A LIMIT EQUATION AND BIFURCATION DIAGRAMS OF SEMILINEAR ELLIPTIC EQUATIONS WITH GENERAL SUPERCritical GROWTH

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Abstract. We study radial solutions of the semilinear elliptic equation

$$\Delta u + f(u) = 0$$

under rather general growth conditions on $f$. We construct a radial singular solution and study the intersection number between the singular solution and a regular solution. An application to bifurcation problems of elliptic Dirichlet problems is given. To this end, we derive a certain limit equation from the original equation at infinity, using a generalized similarity transformation. Through a generalized Cole-Hopf transformation, all the limit equations can be reduced into two typical cases, i.e., $\Delta u + u^p = 0$ and $\Delta u + e^u = 0$.

1. Introduction and Main results

Let $N \geq 3$ and $r := |x|$. In this paper we construct a radial singular solution $u^*(r)$ of the elliptic equation

$$\Delta u + f(u) = 0$$

(1.1)

under rather general growth conditions, and study the intersection number of two radial solutions $Z_{(0,\infty)}[u(\cdot, \rho) - u^*(\cdot)]$. Here, $u(r, \rho)$ is the classical radial solution of (1.1), which satisfies

$$\begin{cases} u'' + \frac{N-1}{r} u' + f(u) = 0, & r > 0, \\ u(0) = \rho, \\ u'(0) = 0, \end{cases}$$

(1.2)

and $Z_I[u_0(\cdot) - u_1(\cdot)]$ denotes the intersection number of the two functions $u_0(r)$ and $u_1(r)$ defined in an interval $I \subset \mathbb{R}$, i.e., $Z_I[u_0(\cdot) - u_1(\cdot)] = \sharp\{r \in I; u_0(r) = u_1(r)\}$. By a radial singular solution $u^*(r)$ of (1.1) we mean that $u^*(r)$ is a classical solution of the equation

$$u'' + \frac{N-1}{r} u' + f(u) = 0$$

(1.3)

on $(0, r_0)$ for some $r_0 > 0$ and $\lim_{r \to 0} u^*(r) = \infty$. We give two applications of the intersection number.

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By \( F(u) \) we define

\[
F(u) := \int_u^\infty \frac{dt}{f(t)}.
\]

We assume the following:

(f1) One of the following (f1-1) or (f1-2) holds:

(f1-1) (a generalization of \( u^p \)) \( f(u) \in C^1[0, \infty), \ f(u) > 0 \) for \( u > 0 \), \( f(0) = 0 \),

\[
f(u) \in C^2(u_0, \infty) \text{ for some } u_0 > 0, \ \lim_{u \downarrow 0} F(u) = \infty, \text{ and } \lim_{u \to \infty} F(u) = 0,
\]

(f1-2) (a generalization of \( e^u \)) \( f(u) \in C^1(\mathbb{R}), \ f(u) > 0 \) for \( u \in \mathbb{R} \),

\[
f(u) \in C^2(u_0, \infty) \text{ for some } u_0 > 0, \ \lim_{u \to -\infty} F(u) = \infty, \text{ and } \lim_{u \to \infty} F(u) = 0.
\]

(f2) There exists the limit

\[
q := \lim_{u \to \infty} \frac{f'(u)^2}{f(u)f''(u)},
\]

which is denoted by \( q \) throughout the present paper, and this limit is in \((0, \infty)\).

Note that the inverse function of \( F \), which is denoted by \( F^{-1}(u) \), can be defined for \( u > 0 \), because of (f1). We define the growth rate of \( f \) by \( p := \lim_{u \to \infty} uf'(u)/f(u) \). By L’Hospital’s rule we have

\[
p = \lim_{u \to \infty} \frac{u}{f(u)f'(u)} = \lim_{u \to \infty} \frac{1}{f(u)f''(u)} = \frac{q}{q-1}, \text{ and hence } \frac{1}{p} + \frac{1}{q} = 1.
\]

The exponent \( q \) represents the Hölder conjugate of the growth rate of \( f \). We will see in Section 2 that \( q \geq 1 \). Let \( q_S \) and \( q_{JL} \) denote the Hölder conjugates of the critical Sobolev exponent and the so-called Joseph-Lundgren exponent, respectively, i.e.,

\[
q_S := \frac{N + 2}{4} \quad \text{and} \quad q_{JL} := \frac{N - 2\sqrt{N - 1}}{4}.
\]

The exponents \( q_S \) and \( q_{JL} \) can be formally defined for all \( N \geq 1 \). However, \( q_S > 1 \) (resp. \( q_{JL} > 1 \)) if and only if \( N \geq 3 \) (resp. \( N \geq 11 \)). In Section 2 we give five examples of \( f \). In particular, \( q = p/(p-1) \) if \( f(u) = u^p \), and \( q = 1 \) if \( f(u) = e^u \). The case \( q = 1 \) includes rapidly growing nonlinearities, e.g.,

\[
\exp(u^p) (p \geq 1), \ \exp(\exp(\cdots \exp(u) \cdots)), \text{ and } (u + 2)^{\left(\frac{u+2}{n}\right)} (n \geq 2).
\]

When \( q > 1 \), the principal term of \( f \) is not necessarily \( u^{q/(q-1)} \), e.g., \( f(u) = u^{q/(q-1)}(\log(u+1))^\gamma (\gamma > 0) \).

The first main result of the paper is the following:
Theorem A. Suppose that $N \geq 3$ and (11) and (12) hold. Let $u(r, \rho)$ be the solution of (1.2). Then the following hold:

(i) If $q < q_S$, then there is $r_0 > 0$ such that (1.5) has a singular solution $u^*(r) \in C^2(0, r_0)$ and

$$u^*(r) = F^{-1}[k^{-1}r^2(1 + \theta(r))].$$

Here, $\theta(r) \in C^2(0, r_0)$, $\theta(r) \to 0$ ($r \downarrow 0$), and

$$k := 2N - 4q.$$

(ii) If $q_{IL} < q < q_S$, then for each $r_1 \in (0, r_0]$, $Z_{(0, r_1)}[u(\cdot, \rho) - u^*(\cdot)] \to \infty$ as $\rho \to \infty$. Here $r_0$ is given in (i).

In order to prove Theorem A, we use the following generalized similarity transformation of elliptic type:

$$v(s, \sigma) = F^{-1}[\lambda^{-2}F(u(r, \rho))] \text{ and } s := \frac{r}{\lambda},$$

where $\sigma := F^{-1}[\lambda^{-2}F(\rho)]$. The parabolic version of (1.6) was introduced by Fujishima [10], the equation (11)]. We show in Section 4 that a certain limit equation for $v$ becomes

$$v'' + \frac{N - 1}{s}v' + f(v) + \frac{q - F(v)f'(v)}{F(v)f(v)}v^2 = 0 \quad \text{when } N \geq 3 \text{ and } q < q_S.$$  

If $v(s)$ satisfies (1.7), then $F^{-1}[\lambda^{-2}F(v(\lambda s))]$, $\lambda > 0$, also satisfies (1.7). The equation (1.7) has the exact singular solution

$$v^*(s) := F^{-1}[k^{-1}s^2].$$

Let $v(s, \sigma)$ denote the solution of the problem

$$\begin{cases}
  v'' + \frac{N - 1}{s}v' + f(v) + \frac{q - F(v)f'(v)}{F(v)f(v)}v^2 = 0, & s > 0, \\
  v(0) = \sigma, \\
  v'(0) = 0.
\end{cases}$$

A large solution of (1.3) is approximated by a solution of (1.7). Thus, it is important to study the intersection number of two solutions of (1.7). The second main result is the following:

Theorem B. Suppose that $N \geq 3$ and (11) and (12) hold. Let $0 < \sigma_0 < \sigma_1$. Let $v(s, \sigma_i)$, $i = 0, 1$, be solutions of (1.9) with $\sigma = \sigma_i$, and let $v^*(s)$ be the singular solution given by (1.8). Then the following hold:

(i) If $q = q_S$, then $Z_{(0, \infty)}[v(\cdot, \sigma_0) - v(\cdot, \sigma_1)] = 1$ and $Z_{(0, \infty)}[v(\cdot, \sigma_0) - v^*(\cdot)] = 2$.

(ii) If $q_{IL} < q < q_S$, then $Z_{(0, \infty)}[v(\cdot, \sigma_0) - v(\cdot, \sigma_1)] = \infty$ and $Z_{(0, \infty)}[v(\cdot, \sigma_0) - v^*(\cdot)] = \infty$.

(iii) If $q \leq q_{IL}$, then $Z_{(0, \infty)}[v(\cdot, \sigma_0) - v(\cdot, \sigma_1)] = 0$ and $Z_{(0, \infty)}[v(\cdot, \sigma_0) - v^*(\cdot)] = 0$.

In particular, if $3 \leq N \leq 9$, then $q_{IL} < 1$, and hence (iii) is vacuous, because $q \geq 1$.

When $q = q_S$, $v(s, \sigma)$ can be written explicitly as follows:

$$v(s, \sigma) = F^{-1}\left[F(\sigma)\left(1 + \frac{s^2}{4NF(1)}\right)^2\right].$$

For the case of a quasilinear elliptic equation with power nonlinearity, see [27].
Remark 1.1. When \( q = 1 \), the condition \( q_{JL} < q < q_{S} \) corresponds to \( 3 \leq N \leq 9 \).

