Control of localization in non-Hermitian systems

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Non-Hermitian systems have been actively studied for open and dissipative quantum systems. One of the remarkable features is the non-Hermitian skin effect, the anomalous condensation of the bulk states at the edge resulting from asymmetric hopping magnitudes. From both theoretical and experimental points of view, it has been studied based on the Hatano-Nelson argument and topological exceptional points. Beyond such non-Hermitian systems, however, different types of non-Hermiticity and their characteristics remain elusive. In this work, we focus on a non-Hermitian system where the hopping phase exists non-reciprocally and discuss the evolution of localization characteristics of the quantum states. We emphasize that the non-Hermiticity encoded in the hopping phase factor gives rise to the delocalization of the states in contrast to the non-Hermitian skin effect. Furthermore, by quantifying the localization in the spectrum via inverse participation ratio and fractal dimension, we demonstrate that the non-Hermitian hopping phase results in delicate controllability of the localization characteristics of quantum states. Our work offers new types of non-Hermitian systems which can control wave localization, and finally we also discuss the relevant experimental applications.

Introduction — Controlling localization characteristics of the states are one of the central issues in quantum systems, determining conductivity, optical transparency and etc[1–9]. To classify such localization and its effective control, one requires to identify the scaling behavior of the wave functions[10–13]. In a periodic crystal, for instance, most of the wave functions are either exponentially decaying or uniformly extending[1]. In contrary, the quasicrystals, self-similar ordered structures without any periodic length scale, allow the critical states, which exhibit novel scaling behaviors, neither exponential decay nor sinusoidal oscillation[5, 13–17]. For explicit characterization of localization, concept of the fractal dimension is adopted, where its value ranges continuously between 0 to 1[16]. The values 0 and 1 correspond to the case of the localized and extended states, respectively, and the values between them correspond to the critical states with power-law scaling behavior[16–20]. To utilize various applications of the critical, extended and localized states, the controllability between them is an important task[20, 21].

In other sides, change of the state localization has been discussed in the non-Hermitian systems[22–33]. In such non-Hermitian systems, one of the exotic phenomena is the non-Hermitian skin effect (NHSE), the massive condensation of bulk modes at the edge[28–36]. In detail, with an open boundary condition, macroscopically many bulk states are pushed away to one of the boundaries of the system, resulting from the asymmetric hopping magnitudes of the kinetic terms[25, 37–39]. At the beginning, it has been understood by the Hatano-Nelson argument, using the imaginary Gauge potential and relevant Gauge transformation of the state[30, 40]. Nowadays, NHSE has been understood in terms of the topological nature of the exceptional points (EPs), where the coalescence of the eigenstates occurs[26, 36, 41–44]. However, since NHSE is originated from the directional preference of the kinetics due to the asymmetric hopping magnitudes, it gives rise to a sudden and instantaneous exponential localization of the states by pushing every state to the edge[22, 30, 39, 40]. Thus, in terms of localization control, NHSE exhibits poor controllability, always resulting in localized state at the edge. Beyond NHSE, therefore, the high controllability of the localization characteristics in between critical, extended and localized state is demanded to utilize the novel quantum transports. So, one could ask for novel ways to delicately control the localization characteristics with non-Hermiticity.

In this paper, to answer the above question, we offer the non-Hermitian system where the non-Hermiticity is taken into account as the non-reciprocal phase of the hopping parameters whose magnitude is uniform. In contrast to the traditional non-Hermitian models explained by the Hatano-Nelson argument[30], such non-Hermiticity does not have any directional preference. Nevertheless, the non-reciprocal hopping phase gives rise to general delocalization of the states. To quantify it, we adopt the inverse participation ratio and fractal dimension[19]. We show that the localization strength in the spectrum decreases as the strength of the non-Hermiticity increases. Importantly, we demonstrate that the non-reciprocal hopping phase indeed manipulates the localization characteristics of the states from the exponentially localized to the extended state with higher controllability compared to the NHSE, where only extreme condensation of the states at the boundary occurs[22, 30]. It turns out that the interference originating from the non-reciprocal phases of the hopping parameters makes the wave function spread out, resulting in the delocalization of the states. Based on our theory, we also suggest an experimental setup to demonstrate control of the localization in terms of the non-Hermitian hopping phases.
from NHSE, our work suggests novel ways to control the localization-delocalization using the non-Hermiticity of the system.

