Methods for deriving functional equations for Feynman integrals

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Abstract. We present short review of two methods for obtaining functional equations for Feynman integrals. Application of these methods for finding functional equations for one- and two-loop integrals is described in detail. It is shown that with the aid of functional equations Feynman integrals in general kinematics can be expressed in terms of simpler integrals. Similarities between Feynman functional equations for Feynman integrals and addition theorem for Abel integrals are shortly discussed.

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1. Introduction
Functional equations (FE) for Feynman integrals for the first time were proposed in Ref. [1]. Details of derivation of such equations for one-loop integrals were described in Refs. [2], [3]. In Ref.[4] a new method for derivation of functional equations was proposed. This method is based on finding algebraic relations between products of propagators. By integrating different combinations of these algebraic relations one can easily obtain functional equations for multiloop integrals. Our functional relations can be used for expressing integrals with nontrivial kinematical dependence in terms of much simpler integrals. Also functional equations can be exploited for analytic continuation of Feynman integrals into different kinematical domains. In the next sections we will explain how to derive FE and will present several illustrative examples.

2. Derivation of FE from recurrence relations
The method proposed in Ref.[1] is based on the use of recurrence relations between integrals. The most general recurrence relation for Feynman integrals can be written as

\[ \sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}, \]  

(1)
where $Q_j, R_k$ are polynomials in masses, scalar products of external momenta, space-time dimension $d$, and powers of propagators. $I_{kr}$ - are integrals with $r$ external lines. The index $k$ just labels different integrals in a recurrence relation. In recurrence relations some integrals are more complicated than the others. The complexity of the integral depends on the number of their arguments and number of loops. The main idea of deriving functional equation from a recurrence relation can be formulated as follows. By choosing kinematical variables, masses, indices of propagators remove most complicated integrals from the relation, i.e. impose conditions:

$$Q_j = 0,$$  

keeping at least some other coefficients $R_k \neq 0$. 

To demonstrate how this idea works in practice let’s consider one-loop $n$-point integrals $I_n^{(d)}$. These integrals satisfy the so-called generalized recurrence relations [5]:

$$G_{n-1} \nu_j j^± I_n^{(d+2)} - (\partial_j \Delta_n) I_n^{(d)} = \sum_{k=1}^n (\partial_j \partial_k \Delta_n) k^{-2} I_n^{(d)},$$

where $j^±$ shifts indices $\nu_j \rightarrow \nu_j \pm 1$, $\partial_j \equiv \frac{\partial}{\partial m_j}$.

$$G_{n-1} = -2^n \begin{vmatrix} p_1p_1 & p_1p_2 & \cdots & p_1p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1p_{n-1} & p_2p_{n-1} & \cdots & p_{n-1}p_{n-1} \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{vmatrix}, \quad Y_{ij} = m_i^2 + m_j^2 - p_{ij}, \quad p_{ij} = (p_i - p_j)^2.$$  

2.1. One-loop propagator type integral

Setting $n = 3$, $j = 1$, $m_3^2 = 0$ in Eq.(3) leads to an expression connecting integrals $I_3^{(d+2)}$, $I_3^{(d)}$ and $I_2^{(d)}$. In order to get rid of integrals $I_3^{(d+2)}$, $I_3^{(d)}$ we must impose conditions:

$$G_2 = 0, \quad \Delta_3 = 0.$$  

Solving this system for $p_{13}$, $p_{23}$ yields FE for the integral $I_2^{(d)}$:

$$I_2^{(d)}(m_1^2, m_2^2, p_{12}) = \frac{p_{12} + m_1^2 - m_2^2 - \alpha_{12}}{2p_{12}} I_2^{(d)}(m_1^2, 0, s_{13}) + \frac{p_{12} - m_1^2 + m_2^2 + \alpha_{12}}{2p_{12}} I_2^{(d)}(0, m_2^2, s_{23}),$$

where

$$s_{13} = \frac{\Delta_{12} + 2p_{12}m_1^2 - (p_{12} + m_1^2 - m_2^2)\alpha_{12}}{2p_{12}},$$

$$s_{23} = \frac{\Delta_{12} + 2p_{12}m_2^2 + (p_{12} - m_1^2 + m_2^2)\alpha_{12}}{2p_{12}},$$

$$\alpha_{12} = \pm \sqrt{\Delta_{12}}.$$  

$$\Delta_{ij} = p^2_{ij} + m_i^4 + m_j^4 - 2p_{ij}m_i^2 - 2p_{ij}m_j^2 - 2m_i^2m_j^2.$$  

