A $q$-Supercongruence from the $q$-Saalschütz theorem

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Abstract. In terms of the $q$-Saalschütz theorem and the Chinese remainder theorem for coprime polynomials, we derive a $q$-supercongruence modulo the third power of a cyclotomic polynomial. In particular, we give a $q$-analogue of a formula due to Long and Ramakrishna [Adv. Math. 290 (2016), 773–808].

Keywords: basic hypergeometric series; $q$-Saalschütz theorem; $q$-supercongruence

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1 Introduction

For any complex variable $x$, define the shifted-factorial to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x + 1) \cdots (x + n - 1) \quad \text{when} \quad n \in \mathbb{N}.$$ 

In 2006, Long and Ramakrishna [10, Proposition 25] proved that, for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \frac{(1/3)_k^3}{k!^3} \equiv \begin{cases} 
\Gamma_p(1/3)^6 \pmod{p^3}, & \text{if } p \equiv 1 \pmod{6}, \\
-p^2/3 \Gamma_p(1/3)^6 \pmod{p^3}, & \text{if } p \equiv 5 \pmod{6},
\end{cases}$$

(1.1)

where $\Gamma_p(x)$ is the $p$-adic Gamma function.

For any complex numbers $x$ and $q$, define the $q$-shifted factorial as

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}) \quad \text{when} \quad n \in \mathbb{N}.$$ 

For simplicity, we also adopt the compact notation

$$(x_1, x_2, \ldots, x_m; q)_n = (x_1; q)_n(x_2; q)_n \cdots (x_m; q)_n.$$ 

Following Gasper and Rahman [2], define the basic hypergeometric series $\phi_r$ by

$$\phi_r \left[ \begin{array}{c} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{array} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, b_2, \ldots, b_r; q)_k} z^k.$$ 

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Then the $q$-Saalschütz theorem (cf. [1, Appendix (II.12)]) can be stated as

\[
\begin{equation}
\left(\begin{array}{c}
q^{-n}, a, b \\
\phi_2 \cdot c, q^{1-n} \frac{ab}{c} ; q, q
\end{array}\right) = \frac{(c/a, c/b ; q)_n}{(c, c/ab ; q)_n}.
\end{equation}
\] (1.2)

Recently, Guo [2, Theorem 1.1] gave that: for any positive integer $n$ with $n \equiv -r \pmod{d}$ and $n \geq d - r$,\n
\[
\sum_{k=0}^{n-1} \frac{(q^r ; q^d)_k}{(q^d ; q^d)_k} q^{dk} \equiv 0 \pmod{\Phi_n(q)^2},
\] (1.3)

where $d \geq 2$ is an integer and $r \leq d - 2$ is an integer such that $\gcd(d, r) = 1$. Here and throughout the paper, $\Phi_n(q)$ stands for the $n$-th cyclotomic polynomial in $q$:

\[\Phi_n(q) = \prod_{1 \leq k \leq n \atop \gcd(k, n) = 1} (q - \zeta^k),\]

where $\zeta$ is an $n$-th primitive root of unity. For more $q$-analogues of supercongruences, we refer the reader to [3–9, 12, 15, 16].

Motivated by the work just mentioned, we shall establish the following Theorem.

**Theorem 1.1.** Let $d \geq 2$, $n$ be positive integers, $t \in \{1, d - 1\}$ and $n \equiv t \pmod{d}$. Then, modulo $\Phi_n(q)^3$,

\[
\sum_{k=0}^{(tn-1)/d} \frac{(q ; q^d)_k}{(q^d ; q^d)_k} q^{dk} \equiv q^{(tn-1)/d} \frac{(q^2 ; q^d-1 ; q^d)}{(q^3 ; q^d ; q^d)} q^{(tn-1)/d} (q^2 ; q^d ; q^d)_{(tn-1)/d}
\]

\[
+ \left[tn\right]^2 \frac{(q^2 ; q^d)^2}{(q ; q^3 ; q^d)_{(tn-1)/d}} \frac{(tn-1)/d}{[di - d + 2]^2}
\]

where $[n] = (1 - q^n)/(1 - q)$ and the denominators on both sides are coprime with $\Phi_n(q)$ when $n \equiv d - 1 \pmod{d}$.

We can deduce the following $q$-supercongruence from Theorem 1.1.

