Algebro-Geometric Solutions and Their Reductions for the Fokas-Lenells Hierarchy

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Abstract

This paper is dedicated to provide theta function representations of algebro-geometric solutions for the Fokas-Lenells (FL) hierarchy through studying an algebro-geometric initial value problem. Further, we reduce these solutions into \( n \)-dark solutions through the degeneration of associated Riemann surfaces.

1 Introduction

In the past few decades, the celebrated nonlinear Schrödinger (NLS) equation has been widely studied in various aspects. In ref. [5], Fokas proposed an integrable generalization of the NLS equation,

\[
iv_t - \nu u_{tx} + \gamma u_{xx} + \sigma |u|^2 (u + ivu_x) = 0, \quad \sigma = \pm 1, \quad x, t \in \mathbb{R}, t > 0, \quad \nu, \gamma, \rho \equiv \text{constant} \in \mathbb{R},
\]

(1.1)

which is known as Fokas equation later, using bi-Hamiltonian methods. Just like the bi-Hamiltonian structure of the well-known Korteweg-de Vries equation can be perturbed to yield the integrable Camassa-Holm equation, the same mathematical trick applied to the two Hamiltonian operators associated with the NLS equation yields the Fokas equation. Under the simple transformation

\[
u \rightarrow \beta \sqrt{\alpha} e^{i\beta x} u, \quad \alpha = \gamma / \nu, \quad \beta = 1 / \nu, \quad \sigma = -\sigma,
\]

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the Fokas equation (1.1) changes to the Forkas-Lenells (FL) equation

\[
\dot{u} + \alpha \beta^2 u - 2i \alpha \beta u_x - \alpha u_{xx} + \sigma i \alpha \beta^2 |u|^2 u_x = 0, \quad \sigma = \pm 1. \tag{1.2}
\]

In the context of nonlinear optics, the FL equation models the propagation of nonlinear light pulses in monomode optical fibers when certain higher-order nonlinear effects are taken into account [16]. In contrast to the case of the NLS equation which comes in a focusing as well as in a defocusing version depending on the values of the parameters, and solitons only exist in the focusing regime, all versions of equation (1.2) are mathematically equivalent up to a change of variables. The transformation

\[
u \rightarrow \sqrt{ab} e^{i(bx + 2abt)} u, \quad a = \gamma / \nu > 0, \quad \xi = x + at, \quad \tau = -ab^2 t.
\]

transforms the Fokas equation (1.1) into

\[
\dot{u} + \sigma |u|^2 u_x = 0, \quad \sigma = \pm 1, \tag{1.3}
\]

the so-called Forkas-Lenells derivative nonlinear Schrödinger equation (FDNS) in some references [22, 25]. An important feature of equation (1.3) is that it describes the first negative flow of the integrable hierarchy associated with the derivative nonlinear Schrödinger (DNLS) equation [5, 12, 13, 16].

In this paper, we start from the following coupled form

\[
\begin{align*}
q_{xt} - q_{xx} + iqq_x r - 2i q_x + q &= 0, \\
r_{xt} - r_{xx} - iqr r_x + 2i r_x + r &= 0,
\end{align*} \tag{1.4}
\]

which are exactly reduced to the FL equation (\(\alpha = \beta = 1\) in (1.2))

\[
q_{xt} - q_{xx} \mp i |q|^2 q_x - 2i q_x + q = 0, \tag{1.5}
\]

for \(r = \pm 7\). Related results can also be directly applied to (1.1) and (1.3) since the existence of these simple transformations among them. It is shown that (1.4) / (1.5) is a completely integrable nonlinear partial differential equation possessing Lax pair, bi-Hamiltonian structure, and soliton solutions [3, 16, 17, 18]. One of the most remarkable feature of the FL equation is that it possesses various kinds of exact solutions such as solitons, breathers, etc.. The bright solitons under vanishing boundary condition have been constructed by inverse scattering transform (IST) method [17], dressing method [18] and Hirota method [21]. The lattice representation and the
$n$-dark solitons of the FL equation have been presented in [25], where a relationship is also established between the FL equation and other integrable models including the NLS equation, the Merola-Ragnisco-Tu equations and the Ablowitz-Ladik equation. In [22], the author has dealt with a sophisticated problem on the dark soliton solutions with a plane wave boundary condition using Hirota method. The breather solutions of the FL equation have also been constructed via a dressing-Bäcklund transformation related to the Riemann-Hilbert problem formulation of the inverse scattering theory [26]. Recently, the authors of [8] has investigated $n$-order rogue waves solutions of FL equation using Darboux transformation method.

The algebro-geometric solution, parameterized by compact Riemann surface of finite genus, is a kind of important solutions in soliton theory. This kind of solutions was originally studied on the KdV equation based on the inverse spectral theory, developed by pioneers such as the authors in [1, 3, 4, 11, 15, 19, 23] and further developed by the authors in [2, 6, 14, 20], etc. In a degenerated case of the algebro-geometric solution, the multi-soliton solution and periodic solution in elliptic function type may be obtained [3].

The purpose of this paper is to analyze the quasi-periodic solutions and dark soliton solutions of the FL hierarchy using the algebro-geometric method [7]. This systematic approach, proposed by Gesztesy and Holden to construct algebro-geometric solutions for integrable equations, has been extended to the whole (1+1) dimensional integrable hierarchy, such as the AKNS hierarchy, the CH hierarchy etc. Recently, we investigated algebro-geometric solutions for the the Degasperis-Procesi hierarchy and Hunter-Saxton hierarchy [9, 10] using this method.

In the present paper, we consider a Cauchy problem (4.1), (4.2) of FL hierarchy with a quasi-periodic initial condition $q, r$ (cf. (3.71), (3.72)) and search for its exact solutions. We will prove the solution of this cauchy problem is unique (cf. Lemma 4.3) and give the explicit form of $q, r$ (cf. Theorem 4.6). We also find that the quasi-periodic solutions obtained in Theorem 4.6 can be linked with the dark solitons of FL hierarchy. Especially, for the FL equation (1.4)/(1.5), the results of [25] about the $n$-dark solitons can be obtained from a different standpoint. As shown in [21, 22], the bright solitons and dark solitons correspond to the vanishing boundary condition and non-vanishing boundary condition, respectively. Hence the authors are confident that there exists another kind of quasi-periodic solutions which may degenerate to the bright solitons. Obviously, this depends on what kinds of Cauchy problem we will investigate.

This paper is organized as follows. In section 2, we construct the FL hierarchy and derive its algebro-geometric solutions. In section 3, we prove the quasi-periodic solutions obtained in this paper are unique and give the explicit form of them. Finally, in section 4, we investigate the dark soliton solutions of FL hierarchy using the algebro-geometric method.
hierarchy using a zero-curvature approach and a polynomial recursion formalism. Moreover, the hyperelliptic curve \( \mathcal{K}_n \) of genus \( n \) associated with the FL zero-curvature pairs is introduced with the help of the characteristic polynomial of Lax matrix \( V_n \) for the stationary FL hierarchy. In section 3, we treat the stationary FL hierarchy and its quasi-periodic solutions. Using these stationary quasi-periodic solutions as initial values, we solve the Cauchy problem and obtain the quasi-periodic solutions of FL hierarchy. In section 4, In section 5, we consider the soliton limit of these quasi-periodic solutions given in section 4 and finally derive the \( n \)-dark solitons of the FL hierarchy.

2 The Fokas-Lenells Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section, we provide the construction of FL hierarchy and derive the corresponding sequence of zero-curvature pairs using a polynomial recursion formalism. Moreover, we introduce the underlying hyperelliptic curve in connection with the stationary FL hierarchy.

Throughout this section, we make the following hypothesis.

**Hypothesis 2.1.** In the stationary case we assume that

\[
q, r \in C^\infty(\mathbb{R}), \quad q(x), q_x(x), r_x(x) \neq 0, \quad x \in \mathbb{R}. \tag{2.1}
\]

In the time-dependent case we suppose

\[
q(\cdot, t), r(\cdot, t) \in C^\infty(\mathbb{R}), \quad t \in \mathbb{R}, \quad q(x, \cdot), r(x, \cdot) \in C^1(\mathbb{R}), \quad x \in \mathbb{R},
q(x, t), q_x(x, t), r_x(x, t) \neq 0, \quad (x, t) \in \mathbb{R}^2. \tag{2.2}
\]

We first introduce the basic polynomial recursion formalism. Define \( \{f_{\ell, \pm}\}_{\ell \in \mathbb{N}_0}, \{g_{\ell, \pm}\}_{\ell \in \mathbb{N}_0} \) and \( \{h_{\ell, \pm}\}_{\ell \in \mathbb{N}_0} \) recursively by

\[
g_{0, +} = -1, \quad h_{0, +} = r_x, \quad f_{0, +} = -q_x, \tag{2.3}
\]

\[
ig_{\ell, +, x} = r_x f_{\ell, +} + q_x h_{\ell, +}, \quad \ell \in \mathbb{N}_0, \tag{2.4}
\]

\[
f_{\ell, +, x} = 2iq_x g_{\ell+1, +} - 2if_{\ell+1, +}, \quad \ell \in \mathbb{N}_0, \tag{2.5}
\]

\[
h_{\ell, +, x} = 2ih_{\ell+1, +} + 2ir_x g_{\ell+1, +}, \quad \ell \in \mathbb{N}_0. \tag{2.6}
\]
and
\[ g_{0,-} = -1/4, \quad h_{0,-} = -ir/2, \quad f_{0,-} = -iq/2, \quad (2.7) \]
\[ ig_{\ell+1,-,x} = q_x h_{\ell,-} + r_x f_{\ell,-}, \quad \ell \in \mathbb{N}_0, \quad (2.8) \]
\[ f_{\ell+1,-,x} = 2iq_x g_{\ell+1,-} - 2if_{\ell,-}, \quad \ell \in \mathbb{N}_0, \quad (2.9) \]
\[ h_{\ell+1,-,x} = 2ih_{\ell,-} + 2ir_x g_{\ell+1,-}, \quad \ell \in \mathbb{N}_0, \quad i = \sqrt{-1}, \quad (2.10) \]

where \( f_{\ell,\pm,x}, g_{\ell,\pm,x} \) and \( h_{\ell,\pm,x}, \ell \in \mathbb{N}_0 \), denote the derivative of \( f_{\ell,\pm}, g_{\ell,\pm}, h_{\ell,\pm} \) with respect to the space variable \( x \), respectively. Explicitly, one obtains
\[ g_{0,+} = -1, \]
\[ g_{1,+} = \frac{1}{2} q_x r_x - c_{1,+}, \]
\[ f_{0,+} = -q_x, \]
\[ f_{1,+} = \frac{1}{2i} q_{xx} - \frac{1}{2} q_x^2 r_x - c_{1,+} g_x, \]
\[ h_{0,+} = r_x, \]
\[ h_{1,+} = \frac{1}{2i} r_{xx} + \frac{1}{2} q_x r_x^2 + c_{1,+} r_x, \]
\[ g_{0,-} = -\frac{1}{4} i, \quad (2.11) \]
\[ g_{1,-} = -\frac{1}{2} q r - \frac{1}{4} c_{1,-}, \]
\[ f_{0,-} = -\frac{1}{2} iq, \]
\[ f_{1,-} = -\int_{x_0}^x (q + iq_x r) \, dx - \frac{1}{2} ic_{1,-} q, \]
\[ h_{0,-} = -\frac{1}{2} ir, \]
\[ h_{1,-} = \int_{x_0}^x (r - iqr_x) \, dx - \frac{1}{2} ic_{1,-} r, \quad \text{etc.} \]

Here \( \{c_{\ell,\pm}\}_{\ell \in \mathbb{N}} \) denote summation constants which naturally arise when solving the differential equations for \( g_{\ell,\pm}, f_{\ell,\pm}, h_{\ell,\pm} \) in (2.3)-(2.10).

We first consider the stationary case. To construct the stationary Fokas-Lenells hierarchy we introduce the following \( 2 \times 2 \) matrix
\[ U(\xi, x) = \begin{pmatrix} -iz & q_x \xi \\ r_x \xi & iz \end{pmatrix}, \quad z = \xi^2 \quad (2.12) \]
and make the ansatz

\[ V_n(\xi, x) = \begin{pmatrix} iG_n(\xi, x) & -F_n(\xi, x) \\ H_n(\xi, x) & -iG_n(\xi, x) \end{pmatrix}, \quad n = (n_-, n_+) \in \mathbb{N}_0^2, \tag{2.13} \]

where \( G_n, F_n \) and \( H_n \) are chosen as Laurent polynomials, namely

\[ G_n(\xi, x) = \sum_{\ell=0}^{n_-} \xi^{-2\ell} g_{n_- - \ell, -}(x) + \sum_{\ell=1}^{n_+} \xi^{2\ell} g_{n_+ - \ell, +}(x), \tag{2.14} \]

\[ F_n(\xi, x) = \sum_{\ell=0}^{n_-} \xi^{-2\ell+1} f_{n_- - \ell, -}(x) + \sum_{\ell=0}^{n_+} \xi^{2\ell} f_{n_+ - \ell, +}(x), \]

\[ H_n(\xi, x) = \sum_{\ell=0}^{n_-} \xi^{-2\ell+1} h_{n_- - \ell, -}(x) + \sum_{\ell=0}^{n_+} \xi^{2\ell} h_{n_+ - \ell, +}(x), \]

and \( g_{\ell, \pm}, f_{\ell, \pm}, h_{\ell, \pm} \) are defined by (2.3)-(2.10). The linear system

\[ \psi_x = U(\xi, x)\psi, \quad \psi_t = V_n(\xi, x)\psi, \quad \psi = (\psi_1, \psi_2)^T \tag{2.15} \]

yields the stationary zero-curvature equation

\[ -V_n x(\xi, x) + [U(\xi, x), V_n(\xi, x)] = 0. \tag{2.16} \]

Inserting (2.12) and (2.13) into (2.16), one easily finds

\[ \xi q_x(x) H_n(\xi, x) + \xi r_x(x) F_n(\xi, x) = iG_{n,x}(\xi, x), \tag{2.17} \]

\[ 2iz F_n(\xi, x) - 2ix q_x(x) G_n(\xi, x) = -F_{n,x}(\xi, x), \tag{2.18} \]

\[ 2iz H_n(\xi, x) + 2ir_x(x) \xi G_n(\xi, x) = H_{n,x}(\xi, x). \tag{2.19} \]

Insertion of (2.14) into (2.17)-(2.19) then yields

\[ f_{n_+, -1, +, x} - 2i q_x g_{n_+, -} - 2if_{n_-, -1, -} = 0, \tag{2.20} \]

\[ -h_{n_+, -1, +, x} + 2i h_{n_-, -1, -} + 2ir_x g_{n_-, -} = 0. \tag{2.21} \]

Thus, varying \( n_\pm \in \mathbb{N}_0 \), equations (2.20) and (2.21) give rise to the stationary Fokas-Lenells (FL) hierarchy which we introduce as follows

\[ s-FL_n(q, r) = \begin{pmatrix} f_{n_+, -1, +, x} - 2i q_x g_{n_+, -} + 2if_{n_-, -1, -} \\ -h_{n_+, -1, +, x} + 2i h_{n_-, -1, -} + 2ir_x g_{n_-, -} \end{pmatrix} = 0, \tag{2.22} \]

\[ n = (n_-, n_+) \in \mathbb{N}_0^2. \]
We record the first few equations in FL hierarchy \((2.22)\) explicitly,

\[
\text{s-FL}_{(0,0)}(q,r) = \left(\frac{1}{2}i q_x - \frac{1}{2}i r_x\right) = 0, \quad (2.23)
\]

\[
\text{s-FL}_{(1,1)}(q,r) = \left(-q_{xx} + i q x r - 2i c_{1,-} q_x + q - r_{xx} - i q r r_x + 2i c_{1,-} r_x + r\right) = 0. \quad (2.24)
\]

In the special case \(c_{1,-} = 1\) in \((2.24)\), one obtains the stationary version of

the Fokas-Lenells system \((1.4)\).

