Approximate Controllability of a Class of Partial Integro–Differential Equations of Parabolic Type

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Abstract

In this paper, we discuss the distributed control problem governed by the following parabolic integro-differential equation (PIDE) in the abstract form

\[
\frac{\partial y}{\partial t} + Ay = \int_0^t B(t, s)y(s)ds + Gu, \quad t \in [0, T],
\]

\[
y(0) = y_0 \in X,
\]

where, \( y \) denotes the state space variable, \( u \) is the control variable, \( A \) is a self adjoint, positive definite linear (not necessarily bounded) operator in a Hilbert space \( X \) with dense domain \( D(A) \subset X \), \( B(t, s) \) is an unbounded operator, smooth with respect to \( t \) and \( s \) with \( D(A) \subset D(B(t, s)) \subset X \) for \( 0 \leq s \leq t \leq T \) and \( G \) is a bounded linear operator from the control space to \( X \).

Assuming that the corresponding evolution equation \((B \equiv 0 \text{ in } (*) )\) is approximately controllable, it is shown that the set of approximate controls of the distributed control problem \((*)\) is nonempty. The problem is first viewed as constrained optimal control problem and then it is approximated by unconstrained problem with a suitable penalty function. The optimal pair of the constrained problem is obtained as the limit of optimal pair sequence of the unconstrained problem. The approximation theorems, which guarantee the convergence of the numerical scheme to the optimal pair sequence, are also proved.

Keywords: approximate controllability; parabolic integro–differential equation; \( C^0 \)-semigroup; optimal control; penalty function; Hammerstein equation; approximation theorems, finite element method, numerical experiment.

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1 Introduction

Consider the following parabolic integro-differential equation with distributed control

$$\frac{\partial y}{\partial t} + Ay(t) = \int_0^t B(t, s)y(s)ds + Gu(t), \quad t \in [0, T],$$

(1.1)

$$y(0) = y_0 \in X,$$

where $X$ denotes a real Hilbert space, $y$ is a state variable, $u$ represents a control variable, $A$ is a self adjoint, positive definite linear operator in $X$ with dense domain $D(A) \subset X$, $B(t, s)$ is also a linear and unbounded operator with $D(A) \subset D(B(t, s)) \subset X$ for $0 \leq s \leq t \leq T$ and $G$ is a bounded linear map from the control space $U$ to $X$.

For an example, let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial \Omega$. For fixed $T < \infty$, let $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial \Omega$. Further, set $A$ as a second-order linear self-adjoint elliptic partial differential operator defined by

$$A = -\sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + a_0(x)I,$$

(1.2)

where the matrix $(a_{ij}(x))$ is symmetric and positive definite, $a_0 \geq 0$ on $\bar{\Omega}$ and $B(t, s)$ is a general second-order partial differential operator of the form

$$B(t, s) = -\sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( b_{i,j}(t, s; x) \frac{\partial}{\partial x_i} \right) + \sum_{j=1}^d b_j(t, s; x) \frac{\partial}{\partial x_j} + b_0(t, s; x)I,$$

(1.3)

with smooth coefficients $b_{i,j}, b_j$ and $b_0$. Let $X = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $D(B) = H^2(\Omega)$, then the following problem

$$\frac{\partial y(t, x)}{\partial t} + A(x)y(t, x) = \int_0^t B(t, s)y(s)ds + Gu(t, x) \quad \text{in} \, Q,$$

$$y(t, x) = 0 \quad \text{on} \, \Sigma,$$

$$y(0, x) = y_0(x) \quad \text{in} \, \Omega,$$

(1.4)

becomes a particular case of the abstract problem (1.1), where $y_0 \in X$.

Parabolic integro-differential equations of the type (1.1) occur in many applications such as heat conduction in materials with memory, compression of poroviscoelastic media, nuclear reactor dynamics, etc. (see, Cushman et al. [3], Dagan [6], Renardy et al. [25]).

For control problems of the heat equation with memory, that is, when $A = -\Delta$ and $B(t, s) = -a(t - s)\Delta u(s)$ in (1.1), where $a(\cdot)$ is a completely monotone convolution kernel, Barbu and Iannelli [1] have discussed approximate controllability using Carleman estimates. Later on, Pandolfi [22] has considered Dirichlet boundary controllability of heat equation with memory in one space dimension by employing Riesz systems. There are several negative results like lack of controllability of such systems, see, [11], [10] and [9]. Subsequently, Fu et al. [7] have established controllability and observability results for a heat equation with hyperbolic memory kernel under general geometric conditions and using Carleman estimates. Further, Pandolfi [21] has employed cosine operator approach to discuss the exact controllability results for the Dirichlet boundary.
control of the Gurtin-Pipkin model which displays a hyperbolic behaviour. On second order integro-differential equations, Kim [13, 14] has established reachability results using continuation arguments and multiplier techniques combined with compactness property. Wang and Wei [29] have proved some sufficient conditions for the controllability of parabolic integro-differential systems in a Banach space. A result in the direction of approximate controllability of integro-differential equations (IDE) using Carleman estimates and continuation argument has been proved by Lefter et. al. in [18]. Loreti and Sfrorza [20] have analyzed reachability problems for a class of IDE using Hilbert uniqueness results. In the present article, an attempt has been made to discuss approximate controls of a distributed control problem for a general class of partial integro-differential equations of parabolic type \(1.1\), under the assumption that the corresponding parabolic equation is approximately controllable. Firstly, the control problem is viewed as an optimal control problem, and using operator theoretic form, an optimal pair of solution is derived, which, in turn, provides a proof for the approximate controllability. The present proof is constructive in its approach and avoids of using Carleman estimates and continuation of argument, etc. Finally, some approximate theorems are established and numerical experiments using finite element method are conducted to confirm our theoretical findings.

Numerical solution by means of finite element methods has been invested by several authors, when \(u\) is a given function and \(G = I\). In [28], Thomée et. al. have considered backward Euler methods and obtained related error estimates for non-smooth data. Pani et. al. [23] have used energy arguments and the duality technique to obtain error estimates for time dependent parabolic integro-differential equations with smooth and non-smooth initial conditions. Lasiecka [16, 17] have considered optimal control problems for linear parabolic equations, which are approximated by a semidiscrete finite element method or Ritz-Galerkin scheme and then the convergence of optimal controls are derived. Moreover, Shen et. al. [26] have developed the finite element and backward Euler scheme for the space and time approximation of a constrained optimal control problem governed by a parabolic integro-differential equation. Further, in [27] Shen et. al. have discussed mathematical formulation and optimality conditions for a quadratic optimal control problems for a quasi-linear integral differential equation and some a priori error estimates are also discussed.

In order to motivate our main results, we first define the operator \(\hat{B}\) as

\[
(\hat{B}y)(t) = \int_{0}^{t} B(t, \tau)y(\tau)d\tau.
\]

Since \(A\) generates a \(C_{0}\)-semigroup \(\{S(t)\}_{t \geq 0}\) of bounded linear operators on \(X\), then for a given \(u \in U\) and \(y_{0} \in D(A)\), the mild solution for the system \((1.1)\) is given by

\[
y(t) = S(t)y_{0} + \int_{0}^{t} S(t - \tau)\hat{B}y(\tau)d\tau + \int_{0}^{t} S(t - \tau)Gu(\tau)d\tau \quad (1.5)
\]

(refer, Pazy [24]). This correspondence which assigns a unique \(y \in Z = L^{2}(0,T;X)\) to a given \(u \in U\), will be denoted by a solution operator, say \(W i.e. W u = y\). Also, set \(Y = L^{2}(0,T;U)\).

The system \((1.1)\) is said to be approximately controllable if for given functions \(y_{0}, \hat{y} \in X\) and a \(\delta > 0\), there exists a control \(u \in U\) such that the corresponding solution \(y\) of system \((1.1)\) also satisfies \(\|y(T) - \hat{y}\|_{X} \leq \delta\).

In view of \((1.5)\), for such control \(u\), we arrive at

\[
\hat{y} = S(T)y_{0} + \int_{0}^{T} S(T - \tau)\hat{B}y(\tau)d\tau + \int_{0}^{T} S(T - \tau)Gu(\tau)d\tau, \quad (1.6)
\]
where \( \hat{y} = y(T) \). Setting the operator \( L : U \to X \) as

\[
Lu = \int_0^T S(T - \tau)u(\tau)d\tau,
\]

then the last term on the right hand side of (1.6) becomes \( LGu \). Now, the adjoint operator \( L^* : X \to Z \) of \( L \) becomes

\[
(L^*z)(\tau) = S(T - \tau)z, \quad \tau \in [0, T] \quad \text{and} \quad z \in X.
\]

If \( G^* \) is the adjoint operator of the operator \( G \), then it follows that

\[
(G^*L^*z)(\tau) = G^*S(T - \tau)z, \quad \tau \in [0, T] \quad \text{and} \quad z \in X.
\]

Thus, the equation (1.6) can be written equivalently as an operator equation

\[
\hat{z} = L\tilde{B}y + LGu,
\]

where \( \hat{z} = \hat{y} - S(T)y_0 \).

