Statefinder parameters for interacting dark energy

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Abstract

We argue that the recently introduced “statefinder parameters” (Sahni et al., JETP Lett. 77, 201 (2003)), that include the third derivative of the cosmic scale factor, are useful tools to characterize interacting quintessence models. We specify the statefinder parameters for two classes of models that solve, or at least alleviate, the coincidence problem.
I. INTRODUCTION

In the last years the conviction that our Universe is undergoing a phase of accelerated expansion has gained further ground among cosmologists [1] albeit the nature of the cosmic substratum (usually called dark energy) behind this acceleration remains as elusive as ever [2]. While several candidates have been proposed [3] there is no agreement on which of them should be considered as the favored one. By all accounts, much more observational input is needed before this point might be settled.

The acceleration is evaluated by the deceleration parameter \( q = -\frac{\ddot{a}}{aH^2} \), where \( a(t) \) stands for the scale factor of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric and \( H = \frac{\dot{a}}{a} \) for the Hubble parameter. As mentioned above, the present state of the Universe seems to be characterized by a negative \( q \), but it is hard to determine its value observationally. Therefore, since many models predict acceleration some further information should be welcome.

Among others, there are cosmological models whose evolution is dominated by interacting components -say, dark matter and dark energy. Models in which the main energy components do not evolve separately but interact with each other bear a special interest since they may alleviate or even solve the “coincidence problem” that afflicts many approaches to late acceleration [4]. This problem can be summarily stated as “why now?”, that is to say: “Why the energy densities of the two main components happen to be of the same order today?” In this paper we focus on a Universe filled with two components, namely, non-relativistic matter (subscript \( m \)) with negligible pressure, i.e., \( p_m \ll \rho_m \) and dark energy (subscript \( x \)) -with equation of state \( p_x = w\rho_x \) where \( w < 0 \) -, such that the latter decays into the former component according to

\[
\begin{align*}
\dot{\rho}_m + 3H\rho_m &= Q, \\
\dot{\rho}_x + 3H(1+w)\rho_x &= -Q,
\end{align*}
\]

where \( Q \geq 0 \) measures the strength of the interaction. For later convenience we will write it as \( Q = -3\Pi H \) where the new quantity \( \Pi \) has the dimension of a pressure.

The Einstein field equations for spatially flat FLRW cosmologies are
\[ H^2 = \frac{8 \pi G}{3} \rho , \]
\[ \dot{H} = -\frac{8 \pi G}{2} (\rho + p_x) , \]

where \( \rho = \rho_m + \rho_x \) is the total energy density, and we have set \( c = 1 \). The quantity \( \dot{H} \) is related to the deceleration parameter \( q \) by \( q = -1 - (\dot{H}/H^2) = (1 + 3w\Omega_x)/2 \), where \( \Omega_x \equiv 8 \pi G \rho_x/(3H^2) \). It is obvious, that the deceleration parameter does not depend on whether or not both components are interacting. However, differentiating \( \dot{H} \) again, we obtain

\[ \frac{\ddot{H}}{H^3} = \frac{9}{2} \left( 1 + \frac{p_x}{\rho} \right) + \frac{9}{2} \left[ w(1 + w) \frac{\rho_x}{\rho} - w\frac{\Pi}{\rho} - \frac{\dot{w}}{3H} \frac{\rho_x}{\rho} \right] . \]

At variance with \( H \) and \( \dot{H} \), the second derivative \( \ddot{H} \) does depend on the interaction between the components. Consequently, to discriminate between models with different interactions or between interacting and non-interacting models it is desirable to characterize the cosmological dynamics additionally by parameters that depend on \( \ddot{H} \). Recently, Sahni et al. \[5\] and Alam et al. \[6\] have introduced a pair of new cosmological parameters (the so-called “statefinder parameters”) that seem to be promising candidates for this purpose. These are:

\[ r = \frac{\ddot{a}}{aH^3} , \quad s = \frac{r - 1}{3(q - \frac{1}{2})} . \]

In the present context of interacting fluids they take the form

\[ r = 1 + \frac{9}{2} \frac{w}{1 + \kappa} \left[ 1 + w - \frac{\Pi}{\rho_x} - \frac{\dot{w}}{3wH} \right] , \]

where \( \kappa \equiv \rho_m/\rho_x \), and

\[ s = 1 + w - \frac{\Pi}{\rho_x} - \frac{\dot{w}}{3wH} . \]

For non-interacting models i.e., for \( \Pi = 0 \), these parameters reduce to the expressions studied
The target of this short communication is to illustrate how the statefinder parameters may be of help when exploring cosmological models whose evolution is dominated by interacting components. While we focus ourselves on interacting cosmologies, we mention that the third derivative of the scale factor is generally necessary to characterize any variation in the overall equation of state of the cosmic medium. This becomes obvious from the general relation

\[ r - 1 = \frac{9}{2} \frac{\dot{\rho}}{\rho} , \]

where \( \rho \) is the total pressure of the cosmic medium which in our case reduces to \( P \approx \rho_x \).