**Non-Hermitian Hamiltonian beyond the Hatano-Nelson model**— Let us consider one-dimensional tight-binding Hamiltonian with \( N \)-sites

\[ H = H_V + H_T, \]

\[ H_V = \sum_{i=1}^{N} V_i \left| i \right\rangle \langle i \right|, \]

\[ H_T = \sum_{i=1}^{N-1} \left( t_L |i \rangle \langle i + 1| + t_R |i + 1 \rangle \langle i| \right), \]

where \( V_i \) are the real-valued local potentials. \( t_L \) and \( t_R \) are the uniform complex-valued left-moving and right-moving hopping parameters, respectively. \( |i\rangle \) represents the particle placed on the \( i \)-th site. Here, we consider the open boundary condition (OBC). The hopping parameters are uniform complex-valued, \( t_L = t_R = T e^{i\theta} \) where \( T \) and \( \theta \) are positive reals. The strength of the non-Hermiticity is given by \( T\sin \theta \), which is maximized when \( \theta = \pi/2 \).

Remarkably, the Hamiltonians in Eq. ([1]) is beyond the Hatano-Nelson argument in a sense that there is no specific directional preference originating from the hopping magnitude \([30, 37]\). Thus, there is no NHSE for the Hamiltonian in this case\([25]\). Nevertheless, we will show that the non-Hermiticity encoded in the non-reciprocal hopping phase changes the localization characteristics of the states compared to the Hermitian counterparts given by \( \theta = 0, \pi \).

**Controls of state localization**— Given a finite hopping magnitude, \( T \), we can induce the delocalization of the states as \( \theta \) approaches \( \pi/2 \), where the strength of the non-Hermiticity becomes maximum. To understand how the non-Hermitian hopping phase factor enables to change the localization of the states, let us consider a localized mode confined at \( i \)-th site due to the strong local potential difference, \( V_i \gg T \). Then, we can treat \( H_T \) as a perturbation. The most notable correction of the wave function is the second order. From the Hamiltonian in Eq. ([1]), the second order contribution can be calculated with \( H_T^2 \),

\[ H_T^2 = t_L t_R \left( 2 \sum_{i=2}^{N-1} |i \rangle \langle i| + |1 \rangle \langle 1| + |N \rangle \langle N| \right) \]

\[ + t_L^2 \sum_{i=1}^{N-2} |i \rangle \langle i + 2| + t_R^2 \sum_{i=1}^{N-2} |i + 2 \rangle \langle i|. \]

Here, the significant contribution for the change of the localization arises from the first term in Eq. ([2]). In the second order, the wave function on the \( i \)-th site results from the interference between the potential and hopping contributions given by \( V_i \) and \( t_L t_R / V_i \), respectively. Contrary to the Hermitian case, where the hopping contribution, \( t_L t_R \), is always positive, the non-Hermitian cases allow arbitrary complex-valued hopping contributions, which lead to destructive interference. It turns out that the destructive interference suppresses the probability amplitude of the \( i \)-th site by spreading out the wave function to the other sites as \( \theta \) approaches \( \pi/2 \). In such a way, the localized modes become delocalized due to the non-Hermiticity. See Appendix A for detailed discussion for the specific case with \( N = 3 \).

Importantly, a non-reciprocal hopping phase provides more controllability of the localization strength, compared to the NHSE where only extreme localization at the boundary occurs\([30, 35, 40]\). In terms of the hopping phase, one can explore various localization characteristics not only exponentially localized or uniformly extended but power-law decaying critical states depending on \( V_i \). To show high controllability of the localization, we consider the quasi-periodic system, particularly the Fibonacci quasicrystal, where both exponentially localized and critical wave functions exist, and discuss the effect of non-hermiticity induced by non-reciprocal hopping phase. We emphasize that our discussion can be generalized in other quasicrystalline systems or the random disordered systems \([45–47]\) (See Appendix B for detailed information). For a concrete argument, let us consider the real-valued on-site potentials which are arranged as the Fibonacci quasicrystal\([20, 21]\). In detail, the Fibonacci quasicrystal is comprised of two different atoms, \( A \) and \( B \) which have the onsite potentials \( V_A \) and \( V_B \), respectively. By using the successive substitution maps, \( A \rightarrow AB \) and \( B \rightarrow A \), one gets the Fibonacci arrangement of the atoms such as \( ABAABABABA \cdots \) \([48, 49]\). For the Hermitian system under the OBC, it is known that there are both exponentially localized states and critical states in the Fibonacci quasicrystal, regardless of the finite hopping magnitude, \( T\)\([50–52]\). Hence, one could ask if the non-Hermiticity with a uniform complex-valued hopping parameter enhances the delocalization of the states in the Fibonacci quasicrystal, and eventually gives rise to the extended states for a finite hopping magnitude under the OBC.

