Bosonization Rules for Electron-Hole Systems - II

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Here we write down and prove closed commutation rules for Fermi bilinears in a two-component Fermi system in terms of the relevant sea-bosons. We show how the commutation rules come out correctly within the RPA-approximation as do the zero temperature correlation functions. We write down hamiltonians of nonrelativistic interacting electron-hole systems and point out several attractive features such as the natural roles excitons play in this language. We then use this language to derive a set of equations analogous to the Semiconductor Bloch Equations in the presence of phonons and with external fields such as those present in pump-probe experiments. We solve these equations using parameters of interesting and topical materials such as GaN and compare with some recent experimental data.

I. BILINEAR SEA-BOSON CORRESPONDENCE

Here we write down the formal correspondence between Fermi bilinears and sea-bosons suitably generalised to two-component systems. The correspondence presented here is necessarily approximate but one that conforms to the spirit of the random-phase approximation, which in the two-component system corresponds to the exciton approximation. However, just as in the one-component case, new physics could be extracted by suitably generalising the notion of the random-phase approximation, we find that here too we may extract new physics by generalising the notion of the exciton approximation so that one goes beyond the dilute limit and includes experimentally relevant situations such as those in which a large number of real carriers are created by external fields. The discussion presented here follows closely the discussion in the response to the comment on our previous work\(^1\). Let us write the Fermi bilinear sea-boson correspondence with spin. (\(q \neq 0\))

\[
\begin{align*}
\hat{c}_{k+q/2,\sigma}^{\dagger} & = \Lambda_{k,\sigma}(q, \sigma') a_{k,\sigma}(-q, \sigma') + a_{k,\sigma'}^{\dagger}(q, \sigma)a_{k,\sigma}(-q, \sigma) \\
& + \sum_{q_1,\sigma_1} a_{k+q/2-q_1/2,\sigma_1}^{\dagger}(q_1 \sigma_1) a_{k-q_1/2,\sigma_1}(q_1 - q \sigma') \\
& - \sum_{q_1,\sigma_1} a_{k-q/2+q_1/2,\sigma_1}^{\dagger}(q_1 \sigma_1) a_{k+q_1/2,\sigma_1}(q_1 - q \sigma_1)
\end{align*}
\]

(1)

Here,

\[
\Lambda_{k,\sigma}(q, \sigma') = \sqrt{n_{k+q/2,\sigma}(1 - n_{k-q/2,\sigma'})}
\]
Let us make the following identifications,

\[ c_{k\uparrow} = c_k \]  
\[ c_{k\downarrow} = d^\dagger_{-k} \]  

Taking a cue from the one-component case let us now argue that (for both \( q = 0 \) and \( q \neq 0 \))

\[ d_{-k-q/2}c_{k-q/2} \approx \sqrt{(1 - \bar{n}_e(k - q/2))(1 - \bar{n}_h(-k - q/2))}a_{k\downarrow}(-q \uparrow) \]

\[ + \sqrt{\bar{n}_e(k - q/2)\bar{n}_h(-k - q/2)}a_{k\uparrow}(q \downarrow) \]  
\[ \Lambda_1(k, q) = \sqrt{(1 - \bar{n}_e(k - q/2))(1 - \bar{n}_h(-k - q/2))} \]  
\[ \Lambda_2(k, q) = \sqrt{\bar{n}_e(k - q/2)\bar{n}_h(-k - q/2)} \]

The above form in Eq. (9) automatically satisfies,

\[ [d_{-k-q/2}c_{k-q/2}, d_{-k'-q'/2}c_{k'-q'/2}] = 0 \]  

Next we would like the following to happen (for both \( q = 0 \) and \( q \neq 0 \), as it does in the Fermi language,

\[ [d_{-k-q/2}c_{k-q/2}, c^\dagger_{k'-q'/2}d^\dagger_{-k'-q'/2}] = \]

\[ \delta_{k,k'}\delta_{q,q'} - d^\dagger_{-k'-q'/2}d_{-k-q/2}\delta_{k'-q'/2}c_{k'-q'/2} - c^\dagger_{k'-q'/2}c_{k-q/2}\delta_{k+q/2,k'+q'/2} \approx \delta_{k,k'}\delta_{q,q'}(1 - \langle d^\dagger_{-k-q/2}d_{-k-q/2}\rangle - \langle c^\dagger_{k-q/2}c_{k-q/2}\rangle) \]

\[ (1 - \bar{n}_e(k - q/2))(1 - \bar{n}_h(-k - q/2)) - \bar{n}_e(k - q/2)\bar{n}_h(-k - q/2) = 1 - (d^\dagger_{-k-q/2}d_{-k-q/2}) - (c^\dagger_{k-q/2}c_{k-q/2}) \]

The expectation value is with respect to the full interacting ground state including (especially including) the external fields that allow for significant real populations to be generated. Let us now argue (again inspired by the one-component system),

\[ d^\dagger_{-k}d_{-k} = n^{(0)}_{\bar{c}}(k) - \sum_{\mathbf{q}_1\sigma_1} a^\dagger_{\mathbf{k} - \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \downarrow)a_{\mathbf{k} - \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \downarrow) + \sum_{\mathbf{q}_1\sigma_1} a^\dagger_{\mathbf{k} + \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \sigma_1)a_{\mathbf{k} + \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \sigma_1) \]  

and,

\[ c^\dagger_{k}c_{k} = n^{(0)}_{\bar{c}}(k) + \sum_{\mathbf{q}_1\sigma_1} a^\dagger_{\mathbf{k} - \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \uparrow)a_{\mathbf{k} - \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \uparrow) - \sum_{\mathbf{q}_1\sigma_1} a^\dagger_{\mathbf{k} + \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \sigma_1)a_{\mathbf{k} + \mathbf{q}_1/2\sigma_1}(\mathbf{q}_1 \sigma_1) \]
Here $n^{(0)}_h(k)$ and $n^{(0)}_e(k)$ account for possible doping in the system. In other words, the presence of excess charge. These two put together obey the attractive identity (charge conservation),

$$\sum_k c^\dagger_k c_k - \sum_k d^\dagger_{-k} d_{-k} = Q = \sum_k [n^{(0)}_e(k) - n^{(0)}_h(k)]$$ (13)

Now we move on to the off-diagonal (in the indices) parts ($q \neq 0$),

$$d^\dagger_{-k+q/2} d_{-k-q/2} = -\Lambda_e(k, q) a_{k\downarrow}(q \downarrow) - \Lambda_h(k, q) a_{k\uparrow}(q \uparrow)$$ (14)

$$c^\dagger_{k+q/2} c_{k-q/2} = \Lambda_e(k, q) a_{k\uparrow}(q \uparrow) + \Lambda_e(k, q) a_{k\downarrow}(q \downarrow)$$ (15)

Assuming that the ground states (of the non-interacting system) are annihilated by the sea-bosons $a_{k\downarrow}(q \downarrow)$ and $a_{k\uparrow}(q \uparrow)$

$$a_{k\downarrow}(q \downarrow)|\text{Free}\rangle = 0, \quad a_{k\uparrow}(q \uparrow)|\text{Free}\rangle = 0$$ (16)

This means,

$$\langle c^\dagger_{k+q/2} c_{k-q/2} c^\dagger_{k+q/2} c^{\dagger}_{k+q/2} \rangle = \bar{n}_e(k + q/2)(1 - \bar{n}_e(k - q/2)) = \Lambda^2_e(k, q)$$ (17)

and similarly for the holes.

$$\Lambda_e(k, q) = \sqrt{\bar{n}_e(k + q/2)(1 - \bar{n}_e(k - q/2))}$$ (18)

$$\Lambda_h(k, q) = \sqrt{\bar{n}_h(-k + q/2)(1 - \bar{n}_h(-k - q/2))}$$ (19)

Then we make a leap of faith and suggest that the same should hold even when there are interactions present and even when external fields are present. That is, the $\bar{n}_h(k) = \langle d^\dagger_{-k} d_{-k} \rangle$ now represents the expectation value with respect to the full interacting ground state. Lastly we would like to point out the internal self-consistency of this approach by computing the commutator between the diagonal and the off-diagonal bilinears. We find much to our relief that no matter what the choices for the coefficients $\Lambda_1, \Lambda_2, \Lambda_e$ and $\Lambda_h$ are, we recover the following exact identities.

$$[d_{-k-q/2} c_{k-q/2}, c^\dagger_{p} c_{p}] = d_{-k-q/2} c_{k-q/2} \delta_{p,k-q/2}$$ (20)

$$[d_{-k-q/2} c_{k-q/2}, d^\dagger_{-p} d_{-p}] = d_{-k-q/2} c_{k-q/2} \delta_{p,k+q/2}$$ (21)

$$[c^\dagger_{k+q/2} c_{k-q/2}, c^\dagger_{p} c_{p}] = c^\dagger_{k+q/2} c_{k-q/2} \delta_{p,k+q/2} - \delta_{p,k+q/2}$$ (22)

$$[d^\dagger_{-k+q/2} d_{-k-q/2}, d^\dagger_{-p} d_{-p}] = d^\dagger_{-k+q/2} d_{-k-q/2} \delta_{p,k+q/2} - \delta_{p,k+q/2}$$ (23)
Thus we have written down a potentially useful set of identities. Let us write down the hamiltonian of free electrons and holes. In the Fermi language it is,

$$H_{free} = \sum_k \varepsilon^c(k)c^\dagger_k c_k + \sum_k \varepsilon^h(k)d^\dagger_{-k} d_{-k} + \sum_q \Omega_{LO} b^\dagger_q h_q$$

(24)

The kinetic energy of the LO-phonon modes is also included. In the sea-boson language it may be expressed as follows:

$$H_{free} = -\sum_{kq \neq 0} (\varepsilon^c(k - q/2) + \varepsilon^h(k + q/2))a^\dagger_{k\uparrow}(q \downarrow)a_{k\uparrow}(q \downarrow)$$

$$-\sum_k \left(\frac{k^2}{2\mu} + E_g\right) a^\dagger_{k\uparrow}(0 \downarrow)a_{k\uparrow}(0 \downarrow) + \sum_{kq \neq 0} (\varepsilon^h(k - q/2) - \varepsilon^h(k + q/2))a^\dagger_{k\downarrow}(q \downarrow)a_{k\downarrow}(q \downarrow)$$

$$+ \sum_{kq \neq 0} (\varepsilon^c(k - q/2) + \varepsilon^c(k + q/2))a^\dagger_{k\uparrow}(q \uparrow)a_{k\uparrow}(q \uparrow) + \sum_k \left(\frac{k^2}{2\mu} + E_g\right) a^\dagger_{k\downarrow}(0 \uparrow)a_{k\downarrow}(0 \uparrow)$$

$$+ \sum_{kq \neq 0} (\varepsilon^c(k + q/2) - \varepsilon^c(k - q/2))a^\dagger_{k\uparrow}(q \uparrow)a_{k\uparrow}(q \uparrow)$$

(25)

Again it may be noted that objects such as $a_{k\uparrow}(0 \uparrow)$ and $a_{k\downarrow}(0 \downarrow)$ are omitted from the formalism. The fact that the Fermi bilinears all evolve properly with respect to this hamiltonian is apparent without the need to perform any calculations. This is a strong indication that we are on the right track. It may puzzle the reader that we have included an object such as $a_{k\downarrow}(0 \uparrow)$ in the above formula. This is due to the following reason. The commutation rule $[d_{-k}c_k, c^\dagger_k, d^\prime_{-k\prime}]$ does not come out right if we don’t. Let us now write down some typical interaction terms.

$$H_{e-h} = -\sum_{q \neq 0} V_{eh}(q) \sum_{k,k'} e^c_{k+q/2} d^\dagger_{-k'-q/2} d^\prime_{-k'-q/2} c_k$$

(26)

