EXISTENCE THEOREM FOR A CLASS OF SEMILINEAR TOTALLY CHARACTERISTIC ELLIPTIC EQUATIONS INVOLVING SUPERCRITICAL CONE SOBOLEV EXPOIENTS

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Abstract. In this paper, we prove the existence of bounded positive solutions for a class of semilinear degenerate elliptic equations involving supercritical cone Sobolev exponents. We also obtain the existence of multiple solutions by the Ljusternik-Schnirelman theory.

1. Introduction. In this paper, we consider the existence of positive solutions for the following Dirichlet problem

\[ \begin{cases} -\Delta_B u = Q(x)|u|^{q-2}u + \lambda P(x)|u|^{p-2}u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \]

where \(2 < q < 2^* \leq p\), \(2^* = \frac{2N}{N-2}\) is the critical cone Sobolev exponents, \(N \geq 3\) and \(\lambda > 0\) is a parameter. Here the domain \(B\) is \([0,1) \times X\) for \(X \subset \mathbb{R}^{N-1}\) compact, which is regarded as the local model near the conical points on manifolds with conical singularities and \(\{0\} \times X \subset \partial B\). Moreover, the operator \(\Delta_B\) in \((P_\lambda)\) is defined by \((x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \ldots + \partial_{x_N}^2\), which is an elliptic operator with totally characteristic degeneracy on the boundary \(x_1 = 0\) (we also call it Fuchsian type Laplace operator), and the corresponding gradient operator is denoted by \(\nabla_B := (x_1^2\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N})\).

Near \(\partial B\) we will often use coordinates \((x_1, x') = (x_1, x_2, \ldots, x_N)\) for \(0 \leq x_1 < 1, x' \in X\). The main purpose of this article is to establish the existence theorem for the problem \((P_\lambda)\) in the cone Sobolev space \(H^{1, \frac{2}{N-2}}_2(B)\). The space \(H^{1, \frac{2}{N-2}}_2(B)\) was introduced in [5] and will be explained later in Section 2 for the completeness. We always assume that \(P(x)\) and \(Q(x)\) are positive and continuous on \(\overline{B}\) and there exist \(\delta_1 > 0\) and \(\delta_2 > 0\) such that

\(\begin{align*}
\lim_{x_1 \to 0} \frac{Q(x_1, x')}{|x_1|^{\delta_1}} &= \theta_1 \\
\lim_{x_1 \to 0} \frac{P(x_1, x')}{|x_1|^{\delta_2}} &= \theta_2
\end{align*}\)

where \(\theta_1\) and \(\theta_2\) are two positive constants.

The analysis on manifolds with conical singularities has been studied in many references, see [11, 12, 17, 18, 19]. The foundation of this analysis has been developed...
through the fundamental works by Schulze [12], and subsequently further expended by him and his collaborators. Recently, H. Chen, X. Liu and Y. Wei established the corresponding Sobolev inequality and Poincaré inequality on the cone Sobolev spaces in [5]. For related nonlinear problems with totally characteristic degeneracy, these inequalities seem to be of fundamental importance to obtain the existence and multiplicity results. Consider the following semi-linear boundary value problem

\begin{equation}
\begin{cases}
-\Delta_B u = f(x, u) & \text{in } B, \\
u = 0 & \text{on } \partial B.
\end{cases}
\end{equation}

For \(f(x, u) = Q(x)|u|^{p-2}u\) in (1.1), the authors in [5] consider the following Dirichlet problem

\((P_0)\)

\begin{equation}
\begin{cases}
-\Delta_B u = Q(x)|u|^{q-2}u & \text{in } B, \\
u = 0 & \text{on } \partial B.
\end{cases}
\end{equation}

and obtained the existence result for \(2 < q < 2^* = \frac{2N}{N-2}\) with \(Q(x)\) satisfying some appropriate conditions. For \(f(x, u) = \lambda u + |u|^{2^*-2}u\), the authors in [6] proved the Dirichlet problem

\begin{equation}
\begin{cases}
-\Delta_B u = \lambda u + |u|^{2^*-2}u & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\end{equation}

possesses at least one positive solution for \(N \geq 4, \lambda \in (0, \lambda_1)\) where \(\lambda_1\) is the first eigenvalue of \(-\Delta_B\) and admits infinitely many solutions for \(N \geq 7, \lambda > 0\). They also studied problem (1.1) for general nonlinearities of subcritical and critical cone Sobolev exponents. Fan and Liu in [15] obtained the existence of multiple positive solutions to (1.1) with \(f(x, u) = f_\lambda |u|^{q-2}u + g(x)|u|^{2^*-2}u\), where \(1 < q < 2, N \geq 3, f_\lambda\) and \(g(x)\) satisfies some suitable conditions. Also the authors in [14] considered the degenerate elliptic equations with singularity and critical cone Sobolev exponents

\begin{equation}
\begin{cases}
-\Delta_B u = |u|^{2^*-2}u + \lambda f(x)u^{-\gamma} & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\end{equation}

and obtained the existence of multiple positive solutions for \(\lambda > 0\) and \(\gamma \in (0, 1)\). Other problems were investigated in [4, 6, 7, 8, 9, 10, 13, 14, 16] and the references therein.

To the authors’ knowledge, there are very few results for problem (1.2) involving supercritical Sobolev exponent. J. Chabrowski and J. Yang in [3] studied the situation of classical Laplacian. Inspired by [3], we investigate problem (\(P_\lambda\)) with supercritical exponent.

In order to solve problem (\(P_\lambda\)) we first consider a truncated problem which involves only a subcritical cone Sobolev exponent. It turns out that weak solutions of the truncated problem are bounded. This allows us to choose a suitable parameter \(\lambda\) and a truncation so that a solution of truncated problem in fact satisfies problem (\(P_\lambda\)). We use the cone Sobolev inequality and Poincaré inequality to prove the existence theorem for the truncated problem. A solution of the truncated problem will be obtained by the mountain-pass theorem.

Our main results are as follows.

**Theorem 1.1.** Suppose (H) holds. Then there exists a constant \(\lambda_0 > 0\) such that for each \(0 < \lambda \leq \lambda_0\) problem \((P_\lambda)\) has at least one positive solution in \(H^{1,N}_{2,0}(B)\).
Theorem 1.2. Suppose (H) holds. Then for any $m \in \mathbb{N}^+$, there exists $\lambda_m > 0$ such that for each $0 < \lambda \leq \lambda_m$, problem $(P_\lambda)$ has at least $m$ distinct pairs of solutions $(-u_j, u_j)$ in $H^{1, \infty}_{2, 0} (\mathbb{B})$, $j = 1, \ldots, m$.

Theorem 1.3. Suppose (H) holds. Then there exists a sequence $\lambda_n \to 0$ such that $u_{\lambda_n} \to u_0 \neq 0$ in $H^{1, \infty}_{2, 0} (\mathbb{B})$ and $u_0$ is a solution of the subcritical problem $(P_0)$.

This paper is organized as follows. In Section 2 we recall the cone Sobolev spaces and some necessary propositions. In Section 3, we prove the existence of positive solutions for the truncated problem corresponding to $(P_{\lambda})$. In Section 4, we give the proof of Theorem 1.1 by Moser iteration. We will prove Theorem 1.2 and Theorem 1.3 in Section 5.

2. Preliminaries. In this section, we introduce a totally characteristic operator and its corresponding cone Sobolev spaces and their properties. For more details we refer to [5].

