Stabilizing a spherical pendulum on a quadrotor

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Abstract
In this article, we design a backstepping control law based on geometric principles to swing up a spherical pendulum mounted on a moving quadrotor. The available degrees of freedom in the control vector also permit us to position the plane of the quadrotor parallel to the ground. The problem addressed here is, indeed, novel and has many practical applications which arise during the transport of a payload mounted on top of a quadrotor. The modeling and control law are coordinate-free and thus avoid singularity issues. The geometric treatment of the problem greatly simplifies both the modeling and control law for the system. The control action is verified and supported by numerical experiments for aggressive maneuvers starting very close to the downward stable equilibrium position of the pendulum.

KEYWORDS
backstepping, nonlinear geometric control, swing up spherical pendulum

1 INTRODUCTION

The problem of achieving an arbitrary orientation and tracking of suitable trajectory for a quadrotor is well studied in literature [6,5]. A quadrotor consists of four arms with rotors attached to them. The propeller attached to the rotors thus provides four independent directions of actuation. Three of these can be utilized to achieve an arbitrary position in \( \mathbb{R}^3 \). In Lee et al. [5], the fourth actuation is used to track a heading direction of the quad. Most commercially available unmanned aerial vehicle (UAVs) can be modeled as quadrotors. The availability of sophisticated and affordable sensors in the recent years has led to large scale manufacturing of UAVs. This has led to their utilization in transporting load over reasonably long distances. The load is usually considered to be suspended by a cable attached to the center of the quadrotor. Cable suspended systems are underactuated, and therefore, there has been an increased effort in the robotics community to study the various control objectives which can be realized by such systems.

In this article, we aim to balance an inverted pendulum mounted on the center of mass of a quad through a universal joint. This mechanical system, called flying inverted pendulum was first introduced in Hehn and D’Andrea [2]. The flying pendulum is the simplest model for a payload mounted on a quadrotor. Therefore, the stability of such a mechanical system is a potentially important problem which has not been addressed in the literature. The problem of a payload suspended through cables from multiple quadrotors has recently gained a lot of attention and is well studied in Lee et al. [7], Lee [4], and Wu and Sreenath [11]. In such a payload mounting, there can be issues of damage to the payload during landing of the quad. In the proposed model, a more practical method of transporting the payload is achieved by mounting it on top of the quad.

A linearization approach was used in Hehn and D’Andrea [2] to stabilize the pendulum on the quadrotor. Specifically, nominal trajectories of the system were considered, and the system was stabilized around these dynamical equilibria. This is a fairly restrictive treatment
as the nonlinearities in the quadrotor can lead to failure of the control action if the initial position of the pendulum is far away from the equilibria, for example, for an initial condition close to the inverted equilibrium position. In this article, the quadrotor and pendulum are modeled as Lagrangian systems, and the control law proposed admits convergence to the desired state from large set of initial conditions thereby allowing aggressive maneuvers. Moreover as we shall see, the pendulum–quadrotor assembly considered in this paper matches that of the experimental setup in Hehn and D’Andrea [2]. The salient advantage of the control action we propose is that it is not only nonlinear and continuous but also coordinate free. In Hehn and D’Andrea [2], the orientation of the quad is modeled using Euler angles which suffer from singularity issues and hence do not allow the control action to achieve aggressive maneuvers. In this article, however, we employ a purely geometric model: The orientation of the quad is modeled as a rotation matrix, and the pendulum is modeled as a point mass on the $2$–sphere, thereby accounting for all possible configurations which the system may assume.

The contributions of this article are (1) explicit control laws in continuous time for the swing up of an inverted pendulum mounted using a spherical joint on a quadrotor; (2) geometric modeling for the study of dynamics and control of the system which allows a large set of initial conditions from which stability is guaranteed; (3) strict feedback form of the part of the dynamics which has to be controlled is shown to exist which allows backstepping control to be applied; (4) backstepping control is used in a purely geometric setting by choosing appropriate Lyapunov functions.

