ds = \int_{\mathcal{E}_h} \{\mathbf{q}\} : [v] \, ds + \int_{\mathcal{E}_h} [\mathbf{q}] \cdot \{v\} \, ds,$$

namely,

$$\langle v, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = \langle \{\mathbf{q}\}, [v] \rangle_{\mathcal{E}_h} + \langle [\mathbf{q}], \{v\} \rangle_{\mathcal{E}_h^i}. \quad (2.3)$$

Further,

$$\{\mathbf{q}\} : [v] = \|\mathbf{q}\| \{v\} \quad \forall e \in \mathcal{E}_h. \quad (2.4)$$

Before discussing various Galerkin methods, we need to introduce the finite element spaces associated with the triangulation $\mathcal{T}_h$. For $k \geq 0$, we first define the spaces as follows

$$V_h^k = \{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

$$Q_h^k = \{p_h \in L^2(\Omega) : p_h|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

$$Q_h^{k,RT} = \{p_h \in L^2(\Omega) : p_h|_K \in \mathcal{P}_k(K) + x\mathcal{P}_k(K), \forall K \in \mathcal{T}_h\},$$

where $\mathcal{P}_k(K)$ is the space of polynomial functions of degree at most $k$ on $K$. We also use the following spaces associated with $\mathcal{E}_h$

$$\hat{Q}_h = \{\hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e) \mathbf{n}_e, \forall e \in \mathcal{E}_h\},$$

$$\hat{Q}_h = \{\hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e), \forall e \in \mathcal{E}_h\},$$

$$\hat{V}_h = \{\hat{v}_h : \hat{v}_h|_e \in \hat{V}(e) \mathbf{n}_e, \forall e \in \mathcal{E}_h, \hat{v}_h|_{\mathcal{E}_h} = 0\},$$

$$\hat{Q}_h^k = \{\hat{p}_h \in L^2(\mathcal{E}_h) : \hat{p}_h|_e \in \mathcal{P}_k(e), \forall e \in \mathcal{E}_h\},$$

$$\hat{V}_h^k = \{\hat{v}_h \in L^2(\mathcal{E}_h) : \hat{v}_h|_e \in \mathcal{P}_k(e), \forall e \in \mathcal{E}_h, \hat{v}_h|_{\mathcal{E}_h} = 0\},$$

where $\hat{Q}(e), \hat{V}(e)$ are some local spaces on $e$ and $\mathcal{P}_k(e)$ is the space of polynomial functions of degree at most $k$ on $e$.

### 2.2 Some properties of DG-derivatives

Are the DG-gradient $\nabla_{dg}$ and DG-divergence $-\text{div}_{dg}$ operators dual operators with each other like the classical gradient and divergence operators? Generally, the answer is no. However, by the definition of DG-derivatives (1.4), (1.5), we have the following relationship:

$$\langle -\text{div}_{dg}^* \hat{u}_h, p_h \rangle = -\langle \hat{u}_h, \text{div}_{dg} p_h \rangle$$

$$= \langle \nabla_{dg} u_h, p_h \rangle_{\mathcal{T}_h} + \langle \hat{u}_h - u_h, p_h \cdot n \rangle_{\partial \mathcal{T}_h}, \quad (2.7)$$

and

$$\langle \nabla_{dg}^* \hat{p}_h, u_h \rangle = \langle \hat{p}_h, \nabla_{dg} u_h \rangle$$

$$= -\langle \text{div}_{dg} p_h, u_h \rangle_{\mathcal{T}_h} + \langle (p_h - \hat{p}_h) \cdot n, u_h \rangle_{\partial \mathcal{T}_h}, \quad (2.8)$$

By direct calculation, we have the following lemma.

**Lemma 2.1** Assume $\hat{u}_h = 0$ on $\mathcal{E}_h^0$. Then, $\nabla_{dg} = -\text{div}_{dg}^*$ if one of the following conditions holds:
(i) $p_h \cdot n_K|_{\partial \Omega} = \hat{p}_h \cdot n_K$;

(ii) $u_h|_{\partial \Omega} = \hat{u}_h$;

(iii) $\hat{p}_h \cdot n_K = \{p_h\} \cdot n_K$ and $\hat{u}_h = \{u_h\}$.

Therefore, we say $\nabla_{dg}$ and $-\text{div}_{dg}$ are conditionally dual with each other. Based on this observation, we find that the DG-derivatives are good approximations for classical weak derivatives when one of the above conditions is satisfied. Furthermore, in the following subsections, we see that the (stabilized) hybrid primal and (stabilized) hybrid mixed methods weakly satisfy condition (i) or (ii), and that the DG methods adopt condition (iii) approximately.

| DG-derivatives | $\nabla_{dg}$ | $\nabla_{dg}^*$ | $\text{div}_{dg}$ | $\text{div}_{dg}^*$ |
|----------------|--------------|----------------|------------------|------------------|
| Weak derivatives [65, 109] | $-\text{div}_{w}$ | $-\text{div}_{w}$ | $-\text{div}_{w}^*$ | $-\text{div}_{w}^*$ |

Table 2.1: Relationship of DG-derivatives and weak derivatives when $\hat{u}$ and $\hat{p}$ are single-valued

The DG-derivatives introduced in this paper are essentially the same as the weak derivatives first introduced by Wang and Ye [109], where the weak derivatives are locally defined at the first place. Weak derivatives then can be defined globally element-by-element. To distinguish the original definition of weak derivatives by Wang and Ye [109], we introduce the different names DG-derivatives. If $\hat{u}$ and $\hat{p}$ are single-valued globally, then the relationship between DG-derivatives and weak derivatives is shown in Table 2.1. We introduce DG-derivatives mainly due to their consistency with the classical weak derivatives on the conforming spaces as shown in the following lemma.

**Lemma 2.2** It holds that

$$\nabla_{dg} = \nabla_h \quad \text{if} \quad V_h \subset V_{\text{non}} = \{u_h \in V_h : \langle u_h, q_h \cdot n_K \rangle_{\partial \Omega} = 0, \forall q_h \in \tilde{Q}_h\}. \tag{2.9}$$

Further,

$$\nabla_{dg} = \nabla \quad \text{if} \quad V_h \subset H^1(\Omega), \quad \text{and} \quad \text{div}_{dg} = \text{div} \quad \text{if} \quad Q_h \subset H(\text{div}, \Omega). \tag{2.10}$$

**Proof.** If $u_h \in V_{non}$, for any $\tilde{q}_h \in \tilde{Q}_h$, we have

$$\langle \nabla_{dg} u_h, q_h \rangle = -(u_h, \text{div} q_h)_{\tau_h} + \langle u_h, (q_h - \hat{q}_h) \cdot n_K \rangle_{\partial \Omega}$$

$$= -(u_h, \text{div} q_h)_{\tau_h} + \langle u_h, q_h \cdot n \rangle_{\partial \Omega}$$

$$= (\nabla_h u_h, q_h)_{\tau_h},$$

which gives rise to the first consistency relationship (2.9). The second can be proven in a similar way.  

### 3 Basic setup

Now we start with the second-order elliptic equation (1.1) and set $p = -\alpha \nabla u$ to obtain the following form

$$\begin{cases}
    cp + \nabla u = 0 & \text{in } \Omega, \\
    \text{div} p = f & \text{in } \Omega.
\end{cases} \tag{3.1}$$
Multiplying the first and second equations by \( q_h \in Q_h \) and \( v_h \in V_h \), respectively, then integrating on an element \( K \in T_h \) we get
\[
\begin{cases}
(c, q_h)_K - (u, \text{div} q_h)_K + \langle u, q_h \cdot n_K \rangle_{\partial K} = 0 & \forall q_h \in Q_h, \\
(p, \nabla v_h)_K - \langle p \cdot n_K, v_h \rangle_{\partial K} = -(f, v_h)_K & \forall v_h \in V_h.
\end{cases}
\]
(3.2)

Summing on all \( K \in T_h \), we have
\[
\begin{cases}
(c, q_h)_T - (u, \text{div} q_h)_T + \langle u, q_h \cdot n_T \rangle_{\partial T} = 0 & \forall q_h \in Q_h, \\
(p, \nabla v_h)_T - \langle p \cdot n, v_h \rangle_{\partial T} = -(f, v_h)_T & \forall v_h \in V_h.
\end{cases}
\]
(3.3)

Now, we approximate \( u \) and \( p \) by \( \hat{u}_h, \hat{p}_h \in Q_h \), respectively, and the trace of \( u \) and the flux \( p \cdot n \) on \( \partial K \) by \( \check{u}_h, \check{p}_h \cdot n \) (see Figure 3.1). Hence, we have
\[
\begin{cases}
(c, q_h)_T - (u_h, \text{div} q_h)_T + \langle \check{u}_h, q_h \cdot n_T \rangle_{\partial T} = 0 & \forall q_h \in Q_h, \\
(p_h, \nabla v_h)_T - \langle \check{p}_h \cdot n, v_h \rangle_{\partial T} = -(f, v_h)_T & \forall v_h \in V_h.
\end{cases}
\]
(3.4)

Next, we derive appropriate equations for the variables of \( \check{u}_h \) and \( \check{p}_h \). There are three different approaches. The starting point of the first two approaches is the following relationship:
\[
\check{p}_h \cdot n_K + \tau \check{u}_h = p_h \cdot n_K + \tau \hat{P}_h(u_h), \quad \check{p}_h = \check{p}_h n_e.
\]
(3.5)

Here, \( \hat{P}_h \) is an operator which will be specified later. Note that we only use one of \( \check{p}_h \) and \( \check{u}_h \) as an unknown and then use (3.5) to determine the other. The third approach is to define \( \check{p}_h \) and \( \check{u}_h \) in terms of \( u_h \) and \( p_h \). The details are given below.

**First approach: Stabilized hybrid primal (WG)** We first set \( \hat{P}_h \) to be an identity operator in (3.5) and \( \check{p}_h := \check{p}_h = \check{p}_h n_e \in \check{Q}_h \) to be a single-valued unknown. The “continuity” of \( \check{u}_h \) is then enforced weakly as follows:
\[
\langle \check{u}_h, \check{q}_h \cdot n \rangle_{\partial T_h} = 0 \quad \forall \check{q}_h \in \check{Q}_h,
\]
(3.6)

where \( \check{u}_h \) is again given by (3.5). From the identity (2.2) and the fact that \( [\check{q}_h] = 0 \), a straightforward calculation shows that (3.6) can be rewritten as
\[
\langle [\check{u}_h], \check{q}_h \rangle_{\check{E}_h} := \sum_{e \in \check{E}_h} \langle [\check{u}_h], \check{q}_h \rangle_e = 0 \quad \forall \check{q}_h \in \check{Q}_h.
\]
(3.7)

From Definition 1.1 for the DG-gradient, WG methods can be rewritten exactly as (1.10). We should note that the stabilization parameter \( \eta \) in (1.11) is in fact \( \tau^{-1} \), where \( \tau \) is the parameter shown in (3.5). As a special case, when \( \eta = 0 \), we obtain the hybrid primal methods [94, 116, 96], namely (1.9).
As a further special case, when \( \eta = 0 \) and \( \bar{Q}_h = \{0\} \), we obtain the primal methods. Then, under certain “continuity” properties pertaining to \( V_h \) and \( Q_h \), the operator \( \nabla_{\text{div}} \) reduces to \( \nabla_h \) and the operator \( \nabla_{\text{div}_{\text{dg}}} \) reduces to \( \nabla_h \) (see Lemma 2.2). Hence, the primal methods read as (1.8). For instance, if we choose \( V_h \) satisfying the “continuity” condition \( V_h \subset H^1_0(\Omega) \) and \( Q_h \) such that \( \nabla V_h \subset cQ_h \), we obtain the conforming FEMs [1, 103, 82, 29, 90, 62]. And if we choose the \( V_h \) satisfying the weak “continuity” condition

\[
V_h = V_{\text{non}}^{k+1} = \{ v_h \in V_h^{k+1}, \int_e [v_h] \cdot \hat{q}_h \, ds = 0, \forall \hat{q}_h \in \bar{Q}_h, \forall e \in E_h^k \},
\]

we obtain the Crouzeix-Raviart (CR) nonconforming element [58] when \( k = 0 \) and \( \nabla_h V_h \subset cQ_h \).

