ARTIN \( L \)-FUNCTIONS ON \( \text{PGL}_3 \)

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Abstract. We study Artin \( L \)-functions on a finite 2-dimensional complex \( X_\Gamma \) arising from \( \text{PGL}_3 \) attached to finite-dimensional representations \( \rho \) of its fundamental group. Some key properties, such as rationality, functional equation, and invariance under induction, of these functions are proved. Moreover, using a cohomological argument, we establish a connection between the Artin \( L \)-functions and the \( L \)-function of \( \rho \), extending the identity on zeta functions of \( X_\Gamma \) obtained in [KL, KLW].

1. Introduction

Let \( F \) be a nonarchimedean local field with the ring of integers \( \mathcal{O} \) and a uniformizer \( \pi \) such that its residue field \( \kappa = \mathcal{O}/\pi \mathcal{O} \) has cardinality \( q \). The building attached to the group \( G = \text{PGL}_3(F) \) is a contractible 2-dimensional simplicial complex \( X \), whose vertices are homothety classes of \( \mathcal{O} \) lattices of rank-3 in the 3-dimensional vector space \( F^3 \). The group \( G \) acts transitively on \( X \) as automorphisms.

Fix a discrete cocompact torsion-free subgroup \( \Gamma \) of \( G \) so that \( X_\Gamma = \Gamma \backslash X \) is a finite complex locally isomorphic to \( X \). Then \( X_\Gamma \) has \( X \) as its universal cover and \( \Gamma \) as its fundamental group. Two kinds of zeta functions, the edge zeta function and the chamber zeta function, on \( X_\Gamma \) were considered in [KL, KLW]. They extend the Ihara zeta functions for graphs to finite two-dimensional complexes. Analogous to the two expressions for the Ihara zeta function in terms of the vertex adjacency operator by Ihara [Ih] and edge adjacency operator by Hashimoto [Ha1], the edge and chamber zeta functions for \( X_\Gamma \) are shown to be rational functions and they satisfy an identity involving operators on the vertices, edges and chambers, resembling the zeta function for a smooth irreducible projective surface defined over a finite field. Along a similar vein, the edge and chamber zeta functions for finite quotients of the building of another rank two group \( \text{Sp}_4(F) \) have been investigated in [FLW], where rationality and identities for the zeta functions are also obtained, but they are more complicated due to the nature of the group.

Let \( \rho \) be a \( d \)-dimensional representation of \( \Gamma \) acting on the space \( V_\rho \) over \( \mathbb{C} \). In this paper we study the \( i \)-th Artin \( L \)-function of \( X_\Gamma \) attached to \( \rho \) for \( i = 1, 2 \) defined by

\[
L_i(X_\Gamma, \rho, u) = \prod_{p} \det \left( I - \rho(Frob_p)u^{l_A(p)} \right)^{-1},
\]

where \( I \) is the identity \( d \times d \) matrix, \( |p| \), which plays the role of an \( i \)-dimensional prime, runs through all equivalence classes of primitive uni-type closed \( i \)-dimensional geodesics \( p \) in \( X_\Gamma \), \( l_A(p) \) is the algebraic length of \( p \), and \( Frob_p \) is a conjugacy class in \( \Gamma \) associated to the “prime” \( |p| \). See §3.3 and §4.4 for detailed definitions. When \( \rho \) is the trivial representation of \( \Gamma \), the above \( L \)-functions coincide with the edge and chamber zeta functions studied in [KL, KLW].

Denote by \( N_i \) the number of \( i \)-dimensional simplices in \( X_\Gamma \). The Euler characteristic \( \chi(X_\Gamma) \) of the complex \( X_\Gamma \) is equal to \( N_0 - N_1 + N_2 \). We summarize the main properties of these Artin \( L \)-functions.

Theorem 1.0.1. \( L_1(X_\Gamma, \rho, u) \) converges absolutely for \( |u| \) small enough to a rational function of the form

\[
L_1(X_\Gamma, \rho, u) = \frac{1}{\det(I - A_E(\rho, u))},
\]

where \( A_E(\rho, u) \) is an edge adjacency operator acting on a free \( \mathbb{C}[u] \)-module of rank \( 2dN_1 \). Consequently, \( L_1(X_\Gamma, \rho, u) \) has a meromorphic continuation to the whole \( u \)-plane, with reciprocal equal to a polynomial of degree \( 2dN_1 \).

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4.3.1 shows its connection with the two vertex adjacency operators on $L$ of irreducible subrepresentations. Define the (unramified) $L$-function, where $\sigma$ is an irreducible representation of $G$. Since $\Gamma$ is discrete and cocompact, the induced representation $\text{Ind}_G^G$ is the identity operator.

With the above notation, for a point $u \in X_\Gamma$, the $L$-function of $\text{Ind}_G^G\rho$ can be decomposed into a direct sum of irreducible subrepresentations. Define the (unramified) $L$-function of $\text{Ind}_G^G\rho$ to be

$$L(\text{Ind}_G^G\rho, u) = \prod_{\sigma} L(\sigma, u)^{m(\sigma)},$$

where $\sigma$ runs through all unramified irreducible representations of $G$ and $m(\sigma)$ is the multiplicity of $\sigma$ in $\text{Ind}_G^G\rho$. It is shown in §2 that the reciprocal of $L(\text{Ind}_G^G\rho, u)$ is a polynomial of degree $3dN_0$, and Proposition 4.3.1 shows its connection with the two vertex adjacency operators on $X_\Gamma$.

The main purpose of this paper is to prove the following identity on $L$-functions.

**Theorem 1.0.2.** Let $L_2(X_\Gamma, \rho, u)$ converges absolutely for $|u|$ small enough to a rational function of the form

$$L_2(X_\Gamma, \rho, u) = \frac{1}{\det(I - A_C(\rho, u))},$$

where $A_C(\rho, u)$ is a chamber adjacency operator acting on a free $\mathbb{C}[u]$-module of rank $3dN_2$. Consequently, $L_2(X_\Gamma, \rho, u)$ has a meromorphic continuation to the whole $u$-plane, with reciprocal equal to a polynomial of degree $3dN_2$.

These zeta functions encode the geometric information of $\Gamma$. The spectral information of $\Gamma$ is characterized by the (local) $L$-function which we now explain. The Satake parameter attached to an irreducible unramified representation $\sigma$ of $G$ is a semisimple conjugacy class $s(\sigma)$ in the complex dual group $\hat{G}(\mathbb{C}) \cong \text{SL}_3(\mathbb{C})$ of $G$. The $L$-function of $\sigma$ attached to the standard representation of $\text{SL}_3(\mathbb{C})$ is

$$L(\sigma, u) = \det(I - s(\sigma)u)^{-1}.$$

Since $\Gamma$ is discrete and cocompact, the induced representation $\text{Ind}_G^G\rho$ can be decomposed into a direct sum of irreducible subrepresentations. Define the (unramified) $L$-function of $\text{Ind}_G^G\rho$ to be

$$L(\text{Ind}_G^G\rho, u) = \prod_{\sigma} L(\sigma, u)^{m(\sigma)},$$

where $\sigma$ runs through all unramified irreducible representations of $G$ and $m(\sigma)$ is the multiplicity of $\sigma$ in $\text{Ind}_G^G\rho$. It is shown in §2 that the reciprocal of $L(\text{Ind}_G^G\rho, u)$ is a polynomial of degree $3dN_0$, and Proposition 4.3.1 shows its connection with the two vertex adjacency operators on $X_\Gamma$.

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**Theorem 1.0.3.** With the above notation, for a $d$-dimensional representation $\rho$ of $\Gamma$ we have

$$(1 - u^3)\chi(X_\Gamma)dL(\text{Ind}_G^G\rho, qu) = \frac{L_1(X_\Gamma, \rho, u)}{L_2(X_\Gamma, \rho, -u)}.$$

This theorem is proved by a cohomological argument. More precisely, we define a cochain complex $C^*$ whose $i$-th cochain group $C^i(X_\Gamma, \rho)$ for $0 \leq i \leq 2$ consists of $V_\rho \otimes_{\mathbb{C}} \mathbb{C}[u]$-valued functions on pointed $i$-simplicies on which the group $\Gamma$ acts via $\rho$, and the coboundary maps are suitable deformations (involving the variable $u$) of the usual coboundary maps. See §4.2 and §5.1 for details. We show in §6 that there exist cochain endomorphisms $\Phi_i = \Phi_i(u)$ on $C^i(X_\Gamma, \rho)$ for $i = 0, 1$ and 2 whose determinants interpret the $L$-functions introduced above:

**Theorem 1.0.4.**

1. $\det(\Phi_0 | C^0(X_\Gamma, \rho)) = L(\text{Ind}_G^G\rho, qu)^{-1}$.
2. $\det(\Phi_1 | C^1(X_\Gamma, \rho)) = (1 - u^3)dN_1L_1(X_\Gamma, \rho, u)^{-1}$.
3. $\det(\Phi_2 | C^2(X_\Gamma, \rho)) = (1 - u^3)^{2dN_2}L_2(X_\Gamma, \rho, -u)^{-1}$.

The desired identity in Theorem 1.0.3 then follows from the fact that for each $i$, the cochain map $\Phi_i$ on $C^i(X_\Gamma, \rho)$ is homotopically equivalent to the cochain map multiplication by $1 - u^3$.

