CROSSED PRODUCT C*-ALGEBRAS OF MINIMAL DYNAMICAL SYSTEMS
ON THE PRODUCT OF THE CANTOR SET AND THE TORUS

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ABSTRACT. This paper studies the relationship between minimal dynamical systems on the product of the Cantor set \((X)\) and torus \((\mathbb{T}^2)\) and their corresponding crossed product \(C^*\)-algebras.

For the case when the cocycles are rotations, we studied the structure of the crossed product \(C^*\)-algebra \(A\) by looking at a large subalgebra \(A_x\). It is proved that, as long as the cocycles are rotations, the tracial rank of the crossed product \(C^*\)-algebra is always no more than one, which then indicates that it falls into the category of classifiable \(C^*\)-algebras.

If a certain rigidity condition is satisfied, it is shown that the crossed product \(C^*\)-algebra has tracial rank zero. Under this assumption, it is proved that for two such dynamical systems, if \(A\) and \(B\) are the corresponding crossed product \(C^*\)-algebras, and we have an isomorphism between \(K_i(A)\) and \(K_i(B)\) which maps \(K_i(\mathbb{C}(X \times \mathbb{T}^2))\) to \(K_i(\mathbb{C}(X \times \mathbb{T}^2))\), then these two dynamical systems are approximately \(K\)-conjugate. The proof also indicates that \(C^*\)-strongly flip conjugacy implies approximate \(K\)-conjugacy in this case.

0. Introduction and Notation

In this section, an introduction of the background is given, and the notations used in this paper are also introduced.

Let \(X\) be a compact infinite metric space, and let \(\alpha \in \text{Homeo}(X)\) be a minimal homeomorphism of \(X\). We can construct the crossed product \(C^*\)-algebra from the minimal dynamical system \((X, \alpha)\), denoted by \(C^*(\mathbb{Z}, X, \alpha)\).

One interesting question is how properties of the dynamical system \((X, \alpha)\) determine properties of the crossed product \(C^*\)-algebra, and how properties of the crossed product \(C^*\)-algebras shed some light on properties of the dynamical system \((X, \alpha)\).

For minimal Cantor dynamical systems, Giodano, Putnam and Skau found (in [GPS]) that for two minimal Cantor dynamical systems, the corresponding crossed product \(C^*\)-algebras are isomorphic if and only if the minimal Cantor dynamical systems are strongly orbit equivalent.

Lin and Matui studied this problem when the base space is the product of the Cantor set and the circle (see [LM1], [LM2]), and they discovered that in the rigid cases (see Definition 3.1 of [LM1]), for two crossed product \(C^*\)-algebras to be isomorphic, the dynamical systems must be approximately \(K\)-conjugate (a “strengthened” version of weak approximate conjugacy, in the sense that it is compatible with the \(K\)-data).

We studied minimal dynamical systems on the product of the Cantor set and the torus. For the case that the cocycles take values in the rotation group, similar results are found for the relationship between \(C^*\)-algebra isomorphisms and approximate \(K\)-conjugacy between two dynamical systems. It is also shown that the tracial rank of the crossed product \(C^*\)-algebra is no more than one.
For the case that the cocycles are Furstenberg transformations, a necessary condition for weak approximate conjugacy between two minimal dynamical systems (via conjugacy maps whose cocycles are Furstenberg transformations) is given.

In section 1, structure of the subalgebra $A_x$ is studied. In section 2, we studied the crossed product $C^*$-algebra and concluded that its tracial rank is always no more than one. In section 3, we give an if and only if condition for when two such rigid (as defined in Definition 2.20) minimal dynamical systems are approximately $K$-conjugate.

Some notations used in this paper are listed below.

Let $(X, \alpha)$ be a minimal dynamical system, by $\alpha$-invariant probability measure $\mu$, we mean such a probability measure $\mu$ on $X$ satisfying $\mu(D) = \mu(\alpha(D))$ for every $\mu$-measurable subset $D$. Following the Markov-Kakutani fixed point theorem, it is shown that the set of $\alpha$-invariant probability measures is not empty (see Lemma 1.9.18 and Theorem 1.9.19 of [Mun] for details).

Let $\mu$ be a measure on $X$. For $f \in C(X)$, we use $\mu(f)$ to denote $\int_X f(x) \, d\mu$.

For a minimal dynamical system $(X, \alpha)$ we use $C^*(\mathbb{Z}, X, \alpha)$ to denote $C(X) \rtimes_\alpha \mathbb{Z}$, the crossed product $C^*$-algebra of the dynamical system $(X, \alpha)$.

In a topological space $X$, we say a subset $D$ is clopen, if $D$ is both closed and open.

For a compact Hausdorff space $Y$, Homeo($Y$) is used to denote the set of all the homeomorphisms of $Y$.

As the Cantor set $X$ is totally disconnected, we can write a homeomorphism of $X \times \mathbb{T}^2$ as $\alpha \times \varphi$ (the skew product form), with $\alpha \in \text{Homeo}(X)$ and $\varphi : X \to \text{Homeo}(\mathbb{T}^2)$ being continuous, and

$$\alpha \times \varphi : X \times \mathbb{T}^2 \to X \times \mathbb{T}^2 \text{ defined by } (x, t_1, t_2) \mapsto (\alpha(x), \varphi(x)(t_1, t_2)).$$

For the case that the cocycles take values in rotation groups, we can further express $\alpha \times \varphi$ as $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathbb{R}_\xi \times \mathbb{R}_\eta)$, with $\xi, \eta : X \to \mathbb{T}$ continuous, and

$$\alpha \times \mathbb{R}_\xi \times \mathbb{R}_\eta : X \times \mathbb{T}^2 \to X \times \mathbb{T}^2 \text{ defined by } (x, t_1, t_2) \mapsto (\varphi(x), t_1 + \xi(x), t_2 + \eta(x)).$$

We use $A$ to denote the corresponding crossed product $C^*$-algebra. For $x \in X$, the subalgebra $A_x$ is defined as below.

**Definition 0.1.** For a minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathbb{R}_\xi \times \mathbb{R}_\eta)$, $A_x$ is defined to be the subalgebra of the crossed product $C^*$-algebra generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and $u \cdot C_0((X \setminus \{x\}) \times \mathbb{T} \times \mathbb{T})$, with $u$ being the implementing unitary in $A$ satisfying $u^* fu = f \circ (\alpha \times \mathbb{R}_\xi \times \mathbb{R}_\eta)^{-1}$.

**Remark:** The idea to define such a sub-algebra in the crossed product can be traced to Putnam’s work (see [Putnam]). From the definition, if $D$ is a clopen subset of the Cantor set $X$, and $1_{D \times \mathbb{T}^2}$ is the characteristic function of $D \times \mathbb{T}^2$, then $u 1_{D \times \mathbb{T}^2} u^* = 1_{D \times \mathbb{T}^2} \circ (\alpha \times \mathbb{R}_\xi \times \mathbb{R}_\eta) = 1_{\alpha^{-1}(D) \times \mathbb{T}^2}$.

Let $\{P_n : n \in \mathbb{N}\}$ be as in the Bratteli-Vershik model of the minimal Cantor dynamical system $(X, \alpha)$ (see [HPS] Theorem 4.2), and let $Y_n$ be the roof of $P_n$ (denoted as $R(P_n)$). Then $\{Y_n\}$ will be a decreasing sequence of clopen sets such that $\bigcap_{n=1}^\infty Y_n = \{x\}$. Use $A_n$ to denote the subalgebra generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and $u \cdot C_0((X \setminus \{y\}) \times \mathbb{T} \times \mathbb{T})$. 


In a C*-algebra $A$, for $a, b \in A$, $a \approx b$ just means $\|a - b\| \leq \varepsilon$. By $a \approx_{\varepsilon_1} b \approx_{\varepsilon_2} c$, we mean $\|a - b\| \leq \varepsilon_1$ and $\|b - c\| \leq \varepsilon_2$. It is clear that $a \approx_{\varepsilon_1} b \approx_{\varepsilon_2} c$ implies $a \approx_{\varepsilon_1 + \varepsilon_2} c$.

In a C*-algebra $A$, $[a, b]$ (the commutator) is defined to be $ab - ba$.

For a C*-algebra $A$ we use $T(A)$ to denote the convex set of all the tracial states on $A$, and $\text{Aff}(T(A))$ to denote all the affine linear functions from $T(A)$ to $\mathbb{R}$.

In a C*-algebra $A$, for $a \in A_+$, we use $\text{Her}(a)$ to denote the smallest hereditary subalgebra that contains $a$.

For a C*-algebra $A$, we use $\text{TR}(A)$ to denote the tracial rank of $A$ (see [Lin4, Definition 3.6.2]). We use $\text{RR}(A)$ to denote the real rank of $A$ ([Lin4, Definition 3.1.6]) and $\text{tsr}(A)$ to denote the stable rank of $A$ ([Lin4, Definition 3.1.1]).

**Definition 0.2.** Let $A$ be a C*-algebra. Let $p$ be a projection of $A$ and let $a \in A_+$. We say that $p \preceq a$ if $p$ is Murray-von Neumann equivalent to a projection $q \in \text{Her}(a)$.

Let $A$ be a C*-algebra. We use $U(A)$ to denote the group of all the unitary elements in $A$. We use $CU(A)$ to denote the norm closure of the group generated by the commutators of $U(A)$. In other words, $CU(A)$ is the norm closure of the group generated by elements in $\{uv^*u^*: u, v \in U(A)\}$. One can check that $CU(A)$ is a normal subgroup of $U(A)$ and $U(A)/CU(A)$ is an abelian group.

**Definition 0.3.** Let $\varphi : A \to B$ be a C*-algebra homomorphism. We define $\varphi^* : U(A)/CU(A) \to U(B)/CU(B)$ to be the map induced by $\varphi$ which maps $[u] \in U(A)/CU(A)$ to $[\varphi(u)] \in U(B)/CU(B)$.

1. **The subalgebra $A_x$**

In this section, we study properties of a “large” subalgebra of $A$, namely $A_x$. The idea of the construction of $A_x$ was first given by Putnam, but the construction here is a bit different from that in the sense that we are removing one fiber $\{x\} \times T \times T$ instead of one point. In other words, we define $A_x$ to be the subalgebra generated by $C(X \times T \times T)$ and $u \cdot C_0((X \setminus \{x\}) \times T \times T)$, with $u$ being the implementing unitary in $A$ (as defined in Section 3).

The following lemma gives the basic structure of $A_x$, which is used to study the structure of $A$.

**Lemma 1.1.** If $(X \times T \times T, \alpha \times R_\xi \times R_\eta)$ is minimal, then for any $x \in X$ there are $k_1, k_2, \ldots \in \mathbb{N}$ and $d_{s,n} \in \mathbb{N}$ for $n \in \mathbb{N}$ such that $A_x \cong \varinjlim_n \bigoplus_{s=1}^{k_n} M_{d_{s,n}}(C(T^2))$.

**Proof.** As $\alpha \times R_\xi \times R_\eta$ is minimal, it follows that $(X, \alpha)$ is also minimal. For $x \in X$, let $\mathcal{P} = \{X(n, v, k): v \in V_n, k = 1, 2, \ldots, h_n(v)\}$ be as in the Bratteli-Vershik model ([HPS, Theorem 4.2]) for $(X, \alpha)$. Let $R(\mathcal{P}_n)$ be the roof set of $\mathcal{P}_n$, defined by $R(\mathcal{P}_n) = \bigcup_{v \in V_n} X(n, v, h_n(v))$. We can assume that the roof sets satisfy

$$\bigcap_{n \in \mathbb{N}} R(\mathcal{P}_n) = \{x\}.$$
Let $A_n$ be the subalgebra of the crossed product $C^*$-algebra $A$ such that $A_n$ is generated by $C(X \times T \times T)$ and $u \cdot C_0((X \setminus R(\mathcal{P}_n)) \times T \times T)$, with $u$ being the implementing unitary element satisfying $ufu^* = f \circ (\alpha \times R_x \times R_y)$ for all $f \in C(X \times T \times T)$. Then it is clear that $A_1 \subset A_2 \subset \cdots$. As we can approximate $f \in C_0((X \setminus \{x\}) \times T \times T)$ with 

$$f_n \in C_0((X \setminus R(\mathcal{P}_n)) \times T \times T) = C((X \setminus R(\mathcal{P}_n)) \times T \times T),$$

we have $\lim_{n \to \infty} f_n = f$ in $C(X \times T \times T)$. 

For $C(X \setminus R(\mathcal{P}_n)_x)$, it is clear that we have

$$C((X \setminus R(\mathcal{P}_n)) \times T \times T) \cong \bigoplus_{v \in V_n 1 \leq k \leq h_n(r) - 1} C\left(\begin{array}{c} X(n, v, k) \times T^2 \end{array}\right).$$

We will show that $A_n \cong \bigoplus_{v \in V_n} M_{h_n(v)}(C(X(n, v, 1)) \otimes C(T^2))$.

Let $e_{i,j}^v = 1_{X(n,v,i)} \cdot u^{i-j}$. Then $e_{i,j}^v \cdot e_{i,j}^{v'} = 0$ if $v \neq v'$. Note that

$$e_{i,j}^v \cdot e_{i,j}^v = 1_{X(n,v,i)} \cdot u^{i-j} \cdot 1_{X(n,v,k)} \cdot u^{k-s} = 1_{X(n,v,i)} \cdot 1_{X(n,v,k+i-j)} \cdot u^{i-j+k-s} = \delta_{k,j} \cdot e_{i,j}^v.$$

In other words, $\{e_{i,j}^v\}_{i,j=1}^{h(v)}$ is a system of matrix units.

As $A_n$ is generated by

$$\{e_{i,j}^v \otimes C\left(\begin{array}{c} X(n, v, 0) \otimes C(T^2) \end{array}\right) : v \in V_n, 1 \leq i, j \leq h(v)\},$$

it follows that

$$A_n \cong \bigoplus_{v \in V_n} M_{h_n(v)}(C(X(n, v, 1)) \otimes C(T^2)).$$

Let $B_n = \bigoplus_{v \in V_n} M_{h_n(v)}(C \otimes C(T^2))$. Then it is clear that $B_n$ can be regarded as a subalgebra of $A_n$.

As for the canonical embedding $\phi_{n,n+1} : A_n \to A_{n+1}$, consider

$$a \in A_n \cong \bigoplus_{v \in V_n} M_{h_n(v)}(C(X(n, v, 1)) \otimes C(T^2))$$

such that $a = (f \otimes g) \cdot u^{i-j} \in e_{i,j}^v \otimes C(X(n, v, 1) \otimes C(T^2))$, with $f \in C(X(n, v, 1)) \cong C(X(n, v, 1))$ and $g \in C(T^2)$.

Note that the Kakutani-Rokhlin partition of $A_{n+1}$ is finer than that of $A_n$. We can write

$$f = \sum_{X(n+1,v,s,k) \subset X(n,k)} f_{s,k} \quad \text{with} \quad f_{s,k} \in C(X(n+1,v,k)).$$

It follows that

$$\phi_{n,n+1}(f \otimes g) = \sum_{X(n+1,v,s,k) \subset X(n,v,k)} f_{s,k} \otimes g.$$

Then we have

$$\phi_{n,n+1}(a) = \left( \sum_{X(n+1,v,s,k) \subset X(n,v,k)} f_{s,k} \otimes g \right) \cdot u^{i-j}$$

$$= \sum_{X(n+1,v,s,k) \subset X(n,v,k)} (f_{s,k} \otimes g) \cdot u^{i-j},$$
with \( \sum_{X(n+1,v,k) \subseteq X(n,v,k)} (f_{s,k} \otimes g) \cdot u^{-j} \) being an element in \( A_{n+1} \). It is then clear that \( \phi_{n,n+1}(B_n) \subseteq B_{n+1} \) if we regard \( B_n \) as a subalgebra of \( A_n \) and \( B_{n+1} \) as a subalgebra of \( A_{n+1} \).

Just abuse notation and use \( \phi_{n,n+1} \) to denote the canonical embedding from \( B_n \) to \( B_{n+1} \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\cdots & \longrightarrow & B_n \\
\downarrow j_n & \ & \downarrow j_{n+1}
\end{array}
\begin{array}{ccc}
\phi_{n,n+1} & \longrightarrow & \phi_{n+1,n+2} \\
B_n & \longrightarrow & B_{n+2}
\end{array}
\begin{array}{ccc}
\cdots & \longrightarrow & A_n \\
\downarrow j_n & \ & \downarrow j_{n+1}
\end{array}
\begin{array}{ccc}
\phi_{n,n+1} & \longrightarrow & \phi_{n+1,n+2} \\
A_n & \longrightarrow & A_{n+2}
\end{array}
\cdots
\]

For every \( a \in A_x = \lim_{n \to \infty} (A_n, \phi_{n,n+1}) \) and every \( \varepsilon > 0 \), there exists \( a_n \in A_n \) such that \( \| a - a_n \| < \frac{\varepsilon}{2} \) if we identify \( a_n \) with \( \phi_{n,\infty}(a_n) \in A_x \). Without loss of generality, we can assume that

\[
a_n = \sum_{k=1}^{L} \sum_{v \in V_n} \sum_{i,j=1}^{h_n(v)} (f_{k,v,i,j} \otimes g_{k,v,i,j}) \cdot e_{i,j}^{v},
\]

with \( f_{k,v,i,j} \in C(X(n,v,0)) \) and \( g_{k,v,i,j} \in C(\mathbb{T}^2) \).

Let \( M = \max_{k,v,i,j} \| g_{k,v,i,j} \| \). For all \( k, v, i, j \) as above, we can find \( \delta > 0 \) such that for \( x, y \in X \), if \( \text{dist}(x, y) < \delta \), then

\[
\| f_{k,v,i,j}(x) - f_{k,v,i,j}(y) \| < \frac{\varepsilon}{2 \cdot M \cdot L \cdot |V_n| \cdot h_n(v)^2}.
\]

According to the Bratteli-Vershik model, \( \bigcap_{n \in \mathbb{N}} R(P_n) = \{ x \} \). We may further require that for all \( n \in \mathbb{N} \), every block \( X(n,v,k) \) in \( P_n \) satisfies \( \text{diam}(X(n,v,k)) < 1/n \). Then we can choose \( N \in \mathbb{N} \) such that \( \text{diam}(R(P_N)) < \delta \). Without loss of generality, we can assume that \( N \geq n \).

In \( P_N \), for every \( X(N,v,k) \), choose \( w_{N,v,k} \in X(N,v,k) \). For \( k = 1, \ldots, L, v \in V_n, i, j = 1, \ldots, h_n(v) \), define

\[
\tilde{f}_{k,v,i,j} = \sum_{X(N,v',k') \subseteq X(n,v,k)} f_{k,v,i,j}(w_{N,v',k'}) \cdot 1_{X(N,v',k')}.
\]

According to our choice of \( N \), it is clear that \( \| f_{k,v,i,j} - \tilde{f}_{k,v,i,j} \| < \frac{\varepsilon}{2 \cdot M \cdot L \cdot |V_n| \cdot h_n(v)^2} \).

For the \( a_n \) given above, define \( \tilde{a}_n \in A_n \) by

\[
\tilde{a}_n = \sum_{k=1}^{L} \sum_{v \in V_n} \sum_{i,j=1}^{h_n(v)} (\tilde{f}_{k,v,i,j} \otimes g_{k,v,i,j}) \cdot e_{i,j}^{v}.
\]

As

\[
\| f_{k,v,i,j} - \tilde{f}_{k,v,i,j} \| < \frac{\varepsilon}{2 \cdot M \cdot L \cdot |V_n| \cdot h_n(v)^2},
\]

it follows that \( \| a_n - \tilde{a}_n \| < \varepsilon/2 \).
As $\widehat{f_{k,v,i,j}}$ is constant on $X(N,v',k')$, it follows that $\phi_{n,N}(\widehat{a_n}) \in B_N$. It is clear that
\[
\|\phi_{n,N}(\widehat{a_n}) - a\| \leq \|\phi_{n,N}(\widehat{a_n}) - a_n\| + \|a - a_n\|
\]
\[
= \|\widehat{a_n} - a_n\| + \|a - a_n\|
\]
\[
\leq \varepsilon/2 + \varepsilon/2
\]
\[
= \varepsilon.
\]
Note that $a \in A_x$ and $\varepsilon > 0$ are arbitrary. It follows that $\bigcup_{n \in \mathbb{N}} \phi_{n,\infty}(B_n)$ is dense in $A_x$. In other words, we have $\lim_{n \to \infty} (B_n, \phi_{n,n+1}) \cong A_x$. As $B_n = \bigoplus_{v \in V_n} M_{h_n(v)}(\mathbb{C} \otimes C(T^2))$, we conclude that
\[
A_x \cong \varinjlim_{n \to \infty} \bigoplus_{s=1}^{k_n} M_{d_{s,n}}(C(T^2)).
\]

**Lemma 1.2.** Let $A_x$ be defined as above. If $\alpha \times R_\xi \times R_\eta$ is minimal, then $A_x$ is simple.

**Proof.** This proof is essentially the same as that of Proposition 3.3 (5) in [LM1]. It works like this:

Note that $X \times T \times T$ is compact and $\alpha \times R_\xi \times R_\eta$ is minimal. It is clear that the positive orbit (under $\alpha \times R_\xi \times R_\eta$) of $(x, t_1, t_2)$ is dense in $X \times T \times T$.

The C*-algebra $A$ corresponds to the groupoid C*-algebra associated with the equivalence relation
\[ R = \{((x, t_1, t_2), (\alpha \times R_\xi \times R_\eta)^k(x, t_1, t_2)) : (x, t_1, t_2) \in X \times T \times T\}, \]
and the C*-subalgebra $A_x$ corresponds to the groupoid C*-algebra associated with the equivalence relation
\[ R_x = R \setminus \{((\alpha \times R_\xi \times R_\eta)^k(x, t_1, t_2), (\alpha \times R_\xi \times R_\eta)^l(x, t_1, t_2)) : (t_1, t_2) \in T \times T, k \geq 0, l \leq 0 \text{ or } k \leq 0, l \geq 0\}. \]

As the positive orbit of any $(x, t_1, t_2)$ is dense in $X \times T \times T$, it follows that each equivalence class of $R_x$ is dense in $X \times T \times T$. According to [Renault], Proposition 4.6], this is equivalent to the simplicity of $A_x$.

Now we study the $K$-theory of $A_x$ using its direct limit structure.

**Lemma 1.3.** The group $K_0(C(T^2))$ is order isomorphic to $\mathbb{Z}^2$ with the unit element identified with $(1,0)$ and the positive cone $D$ being $\{(m,n) : m > 0\} \cup \{(0,0)\}$, and the group $K_1(C(T^2))$ is isomorphic to $\mathbb{Z}^2$.

**Proof.** It follows from the Künneth Theorem that $K_0(C(T^2)) \cong \mathbb{Z}^2$ and $K_1(C(T^2)) \cong \mathbb{Z}^2$.

From algebraic topology, we know that the complex vector bundles on $T^2$ are generated by the 1-dimensional trivial bundle, and the rank is determined by the rank of the trivial bundle, this will give the positive cone of $K_0(C(T^2))$ as $\{(m, n) : (m, n) \in \mathbb{Z}^2, m > 0\} \cup \{(0,0)\}$.

**Lemma 1.4.** There is an isomorphism $\iota : K_0(C(X \times T^2)) \to C(X, \mathbb{Z}^2)$ which sends $[1]$ to the constant function with value $(1,0)$. Furthermore, $\iota$ maps $K_0(C(X \times T^2)_+)$ onto $C(X, D)$, with $D$ as defined in Lemma 1.3.

Moreover, for a clopen set $U$ of $X$ and a projection $\eta \in M_k(C(T^2))$ such that $[\eta] \in K_0(C(T^2))$ corresponds to $(a, b)$ as in Lemma 1.3 $\iota([\text{diag}(1_U, \ldots, 1_U) \cdot \eta]) = (1_U \cdot a, 1_U \cdot b)$. 


Proof. For $D$ as in Lemma 1.3 define
\[
\varphi: C(X, D) \to (K_0(C(X \times \mathbb{T}^2)))_+
\]
by
\[
\varphi(f) = \sum_{(m,n) \in D} \left[ \left( \frac{1}{d_{m,n}} \right) f_{m,n} \right] \cdot \eta_{m,n},
\]
where $\eta_{m,n}$ is a projection in $M_{d_{m,n}}(C(\mathbb{T}^2))$ which is identified with $(m, n)$ as in Lemma 1.3.

If we can show that $\varphi$ is one-to-one, preserves addition, and maps the constant function with value $(1,0)$ to $[1_{C(X \times \mathbb{T}^2)}]$, then we can extend $\varphi$ to a group isomorphism from $C(X, \mathbb{Z}^2)$ to $K_0(C(X \times \mathbb{T}^2))$.

It is easy to check that $\varphi((1,0)) = [1_{C(X \times \mathbb{T}^2)}]$. From the definition, it follows that $\varphi$ preserves addition. We just need to show that $\varphi$ is one-to-one.

Injectivity of $\varphi$:
If $\varphi(f) = 0$ for some $f \in C(X, D)$, then
\[
\sum_{(m,n) \in D} \left[ \left( \frac{1}{d_{m,n}} \right) f_{m,n} \right] \cdot \eta_{m,n} = 0
\]
in $(K_0(C(X \times \mathbb{T}^2)))_+$. As
\[
K_0(C(X \times \mathbb{T}^2)) \cong \bigoplus_{(m,n) \in D} K_0(C(f^{-1}((m, n)) \times \mathbb{T}^2)),
\]
we get that
\[
\left[ \left( \frac{1}{d_{m,n}} \right) f_{m,n} \right] \cdot \eta_{m,n} = 0
\]
in $K_0(C(f^{-1}((m, n)) \times \mathbb{T}^2))$ for all $(m, n) \in D$.

That is, there exists $k \in \mathbb{N}$ such that
\[
\left( \frac{1}{d_{m,n}} \right) f_{m,n} \oplus \text{diag}(1_{C(f^{-1}((m, n)) \times \mathbb{T}^2)}, \ldots, 1_{C(f^{-1}((m, n)) \times \mathbb{T}^2)})
\]
is Murray-von Neumann equivalent to $\text{diag}(1_{C(f^{-1}((m, n)) \times \mathbb{T}^2)}, \ldots, 1_{C(f^{-1}((m, n)) \times \mathbb{T}^2)})$.

Let $s \in M_{d_{m,n}+k}(f^{-1}((m, n)) \times \mathbb{T}^2)$ be the partial isometry corresponding to the Murray-von Neumann equivalence above. Choose $x \in f^{-1}((m, n))$. Then $s(x)$ can be regarded as an element in $M_{d_{m,n}+k}(\mathbb{T}^2)$ that gives a Murray-von Neumann equivalence between
\[
\eta_{m,n} \oplus \text{diag}(1_{C(\mathbb{T}^2)}, \ldots, 1_{C(\mathbb{T}^2)}) \text{ and diag}(1_{C(\mathbb{T}^2)}, \ldots, 1_{C(\mathbb{T}^2)}).
\]

It then follows that $\eta_{m,n} = 0$, which proves injectivity.

Surjectivity of $\varphi$:
For every projection $p \in M_\infty(C(X \times \mathbb{T}^2))$, we can find a partition $X = \bigsqcup_{i=1}^M X_i$ such that $\|p(x) - p(y)\| < 1$ for all $x, y \in X_i$. Choose $x_i \in X_i$ for $i = 1, \ldots, M$, and identify $M_\infty(C(X \times \mathbb{T}^2))$ with $C(X, M_\infty(C(\mathbb{T}^2)))$. Define $p' \in C(X, M_\infty(C(\mathbb{T}^2)))$ by $p'|_{X_i} = p(x_i)$. It is clear that we can regard $p'|_{X_i}$ as an element in $M_\infty(C(\mathbb{T}^2))$. 

Use \((a_i, b_i)\) to denote the corresponding element in \(K_0(C(T^2))\) as identified in Lemma 1.3 and let \(f = \sum_{i=1}^{M} 1_{X_i} \cdot (a_i, b_i)\). Then we can check that \(\varphi(f) = [p']\) in \((K_0(C(X \times T^2)))_+\). As \([p] = [p']\), we have proved surjectivity of \(\varphi\).

As \(\varphi\) is unital, one-to-one and preserves addition, we can extend it to an ordered group isomorphism \(\tilde{\varphi} : C(X, \mathbb{Z}^2) \to K_0(C(X \times T^2))\). Let \(\iota = \tilde{\varphi}^{-1}\), and we have finished the proof. \(\square\)

**Lemma 1.5.** There is an isomorphism
\[
\gamma_n : A_n \to \bigoplus_{v \in V_n} M_{h_n(v)} \left( C(X(n, v, 1)) \otimes C(T^2) \right),
\]
such that for every clopen set \(U\) in \(X\),
\[
\gamma_n(1_{U \times T^2}) = \bigoplus_{v \in V_n} \text{diag} \left( 1_{X(n, v, 1) \cap U}, \ldots, 1_{X(n, v, h(v)) \cap U} \right).
\]

**Proof.** The proof is essentially the same as that of [Putnam, Lemma 3.1]. It can also be obtained as a K-theory version of part of the proof of Lemma 1.1. \(\square\)

**Lemma 1.6.** There is a group isomorphism
\[
\phi : \bigoplus_{v \in V_n} C(X(n, v, 1), \mathbb{Z}^2) \to C(X, \mathbb{Z}^2) / \{ f - f \circ \alpha^{-1} : f |_{Y_n} = 0 \}
\]
such that
\[
\phi \left( (f_1, \ldots, f_{|V_n|}) \right) = \sum_{v \in V_n} [1_{X(n,v,1)} : f_v]
\]
for \((f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C(X(n, v, 1), \mathbb{Z}^2)\).

Furthermore, if we define \(D\) to be
\[
\{ (m,n) \in \mathbb{Z}^2 : m > 0 \} \cup \{ (0,0) \},
\]
and if we define the positive cone of \(\bigoplus_{v \in V_n} C(X(n, v, 1), \mathbb{Z}^2)\) to be \(\bigoplus_{v \in V_n} C(X(n, v, 1), D)\) and the positive cone of \(C(X, \mathbb{Z}^2) / \{ f - f \circ \alpha^{-1} : f |_{Y_n} = 0 \}\) to be \(C(X, D) / \{ f - f \circ \alpha^{-1} : f |_{Y_n} = 0 \}\), then both \(\phi\) and \(\phi^{-1}\) are order preserving.