Let $X \subset \mathbb{R}^{N-1}$ be a closed, compact, $C^\infty$ manifold and $\mathbb{B} = [0, 1) \times X$. We consider the following Riemannian metric
\[
g := \frac{1}{x_1^2} dx_1^2 + dx_2^2 + \ldots + dx_N^2
\]
on manifold $\mathbb{B}$. Then the gradient operator with respect to the metric $g$ is first order differential operator. By direct calculation, we know the corresponding gradient operator is $\nabla_{\mathbb{B}} := (x_1^2 \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N})$. Moreover, the Laplace-Beltrami operator corresponding to the metric $g$ is then of the form
\[
\Delta_{\mathbb{B}} u = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} (\sqrt{\det g} g^{ij} \frac{\partial u}{\partial x_j}) = (x_1 \partial_{x_1})^2 u + \partial_{x_2}^2 u + \ldots + \partial_{x_N}^2 u.
\]

Definition 2.1. For $(x_1, x') \in \mathbb{R}_+ \times \mathbb{R}^{N-1}$, we say that $u(x_1, x') \in L_p(\mathbb{R}_+^N, \frac{dx_1}{x_1})$ if
\[
\|u\|_{L_p} = \left( \int_{\mathbb{R}_+^N} x_1^N |u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} < +\infty.
\]
Moreover, the weighted $L_p$-spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}_+^N, \frac{dx_1}{x_1})$, namely, if $u(x_1, x') \in L_p^\gamma(\mathbb{R}_+^N, \frac{dx_1}{x_1})$, then $x_1^{-\gamma} u(x_1, x') \in L_p(\mathbb{R}_+^N, \frac{dx_1}{x_1})$, and
\[
\|u\|_{L_p^\gamma} = \left( \int_{\mathbb{R}_+^N} x_1^N |x_1^{-\gamma} u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} < +\infty.
\]

Now we can define the weighted Sobolev space for $1 \leq p < +\infty$.

Definition 2.2. For $m \in \mathbb{N}$, and $\gamma \in \mathbb{R}$, the spaces
\[
\mathcal{H}^{m, \gamma}_p(\mathbb{R}_+^N) := \{ u \in \mathcal{D}'(\mathbb{R}_+^N) : x_1^{-\gamma} (x_1 \partial_{x_1})^\alpha \partial_\beta u \in L_p(\mathbb{R}_+^N, \frac{dx_1}{x_1} dx') \},
\]
for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^{N-1}$, and $|\alpha| + |\beta| \leq m$. In other words, if $u(x_1, x') \in \mathcal{H}^{m, \gamma}_p(\mathbb{R}_+^N)$, then $(x_1 \partial_{x_1})^\alpha \partial_\beta u \in L_p(\mathbb{R}_+^N, \frac{dx_1}{x_1} dx')$.

It is easy to see that $\mathcal{H}^{m, \gamma}_p(\mathbb{R}_+^N)$ is a Banach space with norm
\[
\|u\|_{\mathcal{H}^{m, \gamma}_p(\mathbb{R}_+^N)} = \sum_{|\alpha| + |\beta| \leq m} \left( \int_{\mathbb{R}_+^N} x_1^N |x_1^{-\gamma} (x_1 \partial_{x_1})^\alpha \partial_\beta u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}}.
\]
We will always denote \( \omega(x_1) \in C_0^\infty(0, 1) \) such that \( \omega(x_1) = 1 \) when \( x_1 \leq \varepsilon \), and \( \omega(x_1) = 0 \) when \( x_1 > a \) for some \( 0 < \varepsilon < a < 1 \).

**Definition 2.3.** Let \( B \) be the stretched manifold to a manifold \( B \) with conical singularities. Then \( H^{m,\gamma}_p(B) \) for \( m \in \mathbb{N}, \gamma \in \mathbb{R} \) denotes the subspace of all \( u \in W^{m,p}_0(\text{int } B) \), such that
\[
H^{m,\gamma}_p(B) = \{ u \in W^{m,p}_0(\text{int } B) \mid \omega u \in H^{m,\gamma}_p(X^\wedge) \}.
\]
Moreover, the subspace \( H^{m,\gamma}_{p,0}(B) \) of \( H^{m,\gamma}_p(B) \) is defined as follows:
\[
H^{m,\gamma}_{p,0}(B) = [\omega]H^{m,\gamma}_p(X^\wedge) + [1 - \omega]W^{m,p}_0(\text{int } B),
\]
where \( X^\wedge = \mathbb{R}_+ \times X \) is the open stretched cone with the base \( X \) and \( W^{m,p}_0(\text{int } B) \) denotes the closure of \( C_0^\infty(\text{int } B) \) in Sobolev spaces \( W^{m,p}(\tilde{X}) \) when \( \tilde{X} \) is a closed compact \( C^\infty \) manifold of dimension \( N \) that containing \( B \) as a submanifold with boundary.

**Proposition 2.1** (Cone Sobolev Inequality). Assume that \( 1 \leq p < N, \frac{1}{p'} = \frac{1}{p} - \frac{1}{N} \), and \( \gamma \in \mathbb{R} \). Let \( B^N := \mathbb{R}_+ \times \mathbb{R}^{N-1} \), \( x_1 \in \mathbb{R}_+ \) and \( x' = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1} \). The following estimate
\[
\|u\|_{L^p(\mathbb{R}_+)} \leq c_1\|x_1\|_{L^p(\mathbb{R}_+)} + c_2\|\nabla u\|_{L^p(\mathbb{R}_+)}
\]
holds for all \( u \in C^\infty(\mathbb{R}_+^N) \), where \( \gamma^* = \gamma - 1, c_1 = \frac{(N-1)p}{N(N-p)}, \) and \( c_2 = \frac{|N_0 - (\gamma - 1)(N-1)|}{N_0} \).

Moreover, if \( u \in H^{1,\gamma}_{p,0}(\mathbb{R}_+) \), we have
\[
\|u\|_{L^p(\mathbb{R}_+^N)} \leq c\|u\|_{H^{1,\gamma}(\mathbb{R}_+^N)},
\]
where the constant \( c = c_1 + c_2 \), and \( c_1 \) and \( c_2 \) are given in (2.2).

**Proposition 2.2** (Poincaré Inequality). Let \( B = (0, 1) \times X \) be a bounded subset in \( \mathbb{R}_+^N \) with \( X \subset \mathbb{R}^{N-1} \), and \( 1 < p < +\infty, \gamma \in \mathbb{R} \). If \( u(x_1, x') \in H^{1,\gamma}_{p,0}(B) \), then
\[
\|u(x_1, x')\|_{L^p(B)} \leq c\|\nabla u(x_1, x')\|_{L^p(B)},
\]
where \( \nabla_B = (x_1^2\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}) \), and the constant \( c \) depending only on \( B \) and \( p \).

The following compactness result was first proved by H. Chen, X. Liu and Y. Wei in [5] and later corrected by themselves.

**Proposition 2.3.** For \( 1 \leq p < 2^* = \frac{2N}{N-2} \), the embedding \( H^{1,\gamma}_{2,0}(B) \hookrightarrow H^{0,\gamma}_{p,0}(B) \) is continuous if \( \frac{N}{p} - \gamma_1 \geq \frac{N}{2} - \gamma_2 \) and compact if \( \frac{N}{p} - \gamma_1 > \frac{N}{2} - \gamma_2 \).

