A quasiparticle description of various condensed media is a very popular tool in study of their transport and thermodynamic properties. I present here a microscopic theory for the description of diffusion processes in two-component gas of quasiparticle systems with arbitrary dispersion law and statistics. Particularly, I analyze the role of interaction within each subsystem (i.e., between identical quasiparticles) in relaxation of the whole system. The approach for solution of such kinetic problem allows to study the most important limiting cases and to clarify their physical sense. Classical results for diffusion coefficient of light particles in a massive gas (Lorentz model) and of massive particles in a light gas (Rayleigh model) are obtained directly from the general solution without using artificial approaches, as it was done earlier. This particularly provide a possibility to generalize these popular models on quasiparticle systems.

I. INTRODUCTION

It is well known that variety of properties of some condensed media can be described by the interaction processes in quasiparticle gases. These are, say, transverse and longitudinal phonons in solids, phonons and magnons in magnetically ordered materials, phonons and rotons in superfluid helium, conduction electrons and holes in semiconductors etc. As a rule, to study the dissipative properties of such systems investigators use either classical kinetic theory in its simplest form or some semi-intuitive models, which lead sometimes to quite ambiguous results in case of quasiparticle systems. Here I present the theory of diffusion processes in two-component quasiparticle systems, which in general is independent on particular quasiparticles statistics and dispersion law. Let me first briefly review the present state of classical kinetic theory and analyze its pitfalls in description of quantum quasiparticle systems of condensed media.

The kinetic theory of gases in modern understanding takes its beginning from pioneering work by Maxwell in which he has proved the law for distribution of velocities of molecules in homogeneous equilibrium gas (so-called Maxwell velocities distribution) and law of an equidistribution of average energy of molecules in a mixture of gases. These his results were updated and improved in further works devoted to the theory of inhomogeneous gases (about a history of a problem see Ref. 2). However, as basis of all mathematical methods of the modern kinetic theory it is necessary to consider apparently the basic works by Boltzmann, in which both the H-theorem was proved and the classical Boltzmann equation was introduced.

Boltzmann equation is the integro-differential equation describing the collisional behaviour of rareified gas. Until now it remains to be a basis of the kinetic theory of gases and it appears to be very fruitful not only for a research of classical gases, which Boltzmann himself kept in mind, but - with an appropriate generalization - for study of electron transport in solids and plasma, transport of neutrons in nuclear reactors, phonon and roton transport in superfluid liquids, and transport of a radiation in atmospheres of stars and planets. For past 130 years these researches have led to significant achievements both in new areas, and in old one.

Generally, kinetic equation of the Boltzmann type (the equation describing the evolution of a single-particle distribution function in the phase space) represents the integro-differential equation, which remarkable property is the nonlinearity of collision term. Just this fact makes a main obstacle in construction of methods for solution of the kinetic equation. The monographies are devoted to a detail exposition of such methods in case of classical gas systems.

In majority of experimental problems there is no necessity to use the detail microscopic description of gas systems at a level of distribution functions. As a rule, investigation of physical processes in macroscopic systems is carried out at a less detail level of hydrodynamic variables. Since these variables are determined through the moments of a distribution function, then, as a rule, the detail study of main moments of distribution function appropriate to collisions invariants is required, but not the distribution function itself. Thus, the connection between the kinetic theory and hydrodynamics appears to be one of the main problem. In particular, one of the main aspects of this problem is the determination of transport coefficients, such as diffusion coefficient, viscosity (first and second), thermal conductivity appeared in equations of hydrodynamics of a viscous liquid.

In spite of so long history of physical kinetics, today we have rather small number of approaches to a solution of the kinetic equations. All these approaches and methods were formed depending on concrete problems, on which were directed. Most general methods of research of nonequilibrium state of classical and quantum gases were directed rather on demonstrating of a mathematical re-
solvability (or insolubility) of certain basic problems in principle, than on construction of the serviceable theories, suitable for a solution of concrete physical problems.

The classical methods of a solution of the kinetic equations allow to derive the kinetic coefficients as series expansions on an infinite set of orthogonal polynomials. However, it appeared to be very difficult to use these classical results for numerical calculation and analysis of physical processes in real systems. In particular, it is caused by imposibility of selection of the contribution from different types of interactions to various kinetic coefficients, while many physical systems behave qualitatively differently for different ratio between a speed of a relaxation inside each subsystem and between subsystems.

In classical gases with point-like interaction this problem is less important because the speed of relaxation inside subsystems unambiguously determined by their mass and concentration. In the quantum case, when we talk about quasiparticles, the situation becomes more complicated. The mechanism of interaction between quasiparticles can be any independently on simple macroscopic parameters. For example, in some cases such notation as mass cannot be well defined at all (say, what is the mass of phonon?).

The mathematical theory of transport processes is most sequentially advanced for mixtures of classical gases, the evolution of which is described by a set of Boltzmann equations. A basis of classical methods for solution of the Boltzmann equation in case of one-component gas is the formal expansion of distribution function in power series of some parameter $\sigma$ in the form $f = f^{(0)} + \sigma f^{(1)} + \sigma^2 f^{(2)} + \ldots$, so that function $f^{(0)}$ corresponds to statistical equilibrium. In this case parameter $\sigma$ is some scale factor for density, the physical sense of which can be different depending on a particular problem. As a rule, this parameter is formally considered to be small, so that the solution of the kinetic equation represents a problem of singular perturbation.

The most successful methods of a solution of the kinetic equation, such as the Hilbert method and the Chapman–Enskog method, are based on this principle. In spite of the successes of the Chapman–Enskog method in the description of connection of the kinetic theory with the equations of hydrodynamics (the Navier-Stokes equations appear already in the first order in parameter $\sigma$), the explicit expressions for kinetic coefficients have a rather complicated form. Main defect of these expressions is that already in the first order in parameter $\sigma$ their analysis becomes practically impossible. The situation becomes more problematic in the case of two-component gas. The infinite series of integral brackets containing Sonin polynomials does not allow to select explicitly the contributions from different types of interactions in a system to various dissipative coefficients. This frequently leads to necessity to use various ungrounded approximations, such as the Chapman–Cowling approximation or Kihara approximation.

