On the Local Cohomology of Reflexive Modules of Rank One over Normal Semigroup Rings

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Abstract

In this work we describe the local cohomology of reflexive modules of rank one over normal semigroup rings with respect to monomial ideals. Using our description we show that the problem of classifying maximal Cohen-Macaulay modules of rank one can be rephrased in terms of finding integral solutions to certain sets of linear inequalities.

1 Introduction

The motivation for this paper stems from my earlier and ongoing work on equivariant sheaves over toric varieties (see [Per04a], [Per03] and [Per04b]). The main theme of this theory is the interplay between the combinatorics of toric geometry and non-combinatorial aspects from linear algebra. In a sense, this theory extends the combinatorial theory of toric varieties to a semi-combinatorial theory over toric varieties. It turns out that an important building block which we should understand are the reflexive sheaves of rank one. This paper has been written in order to clarify at least a few aspects of these sheaves.

In the case of affine toric varieties, these sheaves correspond to reflexive modules of rank one over a normal semigroup rings $k[\sigma_M]$, where $k$ is an algebraically closed field and $\sigma_M$ a normal subsemigroup of some lattice $M \cong \mathbb{Z}^d$. The main results of this work are related to the following problems:

(i) the computation of local cohomology modules of reflexive modules of rank one over $k[\sigma_M]$ with respect to monomial ideals,
(ii) the classification of maximal Cohen-Macaulay (MCM) modules of rank one over $k[\sigma_M]$.

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Recall that a normal semigroup is of the form

\[ \sigma_M = \{ m \in M \mid \langle m, n(\rho) \rangle \geq 0 \text{ for all } \rho \in \sigma(1) \}, \]

where the \( n(\rho) \) are linear forms in \( N = M^* \), the module dual to \( M \), and the brackets \( \langle , \rangle : M \times N \to \mathbb{Z} \) denote the canonical pairing. The \( n(\rho) \) over \( \mathbb{R}_{\geq 0} \) span a strictly convex polyhedral cone \( \sigma \) in the vector space \( N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \) (see also section 2). Denote \( T = \text{Hom}(M, k^*) \) the algebraic torus acting on the affine toric variety \( U_\sigma \). The \( T \)-invariant divisor class group is isomorphic to the free group \( \mathbb{Z}^{\sigma(1)} \). By a well-known correspondence, there is a one-to-one correspondence between \( T \)-invariant divisors \( D \in \mathbb{Z}^{\sigma(1)} \) and \( M \)-graded reflexive module of rank one over \( k[\sigma_M] \), denoted \( R^D \).

Local cohomology. Using this representation of \( R^D \), the main technical result of this paper will be a characterization in terms of simplicial cohomology of the local cohomology of \( R^D \) with respect to a monomial ideal. For any ring \( S \), an ideal \( B \) of \( S \) and any \( S \)-module \( F \), there is the functor

\[ \Gamma_B F := \lim_{\to n} \text{Hom}(S/B^n, F). \]

The local cohomology modules \( H^i_B F \) are defined as the right derived functors of \( \Gamma_B \) and have the following characterization:

\[ H^i_B F = \lim_{\to n} \text{Ext}^i(S/B^n, F). \]

In general, an explicit description of the modules \( H^i_B F \) is very difficult. However, in our case \( B \) is an ideal in \( k[\sigma_M] \) generated by monomials, and the modules \( H^i_B R^D \) are in a natural way \( M \)-graded and admit an explicit combinatorial description. For this, we construct an \( M \)-graded resolution of \( R^D \) (Proposition 5.1):

\[ \mathbf{D} : 0 \to R^D \to A_0 \to A_1 \to \cdots \to A_{|\sigma(1)|} \to 0, \]

where the \( A_i \) are \( \Gamma_B \)-acyclic \( k[\sigma_M] \)-modules. Applying \( \Gamma_B \) to \( \mathbf{D} \) then yields an isomorphism \( H^i_B R^D \cong H^i(\Gamma_B \mathbf{D}) \). To analyze this cohomology a bit deeper, we consider the support and cosupport of \( B \). The support of \( B \) is defined to be set of faces \( \tau \) of \( \sigma \) such
that the corresponding orbit orb(σ) is contained in the variety \( V(B) \). To introduce the cosupport, we denote \( Σ \) the simplex spanned by \( σ(1) \), i.e. the set underlying \( Σ \) coincides with \( σ(1) \), but we choose an order on this set, such that \( Σ \) becomes an oriented combinatorial simplex. Then the cosupport \( Ξ_B \) of \( B \) is the set of those \( Π \subset Σ \) such that the minimal face \( τ \) of \( σ \) where \( τ(1) \) contains \( Π \), is not contained in the support of \( B \). Then \( Ξ_B \) in a natural way can be considered as a simplicial subcomplex of \( Σ \). Now, for any \( m \in M \), let \( Σ_m := \{ ρ \in Σ \mid \langle m, n(ρ) \rangle < 0 \} \). Then we have for the \( m \)-th graded component of the complex \( Γ_B D \) (Corollary 5.3):

\[
(H_B^i R^D)_m = (H^i(Γ_B D)_m) = \tilde{H}^{i-2}(Ξ_B ∩ Σ_m; k);
\]

i.e. we identify the graded components \( (H_B^i R^D)_m \) with the \((i-2)\)-th reduced cohomology of the simplicial complex \( Ξ_B ∩ Σ_m \). Note that this kind of identification is not new but has been applied in the literature several times to study the local cohomology of (not necessarily normal) semigroup rings (see, for instance, [GW78], [TH86], [Mus00]). In fact, the constructions in this work are a quite straightforward adaption of the methods of Trung and Hoa [TH86]. The new aspect here is that we apply this technique to the study of more general \( k[σ_M] \)-modules. We remark that local cohomology of general \( M \)-graded modules has been studied before (see [HM04], [HM03]), but here, as will be explained below, we will arrive at a more explicit combinatorial picture for the case of the modules \( R^D \).

Maximal Cohen-Macaulay modules of rank one. By a classical theorem of Hochster, the rings \( k[σ_M] \) are Cohen-Macaulay and it is a natural problem to classify (maximal) Cohen-Macaulay modules over \( k[σ_M] \). Although there exists a huge amount of literature concerning the classification of MCM modules in various contexts, to my knowledge, there has not yet been done much work for the case of rings \( k[σ_M] \). The only references I am aware of and which are explicitly devoted to this topic, are [BG03] and [BG02]. One of the main results of [BG03] (Corollary 5.2) — from the perspective of this paper, at least — is that over a normal semigroup ring there exist only finitely many isomorphism classes of MCM modules of rank one.

However, despite of this finiteness result, it seems that a complete classification is very difficult, if not impossible, to achieve. The relevant combinatorics behind such a classification is given by the hyperplane arrangement defined by the real hyperplanes \( H_ρ = \{ m ∈ M ⊗_Z R \mid \langle m, n(ρ) \rangle = -n_ρ \} \) in \( M ⊗_Z R \). To this hyperplane arrangement one can associate its combinatorial type which is represented by the so-called matroid of flats; this structure essentially captures the information on intersections of the hyperplanes \( H_ρ \). It is a well-known fact that \( R^D \) is MCM if and only if \( H_{m_i} R^D = 0 \) for all \( 0 ≤ i < d \), where \( m \) is the maximal \( M \)-graded ideal in \( k[σ_M] \). So, by the results above, \( R^D \) is MCM if and only if for every subset \( Π ⊂ Σ \) such that \( \tilde{H}^{i-2}(Π ∩ Ξ_m, k) ≠ 0 \) for some \( i < d \), the
system of linear inequalities

\[ \langle m, n(\rho) \rangle < -n_\rho \quad \text{for } \rho \in \Pi \]
\[ \langle m, n(\rho) \rangle \geq -n_\rho \quad \text{for } \rho \in \Sigma \setminus \Pi \]

has no solution in \( M \). So, we can phrase the problem of classifying MCM modules of rank one over \( k[\sigma_M] \) as follows. Given the matroid of flats of a hyperplane arrangement \( \bigcup_{\rho \in \Delta(1)} H_\rho \), then, how many possibilities are there to realize (up to translation) combinatorially equivalent hyperplane arrangements such that a certain subset of the open cells of \( M_R \setminus \bigcup_{\rho \in \Delta(1)} H_\rho \) (and part of their boundaries, respectively) does not intersect the lattice \( M \)? A definitive general solution by now seems to be out of reach.