As we will see in Subsections 2.1 and 2.2, Theorem \( \text{[3]} \) is a generalization of the well-known result about intersection numbers for the case \( f(u) = u^p \) or \( e^u \). In this paper two applications of Theorems \( \text{[A]} \) and \( \text{[3]} \) are given in Corollaries 1.2 and 1.3 below.

The first application is about the Morse index of a singular solution. When \( \text{[1]} \) and \( \text{[2]} \) hold, \( \text{[3]} \) has a singular solution \( u^*(r) \) given by Theorem \( \text{[A]} \). The solution \( u^*(r) \) is defined near the origin. We extend the domain of \( u^*(r) \). We assume that \( u^*(r) \) has a first positive zero \( r^*_0 > 0 \), i.e., \( u^*(r) > 0 \) for \( 0 < r < r^*_0 \) and \( u^*(r^*_0) = 0 \). For example, if \( f(u) > \delta > 0 \) for \( u \geq 0 \), then \( u^*(r) \) has the first positive zero. The function \( u^*(r) \) is a singular solution of the Dirichlet problem

\[
\begin{align*}
\Delta u + f(u) &= 0, \quad \text{in } B(r^*_0), \\
u &= 0, \quad \text{on } \partial B(r^*_0), \\
u > 0, \quad \text{in } B(r^*_0),
\end{align*}
\]

(1.10)

where \( B(r^*_0) \subset \mathbb{R}^N \) is a ball with radius \( r^*_0 \). The Morse index of \( u^*(r) \) is defined by

\[
m(u^*) := \text{dim } X; \quad X \text{ is a subspace of } H^1_{0, \text{rad}}(B(r^*_0)), \quad H[\phi] < 0 \text{ for all } \phi \in X \setminus \{0\},
\]

where \( H^1_{0, \text{rad}}(B(r^*_0)) := \{ u(x) \in H^1(B(r^*_0)); \quad u(x) = u(|x|), \quad u(r^*_0) = 0 \} \) and

\[
H[\phi] := \int_{B(r^*_0)} (|\nabla \phi|^2 - f'(u^*) \phi^2) \, dx.
\]

Corollary 1.2 (Morse index of the singular solution). Suppose that \( N \geq 3 \) and \( \text{[1]} \) and \( \text{[2]} \) hold. Let \( u^*(r) \) be the singular solution of (1.10) constructed as above. Then the following hold:

(i) If \( 3 \leq N < 9 \), then \( m(u^*) = \infty \).

(ii) If \( 3 \leq N \leq 9 \), then \( m(u^*) < \infty \).

Remark 1.1. When \( f(u) = u^p + c_0 u \) with \( p > p_S \), a singular solution \( u^* \) of (1.10) was constructed by Merle-Peletier [23]. Guo-Wei [15] studied the Morse index of this singular solution \( u^* \). In [15] it was shown that if \( p_S < p < p_{JL} \) (resp. \( p \geq p_{JL} \)), then \( m(u^*) = \infty \) (resp. \( m(u^*) < \infty \)). Note that their result includes the case \( p = p_{JL} \) which corresponds to the case \( q = q_{JL} \). In author’s previous papers [24, 25] singular solutions of (1.10) were constructed for the case \( f(u) = u^p + g_0(u) \) and \( f(u) = e^u + g_1(u) \), where \( g_0 \) and \( g_1 \) are lower order terms. Moreover, Corollary 1.2 was proved for these two cases. When \( f(u) = \exp(u^p) \) (\( p > 0 \)), Kikuchi-Wei [19] constructed a singular solution \( u^* \) of (1.10) and showed that \( m(u^*) < \infty \) if \( N \geq 11 \).
We have to mention the uniqueness of a radial singular solution in the supercritical case. The uniqueness was proved by several authors. See [4, 9, 16, 21, 26, 29]. However, the equations treated by them are \( \Delta u + a_0(r)u^p + g_0(u, r) = 0 \) and \( \Delta_m u + a_1(r)u^p + g_1(u, r) = 0 \) (\( 1 < m \leq 2 \)), where \( \Delta_m \) denotes the \( m \)-Laplace operator and \( g_0 \) and \( g_1 \) are lower order terms. For other supercritical nonlinearities the uniqueness problem remains open.

The second application is a bifurcation problem. Let \( B \subset \mathbb{R}^N \) denote the unit ball. We consider the problem

\[
\begin{align*}
\Delta U + \mu f(U) &= 0, \quad \text{in } B, \\
U &= 0, \quad \text{on } \partial B, \\
U > 0, \quad \text{in } B,
\end{align*}
\]

where \( \mu > 0 \). We assume the following:

\( f \in C^1[0, \infty) \cap C^2(1, \infty), \ f(u) > 0 \text{ for } u \geq 0, \ \text{and} \ \lim_{u \to \infty} F(u) = 0. \)

When (H1) holds, the domain of \( f \) can be extended to \( \mathbb{R} \) such that (H-2) holds. See Section 8 for details. By the symmetry result of Gidas-Ni-Nirenburg [12] every classical solution of (1.11) is radial. Hence, (1.11) can be reduced to an ODE. It is well known that the solution set of (1.11) can be described as \( \{ (\mu(\rho), U(R, \rho)) \} \) and \( \rho := \| U \|_{L^\infty(B)} \) and that \( \mu(0) = 0. \) Hence, the solution set is a curve emanating from \( (\lambda, U) = (0, 0) \). See [20] for example.

**Corollary 1.3** (Bifurcation diagram). Suppose that \( N \geq 3, \ q < q_S, \) and (H1) and (H2) hold. Then, (1.11) has a singular solution \( (\mu^*, U^*) \) and the following hold:

(i) If \( q_{JL} < q < q_S, \) then the curve \( \{ (\mu(\rho), U(R, \rho)); \ \rho > 0 \} \) has infinitely many turning points around \( \mu^*. \) In particular, (1.11) has infinitely many classical solutions for \( \mu = \mu^*. \)

(ii) If \( q \leq q_{JL}, \ f''(u) > 0 \text{ for } u \geq 0, \) and \( q \leq F(u)f'(u) \leq (N - 2)^2/(8(N - 2q)) \) for \( u \geq 0, \) then \( \mu(\rho) \) is strictly increasing, and hence it has no turning point. The curve can be parametrized by \( \mu \) and it blows up at some \( \mu > 0. \) Therefore, (1.11) has a unique classical solution for each \( 0 < \mu < \bar{\mu}. \)

The intersection number is used in the proof of Corollary 1.3 (i) and (ii). If \( 3 \leq N \leq 9, \) (H1) and (H2) hold, and \( q = 1, \) then \( q_{JL} < q < q_S, \) and hence the conclusions of Corollary 1.3 (i) hold. When \( f(u) = (u + 1)^p \) \((p_S < p < p_{JL})\) and \( f(u) = e^u \) \((3 \leq N \leq 9)\), Joseph-Lundgren [18] proved Corollary 1.3 by phase plane analysis. See Jacobsen-Schmitt [17] for quasilinear equations with \( f(u) = e^u. \) When \( f(u) = u^p + c_0u \) \((p_S < p < p_{JL})\), the existence of infinitely many turning points was numerically shown by Budd-Norbury [2], and later proved by Delbeault-Flores [7] and Guo-Wei [15]. In [24, 25] Corollary 1.3 (i) was proved in the case \( f(u) = u^p + g_0(u) \) \((p_S < p < p_{JL})\) and \( f(u) = e^u + g_1(u) \) \((3 \leq N \leq 9)\), where \( g_0 \) and \( g_1 \) are lower order terms. When \( f(u) = \exp(u^p) \) \(p > 0, 3 \leq N \leq 9, \) and the domain is not necessarily a ball, Dancer [6] proved the existence of infinitely many turning points. However, the locations of the turning points are not determined, and hence the existence of infinitely many positive solutions for some \( \mu > 0 \) remains open. In the ball case Kikuchi-Wei [19] proved Corollary 1.3 (i), using another similarity transformation introduced by [6]. On the other hand,
a non-existence of a turning point was studied by Brezis-Vázquez in a general domain. They gave a necessary and sufficient condition, using a singular solution. See [24, 25, 26] for the cases $f(u) = u^p + g_0(u)$ and $f(u) = e^u + g_1(u)$ in the ball.

