I. INTRODUCTION

There are increasing experimental evidences that multiple parton interactions (MPI) may occur within the same hadronic collision, as a result of the composite and extended nature of the colliding hadrons. After preliminary investigations [1–5], the inclusion of MPI has been proven to be required to obtain a proper description of the multiplicity and topology of the hadronic final state of $pp$ collisions at collider energies [7, 8]. Whereas MPI are characterized by soft and semi-hard components, in the present letter we will focus on double parton scattering (DPS), in which both scatterings involve a large momentum transfer, of the order of few GeV, so that short distance cross sections are perturbatively calculable. DPS could unveil parton correlations in the hadron structure not accessible in single parton scattering (SPS). Such correlations are encoded in novel distributions, i.e. double Parton Distribution Functions (dPDFs) which appear in the DPS cross section. The latter are interpreted as the number densities of a parton pair with a given transverse distance $b_\perp$ and carrying longitudinal momentum fractions $(x_1, x_2)$ of the parent hadron [9–15]. Despite the ongoing theoretical efforts to investigate dPDFs [16–21], the structure of the DPS cross section and its factorization properties [22–28], rather limited knowledge has been accumulated so far and DPS measurements have provided information mainly on $\sigma_{eff}$ in $pp$ collisions [29] and recently in $pA$ collisions [30]. This dimensionless parameter controls the magnitude of DPS contribution under the simplifying assumptions of two uncorrelated hard scatterings and full factorisation of dPDFs in terms of ordinary PDFs and model-dependent distribution in transverse position space. It has been shown in Refs. [31, 32] that the knowledge of $\sigma_{eff}$ can provide information on the proton structure, complementary to that obtained from generalized parton distribution functions. For the final state relevant to this analysis, i.e. four-jet production in $pp$ collisions, recent experimental results are in the range $\sigma_{eff} \sim 8 - 35$ mb [33, 34].

We propose here a strategy to extract novel information on the partonic structure of the proton by considering DPS processes in photon-proton interactions. In such a process, the impact of MPI has been studied in Ref. [35] via Monte Carlo simulations whereas the DPS contribution has been considered in Ref. [36] in the direct photon kinematics, in which the DPS processes is initiated by a $cc$ pair originating from the perturbative splitting of the photon. It is well known that in high-energy reactions a quasi-real photon exhibits a rather complex hadronic structure [37]. It can interact as a point-like particle with partons in the hadronic target, but it can also resolve into a hadronic structure and its partonic constituents could participate in the hard scattering. Additionally, as far as the (low) virtuality $Q^2$ of the photon can be measured by tagging the photon-emitter particles, the average transverse size of the $q\bar{q}$ pair fluctuation, $\langle b_\perp^2 \rangle_\gamma$, can be controlled, since it scales as $1/Q^2$ [38, 39]. Therefore, we address the intriguing question whether a DPS contributions could be observed in quasi-real photon-proton interactions, in full analogy with hadronic ones. In the present letter we will consider $ep$ collisions where the electron is the photon emitter. The generalization to other reactions involving nucleon and/or nuclei, only requires the use of the appropriate photon flux factors. In such a favourable environment
therefore the DPS mechanism, which is especially sensitive to parton pair correlations in transverse plane of the colliding particles, could be studied with a projectile of variable and controllable transverse size. Since a complete formulation of photon dPDFs capturing its longitudinal and transverse structure accommodating both its electromagnetic and hadronic components is missing at the moment, in the present paper we elaborate on a much simpler quantity, $\sigma_{\gamma p}^{\text{eff}}$.

With those results at hand, we present a first estimate of the DPS cross section for the photo-production of four-jet in HERA kinematics accompanied by its main background, i.e. the SPS four-jet photo-production cross-section, which, to the best of our knowledge, has never been discussed in the literature. A reliable estimate of the latter gives, in fact, a limit on the DPS contribution and thus constrains the corresponding models both of the photon and of the proton. We then show that if the $Q^2$ dependence of $\sigma_{\gamma p}^{\text{eff}}$ could be measured, then a first estimate of the mean transverse distance between partons in the proton could be obtained. Such a procedure avoids the intrinsic limitations in the extraction of this quantity from $\sigma_{\gamma p}^{\text{eff}}$ which are discussed in Refs. [31, 32]. Moreover we derived lower limits on the necessary integrated luminosity to observe the predicted $Q^2$ dependence of the DPS cross sections in HERA kinematics.

II. EFFECTIVE CROSS SECTION FOR $\gamma p$ DPS

The $\gamma p$ DPS cross section, $\sigma_{\gamma p}^{\text{DPS}}$, can be written in full analogy with the one appearing in the $pp$ case [7]. The former does depend on both the proton and photon dPDFs, $D_{q_i/p}(x_i, x_j, k_\perp)$ and $D_{\bar{q}_j/\gamma}(x_k, x_l, k_\perp)$, respectively where $i j$ and $kl$ are the flavours of the interacting partons, $k_\perp$ is the momentum imbalance, Fourier conjugate variable to the partonic transverse distance, $b_\perp$, and $x_i$’s are the longitudinal momentum fractions carried by each parton. The photon contribution to the cross section can be formally written similarly to that of meson dPDFs [10] [11]. In particular, the Light-Front (LF) wave functions of the photon can be generally treated as that of a vector meson. With this respect, the dPDFs of the $\rho$ meson have been investigated in Ref. [42]. We leave for future analyses the study of the rich spin structure of these vector systems which could give access to new double spin correlations in the proton.

