Quantal-classical fluctuation relation and the second law of
thermodynamics: The quantum linear oscillator

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Abstract

In this work, we study the fluctuation relation and the second law of thermodynamics within
a quantum linear oscillator externally driven over the period of time \( t = \tau \). To go beyond the
standard approach (the two-point projective measurement one) to this subject and also render it
discussed in both quantum and classical domains on the single footing, we recast this standard
approach in terms of the Wigner function and its propagator in the phase space \((x, p)\). With the
help of the canonical transformation from \((x, p)\) to the angle-action coordinates \((\phi, I)\), we can then
derive a measurement-free (classical-like) form of the Crooks fluctuation relation in the Wigner
representation. This enables us to introduce the work \( W_{(I_0, I_\tau)} \) associated with a single run from
\((I_0)\) to \((I_\tau)\) over the period \( \tau \), which is a quantum generalization of the thermodynamic work with
its roots in the classical thermodynamics. This quantum work differs from the energy difference
\( e_{(I_0, I_\tau)} = e(I_\tau) - e(I_0) \) unless \( \beta, \hbar \to 0 \). Consequently, we will obtain the quantum second-law
inequality \( \Delta F_\beta \leq \langle W \rangle_p \leq \langle e \rangle_p = \Delta U \), where \( \mathbb{P} \), \( \Delta F_\beta \), and \( \langle W \rangle_p \) denote the work (quasi)-probability
distribution, the free energy difference, and the average work distinguished from the internal energy
difference \( \Delta U \), respectively, while \( \langle W \rangle_p \to \Delta U \) in the limit of \( \beta, \hbar \to 0 \) only. Therefore, we can also
introduce the quantum heat \( Q_q = \Delta U - W \) even for a thermally isolated system, resulting from the
quantum fluctuation therein. This is a more fine-grained result than \( \langle W \rangle_p \equiv \Delta U \) obtained from the
standard approach. Owing to the measurement-free nature of the thermodynamic work \( W_{(I_0, I_\tau)} \),
our result can also apply to the (non-thermal) initial states \( \hat{\rho}_0 = (1 - \gamma) \hat{\rho}_\beta + \gamma \hat{\sigma} \) with \( \hat{\sigma} \neq \hat{\rho}_\beta \).

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I. INTRODUCTION

Fluctuation relations such as Jarzynski’s equality and Crooks’ theorem have attracted a great deal of interest owing to their nature of the link between non-equilibrium fluctuations and thermal equilibrium properties of small systems (either classical or quantal) [1, 2]. Here, the stochastic nature of the thermodynamic work performed on a given system emerges through (infinitely) many runs of the external driving of the system; in the classical case, this nature is associated solely with a random sampling of individual microstates from the initially prepared (canonically thermal) state of the system, because after such a sampling, the system becomes thermally isolated and evolves deterministically under Hamilton’s equations. In the quantum case, on the other hand, such a stochastic nature is associated not only with the random sampling from the thermal initial state (as a source of the thermal fluctuation) but also with the quantum fluctuation existing even during the external driving. In fact, an appropriate determination of the work and its probability distribution associated with both thermal and quantum fluctuations, required for a legitimate form of the quantum fluctuation relation, has been one of the central issues in the field of quantum thermodynamics.

The standard approach to the quantum fluctuation relation has been made in the so-called two-point projective measurement (TPM) framework [3–12]: An isolated quantum system is initially prepared in the thermal state \( \hat{\rho}_0 = \hat{\rho}_\beta \) (with \( \beta = 1/k_B T \)) and then undergoes an external driving (denoted by a time-dependent Hamiltonian parameter \( \lambda_t \)). The probability distribution of the single-run work \( w \) for the system in a forward process of the external driving is then given by

\[
P^{(f)}(w) = \sum_{n,m} \delta(w - \Delta e_{nm}) \times P_{nm}^{(f)},
\]

where the energy-eigenvalue difference \( \Delta e_{nm} = e_{m(\tau)} - e_{n(0)} \), between the two outcomes \( e_{n(0)} \) and \( e_{m(\tau)} \) found from the initial \( (t = 0) \) and final \( (t = \tau) \) measurements, and its probability \( P_{nm}^{(f)} = P[m(\tau)|n(0)] P[n(0)] \) consisting of both initial probability \( P[n(0)] = e^{-\beta e_{n(0)}}/Z_\beta(\lambda_0) \) of the \( n \)th energy eigenstate, with the partition function \( Z_\beta(\lambda_0) \) for the initial state \( \hat{\rho}_\beta(\lambda_0) \), and the conditional probability \( P[m(\tau)|n(0)] = |\langle m(\tau)|\hat{U}|n(0)\rangle|^2 \) for the run \( |n(0)\rangle \rightarrow |m(\tau)\rangle \) with the unitary operator \( \hat{U} \). Here, the probabilistic nature of finding those two measurement outcomes gives rise to the stochastic nature of the work. As such, the single-run work is given by \( w_{nm} \equiv \Delta e_{nm} \).
Likewise, the work probability distribution for the system in a backward process starting from $\hat{\rho}_\beta(\lambda_r)$ is given by

$$P^{(b)}(w) = \sum_{n,m} \delta(w + w_{nm}) \times P_{nm}^{(b)},$$

(2)

where $P_{nm}^{(b)} = P[n(0)|m(\tau)] P[m(\tau)]$ with $P[m(\tau)] = e^{-\beta e_n(\tau)} / Z_\beta(\lambda_r)$ and $P[n(0)|m(\tau)] = P[m(\tau)|n(0)]$. It is then straightforward to obtain the Crooks fluctuation theorem

$$P_{nm}^{(f)} = P_{nm}^{(b)} \exp\{\beta (w_{nm} - \Delta F^{(f)}_\beta)\},$$

(3)

in which the free energy difference is $\Delta F^{(f)}_\beta(\lambda_r) = F_\beta(\lambda_r) - F_\beta(\lambda_0)$. With the help of Eqs. (1) and (2), this will result in the quantum Jarzynski equality in its known form

$$\langle e^{-\beta w} \rangle_{\rho(t)} = \int dw \, e^{-\beta w} \, P^{(f)}(w) = e^{-\beta \Delta F^{(f)}_\beta}.$$

(4)

Therefore, the free energy difference and the non-equilibrium fluctuating work can be exactly linked. With the help of the Jensen inequality, this Jarzynski equality gives rise to

$$\Delta F^{(f)}_\beta(\lambda_r) \leq \langle w \rangle_{\rho(t)} \equiv \Delta U(\tau)$$

(5)

as an expression of the second law of thermodynamics in the quantum domain [cf. (46)].

By construction, the non-equilibrium average work $\langle w \rangle_{\rho(t)}$ is identically equal to the internal energy difference $\Delta U(\tau) = \langle \hat{H}(\lambda_r) \rangle_{\rho_t} - \langle \hat{H}(\lambda_0) \rangle_{\rho_0}$ between the initial and final instants of time, where $\langle \hat{H}(\lambda_r) \rangle_{\rho_t} = \text{Tr}\{\hat{H}(\lambda_r) \hat{\rho}(t)\}$.

In spite of its great usefulness, the TPM framework has a conceptual issue when it comes to its generalization: It is nonlegitimate to apply the same form of the work probability to the processes starting from the non-thermal states $\hat{\rho}_0 \neq \hat{\rho}_\beta$ with coherence in the energy basis; because the initial projective measurement then destroys the initial coherence and so produces extra entropy, thus leading to disturbing the original time evolution of the system. As such, this standard approach is not fully quantum-mechanical. Moreover, we also note that an individual external driving ($\lambda_t$) itself, described by unitary dynamics, produces no entropy at all, regardless of the initial outcomes $e_n(0)$; but an appearance of the entropy production, achieved through the (classical) mixture over many runs, is due to the (non-unitary) final projective measurement (thus viewed as an extra non-equilibrium work). Consequently, it still remains an open question to introduce a generalized form of quantum work legitimate for the non-thermal initial condition and the external driving only.

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To go beyond such a limitation, we intend to introduce in this paper an alternative
definition of the quantum work (≠ w) and its distribution formulated without the projective
measurements. For this purpose, we will resort to the classical phase space (x, p), in which
both quantum and classical fluctuation relations can be discussed on the single footing, for
example, by making use of the Wigner function and its propagator [cf. Eqs. (6)-(7)]. In
fact, the Wigner representation is known to be the most classical-like in propagation among
different phase-space representations [13]. Then we will formulate a Crooks fluctuation
theorem in its measurement-free (classical-like) form by recasting the characteristic function
of the TPM framework in the Wigner representation (cf. e.g., [14] for a different phase-
space approach without such a link with the TPM approach). This will finally give a new
definition of the work \( W \) as the direct quantum counterpart to the thermodynamic work
which has its original roots in the classical thermodynamics. For this formulation, we will
also employ the angle-action coordinates \((\phi, I)\) [cf. Eq. (21)]; as is well-known, this pair is
well-defined for the separable systems (e.g., the generic one-dimensional ones) and useful for
the semiclassical analysis [15–19]. To our best knowledge, these coordinates have not been
applied extensively for the study of quantum fluctuation relations. For the sake of an explicit
treatment with analytical rigor, we will restrict our analysis here to a linear oscillator with
its time-dependent frequency \( \lambda_t = \omega(t) \) (cf. [8] for an analysis of this system in the TPM
framework); our methodology will also apply to a more generic class of systems.

Further, as is well-known, the Wigner function can be negative-valued, which reflects
the quantum fluctuation. This will result in the work distribution \( P(W) \) with its negativity.
Therefore, our concern lies in the average values only (over many runs) that have the physical
meaning (cf. the unavoidable negativity of the work distribution has already been studied
in the extended TPM framework where the initial state is non-diagonal in the energy basis,
e.g., [20, 21]). Then, it will be shown that the average work \( \langle W \rangle_p \) is distinguished from
the internal energy difference \( \Delta U \) which remains unaffected under this transformation of
the representation. For comparison, we also point out that it is impossible to consider the
same scenario (free from the projective measurements) for such an alternative definition of
the classical work, because in the classical setup, no projective measurements are required
anyway. Therefore, our alternative approach to the quantum work \( \langle W \rangle_p \) will produce, in
the classical limit, no difference from the TPM approach \( \Delta U \). This will also enable us
to introduce the quantum heat \( Q_q = \Delta U - \langle W \rangle_p \geq 0 \) (with no classical counterpart) for a
thermally isolated system such that \( Q_d + Q_q = \Delta U - \Delta F_\beta \); here, the thermal heat \( Q_d = \langle W \rangle_P - \Delta F_\beta \geq 0 \) corresponds to the dissipative heat which will go out to the environment until the system-environment equilibrium will be achieved, if additional heat exchange between system and environment is carried out after completing the external driving. Consequently, we will acquire, as one of our main findings, the quantum second-law inequality (46) in a more fine-grained form than the inequality \( 5 \). Owing to the measurement-free nature of \( \mathcal{W} \) and \( \mathcal{P}(\mathcal{W}) \), our result for the thermodynamic work in the quantum regime will be further generalized to the initial states being partially thermal in the form of \( \hat{\rho}_0 = (1 - \gamma) \hat{\rho}_\beta + \gamma \hat{\sigma} \) with \( \hat{\sigma} \neq \hat{\rho}_\beta \).

The general layout of this paper is as follows: In Sec. II we provide the phase-space formulation needed for an introduction of our quantum work and its distribution. In Sec. III we derive the quantal-classical Crooks fluctuation theorem in our framework and then discuss the second law of thermodynamics and its implications. In Sec. IV our framework is generalized to the partially thermal initial states and then several examples of those initial states (with coherence in the energy basis) are explicitly considered. Finally, we provide concluding remarks in Sec. V.