At the same time, in spite of some successes, there is no yet consistent mathematical theory for deriving the dissipative coefficients in gases of quasiparticles. And, naturally, the problem of distinguishability of contribution from interactions between identical quasiparticles and between different subsystems has not been solved yet. This problem has his own history. In practice, when analyzing particular physical systems, many physicists use some model approximations for collision integrals. The most popular one is the so-called BGK–approximation. In its simple form BGK–approximation leads sometimes to quite confusing results. Really, let us have several relaxation mechanisms in the system, and, therefore several characteristic times depending on momenta or energies of quasiparticles. The final observed quantities should be obtained by averaging this times in some manner. In particular problems the following question frequently appears: What must be averaged, i.e. the time or the rate (inverse time)? and how to obtain the real relaxation time, i.e. by summation of the times or by summation of the rates? This uncertainty led to many confusing situations. For example, more than for 15 years there were two different theories for the mobility of two-dimensional electron gas localized over the free surface of liquid helium. The first theory assumes the mobility to depend on the averaged characteristic time of electron–ripplon interaction (ripplons are the quantized surface waves of liquid helium). This theory well describes the experimental data for small electron density. Another theory assumes the mobility to be determined by the inverse averaged rate of the same interaction. These theoretical results well fit another experiment with large electron density. The problem of relationship between these two results has naturally appeared. Similar confusion was in the theory of dissipative processes in superfluid. The analogous situation took place for some time in theory of thermal conductivity in solids.

The aim of this paper is to present alternative approach for solution of the system of linearized kinetic equations for two–component gas of quasiparticles with arbitrary statistics and dispersion law. The theory explicitly accounts the all types of interactions in the system. This allows to analyze the contribution of interaction between identical particles into the relaxation of the whole system. I do not restrict myself here in the frame of particular system. So, the results obtained here can be applied to any quantum system, whose dissipative properties are determined by the processes in two–component gas of quasiparticles.

The paper is constructed as follows. In Section II I formulate the problem mathematically and carry out the linearization procedure. Section III is devoted to the procedure of inversing of collision operator by the use of projection operators method. In Section IV I derive the exact solution for characteristic diffusion time and analyze all limiting cases. The classical systems of Lorentz and Rayleigh gases and their generalization to the quasipar-
ticle systems are considered in Section V. In Section VI I consider the generalization of the Kihara approximation on the quasiparticle quantum systems. The outlines and conclusions are given in Section VII.

II. GENERAL EXPRESSIONS

So, consider stationary nonequilibrium state of gaseous mixture of quasiparticles of two species. The most interesting relaxation process in such a system is a diffusion, so I will concentrate on the diffusive processes.

One of the most essential advantage of the offered theory undoubtedly is the fact, that it is correct for quantum gases with any statistics and any dispersion of quasiparticles. All main outcomes remain applicable for both systems with nonzero chemical potential, and with chemical potential equal to zero (when the number of quasiparticles is not conserved). The constructed theory does not meet principal difficulties in generalization on multicomponent systems (i.e. on systems with number of active components exceeding two ones).

The evolution of distribution functions \( f_k \) (subscript \( k = 1, 2 \) numbers the component of the mixture) can be described by the following set of kinetic equations

\[
\frac{\partial f_k}{\partial t} = \sum_{j=1,2} C_{kj}(f_k, f_j), \quad (k = 1, 2),
\]

(1)

where \( \mathbf{v}_k = \partial \epsilon_k / \partial \mathbf{p}_k \) is the velocity of quasiparticle of \( k \)th type; \( \epsilon_k \) and \( \mathbf{p}_k \) are its energy and momentum, respectively; \( \mathbf{r} \) is the coordinate, \( C_{kj}(f_k, f_j) \) is the collision integral, which is a functional of the distribution functions of mixture components 1 and 2. The particular form of these collision integrals depends on the concrete physical problem. To find the diffusion coefficient let us consider the stationary nonequilibrium state of two-components 1 and 2 of the mixture, in which the quasiparticles number densities are functions of coordinate \( \mathbf{r} \). In particular, for the gas of thermal excitations such situation can be realized by creation of constant temperature gradient.

Under the considered conditions there are stationary gradients of partial pressure of components, which result in flow of quasiparticles. This flow is determined by momentum current density

\[
\mathbf{j}_k = - \sum_{j=1,2} \frac{\partial \rho_k}{\partial \mathbf{r}} d_{kj} \frac{\partial P_j}{\partial \mathbf{r}}, \quad (k = 1, 2),
\]

(2)

where \( \rho_k \) is the normal density of \( k \)th component of the system, \( \rho = \rho_1 + \rho_2 \) is the total density, \( d_{kj} \) is the matrix of diffusion times. The partial pressure of quasiparticles is determined in standard manner

\[
P_j = \frac{1}{3} \int \mathbf{p}_j \mathbf{v}_j f_j d\Gamma_j,
\]

(3)

where \( d\Gamma_j \) is the measure in phase space. The density of \( k \)th component can be written in the universal form

\[
\rho_k = -\frac{1}{3} \int \mathbf{p}_k^2 f'_k d\Gamma_k,
\]

(4)

where \( f'_k = \partial f_k / \partial \epsilon_k \). Relations (2)–(4) are suitable for the quasiparticles with arbitrary dispersion law, statistics, and chemical potential (I mean both for zero chemical potential and for nonzero one). Note, the definition of normal density (4) does not depend explicitly on such notations as mass or number density.