This may be recast in the sea-boson language as follows,

$$H_{e-h} = -\sum_{q \neq 0} V_{eh}(q) \sum_{k,k'} [\Lambda_1(k+2+k'/2+q/2, k'-k) a^\dagger_{k+2+k'/2+q/2}(k-k')] + \Lambda_2(k+2+k'/2+q/2, k'-k) a_{k+2+k'/2}$$

$$\Lambda_1(k+2+k'/2-q/2, k'-k) a_{k+2+k'/2-q/2}(k-k')]$$

(27)

Let us now try and write down the e-e/h-h repulsion terms(let us now focus on an undoped system),

$$H_{e-e} = \sum_{q \neq 0} \frac{\nu(q)}{2V} \rho^c(q) \rho^c(-q)$$

(28)

$$H_{h-h} = \sum_{q \neq 0} \frac{\nu(q)}{2V} \rho^h(q) \rho^h(-q)$$

(29)
\[
\rho^e(q) = \sum_k \Lambda_e(k, q) a_{k \uparrow}(-q \uparrow) + \sum_k \Lambda_e(k, -q) a_{k \uparrow}^\dagger(q \uparrow) \quad (30)
\]

\[
\rho^h(q) = -\sum_k \Lambda_h(k, q) a_{k \downarrow}(-q \downarrow) - \sum_k \Lambda_h(k, -q) a_{k \downarrow}^\dagger(q \downarrow) \quad (31)
\]

The coupling to phonons may be written as follows,

\[
H_{ph} = \sum_{q \neq 0} \frac{M_q}{\sqrt{V}} (b_q + b_{-q}^\dagger)(\rho^e(q) - \rho^h(q)) \quad (32)
\]

It may be seen that only in the presence of real charge distributions do the electron-electron/hole-hole repulsion and coupling to phonons contribute appreciably to the Hamiltonian. This means that in the undoped case in the absence of external fields we expect only the excitonic contribution broadened perhaps only via coupling to photons (which is ignored here). External fields, especially pump fields above the band gap cause significant real populations of carriers and these in turn relax by emitting phonons and through Coulomb scattering. Thus the formalism we have written down is simple and ideal for the study of these systems. The coupling to external fields may be written as,

\[
H_{ext}(t) = \left(\frac{|e|}{\mu c}\right) \vec{A}_{ext}(t) \cdot \vec{p}_{vc} \sum_k \left[ \Lambda_1(k, 0) a_{k \downarrow}(0 \uparrow) + \Lambda_2(k, 0) a_{k \uparrow}^\dagger(0 \downarrow) \right] + \left(\frac{|e|}{\mu c}\right) \vec{A}_{ext}(t)^\dagger \cdot \vec{p}_{vc} \sum_k \left[ \Lambda_1(k, 0) a_{k \uparrow}^\dagger(0 \uparrow) + \Lambda_2(k, 0) a_{k \downarrow}(0 \downarrow) \right] \quad (33)
\]

Let us now write down the various equations of motion of this system.

\[
\frac{\partial}{\partial t} a_{k \downarrow}(0 \uparrow) = (\epsilon^e(k) + \epsilon^h(k)) a_{k \downarrow}(0 \uparrow) - \sum_{Q \neq 0} \frac{\nu_{eh}(Q)}{V} a_{k - Q \downarrow}(0 \uparrow)
\]

\[+ \sum_{Q \neq 0} \frac{\nu_{eh}(Q)}{V} [(1 - \Lambda_1(k, 0)\Lambda_1(k - Q, 0)) a_{k - Q \downarrow}(0 \uparrow) - \Lambda_1(k, 0)\Lambda_2(k - Q, 0) a_{k \uparrow}^\dagger(0 \downarrow)] + \left(\frac{|e|}{\mu c}\right) \vec{A}_{ext}(t) \cdot \vec{p}_{vc} \Lambda_1(k, 0) \quad (34)
\]
\[
\begin{align*}
    i \frac{\partial}{\partial t} a_{k\uparrow}(0 \downarrow) &= - (\epsilon^e(k) + \epsilon^h(k)) a_{k\uparrow}(0 \downarrow) \\
    &\quad - \sum_{Q \neq 0} \frac{V_{eh}(Q)}{V} \Lambda_2(k,0)[\Lambda_1(k + Q,0) a_{k+Q\uparrow}(0 \uparrow) + \Lambda_2(k + Q,0) a_{k+Q\downarrow}(0 \downarrow)] \\
    &\quad + \left( \frac{|e|}{\mu c} \right) \bar{A}_{ext}(t). \hat{\rho}_e \Lambda_2(k,0) \\
    i \frac{\partial}{\partial t} a_{k\uparrow}(q \uparrow) &= \frac{k \cdot q}{m_e} a_{k\uparrow}(q \uparrow) + \frac{v(q)}{V} \Lambda_e(k, -q) \rho^{(e)}(-q) + \frac{M_q}{\sqrt{V}} X_q \Lambda_e(k, -q) \\
    i \frac{\partial}{\partial t} a_{k\downarrow}(q \downarrow) &= \frac{k \cdot q}{m_h} a_{k\downarrow}(q \downarrow) + \frac{v(q)}{V} \Lambda_h(k, -q) \rho^{(h)}(-q) \quad - \frac{M_q}{\sqrt{V}} X_q \Lambda_e(k, q) \\
    i \frac{\partial}{\partial t} a_{k\uparrow}^\dagger(-q \uparrow) &= \frac{k \cdot q}{m_e} a_{k\uparrow}^\dagger(-q \uparrow) - \frac{v(q)}{V} \Lambda_e(k, q) \rho^{(e)}(-q) \quad - \frac{M_q}{\sqrt{V}} X_q \Lambda_e(k, q) \\
    i \frac{\partial}{\partial t} a_{k\downarrow}^\dagger(-q \downarrow) &= \frac{k \cdot q}{m_h} a_{k\downarrow}^\dagger(-q \downarrow) + \frac{v(q)}{V} \Lambda_h(k, q) \rho^{(h)}(-q) \quad - \frac{M_q}{\sqrt{V}} X_q \Lambda_h(k, q) \\
    i \frac{\partial}{\partial t} X_q &= (2i\Omega_{LO}) P_{-q} \\
    i \frac{\partial}{\partial t} P_{-q} &= \frac{\Omega_{LO}}{2\hbar} X_q - i \frac{M_q}{\sqrt{V}} (\rho^{(e)}(-q) - \rho^{(h)}(-q)) 
\end{align*}
\]

The last four equations of motion only affect the electron and hole populations but do not impact directly upon the polarization or induced currents. This means that electron-electron and hole-hole repulsion and electron-phonon interaction change the distributions of electrons and holes and the electron-hole attraction determines the absorption spectrum. Let us now write down the electron and hole populations,

\[
\begin{align*}
    \bar{n}_h(k) &= -\langle a_{k\uparrow}^\dagger(0 \downarrow) a_{k\uparrow}(0 \downarrow) \rangle + \langle a_{k\downarrow}^\dagger(0 \uparrow) a_{k\downarrow}(0 \uparrow) \rangle \\
    &= \sum_{q \neq 0} \langle a_{k-q/2\uparrow}^\dagger(q \downarrow) a_{k-q/2\downarrow}(q \downarrow) \rangle + \sum_{q \neq 0} \langle a_{k+q/2\uparrow}^\dagger(q \downarrow) a_{k+q/2\downarrow}(q \downarrow) \rangle \\ \\
    \bar{n}_e(k) &= \langle a_{k\downarrow}^\dagger(0 \uparrow) a_{k\downarrow}(0 \uparrow) \rangle - \langle a_{k\uparrow}^\dagger(0 \downarrow) a_{k\uparrow}(0 \downarrow) \rangle \\
    &= \sum_{q \neq 0} \langle a_{k-q/2\uparrow}^\dagger(q \uparrow) a_{k-q/2\downarrow}(q \uparrow) \rangle - \sum_{q \neq 0} \langle a_{k+q/2\uparrow}^\dagger(q \uparrow) a_{k+q/2\downarrow}(q \uparrow) \rangle 
\end{align*}
\]

In the above sets of equations we have ignored the contribution from objects such as \( a_{k\uparrow}(q \downarrow) \) with \( q \neq 0 \). The reason being that these contributions are difficult to deal with. The practical consequences of this assumption means that we have to restrict our attention to large \( k \). Namely that we must ensure that \( k \) in the above equation is much larger than any inverse length-scale in the problem. Thus we expect our theory to be poor for \( k \) small. This is in fact the case as we shall soon find out. The analysis including
and the whole system proceeds to evolve accordingly. Let us now introduce several Green functions.

However once relaxation processes begin, the distributions respond appropriately and the system must be solved self-consistently. A quantity such as \( a_{k\sigma}(q, t) a_{k'\sigma}(q, 0) \) is zero when the electrons and holes have "ideal" momentum distributions (that is, identically zero for undoped systems). However, when they start to acquire non-zero values due to external fields the above quantity also begins to acquire a non-zero value and the whole system proceeds to evolve accordingly. Let us now introduce several Green functions.

\[
G_{11}(k, k'; q) = -i \langle T a^\dagger_{k\uparrow}(q, t) a_{k'\uparrow}(q, 0) \rangle 
\]

\[
G_{12}(k, k'; q) = -i \langle T a_{k\uparrow}^\dagger(-q, t) a_{k'\uparrow}(q, 0) \rangle 
\]

\[
G_{21}(k, k'; q) = -i \langle T a_{k\downarrow}^\dagger(-q, t) a_{k'\downarrow}(q, 0) \rangle 
\]

\[
G_{22}(k, k'; q) = -i \langle T a_{k\downarrow}(q, t) a_{k'\downarrow}(q, 0) \rangle 
\]

\[
G_{X\sigma}(k; q) = -i \langle X_q(t) a_{k\sigma}^\dagger(q, 0) \rangle 
\]

\[
G_{P\sigma}(k; q) = -i \langle P_{-q}(t) a_{k\sigma}^\dagger(q, 0) \rangle 
\]

\[
i \frac{\partial}{\partial t} G_{11}(k, k'; q, t) = \delta(t) \delta_{k,k'} + \frac{k_q}{m_e} G_{11}(k, k'; q) - i \frac{v(q)}{V} \Lambda_e(k, -q) \langle T \rho^{(c)}(-q, t) a_{k'\uparrow}^\dagger(q, 0) \rangle 
\]

\[
+ \frac{M_q}{\sqrt{V}} \Lambda_e(k, -q) G_{X\uparrow}(k'; q, t) 
\]

\[
i \frac{\partial}{\partial t} G_{12}(k, k'; q, t) = \frac{k_q}{m_e} G_{12}(k, k'; q, t) + i \frac{v(q)}{V} \Lambda_e(k, q) \langle T \rho^{(c)}(-q, t) a_{k'\uparrow}^\dagger(q, 0) \rangle 
\]

\[
- \frac{M_q}{\sqrt{V}} \Lambda_e(k, q) G_{X\uparrow}(k'; q, t) 
\]
\[
    i \frac{\partial}{\partial t} G_{22}(k, k'; q, t) = \delta(t) \delta_{k,k'} - \frac{k \cdot q}{m_h} G_{22}(k, k'; q, t) + i \frac{\nu(q)}{V} \Lambda_{h}(k, -q) \langle T \rho^{(h)}(-q, t) a_{k' \downarrow}^\dagger(q \downarrow, 0) \rangle \\
    + \frac{M_q}{\sqrt{V}} \Lambda_{h}(k, -q) G_{X \downarrow}(k'; q, t)
\]

(52)

\[
    i \frac{\partial}{\partial t} G_{21}(k, k'; q, t) = -\frac{k \cdot q}{m_h} G_{21}(k, k'; q, t) - i \frac{\nu(q)}{V} \Lambda_{h}(k, q) \langle T \rho^{(e)}(-q, t) a_{k' \uparrow}^\dagger(q \uparrow, 0) \rangle \\
    - \frac{M_q}{\sqrt{V}} \Lambda_{h}(k, q) G_{X \uparrow}(k'; q, t)
\]

