Analysis of Research Parameters for Cascading Failures of Interdependent Network Based on the Giant Component

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Abstract. With the deepening of the research on complex network, the cascading failures problem of the interdependent network is one of the hot research issues in the field. Setting reasonable parameters in the cascading failures analysis of interdependent network is of great significance for subsequent research. The mechanism of common models generation is analyzed, and its formula expression form is constructed. Starting from the single network cascading failures pattern, combined with the interdependent network theory research, a cascading failures model of interdependent network based on the giant component is established. The robustness analysis of interdependent networks with different network scale and different average degree is carried out for three common interdependent networks: BA-BA, WS-WS and ER-ER. The best condition for the robustness study of the interdependent network is obtained. That is, the sub-network node scale \( N_S = N_C \geq 100 \) and the average degree \( k = 6 \). The conclusions obtained in this paper can provide a reference for the study of cascading failures in power, communication and other interdependent networks.

1. Introduction
After decades of development, the theoretical system of complex network research has been gradually established and improved, but these theories still have certain limitations. Most of the current research focuses on the cascading failures of a single network, but most real systems do not exist in isolation in real life. The normal operation of nodes in one system may be interdependent with some nodes in other systems, and its failures may cause other nodes to fail to operate normally, which may lead to large-scale cascading failures problems. A typical case is the Italian blackout that occurred on September 28, 2003. Due to the close coupling between the power network and the communication support network, the collapse of a power station caused a large number of nodes in the data acquisition and monitoring communication network to fail, causing more power plant failures, which in turn led to large-scale power outages. For such a coupled network in which the power network and the communication network have mutual dependencies, it is called interdependent network.

The research on the robustness of interdependent network is a new hotspot in the research of complex networks in recent years. Many scholars at home and abroad have applied this model to the structural modeling and robustness research of power networks, communication networks and transportation networks. In 2010, Buldyrev et al.\textsuperscript{[1]} first proposed a theoretical framework for the problem of cascading failures of interdependent network under random failures of nodes, which
opened a new direction for the study of cascading failures in interdependent network. In order to study the influence of coupling probability on interdependent network, Parshani et al.[2] proposed a set of classical theoretical methods. Cheng et al.[3] studied the effects of different modes between interdependent network on network cascading failures. Rahnamay-Naeini et al.[4] proposed a new interdependent Markov chain framework, which can extract the dependence relationship between two critical infrastructures, and then predict the resilience of cascading failures, and describe the dependence of the relationship on system reliability influences. Wang et al.[5] analyzed the cascading failures problem of interdependent weighted networks based on seepage theory and local weighted traffic redistribution rules. Gao et al.[6] analyzed the cascading failures and robustness of four interdependent network under different coupling modes under six different attack strategies. At present, scholars in various fields have done different levels of research on the problem of cascading network failures[7-12].

There are various types of interdependent networks in real life, and the network scale, connection relationship, and degree of association between nodes are not the same. In the study of interdependent network cascading faults, the different parameters of the network may also lead to the difference of the simulation results, resulting in the phenomenon that the data is not objective or the difference is not obvious. The number of network nodes exists from tens, thousands to millions, but most of the literatures on the cascading failures of interdependent network are not uniform for the scale of the network. For example, the number of sub-network nodes is 500 in literature [13], the literature [1] is 1000, and the literature [14] is 100. Generally speaking, in the simulation experiment, the small-scale network will not be able to objectively reflect the intrinsic characteristics of the real-dependent network. However, the large-scale network is close to most real-life networks, which is bound to bring about a geometric increase in the amount of calculation. The average degree of network has a great impact on the robustness of interdependent network. A network with a small average degree is very vulnerable in interdependent network cascading failures, which may cause the network to encounter any fault. The greater the average degree, the better the robustness of the network, and the inability to exhibit the objective phenomenon of cascading failures and provide a reference for practical research.

Therefore, setting reasonable network parameters in the cascading failures analysis of interdependent network is of great significance for subsequent research. This paper analyzes the principle of cascading failures of interdependent network generation, and uses the appropriate robustness measure to study the influence of different parameters on the robustness of the interdependent network from the perspective of the number of nodes and the average degree, and then find the best research parameters.