**Proof.** According to Definition 2.3, we can write \( H^{1,\gamma}_{2,0}(B) \) and \( H^{0,\gamma}_{p,0}(B) \) as follows:
\[
H^{1,\gamma}_{2,0}(B) = [\omega]H^{1,\gamma}_2(X^\wedge) + [1 - \omega]H_0^1(\text{int } B),
\]
\[
H^{0,\gamma}_{p,0}(B) = [\omega]H^{0,\gamma}_p(X^\wedge) + [1 - \omega]L^p(\text{int } B).
\]
With the help of Sobolev inequality, we know that the embedding \( [1 - \omega]H^1_0(\text{int } B) \hookrightarrow [1 - \omega]L^p(\text{int } B) \) is compact for \( 1 \leq p < 2^* \). So it is sufficient to show that the embedding \( [\omega]H^{1,\gamma}_{2,0}(X^\wedge) \hookrightarrow [\omega]H^{0,\gamma}_{p,0}(X^\wedge) \) is compact for \( 1 \leq p < 2^* \). For any \( v(x_1, x') \in H^{m,\gamma}_{p,0}(X^\wedge) \), we define
\[
(S_{\frac{N}{p} - \gamma})v(r, x') = e^{-r(\frac{N}{p} - \gamma)}v(e^{-r}x').
\]
Then \( S_{p,\gamma} \) induces an isomorphism
\[
S_{p,\gamma} : [\omega]H^{0,\gamma}_p(X^\wedge) \to [\tilde{\omega}]W^{m,q}(\mathbb{R} \times X)
\]
with a cut-off function \( \tilde{\omega}(r) = \omega(e^{-r}) \in C_0^\infty(\mathbb{R}_+) \). For \( u_1(x_1, x') \in H^{0,\gamma_1}_{p,0}(X^\wedge) \), we have
\[
S_{p,\gamma_1}(\omega(x_1)u_1(x_1, x')) = \omega(e^{-r})e^{-r(\frac{\gamma_1}{p} - \gamma_1)}u_1(e^{-r}, x')
\]
and it induces an isomorphism
\[
S_{p,\gamma_1} : [\omega]H^{1,\gamma_1}_{p,0}(X^\wedge) \to [\tilde{\omega}]L^p(\mathbb{R} \times X).
\]
For \( u_2(x_1, x') \in H^{1,\gamma_2}_{2,0}(X^\wedge) \), we get again
\[
S_{p,\gamma_2}(\omega(x_1)u_2(x_1, x')) = \omega(e^{-r})e^{-r(\frac{\gamma_1}{p} - \gamma_1)}u_2(e^{-r}, x')
\]
with \( \delta := (\frac{\gamma_2}{p} - \gamma_1) - (\frac{\gamma_2}{p} - \gamma_1) \). On the one hand, if \( \delta = 0 \), then
\[
S_{p,\gamma_1} : [\omega]H^{1,\gamma_2}_{2,0}(X^\wedge) \to [\tilde{\omega}]H^1_0(\mathbb{R} \times X)
\]
is also an isomorphism. Since the embedding \([\tilde{\omega}]H^1_0(\mathbb{R} \times X) \hookrightarrow [\tilde{\omega}]L^p(\mathbb{R} \times X)\) is continuous, then the embedding \([\omega]H^{1,\gamma_2}_{2,0}(X^\wedge) \hookrightarrow [\omega]H^{0,\gamma_1}_{p,0}(X^\wedge)\) is continuous. On the other hand, if \( \delta > 0 \), we set \( \varphi(r) = e^{-\delta \cdot r^s} \). And then all derivatives of \( \varphi(r) \) are uniformly bounded on \( \text{supp} \tilde{\omega} \) for every \( s > 0 \). Then it follows
\[
[\tilde{\omega}]e^{-\delta H^1_0}(\mathbb{R} \times X) = [\tilde{\omega}]\varphi(r)r^{-s}H^1_0(\mathbb{R} \times X)
\]
\[
\hookrightarrow [\tilde{\omega}]r^{-s}H^1_0(\mathbb{R} \times X)
\]
\[
\hookrightarrow [\tilde{\omega}]L^p(\mathbb{R} \times X),
\]
where the last embedding is compact. This completes the proof. \( \square \)

For the completeness, we introduce the well-known Mountain Pass Lemma (see [1, 22, 23]).

**Definition 2.4.** We say that a functional \( I \) satisfies the \((PS)_c\) condition, if for any sequence \( \{u_n\} \subset X \) with the properties:
\[
I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{X'} \to 0,
\]
there exists a subsequence which is convergent, where \( X' \) is the dual space of \( X \).

**Proposition 2.4.** Let \( X \) be a Banach space and \( I \in C^1(X, \mathbb{R}) \). Suppose \( I(0) = 0 \) and it satisfies
(i) there exist \( \alpha > 0, \rho > 0 \) such that if \( \|u\|_X = \rho \), then \( I(u) \geq \alpha \);
(ii) there exists \( e \in X \) such that \( \|e\|_X > \rho \) and \( I(e) < 0 \).
If $I$ satisfies $(PS)_c$ condition with 
\[ c = \inf_{u \in \Gamma} \max_{h \in F} I(u), \]
where 
\[ \Gamma = \{ h \in C([0,1]; X) : h(0) = 0 \text{ and } h(1) = e \}, \]
then $c$ is a critical value of $I$ and $c \geq \alpha$.

3. The truncated problem. In this section, we consider the truncated problem 
\[ \begin{cases} -\Delta_B u = Q(x)|u|^{q-2}u + \lambda P(x)h(u) & \text{in } B, \\ u > 0 & \text{in } \text{int } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.1} \]
where 
\[ h(u) = \begin{cases} 0 & \text{for } u < 0, \\ u^{p-1} & \text{for } 0 \leq u \leq K, \\ K^{p-q}u^{q-1} & \text{for } u > K, \end{cases} \]
for some positive constant $K$ and it will be determined later. The idea of truncation was used in [3, 21]. Let 
\[ H(u) = \int_0^u h(s)ds. \]
Then we have 
\[ h(u) \leq K^{p-q}u^{q-1} \quad \text{for } u \geq 0, \quad \text{(3.2)} \]
and 
\[ H(u) \leq \frac{K^{p-q}}{q}u^q \quad \text{for } u \geq 0. \quad \text{(3.3)} \]
Either in the case $0 \leq u \leq K$ or in the case $u > K$, it is easy to check that 
\[ H(u) \leq \frac{1}{q}h(u)u \quad \text{for } u \geq 0. \quad \text{(3.4)} \]

We define the following functional $I_\lambda$ on $H^{1,q}_{2,0}(B)$, i.e., 
\[ I_\lambda(u) = \frac{1}{2} \int_B |\nabla_B u|^2 \frac{dx_1}{x_1} dx' - \frac{1}{q} \int_B Q(x)|u|^{q-1} \frac{dx_1}{x_1} dx' - \lambda \int_B P(x)H(u) \frac{dx_1}{x_1} dx' \]
\[ = \frac{1}{2}||u||^2 - \frac{1}{q} \int_B Q(x)|u|^{q-1} \frac{dx_1}{x_1} dx' - \lambda \int_B P(x)H(u) \frac{dx_1}{x_1} dx', \tag{3.5} \]
where 
\[ ||u|| = \left( \int_B |\nabla_B u|^2 \frac{dx_1}{x_1} dx' \right)^{\frac{1}{2}}, \]
and it is easy to see that 
\[ ||\nabla_B u||^2 = (\nabla_B u, \nabla_B u) = g^j_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = |x_1 \partial_{x_1} u|^2 + |\partial_{x_2} u|^2 + \ldots + |\partial_{x_N} u|^2. \]

In fact, by Proposition 2.2, $H^{1,q}_{2,0}(B)$ can be regarded as the closure of $C^\infty_0(B)$ with respect to the norm $||u||$. Then $I_\lambda(u)$ is well-defined and belongs to $C^1(H^{1,q}_{2,0}(B), \mathbb{R})$.

