Quark Propagation in the Random Instanton Vacuum

E.V. Shuryak and J.J.M. Verbaarschot

Department of Physics
SUNY, Stony Brook, New York 11794

Abstract

This is the first of a series of papers devoted to a systematic study of QCD correlation functions in a framework of 'instanton vacuum' models. The topic of this paper is to work out approximate formulae for quark propagators in a multi-instanton environment. As an application, and also as a necessary step toward understanding the correlation functions, we study the propagators of scalar and spinor quarks, using the simplest possible model, the so called 'random instanton vacuum' (RIV). Results related to heavy-light mesons, are found to be very consistent with phenomenology.
1. Introduction

The understanding of non-perturbative phenomena in the QCD vacuum remains to be a challenging problem of contemporary physics. One particular way to obtain insights in the complicated quark-quark and quark-anti-quark interactions is provided by point-to-point correlation functions, which, at medium distances of 1/3-1 fm, display strong deviations from free quark propagation. These deviations are strongly channel dependent and change widely in both sign and magnitude (see [1] for a recent review).

Although the hadronic spectrum can be described more or less in terms of a universal confining force and perturbative spin forces, such a simplified picture definitely fails to describe the point-to-point correlation functions. The actual situation is much more complex, and additional theoretical inputs are definitely needed.

Tunnelling phenomena in the QCD vacuum, described semiclassically by the so called instanton solutions were suggested as explanations of at least some of these effects (or even most of them). Classical papers of the one-instanton era describe the discovery of the instanton solution [2], its physical interpretation as tunneling and the relation to chiral anomalies [3], and the semiclassical integration of the quantum fluctuations [4]. The first applications of instantons to QCD problems [5], based on the so called dilute gas approximation, attracted a lot of attention in the late seventies. However, the absence of an explanation for the diluteness and the semiclassical nature of the instanton ensemble led to such a pessimism that most people left the field around 1980.

However, at the same time, phenomenological studies of instanton-related effects [6, 7] have shown that the induced effective interaction of light quarks [8] has exactly the properties needed to explain several puzzles of the hadronic world. Eventually, this development has resulted in the so called 'instanton liquid' model [8, 9], and its phenomenological success revived the hopes of describing the QCD vacuum by a dilute system of semiclassical fluctuations.

Multiple attempts [10, 11] [12, 13] [14, 15] were made to create a quantitative theory of the 'instanton component' of the QCD vacuum. The problem appears to be very difficult: even if one disregards all other non-perturbative phenomena and considers only instantons, it is complicated both by the gauge-induced and, especially, by the fermion-
induced interaction between them. The theory of these phenomena, which we refer to as the *interacting instanton approximation* (IIA)\(^1\), has been developed during the past decade, and the present series paper can be considered as an attempt to clarify its relation to experimental information on correlators in a quantitative way.

So far, the only numerical evaluation of correlators in a multi-instanton environment was made in a series of works by one of us [16, 17]. The first results on a set of correlation functions have well reproduced the data, sometimes in surprising detail. However, the accuracy of the calculations was not high, and both the statistical and the systematic errors should be significantly reduced. This is one of the tasks addressed by the present series of papers.

Recently, the first results for point-to-point correlators from lattice simulations [18] were reported. Remarkably, inside uncertainties they generally agree both with experiment and IIA predictions. Whether these results are or are not related to instantons can be clarified by further lattice studies. In the long run much better lattice data will be obtained, so that a more detailed comparison with experiment becomes possible. In any case, this recent development makes the situation in the field very exciting.

The present paper is the first in a series of three works. The objective is to perform a *systematic* calculation of *all major* correlation functions at distances of the order of 1 fm. A systematic comparison of the results with experimental data is made whenever possible. We also compare our results with predictions of the operator product expansion (OPE), on which the so called QCD sum rules are based. As this task happens to be quite formidable by itself, merely because of the literally dozens of different correlation functions *etc.*, we feel that pushing forward the theory of interacting instantons at the same time would be very confusing. Instead, we have adopted the idea to establish a kind of 'benchmark calculation' for all correlators, using an as simple input model as possible. The particular framework used below will be called the 'random instanton vacuum' (RIV). It is nothing else but the 'instanton liquid model', originally suggested by one of us [8], supplemented by the simplest assumption of a random distribution of the collective coordinates. We do not fit any parameters, leaving them as they were proposed a decade ago at the one-instanton level. It is relatively easy to perform high-statistics numerical studies of this

\(^1\)For a recent review and references on this subject we refer to [1].
simple model, and, as we demonstrate below and in the next papers of this series [19, 20], it reproduces most of the experimental observations amazingly well.

The present paper is organized as follows. In section 2 we discuss the main ingredients of our model, and we pay particular attention to the zero mode sector of the Dirac operator. Numerical results for its spectrum will be presented. The general structure of the quark propagator and its numerical evaluation will be discussed in section 3. In section 4 we discuss the propagation of a scalar quark in the RIV. The more complicated spinor propagator is discussed in section 5, and the results are applied to heavy-light systems in section 6. Concluding remarks are made in section 7.

The mesonic [19] and baryonic [20] correlation functions are discussed in two subsequent papers of this series. After such benchmark for all correlators is established, we are planning to return to their discussion in the presence of instanton interactions with correlations among the positions and orientations. Those studies are subject of future investigations.

2. The model and the zero mode zone

The general reason why instantons are so important for the physics of light quarks is related to the so called fermionic zero mode, which exists for any topologically non-trivial gauge field. In order to explain this let us consider the Euclidean quark propagator defined by

\[ S = - \left< \frac{1}{iD_\mu \gamma_\mu + im} \right>, \]  

(2.1)

where \( iD_\mu = i\partial_\mu + A_\mu \) is the covariant derivative containing the external gauge field, which should be averaged over with the proper weight following from the QCD partition function. The operator may be inverted by first diagonalizing it,

\[ iD_\mu \gamma_\mu \phi_\lambda(x) = \lambda \phi_\lambda(x), \]  

(2.2)

which leads to the general expression

\[ S(x, y) = - \sum_\lambda \frac{\phi_\lambda(x)\phi_\lambda^\dagger(y)}{\lambda + im}, \]  

(2.3)
for the propagator in the gauge field $A_\mu$. It is clear that in the chiral limit ($m \to 0$) the eigenvalue spectrum at small $\lambda$ has important effects on the propagation of hadrons. In particular, one can easily derive the following general formula \[21\] for the Euclidean quark condensate

$$<\bar{\psi}\psi>_{E} = \frac{\pi i}{V} \rho(\lambda = 0),$$  \hspace{1cm} (2.4)

where $\rho(\lambda)$ is the spectral density$^2$.