Since

\[ \left( \frac{P}{\rho} \right)' = \frac{\dot{P}}{\rho} \left[ \frac{\dot{\rho}}{\rho} - \frac{P}{\rho} \right] , \]

it is evident, that an interaction term in \( \dot{P} \approx \dot{\rho}_x = \dot{w}_x \rho_x + w \dot{\rho}_x \) according to (1) will additionally change the time dependence of the overall equation of state parameter \( P/\rho \).

Interacting models allow a dynamical approach to the coincidence problem. The central quantity here is the density ratio \( \kappa \) introduced beneath Eq. (6). This parameter should be a constant of the order of unity at late times for the coincidence problem to be strictly solved. At least it should, however, vary slowly over a time of the order of \( H^{-1} \). The ratio \( \kappa \) is governed by the evolution equation (cf. Eqs. (1))

\[ \dot{\kappa} = -3H \left[ \left( \frac{\rho_x + \rho_m}{\rho_m \rho_x} \right) \Pi - w \right] \kappa . \]

Below we study the statefinder parameters for different solutions of this equation, corresponding to two broad classes of matter–quintessence interacting models.

II. SCALING SOLUTIONS

In a recent paper, the authors showed that scaling solutions, i.e., solutions of the form \( \rho_m/\rho_x \propto a^{-\xi} \), where \( \xi \) denotes a constant parameter in the range \([0, 3]\), can be obtained
when the dark energy component decays into the pressureless matter fluid -Eqs. (11). These solutions are interesting because they alleviate the coincidence problem [9]. Indeed, a model with $\xi = 3$ amounts to the $\Lambda$CDM model with $w = -1$ and $\Pi = 0$. For the opposite extreme value $\xi = 0$ the Universe dynamics admits a stable, stationary solution $\kappa = \text{constant}$, thereby no coincidence problem arises [10]. Hence, the deviation of the parameter $\xi$ from $\xi = 0$ quantifies the severity of the problem whereby any solution that differs from $\xi = -3w$ represents a testable, non-standard cosmological model and any solution with $\xi < 3$ renders the coincidence problem less acute. In that scheme, with $w = \text{constant}$, it can be shown that the interactions which produce scaling solutions are given by

$$\frac{\Pi}{\rho_x} = \left(w + \frac{\xi}{3}\right) \frac{\kappa_0(1 + z)^\xi}{1 + \kappa_0(1 + z)^\xi}, \quad (11)$$

where $\kappa_0$ denotes the present energy density ratio and $z = (a_0/a) - 1$ is the redshift. Inserting this expression in Eqs. (6) and (7) we get for the statefinder parameters

$$r = 1 + \frac{9}{2} \frac{w}{1 + \kappa_0(1 + z)^\xi} \left[1 + w - \left(w + \frac{\xi}{3}\right) \frac{\kappa_0(1 + z)^\xi}{1 + \kappa_0(1 + z)^\xi}\right], \quad (12)$$

and

$$s = 1 + w - \left(w + \frac{\xi}{3}\right) \frac{\kappa_0(1 + z)^\xi}{1 + \kappa_0(1 + z)^\xi}. \quad (13)$$

Figure 1 depicts the function $r(s)$ for different values of $\xi$. The lower the value of $\xi$, the lower the value of the corresponding curve in the $s$-$r$ plane and the less acute results the coincidence problem. Note that qualitatively these curves remain similar to those of non-interacting models (see Fig.1 of Ref. [3]). For the sake of comparison, we note that the $\Lambda$CDM model ($\Pi = 0, w = -1$) corresponds to the point (not shown in the figure) $s = 0, r = 1$.

From an observational point of view and for discriminating between different models, the current values $r_0$ and $s_0$ of the statefinder parameters are of particular relevance. For the scaling models we have

$$r_0 = 1 + \frac{9}{2} \frac{w}{1 + \kappa_0} s_0, \quad \text{and} \quad s_0 = 1 + w - \left(w + \frac{\xi}{3}\right) \frac{\kappa_0}{1 + \kappa_0}, \quad (14)$$
FIG. 1: Selected curves $r(s)$ in the redshift interval $[0, 6]$ (from left to right) with $w = -0.95$ and $\kappa_0 = 3/7$ for three different values of the parameter $\xi$, viz (a) 2.5; (b) 1.5; (c) 0.5.

