The structure of product bases of $\mathbb{C}^2 \otimes \mathbb{C}^n$

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Abstract. In this paper, we mainly characterize the structure of product bases of the complex vector space $\mathbb{C}^2 \otimes \mathbb{C}^n$. It gives an answer to the conjecture in case of $d = 2n$ proposed by McNulty et al in 2016. As the application of the result, we obtain all the product bases of a bipartite system $\mathbb{C}^2 \otimes \mathbb{C}^n$. It is helpful to review the structure of all the product bases of $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$, which given by McNulty et al.

1 Introduction and preliminaries

Einstein, Podolsky and Rosen (EPR) [1] first highlighted the important feature of quantum mechanics which we now call entanglement, that is, a quantum state $|\varphi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$ can not be represented in the form $|\varphi\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$, where $|\varphi_A\rangle \in \mathbb{H}_A$, $|\varphi_B\rangle \in \mathbb{H}_B$. Otherwise, a quantum state is unentangled in a bipartite system, we call it a separable state (or product state).

Definition 1.1. (see [2]) A vector $|v\rangle$ in the tensor product $\bigotimes_{i=1}^{m} \mathbb{C}^{d_i}$ is called a pure product vector if it is a vector of the form $|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_m\rangle$, where $|v_i\rangle \in \mathbb{C}^{d_i}$, $i = 1, 2, \ldots, m$.

Definition 1.2. (Definition 1 [3]) An orthonormal basis $B$ of a complex vector space $\mathbb{C}^d = \bigotimes_{i=1}^{m} \mathbb{C}^{d_i}$ with dimension $d = d_1d_2\cdots d_m$ is a product basis if each element in $B$ takes a pure product vector.

Two orthonormal bases $\{|a_i\rangle \mid i = 1, 2, \ldots, d\}$ and $\{|b_j\rangle \mid j = 1, 2, \ldots, d\}$ of a complex vector space $\mathbb{C}^d$ are mutually unbiased (MU) if $|\langle a_i | b_j \rangle|^2 = \frac{1}{d}$ for all $i, j \in \{1, 2, \ldots, d\}$. The study for MU bases is attractive in recent years since MU bases play important roles in quantum communications. It is known that

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the number of MU bases of the complex vector space $\mathbb{C}^d$ is less than or equal to $d + 1$ \cite{4}. In particular, the maximum number can be reached if $d = p^n$ \cite{5}, where $p$ is a prime number, $n \in \mathbb{N}_+$ and $\mathbb{N}_+$ denotes the set of positive integers. However, whether the bound can be reached for a composite number $d$ is still an open problem if $d \geq 6$. In particular, it is not known if there exist more than three MUBs in dimension 6. Besides, the three MUBs in dimension 6 are the forms of product bases. Based on this situation, many researchers began to study the MU product bases (MUPBs) \cite{3,6,7}. DiVincenzo and Terhal \cite{8} introduced the product bases. McNulty and Weigert \cite{6} discussed all the product bases in $d = 4, 6$. Clearly, a basis of subsystem $\mathbb{H}_A$ tensors a basis of subsystem $\mathbb{H}_B$ constitutes a basis of the bipartite system $\mathbb{H}_A \otimes \mathbb{H}_B$. Naturally, we want to know whether the product bases of a bipartite system can be grouped into the bases of two subsystems. McNulty et al \cite{3} proposed a conjecture about the structure of product bases of a bipartite system as follows.

**Conjecture 1.3.** The set $B = \{|a_i, b_i\} \mid i = 1, 2, \ldots, d = d_1d_2\}$ is an orthonormal product basis of the space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ if and only if the $d$ vectors $|a_i\rangle \in \mathbb{C}^{d_1}$ ($i = 1, 2, \ldots, d$) and the $d$ vectors $|b_i\rangle \in \mathbb{C}^{d_2}$ ($i = 1, 2, \ldots, d$) can be grouped into $d_2$ orthonormal bases of $\mathbb{C}^{d_1}$ and $d_1$ orthonormal bases of $\mathbb{C}^{d_2}$.

