CRITICAL EXPONENTS OF THE $N$-VECTOR MODEL

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ABSTRACT

Recently the series for two RG functions (corresponding to the anomalous dimensions of the fields $\phi$ and $\phi^2$) of the 3D $\phi^4$ field theory have been extended to next order (seven loops) by Murray and Nickel. We examine here the influence of these additional terms on the estimates of critical exponents of the $N$-vector model, using some new ideas in the context of the Borel summation techniques. The estimates have slightly changed, but remain within errors of the previous evaluation. Exponents like $\eta$ (related to the field anomalous dimension), which were poorly determined in the previous evaluation of Le Guillou–Zinn-Justin, have seen their apparent errors significantly decrease. More importantly, perhaps, summation errors are better determined.

The change in exponents affects the recently determined ratios of amplitudes and we report the corresponding new values.

Finally, because an error has been discovered in the last order of the published $\varepsilon = 4 - d$ expansions (order $\varepsilon^5$), we have also reanalyzed the determination of exponents from the $\varepsilon$-expansion.

The conclusion is that the general agreement between $\varepsilon$-expansion and 3D series has improved with respect to Le Guillou–Zinn-Justin.

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1 Introduction and summary of results

Recently the perturbative expansions of the anomalous dimensions of the fields $\phi$ and $\phi^2$ for the $O(N)$ symmetric ($\phi^2_{d=3}$) field theory have been extended to next order (seven loops) in the case $N = 0, \cdots, 3$ by Murray and Nickel [46]. This rather impressive result has led us to reexamine the determinations of the critical exponents for $N = 0$ (polymers), $N = 1$ (Ising like systems), $N = 2$ (superfluid Helium) and $N = 3$ (real ferromagnets). For completeness we have added results (at six loops) for $N = 4$ which correspond to the Higgs sector of the Standard Model at finite temperature. A limitation of the present work is that the series for the RG $\beta$-functions have not been extended (they remain at six loops) and for several exponents this now is the main source of error.

Critical exponents have also been calculated in the form of $\varepsilon = 4 - d$ expansions, up to five loops [11]. Recently a slight error in the previously published series has been corrected [12], and this has motivated us to also reexamine the corresponding estimates (again adding $N = 4$ results).

For the reader who is not interested in details we summarize our main results for $N = 0, \cdots, 3$ in Table 1 ($d = 3$) and in Table 2 ($\varepsilon$-expansion) while $N = 4$ results for both methods can be found in Table 3. We have chosen central values which satisfy all scaling relations, but the apparent errors for $\gamma, \nu, \beta, \eta$ in general have been determined independently. For the $d = 3$ IR fixed point value $g^*$ we give results both in the usual field theory normalization (Eqs. (2.3)) and in the normalization used by Nickel [7],

$$\tilde{g} = \frac{N + 8}{48\pi} g,$$

which is such that the fixed point value is close to 1.

Note that in Table 1 in addition to the plain $\varepsilon$-expansion results (denoted as ”free”) we report some additional results denoted as ”bc” (i.e. with boundary condition) that try to incorporate the knowledge of the exact $d = 2$ values by summing the series $(f(\varepsilon) - f(2))/(2 - \varepsilon)$, where $f(\varepsilon)$ is an exponent with known 2D value. In the case of the exponent $\nu$ for $N = 0$ the $d = 1$ value is also known. We have checked that incorporating this additional piece of information has no significant impact on the final result.

For $N \geq 2$ the analysis of the series with boundary conditions is quite difficult. Therefore we present here only central values, but no error estimates. Values and errors of the corresponding free estimates give some indication.

Let us finally emphasize that we have no real knowledge about the analytic properties of exponents when $d$ approaches 2. Therefore the bc values could be affected by systematic effects.

The article then is organized as follows: in section 2 we summarize a few ideas about perturbative expansion at fixed $d = 3$ dimension and $\varepsilon$-expansion. In section 3 we briefly recall the Borel summation method based on a conformal
mapping of the complex cut plane. Several new variations of the practical implementation of the general method are explained. In section 4 we recall the idea of the pseudo-epsilon expansion and introduce the exponents’ correlation analysis, which consists in eliminating the coupling constant between different exponents. Section 5 contains a discussion of the numerical results. Finally the new values of exponents slightly affect the recently published results [13] for the equation of state of the 3D Ising model, and we present the new determination in section 6 (as well as a revised version of ε-expansion predictions).

Table 1

Critical exponents of the O(N) models from d = 3 expansion (present work).

| N   | 0       | 1       | 2       | 3       |
|-----|---------|---------|---------|---------|
| g_N | 1.413 ± 0.006 | 1.411 ± 0.004 | 1.403 ± 0.003 | 1.390 ± 0.004 |
| g^* | 26.63 ± 0.11  | 23.64 ± 0.07  | 21.16 ± 0.05  | 19.06 ± 0.05  |
| γ   | 1.1596 ± 0.0020 | 1.2396 ± 0.0013 | 1.3169 ± 0.0020 | 1.3895 ± 0.0050 |
| ν   | 0.5882 ± 0.0011 | 0.6304 ± 0.0013 | 0.6703 ± 0.0015 | 0.7073 ± 0.0035 |
| η   | 0.0284 ± 0.0025 | 0.0335 ± 0.0025 | 0.0354 ± 0.0025 | 0.0355 ± 0.0025 |
| β   | 0.3024 ± 0.0008 | 0.3258 ± 0.0014 | 0.3470 ± 0.0016 | 0.3662 ± 0.0025 |
| α   | 0.235 ± 0.003   | 0.109 ± 0.004   | −0.011 ± 0.004  | −0.122 ± 0.010 |
| ω   | 0.812 ± 0.016   | 0.799 ± 0.011   | 0.789 ± 0.011   | 0.782 ± 0.0013 |
| θ = ων | 0.478 ± 0.010 | 0.504 ± 0.008 | 0.529 ± 0.009 | 0.553 ± 0.012 |

Table 2

Critical exponents of the O(N) models from ε-expansion (present work).