Define for \( \delta > 0 \), the set \( U_\delta \subset Y \) of admissible controls of (1.1) by

\[
U_\delta = \left\{ u \in Y : \| L\tilde{B}y + LGu - \hat{z} \|_X \leq \delta \right\}.
\]

It is a closed, convex and bounded (possibly empty) subset of \( Y \).

**Definition 1.1** The problem (1.1) is approximately controllable if for every \( y_0, \hat{y} \in X \) and \( \delta > 0 \), there exists \( u \in Y \) such that \( U_\delta \neq \emptyset \).

We now define our main problem as

**Main Problem.** Find

(i) if \( U_\delta \neq \emptyset \) for each \( \delta > 0 \) and

(ii) if so determine \( u^*_\delta \in U_\delta \) such that

\[
J(u^*_\delta) = \inf_{u \in U_\delta} J(u)
\]

where \( J(u) = \frac{1}{2}\| u \|^2_Y \).

**Definition 1.2** For a given \( \delta > 0 \), let \( u^*_\delta \in U_\delta \) is a solution of the problem (1.9) with \( y^*_\delta \in X \) as the corresponding mild solution of the system (1.1), then the pair \( (u^*_\delta, y^*_\delta) \) is called optimal pair of the constrained optimal control problem (1.9).

Our main thrust is to establish the existence of the optimal pair \( (u^*_\delta, y^*_\delta) \) of the constrained optimal control problem (1.9) and thereafter present a numerical scheme for approximating the optimal pair. Under the assumption \( B \equiv 0 \), in section 2 we first show that the set \( U_\delta \) of admissible controls is nonempty. Then the optimal pair \( (u^*_\delta, y^*_\delta) \) is obtained as a limit of the sequence of an optimal pair \( (u^*_\epsilon, y^*_\epsilon) \), where \( u^*_\epsilon \) minimizes the unconstrained functional \( J_\epsilon(u) \) over the whole space \( Y \) defined by

\[
J_\epsilon(u) = J(u) + \frac{1}{2\epsilon} \left\| Lu + L\tilde{B}Wu - \hat{z} \right\|^2_X,
\]
where $W$ is the operator which assigns to each control $u^*_\epsilon$ the solution $y^*_\epsilon$ of (1.1). We shall refer to $(u^*_\epsilon, y^*_\epsilon)$ as the optimal pair corresponding to the unconstrained problem.

The plan of this paper is as follows: In Section 2, we have shown that the set of admissible control $U_\delta$ is nonempty under the assumption that the corresponding linear system is approximately controllable. The optimal pair $(u^*_\delta, y^*_\delta)$ of the constrained problem (1.9) is obtained as a limit of the optimal pair sequence $(u^*_\epsilon, y^*_\epsilon)$, where $u^*_\epsilon$ minimizes the unconstrained functional $J_\epsilon(u)$ defined by (1.10). We present approximation theorems which guarantee the convergence of the numerical scheme to the optimal pair in Section 3. Error estimates are derived for the final state of the problem in section 4 with an application. In section 5, we conclude this paper by providing some numerical experiments to demonstrate the applicability of our results.

2 Existence of optimal control and convergence to the control problem

In this section, we first show that the set $U_\delta$ of admissible controls is nonempty. Here, we first make the following assumptions for the problem (1.1):

(A1) The set $\{S(t)\}_{t \geq 0}$ of $C_0$-semigroup of bounded linear operators on $X$, generated by $(-A)$ is uniformly bounded, that is, there exists $\beta > 0$ such that $\|S(t)\|_X \leq \beta$, for all $t \in [0, T]$.

(A2) The operator $B(t, \tau)$ is dominated by $A$ together with certain derivatives with respect to $t$ and $\tau$, that is, $\|A^{-1}B(t, \tau)\phi\| \leq \alpha \|\phi\| \quad \forall \quad \phi \in \mathcal{D}(B(t, \tau)), \quad 0 \leq \tau \leq t \leq T$.

(A3) The system (1.1) with $B \equiv 0$ is approximately controllable.

(A4) The operator $G : L^2(0, t; U) \to L^2(0, T; X)$ is a bounded linear operator.

The following lemma is related to the assumption (A3).

**Lemma 2.1** The system (1.1) with $B \equiv 0$ is approximately controllable on $[0, T]$ if and only if one of the following statement holds:

(i) $\overline{\text{Range}(LG)} = X$.

(ii) $\text{Kernel}(G^*L^*) = \{0\}$.

(iii) For all $z \in X$, there holds for $\delta \in (0, 1)$

\[ LGu_\delta = z - \delta \left(\delta I + LGG^*L^*\right)^{-1}z, \]

where $u_\delta := G^*L^*\left(\delta I + LG^*L^*\right)^{-1}z$.

(iv) $\lim_{\delta \to 0^+} \delta \left(\delta I + LG^*L^*\right)^{-1}z = 0$.

For a proof, we refer to Curtain et. al. [4, 5].

As a consequence, it is observed that

$$\lim_{\delta \to 0^+} LGu_\delta = z.$$
and the error $e_\delta z$ due to this approximation is given by

$$e_\delta z = \delta \left( \delta I + LGG^* L^* \right)^{-1} z = 0.$$ 

For approximate controllability of the problem (1.1), we rewrite its controllability equation as

$$u_\delta := G^* L^* \left( \delta I + LGG^* L^* \right)^{-1} \left( \tilde{z} - L\tilde{B}y \right), \quad (2.1)$$

where $\tilde{z} = y(T) - S(T)y_0$.

Now for a fixed $z \in Z$, consider the following linear parabolic integro–differential system which is indexed by $z$

$$\frac{\partial y_z}{\partial t} + Ay_z = \int_0^t B(t, \tau) z(\tau) d\tau + Gu_z, \quad t \in [0, T],$$

$$y_z(0) = y_0 \in X. \quad (2.2)$$

The mild solution $y_z \in Z$ of the above system is given by

$$y_z(t) = S(t)y_0 + \int_0^t S(t-\tau) \tilde{B}z(\tau) d\tau + \int_0^t S(t-\tau) Gu_z(\tau) d\tau, \quad (2.3)$$

and hence,

$$\tilde{y}_z \equiv y_z(T) - S(T)y_0 = \int_0^T S(T-\tau) \tilde{B}z(\tau) d\tau + \int_0^T S(T-\tau) Gu_z(\tau) d\tau. \quad (2.4)$$

In operator theoretic form, (2.4) reduces to the operator equation

$$LGu_z = \tilde{y}_z - L\tilde{B}z, \quad (2.5)$$

for each fixed $z$.

Now under the assumption (A3), the system (1.1) with $B \equiv 0$ is approximately controllable, and hence, Lemma [2.1] implies that $(\delta I + G^* L^* L G)$ is boundedly invertible. For approximate controllability of (2.2), we observe that for a given state $\tilde{z} = \hat{y} - S(T)y_0 \in X$, and for $\delta > 0$, a control $u_{\delta, z}$ solves

$$u_{\delta, z} = G^* L^* (\delta I + LG G^* L^*)^{-1} \left[ \tilde{z} - L\tilde{B}z \right]. \quad (2.6)$$

To keep the notation simple and where there is no confusion, we write $u_{\delta, z}$ simply by $u_z$.

Denote the operator $M_z := G^* L^* (\delta I + LGG^* L^*)^{-1} \left[ \tilde{z} - L\tilde{B}z \right]$ and consider for $\delta \in (0, 1]$ the family of operators $R_\delta : Z \to Z$, which assigns a solution $y_z$ of (1.1) (given by (2.3)), corresponding to $z \in Z$, that is,

$$R_\delta z(t) = S(t)y_0 + \int_0^t S(t-\tau) \left( \tilde{B}z(\tau) + LGM_z(\tau) \right) d\tau. \quad (2.7)$$

Define the operator $K$ by

$$(K y)(t) = \int_0^t S(t-\tau)y(\tau) d\tau. \quad (2.8)$$
After integrations by parts, we arrive at

\[ R_\delta z(t) = S(t)y_0 + KBz(t) + KG_M z(t). \]  \tag{2.9} 

First of all, we need to prove that for each fixed \( \delta \in (0,1] \) the operator \( R_\delta \) has fixed point, say, \( z_\delta \).

Now, the following lemmas deals with some properties of \( K \), and \( L \).