The article is organized as follows: In Section 2, we introduce the notations and the dynamical model; in Section 3 we propose the control strategy by establishing the strict feedback form of the dynamical equations; in Section 4, we present numerical experiments with model parameters chosen close to an experimental setup by choosing several non trivial initial conditions; and finally, we conclude the paper in Section 5.

2 | NOTATION AND DYNAMIC MODELING

2.1 | Notation

Figure 1 represents the quadrotor with a pendulum mounted on it.

The following notation is employed:

- $S$: the spatial frame specified by vectors $\{e_1, e_2, e_3\}$ where $e_1 = (1\ 0\ 0)^\top$, $e_2 = (0\ 1\ 0)^\top$, $e_3 = (0\ 0\ 1)^\top$.
- $B$: the frame fixed to the center of mass of the quadrotor (pivot) specified by vectors $\{b_1, b_2, b_3\}$.
- $P$: the frame fixed to the pivot specified by vectors $\{p_1, p_2, p_3\}$.
- $m$ and $M$: mass of the pendulum and quadrotor, respectively.
- $I$: moment of inertia of the quadrotor in $B$ frame.
- $l$: length of the inextensible cable connecting the center of mass of the quadrotor (pivot) to the mass $m$.
- $d$: distance from the pivot to the rotors.
- $x$: location of the pivot in the $S$ frame.
- $q$: unit vector from the pivot along $-p_1$ in the $B$ frame.
- $R$: rotation matrix expressing the transformation from the $B$ frame to the $S$ frame, $R b_1 = e_1$.
- $y$: unit vector from the pivot along the pendulum in the $S$ frame, $y = R q$.

![Figure 1: Pendulum on quadrotor](image-url)
• $\omega$: angular velocity of the quadrotor expressed in the quadrotor body in $B$ frame
• $g$: gravity vector in the $-e_3$ direction
• $f_i$: magnitude of thrust generated by $i$-th propeller along $-b_3$
• $f$: magnitude of total thrust, $f = \sum_{i=1}^{4} f_i$
• $r_i$: torque generated by the $i$th propeller about the $b_3$ axis

All the four rotors are in the $b_1 - b_2$ plane. We consider a universal (spherical) joint at the pivot and the bob of the pendulum to be symmetric. Therefore, $q$ completely specifies the $P$ frame as we ignore the rotation about $p_1$ axis. By definition, the total thrust is $-f R e_3$ in the inertial frame. It is assumed that the first and the third propellers rotate clockwise, and the second and the fourth propellers rotate counterclockwise; when they are generating a positive thrust $f_i$, the torque generated by the $i$th propeller can be written as $r_i = (-1)^i c f_i$ for a fixed constant $c$. Under these fairly common assumptions (see Lee [4] and Lee et al. [5]), the total thrust $f$ and the total moment $\mu = (\mu_1 \mu_2 \mu_3)^T$ acting at the pivot can be transformed as

$$
\begin{pmatrix}
  f \\
  \mu_1 \\
  \mu_2 \\
  \mu_3 \\
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & 1 & 1 & f_1 \\
  0 & -d & 0 & d & f_2 \\
  d & 0 & -d & 0 & f_3 \\
  -c & c & -c & c & f_4 \\
\end{pmatrix}.
$$

(1)

The variation $\delta L$ of the Lagrangian is given by the following:

$$
\delta L = \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) + \left( \frac{\partial L}{\partial \omega} \delta \omega \right) + \left( \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} \right).
$$

The variations of the individual terms are

- $\dot{x} : \delta \ddot{x} = \frac{d}{dt} (\delta x)$,
- $R : \dot{\Sigma} = -R^T \delta RR^T \dot{R} + R^T \delta \dot{R}$,
- $\omega : \delta \ddot{\omega} = -R^T \delta \dot{R} R^T \dot{R} + R^T \delta \dot{R}$,
- $y : \delta y = \delta Rq + R \delta q = R \dot{\Sigma} R^T y + \gamma \times y$,
- $\dot{y} : \delta \ddot{y} = \frac{d}{dt} (R \dot{\Sigma} R^T y) + \gamma \times y + \gamma \times \dot{y} = R (\dot{\omega}, \dot{\Sigma}) R^T y + R \dot{\Sigma} R^T \dot{y} + \gamma \times y + \gamma \times \dot{y}$

for any $\gamma \in \mathbb{R}^3$ t.s.t. $\gamma^T y = 0$.

