A log canonical threshold test

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Abstract

In terms of log canonical threshold, we characterize plurisubharmonic functions with logarithmic asymptotical behaviour.

1 Introduction and statement of results

Let \( u \) be a plurisubharmonic function on a neighborhood of the origin of \( \mathbb{C}^n \). Its log canonical threshold at 0,

\[
c_u = \sup \{ c > 0 : e^{-c u} \in L^2_{\text{loc}}(0) \},
\]

is an important characteristic of asymptotical behavior of \( u \) at 0. The log canonical threshold \( c(\mathcal{I}) \) of a local ideal in \( \mathcal{I} \subset \mathcal{O}_0 \) can be defined as \( c_u \) for the function \( u = \log |F| \), where \( F = (F_1, \ldots, F_p) \) with \( \{F_j\} \) generators of \( \mathcal{I} \). (Surprisingly, the latter notion was introduced later than its plurisubharmonic counterpart.) For general results on log canonical thresholds, including their computation and applications, we refer to [9], [16], [17].

A classical result due to Skoda [22] states that

\[
c_u \geq \nu_u^{-1},
\]

where \( \nu_u \) is the Lelong number of \( u \) at 0. A more recent result is due to Demailly [8]: if 0 is an isolated point of \( u^{-1}(-\infty) \), then

\[
c_u \geq F_n(u) := n e_n(u)^{-1/n}.
\]

Here \( e_k(u) = (dd^c u)^k \wedge (dd^c \log |z|)^{n-k}(0) \) are the Lelong numbers of the currents \( (dd^c u)^k \) at 0 for \( k = 1, \ldots, n \), and \( d = \partial + \bar{\partial} \), \( d^c = (\partial - \bar{\partial})/2\pi i \); note that \( e_1(u) = \nu_u \). This was extended by Zeriahi [23] to all plurisubharmonic functions with well-defined Monge-Ampère operator near 0.

In [19], inequality (2) was used to obtain the ‘intermediate’ bounds

\[
c_u \geq F_k(u) := k e_k(u)^{-1/k}, \quad 1 \leq k \leq l,
\]

\( l \) being the codimension of an analytic set \( A \) containing the unbounded locus \( L(u) \) of \( u \). None of the bounds for different values of \( k \) can be deduced from the others.

It is worth mentioning that relation (2) was proved in [8] on the base of a corresponding result for ideals [1] obtained in [6]:

\[
c(\mathcal{I}) \geq n e(\mathcal{I})^{-1/n},
\]

where \( e(\mathcal{I}) \) is the Hilbert-Samuel multiplicity of the (zero-dimensional) ideal \( \mathcal{I} \). Furthermore, it was shown in [6] that an equality in (4) holds if and only if the integral closure of \( \mathcal{I} \) is a

\[1\] A direct proof was given later in [2].
power of the maximal ideal $\mathfrak{m}_0$. Accordingly, the question of equality in (2) has been raised in [3], where it was conjectured that, similarly to the case of ideals, the extremal functions would be those with logarithmic singularity at 0.

The conjecture was proved in [20] where it was shown that

$$c_u = F_n(u)$$

if and only if the greenification $g_u$ of $u$ has the asymptotics $g_u(z) = e_1(u) \log |z| + O(1)$ as $z \to 0$. Here the function $g_u$ is the upper semicontinuous regularization of the upper envelope of all negative plurisubharmonic functions $v$ on a bounded neighborhood $D$ of 0, such that $v \leq u + O(1)$ near 0, see [18]. Note that if $u = \log |F|$, then $g_u = u + O(1)$ [21, Prop. 5.1].