**Lemma 2.2** Let the assumptions (A1) and (A2) be satisfied and let the operators \( L : U \to X \) and \( K : Z \to Z \) be defined by \[ \text{1.7} \] and \[ \text{2.8} \] respectively. Then the following estimates hold

\[ \| (K\tilde{B}y)(t) \|_X \leq C \int_0^t \| y(s) \|_X ds, \]

and

\[ \| L\tilde{B}y \|_X \leq C \int_0^T \| y(s) \|_X ds. \]

**Proof.** From the definition of \( K \) and \( \tilde{B} \), we rewrite using semigroup property to get

\[
(K\tilde{B}y)(t) = \int_0^t S(t-\tau) \int_0^\tau B(\tau, s)y(s)dsd\tau \\
= \int_0^t S(t-\tau)AA^{-1} \int_0^\tau B(\tau, s)y(s)dsd\tau \\
= -\int_0^t \frac{d}{d\tau} S(t-\tau) \int_0^\tau A^{-1}B(\tau, s)y(s)dsd\tau.
\]

After integrations by parts, we arrive at

\[
(K\tilde{B}y)(t) = \left[ S(t-\tau) \int_0^\tau A^{-1}B(\tau, s)y(s)ds \right]_0^t - \int_0^t S(t-\tau) \int_0^\tau A^{-1}B_\tau(\tau, s)y(s)dsd\tau \\
- \int_0^t S(t-\tau)A^{-1}B_\tau(\tau, \tau)y(\tau)d\tau \\
= \int_0^t A^{-1}B(t, s)y(s)ds - \int_0^t S(t-\tau) \int_0^\tau A^{-1}B_\tau(\tau, s)y(s)dsd\tau \\
- \int_0^t S(t-\tau)A^{-1}B_\tau(\tau, \tau)y(\tau)d\tau.
\]

We note that

\[
\| (K\tilde{B}y)(t) \|_X \leq \int_0^t \| A^{-1}B(t, s)y(s) \| ds + \int_0^t \| S(t-\tau) \| \int_0^\tau \| A^{-1}B_\tau(\tau, s)y(s) \| dsd\tau
\]

\[ + \int_0^t \| S(t-\tau) \| \| A^{-1}B_\tau(\tau, \tau)y(\tau) \| d\tau, \]

and hence,

\[
\| (K\tilde{B}y)(t) \|_X \leq \alpha \int_0^t \| y(s) \|_X ds + \alpha \| A^{-1} \| (1 + \beta) \int_0^t \| y(\tau) \| X d\tau + \alpha \beta \int_0^t \| y(\tau) \|_X d\tau
\]

\[ \leq \alpha(1 + \beta)(1 + \| A^{-1} \|) \int_0^t \| y(\tau) \|_X d\tau. \]
Thus, we now arrive at
\[ \|(K\hat{B}y)(t)\|_X \leq C \int_0^t \|y(\tau)\|_X d\tau, \]
where \( C \) is a generic constant which depend on \( \alpha, \beta \) and \( \|A^{-1}\| \). Similarly we can show by using the definition of \( L \) and \( \tilde{B} \) that
\[ \|L\tilde{B}y\|_X \leq C \int_0^T \|y(\tau)\|_X d\tau. \]
This completes the proof of the lemma.

On the lines of Lemma 2.2, we have the following result.

**Lemma 2.3** Under the assumptions \((A1), (A2)\) and \((A4)\), the following estimate holds
\[ \left\| \left( KLG_M y \right)(t) \right\|_X \leq C_1 \left( \|\hat{z}\| + C \int_0^t \|y(\tau)\|_X d\tau \right), \]
where \( C_1 \) depends on \( T, \beta, \|L\|, \|G^*L^*\| \) and \( \|\delta I + LGG^*L^*\|^{-1} \).

A variation of Banach contraction mapping principle will help in the proof of the following theorem, which provides the approximate controllability of the system (1.1).

**Theorem 2.1** Under the assumption \((A1)-(A4)\), the operator \( R^n_\delta \) is a contraction on the space \( Z \) for some positive integer \( n \). Moreover, for any arbitrary \( z_0 \in X \), the sequence of iterates \( \{z_{\delta,k}\} \), defined by
\[ z_{\delta,k+1} = R^n_\delta z_{\delta,k}, \quad k = 0, 1, 2, \ldots \tag{2.10} \]
with \( z_{\delta,0} = y_0 \) converges to \( y^*_\delta \), which is a mild solution of the system (1.1). Further, \( u_{\delta,k} = Mz_{\delta,k} \) is such that \( u_{\delta,k} \) converges to \( u^*_\delta = My^*_\delta \), and the system (1.1) is approximately controllable.

**Proof.** Let \( z_1, z_2 \in Z \), then use of (2.9) yields
\[ (R_\delta z_1 - R_\delta z_2)(t) = K\hat{B}(z_1(t) - z_2(t)) + KGM(z_1(t) - z_2(t)). \]
Using Lemma 2.2 and 2.3 we now arrive at
\[ \|(R_\delta z_1 - R_\delta z_2)(t)\|_X \leq C \int_0^t \|z_1(\tau) - z_2(\tau)\|_X d\tau, \]
where \( C \) depends on \( \beta, \alpha, T, \|L\|, \|G^*L^*\| \) and \( \|\delta I + LGG^*L^*\|^{-1} \) and hence,
\[ \|R_\delta z_1 - R_\delta z_2\|_Z \leq CT \sqrt{2} \|z_1 - z_2\|_Z. \]
Proceeding inductively, we obtain that there exists a constant \( \gamma_n = \frac{(2CT)^n}{\sqrt{2n(3\delta\cdots2n-1)}} \), such that
\[ \|R^n_\delta z_1 - R^n_\delta z_2\|_Z \leq \gamma_n \|z_1 - z_2\|_Z. \]
Choose \( n \) large enough (independent of \( T \) and \( C \)) such that \( \gamma_n < 1 \), and hence, \( R^n_\delta \) is a contraction. Therefore, by Banach contraction mapping theorem, \( R^n_\delta \) has a unique fixed point, say \( y^*_\delta \).
which is the limit of the sequence defined by (2.10). This \( y_\delta^* \) is also the unique fixed point of the operator \( R_\delta \), for fixed \( \delta \in (0, 1) \).

Next to show that \( Mz_{\delta,k} \to My_\delta^* \). Setting \( u_{\delta,k} = Mz_{\delta,k} \), where \( z_{\delta,k} \) is the mild solution of the system \( (2.2) \) with control \( u_{\delta,k} \). Then, we obtain

\[
\left\| (Mz_{\delta,k} - My_\delta^*) (t) \right\|_X \leq CT^{1/2} \| G^*(\varepsilon I + LGG^*)^{-1} \| \| z_{\delta,k} - y_\delta^* \|_Z,
\]

and hence,

\[
\left\| (Mz_{\delta,k} - My_\delta^*) \right\|_Z \leq CT \| G^*(\varepsilon I + LGG^*)^{-1} \| \| z_{\delta,k} - y_\delta^* \|_Z.
\]

Since for each fixed \( \delta \in (0, 1) \), the sequence \( z_{\delta,k} \to y_\delta^* \) in \( Z \), this implies that \( Mz_{\delta,k} \to My_\delta^* = u_\delta^* \).

Since \( y_\delta^* \) is the mild solution of the system \( (1.1) \) with control \( u_\delta^* \). As \( z_{\delta,k} \to y_\delta^* \), it follows that \( R_\delta z_{\delta,k} \to R_\delta y_\delta^* = y_\delta^* \). Using the definition of \( R_\delta \) and with similar arguments as earlier, we find that

\[
R_\delta z_{\delta,k} (t) = S(t)y_0 + \int_0^t S(t - \tau) \hat{B}z_{\delta,k} (\tau) \, d\tau + \int_0^t S(t - \tau) Gu_{\delta,k} (\tau) \, d\tau.
\]

As \( k \to \infty \), we obtain

\[
y_\delta^* (t) = S(t)y_0 + \int_0^t S(t - \tau) \hat{B} y_\delta^* (\tau) \, d\tau + \int_0^t S(t - \tau) Gu_\delta^* (\tau) \, d\tau,
\]

and \( y_\delta^* \) is the mild solution of the system \( (1.1) \), corresponding to control \( u_\delta^* \) given by

\[
u_\delta^* = My_\delta^* = G^*(\delta I + LGG^*)^{-1} \left[ \hat{z} - L\hat{B} y_\delta^* \right].
\]

It remains to show that the problem \( (1.1) \) is approximately controllable. To this end observe that

\[
LGu_\delta^* = LGG^* (\delta I + LGG^*)^{-1} \left[ \hat{z} - L\hat{B} y_\delta^* \right]
\]

\[
= \left( (\delta I + LGG^* L^*) - \delta I \right) (\delta I + LGG^* L^*)^{-1} \left[ \hat{z} - L\hat{B} y_\delta^* \right]
\]

\[
= \left[ \hat{z} - L\hat{B} y_\delta^* \right] - \delta \left( \delta I + LGG^* L^* \right)^{-1} \left[ \hat{z} - L\hat{B} y_\delta^* \right].
\]