Overview of the paper. In section 2 we introduce some general facts and notation from toric geometry, which will be used throughout the paper. In section 3 we introduce the simplex over the set of rays of the fan \( \sigma \) and recall some elementary and basic facts on simplicial chain complexes. In section 4 we construct a class of \( k[\sigma_M] \)-modules which are acyclic with respect to local cohomology functors of monomial ideals. We use these modules to construct acyclic resolutions for the modules \( R^D \) in section 5. In that section we also characterize the graded components of local cohomology modules in terms of reduced cohomology of certain cell complexes. Section 6 presents some easy observations and examples concerning the chambers of \( M_R \setminus \bigcup_{\rho \in \Delta(1)} H_\rho \) and the vanishing and nonvanishing of local cohomology in certain degrees. Section 7 is a short insertion to give the clear statement about the conditions on \( R^D \) to be MCM. In section 8 we collect some more facts about the depth of the modules \( R^D \).

2 Toric preliminaries

General notions. We introduce some notation from the theory of toric varieties. For general overview on toric varieties we refer to [Oda88], [Ful93]. We will always assume that \( k \) is an algebraically closed field. \( \mathbb{U}_\sigma \) will denote an affine toric variety over \( k \) defined by a strongly convex rational polyhedral cone \( \sigma \) contained in the real vector space \( N_R \cong N \otimes_{\mathbb{Z}} \mathbb{R} \) over a lattice \( N \cong \mathbb{Z}^d \). We always assume that \( \dim \sigma = \text{rk}_\mathbb{Z} N \).

Let \( M \) be the lattice dual to \( N \) and let \( \langle , \rangle : M \times N \to \mathbb{Z} \) be the canonical pairing. This pairing extends in a natural way to the scalar extensions \( M_R := M \otimes_{\mathbb{Z}} \mathbb{R} \) and \( N_R \).

Elements of \( M \) are denoted by \( m, m' \), etc. if written additively, and by \( \chi(m) \), \( \chi(m') \), etc. if written multiplicatively, i.e. \( \chi(m + m') = \chi(m)\chi(m') \). The lattice \( M \) is identified with the group of characters of the torus \( T = \text{Hom}(M, k^*) \cong (k^*)^d \) acting on \( \mathbb{U}_\sigma \). For any cone \( \sigma \) we will use the following notation:

- faces of \( \sigma \) are denoted by small Greek letters such as \( \rho, \tau \), etc., the natural order among faces is denoted by \( \rho < \tau \);
for any set $F$ of faces of $\sigma$, we call the set 
\[ \text{star}(F) := \{ \eta \mid \tau \prec \eta \text{ for some } \tau \in F \} \]
the *star* of $F$; if $\text{star}(F) = F$, then $F$ is called *star closed*;

- $\sigma(1) := \{ \rho \prec \sigma \mid \dim \rho = 1 \}$, the faces of dimension 1, called *rays*; below we will often find it more convenient to denote this set $\Sigma$ (by abuse of notation);
- $n(\rho)$ denotes the primitive lattice element spanning the ray $\rho$;
- $D_\rho$ denotes the the irreducible $T$-invariant Weil divisor on $U_\sigma$ associated to $\rho \in \sigma(1)$;
- $\bar{\sigma} := \{ m \in \mathbb{M} \mid \langle m, n(\rho) \rangle \geq 0 \text{ for all } \rho \in \sigma(1) \}$ is the cone *dual* to $\sigma$;
- $\bar{\sigma}^\perp := \{ m \in \mathbb{M} \mid \langle m, n \rangle = 0 \text{ for all } n \in \sigma \}$;
- $\sigma_M := \bar{\sigma} \cap M$ is the subsemigroup of $M$ associated to $\sigma$;
- $\sigma_M^\perp := \bar{\sigma}^\perp \cap M$ is the unique maximal subgroup of $M$ contained in $\sigma_M$;
- the semigroup ring $k[\sigma_M] \cong \bigoplus_{m \in \sigma_M} k \cdot \chi(m)$ is identified with the coordinate ring of $U_\sigma$, and the group ring $k[M]$ is identified with the coordinate ring of $T$;
- let $\tau \prec \sigma$, then $\text{orb}(\tau)$ denotes the orbit associated to $\tau$ in $U_\sigma$;
- let $\tau \prec \sigma$, then $N_\tau$ is $N$ intersected with the subvector space of $N$ spanned by $\tau$ over $\mathbb{R}$, $M_\tau$ is the dual module of $N_\tau$, where there is a canonical identification $M_\tau = M^\perp \cap M_{\bar{\tau}}$; moreover, we there is the canonical splitting $M = M^\perp \times M_{\bar{\tau}}$;
- $T_\tau$ denotes the stabilizer subgroup of $T$ over $T_\tau$; $T^\tau := T/T_\tau$, note that $T^\tau \cong \text{orb}(\tau)$.

Recall that dualizing $\sigma$ via $\tau \mapsto \tau^\perp \cap \bar{\sigma}$ induces an order-reversing one-to-one correspondence between faces of $\sigma$ and faces of $\bar{\sigma}$.

**Reflexive modules of rank one.** There exists a short exact sequence of $\mathbb{Z}$-modules:

\[ 0 \rightarrow M \rightarrow \mathbb{Z}^{\sigma(1)} \rightarrow A_{d-1}(U_\sigma) \rightarrow 0, \]

where $A_{d-1}(U_\sigma)$ is the $(d-1)$-st Chow group of $U_\sigma$, i.e. the group of rational equivalence classes of Weil divisors on $U_\sigma$, and $\mathbb{Z}^{\sigma(1)}$ the group which is freely generated over the $T$-invariant irreducible Weil divisors of $U_\sigma$. It was observed by Reid [Rei80] that there is a one-to-one correspondence of classes $\alpha \in A_{n-1}(U_\sigma)$ and isomorphism classes of reflexive sheaves $O(D)$ of rank one over $U_\sigma$, where $D$ is some representative for $\alpha$. Every reflexive module of rank one over $k[\sigma_M]$ is isomorphic to $\Gamma(U_\sigma, O(D))$ for some Weil divisor $D$, and in fact, there is a one-to-one correspondence between rational equivalence classes of Weil divisors and reflexive $k[\sigma_M]$-modules of rank one. The above short exact sequence implies that every class $\alpha$ can be represented by a $T$-invariant Weil divisor $D = D_{\underline{n}}$, where $\underline{n} = (n_\rho) \in \mathbb{Z}^{\sigma(1)}$ and $D = \sum_{\rho \in \sigma(1)} n_\rho D_\rho$. By $T$-invariance, to $D$ there corresponds a reflexive $k[\sigma_M]$-module of rank one, denoted $R^D$, which has a natural $M$-graded structure together with a natural $M$-graded embedding $R^D \hookrightarrow k[M]$. Namely,
denote $M^D_\rho := \{ m \in M \mid \langle m, n(\rho) \rangle \geq -n_\rho \}$, which corresponds to the shifted half space $\rho_M + m$, where $m \in M$ such that $\langle m, n(\rho) \rangle = -n_\rho$. We define $M^D := \bigcap_{\rho \in \sigma(1)} M^D_\rho$, then $R^D := k[M^D]$ is a well-defined $M$-graded reflexive $k[\sigma_M]$-submodule of $k[M]$.