### 2.2 Dynamic model

In this section, we elaborate on derivation of the dynamical equations of the of the quadrotor–pendulum assembly in Figure 1 by variational principles considering the lagrangian in $L$ in (2). The method we employ is truly coordinate free as all configuration variables are considered to lie in an ambient Euclidean space and their corresponding variations are constrained by the geometry of the manifold on which the configuration variables evolve.

The partial derivatives of the Lagrangian $L$ are

$$
\left( \frac{\partial L}{\partial x} \right)^T = (M + m) g e_3; \quad \left( \frac{\partial L}{\partial \dot{x}} \right)^T = (m + M) \ddot{x} + m \ddot{y}
$$

$$
\left( \frac{\partial L}{\partial \omega} \right)^T = I \ddot{\omega}; \quad \left( \frac{\partial L}{\partial y} \right)^T = m g e_3;
$$

$$
\left( \frac{\partial L}{\partial \dot{y}} \right)^T = m \ddot{x} + m \ddot{y}.
$$

According to the variational principle for nonconservative systems,

$$
\int_0^T \delta L(\Gamma, \dot{\Gamma}) dt = \int_0^T F(\Gamma, \dot{\Gamma}) s(\Gamma) dt
$$

(6)

for curves $\Gamma : [0, T] \rightarrow \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3$ with fixed end points: $\Gamma(0) = \Gamma_0$ and $\Gamma(T) = \Gamma_T$, and for generalized forces along the curves given by $F(\Gamma, \dot{\Gamma})$, yields the equations of motion. The RHS of (6) is the time integral of the virtual work over the interval $[0, T]$. The details of this derivation are found in Nayak et al. [10].
Denoting the vector \( z := Re_3 \), the equations of motion are the following:
\[
-(m + M)x - ml\dot{y} + (M + m)ge_3 = -fz. \tag{7}
\]
\[
\|\dot{\omega} + (\omega \times \|\omega\|) = -\mu \tag{8}
\]
\[
y + \|\dot{y}\|^2\dot{y} = \frac{f}{M_l}(\dot{y})^2z \tag{9}
\]

3 | CONTROL STRATEGY

The objective is to choose \( \mu \) and \( f \) so as to stabilize the pendulum in the inverted upright position and position the quadrotor plane parallel to the ground. Mathematically stated, the first and second requirements translate to
\[
\lim_{t \to \infty} y(t) = e_3, \quad \lim_{t \to \infty} z(t) = \lim_{t \to \infty} R(t)e_3 = e_3.
\]

From the definition of \( z \) and differentiating twice, we have \( \ddot{z} = R\dot{\omega}^2 + \ddot{\omega} \dot{e}_3 \). The dynamical equations for the purpose of control design, with \( \mu \) and \( f \) being the control variables, are as follows:

\[
\dot{R} = R\dot{\omega}, \quad \|\dot{\omega} + (\omega \times \|\omega\|) = -\mu
\]

\[
\ddot{z} = R(\dot{\omega}^2 + \{\|^{-1}(\omega \times \omega) - \|^{-1}\mu\})e_3 \quad \text{Quad Eqn} \tag{10}
\]

\[
y + \|\dot{y}\|^2\dot{y} = \frac{f}{M_l}(\dot{y})^2z \quad \text{Pendulum Eqn} \tag{11}
\]

Assumption 1. The controller has full access to the state at all times using appropriate sensors on board the quadrotor.