| N   | 0       | 1       | 2       | 3       |
|-----|---------|---------|---------|---------|
| γ   | 1.1571 ± 0.0030 | 1.2355 ± 0.0050 | 1.3110 ± 0.0070 | 1.3820 ± 0.0090 |
| ν   | 0.5875 ± 0.0025 | 0.6290 ± 0.0025 | 0.6680 ± 0.0035 | 0.7045 ± 0.0055 |
| η   | 0.0300 ± 0.0050 | 0.0360 ± 0.0050 | 0.0380 ± 0.0050 | 0.0375 ± 0.0045 |
| β   | 0.3025 ± 0.0025 | 0.3257 ± 0.0025 | 0.3465 ± 0.0035 | 0.3655 ± 0.0035 |
| ω   | 0.828 ± 0.023   | 0.814 ± 0.018   | 0.802 ± 0.018   | 0.794 ± 0.018   |
| θ   | 0.486 ± 0.016   | 0.512 ± 0.013   | 0.536 ± 0.015   | 0.559 ± 0.017   |
Table 3
Critical exponents in the $O(4)$ models from $d = 3$ and $\varepsilon$-expansion (present work).

| Parameter | $d = 3$ | $\varepsilon$: free, bc |
|-----------|---------|-------------------------|
| $\tilde{g}_{N_i}^*$ | $1.377 \pm 0.005$ | $1.448 \pm 0.015 , 1.460$ |
| $g^*$ | $17.30 \pm 0.06$ | $0.737 \pm 0.008 , 0.742$ |
| $\gamma$ | $1.456 \pm 0.010$ | $0.036 \pm 0.004 , 0.033$ |
| $\nu$ | $0.741 \pm 0.006$ | $0.380 \pm 0.0025$ |
| $\eta$ | $0.0350 \pm 0.0045$ | $0.795 \pm 0.030$ |
| $\beta$ | $0.3830 \pm 0.0045$ | $0.774 \pm 0.020$ |
| $\alpha$ | $-0.223 \pm 0.018$ | $-0.211 \pm 0.024$ |
| $\omega$ | $0.774 \pm 0.020$ | $0.574 \pm 0.020$ |
| $\theta$ | $0.574 \pm 0.020$ | $0.586 \pm 0.028$ |

2 Renormalized $\phi^4$ field theory: $\varepsilon$-expansion and 3D perturbation series

In this article the general framework is the $(\phi^2)^2$, $O(N)$ symmetric, quantum field theory whose bare action is:

$$\mathcal{H}(\phi) = \int \left\{ \frac{1}{2} [\partial_{\mu} \phi(x)]^2 + \frac{1}{2} \lambda_2 \phi^2(x) + \frac{1}{4!} \lambda_4 [\phi^2(x)]^2 \right\} d^d x. \quad (2.1)$$

We recall that near the critical temperature $T_c$ $\lambda_2$ is a linear measure of the temperature. If we denote by $\lambda_{2c}$ the value for which the theory becomes massless ($T = T_c$) then the parameter $t$

$$t = \lambda_2 - \lambda_{2c} \propto T - T_c, \quad (2.2)$$

characterizes the deviation from the critical temperature.

The $(\phi^2)^2$ field theory is renormalizable in four dimensions, and to eliminate UV divergences (for $d < 4$ the theory is super-renormalizable) one introduces renormalized correlation functions. This involves choosing a renormalization scheme and then trading the bare parameters $\lambda_2, \lambda_4$ for a (scheme-dependent) renormalized mass $m$ and dimensionless coupling $g$. The mass parameter $m$ is proportional to the physical mass, or inverse correlation length, of the high temperature phase. It behaves for $t \propto T - T_c \rightarrow 0_+$ as $m \propto t^\nu$, where $\nu$ is the correlation length exponent (see [3] for details).

Renormalization group (RG) arguments tell us that the long distance properties of the massless (critical) theory are governed by non-trivial IR fixed points $g^*$, solution of the equation

$$\beta(g^*) = 0, \quad \text{with } \beta'(g^*) = \omega > 0.$$
The anomalous dimensions $\eta(g)$ and $\eta_2(g)$, of the renormalized field $\phi_r = \phi/\sqrt{Z}$ and of the renormalized composite operator $\langle\phi^2\rangle_r = (Z_2/Z)\phi^2$ respectively, evaluated at $g = g^*$ then yield the two independent combinations of critical exponents (e.g. $\eta = \eta(g^*)$). The explicit forms of the RG functions $\beta(g), \eta(g), \eta_2(g)$ depend on the specific renormalization scheme.

The space dimension relevant for statistical physics is $d = 3$ (occasionally $d = 2$). In this case one faces a serious problem: ordinary perturbative expansion in $g$ in the massless theory is IR divergent for any fixed dimension $d, d < 4$. A solution to this problem was first provided by Wilson–Fisher’s $\epsilon = 4 - d$-expansion. The idea is to avoid IR problems by expanding in $\epsilon = 4 - d$ as well as in the coupling constant $g$. IR singularities are then only logarithmic and can be dealt with. The expansion to the highest order presently available have been performed within the minimal subtraction $\overline{\text{MS}}$ scheme, [1,12]. In this scheme the $d$-dimensional RG beta function takes the exact form

$$\beta_{\overline{\text{MS}}} (g_{\overline{\text{MS}}}, \epsilon) = -\epsilon g_{\overline{\text{MS}}} + f(g_{\overline{\text{MS}}}) = -\epsilon g_{\overline{\text{MS}}} + O(g_{\overline{\text{MS}}}^2).$$

The fixed point equation

$$\beta_{\overline{\text{MS}}} (g^*_{\overline{\text{MS}}}, \epsilon) = 0,$$

can be solved in the form of an $\epsilon$-expansion. The $L$-loop expansion of the $\beta$-function then yields $g^*_{\overline{\text{MS}}}$ up to order $\epsilon^L$. By replacing $g_{\overline{\text{MS}}}$ by $g^*_{\overline{\text{MS}}}$ in the perturbative expansion of the anomalous dimensions $\eta, \eta_2$ one finally obtains the $\epsilon$-expansion of critical exponents. Note that while $g^*$ is scheme-dependent, the $\epsilon$-expansion for universal quantities is scheme-independent.

While this method directly yields a formal expansion for exponents a practical problem arises when one wants to determine exponents for a physical value of $\epsilon$ like $\epsilon = 1$ ($d = 3$). Indeed the $\epsilon$-expansion is divergent as has been first empirically noted in [3] and later confirmed by the large order behaviour analysis. A summation method is therefore required to obtain accurate results.

Following Parisi’s suggestion [3] perturbation series have also been calculated directly in three dimensions in the framework of the massive renormalized theory where correlation functions $\Gamma_{\tau}^{(n)}$ of the renormalized field $\phi_r$ are fixed by the normalization conditions

$$\Gamma_{\tau}^{(2)} (p; m, g) = m^2 + p^2 + O(p^4), \quad (2.3a)$$
$$\Gamma_{\tau}^{(4)} (p_i = 0; m, g) = mg. \quad (2.3b)$$

One may be surprised by the introduction of coupling and field renormalizations in a super-renormalizable theory. The reasons are simple, the bare coupling constant becomes infinite when the physical mass goes to zero. Simultaneously the field renormalization also diverges (see [3,4]).