To exclude convective transport in a quasiparticle system, i.e. to investigate only dissipative processes, hereinafter I shall consider the sum of partial pressure of different components of a mixture to be a constant, so put \( P = P_1 + P_2 = \text{const} \). The analytical relation between matrix elements of a matrix of diffusion times \( d_{kj} \) and usual diffusion coefficient \( D \) of a binary mixture can be obtained by a direct comparison of expression (3) with definition of diffusion coefficient. Thus, one can use the ordinary gas-dynamic definition of current density of \( k \)th component of a mixture (see e.g. Ref. 5)

\[
\mathbf{j}_k = \int \mathbf{p}_k f_k d\Gamma_k.
\]

(5)

It appears that for various both quantum and classical physical systems diffusion coefficient can be written in the most general form

\[
D = u_D^2 \tau_D,
\]

(6)

where \( u_D \) is the characteristic velocity, whose analytic form depends on the particular dispersion law and statistics of quasiparticles, \( \tau_D \) is the characteristic diffusion time to be determined.

According to the relations (4) and (5), for deriving the diffusion coefficient (6) it is necessary to solve a system of the kinetic equations (1). Below for definiteness the diffusion in a system with conserved number of quasiparticles (and, therefore nonzero chemical potential) will be considered. The calculation for case with not conserved number of quasiparticles can be carried out within the similar framework.

Since we are interested in the theory within linear response approximation, let us assume the deviation of distribution functions \( f_k \) from their locally equilibrium values \( f_k^{(0)} \) to be small. So, put as usual

\[
f_k = f_k^{(0)} + \delta f_k, \quad |\delta f_k| \ll f_k^{(0)}. \]

(7)

The small deviation \( \delta f_k \) can be conveniently rewritten in the form

\[
\delta f_k = \frac{\partial f_k^{(0)}}{\partial \epsilon_k} g_k
\]

(8)

with unknown functions \( g_k \). Linearizing the system of kinetic equations (1) we come to the system of linear integro-differential equations for unknown quantities \( g_k \), which determine the degree of system perturbation
\[
\frac{\mathbf{v}_k \partial P_k}{n_k \partial T} = C_{kk} g_k + C_{kj} (g_k + g_j) \quad (k, j = 1, 2; k \neq j).
\]

Here \( n_k \) is the number density of \( k \)th component, \( C_{kj} \) are the linearized collision operators for the collisions within each component \( (k = j) \) and for the collisions between different quasiparticles \( (k \neq j) \). The acting of this operators on an arbitrary function of momentum, say \( \xi(p_{k,j}) \), is determined by the particular form of collision integrals appeared in equation (9). If we deal with the ordinary binary collision integral with probability density function \( W_{kj}(p_k p_j | p_k' p_j') \) then we obtain [10]

\[
C_{kj} \xi(p_{k,j}) = \int W_{kj}(p_k p_j | p_k' p_j') f_j^{(0)}(p_j) \times \left\{ 1 \pm f_k^{(0)}(p_k) \right\}^{-1} \left\{ 1 \pm f_j^{(0)}(p_j') \right\} \left\{ 1 \pm f_k^{(0)}(p_k') \right\} \times \left[ \xi(p_{k,j}) - \xi(p_{k,j}) \right] d\Gamma_j d\Gamma'_j d\Gamma'_j
\]

(10)

for \( k \neq j \), and

\[
C_{kk} \xi(p_k) = \int W_{kk}(p_k p_k | p_k' p_k') f_k^{(0)}(p) \times \left\{ 1 \pm f_k^{(0)}(p_k) \right\}^{-1} \left\{ 1 \pm f_k^{(0)}(p_k') \right\} \left\{ 1 \pm f_k^{(0)}(p_k') \right\} \times \left[ \xi(p_k) + \xi(p_k') - \xi(p_k) - \xi(p_k') \right] d\Gamma d\Gamma' d\Gamma'
\]

(11)

for \( k = j \). The plus and minus in eqs. (10) and (11) signs correspond to bosons and fermions, respectively.

Note that in fact our general approach allows to account in such a manner not only binary collisions, but a variety of more specific types of interaction, such as decay or conversion processes, creation or annihilation of quasiparticles, interaction with boundaries and point defects etc.

### III. INVERSING OF COLLISION OPERATOR

According to the relation (8), the gradient of partial pressure for quasiparticles with nonzero chemical potential can be written as

\[
\frac{\partial P_k}{\partial T} = n_k \left( \frac{\partial \mu_k}{\partial T} \right)_{T=\text{const}},
\]

(12)

where \( \mu_k \) is the chemical potential of \( k \)th subsystem, \( T \) is the mixture temperature.

For further calculations it is convenient to present system (8) in the compact matrix form

\[
\sum_{k=1,2} |\psi_k\rangle \frac{\partial P_k}{\partial T} = \mathcal{C} |g\rangle,
\]

(13)

where

\[
|\psi_1\rangle = \begin{bmatrix} \mathbf{v}_1 n_1^{-1} \\ 0 \end{bmatrix}, \quad |\psi_2\rangle = \begin{bmatrix} 0 \\ \mathbf{v}_2 n_2^{-1} \end{bmatrix}, \quad |g\rangle = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix},
\]

(14)

are the two-component ket-vectors, defined in infinite Hilbert space to be specified. The collisional operator matrix \( \mathcal{C} \) can be conveniently written as a sum

\[
\mathcal{C} = \hat{J} + \hat{S},
\]

(15)

where the operator matrix

\[
\hat{J} = \begin{bmatrix} C_{12} & C_{12} \\ C_{21} & C_{21} \end{bmatrix}
\]

(16)

contains only the quantities corresponding to interactions between quasiparticles from different subsystems, and

\[
\hat{S} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}
\]

(17)

corresponds to the relaxation inside each subsystem. To define completely the Hilbert space I am working in, let us introduce the scalar product of two-dimensional bra-vector \( \langle \zeta | \zeta_1(p_1), \zeta_2(p_2) \rangle \) and ket-vector \( |\chi\rangle = (|\chi\rangle)^\dagger \) in the following manner

\[
\langle \zeta | \chi \rangle = \sum_{k=1,2} \langle \zeta_k | \chi_k \rangle = - \sum_{k=1,2} \int \zeta_k^* \chi_k \frac{\partial f_k^{(0)}}{\partial \epsilon_k} d\Gamma_k,
\]

(18)

where \( \langle \zeta_k | \) and \( |\chi_k\rangle \) are the corresponding one-component bra-vector and ket-vector, respectively. It is easy to verify that with such a choice of scalar product (15) the collision operator \( \mathcal{C} \) becomes hermitian.