We say that $u \in H^{1,q}_{2,0}(B)$ is a weak solution of problem $(T_\lambda)$, if for any $\varphi \in C^0_0(B)$ one has 
\[ \int_B \nabla_B u \nabla_B \varphi \frac{dx_1}{x_1} dx' - \int_B Q|u|^{q-2}u \varphi \frac{dx_1}{x_1} dx' - \lambda \int_B Ph(u) \varphi \frac{dx_1}{x_1} dx' = 0. \tag{3.6} \]
Therefore, a critical point of the variational functional $I_\lambda$ is a weak solution of problem (T). The Mountain Pass Lemma (Proposition 2.4) will be applied to find a critical point of $(T)$. 

**Lemma 3.1.** The functional $I_\lambda$ satisfies two properties as follows

1. there exist two positive constants $\alpha_0$ and $\rho_0$ such that

   $I_\lambda(u) \geq \alpha_0$, for all $u \in \mathcal{H}_{2,0}^{1,n} (\mathbb{B})$ with $\|u\| = \rho_0$;

2. there exist $e_0 \in \mathcal{H}_{2,0}^{1,n} (\mathbb{B})$ such that $\|e_0\| > \rho$ and $I_\lambda(e_0) < 0$.

**Proof.** (1) By the condition $(H)$, we may denote

$$M = \max \{ \sup_{x \in \mathbb{B}} Q(x), \sup_{x \in \mathbb{B}} P(x) \}. \quad (3.7)$$

For any $u \in \mathcal{H}_{2,0}^{1,n} (\mathbb{B})$, by Proposition 2.3, we have

$$I_\lambda(u) = \frac{1}{2} \|u\|^2 - \frac{1}{q} \int_\mathbb{B} Q|u|^q \frac{dx_1}{x_1} dx' - \lambda \int_\mathbb{B} PH(u) \frac{dx_1}{x_1} dx'$$

$$\geq \frac{1}{2} \|u\|^2 - \frac{1}{q} \int_\mathbb{B} Q|u|^q \frac{dx_1}{x_1} dx' - \frac{\lambda K^{p-q}}{q} \int_\mathbb{B} P|u|^q \frac{dx_1}{x_1} dx'$$

$$\geq \frac{1}{2} \|u\|^2 - \frac{M}{q} (1 + \lambda K^{p-q}) \int_\mathbb{B} |u|^q \frac{dx_1}{x_1} dx'$$

$$= \frac{1}{2} \|u\|^2 - \frac{M}{q} (1 + \lambda K^{p-q}) \|u\|_{L_q^\infty (\mathbb{B})}^{q}$$

$$\geq \frac{1}{2} \|u\|^2 - \frac{CM}{q} (1 + \lambda K^{p-q}) \|u\|^{q}.$$ 

Therefore there exist $\alpha_0 > 0$ and $\rho_0 > 0$ such that $I_\lambda(u) \geq \alpha_0 > 0$ for all $u \in \mathcal{H}_{2,0}^{1,n} (\mathbb{B})$ with $\|u\| = \rho_0$. Thus, the conclusion (1) holds.

(2) Let $u_0 \in \mathcal{H}_{2,0}^{1,n} (\mathbb{B}) \setminus \{0\}$ with $u_0 \geq 0$. We have

$$I_\lambda(tu_0) = \frac{t^2}{2} \int_\mathbb{B} |\nabla_B u_0|^2 \frac{dx_1}{x_1} dx' - \frac{t^q}{q} \int_\mathbb{B} Q|u_0|^q \frac{dx_1}{x_1} dx' - \lambda \int_\mathbb{B} PH(tu_0) \frac{dx_1}{x_1} dx',$$

for $t \geq 0$. Since $2 < q < 2^*$, there exists a $t_0$ large enough such that $\|e_0\| > \rho_0$ and $I_\lambda(e_0) < 0$ with $e_0 = t_0 u_0$. The conclusion (2) holds. \hfill \square

**Lemma 3.2.** The functional $I_\lambda$ satisfies the $(PS)_c$ condition on $\mathcal{H}_{2,0}^{1,n} (\mathbb{B})$.

**Proof.** Let $\{u_n\}_{n \in \mathbb{N}^+} \subset \mathcal{H}_{2,0}^{1,n} (\mathbb{B})$ be a $(PS)_c$ sequence of $I_\lambda$. Namely, we have

$$I_\lambda(u_n) \to c \quad \text{and} \quad \|I_\lambda'(u_n)\|_{\mathcal{H}_{2,0}^{-1,n} (\mathbb{B})} \to 0,$$

where $\mathcal{H}_{2,0}^{-1,n} (\mathbb{B})$ is the dual space of $\mathcal{H}_{2,0}^{1,n} (\mathbb{B})$. First, by (3.4) we get

$$c + 1 > I_\lambda(u_n) - \frac{1}{q} \langle I_\lambda'(u_n), u_n \rangle$$

$$= \left( \frac{1}{2} - \frac{1}{q} \right) \|u_n\|^2 + \lambda \left( \int_\mathbb{B} \frac{1}{q} Ph(u_n) u_n \frac{dx_1}{x_1} dx' - \int_\mathbb{B} PH(u_n) \frac{dx_1}{x_1} dx' \right)$$

$$\geq \left( \frac{1}{2} - \frac{1}{q} \right) \|u_n\|^2.$$
for $n$ large enough. This implies that $\{u_n\}$ is bounded in $H^{1,\frac{N}{q}}_{2,0}(B)$. Thus there exist $u \in H^{1,\frac{N}{q}}_{2,0}(B)$ and a subsequence which is still denoted by $\{u_n\}$, such that 

$$u_n \rightharpoonup u \text{ in } L^q_\delta(B)$$

for $q \in (2,2^*)$ and $\frac{N}{q} - \delta \leq \gamma < \frac{N}{q}$, where $\delta = \min\{\delta_1, \delta_2\}$. Moreover, we obtain

$$o(1) = \langle I_\lambda'(u_n) - I_\lambda'(u), u_n - u \rangle = \int_B |\nabla_B u_n - \nabla_B u|^2 \frac{dx_1}{x_1} dx' + A_{1,n} + \lambda A_{2,n},$$

where

$$A_{1,n} = \int_B Q(|u|^{q-2}u - |u_n|^{q-2}u_n)(u_n - u) \frac{dx_1}{x_1} dx'$$

and

$$A_{2,n} = \lambda \int_B (P(h(u) - h(u_n))(u_n - u) \frac{dx_1}{x_1} dx'.$$

By the condition $(H)$, there exist $\bar{C}_1, \bar{C}_2 > 0$ such that

$$|Q(x)| \leq \bar{C}_1 x_1^{\bar{\delta}_1} \text{ and } |P(x)| \leq \bar{C}_2 x_1^{\bar{\delta}_2} \text{ uniformly for } x' \in X. \quad (3.9)$$

Then it follows

$$|A_{1,n}| \leq \bar{C}_1 \int_B x_1^{\delta_1} |u_n|^{q-1} |u_n - u| \frac{dx_1}{x_1} dx' + \bar{C}_1 \int_B x_1^{\delta_1} |u|^{q-1} |u_n - u| \frac{dx_1}{x_1} dx'$$

and

$$A_{11,n} = \bar{C}_1 \int_B x_1^{\delta_1} |u_n|^{q-1} |u_n - u| \frac{dx_1}{x_1} dx', \quad A_{12,n} = \bar{C}_1 \int_B x_1^{\delta_1} |u|^{q-1} |u_n - u| \frac{dx_1}{x_1} dx'.$$