Since \( \| \hat{z} - L\hat{B} y_\delta^* \| \) is bounded, a use of Lemma 2.1 (iv) yields

\[
\lim_{\delta \to 0^+} \left\| -\delta \left( \delta I + LGG^* L^* \right)^{-1} \left[ \hat{z} - L\hat{B} y_\delta^* \right] \right\| = 0,
\]

and hence,

\[
\lim_{\delta \to 0^+} \| LGu_\delta^* + L\hat{B} y_\delta^* - \hat{z} \| = 0,
\]

that is, for given any given \( \delta_1 > 0 \), there exists a \( \delta_0 > 0 \) such that for \( 0 < \delta \leq \delta_0 \)

\[
\| LGu_\delta^* + L\hat{B} y_\delta^* - \hat{z} \| < \delta_1.
\]

Hence, the system \( (1.1) \) is approximately controllable. This completes the rest of the proof. ■
Remark 2.1 Note that $U_\delta \neq \emptyset$. Further, the error of the approximation in this case is given by

$$e_\delta z_\delta^* = \delta \left( \delta I + LGG^*L^* \right)^{-1} \left[ \dot{z} - L\tilde{y}_\delta^* \right].$$

Remark 2.2 Under assumption (A1)-(A4), the Theorem 2.1 implies that the system (1.1) is controllable without any inequality constraint on $T$.

Thus, we have $U_\delta \neq \emptyset$. The pair $(u^*_\delta, y^*_\delta)$ so obtained need not be an optimal pair satisfying (1.9), and hence, the problem (1.9) remains unanswered.

We now change our strategy and examine the process of obtaining the optimal pair of the constrained problem through a sequence of optimal pairs of the unconstrained problems, as indicated in the Section 1. For this purpose, we first define a sequence of functionals $\{J_\epsilon\}$ with $\epsilon > 0$ as

$$J_\epsilon(u) = \frac{1}{2} J(u) + \frac{1}{2\epsilon} P(u), \; u \in Y,$$  \hfill (2.14)

where penalty function $P(u)$ is of the form

$$P(u) = \left\| LGu + L\tilde{B}Wu - \hat{z} \right\|_X^2, \; u \in Y.$$  \hfill (2.15)

Now the problem under investigation is to seek $u^*_\epsilon \in U$ such that

$$J_\epsilon(u^*_\epsilon) = \inf_{u \in Y} J_\epsilon(u).$$  \hfill (2.16)

As in [8], roughly speaking, the approximate controllability can be viewed as the limit of a sequence of optimal control problems (2.16).

We now make further assumption that

(A5) The solution operator $W : U \to Z$ is completely continuous.

Remark 2.3 One of the sufficient condition for $W$ to be completely continuous is that the semigroup $\{S(t)\}$ is compact.

Denote by $E$ the operator $Eu = LGu + L\tilde{B}Wu$, where the operators $L, \tilde{B}$ and $W$ are as defined before. Then, the functional $J_\epsilon$ defined through (2.14) can be written as

$$J_\epsilon(u) = \frac{1}{2} \| u \|_Y^2 + \frac{1}{2\epsilon} \| E u - \hat{z} \|_X^2.$$  \hfill (2.17)

Note that the operator $E$ is a sum of linear continuous operator $L$ and a completely continuous operator $W$ and hence, it is a weakly continuous operator.

Theorem 2.2 Under assumptions (A1)-(A5), the unconstrained optimal control problem (2.16) has an optimal pair $(u^*_\epsilon, y^*_\epsilon)$ such that $u^*_\epsilon \in U$ minimizes $J_\epsilon(u)$ and $y^*_\epsilon$ solves (1.1) corresponding to the control $u^*_\epsilon$. 
Proof. We first prove the weakly lower semicontinuity of the functional \( J_\epsilon \). Let \( u^n_\epsilon \to u^*_\epsilon \) in \( Y \), then, it follows that

\[
\liminf_{n \to \infty} J_\epsilon(u^n_\epsilon) = \liminf_{n \to \infty} \left[ \frac{1}{2} \|u^n_\epsilon\|_Y^2 + \frac{1}{2\epsilon} \|E u^n_\epsilon - \hat{z}\|_X^2 \right] 
\geq \liminf_{n \to \infty} \frac{1}{2} \|u^n_\epsilon\|_Y^2 + \liminf_{n \to \infty} \frac{1}{2\epsilon} \|E u^n_\epsilon - \hat{z}\|_X^2.
\]

Observe that

\[
\|E u^n_\epsilon - \hat{z}\|_X^2 = \|L Gu^n_\epsilon\|_X^2 + \|LBW u^n_\epsilon - \hat{z}\|_X^2 + 2 \left\langle Lu^n_\epsilon, LBW u^n_\epsilon - \hat{z} \right\rangle_X,
\]

and hence,

\[
\liminf_{n \to \infty} \|E u^n_\epsilon - \hat{z}\|_X^2 \geq \liminf_{n \to \infty} \|L Gu^n_\epsilon\|_X^2 + \liminf_{n \to \infty} \|LBW u^n_\epsilon - \hat{z}\|_X^2 
+ 2 \liminf_{n \to \infty} \left\langle L Gu^n_\epsilon, LBW u^n_\epsilon - \hat{z} \right\rangle_X.
\]

From Lemma 2.2, we arrive at

\[
\|LB (W u^n_\epsilon - W u^*_\epsilon)\|_X \leq C \int_0^T \|(W u^n_\epsilon - W u^*_\epsilon)\|_X ds
\leq CT^{1/2} \|W u^n_\epsilon - W u^*_\epsilon\|_Z.
\]

Since \( u^n_\epsilon \to u^*_\epsilon \) in \( Y \), and \( W \) is completely continuous, this implies that \( W u^n_\epsilon \to W u^*_\epsilon \) in \( Z \) and hence, \( LBW u^n_\epsilon - \hat{z} \to LBW u^*_\epsilon - \hat{z} \) in \( X \). Using the fact that \( L \) is weakly continuous and \( W \) is completely continuous, we obtain \( L Gu^n_\epsilon \to L Gu^*_\epsilon \), \( LBW u^n_\epsilon - \hat{z} \to LBW u^*_\epsilon - \hat{z} \) and \( \langle L Gu^n_\epsilon, LBW u^n_\epsilon - \hat{z} \rangle \to \langle L Gu^*_\epsilon, LBW u^*_\epsilon - \hat{z} \rangle \) and along with the fact that the norm is weakly lower semicontinuous functional, we find that

\[
\liminf_{n \to \infty} J_\epsilon(u^n_\epsilon) \geq \frac{1}{2} \|u^*_\epsilon\|_Y^2 + \frac{1}{2\epsilon} \|E u^*_\epsilon - \hat{z}\|_X^2.
\]

This proves the weakly lower semi-continuity of \( J_\epsilon \).

Let \( \{u^n_\epsilon\} \) be a minimizing sequence for the functional \( J_\epsilon \), that is, \( \inf_{u \in Y} J_\epsilon(u) = \lim_{n \to \infty} J_\epsilon(u^n_\epsilon) \). Since \( J_\epsilon \) is coercive, the sequence \( \{u^n_\epsilon\} \) is bounded in \( Y \). Then, there exists a subsequence which is also denoted by \( \{u^n_\epsilon\} \) such that \( u^n_\epsilon \to u^*_\epsilon \) weakly in \( Y \). Since the functional \( (2.17) \) is weakly lower semicontinuous in \( Y \), we arrive at

\[
\inf_{u \in Y} J_\epsilon(u) = \lim_{n \to \infty} J_\epsilon(u^n_\epsilon) = \liminf_{n \to \infty} J_\epsilon(u^n_\epsilon) \geq J_\epsilon(u^*_\epsilon).
\]

Therefore, we obtain

\[
J_\epsilon(u^*_\epsilon) = \inf_{u \in Y} J_\epsilon(u).
\]

As \( y^n_\epsilon = W u^n_\epsilon \) and \( u^n_\epsilon \to u^*_\epsilon \), the complete continuity of \( W \) implies \( y^n_\epsilon \to y^*_\epsilon \), where \( y^*_\epsilon = W u^*_\epsilon \). Thus, \( (u^*_\epsilon, y^*_\epsilon) \) is the optimal pair for the unconstrained optimal control problem \( (2.16) \) and this completes the proof of the theorem.