The sufficiency is not true by the following:

**Example 1.4.** Let $B = \{|a_1\rangle \otimes |b_1\rangle, |a_1^+\rangle \otimes |b_1^+\rangle, |a_2\rangle \otimes |b_2\rangle, |a_2^+\rangle \otimes |b_2^+\rangle\}$ be a subset of $\mathbb{C}^2 \otimes \mathbb{C}^2$, where $\text{rank}\{|x\rangle, |y\rangle\} = 2$ for all distinct $|x\rangle, |y\rangle \in \{|a_1\rangle, |a_2\rangle, |a_1^+\rangle, |a_2^+\rangle\}$ or $|x\rangle, |y\rangle \in \{|b_1\rangle, |b_2\rangle, |b_1^+\rangle, |b_2^+\rangle\}$. Obviously, the set $B$ satisfies the condition of Conjecture 1.3. However, $B$ is not an orthonormal product basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The remainder of this paper is organized as follows. In Section 2, we characterize the structure of product bases of $\mathbb{C}^2 \otimes \mathbb{C}^n$ and obtain the Main theorem. This shows that the first components and the second components of product bases of $\mathbb{C}^2 \otimes \mathbb{C}^n$ can be grouped into $n$ orthonormal bases of $\mathbb{C}^2$ and 2 orthonormal bases of $\mathbb{C}^n$ respectively, which answers the modified conjecture in case of $d = 2n$.

In Section 3, we review all the product bases in $d = 4$ and $d = 6$ as application of the Main theorem. At last, we investigate that all product bases of $\mathbb{C}^2 \otimes \mathbb{C}^n$ and obtain at least $p(n) + 1$ types in the last section. On the base of this paper, we will characterize the structure of product bases of $\mathbb{C}^m \otimes \mathbb{C}^n$ in sequel.
2 The structure of product bases of $\mathbb{C}^2 \otimes \mathbb{C}^n$

In this section, we show that the first components and the second components of product bases of the bipartite system $\mathbb{C}^2 \otimes \mathbb{C}^n$ can be grouped into $n$ orthonormal bases of $\mathbb{C}^2$ and 2 orthonormal bases of $\mathbb{C}^n$.

**Remark 2.1.** Clearly, the product basis $B = \{|a_i\otimes |b_i| \mid i = 1, 2, \ldots, d = mn\}$ of a complex vector space $\mathbb{C}^n \otimes \mathbb{C}^n$ satisfies the following conditions: $\langle a_i|a_j\rangle = 0$ or $\langle b_i|b_j\rangle = 0$ for any $i, j \in \{1, 2, \ldots, d\}$, $i \neq j$.

**Lemma 2.2.** Let $B = \{|a_i\otimes |b_i| \mid i = 1, 2, \ldots, 2n\}$ be a product basis of $\mathbb{C}^2 \otimes \mathbb{C}^n$. Then for each $|a_i|$, $i \in \{1, 2, \ldots, 2n\}$, there exists $|a_j|$ such that $\langle a_i|a_j\rangle = 0$.

**Proof.** Suppose on the contrary. Then $\langle b_i|b_j\rangle = 0$ for all $j$, $j \neq i$ and $j \in \{1, 2, \ldots, 2n\}$. Let $V$ and $W$ be the subspaces generated by $|b_i|$ and $\{|b_j| \mid j \neq i, j \in \{1, 2, \ldots, 2n\}\}$. Then $W = V^\perp$. Clearly, $|a_i\otimes |b_i| \in \mathbb{C}^2 \otimes V$ and $|a_j\otimes |b_j| \in \mathbb{C}^2 \otimes W$ for all $j$, $j \neq i$ and $j \in \{1, 2, \ldots, 2n\}$. This shows that $\{|a_j\otimes |b_j| \mid j \neq i, j \in \{1, 2, \ldots, 2n\}\}$ is linearly dependent, which is impossible.