'Zero modes', discovered in $^3$, are solutions of the Dirac equation in the field of one instanton with eigenvalue $\lambda = 0$. Their existence explains the so called $U(1)$ chiral anomaly: while tunneling, quarks with one chirality 'dive into the Dirac sea', and those with the opposite chirality 'emerge' from it. In $^[19, 20]$ we will show that these phenomena are responsible for a much wider range of physical effects than was believed previously. In particularly, not only the $\eta'$ channel is affected, but, say correlators in the $\pi$ meson or the nucleon channels can be reproduced as well.

The evaluation of a quark propagator in the multi-instanton background field is a complicated problem. In particular, even a more or less dilute set of instantons (and anti-instantons) is described by 12 collective variables per instanton (position, size and orientation), and the spectrum of the Dirac operator depends on all of them. Roughly speaking, the problem resembles the propagation of a sound wave through a turbulent atmosphere, containing many vortices and anti-vortices.

To explain our approximations, one can use an analogy from solid state physics in which the role of atoms is played by instantons, and the electron bound state is represented by the quark 'zero mode'. The general lesson from condensed matter physics is that at finite density of the atoms, atomic bound states may become collective and form 'zones' of delocalized states$^3$. Analogously, fermionic states in the 'instanton vacuum' form the so called 'zero mode zone' (ZMZ)$^4$.

Furthermore, if the ensemble of instantons is dilute, the off-diagonal matrix elements

\[\text{We do not repeat here the standard arguments on the order of the limits } m \to 0 \text{ and } V \to \infty. \text{ For a very detailed recent discussion of this topic see } [22].\]

\[\text{It is important that such zones exist as well for disordered systems (liquids), although in this case they do not have sharp boundaries.}\]

\[\text{What is important here, is that the mode for an individual 'well' is exactly at zero, due to some topological theorem: therefore all small distortions are irrelevant. We thus deal with a degenerate situation, in which the spectrum is governed by the off-diagonal matrix elements.}\]
are small and the zone should be narrow. If so, this set of states may dominate the spectrum near zero virtuality, and therefore be mainly responsible for $\rho(\lambda = 0)$ and the quark condensate. Direct evidence that this indeed happens in the QCD vacuum has been obtained in lattice studies $[23, 24]$. Much more work in this direction can and should be done in order to clarify the relation between lattice results and the theory of instantons.

Returning to the propagator, one arrives at a comprehensive physical picture of how light quarks can easily propagate over large distances in the QCD vacuum: they can do it by simply jumping from one instanton to the next. It is analogous to what conduction electrons do, in solid or liquid metals. Due to the degeneracy of atomic levels, quite small perturbations can crucially enhance or reduce probability of this to happen.

Continuing this analogy further, we have learned from condensed matter physics that going beyond the one-electron approximation one comes across an important interaction between conduction electrons due to the possibility, that two of them may happen to be at the same atom. The same is true for quarks in the instanton vacuum: the fact that they may meet each other at the same instanton leads to their strong interaction. Technically this interaction is revealed, when one compares the average propagator squared (for mesons) or cubed (for baryons) to the same power of the average propagator. The latter is studied in detail below, while the former quantities will discussed in two subsequent papers.

Having outlined the main physics involved, let us now specify the ZMZ in more detail. In a basis of zero modes $[15]$, the Dirac operator reduces to the following matrix

$$i\hat{D} = \begin{pmatrix} 0 & -T_{II} \\ -T_{II}^\dagger & 0 \end{pmatrix},$$

where the overlap matrix elements are defined by

$$T_{II} = \int d^4x \phi^*_I(x)i\gamma \cdot \partial \phi_I(x).$$

The ordinary derivative appears because the gauge field in the covariant derivative has been eliminated by using the equations of motion for the zero modes. Although it is not very important for what follows, in this paper we will use the overlap matrix elements corresponding to ‘streamline’ gauge field configurations. They are discussed in $[25]$ and
are equal to
\[ T_{II} = \frac{1}{2} \text{Tr} \left( \frac{\sigma^\mu R^\mu U_I^{-1} U_I}{R} \right) \frac{F(\lambda)}{(\rho \rho_I)^{1/2}}, \]  
(2.7)
where \( R = R_I - R_I \) is four-distance between the centers, \( U_I^{-1} U_I \) is the relative orientation of the instantons and \( \lambda \) is a conformally invariant parameter defined by
\[ \lambda = \frac{1}{2} \frac{R^2 + \rho^2_i + \rho^2_i}{\rho_i \rho_I} + \frac{1}{2} \left( \frac{(R^2 + \rho^2_i + \rho^2_i)^2}{\rho^2_i \rho^2_I} - 4 \right)^{1/2}, \]  
(2.8)
Here and below we use the standard color matrices \( \sigma^\pm_\mu = (\vec{\sigma}, \mp i)_{\mu} \). These matrices project the relative orientation matrix onto its upper 2 × 2 block. The scalar function \( F(\lambda) \) can be reduced to a simple one dimensional integral
\[ F(\lambda) = 6 \int_0^\infty \frac{dr r^{3/2}}{(r + 1/\lambda)^{3/2}(r + \lambda)^{5/2}}. \]  
(2.9)
Asymptotically, for large \( R \) the overlap matrix elements behave as \( T \sim \rho^2/R^3 \), and thus correspond to the exchange of a massless quark.

In this set of papers we are going to use a simplified instanton ensemble, the so called random instanton vacuum (RIV). We assume the following: (i) all instantons have the same size \( \rho_0 \); (ii) they have random positions and orientations; (iii) the instanton and anti-instanton densities are both equal to \( N/2V \), where \( V \) is the volume of the Euclidean space time and \( N \) is the total number of pseudoparticles.

Thus, there is essentially only one dimensionless parameter in the model \( f = \pi^2 V/N \rho_0^4 \) which describes ‘diluteness’ of the vacuum. All propagators and correlators to be considered, normalized with respect to their values for free massless quarks, are also given by dimensionless ratios. Those can only depend on \( f \), while the overall distance scale is given by a second parameter, e.g. by the ’average separation’ \( R \) defined by \( (V/N)^{1/4} \).