**Proof.** For \((f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C(X(n, v, 1), \mathbb{Z}^2)\), define
\[
\phi \left( (f_1, \ldots, f_{|V_n|}) \right) = \sum_{v \in V_n} [1_{X(n,v,1)} : f_v].
\]

Injectivity of \(\phi\):
Suppose
\[
(f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C(X(n, v, 1), \mathbb{Z}^2)
\]
and that \(\phi((f_1, \ldots, f_{|V_n|})) = 0\). That is, there exists \(H \in C(X, \mathbb{Z}^2)\) with \(H |_{Y_n} = 0\) such that
\[
\sum_{v=1}^{\left|V_n\right|} f_v = H - H \circ \alpha^{-1}.
\]
It follows that
\[
\left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \right) \cdot \left( \sum_{v=1}^{|V_n|} f_v \right) = \left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \right) \cdot (H - H \circ \alpha^{-1}).
\]
As \( H \mid Y_n = 0 \),
\[
\left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \right) \cdot (H - H \circ \alpha^{-1}) = \left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \cdot H \right) \circ \alpha^{-1}.
\]
It then follows that
\[
\left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \right) \cdot (H - H \circ \alpha^{-1}) = \left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \cdot H \right) - \left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \cdot H \right) \circ \alpha^{-1}.
\]
Use \( H_v \) to denote \( \left( \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \right) \cdot H \). It is clear that \( H_v \) is supported on \( \bigcup_{k=1}^{h(v)} X(n,v,k) \).

Now we have \( f_v = H_v - H_v \circ \alpha^{-1} \). As \( f_v \) is supported on \( X(n,v,1) \), we get \( H_v - H_v \circ \alpha^{-1} = 0 \) on \( X(n,v,k) \) for \( 2 \leq k \leq h(v) \), which implies that for all \( x \in X(n,v,1) \),
\[
H(x) = H_v(\alpha(x)) = \cdots = H_v(\alpha^{h(v)-1}(x)).
\]
As \( \alpha^{h(v)-1}(x) \in Y_n \), it follows that \( H_v(\alpha^{h(v)-1}(x)) = 0 \). Now we can conclude that \( H_v = 0 \). It is then clear that \( f_v = 0 \).

Applying the process to all \( v = 1, \ldots, h(v) \), we get \( H = 0 \). It follows that \( f_i = 0 \) for \( i = 1, \ldots, |V_n| \), which proves the injectivity of \( \phi \).

Surjectivity of \( \phi \):
For every \( g \in C(X, \mathbb{Z}^2) \), we need to find
\[
(f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C(X(n,v,1), \mathbb{Z}^2)
\]
such that
\[
\phi \left( (f_1, \ldots, f_{|V_n|}) \right) - g = h - h \circ \alpha^{-1}
\]
for some \( h \in C(X, \mathbb{Z}^2) \) satisfying \( h \mid Y_n = 0 \).

Write \( g \) as
\[
g = 1 \cdot g = \sum_{v \in V_n} \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \cdot g.
\]
For every \( k \) with \( 2 \leq k \leq h(v) \), consider \( (1_{X(n,v,k)} \cdot g) \circ \alpha \). It is easy to check that \( (1_{X(n,v,k)} \cdot g) \circ \alpha \mid Y_n = 0 \) and
\[
1_{X(n,v,k)} \cdot g + ((1_{X(n,v,k)} \cdot g) \circ \alpha - (1_{X(n,v,k)} \cdot g) \circ \alpha \circ \alpha^{-1})
\]
is supported on \( X(n,v,k-1) \).

By repeating this process, we get \( s \in C(X, \mathbb{Z}^2) \) such that \( 1_{X(n,v,k)} \cdot g + (s - s \circ \alpha) \) is supported on \( 1_{X(n,v,1)} \).
Apply the process for all \(1_{X(n,v,k)} \cdot g\) with \(v \in V_n\) and \(1 \leq k \leq h(v)\). We can find \(H \in C(X,\mathbb{Z}^2)\) such that \(g + (H - H \circ \alpha^{-1})\) is supported on \(\alpha(R(P_n)) = \bigoplus_{v \in V_n} X(n,v,1)\). According to the definition, if we set \(f_v = 1_{X(n,v,1)} \cdot (g + (H - H \circ \alpha^{-1}))\), then \(\phi\) will map \((f_1, \ldots, f_{|V_n|})\) to \(g\).

Positivity of \(\phi\):

As

\[
\phi((f_1, \ldots, f_{|V_n|})) = \sum_{v \in V_n} 1_{X(n,v,1)} \cdot f_v,
\]

for

\[
(f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C(X(n,v,1), \mathbb{Z}^2),
\]

if the range of each \(f_i\) is in the positive cone \(D\), it is clear that \(\sum_{v \in V_n} 1_{X(n,v,1)} \cdot f_v \in C(X,D)\).

Thus \(\phi\) is order preserving.

For \(f \in C(X,D)\), we will show that if there is

\[
(f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C(X(n,v,1),D)
\]

such that

\[
\phi(f_1, \ldots, f_{|V_n|}) = [f],
\]

then \(f_v \in C(X(n,v,1))\) for all \(1 \leq v \leq |V_n|\).

In fact, such an element \((f_1, \ldots, f_{|V_n|})\) can be constructed from \(f\) as in the proof of surjectivity of \(\phi\). The fact that \(f \in C(X,D)\) then implies that for all \(v\) with \(1 \leq v \leq |V_n|\), the image \(f_k\) is in \(D\), which finishes the proof.

**Lemma 1.7.** There is an order isomorphism

\[
\rho_{\eta}: K_0(A_n) \rightarrow C(X,\mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X,\mathbb{Z}^2), f|_{Y_n} = 0\}
\]

with the unit element and positive cone of

\[
C(X,\mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X,\mathbb{Z}^2), f|_{Y_n} = 0\}
\]

being \([1_X,0]\) and

\[
\{[g] \in C(X,\mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X,\mathbb{Z}^2), f|_{Y_n} = 0\} : \forall x \in X, g(x) = (0,0) \text{ or } g(x) = (a,b) \text{ with } a > 0\}.
\]

For a clopen subset \(U\) of \(X\) and \(\eta \in M_{k}(C(\mathbb{T}^2))\) such that \([\eta]\in K_0(C(\mathbb{T}^2))\) corresponds to \((a,b)\) as in Lemma \(1.5\), \(\rho_{n}(\diag(1_{U},\ldots,1_{U})\cdot \eta)\) is exactly \([1_{U} \cdot a,1_{U} \cdot b]\) with \(1_{U}\) denoting the continuous function from \(X\) to \(\mathbb{Z}\) that is \(1\) on \(U\) and \(0\) otherwise.

**Proof.** Consider the isomorphism

\[
\gamma_{n}: A_n \rightarrow \bigoplus_{v \in V_n} M_{h_{n}(v)} \left(C(X(n,v,1)) \otimes C(\mathbb{T}^2)\right)
\]

as in Lemma \(1.5\). It is clear that

\[
(\gamma_{n})_{*0}: K_0(A_n) \rightarrow K_0 \left(\bigoplus_{v \in V_n} M_{h_{n}(v)} \left(C(X(n,v,1)) \otimes C(\mathbb{T}^2)\right)\right)
\]

is an order isomorphism.
We know that
\[
K_0 \left( \bigoplus_{v \in V_n} M_{h(n,v)}(C(X(n,v,1)) \otimes C(T^2)) \right) \cong \bigoplus_{v \in V_n} K_0 \left( M_{h(n,v)}(C(X(n,v,1)) \otimes C(T^2)) \right),
\]
and use
\[
h_n : K_0 \left( \bigoplus_{v \in V_n} M_{h(n,v)}(C(X(n,v,1)) \otimes C(T^2)) \right) \to \bigoplus_{v \in V_n} K_0 \left( M_{h(n,v)}(C(X(n,v,1)) \otimes C(T^2)) \right)
\]
to denote the order isomorphism.

There are natural order isomorphisms
\[
l_{n,v} : K_0(M_{h(n,v)}(C(X(n,v,1)) \otimes C(T^2))) \to K_0(C(X(n,v,1)) \otimes C(T^2)).
\]
By Lemma 1.6 we can find order isomorphisms
\[
s_{n,v} : K_0(C(X(n,v,1)) \otimes C(T^2)) \to C(X(n,v,1), \mathbb{Z}^2)
\]
such that each \( s_{n,v} \) maps \([1_{C(X(n,v,1)) \otimes C(T^2)}]\) to the constant function with value \((1,0)\).

Combining \( l_{n,v} \) and \( s_{n,v} \) for all \( v \), we get an order isomorphism
\[
\varphi : \bigoplus_{v \in V_n} K_0(M_{h(n,v)}(C(X(n,v,1)) \otimes C(T^2))) \to \bigoplus_{v \in V_n} C(X(n,v,1), \mathbb{Z}^2)
\]
with the positive cone of \( \bigoplus_{v \in V_n} C(X(n,v,1), \mathbb{Z}^2) \) being \( \bigoplus_{v \in V_n} C(X(n,v,1), \mathbb{D}) \) \( \mathbb{D} \) as defined in Lemma 1.3. Note that \( \varphi \) is not unital.

According to Lemma 1.6 there is an order isomorphism
\[
\psi : \bigoplus_{v \in V_n} C(X(n,v,1), \mathbb{Z}^2) \to C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \mid_{Y_n} = 0\}.
\]

Let
\[
\rho_n = \psi \circ \varphi \circ h_n \circ (\gamma_n)_{*0}.
\]
Then \( \rho_n \) is a group isomorphism from \( K_0(A_n) \) to
\[
C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2), f \mid_{Y_n} = 0\}
\]
because \( \psi, \varphi, h_n \) and \( (\gamma_n)_{*0} \) are all group isomorphisms.

According to Lemma 1.6
\[
\gamma_n(1_{A_n}) = \bigoplus_{v \in V_n} \text{diag}(1_{X(n,v,1)}, \ldots, 1_{X(n,v,h(v))}).
\]
Thus
\[
(\gamma_n)_{*0}([1_{A_n}]) = \sum_{v \in V_n} \sum_{1 \leq k \leq h(v)} [1_{X(n,v,k)}].
\]
It is then clear that
\[
h_n((\gamma_n)_{*0}([1_{A_n}])) = \left( \sum_{1 \leq k \leq h(1)} [1_{X(n,v,k)}], \ldots, \sum_{1 \leq k \leq h(|V_n|)} [1_{X(n,v,h(k))}] \right).
\]
Note that \([1_{X(n,v,k)}] = [1_{X(n,v,1)}]\) in \(K_0(M_n(X(n,v,1)))\). It follows that
\[
\varphi(h_n((\gamma_n)_0([1_{A_n}]))) = \varphi\left(\sum_{1 \leq k \leq h(1)} [1_{X(n,v,k)}], \ldots, \sum_{1 \leq k \leq h(V_n)} [1_{X(n,v,h(k))}]\right)
= \sum_{v \in V_n} h(v) \cdot [1_{X(n,v,1)}].
\]

According to the definition of \(\phi\) as stated in Lemma 1.6, we get
\[
\psi(\varphi(h_n((\gamma_n)_0([1_{A_n}]))) = \psi\left(\sum_{v \in V_n} h(v) \cdot [1_{X(n,v,1)}]\right) = \sum_{v \in V_n} [f_v]
\]
with \(f_v \in C(X, \mathbb{Z}^2)\) satisfying \(f_v|_{X(n,v,1)} = h(v)\) and \(f_v|_{X \setminus X(n,v,1)} = 0\).

Let
\[
H = \sum_{v \in V_n} \sum_{1 \leq k \leq h(v) - 1} 1_{X(n,v,k)} \cdot (h(v) - k).
\]

Then it is clear that \(H|_{Y_n} = 0\) and
\[
H_v \circ \alpha^{-1} = \sum_{v \in V_n} \sum_{2 \leq k \leq h(v)} 1_{X(n,v,k)} \cdot (h(v) - k + 1).
\]

It is easy to check that
\[
H - H \circ \alpha^{-1} = \sum_{v \in V_n} \left[\sum_{2 \leq k \leq h(v)} 1_{X(n,v,k)} \cdot (-1) + 1_{X(n,v,1)} \cdot (h(v) - 1)\right].
\]

In \(C(X, \mathbb{Z}^2)\), it is easy to check that \((\sum_{v \in V_n} f_v) - 1_X = H - H \circ \alpha^{-1}\). In other words, we have
\[
\psi(\varphi(h_n((\gamma_n)_0([1_{A_n}]))) = \sum_{v \in V_n} [f_v] = [1_X],
\]
which implies that \(\rho_n\) is unital.

To show that \(\rho_n\) is order preserving, we just need to show that \(\psi, \varphi, h_n\) and \((\gamma_n)_0\) are all order preserving.

It is clear that \(h_n\) and \((\gamma_n)_0\) are order preserving. According to Lemma 1.6, \(\psi\) is also order preserving. We just need to show that \(\varphi\) is order preserving.

Note that \(\varphi = \bigoplus_{v \in V_n} (s_{n,v} \circ l_{n,v})\). We just need to show that each \(s_{n,v} \circ l_{n,v}\) is order preserving. In fact, \(l_{n,v}\) is order preserving and \(s_{n,v}\) is an order isomorphism. It follows that \(s_{n,v} \circ l_{n,v}\) is order preserving. Thus \(\varphi\) is order preserving.

Now we will show that \(\rho_n\) is order isomorphism. In fact, we just need to show that for every \((a, b) \in \{(m, n): m > 0, n \in \mathbb{Z}\} \cup \{0, 0\}\) and every clopen subset \(U\) of \(X\), if we regard \((1_U \cdot a, 1_U \cdot b)\) as a function in \(C(X, \mathbb{Z}^2)\) defined by
\[
(1_U \cdot a, 1_U \cdot b)(x) = \begin{cases} (a, b) & \text{if } x \in U \\ (0, 0) & \text{if } x \notin U \end{cases}
\]
and we define
\[
\pi: C(X, \mathbb{Z}^2) \to C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2), f|_{Y_n} = 0\}
\]
to be the natural quotient map, then \(\pi((1_U \cdot a, 1_U \cdot b))\) is in the image of \(\rho_n(K_0(A_n)_+)\).
For a clopen subset $U$ of $X$ and $\eta \in M_k(C(\mathbb{T}^2))$ such that $[\eta] \in K_0(\mathbb{T}^2)$ corresponds to the $(a, b)$ above (see Lemma 1.5), we have

$$ \rho_n([\text{diag}(1_U, \ldots, 1_U) \cdot \eta]) = (\phi \circ \varphi \circ h_n \circ (\gamma_n)_*)([\text{diag}(1_U, \ldots, 1_U) \cdot \eta]).$$

According to Lemma 1.5

$$ (h_n \circ (\gamma_n)_*)([\text{diag}(1_U, \ldots, 1_U) \cdot \eta])$$

$$ = (h_n \circ (\gamma_n)_*) \left( \sum_{v \in V_n, 1 \leq k \leq h(v)} [\text{diag}(1_{X(n,v,k) \cap U}, \ldots, 1_{X(n,v,k) \cap U}) \cdot \eta] \right)$$

$$ = \left( \sum_{1 \leq k \leq h(v)} [1_{X(n,v,k) \cap U} \cdot \eta] \right)_{v \in V_n}.$$

Then

$$ (\varphi \circ h_n \circ (\gamma_n)_*)([\text{diag}(1_U, \ldots, 1_U) \cdot \eta])$$

$$ = \left( \sum_{1 \leq k \leq h(v)} (1_{\alpha^{-k-1}}(X(n,v,k) \cap U) \cdot a, 1_{\alpha^{-k-1}}(X(n,v,k) \cap U) \cdot b) \right)_{v \in V_n}$$

which is an element of $\bigoplus_{v \in V_n} C(X(n,v,1), \mathbb{Z}^2)$.

According to the definition of $\phi$ as in Lemma 1.6 it follows that

$$ (\psi \circ \varphi \circ h_n \circ (\gamma_n)_*)([\text{diag}(1_U, \ldots, 1_U) \cdot \eta]) = (\psi)((\varphi \circ h_n \circ (\gamma_n)_*)([\text{diag}(1_U, \ldots, 1_U) \cdot \eta]))$$

$$ = \sum_{v \in V_n} 1_{X(n,v,1)} \cdot f_v$$

with

$$ f_v = \left( \sum_{1 \leq k \leq h(v)} 1_{\alpha^{-k-1}}(X(n,v,k) \cap U) \cdot a, \sum_{1 \leq k \leq h(v)} 1_{\alpha^{-k-1}}(X(n,v,k) \cap U) \cdot b \right).$$

Note that for all $k$ with $1 \leq k \leq h(v) - 1$, we have $1_{X(n,v,k)} | Y_n = 0$. Also, we can check that

$$ 1_{X(n,v,k) \cap U} - 1_{X(n,v,k) \cap U} \circ \alpha^{-1} = 1_{X(n,v,k) \cap U} - 1_{\alpha(X(n,v,k) \cap U)}.$$

It follows that

$$ [1_{X(n,v,k) \cap U}] = [1_{\alpha(X(n,v,k) \cap U)}] \text{ in } C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}), f | Y_n = 0\}$$

for $k = 1, \ldots, h(v)$. We then get that in $C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}), f | Y_n = 0\}$,

$$ \left[ \sum_{1 \leq k \leq h(v)} 1_{\alpha^{-k-1}}(X(n,v,k) \cap U) \right] = \left[ \sum_{1 \leq k \leq h(v)} 1_{X(n,v,k) \cap U} \right].$$
It then follows that
\[
\left[ \sum_{v \in V_n} f_v \right] = \left[ \sum_{v \in V_n} \left( 1_{X(n,v,k) \cap U} \cdot a, 1_{X(n,v,k) \cap U} \cdot b \right) \right] = \left[ (1_U) \cdot a, (1_U) \cdot b \right]
\]
in \( C(X, \mathbb{Z}^2) / \{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2), f \mid Y_n = 0 \} \).

We have proved that \( \rho_n([\text{diag}(1_U, \ldots, 1_U) \cdot \eta]) = \pi((1_U \cdot a, 1_U \cdot b)) \). It then follows that \( \rho_n \) is an order isomorphism, which finishes the proof. \( \square \)

**Corollary 1.8.** Let \( p \) be a projection in \( M_\infty(A_n) \). Then there exists \( p' \in M_\infty(C(X \times \mathbb{T}^2)) \subset M_\infty(A_n) \) such that \( [p] = [p'] \) in \( K_0(A_n) \).

**Proof.** According to Lemma 1.6, we have an isomorphism
\[ \rho_n : K_0(A_n) \to C(X, \mathbb{Z}^2) / \{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2), f \mid Y_n = 0 \}. \]
Let \( \rho_n([p]) = [g] \) for some \( g \in C(X, \mathbb{Z}^2) \). Without loss of generality, we can assume that there is a partition of \( X \) as \( X = \bigsqcup_{i=1}^N X_i \) such that this partition is finer than \( \mathcal{P}_n \) and \( g \mid X_i \) is constant for \( i = 1, \ldots, N \).

As \( [p] \) is in \((K_0(A_n))_+\) and \( \rho_n \) is an order isomorphism, it follows that \( [g] \) is in the positive cone (defined in the statement of Lemma 1.6). For as \( g \) above with \( \rho_n([p]) = [g] \), we can assume that on any given \( X_i \), \( g \mid X_i \) is either \((0,0)\) or \((a_i, b_i) \in \mathbb{Z}^2 \) with \( a_i > 0 \).

According to Lemma 1.3, there exist projections \( \eta_i \in M_d(C(T^2)) \) such that \( [\eta_i] \) in \( K_0(C(T^2)) \) can be identified with \((a_i, b_i) \).

Let
\[ p' = \text{diag} \left( \left( \text{diag}(1_{X_1}, \ldots, 1_{X_N}) \cdot \eta_1, \ldots, \text{diag}(1_{X_1}, \ldots, 1_{X_N}) \cdot \eta_N \right) \right). \]

Then it is clear that \( p' \in M_\infty(C(X \times \mathbb{T}^2)) \).

According to Lemma 1.7, \( \rho_n([p']) = [g] \), so that \( \rho_n([p']) = \rho_n([p]) \). As \( \rho_n \) is an isomorphism (by Lemma 1.6 again), it follows that \( [p] = [p'] \) in \( K_0(A_n) \). \( \square \)

**Lemma 1.9.** Let \( j_n : C(X \times \mathbb{T}^2) \to A_n \) be the canonical embedding, and let \( \iota \) and \( \rho_n \) be as in Lemma 1.4 and Lemma 1.7. Let \( (j_n)_{n:0} : K_0(C(X \times \mathbb{T}^2)) \to K_0(A_n) \) be the induced map on \( K_0 \) and let
\[ \pi_n : C(X, \mathbb{Z}^2) \to C(X, \mathbb{Z}^2) / \{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2), f \mid Y_n = 0 \} \]
be the canonical quotient map. Then the following diagram commutes:
\[
\begin{array}{ccc}
K_0(C(X \times \mathbb{T}^2)) & \xrightarrow{\iota} & C(X, \mathbb{Z}^2) \\
(j_n)_{n:0} \downarrow & & \downarrow \pi_n \\
K_0(A_n) & \xrightarrow{\rho_n} & C(X, \mathbb{Z}^2) / \{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2), f \mid Y_n = 0 \}
\end{array}
\]
Proof. As $K_0(C(X \times \mathbb{Z}^2))$ is generated by its positive cone $(K_0(C(X \times \mathbb{Z}^2)))_+$, we just need to show that $\pi_n \circ \iota = \rho_n \circ (j_n)_0$ on $(K_0(C(X \times \mathbb{Z}^2)))_+$.

For every projection $p \in M_\infty(C(X \times \mathbb{T}^2))$, according to the proof of surjectivity of $\varphi$ in Lemma 1.4, there exist a partition $X = \bigsqcup_{i=1}^M X_i$ and projections $\eta_i \in M_d(C(\mathbb{T}^2))$ for $i = 1, \ldots, M$ such that
\[
[p] = \sum_{i=1}^M [(1_{X_i}, \ldots, 1_{X_i}) \cdot \eta_i].
\]

According to Lemma 1.3, $\eta_i$ can be identified with $(a_i, b_i) \in D$. By Lemma 1.4 we get $\iota([p]) = \sum_{i=1}^M (1_{X_i} \cdot a_i, 1_{X_i} \cdot b_i)$.

By Lemma 1.7
\[
\rho_n((j_n)_0([p])) = \rho_n((j_n)_0(\sum_{i=1}^M [(1_{X_i}, \ldots, 1_{X_i}) \cdot \eta_i]))
\]
\[
= \sum_{i=1}^M [(1_{X_i} \cdot a_i, 1_{X_i} \cdot a_i)].
\]
It is then clear that $(\pi_n \circ \iota)([p]) = (\rho_n \circ (j_n)_0)([p])$. Since $p$ is arbitrary, we have finished the proof. \qed

Corollary 1.10. Let $p, q$ be projections in $M_\infty(C(X \times \mathbb{T}^2)) \subset M_\infty(A_n)$ such that $\iota([p]) - \iota([q]) = h - h \circ \alpha^{-1}$ for some $h \in C(X, \mathbb{Z}^2)$ satisfying $h|_{Y_n} = 0$, with $\iota$ as in Lemma 1.4. Then $(j_n)_0([p]) = (j_n)_0([q])$ in $K_0(A_n)$ with $j_n$ as in Lemma 1.9.

Proof. This follows directly from Lemma 1.9 \qed

Lemma 1.11. For $A_x$ as defined in the beginning of this section,
\[
K_i(A_x) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2)\},
\]
and
\[
K_0(A_x) \cong C(X, D)/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2)\},
\]
with $D$ defined to be $\{(a, b) \in \mathbb{Z}^2: a > 0, b \in \mathbb{Z}\} \cup \{(0, 0)\}$.

Proof. From Lemma 1.7 we know that
\[
K_i(A_n) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2) \text{ and } f|_{Y_n} = 0\}.
\]
As $A_x = \lim_{\rightarrow} A_n$, we get $K_i(A_x) = \lim_{\rightarrow} K_i(A_n)$. Note that the map
\[
(j_n, n+1)_*: K_i(A_n) \to K_i(A_{n+1})
\]
satisfies $(j_{n, n+1})_*([f]) = [f]$ for all $f \in C(X, \mathbb{Z}^2)$. We can conclude that
\[
K_i(A_x) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2) \text{ and } f|_{Y_n} = 0 \text{ for some } n \in \mathbb{N}\}.
\]
As $\bigcap_{n=1}^{\infty} Y_n = \{x\}$, it follows that
\[
\{f \in C(X, \mathbb{Z}^2): f|_{Y_n} = 0 \text{ for some } n \in \mathbb{N}\} = \{f \in C(X, \mathbb{Z}^2): f(x) = 0\}.
\]
Then we have
\[
K_i(A_x) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2) \text{ and } f(x) = 0\}.
\]
For every $g \in C(X, \mathbb{Z}^2)$, define $g_0 = g - g(x)$. It is clear that $g_0 \in \{ f \in C(X, \mathbb{Z}^2) \text{ and } f(x) = 0 \}$.

As $g_0 - g_0 \circ \alpha^{-1} = g - g \circ \alpha^{-1}$, we have

$$K_i(A_x) \cong C(X, \mathbb{Z}^2)/\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \}.$$

Let $j_{n,\infty}: A_n \to A_x$ be the embedding of $A_n$ into $A_x$. Then

$$K_0(A_x)_+ = \bigcup (j_{n,\infty})_0(K_0(A_n)_+).$$

According to Lemma 1.7,

$$K_0(A_n)_+ \cong C(X, D)/\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \text{ and } f|_{Y_n} = 0 \}.$$

Similarly, using the fact that

$$\{ f \in C(X, \mathbb{Z}^2) : f|_{Y_n} = 0 \text{ for some } n \in \mathbb{N} \} = \{ f \in C(X, \mathbb{Z}^2) : f(x) = 0 \},$$

we can conclude that $K_0(A_x)_+ \cong C(X, D)/\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \text{ and } f(x) = 0 \}$.

As

$$\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \text{ and } f(x) = 0 \} = \{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \},$$

we get $K_0(A_x)_+ \cong C(X, D)/\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \}$. \hfill \Box

**Corollary 1.12.** For $A_x$ as in Definition 0.7, $K_i(A_x)$ is torsion free for $i = 0, 1$.

**Proof.** According to Lemma 1.11 we just need to show that $C(X, \mathbb{Z}^2)/\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \}$ is torsion free. A purely algebraic proof is given here.

Suppose we have $g \in C(X, \mathbb{Z}^2)$ and $n \in \mathbb{Z} \setminus \{0\}$ such that $[ng] = 0$ in $C(X, \mathbb{Z}^2)/\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \}$.

If we can show that $[g] = 0$, then we are done. In other words, we need to find $f \in C(X, \mathbb{Z}^2)$ such that $g = f - f \circ \alpha^{-1}$.

As $[ng] = 0$, there exists $F \in C(X, \mathbb{Z}^2)$ such that $ng = F - F \circ \alpha^{-1}$. If $F(x) \in n\mathbb{Z}^2$ for all $x$, just divide both sides by $n$. Then we get $g = (\frac{F}{n}) - (\frac{F}{n}) \circ \alpha^{-1}$ with $\frac{F}{n} \in C(X, \mathbb{Z}^2)$.

Fix $x_0 \in X$, and define $\tilde{F} = F - F(x_0)$. It is clear that $\tilde{F}(x_0) = 0$. As $F - F \circ \alpha^{-1} = ng$, we can easily check that $\tilde{F} - \tilde{F} \circ \alpha^{-1} = ng$. It then follows that

$$\tilde{F}(\alpha(x_0)) = \tilde{F}(x_0) + ng(\alpha(x_0)) = 0 + ng(\alpha(x_0)) \in n\mathbb{Z}^2,$$

$$\tilde{F}(\alpha^2(x_0)) = \tilde{F}(\alpha(x_0)) + ng(\alpha^2(x_0)) \in n\mathbb{Z}^2,$$

... So for every $x \in \text{Orbit}_\mathbb{Z}(x_0)$, we get $\tilde{F}(x) \in n\mathbb{Z}^2$. Note that $\tilde{F}$ is continuous on $X$ and $\text{Orbit}_\mathbb{Z}(x_0)$ is dense in $X$. It follows directly that $\tilde{F}(x) \in n\mathbb{Z}^2$ for all $x \in X$, thus finishing the proof. \hfill \Box

**Corollary 1.13.** For $A_x$ as in Definition 0.7, $\text{TR}(A_x) \leq 1$.

**Proof.** From Lemma 1.1 we know that $A_x$ is a AH algebra with no dimension growth. By Lemma 1.2 $A_x$ is simple. According to Lemma 1.11 $K_i(A_x)$ is torsion free.

As $A_x$ is a simple AH algebra with no dimension growth, it follows that $\text{TR}(A_x) \leq 1$. \hfill \Box
2. THE CROSSED PRODUCT C*-ALGEBRA A

This section contains the main theorem (Theorem 2.18), which states that the tracial rank of the crossed product $C^*(\mathbb{Z}, X \times T \times T, \alpha \times R_x \times R_n)$ has tracial rank no more than one.

We start by showing that for the natural embedding $j : A_x \to A$, the induced homomorphisms $(j_i)_i : K_i(A_x) \to K_i(A)$ are injective for $i = 0, 1$.

**Lemma 2.1.** Let $A$ be $C^*(\mathbb{Z}, X \times T \times T, \alpha \times R_x \times R_n)$ and let $A_x$ be as in Definition 0.1. Let $j : A_x \to A$ be the canonical embedding. Then $j_{*0}$ is an injective order homomorphism from $K_0(A_x)$ to $K_0(A)$.

**Proof.** It is clear that $j_{*0}$ will induce an order homomorphism from $K_0(A_x)$ to $K_0(A)$ and $j_{*0}$ maps $[1_{A_x}]$ to $[1_A]$.