In our subsequent analysis, we need the following properties of the sequence of minimizers \( \{u^n_\epsilon\} \).
Lemma 2.4 Let \( \epsilon > 0 \) be arbitrary and let \( u_\epsilon \in Y \) be a minimizer of \( J_\epsilon(u) \) in \( Y \), where \( J_\epsilon(u) \) as defined by \((2.14)\). For \( \epsilon' < \epsilon \), the followings holds:

(i) \( J_\epsilon(u_\epsilon) \leq J_{\epsilon'}(u_{\epsilon'}) \).

(ii) \( P(u_\epsilon) \geq P(u_{\epsilon'}) \).

(iii) \( J(u_\epsilon) \leq J(u_{\epsilon'}) \).

(iv) \( J(u_\epsilon) \leq J_\epsilon(u_\epsilon) \leq J(u^*) + \frac{\delta_0^2}{2\epsilon} \).

Proof. As \( u_\epsilon \) minimizes \( J_\epsilon \), it follows that

\[
J_\epsilon(u_\epsilon) = J(u_\epsilon) + \frac{1}{2\epsilon}P(u_\epsilon) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon}P(u_{\epsilon'}) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon'}P(u_{\epsilon'}) = J_{\epsilon'}(u_{\epsilon'}).
\]

This proves (i). For (ii), let \( u_\epsilon \) and \( u_{\epsilon'} \) be the minimizers of \( J_\epsilon \) and \( J_{\epsilon'} \), respectively, then, we arrive at

\[
J_\epsilon(u_\epsilon) = J(u_\epsilon) + \frac{1}{2\epsilon}P(u_\epsilon) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon}P(u_{\epsilon'})
\]

and

\[
J_{\epsilon'}(u_{\epsilon'}) = J(u_{\epsilon'}) + \frac{1}{2\epsilon'}P(u_{\epsilon'}) \leq J(u_\epsilon) + \frac{1}{2\epsilon'}P(u_\epsilon)
\]

On adding above inequalities, we get

\[
P(u_\epsilon) \geq P(u_{\epsilon'}).
\]

For (iii), note that

\[
J_\epsilon(u_\epsilon) = J(u_\epsilon) + \frac{1}{2\epsilon}P(u_\epsilon) \leq J(u_{\epsilon'}) + \frac{1}{2\epsilon}P(u_{\epsilon'}).
\]

Hence, using (ii) it follows that

\[
J(u_\epsilon) - J(u_{\epsilon'}) \leq \frac{1}{2\epsilon} (P(u_{\epsilon'}) - P(u_{\epsilon})) \leq 0
\]

and

\[
J(u_\epsilon) \leq J(u_{\epsilon'}).
\]

For (iv), again we observe that

\[
J(u_\epsilon) \leq J_\epsilon(u_\epsilon) = J(u_{\epsilon'}) + \frac{1}{2\epsilon}P(u_{\epsilon'}) \leq J(u^*) + \frac{1}{2\epsilon}P(u^*) \leq J(u^*) + \frac{\delta_0^2}{2\epsilon}.
\]

This completes the rest of the proof. ■

We are now in a position to state the main theorem of this article.

Theorem 2.3 Assume that for a fixed \( \delta > 0 \), \( U_\delta \neq \emptyset \) and assumptions \((A1)-(A4)\) hold. Let \((u^*_\delta, y^*_\delta)\) be an optimal pair of the unconstrained problem \((2.14)\). As \( \epsilon \to 0 \), there exists a subsequence of \((u^*_\epsilon, y^*_\epsilon)\) converges to \((u^*_\delta, y^*_\delta)\), where \((u^*_\delta, y^*_\delta)\) is an optimal pair of the constrained optimal control problem \((1.7)\). Furthermore, if \( U_\delta \) is a singleton then the entire sequence \((u^*_\epsilon, y^*_\epsilon)\) converges to \((u^*_\delta, y^*_\delta)\).
Proof. As for a fixed $\delta > 0$, $U_\delta \neq \emptyset$; the existence of the optimal pair $(u^{*}_\epsilon, y^{*}_\epsilon)$ to the unconstrained problem (2.16) follows from the Theorem 2.2. Let $u^{*}_\epsilon \in U_\delta$. From Lemma 2.4 we have
\[ J_\epsilon(u^{*}_\epsilon) \leq J_\epsilon(u^{*}_{\epsilon'}) \text{ for } \epsilon' < \epsilon \]

Thus, $\{J_\epsilon(u^{*}_\epsilon)\}$ is a monotone decreasing sequence which is bounded below and hence it converges. Similarly, $\{J(u^{*}_\epsilon)\}$ is also a convergent sequence. Now $\frac{1}{\epsilon}P(u^{*}_\epsilon)$, being the difference of two convergent sequence, also converges, which in turn, implies that $P(u^{*}_\epsilon) \to 0$ as $\epsilon \to 0$. Hence,
\[ \lim_{\epsilon \to 0} \|E u^{*}_\epsilon - \tilde{z}\|_X = 0. \]

Since $\{u^{*}_\epsilon\}$ is a uniformly bounded sequence in $Y$, it has a subsequence, again denoted by $\{u^{*}_\epsilon\}$ such that $u^{*}_\epsilon \to u^{*}_\delta$ in $Y$. Weak continuity of $E$ implies that $E u^{*}_\epsilon \to E u^{*}_\delta$. Hence $E u^{*}_\epsilon = \tilde{z}$ and $u^{*}_\delta \in U_\delta$. By the weak lower semicontinuity of the norm functional and Lemma 2.4, we arrive at
\[ \|u^{*}_\delta\|_Y \leq \liminf_{\epsilon \to 0} \|u^{*}_\epsilon\|_Y \leq \limsup_{\epsilon \to 0} \|u^{*}_\epsilon\|_Y \leq \|u^{*}_\delta\|_Y, \]

and hence,
\[ \lim_{\epsilon \to 0} \|u^{*}_\epsilon\|_Y = \|u^{*}_\delta\|_Y. \]

This along with the weak convergence of $u^{*}_\epsilon$ to $u^{*}_\delta$, implies that
\[ u^{*}_\epsilon \to u^{*}_\delta \text{ as } \epsilon \to 0. \]

Again from Lemma 2.4 and weak lower semicontinuity of the norm functional, we obtain the inequality
\[ J(u^{*}_\delta) \leq \liminf_{\epsilon \to 0} J(u^{*}_\epsilon) \leq J(\tilde{u}), \text{ } \tilde{u} \in U_\delta. \]

This, in turn, implies that
\[ J(u^{*}_\delta) \leq J(\tilde{u}) \forall \tilde{u} \in U_\delta. \]

Therefore, $(u^{*}_\delta, y^{*}_\delta)$ is the optimal pair for the constrained optimal problem (1.9). It is also clear that if $U_\delta$ is a singleton then the entire sequence $u^{*}_\epsilon$ converges to $u^{*}_\delta$ in $Y$.

Next, we show the convergence of $y^{*}_\epsilon$ to $y^{*}_\delta$ in $Z$. From (1.5) and Lemma 2.2 we obtain
\[ \|y^{*}_\epsilon(t) - y^{*}_\delta(t)\|_Z \leq C \int_0^t \|y^{*}_\epsilon(\tau) - y^{*}_\delta(\tau)\|_X d\tau + \beta \int_0^t \|u^{*}_\epsilon(\tau) - u^{*}_\delta(\tau)\|_X d\tau \]
\[ \leq C \int_0^t \|y^{*}_\epsilon(\tau) - y^{*}_\delta(\tau)\|_X d\tau + \beta T^{1/2} \|u^{*}_\epsilon - u^{*}_\delta\|_Y. \]

Using Gronwall’s lemma, we arrive that
\[ \|y^{*}_\epsilon(t) - y^{*}_\delta(t)\|_X \leq \beta T^{1/2} \|u^{*}_\epsilon - u^{*}_\delta\|_Y e^{CT}, \]

and hence,
\[ \|y^{*}_\epsilon - y^{*}_\delta\|_Z \leq \beta T e^{CT} \|u^{*}_\epsilon - u^{*}_\delta\|_Y. \]

Since $u^{*}_\epsilon \to u^{*}_\delta$ in $Y$, from (2.18), we obtain $y^{*}_\epsilon \to y^{*}_\delta$ in $Z$ as $\epsilon \to 0$. This completes the rest of the proof. ■
3 Approximation theorems

In our analysis, we are interested in the computation of the optimal control pair for the unconstrained problem. We first begin by establishing some properties of the operator arising from the derivative of the functional $J_\epsilon$, which is defined as follows:

$$J_\epsilon(u) = \frac{1}{2}\|u\|^2_Y + \frac{1}{2\epsilon}\|\mathcal{E}u - \hat{z}\|^2_X,$$  \hfill (3.1)

where $\hat{z}$ is a fixed element in $X$. We first recall the unconstrained optimal control problem

$$J_\epsilon(u) = \inf_{v \in Y} J_\epsilon(v).$$  \hfill (3.2)

**Lemma 3.1** The critical point of the functional $J_\epsilon$ is given by the solution of the operator equation

$$u + \frac{1}{\epsilon}K(\mathcal{E}u - \hat{z}) = 0,$$  \hfill (3.3)

where $K = (LG + L\tilde{B}W)^*$, $\mathcal{E}u = (LG + L\tilde{B}W)u$ and $y = Wu$.