**Remark.** The combinatorial Artin $L$-functions attached to representations were considered by Ihara in [BH], Hashimoto in [Ha2, Ha3], Mizuno and Sato in [MS], and Stark and Terras in [ST2] for graphs. In the case of a finite connected undirected graph $Y$, the equivalence classes of primitive tailless closed geodesics in $Y$, that is, the “primes” for $Y$, naturally correspond to the conjugacy classes of nonidentity primitive elements in the fundamental group of $Y$. As pointed out in [KL], this is no longer the case for our $X_\Gamma$. Namely there are more Frobenius conjugacy classes than conjugacy classes of nonidentity primitive elements in $\Gamma$. Our results are generalizations of those of Hashimoto [Ha2] from graphs to two-dimensional complexes, but our method is different from his. For trivial $\rho$, two different proofs are in the literature: the one in [KL] results from counting the number of desired closed geodesics of given length, while in [KLW] the identity is derived using representation theory. The cohomological method described above is a generalization of the approach by Bass [Ba] and re-interpreted by Hoffman [Ho] for graphs. In particular it provides a third proof of the identities on zeta functions established in [KL, KLW],
Set
\[ \epsilon(\rho, u) = \left( 1 - \left( \frac{u}{q} \right)^3 \right)^{dN_0/2} \left( 1 - (qu)^3 \right)^{dN_0/2}. \]

In §2.2 we show that there is a functional equation relating the L-function of \( \text{Ind}_{\Gamma}^G \rho \) and that of \( \text{Ind}_{\Gamma'}^G \rho' \).

**Theorem 1.0.5.** The following functional equation holds:
\[ \epsilon(\rho, \frac{1}{qu}) L(\text{Ind}_{\Gamma}^G \rho, \frac{1}{qu}) = \epsilon(\rho^*, qu) L(\text{Ind}_{\Gamma'}^G \rho^*, qu). \]

Here \( \rho^* \) is the contragredient representation of \( \rho \).

Combined with Theorem 1.0.3 the above functional equation can be restated in terms of the quotient of the Artin L-functions.

**Theorem 1.0.6.** The following functional equation holds:
\[ \hat{\epsilon}(\rho, \frac{1}{q^2u}) \frac{L_1(X_{\Gamma}, \rho, \frac{1}{q^2u})}{L_2(X_{\Gamma}, \rho, -\frac{1}{q^2u})} = \hat{\epsilon}(\rho^*, u) \frac{L_1(X_{\Gamma}, \rho^*, u)}{L_2(X_{\Gamma}, \rho^*, -u)}. \]

Here \( \hat{\epsilon}(\rho, u) = \epsilon(\rho, qu)(1 - u^3)^{-\chi(X_{\Gamma})d} \).

It follows immediately from the definition that for \( i = 1, 2 \) the Artin L-function decomposes into a product when the representation is a direct sum, that is,
\[ L_i(X_{\Gamma}, \rho_1 \oplus \rho_2, u) = L_i(X_{\Gamma}, \rho_1, u)L_i(X_{\Gamma}, \rho_2, u). \]

In §7 we show that the Artin L-function is invariant under induction, just like the usual Artin L-functions attached to representations of the absolute Galois group of a number field.

**Theorem 1.0.7.** Suppose \( \rho \) is induced from a finite-dimensional representation \( \rho' \) of a finite-index subgroup \( \Gamma' \) of \( \Gamma \). Let \( X_{\Gamma'} = \Gamma' \backslash X \). Then for \( i = 1, 2 \) we have
\[ L_i(X_{\Gamma}, \rho, u) = L_i(X_{\Gamma'}, \rho', u). \]

The corresponding statement for graphs was proved by Hashimoto in [Ha3], where it was proved by counting closed geodesics, using definition of the Artin L-function. Our proof in §7 compares the actions of the edge/chamber adjacency operators on \( X_{\Gamma} \) and \( X_{\Gamma'} \), using Theorems 1.0.1 and 1.0.2.

In particular, when \( \rho' \) is the identity representation of a finite-index normal subgroup \( \Gamma' \), the induced representation \( \text{Ind}_{\Gamma'}^G 1 \) decomposes into the direct sum \( \bigoplus_{\sigma \in \overline{\Gamma'/\Gamma}} m(\sigma)\sigma \), where \( \overline{\Gamma'/\Gamma} \) consists of all irreducible representations of the quotient group \( \Gamma'/\Gamma' \), and the multiplicity \( m(\sigma) \) is equal to the degree of \( \sigma \). Thus
\[ Z_i(X_{\Gamma'}, u) = L_i(X_{\Gamma'}, 1, u) = \prod_{\sigma \in \overline{\Gamma'/\Gamma}} L_i(X_{\Gamma}, \sigma, u)^{m(\sigma)}, \]
for \( i = 1, 2 \). By Theorems 1.0.1 and 1.0.2 the reciprocal of each \( L_i \) above is a polynomial, hence we conclude

**Corollary 1.0.8.** Let \( \Gamma' \) be a normal subgroup of \( \Gamma \) of finite index. Then for \( i = 1 \) and 2, \( Z_i(X_{\Gamma'}, u)^{-1} \) divides \( Z_i(X_{\Gamma'}, u)^{-1} \).

The corresponding statement for graphs was proved in [Ha2].

2. L-functions and Functional Equations

2.1. L-functions. The group \( K = \text{PGL}_3(\mathbb{O}) \) is the standard maximal compact subgroup of \( G = \text{PGL}_3(F) \). The Hecke algebra \( H(G, K) \) is generated by the following two Hecke operators:
\[ A_1 = K \left( \begin{array}{cc} 1 & \pi \\ 0 & 1 \end{array} \right) K \quad \text{and} \quad A_2 = K \left( \begin{array}{cc} 1 & \pi \\ \pi & 1 \end{array} \right) K. \]

The Satake isomorphism \( \phi : H(G, K) \to C[z_1, z_2, z_3]^{S_3}/(z_1z_2z_3 - 1) \) is characterized by
\[ \phi(A_1) = q(z_1 + z_2 + z_3) \quad \text{and} \quad \phi(A_2) = q(z_1z_2 + z_2z_3 + z_1z_3). \]
For an unramified irreducible representation $(\sigma, V_\sigma)$ of $G$ with the Satake parameter $s(\sigma)$ equal to the conjugacy class of $\left( \begin{smallmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{smallmatrix} \right)$, its $K$-fixed subspace $V_\sigma^K$ is one-dimensional on which $I - A_1 u + q A_2 u^2 - q^3 u^3 I$ acts as multiplication by the scalar
\[
\det(I - A_1 u + q A_2 u^2 - q^3 u^3 I \mid V_\sigma^K) = 1 - q(\lambda_1 + \lambda_2 + \lambda_3)u + q^2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3)u^2 - q^3 u^3
\]
\[
= \prod_{\lambda=1}^{3}(1-q\lambda_iu) = L(\sigma, qu)^{-1}.
\]
Here and thereafter $\det(A|W)$ denotes the determinant of the linear operator $A$ on the finite dimensional vector space $W$. The induced representation
\[
\text{Ind}^G_G \rho = \{ f : G \to V_\rho \mid f(\gamma x) = \rho(\gamma)f(x), \text{ for all } \gamma \in \Gamma \text{ and } x \in G \}
\]
decomposes into a direct sum of irreducible representations $\sigma$ of $G$. The total number of unramified $\sigma$’s, counting multiplicity, is equal to the dimension of $(\text{Ind}^G_G \rho)^K$. Hence
\[
L(\text{Ind}^G_G \rho, qu) = \prod_\sigma L(\sigma, qu) = \prod_\sigma \det(I - A_1 u + q A_2 u^2 - q^3 u^3 I \mid V_\sigma^K)^{-1}
\]
\[
= \det(I - A_1 u + q A_2 u^2 - q^3 u^3 I \mid (\text{Ind}^G_G \rho)^K)^{-1}.
\]
We record this in

**Proposition 2.1.1.**

\[
L(\text{Ind}^G_G \rho, qu) = \det(I - A_1 u + q A_2 u^2 - q^3 u^3 I \mid (\text{Ind}^G_G \rho)^K)^{-1}.
\]

Note that the dimension of $(\text{Ind}^G_G \rho)^K$ is equal to the cardinality of the double cosets in $\Gamma \backslash G / K$ times the dimension of $V_\rho$, that is, $N_0 d$. Hence the denominator of $L(\text{Ind}^G_G \rho, qu)$ is a polynomial of degree $3dN_0$.