To quantify the localization characteristics of the wave function, we use both the inverse participation ratio (IPR) and the fractal dimension of the state. Note that the amount of localization for the wave function, \( \Psi \), can be quantified by the IPR defined as \( \sum_{i=1}^{N} |\Psi(x_i)|^4 \) where \( N \) is the system size and \( \Psi(x_i) \) is the wave function at \( x_i\)\([34, 53, 54]\). Although a larger IPR indicates stronger localization, however, the value of IPR alone is insufficient to determine the detailed localization characteristics and scaling behaviors of the wave function since IPR is an averaged quantity over space\([13, 53]\). Thus, we investigate the system size dependence of the IPR, indicating the spatial distribution of the wave function. Specifically, it is known that for sufficiently large system size, \( N \), the IPR shows the scaling behavior as \( N^{-D_2} \) where \( D_2 \) is called the fractal dimension\([13, 16, 19]\). An exponentially localized state shows \( D_2 = 0 \) and a uniformly extended state shows \( D_2 = 1 \). The critical states have
the intermediate fractal dimensions, $0 < D_2 < 1$[13, 19].

First, we consider the states whose IPR is the maximum and minimum value among the eigenstates. Let us call these states as maximally localized and maximally extended states in the spectrum, respectively. Fig.1 (a) and (f) respectively illustrate the landscapes of the fractal dimension of the maximally localized state and maximally extended state, as functions of the magnitude and phase of the hopping parameter, $T$ and $\theta$. Here, the potential difference between atoms $A$ and $B$ is given by $V_A - V_B = 2V$. As $\theta$ approaches $\pi/2$, the fractal dimensions of the maximally localized or extended states increase, and this indicates the delocalization of the states.

In detail, Figs.1 (b)-(d) show that with sufficiently large $T$ ($T > 10V$ in Fig.1 (a)), one can control the localization characteristics of the maximally localized state from the exponentially localized one to the sinusoidally extended one in terms of $\theta$. In the Hermitian case, the fractal dimension of the maximally localized state remains zero corresponding to the exponentially localized state regardless of the magnitude of $T$ (See Fig.1 (a) and (b)). However, in the non-Hermitian cases, the fractal dimension of the maximally localized could be nonzero. In particular, Fig.1 (d) shows that for sufficiently large but finite hopping magnitude, the maximally localized state could become an extended state having $D_2 = 1$. In this case, every eigenstate has $D_2 = 1$. Note that in the Hermitian system, this is asymptotically achieved by infinite hopping magnitude or zero spatial potential gradient limits only[1].

The reason why we can universally achieve $D_2 = 1$ even with finite hopping magnitude in the Fibonacci quasicrystal is as follows. The non-reciprocal hopping phase controls the localization of states by forming non-trivial interference between the hopping, and potential contributions of the state, originating from the hopping parameters and spatial potential gradient, respectively. As $\theta$ approaches $\pi/2$, the interference becomes destructive, and hence the potential contribution could be canceled out by the hopping contribution. Thus, the non-Hermiticity leads to the delocalization of the states by effectively blinding the spatial potential gradient such as the quasiperiodic structures with the destructive interference. This enables us to achieve the uniformly extended state even in the presence of the spatial potential gradient with finite hopping magnitudes.

Although the maximally localized state remains exponentially localized as $D_2 = 0$ regardless of $\theta$ when $T$ is small ($T \lesssim 4V$ in Fig.1 (a)), the non-Hermiticity enhances the localization length of the state. It turns out that the interference originated from non-reciprocal hopping phase induces the penetration of the wave function to the bulk of the system. Thus, even with a small $T$, we can manipulate the localization strength of the state in terms of the non-reciprocal hopping phase. See Appendix C for detailed information related to the control of the localization length.