3.1 | Control on the 2–sphere

Both the subsystems to be controlled, namely the \( z \) and the \( y \) variables, evolve on a two-dimensional sphere (or the 2-sphere). We first present a few preliminaries of control on a sphere.

Definition 3.1. A fully actuated simple mechanical system (an SMS) on the 2–sphere denoted by \( S^2 \) is specified by the 3-tuple \( (S^2, I_3, u) \) where \( I_3 \) is the Euclidean metric on \( S^2 \) and \( u \in \mathbb{R}^3 \) is the control vector. The equations for the controlled SMS \( (S^2, I_3, u) \) are as follows.

\[
\nabla_\phi \dot{\phi} := \ddot{\phi}(t) + \|\dot{\phi}\|^2\dot{\phi}(t) = -(\dot{\phi})^2u \tag{12}
\]

where \( \phi(t) \in S^2 \) is the controlled trajectory and \( V \) is the affine connection corresponding to the Euclidean metric.

Definition 3.2. The SMS \( (S^2, I_3, u) \) is said to be fully actuated if the control forces \( u \) generate the cotangent bundle \( T^*S^2 \).

We now state two useful lemmas for asymptotic stabilization about a set point and asymptotic tracking of a reference trajectory for an SMS on \( S^2 \).

Lemma 3.1. Regulation on \( S^2 \). Consider the fully actuated SMS in (12). The following control law ensures that \( \phi(t) \) is asymptotically stable about \( e_3 \in S^2 \):

\[
u = -k_p\ddot{\phi}e_3 - k_d\dot{\phi}
\]

Proof. Lemma 11.7 in Bullo [1]

Lemma 3.2. Tracking a trajectory on \( S^2 \). Consider the fully actuated SMS in (12). The following control law ensures that \( \phi(t) \) asymptotically tracks a smooth and bounded reference trajectory \( \phi_d(t) \in S^2 \)

\[
u = -k_p\ddot{\phi}e_3 - k_d\dot{\phi} + V_\phi(\tau(\phi, \phi_d)\dot{\phi}_d)
\]

where the velocity error \( \nu_v \in T_\phi S^2 \) is

\[
u_v := \dot{\phi} - \tau(\phi, \phi_d)\dot{\phi}_d
\]

the transport map \( \tau(\phi, \phi_d) : T_\phi S^2 \rightarrow T_\phi S^2 \) is

\[
\tau(\phi, \phi_d)\dot{\phi}_d := (\phi_d \times \dot{\phi}_d) \times \phi
\]

and \( V_\phi(\tau(\phi, \phi_d)\dot{\phi}_d) \) is the feedforward part of the control which simplifies as follows:

\[
V_\phi(\tau(\phi, \phi_d)\dot{\phi}_d) = \langle \phi, \phi_d \times \dot{\phi}_d \rangle (\phi \times \dot{\phi}) + (\phi_d \times \dot{\phi}) \times \phi
\]

Proof. Section 11.3.2 in Bullo [1]

3.2 | Approach to controller design

Feedback regularization [8] refers to the use of feedback to impart the structure of an SMS to a fully actuated mechanical system. Once a system is feedback regularized, a straightforward proportional derivative (PD) control action could then be employed to stabilize the system. It is observed from (11) that the only control variable is \( f \) which is one dimensional. The cotangent bundle of \( S^2 \) can be generated by at least three independent covector fields [9]. Therefore, in order to attain an arbitrary configuration on \( S^2 \), at least three independent directions of control are necessary. From Definition 3.2, it means that the system described by Equation (11) is not fully actuated. Therefore, feedback regularization cannot be utilized to stabilize \( y(t) \) about \( e_3 \).
It is also observed that the output $z(t)$ affects the acceleration of the pendulum. Therefore, if we show that the set of Equations (11) and (10) are in strict-feedback form [3], a backstepping control can be used to stabilize $y(t)$ by choosing an appropriate $z(t)$ as an intermediate control for the Equation (11). Since $\mu \in \mathbb{R}^3$ appears in (10), therefore, there are three independent control directions, and (10) is fully actuated. Therefore, $z(t)$ can reach an arbitrary configuration on $S^2$ using feedback regularization.