(53)

\[
    i \frac{\partial}{\partial t} G_{X \sigma}(k; q, t) = (2i \Omega_{LO}) G_{P \sigma}(k; q, t)
\]

(54)

\[
    i \frac{\partial}{\partial t} G_{P \uparrow}(k; q, t) = \left(\frac{\Omega_{LO}}{2i}\right) G_{X \uparrow}(k; q, t) - \frac{M_q}{\sqrt{V}} \langle T \rho^{(e)}(-q, t) a_{k' \uparrow}^\dagger(q \uparrow, 0) \rangle
\]

(55)

\[
    i \frac{\partial}{\partial t} G_{P \downarrow}(k; q, t) = \left(\frac{\Omega_{LO}}{2i}\right) G_{X \downarrow}(k; q, t) + \frac{M_q}{\sqrt{V}} \langle T \rho^{(h)}(-q, t) a_{k' \downarrow}^\dagger(q \downarrow, 0) \rangle
\]

(56)

In order to solve this system we have to expand the Green functions in terms of Matsubara frequencies. We introduce a temperature just for ease of doing calculations. In the end we shall go to the zero temperature limit as this is the regime when the interpretations are the cleanest (here \( z_n = 2\pi n/\beta \) and \( \beta = 1/k_B T \)).

\[
    (iz_n - \frac{k \cdot q}{m_e}) G_{11}(k, k'; q, z_n) = \frac{1}{-i \beta} \delta_{k,k'} - i \frac{\tilde{\nu}(q, z_n)}{V} \Lambda_{e}(k, -q) \langle T \rho^{(e)}(-q, z_n) a_{k' \uparrow}^\dagger(q \uparrow, 0) \rangle
\]

(57)

\[
    (iz_n - \frac{k \cdot q}{m_e}) G_{12}(k, k'; q, z_n) = i \frac{\tilde{\nu}(q, z_n)}{V} \Lambda_{e}(k, q) \langle T \rho^{(e)}(-q, z_n) a_{k' \uparrow}^\dagger(q \uparrow, 0) \rangle
\]

(58)

\[
    (iz_n + \frac{k \cdot q}{m_h}) G_{22}(k, k'; q, z_n) = \frac{1}{-i \beta} \delta_{k,k'} + i \frac{\tilde{\nu}(q)}{V} \Lambda_{h}(k, -q) \langle T \rho^{(h)}(-q, z_n) a_{k' \downarrow}^\dagger(q \downarrow, 0) \rangle
\]

(59)

\[
    (iz_n + \frac{k \cdot q}{m_h}) G_{21}(k, k'; q, z_n) = -i \frac{\tilde{\nu}(q, z_n)}{V} \Lambda_{h}(k, q) \langle T \rho^{(h)}(-q, z_n) a_{k' \downarrow}^\dagger(q \downarrow, 0) \rangle
\]

(60)

\[
    \tilde{\nu}(q, z_n) = \nu(q) - 2 \frac{M_q^2 \Omega_{LO}}{z_n^2 + \Omega_{LO}^2}
\]

(61)

\[
    -i \langle T \rho^{(e)}(-q, z_n) a_{k' \uparrow}^\dagger(q \uparrow, 0) \rangle = \sum_p \Lambda_{e}(p, -q) G_{11}(p, k'; q, z_n) + \sum_p \Lambda_{e}(p, q) G_{12}(p, k; q, z_n)
\]

(62)

\[
    -i \langle T \rho^{(h)}(-q, z_n) a_{k' \downarrow}^\dagger(q \downarrow, 0) \rangle = - \sum_p \Lambda_{h}(p, -q) G_{22}(p, k'; q, z_n) - \sum_p \Lambda_{h}(p, q) G_{21}(p, k; q, z_n)
\]

(63)
\[
-i\langle T \rho^{(e)}(-q, z_n) a_{k \uparrow}^+(q \uparrow, 0) \rangle = \left( \frac{1}{-i\beta} \right) \Lambda_e(k', -q)
\]
\[
+ i \frac{\tilde{v}(q, z_n)}{V} \sum_p \frac{\Delta_2^e(p, q) - \Delta_2^e(p, -q)}{iz_n - \frac{p \cdot q}{m_e}} \langle T \rho^{(e)}(-q, z_n) a_{k \uparrow}^+(q \uparrow, 0) \rangle
\]  
(64)

\[
\langle T \rho^{(e)}(-q, z_n) a_{k \uparrow}^+(q \uparrow, 0) \rangle = \left( \frac{1}{-i\beta} \right) \frac{i \Lambda_e(k', -q)}{iz_n - \frac{k \cdot q}{m_e}} \frac{1}{\epsilon^{(e)}(q, iz_n)}
\]  
(65)

where,
\[
\epsilon^{(e)}(q, iz_n) = 1 + \frac{\tilde{v}(q, z_n)}{V} \sum_p \frac{\tilde{n}^{(e)}(p + q/2) - \tilde{n}^{(e)}(p - q/2)}{iz_n - \frac{p \cdot q}{m_e}}
\]  
(66)

\[
\langle T \rho^{(h)}(-q, z_n) a_{k \downarrow}^+(q \downarrow, 0) \rangle = \left( \frac{1}{-i\beta} \right) \frac{-i \Lambda_h(k', -q)}{iz_n + \frac{k \cdot q}{m_h}} \frac{1}{\epsilon^{(h)}(q, iz_n)}
\]  
(67)

where,
\[
\epsilon^{(h)}(q, iz_n) = 1 + \frac{\tilde{v}(q, z_n)}{V} \sum_p \frac{\tilde{n}^{(h)}(p + q/2) - \tilde{n}^{(h)}(p - q/2)}{iz_n - \frac{p \cdot q}{m_h}}
\]  
(68)

Therefore,
\[
\langle a_{k \uparrow}^+(q \uparrow) a_{k \uparrow}(q \uparrow) \rangle = \left( \frac{1}{-i\beta} \right) \left( \frac{1}{V} \right) \sum_n \frac{\tilde{v}(q, z_n)}{\epsilon^{(e)}(q, iz_n)} \frac{i \Lambda_2^e(k, -q)}{(iz_n - \frac{k \cdot q}{m_e})^2}
\]  
(69)

\[
\langle a_{k \downarrow}^+(q \downarrow) a_{k \downarrow}(q \downarrow) \rangle = \left( \frac{1}{-i\beta} \right) \left( \frac{1}{V} \right) \sum_n \frac{\tilde{v}(q, z_n)}{\epsilon^{(h)}(q, iz_n)} \frac{i \Lambda_2^h(k, -q)}{(iz_n + \frac{k \cdot q}{m_h})^2}
\]  
(70)

Then when we go to the zero-temperature limit we have to integrate over all \( n \). In Fig.1 we see the pole structure of the above equations. Call \( iz_n = i z \). Then,
\[
\langle a_{k \uparrow}^+(q \uparrow) a_{k \uparrow}(q \uparrow) \rangle = \left( \frac{1}{-i\beta} \right) \left( \frac{1}{V} \right) \int_C dz \frac{\tilde{v}(q, z)}{\epsilon^{(e)}(q, iz)} \frac{i \Lambda_2^e(k, -q)}{(iz - \frac{k \cdot q}{m_e})^2}
\]  
(71)

In Fig.1 we see the pole structure of the above contour integral. Let us assume that \( k \cdot q > 0 \) then the pole \( z = -i k \cdot q/m_e \) is in the lower half-plane. We have to close the contour in such a way that this pole is excluded from consideration, since if it were included we would have a formula for \( \langle a_{k \uparrow}^+(q \uparrow) a_{k \uparrow}(q \uparrow) \rangle \) rather than \( \langle a_{k \uparrow}^+(q \uparrow) a_{k \uparrow}(q \uparrow) \rangle \). Therefore we have to close the contour in the upper half-plane (and \( C = C_1 \)). Now, if we count the number of poles in the integrand we find that first of all, the zeros of the dielectric function that lie on the positive imaginary axis of the \( z \)-plane (how many zeros are there, is an important question which we shall address subsequently) contribute. Then it seems at first sight that even the poles of \( \tilde{v}(q, z) \) contribute. The poles of this function lie at \( \pm i \Omega_{LO} \). However, upon closer examination we find that this is not the case. The poles of \( \tilde{v}(q, z) \) are also poles of \( \epsilon^{(e)}(q, iz) \) and the two cancel. Thus the only
poles that contribute are the zeros of the dielectric function that lie on the positive imaginary axis. How many such zeros are there? If one counts only the collective modes then one arrives at the conclusion that there are only two, one corresponding to the plasmon (modified by phonons) and the other corresponding to phonons (modified by Coulomb interactions). There is another mode that is equally important, indeed it would be a serious mistake to ignore this contribution, namely the particle-hole mode. We have encountered this problem before. In our earlier article we presented an argument that shows how one may incorporate the particle-hole mode. In retrospect it seems that the approach presented there is not a good one, although it serves well to illustrate the importance of the particle-hole mode. Here we shall take the point of view that all energies are allowed as zeros of the dielectric function (for each \( q \)) but each comes with a weight corresponding to the strength of the dynamical structure factor at that energy. Thus for small \( q \) we recover naturally the collective modes but for larger \( q \) we start summing the particle-hole modes as well. There is really is no rigorous justification for this point of view except that it is physically well-motivated.

\[
\langle a_{k\uparrow}(q \uparrow) a_{k\downarrow}(q \uparrow) \rangle = \frac{1}{V} \int \frac{dz}{2\pi i} \frac{\tilde{v}(q, z)}{\epsilon^{(e)}(q, iz)} \frac{\Lambda^2(k, -q)}{(iz - \frac{k \cdot q}{m_c})^2}
\]

(72)

The poles are \( z = i \omega^{(e)}_I(q) \), \( \omega^{(e)}_I(q) > 0 \) satisfies \( \epsilon^{(e)}(q, \omega^{(e)}_I) = 0 \).

\[
\langle a_{k\uparrow}(q \uparrow) a_{k\uparrow}(q \uparrow) \rangle = \frac{1}{V} \sum_k \frac{\tilde{v}(q, i \omega^{(e)}_I)}{\epsilon^{(e)}(q, iz)} \frac{\Lambda^2(k, -q)}{(iz - \frac{k \cdot q}{m_c})^2}
\]

(73)

Let us first evaluate this. Similarly, one may write for holes,

\[
\langle a_{k\downarrow}(q \downarrow) a_{k\downarrow}(q \downarrow) \rangle = \frac{1}{V} \sum_k \frac{\tilde{v}(q, i \omega^{(h)}_I)}{\epsilon^{(h)}(q, iz)} \frac{\Lambda^2(k, -q)}{(iz - \frac{k \cdot q}{m_h})^2}
\]

(74)

Let us now evaluate these quantities more explicitly.

\[
\frac{\partial}{\partial z}|_{z=i\omega_I} \epsilon(q, iz) = -P(q, iz)|_{z=i\omega_I} \frac{\partial}{\partial z}|_{z=i\omega_I} \tilde{v}(q, z) - \tilde{v}(q, z)|_{z=i\omega_I} \frac{\partial}{\partial z}|_{z=i\omega_I} P(q, iz)
\]

\[
= -\frac{1}{\tilde{v}(q, z)} \frac{\partial}{\partial z}|_{z=i\omega_I} \tilde{v}(q, z) - \tilde{v}(q, z)|_{z=i\omega_I} \frac{\partial}{\partial z}|_{z=i\omega_I} P(q, iz)
\]

Since,

\[
\tilde{v}(q, z) = v(q) - \frac{2 \Omega_{LO} M^2_a}{z^2 + \Omega_{LO}^2}
\]