Let us discuss the significance of the non-Hermitian characteristics in terms of the change for the scaling behavior of the maximally localized state. To capture it, we compute the phase rigidity of the maximally localized state, given by $r(\Psi) = |\langle \Psi_L | \Psi_R \rangle|$, where $\Psi$ is the maximally localized state[41, 43]. The subscripts $L$ and $R$ indicate the left and right eigenstates. Note that $r(\Psi) = 1$ for the Hermitian case, while $r(\Psi) \leq 1$ for the non-Hermitian case because the right eigenstate could be non-orthogonal to each other. At the EPs where the multiple right eigenstates coalesce, the phase rigidity vanishes[42, 43]. Fig.1 (e) illustrates the landscape for $\log[r(\Psi)]$. Comparing Fig.1 (a) and (e), the phase rigidity suddenly drops at the boundaries where the change of the localization characteristics happens. Thus, when the localization characteristics change, the maximally localized state strongly hybridizes with other critical or extended states by passing through the EPs[43].

Now let us consider the maximally extended state. In the Hermitian system, the maximally extended state has the fractal dimension, $D_2 < 1$ for any finite $T$ due to the fractal structure of the Fibonacci quasicrystal[50]. However, Fig.1 (f) shows that the non-Hermiticity can enhance the fractal dimension of the maximally extended state as $D_2 = 1$. Thus, uniformly extended states can emerge in the non-Hermitian systems even for quasicrystals (See Fig.1 (h)). Comparing Figs.1 (g) and (h), which illustrate the maximally extended states where $\theta = 0$ and $\pi/2$, respectively for the same $T$, one can see that not only the delocalization happens but the fractality of the wave function disappears due to the non-Hermiticity. It turns out that the non-reciprocal phase factor of the hopping parameters induces delocalization by screening the detailed structure of the lattice such as the Fibonacci pattern by the interference effect rather than just increasing the mobility of the particle.

One could ask how the localization strength of other states evolves as the strength of the non-Hermiticity increases. For general states, the interference effect can enhance the localization strength in terms of the IPR, distinct from the localized modes. See Appendix D for detailed information about state-dependent controls of the localization strength. Nevertheless, it is important to point out that the non-Hermiticity leads to the trend of delocalization in the spectrum as $\theta$ approaches $\pi/2$. One quantifies the localization strength in the spectrum in terms of the mean value of the IPR of the eigenstates[34]. For a given $T$, we indeed observe that the mean value of the IPR decreases as $\theta$ approaches $\pi/2$. Hence, the non-Hermiticity gives rise to the trend of delocalization of the states with an OBC, distinct from NHSE. See Appendix E for detailed information.

**Discussion and Conclusion** — We briefly discuss the experimental setup which enables us to demonstrate our results. Non-Hermitian complex-valued hopping terms could be simulated with an optical system illustrated in Fig.2[55, 56]. The optical resonators (blue and red) drawn as circles in Fig.2 with the resonance frequencies ($-V$ and $+V$, respectively) are arranged as the Fi-
FIG. 1. The landscape of the fractal dimensions of the states for (a) maximum value of IPR and (f) minimum value of IPR, which corresponds to the (red) maximally localized and (blue) extended state, respectively. For $T = 13V$, the circle, square and triangle indicate $\theta = 0, 85^\circ$ and $90^\circ$ respectively. (b-d) Evolution of the localization characteristics of the maximally localized state for (b) $\theta = 0$, (c) $\theta = 85^\circ$ and (d) $\theta = 90^\circ$, respectively, with $T = 13V$. The fractal dimensions are (b) $D_2 = 0$, (c) $D_2 = 0.411$ and (d) $D_2 = 1$, respectively. As the phase approaches the maximal value of non-Hermiticity, $\theta = 90$ degree, the state evolves from the exponentially localized state (b) to the extended state (d) where intermediate power-law decaying critical states (c) emerge at intermediate $\theta$. (e) Log of phase rigidity of the maximally localized state. (g-h) Evolution of the localization characteristics of the maximally extended state for (g) $\theta = 0$ and (h) $\theta = 90^\circ$, respectively, with $T = 13V$. The fractal dimensions are (g) 0.915 and (h) 1, respectively. The localization characteristics evolve from the self-similar critical state (g) for Hermitian case to the uniformly oscillating extended state (h) for maximally non-Hermitian case. See the main text for more details.

bonacci quasicrystal. Here, we use the optical diodes to manipulate the phase of the right-moving and left-moving waves separately.[57–59]. The phases of right/left-moving waves are uniformly controlled by multiplying $e^{i\theta}$ with identical phase shifters represented by yellow films in Fig.2.