As suggested a decade ago in [8], the parameters are chosen such that several bulk properties of the QCD vacuum are reproduced. In order to obtain the traditional value for gluonic condensate one has to take \( R = 1.0 \, fm \) (inside uncertainties), whereas the quark condensate (and several other phenomenological parameters) are reproduced for \( \rho_0 = 1/3 \, fm \).

Somewhat surprisingly, already this simple model leads to a very reasonable description of many correlation functions. This does not mean that we have a full description of the QCD vacuum, and a number of disclaimers have to be made:
(i) First of all, the model does not include neither the (so far mysterious) field fluctuations leading to confinement, nor even perturbative fields, leading to phenomena like the Coulomb interaction or radiative corrections to ’asymptotically free’ propagation. The confining fields are supposed to modify our results substantially for quarks travelling far apart, while perturbative corrections are of \( O(\alpha_s(x)/\pi) \sim 10-20\% \) at relevant distances.

(ii) The overall good performance of the model does not mean that all results are good. For example, the model definitely ’overshoots’ the repulsion in the isovector scalar mesonic channel [19], predicting a negative correlator in some window of distances. It violates positivity and is obviously wrong.

(iii) Another sector, in which RIV does not perform correctly, is related to large-scale fluctuations of the topological charge. Obviously, a random distribution of instantons implies that in any volume \( V \) one expects to have excess of charge \( \delta Q \sim V^{1/2} \). In other words, the topological susceptibility \( \chi = <Q^2>/V \) is non-zero. This feature is known to be wrong in the case of massless fermions, where \( \chi \) should vanish for large volumes. Thus, one expects that the \( \eta' \) correlator evaluated in the RIV ’overshoots’ the right behaviour, which is indeed what was found [19].

The strong fluctuations of the topological charge can also be seen directly in the spectrum of the Dirac operator and in the quark condensate. The generic argument for the eigenvalue distribution goes as follows. If the limit \( N_c \to \infty \) is taken before the thermodynamic limit, the matrix elements behave as independent random variables resulting in a semi-circular eigenvalue distribution [15, 26]. However, if the thermodynamic limit is taken at finite \( N_c \), as it should, the number of matrix elements is much larger than the number of independent collective variables, leading to strong correlations between them. In such case one expects the resulting eigenvalue distribution to be gaussian [14, 20]. On the other hand, an excess \( \delta Q \) of topological charge may create extra quasizero modes, but the naive argument giving rise to a divergent spectral level density \( \rho(\lambda = 0) \) disagrees with our numerical simulations of the RIV. This is hardly surprising since the eigenvalues depend in a very nonlinear way on the matrix elements.

Numerical results for an ensemble of 2560 configurations of 128 instantons (dotted line)

---

5If one ignores the interaction with the zero modes of the ’\( Q = 0 \) background’, one finds for a topological charge excess of \( \delta Q \sim \sqrt{V} \) and corresponding overlap overlap matrix elements of \( T \sim \rho^2/V^{3/4} \) a level spacing of \( \sim 1/V^{5/4} \) leading to a divergent spectral density.
and 160 configurations of 512 instantons (full line) are shown in Fig. 1. In both cases the
density of instantons $N/V = 1$. The two spectra nearly coincide and are indistinguishable
for small $\lambda$, even if the bin size is decreased by an order of magnitude. This suggests that
the thermodynamic limit will not alter our results. The dashed line is the corresponding
gaussian distribution with variance given by $\sigma^2 = 2 < \text{Tr}(TT^\dagger)/N >$, where $< \cdots >$
denotes ensemble averaging. At small $\lambda$ our results suggest a strong departure from the
gaussian distribution, which is related to the strong fluctuations of the topological charge.
Although the spectrum itself does not become infinite, it seems to show an infinite slope
$\rho'(\lambda)$ at $\lambda = 0$. In terms of the normalized cumulants $\kappa_4/\kappa_2^2$ and $\kappa_6/\kappa_2^3$ with values of
0.180 and $-0.910$, respectively, the deviations from the gaussian distribution are much
less visible. We also have checked that a semi-circular distribution is obtained when the
overlap matrix elements are distributed as independent gaussian random variables with
zero mean.

The quark condensate, evaluated from the spectral density at $\lambda = 0$, is enhanced by
a significant factor, and fluctuations of the quark condensate, invoking the fluctuation-
dissipation theorem, diverge in the chiral limit. This phenomenon should not take place
if the inter-instanton interaction due to massless quarks is present. Indeed, simulations of
an ensemble of interacting instantons do not show this enhancement at $\lambda = 0$. A similar
artificial peak at $\lambda = 0$ should be observed in quenched lattice calculations, but it should
be absent in simulations with dynamical fermions. However, existing lattices are far too
small to contain hundreds of instantons and to reveal this phenomenon clearly.

In practice, the effect of this enhancement is small due to the finite quark masses
which smear out the small eigenvalues. Considering the peak at $\lambda = 0$ as an artifact
of the quenched approximation we will compensate for this by taking the quark masses
somewhat larger than their phenomenological values, see details below.

3. Quark propagators: generalities

In this section we discuss the quark propagator in a multi-instanton gauge field. Before

In the case of dynamical quarks we have another reason for not taking the quark masses too small:
at finite volume the smallest eigenvalues are $\sim 1/V$ leading to a dip in the spectrum at $\lambda = 0$. 
going into details, let us clarify the following important point. Strictly speaking, the propagator is not a physical quantity because it is \textit{gauge-dependent}. As discussed in detail in \textit{e.g.} [17], this problem may be circumvented \textit{à la} Schwinger by complementing its definition with the path ordered exponential $P \exp[(ig/2) \int_x^y A^a \sigma^a_{\mu} dx_{\mu}]$. Physically, this corresponds to the addition of a static anti-quark: thus one naturally proceeds to the discussion of heavy-light mesons (see below).

Although propagators are \textit{not} physical quantities, the specific feature of the 'instanton vacuum' (which contains neither perturbative nor confining vacuum fields!) is that this extra Schwinger factor is actually very close to one: static quarks essentially \textit{ignore} instantons. This statement was demonstrated explicitly in [17], and a related analytical argument was given in [27]. In this sense, one may say that our data on propagators are 'practically gauge invariant', and therefore are not purely academic quantities\footnote{Readers not satisfied by this argument should note that the main objects of these papers, the correlation functions, are manifestly gauge invariant by construction.}.

