A Jensen Inequality for a Family of Analytic Functions

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Abstract

We improve an estimate obtained in [Br] for the average number of limit cycles of a planar polynomial vector field situated in a neighbourhood of the origin provided that the field in a larger neighbourhood is close enough to a linear center. The result follows from a new distributional inequality for the number of zeros of a family of univariate holomorphic functions depending holomorphically on a parameter.

1. Introduction.

1.1. Let \( f := \{f_v : v \in B(s, R)\}, R > 1, \) be a family of holomorphic (in the open unit disk \( \mathbb{D}_1 \)) functions depending holomorphically on a parameter \( v \) varying in the open Euclidean ball \( B(s, R) \subset \mathbb{C}^s \). Hereafter \( B(s, R) \) and \( E(s, R) \) denote the open Euclidean balls in \( \mathbb{C}^s \) and \( \mathbb{R}^s \) centered at 0 of radius \( R \), and \( \mathbb{D}_t := B(1, t) \subset \mathbb{C} \). Further, set

\[
N_{f, \rho}(v) := \# \{ z \in \mathbb{D}_\rho ; \ f_v(z) = 0 \}; \quad (1.1)
\]

in addition, \( N_{f, \rho}(v) = +\infty \), if \( f_v = 0 \) identically on \( \mathbb{D}_1 \). In [Br] we proved

**Theorem.** Assume that the family \( f \) satisfies

\[
\sup_{v \in B(s, R)} \sup_{z \in \mathbb{D}_1} |f_v(z)| < M < \infty .
\]

Then for every \( T \geq 0 \), the inequality

\[
|\{ v \in E(s, 1) ; \ N_{f, \rho}(v) \geq T \}| \leq c_1 e^{-c_2 T / \log M} \cdot |E(s, 1)|
\]

holds with constants \( c_1, c_2 \) depending only on \( \rho, R \). Here \( |E| \) denotes the Lebesgue measure of \( E \subset \mathbb{R}^s \).

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We applied this result to estimate the average number of isolated closed trajectories (limit cycles) of a planar polynomial vector field situated in a neighbourhood of the origin provided that the field in a larger neighbourhood is close enough to a linear center. Central to this subject is the second part of Hilbert’s sixteenth problem asking whether the number of limit cycles of a planar polynomial vector field is always bounded in terms of its degree. This is one of the few Hilbert problems which remain unsolved (for the main results see e.g. [B], [E], [FY], [I] and references there in). According to Smale (see [S]) the global estimate should be polynomial in the degree \( d \) of the components of the vector field. The mean estimate obtained in [Br] is of the order \((\log d)^2\) which is substantially better than Smale’s conjecture.

**Theorem 1.1** ([Br, Th.A]) Consider the equation

\[
\begin{align*}
\dot{x} &= -y + F(x, y), \\
\dot{y} &= x + G(x, y), \\
F(x, y) &= \sum_{1 \leq i+k \leq d} a_{ki}x^k y^i, \\
G(x, y) &= \sum_{1 \leq i+k \leq d} b_{ki}x^k y^i,
\end{align*}
\]  

\( (1.2) \)

\[
\sum_{i,k} |a_{ki}|^2 + \sum_{i,k} |b_{ki}|^2 \leq N^2. \tag{1.3}
\]

Let \( s := d(d+3), d \geq 2, \) be the dimension of the space of real coefficients of equation \((1.2)\). Let \( v \) be a point in this space, and \( C(v) \) be the number of limit cycles of the corresponding equation \((1.2)\) in the disk \( D_{1/2} := \{(x, y) \in \mathbb{R}^2; |x|^2 + |y|^2 \leq 1/4\}\). Inequality \((1.3)\) implies that \( v \in E(s, N) \). Assume that

\[
N \leq \frac{1}{40 \pi \sqrt{d}}. \tag{1.4}
\]

Then there exist absolute positive constants \( C_1, C_2 \) such that for any \( T \geq 0 \)

\[
|\{ v \in E(s, N); \, C(v) \geq T \}| \leq C_1 s e^{-C_2 T/\log d} \cdot |E(s, N)|.
\]

As a corollary we obtain

**Corollary 1.2** Under assumptions of Theorem \([L.2]\) the average number of limit cycles of vector fields \((L.2)\) in the disk \( D_{1/2} \) is bounded from above by \( c(\log d)^2 \) with an absolute constant \( c > 0 \).