Let us explain technical details. The exponent $q$, which is the Hölder conjugate of the growth rate of $f$, was introduced in Dupaigne-Farina. They gave sufficient conditions for (1.1) to have a bounded stable nonnegative solution in $\mathbb{R}^N$, using $q$. Fujishima-Ioku used the exponent $A := \lim_{u \to \infty} F(u)f'(u)$ to study the solvability of semilinear parabolic equations. When $f$ is a $C^2$-function, $A$ is equal to $q$. Indeed, by L'Hospital’s rule we have

$$A = \lim_{u \to \infty} F(u)f'(u) = \lim_{u \to \infty} \frac{F'(u)}{1/f'(u)} = \lim_{u \to \infty} \frac{f'(u)^2}{f(u)f''(u)} = q.$$ 

One of the advantages of using $q$ is that $q$ can deal with the exponential and super-exponential growth ($q = 1$), while $p = q/(q - 1)$ cannot. A singular solution of Theorem A (i) is constructed by a standard method with the contraction mapping theorem. However, our method can be applied to rather general nonlinearities owing to the expression of the singular solution (1.4). In the proof of Theorem A (ii) we rescale a regular and singular solutions of (1.3), using the generalized similarity transformation (4.1). Then these two functions locally uniformly converge to a regular and singular solutions of (1.7), respectively. Hence, (1.7) can be considered as a limit equation of (1.3). By $f_q(u)$ we define

$$f_q(u) := \begin{cases} 
  u^p & \text{if } q > 1, \\
  e^u & \text{if } q = 1,
\end{cases}$$

where $p := q/(q - 1)$ provided that $q > 1$. Let $v(s, \sigma)$ be the solution of (1.9). We use a generalized Cole-Hopf transformation:

$$w(s, \tau) := F_q^{-1}[F[v(s, \sigma)]] \quad \text{and} \quad \tau := F_q^{-1}[F[\sigma]],$$

where

$$F_q[v] := \int_v^\infty \frac{dt}{f_q(t)} = \begin{cases} 
  \frac{1}{p-1} v^{p+1} & \text{if } q > 1, \\
  e^{-v} & \text{if } q = 1
\end{cases}$$

and $F_q^{-1}$ is the inverse function of $F_q$. Specifically,

$$w(s, \tau) := \begin{cases} 
  (p - 1) \frac{1}{p-1} (F[v(s, \sigma)])^{\frac{1}{p-1}} & \text{if } q > 1, \\
  -\log F[v(s, \sigma)] & \text{if } q = 1,
\end{cases}$$

where $p := q/(q - 1)$ if $q > 1$. This transformation was introduced by Fujishima-Ioku and used in the study of the existence of a solution for semilinear parabolic equations. By Lemma 5.1 we see that $w$ satisfies $\Delta w + f_q(w) = 0$, i.e.,

$$(1.13) \quad \begin{cases} 
  \Delta w + w^p = 0 & \text{if } q > 1, \\
  \Delta w + e^w = 0 & \text{if } q = 1.
\end{cases}$$

It is interesting that all the limit equations (1.7) can be classified into two typical cases (1.13) and that all the limit equations become a one-parameter family of equations $\{\Delta w + f_q(w) = 0; \ q \geq 1\}$ in spite of arbitrariness of $f$. See Figure 1 which shows a fundamental...
\[ \Delta v + f(v) + \frac{2-F(v)f'(v)}{F(v)f(v)}|\nabla v|^2 = 0 \quad \Rightarrow \quad \Delta w + f_q(w) = 0 \]

\[ \Delta u + f(u) = 0 \quad \text{with} \ 1 \leq q < q_S \]

\[ \Delta u + f_q(u) = 0 \]

Figure 1. Relation between the original equation and limit equation

strategy in the paper. Intersection properties of the two typical equations are well known. Theorem B follows from them.

Corollary 1.2 is a simple application of Theorem A. In the proof of Corollary 1.2 we use a convexity of \( f \). In the proof of Corollary 1.3 (i) we use Theorem A (ii). Specifically, the existence of infinitely many turning points corresponds to \( Z_{0, r_1}[u(\cdot, \rho) - u^*(\cdot)] \to \infty \) as \( \rho \to \infty \). In the proof of Corollary 1.3 (ii) we use a comparison theorem devised by Gui [13, 14]. We can compare radial solutions of (1.7) and \( \Delta u + f_q(u) = 0 \). Then we show that the first (and hence every) eigenvalue is strictly positive, using an argument similar to the one used in Brezis-Vázquez [3]. Therefore, the curve has no turning point. These methods were used in [19, 24, 25, 26] although nonlinearities are restricted.

This paper consists of eight sections. In Section 2 we show that \( q \geq 1 \). Five examples of nonlinearities are given. In Section 3 we construct a singular solution. In Section 4 we prove the convergence to solutions of the limit equation of (1.3). In Section 5 and 6 we prove Theorem B and A, respectively. In Sections 7 and 8 we prove Corollaries 1.2 and 1.3, respectively.

2. Preliminaries and five nonlinearities

The following lemma is a fundamental property of the exponent \( q \), which was found by [11]. See [11, Remark 1.1].

Lemma 2.1. Suppose that \( (f1) \) and \( (f2) \) hold. Then \( q \geq 1 \).

We show the proof for readers’ convenience.

Proof. By L’Hospital’s rule we have

\[ \lim_{u \to \infty} F(u)f'(u) = \lim_{u \to \infty} \frac{F(u)}{1/f'(u)} = \lim_{u \to \infty} \frac{f'(u)^2}{f(u)f''(u)} = q. \]

We show by contradiction that \( q \geq 1 \). Suppose that \( 0 \leq q < 1 \). There exists \( \kappa \in (0, 1) \) such that \( F(u)f'(u) < \kappa \) for large \( u > 0 \). Since \( f'(u) = F''(u)F(u)^{-2} \), there is \( u_0 \in \mathbb{R} \) such that \( F(u)F''(u)F(u)^{-2} < \kappa \) for \( u > u_0 \). Note that \( f'(u) < 0 \). Solving the differential inequality, we have

\[ F(u)^{1-\kappa} < F(u_0)^{1-\kappa} - (1-\kappa)\frac{F'(u_0)}{F(u_0)^\kappa}(u - u_0) \quad \text{for} \quad u > u_0. \]

We see that \( F(u) < 0 \) for large \( u > 0 \). We obtain a contradiction, because \( F(u) > 0 \).
2.1. Power nonlinearity, \( q > 1 \). Let \( f_a(u) := (u + a)^p \), \( p > 1 \). Then, (1.1) (resp. (1.1)) holds if \( a > 0 \) (resp. \( a = 0 \)). Since \( F^{-1}[\lambda^{-2}F(u(\lambda s))] = \lambda^{2/(p-1)}(u(\lambda s) + a) - a \), the transformation \( u(s) \mapsto F^{-1}[\lambda^{-2}F(u(\lambda s))] \) is the usual similarity transformation in the power case when \( a = 0 \). Since \( f_a''(u)/f_a''(u) = p/(p-1) \) for \( u \geq 0 \), we see that \( q = p/(p-1) \), and hence (1.2) holds. When \( a = 0 \), Theorem 3 recovers the well-known intersection property studied by [18]. Since \( F(u)f_a''(u) \equiv q \) for \( u \geq 0 \), the limit equation (1.7) is the original equation (1.3). Therefore, if \( u(r) \) is a solution of (1.3), then \( F^{-1}[\lambda^{-2}F(u(\lambda r))] \) is also a solution of (1.3). When \( a > 0 \), Corollary 1.2 is applicable. If \( p_S < p < p_{JL} \), then \( q_{JL} < q < q_S \) and Corollary 1.3 (i) is applicable. If \( p \geq p_{JL} \), then \( q \leq q_{JL} \) and Corollary 1.3 (ii) is applicable.

2.2. Exponential nonlinearity, \( q = 1 \). Let \( f(u) := e^u \). Then (1.1-2) holds. Since \( F^{-1}[\lambda^{-2}F(u(\lambda s))] = u(\lambda s) + 2 \log \lambda \), the transformation \( u(s) \mapsto F^{-1}[\lambda^{-2}F(u(\lambda s))] \) is the usual transformation in the exponential case. Since \( f'(u)^2/(f(u)f''(u)) \equiv 1 \) for \( u \in \mathbb{R} \), we see that \( q = 1 \), and hence (1.2) holds. Theorem 3 recovers the intersection property for the case \( f(u) = e^u \), which was studied by [18]. Since \( F(u)f(u) \equiv 1 \) for \( u \geq 0 \), the limit equation (1.7) is the original equation (1.3). Therefore, if \( u(r) \) is a solution of (1.3), \( F^{-1}[\lambda^{-2}F(u(\lambda r))] \) is also a solution of (1.3). Corollary 1.2 is applicable. If \( 3 \leq N \leq 9 \), then \( q_{JL} < q < q_S \) and Corollary 1.3 (i) is applicable. If \( N \geq 10 \), then \( q \leq q_{JL} \) and Corollary 1.3 (ii) is applicable.

2.3. Log-convex or log-concave function, \( q = 1 \). We consider the case \( f(u) := \exp(g(u)) \) and \( g'(u) > 0 \) (\( u \geq 0 \)). Since

\[
F(u)f'(u) = 1 - g'(u)e^{g(u)} \int_0^\infty \frac{g''(t)}{g'(t)^2}e^{-g(t)}dt
\]

and

\[
\frac{f'(u)^2}{f(u)f''(u)} = \frac{1}{1 + \frac{g''(u)}{g'(u)^2}}
\]

we can check that if \( g''(u) < 0 \), \( g'(u)^2 + g''(u) > 0 \), \( \lim_{u \to \infty} g''(u)/g'(u)^2 = 0 \), and \( N > 0 \) is large, then Corollary 1.3 (ii) is applicable.

Next we consider the case \( f(u) := \exp(g(u)) \) and \( g'(u) > 0 \) (\( u \geq 0 \)) and \( g''(u) > 0 \) (\( u \geq 0 \)). The following lemma holds:

**Lemma 2.2.** If \( f \) satisfies (1.2), then \( q = 1 \).

**Proof.** We see by (2.1) that \( \lim_{u \to \infty} f'(u)^2/f(u) \cdot f''(u) \leq 1 \), because the limit exists. By Lemma 2.1 we see that \( q = 1 \). \( \square \)

We can easily construct an example of \( g(u) \) such that \( g'(u) > 0 \), \( g''(u) > 0 \), and \( g''(u)/g'(u)^2 \) oscillates as \( u \to \infty \). Therefore, the limit \( \lim_{u \to \infty} g''(u)/g'(u)^2 \) may not exist even if \( g'(u) > 0 \) and \( g''(u) > 0 \). The assumption (1.2) in Lemma 2.2 cannot be removed.

We give some examples such that the limit exists and Corollaries 1.2 and 1.3 (i) are applicable to \( f(u) = \exp(g(u)) \).