**Philosophy of control design:** Consider the two Equations (10) and (11).

- **Pendulum stabilization:** In order to apply a backstepping technique to this system of equations, we first choose a desired vector $z_d(y(t), y'(t)) \in S^2$ which acts as the feedback control to the pendulum equation so that $y(t)$ is asymptotically stable about $e_3$. To do so, we define an intermediate control variable $f_p$ as follows:
  \[
  f_p := k_p e_3 + k_d y', \quad z_d(y, y') = \frac{f_p}{\|f_p\|},
  \]
  and set the control $f$ as
  \[
  f := -Ml \frac{f_p}{\|f_p\|},
  \]
  which renders
  \[
  \ddot{y} + \|y\|^2 y = -(\dot{y})^2 f_p = -((\dot{y})^2 k_p e_3 + k_d y')
  \]

- **Quadrotor stabilization:** In the next step, we wish to make the quadrotor equation track the trajectory $z_d(t)$. The choice of $z_d(y(t), y'(t))$ is to be made so that
  1. The feedback control for stabilization of an SMS (given by 13 in Lemma 3.1) is introduced through for stabilization of $y(t)$ at $e_3$.
  2. $z_d(e_3, 0) = e_3$ so that $z(t)$ is simultaneously stabilized about $e_3$ along when $(y(t), y'(t)) = (e_3, 0)$.

  The choice of $\mu$ in (10) is to be made such that the error variable $(z - z_d(t))$ is driven to zero in two steps. In the first step, the structure of an SMS on $S^2$ is imparted to (10), and in the second, the tracking control in Lemma 3.2 is employed so that $z(t)$ asymptotically tracks the previously chosen $z_d(t)$.

  $k_p$ is chosen such that $\dot{x} = 0$ after the control objective is achieved.

### 3.3 Main result

In the following theorem, using appropriate Lyapunov functions, we show that the system of Equations (11) and (10) is asymptotically stable about $(y(t), y'(t), z(t), \dot{z}(t)) = (e_3, 0, e_3, 0)$ for a suitable choice of $z_d(t)$, $\mu$ and $f$.

**Theorem 1.** The following control thrust $f$ and moment $\mu$ ensures that $\lim_{t \to \infty} y(t) = e_3$ and $\lim_{t \to \infty} z(t) = e_3$

\[
\begin{align*}
  f & := -Ml \frac{f_p}{\|f_p\|} \quad \text{and},
  \\
  \langle z^{-1} \mu \rangle e_3 & := (\omega^2 + \langle \omega^2 \rangle) e_3 + R^T ||z||^2 (z - u_{fb}),
\end{align*}
\]

where

\[
\begin{align*}
  f_p & := k_p e_3 + k_d y', \\
  k_p & = \frac{(M + m)g}{Ml \langle e_3 \rangle},
  \\
  u_{fb} & = -k_1 (\dot{z})^2 z_d - k_2 v_e + \nabla \dot{z} (\tau(z, z_d) \dot{z}_d) - (\dot{z})^2 \beta,
  \\
  \tau(z, z_d) \dot{z}_d & := (z_d \times \dot{z}_d) \times z v_e := \dot{z} - \tau(z, z_d) \dot{z}_d,
  \\
  z_d(y, y') & = \frac{f_p}{\|f_p\|},
\end{align*}
\]

$k_1, k_2, k_d$ are positive constants, and $\beta(t) \in \mathbb{R}^3$ is defined as

\[
\beta = -\frac{f_p}{\|f_p\|} A^T (z - z_d)
\]

and is the minimum norm solution to $Av_e = \dot{y}$, where $A : T_z S^2 \to T_y S^2$.

