Baryon wave functions in QCD and integrable models

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Abstract

A new theoretical framework is proposed for the description of baryon light-cone distribution amplitudes, based on the observation that their scale dependence to leading logarithmic accuracy is described by a completely integrable model. The physical interpretation is that one is able to find a new quantum number that distinguishes partonic components in the nucleon with different scale dependence.

1 Introduction

The notion of baryon distribution amplitudes refers to the valence component of the Bethe-Salpeter wave function at small transverse separations and is central for the theory of hard exclusive reactions involving baryons [1]. As usual for a field theory, extraction of the asymptotic behavior (here: zero transverse separation) introduces divergences that can be studied by the renormalization-group (RG) method. The distribution amplitude \( \phi \) thus becomes a function of the three quark momentum fractions \( x_i \) and the scale that serves as a UV cutoff in the allowed transverse momenta. Solving the corresponding RG equations one is led to the expansion

\[
\phi(x_i, Q^2) = 120 x_1 x_2 x_3 \sum_{N,q} \phi_{N,q}(Q_0^2) P_{N,q}(x_i) \left( \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right)^{\gamma_{N,q}/b}
\]

where the summation goes over all multiplicatively renormalizable operators built of three quarks and \( N \) derivatives. The polynomials \( P_{N,q}(x_i) \) and anomalous dimensions \( \gamma_{N,q} \) are obtained by the diagonalization of the mixing matrix for the three-quark operators.
Fig. 1. The spectrum of anomalous dimensions \( \gamma_N \equiv (1 + 1/N_c)E_N + 3/2C_F \) for the baryon distribution amplitudes with helicity \( \lambda = 1/2 \). The lines of the largest and the smallest eigenvalues for \( \lambda = 3/2 \) are indicated by dots for comparison.

\[
B_{k_1,k_2,k_3} = (D_{\pm}^{k_1}q)(D_{\pm}^{k_2}q)(D_{\pm}^{k_3}q); \quad k_1 + k_2 + k_3 = N
\]

and \( \phi_{N,q}(Q_0^2) \) are the corresponding (nonperturbative) matrix elements.

As well known, conformal symmetry allows one to resolve the mixing with operators containing total derivatives [2]–[9] and requires that the ‘eigenfunctions’ \( P_{N,q} \) corresponding to different values of \( N \) are mutually orthogonal with the weight function \( 120 x_1 x_2 x_3 \) that plays the role of the asymptotic wave function. In difference to mesons, the conformal symmetry is not sufficient, however, to solve the RG equations: for each \( N \) there exist \( N + 1 \) independent operators with the same conformal spin which mix with each other. This mixing produces a nontrivial spectrum of anomalous dimensions, see Fig. 1, about which little was known until recently. In what follows, I describe a new approach to the scale dependence of baryon distribution amplitude developed in [11,12]. The main result is that one is able to identify the second summation index \( q \) in (1) with an eigenvalue of a certain conserved charge. The physical interpretation is that one is able to find a new ‘hidden’ quantum number that distinguishes between partonic components in the proton with different scale dependence.
Hamiltonian formulation of the evolution equations

In the usual formulation of the evolution equations the conformal invariance is not explicit. It can be made manifest, however, if the evolution kernels are rewritten in terms of the generators of the SL(2) collinear subgroup of conformal transformations. The scale dependence is different for helicity \( \lambda = 3/2 \) and \( \lambda = 1/2 \) operators and the corresponding evolution kernels can be written in the following compact form [11,12]:

\[
H_{3/2} = 2 \left( 1 + \frac{1}{N_c} \right) \sum_{i<k} \left[ \psi(J_{ik}) - \psi(2) \right] + \frac{3}{2} C_F ,
\]

\[
H_{1/2} = H_{3/2} - 2 \left( 1 + \frac{1}{N_c} \right) \left[ \frac{1}{J_{12}(J_{12} - 1)} + \frac{1}{J_{23}(J_{23} - 1)} \right] .
\]