From \((2.17)-(2.19)\) one infers that

\[
\frac{d}{dx} \det(V_n(\xi,x)) = \frac{d}{dx} \left( G^2_n(\xi,x) + F_n(\xi,x)H_n(\xi,x) \right) = 0, \quad (2.25)
\]

and hence

\[
G^2_n(\xi,x) + F_n(\xi,x)H_n(\xi,x) = R_n(\xi), \quad (2.26)
\]

where the Laurent polynomial \(R_n\) is \(x\)-independent. One may write \(R_n\) as

\[
R_n(\xi) = z^{-2n} \prod_{m=0}^{2n+1} (\xi - E_m), \quad (E_m)^{2n+1}_{m=0} \subset \mathbb{C},
\]

\[
n = 2n_+ + 2n_- - 1 \in \mathbb{N}_0. \quad (2.27)
\]

Moreover, \((2.26)\) also implies

\[
\lim_{\xi \to 0} z^{2n} R_n(\xi) = \prod_{m=0}^{2n+1} E_m
\]

and hence

\[
\prod_{m=0}^{2n+1} E_m = \frac{1}{16}. \quad (2.28)
\]

Relation \((2.26)\) allows one to introduce a hyperelliptic curve \(K_n\) of arithmetic genus \(n = 2n_+ + 2n_- - 1\) (possibly with a singular affine part), where

\[
K_n : \mathcal{F}(\xi,y) = y^2 - z^{2n} R_n(\xi) = y^2 - \prod_{m=0}^{2n+1} (\xi - E_m) = 0,
\]

\[
n = 2n_+ + 2n_- - 1 \in \mathbb{N}_0. \quad (2.29)
\]

Next we turn to the time-dependent Fokas-Lenells hierarchy. For that purpose the coefficients \(q\) and \(r\) are now considered as functions of both the
space and time. For each system in this hierarchy, that is, for each \( n \), we introduce a deformation (time) parameter \( t_n \in \mathbb{R} \) in \( q, r \), replacing \( q(x), r(x) \) by \( q(x, t_n), r(x, t_n) \). Moreover, the definitions (2.12), (2.13) and (2.14) of \( U, V \) and \( F_n, G_n, H_n \), respectively, still apply by adding a parameter \( t_n \in \mathbb{R} \), that is,

\[
U(\xi, x, t_n) = \begin{pmatrix}
-iz \\
r_x(x, t_n)z \\
iz
\end{pmatrix} q_x(x, t_n)\xi, \quad z = \xi^2, \tag{2.30}
\]

\[
V_n(\xi, x, t_n) = \begin{pmatrix}
ig_n(\xi, x, t_n) \\
-ig_n(\xi, x, t_n)
\end{pmatrix} F_n(\xi, x, t_n) - \begin{pmatrix}
-izq_x(\xi, x, t_n) \\
r_x(\xi, x, t_n)z
\end{pmatrix} G_n(\xi, x, t_n), \quad n = (n_-, n_+) \in \mathbb{N}_0^2, \tag{2.31}
\]

\[
G_n(\xi, x, t_n) = \sum_{\ell=0}^{n_-} \xi^{-2\ell} g_{n_-, \ell, -}(x, t_n) + \sum_{\ell=1}^{n_+} \xi^{2\ell} g_{n_+, \ell, +}(x, t_n), \tag{2.32}
\]

\[
F_n(\xi, x, t_n) = \sum_{\ell=0}^{n_-} \xi^{-2\ell+1} f_{n_-, \ell, -}(x, t_n) + \sum_{\ell=0}^{n_+} \xi^{2\ell} f_{n_+, \ell, +}(x, t_n), \tag{2.33}
\]

\[
H_n(\xi, x, t_n) = \sum_{\ell=0}^{n_-} \xi^{-2\ell+1} h_{n_-, \ell, -}(x, t_n) + \sum_{\ell=0}^{n_+} \xi^{2\ell} h_{n_+, \ell, +}(x, t_n) \tag{2.34}
\]

with \( g_{\ell, \pm}, f_{\ell, \pm}, h_{\ell, \pm} \) defined by (2.3)-(2.10). Equation (2.16) now needs to be changed to

\[
0 = U_{t_n}(\xi, x, t_n) - V_n(\xi, x, t_n) + [U(\xi, x, t_n), V_n(\xi, x, t_n)] = 0. \quad n \in \mathbb{N}_0^2. \tag{2.35}
\]

Insertion of (2.3)-(2.10), (2.30)-(2.34) into (2.35) then yields

\[
0 = U_{t_n}(\xi, x, t_n) - V_n(\xi, x, t_n) + [U(\xi, x, t_n), V_n(\xi, x, t_n)] = \begin{pmatrix}
-ig_{n_+, x} + q_x H_n + r_x H_n \\
r_x \xi - H_n + 2ir_x \xi G_n + 2iz H_n \quad iG_{n_+, x} - q_x H_n - r_x H_n F_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
x(q_{st_n} + f_{n_+, -1, +} + 2i q_x g_{n_-}, -) + 2if_{n_- - 1, -} \\
x(r_{st_n} - h_{n_+, -1, +} + 2h_{n_- - 1, -} + 2ir_x g_{n_- - 1, -})
\end{pmatrix}. \tag{2.36}
\]
Equation (2.36) gives rise to two equivalent forms of (2.35),

\[ 0 = -iG_{n,x}(\xi, x, t_n) + q_x(x, t_n)\xi H_{n}(\xi, x, t_n) + r_x(x, t_n)\xi F_{n}(\xi, x, t_n), \]

(2.37)

\[ q_{xt}(x, t_n)\xi = -F_{n,x}(\xi, x, t_n) - 2izF_n(\xi, x, t_n) + 2iq_x(x, t_n)\xi G_n(\xi, x, t_n), \]

(2.38)

\[ r_{xt}(x, t_n)\xi = H_{n,x}(\xi, x, t_n) - 2ir_x(x, t_n)\xi G_n(\xi, x, t_n) - 2izH_n(\xi, x, t_n), \]

(2.39)

and

\[ q_{xt} + f_{n,1,+}x - 2iq_xg_{n,-} + 2ifi_{n,-} = 0, \]

\[ r_{xt} - h_{n,1,+}x + 2ih_{n,-} + 2ir_xg_{n,-} = 0. \]

Varying \( n \in \mathbb{N}_0^2 \), the collection of evolution equations

\[
\text{FL}_n(q, r) = \left( \begin{array}{c}
q_{xt} + f_{n,1,+}x - 2iq_xg_{n,-} + 2ifi_{n,-} \\
r_{xt} - h_{n,1,+}x + 2ih_{n,-} + 2ir_xg_{n,-}
\end{array} \right) = 0,
\]

(2.40)

\( t_n \in \mathbb{R}, \ n = (n_-, n_+) \in \mathbb{N}_0^2 \)

then defines the time-dependent Fokas-Lenells hierarchy. Explicitly,

\[
\text{FL}_{(0,0)}(q, r) = \left( \begin{array}{c}
q_{xt} + \frac{1}{2}iq_x \\
r_{xt} - \frac{1}{2}ir_x
\end{array} \right) = 0,
\]

(2.41)

\[
\text{FL}_{(1,1)}(q, r) = \left( \begin{array}{c}
q_{xt} - qx - iq_xr - 2ic_1q_x + q \\
r_{xt} - qx - 2ic_1r_x + r
\end{array} \right) = 0, \quad \text{etc.},
\]

(2.42)

represent the first few equations of the time-dependent Fokas-Lenells hierarchy. The special case \( n = (1, 1) \), and \( c_{1,-} = 1 \), that is,

\[
\left( \begin{array}{c}
q_{xt} - qx - iq_xr - 2iq_x + q \\
r_{xt} - qx - 2ic_1r_x + r
\end{array} \right) = 0
\]

represents the Fokas-Lenells system (1.4).

Finally, it will also be useful to work with the corresponding homogeneous coefficients \( \hat{f}_{\ell,\pm}, \hat{g}_{\ell,\pm}, \) and \( \hat{h}_{\ell,\pm} \), defined by the vanishing of the integration constants \( c_k \) for \( k = 1, \ldots, \ell \), and choosing \( c_{0,\pm} = 1 \),

\[
\hat{f}_{0,+} = f_{0,+} = -q_x, \quad \hat{f}_{0,-} = f_{0,-} = -\frac{1}{2}iq, \quad \hat{f}_{\ell} = f_{\ell}|_{c_k=0, k=1,\ldots,\ell,} \quad \ell \in \mathbb{N},
\]

\[
\hat{g}_{0,+} = g_{0,+} = -1, \quad \hat{g}_{0,-} = g_{0,-} = -\frac{1}{4}, \quad \hat{g}_{\ell,+} = g_{\ell,+}|_{c_k=0, k=1,\ldots,\ell,} \quad \ell \in \mathbb{N},
\]

\[
\hat{h}_{0,+} = h_{0,+} = r_x, \quad \hat{h}_{0,-} = h_{0,-} = -\frac{1}{2}ir, \quad \hat{h}_{\ell,+} = h_{\ell,+}|_{c_k=0, k=1,\ldots,\ell,} \quad \ell \in \mathbb{N}.
\]

(2.43)
By induction one infers that

\[ f_{\ell, \pm} = \sum_{k=0}^{\ell} c_{\ell-k} \hat{f}_{k, \pm}, \quad g_{\ell, \pm} = \sum_{k=0}^{\ell} c_{\ell-k} \hat{g}_{k, \pm}, \quad h_{\ell, \pm} = \sum_{k=0}^{\ell} c_{\ell-k} \hat{h}_{k, \pm}, \quad \ell \in \mathbb{N}_0. \]

(2.44)

In a slight abuse of notation we will occasionally stress the dependence of \( f_{\ell, \pm}, g_{\ell, \pm}, h_{\ell, \pm} \) on \( q, r \) (or \( x, t \)) by writing \( f_{\ell, \pm}(q, r), g_{\ell, \pm}(q, r), h_{\ell, \pm}(x, t) \). Similarly, with \( F_{\ell, +}, G_{\ell, +}, H_{\ell, +} \) denoting the polynomial parts of \( F_\ell, G_\ell, H_\ell \), respectively, and \( F_{\ell, -}, G_{\ell, -}, H_{\ell, -} \) denoting the Laurant parts of \( F_\ell, G_\ell, H_\ell \), such that

\[ F_\ell(\xi) = F_{\ell, -}(\xi) + F_{\ell, +}(\xi), \quad G_\ell(\xi) = G_{\ell, -}(\xi) + G_{\ell, +}(\xi), \quad H_\ell(\xi) = H_{\ell, -}(\xi) + H_{\ell, +}(\xi), \]

one finds that

\[ F_{\ell, \pm} = \sum_{k=1}^{\ell_+} c_{\ell_+ - k} \pm \hat{F}_{k, \pm}, \quad G_{\ell, -} = \sum_{k=0}^{\ell_-} c_{\ell_- - k} \pm \hat{G}_{k, +}, \quad G_{\ell, +} = \sum_{k=0}^{\ell_+} c_{\ell_+ - k} \pm \hat{G}_{k, -}, \quad H_{\ell, \pm} = \sum_{k=1}^{\ell_-} c_{\ell_- - k} \pm \hat{H}_{k, \pm}, \]

where \( \hat{F}_{k, \pm}, \hat{G}_{k, \pm}, \hat{H}_{k, \pm} \) are corresponding homogeneous quantities of \( F_{k, \pm}, G_{k, \pm}, H_{k, \pm} \).

3 Stationary Fokas-Lenells formalism

This section is devoted to a detailed study of the stationary Fokas-Lenells hierarchy. We first define a fundamental meromorphic function \( \phi(P, x) \) on the hyperelliptic curve \( K_n \), using the polynomial recursion formalism described in section 2, and then study the properties of the Baker-Akhiezer function \( \psi(P, x, x_0) \), Dubrovin-type equations, trace formulas and theta function representations of \( \phi, \psi_1, \psi_2, q, r \).

For major parts of this section we suppose \( (2.1), (2.2), (2.3)-(2.10), (2.12)-(2.22) \), keeping \( n \in \mathbb{N}_0 \) fixed.

We recall the hyperelliptic curve

\[ K_n : \mathcal{F}(\xi, y) = y^2 - z^{2n_+} R_\xi(\xi) = y^2 - \prod_{m=0}^{2n+1} (\xi - E_m) = 0, \]

(3.1)

\[ R_\xi(\xi) = z^{-2n_+} \prod_{m=0}^{2n+1} (\xi - E_m), \quad \{E_m\}_{m=0}^{2n+1} \subset \mathbb{C}, \quad n = 2n_+ + 2n_- - 1 \in \mathbb{N}_0, \]
as introduced in [22]. Throughout this section we assume $\mathcal{K}_n$ to be non-singular, that is, we suppose that
\[ E_m \neq E_{m'} \text{ for } m \neq m', \ m, m' = 0, 1, \cdots, 2n + 1. \] (3.2)

$\mathcal{K}_n$ is compactified by joining two points at infinity $P_{\infty_+}, P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$. Points $P$ on
\[ \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\} \]
are represented as pairs $P = (\xi, y(P))$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_n$ satisfying
\[ \mathcal{F}_n(\xi, y(P)) = 0. \]

The complex structure on $\mathcal{K}_n$ is defined in the usual way by introducing local coordinates
\[ \zeta_{Q_0} : P \to (\xi - \xi_0) \]
near points $Q_0 = (\xi_0, y(Q_0)) \in \mathcal{K}_n$, which are neither branch nor singular points of $\mathcal{K}_n$; near the points $P_{\infty_{\pm}} \in \mathcal{K}_n$, the local coordinates are
\[ \zeta_{P_{\infty_{\pm}}} : P \to \xi^{-1}, \]
and similarly at branch and singular points of $\mathcal{K}_n$. Hence $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of topological genus $n$ in a standard manner.

The holomorphic map $\ast$, changing sheets, is defined by
\[
\begin{align*}
\ast : & \mathcal{K}_n \to \mathcal{K}_n, \\
P = (\xi, y_j(\xi)) \to P^* = (z, y_{j+1(\text{mod } 2)}(\xi)), & j = 0, 1, \\
P^{**} := (P^*)^*, & \text{etc.,}
\end{align*}
\] (3.3)

where $y_j(\xi), j = 0, 1$, denote the two branches of $y(P)$ satisfying $\mathcal{F}_n(\xi, y) = 0$, namely
\[ (y - y_0(\xi))(y - y_1(\xi)) = y^2 - z^{-2n_{-}}R_{2n+2}(\xi) = 0. \] (3.4)

Taking into account (3.4), one easily derives
\[
\begin{align*}
y_0(\xi) + y_1(\xi) &= 0, \\
y_0(\xi)y_1(\xi) &= -z^{-2n_{-}}R_{2n+2}(\xi), \\
y_0^2(\xi) + y_1^2(\xi) &= 2z^{-2n_{-}}R_{2n+2}(\xi).
\end{align*}
\] (3.5)
Positive divisors on $K_n$ of degree $n$ are denoted by

$$
D_{P_1,\ldots,P_n} : \begin{cases}
K_n \rightarrow \mathbb{N}_0, \\
P \rightarrow D_{P_1,\ldots,P_n} = \begin{cases}
k & \text{if } P \text{ occurs } k \text{ times in } \{P_1,\ldots,P_n\}, \\
0 & \text{if } P \notin \{P_1,\ldots,P_n\}.
\end{cases}
\end{cases}
$$

(3.6)

Moreover, for a nonzero, meromorphic function $f$ on $K_n$, the divisor of $f$ is denoted by $(f)$.

For notational simplicity we will usually assume that $n \in \mathbb{N}$ and hence $n \in \mathbb{N}^2 \setminus \{(0,0)\}$. (The trivial case $n = (0,0)$ is excluded in our discussion since the "genus" of corresponding curve is $-1 < 0$.)

We denote by $\{\mu_j(x)\}_{j=1,\ldots,n}$ and $\{\nu_j(x)\}_{j=1,\ldots,n}$ the zeros of $(\cdot)^{2n-1}F_n(\cdot,x)$ and $(\cdot)^{2n-1}H_n(\cdot,x)$, respectively. Thus we may write

$$
F_n(\xi,x) = -q(x)\xi^{-2n+1}\prod_{j=1}^n (\xi - \mu_j(x)),
$$

(3.7)

$$
H_n(\xi,x) = r(x)\xi^{-2n+1}\prod_{j=1}^n (\xi - \nu_j(x)).
$$

(3.8)

We now introduce $\{\hat{\mu}_j\}_{j=1,\ldots,n} \subset K_n$ and $\{\hat{\nu}_j\}_{j=1,\ldots,n} \subset K_n$ by

$$
\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x)^{2n-1}G_n(\mu_j(x),x)), \quad j = 1,\ldots,n,
$$

(3.9)

and

$$
\hat{\nu}_j(x) = (\nu_j(x), \nu_j(x)^{2n-1}G_n(\nu_j(x),x)), \quad j = 1,\ldots,n.
$$

(3.10)

We also introduce the points $P_0,\pm$ by $P_{0,\pm} = (0, \pm \frac{1}{2}) \in K_n$ (cf. (2.28)).

Next we define the fundamental meromorphic function on $K_n$ by

$$
\phi(P,x) = -\frac{i\xi^{-2n-1} - iG_n(\xi,x)}{F_n(\xi,x)} = \frac{H_n(\xi,x)}{i\xi^{-2n-1} + iG_n(\xi,x)},
$$

(3.11)

with divisor of $\phi(\cdot,x)$ given by

$$
(\phi(\cdot,x)) = D_{P_0,-\hat{\nu}(x)} - D_{P_\infty,\hat{\mu}(x)},
$$

(3.12)

using (3.7) and (3.8). Here we abbreviated

$$
\hat{\mu}(x) = \{\hat{\mu}_1(x) = P_{0,+}, \hat{\mu}_2(x), \ldots, \hat{\mu}_n(x)\},
$$

$$
\hat{\nu}(x) = \{\hat{\nu}_1(x) = P_{\infty,-}, \hat{\nu}_2(x), \ldots, \hat{\nu}_n(x)\}.
$$

(3.13)
Given \( \phi(\cdot, x) \), the stationary Baker-Akhiezer function \( \psi \) is then defined by

\[
\psi(P, x, x_0) = \left( \psi_1(P, x, x_0) \quad \psi_2(P, x, x_0) \right),
\]

\[
\psi_1(P, x, x_0) = \exp \left( \int_{x_0}^{x} dx' \left( -iz + q_x(x') \xi \phi(P, x') \right) \right), \quad (3.14)
\]

\[
\psi_2(P, x, x_0) = \phi(P, x) \exp \left( \int_{x_0}^{x} dx' \left( r_x(x') \xi \phi^{-1}(P, x') + iz \right) \right). \quad (3.15)
\]

Basic properties of \( \phi \) and \( \psi \) are summarized in the following result.