II. PHASE-SPACE FORMULATION AND QUANTUM WORK DISTRIBUTION

A. Wigner function and its Propagator

To take into consideration the phase-space counterpart to the forward work distribution in Eq. (1), we will make use of the Wigner function (representing the initial distribution corresponding to \( P[n(0)] \) therein) \[ (2\pi\hbar)^{-1/2} \int d\xi \left\langle x + \frac{\xi}{2} \right| \hat{\rho} \left| x - \frac{\xi}{2} \right\rangle \exp \left( -\frac{i}{\hbar} p\xi \right) \] (6)

(for \( \hat{\rho} = \hat{\rho}_\beta \)) and its propagator (corresponding to the conditional probability \( P[m(\tau)|n(0)] \)) \[ (2\pi\hbar)^{-1} \int d\xi d\xi' \exp \left[ -\frac{i}{\hbar} (p\xi + p'\xi') \right] K \left( x + \frac{\xi}{2}; \tau \right| x' - \frac{\xi'}{2}; 0 \right) K^* \left( x - \frac{\xi}{2}; \tau \right| x' + \frac{\xi'}{2}; 0 \right), \] (7)

\[ T^{(f)}_\mathcal{W}(x, p; m|n(0)) := (2\pi\hbar)^{-1} \text{Tr} \left\{ \hat{\Delta}(x, p) \hat{U}(\tau) \hat{\Delta}(x', p') \hat{U}^\dagger(\tau) \right\} = \]

\[ (2\pi\hbar)^{-1} \int d\xi d\xi' \exp \left[ -\frac{i}{\hbar} (p\xi + p'\xi') \right] K \left( x + \frac{\xi}{2}; \tau \right| x' - \frac{\xi'}{2}; 0 \right) K^* \left( x - \frac{\xi}{2}; \tau \right| x' + \frac{\xi'}{2}; 0 \right), \]
in which the operator \( \hat{\Delta}(x,p) = \int_{-\infty}^{\infty} d\xi \left[ x - \xi/2 \right] \left[ x + \xi/2 \right] e^{-i p \xi / \hbar} \) and the usual propagator 
\( K(x;\tau|x';0) = \langle x|\hat{U}(\tau)|x'\rangle \); e.g., for the sudden switch \( (\hat{U}_s = \mathbb{1}) \), we can easily obtain

\[
T^{(t)}_{W,s}(x,p;\tau|x',p';0) = \exp \left\{ \frac{2i}{\hbar} (x-x') p \right\} \delta(x-x') \delta(p-p') .
\] (8)

As is well-known, the Wigner function satisfies its marginal probability distributions

\[
\int dp \ W_\rho(x,p) = \langle x|\hat{\rho}|x \rangle ; \quad \int dx \ W_\rho(x,p) = \langle p|\hat{\rho}|p \rangle
\] (9)

and gives the expectation value

\[
\text{Tr}(\hat{\rho} \hat{A}) = \int dx \int dp \ W_\rho(x,p) \ A(x,p)
\] (10)

together with the Weyl-Wigner c-number representation of the observable \( \hat{A} \) given by

\[
A(x,p) = \int_{-\infty}^{\infty} d\xi \ \exp \left( -\frac{i}{\hbar} p \xi \right) \left\langle x + \frac{\xi}{2} \right| \hat{A} \left| x - \frac{\xi}{2} \right\rangle .
\] (11a)

\[
\hat{A} = (2\pi\hbar)^{-1} \int d\xi \int dx dp \left( x + \frac{\xi}{2} \right) A(x,p) \ \exp \left( \frac{i}{\hbar} p \xi \right) \left| x - \frac{\xi}{2} \right\rangle .
\] (11b)

Similarly, we have

\[
\text{Tr}(\hat{\rho}_1 \hat{\rho}_2) = 2\pi\hbar \int dx \int dp \ W_{\rho_1}(x,p) W_{\rho_2}(x,p).
\] (12)

And it is the propagator \( T^{(t)}_{W}(x,p;\tau|x',p';0) \) that generates the trajectory running from the position \( (x',p') \) at \( t = 0 \) to \( (x,p) \) at \( t = \tau \). In the limit of \( \hbar \to 0 \), those Wigner trajectories exactly reduce to the classical trajectories. It is also easy to verify that

\[
\int dx' dp' T^{(t)}_{W}(x,p;\tau|x',p';0) = \int dx dp T^{(t)}_{W}(x,p;\tau|x',p';0) = 1
\] (13)

and

\[
\int dp dp' T^{(t)}_{W}(x,p;\tau|x',p';0) = (2\pi\hbar) |K(x;\tau|x';0)|^2 \geq 0
\] (14a)

\[
\int dx dx' T^{(t)}_{W}(x,p;\tau|x',p';0) = (2\pi\hbar) |\tilde{K}(p;\tau|p';0)|^2 \geq 0 ,
\] (14b)

where \( \tilde{K}(p;\tau|p';0) = \langle p|\hat{U}(\tau)|p' \rangle \). The time-evolution \( \hat{\rho}_r = \hat{U}(\tau) \hat{\rho}_0 \hat{U}^\dagger(\tau) \) is then rewritten in the Wigner representation as

\[
W_{\rho_r}(x,p) = \int dx' dp' T^{(t)}_{W}(x,p;\tau|x',p';0) W_{\rho_0}(x',p') .
\] (15)
where both quantities $X$ and $\omega(t)$ through infinitely many trajectories. Likewise, the Wigner propagator for the backward process [cf. Eq. (2)] is given by $T_w^{(b)}(x', p'; \tau | x, p; 0) := (2\pi \hbar)^{-1} \text{Tr} \{ \hat{\Delta}(x', p') \hat{U}^\dagger(\tau) \hat{\Delta}(x, p) \hat{U}(\tau) \}$, which is easily shown to be identical to $T_w^{(f)}(x, p; \tau | x', p'; 0)$.

For a quantum linear oscillator with a time-dependent frequency $\omega(t)$, we have the propagator in the Gaussian form

$$K(x; t|x'; 0) = \left( \frac{m}{2\pi i\hbar X} \right)^{1/2} \exp \left[ \frac{i m}{2\hbar X} \left\{ x^2 \dot{X} - 2xx' + (x')^2 Y \right\} \right],$$

where both quantities $X = X(t)$ and $Y = Y(t)$, with $(X(0) = 0, \dot{X}(0) = 1)$ and $(Y(0) = 1, \dot{Y}(0) = 0)$, are the solutions to the classical equation of motion $\ddot{X} + \{\omega(t)\}^2 X = 0$. Then it is straightforward that with the help of Eq. (16), the Wigner propagator in (7) will be evaluated explicitly.

**B. Quantum Work Distribution and Angle-Action Coordinates**

In the classical scenario, on the other hand, the work distribution for a thermally isolated system in a forward process starting from the thermal state can be expressed as

$$(P_c)^{(f)}(w) = \int \int dE_\tau dE_0 \delta(w - W_{0\tau}) \times (P_c)^{(f)}_{0\tau},$$

in which the energy difference $W_{0\tau} = E_\tau - E_0$ as a single-run work, and its probability density $(P_c)_{0\tau}^{(f)} = P_c(E_\tau | E_0) P_c(E_0)$; here, we have the instantaneous energy $E_t = H(z_t; \lambda_t)$ with its trajectory $z_t = (x_t, p_t)$ evolving from $z_0$ under Hamilton’s dynamics, and the initial probability density $P_c(E_0) = \{e^{-\beta E_0}/Z_{\beta,c}(\lambda_0)\} g(E_0)$ with the classical partition function $Z_{\beta,c}(\lambda_0) = e^{-\beta E_{\beta,c}(\lambda_0)}$ and the density of states $g(E_0)$, and the conditional probability density $P_c(E_\tau | E_0)$ for $E_0 \to E_\tau$. As such, this expression of the classical work distribution directly shows the formal similarity to the quantum-mechanical result in Eq. (11).

Motivated by such an analogy between the quantum and classical work distributions, we begin by rewriting the Fourier transform of Eq. (11)

$$A^{(f)}(u) := \int dw P^{(f)}(w) e^{iux} = \sum_{n,m} P[m(\tau)|n(0)] P[n(0)] e^{iux} e_{m(\tau)} - e_{n(0)}$$

into its phase-space counterpart: By using Eqs. (6)-17 with (12) and (15), we can acquire

$$A^{(f)}(u) = \int dx dp \int dx' dp' \Xi^{(f)}_n(x', p'; x, p) T_w^{(f)}(x, p; \tau | x', p'; 0),$$

$$\Xi^{(f)}_n(x', p'; x, p) = \int \left[ \hat{D}(x', p') \hat{U}^\dagger(\tau) \hat{D}(x, p) \hat{U}(\tau) \right]_n.$$
where the factor
\[ \Xi^{(f)}_{n}(x', p'; x, p) = \frac{2\pi\hbar}{Z_{\beta}(\lambda_{0})} \sum_{n,m} W_{n(0)}(x', p'; \lambda_{0}) e^{-i\nu_{n}(0)} W_{m}(x, p; \lambda_{r}) e^{i\nu_{m}(t)}. \]

With the help of the relation \( \sum_{m} W_{m}(x, p) = (2\pi\hbar)^{-1} \), it is easy to observe here that
\[ \Xi^{(f)}_{0}(x', p'; x, p) = W_{\beta}(x', p'; \lambda_{0}). \tag{20} \]

Now we restrict our discussion to a driven linear oscillator with \( \lambda_{t} = \omega_{t} \). To obtain the quantum work distribution in the Wigner representation taking the form of its classical counterpart in Eq. (17), we employ the change of coordinates from \( (x', p'; x, p) \) to the angle-action pairs \( (\phi_{0}, \mathbb{I}_{0}; \phi_{r}, \mathbb{I}_{r}) \) associated with the initial position \( (\phi_{0}, \mathbb{I}_{0}) \) and the final position \( (\phi_{r}, \mathbb{I}_{r}) \); here \[ \mathbb{I} = \frac{1}{2\pi} \oint p dx \geq 0, \tag{21} \]
where the symbol \( \oint \) denotes the integral which runs over a single period in the phase space. For a linear oscillator, it follows that \( x = (2\mathbb{I}_{t}/(m\omega_{t}))^{1/2} \sin \phi_{t} \) and \( p = (2m\omega_{t}\mathbb{I}_{t})^{1/2} \cos \phi_{t} \) with \( E_{t} = \omega_{t}\mathbb{I}_{t} \). This enables us to rewrite Eq. (18b) as
\[ A^{(f)}(u) = \int_{0}^{\infty} \int_{0}^{\infty} d\mathbb{I}_{0} d\mathbb{I}_{r} \Xi^{(f)}_{u}(\mathbb{I}_{0}, \mathbb{I}_{r}) \times B^{(f)}(\mathbb{I}_{r}|\mathbb{I}_{0}), \tag{22} \]
where the two-state quantity [cf. (19)] and the conditional distribution are
\[ \Xi^{(f)}_{u}(\mathbb{I}_{0}, \mathbb{I}_{r}) = \frac{(2\pi\hbar) Z_{\beta}(\omega_{0}) Z_{\beta}(\omega_{r})}{Z_{\beta}(\omega_{0})} W_{\beta}(\omega_{0}) W_{\beta}(\omega_{r}) \tag{23a} \]
\[ B^{(f)}(\mathbb{I}_{r}|\mathbb{I}_{0}) = \int_{0}^{2\pi} d\phi \int_{0}^{2\pi} d\phi_{0} \tilde{T}_{W}^{(f)}(\phi_{r}, \mathbb{I}_{r}; \tau|\phi_{0}, \mathbb{I}_{0}; 0), \tag{23b} \]
respectively [cf. (14a)-(14b)]; the propagator \( T_{W}^{(f)}(x, p; \tau|x', p'; 0) \rightarrow \tilde{T}_{W}^{(f)}(\phi_{r}, \mathbb{I}_{r}; \tau|\phi_{0}, \mathbb{I}_{0}; 0) \).

Here we adopted the Wigner function of the \( n \)th energy eigenstate \[ W_{n}(x, p) = \frac{(-1)^{n}}{\pi\hbar} e^{-|\eta(x, p)|^{2}} L_{n}(4|\eta(x, p)|^{2}), \tag{24} \]
in which the \( n \)th Laguerre Polynomial \( L_{n}(A) \) and \( \eta = 2^{-1/2}\{\kappa x + ip(h\kappa)^{-1}\} \) with \( \kappa = (m\omega/\hbar)^{1/2} \), and then applied the identity \( \sum_{n=0}^{\infty} L_{n}(A) z^{n} = (1 - z)^{-1} e^{A z/(z-1)} \) \[ 32 \], giving rise to the relation \( \sum_{n=0}^{\infty} W_{n}(x, p) e^{-\beta\epsilon_{n}} = Z_{\beta} W_{\beta}(x, p) \) indeed; the thermal Wigner function is given by the Gaussian form \[ 33 \]
\[ W_{\beta}(x, p; \omega) = \frac{\text{sech}(\beta\hbar \omega/2)}{(2\pi\hbar) Z_{\beta}(\omega)} \exp \left[ -\left( \frac{\beta\hbar \omega}{2} \right) \left\{ (\kappa x)^{2} + \frac{p^{2}}{(h\kappa)^{2}} \right\} \right] \geq 0 \tag{25a} \]
\[ \rightarrow W_{\beta}(\mathbb{I}; \omega) = \frac{\text{sech}(\beta\hbar \omega/2)}{(2\pi\hbar) Z_{\beta}(\omega)} \exp \left\{ -\frac{2}{h} \tan \left( \frac{\beta\hbar \omega}{2} \right) \right\} \geq 0, \tag{25b} \]
where the partition function is given by \( Z_\beta(\omega) = 2^{-1} \cosh(\beta \hbar \omega/2) \).

Here we take into consideration three particular cases for Eq. (22); first, the case of \( u = 0 \), in which \( A^{(f)}(0) = 1 \). Then we can introduce, with the help of Eq. (20), the joint (quasi)probability distribution associated with a single motion from \((\mathbb{I}_0)\) at \( t = 0 \) to \((\mathbb{I}_\tau)\) at \( t = \tau \)

\[
P^{(f)}(\mathbb{I}_0, \mathbb{I}_\tau) = B^{(f)}(\mathbb{I}_\tau|\mathbb{I}_0) \ W_\beta(\mathbb{I}_0; \omega_0) \geq 0
\]

with its normalization \( \int_0^\infty \int_0^\infty d\|0\| d\|\tau\| \ P^{(f)}(\mathbb{I}_0, \mathbb{I}_\tau) = 1 \); the non-negative nature of \( B^{(f)}(\mathbb{I}_\tau|\mathbb{I}_0) \) is verified in Appendix A. Likewise, the joint (quasi)probability distribution for the backward process can also be acquired

\[
P^{(b)}(\mathbb{I}_\tau, \mathbb{I}_0) = B^{(b)}(\mathbb{I}_0|\mathbb{I}_\tau) \ W_\beta(\mathbb{I}_\tau; \omega_\tau) \geq 0, \tag{27}
\]

where \( B^{(b)}(\mathbb{I}_0|\mathbb{I}_\tau) = B^{(f)}(\mathbb{I}_\tau|\mathbb{I}_0) \); note the discussion after Eq. (15). As a result, we observe that the (quasi)probability distributions \( P^{(f)}(\mathbb{I}_0, \mathbb{I}_\tau) \) and \( P^{(b)}(\mathbb{I}_\tau, \mathbb{I}_0) \) are the counterparts to \( P^{(f)}_{nm} \) and \( P^{(b)}_{nm} \) in Eqs. (1) and (2), respectively.