Therefore, we obtain our result. \(\square\)

By this lemma, the product basis $B = \{|a_i\otimes |b_i| \mid i = 1, 2, \ldots, 2n\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^n$ can be rearranged in the following form:

\[
\{|a_i\otimes A(a_i), |a_i^+\otimes A(a_i^+) \mid i = 1, 2, \ldots, r\},
\]

where

\[
A(a_i) = \{|b_j| \mid |a_i\otimes |b_j| \in B, j = 1, 2, \ldots, 2n\},
\]

\[
A(a_i^+) = \{|b_j| \mid |a_i^+\otimes |b_j| \in B, j = 1, 2, \ldots, 2n\}.
\]

**Theorem 2.3.** Given a product basis $B = \{|a_i\otimes A(a_i), |a_i^+\otimes A(a_i^+) \mid i = 1, 2, \ldots, r\}$ as above. Then the cardinalities of $A(a_i)$ and $A(a_i^+)$ are equal, i.e., $|A(a_i)| = |A(a_i^+)| = m_i$, $i = 1, 2, \ldots, r$ and $\sum_{i=1}^{r} m_i = n$.

**Proof.** According to the orthogonality, we find that the elements of $A(a_i)$ are orthogonal to the elements of $A(a_j)$ and the elements of $A(a_i^+)$ for any $i, j = 1, 2, \ldots, r$ and $i \neq j$. Besides, the elements of $A(a_i)$ are mutually orthogonal and the elements of $A(a_i^+)$ are also mutually orthogonal. Let $|A(a_i)| = m_i$, $|A(a_i^+)| = n_i$ for any $i = 1, 2, \ldots, r$. So, we obtain that $\sum_{i=1}^{r} m_i \leq n$, $\sum_{i=1}^{r} n_i \leq n$, $n_j + \sum_{i \neq j} m_i \leq n$. [3]
and \( m_j + \sum_{i \neq j} n_i \leq n \) for every \( j = 1, 2, \ldots, r \). On the other hand, \( \sum_{i=1}^{r} m_i + \sum_{i=1}^{r} n_i = 2n \). Thus, \( \sum_{i=1}^{r} m_i = \sum_{i=1}^{r} n_i = n \). Let \( U_i \) and \( V_i \) be the subspaces generated by \( A(a_i) \) and \( A(a_i^+) \) respectively. Then \( \mathbb{C}^n = \bigoplus_{i=1}^{r} U_i = \bigoplus_{i=1}^{r} V_i \), where the symbol \( \bigoplus U_i \) means that \( U_i \) are mutually orthogonal. From the orthogonality, we obtain \( U_i \subseteq V_i \). Dually, \( V_i \subseteq U_i \). So, \( U_i = V_i \) for all \( i = 1, 2, \ldots, r \). Consequently, \( m_i = n_i \) for all \( i = 1, 2, \ldots, r \).

Conversely, let \( \mathbb{C}^n = \bigoplus_{i=1}^{r} V_i \), \( A_i \) and \( B_i \) are the orthonormal bases of \( V_i \), \( |a_i\rangle \) and \( |a_i^+\rangle \) are the states in \( \mathbb{C}^2 \), where \( |a_i^+\rangle \) is the unique state orthogonal to \( |a_i\rangle \), \( i = 1, 2, \ldots, r \). Then, it is easy to check that the set \( \{|a_i\rangle \otimes A_i; |a_i^+\rangle \otimes B_i \mid i = 1, 2, \ldots, r \} \) constitutes an orthonormal basis of \( \mathbb{C}^2 \otimes \mathbb{C}^n \). Combining Lemma 2.2 with Theorem 2.3, we have the following result.