Series up to six loops obtained in this scheme in Ref. [7] for $N = 0 \cdots 3$ have been generalized in Ref. [36] to any $N$. Only recently in [46] the results for $\eta$
and $\eta_2$ (but not for $\beta(g)$) have been extended to seven loops for $N = 0 \cdots 3$ (see Appendix A1). One problem here is that the value $g^*$ of the fixed point coupling is affected by summation errors on the $\beta$-function. Errors on $g^*$ then induces systematic errors for all critical exponents (see section 3).

3 Series summation

Perturbative quantum field theory generates divergent series. Summing such series by simply adding successive terms is meaningful only as long as coupling constants remain small enough (like in QED). Here however the expansion parameter, the fixed point value $g^*$, is a number of order 1: one therefore faces the problem of evaluating the sum of divergent series in a non-trivial regime.

In this article the Borel–Leroy transformation has been used, followed by a conformal mapping [33] (a new version of the method developed in [8] for critical exponents) to sum the series. We recall that the Borel summability of the $\phi^4$ theory in two and three dimensions has been established in [32].

Let $S(z)$ be any (Borel summable) function whose series has to be summed. We transform the series:

$$S(z) = \sum_{k=0}^{\infty} S_k z^k,$$  \hspace{1cm} (3.1)

into:

$$S(z) = \sum_{k=0}^{\infty} B_k(b) \int_0^{\infty} t^b e^{-t} u^k(zt) dt,$$  \hspace{1cm} (3.2)

with:

$$u(s) = \frac{\sqrt{1 + as} - 1}{\sqrt{1 + as} + 1}. \hspace{1cm} (3.3)$$

The coefficients $B_k$ are calculated by expanding in powers of $z$ the r.h.s. of equation (3.2) and identifying with expansion (3.1). The constant $a$ has been determined by the large order behaviour analysis. The explicit values are

$$a = 0.147774232 \times \frac{9}{N + 8}, \hspace{1cm} (3.4)$$

for the perturbative expansion in $d = 3$ dimensions and

$$a = \frac{3}{N + 8}, \hspace{1cm} (3.5)$$

for the $\varepsilon = 4 - d$ expansion. We map the Borel plane, cut at the instanton singularity $s = -1/a$, onto a circle in the $u$-plane in such a way to enforce maximal analyticity and thus to optimize the rate of convergence (for details see e.g. [8]).
Additional technical details. Following an idea introduced in \cite{8} for the summation of the $\varepsilon$-expansion we have in addition made a homographic transformation on the coupling constant $z$ to displace possible singularities in the complex $z$-plane:

\[ z = z'/(1 + qz'). \]

We have looked for values of the parameters $q$ and $b$ for which the results were specially insensitive to the order $k$: in practice the absolute differences of results corresponding to three successive orders have been minimized. When several solutions were found the less sensitive solution was chosen. Moreover the value of $b$ had to stay within a reasonable range around the value predicted by the large order behaviour.

For each series $S(z)$ we have applied the summation procedure both to $S$ and $1/S$. Finally we have introduced “shifts”, for each series summing

\[ S_s(z) = \left( S(z) - \sum_{k=0}^{s-1} S_k z^k \right) / z^s, \]

In practice only the cases $s = 0$ (no subtractions) and $s = 1, 2$ have proven useful. Thus for each exponent we have obtained six results whose spread gives an indication of summation errors.

In some examples (in particular $g^*$) shifts have produced strongly oscillating results. It has appeared that it would be useful to somehow interpolate between shifted series. An idea, new to this work, has been to consider the combination

\[ S' \equiv (1 + rg)S, \]

where $r$ has been used as a third variational parameter (dividing of course the final result by the factor $(1 + rg)$). The precise value of $r$ has been obtained by minimizing the dependence in $b$. This new additional parameter has proven quite useful: it has allowed, as expected, to obtain series with better apparent convergence as well as better general consistency. It has also revealed that in a few cases the apparent convergence at $r = 0$ was deceptive, the results being unstable with respect to a variation of $r$. These cases had already been singled out by the extreme values of the optimal $b, q$ parameters.

The main consequence of this new approach has been a decrease in the values of $g^*$ though the length of the series has not changed (better agreement between different shifts at nonzero $r$, see Fig.1), and of $\gamma$ for $N = 0$ and $d = 3$ (best $r = 0$ values of the exponent revealed to be unstable).

Errors. The summation error for any quantity $S(z)$ has been estimated by looking at differences between successive orders, sensitivity to the parameters and spread between all results concerning the same exponent (this has also involved checking scaling relations).
In the case of the 3D perturbative expansion, the total error for each exponent $S$ is the sum of the intrinsic summation error at fixed $\tilde{g}^*$, $\Delta S$, and the error induced by the error in $\tilde{g}^*$, $\Delta \tilde{g}^*$:

$$S = S^* \pm \Delta S \pm \left(\frac{dS}{d\tilde{g}}\right)_{\tilde{g}^*} \Delta \tilde{g}^*. \tag{3.8}$$

We thus also give the derivatives of exponents with respect to $\tilde{g}^*$. Two derivatives are displayed in Table 4, all other ones can be deduced from scaling relations. The reader can thus infer the sensitivity of exponents to a change in the values and errors of $g^*$.

**Table 4**

*Critical exponents: Sensitivity to $\tilde{g}^*$ determination.*

| $N$ | 0   | 1   | 2   | 3   | 4   |
|-----|-----|-----|-----|-----|-----|
| $d\gamma/d\tilde{g}^*$ | 0.10 | 0.18 | 0.28 | 0.39 | 0.50 |
| $d\nu/d\tilde{g}^*$   | 0.069 | 0.11 | 0.17 | 0.22 | 0.29 |

Let us stress here that we quote in our tables the total combined error (as well in [1,2]) while only the intrinsic summation error is reported in the Table 10 for the alternative result of [46].

For what concerns the $\varepsilon$-expansion the total error is directly given by the intrinsic summation error of each exponent and the situation is in principle more favourable: the only problem then is that the available series are shorter (they are technically more difficult to obtain) and the summation error is then bigger!

**Remarks.** The comparison between results coming from direct $d = 3$ series and $\varepsilon$-expansion is useful not only to test the accuracy of our numerical methods. Their consistency is also important to test various assumptions or properties.