The second case is given by \(-i\partial_u A^{(f)}(u)|_{u=0} = \langle w \rangle_{\rho^{(f)}} [\text{cf. (18a)}] \), which equals the internal energy difference \( \Delta U(\tau) \). This quantum-mechanical average value can now be expressed as

\[
\Delta U(\tau) = \int_0^\infty \int_0^\infty d\|0\| d\|\tau\| \Delta e_{(0,\tau)} \times P^{(f)}(\mathbb{I}_0, \mathbb{I}_\tau), \tag{28}
\]

in which the (single-motion) energy difference between the initial position \( \mathbb{I}_0 \) and the final position \( \mathbb{I}_\tau \) is given by

\[
\Delta e_{(0,\tau)} = (\omega_\tau \mathbb{I}_\tau) - (\omega_0 \mathbb{I}_0) \left\{ \text{sech}(\beta \hbar \omega_0/2) \right\}^2 - (\hbar \omega_0/2) \left\{ \tanh(\beta \hbar \omega_0/2) \right\}. \tag{29}
\]

To explicitly evaluate Eq. (28), we obtain the first moment (cf. Appendix A)

\[
\langle \mathbb{I}_\tau \rangle_{\rho^{(f)}} = \int_0^\infty \int_0^\infty d\|0\| d\|\tau\| \mathbb{I}_\tau \times P^{(f)}(\mathbb{I}_0, \mathbb{I}_\tau) = \langle \mathbb{I}_0 \rangle_{\rho^{(f)}} \mathbb{K}_\tau, \tag{30}
\]

where \( \langle \mathbb{I}_0 \rangle_{\rho^{(f)}} = (\hbar/2) \coth(\beta \hbar \omega_0/2) \) and the (classical) dimensionless quantity

\[
\mathbb{K}_t = \frac{1}{2} \left\{ \frac{(\dot{X}_t Y_t - 1)^2}{\omega_0 \omega_t} + \omega_0 \omega_t (X_t)^2 + \frac{\omega_0 (\dot{X}_t)^2}{\omega_t} + \frac{\omega_t (Y_t)^2}{\omega_0} \right\} \tag{31}
\]

with \( \mathbb{K}_0 = 1 \). By using the inequality of arithmetic and geometric means, it is easy to see that \( \mathbb{K}_t \geq 1 \); e.g., for the sudden switch in Eq. (8), we find that \( \mathbb{K}_{t,s} = (\omega_0/\omega_t + \omega_t/\omega_0)/2 \). If
the process is carried out adiabatically, it turns out that $K_t = 1$ and so $\langle \mathbb{I}_\tau \rangle_{\psi(t)}$ is invariant. Eqs. (29) and (30) finally give the internal energy difference, which is, in fact, identical to

$$\Delta U(\tau) = \omega_\tau \langle \mathbb{I}_\tau \rangle_{\psi(t)} - \omega_0 \langle \mathbb{I}_0 \rangle_{\psi(t)} = \langle \mathbb{I}_\tau \rangle_{\psi(t)} (\omega_\tau K_\tau - \omega_0),$$

as required [cf. Eqs. (10) and (A15)-(A16)]. This reduces to $\beta^{-1} \{\langle \omega_\tau / \omega_0 \rangle K_\tau - 1\}$ in the classical limit.

The third case is given by $(-i\partial_u)^2 A^{(1)}(u)|_{u=0} = \langle w^2 \rangle_{\psi(t)} = \langle (e_m(\tau) - e_n(0))^2 \rangle_{nm}$. This is shown to differ from $\langle (\omega_\tau \mathbb{I}_\tau - \omega_0 \mathbb{I}_0)^2 \rangle_{\psi(t)}$, though; in fact, with the help of the second-moment relations

$$\langle (\mathbb{I}_\tau)^2 \rangle_{\psi(t)} = \int_0^\infty \int_0^\infty d\mathbb{I}_0 d\mathbb{I}_\tau (\mathbb{I}_\tau)^2 \times \mathbb{P}^{(2)}(\mathbb{I}_0, \mathbb{I}_\tau) = \{3 (K_\tau)^2 - 1\} \{\langle \mathbb{I}_0 \rangle_{\psi(t)}\}^2$$

$$\langle (\mathbb{I}_0)^2 \rangle_{\psi(t)} = \int_0^\infty \int_0^\infty d\mathbb{I}_0 d\mathbb{I}_\tau (\mathbb{I}_\tau)^2 \times \mathbb{P}^{(2)}(\mathbb{I}_0, \mathbb{I}_\tau) = 2 \{\langle \mathbb{I}_0 \rangle_{\psi(t)}\}^2$$

$$\langle \mathbb{I}_0 \mathbb{I}_\tau \rangle_{\psi(t)} = \int_0^\infty \int_0^\infty d\mathbb{I}_0 d\mathbb{I}_\tau \mathbb{I}_0 \mathbb{I}_\tau \times \mathbb{P}^{(2)}(\mathbb{I}_0, \mathbb{I}_\tau) = 2 K_\tau \{\langle \mathbb{I}_0 \rangle_{\psi(t)}\}^2$$

(cf. Appendix A), we can verify that

$$\langle w^2 \rangle_{\psi(t)} = \langle (\omega_\tau \mathbb{I}_\tau - \omega_0 \mathbb{I}_0)^2 \rangle_{\psi(t)} + \frac{\hbar^2 \omega_0 \omega_\tau}{2} K_\tau - \frac{\hbar^2}{4} \{(\omega_0)^2 + (\omega_\tau)^2\}$$

$$= \{\Delta e_{(\omega_0, \omega_\tau)} \}^2_{\psi(t)} + \left\{ \left( \frac{\hbar \omega_0}{\hbar} \right) \text{sech} \left( \frac{\beta \hbar \omega_0}{2} \right) \right\}^2 - \left( \frac{\hbar \omega_\tau}{2} \right)^2,$$

in which $\langle (\omega_\tau \mathbb{I}_\tau - \omega_0 \mathbb{I}_0)^2 \rangle_{\psi(t)} = \{3 (\omega_\tau)^2 (K_\tau)^2 - 4 \omega_0 \omega_\tau K_\tau + 2 (\omega_0)^2 - (\omega_\tau)^2\} \{\langle \mathbb{I}_0 \rangle_{\psi(t)}\}^2$ [cf. (29)]. In fact, the quantum-mechanical expectation value (in the form of the first moment), such as $\Delta U(\tau)$, is identically evaluated in both TPM and Wigner frameworks; however, it can be shown that such a framework-independent behavior is not available any longer for all higher-order moments $\langle (\mathbb{I}_\tau)^n (\mathbb{I}_0)^m \rangle_{\psi(t)}$. At this point, we also remind that differing from the energy operator $\hat{H}$, the quantum work $w$ performed by an external agent is not a quantum-mechanical observable $\mathbb{I}$.

## III. QUANTUM CROOKS FLUCTUATION THEOREM AND THE SECOND LAW

### A. Crooks Theorem in the Wigner Representation

We are ready to consider a quantum Crooks fluctuation theorem in the classical phase space: By combining Eqs. (26) and (27), leading to

$$\frac{\mathbb{P}^{(2)}(\mathbb{I}_0, \mathbb{I}_\tau)}{\mathbb{P}^{(0)}(\mathbb{I}_\tau, \mathbb{I}_0)} = \frac{W_\beta(\mathbb{I}_0; \omega_0)}{W_\beta(\mathbb{I}_\tau; \omega_\tau)},$$

(35)
we can easily obtain this fluctuation theorem given by

\[ \mathbb{P}^{(f)}(\mathbb{I}_0, \mathbb{I}_\tau) = \mathbb{P}^{(b)}(\mathbb{I}_\tau, \mathbb{I}_0) \exp \left\{ \beta \left( \mathbb{W}_{(I_0,I_\tau)} - \Delta F^{(f)}_\beta \right) \right\}. \]  

(36)

Here, the free energy difference \( \Delta F^{(f)}_\beta \) for the forward process is explicitly given by \( F_\beta(\omega_\tau) - F_\beta(\omega_0) = \beta^{-1} \ln \left[ \frac{\sinh(\beta h\omega_\tau/2)\sinh(\beta h\omega_0/2)}{\sinh(\beta h\omega_0/2)\sinh(\beta h\omega_0/2)} \right] \), and the single-motion work associated with the transformation from the initial position \( \mathbb{I}_0 \) to the final position \( \mathbb{I}_\tau \) is identified as

\[ \mathbb{W}_{(I_0,I_\tau)} = -\frac{1}{\beta} \ln \left\{ \frac{W_\beta(\mathbb{I}_\tau; \omega_\tau)}{W_\beta(\mathbb{I}_0; \omega_0)} \right\} + \Delta F^{(f)}_\beta(\omega_\tau) \]

\[ = (\omega, \mathbb{I}_\tau) \frac{\tanh(\beta h\omega_\tau/2)}{\beta h\omega_\tau/2} - (\omega_0, \mathbb{I}_0) \frac{\tanh(\beta h\omega_0/2)}{\beta h\omega_0/2} + \frac{1}{\beta} \ln \left\{ \frac{\cosh(\beta h\omega_\tau/2)}{\cosh(\beta h\omega_0/2)} \right\}. \]

(37)

By construction, this form of the thermodynamic work in the quantum regime, linked to \( \Delta F^{(f)}_\beta \), was derived from requiring the Crooks theorem in the Wigner representation. Taking Eqs. \( (11a)-(11b) \) into consideration, we find that this quantum work, expressed in terms of the action coordinates and formulated without resorting to any projective measurements, is evidently not a quantum-mechanical observable. Here, we also observe that this work and \( \Delta e_{(I_0,I_\tau)} \) in Eq. \( (29) \) differ from each other, while both become identical \( (\omega, \mathbb{I}_\tau - \omega_0, \mathbb{I}_0) \) in the limit of \( \beta, \hbar \to 0 \) where the quantum Crooks theorem in Eq. \( (36) \) reduces to its classical counterpart in its known form.

Then we can introduce the quantum work distribution for the forward process in the Wigner representation

\[ \mathbb{P}^{(f)}(\mathbb{W}) = \int_0^\infty \int_0^\infty d\mathbb{I}_0 d\mathbb{I}_\tau \delta(\mathbb{W} - \mathbb{W}_{(I_0,I_\tau)}) \mathbb{P}^{(f)}(\mathbb{I}_0, \mathbb{I}_\tau) \]

[cf. Eq. \( (1) \)], which is valid in the entire quantum regime and may also be viewed as the quantum generalization of the classical work distribution in Eq. \( (17) \). Likewise, the work distribution for the backward process turns out to be

\[ \mathbb{P}^{(b)}(\mathbb{W}) = \int_0^\infty \int_0^\infty d\mathbb{I}_\tau d\mathbb{I}_0 \delta(\mathbb{W} + \mathbb{W}_{(I_0,I_\tau)}) \mathbb{P}^{(b)}(\mathbb{I}_\tau, \mathbb{I}_0) \]

[cf. Eq. \( (2) \)]. With the help of Eqs. \( (38) \) and \( (39) \), the integration of Eq. \( (36) \) over \( \mathbb{I}_0 \) and \( \mathbb{I}_\tau \) will result in the quantum Jarzynski equality

\[ \langle e^{-\beta \mathbb{W}} \rangle^{(f)} = \int d\mathbb{W} e^{-\beta \mathbb{W}} \mathbb{P}^{(f)}(\mathbb{W}) = e^{-\beta \Delta F^{(f)}_\beta}. \]

(40)

Employing Eqs. \( (37) \) and \( (38) \) with \( (30) \), we can also evaluate the average work such that

\[ \langle \mathbb{W} \rangle^{(f)} = \int d\mathbb{W} \mathbb{W} \times \mathbb{P}^{(f)}(\mathbb{W}) = Q^{(f)} + \Delta F^{(f)}_\beta(\omega_\tau). \]

(41)
Here, the dissipative heat is explicitly given by

$$Q_d^{(f)} = \frac{1}{\beta} \left\{ \frac{K_{\tau}}{\tilde{K}_{\tau}} - 1 + \ln(\tilde{K}_{\tau}) \right\} \geq 0,$$

where $\tilde{K}_{\tau} = \{ \coth(\beta\hbar\omega_0/2) \}/\{ \coth(\beta\hbar\omega/2) \}$; the minimum value of $Q_d^{(f)}$ is achieved if the entire process is carried out adiabatically ($K_{\tau} = 1$). With the help of the inequality given by $y \geq \ln(y) + 1$ (with $y = 1/\tilde{K}_{\tau}$), it is then easy to verify that this minimum value is non-negative, and so the second law of thermodynamics is met in Eq. (41); cf. Eq. (53) for the same discussion in other phase-space representations.