### 2.2. A functional equation.**

In this subsection we prove the functional equation satisfied by $L(\text{Ind}^G_G \rho, qu)$ as stated in Theorem 1.0.6. Given an irreducible unramified representation $(\sigma, V)$ with the Satake parameter $s_\rho = \left( \begin{smallmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{smallmatrix} \right)$, its contragredient representation $(\sigma^*, V^*)$ has the Satake parameter equal to $s_\rho^{-1}$. Since $\lambda_1 \lambda_2 \lambda_3 = 1$, we have
\[
A_1|_V = q(\lambda_1 + \lambda_2 + \lambda_3) = q(\lambda_2^{-1} \lambda_3^{-1} + \lambda_1^{-1} \lambda_3^{-1} + \lambda_1^{-1} \lambda_2^{-1}) = A_2|_V.
\]
and
\[
A_2|_V = q(\lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_1 \lambda_2) = q(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}) = A_1|_{V^*}.
\]
Therefore,
\[
L(\sigma^*, qu) = \det(1 - A_2 u + q A_1 u^2 - q^3 u^3 | V)^{-1}
\]
and
\[
L(\text{Ind}^G_G \rho, \frac{1}{qu}) = \det \left( 1 - A_1 \frac{1}{q^2 u} + A_2 \frac{1}{q^4 u^2} - \frac{1}{q^6 u^3} \mid (\text{Ind}^G_G \rho)^K \right)^{-1}
\]
\[
= (-q^3 u^3)^{dN_0} \det(1 - A_2 u + q A_1 u^2 - q^3 u^3 \mid (\text{Ind}^G_G \rho)^K)^{-1}
\]
\[
= (-q^3 u^3)^{dN_0} L(\text{Ind}^G_G \rho^*, qu).
\]
Let
\[
\epsilon(\rho, u) = \left( 1 - \frac{u}{q} \right)^{dN_0/2} \left( 1 - (qu)^3 \right)^{dN_0/2}.
\]
It is easy to verify that
\[
\epsilon(\rho, \frac{1}{qu}) L(\text{Ind}^G_G \rho, \frac{1}{qu}) = \epsilon(\rho^*, qu) L(\text{Ind}^G_G \rho^*, qu),
\]
which proves Theorem 1.0.6.
3. Paths and galleries on the simplicial complex $X$

These were discussed in detail in [KL]. In this section we recall them and set up notation to be used later.

3.1. The building $X$ of $\text{PGL}_3(F)$. The vertices of the building $X$ of $\text{PGL}_3(F)$ are the homothety classes of $\mathcal{O}$-lattices $a$ in $F^3$. Given an inclusion relation of lattices $a_1 \supseteq a_2 \supseteq \cdots \supseteq a_r$, denote by $[a_1 \supseteq a_2 \supseteq \cdots \supseteq a_r]$ the homothety class of this relation. Hence the vertices of $X$, also called the (pointed) 0-simplices of $X$, are denoted by $[a]$. Two vertices $[a_1]$ and $[a_2]$ form an edge (or 1-simplex) $E = \{[a_1], [a_2]\}$ of $X$ if there exist representatives $a_1$ and $a_2$ so that $\pi^{-1} a_2 \supseteq a_1 \supseteq a_2(\supseteq \pi a_1)$. In this case, $a_1/a_2$ is a proper subspace of $\pi^{-1} a_2/a_2 \cong (F_q)^3$ with dimension

$$|a_1/a_2| := \dim F_q a_1/a_2 = 1 \text{ or } 2, \quad \text{and} \quad |a_2/\pi a_1| = \dim F_q a_2/\pi a_1 = 3 - |a_1/a_2|.$$ 

To $E$, we associate two pointed edges: $[\pi^{-1} a_2 \supseteq a_1 \subsetneq a_2]$ of type $[a_1/a_2]$, and $[a_1 \supseteq a_2 \supseteq \pi a_1]$ of type $[a_2/\pi a_1]$. Define the algebraic length of a pointed edge to be its type. Three vertices, $[a_1], [a_2]$ and $[a_3]$ form a chamber (or 2-simplex) $C = \{[a_1], [a_2], [a_3]\}$ if there exist representatives $a_1, a_2$ and $a_3$ so that $a_1 \supseteq a_2 \supseteq a_3 \supseteq a_1(\supseteq \pi a_2 \supseteq \pi a_3)$. In this case, we associate to $C$ three pointed chambers $[\pi^{-1} a_3 \supseteq a_1 \supseteq a_2]$, $[a_1 \supseteq a_2 \supseteq a_3 \supseteq a_2]$, and $[a_1 \supseteq a_2 \supseteq a_3 \supseteq \pi a_3]$. The algebraic length of a pointed chamber $[a_1 \supseteq a_2 \supseteq a_3 \supseteq a_1]$ is defined to be the type of the pointed edge $[a_1 \supseteq a_2 \supseteq a_3 \supseteq a_1]$, which is always equal to 1.

An element $g \in G$ acts on the vertices of $X$ by sending $[a]$ to $[ga]$. It preserves edges and chambers, and hence $G$ acts on $X$ as automorphisms. Note that $K$ is the stabilizer of the vertex represented by the lattice spanned by the standard basis of $F^3$. As $G$ acts transitively on vertices of $X$, the coset space $G/K$ parametrizes the vertices of $X$. Furthermore, $G$ also acts transitively on pointed edges and pointed chambers and these two sets can be parametrized by cosets of certain parahoric subgroup and Iwahoric subgroup of $G$, respectively. See [KL] for details.

3.2. Out-neighbors. The out-neighbors of a pointed edge $[\pi^{-1} a_2 \supseteq a_1 \subsetneq a_2]$ of type $[a_1/a_2]$ are the pointed edges $[\pi^{-1} a_3 \supseteq a_2 \supseteq a_3]$ with type $[a_2/a_3] = [a_1/a_2]$ such that $[a_1], [a_2], [a_3]$ do not form a chamber. In this case we have two relations $a_1 \supseteq a_2 \supseteq \pi a_1 \supseteq \pi a_2$ and $a_1 \supseteq a_2 \supseteq a_3 \supseteq \pi a_2$. The condition $[a_1/a_2] = [a_2/a_3]$ implies that one of $a_3/\pi a_2$ and $\pi a_1/\pi a_2$ is a one-dimensional subspace of $a_2/\pi a_2 \cong F_q^3$ and the other is two-dimensional. Denote by $N(e)$ the collection of out-neighbors of a pointed edge $e$. Therefore, we obtain a criterion for out-neighbors of a pointed edge:

$$[\pi^{-1} a_3 \supseteq a_2 \supseteq a_3] \in N([\pi^{-1} a_2 \supseteq a_1 \supseteq a_2]) \iff [a_1/a_2] = [a_2/a_3], a_3 \not\supseteq \pi a_1, \pi a_1 \not\supseteq a_3 \iff [a_1/a_2] = [a_2/a_3], a_3 + \pi a_1 = a_2,$$

where $a_3 + \pi a_1$ is the lattice generated by $a_3$ and $\pi a_1$. Observe that a pointed edge has $q^2$ out-neighbors.

For a pointed chamber $c = [\pi^{-1} a_3 \supseteq a_1 \supseteq a_2 \supseteq a_3]$, its out-neighbors are pointed chambers $[\pi^{-1} a_4 \supseteq a_2 \supseteq a_3 \supseteq a_4]$ with $[a_4] \neq [a_1]$; denote the collection by $N(c)$. In terms of lattices, we have

$$[\pi^{-1} a_4 \supseteq a_2 \supseteq a_3 \supseteq a_4] \in N([\pi^{-1} a_3 \supseteq a_1 \supseteq a_2 \supseteq a_3]) \iff a_4 \not\supseteq \pi a_1 \iff a_4 + \pi a_1 = a_3.$$ 

Hence a pointed chamber has $q$ out-neighbors.

3.3. Paths and galleries. An edge path $p$ of $X$ is a sequence $e_1 \rightarrow e_2 \rightarrow \cdots \rightarrow e_n$ of pointed edges in the 1-skeleton of $X$ such that $e_{i+1}$ is an out-neighbor of $e_i$ for $i = 1, \ldots, n - 1$; all pointed edges in $p$ have the same type $j$, equal to 1 or 2, called the type of the path. We define the geometric length $l_G(p)$ of $p$ to be $n$ and the algebraic length $l_A(p)$ to be $jn$. Note that a path in $X$ is a directed straight line segment in an apartment.

A type 1 gallery $g$ in $X$ is a sequence of pointed chambers $c_1 \rightarrow \cdots \rightarrow c_n$ in $X$ so that $c_{i+1}$ is an out-neighbor of $c_i$ for $i = 1, \ldots, n - 1$. In other words, there exists a sequence of lattices $a_1 \supseteq \cdots \supseteq a_{n+2}$ so that $c_i = [\pi^{-1} a_{i+2} \supseteq a_1 \supseteq a_{i+1} \supseteq a_{i+2}]$ for $1 \leq i \leq n$. We define both the geometric length $l_G(g)$ and the algebraic length $l_A(g)$ of $g$ to be $n$. Geometrically a type one gallery is a directed straight gallery in an apartment.

For convenience, a type 1 gallery in $X$ is called a uni-type 2-dimensional geodesic, and an edge path contained in the 1-skeleton of $X$ is called a uni-type 1-dimensional geodesic.
4. Artin $L$-functions attached to representations of $\Gamma$

4.1. The finite quotient $X_\Gamma$. Let $\Gamma$ be a discrete cocompact torsion-free subgroup of $G$ so that $X_\Gamma := \Gamma \backslash G$ is a finite simplicial complex locally isomorphic to $X$. Since $X$ is contractible, $\Gamma$ is isomorphic to the fundamental group of $X_\Gamma$. Explicit constructions of such finite complexes can be found in [Sar] for instance, in which the 1-skeleton of the complexes may be described as Cayley graphs on subgroups of $\text{PGL}_3(\mathbb{F}_q)$ containing $\text{PSL}_3(\mathbb{F}_q)$.

