Diffusion accompanying noise induced transport in frictional ratchets

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We study the noise induced transport of an overdamped Brownian particle in frictional ratchet systems in the presence of external Gaussian white noise fluctuations. The analytical expressions for current and diffusion coefficient are derived and the reliability or coherence of transport are discussed by means of their ratio. We show that frictional ratchets exhibit larger coherence as compared to the flashing and rocking ratchets.

PACS numbers: 05.40.Jc, 05.40.Ca, 02.50.Ey
Keywords: Ratchets, Brownian motors, Noise, Transport coherence

I. INTRODUCTION

The phenomenon of noise induced transport (thermal ratchets) are being discussed extensively in recent years. Ratchets are systems that exploit the nonequilibrium fluctuations present in the medium to generate a directed flow of Brownian particles in the absence of a bias 1, 2. Such ratchet models are found to have wide range of applications in biological systems and in nanotechnology 3.

Several physical models like flashing ratchets 4, rocking ratchets 5 etc., where potential has been taken to be asymmetric in space, have been developed. In these models to generate noise induced directional transport the nonequilibrium fluctuations need to be correlated in time. One can also generate unidirectional currents in the presence of symmetric ratchet potentials, however it has to be driven by correlated time asymmetric force 6. There is yet another class of ratchets, namely the frictional ratchets which we consider here, where the friction coefficient varies in space 7, 8. In such ratchet systems it is possible to get unidirectional currents even in a symmetric potential in the absence of a net bias. Moreover, external fluctuations need not be correlated in time. In the presence of external parametric noise the particle on an average absorbs energy from the noise source. The particle spends larger time in the region of space where the friction is higher and hence the energy absorption from the noise source is higher in these regions. Thus the particle in the higher friction regions feel effectively higher temperatures. Hence the problem of particle motion in an inhomogeneous medium in presence of an external noise becomes equivalent to the problem in a space dependent temperature 9, 10. Such systems are known to generate unidirectional currents. This follows as a corollary to Landauer’s blow torch theorem that the notion of stability changes dramatically in the presence of temperature inhomogeneities 11. In such cases the notion of local stability, valid in equilibrium systems, does not hold.

Frictional inhomogeneities are common in superlattice structures, semiconductors or motion in porous media. Frictional inhomogeneity changes the dynamics of the particle nontrivially as compared to the homogeneous case. This in turn has been shown to give rise to many counter intuitive phenomena like noise induced stability, stochastic resonance, enhancement in efficiency etc., in driven non-equilibrium systems 12, 13.

Extensive studies have been done on the nature of currents and their reversals, stochastic energetics (thermodynamic efficiency) etc., in different class of ratchet models. In contrast, there are very few studies which addresses the question of diffusion accompanying transport in ratchet systems 14, 15, 16. This is intimately related to the question of reliability of transport. Diffusion infact detriments the quality of transport. In our present work we focus on the transport coherence in frictional ratchets in the presence of external Gaussian...
white noise fluctuations.

Transport of Brownian particles are always associated with a diffusive spread. When a particle on an average moves a distance \( L \) due to its velocity, there will always be an accompanying diffusive spread. If this diffusive spread is much smaller than the distance travelled, then the particle motion is considered to be coherent or optimal or reliable. This is in turn quantified by a dimensionless quantity, Péclet number \( Pe \), which is the ratio of current to the diffusion constant. Higher the Péclet number greater than 2 implies coherence in the transport. Péclet numbers for some of the models like flashing and rocking ratchets show low coherence of transport with \( Pe \sim 0.2 \) and \( Pe \sim 0.6 \) respectively. Another study on symmetric periodic potentials along with spatially modulated white noise showed a coherent transport with Péclet number less than 3. In the same study a special kind of strongly asymmetric potential is found to increase \( Pe \) to 20 in some range of physical parameters. Experimental studies on molecular motors showed more reliable transport with Péclet number ranging from 2 to 6.