(75)

\[
\frac{\partial}{\partial z}|_{z=i\omega_I} P(q, iz) = \frac{i}{V} \sum_k \frac{\bar{n}_{k-q/2} - \bar{n}_{k+q/2}}{(\omega_I - \frac{k \cdot q}{m})^2}
\]

(76)

\[
\frac{\partial}{\partial z}|_{z=i\omega_I} \tilde{v}(q, z) = \frac{4 i \omega_{LO} M^2_a}{(\omega_I - \Omega_{LO}^2)^2}
\]

(77)

\[
\frac{\partial}{\partial z}|_{z=i\omega_I} \epsilon(q, iz) = -\frac{1}{\tilde{v}(q, iz)} \left\{ \frac{M^2_a}{V} \frac{4 \Omega_{LO} \omega_I}{(\omega_I - \Omega_{LO}^2)^2} \left( \frac{\tilde{v}(q, iz)}{V} \right)^2 + \sum_k \frac{\bar{n}_{k-q/2} - \bar{n}_{k+q/2}}{(\omega_I - \frac{k \cdot q}{m})^2} \right\}
\]

(78)
As we pointed our just a while ago, it is necessary that we interpret the sum over I in a special manner so that we are able to recover both the collective as well as the particle-hole modes. The way this is done is through the following identification,

\[
\sum_{I} f_{I}(q, \omega I) = \sum_{q} \int_{0}^{\infty} d\omega \ W(q, \omega) f(q, \omega)
\]  

where the weight is the dynamical structure factor normalised to unity.

\[
W(q, \omega) = \frac{S(q, \omega)}{\int_{0}^{\infty} d\omega \ S(q, \omega)}
\]

The dynamical structure factor is defined to be the the dynamical density-density correlation function fourier-transformed divided by the total number of particles. Let us first write down,

\[
\langle T \rho^{(e)}(-q, t) a_{k'}^{\dag} (q \uparrow, 0) \rangle = \left( \frac{1}{-i\beta} \right) \sum_{n} e^{\varepsilon_{n}t} \frac{i\Lambda_{e}(k', -q)}{iz_{n} - \frac{k'q}{m_{e}}} \frac{1}{\epsilon^{(e)}(q, iz_{n})}
\]

Since Im$(t) \in [0, -\beta]$, if Im$(t) < 0$ then,

\[
\langle \rho^{(e)}(-q, t) a_{k'}^{\dag} (q \uparrow, 0) \rangle = \left( \frac{1}{-i\beta} \right) \int_{C_{-}} dz \ e^{zt} \frac{i\Lambda_{e}(k', -q)}{iz - \frac{k'q}{m_{e}}} \frac{1}{\epsilon^{(e)}(q, iz)}
\]

\[
= \left( \frac{i}{2\pi} \right) \int_{-\infty}^{+\infty} dx \ e^{xt} \frac{i\Lambda_{e}(k', -q)}{ix - \frac{k'q}{m_{e}}} \frac{1}{\epsilon^{(e)}(q, ix)}
\]

If Im$(t) > 0$ then,

\[
\langle a_{k'}^{\dag} (q \uparrow, 0) \rho^{(e)}(-q, t) \rangle = \left( \frac{1}{-i\beta} \right) \int_{C_{+}} dz \ e^{zt} \frac{i\Lambda_{e}(k', -q)}{iz - \frac{k'q}{m_{e}}} \frac{1}{\epsilon^{(e)}(q, iz)}
\]

\[
= \left( \frac{i}{2\pi} \right) \int_{-\infty}^{+\infty} dx \ e^{xt} \frac{i\Lambda_{e}(k', -q)}{-ix - \frac{k'q}{m_{e}}} \frac{1}{\epsilon^{(e)}(q, ix)}
\]

Here $C_{+}(C_{-})$ is the semi-circle in the upper(lower) half plane. Let us now take the complex conjugate of the above equation. Im$(t) > 0$ implies Im$(t^{*}) < 0$. Therefore if Im$(t^{*}) < 0$,

\[
\langle \rho^{(e)}(q, t^{*}) a_{k'}^{\dag} (q \uparrow, 0) \rangle = \left( \frac{i}{2\pi} \right) \int_{-\infty}^{+\infty} dx \ e^{xt^{*}} \frac{-i\Lambda_{e}(k', -q)}{-ix - \frac{k'q}{m_{e}}} \frac{1}{\epsilon^{(e)}(q, ix)}
\]

Define,

\[
\rho^{(e,a)}(q, 0) = \sum_{k'} \Lambda_{e}(k', -q) a_{k'}^{\dag} (q \uparrow, 0)
\]

\[
\rho^{(e,b)}(q, 0) = \sum_{k'} \Lambda_{e}(k', q) a_{k'}^{\dag} (-q \uparrow, 0)
\]

therefore,
\[
\rho^{(c)}(\mathbf{q}, 0) = \rho^{(c,a)}(\mathbf{q}, 0) + \rho^{(c,b)}(\mathbf{q}, 0)
\]  

(87)

Then we have \(Im(t) < 0\),

\[
\langle \rho^{(c)}(-\mathbf{q}, t)\rho^{(c)}(\mathbf{q}, 0) \rangle = \left( \frac{i}{2\pi} \right) \int_{-\infty}^{+\infty} dx \, e^{xt} \sum_{\mathbf{k}'} \frac{i(\bar{n}_{c}(\mathbf{k}' - \mathbf{q}/2) - \bar{n}_{c}(\mathbf{k}' + \mathbf{q}/2))}{i x - k' \frac{a}{m_v}} \frac{1}{\epsilon^{(c)}(\mathbf{q}, ix)}
\]  

(88)

For \(Im(t^*) > 0\),

\[
\langle \rho^{(c)}(\mathbf{q}, 0)\rho^{(c)}(-\mathbf{q}, t^*) \rangle = \left( \frac{i}{2\pi} \right) \int_{-\infty}^{+\infty} dx \, e^{xt^*} \sum_{\mathbf{k}'} \frac{i(\bar{n}_{c}(\mathbf{k}' + \mathbf{q}/2) - \bar{n}_{c}(\mathbf{k}' - \mathbf{q}/2))}{-ix + k' \frac{a}{m_v}} \frac{1}{\epsilon^{(c)}(\mathbf{q}, ix)}
\]  

(89)

Combining these two,

\[
(T \rho^{(c)}(-\mathbf{q}, t)\rho^{(c)}(\mathbf{q}, 0)) = \left( \frac{i}{2\pi} \right) \int_{-\infty}^{+\infty} dx \, e^{xt} \sum_{\mathbf{k}'} \frac{i(\bar{n}_{c}(\mathbf{k}' - \mathbf{q}/2) - \bar{n}_{c}(\mathbf{k}' + \mathbf{q}/2))}{ix - k' \frac{a}{m_v}} \frac{1}{\epsilon^{(c)}(\mathbf{q}, ix)}
\]  

(90)

Define the Green function\(^\text{1}\),

\[
\mathcal{D}(\mathbf{q}, t) = -\langle T \rho^{(c)}(-\mathbf{q}, t)\rho^{(c)}(\mathbf{q}, 0) \rangle
\]  

(91)

then,

\[
\mathcal{D}_{\text{ref}}(\mathbf{q}, \omega) = V \frac{P^{\text{ref}}(\mathbf{q}, \omega)}{\epsilon^{(c)}(\mathbf{q}, \omega)}
\]  

(92)

The corresponding spectral function is the dynamical structure factor,

\[
N_c \, S(\mathbf{q}, \omega) = -2Im(\mathcal{D}_{\text{ref}}(\mathbf{q}, \omega))
\]  

(93)

\[
\epsilon^{(c)}_{+}(\mathbf{q}, \omega) = 1 - v^{r}(\mathbf{q}, \omega)P^{r}_{c}(\mathbf{q}, \omega) + v^{i}(\mathbf{q}, \omega)P^{i}_{c}(\mathbf{q}, \omega)
\]  

(94)

\[
\epsilon^{(c)}_{\downarrow}(\mathbf{q}, \omega) = -v^{r}(\mathbf{q}, \omega)P^{r}_{c}(\mathbf{q}, \omega) - v^{i}(\mathbf{q}, \omega)P^{i}_{c}(\mathbf{q}, \omega)
\]  

(95)

This procedure ensures that we correctly incorporate both the collective (for small \(\mathbf{q}\)) and the particle-hole modes. There is an alternative approach that comes to mind. That is the method of exact diagonalisation. Consider the hamiltonian,

\[
H' = \sum_{\mathbf{k}, \mathbf{q}} k_{\mathbf{k}} b_{\mathbf{k} \uparrow}^{\dagger} a_{\mathbf{k} \uparrow} + \sum_{\mathbf{k}, \mathbf{q}} k_{\mathbf{k}} b_{\mathbf{k} \downarrow}^{\dagger} a_{\mathbf{k} \downarrow} + \sum_{\mathbf{q}} \Omega_{LO} b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}} + \sum_{\mathbf{q} \neq 0} \frac{v(\mathbf{q})}{2V} \sum_{\mathbf{k}, \mathbf{k}'} [\Lambda_{c}(\mathbf{k}, \mathbf{q}) a_{\mathbf{k} \uparrow}(\mathbf{q} \uparrow) + \Lambda_{c}(\mathbf{k}, -\mathbf{q}) a_{\mathbf{k} \downarrow}(\mathbf{q} \downarrow)]\Lambda_{c}(\mathbf{k}', -\mathbf{q}) a_{\mathbf{k}' \uparrow}(\mathbf{q} \uparrow) + \Lambda_{c}(\mathbf{k}', \mathbf{q}) a_{\mathbf{k}' \downarrow}(\mathbf{q} \downarrow)
\]
Therefore,

$$+ \sum_{q \neq 0} \frac{v(q)}{2V} \sum_{k,k'} [\Lambda_h(k,q)a_{k\downarrow}(-q \downarrow) + \Lambda_h(k,-q)a_{k\uparrow}^\dagger (q \downarrow)] [\Lambda_h(k',-q)a_{k'\downarrow}^\dagger (q \downarrow)] + \Lambda_h(k',q)a_{k'\uparrow}^\dagger (-q \downarrow)]$$

$$+ \sum_{q \neq 0} \frac{M}{\sqrt{V}} (b_q + b_{-q}^\dagger) [\Lambda_e(k,q)a_{k\uparrow}(-q \uparrow) + \Lambda_e(k,-q)a_{k\downarrow}^\dagger (q \uparrow)] + \Lambda_h(k,q)a_{k\downarrow}(-q \downarrow) + \Lambda_h(k,-q)a_{k\uparrow}^\dagger (q \downarrow)]$$

(96)

In order to diagonalise this we proceed as follows. Let us postulate the existence of dressed sea-bosons $d_{Iσ}(q)$ such that,

$$H' = \sum_{I,q,σ} \omega_{Iσ}(q)d_{Iσ}^\dagger(q)d_{Iσ}(q)$$

(97)

Now for some notation. $σ = e, h$ (correspondingly $σ = \uparrow, \downarrow$, furthermore, $S(σ)$ is such that $S(\uparrow) = +1$ and $S(\downarrow) = -1$).

$$a_{kσ}(qσ) = \sum_I [a_{kσ}(qσ), d_{Iσ}^\dagger(q)]d_{Iσ}(q) - \sum_I [a_{kσ}(qσ), d_{Iσ}(-q)]d_{Iσ}^\dagger(-q)$$

(98)

$$b_q = \sum_{I,σ} [b_q, d_{Iσ}^\dagger(q)]d_{Iσ}(q) - \sum_{I,σ} [b_q, d_{Iσ}(-q)]d_{Iσ}^\dagger(-q)$$

(99)