This equation represent integral with arbitrary masses $m_1, m_2$ and arbitrary external momentum in terms of integrals with one propagator massless. $I_2^{(d)}$ integrals with one massless propagator can be expressed in terms of Gauss’ hypergeometric function $2F_1$ [6], [7]. Evaluation of such integrals is much easier than evaluation of integrals with two masses.
2.2. FE for one-loop vertex type integral

At $n = 4$, $j = 1$, $m_4 = 0$, in order to get rid of integrals $I_4^{(d)}$, $I_4^{(d+2)}$ and also one vertex integral $I_3^{(d)}$ in Eq. (3), we impose conditions

$$G_3 = 0, \quad \partial_1 \Delta_4 = 0, \quad \partial_1 \partial_2 \Delta_4 = 0.$$  (7)

This system of equations depends on 10 variables $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}, m_1^2, m_2^2, m_3^2, m_4^2$ and it can be solved by excluding, for example, $p_{14}, p_{24}$ and $p_{34}$. Substituting solution for $p_{14}, p_{24}, p_{34}$ into Eq. (3) leads to the following FE:

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2, s_{23}, s_{13}, s_{12}) =$$

$$\frac{s_{13} + m_3^2 - m_1^2 + \alpha_{13}}{2s_{13}} I_3^{(d)}(m_2^2, m_3^2, 0, s_{34}^{(13)}, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{23})$$

$$+ \frac{s_{13} - m_3^2 + m_1^2 - \alpha_{13}}{2s_{13}} I_3^{(d)}(m_1^2, m_2^0, 0, s_{24}(m_1^2, m_3^2, s_{23}, s_{13}, s_{12}), s_{14}^{(13)}, s_{12}),$$  (8)

where

$$p_{14} = s_{14}^{(13)},$$
$$p_{24} = s_{24}(m_1^2, m_2^2, p_{23}, p_{13}, p_{12}) = \frac{(p_{12} + p_{23} - m_1^2 - m_2^2)p_{13} + (p_{12} - p_{23} - m_1^2 + m_2^2)(m_3^2 - m_1^2 + \alpha_{13})}{2p_{13}},$$
$$p_{34} = s_{34}^{(13)},$$  (9)

and

$$s_{14}^{(ij)} = \frac{\Delta_{ij} + 2m_i^2 p_{ij} - (p_{ij} + m_i^2 - m_j^2)\alpha_{ij}}{2p_{ij}},$$
$$s_{34}^{(ij)} = \frac{\Delta_{ij} + 2m_j^2 p_{ij} + (p_{ij} + m_j^2 - m_i^2)\alpha_{ij}}{2p_{ij}},$$
$$\alpha_{ij} = \pm \sigma(p_{ij} - m_i^2 + m_j^2)\sqrt{\Delta_{ij}}.$$  (10)

Again as it was for integral $I_2^{(d)}$ integral $I_3^{(d)}$ with arbitrary arguments can be expressed in terms of integrals with at least one propagator massless. By applying formula (8) to integrals in the right-hand side of this equation several times one can express $I_3^{(d)}$ with arbitrary arguments in terms of integrals with two massless propagators and one line with massive propagator. Also one of kinematical arguments in these integrals will be zero and other arguments will be functions of initial kinematical arguments and masses. This integral can be expressed in terms of Appell hypergeometric function $F_1$ and Gauss’ hypergeometric function $2F_1$ [1].

