ESTIMATES OF NORMS OF LOG-CONCAVE RANDOM MATRICES
WITH DEPENDENT ENTRIES

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Abstract. We prove estimates for $E\|X : \ell_p^n \to \ell_q^m\|$ for $p, q \geq 2$ and any random matrix $X$ having the entries of the form $a_{ij}Y_{ij}$, where $Y = (Y_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ has i.i.d. isotropic log-concave rows. This generalises the result of Guédon, Hinrichs, Litvak, and Prochno for Gaussian matrices with independent entries. Our estimate is optimal up to logarithmic factors. As a byproduct we provide the analogue bound for $m \times n$ random matrices, which entries form an unconditional vector in $\mathbb{R}^{mn}$. We also prove bounds for norms of matrices which entries are certain Gaussian mixtures.

1. Introduction and main result

A classical result regarding spectra of random matrices is Wigner’s Semicircle Law, which describes the limit of empirical spectral measures of a random matrix with independent centred entries with equal variance. Theorems of this type say nothing about the largest eigenvalue (i.e. the operator norm). However, Seginer proved in [17] that for a random matrix $X$ with i.i.d. symmetric entries $E\|X\|_{2,2}$ (by $\|A\|_{p,q}$ we denote the operator norm of the matrix $A$ from $\ell_p$ to $\ell_q$) is of the same order as the expectation of the maximum Euclidean norm of rows and columns of $X$. The same holds true for the structured Gaussian matrices (i.e. when $X_{ij} = a_{ij}g_{ij}$ and $g_{ij}$ are i.i.d. standard Gaussian variables), as was recently shown by Latala, van Handel, and Youssef in [14], and up to a logarithmic factor for any $X$ with independent centred entries, see [16]. The advance of the two latest results is that they do not require that the entries of $X$ are equally distributed (nor that they have equal variances).

Another upper bound for $E\|X\|_{2,2}$ also does not require equal distributions but only the independence of entries: by [9] we know that

$$E\|X\|_{2,2} \lesssim \max_i \left( \sum_j E X^2_{ij} \right)^{1/2} + \max_j \left( \sum_i E X^2_{ij} \right)^{1/2} + \left( \sum_{i,j} E X^4_{ij} \right)^{1/4}.$$ 

This bound is dimension free, but in some cases is worse than the one from [16].

Upper bounds for the expectation of other operator norms were investigated in [2] in the case of independent centred entries bounded by 1. For $q \geq 2$ and $m \times n$ matrices the authors proved that $E\|X\|_{2,q} \lesssim \max\{m^{1/q}, \sqrt{n}\}$. In [6] Guédon, Hinrichs, Litvak, and

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Prochno proved that for a structured Gaussian matrix $X = (a_{ij}X_{ij})_{i \leq m, j \leq n}$ and $p, q \geq 2$, 
\[
\mathbb{E}\|X\|_{p', q} \leq C(p, q) \left[ (\log m)^{1/q} \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p} + \max_{1 \leq j \leq n} \left( \sum_{i=1}^{m} |a_{ij}|^q \right)^{1/q} \right]^q.
\]
This estimate is optimal up to logarithmic factors (see Remark 1.2 below). Note that in the case $(p, q) \neq (2, 2)$ moment method fails in estimating $\mathbb{E}\|X\|_{p', q}$ (as it gives information only on the spectrum of $X$).

All the mentioned results require the independence of entries of $X$. In this article we will see how to generalise the main result of [6] to a wide class of random matrices with independent uncorrelated log-concave rows, following the scheme of proof of the original theorem from [6]. In order to obtain the key estimates for log-concave vectors needed in the proof we use the comparison of weak and strong moments of $\ell_p$-norm of $X$ from [11] and a Sudakov minoration-type bound from [10].

Our estimate is optimal (for fixed $p, q \geq 2$) up to a factor depending logarithmically on the dimension. Let us stress that we do not require the rows of $X$ to have independent, but only uncorrelated coordinates (and to be log-concave) — we require the independence only between the rows.

Before we state our main results, let us say a few words about log-concave vectors. We say that a random vector $X$ in $\mathbb{R}^n$ is log-concave, if for any compact nonempty sets $K, L \subset \mathbb{R}^n$ and $\lambda \in [0, 1]$, 
\[
\mathbb{P}(X \in \lambda K + (1 - \lambda)L) \geq \mathbb{P}(X \in K)^\lambda \mathbb{P}(X \in L)^{1-\lambda}.
\]
The class of log-concave vectors is closed under linear transformations, convolutions and weak limits. By the result of Borell [3] an $n$-dimensional vector with a full dimensional support is log-concave if and only if it has a log-concave density, i.e. has a density of the form $e^{-h}$, where $h$ is a convex function with values in $(-\infty, \infty]$.

Log-concave vectors are a natural generalisation of vectors distributed uniformly over convex bodies. Moreover, distribution of any log-concave vector can be obtained as a weak limit of projections of uniform measures over (higher dimensional) convex bodies (see for example [1]). Other results and conjectures about log-concave vectors are discussed in monograph [4].

We say that a vector $X$ in $\mathbb{R}^n$ is isotropic if $\text{Cov} X = \text{Id}$. If $X$ is a log-concave random vector in $\mathbb{R}^n$ with full dimensional support, then there exists a linear transformation $T$ such that $\text{Cov}(TX) = \text{Id}$, so the isotropicity is only a matter of normalisation.

To make the notation more clear, if $A = (A_{ij})_{i \leq m, j \leq n}$ is an $m \times n$ matrix, we denote by $A_i \in \mathbb{R}^n$ its $i$-th row and by $A^{(j)} \in \mathbb{R}^m$ we denote its $j$-th column. We are ready now to present the main theorem.

**Theorem 1.1.** Let $m \geq 2$, let $Y_1, \ldots, Y_m$ be i.i.d. isotropic log-concave vectors in $\mathbb{R}^n$, and let $A = (A_{ij})$ be an $m \times n$ (deterministic) matrix. Consider a random matrix $X$ with entries $X_{ij} = A_{ij}Y_{ij}$ for $i \leq m, j \leq n$, where $Y_{ij}$ is the $j$-th coordinate of $Y_i$. Then
for every \( p, q \geq 2 \) we have
\[
\mathbb{E}\|X\|_{p',q} \leq C(p,q) \left( (\log m)^{1/q} \max_{1 \leq i \leq m} \|A_i\|_p + \max_{1 \leq j \leq n} \|A^{(j)}\|_q + (\log m)^{1+\frac{2}{q}} \mathbb{E} \max_{1 \leq i \leq m} |X_{ij}| \right),
\]
where \( C(p,q) \) depends only on \( p \) and \( q \).

Remark 1.2. Note that the bound from Theorem 1.1 is optimal up to a constant depending on \( p, q \) and logarithmically on the dimension. Indeed, since \( Y_{ij} \) is log-concave we have by the regularity of \( Y_{ij} \) (see (2.1) below) that \( \mathbb{E}|Y_{ij}| \geq (2C_1)^{-1} (\mathbb{E}Y_{ij}^2)^{1/2} = (2C_1)^{-1} \).