With the help of Hölder inequality, we have

$$A_{11,n} \leq \bar{C}_1 \left( \int_B \left[ \frac{x_1^{\delta_1}}{x_1^{\gamma}} (u_n - u) \right]^{q-1} \frac{dx_1}{x_1} dx' \right)^{\frac{q}{q-1}} \left( \int_B \left[ \frac{x_1^{\delta_1} |u_n|^{q-1} |u_n - u| \frac{dx_1}{x_1} dx' \right]^{\frac{q}{q-1}} \right)^{\frac{q}{q-1}}$$

$$= \bar{C}_1 \|u_n - u\|_{L^q_{\delta}(B)} \|u_n\|_{L^q_{\delta}(B)}^{q-1},$$

where $\gamma_1 = \frac{N}{q} - \frac{1}{q-1}(\delta_1 - (\frac{N}{q} - \gamma))$. Since $\gamma \in \left[ \frac{N}{q} - \delta, \frac{N}{q} \right]$ and $\frac{N}{q} - \gamma_1 = \frac{1}{q-1}(\delta_1 - (\frac{N}{q} - \gamma)) \geq 0$, we know that $H^{1,\frac{N}{q}}_{2,0}(B) \rightharpoonup L^q_{\delta}(B)$, which implies $\{u_n\}$ is bounded in $L^q_{\delta}(B)$. Therefore, by (3.8), we have $A_{11,n} \to 0$ as $n \to \infty$. Analogously we get $A_{12,n} \to 0$ as $n \to \infty$, which means that $A_{1,n} \to 0$ as $n \to \infty$. On the other hand, by (3.2) and (3.9), we obtain

$$|A_{2,n}| \leq \int_B P[h(u_n)] \cdot |u_n - u| \frac{dx_1}{x_1} dx' + \int_B P[h(u)] \cdot |u_n - u| \frac{dx_1}{x_1} dx'$$

$$\leq K^{p-q} \int_B x_1^{\delta_2} |u_n|^{q-1} |u_n - u| \frac{dx_1}{x_1} dx' + K^{p-q} \int_B x_1^{\delta_2} |u|^{q-1} |u_n - u| \frac{dx_1}{x_1} dx'$$

$$\to 0,$$

as $n \to \infty$.

Summing up, we get

$$\|u_n - u\|^2 = \langle I_\lambda'(u_n) - I_\lambda'(u), u_n - u \rangle - o(1) = o(1).$$

It shows that $u_n \to u$ strongly in $H^{1,\frac{N}{q}}_{2,0}(B)$ as $n \to \infty$. We get the assertion. \hfill \Box

Combining Lemma 3.1 and Lemma 3.2, we have the following existence result.
Proposition 3.1. Suppose (H) holds, for any $\lambda > 0$, the truncated problem $(T_\lambda)$ possesses a positive solution $u_\lambda \in H_{2,0}^{1,\frac{N}{q}}(\mathbb{B})$. Moreover, the mountain pass level $c_\lambda = I_\lambda(u_\lambda)$.

The positivity of a solution can be ensured by replacing $I_\lambda$ to $I_\lambda^+$ defined by

$$I_\lambda^+(u) = \frac{1}{2} \int_{\mathbb{B}} |\nabla u|^2 \frac{dx_1}{x_1} dx' - \frac{1}{q} \int_{\mathbb{B}} Q(x)(u^+)^q \frac{dx_1}{x_1} dx' - \lambda \int_{\mathbb{B}} P(x)H(u) \frac{dx_1}{x_1} dx'.$$

Moreover, we may apply the cone maximum principle in order to obtain $u_\lambda > 0$ in int $\mathbb{B}$.

4. Proof of Theorem 1.1. In this section, we prove that a solution $u_\lambda$ of the problem $(T_\lambda)$ obtained in Proposition 3.1 must be bounded, namely $\|u_\lambda\|_{L^\infty(\mathbb{B})} \leq K$ provided that $K$ and $\lambda$ are appropriately chosen. For this purpose, we will use the Moser iteration method [20]. This implies obviously that such $u_\lambda$ solves problem $(P_\lambda)$.

Let

$$S(\mathbb{B}) = \inf_{u \in H_{2,0}^{1,\frac{N}{q}}(\mathbb{B}) \setminus \{0\}} \left( \int_{\mathbb{B}} |u|^2 \frac{dx_1}{x_1} dx' \right)^{\frac{1}{2}}$$

be the best constant corresponding to the Sobolev embedding $H_{2,0}^{1,\frac{N}{q}}(\mathbb{B}) \hookrightarrow L_1^{\frac{N}{q}}(\mathbb{B})$. Then for any $\mathbb{B}$, we know $S(\mathbb{B}) = S(\mathbb{R}^N)$ (see [15]). Thus we denote $S := S(\mathbb{B}) = S(\mathbb{R}^N_+)$ for simplicity, and $S$ is achieved by the function

$$U(x_1,x') = \frac{C}{(1 + |\ln x_1|^2 + |x'|^2)^\frac{N+2}{2}}.$$ 

We introduce the functional $I_\ast : H_{2,0}^{1,\frac{N}{q}}(\mathbb{B}) \to \mathbb{R}$ defined by

$$I_\ast(u) = \frac{1}{2} \int_{\mathbb{B}} |\nabla u|^2 \frac{dx_1}{x_1} dx' - \frac{1}{q} \int_{\mathbb{B}} Q|u|^q \frac{dx_1}{x_1} dx'.$$ (4.2)

If we denote $c_0$ is the mountain pass level related to $I_\ast$, which is associated to the problem in [5]

$$(P_0) \begin{cases} -\Delta_\ast u = Q(x)|u|^{q-2}u & \text{in } \mathbb{B}, \\
 u = 0 & \text{on } \partial \mathbb{B}. \end{cases}$$

We conclude that $I_\lambda(u) \leq I_\ast(u)$ for each $u \in H_{2,0}^{1,\frac{N}{q}}(\mathbb{B})$ since $P(x)H(u) \geq 0$ in $\mathbb{B}$. Thus, we can find $e \in H_{2,0}^{1,\frac{N}{q}}(\mathbb{B})$ such that $I_\ast(e) < 0$ and $I_\lambda(e) < 0$ obviously. Hence $c_\lambda \leq c_0$.