(i) \( f(u) = \exp(u^p) \) (\( p > 1 \)),

(ii) \( f(u) = \exp(e^u) \),
Lemma 3.1. Suppose that \( g_n(u) \) satisfies the following:

\[
(2.2) \quad \text{For } u \geq 0, g_n'(u) > 0 \text{ and } g_n''(u) > 0, \text{ and } \lim_{u \to \infty} \frac{g_n''(u)}{g_n'(u)^2} = 0,
\]

then \( g_{n+1}(u) := \exp(g_n(u)) \) also satisfies (2.2). Thus, \( f_n(u) := \exp(g_n(u)), n \geq 1 \), satisfies (11) and (12) if (2.2) holds for \( n = 1 \). A typical example is the case \( g_2(u) := e^u \). Then \( f_n \) is the \( n \)-th tetration function

\[
f_n(u) := \exp(\exp(\cdots \exp(u) \cdots)) \quad (n \geq 2).
\]

2.4. Product of power and log, \( q > 1 \). Let \( f(u) := (u + a)^p(\log(u + a))^{\gamma}, p > 1, \gamma \geq 0, \) and \( a > 1 \). Then (11) holds. By direct calculation we see that \( q = \lim_{u \to \infty} f'(u)^2/(f(u)f''(u)) = p/(p - 1) \). Corollaries 1.2 and 1.3 (i) are applicable.

2.5. Tetration, \( q = 1 \). Let \( f_{n+1}(u) := (u + a)^{f_n(u)} \) \((a > 1)\). Suppose the following:

\[
(2.3) \quad \text{For } u \geq 0, f_n(u) > 1, f_n'(u) > 0, \text{ and } f_n''(u) > 0 \text{ and } \lim_{u \to \infty} \frac{f_n'(u)^2}{f_n(u)f_n''(u)} = 1.
\]

Then we easily see that \( \lim_{u \to \infty} f_n(u) = \infty \). Using this limit, we can show that \( f_{n+1}(u) \) also satisfies (2.3). By (2.3) and the definition of \( f_{n+1}(u) \) we have that

\[
(2.4) \quad \frac{f_{n+2}'(u)^2}{f_{n+2}(u)f_{n+2}''(u)} = \left[ 1 + \frac{f_{n+1}'(u)\log(u + a) + 2f_{n+1}'(u)(u + a)^{-1} - f_{n+1}(u)(u + a)^{-1}}{f_{n+1}'(u)\log(u + a) + f_{n+1}(u)(u + a)^{-1}} \right]^{-1} \leq 1,
\]

because

\[
f_{n+1}''(u)\log(u + a) + \frac{2f_{n+1}'(u)}{u + a} - \frac{f_{n+1}'(u)}{(u + a)^2} \geq 0 \quad \text{for } u \geq 0.
\]

By (2.3) and (2.4) we see that if \( f_1(u) \) satisfies (2.3), then \( f_n(u) \) satisfies (11) and (12) for \( n \geq 1 \).

Let \( f_2(u) := (u + a)^{u + a} \) \((a > 1)\) and \( f_{n+1} := (u + a)^{f_n(u)} \) \((n \geq 2)\). Then,

\[
f_n(u) = (u + a)^{\underbrace{\cdots (u + a) \cdots}_{(u + a)}}
\]

which is called the \( n \)-th tetration of \((u + a)\). Corollaries 1.2 and 1.3 (i) are applicable to \( f_n(u), n \geq 2 \).

3. Singular solution

The goal of this section is to prove the following:

**Lemma 3.1.** Suppose that \( N \geq 3 \) and (11) and (12) hold. Let \( k \) be defined by (1.3). If \( q < q_s \), then there are a small \( r_0 > 0 \) and \( \theta(r) \in C^2(0, r_0) \) such that \( u^*(r) := F^{-1}[k^{-1}r^2(1 + \theta(r))] \) satisfies (1.3) on \((0, r_0)\) and \( \lim_{r \to 0} \theta(r) = 0 \). As a consequence, \( \lim_{r \to 0} u^*(r) = \infty \), and hence \( u^*(r) \) is a singular solution of (1.3) near \( r = 0 \).
Proof. We find a singular solution of the form (1.3). Let \( T < 0 \) be negatively large, and let \( x(t) \in C^2(-\infty, T+1) \). We assume that \( \|x(t)\|_{L^\infty(-\infty, T+1)} \) is small. We set \( x(t) := \theta(r) \) and \( t := \log r \). Then

\[
(3.1) \quad u^*(r) = F^{-1}[k^{-1}e^{2t}(1 + x(t))].
\]

Substituting \( u^*(r) \) into (1.3), we have

\[
x'' + (N + 2)x' + 2Nx + 4q - q\frac{(x' + 2x + 2)^2}{x + 1} - (F(u^*)f'(u^*) - q)\frac{(x' + 2x + 2)^2}{x + 1} = 0.
\]

The equation is equivalent to

\[
x'' + (N + 2 - 4q)x' + (2N - 4q)x - q\frac{x'^2}{x + 1} - (F(u^*)f'(u^*) - q)\frac{x' + 2x + 2}{x + 1} = 0.
\]

We construct a solution such that \( x(t) \to 0 \) as \( t \to -\infty \). Let \( y(t) := x'(t) \). Then

\[
\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = -A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f_0(x, y) + f_1(x, y, t) \end{pmatrix},
\]

where

\[
A := \begin{pmatrix} 0 & -1 \\ 2N - 4q & N + 2 - 4q \end{pmatrix},
\]

\[
f_0(x, y) := \frac{qy^2}{x + 1},
\]

\[
f_1(x, y, t) := (F(u^*)f'(u^*) - q)\frac{(y + 2x + 2)}{x + 1}.
\]

We show that the Lipschitz constants of \( f_0 \) and \( f_1 \) are small. Let \( X := C([-\infty, T], \mathbb{R}^2) \), and let \( \varepsilon > 0 \) be small. Here, \( T < 0 \) and \( \varepsilon > 0 \) are chosen later. We define \( B_\varepsilon := \{(x, y) \in X; \|x\|_{L^\infty(-\infty, T)} + \|y\|_{L^\infty(-\infty, T)} < \varepsilon\} \). Let \( (x_1, y_1), (x_2, y_2) \in B_\varepsilon \). We have

\[
|f_0(x_2, y_2) - f_0(x_1, y_1)| \leq q\left|\frac{y_1 + y_2}{x_2 + 1}\right| |y_2 - y_1| + \frac{q|y_1|^2}{|(x_1 + 1)(x_2 + 1)|} |x_2 - x_1|
\]

\[
\leq C\varepsilon(|x_2 - x_1| + |y_2 - y_1|).
\]

By \( u_i(r), i = 1, 2 \), we define \( u_i(r) := F^{-1}[k^{-1}e^{2t}(1 + x_i(t))] \). We have

\[
|f_1(x_2, y_2, t) - f_1(x_1, y_1, t)| \leq |F(u_1)f'(u_1) - q| \left|\frac{(2x_2 + y_2 + 2)^2}{x_2 + 1} - \frac{(2x_1 + y_1 + 2)^2}{x_1 + 1}\right|
\]

\[
+ |F(u_2)f'(u_2) - F(u_1)f'(u_1)| \left|\frac{(2x_2 + y_2 + 2)^2}{x_2 + 1}\right|.
\]

Since

\[
\left|\frac{(2x_2 + y_2 + 2)^2}{x_2 + 1} - \frac{(2x_1 + y_1 + 2)^2}{x_1 + 1}\right| \leq \frac{|2x_2 + y_2 + 2|^2}{|x_1 + 1|(x_2 + 1)} |x_2 - x_1|
\]

\[
+ \frac{|2x_1 + 2x_2 + y_1 + y_2 + 4|}{|x_1 + 1|} |2(x_2 - x_1) + (y_2 - y_1)|
\]

\[
\leq C(|x_2 - x_1| + |y_2 - y_1|) \quad \text{and}
\]