Consequently, the average work $\langle W \rangle_{p(t)}$ is distinguished from the internal energy difference in (32); at zero temperature, we have $\Delta U(\tau) \rightarrow (\hbar/2) (\omega_\tau K_{\tau} - \omega_0)$ but $\langle W \rangle_{p(t)} = \Delta F_{p}^{(f)} \rightarrow (\hbar/2) (\omega_\tau - \omega_0)$ while in the high-temperature regime ($\beta \rightarrow 0$), the two first moments become identical. Therefore, it is legitimate to say that this difference should be ascribed to the (non-thermal) quantum fluctuation. In fact, Figs. 1 and 2 show that

$$\langle W \rangle_{p(t)} \leq \Delta U(\tau) = \langle w \rangle_{p(t)}.$$

It is also tempting to examine more rigorously this inequality for its validity. To do so, we restrict ourselves to the periodic external drivings ($\omega_\tau = \omega_0$) for arbitrary pairs of $(\omega_t, \tau)$. Then, it follows from Eqs. (32) and (41) that the net external work on the system and the net internal energy difference are

$$\langle W \rangle_{p(t)},p = \beta^{-1} \left\{ (K_{\tau})_p - 1 \right\}, \quad \Delta U_p^{(f)}(\tau) = \langle W \rangle_{p(t),p} y \coth(y),$$

respectively, where $y = \beta\hbar\omega_0/2$. Because the factor $y \coth(y)$ monotonically increases for $y \geq 0$, it is easy to see that

$$\langle W \rangle_{p(t),p} \leq \Delta U_p^{(f)}(\tau).$$

Now we are in a position to discuss the quantum second law associated with $\langle W \rangle_{p(t)}$: The inequality (43), together with the Jensen inequality resulting from Eq. (40), yields the quantum-thermodynamic inequality

$$\Delta F_{p}^{(f)}(\omega_\tau) \leq \langle W \rangle_{p(t)} \leq \Delta U(\tau)$$

as one of our main findings. This represents a more fine-grained result than the inequality (5) obtained from the TPM framework; the first inequality reduces to the equality if the
thermodynamic quasi-static process with $Q_d^{(f)} = 0$ (obtained from both $K_\tau = 1$ and $\tilde{K}_\tau = 1$) is carried out (differing from the adiabatic process with $K_\tau = 1$). This can be implemented if additional heat exchange between system and environment is undergone infinitely slowly over the entire process. On the other hand, the second inequality reduces to the equality in the limit of $\beta, \hbar \to 0$ only; therefore, we can introduce the quantum heat $Q_q^{(f)} := \Delta U(\tau) - \langle \mathcal{W} \rangle_{\rho(t)} \geq 0$ (even for a thermally isolated system) which vanishes in the classical limit. This extra heat $Q_q^{(f)}$, different from the dissipative heat $Q_d^{(f)}$, is accordingly interpreted as a “built-in” quantity induced by the (non-thermal) quantum fluctuation. In Fig. 3, the behaviors of $Q_q^{(f)}$ and $Q_d^{(f)}$ are explicitly compared (note that $Q_q^{(f)} / Q_d^{(f)} > 1$).

It is also interesting to discuss the difference between two second moments $\langle \mathcal{W}^2 \rangle_{\rho(t)}$ and $\langle w^2 \rangle_{\rho(t)}$ as a next step to the first-moment inequality (43): With the help of Eqs. (33a)-(33c), we can acquire the relative variance

$$\left\langle (\Delta \mathcal{W})^2 \right\rangle_{\rho(t)} = \frac{\langle \mathcal{W}^2 \rangle_{\rho(t)} - \{ \langle \mathcal{W} \rangle_{\rho(t)} \}^2}{\langle \mathcal{W}^2 \rangle_{\rho(t)}}$$

$$= \left[ 1 + \frac{\beta^2}{4} \left\{ \frac{2 (K_\tau)^2 - 1}{\{ \coth(\beta \hbar \omega_\tau / 2) \}^2} - \frac{K_\tau \langle \mathbb{I}_0 \rangle_{\rho(t)} / \hbar}{\coth(\beta \hbar \omega_\tau / 2)} + \frac{1}{4} \right\} \right]^{-1} \langle \mathcal{W}^2 \rangle_{\rho(t)}$$

which is less than its counterpart $\langle (\Delta w)^2 \rangle_{\rho(t)} = 1 - (\Delta U)^2 / \langle w^2 \rangle_{\rho(t)}$ [cf. Eq. (44a)]; at zero temperature, $\langle (\Delta \mathcal{W})^2 \rangle_{\rho(t)}$ identically vanishes while $\langle (\Delta w)^2 \rangle_{\rho(t)} \to 2 (\omega_\tau)^2 \{ (K_\tau)^2 - 1 \} \left\{ 3 \omega_\tau^2 (K_\tau)^2 - 2 \omega_0 \omega_\tau K_\tau + (\omega_0)^2 - 2 (\omega_\tau)^2 \right\}^{-1}$. Again, it is the (non-thermal) quantum fluctuation contribution to $\langle (\Delta w)^2 \rangle_{\rho(t)}$ that gives this difference. On the other hand, in the high-temperature regime ($\beta \to 0$), these two quantities become identical. Fig. 4 demonstrates this variance difference.

Finally, we point out that the above discussion signifies that unlike the internal energy difference, the average quantum work (not viewed as a quantal expectation value) may be contingent upon the representation in consideration; in fact, there has thus far been no broadly agreed-upon “textbook” definition of quantum work [12]. Therefore, the choice of an appropriate representation for the average thermodynamic work in the (entire) quantum regime with its direct classical counterpart on the same footing will also be a significant issue.
B. Why only the Wigner Representation for the work distribution?

In fact, in addition to the Wigner function $W_\rho(x, p)$, one has several other quasi-probability distributions such as the Husimi function $Q_\rho(x, p) \geq 0$, the Glauber-Sudarshan function $P_\rho(x, p)$, and the standard-ordered function $F_\rho^{(s)}(x, p)$ [24]. Therefore, it is also tempting to discuss the quantum work (quasi)probability distribution in these additional representations. First, we point out that the two functions $K_\rho(x, p)$ and $F_\rho^{(s)}(x, p)$ are not always real-valued and therefore will not be under our consideration here. Therefore, we focus on the two real-valued functions $Q_\rho(x, p)$ and $P_\rho(x, p)$ only, which do not fulfill the marginal-distribution condition, like in Eq. (9) for $W_\rho(x, p)$, though. In fact, the Husimi function can simply be understood as the convolution of the Wigner function with a Gaussian filter such that

$$Q_\rho(x, p) = \frac{1}{\pi \hbar} \int dx' dp' W_\rho(x', p') \exp\left\{ - \left( \kappa (x' - x) \right)^2 + \left( \frac{p' - p}{\hbar \kappa} \right)^2 \right\}. \quad (48)$$

Likewise, the relation between the Husimi and Glauber-Sudarshan functions is

$$Q_\rho(x, p) = \frac{1}{2\pi \hbar} \int dx' dp' P_\rho(x', p') \exp\left\{ - \frac{1}{2} \left( \kappa (x' - x) \right)^2 + \left( \frac{p' - p}{\hbar \kappa} \right)^2 \right\}. \quad (49)$$

As an example, the thermal state for a linear oscillator is given by the Gaussian form

$$Q_\beta(x, p; \omega_t) = \frac{\cosh(\beta \hbar \omega_t/2) + a_\tau \sinh(\beta \hbar \omega_t/2)}{(2\pi \hbar) Z_\beta(\omega_t)^{-1}} \times \exp\left[ - \left( \coth \frac{\beta \hbar \omega_t}{2} + a_\tau \right)^{-1} \left( \kappa_t x \right)^2 + \frac{p^2}{(\hbar \kappa_t)^2} \right] \geq 0 \quad (50)$$

with $a_\tau = 1$, and $P_\beta(x, p; \omega_t)$ obtained from Eq. (50) but with $a_\tau = -1$; cf. $W_\beta(x, p; \omega_t)$ with $a_\tau = 0$.

The propagators for all other phase-space representations have been discussed in [13]; in fact, they can be expressed in terms of the Wigner propagator $T_w(x, p; \tau|x', p'; 0)$ and the evolution kernels $G$’s such that

$$T_\tau(x, p; \tau|x', p'; 0) = \int dx_1 \int dp_1 \int dx_2 \int dp_2 G_{w \rightarrow \tau}(x_1 - x, p_1 - p) \times T_w(x_1, p_1; \tau|x_2, p_2; 0) G_{\tau \rightarrow w}(x' - x_2, p' - p_2), \quad (51)$$

in which the symbols $Y = W, Q, P, K, F^{(s)}$ denote the respective phase-space representations; the kernels are explicitly given in [13], e.g., $G_{w \rightarrow w}(x, p) = \delta(x) \delta(p)$. This means that these
propagators are equivalent to transforming first into the Wigner representation \((G_{\gamma \rightarrow \text{W}})\) and then propagating with \(T_{\text{W}}\), followed by transforming into the original representation \((G_{\text{W} \rightarrow \gamma})\). Further, it has been shown that free propagation in the Wigner representation is completely classical-like such that \(T_{\text{W}}(x, p; \tau | x', p'; 0) \rightarrow \delta(p - p') \delta(x' + t p/m - x)\); however, in all other representations such a simple one-to-one correspondence between the initial and final phase-space points is unavailable even for free propagation. This implies that propagations in all other representations will in general possess the non-classical features in more complicated form than the Wigner propagation. Further, unlike \(T_{\text{W}}\), both propagators \(T_{Q}\) and \(T_{P}\) under our consideration have been shown to involve divergent evolution kernels \(G'\)’s in Eq. (51). This divergence for \(\gamma = Q, P\) will make it obscure to have the symmetry given by \(T_{\gamma}(f) = T_{\gamma}(b)\) between the forward and backward processes, which, for \(\gamma = W\), led to the result like in Eq. (55) and then the quantal-classical Crooks theorem in its compact form.

Nevertheless, it is still instructive to consider Eq. (55) but now expressed in terms of either \(Q_{\beta}\)’s or \(P_{\beta}\)’s in place of \(W_{\beta}\)’s. Then, it will be straightforward to introduce the “work” in the Husimi representation given by

\[
\mathbb{W}_{Q_{\beta}, (I_0, I_\tau)} = -\frac{1}{\beta} \ln \left\{ \frac{Q_{\beta}(I_\tau; \omega_\tau)}{Q_{\beta}(I_0; \omega_0)} \right\} + \Delta F_{\beta}^{(f)}(\omega_\tau)
\]

\((\neq \mathbb{W}_{(I_0, I_\tau)} \text{ in } (37))\) and likewise the “work” \(\mathbb{W}_{P_{\beta}, (I_0, I_\tau)}\) in the Glauber-Sudarshan representation. By applying the same techniques as for Eqs. (41)-(42), we can finally arrive at the “average work”

\[
\langle \mathbb{W}_{\gamma} \rangle_{\gamma(f)} = Q_{\gamma, d}(f) + \Delta F_{\beta}^{(f)}(\omega_\tau)
\]

\((\neq \langle \mathbb{W} \rangle_{\beta(f)} \text{ in } (41))\) expressed in terms of the “dissipative heat”

\[
Q_{\gamma, d}(f) = \frac{1}{\beta} \left\{ \frac{K_{\gamma}}{K_{\gamma} + a_{\gamma} \tanh(\beta \hbar \omega_0/2)} - \frac{1}{1 + a_{\gamma} \tanh(\beta \hbar \omega_0/2)} + \ln \frac{K_{\gamma} + a_{\gamma} \tanh(\beta \hbar \omega_0/2)}{1 + a_{\gamma} \tanh(\beta \hbar \omega_0/2)} \right\}.
\]

Then it follows that \(Q_{Q, d}(f) \neq 0\) displays the violation of the second law, and \(Q_{P, d}(f)\) incorrectly behaves in the low-temperature regime (cf. Fig. 5).

As a result, we see that although the internal energy and its difference can be uniquely determined regardless of the phase-space representations under consideration, it is the Wigner representation only that propagates in the most classical way among them. Therefore, it is legitimate to say that the Wigner representation is the most appropriate choice for the
study of the average thermodynamic work in the quantum regime equipped with the canonical transition to its classical counterpart $\langle W \rangle = \int_0^\tau dt \dot{\lambda} \partial_\lambda H(z_t; \lambda_t)$ in the simplest way. In fact, thermodynamics originally appeared from the classical domain. Therefore, defining quantum work in the Wigner representation is a consistent step toward a generalization of the classical work.

C. Comments on our results

Several additional comments are deserved here. First, it is instructive to compare our exact analysis with the semiclassical analysis, carried out in [12], in the context of the quantum-classical correspondence principle. In their analysis, the classical work distribution [cf. Eq. (17)], built from the classical trajectories that connect the initial and final energies, has shown an excellent approximation to its TPM counterpart in the semiclassical regime. On the other hand, our work distribution in the quantum regime, built from the Wigner trajectories that connect the initial and final action values, renders the average quantum work distinguished from the internal energy difference; however, in the classical limit, as discussed above, these trajectories exactly reduce to the classical ones and this distinction between the average work and internal energy difference goes away. Consequently, we may say that our quantal-classical analysis consistently accommodates this semiclassical analysis of the TPM framework.