Thus every \( j \leq n \), (we take \( u = e_j \), use the unconditionality of \( \| \cdot \|_q \) and the Jensen inequality)
\[
\mathbb{E}\|X\|_{p',q} = \mathbb{E} \sup_{u \in \ell_p^n} \| Xu \|_q \geq \mathbb{E}\|X^{(j)}\|_q = \mathbb{E}|(Y_{ij}, A_{ij})|_q \geq (2C_1)^{-1} \|A^{(j)}\|_q.
\]
Since \( \|X\|_{p',q} = \|X^T\|_{q',p} \), we also have \( \mathbb{E}\|X\|_{p',q} \geq (2C_1)^{-1} \|A_i\|_p \) for all \( i \leq m \). Moreover, for all \( i \leq m \) and \( j \leq n \), (we take \( v = e_i \) and \( u = e_j \) \( \text{sgn} X_{ij} \))
\[
\|X\|_{p',q} = \sup_{u \in \ell_p^n} \sup_{v \in \ell_q^n} v^T X u \geq |X_{ij}|.
\]
Therefore
\[
\mathbb{E}\|X\|_{p',q} \geq (4C_1 + 1)^{-1} \left[ \max_{1 \leq i \leq m} \|A_i\|_p + \max_{1 \leq j \leq n} \|A^{(j)}\|_q + \mathbb{E} \max_{1 \leq i \leq m} |X_{ij}| \right],
\]
what yields the claim.

The next corollary is a version of Theorem 1.1 in the spirit of the aforementioned results from [17, 14, 16]. It follows directly from (1.3), and the Jensen inequality.

Corollary 1.3. Under the assumptions of Theorem 1.1 we have
\[
\mathbb{E}\|X\|_{p',q} \leq C(p,q) \left( (\log m)^{1+\frac{2}{q}} \mathbb{E} \max_{1 \leq i \leq m} \left( \sum_{j=1}^n |X_{ij}|^p \right)^{1/p} + \mathbb{E} \max_{1 \leq j \leq n} \left( \sum_{i=1}^m |X_{ij}|^q \right)^{1/q} \right).
\]

Remark 1.4. If the rows and columns of \( Y \) are isotropic and log-concave (we do not require independence), and \( p, q \geq 1 \), then
\[
\mathbb{E} \max_{1 \leq i \leq m} \left( \sum_{j=1}^n |A_{ij}Y_{ij}|^p \right)^{1/p} + \mathbb{E} \max_{1 \leq j \leq n} \left( \sum_{i=1}^m |A_{ij}Y_{ij}|^q \right)^{1/q} \leq C \left( p^2 \max_{1 \leq i \leq m} \|A_i\|_p + q^2 \max_{1 \leq j \leq n} \|A^{(j)}\|_q + (p + q) \log(m \vee n) \mathbb{E} \max_{1 \leq i \leq m} |A_{ij}Y_{ij}| \right),
\]
what means that the bound we used in the proof of Corollary 1.3 (the one which uses the Jensen inequality) may be reversed (in the log-concave setting) up to a logarithmic factor and constants depending only on \( p \) and \( q \). Therefore the estimates from Theorem 1.1 and Corollary 1.3 are equivalent up to a logarithmic factor. Inequality (1.2) follows directly from the following proposition.
Proposition 1.5. Let $Y$ be an $m \times n$ random matrix, with isotropic and log-concave rows, let $B$ be a deterministic $m \times n$ matrix, and let $p \geq 1$. Then

$$\mathbb{E} \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |B_{ij}Y_{ij}|^p \right)^{1/p} \lesssim p^2 \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |B_{ij}|^p \right)^{1/p} + p \log(m \vee n) \mathbb{E} \max_{1 \leq i \leq m} |B_{ij}Y_{ij}|.$$ 

It turns out that instead of assuming the log-concavity, we may assume the unconditional性, i.e. that an $m \times n$ random matrix we consider, treated as an $(mn)$-dimensional vector, is unconditional (we no longer assume the independence of rows). Recall that we say that a random vector $Z$ in $\mathbb{R}^d$ is unconditional, if for every choice of signs $\eta \in \{-1,1\}^d$ the vectors $Z$ and $(\eta_i Z_i)_{i \leq d}$ are equally distributed (or, equivalently, that $Z$ and $(\varepsilon_i Z_i)_{i \leq d}$ are equally distributed, where $\varepsilon_1, \ldots, \varepsilon_d$ are i.i.d. symmetric Bernoulli variables, independent of $Z$). The assertion of the next corollary is expressed in the spirit of Corollary 1.3, which is more natural in the non log-concave setting (without the assumption of log-concavity the assertions of Theorem 1.1 and Corollary 1.3 are no longer equivalent).

Corollary 1.6. Assume that $X$ is a random matrix such that the $(mn)$-dimensional vector $(X_{1,1}, \ldots, X_{1,n}, X_{2,1}, \ldots, X_{2,n}, X_{m,1}, \ldots, X_{mn})$ is unconditional. Then for every $p, q \geq 2$ we have

$$(1.3) \quad \mathbb{E}\|X\|_{p',q} \leq C(p,q)\left( (\log m)^{2 + \frac{1}{p'}} \mathbb{E} \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |X_{ij}|^p \right)^{1/p} + \sqrt{\log n} \mathbb{E} \max_{1 \leq j \leq n} \left( \sum_{i=1}^{m} |X_{ij}|^q \right)^{1/q} \right),$$

where $C(p,q)$ depends only on $p$ and $q$.

The rest of this note is organised as follows. Section 2 contains results from other articles, which will be used in a sequel. Section 3 contains generalisations of Lemmas 3.1 and 3.2 from [6] to the log-concave setting and the proof of Theorem 1.1. In Section 4 we will show how to deduce an analogue of Theorem 1.1 for Gaussian mixtures (see Corollary 4.2) and we will provide a proof of Proposition 1.5. Section 5 is devoted to the proof of Corollary 1.6.

Notation. By $C$ we denote universal constants. If a constant $C$ depends on a parameter $\alpha$, we express it as $C(\alpha)$. The value of $C, C(\alpha)$ may differ at each occurrence. Whenever we want to fix the value of an absolute constant we use letters $C_1, C_2, \ldots$. We may always assume that $C, C_i \geq 1$. For two quantities $a, b$ we write $a \lesssim b$ if there exists a constant $C$, such that $a \leq Cb$, and $a \sim b$, if $a \lesssim b$ and $b \lesssim a$. For two numbers $a$ and $b$ we write $a \asymp b$ instead of $\max\{a,b\}$.