Denote by $X_i$ the set of pointed $i$-simplices of $X$ for $i = 0, 1, 2$. The group $\Gamma$ acts freely and transitively on $X_i$ by left translation. Fix a choice of a subset $S_i$ of $X_i$ representing the orbit space $\Gamma \backslash X_i$. Then the elements in $X_i$ can be labeled by $\Gamma S_i$. Geometrically the building $X$ is a maximal unramified cover of $X_\Gamma$ with covering group $\Gamma$. The fibre of an $i$-simplex of $X_\Gamma$ represented by $s \in S_i$ is $\Gamma S$. For the convenience of later discussions, we require that if a pointed 2-simplex $[\pi^{-1} a_2 \supseteq a_0 \supseteq a_1 \supseteq a_2]$ lies in $S_2$, so do $[a_0 \supseteq a_1 \supseteq a_2 \supseteq \pi a_0]$ and $[a_1 \supseteq a_2 \supseteq \pi a_0 \supseteq \pi a_1]$; and if a pointed 1-simplex $[\pi^{-1} a_2 \supseteq a_1 \supseteq a_2]$ lies in $S_1$, then so does its opposite $[a_1 \supseteq a_2 \supseteq \pi a_1]$. For $i = 0, 1, 2$, the cardinality of $S_i$ is $(i+1)N_i$, where $N_i$ is the number of pointed $i$-simplices in $X_\Gamma$.

4.2. Cochain groups. Let $V_\rho[u]$ denote the tensor product $V_\rho \otimes \mathbb{C}[u]$ of $V_\rho$ with the polynomial ring $\mathbb{C}[u]$. It is a free $\mathbb{C}[u]$-module of rank $d$ admitting the action by $\Gamma$ on $V_\rho$. For each $i \in \{0, 1, 2\}$ denote by $C^i(X_\Gamma, \rho) = C^i(X_\Gamma, V_\rho[u])$ the space

$$C^i(X_\Gamma, \rho) = \{ f : X_i \to V_\rho[u] \mid f(\gamma x_i) = \rho(\gamma)f(x_i) \text{ for all } \gamma \in \Gamma \text{ and } x_i \in X_i \}.$$ 

Note that functions in $C^i(X_\Gamma, \rho)$ are determined by their values on $S_i$, hence it is a free module over $\mathbb{C}[u]$ of rank $d(i+1)N_i$.

4.3. Vertex Adjacency operators. Let $A_1$ and $A_2$ be the vertex adjacency operators on $C^0(X_\Gamma, \rho)$ given by

$$A_i f([a_0]) = \sum_{a_0 \supseteq b \supseteq \pi a_0, |a_0 / b| = i} f([b]).$$

Since the vertices of $X$ can be parametrized by the cosets $G/K$, functions in $C^0(X_\Gamma, \rho)$ as described above are precisely the functions in the space $\text{Ind}_i G^0 \rho$ which are right invariant by $K$. Therefore we may identify $C^0(X_\Gamma, \rho)$ with the set $(\text{Ind}_i G^0 \rho)_K \otimes \mathbb{C}[u]$. Under this identification, the adjacency operators $A_1$ and $A_2$ defined above coincide with the Hecke operators $A_1$ and $A_2$ in [2]. In view of Proposition 4.3.1. we conclude

Proposition 4.3.1.

$$L(\text{Ind}_i G^0 \rho, qu) = \frac{1}{\det(I - A_1 u + qA_2 u^2 - q^3 I u^3 \mid C^0(X_\Gamma, \rho))}.$$ 

4.4. Artin $L$-functions attached to representations of $\Gamma$. For $i = 1, 2$, two closed $i$-dimensional paths in $X_\Gamma$ are called equivalent if one can be obtained from the other by changing the starting simplex. Denote by $[c]$ the equivalence class of a closed $i$-dimensional path $c$. A closed $i$-dimensional path $c$ in $X_\Gamma$ is called primitive if it is not obtained by repeating a shorter path more than once; it is called a uni-type geodesic if it lifts to a uni-type geodesic in $X$.

Denote by $P_i^{(n)}$ the set of all $i$-dimensional uni-type closed geodesics in $X_\Gamma$ with geometric length $n$, and by $P_i$ the union of $P_i^{(n)}$ for $n \geq 1$. Let $P_i^{pr}$ be the subset of primitive paths in $P_i$, and $[P_i^{pr}]$ be the set of equivalence classes of paths in $P_i^{pr}$. The elements in $[P_i^{pr}]$ play the role of primes for the $i$-th zeta and Artin $L$-functions.

Given an element $p$ in $P_i^{(n)}$, its starting pointed edge is represented by a unique $s_0 \in S_1$ and $p$ can be uniquely lifted to a uni-type path $\tilde{p} : s_0 \to s_1 \to \cdots \to s_n = \gamma p s_0$ in $X$, where $\gamma p \in \Gamma$. If $p$ is lifted to a path $\tilde{p}'$ in $X$ starting at $s_0' = \gamma a s_0$, then $\tilde{p}'$ ends at $\gamma \gamma p \gamma a^{-1} s_0$.

Hence to $p$ in $P_i$ we can associate an element $\gamma p \in \Gamma$ which is unique up to conjugation. Note that if $s_0' = \gamma p s_0$, then $\tilde{p}' = \gamma p \tilde{p}$. Thus $p$ repeated twice is lifted to the path $s_0 \to s_1 \to \cdots \to s_n = \gamma p s_0 \to \gamma p s_1 \to \cdots \to \gamma p s_n = \gamma^2 p s_0$. The projection to $X_\Gamma$ of the sub-paths $s_j \to \cdots \to s_{j+1}$ for $1 \leq j \leq n - 1$ runs through the paths equivalent to $p$. This shows that the conjugacy class $[\gamma p]$ in $\Gamma$ of $\gamma p$ depends only on the equivalence class $[p]$ of $p$. When $p$ is primitive, call $[\gamma p]$ the “Frobenius at the prime $p$” and denote it by $\text{Frob}_{[p]}$. 

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In a similar manner, to $g$ in $P_2$ we associate the conjugacy class $[g]$ and define $\text{Frob}_g$ for each prime $[g] \in [P_2^p]$. Observe that if $p$ in $P_2$ is obtained from the path $p'$ by repeating it $k$-times, then $l_A(p) = k \cdot l_A(p')$, $l_G(p) = k \cdot l_G(p')$ and $\gamma_p = (\gamma_{p'})^k$. 

Now fix a $d$-dimensional representation $(\rho, V_{\rho})$ of $\Gamma$. For $i = 1, 2$ define the $i$-th Artin $L$-function of $X_\Gamma$ associated to $\rho$ to be

$$L_i(X_\Gamma, \rho, u) = \prod_{[c] \in [P_2^p]} \frac{1}{\det (I - \rho(\text{Frob}_c)u^{l_A(c)})}.$$ 

Note that when $\rho$ is the trivial representation of $\Gamma$, the $i$th Artin $L$-function coincides with the zeta function $Z_i(X_\Gamma, u)$ defined in [KL, KLW]. Since the determinant of a matrix is invariant under conjugation and two equivalent paths have the same algebraic length, the Artin $L$-function above is well-defined. We shall show in §4.6 that it converges absolutely for $|u|$ small to the reciprocal of a polynomial.

4.5. Edge adjacency operator. Define the edge adjacency operator $A_E(\rho, u)$ on $C^1(X_\Gamma, \rho)$ by sending $f \in C^1(X_\Gamma, \rho)$ to $A_E(\rho, u)f$ whose value at $e \in X_1$ is given by

$$A_E(\rho, u)f(e) = u^{l_A(e)} \sum_{e' \in N(e)} f(e').$$

We proceed to represent $A_E(\rho, u)$ by a block matrix $M_E(\rho, u)$ whose rows and columns are parametrized by the set $S_1$ representing the pointed 1-simplices in $X_\Gamma$. Given $s \in S_1$, consider the above definition at $e = s$. Then $e' \in N(e)$ lies in the $\Gamma$-orbit of some $s' \in S_1$, and there is a unique $\gamma_{ss'} \in \Gamma$ such that $e' = \gamma_{ss'}s'$ and hence $f(e') = \rho(\gamma_{ss'})f(s')$. The $ss'$-entry of $M_E(\rho, u)$ is the $d \times d$ matrix $\rho(\gamma_{ss'})u^{l_A(s)}$ if $\gamma_{ss'}$ is an out-neighbor of $\Gamma s$ in $X_\Gamma$, and the zero $d \times d$ matrix otherwise. The block matrix $M_E(\rho, u)$ representing $A_E(\rho, u)$ depends on the choice of the set $S_1$. For a different choice of $S_1$, the matrix is replaced by a conjugation.