**Monomial ideals.** Let $I \subset \sigma_M$ be a semigroup ideal, then $B := \bigoplus_{m \in I} k \cdot \chi(m)$ is an ideal in $k[\sigma_M]$. On the other hand, for every $M$-graded ideal $B$ the set $I = \{ m \mid B_m \neq 0 \}$ forms a semigroup ideal in $\sigma_M$. We call the class of ideals coming from semigroup ideals the *monomial ideals*.

**Definition 2.1:** Let $B$ be a monomial ideal, then its support is defined as
$$\text{supp}(B) := \{ \tau \prec \sigma \mid I \cap \tau \perp \cap \tilde{\sigma} = \emptyset \}.$$ Note that $\text{supp}(B)$ is star-closed and the variety $V(B)$ coincides with $\bigcup_{\tau \in \text{supp}(B)} \text{orb} (\tau) \subset U_\sigma$. Moreover, in the particular case where $B$ is the unique maximal homogeneous ideal of $k[\sigma_M]$, we have $\text{supp}(B) = \{ \sigma \}$ and $V(B) = \text{orb}(\sigma)$.

### 3 The simplex spanned by $\sigma$

We denote $\Sigma$ the *simplex* spanned by $\sigma$, i.e. the set underlying $\Sigma$ coincides with $\sigma(1)$, but we choose a total ordering on the elements of $\Sigma$, i.e. $\Sigma = \{ \rho_1 < \rho_2 < \cdots < \rho_n \}$. Any subset $\Pi \subset \sigma(1)$ corresponds to a face of $\Sigma$ with orientation induced by the ordering of $\Sigma$. In what follows we will find it convenient to identify $\sigma(1)$ and $\Sigma$ as sets and to write $\Sigma$ instead of $\sigma(1)$.

For any $\Pi \subseteq \Sigma$ with $|\Pi| = r$, we consider the augmented cochain complex:
$$\Phi_{\Pi} : 0 \longrightarrow \bigoplus_{\rho \in \Pi} k \cdot \rho \longrightarrow \bigoplus_{\Gamma \subseteq \Pi, |\Gamma| = 2} k \cdot \Gamma \longrightarrow \bigoplus_{\Gamma \subseteq \Pi, |\Gamma| = 1} k \cdot \Gamma \longrightarrow 0,$$
which is an exact sequence of $\mathbb{Z}$-modules, as $\Pi$ is contractible. To avoid to mention the case $\Pi = \emptyset$ repeatedly as a special case in exactness arguments, we adopt the convention that for $\Pi = \emptyset$ the corresponding augmented cochain complex is $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \rightarrow 0$. For any two subsets $\Pi \subset \Upsilon \subset \Sigma$, the canonical projections
$$
\bigoplus_{\Gamma \subseteq \Upsilon, |\Gamma| = i} \mathbb{Z} \cdot \Gamma \longrightarrow \bigoplus_{\Gamma \subseteq \Pi, |\Gamma| = i} \mathbb{Z} \cdot \Gamma
$$
for every $i = 1, \ldots, |\Upsilon|$, induce a surjective chain map $\Phi_{\Upsilon} \rightarrow \Phi_{\Pi}$.

**Definition 3.1:** Let $\Pi \subset \Sigma$, then we denote $$\tau_{\Pi} := \min \{ \tau \prec \sigma \mid \Pi \subset \tau(1) \},$$ i.e. $\tau_{\Pi}$ is the minimal face of $\sigma$ such that $\Pi \subset \tau(1)$. If $\Pi = \emptyset$, then $\tau_{\Pi}$ is the zero cone.
Definition 3.2: Let $F$ be any set of faces of $\sigma$, then we set

$$\Xi_F := \{\Pi \subset \Sigma \mid \tau \notin \text{star}(F)\},$$

which we consider as a simplical subcomplex of $\Sigma$. If $F = \{\tau\}$ for some $\tau \prec \sigma$, then we write $\Xi_{\{\tau\}}$ instead of $\Xi_{\tau}$, and we denote $\Xi_{\sigma}$ simply by $\Xi$. If $B \subset k[\sigma_M]$ is a monomial ideal and $F = \text{supp}(B)$, then we also write $\Xi_B$ instead of $\Xi_{\text{supp}(B)}$. $\Xi_B$ is called the cosupport of $B$ in $\Sigma$.

Clearly, for every $\Upsilon \subset \Pi \subset \Sigma$, $\Pi \in \Xi_F$ implies $\Upsilon \in \Xi_F$, and $\Xi_F$ is a simplicial subcomplex of $\Sigma$.

Now let $\Pi \subset \Sigma$ be any subset which we consider as subsimplex and let $F$ any set of faces of $\sigma$, then we set $\Pi \cap \Xi_F := \{\Upsilon \mid \Upsilon \subset \Pi \text{ and } \Upsilon \in \Xi_F\}$, which is a subcomplex of both $\Pi$ and $\Xi_F$. Therefore there exists a subcomplex $\Phi_{\Pi,\Xi_F}$ of $\Phi_{\Pi}$ which is build of the terms

$$\Phi_{\Pi,\Xi_F}^i := \bigoplus_{\Gamma \subset \Pi \cap \Xi_F, |\Pi| = i} \mathbb{Z} \cdot \Gamma,$$

The relative reduced cohomology $\tilde{H}^i(\Pi, \Pi \cap \Xi_F; \mathbb{Z})$ then is the homology of the quotient complex $\Phi_{\Pi}/\Phi_{\Pi,\Xi_F}$. Note that because $\Pi$ is contractible, there are isomorphisms $\tilde{H}^i(\Pi, \Pi \cap \Xi_F; \mathbb{Z}) \cong \tilde{H}^{i-1}(\Pi \cap \Xi_F; \mathbb{Z})$ for every $i \in \mathbb{Z}$. Likewise, for our chosen field $k$, we obtain the relative reduced cohomology with coefficients in $k$ as the homology of $\Phi_{\Pi} \otimes \mathbb{Z} k/(\Phi_{\Pi,\Xi_F} \otimes \mathbb{Z} k)$ and $\tilde{H}^i(\Pi, \Pi \cap \Xi_F; k) \cong \tilde{H}^{i-1}(\Pi \cap \Xi_F; k)$ for every $i \in \mathbb{Z}$.

4 $\Gamma_B$-Acyclic Modules

Definition 4.1: Let $\Pi \subset \Sigma$ be a nonempty subset, then we set

$$H_D^\Pi := M \setminus \bigcup_{\rho \in \Pi} M_D^\rho$$

and define

$$k[H_D^\Pi] := \text{coker} \left( \bigoplus_{\rho \in \Pi} k[M_D^\rho] \rightarrow k[M] \right).$$

In the special case where $\Pi = \{\rho\}$, we write $H_D^\rho$ and $k[H_D^\rho]$, respectively.