In our present work we show that system inhomogeneities help in enhancing the coherence in the transport depending sensitively on the physical parameters. For this we consider a simple spatially symmetric sinusoidal potential. \( Pe \) of the order of 3 can be readily obtained. The noise strength of the external parametric white noise fluctuations play a constructive role in enhancing the coherence in transport. As opposed to this, temperature (internal fluctuations) degrades the coherence in transport.

II. MODEL:

We consider the overdamped dynamics of a Brownian particle moving in a medium with spatially varying frictional coefficient \( \eta(q) \) at temperature \( T \). Using a microscopic treatment the Langevin equations for the Brownian particle in a space dependent frictional medium has been obtained earlier. The corresponding overdamped Langevin equation of motion is given by

\[
\dot{q} = -\frac{V'(q)}{\eta(q)} - \frac{k_B T \eta'(q)}{2[\eta(q)]^2} + \sqrt{\frac{k_B T}{\eta(q)}} f(t)
\]

with \( f(t) \) having mean \( \langle f(t) \rangle = 0 \) and \( f(t)f(t') = 2\delta(t - t') \) where \( \delta \) denotes the ensemble average and \( q \) the coordinate of the particle.

The system is then subjected to an external parametric additive white noise fluctuating force \( \xi(t) \), so that the equation of motion becomes

\[
\dot{q} = -\frac{V'(q)}{\eta(q)} - \frac{k_B T \eta'(q)}{2[\eta(q)]^2} + \sqrt{\frac{k_B T}{\eta(q)}} f(t) + \xi(t)
\]

For periodic functions \( V(q) \) and \( \eta(q) \) with periodicity \( L \), one can readily obtain analytical expression for particle velocity and is given by

\[
v = L \frac{1 - \exp[-2 \pi \delta]}{\int_0^{2\pi} dy \exp[-\psi(y)]} \int_0^{2\pi} dx \exp\left[\frac{\psi(x)}{A(x)}\right]
\]

with the generalized potential \( \psi(q) \) as

\[
\psi(q) = \int dx \frac{V'(x)}{k_B T + \Gamma \eta(x)}
\]

and \( A(q) \) as

\[
A(q) = \frac{k_B T + \Gamma \eta(q)}{\eta(q)}
\]

with

\[
\delta = \psi(q) - \psi(q + 2\pi)
\]

which in turn determines the effective slope of the generalized potential \( \psi(q) \). Hence the sign of \( \delta \) gives the direction of current which follows from Eqn.

In our present work we have taken the potential \( V(q) = V_0 \sin(q) \) and \( \eta(q) = \eta_0 [1 - \lambda \sin(q - \phi)] \), \( 0 < \lambda < 1 \). The phase lag \( \phi \) between \( V(q) \) and \( \eta(q) \) brings in the intrinsic asymmetry in the dynamics of the system. The effective potential \( \psi(q) \), \( \delta \) and \( A(q) \) are
obtained from Eqs. [8] and [7]. Following references [19, 20], one can obtain exact analytical expressions for the diffusion coefficient $D$ and the average velocity $v$ as

$$D = \frac{\int_{q_0}^{q_0+L} \frac{dx}{L} A(x) [I_+(x)]^2 I_-(x)}{\left[\int_{q_0}^{q_0+L} \frac{dx}{L} I_+(x)\right]^3}$$  \tag{8}

$$v = \frac{L(1 - \exp[-L\delta])}{\int_{q_0}^{q_0+L} \frac{dx}{L} I_+(x)}$$  \tag{9}

where $I_+(x)$ and $I_-(x)$ are as given below

$$I_+(x) = \frac{1}{A(x)} \exp[\psi(x)] \int_{x-L}^{x} dy \exp[-\psi(y)]$$  \tag{10}

$$I_-(x) = \exp[-\psi(x)] \int_{x}^{x+L} dy \frac{1}{A(y)} \exp[\psi(y)]$$  \tag{11}

$L$ here represents the period of the potential ($= 2\pi$ in our case). Now, the time taken for a Brownian particle to travel a distance $L$ is given as $\tau = L/v$ and the spread of the particle in the same time is given as $< (\Delta q)^2 > = 2D\tau$. For a reliable transport we require $< (\Delta q)^2 > < 2D\tau < L^2$. This in turn implies that $Pe = Lv/D > 2$ for coherent transport.