The pair production amplitude, at a given photon virtuality $Q^2$, can be described within a Light-Front formalism in terms of the LF wave function, $\psi^{\gamma_5}$ [10]:

$$D_{q_i/\gamma}(x, k_\perp, Q^2) = \int d^2k_\perp \phi_{q_i}(x, k_\perp, Q^2) \psi_\gamma^{\gamma_5}(x, k_\perp, Q^2) \times \psi^{\gamma_5}(x, k_{\perp,1} + k_\perp; Q^2).$$

In the above equation we take into account the lowest Fock components. This in turn implies that, being a two-particle state, the longitudinal momentum of the second parton is given by $1 - x$. The integration runs over the transverse momentum of one parton of the pair, $k_{\perp,1}$, with $k_{\perp,2} = -k_{\perp,1}$.

Given the LF description of the unpolarized dPDFs and PDFs [10], one can derive the expression of the effective cross section in terms of effective form factors (eff) [13]. The latter, for the photon, reads:

$$F_2^\gamma(k_\perp) = \frac{\sum q \int dx D_{q_i/\gamma}(x, k_\perp)}{\sum q \int dx D_{\bar{q}_j/\gamma}(x, k_\perp = 0)},$$

in which the summation and integration, at variance with proton case, run over the indexes of only one parton of the pair. Such a definition of the eff relies on the approximation, frequently assumed in phenomenological analyses in the $pp$ case, that momentum correlations and parton flavor dependence are neglected. Moreover it guarantees that $F_2^\gamma(k_\perp = 0) = 1$, as required by the probabilistic interpretation of double parton distributions in coordinate space and their corresponding normalization. In terms of these quantities, the $\gamma p$ effective cross sections can be written as:

$$\sigma_{\gamma p}^{\text{eff}}(Q^2) = \left[ \int d^2k_\perp \frac{F_2^\gamma(k_\perp)}{\bar{F}_2^{\gamma_5}(k_\perp; Q^2)} \right]^{-1}.$$  

Under the additional assumption that double PDFs can be written, at any perturbative scale, as product of ordinary PDFs, the $\gamma p$ DPS cross section for the production of the final state $A + B$ is rearranged in a pocket formula $\sigma_{\gamma p}^{\text{DPS}} \sim \sigma_{\gamma p}^{\text{eff}}(A/B)$. Such an approximation allows an estimation of $\sigma_{\gamma p}^{\text{DPS}}$ by making use of known calculations of single parton scattering (SPS) cross sections $A(B)$ with $A(B)$ final states. It is worth to remark that such a procedure, largely used in DPS in $pp$ collisions and also adopted here, neglects any type of perturbative and non-perturbative correlations in double PDFs.

III. NUMERICAL RESULTS FOR $\sigma_{\gamma p}^{\text{eff}}$

The evaluation of $\sigma_{\gamma p}^{\text{eff}}(Q^2)$ requires the knowledge of the photon effective form factor for which we use phenomenological parametrizations. In particular we consider the dipole one of Ref. [15], which we address as model “S”:

$$F_2^\gamma(k_\perp) = \left( 1 + \frac{k_\perp^2}{m_\rho^2} \right)^{-4},$$

with $m_\rho^2 = 1.1$ GeV$^2$. Such a model returns a $\sigma_{\gamma p}^{\text{DPS}} \sim 30$ mb. In addition, in order to explore the dependence of our results on the functional dependence of the proton eff, we also considered a Gaussian ansatz [29] of the type

$$F_2^\gamma(k_\perp) = e^{-\alpha_i k_\perp^2}, \quad i = 1, 2.$$
The parameter $\alpha$ is fixed to $\alpha_1 = 1.53$ GeV$^{-2}$ which returns $\sigma_{\text{eff}}^{p\gamma} = 15$ mb ("G$^1_1$" model) and to $\alpha_2 = 2.56$ GeV$^{-2}$ which returns $\sigma_{\text{eff}}^{p\gamma} = 25$ mb ("G$^2_2$" model). All considered proton effs satisfy the normalization condition $F_2^p(k_{\perp} = 0) = 1$.