Second, we add remarks upon the adiabatic process ($K_\tau = 1$): If an external driving acts infinitely slowly, there are no transitions between different eigenstates (cf. [16, 34] for the quantum adiabatic theorem) such that the conditional probability $P[m(\tau)|n(0)] = \delta_{nm}$, and thus no additional quantum fluctuation is produced over the driving. Therefore, while in the non-adiabatic process the quantum work $w_{nm}$ of the TPM framework has no independent physical reality until completion of the final measurement [12], in the adiabatic process this work has the physical reality over the driving indeed; therefore, in this case, the stochastic nature of the quantum work becomes associated solely with the random nature of the initial state, like in the classical setup. Then, the quantum heat $Q_q^{(0)}$ in the Wigner representation also appears solely from the quantum fluctuation in the initial state.

Third, it is also instructive to emphasize that we resorted to the action variable ($\mathbb{I}$) in our discussion, not to the EBK quantization rule given by $\mathbb{I}_n = \hbar (n + \alpha/4)$ [15, 16]: For a
linear oscillator with the Maslov index $\alpha = 2$, this semiclassical quantization is exactly valid over the entire quantum regime such that the energy eigenvalue $E_n = \omega I_n$. On the other hand, our approach simply made use of the continuous nature of the action variable, which underlies the continuous nature of the thermodynamic work $W_{(q,r)}$ in the quantum regime.

Finally, we remind that the work distribution in the Wigner representation is positive valued for a linear oscillator because of the Gaussian nature of the initial thermal state and its time evolution [cf. Eqs. (26)-(27)]. However, this is not the case any longer for generic systems such as a single particle confined by a one-dimensional infinite potential. Further, in our framework of quantum thermodynamics formulated without resorting to any projective measurements, the measurability of single-motion values $\Delta e_{(q,r)}$ and $W_{(q,r)}$ is inherently abandoned even for the process starting from the thermal state because of the quasi-probabilistic nature of the Wigner function; instead, our concern lies in the average values $\Delta U(\tau)$ and $\langle W \rangle_p$ only that reveal the more fine-grained form of the quantum second law in (46) with the canonical transition to its classical counterpart. In fact, the TPM framework is viewed as the special case only that the measurability of single-run values $\Delta e_{nm}$ is available. However, it is such a measurement-free nature of our framework that enables us to be free from a determination of the energy eigenvalues (required for the TPM framework) and straightforwardly generalize our findings to the processes starting from the non-thermal initial states with coherence in the energy basis. This subject will explicitly be covered by the following section.

IV. PARTIALLY THERMAL INITIAL STATES FOR FLUCTUATION RELATIONS

Now we generalize the Crooks fluctuation theorem in Eq. (36) by considering the partially thermal states (as a particular class of non-equilibrium initial states) such that $\hat{\rho}_0 = (1 - \gamma) \hat{\rho}_\beta(\omega_0) + \gamma \hat{\sigma}(\omega_0)$ for the forward process and $\hat{\rho}_r = (1 - \gamma) \hat{\rho}_\beta(\omega_\tau) + \gamma \hat{\sigma}(\omega_\tau)$ for the backward process. Here, the symbols $\hat{\sigma}$ and $\gamma$ (with $0 \leq \gamma < 1/2$) denote a non-thermal state and an imperfection in preparing the thermal state $\hat{\rho}_\beta$, respectively. To do so, we first modify $A^{(f)}(0)$ in Eq. (18b), simply by replacing $W_\beta$ with $W_{\rho_0}$, into

$$A^{(f)}_{\rho_0}(0) = \int dx dp \int dx' dp' W_{\rho_0}(x', p'; \omega_0) T_{W}^{(f)}(x, p; \tau | x', p'; 0) , \quad (55)$$
being unity. Then it is straightforward to rewrite this as \( A_{\rho_0}^{(f)}(0) = \int_0^\infty \int_0^\infty d\Pi_0 d\Pi_r \, \mathbb{P}_{\rho_0}^{(f)}(\Pi_0, \Pi_r) \), in which the joint distribution for the forward process is given by [cf. Eq. (26)]

\[
\mathbb{P}_{\rho_0}^{(f)}(\Pi_0, \Pi_r) = (1 - \gamma) \, \mathbb{P}^{(f)}(\Pi_0, \Pi_r) + \gamma \, G_{\sigma}^{(f)}(\Pi_0, \Pi_r)
\]

Here, the second term on the right-hand side will be determined explicitly for a given state \( \hat{\sigma} \). Likewise, we have for the backward process [cf. Eq. (27)]

\[
\mathbb{P}_{\rho_r}^{(b)}(\Pi_r, \Pi_0) = (1 - \gamma) \, \mathbb{P}^{(b)}(\Pi_r, \Pi_0) + \gamma \, G_{\sigma}^{(b)}(\Pi_r, \Pi_0)
\]

These two distributions can, in general, be negative valued and will be used for the generalized Crooks theorem in the Wigner representation. Several particular cases for \( \hat{\sigma} \) will be under consideration below.

### A. Mixture of thermal and eigen-energy states

Let \( \hat{\sigma}_1 = |n\rangle \langle n| \) with \( W_{\sigma_1}(x, p) \). Then, we need the Wigner function of the nth energy eigenstate in Eq. (24); e.g.,

\[
W_0(\Pi) = \frac{1}{\pi \hbar} e^{-2t/\hbar} ; \quad W_1(\Pi) = \frac{1}{\pi \hbar} e^{-2t/\hbar} \left( \frac{4 \Pi}{\hbar} - 1 \right) ; \quad W_2(\Pi) = \frac{1}{\pi \hbar} e^{-2t/\hbar} \left( \frac{8 \Pi^2}{\hbar^2} - \frac{8 \Pi}{\hbar} + 1 \right).
\]

The two quasi-probability distributions \( W_1(\Pi) \) and \( W_2(\Pi) \) can be negative valued indeed. By combining Eqs. (56) and (57) with \( G_{\sigma}^{(f,b)} \rightarrow W_n^{(f,b)} \) \( B^{(f)} \) in this case, it is straightforward to acquire the generalized Crooks fluctuation theorem

\[
\frac{\mathbb{P}^{(f)}(\Pi_0, \Pi_r)}{\mathbb{P}^{(b)}(\Pi_r, \Pi_0)} = \frac{(1 - \gamma) \, W_\beta(\Pi_r; \omega_r) + \gamma \, W_n(\Pi_r)}{(1 - \gamma) \, W_\beta(\Pi_0; \omega_0) + \gamma \, W_n(\Pi_0)} = \exp \left\{ \beta \left( \mathbb{W}_{\perp}(\Pi_0, \Pi_r) - \Delta F^{(f)}_\beta \right) \right\} \geq 0
\]

([\( \mathbb{P}_{\rho_0,1} \rightarrow \mathbb{P}_{\perp}^{(f)} \) and \( \mathbb{P}_{\rho_r,1} \rightarrow \mathbb{P}_{\perp}^{(b)} \) in notation]; here, the generalized work is identified as

\[
\mathbb{W}_{\perp}(\Pi_0, \Pi_r) = \mathbb{W}_{\perp}(\Pi_0, \Pi_r) - \frac{1}{\beta} \ln \left[ \frac{1 + \{ \gamma/(1 - \gamma) \} \, W_n(\Pi_0) / W_\beta(\Pi_r; \omega_r)}{1 + \{ \gamma/(1 - \gamma) \} \, W_n(\Pi_0) / W_\beta(\Pi_0; \omega_0)} \right]
\]

for a given value of \( \gamma \) with \( \gamma/(1 - \gamma) < 1 \) [cf. Eq. (37)]. As such, the second term is not linear in \( \Pi_0 \) and \( \Pi_r \) any longer. We also find that the initial state \( W_{\rho_0,1}(x, p; \omega_0) \) and so the distribution \( \mathbb{P}_{\perp}^{(f)}(\Pi_0, \Pi_r) \) can be negative valued indeed if \( n \neq 0 \) and \( \gamma_{th_{\perp}} < \gamma < 1/2 \), where the threshold value \( \gamma_{th_{\perp}} \) should be determined for the respective initial state (cf. Fig. 6).
Then, Eq. (59) will yield the Jarzynski equality
\[
\langle e^{-\beta W} \rangle_{\beta}^{(f)} = \int dW e^{-\beta W} P_{\beta}^{(f)}(W) = e^{-\beta \Delta F_{\beta}}
\]  
(61)
where the work distribution is
\[
P_{\beta}^{(f)}(W) = \int_{0}^{\infty} \int_{0}^{\infty} dI_0 dI_\tau \, \delta(W - W_1(I_0, I_\tau)) P_{\beta}^{(f)}(I_0, I_\tau).
\]  
(62)
Eq. (61) will finally give rise to the generalized second-law inequality
\[
\Delta F_{\beta}^{(f)} \leq \langle W \rangle_{\beta}^{(f)} \leq \Delta U_1(\tau)
\]  
(63)
[cf. (46) for \( r = 0 \)]. Here, the internal energy difference
\[
\Delta U_1(\tau) = \omega_\tau \langle I_\tau \rangle_{\beta}^{(f)} - \omega_0 \langle I_0 \rangle_{\beta}^{(f)} \{ (1-r) \coth(\beta \hbar \omega_0/2) + r(2n+1) \}
\]  
(64)
is still expressed in terms of both first moments [cf. (52)]. On the other hand, the average work \( \langle W \rangle_{\beta}^{(f)} \) will be expressed in terms of the higher-order moments in addition to the first moments; cf. Eqs. (33a)-(33c) and (A14a)-(A14b) are useful also for \( r \neq 0 \) in this case. The second inequality in (63) can be verified, as in (43) for \( r = 0 \). These behaviors of \( \langle W \rangle_{\beta}^{(f)} \) are demonstrated in Fig. 7.

B. Mixture of thermal and energy-superposed states: Case 1

Let \( \hat{\sigma}_2 = |n\rangle + |n+1\rangle(\langle n\rangle + \langle n+1\rangle)/2 \), which possesses the energy coherence. Then, we also need the Moyal functions [25]
\[
W_{|m\rangle\langle n|}(x, p) = \frac{(-1)^n}{\pi \hbar} \left( \frac{n!}{m!} \right)^{1/2} \{ 2 \eta^*(x, p) \}^{m-n} e^{-2|\eta(x, p)|^2} L_n^{(|m-n|)}(4 \eta^2(x, p))
\]  
(65)
for \( n \leq m \), where \( L_n^{(k)}(\cdots) \) denotes the associated Laguerre polynomial, and \( W_{|n\rangle\langle m|}(x, p) = \{ W_{|m\rangle\langle n|}(x, p) \}^* \); e.g.,
\[
W_{|1\rangle\langle 0|}(x, p) = \frac{\sqrt{2}}{\pi \hbar} \left( \frac{1}{\hbar} \right)^{1/2} \exp \left[- \left\{ (\kappa x)^2 + \left( \frac{p}{\hbar} \right)^2 \right\} \right]
\]  
(66a)
\[
W_{|2\rangle\langle 1|}(x, p) = \frac{2}{\pi \hbar} \left( \frac{1}{\hbar} \right)^{1/2} \left\{ (\kappa x)^2 + \left( \frac{p}{\hbar} \right)^2 - 1 \right\} \exp \left[- \left\{ (\kappa x)^2 + \left( \frac{p}{\hbar} \right)^2 \right\} \right]
\]  
(66b)
\[
W_{|2\rangle\langle 0|}(x, p) = \frac{\sqrt{2}}{\pi \hbar} \left\{ (\kappa x)^2 - \left( \frac{p}{\hbar} \right)^2 - 2i \xi p \right\} \exp \left[- \left\{ (\kappa x)^2 + \left( \frac{p}{\hbar} \right)^2 \right\} \right].
\]  
(66c)
We see that the angle coordinate $\phi$ will also be needed for the Moyal functions $W_{|m\rangle\langle n|}(\phi, \Pi)$. Employing Eqs. (24) and (65), we can obtain the Wigner function

$$W_{\sigma_2}(\phi, \Pi; n) = \frac{(-1)^n}{2\pi \hbar} e^{-2\pi^2/h} \left\{ L_n(4 \Pi/h) - L_{n+1}(4 \Pi/h) + 4 \sqrt{\hbar/(n+1)} (\sin \phi) L_{n}^{(1)}(4 \Pi/h) \right\}.$$  

Then, we can find from Eq. (56) that the joint distribution for the forward process is

$$P^{(f)}_2(\Pi_0, \Pi_r) = B^{(f)}(\Pi_r|\Pi_0) \left\{ 1 - \gamma \right\} W_\beta(\Pi_0; \omega_0) + \gamma \frac{W_n(\Pi_0) + W_{n+1}(\Pi_0)}{2} +$$

$$\gamma \frac{2}{\pi \hbar} C^{(f)}(\Pi_r|\Pi_0) \left\{ \frac{\Pi_0}{\hbar (n+1)} \right\}^{1/2} e^{-2\pi \omega_0/h} L_n^{(1)}(4 \Pi_0/h),$$

where the second conditional distribution is given by

$$C^{(f)}(\Pi_r|\Pi_0) = \text{Re} \int_0^{2\pi} \int_0^{2\pi} d\phi_r d\phi_0 (\sin \phi_0) \tilde{T}^{(f)}_W(\phi_r, \Pi_r; \tau|\phi_0, \Pi_0; 0).$$