**Lemma 3.1.** Assume (3.11), (3.14), (3.15), \( P = (z, y) \in \mathcal{K}_n \setminus \{ P_{\infty, \pm}, P_{0, \pm} \} \), and let \( (\xi, x, x_0) \in \mathbb{C} \times \mathbb{R}^2 \). Then

(i) \( \phi(P, x) \) satisfies the Riccati-type equation

\[
\phi_x(P, x) = r_x \xi + 2i z \phi(P, x) - q_x \xi \phi^2(P, x) \quad (3.16)
\]

and

\[
\phi(P, x) \phi(P^*, x) = \frac{H_n(\xi, x)}{F_n(\xi, x)}, \quad (3.17)
\]

\[
\phi(P, x) + \phi(P^*, x) = \frac{2i G_n(\xi, x)}{F_n(\xi, x)} \quad (3.18)
\]

\[
\phi(P, x) - \phi(P^*, x) = -\frac{2i \xi^{-2n} - y(P)}{F_n(\xi, x)} \quad (3.19)
\]

(ii) \( \psi(P, x, x_0) \) satisfies the first-order system

\[
\psi_x(P, x, x_0) = U(\xi, x) \psi, \quad (3.20)
\]

\[
V_n(\xi, x) \psi(P, x, x_0) = i \xi^{-2n} - y(P) \psi(P, x, x_0). \quad (3.21)
\]

Moreover,

\[
\psi_1(P, x, x_0) = \sqrt{\frac{H_n(\xi, x)}{F_n(\xi, x_0)}} \exp \left( - \int_{x_0}^{x} dx' \left( \frac{q_x(x') \xi^{-2n-1} y(P)}{F_n(\xi, x')} \right) \right), \quad (3.22)
\]

and

\[
\psi_1(P, x, x_0) \psi_1(P^*, x, x_0) = \frac{H_n(\xi, x)}{F_n(\xi, x_0)}, \quad (3.23)
\]

\[
\psi_2(P, x, x_0) \psi_2(P^*, x, x_0) = \frac{H_n(\xi, x)}{F_n(\xi, x_0)}, \quad (3.24)
\]

\[
\psi_1(P, x, x_0) \psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0) \psi_2(P, x, x_0)
\]

\[
= \frac{2i G_n(\xi, x)}{F_n(\xi, x_0)}. \quad (3.25)
\]

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(iii) $\phi, \psi$ satisfy
\[
\psi_2(P, x, x_0) = \phi(P, x)\psi_1(P, x, x_0). \tag{3.26}
\]

Proof. To prove (3.16) one uses the definition (3.11) of $\phi$ and equations (2.17)-(2.19) to obtain
\[
\phi_x(P, x) = -\left(\frac{i\xi^{-2n}y(P) - iG_n(\xi, x)}{F_n(\xi, x)}\right) x
= \frac{iG_n(x, \xi)}{F_n(\xi, x)} + \phi(P, x) \frac{F_n}{F_n(\xi, x)}
= \frac{\xi q_x H_n(\xi, x) + \xi r_x F_n(\xi, x)}{F_n(\xi, x)} + \phi(P, x) \frac{2iz F_n(\xi, x) - 2iz q_x G_n(\xi, x)}{F_n(\xi, x)}
= \frac{i\xi^{-2n}y(P) + iG_n(\xi, x)}{F_n(\xi, x)} + \frac{-2iz q_x G_n(\xi, x)}{F_n(\xi, x) + 2iz\phi(P, x) + r_x \xi}
= r_x \xi + 2iz\phi(P, x) - q_x \phi^2(P, x).
\]

Equations (3.17)-(3.19) are clear from the definitions of $\phi$ and $y$. By definitions of $\psi$,
\[
\psi_{1,x}(P, x, x_0) = (-iz + q_x \phi(P, x))\psi_{1,x}(P, x, x_0), \tag{3.27}
\psi_{2,x}(P, x, x_0) = (iz + r_x \phi(P, x)^{-1})\psi_{2}(P, x, x_0), \tag{3.28}
\]
the function $\psi_2(P, x, x_0)/\psi_1(P, x, x_0)$ satisfies the first-order linear equation
\[
\frac{dL(P, x, x_0)}{dx} = (r_x \phi^{-1}(P, x) + 2iz - q_x \phi(P, x)) L(P, x, x_0).
\]

Since $\psi_2(P, x, x_0)/\psi_1(P, x, x_0)$ and $\phi(P, x)$ take the same value at $x = x_0$, that is, $\psi_2(P, x, x_0)/\psi_1(P, x, x_0) = \phi(P, x_0)$, one derives (3.26), (3.20), (3.21) are clear from (3.27), (3.28) and (3.26). (3.23)-(3.25) follow from (3.14), (3.15), (3.17)-(3.19). Finally, by (2.18), (3.11), (3.14),
\[
\psi_1(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx' (-iz + q_x(x')\xi \phi(P, x'))\right)
= \exp\left(\int_{x_0}^{x} dx' (-iz + q_x(x')\xi \frac{i\xi^{-2n}y(P) - iG_n(\xi, x')}{-F_n(\xi, x')}\right)
= \exp\left(\int_{x_0}^{x} dx' \left(\frac{F_n(x', \xi)}{2F_n(\xi, x')} \frac{-i\xi^{-2n+1}q_x(x')y(P)}{F_n(\xi, x')}\right)\right), \tag{3.29}
\]

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which proves \(3.22\).

Concerning the dynamics of the zeros \(\mu_j(x)\) and \(\nu_j(x)\) of \(F_n(\xi,x)\) and \(H_n(\xi,x)\) one obtains the following Dubrovin-type equations.

**Lemma 3.2.** Suppose \(2.1\) and the \(n\)th stationary Fokas-Lenells equation \(2.22\) holds subject to the constraint \(3.2\) on an open interval \(\bar{\Omega}_\mu \subseteq \mathbb{R}\). Suppose that the zeros \(\{\mu_j(x)\}_{j=0,...,n}\) of \(\xi^{2n-1}F_n(\xi,x)\) remain distinct and nonzero for \(x \in \bar{\Omega}_\mu\). Then \(\{\hat{\mu}_j(x)\}_{j=0,...,n}\) defined by \(3.9\), satisfies the following first-order system of differential equations

\[
\mu_{j,x}(x) = \frac{-2iy(\hat{\mu}_j(x))}{\prod_{k=1,k\neq j}^{n}(\mu_j(x) - \mu_k(x))}, \quad j = 1,\ldots,n, \quad x \in \bar{\Omega}_\mu. \quad (3.30)
\]

Next, assume \(\mathcal{K}_n\) to be nonsingular and introduce initial condition

\[
\{\hat{\mu}_j(x_0)\}_{j=1,...,n} \subset \mathcal{K}_n \quad (3.31)
\]

for some \(x_0 \in \mathbb{R}\), where \(\mu_j(x_0) \neq 0, j = 1,\ldots,n\), are assumed to be distinct. Then there exists an open interval \(\Omega_\mu \subseteq \mathbb{R}\), with \(x_0 \in \Omega_\mu\), such that the initial value problem \(3.30, 3.31\) has a unique solution \(\{\hat{\mu}_j\}_{j=1,...,n} \subset \mathcal{K}_n\) satisfying

\[
\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 0,\ldots,n, \quad (3.32)
\]

and \(\mu_j, j = 1,\ldots,n\), remain distinct and nonzero on \(\Omega_\mu\).

For the zeros \(\{\nu_j(x)\}_{j=1,...,n}\) of \(\xi^{2n-1}H_n(\xi,x)\) similar statements hold with \(\mu_j\) and \(\Omega_\mu\) replaced by \(\nu_j\) and \(\Omega_\nu\), etc. In particular, \(\{\hat{\nu}_j\}_{j=1,...,n}\), defined by \(3.10\), satisfies the system

\[
\nu_{j,x}(x) = \frac{-2iy(\hat{\nu}_j(x))}{\prod_{k=1,k\neq j}^{n}(\nu_j(x) - \nu_k(x))}, \quad j = 1,\ldots,n, \quad x \in \bar{\Omega}_\mu. \quad (3.33)
\]

**Proof.** We only prove equation \(3.30\) since the proof of \(3.33\) follows in an identical manner. Inserting \(\xi = \mu_j\) into equation \(2.18\), one concludes from \(3.9\),

\[
F_n(x) = q_x\mu_j^{-2n-1} \prod_{k=1}^{n}(\mu_j - \mu_k) = 2iq_x\mu_j G_n(\mu_j)
\]

\[
= -2iq_x\mu_j^{-2n-1} y(\hat{\mu}_j), \quad (3.34)
\]

proving \(3.30\). The smooth assertion \(3.32\) is clear as long as \(\hat{\mu}_j\) stays away from the branch points \((E_m, 0)\). In case \(\hat{\mu}_j\) hits such a branch point, one can
use the local chart around \((E_m, 0)\) (with the local chart \(\zeta = \sigma(\xi - E_m)^{1/2}, \sigma \in \{1, -1\}\)) to verify (3.32).

Next, we turn to the trace formulas of the FL invariants, that is, expressions of \(f_{\ell, \pm}\) and \(h_{\ell, \pm}\) in terms of symmetric functions of the zeros \(\mu_j\) and \(\nu_{\ell}\) of \((\cdot)^{2n-1}F_{\ell}(\cdot)\) and \((\cdot)^{2n-1}H_{\ell}(\cdot)\), respectively. For simplicity we just record the simplest case.

**Lemma 3.3.** Suppose \((2.1)\) and the \(n\)th stationary Fokas-Lenells system \((2.22)\) holds and let \(x \in \mathbb{R}\). Then

\[
\frac{q_{xx}}{2iq_x} - \frac{1}{2}q_x r_x - c_{1,+} = -\sum_{j=1}^{n} \mu_j, \quad (3.35) \\
\frac{r_{xx}}{2ir_x} + \frac{1}{2}q_x r_x + c_{1,+} = -\sum_{j=1}^{n} \nu_j, \quad (3.36) \\
\frac{iq}{2q_x} = (-1)^n \prod_{j=1}^{n} \mu_j, \quad (3.37) \\
\frac{ir}{2r_x} = (-1)^{n-1} \prod_{j=1}^{n} \nu_j. \quad (3.38)
\]

**Proof.** \((3.35)-(3.38)\) follow by comparison powers of \(\xi\) substituting \((3.7)\) and \((3.8)\) into \((2.14)\) taking into account \((2.11)\).

Next we turn to the asymptotic properties of \(\phi\) and \(\phi_j, j = 1, 2\).

**Lemma 3.4.** Suppose \((2.1)\) and the \(n\)th stationary Fokas-Lenells system \((2.22)\) holds and let \(P \in K_n \setminus \{P_{\infty, \pm}, P_{0, \pm}\}, x \in \mathbb{R}\). Then

\[
\phi(P, x) = \begin{cases} 
2i[q_x(x)]^{-1} \xi^{-1} + O(\xi), & P \to P_{\infty, +}, \quad \zeta = \xi^{-1}, \\
\frac{1}{2}[r_x(x)/2] \xi + O(\xi^3), & P \to P_{\infty, -}
\end{cases} \quad (3.39)
\]

\[
\phi(P, x) = \begin{cases} 
q(x)^{-1} \xi^{-1} + O(\xi), & P \to P_{0, +}, \quad \zeta = \xi, \\
r(x) \xi + O(\xi^3), & P \to P_{0, -}
\end{cases} \quad (3.40)
\]

\[
\psi_1(P, x) = \begin{cases} 
e^{i(x-x_0)\zeta^{-2} + O(1)}, & P \to P_{\infty, +}, \quad \zeta = \xi^{-1}, \\
e^{-i(x-x_0)\zeta^{-2} + O(1)}, & P \to P_{\infty, -}
\end{cases} \quad (3.41)
\]

\[
\psi_1(P, x) = \begin{cases} 
\frac{q(x)}{q(x_0)}(1 + O(\xi)), & P \to P_{0, +}, \quad \zeta = \xi, \\
1 + O(\xi), & P \to P_{0, -}
\end{cases} \quad (3.42)
\]
ψ_2(P, x) = \begin{cases} \left( \frac{2i}{q_x(x)} \zeta^{-1} + O(\zeta) \right) e^{i(x-x_0)\zeta^{-2} + O(1)}, & P \to P_{\infty^+}, \zeta = \xi^{-1}, \\ \left( \frac{1}{q_x(x)} \zeta^{-1} + O(\zeta) \right) e^{-i(x-x_0)\zeta^{-2} + O(1)}, & P \to P_{\infty^-}, \zeta = \xi. \end{cases} \quad (3.43)

\psi_2(P, x) = \begin{cases} \frac{i}{q_x(x)} \zeta^{-1} + O(\zeta), & P \to P_{0^+}, \zeta = \xi, \\ r(x) \zeta + O(\zeta^2), & P \to P_{0^-}, \zeta = \xi. \end{cases} \quad (3.44)

**Proof.** The existence of the asymptotic expansions of φ in terms of the appropriate local coordinates ζ = ξ^{-1} near P_{\infty^+} and ζ = ξ near P_{0,\pm} is clear from its explicit expression in (3.11). Next, we compute these explicit expansions coefficients in (3.39) and (3.40). Inserting each of the following asymptotic expansions

φ = φ_0 \zeta^{-1} + φ_1 + O(\zeta), \quad \text{as } P \to P_{\infty^+}, \quad (3.45)
φ = φ_0 \zeta + φ_1 \zeta^2 + O(\zeta^2), \quad \text{as } P \to P_{\infty^-}, \quad (3.46)
φ = φ_0 \zeta^{-1} + φ_1 + O(\zeta), \quad \text{as } P \to P_{0^+}, \quad (3.47)
φ = φ_0 \zeta + φ_1 \zeta^2 + O(\zeta^2), \quad \text{as } P \to P_{0^-} \quad (3.48)

into the Riccati-type equation (3.16) and, upon comparing coefficients of powers of ξ, which determines the expansion coefficients of φ_k in (3.39) and (3.40), one concludes (3.39) and (3.40). (3.41), (3.42) are clear from (3.14), (3.39) and (3.40). (3.43) and (3.44) follow by (3.39)-(3.42) and (3.26). \[\square\]

Next, we introduce the holomorphic differentials η_{\ell}(P) on K_n

η_{\ell}(P) = \frac{\xi^{-\ell} - 1}{y(P)} d\xi, \quad \ell = 1, \ldots, n, \quad (3.49)

and choose a homology basis \{a_j, b_j\}_{j=1}^n on K_n in such a way that the intersection matrix of the cycles satisfies

\begin{align*}
a_j \circ b_k &= \delta_{j,k}, & a_j \circ a_k &= 0, & b_j \circ b_k &= 0, & j, k = 1, \ldots, n.
\end{align*}

Associated with K_n one introduces an invertible matrix E \in GL(n, \mathbb{C})

\begin{align*}
E &= (E_{j,k})_{n \times n}, & E_{j,k} &= \int_{a_k} \eta_j, \\
ξ(k) &= (c_1(k), \ldots, c_n(k)), & c_j(k) &= (E^{-1})_{j,k}, \quad (3.50)
\end{align*}

and the normalized holomorphic differentials

\begin{align*}
ω_j &= \sum_{\ell=1}^n c_j(\ell) \eta_\ell, & \int_{a_k} ω_j &= δ_{j,k}, & \int_{b_k} ω_j &= τ_{j,k}, & j, k = 1, \ldots, n. \quad (3.51)
\end{align*}
Apparently, the Riemann matrix $\tau = (\tau_{i,j})$ is symmetric and has a positive-definite imaginary part. Associated with $\tau$ one defines the period lattice $L_n$ in $\mathbb{C}^n$ by

$$L_n = \{ z \in \mathbb{C}^n | z = N + \tau M, \ N, M \in \mathbb{Z}^n \}.$$  

The Riemann theta function associated with Riemann surface $K_n$ and the homology basis $\{a_j, b_j\}_{j=1,\ldots,n}$ is given by

$$\theta(z) = \sum_{n \in \mathbb{Z}^n} \exp \left( 2\pi i (n, z) + \pi i (n, \tau n) \right), \ z \in \mathbb{C}^n,$$

where $(A, B) = \sum_{j=1}^{n} A_j B_j$ denotes the inner product in $\mathbb{C}^n$. Then the Jacobi variety $J(K_n)$ of $K_n$ is defined by

$$J(K_n) = \mathbb{C}^n / L_n,$$

and the Abel maps are defined by

$$A_{Q_0} : \mathcal{K}_n \rightarrow J(K_n),$$

$$P \mapsto A_{Q_0}(P) = (A_{Q_0,1}(P), \ldots, A_{Q_0,n}(P))$$

$$= \left( \int_{Q_0} \omega_1, \ldots, \int_{Q_0} \omega_n \right) \mod L_n \quad (3.53)$$

and

$$\alpha_{Q_0} : \text{Div}(K_n) \rightarrow J(K_n),$$

$$\mathcal{D} \mapsto \alpha_{Q_0}(\mathcal{D}) = \sum_{P \in K_n} \mathcal{D}(P) A_{Q_0}(P) \quad (3.54)$$

$$\triangleq (\alpha_{Q_0,1}(\mathcal{D}), \ldots, \alpha_{Q_0,n}(\mathcal{D})),$$

where $Q_0$ is a fixed base point and the same path is chosen from $Q_0$ to $P$ in $(3.53)$ and $(3.54)$.