System (13) is the system of non-uniform linear integral equations. According to the general theory of integral equations the sought solution \( |g\rangle \) of system (13) must be orthogonal to the solution of corresponding uniform equations

\[
\mathcal{C} |\phi_1\rangle = 0.
\]

(19)

The normalized solution of equation (19) can be written in the following form

\[
|\phi_1\rangle = \frac{1}{\sqrt{\mathcal{D}}} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix},
\]

(20)

This vector \( |\phi_1\rangle \) corresponds to the total momentum of our two-component quasiparticle system. In general, equation (19) has another solutions, corresponding to conservation of energy, particle number etc. I account here only the momentum conservation law because the sought solution can depend only on the quasiparticle momenta (see initial equations (13)). It is convenient to write the formal solution of equation (13) so that the orthogonality condition

\[
\mathcal{C} |\phi_1\rangle = 0.
\]
\[ \langle g | \phi_1 \rangle = 0 \] (21)

is contained explicitly in the solution. For this purpose let us define the projection operator \( P_n \) onto the subspace orthogonal to the vector \( | \phi_1 \rangle \),

\[ P_n = 1 - P_c, \quad P_c = | \phi_1 \rangle \langle \phi_1 |. \] (22)

As a result, the formal solution of equation (13) can be written in the form

\[ |g\rangle = P_n (\hat{\mathbf{c}}^{-1}) P_n \sum_{k=1,2} |\psi_k\rangle \frac{\partial P_k}{\partial \tau} \] (23)

Further we must insert the solution (23) into the expression for current density \( \mathbf{J} \) keeping in mind the relations (2) and (4). Comparing the obtained result with definition (2) for a matrix of diffusion times we come to

\[ d_{11} = \frac{\rho_2}{\rho_1} \tau_D, \quad d_{12} = d_{21} = \tau_D, \quad d_{22} = \frac{\rho_1}{\rho_2} \tau_D, \] (24)

where

\[ \tau_D = -\langle \phi_2 | \hat{\mathbf{c}}^{-1} | \phi_2 \rangle \] (25)

is the characteristic diffusion time, and

\[ |\phi_2\rangle = \frac{1}{\sqrt{3 \rho_1 \rho_2}} \begin{pmatrix} \rho_2 \rho_1 \\ -\rho_1 \rho_2 \end{pmatrix} \] (26)

is the characteristic diffusion vector, which is orthogonal to \( | \phi_1 \rangle \).

Now the problem is reduced to calculation of the matrix element (23), which contains the inverse matrix operator determined by integral collision operators.

### IV. EXACT AND LIMITING EXPRESSIONS FOR THE DIFFUSION TIME

To derive an exact analytical expression for the unknown quantity (25) it is necessary to introduce a full system of orthonormal two-dimensional vectors \( | \phi_n \rangle \) (here \( n = 1, 2, 3 \ldots \)) belong to the infinite-dimensional Hilbert space with a scalar product (18). The concrete choice of a system of basis vectors in many respects depends on convenience of calculations within the framework of a concrete physical problem (see, e.g. Refs. 5 and 8). In our problem it is convenient to take vector (24) as the first of them, and (26) as the second. The remaining vectors can be arbitrary (for example, such vectors can be built on the basis of Sonin polynomials in classical case [4], or Akhiezer-Aleksin-Khoudosov polynomials in quantum systems), but should satisfy to the completeness and orthogonality conditions

\[ \sum_{m=1}^{\infty} | \phi_m \rangle \langle \phi_m | = 1, \quad \langle \phi_m | \phi_n \rangle = \delta_{mn}. \] (27)

In a constructed full system of vectors an exact expression for the diffusion time

\[ \tau_D = -\left\{ \left( \hat{J} + \hat{S} \right)^{-1} \right\}_{22} \] (28)

can be reduced to the visual analytical form allowing simple physical interpretation and providing a possibility to carry out the in-depth qualitative analysis of the obtained result in various limiting cases, corresponding to different mechanisms of equilibration in the system of quasiparticles:

\[ \tau_D = -\left\{ I_{22} - \sum_{n,m=3}^{\infty} I_{2n} \left[ (I + S)^{-1} \right]_{nm} I_{m2} \right\}^{-1} \] (29)

Here the square matrices \( I \) and \( S \) contains the following matrix elements

\[ \|I\|_{nm} = I_{nm} = \langle \phi_n | \hat{J} | \phi_m \rangle, \] (30)

\[ \|S\|_{nm} = I_{nm} = \langle \phi_n | \hat{S} | \phi_m \rangle. \] (31)

To obtain the formal result (29) I have used the relations (19) and (27).

The matrices (30) and (31) are infinite-dimensional and non-diagonal. Therefore the exact solution (29) does not allow to obtain the closed analytical expression for characteristic diffusion time. However it is necessary to emphasize, that the result (29) contains explicitly not only the quantities responsible for interaction between quasiparticles of different types, but also the matrix elements appropriate to collisions between the quasiparticles within each subsystem. It allows to investigate various limiting cases, to find minimum and maximum values of diffusion time, to construct correct interpolation formulae and useful analytical models, to make calculations on computers for concrete physical systems.