In the case of the $d = 3$ expansion we assume more analyticity in the Borel plane as has been rigorously proven. Semiclassical instanton analysis indicates that our assumption is quite plausible but this is not a proof. Moreover several authors (see e.g. [13,70]) have argued that RG functions are not regular at $g = g^*$. We have of course checked that these singularities, if they exist, are weak. Numerical evidence is that all RG functions are at least differentiable at $g = g^*$ (including $\beta'(g)$ which yields $\omega$). We cannot of course exclude the situation where these singularities are so weak as to escape detection, but strong enough to influence results at the level of accuracy at which exponents are determined. Our apparent errors could then be underestimated. Nevertheless it should be emphasized that if the hypothesis of analyticity in the cut Borel plane holds,
the Borel summation should anyway converge asymptotically even in presence of confluent singularities.

For what concerns the $\epsilon$-expansion problems are more serious, since Borel summability has not even be proven. Moreover there are indications that $UV$-renormalon singularities could prevent Borel summability, [66]. These singularities are related to the large momentum behaviour of renormalized perturbation theory (the “Landau ghost” problem). A plausible conjecture is that quantities related to the massless theory are renormalon-free since they can be calculated in the theory with UV cut-off. This in particular applies to critical exponents. Instead the question remains open for quantities only defined in the massive renormalized theory, like the fixed point coupling constant $g^*(\epsilon)$ defined by (2.3.)

Note that, because the $\epsilon$ series are rather short, empirical evidence is weak.

4 Pseudo-epsilon expansion and exponents’ correlation analysis

In [2], a method was introduced to try to circumvent the problem of systematic errors induced by an error in the determination of $g^*$: the so-called pseudo-epsilon expansion. The idea is to mimic the $\epsilon$-expansion and introduce a new parameter $\tilde{\epsilon}$ in terms of which $\tilde{g}^*$ is expanded as well as all critical exponents.

The $d=3$ beta-function in the scheme Eqs. (2.3) has the form:

$$\beta(g) = -g + \beta_2(g)$$

where $\beta_2$ begins at order $g^2$ with a positive coefficient of order 1. We then replace the $\beta$-function by a new function $\beta(g, \tilde{\epsilon})$

$$\beta(g, \tilde{\epsilon}) = -\tilde{\epsilon}g + \beta_2(g),$$

and expand $g^*(\tilde{\epsilon})$, the solution to $\beta(g, \tilde{\epsilon}) = 0$, in powers of $\tilde{\epsilon}$. Eventually we have to sum the series for the value of $\tilde{\epsilon} = 1$ to recover the initial equation.

This method has been systematically used in [4], and this explains why some of the new values of exponents we obtain in this work differ less from the previous values of [2] than the change in $g^*$ would lead to expect.

To apply the same method here, we face the problem that the series for the $\beta$-function have not been extended to seven loops, and therefore for the exponents $\gamma$ or $\nu$, for example, the information of the additional seven loops term cannot be used. However since $\eta(g)$ starts only at order $g^2$, to determine $\eta$ at loop order $L$, $g^*(\tilde{\epsilon})$ is required only at loop order $L - 1$. This also applies to the exponent $\delta$ which only depends on $\eta$, and to which we have equally applied the summation procedures.

It follows that for $N = 0, 1, 2, 3$ the pseudo-epsilon expansion yields genuine seven loop information on $\eta, \delta$, with apparent errors much smaller compared to six-loop results.
Still to try to circumvent the problem of shorter $\beta(g)$ series and $g^*$ determination, we have in this work introduced another idea. We have directly eliminated the coupling constant between a pair of independent exponents. For example we have inverted the relation $g \mapsto 2 - 1/\nu$

$$g(\nu) = \sum g_k (2 - 1/\nu)^k,$$

and then expressed other exponents as series in $2 - 1/\nu$. In this way we have obtained correlation curves between exponents, which all eventually can be translated into relations $\eta(\nu)$. We have applied the same idea starting from the exponents $\gamma, \beta$, expanding in powers of $1 - 1/\gamma$ and $4 - 1/\beta$.

With this in mind it is interesting to consider the derivatives $d\eta/d\nu$ at the fixed point, which we thus display in Table 5. Other derivatives can be deduced, using scaling relations. The correlation line can be fixed by taking a point from the list of Table 1.

Finally let us note that we can push this idea up to expanding the RG $\beta$-function in powers of for example $2 - 1/\nu$ and solving directly the fixed point equation $\beta(\nu^*) = 0$.

We have tried this idea but the main problem we have faced is that the general structure of series generated by this set of transformations is rather complicated and therefore the apparent errors are quite large (a problem which already limits the accuracy of the pseudo-epsilon expansion). Therefore the method has mainly be used as a check of consistency among the data generated by more direct summation. It is possible that more accurate constraints could be obtained with more work to better understand the convergence of these new series, but we have eventually generated so many data that it became difficult to analyze all of them with the same care.

Table 5

| $N$  | 0  | 1  | 2  | 3  | 4  |
|------|----|----|----|----|----|
| $d\eta/d\nu$ $(d = 3)$ | 0.83 | 0.59 | 0.43 | 0.32 | 0.27 |

5 Numerical results

Let us first consider $d = 3$ results. The values of $g^*$ have been obtained by looking for the zeros of the summed RG function $\beta(g)$. The various methods explained in section 3 have been used, shifts 0, 1, 2, generating three set of values for each $N$, depending on three parameters $b, q, r$. Quoted errors for $g^*$ reflect the apparent
convergence with the order $k$ ($k \leq 7$), the sensitivity of $g^*$ to a variation of the parameters $b, q, r$ around optimal values as well as the spread between different shifts. As additional checks we have looked for the zeros of the function $\nu(g)\beta(g)$ (a rather arbitrary choice with the weak motivation that the derivative yields the exponent $\theta = \omega \nu$) and calculated $g^*$ from the pseudo-expansion (see [2] for details). Final results are reported in Table 1 (Table 3 for $N = 4$).

For what concerns the values of exponents, we have summed the seven (six for $N = 4$) loop series at the values of $g^*$ determined before. We have summed independently the five exponents $\gamma, \beta, \nu, \delta, \eta$ by using the three parameters $b, q, r$ for each exponent (and its inverse) and shifts 0, 1 (shift 2 was considered only as a check). Again, errors have been estimated by decreasing the order and looking at the spread between summation of different equivalent series, as explained in section 3. Additional checks have been derived from pseudo-expansion and exponents’ correlation analysis, introduced in section 4. Table 1 and Table 3 report the results of the analysis.