Moreover, by (3.4), we have

$$c_0 \geq c_\lambda = \frac{1}{2} \|u_\lambda\|^2 - \frac{1}{q} \int_{\mathbb{B}} Qu_\lambda \frac{dx_1}{x_1} dx' - \lambda \int_{\mathbb{B}} PH(u_\lambda) \frac{dx_1}{x_1} dx'$$

$$\geq \frac{1}{2} \|u_\lambda\|^2 - \frac{1}{q} \int_{\mathbb{B}} Qu_\lambda \frac{dx_1}{x_1} dx' - \frac{\lambda}{q} \int_{\mathbb{B}} Ph(u_\lambda) u_\lambda \frac{dx_1}{x_1} dx'$$

$$= (\frac{1}{2} - \frac{1}{q}) \|u_\lambda\|^2 - \frac{q-2}{2q} \|u_\lambda\|^2.$$ (4.3)

Therefore, by (4.1) it yields that

$$\|u_\lambda\|_{L_1^{\frac{N}{q}}(\mathbb{B})} \leq S^{-\frac{1}{2}} \|u_\lambda\| \leq S^{-\frac{1}{2}} (\frac{2q}{q-2} c_0)^{\frac{1}{2}} =: S^{-\frac{1}{2}} c_0.$$ (4.4)
Proof of Theorem 1.1. We first prove that there exists a constant \( \lambda_0 > 0 \) such that for \( 0 < \lambda \leq \lambda_0 \) the positive solution \( u_\lambda \) of \((T_\lambda)\) satisfies
\[
\|u_\lambda\|_{L^\infty(B)} \leq K. \tag{4.5}
\]
For simplicity, we set \( u = u_\lambda \). For \( L \geq K \), let us define the following functions
\[
u_L = \begin{cases}
  u & \text{for } u \leq L, \\
  L & \text{for } u > L,
\end{cases}
\]
and
\[
\phi = uu_L^{2(\beta-1)}, \quad w_L = uu_L^{\beta-1},
\]
where \( \beta \geq 1 \) will be determined later. Taking \( \phi \) as a test function in \((T_\lambda)\), we get
\[
\int_B \left( u_L^{2(\beta-1)}|\nabla_B u|^2 + 2(\beta - 1)u_L^{2\beta-3}u|\nabla_B u|\nabla_B u_L \right) \frac{dx_1}{x_1} \, dx' \\
= \int_B Q u^q u_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx' + \lambda \int_B Ph(u)uu_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx'.
\]
By the definition of \( u_L \), we have
\[
\int_B u_L^{2\beta-3}u|\nabla_B u|\nabla_B u_L \frac{dx_1}{x_1} \, dx' = \int_{\{u \leq L\}} u^{2(\beta-1)}|\nabla_B u|^2 \frac{dx_1}{x_1} \, dx' \geq 0. \tag{4.7}
\]
Therefore, we obtain by (3.2) and (4.7)
\[
\int_B u_L^{2(\beta-1)}|\nabla_B u|^2 \frac{dx_1}{x_1} \, dx' \leq \int_B Q u^q u_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx' + \lambda K^{p-q} \int_B Ph(u)uu_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx' \\
\leq M_1 \int_B u_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx',
\]
where \( M_1 = M(1 + \lambda K^{p-q}) \). On the other hand, we have
\[
|\nabla_B u_L|^2 = u_L^{2(\beta-1)}|\nabla_B u|^2 + (\beta - 1)^2 u_L^{2\beta-3} |\nabla_B u_L|^2 + 2(\beta - 1)u_L^{2\beta-3} |\nabla_B u|\nabla_B u_L \\
\leq 2(u_L^{2(\beta-1)}|\nabla_B u|^2 + (\beta - 1)^2 u_L^{2(\beta-2)} |\nabla_B u_L|^2) \tag{4.9}
\]
and
\[
\int_B (\beta - 1)^2 u_L^{2(\beta-2)} |\nabla_B u|\nabla_B u_L \frac{dx_1}{x_1} \, dx' = \int_{u \leq L} (\beta - 1)^2 u_L^{2(\beta-1)} |\nabla_B u|^2 \frac{dx_1}{x_1} \, dx' \\
\leq \int_B (\beta - 1)^2 u_L^{2(\beta-1)} |\nabla_B u|^2 \frac{dx_1}{x_1} \, dx'. \tag{4.10}
\]
Hence, by (4.8)-(4.10), we obtain
\[
\int_B |\nabla_B u_L|^2 \frac{dx_1}{x_1} \, dx' \leq 2(1 + (\beta - 1)^2) \int_B |\nabla_B u|^2 u_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx' \\
\leq 4\beta^2 \int_B |\nabla_B u|^2 u_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx' \tag{4.11} \\
\leq 4\beta^2 M_1 \int_B u^q u_L^{2(\beta-1)} \frac{dx_1}{x_1} \, dx'.
\]
Applying (4.1), (4.11) and Hölder inequality, we have

\[
\left( \int_B |w_L|^2 \frac{dx'}{x_1} \right)^\frac{1}{2} \\
\leq 4S^{-\frac{1}{2}} \beta^2 M_1 \int_B u^\alpha w_L^{2(\beta-1)} \frac{dx_1}{x_1} \\
= 4S^{-\frac{1}{2}} \beta^2 M_1 \left( \int_B u^{q-2} w_L^{\frac{2q}{q-2}} \frac{dx_1}{x_1} \right)^\frac{q-2}{q} \left( \int_B w_L^{\frac{2q}{q-2}} \frac{dx_1}{x_1} \right)^{\frac{q-2}{q}}. 
\]

(4.12)

We set \( \alpha^* = \frac{2q}{2q-(q-2)} \) and then \( 2 < \alpha^* < 2^\star \). Therefore, by (4.4) we get

\[
\|w_L\|^2_{L_w^\infty(B)} \leq 4S^{-\frac{1}{2}} \beta^2 M_1 c_0^{\alpha^*-2} \|w_L\|^2_{L_w^\infty(B)}. 
\]

(4.13)

Since \( w_L = w_0^{\beta-1} \leq L^\beta - 1 \in L_{w_0}^\infty(B) \), we have \( w_L \in L_w^\infty(B) \). If \( u^\beta \in L_w^{\infty}(B) \), with the Lebesgue’s Dominated Convergence Theorem and Fatou’s Lemma, then we obtain

\[
\left( \int_B u^{\beta} \frac{dx_1}{x_1} \right)^\frac{1}{\beta} = \left( \int_B \lim_{L \to \infty} u^\alpha w_L^{\frac{2q}{q-2}} \frac{dx_1}{x_1} \right)^\frac{1}{\beta} \leq \lim_{L \to \infty} \left( \int_B u^\alpha w_L^{\frac{2q}{q-2}} \frac{dx_1}{x_1} \right)^\frac{1}{\beta} \\
\leq 4S^{-\frac{1}{2}} \beta^2 M_1 c_0^{\alpha^*-2} \lim_{L \to \infty} \left( \int_B u^\alpha \frac{dx_1}{x_1} \right)^\frac{1}{\beta} \\
= 4S^{-\frac{1}{2}} \beta^2 M_1 c_0^{\alpha^*-2} \left( \int_B \lim_{L \to \infty} u^\alpha \frac{dx_1}{x_1} \right)^\frac{1}{\beta} \\
= 4S^{-\frac{1}{2}} \beta^2 M_1 c_0^{\alpha^*-2} \left( \int_B u^\alpha \frac{dx_1}{x_1} \right)^\frac{1}{\beta}. 
\]

That is

\[
\|u^\beta\|_{L_w^{\infty}(B)} \leq (4S^{-\frac{1}{2}} \beta \beta^2 M_1 c_0^{\alpha^*-2})^{\frac{1}{2}} \beta \|u^\beta\|_{L_w^{\infty}(B)}. 
\]

We denote \( M_2 = (4S^{-\frac{1}{2}} \beta \beta^2 M_1 c_0^{\alpha^*-2})^{\frac{1}{2}} \beta \) and rewrite the above inequality in the form

\[
\|u\|_{L_w^{\infty}(B)} \leq M_2 \beta \|u\|_{L_w^{\infty}(B)}. 
\]

(4.14)

Let \( \chi = \frac{2}{\alpha^*} \) and then \( \beta \cdot 2^\star = \beta \chi \alpha^* \). Then \( \chi > 1 \) and (4.14) can be written as

\[
\|u\|_{L_w^{\infty}(B)} \leq \left( \sum_{m=0}^{\infty} \chi^{-m} \right)^{\frac{1}{\alpha^*}} \|u\|_{L_w^{\infty}(B)}. 
\]