To carry out the detailed analysis of the formal solution (29), it is convenient to concretize slightly the vector basis \( | \phi_n \rangle \) for \( n > 2 \). Namely, let us choose the remaining vectors in the following form:

\[ |\phi_{2n+1}\rangle = \frac{1}{\lambda_n^{\alpha}} \begin{pmatrix} F^{(\alpha)}(p_1) \\ 0 \end{pmatrix}, \] (32)

\[ |\phi_{2n+2}\rangle = \frac{1}{\lambda_n^{\alpha}} \begin{pmatrix} 0 \\ F^{(\alpha)}(p_2) \end{pmatrix}, \] (33)

where \( \alpha = 1, 2, 3, \ldots, F^{(\alpha)}(p_k) \) is the system of properly chosen orthogonal polynomials with the norm \( \lambda_k^{\alpha} = (F^{(\alpha)}(p_k) F^{(\alpha)}(p_k))^{1/2} \), such that \( \langle F^{(\alpha)}(p_k) | F^{(\alpha)}(p_k) \rangle = 0 \) (it is clear that such polynomials cannot contain first degree of momentum). Such a choice (32), (33) provide a possibility to separate in some manner the component of the mixture. The matrix \( \hat{J} + \hat{S} \) takes now the following form
\[ J + S = -\left( \begin{array}{cccc}
\nu_{11}^{(1)} & \nu_{12}^{(1)} & \nu_{12}^{(12)} & \ldots \\
\nu_{11}^{(2)} & \nu_{22}^{(1)} & \nu_{22}^{(12)} & \ldots \\
\nu_{11}^{(2)} & \nu_{21}^{(21)} & \nu_{21}^{(22)} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{array} \right) \]

\[ -\left( \begin{array}{cccc}
\omega_1^{(1)} & 0 & \omega_1^{(12)} & \ldots \\
0 & \omega_2^{(11)} & 0 & \ldots \\
\omega_1^{(21)} & 0 & \omega_1^{(22)} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{array} \right), \quad (34) \]

where I have introduced the following "higher" interaction rates:

\[ \nu_{jk}^{(\alpha\beta)} = \frac{\langle F^{(\alpha)}(P_j)|C_{jk}|F^{(\beta)}(P_k) \rangle}{\mathcal{N}_j^{(\alpha)}\mathcal{N}_k^{(\beta)}} \quad (35) \]

and

\[ \nu_{j}^{(\alpha\beta)} = -\frac{\langle F^{(\alpha)}(P_j)|C_{jj}|F^{(\beta)}(P_j) \rangle}{\mathcal{N}_j^{(\alpha)}\mathcal{N}_j^{(\beta)}} \quad (36) \]

for collisions of different quasiparticles, and

\[ \omega_{j}^{(\alpha\beta)} = -\frac{\langle F^{(\alpha)}(P_j)|C_{jj}|F^{(\beta)}(P_j) \rangle}{\mathcal{N}_j^{(\alpha)}\mathcal{N}_j^{(\beta)}} \quad (37) \]

corresponding to relaxation within each subsystem. The representation (34) helps us to understand the explicit structure of the formal solution (28). This particularly provide a possibility to study the most important limiting cases and to reveal their physical matter.

So, in case of infinitely fast establishment of an equilibrium between quasiparticles of identical type (so-called complete control regime (28)), when the strong inequalities

\[ \omega_{j}^{(\alpha\beta)} \gg \nu_{j}^{(\alpha\beta)}, \nu_{jk}^{(\alpha\beta)}, \quad (j, k = 1, 2) \quad (38) \]

take a place, the second term in the brackets of general result (28) vanishes and diffusion time (28) is given by the following simple formula (38):

\[ \tau_{(cc)}^{(D)} = \tau_{(min)}^{(D)} = -\frac{1}{\tau_{22}} = \left( \frac{1}{\tau_{12}} + \frac{1}{\tau_{21}} \right)^{-1}, \quad (39) \]

where

\[ \tau_{kj}^{(0)} = -\langle C_{kj} \rangle^{-1}, \quad (k, j = 1, 2; k \neq j). \quad (40) \]

Here and thereafter the brackets \( \langle \cdots \rangle \) stand for a normalized average defined by the relation

\[ \langle A \rangle_k = \frac{1}{3\rho_k} \langle p_k | A | p_k \rangle, \quad (k = 1, 2). \quad (41) \]

According to the momentum conservation law \( C_{kj}|p_k\rangle = -C_{jk}|p_j\rangle \) the relation between "basic" interaction rates in equation (39) can be found as follows

\[ \tau_{12}^{(0)} = \rho_2 \tau_{21}^{(0)} \quad \Rightarrow \quad \tau_{12} = \rho_2 \tau_{21}. \quad (42) \]

Proceeding from hermiticity and negativity of operators \( S \) and \( J \) with the help of well–known Cauchy-Bunyakovskiy inequality it is possible to show that the following inequality is always valid

\[ \tau_D \geq \tau_{D}^{(min)}, \quad (43) \]

where \( \tau_{D}^{(min)} \) is determined by eq. (38).

In the opposite limiting case of extremely slow establishment of equilibrium between identical quasiparticles, i.e. when

\[ \omega_{j}^{(\alpha\beta)} \ll \nu_{j}^{(\alpha\beta)}, \nu_{jk}^{(\alpha\beta)}, \quad (j, k = 1, 2), \quad (44) \]

the second matrix in eq. (44) can be neglected and the whole formula (29) can be converted so that the diffusion time \( \tau_D \) runs up to its maximum value

\[ \tau_{D}^{(max)} = -\langle \phi_2 | \dot{J}^{-1} | \phi_2 \rangle. \quad (45) \]

Comparing the main limiting results (39) and (45) we come to the important conclusion: The main qualitative difference between expressions for diffusion time in case of a fast and slow relaxation inside each subsystem of a mixture consists in a method of averaging of operator matrices corresponding to interaction between different subsystems.