For what concerns the $\varepsilon$-expansion the procedure is the same as for the $d = 3$ series, apart from the fact that the $g^*$ step is bypassed, the series being summed at $\varepsilon = 1$ for the physical dimension three.

More precisely we have summed the genuine $\varepsilon$ series for the exponents (called “free” in Table 2) and we summed as well the modified series

$$\tilde{S}(d) \equiv \frac{S(d) - S(2)}{d - 2}$$

in which for each exponent is imposed the exact value at $d = 2$ (referred to “bc”, with boundary conditions, in Table 2). For $N = 0, 1$ the $d = 2$ exact exponents are obtained from the underlying conformal theories, for $N = 2$ from the identification with the Kosterlitz–Thouless transition, while for $N > 2$ the behaviour near $d = 2$ can be obtained from the $O(N)$ non-linear $\sigma$-model.

The general conclusions are the following: by imposing boundary conditions, we decrease the apparent errors for $N = 0$. For $N = 1$ apparent errors remain about the same but the central values are slightly modified. For $N \geq 2$ the convergence of the series with boundary condition is worse and we report in Table 2 only central values for the exponents for which the convergence seems reasonable. Errors can approximately been inferred from the difference with the free values and the corresponding apparent errors.

6 Updated values for the $N = 1$ equation of state and critical exponents

The new values of the critical exponents obtained in this work directly affect the determination of the scaling equation of state for the $N = 1, d = 3$ case, by the method presented in [13]. We thus report here the new estimates. The results for the $\varepsilon$-expansion have been revised too (in particular the errors on amplitude ratios have been reconsidered).
We recall that our starting point was an estimation of the values of coefficients $F_k$ of small magnetization expansion for the derivative of the effective potential $V$ (free energy) with respect to the scaled renormalized field $z$ (magnetization)

$$
\frac{\partial V}{\partial z} = z + \frac{1}{6} z^3 + \sum_{k=2} \left( F_{2k+1}(g) z^{2k+1} \right). \tag{6.1}
$$

These coefficients $F_k$ have been summed in [13] by using the available series up to five loops [25,26,27] and are reported in Table 6, compared with results of other techniques (a misprint in the last digit of the value of $\tilde{g}^*$ has been corrected).

A uniform approximation for the equation of state has then be provided by the determination of the auxiliary function $h(\theta)$ defined by the reparametrization (see also [28,22,23]):

$$
z = \rho \theta / (1 - \theta^2)^\beta \tag{6.2}
$$

$$
h(\theta) = \rho^{-1} (1 - \theta^2)^{\beta \delta} F(z(\theta)). \tag{6.3}
$$

The Order Dependent Mapping technique [67], has been used to improve convergence of the small $\theta$-expansion by an optimal choice of the parameter $\rho$.

The new result coming from the revised values of $\gamma, \beta$ is

$$
h(\theta)/\theta = 1 - 0.762(3) \theta^2 + 0.0082(10) \theta^4, \tag{6.4}
$$

that is obtained from $\rho^2 = 2.86$. This expression of $h(\theta)$ has a zero at

$$
\theta_0^2 = 1.33, \tag{6.5}
$$

to which corresponds the value of the complex root $z_0$ of $F(z)$, $|z_0| = 2.80$ (the phase, given by Eq. (6.2), is $-i\pi \beta$).

The revised $\varepsilon$-expansion estimations of $F_k$ (in Table 6) and of critical amplitudes presented in this paper are obtained using the revised $\gamma, \beta$ of Table 2 and the following expression of $h(\theta)$ (summed at $\varepsilon = 1$):

$$
h(\theta)/\theta = 1 - 0.72(6) \theta^2 + 0.0136(20) \theta^4. \tag{6.6}
$$

It should be emphasized that while critical amplitudes and the equation of state are universal quantities, $h(\theta)$ is not a universal function; in particular the variable $\theta$ of $\varepsilon$-expansion should not be identified to the corresponding variable of the $d = 3$ analysis, because they are defined from a different mapping Eq. (6.2), (different $\rho$ and $\beta$). It follows that $h(\theta)$ of the two methods (and their errors) cannot be compared directly.

Our value of $g^*$ from $\varepsilon$-expansion (in Table 6) has been obtained from our own analysis of $O(\varepsilon^4)$ series of [70]. We report in Table 6 also the recent results
of $O(\varepsilon^3)$, obtained by a direct summation of $O(\varepsilon^3)$ series for $F_k$ (improved by imposing boundary conditions at smaller dimensions).

Widom’s scaling function $f(x)$ (with $f(-1) = 0$ and $f(0) = 1$) can easily be derived by (numerically) solving the following system:

$$
\begin{align*}
  f(x) &= \theta^{-\delta} h(\theta)/h(1) \\
  x &= \left( \frac{1 - \theta^2}{1 - \theta_0^2} \right) \left( \frac{\theta_0}{\theta} \right)^{1/\beta}
\end{align*}
$$

(6.7)

Table 6
Equation of state.

| $\varepsilon$-exp., this work | $g^*$ | $F_3$ | $F_7 \times 10^4$ | $F_9 \times 10^5$ |
|-------------------------------|-------|-------|-----------------|-----------------|
| $\varepsilon$-exp., [37, 38]  | 23.3  | 0.0177 ± 0.0010 | 4.8 ± 0.6 | −3.3 ± 0.3 |
| $d = 3$, this work | 23.64 ± 0.07 | 0.01711 ± 0.00007 | 4.9 ± 0.5 | −7 ± 5 |
| $d = 3$ [39] | 23.71 | 0.01703 | 10 |
| HT [40] | 23.72 ± 1.49 | 0.0205 ± 0.0052 | |
| HT [37] | 24.45 ± 0.15 | 0.017974 ± 0.00015 | |
| HT [41] | 23.69 ± 0.10 | 0.0168 ± 0.0012 | 5.4 ± 0.7 | −2.3 ± 1.1 |
| MC [42] | 23.3 ± 0.5 | 0.0227 ± 0.0026 | |
| MC [43] | 24.5 ± 0.2 | 0.027 ± 0.002 | 23.6 ± 4 |
| ERG [44] | 28.9 | 0.016 | 4.3 |
| ERG [47] | 20.72 ± 0.01 | 0.01719 ± 0.00004 | 4.9 ± 0.1 | −5.2 ± 0.3 |