The inverse relation is,

$$d_{Iσ}(q) = \sum_k [d_{Iσ}(q), a_{kσ}^\dagger(qσ)]a_{kσ}(qσ) - \sum_k [d_{Iσ}(q), a_{kσ}(-qσ)]a_{kσ}^\dagger(-qσ)$$

$$+ [d_{Iσ}(q), b_q^\dagger]b_q - [d_{Iσ}(q), b_q]b_{-q}^\dagger$$

(100)

Therefore,

$$\omega_{Iσ}(q)d_{Iσ}(q) = \sum_k S(σ) \frac{k}{m_σ} [d_{Iσ}(q), a_{kσ}^\dagger(qσ)]a_{kσ}(qσ) - S(σ) \sum_k \frac{k}{m_σ} [d_{Iσ}(q), a_{kσ}(-qσ)]a_{kσ}^\dagger(-qσ)$$

$$+ \frac{v(q)}{V} \sum_{k,k'} [\Lambda_e(k,q)[d_{Iσ}(q), a_{kσ}(-qσ)] + \Lambda_e(k,-q)[d_{Iσ}(q), a_{kσ}^\dagger(qσ)]][\Lambda_e(k',-q)a_{k'σ}(qσ) + \Lambda_e(k',q)a_{k'σ}^\dagger(-qσ)]$$

$$+ \Omega_{LO}[d_{Iσ}(q), b_q^\dagger]b_q + \Omega_{LO}[d_{Iσ}(q), b_q]b_{-q}^\dagger$$

$$+ \frac{M}{\sqrt{V}} [d_{Iσ}(q), b_{-q}] + [d_{Iσ}(q), b_q^\dagger] \sum_k [\Lambda_e(k,-q)a_{kσ}(qσ) + \Lambda_e(k,q)a_{kσ}^\dagger(-qσ)]$$

$$+ \frac{M}{\sqrt{V}} (b_q + b_{-q}^\dagger) \sum_k (\Lambda_e(k,q)[d_{Iσ}(q), a_{kσ}(-qσ)] + \Lambda_e(k,-q)[d_{Iσ}(q), a_{kσ}^\dagger(qσ)])$$

(101)
\begin{align}
(\omega_{I\sigma}(q) - S(\sigma) \frac{k,q}{m_\sigma})[d_{I\sigma}(q), a_{k\sigma}^\dagger(q\sigma)] &= S(\sigma) \frac{\hat{v}(q, I\sigma)}{V} \Lambda_{\sigma}(k, -q) \rho(\sigma)(q, I) \\
(\omega_{I\sigma}(q) - S(\sigma) \frac{k,q}{m_\sigma})[d_{I\sigma}(q), a_{k\sigma}(-q\sigma)] &= -S(\sigma) \frac{\hat{v}(q, I\sigma)}{V} \Lambda_{\sigma}(k, q) \rho(\sigma)(q, I) \\
\rho^{(e)}(q, I) &= \sum_k \Lambda_e(k, q)[d_I(q), a_k(-q\uparrow)] + \sum_k \Lambda_e(k, -q)[d_I(q), a_k^\dagger(q\uparrow)] \\
\rho^{(h)}(q, I) &= -\sum_k \Lambda_h(k, q)[d_I(q), a_k(-q\downarrow)] - \sum_k \Lambda_h(k, -q)[d_I(q), a_k^\dagger(q\downarrow)] \\
(\omega_{I\sigma}(q) - \Omega_{LO})[d_{I\sigma}(q), b_q^\dagger] &= S(\sigma) \frac{M_q}{\sqrt{V}} \rho(\sigma)(q, I) \\
(\omega_{I\sigma}(q) + \Omega_{LO})[d_{I\sigma}(q), b_{-q}] &= -S(\sigma) \frac{M_q}{\sqrt{V}} \rho(\sigma)(q, I)
\end{align}

From this we have the following fact that \( \omega_I \) are zeros of the dielectric function,

\begin{align}
\epsilon^{(e)}(q, \omega_I^{(e)}) &= 0 \\
\epsilon^{(h)}(q, \omega_I^{(h)}) &= 0
\end{align}

\begin{equation}
\epsilon^{(e,h)}(q, \omega) = 1 - v(q, \omega) P^{(e,h)}(q, \omega)
\end{equation}

and,

\begin{equation}
v(q, \omega) = v(q) + \frac{2\Omega_{LO} M_q^2}{\omega^2 - \Omega_{LO}^2}
\end{equation}

and \( P^{(e,h)}(q, \omega) \) is the usual RPA-polarization bubble. If we now make use of the fact that \([d_{I\sigma}(q), d_{I\sigma}^\dagger(q)] = 1\) then we have,

\begin{align}
\sum_k ||[d_{I\sigma}(q), a_{k\sigma}^\dagger(q\sigma)]|^2 - \sum_k ||[d_{I\sigma}(q), a_{k\sigma}(-q\sigma)]|^2 \\
+ ||[d_{I\sigma}(q), b_q^\dagger]|^2 - ||[d_{I\sigma}(q), b_{-q}]|^2 &= 1 \\
\sum_k \Lambda_e^2(k, -q) - \Lambda_e^2(k, q) \rho(\sigma)(q, I)^2 \left( \frac{v(q, I\sigma)}{V} \right)^2 \\
+ M_q^2 \left( \frac{\rho(\sigma)(q, I)}{V} \right)^2 4\omega_{I\sigma}(q) \Omega_{LO} \\
\left( \frac{\omega_{I\sigma}(q) - \Omega_{LO}^2}{\Omega_{LO}^2} \right) &= 1
\end{align}

Therefore,
\[ \rho^{(e)}(q, I) = \left( \frac{V(q, I, e)}{V} \right)^2 \sum_k \tilde{n}^{(e)}(k - q/2) - \tilde{n}^{(e)}(k + q/2) + \frac{M_q^2}{V} \left( \frac{4\omega_{j,e}(q)\Omega_{LO}}{(\omega_{j,e}(q) - \Omega_{LO}^2)^2} \right) \frac{1}{2} \] (114)

\[ \rho^{(h)}(q, I) = \left( \frac{V(q, I, h)}{V} \right)^2 \sum_k \tilde{n}^{(h)}(k + q/2) - \tilde{n}^{(h)}(k - q/2) + \frac{M_q^2}{V} \left( \frac{4\omega_{j,h}(q)\Omega_{LO}}{(\omega_{j,h}(q) - \Omega_{LO}^2)^2} \right) \frac{1}{2} \] (115)

Therefore,

\[ \langle a_{k\sigma}^\dagger(q\sigma)a_{k\sigma}(q\sigma) \rangle = \sum_I \langle [a_{k\sigma}(q\sigma), d_{I\sigma}(-q)] \rangle^2 = \sum_I \left( \frac{V(q, I, e)}{V} \right)^2 \Lambda_\sigma^2(k, -q) - \frac{\rho^{2\sigma}(-q, I)}{(\omega_{I,e}(q) + S(\sigma)\frac{k_\sigma}{m_\sigma})^2} \] (116)

\[ \langle a_{k\uparrow}^\dagger(q\uparrow)a_{k\downarrow}(q\downarrow) \rangle = \sum_I \left( \frac{V(q, I, e)}{V} \right)^2 \Lambda^2_\uparrow(k, -q) - \frac{\rho^{(e)}(-q, I)}{(\omega_{I,e}(q) + \frac{k_\uparrow}{m_\uparrow})^2} \] (117)

\[ \langle a_{k\downarrow}^\dagger(q\downarrow)a_{k\uparrow}(q\uparrow) \rangle = \sum_I \left( \frac{V(q, I, h)}{V} \right)^2 \Lambda^2_\downarrow(k, -q) - \frac{\rho^{(h)}(-q, I)}{(\omega_{I,h}(q) + \frac{k_\downarrow}{m_\downarrow})^2} \] (118)

After some algebra it is clear that Eqs.(72) and (74) are identical to Eqs.(117) and (118) respectively.

Let us now solve the fundamental equations namely Eq.(34) and Eq.(35). For this we first would like to decompose the various fields in the exciton basis.

\[ a_{k\downarrow}(0 \uparrow) = \sum_I \tilde{\varphi}_I(k)e^{-i\epsilon_I} t \tilde{D}_I \] (119)

\[ a_{k\uparrow}(0 \downarrow) = e^{i(\epsilon(k) + \epsilon^*(k))} t \tilde{a}_{k\uparrow}(0 \downarrow) \] (120)

This means we may rewrite these equations as follows:

\[ i \frac{\partial}{\partial t} \tilde{D}_I(t) = \left| \frac{e|}{\mu c} \right| e^{i\epsilon_I} t A^{\ast}_{ext}(t)\tilde{p}_{ec} \sum_k \Lambda_1(k, 0)\tilde{\varphi}_I^\dagger(k) \] (121)

\[ + \sum_{Q \neq 0} \frac{v_{eh}(Q)}{V} \sum_{k,j} \tilde{\varphi}_I^\dagger(k)(1 - \Lambda_1(k, 0)\Lambda_2(k - Q, 0))\tilde{\varphi}_J(k - Q)e^{i(\epsilon_I - \epsilon_J)} t \tilde{D}_J(t) \]

\[ - \sum_k \tilde{\varphi}_I^\dagger(k)\Lambda_1(k, 0)\Lambda_2(k - Q, 0)e^{i\epsilon_I} t e^{-i(\epsilon^*(k - Q) + \epsilon^*(k - Q))} t \tilde{a}_{k - Q\uparrow}(0 \downarrow) \]

\[ \frac{\partial}{\partial t} \tilde{a}_{k\uparrow}(0 \downarrow) = - \sum_{Q \neq 0} \frac{v_{eh}(Q)}{V} \Lambda_2(k, 0)[\Lambda_1(k + Q, 0)\tilde{\varphi}_J^\dagger(k + Q)e^{i\epsilon_J} t e^{-i(\epsilon^*(k) + \epsilon^*(k))} t \tilde{D}_J^t] \]

\[ + \Lambda_2(k + Q, 0)e^{-i(\epsilon^*(k) + \epsilon^*(k))} t e^{i(\epsilon^*(k) + \epsilon^*(k))} t \tilde{a}_{k + Q\uparrow}(0 \downarrow) + \left| \frac{e|}{\mu c} \right| A^{\ast}_{ext}(t)\tilde{p}_{ec} \Lambda_2(k, 0)e^{-i(\epsilon^*(k) + \epsilon^*(k))} t \] (122)
\[ i \frac{\partial}{\partial t} \tilde{D}_0(t) = \left( \frac{|e|}{\mu_c} \right) e^{i \omega t} A^{*}_{ext}(t) \bar{\rho}_{ec} \sum_k \Lambda_1(k, 0) \tilde{\varphi}_0(k) \]

\[ + \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} \tilde{\varphi}_0^*(k) (1 - \Lambda_1(k, 0) \Lambda_1(k', 0)) \tilde{\varphi}_0(k') \tilde{D}_0(t) \]

\[ + \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} [\tilde{\varphi}_0^*(k) (1 - \Lambda_1(k, 0) \Lambda_1(k', 0)) e^{i(\epsilon_{k} - \epsilon_{k'})} t \tilde{D}_k(t)] \]

\[ - \tilde{\varphi}_0^*(k) \Lambda_1(k, 0) \Lambda_1(k', 0) e^{i(\epsilon_{k} - \epsilon_{k'})} t \tilde{A}_k^\dagger(0 \downarrow) \]  

\[ i \frac{\partial}{\partial t} \tilde{D}_k(t) = \left( \frac{|e|}{\mu_c} \right) e^{i \omega t} A^{*}_{ext}(t) \bar{\rho}_{ec} \Lambda_1(k, 0) \]

\[ + \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} (1 - \Lambda_1(k, 0) \Lambda_1(k', 0)) \tilde{\varphi}_0(k') e^{i(\epsilon_{k} - \epsilon_{k'})} t \tilde{D}_0(t) \]

\[ + \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} [(1 - \Lambda_1(k, 0) \Lambda_1(k', 0)) e^{i(\epsilon_{k} - \epsilon_{k'})} t \tilde{D}_k(t)] \]

\[ \tilde{\varphi}_0(k) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{a_X}} \frac{2}{\sqrt{1/a_X^2 + k^2}} \]  

