Integration of the SL(2,\(\mathbb{R}\))/U(1) Gauged WZNW Model with Periodic Boundary Conditions

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Abstract

Gauged WZNW models are integrable conformal field theories. We integrate the classical SL(2,\(\mathbb{R}\))/U(1) theory with periodic boundary conditions, which describes closed strings moving in a curved target-space geometry. We calculate its Poisson bracket structure by solving an initial state problem. The results differ from previous field-theoretic calculations due to zero modes. For a future exact canonical quantization the physical fields are (non-locally) transformed onto canonical free fields.

Key words: conformal field theory, integrability, black hole

1 Introduction

The SL(2,\(\mathbb{R}\))/U(1) gauged WZNW model has attracted much interest in the past \([1,2,3,4,5,6,7,8]\) before it was recognized that this non-linear theory is classically integrable \([9]\). More generally, we could prove that integrability holds for any gauged WZNW theory \([10]\). This was known for nilpotent gauging only, which yields Toda theories \([11,12]\). So far we have completely integrated the non-linear equations of motion of the classical SL(2,\(\mathbb{R}\))/U(1) model for a field theoretic case with asymptotic boundary conditions \([10]\).

In this paper we solve the classical SL(2,\(\mathbb{R}\))/U(1) theory for periodic boundary conditions. As a conformal field theory this model describes a
closed string moving in the background of a black hole target-space metric \[ \mathcal{M} \]. It is especially interesting for quantization; quantum mechanical deformations of its metric and a correlated dilaton \[ \phi \] were obtained in some perturbative manner \[ \cite{4, 5, 6} \]. However, these calculations were based on an incomplete effective action \[ \cite{13, 14, 10} \].

Our intention is to provide a different understanding of such quantum mechanical results. Starting with an entirely classical approach \[ \cite{1, 10} \], we expect that the exact classical solution of this theory will facilitate its exact canonical quantization. We calculate, as in the field theoretic case, the Poisson bracket structure of the theory by solving an initial state problem and look for a canonical transformation of the physical fields onto canonical free fields. But the results for periodic boundary conditions cannot simply be inferred from the field theoretic ones because additional zero modes become important.

To make this paper self-contained, we mention in section 2 some of our earlier results; more details are found in ref. \[ \cite{10} \]. First we define the theory, give its general solution and inspect the symmetry properties. Section 3 solves an initial state problem which allows us to calculate in section 4 the Poisson bracket structure of the model. A free-field realization of these brackets is given in section 5. The summary provides a canonical transformation of the physical fields onto the free fields. Some technical details are found in two appendices.

2 The SL(2,\( \mathbb{R} \))/U(1) theory

The exact action of the SL(2,\( \mathbb{R} \))/U(1) gauged WZNW theory written in light-cone coordinates \( z = \tau + \sigma, \bar{z} = \tau - \sigma \)

\[
S[r, t] = \frac{1}{\gamma^2} \int_M \left( \partial_z r \partial_\bar{z} r + \tanh^2 r \partial_\bar{z} t \partial_\bar{z} t \right) dz d\bar{z}
\]

was derived entirely classically and in a gauge invariant manner \[ \cite{1, 10} \]. \( M \) has cylindrical topology where the space-like submanifolds are topologically equivalent to a circle

\[
M = \mathbb{R} \times S^1, \quad \text{i.e.} \quad 0 \leq \sigma \leq 2\pi, \quad -\infty < \tau < \infty.
\]

The physical fields \( r(\sigma, \tau), t(\sigma, \tau) \), which represent the position of a closed bosonic string in the target-space at proper time \( \tau \), are subject to the bound-
ary conditions

$$r(\sigma + 2\pi, \tau) = r(\sigma, \tau), \quad t(\sigma + 2\pi, \tau) = t(\sigma, \tau) + 2\pi w, \quad w \in \mathbb{Z}.$$  \hspace{1cm} (3)

The $t$ coordinate is an angular variable given modulo $2\pi$ only, and the winding number $w$ tells us how often the string surrounds the coordinate origin. The string moves in the curved metric of a Euclidean black hole \[3\]

$$ds^2 = dr^2 + \tanh^2 r \, dt^2,$$  \hspace{1cm} (4)

and the dynamics is given by the equations of motion

$$\partial_z \partial\bar{z} r = \frac{\sinh r}{\cosh^3 r} \partial_z t \partial\bar{z} t,$$

$$\partial_z \partial\bar{z} t = -\frac{1}{\sinh r \cosh r} (\partial_z r \partial\bar{z} t + \partial_z t \partial\bar{z} r).$$  \hspace{1cm} (5)

These equations are integrable because they have a Lax pair representation

$$[\partial_z - \bar{C}, \partial_z - C] = \partial_z \bar{C} - \partial_z C - [C, \bar{C}] = 0.$$  \hspace{1cm} (6)

$C$ and $\bar{C}$ take values in the Lie algebra of the group $SL(2,\mathbb{R})$ [10]

$$C = C_a T^a, \quad \bar{C} = \bar{C}_a T^a, \quad (a = 1, 2, 3)$$  \hspace{1cm} (7)

with

$$T^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^3 = T^1 T^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (8)

One can check that

$$C_1 = -\frac{1}{2} \tanh^2 r \partial_z t, \quad C_2 = C_3 = 0, \quad \bar{C}_1 = \frac{1}{2} \tanh^2 r \partial\bar{z} t,$$

$$\bar{C}_2 = -\frac{1}{\cosh r} \partial\bar{z} (\sinh r \cos t), \quad \bar{C}_3 = \frac{1}{\cosh r} \partial\bar{z} (\sinh r \sin t)$$  \hspace{1cm} (9)

makes the flatness condition (6) equivalent to the equations of motion (5). But unlike Toda theories \[15\] there is at present no general method to integrate a Lax pair following from a non-nilpotent gauged WZNW model like (6) directly. We found the general solution of (5) in \[8\] by analysing non-abelian Toda theories \[7, 8\] as

$$\sinh^2 r = X \bar{X}, \quad t = i(B - \bar{B}) + \frac{i}{2} \ln \frac{X}{\bar{X}}.$$  \hspace{1cm} (10)
with the definitions
\[ X = A + \frac{B'}{A'}(1 + A\bar{A}), \quad \bar{X} = \bar{A} + \frac{B'}{A'}(1 + A\bar{A}). \] (11)

\[ A = A(z), \quad B = B(z), \quad \bar{A} = \bar{A}(\bar{z}) \quad \text{and} \quad \bar{B} = \bar{B}(\bar{z}) \]
are complex (anti-) chiral parameter functions and \( A'(z) \) etc. derivatives. However, as we shall see we must restrict this solution in order to render \( r \) and \( t \) real. Straightforward substitution shows that the solution (10) fulfills the equations of motion (5). But it will become obvious from the investigation of initial-value problems of section 3 that the solution (10) exhausts the entire solution space (excluding singular solutions).