From Eq. (6.7) and the revised values of the critical exponents we can calculate various critical amplitude ratios that are reported in Tables 7 and 8 and are compared with other theoretical and experimental results. The reader can find all definitions and more details in [13]. (See also [31] for a report on the subject)

Table 8
Other amplitude ratios.

| $\varepsilon$-exp., this work | $R_0$ | $R_3$ | $C_4^+ / C_4^-$ |
|-------------------------------|-------|-------|-----------------|
| $d = 3$, this work | 0.1275 ± 0.0003 | 6.4 ± 0.2 | −9.0 ± 0.3 |
| $\varepsilon$-expansion, this work | 0.12584 ± 0.00013 | 6.08 ± 0.06 | −9.1 ± 0.6 |
| HT series [47] | 0.127 ± 0.002 | 6.07 ± 0.19 | −8.6 ± 1.5 |
Table 7
Amplitude ratios.

|                                   | $A^+/A^-$ | $C^+/C^-$ | $R_c$          | $R_X$          |
|-----------------------------------|-----------|-----------|----------------|----------------|
| $\varepsilon - \exp.$, [24,23]    | 0.524 ± 0.010 | 4.9       | 0.0569 ± 0.0035 | 1.648 ± 0.036  |
| $\varepsilon - \exp.$, this work | 0.527 ± 0.037 | 4.73 ± 0.16 | 0.0594 ± 0.001  | 1.7            |
| $d = 3$, [25]                     | 0.541 ± 0.014 | 4.77 ± 0.30 | 0.0574 ± 0.0020 | 1.669 ± 0.018  |
| $d = 3$, this work                | 0.537 ± 0.019 | 4.79 ± 0.10 | 0.0581 ± 0.0010 | 1.75           |
| HT [29,30]                        | 0.523 ± 0.009 | 4.95 ± 0.15 | 0.050 ± 0.015   | 1.75 ± 0.30    |
| MC [68]                           | 0.560 ± 0.010 | 4.75 ± 0.03 | 0.047 ± 0.010   | 1.69 ± 0.14    |
| bin. mix.                         | 0.56 ± 0.02  | 4.3 ± 0.3  | 0.050 ± 0.015   | 1.75 ± 0.30    |
| liqu. – vap.                      | 0.48–0.53    | 4.8–5.2    | 0.047 ± 0.010   | 1.69 ± 0.14    |
| magn. syst.                       | 0.49–0.54    | 4.9 ± 0.5  |                |                |

7 Conclusions

Before discussing our results, let us review the results for critical exponents obtained by other theoretical methods or as well as experiments.

The previous most accurate determinations of the critical exponents of the $O(N)$ vector model, from quantum field theory and renormalization group, have been reported in [1,2] and are shown in Table 9 (we refer to these results as LG–ZJ). In Table 10 we report the Murray–Nickel (M–N) predictions (direct fit of $g$ series) with the authors’ preferred choice of $\tilde{g}^*$ (they report only summation errors; errors from $\tilde{g}^*$ should be added). In Table 11 we list some values for $N = 4$ obtained from Padé Borel summation of $d = 3$ series up to six loops (see [36] where results for many values of $N > 3$ are given). An analysis based on Order Dependent Mapping [67] of $d = 3$ series can be found in [69]. In Table 12 we report the previous analysis of $\varepsilon$-expansion while in Table 13 we quote some recent results. Other available theoretical predictions come from the analysis of High Temperature (HT) series in lattice models, Table 14, and Monte-Carlo (MC) simulations Table 15. Finally in Table 16 we report for completeness some estimates from the truncated “Exact Renormalization Group” approach.

Table 11
Critical exponents: results of Padé Borel summation for $N = 4$, [36].

| $N$, Ref. | $\tilde{g}^*$ | $\gamma$ | $\nu$ | $\eta$ |
|-----------|--------------|----------|-------|-------|
| 4, [36]   | 1.369        | 1.449    | 0.738 | 0.036 |
For what concerns experimental determinations of critical exponents a few significant results are displayed in Table 17.

3D series. In general the new estimates displayed in Table 1 are more accurate
than the previous LG-ZJ results. They are compatible within errors with the previous analysis. A closer inspection shows however some significant changes which require discussion.

The main effect comes from the new (and smaller) values of the fixed point coupling constant for $N < 3$. The changes are a direct consequence of the new techniques we have introduced. In the old calculation LG–ZJ had noticed two puzzling features: the optimal values of the parameter $b$ were somewhat large, compared to what large order behaviour did suggest. Moreover the three shifts $0, 1, 2$ gave strongly oscillating results.

Introduction of the new parameter $r$ (see Eq. (3.7)) has shown that the old

| $N$ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| $\tilde{g}^*(O(\varepsilon^1))$ | $1.390 \pm 0.017$ | $1.397 \pm 0.008$ | $1.413 \pm 0.013$ | $1.387 \pm 0.007$ |
| $\gamma$ | $1.1559 \pm 0.0010$ | $1.240 \pm 0.005$ | $1.304 \pm 0.007$ | $1.372 \pm 0.006$ |
| $\nu$ | $0.5882 \pm 0.0011$ | $0.631 \pm 0.003$ | $0.664 \pm 0.003$ | $0.699 \pm 0.004$ |

| $N$, Ref. | $\gamma$ | $\nu$ | $\alpha$ | $\theta = \omega \nu$ |
|-----------|---------|-------|---------|----------|
| 0, [4]    | $1.1595 \pm 0.0012$ | $0.588 \pm 0.001$ |         |          |
| 0, [5]    | $1.1613 \pm 0.0001$ | $0.588 \pm 0.001$ |         |          |
| 1, [6]    | $1.239 \pm 0.002$ | $0.631 \pm 0.003$ |         |          |
| 1, [7]    | $1.2385 \pm 0.0025$ | $0.6305 \pm 0.0015$ |         |          |
| 1, [8]    | $1.239 \pm 0.003$ | $0.631 \pm 0.004$ |         |          |
| 1, [9]    | $1.2395 \pm 0.0004$ | $0.632 \pm 0.001$ | $0.105 \pm 0.007$ | $0.54 \pm 0.05$ |
| 1, [10]   | $1.239 \pm 0.003$ | $0.632 \pm 0.003$ | $0.101 \pm 0.004$ |          |
| 1, [11]   | $1.237 \pm 0.002$ | $0.630 \pm 0.0015$ |         |          |
| 1, [12]   | $1.2385 \pm 0.0005$ | $0.6310 \pm 0.0005$ |         | $0.52 \pm 0.03$ |
| 1, [13]   | $1.237 \pm 0.004$ |         | $0.104 \pm 0.004$ | $0.108 \pm 0.005$ |
| 2, [14]   | $1.323 \pm 0.003$ | $0.674 \pm 0.003$ |         |          |
| 2, [15]   | $1.323 \pm 0.0015$ | $0.670 \pm 0.007$ |         |          |
| 3, [16]   | $1.402 \pm 0.003$ | $0.714 \pm 0.002$ |         |          |
| 3, [17]   | $1.40 \pm 0.03$ | $0.72 \pm 0.01$ |         |          |
| 4, [18]   | $1.474 \pm 0.004$ | $0.750 \pm 0.003$ |         |          |
Table 15