Furthermore,

\[ \tilde{n}_e(k) = \tilde{n}_0(k) + \sum_{l, q} \frac{v(q, I, e)}{V} n_e(k - q) (1 - \tilde{n}_e(k)) \frac{(\rho(e)(-q, I))^2}{(\omega_{I,e}(q) + \frac{kq}{m_e} + \frac{q^2}{2m_e})^2} \]

\[ - \sum_{l, q} \frac{v(q, I, e)}{V} n_e(k) (1 - \tilde{n}_e(k + q)) \frac{(\rho(e)(q, I))^2}{(\omega_{I,e}(q) + \frac{kq}{m_e} + \frac{q^2}{2m_e})^2} \]  

\[ \]
\[
\tilde{n}_e(k) = \tilde{n}_0(k) - \sum_{I, q} \left( \frac{v(q, I, h)}{V} \right)^2 \tilde{n}_h(k)(1 - \tilde{n}_h(k - q)) \frac{(\rho^{(h)}(-q, I))^2}{(\omega_{I, h}(q) - \frac{kq}{m_h} + \frac{q^2}{2m_h})^2} \\
+ \sum_{I, q} \left( \frac{v(q, I, h)}{V} \right)^2 \tilde{n}_h(k + q)(1 - \tilde{n}_h(k)) \frac{(\rho^{(h)}(-q, I))^2}{(\omega_{I, h}(q) - \frac{kq}{m_h} + \frac{q^2}{2m_h})^2} 
\]

or,

\[
\tilde{n}_0(k) = |\phi_0(k)|^2 \langle \tilde{D}_0^\dagger(t) \tilde{D}_0(t) \rangle + \langle \tilde{D}_k^\dagger(t) \tilde{D}_k(t) \rangle \\
+ \phi_0(k)e^{i(\epsilon_k - \epsilon_0)t} \langle \tilde{D}_k^\dagger(t) \tilde{D}_0(t) \rangle + \phi_0(k)e^{-i(\epsilon_k - \epsilon_0)t} \langle \tilde{D}_0^\dagger(t) \tilde{D}_k(t) \rangle - \langle \tilde{a}_{k\uparrow}(0 \downarrow) \tilde{a}_{k\uparrow}(0 \downarrow) \rangle
\]

\[
\tilde{n}_e(k) = \tilde{n}_0(k) A_e(k) + (1 - \tilde{n}_0(k)) B_e(k) 
\]

\[
\tilde{n}_h(k) = \tilde{n}_0(k) A_h(k) + (1 - \tilde{n}_0(k)) B_h(k) 
\]

\[
A_e(k) = \frac{1}{1 + \frac{T_e^e(k)}{T^e_1(k)}} 
\]

\[
B_e(k) = \frac{1}{1 + \frac{1 + T_e^e(k)}{T^e_1(k)}} 
\]

\[
T_e^e(k) = \sum_{I, q} \left( \frac{v(q, I, e)}{V} \right)^2 \tilde{n}_e(k - q) \frac{(\rho^{(e)}(-q, I))^2}{(\omega_{I, e}(q) + \frac{kq}{m_e} - \frac{q^2}{2m_e})^2} 
\]

\[
T_e^h(k) = \sum_{I, q} \left( \frac{v(q, I, e)}{V} \right)^2 (1 - \tilde{n}_e(k + q)) \frac{(\rho^{(e)}(-q, I))^2}{(\omega_{I, e}(q) + \frac{kq}{m_e} + \frac{q^2}{2m_e})^2} 
\]

\[
A_h(k) = \frac{1}{1 + \frac{T^h_e(k)}{T^h_1(k)}} 
\]

\[
B_h(k) = \frac{1}{1 + \frac{1 + T^h_e(k)}{T^h_1(k)}} 
\]
\[ T_1^h(k) = \sum_{l,q} \left( \frac{v(q, I, h)}{V} \right)^2 (1 - \bar{n}_h(k - q))(\frac{(\rho(h) - q I)^2}{(\omega_{l,h}(q) - \frac{k}{m_h} + \frac{q^2}{2m_h})^2} \right) \]  

\[ T_2^h(k) = \sum_{l,q} \left( \frac{v(q, I, h)}{V} \right)^2 \bar{n}_h(k + q)(\frac{(\rho(h) - q I)^2}{(\omega_{l,h}(q) - \frac{k}{m_h} + \frac{q^2}{2m_h})^2} \right) \]  

\[ \frac{\partial}{\partial t} \bar{D}_0^i(t) = -\frac{|e|}{\mu c} A_x(\epsilon_0, t)p_{ec} \sum_k \Lambda_1(k, 0) \bar{\varphi}_0(k) \]  

\[ + \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} \bar{\varphi}_0(k)(1 - \Lambda_1(k, 0)\Lambda_1(k', 0))\bar{\varphi}_0(k') \bar{D}_0^0(t) \]  

\[ + \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} \sin((\epsilon_0 - \epsilon_{k'}) t) [\bar{\varphi}_0(k)(1 - \Lambda_1(k, 0)\Lambda_1(k', 0))\bar{D}_k^i(t) \]  

\[ - \bar{\varphi}_0(k)\Lambda_1(k, 0)\Lambda_2(k', 0)\bar{a}_{k' \uparrow}^i(0 \downarrow) \]  

\[ + \bar{\varphi}_0(k)\Lambda_1(k, 0)\Lambda_2(k', 0)\bar{a}_{k' \downarrow}^i(0 \downarrow) \]  

\[ \frac{\partial}{\partial t} \bar{D}_0^i(t) = -\frac{|e|}{\mu c} A_x(\epsilon_0, t)p_{ec} \sum_k \Lambda_1(k, 0) \bar{\varphi}_0(k) \]  

\[ - \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} \bar{\varphi}_0(k)(1 - \Lambda_1(k, 0)\Lambda_1(k', 0))\bar{\varphi}_0(k') \bar{D}_0^0(t) \]  

\[ + \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} \cos((\epsilon_0 - \epsilon_{k'}) t) [\bar{\varphi}_0(k)(1 - \Lambda_1(k, 0)\Lambda_1(k', 0))\bar{D}_k^i(t) \]  

\[ - \bar{\varphi}_0(k)\Lambda_1(k, 0)\Lambda_2(k', 0)\bar{a}_{k' \downarrow}^i(0 \downarrow) \]  

\[ + \bar{\varphi}_0(k)\Lambda_1(k, 0)\Lambda_2(k', 0)\bar{a}_{k' \uparrow}^i(0 \downarrow) \]  

\[ \frac{\partial}{\partial t} \bar{D}_k^i(t) = -\frac{|e|}{\mu c} A_x(\epsilon_k, t)p_{ec} \Lambda_1(k, 0) \]  

\[ \left[ \sum_{k' \neq k} \frac{v_{eh}(k - k')}{V} \sin((\epsilon_0 - \epsilon_{k'}) t) [\bar{\varphi}_0(k)(1 - \Lambda_1(k, 0)\Lambda_1(k', 0))\bar{D}_k^i(t) \]  

\[ + \bar{\varphi}_0(k)\Lambda_1(k, 0)\Lambda_2(k', 0)\bar{a}_{k' \downarrow}^i(0 \downarrow) \]  

\[ \frac{\partial}{\partial t} \bar{D}_k^i(t) = -\frac{|e|}{\mu c} A_x(\epsilon_k, t)p_{ec} \Lambda_1(k, 0) \]
In order to simplify the calculations further, let us define,

\[ \partial_t \tilde{\partial}_i^r \tilde{\partial}_j^r + \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} (1 - \Lambda_1(k,0)\Lambda_1(k',0)) \tilde{\partial}_0^r(k') \{ \sin((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) + \cos((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) \} + \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} (1 - \Lambda_1(k,0)\Lambda_1(k',0)) \tilde{\partial}_0^r(k') \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) - \sin((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) \} - \Lambda_1(k,0)\Lambda_2(k',0) \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\downarrow}^r(0 \downarrow) - \sin((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\downarrow}^r(0 \downarrow) \} \] (143)

\[ -\frac{\partial}{\partial t} \tilde{D}_k^i(t) = (\frac{|e|}{\mu c}) A_X^r(\epsilon_k, t) p_{vc} \Lambda_1(k,0) \]

\[ + \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} (1 - \Lambda_1(k,0)\Lambda_1(k',0)) \tilde{\partial}_0^r(k') \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) - \sin((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) \} + \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} (1 - \Lambda_1(k,0)\Lambda_1(k',0)) \tilde{\partial}_0^r(k') \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) + \sin((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) \} - \Lambda_1(k,0)\Lambda_2(k',0) \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\uparrow}^r(0 \downarrow) + \sin((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\uparrow}^r(0 \downarrow) \} \] (144)

\[ \frac{\partial}{\partial t} \tilde{a}_{k',\downarrow}^r(0 \downarrow) = \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} \Lambda_2(k,0)\Lambda_1(k',0) \tilde{\partial}_0^r(k') \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) + \sin((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) \} + \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} \Lambda_2(k,0)\Lambda_1(k',0) \tilde{\partial}_0^r(k') \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) - \sin((\epsilon_k - \epsilon_0) t)\tilde{D}_0^r(t) \} - \Lambda_2(k',0) \{ \cos((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\uparrow}^r(0 \downarrow) - \sin((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\uparrow}^r(0 \downarrow) \} \] (145)

\[ \frac{\partial}{\partial t} \tilde{a}_{k',\uparrow}^r(0 \downarrow) = \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} \Lambda_2(k,0)\Lambda_1(k',0) \tilde{\partial}_0^r(k') \{ -\sin((\epsilon_0 - \epsilon_k) t)\tilde{D}_0^r(t) + \cos((\epsilon_0 - \epsilon_k) t)\tilde{D}_0^r(t) \} + \sum_{k' \neq k} \frac{v_{ch}(k-k')}{V} \Lambda_2(k,0)\Lambda_1(k',0) \tilde{\partial}_0^r(k') \{ -\sin((\epsilon_0 - \epsilon_k) t)\tilde{D}_0^r(t) - \cos((\epsilon_0 - \epsilon_k) t)\tilde{D}_0^r(t) \} + \Lambda_2(k',0) \{ \sin((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\downarrow}^r(0 \downarrow) - \cos((\epsilon_k - \epsilon_0) t)\tilde{a}_{k',\downarrow}^r(0 \downarrow) \} \] (146)
\[
P_i(q, \omega) = \left( \frac{1}{4\pi} \right) \int_0^\infty dk \, k^2 \bar{n}(k)(-\frac{m}{|k||q|})[\theta(\omega - \frac{|k||q|}{m} - \frac{q^2}{2m}) - \theta(\omega + \frac{|k||q|}{m} - \frac{q^2}{2m})]
\]

\[
P_r(q, \omega) = \int_0^\infty \frac{d\omega'}{\pi} P_i(q, \omega')(\frac{2\omega'}{\omega'^2 - \omega^2})
\]

\[
S(q, \omega) \sim \frac{P_i(q, \omega)}{(1 - v(q, \omega)P_r(q, \omega))^2 + v^2(q, \omega)(P_i(q, \omega))^2}
\]

\[
\rho(q, \omega) = \{-\frac{v(q, \omega)}{V}\}^2V \int_0^\infty \frac{d\omega'}{\pi} P_i(q, \omega')(\frac{4\omega\omega'}{(\omega'^2 - \omega^2)^2} + \frac{M^2_q}{V} \frac{4\omega\Omega_{LO}}{(\omega^2 - \Omega_{LO}^2)^2})^{-\frac{1}{2}}
\]