The other input appearing in Eq. (3) is the photon eff. The latter is calculated making use of the photon wave functions presented in Refs. [44, 45]. Among those presented in Ref. [44], we make use of the wave functions corresponding to the so called "spectral quark model". In Ref. [45], those quantities were evaluated to lowest order QED in momentum space. As it is well known [39], the modulus squared of the corresponding wave function is logarithmically divergent at large parton transverse momentum $k_{\perp}$ in Eq. (3). One option to regulate it is by introducing an upper physical cut-off on the $k_{\perp}$ integration, possibly of the order of the hard scale entering the scattering process. In the present work we pursued instead the idea of considering a large $k_{\text{cut}}$ for the reason to be detailed hereafter. When increasingly higher cut-offs are used in the evaluation of $F_2^p$, its tail at large $k_{\perp}$ shows, as expected, a sensitivity to $k_{\text{cut}}$ and it approaches a constant value of 1 for asymptotically large values of $k_{\text{cut}}$, i.e. the form factor of a structureless photon. However, the corresponding variations on the $\gamma p$ effective cross-section evaluated via Eq. (3) are much reduced since the fast falling behaviour of the proton eff at high $k_{\perp}$ [31, 32] effectively regulates the tail of the photon eff in the convolution integral in Eq. (3) and grants its convergence. Although a residual dependence of $\sigma_{\text{eff}}^{p\gamma}$ on $k_{\text{cut}}$ is still present, we have numerically verified that the effective cross-section varies no more than $\sim 1$ mb when $k_{\text{cut}}$ is raised from $k_{\text{cut}} = 50$ GeV to $k_{\text{cut}} = 10^3$ GeV, with the latter being used as default value in the following. See further details on this topic in the appendix [A].

Given these observations, the advantage of using a large cutoff is twofold: firstly one avoids to introduce a prescription for setting a physical cut-off and the resulting arbitrariness; secondly one obtains a lower limit on the $\gamma p$ effective cross-section, which implies that our estimate of the DPS cross section given by QED contribution should be considered as its upper limit. This procedure is analogous to the one adopted in Ref. [39] in coordinate space, where the square modulus of the photon wave function, divergent in the small $b_{\perp}$ limit, is de facto regularized by the so-called dipole cross section which vanishes as $b_{\perp}$ goes to zero. In the case of hadronic model of the photon presented in Ref. [44] such an issue is not present, since the corresponding wave function is properly normalized from the beginning.

We present our numerical estimates for $\sigma_{\text{eff}}^{p\gamma}(Q^2)$ in Fig. 1. Since in the present paper we will consider photoproduction in $ep$ collisions, the lower limit on $Q^2$ is set of the order of $m_e^2$, the mass of the electron, appearing in the Weizsäcker-Williams approximation for the spectrum of the exchanged photon. One may notice that the hadronic models of Ref. [44] returns a systematically higher $\sigma_{\text{eff}}^{p\gamma}$ with respect to the pure electromagnetic one [45]. The spread between the curves pertinent to the same photon model indicates a rather large sensitivity to the proton effective form factor. Both models display a peculiar pattern of the $Q^2$ dependence: both start from a plateau at low $Q^2$ and decrease at larger $Q^2$, with the onset of the decrease occurring at rather different values of $Q^2$. The shape of the distribution is replicated irrespectively of the adopted proton eff. We observe that, in the limit of high photon virtuality, the value of $\sigma_{\text{eff}}^{p\gamma}$ can be predicted in complete analogy with the gluon splitting case elaborated in Ref. [46]. In fact, the $qq$ pair, originated by the electromagnetic splitting of a highly virtual photon, is characterized by quite small transverse distance, $\langle b_{\perp}^2 \rangle_\gamma \propto 1/Q^2$, and asymptotically one has $(\sigma_{\text{eff,asy}}^{p\gamma})^{-1} = \int d^2 k_{\perp}/(2\pi)^2 F_2^p(k_{\perp}) = \tilde{F}_2^p(b_{\perp} = 0)$, where $\tilde{F}_2^p(b_{\perp})$ is the Fourier transform of the eff and $b_{\perp}$ is the conjugate variable to $k_{\perp}$. Adopting the $G_1$ model for the proton eff, the calculation with the photon wave function of Ref. [45] returns a value $\sigma_{\text{eff}}^{p\gamma}(Q^2 = 100$ GeV$^2) \sim 7.52$ mb while the predictions from Ref. [46] would give $\sigma_{\text{eff,asy}}^{p\gamma} \sim 7.5$ mb, showing remarkable agreement. We point out that the analysis of Ref. [36] is performed in the approximation described above, where the perturbative splitting of the photon into a $ee$ pair probes the proton eff (model S) at zero transverse distance, which makes that analysis complementary to the one discussed here, where the quasi-real photon develops a partonic structure at larger transverse distances.
IV. THE FOUR-JET PHOTO-PRODUCTION CROSS-SECTION

The four-jet final states have been measured in photo-production at HERA by the ZEUS collaboration [17]. In that analysis, they considered jets with transverse energy $E_T > 6$ GeV and laboratory pseudorapidity $|\eta_{jet}| < 2.4$, in the kinematic region $Q^2 < 1$ GeV$^2$ and the energy fraction transferred from the lepton to the photon, $y = E_\gamma/E_l$, in the range $0.2 \leq y \leq 0.85$. The comparison with leading-logarithmic parton-shower Monte Carlo models [35, 48, 49] showed that the inclusion in the simulation of multi-parton interactions significantly improve models. The DPS contribution to that final state by adopting kinematical cuts of Ref. 17. The expression for the differential cross section initiated by a quasi-real photon is then generalized according to the results presented in Ref. [46]:

$$d\sigma^{4i}_{DPS} = \frac{1}{2} \sum_{ab,cd} \int dy \, dQ^2 \frac{f_{\gamma/e}(y, Q^2)}{\int_{Q^2}^{\infty} f_{\gamma/e}(Q^2)} \times$$

$$\times \int dx_p dx_{q_1} f_{ij}(x_{\gamma_1}) d\sigma^{2i}_{ab}(x_{\gamma_1}, x_{\gamma_2})$$

$$\times \int dx_p dx_{q_2} f_{ij}(x_{\gamma_2}) d\sigma^{2i}_{cd}(x_{\gamma_1}, x_{\gamma_2}) .$$