**Theorem 1.2.** Let $d \geq 2$, $n$ be positive integers, $t \in \{1, 2\}$ and $n \equiv t \pmod{3}$. Then, modulo $\Phi_n(q)^3$,

\[
\sum_{k=0}^{(tn-1)/3} \frac{(q ; q^3)_k}{(q^3 ; q^3)_k} q^{3k} \equiv q^{(tn-1)/3} \frac{(q^2 ; q^3)^2}{(q^3 ; q^3)_{(tn-1)/3}} q^{(tn-1)/3} \frac{(tn-1)/3}{[3i - 1]^2}
\]

\[
\times \left\{1 + (-1)^{tn-1/3} q^{tn(tn-1)/6} \left[tn\right]^2 \sum_{i=1}^{(tn-1)/3} \frac{q^3i - 1}{[3i - 1]^2}\right\}.
\]
It is not difficult to understand that Theorem 1.2 implies the following $q$-analog of (1.1).

**Corollary 1.3.** Let $n$ be a positive integer and $t \in \{1, 2\}$. Then, modulo $\Phi_n(q)^3$,
\[
\sum_{k=0}^{(tn-1)/3} \frac{(q; q^3)^3_k}{(q^3; q^3)^3_k} q^{3k} \equiv q^{(tn-1)/3} \frac{(q^2; q^3)^2_{(tn-1)/3}}{(q^3; q^3)^2_{(tn-1)/3}} \left\{ 1 + (-1)^{t-1} [tn]^2 \sum_{i=1}^{(tn-1)/3} \frac{q^{3i-1}}{[3i-1]^2} \right\},
\]
where $t = 1$ when $n \equiv 1 \pmod{6}$ and $t = 2$ when $n \equiv 5 \pmod{6}$.

Letting $n = p$ be a prime and taking $q \to 1$ in this theorem, we obtain the conclusion.

**Corollary 1.4.** Let $p$ be a prime and $t \in \{1, 2\}$. Then
\[
\sum_{k=0}^{(tp-1)/2} \frac{(1/3)^3_k}{k!^3} \equiv \frac{(2/3)^2_{(tp-1)/3}}{(1)^2_{(tp-1)/3}} \left\{ 1 + (-1)^t (tp)^2 \sum_{i=1}^{(tp-1)/3} \frac{1}{(3i-1)^2} \right\} \pmod{p^3},
\]
where $t = 1$ when $p \equiv 1 \pmod{6}$ and $t = 2$ when $p \equiv 5 \pmod{6}$.

For the sake of explaining the equivalence of (1.1) and (1.4), we need to verify the following relations.

**Proposition 1.5.** Let $p > 3$ be a prime. Then
\[
\frac{(2/3)^2_{(p-1)/3}}{(1)^2_{(p-1)/3}} \left\{ 1 + p^2 \sum_{i=1}^{(p-1)/3} \frac{1}{(3i-1)^2} \right\} \equiv \Gamma_p(1/3)^6 \pmod{p^3},
\]
where $p \equiv 1 \pmod{6}$,
\[
\frac{(2/3)^2_{(2p-1)/3}}{(1)^2_{(2p-1)/3}} \left\{ 1 - 4p^2 \sum_{i=1}^{(2p-1)/3} \frac{1}{(3i-1)^2} \right\} \equiv -\frac{p^2}{3} \Gamma_p(1/3)^6 \pmod{p^3},
\]
where $p \equiv 5 \pmod{6}$.

The rest of the paper is arranged as follows. By means of (1.2) and the Chinese remainder theorem for coprime polynomials, a $q$-supercongruence modulo $(1 - aq^{tn})(a - q^{tn})(b - q^{tn})$ will be derived in Section 2. Then it is used to provide the proofs of Theorems 1.1 and 1.2 in the same section. Finally, the proof of Proposition 1.5 will be displayed in Section 3.

## 2 Proofs of Theorems 1.1 and 1.2

In order to prove Theorem 1.1, we require the following parameter extension of it.
Theorem 2.1. Let $d \geq 2$, $n$ be positive integers, $t \in \{1, d-1\}$ and $n \equiv t \pmod{d}$. Then, modulo $(1-aq^tn)(a-q^tn)(b-q^tn)$,

$$\sum_{k=0}^{(tn-1)/d} \frac{(aq, q/a, q/b; q^d)_k}{(q^d, c, q^{d+3}/bc; q^d)_k} q^{dk} \equiv \left(1-aq^tn\right)(a-q^tn) \frac{(c/qa, ac/q; q^d)_{(tn-1)/d}}{(a-b)(1-ab)} \frac{(c, c/q^2; q^d)_{(tn-1)/d}}{(c, c/q^2; q^d)_{(tn-1)/d}} \frac{(b-q^tn)(a^2-1+aq^tn)}{(a-b)(1-ab)} \frac{(q/b)_{(tn-1)/d} (bc/q, q^d+2/c; q^d)_{(tn-1)/d}}{(c, q^{d+3}/bc; q^d)_{(tn-1)/d}}. \quad (2.1)$$