The theory is also characterized by conservation laws. The equations of motion (5) guarantee, in particular, conservation and chirality of the energy-momentum tensor (we shall omit the anti-chiral parts whenever possible)
\[ T \equiv T_{zz} = \frac{1}{\gamma^2} \left( (\partial_z r)^2 + \tanh^2 r \ (\partial_z t)^2 \right), \quad T_{\bar{z}z} = 0, \] (12)
and in addition of parafermionic observables [16, 1, 9]
\[ V_\pm = \frac{1}{\gamma^2} e^{\pm i\nu} \left( \partial_z r \pm i \tanh r \ \partial_z t \right), \] (13)
where \( \nu \) is defined by
\[ \partial_z \nu = (1 + \tanh^2 r) \ \partial_z t, \quad \partial_{\bar{z}} \nu = (1 - \tanh^2 r) \ \partial_{\bar{z}} t. \] (14)

Since the integrability condition of these equations corresponds to one of the equations of motion (5), the general solution (10) integrates eqs. (14) to
\[ \nu = t + i(B + \bar{B}) + i \ln(1 + A\bar{A}) - \frac{i}{2} \ln(1 + XX) + \nu_0. \] (15)

The main purpose of this paper is to calculate the Poisson bracket structure of the model, assuming canonical Poisson brackets for the physical fields \( r,t \) and their conjugate momenta. Therefore, we have to find \( A, B, \bar{A} \) and \( \bar{B} \) as functions of these physical variables. In principle, they are given by solving an initial state problem defined by a second order differential equation of the Gelfand-Dikii type
\[ y'' - (\partial_z V_+/V_-)y' - \gamma^2 Ty = 0, \] (16)
because its two independent solutions $y_1$, $y_2$ are related to our parameter functions
\[ y_1 = e^B, \quad y_2 = Ae^B, \quad (17) \]
and the coefficients of (16) are functions of $r$, $t$ and their derivatives. This differential equation simply follows from the conserved quantities (12), (13) and (17) as an identity.

But the functions $A$, $B$, $\tilde{A}$ and $\tilde{B}$ are not uniquely determined by this procedure. Because the solution (14) is invariant under the GL(2,$\mathbb{C}$) transformations
\[
\begin{align*}
A & \rightarrow T[A] = \frac{aA - b}{cA + d}, \\
B & \rightarrow T[B] = B + \ln(cA + d), \\
\tilde{A} & \rightarrow T[\tilde{A}] = \frac{d\tilde{A} - c}{b\tilde{A} + a}, \\
\tilde{B} & \rightarrow T[\tilde{B}] = \tilde{B} + \ln(b\tilde{A} + a),
\end{align*}
(18)
\]
they are only given by the physical fields up to four complex constants. We shall fix this arbitrariness in the next section.

The monodromy properties of the functions $A$, $B$, $\tilde{A}$, $\tilde{B}$ are, as well, determined by GL(2,$\mathbb{C}$) transformations
\[
\begin{align*}
A(z + 2\pi) &= T'[A(z)] = \frac{pA(z) - q}{rA(z) + s}, \\
B(z + 2\pi) &= T'[B(z)] = B(z) + \ln(rA(z) + s), \\
\tilde{A}(\bar{z} - 2\pi) &= T'[	ilde{A}(\bar{z})] = \frac{s\tilde{A}(\bar{z}) - r}{q\tilde{A}(\bar{z}) + p}, \\
\tilde{B}(\bar{z} - 2\pi) &= T'[	ilde{B}(\bar{z})] = \tilde{B}(\bar{z}) + \ln(q\tilde{A}(\bar{z}) + p),
\end{align*}
(19)
\]
\[
\begin{pmatrix} \frac{p}{r} & -q \\ c & d \end{pmatrix} \in \text{GL}(2,\mathbb{C}).
\]

We should remark here that the GL(2,$\mathbb{C}$) transformations act, indeed, in two different manners. We also find that $\nu$ is not periodic modulo $2\pi$
\[
\nu(\sigma + 2\pi, \tau) = \nu(\sigma, \tau) + 2\pi w + i \ln(ps + qr).
(20)
Therefore, the conserved quantities $V_{\pm}$ and $\bar{V}_{\pm}$ are not periodically defined. We can describe their periodicity behaviour by the conserved total momentum of the field $t$

$$P_t = \frac{1}{\gamma^2} \int_0^{2\pi} \tanh^2 r \, t \, d\sigma' = \int_0^{2\pi} \pi_t(\sigma', \tau) \, d\sigma'.$$

(21)

Using the eqs. (14), instead of (20) we obtain for $\nu$ and $\bar{\nu}$ the periodicity relations in terms of $P_t$

$$\nu(\sigma + 2\pi, \tau) - \nu(\sigma, \tau) = \int_\sigma^{\sigma+2\pi} \nu'(\sigma', \tau) \, d\sigma' = 2\pi w + \gamma^2 P_t,$$

$$\bar{\nu}(\sigma + 2\pi, \tau) - \bar{\nu}(\sigma, \tau) = \int_\sigma^{\sigma+2\pi} \bar{\nu}'(\sigma', \tau) \, d\sigma' = 2\pi w - \gamma^2 P_t.$$  

(22)