Clearly, we have $H_D^\Pi = \bigcap_{\rho \in \Pi} H_D^\rho$ and $k[H_D^\Pi] = \bigoplus_{m \in H_D^\Pi} k \cdot \chi(m)$ a $k[\sigma_M]$-module. Let $\rho \in \Sigma$ and any $m \in H_D^\rho$, then for any subset $M' \subset \rho_M^\perp$ the set $m + M'$ is contained in $H_D^\rho$. This is in particular true when $M'$ is a subgroup of $\rho_M^\perp$. Thus the following lemma is immediate:

Lemma 4.2: Let $M' \subset M$ be any subgroup and $\Pi \subset \Sigma$. Then for any $m \in H_D^\Pi$ the set $m + M'$ is contained in $H_D^\Pi$ if and only if $M' \subset \bigcap_{\rho \in \Pi} \rho_M^\perp$. 


Proposition 4.3: Let $B \subset k[\sigma_M]$ be a monomial ideal, $D \in \mathbb{Z}^{\sigma(1)}$ and $\Pi \subseteq \Sigma$, then:

(i) $\Gamma_B k[H^D_\Pi] = \begin{cases} k[H^D_\Pi] & \text{if } \Pi \not\subseteq \Xi_B \\ 0 & \text{else.} \end{cases}$

(ii) The module $k[H^D_\Pi]$ is $\Gamma_B$-acyclic.

Proof. Let first $\Pi \subseteq \Xi_B$, such that $\tau_\Pi \notin \text{supp}(B)$. So there exists an integral element $m$ in the relative interior of $\tau_\Pi \cap \sigma$ such that the monomial $\chi(m)$ is contained in $B$. As the group $M' := \bigcap_{\rho \in \Pi} \sigma_M^\perp$ contains $\tau_\Pi^\perp \cap \sigma_M$ (note that $\tau_\Pi^\perp \cap \sigma_M = M' \cap \sigma_M$) and thus $m \in M'$, we have $m + M' = M'$. This implies that any power of $\chi(m)$ is a nonzero divisor of the module $k[H^D_\Pi]$, and moreover, multiplication by $\chi(m)$ even represents an automorphism of the module $k[H^D_\Pi]$. So, any power of $\chi(m)$ also acts as an automorphism on the local cohomology modules $H^i_B k[H^D_\Pi]$ for every $i \geq 0$. But because the (ring theoretic) support of $k[H^D_\Pi]$ is contained in the support of $B$, for every element $x \in H^i_B k[H^D_\Pi]$ there exists some $n > 0$ such that $\chi(m)^n x = 0$, hence $H^i_B k[H^D_\Pi] = 0$ for every $i \geq 0$. So for $\Pi \subseteq \Xi_B$, (i) and (ii) are true.

Now consider the case $\Pi \not\subseteq \Xi_B$, i.e $\tau_\Pi \in \text{supp}(B)$. In that case, $B$ contains no monomial whose degree is contained in $\tau_\Pi^\perp \cap \sigma_M$, and thus not in $M'$, so for every $x \in k[H^D_\Pi]$, there exists some $n > 0$ such that $B^n x = 0$. So, the support of $k[H^D_\Pi]$ is contained in the support of $B$, and thus $\Gamma_B k[H^D_\Pi] = k[H^D_\Pi]$ and $H^i_B k[H^D_\Pi] = 0$ for every $i > 0$. \hfill \Box

5 A $\Gamma_B$-Acyclic Resolution

Let $m \in M$ and define $\Sigma_m := \{\rho \in \Sigma \mid m \in H^D_\rho\}$. Then we can consider the exact sequence $\Phi_{\Sigma_m}$, respectively $\Phi_{\Sigma_m} \otimes k$. For every $-1 \leq i \leq |\Sigma_m|$, we can identify the vector space at the $i$-th position of the complex $\Phi_{\Sigma_m} \otimes k$ as follows:

$$\bigoplus_{\Pi \subseteq \Sigma_m \atop |\Pi| = i} k \cdot \Pi \cong \left( \bigoplus_{\Pi \subseteq \Sigma_m \atop |\Pi| = i} k[H^D_\Pi] \right)_m$$

With this identification for every $m \in M$, we obtain an exact sequence of vector spaces:

$$D^{-1} : 0 \rightarrow R^D \rightarrow k[M] \xrightarrow{\Phi_0} \bigoplus_{\rho \in \Sigma} k[H^D_\rho] \xrightarrow{\Phi_1} \bigoplus_{\Pi \subseteq \Sigma \atop |\Sigma| = 2} k[H^D_\Pi] \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_{n-1}} k[H^D_\Sigma] \rightarrow 0,$$

where the maps $\Phi_i$ are given by the direct sum of the cochain maps of the complexes $\Phi_{\Sigma_m}$ for every $m \in M$.

Proposition 5.1: For any monomial ideal $B \subset k[\sigma_M]$, the complex $D^{-1}$ is a $\Gamma_B$-acyclic resolution of $R^D$. 

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Proof. The exactness of $D$ already follows from its exactness as complex of $k$-vector spaces. The $\Gamma_B$-acyclicity of the $k[H^D_{\Pi}]$ has already been considered in the previous section, so it remains only to show that $D$ indeed is a complex of $k[\sigma_M]$-modules. For this, by $k$-linearity it suffices to show that $\Phi_i \circ \chi(m) = \chi(m) \circ \Phi_i$ for any $-1 \leq i \leq |\Sigma|$ and $m \in \sigma_M$, where $\chi(m)$ is considered as $k$-linear homomorphism. Consider any $m' \in M$, then we have inclusions $\Sigma_{m' + m} \subset \Sigma_{m'} \subset \Sigma$, and we observe that multiplication with $\chi(m)$ just results in the chain map $\Phi_{\Sigma_{m'}} \otimes Z \to \Phi_{\Sigma_{m'} + m} \otimes Z$, which yields the desired result.

We define $D_{\Xi_B}$ as the subcomplex of $D$ which is built of the terms

$$D^i_{\Xi_B} := \bigoplus_{\Gamma \in \Xi_B} k[H^D_{\Gamma}].$$

It is straightforward to see that degree-wise, for every $m \in M$, this complex coincides with the complex $\Phi_{\Sigma_m, \Xi_B} \otimes Z$.

**Proposition 5.2:** $H^B_R \frac{D}{D_{\Xi_B}} = H^i(D_{\Xi_B})$ for every $i \geq 0$.

**Proof.** Because $D$ is acyclic, we have that $H^B_R \frac{D}{D_{\Xi_B}} = H^i(\Gamma_B(D))$. By proposition 4.3, $\Gamma_B(k[H^D_{\Pi}]) = 0$ if $\Pi \subset \Xi_B$ and $\Gamma_B(k[H^D_{\Pi}]) = k[H^D_{\Pi}]$ if $\Pi \not\subset \Xi_B$, so the claim follows.

**Corollary 5.3:** For every $m \in M$ and every $i \geq 0$:

$$\left(H^B_R \frac{D}{D_{\Xi_B}}\right)_m = \tilde{H}^{i-2}(\Xi_B \cap \Sigma_m; k).$$

**Proof.** Degree-wise, for every $m \in M$, the complex $D_{\Xi_B}$ is the augmented complex of relative cohomology with coefficients in $k$ of the pair $(\Sigma_m, \Xi_B \cap \Sigma_m)$, shifted by 1. So, in every degree, we have an identification of the cohomology groups $(H^i \frac{D}{D_{\Xi_B}})_m = \tilde{H}^{i-1}(\Sigma_m, \Xi_B \cap \Sigma_m; k)$. Evaluating the long exact cohomology sequence, using that $\tilde{H}^i(\Sigma_m; k) = 0$ for all $i \in \mathbb{Z}$, we obtain that $\tilde{H}^{i-1}(\Sigma_m, \Xi_B \cap \Sigma_m; k) \cong \tilde{H}^{i-2}(\Xi_B \cap \Sigma_m; k)$ for every $i \in \mathbb{Z}$.