**Second approach: Stabilized hybrid mixed (HDG)** We set \( \hat{u}_h := \hat{u}_h \in \hat{V}_h \) to be a single-valued unknown. The “continuity” of \( \hat{p}_h \) is then enforced weakly as follows:

\[
(\hat{p}_h \cdot n, \hat{v}_h)_{\partial T_h} = 0 \quad \forall \hat{v}_h \in \hat{V}_h, \tag{3.9}
\]

where \( \hat{p}_h \) is given by (3.5). From the identity (2.2) and the fact that \( [\hat{v}_h] = 0 \), a straightforward calculation shows that (3.9) can be rewritten as

\[
([\hat{p}_h], \hat{v}_h)|_{E_h^i} := \sum_{e \in E_h^i} ([\hat{p}_h], \hat{v}_h)_e = 0 \quad \forall \hat{v}_h \in \hat{V}_h. \tag{3.10}
\]

If \( \hat{P}_h \) is an identity operator, using the DG-divergence from Definition 1.1, we can rewrite the standard HDG method as (1.14). If \( \hat{P}_h \) is a local \( L^2 \) projection, namely

\[
\hat{P}_h|_{\partial K} := \hat{P}_{\partial K} : L^2(\partial K) \to \bar{Q}(\partial K),
\]

where \( \bar{Q}(\partial K) \) is the trace space on \( \partial K \), namely \( \bar{Q}(\partial K) = \bigcup_{e \in \partial K} \bar{Q}(e) \), we obtain the modified HDG methods with reduced stabilization [88].

As a special case, when \( \tau = 0 \), we obtain the hybrid mixed methods [3, 21, 41], namely (1.13). As a further special case, when \( \tau = 0 \) and \( \bar{V}_h = \{0\} \), we obtain the mixed methods. Then, under certain “continuity” properties pertaining to \( Q_h \) and \( V_h \), the operator \( \text{div}_{\text{dg}} \) reduces to \( \text{div}_h \). Hence, the mixed methods [95, 86, 19, 17, 21, 11] read as (1.12).

**Third approach: DG** We define \( \hat{p}_h = \hat{p}_h \) and \( \hat{u}_h = \hat{u}_h \) in terms of \( u_h \) and \( p_h \), namely

\[
\begin{aligned}
\hat{p}_h &= \hat{p}(p_h, u_h) \quad \text{on } E_h, \\
\hat{u}_h &= \hat{u}(p_h, u_h) \quad \text{on } E_h.
\end{aligned} \tag{3.11}
\]

These three different approaches can be summarized in Table 3.1.

4 Stability and convergence analysis of stabilized hybrid primal (WG) methods

The stabilized hybrid primal (WG) methods read: Find \( (\hat{p}_h, u_h) \in \bar{Q}_h \times V_h \) such that for any \( (\bar{q}_h, v_h) \in \bar{Q}_h \times V_h \),

\[
\begin{aligned}
a_w(\hat{p}_h, \bar{q}_h) + b_w(u_h, \bar{q}_h) &= 0, \\
b_w(\hat{p}_h, v_h) &= -(f, v_h)|_{T_h}.
\end{aligned} \tag{4.1}
\]
In this subsection, we shall show the well-posedness of WG methods. Here, $\eta > 0$ is the stabilized parameter.

The following lemma shows the consistency property of stabilized hybrid primal (WG) methods.

**Lemma 4.1** Let $f \in L^2(\Omega)$ and $(p, u)$ be the solution of (1.2) or (1.3), then $(p, u)$ satisfies the following consistency property

\[
\begin{align*}
\forall q_h \in \bar{Q}_h, \\
b_w(p, v_h) = - \langle f, v_h \rangle_{\mathcal{T}_h}.
\end{align*}
\] (4.3)

**4.1 Gradient-based uniform inf-sup condition**

In this subsection, we shall show the well-posedness of WG methods (4.2) when choosing the parameter $\eta$ as $\eta = \rho h_K$ for $\rho > 0$. More precisely, we will give the uniform inf-sup condition under the following parameter-dependent norms

\[
\begin{align*}
\| \tilde{p}_h \|^2_{0,\rho,h} &= (cp_h, p_h)_{\mathcal{T}_h} + \rho \sum_{K \in \mathcal{T}_h} h_K \langle (p_h - \bar{p}_h) \cdot n_K, (p_h - \bar{p}_h) \cdot n_K \rangle_{\partial K}, \\

\| v_h \|^2_{1,\rho,h} &= \| \nabla v_h \|^2 + \rho^{-1} \sum_{e \in \mathcal{E}_h} h_e^{-1} \| \tilde{Q}_e([v_h]) \|^2_{0,e},
\end{align*}
\] (4.4)

where $\tilde{Q}_e$ is the $L^2$ projection from $L^2(e)$ to $\tilde{Q}(e)$. We point out that $\| v_h \|_{1,\rho,h}$ is indeed a norm on $V_h$ if $\tilde{Q}_h \subset \bar{Q}_h$, namely $\tilde{Q}_h$ contains the piecewise constant space on $\mathcal{E}_h$.

Using these parameter-dependent norms (4.4), we have the following results, whose details will be reported in [74, 73].

**Theorem 4.2** ([74, 73]) For any $0 < \rho \leq 1$, and for any $\tilde{p}_h, \tilde{q}_h \in \bar{Q}_h, v_h \in V_h$, we have

\[
\begin{align*}
\| a_w(\tilde{p}_h, \tilde{q}_h) \| &\leq \| \tilde{p}_h \|^2_{0,\rho,h} \| \tilde{q}_h \|^2_{0,\rho,h}, \\
b_w(\tilde{p}_h, v_h) &\leq C_w \| \tilde{p}_h \|^2_{0,\rho,h} \| v_h \|^2_{1,\rho,h}, \\
a_w(\tilde{p}_h, \tilde{p}_h) &\geq \| \tilde{p}_h \|^2_{0,\rho,h},
\end{align*}
\] (4.5)

where $C_w$ is independent of both mesh size $h$ and $\rho$. 

---

Table 3.1: Summary of WG, HDG and DG methods

| Type of Methods | WG | HDG | DG |
|----------------|----|-----|----|
| Volume Equations | $(c\tilde{p}_h, \tilde{q}_h)_{\mathcal{T}_h} + \langle u_h, \text{div} \tilde{q}_h \rangle_{\mathcal{T}_h} + \langle \bar{u}_h, \tilde{q}_h \rangle_{\partial \mathcal{T}_h} = 0,$ | $\forall \tilde{q}_h \in \bar{Q}_h,$ | $\forall \tilde{q}_h \in \bar{Q}_h,$ |
| Interelement Equations | $\langle \bar{p}_h \cdot n_K + \tau \tilde{v}_h \rangle = \langle \bar{p}_h \cdot n_K \rangle_{\partial K}$ | $\langle \bar{p}_h \cdot n_K \rangle_{\partial K}$ | $\langle \bar{p}_h \cdot n_K \rangle_{\partial K}$ |
| $\langle \bar{v}_h, [\tilde{p}_h] \rangle_{\mathcal{E}_h} = 0$ | $\forall \bar{v}_h \in \bar{Q}_h$ | $\forall \bar{v}_h \in \bar{Q}_h$ | $\forall \bar{v}_h \in \bar{Q}_h$ |

---

\[\rho > \rho^*\text{ independent of } h\]
Theorem 4.3 ([74, 73]) Assume that $\nabla_h V_h \subset Q_h$, then for any $0 < \rho \leq 1$, we have
\[
\inf_{v_h \in V_h} \sup_{\tilde{p}_h \in Q_h} \frac{b_w(\tilde{p}_h, v_h)}{\|v_h\|_{1,\rho,h} \|\tilde{p}_h\|_{0,\rho,h}} \geq \beta_{w,0},
\]
where $\beta_{w,0}$ is independent of both mesh size $h$ and $\rho$.

Corollary 4.4 ([74, 73]) Assume that $\nabla_h V_h \subset Q_h$. Then there exists a unique solution $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ satisfying (4.1) with $\eta = \rho h_K$. Further, for any $0 < \rho \leq 1$ the following estimate holds
\[
\|\tilde{p}_h\|_{0,\rho,h} + \|u_h\|_{1,\rho,h} \leq C_{w,1} \|f\|_{-1,\rho,h},
\]
where $C_{w,1}$ is a uniform constant with respect to both $\rho$ and $h$.

Theorem 4.5 ([74]) Let $(p, u)$ be the solution of (1.2) and assume that $p \in H^1(\Omega)$. Further, let $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ be the solution of (4.1) with $\eta = \rho h_K$. If we choose the spaces $Q_h \times V_h$ such that $\nabla_h V_h \subset Q_h$, then for any $0 < \rho \leq 1$ the following estimate holds
\[
\|p - \tilde{p}_h\|_{0,\rho,h} + \|u - u_h\|_{1,\rho,h} \leq C_{r,1}^{1/2} \inf_{\tilde{q}_h \in Q_h, v_h \in V_h} (\|p - \tilde{q}_h\|_{0,\rho,h} + \|u - v_h\|_{1,\rho,h}),
\]
where $C_{r,1}$ is a uniform constant with respect to both $\rho$ and $h$.

Corollary 4.6 ([74]) Let $(p, u)$ be the solution of (1.2) and $p \in H^{k+1}(\Omega)$, $u \in H^{k+2}(\Omega)$, and $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ be the solution of (4.1) with $\eta = \rho h_K$. If we choose the spaces $V_h \times Q_h \subset \tilde{Q}_h \times Q_h \subset V_h$ as $V_h \times Q_h \subset Q_h$ and $Q_h$, then for any $0 < \rho \leq 1$ the following estimate holds
\[
\|p - \tilde{p}_h\|_{0,\rho,h} + \|u - u_h\|_{1,\rho,h} \leq C_{r,1}^{1/2} (\|p\|_{k+1,h} + \|u\|_{k+2,h}),
\]
where $C_{r,1}$ is independent of both $\rho$ and $h$.

The above three theorems improve the results of [110], where the inf-sup condition for a given constant $\rho$ was proven.

4.2 Divergence-based uniform inf-sup condition

Next, we shall show the well-posedness of WG methods under another pair of the parameter-dependent norms. We choose the parameter $\eta$ as $\eta = \rho^{-1} h_K^{-1}$ in (4.2) and define the norms as follows
\[
\|\tilde{p}_h\|_{\text{div},\rho,h}^2 = (cp_h, p_h)_{\mathcal{T}_h} + (\text{div} p_h, \text{div} p_h)_{\mathcal{T}_h} + \rho^{-1} \sum_{K \in \mathcal{T}_h} h_K^{-1} ((p_h - \tilde{p}_h) \cdot n_K, (p_h - \tilde{p}_h) \cdot n_K)_{\partial K},
\]
\[
\|u_h\|^2 = (u_h, u_h)_{\mathcal{T}_h}.
\]
We have the uniform inf-sup condition for the following formulation
\[
A_w((\tilde{p}_h, u_h), (\tilde{q}_h, v_h)) = a_w(\tilde{p}_h, \tilde{q}_h) + b_w(\tilde{q}_h, u_h) + b_w(\tilde{p}_h, v_h).
\]

Theorem 4.7 ([74, 73]) Let $R_h \subset H(\text{div}, \Omega) \cap Q_h$ be the Raviart-Thomas finite element space. Assume that $\|R_h\| \subset \tilde{Q}_h$ and $V_h = \text{div}_h Q_h$. Then, we have
\[
\inf_{(\tilde{p}_h, u_h) \in Q_h \times V_h} \sup_{(\tilde{q}_h, v_h) \in Q_h \times V_h} \frac{A_w((\tilde{p}_h, u_h), (\tilde{q}_h, v_h))}{\|\tilde{p}_h\|_{\text{div},\rho,h} + \|\tilde{q}_h\|_{\text{div},\rho,h}} \geq \beta_{w,1},
\]
where $\beta_{w,1}$ is independent of both $\rho$ and mesh size $h$. 
Corollary 4.8 ([74, 73]) Assume that the spaces $\tilde{Q}_h \times V_h$ satisfy the conditions of Theorem 4.7. Then there exists a unique solution $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ satisfying (4.1) with $\eta = \rho^{-1}h_K^{-1}$, and for any $0 < \rho \leq 1$ the following estimate holds

$$\|\tilde{p}_h\|_{\text{div}, p, h} + \|u_h\| \leq C_{\text{w}, 2}\|f_h\|,$$

(4.13)

where $C_{\text{w}, 2}$ is a uniform constant with respect to both $\rho$ and $h$.

Theorem 4.9 ([74]) Let $(p, u)$ be the solution of (1.3) and assume that $p \in H^1(\Omega)$. Let $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ be the solution of (4.1). If we choose the spaces $\tilde{Q}_h \times V_h$ such that the inf-sup condition (4.12) is satisfied, then for any $0 < \rho \leq 1$ the following estimate holds

$$\|p - \tilde{p}_h\|_{\text{div}, p, h} + \|u - u_h\| \leq C_{e, 2} \inf_{q_h \in Q_h, v_h \in V_h} \left(\|p - \tilde{q}_h\|_{\text{div}, p, h} + \|u - v_h\|\right),$$

(4.14)

where $C_{e, 2}$ is a uniform constant with respect to both $\rho$ and $h$.