Next, let $\Omega^{(3)}_{P_0, -P_\infty, -P_\infty}$ be the normal differential of the third kind holomorphic on $K_n \setminus \{P_0, -P_\infty, -P_\infty\}$ with simple poles at $P_0$, $-P_\infty$, and $P_\infty$, and residues 1 and $-1$, respectively. Explicitly, one writes $\Omega^{(3)}_{P_0, -P_\infty}$ as

$$\Omega^{(3)}_{P_0, -P_\infty} = \left( \frac{y - 1/2}{\xi} - \prod_{j=1}^{n} (\xi - \lambda'_j) \right) \frac{d\xi}{2y},$$

(3.55)
where the constants \( \{ \lambda_j \}_{j=1, \ldots, n} \subset \mathbb{C} \) are uniquely determined by employing the normalization
\[
\int_{a_j} \Omega_{P_0, -P_\infty}^{(3)} = 0, \quad j = 1, \ldots, n.
\]
The explicit formula (3.55) then implies the following asymptotic expansion
\[
\int_{Q_0}^{P} \Omega_{P_0, -P_\infty}^{(3)} = \begin{bmatrix} \frac{1}{\ln(\zeta)} \\ \omega_0^{0, \pm} + O(\zeta) \end{bmatrix}, \quad \omega_0^{0, \pm} \in \mathbb{C}, \quad P \to P_0,\pm
\]
Moreover, the Abelian differential of the second kind \( \Omega_{P_\infty, 1}^{(2)} \) are chosen such that
\[
\Omega_{P_\infty, 1}^{(2)} = \begin{bmatrix} \zeta^{-3} + O(1) \end{bmatrix} d\zeta, \quad P \to P_\infty
\]
In the following it will be convenient to introduce the abbreviations
\[
\xi(P, Q) = \Xi Q_0 - A Q_0(P) + \omega Q_0(D_Q), \quad P \in K_n, \quad Q = (Q_1, \ldots, Q_n) \in \text{Sym}^n(K_n),
\]
where \( \Xi Q_0 \) is the vector of Riemann constants (cf. (A.45) [7]). It turns out that \( \xi(\cdot, Q) \) is independent of the choice of base point \( Q_0 \) (cf. (A.52), (A.53) [7]).

Given these preparations, the theta function representations of \( \phi, \psi_1, \phi_2, q \) and \( r \) then read as follows.
Theorem 3.5. Suppose (2.1) and the \( n \)th stationary Fokas-Lenells equation (2.22) holds subject to the constraint (3.2) on an open interval \( \Omega \subseteq \mathbb{R} \). Moreover, let \( P \in K_n \backslash \{ P_{0-}, P_{\infty+} \} \) and \( x \in \Omega \). In addition, suppose that \( D_{\hat{\mu}(x)} \), or equivalently, \( D_{\hat{\mu}(x)} \) is nonspecial for \( x \in \Omega \). Then, \( \phi, \psi_1, \psi_2, q, r \) admit the following representations

\[
\phi(P, x) = C(x) \frac{\theta(\xi(P, \hat{\mu}(x)))}{\theta(\xi(P, \hat{\mu}(x)))} \exp \left( \int_{Q_0}^{P_{0-}, P_{\infty+}} \Omega_{0-}^{(3)} \right), \tag{3.64}
\]

\[
\psi_1(P, x) = C(x, x_0) \frac{\theta(\xi(P, \hat{\mu}(x)))}{\theta(\xi(P, \hat{\mu}(x)))} \exp \left( -i(x - x_0) \int_{Q_0}^{P_{0-}, P_{\infty+}} \Omega_{0-}^{(2)} \right), \tag{3.65}
\]

\[
\psi_2(P, x) = C(x) C(x, x_0) \frac{\theta(\xi(P, \hat{\mu}(x)))}{\theta(\xi(P, \hat{\mu}(x)))} \times \exp \left( \int_{Q_0}^{P_{0-}, P_{\infty+}} -i(x - x_0) \int_{Q_0}^{P_{0-}, P_{\infty+}} \Omega_{0-}^{(2)} \right), \tag{3.66}
\]

where

\[
C(x) = \frac{1}{q(x_0)} \frac{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))}{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))} \frac{\theta(\xi(P_{0+}, \hat{\mu}(x)))}{\theta(\xi(P_{0+}, \hat{\mu}(x)))} \times e^{-i(x - x_0)(e_{0-} - e_{0+}) - \omega_0^{0+}}, \tag{3.67}
\]

\[
C(x, x_0) = \frac{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))}{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))} e^{i(x - x_0)e_{0-}}. \tag{3.68}
\]

The Abel map linearizes the divisors \( D_{\hat{\mu}(x)} \) and \( D_{\hat{\mu}(x)} \) in the sense that

\[
\omega_{Q_0}(D_{\hat{\mu}(x)}) = \omega_{Q_0}(D_{\hat{\mu}(x)}) - iU_{0-}^{(2)}(x - x_0), \tag{3.69}
\]

\[
\omega_{Q_0}(D_{\hat{\mu}(x)}) = \omega_{Q_0}(D_{\hat{\mu}(x)}) - iU_{0-}^{(2)}(x - x_0). \tag{3.70}
\]

Moreover, one derives

\[
q(x) = q(x_0) \frac{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))}{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))} \frac{\theta(\xi(P_{0+}, \hat{\mu}(x)))}{\theta(\xi(P_{0+}, \hat{\mu}(x)))} e^{i(x - x_0)(e_{0-} - e_{0+})}, \tag{3.71}
\]

\[
r(x) = r(x_0) \frac{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))}{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))} \frac{\theta(\xi(P_{0+}, \hat{\mu}(x)))}{\theta(\xi(P_{0+}, \hat{\mu}(x)))} e^{-i(x - x_0)(e_{0-} - e_{0+})}, \tag{3.72}
\]

\[
q(x_0)r(x_0) = \frac{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))}{\theta(\xi(P_{0-}, \hat{\mu}(x_0)))} \frac{\theta(\xi(P_{0+}, \hat{\mu}(x)))}{\theta(\xi(P_{0+}, \hat{\mu}(x)))} e^{\omega_0^{0-} - \omega_0^{0+}}. \tag{3.73}
\]

Proof. First, we temporarily assume that

\[
\mu_j(x) \neq \mu_j'(x), \quad v_k(x) \neq v_k'(x), \quad \text{for } j \neq j', k \neq k' \text{ and } x \in \tilde{\Omega}, \tag{3.74}
\]

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for appropriate \( \Omega \subseteq \Omega \). Since by (3.12), \( D_{\hat{P}_0^{-}} \sim D_{P_{\infty}^{-} \hat{\mu}} \), and \((P_{0,-})^* \notin \{ \hat{\nu}_1, \cdots, \hat{\nu}_n \} \) by hypothesis, one can use Theorem A.31 in \([7]\) to conclude that \( D_{\hat{\mu}} \in \text{Sym}^n(K_n) \) is nonspecial. This argument is of course symmetric with respect to \( \hat{\mu} \) and \( \hat{\nu} \). Thus, \( D_{\hat{\mu}} \) is nonspecial if and only if \( D_{\hat{\nu}} \) is.

Next we define the right-hand side of (3.65) to be \( \tilde{\psi}_1 \). We intend to prove \( \psi_1 = \tilde{\psi}_1 \), with \( \psi_1 \) given by (3.14). For that purpose we first investigate the divisor of \( \psi_1 \). Since the zeros and poles can only come from zeros of \( F_{\hat{\mu}}(\xi, x) \) in (2.14), one computes using (3.9), the definition (3.11) of \( \phi \), and the Dubrovin equations (3.30),

\[
q_{x'}(x') \xi \phi(P, x') = -q_{x'}(x') \mu_j(x') \frac{2iy(P)}{-q_{x'}(x') \mu_j(x')} \prod_{k=1, k \neq j} \mu_k(x') - O(1)
\]

\[
\times \frac{1}{\xi - \mu_j(x')} + O(1)
\]

\[
\psi_1(P, x, x_0) = \begin{cases} 
(\xi - \mu_j(x))O(1), & \text{as } P \to \hat{\mu}_j(x) \neq \hat{\nu}_j(x_0), \\
O(1), & \text{as } P \to \hat{\nu}_j(x) = \hat{\mu}_j(x_0), \\
(\xi - \mu_j(x_0))^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0) \neq \hat{\nu}_j(x), 
\end{cases}
\]

(3.76)

Together with (3.14) this yields

\[
\psi_1(P, x, x_0) = \begin{cases} 
(\xi - \mu_j(x))O(1), & \text{as } P \to \hat{\mu}_j(x) \neq \hat{\nu}_j(x_0), \\
O(1), & \text{as } P \to \hat{\nu}_j(x) = \hat{\mu}_j(x_0), \\
(\xi - \mu_j(x_0))^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0) \neq \hat{\nu}_j(x), 
\end{cases}
\]

(3.76)

with \( O(1) \neq 1 \). Consequently, \( \psi_1 \) and \( \tilde{\psi}_1 \) have identical zeros and poles on \( K_n \backslash \{ P_{\infty} \} \), which are all simple by hypothesis (3.74). Next, comparing the behavior of \( \psi_1 \) and \( \tilde{\psi}_1 \) near \( P_{\infty} \), taking into account (3.14) and (3.61), the expression (3.65) for \( \psi_1 \), and (3.11), then shows that \( \psi_1 \) and \( \tilde{\psi}_1 \) have identical exponential behavior up to order \( O(1) \) near \( P_{\infty} \). Thus, \( \psi_1 \) and \( \tilde{\psi}_1 \) share the same singularities and zeros, and the Riemann-Roch-type uniqueness result (cf. Lemma 3.4 [8]) then proves that \( \psi_1 \) and \( \tilde{\psi}_1 \) coincide up to normalization. By (3.62) one infers from the right-hand side of (3.65) that

\[
\tilde{\psi}_1(P, x, x_0) = C(x, x_0) \frac{\theta(\xi(P_{0,-}, \hat{\mu}(x)))}{\theta(\xi(P_{0,-}, \hat{\nu}(x_0)))} e^{-i(x-x_0)\epsilon_{0,-} + O(\zeta)}
\]

as \( P \to P_{0,-} \), (3.77)

\[
\tilde{\psi}_1(P, x, x_0) = C(x, x_0) \frac{\theta(\xi(P_{0,+}, \hat{\mu}(x)))}{\theta(\xi(P_{0,+}, \hat{\nu}(x_0)))} e^{-i(x-x_0)\epsilon_{0,+} + O(\zeta)}
\]

as \( P \to P_{0,+} \). (3.78)
A comparison of (3.42), (3.77), (3.78) then yields (3.68), (3.71) subject to (3.74). By (3.12), one infers that 
\[ \phi(P, x) \exp\left( - \int_{Q_0} \Omega_{P_{0,-}, P_{\infty}}^{(3)} \right) \]
must be of the type
\[ \phi(P, x) \exp\left( - \int_{Q_0} \Omega_{P_{0,-}, P_{\infty}}^{(3)} \right) = C(x) \frac{\theta(\xi(P, \hat{\nu}(x)))}{\theta(\xi(P, \hat{\mu}(x)))} \]
for some function \( C(x), x \in \mathbb{C} \). A comparison of (3.79) and asymptotic relations (3.40) then yields, with the help of (3.56), the following expressions
\[ \frac{1}{q(x)} = C(x) \frac{\theta(\xi(P_{0,+}, \hat{\nu}(x)))}{\theta(\xi(P_{0,+}, \hat{\mu}(x)))} e^{\omega_{0,+}}, \quad (3.80) \]
\[ r(x) = C(x) \frac{\theta(\xi(P_{0,-}, \hat{\nu}(x)))}{\theta(\xi(P_{0,-}, \hat{\mu}(x)))} e^{\omega_{0,-}}, \quad (3.81) \]
\[ q(x)r(x) = \frac{\theta(\xi(P_{0,-}, \hat{\nu}(x)))}{\theta(\xi(P_{0,+}, \hat{\mu}(x)))} \theta(\xi(P_{0,+}, \hat{\nu}(x))) \theta(\xi(P_{0,-}, \hat{\mu}(x))) e^{\omega_{0,+} - \omega_{0,-}}. \quad (3.82) \]

Taking into account (3.68), (3.71), (3.80)-(3.82), one easily derives (3.67), (3.71)-(3.73), (3.66) follows by (3.26), (3.64) and (3.65). Next we only prove the linearity of the Abel map with respect to \( x \) in (3.69) since the proof for (3.70) can be derived in an identical fashion. Using the Dubrovin equations (3.30), expression (3.51), and Lagrange’s interpolation formula
\[ \sum_{j=1}^{n} \mu_{j}^{t-1} = \begin{cases} 0 & \ell \neq n \\ 1 & \ell = n \end{cases}, \mu_j \in \mathbb{C}, \ell, j = 1, \ldots, n, \]
on one infers
\[ \partial_x \alpha_{Q_0}(D_{\hat{\mu}(x)}) = \partial_x \left( \sum_{j=1}^{n} \int_{Q_0} \mu_j(x) \right) \]
\[ = \sum_{j=1}^{n} \mu_j(x) \sum_{\ell=1}^{n} \frac{\mu_j^{t-1}(x)}{y(\mu_j(x))} \]
\[ = \sum_{j=1}^{n} \sum_{\ell=1}^{n} \frac{-2i\mu_j^{t-1}(x)\xi(\ell)}{\prod_{k=1, k \neq j}^{n} (\mu_j(x) - \mu_\ell(x))} \]
\[ = -2i \xi(n) = -i \xi^{(2)}, \quad (3.83) \]
which proves (3.69). The extension of all these results from \( \tilde{\Omega} \) to \( \Omega \) then simply follows from the continuity of \( \alpha_{Q_0} \) and the hypothesis of \( D_{\hat{\mu}(x)} \) being nonspecial on \( \Omega \).
4 Quasi-periodic Solutions

In this section, we extend the the algebro-geometric analysis of Section 2.3 to the time-dependent FL hierarchy.

Throughout this section we assume (2.2) holds.

The time-dependent algebro-geometric initial value problem of the FL hierarchy is to solve the time-dependent \( r \)th FL flow with a stationary solution of the \( n \)th equation as initial data in the hierarchy. More precisely, given \( n \in \mathbb{N}_0 \setminus \{(0,0)\} \), based on the solution \( q^{(0)}, r^{(0)} \) of the \( n \)th stationary HS equation \( s \)–FL\( _n \{q^{(0)}, r^{(0)}\} = 0 \) associated with \( \mathcal{K}_n \) and a set of integration constants \( \{c_{r, \pm}^{(0)}\}_{\ell = 1, \ldots, n} \subset \mathbb{C} \), we want to build up a solution \( q, r \) of the \( r \)th FL flow \( \text{FL}_r(q, r) = 0 \) such that \( q(t_{0, \underline{r}}) = q^{(0)}, r(t_{0, \underline{r}}) = r^{(0)} \) for some \( t_{0, \underline{r}} \in \mathbb{R}, \underline{r} \in \mathbb{N}_0 \setminus \{(0,0)\} \). To emphasize that the integration constants in the definitions of the stationary and the time-dependent FL equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation \( \tilde{V}_r, \tilde{F}_r, \tilde{G}_r, \tilde{H}_r, \tilde{f}_r, \tilde{g}_r, \tilde{h}_r, \tilde{c}_r \) in order to distinguish them from \( V_r, F_r, G_r, H_r, f_r, g_r, h_r, c_r \) with respect to \( \xi \) in the following. In addition, we mark the individual \( r \)th FL flow by a separate time variable \( t_{r, \underline{r}} \in \mathbb{R} \).