**Proof.** Denote the desired equilibrium of (11) $(e_3, 0) = (y^*, 0)$. On substituting $f$ from (15) and $z = z_d$ from (20) in (11), we obtain

\[
\ddot{y} + \|y\|^2 y = -k_p (\dot{y})^2 e_3 - k_d y
\]

as $(\dot{y})^2 \dot{y} = \dot{y}$. From Lemma 3.1, it is observed that 22 is asymptotically stable about $(y^*, 0)$ for all $t$ such that $z(t) = z_d(t)$. Subtracting $f / Ml \langle \dot{y} \rangle^2 z_d$ to both sides of (11) yields

\[
\nabla \dot{y} = -k_p (\dot{y})^2 e_3 - k_d y + \frac{f}{Ml} \langle \dot{y} \rangle^2 (z - z_d).
\]

Consider the Lyapunov function for the pendulum Equation (23) $V_1 = k_p (1 - \dot{y}^2 e_3) + 1/2 \|\dot{y}\|^2$. Therefore,
Next we look at the quadrotor subsystem in (10). The desired equilibrium is \( z_d^* = e_3 \) and by choice, \( z_d(y^*, 0) = z_d^* \). Therefore, both the dynamical Equations (10) and (11) attain their respective equilibria simultaneously. We choose \( \mu \) so that \( z(t) \) tracks \( z_d(t) \). Substituting
\|^{-1} \mu \| e_3 \text{ from (16) in (10)},

\dot{z} = -\| \dot{z} \| z + u_f, \text{ which means } \nabla \dot{z} = u_f. \quad (24)

Since \( z(t) \in S^2 \), from Lemma 3.2, we know that the feedback control that must be introduced to track \( z_d(t) \) is \( u_f \) defined in (19). The Lyapunov function for the \( z \) subsystem is chosen as follows:

\[ V_2 = k_1(1 - z^T z_d) + \frac{1}{2} \| \dot{z} \| - \tau(z, z_d) \dot{z}_d^2 \]

\[ = k_1(1 - z^T z_d) + \frac{1}{2} \| v_e \|^2 \]

The transport map \( \tau(z, z_d) : T_z S^2 \rightarrow T_{z_d} S^2 \) is compatible with the potential function \( k_1(1 - z^T z_d) \) (as defined in Theorem 11.19 in Bullo [1]); therefore, \( \frac{d}{dt} k_1 \)

\[ \frac{d}{dt} V_2 = \langle k_1(\dot{z})^2 z_d, v_e \rangle + \langle \nabla \dot{z}, v_e \rangle \]

\[ = \langle k_1(\dot{z})^2 z_d, v_e \rangle + \langle \nabla \dot{z} - \tau(z, z_d) \dot{z}_d, v_e \rangle \]

\[ = \langle k_1(\dot{z})^2 z_d, v_e \rangle + \langle \nabla \dot{z}, v_e \rangle - \langle \nabla \dot{z} \tau(z, z_d) \dot{z}_d, v_e \rangle \]

\[ = -\langle k_2 v_e, v_e \rangle - \langle \dot{z}^2 \beta, v_e \rangle. \quad (25) \]

Define a Lyapunov function \( V := V_1(y, \dot{y}) + V_2(z, \dot{z}, z_d, \dot{z}_d) \) for the entire system of Equations (22–24), with \( u_f \) defined in (19). By choosing \( \beta \) according to (21), we ensure that the total Lyapunov function is nonincreasing, as follows.

\[ \frac{d}{dt} V = \frac{d}{dt} V_1 + V_2 \leq \langle k_2 \dot{y}, \dot{y} \rangle - \langle k_2 v_e, v_e \rangle \leq 0 \]

Remark 1. In the expression for \( \beta \) in (19), \( A : T_z S^2 \rightarrow T_y S^2 \) is a transport map. \( \beta \) is introduced as a

FIGURE 5  Swing up for fourth set of initial conditions

FIGURE 6  Swing up for fifth set of initial conditions
feedback force to cancel the effect of the error between desired trajectory \( z_d(t) \) and the state trajectory \( z(t) \), which appears as \( f/Ml(\dot{y})^2(z - z_d) \) in (23). Therefore, \( \beta(t) \) vanishes for all \( t \) such that \( z(t) = z_d(t) \).