Corollary 4.10 ([74]) Let $(p, u)$ be the solution of (1.3) and assume that $p \in H^{k+1}(\Omega)$, $\text{div} p \in H^{k+1}(\Omega)$, $u \in H^{k+1}(\Omega)$. Let $(\tilde{p}_h, u_h) \in \tilde{Q}_h \times V_h$ be the solution of (4.1) with $\eta = \rho^{-1}h_K^{-1}$. If we choose the spaces $V_h \times Q_h \times \tilde{Q}_h$ as $V_h^k \times Q_h^{k, RT} \times \tilde{Q}_h^k$, then for any $0 < \rho \leq 1$ the following estimate holds

$$\|p - \tilde{p}_h\|_{\text{div}, p, h} + \|u - u_h\| \leq C_{r, 2}h^{k+1} + \|\text{div} p\|_{k+1} + \|u\|_{k+1},$$

(4.15)

where $C_{r, 2}$ is independent of both $h$ and $\rho$.

We show here some convergence results which are uniform with respect to the stabilization parameter, which, again, improve the convergence results in [110, 30]. These are mainly based on the uniform inf-sup conditions we present in Theorems 4.3 and 4.7.

5 Stability and convergence analysis of stabilized hybrid mixed (HDG) methods

The stabilized hybrid mixed (HDG) methods read: Find $(p_h, \hat{u}_h) \in Q_h \times \tilde{V}_h$ such that for any $(q_h, \hat{v}_h) \in Q_h \times \tilde{V}_h$,

$$\begin{cases}
a_h(p_h, q_h) + b_h(q_h, \hat{u}_h) = 0, \\
b_h(p_h, \hat{v}_h) + c_h(\hat{u}_h, \hat{v}_h) = -(f, \hat{v}_h),
\end{cases}$$

(5.1)

Here, the bilinear forms are defined as follows

$$\begin{align*}
a_h(p_h, q_h) &= (cp_h, q_h), \\
b_h(q_h, \hat{u}_h) &= -(u_h, \text{div} q_h) + \langle \hat{u}_h, q_h \cdot \hat{n}\rangle_{\partial T_h}, \\
c_h(\hat{u}_h, \hat{v}_h) &= -\langle \tau(\hat{P}_h(u_h) - \hat{u}_h), \hat{P}_h(v_h) - \hat{v}_h\rangle_{\partial T_h},
\end{align*}$$

(5.2)

and $\tau > 0$ is the stabilization parameter. If $\hat{P}_h$ is an identity operator, then we obtain the standard HDG method. If $\hat{P}_h$ is a local $L^2$ projection, i.e.

$$\hat{P}_h|_{\partial K} := \hat{P}_{\partial K} : L^2(\partial K) \rightarrow \bar{Q}(\partial K),$$

(5.3)

where $\bar{Q}(\partial K)$ is the trace space on $\partial K$, namely $\bar{Q}(\partial K) = \bigcup_{e \in \partial K} \bar{Q}(e)$, we obtain the modified HDG methods with reduced stabilization [88].

The following lemma shows the consistency property of stabilized hybrid mixed (HDG) methods.
Lemma 5.1 Let $f \in L^2(\Omega)$, and $(p, u)$ be the solution of (1.2) or (1.3), then $(p, u)$ satisfies the following consistency property
\[
\begin{align*}
  a_h(p, q_h) + b_h(q_h, u) &= 0 \quad \forall q_h \in Q_h, \\
  b_h(p, \tilde{v}_h) + c_h(u, \tilde{v}_h) &= -(f, v_h)_T_h \quad \forall \tilde{v}_h \in V_h.
\end{align*}
\] (5.4)

5.1 Divergence-based uniform inf-sup condition

In this subsection, we will give the uniform inf-sup condition for (5.2) when $\tau = \rho h^k$ under the following parameter-dependent norms
\[
\|p_h\|_{\text{div}, p, h}^2 = (cp_h, p_h)_T_h + (\text{div} p_h, \text{div} p_h)_T_h + \rho^{-1} \sum_{e \in E_h} h_e^{-1} \langle \hat{P}_e([p_h]), \hat{P}_e([p_h]) \rangle_e,
\]
\[
\|\tilde{v}_h\|_{0, p, h} = (v_h, v_h)_T_h + \rho \sum_{e \in E_h} h_e \langle \tilde{v}_h, \tilde{v}_h \rangle_e.
\] (5.5)

where $\hat{P}_e : L^2(e) \mapsto \hat{V}(e)$ is the $L^2$ projection.

Using the parameter-dependent norms (5.5), we have the following results, whose details will be reported in [74, 73].

Theorem 5.2 For any $0 < \rho \leq 1$, the boundedness of $a_h(\cdot, \cdot), b_h(\cdot, \cdot)$ and $c_h(\cdot, \cdot)$ is as follows [74]
\[
\begin{align*}
  |a_h(p_h, q_h)| &\leq \|p_h\|_{\text{div}, p, h} \|q_h\|_{\text{div}, p, h}, \\
  |b_h(q_h, \tilde{u}_h)| &\leq \|q_h\|_{\text{div}, p, h} \|\tilde{u}_h\|_{0, p, h}, \\
  |c_h(\tilde{u}_h, \tilde{v}_h)| &\leq C \|\tilde{u}_h\|_{0, p, h} \|\tilde{v}_h\|_{0, p, h},
\end{align*}
\] (5.6)

where $C$ is independent of both mesh size $h$ and $\rho$.

Denote
\[\text{Ker}(B) := \{q_h \in Q_h : b_h(q_h, \tilde{u}_h) = 0, \forall \tilde{u}_h \in V_h\}.\]

Then we have the coercivity of $a_h(\cdot, \cdot)$ on the Ker($B$) and the inf-sup condition of $b_h(\cdot, \cdot)$ as follows.

Theorem 5.3 ([74, 73]) Assume that $\text{div} Q_h \subset V_h$. Then
\[a_h(p_h, p_h) \geq \|p_h\|_{\text{div}, p, h}^2 \quad \forall p_h \in \text{Ker}(B).\] (5.7)

Theorem 5.4 ([74, 73]) For $k \geq 0$, if $Q_h = Q_h^{k+1}, V_h = V_h^k, \tilde{V}_h = \tilde{V}_h^r$ where $0 \leq r \leq k + 1$, or $Q_h = \tilde{Q}_h^{k,RT}, V_h = V_h^k, \tilde{V}_h = \tilde{V}_h^r$ where $0 \leq r \leq k$, then we have
\[
\inf_{\tilde{u}_h \in \tilde{V}_h} \sup_{q_h \in Q_h} \frac{b_h(q_h, \tilde{u}_h)}{\|q_h\|_{\text{div}, p, h} \|\tilde{u}_h\|_{0, p, h}} \geq \beta_0,
\] (5.8)

where $\beta_0$ is a constant independent of both $\rho$ and mesh size $h$.

Remark 5.5 When $\tau = 0$, we can also have the stability result as Theorem 5.4, when choosing the following norms for any $\tilde{v}_h \in \tilde{V}_h$ and $p_h \in Q_h$
\[
\|p_h\|_{\text{div}, 1, h}^2 = (cp_h, p_h)_T_h + (\text{div} p_h, \text{div} p_h)_T_h + \sum_{e \in E_h} h_e^{-1} \langle \hat{P}_e([p_h]), \hat{P}_e([p_h]) \rangle_e,
\]
\[
\|\tilde{v}_h\|_{0, 1, h}^2 = (v_h, v_h)_T_h + \sum_{e \in E_h} h_e \langle \tilde{v}_h, \tilde{v}_h \rangle_e.
\]
Theorem 5.6 ([74]) Let \((p, u)\) be the solution for (1.3) and \((p_h, \tilde{u}_h)\) ∈ \(Q_h × \tilde{V}_h\) be the solution for (5.1) with \(τ = ph_K\). If we choose the spaces \(Q_h × \tilde{V}_h\) that satisfy the condition in Theorem 5.4, then for any \(0 < \rho ≤ 1\) the following estimate holds

\[
\|p - p_h\|_{\text{div}, \rho, h} + \|u - \tilde{u}_h\|_{0, \rho, h} ≤ C_{e,3} \inf_{q_h ∈ Q_h, \tilde{v}_h ∈ \tilde{V}_h} (\|p - q_h\|_{\text{div}, \rho, h} + \|u - \tilde{v}_h\|_{0, \rho, h}),
\]

(5.9)

where \(C_{e,3}\) is a uniform constant with respect to both \(\rho\) and \(h\).

Corollary 5.7 ([74]) Let \((p, u)\) be the solution of (1.3) and \(p ∈ H^{k+1}(Ω), \text{div} p ∈ H^{k+1}(Ω), u ∈ H^{k+1}(Ω),\) and \((p_h, \tilde{u}_h)\) ∈ \(Q_h × \tilde{V}_h\) be the solution of (5.1) with \(τ = ph_K\). If we choose the spaces \(V_h × Q_h × \tilde{V}_h\) as \(V_h^k × Q_h^{RT} × \tilde{V}_h\), then the following estimate holds

\[
\|p - p_h\|_{\text{div}, \rho, h} + \|u - \tilde{u}_h\|_{0, \rho, h} ≤ C_{r,3} h^k(\|p\| + \|\text{div} p\| + \|u\|),
\]

(5.10)

where \(C_{r,3}\) is independent of both \(h\) and \(ρ\).

5.2 Gradient-based uniform inf-sup condition

Next, we shall present the well-posedness of stabilized hybrid mixed (HDG) methods under another pair of parameter-dependent norms. We choose \(τ = ph^{-1}K\) in (5.2) and define for any \(q_h ∈ Q_h\) and \(\tilde{v}_h ∈ \tilde{V}_h\)

\[
\|q_h\|_h^2 = (c_q, q_h)_T_h,
\]

\[
\|\tilde{v}_h\|_{1, \rho, h}^2 = (\nabla_h v_h, \nabla_h \tilde{v}_h)_T_h + \rho^{-1} h^{-1} (\tilde{P}_h(v_h) - \tilde{v}_h, \tilde{P}_h(v_h) - \tilde{v}_h)_{\partial K},
\]

(5.11)

where \(\tilde{P}_h\) is either an identity operator or a local projection as illustrated in (5.3). A straightforward calculation shows that

\[
\langle \tilde{P}_h(v_h) - \tilde{v}_h, \tilde{P}_h(v_h) - \tilde{v}_h \rangle_{\partial K} = 2\langle \{\tilde{P}_h(v_h) - \tilde{v}_h\}, \{\tilde{P}_h(v_h) - \tilde{v}_h\} \rangle_{\mathcal{E}_h} + \frac{1}{2} \langle \|\tilde{P}_h(v_h) - \tilde{v}_h\|, \|\tilde{P}_h(v_h) - \tilde{v}_h\| \rangle_{\mathcal{E}_h}.
\]

Hence, if \(\tilde{P}_h\) is the identity operator, then \(\|\tilde{v}_h\|_{1, \rho, h}\) is indeed a norm on \(\tilde{V}_h\). Moreover, if \(\tilde{P}_h\) is the local projection defined in (5.3), then \(\|\tilde{v}_h\|_{1, \rho, h}\) is indeed a norm on \(\tilde{V}_h\) when \(\tilde{V}_h^0 ⊂ \tilde{V}_h\), i.e. \(\tilde{V}_h\) contains the piecewise constant space on \(\mathcal{E}_h\).

We have the uniform inf-sup condition for the following formulation

\[
A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h)) = a_h(p_h, q_h) + b_h(q_h, \tilde{u}_h) + b_h(p_h, \tilde{v}_h) + c_h(\tilde{u}_h, \tilde{v}_h).
\]

(5.12)

Theorem 5.8 ([74, 73]) Assume that \(\nabla_h V_h ⊂ Q_h\). Then there exists a positive constant \(ρ_0\) which only depends on the shape regularity of the mesh, such that for any \(0 < ρ ≤ ρ_0\), we have

\[
\inf_{(p_h, \tilde{u}_h) ∈ Q_h × \tilde{V}_h} \sup_{q_h, \tilde{v}_h ∈ Q_h × \tilde{V}_h} \frac{A_h((p_h, \tilde{u}_h), (q_h, \tilde{v}_h))}{(\|\tilde{u}_h\|_{1, \rho, h} + \|P_h\|)(\|\tilde{v}_h\|_{1, \rho, h} + \|q_h\|)} ≥ \beta_1,
\]

(5.13)

where \(\beta_1\) is independent of both \(\rho\) and mesh size \(h\).