(4.15)

The result is now obtained by iteration of estimate (4.15). Taking \( \beta = \chi^m \) \((m = 0, 1, 2, \ldots)\), we have that, for any \( k \in \mathbb{N} \),

\[
\|u\|_{L_w^{\infty}(B)} \leq \prod_{m=0}^{k-1} \left( \sum_{m=0}^{\infty} \chi^{-m} \right)^{\frac{1}{\alpha^*}} \|u\|_{L_w^{\infty}(B)} \leq M_2 \chi^m \|u\|_{L_w^{\infty}(B)} \leq M_2 \chi^m \|u\|_{L_w^{\infty}(B)} \\
\leq M_2 \chi^{\tau_2} \|u\|_{L_w^{\infty}(B)}, 
\]

where \( \sum_{m=0}^{\infty} \chi^{-m} \leq \frac{1}{\chi^m} : = \tau_1 \) and \( \tau_2 = \sum_{m=0}^{\infty} m \chi^{-m} \). This estimate combined with the cone Sobolev embedding inequality and (4.3) yields

\[
\left( \int_B u^{k \alpha^*} \frac{dx_1}{x_1} \right)^{\frac{1}{k \alpha^*}} \leq M_2 \chi^{\tau_2} C_{\alpha^*} \|u\|_{L_w^{\infty}(B)} \leq M_2 \chi^{\tau_2} C_{\alpha^*} c_0, 
\]

(4.16)
Moreover, \( \lambda \) we choose for which \( E \). Taking \( \lambda \) as increasing as \( \lambda \) goes closer to 0. In order to get multiple solutions, the Ljusternik-Schnirelman theory of critical points ([1, 2, 3]) will be applied and the function \( h \) getting closer to 0. In order to get multiple solutions, the Ljusternik-Schnirelman theory of critical points ([1, 2, 3]) will be applied and the function \( h \) will be extended as an odd function, i.e.,

\[
h(u) = \begin{cases} 
  u^{p-1} & \text{for } 0 \leq u \leq K, \\
  K^{p-q} u^{q-1} & \text{for } u > K, \\
  -(-u)^{p-1} & \text{for } -K \leq u < 0, \\
  -K^{p-q} (-u)^{q-1} & \text{for } u < -K.
\end{cases}
\]

Next, let us state a variant of the dual variational principle of A. Ambrosetti and P. Rabinowitz [1].

Let \( E \) be a Banach space and \( B_r = \{ u \in E : ||u|| < r \} \). A set \( Y \subset E \) is called symmetric if \( u \in Y \) implies \( -u \in Y \), and the class of closed and symmetric subsets of \( E \setminus \{0\} \) is denoted by \( \Sigma(E) \). For \( Y \in \Sigma(E) \), \( \nu(Y) \) denotes the genus of \( Y \), i.e.,

\[
\nu(Y) = \min \{ m \in \mathbb{N}^+ | \text{there exists } \phi \in C(Y, \mathbb{R}^m \setminus \{0\}) \}.
\]

Moreover, \( \nu(Y) = \infty \) if there exists no such finite \( m \), and \( \nu(\emptyset) = 0 \).
For $I \in C^1(E, \mathbb{R})$, we set $E_+ = \{ u \in E : I(u) \geq 0 \}$ and

$$H = \{ h \in C(E, E) : h \text{ is odd homeomorphism and } h(B_1) \subset E_+ \},$$

$\Gamma_m = \{ A \subset E : A \text{ is compact, symmetric and } \nu(A \cap h(\partial B_1)) \geq m \text{ for any } h \in H \}$.

We introduce the following lemma (Lemma 3.1 in [2]), where the proof is exactly the same as that in [1].

**Lemma 5.1.** If $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$ and

(S1) there exist $\rho, \alpha > 0$ such that $I(u) > 0$ for all $u \in B_{\rho} \setminus \{0\}$ and $I(u) \geq \alpha$ for all $u \in \partial B_{\rho}$;

(S2) $I(u) = I(-u)$;

(S3) for any finite dimensional subspace $E^m \subset E$, $E^m \cap E_+$ is bounded.

Then

(i) $\Gamma_m \neq \emptyset$ for all $m = 1, 2, \ldots$, and $0 < \alpha \leq b_m \leq b_{m+1}$;

(ii) $b_m$ is a critical value of $I$ if $I$ satisfies (PS)$_c$ condition for $c = b_m$.

Moreover, if $b_{m+1} = \ldots = b_{m+t} = b$, then $\nu(K_b) \geq 1$, where $K_b = \{ u \in E : I(u) = b \text{ and } I'(u) = 0 \}$.

Lemma 5.1 implies also that for each $m \in \mathbb{N}^+$ there exist at least $m$ distinct pairs of critical points for $I$ if $I$ satisfies the conditions in Lemma 5.1, whatever the critical values $b_j \ (j = 1, \ldots, m)$ are equal or not.

In what follows, we always take $E = H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B})$ and set

$$E_\lambda = \{ u \in H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B}) : I_\lambda(u) \geq 0 \}, \quad E_* = \{ u \in H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B}) : I_*(u) \geq 0 \},$$

$$H_\lambda = \{ h \in C(H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B}), H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B})) : h \text{ is odd, homeomorphism and } h(B_1) \subset E_\lambda \},$$

and

$$H_* = \{ h \in C(H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B}), H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B})) : h \text{ is odd, homeomorphism and } h(B_1) \subset E_* \}.$$ Let

$$\Gamma^m_\lambda = \{ A \subset H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B}) : A \text{ is compact, symmetric and } \nu(A \cap h(\partial B_1)) \geq m \text{ for any } h \in H_\lambda \}$$

and

$$\Gamma^m_* = \{ A \subset H^{1, \frac{N}{2}}_{2, 0}(\mathbb{B}) : A \text{ is compact, symmetric and } \nu(A \cap h(\partial B_1)) \geq m \text{ for any } h \in H_* \}.$$ Since $E_\lambda \subset E_*$, we have $\Gamma^m_* \subset \Gamma^m_\lambda$ for $m = 1, 2, \ldots$ We now define min-max levels for $I_\lambda$ and $I_*$ as follows,

$$c^m_\lambda = \inf_{A \in \Gamma^m_\lambda} \max_{u \in A} I_\lambda(u) \quad \text{and} \quad c^m_* = \inf_{A \in \Gamma^m_*} \max_{u \in A} I_*(u).$$

**Proof of Theorem 1.2.** By the conclusion $\Gamma^m_* \subset \Gamma^m_\lambda$ and the definitions of $c^m_\lambda$ and $c^m_*$, we have

$$c^m_* \leq \inf_{A \in \Gamma^m_*} \max_{u \in A} I_*(u) \leq \inf_{A \in \Gamma^m_*} \max_{u \in A} I_*(u) = c^m_* \quad (5.2)$$
for \( m = 1, 2, \ldots \) By Lemma 3.2, \( I_\lambda \) satisfies the \((PS)_c\) condition for each \( c = c^m_\lambda \). Similarly, it is easy to check that \( I_* \) satisfies the \((PS)_c\) condition for each \( c = c^m_* \). It is easy to show that \( I_* \) and \( I_\lambda \) satisfy all conditions in Lemma 5.1. Therefore, \( c^m_\lambda \) and \( c^m_* \) are critical values of \( I_\lambda \) and \( I_* \) respectively, \( m = 1, 2, \ldots \) That implies, for fixed \( m \in \mathbb{N}^+ \), there exist \( m \) distinct pairs of solutions \((-u^j_\lambda, u^j_\lambda)\) \( (j = 1, \ldots, m) \) of \((T_\lambda)\) for each \( \lambda > 0 \) and \( m \) distinct pairs of solutions \((-u^j_0, u^j_0)\) \( (j = 1, \ldots, m) \) of \((P_0)\).