3. Derivation of FE by using algebraic relations for propagators

Analyzing FE for one-loop Feynman integrals one can observe that integrands are rather similar and differ only by one propagator. In the relation for one-loop propagator type integral integrands are

$$\frac{1}{D_1 D_2}, \quad \frac{1}{D_0 D_2}, \quad \frac{1}{D_1 D_0}.$$
and integrands for the one-loop vertex type integrals are

\[
\frac{1}{D_1D_2D_3}, \quad \frac{1}{D_0D_2D_3}, \quad \frac{1}{D_1D_0D_3}, \quad \frac{1}{D_1D_2D_0}
\]

where

\[
D_0 = (k_1 - p_0)^2 - m_0^2 + i\epsilon, \quad D_1 = (k_1 - p_1)^2 - m_1^2 + i\epsilon, \\
D_2 = (k_1 - p_2)^2 - m_2^2 + i\epsilon, \quad D_3 = (k_1 - p_3)^2 - m_3^2 + i\epsilon,
\]

and \(i\epsilon\) is traditional causal prescription. Here \(k_1\) is integration momentum and on scalar products of \(p_1, p_2, \ldots\) some restrictions, like \(G_n = 0\), are imposed. Such restrictions effectively lead to linear dependence of these vectors.

### 3.1. Algebraic relations for products of propagators

One can raise the question: would it be possible to find algebraic relations for products of propagators and derive FE from such relations? From the explicit FE for one-loop propagator integral we can try to find relation of the form:

\[
\frac{1}{D_1D_2} = \frac{x_1}{D_0D_2} + \frac{x_2}{D_1D_0}
\]

where we assume that \(x_1, x_2, p_0, p_1\) and \(p_2\) are independent of \(k_1\). Putting all terms over the common denominator we get

\[
D_0 = x_1D_1 + x_2D_2.
\]

By differentiating (12) with respect to \(k_1\) we obtain two equations:

\[
x_1 + x_2 = 1, \quad p_0 = x_1p_1 + x_2p_2.
\]

Taking into account these relationships we get from (12) an additional equation:

\[
x_1x_2s_{12} + m_0^2 - x_1m_1^2 - x_2m_2^2 = 0,
\]

where \(s_{12} = (p_1 - p_2)^2\). From Eqs. (13), (14) one get expressions for \(x_1, x_2\):

\[
x_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} - \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}}, \\
x_2 = \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} + \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}},
\]

and

\[
\Lambda_2 = s_{12}^2 + m_1^4 + m_2^4 - 2s_{12}(m_1^2 + m_2^2) - 2m_1^2m_2^2.
\]

Integration of the obtained algebraic relation with respect to \(k_1\) gives the following FE:

\[
I^{(d)}_2(m_1^2, m_2^2, s_{12}) = \frac{s_{12} + m_1^2 - m_2^2 + \lambda}{2s_{12}} I^{(d)}_2(m_1^2, m_0^2, s_{13}(m_1^2, m_2^2, m_0^2, s_{12})) \\
+ \frac{s_{12} - m_1^2 + m_2^2 - \lambda}{2s_{12}} I^{(d)}_2(m_2^2, m_0^2, s_{23}(m_1^2, m_2^2, m_0^2, s_{12})),
\]

Integration of the obtained algebraic relation with respect to \(k_1\) gives the following FE:
where
\[ s_{13} = \frac{\Lambda^2 + 2s_{12}(m_2^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} \lambda \]
\[ s_{23} = \frac{\Lambda^2 + 2s_{12}(m_2^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 - s_{12}}{2s_{12}} \lambda. \]
\[ \lambda = \sqrt{\Lambda^2 + 4s_{12}m_0^2}. \]

Parameter \( m_0 \) is arbitrary and can be taken at will. The same equation was obtained from recurrence relations by imposing conditions on Gram determinants.