6 The Chambers in $M_{\mathbb{R}}$ Determined by $R^D$

In this section we assume the divisor $D = \sum_{\rho \in \Sigma} n\rho D_{\rho}$ to be fixed, except where explicitly stated otherwise. For understanding the local cohomology modules $H^B_R D$, it is important to know whether for some $\Pi \subset \Sigma$ a system of inequalities

$$\langle m, n(\rho) \rangle < -n_{\rho}, \text{ for } \rho \in \Pi$$

$$\langle m, n(\rho) \rangle \geq -n_{\rho}, \text{ for } \rho \in \Sigma \setminus \Pi$$


has integral solutions or not. For every \( \rho \in \Sigma \), the linear equation \( \langle m, n(\rho) \rangle = -n(\rho) \) defines a hyperplane \( H_{\rho} \) in \( M \), and the set of hyperplanes \( H_{\rho}, \rho \in \Sigma \) forms a hyperplane arrangement in \( M_\mathbb{R} \). The set of inequalities above determines a chamber \( C^s_{\Pi} \) in the complement of this hyperplane arrangement, i.e., if nonempty, the closure of the set of points \( m \) fulfilling these inequalities form a polyhedron bounded by the hyperplanes \( H_{\rho} \).

To be more precise, we define \( C^s_{\Pi} \) to be the set of points fulfilling the strict inequalities

\[
\langle m, n(\rho) \rangle < -n(\rho), \text{ for } \rho \in \Pi \\
\langle m, n(\rho) \rangle > -n(\rho), \text{ for } \rho \in \Sigma \setminus \Pi
\]

and \( C_{\Pi} \) analogously, but allowing equality in both types of equations. Note that 'ss' above stands for semi-strict inequalities. Moreover, note that we have for simpler notation omitted any reference to the divisor \( D \). The complement \( M_\mathbb{R} \setminus \bigcup_{\rho \in \Sigma} H_{\rho} \) then equals the set \( \bigcup_{\Pi \subset \Sigma} C^s_{\Pi} \). We have the following:

**Lemma 6.1:** The chambers \( C^s_{\Pi} \) and \( C_{\Pi} \) are in one-to-one correspondence.

**Proof.** We show that if some point \( m \) is contained in \( C^s_{\Pi} \), then there exists a point \( m' \) which is contained in \( C^s_{\Pi} \) and vice versa. Let first \( m \in C^s_{\Pi} \) for some \( \Pi \subset \Sigma \), then it is clearly contained in \( C^s_{\Pi} \). Now let \( m \in C^s_{\Pi} \) for some \( \Pi \subset \Sigma \). Let \( \Gamma \subset \Sigma \setminus \Pi \) given by precisely those \( \rho \) such that \( \langle m, n(\rho) \rangle = 0 \). Now, as the strict inequalities form an open condition, we can choose an \( \varepsilon \)-neighbourhood \( U_\varepsilon(m) \) in \( M_\mathbb{R} \) such that for all points \( x \in U_\varepsilon(m) \) the same strict inequalities hold as for \( m \). Now, the inequalities \( \langle m, n(\rho) \rangle \geq 0 \) for \( \rho \in \Gamma \) determine a convex, unbounded, polyhedron in \( M_\mathbb{R} \) which is not contained in a proper subspace of \( M_\mathbb{R} \), and \( m \) is located in its boundary. Thus \( U_\varepsilon(m) \) must intersect the interior of this polyhedron, and we can choose some \( m' \) from the intersection of \( U_\varepsilon \) and the interior of the polyhedron. Then it follows that \( m' \in C^s_{\Pi} \). \( \square \)

To better understand the chambers type \( C^s_{\Pi} \), we have also to consider the chambers of type \( C_{\Pi} \), which are closed polyhedra in \( M_\mathbb{R} \). Denote \( \sigma_{\Pi} \) the convex polyhedral cone generated over \( \mathbb{R}_{\geq 0} \) by the lattice vectors \(-n(\rho)\) for \( \rho \in \Pi \) and by \( n(\rho) \) for \( \rho \in \Sigma \setminus \Pi \). Then every \( C_{\Pi} \) can be written as Minkowski sum \( P_{\Pi} + \sigma_{\Pi} \), where \( P_{\Pi} \) is a compact polyhedron and \( \sigma_{\Pi} \) is the dual cone of \( \sigma_{\Pi} \). Our first observation is that the polyhedra \( C_{\Pi} \) do not have lineality spaces (see also [Zie95], §1.5):

**Lemma 6.2:** The cones \( \sigma_{\Pi} \) have dimension \( d \) in \( N_\mathbb{R} \).

**Proof.** Assume the vectors \(-n(\rho)\) for \( \rho \in \Pi \) and \( n(\rho) \) for \( \rho \in \Sigma \setminus \Pi \) span a proper subvector space of \( N_\mathbb{R} \), then also the vectors \( n(\rho) \), for \( \rho \in \Sigma \), span a proper subvector space, but this is not possible, since \( \dim \sigma = d \). \( \square \)

So, the cone \( \sigma_{\Pi} \) can be identified with the recession cone (sometimes also called the characteristic cone) of the polyhedron \( C_{\Pi} \). In general, \( \sigma_{\Pi} \) is not \( d \)-dimensional. \( \sigma_{\Pi} \)
being $d$-dimensional is equivalent to that $\sigma_\Pi$ does not contain a nonzero subvector space of $N_\mathbb{R}$. A general criterion for $\sigma_\Pi$ not containing a nonzero subvector space is obtained by checking the intersection of the two cones $\sigma_\Pi^1$ and $\sigma_\Pi^2$, where $\sigma_1$ is spanned over $\mathbb{R}_{\geq 0}$ by $n(\rho)$, $\rho \in \Pi$ and $\sigma_2$ is spanned over $\mathbb{R}_{\geq 0}$ by $n(\rho)$, $\rho \in \Sigma \setminus \Pi$.

**Lemma 6.3:** $\sigma_\Pi$ contains a nonzero subvector space if and only if $\sigma_\Pi^1 \cap \sigma_\Pi^2 \neq \{0\}$.

**Proof.** Assume first that $\sigma_\Pi$ contains a nonzero subvector space $V$ and let $n \in V$. Then we can write $n = \sum_{\rho \in \Pi} \alpha_\rho (n(\rho)) + \sum_{\rho \in \Sigma \setminus \Pi} \beta_\rho n(\rho)$, where $\alpha_\rho, \beta_\rho \geq 0$. Because also $-n \in V$, we have $-n = \sum_{\rho \in \Pi} \gamma_\rho (n(\rho)) + \sum_{\rho \in \Sigma \setminus \Pi} \delta_\rho n(\rho)$, for $\gamma_\rho, \delta_\rho \geq 0$. Summing up, we obtain $n - n = 0 = \sum_{\rho \in \Pi} (\alpha_\rho + \gamma_\rho) (n(\rho)) + \sum_{\rho \in \Sigma \setminus \Pi} (\beta_\rho + \delta_\rho) n(\rho)$ where not all of the $\alpha_\rho, \gamma_\rho$ and not all of the $\beta_\rho, \delta_\rho$ are zero, and thus the nonzero element $\sum_{\rho \in \Pi} (\alpha_\rho + \gamma_\rho) n(\rho) = \sum_{\rho \in \Sigma \setminus \Pi} (\beta_\rho + \delta_\rho) n(\rho)$ is contained in $\sigma_\Pi^1$ and $\sigma_\Pi^2$. In the other direction, let $0 \neq n \in \sigma_\Pi^1 \cap \sigma_\Pi^2$, i.e. $n = \sum_{\rho \in \Pi} \alpha_\rho n(\rho) = \sum_{\rho \in \Sigma \setminus \Pi} \beta_\rho n(\rho)$, where $\alpha_\rho, \beta_\rho \geq 0$ not all zero. Then $n \in \sigma_\Pi$ and $-n = \sum_{\rho \in \Pi} \alpha_\rho (-n(\rho)) \in \sigma_\Pi$ and thus $\sigma_\Pi$ contains the subvector space spanned by $n$. 