So it holds that

$$V_{\pm}(z + 2\pi) = e^{\pm i\gamma^2 P_t} V_{\pm}(z), \quad \bar{V}_{\pm}(\bar{z} - 2\pi) = e^{\pm i\gamma^2 P_t} \bar{V}_{\pm}(\bar{z}),$$

(23)

and we can define, up to a constant normalization, new periodic conserved quantities

$$W_{\pm} \equiv e^{\pm i\gamma^2 P_t z/(2\pi)} V_{\pm}, \quad \bar{W}_{\pm} \equiv e^{\pm i\gamma^2 P_t \bar{z}/(2\pi)} \bar{V}_{\pm}.$$  

(24)

Comparing (20) and (22) the real-valued momentum $P_t$ becomes

$$\gamma^2 P_t = i \ln(ps + qr),$$

(25)

so that the monodromy transformations (19) are restricted to those with unit determinant

$$|ps + qr| = 1.$$  

(26)

3 The solution of initial value problems

It is advantageous to use in the following calculations Kruskal coordinates

$$u = \sin hr \, e^{it}, \quad \bar{u} = \sin hr \, e^{-it}.$$  

(27)
The general solution (10) can then be parameterized most symmetrically by the solution of the Gelfand-Dikii equations $y_k(z)$, $\bar{y}_k(\bar{z})$

$$\begin{align}
u &= \frac{\bar{y}_1 y_1' + y_2 y_2'}{y_1 y_2' - y_1' y_2}, \quad \bar{u} = \frac{y_1 \bar{y}_1' + y_2 \bar{y}_2'}{y_1 y_2' - \bar{y}_1' \bar{y}_2}.
\end{align}$$

(28)

We shall restrict ourselves, furthermore, to regular solutions

$$y_1 y_2' - y_1' y_2 \neq 0, \quad \bar{y}_1 \bar{y}_2' - \bar{y}_1' \bar{y}_2 \neq 0 \quad \forall z, \bar{z},$$

(29)

which guarantee that the Gelfand-Dikii equations are not singular, because their coefficients, determined by

$$T = \frac{1}{\gamma^2} \frac{y_1'' y_2' - y_1' y_2''}{y_1 y_2' - y_1' y_2}, \quad V_- = \frac{e^{-i\nu_0}}{\gamma^2} (y_1 y_2' - y_1' y_2),$$

(30)

are non-singular. The GL$(2, \mathbb{C})$ invariance of the general solution (28) now takes the form

$$\begin{align}
\begin{pmatrix} y_2 \\ y_1 \end{pmatrix} &\rightarrow \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}, \\
\begin{pmatrix} \bar{y}_2 \\ \bar{y}_1 \end{pmatrix} &\rightarrow \begin{pmatrix} d & -c \\ b & a \end{pmatrix} \begin{pmatrix} \bar{y}_2 \\ \bar{y}_1 \end{pmatrix},
\end{align}$$

(31)

$$\begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}).$$

It determines the solutions $y_k(z)$, $\bar{y}_k(\bar{z})$ in terms of the physical fields $u$, $\bar{u}$, just as before, up to four indeterminate integration constants, provided we have chosen for the physical fields the initial values at ‘time’ $\tau_0$

$$u(\sigma, \tau_0) = u_0(\sigma), \quad \bar{u}(\sigma, \tau_0) = \bar{u}_0(\sigma), \quad \dot{u}(\sigma, \tau_0) = u_1(\sigma), \quad \dot{\bar{u}}(\sigma, \tau_0) = \bar{u}_1(\sigma).$$

(32)

But in contrast to that, the chiral and anti-chiral second order Gelfand-Dikii differential equations allow together eight integration constants for their four independent solutions. This puzzle can be solved as follows: differentiating the general solution (28), four first order differential equations result

$$\begin{align}
y_1' &= \frac{\partial_z \bar{u}}{1 + u \bar{u}} (uy_1 - \bar{y}_2), \quad y_2' = \frac{\partial_z \bar{u}}{1 + u \bar{u}} (uy_2 + \bar{y}_1), \\
\bar{y}_1' &= \frac{\partial_z u}{1 + u \bar{u}} (\bar{u}y_1 - y_2), \quad \bar{y}_2' = \frac{\partial_z u}{1 + u \bar{u}} (\bar{u} \bar{y}_2 + y_1).
\end{align}$$

(33)

The elimination of the anti-chiral functions $\bar{y}_k(\bar{z})$ yield, again, the Gelfand-Dikii equations

$$y_k'' - (\partial_z V_- / V_-) y_k' - \gamma^2 Ty_k = 0,$$

(34)
and correspondingly the anti-chiral equations result. In case, we look now for solutions of these equations which fulfill besides the initial state conditions (32) the linear differential equations (33) too, the number of integration constants is reduced from eight to the four of the GL(2,C) invariance group. Fixing this GL(2,C) invariance the parameter functions \(y_k, \bar{y}_k\) are determined, in principle, from the physical fields \(u, \bar{u}\) uniquely.

Since the coefficients of the Gelfand-Dikii equations (34) are periodic functions of \(z\), apart from \(y_k(z)\) also the functions \(y_k(z + 2\pi)\) are solutions of these equations. They are given by (19) as linear combinations of the \(y_k(z)\)

\[
\begin{pmatrix} y_2(z + 2\pi) \\ y_1(z + 2\pi) \end{pmatrix} = M \begin{pmatrix} y_2(z) \\ y_1(z) \end{pmatrix}, \quad M = \begin{pmatrix} p & -q \\ r & s \end{pmatrix} \in \text{GL}(2,\mathbb{C}).
\]

Under the GL(2,C) transformation \(N\), the monodromy transformation \(M\) changes according to

\[
M \rightarrow NMN^{-1}.
\]  

In case that

(1) \((\text{tr} M)^2 \neq 4 \det M\) \text{ or } (2) \(M = aI_2\),

(37)

\(M\) can be brought to diagonal form

\[
M = \begin{pmatrix} e^{-\alpha'} & 0 \\ 0 & e^{\alpha'} \end{pmatrix}, \quad \bar{M} = \begin{pmatrix} e^{-\alpha'} & 0 \\ 0 & e^{\alpha'} \end{pmatrix},
\]

(38)

(the non-diagonalizable cases can be obtained by a limiting procedure).