To show that $j_{*0}$ is injective, we need to show that whenever $p, q \in M_\infty(A_x)$ are projections such that $j_{*0}([p]) = j_{*0}([q])$ in $K_0(A)$, we have $[p] = [q]$ in $K_0(A_x)$. As in Lemma 1.9, we can find $m, n \in \mathbb{N}$ and projections $e, f \in M_\infty(A_x)$ such that $[e] = [p]$ and $[f] = [q]$ in $K_0(A_x)$. Let $j : A_x \to A$ be as in Definition 0.1. We need to show that if $j_{*0}([p]) = j_{*0}([q])$ in $K_0(A)$, then $[p] = [q]$ in $K_0(A_x)$. In fact, if $j_{*0}([p]) = j_{*0}([q])$, then we have $j_{*0}([p]) = j_{*0}([q])$, which implies that $j_{*0}([e]) = j_{*0}([f])$ in $K_0(A)$.

By Proposition 1.8, we have the six-term exact sequence in our situation reads as follows:

$$K_0(C(X \times T^2)) \xrightarrow{id_{*0} + \alpha_{*0}} K_0(C(X \times T^2)) \xrightarrow{j_{*0}} K_0(A) \xrightarrow{id_{*1} + \alpha_{*1}} K_0(C(X \times T^2))$$

As $j_{*0}([p_n]) = j_{*0}([q_n])$, by the exact sequence above, $[p_n'] - [q_n']$ is in the image of $(id_{*0} + \alpha_{*0})$. That is, there exists $x$ in $K_0(C(X \times T^2))$ such that $[p_n'] - [q_n'] = x - \alpha_{*0}(x)$. Apply $i$ as defined in Lemma 1.4 on both sides. We get

$$i([p_n']) - i([q_n']) = i(x) - i(\alpha_{*0}(x)) \in C(X, \mathbb{Z}).$$

Note that $i(\alpha_{*0}(x)) = i(x) \circ \alpha$. We get $i([p_n']) - i([q_n']) = (-i(x) \circ \alpha) - (-i(x) \circ \alpha) \circ \alpha^{-1}$. We can choose $N \in \mathbb{N}$ such that for all $k > N$, $(-i(x) \circ \alpha)$ restricted to $Y_k$ will be a constant function, say $c \in \mathbb{Z}$. It is clear that

$$i([p_n']) - i([q_n']) = (-i(x) \circ \alpha - c) - (-i(x) \circ \alpha - c) \circ \alpha^{-1}.$$ 

Choose $m \in \mathbb{N}$ such that $m > \max(n, N)$. Then $(-i(x) \circ \alpha - c)|_{Y_m} = 0$. According to Corollary 1.4, we have $i([p_{n+1}]) = i([q_{n+1}])$ with $n_{m+1}$ as in Lemma 1.9.

We have shown that $[p_n'] = [q_n']$ in $K_0(A_m)$. Note that $[p_n'] = [p_n]$ and $[q_n'] = [q_n]$ in $K_0(A_n)$ and $m > n$. It follows that $[p_n'] = [p_n]$ and $[q_n'] = [q_n]$ in $K_0(A_m)$. We then have that $[p_n] = [q_n]$ in $K_0(A_m)$, so that $[p_n] = [q_n]$ in $K_0(A_x)$.

Note that $[p_n] = [p]$ and $[q_n] = [q]$ in $K_0(A_x)$. It then follows that $[p] = [q]$ in $K_0(A_x)$, which finishes the proof.

**Lemma 2.2.** Let $A$ be $C^*(\mathbb{Z}, X \times T \times T, \alpha \times R_x \times R_n)$ and let $A_x$ be as in Definition 0.1. Let $j : A_x \to A$ be the canonical embedding. Then $j_{*1}$ is an injective homomorphism from $K_1(A_x)$ to $K_1(A)$. 

\[\ ]
Proof. The proof is similar to the proof of Lemma 2.1.

For any two unitaries \( x, y \in A_n \) such that \( j_{n+1}(x) = j_{n+1}(y) \) in \( K_1(A) \), we need to show that \( [x] = [y] \). For \( x, y \) as above, we can find \( n \in \mathbb{N} \) and \( x', y' \in M_\infty(A_n) \) such that \( [x] = [x'] \) and \( [y] = [y'] \) in \( K_1(A_x) \).

From Lemma 1.3, we get the structure of \( A_n \), which then implies the fact that

\[
K_1(A_n) \cong C(X, \mathbb{Z}^2)/\{ f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2) \text{ and } f |_{\gamma_n} = 0 \}.
\]

Similar to the analysis of the Pimsner-Voiculescu six-term exact sequence as in the proof of Lemma 2.1, we get \( [x'] = [y'] \) in \( K_1(A_m) \) for \( m \) large enough. Then it follows that \( [x'] = [y'] \) in \( K_1(A_x) \), which implies that \( [x] = [y] \) in \( K_1(A_x) \).

The following result is a known fact, and it is used later to show approximate unitary equivalence.

**Proposition 2.3.** Let \( A \) be an infinite dimensional simple unital AF algebra and let \( CU(A) \) be as in Section 2. Then \( U(A) = CU(A) \).

**Proof.** For every unitary \( u \in A \) and every \( \varepsilon > 0 \), we will show that \( \text{dist}(u, CU(A)) < \varepsilon \).

As \( A \) is unital and infinite dimensional, we can assume that \( A \cong \varinjlim A_n \) with each \( A_n \) being a finite dimensional C*-algebra and each map \( j_{n,n+1} : A_n \rightarrow A_{n+1} \) being unital. Write

\[
A_n \cong \bigoplus_{k=1}^{d_n} M_{d_n,k}(\mathbb{C})
\]

with \( d_{n;1} \leq d_{n;2} \leq \cdots \leq d_{n;s_n} \).

Let \( d'_n = \min\{d_{n;1}, \ldots, d_{n;s_n}\} \). As \( A \) is simple, we have \( \lim_{n \to \infty} d'_n = \infty \).

For \( u \) and \( \varepsilon \) as given above, we can choose \( n \) large enough such that \( d'_n > \frac{2\varepsilon}{\varepsilon} \) and there exists \( v \in U(A_n) \) satisfying \( \|u - v\| < \varepsilon/2 \). Let \( \pi_{n,k} \) be the canonical projection from \( A_n \) to \( M_{d_n,k}(\mathbb{C}) \). It is known that for any \( w \in U(A) \), we have \( w \in CU(A_n) \) if and only if \( \det(\pi_{n,k}(w)) = 1 \) for \( k = 1, \ldots, s_n \). Without loss of generality, we can assume that

\[
\pi_{n,k}(u_n) = \text{diag}(\lambda_{k,1}, \ldots, \lambda_{k,d_n,k}), \quad \text{with } |\lambda_{k,d_{n,k}}| = 1.
\]

Choose \( L_k \) such that \( -\pi \leq L_k < \pi \) and \( \det(\pi_{n,k}(u_n)) = e^{iL_k} \). For \( k = 1, \ldots, s_n \), define

\[
v'_k = \text{diag}(\lambda_{k,1}, e^{-iL/d_n,k}, \ldots, \lambda_{k,d_n,k}, e^{-iL/d_n,k}).
\]

Let \( v' = \text{diag}(v'_1, \ldots, v'_{s_n}) \). It is then clear that \( \|u_n - u'_n\| \leq \pi/d'_n \). It is easy to check that \( \det(\pi_{n,k}(v')) = 1 \) for all \( k = 1, \ldots, s_n \), which then implies that \( v' \in CU(A_n) \subset CU(A) \).

Note that \( d'_n > \frac{2\varepsilon}{\varepsilon} \). We have

\[
\text{dist}(u, CU(A)) \leq \|u - v'\| \leq \|u - v\| + \|v - v'\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

As \( \varepsilon \) can be chosen to be arbitrarily small, it follows that \( u \in CU(A) \). \( \square \)

We will need the fact that a cut-down of the crossed product C*-algebra by a projection in \( C(X) \) is similar to the original crossed product C*-algebra, and can be regarded as a crossed product C*-algebra of the induced action. Some definitions and facts will be given here.

Let \( (X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathbb{R}_\xi \times \mathbb{R}_\eta) \) be a minimal topological dynamical system as defined in Section 2. Let \( D \) be a clopen subset of \( X \), and let \( x \in D \). For simplicity, we use \( \varphi \) to denote \( \alpha \times \mathbb{R}_\xi \times \mathbb{R}_\eta \).
Define $\tilde{\varphi}: D \times T \times T \to D \times T \times T$ by $\tilde{\varphi}(y,t_1,t_2) = \varphi^f((y,t_1,t_2))$, where $f(x)$ is the first return time function defined by

$$f(x) = \min\{n \in \mathbb{N}: \ n > 0, \ \varphi^n(x) \in U\}.$$  

As $\varphi$ is minimal on $X \times T \times T$, for every $x \in X$, the orbit of $x$ under $\varphi$ is dense in $X$. It then follows that the intersection of this orbit with $D$ is dense in $D$, which implies that $\tilde{\varphi}$ is also minimal on $D \times T \times T$. As the composition of rotations on the circle is still a rotation on the circle, we can find maps $\tilde{\xi}, \tilde{\eta}: D \to T$ such that $\tilde{\varphi} = \tilde{\alpha} \times R_{\tilde{\xi}} \times R_{\tilde{\eta}}$ with $\tilde{\alpha}(x) = \alpha f(x)(x)$ for $f$ as defined above.

It follows that $\tilde{\xi}$ and $\tilde{\eta}$ are both continuous functions. In fact, as $D$ is clopen, we have that $f$ is continuous, which then implies that $\tilde{\xi}$ and $\tilde{\eta}$ are continuous.

As $(D \times T \times T, \tilde{\varphi})$ is a minimal dynamical system, the corresponding crossed product C*-algebra $C^*(\mathbb{Z}, D \times T \times T, \tilde{\varphi})$ is simple. Use $\tilde{u}$ to denote the implementing unitary in $C^*(\mathbb{Z}, D \times T \times T, \tilde{\varphi})$.

Define $\tilde{A}_z$ to be the subalgebra of $C^*(\mathbb{Z}, D \times T \times T, \tilde{\varphi})$ generated by $C(D \times T \times T)$ and $\tilde{u} \cdot C_0(D \setminus \{x\}) \times T \times T$.

The lemma below shows that the cut down of the original crossed product C*-algebra is isomorphic to the crossed product C*-algebra of the induced homeomorphism.

**Lemma 2.4.** Let $\varphi$ and $\tilde{\varphi}$ be defined as above. There is a C*-algebra isomorphism from $C^*(\mathbb{Z}, D \times T \times T, \tilde{\varphi})$ to $1_{D \times T \times T} \cdot A \cdot 1_{D \times T \times T}$.

**Proof.** Let $f: D \to \mathbb{N}$ be the first return time function. As $D$ is clopen, $f$ is continuous. As $X$ is compact and $D$ is closed in $X$, $D$ is also compact. Continuity of $f$ then implies that $f(D)$ is a compact set, that is, a finite subset of $\mathbb{N}$. Write $f(D) = \{k_1, \ldots, k_N\}$ with $N, k_1, \ldots, k_N \in \mathbb{N}$ and set $D_i = f^{-1}(k_i)$.

In $1_{D \times T \times T} \cdot A \cdot 1_{D \times T \times T}$, let $w = \sum_{i=1}^N 1_{D_i \times T \times T} \cdot u^{k_i}$. Then we have

$$ww^* = \left(\sum_{i=1}^N 1_{D_i \times T \times T} \cdot u^{k_i}\right) \cdot \left(\sum_{i=1}^N 1_{D_i \times T \times T} \cdot u^{k_i}\right)^*$$

$$= \left(\sum_{i=1}^N 1_{D_i \times T \times T} \cdot u^{k_i}\right) \cdot \left(\sum_{j=1}^N u^{-k_j} 1_{D_j \times T \times T}\right)$$

$$= \sum_{i,j=1}^N 1_{D_i \times T \times T} \cdot u^{k_i} \cdot u^{-k_j} 1_{D_j \times T \times T}$$

$$= \sum_{i,j=1}^N 1_{D_i \times T \times T} \cdot u^{k_i-k_j} 1_{D_j \times T \times T}$$

$$= \sum_{i,j=1}^N 1_{D_i \times T \times T} \cdot (1_{D_j \times T \times T} \circ (\alpha \times R_{\tilde{\xi}} \times R_{\tilde{\eta}})^{k_i-k_j}) \cdot u^{k_i-k_j}.$$  

We need the following claim to get that $ww^* = 1_D$.

**Claim 2.5.** For $D_i, k_i$ as above,

$$1_{D_i \times T \times T} \cdot (1_{D_j \times T \times T} \circ (\alpha \times R_{\tilde{\xi}} \times R_{\tilde{\eta}})^{k_i-k_j}) = \begin{cases} 1_{D_i \times T \times T} & i = j \\ 0 & i \neq j \end{cases}.$$  

**Proof of claim:**
If $k_j > k_i$, then $\alpha^{k_j-k_i}(D_j) \subset X \setminus D$. Thus $D_i \cap \alpha^{k_j-k_i}(D_j) = \emptyset$.

If $k_j < k_i$, we claim that $D_i \cap \alpha^{k_j-k_i}(D_j) = \emptyset$. If not, choose $s \in D_i \cap \alpha^{k_j-k_i}(D_j)$. We can assume $s = \alpha^{k_j-k_i}(y)$ for some $y \in D_j$. It is then clear that $\alpha^{k_i-k_j}(s) = y \in D_j \subset D$, contradicting the fact that the first return time of $s$ (in $D_j$) is $k_i$.

If $k_j = k_i$, it is clear that $1_{D_i} \cdot (1_{D_j} \circ \alpha^{k_i-k_j}) = 1_{D_i}$.

This proves the claim.

Using the claim, we get

$$ww^* = \sum_{i,j=1}^{N} 1_{D_i \times \mathbb{T} \times \mathbb{T}} \cdot (1_{D_j \times \mathbb{T} \times \mathbb{T}} \circ (\alpha \times R_{\xi} \times R_{\eta})^{k_i-k_j}) \cdot u^{k_i-k_j}$$

$$= \sum_{i=1}^{N} 1_{D_i \times \mathbb{T} \times \mathbb{T}}$$

$$= 1_{D \times \mathbb{T} \times \mathbb{T}}.$$

Now we calculate $w^*w$. It is clear that

$$w^*w = \left( \sum_{i=1}^{N} 1_{D_i \times \mathbb{T} \times \mathbb{T}} \cdot u^{k_i} \right)^* \cdot \left( \sum_{i=1}^{N} 1_{D_i \times \mathbb{T} \times \mathbb{T}} \cdot u^{k_i} \right)$$

$$= \left( \sum_{j=1}^{N} u^{-k_j} \cdot 1_{D_j \times \mathbb{T} \times \mathbb{T}} \right)^* \cdot \left( \sum_{i=1}^{N} 1_{D_i \times \mathbb{T} \times \mathbb{T}} \cdot u^{k_i} \right)$$

$$= \sum_{i,j=1}^{N} u^{-k_j} \cdot 1_{D_j \times \mathbb{T} \times \mathbb{T}} \cdot 1_{D_i \times \mathbb{T} \times \mathbb{T}} \cdot u^{k_i}$$

$$= \sum_{i=1}^{N} u^{-k_i} \cdot 1_{D_i \times \mathbb{T} \times \mathbb{T}} \cdot u^{k_i}$$

$$= \sum_{i=1}^{N} 1_{D_i \times \mathbb{T} \times \mathbb{T}} \circ (\alpha \times R_{\xi} \times R_{\eta})^{-k_i}$$

$$= \sum_{i=1}^{N} 1_{D_i \times \mathbb{T} \times \mathbb{T}} \cdot (\alpha \times R_{\xi} \times R_{\eta})^{k_i}(D_i \times \mathbb{T} \times \mathbb{T})$$

$$= \sum_{i=1}^{N} 1_{\tilde{\varphi}(D_i \times \mathbb{T} \times \mathbb{T})}$$

$$= 1_{D \times \mathbb{T} \times \mathbb{T}}.$$

So far, we have shown that $w$ is a unitary in $1_{D \times \mathbb{T} \times \mathbb{T}} \cdot A \cdot 1_{D \times \mathbb{T} \times \mathbb{T}}$.

Define a map

$$\gamma: C^*(\mathbb{Z}, D \times \mathbb{T} \times \mathbb{T}, \tilde{\varphi}) \to 1_{D \times \mathbb{T} \times \mathbb{T}} \cdot A \cdot 1_{D \times \mathbb{T} \times \mathbb{T}}$$

by

$$\gamma(f) = f \text{ for all } f \in C(D \times \mathbb{T} \times \mathbb{T}) \text{ and } \gamma(\tilde{w}) = w.$$
We will check that $\gamma$ is well-defined and gives the desired isomorphism between $C^*(\mathbb{Z}, D \times T \times T, \tilde{\varphi})$ and $1_{D \times T} \cdot A \cdot 1_{D \times T}$. In fact, for all $f \in C(D \times T \times T)$, we have

$$\gamma(\tilde{u}^* f \tilde{u}) = \gamma(f \circ \tilde{\varphi}^{-1})$$

$$= f \circ \tilde{\varphi}^{-1}.$$ We also have

$$\gamma(\tilde{u}^* f \tilde{u}) = \gamma(\tilde{u}^*) \cdot \gamma(f) \cdot \gamma(\tilde{u})$$

$$= u^* \cdot f \cdot w$$

$$= \left( \sum_{j=1}^{N} 1_{D_j} \cdot u^{k_j} \right)^* \cdot \left( f \cdot \sum_{i=1}^{N} 1_{D_i} \right) \cdot \left( \sum_{i=1}^{N} 1_{D_i} \cdot u^{k_i} \right)$$

$$= \sum_{i,j,k=1}^{N} u^{-k_j} \cdot 1_{D_j} \cdot f \cdot 1_{D_i} \cdot 1_{D_i} \cdot u^{k_i}$$

$$= \sum_{i=1}^{N} u^{-k_i} \cdot (f \cdot 1_{D_i}) \cdot u^{k_i}$$

$$= f \circ \tilde{\varphi}^{-1},$$

which then implies that $\gamma$ is really a homomorphism.

To show that $\gamma$ is surjective, we will show that for every $g \in C(X \times T \times T)$ and $n \in \mathbb{N}$, $1_{D \times T} \cdot (g u^n) \cdot 1_{D \times T} \times T$ is in the image of $\gamma$. Note that

$$1_{D \times T} \cdot (g u^n) \cdot 1_{D \times T} \times T = (1_{D \times T} \cdot g) \cdot (u^n \cdot 1_{D \times T})$$

$$= (1_{D \times T} \cdot g) \cdot 1_{\alpha^{-n}(D \times T \times T)} \cdot u^n.$$ Without loss of generality, we assume that

$$D \cap \alpha^{-n}(D) \neq \emptyset.$$ Note that there is $s$ with $1 \leq s \leq N$ such that $D \cap \alpha^{-n}(D) = D_s$, $n = k_s$ and $D_s$ is exactly $f^{-1}(n)$. It follows that

$$1_{D \times T} \cdot (g u^n) \cdot 1_{D \times T} \times T = (g \cdot 1_{D_s \times T \times T}) \cdot u^n$$

$$= (g \cdot 1_{D_s \times T \times T}) \cdot (1_{D_s \times T \times T} \cdot u^n).$$

It is clear that we can identify $g \cdot 1_{D_s \times T \times T}$ with a function in $C(D \times T \times T)$. Note that $w = \sum_{i=1}^{N} 1_{D_i \times T \times T} \cdot u^{k_i}$. We have

$$\gamma((g \cdot 1_{D_s \times T \times T}) \cdot (\tilde{u})) = \gamma((g \cdot 1_{D_s \times T \times T})) \cdot \gamma(\tilde{u})$$

$$= (g \cdot 1_{D_s \times T \times T}) \cdot \left( \sum_{i=1}^{N} 1_{D_i \times T \times T} \cdot u^{k_i} \right)$$

$$= g \cdot 1_{D_s \times T \times T} \cdot u^{k_s}$$

$$= g \cdot 1_{D_s \times T \times T} \cdot u^n$$

$$= 1_{D \times T} \cdot (g u^n) \cdot 1_{D \times T} \times T.$$ Then we have proved that $\gamma$ is surjective. As $C^*(\mathbb{Z}, D \times T \times T, \tilde{\varphi})$ is a simple $C^*$-algebra, it follows that $\gamma$ is a $C^*$-algebra isomorphism. □
The idea of topological full group of the Cantor set is needed in the next lemma, and a definition is given below.

**Definition 2.6.** Let $X$ be the Cantor set and let $\alpha$ be a minimal homeomorphism of $X$. We say that $\beta \in \text{Homeo}(X)$ is in the full group of $\alpha$ if $\beta$ preserves the orbit of $\alpha$. That is, for any $x \in X$, $\beta(\{\alpha^n(x)\}_{n \in \mathbb{Z}}) = \{\alpha^n(x)\}_{n \in \mathbb{Z}}$. In this case, there exists a unique function $n: X \to \mathbb{Z}$ such that $\beta(x) = \alpha^n(x)$ for all $x \in X$.

We say that $\beta \in \text{Homeo}(X)$ is in the topological full group of $\alpha$ if the function $n$ above is continuous.

We use $[\alpha]$ to denote the full group of $\alpha$, and use $[[\alpha]]$ to denote the topological full group of $\alpha$.

**Lemma 2.7.** Let $X$ be the Cantor set and let $\alpha$ be a minimal homeomorphism of $X$. Let $Y$ and $U$ be two clopen subsets of $X$ such that $U \subset Y$. If there exists $\beta \in [[\alpha]]$ such that $\beta(U) \subset Y$ and $U \cap \beta(U) = \emptyset$, then there exists $\gamma \in [[\alpha]]$ such that $\gamma(Y) = Y$, $\gamma|_U = \beta|_U$ and $\gamma|_{X \setminus Y} = \text{id}|_{X \setminus Y}$.

**Proof.** As $\beta \in [[\alpha]]$, there exists a continuous function $n_1: X \to \mathbb{Z}$ such that $\beta(x) = \alpha^{n_1(x)}(x)$ for all $x \in X$. Let $U_j = U \cap n_1^{-1}(j)$ for $j \in \mathbb{Z}$. As the sets $n_1^{-1}(j)$ are mutually disjoint for $j \in \mathbb{Z}$, so are the sets $U_j$. Now we have $\beta(U) = \bigsqcup_{j=-\infty}^{\infty} \alpha^j(D_j)$. Define $\gamma \in \text{Homeo}(X)$ by $\gamma(x) = \alpha^{n_2(x)}(x)$, with

$$n_2(x) = \begin{cases} n_1(x) & x \in U \\ -j & x \in \alpha^j(U_j) \\ 0 & x \notin U \text{ and } x \notin \beta(U) \end{cases}$$

As $U \cap \beta(U) = \emptyset$, we get $U \cap \alpha^j(U_j) = \emptyset$ for all $j \in \mathbb{Z}$. Thus $n_2$ is a well-defined function. Then we can check that $\gamma|_U = \beta|_U$ as $n_1|_U = n_2|_U$. It is also obvious that $\gamma(\beta(U)) = U$ and $\gamma|_{Y \setminus (U \cup \beta(U))} = \text{id}|_{Y \setminus (U \cup \beta(U))}$. It follows that $\gamma(Y) = Y$. As $n_2(x) = 0$ when $x \notin Y$, we get $\gamma|_{X \setminus Y} = \text{id}|_{X \setminus Y}$.

**Lemma 2.8.** Let $X$ be the Cantor set. Let $\alpha$ be a minimal homeomorphism of $X$, and let $x \in X$. Let $A$ be the crossed product C*-algebra of the dynamical system $(X, \alpha)$. Use $A_x$ to denote the subalgebra generated by $C(X)$ and $u \cdot C_0(X \setminus \{x\})$. Let $D$ be a clopen subset of $X$ and let $n \in \mathbb{N}$ be such that $x \notin \bigcup_{k=0}^{n-1} \alpha^k(D)$. In $A_x$, the element $s = u \cdot 1_{\alpha^{-n-1}(D)} \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D$ is a partial isometry such that $s^*s = 1_D$ and $ss^* = 1_{\alpha^n(D)}$.

**Proof.** We just need to check $ss^* = 1_{\alpha^n(D)}$, $s^*s = 1_D$, and $s \in A_x$.

In fact,

$$ss^* = (u \cdot 1_{\alpha^{-n-1}(D)} \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D) \cdot (u \cdot 1_{\alpha^{-n-1}(D)} \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D)^*$$
$$= u \cdot 1_{\alpha^{-n-1}(D)} \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D \cdot u^* \cdot 1_{\alpha(D)} \cdot u^* \cdots 1_{\alpha^{-n-1}(D)} \cdot u^*$$
$$= 1_{\alpha^n(D)},$$

and

$$s^*s = (u \cdot 1_{\alpha^{-n-1}(D)} \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D)^* \cdot (u \cdot 1_{\alpha^{-n-1}(D)} \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D)$$
$$= 1_D \cdot u^* \cdot 1_{\alpha(D)} \cdot u^* \cdots 1_{\alpha^{-n-1}(D)} \cdot u^* \cdot u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D$$
$$= 1_D.$$
As } x \notin \bigcup_{k=0}^{n-1} \alpha^k(D) \text{, it follows that } u \cdot 1_{\alpha^k(D)} \in A_x \text{ for } k = 0, \ldots, n - 1. \text{ Thus } s, s^* \in A_x. \tag*{$\square$}

**Remark:** It is easy to check that } s = u^n \cdot 1_D \text{ and } s^* = (u^n \cdot 1_D)^* = 1_D \cdot u^{-n}.

**Lemma 2.9.** Let } X \text{ be the Cantor set and let } \alpha \text{ be a minimal homeomorphism of } X. \text{ Let } u \text{ be the implementing unitary of the crossed product C*-algebra } C^*(\mathbb{Z}, X, \alpha). \text{ For } \gamma \in [\alpha][\alpha], \text{ there exist mutually disjoint clopen sets } X_1, \ldots, X_N \text{ and } n_1, \ldots, n_N \in \mathbb{N} \text{ such that } X = \bigsqcup_{i=1}^N X_i \text{ and } \gamma(x) = \alpha^{n_i}(x) \text{ for } x \in X_i. \text{ Furthermore, } w = \sum_{i \in \mathbb{N}} 1_{X_i} \cdot u^{n_i} \text{ is a unitary element in } C^*(\mathbb{Z}, X, \alpha) \text{ satisfying } w^*f = f \circ \gamma^{-1} \text{ for all } f \in C(X).

**Proof.** As } \gamma \in [\alpha][\alpha], \text{ there exists a continuous function } n: X \to \mathbb{Z} \text{ such that } \gamma(x) = \alpha^n(x)(x) \text{ for all } x \in X. \text{ As } X \text{ is compact and } n \text{ is continuous, the range } n(X) \text{ must be finite.}

Define

\[
 w = \sum_{k \in n(X)} 1_{Y_k} \cdot u^k
\]

where } Y_k = n^{-1}(k). \text{ As } n(X) \text{ is finite, we have finitely many sets } Y_k. \text{ As } \gamma \text{ is a homeomorphism, it follows that } \alpha^k(Y_k) \cap \alpha^j(Y_j) = \emptyset \text{ if } k \neq j.

We will check that } w w^* = 1 \text{ and } w^* w = 1.

Note that

\[
 w w^* = \left( \sum_{k,j \in \mathbb{Z}} 1_{Y_k} \cdot u^k \right) \left( \sum_{j \in \mathbb{Z}} 1_{Y_j} \cdot u^{-j} \right)^* = \sum_{k,j \in \mathbb{Z}} 1_{Y_k} \cdot u^k \cdot u^{-j} \cdot 1_{Y_j} = \sum_{k,j \in \mathbb{Z}} 1_{Y_k} \cdot (1_{Y_j} \circ \alpha^{k-j}) \cdot u^{k-j} = \sum_{k,j \in \mathbb{Z}} 1_{Y_k} \cdot 1_{\alpha^{j-k}(Y_j)} \cdot u^{k-j}.
\]

As } \alpha^k(Y_k) \cap \alpha^j(Y_j) = \emptyset \text{ if } k \neq j \text{, it follows that } \alpha^{j-k}(Y_j) \cap Y_k = \emptyset \text{ if } k \neq j. \text{ Then we get}

\[
 w w^* = \sum_{k,j \in \mathbb{Z}} 1_{Y_k} \cdot 1_{\alpha^{j-k}(Y_j)} \cdot u^{k-j} = \sum_{k} 1_{Y_k} = 1.
\]

As } C^*(\mathbb{Z}, X, \alpha) \text{ has stable rank one, it is finite. It then follows that } w^* w = 1. \text{ So far, we have shown that } w \text{ is a unitary element in } C^*(\mathbb{Z}, X, \alpha).

To show that } w^* f = f \circ \gamma^{-1}, \text{ we just need to show that for each } i \text{ and every clopen set } D \subset Y_i, \text{ we have } w^* 1_D w = 1_D \circ \gamma^{-1}. \text{ As } C(X) \text{ is generated by}

\[
 \{ 1_D : D \text{ is a clopen set of } Y_i \text{ for some } i \in \mathbb{Z} \},
\]

that will imply } w^* f = f \circ \gamma^{-1} \text{ for all } f \in C(X).
For a clopen set $D \subset Y$, it is clear that
\[
\phi^* 1_D \phi = \left( \sum_{j \in \mathbb{Z}} 1_{Y_j} \cdot u^j \right)^* \cdot 1_D \cdot \left( \sum_{k \in \mathbb{Z}} 1_{Y_k} \cdot u^k \right)
\]
\[
= \sum_{j,k \in \mathbb{Z}} u^{-j} \cdot 1_{Y_j} \cdot 1_D \cdot 1_{Y_k} \cdot u^k
\]
\[
= u^{-i} \cdot 1_D \cdot u^i
\]
\[
= 1_D \circ \alpha^{-i}
\]
\[
= 1_D \circ \gamma^{-1},
\]
which finishes the proof. \hfill \Box

Some facts about Cantor dynamical systems that will be needed are given below.

**Lemma 2.10.** Let $(X, \alpha)$ be a minimal Cantor dynamical system and let $x \in X$. Let $U$ and $V$ be two clopen subsets of $X$. Let $A$ be the crossed product $C^*$-algebra of $(X, \alpha)$ and let $A_x$ be the subalgebra generated by $C(X)$ and $u \cdot C_0(X \setminus \{x\})$, with $u$ being the implementing unitary element in $A$ satisfying $ufu^* = f \circ \alpha^{-1}$ for all $f \in C(X)$. If there exists an integer $n \geq 1$ such that $\alpha^n(U) = V$ and $x \notin \bigcup_{k=0}^{n-1} \alpha^k(U)$, then there exists $w \in A_x$ such that $w \cdot 1_U \cdot w^* = 1_V$.