Very interesting situation can be realized when one, say first, subsystem equilibrates very slowly \( C_{11} \to 0 \), but relaxation in another one is extremely fast \( C_{22} \to \infty \), so that

\[ \omega_{1}^{(\alpha\beta)} \ll \nu_{1}^{(\alpha\beta)}, \nu_{12}^{(\alpha\beta)}, \quad \omega_{2}^{(\alpha\beta)} \gg \nu_{2}^{(\alpha\beta)}, \nu_{21}^{(\alpha\beta)}. \quad (46) \]

In this case the results (29) and (34) can be converted as follows

\[ \tau_D \approx -\langle C_{12}^{-1} \rangle_{1} - \langle C_{21}^{-1} \rangle_{2}. \quad (47) \]

Let us pay attention to the principal difference between first and second terms in relation (47). Again, in the first term we average the time, while in the second one we average the rate of interaction between quasiparticles of different species. According to the Cauchy–Bunyakovskiy inequality the characteristic time \( \langle C_{jk}^{-1} \rangle \) is always less than the time \( \langle C_{kj}^{-1} \rangle \) for any momentum dependence of \( C_{jk} \). If the collision operator \( C_{jk} \) does not depend on the quasiparticles momenta, then \( \tau_D \equiv \tau_{D}^{(min)} \). If \( C_{jk} \) depends on momenta, then speed of equilibration of a system depends on how \( \langle C_{jk}^{-1} \rangle \) the equilibration between identical quasiparticles is. Such situation brightly manifests in a phonon–impuriton system of superfluid mixtures of helium isotopes and in phonon systems in solids. There is a two–stage mechanism of relaxation in these systems. At the first stage quasiparticles of the second type interact only with those quasiparticles of first
type, whose momenta correspond to maximum of collision operator $C_{12}$. At the second stage establishment of an equilibrium in a system is determined by the interaction of quasiparticles of 1st type with each other. This occurs because the quasiparticles of first type with minimum $C_{12}$ prefer to interact with such quasiparticles within their subsystem, which are already at equilibrium with second subsystem.

The significance of manner of averaging of collision operators can be illustrated on the simple example of lattice thermal conductivity in solids. The interaction rate between phonon and scattering center $\nu_{\text{phi}} \propto C_{\text{phi}}$ (say, point defect or impurity) is proportional to $p_{\text{ph}}$ (so-called Rayleigh scattering). So, the integral $\langle \_ \_ \_ \rangle$ diverges at zero momentum and the corresponding relaxation time tends to zero. This means that nonequilibrium long wavelength phonon simply "does not see" an impurity. At the same time the quantity $\langle C_{\text{phi}} \rangle_{\text{ph}}$ is finite and leads to finite thermal current. So, the mechanism of equilibration of such system can be drawn as follows: at first all phonons come to quasi-equilibrium in their own subsystems, which corresponds to some stationary flux of phonons, then they begin to scatter on the impurities and point defects. As it can be seen from the results [24], [34], [38], [45], and [17], analogous competition mechanisms can occur in any two-component quasiparticle system.

Calculation of $\langle C_{jk} \rangle_{j}$ for particular physical systems does not meet any difficulties. However, to calculate $\langle C_{-1} \rangle_{j}$ we must inverse the collision operator $C_{jk}$, which is not a straightforward operation. As a rule, it can be done by replacing $C_{jk}$ with some characteristic rate of interaction between quasiparticles $\nu_{jk} = \nu_{jk}(p_k)$, and then by straightforward averaging the value $\nu_{jk}(p_k)^{-1}$, which is simply a multiplying operator. In the next Section I consider the most popular models in various problems associated with two-component classical gases or with condensed media whose transport properties are determined by the processes in two-component quasiparticle systems.

**V. THE RAYLEIGH AND LORENTZ MODELS FOR TWO-COMPONENT GASEOUS MIXTURES**

Any theory claiming for a solution of some complicated special problems, should first of all be agreed necessarily with some fundamental results in the most simple limiting cases. In the kinetic theory of classical gases the diffusion in Lorentz gas (diffusion of light very rarefied component in a gas of massive slow particles) and Rayleigh gas (diffusion of massive particles with small concentration in a light gas) traditionally are considered.

Both classical models, i.e. the Rayleigh and Lorentz gases correspond in fact to the limiting case [16], but are more restricted in particles characteristics. Let us start with Lorentz model. This is the mixture of light component with very small concentration and the massive gas (buffer component). Let the light component be 1st one. Let me briefly review the classical approach to a problem of diffusion in such a mixture [24]. The strong inequality

$$n_1/n_2 \ll 1$$

was required for light particles to interact only with a massive component, but not with one another. Large difference in masses of particles

$$m_1/m_2 \ll 1$$

ensured an elasticity of scattering of particles of a light component on massive particles and large difference in thermal velocities of particles of different types. Thus, particles of the massive component can be treated as fixed and described by their equilibrium distribution function. With the purpose of calculation of a diffusion coefficient in such a system the strong inequalities [48] and [19] are usually used for simplification of initial kinetic equation, which can be reduced to the so-called Boltzmann-Lorentz equation [12]. The consequent solution of this equation gives explicit expression for a diffusion coefficient.