4.6. A proof of Theorem 1.0.1. A closed path $p$ in $P_1^{(n)}$ is a sequence of pointed edges $e_0 \rightarrow e_1 \cdots \rightarrow e_n = e_0$ of length $n$, where each $e_i$ is represented by a unique $s_i \in S_1$ and $e_{i+1}$ is an out-neighbor of $e_i$ in $X_\Gamma$ for $0 \leq i \leq n - 1$. Its lifting $\bar{p}$ in $X$ starting at $s_0$ is

$$\bar{p} : s_0 \rightarrow \gamma_{s_0s_1}s_1 \rightarrow (\gamma_{s_0s_1}\gamma_{s_1s_2})s_2 \rightarrow \cdots \rightarrow (\gamma_{s_0s_1} \cdots \gamma_{s_{n-1}s_n})s_n = \gamma_p s_0$$

so that the associated $\gamma_p$ explained in §4.4 is $\gamma_p = \gamma_{s_0s_1} \cdots \gamma_{s_{n-1}s_n}$. Thus we have

$$\rho(\gamma_p) = \rho(\gamma_{s_0s_1}) \cdots \rho(\gamma_{s_{n-1}s_n}).$$

On the other hand, the $s_0s_0$ entry of $M_E(\rho, u)^n$ is the sum of all possible products of $n$ entries of $M_E(\rho, u)$ of the form $\rho(\gamma_{s_0s_1})\rho(\gamma_{s_1s_2}) \cdots \rho(\gamma_{s_{n-1}s_n})u^{l_A(s_0)+l_A(s_1)+\cdots+l_A(s_{n-1})}$, in which $s_n = s_0$, and for $0 \leq i \leq n - 1$, each $\Gamma s_{i+1}$ is an out-neighbor of $\Gamma s_i$. In other words, the $s_0s_0$ entry of $M_E(\rho, u)^n$ is the sum of $\rho(\gamma_p)u^{l_A(p)}$ over elements $p$ in $P_1^{(n)}$ starting at $\Gamma s_0$. This shows that

$$\text{Tr}(M_E(\rho, u)^n) = \sum_{p \in P_1^{(n)}} \text{Tr}(\rho(\gamma_p))u^{l_A(p)}.$$ 

To proceed, we shall use the following well-known facts in linear algebra.

**Proposition 4.6.1.** Let $A$ be a square matrix over $\mathbb{C}$ with norm less than 1. Then

$$\log(I - A) = -\sum_{n=1}^{\infty} \frac{A^n}{n} \text{ converges and } \text{Tr}(\log(I - A)) = \log(\det(I - A)).$$

Order the pointed edges in $S_1$ so that those of type 1 are before those of type 2. Then $M_E(\rho, u)$ is of the form

$$M_E(\rho, u) = \begin{pmatrix} B_1u & 0 \\ 0 & B_2u^2 \end{pmatrix}.$$
for some \(dN_1 \times dN_1\) complex matrices \(B_1\) and \(B_2\). The norms of \(B_1\) and \(B_2\) are bounded and depend only on \(A_E(\rho, u)\). It follows from Proposition 4.6.1 and 4.2 that, for \(|u| < \min\{\|B_1\|^{-1}, \sqrt{\|B_2\|^{-1}}\}\), there holds

\[
\log \det(I - A_E(\rho, u)) = \log \det(I - M_E(\rho, u)) = \text{Tr}(\log(I - M_E(\rho, u))) = - \sum_{n=1}^{\infty} \frac{\text{Tr}(M_E(\rho, u)^n)}{n}
\]

\[
= - \sum_{n=1}^{\infty} \sum_{p \in \mathcal{P}_n} \frac{\text{Tr}(\rho(p))u^A(p)}{n} = - \sum_{p \in \mathcal{P}_1} \frac{\text{Tr}(\rho(p))u^A(p)}{l_G(p)}
\]

\[
= - \sum_{m=1}^{\infty} \sum_{p \in \mathcal{P}_m} \frac{\text{Tr}(\rho(p))u^A(p^m)}{l_G(p^m)} = - \sum_{m=1}^{\infty} \sum_{p \in \mathcal{P}_m} \frac{\text{Tr}(\rho(p))u^A(p)^m}{m l_G(p)}.
\]

Since the number of closed paths equivalent to a primitive cycle is equal to its geometric length, the above can be rewritten as

\[
\log \det(I - A_E(\rho, u)) = - \text{Tr} \left( \sum_{[p] \in [\mathcal{P}_m]} \sum_{m=1}^{\infty} (\rho(\text{Frob}_p))u^A(p)^m \right)
\]

\[
= \text{Tr} \left( \sum_{[p] \in [\mathcal{P}_m]} \log \left( I - \rho(\text{Frob}_p)u^A(p) \right) \right)
\]

\[
= \log \left( \prod_{[p] \in [\mathcal{P}_m]} \det \left( I - \rho(\text{Frob}_p)u^A(p) \right) \right) = - \log L_1(X_\Gamma, \rho, u).
\]

Exponentiating both sides proves Theorem 1.0.1.

4.7. Chamber adjacency operator and a proof of Theorem 1.0.2. Define the chamber adjacency operator \(A_C(\rho, u)\) on \(C^2(X_\Gamma, \rho)\) by sending \(f \in C^2(X_\Gamma, \rho)\) to \(A_C(\rho, u)f\) whose value at \(c \in X_2\) is given by

\[
A_C(\rho, u)f(c) = u^A(c) \sum_{c' \in N(c)} f(c') = u \sum_{c' \in N(c)} f(c')
\]

because all pointed 2-simplices have algebraic length equal to 1. Similar to the edge adjacency operator, the chamber adjacency operator \(A_C(\rho, u)\) can be represented by a block matrix \(M_C(\rho, u)\) whose rows and columns are parametrized by the set \(S_2\) representing the pointed 2-simplices in \(X_\Gamma\). The \(cc'\) entry of \(M_C(\rho, u)\) is the \(d \times d\) matrix \(\rho(\gamma_{cc'})u\) if \(\gamma_{cc'}\) is an out-neighbor of \(c\) in \(X_\Gamma\), and the zero \(d \times d\) matrix otherwise.

By an argument similar to the previous subsection, we have that, for \(|u|\) small enough,

\[
\log \det(I + A_C(\rho, u)) = - \log L_2(X_\Gamma, \rho, -u),
\]

and hence Theorem 1.0.2 holds.

5. Pointed simplicial cohomology groups. For \(i = 0, 1, 2\) denote by \(C^i(X)\) the free \(\mathbb{C}[u]\)-module of functions \(f_i : X_i \to V_p[u]\). The action of \(\Gamma\) on \(X_i\) yields the action of \(\Gamma\) on \(C^i(X)\) given by \((\gamma f_i)(x) = f_i(\gamma x)\) for all \(\gamma \in \Gamma\), \(f_i \in C^i(X)\) and \(x \in X_i\). Then \(C^i(X_\Gamma, \rho)\) consists of the functions in \(C^i(X)\) on which the action of \(\Gamma\) is given by \(\rho\). Define the map \(d_i : d_i(u) : C^i(X) \to C^{i+1}(X)\) by

\[
(d_0f)([\pi^{-1}a_1 \supseteq a_0 \supseteq a_1]) = u^{[a_0/a_1]}f_0([a_1]) - f_0([a_0]),
\]

\[
(d_1f)([\pi^{-1}a_2 \supseteq a_0 \supseteq a_1 \supseteq a_2]) = u f_1([\pi^{-1}a_2 \supseteq a_1 \supseteq a_2]) - f_1([\pi^{-1}a_2 \supseteq a_0 \supseteq a_2]) + f_1([\pi^{-1}a_1 \supseteq a_0 \supseteq a_1]),
\]

and all other \(d_j\) to be the zero map. It follows from

\[
d_1(d_0f)([\pi^{-1}a_2 \supseteq a_0 \supseteq a_1 \supseteq a_2]) = u d_0f([\pi^{-1}a_2 \supseteq a_1 \supseteq a_2]) - d_0f([\pi^{-1}a_2 \supseteq a_0 \supseteq a_2]) + d_0f([\pi^{-1}a_1 \supseteq a_0 \supseteq a_1])
\]

\[
= u(u f_0([a_2]) - f_0([a_1])) - (u^2 f_0([a_2]) - f_0([a_0])) + (u f_0([a_1]) - f_0([a_0])) = 0
\]
that the $d_i$’s are coboundary maps. Note that when $u = 1$, $d_0$ and $d_1$ are the usual coboundary maps. As $d_i$ commutes with the action of $\Gamma$, it defines a coboundary map $d_i : C^i(X, \rho) \to C^{i+1}(X, \rho)$. This gives rise to the $i$-th pointed simplicial cohomology group

$$H^i(X, \rho) = \ker(d_i)/\text{Im}(d_{i-1})$$

for $i = 0, 1, 2$, which measures the failure of exactness at $C^i(X, \rho)$ to the $i$-th pointed simplicial cohomology group

$$C^* : 0 \to C^0(X, \rho) \xrightarrow{d_0} C^1(X, \rho) \xrightarrow{d_1} C^2(X, \rho) \to 0.$$ 

For $i = 1, 2$ define the map $\delta_i = \delta_i(u) : C^i(X) \to C^{i-1}(X)$ which sends $f_i \in C^i(X)$ to $C^{i-1}(X)$ given by

$$(\delta_1 f_1)([a_0]) = \sum_{a_0 \geq b \geq a_0/b = 1} u f_1([a_0 \geq b \geq \pi a_0]) - \sum_{a_0 \geq b \geq a_0/b = 2} q u^2 f_1([a_0 \geq b \geq \pi a_0])$$

and

$$(\delta_2 f_2)([\pi^{-1} a_1 \geq a_0 \geq 1]) = \sum_{a_0 \geq b \geq a_1} - u f_2([a_0 \geq b \geq a_1 \geq \pi a_0]) + \sum_{a_1 \geq b \geq a_0} u^2 f_2([a_1 \geq b \geq \pi a_0 \geq \pi a_1]).$$

Note that in $\delta_2 f_2$ only the first or the second sum is nonempty according as $|a_0/a_1| = 2$ or 1. Since $\delta_i$ commutes with the action of $\Gamma$, it defines a map $\delta_i(u) : C^i(X, \rho) \to C^{i-1}(X, \rho)$.