**Proof.** As $x \notin \bigcup_{k=0}^{n-1} \alpha^k(U)$, we can find a Kakutani-Rokhlin partition $P$ of $X$ with respect to $\alpha$ such that the roof set $R(P)$ is a clopen set containing $x$ and $R(P) \cap (\bigcup_{k=0}^{n-1} \alpha^k(U)) = \emptyset$.

Write
\[
P = \bigcup_{1 \leq s \leq N, 1 \leq k \leq h(s)} X(s, k)
\]
with $\alpha(X(s, k)) = X(s, k + 1)$ for all $k = 1, \ldots, h(s) - 1$ and $\alpha(R(P)) \subset \bigcup_{1 \leq s \leq N} X(s, 1)$.

Use $A_P$ to denote the subalgebra generated by $C(X)$ and $u \cdot C_0(X \setminus R(P))$. Then
\[
A_P \cong \bigoplus_{s=1}^N M_{h(s)}(C(X(s, 1))).
\]

In other words, there exists a $C^*$-algebra isomorphism
\[
\varphi: A_P \longrightarrow \bigoplus_{s=1}^N M_{h(s)}(C(X(s, 1)))
\]
satisfying
\[
\varphi(1_{X(s,k)}) = \text{diag}(0, \ldots, 0, 1, 0, \ldots) \in M_{h(s)}(C(X, 1))
\]
with the $k$-th diagonal element being $1_{X(s,k)}$.

It is clear that $1_U = \sum_{s,k} 1_{U \cap X(s,k)}$ and $1_V = \sum_{s,k} 1_{V \cap X(s,k)}$. Define $U_s$ to be $\bigcup_k (U \cap X(s,k))$ and $V_s$ to be $\bigcup_k (V \cap X(s,k))$. It is clear that $1_U = \sum_s 1_{U_s}$ and $1_V = \sum_s 1_{V_s}$.

Recall the isomorphism $\varphi$ above. By abuse of notation, we can regard $1_{U_s}$ and $1_{V_s}$ as two diagonal matrices in $M_{h(s)}(C(X(s,1)))$. 
If we can find unitary elements $w_s \in M_{h(s)}(C(X,s,1))$ such that $w_s \cdot 1_U \cdot w_s^* = 1_{V_i}$, by setting $w = w_1 + \cdots + w_s$, it is then clear that $w$ is unitary element in $\bigoplus_{s=1}^N M_{h(s)}(C(X(s,1)))$ such that $w \cdot 1_U \cdot w^* = 1_V$, which is equivalent to the existence of a unitary in $A_P$ conjugating $1_U$ to $1_V$. As $x \in R(P)$, we can regard $A_P$ as a subalgebra of $A_x$. Then the unitary $w$ in $A_P$ is also a unitary in $A_x$.

Let $w_s$ be a unitary element in $M_{h(s)}(C(X,s,1))$ satisfying

\[ w_s \cdot E_{i,i} \cdot w_s^* = E_{i+1,i+1} \]

for $i = 1, \ldots, h(s) - 1$ and

\[ w_s E_{h(s),h(s)} w_s^* = E_{1,1}, \]

with $(E_{i,j})$ being the standard system of matrix units. It follows that $w_s \cdot 1_U \cdot w_s^* = 1_{V_i}$, which finishes the proof. \hfill \Box

**Lemma 2.11.** Let $(X, \alpha)$ be a minimal Cantor dynamical system and let $U, V$ be two clopen subsets of $X$ satisfying $\alpha^n(U) = V$ for some $n \in \mathbb{N}$. Then there exists a partition of $U$, say $U = \bigcup_{i=1}^m U_i$ with each $U_i$ clopen such that for all $k = 1, \ldots, n$ and $i, j = 1, \ldots, m$ with $i \neq j$, we have $\alpha^k(U_i) \cap \alpha^k(U_j) = \varnothing$.

**Proof.** We just need to find a partition of $U$ into $U = \bigcup_{i=1}^m U_i$ such that for every given $i$ with $1 \leq i \leq m$, the clopen sets $\alpha^1(U_i), \ldots, \alpha^n(U_i)$ are mutually disjoint.

For every $y \in U$, as $\alpha$ is a minimal homeomorphism, we can find a clopen set $D_y \subset U$ such that $\alpha^1(D_y), \ldots, \alpha^n(D_y)$ are mutually disjoint. As $U$ is compact, there exists a finite subset of $U$, say $\{y_1, \ldots, y_N\}$, such that $\bigcup_{s=1}^N D_{y_s} = U$.

As the intersection of two clopen sets is still clopen, without loss of generality, we may assume that the sets $D_{y_1}, \ldots, D_{y_N}$ are mutually disjoint. That is, $U = \bigcup_{i=1}^m D_{y_i}$. It is then clear that for any given $s$ with $1 \leq s \leq N$, $\alpha^k(D_{y_s})$ are mutually disjoint for $k = 1, \ldots, n$, which finishes the proof. \hfill \Box

The lemma below is the strengthened version of Lemma 2.10 in the sense that we no longer require $x \notin \bigcup_{k=0}^{n-1} \alpha^k(U)$.

**Lemma 2.12.** Let $X$ be the Cantor set and let $x \in X$. Let $\alpha$ be a minimal homeomorphism of $X$ and let $A_x$ be defined as in Lemma 2.10. For every $n \in \mathbb{N}$ and clopen subset $U \subset X$, there exists a unitary element $w \in A_x$ such that

\[ w = \sum_{j \in \mathbb{Z}} 1_{D_j} w^j \text{ and } w \cdot 1_U \cdot w^* = 1_{\alpha^n(U)}, \]

where $D_j$ for $j \in \mathbb{Z}$ are mutually disjoint clopen subsets of $X$ satisfying $X = \bigcup_{j \in \mathbb{Z}} D_j$, and all but finitely many $D_j$ are empty.
Proof. Let $d$ be the metric on $X$. As $(X, \alpha)$ is a minimal dynamical system, $x, \alpha(x), \ldots, \alpha^n(x)$ are distinct from each other.

Let

$$R = \frac{1}{2} \min_{0 \leq i, j \leq n, i \neq j} d(\alpha^i(x), \alpha^j(x)).$$

It is clear that $R > 0$.

For $k$ with $0 \leq k \leq n$, if $x \in \alpha^k(U)$, as $\alpha^k(U)$ is clopen, there exists $r_k > 0$ such that the open set \{ $y \in X : d(x, y) < r_k$ \} is a subset of $\alpha^k(U)$. If $x \notin \alpha^k(U)$, as $\alpha^k(U)$ is compact, $\inf_{y \in \alpha^k(U)} d(x, y) = d(x, y')$ for some $y' \in \alpha^k(U)$. In this case, let $r_k = \inf_{y \in \alpha^k(U)} d(x, y)$.

Let

$$r = \min(R, r_0, r_1, \ldots, r_n) > 0$$

and define $E'$ to be

$$\{ y \in X : d(x, y) < r \}.$$  

Then $E'$ is an open subset of $X$. As the topology of the Cantor set $X$ is generated by clopen sets, we can find a clopen subset $E \subset E'$ such that $x \in E$.

According to the definition of $r$, it follows that for $k = 0, 1, \ldots, n$, either $E' \subset \alpha^k(U)$ or $E' \cap \alpha^k(U) = \emptyset$. The fact that $E \subset E'$ implies that for $k = 0, 1, \ldots, n$, either $E \subset \alpha^k(U)$ or $E \cap \alpha^k(U) = \emptyset$.

Let $P$ be a Kakutani-Rokhlin tower such that the roof set is $E$. As $E$ is the roof set and $E, \alpha(E), \ldots, \alpha^n(E)$ are mutually disjoint, it follows that the height of each tower in $P$ is greater than $n + 1$.

Use $X(N, v, s)$ to denote the clopen subset of the partition $P$ at the $v$-th tower, with height $s$. Then

$$X = \bigcup_{v \in V, 1 \leq k \leq h(v)} X(n, v, s),$$

where $h(v)$ is the height of the $v$-th tower.

Let $U_{v, k} = U \cap X(N, v, k)$. Then

$$U = \bigcup_{v \in V, 1 \leq k \leq h(v)} U_{v, k}.$$  

For every $v, k$ such that $U_{v, k} \neq \emptyset$, if there exists $m \in \mathbb{N}$ such that $1 \leq m \leq n$ and $\alpha^m(U_{v, k}) \subset \alpha(E)$, then $E \cap \alpha^{m-1}(U) \neq \emptyset$. According to our choice of $E$, for all $s$ with $1 \leq s \leq n$, either $E \subset \alpha^s(U)$ or $E \cap \alpha^s(U) = \emptyset$. By assumption, we have $\alpha^m(U_{v, k}) \subset \alpha(E)$ and $U_{v, k} \neq \emptyset$. Then

$$E \cap \alpha^{m-1}(U) \supset E \cap \alpha^{m-1}(U_{v, k}) = \alpha^{m-1}(U_{v, k}) \neq \emptyset,$$

which implies that $E \subset \alpha^{m-1}(U)$.

Let $A_E$ be the subalgebra of $A$ generated by $C(X)$ and $u \cdot C_0(X \setminus R(P))$, with $u$ being the implementing unitary of $A$. We will show that there exists a unitary element $w \in A_E$ such that

$$w = \sum_{j \in \mathbb{Z}} 1_{D_j} w^j,$$

with all the sets $D_j$ for $j \in \mathbb{Z}$ being mutually disjoint and $w \cdot 1_U \cdot w^* = 1_{\alpha^n(U)}$. As $A_E$ can be regarded as a subalgebra of $A_x$, that is enough to prove the lemma if we can find the unitary $w$ as described above.
If \( k + n \leq h(v) \), this is the case that \( x \notin \bigcup_{j=0}^{n-1} \alpha^j(U_{v,k}). \) According to Lemma 2.8 there exists a partial isometry \( s_{v,k} \in A_x \) such that \( s_{v,k}^*s_{v,k} = 1_{U_{v,k}} \) and \( s_{v,k}^*s_{v,k} = 1_{\alpha^n(U_{v,k})} = 1_{U_{v,k+n}}. \) According to the remark after Lemma 2.8 we have \( s_{v,k} = u^n \cdot 1_{U_{v,k}}. \)

If there is a nonempty \( U_{v,k} \) such that \( k + n > h(v) \), then

\[
\alpha^{h(v)-k}(U) \cap E \supset \alpha^{h(v)-k}(U_{v,k}) \cap E \neq \emptyset.
\]

According to the construction of \( E \), it follows that \( E \subset \alpha^{h(v)-k}(U) \), which then implies that \( \alpha^{-(h(v)-k)}(E) \subset U. \) Intersecting both sets with

\[
\alpha^{-(h(v)-k)}(E) = \bigcup_{v' \in V} X(n, v', h(v') - (h(v) - k)),
\]

we get

\[
\bigcup_{v' \in V} X(n, v', h(v') - (h(v) - k)) = \alpha^{-(h(v)-k)}(E) \cap \bigcup_{v' \in V} X(n, v', h(v') - (h(v) - k))
\]

\[
\subset U \cap \bigcup_{v' \in V} X(n, v', h(v') - (h(v) - k))
\]

\[
\subset \bigcup_{v' \in V} X(n, v', h(v') - (h(v) - k)),
\]

which implies that

\[
U \cap \bigcup_{v' \in V} X(n, v', h(v') - (h(v) - k)) = \bigcup_{v' \in V} X(n, v', h(v') - (h(v) - k)).
\]

In other words,

\[
U_{v', h(v') - (h(v) - k)} = X(n, v', h(v') - (h(v) - k)) \text{ for all } v \in V'.
\]

Now we have

\[
\alpha^{-(h(v)-k)}(E) = \bigcup_{v' \in V} U_{v', h(v') - (h(v) - k)} = \bigcup_{v' \in V} X_{v', h(v') - (h(v) - k)}.
\]

It follows that

\[
\alpha^n \left( \bigcup_{v' \in V} U_{v', h(v') - (h(v) - k)} \right) = \alpha^n \left( \bigcup_{v' \in V} X_{v', h(v') - (h(v) - k)} \right) = \bigcup_{v' \in V} X_{v', n-(h(v) - k)}.
\]

By Lemma 2.8 there exists a partial isometry \( s'_{v,k} \) such that

\[
s'_{v,k}^*s'_{v,k} = 1_{U(v', h(v') - (h(v) - k))}
\]

and

\[
s'_{v,k}^*s'_{v,k} = 1_{\alpha^n(U(v', h(v') - (h(v) - k))}
\]

\[
= 1_{U(v', h(v') + n - (h(v) - k)) - h(v')}
\]

Furthermore, according to the remark after Lemma 2.8 \( s'_{v,k} \in A_E. \)

For every non-empty \( U_{v,k} \), either \( k + n \leq h(v) \) or \( U \supset \alpha^{-(h(v)-k)}(R(P)) \). Thus the above two cases will give a partial isometry \( s \in A_E \) such that \( ss^* = 1_U \) and \( s^*s = 1_{\alpha^n(U)}. \)

There exists a partial isometry \( \tilde{s} \in A_E \) such that

\[
\tilde{s}\tilde{s}^* = 1_{X\setminus U} \text{ and } \tilde{s}^*\tilde{s} = 1_{X\setminus \alpha^n(U)}.
\]
Let \( w = s + \tilde{s} \). Then \( w \) is a unitary element in \( A_E \) satisfying \( w \cdot 1_E \cdot w^* = 1_{\alpha^n(t)} \), which finishes the proof. \( \square \)

**Lemma 2.13.** Let \( X \) be the Cantor set and let \( x \in X \). Let \( D \) be a clopen subset of \( X \) satisfying \( x \in D \), and use \( X \times \mathbb{T}_1 \times \mathbb{T}_2 \) to denote the product of the Cantor set and two dimensional torus. Let \( A \) be the crossed product \( \mathbb{C}^\ast\)-algebra \( C^\ast(X, \mathbb{T}_1 \times \mathbb{T}_2, \omega_1 \times \mathbb{R}_\xi \times \mathbb{R}_\eta) \) and let \( u \) be the implementing unitary of \( A \). Let \( z_1 \in C(\mathbb{T}_1, \mathbb{C}) \) be defined by \( z_1(t) = t \) and let \( z_2 \in C(\mathbb{T}_2, \mathbb{C}) \) be defined by \( z_2(t) = t \). By abuse of notation, we identify \( M = C(X, \mathbb{T}_1 \times \mathbb{T}_2) \) be the crossed product \( \mathbb{C}^\ast\)-algebra \( \mathbb{C}(\mathbb{T}_1 \times \mathbb{T}_2, \omega_1 \times \mathbb{R}_\xi \times \mathbb{R}_\eta) \) and let \( z \in \mathbb{C}(\mathbb{T}_1 \times \mathbb{T}_2) \) be defined by \( z_1 \). Then there exists a partial isometry \( w \in A_x \) (with \( A_x \) as defined in Lemma 2.10) such that \( w^*w = p \), \( ww^* = q \) and \( \|wz^*pu^M - z_q\| < \varepsilon \) for \( i = 1, 2 \).

**Proof.** According to Lemma 2.12, we can find a unitary element \( w_1 \in A_x \) such that

\[
0 = \sum_{k \in \mathbb{Z}} u^k 1_n^{-1}(k)
\]

for some \( n \in C(X, \mathbb{Z}) \) and

\[
w_1p\alpha^1 = q.
\]

Let

\[
j_0: C(\mathbb{T}_1 \times \mathbb{T}_2) \to C(D \times \mathbb{T}_1 \times \mathbb{T}_2)
\]

be defined by \( j_0(f) = 1_D \otimes f \) for all \( f \in C(\mathbb{T}_1 \times \mathbb{T}_2) \). Then it is clear that \( j \) is an injective homomorphism.

As \( C(D \times \mathbb{T}_1 \times \mathbb{T}_2) \subset pA_xp \) (with \( p = 1_D \)), we hence get the canonical inclusion map

\[
\phi_0: C(D \times \mathbb{T}_1 \times \mathbb{T}_2) \to pA_xp.
\]

Define

\[
\phi_1: C(D \times \mathbb{T}_1 \times \mathbb{T}_2) \to pA_xp
\]

by

\[
\phi_1(g) = w_1^* \cdot u^M \cdot g \cdot u^{-M} \cdot w_1 \text{ for all } g \in C(D \times \mathbb{T}_1 \times \mathbb{T}_2).
\]

As \( q = u^Mpu^{-M} \) and \( p = 1_D \), it follows that \( u^M \cdot g \cdot u^{-M} \in qC(X \times \mathbb{T}_2)^q \subset qA_xq \). The fact that \( w_1p\alpha^1 = q \) implies that \( w_1^*qA_xqw_1 = pA_xq \). So far, we have shown that \( \phi_1 \) is really a homomorphism from \( C(D \times \mathbb{T}_2) \) to \( pA_xp \). As \( \|\phi_1(g)\| = \|g\| \), it is clear that \( \phi_1 \) is injective.

Define \( \varphi_0 = \phi_0 \circ j_0 \) and \( \varphi_1 = \phi_1 \circ j_0 \). Then \( \varphi_0, \varphi_1 \) are two injective homomorphisms from \( C(\mathbb{T}_2) \) to \( pA_xp \).

Let

\[
j: pA_xp \to pAp
\]

be the canonical embedding.

By Lemmas 2.1 and 2.2

\[
j_{\ast i}: K_i(pA_xp) \to K_i(pAp)
\]

will induce an injective embedding of \( K_i(pA_xp) \) into \( K_i(pAp) \) for \( i = 0, 1 \).

Consider \( (\varphi_0)_{\ast i} \) and \( (\varphi_1)_{\ast i}: K_i(C(D \times \mathbb{T}_2)) \to K_i(pA_xp) \) for \( i = 0, 1 \). As \( \varphi_1(f) = w_1^*u^Mfu^{-M}w_1 \), it is clear that \( (\varphi_0)_{\ast i}(a) = (\varphi_1)_{\ast i}(a) \) in \( K_i(pAp) \) for all \( a \in K_i(D \times \mathbb{T}_2) \). Since we know that
For a $C^*$-algebra $B$, recall from Section 1.3 that $T(B)$ denotes the convex set of all tracial states on $B$. For all $\tau \in T(pAp)$ and $g \in C(D \times \mathbb{T}_1 \times \mathbb{T}_2)$, it is clear that

$$\tau(w_1^a u^M gu^{-M} w_1) = \tau(g).$$

As $T(pAp) = T(pA_x p)$, it follows that for every tracial state $\tau' \in T(pA_x p)$, we have

$$\tau'(w_1^a u^M gu^{-M} w_1) = \tau'(g).$$

It is then clear that for all $\tau' \in T(pA_x p)$ and $f \in C(\mathbb{T}_1 \times \mathbb{T}_2)$,

$$\tau'(\varphi_0(f)) = \tau'(\varphi_1(f)).$$
So far, we have shown that \( \varphi_0^\circ(z_1 \otimes 1_{T_2}) = \varphi_1^\circ(z_1 \otimes 1_{T_2}) \). In the same way, it follows that \( \varphi_0^\circ(1_{T_1} \otimes z_2) = \varphi_1^\circ(1_{T_1} \otimes z_2) \).

According to [Lin1] Theorem 10.10], we conclude that \( \varphi_0 \) and \( \varphi_1 \) are approximately unitarily equivalent. Then there exists a unitary \( w_2 \in pA_x p \) such that

\[
\| w_1^* w_2^i u^M z_i u^{-M} w_1 - w_2 z_i w_1 \| < \varepsilon - \| u^M z_i u^{-M} - z_i q \|. 
\]

Let \( w = w_1 w_2 \). Then

\[
\| u^M z_i u^{-M} - z_i q \| < \varepsilon \text{ for } i = 1, 2.
\]

We can easily check that

\[
w^* w = w_2^* w_1^* w_1 w_2 = w_2^* w_2 = p \quad \text{and} \quad \quad w w^* = w_1 w_2 w_2^* w_1 = w_1 p w_1^* = q,
\]

which finishes the proof. \( \square \)

**Lemma 2.14.** We write \( X \times T \times T \) as \( X \times T_1 \times T_2 \) to distinguish the factors. Let \( A \) be the crossed product C*-algebra \( C^*(\mathbb{Z}, X \times T_1 \times T_2, \alpha \times R_T \times R_T) \) and let \( u \) be the implementing unitary of \( A \).

Let \( z \in X \). For any \( N \in \mathbb{N} \), any \( \varepsilon > 0 \) and any finite subset \( G \subset C(X \times T \times T) \), we have a natural number \( M > N \), a clopen neighborhood \( U \) of \( x \) and a partial isometry \( w \in A_x \) (with \( A_x \) defined as in Lemma [2.10]) satisfying the following:

1. \( \alpha^{-N+1}(U), \alpha^{-N+2}(U), \ldots, U, \alpha(U), \ldots, \alpha^M(U) \) are mutually disjoint, and \( \mu(U) < \varepsilon/M \) for all \( \alpha \)-invariant probability measure \( \mu \),
2. \( w^* w = 1_U \) and \( w w^* = 1_{\alpha^{-M}(U)} \),
3. \( u^{-i} w u^i \in A_x \) for \( i = 0, 1, \ldots, M - 1 \),
4. \( \| w f - f w \| < \varepsilon \) for all \( f \in G \).

**Proof.** By abuse of notation, we identify \( f \in C(X) \) with \( f \otimes \text{id}_{T_1} \otimes \text{id}_{T_2}, g \in C(T_1) \) with \( \text{id}_X \otimes g \otimes \text{id}_{T_2} \) and \( h \in C(T_2) \) with \( \text{id}_X \otimes \text{id}_{T_1} \otimes h \).

Without loss of generality, we can assume that

\[ G = \{ f_1, \ldots, f_k, z_1, z_2 \}, \]

where \( f_i \in C(X) \subset C(X \times T_1 \times T_2) \) for \( i = 1, \ldots, k \) and \( z_i(t_i) = t_i \) for \( t_i \in T_i, i = 1, 2 \).

There exists a neighborhood \( E \) of \( x \) such that

\[ |f_i(x) - f_i(y)| < \varepsilon/2 \]

for all \( y \in E \) and \( i = 1, \ldots, k \). It then follows that for any \( y_1, y_2 \in E \) and \( i \) such that \( 1 \leq i \leq k \), we have

\[ |f_i(y_1) - f_i(y_2)| < \varepsilon. \]

As \((X, \alpha)\) is minimal, there exists \( M > N \) such that \( \alpha^M(x) \in E \). Let

\[ K = \max \left\{ M, \frac{M}{\varepsilon} + 1 \right\}. \]

It is clear that the points \( \alpha^{-N+1}(x), \alpha^{-N+2}(x), x, \alpha(x), \ldots, \alpha^K(x) \) are distinct. Then there exists a clopen set \( U \) containing \( x \) such that \( U \subset E, \alpha^M(U) \subset E \) and \( \alpha^{-N+1}(U), \alpha^{-N+2}(U), U, \alpha(U), \ldots, \alpha^K(U) \) are disjoint.
As $\alpha^{-N+1}(U), \alpha^{-N+2}(U), U, \alpha(U), \ldots, \alpha^K(U)$ are disjoint, for every $\alpha$-invariant probability measure $\mu$, we have $\mu(U) < \varepsilon/M$.

By Lemma 2.13 there exists a partial isometry $w \in A_x$ such that $w^*w = 1_U$ and $ww^* = 1_{\alpha^M(U)}$.

As $U \subset E$ and $\alpha^M(U) \subset E$, it follows that $\|w_f - f_\alpha w\| < \varepsilon$ for $0 \leq i \leq k$. The fact that $\|u^M z_i p a^{-M} - z_i q\| < \varepsilon$ implies $\|w z_i - z_i w\| < \varepsilon$ for $i = 1, 2$. So far, (4) is checked.

From our construction of $U$, we have (1). The assertion (2) follows from our construction of $w$. Note that $U, \alpha(U), \ldots, \alpha^M(U)$ are mutually disjoint. We can check that $u^{-1}w u \in A_x$ for $i = 0, \ldots, m - 1$, thus finishing the proof. $\square$

**Definition 2.15.** Let $\mathcal{C}$ be a category of unital separable $C^*$-algebras. A separable simple $C^*$-algebra $A$ is called $\mathcal{C}$-Popa if for every finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a nonzero projection $p \in A$ and a unital subalgebra $B$ of $p A p$ (with $1_B = p$) such that $B \in \mathcal{C}$ and

1) $\|\{x, p\}\| \leq \varepsilon$ for all $x \in \mathcal{F}$,
2) $p \cdot x \cdot p \in B$ for all $x \in \mathcal{F}$.

**Lemma 2.16.** Let $\mathcal{C}$ be a category of unital separable $C^*$-algebras. Let $A$ be a separable simple $C^*$-algebra. If for every finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a nonzero projection $p \in A$ and a unital subalgebra $B$ of $p A p$ such that $B$ is $\mathcal{C}$-Popa and

1) $\|\{x, p\}\| \leq \varepsilon$ for all $x \in \mathcal{F}$,
2) $p \cdot x \cdot p \in B$ for all $x \in \mathcal{F}$,

then $A$ is $\mathcal{C}$-Popa.

**Proof.** For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, we can find a subalgebra $B$ such that $B$ is $\mathcal{C}$-Popa and

1) $\|\{x, 1_B\}\| \leq \varepsilon$ for all $x \in \mathcal{F}$,
2) $1_B \cdot x \cdot 1_B \in B$ for all $x \in \mathcal{F}$.

Use $1_B \mathcal{F} 1_B$ to denote the set $\{1_B x 1_B : x \in \mathcal{F}\}$. As $1_B \cdot x \cdot 1_B \in B$, for every $x \in \mathcal{F}$, choose an element $y_x \in B$ satisfying $\|y_x - 1_B \cdot x \cdot 1_B\| \leq \varepsilon$. Use $\mathcal{G}$ to denote $\{y_x : x \in \mathcal{F}\}$ with $y_x$ as described.

As $B$ is $\mathcal{C}$-Popa, we can find $E \subset B$ such that $E \in \mathcal{C}$ and

a) $\|\{1_E, y_x\}\| \leq \varepsilon$ for all $y_x \in \mathcal{G}$,
b) $1_E \cdot y_x \cdot 1_E \in E$ for all $y_x \in \mathcal{G}$.

We then check that

\[1_E \cdot x \cdot 1_E \approx 2 \varepsilon \|1_E \cdot 1_B \cdot x \cdot 1_B - 1_B \cdot x \cdot 1_B \cdot 1_E\| \approx 2 \varepsilon \|1_E \cdot x \cdot 1_E\|.

It then follows that

$\|1_E \cdot x - x \cdot 1_E\| \approx 4 \varepsilon \|1_E \cdot y_x - y_x \cdot 1_E\|$.

As $\|\{1_E, y_x\}\| \leq \varepsilon$, we get $\|\{x, 1_E\}\| \leq 5 \varepsilon$.

For any $x \in A$, we have

$\text{dist}(1_E \cdot x \cdot 1_E, E) = \text{dist}(1_E \cdot (1_B \cdot x \cdot 1_B) \cdot 1_E, E)$

$\approx \varepsilon \text{dist}(1_E \cdot y_x \cdot 1_E, E)$

$\approx \varepsilon 0$.

Then it is clear that $1_E \cdot x \cdot 1_E \in 2 \varepsilon E$. 

Thus for every finite subset $F \subset A$ and $\epsilon > 0$, we can find the subalgebra $E$ of $A$ as described above such that $E \in C$ and
1) $\| [x, 1_E] \| \leq 5\epsilon$ for all $x \in F$,
2) $1_E \cdot x \cdot 1_E \in 2E$ for all $x \in F$,
which shows that $A$ is $C$-Popa.

This following is a technical result that will be needed later.

**Proposition 2.17.** Let $A$ be a $C^*$-algebra. For every $a \in A$ such that $\| a - a^2 \| \leq \delta < \frac{1}{4}$, there exists a projection $p \in C^*(a)$ such that $\| p - a \| \leq \sqrt{\delta}$.

**Proof.** Just refer [Lin4, Lemma 2.5.5].

**Theorem 2.18.** Let $X$ be the Cantor set and let $\alpha \times R_{\xi} \times R_{\eta}$ be a minimal action on $X \times \mathbb{T} \times \mathbb{T}$. Use $A$ to denote the crossed product $C^*$-algebra of the minimal system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\eta})$. Then $\text{TR}(A) \leq 1$.

**Proof.** According to [HLX, Lemma 4.3], for simple $C^*$-algebra $A$, if for every $\epsilon > 0$, $c \in A^+ \setminus \{0\}$ and finite subset $F \subset A$, there exists a nonzero projection $p$ and a unital subalgebra $B$ of $pAp$ such that $\text{TR}(B) \leq 1$ and
1) $\| [x, p] \| \leq \epsilon$ for all $x \in F$,
2) $\text{dist}(p \cdot x \cdot p, B) \leq \epsilon$ for all $x \in F$,
3) $1 - p \preceq c$ as in Definition 0.2. That is, $1 - 1_B$ is Murray-von Neumann equivalent to a projection in $\text{Her}(c)$, then it follows that $\text{TR}(A) \leq 1$.

Let $A_x$ be as defined in Lemma 0.1. According to Lemma 1.13, $\text{TR}(A_x) = 1$. If we can find a projection $e \in A_x$ such that $B = eAxe$ satisfies the previous three conditions, then we are done.

As $A$ is generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and the implementing unitary $u$, we can assume that the finite set is $F \cup \{u\}$ with $F \subset C(X \times \mathbb{T} \times \mathbb{T})$.

Choose $N \in \mathbb{N}$ such that $2\pi/N < \epsilon$ and let

$$G = \bigcup_{i=0}^{N-1} u_i^{*} F u_i^{*}.$$

According to Lemma 2.13, with respect to $G$ and $\epsilon$ above, we can find $M > N$, a clopen neighborhood of $x$ and a partial isometry $w \in A_x$ satisfying $w^* w = 1_U$, $ww^* = 1_{\alpha M(U)}$ and $\| [w, f] \| < \epsilon$ for all $f \in F$.