**Proof:** We note that

$$J_\epsilon(u + hv) - J_\epsilon(u) = \frac{1}{2}\langle u + hv, u + hv \rangle$$

$$+ \frac{1}{2\epsilon}\left( (LG + L\tilde{B}W)(u + hv) - \hat{z}, (LG + L\tilde{B}W)(u + hv) - \hat{z} \right)$$

$$- \frac{1}{2}\langle u, u \rangle - \frac{1}{2\epsilon}\left( (LG + L\tilde{B}W)(u) - \hat{z}, (LG + L\tilde{B}W)(u) - \hat{z} \right)$$

$$= h \langle u, v \rangle + \frac{h^2}{2} \langle v, v \rangle + \frac{h}{\epsilon} \left\langle (LGu + L\tilde{B}Wu - \hat{z}, (LG + L\tilde{B}W)(v) \right\rangle$$

$$+ \frac{h^2}{2\epsilon} \left\langle (LG + L\tilde{B}W)(v), (LG + L\tilde{B}W)(v) \right\rangle.$$  

Then, $J'_\epsilon(u)$ is given by

$$J'_\epsilon(u)v = \lim_{h \to 0} \frac{J_\epsilon(u + hv) - J_\epsilon(u)}{h}$$

$$= \langle u, v \rangle + \frac{1}{\epsilon} \left\langle LGu + L\tilde{B}Wu - \hat{z}, (LG + L\tilde{B}W)(v) \right\rangle$$

$$= \langle u, v \rangle + \frac{1}{\epsilon} \left\langle (LG + L\tilde{B}W)^*(LG + L\tilde{B}W)u - \hat{z}, v \right\rangle,$$

and hence,

$$J'_\epsilon(u) = u + \frac{1}{\epsilon}(LG + L\tilde{B}W)^*(LG + L\tilde{B}W)u - \hat{z}).$$

If $u$ is a critical point of $J_\epsilon$, then it follows that

$$u + \frac{1}{\epsilon}\mathcal{K}(LGu + L\tilde{B}Wu - \hat{z}) = 0,$$

where $\mathcal{K} = (LG + L\tilde{B}W)^*$. This concludes the proof. \hfill \blacksquare
Note that, in the literature, the operator equation (3.3) is known as the Hamerstein equation (see, Joshi et al. [12]). Also note that the operator $K$ is bounded linear operator. We first assume that the critical point of $J_\epsilon$ is the unique minimizer of $J_\epsilon$. Then the minimizing problem (3.2) is equivalent to the following solvability problem in the space $Y$:

$$u + \frac{1}{\epsilon} KEu = \hat{w}, \quad (3.4)$$

where $\hat{w} = \frac{1}{\epsilon} K\hat{z}$. We now first begin approximating the main problem in the following way.

Consider a family $\{X_m\}$ of finite dimensional subspaces of $X$ such that

$$X_1 \subset X_2 \subset \ldots \subset X_m \ldots \subset X \quad \text{with} \quad \bigcup_{m=1}^{\infty} X_m = X.$$ 

Let $\{\phi_i\}_{i=1}^{\infty}$ be a basis for $X$. The approximating scheme for the space $Y = L^2(0,T;X)$ is then given by the family of subspaces $Y_m = L^2(0,T;X_m)$ such that

$$Y_1 \subset Y_2 \subset \ldots \subset Y_m \ldots \subset Y \quad \text{with} \quad \bigcup_{m=1}^{\infty} Y_m = Y.$$ 

Note that, the solution of the system (1.1) is given by $y(t) = \sum_{i=1}^{\infty} \alpha_i \phi_i$ with the control $u(t) = \sum_{i=1}^{\infty} \beta_i \phi_i$.

Let $P_m : X \to X_m$ be the projection given by

$$P_m[y(t)] = \sum_{i=1}^{m} \alpha_i \phi_i, \quad t \in [0,T],$$

where $X_m = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$. Then, this induces in a natural way the projection $\tilde{P}_m : Y \to Y_m$ given by

$$(\tilde{P}_m y)(t) = P_m y(t).$$

The projections $P_m$ and $\tilde{P}_m$ generate the approximating operators $K_m$ and $E_m$ defined by $K_m = \tilde{P}_m K$ and $E_m u = P_m E u$. Then, the approximated minimization problem is stated as: Find $u_m \in Y_m$ such that

$$J_{\epsilon,m}(u_m) = \inf_{u \in Y_m} \left[ J_{\epsilon,m}(u) = \frac{1}{2} \|\tilde{P}_m u\|_{Y_m}^2 + \frac{1}{2\epsilon} \|P_m E P_m u - P_m \hat{z}\|_{X_m}^2 \right]. \quad (3.5)$$

As in the case of problem (2.16), one can show that the problem (3.5) has a solution $u_m \in Y_m$, and hence, its critical point satisfying the operator equation in the approximating space $Y_m$ as

$$u_m + \frac{1}{\epsilon} K_m (E_m u_m - P_m \hat{z}) = 0. \quad (3.6)$$

Following theorem shows that the solution for the problem (3.6) is uniformly bounded in $Y_m$ and the approximating pair $(u_m^*, y_m^*)$ converges to $(u^*, y^*)$, where $(u^*, y^*)$ is an optimal pair of the constrained problem (1.9).
Theorem 3.1  Let $U_\delta \neq \emptyset$ and $u^*_m$ be the solution to the problem \([3.3]\). Then \(\{u^*_m\}\) is uniformly bounded in $Y_m$. If in addition, $J_\varepsilon$ possesses a unique minimizer in $Y$ which is also the only critical point of $J_\varepsilon$, then \([3.3]\) has an optimal pair $\left( u^*_m, y^*_m \right)$ which converges to $(u^*, y^*)$ in $Y$, where $(u^*, y^*)$ is an optimal pair of the constrained problem \([1.3]\).

Proof: Existence of the optimal pair $(u^*_m, y^*_m)$ to the optimal control problem \([3.5]\) follows from Theorem \([2.2]\).

Let $u^* \in U_\delta$, then from the definition of $U_\delta$, we have $\|E u^* - \hat{z}\| \leq \delta$. Define $u^*_m = \hat{P}_m u^*$. Then

\[
\frac{1}{2} \|u^*_m\|^2 \leq J_{\varepsilon, m}(u^*_m) = \frac{1}{2} \|\hat{P}_m u^*\|^2 + \frac{1}{2\varepsilon} \|P_m E u^*_m - P_m \hat{z}\|^2 \\
\leq \frac{1}{2} \|\hat{P}_m u^*\|^2 + \frac{1}{2\varepsilon} \|P_m\|^2 \|E u^*_m - \hat{z}\|^2 \\
\leq \frac{1}{2} \|\hat{P}_m\|^2 \|u^*\|^2 + \frac{1}{e} \|P_m\|^2 \left( \|E u^*_m - E u^*\|^2 + \|E u^* - \hat{z}\|^2 \right).
\]

Since $u^*_m = \hat{P}_m u^* \rightarrow u^*$ and $E$ is weakly continuous, we have $E u^*_m \rightarrow E u^*$. Hence, both the term on right hand side is bounded. Therefore \(\{u^*_m\}\) is uniformly bounded.

Since $\{u^*_m\}$ is uniformly bounded, it has a subsequence, still denoted by $u^*_m$, which converges weakly to $u^*$ in $Y$. Then from the weak lower semicontinuity of the norm functional and Lemma \([2.4]\), we arrive at

\[
\|u^*\| \leq \liminf_{m \rightarrow \infty} \|u^*_m\| \leq \limsup_{m \rightarrow \infty} \|u^*_m\| \leq \|u^*\|.
\]

This implies

\[
\lim_{m \rightarrow \infty} \|u^*_m\| = \|u^*\|.
\]

Together with the fact that $u^*_m \rightarrow u^*$ in $Y$, we obtain

\[
u^*_m \rightarrow u^* \quad \text{in} \quad Y, \quad \text{as} \quad m \rightarrow \infty.
\]

As $y^*_m = W u^*_m$ and $u^*_m \rightarrow u^*$, then continuity of the solution operator $W$ implies that $y^*_m \rightarrow y^* = W u^*$. This now completes the rest of the theorem. 