Let

$$\Delta_0(u) = \delta_1(u) d_0(u), \quad \Delta_1(u) = \delta_2(u) d_1(u) + d_0(u) \delta_1(u) \quad \text{and} \quad \Delta_2(u) = d_1(u) \delta_2(u).$$

Observe that $\Delta_i(u)$ is a cochain endomorphism on $C^i(X, \rho)$ and

$$\Delta_i(u) = 0 \quad \text{on} \quad H^i(X, \rho).$$

For $i = 0, 1$ and 2, define

$$\Phi_i(u) = \Delta_i(u) + (1 - u^3) I,$$

which is also a cochain endomorphism on $C^i(X, \rho)$.

Assuming Theorem 1.0.4 which will be proved in the next section, we establish Theorem 1.0.3. By setting $u = 0$, it is obvious that $\Phi_i(u)$ on $C^i(X, \rho)$ has nonzero determinant for $i = 0, 1, 2$. By a general theory on cohomology groups, we have

$$\prod_{i=0}^{\delta} \det(\Phi_i(u) \mid C^i(X, \rho))^{i-1} = \prod_{i=0}^{\delta} \det(\Phi_i(u) \mid H^i(X, \rho))^{i-1} = \prod_{i=0}^{\delta} \det((1 - u^3) I \mid H^i(X, \rho))^{i-1},$$

$$= \prod_{i=0}^{\delta} \det((1 - u^3) I \mid C^i(X, \rho))^{i-1} = (1 - u^3)^{d(N_0 - 2N_1 + 3N_2)}.$$ 

On the other hand, by Theorem 1.0.4 we also have

$$\prod_{i=0}^{\delta} \det(\Phi_i(u) \mid C^i(X, \rho))^{i-1} = \frac{\det(I - A_1(\rho) u + q A_2(\rho) u^2 - q^3 u^3 I) (1 - u^3)^{2dN_2} \det(I + A_C(\rho, u))}{(1 - u^3)^{2dN_2} \det(I - A_E(\rho, u))} \frac{L_1(X, \rho, u)}{L(\text{Ind}^G_\rho, qu) L_2(X, \rho, -u)},$$

Comparing the above two expressions of the alternating product of $\det(\Phi_i(u))$, we obtain

$$(1 - u^3)^{\chi(X, \rho)d} L(\text{Ind}^G_\rho, qu) = \frac{L_1(X, \rho, u)}{L_2(X, \rho, -u)},$$

which is Theorem 1.0.3.
6.1. The operator $\Phi_0(u)$. Recall that for a lattice $a_0$, $a_0/\pi a_0 \cong \mathbb{F}_q^3$. In this 3-dimensional vector space over $\mathbb{F}_q$ there are $q^2 + q + 1$ lines and the same number of planes. Further, a line in this space is contained in $q + 1$ planes. These results are restated in terms of lattices as follows.

**Proposition 6.1.1.**

(a) Given a lattice $a_0$, the number of pointed edges $[\pi^{-1}b \supseteq a_0 \supseteq b]$ in $X_1$ of type $i$ is equal to $q^2 + q + 1$ for $i = 1$ or 2.

(b) Given a type 1 pointed edge $[a_0 \supseteq c \supseteq \pi a_0]$, there are $q+1$ pointed chambers of the form $[a_0 \supseteq b \supseteq \pi a_0]$.

Using Proposition 6.1.1 (a), we compute $\Phi_0 f$ for $f \in C^0(X)$:

\[
\Phi_0(u)f([a_0]) = (\delta_1 d_0 + 1 - u^3) f([a_0]) = (1 - u^3) f([a_0]) + \sum_{a_0 \supseteq b \supseteq \pi a_0} (-q)^{|a_0/b| - 1} u^{|a_0/b|} (d_0 f)([a_0 \supseteq b \supseteq \pi a_0])
\]

\[
= (1 - u^3) f([a_0]) + \sum_{a_0 \supseteq b \supseteq \pi a_0} (-q)^{|a_0/b| - 1} u^{|a_0/b|} \left( u^{|b/\pi a_0|} f([\pi a_0]) - f([b]) \right)
\]

\[
= (1 - u^3) f([a_0]) + (q^2 + q - q(q^2 + q + 1)) u^3 f([a_0]) - uA_1 f([a_0]) + qu^2(A_2 f)([a_0])
\]

\[
= (I - A_1 u + qA_2 u^2 - q^3 u^3 I) f([a_0]).
\]

In other words, $\Phi_0(u) = I - A_1 u + qA_2 u^2 - q^3 u^3 I$ and hence $\det(\Phi_0(u) | C^0(X_1, \rho)) = L(\text{Ind}^\gamma \rho, qu)^{-1}$ by Proposition 4.3.1. This proves Theorem 1.0.4 (1).

6.2. The operator $\Phi_1(u)$. Introduce the following two operators on $C^1(X)$:

\[
Q(u)f([\pi^{-1}a_1 \supseteq a_0 \supseteq a_1]) = \sum_{a_0 \supseteq b \supseteq \pi a_0} u^{|a_0/b|} f([\pi^{-1}a_1 \supseteq b \supseteq a_1])
\]

and

\[
J_E(u)f([\pi^{-1}a_1 \supseteq a_0 \supseteq a_1]) = u^{|a_0/a_1|} f([a_0 \supseteq a_1 \supseteq \pi a_0]).
\]

They preserve the subspace $C^1(X_1, \rho)$, and will be viewed as operators on this space. As such, $J_E = J_E(u)$ is an involution up to scalar, more precisely, $J_E$ is multiplication by $u^3$. A straightforward computation shows that

\[
J_E A_E J_E^{-1} f([\pi^{-1}a_1 \supseteq a_0 \supseteq a_1]) = u^{|a_1/\pi a_0|} \sum_{a_0 \supseteq b \supseteq \pi a_0, [b/\pi a_0] = [a_0/a_1], a_0 \supseteq b \supseteq a_1} f([a_0 \supseteq b \supseteq \pi a_0]).
\]

Furthermore, $Q = Q(u)$ is unipotent with determinant 1 on $C^1(X_1, \rho)$, hence it is an automorphism there. Under $Q$, the action of $\delta_1$ is much simplified. More precisely, for $f_1 \in C^1(X_1, \rho)$ we have

\[
(\delta_1 Q)f_1([a_0]) = \sum_{a_0 \supseteq b \supseteq \pi a_0, [a_0/b] = 1} uQf_1([a_0 \supseteq b \supseteq \pi a_0]) - \sum_{a_0 \supseteq b \supseteq \pi a_0, [a_0/b] = 2} qu^2 Qf_1([a_0 \supseteq b \supseteq \pi a_0])
\]

\[
= \sum_{a_0 \supseteq b \supseteq \pi a_0, [a_0/b] = 1} u^{|a_0/b|} \sum_{b \supseteq c \supseteq \pi a_0} u^{|b/c|} f_1([a_0 \supseteq c \supseteq \pi a_0])
\]

\[
- \sum_{a_0 \supseteq b \supseteq \pi a_0, [a_0/b] = 2} qu^2 f_1([a_0 \supseteq b \supseteq \pi a_0])
\]

\[
= \sum_{a_0 \supseteq b \supseteq \pi a_0, [a_0/c] = 1} u^{|a_0/c|} f_1([a_0 \supseteq c \supseteq \pi a_0]) + \sum_{a_0 \supseteq b \supseteq \pi a_0, [a_0/c] = 2} (q + 1) u^2 f_1([a_0 \supseteq c \supseteq \pi a_0])
\]

\[
- \sum_{a_0 \supseteq b \supseteq \pi a_0, [a_0/b] = 2} qu^2 f_1([a_0 \supseteq b \supseteq \pi a_0])
\]

\[
= \sum_{a_0 \supseteq b \supseteq \pi a_0} u^{|a_0/c|} f_1([a_0 \supseteq c \supseteq \pi a_0]).
\]
The operator 
\[ W(u) = I + J_E(u) \]
also simplifies our computations. Write \( W \) for \( W(u) \) for short. Indeed, for \( f_0 \in C^0(X_{\Gamma}, \rho) \), we have

\[(6.3) \quad (Wd_0)f_0(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) = d_0f_0(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) + u[a_0/a_1]d_0f_0([a_0 \supseteq a_1 \supseteq \pi a_0])
= u[a_0/a_1]f_0([a_1]) - f_0([a_0]) + u[a_0/a_1](u[a_1/\pi a_0]f_0([a_0]) - f_0([a_1]))
= -(1-u^3)f_0([a_0]).\]

Further, for \( f_2 \in C^2(X_{\Gamma}, \rho) \) and a pointed edge \( \pi^{-1}a_1 \supseteq a_0 \supseteq a_1 \in S_1 \) with \( |a_0/a_1| = 1 \), we have

\[(6.4) \quad (W\delta_2)f_2(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) = \delta_2f_2(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) - u\delta_2f_2([a_0 \supseteq a_1 \supseteq \pi a_0]) = 0\]
and

\[(6.5) \quad (W\delta_2)f_2([a_0 \supseteq a_1 \supseteq \pi a_0]) = \delta_2f_2([a_0 \supseteq a_1 \supseteq \pi a_0]) - u^2\delta_2f_2(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1)
= (1-u^3)\delta_2f_2([a_0 \supseteq a_1 \supseteq \pi a_0]).\]

Given \( f \in C^1(X_{\Gamma}, \rho) \) we apply the above results to compute \( \frac{1}{\pi}(W\Phi_1)f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) \) according to the type \( |a_0/a_1| \) of the pointed edge.