The total spinor propagator

$$S(x,y) = \langle x| -\frac{1}{i\gamma_\mu D_\mu + im} |y \rangle,$$

will be approximated by the sum

$$\sum_{I,J} \langle x|I\rangle \frac{-1}{i\gamma_\mu D_\mu + im} |J\rangle \langle J|y \rangle + \sum_{P,Q} \langle x|P\rangle \frac{-1}{i\gamma_\mu D_\mu + im} |Q\rangle \langle Q|y \rangle,$$

where $I$ and $J$ are zero modes, or the 'bound states' in terms of the analogy put forward in chapter 2, and $P$ and $Q$ are nonzero modes which can be thought of as 'scattering states'. The latter can only be numerated by a continuum of eigenvalues, \textit{e.g.} the momenta at large distances which makes the complete diagonalization of the Dirac operator a very complicated task. The simplifying assumption we have made is that the cross term between both type of modes is small and can be neglected. The first term follows from the Dirac operator in the space of zero modes (see eq. (2.5)) and is given by

$$S^{ZM}(x,y) = \phi_I(x) \left( \frac{1}{T-im} \right)_{IJ} \phi_J^*(y).$$

The second term is written as the sum of two contributions

$$\langle x| \frac{1}{-(\gamma \cdot D)^2 + m^2} i\gamma \cdot D |y \rangle' + im \langle x| \frac{1}{-(\gamma \cdot D)^2 + m^2} |y \rangle',$$
where the ′ indicates that the zero mode contribution should be excluded in the calculation of the expectation value. In the second term we will neglect the interaction of the spin-induced gluomagnetic moment with the field. This amounts to the replacement of $(\gamma \cdot D)^2 \rightarrow D^2$, and is exact for some correlators [28]. What remains is the propagator of a scalar quark which will be discussed in section 4. The first term is the most complicated one and we postpone its discussion until section 5.

The propagator in the approximation (3.1) still contains all possible gamma matrix structures. In other words, if we write

$$S = \sum a_i \Gamma_i \quad (3.5)$$

$$\Gamma_i = 1, \gamma_5, \gamma_\mu, i\gamma_5\gamma_\mu, i\gamma_\mu\gamma_\nu (\mu \neq \nu), \quad (3.6)$$

all coefficients $a_i$, which are $SU(3)$-matrices, are nonzero. However, only two coefficients have a nonzero average value: $a_1$ and $a_{\gamma_0}$ (we assume that the propagation takes place in time direction). On the other hand, the average of the combination $\text{Tr} a_i a_i^\dagger \neq 0$ for all structures. In fact, linear combinations of these quantities are nothing else but mesonic correlation functions which will be discussed in [19].

The ensemble average of the propagator will be evaluated numerically via a Monte-Carlo simulation of 256 instantons in a box of $3.36^3 \times 6.72 fm$. The distribution of the position of the instantons will be taken uniform, whereas the size is kept fixed at $0.35 fm$. The orientations are sampled from the invariant group measure. The average propagators $\langle S(x + \tau, x) \rangle$ are calculated by averaging over an ensemble of 50 configurations and over 100 randomly chosen initial points $x$ for each configuration and for each value of the separation $\tau$ from the initial point. To $u$ and $d$ quark masses are taken equal to $10 MeV$, which is still larger than the physical value. As discussed in section 2, the effect of the anomalous large number of small eigenvalues is suppressed this way (see Fig. 1). The strange mass is taken equal to the 'common sense' value of $140 MeV$.

4. Scalar quarks: propagators and mesons

In this section we study the propagator of a massless scalar quark (or squark, for
brevity). Its analytical expression for one instanton is known\[29\]:

\[
D(x, y) = \frac{1}{4\pi^2(x-y)^2} \frac{1}{\sqrt{1 + \rho^2/(x-z)^2}} \left[ 1 + \frac{\rho^2 \sigma^- \cdot (x-z) \sigma^+ \cdot (y-z)}{(x-z)^2(y-z)^2} \right] \frac{1}{\sqrt{1 + \rho^2/(y-z)^2}},
\]

(4.1)

where \(z\) denotes the position of the center of the instanton. The propagator in the field of an anti-instanton is obtained by interchanging \(\sigma^+\) and \(\sigma^-\). If the instanton is rotated in color space by the matrix \(R\), the propagator should be rotated by the matrix \(R^{ab} \sigma^b\).

The first factor \(D_0 = 1/4\pi^2(x-y)^2\) is the free scalar propagator. When all distances involved exceed the instanton size \(\rho\), the remaining factors approach 1, leaving only the free propagator. Slightly less trivial is the short-distance limit \((x-y)^2 \to 0\). Expanding in powers of \((x-y)^2\) we obtain

\[
D_I(x, y) = \frac{1}{4\pi^2(x-y)^2} + \frac{i\rho^2 \eta_{\mu\nu}^a (x-z)_{\mu} (y-z)_{\nu} \sigma_a}{4\pi^2(x-y)^2(x-z)^2(y-z)^2} - \frac{\rho^2}{8\pi^2(x-z)^2(y-z)^2} + \cdots \tag{4.2}
\]

and as the numerator of the second last term is proportional to \((x-y)\), the free propagator \(D_0\) dominates at small distances as well.

The very complicated problem of propagation in a multi-instanton environment is significantly simplified at not too large distances, where only one or a few close instantons contribute. Analyzing the expression given above, one realizes that significant corrections only appear if either \(x\) or \(y\) or both are well within an instanton, i.e. at a distance no more than \(\approx \rho\) from its center. In \[8\] only the effect of the 'closest' instanton was taken into account.

One might think that one could improve on this by treating the effect of the other instantons as small perturbations and simply adding their contributions,

\[
D_{sc} = D_0 + \sum_I (D_I - D_0). \tag{4.3}
\]

As discussed in \[28\], this correction is the first term of a 'rescattering' series. However, there is a serious problem: the integrated contribution of distant instantons diverges because of the term \(\int d^4z/(x-z)^2(y-z)^2\).

\[8\] Here and below the instanton field is assumed to be in the singular gauge, \(A^a_\mu = 2\eta^a_{\mu\nu} x_\nu \rho^2/[x^2(x^2+\rho^2)]\), where \(\eta^a_{\mu\nu}\) is the 't Hooft symbol.