The periodicity conditions (34) then simplify to

\[
y_1(z + 2\pi) = e^{\alpha'}y_1(z), \quad y_2(z + 2\pi) = e^{-\alpha'}y_2(z),
\]

\[
\bar{y}_1(\bar{z} - 2\pi) = e^{-\alpha'}\bar{y}_1(\bar{z}), \quad \bar{y}_2(\bar{z} - 2\pi) = e^{\alpha'}\bar{y}_2(\bar{z}).
\]

These properties already restrict the possible GL(2,C) transformations (31) to scalings with only two free parameters \(a\) and \(d\)

\[
\begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \rightarrow \begin{pmatrix} ay_2 \\ dy_1 \end{pmatrix}, \quad \begin{pmatrix} \bar{y}_2 \\ \bar{y}_1 \end{pmatrix} \rightarrow \begin{pmatrix} d\bar{y}_2 \\ a\bar{y}_1 \end{pmatrix}.
\]

(40)

This means, we have implicitly fixed two of the four integration constants. Any function invariant under the scalings (40) can now be determined uniquely by (31), in particular, the periodic quotients

\[
\eta_k(z) \equiv \frac{y_k(z)}{y_k(z)}, \quad k = 1, 2.
\]  

(41)
They determine the solution of (34) by
\[
\ln y_1(z) = \frac{1}{2} \int_0^{2\pi} \eta_1(z') h(z - z') dz' + \frac{z}{2\pi} \alpha_1 + \frac{\bar{\psi} Q}{2},
\]
\[
\ln y_2(z) = \frac{1}{2} \int_0^{2\pi} \eta_2(z') h(z - z') dz' + \frac{z}{2\pi} \alpha_2 + Q_\lambda - \frac{\psi Q}{2}.
\] (42)

\(Q_\lambda, \psi Q\) are integration constants defined by the eqs. (32) of Appendix A, and \(h(z)\) denotes the periodic saw-tooth function
\[
h(z) = \epsilon_{2\pi}(z) - \frac{z}{\pi} = 2n + 1 - \frac{z}{\pi}\text{ for } 2\pi n < z < 2\pi(n + 1), \quad n \in \mathbb{Z}.\] (43)

Here \(\epsilon_{2\pi}(z)\) is the stair-step function
\[
\epsilon_{2\pi}(z) = 2n + 1 \text{ for } 2\pi n < z < 2\pi(n + 1), \quad n \in \mathbb{Z},
\] (44)
and
\[
\alpha_k = \int_0^{2\pi} \eta_k(z) dz
\] (45)
are the zero modes of the fields \(\eta_k(z)\). (In passing we mention that (33), (42) imply \(e^{\alpha'} = e^{\alpha_1}\), and we define \(\alpha' \equiv \alpha_1\).) But we have to stress here especially that this result does not deliver \(\eta_k\) or \(y_k\) explicitly as functions of \(u, \bar{u}\). However, it will be sufficient in order to calculate their Poisson brackets.

4 The Poisson brackets

We calculate Poisson brackets by assuming canonical Poisson brackets of the physical fields, which are obtained from the action (3). For the Kruskal coordinates we get the following non-vanishing expressions
\[
\{u(\sigma, \tau), \dot{u}(\sigma', \tau)\} = \{\bar{u}(\sigma, \tau), \dot{\bar{u}}(\sigma', \tau)\} = 2\gamma^2(1 + u\bar{u})\delta_{2\pi}(\sigma - \sigma'),
\]
\[
\{\dot{u}(\sigma), \dot{u}(\sigma')\} = 2\gamma^2(\dot{u}\bar{u} - u\dot{\bar{u}})\delta_{2\pi}(\sigma - \sigma'),
\] (46)
where \(\delta_{2\pi}\) is the periodic \(\delta\)-function defined by
\[
\delta_{2\pi}(\sigma - \sigma') \equiv \sum_{n=-\infty}^{\infty} \delta(\sigma - \sigma' + 2\pi n). \] (47)
This allows us to calculate Poisson brackets of all quantities explicitly expressed in terms of the physical fields. Here we want to determine the Poisson brackets of the parameter functions \( y_k(z) \). We saw in the preceding section, that the \( \eta_k \) are uniquely defined by the initial state conditions, but they were not given explicitly as functions of the physical fields. We shall show that the Poisson brackets of the \( \eta_k \), and the \( y_k \) can, nevertheless, be derived.

First, we have to determine the Poisson brackets of the \( \eta_k(z) \), and those of the \( y_k \) then follow by means of (42). We calculate the variations \( \delta \eta_k(z) \) as functions of the varied physical fields and momenta by varying the Gelfand-Dikii equations

\[
\delta y_k'' - (\partial_z V_-/V_-)\delta y_k' - \gamma^2 T \delta y_k = \delta(\partial_z V_-/V_-) y_k' + \gamma^2 \delta T y_k. \tag{48}
\]

We vary, as well, the boundary conditions (39), eliminate \( \delta \alpha' \) and \( \delta \bar{\alpha}' \), and get subsidiary conditions for the equations (48)

\[
y_k'(z)\delta y_k(z + 2\pi) - y_k(z)\delta y_k'(z + 2\pi) = y_k'(z + 2\pi)\delta y_k(z) - y_k(z + 2\pi)\delta y_k'(z). \tag{49}
\]

They restrict the general solution of (48), which is defined by a special solution of (48) and the general solution of (34), to

\[
\delta y_k(z) = \int_0^{2\pi} \Omega_k(z, z') \left( \delta(\partial V_- / V_-)(z') y_k'(z') + \gamma^2 \delta T(z') y_k(z') \right) dz' + \delta C_k y_k(z),
\]

\[
\Omega_1(z, z') \equiv \frac{y_2(z') y_1(z)}{y_1(z') y_2(z') - y_2(z') y_1(z')} \frac{E(z, z') - \epsilon_2 \pi (z - z')}{2},
\]

\[
\Omega_2(z, z') \equiv \frac{y_1(z') y_2(z)}{y_1(z') y_2(z') - y_2(z') y_1(z')} \frac{E(z', z) + \epsilon_2 \pi (z - z')}{2},
\]

\[
E(z, z') \equiv \frac{\exp \left( \frac{\alpha_1 - \alpha_2}{2} \epsilon_2 \pi (z - z') \right)}{\sinh \frac{\alpha_1 - \alpha_2}{2}} \frac{y_2(z) y_1(z')}{y_1(z) y_2(z')}. \tag{50}
\]