Let $p = 1_U$ and $q = 1_{\alpha M(U)}$. For $t \in [0, \pi/2]$, define

$$P(t) = p \cos^2 t + \sin t \cos(t(w + w^*) + q \sin^2 t).$$

As $pq = 0$ and $p, q$ are Murray-von Neumann equivalent via $w$, it follows that $t \mapsto P(t)$ is a path of projections with $P(0) = p$ and $P(\pi/2) = q$.

Define

$$e = 1 - \left( \sum_{i=0}^{M-N} u_i^{*} p u_i + \sum_{i=1}^{N-1} u_i^{-1} P(i\pi/2N) u_i \right).$$
According to Lemma 2.14, \( u^{-i}wu^i \in A_x \) for \( i = 0, \ldots, m - 1 \). It is clear that \( e \in A_x \). It follows that \( e \) is a projection.

We first show that for \( e \in A_x \) above, the following hold.

1) \( \|x, e\| \leq \varepsilon \) for all \( x \in \mathcal{F} \cup \{u\} \); (C1)

2) dist(\( exe, eA_xe \)) \( \leq \varepsilon \) for all \( x \in \mathcal{F} \cup \{u\} \). (C2)

For the part of (C1) involving \( u \), note that

\[
ueu^* - e = 1 - u \left( \sum_{i=0}^{M-N} u^i pu^{-i} + \sum_{i=1}^{N-1} u^{-i} P(i\pi/2N)u^i \right) u^*
\]

\[
= - \sum_{i=0}^{M-N+1} u^i pu^{-i} + \sum_{i=1}^{M-N} u^i pu^{-i} + \sum_{i=1}^{N-1} u^{-i} P(i\pi/2N)u^i
\]

\[
- \sum_{i=0}^{N-2} u^{-i} P((i + 1)\pi/2N)u^i
\]

\[
= p - u^{M-N+1} p(u^*)^{M-N+1} + (u^*)^{N-1} P((N-1)\pi/2N)u^{N-1} - P(\pi/2N)
\]

\[
+ \sum_{i=1}^{N-2} u^{-i} (P(i\pi/2N) - P((i + 1)\pi/2N))u^i
\]

\[
= p - P(\pi/2N) + u^{-(N-1)} P((N-1)\pi/2N)u^{N-1} - u^{M-N+1} pu^{-(M-N+1)}
\]

\[
+ \sum_{i=1}^{N-2} u^{-i} (P(i\pi/2N) - P((i + 1)\pi/2N))u^i.
\]

As \( 2\pi/N < \varepsilon \), we get \( \|ueu^* - e\| < \varepsilon \). It then follows that \( \|ue - eu\| < \varepsilon \). By Lemma 2.14, \( \|fe - ef\| < \varepsilon \) for all \( f \in \mathcal{F} \). So far, we have checked (C1).

For \( f \in \mathcal{F} \subset C(X \times \mathbb{T} \times T) \), as \( f \in A_x \), we get \( ef(e) \in eA_xe \). As \( eu \in A_x \), it is clear that \( eue = eu(eu) \in eA_xe \). Thus we have checked (C2).

Let \( C \) be the set of all the unital separable C*-algebras \( C \) such that there exist \( N \in \mathbb{N} \) and one dimensional finite CW complexes \( X_i \) and \( d_i \in \mathbb{N} \) with \( 1 \leq i \leq N \) and

\[ C \cong \bigoplus_{i=1}^{N} M_{d_i} \cdot (C(X_i)). \]

Note that \( \varepsilon \) can be chosen to be arbitrarily small, and also note that \( eA_xe \) has tracial rank no more than one, which implies that \( eA_xe \) is \( C \)-Popa.

By Lemma 2.10, \( A \) is also \( C \)-Popa. According to [Lin13 Lemma 3.6.6], \( A \) has property (SP). For the given element \( c \in A_x \), there exists a non-zero projection \( q \in \text{Her}(c) \). Let \( \delta_0 = \inf\{\tau(q) : \tau \in T(A)\} \). As \( A \) is simple and \( q \neq 0 \), we get \( \tau(q) > 0 \) for all \( \tau \in T(A) \). As \( T(A) \) is a weak* closed subset of the unit ball of \( A^* \), noting that the unit ball of \( A^* \) is weak* compact by Alaoglu’s Theorem, it follows that \( T(A) \) is also compact. Thus \( \delta_0 \geq 0 \).

Without loss of generality, we can assume that \( \varepsilon < \min\{1, \frac{1}{8}\delta_0, \frac{1}{(4d_0)^N}\} \) and \( q \in \mathcal{F} \).

It remains to show that \( 1 - e \) is Murray-von Neumann equivalent to a projection in \( \text{Her}(c) \).
As \( q \in \mathcal{F} \), we have
\[ \| [q, e] \| \leq \varepsilon \text{ and } \text{dist}(eqe, eA_e) \leq \varepsilon. \]
We can find \( b \in (eA_e)_s \) such that \( \|eqe - b\| \leq \varepsilon \). Note that \( \| [q, e] \| \leq \varepsilon \) implies that \( \|(eqe)^2 - eqe\| \leq \varepsilon \). According to Proposition 2.17, there exists a projection \( q' \in A \) such that \( \|q' - eqe\| \leq \sqrt{\varepsilon} \) and \( q' \preceq eqe \) as in Definition 0.2.

Note that we have
\[
\|b^2 - b\| \leq \|b^2 - (eqe)^2\| + \|(eqe)^2 - eqe\| + \|eqe - b\|
\leq 3\varepsilon + \varepsilon + \varepsilon
= 5\varepsilon.
\]

By Proposition 2.17 again, there exists a projection \( p \in eA_e \) such that
\[ \|p - b\| \leq \sqrt{5\varepsilon} \text{ and } |p| \leq |b|. \]
As
\[ \|p - q'\| \leq \|p - b\| + \|b - eqe\| + \|eqe - q'\| \leq \sqrt{5\varepsilon} + \varepsilon + \sqrt{\varepsilon}, \]
it follows that \( |p| = |q'| \). As \( q' \preceq eqe \) and \( eqe \preceq q \),
we conclude that \( p \preceq q \) in \( A \).

Note that
\[ q = eqe + (1 - e)qe + eq(1 - e) + (1 - e)q(1 - e). \]
For every \( \tau \in T(A) \), we have
\[ \tau(q) = \tau(eq) + \tau((1 - e)q(1 - e)) + \tau((1 - e)qe + eq(1 - e)). \]
According to (C1) and our choice of \( \varepsilon \), we have
\[ \tau(eq) + \tau((1 - e)q(1 - e)) > \tau(q) - \varepsilon > \frac{1}{2} \tau(q). \]
As \( \tau \) is a tracial state and \( e \) is a projection,
\[ \tau((1 - e)q(1 - e)) \leq \tau((1 - e)1(1 - e)) = \tau(1 - e). \]
Note that \( \tau(1 - e) < \frac{1}{4} \tau(q) \) for all \( \tau \in T(A) \) (because \( \tau(1 - e) < \frac{1}{4} \delta_0 \)). We can conclude that
\[ \tau(eq) > \frac{1}{2} \tau(q) - \tau((1 - e)q(1 - e)) \geq \frac{1}{2} \tau(q) - \tau(1 - e) > \frac{1}{4} \tau(q) \geq \frac{1}{4} \delta_0 > 0. \]

In our construction, note that
\[ \|p - eqe\| \leq \|p - b\| + \|b - eqe\| \leq \sqrt{5\varepsilon} + \varepsilon. \]
It follows that
\[ \tau(p) \geq \frac{1}{4} \delta_0 - (\sqrt{5\varepsilon} + \varepsilon) \geq \frac{1}{8} \delta_0 \text{ for all } \tau \in T(A). \]
According to our construction, we have
\[ \tau(1 - e) < M \cdot \varepsilon = \varepsilon \leq \frac{1}{8} \delta_0 \leq \tau(p) \]
for all \( \tau \in T(A) \), which then implies that \( 1 - e \preceq p \). As \( |p| \leq |c| \) (as in Definition 0.2), we get \( [1 - e] \leq |c| \) (as in Definition 0.2), which finishes the proof. \( \square \)
The following result on the $K$-theory of the crossed product C*-algebra above follows from Pimsner-Voiculescu six-term exact sequence.

**Proposition 2.19.** Let $A$ be the crossed product C*-algebra of the minimal dynamical system $(X \times \mathbb{T} \times T, \alpha \times R_\xi \times R_n)$. Then

$$K_0(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2$$

and

$$K_1(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2.$$

**Proof.** Use $j : C(X \times \mathbb{T}^2) \to A$ to denote the canonical embedding of $C(X \times \mathbb{T}^2)$ into $A$. We have the Pimsner-Voiculescu six-term exact sequence:

$$
\begin{array}{cccc}
K_0(C(X \times \mathbb{T}^2)) & \overset{id_{\alpha_0} - \alpha_0}{\longrightarrow} & K_0(C(X \times \mathbb{T}^2)) & \overset{j \circ 0}{\longrightarrow} & K_0(A) \\
& & j & & \\
K_1(A) & \overset{j_1}{\longrightarrow} & K_1(C(X \times \mathbb{T}^2)) & \overset{id_{\alpha_1} - \alpha_1}{\longrightarrow} & K_1(C(X \times \mathbb{T}^2)).
\end{array}
$$

We know that

$$K_0(C(T^2)) \cong \mathbb{Z}^2, \quad K_1(C(T^2)) \cong \mathbb{Z}^2$$

and

$$K_0(C(X)) \cong C(X, \mathbb{Z}), \quad K_1(C(X)) = 0.$$ 

According to the Künneth theorem, $K_0(C(X \times \mathbb{T}^2)) \cong C(X, \mathbb{Z}^2)$ and $K_1(C(X \times \mathbb{T}^2)) \cong C(X, \mathbb{Z}^2)$. For $i = 0, 1$, consider the image of $id_{\alpha_i} - \alpha_{\alpha_i}$. They are both isomorphic to

$$\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2)\}.$$

The kernel of $id_{\alpha_i} - \alpha_{\alpha_i}$ for $i = 0, 1$ is

$$\{f \in C(X, \mathbb{Z}^2) : f = f \circ \alpha\}.$$

Assume that $f$ is in the kernel of $id_{\alpha_i} - \alpha_{\alpha_i}$ for $i = 0, 1$. Fix $x_0 \in X$. We have $f(\alpha^n(x_0)) = f(x_0)$ for all $n \in \mathbb{Z}$. As $\alpha$ is a minimal homeomorphism of the Cantor set $X$ and $f$ is continuous, $f$ must be a constant function from $X$ to $\mathbb{Z}^2$. Now we conclude that

$$\ker(id_{\alpha_i} - \alpha_{\alpha_i}) \cong \mathbb{Z}^2.$$

As the six-term sequence above is exact, we have the short exact sequence:

$$0 \longrightarrow \text{coker}(id_{\alpha_0} - \alpha_{\alpha_0}) \longrightarrow K_0(A) \longrightarrow \ker(id_{\alpha_1} - \alpha_{\alpha_1}) \longrightarrow 0.$$ 

As $\ker(id_{\alpha_i} - \alpha_{\alpha_i}) \cong \mathbb{Z}^2$ and $\mathbb{Z}^2$ is projective, it follows that

$$K_0(A) \cong \text{coker}(id_{\alpha_0} - \alpha_{\alpha_0}) \oplus \mathbb{Z}^2.$$

As $\text{coker}(id_{\alpha_0} - \alpha_{\alpha_0}) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha : f \in C(X, \mathbb{Z}^2)\}$, we get

$$K_0(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha : f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2.$$ 

Similarly, we get that $K_1(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha : f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2.$ \qed
If we require a certain “rigidity” condition on the dynamical system \((X \times T \times T, \alpha \times R_\xi \times R_\eta)\), then the tracial rank of the crossed product will be zero.

**Definition 2.20.** Let \((X \times T \times T, \alpha \times R_\xi \times R_\eta)\) be a minimal dynamical system. Let \(\mu\) be an \(\alpha \times R_\xi \times R_\eta\)-invariant probability measure on \(X \times T \times T\). It will induce an \(\alpha\)-invariant probability measure on \(X\) defined by \(\pi(u)(D) = \mu(D \times T \times T)\) for every Borel set \(D \subset X\). We say that \((X \times T \times T, \alpha \times R_\xi \times R_\eta)\) is rigid if \(\pi\) gives a one-to-one map between the \(\alpha \times R_\xi \times R_\eta\)-invariant probability measures and the \(\alpha\)-invariant probability measures.

**Remark:** For minimal actions on \(X \times T \times T\) of the type \(\alpha \times R_\xi \times R_\eta\), it is easy to see that \(\pi\) always maps the set of \(\alpha \times R_\xi \times R_\eta\)-invariant probability measures over \(X \times T \times T\) onto the set of \(\alpha\)-invariant measures over \(X\).

According to Theorem 4.6 in [Lin-Phillips], the “rigidity” condition defined above implies that the crossed product C*-algebra has tracial rank zero.

**Proposition 2.21.** Let \((X \times T \times T, \alpha \times R_\xi \times R_\eta)\) be a minimal dynamical system. If it is rigid, then the corresponding crossed product C*-algebra \(C^*(Z, X \times T \times T, \alpha \times R_\xi \times R_\eta)\) has tracial rank zero.

**Proof.** Use \(A\) to denote \(C^*(Z, X \times T \times T, \alpha \times R_\xi \times R_\eta)\). We will show that

\[
\rho: K_0(A) \to \text{Aff}(T(A))
\]

has a dense range, which will then imply that \(\text{TR}(A) = 0\) according to [Lin-Phillips] Theorem 4.6.

For the crossed product C*-algebra \(B = C^*(Z, X, \alpha)\), we know that \(B\) has tracial rank zero and \(\rho_B: K_0(B) \to T(B)\) has the dense range. According to [Putnam] Theorem 1.1, \(K_0(A) \cong C(X, Z)/\{f - f \circ \alpha^{-1}\}\). For every \(x \in K_0(A)\), we can find \(f \in C(X, Z)\) such that \(\hat{\tau}(\tau) := \tau(x)\) equals \(\tau(f) = \int_X f \, d\mu_x\).

As \(\alpha \times R_\xi \times R_\eta\) is rigid, there is a one-to-one correspondence between \((\alpha \times R_\xi \times R_\eta)\)-invariant measures and \(\alpha\)-invariant measures. In other words, \(T(A)\) is homeomorphic to \(T(B)\) (as two convex compact sets). Let \(h \in C(X)\) be a projection. Then \(h \otimes 1_{C(T \times T)}\) is a projection in \(A\).

As \(\rho_B\) has a dense range in \(\text{Aff}(T(B))\), we have that \(\rho\) has dense range in \(\text{Aff}(T(A))\). As \(X \times T \times T\) is an infinite finite dimensional metric space and \(\alpha \times R_\xi \times R_\eta\) is minimal, according to [Lin-Phillips] Theorem 4.6, \(C^*(Z, X \times T \times T, \alpha \times R_\xi \times R_\eta)\) has tracial rank zero. \(\square\)

3. Examples

This section contains examples of minimal dynamical systems of type \((X \times T \times T, \alpha \times R_\xi \times R_\eta)\) that is rigid. It also contains a concrete example of a minimal dynamical system of the same type but is not rigid.

We start with a criterion for determining whether a dynamical system of \((X \times T \times T, \alpha \times R_\xi \times R_\eta)\) is minimal or not. This result is a special case of the remark of page 582 in [Furstenberg]. The proof here essentially follows that of Lemma 4.2 of [LMT].
Lemma 3.1. Let $Y$ be a compact metric space, and let $\beta \times R_\eta$ be a skew product homeomorphism of $Y \times T$ with $\beta \in \text{Homeo}(Y)$, $\eta : Y \to T$ and

\[ (\beta \times R_\eta)(y, t) = (\beta(y), t + \eta(y)) \]

with $T$ identified with $\mathbb{R}/\mathbb{Z}$.

Then $\beta \times R_\eta$ is minimal if and only if $(Y, \beta)$ is minimal and there exist no $f \in C(Y, T)$ and non-zero integer $n$ such that

\[ n\eta = f \circ \beta - f. \]

Proof. Proof of the “if” part:

If $(Y, \beta)$ is minimal and there exist no $f \in C(Y, T)$ and non-zero integer $n$ such that $n\eta = f \circ \beta - f$, we will prove that $\beta \times R_\eta$ is minimal.

If $\beta \times R_\eta$ is not minimal, then there exists a proper minimal subset $E$ of $Y \times T$. Let $\pi_Y : Y \times T \to Y$ be the canonical projection onto $Y$. Note that $\pi_Y \circ (\beta \times R_\eta) = \beta \circ \pi_Y$. It follows that $\pi_Y(E)$ is an invariant subset of $Y$. As $Y$ is compact, so is $\pi_Y(E)$. Since $(Y, \beta)$ is minimal, the closed invariant set $\pi_Y(E)$ must be $Y$.

Let’s consider

\[ D := \{ t \in T : (id_Y \times R_t)(E) = E \}. \]

As $(id_Y \times id_T)(E) = E$, the set $D$ is not empty. Note that $D$ is a subgroup of $T$. It follows that $D$ is a non-empty subgroup of $T$ (with $T$ identified with the quotient group $\mathbb{R}/\mathbb{Z}$).

If we have $\{t_n\}_{n \in \mathbb{N}} \subset D$ such that $t_n \to t$, then for any $\omega \in E$, we have $(id \times R_{t_n})\omega \in E$. Then $t_n \to t$ implies that $(id \times R_{t_n})w \to (id \times R_t)w$. As $E$ is closed, $(id \times R_t)w \in E$.

So far, we have shown that if $t_n \in D$ for $n \in \mathbb{N}$ and $t_n \to t$, then $t \in D$. Note that $\{t_n\}_{n \in \mathbb{N}} \subset D$ and $t_n \to t$ is equivalent to $\{-t_n\}_{n \in \mathbb{N}} \subset D$ and $-t_n \to -t$. It follows that $-t \in D$. In other words, we have

\[ (id \times R_t)(E) \subset E \quad \text{and} \quad (id \times R_{-t})(E) \subset E. \]

Then we get

\[ E = (id \times R_t)((id \times R_{-t})(E)) \subset (id \times R_t)(E) \subset E, \]

which implies that $(id \times R_t)E = E$. In other words, $D$ is closed.

As $E$ is a proper subset and $\pi_Y(E) = Y$, $D$ must be a proper subgroup of $T$. Otherwise, for any $(y, t) \in Y \times T$, as $\pi_Y(E) = Y$, there exists $t' \in T$ such that $(y, t') \in E$. Since $t - t' \in D = T$, $(y, t + (id \times R_{-t'})(y, t')) \in E$, which indicates that $E = Y \times T$, contradicting the fact that $E$ is a proper subset.

As a proper closed subgroup of $T$, $D$ must be

\[ \left\{ \frac{k}{n} \right\}_{0 \leq k \leq n - 1} \quad \text{with} \quad n = |D|. \]

Let $\pi_T$ be the canonical projection from $Y \times T$ onto $T$. For $y \in Y$, use $E_y$ to denote $\pi_T(E \cap \pi_Y^{-1}\{\{y\}\})$.

Using the fact that $E$ is a minimal subset of $(\beta, R_\eta)$, we will show that $E_y$ must be $n$ points distributed evenly on the circle for all $y \in Y$.

We claim that if $t, t' \in E_y$, then for any $m \in \mathbb{Z}$, $t + m(t' - t)$ must be in $E_y$. To prove this claim, if $t, t' \in E_y$, then there exists $\{k_n\}_{n \in \mathbb{N}}$ such that $k_n \to \infty$ and $\text{dist}((\beta \times R_\eta)^{k_n}(y, t), (y, t')) \to 0$. Note that

\[ \text{dist}((\beta \times R_\eta)^{k_n}(y, t), (y, t')) = \text{dist}((\beta \times R_\eta)^{k_n}(y, t), (y, t + 2(t' - t))). \]

It follows that $(y, t + 2(t' - t)) \in \text{Orbit}_{\beta \times R_\eta}(y, t)$. By induction, we conclude that if $t, t' \in E_y$, then for any $m \in \mathbb{Z}$, $t + m(t' - t)$ is also in $E_y$, proving the claim.
For any $y \in Y$, consider $E_y$, which is a non-empty closed subset of $T$. Let

$$l_y = \inf_{t_1, t_2 \in E_y} \text{dist}(t_1, t_2).$$

Note that if $t, t' \in E_y$, then $t + m(t' - t) \in E_y$. The fact that $E_y \subseteq T$ implies that $l_y > 0$. It is then clear that $E_y$ is made up of $1/l_y$ points distributed evenly on $T$.

**Claim:** For every $y \in Y$, $1/l_y = |D|$. For given $y \in Y$, as $(\text{id} \times R_t)(E) = E$ for all $t \in D$, we get that $E_y$ is invariant under $R_t$ for all $t \in D$. It then follows that $1/l_y = kn$ with $k \in \mathbb{N}$ and $n = |D|$.

If $k > 1$, write

$$E_y = \{(y, t_1), \ldots, (y, tkn)\}.$$ 

Use $\text{Orbit}_{\beta \times R_n}(E_y)$ to denote $\bigcup_{m=1}^{\infty} (\beta \times R_n)^m(E_y)$.

As $\beta$ is minimal, for every $y' \in Y$, there is a sequence $(m_k)_{k \in \mathbb{N}}$ such that

$$\beta^{m_k}(y) \to y'.$$

The fact that $\text{Orbit}_{\beta \times R_n}(E_y)$ is dense implies that there exists $t' \in T$ such that $(y', t')$ is in the closure of $\text{Orbit}_{\beta \times R_n}(E_y)$. Note that for every $m \in \mathbb{N}$, $(\beta \times R_n)^m(E_y)$ consists of $kn$ points distributed evenly on the circle. It follows that $E_y$ contains at least $nk$ points distributed evenly on the circle.

Now we have shown that for every $a \in Y$, $E_a$ is made up of at least $nk$ evenly distributed points on the circle, which then implies that $D$ contain at least $nk$ elements. The assumption that $k > 1$ gives a contradiction.

We then conclude that $k = 1$, which proves the claim.

By the claim above, for all $y \in Y$, the set $E_y$ is made up of $n$ points distributed evenly on $T$. If we define

$$nE = \{(x, nt): (x, t) \in E\},$$

then $nE$ is the graph of some continuous map $g: Y \to T$. As $E$ is closed, so is $nE$, which implies that $g$ is continuous. As $E$ is $(\beta \times R_n)$-invariant, for every $(x, t) \in E$, it follows that

$$(\beta \times R_n)(x, t) = (\beta(x), t + n(x)) \in E.$$ 

In other words, we have $n(t + \eta(x)) = g(\beta(x))$. As $nt = g(x)$, it follows that $n\eta = g \circ \beta - g$, which finishes the proof of “if” part.

Proof of the “only if” part:

Suppose $\beta \times R_\eta$ is minimal. Then it is clear that $(Y, \beta)$ is a minimal system.

Suppose that there exists nonzero $n \in \mathbb{Z}$ such that $n\eta = g \circ \beta - g$ for some $g \in C(X, T)$. Let

$$E = \{(y, t) \in Y \times T: nt = g(y)\}.$$ 

For $(y, t) \in E$, we have $(\beta \times R_\eta)(y, t) = (\beta(y), t + \eta(y))$. As

$$n(t + \eta(y)) = nt + n\eta(y) = g(y) + n\eta(y) = g(\beta(y)),$$

it follows that $E$ is $(\beta \times R_\eta)$-invariant.

As $g$ is continuous, $E$ is closed. And it is clear that $E$ is a proper subset of $Y \times T$. Now we have a proper closed $(\beta \times R_\eta)$-invariant set in $Y \times T$, contradicting the minimality of $\beta \times R_\eta$.  

Lemma 3.1 provides an inductive approach to determine the minimality of some dynamical systems. Following this lemma, we get the proposition below.

**Proposition 3.2.** Let $\alpha \times R_\xi \times R_\eta$ be a homeomorphism of $X \times T \times T$. Then $\alpha \times R_\xi \times R_\eta$ is minimal if and only if

i) $(X, \alpha)$ is minimal,

ii) $\xi$ is not a torsion element in $C(X, \mathbb{T})/\{f \circ \alpha - f\}$,

iii) For $\tilde{\eta} \in C(X \times T, \mathbb{T})$ defined by $\tilde{\eta}(x, t) = \eta(x)$, the map $\tilde{\eta}$ is not a torsion element in $C(X \times T, \mathbb{T})/\{f \circ (\alpha \times R_\xi) - f : f \in C(X \times T, \mathbb{T})\}$.

**Proof.** The “if” part:

Note that $(X \times T \times T, \alpha \times R_\xi \times R_\eta)$ is a skew product of $(X \times T, \alpha \times R_\xi)$ and $\tilde{\eta}$, where $\tilde{\eta}$ is defined by

$$R_\eta \colon X \times T \to \text{Homeo}(\mathbb{T}), \text{ with } (\tilde{\eta}(x, t))(t') = t' + \eta(x).$$

From i) and ii), using Lemma 4.2 of [LM1], $(X \times T, \alpha \times R_\xi)$ is minimal. According to Lemma 3.1 and by iii), we conclude that $\alpha \times R_\xi \times R_\eta$ is minimal.

The “only if” part:

As $(X \times T \times T, \alpha \times R_\xi \times R_\eta)$ is the skew product of $(X \times T, \alpha \times R_\xi)$ and $\tilde{\eta}$, with $\tilde{\eta}$ defined as above, the minimality of $(X \times T \times T, \alpha \times R_\xi \times R_\eta)$ implies the minimality of $(X \times T, \alpha \times R_\xi)$. By Lemma 4.2 of [LM1], that implies (i) and (ii).

For (iii), suppose that $\tilde{\eta}$ is a torsion element, that is, there is non-zero $n \in \mathbb{Z}$ and $f \in C(X \times T, \mathbb{T})$ such that $n\tilde{\eta} = f \circ (\alpha \times R_\xi) - f$. By Lemma 3.1 it follows that $(X \times T \times T, \alpha \times R_\xi \times R_\eta)$ is not minimal, a contradiction. \hfill \qedsymbol

Proposition 3.2 enables us to construct minimal dynamical systems on $X \times T \times T$ inductively. In fact, we have the following lemma.

**Lemma 3.3.** Given any minimal dynamical system $(X \times T, \alpha \times R_\xi)$, there exist uncountably many $\theta \in [0, 1]$ such that if we use $\theta$ to denote the constant function in $C(X, \mathbb{T})$ defined by $\theta(x) = \theta$ for all $x \in X$ (identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$), then the dynamical system $(X \times T \times T, \alpha \times R_\xi \times R_\theta)$ is still minimal.

**Proof.** Note that the dynamical system $(X \times T, \alpha \times R_\xi)$ is minimal. According to Lemma 3.1 $(X, \alpha)$ must be a minimal dynamical system, and $\xi$ is not a torsion element in

$$C(X, \mathbb{T})/\{f - f \circ \alpha : f \in C(X, \mathbb{T})\}.$$

This implies that conditions i) and ii) in Proposition 3.2 are already satisfied.

According to Proposition 3.2, for $(X \times T \times T, \alpha \times R_\xi \times R_\theta)$ to be minimal, we just need to find $\theta \in \mathbb{R}$ such that for every $n \in \mathbb{Z} \setminus \{0\}$ and $f \in C(X \times T, \mathbb{T})$, we have

$$n\theta \neq f - f \circ (\alpha \times R_\xi).$$

If this is not true, then we have

$$n\theta = f - f \circ (\alpha \times R_\xi).$$
Let $F : X \times \mathbb{T} \to \mathbb{R}$ be a lifting of $f$. That is, $F \in C(X \times \mathbb{T}, \mathbb{R})$ and the following diagram commutes:

$$
\begin{array}{ccc}
X \times \mathbb{T} & \xrightarrow{F} & \mathbb{R} \\
\downarrow{f} & & \downarrow{\pi} \\
\mathbb{T} & & \\
\end{array}
$$

with $\pi(t) = t$ for all $t \in \mathbb{R}$ (identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$).

Using $[F]$ to denote $\pi \circ F$, it follows that

$$
n\theta = [F] - [F \circ (\alpha \times \rho)]
= [F - F \circ (\alpha \times \rho)].
$$

In other words, there exists $g \in C(X \times \mathbb{T}, \mathbb{Z})$ such that

$$
n\theta - (F - F \circ (\alpha \times \rho)) = g.
$$

For every $(\alpha \times \rho)$-invariant probability measure $\mu$, we have $\mu(n\theta) = \mu(g)$, with $\mu(n\theta) = \int_{X \times \mathbb{T}} n\theta \, d\mu$
and $\mu(g) = \int_{X \times \mathbb{T}} g \, d\mu$.

Since $\mu(n\theta) = n\mu(\theta)$, it follows that

$$
\mu(\theta) = \frac{\mu(g)}{n} = \mu\left(\frac{g}{n}\right).
$$

Let $A$ be the crossed product C*-algebra of $(X \times \mathbb{T}, \alpha \times \rho)$. Define

$$
\rho : A_{sa} \rightarrow \text{Aff}(T(A))
$$

by $\rho(a)(\tau) = \tau(a)$ for all $a \in A_{sa}$ and $\tau \in T(A)$. Then we have $\rho(\theta) = \rho\left(\frac{g}{n}\right)$ in $\text{Aff}(T(A))$.

Now we have shown that if $\theta$ (as a constant function) is a torsion element in

$$
C(X \times \mathbb{T}, \mathbb{T})/\{f - f \circ \alpha : f \in C(X \times \mathbb{T}, \mathbb{T})\}
$$

with order $n$, then there exists $g \in C(X \times \mathbb{T}, \mathbb{Z})$ such that $\rho(\theta) = \rho\left(\frac{g}{n}\right)$.

As $\mathbb{T}$ is connected, we have $C(X \times \mathbb{T}, \mathbb{Z}) \cong C(X, \mathbb{Z})$. Note that the set

$$
\left\{\frac{g}{n} : g \in C(X \times \mathbb{T}, \mathbb{Z}) \cong C(X, \mathbb{Z}), n \in \mathbb{Z} \setminus \{0\}\right\}
$$

contains countably many elements. It follows that its image under $\rho$ contains at most countably many elements. The fact that $[0, 1]$ contains uncountably many elements and $\rho(\theta) = 0$ if and only if $\theta = 0$ implies that there exists (uncountably many, in fact) $\theta \in \mathbb{R}$ such that $\theta$ (as a constant function) is not a torsion element in

$$
C(X \times \mathbb{T}, \mathbb{T})/\{f - f \circ \alpha : f \in C(X \times \mathbb{T}, \mathbb{T})\},
$$

which then implies that $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\theta})$ is still minimal. \qed

We now give examples of rigid and non-rigid minimal actions of on $X \times \mathbb{T} \times \mathbb{T}$.