It would be nice if one could determine the dimension of the recession cone solely by the combinatorics involved with the $\Pi$s, but in general this seems not to be possible, as example 6.9 below will show. However, at least in the two extremal cases there are some tools available. For the case of vanishing recession cones, the corresponding chambers $C_\Pi^n$ are bounded, and these bounded chambers can at least be counted by means of the *matroid of flats* associated to the hyperplane arrangement $\bigcup_{\rho \in \Sigma} H_\rho$. For this, we refer to the book [BLS+93], chapter 4, in particular Corollary 4.6.8. The theory developed there also yields a formula for counting all chambers $C_\Pi^n$.

The other extremal case is that of the recession cone $\check{\sigma}_\Pi$ having dimension $d$. For these we can make use of the connection to local cohomology as developed in the previous sections.

**Proposition 6.4:** Let $\check{\sigma}_\Pi$ be $d$-dimensional. Then either $\Pi = \Sigma$ or $\Pi \cap \Xi$ is contractible.

**Proof.** First note that $\Pi = \Sigma$, then $\sigma_\Pi$ is just the negative cone of $\sigma$, so we may assume that $\Pi \neq \Sigma$. Let $\check{\sigma}_\Pi$ be $d$-dimensional and assume that $D = 0$, i.e. $R^D = k[\sigma_M]$. Then the corresponding hyperplane arrangement $\bigcup_{\rho \in \Sigma} H_\rho$ is a central arrangement and the interior of $\check{\sigma}_\Pi$ coincides with the chamber $C_\Pi^0$ in the complement of this arrangement, which therefore is nonempty. Moreover, $C_\Pi^0$ has nonempty intersection with $M$. Choosing some $m \in M \cap C_\Pi^0$, we can compute the local cohomology $H^i_{m} k[\sigma_M]$ in degree $m$, where $m$ is the maximal homogeneous ideal of $k[\sigma_M]$. A well-known result states that the ring $k[\sigma_M]$ is Cohen-Macaulay, and thus all these local cohomology modules for $0 \leq i < d$ vanish. This in particular implies by Corollary 5.3 that the reduced cohomology groups $\tilde{H}^{i-2}(\Pi \cap \Xi, k)$ vanish for $0 \leq i < d$. So, because $\Pi \neq \Sigma$, this implies that $\Pi \cap \Xi$ is a contractible topological space.
The case $d = 3$. We show that in the case $d = 3$, there are possible only two types of recession cones $\sigma_\Pi$, namely either $\sigma_\Pi$ is strictly convex, or $\sigma_\Pi = N_{\mathbb{R}}$. For $d = 3$, the cell complex $\Xi$ is a topological 1-dimensional sphere which can explicitly be realized as follows. Choose a hyperplane $H$ of $N_{\mathbb{R}}$ such that $\sigma \cap H =: P$ is a bounded polyhedron. Then the vertices of $P$ are given by $H \cap \rho$ and the facets of $P$ coincide with the facets of $\sigma$ intersected with $H$. The set $\Xi$ then is geometrically realized as the union of all 1- and 2-dimensional faces of $\sigma$ intersected with $H$, i.e., $\Xi$ has an explicit realization as the boundary of a convex polytope in the plane $H$. Moreover, $\Pi \neq \Sigma$ can be identified with a union of closed intervals in $\Xi$. It is straightforward to see that the two cones $\sigma^1_\Pi$ and $\sigma^2_\Pi$ intersect nontrivially if and only if the sets $P_1$, $P_2$ intersect, where $P_1$ and $P_2$ are convex hulls of the points $\rho \cap H$, $\rho \in \Pi$ and of the points $\rho \cap H$, $\rho \in \Sigma \setminus \Pi$, respectively.

We have the following

**Proposition 6.5:** The cones $\sigma^1_\Pi$ and $\sigma^2_\Pi$ intersect nontrivially if and only if $\Pi$ consists of more than one interval. In that case $\sigma_\Pi = N_{\mathbb{R}}$.

**Proof.** First note that, because $\Xi$ is a circle, $\Pi$ consists of as many intervals as $\Sigma \setminus \Pi$. It follows from elementary geometric considerations that in the case $\Pi$ consists of one interval, the polytopes $P_1$ and $P_2$ can not intersect, and in the case where $\Pi$ consists of more than one interval, one can choose vertices $p_1, p_2 \in P_1$ and $q_1, q_2 \in P_2$ such that the lines $l_1 = p_1 + r(p_2 - p_1), r \in \mathbb{R}$, $l_2 = q_1 + s(q_2 - q_1), s \in \mathbb{R}$, intersect in some point $a$ different from $p_1, p_2, q_1, q_2$. By arguments used before, this implies that $\sigma_\Pi$ contains the subvector space spanned by $a$, and because $a$ lies in the relative interior of the cone spanned by $q_1, q_2$, $\sigma_\Pi$ also contains the 2-dimensional vector space spanned by $q_1, q_2$ over $\mathbb{R}$. Moreover, the points $p_1, p_2$ are contained respectively on both, the positive and the negative side of this vector space, so $\sigma_\Pi = N_{\mathbb{R}}$. \hfill \Box

Now we can prove:

**Theorem 6.6:** In the case $d = 3$, for $\Pi \subset \Sigma$, there are the following possibilities:

(i) $\bar{H}^0(\Pi \cap \Xi, k) = 0$ and $C_\Pi$ has $d$-dimensional recession cone $\sigma_\Pi$,

(ii) $\bar{H}^0(\Pi \cap \Xi, k) \neq 0$ and $C^{ss}_\Pi$ is either bounded or empty.

**Proof.** We only observe that for $\Pi \neq \Sigma$, $\bar{H}^0(\Pi \cap \xi, k) = 0$ is equivalent to that $\Pi$ is contractible and apply propositions 6.4 and 6.5 \hfill \Box

We give the easiest example for the case $d = 3$:

**Example 6.7:** Let $\sigma$ be spanned over $\mathbb{R}_{\geq 0}$ by the primitive vectors $n_1 = (1, 0, 0)$, $n_2 = (0, 1, 0)$, $n_3 = (-1, 1, 1)$, $n_4 = (0, 0, 1)$ and we consider the divisor $D = -kD_2$ for some $k > 0$. For simplicity, we write here and in the examples below $n_i$ instead of $n(\rho_i)$
and $D_i$ instead of $D_p$. The hyperplane arrangement determined by $D$ realizes precisely 15 nonempty chambers out of 16 possible choices $\Pi \subset \Sigma$, with a unique bounded chamber for $\Pi = \{\rho_1, \rho_3\}$. We have $\dim \widetilde{H}^0(\Pi \cap \Xi, k) = 1$, and so the local cohomology module $H^2_{\mathbb{P}^3}R^D$ does not vanish if $C_{\Pi}^{ss} \cap M \neq \emptyset$. This is the case for every $k > 1$. For $k = 1$, the $C_{\Pi}^{ss} \cap M = \emptyset$, and thus $R^D$ is a Cohen-Macaulay module. We obtain another Cohen-Macaulay module by analogous considerations for $D = -D_1$, where this time the unique bound chamber is realized for $\Pi = \{\rho_2, \rho_3\}$. Altogether, in this example there are three isomorphism classes of maximal Cohen-Macaulay modules, represented by $k[\sigma_M]$ itself, $R^{-D_1}$, and $R^{-D_2}$.