The main purpose of the present paper is to improve the above estimates.

Let \( F_0(x, y) := a_{10} x + a_{01} y \) and \( G_0(x, y) := b_{10} x + b_{01} y \) be the linear parts of \( F \) and \( G \). We identify \( \mathbb{R}^4 \) with the subspace of \( \mathbb{R}^s \) consisting of coefficients of the pair \( F_0, G_0 \). Let \( \pi : \mathbb{R}^s \to \mathbb{R}^4 \) be the natural projection in the space of coefficients and \( V \subset \mathbb{R}^s \) be a convex set such that

\[
\frac{|E(4, N)|}{|\pi(V)|} \leq \delta < \infty. \tag{1.5}
\]

In what follows \( |E| \) for \( E \subset V \) denotes the Lebesgue measure of \( E \) in \( A \), where \( A \subset \mathbb{R}^s \) is the real affine hull of \( V \).

Under the assumptions of Theorem \([L.1]\) and \([L.3]\) we prove
Theorem 1.3 There exist absolute positive constants $C_1, C_2, C_3$ such that for any $T \geq 0$
$$|\{v \in V; \ C(v) \geq T\}| \leq C_1 \delta d^{C_2} e^{-C_3 T} \cdot |V|$$

Corollary 1.4 There is an absolute positive constant $c > 0$ such that
$$\frac{1}{|V|} \int_V C(v) dv \leq c (\log \delta + \log d).$$

In particular, for $\delta = 1$ we obtain that $(\log d)^2$ in Corollary 1.2 can be replaced by
$\log d$. In view of Smale’s conjecture, the above inequalities are especially interesting
when $\delta$ is a polynomial function of $d$.

1.2. We deduce Theorem 1.3 from a new distributional inequality for families of
univariate holomorphic functions depending holomorphically on a parameter. To
formulate this result, we first recall some definitions from [Br1].

Let $O_R$ denote the set of holomorphic functions defined on $B(N, R) \subset C^N$. In
[Br1, Th.1.1] we proved the following statement.

Let $f \in O_R$, $R > 1$, and $I$ be a real interval situated in $B(N, 1)$. (Hereafter we
identify $C^N$ with $\mathbb{R}^{2N}$.) There is a constant $d = d(f, R) > 0$ such that for any $I$ and
any measurable subset $\omega \subset I$
$$\sup_I |f| \leq \left(\frac{4|I|}{|\omega|}\right)^{\frac{d}{d_f}} \sup_\omega |f|. \tag{1.6}$$

The optimal constant in (1.6) is called the Chebyshev degree of $f \in O_R$ in $B(N, 1)$
and is denoted by $d_f(R)$. For instance, according to the classical Remez inequality
$d_f(R)$ does not exceed the (total) degree of $f$, provided that $f$ is a polynomial.
Example 1.14 in [Br1] shows that even in this case it can be essentially smaller than
the degree. Note also that $d_f(R)$ can be estimated in terms of the valency of $f$ (see
[Br1, Prop.1.7]).

We are ready to formulate our second result.