The next step is to discretize in the direction of $t$. This leads to finite dimensional subspaces $Y^k_m$ of each fixed $Y_m$ as follows

\[
Y^k_m = \left\{ y^k_m \in \mathcal{P}_0 : y^k_m|_{[t_{l-1}, t_l]} = y^l_m, \ t_0 = 0, \ t_k = 1, \ t_l = l \Delta t, \Delta t = 1/k, \ 1 \leq l \leq k \right\}
\]

where $\mathcal{P}_0$ is the space of piecewise constant polynomials. It is clear that $Y^k_m$ satisfies the following property

\[
Y^1_m \subset Y^2_m \subset \ldots \subset Y^k_m \subset \ldots \subset Y_m \quad \text{with} \quad \bigcup_{k=1}^{\infty} Y^k_m = Y_m.
\]

We denote by $Q^k_m$, the orthogonal projection from $Y_m$ to $Y^k_m$. This induces the operators $\mathcal{K}_m = \mathcal{Q}^k_m \mathcal{K}_m$ and $\mathcal{E}_m u = \mathcal{Q}^k_m \mathcal{E}_m u$.

For a fixed $m$, we approximate the minimization problem \([3.5]\) by the following minimization problem in the finite dimensional subspace $Y^k_m$ of $Y_m$. 


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Find $u_m^k \in Y_m^k$ such that

$$\Phi(u_m^k) = \inf_{u_m \in Y_m^k} \left[ J_{k,m}(u_m) = \frac{1}{2} \| Q_m^k u_m \|_{Y_m^k}^2 + \frac{1}{2\epsilon} \| Q_m^k e_m u_m - \dot{z}_m \|_{X_m^k}^2 \right].$$ (3.7)

The unique minimizer of the problem (3.7) is given by the critical point of $\Phi$, which is equivalent to the following solvability problem in the space $Y_m^k$.

$$u_m^k + \frac{1}{\epsilon} k_m^k \left( e_m^k u_m - P_m \dot{z} \right) = 0.$$ (3.8)

On the lines of Theorem 3.1 we have the following theorem giving the convergence of the approximation optimal pair $(u_m^k, y_m^k)$ as $k \to \infty$ with $m$ fixed.

**Theorem 3.2** Let $\{u_m^k\}$ be the solution of the problem (3.8). Then the approximating optimal pair $(u_m^k, y_m^k)$ converges to $(u_m^*, y_m^*)$ in $Y_m$.

### 4 Application

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial \Omega$. For fixed $T > 0$, let $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial \Omega$. Let $A$ be a second order uniformly elliptic differential operator given by (1.2). Further, assume that the operator $B(t, s)$ is an unbounded partial differential operator of order $\beta \leq 2$ given by (1.3).

Set $X = L^2(\Omega)$, $V = H^1_0(\Omega)$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $D(B) = H^2(\Omega)$. Then the weak formulation of the problem (1.1) is given by

$$(y_t, \phi) + A(y, \phi) = \int_0^t B(t, s; y(s), \phi) ds + (u, \phi) \quad \forall \phi \in V, \quad t \in [0, T]$$ (4.1)

$$y(0) = y_0,$$

where $A(\cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ and $B(t, s; \cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ are the continuous bilinear forms corresponding the operators $A$ and $B(t, s)$ respectively, that is

$$A(y, \phi) = \int_{\Omega} \left( \sum_{i,j=1}^d a_{ij}(x) \frac{\partial y}{\partial x_j} \frac{\partial \phi}{\partial x_i} + c(x) y \phi \right) dx,$$

and

$$B(t, s; y, \phi) = \int_{\Omega} \left( \sum_{i,j=1}^d b_{ij}(t, s, x) \frac{\partial y}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \sum_{j=1}^d b_j(t, s, x) \frac{\partial y}{\partial x_j} \phi + b_0(t, s, x) y \phi \right) dx.$$

Here, $(a_{ij}(x))$ is a symmetric and positive definite matrix for all $x \in \bar{\Omega}$ and $a_0(x) \geq 0$. From Lumer–Phillips theorem (see, Pazy [24]), $(-A)$ generates a $C_0$–semigroup. For $y_0 \in D(A)$, the unique mild solution for the system (1.1) is given by

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)B\dot{y}(s)ds + \int_0^t S(t-\tau)u(s)ds.$$ (4.2)
For final time \( t = T \), we obtain
\[
y(T) = S(T)y_0 + L\tilde{B}y + Lu
\]
where the operator \( \tilde{B} \) and \( L \) are defined as before.

Since all the hypotheses (A1-A4) are satisfied, an appeal to Theorem 2.1 ensures the approximate controllability of (4.1). Also, set \( U = L^2(\Omega) \) and \( Y = L^2(0, T, U) \), the solution operator \( W : U \to Y \) is compact and an application to Theorem 2.2 and 2.3 shows the existence of optimal control.

Let \( \{ J_h \} \) be a family of regular triangulation of \( \Omega \) with \( 0 < h < 1 \). For \( K \in J_h \), set \( h_K = \text{diam}(K) \) and \( h = \max(h_K) \). Let
\[
V_h = \left\{ v_h \in C^0(\bar{\Omega}) : v_h|_K \in P_1(K), K \in J_h, v_h = 0 \text{ on } \partial \Omega \right\},
\]
where \( P_1(K) \) is the space of linear polynomials on \( K \). Then, the semidiscrete Galerkin approximation of (4.1) is defined by
\[
(y_h(t), \chi) + A(y_h, \chi) = \int_0^t B(t, s; y_h(s), \chi) \, ds + (u, \chi) \quad \forall \chi \in V_h, \quad t \in [0, T]
\]
(4.4)
\[
y_h(0) = y_{0h},
\]
where, \( y_{0h} \) is the approximation of \( y_0 \) in \( V_h \).

Let \( \{ \varphi_i \}_{i=1}^{N_h} \) be a bases of the finite element space \( V_h \). Since \( y_h(t) \in V_h \), we write
\[
y_h(t) = \sum_{i=1}^{N_h} \alpha_i(t) \varphi_i(x),
\]
where \( \{ \alpha_i \}_{i=1}^{N_h} \) satisfies
\[
\sum_{i=1}^{N_h} \left[ (\varphi_i, \varphi_j) \alpha_i'(t) + A(\varphi_i, \varphi_j) \alpha_i(t) - \int_0^t B(\varphi_i, \varphi_j) \alpha_i(s) \, ds \right] = (u(t), \varphi_j), \quad j = 1, 2, \ldots, N_h, \quad \alpha_i(0) = \gamma_i.
\]
(4.5)

Here, \( \gamma_i \) is the coefficient of \( \varphi_i(x) \) in the representation of \( y_{0h} \), that is, \( y_{0h} = \sum_{i=1}^{N_h} \gamma_i \varphi_i(x) \). This is the first order system of ordinary differential equations.

In matrix form, system (4.5) can be written as follows
\[
M\alpha' + A\alpha - \int_0^t B\alpha(s) \, ds = U
\]
(4.6)
where \( M = [M_{ij}] \) with \( M_{ij} = (\phi_i, \phi_j) \), \( A = [A_{ij}] \) with \( A_{ij} = A(\phi_i, \phi_j) \), \( B = B_{ij} \) with \( B_{ij} = B(\phi_i, \phi_j) \) and \( U = [U_j] \) with \( U_j = (u, \phi_j) \). Note that the system (4.6) leads to a system
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of ordinary integro–differential equations and since the mass matrix $M$ is invertible, the system \([4.0]\) is uniquely solvable in $C^1(0,T)$.