The variations \( \delta C_k \) correspond to the undetermined scalings (40). They cannot simply be set to zero because we are integrating (50) over non-periodic functions of \( z' \), and without the term \( \delta C_k y_k(z) \) these integrals would depend on a shift of the integration range.
Since the functions $\eta_k(z)$ do not depend on the scalings (40), their variations
\[
\delta \eta_k(z) = \frac{\delta y_k'(z)}{y_k(z)} = \frac{\delta y_k'(z)}{y_k(z)} - \frac{y_k'(z)\delta y_k(z)}{y_k(z)^2}
\] (51)
show, indeed, the cancelation of $\delta C_k$ in
\[
\delta \eta_k(z) = \int_0^{2\pi} \omega_k(z, z') \left( \delta(\partial V_-/V_-)(z')\eta_k'(z') + \gamma^2 \delta T(z')\eta_k(z') \right) dz',
\]
\[
\omega_1(z, z') \equiv \frac{1}{2} E(z, z') \frac{\eta_1(z) - \eta_2(z)}{\eta_1(z') - \eta_2(z')},
\]
\[
\omega_2(z, z') \equiv -\frac{1}{2} E(z', z) \frac{\eta_1(z) - \eta_2(z)}{\eta_1(z') - \eta_2(z')}.
\] (52)
The integrands of (52) are periodic functions of the integration variable, and (52) determines the Poisson brackets of $\eta_k(z)$. The non-vanishing ones are given by
\[
\{\eta_1(z), \eta_2(z')\} = \frac{\gamma^2}{2} (\eta_1(z) - \eta_2(z)) E(z, z')(\eta_1(z') - \eta_2(z')) - \gamma^2 (\eta_1(z) - \eta_2(z)) \delta_{2\pi}(z - z').
\] (53)
The Poisson brackets of the functions $\ln y_k$ result, finally, by means of (12)
\[
\{\ln y_1(z), \ln y_1(z')\} = 0,
\]
\[
\{\ln y_1(z), \ln y_2(z')\} = \frac{\gamma^2}{2} \left( e_{2\pi}(z - z') - \frac{z - z'}{2\pi} \right) - \frac{\gamma^2}{2} E(z, z') + \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz E(z, z'),
\]
\[
\{\ln y_1(z), \ln \bar{y}_1(z')\} = -\frac{\gamma^2}{4\pi} (z - \bar{z'}),
\]
\[
\{\ln \bar{y}_1(z), \ln y_2(z')\} = -\frac{\gamma^2}{8\pi} \int_0^{2\pi} d\bar{z} \eta_2(z') E(z, z'),
\] (54)
\[
\{\ln y_2(z), \ln y_2(z')\} = -\frac{\gamma^2}{8\pi} \int_0^{2\pi} d\bar{z} \eta_2(z') E(z, z') + \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz \eta_2(z') E(z', z),
\]
\{ \ln y_2(z), \ln \bar{y}_2(z') \} = -\frac{\gamma^2}{4\pi} (z - z') - \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz' E(z', z) + \frac{\gamma^2}{8\pi} \int_0^{2\pi} d\bar{z}' \bar{E}(\bar{z}, \bar{z}')

In distinction to the field theoretic results of ref. [10] we observe here a structurally changed non-local realization of the algebra. This is due to zero modes which arise additionally in the periodic case. It might be surprising that the algebra treats \( y_1 \) and \( y_2 \) (as well as \( \bar{y}_1 \) and \( \bar{y}_2 \)) asymmetrically. A symmetric treatment of the functions \( y_k \) and \( \bar{y}_k \) is presented in appendix B. But it turns out that the algebra (54) is more appropriate for transforming the \( y_k, \bar{y}_k \) onto canonical free fields.

5 The canonical transformation onto periodic free fields

There are several methods to find relations between \( y_k(z), \bar{y}_k(\bar{z}) \) and chiral, respectively anti-chiral components \( \phi_k(z), \bar{\phi}_k(\bar{z}) \) of canonical free fields \((k = 1, 2)\)

\[ \psi_k(\sigma, \tau) = \phi_k(z) + \bar{\phi}_k(\bar{z}). \] (55)

Sometimes it will be useful to have the mode expansions in mind, e.g.,

\[ \phi_k(z) = \frac{q_k}{2} + \left( \frac{p_k}{4\pi} + \frac{w'}{2\gamma} \delta_{k,2} \right) z + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{a_n^{(k)}}{n} e^{-inz}. \] (56)

\( w' \) is an integer ‘winding number’.

Here we simply assume that there is a free-field theory with a corresponding free-field energy-momentum tensor which is canonically related to our \( \text{SL}(2, \mathbb{R}) \) model. Then the easiest and most straightforward approach identifies the energy-momentum tensors of both theories

\[ T(z) = (\partial_z \phi_1)^2 + (\partial_z \phi_2)^2 = \frac{1}{\gamma^2} \frac{y_1'' y_2' - y_1' y_2''}{y_1 y_2 - y_1' y_2'}. \] (57)

Furthermore, we assume that the free fields \( \psi_1, \psi_2 \) are local expressions of the parameter functions \( y_k \).