Corollary 5.9 ([74, 73]) Assume that \(\nabla_h V_h ⊂ Q_h\). Then there exists a unique solution \((p_h, \tilde{u}_h) ∈ Q_h × \tilde{V}_h\) satisfying (5.1) with \(τ = ph^{-1}K\). Further, there exists a positive constant \(ρ_0\) such that for any \(0 < ρ ≤ ρ_0\) the following estimate holds

\[
\|p_h\| + \|\tilde{u}_h\|_{1, \rho, h} ≤ C_{d,2} \|f\|_{-1, \rho, h},
\]

(5.14)

where \(C_{d,2}\) is a uniform constant with respect to both \(\rho\) and \(h\) and \(\|f\|_{-1, \rho, h} = \sup_{\tilde{v}_h ∈ \tilde{V}_h} \frac{(f, \tilde{v}_h)_{\mathcal{T}_h}}{\|\tilde{v}_h\|_{1, \rho, h}}\).
From the above corollary and the discrete Poincaré–Friedrichs inequalities for piecewise $H^1$ functions [12], i.e., $\|v_h\| \lesssim \|\nabla_h v_h\| + \sum_{e \in E_h} h_e^{-1}\|v_h\|_{0,e}$, we further have $\|p_{h}\| + \|u_{h}\|_{1,\rho,h} \leq C_{d,3}\|f_h\|$. The stability and consistency results of the stabilized hybrid mixed (HDG) methods lead to the following quasi-optimal approximation.

**Theorem 5.10** ([74]) Let $(p, u)$ be the solution of (1.2) and $(p_{h}, \tilde{u}_{h}) \in Q_{h} \times \tilde{V}_{h}$ be the solution of (5.1) with $\tau = \rho^{-1} h_{K}^{-1}$. If we choose the spaces $Q_{h} \times \tilde{V}_{h}$ such that $\nabla_h V_h \subset Q_{h}$, then there exists a constant $\rho_{0}$ such that for any $0 < \rho \leq \rho_{0}$ the following estimate holds

$$\|p - p_{h}\| + \|u - \tilde{u}_{h}\|_{1,\rho,h} \leq C_{e,4} \inf_{\bar{q}_{h} \in Q_{h}, \tilde{v}_{h} \in \tilde{V}_{h}} \left(\|p - q_{h}\| + \|u - \tilde{v}_{h}\|_{1,\rho,h}\right),$$

where $C_{e,4}$ is a uniform constant with respect to both $\rho$ and $h$.

**Corollary 5.11** ([74]) Let $(p, u)$ be the solution of (1.2) and $p \in H^{k+1}(\Omega), u \in H^{k+2}(\Omega)$, $(p_{h}, \tilde{u}_{h}) \in Q_{h} \times \tilde{V}_{h}$ be the solution of (5.1) with $\tau = \rho^{-1} h_{K}^{-1}$, and $\tilde{P}_{h}$ be an identity operator. If we choose the spaces $V_h \times Q_h \times \tilde{V}_h$ as $V_h^{k+1} \times Q_h^{k} \times \tilde{V}_h^{k+1}$, then the following estimate holds

$$\|p - p_{h}\| + \|u - \tilde{u}_{h}\|_{1,\rho,h} \leq C_{r,4} h^{k+1}(\|p\|_{k+1} + \|u\|_{k+2}),$$

where $C_{r,4}$ is independent of both $h$ and $\rho$.

**Corollary 5.12** ([74]) Let $(p, u)$ be the solution of (1.2) and $p \in H^{k+1}(\Omega), u \in H^{k+2}(\Omega)$, $(\tilde{p}_{h}, u_{h}) \in Q_{h} \times \tilde{V}_{h}$ be the solution of (5.1) with $\tau = \rho^{-1} h_{K}^{-1}$ and $\tilde{P}_{h}$ be a local $L^2$ projection illustrated in (5.2). If we choose the spaces $V_h \times Q_h \times \tilde{V}_h$ as $V_h^{k+1} \times Q_h^{k} \times \tilde{V}_h^{k}$, then the following estimate holds

$$\|p - p_{h}\| + \|u - \tilde{u}_{h}\|_{1,\rho,h} \leq C_{r,5} h^{k+1}(\|p\|_{k+1} + \|u\|_{k+2}),$$

where $C_{r,5}$ is independent of both $h$ and $\rho$.

In [47], Cockburn, Gopalakrishnan and Sayas established the error analysis for stabilized hybrid mixed (HDG) methods based on a carefully designed projection operator. In this paper, we present several uniform convergence results with respect to the stabilization parameter. As a result, the constants in the error estimates $C_{r,i}(i = 1, 2, \cdots, 5)$ are independent of $\rho$.

### 6 Discontinuous Galerkin methods

In recent years, DG methods have been applied to the solution of various differential equations due to their flexibility in constructing feasible local-shape function spaces and their advantage in capturing non-smooth or oscillatory solutions effectively. Instead of using the Lagrange multiplier technique, a penalty term is added to the bilinear form of the DG method to force the continuity (see [4, 26, 70, 72, 71] and the references therein). With the concept of DG-gradient and DG-divergence defined as in Definition 1.1, most of the DG methods for approximating the elliptic problem can be written as

$$\begin{cases}
\begin{aligned}
cp_{h} - \text{div}^{*}_{\text{dg}} \tilde{u}_{h} &= 0 &\text{in } \Omega, \\
-\nabla^{*}_{\text{dg}} \tilde{p}_{h} &= f_{h} &\text{in } \Omega, \\
\tilde{p}_{h} &= \tilde{p}(p_{h}, u_{h}) &\text{on } E_{h}, \\
\tilde{u}_{h} &= \tilde{u}(p_{h}, u_{h}) &\text{on } E_{h},
\end{aligned}
\end{cases}$$

(6.1a) (6.1b) (6.1c) (6.1d)
where \( \tilde{p} \) and \( \tilde{u} \) are the formulas for defining \( \tilde{p}_h \) and \( \tilde{u}_h \) in the terms of \( p_h \) and \( u_h \), respectively. A crucial feature of DG methods is that \( \tilde{p}_h \) and \( \tilde{u}_h \) are given explicitly in (6.1c) – (6.1d). A basic question is: How do we define \( p_h \) and \( u_h \) in order for the DG schemes to result in good approximations of the original problems? In the DG schemes, the local problems on each element \( K \) are connected through the \( \tilde{p}_h \cdot \mathbf{n}_K \) and \( \tilde{u}_h \). Therefore, in order for make the schemes to be good approximations, \( \tilde{p}_h \) and \( \tilde{u}_h \) should be single-valued on the element edges. Recalling condition (iii) in Lemma 2.1 when \( \nabla_{dK} \) and \( -\nabla_{dK} \) are mutually dual, we see that \( p_h = \{ p_h \} \) and \( u_h = \{ u_h \} \) are natural choices. However, it is known that such choices cannot ensure the stability of the DG schemes. Hence, penalty terms are used to force the continuity of either \( p_h \) or \( u_h \).

Consequently, in general, to define the numerical traces, (6.1c) – (6.1d) can be given as

\[
\begin{aligned}
\tilde{p}_h &= \gamma \{ p_h \} + (1 - \gamma)\{ -\alpha \nabla_h u_h \} - \beta \{ p_h \} + \mu_1(\{ u_h \}) \quad \text{on } \mathcal{E}_h, \\
\tilde{u}_h &= \{ u_h \} + \beta \cdot \{ u_h \} + \mu_2(\{ p_h \}) \quad \text{on } \mathcal{E}_h, \quad \tilde{u}_h = 0 \quad \text{on } \mathcal{E}_h^\partial. \\
\end{aligned}
\]

Here, \( \gamma \) and \( \beta \) are parameters that we can choose, and \( \mu_1(\{ u_h \}) \) and \( \mu_2(\{ p_h \}) \) are penalty terms. The possible choices of numerical fluxes in the literature are summarized in Table 6.1. In [8, 25], \( r_e : L^2(\mathcal{E}_h) \rightarrow Q_h \) is a lifting operator defined by

\[
\int_{\Omega_e} r_e(\mathbf{w}) \cdot \mathbf{q}_h \, dx = - \int_{\mathcal{E}_h} \mathbf{w} \cdot \{ \mathbf{q}_h \} \, ds \quad \forall \mathbf{q}_h \in Q_h.
\]

| Method                      | \( \gamma \) | \( \beta \) | \( \mu_1(\{ u_h \}) \) | \( \mu_2(\{ p_h \}) \) |
|-----------------------------|--------------|--------------|-------------------------|-------------------------|
| IP method \[59, 114\]       | 0            | 0            | \( \eta_e h^{-1}_e \{ u_h \} \) | 0                       |
| LDG method \[54\]          | 1            | \( \mathcal{O}(1) \) | \( \eta_e h^{-1}_e \{ u_h \} \) | 0                       |
| DG Method of Bassi et. al. \[8\] | 0            | 0            | \( \eta_e \{ r_e(\{ u_h \}) \} \) | 0                       |
| DG Method of Brezzi et. al. \[25\] | 0            | 0            | \( \eta_e \{ r_e(\{ u_h \}) \} \) | 0                       |
| Mixed DG method            | 1            | 0            | 0                       | \( \eta_e h^{-1}_e \{ p_h \} \) |

Table 6.1: DG methods: Numerical fluxes, \( \eta_e = \mathcal{O}(1) \)

In the next two subsections, we introduce two classes of DG methods. The first class of DG methods is used to approximate the form (1.2) so a penalty term \( \mu_1(\{ u_h \}) \) is needed to force the continuity of \( u_h \), and we name this class of DG methods primal DG methods. The second class of DG methods, which is named mixed DG methods, is aimed to approximate the mixed form of the elliptic problem. Hence, a penalty term \( \mu_2(\{ p_h \}) \) is added to force the normal continuity of \( p_h \).

### 6.1 Primal discontinuous Galerkin methods

By Definition 1.1, since \( \tilde{u}_h \) and \( \tilde{p}_h \) are single-valued, we establish the following relations using integration by parts and (2.2):

\[
\langle -\nabla_{dK}^\ast \tilde{u}_h, \mathbf{q}_h \rangle = -(u_h, \nabla \mathbf{q}_h)_{\mathcal{T}_h} + \langle \tilde{u}_h, \{ \mathbf{q}_h \} \rangle_{\mathcal{E}_h} \quad (6.4)
\]

\[
= (\nabla_h u_h, q)_{\mathcal{T}_h} - \langle [u_h], \{ \mathbf{q}_h \} \rangle_{\mathcal{E}_h} + \langle \tilde{u}_h - \{ u_h \}, \{ \mathbf{q}_h \} \rangle_{\mathcal{E}_h}, \quad (6.5)
\]

\[
\langle -\nabla_{dK}^\ast \tilde{p}_h, \mathbf{v}_h \rangle = -(p_h, \nabla \mathbf{v}_h)_{\mathcal{T}_h} + \langle \tilde{p}_h, \{ \mathbf{v}_h \} \rangle_{\mathcal{E}_h} \quad (6.6)
\]

\[
= (\nabla p_h, \mathbf{v}_h)_{\mathcal{T}_h} + \langle \tilde{p}_h - \{ p_h \}, \{ \mathbf{v}_h \} \rangle_{\mathcal{E}_h} - \langle [p_h], \{ \mathbf{v}_h \} \rangle_{\mathcal{E}_h}. \quad (6.7)
\]

Motivated by (6.5) and (6.6), most of the existing primal DG methods can be written as (6.1a) – (6.1b) and (6.2) with specific choices of the parameters \( \gamma \), \( \beta \), and the penalty term \( \mu_1(\{ u_h \}) \). Examples of primal
methods are the IP method [59, 114], the LDG method [54], the method of Bassi et al. [8], and the method of Brezzi et al. [25] listed in Table 6.1. Note that all of these DG methods have a penalty term on $u_h$ so they intend to approximate the solution of the primal form. To put it simply, if $\alpha$ is piecewise constant and $\nabla_h u_h \subset Q_h$, then we can eliminate $p_h$ to obtain the DG formulations with $u_h$ solely. We refer to [4] for a detailed discussion of primal DG methods.

**Remark 6.1** As a combination of continuous and discontinuous Galerkin methods, the so-called enriched DG (EDG) methods [9, 102], which are locally conservative, enrich the approximation space of the continuous Galerkin methods with piecewise constant functions. EDG methods adopt the same weak formulation as DG methods, but have a smaller number of degrees of freedom than the DG methods.

### 6.2 Mixed discontinuous Galerkin methods

In this subsection, we derive a new family of mixed DG methods, which can be regarded as the dual form of primal DG methods. In the literature, there are some existing works that discuss mixed DG methods for elliptic problems [22, 27, 33], but all of these schemes are aimed at approximating the primal form (1.2). Alternatively, the mixed DG methods we propose are designed to approximate the mixed form (1.3).