**Theorem 2.4. (Main theorem)** The set \( B = \{|a_i\rangle \otimes |b_i\rangle \mid i = 1, 2, \ldots, d \} \) is an orthonormal product basis of the space \( \mathbb{C}^2 \otimes \mathbb{C}^n \) if and only if the \( d \) vectors \( |a_i\rangle \) \((i = 1, 2, \ldots, d)\) and the \( d \) vectors \( |b_i\rangle \) \((i = 1, 2, \ldots, d)\) can be grouped into \( n \) orthonormal bases \( B_i(2) \) \((i = 1, 2, \ldots, r)\) and \( 2n \) orthonormal bases \( B_i(n) \) \((i = 1, 2)\), respectively, where \( B_i(2) = \{|a_i\rangle, |a_i^+\rangle\} \), \( |B_i(2)| = m_i \) \((i = 1, 2, \ldots, r)\), \( B_1(n) = \bigcup_{i=1}^{r} A(a_i), B_2(n) = \bigcup_{i=1}^{r} A(a_i^+) \), \( A(a_i) \) and \( A(a_i^+) \) are the orthonormal bases of the subspace \( V_i \) and \( \mathbb{C}^n = \bigoplus_{i=1}^{r} V_i \).

**Definition 2.5.** Let \( \mathbb{C}^2 \otimes \mathbb{C}^n \) be a bipartite system. If \( \mathbb{C}^n = \bigoplus_{i=1}^{r} V_i \) and \( \text{dim} V_i = m_i \) for \( i = 1, 2, \ldots, r \), then \((m_1, m_2, \ldots, m_r)\) is said to be a right type of \( \mathbb{C}^2 \otimes \mathbb{C}^n \).

### 3 The product bases of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) and \( \mathbb{C}^2 \otimes \mathbb{C}^3 \)

McNulty and Weigert [6] discussed all the product bases of a complex vector space in dimension 4 and 6. However, it is difficult for us to go further in high dimension spaces by their’s method. Moreover, they also proposed the definition of local equivalent transformations (LETs). LETs are defined by the requirement that they preserve the product structure of all states. In this section, we review the product bases of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) and \( \mathbb{C}^2 \otimes \mathbb{C}^3 \) from the perspective of Main theorem.
3.1 The product bases of $\mathbb{C}^2 \otimes \mathbb{C}^2$

**Lemma 3.1. (Lemma 1 [B])** Any orthonormal product basis of the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is equivalent to a member of one of the families

$$I_0 = \{|j_z, k_z\rangle\},$$

$$I_1 = \{|0_z, k_z\rangle, |1_z, \hat{u}k_z\rangle\},$$

$$I_2 = \{|j_z, 0_z\rangle, |\hat{v}j_z, 1_z\rangle\},$$

where the operators $\hat{u}, \hat{v} \in SU(2)$ act on the space $\mathbb{C}^2$ such that the states $|0_z\rangle$ and $\hat{u}|0_z\rangle$, as well as the states $|0_z\rangle$ and $\hat{v}|0_z\rangle$, are skew.

McNulty and Weigert’s work are a bit not concise. Next, we apply the Main theorem to describe all the product bases in dimension 4. Any orthonormal product basis of the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ must be of the form $\{|a_i\rangle \otimes |b_i\rangle | i = 1, 2, 3, 4\}$. From the Theorem 2.4, we know that $\{|a_i\rangle | i = 1, 2, 3, 4\}$ can be grouped into 2 orthonormal bases of $\mathbb{C}^2$. Therefore, the product basis must be one of the following cases:

**Case 1.** $\{|a_1\rangle \otimes |b_1\rangle, |a_1^+\rangle \otimes |c_1\rangle, |a_2\rangle \otimes |b_2\rangle, |a_2^+\rangle \otimes |c_2\rangle\}$, where $|a_1\rangle \neq |a_2\rangle$ and $\langle a_1 | a_2 \rangle \neq 0$. In this case, from the orthogonality, we obtain that $|l_1\rangle = |l_2^+\rangle$ ($l = b, c$) and $|b_j\rangle = |c_j\rangle$ ($j = 1, 2$). Thus, the product basis is of the form

$$B_0 = \{|a_1\rangle \otimes |b_1\rangle, |a_1^+\rangle \otimes |b_1\rangle, |a_2\rangle \otimes |b_2\rangle, |a_2^+\rangle \otimes |b_2^+\rangle\}.$$