\[
|F(u_1)f'(u_1)| \leq \frac{1}{2} \left(1 + \frac{1}{2}\right) (2x_1 + y_1 + 2)^2.
\]
By (3.4), and (3.8) we have
\[ |F(u_1)f'(u_1) - q| < \varepsilon \] for \( t < T \), we have
\[ (3.4) \quad |F(u_1)f'(u_1) - q|\left| \frac{(2x_2 + y_2 + 2)^2}{x_2 + 1} - \frac{(2x_1 + y_1 + 2)^2}{x_1 + 1} \right| \leq C\varepsilon (|x_2 - x_1| + |y_2 - y_1|) \]
for \( t < T \). Let \( w_i(t) := F(u_i(r)) \), \( i = 1, 2 \). Since
\[
d\frac{d}{dw}(w f'(F^{-1}(w))) = \left( 1 - w f'(F^{-1}(w)) \frac{f'(F^{-1}(w)) f''(F^{-1}(w))}{f'(F^{-1}(w))^2} \right) f'(F^{-1}(w)),
\]
it follows from the mean value theorem that there is \( \bar{w} \) such that
\[
w_2f'(F^{-1}(w_2)) - w_1f'(F^{-1}(w_1)) = \left( 1 - \bar{w} f'(F^{-1}(\bar{w})) \frac{f'(F^{-1}(\bar{w})) f''(F^{-1}(\bar{w}))}{f'(F^{-1}(\bar{w})^2} \right) f'(F^{-1}(\bar{w}))(w_2 - w_1),
\]
and \( \min\{w_1(t), w_2(t)\} \leq \bar{w}(t) \leq \max\{w_1(t), w_2(t)\} \). Let \( \bar{u} := F^{-1}(\bar{w}) \). Then
\[
(3.5) \quad F(u_2)f'(u_2) - F(u_1)f'(u_1) = \left( 1 - F(\bar{u})f'(\bar{u}) \frac{f'(\bar{u}) f''(\bar{u})}{f'(\bar{u})^2} \right) f'(\bar{u})(F(u_2) - F(u_1)),
\]
where \( \bar{u}(r) \) satisfies \( \min\{u_1(r), u_2(r)\} \leq \bar{u}(r) \leq \max\{u_1(r), u_2(r)\} \). Since \( \lim_{r \to 0} \min\{u_1(r), u_2(r)\} = \infty \), \( \lim_{r \to 0} \bar{u}(r) = \infty \). By L'Hospital's rule we have
\[
\lim_{u \to \infty} F(u)f'(u) = \lim_{u \to \infty} \left| \int_u^\infty ds/f(s) \right| = \lim_{u \to \infty} \frac{f'(u)^2}{f(u)f''(u)} = q.
\]
Thus,
\[
\lim_{u \to \infty} F(u)f'(u) \frac{f(u)f''(u)}{f'(u)^2} = 1.
\]
This means that
\[
\left| 1 - F(\bar{u})f'(\bar{u}) \frac{f'(\bar{u}) f''(\bar{u})}{f'(\bar{u})^2} \right| < \varepsilon \text{ for small } r > 0.
\]
Since \( \min\{u_1(r), u_2(r)\} \leq \bar{u}(r) \leq \max\{u_1(r), u_2(r)\} \), there is \( \bar{x}(t) \) such that \( \min\{x_1(t), x_2(t)\} \leq \bar{x}(t) \leq \max\{x_1(t), x_2(t)\} \) and \( \bar{u}(r) = F^{-1}(k^{-1}e^{2t}(1 + \bar{x}(t))) \). Then \( ke^{2t}(1 + \bar{x}) = F(\bar{u}) \). We have
\[
|f'(\bar{u})(F(u_2) - F(u_1))| \leq |f'(\bar{u})ke^{2t}(x_2 - x_1)|
\leq \left| f'(\bar{u})F(\bar{u}) \frac{x_2 - x_1}{\bar{x} + 1} \right|
\leq C|F(\bar{u})f'(\bar{u})||x_2 - x_1|.
\]
By (3.5) and (3.7) we see that
\[
(3.8) \quad |F(u_2)f'(u_2) - F(u_1)f'(u_1)| \leq C\varepsilon |x_2 - x_1| \text{ for } t < T.
\]
By (3.3), (3.4), and (3.8) we have
\[
(3.9) \quad |f_1(x_2, y_2, t) - f_1(x_1, y_1, t)| \leq C\varepsilon (|x_2 - x_1| + |y_2 - y_1|) \text{ for } t < T.
\]
By $\xi(t), G(\xi(t), t), \mathcal{F}(\xi(t))$ we define

$$
\begin{align*}
\xi(t) &:= \left(\begin{array}{c} x(t) \\ y(t) \end{array}\right), \\
G(\xi(t), t) &:= \left(\begin{array}{c} f_0(x, y) \\ 0 \end{array}\right), \\
\mathcal{F}(\xi(t)) &:= \int_{-\infty}^{t} e^{-(t-\tau)A}G(\xi(\tau), \tau)d\tau,
\end{align*}
$$

respectively. We find a solution of the equation $\xi(t) = \mathcal{F}(\xi(t))$ in $B_\varepsilon$ if $\varepsilon > 0$ is small and $T < 0$ is negatively large. Then the solution $\xi(t)$ is corresponding to a solution of $[1.3]$ near $r = 0$. In fact, $u^*(r) = F^{-1}[k^{-1}r^2(1 + x(\log r))]$ becomes a singular solution near $r = 0$. The eigenvalues of $A$

$$
\lambda_\pm := \frac{(N + 2 - 4q) \pm \sqrt{D}}{2}, \text{ where } D := (N + 2 - 4q)^2 - 4(2N - 4q).
$$

Therefore, $\|e^{-tA}\|_{\mathbb{R}^2} \leq C e^{-\mu t}$, where

$$
\mu := \begin{cases} 
\frac{N + 2 - 4q - \sqrt{D}}{2} > 0 & \text{if } 1 < q < q_{JL}, \\
\frac{N + 2 - 4q}{2} - \delta > 0 & \text{if } q = q_{JL}, \\
\frac{N + 2 - 4q}{2} > 0 & \text{if } q_{JL} < q < q_S,
\end{cases}
$$

and $\delta > 0$ is small. If $t < T$, then by (3.2) and (3.9) we have

$$
\begin{align*}
\|\mathcal{F}(\xi_2(t)) - \mathcal{F}(\xi_1(t))\|_{\mathbb{R}^2} &\leq \int_{-\infty}^{t} \|e^{-(t-\tau)A}(G(\xi_2(\tau), \tau) - G(\xi_1(\tau), \tau))\|_{\mathbb{R}^2} d\tau \\
&\leq C \int_{-\infty}^{t} e^{-\mu(t-\tau)}d\tau C\varepsilon \|\xi_2(t) - \xi_1(t)\|_X \\
&= \frac{C\varepsilon}{\mu} \|\xi_2(t) - \xi_1(t)\|_X.
\end{align*}
$$

If $T < 0$ is negatively large, then $\varepsilon > 0$ can be chosen arbitrarily small. Hence, there is $\kappa \in (0, 1)$ such that $\|\mathcal{F}(\xi_2) - \mathcal{F}(\xi_1)\|_X \leq \kappa \|\xi_2 - \xi_1\|_X$ for $\xi_1, \xi_2 \in B_\varepsilon$. Since $f_0(0, 0) = 0$ and $|f_1(0, 0, t)| = o(1)$ as $t \to -\infty$, we see that $\|\mathcal{F}(0)\|_X = o(1)$. Then,

$$
(3.10) \quad \|\mathcal{F}(\xi)\|_X \leq \|\mathcal{F}(\xi) - \mathcal{F}(0)\|_X + \|\mathcal{F}(0)\|_X \leq \kappa \varepsilon + o(1) < \varepsilon
$$

provided that $T < 0$ is negatively large. Hence, $\mathcal{F}$ is a contraction mapping on $B_\varepsilon$. It follows from the contraction mapping theorem that $\mathcal{F}$ has a unique fixed point in $B_\varepsilon$ which is a solution of $\xi = \mathcal{F}(\xi)$. When $T$ is negatively large, $\varepsilon > 0$ can be taken small. By (3.10) and the uniqueness of the fixed point in $B_\varepsilon$ we see that $\|\xi(t)\|_{\mathbb{R}^2} \to 0$ as $t \to -\infty$. Thus, $x(t) \to 0$ as $t \to -\infty$. Let $\theta(r) := x(\log r)$. Then the conclusion holds. \qed
4. Convergence to a solution of the limit problem

Let \( u(r, \rho) \) be the solution of (4.2). By \( \tilde{u}(s) \) we define

\[
\tilde{u}(s) := F^{-1}[\lambda^{-2}F[u(r, \rho)]], \quad s := \frac{r}{\lambda}, \quad \text{and} \quad \lambda := \sqrt{\frac{F(\rho)}{F(1)}}.
\]

Then \( \tilde{u}(s) \) satisfies

\[
\begin{align*}
\tilde{u}''(s) + \frac{N-1}{s} \tilde{u}'(s) + f(\tilde{u}(s)) + \frac{F(u(\lambda s, \rho)) f'(u(\lambda s, \rho)) - F(\tilde{u}(s)) f'(\tilde{u}(s))}{F(\tilde{u}(s)) f'(\tilde{u}(s))} \tilde{u}'(s)^2 &= 0, \quad s > 0, \\
\tilde{u}(0) &= 1, \\
\tilde{u}'(0) &= 0.
\end{align*}
\]

First, we study the limit of \( \tilde{u}(s) \) as \( \rho \to \infty \). Note that \( \lambda \downarrow 0 \) as \( \rho \to \infty \). We need the following proposition:

**Proposition 4.1** ([22, Theorem 2.4]). Let \( h \in C^1(0, \infty) \cap C[0, \infty) \). Assume that \( N \geq 3 \) and there exists \( p > (N+2)/(N-2) \) such that

\[
u h(u) \geq (1 + p) H(u) \quad \text{for large } u > 0,
\]

where \( H(u) := \int_0^u h(t) dt \). Let \( u(r, \rho) \) be the solution of the problem

\[
\begin{align*}
u'' + \frac{N-1}{r} \nu' + h(u) &= 0, \quad r > 0, \\
u(0) &= \rho, \\
u'(0) &= 0,
\end{align*}
\]

and let \( R(\rho) \) be the first positive zero of \( u(\cdot, \rho) \) if it exists. Then there are \( \rho_0 \in \mathbb{R} \) and \( R_0 > 0 \) such that \( R(\rho) > R_0 \) for \( \rho > \rho_0 \).