2. Common complex network model generation mechanism

The model of the network characterizes the connection mode between nodes and nodes in the actual network. Here are some common network models.

2.1 Regular network and completely random network

Commonly used in rule networks are globally coupled networks, nearest neighbor coupled networks, and star networks. A globally coupled network means that any two nodes in the network have connected edges, as shown in Figure 1(a). It has the smallest average path length and the largest clustering factor. The nearest neighbor coupled network means that the nodes in the network are only connected to the (even) nodes around it, as shown in Figure 1(b). The average path length and clustering coefficient are:

$$L \approx \frac{N}{2K} \rightarrow \infty (N \rightarrow \infty)$$  \hspace{1cm} (1)

$$C = \frac{3(K-2)}{4(K-1)} \approx \frac{3}{4}$$  \hspace{1cm} (2)
The star network is characterized by the existence of a central point, and the remaining points are connected to the center point, as shown in Figure 1(c). The average path length and clustering coefficient are:

\[ L \approx 2 \frac{2(N-1)}{N(N-1)} \rightarrow 2(N \rightarrow \infty) \]  

(3)

\[ C = \frac{N-1}{N} \rightarrow 1(N \rightarrow \infty) \]  

(4)

\[ \text{(a) Globally coupled network} \quad \text{(b) Nearest coupled network} \quad \text{(C) Star network} \]

Figure 1. Several common rule networks

The ER random graph theory established by two Hungarian mathematicians Erdos and Renyi is the most classical model in random networks[15]. After the end of the late 1950s, in the 40 years to the end of the 20th century, large-scale networks were mainly described by ER random graphs. The construction method is also relatively simple: given the number of network nodes \( N \), the maximum number of connected edges between all nodes is \( \binom{N}{2} \), and randomly selects \( W \) edges to form a random network. The degree distribution is subject to the binomial distribution:

\[ p(k) = C_{N-1}^k p^k (1-p)^{N-1-k} \]  

(5)

When \( N \) is larger, its degree distribution also approximates Poisson distribution.

\[ p(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!} \]  

(6)

The average shortest path and agglomeration coefficient are:

\[ L \sim \ln N / \ln(pN) \]  

(7)

\[ C \sim p \]  

(8)

In addition to the ER random network, there is a random regular network with the same degree of each node, referred to as a RR Random Network.

2.2 Small World Network

The WS Small World Network is a network between a regular network and a random network. Its construction process starts from the nearest neighbor coupling network with \( N \) nodes. Each node is connected with its nearest \( k \) nodes, and then each link edge is randomly reconnected with probability \( p \), while ensuring there are no self-loops and repetitive edges. Therefore, in the WS small world network, the parameter \( p \) represents the degree of network randomization, and different reconnection probability represents different networks, as shown in Figure 2.
From the above construction process, it can be concluded that the average path length and clustering coefficient change with the change of random probability \( p \), and the analytical expression of the clustering coefficient is:

\[
C(p) = \frac{3(K-1)}{4(K-1)}(1-p)^3
\]  

(9)

So far, there is no exact calculation expression for the average shortest path of the WS small world network, so Newman et al. give an approximate formula:

\[
L(p) = \frac{2N}{K} f(NKp / 2)
\]  

(10)

\[
f(x) \approx \frac{1}{2\sqrt{x^2 + 2x}} \arctan \sqrt{\frac{x}{x+2}}
\]  

(11)

Using the above formula, the average path length and clustering coefficient can be simulated as a function of random probability \( p \), as shown in Figure 3:

![Figure 3. WS small world network average path length and clustering coefficient](image)

When \( p = 0 \), the nearest neighboring coupling network has a larger clustering coefficient \( C \approx 0.75 \) and a longer average shortest path \( L \approx \frac{N}{2K} \). As can be seen from Figure 3, as the degree of randomization \( p \) increases, the average shortest path of the WS small-world network drops rapidly, while the clustering coefficient is basically unchanged. This characteristic is consistent with the two features of smaller average shortest path and larger clustering characteristics. The network with such features is a small world network.