Proof. When $a = q^{-tn}$ or $a = q^tn$, the left-hand side of (2.1) is equal to

$$\sum_{k=0}^{(tn-1)/d} \frac{(q^{1-t}, q^{1+tn}, q/b; q^d)_k}{(q^d, c, q^{d+3}/bc; q^d)_k} q^{dk} = 3\phi_2 \left[ q^{1-t}, q^{1+tn}, q/b; c, q^{d+3}/bc; q^d, q^d \right]. \quad (2.2)$$

Via (1.2), the right-hand side of (2.2) can be expressed as

$$\left(\frac{q/b}{bc/q, q^{d+2}/c; q^d} \right)_{(tn-1)/d} \frac{(bc/q, q^{d+2}/c; q^d)_{(tn-1)/d}}{(c, q^{d+3}/bc; q^d)_{(tn-1)/d}}. \quad (2.3)$$

Since $(1-aq^tn)$ and $(a-q^tn)$ are relatively prime polynomials, we get the following result: Modulo $(1-aq^tn)(a-q^tn)$,

$$\sum_{k=0}^{(tn-1)/d} \frac{(aq, q/a, q/b; q^d)_k}{(q^d, c, q^{d+3}/bc; q^d)_k} q^{dk} \equiv \left(\frac{q/b}{bc/q, q^{d+2}/c; q^d} \right)_{(tn-1)/d} \frac{(bc/q, q^{d+2}/c; q^d)_{(tn-1)/d}}{(c, q^{d+3}/bc; q^d)_{(tn-1)/d}}. \quad (2.4)$$

When $b = q^tn$, the left-hand side of (2.1) is equal to

$$\sum_{k=0}^{(tn-1)/d} \frac{(aq, q/a, q^{1-tn}; q^d)_k}{(q^d, c, q^{d+3-tn}/c; q^d)_k} q^{dk} = 3\phi_2 \left[ aq, q/a, q^{1-tn}; c, q^{d+3-tn}/c; q^d, q^d \right]. \quad (2.4)$$

Via (1.2), the right-hand side of (2.4) can be written as

$$\frac{(c/qa, ac/q; q^d)_{(tn-1)/d}}{(c, c/q^2; q^d)_{(tn-1)/d}}. \quad (2.5)$$

Therefore, we are led to the following conclusion: Modulo $(b-q^tn)$,

$$\sum_{k=0}^{(tn-1)/d} \frac{(aq, q/a, q/b; q^d)_k}{(q^d, c, q^{d+3}/bc; q^d)_k} q^{dk} \equiv \frac{(c/qa, ac/q; q^d)_{(tn-1)/d}}{(c, c/q^2; q^d)_{(tn-1)/d}}. \quad (2.5)$$
It is clear that the polynomials \((1 - aq^m)(a - q^m)\) and \((b - q^m)\) are relatively prime. Noting the \(q\)-congruences
\[
\frac{(b - q^m)(ab - 1 - a^2 + aq^m)}{(a - b)(1 - ab)} \equiv 1 \pmod{(1 - aq^m)(a - q^m)},
\]
\[
\frac{(1 - aq^m)(a - q^m)}{(a - b)(1 - ab)} \equiv 1 \pmod{(b - q^m)}
\]
and employing the Chinese remainder theorem for coprime polynomials, we derive Theorem 1.2 from \((2.3)\) and \((2.5)\).