Here \( \psi(x) \) is the logarithmic derivative of the \( \Gamma \)-function and \( J_{ik}, i, k = 1, 2, 3 \) are defined in terms of the two-particle Casimir operators of the \( SL(2,R) \) group

\[
J_{ik}(J_{ik} - 1) = L_{ik}^2 \equiv (\vec{L}_i + \vec{L}_k)^2 ,
\]

with \( \vec{L}_i \) being the group generators acting on the \( i \)-th quark. In the momentum fraction representation (1) the generators take the form [12]

\[
L_{k,0} P(x_i) = (x_k \partial_k + 1) P(x_i) ,
L_{k,+} P(x_i) = -x_k P(x_i) ,
L_{k,-} P(x_i) = (x_k \partial_k^2 + 2 \partial_k) P(x_i) .
\]

Solution of the evolution equations corresponds in this language to solution of the Schrödinger equation

\[
HP_{N,q}(x_i) = \gamma_{N,q} P_{N,q}(X_i)
\]

with \( \gamma_{N,q} \) being the anomalous dimensions. The \( SL(2,R) \) invariance of the evolution equations implies that the generators of conformal transformations commute with the ‘Hamiltonians’

\[
[H, L^2] = [H, L_\alpha] = 0 ,
\]

where \( L^2 = (\vec{L}_1 + \vec{L}_2 + \vec{L}_3)^2 \) and \( L_\alpha = L_{1,\alpha} + L_{3,\alpha} + L_{3,\alpha} \), so that the polynomials \( P_{N,q}(x_i) \) corresponding to multiplicatively renormalizable operators can be chosen simultaneously to be eigenfunctions of \( L^2 \) and \( L_0 \).
\[ L^2 P_{N,q} = (N + 3)(N + 2)P_{N,q}, \]
\[ L_0 P_{N,q} = (N + 3)P_{N,q}, \]
\[ L_- P_{N,q} = 0. \] (8)

The third condition in (8) ensures that the operators do not contain overall total derivatives.

Main finding of [11] is that the Hamiltonian \( H_{3/2} \) possesses an additional integral of motion (conserved charge):

\[ Q = \frac{i}{2}[L_{12}^2, L_{23}^2] = i(\partial_1 - \partial_2)(\partial_2 - \partial_3)(\partial_3 - \partial_1)x_1x_2x_3, \quad [H_{3/2}, Q] = 0. \] (9)

The evolution equation for baryon distribution functions with maximum helicity is, therefore, completely integrable. The premium is that instead of solving a Schrödinger equation with a complicated nonlocal Hamiltonian, it is sufficient to solve a much simpler equation

\[ QP_{N,q}(x_i) = qP_{N,q}(x_i). \] (10)

Once the eigenfunctions are found, the eigenvalues of the Hamiltonian (anomalous dimensions) are obtained as algebraic functions of \( N, q \).

It is necessary to add that the Hamiltonian in (2) is known as the Hamiltonian describing the so-called XXX\(_{s=-1}\) Heisenberg spin magnet. The same Hamiltonian was also encountered in the theory of interacting reggeons in QCD [13,14].

3 Summary of results: Helicity \( \lambda = 3/2 \) distributions

The equation in (10) cannot be solved exactly, but a wealth of analytic results can be obtained by means of the \( 1/N \) expansion [15,12]. The general structure of the spectrum is illustrated in Fig. 2. It is easy to see that if \( q \) is an eigenvalue of \( Q \), then \( -q \) is also an eigenvalue, whereas the Hamiltonian only depends on the absolute value \( |q| \). It follows that all anomalous dimensions are double degenerate except for the lowest ones for each even \( N \), corresponding to the solution with \( q = 0 \). The corresponding eigenfunctions have a very simple form

\[
x_1x_2x_3P_{N,q=0}(x_i) = x_1(1 - x_1)C_{N+1}^{3/2}(1 - 2x_1) + x_2(1 - x_2)C_{N+1}^{3/2}(1 - 2x_2) + x_3(1 - x_3)C_{N+1}^{3/2}(1 - 2x_3)
\] (11)

and the anomalous dimension is equal to
Fig. 2. The spectrum of eigenvalues for the conserved charge $Q$ (a) and for the helicity-3/2 Hamiltonian $H_{3/2}$ (b), see text. Notation: $h = N + 3$.