The equality situation in [1] (i.e., in (3) with $k = 1$) was first treated in [5] and [11] for the dimension $n = 2$: the functions satisfying $c_u = \nu_u^{-1}$ were proved in that case to be of the form $u = c \log |f| + v$, where $f$ is an analytic function, regular at 0, and $v$ is a plurisubharmonic function with zero Lelong number at 0. In a recent preprint [15], the result was extended to any $n$. This was achieved by a careful slicing technique reducing the general case to the aforementioned two-dimensional result. In addition, it used a regularization result for plurisubharmonic functions with keeping the log canonical threshold (see Lemma 1 below).

Concerning inequalities (3), it was shown in [19] that the only multi-circled plurisubharmonic functions $u(z) = u(|z_1|, \ldots, |z_n|)$ satisfying $c_u = F_l(u)$ are essentially of the form $c \max_{j \in J} \log |z_j|$ for an $l$-tuple $J \subset \{1, \ldots, n\}$. Here we address the question on equalities in the bounds (3) in the general case.

We present an approach that is different from that of [15] and which actually works also for the 'intermediate' equality situations. It is based on a recent result of Demailly and Pham Hoang Hiep [10]: if the complex Monge-Ampère operator $(dd^c u)^n$ is well defined near 0 and $e_1(u) > 0$, then

$$c_u \geq E_n(u) := \sum_{1 \leq j \leq n} \frac{e_{j-1}(u)}{e_j(u)},$$

where $e_0(u) = 1$. In particular, this implies (2) and sharpens, for the case of functions with well-defined Monge-Ampère operator, inequality (1). Moreover, it is this bound that was used in [20] to prove the conjecture from [5] on functions satisfying (5).

Given $1 < l \leq n$, let $\mathcal{E}_l$ be the collection of all plurisubharmonic functions $u$ whose unbounded loci $L(u)$ have zero $2(n - l - 1)$-dimensional Hausdorff measure. For such a function $u$, the currents $(dd^c u)^k$ are well defined for all $k \leq l$ [12]. In particular, $u \in \mathcal{E}_l$ if $L(u)$ lies in an analytic variety of codimension at least $l$. Furthermore, we set $\mathcal{E}_1$ to be just the collection of all plurisubharmonic functions near 0.

Let $c_u(z)$ denote the log canonical threshold of $u$ at $z$ and, similarly, let $e_k(u, z)$ denote the Lelong number of $(dd^c u)^k$ at $z$; in our notation, $c_u(0) = c_u$ and $e_k(u, 0) = e_k(u)$. As is known, the sets $\{z : c_u(z) \leq c\}$ are analytic for all $c > 0$. Our first result describes, in particular, regularity of such a set for $c = c_u$, provided $c_u = F_l(u)$.

For $u \in \mathcal{E}_l$ we set

$$E_k(u) = \sum_{1 \leq j \leq k} \frac{e_{j-1}(u)}{e_j(u)}, \quad k \leq l.$$
Theorem 1 Let \( u \in \mathcal{E}_l \) for some \( l \geq 1 \), and let \( e_1(u) > 0 \). Then

(i) \( c_u \geq E_k(u) \) for all \( k \leq l \);
(ii) \( c_u \geq F_k(u) \) for all \( k \leq l \);
(iii) if \( u \) satisfies \( c_u = F_k(u) \) for some \( k \leq l \), then \( k = l \) and there is a neighborhood \( V \) of the origin such that the set \( A = \{ z : c_u(z) \leq c_u \} \) is an \( l \)-codimensional manifold in \( V \).

Furthermore, \( A = \{ z : e_l(u, z) \geq e_l(u) \} \).

For \( l = 1 \), assertion (iii) re-proves the aforementioned result from [15]. Let \( A = \{ z_1 = 0 \} \), then the function \( u - c_u \log |z_1| \) is locally bounded from above near \( A \) and thus extends to a plurisubharmonic function \( v \); evidently, \( \nu_v = 0 \). On the other hand, all the functions \( u = c_u \log |z_1| + v \) with \( \nu_v = 0 \) satisfy \( c_u = \nu_u \).