In the above equation the one half factor takes into account two identical dijet systems in the final state. The sum runs over partons active in the first scattering ($a, b$) or in the second one ($c, d$), where $d\sigma^{2i}$ represent the differential partonic cross sections. Since $\sigma_{eff}$ depends on $Q^2$, the photon flux, $f_{\gamma/e}$, in its $Q^2$-unintegrated version [50], has been used. The distributions $f_{ij/p}(x_{fp})$ and $f_{ij/q}(x_{\gamma})$ represent the proton and of the photon PDFs for which we use the leading order sets of Ref. [51] and [52], respectively. Dijet cross sections have been evaluated to leading order accuracy in the strong coupling with ALPGEN [53], properly adapted to cope with photoproduction processes, with final state partons identified as jets. Factorization and renormalization scales have been both set to average transverse momentum of the jets. Such a cross section receives two contributions that can be classified by the fractional momentum of partons in the photon, $x_\gamma$, reconstructed by jet kinematics: the one from the resolved photon process, in which the photon behaves like an hadron with its own parton distributions, and the direct one, in which the photon interacts as a point-like particle with partons in the proton target. The former populates the whole $x_\gamma$ range while the latter, at LO, is concentrated at $x_\gamma = 1$. The two mechanisms mix under higher-order corrections [54] and therefore kinematic cuts are used by experimental collaborations [55, 56] to select the resolved-enriched contribution ($x_\gamma < 0.75$) and a direct-enriched one ($x_\gamma > 0.75$). In the present analysis we are interested in the resolved component and therefore the cut $x_\gamma < 0.75$ is enforced on the evaluation of dijet cross sections. Out of this predictions, the DPS cross section is built via the pocket formula in Eq. (6) by enforcing, for consistency, the same cut on the parton pair fractional momenta, $x_{\gamma,1} + x_{\gamma,2} < 0.75$. The main background to the DPS signal is represented by the SPS four-jet photoproduction process. The latter is again calculated with ALPGEN interfaced with the photon flux factor and photon PDFs, enforcing $x_\gamma < 0.75$ and with same settings discussed for the dijet cross sections. The experimental four-jet photoproduction cross section, $(\sigma_{exp})$, that can be inferred from distributions presented in Ref. [17] for $x_\gamma < 0.75$ is 135 pb.

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline $Q^2$ & $\sigma_{DPS}$ [pb] & $\sigma_{SPS}$ [pb] \\
\hline
10$^2$ & 35.1 & 36.6 & 114.1 & 86 & 2.12 \\
10$^3$ & 29.1 & 15.2 & 44.3 & 33 & 1.91 \\
10$^4$ & 26.4 & 13.7 & 40.1 & 30 & 1.93 \\
\hline
\end{tabular}
\caption{Predictions for the LO DPS and SPS cross sections for four-jet photo-production in three ranges of $Q^2$. In the last column, the ratio between the calculated cross-sections to the total one is displayed. In the DPS case, each row corresponds to prediction obtained with a given $pp$ eff ($G_1, G_2, S$) and the photon wave function of Refs. [44] (three upper rows) and Ref. [45] (three bottom rows). In the last column the ratio Eq. (7) is shown.}
\end{table}

We report in Tab. 1 the results for the $\sigma_{DPS}$ and $\sigma_{SPS}$ obtained for three ranges of photon virtualities in HERA kinematics. Predictions are displayed on different row depending on the adopted proton eff and photon wave functions. As far as the comparison with the experimental cross section for $Q^2 < 1$ GeV$^2$ is concerned, the DPS cross section gives a sizeable contribution for all configurations whereas the LO SPS almost saturates the experimental cross section. With this respect, this preliminary investigation already indicates that some configurations, e.g. the LO QED description of the photon combined with the $G_1$ proton eff, are not favorable combinations since the corresponding DPS cross section alone exceeds the experimental one.