For a random variable $X$ by $\|X\|_p$ we denote the $p$-th integral norm of $X$, i.e. the quantity $(\mathbb{E}|X|^p)^{1/p}$ (in the case $X = \|Y\|$ we also call this quantity the $p$-th strong moment of $Y$ associated with the norm $\|\cdot\|$). For a vector $x \in \mathbb{R}^n$ (in particular for a random vector $X$) and $r \geq 1$, by $\|x\|_r$ we denote the $\ell_r$-norm of $x$, i.e. $\|x\| := (\sum_{i=1}^{n} |x_i|^r)^{1/r}$. For $r = 2$ we shall also write $\|\cdot\|$ instead of $\|\cdot\|_2$. It will be always clear from the context, what $\|X\|_q$ means for a random object $X$, so the double meaning of $\|\cdot\|_q$ will not lead to any misunderstanding. Recall that for an $m \times n$ matrix $A$ by $\|A\|_{p,q}$
we denote its norm from $\ell_p^n$ to $\ell_q^m$. For $p \in [1, \infty]$ by $p'$ we denote the Hölder conjugate of $p$, i.e. $1 = \frac{1}{p} + \frac{1}{p'}$.

2. Preliminaries

We will frequently use the regularity of $f(Z)$ for log-concave vectors $X$ and seminorms $f$, i.e.

$$\left(\mathbb{E} f(Z)^p\right)^{1/p} \leq C_1 \frac{p}{q} \left(\mathbb{E} f(Z)^q\right)^{1/q} \quad \text{for } p \geq q \geq 1$$

(see [4, Theorem 2.4.6]).

We will also need the comparison of weak and strong moments for $\ell_p$-norms of log-

concave vectors:

**Theorem 2.1** ([11, Theorem 5]). Let $Z$ be a log-concave vector in $\mathbb{R}^n$, and let $p \in [1, \infty)$. Then

$$\left(\mathbb{E} \|Z\|^q_p\right)^{1/q} \leq C p \left(\mathbb{E} \|Z\|^q_p + \sigma_{p,X}(q)\right) \quad \text{for } q \geq 1,$$

where

$$\sigma_{p,X}(q) := \sup_{t \in B_p^n} \left\| \sum_{i=1}^n t_i Z_i \right\|_q$$

is the $q$-th weak moment of $X$ associated with the $\ell_p$-norm.

We will use the previous theorem also in the tail-bound version:

**Corollary 2.2.** Assume $Z$ is a log-concave vector in $\mathbb{R}^n$, and $p \in [1, \infty)$. Then

$$\mathbb{P}\left(\|Z\| \geq C p(u + \mathbb{E}\|Z\|_p)\right) \leq C_3 \sup_{t \in B_p^n} \mathbb{P}\left(\sum_{i=1}^n t_i Z_i \geq u\right).$$

For the Reader’s convenience we give a proof of this corollary, which goes along the lines of the proof of Corollary 1.3 in [12].

**Proof.** Define a random variable $S := \|Z\|_p$. By the Paley–Zygmund inequality and (2.1) we have for $t \in \mathbb{R}^n$, and $q \geq 1$,

$$\mathbb{P}\left(\sum_{i=1}^n t_i Z_i \geq \frac{1}{2} \left\| \sum_{i=1}^n t_i Z_i \right\|_q\right) = \mathbb{P}\left(\left\| \sum_{i=1}^n t_i Z_i \right\|^q \geq 2^{-q} \mathbb{E}\left|\sum_{i=1}^n t_i Z_i\right|^q\right)$$

$$\geq (1 - 2^{-q})^2 \left(\frac{\sum_{i=1}^n t_i Z_i}{\|\sum_{i=1}^n t_i Z_i\|_2}\right)^{2q} \geq e^{-C_4 q}.$$

(2.3)

In order to show (2.2) we consider 3 cases.

**Case 1.** $2u < \sup_{t \in B_p^n} \left\| \sum_{i=1}^n t_i Z_i \right\|$. Then by (2.3)

$$\sup_{t \in B_p^n} \mathbb{P}\left(\sum_{i=1}^n t_i Z_i \geq u\right) \geq e^{-2C_4}$$

and (2.2) obviously holds if $C_3 \geq \exp(2C_4)$.
Case 2. $\sup_{t \in B_{r'}^n} \left\| \sum_{i=1}^n t_i Z_i \right\|_2 \leq 2u < \sup_{t \in B_{r'}^n} \left\| \sum_{i=1}^n t_i Z_i \right\|_x$. Let us then define

$$q := \sup \left\{ r \geq 2C_4 : \sup_{t \in B_{r'}^n} \left\| \sum_{i=1}^n t_i Z_i \right\|_{r/C_4} \leq 2u \right\}.$$ 

By (2.3) we have

$$\sup_{t \in B_{r'}^n} \mathbb{P} \left( \left| \sum_{i=1}^n t_i Z_i \right| > u \right) \geq e^{-q}.$$ 

By (2.1), Theorem 2.1, and Chebyshev’s inequality we have

$$\mathbb{P}(S \geq C_5 \rho(ES + u)) \leq \mathbb{P}(S \geq \|S\|_q) \leq e^{-q}$$

for $C_5$ large enough. Thus (2.2) holds in this case.

Case 3. $u > \sup_{t \in B_{r'}^n} \left\| \sum_{i=1}^n t_i Z_i \right\|_x = \|S\|_x$. Then $\mathbb{P}(S \geq u) = 0$ and (2.2) holds for any $C_2 \geq 1$.

In the proof of Theorem 1.1 we will use Theorem 2.1 from [6], which is another version of results provided before by Guédon–Rudelson in [8], and by Guédon–Mendelson–Pajor–Tomczak-Jaegerman in [7]:

**Theorem 2.3** ([6, Theorem 2.1]). Let $E$ be a Banach space with modulus of convexity of power type 2 with constant $\lambda$. Let $X_1, \ldots, X_m \in E^*$ be independent random vectors, and let $q \geq 2$. Define

$$u := \sup_{t \in B_E} \left( \sum_{i=1}^m E|\langle X_i, t \rangle|^q \right)^{1/q},$$

and

$$v := \left( \lambda^8 (T_2(E^*))^2 \log m E \max_{1 \leq i \leq m} \|X_i\|_E^q \right)^{1/q},$$

where $T_2(E^*)$ is the Rademacher type 2 constant of $E^*$. Then

$$\left[ \mathbb{E} \sup_{t \in B_E} \left| \sum_{i=1}^m \left( |\langle X_i, t \rangle|^q - E|\langle X_i, t \rangle|^q \right) \right|^{1/q} \right] \leq C(\sqrt{uv} + v) \leq 2C(u + v).$$

We will use Theorem 2.3 with $E = \ell_p^n$. In this case $\lambda$ and $T_2(E^*)$ are known.

3. Proof of Theorem 1.1

The next two lemmas provide estimates of the quantities $u$ and $v$ appearing in Theorem 2.3 in the case $E = B_{r'}^n$.