In summary, we have provided a novel way to control the localization of the quantum states by using the non-Hermiticity beyond the Hatano-Nelson argument. Our non-Hermitian model whose non-Hermiticity exists as the non-reciprocal phase of the uniform hopping parameter does not have a specific directional preference, distinct from the Hatano-Nelson model. It is important to note that, depending on the state, the localization could be enhanced or reduced as a function of non-reciprocal hopping phase. This enables us to control the localization of the states in a very delicate way. Particularly, for exponentially localized or critical states emerging in the quasicrystalline systems, the non-Hermiticity induces delocalization of the states. By exemplifying the Fibonacci quasicrystal, we have shown that non-Hermiticity can indeed change the localization characteristics in between exponentially localized, critical and uniformly extended states, and this is originated from the destructive interference between the potential gradient and hopping contributions of the wave functions. Our work opens the utility of the non-Hermiticity for high controllability of the localization characteristics.

FIG. 2. Schematic figure of the experimental realization of our non-Hermitian model. $V$ is the real-valued resonance frequency of the optical resonators drawn as red and blue circles. The red resonators admit the frequency, $+V$ and the blue resonators admit the frequency, $-V$, where they are arranged as the Fibonacci sequence. Effective hopping terms for right-moving and left-moving waves are respectively controlled by using the optical diodes represented by the green arrows in the figure. The phase factor of the effective hopping parameter would be added by the phase shifter shown as yellow films.

Our non-Hermitian model without directional preference can be generalized to higher-dimensional systems such as two- or three-dimensional lattices. We suspect that similar delocalization phenomena would appear in
the higher-dimensional lattices due to the non-reciprocal phases of the hopping parameter. Since the geometry of the lattice such as square, triangular, and even quasicrystalline structures like Penrose tiling[20, 60] affects the interference of the wave functions in the higher-dimensional systems[19], we expect that anomalous delocalization would be emergent from the interference effect induced by the non-Hermiticity. It is an interesting future work.

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Appendix A: Controls of the localization by the interference due to the non-reciprocal phase of the hopping parameters: The simplest model study

Here, to understand how the complex-valued hopping parameter controls the localization of the wave function without directional preference, we consider the simplest version of the Hamiltonian in Eq.(1) in the main text that is comprised of three sites, \(A, B\) and \(C\) whose real on-site potential energies are \(-V, +V\) and \(-V\), respectively. The Hamiltonian matrix is given by

\[
H = \begin{pmatrix}
-V & t & 0 \\
t & V & t \\
0 & t & -V
\end{pmatrix}, \tag{A1}
\]

where \(t = Te^{i\theta}\) is the uniform complex-valued hopping parameter. Without loss of generality, let us assume that \(0 \leq \theta \leq \pi/2\) and \(V > 0\). The Hamiltonian in Eq.(A1) has the three right eigenvectors, \(\psi_0 = (1.0 - 1)/\sqrt{2}\), \(\psi_\pm = (t, V \pm \sqrt{V^2 + 2t^2}, t)/N\) where \(N\) is the normalization constant. For \(\psi_\pm\), the amount of the localization on the \(B\) site is given by

\[
\gamma_\pm(T, \theta) = \frac{\left|\psi_\pm(B)/\psi_\pm(A)\right|^2}{\frac{V \pm \sqrt{V^2 + 2T^2e^{2i\theta}}}{T}^2}, \tag{A2}
\]