These results, however, should be interpreted with special care. Higher order corrections to the dijet photoproduction cross section induce an increase of theoretical predictions by a factor of 1.3 going from LO to NLO [57, 58] and by a factor 1.05 going from NLO to NNLO [59]. This in turn implies that by using LO estimates for the dijet cross section in the pocket formula, the latter gives a lower limit on the DPS cross section, as far as higher-order corrections are considered. On the other hand, a good theoretical control of higher-order...
corrections to the SPS background is mandatory for a proper extraction of the DPS signal. For example, the large spread in $\sigma_{\text{eff}}$ values reported in the experimental analysis of Ref. [34] reflects the level of uncertainty in the theoretical estimation of the $\sigma_{\text{DPS}}$ peak. With this respect, NLO results for four-jet production in $pp$ collisions at 8 TeV have been first calculated in Ref. [60, 61] showing that NLO predictions are nearly half of the LO estimates. If such a trend should be confirmed also in four-jet photo-production in HERA kinematics, our LO result would represent therefore an upper limit on the SPS background. The scenario concerning the uncertainties connected to higher order corrections appears as follows. Our LO estimates sets a lower limit on the DPS cross section, whereas we have presented arguments showing that the LO SPS could represent its upper bound. Both these findings converge into a conservative scenario for the DPS contribution to four jet cross section. For those reasons we do not provide here a full propagation of theoretical uncertainties to our final results. Nevertheless, it is worth to mention that the largest theoretical uncertainty, and by far dominant over all others, comes from a) models of the proton structure and the spread in the corresponding $\sigma_{\text{eff}}^{pp}$ values and b) the use of a LO QED treatment of the photon in addition to quark models for its hadronic component, without their consistent combination into a photon double PDFs, which in turns generates a wide spread on $\sigma_{\text{eff}}^{pp}$ predictions.

Despite all these warnings, all models predict large DPS fractions suggesting that jets photoproduction in $ep$ collisions could represent an interesting channel to search for the DPS contribution.

V. EXTRACTION OF THE $Q^2$-DEPENDENCE OF $\sigma_{\text{eff}}^{pp}$

Given the rather large uncertainties on the absolute DPS cross sections, it seems to us premature to consider more differential observable at this point. We make only one exception by discussing the $Q^2$-dependence of the DPS cross section. The latter is of primary importance to us since it is linked to the concept of a photon of variable transverse size and, in turn, to a $Q^2$-dependent $\sigma_{\text{eff}}^{pp}$. We investigate whether such a dependence, which adds on top of the one naturally induced by the photon flux, is eventually observable. We perform such an analysis within the HERA settings presented in the previous Section. We set the notation by sketching Eq. [44] as $d\sigma_{\text{DPS}}(\text{bin}) \sim \int_{\text{bin}} dQ^2 g(Q^2)/\sigma_{\text{eff}}^{pp}(Q^2)$, where bin stands for a given interval of integration over $Q^2$ and the function $g$ encodes the flux factor, the PDFs and elementary cross sections. We present in the first two columns of Tab. [1] the DPS cross section integrated over ranges of photon virtualities. Then we define the ratio $R$:

$$R = \frac{d\sigma_{\text{DPS}}(\text{bin1})}{d\sigma_{\text{DPS}}(\text{bin2})}. \quad (7)$$

In the case $\sigma_{\text{eff}}^{pp}$ were a constant, the latter quantity would be: $R \sim \int_{\text{bin}1} dQ^2 g(Q^2)/\int_{\text{bin}2} dQ^2 g(Q^2) \sim 2.1$. Therefore, if the ratio of the DPS cross sections, evaluated in the two bins, results to be different from that number, this fact would directly point to $Q^2$ effects on $\sigma_{\text{eff}}^{pp}$ or possible correlations breaking the pocket formula.

As one can see in the last column of Tab. [1], LO QED and the model encoding non perturbative QCD effects predict deviations of $R$ from the reference value of 2.1. We close this section remarking that this ratio is particular effective since it does not depend on the chosen proton and photon effs, it is sensitive to the dependence of $\sigma_{\text{eff}}^{pp}$ on $Q^2$ and finally, if applied to the SPS four-jet cross section, its value gives a theoretical benchmark without requiring the exact knowledge of the absolute four-jet cross section.

Furthermore, we also developed a procedure to establish the minimum integrated luminosity to experimentally access $Q^2$ effects in $\sigma_{\text{eff}}^{pp}(Q^2)$. We have converted the cross sections in Tab. [2] in expected number of events with a given integrated luminosity. Statistical errors and the corresponding bands are calculated assuming a Poissonian distribution. The results are presented in Fig. 2 where the blue curves indicate results for the $\sigma_{\text{eff}}^{pp}(Q^2)$ while red ones indicate the number of events obtained with a constant, $Q^2$ independent, $\sigma_{\text{eff}}^{pp}$ which reproduces the total cross section for $Q^2 < 1$ GeV$^2$ obtained with $\sigma_{\text{eff}}^{pp}(Q^2)$. Among the various model results listed in Tab. [2], in Fig. 2 it is shown, on purpose, the one with the lower cross section and with a smooth $Q^2$ dependence. We find that the minimal integrated luminosity which makes distinguishable the two models...
VI. THE GEOMETRY OF $\sigma_{eff}^{\gamma p}(Q^2)$