Instead of penalizing $u_h$, we consider a penalty term for $p_h$ to obtain the mixed DG schemes. Let us choose $\gamma = 1, \beta = 0, \mu_1(\|u_h\|) = 0$, and $\mu_2(|p_h|) = \eta e^{-1}|p_h|$ in (6.2), i.e.,

$$
\begin{align*}
\hat{p}_h &= \{p_h\} & \text{on } \mathcal{E}_h, \\
\hat{u}_h &= \{u_h\} + \eta e^{-1}|p_h| & \text{on } \mathcal{E}_h, \\
\hat{u}_h &= 0 & \text{on } \mathcal{E}_h^\partial.
\end{align*}
$$

We can see that this choice is the dual of the simplified LDG method [54] (when $\beta = 0$) in the sense that the definitions $\hat{p}$ and $\hat{u}$ in (6.1c) – (6.1d) are exchanged in the two schemes. The numerical scheme (6.1a) – (6.1d) with such choices can be written as the mixed DG formulation: Find $(p_h, u_h) \in Q_h \times V_h$ such that

$$
\begin{align*}
\begin{cases}
\begin{aligned}
a_h^{\text{MDG}}(p_h, q_h) + b_h^{\text{MDG}}(q_h, u_h) &= 0 & \forall q_h \in Q_h, \\
b_h^{\text{MDG}}(p_h, v_h) &= -\int_{\Omega} f v_h \, dx & \forall v_h \in V_h.
\end{aligned}
\end{cases}
\end{align*}
$$

Here, we choose $\eta_e = O(1)$, and define

$$
\begin{align*}
a_h^{\text{MDG}}(p, q) &= (\mathbf{p}, \mathbf{q})_{\mathcal{T}_h} + \langle \eta e^{-1}|p|, |q| \rangle_{E_h^e} & \forall p, q \in Q_h \cup H(\text{div}; \Omega), \\
b_h^{\text{MDG}}(p, v) &= -(\text{div} v p, v)_{\mathcal{T}_h} + \langle |p|, |v| \rangle_{E_h^e} & \forall p \in Q_h \cup H(\text{div}; \Omega), \forall v \in V_h \cup H^1(\Omega).
\end{align*}
$$

**Remark 6.2** With the choice of the numerical traces: $\gamma = 1, \beta = 0, \mu_1(\|u_h\|) = 0$ and $\mu_2(|p_h|) = \eta_e \{r_e(|p_h|)\}$, we can obtain another mixed DG scheme, which is the dual form of the method of Brezzi et al. [25]. Here, the lifting operator $r_e : L^2(\mathcal{E}_h) \mapsto V_h$ is defined by

$$
\int_{\Omega} r_e(w) v_h \, dx = -\int_{\mathcal{E}_h} w \{v_h\} \, ds & \forall v_h \in V_h.
$$

We are also aware that if $\gamma = 0$ or $\beta \neq 0$, the resulting mixed DG schemes are not symmetric.

Next, we prove the well-posedness of the mixed DG formulation (6.9) when choosing

$$
V_h = V_h^k, \quad Q_h = Q_h^{k+1},
$$

for $k \geq 0$, which leads to the optimal order of convergence in the $L^2$ norm $\|\cdot\|$ for $u$ and the following norm for $p$:

$$
\|q\|_{\text{MDG}, h}^2 := (\mathbf{c} q, q)_{\mathcal{T}_h} + (\text{div}_h q, \text{div}_h q)_{\mathcal{T}_h} + \langle \eta e^{-1}|q|, |q| \rangle_{E_h^e} & \forall q \in Q_h \cup H(\text{div}; \Omega).
$$

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**Boundedness.** A direct calculation shows that $a_{h}^{MDG} (\cdot, \cdot)$ satisfies
\[
a_{h}^{MDG} (p, q) \leq \|p\|_{MDG, h} \|q\|_{MDG, h} \quad \forall p, q \in Q_{h} \cup H(\text{div}; \Omega). \tag{6.15}
\]
From the estimate of lifting operator (see also [4, 26])
\[
\|r_{e}(w)\|_{0, \Omega} \lesssim h^{-1/2}_{e} \|w\|_{0, e},
\]
we have the boundedness of $b_{h}^{MDG} (\cdot, \cdot)$.

**Lemma 6.3** It holds that
\[
\begin{align*}
b_{h}^{MDG} (q, v_{h}) & \lesssim \|q\|_{MDG, h} \|v_{h}\|_{0} \quad \forall q \in Q_{h} \cup H(\text{div}; \Omega), \forall v_{h} \in V_{h}, \tag{6.17} \\
b_{h}^{MDG} (q, v) & \lesssim \|q\|_{MDG, h} (\|v\|_{0} + h|v|_{1, h}) \quad \forall q \in Q_{h} \cup H(\text{div}; \Omega), \forall v \in H^{1} (\Omega). \tag{6.18}
\end{align*}
\]

**Stability.** According to the theory of mixed methods, the stability of the saddle point problem (6.9) is the corollary of the following two conditions [16, 21]:

1. K-ellipticity:
\[
a_{h}^{MDG} (q_{h}, q_{h}) \geq \|q_{h}\|_{MDG, h}^{2} \quad \forall q_{h} \in Z_{h}, \tag{6.19}
\]
where $Z_{h} = \{q_{h} \in Q_{h} \mid b_{h}^{MDG} (q_{h}, v_{h}) = 0, \forall v_{h} \in V_{h}\}$.

2. The discrete inf-sup condition:
\[
\inf_{v_{h} \in V_{h}} \sup_{q_{h} \in Q_{h}} \frac{b_{h}^{MDG} (q_{h}, v_{h})}{\|q_{h}\|_{MDG, h} \|v_{h}\|} \geq 1. \tag{6.20}
\]

**Theorem 6.4** The mixed DG schemes (6.9) are well-posed for $(Q_{h}^{k+1}, \|\cdot\|_{MDG, h})$ and $(V_{h}^{k}, \|\cdot\|)$.

**Proof.** We first show the K-ellipticity (6.19). By the definition of the lifting operator (6.12), we have
\[
b_{h}^{MDG} (q_{h}, v_{h}) = \int_{\Omega} \left( \text{div} q_{h} + \sum_{e \in E_{h}^{k}} r_{e} (q_{h}) \right) v_{h} \, dx,
\]
which implies that
\[
Z_{h} = \{q_{h} \in Q_{h}^{k+1} \mid \text{div} q_{h} + \sum_{e \in E_{h}^{k}} r_{e} (q_{h}) = 0\}.
\]
Let $\eta_{0} = \inf_{e \in E_{h}^{k}} \eta_{e}$ be a positive constant that is independent of the grid size. Then (6.16) implies
\[
a(q_{h}, q_{h}) \geq \|q_{h}\|_{0, \Omega}^{2} + \eta_{0} \sum_{e \in E_{h}^{k}} h_{e}^{-1} \|q_{h}\|_{0, e}^{2} \geq \|q_{h}\|_{MDG, h}^{2} \quad \forall q_{h} \in Z_{h}. \tag{6.21}
\]
The inf-sup condition (6.20) follows from the inf-sup condition for the BDM element. \( \blacksquare \)

**Remark 6.5** A similar argument shows that the penalty term $\langle \eta_{h} h_{e}^{-1} [p_{h}], [q_{h}] \rangle_{E_{h}^{k}}$ can be replaced by $\langle \eta_{e} r_{e} ([\sigma_{h}]), r_{e} ([q_{h}]) \rangle_{E_{h}^{k}}$, and the well-posedness of the corresponding scheme can be proved similarly with the modified norm $\|q\|^{2}_{h} = \langle cq, q \rangle_{h} + (\text{div} q, \text{div} q)_{h} + \langle \eta_{e} r_{e} ([q]), r_{e} ([q]) \rangle_{E_{h}^{k}}$. 

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Lemma 6.6 Assume that the solution \((q, u) \in H(\text{div}; \Omega) \times H^1(\Omega)\). Then we have
\[
\begin{cases}
a_h^{\text{MDG}}(p - p_h, q_h) + b_h^{\text{MDG}}(q_h, u - u_h) = 0 & \forall q_h \in Q_h, \\
b_h^{\text{MDG}}(p - p_h, v_h) = 0 & \forall v_h \in V_h.
\end{cases}
\]
(6.22)

By combining Lemma 6.6 and the well-posedness of the mixed DG formulation (6.9), we have the following a priori error estimates.

Theorem 6.7 Let \((p_h, u_h) \in Q_h^{k+1} \times V_h^k\) be the solution for the mixed DG formulation (6.9), and \((p, u) \in H(\text{div}; \Omega) \times H^1(\Omega)\) be the solution for (1.3). Then we have
\[
\|p - p_h\|_{\text{MDG}, h} + \|u - u_h\| \lesssim \inf_{p_h \in Q_h^{k+1}} \|p - p_h\|_{\text{MDG}, h} + \inf_{v_h \in V_h^k} (\|u - v_h\|_0 + h\|u - v_h\|_1,h).
\]
(6.23)

Using the Scott-Zhang interpolation [99], we have the following theorem.

Theorem 6.8 Let \((p_h, u_h) \in Q_h^{k+1} \times V_h^k\) be the solution of the mixed DG formulation (6.9). Assume that the solution of (1.3) satisfies \((p, u) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)\). Then we have
\[
\|p - p_h\|_{\text{MDG}, h} + \|u - u_h\| \lesssim h^{k+1}(\|p\|_{k+2} + |u|_{k+1}).
\]
(6.24)

Remark 6.9 \(Q_h\) can be chosen as a discontinuous RT finite element, i.e., \(Q_h = Q_h^{k,\text{RT}}\), and the corresponding well-posedness and error estimates can also be obtained similarly.

Remark 6.10 The mixed DG method for linear elasticity and the proof of the well-posedness can also be provided due to the stability analysis in [117], which shows that optimal convergence rates are achieved for both stress and displacement variables.

Numerical examples of mixed DG methods We next illustrate the performance of mixed DG methods for the Poisson problem in 2D. The problem is computed on the unit square \(\Omega = [0, 1]^2\) with a homogeneous boundary condition that \(u = 0\) on \(\partial \Omega\). The coefficient matrix \(\alpha = I_2\), where \(I_2 \in \mathbb{R}^{2 \times 2}\) is the identity matrix. The exact solution satisfies
\[
u = \sin(2\pi x) \sin(\pi y) \quad \text{and} \quad p = \begin{pmatrix} 2\pi \cos(2\pi x) \sin(\pi y) \\ \pi \sin(2\pi x) \cos(\pi y) \end{pmatrix}.
\]
(6.25)

The exact load function \(f\) can be analytically derived for the given \(u\). Non-nested, quasi-uniform unstructured grids with different grid sizes are used in the computation. The parameter \(\eta_e\) is set to be 1. For the unstructured grid, we define \(h = N_{\text{ele}}^{-1/d}\), where \(N_{\text{ele}}\) is the number of elements. From Table 6.2, the optimal convergence can be observed. Moreover, we observe that the \(L^2\) error of \(p\) is of order \(k + 2\), which is one order higher than the error estimate (6.24).

7 Relationship between different methods

In this section, we shall discuss the relationship between different methods.
(a) Poisson problem: $P^{-1}_1 - P^{-1}_0$, unstructured grids

| N_{ele} | $\|u - u_h\|_{0, \Omega}$ | $h^n$ | $\|p - p_h\|_{0, \Omega}$ | $h^n$ | $\|\text{div}_h (p - p_h)\|_{0, \Omega}$ | $h^n$ |
|---------|---------------------|------|---------------------|------|---------------------|------|
| 220     | 0.0772588           | –    | 0.111405            | –    | 3.79529             | –    |
| 976     | 0.0369387           | 0.99 | 0.0255594           | 1.98 | 1.82077             | 0.99 |
| 4054    | 0.0180824           | 1.00 | 0.006156            | 2.00 | 0.89211             | 1.00 |

(b) Poisson problem: $P^{-1}_2 - P^{-1}_1$, unstructured grids

| N_{ele} | $\|u - u_h\|_{0, \Omega}$ | $h^n$ | $\|p - p_h\|_{0, \Omega}$ | $h^n$ | $\|\text{div}_h (p - p_h)\|_{0, \Omega}$ | $h^n$ |
|---------|---------------------|------|---------------------|------|---------------------|------|
| 220     | 0.0066776           | –    | 0.00557092          | –    | 0.332017            | –    |
| 976     | 0.00146501          | 2.04 | 0.000581936         | 3.03 | 0.0726511           | 2.04 |
| 4054    | 0.000357469         | 1.98 | 6.98718e-05         | 2.98 | 0.0177161           | 1.98 |

(c) Poisson problem: $P^{-1}_3 - P^{-1}_2$, unstructured grids

| N_{ele} | $\|u - u_h\|_{0, \Omega}$ | $h^n$ | $\|p - p_h\|_{0, \Omega}$ | $h^n$ | $\|\text{div}_h (p - p_h)\|_{0, \Omega}$ | $h^n$ |
|---------|---------------------|------|---------------------|------|---------------------|------|
| 220     | 0.00389143          | –    | 0.0022748          | –    | 0.0193726           | –    |
| 976     | 4.20827e-05         | 2.99 | 1.16707e-05        | 3.99 | 0.00209901          | 2.99 |
| 4054    | 4.95135e-06         | 3.01 | 6.79925e-07        | 3.99 | 0.000245922         | 3.01 |

Table 6.2: Poisson problem: the convergence order on 2D non-nested unstructured grids