Critical exponents: MC

| $N$, Ref. | $\gamma$ | $\nu$ | $\beta$ | $\eta$ | $\theta = \omega \nu$ |
|-----------|---------|-------|--------|--------|-------------------|
| 0, [53, 54] | 1.1575 ± 0.0006 | 0.5877 ± 0.0006 |        |        | 0.515 ± 0.017 |
| 0, [71] | 0.58758 ± 0.00007 |        |        |        |                  |
| 1, [56] | 0.631 ± 0.001 | 0.3269 ± 0.0006 | 0.038 ± 0.002 |        |                  |
| 1, [57] | 0.6289 ± 0.0008 |        |        |        |                  |
| 1, [58] | 0.625 ± 0.001 | 0.0025 ± 0.006 | .44 |        |                  |
| 1, [59] | 0.6294 ± 0.0009 | 0.0374 ± 0.0014 | .55 ± 0.06 | | |
| 2, [60] | 1.324 ± .001 | 0.664 ± 0.006 |        |        |                  |
| 2, [61] | 1.323 ± .002 | 0.670 ± 0.002 |        |        |                  |
| 2, [62] | 1.316 ± .003 | 0.6721 ± 0.0013 | 0.042 ± 0.002 | .54 ± 0.08 | |
| 3, [63] | 1.3896 ± .0070 | 0.7036 ± .0023 | .362 ± .004 | .0027 ± 0.002 | |
| 3, [64] | 1.396 ± .003 | 0.7128 ± .0014 | 0.041 ± .002 | .51 ± .11 | |
| 4, [65] | 1.476 ± .002 | 0.7525 ± .0010 | 0.038 ± .001 | | |
| 4, [66] | 1.477 ± .018 | 0.748 ± .009 | .3836 ± .0046 | | |

Table 16

Critical exponents: “Exact Renormalization Group” estimates.

| $N$, Ref. | $\tilde{g}^*$ | $\gamma$ | $\nu$ | $\eta$ | $\theta = \omega \nu$ |
|-----------|-------------|-------|-------|--------|-------------------|
| 1, [44] | 1.726 | 1.247 | 0.638 | 0.045 |                  |
| 2, [44] | 1.675 | 1.371 | 0.700 | 0.042 |                  |
| 3, [44] | 1.619 | 1.474 | 0.752 | 0.038 |                  |
| 4, [44] | 1.566 | 1.556 | 0.791 | 0.034 |                  |
| 1, [47] | 0.618 ± 0.014 | .054 | .56 ± .007 | | |
| 1, [55] | 0.6262 ± 0.0013 | | | | |
| 1, [72] | 0.625 ± 0.007 | 0.030 ± 0.005 | 0.48 ± 0.04 | | |

Apparent convergence corresponded to an unstable region of parameters. By varying $r$ we find a region where these problems are solved to a large extent: the results are less sensitive, various shifts agree, and all parameters have more reasonable values. Figure 1 exemplifies this situation for the $N = 0$ case.

Another example exhibited a similar instability at $r = 0$: $\gamma$, $N = 0$ $d = 3$. This (as well as the decrease induced from that of $g^*$) explains the new different value we obtained here.

Finally $N = 3$ values show a consistent effect: the three exponents $\gamma, \nu, \beta$ increase. This simply suggests that $N = 3$ errors had been underestimated in LG–ZJ analysis.
Table 17

Critical exponents: selected recent experiments

| N, Ref. | $\gamma$ | $\nu$       | $\beta$       | $\alpha$       | $\theta = \omega \nu$ |
|-------|---------|-------------|-------------|-------------|-------------------|
| 0, [48] |        | 0.586 ± 0.004 |          |            |                   |
| 1, [73] | 1.25 ± 0.01 | 0.64 ± 0.01 | 0.327 ± 0.002 | 0.57 ± 0.09    |
| 1, [74, 75, 79] | 1.233 ± 0.010 |          |            |            |                   |
| 2, [74] | 0.6708 ± 0.0004 | -0.1285 ± 0.00038 |          |            |                   |
| 2, [76] | 0.6705 ± 0.0006 |          |            |            |                   |
| 3, [79] |        |          |            |            | 0.61 ± 0.06     |

$g_0$ vs $r$, N=0

Fig. 1 Values of $g^*$, $N = 0$, as a function of the parameter $r$ for shifts 0,1,2

Conversely the values of $\nu$, $N = 0,1,2$ are quite stable, even though the corresponding values of $g^*$ have changed. The reasons are a very good apparent convergence of the pseudo-epsilon expansion (on which previous analysis partially relied) at previous order.

The apparent errors have generally been reduced, as should be expected, except for $N = 3$ (see the comment above). The improvement is specially significant for the exponent $\eta$ that was poorly determined before. With a few exceptions the general trend for a given exponent is the increase of the direct summation error with $N$. This effect has a simple explanation: our summation method relies on the large order behaviour analysis, and the asymptotic regime sets in later when $N$ increases [80]. Perhaps a clever use of the knowledge coming from the large $N$ expansion could improve the situation.
Note finally that the term added to two of the three RG series has allowed not only to decrease apparent errors but also to estimate them more reliably.

For several exponents (such as e.g. $\gamma(N = 1)$) errors are now dominated by errors induced by the determination of $g^*$. To further improve the situation it will be necessary to also add a new term to the RG $\beta$-function.

*(Free) $\varepsilon$-expansion and 3D series.* Comparison with the previous LG–ZJ $\varepsilon$ estimates shows no striking effect, small deviations being due to the use of corrected series.

For the exponent $\nu$ the consistency between 3D and $\varepsilon$ results remains very good at all $N$. The situation has markedly improved for the exponent $\gamma$, $N = 0$: there is still a systematic discrepancy in the central values for the exponents (about 0.002), but the difference is reduced by more than a factor two, which is quite encouraging. A similar comment applies to the central values of the exponent $\eta$ $N = 0, \ldots, 3$, where the discrepancy is also reduced by a factor two. For what concerns the $N = 4$ prediction, the agreement with the corresponding $d = 3$ results is quite satisfactory but apparent errors are large.