Realizing that

$$q_0 = \frac{1 + \kappa_0 + 3w}{2(1 + \kappa_0)}, \quad (15)$$

and introducing

$$q_{0\Lambda} \equiv q_0 (w = -1) = \frac{12 - \kappa_0}{2(1 + \kappa_0)} \Leftrightarrow \frac{3\kappa_0}{2(1 + \kappa_0)} = 1 + q_{0\Lambda}, \quad (16)$$

we may classify the different models through their dependence $s_0(q_0)$ which reads

$$s_0 = \frac{2}{3} \left[ q_0 - q_{0\Lambda} + \left( \frac{\xi}{3} - 1 \right)(1 + q_{0\Lambda}) \right]. \quad (17)$$

The first part in the bracket on the right hand side describes the deviation from $w = -1$, the second part accounts for the deviations from the $\Lambda$CDM scaling $\xi = 3$. For models with $w = -1$, e.g., which all have the same deceleration parameter $q_0 = q_{0\Lambda}$, we have $s_0 = \frac{2}{3} \left( \frac{\xi}{3} - 1 \right)(1 + q_{0\Lambda})$. Of course, $\xi = 3$ corresponds to the $\Lambda$CDM model with $s_0 = 0$. Assuming $\kappa_0 = 3/7$, a scaling $\xi = 1$ results in $s_0 (\xi = 1) = -0.2$, while the stationary solution...
ξ = 0 has \( s_0(\xi = 0) = -0.3 \). Similar considerations hold for other values of \( w \). Thus, the parameter \( s_0 \) is able to discriminate between different scaling models, characterized by the same deceleration parameter.

**A. Luminosity distance**

It is interesting to see how the statefinder parameters enter the expression for the luminosity distance. Up to second order in the redshift the Hubble rate is

\[
H(z) = H_0 + \left( \frac{dH}{dz} \right)_{z=0} z + \frac{1}{2} \left( \frac{d^2H}{dz^2} \right)_{z=0} z^2 + \ldots.
\]

By virtue of

\[
\frac{dH}{dz} = \frac{q+1}{z+1} H \quad \text{and} \quad \frac{d^2H}{dz^2} = \frac{r - 1 + 2(1+q) - (1+q)^2}{(1+z)^2} H,
\]

this can be written as

\[
H(z) = H_0 \left\{ 1 + (q_0 + 1) z + \frac{1}{2} \left[ r_0 - 1 + 2(q_0 + 1) - (q_0 + 1)^2 \right] z^2 + \ldots \right\}.
\]

The luminosity distance

\[
d_L = (1+z) \int \frac{dz}{H},
\]

becomes (cf. \( D \))

\[
d_L = H_0^{-1} z \left[ 1 + \frac{1}{2} (1-q_0) z + \frac{1}{6} \left( 3(q_0 + 1)^2 - 5(q_0 + 1) + 1 - r_0 \right) z^2 + \ldots \right].
\]

Since the interaction affects \( r_0 \) but neither \( q_0 \) nor \( H_0 \), it is obvious that the luminosity distances of different interacting as well as of interacting and non-interacting models manifests itself only in third order in the redshift. For the scaling model leading to \( \xi = 1 \), and \( w = -1 \)
we have $q_0 = - (2 - \kappa_0) / [2 (1 + \kappa_0)]$ and $r_0 = 1 - [3 \kappa_0 / (1 + \kappa_0)^2]$, and we recover the previously obtained expression (see Eq. (38) of Ref. [8])

$$d_L \approx H_0^{-1} z \left[ 1 + \frac{1 + \kappa_0}{1 + \kappa_0} z - \frac{1 - \kappa_0}{8 (1 + \kappa_0)^2} (6 + \kappa_0) z^2 \right]. \tag{23}$$

For the $\Lambda$CDM model the factor $(6 + \kappa_0)$ occurring in last expression is replaced by $(10 + \kappa_0)$. On the other hand, for a given $w$ all the scaling models, including $\Lambda$CDM, are degenerate with respect to the deceleration parameter. For all these models the present value of $q$ is

$$q_0 = \frac{1}{2} + \frac{3}{2} \frac{w}{1 + \kappa_0}. \tag{24}$$

This expression also holds true for the asymptotically stable model of the next section. This demonstrates explicitly that discrimination between interacting and non-interacting models or between different interacting models requires the knowledge of the luminosity distance up to the third order in the redshift. In other words, the circumstance that the interaction is felt only by parameters containing the third derivative of the scale factor corresponds to the fact that the luminosity distance of interacting models is affected only in the third order of the redshift.