**Case 2.** $\{|a\rangle \otimes |b_1\rangle, |a^+\rangle \otimes |b_2\rangle\}$, if $\{|b_1\rangle, |b_2^+\rangle\}$ is different from $\{|b_2\rangle, |b_2^+\rangle\}$, we obtain that the product basis is of the form

$$B_1 = \{|a\rangle \otimes |b_1\rangle, |a\rangle \otimes |b_2^+\rangle, |a^+\rangle \otimes |b_2\rangle, |a^+\rangle \otimes |b_2^+\rangle\}.$$

Otherwise, we have the product basis is a direct product basis, which is

$$B_2 = \{|a\rangle \otimes |b\rangle, |a\rangle \otimes |b^+\rangle, |a^+\rangle \otimes |b\rangle, |a^+\rangle \otimes |b^+\rangle\}.$$

The right type is (1,1) in case 1 and the right type is (2,0) in case 2. Because of the same right type of $B_1$ and $B_2$, we can consider the left type similarly. The left type of $B_1$ is (1,1), but the left type of $B_2$ is (2,0). We also know that the
left type of $B_0$ is (2,0). The types of the construction given above are the same as McNulty and Weigert’s work.

From Corollary 1 of [3], we know that, up to local equivalence transformations, there exists a unique triple of MUPBs in dimension 4 as follows,

\[
B'_0 = \{ |j_z^1 \rangle \otimes |j_z^2 \rangle \};
\]
\[
B'_1 = \{ |j_z^1 \rangle \otimes |j_z^3 \rangle \};
\]
\[
B'_2 = \{ |j_y^1 \rangle \otimes |j_y^2 \rangle \};
\]

here $\{ |j_b^j \rangle : j = 0, 1 \}, b = z, x, y,$ are, for each $r = 1, 2,$ the eigenstates of the three Pauli operators in $\mathbb{C}^2$. The three MUPBs are corresponding to the product basis $B_2$.

### 3.2 The product bases of $\mathbb{C}^2 \otimes \mathbb{C}^3$

**Lemma 3.2. (Lemma 2 [6])** Any orthonormal product basis of the space $\mathbb{C}^2 \otimes \mathbb{C}^3$ is equivalent to a member of one of the families

\[
I_0 = \{ |j_z, J_z \rangle \},
\]
\[
I_1 = \{ |0_z, J_z \rangle, |1_z, \hat{U} J_z \rangle \},
\]
\[
I_2 = \{ |j_z, 0_z, |\hat{u}0_z, 1_z \rangle, |\hat{u}0_z, 2_z \rangle, |\hat{u}1_z, \hat{V} 1_z \rangle, |\hat{u}1_z, \hat{V} 2_z \rangle \},
\]
\[
I_3 = \{ |j_z, 0_z, |\hat{w}j_z, 1_z \rangle, |\hat{w}j_z, 2_z \rangle \},
\]

with $j = 0, 1$ and $J = 0, 1, 2$; the operators $\hat{u}, \hat{v}, \hat{w} \in SU(2)$ and $\hat{U}, \hat{V} \in SU(3)$ act on $\mathbb{C}^2$ and $\mathbb{C}^3$, respectively, with $\hat{V}$ leaving the state $|0_z \rangle$ invariant; the parameters of the operators $\hat{u}, \ldots, \hat{V}$ are chosen in such a way that no product basis occurs more than once.