Let us show, that the classical results for diffusion in Lorentz gas can be directly obtained from the general results given by eqs. [24] and [34]. The operations below have certain value itself as a generalization of Rayleigh and Lorentz models on quantum gases of quasiparticles. Such generalization is not trivial because of impossibility to introduce the mass and conserved number density for some quasiparticles. So, we will formulate the problem in terms of normal densities. In this regard for Lorentz gas we replace two strong inequalities with the following

$$\rho_1/\rho_2 \ll 1.$$ (50)

Then we can rewrite vector $|\phi_2\rangle$,

$$|\phi_2\rangle = \frac{1}{\sqrt{\rho_1}} \begin{vmatrix} p_1 \\ 0 \end{vmatrix} \equiv \begin{vmatrix} \chi_1 \\ 0 \end{vmatrix}.$$ (51)

An equilibrium in massive component leads to the following simple expression for operator matrix $\hat{J}$ given in general by eq. (48):

$$\hat{J} = \begin{pmatrix} C_{12} & 0 \\ 0 & 0 \end{pmatrix}. (52)$$

Then, using the explicit expression (34) we can reduce the general result (24) to the following form

$$\tau_D^{(L)} = \left\{ \chi_1 |C_{12}| \chi_1 \right\} - \frac{1}{N_1^{(\alpha)} N_1^{(\beta)}} \times \sum_{\alpha, \beta = 1}^{\infty} \left\{ \chi_1 |C_{12}| F^{(\alpha)}(p_1) \right\} \left\{ ||\phi_{2\alpha+1} |C_{12}| \phi_{2\beta+1}| \right\}^{-1}_{\alpha \beta} \times \left\{ F^{(\beta)}(p_1) |C_{12}| \chi_1 \right\}^{-1}. (53)$$
Vectors $|\varphi_{2\alpha+1}\rangle$ represent a complete set of orthonormal vector-functions in momentum space of first component. This allows us to rewrite relation (63) as follows

$$\tau_D^{(L)} = -\frac{1}{3p_1} (p_1 |C_{12}^{-1} | p_1) \equiv \tau_D^{(\infty)} $$

Further simplification can be achieved if we believe the massive particles to be fixed during a collision, so that

$$v = |v_1 - v_2| \approx |v_1|. $$

This particularly means that differential cross section depends on momenta of light particles only. Moreover, the momentum of a light particle can change only its direction but not an absolute value. With these assumptions we can write collision operator $C_{12}$ as

$$C_{12} = -|v_1| \sigma_1 n_2, \quad (56)$$

which is simply a multiplication operator and, therefore, it can be inversed without any difficulties. Here I introduced the transport cross section

$$\sigma_1 = \int (1 - \cos \theta_1 \sigma_1) d\sigma, \quad (57)$$

where $\theta_1$ is the scattering angle, $d\sigma$ is the ordinary differential cross section. As a result, using the relations (54), (55) and defining the diffusion coefficient in binary gas in usual manner we come to the classical result

$$D_{12}^{(L)} = \frac{1}{3n_1} \int f_1^{(0)} \frac{1}{\sigma_1} |v_1| d\Gamma_1. \quad (58)$$

The obtained expression (58) coincides with analogous formulae obtained in Refs. 5 and 23 by simplification of the initial kinetic equation.

Note that for deriving the formula (58) in the frame of developed here approach it is enough to require fulfilment of strong inequality (60) for normal densities of components of a mixture, absence of equilibrium in first component, equilibrium in second component and elasticity of quasiparticles scattering.

Now I shall consider one more classical example, namely a diffusion in so-called Rayleigh gas, that is diffusion of very rarefied massive gas in the light buffer component with large concentration. In other words, we, as well as in the previous case, have a mixture of light and massive components, but under opposite conditions. So we assume the concentration of a massive component to be so small that

$$\rho_2 / \rho_1 \ll 1, \quad (59)$$

Now I assume, that the light component of a mixture is already in equilibrium, while the particles of the massive component almost do not interact with one another. In this limiting case we again can use the result (58) with a replacement of subsystems 1 ↔ 2. In this case, however, the relative velocity of particles from different components again is determined by a velocity of light particle. Therefore, the appropriate collision operator does not depend on momenta, so that in view of an orthogonality of vectors of selected basis its nongiagonal elements vanish, i.e.

$$\chi_2 |C_{21} |F^{(\beta)} |p_2\rangle = 0, \quad (\beta = 1, 2 \ldots). \quad (60)$$

Thus, proceeding from the relation (53) with a replacement of subscripts of components, we come to the relation

$$\tau_D^{(R)^{-1}} = -\frac{1}{3 \rho_2} (p_2 |C_{21} | p_2) \equiv \tau_D^{(\infty)^{-1}}, \quad (61)$$

which determines diffusion time in a weak solution of massive component in an equilibrium light gas. Further, proceeding from a momentum conservation law, we obtain

$$\tau_D^{(R)^{-1}} = -\frac{1}{3 \rho_2} (p_1 |C_{12}^{-1} | p_1)
= \frac{n_1 n_2}{3 \rho_2 T} \int f_1^{(0)} p_1^2 |v_1| \sigma_1 d\Gamma_1. \quad (62)$$

Expression (62) leads to the well known relation for a diffusion coefficient of massive particles in equilibrium light classical gas. Let me remark, that in Ref. 5 this result was obtained by an indirect method with the use of Einstein relation between diffusion coefficient and mobility. The use of the method developed here has allowed to obtain the formula (62) without engaging artificial approaches, i.e. immediately from a general solution (21), (23). The developed approach allows to generalize the result (62), as well as (58), on quasiparticle systems with arbitrary dispersion law and statistics, in particular, on such systems, in which it is impossible to define notation of a mass in its classical sense.