When \( l > 1 \), there are functions \( u \) such that \( \{ z : c_u(z) \leq c_u \} \) is an \( l \)-codimensional manifold, but \( c_u > F_l(u) \). Indeed, let us take \( u(z_1, z_2, z_3) = \max\{ \log |z_1|, 2 \log |z_2| \} \in \mathcal{E}_2 \). Then \( A = \{ z \in \mathbb{C}^n : c_u(z) \leq c_u \} = \{ z_1 = z_2 = 0 \} \), while \( F_2(u) = \sqrt{2} < 3/2 = c_u \). (Note that \( c_u = E_2(u) \) in this case.)

Furthermore, the same example shows that the equality \((dd^c u)^2 = \delta^2 \) \( [z_1 = z_2 = 0] \) does not imply \( u = \delta \log |(z_1, z_2)| + v \) with plurisubharmonic \( v \) and \( \nu_v = 0 \).

Therefore, in the higher dimensional situation we need to deduce a more precise information on asymptotical behavior of \( u \) near \( A \). By analogy with the case \( l = n \), it is tempting to make the following conjecture.

Let \( u \in \mathcal{E}_l \), then

\[
c_u = F_l(u)
\]

if and only if, for a choice of coordinates \( z = (z', z'') \in \mathbb{C}^l \times \mathbb{C}^{n-l} \), the greenification \( g_u \) of \( u \) near 0 satisfies

\[
g_u = e_1(u) \log |z'| + O(1) \text{ as } z \to 0.
\]

The 'if' direction is obvious in view of \( c_u = c_{g_u} \) [20] and the trivial fact \( c_{\log |z'|} = l \), however the reverse statement might be difficult to prove even in the case \( l = 1 \) because that would imply non-existence of a plurisubharmonic function \( \phi \) with \( e_1(\phi) = 0 \) and \( e_n(\phi) > 0 \), which is a known open problem. Namely, let such a function \( \phi \) exist, and set \( u = \phi + \log |z_1| \). Then

\[
1 = \nu_u \leq c_u \leq e_{\log |z_1|} = 1.
\]

On the other hand, for \( D = \mathbb{D}^n \), \( g_u = g_\phi + \log |z_1| \) and the relation \( e_n(\phi) > 0 \) implies \( g_\phi \neq 0 \) and thus \( \lim \inf (g_u - \log |z_1|) = -\infty \) when \( z \to 0 \).

What we can prove is the following, slightly weaker statement.

Theorem 2 If \( u \in \mathcal{E}_l \) satisfies \((\star)\), then \( e_k(u) = e_1(u)^k \) for all \( k \leq l \) and, for a choice of coordinates \( z = (z', z'') \in \mathbb{C}^l \times \mathbb{C}^{n-l} \), the function \( u \) satisfies \( u \leq e_1(u) \log |z'| + O(1) \) near 0, while the greenification \( g_{u_N} \) of \( u_N = \max\{u, N \log |z|\} \) with any \( N \geq e_1(u) \) satisfies

\[
g_{u_N} = \max\{e_1(u) \log |z'|, N \log |z''|\} + O(1), \quad z \to 0.
\]
Let us fix a neighborhood $D \subset V$ of 0 to be the product of unit balls in $\mathbb{C}^l$ and $\mathbb{C}^{n-l}$ and consider the greenifications with respect to $D$. Then the functions $g_{u_N}$ are equal to max$\{e_1(u) \log |z'|, N \log |z''|\}$ and they converge, as $N \to \infty$, to $e_1(u) \log |z'| \geq g_u$.

Denote, for any bounded neighborhood $D$ of 0 and any $u$ plurisubharmonic in $D$,

$$\tilde{g}_u = \lim_{N \to \infty} g_{u_N},$$

where $u_N = \max\{u, N \log |z|\}$. Evidently, $\tilde{g}_u \geq g_u$.

