Mathematical model of ships convoy in delivery of goods

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Abstract. This paper describes the ship convoys intended for mathematically modelling. The phenomenon essentially imitates the behaviour of a shoal of fish for several advantages. However, a control design is investigated to ascertain the factors influencing the fleet to navigate jointly without diversion or collision. Furthermore, the vessels are assumed to function independently, but are only assigned a common task of delivering goods. The control design process employs an optimal method known as the Pontryagin Maximum Principle, and the model implemented is generally applied to marine vessels. Meanwhile, the control equation utilizes the Nomoto’s model. In addition, the ship model supports the rudder input responsible for the change in the direction of motion. Also, this study exposes a functional cost pattern describing the joint tasks and further models the ship convoys for optimum control. The numerical simulation provides a detailed summary of the applied optimal process to solve the problems encountered in deliveries.

1. Introduction
This paper begins with an explanation of the meaning of a convoy. A convoy is a group of vehicles that move together for mutual support and protection. It is interesting to expose ships convoy, especially its control design for each ship.

The recent publications about multi ships for example, can be found in the following references. In [1] the authors designed the control for multi ship through Newton and Quasi Newton. The multi boat for sea surveillance is exposed in [2]. The multi grid to simulate crowd’s evacuation of the ship is discussed in [3]. Disturbance Rejection is used in [4] to assign inter-vehicle spacing in a convoy. The path tracking for unmanned ships is described in [5]. The biomimetic AUV for multi-vehicle cooperation is discussed in [6]. The problem optimization in multi-modal transportation network is described in [7]. The naval convoy operations is exposed in [8]. If multi boats in the accident, search and rescue boats is considered in [9]. While, the cooperative control among the vehicles is exposed in [10]. Next, the system for multi-role mission ships is exposed in paper [11]. While, the multi-ship collision avoidance with iterative observation and inference can be found in [12], [13]. After that, a multi-ship following model for icebreaker discussed in [14]. An improved time discretized non-linear velocity obstacle method for multi-ship encounter detection can be found in [15]. Next, the multi-ship following operation in ice-covered waters with consideration of inter-ship communication is published in [16].

The next question is why the ships move in convoy? The example of publication that exposes the reason for the ships move in convoy is reference [17]. The reason for maritime security is one of the reasons for ship convoys. Naval convoy is a traditional maritime security system which has been used for a long time. In a convoy naval, ships that are vulnerable to being attack when sailing the ocean will sail in groups to provide mutual protection. If sailing alone the ship will easily be captured or destroyed.
by the opponent, but if sailing together, the opponent will have difficulty targeting one specific ship and is less likely to destroy all ships in the convoy. Ships that sail in convoys will also be easier to control and are escorted by armed warships in smaller numbers.

After considering the newest research in multi ships and the benefit of the ships convoy, the author tries to explain what is the novelty in this paper? The novelty of this paper can be explained as follows. This paper uses optimal control to model the convoy of the ships. The movement of each ship is described in a dynamical model in which the convoy form a system of nonlinear differential equations. The duty of the ship convoy described in the objective cost functional model. The other novelty in this paper is the ship's linkage one to the other, just in the cost functional. A ship cannot influence the other ships. The role of each ship is equal.

2. Problem statement
Delivery of goods with large cargo vessels carrying many goods at once, creates problems if the destination does not have a deep port. This problem was resolved by a small ship's convoy, which together moved to carry goods. The convoy control model of the ship carrying this item uses the optimal control approach.

In other words, optimal control approach is used to solve the main problem of this paper i.e. controlling the convoy of the ships. In the optimal control problem, there are two important models. The first is the ship dynamical model, and the second one is the cost functional model. As an optimal control problem, which in this case controlling multi ships is how to solve the optimal control with dynamical model represent ship mathematical model and the cost functional that must be minimized. The cost functional describes the convoy of ships. The type of optimal control problem used in this paper, is optimal control problem with fixed time and fixed end point.

3. Method for solving the problem and Mathematical Model
The motion equation of the \(i\)th ship follows the general model of vehicle in the form of a nonlinear differential equation system as in reference [18].