VI. KIHARA APPROXIMATION

In the previous sections I considered only the most physically interesting limiting cases. However, in practice the following problem can appear: How to calculate some dissipative coefficient more precisely in the intermediate case, i.e. when the considered limiting situations do not take a place. Of course, the most straightforward way is to compute it numerically using the formulae (23)–(24). In that case we are forced to restrict ourselves with some finite matrices in eq. (28). This is an analog of the Chapman–Cowling approximation in classical gaseous mixtures. In 1949 Kihara proposed another approximation, which is in fact simpler than Chapman–Cowling approximation. He simply proposed to neglect the nondiagonal integral brackets (see Ref. 2), which is exact for Maxwell molecules. Unfortunately, this approximation is unproved until now and it can be partially
justified only by experience in numerical calculations for classical gases. In our theory such an approximation can be introduced by neglecting all the nondiagonal matrix elements in (34). After this procedure the inverse matrix \((I + S)^{-1}\) takes the form

\[
(I + S)^{-1} = \begin{pmatrix}
t_{1}^{(11)} & 0 & 0 & \cdots \\
0 & t_{2}^{(11)} & 0 & \cdots \\
0 & 0 & t_{1}^{(22)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \tag{63}
\]

where I introduced the diagonal characteristic times

\[
t_{j}^{(\alpha\beta)} = \left[\nu_{j}^{(\alpha\beta)} + \omega_{j}^{(\alpha\beta)}\right]^{-1}. \tag{64}
\]

In view of eqs. (63) and (64) we can rewrite the result (29) in the following form

\[
\tau_{D}^{-1} \approx -\langle \phi_{2} | C | \phi_{2} \rangle + \frac{\langle \phi_{2} | C^{\dagger} | \phi_{3} \rangle \langle \phi_{3}^{\dagger} | C^{\dagger} | \phi_{2} \rangle}{\langle \phi_{3}^{\dagger} | C | \phi_{3} \rangle} + \frac{\langle \phi_{2}^{\dagger} | C^{\dagger} | \phi_{4} \rangle \langle \phi_{4}^{\dagger} | C^{\dagger} | \phi_{2} \rangle}{\langle \phi_{4}^{\dagger} | C | \phi_{4} \rangle} + \ldots \tag{65}
\]

The series expansion (65) gives the correction to the limiting result (39). In practice we should estimate numerically an appropriate radius of convergence in series (65) and keep necessary number of terms.

In general it is clear that the Kihara approximation works well when the momentum dependence of collision operators is weak enough. But sometimes the series like (65) have infinite radius of convergence. This indicates simply that it is better to start with the opposite limiting formula (45) as a zero order approximation.

\section{VII. Conclusions}

In the present work I present the general theory providing a possibility to investigate diffusion processes in two-component gas of quasiparticles with arbitrary statistics and dispersion. The obtained main equations of the theory are correct for both systems with conserved and not conserved number of quasiparticles, which is mathematically expressed in not equality or in equality of a chemical potential to zero, accordingly. The proposed theory can be generalized on the cases of classical and quantum gaseous mixtures with arbitrary number of components.

To solve the formulated problem I start with the system of kinetic equations \(\{1\}\) driving the evolution of corresponding distribution functions of components of a mixture. After a standard procedure of a linearization \(\{2\}\) of a kinetic problem, I have chosen the basis in infinite-dimensional two-parameter Hilbert space with a scalar product \(\{3\}\) selected so that operator matrix of collision integrals \(\{4\} \ldots \{7\}\) becomes Hermitian. As is known, the inverse matrix of collision integrals does not exist because of the moments of collision integrals. However, by projecting on the nucleus of an integral operator of collisions it is possible to define somewhat inverse matrix. In the present work this procedure has been made with the help of introducing of projection operator \(\{22\}\) corresponding to conservation of total momentum of a quasiparticle system. As an outcome it allowed me to obtain the general expression \(\{29\}\) for characteristic time (see also eq. \(\{64\}\)), determining speed of a diffusion relaxation in a system.

The obtained general result contains explicitly quantities responsible for interactions between both quasiparticles from different components and between identical quasiparticles. It allows to analyze qualitative difference between mechanisms of equilibration of a whole system in various limiting cases. So, if the relaxation inside each component of a system is instantaneous, the diffusion time is determined by an inverse average of collision operator, describing interaction between quasiparticles of different types. In the opposite limiting case, when the equilibrium in a system occurs infinitely longly, the diffusion time is equal to an average of the inverse collision operator. The principal difference between these two limiting results can be easily understood using an example of phonon thermal conductivity of solids. The thermal conductivity in this case is simply a diffusion of phonons in a system of fixed ”scatterers” (impurities, boundaries, defects etc.). If the phonons did not come yet to equilibrium with one another, the thermal conductivity is determined by average of inverse frequency of scattering of phonons on scatterers. In case of a long wavelength phonon, such frequency is proportional to the fourth degree of momentum (Rayleigh scattering). Therefore when averaging the integral simply diverges at zero momentum. That is the long wavelength phonon simply “does not feel” the scatterer. On the contrary, in case when the phonons came in an equilibrium with one another, the magnitude of diffusion time appears to be finite, because now the frequency is averaged, instead of time. This corresponds to the so-called two-stage mechanism of a relaxation. At first an equilibrium in phonon gas appropriate to some stationary phonon flux is established. Then phonons scatter on scatterers. Just this process results in a finite heat flux.

The general results, obtained in the work, give correct expressions for diffusion coefficients of a light gas in a massive one (Lorentz gas) \(\{58\}\) and a massive gas in a light one (Rayleigh gas) \(\{24\}\). Earlier, these classical results were obtained only with use of artificial methods based on simplification of the initial kinetic equation. Furthermore these two most popular model systems can be easily generalized to describe quasiparticle systems as well as classical gases.
ACKNOWLEDGMENTS

I am deeply grateful to my colleagues Professor I. N. Adamenko and Dr. K. É. Nemchenko, collaboration with which had initiated solution of the given problem when we were working on the problems concerning dissipative processes in superfluids. I do appreciate their contribution to the formulation of the problem and discussion of the main obtained results.

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