**Lemma 3.1.** Assume $p, q, X$, and $Y$ are as in Theorem 1.1. Then

$$\left( E \max_{1 \leq i \leq m} \|X_i\|^2 \right)^{1/q} \leq C(p, q) \left[ \max_{1 \leq i \leq m} \|A_i\|_p + \log m E \max_{1 \leq i \leq m} |X_{ij}| \right],$$

where $C(p, q)$ depends only on $p$ and $q$. 
Lemma 3.2. Assume $p, q, X,$ and $Y$ are as in Theorem 1.1. Then

$$
(3.1) \sup_{t \in B_p^n} \left( \sum_{i=1}^{m} E|X_i, t|^q \right)^{1/q} \leq C_1 q \max_{1 \leq j \leq n} \|A^{(j)}\|_q.
$$

In the proof of Lemma 3.1 we will also need the following estimate:

Lemma 3.3. Assume that $Z$ is an isotropic log-concave vector in $\mathbb{R}^m$. Then for all $1 \leq k \leq m$ and all $a \in \mathbb{R}^m$ we have

$$
\mathbb{E} \max_{1 \leq i \leq m} |a_i Z_i| \geq \frac{1}{C_k} \max_{k \leq m} (a_i^* \min_{k \leq m} \|Z_i\|_{\log(k+1)}),
$$

where $(a_i^*)_{i=1}^m$ denotes the non-increasing rearrangement of $(|a_i|)_{i=1}^m$.

In order to prove Theorem 1.1, we repeat the proof scheme from [6].

Proof of Theorem 1.1. We use Theorem 2.3 for $E = \ell_p^n$. Then $\lambda \sim p$ (see [15, Theorem 5.3]) and $T_3(E^*) \sim \sqrt{\lambda}$. Let $u$ and $v$ be given by formulas (2.4) and (2.5). The triangle inequality, Theorem 2.3, Lemma 3.1, and Lemma 3.2 yield

$$
\mathbb{E} X_{p,q} \leq \left( \mathbb{E} X_{p,q} \frac{1}{q} \right)^{1/q} = \left[ \mathbb{E} \sup_{t \in B_p^n} \sum_{i=1}^{m} \langle t, X_i \rangle \right]^{1/q} \\
\leq \left[ \mathbb{E} \sup_{t \in B_p^n} \left( \sum_{i=1}^{m} \left( \langle X_i, t \rangle - \mathbb{E} \langle X_i, t \rangle \right)^q \right) \right]^{1/q} + \sup_{t \in B_p^n} \left( \mathbb{E} \sum_{i=1}^{m} \langle t, X_i \rangle \right)^{1/q} \\
\leq C \cdot (u + v) \\
\leq C(p, q) \left( \log m \right)^{1/q} \max_{1 \leq i \leq m} \|A_i\|_p + \max_{1 \leq j \leq n} \|A^{(j)}\|_q + \left( \log m \right)^{\frac{1}{p} + 1} \mathbb{E} \max_{1 \leq i \leq m} \|X_{ij}\|.
$$

The main contribution of this article lies in the proofs of Lemmas 3.1, 3.2, and 3.3.

Proof of Lemma 3.3. We may and do assume that $a_1 \geq a_2 \geq \ldots \geq a_m \geq 0$, i.e. $a_i^* = a_i$ for $i \leq m$. By [10, Proposition 3.3] we have for all $k \leq m$,

$$
\mathbb{E} \max_{1 \leq i \leq k} |a_i Z_i| \geq C^{-1} \min_{1 \leq i \leq k} \|a_i Z_i\|_{\log(k+1)} \geq C^{-1} a_k \min_{1 \leq i \leq m} \|Z_i\|_{\log(k+1)}.
$$

Thus

$$
\mathbb{E} \max_{1 \leq i \leq m} |a_i Z_i| = \max_{1 \leq k \leq m} \mathbb{E} \max_{1 \leq i \leq k} |a_i Z_i| \geq C^{-1} \min_{1 \leq k \leq m} \left( a_k \min_{1 \leq i \leq m} \|Z_i\|_{\log(k+1)} \right). \quad \square
$$

Proof of Lemma 3.1. We may and do assume that $m \geq 2$.

Since we may approximate $A_{ij}$ by nonzero numbers, we may and do assume that $a_{ij} \neq 0$ for all $i, j$. Let $C_2, C_3$ be the constants from (2.2), let $C_6$ be the constant from Lemma 3.3, and recall that $C_1$ is the constant from (2.1). We may assume that all these constants are greater than 1.
Note that for any $a, b \in \mathbb{R}$ we have $a = (a - b) + a \wedge b$. Thus, by the triangle inequality,

\[(3.2) \quad (\mathbb{E} \max_{1 \leq i \leq m} \|X_i\|_p^q)^{1/q} \leq \left( \mathbb{E} \max_{1 \leq i \leq m} \left( \|X_i\|_p - C_2p\mathbb{E}\|X_i\|_p\right)^q 1_{\{\|X_i\|_p \geq C_2p\mathbb{E}\|X_i\|_p\}} \right)^{1/q} + C_2p \max_{1 \leq i \leq m} \mathbb{E}\|X_i\|_p.\]

Moreover, for every $1 \leq i \leq m$ we have by (2.1) and the isotropicity of $Y_i$, that

\[(3.3) \quad \mathbb{E}\|X_i\|_p \leq \left( \sum_{j=1}^{n} \mathbb{E}[|Y_{ij}|^p | A_{ij}]^p \right)^{1/p} \leq \max_{j \leq n} \|Y_{ij}\|_p A_i \leq C_1p A_i \|
\]

Now we pass to the estimation of the first term of (3.2). Let

\[B := C_1^2C_6 \log(m+1) \mathbb{E} \max_{1 \leq i \leq m} |X_{ij}| \quad \text{and} \quad \sigma := \left( \max_{1 \leq i \leq m} \sigma_{p,X_i}(2) \right) \vee B.\]

By (2.2) we have

\[(3.4) \quad \mathbb{E} \max_{1 \leq i \leq m} \left[ \left(\|X_i\|_p - C_2p\mathbb{E}\|X_i\|_p\right)^q 1_{\{\|X_i\|_p \geq C_2p\mathbb{E}\|X_i\|_p\}} \right] \leq (2C_2p\sigma)^q + \int_{2C_2p\sigma}^{\infty} qu^{q-1}\mathbb{P}\left( \max_{1 \leq i \leq m} \left(\|X_i\|_p - C_2p\mathbb{E}\|X_i\|_p\right) \geq u \right) du
\]

\[\leq (2C_2p\sigma)^q + (C_2p)^q \sum_{i=1}^{m} \int_{2C_2p\sigma}^{\infty} qu^{q-1}\mathbb{P}\left(\|X_i\|_p - C_2p\mathbb{E}\|X_i\|_p \geq C_2pu \right) du
\]

\[\leq (2C_2p\sigma)^q + (C_2p)^q C_3 \sum_{i=1}^{m} \int_{2C_2p\sigma}^{\infty} qu^{q-1} \sup_{|t| \leq 1} \mathbb{P}\left( \sum_{j=1}^{\infty} t_j X_{ij} \right) \geq u \right) du.
\]

For $u \geq \sup_{|t| \leq 1} \|\sum_{j=1}^{\infty} t_j X_{ij}\|_\infty$ the function we integrate vanishes, so from now on we will consider only $i$'s for which $u < \sup_{|t| \leq 1} \|\sum_{j=1}^{\infty} t_j X_{ij}\|_\infty$.