12
In order to say more about of the origin of this term let us look at the expansion (4.2) in greater detail. When we allow for the possibility of a nonzero current squark mass the expansion of $1/(-D^{2}_\mu + m^2)$ in powers of the gauge field results in

$$
\frac{1}{-D^{2}_\mu + m^2}(x, y) = D_m(x, y) + \int d^4u D_m(x, u) i\sigma^a A^a_\mu (u-z) \partial_\mu D_m(u, y) + \cdots \quad (4.4)
$$

This expansion which also was considered in \[30\] provides us with the corrections of $O(\rho^2)$.

The interpretation of the correction in this formula is that the quark propagates to some point $u$ in space time, then exchanges 2 or more gluons with the center of the instanton (obtained from the expansion of the instanton profile in inverse powers of $(u-z)^2$) and finally proceeds to its end point. Therefore the factors $1/(x-z)^2$ and $1/(x-y)^2$, obtained after integration over $u$ do not have an unambiguous interpretation as the propagator of a gluon or of a squark. Both gluons and squarks acquire a mass due to non-perturbative effects, and therefore do not propagate far. Writing the propagator as

$$
D_I(x, y) = D_0(x, y) + \frac{1}{4\pi^2(x-z)^2} \delta D \frac{1}{4\pi^2(y-z)^2}, \quad (4.5)
$$

suggests a cure to our ‘infrared problem’: one should substitute the massless propagators by the massive propagators $D_M(x, z)$ and $D_M(y, z)$, which makes the integral convergent.

Since the divergence is logarithmic, the exact form of the cutoff is not very important. The value of the mass is certainly larger than the current quark masses. The natural scale for this cut-off mass $M \approx \Lambda_{QCD}$. In the case of the strange quark one might argue to add its mass to the cutoff mass, but since we are already considering a relative small correction, we will refrain from this kind of fine tuning.

Results for $\text{Tr } D(x, 0)$ as a function of the distance are shown in Fig. 2a. For convenience, $\text{Tr } D(x, 0)$ is normalized to the free massless scalar propagator $D_0 = 1/4\pi^2 x^2$, so the deviation from 1 is caused by the deflection of the squark in the instanton gauge field. The solid line corresponds to a massive scalar propagator, with a fitted mass value of $M_{\text{squark}} = 140 \text{ MeV}$. Note, that this value is roughly half the ‘constituent quark mass’, expected for spinor quarks from phenomenological considerations.

In Fig. 2b we show the simplest mesonic correlation function, obtained by averaging...

---

\footnote{This solution to our infrared problem is in fact similar to the well known diagram resummation in plasma physics \[31\], which takes care of Debye screening.}
the square of the squark propagator

\[ K(x) = \langle \text{Tr}(D(x,0)D(0,x)) \rangle. \]  

(4.6)

As usual, the data shown are normalized with respect to the perturbative correlator, and should be close to 1 at small \( x \). The solid line corresponds to the square of the massive propagator, fitted to data in Fig. 2a. If two quarks would propagate independently, the data would be in agreement with it. However, all measured points are above this curve, which means that scattering on instantons not only generates an effective squark mass, but also an attractive interaction. The fitted value of the meson mass is \( \approx 150 \text{MeV} \), which is less than \( 2M_{\text{squark}} \approx 280 \text{MeV} \). Of course, it is natural that due to such attractive interaction scalar quarks form mesons.

Note, that the origin of this attraction at small distances can be traced to the second term in the r.h.s. of (4.2): its average (e.g. over orientations) is zero, but not the average of its square. It is the simplest example of 'hidden' components of the quark propagators to be discussed in the next paper of this series [19].

Concluding this section, we have shown that the instanton vacuum can produce an effective squark mass and an interaction between them. This implies that similar effects can be expected for spinor quarks as well.

5. Effect of non-zero modes for spinor quarks

The main special feature of the spinor propagator is that it also receives a contribution from the zero modes. However, before discussing the complete propagator we first study the part of non-zero mode contribution that is given by the first term in eq. (3.4). It is known analytically for a massless quark in the field of a single instanton [29]:

\[ S_I(x,y) = \frac{1}{\sqrt{1 + \rho^2/x^2}} \frac{1}{\sqrt{1 + \rho^2/y^2}} \left( S_0(x,y) (1 + \frac{\rho^2 \sigma^- \cdot x \sigma^+ \cdot y}{x^2 y^2}) - D_0(x,y) \frac{\rho^2}{x^2 y^2} \left( \frac{\sigma^- \cdot x \sigma^+ \cdot \gamma \sigma^- \cdot \Delta \sigma^+ \cdot y}{\rho^2 + x^2} \gamma_5^+ + \frac{\sigma^- \cdot x \sigma^+ \cdot \Delta \sigma^- \cdot \gamma \sigma^+ \cdot y}{\rho^2 + y^2} \gamma_5^- \right) \right), \]  

(5.1)

where we have introduced the projectors \( \gamma_5^\pm = (1 \pm \gamma_5)/2 \). The massless free quark
propagator is denoted by \( S_0(x, y) \), and the free massless scalar propagator, \( D_0(x, y) \) is
given by the first term in the r.h.s. of eq. (4.6). The expression for anti-instantons
is obtained by interchanging \( \sigma^+ \) and \( \sigma^- \). The \( SU(3) \)–extension of the propagator is
obtained by embedding this propagator in the upper 2 \( \times \) 2 block of a 3 \( \times \) 3 matrix, and
substituting the free propagator in the remaining diagonal matrix element. For arbitrary
orientation, \( U \), of the instanton \( S_I(x, y) \) has to be rotated covariantly.

We generalize \( S_I(x, y) \) to many instantons, in the same approximate way as in the
previous section for squarks, namely by summing all deviations from the free propagator:

\[
S_\gamma(x, y) = S_0(x, y) + \sum_I (S_I(x, y) - S_0(x, y)).
\] (5.2)

Although in this case the contribution of distant instantons is not divergent, we still think
it should be excluded on the basis that gluons cannot propagate that far. If the NZM-
induced effects are treated as a correction, it is logical to also include here an effective
mass to cut off the contribution of distant instantons.