In fact, we can show that $C^{(f)}(\Pi_r|\Pi_0) \equiv 0$ (cf. Appendix B). Consequently, we see that the off-diagonal terms of the initial state $\hat{\rho}_{0,2}$ do not contribute to $P^{(f)}_2(\Pi_0, \Pi_r)$, meaning that the diagonal form $\hat{\sigma}_2 = (|n\rangle\langle n| + |n+1\rangle\langle n+1|)/2$, in place of $\hat{\sigma}_2$, will give the same result for $P^{(f)}_2(\Pi_0, \Pi_r)$. Similarly, we can acquire the joint distribution for the backward process

$$P^{(b)}_2(\Pi_r, \Pi_0) = B^{(b)}(\Pi_r|\Pi_0) \left\{ 1 - \gamma \right\} W_\beta(\Pi_r; \omega_r) + \gamma \frac{W_n(\Pi_r) + W_{n+1}(\Pi_r)}{2}.$$ 

Combining Eqs. (68) and (70), it is straightforward to acquire the Crooks fluctuation theorem in the form of Eq. (59) and the Jarzynski equality in the form of (61) as well as the second-law inequality in the form of (55), where the pertinent work $\tilde{W}_2(\Pi_0, \Pi_r)$ is accordingly given by Eq. (61) but with $W_n(\Pi_r) \to \{W_n(\Pi_r) + W_{n+1}(\Pi_r)\}/2$, and its distribution is then

$$P^{(f)}_2(\tilde{W}) = \int_0^{\infty} \int_0^{\infty} d\Pi_0 d\Pi_r \delta(\tilde{W} - \tilde{W}_2(\Pi_0, \Pi_r)) P^{(f)}_2(\Pi_0, \Pi_r).$$

Also, the internal energy difference $\Delta U_2(\tau) \to \Delta U_2(\tau)$ such that

$$\Delta U_2(\tau) = \omega_r \langle \Pi_r \rangle^{(f)}_2 - \omega_0 \langle \Pi_0 \rangle^{(f)}_2 = (\hbar/2) (\omega_r \mathbb{K}_r - \omega_0) \{(1 - r) \coth(\beta \hbar \omega_0/2) + 2r(n + 1)\}.$$ 

C. Mixture of thermal and energy-superposed states: Case 2

Let $\hat{\sigma}_3 = (|n\rangle + |n+1\rangle + |n+2\rangle)(\langle n| + \langle n+1| + \langle n+2|)/3$. Then, we have

$$W_{\sigma_3}(\phi, \Pi) = \frac{2}{3} \left\{ W_{\sigma_2}(\phi, \Pi; n) + W_{\sigma_2}(\phi, \Pi; n+1) \right\} + \frac{1}{3} \left\{ -W_{n+1}(\phi, \Pi) + F_{n,n+2}(\phi, \Pi) \right\}$$
[cf. Eq. (67)], where the last term on the right-hand side is

\[ F_{n+2}(\phi, \Pi) = W_{n+2}(\phi, \Pi) + W_{n}(\phi, \Pi) \]

\[ = \frac{8(-1)^n}{\pi \hbar} \sqrt{\{(n+1)(n+2)\}^{-1}} e^{-2^1/\hbar} (\Pi/\hbar) \{2(\sin \phi)^2 - 1\} L_n^{(2)}(4\Pi/\hbar). \]  

(74)

We can obtain from Eq. (56) the joint distribution for the forward process

\[ \mathbb{P}^{(f)}_{\Delta}(\Pi_0, \Pi_r) = (1 - \gamma) B^{(f)}(\Pi_r|\Pi_0) W_\beta(\Pi_0; \omega_0) + \gamma G^{(f)}_{\Delta}(\Pi_0, \Pi_r), \]

(75)

where

\[ G^{(f)}_{\Delta}(\Pi_0, \Pi_r) = G^{(f)}_{\Delta}(\Pi_0, \Pi_r) + \frac{8(-1)^n}{3\pi \hbar} \sqrt{\{(n+1)(n+2)\}^{-1}} \{2 D^{(f)}(\Pi_r|\Pi_0) - B^{(f)}(\Pi_r|\Pi_0)\} (\Pi_0/\hbar) e^{-2\pi/\hbar} L_n^{(2)}(4\Pi/\hbar). \]  

(76)

Here, the diagonal-term contribution and the third conditional distribution are

\[ G^{(f)}_{\Delta}(\Pi_0, \Pi_r) = B^{(f)}(\Pi_r|\Pi_0) \frac{W_n(\Pi_0) + W_{n+1}(\Pi_0) + W_{n+2}(\Pi_0)}{3} \]  

(77a)

\[ D^{(f)}(\Pi_r|\Pi_0) = \text{Re} \int_0^{2\pi} \int_0^{2\pi} d\phi_r d\phi_0 (\sin \phi_0)^2 \tilde{T}^{(f)}(\phi_r, \Pi_r; \tau|\phi_0, \Pi_0; 0), \]  

(77b)

respectively (cf. Appendix B). We see that the joint distribution \( \mathbb{P}^{(f)}_{\Delta} \) differs from its counterpart \( \mathbb{P}^{(f)}_{\Delta} \) given by Eq. (45) but with \( G^{(f)}_{\Delta} \rightarrow G^{(f)}_{\Delta} \) (without coherence), obtained from the diagonal form \( \tilde{\sigma}_3 = (|n\rangle\langle n| + |n+1\rangle\langle n+1| + |n+2\rangle\langle n+2|)/3 \) in place of \( \tilde{\sigma}_3 \). Likewise, the joint distribution for the backward process is given by

\[ \mathbb{P}^{(b)}_{\Delta}(\Pi_r, \Pi_0) = (1 - \gamma) B^{(b)}(\Pi_r|\Pi_0) W_\beta(\Pi_r; \omega_r) + \gamma G^{(b)}_{\Delta}(\Pi_r, \Pi_0), \]

(78)

in which \( G^{(b)}_{\Delta}(\Pi_r, \Pi_0) \) is given by Eq. (76) but with \( \Pi_0 \rightarrow \Pi_r \) while \( B^{(b)}(\Pi_r|\Pi_0) \) remains unaffected, and \( D^{(b)}(\Pi_0|\Pi_r) \) is given by Eq. (77b) but with sin \( \phi_0 \rightarrow \sin \phi_r \).

Then it is easy to acquire the Crooks fluctuation theorem, the Jarzynski equality and the second-law inequality in the form of Eqs. (59), (61) and (63), respectively. Here, the pertinent work \( \mathbb{W}_{\Delta_{l_{\langle 0,1 \rangle}} \rightarrow l_{\langle 0,1 \rangle}} \) is explicitly given by Eq. (60) but with \( W_n(\Pi_0) \rightarrow G^{(f)}_{\Delta}(\Pi_0, \Pi_r)/B^{(f)}(\Pi_r|\Pi_0) \) and \( W_n(\Pi_r) \rightarrow G^{(b)}_{\Delta}(\Pi_r, \Pi_0)/B^{(b)}(\Pi_r|\Pi_0) \), as well as its distribution is

\[ \mathbb{P}^{(f)}_{\Delta}(\mathbb{W}) = \int_0^{\infty} d\Pi_0 d\Pi_r \delta(\mathbb{W} - \mathbb{W}_{\Delta_{l_{\langle 0,1 \rangle}} \rightarrow l_{\langle 0,1 \rangle}}) \mathbb{P}^{(f)}_{\Delta}(\Pi_0, \Pi_r). \]  

(79)

Also, the internal energy difference \( \Delta U_{\Delta}(\tau) \rightarrow \Delta U_{\Delta}(\tau) \). In Fig. 8, the behaviors of \( \langle \mathbb{W} \rangle^{(f)}_{\Delta} \) are explicitly compared with those of its counterpart \( \langle \mathbb{W} \rangle^{(f)}_{\Delta} \) (obtained from \( \mathbb{P}^{(f)}_{\Delta} \)); as shown,
the quantities $\langle \mathcal{W} \rangle_{p(t)}$ and $\Delta U_3(\tau)$ with coherence are greater than their counterparts $\langle \mathcal{W} \rangle_{p(t)' \neq 0}$ and $\Delta U_3(\tau)$ without coherence, respectively, where for $n = 0$

$$\Delta U_3(\tau) = \omega_\tau \langle \mathcal{I}_\tau \rangle_{p(t)_{\neq 0}} - \omega_0 \langle \mathcal{I}_0 \rangle_{p(t)_{\neq 0}} = \omega_\tau \left[(1 - r) \langle \mathcal{I}_\tau \rangle_{p(t)} + \frac{\gamma}{3} \left\{ \frac{8}{\hbar^2} \langle \mathcal{I}_\tau (\mathcal{I}_0)^2 \rangle_{p(t)} + \frac{4}{\hbar} (2^{1/2} - 1) \langle \mathcal{I}_\tau \mathcal{I}_0 \rangle_{p(t)} + \langle \mathcal{I}_\tau \rangle_{p(t)} \right\}_{\beta \to \infty} \right] - \omega_0 [(\omega_\tau \to \omega_0)]$$

$$= \frac{\hbar}{2} (\omega_\tau \mathcal{K}_\tau - \omega_0) \left\{ (1 - r) \coth(\beta \omega_0/2) + \frac{r}{3} (9 + 4 \sqrt{2}) \right\}$$

(80a)

[cf. (A14a)-(A14b)], and

$$\Delta U_3(\tau) = \omega_\tau \langle \mathcal{I}_\tau \rangle_{p(t)_{\neq 0}} - \omega_0 \langle \mathcal{I}_0 \rangle_{p(t)_{\neq 0}} = (\hbar/2) (\omega_\tau \mathcal{K}_\tau - \omega_0) \left\{ (1 - r) \coth(\beta \omega_0/2) + r (2n + 3) \right\}.$$  

(80b)

D. Comments on our results

Now we give the interpretation of our findings in the present section. Differing from the thermal initial state (with $\gamma = 0$), the partially thermal initial state (with $\gamma > 0$) results in the fact that the internal energy difference is still given by the first moments, but the average work, obtained from the generalized work in non-linear form, necessarily contains the higher-order moment contributions. Remarkably enough, such a generalized work can also be linked to the fully thermodynamic quantity $\Delta F_\beta$ operationally through our generalized Crooks theorem (and the resulting Jarzynski equality), finally giving rise to the second-law inequality associated with the average work. If the parameter $\gamma$ continues to increase such that it becomes greater than its threshold value $\gamma_{th}$ (like in Fig. 6), then a dominance of the quantum fluctuation over the thermal fluctuation will be found for the initial state in non-Gaussian form (e.g., $\hat{\sigma}_1 = |n\rangle \langle n|$ with $n \neq 0$) so that the initial Wigner function and the work distribution can be negative valued. In the classical scenario, on the other hand, this exact link between the generalized work and the free energy difference is well-defined (i.e., with no negativity of the work probability distribution) also for the single-motion values through our generalized Crooks theorem in the classical limit. In fact, a perfect preparation of the thermal state (with $r = 0$) could be a formidable task in reality.

It is also instructive to point out that our result obtained from the phase-space framework, free from the projective measurement, is consistent with the result obtained from the histories
framework (as a different generalization of the TPM framework) in [21], which employed
the time-reversal symmetrized work distributions for non-thermal initial states, concluding
that thermodynamic work in the quantum regime cannot be determined by the projective
measurements.

V. CONCLUSION

We studied the quantum fluctuation relations in the Wigner representation. To make our
analysis as exact as possible, we restricted our discussion here to a driven quantum linear
oscillator. Then we obtained the single-motion work (in closed form) in the quantum regime
and its distribution expressed in terms of the action coordinates only, without resorting to
any projective measurements, such that the quantum and classical setups can be analyzed
on the single footing. This enabled us to derive the quantum Crooks fluctuation theorem,
the quantum Jarzynski equality and the second-law inequality in more fine-grained form
than their counterparts in the two-point projective measurement (TPM) framework, in that
the resulting average work \( \langle W \rangle \) in our framework notably differs from the internal energy
difference \( \Delta U(\tau) \) between the initial and final states, thus rendering the quantum heat
\( Q_q = \Delta U(\tau) - \langle W \rangle \geq 0 \) introduced. Such a discrepancy between \( \Delta U(\tau) \) and \( \langle W \rangle \) was shown
to disappear gradually in the semiclassical regime. This result contrasts with \( \langle W \rangle \equiv \Delta U(\tau) \)
in the standard TPM framework. We also provided a justification for the choice of the
Wigner representation for our analysis rather than any other phase-space representations.
We showed that it is the Wigner representation that behaves in the most classical-like way
and is most appropriate for the quantum work with the canonical transition to its classical
counterpart, in that thermodynamics originally took its shape from the classical scenario.

Our findings were straightforwardly generalized to the processes starting from a partic-
ular class of non-thermal initial states including the states with quantum coherence in the
eigen-energy basis. Here, we introduced the generalized quantum work with its non-linear
nature and the work distribution with its negativity. As a matter of fact, the unavoidable
negativity of the work distribution has been well-known also in the extended TPM frame-
work where the initial state is non-diagonal in the energy basis. In this paper, on the other
hand, we used such a negativity resulting from the quasi-probabilistic nature of the Wigner
function as our stating point for a new framework free from the projective measurements,
which led to achieving the aforementioned fine-grained results in the quantum thermodynamics. As a result, it is legitimate to claim that our results, covering the genuine quantum to (semi)classical regimes, can provide a more sophisticated discussion of the second law of thermodynamics associated with the average work within an isolated quantum system. Finally, as long as the angle-action coordinates are well-defined, our methodology will continue to apply to different quantum systems including the generic one-dimensional systems.