It is appropriate to introduce complex free fields

\[ \psi = \psi_1 + i\psi_2, \quad \bar{\psi} = \psi_1 - i\psi_2, \] (58)
which factorize the components of the energy-momentum tensor

\[ T(z) = \partial_z \psi \partial_z \bar{\psi}, \quad \bar{T}(z) = \partial_z \bar{\psi} \partial_z \psi. \]  

(59) gives a corresponding chiral decomposition of \( \psi \) and \( \bar{\psi} \)

\[ \psi(\sigma, \tau) = \phi(z) + \bar{\chi}(\bar{z}), \quad \bar{\psi}(\sigma, \tau) = \chi(z) + \bar{\phi}(\bar{z}), \]  

(60)

with

\[ \phi(z) = \phi_1(z) + i\phi_2(z), \quad \bar{\phi}(\bar{z}) = \bar{\phi}_1(\bar{z}) - i\bar{\phi}_2(\bar{z}), \]

\[ \chi(z) = \phi_1(z) - i\phi_2(z), \quad \bar{\chi}(\bar{z}) = \bar{\phi}_1(\bar{z}) + i\bar{\phi}_2(\bar{z}). \]  

(61)

The most general solution of this problem depends on several complex constants, and it is given by (54)

\[ \phi = \frac{1}{\gamma C} \left( \ln \frac{\alpha y'_1 + \beta y'_2}{y_1 y'_2 - y'_1 y_2} + D \right), \quad \chi = \frac{C}{\gamma} \ln \left( \frac{\alpha y_1 + \beta y_2}{y_1 y_2} \right), \]

\[ \bar{\phi} = \frac{1}{\gamma \bar{C}} \left( \ln \frac{\bar{\alpha} \bar{y}'_1 + \bar{\beta} \bar{y}'_2}{\bar{y}_1 \bar{y}'_2 - \bar{y}'_1 \bar{y}_2} + \bar{D} \right), \quad \bar{\chi} = \frac{\bar{C}}{\gamma} \ln \left( \frac{\bar{\alpha} \bar{y}_1 + \bar{\beta} \bar{y}_2}{\bar{y}_1 \bar{y}_2} \right). \]  

(62)

But the constants can be further restricted. Taking into consideration the invariance of (55) under \( \phi \to e^{i\delta} \phi, \chi \to e^{-i\delta} \chi \), we can choose \( C \) real positive. Of course, the physics should not depend on the choice of the branch of the logarithm. This implies \( C = 1 \). Furthermore, \( \phi_2 \) is defined modulo \( 2\pi/\gamma \) only (i.e. \( \phi_2 \) takes values on a circle with radius \( 1/\gamma \))

\[ \phi_2 \equiv \phi_2 + \frac{2\pi}{\gamma}, \quad \bar{\phi}_2 \equiv \bar{\phi}_2 + \frac{2\pi}{\gamma}. \]  

(63)

Up to winding contributions the free fields \( \psi_k \) are assumed periodic

\[ \psi_1(\sigma + 2\pi, \tau) = \psi_1(\sigma, \tau), \quad \psi_2(\sigma + 2\pi, \tau) = \psi_2(\sigma, \tau) + \frac{2\pi w'}{\gamma}, \quad w' \in \mathbb{Z}, \]  

(64)

and we obtain

\[ \phi_k(z + 2\pi) - \phi(z) = \frac{p_k}{2} + \delta_{k,2} \frac{\pi w'}{2\gamma}, \]

\[ \bar{\phi}_k(\bar{z} + 2\pi) - \bar{\phi}(\bar{z}) = \frac{p_k}{2} - \delta_{k,2} \frac{\pi w'}{2\gamma}, \quad p_k \in \mathbb{R}. \]  

(65)
But this is consistent with (62) and (39) only if one of the pairs \((\alpha, \bar{\alpha})\) and \((\beta, \bar{\beta})\) is \((0, 0)\). Choosing \(\beta = \bar{\beta} = 0\), the rescaling (40) now allows us to set \(\alpha = \bar{\alpha} = 1\), and considering the invariance under \(\psi_k \rightarrow \psi_k + \text{const}\), we can without loss of generality fix \(D = \bar{D} = 0\). Thus the solution (62) simplifies finally to
\[
\phi = \frac{1}{\gamma} \ln \frac{y_1'}{y_1 y_2 - y_1' y_2}, \quad \chi = \frac{1}{\gamma} \ln y_1, \\
\bar{\phi} = \frac{1}{\gamma} \ln \frac{\bar{y}_1'}{\bar{y}_1 \bar{y}_2 - \bar{y}_1' \bar{y}_2}, \quad \bar{\chi} = \frac{1}{\gamma} \ln \bar{y}_1. \quad (66)
\]
As expected, from the non-local Poisson bracket relations (54) we get for the fields \(\phi_k, \bar{\phi}_k\), indeed, the local free-field Poisson brackets
\[
\{\phi_k (\tau + \sigma), \phi_l (\tau + \sigma')\} = -\frac{\delta_{kl}}{4} \left( \epsilon_{2\pi} (\sigma - \sigma') - \frac{\sigma - \sigma'}{2\pi} \right), \\
\{\bar{\phi}_k (\tau - \sigma), \bar{\phi}_l (\tau - \sigma')\} = \frac{\delta_{kl}}{4} \left( \epsilon_{2\pi} (\sigma - \sigma') - \frac{\sigma - \sigma'}{2\pi} \right), \\
\{\phi_k (\tau + \sigma), \bar{\phi}_l (\tau - \sigma')\} = -\frac{\delta_{kl}}{8\pi} (\sigma + \sigma'). \quad (67)
\]
Solving now (66) for \(y_k, \bar{y}_k\), their non-local free-field representation result
\[
y_1(z) = \exp (\gamma \chi(z)), \\
y_2(z) = -\frac{\exp (\gamma \chi(z))}{2 \sinh(\gamma p_1/2)} \int_0^{2\pi} dz' \gamma \chi'(z') \exp \left(-\frac{\gamma p_1}{2} \epsilon_{2\pi} (z - z') - 2\gamma \phi_1(z')\right), \\
\bar{y}_1(\bar{z}) = \exp (\gamma \bar{\chi}(\bar{z})), \quad (68) \\
\bar{y}_2(\bar{z}) = -\frac{\exp (\gamma \bar{\chi}(\bar{z}))}{2 \sinh(\gamma p_1/2)} \int_0^{2\pi} d\bar{z}' \gamma \bar{\chi}'(\bar{z}') \exp \left(-\frac{\gamma p_1}{2} \epsilon_{2\pi} (\bar{z} - \bar{z}') - 2\gamma \bar{\phi}_1(\bar{z}')\right),
\]
where the zero mode momentum is given by
\[
p_1 = \int_0^{2\pi} \dot{\psi}_1(\tau, \sigma) d\sigma. \quad (69)
\]
We have checked that the free-field Poisson brackets (67) yield, conversely, the non-local Poisson brackets of the \(y_k(z), \bar{y}_k(\bar{z})\) (54), and we could show
that these results also follow from the Gelfand-Dikii equations (34), in case, their coefficients are expressed in terms of the free fields and the initial state problem is solved anew.