As discussed in Refs. [31,32], the rather limited knowledge of the proton eff, entering the definition of $\sigma_{eff}^{\gamma p}$, prevents a precise extraction of $\langle b_2^2 \rangle_p$. In fact, in $pp$ or $pA$ collisions, the integrand defining the relative $\sigma_{eff}$, see e.g. Eq. (3), is completely unknown. Therefore, without assuming some model constraints, one cannot directly relate $\langle b_2^2 \rangle_p$ to unique data. The possibility to have in $\gamma p$ interactions a projectile of variable $Q^2$, can provide a unique chance to extract $\langle b_2^2 \rangle_p$. Without specifying any peculiar photon w.f., one may define the Fourier Transform of the eff, $\tilde{F}_2(b_\perp)$, which is interpreted as the probability distribution of finding two partons at a given transverse distance $b_\perp$. Once this quantity has been evaluated within some model of the photon structure, it can be power expanded as:

$$\tilde{F}_2(b_\perp;Q^2) = \sum_n C_n(b_\perp;Q^2)(b_\perp - \bar{b}_\perp)^n,$$

and freedom is left in the choice in the expansion point $\bar{b}_\perp$. Eq. (3) can now be rewritten as:

$$\sigma_{eff}^{\gamma p}(Q^2)^{-1} = \int db_\perp \tilde{F}_2^\ast(b_\perp)\tilde{F}_2(b_\perp;Q^2) = \sum_n C_n(b_\perp;Q^2)\langle (b_\perp - \bar{b}_\perp)^n \rangle_p.$$

A realistic description of $C_n(b_\perp;Q^2)$, together with data on the $Q^2$ dependence of $\sigma_{eff}^{\gamma p}(Q^2)$, will allow to access the transverse distance of partons in the proton. In fact, for a given specific dependence of $C_n$ on $Q^2$, one can identify an operator, $O_{Q^2}^m$, such that

$$O_{Q^2}^m[\sigma_{eff}^{\gamma p}(Q^2)]^{-1} = O_{Q^2}^m C_m(b_\perp, Q^2)\langle (b_\perp - \bar{b}_\perp)^m \rangle_p,$$

and then one can select and extract $\langle (b_\perp - \bar{b}_\perp)^m \rangle_p$, i.e. the relevant information on the proton structure. For the moment being we do not specify any functional expression of $O_{Q^2}^m$. This quantity, related to the explicit expressions of $C_m(b_\perp, Q^2)$, could be, e.g., a proper differential operator on $Q^2$. We have successfully tested the procedure both analytically and numerically with different proton and photon eff models. The identification of the correct operator is however not unique and freedom is left in the choice of the expansion point. Such a flexible feature can be useful for possible experimental applications. The only practical limitation is represented by the accuracy with which the dependence of $\sigma_{eff}^{\gamma p}$ on $Q^2$ could be eventually measured. Therefore, this procedure should be properly optimised along the experimental extraction conditions. In closing this section, we stress again that the procedure can be used only by considering realistic description of the photon splitting mechanism by taking into account higher order QED effects. Examples of application of this procedure are discussed in the appendix B.

This relation is one of the main goals of the present analysis and constitutes motivation to suggest this type of measurements at facilities where the photon virtuality can be experimentally measured such as the future Electron Ion Collider [62].

VII. SUMMARY

In the present analysis we have derived effective cross sections for photon induced processes which are essential ingredients in the predictions of DPS cross sections in quasi-real photon proton interactions. The latter have been obtained with the help of electromagnetic and hadronic model of the photon formulated in terms of light cone wave functions. For the four-jet final state in HERA kinematics we found a sizeable DPS contribution. This conclusion persists after considering estimates of higher order corrections, both to the DPS and SPS processes, taken from the literature. In the case the photon virtuality $Q^2$ could be measured, we have investigated the dependence of $\sigma_{eff}^{\gamma p}$ on such a parameter, which is directly related to the size of the dipole originated by the photon fluctuation. We set lower limits on the integrated luminosity needed to observe such an effect and we present, for such a case, a novel procedure which would allow to extract new information on the proton structure.

This work was supported: i) in part by the STRONG-2020 project of the European Union Horizon 2020 research and innovation programme under grant agreement No 824093; ii) by the European Research Council under the European Union’s Horizon 2020 research and innovation program (Grant agreement No. 804840); iii) by the project “Photon initiated double parton scattering: illuminating the proton parton structure” on the FRB of the University of Perugia.
Appendix A: Cut-off dependence of the results

In this section we discuss the dependence of the outcomes of the present analysis on the values of $k_{\text{cut}}$. Let us remind that the modulus squared of the photon QED wave function is logarithmically divergent at large parton transverse momentum $k_{\perp}$ in Eq. (1). Therefore, a regulating procedure is required. In turn, it is important to discuss the impact of such a strategy on the results of the calculations. In fact, several possibilities are available, e.g., we can integrate up to a physical cut-off, possibly of the order of the hard scale entering the scattering process or related to other prescriptions. In the present work we pursued instead the idea of considering a large $k_{\text{cut}}$ for the reason to be detailed hereafter. When $F_2^\gamma$ is evaluated with increasingly higher cut-offs, as shown in Fig. 3, its tail at large $k_{\perp}$ shows, as expected, a sensitivity to $k_{\text{cut}}$ and it approaches a constant value of 1 for asymptotically large values of $k_{\text{cut}},$ i.e. the form factor of a structureless photon.