Theorem 1.5 Let $f := \{f_v \in O_R, R > 1\}$ be a family of holomorphic in \(\mathbb{D}_1\)
functions depending holomorphically on $v$. Define $f_0(v) := f_v(0) \in O_R$. Let
$V \subset B(s, 1) \subset C^s(\mathbb{R}^{2s})$ be a convex set of real dimension $k$. Assume that
$$\sup_{v \in V} \sup_{z \in \mathbb{D}_1} |f_v(z)| \leq M_1 < \infty \quad \text{and} \quad \sup_{v \in V} |f_0(v)| \geq M_2 > 0.$$ 

Then for $\rho < 1$ and every $T \geq 0$,
$$|\{v \in V; \ N_{f, \rho}(v) \geq T\}| \leq 4k \left(\frac{M_1}{M_2}\right)^{1/d_{f_0}} \cdot e^{-\left(\log(1/\rho) / d_{f_0}(R)\right)T} \cdot |V|. \tag{3}$$

As a corollary we obtain

Corollary 1.6 Under the assumptions of Theorem 1.5,
$$\frac{1}{|V|} \int_V N_{f, \rho}(v) dv \leq \frac{d_{f_0}(R) \log(4e k) + \log(M_1/M_2)}{\log(1/\rho)}. \tag{4}$$
2. Proof of Theorem 1.5 and Corollary 1.6.

**Proof of Theorem 1.5.** We first recall the classical Jensen inequality (see e.g. [R,p.299]).

Let $f$ be a holomorphic function in $D$ continuous on $S$, $0 < r < 1$, such that $f(0) \neq 0$. Let $n_f(r)$ be the number of zeros of $f$ in $D_r, 0 < r < 1$, counted with their multiplicities, and $M_f := \sup_{S} |f|$. Then

$$n_f(r) \leq \frac{\log(M_f/|f(0)|)}{\log(1/r)}. \quad (2.1)$$

Let us denote $\omega(T) := \{v \in V : \mathcal{N}_{f,\rho}(v) \geq T\}$. Then exactly as in [Br, Prop.2.1] one can check that the function $\mathcal{N}_{f,\rho}$ is upper semicontinuous on $V \setminus S$, where $S$ is a certain closed subset of $V$ and $|S| = 0$. In particular, $\omega(T)$ is measurable.

Without loss of generality we will assume that there is an $x \in \omega(T)$ such that $\sup_{\omega(T)} |f_0| = |f_0(x)|$. Then from (2.1) and [Br1,Th.1.9] we have

$$T \leq \mathcal{N}_{f,\rho}(x) \leq \frac{\log(M_f/|f_0(x)|)}{\log(1/\rho)}$$

and

$$M_2 \leq \sup_{V} |f_0| \leq \left(\frac{4k|V|}{|\omega(T)|}\right)^{d_{f_0}(R)} \cdot \sup_{\omega(T)} |f_0|.$$

From these inequalities we obtain

$$T \leq \frac{1}{\log(1/\rho)} \cdot \log \left(\frac{M_1}{M_2} \cdot \left(\frac{4k|V|}{|\omega(T)|}\right)^{d_{f_0}(R)}\right).$$

The latter is equivalent to the required inequality. □

**Proof of Corollary 1.6.** Let

$$K := \frac{\log[(M_1/M_2) \cdot (4k)^{d_{f_0}(R)}]}{\log(1/\rho)}.$$

Then a well-known formula and the inequality of Theorem 1.5 imply

$$\frac{1}{|V|} \int_{V} \mathcal{N}_{f,\rho}(v) dv = \int_{0}^{\infty} \omega(T) dT \leq$$

$$\int_{0}^{\infty} \min \left\{ 1, 4k \left(\frac{M_1}{M_2}\right)^{1/d_{f_0}(R)} \cdot e^{-[\log(1/\rho)/d_{f_0}(R)]T} \right\} dT =$$

$$\int_{0}^{K} dT + 4k \left(\frac{M_1}{M_2}\right)^{1/d_{f_0}(R)} \int_{K}^{\infty} e^{-[\log(1/\rho)/d_{f_0}(R)]T} dT = K + d_{f_0}(R)/\log(1/\rho) =$$

$$\frac{d_{f_0}(R) \log(4ek) + \log(M_1/M_2)}{\log(1/\rho)}. \quad \square$$
3. Proof of Theorem 1.3 and Corollary 1.4.

For completeness of the proof we repeat some arguments presented in [Br].