**Theorem 3** Let $u \in \mathcal{E}_1$ be such that $\tilde{g}_u = g_u$. Then it satisfies (10) if and only if, for a choice of coordinates $z = (z', z'') \in \mathbb{C}^l \times \mathbb{C}^{n-l}$, $g_\alpha = e_1(u) \log |z'| + O(1)$ as $z \to 0$.

In particular, this is true for $u = \alpha \log |F| + O(1)$, where $F$ is a holomorphic mapping, $F(0) = 0$. Moreover, in this case we also have $u = e_1(u) \log |z'| + O(1)$.

The statement on $\alpha \log |F|$ can be reformulated in algebraic terms as follows. Let $\mathcal{I}$ be an ideal of the local ring $\mathcal{O}_0$, and let $V(\mathcal{I})$ be its variety: $V(\mathcal{I}) = \{z : f(z) = 0 \forall f \in \mathcal{I}\}$. If $\text{codim}_0 V(\mathcal{I}) \geq k$, then the mixed Rees’ multiplicity $e_k(\mathcal{I}, m_0)$ of $k$ copies of $\mathcal{I}$ and $n-k$ copies of the maximal ideal $m_0$ is well defined [4]. If $k = n$, then, as shown in [8], the Hilbert-Samuel multiplicity $e(\mathcal{I})$ of $\mathcal{I}$ equals $e_n(u)$, where, as before, $u = \log |F|$ for generators $\{F_p\}$ of $\mathcal{I}$. By the polarization formula, $e_k(\mathcal{I}, m_0) = e_k(u)$ for all $k$; by a limit transition, this holds true for all $k \leq l$ if $\text{codim}_0 V(\mathcal{I}) = l$.

Bounds [3] specify for this case as

$$c(\mathcal{I}) \geq k e_k(\mathcal{I}, m_0)^{-1/k}, \quad 1 \leq k \leq l;$$

from Theorems [1] and [3] we thus derive

**Corollary 1** If $\text{codim}_0 V(\mathcal{I}) = l$ and $c(\mathcal{I}) = k e_k(\mathcal{I}, m_0)^{-1/k}$ for some $k \leq l$, then $k = l$, $V(\mathcal{I})$ is an $l$-codimensional hypersurface, regular at 0, and there exists an ideal $n_0$ generated by coordinate (smooth transversal) germs $f_1, \ldots, f_l \in \mathcal{O}_0$ such that $\mathcal{I} = n_0^s$ for some $s \in \mathbb{Z}_+$.

## 2 Proofs

In what follows, we will use the mentioned regularization result by Qi’an Guan and Xiangyu Zhou. Note that its proof rests on the strong openness conjecture from [9], proved in [13] and [14], see also [3].

**Lemma 1** [15] Prop. 2.1 Let $u$ be a plurisubharmonic function near the origin, $\sigma_u = 1$. Then there exists a plurisubharmonic function $\tilde{u} \geq u$ on a neighborhood of 0 such that $e^{-2u} - e^{-2\tilde{u}}$ is integrable on $V$ and $\tilde{u}$ is locally bounded on $V \setminus \{z : c_u(z) \leq 1\}$.

We will also refer to the following uniqueness theorem.

**Lemma 2** ([18] Lem. 6.3 and [20] Lem. 1.1) If $u$ and $v$ are two plurisubharmonic functions with isolated singularity at 0, such that $u \leq v + O(1)$ near 0 and $e_n(u) = e_n(v)$, then their greenifications coincide.

\footnote{For the general case of non-isolated singularities, see [1] Thm. 3.7}
Proof of Theorem 1. Since all the functionals \( u \mapsto c_\nu, E_k(u), F_k(u) \) are positive homogeneous of degree \(-1\), we can assume \( c_\nu = 1 \).

Let \( \tilde{u} \) be the function from Lemma \([1]\) Its unbounded locus \( L(\tilde{u}) \) is contained in the analytic variety \( A = \{ z : c_\nu(z) \leq 1 \} \). Since \( A \subset L(u) \) and \( u \in \mathcal{E}_l \), \( \operatorname{codim} A \geq l \).