Let $P_h : V \to V_h$ be the $L^2$–projection and let \(\{S_h(t)\}\) denote the finite element analogue of $S(t)$, defined by the semidiscrete equation \([4.4]\) with $u = 0$, $B = 0$. This operator on $V_h$ may be defined as the semigroup generated by the discrete analogue $A_h : V_h \to V_h$ of $A$, where

$$ (A_h v, \chi) = A(v, \chi) \quad \forall v, \chi \in V_h. $$

Define the discrete analogue $B_h = B_h(t,s) : V_h \to V_h$ of $B = B(t,s)$ by

$$ (B_h(t,s)v, \chi) = B(t,s;v,\chi) \quad \forall v, \chi \in V_h, \quad 0 \leq s \leq t \leq T. $$

Now we write the semidiscrete problem \([4.4]\) in an abstract form

$$ y_{h,t} + A_h y_h = \int_0^t B_h(t,s)y_h(s)ds + P_h u \equiv \tilde{B}_h y_h + P_h u, \quad \text{for } t \in [0,T], $$

$$ y_h(0) = P_h y_0. \quad \text{(4.7)} $$

Using Duhamel’s principle, the solution $y_h$ of the semidiscrete problem \([4.7]\) may be written as

$$ y_h(t) = S_h(t)P_h y_0 + \int_0^t S_h(t-s)\tilde{B}_h y_h(s)ds + \int_0^t S_h(t-s)P_h u(s)ds. \quad \text{(4.8)} $$

At time $t = T$, equation \([4.8]\) becomes

$$ y_h(T) = S_h(T)P_h y_0 + L_h \tilde{B}_h y_h + L_h P_h u, \quad \text{(4.9)} $$

where $L_h$ is defined by

$$ L_h v_h = \int_0^T S_h(T-\tau)v_h(\tau)d\tau. \quad \text{(4.10)} $$

Setting $e = y_h - y$, we have

$$ e(T) = (S_h(T)P_h - S(T)) y_0 $$

$$ + \left( \int_0^T S_h(T-s)\tilde{B}_h y_h(s)ds - \int_0^T S(T-s)\tilde{B} y(s)ds \right) $$

$$ + \left( \int_0^T S_h(T-s)P_h u(s)ds - \int_0^T S(T-s)u(s)ds \right) $$

$$ = F_h(T)y_0 + \int_0^T F_h(T-s)\tilde{B} y(s)ds $$

$$ + \int_0^T S_h(T-s)\left( \tilde{B}_h y_h(s) - P_h \tilde{B} y(s) \right)ds + \int_0^T F_h(T-s)u(s)ds $$

$$ = I_1 + I_2 + I_3 + I_4, \quad \text{(4.11)} $$

where the operator $F$ is defined as $F_h((t) = S_h(t)P_h - S(t)$. For $F_h$, it is well known that (see, Theorem 3.1 of Bromle et. al. \([2]\))

$$ \|F_h(t)v\| \leq ch^s t^{-(s-m)/2} |v|_m, \quad 0 \leq m \leq s \leq 2. \quad \text{(4.12)} $$
For $I_1$, using estimates (4.12) for $v = y_0$, we get
\[ I_1 \leq \|F_h(T)y_0\| \leq ch^sT^{-(s-m)/2}|y_0|_m, \quad 0 \leq m \leq s \leq 2. \] (4.13)

Now for $I_4$,
\[ I_4 = \int_0^T F_h(T-s)u(s)ds \leq \int_0^T \|F_h(T-s)u(s)\|ds \]
\[ \leq ch^s \int_0^T (T-s)^{-(s-m)/2}|u(s)|_m ds \]
\[ \leq ch^s \left( \int_0^T (T-s)^{-(s-m)} ds \right)^{1/2} \left( \int_0^T |u(s)|^2_m ds \right)^{1/2} \]

Here, take $0 < s - m < 1$ with $0 \leq m \leq s \leq 2$. For the estimates of $I_2$, a use of Lemma 4.3 in Zhang [30] (pp. 135) yields
\[
\|I_2\| \leq \|\tilde{F}_h(B)(T)\| \leq \|\int_0^T F_h(T-s)\tilde{B}y(s)ds\| \\
\leq ch^2 \int_0^T (T-s)^{-1/2}\|\tilde{B}y(s)\|ds \\
\leq ch^2 \int_0^T (T-s)^{-1/2} \left( \int_0^s \|B(T, \tau)y(\tau)\|d\tau \right) ds \\
\leq ch^2 \int_0^T (T-s)^{-1/2} \left( \int_0^s \|y(\tau)\|_{2\tau} d\tau \right) ds \\
\leq ch^2 \int_0^T (T-s)^{-1/2} \left( \int_0^T \|y(\tau)\|_{2\tau} d\tau \right) ds \\
\leq c(T)h^2 \int_0^T \|y(\tau)\|_{2\tau} d\tau \\
\leq c(T)h^2 (\|y_0\| + \|u\|_{L^2(L^2)})
\]

For the term $I_3$, we again follow the idea of the proof of [30], that is, following the existence of $e_2$ term in ([30], pp 135-138) to conclude the estimates of $I_3$ as
\[ |S_h(t)\chi|_{q,h} \leq ct^{-(p-q)/2}|\chi|_{p,h}. \] (4.14)

For $m \leq 1$, we have
\[
\langle I_3, \chi \rangle = \left\langle \int_0^T \int_0^s S_h(T-s)(B_h y_h(\tau) - P_h B y(\tau))d\tau ds, \chi \right\rangle \\
= \int_0^T \left\langle \int_0^s (B_h y_h(\tau) - P_h B y(\tau))d\tau, S_h(T-s)\chi \right\rangle ds \\
= \int_0^T \int_0^s B(s, \tau, e(\tau), S_h(T-s)\chi)d\tau ds.
\]

Using (4.13) of [30] and (4.14), we obtain
\[ |\langle I_3, \chi \rangle| \leq c(T) \int_0^T \|e(\tau)\|d\tau \|\chi\|, \]

and hence
\[ \|I_3\| = \sup_{0 \neq \chi \in L^2} \frac{|\langle I_3, \chi \rangle|}{\|\chi\|} \leq c(T) \int_0^T \|e(T)\| d\tau. \]

On substitution in (4.11), we arrive at
\[ \|e(T)\| \leq c(h^2 T^{-1/2}\|y_0\| + h^{1-\delta_0} \left( \int_0^T (T-s)^{1-\delta_0} ds \right)^{1/2} \left( \int_0^T \|u(s)\|^2 ds \right)^{1/2} + h^2 (\|y_0\| + \|u\|_{L^2(L^2)}) + c(T) \int_0^T \|e(\tau)\| d\tau \] (for small $\delta_0$).

By Gronwall’s lemma, we obtain for small $\delta_0$,
\[ \|e(T)\| \leq ch^{1-\delta_0} (\|y_0\| + \|u\|_{L^2(L^2)}). \]

**Full Discretization:** Let $k$ be the step size in time, $t_n = nk$, $n = 0, 1, 2, \cdots$, $N = T/k$ and let $u^n = u(t_n)$. For $\phi \in C[0,T]$ set
\[ \partial_t \phi(t_n) = \frac{\phi(t_n) - \phi(t_{n-1})}{\Delta t}. \]

The approximation $y_h^n \in V_h$ of $y_h$ at time $t = t_n$ is now defined as a solution of
\[ \left( \partial_t y_h^n, \chi \right) + A(y_h^n, \chi) = Q^n(B(y_h^n), \chi) + (u^n, \chi), \quad \chi \in V_h, \quad n = 1, 2, \cdots, N, \]
where we have used the left rectangular rule
\[ Q^n(y) = \sum_{i=0}^{n-1} k y(t_j) \approx \int_0^{t_n} y(s) ds \]
to discretize the Volterra integral term.

As a consequence of Theorem 3.2 it is possible to show that an approximate pair $\{u_h^n, y_h^n\}$ converges to the optimal pair $\{u^*, y^*_h\}$.

## 5 Numerical experiment

In this section, we present a numerical experiment to illustrate the computation of the minimizer $u^*$. We consider the following one dimensional initial–boundary value problem
\[ \begin{align*}
\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} &= \int_0^t B(t,s) y(s) ds + u(t,x), \quad \text{on } (0,T) \times (0,1) \\
y(t,x) &= y_0(x), & x \in (0,1) \\
y(t,0) &= 0 = y(1,t), & t \in [0,T]
\end{align*} \tag{5.1} \]

Set $T = 1$, $\Omega = (0,1) \subset \mathbb{R}$ with $B(t,s) = \exp(-\pi^2 (t-s)) I$, $y_0(x) = \sin(\pi x)$ and $\dot{y} = \exp(-\pi^2) \sin(\pi x)$. For this system $\dot{y} \in R(T,y_0)$, since $y(t,x) = \exp(-\pi^2 t) \sin(\pi x)$ is an exact solution of the system (5.1) corresponding to the control function $u(t,x) = -t \exp(-\pi^2 t) \sin(\pi x)$ with $y(T,x) = \exp(-\pi^2) \sin(\pi x)$.
Here, we choose $\Delta t$, $h$ and $N = 1/\Delta t$. Using MOA algorithm (see, Joshi et. al. [15]), we compute $u^n, n = 1, 2, \ldots, N$ and then plot the graph of numerical results for $N = 40$. In Figure 1, we plot the graph of the approximated state at time $T = 1$ and the given final state $\hat{y} = \exp(-\pi^2)\sin(\pi x)$ corresponding to the approximated optimal control $u^*$. The approximated optimal control $u^*$ has been shown in Figure 2. Figure 3(i) shows the surface of the computed state corresponding to the optimal control $u^*$, whereas Figure 3(ii) shows the surface of the exact solution of the system (5.1).

![Graph](image)

Figure 1: Comparison between $y(T)$ and $\hat{y}$.

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Figure 2: Graph of the computed optimal control $u^*$.

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Figure 3: Graph of (i) the computed state corresponding to the optimal control $u^*$, and (ii) exact solution.

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