Note that if $1 \leq i \leq m$ and $\sup_{|t| \leq 1} \|\sum_{j=1}^{\infty} t_j X_{ij}\|_\infty > u \geq e\sigma \geq e\sigma_{p,X_i}(2)$, then

\[r := r(i) := \sup\{s \geq 2 : \sigma_{p,X_i}(s) \leq u/e \in [2, \infty)\}
\]

and $\sigma_{p,X_i}(r) = u/e$. Therefore

\[(3.5) \quad \sup_{|t| \leq 1} \mathbb{P}\left( \sum_{j=1}^{\infty} t_j X_{ij} \right) \geq u \leq \frac{\sup_{|t| \leq 1} \| \langle t, X_i \rangle \|_r}{u^r} = e^{-r}.
\]

Now we will estimate $r$ from below. For $t \geq 2$ let

\[\psi(t) = t \min_{1 \leq j \leq n} |Y_{ij}|.\]
Since $Y_1$’s are identically distributed, $\varphi$ does not depend on $i$. By (2.1), and the isotropy of $Y$ we have

$$
\sigma_{p,X_i}(t) \leq \sigma_{2,X_i}(t) \leq C_1 t \max_{|x| \leq 1} \left( \mathbb{E} \left( \sum_{j=1}^{n} A_{ij} X_j x_j \right)^2 \right)^{1/2}
$$

$$
= C_1 t \max_{|x| \leq 1} \left( \mathbb{E} \left( \sum_{j=1}^{n} A_{ij}^2 x_j^2 \right)^2 \right)^{1/2}
$$

(3.6)

$$
C_1 \max_{1 \leq j \leq n} |A_{ij}| \cdot \|Y_j\|_2 \leq C_1 \varphi(t) \max_{1 \leq j \leq n} |A_{ij}|.
$$

Since we can permute the rows of $A$, we may and do assume that

$$
\max_{1 \leq j \leq n} |A_{1j}| \geq \ldots \geq \max_{1 \leq j \leq n} |A_{mj}|.
$$

Let $j(i) \leq n$ be such an index that $|A_{ij(i)}| = \max_{1 \leq j \leq n} |A_{ij}|$. Lemma 3.3 applied to $Z_t = Y_{ij(i)}$ and the non-increasing sequence $a_t = |A_{ij(i)}|$ implies

$$
\mathbb{E} \max_{1 \leq i \leq m \atop 1 \leq j \leq n} |X_{ij}| \geq \mathbb{E} \max_{1 \leq i \leq m \atop 1 \leq j \leq n} |A_{ij(i)}| Y_{ij(i)} | \geq C_6^{-1} (\log(m + 1))^{-1} \max_{1 \leq i \leq m} \left( \varphi(\log(i + 1)) |A_{ij(i)}| \right),
$$

so for all $i \leq m$ we have

$$
B \geq C_1^2 \varphi(\log(i + 1)) |A_{ij(i)}| = C_1^2 \varphi(\log(i + 1)) \max_{1 \leq j \leq n} |A_{ij}|.
$$

Note that by (2.1) for all $r \geq \lambda \geq 2$ we have $\sigma_{p,X_i}(r/\lambda) \geq \sigma_{p,X_i}(r)/(C_1 \lambda)$. Take $\lambda = \sigma_{p,X_i}(r)/B = u/(Be) \geq 2$. Then by a calculation similar to the one above we get

$$
u = \sigma_{p,X_i}(r) \leq C_1 \frac{r}{2} \max_{1 \leq j \leq n} |A_{ij}| \leq C_1^2 r \max_{1 \leq i \leq m \atop 1 \leq j \leq n} |A_{ij}| \|Y_{ij}| \leq C_1^2 r \max_{1 \leq i \leq m \atop 1 \leq j \leq n} |X_{ij}| \leq Br,
$$

so indeed $r \geq \lambda \geq 2$.

Therefore for all $i \leq m$ we have

(3.7) $\frac{B}{C_1} = \frac{1}{\lambda C_1} \sigma_{p,X_i}(r) \leq \sigma_{p,X_i}(r/\lambda) \leq C_1 \varphi(\frac{r}{\lambda}) \max_{1 \leq j \leq n} |A_{ij}| \leq \frac{B \varphi(\frac{u}{B})}{C_1 \varphi(\log(i + 1))}.$

Since the function $\varphi$ is strictly increasing, the previous inequality yields $r \geq \lambda \log(i + 1)$. This together with (3.5) implies that (recall that $\lambda = \frac{u}{Be} \geq 2$)

(3.8) $\sum_{i=1}^{m} \sup \left\{ \mathbb{P} \left( \sum_{j=1}^{n} t_j X_{ij} \right) \geq u \right\} \leq \sum_{i=1}^{m} (i + 1) - \frac{u}{2} \leq 2 - \frac{u}{2} + \int_{2}^{\infty} x^{-\frac{u}{2}} dx \leq 3 \cdot 2 - \frac{u}{2}.$

Inequalities (3.4), (3.8), and the Stirling formula yield that

(3.9) $\left( \mathbb{E} \left[ \max_{1 \leq i \leq m} \left( \|X_i\|_p - C_2 \mathbb{E} \|X_i\|_p \right)^q \mathbb{I}_{\{ \|X_i\|_p \geq C_2 \mathbb{E} \|X_i\|_p \}} \right] \right)^{1/q} \leq C_2 C_3^{1/q} \sigma pq.$

Moreover, by (2.1)

$$
\max_{1 \leq i \leq m} \sigma_{p,X_i}(2) \leq 2C_1 \max_{1 \leq i \leq m} \sigma_{p,X_i}(1) \leq 2C_1 \max_{1 \leq i \leq m} \mathbb{E} \|X_i\|_p,
$$

where the second inequality holds since the weak first moment is bounded above by the strong first moment. This together with (3.2), (3.3), and (3.9) gives the assertion. \( \square \)
Proof of Lemma 3.2. Note that if $0 \leq r \leq s$, then for every $x \in \mathbb{R}^n$ we have $|x|_s \leq |x|_r$, so we may and do assume $p = 2$. By (2.1), the isotropicity of $Y$, and the Jensen inequality we have

$$
\sup_{t \in B^n_2} \left( \sum_{i=1}^m \mathbb{E}|\langle X_i, t \rangle|^q \right)^{1/q} \leq C_1 q \sup_{||t||_2 \leq 1} \left( \sum_{i=1}^m \left( \mathbb{E}|\langle X_i, t \rangle|^2 \right)^{q/2} \right)^{1/q} = C_1 q \sup_{||t||_2 \leq 1} \left( \sum_{i=1}^m \left( \sum_{j=1}^n A_{ij}^2 t_j^2 \right)^{q/2} \right)^{1/q} \leq C_1 q \sup_{||t||_2 \leq 1} \left( \sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^q t_j^2 \right)^{1/q} = C_1 q \left( \sup_{||t||_2 \leq 1} \left( \sum_{j=1}^n \|A(j)\|_q^q t_j^2 \right)^{1/q} \right) = C_1 q \max_{1 \leq j \leq n} \|A(j)\|_q. \quad \square
$$

Remark 3.4. By the same reasoning as in the log-concave case, we may prove (using [12, Corollary 1.3], [13, Theorem 2.1], and the claim below instead of (2.2), Lemma 3.3 and the previous estimates on $\sigma_{p, X_i}(s)$, respectively) the following.