**Lemma 4.2.** Suppose that \( N \geq 3 \) and (17) and (12) with \( q < q_S \) holds. Let \( v(s, 1) \) be the solution of (1.9) with \( \sigma = 1 \). Then

\[
\tilde{u}(s) \to v(s, 1) \text{ in } C_{\text{loc}}[0, \infty) \quad \text{as } \rho \to \infty.
\]

**Proof.** Let \( s_0 > 0 \) be fixed. We show that

\[
u(\lambda s, \rho) \to \infty \quad \text{uniformly in } s \in [0, s_0] \quad \text{as } \rho \to \infty.
\]

Let \( M > 0 \) be large. Let \( u_M(r) := u(r, \rho) - M \). Then \( u_M \) satisfies

\[
\begin{align*}
u'' + \frac{N-1}{r} \nu' + f_M(u_M) &= 0, \quad r > 0, \\
u_M(0) &= \rho - M, \\
u_M'(0) &= 0,
\end{align*}
\]

where \( f_M(u) := f(u + M) \). Because of (11) and (12), we easily see that \( f(u) \to \infty \) as \( u \to \infty \). There is \( c > 0 \) such that \( f_M(u) > c \) for \( u \geq 0 \). Hence, \( u_M \) has the first positive zero when \( \rho > M \). Let \( F_0(u) := \int_u^0 f(t) dt \). We consider the case \( 1 < q < q_S \). Using L'Hospital rule twice, we have

\[
\lim_{u \to \infty} \frac{uf(u)}{F_0(u)} = \lim_{u \to \infty} \left( 1 + \frac{u}{f(u)} \right) = \lim_{u \to \infty} \left( 1 + \frac{1}{1 - \frac{f(u)}{F(u)}} \right) = 1 + \frac{q}{q - 1}.
\]
Since $1 < q < q_S$, we see that $1 + q/(q - 1) > 1 + (N + 2)/(N - 2)$. Therefore, $f_M$ satisfies (4.3) in Proposition 4.4 if $M > 0$ is large. We consider the case $q = 1$. Then
\[ \lim_{u \to \infty} u f(u)/F_0(u) = \infty, \]
and hence $f_M$ satisfies (4.3). In both cases we see by Proposition 4.4 that there are $\rho_M > 0$ and $r_M > 0$ such that if $\rho > \rho_M$, then $u(r, \rho) \geq M$ for $0 \leq r \leq r_M$. If $\rho > 0$ is sufficiently large, then $u(\lambda s, \rho) \geq M$ for $0 \leq s \leq s_0$, because $\lim_{\rho \to \infty} \lambda = 0$ and $s_0 < r_M/\lambda$. Since $M > 0$ can be chosen arbitrarily large, (4.4) is proved, and hence,
\[ F(u(\lambda s, \rho)) f'(u(\lambda s, \rho)) \to q \quad \text{uniformly in } s \in [0, s_0] \text{ as } \rho \to \infty. \]
Because of the continuity of $\tilde{u}(s) \in C([0, s_0])$ with respect to the nonlinearity in (1.2), we see that $\tilde{u}(s) \to v(s, 1)$ in $C([0, s_0])$ as $\rho \to \infty$. Since $s_0 > 0$ can be chosen arbitrarily, we obtain the conclusion.

Because of Lemma 4.2, the limit equation of (1.3) as $\rho \to \infty$ becomes (1.7) when $u$ and $v$ are related by (1.6) with $\lambda = \sqrt{F(\rho)/F(1)}$.

Second, we study the limit of the rescaled singular solution as $\lambda \downarrow 0$.

**Lemma 4.3.** Suppose that $N \geq 3$ and (f1) and (f2) with $q < q_S$ hold. Let $u^*(r)$ be the singular solution given by Lemma 3.7 and let $s$ and $\lambda$ be defined by (4.1). Let $\tilde{u}^*(s) := F^{-1}[\lambda^{-2}F(u^*(r))]$. Then
\[ \tilde{u}^*(s) \to v^*(s) \quad \text{in } C_{\text{loc}}(0, \infty) \quad \text{as } \rho \to \infty, \]
where $v^*(s) = F^{-1}[k^{-1}s^2]$ which is defined by (1.8).

**Proof.** Since $u^*(r) = F^{-1}[k^{-1}r^2(1 + \theta(r))]$, we have
\[ \tilde{u}^*(s) = F^{-1}[\lambda^{-2}F(F^{-1}[k^{-1}(\lambda s)^2(1 + \theta(\lambda s))])]. \]
Let $0 < s_0 < s_1$ be fixed. Since $\theta(r) \to 0$ as $r \downarrow 0$, we see that $\theta(\lambda s) \to 0$ uniformly in $s \in [s_0, s_1]$ as $\rho \to \infty$. Thus, $\tilde{u}^*(s) \to F^{-1}[k^{-1}s^2]$ uniformly in $s \in [s_0, s_1]$ as $\rho \to \infty$. Since $s_0$ and $s_1$ can be chosen arbitrarily under the condition $0 < s_0 < s_1$, we obtain the conclusion.

Because of Lemmas 4.2 and 4.3, the limit functions of $\tilde{u}(s)$ and $\tilde{u}^*(s)$ satisfy (1.7).

### 5. Proofs of Theorem B

Let $v(s, \sigma)$ be the solution of (1.9), and let $w(s, \tau)$ and $\tau$ be defined by (1.12). Note that $\tau = F^{-1}_q[F[\sigma]]$ and $\tau$ is strictly increasing in $\sigma$.

**Lemma 5.1.** Suppose that $N \geq 3$. The following two statements hold:
(i) If $q > 1$, then $w$ satisfies
\[ \begin{cases} \\
 w'' + \frac{N-1}{s} w' + w^p = 0, \quad s > 0, \\
 w(0) = \tau, \\
 w'(0) = 0 \\
\end{cases} \]
and
\[ w^*(s) := F^{-1}_q[F[v^*(s)]] = \left\{ \frac{2}{p-1} \left( N-2 - \frac{2}{p-1} \right) \right\}^{1/(p-1)} s^{-\frac{2}{p-1}} \]
is a singular solution of the equation in (5.3). Here, \( p := q/(q - 1) \).

(ii) If \( q = 1 \), then \( w \) satisfies

\[
\begin{align*}
w'' + \frac{N-1}{s} w' + e^w &= 0, \quad s > 0, \\
w(0) &= \tau, \\
w'(0) &= 0
\end{align*}
\]

and

\[
w^*(s) := F_1^{-1}[F[u^*(s)]] = -2 \log s + \log 2(N - 2)
\]

is a singular solution of the equation in (5.3).

Proof. By direct calculation we can obtain the conclusions. We omit the proof. \( \Box \)

The following two propositions are well known:

**Proposition 5.2.** Suppose that \( N \geq 3 \). Let \( 0 < \tau_0 < \tau_1 \). Let \( w(s, \tau_i), i = 0, 1, \) be solutions of (5.1) with \( \tau = \tau_i \), and let \( w^*(s) \) be the singular solution given by (5.2). Then the following holds:

(i) If \( p = p_S \), then \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w(\cdot, \tau_1)] = 1 \) and \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w^*(\cdot)] = 2 \).

(ii) If \( p_S < p < p_{JL} \), then \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w(\cdot, \tau_1)] = \infty \) and \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w^*(\cdot)] = \infty \).

(iii) If \( p \geq p_{JL} \), then \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w(\cdot, \tau_1)] = 0 \) and \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w^*(\cdot)] = 0 \).

See [18, 27, 28] for example.

**Proposition 5.3.** Suppose that \( N \geq 3 \). Let \( \tau_0 < \tau_1 \). Let \( w(s, \tau_i), i = 0, 1, \) be solutions of (5.3) with \( \tau = \tau_i \), and let \( w^*(s) \) be the singular solution given by (5.4). Then the following holds:

(i) If \( 3 \leq N \leq 9 \), then \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w(\cdot, \tau_1)] = \infty \) and \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w^*(\cdot)] = \infty \).

(ii) If \( N \geq 10 \), then \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w(\cdot, \tau_1)] = 0 \) and \( Z_{(0,\infty)}[w(\cdot, \tau_0) - w^*(\cdot)] = 0 \).

See [17, 18] for example.

**Proof of Theorem A** We consider the case \( q > 1 \). The cases \( q = q_S, q_{JL} < q < q_S \), and \( q \leq q_{JL} \) correspond to \( p = p_S, p_S < p < p_{JL} \), and \( p \geq p_{JL} \), respectively. The conclusion of Theorem A follows from Proposition 5.2. We consider the case \( q = 1 \). The cases \( q_{JL} < q < q_S \) and \( q \leq q_{JL} \) correspond to \( 3 \leq N \leq 9 \) and \( N \geq 10 \), respectively. The conclusion of Theorem A follows from Proposition 5.3. Note that the case \( q = q_S \) corresponds to \( N = 2 \). The proof is complete. \( \Box \)

**6. Proof of Theorem A**

**Proof of Theorem A**

(i) follows from Lemma 3.1.