II. COMPARISON WITH SEMICONDUCTOR BLOCH EQUATIONS

The equations presented in the previous section namely Eqs. (121) and (122) are intended as alternatives to the usual Semiconductor Bloch equations used to study semiconductors. Let us now try to solve this system with a pulse field. That is, we apply an external field with central frequency \(\omega_X\) and assume it lasts for a time \(\tau_X\) and therefore we have a spread in frequency \(\Gamma_X = \frac{2\pi}{\tau_X}\). We would like to see the evolution of the polarization and populations in this case. This exercise also enables us to compare our results with those of the SBEs and ascertain where the differences lie. To this end let us set,

\[
\vec{A}_{ext}(t) = \hat{p}_{vc} A_X(0, t)
\]

where,

\[
A_X(E, t) = A_0 \int_{-\infty}^\infty d\omega \frac{\Gamma_X/\pi}{(\omega - \omega_X)^2 + \Gamma_X^2}e^{i(\omega - E) t}
\]

For comparison the SBE is reproduced below.

\[
g_{hh}(kt) = i\bar{n}_h(k) = i(d_{-k}^d c_{-k}), \quad = i \, f(k) g_{ee}(kt) = i\bar{n}_e(k) = i(c_k^d c_{-k})
\]

\[
g_{he}(kt) = i(d_{-k}(t)c_{k}(t)) = i \, p(k)
\]

\[
i \frac{\partial}{\partial t} g_{hh}(kt) = 2 \, Re(\Omega(kt)g_{he}(kt)) + R_{hh}(kt)
\]

\[
i \frac{\partial}{\partial t} g_{he}(kt) = -\Omega(kt)(i - 2g_{hh}(kt)) + (\epsilon_h(k) + \epsilon_e(k) - 2\Sigma(kt))g_{he}(kt) + R_{he}(kt)
\]

\[
\Omega(kt) = \frac{|e|}{\mu c} \vec{A}_{ext}(t) . \vec{P}_{vc} - i \sum_{k'} v_{k-k'} g_{he}(k't)
\]
\( \Sigma(\mathbf{k} t) = -i \sum_{\mathbf{k}'} v_{\mathbf{k}-\mathbf{k}'} \, g_{hh}(\mathbf{k}' t) \) (153)

Define,

\[ \Omega(\mathbf{k}, t) = \hat{\Omega}(\mathbf{k}, t) e^{-i(k^2/2\mu + E_g)t}, \quad p(\mathbf{k}, t) = \tilde{p}(\mathbf{k}, t) e^{-i(k^2/2\mu + E_g)t} \] (154)

\[ \frac{\partial f(\mathbf{k})}{\partial t} = 2 \tilde{\Omega}_{R}(\mathbf{k}, t) \tilde{p}_{I}(\mathbf{k}) - 2 \tilde{\Omega}_{I}(\mathbf{k}, t) \tilde{p}_{R}(\mathbf{k}) \] (155)

\[ \frac{\partial \tilde{p}_{R}(\mathbf{k})}{\partial t} = \hat{\Omega}_{R}(\mathbf{k}, t)(1 - 2f(\mathbf{k})) - 2\Sigma(\mathbf{k}, t) \tilde{p}_{I}(\mathbf{k}, t) \] (156)

\[ \frac{\partial \tilde{p}_{I}(\mathbf{k})}{\partial t} = \hat{\Omega}_{I}(\mathbf{k}, t)(1 - 2f(\mathbf{k})) + 2\Sigma(\mathbf{k}, t) \tilde{p}_{R}(\mathbf{k}, t) \] (157)

\[ \hat{\Omega}_{R}(\mathbf{k}, t) = \left( \frac{|e|}{\mu c} \right) \tilde{A}_{ext}(t) p_{vc} \cos((k^2/2\mu + E_g - \omega_X)t) + \sum_{\mathbf{k}', \mathbf{k} \neq \mathbf{k}'} v_{\mathbf{k}-\mathbf{k}'} [\tilde{p}_{R}(\mathbf{k}') \cos((k^2 - k'^2)t/2\mu) - \tilde{p}_{I}(\mathbf{k}') \sin((k^2 - k'^2)t/2\mu)] \] (158)

\[ \hat{\Omega}_{I}(\mathbf{k}, t) = \left( \frac{|e|}{\mu c} \right) \tilde{A}_{ext}(t) p_{vc} \sin((k^2/2\mu + E_g - \omega_X)t) + \sum_{\mathbf{k}', \mathbf{k} \neq \mathbf{k}'} v_{\mathbf{k}-\mathbf{k}'} [\tilde{p}_{I}(\mathbf{k}') \cos((k^2 - k'^2)t/2\mu) + \tilde{p}_{R}(\mathbf{k}') \sin((k^2 - k'^2)t/2\mu)] \] (159)

\[ \Sigma(\mathbf{k}, t) = \sum_{\mathbf{k}', \mathbf{k} \neq \mathbf{k}'} v_{\mathbf{k}-\mathbf{k}'} f(\mathbf{k}') \] (160)

III. OPTICAL CONDUCTIVITY

First define the total polarization,

\[ P(t) = \sum_{\mathbf{k}} \langle d_{-\mathbf{k}c_{\mathbf{k}}} \rangle \] (161)

From this we may obtain the Fourier component,

\[ \tilde{P}(\omega) = \int_{-\infty}^{\infty} dt \, P(t) \, e^{i\omega t} \] (162)

Now since,

\[ A_\tau(t) = A_\tau e^{-i \omega_X \tau \delta(t - \tau)} \] (163)

\[ E(t) = -\frac{\partial A_\tau(t)}{\partial t}, \quad -A_\tau e^{-i \omega_X \tau \delta'(t - \tau)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega t} E(\omega) \] (164)
\[ E(\omega) = (i\omega)A_\tau e^{i(\omega - \omega x)\tau} \]  

(165)

Since,

\[ j(\omega) = p_{vc}\tilde{P}(\omega) \]  

(166)

We have

\[ \sigma(\omega) = p_{vc}e^{-i(\omega - \omega x)\tau}\lim_{A_\tau \to 0} \frac{(\tilde{P}(\omega, A_\tau) - \tilde{P}(\omega, 0))}{(i\omega)A_\tau} \]  

(167)
Here,

\[ \tilde{P}(\omega) = \sum_k \int_{-\infty}^{\tau} dt \sqrt{(1 - \tilde{n}_e(k))(1 - \tilde{n}_h(k))} [\tilde{\varphi}_0(k)\tilde{D}_0^{(-)}(t)e^{i(\omega - \epsilon_0)t} + \tilde{D}_k^{(-)}(t)e^{i(\omega - \epsilon_k)t}] \]

\[ + \sum_k \int_{-\infty}^{\tau} dt \sqrt{\tilde{n}_e(k)\tilde{n}_h(k)}e^{i(\omega - \epsilon_k)t}\tilde{a}^\dagger_k(0, \downarrow, -) \]

\[ + \sum_k \int_{\tau}^{\infty} dt \sqrt{(1 - \tilde{n}_e(k))(1 - \tilde{n}_h(k))} [\tilde{\varphi}_0(k)\tilde{D}_0^{(+)}(t)e^{i(\omega - \epsilon_0)t} + \tilde{D}_k^{(+)}(t)e^{i(\omega - \epsilon_k)t}] \]

\[ + \sum_k \int_{\tau}^{\infty} dt \sqrt{\tilde{n}_e(k)\tilde{n}_h(k)}e^{i(\omega - \epsilon_k)t}\tilde{a}^\dagger_k(0, \downarrow, +) \] (168)

\[ i(\tilde{D}_0^{(+)}(\tau) - \tilde{D}_0^{(-)}(\tau)) = \left| \frac{e}{\mu c} \right| e^{i(t_0 - \omega x)\tau} A_{r \tau \omega} \sum_k \tilde{\varphi}_0(k)\Lambda_1(k, 0) \] (169)

\[ i(\tilde{D}_k^{(+)}(\tau) - \tilde{D}_k^{(-)}(\tau)) = \left| \frac{e}{\mu c} \right| e^{i(\frac{k^2}{2\mu} + E_0 - \omega x)\tau} A_{r \tau \omega} \Lambda_1(k, 0) \] (170)

\[ i(\tilde{a}_k^\dagger(0, \downarrow, +) - \tilde{a}_k^\dagger(0, \downarrow, -)) = \left| \frac{e}{\mu c} \right| e^{-i(\frac{k^2}{2\mu} + E_0 - \omega x)\tau} A_{r \tau \omega} \Lambda_2(k, 0) \] (171)

\[ \tilde{P}_R(\omega) = \int_{\tau}^{\infty} \frac{4\pi k^2}{(2\pi)^3} dk \int_{-\infty}^{\tau} dt \sqrt{(1 - \tilde{n}_e(k))(1 - \tilde{n}_h(k))} \]

\[ [\tilde{\varphi}_0(k)\tilde{D}_0^{(-), R}(t)\cos((\omega - \epsilon_0)t) - \tilde{\varphi}_0(k)\tilde{D}_0^{(-), I}(t)\sin((\omega - \epsilon_0)t) + \tilde{D}_k^{(-), R}(t)\cos((\omega - \epsilon_k)t) - \tilde{D}_k^{(-), I}(t)\sin((\omega - \epsilon_k)t)] \]

\[ + \int_0^{\infty} \frac{4\pi k^2}{(2\pi)^3} dk \int_{-\infty}^{\tau} dt \sqrt{\tilde{n}_e(k)\tilde{n}_h(k)}\cos((\omega - \epsilon_k)t)\tilde{a}^\dagger_k(0, \downarrow, -) \]

\[ + \int_0^{\infty} \frac{4\pi k^2}{(2\pi)^3} dk \int_{-\infty}^{\tau} dt \sqrt{\tilde{n}_e(k)\tilde{n}_h(k)}\sin((\omega - \epsilon_k)t)\tilde{a}^\dagger_k(0, \downarrow, -) \]

\[ + \int_0^{\infty} \frac{4\pi k^2}{(2\pi)^3} dk \int_{\tau}^{\infty} dt \sqrt{(1 - \tilde{n}_e(k))(1 - \tilde{n}_h(k))} \]

\[ [\tilde{\varphi}_0(k)\tilde{D}_0^{(+), R}(t)\cos((\omega - \epsilon_0)t) - \tilde{\varphi}_0(k)\tilde{D}_0^{(+), I}(t)\sin((\omega - \epsilon_0)t) + \tilde{D}_k^{(+), R}(t)\cos((\omega - \epsilon_k)t) - \tilde{D}_k^{(+), I}(t)\sin((\omega - \epsilon_k)t)] \]

\[ + \int_0^{\infty} \frac{4\pi k^2}{(2\pi)^3} dk \int_{\tau}^{\infty} dt \sqrt{\tilde{n}_e(k)\tilde{n}_h(k)}\cos((\omega - \epsilon_k)t)\tilde{a}^\dagger_k(0, \downarrow, +) \]

\[ + \int_0^{\infty} \frac{4\pi k^2}{(2\pi)^3} dk \int_{\tau}^{\infty} dt \sqrt{\tilde{n}_e(k)\tilde{n}_h(k)}\sin((\omega - \epsilon_k)t)\tilde{a}^\dagger_k(0, \downarrow, +) \] (172)
\[
\hat{P}_1(\omega) = \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \int_{-\infty}^\tau dt \sqrt{(1-\bar{n}_e(k))(1-\bar{n}_h(k))} \\
[\varphi_0(k) \tilde{D}_0^{(-)}R(t) \sin((\omega-\epsilon_0)t) + \varphi_0(k) \tilde{D}_0^{(-)}I(t) \cos((\omega-\epsilon_0)t) + \tilde{D}_k^{(+)}R(t) \sin((\omega-\epsilon_k)t) + \tilde{D}_k^{(+)}I(t) \cos((\omega-\epsilon_k)t)] 
\]