Since each \( c^m_* \) is independent of \( K \) and \( \lambda \), following the proof of Lemma 4.1, we can obtain that its assertion continues to hold for any solution \( u \in \mathcal{H}^{1, N}_{2,0} (\mathbb{B}) \) of problem \((T_\lambda)\) with the odd truncation \( h \), by redefining

\[
\left\{ \begin{array}{ll}
u_L = u & \text{for } |u| \leq L \\
L & \text{for } |u| > L
\end{array} \right.
\]

and applying the conclusion

\[ h(u)u \leq K^{p-q}|u|^q. \]

Therefore, by \((4.4)\), \((4.16)-(4.17)\) and \((5.2)\) we can show that for any \( m \in \mathbb{N}^+ \), each critical point \( u^m_\lambda \) of \( I_\lambda \) at the level \( c^m_\lambda \) satisfies the estimate

\[ \|u^m_\lambda\|_{L^\infty(\mathbb{B})} \leq \left( 4S^{-\frac{q}{2}}(c^m_\lambda)^{q-2}M(1 + \lambda K^{p-q}) \right)^{\frac{1}{q'}} \chi^{q/2} C_\alpha^* c^m_\lambda, \]

where \( \bar{c}^m_\lambda = \left( \frac{2q}{q-2}c^m_\lambda \right)^{\frac{1}{2}} \). We now fix \( K \) such that

\[ K > \left( 4M \right)^{\frac{q}{2}} C_\alpha^* S^{-\frac{q}{q+1}} \chi^{q/2} (c^m_\lambda)^{\left( \frac{q-2}{q} \right) + 1}. \]

Choose \( \lambda_m \) satisfying

\[ \lambda_m \leq \left( \frac{S^\frac{q}{2}K^{\frac{q}{q+1}}}{4MC_\alpha^{\frac{2}{q+1}} \chi^{\frac{2}{q+1}} \left( c^m_\lambda \right)^{q-2 + \frac{2}{q}}} - 1 \right) K^{q-p}, \]

and we get that each critical point \( u^m_\lambda \) corresponding to \( c^m_\lambda \) for \( 0 < \lambda \leq \lambda_m \) satisfies \( \|u\|_{L^\infty(\mathbb{B})} \leq K \). For any given \( m \in \mathbb{N}^+ \) above and any \( 0 < \lambda \leq \lambda_m \), we have that each critical point \( u^j_\lambda \) of \( I_\lambda \) at the level \( c^j_\lambda \) \( (1 \leq j \leq m - 1) \) satisfies

\[
\|u^j_\lambda\|_{L^\infty(\mathbb{B})} \leq \left( 4S^{-\frac{q}{2}}(c^j_\lambda)^{q-2}M(1 + \lambda K^{p-q}) \right)^{\frac{1}{q'}} \chi^{q/2} C_\alpha c^j_\lambda \leq \left( 4S^{-\frac{q}{2}}(c^m_\lambda)^{q-2}M(1 + \lambda_m K^{p-q}) \right)^{\frac{1}{q'}} \chi^{q/2} C_\alpha c^m_\lambda \leq K.
\]

Here we apply Lemma 4.1 and the fact that \( \bar{c}^j_\lambda = \left( \frac{2q}{q-2}c^j_\lambda \right)^{\frac{1}{2}} \leq c^m_\lambda \) for \( 1 \leq j \leq m - 1 \). Therefore, for each \( 0 < \lambda \leq \lambda_m \), the problem \((P_\lambda)\) has at least \( m \) distinct pairs of solutions \((-u_j, u_j)\) \( (j = 1, \ldots, m) \). We finish the proof.

Next, we show that there exists a sequence \( \lambda_n \to 0 \) such that \( u_{\lambda_n} \to u_0 \neq 0 \) in \( \mathcal{H}^{1, N}_{2,0} (\mathbb{B}) \), where \( \{u_{\lambda_n}\} \) \( (0 < \lambda_n \leq \lambda_0) \) is a family of mountain pass solutions of problems \((P_{\lambda_n})\) \( (0 < \lambda_n \leq \lambda_0) \), and \( u_0 \) is a solution of the subcritical problem \((P_0)\).
**Proof of Theorem 1.2.** Since \( \|u_\lambda\|_{L^\infty(B)} \leq K \) for \( 0 < \lambda \leq \lambda_0 \) and \( c_\lambda = I_\lambda(u_\lambda) \leq c_0 \), we have

\[
I_\lambda(u_\lambda) \geq \frac{1}{2} \int_B |\nabla_B u_\lambda|_1^2 \, dx' - \frac{1}{q} \int_B Q|u_\lambda|^q \, dx' - \frac{\lambda}{q} \int_B P|u_\lambda|^p \, dx' = \left( \frac{1}{2} - \frac{1}{q} \right) \|u_\lambda\|_1^2.
\]

This implies that \( \{u_{\lambda_n}\} (0 < \lambda_n \leq \lambda_0) \) is bounded in \( H_{2,0}^{1,n}(B) \). Therefore, we may assume that there exists a sequence \( \lambda_n \to 0 \) such that \( u_{\lambda_n} \to u_0 \neq 0 \) in \( H_{2,0}^{1,n}(B) \) as \( n \to \infty \). It is easy to see that \( u_0 \) is a weak solution of \((P_0)\). By Proposition 2.3, we have

\[
u_{\lambda_n} \to u_0 \quad \text{in} \quad L^{\frac{n-1}{n}}_\delta(B)
\]

for \( q \in (2, 2^*) \). Moreover, we deduce that

\[
\left| \left( \int_B Qu_{\lambda_n}^q \frac{dx_1}{x_1} \right)^{\frac{1}{q}} - \left( \int_B Qu_0^q \frac{dx_1}{x_1} \right)^{\frac{1}{q}} \right| \leq \left( \int_B Qu_{\lambda_n}^q - Qu_0^q \right)^{\frac{1}{q}} \left( \int_B Qu_0^q \frac{dx_1}{x_1} \right)^{\frac{1}{q}} \leq C_1 \left( \int_B |u_{\lambda_n} - u_0| \frac{dx_1}{x_1} \right)^{\frac{1}{q}} = C_1 \|u_{\lambda_n} - u_0\|_{L^{\frac{n-1}{n}}_\delta(B)} \to 0,
\]

which implies that

\[
\lim_{n \to \infty} \int_B Qu_{\lambda_n}^q \frac{dx_1}{x_1} \, dx' = \int_B Qu_0^q \frac{dx_1}{x_1} \, dx'.
\]

Therefore,

\[
\lim_{n \to \infty} \int_B |\nabla_B u_{\lambda_n}|_1^2 \, dx' = \lim_{n \to \infty} \int_B Qu_{\lambda_n}^q \frac{dx_1}{x_1} \, dx' + \lim_{n \to \infty} \lambda_n \int_B Pu_{\lambda_n}^p \frac{dx_1}{x_1} \, dx' = \int_B Qu_0^q \frac{dx_1}{x_1} \, dx' = \int_B |\nabla_B u_0|^2 \frac{dx_1}{x_1} \, dx',
\]

since \( u_0 \) is a weak solution of the equation in \((P_0)\). The proof is completed.

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