$$
\gamma_{N,q=0} = (1 + 1/N_c) \left[ 4\psi(N + 3) + 4\gamma_E - 6 \right] + 3/2C_F. 
$$

Furthermore, the eigenvalues of $Q$ lie on trajectories (see Fig. 2) corresponding to the semi-classically quantized soliton waves [16]. The corresponding trajectories for the anomalous dimensions have a rather peculiar form. Each of them can be considered as a separate partonic component in the nucleon wave function, in the same spirit as the leading-twist parton distribution functions in deep-inelastic scattering arises from the analytic continuation of the anomalous dimensions, giving rise to the Altarelli-Parisi splitting function. The asymptotic expansions for the charge $q$ and the anomalous dimensions at large $N$ are available to the order $1/N^8$ [15,12] and give very accurate results. The algebraic structure of the spectrum is very complicated. As an example, I present
the expression for the anomalous dimension $\gamma_{N,q} = (1 + 1/N_c)E_{N,q} + 3/2C_F$ as a function of the charge $q$: [15]

$$E_{N,q} = 2 \ln 2 - 6 + 6\gamma_E + 2\text{Re} \sum_{k=1}^{3} \psi(1 + i\eta^3 \delta_k) + O(\eta^{-6}),$$  \hspace{1cm} (13)

where $\eta = \sqrt{(N + 3)(N + 2)}$ and $\delta_k, k = 1, 2, 3$ are the three roots of the cubic equation

$$2\delta_k^3 - \delta_k - q/\eta^3.$$  \hspace{1cm} (14)

Accuracy of (13) is excellent, as illustrated in Fig. 3.

4 Helicity $\lambda = 1/2$ distributions

The additional term in $H_{1/2}$ spoils integrability but can be considered as a small correction for the most part of the spectrum [12]. Its effect on the two lowest levels is drastic, however. To illustrate this, consider the flow of energy levels for the Hamiltonian $H(\epsilon) = \sum_{i<k} \left[ \psi(J_{ik}) - \psi(2) \right] - \epsilon \left[ 1/L_{12}^2 + 1/L_{23}^2 \right]$ (cf. (2), (3)) as a function of an auxiliary parameter $\epsilon$, see Fig. 4. It is seen that the two lowest levels decouple from the rest of the spectrum and are
Fig. 4. The flow of energy eigenvalues for the Hamiltonian $H(\epsilon)$ for $N = 30$. The solid and the dash-dotted curves show the parity-even and parity-odd levels, respectively. The two vertical dashed lines indicate $H_{3/2} \equiv H(\epsilon = 0)$ and $H_{1/2} \equiv H(\epsilon = 1)$, respectively (up to the color factors). The horizontal dotted line shows position of the ‘ground state’ given by Eq. (12).

separated from it by a finite mass gap. As shown in [12], this phenomenon can be interpreted as binding of the two quarks with opposite helicity and forming a scalar “diquark”. The effective Hamiltonian for the low-lying levels can be constructed and turns out to be a generalization of the famous Kronig-Penney problem for a particle in a $\delta$-function type periodic potential. The value of the mass gap between the lowest and the next-to-lowest anomalous dimensions at $N \to \infty$ can be calculated combining the small-$\epsilon$ and the large-$\epsilon$ expansions and is equal to

$$\Delta \gamma = 0.32 \cdot (1 + 1/N_c) \quad (15)$$

in agreement with the direct numerical calculations.