### 2.3 Scale-free Network

In 1999, A. L. Barabasi and R. Albert proposed another network that is more widespread in real life – scale-free network. The characteristic of this network is that its degree distribution function appears as a power law distribution, and its node connection degree has no specific length. The construction
algorithm of BA scale-free network is mainly divided into two steps: ① Growth: starting from a network with an initial number of \( m_0 \) nodes, each time a new node is introduced, and connected to \( m(m < m_0) \) existing nodes, a total of \( t \) is added. Nodes. ② Priority connection: The newly added node is connected to the original node \( i \) by probability \( P_i = \frac{k_i}{\sum_j k_j} \), where \( k_i \) is the degree of node \( i \). Obviously, the average degree of BA scale-free networks is \( 2m \).

The average shortest path and clustering coefficient of the network are:

\[
L \sim \frac{\log N}{\log \log N}
\]

\[
C = \frac{m^2(m+1)^2}{4(m-1)} \left[ \ln\left(\frac{m+1}{m}\right) - \frac{1}{m+1} \right] \left[ \ln(t) \right]^2
\]

(12)

(13)

Its degree distribution function is shown in Figure 4:

![Figure 4. Degree distribution function of BA Scale-freeNetwork](image)

**3. Simulation analysis method for cascading failure of interdependent network**

### 3.1 Analysis of cascading failures of single network

First, the process of single complex network cascading failures is introduced. The robustness of a single complex network refers to the ability of the entire network to maintain its structural and functional integrity after the node is destroyed.

Therefore, the basic idea of complex network cascading failures theory and simulation analysis is to change the number or proportion of nodes destroyed, and then study the ability of the network to maintain robustness at different scales.

After some nodes in the general network are destroyed, they will show two forms. One is that there is a giant component, then the network structure is considered to be functional, as shown in Figure 5(a); the other is that there is no giant component. The network is considered to be completely paralyzed, as shown in Figure 5(b).
The parameter that measures the robustness of a complex network is generally the scale of the giant component remaining after the cascading failure. As shown in Figure 5(a), there are 16 nodes in the network. Now remove 3 red nodes. The remaining giant components in the network are the yellow shaded parts in Figure 6. If the proportion of nodes removed in the network is defined as $p$, then the proportion of reserved nodes is $1 - p$, and the proportion of the remaining giant components to the total number of nodes in the network after the cascading failures occurs is $P_\infty$. For a complex network with an infinite number of nodes, when changing the proportion $p$ of nodes removed in the network, the change of the proportion of nodes remaining in the complex network is shown in Figure 5(b). It can be seen that only when the proportion $p \leq p_c$ of the nodes is removed, the network giant components are retained, and the system presents a second-order phase transition. This key point $p_c$ in the network giant component is called the seepage threshold and is another important parameter to measure the robustness of the network. When the value of the seepage threshold is larger, the robustness of the network is better, and it can resist a large number of node deletions; when the value of the seepage threshold is small, the robustness of the network is worse.

Therefore, there are two main parameters for measuring the robustness of the equipment system network based on the scale of the remaining giant components of the network. One is the scale of the giant component in the network after the cascading failure, and the other is the seepage threshold. The larger the value of the former, the smaller the value of the seepage threshold, the better the robustness of the network, and the higher the reliability of the equipment system. It has a strong correlation with the giant component, so this paper studies the interdependent network cascading failures problem based on the giant component.

### 3.2 Interdependent network generation principles

In general, an interdependent network is formed by two or more sub-networks coupled together. The edge inside each subnet is called the inner connection edge, and the edge between the sub network and the sub network is called the coupling interdependent edge. Depending on the connection of the
coupled interdependent edges, the interdependent network can be divided into four forms: one-to-one interdependent network, multiple corresponding interdependent network, directed corresponding interdependent network, and partially corresponding interdependent network, as shown in Figure 7.