Summing up, we are interested in solutions \( q, r \) of the time-dependent algebro-geometric initial value problem

\[
\begin{align*}
\tilde{\text{FL}}_n \{q, r\} &= \left( \begin{array}{l}
q_{t_{r, \underline{r}}} + f_{r, -1, +} + x - 2i q_x g_{r, +} - 2i f_{r, -1, -} + 2i f_{r, -1, -} + 2i g_{r, -1, -} + 2i r_x g_{r, -1, -} \\
\end{array} \right) = 0, \\
(q, r)|_{t_{r, \underline{r}} = t_{0, \underline{r}}} &= (q^{(0)}, r^{(0)}), \\
\text{s-FL}_n \{q^{(0)}, r^{(0)}\} &= \left( \begin{array}{l}
f_{n, -1, +, +} - 2i q_x^{(0)} g_{n, -1, -} + 2i f_{n, -1, -} + 2i f_{n, -1, -} + 2i g_{n, -1, -} + 2i r_x^{(0)} g_{n, -1, -} \\
\end{array} \right) = 0,
\end{align*}
\]

(4.1)

(4.2)

for some \( t_{0, \underline{r}} \in \mathbb{R} \), where \( q = q(x, t_{\underline{r}}), r = r(x, t_{\underline{r}}) \) satisfy (2.2) and a fixed curve \( \mathcal{K}_n \) is associated with the stationary solution \( q^{(0)}, r^{(0)} \) in (1.2). Here

\[
\underline{n} = (n_+, n_-) \in \mathbb{N}_0^2, \underline{r} = (r_+, r_-) \in \mathbb{N}_0^2, n = 2n_+ + 2n_- - 1 \in \mathbb{N}.
\]

Noticing that the FL flows are isospectral, we further assume that (1.2) holds not only for \( t_{\underline{r}} = t_{0, \underline{r}} \), but also for all \( t_{\underline{r}} \in \mathbb{R} \). In terms of Lax pairs this amounts to solving the zero-curvature equations

\[
\begin{align*}
U_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) - \tilde{V}_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x, t_{\underline{r}}), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0, \quad (4.3) \\
- V_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0, \quad (4.4)
\end{align*}
\]

\begin{align*}
U_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) - \tilde{V}_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x, t_{\underline{r}}), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0, \\
- V_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0,
\end{align*}

\begin{align*}
U_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) - \tilde{V}_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x, t_{\underline{r}}), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0, \\
- V_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0,
\end{align*}

\begin{align*}
U_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) - \tilde{V}_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x, t_{\underline{r}}), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0, \\
- V_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0,
\end{align*}

\begin{align*}
U_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) - \tilde{V}_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x, t_{\underline{r}}), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0, \\
- V_{t_{\underline{r}}} (\xi, x, t_{\underline{r}}) + [U(\xi, x), \tilde{V}_r (\xi, x, t_{\underline{r}})] &= 0,
\end{align*}
where

\[
U(\xi, x, t) = \begin{pmatrix}
-iz \\
r_x(x, t) \xi \\
iz
\end{pmatrix}
\begin{pmatrix}
q_x(x, t) \xi \\
q_x(x, t) \xi
\end{pmatrix},
\]
and

\[
V_n(\xi, x, t) = \begin{pmatrix}
iG_n(\xi, x, t) \\
H_n(\xi, x, t)
\end{pmatrix},
\]

\[
\bar{V}_n(\xi, x, t) = \begin{pmatrix}
i\bar{G}_n(\xi, x, t) \\
\bar{H}_n(\xi, x, t)
\end{pmatrix},
\]

for fixed \( n, r \in \mathbb{N}_0 \setminus \{(0,0)\} \). Here \( f_{\ell, \pm}, g_{\ell, \pm}, h_{\ell, \pm}, f, g, \) and \( h \) are defined as in (2.8)–(2.10), with \( q(x) \) replaced by \( q(x, t) \), etc, and with appropriate integration constants \( c_{\ell, \pm}, \ell \in \mathbb{N}, \) and \( \tilde{c}_{s, \pm}, s \in \mathbb{N} \. \) Explicitly, (4.5) and (4.6) are equivalent to

\[
0 = -iG_n(x, t),
\]

\[
q_x(x, t) \xi = -F_{n,x}(\xi, x, t),
\]

\[
r_x(x, t) \xi = H_{n,x}(\xi, x, t),
\]

\[
0 = -iG_n(x, t) + q_x(x, t) H_n(x, t) + q_x(x, t) G_n(x, t),
\]

\[
0 = -F_{n,x}(\xi, x, t) + 2izF_n(\xi, x, t) - 2izq_x(x, t) G_n(\xi, x, t),
\]

\[
0 = F_{n,x}(\xi, x, t) + 2iG_n(\xi, x, t) F_n(\xi, x, t) + 2izq_x(x, t) G_n(\xi, x, t),
\]

\[
0 = -H_{n,x}(\xi, x, t) + 2izH_n(\xi, x, t) + 2izq_x(x, t) G_n(\xi, x, t).
\]
Equation (4.12) then yields

\[
\frac{d}{dx} \det(V_n(\xi, x, t_r)) = \frac{d}{dx} \left( G_n^2(\xi, x, t_r) + F_n(\xi, x, t_r)H_n(\xi, x, t_r) \right) = 0,
\]

(4.13)

and meanwhile (cf. Lemma 4.2)

\[
\frac{d}{dt_r} \det(V_n(\xi, x, t_r)) = \frac{d}{dt_r} \left( G_n(z, x, t_r)^2 + F_n(\xi, x, t_r)H_n(\xi, x, t_r) \right) = 0.
\]

(4.14)

Hence, \( G_n(\xi)^2 + zF_n(\xi)H_n(\xi) \) is independent of variables both \( x \) and \( t_r \), which implies the basic identity (2.26)

\[
G_n(\xi, x, t_r)^2 + F_n(\xi, x, t_r)H_n(\xi, x, t_r) = R_n(\xi)
\]

(4.15)

holds and the hyperelliptic curve \( \mathcal{K}_n \) is still given by (2.29).

As in the stationary context (3.9), (3.10) we introduce

\[
\hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), -\mu_j(x, t_r)^{2n} - G_n(\mu_j(x, t_r), x, t_r)), \quad j = 1, \ldots, n,
\]

(4.16)

and

\[
\hat{\nu}_j(x, t_r) = (\nu_j(x, t_r), \nu_j(x, t_r)^{2n} - G_n(\nu_j(x, t_r), x, t_r)), \quad j = 1, \ldots, n.
\]

(4.17)

In analogy to (3.11), one defines the following meromorphic function

\[
\phi(P, x, t_r) = -\frac{i\xi^{-2n} - y - iG_n(\xi, x, t_r)}{F_n(\xi, x, t_r)}
\]

\[
= \frac{i\xi^{-2n} - y + iG_n(\xi, x, t_r)}{H_n(\xi, x, t_r)},
\]

(4.18)

with divisor of \( \phi(\cdot, x, t_r) \) given by

\[
(\phi(\cdot, x, t_r)) = D_{P_0, \xi}(x, t_r) - D_{P_{\infty}, \mu}(x, t_r).
\]

(4.19)

The time-dependent Baker-Ahiezer function \( \psi \) is then defined in terms of \( \phi \)
ψ(P, x, x₀, tᵣ, t₀) = \left( \begin{array}{c} ψ₁(P, x, x₀, tᵣ, t₀) \\ ψ₂(P, x, x₀, tᵣ, t₀) \end{array} \right),

ψ₁(P, x, x₀, tᵣ, t₀) = \exp \left( \int_{t₀}^{tᵣ} ds(i\tilde{G}_r(ξ, x₀, s) - \tilde{F}_r(ξ, x₀, s)\phi(P, x₀, s)) + \int_{x₀}^{x} dx' (-iz + qₓ(x', tᵣ)\xiφ(P, x', tᵣ)) \right),

ψ₂(P, x, x₀, tᵣ, t₀) = φ(P, x, tᵣ)ψ₁(P, x, x₀, tᵣ, t₀),

P = (ξ, y) ∈ K \backslash \{P₀, P∞\}, (x, tᵣ) ∈ R².

with fixed (x₀, t₀, r) ∈ R².

The following lemma records basic properties of φ and ψ in analogy to the stationary case discussed in Lemma 3.1.

Lemma 4.1. Assume (2.22) and suppose that (4.3), (4.4) hold.

(i) Let P = (ξ, y) ∈ K \backslash \{P₀, P∞\} and (x, x₀, tᵣ, t₀) ∈ R⁴. Then φ satisfies

φₓ(P) = rₓξ + 2izφ(P) - qₓξφ²(P),

and

(qₓξφ(P))ₜᵣ = \left( -\tilde{F}_rφ(P) + i\tilde{G}_r \right)ₜᵣ,

φₜᵣ(P) = (qₓξ)⁻¹ \left( 2iz\tilde{F}_rφ(P) - \tilde{F}_rφₓ(P) + i\tilde{G}_rφₓ,φ(P) \right) - 2i\tilde{G}_rφ(P)

= \tilde{H}_r - 2i\tilde{G}_rφ(P) + \tilde{F}_rφ²(P),

φ(P)φ(P*) = \frac{Hₙ(ξ)}{Fₙ(ξ)},

φ(P) + φ(P*) = \frac{2iGₙ(ξ)}{Fₙ(ξ)},

φ(P) - φ(P*) = - \frac{2ix² - y(P)}{Fₙ(ξ)}.

(ii) Assuming P = (z, y) ∈ K \backslash \{P₀, P∞\}, then ψ satisfies

ψₓ(P) = U(ξ)ψ(P),

Vₙ(ξ, x)ψ(P) = iξ² - y(P)ψ(P),

ψₜᵣ(P) = \tilde{V}_r(ξ)ψ(P).
and one derives

\[
\psi_1(P, x, x_0, t_r, t_0, \xi) = \sqrt{\frac{F_n(\xi, x, t_r)}{F_n(\xi, x_0, t_0, \xi)}} \exp\left( \int_{t_0}^{t_r} ds \left( \frac{i \xi_{2n-1} y\bar{F}_r(\xi, x_0, s)}{F_n(\xi, x_0, s)} \right) \right) - \int_{x_0}^{x} dx' \left( \frac{q_x(x') \xi^{-2n+1} y(P)}{F_n(\xi, x')} \right),
\]

(4.31)

and

\[
\psi_1(P, x, x_0, t_r, t_0, \xi) \psi_1(P^*, x, x_0, t_r, t_0, \xi) = \frac{F_n(\xi, x, t_r)}{F_n(\xi, x_0, t_0, \xi)}, \quad (4.32)
\]

\[
\psi_2(P, x, x_0, t_r, t_0, \xi) \psi_2(P^*, x, x_0, t_r, t_0, \xi) = \frac{H_n(\xi, x, t_r)}{F_n(\xi, x_0, t_0, \xi)}, \quad (4.33)
\]

\[
\psi_1(P, x, x_0, t_r, t_0, \xi) \psi_2(P^*, x, x_0, t_r, t_0, \xi) + \psi_1(P^*, x, x_0, t_r, t_0, \xi) \psi_2(P, x, x_0, t_r, t_0, \xi) = 2iG_n(\xi, x, t_r) \quad (4.34)
\]

In addition, as long as the zeros of \( F_n(\cdot, x, t_r) \) are all simple for \((x, t_r) \in \Omega, \Omega \subseteq \mathbb{R}^2 \) open and connected, \( \psi(P, x_0, t_r, t_0, \xi) \) is meromorphic on \( \mathcal{K}_n \backslash \{P_0, \pm \} \) for \((x, t_r), (x_0, t_0, \xi) \in \Omega \).

Proof. Equation (4.22), (4.25) - (4.27), (4.32) - (4.34) are proved as in Lemma 3.1. To prove (4.23) and (4.24) one first observes that

\[
(q_x \phi)_{t_r} + \xi^{-1} (\bar{F}_r \phi - i\bar{G}_r)_x C \exp\left( \int_{x_0}^{x} dx' (2iz - 2q_x \xi \phi + \frac{q_{xx}}{q_x}) \right),
\]

(4.36)

where the left-hand side is meromorphic in a neighborhood of \( P_{\infty^+} \), while the right-hand side is not meromorphic near \( P_{\infty^-} \) only if \( C = 0 \). This proves (4.23). Equation (4.24) is an immediate consequence of (4.12) and (4.23). Relations (4.28) - (4.30) are clear from (4.20), (4.21), (4.22) and (4.24). (4.31) follows by (4.12), (4.13), (4.20) and (4.39). That \( \psi_1(\cdot, x, x_0, t_r, t_0, \xi) \) is meromorphic on \( \mathcal{K}_n \backslash \{P_0, \pm \} \) if \( F_n(\cdot, x, t_r) \) has only simple zeros is a consequence of

\[
-iz + q_x \xi \phi(P, x', t_r) \bigg|_{P \rightarrow \tilde{P}_r(x', t_r)} \partial_{x'} \ln(\tilde{F}_n(\xi, x', t_r)) + O(1)
\]

(4.37)
as \( \xi \rightarrow \mu_j(x', t_\underline{\omega}) \), using (4.12), (4.16) and (4.18) and

\[
- \tilde{F}_\underline{\omega}(\xi, x_0, s)\phi(P, x_0, s) = \partial_t \ln(F_{\underline{\omega}}(\xi, x_0, s)) + O(1),
\]

using (4.12), (4.16) and (4.39). This follows from (4.20) by restricting \( P \) to a sufficiently small neighborhood \( \mathcal{U}_j(x_0) \) of \( \{ \tilde{\mu}_j(x_0, s) \in \mathcal{K}_n | x_0, s \in \Omega, s \in [t_0, t_\underline{\omega}] \} \) such that \( \tilde{\mu}_k(n_0, s) \in \mathcal{U}_j(x_0) \) for all \( s \in [t_0, t_\underline{\omega}] \) and for all \( k \in \{ 1, \ldots, n \} \setminus \{ j \} \) and by simultaneously restricting \( P \) to a sufficiently small neighborhood \( \mathcal{U}_j(t_\underline{\omega}) \) of \( \{ \tilde{\mu}_j(x', t_\underline{\omega}) \in \mathcal{K}_n | (x', t_\underline{\omega}) \in \Omega, x' \in [x_0, x] \} \) such that \( \tilde{\mu}_k(x', t_\underline{\omega}) \notin \mathcal{U}_j(t_\underline{\omega}) \) for all \( x' \in [x_0, x] \) and all \( k \in \{ 1, \ldots, n \} \setminus \{ j \} \).

Next we consider the \( t_\underline{\omega} \)-dependence of \( F_{\underline{\omega}}, G_{\underline{\omega}}, H_{\underline{\omega}} \).

**Lemma 4.2.** Assume (2.22) and suppose that (4.33), (4.34) hold. Then

\[
F_{n,t_\underline{\omega}} = 2i(\tilde{G}_n F_{\underline{\omega}} - G_{\underline{\omega}}\tilde{F}_n),
\]

\[
G_{n,t_\underline{\omega}} = i(\tilde{F}_n H_{\underline{\omega}} - F_{\underline{\omega}}\tilde{H}_n),
\]

\[
H_{n,t_\underline{\omega}} = 2i(G_{\underline{\omega}}\tilde{H}_n - \tilde{G}_n H_{\underline{\omega}}).
\]

In addition, (4.39)-(4.41) are equivalent to

\[
- V_{n,t_\underline{\omega}} + [\tilde{V}_\underline{\omega}, V_{\underline{\omega}}] = 0,
\]

and hence (4.44) holds.

**Proof.** We proves (4.39) by using (4.27) which shows that

\[
(\phi(P) - \phi(P^*))_{t_\underline{\omega}} = \frac{2i\xi^{-2n}y F_{n,t_\underline{\omega}}}{F_{\underline{\omega}}^2}.
\]

However, the left-hand side of (4.43) also equals

\[
(\phi(P) - \phi(P^*))_{t_\underline{\omega}} = \frac{2i\xi^{-2n}y (\tilde{G}_n F_{\underline{\omega}} - \tilde{F}_n G_{\underline{\omega}})}{F_{\underline{\omega}}^2},
\]

using (4.21), (4.26) and (4.27). Combing (4.43) and (4.44) proves (4.39). Similarly, starting from (4.26)

\[
(\phi(P) + \phi(P^*))_{t_\underline{\omega}} = 2iF_{n,t_\underline{\omega}}^2(G_{n,t_\underline{\omega}} F_{\underline{\omega}} - F_{n,t_\underline{\omega}} G_{\underline{\omega}})
\]

yields (4.40). (4.41) is a consequence of (4.12), (4.39) and (4.40). Finally, differentiating \( G_{\underline{\omega}}(\xi, x, t_\underline{\omega})^2 + F_{\underline{\omega}}(\xi, x, t_\underline{\omega})H_{\underline{\omega}}(\xi, x, t_\underline{\omega}) \) with respect to \( t_\underline{\omega} \), and using (4.39)-(4.41) then yields \( R_{n,t_\underline{\omega}} = 0 \), or equivalently, (4.44).

Next we turn to the Dubrovin-type equations, which governs the dynamics of \( \mu_j \) and \( \nu_j \) with respect to variations of \( x \) and \( t_\underline{\omega} \).
Lemma 4.3. Suppose (2.2), (4.3), (1.4) on an open and connected interval \( \tilde{\Omega}_\mu \subseteq \mathbb{R}^2 \). Suppose that the zeros \( \{ \mu_j(\cdot) \}_{j=0, \ldots, n} \) of \( \xi^{2n-1}F_n(\cdot) \) remain distinct and nonzero on \( \tilde{\Omega}_\mu \). Then \( \{ \tilde{\mu}_j(x) \}_{j=0, \ldots, n} \) defined by (4.16), satisfies the following first-order system of differential equations

\[
\mu_{j,x}(x, t_r) = \frac{-2iy(\tilde{\mu}_j(x, t_r))}{\prod_{k=1, k \neq j}^{n}(\mu_j(x, t_r) - \mu_k(x, t_r))},
\]

(4.46)

\[
\mu_{j,t_r}(x, t_r) = \frac{-2iy(\tilde{\mu}_j(x, t_r))\tilde{F}_n(\mu_j(x, t_r))}{q_x(x, t_r)\mu_j(x, t_r)\prod_{k=1, k \neq j}^{n}(\mu_j(x, t_r) - \mu_k(x, t_r))},
\]

(4.47)

Next, assume \( \mathcal{K}_n \) to be nonsingular and introduce initial condition

\[
\{ \tilde{\mu}_j(x_0, t_{0,r}) \}_{j=1, \ldots, n} \subset \mathcal{K}_n
\]

(4.48)

for some \( (x_0, t_{0,r}) \in \mathbb{R}^2 \), where \( \mu_j(x_0, t_{0,r}) \neq 0, j = 1, \ldots, n \), are assumed to be distinct. Then there exists an open interval \( \Omega_\mu \subseteq \mathbb{R} \), with \( x_0 \in \Omega_\mu \), such that the initial value problem (4.46), (4.48) has a unique solution \( \{ \tilde{\mu}_j \}_{j=1, \ldots, n} \subset \mathcal{K}_n \) satisfying

\[
\tilde{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 0, \ldots, n
\]

(4.49)

and \( \mu_j, j = 1, \ldots, n \), remain distinct and nonzero on \( \Omega_\mu \).