III. ASYMPTOTICALLY STABLE SOLUTIONS

For the special case that the interaction term is assumed to obey $\Pi = -c^2 \rho$ with $c^2$ = constant $< 1$, the evolution equation (10) has two stationary solutions for $w = \text{constant}$, namely, $\kappa_s^+$ and $\kappa_s^- = 1 / \kappa_s^+$ (with $\kappa_s^+ > 1$), given by

$$\kappa_s^\pm = - \left[ 1 + \frac{1}{2 c^2} \left( w \mp \sqrt{w(w + 4 c^2)} \right) \right], \tag{25}$$

-see Ref. [11] for details. It can be shown that whereas the solution $\kappa_s^-$ is stable the solution $\kappa_s^+$ is unstable. There exists a solution

$$\kappa = \kappa_s^- \frac{1 + y \kappa_s^+}{1 + y \kappa_s^-}, \tag{26}$$
with $y = (a_{eq}/a)^{\lambda}$ where $a_{eq}$ is the scale factor at which the energy density of dark energy equals the energy density of dark matter and $\lambda = -3w(1 - \kappa_{s}^{-})/(1 + \kappa_{s}^{+})$, according to which $\kappa$ evolves from a matter dominated phase ($\kappa_{s}^{+} > 1$) for $a \ll a_{eq}$ to a dark energy dominated phase ($\kappa_{s}^{-} < 1$) for $a \gg a_{eq}$ as the Universe expands. In this case we have

$$\frac{\Pi}{\rho_{x}} = -c^{2}(1 + \kappa_{s}^{-})\frac{1 + y_{0}(1 + z)^{\lambda}}{1 + y_{0}(1 + z)^{\lambda} \kappa_{s}^{-}} , \quad (27)$$

with $y_{0} = \kappa_{s}^{+}(\kappa_{0} - \kappa_{s}^{-})/(\kappa_{s}^{+} - \kappa_{0})$.

Accordingly, we readily obtain that the statefinder parameters of this model reduce to

$$r = 1 + \frac{9}{2}w \left[ \frac{1 + w}{1 + \kappa_{s}^{-}} \frac{1 + y_{0}(1 + z)^{\lambda} \kappa_{s}^{-}}{1 + y_{0}(1 + z)^{\lambda} \kappa_{s}^{-}} + c^{2} \right] , \quad (28)$$

and

$$s = 1 + w + c^{2}(1 + \kappa_{s}^{-})\frac{1 + y_{0}(1 + z)^{\lambda}}{1 + y_{0}(1 + z)^{\lambda} \kappa_{s}^{-}} , \quad (29)$$

where $c^{2} = -w\kappa_{s}^{-}/(1 + \kappa_{s})^{2}$ is valid.

Figure 2 shows some graphs of the function $r(s)$ for different choices of $\kappa_{s}^{-}$. At variance with graphs of the scaling model, the location of the curves in the plane $(s, r)$ is unrelated to the alleviation or solution of the coincidence problem since $\kappa = \kappa_{s}^{-} = \text{constant}$ at late times for all the cases of this model.

For the present value of the parameter $s$ this model yields

$$s_{0} = 1 + w + c^{2}(1 + \kappa_{0}) , \quad \text{where} \quad 1 + \kappa_{0} = (1 + \kappa_{s}^{-})\frac{1 + y_{0}}{1 + y_{0} \kappa_{s}^{-}} . \quad (30)$$

A parallel study to the case of the scaling solutions leads to

$$s_{0} = \frac{2}{3} \left[ q_{0} - q_{0}\Lambda + \frac{3}{2}c^{2} \right] (1 + \kappa_{0}) . \quad (31)$$

Again, the first two terms in the bracket account for the difference to models with $w = -1$. 


FIG. 2: Selected curves $r(s)$ in the redshift interval $[0, 6]$ (from left to right) with $w = -0.95$ and $\kappa_0 = 3/7$, for different values of $\kappa_s^-$, viz (a) 0.3; (b) 0.35; (c) 0.4.

for which $q_0 = q_0(w = -1) \equiv q_0\Lambda$. The $c^2$ term describes the impact of the interaction on the parameter $s_0$.

IV. CONCLUDING REMARKS

The statefinder parameters introduced in [5] and [6] are expected to be useful tools in testing interacting cosmologies that solve or at least alleviate the coincidence problem which besets many approaches to late acceleration. It is manifest that while the deceleration parameter does not feel the interaction between the dark energy and dark matter the statefinder pair $(r, s)$ does.

We hope that in some not distant future we will have at our disposal observational techniques capable of determining these parameters. These are bound to shed light on the nature of dark energy and dark matter.
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