This proves that these free-field transformations of the physical fields \( r, t, \) or \( u, \bar{u} \) are canonical transformations, and they are one to one.

6 Summary

We have completely integrated the periodic \( \text{SL}(2, \mathbb{R})/\text{U}(1) \) gauged WZNW theory and calculated its symplectic structure. This allows us to relate this model canonically to a free-field theory. The results could be summarized in terms of local Bäcklund transformations which are identical to those of the non-periodic case \([10]\). Instead, we give here the complete canonical transformation of the fields \( u(\sigma, \tau), \bar{u}(\sigma, \tau) \) onto the free fields

\[
\begin{align*}
u &= e^{\gamma(\phi + \bar{\chi})} \left(1 + \Phi \bar{\Phi} \right) - \frac{1}{4} e^{-\gamma(\phi + \bar{\chi})} + \frac{i}{2} \left(e^{\gamma(\phi - \bar{\phi})} \Phi + e^{-\gamma(\chi - \bar{\chi})} \bar{\Phi} \right), \\
\bar{\nu} &= e^{\bar{\gamma}(\bar{\phi} + \chi)} \left(1 + \Phi \bar{\Phi} \right) - \frac{1}{4} e^{-\bar{\gamma}(\bar{\phi} + \chi)} - \frac{i}{2} \left(e^{\bar{\gamma}(\chi - \bar{\chi})} \Phi + e^{-\bar{\gamma}(\phi - \bar{\phi})} \bar{\Phi} \right). \quad (70)
\end{align*}
\]

This transformation is non-locally defined by

\[
\begin{align*}
\Phi(z) &= -\frac{1}{2 \sinh(\gamma p_1/2)} \int_0^{2\pi} \mathrm{d}z' \gamma \phi'_2(z') \exp \left(-\frac{\gamma p_1}{2} \epsilon_{2\pi}(z - z') - 2\gamma \phi_1(z') \right), \\
\bar{\Phi}(\bar{z}) &= -\frac{1}{2 \sinh(\gamma p_1/2)} \int_0^{2\pi} \mathrm{d}\bar{z}' \gamma \bar{\phi}'_2(\bar{z}') \exp \left(-\frac{\gamma p_1}{2} \epsilon_{2\pi}(\bar{z} - \bar{z}') - 2\gamma \bar{\phi}_1(\bar{z}') \right). \quad (71)
\end{align*}
\]

As in Liouville theory this structure might require quantum mechanical deformations. But we might be confronted, as well, with further unusual problems related to the quantization of the parafermionic structure of the theory, which is classically defined by non-linear Poisson brackets. Using the freedom of normalization of \([24]\) (here we take into consideration the full \( q \) zero mode of the free field) for the periodic case the parafermions fulfil

\[
\begin{align*}
\{W_\pm(z), W_\pm(z')\} &= \gamma^2 W_\pm(z) W_\pm(z') \delta(z - z'), \\
\{W_\pm(z), W_{\mp}(z')\} &= -\gamma^2 W_\pm(z) W_{\mp}(z') \delta(z - z') + \frac{1}{\gamma^2} \left(\partial_z + i\gamma p_2 \right) \delta_{2\pi}(z - z'), \\
\{p_2, W_\pm(z')\} &= \mp 2i\gamma W_\pm(z'), \quad (72)
\end{align*}
\]
In our opinion this may be a good starting point for a quantization. Therefore, it remains a challenge to implement the exact canonical quantization of the SL(2,R)/U(1) model on the basis of our results.

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A The definition of the integration constants

Here we explain the relations (42) in more detail.

Using chirality, the functions \( \eta_k(z) \) of (41) can be integrated to

\[
\ln y_k(z) = \frac{1}{2} \int_0^{2\pi} \eta_k(z')h(z - z')dz' + \frac{z}{2\pi} \alpha_k + D_k. \tag{73}
\]

\( h(z) \) and \( \alpha_k \) are defined by (43), (45). \( D_k \) are the integration constants under discussion. Let us consider the canonical free field (58), (66)

\[
\psi(\sigma, \tau) = \frac{1}{\gamma} \ln \left( \frac{y_1'(z)}{y_1(z)y_2'(z) - y_1'(z)y_2(z)} \right) + \frac{1}{\gamma} \ln \bar{y}_1(\bar{z}). \tag{74}
\]

Equations (41) and (33) allow one to replace the functions \( y_1'(z), y_2(z) \) and \( \bar{y}_1(\bar{z}) \) by \( \eta_k(z) \). Then, \( \psi \) is completely given in terms of \( \eta_k(z) \) and the physical fields \( u, \bar{u} \)

\[
\psi(\sigma, \tau) = \frac{1}{\gamma} \ln \left( \frac{u\partial_z \bar{u} - \bar{y}_2(1 + u\bar{u})\partial_z \bar{u}}{\bar{y}_1 - \eta_1 - \eta_2} \right). \tag{75}
\]

Similarly, eqs. (33) yield

\[
\eta_{1,2} = \frac{\partial_z u - \bar{y}_2, u}{u\partial_z u - \bar{y}_2, u(1 + u\bar{u})} \partial_z \bar{u}, \tag{76}
\]

which shows that we could express \( \psi \), as well, in terms of \( \eta_k(\bar{z}) \) and \( u, \bar{u} \).