We conclude that in the case $d = 3$, for computing local cohomology of the modules $R^D$ it suffices to check the disconnected subsets $\Pi \subset \Sigma$. In general, this may not be so simple, as we will see in the following examples. The first example shows that even for topologically nontrivial $\Pi$, in general, the chamber $C_{\Pi}^{ss}$ must not be bounded.

**Example 6.8:** Let $\sigma$ be spanned over $\mathbb{R}_{\geq 0}$ by the primitive vectors $n_1 = (1, 0, 0, 0), n_2 = (0, 1, 0, 0), n_3 = (-1, 1, 1, 0), n_4 = (0, 0, 1, 0), n_5 = (0, 0, 0, 1)$. This is the cone from the previous example extended by one ray in four-dimensional direction. We choose $D = -kD_2$ for some $k > 0$, and it is straightforward to see that, because $n_5$ is orthogonal to the other $n_i$, the number of chambers is double the number of chambers of the previous example. Moreover, it is easy to see that this time there are no bounded chambers, but still, for $\Pi := \{\rho_1, \rho_4\}$, we have that $\Pi \cap \Xi$ consists of two points, and hence $\dim \widetilde{H}^0(\Pi \cap \Xi, k) = 1$.

The next example shows that the contractibility of $\Pi$ does not imply that the recession cone $\sigma_{\Pi}$ is strictly convex. Moreover, the example shows that the strict convexity of $\sigma_{\Pi}$ cannot depend on the combinatorics of $\Pi$ in a simple way, but also depends on the concrete embedding of the cone $\sigma$ in $N_{\mathbb{R}}$.

**Example 6.9:** Consider the four-dimensional cone $\sigma$ spanned by $n_1 = (0, 0, 0, 1), n_2 = (1, 0, 0, 1), n_3 = (0, 1, 0, 1), n_4 = (0, 0, 1, 1), n_5 = (1, 1, 0, 1), n_6 = (1, 0, 1, 1), n_7 = (0, 1, 1, 1), n_8 = (1, 1, 1, 1)$, i.e. $\sigma$ is spanned over the three-dimensional unit cube shifted to the hyperplane $x_4 = 1$. Set $\Pi := \{\rho_1, \rho_3, \rho_4, \rho_6\}$. Then $\Pi \cap \Xi$ is contractible and we have $\widetilde{H}^i(\Pi \cap \Xi, k) = 0$ for all $i$. But, we have $n_3 + n_6 = n_2 + n_7$, so the cones $\sigma_{\Pi}^1$ and $\sigma_{\Pi}^2$ intersect, and the recession cone $\sigma_{\Pi}$ is of dimension smaller than 4.

Now consider the cone $\sigma'$ which is spanned by the same $n_1, \ldots, n_8$ as $\sigma$, except that $n_4$ and $n_6$ are replaced by $n'_4 = (0, -1, 1, 1)$ and $n'_6 = (1, -1, 1, 1)$. $\sigma'$ is combinatorially equivalent to $\sigma$, but by straightforward computation one finds that $(\sigma')^1$ and $(\sigma')^2$ do not intersect, and thus $\sigma_{\Pi}$ is a $d$-dimensional recession cone of $C_{\Pi}$ and thus $C_{\Pi}^{ss}$ is nonempty for every module $R^D$ over $k[\sigma'_{M}]$. 
7 Maximal Cohen-Macaulay Modules of Rank One

By the results of section 5, the problem of classifying maximal Cohen-Macaulay modules (MCMs) of rank one now has essentially become a problem of integer programming. To see this more clearly, let us reformulate the results for this case. For $R^D$ being an MCM is equivalent to that all local cohomology modules $H^i_m R^D$ vanish for $i < d$, where $m$ is the maximal homogeneous ideal of $k[\sigma_M]$. This in particular is equivalent to the vanishing of the cohomology groups $\tilde{H}^{i-2}(\Sigma_m \cap \Xi, k)$ for every $i < d$ and every $m \in M$. Now, as we have seen in example 6.7, not every $\Pi$ such that $C^{ss}_\Pi$ is nonempty in $M_\mathbb{R}$ equals $\Sigma_m$ for some $m \in M$, i.e. not every $C^{ss}_\Pi$ has nonempty intersection with $M$ although it is realized in the complement of the arrangement $\bigcup_{\rho \in \Sigma} H_\rho$. So let us state the MCM-condition for $R^D$ as a theorem:

**Theorem 7.1:** $R^D$ is an MCM if and only if for every $\Pi \subset \Sigma$ one of the following two conditions holds:

1. $\tilde{H}^{i-2}(\Pi \cap \Xi, k) = 0$ for all $i < d$,
2. $\tilde{H}^{i-2}(\Pi \cap \Xi, k) \neq 0$ for some $i < d$ and the chamber $C^{ss}_\Pi$ is either empty or has empty intersection with $M$.

We also state another, equivalent formulation:

**Theorem 7.2:** $R^D$ is an MCM if and only if for every $\Pi \subset \Sigma$ one of the following two conditions holds:

1. $\tilde{H}^{i-2}(\Pi \cap \Xi, k) = 0$ for all $i < d$,
2. $\tilde{H}^{i-2}(\Pi \cap \Xi, k) \neq 0$ for some $i < d$ and the system of inequalities

$$\langle m, n(\rho) \rangle < -n_\rho \text{ for } \rho \in \Pi$$
$$\langle m, n(\rho) \rangle \geq -n_\rho \text{ for } \rho \in \Sigma \setminus \Pi$$

has no integral solution.

One can relate the classification problem for MCMs of rank one to the problem of understanding hyperplane arrangements in $M_\mathbb{R}$ induced by the hyperplanes $H_\rho$, which are shifts of hyperplanes $\rho^\perp$ corresponding to some cone $\sigma \in N_\mathbb{R}$ (such that, in particular, the hyperplanes $H_\rho$ are rational). If one fixes the combinatorial type of the hyperplane arrangement $\bigcup_{\rho \in \Sigma} H_\rho$, say, its matroid of flats, then, in how many ways can this hyperplane arrangement be realized by shifting hyperplanes $H_\rho$, while keeping the combinatorial type, such that the cells $C^{ss}_\Pi$ with some nonvanishing cohomology group do not intersect $M$?
8 Singularity Sets

In order to actually proof that some module $R^D$ is an MCM, one effectively has to check the inequalities of theorem 7.2 for nearly all possible sets $\Pi \subset \Sigma$ which in general is a quite expensive task. In practice, however, it might be a better strategy to check that some given $R^D$ is not an MCM. In the rest of this paper we will collect some general results which can be helpful for this purpose.