where \(\psi_\pm(x)\) is the wave function of the \(\psi_\pm\) at the site \(x\). Thus, for \(\theta = 0\), \(\psi_+\) has larger localization on the \(B\) site than \(\psi_-\). Since \(\gamma_+\) in Eq.(A2) is the decreasing function on \(0 \leq \theta \leq \pi/2\) for given \(T\), the wave function of the \(\psi_+\) spreads out to the \(A\) and \(C\) sites as \(\theta\) increases. This is originated from the destructive interference on the \(B\) site by the nontrivial phase factor from the right-moving and left-moving hopping parameters. Specifically, the wave function on the \(B\) site, \(\psi_+(B)\) could be interpreted as the interference between the potential contribution \(V\) and the hopping contribution \((t^2/V)\) which have traveled the \(A\) or \(C\) sites. If we write the right-moving and left-moving hopping parameters as \(t_R\) and \(t_L\), respectively, then \(t^2\) should be written as \(t_Rt_L\). For Hermitian case, since the phase factor of the right-moving and left-moving hopping parameters are reciprocal, \(t_Rt_L\) is always positive real. Thus, the interference between the potential contribution and the hopping contribution always gives rise to the constructive interference. On the other hand, for non-Hermitian hopping parameters, \(t_Rt_L\) could be arbitrary complex number. Particularly, if both \(t_R\) and \(t_L\) are purely imaginary, then \(t_Rt_L\) becomes negative which maximally suppresses the amount of the localization on the \(B\) site as the destructive interference. In such a way, the non-Hermiticity without directional preference could control the localization of the wave function.

Appendix B: Control of localization in the uniformly random disordered chain

Here, we consider the random disordered chain where the on-site energies have 50% level of disorder. Specifically, the random on-site energies are between -1.5\(V\) to \(-0.5V\). Figure.S1 demonstrates that the the localization is suppressed as the non-Hermiticity increases even for the random disordered system. The MIPR decreases as \(\theta\) becomes \(\pi/2\) which corresponds to the maximal strength of the non-Hermiticity for given hopping magnitude \(T\). Thus, the delocalization is induced by the non-reciprocal hopping phases in the random disordered chain system.

Appendix C: Localization length evolution in the Fibonacci quasicrystal with small hopping magnitude regime

Here, we consider the Fibonacci quasicrystal with the small hopping magnitude, \(T\) regime. Despite the fractal dimension is constantly zero for the maximally localized state having the maximum value of the IPR where small \(T\) regime, the localization length is drastically controlled via the strength of the non-Hermiticity. In detail, let us define the localization length, \(\xi\) of the exponentially localized state as

\[
\xi = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} \tag{C1}
\]

where \(\hat{x}\) is the position operator and \(\langle \hat{O} \rangle\) is the expectation value of the operator, \(\hat{O}\). For \(T = 3V\), we demonstrate the evolution of the localization length as the function of the non-reciprocal phase of the hopping parameter, \(\theta\). Remind that the uniform non-Hermitian hopping parameter is given by \(t = Te^{i\theta}\). Fig.S2 (a) shows that the localization length
FIG. S1. MIPR as the function of the phase of non-Hermitian hopping parameter, $\theta$ in the random disordered system. The level of the disorder is 50%. The system size, $N = 233$ and the magnitude of the hopping parameter, $T = 4V$. 

is drastically increases as $\theta$ approaches to $\pi/2$. Hence, although the localization characteristics of the maximally localized state is exponentially decaying for every $\theta$ with small $T$, the localization length could be manipulated in terms of the non-reciprocal hopping phase. Fig.S2 (b) compares the probability distribution of the maximally localized states for $\theta = 0$ (red) and $\theta = \pi/2$ (blue) in the log scale. The linear scalings in Fig.S2 (b) indicate the identical exponentially decaying behavior, while the different slopes show that the wave can more penetrate to the bulk for the non-Hermitian case i.e. delocalization emerges.

Appendix D: State-dependent controls of the localization

Unlike the NHSE condensing every states to the edges, the non-reciprocal phase factor of the hopping parameters could either reduce or enhance the amount of the localization of each states. For instance, in the example of previous section, $\gamma_-$ which is the amount of the localization on the $B$ site of $\psi_-$, is an increasing function of $\theta$. Hence, depending on the energy level we are interested in, the localization is either enhanced or reduced as a function of the strength of the non-Hermiticity. This leads to the state-dependent controls of the localization, distinct from NHSE. To demonstrate the state-dependent controls of the localization strength, we explore the evolution of the inverse participation ratio (IPR) as the function of the magnitude and phase of the hopping parameters. Interestingly, unlike the hopping magnitude which universally reduces the IPR, the non-Hermiticity originated from the non-reciprocal hopping phase can either increase or decrease the IPR depending on the states. Specifically, Fig.S3 (a) shows that the IPR of the $\psi_+$ decreases as the non-Hermiticity becomes strong, while Fig.S3 (b) shows that the IPR of the $\psi_-$ can be increased. Thus, it gives more controllability of the localization strength depending on the specific states, compared to the NHSE where only extreme localization at the boundary emerges.