Buldyrev et al.[1] first proposed a theoretical framework for the problem of cascaded faults in a network with random failures, which opens up a new direction for the study of cascaded network cascading faults. This paper assumes that there are two sub-networks A and B with the same node, and the form of the coupled interdependent edges is a one-to-one correspondence. That is, there is a coupling interdependent edge between node $A_i$ in network A and node $B_i$ in network B. When node $A_i$ in network A fails, the failure is transmitted to network B through the coupling interdependent edge, and node $B_i$ also fails; otherwise, when network B When node $B_i$ fails, the failure is passed to network A through the coupling interdependent edge, and node $A_i$ also fails. Within each subnet, the model considers only the nodes belonging to the largest connected subgroup to be valid, and the remaining nodes will fail, which in turn is passed to the other network through the coupled interdependent edge. The specific cascading failures process is shown in Figure 8.

Figure 7. Four different ways of interdependent network
A node in network A is removed.

The first step of the cascade failure process, network A splits into 3 subgroups.

The second step of the cascade failure process, in the remaining huge components of the dependent network, the network B splits into two subgroups.

End of cascading failure.

Figure 8. Interdependent network cascading failures process

From the above model analysis, the interdependent network is much more vulnerable than the single network in resisting the cascading failures robustness. Theoretical analysis also shows that the cascading failures process of the one-to-one correspondence interdependence network appears as a first-order phase transition, while the cascading failures process of a single network appears as a second-order phase transition, as shown in Figure 9.

Figure 9. Two different types of phase transitions depending on interdependent network level

In order to improve the robustness of the interdependent network, the number of coupled interdependent edges can be reduced, so that some nodes have no interdependent edges, which can make the seepage phase transition change from the first phase transition to the second phase transition. By reducing the proportion of interdependent nodes, we can clearly see that the seepage phase transition changes from a first-order phase transition to a second-order phase transition, as shown in Figure 10.
3.3 Interdependent network cascading failures model based on giant component

The coupled interdependent network studied in this section consists of two interdependent sub-networks $S$ and $C$. Suppose two networks have the same number of nodes, i.e. $N_S = N_C$. And assuming that the sub-networks have the same topology, three different network structures of Watts-Strogatz (WS) small world network, Barabasi-Albert (BA) scale-free network and Erdos-Renyi (ER) random network are studied.

When constructing the WS small world network, take the probability $q = 0.5$; when constructing the ER random network, take the probability $q = 1$. At the same time, the number of neighbors $K$ of closest the coupled network is the average degree of the WS small world network and the ER random network. It was analyzed in section 2.3 that the average degree of BA scale-free networks is $2N$.

A necessary and sufficient condition for any point $u \in V_s$ in the sub-network $S$ to be a valid node is that the node belongs to the giant component of the sub-network $S$ and at least one interdependent edge of the node connects the valid nodes in the sub-network $C$. This is similar for any node in the sub-network $C$. In order to trigger the cascading failures of the interdependent network, we remove the node of the sub-network $S$ with a ratio of $p$ by some rule (randomly or deliberately), and then for the sub-network $S$, we need to rediscover the giant components in the network. All nodes that are not connected to the giant component will be removed; then, remove the node in sub-network $C$ that does not have an interdependent edge; then, the giant components in sub-network $C$ need to be redefined. Nodes that are not part of the giant component will be removed again; the removed node will be passed to the network $S$,..., which triggers a wider range of cascading failures.

In order to characterize the robustness of the entire interdependent network, we define $G$ as the proportion of nodes remaining in the interdependent network after the cascading failures occur, i.e:

$$G = \frac{N'_S + N'_C}{N_S + N_C}$$

(14)

$N'_S$ and $N'_C$ represent the number of remaining nodes in the sub-network $S$ and the sub-network $C$, respectively. Obviously, the larger the proportion $p$ of the removed nodes, the smaller the proportion $G$ of nodes remaining in the interdependent network.

In order to eliminate the influence of random factors, the process of the above cascading failures is simulated $M$ times. Therefore, the average number $\overline{G}$ can be used to characterize the robustness of the interdependent network, as follows:

$$\overline{G} = \frac{G_1 + G_2 + \ldots + G_M}{M}$$

(15)
In this simulation calculation, the number of simulations time is taken as $M = 1000$. It can be seen from the above analysis that the greater the average number of remaining nodes $\overline{G}$ of the network after the cascading failures occurs, the better the robustness of the network, and vice versa.