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Appendix A: Evaluations of Eqs. (30) and (33a)-(33c)

We begin by showing the non-negativity of the conditional distribution \( B^{(f)}(I|I_0) \) for all \((I_0, I)\) in (26): We first substitute Eq. (16) into Eq. (7) with \( x = \{2\bar{I}/(m\omega)\}^{1/2} \sin \phi \) and \( p = (2m\omega\bar{I})^{1/2} \cos \phi \), followed by executing the integrals over \( \xi_1 \) and \( \xi_2 \). This will transform Eq. (23b) into

\[
B^{(f)}(f)(I|I_0) = m\hbar^2 |X_\tau| \int d\xi_1 d\xi_2 J_0(b_\tau \sqrt{I_\tau}) J_0(b_0 \sqrt{I_0}),
\]

where

\[
g_1 = \left( \frac{2m\bar{I}}{\omega_0} \right)^{1/2} \frac{1}{X_\tau} \sin(\phi_0) + (2m\omega_0 I_\tau)^{1/2} \cos(\phi_\tau) - \left( \frac{2mI_\tau}{\omega_\tau} \right)^{1/2} \frac{\dot{X}_\tau}{X_\tau} \sin(\phi_\tau),
\]

and \( g_2 \) is given by Eq. (A2) with \((\phi_0 \leftrightarrow \phi_\tau), (I_0 \leftrightarrow I_\tau), (\omega_0 \leftrightarrow \omega_\tau) \) and \((\dot{X}_\tau \rightarrow Y_\tau)\). With the help of the identity \( \delta\{g(y)\} = \sum_j \delta(y - y_j)/|g'(y_j)| \) with \( g(y_j) = 0 \), we can finally observe that \( B^{(f)}(I|I_0) \geq 0 \) [cf. Eqs (14a)-(14b)].

Now we explicitly evaluate the first moments \( \langle I_0 \rangle_{\phi(t)} \) and \( \langle I_\tau \rangle_{\phi(t)} \) in Eq. (30): For later purposes, we begin by verifying that \( A^{(f)}(0) = \langle (I_\tau)^0 \rangle_{\phi(t)} = 1 \) in (22). With the help of the relation \((2\pi) J_0(\sqrt{A^2 + B^2}) = \int_0^{2\pi} d\phi \ e^{i\{A(\sin \phi)+B(\cos \phi)\}} \) obtained from the identity \( J_0(y) = (1/\pi) \int_0^{\pi} d\phi \ \cos(y \sin \phi) \) for the Bessel function \( J_0(y) \), we can transform Eq. (23b) into

\[
B^{(f)}(I|I_0) = \frac{m}{\hbar^2 |X_\tau|} \int d\xi_1 d\xi_2 J_0(b_\tau \sqrt{I_\tau}) J_0(b_0 \sqrt{I_0}),
\]
in place of (A11), where

\[ b_\tau = \left\{ \frac{2m}{(\hbar X_\tau)^2 \omega_\tau} \left( \dot{X}_\tau \xi_1 + \xi_2 \right)^2 + \frac{2m \omega_\tau}{\hbar^2} (\xi_1)^2 \right\}^{1/2}, \]  

(A4)

and \( b_0 \) is given by (A4) with \( (\xi_1 \leftrightarrow \xi_2), (\omega_0 \leftrightarrow \omega_\tau) \) and \( (\dot{X}_\tau \rightarrow Y_\tau) \). Eqs. (25H) and (A3) give

\[ \left\langle \left( \int_{\tau} \right)^0 \right\rangle_{\rho(t)} = \frac{m}{(2\pi \hbar^3)} \frac{Z_\beta(\omega_0)|X_\tau|}{\sech \left( \frac{\beta \hbar \omega_0}{2} \right)} \int \int d\xi_1 \, d\xi_2 \, \Lambda_0(\xi_1, \xi_2), \]  

(A5)

where

\[ \Lambda_n(\xi_1, \xi_2) = \lim_{a_\tau \to 0} \int_0^\infty d\Pi_\tau (\Pi_\tau)^n \times J_0(b_\tau \sqrt{\Pi_\tau}) \, e^{-(a_\tau)^2 \tau} \int_0^\infty d\Pi_0 J_0(b_0 \sqrt{\Pi_0}) \, e^{-(a_0)^2 \tau} \]  

(A6)

with \((a_0)^2 = (2/\hbar) \tanh(\beta \hbar \omega_0/2)\). We now apply the identity

\[ \int_0^\infty dt \, t^{\mu-1} \exp(-a^2 t^2) \, J_\nu(bt) = \frac{(b/2)^\nu}{2(a^{\mu+\nu})} \Gamma(\nu+1) \, 1F_1 \left( \frac{\mu+\nu}{2}; \nu+1; -\frac{b^2}{4a^2} \right) \]  

(A7)

with \( (\mu = 2, \nu = 0) \) and \( 1F_1(1; 1; z) = e^z \) to Eq. (A6) with \( n = 0 \) such that

\[ \Lambda_0(\xi_1, \xi_2) = \lim_{a_\tau \to 0} \frac{1}{(a_\tau)^2} e^{-\gamma(2a_\tau)^2} \left| \right|_{\gamma=1} \frac{1}{(a_0)^2} e^{-(b_0)^2(2a_0)^{-2}}. \]  

(A8)

The integration of this over \( \xi_1 \) and \( \xi_2 \) in Eq. (A5) will give rise to

\[ \int \int d\xi_1 \, d\xi_2 \, \Lambda_0(\xi_1, \xi_2) = \lim_{a_\tau \to 0} \frac{2\pi \hbar^2/m}{(a_\tau)^2} \left. F_\tau(\gamma) \right|_{\gamma=1} \frac{2\pi \hbar^2/m}{(a_0)^2} |X_\tau|, \]  

(A9)

where

\[ F_\tau(\gamma) = \left\{ \frac{\gamma(\dot{X}_\tau)^2}{(a_\tau X_\tau)^2 \omega_\tau} + \frac{\omega_\tau}{(a_\tau)^2} + \frac{1}{(a_0 X_\tau)^2 \omega_0} \right\}^{-1/2}; \quad G_\tau(\gamma) = \left\{ \frac{\gamma(\dot{X}_\tau)^2}{(a_\tau X_\tau)^2 \omega_\tau} + \frac{\omega_0}{(a_\tau)^2} + \frac{(Y_\tau)^2}{(a_0 X_\tau)^2 \omega_0} - [F_\tau(\gamma)]^2 \left( \frac{\gamma(\dot{X}_\tau)^2}{(a_\tau X_\tau)^2 \omega_\tau} + \frac{Y_\tau}{(a_0 X_\tau)^2 \omega_0} \right) \right\}^{-1/2}. \]  

(A10)

Finally, it follows that \( \left\langle \left( \int_{\tau} \right)^0 \right\rangle_{\rho(t)} = 1 \) indeed.

Then, it is straightforward to obtain the first moment [cf. Eq. (A5)]

\[ \left\langle \int_{\tau} \right\rangle_{\rho(t)} = \frac{m}{(2\pi \hbar^3)} \frac{Z_\beta(\omega_0)|X_\tau|}{\sech \left( \frac{\beta \hbar \omega_0}{2} \right)} \int \int d\xi_1 \, d\xi_2 \, \Lambda_1(\xi_1, \xi_2). \]  

(A11)

We now apply (A7) with \( (\mu = 4, \nu = 0) \) and \( 1F_1(2; 1; z) = (1 + z) e^z = (1 + \partial_\gamma) e^{\gamma z} |_{\gamma=1} \) to (A6) with \( n = 1 \) such that

\[ \Lambda_1(\xi_1, \xi_2) = \lim_{a_\tau \to 0} \frac{1}{(a_\tau)^4} \left| 1 + \partial_\gamma \right| e^{-\gamma(2a_\tau)^2} \left|_{\gamma=1} \frac{1}{(a_0)^2} e^{-(b_0)^2(2a_0)^{-2}}. \]  

(A12)
Next, similarly to Eq. (A9), we can obtain

$$\int \int d\xi_1 d\xi_2 \Lambda_1(\xi_1, \xi_2) = \lim_{\alpha_0 \to 0} \frac{2\pi \hbar^2/m}{(\alpha_0)^2 (\alpha_0)^4} (1 + \partial_\gamma) F_\tau(\gamma) G_\tau(\gamma)|_{\gamma = 1}. \quad (A13)$$

This finally simplifies to \((2\pi \hbar^2)|X_\tau|K_\tau/\{m(\alpha_0)^4\}\) [cf. (31)]. Therefore, Eq. (A11) reduces to (30). Likewise, we can also evaluate the first moment \(\langle \hbar \rangle_{\tau(t)}\) with the help of \((a_0 \leftrightarrow a_\tau)\) and \((b_0 \leftrightarrow b_\tau)\) in (A12).

We can further evaluate the higher-order moments in the form of \(\langle (\hbar_\tau)^n (\hbar_0)^m \rangle_{\tau(t)}\) by applying the same techniques with the help of Eqs. (A6), (A7) and (A10) as well as the recurrence relation \((b - a) \int_0^1 F_1(a - 1; b; z) + (2a - b + z) \int_0^1 F_1(a; b; z) - a \int_0^1 F_1(a + 1; b; z)\) \([35]\); e.g., for the second moment \(\langle (\hbar_\tau)^2 \rangle_{\tau(t)}\), we have (A7) with \((\mu = 6, \nu = 0)\) and \(\int_0^1 F_1(3; 1; z) = (z^2 + 4z + 2)e^z/2\) [cf. Eqs. (33a)-(33c)]. Likewise, the third moments can be evaluated

$$\langle (\hbar_\tau)^3 \rangle_{\tau(t)} = \{15 (\hbar_\tau)^2 - 9\} K_\tau \{(\hbar_0)^3\}_{\tau(t)} ; \quad \langle (\hbar_\tau)^3 \rangle_{\tau(t)} = 6 \{(\hbar_0)^3\}_{\tau(t)} \quad (A14a)$$

$$\langle (\hbar_\tau)^2 (\hbar_0) \rangle_{\tau(t)} = \{9 (\hbar_\tau)^2 - 3\} K_\tau \{(\hbar_0)^3\}_{\tau(t)} ; \quad \langle (\hbar_\tau) (\hbar_0)^2 \rangle_{\tau(t)} = 6 K_\tau \{(\hbar_0)^3\}_{\tau(t)} \quad (A14b)$$

Finally, we verify that the internal energy difference as a picture-independent quantity can be evaluated also in the TPM framework: To do so, we consider the expectation value \(\langle \hat{H}(\omega_\tau) \rangle_{\rho_\tau} = \text{Tr}\{\hat{H}(\omega_\tau) \hat{U}(\tau) \hat{\rho}_0 \hat{U}^\dagger(\tau)\}\) with \(\hat{\rho}_0 = \hat{\rho}_\beta\). This can be rewritten as

$$\int dy dy' dx dx' \left\langle y \left| \left( \frac{\hat{p}^2}{2m} + \frac{m \omega_\tau^2 \hat{x}^2}{2} \right) \right| x \right\rangle K(x; \tau|x'; 0) K^*(y; \tau|y'; 0) \langle x'|\hat{\rho}_\beta|y'\rangle. \quad (A15)$$

where the propagator \(K(\cdots)\) in Eq. (16) and

$$\langle x|\hat{\rho}_\beta|y\rangle = \left\{ \left( \frac{m \omega_\beta}{\pi \hbar} \right) \tanh(\beta \hbar \omega_0/2) \right\}^{1/2} \times$$

$$\exp \left[ -\frac{m \omega_\beta}{4\hbar} \left( (x + y)^2 \tanh(\beta \hbar \omega_0/2) + (x - y)^2 \coth(\beta \hbar \omega_0/2) \right) \right]$$

[33]. Then, it is straightforward to find that \(\langle \hat{H}(\omega_\tau) \rangle_{\rho_\tau} = \omega_\tau \langle \hbar_\tau \rangle_{\tau(t)}\).

**Appendix B: Evaluations of Eqs. (69) and (77b)**

We follow the steps similar to those for the derivation of Eq. (A3): By employing the identity

$$\int_0^{2\pi} d\phi e^{i\{A(\sin \phi) + B(\cos \phi)\}} (\sin \phi) = \frac{\partial A}{i} \int_0^{2\pi} d\phi e^{i\{A(\sin \phi) + B(\cos \phi)\}} = -2\pi i \partial_A J_0(\sqrt{A^2 + B^2})$$

(B1)
with \( dJ_0(z)/dz = -J_1(z) \), we can finally obtain

\[
\int_0^{2\pi} \int_0^{2\pi} d\phi \, d\phi_0 \, \tilde{T}_w^{(f)}(\phi_\tau, \|\tau; \phi_0, \|0; 0) \left( \sin \phi_0 \right) = \frac{-i m}{\hbar^2 |X(\tau)|} \int \int d\xi_1 d\xi_2 \, J_0(b_\tau \sqrt{\|\tau}) \, J_1(b_0 \sqrt{\|0}) \left( A_0/b_0 \right) \tag{B2}
\]

with \( A_0 = \sqrt{2m/\omega_0} (\hbar |X_\tau|)^{-1} (Y_\tau \xi_2 + \xi_1) \), which is purely imaginary. Therefore, its real part \( C^{(f)}(\|\tau; \|0) \) becomes zero indeed.