Passing to polar coordinates in equation (1.2) we get
\[
\frac{dr}{d\phi} = \frac{P}{1 + Q} r
\]  
where \( P(r, \phi) := \frac{xF(x, y) + yG(x, y)}{r^2}, \ Q(r, \phi) := \frac{xG(x, y) - yF(x, y)}{r^2}, \ x = r \cos \phi, \ y = r \sin \phi. \)

Let us complexify \( r \): \( r \in \mathbb{D}_1 = \{ z \in \mathbb{C}; |z| < 1 \}. \) Consider equation (1.2) with complex coefficients that satisfy (1.3). In the domain \( U = \mathbb{D}_1 \times [0, 2\pi] \) we have
\[
\sup_U \left| \frac{F(x, y)}{r} \right| \leq \sqrt{\sum_{1 \leq i+k \leq d} |a_{ki}|^2 \sum_{1 \leq i+k \leq d} (\cos \phi)^{2k} (\sin \phi)^{2i} \leq \sqrt{\sum_{1 \leq i+k \leq d} |a_{ki}|^2} \cdot \sqrt{d}. \]

Similarly,
\[
\sup_U \left| \frac{G(x, y)}{r} \right| \leq \sqrt{\sum_{1 \leq i+k \leq d} |b_{ki}|^2} \cdot \sqrt{d}. \]

Hence, by (1.3),
\[
\sup_U \left| \frac{P}{1 + Q} \right| \leq \frac{N \sqrt{d}}{1 - N \sqrt{d}} =: \delta_N. \]

For the \( N \) of (1.4),
\[
\delta_{2N} < 3N \sqrt{d} \leq \frac{3}{40\pi}. \tag{3.2}
\]

**Proposition 3.1** Consider the equation
\[
\frac{dz}{d\phi} = H(z, \phi) \cdot z, \quad (z, \phi) \in U, \quad \sup_U |H| \leq \delta_{2N}, \tag{3.3}
\]
where \( \delta_{2N} \) satisfies (3.2). Then any solution \( z(\phi) \) of (3.3) with initial condition \( z(0) \in \mathbb{D}_{3/4} \) may be extended to \([0, 2\pi]\), and \(|z(\phi) - z(0)| \leq 8\pi N \sqrt{d} |z(0)|\).

**Proof.** We have \( \frac{d}{d\phi} (\log z) = H. \) Hence, while \(|z(\phi)| \leq 1, \phi \in [0, 2\pi]\), we have
\[
|z(\phi)| \leq e^{2\pi \delta_{2N}} |z(0)|.
\]

This follows from the Lagrange inequality. But \( e^{2\pi \delta_{2N}} < e^{\frac{3}{4}} < 4/3; \) hence, \(|z(\phi)| \leq \frac{4}{3} |z(0)| < 1. \) In particular, \((z(\phi), \phi) \in U\) and
\[
z(\phi) = z(0)e^{\int_0^\phi H(z(t), t)dt}.
\]

Thus we have
\[
|z(\phi) - z(0)| \leq |z(0)| \cdot |e^{\int_0^\phi H(z(t), t)} - 1| \leq |z(0)| e^{2\pi \delta_{2N}} 2\pi \delta_{2N} < 8\pi N \sqrt{d} |z(0)|.
\]
Now let $P_v$ be the Poincaré map corresponding to equation (3.1) obtained from a vector field $v$ of the type (1.2) with complex coefficients. Consider the function $g_v(z) = \frac{P_v(z)}{z} - 1$. According to the method of successive approximations to solutions of (3.1), $g_v(z)$ is holomorphic on $B(s, 1/(20\pi \sqrt{d})) \times \mathbb{D}_{3/4}$. The limit cycles of (1.2) located in $D_{1/2}$ correspond to certain zeros of $g_v(z)$ in $\mathbb{D}_{1/2}$. For $|z| < 3/4$ and $v \in B(s, 2N)$ we have $|g_v(z)| \leq 8\pi N \sqrt{d}$ by Proposition 3.1.