Let $X$ be an $m \times n$ random matrix with entries $X_{ij} = A_{ij} Y_{ij}$, where $Y_{ij}$ are independent symmetric random variables such that $\mathbb{E} Y_{ij}^2 = 1$. Assume that for any $r \geq 2$ and any $1 \leq i \leq m$, $1 \leq j \leq n$ we have $\frac{m}{2} \leq \|Y_{ij}\|_r \leq L r^\beta$ with $\beta \in [\frac{1}{r}, 1]$. Then for every $p, q \geq 2$ we have

$$
\mathbb{E}\|X\|_{p', q} \leq C(p, q, L) \left( (\log m)^{1/q} \max_{1 \leq i \leq m} \|A_i\|_p + \max_{1 \leq j \leq n} \|A(j)\|_q + (\log m)^{1/q} \mathbb{E} \max_{1 \leq i \leq m} |X_{ij}| \right),
$$

where $C(p, q, L)$ depends only on $p, q,$ and $L$. At the end of Section 4 we provide another result concerning this type of random matrices (see Corollary 4.5).

As we mentioned, it suffices to prove the claim:

$$(3.10) \quad \left\| \sum_{j=1}^n t_j Y_{ij} \right\|_r \leq C L r^\beta \left\| \sum_{j=1}^n t_j Y_{ij} \right\|_2 = C L r^\beta \|t\|_2,$$

where $C$ is an absolute constant, and repeat the proof of Theorem 1.1.

Proof of the claim. It suffices to consider $r = 2k$, where $k$ is an integer. Let us denote

$$
c_{i_1, \ldots, i_n} := \binom{i_1 + \ldots + i_n}{i_1} \binom{i_2 + \ldots + i_n}{i_2} \ldots \binom{i_n}{i_n}.
$$

Let $G = (G_j)_{j=1}^n$ be the standard $n$-dimensional Gaussian vector. Recall that for any $t \in \mathbb{R}^n$ and $r \geq 1$ we have $\|\sum_{j=1}^n t_j G_j\|_r = \|t\|_2 \|G_1\|_r \sim \|t\|_2 \sqrt{r} = \sqrt{r} \|\sum_{j=1}^n t_j Y_{ij}\|_2.$
Let \( G \) variables. Let

\[
\begin{align*}
4.1 \quad \text{Definition}
\end{align*}
\]

of \( r \)

Let \( Z \) is log-concave and isotropic (considered as a random vector in \( p \)

clear from the proof, that the corollary below is true also for another type of matrices:

Corollary 4.2. Let \( m, n \geq 2 \), let \( \gamma \geq 0 \), let \( B = (B_{ij}) \) be a deterministic \( m \times n \) matrix, and let \( G = (G_{ij})_{i \leq m, j \leq n} \) be a random matrix which entries are i.i.d. standard Gaussian variables. Let \( X_{ij} = |Z_{ij}|^\gamma B_{ij} G_{ij} \), where \( Z = (Z_{ij})_{i \leq m, j \leq n} \) is a log-concave and isotropic

|l_{ij} 

what finishes the proof of (3.10). \( \square \)

By the claim we get

\[
\sigma_p c_{Y_1}(q) \leq CLq^{\beta} \sup_{s \in \mathbb{B}_p} \sqrt{\sum_{j=1}^n s_j^2 c_j^2} = CLq^{\beta} \max_{1 \leq j \leq n} |c_j| \leq C l_{ij} \max_{1 \leq j \leq n} |c_j|,
\]

what allows us to obtain a version of (3.6) for \( \varphi(t) := \min_{1 \leq j \leq n} l_{ij} \).}

what finishes the proof of (3.10). \( \square \)

By the claim we get

\[
\sigma_p c_{Y_1}(q) \leq CLq^{\beta} \sup_{s \in \mathbb{B}_p} \sqrt{\sum_{j=1}^n s_j^2 c_j^2} = CLq^{\beta} \max_{1 \leq j \leq n} |c_j| \leq C l_{ij} \max_{1 \leq j \leq n} |c_j|,
\]

what allows us to obtain a version of (3.6) for \( \varphi(t) := \min_{1 \leq j \leq n} l_{ij} \).

4. Estimates of norms of matrices in the case of Gaussian mixtures

Let us recall the definition from [5], where the significance of Gaussian mixtures is also described.

\textbf{Definition 4.1.} A random variable \( X \) is called a (centred) Gaussian mixture if there exists a positive random variable \( r \) and a standard Gaussian random variable \( y \), independent of \( r \), such that \( X \) has the same distribution as \( r y \).

We will work with matrices of the form \((R_{ij}B_{ij}G_{ij})_{i \leq m, j \leq n}\) which entries are Gaussian mixtures. We additionally assume that \( R_{ij} = |Z_{ij}|^\gamma \), where \( \gamma \geq 0 \), and that the matrix \( Z \) is log-concave and isotropic (considered as a random vector in \( \mathbb{R}^{mn} \)). It will be clear from the proof, that the corollary below is true also for another type of matrices; \((R_{ij}B_{ij}G_{ij})_{i \leq m, j \leq n}\), where \( R_i = |Z_i|^\gamma \), and \((Z_1, \ldots, Z_m)\) is an arbitrary isotropic log-concave random vector.