**Proof of Theorem 1.2.** Letting \(b \to 1, c \to q^3\) in Theorem 2.1, we arrive at the formula: Modulo \(\Phi_n(q)(1 - aq^m)(a - q^m),\)
\[
\sum_{k=0}^{(tn-1)/d} \frac{(aq, q/a; q^d)_{\frac{d}{k}}}{(q^d, q^d; q^d)_{\frac{d}{k}}} q^d_{\frac{d}{k}}
\equiv q^{(tn-1)/d} \frac{(q^2, q^{d-1}; q^d)_{\frac{d}{(tn-1)/d}}}{(q^3, q^d; q^d)_{\frac{d}{(tn-1)/d}}} + \frac{(1 - aq^m)(a - q^m)}{(1 - a)^2}
\times \left\{ q^{(tn-1)/d} \frac{(q^2, q^{d-1}; q^d)_{\frac{d}{(tn-1)/d}}}{(q^3, q^d; q^d)_{\frac{d}{(tn-1)/d}}} \frac{(aq^2, q^2/a; q^d)_{\frac{d}{(tn-1)/d}}}{(q, q^3; q^d)_{\frac{d}{(tn-1)/d}}} \right\}.
\]
Through the relation:
\[
q^{(tn-1)/d} \frac{(q^2, q^{d-1}; q^d)_{\frac{d}{(tn-1)/d}}}{(q^3, q^d; q^d)_{\frac{d}{(tn-1)/d}}} \equiv \frac{(q^2; q^d)_{\frac{d}{(tn-1)/d}}}{(q; q^d)_{\frac{d}{(tn-1)/d}}} \pmod{\Phi_n(q)},
\]
we are led to the conclusion: Modulo \(\Phi_n(q)(1 - aq^m)(a - q^m),\)
\[
\sum_{k=0}^{(tn-1)/d} \frac{(aq, q/a; q^d)_{\frac{d}{k}}}{(q^d, q^d; q^d)_{\frac{d}{k}}} q^d_{\frac{d}{k}}
\equiv q^{(tn-1)/d} \frac{(q^2, q^{d-1}; q^d)_{\frac{d}{(tn-1)/d}}}{(q^3, q^d; q^d)_{\frac{d}{(tn-1)/d}}} + \frac{(1 - aq^m)(a - q^m)}{(1 - a)^2}
\times \left\{ \frac{(q^2, q^2; q^d)_{\frac{d}{(tn-1)/d}}}{(q, q^3; q^d)_{\frac{d}{(tn-1)/d}}} \frac{(aq^2, q^2/a; q^d)_{\frac{d}{(tn-1)/d}}}{(q, q^3; q^d)_{\frac{d}{(tn-1)/d}}} \right\}.
\]
By the L’Hôpital rule, we have
\[
\lim_{a \to 1} \frac{(1 - aq^m)(a - q^m)}{(1 - a)^2} \left\{ \frac{(q^2; q^2; q^d)_{\frac{d}{(tn-1)/d}}}{(q, q^3; q^d)_{\frac{d}{(tn-1)/d}}} - \frac{(aq^2, q^2/a; q^d)_{\frac{d}{(tn-1)/d}}}{(q, q^3; q^d)_{\frac{d}{(tn-1)/d}}} \right\}
\]
\[
= \left|tn\right|^2 \frac{(q^2; q^d)_{\frac{d}{(tn-1)/d}}}{(q, q^3; q^d)_{\frac{d}{(tn-1)/d}}} \sum_{i=1}^{(tn-1)/d} \frac{q^{di-d+2}}{|di - d + 2|^2}.
\]
Letting \( a \to 1 \) in (2.6) and utilizing the above limit, we complete the proof of Theorem 1.1.

\[\Box\]

**Proof of Theorem 1.2.** It is routine to verify the relation:

\[(q; q^3)_{(m-1)/3} = (1 - q)(1 - q^4) \cdots (1 - q^{tn-3}) \equiv (1 - q^{1-tn})(1 - q^{4-tn}) \cdots (1 - q^{-3}) \]

\[= (-1)^{(tn-1)/3} q^{-(tn-1)(n+2)/6} (q^3; q^3)_{(tn-1)/3} \pmod{\Phi_n(q)}, \quad (2.7)\]

where \( t \in \{1, 2\} \) and \( n \equiv t \pmod{3} \). Setting \( d = 3 \) in Theorem 1.1 we obtain the result: Modulo \( \Phi_n(q)^3 \),

\[\sum_{k=0}^{(tn-1)/3} \frac{(q; q^3)_k^3}{(q^3; q^3)_k^3} t^{3k} \equiv q^{(tn-1)/3} \frac{(q^2; q^3)_{(tn-1)/3}^2}{(q^3; q^3)_{(tn-1)/3}^2} \]

\[+ [tn]^2 \frac{(q^2; q^3)_{(tn-1)/3}^2}{(q, q^2; q^3)_{(tn-1)/3}^2} \sum_{i=1}^{(tn-1)/3} \frac{q^{3i-1}}{[3i - 1]^2}, \quad (2.8)\]

where \( t \in \{1, 2\} \) and \( n \equiv t \pmod{3} \). Substituting (2.7) into (2.8), we get Theorem 1.2.