(ii) Let \( \tilde{u}(s) := F^{-1}[\lambda^{-2}F[u(r, \rho)]] \), \( \tilde{u}^*(s) := F^{-1}[\lambda^{-2}F[u^*(r)]] \), \( s := r/\lambda \), and \( \lambda := \sqrt{F(\rho)/F(1)} \). By Lemmas 1.2 and 1.3 we see that as \( \rho \to \infty \),

\[
\tilde{u}(s) \to v(s, 1) \quad \text{in} \quad C_{loc}[0, \infty),
\]

\[
\tilde{u}^*(s) \to v^*(s) \quad \text{in} \quad C_{loc}(0, \infty).
\]
Since \( q_{1L} < q < q_S \), Theorem 13(ii) says that
\[
(6.3) \quad \mathcal{Z}_{(0,\infty)}[v(\cdot, 1) - v^*(\cdot)] = \infty.
\]
Let \( r_1 > 0 \) be given by the assumption in Theorem A. Since the same transformation is applied to \( \tilde{u}(s) \) and \( \tilde{u}^*(s) \), we see that \( \mathcal{Z}_{(0,r_1)}[u(\cdot, \rho) - u^*(\cdot)] = \mathcal{Z}_{(r_1/r, \infty)}[\tilde{u}(\cdot) - \tilde{u}^*(\cdot)] \). For each \( M > 0 \), there is \( s_M > 0 \) and \( \rho_M > 0 \) such that \( \mathcal{Z}_{(0,s_M)}[\tilde{u}(\cdot) - \tilde{u}^*(\cdot)] \geq M \) for \( \rho > \rho_M \), because of (6.1), (6.2) and (6.3). If \( \rho > 0 \) is large, then \((0,s_M) \subset (0, r_1/\lambda)\), and hence
\[
\mathcal{Z}_{(0,r_1)}[u(\cdot, \rho) - u^*(\cdot)] = \mathcal{Z}_{(0,r_1/\lambda)}[\tilde{u}(\cdot) - \tilde{u}^*(\cdot)] \geq M.
\]
Since \( M \) can be chosen arbitrarily large, \( \mathcal{Z}_{(0,r_1)}[u(\cdot, \rho) - u^*(\cdot)] \to \infty \) as \( \rho \to \infty \). \( \square 

7. Morse Index of a Singular Solution

Proof of Corollary 14 (i) Because of (12), \( f''(u) > 0 \) for large \( u > 0 \), and hence \( f(u) \) is convex for large \( u \). There is \( r_2 \in (0, r_0) \) such that \( f(u) \) is convex for \( u \in \{ u^*(r) \in \mathbb{R} : 0 < r < r_2 \} \). Here, \( r_0 \) is given in Theorem A (i). It follows from Theorem A (ii) that, for each \( n \geq 1 \), there is a large \( \rho > 0 \) such that \( u^*(\cdot) - u(\cdot, \rho) \) has at least \( 2n + 1 \) zeros in \((0, r_2)\). Then \( \{ z_i \}_{i=1}^{2n+1}, z_1 < z_2 < \cdots < z_{2n+1} \), denotes the first \( 2n + 1 \) zeros. By \( \phi_i(r), i = 1, 2, \ldots, n \), we define
\[
\phi_i(r) := \begin{cases} u^*(r) - u(r, \rho) & \text{if } r \in (z_{2i}, z_{2i+1}), \\ 0 & \text{if } r \notin (z_{2i}, z_{2i+1}). \end{cases}
\]
Since \( f(u) \) is convex for large \( u \) and \( u(r, \rho) < u^*(r) \) for \( r \in (z_{2i}, z_{2i+1}) \), we have that
\[
\frac{f(u^*(r)) - f(u(r, \rho))}{u^*(r) - u(r, \rho)} < f'(u^*(r)) \quad \text{for } r \in (z_{2i}, z_{2i+1}).
\]
Let
\[
V(r) := \begin{cases} \frac{f(u^*(r)) - f(u(r, \rho))}{u^*(r) - u(r, \rho)} & \text{if } u^*(r) \neq u(r, \rho), \\ f'(u^*(r)) & \text{if } u^*(r) = u(r, \rho). \end{cases}
\]
Since \( \phi_i(z_{2i}) = \phi_i(z_{2i+1}) = 0 \), we have that
\[
\int_{B(r_0^i)} (|\nabla \phi_i|^2 - f'(u^*) \phi_i^2) \, dx = \int_{\{z_2 < r < z_{2i+1}\}} (|\nabla \phi_i|^2 - f'(u^*) \phi_i^2) \, dx \
< \int_{\{z_2 < r < z_{2i+1}\}} (|\nabla \phi_i|^2 - V \phi_i^2) \, dx \
= - \int_{\{z_2 < r < z_{2i+1}\}} \phi_i (\Delta \phi_i + V \phi_i) \, dx \
+ \int_{\{r = z_{2i+1}\}} \phi_i \frac{\partial}{\partial n} \phi_i \, d\sigma + \int_{\{r = z_{2i}\}} \phi_i (-\frac{\partial}{\partial n} \phi_i) \, d\sigma \
= 0,
\]
where \( \partial/\partial n \) denotes the outer normal derivative, and we use \( \Delta \phi_i + V \phi_i = 0 \) in \( \{z_2 < r < z_{2i+1}\} \). Since the supports of \( \phi_i \) and \( \phi_j, j \neq i \), are disjoint and \( n \) can be chosen arbitrarily large, (7.1) indicates that \( m(u^*) = \infty \).
(ii) Let $\varepsilon > 0$ be fixed. We determine $\varepsilon$ later. Since $\lim_{u \to \infty} F(u) f'(u) = q$ (See (3.6)), we see that $F(u^*(r)) f'(u^*(r)) < q + \varepsilon$ for small $r > 0$. By Theorem A (i) we have $F(u^*(r)) = k^{-1} r^2 (1 + \theta(r))$. Therefore, $f'(u^*(r)) < 2(N - 2q)(q + \varepsilon)/(r^2(1 + \theta(r)))$ for small $r > 0$. Since $(q + \varepsilon)/(1 + o(1)) < q + 2\varepsilon$ for small $\varepsilon > 0$, there is $r_3 > 0$ such that

$$f'(u^*(r)) < \frac{2(N - 2q)(q + 2\varepsilon)}{r^2} \quad \text{for} \quad 0 < r < r_3. \tag{7.2}$$

Since $q < q_{\text{IL}}$, we see that $2(N - 2q)q < (N - 2)^2/4$. We can choose $\varepsilon > 0$ small enough so that $2(N - 2q)(q + 2\varepsilon) < (N - 2)^2/4$. Then there is $\delta \in (0, 1)$ such that

$$2(N - 2q)(q + 2\varepsilon) = (1 - \delta) \frac{(N - 2)^2}{4}. \tag{7.3}$$

We define

$$\chi_0(r) := \begin{cases} 1 & \text{if } 0 \leq r < r_3/2, \\ 0 & \text{if } r_3 < r, \end{cases}$$

where $0 \leq \chi_0(r) \leq 1$ and $\chi_0(r) \in C^1$. Let $\chi_1(r) := 1 - \chi_0(r)$. By (7.2) and (7.3) we see that for $\phi \in H^1_{0, \text{rad}}(B(r_0^*))$,

$$\int_{B(r_0^*)} (|\nabla \phi|^2 - f'(u)\phi^2) \, dx = \int_{B(r_0^*)} \left\{ (1 - \delta) |\nabla \phi|^2 - \chi_0 f'(u^*)\phi^2 \right\} \, dx$$

$$+ \int_{B(r_0^*)} \left\{ \delta |\nabla \phi|^2 - \chi_1 f'(u^*)\phi^2 \right\} \, dx$$

$$\geq (1 - \delta) \int_{B(r_0^*)} \left( |\nabla \phi|^2 - \frac{(N - 2)^2}{4} \phi^2 \right) \, dx$$

$$+ \delta \int_{B(r_0^*)} \left( |\nabla \phi|^2 - \frac{\chi_1}{\delta} f'(u^*)\phi^2 \right) \, dx$$

$$\geq \delta \int_{B(r_0^*)} \left( |\nabla \phi|^2 - \frac{\chi_1}{\delta} f'(u^*)\phi^2 \right) \, dx,$$ \tag{7.4}$$

where we used Hardy’s inequality. Since $|\chi_1 f'(u^*)|/\delta$ is bounded on $B(r_0^*)$, the operator $-\Delta - \chi_1 f'(u^*)/\delta$ with the Dirichlet boundary condition has at most finitely many negative eigenvalues, i.e., dim $X_1 < \infty$. Here,

$$X_1 := \left\{ \phi \in H^1_{0, \text{rad}}(B(r_0^*)); \int_{B(r_0^*)} \left( |\nabla \phi|^2 - \frac{\chi_1}{\delta} f'(u^*)\phi^2 \right) \, dx < 0 \right\} \cup \{0\}.$$ \hspace{1cm}$$

We prove $m(u^*) < \infty$ by contradiction. Suppose that $m(u^*) = \infty$. Then

$$\dim \left( \left\{ \phi \in H^1_{0, \text{rad}}(B(r_0^*)); \int_{B(r_0^*)} \left( |\nabla \phi|^2 - f'(u^*)\phi^2 \right) \, dx < 0 \right\} \cup \{0\} \right) = \infty.$$ \hspace{1cm}$$

Because of (7.4), we see that dim $X_1 = \infty$. We obtain a contradiction. Thus, $m(u^*) < \infty$. \hspace{1cm} \square
8. Bifurcation diagram

Suppose that \( N \geq 3 \) and (1.1) and (1.2) with \( q < q_5 \) hold. The problem (1.1) is equivalent to the following problem

\[
\begin{cases}
U'' + \frac{N-1}{R} U' + \mu f(U) = 0 & 0 < R < 1, \\
U(1) = 0, \\
U(R) > 0, & 0 < R < 1.
\end{cases}
\]

Let \( u(r) := U(R) \) and \( r := \sqrt{\mu} R. \) Then \( u \) satisfies

\[
\begin{cases}
\mu u'' + \frac{N-1}{r} u' + f(u) = 0, & 0 < r < \sqrt{\mu}, \\
u(\sqrt{\mu}) = 0, \\
u(r) > 0, & 0 < r < \sqrt{\mu}.
\end{cases}
\]

We consider (1.2). Let \( u(r, \rho) \) be the solution of (1.2). Because of (1.1), there is \( \delta > 0 \) such that

\[(8.1) \quad f(u) > \delta \quad \text{for} \quad u \geq 0.\]

It is well known that \( u(\cdot, \rho) \) has the first positive zero \( r_0(\rho) \). Let \( \mu(\rho) := r_0(\rho)^2 \). Because of Theorem A (i), (1.3) has a singular solution \( u^*(r) \) near \( r = 0 \). We extend the domain of \( u^* \). By (8.1) we see that \( u^*(r) \) also has the first positive zero \( r_0^* \). Then \( (\mu^*, U^*(R)) := ((r_0^*)^2, u^*(r_0^* R)) \) is a singular solution of (1.1).