7.1 From stabilized hybrid methods to the LDG methods

Let us show that the stabilized hybrid mixed methods can deduce the LDG scheme if we set $\hat{u}_h = \{u_h\} + \beta \cdot \{\tilde{u}_h\}$ and $\hat{v}_h = \{v_h\} + \beta \cdot \{\tilde{v}_h\}$. First, we can see that the first equation for the stabilized hybrid mixed (HDG) methods (1.14) is equivalent to (6.1a) formally. Further, the left hand of the second equation of (1.14) is

$$-(\text{div} p_h, v_h)_{\Omega} + \langle p_h \cdot n, \hat{v}_h \rangle_{\partial \Omega} - \langle \tau (u_h - \hat{u}_h), v_h - \hat{v}_h \rangle_{\partial \Omega}$$

$$= (p_h, \nabla v_h)_{\Omega} + \langle p_h \cdot n, \hat{v}_h - v_h \rangle_{\partial \Omega} - \langle \tau (u_h - \hat{u}_h), v_h - \hat{v}_h \rangle_{\partial \Omega}$$

$$= (p_h, \nabla v_h)_{\Omega} + \langle \{p_h\}, \{\hat{v}_h - v_h\} \rangle_{\Omega} + \langle [p_h], \{\hat{v}_h - v_h\} \rangle_{\Omega}$$

(by (2.3))

$$- 2 \langle \tau (u_h - \hat{u}_h), v_h - \hat{v}_h \rangle_{\Omega} - \frac{1}{2} \langle \tau \|u_h - \hat{u}_h\|_h, \|v_h - \hat{v}_h\|_h \rangle_{\Omega}$$

(7.1)

which is same as (6.6) under $\hat{p}_h = \{p_h\} - \beta \{p_h\} + \frac{1}{2} \tau (1 + 4|\beta \cdot n_e|^2) \|u_h\|_h$. This is exactly the definition of the numerical trace $\hat{p}_h$ of the primal LDG methods (see the LDG methods in Table 6.1) when $\tau = O(h^{-1})$. Therefore, primal LDG methods can be formally deduced from stabilized hybrid mixed (HDG) methods when choosing the space

$$\tilde{V}_h = \{(v_h, \hat{v}_h) : v_h \in V_h, \hat{v}_h = \{v_h\} + \beta \cdot \{\tilde{v}_h\}\}.$$ (7.2)

On the other hand, when considering the DG scheme (6.1) with $\hat{p}_h = \{p_h\} + \eta_h h^{-1} \|u_h\|_h$, we obtain the
following from (7.1) that

\[
\langle \nabla_{\partial T} \hat{p}_h, v_h \rangle = (p_h, \nabla v_h)_{\mathcal{T}_h} - \langle v_h, \hat{p}_h \cdot n \rangle_{\partial \mathcal{T}_h} = (p_h, \nabla v_h)_{\mathcal{T}_h} - \langle \{p_h\} + \eta h^{-1} \{u_h\}, [v_h] \rangle_{\mathcal{E}_h}
\]

\[
= -(\text{div} \, p_h, v_h)_{\mathcal{T}_h} + \langle p_h \cdot n, \{v_h\} \rangle_{\partial \mathcal{T}_h} - \sum_{K \in \mathcal{T}_h} 2\eta h^{-1} \langle u_h - \{u_h\}, v_h - \{v_h\} \rangle_{\partial K}
\]

\[
= -(\text{div}_{\partial T} p_h, (v_h, \{v_h\})) - \sum_{K \in \mathcal{T}_h} 2\eta h^{-1} \langle u_h - \{u_h\}, v_h - \{v_h\} \rangle_{\partial K}.
\]

Hence, the (simplified) primal LDG will formally return to the stabilized hybrid mixed methods (1.14) by replacing \( \{u_h\} \) and \( \{v_h\} \) with the new trial variable \( \hat{u}_h \) and test variable \( \hat{v}_h \), respectively. In the same way, it is readily seen that mixed LDG methods (6.8) can be formally deduced from stabilized hybrid primal (WG) methods (1.10) if the space is specified as

\[
\tilde{Q}_h = \{ (q_h, \hat{q}_h) : q_h \in Q_h, \hat{q}_h = \{q_h\} \}.
\] (7.3)

And the mixed LDG methods (6.8) can also formally return to the WG methods (1.10) by replacing \( \{p_h\} \) and \( \{q_h\} \) with the new trial variable \( \hat{p}_h \) and test variable \( \hat{q}_h \), respectively.

**Remark 7.1** In order to derive the other DG schemes in a similar fashion, we need to introduce another stabilization to the hybrid method. For instance, instead of \( S^+_c \), we can introduce another non-symmetric stabilization term in the stabilized hybrid primal methods (1.15), i.e.,

\[
\langle \tau (\{r_c(u_h n)\} \cdot n - \hat{u}_h), v_h - \hat{v}_h \rangle_{\partial \mathcal{T}_h}.
\]

In light of (7.1), when choosing the special space with \( \beta = 0 \), we obtain

\[
- (\text{div} \, p_h, v_h)_{\mathcal{T}_h} + \langle p_h \cdot n, \hat{v}_h \rangle_{\partial \mathcal{T}_h} - \langle \tau (\{r_c(u_h n)\} \cdot n - \hat{u}_h), v_h - \hat{v}_h \rangle_{\partial \mathcal{T}_h}
\]

\[
= (p_h, \nabla v_h)_{\mathcal{T}_h} + \langle \{p_h\}, [\hat{v}_h - v_h] \rangle_{\mathcal{E}_h} + \langle \{p_h\}, \{\hat{v}_h - v_h\} \rangle_{\mathcal{E}_h} - \frac{1}{2} \langle \tau [r_c(u_h n) \cdot n - \hat{u}_h], [v_h - \hat{v}_h] \rangle_{\mathcal{E}_h} (7.4)
\]

\[
= (p_h, \nabla v_h)_{\mathcal{T}_h} - \langle \{p_h\} + \frac{1}{2} \tau [r_c([u_h n])], [v_h] \rangle_{\mathcal{E}_h},
\]

which gives rise to the \( \hat{p}_h \) by Brezzi et al. [25] when \( \tau = O(1) \) (see the DG Method of Brezzi et al. in Table 6.1). Similarly, the following non-symmetric stabilization term can be adopted for (1.11) in place of \( S^+_p \)

\[
\langle \eta (\{r_e(p_h)\} - \hat{p}_h) \cdot n, (q_h - \hat{q}_h) \cdot n \rangle_{\partial \mathcal{T}_h}.
\]

Then the mixed DG method of Brezzi et. al (see Remark 6.2) can be derived when \( \eta = O(1) \).

### 7.2 Mixed methods as the limiting case of WG methods

For a given mesh, we will now try to prove the convergence of WG methods (1.10) to mixed methods (1.12) when \( \rho \to 0 \), where the stabilization parameter is set as \( \eta = \rho^{-1} h^{-1} \).

Consider the \( H(\text{div}) \) conforming subspace \( Q_h^c := Q_h \cap H(\text{div}, \Omega) \subset Q_h \), the mixed methods (1.12) in variational form are written as: Find \( (p_h^c, u_h^c) \in Q_h^c \times V_h \) such that

\[
\begin{cases}
(p_h^c, q_h^c)_{\mathcal{T}_h} - (u_h^c, \text{div} q_h^c)_{\mathcal{T}_h} = (g_1, q_h^c)_{\mathcal{T}_h} \\
(\text{div} p_h^c, v_h)_{\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h} + (g_2, v_h)_{\partial \mathcal{T}_h} & \forall q_h^c \in Q_h^c, \forall v_h \in V_h,
\end{cases}
\] (7.5)
where \( g_1 = 0 \) and \( g_2 = 0 \) when applied to the Poisson equation (1.3). Then, by \( V_h \subset \text{div}Q_h \subset \text{div}_h Q_h \subset V_h \), the well-posedness of the mixed methods (cf. [21, 11]) implies that

\[
\| \mathbf{p}_h \|_{H(\text{div})} + \| v_h \| \leq C_M \left( \| f \| + \sup_{q_h \in Q_h} \frac{(g_1, q_h)_{\mathcal{T}_h}}{\| q_h \|_{H(\text{div})}} + \sup_{v_h \in V_h} \frac{(g_2, v_h)_{\partial \mathcal{T}_h}}{\| v_h \|} \right),
\] (7.6)

Recall that the spaces defined on \( \mathcal{E}_h \) (see (2.6)) of WG methods are given by

\[
\hat{Q}_h = \{ \hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e) n_e, \forall e \in \mathcal{E}_h \}, \quad \hat{Q}_h = \{ \hat{p}_h : \hat{p}_h|_e \in \hat{Q}(e), \forall e \in \mathcal{E}_h \}.
\]

We make the following assumption on the finite element spaces of WG methods.

**Assumption 7.2** Assume that the spaces \( Q_h, \hat{Q}_h \) and \( V_h \) satisfy

1. \( \text{div}_h Q_h = V_h \);
2. \( \| Q_h \|_e \subset \hat{Q}(e), \forall e \in \mathcal{E}_h \);
3. There exists a constant \( C_M \) independent of \( h \), such that for any \( p_h \in Q_h \),

\[
\inf_{p_h' \in Q_h} (\| p_h' - p_h \| + \| \text{div}_h (p_h' - p_h) \|) \leq C_M \sum_{e \in \mathcal{E}_h} h_e^{-1/2} \| [p_h] \|_{0,e},
\] (7.7)

where \( Q_h^e = Q_h \cap H(\text{div} ; \Omega) \).

We note that the first assumption in Assumption 7.2 ensures well-posedness of the mixed methods (7.5). Several examples are given below.

**Example 7.3** Raviart-Thomas type: \( Q_h = Q_h^{k,RT}, \hat{Q}(e) = P_k(e), V_h = V_h^k, \) for \( k \geq 0 \).

**Example 7.4** Brezzi-Douglas-Marini type: \( Q_h = Q_h^{k+1}, \hat{Q}(e) = P_{k+1}(e), V_h = V_h^k, \) for \( k \geq 0 \).

**Lemma 7.5** If we choose the spaces as in Example 7.4 or Example 7.3, then Assumption 7.2 holds.

**Proof.** We only sketch the proof of (7.7) in Assumption 7.2. Denote the set of degrees of freedom of RT or BDM element by \( D \), see [21, 11]. We then define \( p_h' \) as

\[
d(p_h') = \frac{1}{|T_d|} \sum_{k \in T_d} d(p_h'|_{T_d}) \quad \forall d \in D,
\]

where \( T_d \) denotes the set of elements that share the degrees of freedom \( d \) and \( |T_d| \) denotes the cardinality of this set. By the standard scaling argument,

\[
\sum_{k \in T_h} \| p_h' - p_h \| \lesssim \sum_{e \in \mathcal{E}_h} h_e^{1/2} \| [p_h] \|_{0,e}.
\]

Then (7.7) follows from the inverse inequality.

We rewrite the WG methods (1.10) in the variational form as: Find \( (p_h^n, u_h^n, \hat{p}_h^n) \in Q_h \times V_h \times \hat{Q}_h \) such that for any \( (q_h, v_h, \hat{q}_h) \in Q_h \times V_h \times \hat{Q}_h \)

\[
\begin{aligned}
\langle (p_h^n, q_h)_{\mathcal{T}_h} + \rho^{-1}(h^{-1}(p_h^n - \hat{p}_h) \cdot n, (q_h - \hat{q}_h) \cdot n)_{\partial \mathcal{T}_h} + (\nabla u_h, q_h)_{\mathcal{T}_h} - \langle u_h, \hat{q}_h \cdot n \rangle_{\partial \mathcal{T}_h} = 0, \\
-\langle p_h^n, \nabla v_h \rangle_{\partial \mathcal{T}_h} + \langle \hat{p}_h^n \cdot n, v_h \rangle_{\partial \mathcal{T}_h} = (f, v_h).
\end{aligned}
\] (7.8)

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Theorem 7.6 Under the Assumption 7.2, WG methods (1.10) converge to the mixed methods (1.12) as \( \rho \to 0 \) with \( \eta = \rho^{-1} h^{-1} \). More precisely, we have
\[
\| p_h^n - p_h^n \|_{H_0(\text{div})} + \| u_h^\eta - u_h^\eta \| \leq C_{w,3} \rho^{1/2} \| f \|, 
\] (7.9)
where \( C_{w,3} \) is independent of both mesh size \( h \) and \( \rho \).