### III. RESULTS AND DISCUSSIONS

The velocity ($v$), diffusion constant ($D$) and the Pe\c{c}let number ($Pe$) are studied as a function of different physical parameters. All the physical quantities are taken in dimensionless form. In particular, velocity and diffusion are normalized by $(V_0/\eta_0)L$ and $(V_0/\eta_0)$ respectively. Throughout our work we have set $V_0$ and $\eta_0$ to be unity. Similarly $\Gamma$ and $T$ are scaled with respect to $V_0$ and $V_0/\eta_0$ respectively.

In Fig. 1 we have plotted the effective potential $\psi(q)$ to emphasize this. The effective potential is scaled with respect to temperature. The nature of currents (in particular the direction of current) can be readily inferred from the plot of effective potential $\psi(q)$ as a function of $q$ for $\phi = 0.3\pi$ and $\phi = 1.3\pi$ for various values of physical parameters as mentioned in the figure caption. For $\phi$ values between 0 to $\pi$ the effective potential will be tilted downward and hence the current direction is positive. For $\phi$ values between $\pi$ and $2\pi$ the current is in the negative direction. The barrier heights of the effective potential decreases with increase in the strength of the external noise $\Gamma$. This in turn will lead to an enhancement in the noise induced velocity which will be emphasized in later discussions.

In Fig. 2 we plot $v$, $D$ and $Pe$ as a function of the phase difference $\phi$ between the periodic functions $V(q)$ and $\eta(q)$ for a fixed noise strength, $\Gamma$ and $\lambda$, the amplitude of the periodic modulation of $\eta(q)$. We have evaluated these quantities by numerically integrating Eqs. [8] & Eqn. [10] using quadrature methods. All the physical parameters are mentioned in the figure caption. As expected, all the quantities are periodic functions of phase $\phi$ with the velocity $v$ being zero at $\phi = 0, \pi$ and $2\pi$. Fig. 2 [21, 22]. In the range between 0 to $\pi$, the effective potential $\psi(q)$ will be tilted down in the forward direction and the current hence is positive. When the phase difference is between $\pi$ and $2\pi$ the effective potential will be tilted in the opposite direction and hence the current is in the negative direction as discussed above. This also follows from the fact that in the region of $\phi$ between 0 to $\pi$ higher friction regions lie between the potential minima and the corresponding neighbouring maxima on the right and hence the particle on the average gains more energy in this region as discussed in the introduction. As a consequence the particle in a well finds it easier to cross the peak of the potential and go over to the right side of the well as compared to crossing over to the other side (left of the well). Hence current in the positive direction is assured. It should also be noted that when current is zero diffusion constant is finite (at the values of $\phi = 0, \pi$ and $2\pi$) [21, 22]. For the parameters chosen it appears as if $v$ and $D$ are symmetric around $\pi/2$. However it is not so. This has been
verified with other parametric values. Also, between 0 and $\pi$ and $\pi$ and $2\pi$ pe´clet number exhibits a maximum with the value being around 2.5. Both the velocity and diffusion constant exhibit a maxima between 0 and $\pi$. However, the range of variation of the amplitude of velocity is higher than that of diffusion resulting in a maxima in pe´clet number.

![Figure 2: Plot of $v$, $D$, and $Pe$ vs $\phi$ for $\Gamma = 1$, $\lambda = 0.9$ and $T = 0.01$](image)