Let $\varphi_0 : \mathbb{T} \to \mathbb{T}$ be a Denjoy homeomorphism (see [PSS] Definition 3.3 or [KatokHasseblatt] Prop 12.2.1) with rotation number $r(\gamma) = \theta$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. It is known that $\varphi_0$ has a unique proper invariant closed subset of $\mathbb{T}$, which is a Cantor set, and that $\varphi_0$ restricted on this Cantor set is minimal. Let $X$ be the Cantor set and use $\varphi : X \to X$ to denote the restriction of $\varphi_0$ to $X$. 

According to the Poincare Classification Theorem (see KatokHasseblatt Theorem 11.2.7), there is a non-invertible continuous monotonic map \( h: T \to T \) such that the following diagram commutes:

![Diagram](image)

Using the restriction of \( \varphi \) to the invariant subset (which is the Cantor set \( X \)), we get a commutative diagram:

![Diagram](image)

It is known that for a Denjoy homeomorphism, \( h|_X \) maps \( X \) onto \( T \).

Recall that for \( \xi, \eta: T \to T \), the action

\[
\gamma: (s, t_1, t_2) \mapsto (s + \theta, t_1 + \xi(s), t_2 + \eta(s))
\]

is called a Furstenberg transformation. Consider the action

\[
\alpha \times R_{\xi h} \times R_{\eta h}: X \times T \times T \to X \times T \times T.
\]

It is clear that we have the commutative diagram below:

![Diagram](image)

In this case, if \( \gamma \) is minimal, then \( \alpha \times R_{\xi h} \times R_{\eta h} \) is also minimal, as will be shown in the next proposition.

**Proposition 3.4.** For the minimal dynamical systems as in diagram (1), if \( (T \times T \times T, \gamma) \) is a minimal dynamical system, then \( (X \times T \times T, \alpha \times R_{\xi h} \times R_{\eta h}) \) is also a minimal dynamical system.

**Proof.** Assume that \( (T \times T \times T, \gamma) \) is minimal and \( (X \times T \times T, \alpha \times R_{\xi h} \times R_{\eta h}) \) is not minimal. It then follows that there exist \( (x, t_1, t_2) \in X \times T \times T \), nonempty open subset \( D \subseteq X \) and open subsets \( U, V \subseteq T \) such that

\[
\{ (\alpha \times R_{\xi h} \times R_{\eta h})^n(x, t_1, t_2) \}_{n \in \mathbb{N}} \cap (D \times U \times V) = \emptyset.
\]

Define

\[
\pi_1, \pi_2: X \times T \times T \to T \times T
\]

by

\[
\pi_1(x, t_1, t_2) = t_1 \quad \text{and} \quad \pi_2(x, t_1, t_2) = t_2.
\]

As \( \alpha \) is a minimal action on the Cantor set \( X \), the statement implies that for every \( k \in \mathbb{N} \) such that \( \alpha^k(x) \in D \), we have

\[
\pi_1((\alpha \times R_{\xi h} \times R_{\eta h})^k(x)) \notin U \quad \text{and} \quad \pi_2((\alpha \times R_{\xi h} \times R_{\eta h})^k(x)) \notin V.
\]
Note that if we regard the Cantor set $X$ as a subset of $\mathbb{T}$, then $h|_X : X \to \mathbb{T}$ is a noninvertible continuous monotone function. For the open set $D \subset X$, without loss of generality, we can assume that (by identifying $X$ as a subset of $\mathbb{T}$ and identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$)
$$D = (a, b) \cap X \text{ with } a, b \in (0, 1) \text{ and } a < b.$$ It then follows that there exists $c, d \in (0, 1)$ with $c < d$ (without loss of generality, we can assume that $0 \notin h|_X (D)$ such that $h|_X (D)$ is one of the following:

\begin{itemize}
  \item $(c, d)$,
  \item $(c, d]$, 
  \item $[c, d)$ or 
  \item $[c, d]$.
\end{itemize}

In either case, there exists $c', d' \in (0, 1)$ with $c' < d'$ such that $(c', d') \subset h|_X (D)$.

Let $t_x = h|_X (x)$. It is then clear that
$$h|_X ((\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}})^k(x, t_1, t_2)) = \gamma^n(t_x, t_1, t_2)$$ for all $n \in \mathbb{N}$. As $h|_X (D)$ is monotone, for every $k \in \mathbb{N}$, if $R^k_\alpha (t_x) \in (c', d')$, then we have $\alpha^k(x) \in D$, which implies (see \ref{3}) that
$$\pi_1 ((\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}})^k(x, t_1, t_2)) \notin U \text{ and } \pi_2 ((\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}})^k(x, t_1, t_2)) \notin V.$$ Define
$$\rho_1, \rho_2 : T \times T \times T \to T \times T$$ by $\rho_1(t_0, t_1, t_2) = t_1$ and $\rho_2(t_0, t_1, t_2) = t_2$. It is easy to check that for all $n \in \mathbb{N}$, we have
$$\pi_1 ((\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}})^k(x, t_1, t_2)) = \rho_1 (\gamma^k(t_x, t_1, t_2)).$$ Then we have that for every $k \in \mathbb{N}$ such that $R^k_\alpha (t_x) \in (c', d')$,
$$\rho_1 (\gamma^k(t_x, t_1, t_2)) \notin U \text{ and } \rho_2 (\gamma^k(t_x, t_1, t_2)) \notin V.$$ According to the definition of the Furstenberg transformation $\gamma$, it follows that
$$\{\gamma^n(t_x, t_1, t_2)\}_{n \in \mathbb{N}} \cap (c', d') \times U \times V = \emptyset,$$ contradicting the minimality of $\gamma$, which finishes the proof. \hfill $\square$

The proposition below shows that if the two dynamical systems in Prop \ref{3.4} are minimal, then there is a one-to-one correspondence between the invariant measures on them.

\begin{proposition}
If the dynamical systems $(\mathbb{T} \times \mathbb{T} \times \mathbb{T}, \gamma)$ and $(X \times T \times T, \alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}})$ (as in diagram \textbf{2}) are minimal, then there is a one-to-one correspondence between the $\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}}$-invariant probability measures and the $\gamma$-invariant probability measures.
\end{proposition}

\begin{proof}
First of all, we will define the correspondence between the $\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}}$-invariant probability measures and the $\gamma$-invariant probability measures.

For simplicity, we use $H$ to denote the function $h|_X$ in diagram \textbf{1}. We use $M_{\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}}}$ to denote the set of $\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}}$-invariant probability measures on $X \times T \times T$ and $M_\gamma$ to denote the set of $\gamma$-invariant probability measures on $T \times T \times T$.

Define \[ \varphi : M_{\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}}} \to M_\gamma \] and \[ \psi : M_\gamma \to M_{\alpha \times R_{\xi_{coh}} \times R_{\eta_{coh}}} \] by
$$\varphi(\mu)(D) = \mu (|(H \times \text{id}_T \times \text{id}_T)|^{-1}(D)) \text{ and } \psi(\nu)(E) = \nu ((H \times \text{id}_T \times \text{id}_T)(E))$$
for all Borel subsets $D$ of $\mathbb{T} \times \mathbb{T} \times \mathbb{T}$, Borel subsets $E$ of $X \times \mathbb{T} \times \mathbb{T}$, $\mu \in M_\alpha \times R_{\xi \mathbb{T}} \times R_{\eta \mathbb{T}}$ and $\nu \in M_\gamma$.

We need to show that the $\varphi$ and $\psi$ above are well-defined.

As every $\mu \in M_\alpha \times R_{\xi \mathbb{T}} \times R_{\eta \mathbb{T}}$ is a probability measure, it follows that $\varphi(\mu)(\mathbb{T} \times \mathbb{T} \times \mathbb{T}) = 1$.

For every Borel subset $D \subset \mathbb{T} \times \mathbb{T} \times \mathbb{T}$, as both $\alpha \times R_{\xi \mathbb{T}} \times R_{\eta \mathbb{T}}$ and $\gamma$ are homeomorphisms, it follows that

$$(H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(\gamma(D)) = (\alpha \times R_{\xi \mathbb{T}} \times R_{\eta \mathbb{T}})((H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(D)),$$

which implies that $\varphi(\mu)$ is $\gamma$-invariant.

For a sequence of Borel subsets $D_1, D_2, \ldots$ of $\mathbb{T} \times \mathbb{T} \times \mathbb{T}$ such that $D_i \cap D_j = \emptyset$ if $i \neq j$, it is clear that $(H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})^{-1}(D_1), (H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})^{-1}(D_2), \ldots$ are Borel subsets of $X \times \mathbb{T} \times \mathbb{T}$ (as $H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T}$ is continuous) satisfying $(H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})^{-1}(D_i) \cap (H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})^{-1}(D_j) = \emptyset$ if $i \neq j$. Then we have that

$$\varphi(\mu)\left(\bigcup_{n=1}^\infty D_n\right) = \sum_{n=1}^\infty \varphi(\mu)(D_n).$$

So far, we have shown that $\varphi$ is a well-defined map from $M_\alpha \times R_{\xi \mathbb{T}} \times R_{\eta \mathbb{T}}$ to $M_\gamma$.

Now we will check the map $\psi$.

As every $\nu \in M_\gamma$ is a probability measure, it follows that

$$\psi(\nu)(X \times \mathbb{T} \times \mathbb{T}) = \nu(\mathbb{T} \times \mathbb{T} \times \mathbb{T}) = 1.$$

For every Borel subset $E \subset X \times \mathbb{T} \times \mathbb{T}$, we will show that $\psi(\nu)(E)$ is well-defined. According to the definition of $\psi(\nu)$, we just need to show that $(H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(E)$ is $\nu$-measurable.

For any two open subsets $S_1$ and $S_2$ of $X \times \mathbb{T} \times \mathbb{T}$, we have

$$(H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(S_1 \cup S_2) = (H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(S_1) \cup (H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(S_2),$$

$$(H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(S_i^c) = ((H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(S_i))^c \text{ for } i = 1, 2.$$
is of measure zero for all \( \gamma \)-invariant measure \( \nu \). Note that \( F' \) is a Borel set. For every \( \gamma \)-invariant measure \( \nu \), \( F' \) is both \( \nu \)-measurable. It then follows that \((H \times \text{id}_T \times \text{id}_T)(F)\) is measurable. Recall that

\[
\psi(\nu)(F) = \nu((H \times \text{id}_T \times \text{id}_T)(F)).
\]

It follows that for \( \psi(\nu) \) is well-defined on all the Borel subsets of \( X \times T \times T \).

For a sequence of Borel subsets \( E_1, E_2, \ldots \) of \( X \times T \times T \) such that \( D_i \cap D_j = \emptyset \) if \( i \neq j \), and for every \( \gamma \)-invariant probability measure \( \nu \), we will show that

\[
\psi(\nu) \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \psi(\nu)(E_n).
\]

According to the definition, we have

\[
\psi(\nu) \left( \bigcup_{n=1}^{\infty} E_n \right) = \nu \left( (H \times \text{id}_T \times \text{id}_T) \left( \bigcup_{n=1}^{\infty} E_n \right) \right)
\]

Note that

\[
(H \times \text{id}_T \times \text{id}_T) \left( \bigcup_{n=1}^{\infty} E_n \right) = \bigcup_{n=1}^{\infty} (H \times \text{id}_T \times \text{id}_T)(E_n)
\]

and

\[
(H \times \text{id}_T \times \text{id}_T)(E_i) \cap (H \times \text{id}_T \times \text{id}_T)(E_j) \subset H(X_0) \times T \times T \text{ for } i \neq j.
\]

Recall that \( H(X_0) \times T \times T \) is a set of measure zero for every \( \gamma \)-invariant probability measure. It follows that

\[
\psi(\nu) \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \psi(\nu)(E_n).
\]

For every Borel subset \( E \subset X \times T \times T \), according to the commutative diagram \( \blacksquare \), we have

\[
(\gamma \circ (H \times \text{id}_T \times \text{id}_T))E = ((H \times \text{id}_T \times \text{id}_T) \circ (\alpha \times R_{\xi_{0h}} \times R_{\eta_{0h}}))(E).
\]

It then follows that

\[
\psi(\nu)(E) = \nu((H \times \text{id}_T \times \text{id}_T)E)
\]

\[
= \nu(\gamma ((H \times \text{id}_T \times \text{id}_T)E))
\]

\[
= \nu ((H \times \text{id}_T \times \text{id}_T)((\alpha \times R_{\xi_{0h}} \times R_{\eta_{0h}})E))
\]

\[
= \psi(\nu)((\alpha \times R_{\xi_{0h}} \times R_{\eta_{0h}})E),
\]

which implies that \( \psi(\nu) \) is \( \alpha \times R_{\xi_{0h}} \times R_{\eta_{0h}} \)-invariant.

So far, we have shown that \( \psi \) is a well-defined map from \( M_{\gamma} \) to \( M_{\alpha \times R_{\xi_{0h}} \times R_{\eta_{0h}}} \).

Now we will show that for every \( \alpha \times R_{\xi_{0h}} \times R_{\eta_{0h}} \)-invariant measure \( \mu \) and \( \gamma \)-invariant measure \( \nu \), we have

\[
(\varphi \circ \psi)(\nu) = \nu \text{ and } (\psi \circ \varphi)(\mu) = \mu.
\]

In fact, we just need to show that for every Borel subset \( D \) of \( T \times T \times T \) and every Borel subset \( E \) of \( X \times T \times T \),

\[
\nu((H \times \text{id}_T \times \text{id}_T)((H \times \text{id}_T \times \text{id}_T)^{-1}(D)) \triangle D) = 0
\]

and

\[
\mu((H \times \text{id}_T \times \text{id}_T)^{-1}((H \times \text{id}_T \times \text{id}_T)(E)) \triangle E) = 0.
\]
As
\[(H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})((H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})^{-1}(D)) = D,\]
the equation holds.

Note that
\[((H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})^{-1}((H \times \text{id}_\mathbb{T} \times \text{id}_\mathbb{T})(E)) \triangle E) \subset X_0 \times \mathbb{T} \times \mathbb{T}.

The fact that \(X_0\) consists of countably many points and the minimal action \(\alpha \times R_{\xi_{\text{gh}}} \times R_{\eta_{\text{gh}}}\) has skew product structure implies that
\[\mu(X_0 \times \mathbb{T} \times \mathbb{T}) = 0.\]

It then follows that the equation holds, which finishes the proof. \(\square\)

By Proposition above, there is a one-to-one correspondence between the \(\alpha \times R_{\xi_{\text{gh}}} \times R_{\eta_{\text{gh}}}-\)invariant probability measures and the \(\gamma\)-invariant probability measures (because if two measures coincide on all the Borel sets, they must be the same measure).

It follows that a minimal Furstenberg transformation on \(\mathbb{T}^3\) that is uniquely ergodic will yield an example of a rigid minimal action on \(X \times \mathbb{T} \times \mathbb{T}\), and a minimal transformation on \(\mathbb{T}^3\) that is not uniquely ergodic will yield an example of a non-rigid minimal action on \(X \times \mathbb{T} \times \mathbb{T}\).

**Example 3.6.** This is an example of rigid minimal dynamical system \((X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\eta})\).

Let \((X, \alpha)\) be a Denjoy homeomorphism with rotation number \(\theta_1 \in \mathbb{R} \setminus \mathbb{Q}\).

Choose \(\theta_2, \theta_3\) such that \(1, \theta_1, \theta_2, \theta_3 \in \mathbb{R}\) are linearly independent over \(\mathbb{Q}\). That is, if \(\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}\) and satisfy
\[\lambda_0 + \lambda_1 \theta_1 + \lambda_2 \theta_2 + \lambda_3 \theta_3 = 0,\]
then \(\lambda_i = 0\) for \(i = 0, \ldots, 3\).

The dynamical system \((\mathbb{T} \times \mathbb{T} \times \mathbb{T}, R_{\theta_1} \times R_{\theta_2} \times R_{\theta_3})\) is minimal and uniquely ergodic.

Define \(\varphi : X \to \text{Homeo}(\mathbb{T}^2)\) by
\[\varphi(x)(z_1, z_2) = (z_1 e^{2\pi i \theta_2}, z_2 e^{2\pi i \theta_3}).\]

As \((\mathbb{T} \times \mathbb{T} \times \mathbb{T}, R_{\theta_1} \times R_{\theta_2} \times R_{\theta_3})\) is uniquely ergodic, so is \((X \times \mathbb{T}^2, \alpha \times \varphi)\). This gives an example of a rigid minimal dynamical system \((X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\eta})\).

**Example 3.7.** We will give an example of minimal dynamical system \((X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\eta})\) such that it is not rigid.

According to Furstenberg (see page 585), there exists a minimal a Furstenberg transformation \(\gamma_0 : \mathbb{T}^2 \to \mathbb{T}^2\) such that
\[\gamma_0(z_1, z_2) = (z_1 e^{2\pi i \theta}, f(z_1)z_2)\]
for some \(\theta \in \mathbb{R} \setminus \mathbb{Q}\) and contractible \(f \in C(\mathbb{T}, \mathbb{T})\), and \(\gamma_0\) is not uniquely ergodic.
Let \((\mathbb{T}, \varphi)\) be a Denjoy homeomorphism with rotation number \(\theta\). Let \((X, \alpha)\) be the minimal Cantor dynamical system derived from \((\mathbb{T}, \varphi)\) which factors through \((\mathbb{T}, R_{\theta})\). In other words, \(\alpha = \varphi|_X\) and we have the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{T} & \xrightarrow{R_{\theta}} & \mathbb{T}
\end{array}
\]

with \(\pi: X \to \mathbb{T}\) being a surjective map.

Define \(\xi: X \to \text{Homeo}(\mathbb{T})\) by \(\xi(x)(z) = f(\pi(x))z\). We can then check that the following diagram commutes:

\[
\begin{array}{ccc}
X \times \mathbb{T} & \xrightarrow{\alpha \times R_{\xi}} & X \times \mathbb{T} \\
\pi \times \text{id}_\mathbb{T} \downarrow & & \pi \times \text{id}_\mathbb{T} \downarrow \\
\mathbb{T}^2 & \xrightarrow{\gamma_0} & \mathbb{T}^2
\end{array}
\]

As \(\pi\) is surjective, so is \(\pi \times \text{id}_\mathbb{T}\). Minimality of \(\gamma_0\) then implies minimality of \(\alpha \times R_{\xi}\). As \(\gamma_0\) is not uniquely ergodic, similarly to the proof of Proposition 3.5, it follows that \((X \times \mathbb{T}, \alpha \times R_{\xi})\) is not uniquely ergodic.

In the commutative diagram (6), note that \(\pi\) is onto, and \((\mathbb{T}, R_{\theta})\) is uniquely ergodic. It follows that \((X, \alpha)\) is also uniquely ergodic.

As \((X \times \mathbb{T}, \alpha \times R_{\xi})\) is not uniquely ergodic, there exist more than one \((\alpha \times R_{\xi})\)-invariant probability measure. Let \(\mu\) and \(\nu\) to be two such measures on \(X \times \mathbb{T}\) that are different from each other.

According to Lemma 5.3, there exists \(\theta \in \mathbb{R}\) such that if we use \(R_{\theta}\) to denote the function in \(C(X, \text{Homeo}(\mathbb{T}))\) defined by

\[R_{\theta}(x)(z) = xe^{2\pi i \theta} \text{ for all } x \in X \text{ and } z \in \mathbb{T},\]

then the dynamical system \((X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\theta})\) is still minimal.

Use \(m\) to denote the Lebesgue measure on \(\mathbb{T}\). For the \((\alpha \times R_{\xi})\)-invariant probability measures \(\mu\) and \(\nu\), as \(R_{\theta}\) is a rotation of the circle, we can check that both \(\mu \times m\) and \(\nu \times m\) are \((\alpha \times R_{\xi} \times R_{\theta})\)-invariant probability measures on \(X \times \mathbb{T} \times \mathbb{T}\).

As \(\mu\) and \(\nu\) are different measures, it is clear that \(\mu \times m\) is different from \(\nu \times m\).

Now we have at least two \((\alpha \times R_{\xi} \times R_{\theta})\)-invariant measures. Note that \((X, \alpha)\) is uniquely ergodic. We have that the dynamical system \((X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\theta})\) is not uniquely ergodic.

**Remark:** For this example, the corresponding crossed product C*-algebra has tracial rank one and the dynamical system \((X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi} \times R_{\theta})\) is not rigid. The reason is as follows.

Consider the dynamical system \((X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi})\). It is not uniquely ergodic. As \((X, \alpha)\) is uniquely ergodic, it follows that \((X \times \mathbb{T}, \alpha \times R_{\xi})\) is not rigid.

Use \(A\) to denote the crossed product C*-algebra \(C^*(\mathbb{Z}, X \times \mathbb{T}, \alpha \times R_{\xi})\). According to Theorem 4.3 of [LM2], the algebra \(A\) has tracial rank one. By Proposition 1.10 (1) of [Ph2], \(\rho_A(K_0(A))\) is not dense in \(\text{Aff}(T(A))\).

Note that \(A\) is an AT-algebra. According to Theorem 2.1 of [EL1], \(A\) is approximately divisible. By Theorem 1.4 (e) of [BK2], and noting that real rank of \(A\) is not zero (as tracial rank of \(A\) is one and \(A\) is AT-algebra), we have that the projections in \(A\) does not separate traces of \(A\). In other
words, there exist two \((\alpha \times \mathbb{R}_\xi)\)-invariant measures \(\mu\) and \(\nu\) such that
\[
\mu \neq \nu, \text{ and } \mu(x) = \nu(x) \text{ for all } x \in K_0(A).
\]

Define measures \(\mu_X, \nu_X\) by
\[
\mu_X(D) = \mu(D \times \mathbb{T}) \quad \text{and} \quad \nu_X(D) = \nu(D \times \mathbb{T})
\]
for all Borel sets \(D \subset X\). It is clear that both \(\mu_X\) and \(\nu_X\) are \(\alpha\)-invariant probability measures on \(X\).

Note that \(C(X, \mathbb{Z})\) is generated by the projections in \(C(X)\). Also note that the \(\mathbb{C}\)-linear span of \(C(X, \mathbb{Z})\) is dense in \(C(X, \mathbb{R})\). The fact that the projections in \(A\) do not separate \(\mu\) and \(\nu\) implies that \(C(X, \mathbb{Z})\) do not separate \(\mu_X\) and \(\nu_X\), which then implies that \(\mu_X = \nu_X\).

Use \(B\) to denote \(C^*(\mathbb{Z}, X \times \mathbb{T}_1 \times \mathbb{T}_2, \alpha \times \mathbb{R}_\xi \times \mathbb{R}_\theta)\). Let \(m\) be the Lebesgue measure on \(\mathbb{T}\). It is clear that \(\mu \times m\) and \(\nu \times m\) are two \((\alpha \times \mathbb{R}_\xi \times \mathbb{R}_\theta)\)-invariant probability measures.

We will show that the projections in \(B\) do not separate \(\mu \times m\) and \(\nu \times m\).

From Proposition 2.11,
\[
K_0(B) \cong C(X, \mathbb{Z}^2)/\{(f, g) - (f, g) \circ \alpha^{-1}: f, g \in C(X, \mathbb{Z})\} \oplus \mathbb{Z} \oplus \mathbb{Z}.
\]

The two copies of \(\mathbb{Z}\) correspond to the two generalized Rieffel projections \(e_1\) and \(e_2\), given by \(e_1 = g_1 u^* f_1 + u g_1\), and \(e_2 = g_2 u^* f_2 + u g_2\), where \(e_i, f_i, g_i\) are defined similarly to the functions defined in Section 6 of [LM]. \(f_1(x, z_1, z_2) = f_1(x, z_1, z_2')\) and \(f_2(x, z_1, z_2) = f_1(x, z_1', z_2')\) for all \(z_1, z_1', z_2, z_2' \in \mathbb{T}_2\).

As the projections in \(A\) do not distinguish \(\mu\) and \(\nu\), it follows that the elements in \(K_0(B)\) that correspond to the first two summands of \(\mathbb{Z}\) do not separate \(\mu \times m\) and \(\nu \times m\).

For the generalized Rieffel projection \(e_2\), as \(f_2(x, z_1, z_2)\) is independent of \(z_1\), we have
\[
f(x, z_1, z_2) = F_2(x, z_2) \text{ for some } F \in C(X \times \mathbb{T}_2, \mathbb{R}).
\]

Recall that for a measure \(\sigma\) on \(X\) and \(f \in C(X)\), we use \(\sigma(f)\) to denote \(\int_X f(x) \, d\mu\) (see Section 6). We check that
\[
(\mu \times m)(e_2) = (\mu \times m)(f_2)
\]
\[
= \int_{(X \times \mathbb{T}_1) \times \mathbb{T}_2} f_2(x, z_1, z_2) \, d(\mu \times m)
\]
\[
= \int_{X \times \mathbb{T}_2} F_2(x, z_2) \, d(\mu_X \times m)
\]
\[
= \int_{X \times \mathbb{T}_2} F_2(x, z_2) \, d(\nu_X \times m)
\]
\[
= \int_{(X \times \mathbb{T}_1) \times \mathbb{T}_2} f_2(x, z_1, z_2) \, d(\nu \times m)
\]
\[
= (\nu \times m)(f_2)
\]
\[
= (\nu \times m)(e_2).
\]

Then we have shown that \(e_2\) does not separate \(\mu \times m\) and \(\nu \times m\) either, which then implies that the projections in \(B\) cannot separate traces of \(B\).

According to Theorem 1.4 of [BKR], the real rank of \(B\) is not zero. Then it follows that the tracial rank of \(B\) is not zero.

By Theorem 2.18, the tracial rank of \(B\) must be one.

According to Proposition 2.21, the dynamical system \((X \times \mathbb{T}, \alpha \times \mathbb{R}_\xi \times \mathbb{R}_\theta)\) is not rigid.
4. Approximate Conjugacies

In this section, we start with a sufficient condition for approximate K-conjugacy between two minimal dynamical systems \((X \times T \times T, \alpha \times R \times R \eta_1)\) and \((X \times T \times T, \beta \times R \times R \eta_2)\). Then we give an if and only if condition for weak approximate conjugacy of these two dynamical systems, showing that weak approximate conjugacy just depends on \(\alpha\) and \(\beta\). In Theorem 4.12 an if and only if condition for approximate K-conjugacy between these two dynamical systems is given.

In \[LM3\], several notions of approximate conjugacy between dynamical systems are introduced. In \[LM1\], it is shown that for rigid minimal systems on 
\(X \times T\) (with \(X\) being the Cantor set and \(T\) being the circle; see Definition 3.1 of \[LM1\]), the corresponding crossed product C*-algebras are isomorphic if and only if the dynamical systems are approximately K-conjugate.

For two minimal rigid dynamical systems \((X \times T \times T, \alpha \times R \times R \eta)\) and \((X \times T \times T, \beta \times R \times R \eta)\), we study the relationship between approximate K-conjugacy and the isomorphism of crossed product C*-algebras.

We start with basic definitions and facts about conjugacy and approximate conjugacy.

**Definition 4.1.** Let \(X, Y\) be two compact metric spaces, and let \(\alpha \in \text{Homeo}(X)\) and \(\beta \in \text{Homeo}(Y)\) be two minimal actions. We say that \((X, \alpha)\) and \((Y, \beta)\) are conjugate if there exists \(\sigma \in \text{Homeo}(X, Y)\) such that \(\sigma \circ \alpha = \beta \circ \sigma\). We say that \((X, \alpha)\) and \((Y, \beta)\) are flip conjugate if \((X, \alpha)\) is conjugate to \((Y, \beta)\) or \((Y, \beta^{-1})\).

**Definition 4.2.** Let \(X, Y\) be two compact metric spaces, and let \(\alpha \in \text{Homeo}(X)\) and \(\beta \in \text{Homeo}(Y)\) be two minimal actions. We say that \((X, \alpha)\) and \((Y, \beta)\) are weakly approximately conjugate if there exist \(\sigma_n \in \text{Homeo}(X, Y)\) and \(\gamma_n \in \text{Homeo}(Y, X)\) for \(n \in \mathbb{N}\) such that

\[
\text{dist}(f \circ \sigma_n \circ \alpha, f \circ \beta \circ \sigma_n) \to 0 \quad \text{and} \quad \text{dist}(g \circ \alpha \circ \gamma_n, g \circ \gamma_n \circ \beta) \to 0 \quad \text{as} \quad n \to \infty
\]

for all \(f \in C(X)\) and \(g \in C(Y)\), where \(\text{dist}(f_1, f_2)\) is defined to be \(\sup_{x \in D} \text{dist}(f_1(x), f_2(x))\) for all continuous functions \(f_1, f_2\) on the metric space \(D\).

It is clear that if two minimal dynamical systems are conjugate, then they are weakly approximately conjugate. Generally speaking, the inverse implication does not hold.

Now we will recall the definition of C*-strong approximate conjugacy (which is defined by Huaxin Lin in \[Lin4\]).

Given minimal dynamical systems \((X, \alpha)\) and \((Y, \beta)\), if they are flip conjugate, then it is easy to check that the corresponding crossed product C*-algebras \(C^*(Z, X, \alpha)\) and \(C^*(Z, Y, \beta)\) are isomorphic.

According to \[Tomiyama\] (Corollary of Theorem 2), for two minimal dynamical systems \((X, \alpha)\) and \((Y, \beta)\), there exists an isomorphism

\[
\varphi: C^*(Z, X, \alpha) \longrightarrow C^*(Z, Y, \beta)
\]

satisfying \(\varphi(C(X)) = C(Y)\) if and only if these two dynamical systems are flip conjugate.