\[\Box\]

**3 Proof of Proposition 1.5**

Let \( \Gamma'_p(x) \) and \( \Gamma''_p(x) \) respectively be the first derivative and second derivative of \( \Gamma_p(x) \). In terms of the properties of the \( p \)-adic Gamma function, we arrive at

\[
\frac{(2/3)^2(p-1)/3}{(1)^2(p-1)/3} = \left\{ \frac{\Gamma_p((1+p)/3)\Gamma_p(1)}{\Gamma_p(2/3)\Gamma_p(2+p)/3} \right\}^2
\]

\[= \left\{ \Gamma_p(1/3)\Gamma_p((1+p)/3)\Gamma_p((1-p)/3) \right\}^2
\]

\[\equiv \Gamma_p(1/3)^2 \left\{ \Gamma_p(1/3) + \Gamma'_p(1/3)\frac{p}{3} + \Gamma''_p(1/3)\frac{p^2}{18} \right\}^2
\]

\[\times \left\{ \Gamma_p(1/3) - \Gamma'_p(1/3)\frac{p}{3} + \Gamma''_p(1/3)\frac{p^2}{18} \right\}^2
\]

\[\equiv \Gamma_p(1/3)^6 \left\{ 1 - \frac{2p^2}{9} G_1(1/3)^2 + \frac{2p^2}{9} G_2(1/3) \right\} \pmod{p^3}, \quad (3.1)\]

where \( G_1(x) = \Gamma'_p(x)/\Gamma_p(x) \) and \( G_2(x) = \Gamma''_p(x)/\Gamma_p(x) \).

Let

\[S_n^{(0)}(p) = 1, \quad S_n^{(l)}(p) = \sum_{1 \leq k_1 < k_2 \cdots < k_l \leq n} \frac{1}{k_1 k_2 \cdots k_l}, \quad H_n^{(l)}(p) = \sum_{1 \leq k \leq n} \frac{1}{k^l},\]

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where $\ell \in \mathbb{N}$. In terms of the two relations from H. Pan, Tauraso and Wang [11, Theorem 4.1]:

\[
G_1(1/3) \equiv G_1(0) + H_{(2p-2)/3} \pmod{p},
\]

\[
G_2(1/3) \equiv G_2(0) + 2G_1(0)H_{(2p-2)/3} + 2\delta^{(2)}_{(2p^2-2)/3}(p) \pmod{p^2}
\]

and the equation (cf. [14, Lemma 4.3]):

\[
G_2(0) = G_1(0)^2,
\]

we are led to

\[
G_2(1/3) - G_1(1/3)^2 \equiv 2\delta^{(2)}_{(2p^2-2)/3}(p) - H_{(2p-2)/3}^2
\]

\[
\equiv -H_{(2p^2-2)/3}(p)
\]

\[
\equiv - \sum_{i=1}^{(2p-2)/3} \frac{1}{(i + \frac{2p^2-2p}{3})^2}
\]

\[
\equiv -H_{(2p^2-2)/3}^2 \pmod{p}. \quad (3.2)
\]

Via (3.1) and (3.2), we can proceed as follows:

\[
\frac{(2/3)^2_{(p-1)/3}}{(1)^2_{(p-1)/3}} \left\{ 1 + p^2 \sum_{i=1}^{(p-1)/3} \frac{1}{(3i-1)^2} \right\}
\]

\[
\equiv \Gamma_p(1/3)^6 \left\{ 1 - \frac{2p^2}{9} H_{(2p-2)/3}^{(2)} \right\} \left\{ 1 + p^2 \sum_{i=1}^{(p-1)/3} \frac{1}{(3i-1)^2} \right\}
\]

\[
\equiv \Gamma_p(1/3)^6 \left\{ 1 - \frac{2p^2}{9} H_{(2p-2)/3}^{(2)} + p^2 \sum_{i=1}^{(p-1)/3} \frac{1}{(3i-1)^2} \right\}
\]

\[
\equiv \Gamma_p(1/3)^6 \pmod{p^3}.
\]

This proves the supercongruence (1.5). The proof of (1.6) admits a similar process. The corresponding detail has been omitted.

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