Likewise, we can simplify Eq. (77b) into

\[
D^{(f)}(\|\tau; \|0) = \frac{m}{\hbar^2 |X_\tau|} \int \int d\xi_1 d\xi_2 \, J_0(b_\tau \sqrt{\|\tau}) \left\{ \frac{J_1(b_0 \sqrt{\|0})}{b_0 \sqrt{\|0}} - \frac{(A_0)^2 J_2(b_0 \sqrt{\|0})}{(b_0)^2} \right\}, \tag{B3}
\]

which is real-valued. Here we also used \( dJ_1(z)/dz = J_0(z) - J_1(z)/z \) and \( J_0(z) + J_2(z) = 2 J_1(z)/z \).

[1] C. Jarzynski, *Nonequilibrium Equality for Free Energy Differences*, Phys. Rev. Lett. 78, 2690 (1997).

[2] G. E. Crooks, *Nonequilibrium Measurements of Free Energy Differences for Microscopically Reversible Markovian Systems*, J. Stat. Phys. 90, 1481 (1998).

[3] P. Talkner, E. Lutz and P. Hänggi, *Fluctuation theorems: work is not an observable*, Phys. Rev. E 75, 050102 (2007).

[4] J. Kurchan, *A Quantum Fluctuation Theorem*, arXiv:cond-mat/0007360v2.

[5] H. Tasaki, *Jarzynski Relations for Quantum Systems and Some Applications*, arXiv:cond-mat/0009244v2.

[6] S. Mukamel, *Quantum Extension of the Jarzynski Relation: Analogy with Stochastic Dephasing*, Phys. Rev. Lett. 90, 170604 (2003).

[7] P. Talkner, P. Hänggi and M. Morillo, *Microcanonical quantum fluctuation theorems*, Phys. Rev. E 77, 051131 (2008).

[8] S. Deffner and E. Lutz, *Nonequilibrium work distribution of a quantum harmonic oscillator*, Phys. Rev. E 77, 021128 (2008).

[9] M. Esposito, U. Harbola and S. Mukamel, *Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems*, Rev. Mod. Phys. 81, 1665 (2009).
[10] M. Campisi, P. Hänggi and P. Talkner, *Colloquium: Quantum fluctuation relations: Foundations and applications*, Rev. Mod. Phys. **83**, 771 (2011); *Erratum*, Rev. Mod. Phys. **83**, 1653 (2011).

[11] P. Hänggi and P. Talkner, *The other QFT*, Nat. Phys. **11**, 108 (2015).

[12] C. Jarzynski, H. T. Quan and S. Rahav, *Quantum-Classical Correspondence Principle for Work Distributions*, Phys. Rev. X **5**, 031038 (2015).

[13] B. Segev, *Causality and propagation in the Wigner, Husimi, Glauber, and Kirkwood phase-space representations*, Phys. Rev. A **63**, 052114 (2001).

[14] S. Deffner, *Quantum entropy production in phase space*, EPL **103**, 30001 (2013).

[15] M. Brack and R. K. Bhaduri, *Semiclassical Physics* (Addison-Wesley, New York, 1997).

[16] W. Dittrich and M. Reuter *Classical and Quantum Dynamics From Classical Paths to Path Integrals*, 4th ed. (Springer, New York, 2016).

[17] A. Lahiri, G. Ghosh, T. K. Kar, *Action-angle variables in quantum mechanics*, Phys. Lett. A **238**, 239 (1998).

[18] W. H. Miller, *Classical-limit quantum mechanics and the theory of molecular collisions*, in *Advances in Chemical Physics*, Vol. 25, edited by I. Prigogine and S. A. Rice (John Wiley & Sons, New York, 1974).

[19] W. H. Miller and S. J. Cotton, *Communication: Wigner functions in action-angle variables, Bohr-Sommerfeld quantization, the Heisenberg correspondence principle, and a symmetrical quasi-classical approach to the full electronic density matrix*, J. Chem. Phys. **145**, 081102 (2016).

[20] A. E. Allahverdyan, *Nonequilibrium quantum fluctuations of work*, Phys. Rev. E **90**, 032137 (2014).

[21] H. J. D. Miller and J. Anders, *Time-reversal symmetric work distributions for closed quantum dynamics in the histories framework*, New J. Phys. **19**, 062001 (2017).

[22] E. P. Wigner, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev. **40**, 749 (1932).

[23] M. Hillery, R. F. O’Connell, M. O. Scully, and E. P. Wigner, *Distribution Functions in Physics: Fundamentals*, Phys. Rep. **106**, 121 (1984).

[24] H.-W. Lee, *Theory and Application of the Quantum Phase-Space Distribution Functions*, Phys. Rep. **259**, 147 (1995).
[25] V. Bužek and P. L. Knight, *Quantum Interference, Superposition States of Light, and Nonclassical Effects*, Prog. Opt. **34**, 1 (1995).

[26] W. P. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, Berlin, 2001).

[27] C. K. Zachos, D. B. Fairlie and T. L. Curtright, *Quantum Mechanics in Phase Space* (World Scientific, Singapore, 2005).

[28] B. Leaf, *Weyl Transform in Nonrelativistic Quantum Dynamics*, J. Math. Phys. **9**, 769 (1968).

[29] S. R. de Groot and L. G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972).

[30] T. B. Smith, *Semiclassical approximation in the Weyl picture by path summation*, J. Phys. A: Math. Gen. **11**, 2179 (1978).

[31] K. Husimi, *Miscellanea in Elementary Quantum Mechanics, II*, Prog. Theor. Phys. **9**, 381 (1953).

[32] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Academic Press, San Diego, 2007).

[33] D. J. Tannor, *Introduction to Quantum Mechanics: A Time-Dependent Perspective* (University Science Books, Sausalito/CA, 2007).

[34] M. Born and V. A. Fock, *Beweis des Adiabatensatzes*, Z. Phys. A **51**, 165 (1928).

[35] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
Fig. 1: (Color online) The (rescaled) dimensionless quantity $z_1 = \left\{ \Delta U(\tau) - \langle \mathbb{W}_{s(t)} \rangle \right\}/(12\hbar \omega_0)$ in (43) versus $(x = \omega_{\tau}/\omega_0, y = K_{\tau})$. The dimensionless inverse temperature $\beta \hbar \omega_0 = 8$ (the low-temperature regime); cf. Fig. 2.
Fig. 2: (Color online) The (rescaled) dimensionless quantity $z_2 = \{\Delta U(\tau) - \langle W \rangle_{\tau(0)}\}/(12\hbar\omega_0)$ versus $(x = \omega_\tau/\omega_0, y = K_\tau)$. The dimensionless inverse temperature $\beta \hbar \omega_0 = 4$ (the high-temperature regime). For comparison with Fig. 1 we see that $z_2$ is smaller than $z_1$. 
Fig. 3 (Color online) The ratio $y = \frac{Q_{\text{d}}(f)}{Q_{\text{q}}(f)}$ after Eq. (46) versus $x = K_{r}$. Let both dimensionless quantities $\tilde{\beta} = \beta \hbar \omega_0$ and $\omega = \omega_r / \omega_0$. From top to bottom: (high temperature $\tilde{\beta} = 5, \omega = 2$, black); (high temperature $\tilde{\beta} = 5, \omega = 3$, red); (low temperature $\tilde{\beta} = 7, \omega = 2$, blue); (low temperature $\tilde{\beta} = 7, \omega = 3$, brown). This result consists with the fact that if $K_{r}$ increases (i.e., more non-adiabatic), then $Q_{\text{d}}(f)$ increases. From $y < 1$, it is also noted that the quantum heat $Q_{\text{q}}(f)$ is large enough (as compared with the dissipative heat $Q_{\text{d}}(f)$) in such a thermally isolated system, particularly in the low-temperature regime. Therefore, the quantum heat should be treated separately without being neglected, as in our framework.
Fig. 4: (Color online) Solid: The relative variance $y_1 = \langle (\Delta \overline{W})^2 \rangle_{\nu(\theta)}$ in Eq. (47) versus $x = K_x$. Let both dimensionless quantities $\tilde{\beta} = \beta \hbar \omega_0$ and $\tilde{\omega} = \omega \tau / \omega_0$. From top to bottom (at $x = 1.01$): (high temperature $\tilde{\beta} = 0.2$, $\tilde{\omega} = 2$, brown); (high temperature $\tilde{\beta} = 0.2$, $\tilde{\omega} = 3$, blue); (low temperature $\tilde{\beta} = 2$, $\tilde{\omega} = 2$, red); (low temperature $\tilde{\beta} = 2$, $\tilde{\omega} = 3$, black). Dash: The relative variance $y_2 = \langle (\Delta \overline{W})^2 \rangle_{\nu(\theta)}/(\hbar \omega_0)^2$ in Eq. (34a). From top to bottom: The same as for $y_1$. We see that $y_1$ and $y_2$ are almost identical in the high-temperature regime ($\tilde{\beta} = 0.2$).
Fig. 5: (Color online) The dimensionless “dissipative heat” \( y = \frac{Q^{(f)}_{\tau, d}}{\hbar \omega_0} \) in Eq. (54) versus the dimensionless temperature \( x = (\beta \hbar \omega_0)^{-1} \). Let \( \omega_r/\omega_0 = 1.3 \). From top to bottom (solid): \( \Upsilon = P \), purple); \( \Upsilon = W \), brown); \( \Upsilon = Q \), blue). For comparison, \( y = 0 \) (dash, red). We see that the \( P \)-curve shows its minimum value \( y = 0.386 \) (dashdot, black) at \( x = 2.776 \), which is physically inconsistent, and the \( Q \)-curve can be negative valued. The \( y \)-values of all three curves increase with the temperature in the high-temperature regime. Therefore, it is the \( W \)-curve only that consists with the thermodynamics.
Fig. 6: (Color online) The dimensionless Wigner function $y = \pi \hbar \{ (1-\gamma) W_\beta (\mathbb{I}) + \gamma W_n (\mathbb{I}) \}$ with $n = 1$ after Eq. (60) versus the dimensionless action $x = \mathbb{I}/\hbar$. Let the dimensionless inverse temperature $\tilde{\beta} = \beta \hbar \omega_0$. Dash: From top to bottom at $x = 0.5$, ($\tilde{\beta} = 2.5$, black) with $\gamma = 0.460$, which is its threshold value $(\gamma_{th,\perp})_\beta$; ($\tilde{\beta} = 1$, red) with $\gamma = 0.316 = (\gamma_{th,\perp})_\beta$. Solid: From top to bottom at $x = 0.5$, ($\tilde{\beta} = 2.5$, blue) with $\gamma = 0.495 > (\gamma_{th,\perp})_\beta$ thus showing its negativity; ($\tilde{\beta} = 1$, purple) with $\gamma = 0.490 > (\gamma_{th,\perp})_\beta$ thus showing its negativity. For comparison, $y = 0$ (dashdot, brown).
Fig. 7: (Color online) A periodic external driving with $\omega_\tau = \omega_0$ and the partially thermal initial state with $\hat{\sigma}_1 = |1\rangle\langle 1|$. Let the dimensionless inverse temperature $\beta \hbar \omega_0 = 2$. The dimensionless internal energy difference $y_1 = \Delta U_1(\tau)/\hbar \omega_0$ in (63) versus $x = K_\tau$ (dash). From top to bottom: ($\gamma = 0.1$, black); ($\gamma = 0$, red). The dimensionless average work $y_2 = \langle W \rangle_{\hat{\sigma}_1}^{(f)} / \hbar \omega_0$ (solid). From top to bottom: ($\gamma = 0.1$, blue); ($\gamma = 0$, purple). The dimensionless free energy difference $y_3 = \Delta F_{\beta}^{(f)} / \hbar \omega_0 = 0$ (dashdot, brown). For the evaluation of the curve $y_2$, we used $\ln(1 + z) \approx z$ for $|z| \ll 1$ (i.e., $\gamma = 0.1 \ll 1$).
FIG. 8: (Color online) A periodic external driving with \( \omega_\tau = \omega_0 \) and the partially thermal initial state with \( \hat{\sigma}_3 = (|0\rangle + |1\rangle + |2\rangle)(\langle 0| + \langle 1| + \langle 2|)/3 \). Let the dimensionless inverse temperature \( \beta \hbar \omega_0 = 2 \). The dimensionless internal energy difference \( y_1 = \Delta U_3(\tau)/\hbar \omega_0 \) in Eq. (80a) versus \( x = K_\tau \) (dash). From top to bottom: \( (\gamma = 0.17, \text{red}) \) with coherence; \( (\gamma = 0.17, \text{brown}) \) without coherence, i.e., \( y_1 \to \Delta U_3(\tau)/\hbar \omega_0 \) in Eq. (80b); \( (\gamma = 0, \text{blue}) \). The dimensionless average work \( y_2 = \langle W \rangle_{\beta}^{(f)}/\hbar \omega_0 \) after Eq. (79) (solid). From top to bottom: \( (\gamma = 0.17, \text{green}) \) with coherence; \( (\gamma = 0.17, \text{black}) \) without coherence, i.e., \( y_2 \to \langle W \rangle_{\beta}^{(f)}/\hbar \omega_0; \) (dashdot, khaki). For the evaluation of the curve \( y_2 \), we used \( \ln(1 + z) \approx z \) for \( |z| \ll 1 \) (i.e., \( \gamma = 0.17 \ll 1 \)).