In view of Tomiyama’s result above, C*-strong approximate flip conjugacy is defined as below.
Definition 4.3 (See [Lin4].) Let \((X, \alpha)\) and \((X, \beta)\) be two minimal dynamical systems such that 
\[\text{TR}(C^*(\mathbb{Z}, X, \alpha)) = \text{TR}(C^*(\mathbb{Z}, X, \beta)) = 0,\]
we say that \((X, \alpha)\) and \((X, \beta)\) are \(C^*\)-strongly approximately flip conjugate if there exists a sequence of isomorphisms
\[
\varphi_n : C^*(\mathbb{Z}, X, \alpha) \to C^*(\mathbb{Z}, X, \beta), \quad \psi_n : C^*(\mathbb{Z}, X, \beta) \to C^*(\mathbb{Z}, X, \alpha)
\]
and a sequence of isomorphisms \(\chi_n, \lambda_n : C(X) \to C(X)\) such that
1) \(\lim_{n \to \infty} [\varphi_n] = [\varphi_m] = [\psi_n^{-1}]\) in \(KL(C^*(\mathbb{Z}, X, \alpha), C^*(\mathbb{Z}, X, \alpha))\) for all \(m, n \in \mathbb{N}\),
2) \(\lim_{n \to \infty} \|\varphi_n \circ j_\alpha(f) - j_\beta \circ \chi_n(f)\| = 0\) and \(\lim_{n \to \infty} \|\psi_n \circ j_\beta(f) - j_\alpha \circ \lambda_n(f)\| = 0\) for all \(f \in C(X)\),
with \(j_\alpha, j_\beta\) being the injections from \(C(X)\) into \(C^*(\mathbb{Z}, X, \alpha)\) and \(C^*(\mathbb{Z}, X, \beta)\).

Some notation will be introduced before the next result about \(C^*\)-strong approximate conjugacy.

Let \(A\) be a separable amenable \(C^*\)-algebra that satisfies Universal Coefficient Theorem. For \(\theta \in KL(A, B)\), there are induced homomorphisms \(\Gamma(\theta)_i : K_i(A) \to K_i(B)\) for \(i = 0, 1\). Define 
\[
\rho_A : A_{sa} \to \text{Aff}(T(A)) \quad \text{by} \quad \rho_A(a)(\tau) = \tau(a) \quad \text{for all} \quad a \in A_{sa} \quad \text{and} \quad \tau \in T(A).
\]
Suppose \(A\) and \(B\) are two unital simple \(C^*\)-algebras with tracial rank zero and \(\gamma : K_0(A) \to K_0(B)\) is an order preserving homomorphism. As \(A\) has real rank zero, \(\gamma\) will induce a positive homomorphism \(\gamma_\rho : \text{Aff}(T(A)) \to \text{Aff}(T(B))\).

The theorem below ([Lin4. Theorem 2.5]) gives one necessary condition for \(C^*\)-strong approximate flip conjugacy between two crossed product \(C^*\)-algebras.

**Theorem 4.4.** Let \((X, \alpha)\) and \((X, \beta)\) be two minimal dynamical systems such that the corresponding crossed product \(C^*\)-algebras \(A_\alpha\) and \(A_\beta\) both have tracial rank zero. Then \(\alpha\) and \(\beta\) are \(C^*\)-strongly approximately flip conjugate if the following holds: There is an isomorphism \(\chi : C(X) \to C(X)\) and there is \(\theta \in KL(A_\alpha, A_\beta)\) such that \(\Gamma(\theta)\) gives an isomorphism
\[
\Gamma(\theta) : (K_0(A_\alpha), K_0(A_\alpha)_{+}, [1], K_1(A_\alpha)) \to (K_0(A_\beta), K_0(A_\beta)_{+}, [1], K_1(A_\beta)),
\]
and such that
\[
[j_\alpha] \times \theta = [j_\beta \circ \chi] \quad \text{in} \quad KL(C(X), A_\beta)
\]
and
\[
\rho_{A_\beta} \circ j_\beta \circ \chi(f) = ((\Gamma(\theta)_0)_\rho) \circ \rho_{A_\alpha} \circ j_\alpha(f)
\]
for all \(f \in C(X)_{sa}\).

If \(K_i(C(X))\) is torsion free, then a simplified version of this result holds ([Lin4. Corollary 2.6]).

**Corollary 4.5.** Let \(X\) be a compact metric space with torsion free \(K\)-theory. Let \((X, \alpha)\) and \((X, \beta)\) be two minimal dynamical systems such that \(\text{TR}(A_\alpha) = \text{TR}(A_\beta) = 0\). Suppose that there is an order isomorphism that maps \([1_{A_\alpha}]\) to \([1_{A_\beta}]\): 
\[
\gamma : (K_0(A_\alpha), K_0(A_\alpha)_{+}, [1_{A_\alpha}], K_1(A_\alpha)) \to (K_0(A_\beta), K_0(A_\beta)_{+}, [1_{A_\beta}], K_1(A_\beta)),
\]
such that there exists an isomorphism \(\chi : C(X) \to C(X)\) satisfying
\[
\gamma \circ (j_\alpha)^i = (j_\beta \circ \chi)^i \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad \gamma_\rho \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi \quad \text{on} \quad C(X)_{sa}.
\]
Then \((X, \alpha)\) and \((X, \beta)\) are \(C^*\)-strongly approximately flip conjugate.
In the rest of this section, for a minimal homeomorphism $\alpha$ on the Cantor set $X$, we will use $K^0(X, \alpha)$ to denote the ordered group

$$C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2)\}$$

with the positive cone being (denoted by $K^0(X, \alpha)_+$)

$$C(X, D)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2)\}$$

where $D$ is as defined in Lemma 1.11. In $K^0(X, \alpha)$, we define the unit element to be

$$[(1, 0)_{C(X, \mathbb{Z}^2)}] \in C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2)\},$$

with $(1, 0)_{C(X, \mathbb{Z}^2)}$ being the constant function in $C(X, \mathbb{Z}^2)$ that maps every $x \in X$ to $(1, 0) \in \mathbb{Z}^2$. We use $1_{K^0(X, \alpha)}$ to denote this unit element.

\textbf{Lemma 4.6.} Let $X$ be the Cantor set. For every minimal action $\alpha \in \text{Homeo}(X)$, if there is an order isomorphism

$$\varphi: (K^0(X, \alpha), K^0(X, \alpha)_+, 1_{K^0(X, \alpha)}) \rightarrow (K^0(X, \beta), K^0(X, \beta)_+, 1_{K^0(X, \beta)}),$$

then there is an order isomorphism

$$\widetilde{\varphi}: (C(X, \mathbb{Z}^2), C(X, D), (1, 0)_{C(X, \mathbb{Z}^2)}) \rightarrow (C(X, \mathbb{Z}^2), C(X, D), (1, 0)_{C(X, \mathbb{Z}^2)})$$

such that the following diagram commutes:

\begin{equation}
\begin{array}{ccc}
(C(X, \mathbb{Z}^2), C(X, D)) & \xrightarrow{\widetilde{\varphi}} & (C(X, \mathbb{Z}^2), C(X, D)) \\
\pi_\alpha \downarrow & & \pi_\beta \\
(K^0(X, \alpha), K^0(X, \alpha)_+) & \xrightarrow{\varphi} & (K^0(X, \beta), K^0(X, \beta)_+)
\end{array}
\end{equation}

where $\pi_\alpha, \pi_\beta$ are the canonical projections from $C(X, \mathbb{Z}^2)$ to $K^0(X, \alpha)$ and $K^0(X, \beta)$. In fact, there exists $\sigma \in \text{Homeo}(X)$ such that $\varphi(F) = F \circ \sigma^{-1}$ for all $F \in C(X, \mathbb{Z}^2)$.

\textbf{Proof.} The proof is based on \textbf{LM3} Theorem 2.6.

Define $K^0(X, \alpha)$ to be

$$C(X, \mathbb{Z})/\{g - g \circ \alpha^{-1} : g \in C(X, \mathbb{Z})\}$$

and $K^0(X, \alpha)_+$ to be

$$C(X, \mathbb{Z}^+ \cup \{0\})/\{g - g \circ \alpha^{-1} : g \in C(X, \mathbb{Z})\}.$$  

We can check that $(\overline{K^0(X, \alpha)}, \overline{K^0(X, \alpha)_+})$ gives an ordered group with order unit. Define

$$h: K^0(X, \alpha) \rightarrow \overline{K^0(X, \alpha)} \text{ by } h([f]) = [f_1]$$

for every $f = (f_1, f_2) \in C(X, \mathbb{Z}^2)$, with $f_1, f_2 \in C(X, \mathbb{Z})$.

From the definition, we can check that $h$ is surjective and $h(K^0(X, \alpha)_+) = \overline{K^0(X, \alpha)_+}$. For the isomorphism

$$\varphi: (K^0(X, \alpha), K^0(X, \alpha)_+) \rightarrow (K^0(X, \beta), K^0(X, \beta)_+),$$

define

$$\varphi_0: \overline{K^0(X, \alpha)} \rightarrow \overline{K^0(X, \beta)} \text{ by } \varphi_0([f]) = h(\varphi([f, 0]))$$

for all $f \in C(X, \mathbb{Z})$. 
Suppose that there exist $f_1, f_2, g \in C(X, Z)$ such that $f_1 - f_2 = g - g \circ \alpha^{-1}$. Then it follows that $(f_1, 0) - (f_2, 0) = (g, 0) - (g, 0) \circ \alpha^{-1}$, which implies that $\varphi((f_1, 0)) = \varphi((f_1, 0))$. It is now clear that $\varphi_0$ is well-defined.

Note that $\varphi_0([1_{C(X,Z)}]) = h(\varphi([1_{C(X,Z)}]))$. As $\varphi$ is unital, $\varphi(1_{K_0(\alpha)}) = 1_{K_0(\beta)}$, which then implies that $\varphi_0([1_{C(X,Z)}]) = h([1_{C(X,Z)}]) = [1_{C(X,Z)}]$. We can now claim that $\varphi_0$ is unital.

For any $f \in C(X, Z^+ \cup \{0\}), \varphi_0([f]) = h(\varphi(([f, 0]))$. As both $\varphi$ and $h$ are order preserving, $\varphi_0$ is also order preserving.

So far, we have that $\varphi_0: K_0^0(X, \alpha) \to K_0^0(X, \beta)$ is unital and order preserving. According to [LM3] Theorem 2.6, there exists a continuous order preserving map

$$\widetilde{\varphi}_0: (C(X, Z), C(X, Z)_+, 1_{C(X,Z)}) \to (C(X, Z), C(X, Z)_+, 1_{C(X,Z)}),$$

such that the following diagram commutes:

$$
\begin{array}{c}
(C(X, Z), C(X, Z)_+) \xrightarrow{\varphi_0} (C(X, Z), C(X, Z)_+) \\
\downarrow \pi_0 \\
(K_0^0(X, \alpha), K_0^0(X, \alpha)_+) \xrightarrow{\varphi_0} (K_0^0(X, \beta), K_0^0(X, \beta)_+)
\end{array}
$$

Now we need to construct the unital positive linear map

$$\widetilde{\varphi}: (C(X, Z^2), C(X, D)) \to (C(X, Z^2), C(X, D)),$$

such that diagram \[5\] commutes.

For the $\widetilde{\varphi}_0$ we get, note that $\widetilde{\varphi}_0$ is a unital positive isomorphism from $K_0^0(X, \alpha)$ to $K_0^0(X, \beta)$. As $C(X)$ is a unital AF-algebra, by the existence theorem of classification of unital AF-algebras, there exists an isomorphism $\psi: C(X) \to C(X)$ such that (identifying $K_0(C(X))$ with $C(X, Z)$ and $K_0(C(X)_+)$ with $C(X, Z)_+$)

$$\psi_0: (C(X, Z), C(X, Z)_+, [1]) \to (C(X, Z), C(X, Z)_+, [1])$$

coincides with $\widetilde{\varphi}_0$.

As $\psi$ is an isomorphism, there exists $\sigma: X \to X$ such that $\psi(f) = f \circ \sigma^{-1}$ for all $f \in C(X)$.

Define $\tilde{\varphi}: C(X, Z^2) \to C(X, Z^2)$ by $\tilde{\varphi}((f, g)) = (\psi(f), \psi(g))$ for all $f, g \in C(X, Z)$. In other words, $\tilde{\varphi}((f, g)) = (f, g) \circ \sigma^{-1}$ for all $(f, g) \in C(X, Z^2)$.

For the $\tilde{\varphi}$ above-defined, it is easy to check that it is unital and linear. It remains to show that $\tilde{\varphi}$ maps positive cone to positive cone, and makes the diagram commute.

For every $(f, g) \in C(X, D)$, we get $\tilde{\varphi}((f, g)) = (f, g) \circ \sigma^{-1}$. As $(f, g) \in C(X, D)$, it is clear that $(f, g) \circ \sigma^{-1} \in C(X, D)$. So far, we proved that $\tilde{\varphi}$ is a positive map.

We can check that

$$\pi_\beta \circ \tilde{\varphi}((f, g)) = \pi_\beta(h(f), h(g))$$

$$= \pi_\beta(\tilde{\varphi}_0(f), \tilde{\varphi}_0(g))$$

$$= \pi_\beta(\varphi_0(f), 0) + \pi_\beta(0, \varphi_0(g))$$

$$= \pi_\beta'(\varphi_0'(f), 0) + (0, \pi_\beta'(\varphi_0'(g))$$

$$= (\varphi_0 \circ \pi_\alpha(f), 0) + (0, \varphi_0 \circ \pi_\alpha'(g))$$

$$= \varphi \circ \pi_\alpha((f, 0)) + \varphi \circ \pi_\alpha((0, g))$$

$$= \varphi \circ \pi_\alpha((f, g)),$$

which implies the commutativity of diagram \[5\].
As \( \varphi((f, g)) = (f, g) \circ \sigma^{-1} \) for all \( f, g \in C(X, \mathbb{Z}) \), we get that \( \varphi \) is an isomorphism, which finishes the proof.

\[
\begin{array}{c}
\text{Diagram:}
\end{array}
\]

**Theorem 4.7.** Let \( (X \times \mathbb{T} \times T, \alpha \times R_{\xi}, \beta \times R_{\mu}) \) and \( (X \times \mathbb{T} \times T, \beta \times R_{\xi} \times R_{\mu}) \) be two minimal rigid Cantor dynamical systems. Use \( A, B \) to denote the two corresponding crossed product \( C^* \)-algebras. According to Proposition 2.19, \( K^0(X, \alpha) \) is a direct summand of \( K_0(A) \) and \( K^0(X, \beta) \) is a direct summand of \( K_0(B) \). Let

\[
j_A : K^0(X, \alpha) \to K_0(A) \cong K^0(X, \alpha) \oplus \mathbb{Z}^2 \quad \text{and} \quad j_B : K^0(X, \beta) \to K_0(B) \cong K^0(X, \alpha) \oplus \mathbb{Z}^2
\]

be defined by

\[
j_A(x) = (x, 0) \quad \text{and} \quad j_B(x) = (x, 0).
\]

If there is an order preserving isomorphism \( \rho \) from \( K_0(A) \) to \( K_0(B) \) that maps \( K^0(X, \alpha) \) onto \( K^0(X, \beta) \), then these two dynamical systems are \( C^* \)-strongly approximately conjugate.

**Proof.** We have the following commutative diagram:

\[
\begin{array}{ccc}
K_0(A) & \overset{\rho}{\longrightarrow} & K_0(B) \\
\downarrow j_A & & \downarrow j_B \\
K^0(X, \alpha) & \overset{\rho|_{K^0(X, \alpha)}}{\longrightarrow} & K^0(X, \beta)
\end{array}
\]

According to Lemma 4.6, we can lift

\[
\rho|_{K^0(X, \alpha)} : K^0(X, \alpha) \to K^0(X, \beta)
\]

to

\[
\tilde{\rho} : C(X, \mathbb{Z}^2) \to C(X, \mathbb{Z}^2),
\]

which will yield the commutative diagram

\[
\begin{array}{ccc}
K_0(A) & \overset{\rho}{\longrightarrow} & K_0(B) \\
\downarrow j_A & & \downarrow j_B \\
K^0(X, \alpha) & \overset{\rho|_{K^0(X, \alpha)}}{\longrightarrow} & K^0(X, \beta) \\
\downarrow \pi_\alpha & & \downarrow \pi_\beta \\
C(X, \mathbb{Z}^2) & \overset{\tilde{\rho}}{\longrightarrow} & C(X, \mathbb{Z}^2)
\end{array}
\]

In fact, according to Lemma 4.6, there exists \( \sigma \in \text{Homeo}(X) \) such that \( \tilde{\rho}(F) = F \circ \sigma^{-1} \). Define

\[
\chi : C(X \times \mathbb{T}^2) \to C(X \times \mathbb{T}^2)
\]

by \( \chi(f) = f \circ (\sigma \times \text{id}_{\mathbb{T}^2}) \) for all \( f \in C(X \times \mathbb{T}^2) \).

According to the Künneth Theorem, we get that \( K_0(C(X \times \mathbb{T}^2)) \cong C(X, \mathbb{Z}^2) \). By Lemma 4.3, if we identify \( K_0(C(X \times \mathbb{T}^2)) \) with \( C(X, \mathbb{Z}^2) \), the positive cone will be identified with \( C(X, D) \), with \( D \) as defined in Lemma 4.3. Choose \( x \in X \). According to Lemma 4.11 we know that \( K_0(A_x) \cong K^0(X, \alpha) \) and \( K_0(B_x) \cong K^0(X, \beta) \), with \( A_x, B_x \) being the subalgebras of \( A \) and \( B \), as in Definition 1.1.
Now we have the commutative diagram

$$
\begin{array}{ccc}
K_0(A) & \xrightarrow{\rho} & K_0(B) \\
\downarrow{(j_\alpha)_0} & & \downarrow{(j_\beta)_0} \\
K_0(C(X \times T^2)) & \xrightarrow{\tilde{\rho}} & K_0(C(X \times T^2))
\end{array}
$$

Note that $\tilde{\rho}$ is induced by the $\chi: C(X \times T^2) \to C(X \times T^2)$ defined above. We have shown that $\rho \circ (j_\alpha)_i = (j_\beta \circ \chi)_i$, $i = 0, 1$.

We will show that $\gamma_\rho \circ j_\alpha = \rho A_\beta \circ j_\beta \circ \chi$ on $C(X)_a$. For every tracial state $\tau \in T(C^*(Z, X, \beta))$, we know that it corresponds to a $\beta$-invariant probability measure $\mu_B$ in such sense that $\tau(a) = \mu(E(a))$, with $E$ being the conditional expectation from $C^*(Z, X, \beta)$ to $C(X))$.

For any $\beta$-invariant probability measure $\mu_B$ on $X$, if we use $v$ to denote standard Lebesgue measure on $T$, it is then clear that $\mu_B \times v \times v$ is $\beta \times \sigma T \times \sigma T$-invariant. As the dynamical system $(X \times T \times T, \beta \times \sigma T \times \sigma T)$ is rigid, for every $\beta \times \sigma T \times \sigma T$-invariant probability measure, it must be $\mu \times v \times v$, with $\mu$ being an $\beta$-invariant probability measure and $v$ being the Lebesgue probability measure.

Note that $A$ denotes $C^*(Z, X \times T \times T, \alpha \times R_{\xi_2} \times R_{\eta_2})$ and $B$ denotes $C^*(Z, X \times T \times T, \beta \times R_{\xi_1} \times R_{\eta_1})$.

According to Proposition 2.19, the fact that $K_0(A)$ is isomorphic to $K_0(B)$ implies that $K_1(A)$ is also isomorphic to $K_1(B)$. According to Proposition 2.21, the tracial rank of $A$ and $B$ are both zero, thus classifiable via the K-data.

Let $\varphi: A \to B$ be the $C^*$-algebra isomorphism such that

$$
\varphi_a: K_0(A) \to K_0(B)
$$

coincides with the $\rho$ in the statement. Define

$$
\varphi^*: T(B) \to T(A)
$$

as $\varphi^*(\tau_B)(a) = \tau_B(\varphi(a))$ for all $a \in A$ and $\tau_B \in T(B)$.

Note that a $C^*$-algebra with tracial rank zero must have real rank zero. We can now claim that for every $a \in C^*(Z, X, \alpha)_a$ and $\tau_B \in T(B)$ given by $\mu_B \times v \times v$,

$$(\gamma_\rho \circ j_\alpha)(a)(\tau_B) = \varphi^*(\tau_B)(a).$$

Consider

$$a = f \otimes g \otimes h \in C(X \times T \times T)_a \subset A_a$$

with $f \in C(X)_a$, $g \in C(T)_a$ and $h \in C(T)_a$, and use $\tau_A$ to denote $\varphi^*(\tau_B)$. As $\alpha \times R_{\xi_1} \times R_{\eta_1}$ is rigid, there exists an $\alpha$-invariant measure $\mu_A$ such that $\tau_A(a) = (\mu_A \times v \times v)(E(a))$, with $E$ being the conditional expectation from $A$ to $C(X \times T \times T)$ and $v$ being the Lebesgue measure on the circle. It follows that $(\gamma_\rho \circ j_\alpha)(a)(\tau_B) = \tau_A(a) = \mu_A(f \cdot v(g) \cdot v(h))$.

As for $((\rho A_\beta \circ j_\beta \circ \chi)(a))(\tau_B)$, we know from the definition that

$$((\rho A_\beta \circ j_\beta \circ \chi)(a))(\tau_B) = \tau_B(\chi(f \otimes g \otimes h)) = (\mu_B \times v \times v)(\chi(f \otimes g \otimes h)).$$

Recall the definition of $\chi$. We have

$$(\mu_B \times v \times v)(\chi(f \otimes g \otimes h)) = \mu_B(f \circ \sigma^{-1}) \cdot v(g) \cdot v(h).$$

If we can show that $\mu_B(f \circ \sigma^{-1}) = \mu_A(f)$, then it follows that

$$(\mu_B \times v \times v)(\chi(f \otimes g \otimes h)) = \mu_A(f) \cdot v(g) \cdot v(h) = (\mu_A \times v \times v)(f \otimes g \otimes h).$$
and we can then get
\[ \gamma_\rho \circ j_\alpha = \rho_{AB} \circ j_\beta \circ \chi \text{ on } C(X \times T^2)_{sa}. \]

We will show that for all \( f \in C(X, Z) \) and \( \mu_A, \mu_B \) as given above, we have \( \mu_B(f \circ \sigma^{-1}) = \mu_A(f) \).

As both dynamical systems \( \alpha \times \mathbb{R}_{\xi_1} \times \mathbb{R}_{\eta_1} \) and \( \beta \times \mathbb{R}_{\xi_2} \times \mathbb{R}_{\eta_2} \) are rigid, by Proposition 2.21 we have \( \text{TR}(A) = \text{TR}(B) = 0 \). According to Corollary 4.5 these two dynamical systems \( (X \times T \times T, \alpha \times \mathbb{R}_{\xi_1} \times \mathbb{R}_{\eta_1}) \) and \( (X \times T \times T, \beta \times \mathbb{R}_{\xi_2} \times \mathbb{R}_{\eta_2}) \) are \( C^* \)-strongly approximately conjugate. \( \Box \)
We studied the weakly approximate conjugacy between to dynamical systems $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$ and give an if and only if condition for the weakly approximate conjugacy.

For minimal homeomorphisms $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$, the following lemma shows that whether they are weakly approximately conjugate or not is determined by $\alpha$ and $\beta$ only, and has nothing to do with $R_{\xi_i}$ and $R_{\eta_i}$ for $i = 1, 2$.

**Lemma 4.8.** Let $(X, \alpha)$ and $(X, \beta)$ be two minimal Cantor dynamical systems. For continuous maps $\xi_1, \xi_2, \eta_1, \eta_2$: $X \to \mathbb{T}$, $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2})$ are weakly approximately conjugate if and only if $(X, \alpha)$ and $(X, \beta)$ are weakly approximately conjugate.

**Proof.** The “if” part:

For every $\varepsilon > 0$, we will show that there exists $\sigma_n \in \text{Homeo}(X \times \mathbb{T} \times \mathbb{T})$ such that

$$\text{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}, \beta) < \varepsilon.$$  

As $(X, \beta)$ is a minimal Cantor dynamical system, there exists a Kakutani-Rokhlin partition

$$\{X_{s,k} : 1 \leq s \leq n, 0 \leq k < h(s)\}$$

such that $h(s) > 5/\varepsilon$, and $\text{diam}(X_{s,j}) < \varepsilon/5$, where $\text{diam}(X_{s,j})$ is defined to be $\sup_{x,y \in X_{s,j}} \text{dist}(x, y)$.

For any two clopen sets $X_{s_1,j_1}$ and $X_{s_2,j_2}$ in the Kakutani-Rokhlin partition, there exists $\delta_{s_1,j_1:s_2,j_2} > 0$ such that if $x, y \in X_{s_1,j_1} \cup X_{s_2,j_2}$ and $\text{dist}(x, y) < \delta_{s_1,j_1:s_2,j_2}$, then either $x, y \in X_{s_1,j_1}$ or $x, y \in X_{s_2,j_2}$.

Let $\delta = \min \delta_{s_1,j_1:s_2,j_2}$, where $X_{s_j,j}$ and $X_{s_j',j'}$ traverse through all pairs of distinct clopen sets in the Kakutani-Rokhlin partition above.

As $(X, \alpha)$ and $(X, \beta)$ are weakly approximately conjugate, there exists $\gamma_n \in \text{Homeo}(X)$ such that

$$\text{dist}(\gamma \circ \alpha \circ \gamma^{-1}(x), \beta(x)) < \delta.$$  

According to the definition of $\delta$, it follows that for every $X_{s,j}$ in the Kakutani-Rokhlin partition above, we have

$$\gamma \circ \alpha \circ \gamma^{-1}(X_{s,j}) = \beta(X_{s,j}).$$

Without loss of generality (replacing $\alpha$ with $\gamma \circ \alpha \circ \gamma^{-1}$), we can assume that $\alpha$ and $\beta$ satisfies

$$\alpha(X_{s,j}) = \beta(X_{s,j}).$$

Identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$, and define $\pi$ by $\pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z}, t \mapsto t + \mathbb{Z}$. For all $x \in X_{s,0}$, define $h(x) = 0$. For $x \in X_{s,k}$ with $0 < k < h(s)$, define

$$f_1(x) = \sum_{j=1}^{k} (\xi_2 - \xi_1)(\alpha^{-j}(x)).$$

As $\xi_1$ and $\xi_2$ are both in $C(X, \mathbb{T})$, it follows that the above defined $f_1$ is a continuous function from $X$ to $\mathbb{T}$.

For $x \in X_{s,k}$, define

$$g_1(x) = \sum_{j=1}^{h(x)} (\xi_2 - \xi_1)(\alpha^{-j}(\alpha^{h(s)-k}(x))).$$

It is also clear that $g_1 \in C(X, \mathbb{T})$.

As $X$ is totally disconnected, we can divide $X$ into $\bigcup_{k=1}^{N} X_k$, with every $X_k$ being a clopen subset of $X$ satisfying $\text{dist}(h(x), h(y)) < \frac{1}{4}$ for $x, y$ in the same $X_k$. For $g_1 |_{X_k}$, we can lift it to continuous function $G_{1,k}: X_k \to [0 - \frac{1}{4}, 1 + \frac{1}{4}]$ satisfying $g_1 |_{X_k} = \pi \circ G_{1,k}$.
Define $G_1: X \to \mathbb{R}$ by setting $G_1(x) = G_{1,k}(x)$ if $x \in X_k$. It is then easy to check that $G_1$ is a lifting of $g_1$ satisfying

$$g_1 = \pi \circ G_1 \text{ and } G_1(x) \in [0 - \frac{1}{4}, 1 + \frac{1}{4}] \text{ for all } x \in X.$$ 

For $x \in X_{s,k}$, define

$$s_1(x) = f_1(x) - \frac{G_1(x) \cdot k}{h(s)} + \mathbb{Z}.$$ 

Similarly, define $f_2(x) = 0$ if $x \in X_{s,0}$ and

$$f_2(x) = \sum_{j=1}^{k} (\eta_2 - \eta_1)(\alpha^{-j}(x))$$

for $x \in X_{s,k}$ with $0 < k < h(s)$. Define

$$g_2(x) = \sum_{j=1}^{h(s)} (\eta_2 - \eta_1) \left(\alpha^{-j} \left(\alpha^{h(s)-k}(x)\right)\right).$$

As $X$ is totally disconnected, we can find a lifting $G_2 \in C(X, \mathbb{R})$ such that

$$g_2 = \pi \circ G_2 \text{ and } G_2(x) \in \left[0 - \frac{1}{4}, 1 + \frac{1}{4}\right]$$

for all $x \in X$.

For $x \in X_{s,k}$, define

$$s_2(x) = f_2(x) - \frac{G_2(x) \cdot k}{h(s)} + \mathbb{Z}.$$ 

For the $s_1$ and $s_2$ we have defined, it is easy to check that they are continuous function from $X$ to $\mathbb{R}/\mathbb{Z}$. According to our identification, we can regard $s_1$ and $s_2$ as functions in $C(X, T)$.

We will show that $(id_X \times R_{s_1} \times R_{s_2})$ will approximately conjugate $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$.

For every $(x, t_1, t_2) \in X \times T \times T$, we have

$$(id_x \times R_{s_1} \times R_{s_2}) \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \circ (id_x \times R_{s_1} \times R_{s_2})^{-1}(x, t_1, t_2)$$

$$= (id_x \times R_{s_1} \times R_{s_2}) \circ (\alpha \times R_{\xi_1} \times R_{\eta_1})(x, t_1 - s_1(x), t_2 - s_2(x))$$

$$= (id_x \times R_{s_1} \times R_{s_2})(\alpha(x), t_1 - s_1(x) + \xi_1(x), t_2 - s_2(x) + \eta_1(x))$$

$$= (\alpha(x), t_1 + \xi_1(x) - s_1(x) + s_1(\alpha(x)), t_2 + \eta_1(x) - s_2(x) + s_2(\alpha(x))).$$

and it is clear that

$$(\beta \times \xi_2 \times \eta_2)(x, t_1, t_2) = (\beta(x), t_1 + \xi_2(x), t_2 + \eta_2(x)).$$

As $\alpha(X_{s,j}) = \beta(X_{s,j})$ and diam($X_{s,j}$) < $\varepsilon/5$, we have $\text{dist}(\alpha(x), \beta(x)) < \varepsilon/5$ for all $x \in X$. Consider the distance between $t_1 + \xi_1(x) - s_1(x) + s_1(\alpha(x))$ and $t_1 + \xi_2(x)$. We get

$$|t_1 + \xi_1(x) - s_1(x) + s_1(\alpha(x)) - (t_1 + \xi_2(x))| = |s_1(\alpha(x)) - s_1(x) + \xi_1(x) - \xi_2(x)|.$$
According to the definition of $s_1$, if $x \in X_{s,h(s)}$ (that is, $x$ is on the roof), then

$$s_1(x) = \sum_{j=1}^{h(s)} (\xi_2 - \xi_1) (\alpha^{-j}(x)) - G_1(x)$$

$$= \sum_{j=1}^{h(s)} (\xi_2 - \xi_1) (\alpha^{-j}(x)) - \sum_{j=0}^{h(s)} (\xi_2 - \xi_1)(\alpha^{-j}(x))$$

$$= -(\xi_2 - \xi_1)(x)$$

$$= 0.$$

We know that $s_1(\alpha(x)) = 0$ as $(\alpha^{-h(s)})(x) \in X_{s,0}$. It is then clear that

$$|s_1(\alpha(x)) - s_1(x) + \xi_1(x) - \xi_2(x)| = 0$$

if $x$ is in the roof set.