On the other hand, substituting (73) into (74) we obtain

\[
\psi(\sigma, \tau) = \frac{1}{2} \int_0^{2\pi} \left[ \eta_1(\tau + \sigma') + \bar{y}_2(\tau - \sigma') \right] h(\sigma - \sigma')d\sigma' +
\]

\[
+ \frac{\tau}{2\pi} (\alpha_1 - \bar{\alpha}_2) + i\sigma m + D_1 - \bar{D}_2 + \lambda(z). \tag{77}
\]
Here
\[ \lambda(z) \equiv \ln \eta_1(z) - \ln(\eta_2(z) - \eta_1(z)), \] (78)
and \( m \), defined by
\[ \alpha_1 + \bar{\alpha}_2 = 2\pi im, \] (79)
is an integer due to the periodicity of \( y_1(z)/\bar{y}_2(z) \) (cp. (73) with the (anti-) chiral (73)). \( D_1 - \bar{D}_2 \) is uniquely determined by the constant zero modes of \( \psi \) and \( \lambda \)
\[ D_1 - \bar{D}_2 = \psi_Q - Q_\lambda. \] (80)
Since the two parts of (74) are the chiral and anti-chiral components of \( \psi \), and distributing the zero mode \( \psi_Q \) half and half to these components, by comparison we obtain
\[ D_1 = \frac{\psi_Q}{2} + Q_\lambda, \quad \bar{D}_1 = \frac{\bar{\psi}_Q}{2} + \bar{Q}_\lambda, \quad D_2 = -\frac{\bar{\psi}_Q}{2}, \quad \bar{D}_2 = -\frac{\psi_Q}{2}. \] (81)
This immediately implies (42). We give, finally, the explicit expressions for the constants \( \psi_Q \) and \( Q_\lambda \) in terms of the \( \eta_k \)
\[ \psi_Q = \frac{1}{2\pi} \int_0^{2\pi} (\psi(\sigma, \tau) - \psi_P(\sigma, \tau)) \, d\sigma, \quad \text{with} \]
\[ \psi_P(\sigma, \tau) = \frac{\sigma}{2\pi} \int_0^{2\pi} \frac{\partial \psi(\sigma', \tau)}{\partial \sigma'} \, d\sigma' + \frac{\tau}{2\pi} \int_0^{2\pi} \frac{\partial \psi(\sigma', \tau)}{\partial \tau} \, d\sigma' \]
\[ Q_\lambda = \frac{1}{2\pi} \int_0^{2\pi} \left( \lambda(z) - \frac{P_\lambda}{2\pi z} \right) \, dz, \quad \text{with} \]
\[ P_\lambda = \int_0^{2\pi} \lambda'(z) \, dz = \int_0^{2\pi} \frac{\eta_1(z)\eta_2'(z) - \eta_1'(z)\eta_2(z)}{\eta_1(z)(\eta_1(z) - \eta_2(z))} \, dz, \] (82)
which also determine their commutation relations.

**B Symmetric Poisson brackets**

In this appendix we define functions \( \tilde{y}_k(z) \) and \( \tilde{\bar{y}}_k(\bar{z}) \) with Poisson brackets which are symmetric under the exchange \( 1 \leftrightarrow 2 \).
Using the shorthand notation

\[
\kappa_Q = \frac{1}{2\pi} \int_0^{2\pi} \left[ \ln y_1(\tau + \sigma) - \ln \tilde{y}_2(\tau - \sigma) - \frac{\tau}{2\pi} (\bar{\alpha}_1 - \alpha_2) - i \sigma \tilde{m} \right] d\sigma,
\]

\[
\tilde{\kappa}_Q = \frac{1}{2\pi} \int_0^{2\pi} \left[ \ln \tilde{y}_1(\tau - \sigma) - \ln y_2(\tau + \sigma) - \frac{\tau}{2\pi} (\bar{\alpha}_1 - \alpha_2) - i \sigma \tilde{m} \right] d\sigma.,
\]

these functions are defined by

\[
\ln \tilde{y}_1(z) = \ln y_1(z) + \frac{\kappa_Q}{2} - \frac{\bar{\psi}_Q}{2}, \quad \ln \tilde{y}_2(z) = \ln y_2(z) + \frac{\bar{\kappa}_Q}{2} - \frac{\psi_Q}{2},
\]

\[
\ln \tilde{y}_1(\tilde{z}) = \ln y_1(\tilde{z}) + \frac{\bar{\kappa}_Q}{2} - \frac{\bar{\psi}_Q}{2}, \quad \ln \tilde{y}_2(\tilde{z}) = \ln y_2(\tilde{z}) + \frac{\kappa_Q}{2} - \frac{\psi_Q}{2}. \quad (83)
\]

The symmetric Poisson brackets are

\[
\{ \ln \tilde{y}_1(z), \ln \tilde{y}_2(z') \} = \frac{\gamma^2}{2} \left( \epsilon_{2\pi}(z - z') - \frac{z - z'}{2\pi} \right) - \frac{\gamma^2}{2} E(z, z') +
\]

\[
+ \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz E(z, z') + \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz' E(z, z') -
\]

\[
- \frac{\gamma^2}{32\pi^2} \int_0^{2\pi} \int_0^{2\pi} dz dz' (E(z, z') - \bar{E}(z', z)),
\]

\[
\{ \tilde{y}_1(z), \tilde{y}_1(z') \} = \{ \tilde{y}_2(z), \tilde{y}_2(z') \} = \{ \tilde{y}_1(z), \tilde{y}_2(z') \} = 0,
\]

\[
\{ \ln \tilde{y}_1(z), \ln \tilde{y}_1(z') \} = -\frac{\gamma^2}{4\pi} (z - z') + \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz' E(z, z') -
\]

\[
- \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz' \bar{E}(z', \tilde{z}) - \frac{\gamma^2}{32\pi^2} \int_0^{2\pi} \int_0^{2\pi} dz dz' (E(z, z') - \bar{E}(z', \tilde{z})),
\]

\[
\{ \ln \tilde{y}_2(\tilde{z}), \ln \tilde{y}_2(z') \} = -\frac{\gamma^2}{4\pi} (\tilde{z} - z') + \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz E(z, z') -
\]

\[
- \frac{\gamma^2}{8\pi} \int_0^{2\pi} dz' \bar{E}(z', \tilde{z}) - \frac{\gamma^2}{32\pi^2} \int_0^{2\pi} \int_0^{2\pi} dz dz' (E(z, z') - \bar{E}(z', \tilde{z})). \quad (85)
\]
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