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = 
\begin{bmatrix}
\cos \Psi \cos \Theta & -\sin \Psi \cos \Phi + \cos \Psi \sin \Theta \sin \Phi & \sin \Psi \cos \Phi + \cos \Psi \cos \Phi \sin \Theta \\
\sin \Psi \cos \Theta & \cos \Psi \cos \Phi + \sin \Psi \sin \Theta \sin \Phi & -\cos \Psi \sin \Phi + \sin \Psi \sin \Phi \cos \Theta \\
-\sin \Theta & \cos \Theta \sin \Theta & \cos \Theta \cos \Phi
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
\]  \( (1.a) \)

\[
\begin{bmatrix}
\dot{r} \\
\dot{\Theta} \\
\dot{\Psi}
\end{bmatrix} = 
\begin{bmatrix}
0 & \sin \Phi \tan \Theta & \cos \Phi \tan \Theta \\
0 & \sin \Phi & -\sin \Phi \\
0 & \sin \Phi / \cos \Theta & \cos \Phi / \cos \Theta
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]  \( (1.b) \)

The input or control of the ship, follows the equation from [19]

\[
\dot{r}(t) = -K \ast r(t) + \delta(t).
\]  \( (1.c) \)

Consider (1.a) and (1.b), the ship moves in the plane \((z=0)\) and by assuming that in the absence of the roll and pitch modes \((\Theta=\Phi=0)\), and the derivative \(u\) with respect to \(t\) and the derivative \(v\) with respect \(t\), in this paper, are assumed as constant function then the states variables of the ship are four \((x, y, \Psi, r)\) and the input or control is \(\delta(t)\). Consider (1.c), \(K\) is a constant. The physical description of the ships model is given as follows, \((x(t), y(t))\) is the position of the ship, \(\Psi(t)\) denotes the heading angle of the ship, next \(r(t)\) and \(\delta(t)\) are the yaw rate dan rudder angle of the ship, respectively. While \(u(t)\) and \(v(t)\) are surge dan sway of the ship, respectively. The input or control of the ship is the ruder of the ship.

The motion model of a ship follows equation (1.a) \(- (1.c)\), while in the convoy the ship will be worked on for \(n\) ships moving together. So if in one ship there are four state variables and one input, then in a convoy involving \(n\) ships, there are \(4n\) state variables. Because the method used is optimum control, the variables involved are \(8n\) variables. In convoys, ships may not move away from each other and collide with each other. According to the mathematical model, the ships moving in this convoy are expressed in the model of common duty.
The common duty of the ships, moving together to carry goods in the convoy, modeled by the cost functional. The cost functional which is used to model the common duty in this paper is

\[
J = \frac{1}{2} \int_0^T \left( \alpha \sum_{i=1}^k \delta_i^2 + \beta Q + \frac{\rho}{q} \right) dt
\]

Consider cost functional equation (2), the cost functional includes three summand. The first is cost control, the second is attractor term and the last one is the repellant term. The first term describes control for each ship that make the direction of ship heading is changing. The second term forces the ships do not move far each other. The last term, forces the ships do not collide one to the other. In (2), \(\alpha, \beta\) and \(\rho\) are constant. The quadratic distance between two ships denoted by \(Q\).

The cost functional besides has relation with the common duty of the ships also has a relation with the existence of the optimal control solution. The convexity of the cost function guarantees this existence of the optimal control solution. The existence of the optimal control problem, formally given in the following theorem.

**Theorem 3.1**

*The optimal control problem with cost functional (2) exists*

**Proof:**

Consider that the cost functional (2), since each summand is convex then the cost functional \(J\) is also convex. Then the existence of the optimal control is guaranteed. 