The last necessary step is to include the effects of a non-zero bare quark mass: this
is especially needed for the discussion of correlators involving strange quarks. Generally
speaking, there are at least three different scales to which the strange quark mass, \( m_s \),
should be compared:

(i) The eigenvalues of the Dirac operator in the ZMZ which are comparable to \( m_s \). There-
fore, one should be careful at this point and not treat \( m_s \) as a small parameter\(^\text{10}\); the
mass in the strange component of the denominator of the ZMZ term in the propagator
will be taken equal to \( m_s \).

(ii) The field strength of the gauge field inside instantons. Its value is very large:
\[
\sqrt{G_{\mu\nu}^2} \sim 8\sqrt{3}/g\rho^2 \sim 2\text{GeV}^2 \gg m_s^2 \sim .02\text{GeV}^2,
\]
and therefore the motion of strange quarks inside instantons is essentially the same as that of massless ones.

(iii) The total (Euclidean) propagation time. In this work this time is comparable to the
strange quark mass (\( \tau \sim 1 - 2\text{fm} \sim m_s^{-1} \)), which makes the strange quark propagators
substantially different from non-strange ones. However, in a reasonably dilute instanton
vacuum most of this motion takes place in relatively empty space, and one may therefore
describe these corrections using the free massive propagator.

\(^{10}\)Note that at this point we deviate from what is done in chiral perturbation theory.
In such a complex situation, the problem is to work out a useful *interpolation formula*, which combines several limiting cases for which an analytical solution is known and presumably provides a reasonable description in the general case. After trying several expressions, we arrived at the following expression in which factorization of the mass-related damping of the propagator from corrections of scattering on instantons is assumed:

\[ S_{\gamma}(m, x, y) = S_{m}(x, y) + D_{M}(x, 0)\delta S(x, y)D_{M}(0, y). \]  

The correction \( \delta S \) is given by

\[ \delta S(x, y) = \frac{S_{\gamma}(x, y) - S_{0}(x, y)}{D_{0}(x, 0)D_{0}(0, y)}, \]  

and \( S_{m}(x, y) \) is the *massive* free fermion propagator. At small distances (inside instantons) the propagation of quarks approaches free propagation, while at large \( |x - y| \) the quarks move as free massive fermions in between successive scatterings on instantons while the correction \( \delta S \) is exponentially suppressed. We will choose \( M = \Lambda_{QCD} \) which we believe to be a natural choice for this cut-off mass.

Finally we proceed to the expression for the total propagator \( S(x, y) \) by adding the contributions discussed is this section and in section 3 and 4. As a result we find

\[ S(x, y) = S_{\gamma}(m, x, y) + \phi_{I}(x) \left( \frac{1}{T - im} \right) \phi_{I}^{*}(y) + imD_{sc}(x, y), \]  

where \( D_{sc} \) is defined in eqs. (4.3) and (4.5) modified as discussed below eq. (4.5).

In Fig. 3 we show the chirality flip \( \text{Tr} S \) (a) and the chirality non-flip \( \text{Tr} \gamma_{0}S \) (b) components of this propagator. The squares represent results for the full propagator, whereas the crosses are for the propagator \( S(x, y) = S_{m}(x, y) + S^{ZM}(x, y) \), which includes only the modification of the propagator due to zero modes. The first trace \( \text{Tr} S \) is normalized with respect to the short distance limit of the massive free quark propagator \( \text{Tr} S_{0}(x, m)/m \) such that its value at \( x = 0 \) is equal to the mass \( m \). The second trace is normalized with respect to the free propagator. One observes that their effect is significant, especially at medium distances \( x \sim 1/2fm \).

The physical question we are going to discuss is whether or not these results can be interpreted as the appearance of some *universal* effective mass \( m_{eff}^{11} \). For this reason we

\[ ^{11} \text{The existence of a momentum dependent effective mass has also been obtained analytically in the large } N_{c}-\text{limit} \]
have plotted the behaviour of the massive fermion propagator for $m = 200, 300, 400 \, MeV$.
At small $x$, the calculated points start from zero, consistent with bare quark propagation
with a small input mass. However, for $x > 0.5 \, fm$ they even somewhat 'overshoot' the
curves, but qualitatively imitate their $x$–dependence. Since none of the curves is par-
ticularly close to the data trend it is not possible to select a 'preferable' mass from the
chirality flip amplitude.

The chirality non-flip component of the propagator, shown in Fig. 3b, tends to 1 at
small $x$ (which corresponds to massless propagation), but at larger $x$ we do find qualitative
agreement with the massive free propagators. The square points follow the pattern given
by the solid line. This means that the net effect of the 'instanton vacuum' on quark
propagation can indeed be represented approximately by the appearance of an effective
mass of the order of $300 \, MeV$. This is about twice larger than for the scalar quark.
Indeed, when only the contribution of the zero modes is included, as shown by the crosses
in Fig. 3b, the 'effective mass' is about twice smaller. This shows that roughly half the
constituent mass is due to the zero modes and the other half due to the non-zero modes.

Note that the curves in Fig. 3 imply that the amplitude of 'bare-to-dressed' quark
transition amplitude (the $Z$-factors introduced in [10]) is exactly 1, so this feature of '100
percent dressing' is approximately reproduced by our propagators as well.

In conclusion, there is some qualitative agreement with the 'constituent quark' model.
However, one should not take it literally. First of all, we remind that agreement is not
really good for the chirality-flip amplitude shown in Fig. 3a. And, last but not least, if two
(or more) quarks propagate together, they show a very strong interaction. Therefore, as
we will show in future publications, none of the mesonic or baryonic correlators measured
are actually reproduced by the naive 'constituent quark' model.

6. Heavy-light mesons

Heavy-light mesonic correlators were first discussed by one of us in [32], where the
concept of 'heavy quark symmetry' between any channels with sufficiently heavy quark
masses was introduced. Indeed, in the heavy quark limit $M_Q \gg \Lambda_{QCD}$ the heavy quark
behaves as a static center (the mesons’ ‘nucleus’), and the corresponding correlation functions essentially reduce to the propagator of the light quark. As the direction of the spin of heavy quark is irrelevant in this static limit, a specific degeneracy of correlation functions follows. For example, the spatial component of the vector correlator coincides with pseudoscalar correlator, and the spatial component of the axial correlator coincides with scalar correlator.