We next describe all the product bases in dimension 6 by the Main theorem. The procedure and results are a slight different from the research by McNulty and Weigert. Consider an orthonormal product basis of the complex vector space $\mathbb{C}^2 \otimes \mathbb{C}^3$:

**Case 1. The right type is (1,1,1):** The form of the product basis must be $\{ |a_1 \rangle \otimes |b_1 \rangle, |a_2^+ \rangle \otimes |c_1 \rangle, |a_2 \rangle \otimes |b_2 \rangle, |a_2^+ \rangle \otimes |c_2 \rangle, |a_3 \rangle \otimes |b_3 \rangle, |a_3^+ \rangle \otimes |c_3 \rangle \}$, where
\( |a_i \neq |a_j \) and \( \langle a_i | a_j \rangle \neq 0 \) for any \( i, j \in \{1, 2, 3\} \) and \( i \neq j \). The orthogonality condition implies that both \{\( |b_1\), \( |b_2\), \( |b_3\)\} and \{\( |c_1\), \( |c_2\), \( |c_3\)\} are orthonormal bases of \( \mathbb{C}^3 \). Since for each \( i, i = 1, 2, 3 \) the subspaces generated by \( |b_i\) and \( |c_i\) are the same one, it follows that \( |b_i\) = \( |c_i\) \).

The form of this kind basis is as follows:

\[ B_0 = \{ |a_1 \rangle \otimes |b\rangle, |a_1^\perp \rangle \otimes |b\rangle, |a_2 \rangle \otimes |b^\perp\rangle, |a_2^\perp \rangle \otimes |b^\perp\rangle, |a_3 \rangle \otimes |b^{\perp\perp}\rangle, |a_3^\perp \rangle \otimes |b^{\perp\perp}\rangle \}. \]

**Case 2. The right type is (2,1):** The form of the product basis must be: 
\( \{ |a_1 \rangle \otimes \{ |b_1\rangle, |b_2\rangle \}, |a_1^\perp \rangle \otimes \{ |b_1\rangle, |b_2\rangle \}, |a_2 \rangle \otimes |b_3\rangle, |a_2^\perp \rangle \otimes |b_3\rangle \} \), where \( \{ |a_1\rangle, |a_1^\perp \rangle \} \) and \( \{ |a_2\rangle, |a_2^\perp \rangle \} \) are different orthonormal bases of a subspace \( V_1 \) in dimension 2, \( |b_3\rangle \) and \( |b_4\rangle \) generate the same subspace \( V_2 \) in dimension 1, \( V_1 \) and \( V_2 \) are orthogonal.

Thus, the product bases is of the following form (consults [6]):

\[ B_1 = \{ |a_1 \rangle \otimes |b\rangle, |a_1 \rangle \otimes |b^\perp\rangle, |a_1^\perp \rangle \otimes \hat{V}|b\rangle, |a_1^\perp \rangle \otimes \hat{V}|b^\perp\rangle, |a_2 \rangle \otimes |b^{\perp\perp}\rangle, |a_2^\perp \rangle \otimes |b^{\perp\perp}\rangle \}, \]

where \( \hat{V} \) is a unitary operator of \( V_1 \) defined by \( \hat{V}|b\rangle = \alpha|b\rangle + \beta|b^\perp\rangle, \hat{V}|b^\perp\rangle = \overline{\beta}|b\rangle - \overline{\alpha}|b^\perp\rangle, |\alpha|^2 + |\beta|^2 = 1. \)

**Case 3. The right type is (3,0):** In this case, the product basis is

\[ \{ |a \rangle \otimes \{ |b_1\rangle, |b_1^\perp\rangle, |b^{\perp\perp}\rangle \}, |a^\perp \rangle \otimes \{ |b_2\rangle, |b_2^\perp\rangle, |b^{\perp\perp}\rangle \} \}. \]