Case I. \( |a_0/a_1| = 1 \). By (6.1)-(6.4) we have

\[(6.6) \quad \frac{1}{1-u^3}(W\Phi_1)f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1)
= \frac{1}{1-u^3}W((1-u^3)I + \delta_2d_1 + d_0\delta_1)Qf(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1)
\]

by (6.2)
\[ WQf(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) = -\delta_1Qf([a_0]) \]

by (6.4)
\[ f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) + u \sum_{a_1 \supseteq b \supseteq \pi a_0} u[a_1/b]f([a_0 \supseteq b \supseteq \pi a_0]) - \sum_{a_0 \supseteq c \supseteq \pi a_0} u[a_0/c]f([a_0 \supseteq c \supseteq \pi a_0])
= f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) - \sum_{a_0 \supseteq b \supseteq \pi a_0, a_1 \supseteq b} u[a_0/b]f([a_0 \supseteq b \supseteq \pi a_0]) \]

by (6.1)
\[ (I - J_{EA}J_{E}^{-1})f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) - \sum_{a_0 \supseteq b \supseteq \pi a_0, |a_0/b| = 1, b \neq a_1} uf([a_0 \supseteq b \supseteq \pi a_0]).\]

Case II. \( |a_0/a_1| = 2 \). Applying (6.3) and (6.2), we obtain

\[(6.7) \quad \frac{1}{1-u^3}(Wd_0\delta_1)\Phi_1f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1) = -\sum_{a_0 \supseteq c \supseteq \pi a_0} u[a_0/c]f([a_0 \supseteq c \supseteq \pi a_0]).\]

On the other hand,

\[(6.8) \quad \frac{1}{1-u^3}(W\delta_2d_1)f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1)
\]

by (6.3)
\[ (\delta_2d_1)f(\pi^{-1}a_1 \supseteq a_0 \supseteq a_1)
= \sum_{a_0 \supseteq b \supseteq a_1} u(d_1)\Phi_1f([a_0 \supseteq b \supseteq a_1 \supseteq \pi a_0])
= \sum_{a_0 \supseteq b \supseteq a_1} -u^2f([a_0 \supseteq a_1 \supseteq \pi a_0]) + uf([a_0 \supseteq b \supseteq \pi a_0]) - uf(\pi^{-1}a_1 \supseteq b \supseteq a_1)
\]

by (6.2)
\[ f([a_0 \supseteq a_1 \supseteq \pi a_0]) + u \sum_{b \supseteq c \supseteq \pi a_0} u[b/c]f([a_0 \supseteq c \supseteq \pi a_0]) - uf([\pi^{-1}a_1 \supseteq b \supseteq a_1]) \]

= \sum_{a_0 \supseteq b \supseteq a_1} \left( -u^2f([a_0 \supseteq a_1 \supseteq \pi a_0]) + u \sum_{b \supseteq c \supseteq \pi a_0} u[b/c]f([a_0 \supseteq c \supseteq \pi a_0]) - uf([\pi^{-1}a_1 \supseteq b \supseteq a_1]) \right)

= \sum_{a_0 \supseteq b \supseteq a_1} \left( \sum_{b \supseteq c \supseteq \pi a_0, c \neq a_1} u[a_0/c]f([a_0 \supseteq c \supseteq \pi a_0]) - \sum_{a_0 \supseteq b \supseteq a_1} u[a_0/b]f([\pi^{-1}a_1 \supseteq b \supseteq a_1]). \right)\]
It leaves invariant the subspace $C$ as stated in Theorem 1.0.4, (2). Adding the above two formulæ yields
\[
\frac{1}{1 - u^3} (W \Delta_1 Q)f((\pi^{-1} b \supseteq a_0 \supseteq a_1))
\]
\[
= - \sum_{a_0 \supseteq b \supseteq \pi a_0 \setminus \{a_0\}} u f((a_0 \supseteq b \supseteq \pi a_0)) - u^2 f((a_0 \supseteq b \supseteq \pi a_0)) - u \sum_{a_0 \supseteq b \supseteq a_1} f((b \supseteq a_0 \supseteq a_1))
\]
\[
= -(J_E A E J^{-1}_E) f((\pi^{-1} a_1 \supseteq a_0 \supseteq a_1)) - J_E Q f((\pi^{-1} a_1 \supseteq a_0 \supseteq a_1)) - (Q - I) f((\pi^{-1} a_1 \supseteq a_0 \supseteq a_1)).
\]
As $\Phi_1 = \Delta_1 + (1 - u^3) I$, the above can be rewritten as
\[
(6.9) \quad \frac{1}{1 - u^3} (W \Phi_1 Q)f((\pi^{-1} a_1 \supseteq a_0 \supseteq a_1)) = (I - J_E A E J^{-1}_E) f((\pi^{-1} a_1 \supseteq a_0 \supseteq a_1)).
\]
Combining (6.6) and (6.9) yields the following identity on operators
\[
(6.10) \quad \frac{1}{1 - u^3} W \Phi_1 Q = I - J_E A E J^{-1}_E - N,
\]
where $N$ is the operator on $C^1(X, \rho)$ sending $f$ to $Nf$ which is zero at pointed type 2 edges, and whose value at a pointed type 1 edge $[\pi^{-1} a_1 \supseteq a_0 \supseteq a_1]$ is given by
\[
Nf((\pi^{-1} a_1 \supseteq a_0 \supseteq a_1)) = \sum_{a_0 \supseteq \pi a_0 \setminus \{a_0\}} u f((a_0 \supseteq \pi a_0)).
\]
Thus $N^2 = 0$. As noted before, $(I - J_E)(I + J_E) = I - J^{-1}_E = (1 - u^3) I$. Hence multiplying both sides of (6.10) by $I - J_E$ on the left gives rise to the identity
\[
\Phi_1 = (I - J_E)(I - J_E A E J^{-1}_E - N)Q^{-1}.
\]
Now we express the determinant of $\Phi_1$ on $C^1(X, \rho)$ in terms of the determinants of the operators on the right hand side on the same space. As remarked before $Q$ and hence $Q^{-1}$ have determinant 1. By pairing off a type 1 pointed edge $[\pi^{-1} a_1 \supseteq a_0 \supseteq a_1] \in S_1$ with its type 2 opposite $[a_0 \supseteq a_1 \supseteq \pi a_0] \in S_1$, we partition the $2N_1$ pointed edges in $S_1$ into $N_1$ pairs and with respect to this basis the operator $I - J_E$ is represented by $N_1$ diagonal block matrices of the form
\[
\left( \begin{array}{cc} I_d & -u I_d \\ -u I_d & I_d \end{array} \right),
\]
where $I_d$ denotes the identity $d \times d$ matrix. Therefore $\det(I - J_E) = (1 - u^3)^{dN_1}$. Finally to compute the determinant of $I - J_E A E J^{-1}_E - N$, we order the pointed edges in $S_1$ by first selecting those of type 1 then followed by those of type 2. With respect to this basis, the operator $J_E A E J^{-1}_E$ is represented by the diagonal block matrix
\[
\left( \begin{array}{cc} B_1 u & 0 \\ 0 & B_2 u^2 \end{array} \right)
\]
and $J_E A E J^{-1}_E + N$ by a lower triangular block matrix
\[
\left( \begin{array}{cc} B_1 u & 0 \\ A & B_2 u^2 \end{array} \right).
\]
Therefore
\[
\det(I - J_E A E J^{-1}_E - N) = \det(I - J_E A E J^{-1}_E) = \det(I - A E(\rho, u)).
\]
Put together, we have shown
\[
\det(\Phi_1(u) \mid C^1(X, \rho)) = (1 - u^3)^{dN_1} \det(I - A E(\rho, u)) = (1 - u^3)^{dN_1} L_1(X, \rho, u)^{-1},
\]
as stated in Theorem 120.3 (2).

6.3. The operator $\Phi_2(u)$. Define the operator $J_C$ on $C^2(X)$ which sends $f \in C^2(X)$ to
\[
J_C f((\pi^{-1} a_2 \supseteq a_0 \supseteq a_1 \supseteq a_2)) = f((a_0 \supseteq a_1 \supseteq a_2 \supseteq \pi a_0)).
\]
It leaves invariant the subspace $C^2(X, \rho)$. Further, for $g \in C^2(X, \rho)$, an easy computation shows
\[
J_C A C J^{-1}_C g((\pi^{-1} a_2 \supseteq a_0 \supseteq a_1 \supseteq a_2)) = \sum_{a_0 \supseteq b \supseteq a_2 \setminus \{a_0\}} u g((a_0 \supseteq b \supseteq a_2 \supseteq \pi a_0)).
\]
We begin with
Proposition 6.3.1. On \( C^2(X_\Gamma, \rho) \) there holds the identity
\[
d_1 \delta_2 = (I + J_C u + J_C^2 u^2)(I + J_C AC J_C^{-1}) - (1 - u^3)I.
\]
Equivalently, \( \Phi_2(u) = d_1(u) \delta_2(u) + (1 - u^3)I = (I + J_C u + J_C^2 u^2)(I + J_C AC(\rho, u)J_C^{-1}). \)