More specifically, we discuss correlators of currents containing one heavy $Q = (c, b, t)$ and one light quark $q$ defined by the expectation value

$$K_\Gamma(x) = < \bar{Q} \Gamma q \bar{q} \Gamma Q >, \tag{6.1}$$

where $\Gamma$ is one of the possible gamma matrix structures. In this section we consider $1$ (S), $\gamma_5$ (P), $\vec{\gamma}$ (V) and $\vec{\gamma}\gamma_5$ (A). The heavy quark propagator that enters (6.1) is given by the nonrelativistic quark propagator times a path ordered exponential. In [17] it was shown that the correction due to the path ordered exponential is small. This allows us to translate the propagator immediately into heavy-light correlation functions. Using that only the large components of the heavy quark spinor are relevant one obtains the correlator

$$K_\Gamma(x) = i \text{Tr}(S(-x) \Gamma \frac{1 + \gamma_0 x/|x|}{2}) \tag{6.2}$$

times a non-relativistic quark propagator. The separation $x$ is chosen along the positive time axis. Only the parity of the current is essential, and for the parity $P = \pm 1$ channels with $\Gamma = 1, \gamma_5$ one obtains the following correlation functions

$$K_{\pm}(x) = i \text{Tr}[\frac{1 \pm \gamma_0}{2} S(-x)]. \tag{6.3}$$

As was noticed in [32], the splitting of these correlators at small distances, normalized to the free quark correlator $K_0$, is simply proportional to the quark condensate

$$K_{\pm}(x)/K_0(x) = 1 \mp \frac{\pi^2 x^3}{6} |< \bar{\psi} \psi >| + \cdots. \tag{6.4}$$

In order to describe the correlator in the whole region, we use a standard parametrization for the spectral function, a resonance plus the perturbative continuum above a certain 'threshold' energy $E_0$, in this case given by [32]:

$$\text{Im}K_{\pm}(E) = 6\pi n \delta(E - E_{\text{res}}) + \theta(E - E_0) \frac{3E^2}{2\pi}, \tag{6.5}$$
where \( n = f_Q^2 M_q / 12 \) is the 3-dimensional density of the light quark at the center. In terms of the space-time correlator, it translates into the following expression:

\[
K_\pm(x) / K_0(x) = 2\pi^2 n x^3 e^{-E_{res} x} + (1 + E_0 x + E_0^2 x^2 / 2) e^{-E_0 x}.
\]  
(6.6)

Our results for the propagator, in the form \( K_\pm(x) / K_0(x) \), are shown in Fig. 4, where the symbols \( V \) and \( A \) refer to the spatial components of the vector and axial currents, respectively. The dashed curves correspond to the three-parameter fit (6.6), with the values of the parameters given in Table 1. For comparison we also give results of previous works using various methods\(^{12}\).

The resonance position can also be compared to the experimental values for \( B \) meson masses, provided that the \( b \) quark mass value is obtained from different sources (e.g. sum rules for upsilons). Assuming that \( m_b \approx 4800 \text{ MeV} \), as follows from such sources, one obtains for \( E_{res} = m_B - m_b \approx 475 \text{MeV} \), which is not too far from our fitted value of 615 MeV.

The splitting of opposite parity states is only known for charmed mesons (such states have not yet been discovered for \( B \) mesons): it is about 450 MeV (see discussion is \([1]\)), to be compared to our estimated difference of 575 MeV. This value is very close to twice the constituent quark mass, a result which also was derived in \([34]\) using completely different methods.

It is instructive to consider the small distance expansion of the correlation function (6.6),

\[
K(x) / K_0(x) = 1 + (2\pi^2 n - \frac{1}{6} E_0^3)x^3 + \left(\frac{1}{8} E_0^4 - 2\pi^2 n E_{res}\right)x^4 + \cdots
\]  
(6.7)

Equating this to (6.4) we find the so called ‘duality relation’,

\[
\pm \frac{\pi^2}{6} < \bar{\psi} \psi > = (2\pi^2 n^\pm - \frac{1}{6} E_0^{3\pm}),
\]  
(6.8)

which should be approximately satisfied. From the r.h.s. we find the value for the quark condensate of \(-(215\text{ MeV})^3\) in case of the \(0^-\) state and \(-(578\text{ MeV})^3\) in case of the \(0^+\) state. The discrepancy between these two numbers is a signature of an early deviation of our fit and, possibly, the true correlator from the predictions of the operator product

\(^{12}\) Comparison of compiled lattice results with results from QCD sum rules, as well as the discussion of \(1/m_Q\) corrections can be found e.g. in \([33]\).
expansion. This also follows from the right hand side of the dispersion relation which can be expanded in powers of $x$ for $x \ll 1/E_0 \sim 0.2 f m$ only.

7. Conclusions and discussion

In this work we report an extensive numerical study of the simplest ensemble of instantons: the random instanton vacuum. This vacuum is characterized by two parameters: (i) the size $\rho_0$ and (ii) the density $N/V$ of instantons (plus anti-instantons) in the QCD vacuum, which were chosen to reproduce the size of the quark condensate and the gluon condensate.

The topological charge density in this model fluctuates strongly on all length scales, which shows up in the form of an excess of small eigenvalues of the Dirac operator above the expected gaussian distribution.

For simplicity, we first studied the propagation of scalar quarks (or squarks), in which case there is only a nonzero mode contribution to the propagator. It appears that the scalar mesonic correlation function can be described remarkably well by a 'constituent quark' model with a mass of about $140 \text{ MeV}$.

The spinor quark propagator has two different components: the chirality non-flip and the chirality flip parts. The first one can again be described by an effective quark mass of $m_{\text{eff}} \approx 300 \text{ MeV}$. The second component is more complicated and corresponds approximately to a free massive propagator at distances larger than about $1/2 f m$. At distances of $x \approx 0.5 - 1.0 f m$ both the zero and non-zero fermion modes contribute a comparable amount to the constituent quark mass.

Observable consequences of these statements are most clearly seen in the spectra of heavy-light mesons. We find a mass difference of $615 \text{ MeV}$ between the $B-$meson and the bottom quark which agrees well with the phenomenological value. The splitting between the mass of opposite parity states was found to be $575 \text{ MeV}$, which is roughly equal to
twice the constituent quark mass.