Proof. From the assumption \( \| Q_h^e \|_{e} \subset \mathcal{Q}(e) \), by taking \( q_h = q_h^e \) and \( \hat{q}_h | e = (q_h^e \cdot n_e) n_e \) in (7.8) and integrating by parts, we see that \( (p_h^n, u_h^\eta) \) satisfies
\[
( c(p_h^n, q_h^e) )_h - (u_h^\eta - u_h^\eta, \text{div} q_h^e)_h = 0 \quad \forall q_h^e \in Q_h^e. 
\] (7.10)
Subtracting (7.5) from (7.10) and the second equation of (7.8), we have
\[
\left\{ \begin{array}{l}
( c(p_h^n - p_h^n, q_h^e) )_h - (u_h^\eta - u_h^\eta, \text{div} q_h^e)_h = 0 \\
( \text{div}(p_h^n - p_h^n), v_h )_h = \langle (p_h^n - \hat{p}_h^n) \cdot n, v_h \rangle_{\partial \Omega_h} + (\text{div}(p_h^n - p_h^n), v_h )_h \\
\end{array} \right. \quad \forall q_h^e \in Q_h^e, \forall v_h \in V_h. 
\] (7.11)
Noting that \( p_h^n \not\in Q_h^e \), we have that, for any \( p_h^n \in Q_h^e \),
\[
\left\{ \begin{array}{l}
( c(p_h^n - p_h^n, q_h^e) )_h - (u_h^\eta - u_h^\eta, \text{div} q_h^e)_h = (c(p_h^n - p_h^n, q_h^e) )_h \\
( \text{div}(p_h^n - p_h^n), v_h )_h = \langle (p_h^n - \hat{p}_h^n) \cdot n, v_h \rangle_{\partial \Omega_h} + (\text{div}(p_h^n - p_h^n), v_h )_h \\
\end{array} \right. \quad \forall q_h^e \in Q_h^e, \forall v_h \in V_h. 
\] (7.12)
Because \( (p_h^n - p_h^n) \in Q_h^e, (u_h^\eta - u_h^\eta) \in V_h \), by the well-posedness of the mixed methods (7.6), trace inequality, inverse inequality and Cauchy inequality, we have
\[
\| p_h^n - p_h^n \|_{H(\text{div})} + \| u_h^\eta - u_h^\eta \| \leq C_M \left( \sup_{q_h^e \in Q_h^e} \frac{(c(p_h^n - p_h^n, q_h^e) )_h}{\| q_h^e \|_{H(\text{div})}} + \sup_{v_h \in V_h} \frac{\langle (p_h^n - \hat{p}_h^n) \cdot n, v_h \rangle_{\partial \Omega_h} + (\text{div}(p_h^n - p_h^n), v_h )_h}{\| v_h \|} \right) 
\]
\[
\lesssim \| p_h^n - p_h^n \| + \| \text{div}_h(p_h^n - p_h^n) \| + \langle h^{-1}(p_h^n - \hat{p}_h^n) \cdot n, (p_h^n - \hat{p}_h^n) \cdot n \rangle_{\partial \Omega_h}^{1/2}.
\]
Hence, by Assumption 7.2 and inequality inverse, we have
\[
\| p_h^n - p_h^n \|_{H(\text{div})} + \| u_h^\eta - u_h^\eta \| \lesssim \langle h^{-1}(p_h^n - \hat{p}_h^n) \cdot n, (p_h^n - \hat{p}_h^n) \cdot n \rangle_{\partial \Omega_h}^{1/2} + \inf_{p_h^n \in Q_h^e} \left( \| p_h^n - p_h^n \| + \| \text{div}_h(p_h^n - p_h^n) \| \right) 
\]
\[
\lesssim \langle h^{-1}(p_h^n - \hat{p}_h^n) \cdot n, (p_h^n - \hat{p}_h^n) \cdot n \rangle_{\partial \Omega_h}^{1/2} + \sum_{e \in E_h} h_e^{-1/2} \| p_h^n \|_{0,e}. 
\]
From the fact that
\[
\langle (p_h^n - \hat{p}_h^n) \cdot n, (p_h^n - \hat{p}_h^n) \cdot n \rangle_{\partial \Omega_h} = 2 \left( \| p_h^n - \hat{p}_h^n \|, \| p_h^n - \hat{p}_h^n \| \right) \varepsilon_h + \frac{1}{2} \left( \| p_h^n \|, \| p_h^n \| \right) \varepsilon_h,
\]
we obtain
\[
\| p_h^n - p_h^n \|_{H(\text{div})} + \| u_h^\eta - u_h^\eta \| \lesssim \langle h^{-1}(p_h^n - \hat{p}_h^n) \cdot n, (p_h^n - \hat{p}_h^n) \cdot n \rangle_{\partial \Omega_h}^{1/2} + \sum_{e \in E_h} h_e^{-1/2} \| p_h^n \|_{0,e} 
\]
\[
\lesssim \langle h^{-1}(p_h^n - \hat{p}_h^n) \cdot n, (p_h^n - \hat{p}_h^n) \cdot n \rangle_{\partial \Omega_h}^{1/2} \lesssim \rho^{1/2} \| f \|,
\]
where we used Corollary 4.8 in the last step. This completes the proof. ■
From WG to RT: \( \eta = \rho^{-1} h^{-1}, k = 0, \) uniform grid

\[
\begin{array}{cccccc}
\rho & \|u_h^\eta - u_h^{\text{RT}}\| & \rho^\alpha & \|p_h^\eta - p_h^{\text{RT}}\|_0 & \rho^\alpha & \|\text{div}_h(p_h^\eta - p_h^{\text{RT}})\| & \rho^\alpha \\
1/4 & 0.003539 & - & 0.025589 & - & 0.101364 & - \\
1/8 & 0.01777 & 0.99 & 0.012850 & 0.99 & 0.050819 & 1.00 \\
1/16 & 0.00890 & 1.00 & 0.006439 & 1.00 & 0.025444 & 1.00 \\
\end{array}
\]

Table 7.1: Convergence rate from WG to RT mixed methods on 2D uniform grids

Numerical examples on the convergence from WG methods to mixed methods

We present some numerical examples to support the theoretical results. We consider the 2D Poisson problem described in (6.25). A uniform grid with \( h = 1/4 \) is fixed for different \( \rho \)'s with \( \eta = \rho^{-1} h^{-1} \). First, we choose the RT-type discrete spaces in WG methods, see Example 7.3. When \( \rho \to 0 \), WG methods do converge to the mixed methods, see Table 7.1. Next, we choose the BDM-type discrete spaces in WG methods, see Example 7.4. When \( \rho \to 0 \), WG methods do converge to BDM, see Table 7.2. Further, we observe the first-order convergence on \( \rho \) in both Table 7.1 and Table 7.2, which is 1/2-order higher than our theoretical finding in Theorem 7.6.

From WG to RT: \( \eta = \rho^{-1} h^{-1}, k = 1, \) uniform grid

\[
\begin{array}{cccccc}
\rho & \|u_h^\eta - u_h^{\text{RT}}\| & \rho^\alpha & \|p_h^\eta - p_h^{\text{RT}}\|_0 & \rho^\alpha & \|\text{div}_h(p_h^\eta - p_h^{\text{RT}})\| & \rho^\alpha \\
1/4 & 0.003681 & - & 0.004955 & - & 0.102957 & - \\
1/8 & 0.001843 & 1.00 & 0.002482 & 1.00 & 0.051582 & 1.00 \\
1/16 & 0.000922 & 1.00 & 0.001242 & 1.00 & 0.025817 & 1.00 \\
\end{array}
\]

Table 7.2: Convergence rate from WG to RT mixed methods on 2D uniform grids

7.3 Primal methods as the limiting case of HDG methods

For a given mesh, we next try to prove that the HDG methods (1.14) converge to primal methods (1.8) when \( \rho \to 0 \) and the stabilization parameter is set to be \( \tau = \rho^{-1} h^{-1} \).

Consider the \( H^1 \) conforming subspace \( V_h^c = V_h \cap H^1_0(\Omega) \subset V_h \), then the primal methods (1.8) in the
ensures the well-posedness of the primal methods (see the conforming relatives in Assumption 7.7). Assume that the spaces

\[ \|p^c_h\| + \|v^c_h\| \leq C_p \left( \|f\|_{-1,h} + \sup_{q_h \in Q_h} \frac{\langle g_1, q_h \rangle_{T_h} + \langle g_2, q_h \cdot n \rangle_{\partial T_h}}{\|q_h\|} \right), \]  

(7.14)

where \( \|f\|_{-1,h} = \sup_{v_h \neq 0, v_h \in V^c_h} \frac{\langle f, v_h \rangle_{T_h}}{\|v_h\|_{1,1}}. \)

Recall that the space \( \mathcal{E}_h \) (see (2.6)) of HDG methods is given by

\[ \hat{\mathcal{V}}_h = \{ \hat{v}_h : \hat{v}_h|_e \in \hat{\mathcal{V}}(e), \forall e \in \mathcal{E}^i_h, \hat{v}_h|_{\mathcal{E}^0_h} = 0 \}. \]

We make the following assumption on the finite element spaces of HDG methods.

**Assumption 7.7** Assume that the spaces \( Q_h, V_h \) and \( \hat{\mathcal{V}}_h \) satisfy

1. \( \nabla_h V_h \subset Q_h; \)
2. \( \{V_h\}|_e \subset \hat{\mathcal{V}}(e), \forall e \in \mathcal{E}^i_h; \)
3. There exists a constant \( C_p^I \) independent of \( h \), such that for any \( u_h \in V_h \),

\[ \inf_{u_h^I \in V^k_h} \left( \|(u_h^I - u_h^c)\| + \|\nabla_h(u_h^I - u_h^c)\| \right) \leq C_p^I \sum_{e \in \mathcal{E}_h} h_e^{-1/2} \|u_h\|_{0,e}. \]  

(7.15)

where \( V^c_h = V_h \cap H^1(\Omega) \).

We note that the first assumption in Assumption 7.7 ensures the well-posedness of the primal methods (7.13). The following example satisfies Assumption 7.7 (see the conforming relatives in [15, 14]).

**Example 7.8** \( Q_h = Q_h^k, V_h = V_h^{k+1}, \hat{\mathcal{V}}(e) = P_{k+1}(e), \) for \( k \geq 0 \).

We rewrite the HDG methods (1.14) in the variational form as: Find \( (p^c_h, u^c_h, \hat{\nu}^c_h) \in Q_h \times V_h \times \hat{\mathcal{V}}_h \) such that for any \( (q_h, v_h, \hat{\nu}_h) \in Q_h \times V_h \times \hat{\mathcal{V}}_h \)

\[ \begin{align*}
(c p^c_h, q_h)_{\mathcal{T}_h} - (u_h^c, \text{div} q_h)_{\mathcal{T}_h} + \langle \hat{\nu}_h^c, q_h \cdot n \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\text{div} p^c_h, v_h)_{\mathcal{T}_h} - \langle p^c_h \cdot n, \hat{\nu}_h \rangle_{\partial \mathcal{T}_h} + \rho^{-1}(u_h^c - \hat{\nu}_h^c, v_h - \hat{\nu}_h)_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}.
\end{align*} \]  

(7.16)

**Theorem 7.9** Under the Assumption 7.7, the HDG methods (1.14) converge to the primal methods (1.8) as \( \rho \to 0 \) with \( \tau = \rho^{-1} h^{-1}_K \). More precisely, we have

\[ \|p^c_h - p^c_h\| + \|u_h^c - u_h^c\|_{1,h} \leq C_{d,3} \rho^{1/2} \|f\|_{-1,\rho,h}, \]  