If \( \{ |b_1\rangle, |b_1^\perp\rangle, |b^{\perp\perp}\rangle \} \neq \{ |b_2\rangle, |b_2^\perp\rangle, |b^{\perp\perp}\rangle \} \), then the product basis is of the form

\[ B_2 = \{ |a \rangle \otimes \{ |b_1\rangle, |b_1^\perp\rangle, |b^{\perp\perp}\rangle \}, |a^\perp \rangle \otimes \{ |b_2\rangle, |b_2^\perp\rangle, |b^{\perp\perp}\rangle \} \}. \]

If it is not so, then the product basis is a direct product basis

\[ B_3 = \{ |a \rangle \otimes \{ |b\rangle, |b^\perp\rangle, |b^{\perp\perp}\rangle \}, |a^\perp \rangle \otimes \{ |b\rangle, |b^\perp\rangle, |b^{\perp\perp}\rangle \} \}. \]

If we look into the left type in case 2, then we find that there exist some difficulties. If \( \hat{V}|b\rangle = |b\rangle, \hat{V}|b^\perp\rangle = |b^\perp\rangle \) or \( \hat{V}|b\rangle = |b^\perp\rangle, \hat{V}|b^\perp\rangle = |b\rangle \), then the left type of \( B_1 \) is (2,0). If \( \hat{V}|b\rangle \) is not so, then the left type of \( \hat{V}|b\rangle \) is (1,1). However, the left type of \( |b^{\perp\perp}\rangle \) is (2,0). In case 3, the left type corresponding \( B_2 \) is not well-defined. Due to this fact, we only consider the right type of \( \mathbb{C}^2 \otimes \mathbb{C}^3 \).  

It is still unknown if there exist more than three MUBs of a complex vector space with dimension 6. A possible choice [7] of three MUBs is to take the products
\[ B_0' = I_6 = I_2 \otimes I_3, \quad B_1' = B_{11} \otimes B_{12}, \quad B_2' = B_{21} \otimes B_{22}, \] where \( I_2 \) and \( I_3 \) denote identity operators and

\[ B_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad B_{12} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \]

\[ B_{21} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad B_{22} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}, \]

with \( \omega = e^{\frac{2\pi i}{3}} \). The three MUBs are corresponding to the product basis \( B_3 \).

4 The product bases of \( \mathbb{C}^2 \otimes \mathbb{C}^n \)

The analysis in Section 2 indicates that the right type of \( \mathbb{C}^2 \otimes \mathbb{C}^n \) is corresponding to a partition of the positive integer \( n \). A partition of a positive integer \( n \) is a representation of writing \( n \) as a sum of several positive integers. That is, if \( n_1, n_2, \ldots, n_k \) are positive integers and \( n_1 \geq n_2 \geq \cdots \geq n_k \), then the representation \( n = n_1 + n_2 + \cdots + n_k \) is called a partition of \( n \) with \( k \) parts. If there have no other restrictions on \( n_i \) and \( k \), then we call it an unrestricted partition or partition simply. Let \( p(n) \) denote the number of unrestricted partitions of \( n \). Then we have the following result.

**Theorem 4.1.** A product basis of the complex vector space \( \mathbb{C}^2 \otimes \mathbb{C}^n \) is corresponding a partition of \( n \) and vice versa.

**Proof.** By the Main theorem, if \( \mathbb{C}^n = \bigoplus_{i=1}^r V_i \), \( A(a_i) \) and \( A(a_i^\perp) \) are the orthonormal bases of \( V_i \) for each \( i \), then \( \{ |a_i\rangle \otimes A(a_i), |a_i^\perp\rangle \otimes A(a_i^\perp) \mid i = 1, 2, \ldots, r \} \) is the product basis of \( \mathbb{C}^2 \otimes \mathbb{C}^n \), where \( |a_i^\perp\rangle \) is the unique state orthogonal to \( |a_i\rangle \) for any \( i = 1, 2, \ldots, r \). We obtain that an orthonormal product basis of the complex vector space \( \mathbb{C}^2 \otimes \mathbb{C}^n \) as follows:

\[ B = \{ |a_i\rangle \otimes A(a_i), |a_i^\perp\rangle \otimes A(a_i^\perp) \mid i = 1, 2, \ldots, r \} \]