Similar to the relation with two propagators one can find relation for three propagators:
\[ \frac{1}{D_1D_2D_3} = \frac{x_1}{D_2D_3D_0} + \frac{x_2}{D_1D_3D_0} + \frac{x_3}{D_1D_2D_0}. \quad (16) \]
Here \( p_0, p_1, p_2 \) and \( p_3 \) are external momenta assumed to be independent on \( k_1 \). Momentum \( k_1 \) will be integration momentum and we assume also that \( x_i \) do not depend on \( k_1 \). Multiplying both sides of equation (16) by the product \( D_1D_2D_3D_0 \) we get
\[ D_0 = x_1D_1 + x_2D_2 + x_3D_3. \quad (17) \]

Differentiation of this equation with respect to \( k_1 \) gives two equations:
\[ x_1 + x_2 + x_3 = 1, \quad p_0 = x_1p_1 + x_2p_2 + x_3p_3. \quad (18) \]

Taking into account these relationships from Eq. (17) we obtain:
\[ x_2x_3p_{23} + x_1x_3p_{13} + x_1x_2p_{12} - x_1m_1^2 - x_2m_2^2 - x_3m_3^2 + m_0^2 = 0. \quad (19) \]
This system has the following solution
\[ x_1 = 1 - \alpha - x_2, \quad x_3 = \alpha, \quad (20) \]
where \( \alpha \) is the root of the quadratic equation
\[ \alpha^2p_{13} + [m_3^2 - m_1^2 - p_{13} + x_2(p_{13} + p_{12} - p_{23})]\alpha + m_1^2 - m_0^2 + (m_2^2 - m_1^2 - p_{12} + p_{12}x_2)x_2 = 0. \]

This root depends on 2 arbitrary parameters - \( m_0 \) and \( x_2 \). By integrating the obtained relation we get the same FE as it was given before.

As was shown in Ref. [4] one can also derive algebraic relations for products of any number of propagators.

Functional relations for Feynman integrals with integrands being rational functions of scalar products of different momenta to some extent resemble Abel’s addition theorem [8], [9], [10]. It would be useful to find similarities between Feynman integrals and Abel integrals.

3.2. Some remarks about Abel integrals

In this subsection we will present some facts about Abelian integrals that may be useful in investigation of functional equations for Feynman integrals. Abelian integral is an integral in the complex plane of the form
\[ \int_{z_0}^{z} R(x, w)dx, \]
where $R(x, w)$ is an arbitrary rational function of the two variables $x$ and $w$. These variables are related by the equation

$$F(x, w) = 0,$$

where $F(x, w)$ is an irreducible polynomial in $w$,

$$F(x, w) \equiv \phi_n(x)w^n + \ldots + \phi_1(x)w + \phi_0(x),$$

whose coefficients $\phi_j(x)$, $j = 0, 1, \ldots n$ are rational functions of $x$. Abelian integrals are natural generalizations of elliptic integrals, which arise when

$$F(x, w) = w^2 - P(x),$$

where $P(x)$ is a polynomial of degree 3 and 4. If degree of the polynomial is greater than 4 then we have hyperelliptic integral. For Abel integrals an important theorem was proven. Let $C$ and $C'$ be plane curves given by the equations

$$C : \quad F(x, y) = 0, \quad C' : \quad \phi(x, y) = 0.$$

These curves have $n$ points of intersections $(x_1, y_1), \ldots, (x_n, y_n)$, where $n$ is the product of degrees of $C$ and $C'$. Let $R(x, y)$ be a rational function of $x$ and $y$ where $y$ is defined as a function of $x$ by the relation $F(x, y) = 0$. Consider the sum

$$I = \sum_{i=1}^{n} \int_{x_{i,0}}^{x_{i,1}} R(x, y) dx. \quad (21)$$

Integrals being taken from a fixed point to the $n$ points of intersections of $C$ and $C'$. If some of the coefficients $a_1, a_2, \ldots, a_k$ of $\phi(x, y)$ are regarded as continuous variables, the points $(x_i, y_i)$ will vary continuously and hence $I$ will be a function, whose form is to be determined, of the variable coefficients $a_1, a_2, \ldots, a_k$.