Let us consider linearization of (1.2)

$$
\begin{align*}
\dot{x} &= -y + F_l(x, y), \\
\dot{y} &= x + G_l(x, y).
\end{align*}
$$

Let $w := (a_{10}, a_{01}, b_{10}, b_{01}) \in \mathbb{C}^4$ be a point in the space of coefficients of (3.4). The Poincaré map for (3.4) can be calculated explicitly:

$$
P_w^l(z) := e^{f(w)} \cdot z;
$$

$$
f(w) := \int_0^{2\pi} \frac{a_{10} \cos^2 \phi + b_{01} \sin^2 \phi + (a_{01} + b_{10}) \sin \phi \cdot \cos \phi}{1 + b_{10} \cos^2 \phi - a_{01} \sin^2 \phi + (b_{01} - a_{10}) \sin \phi \cdot \cos \phi} \, d\phi.
$$

Also it is easy to see that

$$
g_0(v) := g_v(0) = e^{f(\pi(v))} - 1.
$$

Let $v_0$ be the vector field defined by

$$
\dot{x} = -y + \frac{N}{\sqrt{2}} x, \quad \dot{y} = x + \frac{N}{\sqrt{2}} y.
$$

By definition, $v_0 \in E(s, N)$ and

$$
g_0(v_0) = e^{\sqrt{2} \pi N} - 1 > \sqrt{2} \pi N.
$$

Further, $h(v) := g_v(v/(40\pi \sqrt{d})) \in \mathcal{O}_2$. Let $D := d_h(2)$ be the Chebyshev degree of $h$ in the ball $B(s, 1)$. Since $h$ is pullback by $\pi$ of a holomorphic function defined on $B(4, 2) \subset \mathbb{C}^4$, $D$ coincides with the Chebyshev degree of $h|_{B(4, 2)}$ in $B(4, 1)$ and therefore it does not depend on $s$.

Now assume that $V \subset E(4, N)$ satisfies (1.3). Then [Br1, Th.1.9] applied to $g_0|_{E(4, N)}$ implies that

$$
\sup_{\pi(V)} |g_0|_{E(4, N)} \geq (1/16\delta)^D \sup_{E(4, N)} |g_0|_{E(4, N)} \geq (1/16\delta)^D |g_0(v_0)| > \sqrt{2} \pi N (1/16\delta)^D.
$$

But $\sup_{\pi(V)} |g_0|_{E(4, N)} = \sup_V |g_0|$. Thus applying Theorem 1.5 to $g(v, z) := g_v(3z/4)$ with $M_1 = 8\pi N \sqrt{d}$ and $M_2 = \sqrt{2} \pi N (1/16\delta)^D$, and using the fact that $k \leq s < 3d^2$ (for $d \geq 2$), we have

$$
|\{v \in V; C(v) \geq T\}| \leq |\{v \in V; \mathcal{N}_{g,2/3}(v) \geq T\}| \leq C_1 \delta d^C_2 \cdot e^{-C_3T} \cdot |V|
$$

with $C_1 = 6 \cdot 32^{1+1/2D}$, $C_2 = 2 + 1/2D$ and $C_3 = \log(3/2)/D$. \hfill \Box
Remark 3.2 We used in the proof the fact that \( \{ v \in V; C(v) \geq T \} \) is a measurable set. Indeed, \( C(v) \) coincides with the number of nonnegative zeros of \( g_v(z) \). Then a slight modification of the proof of Proposition 2.1 in [Br] shows that outside of a set of measure 0 in \( V \), the function \( C(v) \) is the pointwise limit of a nonincreasing sequence of measurable functions. In particular, \( C(v) \) is measurable.

Proof of Corollary 1.4. The result follows directly from Corollary 1.3. \( \square \)

Remark 3.3 In general the inequality of [Br,Th.B] (see the beginning of this paper) is more powerful than that of Theorem 1.5. The methods presented in this paper and in [Br] can be applied for more general equations of type (1.2), where \( F, G \) are analytic functions of \( x \) and \( y \) depending analytically on a multidimensional parameter.

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