### 3.4 Zero dynamics

In the above section, we use feedback control for the output \( z(t) \) and only one state \( y(t) \) of the system (7–11) by eliminating the other states from the dynamics of \( y \) and \( z \) variables. To understand the zero dynamics, we take a look at the effect of the applied control action on the other states \( x(t) \) and \( R(t) \). At the steady state, \((y(t), \dot{y}(t), z(t), \dot{z}(t)) = (e_3, 0, e_3, 0) \) and \( k_p = (M + m)g/Ml \); therefore, \( v = k_p e_3 \), \( \|v\| = k_p \), \( f = -Ml \), and the controlled trajectory \( x(t) \) evolves as follows:
\[ x = g e_3 - \frac{M l_p}{m + M} e_3 = g e_3 - g e_3 = 0. \]  

(26)

4 | NUMERICAL EXPERIMENTS

A quadrotor is considered with

\[ I = diag([0.0820; 0.0845; 0.1377]) kgm^2 \] and \( M = 0.4 kg. \)

The pendulum has

\[ m = 0.1 m \text{ and } l = 0.5 m. \]

The initial conditions for orientation and position of the quad are

\[ R(0) = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0 \\ 0.48 & 0.64 & 0.60 \end{pmatrix}, \quad \omega(0) = \begin{pmatrix} 0.8 \\ -0.3 \\ 0.5 \end{pmatrix}, \]

\[ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \dot{x}(0) = \begin{pmatrix} 2 \\ 1.5 \end{pmatrix}. \]

The controller gains in (19) and (20) are denoted by \( K = (k_d \quad k_1 \quad k_2) \). The \( \beta \) in (19) is computed using “lsqminnorm” routine in MATLAB for all the experiments. As both \( z(t) \) and \( y(t) \) are in \( S^2 \), we plot the last coordinates which is shown to approach 1 asymptotically thereby showing that both \( z(t) \) and \( y(t) \) approach \( e_3 \) asymptotically. All the experiments are performed for the time \( t \in [0s, 6.5s] \). The initial bob positions, velocities, and gain matrices \( K \) are varied as follows:

| Experiment no. | \( y(0) \) | \( \cdots y(0) \) | \( K \) | Figure number |
|----------------|---------|-----------------|------|--------------|
| 1              | \( 1/\sqrt{2} \) | 0.5 0 8 4       | Figure 2 |
| 2              | \( 1/\sqrt{2} \) | -0.5 4 9 4     | Figure 3 |
| 3              | \( 1/\sqrt{2} \) | 0.7 9 4 4     | Figure 4 |
| 4              | \( 1/\sqrt{2} \) | 0.1 3 1 1     | Figure 5 |
| 5              | \( 1/\sqrt{2} \) | 0.7 1 1 1     | Figure 6 |

Figures 7–9 show the pivot and the bob of the pendulum in \( S^2 \) frame in stop motion for Experiments 2 and 4.

AUTHOR CONTRIBUTIONS

Aradhana Nayak: Conceptualization, formal analysis, investigation, methodology, validation. Ravi N. Banavar: Conceptualization, funding acquisition, supervision, validation. D.H.S. Maithripala: Conceptualization, investigation, supervision, validation.

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**How to cite this article**: A. Nayak, R. N. Banavar, and D. H. S. Maithripala, *Stabilizing a spherical pendulum on a quadrotor*, Asian J Control **24**(2022), 1112–1121. https://doi.org/10.1002/asjc.2577