4. Robust analysis of interdependent network under different parameters

4.1 Robust analysis of different scale network

Using the cascading failures model established above, sub-networks $G_S = (V_S, E_S)$ and $G_C = (V_C, E_C)$ are respectively established for three different networks: WS-WS interdependent network, BA-BA interdependent network and ER-ER interdependent network. We study the robustness of the interdependent network with the number of different network nodes under random attacks. In the model, $N_S = N_C$ and $N$ takes 12 cases of 10, 50, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000 respectively. The specific simulation results are shown in Figure 11~13:

![Figure 11](image1.png)

Figure 11. Robust analysis of WS-WS interdependent network with different scale network

![Figure 12](image2.png)

Figure 12. Robust analysis of ER-ER interdependent network with different scale network
Figure 13. Robust analysis of BA-BA interdependent network with different scale network

It can be seen from Figure11-13 that as the removal proportion $p$ increases, the average remaining node $\overline{G}$ of the interdependent network decreases continuously, and decreases rapidly when $p > 0.3$, indicating that a large-scale cascading failures occurs in the interdependent network at this time. And when there is a key point $p = p_c$, the average number of remaining nodes $\overline{G} \to 0$. At the same time, interdependent networks exhibit different properties when the number of network nodes is different. That is, as the removal proportion continues to increase, after the cascading failures occurs, the average remaining node of the interdependent network when $N_S$ is smaller is significantly larger than when $N_S$ is large. The robustness of a interdependent network is significantly better than $N_S \geq 100$ at $N_S = 50$. Moreover, when $N_S \geq 100$, the average number of remaining nodes is basically the same as the removal proportion $p$, which indicates that the difference in robustness of networks under different network scales is small under different network scale when $N_S \geq 100$, and large-scale interdependent network cascading failures simulation studies can be performed under $N \geq 100$ conditions. In this way, the time cost can be minimized under the condition that the objective actual characteristics of the interdependent network are met.

4.2 Robustness analysis of different average degree

Based on the research conclusions in Section 3.1, three different interdependent networks, WS-WS, BA-BA and ER-ER, are analyzed. Two sub-networks $G_S = (V_S, E_S)$ and $G_C = (V_C, E_C)$ of the interdependent network are established respectively, and the number of network nodes is set to $N_S = N_C = 100$. The problem of the robustness of the network in the case of different network average degree $k$ taking 4, 6, 8 ... 20, respectively, is studied. The specific simulation results are shown in Figure 14~16:
It can be seen from Figure 14-16 that as the average degree of the network increases, the falling speed of the average remaining nodes becomes slower and slower after the cascading failures of the interdependent network, indicating that the robustness is gradually increasing. When the average degree is greater than 8, the robustness is almost the same, and it is close to the ideal proportion of the average remaining nodes. This characteristic that robustness change are close to linear changes can not show cascading effects, nonlinearities relationship of interdependent network, and there is no great
reference significance for the study of interdependent network cascading failures. As can be seen from Figure 14-16, when the average degree of the interdependent network is between 4-6 and 6-8, the robustness changes drastically. Therefore, when the robustness of the interdependent network is studied, setting the average degree of the interdependent network to 6 can better show a series of characteristics after the cascading failures of the interdependent network.

5. Conclusion
Based on the real network, the authors analyze the occurrence process of interdependent network cascading failures, and propose a interdependent network cascading failures model based on giant components.

At the level of simulation analysis, based on the construction of the interdependent network cascading failures model, the robustness analysis of interdependent network with different network scale and different average degree is carried out. After analyzing the three common interdependent networks of BA-BA, WS-WS and ER-ER, the optimal condition for the research on interdependent network is obtained, that is, the sub-network node scale is $N_s = N_c \geq 100$, and the average degree of the interdependent network is $k = 6$.

The conclusions obtained in this paper can provide reference for the subsequent cascading failures research of interdependent network.

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