Abel’s theorem:
The partial derivatives of the sum $I$, with respect to any of the coefficients of the variable curve $\phi(x, y) = 0$, is a rational function of the coefficients and hence $I$ is equal to a rational function of the coefficients of $\phi(x, y) = 0$, plus a finite number of logarithms or arc tangents of such rational functions. What is very important - integrals themselves can be rather complicated transcendental functions but their sum can be simple.

Example: Elliptic integral of the second type [9]:

$$E(k, x) = \int_0^x \frac{(1 - k^2 x)dx}{\sqrt{x(1-x)(1-k^2x)}}.$$

We take as $C$ and $C'$

$$C : \quad y^2 = x(1-x)(1-k^2x), \quad C' : \quad y = ax + b. \quad (22)$$

The elimination of $y$ between two equations will give us as the abscissae $x_1, x_2, x_3$ of the points of intersection the three roots of the equation:

$$\phi(x) = k^2x^3 - (1 + k^2 + a^2)x^2 + (1 - 2ab)x - b^2 = 0. \quad (23)$$
The corresponding sum will be
\[
I(a, b) = \int_{0}^{x_1} R(x, y)dx + \int_{0}^{x_2} R(x, y)dx + \int_{1/k^2}^{x_3} R(x, y)dx,
\]
where
\[
R(x, y) = \frac{1 - k^2x}{y}.
\]
(24) 
Abel’s theorem gives addition formula:
\[
\int_{0}^{x_1} R(x)dx + \int_{0}^{x_2} R(x)dx + \int_{1/k^2}^{x_3} R(x)dx = -2a + \kappa,
\]
(25) where \( \kappa \) is an arbitrary constant.

One can see that FE for Feynman integrals and the relationship (25) have some common features. Arguments of Feynman integrals and arguments of the functions in the left-hand side of Eq. (25) are determined from algebraic equations and integrands in both cases are rational functions of some algebraic functions.

One can find FE for Feynman integrals following closely derivation of relationships for usual algebraic integrals. Deriving relations for propagators we used orthogonality condition \( G_n = 0 \). In fact it is not needed to assume such a relation. For example, to fix parameters in algebraic relations for products of two propagators
\[
R_2(k_1, p_1, p_2, m_1^2, m_2^2, m_0^2) = \frac{1}{D_1 D_2} - \frac{x_1}{D_2 D_0} - \frac{x_2}{D_1 D_0} = 0,
\]
(26) instead of \( G_n = 0 \) we can impose conditions
\[
\frac{\partial x_1}{\partial k_{1\mu}} = \frac{\partial x_2}{\partial k_{1\mu}} = 0,
\]
(27) having in mind that only \( D_j \) factors will depend on momentum \( k_1 \). Multiplying both sides of Eq. (26) by \( D_0 D_1 D_2 \) we get
\[
D_0 - x_1 D_1 - x_2 D_2 = (1 - x_1 - x_2)k_1^2 + 2x_1 k_1 p_1 + 2x_2 k_1 p_2 + \text{terms involving}\ p_1, p_2, m_1^2, m_2^2, m_0^2 = 0.
\]
(28)