To understand the role of spatial asymmetry in the potential we have considered an asymmetric potential of the form $V(q) = V_0[\sin(q) - \mu/4 \sin(2q)]$ with the asymmetry parameter $\mu = 1$. Fig. 3 shows the behaviour of $v$, $D$ and $Pe$ for this simple asymmetric case with the other physical parameters kept the same as in Fig. 2. For this case we notice that the velocity is not zero when $\phi = 0, \pi$ or $2\pi$ as anticipated. Moreover, it is also clear that the presence of this simple asymmetry does not make a significant contribution to increase an increase in $Pe$. Hence we restrict to the simple potential $V = V_0 \sin(x)$ in our further analysis. From these two figures we conclude that for a certain range of $\phi$ depending on system parameters the transport is coher-ent ($Pe \gg 2$) and moreover this range is quite large. In Fig. 4 we plot $v$, $D$ and $Pe$ as a function of the external white noise strength $\Gamma$. The parameters chosen are given in the figure captions. The value of $\phi$ corresponds to the value at which a maximum in $Pe$ is seen for the parameters chosen in Fig. 2. We observe that the velocity increases monotonically with the external white noise and saturates at higher values of $\Gamma$ (not shown in figure). In contrast, the diffusion coefficient keeps on increasing indefinitely. The $Pe$ exhibits a maxima around $\Gamma = 1.8$ and the value being approximately 2.7. The reason behind this maxima is same as that given in [14] for the symmetric potential case.

In Fig. 5 we plot velocity, diffusion and $Pe$ as a function of $T$ for $\Gamma = 1.7$. We observe that the current exhibits a peak and monotonously decreases to zero as a function of $T$. At higher $T$, ($k_B T > \Gamma \eta_0$) this behaviour is expected as the temperature overshadows the effect of inhomogeneity in the effective potential thus suppressing the ratchet effect. If $\Gamma$ is much larger than

![Figure 3: Plot of $v$, $D$, and $Pe$ vs $\phi$ for $\mu = 1$, $\Gamma = 1$, $\lambda = 0.9$ and $T = 0.01$](image)
FIG. 4: Plot of $v$, $D$ and $Pe$ vs $\Gamma$ for $\phi = 0.48\pi$, $\lambda = 0.9$ and $T = 0.01$.

FIG. 5: Plot of $v$, $D$ and $Pe$ vs $T$ for $\phi = 0.48\pi$, $\Gamma = 1.7$ and $\lambda = 0.9$.

FIG. 6: Plot of $v$, $D$ and $Pe$ vs $\lambda$ for $\phi = 0.48\pi$, $\Gamma = 1.7$ and $T = 0.01$.

1 the peak in the current disappears and hence current monotonously decreases with $T$. In contrast, diffusion constant increases monotonously. The $Pe$ decreases monotonously with $T$ thereby suppressing the coherence in the transport.

In Fig. 6 we plot velocity, diffusion and $Pe$ for values of $\phi = 0.48\pi$ and $\Gamma = 1.7$ as a function of the amplitude of oscillation of the friction coefficient $\lambda$ ($0 < \lambda < 1$). We see that the current increases as a function of $\lambda$ whereas the diffusion constant decreases as a function of $\lambda$. In the present case diffusion constant exhibits a minima near $\lambda = 0.8$. However, for other parameter values we see that $D$ need not exhibit a minima. It monotonically decreases. It should be noted that $\lambda$ cannot be greater than or equal to 1 so as to maintain the friction coefficient to be positive. The pe\'clet number increases monotonously as a function of $\lambda$. This is the only case for which we observe a decrease in $D$ while current simultaneously increases as a function of $\lambda$. Thus from our plot we see that the presence of $\lambda$ makes the transport more coherent.
IV. CONCLUSIONS

We have studied the reliability or coherence of transport in frictional ratchet systems in the presence of external white noise fluctuations. Pedlet number greater than 2 can be readily obtained in a wide range of parameter space implying that the transport is coherent. External noise ($\Gamma$) helps in transport coherence whereas the internal noise ($T$) degrades the transport coherence. As a function of $\Gamma$ and $T$ higher drift velocity is also linked with higher diffusion coefficient. It is always desirable to have larger accompanying current with minimal diffusive spread to have reliable transport. For this we should have regions where velocity increases accompanied by a decrease in diffusion coefficient as a function of $\Gamma$ and $T$. This maybe achieved in our model by appropriately choosing strongly asymmetric spatially periodic potential like the one used in [14]. Our frictional ratchet systems exhibits much larger coherence compared to previously studied single particle flashing and rocking ratchet systems.

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