For the zeros \( \{ \nu_j(\cdot) \}_{j=1, \ldots, n} \) of \( \xi^{2n-1}H_n(\cdot) \) similar statements hold with \( \mu_j \) and \( \Omega_\mu \) replaced by \( \nu_j \) and \( \Omega_\nu \), etc. In particular, \( \{ \nu_j \}_{j=1, \ldots, n} \), defined by (1.17), satisfies the system

\[
\nu_{j,x}(x, t_r) = \frac{-2iy(\tilde{\nu}_j(x, t_r))}{\prod_{k=1, k \neq j}^{n}(\nu_j(x, t_r) - \nu_k(x, t_r))},
\]

(4.50)

\[
\nu_{j,t_r}(x, t_r) = \frac{2iy(\tilde{\nu}_j(x, t_r))\tilde{H}_n(\nu_j(x, t_r))}{r_x(x, t_r)\nu_j(x, t_r)\prod_{k=1, k \neq j}^{n}(\nu_j(x, t_r) - \nu_k(x, t_r))},
\]

(4.51)

\( j = 1, \ldots, n, \quad x \in \Omega_\nu \).

Proof. It suffices to prove (1.47) since the argument for (1.50) is analogous and that for (4.46) and (4.51) has been given in the proof of Lemma 3.2.

Inserting \( \xi = \mu_j(x, t_r) \) into (4.39), observing (4.16), yields

\[
\tilde{F}_n(x, t_r)(\mu_j) = -q_x\mu_j^{-2n-1} \prod_{k=1, k \neq j}^{n}(\mu_j - \mu_k) = -2iG_n(\mu_j)\tilde{F}_n(\mu_j),
\]

\[
= 2i\xi^{-2n-y}(\tilde{\mu}_j)\tilde{F}_n(\mu_j).
\]

(4.52)
which indicates \((4.47)\). □

Since the stationary trace formulas for \(f_{\ell,\pm}\) and \(h_{\ell,\pm}\) in terms of symmetric functions of the zeros \(\mu_j\) and \(\nu_\ell\) of \(\ell^{2n-1}F_n(\cdot)\) and \(\ell^{2n-1}H_n(\cdot)\) in Lemma 3.2 extend line by line to the corresponding time-dependent setting, we next record their \(t_\text{r}\)-dependent analogs without proof. For simplicity we again confine ourselves to the simplest cases only.

\textbf{Lemma 4.4.} Assume hypothesis \((2.2)\) and suppose that \((4.3)\) and \((4.4)\) hold. Then,

\[
\frac{q_{xx}}{2iq_x} - \frac{1}{2} q_x r_x - c_{1,+} = -\sum_{j=1}^{n} \mu_j, \tag{4.53}
\]

\[
\frac{r_{xx}}{2ir_x} + \frac{1}{2} q_x r_x + c_{1,+} = -\sum_{j=1}^{n} \nu_j, \tag{4.54}
\]

\[
\frac{iq}{2q_x} = (-1)^{n} \prod_{j=1}^{n} \mu_j, \tag{4.55}
\]

\[
\frac{ir}{2r_x} = (-1)^{n-1} \prod_{j=1}^{n} \nu_j. \tag{4.56}
\]

Next we turn to the asymptotic expansions of \(\phi\) and \(\psi\) in a neighborhood of \(P_{\pm}\) and \(P_{0,\pm}\).

\textbf{Lemma 4.5.} Assume hypothesis \((2.2)\) and suppose that \((4.3)\) and \((4.4)\) hold. Moreover, let \(P = (\xi,y) \in K_n \setminus \{P_{\pm}, P_{0,\pm}\}, (x,t_\text{r}) \in \mathbb{R}^2, (x,x_0,t_0,\text{r},t_\text{r}) \in \mathbb{R}^4\). Then,
It remains to investigate
\[ \psi(P, x, t_x) = \begin{cases} \frac{2i[q_x(x, t_x)]^{-1}}{t} \zeta + O(\zeta^3), & P \to P_{\infty+}, \\ \frac{[ir_x(x, t_x)]/2}{t} \zeta + O(\zeta^3), & P \to P_{\infty-}, \end{cases} \]  
(4.57)

Next we compute the asymptotic expansions of the integrand in (4.61). Focusing on the homogeneous coefficients first, and then using the relations
\[ \tilde{F}_\zeta = \sum_{s=1}^{r_+} \hat{c}_{r_+ - s} F_{s, +} + O(1), \quad \tilde{G}_\zeta = \sum_{s=1}^{r_+} \hat{c}_{r_+ - s} G_{s, +} + O(1), \]
one finds as $P \to P_{\infty \pm}$,
\[
i\tilde{G} \phi = i\tilde{G} + F_\mu F_\mu^{-1} (i\xi - 2n - y) - iG_\mu
\]
\[
= \frac{F_\mu F_\mu^{-1} + i\xi - 2n - y\tilde{F}_\mu}{F_\mu} \quad \zeta \to 0,
\]
\[
= \pm i \sum_{s=1}^{r_+} c_{r+s} \zeta^{-2s} + O(1), \quad \zeta = 1/\xi.
\quad (4.62)
\]

Insertion of (4.62) into (4.61) then proves (4.59) as $P \to P_{\infty \pm}$.

Similarly, as $P \to P_{0 \pm}$,
\[
i\tilde{G} \phi = i\tilde{G} + F_\mu F_\mu^{-1} (i\xi - 2n - y) - iG_\mu
\]
\[
= \frac{F_\mu F_\mu^{-1} + i\xi - 2n - y\tilde{F}_\mu}{F_\mu} \quad \zeta \to 0,
\]
\[
= \pm i \sum_{s=1}^{r_+} c_{r+s} \zeta^{-2s} + O(1), \quad \zeta = \xi.
\quad (4.63)
\]

Insertion of (4.63) into (4.61) then proves (4.60) as $P \to P_{0 \pm}$.

Next, we turn to the principal result of this section, the representation of $\phi, \psi, 1, q, r$ in terms of Riemann theta function associated with $K_n$, assuming $n = (n_- , n_+ ) \in \mathbb{N}_0^2 \backslash \{(0,0)\}$ for the remainder of this section. In addition to (3.55) and (3.58), let $\Omega_{P_{\infty \pm}, k}$ and $\Omega_{P_{0 \pm}, k}$ be the normalized differentials of the second kind with a unique pole at $P_{\infty \pm}$ and $P_{0 \pm}$, respectively, and principal parts
\[
\Omega_{P_{\infty \pm}, k} = (\zeta^{-2k} + O(1)) d\zeta, \quad P \to P_{\infty \pm}, \quad \zeta = \xi^{-1}, \quad k \in \mathbb{N}_0,
\quad (4.64)
\]
\[
\Omega_{P_{0 \pm}, k} = (\zeta^{-2k} + O(1)) d\zeta, \quad P \to P_{0 \pm}, \quad \zeta = \xi^{-1}, \quad k \in \mathbb{N}_0,
\quad (4.65)
\]
with vanishing $a$-periods,
\[
\int_{a_j} \Omega_{P_{\infty \pm}, k} = \int_{a_j} \Omega_{P_{0 \pm}, k} = 0, \quad j = 1, \ldots, n.
\]

Moreover, we define
\[
\Omega_{\Omega} = \left( \sum_{s=1}^{r_-} 2sc_{r+s} (\Omega_{P_{0 \pm}, 2s-1} - \Omega_{P_{0 \pm}, 2s-1}) + \sum_{s=1}^{r_+} 2sc_{r+s} (\Omega_{P_{\infty \pm}, 2s-1} - \Omega_{P_{\infty \pm}, 2s-1}) \right)
\quad (4.66)
\]
and abbreviate
\[
\tilde{\Omega}_L^{\pm} = \lim_{t \to t_{0,\pm}} \left( \int_{Q_0}^P \tilde{\Omega}_L^{(2)} \pm \sum_{s=1}^{r_+ \pm} \tilde{c}_{r_+ \pm s} \zeta^{-2s} \right),
\]
and
\[
\tilde{\Omega}_L^{0,\pm} = \lim_{t \to t_{0,\pm}} \left( \int_{Q_0}^P \tilde{\Omega}_L^{(2)} \pm \sum_{s=1}^{r_- \pm} \tilde{c}_{r_- \pm s} \zeta^{-2s} \right).
\]

The vector of \( b \)-periods of \( \tilde{\Omega}_L^{(2)} \) is denoted by
\[
\tilde{U}_L^{(2)} = (\tilde{U}_L^{(2)}_1, \ldots, \tilde{U}_L^{(2)}_n), \quad \tilde{U}_L^{(2)}_{j} = \frac{1}{2\pi i} \int_{b_j} \tilde{\Omega}_L^{0,\pm}, \quad j = 1, \ldots, n.
\]

**Theorem 4.6.** Assume (2.2) and suppose that (4.3) and (4.4) hold subject to the constraint (3.2) on \( \Omega \), where \( \Omega \subseteq \mathbb{R}^2 \) is open and connected. In addition, let \( P \in \mathcal{K}_n \{ P_{0,\pm} \} \), \( (x,t_x) \in \mathbb{R}^2 \) and \( (x,x_0,t_x,t_{0,x}) \in \mathbb{R}^4 \). Moreover, suppose that \( D_{\tilde{\rho}(x,t_x)} \), or equivalently, \( D_{\tilde{\varphi}(x,t_x)} \), is nonspecial for \((x,t_x) \in \Omega\). Then \( \phi, \psi, q, r \) admit the following representations
\[
\phi(P,x,t_x) = C(x,t_x) \frac{\theta(\xi(P,\tilde{\mu}(x,t_x)))}{\theta(\xi(P,\hat{\mu}(x,t_x)))} \exp \left( \int_{Q_0}^P \tilde{\Omega}_L^{(3)} \right),
\]
and
\[
\psi_1(P,x,x_0,t_x,t_{0,x}) = C(x,x_0,t_x,t_{0,x}) \frac{\theta(\xi(P,\tilde{\mu}(x,t_x)))}{\theta(\xi(P,\hat{\mu}(x_0,t_{0,x})))} \times \exp \left( -i(x-x_0) \int_{Q_0}^P \tilde{\Omega}_L^{(2)} - i(t_x-t_{0,x}) \int_{Q_0}^P \tilde{\Omega}_L^{(2)} \right),
\]
where
\[
C(x,t_x) = \frac{\theta(\xi(P_{0,-},\tilde{\mu}(x,t_x)))}{q(x_0,t_{0,x})} \frac{\theta(\xi(P_{0,-},\hat{\mu}(x_0,t_{0,x})))}{\theta(\xi(P_{0,-},\hat{\mu}(x,t_x)))} \times e^{-i(x-x_0)(e_0,-e_0,+) - i(t_x-t_{0,x})\tilde{\Omega}_L^{0,-}},
\]
and
\[
C(x,x_0,t_x,t_{0,x}) = \frac{\theta(\xi(P_{0,-},\tilde{\mu}(x,t_x)))}{\theta(\xi(P_{0,-},\hat{\mu}(x_0,t_{0,x})))} e^{i(x-x_0)e_0, -i(t_x-t_{0,x})\tilde{\Omega}_L^{0,-}}.
\]

The Abel map linearizes the auxiliary divisors \( D_{\tilde{\varphi}(x,t_x)}, D_{\tilde{\varphi}(x,t_x)} \) in the sense that
\[
\mathcal{A}_Q^{0}(\tilde{D}_{\tilde{\varphi}(x,t_x)}) = \mathcal{A}_Q^{0}(\tilde{D}_{\tilde{\varphi}(x,t_x)}) - \tilde{U}_L^{(2)}(x-x_0) - \tilde{U}_L^{(2)}(t-t_{0,x}),
\]
and
\[
\mathcal{A}_Q^{0}(\tilde{D}_{\tilde{\varphi}(x,t_x)}) = \mathcal{A}_Q^{0}(\tilde{D}_{\tilde{\varphi}(x,t_x)}) - \tilde{U}_L^{(2)}(x-x_0) - \tilde{U}_L^{(2)}(t-t_{0,x}).
\]
Moreover, one derives

\[
q(x, t_\xi) = q(x_0, t_{0,\xi}) \frac{\theta(\xi(P_0, -\xi(x_0, t_0, \xi)))}{\theta(\xi(P_0, -\xi(x, t_\xi)))} \frac{\theta(\xi(P_0, +\xi(x, t_\xi)))}{\theta(\xi(P_0, +\xi(x_0, t_{0,\xi})))} \\
\times e^{i(x-x_0)(\epsilon_0, -\epsilon_0, +i(t_{0,\xi}-t_{0,\xi}))} \tilde{\Omega}_-^{0,0} - \tilde{\Omega}_-^{0,0},
\]  

(4.76)

\[
r(x, t_\xi) = r(x_0, t_{0,\xi}) \frac{\theta(\xi(P_0, -\xi(x, t_\xi)))}{\theta(\xi(P_0, -\xi(x_0, t_{0,\xi})))} \frac{\theta(\xi(P_0, +\xi(x, t_\xi)))}{\theta(\xi(P_0, +\xi(x_0, t_{0,\xi})))} \\
\times e^{-i(x-x_0)(\epsilon_0, -\epsilon_0, +i(t_{0,\xi}-t_{0,\xi}))} \tilde{\Omega}_-^{0,0} + \tilde{\Omega}_-^{0,0},
\]  

(4.77)

\[
\hat{q}(x_0, t_{0,\xi})r(x, t_\xi) = \frac{\theta(\xi(P_0, -\xi(x, t_\xi)))}{\theta(\xi(P_0, -\xi(x_0, t_{0,\xi})))} \frac{\theta(\xi(P_0, +\xi(x, t_\xi)))}{\theta(\xi(P_0, +\xi(x_0, t_{0,\xi})))} \\
\times e^{i\omega_0 - i\omega_0,0}.
\]  

(4.78)

**Proof.** As in the corresponding stationary case we temporarily assume

\[
\mu_j(x, t_\xi) \neq \mu_{j'}(x, t_\xi), \text{ for } j \neq j', \ (x, t_\xi) \in \tilde{\Omega}
\]  

(4.79)

for appropriate \( \tilde{\Omega} \subseteq \Omega \) and define the right-hand side of (4.71) to be \( \tilde{\psi}_1 \). We intend to prove \( \tilde{\psi}_1 = \psi_1 \), where \( \psi_1 \) is given in (4.20). For that purpose we first investigate the local zeros and poles of \( \psi_1 \) and note

\[
\hat{q}(x', t_\xi)\phi(P, x', t_\xi) \bigg|_{P \to \hat{\mu}_j(x', t_\xi)} = \frac{2iy \hat{\mu}_j(x, t_\xi)}{\prod_{k=1,k \neq j}^{n} (\mu_j(x', t_\xi) - \mu_k(x', t_\xi))} \times \frac{1}{\xi - \mu_j(x', t_\xi)} + O(1)
\]  

\[
\bigg|_{P \to \hat{\mu}_j(x', t_\xi)} = \partial^x \ln(\xi - \mu_j(x', t_\xi)) + O(1).
\]  

(4.80)

\[
-\tilde{F}_\xi(x_0, s, s)\phi(P, x_0, s) = \frac{2iy \hat{\mu}_j(x_0, s)}{\prod_{k=1,k \neq j}^{n} (\mu_j(x_0, s) - \mu_k(x_0, s))} \times \frac{1}{\xi - \mu_j(x_0, s)} \times \tilde{F}_\xi(x_0, s) q_x(x_0, s) + O(1)
\]  

\[
= \partial^s \ln(\xi - \mu_j(x_0, s)) + O(1),
\]  

(4.81)
using \([4.16], [4.18], [4.46]\) and [4.47]. Thus
\[
\psi_1(P, x, x_0, t_r, t_0, x) = \begin{cases} 
(\xi - \mu_j(x, t_r))O(1), & \text{as } P \to \hat{\mu}_j(x, t_r) \neq \hat{\mu}_j(x_0, t_0, x), \\
O(1), & \text{as } P \to \hat{\mu}_j(x, t_r) = \hat{\mu}_j(x_0, t_0, x), \\
(\xi - \mu_j(x_0, t_0, x))^{-1}O(1), & \text{as } P \to \hat{\mu}_j(x_0, t_0, x) \neq \hat{\mu}_j(x, t_r), 
\end{cases}
\]
for \(P = (\xi, y) \in \mathcal{K}_n\), (4.82)

with \(O(1) \neq 0\) and hence \(\psi_1\) and \(\tilde{\psi}_1\) have identical zeros and poles on \(\mathcal{K}_n \setminus \{P_{\infty, 0}, P_{0, \pm}\}\) which are all simple. It remain to study the behavior of \(\psi_1\) near \(P_{\infty, 0}, P_{0, \pm}\). One infers from \([1.59], [4.60], [4.71]\) that \(\tilde{\psi}_1\) and \(\psi_1\) have the same essential singularities at \(P_{\infty, 0}, P_{0, \pm}\) and the Riemann-Roch-type uniqueness result [7] proves that \(\psi_1\) and \(\tilde{\psi}_1\) coincide up to normalization. This proves (4.71) for some \(C(x, x_0, t_r, t_0, x) \in C^\infty(\mathbb{R}^4)\). The expression (4.19) for the divisor \(\phi\) then yields
\[
\phi(P, x, t_r) = C(x, t_r) \frac{\theta(\xi(P_0, \hat{\mu}(x, t_r)))}{\theta(\xi(P_0, \hat{\mu}(x, t_r)))} \exp \left( \int_{Q_0}^{P} \Omega \right),
\]
where \(C(x, t_r)\) is in dependent of \(P \in \mathcal{K}_n\). Hence (4.58) implies
\[
\frac{1}{q(x, t_r)} = C(x, t_r) \frac{\theta(\xi(P_0, \hat{\mu}(x, t_r)))}{\theta(\xi(P_0, \hat{\mu}(x, t_r)))} e^{\omega_0^0},
\]
(4.84)

\[
r(x, t_r) = C(x, t_r) \frac{\theta(\xi(P_0, \hat{\mu}(x, t_r)))}{\theta(\xi(P_0, \hat{\mu}(x, t_r)))} e^{\omega_0^0},
\]
(4.85)

\[
q(x, t_r) r(x, t_r) = \frac{\theta(\xi(P_0, \hat{\mu}(x, t_r))) \theta(\xi(P_0, \hat{\mu}(x, t_r)))}{\theta(\xi(P_0, \hat{\mu}(x, t_r)))} e^{\omega_0^0 - \omega_0^0}. \]
(4.86)