![Figure 3: The photon eff as a function of $k_{\perp}$ and evaluated at $Q^2 = 0.25$ GeV$^2$ for different cut-off.](image)

The corresponding variations on the $\gamma p$ effective cross-section are instead much reduced and presented in Table II for three proton models as a function of $k_{\text{cut}}$. Its lowest value is in the range of jet transverse momenta discussed in the phenomenological session, while the maximum is a numerical proxy for infinity.

| $k_{\text{cut}}$ [GeV] | 5 | 10 | 50 | 100 | 1000 | $\Delta_{10-100}$ [mb] | $\Delta_{10-100}$ [mb] | $\Delta_{50-100}$ [mb] |
|------------------------|---|----|----|-----|------|-----------------|-----------------|-----------------|
| G1                     | 11.13 | 10.32 | 9.16 | 8.95 | 8.49 | 2.64 | 1.87 | 0.67 |
| G2                     | 16.93 | 16.46 | 14.63 | 14.44 | 13.85 | 3.08 | 2.61 | 0.78 |
| S                      | 18.57 | 17.87 | 16.81 | 15.8 | 14.89 | 3.65 | 2.98 | 0.92 |

Table II: The photon-proton effective cross-section, in [mb], for $Q^2 = 0.26$ GeV$^2$ and different $k_{\text{cut}}$ and ansatz for the proton eff. Here we define $\Delta_{k_1,k_2} = \sigma_{\gamma p}^{\text{eff}}(Q^2,k_1) - \sigma_{\gamma p}^{\text{eff}}(Q^2,k_2)$.

In all considered cases we stress that the fast falling behaviour of the proton eff at high $k_{\perp}$ effectively regulates the tail of the photon eff in the convolution integral in Eq. (3), granting its convergence. As expected, a residual dependence of $\sigma_{\gamma p}^{\text{eff}}$ on $k_{\text{cut}}$ is visible and quantified as a difference in the last columns of the Table. However, we notice that for increasingly higher $k_{\text{cut}}$ the values of $\sigma_{\gamma p}^{\text{eff}}$ stabilize: it varies no more than $\sim 1$ mb when $k_{\text{cut}}$ is raised from $k_{\text{cut}} = 50$ GeV to $k_{\text{cut}} = 10^3$ GeV. The latter value is therefore used as a default in the following numerical evaluations.

Given these observations, the advantage of using a large cutoff is twofold: 1) one avoids to give a particular prescription for the setting of a physical cut-off 2) one obtains the lower limit on the $\gamma p$ effective-cross section, which implies that our estimate of the DPS cross section given by QED contribution should be considered as its upper limit. We also notice that, even considering physical cut-offs, we obtain reduced but still sizeable DPS contributions, leaving unchanged the main message of the analysis.
Appendix B: Examples of the application of the procedure Eq. (10)

In the first part of this additional document, we provide examples of the application of the procedure developed in Sect. VI of the manuscript. For the sake of clarity, “eq.” refers to an equation present in this document while “Eq.” refers to that shown in the main manuscript.

We start from Eq. (9) and without losing generality, we consider the expansion point \( \bar{b}_\perp = 0 \) and set \( b_\perp \equiv b \).

\[
\left[ \sigma_{\gamma p}^{\gamma p}(Q^2) \right]^{-1} = \sum_n C_n(Q^2) \int d^2b \, b^n \mathcal{F}_2^\gamma(b) = \sum_n C_n(Q^2) \langle b^n \rangle_p; \tag{B1}
\]

where here \( \langle b^n \rangle_p \) is the \( n \)th-moment of transverse distance between two partons inside the proton and it does not depend on \( Q^2 \).

If a realistic calculation of \( C_n(Q^2) \) is available, an operator, depending on \( Q^2 \), could be identified such that, for example:

\[
\hat{O}_{Q^2} \left[ \sigma_{\gamma p}^{\gamma p}(Q^2) \right]^{-1} \propto C_n(Q^2) \langle b^n \rangle_p. \tag{B2}
\]

In other words, it could be possible to select the \( n \) power contribution of the expansion and extract the relevant information on the mean partonic transverse distance. In the following we provide two basic examples of the application of this procedure.

1. Gaussian-Gaussian scenario

Let us assume that the photon effective from factor has a Gaussian form of the type:

\[
\tilde{F}_2^\gamma(b;Q^2) = \frac{Q^2}{\alpha^2 \pi^2} e^{-b^2 Q^2 / \alpha^2}, \tag{B3}
\]

which is properly normalized:

\[
\int d^2b \, \tilde{F}_2^\gamma(b;Q^2) = 1, \tag{B4}
\]

and whose width is controlled by the adjustable parameter \( \alpha \). For such a distribution the main transverse distance between the partons produced by the splitting mechanism is:

\[
\langle b^2 \rangle_\gamma = \int d^2b \, b^2 \tilde{F}_2^\gamma(b;Q^2) = \frac{\alpha^2}{Q^2}. \tag{B5}
\]

Notably, for a highly virtual photon, i.e. at large \( Q^2 \), the mean distance between the two produced partons goes to zero, as expected.