One point should however be stressed: since the series are shorter it is more difficult to assess apparent errors and the errors we quote are thus less reliable than for the $d = 3$ series.

*Free and bc $\varepsilon$-expansion.* For the $\varepsilon$-expansion we report a second set of values, obtained by imposing the exact $d = 2$ values, referred as bc (i.e. with boundary conditions) in Table 2 to distinguish them from the unconstrained values denoted by free. Some remarks are in order about the bc approach. First we do not know the analytic structure of exponents near $d = 2$. The only piece of evidence comes from using the $\varepsilon$-expansion for $d = 2$. For $N = 0$ the agreement with exact results is quite good [8], suggesting that $d = 2$ is a regular point. For $N = 1$ the agreement is less striking, leaving room for some complex behaviour. For values $N > 2$ there are indications (IR-renormalons of the $d = 2$ non-linear $\sigma$-model) that $d = 2$ corresponds to an essential singularity (probably non-Borel summable). It is then easy to construct examples where fitting leads to worse results.

Second in the case $N \geq 2$ the new series have a more complicated structure which makes summation (which is a form of extrapolating series to higher orders) and even more error estimation difficult.

Although the two analyses give compatible results, it happens that bc has the general tendency to give values of $\gamma, \nu$ for $N \geq 2$ larger than free ones and values of $\eta$ for $N > 2$ smaller than free ones. This point is understood by remembering that at $d = 2$ the corresponding $\gamma, \nu, 1/\eta$ are infinite and thus it is reasonable that imposing this behaviour at $d = 2$ tends to increase the value of the exponent at $d = 3$.

Finally it is also remarkable that for $N = 0$ the bc errors are smaller than the free ones.
Other methods. The general agreement between the HT series, the MC results, and the new $d = 3$ determinations is in general improved. This in particular applies to the SAW where recent long simulations provide very accurate estimates.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Comparison between various estimates of the exponent $\nu$, $N=2$: He li from [77], He go from [78], $d3$ gz from present work ($d = 3$), $d3$ lz from LG–ZJ ($d = 3$), ep gz from present work ($\varepsilon$), ep lz from LG–ZJ ($\varepsilon$), ep pv from [70] ($\varepsilon$), mc j from [61], mc b from [62], ht bc from [19].}
\end{figure}

Experiments. The improved agreement of present results with the recent measures on superfluid Helium systems, $N = 2$, is remarkable and is displayed graphically in Figure 2. In spite of our efforts the best experimental value is still much more accurate than the theoretical estimate. In all other cases the agreement with experiments is good as seen from Table 17.

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APPENDICES

A1 Series $d = 3$

We report here the seven loop series for the $O(N)$ symmetric $(\phi^2)^2$ theory, $N = 0, 1, 2, 3$, computed by Murray and Nickel [46]. The functions $\eta(\tilde{g})$ $\eta_2(\tilde{g})$ below are defined by

$$
\eta(\tilde{g}) = m \frac{d \log Z}{dm} \quad \eta_2(\tilde{g}) = m \frac{d \log Z_2}{dm}
$$

where the renormalization constant are defined by $\phi_0 = \sqrt{Z} \phi_r$ and $(\phi^2)_r = \frac{Z}{Z_r}(\phi_0)^2$ (subscript $0, r$ indicate respectively bare and renormalized fields). The critical exponents $\eta, \nu$ can be found by the identification $\eta = \eta(\tilde{g}^*)$ and $\nu = (2 + \eta_2(\tilde{g}^*) - \eta(\tilde{g}^*))^{-1}$. The symmetry number $N$ is reported in square brackets below.

$$
\eta[0] = \frac{\tilde{g}^2}{108} + 0.0007713750 \tilde{g}^3 + 0.0015898706 \tilde{g}^4 - 0.0006606149 \tilde{g}^5
+ 0.0014103421 \tilde{g}^6 - 0.001901867 \tilde{g}^7
$$

$$
\eta[1] = \frac{8 \tilde{g}^2}{729} + 0.0009142223 \tilde{g}^3 + 0.0017962229 \tilde{g}^4 - 0.0006536980 \tilde{g}^5
+ 0.0013878101 \tilde{g}^6 - 0.001697694 \tilde{g}^7
$$

$$
\eta[2] = \frac{8 \tilde{g}^2}{675} + 0.0009873600 \tilde{g}^3 + 0.0018368107 \tilde{g}^4 - 0.0005863264 \tilde{g}^5
+ 0.0012513930 \tilde{g}^6 - 0.001395129 \tilde{g}^7
$$

$$
\eta[3] = \frac{40 \tilde{g}^2}{3267} + 0.0010200000 \tilde{g}^3 + 0.0017919257 \tilde{g}^4 - 0.0005040977 \tilde{g}^5
+ 0.0010883237 \tilde{g}^6 - 0.001111499 \tilde{g}^7
$$

$$
\eta_2[0] = \frac{-\tilde{g}}{4} + \frac{\tilde{g}^2}{16} - 0.0357672729 \tilde{g}^3 + 0.0343748465 \tilde{g}^4 - 0.0408958349 \tilde{g}^5
+ 0.0597050472 \tilde{g}^6 - 0.09928487 \tilde{g}^7
$$
\[ \eta_2[1] = \frac{-\tilde{g}}{3} + \frac{2\tilde{g}^2}{27} - 0.0443102531\tilde{g}^3 + 0.0395195688\tilde{g}^4 - 0.0444003474\tilde{g}^5 \\
+ 0.0603634414\tilde{g}^6 - 0.09324948\tilde{g}^7 \]

\[ \eta_2[2] = \frac{-2\tilde{g}}{5} + \frac{2\tilde{g}^2}{25} - 0.0495134446\tilde{g}^3 + 0.0407881055\tilde{g}^4 - 0.0437619509\tilde{g}^5 \\
+ 0.055575703\tilde{g}^6 - 0.08041336\tilde{g}^7 \]

\[ \eta_2[3] = \frac{-5\tilde{g}}{11} + \frac{10\tilde{g}^3}{121} - 0.0525519564\tilde{g}^3 + 0.0399640005\tilde{g}^4 - 0.0413219917\tilde{g}^5 \\
+ 0.0490929344\tilde{g}^6 - 0.06708630\tilde{g}^7 \]