If $x$ is not in the roof, in other words, for $x \in X_{s,k}$ with $0 \leq k < h(s) - 1$, we have

$$s_1(\alpha(x)) - s_1(x) = (\xi_2 - \xi_1)(x) - \frac{G_1(x)}{h(s)}.$$

As $G_1(x) \in [0 - \frac{1}{5}, 1 + \frac{1}{2}]$ for all $x$, and we have $h(s) > 5/\varepsilon$ for all $s$, it then follows that

$$|s_1(\alpha(x)) - s_1(x) + \xi_1(x) - \xi_2(x)| < 2\varepsilon/5$$

for all $x \in X$.

Similarly, we have

$$|t_2 + \eta_1(x) - s_2(x) + s_2(\alpha(x)) - (t_2 + \eta_2(x))| = |s_2(\alpha(x)) - s_2(x) + \eta_1(x) - \eta_2(x)|$$

and

$$|s_2(\alpha(x)) - s_2(x) + \eta_1(x) - \eta_2(x)| < 2\varepsilon/5$$

for all $x \in X$.

So far, we have proved that

$$\text{dist} \left( (\text{id}_x \times R_{s_1} \times R_{s_2}) \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \circ (\text{id}_x \times R_{s_1} \times R_{s_2})^{-1}, \beta \times R_{\xi_2} \times R_{\eta_2} \right)$$

$$< \varepsilon/5 + 2\varepsilon/5 + 2\varepsilon/5$$

$$= \varepsilon.$$

As we can construct such conjugacy maps for all $\varepsilon > 0$, it follows that $\alpha \times R_{\xi_1} \times R_{\eta_1}$ is weakly approximately conjugate to $\beta \times R_{\xi_2} \times R_{\eta_2}$ if $\alpha$ is weakly approximately conjugate to $\beta$.

The “only if” part.

If a sequence of $\sigma_n$ in Homeo($X \times T^2$) approximately conjugates $\alpha \times R_{\xi_1} \times R_{\eta_1}$ to $\beta \times R_{\xi_2} \times R_{\eta_2}$, as $X$ is totally disconnected, we can write $\sigma_n$ as $\gamma_n \times \varphi$, with $\gamma_n \in \text{Homeo}(X)$ and $\varphi: X \rightarrow \text{Homeo}(T^2)$ being a continuous map.

Let $P: X \times T^2 \rightarrow X$ be defined by $P(x, (t_1, t_2)) = x$ (the canonical projection onto $X$). We can easily check that

$$P((\sigma_n \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \circ \sigma_n^{-1})(x, (t_1, t_2))) = (\gamma_n \circ \alpha \circ \gamma_n^{-1})(x).$$

As $\sigma_n \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \circ \sigma_n^{-1} \rightarrow \beta \times R_{\xi_2} \times R_{\eta_2}$, we have

$$P((\sigma_n \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \circ \sigma_n^{-1})(x, (t_1, t_2))) \rightarrow P((\beta \times R_{\xi_2} \times R_{\eta_2})(x, (t_1, t_2))),$$

which then implies that

$$(\gamma_n \circ \alpha \circ \gamma_n^{-1})(x) \rightarrow \beta(x) \text{ for all } x \in X.$$
From Lemma 4.8 we know that the if and only if condition for \( \alpha \times R_{\xi_1} \times R_{\eta_1} \) and \( \beta \times R_{\xi_2} \times R_{\eta_2} \) to be weakly approximately conjugate is that \( \alpha \) and \( \beta \) are weakly approximately conjugate.

One might be wondering whether we have weak approximate conjugacy between \( \alpha \times R_{\xi_1} \times R_{\eta_1} \) and \( \beta \times R_{\xi_2} \times R_{\eta_2} \), can we expect to have the isomorphism between C*-algebras \( C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1}) \) and \( C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2}) \)?

Generally speaking, weak approximate conjugacy is not enough to imply that the corresponding crossed product C*-algebras are isomorphic. Examples can be found in [Lin4], [LM1], [LM2], [LM3]. Before the definition of approximate K-conjugacy is given, the definition of asymptotic morphism will be given and a technical result needs to be mentioned.

**Definition 4.9.** A sequence of contractive completely positive linear maps \( \{\varphi_n\} \) from C*-algebra \( A \) to C*-algebra \( B \) is said to be an asymptotic morphism, if

\[
\lim_{n \to \infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0 \text{ for all } a, b \in A.
\]

**Proposition 4.10.** [Lin4]

Let \((X, \alpha)\) and \((X, \beta)\) be two dynamical systems. If there exists a sequence of homeomorphisms \( \sigma_n : X \to X \) such that \( \lim_{n \to \infty} \text{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}, \beta) = 0 \), then for a sequence of unitaries \( \{z_n\} \) in \( A_\alpha \) with

\[
\lim_{n \to \infty} \|z_n j_\alpha(f) - j_\alpha(f) z_n\| = 0 \text{ for all } f \in C(X),
\]

there exists a unital asymptotic morphism \( \{\varphi_n^\sigma\} \) from \( A_\beta \) to \( A_\alpha \) such that

\[
\lim_{n \to \infty} \|\psi_n^\sigma(u_\beta) - u_\alpha z_n\| = 0 \text{ and } \lim_{n \to \infty} \|\psi_n^\sigma(fj_\beta(f)) - j_\alpha(f \circ \sigma_n)\| = 0
\]

for all \( f \in C(X) \).

Proof. This is Proposition 3.1 in [Lin4]. \( \square \)

Now we can give the definition of approximate K-conjugacy between two dynamical systems \((X, \alpha)\) and \((X, \beta)\).

**Definition 4.11.** For two minimal dynamical systems \((X, \alpha)\) and \((Y, \beta)\), with \( X \) and \( Y \) being compact metrizable spaces, we say that \((X, \alpha)\) and \((Y, \beta)\) are approximately K-conjugate if there exist homeomorphisms \( \sigma_n : X \to Y \), \( \tau_n : Y \to X \), and an isomorphism \( \rho : K_s(C^*(\mathbb{Z}, Y, \beta)) \to K_s(C^*(\mathbb{Z}, X, \alpha)) \) between K-groups such that

\[
\sigma_n \circ \alpha \circ \sigma_n^{-1} \to \beta, \quad \tau_n \circ \beta \circ \tau_n^{-1} \to \alpha,
\]

and the associated discrete asymptotic morphisms \( \psi_n : B \to A \) and \( \varphi_n : A \to B \) induce the isomorphisms \( \rho \) and \( \rho^{-1} \) respectively.
**Remark:** According to Proposition 4.10, the weak approximate conjugacy maps will induce asymptotic morphisms. But it is not generally true that the asymptotic morphisms will induce a homomorphism of $K_0$ and $K_1$ data. In Definition 4.11, those approximate conjugacies must not only induce a pair of homomorphisms between $K_i(A)$ and $K_i(B)$, in addition, these homomorphisms must be a pair of isomorphisms that are inverses of each other.

For the classical case of minimal Cantor dynamical systems, it is shown in [LM3] that two Cantor minimal dynamical systems are approximately K-conjugate if and only if the corresponding crossed product C*-algebras are isomorphic. For the case of $(X \times \mathbb{T}, \alpha \times R_\xi)$, with $\alpha \in \text{Homeo}(X)$ being minimal homeomorphism and $\xi: X \to \mathbb{T}$ being a continuous map, similar results are obtained in Theorem 7.8 of [LM1].

Based on Theorem 4.7 and Lemma 4.8, we will give an if and only if condition for approximate K-conjugacy between $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$.

**Theorem 4.12.** Let $X$ be the Cantor set. Let $\alpha, \beta \in \text{Homeo}(X)$ be minimal homeomorphisms, and let $\xi_1, \xi_2, \eta_1, \eta_2: X \to \mathbb{T}$ be continuous map such that both $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$ are minimal rigid homeomorphism of $X \times \mathbb{T} \times \mathbb{T}$ (as in Definition 2.20). Use $A$ to denote the crossed product C*-algebra corresponding to the minimal system $(X \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1})$, and $B$ to denote the one corresponding to $(X \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2})$. Use $K^0(X, \alpha)$ to denote $C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2)\}$ and $K^0(X, \beta)$ to denote $C(X, \mathbb{Z})/\{f - f \circ \beta^{-1}: f \in C(X, \mathbb{Z}^2)\}$.

The following are equivalent:
1) $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2})$ are approximately K-conjugate,
2) There is an order isomorphism $\rho: K_0(B) \to K_0(A)$ that maps $K^0(X, \beta)$ to $K^0(X, \alpha)$.

**Proof.** 1) $\Rightarrow$ 2):

If $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2})$ are approximately K-conjugate, according to the definition of approximate K-conjugacy (Definition 4.11), there exists $\sigma_n \in \text{Homeo}(X \times \mathbb{T} \times \mathbb{T})$ such that
$$\text{dist}(\sigma_n \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \circ \sigma_n^{-1}, \beta \times R_{\xi_2} \times R_{\eta_2}) \to 0,$$
and the discrete asymptotic morphism induced by $\{\sigma_n: n \in \mathbb{N}\}$ will yield an isomorphism from $K_*(B)$ to $K_*(A)$.

That is, there exists an isomorphism
$$\phi_0: (K_0(B), K_0(B)_+, [1_B], K_1(B)) \to (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Define $\phi$ to be the restriction of $\phi_0$ on $K_0(A)$. We just need to show that $\phi$ maps $K^0(X, \beta)$ to $K^0(X, \alpha)$.

According to the Pimsner-Voiculescu six-term exact sequence (as in the proof of Proposition 2.19), we have
$$(j_\beta)_*(C(X \times \mathbb{T} \times \mathbb{T})) \cong K^0(X, \beta) = C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z}^2)\}.$$

As $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$ are approximately K-conjugate, for given projection $p \in M_\infty(B)$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we have $[p \circ \sigma_n] = [p \circ \sigma_m]$ in $K_0(A)$.

It is obvious that $[p \circ \sigma_n] \in (j_\alpha)_*(C(X \times \mathbb{T} \times \mathbb{T}))$. Then we can conclude that the isomorphism $\rho$ induced by the conjugacy maps will map $K^0(X, \beta)$ to $K^0(X, \alpha)$.

2) $\Rightarrow$ 1):

It is easy to check that 2) implies the following commutative diagram:
According to Theorem 5.4 of [LM3], \((N, \alpha) \sim \beta\) if \((N, \alpha) \sim (N, \beta)\) are weakly approximately conjugate.

By choosing \(n\) such that \(H(0) = 1\) without loss of generality, we can assume that \(\alpha \times R_{\xi_1} \times R_{\eta_1} \sim \alpha \times R_{\xi_2} \times R_{\eta_2}\) are \(C^*\)-strongly flip conjugate.

The map \(\rho\) above induces an order preserving isomorphism between \(K^0(X, \beta)\) (which is isomorphic to \(C(X, \mathbb{Z})/\{f - f \circ \beta^{-1}\}\), with order described as in Lemma 1.11) and \(K^0(X, \alpha)\) (which is isomorphic to \(C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1}\}\), with order described as in Lemma 1.11). Note that

\[
K_0(C^*(\mathbb{Z}, X, \alpha)) \cong C(X, \mathbb{Z})/\{g - g \circ \alpha^{-1} : g \in C(X, \mathbb{Z})\},
\]

with

\[
K_0(C^*(\mathbb{Z}, X, \alpha))_+ \cong C(X, \mathbb{Z})/\{g - g \circ \alpha^{-1} : g \in C(X, \mathbb{Z}), g \geq 0\}.
\]

It follows that there is an order isomorphism

\[\tilde{\rho} : (K_0(C^*(\mathbb{Z}, X, \beta)), K_0(C^*(\mathbb{Z}, X, \alpha))_+) \to (K_0(C^*(\mathbb{Z}, X, \beta)), K_0(C^*(\mathbb{Z}, X, \alpha))_+),\]

According to Theorem 5.4 of [LM3], \((X, \alpha) \sim (X, \beta)\) are approximately \(K\)-conjugate. Thus they are weakly approximately conjugate.

For any \(\varepsilon > 0\) and any finite subset \(F \subset C(X \times \mathbb{T} \times \mathbb{T})\), as \(\beta\) is minimal, we can find Kakutani-Rokhlin partition

\[P = \{X(s, k) : s \in S, 1 \leq k \leq H(s)\}\]

such that \(H(s) > \frac{32\pi}{\varepsilon}\) for all \(s \in S\) and \(\text{diam}(X(s, k)) < \frac{\varepsilon}{16}\).

As \(C(X \times \mathbb{T}_1 \times \mathbb{T}_2)\) is generated by

\[\{1, z_1, z_2 : D\ \text{is a clopen subset of} \ X, z_i \text{ is the identity function on} \ T_i\}\],

without loss of generality, we can assume that

\[F = \{1_{X(s, k)}, z_1 1_{X(s, k)}, z_2 1_{X(s, k)} : s \in S, 1 \leq k \leq H(s)\}\].

The fact that \((X, \alpha) \sim (X, \beta)\) are approximately \(K\)-conjugate implies that there exist \(\{\sigma_n \in \text{Homeo}(X) : n \in \mathbb{N}\}\) such that

\[\sigma_n \circ \alpha \circ \sigma_n^{-1} \to \beta.\]

By choosing \(n\) large enough, just as in the proof of the “if” part of Theorem 4.8, we get

\[(\sigma_n \circ \alpha \circ \sigma_n^{-1})(X(s, k)) = \beta(X(s, k)) \text{ for} s \in S, 1 \leq k \leq H(s).\]

Without loss of generality, we can assume that

\[\alpha(X(s, k)) = \beta(X(s, k)) \text{ for} s \in S, 1 \leq k \leq H(s).\]

As in the proof of “if” part of Theorem 4.8 there exist maps \(\{\text{id}_X \times R_{g_n} \times R_{h_n}\}_n\) such that

\[(\text{id}_X \times R_{g_n} \times R_{h_n}) \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \circ (\text{id}_X \times R_{g_n} \times R_{h_n})^{-1} \to (\beta \times R_{\xi_2} \times R_{\eta_2}),\]

with all the \(g_n, h_n : X \to \mathbb{T}\) being continuous functions as defined in the proof of Theorem 4.8.

We will show that the conjugacy maps \(\{\text{id}_X \times R_{g_n} \times R_{h_n} : n \in \mathbb{N}\}\) will induce an isomorphism between \(K^*_0(B)\) and \(K^*_0(A)\).
The idea is like this: We know that these two dynamical systems $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$ are $C^*$-strongly flip conjugate. Thus there exists $\psi_n : B \to A$ such that the following diagram approximately commutes:

\[
\begin{array}{c}
B \\
\psi_n \\
\downarrow j_B \\
C(X \times T \times T) \\
\xrightarrow{\chi_n} \\
\uparrow j_A \\
A
\end{array}
\]

As we had assumed that (without loss of generality) $\alpha(X(s,k)) = \beta(X(s,k))$ for $s \in S, k = 1, \ldots, H(s)$, the $\chi_n$ in the diagram above satisfies

\[
\text{dist}(\chi_n(x), x) < \text{diam}(X(s,k)) < \varepsilon/M
\]

for $x \in X(s,k)$. In other words, restricted on $C(X \times T \times T)$, $\chi_n$ is close to the identity map.

Note that $\{\psi_n\}$ are isomorphisms and $[\psi_n] = [\psi_m]$ in $KL(B, A)$ for $m, n$ large enough. If we can find $W_n \in U(A)$ such that $f \circ \sigma_n$ is close to $W_n^* \psi_n(f)W_n$ in $A$, and $W_n^* \psi_n(u_B)W_n$ is close to $u_A z_n$ in $A$, where $z_n$ is a unitary element that “almost” commutes with $C(X \times T \times T)$, then it follows that the conjugacy maps $\{\text{id}_X \times R_{\eta_n} \times R_{\nu_n} : n \in N\}$ will induce an isomorphism between $K_s(B)$ and $K_s(A)$.

The complete proof is as below:

Let $g_1, g_2, f_1, f_2$ be as defined in the proof of Lemma 4.8 and let

\[
\mathcal{F}_1 = \{g_1 \cdot 1_{X(s,k)} \cdot f_1 \cdot 1_{X(s,k)} : s \in S, 1 \leq k \leq H(s)\}.
\]

We can further divide $\alpha^{-1}(X(s,1))$ into the disjoint union of clopen sets $Y(s,1), Y(s,2), \ldots, Y(s,N(s))$, and choose $x_{s,j} \in Y(s,j)$ such that

\[
|f(x) - f(x_{s,j})| < \varepsilon/16 \quad \text{for all } f \in \mathcal{F}_1, 1 \leq j \leq N(s), s \in S.
\]

Let $G_1, G_2$ be the same as the one defined in the proof of Theorem 4.8. That is, $G_1$ is the lifting of $g_1(x) = \sum_{j=1}^{h(s)} (\xi_2 - \xi_1)(\alpha^{-j}(\alpha^{h(s)} - k(x)))$, $G_2$ is the lifting of $g_2(x) = \sum_{j=1}^{h(s)} (\xi_2 - \xi_1)(\alpha^{-j}(\alpha^{h(s)} - k(x)))$, and $G_i(x) \in [0 - \frac{1}{4}, 1 + \frac{1}{4}]$. As both $G_1, G_2$ are path connected to the zero function, it is clear that

\[
[z_i \cdot 1_{Y(s,j)}] = [z_i \cdot e^{-i2\pi G_1/H(s)} \cdot 1_{Y(s,j)}]
\]

in $K_1(A)$ for $i = 1, 2$ and $k = 1, 2$.

Let $\iota_{s,j} : C(1_{Y_{s,j}} \times T \times T) \to 1_{Y_{s,j}} \cdot A \cdot 1_{Y_{s,j}}$ be the inclusion map. Let two homomorphisms

\[
\Delta_{s,j}, \delta_{s,j} : C(T^2) \to C(1_{Y_{s,j}} \times T \times T)
\]

be defined by

\[
\Delta_{s,j}(f) = \text{id}_{Y(s,j)} \otimes f
\]

and

\[
\delta_{s,j}(f)(x, z_1, z_2) = \text{id}_{Y_{s,j}}(x) \cdot f(z_1 \cdot e^{i2\pi G_1(x_{s,j})/H(s)}, z_2 \cdot e^{i2\pi G_2(x_{s,j})/H(s)}).
\]
Lemma 2.5.7, by taking $B$ up to 1 for all $f$ in $F$. It is clear that these two maps are monomorphisms.

By Proposition 2.21, $\text{TR}(A) = 0$, and it follows that $\text{TR}(1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}) = 0$.

As $G_1, G_2$ are contractible, we can claim that

$$[\iota_{s,j} \circ \Delta_{s,j}] = [\iota_{s,j} \circ \delta_{s,j}] \text{ in } KL(C(T^2), 1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}).$$

For every $f \in 1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}$, and for every tracial state $\tau$ on $1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}$, consider $\tau((\iota_{s,j} \circ \delta_{s,j})(f))$ and $\tau((\iota_{s,j} \circ \delta_{s,j})(f))$. By Lemma 2.4, we can regard $1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}$ as the crossed product C*-algebra of the induced minimal homeomorphism of $Y_s,j \times \mathbb{T} \times \mathbb{T}$. As $\alpha \times R_\xi \times R_\eta$ is rigid, it follows that the traces on $1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}$ also corresponds to such measures like $\mu \times v$, with $v$ being the Lebesgue measure on the torus.

Now we have

$$\tau((\iota_{s,j} \circ \Delta_{s,j})(f)) = \tau(\text{id}_{Y_s,j} \otimes f)$$

$$= \mu(Y(s,j)) \cdot \int_{T^2} f((z_1, z_2)) \, dv$$

$$= \mu(Y(s,j)) \cdot \int_{T^2} f(z_1 \cdot e^{i2\pi G_1(x_{s,j})/H(s)} \cdot z_2 \cdot e^{i2\pi G_2(x_{s,j})/H(s)}) \, dv$$

$$= \tau((\iota_{s,j} \circ \delta_{s,j})(f)).$$

As $\text{TR}(1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}) = 0$, $[\iota_{s,j} \circ \Delta_{s,j}] = [\iota_{s,j} \circ \delta_{s,j}]$ and $\tau((\iota_{s,j} \circ \Delta_{s,j})(f)) = \tau((\iota_{s,j} \circ \delta_{s,j})(f))$ for all $\tau \in T(1_{Y_s,j} \cdot A \cdot 1_{Y_s,j})$. According to Theorem 3.4 of Lin2, the two monomorphisms $\iota_{s,j} \circ \Delta_{s,j}$ and $\iota_{s,j} \circ \delta_{s,j}$ are approximately unitarily equivalent. Thus there exists a unitary element $v_{s,j} \in 1_{Y_s,j} \cdot A \cdot 1_{Y_s,j}$ such that

$$\|v_{s,j}^* z_i q_{s,j} v_{s,j} - z_i e^{-i2\pi G_1(x_{s,j})/H(s)} 1_{Y_s,j}\| < \varepsilon/(16K)$$

for all $s \in S, 1 \leq k \leq H(s), 1 \leq j \leq N(s)$.

Let $v_s = \sum_{j=1}^{N(s)} v_{s,j}$. As $Y_{s,1}, Y_{s,2}, \ldots, Y_{s,N(s)}$ are mutually disjoint, we have

$$\|(v_s^k)^* z_i f(x) 1_{\alpha^{-1}(X(s,1))} v_s^k - z_i e^{-2\pi k G_1(x)/H(s)} f(x) 1_{\alpha^{-1}(X(s,1))}\| < \varepsilon/16 + K \varepsilon/(16K) + \varepsilon/16$$

$$< \varepsilon/4.$$ 

for all $f \in F_1, s \in S$.

Let $\mathcal{F}_2 = \mathcal{F} \cup \{1_{Y_s,j} \cdot z_i 1_{Y_s,j} \cdot f 1_{\alpha^{-1}(X(s,1))} \cdot f \in \mathcal{F}_1, s \in S, 1 \leq k \leq H(s)\}.$

As $\alpha \times R_\xi \times R_\eta$ is $C^*$-strongly flip conjugate to $\alpha \times R_\xi \times R_\eta$, for any $\delta > 0$, and for the $\mathcal{F}_2 \subset C(X \times \mathbb{T} \times \mathbb{T}),$ there exists a $C^*$-algebra isomorphism $\psi: B \rightarrow A$ such that

$$\|\psi(j_{\delta}(f)) - j_{\delta}(f)\| < \delta$$

and $\|\psi(u_B)^* j_{\delta}(f) \psi(u_B) - j_{\delta}(f \circ \beta)\| < \delta$ for all $f \in \mathcal{F}_2$.

Note that $1_{X(s,k)},$ for $s \in S$ and $1 \leq k \leq H(s)$, are mutually orthogonal projections and add up to $1_B$, and $\{1_{X(s,k)}: s \in S, 1 \leq k \leq H(s)\} \subset \mathcal{F}_2$. According to the perturbation lemma Lin2 Lemma 2.5.7, by taking $\delta$ to be small enough, the fact that $\|\psi(j_{\delta}(f)) - j_{\delta}(f)\| < \delta$ will imply that there exists $v \in U(A)$ such that

$$v \approx \varepsilon/(16K^2) \psi(u_B)$$
and
\[ v^* 1_{X(s,k)} v = 1_{X(s,k)} \circ \beta \text{ and } \|v^* f v - f \circ \beta\| < \varepsilon/(4K) \text{ for all } f \in F_2. \]

Define \( W = \sum_{s \in S} \sum_{k=1} H(s) 1_{X(s,k)} v^{-k} v_u^k u^k. \) Then we can check that
\[
W^* W = \left( \sum_{s \in S} \sum_{k=1} H(s) 1_{X(s,k)} v^{-k} v_u^k u^k \right)^* \cdot \sum_{s' \in S} \sum_{k'=1} H(s') 1_{X(s',k')} v^{-k'} v_u^{k'} u^{k'}
\]
\[ = \sum_{s \in S} \sum_{k=1} H(s) u^{-k} v_u^{-k} v_k 1_{X(s,k)} 1_{X(s,k)} v^{-k} v_u^k u^k \]
\[ = \sum_{s \in S} \sum_{k=1} H(s) u^{-k} 1_{\alpha^{-1}(X(s,1))} v_k^k u^k \]
\[ = \sum_{s \in S} \sum_{k=1} H(s) 1_{\alpha^k(\alpha^{-1}(X(s,1)))} \]
\[ = \sum_{s \in S} H(s) 1_{X(s,k)} \]
\[ = 1_A. \]

As \( \text{TR}(A) = 0, \) we have \( \text{trs}(A) = 1. \) Thus \( W^* W = 1_A \) implies that \( WW^* = 1_A. \) So far, it is checked that \( W \) is a unitary element in \( A. \)

As
\[ \| (v_u^k)^* z_1 f (x) 1_{\alpha^{-1}(X(s,1))} v_u^k - z e^{-2\pi k G_i(x)/H(s)} f (x) 1_{\alpha^{-1}(X(s,1))} \| < \varepsilon/4 \]
and
\[ \| v^* f v - f \circ \beta \| < \varepsilon/(4K) \text{ for all } f \in F_2 \text{ and for all } f \in F_2, \]
we have
\[
W^* z_1 1_{X(s,k)} W = \left( \sum_{s_1 \in S} \sum_{k_1=1} H(s_1) 1_{X(s_1,k_1)} v^{-k_1} v_u^{k_1} u^{k_1} \right)^* z_1 1_{X(s,k)} \left( \sum_{s_2 \in S} \sum_{k_2=1} H(s_2) 1_{X(s_2,k_2)} v^{-k_2} v_u^{k_2} u^{k_2} \right)
\]
\[ = \left( \sum_{s_1 \in S} \sum_{k_1=1} u^{-k_1} v_u^{-k_1} v^{k_1} 1_{X(s_1,k_1)} 1_{X(s_1,k_1)} u^{-k_1} v_u^{k_1} \right) z_1 1_{X(s,k)} \left( \sum_{s_2 \in S} \sum_{k_2=1} u^{-k_2} v_u^{-k_2} v^{k_2} 1_{X(s_2,k_2)} 1_{X(s_2,k_2)} u^{-k_2} v_u^{k_2} \right)
\]
\[ = u^{-k} v_u^{-k} v^{k} 1_{X(s,k)} z_1 1_{X(s,k)} 1_{X(s,k)} u^{-k} v_u^{k} u^{k}
\]
\[ = u^{-k} v_u^{-k} v^{k} (z_1 1_{X(s,k)}) u^{-k} v_u^{k} u^{k}
\]
\[ \approx \varepsilon/(4K) u^{-k} v_u^{-k} ((z_1 1_{X(s,k)}) \circ \beta^k) v_u^{k} u^{k}
\]
\[ \approx \varepsilon/(4K) + \varepsilon/4 \left( z 1_{X(s,k)} \right) \circ \sigma, \]
where
\[
\sigma(x, t_1, t_2) = \left( x, t_1 + \sum_{j=1}^k \xi_2 (\alpha^{j-1}(\beta^{-k}(x))) - \xi_1 (\beta^{-j}(x)) \right) - kG_1(x)/H(s),
\]
\[
t_2 + \left( \sum_{j=1}^k \eta_2 (\alpha^{j-1}(\beta^{-k}(x))) - \eta_1 (\beta^{-j}(x)) \right) - kG_1(x)/H(s),
\]
for \(x \in X(s, k)\) with \(s \in S\) and \(1 \leq k \leq H(s)\).

Then it follows that
\[
\|W^* z_i 1_{X(s,k)} W - (z_i 1_{X(s,k)}) \circ \sigma\| < K(\varepsilon/4K) + \varepsilon/4 < \varepsilon.
\]

Similar to the proof of Theorem 4.8, we have
\[
\text{dist}(\sigma \circ (\alpha \times R_{\xi_1} \times R_{\eta_1}) \sigma^{-1}, \beta \times R_{\xi_2} \times R_{\eta_2}) < \varepsilon.
\]

Consider the map \(\text{ad}W \circ \psi\), we have that
\[
\|\text{ad}W \circ \psi)(j_\beta(f)) - j_\alpha(f \circ \sigma)\| < \varepsilon + \delta.
\]

If \((\text{ad}W \circ \psi)\) maps \(u_B\) to \(u_A\) or \(u_A \cdot y\) such that \(\|y f - f y\| < \varepsilon\) for all \(f \in F\), then it follows that the K-map induced by approximate conjugacy map \(\sigma\) (restricted to \(F\)) will coincide with \([\text{ad}W \circ \psi] \in KL(B, A)\).

In fact, we can check that
\[
W^* v^* W z_i 1_{X(s,k)} W^* v W \approx_d \varepsilon u_A^* z_i 1_{X(s,k)} u_A,
\]
which then implies that \(\|y f - f y\| < \varepsilon\) if we define \(y = u_A^*(W^* v W) \in U(A)\).

As
\[
(\text{ad}W \circ \psi)(u_B) = W \psi(u_B) W \approx_{\varepsilon/(16K^2)} W^* v W = u_A y,
\]
we may claim that the K-map induced by approximate conjugacy map \(\sigma\) (restricted to \(F\)) will coincide with \([\text{ad}W \circ \psi] \in KL(B, A)\).

As \(C(X \times T \times T)\) is separable, by taking \(F\) to be large enough and \(\varepsilon \to 0\), it follows that the weak approximate conjugacy map \(\sigma\) will induce an isomorphism from \(K_i(B)\) to \(K_i(A)\), which finishes the proof. \(\Box\)

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