Another important remark arising from Fig.S3 is as follows. There is an exceptional point (EP), $(\theta, T/V) = (\pi/2, 1/\sqrt{2})$, where $\psi_+$ and $\psi_-$ coalesce. Surprisingly, even if we increase the hopping magnitude larger than this EP condition keeping $\theta = \pi/2$, the IPR whose value is exactly $3/8$ does not change in both states. Note that this characteristics correspond to the extended states for the case of $V = 0$ limit i.e., the uniform on-site potential. Thus, the non-reciprocal phases of the hopping parameters eventually enhance the delocalization in the spectrum in contrast
FIG. S2. (a) Localization length ($\xi$ defined in Eq. (C1)) evolution as the function of the non-reciprocal phase ($\theta$). The non-Hermiticity induces the delocalization, thus, the localization length increases as the non-Hermiticity becomes stronger. (b) Comparison of the probability distribution of the maximally localized states for (blue) $\theta = \pi/2$ and (red) $\theta = 0$ in the log scale. The linear scaling in the figure indicates the exponentially decaying. The smaller slope indicates that the larger localization length for the non-Hermitian case. The hopping magnitude $T = 3V$. The system size is $N = 987$.

FIG. S3. The landscape of IPR for the states (a) $\psi_+$ and (b) $\psi_-$, as functions of $T/V$ and $\theta$ with the hopping parameter, $T e^{i\theta}$, and onsite potential, $V$. As the magnitude of the hopping parameters increases, the IPR decreases in both states. On the other hand, the IPR can be either enhanced or reduced as a function of $\theta$ depending on the states. Specifically, when $\theta$ approaches $\pi/2$, the IPR of the state $\psi_+$ decreases but the IPR of the state $\psi_-$ increases. The exceptional point $(\theta, T/V) = (\pi/2, 1/\sqrt{2})$, where the IPR is saturated to $3/8$ is highlighted.

to the NHSE. Moreover, in the Hermitian case, such extended states are asymptotically achieved only for $T \gg V$ limit, whereas, one can get such extended states even for the finite hopping magnitude in the non-Hermitian system. It is because the non-Hermitian interference effect induces the delocalization by screening the presence of the spatial potential gradient rather than increasing the mobility of the particles. See the main text for detailed discussions.
FIG. S4. The mean value of the IPR (MIPR) of the spectrum as the function of the strength of the non-Hermiticity given by the phase angle of the hopping parameter, $\theta$. Here, the hopping magnitude is $T = 13V$. At $\theta = 90$ degree, the non-Hermiticity becomes maximum for given $T$. The MIPR which is the amount of the localization in the spectrum decreases as the non-Hermiticity becomes stronger.

Appendix E: Delocalization trend in terms of the non-Hermiticity

In this section, we address how the localization strength of the states other than the maximally localized or extended states evolves in terms of the non-reciprocal hopping phase. To this end, we quantify the localization strength in the spectrum in terms of the mean value of the IPR (MIPR) given by

$$MIPR(T, \theta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\psi_i(x_j)|^4. \quad (E1)$$

Here, $\psi_i(x_j)$ is the wave function of the $i$-th state on the position, $x_j$. For a given $T$, Fig.S4 illustrates the MIPR as a function of the phase of the hopping parameter, $\theta$. The MIPR decreases as $\theta$ approaches $\pi/2$. Thus, the localization strength in the spectrum is suppressed in the Fibonacci quasicrystal due to the non-Hermiticity. Remind that our non-Hermiticity encoded in the phase of the hopping parameter does not have the directional preference, and hence it can either enhance or reduce the localization of each state (See Sec.D.). Nevertheless, the non-Hermiticity gives rise to the trend of delocalization, distinct from NHSE.