We introduce the notion of singularity sets; for the general theory of singularity sets and their relation to local cohomology we refer to the book [ST71]. For a variety $X$ over some algebraically closed field $k$ and some coherent sheaf $\mathcal{F}$, the singularity sets of $\mathcal{F}$ are defined for integers $i \geq 0$ as

$$S_i(\mathcal{F}) := \{ x \in X \mid \text{depth}_{\mathcal{O}_{X,x}} \mathcal{F}_x \leq i \},$$

i.e. the set of points $x$ in $X$ such that the depth of the stalk $\mathcal{F}_x$ does not exceed $i$. The sets $S_i(\mathcal{F})$ are closed subsets of $X$ and every coherent sheaf defines a filtration of $X$ by closed subsets $\emptyset \subset S_0(\mathcal{F}) \subset \cdots \subset S_i(\mathcal{F}) \subset \cdots = X$. This filtration of course becomes stationary for $i \geq \dim X$ with $S_{\dim X}(\mathcal{F}) = X$. We are only interested in the situation where $X = U_\sigma$ is an affine toric variety and $\mathcal{F} = \mathcal{O}(D)$, i.e. the sheafification of the module $R^D$ over $U_\sigma$. Because $D$ is $T$-invariant, the depth of $\mathcal{O}(D)_x$ remains constant over every orbit $\text{orb}(\tau) \subset U_\sigma$. For $\tau' \prec \tau$, his restriction $\Gamma(U_\sigma, \mathcal{O}(D)) \rightarrow \Gamma(U_{\tau'}, \mathcal{O}(D))$ corresponds to the localization $R^D \rightarrow R^D_{X_{\chi(m_{\tau'})}}$, where $m_{\tau'}$ is a lattice element from the relative interior of the cone $\tau^\perp \cap \check{\sigma}$; in particular, $k[\tau_{M_{\tau}}] = k[\tau_{M_{\tau'}}]_{X_{\chi(m_{\tau'})}}$. Denote $\tau_{M_{\tau}} := \tau_M \cap M_{\tau}$, then the semigroup $\tau_M$ splits into a cartesian product $\tau_M = \tau_{M_{\tau}}^\perp \times \tau_{M_{\tau}}$. Correspondingly, the affine toric variety $U_{\tau}$ splits into the cartesian product $U_{\tau} \times U_{\tau'}$, where $U_{\tau'} = \text{spec} k[\tau_{M_{\tau'}}]$ is an affine toric variety of dimension $d - \text{codim} \tau$. The corresponding projection $p : U_{\tau} \rightarrow U_{\tau'}$ is a flat morphism. The following is a well-known fact on equivariant sheaves or $M$-graded modules, respectively, which we present without proof.

**Proposition 8.1:** Every $T$-equivariant coherent sheaf $\mathcal{E}$ over $U_\tau$ is isomorphic to $p^* \mathcal{E}'$ for some $T^{\sigma'}$-equivariant sheaf $\mathcal{E}'$ over $U_{\tau'}$. Equivalently, every finitely generated $M$-graded $k[\tau_M]$-module $E$ is isomorphic to $E' \otimes_{k[\tau_M]} k[\tau_M]$ for some $M_{\tau}$-graded $k[\tau_{M_{\tau}}]$-module $E'$.

In particular, $\mathcal{O}(D) \cong p^* \mathcal{O}(D')$, where $D' = \sum_{\rho \in \tau'(1)} n_{\rho} D_{\rho}$ is a $T^{\sigma'}$-invariant divisor on $U_{\tau'}$ and $\mathcal{O}(D')$ is the sheafification of the $M$-graded $k[\tau_{M_{\tau}}]$-module $R^{D'}$. We obtain:

**Lemma 8.2:** For every $x \in U_{\tau}$, we have $\text{depth}_{\mathcal{O}_{U_{\tau},x}} \mathcal{O}(D)_x = \text{depth}_{\mathcal{O}_{U_{\tau'},p(x)}} \mathcal{O}(D')_{p(x)} + \text{codim} \tau$.

**Proof.** As the morphism $p$ is flat and thus local, we can apply [Mat89], Thm. 23.3 and obtain

$$\text{depth}_{k[\tau_M],x} R^D_x = \text{depth}_{k[\tau_M],x} R^{D'}_{p(x)} \otimes_{k[\tau_{M_{\tau'}}],p(x)} k[\tau_M]_x = \text{depth}_{k[\tau_{M_{\tau'}},p(x)} R^{D'}_{p(x)} +$$

15
depth_{k[\tau_M]}(k[\tau_M]/p_\tau)_x for every point x ∈ U_\tau, where here p_\tau denotes the maximal homogeneous ideal of k[\tau_M].

With help of this lemma, we set:

**Definition 8.3:** Let i ≥ 0, then we set

\[ S_i := \{ \tau < \sigma \mid \text{depth}_{k[\sigma_M]} R^{D'}_x \leq i - \text{codim} \tau \text{ for some point } x \in \text{orb}(\tau) \}. \]

In this definition, we have omitted any explicit reference to D for clearer notation. Note that S_i is star-closed, and by the discussion above, S_i(\mathcal{O}(D)) is equal to \bigcup_{\tau \in S_i} \text{orb}(\tau) for all i ≥ 0. Now observe:

**Lemma 8.4:** R^{D'} is MCM if and only if S_i = ∅ for 0 ≤ i < d.

Denote Ξ^\tau := \{ \pi_\tau \prec \sigma \mid \tau_\pi \notin \text{star}(\sigma) \}, where we consider Ξ^\tau as a subcomplex of the simplex of \tau. The following lemma is immediate:

**Lemma 8.5:** Let \tau \prec \sigma and \Pi \subset \tau(1), then \Pi \cap \Xi^\tau = \Pi \cap \Xi^\tau.

For any subset \Pi \subset \tau(1), the splitting M \cong \tau_M^1 \times M_\tau is compatible with linear inequalities \langle m, n(\rho) \rangle < -n_\rho for \rho \in \Pi, respectively \langle m, n(\rho) \rangle ≥ -n_\rho for \rho \in \tau(1) \setminus \Pi, in the sense that some m ∈ M fulfills these inequalities if and only if every m' with m' − m ∈ \tau_M^1 fulfills these inequalities. The following is the main result of this section:

**Theorem 8.6:** \tau ∈ S_k if and only if \tilde{H}^{i−2}(\Pi \cap \Xi^\tau, k) \neq 0 for some i ≤ k − \text{codim} \tau and some subset \Pi \subset \tau(1) and there exists an integral solution to the system of inequalities

\[
\begin{align*}
\langle m, n(\rho) \rangle &< -n_\rho \text{ for } \rho \in \Pi \\
\langle m, n(\rho) \rangle &\geq -n_\rho \text{ for } \rho \in \tau(1) \setminus \Pi
\end{align*}
\]

in M or equivalently, in M_\tau.

**Proof.** We have \tau ∈ S_k if and only if \tilde{H}^i_{p_\tau} R^{D'} \neq 0 for some i ≤ k − \text{codim} \tau, where p_\tau is the maximal homogeneous ideal of k[\tau_M]. This in turn is equivalent to that there exists some m ∈ M_\tau and some i ≤ k − \text{codim} \tau such that \tilde{H}^i(\Sigma^\tau_m \cap \Xi^\tau, k) \neq 0, where \Sigma^\tau_m = \{ \rho \in \tau(1) \mid \langle m, n(\rho) \rangle < -n_\rho \}. Moreover, by lemma 8.5, \Sigma^\tau_m \cap \Xi^\tau = \Sigma^\tau_m \cap \Xi^\tau. \Box

The theorem can help to reduce the number of inequalities one has to check in order to determine the sets S_i. However, in the case d = 3, this is not of much help.

**Proposition 8.7 ([ST71], Corollary 1.21):** Let X be an irreducible variety of dimension d and let F be a coherent sheaf on X, then F is reflexive if and only if dim S_i(F) ≤ i − 2 for all i < d.

For d = 3 this implies that S_2 is either \{ σ \} or empty.
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