(7.17)

where \( C_{d,3} \) is independent of both mesh size \( h \) and \( \rho \), and \( \|f\|_{-1,\rho,h} = \sup_{v_h \in \mathcal{V}_h} \frac{\langle f, v_h \rangle_{T_h}}{\|v_h\|_{1,\rho,h}}. \)
Proof. From the assumption \( \{ V_h \}_e \subset \tilde{V}(e) \), by taking \( v_h = \tilde{v}_h^e \) and \( \hat{v}_h \) in (7.16) and integrating by parts, we see that
\[
- \langle (p_h^e, \nabla v_h^e) \rangle_{\Gamma_h} = (f, v_h^e)_{\Gamma_h} \quad \forall v_h^e \in V_h^e. \tag{7.18}
\]
Subtracting (7.13) from the first equation of (7.16) and (7.18), we have
\[
\begin{cases}
(c(p_h^e - p_h^e), q_h)_{\Gamma_h} + \langle \nabla u_h^e - \nabla \tilde{u}_h^e, q_h \rangle_{\Gamma_h} = \langle u_h^e - \tilde{u}_h^e, q_h \cdot n \rangle_{\partial \Gamma_h} + \langle \nabla u_h^e + \nabla \tilde{u}_h^e, q_h \rangle_{\Gamma_h} & \forall q_h \in Q_h, \\
- \langle p_h^e - p_h^e, \nabla v_h^e \rangle_{\Gamma_h} = 0 & \forall v_h^e \in V_h^e.
\end{cases} \tag{7.19}
\]
Again, for any \( u_h^e \in V_h^e \), we have
\[
\begin{cases}
(c(p_h^e - p_h^e), q_h)_{\Gamma_h} + \langle \nabla u_h^e - \nabla \tilde{u}_h^e, q_h \rangle_{\Gamma_h} = \langle u_h^e - \tilde{u}_h^e, q_h \cdot n \rangle_{\partial \Gamma_h} + \langle \nabla u_h^e - \nabla \tilde{u}_h^e, q_h \rangle_{\Gamma_h} & \forall q_h \in Q_h, \\
- \langle p_h^e - p_h^e, \nabla v_h^e \rangle_{\Gamma_h} = 0 & \forall v_h^e \in V_h^e.
\end{cases} \tag{7.20}
\]
Because \( p_h^e - p_h^e \in Q_h \) and \( \nabla \tilde{u}_h^e - \nabla \tilde{u}_h^e \in V_h^e \), using (7.14), trace inequality, inverse inequality and Cauchy inequality, we obtain
\[
\| p_h^e - p_h^e \| + \| u_h^e - u_h^e \|_1 \leq \frac{C_p}{h_{\min}} \sup_{e \in E_h} \frac{\langle u_h^e - \tilde{u}_h^e, q_h \cdot n \rangle_{\partial \Gamma_h} + \langle \nabla u_h^e - \nabla \tilde{u}_h^e, q_h \rangle_{\Gamma_h}}{\| q_h \|} \tag{7.21}
\]
\[
\lesssim \| u_h^e - u_h^e \|_1, h + \| h^{-1}(u_h^e - \tilde{u}_h^e), u_h^e - \tilde{u}_h^e \|_{1/2}^{1/2}.
\]
Noting that the local projection \( \hat{P}_h \) in (5.11) is an identity operator as \( \{ V_h \}_e \subset \tilde{V}(e) \), and
\[
\langle u_h^e - \tilde{u}_h^e, u_h^e - \tilde{u}_h^e \rangle_{\partial \Gamma_h} = 2\{\{ u_h^e - \tilde{u}_h^e \}, \{ u_h^e - \tilde{u}_h^e \}\} \varepsilon_h + \frac{1}{2} \varepsilon_h \langle \| u_h^e \|, [u_h^e] \rangle_{\varepsilon_h} \tag{7.22}
\]
Therefore, Assumption 7.7, (7.21), and (7.22) imply that
\[
\| p_h^e - p_h^e \| + \| u_h^e - u_h^e \|_1, h \leq \inf_{u_h^e \in V_h^e} \left( \| p_h^e - p_h^e \| + \| u_h^e - u_h^e \|_1 + \| u_h^e - u_h^e \|_1, h \right)
\]
\[
\lesssim \| h^{-1}(u_h^e - \tilde{u}_h^e), u_h^e - \tilde{u}_h^e \|_{1/2}^{1/2} + \inf_{u_h^e \in V_h^e} \| u_h^e - u_h^e \|_1, h
\]
\[
\lesssim \| h^{-1}(u_h^e - \tilde{u}_h^e), u_h^e - \tilde{u}_h^e \|_{1/2}^{1/2} + \sum_{c \in E_h} h_c^{-1/2} \| [u_h^e] \|_{0, c}
\]
\[
\lesssim \| h^{-1}(u_h^e - \tilde{u}_h^e), u_h^e - \tilde{u}_h^e \|_{1/2}^{1/2} \lesssim \rho^{1/2} \| f \|_{-1, \rho, h},
\]
where Corollary 5.9 was used in the last step. \( \blacksquare \)

Remark 7.10 We have \( \hat{P}_h \) as an identity operator in the definition of \( \| \cdot \|_{1, \rho, h} \) (see (5.11)). Therefore, when \( \rho \leq 1 \), we have
\[
\inf_{v_h^e \in V_h^e} \| \tilde{v}_h \|_{1, \rho, h}^2 = \inf_{v_h^e \in V_h^e} \langle \nabla v_h^e, \nabla v_h^e \rangle_{\Gamma_h} + \sum_{K \in T_h} \rho^{-1} h_K^{-1} \langle v_h^e - \hat{v}_h, v_h^e - \hat{v}_h \rangle_{\partial K}
\]
\[
\lesssim \langle \nabla v_h^e, \nabla v_h^e \rangle_{\Gamma_h} + \rho^{-1} \sum_{c \in E_h} h_c^{-1/2} \| [v_h^e] \|_{0, c} \lesssim \| v_h^e \|_{1, h}.
\]
Hence, when \( \rho \leq 1 \),
\[
\| f \|_{-1, \rho, h} = \sup_{v_h^e \in V_h^e} \frac{(f, v_h^e)_{\Gamma_h}}{\| v_h^e \|_{1, \rho, h}} \leq \sup_{v_h^e \in V_h^e} \frac{\inf_{v_h^e \in V_h^e} \| \tilde{v}_h \|_{1, \rho, h} \langle f, v_h^e \rangle_{\Gamma_h}}{\| v_h^e \|_{1, h}} \lesssim \| f \|,
\]
which means that the solutions of HDG methods converge to the those of primal methods of order \( \rho^{1/2} \) at least.
Numerical examples on the convergence from HDG methods to primal methods  We present some numerical examples to support the theoretical results. We consider the 2D Poisson problem described in (6.25). A uniform grid with $h = 1/4$ is fixed for different $\rho$'s with $\tau = \rho^{-1}h^{-1}$. We choose the discrete spaces in Example 7.8 for HDG methods. We observe the convergence from HDG methods to primal methods in Table 7.3. Similar to the numerical examples from WG to mixed methods, the convergence rate seems to be of order one, which is higher than our theoretical finding in Theorem 7.9.

(a) $\tau = \rho^{-1}h^{-1}$: $k = 0$, uniform grid

| $\rho$  | $\|u_h - u^c\|_0$ | $\rho^\alpha$ | $|u_h - u^c|_{1,h}$ | $\rho^\alpha$ |
|--------|-----------------|--------------|-------------------|--------------|
| 1/4    | 0.307403        | -            | 1.639017          | -            |
| 1/8    | 0.205585        | 0.58         | 1.066058          | 0.62         |
| 1/16   | 0.137903        | 0.58         | 0.727442          | 0.55         |
| 1/32   | 0.088576        | 0.64         | 0.484124          | 0.59         |
| 1/64   | 0.053204        | 0.73         | 0.300815          | 0.69         |
| 1/128  | 0.029933        | 0.83         | 0.173458          | 0.79         |
| 1/256  | 0.016031        | 0.90         | 0.094330          | 0.88         |
| 1/512  | 0.008321        | 0.95         | 0.049389          | 0.93         |

(b) $\tau = \rho^{-1}h^{-1}$, $k = 1$, uniform grid

| $\rho$  | $\|u_h^c - u^c\|_0$ | $\rho^\alpha$ | $|u_h^c - u^c|_{1,h}$ | $\rho^\alpha$ |
|--------|-----------------|--------------|-------------------|--------------|
| 1/4    | 0.037789        | -            | 0.686199          | -            |
| 1/8    | 0.022577        | 0.74         | 0.383381          | 0.84         |
| 1/16   | 0.014676        | 0.62         | 0.240527          | 0.67         |
| 1/32   | 0.009914        | 0.57         | 0.165801          | 0.54         |
| 1/64   | 0.005639        | 0.60         | 0.114649          | 0.53         |
| 1/128  | 0.004046        | 0.69         | 0.074111          | 0.63         |
| 1/256  | 0.002331        | 0.80         | 0.044007          | 0.75         |
| 1/512  | 0.001269        | 0.88         | 0.024376          | 0.85         |

Table 7.3: Convergence rate from HDG to primal methods on 2D uniform grids

7.4 Duality relationship

To make DG-derivatives good approximations of classical weak derivatives, the dual relationship between gradient and divergence operators should be preserved as shown in Lemma 2.1. The (stabilized) hybrid primal methods and (stabilized) hybrid mixed methods approximately satisfy conditions (i) and (ii) in Lemma 2.1, respectively, and the DG methods adopt condition (iii) approximately. In this subsection, we discuss the duality relationship of various Galerkin methods in the context of convex optimization.

To begin with, we know that the primal form (1.2) can be characterized as the following saddle point problem: Find $(p, u) \in L^2(\Omega) \times H^1_0(\Omega)$ such that

$$L(p, u) = \inf_{v \in H^1_0(\Omega)} \sup_{q \in L^2(\Omega)} L(q, v),$$

where

$$L(q, v) := -(q, \nabla v)_\Omega - \frac{1}{2}(cq, q)_\Omega - (f, v)_\Omega.$$  

In contrast, the mixed form of the elliptic problem is equivalent to the following saddle point problem: Find $(p, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$L^*(p, u) = \sup_{q \in H(\text{div}, \Omega)} \inf_{v \in L^2(\Omega)} L^*(q, v),$$

where

$$L^*(q, v) := (\text{div} q, v)_\Omega - \frac{1}{2}(cq, q)_\Omega - (f, v)_\Omega.$$  

The above two saddle point problems are dual with each other from the point of view of duality in convex optimization [11, 60].
To mimic the above duality at the discrete level, we define the following optimization target

\[ L_h(\tilde{q}_h, v_h) = -\langle \nabla_d g v_h, \tilde{q}_h \rangle_\Omega - \frac{1}{2} \langle c q_h, q_h \rangle_\Omega - \langle f_h, v_h \rangle_\Omega, \]

and the hybrid primal methods (1.9) can be derived from the optimization problem

\[
\inf_{v_h \in V_h} \sup_{\tilde{q}_h \in \tilde{Q}_h} L_h(\tilde{q}_h, v_h). \tag{7.23}
\]

Furthermore, the hybrid mixed methods (1.13) can be derived from the optimization problem

\[
\sup_{q_h \in Q_h} \inf_{\tilde{v}_h \in \tilde{V}_h} L^*_h(q_h, \tilde{v}_h), \tag{7.24}
\]

where

\[ L^*_h(q_h, \tilde{v}_h) := (\text{div} g q_h, \tilde{v}_h)_\Omega - \frac{1}{2} \langle c q_h, q_h \rangle_\Omega - \langle f_h, v_h \rangle_\Omega. \]

We immediately see that hybrid primal and hybrid mixed methods are dual to each other as (7.23) and (7.24) are mutually dual in the context of convex optimization.

A similar argument can be applied to the stabilized methods. Given the stabilized hybrid primal (WG) methods (1.10), we can prove that it is equivalent to the optimization problem

\[
\inf_{v_h \in V_h} \sup_{\tilde{q}_h \in \tilde{Q}_h} L_h(\tilde{q}_h, v_h) - \frac{1}{2} \langle \eta (p_h - \tilde{p}_h) \cdot n, (p_h - \tilde{p}_h) \cdot n \rangle_{\partial \Omega_h}, \tag{7.25}
\]

The stabilized hybrid mixed (HDG) methods (1.14) is equivalent to the optimization problem

\[
\sup_{q_h \in Q_h} \inf_{\tilde{v}_h \in \tilde{V}_h} L^*_h(q_h, \tilde{v}_h) + \frac{1}{2} \langle \tau (v_h - \tilde{v}_h), v_h - \tilde{v}_h \rangle_{\partial \Omega_h}. \tag{7.26}
\]

Since primal DG and mixed DG can be deduced formally from stabilized hybrid mixed methods and stabilized hybrid primal methods, respectively, by taking \( \hat{u}_h = \{u_h\} \) and \( \hat{p}_h = \{p_h\} \), respectively, they are formally dual with each other as well.

### 8 Summary

In this paper, we present a unified study for the design of various finite element methods through the concept of DG-derivatives. Then we compare these methods and show their relationships in Table 1.1. We find that the schemes of stabilized hybrid mixed methods and stabilized hybrid primal methods are mutually dual, and hybrid primal and hybrid mixed are dual with each other as well.

Furthermore, we see that each finite element method approximates either a primal or mixed form of the problem. Continuity of \( u \) is needed for the primal form, and on the other hand, \( H(\text{div}) \) continuity of \( p \) is required for the mixed form. To design finite element methods, we have to use certain mechanics to make the numerical approximations \( u_h \) or \( p_h \) satisfy certain continuity requirements. There are five approaches: (1) choosing a finite element space with strongly continuity (conforming FEMs); (2) choosing a finite element space with weakly continuity (nonconforming FEMs); (3) using the Lagrange multiplier to force the continuity (hybrid methods); and (4) using the Lagrange multiplier and stabilization to force the continuity (HDG and WG methods); (5) adding a penalty term in the weak form (DG methods).
Through this study, we derive the mixed DG methods. The well-posedness of this method is proven, and optimal error estimates are obtained. We also present rigorous proofs of the convergence from WG to mixed methods as well as the convergence from HDG to primal methods, as the stabilization parameter goes to infinity.

There are some other important FEMs that are not covered by our framework, such as finite volume methods [61], mimetic finite difference methods [24, 23], and virtual element methods [10, 20]. Our study is mainly done for second-order elliptic boundary value problems. Extension of this study to higher-order problems is currently under investigation and will be reported in a future work.

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