\[
+ \int_0^\tau dt \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \sqrt{n_e(k)\bar{n}_h(k)} \sin((\omega-\epsilon_k)t) \hat{a}_{k1}^R(0,\downarrow,-) 
\]

\[
- \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \int_{-\infty}^\tau dt \sqrt{n_e(k)\bar{n}_h(k)} \cos((\omega-\epsilon_k)t) \hat{a}_{k1}^I(0,\downarrow,-) 
\]

\[
+ \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \int_0^\tau dt \sqrt{(1-\bar{n}_e(k))(1-\bar{n}_h(k))} 
\]

\[
[\varphi_0(k) \tilde{D}_0^{(+)}R(t) \sin((\omega-\epsilon_0)t) + \varphi_0(k) \tilde{D}_0^{(+)}I(t) \cos((\omega-\epsilon_0)t) + \tilde{D}_k^{(+)}R(t) \sin((\omega-\epsilon_k)t) + \tilde{D}_k^{(+)}I(t) \cos((\omega-\epsilon_k)t)] 
\]

\[
+ \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \int_0^\tau dt \sqrt{n_e(k)\bar{n}_h(k)} \sin((\omega-\epsilon_k)t) \hat{a}_{k1}^R(0,\downarrow,+ ) 
\]

\[
- \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \int_0^\tau dt \sqrt{n_e(k)\bar{n}_h(k)} \cos((\omega-\epsilon_k)t) \hat{a}_{k1}^I(0,\downarrow,+ ) 
\]

\[
(173) 
\]

\[
\tilde{D}_0^{(+)}R(\tau) - \tilde{D}_0^{(-)}R(\tau) = \frac{|e|}{\mu c} A_{p_{ee}} \sin((\epsilon_0 - \omega_X)\tau) \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \varphi_0(k) \Lambda_1(k,0) 
\]

\[
(174) 
\]

\[
\tilde{D}_0^{(+)}I(\tau) - \tilde{D}_0^{(-)}I(\tau) = -\frac{|e|}{\mu c} A_{p_{ee}} \cos((\epsilon_0 - \omega_X)\tau) \int_0^\infty \frac{4\pi k^2}{(2\pi)^3} dk \varphi_0(k) \Lambda_1(k,0) 
\]

\[
(175) 
\]

\[
\tilde{D}_k^{(+)}R(\tau) - \tilde{D}_k^{(-)}R(\tau) = \frac{|e|}{\mu c} A_{p_{ee}} \sin((\epsilon_k - \omega_X)\tau) \Lambda_1(k,0) 
\]

\[
(176) 
\]

\[
\tilde{D}_k^{(+)}I(\tau) - \tilde{D}_k^{(-)}I(\tau) = -\frac{|e|}{\mu c} A_{p_{ee}} \cos((\epsilon_k - \omega_X)\tau) \Lambda_1(k,0) 
\]

\[
(177) 
\]

\[
\tilde{a}_{k1}^R(0,\downarrow,+ ) - \tilde{a}_{k1}^R(0,\downarrow,- ) = -\frac{|e|}{\mu c} A_{p_{ee}} \sin((\epsilon_k - \omega_X)\tau) \Lambda_2(k,0) 
\]

\[
(178) 
\]

\[
\tilde{a}_{k1}^I(0,\downarrow,+ ) - \tilde{a}_{k1}^I(0,\downarrow,- ) = -\frac{|e|}{\mu c} A_{p_{ee}} \cos((\epsilon_k - \omega_X)\tau) \Lambda_2(k,0) 
\]

\[
(179) 
\]

\[
\hat{P}(\omega) = \sum_k \int_{-\infty}^\tau dt \sqrt{(1-\bar{n}_e(k))(1-\bar{n}_h(k))} [\varphi_0(k) \tilde{D}_0^{(-)}(t) e^{i(\omega-\epsilon_0)t} + \tilde{D}_0^{(-)}(t) e^{i(\omega-\epsilon_0)t}] 
\]

\[
+ \sum_k \int_{-\infty}^\tau dt \sqrt{n_e(k)\bar{n}_h(k)} e^{i(\omega-\epsilon_k)t} \hat{a}_{k1}^R(0,\downarrow,-) 
\]

\[
+ \sum_k \int_{-\infty}^\tau dt \sqrt{n_e(k)\bar{n}_h(k)} e^{i(\omega-\epsilon_k)t} \hat{a}_{k1}^I(0,\downarrow,+ ) 
\]

\[
(180) 
\]
The Optical Conductivity

\[ \text{Re}(\sigma(\omega)) = |e| LT_{A_{\tau} \rightarrow 0} \cos((\omega - \omega_X)\tau) \hat{P}(\omega, A_{\tau}) - \sin((\omega - \omega_X)\tau) \hat{P}_R(\omega, A_{\tau}) - \cos((\omega - \omega_X)\tau) \hat{P}(\omega, 0) + \sin((\omega - \omega_X)\tau) \hat{P}_R(\omega, 0) \]

\[ \omega A_{\tau} \] 

(181)

A quick check of dimensions. In units of \( \hbar = c = 1 \), all masses are of dimension inverse length we take to be centimeter. All times are in centimeters. Charge is dimensionless. Since \( \frac{\omega}{p_{\nu c}} A_{\tau} \) is dimensionless, it follows that \([A_{\tau}] = L^{-1}\). It is easy to check that \( [\text{Re}(\sigma(\omega))] = L^{-1} \) as it should be.

IV. RESULTS AND DISCUSSION

The equations written down in the previous sections have to be solved numerically. It is worthwhile to point out some pitfalls and problems. Let us first focus on the SBEs. As we pointed out in our earlier work involving the SBE’s the sums over \( k' \) have to be carried out in a special manner so as to avoid potential divergences.

\[ \sum_{k \neq k'} \frac{v_{k-k'}}{V} f(k') = \frac{(4\pi \hbar^2)}{(2\pi)^2} \int_0^{k_{\text{max}}} dk' \left( \frac{k'}{k} \right) \ln \left( \frac{(k' + k)}{(k' - k)} \right) (f(k') - f(k)) \]

\[ + \left( \frac{(4\pi \hbar^2)}{(2\pi)^2} f(k) \right) \frac{1}{k} \left( k^2 - k_{\text{max}}^2 \right) \ln \left( \frac{(k_{\text{max}} + k)}{(k_{\text{max}} - k)} \right) + \frac{1}{k_{\text{max}}} \] 

(182)

Further, it was suggested by Binder et.al. that we should use a momentum cutoff \( k_{\text{max}} = 12/a_{\text{Bohr}} \), where \( a_{\text{Bohr}} \) is the exciton Bohr radius. The justification for this stems from the fact that beyond this cutoff the probability of the electron existing is negligible. This assertion is true only if the pump field frequency is below or equal the band-gap. When the frequency is well above the band gap the situation is less clear and care must be taken inorder not to lose features that may be present at high momenta. Unfortunately we have found that even when the pump field has a frequency equal to the band gap the prescription of Binder et.al. has some problems. In particular, we have found that if we try and sneak a peek at the form of the distribution for \( k >> 12/a_{\text{Bohr}} \), we find a periodic pattern suggesting therefore that electrons can exist at (arbitrarily) high momenta long after a pump field whose frequency is at the band gap is switched off (after a time long enough so that we may still meaningfully talk of a well-defined frequency). This is a paradoxical and counterintuitive result that has been glossed over by the pioneers. The arbitrary cutoff of Binder et.al. should not be taken too seriously. In order to make more sense out of all this we have to claim that the SBE’s produce the correct momentum distributions only for small enough \( k \), and we have to use some judgement as to where we should cutoff the distributions.

The sea-boson analogs of the SBE’s written down above have their own numerical problems. First is the fact that even in the two-component case the sea-boson technique works well only when \( q << k \). Since we have chosen to study only \( q = 0 \), parts of the hamiltonian, it seems that we are in good shape. However we
find that even then there is a cutoff small $k = k_{min}$ cutoff below which the momentum distributions become unphysical (larger than unity). This is true if we use the formula in Eq. (130) $\tilde{n}_e(k) = \tilde{n}_h(k) = \tilde{n}_0(k)$. Further we find that this identification is the analog of the SBE. It is comforting to know that the sea-boson technique is equivalent to the SBE in some limit. In the SBEs, the momentum distributions of the electron and holes are identical even if the effective masses are very different. This is due to the fact that the SBEs neglect the collision terms responsible for the asymmetry that we would otherwise expect. Similarly, the sea-boson equations at the level of Eq. (130) neglect the repulsion and phonon terms. However the SBEs do include repulsion at the Hartree-Fock level, therefore the analogy between the two is not exact. In Fig. 1 we see how far we may take this analogy between the SBEs and the sea-boson equations. The approach toward unphysical behavior for the small $k$ limit of the momentum distribution obtained using the sea-boson equations is also seen. When we include the effects of repulsion and phonons, the answers change quite dramatically. In fact they are so very different from the SBE results that we have decided not to publish them. It will take some more time before a thorough analysis is completed and all the ramifications are explored. For now we shall assume that the momentum distribution is that given by $\tilde{n}_0(k)$ or that given by the SBEs. Let us first write down some formulas that relate the real part of the conductivity to the absorption coefficient. We may expect the two to have qualitatively similar features. However just to be sure and so that we don’t make any mistakes having gotten this far, let us write down the formulas. They are a combination of the formulas from the text by Haug and Koch and the one by Manah. The transverse dielectric function may be decomposed as follows.

$$\epsilon(\omega) = \epsilon_1(\omega) + i \epsilon_2(\omega)$$  \hspace{1cm} (183)

The absorption coefficient, refractive index and the real part of the conductivity are given by (in units $\hbar = c = 1$),

$$\alpha(\omega) = \frac{\omega}{n(\omega)} \epsilon_2(\omega)$$  \hspace{1cm} (184)

$$n(\omega) = \left\{ \frac{1}{2} (\epsilon_1(\omega) + \sqrt{\epsilon_1^2(\omega) + \epsilon_2^2(\omega)}) \right\}^{\frac{1}{2}}$$  \hspace{1cm} (185)

The real part of the conductivity is,

$$Re(\sigma(\omega)) = \frac{\omega}{4\pi} \epsilon_2(\omega)$$  \hspace{1cm} (186)

It is better not to use the Kramers-Kronig relations as our answer for the real part of the conductivity is undetermined up to a factor (actually it has no reason to, it just so happens that the magnitude does not agree with observations). We may write down a formula for the imaginary part of the conductivity just as we did the real part.

$$Im(\sigma(\omega)) = |e| L_{A_x \rightarrow 0} \frac{-sin((\omega - \omega_X)\tau)\tilde{P}_I(\omega, A_x) - cos((\omega - \omega_X)\tau)\tilde{P}_R(\omega, A_x) + sin((\omega - \omega_X)\tau)\tilde{P}_I(\omega, 0) + cos((\omega - \omega_X)\tau)\tilde{P}_R(\omega, 0)}{\omega A_r}$$  \hspace{1cm} (187)
\[ \text{Im}(\sigma(\omega)) = -\frac{\omega}{4\pi}(\epsilon_1(\omega) - 1) \] (188)

Therefore we may deduce the optical dielectric function and hence the absorption coefficient. All this would not be necessary if the refractive index was close to unity and then \( \epsilon_2 \) is negligible in comparison with \( \epsilon_1 \) for all frequencies. Then the real part of the conductivity would be proportional to the absorption coefficient. Let us compare the experimental magnitude of the absorption coefficient and the energy \( \omega \). We find according to the experiments of Song’s group, \( |\alpha| \approx 10^5 \text{cm}^{-1} \) whereas \( \omega \approx \frac{2.0\pi}{(352 \times 10^{-7} \text{cm})} = 1.78 \times 10^5 \text{cm}^{-1} \). we can see that these two quantities are comparable to each other suggesting thereby that the real part of the dielectric function is not close to unity.

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