\textbf{Corollary 4.2.} Let \( m, n \geq 2 \), let \( \gamma \geq 0 \), let \( B = (B_{ij}) \) be a deterministic \( m \times n \) matrix, and let \( G = (G_{ij})_{i \leq m, j \leq n} \) be a random matrix which entries are i.i.d. standard Gaussian variables. Let \( X_{ij} = |Z_{ij}|^\gamma B_{ij} G_{ij} \), where \( Z = (Z_{ij})_{i \leq m, j \leq n} \) is a log-concave and isotropic
random matrix independent of $G$. Then for every $p, q \geq 2 \vee \frac{1}{\gamma}$ we have
\[
\mathbb{E}\|X\|_{p', q} \leq C(p, q, \gamma) \left( (\log m)^{\frac{\gamma}{2} + \gamma} \max_{1 \leq i \leq m} \|B_i\|_p + (\log n)^{\gamma} \max_{1 \leq j \leq n} \|B^{(j)}\|_q + (\log m)^{1 + \frac{1}{q}} \mathbb{E} \max_{1 \leq i \leq m} \|X_{ij}\| \right).
\]

**Proof.** Theorem 1.1 applied to $Y = G$ and $A_{ij} = |Z_{ij}|^\gamma B_{ij}$ yields
\[
\mathbb{E}\|X\|_{p', q} \leq C(p, q) \left( (\log m)^{1/q} \mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_p + \mathbb{E} \max_{1 \leq j \leq n} \|B_{ij}|Z_{ij}|^\gamma\|_q \right)
+ (\log m)^{1 + \frac{1}{q}} \mathbb{E} \max_{1 \leq i \leq m} \|X_{ij}\|,
\]
so it suffices to prove that
\[
\mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\| \leq C(p, \gamma)(\log m)^\gamma \max_{1 \leq i \leq m} \|B_i\|_p
\]
and
\[
\mathbb{E} \max_{1 \leq j \leq n} \|B_{ij}|Z_{ij}|^\gamma\| \leq C(q, \gamma)(\log n)^\gamma \max_{1 \leq j \leq n} \|B^{(j)}\|_q
\]
for $p \geq 1 \vee \frac{1}{\gamma}$. By the symmetry of assumptions we need only to show (4.1).

If $\gamma < 1$, then
\[
\mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\| \leq \left( \mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_p \right)^{\gamma},
\]
and
\[
\|(B_{ij}|Z_{ij}|^\gamma)_j\|_p = \|B_i\|_p
\]
so it suffices to consider only $\gamma \geq 1$ (we used here the assumption that $p \geq \frac{1}{\gamma}$).

Note that for any $u \geq 1$ we have
\[
\mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_p \leq \left( \mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_p \right)^{1/u}
\]
and
\[
\mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_p \leq \left( \mathbb{E} \max_{1 \leq i \leq m} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_p \right)^{1/u}
\]
(4.2)

Fix $i \leq m$. By Theorem 2.1 applied to $p = p\gamma$, $q = u\gamma$ (recall that $\gamma \geq 1$, so $u\gamma, p\gamma \geq 1$), and $Z_j = |B_{ij}|^{1/\gamma} Z_{ij}$ we have
\[
(C p\gamma)^{-\gamma} \left( \mathbb{E} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_{p\gamma} \right)^{1/u} \leq \left( \mathbb{E} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_{p\gamma} + \sup_{t \in B_{p\gamma}^u} \left\| \sum_{j=1}^n |B_{ij}|^{1/\gamma} Z_{ij} t_j \right\|_{u\gamma} \right)^{\gamma}
\]
(4.3)
\[
\leq 2^{\gamma-1} \left( \mathbb{E} \|(B_{ij}|Z_{ij}|^\gamma)_j\|_{p\gamma} + \sup_{t \in B_{p\gamma}^u} \left\| \sum_{j=1}^n |B_{ij}|^{1/\gamma} Z_{ij} t_j \right\|_{u\gamma} \right)^{\gamma}.
\]
Let us use (2.1) and the assumption \( \mathbb{E} Z_{ij}^2 = 1 \) to estimate the first term in (4.3):

\[
(4.4) \quad \mathbb{E} \left( \sum_{j=1}^{n} |B_{ij}|^p |Z_{ij}|^{p\gamma} \right)^{1/p} \leq \left( \sum_{j=1}^{n} |B_{ij}|^p E |Z_{ij}|^{p\gamma} \right)^{1/p} \leq (C_1 p\gamma)^\gamma \|B_i\|_p.
\]

Recall that \( B'_{ij} \subset B_{ij}^2 \). We use again (2.1) and the isotropicity of \( Z_i \) to estimate the second term in (4.3):

\[
\sup_{t \in B_{ij}'} \left\| \sum_{j=1}^{n} |B_{ij}|^{1/\gamma} Z_{ij} t_j \right\|_{\gamma} \leq (C_1 u\gamma)^\gamma \sup_{t \in B_{ij}^2} \left\| \sum_{j=1}^{n} |B_{ij}|^{1/\gamma} Z_{ij} t_j \right\|_{\gamma} = (C_1 u\gamma)^\gamma \max_{1 \leq j \leq n} |B_{ij}| \leq (C_1 u\gamma)^\gamma \|B_i\|_p.
\]

Take \( u = \log m \) and put together (4.2), (4.3), and (4.4) to get the assertion. \( \square \)

**Remark 4.3.** Using [6, Theorem 1.1] instead of Theorem 1.1 in the proof above yields a slightly better estimate:

\[
\mathbb{E} \|X\|_{p', q} \leq C(p, q) \left( (\log m)^{1/2} + \gamma \max_{1 \leq i \leq m} \|B_i\|_p + (\log n)^\gamma \max_{1 \leq j \leq n} \|B^{(j)}\|_q \right).
\]

**Remark 4.4.** It is clear from the proof of Corollary 4.2 that in the case \( Z_{ij} = G'_{ij} \) where \( G'_{ij} \) are i.i.d. standard Gaussian variables, inequality (4.1) may be slightly improved:

\[
(4.6) \quad \mathbb{E} \max_{1 \leq i \leq m} \|\langle B_{ij}, G'_{ij} \rangle^\gamma \|_p \leq C(p, \gamma)(\log m)^{\gamma/2} \max_{1 \leq i \leq m} \|B_i\|_p.
\]

In order to obtain this improvement one should use \( \|\langle t, G_i \rangle\|_{u\gamma} \leq \sqrt{w\gamma} \|\langle t, G_i \rangle\|_2 \) instead of \( \|\langle t, Z_i \rangle\|_{u\gamma} \leq u\gamma \|\langle t, Z_i \rangle\|_2 \). Therefore, if we additionally use Remark 4.3, the assertion of Corollary 4.2 in the case \( Z_{ij} = G'_{ij} \) (where \( G' \) is independent of \( G \)) will state that

\[
(4.7) \quad \mathbb{E} \|X\|_{p', q} \leq C(p, q, \gamma) \left( (\log m)^{1/2} + \gamma \max_{1 \leq i \leq m} \|B_i\|_p + (\log n)^{\gamma/2} \max_{1 \leq j \leq n} \|B^{(j)}\|_q \right).
\]

**Proof of Proposition 1.5.** We begin similarly as in the proof of (4.1) (in the case \( \gamma = 1 \)), but we estimate the second term on the right-hand side of (4.3) in a slightly different way, using (2.1):

\[
\sup_{t \in B_{ij}'} \left\| \sum_{j=1}^{n} B_{ij} Y_{ij} t_j \right\|_u \leq n^{1/u} \sup_{t \in B_{ij}'} \left( \mathbb{E} \max_{1 \leq j \leq n} |t_j B_{ij} Y_{ij}| u \right)^{1/u} \leq n^{1/u} C_1 u \mathbb{E} \max_{1 \leq j \leq n} |B_{ij} Y_{ij}|.
\]

We take \( u = \log(m \vee n) \) to get the assertion. \( \square \)
Then for all $p, q, r$

Assume that for any $w$ yielding the assertion, since 

worse constants than in Remark 3.4. The proof is based on the fact, that variables $Y_{ij}$ satisfying the moment assumption from Remark 3.4 are comparable with a certain Gaussian mixtures.