where \( |A(a_i)| = |A(a_i^\perp)| = n_i \) and \( \sum_{i=1}^r n_i = n, 1 \leq r \leq n \). In other words, given a partition of \( n \), there exists a corresponding product basis. \( \square \)
Clearly, \( n_1 = n \) is a partition of \( n \). If we chose two orthogonal bases \( B_1 \) and \( B_2 \) of the complex vector space \( \mathbb{C}^n \), then \( B_1 = B_2 \) or \( B_1 \neq B_2 \). The first case yields a direct product basis of \( \mathbb{C}^2 \otimes \mathbb{C}^n \). So, we have

**Theorem 4.2.** There are at least two product bases of the complex vector space \( \mathbb{C}^2 \otimes \mathbb{C}^n \) whose right type is \((n,0)\).

Furthermore, we have

**Theorem 4.3.** Let \((m_1, m_2, \ldots, m_r)\) be a right type of \( \mathbb{C}^2 \otimes \mathbb{C}^n \) and \( \mathbb{C}^n = \bigoplus_{i=1}^{r} V_i \). If \( m_i \geq 1 \), then there exist two product bases \( A_i^{k_i} \) \((k_i = 1,2)\) of the complex vector space \( \mathbb{C}^2 \otimes V_i \) such that \( \{ A_i^{k_i} \mid i = 1, 2, \ldots, r \} \) forms a product basis of \( \mathbb{C}^2 \otimes \mathbb{C}^n \).

The analysis indicates that there is no product basis of \( \mathbb{C}^2 \otimes \mathbb{C}^n \) whose the left type is \((1,1)\) and the right type is \((1,1,\ldots,1)\). That is, if all of \( n \) bases are different, then

\[
B = \{ |a_i\rangle \otimes |A^{(i)}\rangle, |a_i^\perp\rangle \otimes |B^{(i)}\rangle \mid i = 1, 2, \ldots, n \},
\]

where \( |a_i^\perp\rangle \) is the unique state orthogonal to \( |a_i\rangle \) for \( i = 1, 2, \ldots, n \), both \( \{ |A^{(i)}\rangle \mid i = 1, 2, \ldots, n \} \) and \( \{ |B^{(i)}\rangle \mid i = 1, 2, \ldots, n \} \) are the orthonormal bases of \( \mathbb{C}^n \). From the orthogonal conditions, we obtain that \( \{ |A^{(i)}\rangle \mid i = 1, 2, \ldots, n \} = \{ |B^{(i)}\rangle \mid i = 1, 2, \ldots, n \} \). Therefore, if we consider the left type of \( B \), then the left type of \( B \) must be \((2,0)\). In other words, there must exist product bases of the form of the left type is \((2,0)\) and the right type is \((1,1,\ldots,1)\).

How many different product bases are there in \( \mathbb{C}^2 \otimes \mathbb{C}^n \)? A rough estimate answer is, there are at least \( p(n) + 1 \) types.

From Corollary 4 of [3], we know that any triple of MUPBs of the complex vector space \( \mathbb{C}^2 \otimes \mathbb{C}^n \) must have the following form

\[
B_0' = \{ |j_z\rangle \otimes |G(j_z)\rangle \},
\]

\[
B_1' = \{ |j_x\rangle \otimes |G(j_x)\rangle \},
\]

\[
B_2' = \{ |j_y\rangle \otimes |G(j_y)\rangle \},
\]

up to local equivalence transformations, here \( \{ |j_b\rangle \mid j = 0,1 \} \), \( b = z,x,y \), are the eigenstates of the three Pauli operators of \( \mathbb{C}^2 \), and \( G(j_b) \) are bases of \( \mathbb{C}^n \) for each
such that the three set \( \{G(j_b) \mid j = 0, 1\} \) are mutually unbiased. The three MUPBs are corresponding to the direct product basis.

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