For \( \tilde{u} \), statement (i) is proved in [20 Thm. 1.4]. Note that the relation \( u \leq \tilde{u} \) implies \( e_k(u) \geq e_k(\tilde{u}) \) for all \( k \leq l \) and thus \( E_l(u) \leq E_l(\tilde{u}) \) \([10]\). Since \( c_\nu = c_{\tilde{u}} \), this gives us (i).

Assertion (ii) follows from (i) by the arithmetic-geometric mean theorem.

To prove (iii), we first note that (i) implies \( c_\nu \geq E_l(u) > E_k(u) \geq F_k(u) \) for any \( k < l \), so we cannot have \( c_\nu = F_k(u) \) unless \( k = l \).

Next, if the analytic variety \( A \) has codimension \( m > l \), then \( \tilde{u} \in \mathcal{E}_m \), so \( c_\nu = c_{\tilde{u}} \geq E_m(\tilde{u}) > E_l(\tilde{u}) \geq E_l(u) \geq F_l(u) \), which contradicts the assumption, so \( \operatorname{codim} A = l \).

Now we prove that \( 0 \) is a regular point of the variety \( A \). By Siu’s representation formula, \( (dd^c u)_l = \sum p_j[A_j] + R \) on a neighborhood \( V \) of \( 0 \), where \( p_j > 0 \), \( [A_j] \) are integration currents along \( l \)-codimensional analytic varieties containing \( 0 \), and \( R \) is a closed positive current such that for any \( a > 0 \) the analytic variety \( \{ z \in V : \nu(R, z) \geq a \} \) has codimension strictly greater than \( l \). If \( \nu(R, 0) > 0 \), then for almost all points \( z \in A \) we have \( e_l(u, z) < e_l(u) \). This implies, by (ii), \( c_{\nu}(z) > c_{\nu} \) for all such points \( z \), which is impossible. The same argument shows that the collection \( \{ A_j \} \) consists of at most one variety and \( 0 \) is its regular point. \( \square \)

Proof of Theorem 2. By the arithmetic-geometric mean theorem, the condition \( c_\nu = F_l(u) \) implies, in view of the inequality \( c_\nu \geq E_l(u) \), the relations

\[
e_k(u) = \frac{e_{k-1}(u)}{e_k(u)} \frac{e_{j-1}(u)}{e_j(u)}
\]

for any \( k, j \leq l \), which gives us \( e_k(u) = [e_1(u)]^k \) for all \( k \leq l \).

Since relation (7) for \( e_1(u) = 0 \) is obvious (in this case \( g_{u_N} = 0 \)), we can assume \( e_1(u) = 1 \).

Note that for any \( z \), we have \( e_k(u, z) \geq [e_1(u, z)]^k \). As follows from the proof of (iii), the relation \( c_\nu = F_l(u) \) implies then, on a neighborhood \( V \) of \( 0 \),

\[
A \cap V = \{ z \in V : c_\nu(z) \leq 1 \} = \{ z \in V : F_l(u, z) \leq 1 \} = \{ z \in V : e_k(u, z) \geq 1 \}
\]

for all \( k \leq l \). Moreover, we have \( e_k(u, z) = e_1(u, z) )^k \) for almost all \( z \in A \cap V \).

Let us choose, according to Theorem \([1]\) a coordinate system such that \( A \cap V = \{ z \in V : z_k = 0, 1 \leq k \leq l \} \). Denote \( \nu(z) = \log |z'|, z = (z', z'') \in \mathbb{C}^l \times \mathbb{C}^{n-l} \), then \( A \cap V = \{ z : e_k(u, z) \geq e_k(u, z) \} \), with equalities almost everywhere.