4. **Convergence analysis**

Since the main problem in this paper is solved by optimal control, especially the optimal control approach with fixed time and fixed endpoint, the values for costate variables are needed. If the determination of initial value for costate variables uses the trial and error method, then the values for state variables, in the final time, maybe very different from determining values of state variables. The difficulty in the optimal control approach is the determination of the initial value of the costate variables. From the initial guess for initial values of costate, the value of state variables in the final time is obtained. Next, the computed value for state variables in the final time and the final values are compared. If the error value is less than or equal to the specified error value, the proof is complete. If the error value is still greater, then improvement is made from the value for the initial conditions of the costate variables. The repeated processes are done, until the minimum error obtained or satisfy the determined error. The convergence analysis describes the values from the guess value of the initial condition for the costate variables to the suitable value of the costate variables. From any initial values for costate variables can give value for state variables in the final time that may be different from determining final state value.

The convergence analysis of the sequence values for initial values for costate variables given as follows. The sequence of the values goes from an initial guess value to the suitable initial condition for the costate variables. To make the proof simpler, denote the system of differential equation with \([x\ lambda]\). Symbol \(x\) denotes state variables and \(\lambda\) denotes costate variables. The system of the differential equation which contains \([x\ lambda]\) assumed satisfy Lipschitz condition. The initial and final condition for the state variables \(x\) denoted by \(x_0\) and \(x_f\) respectively. There are no requirements for the costate variables or \(\lambda\). First, take \(\lambda\) null for the initial condition for \(\lambda\) as a guess. View the problem as an initial value problem with \(x_0\) and \(\lambda_0\) null as the initial condition. From this initial value problem, we obtain computed \(x\)-final or we denote it as \(\text{cx}\). Then compute \(|\text{cx}_{i-1} - x_f|\). Next, we change the value of \(\lambda\) null with \(\lambda\) null _1_. The process determination of \(\lambda\) null to \(\lambda\) null _1_ uses a finite difference approach. This approach can be done for this problem, because the system of differential equations satisfies Lipschitz condition. and with a similar process, we have \(\text{cx}_{i-1}\). With the same process, but we apply a condition for \(\lambda\) null _2_, then \(\text{cx}_{i-2}\) will be obtained. We do the process repeatedly, so we obtained sequence \{\(\lambda\) null _1_\} and sequence \{\(\text{cx}_{i-1}\)\}. If the sequence \{\(\text{cx}_{i-1}\)\} converges to some \(\text{cx}\) that makes \(|\text{cx} - x_f|\)→0 then \{\(\lambda\) null _i_\}→suitable initial condition for costate variables. The process can be done because the system of differential equation satisfies Lipschitz condition so the process of changing the initial value for co-state variables...
successfully done and the sequence of the values converges to the suitable initial condition for costate that make the computed final state value converges to the final state value in determined time. The proof of convergence end.

5. Simulation results
The simulation scenario is given as follows. The ships move from their initial position and orientation. In a certain time, the ships must arrive in certain positions and orientations.

![Figure 1](image1.png)

**Figure 1.** values of each state variables versus time.

The given values for the state variables $\Psi, r, x$ and $y$ in the initial and final time are plotted in Figure 1. With the values of the costate variable that earned, the optimal path of the ships can be plotted in Figure 2. During the movement the ships must move together in groups in convoys to carry goods, must not stay away and may not collide with each other. One ship, does not affect other ships. The ships were only bound to one another by a common goal of moving together to carry goods in the convoy.
Figure 2. Optimal path of each ship.

Simulation made to show that the method succesfully to control the convoy. The main simulation result gives in Figure 2.

6. Conclusion
From the discussion above, it can be concluded that the design of control for each cargo ship conducting a convoy of ships was successfully carried out. Ships sending goods are modelled with general models of vehicles and equations for ship control in the form of rudder ships following the Nomoto’s model. Furthermore, with the optimum control method approach, the problem of controlling ship convoys is worked on. The ships carrying out the convoy managed to move without moving away from each other and not colliding with each other. In the simulation, a convoy of five moving ships was successfully controlled. Each ship with four state variables, so with optimal control, each ship has eight variables and because it is worked on for five ships, then the number of variables is 40 variables. The optimal path results for each ship are obtained and reported in the simulation results. In the next, problem as a suggest for the other author, the convoy can made avoid the avoidance.

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