**Corollary 4.5.** Let $m, n \geq 2$, $\gamma \geq \frac{1}{2}$, and let $X$ be an $m \times n$ random matrix with entries $X_{ij} = A_{ij}Y_{ij}$, where $Y_{ij}$ are independent symmetric random variables such that $EY_j^2 = 1$. Assume that for any $r \geq 2$ and any $1 \leq i \leq m$, $1 \leq j \leq n$ we have $\frac{\alpha}{2} \leq \|Y_{ij}\|_r \leq Lr^\beta$. Then for all $p, q \geq 2$,

$$
\w E\|X\|_{p', q}' \leq C(p, q, L, \beta)\left[ (\log m)^{\beta + \frac{1}{2}} \max_{1 \leq i \leq m} \|A_{ii}\|_p + (\log n)^{\beta} \max_{1 \leq j \leq n} \|A^{(j)}\|_q \\
+ (\log m)^{1/q} \sqrt{\log mn} \w E \max_{1 \leq i \leq m} |X_{ij}| \right].
$$

**Proof.** Let $G_{ij}, G'_{ij}, i \leq m, j \leq n$, be i.i.d. standard Gaussian variables. Let $(\varepsilon_{ij})$ be i.i.d. symmetric Bernoulli random variable, independent of $G$ and $G'$. Note that $Y_{ij}' := |G_{ij}|^{2\beta} \varepsilon_{ij}$ satisfies $\frac{\alpha}{2} \leq \|Y_{ij}'\|_r \leq L'r^{\beta}$ for all $r \geq 2$, with a universal constant $L'$, since $\|G_{ij}\|_r \sim \sqrt{s}$ for $s \geq 1$. Let $X' = (X_{ij}')$ be the $m \times n$ random matrix with entries $X_{ij}' = A_{ij}Y_{ij}'$. By [14, Lemma 4.7] we know that

$$
\frac{1}{C(L, L', \beta)} \w E\|X\|_p \leq \w E\|X\|_p \leq C(L, L', \beta) \w E\|X'\|
$$

for any norm $\| \cdot \|$ on $m \times n$ real matrices. In particular

$$
\w E\|X\|_{p', q}' \leq C(L, \beta) \w E\|X'\|_{p', q}, \quad \text{and} \quad \w E \max_{1 \leq i \leq m} |X_{ij}'| \leq C(L, \beta) \w E \max_{1 \leq j \leq n} |X_{ij}|.
$$

Moreover, by the Jensen inequality and by (4.7) applied with $\gamma = 2\beta$ we have

$$
\w E\|(X_{ij}')\|_{p', q}' = \w E\|(\varepsilon_{ij}A_{ij}|G_{ij}'|^{2\beta})\|_{p', q}' = \sqrt{\frac{\pi}{2}} \w E\|(E|G_{ij}|\varepsilon_{ij}A_{ij}|G_{ij}'|^{2\beta})\|_{p', q}' \\
\leq \sqrt{\frac{\pi}{2}} \w E\|(G_{ij}|G_{ij}'|^{2\beta})\|_{p', q}' = \sqrt{\frac{\pi}{2}} \w E_X\w E_G\|(A_{ij}G_{ij}|G_{ij}'|^{2\beta})\|_{p', q}' \\
\leq C(p, q)\left[ (\log m)^{\beta + \frac{1}{2}} \max_{1 \leq i \leq m} |A_{ii}|_p + (\log n)^{\beta} \max_{1 \leq j \leq n} |A^{(j)}|_q \\
+ (\log m)^{1/q} \max_{1 \leq i \leq m} |A_{ij}G_{ij}| \cdot |G_{ij}'|^{2\beta} \right] \\
\leq C(p, q)\left[ (\log m)^{\beta + \frac{1}{2}} \max_{1 \leq i \leq m} |A_{ii}|_p + (\log n)^{\beta} \max_{1 \leq j \leq n} |A^{(j)}|_q \\
+ (\log m)^{1/q} \max_{1 \leq i \leq m} |G_{ij}| \w E \max_{1 \leq j \leq n} |X_{ij}'| \right],
$$

what yields the assertion, since $\w E \max_{1 \leq i \leq m} |G_{ij}| \sim \sqrt{\log mn}$. □
5. The case of unconditional entries

Proof of Corollary 1.6. Since $X$ is unconditional, it has the same distribution as the matrix $(\varepsilon_{ij}X_{ij})_{1 \leq i, j \leq n}$, where $\varepsilon_{ij}$ are i.i.d. symmetric Bernoulli variables independent of $X$. Let $G_{ij}$ be i.i.d. standard Gaussian variables independent of $X$ and $(\varepsilon_{ij})_{1 \leq i, j \leq n}$. Then

$$
\mathbb{E}\|X_{ij}\|_{p', q} = \mathbb{E}\|(\varepsilon_{ij}X_{ij})\|_{p', q} = \sqrt{\frac{\pi}{2}} \mathbb{E}\|(\varepsilon_{ij}X_{ij}|G_{ij})\|_{p', q}
\leq \sqrt{\frac{\pi}{2}} \mathbb{E}\|(\varepsilon_{ij}X_{ij}|G_{ij})\|_{p', q} = \sqrt{\frac{\pi}{2}} \mathbb{E}_{X}\|X_{ij}|G_{ij}\|_{p', q}
\leq C(p, q) \left( (\log m)^{1+\frac{1}{p'}} \mathbb{E}_{X}\mathbb{E}_{G} \max_{1 \leq i \leq m} \left( \sum_{j=1}^{m} |X_{ij}G_{ij}|^p \right)^{1/p}
+ \mathbb{E}_{X}\mathbb{E}_{G} \max_{1 \leq j \leq n} \left( \sum_{i=1}^{m} |X_{ij}G_{ij}|^q \right)^{1/q} \right),
$$

where in the last step we used Corollary 1.3 to estimate the mean with respect to $G$. We use (4.6) with $\gamma = 1$ (to $\mathbb{E}_{G}$ in each term above separately) to get the assertion.

Remark 5.1. Using [6, Theorem 1.1] instead of Theorem 1.1 in the proof above yields a slightly better estimate in Theorem 1.6:

$$
(5.1) \quad \mathbb{E}\|X\|_{p', q} \leq C(p, q) \left( (\log m)^{1+\frac{1}{p'}} \mathbb{E}_{X}\mathbb{E}_{G} \max_{1 \leq i \leq m} \left( \sum_{j=1}^{m} |X_{ij}|^p \right)^{1/p}
+ \sqrt{\log n}\mathbb{E}_{X}\mathbb{E}_{G} \max_{1 \leq j \leq n} \left( \sum_{i=1}^{m} |X_{ij}|^q \right)^{1/q} \right).
$$

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