Proof. We compare both sides evaluated at \( f \in C^2(X_\Gamma, \rho) \). The left hand side is
\[
d_1(\delta_2 f)(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2) = u(\delta_2 f)(\pi^{-1} a_2 \ni a_1 \ni a_2) + u^3 \sum_{a_0, b_2 \ni a_0 \ni a_2} f([a_0 \ni b \ni \pi a_0 \ni a_1]) + u^2 \sum_{a_0, b_2 \ni a_2 \ni a_1} f([a_0 \ni b \ni \pi a_0 \ni a_1]) + u^3 \sum_{a_2 \ni a_0 \ni a_1} f([a_2 \ni b \ni \pi a_0 \ni a_1]).
\]
For the right hand side, we first compute
\[
(I + J_C u + J_C^2 u^2)(I + J_C AC J_C^{-1}) f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2) = (I + J_C u + J_C^2 u^2) f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2)
\]
\[
= (I + J_C AC J_C^{-1}) f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2) + u(I + J_C AC J_C^{-1}) f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2) + u^2 f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2) + u^3 \sum_{a_0, b_2 \ni a_2 \ni a_1} f([a_0 \ni b \ni \pi a_0 \ni a_1])
\]
\[
= (d_1 \delta_2 f)(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2).
\]
Therefore the right hand side is equal to
\[
(I + J_C u + J_C^2 u^2)(I + J_C AC J_C^{-1}) f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2) - (1 - u^3) f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2) = d_1 \delta_2 f(\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2).
\]
This proves the proposition. \( \square \)

Now we compute the determinant of \( \Phi_2 \). For \( c = [\pi^{-1} a_2 \ni a_0 \ni a_1 \ni a_2] \) in \( S_2 \), write \( c' \) for the pointed chamber \([a_0 \ni a_1 \ni a_2] \ni \pi a_0 \ni a_2 \ni a_2 \) and \( c'' \) for \([a_1 \ni a_2 \ni a_2] \ni a_0 \ni a_1 \ni a_2 \). By our choice of \( S_2 \), \( c' \) and \( c'' \) are all in \( S_2 \), and the pointed chambers in \( S_2 \) can be partitioned into \( N_2 \) disjoint triples \([c,c',c'']\). With respect to each triple, the operator \( I + J_C u + J_C^2 u^2 \) is represented by the \( 3d \times 3d \) matrix \[
\begin{pmatrix}
I_d & u I_d & u^2 I_d \\
u^2 I_d & I_d & u I_d \\
u I_d & u^3 I_d & I_d
\end{pmatrix},
\]
which has determinant \( (1 - u^3)^{3d} \). Here \( I_d \) is the \( d \times d \) identity matrix. Thus \( \det(I + J_C u + J_C^2 u^2 | C^2(X_\Gamma, \rho)) = (1 - u^3)^{3d N_2} \). Combined with
\[
\det(I + J_C AC(\rho, u)J_C^{-1}) = \det(I + AC(\rho, u)) = \det(I - AC(\rho, -u)) = L_2(X_\Gamma, \rho, -u)^{-1},
\]
we get \( \det(\Phi_2(u) | C^2(X_\Gamma, \rho)) = (1 - u^3)^{2d N_2} L_2(X_\Gamma, \rho, -u)^{-1} \), as claimed in Theorem 1.0.13 (3).

6.4. Cohomological interpretation of the proof. The computations in §6.1-§6.3 can be rephrased as follows. We define two homomorphisms \( \Psi_1 = \{ \Psi_{1,i} | i = 0, 1, 2 \} \) and \( \Psi_2 = \{ \Psi_{2,i} | i = 0, 1, 2 \} \) from the complex \( C^* \) to itself as follows. For the first, \( \Psi_{1,i} : C^i(X_\Gamma, \rho) \to C^i(X_\Gamma, \rho) \) is multiplication by \( 1 - u^3 \) for each
0 ≤ i ≤ 2. It is clear that \( \Psi_{1,i+1}d_i = d_i\Psi_{1,i} \) for \( i = 0, 1 \). Hence \( \Psi_1 \) is an endomorphism of the complex \( C_* \). For the second map, \( \Psi_{2,i} : C^i(X_\Gamma, \rho) \to C^i(X_\Gamma, \rho) \) are defined as

\[
\begin{align*}
\Psi_{2,0} &= I - A_1u + \frac{1}{2}A_2u^2 - u^3I \\
\Psi_{2,1} &= (I - J_E)(I - J_EA_E(\rho, u)J_E^{-1} - N)Q^{-1} \\
\Psi_{2,2} &= (I + J_Eu^2 + J_E^2u^3)(I + J_CAC(\rho, u)J_C^{-1}).
\end{align*}
\]

That \( \Psi_2 \) is also an endomorphism of the complex \( C_* \) follows from the fact that each \( \Psi_{2,i} = \Phi_i \) and \( \Phi_i \) have the desired property. Further, the relation \( \Psi_{2,i} - \Psi_{1,i} = \Delta_i = d_{i-1}\delta_i + \delta_{i+1}d_i \) for each \( i \) shows that \( \Psi_1 \) and \( \Psi_2 \) are homotopically equivalent. Therefore Theorem 1.0.3 holds.

### 7. A proof of Theorem 1.0.7

For a finite-dimensional representation \( (\rho', V_{\rho'}) \) of a finite-index subgroup \( \Gamma' \) of \( \Gamma \), regard the space \( V_\rho \) of the induced representation \( \rho = \text{Ind}_{\gamma}^{\Gamma} \rho' \) as the set

\[
V_\rho = \{ f : \Gamma \to V_{\rho'} : f(\gamma') = \rho'(\gamma')f(\gamma) \text{ for all } \gamma' \in \Gamma', \gamma \in \Gamma \}
\]
on which \( \rho \) acts by right translation \( \rho(\gamma)f(\gamma) = f(\gamma\gamma) \). Let \( i \in \{1, 2\} \). Given \( g \in C^i(X_\Gamma, \rho) \), define the function \( f_g \) on \( X_i \) such that its value at \( x_i \in X_i \) is a function \( f_g(x_i) : \Gamma \to V_{\rho'}[u] \) given by

\[
f_g(x_i)(\gamma) := g(\gamma x_i) \quad \text{for all } \gamma \in \Gamma.
\]

Then for \( \gamma' \in \Gamma' \), it follows from the definition that

\[
f_g(x_i)(\gamma' \gamma) = g(\gamma' \gamma x_i) = \rho'(\gamma')g(\gamma x_i) = \rho'(\gamma')f_g(x_i)(\gamma),
\]

which shows that \( f_g(x_i) \) lies in \( V_{\rho'}[u] \). Moreover, for \( \gamma \in \Gamma \),

\[
f_g(\gamma x_i)(\gamma) = g(\gamma x_i) = f_g(x_i)(\gamma) = \rho(\gamma)f_g(x_i)(\gamma)
\]

implies that \( f_g \in C^i(X_\Gamma, \rho) \). Hence \( g \mapsto f_g \) defines a homomorphism \( \phi_i : C^i(X_{\Gamma'}, \rho') \to C^i(X_\Gamma, \rho) \) as \( \mathbb{C}[u] \)-modules. Conversely, for \( f \in C^i(X_\Gamma, \rho) \), set

\[
g_f(x_i) := f(x_i)(1) \quad \text{for all } x_i \in X_i.
\]

Then for \( \gamma' \in \Gamma' \),

\[
g_f(\gamma' x_i) = f(\gamma' x_i)(1) = \left( \rho(\gamma')f(x_i) \right)(1) = f(x_i)(\gamma') = \rho'(\gamma')(f(x_i)(1)) = \rho'(\gamma')g_f(x_i).
\]

Thus \( g_f \) lies in \( C^i(X_{\Gamma'}, \rho') \). It is easy to see that \( f \mapsto g_f \) defines the inverse map of \( \phi_i \) which implies that \( \phi_i \) is an isomorphism.

Next we claim that the diagram

\[
\begin{array}{ccc}
C^1(X_{\Gamma'}, V_{\rho'}[u]) & \xrightarrow{\phi_1} & C^1(X_\Gamma, V_\rho[u]) \\
\bigg| & A_E(\rho', u) & \bigg| \\
A_E(\rho', u) & \xrightarrow{\phi_1} & A_E(\rho, u) \\
\bigg| & C^1(X_{\Gamma'}, V_{\rho'}[u]) & \xrightarrow{\phi_1} \\
\bigg| & C^1(X_\Gamma, V_\rho[u]) & \xrightarrow{\phi_1}
\end{array}
\]

commutes. If so, then combined with Theorem 1.0.3, this gives

\[
L_1(X_{\Gamma'}, \rho', u) = \frac{1}{\det(I - A_E(\rho', u))} = \frac{1}{\det(I - A_E(\rho, u))} = L_1(X_\Gamma, \rho, u).
\]
To prove the claim, given $g \in C^1(X_{\Gamma'}, \rho')$, $e \in X_1$, and $\gamma \in \Gamma$, we examine
\[
f_{A_E(\rho', u)g}(e)(\gamma) = A_E(\rho', u)g(\gamma e) = \sum_{e' \in N(e)} u^{A(\gamma e)}(\gamma e') \quad \text{since } N(\gamma e) = \gamma N(e) = \sum_{e' \in N(e)} u^{A(\gamma e)}(\gamma e') \quad \text{since } \Gamma \text{ preserves the type of pointed edges}
\]
that is, $f_{A_E(\rho', u)g} = A_E(\rho, u)f_g$, as claimed.
A similar argument proves $L_2(X_{\Gamma'}, \rho', u) = L_2(X_{\Gamma}, \rho, u)$.

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