Acknowledgements

The reported work was partially supported by the US DOE grant DE-FG-88ER40388. We acknowledge the NERSC at Lawrence Livermore where most of the computations presented in this paper were performed.
References

[1] E. Shuryak, Rev. Mod. Phys. (1993) (in press).

[2] A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys.Lett. 59B, 85 (1975).

[3] G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976).

[4] G. ’t Hooft, Phys.Rev. D14, 3432 (1976).

[5] C. Callan, R. Dashen, and D. Gross, Phys. Rev. D17, 2717 (1978).

[6] B. Geshkenbein and B. Ioffe, Nucl. Phys. B166 (1980) 340.

[7] V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, Nucl.Phys. B191, 301 (1981).

[8] E. Shuryak, Nucl. Phys. B203, 93, 116, 140 (1982).

[9] E. Shuryak, Nucl. Phys. B214, 237 (1983).

[10] C. Callan, R. Dashen, and D. Gross, Phys. Rev. D19, 1826 (1979).

[11] R. Carlitz and D. Creamer, Ann. Phys. (NY) 118, 429 (1979).

[12] E. Ilgenfritz and M. Mueller-Preussker, Nucl. Phys. B184 (1981) 443.

[13] C. Carneiro and N. McDougall, Nucl. Phys. B245 (1984) 293.

[14] D. Diakonov and V. Petrov, Nucl. Phys. B245, 259 (1984).

[15] D. Diakonov and V. Petrov, Nucl. Phys. B272 (1986) 457.

[16] E. Shuryak, Nucl. Phys. B319 (1989) 521, 541.

[17] E. Shuryak, Nucl. Phys. B328 (1989) 85, 102.

[18] M. Chu, J. Grandy, S. Huang, and J. Negele, MIT Preprint CTP-2113 (1992).

[19] E. Shuryak and J. Verbaarschot, Mesonic correlators in a random instanton vacuum, Stony Brook preprint NTG-92/40 (1992).
[20] T. Schaefer, E. Shuryak and J. Verbaarschot, *Baryonic correlators in a random instanton vacuum*, Stony Brook preprint NTG-92/41 (1992).

[21] T. Banks and A. Casher, Nucl. Phys. B**169** (1980) 103.

[22] H. Leutwyler and A. Smilga, Phys. Rev. D (1992) (in press).

[23] S. Hands and M. Teper, Nucl. Phys. B**347** (1990) 819.

[24] M. Chu and S. Huang, Caltech Preprint MAP-138 (1992).

[25] E. Shuryak and J. Verbaarschot, Phys. Rev. Lett. **68** (1992) 2576.

[26] T. Brody et al., Rev. Mod. Phys. **53** (1981) 385.

[27] D. Diakonov, V. Petrov, and P. Pobylitsa, Phys. Lett. B**226** (1989) 387.

[28] N. Andrei and D. Gross, Phys. Rev. D**18D**, 468 (1978).

[29] L. Brown, R. Carlitz, D. Creamer, and C. Lee, Phys. Rev. D**17** (1978) 1583.

[30] H. Levine and G. Yaffe, Phys. Rev. D**19** (1979) 1225.

[31] M. Gell-Mann and K. Brueckner, Phys. Rev. D**19** (1957) 364.

[32] E. Shuryak, Nucl. Phys. B**198** (1982) 83.

[33] V. Eletsky and E. Shuryak, Phys. Rev. Lett. (1992).

[34] M. Nowak and I. Zahed, Stony Brook Preprint SUNY-NTG-92/27 (1992)

[35] G. Martinelli, Rome University preprint ROME-799-1991 (1991).

[36] L. Maiani, Helv. Phys. Acta **64** (1991) 853.
\[ n = f^2 M_Q / 12, \quad [fm^{-3}] \]

| channel $J^P$ | $E_{\text{res}}$ [MeV] | $E_0$ [MeV] | ref. | comment |
|---------------|----------------------|-------------|------|---------|
| $0^-, 1^-$    | 615                  | 1010        | this work | random instantons |
| 400±100       | 1±.5                 | 900±100     | 32   | QCD sum rules |
| 480±80        | 1±.3                 | 700         | 34   | IIA |
| 500±30        | 2±.5                 | 1100±100    | 38   | sum rules |
| - - -          | 4.5±1                | - - -       | 35, 36 | lattice, ’static’ |
| - - -          | 2.5±1.               | - - -       | 35, 36 | lattice, ’dynamic’ |
| $0^+, 1^+$    | 1190                 | 1745        | this work | random instantons |
| 1200±200      | 6.3                  | 1800±200    | 32   | QCD sum rules |
| 1000±200      | .7±.5                | 1200±200    | 17   | IIA |

Table 1.

Fitted parameters of heavy-light mesons, compared to those derived in previous works. $E_{\text{res}}$, $n$, $E_0$ are the resonance mass (counted from the heavy quark mass), the density of the light quark at the center and the ’asymptotic freedom threshold’ (see text). For comparison, we give some lattice results based on a static heavy quark and a $1/m_Q$ expansion, as well as results based on the extrapolation of simulations with dynamical quarks.
Figure Captions.

Fig 1. The eigenvalue density $n(\lambda)$ of the Dirac operator in the space of zero modes for an ensemble of 512 and 128 instantons. The normalization $\int_{-\infty}^{\infty} d\lambda \rho(\lambda) = 1$ and the bin size used to obtain the histograms was 0.025. The variance of the gaussian curve is equal to the variance of $n(\lambda)$ for $N = 512$.

Fig. 2. The average propagator of a scalar quark (a) and its averaged square (b) (corresponding to the scalar meson), normalized to the propagator of a massless quark. The solid lines in (a) and (b) correspond to a propagator with a fitted mass value of 140 $MeV$.

Fig. 3. The chirality-flip (a) and non-flip (b) components of the quark propagator versus the distance $x$ (in $fm$). The normalization, indicated in the figure, is discussed in the text. Crosses in (b) correspond to the effects of zero-modes only, while the squares give the complete result. Three lines, the short-dashed, solid and long-dashed correspond to a massive free propagator with a mass of 200, 300 and 400 $MeV$, respectively.

Fig. 4. The correlation functions for negative ($P =$ pseudoscalar, $V =$ vector (spatial)) and positive ($S =$ scalar, $A =$ axial (spatial)) parity heavy-light mesons. The data points are our results, while the dashed curves are fits, discussed in the text and in Table 1.