5 Further developments

The approach developed in [11,12] turns out to be general and is applicable to the analysis of all three-parton systems in QCD, albeit with some modifications. The extension to three three-gluon operators is straightforward but tedious, since elementary fields have a higher conformal spin [17]. Probably more interesting from both phenomenological and mathematical point of view are the applications to quark-antiquark-gluon systems. The corresponding evolution equations are integrable in the limit of large number of colors [11] and reduce to a different type of integrable models - the so-called open chains. A rather detailed analysis of integrable quark-antiquark-gluon operators has
been given in [11,18,19] and, most recently, a concrete phenomenological application of this method to the structure function \( g_2(x, Q^2) \) was considered in work [20].

To summarize, a powerful mathematical framework has been developed for the description of the evolution of baryon distribution amplitudes. The mathematical structure of evolution equations is very elegant and reveals certain qualitative features of the distribution amplitudes. A lot of analytical results is obtained, in different limits. The formalism is general and was applied already to the other existing three-parton distributions. In a more general context, the integrability of evolution equations reveals an additional symmetry of QCD and its close relation to exactly solvable statistical models. Remarkably enough, the same symmetry has been observed in the studies of the Regge asymptotics of three-gluon distributions. All these features are not seen at the level of QCD Lagrangian and their origin has to be understood better.

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**References**

[1] V.A. Avdeenko, V.L. Chernyak and S.A. Korenblit, Yad. Fiz. **33** (1981) 481;
G.P. Lepage and S.J. Brodsky, Phys. Rev. Lett. **43** (1979) 545, 1625 (E);
S.J. Brodsky, G.P. Lepage and A.A. Zaidi, Phys. Rev. **D23** (1981) 1152;
S.J. Brodsky and G.P. Lepage, Phys. Rev. **D24** (1981) 2848;
A.I. Milshtein and V.S. Fadin, Yad. Fiz. **35** (1982) 1603.

[2] A.V. Radyushkin, Preprint JINR-P2-10717, Jun 1977;
A.V. Efremov and A.V. Radyushkin, Phys. Lett. **B94** (1980) 245; Teor. Mat. Fiz. **42** (1980) 147.

[3] S.J. Brodsky et al., Phys. Lett. **B91** (1980) 239; Phys. Rev. **D33** (1986) 1881.

[4] Yu.M. Makeenko, Sov. J. Nucl. Phys. **33** (1981) 440.

[5] Th. Ohrndorf, Nucl. Phys. **B198** (1982) 26.

[6] G.P. Lepage and S.J. Brodsky, Phys. Rev. Lett. **43** (1979) 545; Erratum-ibid. **43** (1979) 1625.

[7] M.E. Peskin, Phys. Lett. **B88** (1979) 128.
[8] K. Tesima, Nucl. Phys. B202 (1982) 523.
[9] Su-Long Nyeo, Z. Phys. C54 (1992) 615.
[10] N.G. Stefanis, Acta Phys. Polon. B25 (1994) 1777; Dissertation, JINR, 1997 (unpublished).
[11] V.M. Braun, S.E. Derkachov and A.N. Manashov, Phys. Rev. Lett. 81 (1998) 2020.
[12] V.M. Braun, S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, Nucl. Phys. B553 (1999) 355.
[13] L.D. Faddeev and G.P. Korchemsky, Phys. Lett. B342 (1995) 311.
[14] L.N. Lipatov, JETP Lett. 59 (1994) 596.
[15] G.P. Korchemsky, Nucl. Phys. B443 (1995) 255; ibid. B462 (1996) 333; ibid. B498 (1997) 68; Preprint LPTHE-Orsay-97-62 [hep-ph/9801377].
[16] G.P. Korchemsky and I.M. Krichever, Nucl. Phys. B505 (1997) 387.
[17] A. V. Belitsky, Nucl. Phys. B574 (2000) 407.
[18] A. V. Belitsky, Nucl. Phys. B558 (1999) 259.
[19] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, Nucl. Phys. B566 (2000) 203.
[20] V. M. Braun, G. P. Korchemsky and A. N. Manashov, Phys. Lett. B476 (2000) 455.