The asymptotic behavior (4.60) of \(\psi_1\) near \(P_{0, \pm}\) then yields
\[
\tilde{\psi}_1(P, x, x_0, t_r, t_0, x) \approx C(x, x_0, t_r, t_0, x) \frac{\theta(\xi(P_0, \hat{\mu}(x, t_r)))}{\theta(\xi(P_0, \hat{\mu}(x, t_r)))} e^{-i(x - x_0) \omega_0^0 - i(t_r - t_0, t_r)} + O(\zeta) \quad \text{as } P \to P_{0, -},
\]
(4.87)

\[
\tilde{\psi}_1(P, x, x_0, t_r, t_0, x) \approx C(x, x_0, t_r, t_0, x) \frac{\theta(\xi(P_0, \hat{\mu}(x, t_r)))}{\theta(\xi(P_0, \hat{\mu}(x, t_r)))} e^{-i(x - x_0) \omega_0^0 - i(t_r - t_0, t_r)} + O(\zeta) \quad \text{as } P \to P_{0, +}.
\]
(4.88)

A comparison of (4.60), (4.87) and (4.88) then yields (4.73) and (4.76). (4.72) follows from (4.76) and (4.84). (4.77) is a consequence of (4.72) and
The linearization property of the Abel map in (4.74) and (4.75) a standard investigation of the differentials \( \Omega_i(x, x_0, t_r, t_0, r) = d \ln(\psi_i(\cdot, x, x_0, t_r, t_0, r)) \), \( i = 1, 2 \) (c.f. [23]).

5 \( n \)-Dark Solitons

In this section, we will link the quasi-periodic solutions of FL hierarchy derived in section 4 with the \( n \)-dark solitons through a limiting procedure.

It is known that the solutions obtained after degeneration of the hyperelliptic spectral curve depend on the ramification points of \( K_n \) and different choices may lead to different solutions such as solitons, cusps or peakons, breathers, etc. in some other integrable models. To derive the \( n \)-dark solitons of FL hierarchy, we degenerate the hyperelliptic curve \( K_n \) of genus \( n \) into a genus zero algebraic curve by pinching all \( a_j \)-cycles of the associated Riemann surface (cf. [3]). We assume that the ramification points \( E_m \) are ordered according to

\[
\text{Re}(E_j) \leq \text{Re}(E_k), \quad j < k, \quad j, k = 0, \ldots, 2n + 1,
\]

and consider the limit

\[
E_{2m-1}, E_{2m} \to \alpha_m, \quad m = 1, \ldots, n, \quad (5.1)
\]

where \( \alpha_m \neq \alpha_k \) for \( m \neq k \). Putting \( E_0 = -\beta, E_{2n+1} = \beta \) with \( \beta > 0 \), one finds

\[
K_n \to K_n^0: y^2 = (\xi^2 - \beta^2) \prod_{j=1}^{n} (\xi - \alpha_j)^2, \quad (5.2)
\]

where \( \beta \neq \alpha_j, j = 1, \ldots, n \). Then the holomorphic differentials \( \omega_j \) (cf. [3.51]),

\[
\omega_j = \sum_{\ell=1}^{n} c_j(\ell) \xi^{\ell-1} \left( \prod_{m=0}^{2n+1} (\xi - E_m) \right)^{-1/2},
\]

\[
\to \sum_{\ell=1}^{n} c_j(\ell) \xi^{\ell-1} \left( (\xi^2 - \beta^2) \prod_{m=1}^{n} (\xi - \alpha_m)^2 \right)^{-1/2}. \quad (5.3)
\]
Using the normalization condition \( \int_{a_k} \omega_j = \delta_{jk} \) and

\[
\int_{a_k} \omega_j \to \int_{a_k} \sum_{\ell=1}^n c_0^j(\ell) \xi^{\ell-1} \left( \sqrt{\xi^2 - \beta^2} \prod_{m=1}^n (\xi - \alpha_m) \right)^{-1} = 2\pi i \varphi_j(\alpha_k) \left( \sqrt{\alpha_k^2 - \beta^2} \prod_{m=1, m \neq k}^n (\alpha_k - \alpha_m) \right)^{-1},
\]

(5.4)

one concludes

\[
\varphi_j(\alpha_k) = \frac{1}{2\pi i} \delta_{jk} \sqrt{\alpha_k^2 - \beta^2} \prod_{m=1, m \neq k}^n (\alpha_k - \alpha_m).
\]

(5.5)

Here we employ the notation

\[
\varphi_j(\xi) = \sum_{\ell=1}^n c_0^j(\ell) \xi^{\ell-1}.
\]

(5.6)

Especially, one obtains

\[
c_0^j(n) = \sqrt{\alpha_j^2 - \beta^2} \frac{1}{2\pi i}, \quad \varphi_j(\xi) = c_0^j(n) \prod_{m=1, m \neq j}^n (\xi - \alpha_m)
\]

(5.7)

from (5.5). Comparing the coefficients (5.6) and (5.7) yields

\[
c_0^j(n-1) = -c_0^j(n) \sum_{m \neq j}^n \alpha_m = -\sqrt{\alpha_j^2 - \beta^2} \frac{1}{2\pi i} \sum_{m \neq j}^n \alpha_m,
\]

\[
c_0^j(n-2) = c_0^j(n) \sum_{m \neq j}^n \alpha_m \alpha_n = \sqrt{\alpha_j^2 - \beta^2} \frac{1}{2\pi i} \sum_{m \neq j}^n \alpha_m \alpha_n,
\]

etc.

Then using (5.7), one finds

\[
\omega_j \to \omega_j^0 = \sqrt{\alpha_j^2 - \beta^2} \left( \sqrt{\xi^2 - \beta^2} (\xi - \alpha_j) \right)^{-1}.
\]

(5.8)

The elements of Riemann matrix \( \tau = (\tau_{jk}) \)

\[
\tau_{jk} = \int_{b_j} \omega_k \to 2 \int_{\alpha_j}^{\beta} \omega_k^0 = \frac{i}{\pi} \ln \left| \frac{\eta_j + \eta_k}{\eta_j - \eta_k} \right| \equiv \tau_{jk}^0
\]

(5.9)
with \( \eta_k = (\alpha_k - \beta)^{1/2}(\alpha_k + \beta)^{-1/2} \). So for the diagonal elements of \( \tau \), \( \text{Re}(i\tau_{kk}) \to -\infty \) in the limit (5.1). Then one can rewrite the Riemann theta function (3.52) as

\[
\theta(z) = \sum_{k \in \mathbb{Z}^n} \exp \left( 2\pi i \sum_{j=1}^n k_j z_j + 2\pi i \sum_{j<m} \tau_{jm} k_j k_m + \pi i \sum_{j=1}^n \tau_{jj} k_j^2 \right)
\]

\[
= \sum_{k \in \mathbb{Z}^n} \exp \left( 2\pi i \sum_{j=1}^n k_j \left( z_j + \frac{1}{2} \tau_{jj} \right) + 2\pi i \sum_{j<m} \tau_{jm} k_j k_m + \pi i \sum_{j=1}^n \tau_{jj} (k_j - 1) \right)
\]

\[
\sim \sum_{k \in \{0,1\}^n} \exp \left( 2\pi i \sum_{j=1}^n k_j \left( z_j + \frac{1}{2} \tau_{jj} \right) + 2\pi i \sum_{j<m} \tau_{jm} k_j k_m \right), \quad k = (k_1, \ldots, k_n).
\]

(5.10)

**Theorem 5.1.** The vectors \( \vec{U}_0^{(2)}, \vec{U}_r^{(2)} \) in (4.74), (4.75) have an alternative description:

\[
\vec{U}_0^{(2)} = (U_{0,1}^{(2)}, \ldots, U_{0,n}^{(2)}), \quad U_{0,j}^{(2)} = 2 \sum_{\ell=1}^n c_j(\ell) \hat{c}_{\ell+1-n}(E), \quad (5.11)
\]

\[
\vec{U}_r^{(2)} = (\vec{U}_{0,1}^{(2)}, \ldots, \vec{U}_{0,n}^{(2)}), \quad \vec{U}_{0,j}^{(2)} = 2(-1)^n \prod_{m=0}^{2n+1} E_m^{-1/2} \sum_{s=1}^{r_-} \sum_{\ell=1}^{2s} c_j(\ell) \hat{c}_{2s-\ell}(E^{-1})
\]

\[
+ 2 \sum_{s=1}^{r_+} \hat{c}_{r_+-s} \sum_{\ell=1}^n c_j(\ell) \hat{c}_{\ell+2s-n-1}(E). \quad (5.12)
\]

Accordingly, in the limit (5.1)

\[
U_{0,j}^{(2)} \to \left[ U_{0,j}^{(2)} \right] = 2 \sum_{\ell=1}^n c_j(\ell) \hat{c}_{\ell+1-n}(E^0), \quad (5.13)
\]

\[
\vec{U}_{0,j}^{(2)} \to \left[ \vec{U}_{0,j}^{(2)} \right] = 2(-1)^{n+1} \beta^2 \prod_{m=1}^n \alpha_m^{-1} \sum_{s=1}^{r_-} \sum_{\ell=1}^{2s} c_j(\ell) \hat{c}_{2s-\ell}(E^0)^{-1}
\]

\[
+ 2 \sum_{s=1}^{r_+} \hat{c}_{r_+-s} \sum_{\ell=1}^n c_j(\ell) \hat{c}_{\ell+2s-n-1}(E^0). \quad (5.14)
\]

where we use the notation

\[
E^0 = (\beta, -\beta, \alpha_1, \ldots, \alpha_n). \quad (5.15)
\]
Proof. We only prove (5.12) and the proof for (5.11) is similar (or cf. [7]).

One computes
\[
\omega_j = \sum_{\ell=1}^{n} c_j(\ell) \eta_{\ell} = \sum_{\ell=1}^{n} c_j(\ell) \frac{\xi^{\ell-1}}{y(\xi)} d\xi
\]
\[
= \sum_{\ell=1}^{n} c_j(\ell) \xi^{\ell-1} \sum_{k=0}^{\infty} \bar{c}_k(E) \zeta^k d\zeta
\]
\[
= (-1)^n \prod_{m=0}^{2n+1} E_m^{-1/2} \sum_{k=0}^{2s} \left( \sum_{\ell=1}^{k+1} c_j(\ell) \bar{c}_{k+1-\ell}(E) \zeta^k \right) d\zeta
\]
(5.16)

where
\[
\bar{c}_j(E) = \dot{c}_j(E^{-1}).
\]

Then the Bilinear Riemann Relation shows that
\[
\frac{1}{2\pi i} \int_{b_j} \Omega_{F_{0\nu},2s-1}^{(2)} = \frac{1}{2\pi} \sum_{\ell=1}^{n} c_j(\ell) \dot{\epsilon}_{\ell+2s-n-1}(E),
\]
(5.17)
\[
\frac{1}{2\pi i} \int_{b_j} \Omega_{F_{0\nu},2s-1}^{(2)} = -\frac{1}{2\pi} \sum_{\ell=1}^{n} c_j(\ell) \dot{\epsilon}_{\ell+2s-n-1}(E),
\]
(5.18)
\[
\frac{1}{2\pi i} \int_{b_j} \Omega_{F_{0\nu},2s-1}^{(2)} = \frac{1}{2\pi} (-1)^n \prod_{m=0}^{2n+1} E_m^{-1/2} \sum_{\ell=1}^{2s} c_j(\ell) \bar{c}_{2s-\ell}(E),
\]
(5.19)
\[
\frac{1}{2\pi i} \int_{b_j} \Omega_{F_{0\nu},2s-1}^{(2)} = -\frac{1}{2\pi} (-1)^n \prod_{m=0}^{2n+1} E_m^{-1/2} \sum_{\ell=1}^{2s} c_j(\ell) \bar{c}_{2s-\ell}(E),
\]
(5.20)

and hence
\[
\frac{1}{2\pi i} \int_{b_j} \Omega_{F_{0\nu},2s-1}^{(2)} = -2(-1)^n \prod_{m=0}^{2n+1} E_m^{-1/2} \sum_{s=1}^{2s} c_j(\ell) \bar{c}_{r+s-\ell}(E)
\]
\[
+ 2 \sum_{s=1}^{r} \bar{c}_{r+s-\ell} + \sum_{\ell=1}^{n} c_j(\ell) \dot{\epsilon}_{\ell+2s-n-1}(E) \in \mathbb{R}, \quad j = 1, \ldots, n.
\]
(5.21)

Finally, the expressions (5.13) (5.14) follow by (5.11), (5.12). □

In the following we calculate the limit values of the constants \( \Omega_{\nu,0}^{(0)} \), \( \epsilon_{0,+} \).
We introduce the notations
\[ L_n = \frac{(2n-1)!!}{2^n n!}, \quad M_0^m = 1, \]
\[ M_1^m = -\sum_{j=1}^{2s} m_j, \quad M_n^m = (-1)^n \sum_{1 \leq i_1 < \ldots < i_n \leq 2s} m_{i_1} \ldots m_{i_n}, \]
\[ m_j \in \mathbb{C}, \quad 1 \leq j \leq 2s, \quad s \in \mathbb{N}, \]
and the Abel differentials \( \Omega_{P_{b, \pm 2s-1}}^{(2)}, \Omega_{P_{\infty, \pm 2s-1}}^{(2)} \) of second kind are explicitly defined by
\[ \Omega_{P_{\infty, \pm 2s-1}}^{(2)} = \pm \sum_{j=0}^{2s} c_j(E) \xi^{n+2s-j} y(\xi) + \sum_{i=1}^{n} c_j^{(2s-1)} \omega_j, \]
\[ = \pm \prod_{j=1}^{n+2s} (\xi - m_j^s) y(\xi), \]
\[ \Omega_{P_{b, \pm 2s-1}}^{(2)} = \pm (-1)^{n+1} \prod_{m=0}^{2n+1} E_m^{1/2} \sum_{j=0}^{2s} c_j(E) \xi^{-(2s+1)+j} y(\xi) \]
\[ + \sum_{i=1}^{n} c_j^{(2s-1)} \omega_j, \]
\[ = \pm \frac{\xi^{-(2s+1)} \prod_{j=1}^{n+2s} (\xi - \bar{m}_j^s)}{y(\xi)} d\xi, \]
where the constants \( c_j^{(2s-1)}, c_j^{(2s-1)}, m_j, \bar{m}_j \) are determined by the normalization conditions
\[ \int_{a_j} \Omega_{P_{\infty, \pm 2s-1}}^{(2)} = 0, \quad j = 1, \ldots, n, \]
\[ \int_{a_j} \Omega_{P_{b, \pm 2s-1}}^{(2)} = 0, \quad j = 1, \ldots, n. \]

Then we get
\[ 0 = \int_{a_j} \Omega_{P_{\infty, \pm 2s-1}}^{(2)} = \int_{a_j} \frac{\prod_{j=1}^{n+2s} (\xi - m_j^s)}{y(\xi)} d\xi \]
\[ \Rightarrow \int_{a_j} \frac{\prod_{j=1}^{n+2s} (\xi - [m_j^s]_0)}{\sqrt{\xi^2 - \beta^2} \prod_{j=1}^{n+2s} (\xi - \alpha_j)} d\xi. \]
Hence $\alpha_j, j = 1, \ldots, n$ are the roots of polynomials $\prod_{j=1}^{n+2s}(\xi - [m_j^{s}]^0)$ using a standard residue formula. Keeping $s \in \mathbb{N}$ fixed and assuming $[m_{2s+j}^{s}]^0 = \alpha_j, j = 1, \ldots, n,$ and $M_p = 0$ for $p \in \mathbb{Z}\setminus[1,2s]$, one obtains that

$$
\int_{a_j}^{\infty} \Omega_{P_{\infty},2s-1}^{(2)} \frac{\prod_{j=1}^{2s}(\xi - [m_j^{s}]^0)}{\sqrt{\xi^2 - \beta^2}} d\xi \\
\zeta \rightarrow 0 \int_{a_j}^{\infty} \zeta^{-(2s+1)} \left( \sum_{i=1}^{2s} M_i^{[m_j^{s}]^0} \xi^i \right) \left( \sum_{j=0}^{\infty} L_j \beta^{2j} \zeta^{2j} \right) d\zeta \\
= \pm \int_{a_j}^{\infty} \zeta^{-(2s+1)} \left( \sum_{j=0}^{\infty} \sum_{t=0}^{\infty} M_t^{[m_j^{s}]^0} L_j \beta^{2j} \right) \zeta^t d\zeta, \\
= 0, \quad \xi = \zeta^{-1}, \quad P \rightarrow P_{\infty}.
$$

This leads to

$$
\sum_{j=0}^{\infty} M_t^{[m_j^{s}]^0} L_j \beta^{2j} = 0, \quad t = 1, \ldots, 2s.
$$

Therefore, $M_j^{[m_j^{s}]^0}, j = 1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor$, can be derived from the recursion relation above. Explicitly,

$M_1^{[m_j^{s}]^0} = 0,$

$M_2^{[m_j^{s}]^0} = -L_1 \beta^2$

$M_3^{[m_j^{s}]^0} = 0,$

$M_4^{[m_j^{s}]^0} = (L_1^2 - L_2) \beta^4$, etc.