Differentiating this relation with respect to \( k_{1\mu} \), contracting with \( k_1, p_1, p_2, p_0 \) and taking into account (27) gives several equations:
\[
-2x_1(k_1^2 - k_1 p_1) - 2x_2(k_1^2 - k_1 p_2) + 2k_1^2 - 2k_1 p_0 = 0,
\]
\[
2(1 - x_1 - x_2)k_1 p_1 + 2x_1 p_1^2 + 2x_2 p_1 p_2 - 2p_1 p_0 = 0,
\]
\[
2(1 - x_1 - x_2)k_1 p_2 + 2p_1 p_2 x_1 + 2x_2 p_2^2 - 2p_2 p_0 = 0,
\]
\[
2(1 - x_1 - x_2)k_1 p_0 + 2x_1 p_1 p_0 + 2x_2 p_2 p_0 - 2p_0^2 = 0.
\]
(29) 
They can be used to express \( k_1 p_0, p_1 p_0, p_2 p_0, x_1, x_2 \) in terms of \( k_1^2, k_1 p_1, k_1 p_2, p_1^2, p_1 p_2, p_2^2 \) considered to be independent variables. For example, we get:
\[
k_1 p_0 = x_1 k_1 p_1 - x_1 k_1 p_2 + k_1 p_2 + \frac{x_1}{2}(m_1^2 - m_2^2 - p_1^2 + p_2^2) + \frac{1}{2}(m_2^2 - p_2^2 - m_0^2 + p_0^2),
\]
(30)
and similar expressions for other scalar products of $p_0$. Solution for $x_1, x_2$ is the same as in Eq. (15) and as a result we obtained the same relation between products of two propagators.

Solution of the above system of equations is rather similar to finding intersections of two plane curves considered in Abel’s theorem. Unfortunately extension of Abel’s theorem for integrals of algebraic functions of several variables is not an easy task. In the case of functions of two variables some results were obtained long time ago in Refs. [11], [12].

Similar to usual algebraic integrals of one variable we can construct various integrands out of our different relationships for products of propagator. These integrands will be rational functions in independent variables. Integrations should be done over $d$ dimensional space. Rational function must resemble integrands for Feynman integrands.

For example, multiplying $R_2(k_2, p_2, p_4, m_2^2, m_4^2, \tilde{m}_0^2)$ by

$$
\frac{1}{[(k_1 - p_1)^2 - m_1^2]^{\nu_1} (k_1 - p_3)^2 - m_3^2]^{\nu_3} (k_1 - k_2)^2 - m_5^2]^{\nu_5}},
$$

and integrating this product with respect to $k_1, k_2$ leads to the following FE:

$$
\int \frac{d^dk_1 d^dk_2}{[(k_1 - p_1)^2 - m_1^2]^{\nu_1} (k_1 - p_3)^2 - m_3^2]^{\nu_3} (k_1 - k_2)^2 - m_5^2]^{\nu_5}} R_2(k_2, p_2, p_4, m_2^2, m_4^2, \tilde{m}_0^2) = 0,
$$

Integrals in this equation correspond to the diagram given in Fig.1.

![Fig.1. FE for the two-loop box integral.](image)

By integrating product of the relationship

$$
R_3(k_1, p_1, p_2, p_3, m_1^2, m_2^2, m_3^2, m_0^2)
= \frac{1}{D_1D_2D_3} - \frac{x_1}{D_2D_3D_0} - \frac{x_2}{D_1D_3D_0} - \frac{x_3}{D_1D_2D_0} = 0,
$$

and one loop-propagator integral

$$
\int \int \frac{d^dk_1 d^dk_2}{[(k_1 - p_1)^2 - m_1^2]^{\nu_1} (k_1 - k_2)^2 - m_5^2]^{\nu_5}} R_3(k_2, p_2, p_3, p_4, m_2^2, m_3^2, m_4^2, m_0^2) = 0,
$$

we obtain the FE for the integral corresponding to the following diagram with arbitrary $\nu_1, \nu_5$, and arbitrary momenta and masses

![Fig.2. FE for the two-loop integral with propagator insertion.](image)

By integrating different products of $R_2, R_3$ and products of different propagators with respect to selected momenta one can easily derive FE for various multiloop Feynman integrals.
4. Concluding remarks
We described two different methods for finding FE for Feynman integrals with any number of loops and external legs. FE reduce integrals with complicated kinematics to simpler integrals. FE can be used for analytic continuation of Feynman integrals without knowing explicit analytic result. At the present time application of these methods for some two- and three-loop integrals is in progress. Systematic investigation of FE for Feynman integrals based on algebraic geometry and group theory is needed. Some improvements of these methods can be done by exploiting known methods for algebraic integrals. The methods can be extended for finding functional equations among hypergeometric as well as holonomic functions.

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