In particular, we have \( u(z) \leq \log |z - (0, \zeta''')| + C(\zeta''') \) as \( z \to (0, \zeta''') \) for all \( z \in \mathbb{C}^n \) and \( \zeta'''' \in \mathbb{C}^{n-l} \) that are close enough to \( 0 \). Assuming \( u(z) \leq 0 \) for all \( z \) with \( \max |z_k| < 2 \), we get \( u(z) \leq \log |z - (0, \zeta''')| \) for all \( z \in V \) and \( \zeta'''' \in \mathbb{C}^{n-l} \) with \( (0, \zeta''') \in V \). By choosing \( \zeta'''' = z'''' \) this gives us \( u(z) \leq v(z) \) on \( V \).

Let \( u_N = \max \{ u, N \log |z| \} \) and \( v_N = \max \{ v, N \log |z| \} \). Then \( u_N \leq v_N \), while for \( N \geq 1 \) we get, by Demailly’s comparison theorem for the Lelong numbers \([7]\),

\[
e_n(u_N) \leq (dd^c u)_l \wedge (dd^c N \log |z|)^{n-l}(0) = N^{n-l} e_l(u) = N^{n-l} = e_n(v_N).
\]
By Lemma 2 \( g_{uN} = g_{vN} \). \( \square \)

**Proof of Theorem 3.** The only part to prove is the one concerning \( u = \alpha \log |F| + O(1) \); we assume \( \alpha = 1 \). As follows from Theorem 2, one can choose coordinates such that the zero set \( Z_F \) of \( F \) is \( \{ z : z' = 0 \} \cap V \subset \{ 0 \} \times \mathbb{C}^{n-I} \). Observe that for such a function \( u \) we have \( e_k(u, z) = e_1(u)^k \) for all \( z \in Z_F \) near 0.

Let \( I \) be the ideal generated by the components of the mapping \( F \). Then, as mentioned in Section 1, \( e_l(u) \) equals \( e_l(I, m_0) \), the mixed multiplicity of \( l \) copies of the ideal \( I \) and \( n-l \) copies of the maximal ideal \( m_0 \). By [4, Prop. 2.9], \( e_l(I, m_0) \) can be computed as the multiplicity \( e(J) \) of the ideal \( J \) generated by generic functions \( \Psi_1, \ldots, \Psi_l \in I \) and \( \xi_1, \ldots, \xi_{n-l} \in m_0 \). Since \( e(J) = e_l(w) \), where \( w = \log |\Psi| \), we have \( e_l(u) = e_l(w) \).

Let now \( v = e_1(u) \log |z'| \), \( w_N = \max \{ w, N \log |z'| \} \), and \( v_N = \max \{ v, N \log |z''| \} \). Since \( w \leq \log |F| + O(1) \), we have from Theorem 2 the inequality \( w \leq v + O(1) \) and thus \( w_N \leq v_N + O(1) \). Note that the mapping \( \Psi \) satisfies the Lojasiewicz inequality \( |\Psi_0(z)| \geq |z'|^M \) near 0 for some \( M > 0 \). Therefore, for sufficiently big \( N \) we have \( w_N = w'_N = \max \{ w, N \log |z| \} \).

Then, as in the proof of Theorem 2 we compute

\[
e_n(w_N) = e_n(w'_N) - (dd^c w')^l (dd^c N \log |z|)^{n-l}(0) = N^{n-l} e_l(w) = N^{n-l} e_l(u) = e_n(v_N),
\]

which, by Lemma 2 implies \( g_{w_N} = g_{v_N} \) for the greenifications on a bounded neighborhood \( D \) of 0.

We can assume \( D = \{ |z'| < 1 \} \times \{ |z''| < 1 \} \), then \( g_{v_N} = v_N \), while \( g_{w_N} \leq w_N \) because the latter function is maximal on \( D \) and nonnegative on \( \partial D \). Letting \( N \to \infty \) we get \( w \geq v \).

Since \( w \leq u + O(1) \), we have, in particular, \( u \geq v + O(1) \), which, in view of Theorem 2 completes the proof. \( \square \)

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