Solving the algebraic equation $\prod_{j=0}^{2s} M_j^{[m_j^{s}]^0} z^{2s-j} = 0$, one gets the explicit expressions for $[m_j^{s}]^0, j = 1, \ldots, 2s$, which depend on the parameter $\beta$. Similarly,

$$
\int_{a_j}^{\infty} \Omega_{P_{0,\pm},2s-1}^{(2)} \frac{\xi^{-(n+2s)} \prod_{j=1}^{2s}(\xi - [m_j^{s}]^0)}{\sqrt{\xi^2 - \beta^2}} d\xi \\
\zeta \rightarrow 0 \int_{a_j}^{\infty} \zeta^{-(2s+1)} \left( \sum_{i=1}^{2s} M_i^{[\tilde{m}_j^{s}]^0} \xi^i \right) \left( \sum_{j=0}^{\infty} L_j \beta^{2j} \zeta^{2j} \right) d\zeta \\
= \pm \frac{\prod_{m=1}^{2s}[\tilde{m}_j^{s}]^0}{\sqrt{-\beta^2}} \int_{a_j}^{\infty} \zeta^{-(2s+1)} \sum_{t=0}^{\infty} \left( \sum_{j=0}^{\infty} M_t^{[\tilde{m}_j^{s}]^0} L_j \beta^{-2j} \right) \zeta^t d\zeta, \\
= 0, \quad \xi = \zeta, \quad P \rightarrow P_{0,\pm}.
$$

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gives rise to
\[
\sum_{j=0}^{\infty} \mathcal{M}_{1-2j}^{\tilde{m}_j^0} \mathcal{L}_j \beta^{-2j} = 0, \quad t = 1, \ldots, 2s. \tag{5.31}
\]
and the explicit constants for \([\tilde{m}_j^0], j = 1, \ldots, 2s.\)

In summary, we have the following conclusions for the constants \(e_0, \tilde{\Omega}_+, \tilde{\Omega}_-\) and related limiting process.

**Theorem 5.2.** In the limit \(\tilde{\mathcal{L}}\),
\[
\Omega_{P_{\infty, 2s-1}}^{(2)} \rightarrow \Omega_{P_{\infty, 2s-1}}^{(2)} = \pm \frac{\Pi_{j=1}^{2s} (\xi - [m_j^0])}{\sqrt{\xi^2 - \beta^2}} d\xi, \tag{5.32}
\]
\[
\Omega_{\tilde{\mathcal{L}}, 2s+1}^{(2)} \rightarrow \Omega_{\tilde{\mathcal{L}}, 2s+1}^{(2)} = \pm \frac{\xi^{-(n+2s)} \Pi_{j=1}^{2s} (\xi - [\tilde{m}_j^0])}{\sqrt{\xi^2 - \beta^2}} d\xi, \tag{5.33}
\]
where \([m_j^0], [\tilde{m}_j^0]\) are \(2s\) roots of polynomials
\[
\prod_{j=0}^{2s} \mathcal{M}_j^{m_j^0} z^{2s-j} = 0, \quad \prod_{j=0}^{2s} \mathcal{M}_j^{\tilde{m}_j^0} z^{2s-j} = 0, \tag{5.34}
\]
and \(\mathcal{M}_j^{m_j^0}, \mathcal{M}_j^{\tilde{m}_j^0}\) are defined by \(5.26\) and \(5.31\), respectively. Let \(Q_0 = (\beta, 0) \in \mathcal{K}_n\) and \(Q_1 = (\beta_1, y(\beta_1))\) be a point near \(P_{0, +}\). Then
\[
\tilde{\Omega}_+ \rightarrow [\tilde{\Omega}_+] = -4 \sum_{s=1}^{r_+} s \tilde{c}_{r-s} \frac{\Pi_{j=1}^{2s} [\tilde{m}_j^0]}{\sqrt{-\beta^2}} \sum_{t=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{t+1} \mathcal{M}_j^{(m_j^0)^{-1}} \mathcal{L}_j \beta^{-2j} \beta_1^{t+1}
\]
\[
+ 4 \sum_{s=1}^{r_+} s \tilde{c}_{r-s} \frac{\Pi_{j=1}^{2s} [\tilde{m}_j^0]}{\sqrt{-\beta^2}} \sum_{t=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{t+1} \mathcal{M}_j^{\tilde{m}_j^0} \mathcal{L}_j \beta^{-2j} \frac{1}{\beta_1}^{t+1}, \tag{5.35}
\]
\[
\tilde{\Omega}_- \rightarrow [\tilde{\Omega}_-] = - [\tilde{\Omega}_+]^0, \tag{5.36}
\]
\[
e_0^+ \rightarrow [e_0^+] = 2 \sum_{j=0}^{\infty} \sum_{t=0}^{\infty} \frac{1}{t+1} \mathcal{M}_j^{m_j^0} \mathcal{L}_j \beta^{-2j} \frac{1}{\beta_1}^{t+1}, \tag{5.37}
\]
\[
e_0^- \rightarrow [e_0^-] = - [e_0^+] = 0. \tag{5.38}
\]
Proof. (5.32)-(5.34) follow by (5.28)-(5.31). We only need to consider the asymptotic behavior near \(P_{0,+}, P_{\infty+}\) since

\[
\int_{Q_0}^P \Omega^{(2)}_{P_{\infty+},2s-1} = - \int_{Q_0}^{P^*} \Omega^{(2)}_{P_{\infty+},2s-1}, \quad \int_{Q_0}^P \Omega^{(2)}_{P_{0,+},2s-1} = - \int_{Q_0}^{P^*} \Omega^{(2)}_{P_{0,+},2s-1},
\]

(5.39)

\(Q_0\) is a branch point). Using (5.28)-(5.34), one obtains

\[
\int_{Q_0}^P [\Omega^{(2)}_{P_{\infty+},2s-1}]^0 = \pm \int_{Q_0}^P \frac{\prod_{j=1}^{2s} (\xi - [m_j])^0}{\sqrt{\xi^2 - \beta^2}} d\xi
\]

\[
\zeta \equiv 0 = \frac{1}{2s} \zeta^{-2s} + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{k} M_{2s+k-2j} \mathcal{L}_j \beta^{2j-k} + O(\zeta),
\]

\[
\xi = \zeta^{-1}, P \to P_{\infty+},
\]

(5.40)

\[
\int_{Q_0}^P [\Omega^{(2)}_{P_{\infty+},2s-1}]^0 = \pm \int_{Q_0}^P \frac{\prod_{j=1}^{2s} (\xi - [m_j])^0}{\sqrt{\xi^2 - \beta^2}} d\xi
\]

\[
\zeta \equiv 0 = \frac{1}{\beta} \prod_{j=1}^{2s} [m_j]^0 \sum_{t=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{t+1} M_{t-2j} \mathcal{L}_j \beta^{-2j} \frac{1}{\beta_1} \beta^{t+1}
\]

\[
- \int_{\beta}^{\beta_1} \prod_{j=1}^{2s} (\xi - [m_j])^0 d\xi + O(\zeta),
\]

\[
\xi = \zeta, P \to P_{0,+},
\]

(5.41)

\[
\int_{Q_0}^P [\Omega^{(2)}_{P_{0,+},2s-1}]^0 = \pm \int_{Q_0}^P \frac{\xi^{-2(s+1)} \prod_{j=1}^{2s} (\xi - [\tilde{m}_j])^0}{\sqrt{\xi^2 - \beta^2}} d\xi
\]

\[
\zeta \equiv 0 = \sum_{t=0}^{\infty} \frac{1}{t+2} \left( \frac{1}{\beta} \right)^{t+2} + O(\zeta),
\]

\[
\xi = \zeta^{-1}, P \to P_{\infty+},
\]

(5.42)

\[
\int_{Q_0}^P [\Omega^{(2)}_{P_{0,+},2s-1}]^0 = \pm \int_{Q_0}^P \frac{\xi^{-2(s+1)} \prod_{j=1}^{2s} (\xi - [\tilde{m}_j])^0}{\sqrt{\xi^2 - \beta^2}} d\xi
\]

\[
\zeta \equiv 0 = \frac{1}{2s} \zeta^{-2s} + \sum_{m=1}^{2s} \sum_{j=0}^{\infty} \frac{1}{t+1} M_{t-2j}^{(m_0)} \mathcal{L}_j \beta^{-2j} \beta_1^{t+1}
\]

\[
+ \int_{\beta}^{\beta_1} \frac{\xi^{-2(s+1)} \prod_{j=1}^{2s} (\xi - [\tilde{m}_j])^0}{\sqrt{\xi^2 - \beta^2}} d\xi + O(\zeta),
\]

\[
\xi = \zeta, P \to P_{0,+}.
\]

(5.43)
\( (5.35) \) is a consequence of \( (4.66), (4.68), (5.41) \) and \( (5.43) \). \( (5.37) \) follows from \( (3.60), (3.62) \) and \( (5.41) \). Also \( (5.36) \) and \( (5.38) \) hold by \( (5.41) \).

Let \( \tau_{Q_0 P_0^+} \) be the integration path from \( Q_0 \) to \( P_0^+ \) which completely lies on the sheet 1 (containing \( P_0^+ \)) of the Riemann surface \( K_n \). Then

\[
\alpha_{Q_0,j}(P_{0,+}) = \int_{\tau_{Q_0 P_0^+}} \omega_j \\
= \frac{i}{2\pi} \ln \left| \frac{i + \eta_j}{i - \eta_j} \right| \equiv \left[ \alpha_{Q_0,j}(P_{0,+}) \right]^0, \tag{5.44}
\]
or more generally, for any \( P = (\xi_0, y(\xi_0)), \xi_0 \in \mathbb{C} \) on sheet 1,

\[
\alpha_{Q_0,j}(P) = \int_{Q_0}^P \omega_j \\
= \frac{\sqrt{\alpha_j^2 - \beta^2}}{2\pi i} \int_\beta^{\xi_0} \frac{1}{\sqrt{\xi^2 - \beta^2(\xi - \alpha_j)}} d\xi \\
= \frac{i}{2\pi} \ln \left| \frac{\sqrt{\frac{\xi_0 - \beta}{\xi_0 + \beta} + \eta_j}}{\sqrt{\frac{\xi_0 - \beta}{\xi_0 + \beta} - \eta_j}} \right| \equiv [\alpha_{Q_0,j}(P)]^0. \tag{5.45}
\]

To obtain reasonable solutions we assume

\[
\Xi_{Q_0,j} = \frac{1}{2} B_{jj} + \varepsilon_j \tag{5.46}
\]
to hold where \( \varepsilon_j, j = 1, \ldots, n \) are supposed to be chosen arbitrarily but to be invariant with respect to variations of \( E_j \), for example,

\[
\varepsilon_j = \frac{1}{2} - 2\pi i \sum_{\ell = 1, \ell \neq j}^n \text{Res}_{\xi = \alpha_\ell} \left( \int_\beta^\xi \frac{1}{\sqrt{(\xi^2 - \beta^2)((\xi')^2 - \beta^2)(\xi - \alpha_\ell)(\xi' - \alpha_j)}} d\xi' \right). \tag{5.47}
\]

Similar to the Cauchy problem discussed in section 4, we are interested in soliton solutions \( q, r \) of

\[
\ddot{F}_L(q, r) = 0, \quad (q, r)|_{\xi = \tau_0, z} = (q^{(0)}, r^{(0)}), \tag{5.48}
\]

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with \( q^{(0)}, r^{(0)} \) satisfying
\[
s\text{-FL}_{\bar{m}}(q^{(0)}, r^{(0)}) = 0, \tag{5.49}
\]
or equivalently,
\[
q^{(0)}(x) = q(x_0) \frac{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_0^j(P_{0,+}) + \Lambda_0^j(P_{0,-}))})}{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_0^j(P_{0,+}) + \Lambda_0^j(P_{0,-}))})} e^{i(x-x_0)([e_0,-][e_0,+]^0)},
\tag{5.50}
\]
\[
r^{(0)}(x) = r(x_0) \frac{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_0^j(P_{0,+}) + \Lambda_0^j(P_{0,-}))})}{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_0^j(P_{0,+}) + \Lambda_0^j(P_{0,-}))})} e^{-i(x-x_0)([e_0,-][e_0,+]^0)}. \tag{5.51}
\]

Here and thereafter, we denote
\[
\Lambda_0^j(P) = -\varepsilon_j + [\alpha_{Q_0,j}(P)]^0 - [\alpha_{Q_0,j}(D_{\bar{m}}(x_0))]^0 - i[U_{\bar{m}}^{(2)}]^0(x - x_0),
\]
\[
\Lambda_j(P) = -\varepsilon_j + [\alpha_{Q_0,j}(P)]^0 - [\alpha_{Q_0,j}(D_{\bar{m}}(x_0, t_0))]^0 - i[U_{\bar{m}}^{(2)}]^0(x - x_0)
- i[U_{\bar{m}}^{(2)}]^0(t_{\Sigma} - t_{0,\Sigma}),
\]
for \( \forall P \in \mathcal{K}_n \setminus \{Q_0\} \). Then we have the following result.

**Theorem 5.3.** Assume \((2.1), (2.2)\) and suppose that (1.1) and (1.2) hold with respect to the constraint (3.2) on \( \Omega \), where \( \Omega \subseteq \mathbb{R}^2 \) is open and connected. Moreover, suppose that \( D_{\bar{m}}(x, t_{\Sigma}) \), or equivalently, \( D_{\bar{m}}(x, t_{\Sigma}) \), is nonspecial for \((x, t_\Sigma) \in \Omega \). Then for the Cauchy problem of FL hierarchy (5.48) we obtain the following n-dark soliton solutions
\[
q(x, t_{\Sigma}) = q(x_0, t_{0,\Sigma}) \frac{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_j(P_{0,+}) + \Lambda_k(P_{0,+}))})}{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_j(P_{0,-}) + \Lambda_k(P_{0,-}))})} \times e^{i(x-x_0)([e_0,-][e_0,+]^0) + i(t_{\Sigma}-t_{0,\Sigma})([\tilde{\Omega}_{\Sigma}^{-1}][\tilde{\Omega}_{\Sigma}^{0}])} \tag{5.52},
\]
\[
r(x, t_{\Sigma}) = r(x_0, t_{0,\Sigma}) \frac{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_j(P_{0,+}) + \Lambda_k(P_{0,+}))})}{\det(\delta_{ik} + \frac{2n_i + n_k}{n_j + n_k} e^{\pi i(\Lambda_j(P_{0,-}) + \Lambda_k(P_{0,-}))})} \times e^{-i(x-x_0)([e_0,-][e_0,+]^0) - i(t_{\Sigma}-t_{0,\Sigma})([\tilde{\Omega}_{\Sigma}^{-1}][\tilde{\Omega}_{\Sigma}^{0}])} \tag{5.53}.
\]
Proof. It suffices to consider the limit (5.1) of (4.76), (4.77). Using (5.10), the symmetric property $\theta(z) = \theta(-z)$ and the formula

$$
\sum_{k \in \{0,1\}^n} \exp \left( 2\pi i \sum_{j=1}^{2n} k_j z_j + 2\pi i \sum_{j<m} \tau_{jm} k_j k_m \right) = \det B,
$$

$$
B = (b_{ik})_{n \times n}, \quad b_{ik} = \delta_{ik} + \frac{2\eta_i}{\eta_i + \eta_k} e^{\pi i \eta_i + \eta_k},
$$
onelone concludes (5.52), (5.53). □

Remark 5.4. (i) Taking fixed $r = (1,1)$ and varying $n \in \mathbb{N}^2 \backslash \{(0,0)\}$, one finally derives the $n$-dark soliton solutions of FL equation (1.4), which are consistent with those by Darboux transformation method [25].

(ii) In Theorem 5.1, taking some fixed $r \in \mathbb{N}^2 \backslash \{(0,0)\}$, and varying $n \in \mathbb{N}^2 \backslash \{(0,0)\}$, we obtains the $n$-dark soliton solutions of the $r$th equation in the FL hierarchy.

(iii) The $n$-dark solitons of FL hierarchy in fact depend on $2n+2$ parameters $\beta, \alpha_1, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \varepsilon_n$.

Acknowledgments

The work described in this paper was supported by grants from the National Science Foundation of China (Project No.11271079), Doctoral Programs Foundation of the Ministry of Education of China, and the Shanghai Shuguang Tracking Project (project 08GG01).

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