Within this choice, we can expand \( \tilde{F}_2^\gamma(b;Q^2) \) as

\[
\tilde{F}_2^\gamma(b;Q^2) \sim \frac{Q^2}{\pi \alpha^4} - \frac{Q^4}{\pi \alpha^4} b^2 + O(b^4), \tag{B6}
\]

where \( C_0(Q^2) = \frac{Q^2}{\pi \alpha^2} \) and \( C_2(Q^2) = -\frac{Q^4}{\pi \alpha^4} \). In this scenario, one can chose the relevant operator which isolates \( \langle b^2 \rangle_p \):

\[
\hat{O} = d/(Q^2 dQ)|_{Q^2=0} \langle b^2 \rangle_p. \tag{B7}
\]

Another possibility could be \( \hat{O} = d^4/(d^4Q)|_{Q^2=0} \). In order to show how the procedure works, we further assume that the proton eff has the form

\[
\tilde{F}_2^p(b) = e^{-b^2 \beta^2 / \pi}, \tag{B8}
\]
which obeys the normalization condition:
\[ \int d^2b \, \tilde{F}_2(b) = 1, \] (B9)

and for which the partonic mean transverse distance reads
\[ \int d^2b \, b^2 \tilde{F}_2(b) = \frac{1}{\beta^2}. \] (B10)

From these models of the photon and the proton effs, we may calculate the $\gamma p$ effective cross section as:
\[ [\sigma_{\gamma p}^{\text{eff}}(Q^2)]^{-1} = \frac{\beta^2 Q^2}{\pi (\alpha^2 \beta^2 + Q^2)}, \] (B11)

and one gets:
\[ \frac{d}{Q^2 dQ} \left( [\sigma_{\gamma p}^{\text{eff}}(Q^2)]^{-1} - C_0(Q^2) \right) \bigg|_{Q^2=0} = -\frac{4}{\pi \alpha^4 \beta^2}, \] (B12)
\[ \frac{d}{Q^2 dQ} \left( C_2(Q^2) \right) \bigg|_{Q^2=0} = -\frac{4}{\pi \alpha^4}, \] (B13)

which can be combined to give
\[ \langle b^2 \rangle_p = \frac{d^4}{dQ^4} \left( [\sigma_{\gamma p}^{\text{eff}}(Q^2)]^{-1} - C_0(Q^2) \right) \bigg|_{Q^2=0} = \frac{1}{\beta^2}, \] (B14)

which reproduces the analytic result obtained in eq.(10).

2. Gaussian-Dipole scenario

Here we consider the photon distribution of Eq. (3) for $\alpha = 1$ (this choice is driven by the fact that in this case the procedure requires a numerical integration) while for the proton one we consider the Fourier Transform of the eff described by the dipole profile function of Ref. [15]:

\[ F^p_2(k_{\perp}) = \left( 1 + \frac{k_{\perp}^2}{1.1 \text{ GeV}^2} \right)^{-2}. \] (B15)

We change here the operator to $O^2_2 = d^4/(dQ^4)|_{Q^2=0}$ in order to remark that its choice can be optimized depending on the mathematical difficulties in the procedure. Indeed the final value of the extracted $\langle b^n \rangle_p$ should not depend on its particular choice. distance should not.

With these settings, the right hand side of the equivalent of eq. (7) becomes:
\[ \frac{d^4}{dQ^4} \left( C_2(Q^2) \right) \bigg|_{Q^2=0} = -\frac{24}{\pi}. \] (B16)

At this point we should estimate the corresponding $\sigma_{\gamma p}^{\text{eff}}(Q^2)$. In this case, an analytic expression for this quantity can not be obtained. Therefore we numerically evaluate $\sigma_{\gamma p}^{\text{eff}}(Q^2)$ and then find a good fitting function of it. In particular, since the chosen operator requires to take the limit $Q^2 \to 0$, it is mandatory to provide a realistic fitting function of $[\sigma_{\gamma p}^{\text{eff}}(Q^2)]^{-1}$ in this region. For example, as shown in Fig. 4 a good functional form is:
\[ [\sigma_{\gamma p}^{\text{eff}}(Q^2)]^{-1} \sim 14.7185 Q^6 - 8.18619 Q^5 - 2.315 Q^4 + 1.42751 Q^2 + 0.0533711 Q. \] (B17)

where it is understood that the numerical coefficients have the correct dimension to reproduce the dimension on the left hand side. Within this choice:
Figure 4: The calculation of $\sigma^p_{\gamma f}(Q^2)^{-1}$ within the eff eq. (15) and photon Gaussian distribution eq.(3) for $\alpha = 1$. Dots represent the numerical calculation and the full line is the fitting function eq. (17).

$$\frac{d^4}{dQ^4}\left[\sigma^p_{\gamma f}(Q^2)^{-1}\right]_{Q^2=0} = -55.56 \text{ GeV}^{-2}.$$  \hfill (B18)

By comparing the eq. (16) to eq. (18) we get: $\langle b_{\perp}^2 \rangle_p \sim 7.2728 \text{ GeV}^{-2}$. The value extracted from the proton eff eq. (15) following the procedure presented in Refs. [31-32] is given by

$$\langle b_{\perp}^2 \rangle_p = -\left. \frac{d}{dk_{\perp}dk_{\perp}} F^p(k_{\perp}) \right|_{k_{\perp}=0} = 7.2727 \text{ GeV}^{-2}.$$  \hfill (B19)

Those two values are rather close and this fact let us positively conclude on the effectiveness of the proposed procedure.

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