We extend the domain of \( f(u) \). We can assume that \( f \in C^1(\mathbb{R}), f(u) > 0 \) for \( u \in \mathbb{R}, \) and \( f(u) = \delta/2 \) for \( u < -1 \). Then (1.1-2) holds, and hence Theorem A is applicable.

**Lemma 8.1.** Suppose that \( N \geq 3 \) and (1.1) and (1.2) hold. Let \( u(r, \rho) \) be the solution of (1.2) and \( \mu^* := (r_0^*)^2 \). Let \( \mu(\rho) \) be as above. If \( q_{1L} < q < q_5 \), then \( \mu(\rho) \) oscillates infinitely many times around \( \mu^* \) as \( \rho \to \infty \).

**Proof.** Let \( z(\rho) := Z_{(0, \min\{r_0(\rho), r_0^*\})}[u(\cdot, \rho) - u^*(\cdot)] \) and let \( I := (0, \min\{r_0(\rho), r_0^*\}) \). It is clear that \( \{u(r, \rho) - u^*(\cdot) = 0\} \) does not have an accumulation point. Hence, \( z(\rho) < \infty \). We see that each zero of \( u(r, \rho) - u^*(r) \) is simple and \( u(r, \rho) - u^*(r) \) is a \( C^1 \)-function of \( (r, \rho) \). It follows from the implicit function theorem that each zero of \( u(r, \rho) - u^*(r) \) continuously depends on \( \rho \). Because \( z(\rho) \) does not change in a neighborhood of each fixed \( \rho \), \( z(\rho) \) does not change if another zero does not enter \( I \) from \( \partial I \) or a zero in \( I \) goes out of \( I \). We prove the conclusion of the lemma by contradiction. Suppose that there is \( \rho_0 > 0 \) such that \( \mu(\rho) < \mu^* \) for \( \rho > \rho_0 \). Let \( \tilde{\rho} := \min\{r_0(\rho_0), r_0^*\} \). We see that \( u(0, \rho) - u^*(0) = -\infty \) and \( u(\tilde{\rho}, \rho) - u^*(\tilde{\rho}) < 0 \). Thus, another zero cannot enter, and \( z(\rho) \) is bounded for \( \rho > 0 \). This contradicts Theorem A (ii). Similarly, we obtain the contradiction in the case where \( \mu(\rho) > \mu^* \) for \( \rho > \rho_0 \). As a consequence, \( \mu(\rho) \) has to oscillate infinitely many times around \( \mu^* \) as \( \rho \to \infty \).  

Corollary 1.3 (i) immediately follows from Lemma 8.1.

We prove Corollary 1.3 (ii). We apply a generalized Cole-Hopf transformation, which is mentioned in Section 1, to \( u(r, \rho) \). Let

\[(8.2) \quad \tilde{u}(r, \sigma) := F_{q}^{-1}[F[u(r, \rho)]] \quad \text{and} \quad \sigma := F_{q}^{-1}[F(\rho)].\]
Note that $\sigma$ is strictly increasing in $\rho$. The function $\tilde{u}$ satisfies
\[ \Delta \tilde{u} + f_q(\tilde{u}) + (F(u)f'(u) - q)|\nabla \tilde{u}|^2 = 0. \]

First we show the following:

**Lemma 8.2.** Suppose that $N \geq 3$ and with $q \leq q_{JL}$ hold, and $F(u)f'(u) \geq q$ for $u \geq 0$. Then,
\[ (8.3) \quad \tilde{u}(r, \sigma) \leq F_q^{-1}[k^{-1}r^2] \text{ for } r \geq 0. \]

**Proof.** Let $\alpha_1 > \alpha_0(> \sigma)$ and let $\tilde{u}_i$, $i = 0, 1$, be the solution of the problem
\[
\begin{cases}
\tilde{u}''_i + \frac{N-1}{r} \tilde{u}'_i + f_q(\tilde{u}_i') = 0, & r > 0, \\
\tilde{u}_i(0) = \alpha_i, \\
\tilde{u}_i'(0) = 0.
\end{cases}
\]

Since $q \leq q_{JL}$, Propositions 5.2 and 5.3 say that $\tilde{u}_1(r) > \tilde{u}_0(r)$ for $r \geq 0$. Let $w_1(r) := \tilde{u}_1(r) - \tilde{u}_0(r)$. Then
\[
\begin{cases}
\Delta w_1 + V_1w_1 = 0 & \text{in } \mathbb{R}^N, \\
w_1 > 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]
where
\[ V_1 := \frac{f_q(\tilde{u}_1(r)) - f_q(\tilde{u}_0(r))}{\tilde{u}_1(r) - \tilde{u}_0(r)}. \]

We show by contradiction that
\[ (8.4) \quad \tilde{u}(r, \sigma) < \tilde{u}_0(r) \quad \text{for } r > 0. \]

Suppose the contrary, i.e., there is $r_0 > 0$ such that $\tilde{u}(r, \sigma) < \tilde{u}_0(r)$ for $0 < r < r_0$ and $\tilde{u}(r_0, \sigma) = \tilde{u}_0(r_0)$. Let $w_0(r) := \tilde{u}_0(r) - \tilde{u}(r, \sigma)$. Then
\[
\begin{cases}
\Delta w_0 + V_0w_0 = (F(u)f'(u) - q)|\nabla \tilde{u}|^2 \geq 0 & \text{in } B(r_0), \\
w_0 > 0 & \text{in } B(r_0),
\end{cases}
\]
where $B(r_0)$ is an open ball with radius $r_0$ and
\[ V_0 := \begin{cases}
\frac{f_q(\tilde{u}_0(r)) - f_q(\tilde{u}(r, \sigma))}{\tilde{u}_0(r) - \tilde{u}(r, \sigma)} & \text{if } \tilde{u}_0(r) \neq \tilde{u}(r, \sigma), \\
f_q'(\tilde{u}_0(r)) & \text{if } \tilde{u}_0(r) = \tilde{u}(r, \sigma).
\end{cases} \]

Since $f_q$ is strictly convex, we see that $V_1 > V_0$. Let $\omega_N$ denote the surface area of the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Since $w_0'(r_0) \leq 0$ and $w_0(r_0) = 0$,
\[
0 > - \int_{B(r_0)} (V_1 - V_0)w_0w_1 + w_1(F(u)f'(u) - q)|\nabla \tilde{u}|^2dx
\]
\[
= \int_{B(r_0)} (w_0\Delta w_1 - w_1\Delta w_0)dx
\]
\[
= \omega_Nr_0^{N-1}(w_0(r_0)w_1'(r_0) - w_1(r_0)w_0'(r_0)) \geq 0,
\]
which is a contradiction. Thus, (8.4) holds. Let $\tilde{u}^*(r) := F_q^{-1}[k^{-1}r^2]$. Then $\tilde{u}^*$ is a singular solution of $\Delta \tilde{u}^* + f_q(\tilde{u}^*) = 0$. Since $q \leq q_{JL}$, Propositions 5.2 and 5.3 say that $Z_{(0, \infty]}[\tilde{u}_0(\cdot) - \tilde{u}^*(\cdot)] = 0$, and hence $\tilde{u}_0(r) < \tilde{u}^*(r)$ for $r > 0$. Thus, (8.3) holds. □
Proof of Corollary 1.3. Since $f(0) > 0$, the bifurcation curve starts from $(0, 0)$ and consists of minimal solutions near $(0, 0)$. Since $f' > 0$ and $f'' > 0$, there is $\bar{\mu} > 0$ such that either the curve blows up at $\bar{\mu}$ or the curve has a turning point at $\bar{\mu}$. See [11, 5]. We show by contradiction that the curve blows up at $\bar{\mu}$. Suppose the contrary, i.e., $(\bar{\mu}, U(\bar{R}))$ is a turning point. Let $u(r) := U(\bar{R})$ and $r := \sqrt{\bar{\mu}} R$, and let $\tilde{u}$ be defined by (8.2). Because of Lemma 8.2, $\tilde{u}(r, \sigma) \leq F^{-1} \left\{ \frac{k-1}{4r^2} \right\}$. Hence, $k^{-1} r^2 \leq F^{-1} \left\{ \tilde{u} \right\} = F[u]$. We have that $u \leq F^{-1} \left\{ \frac{k-1}{4r^2} \right\}$. Using $f'' \geq 0$ and the assumption of Corollary 1.3, we have
\[
f'(u) \leq f'(F^{-1}[k^{-1}r^2]) \leq \frac{(N-2)^2}{8(N-2q)F[F^{-1}[k^{-1}r^2]]} = \frac{(N-2)^2}{4r^2}.
\]
By Hardy’s inequality we have
\[
\int_{B_0} (|\nabla \phi|^2 - f'(u)\phi^2) \, dx \geq \int_{B_0} \left( |\nabla \phi|^2 - \frac{(N-2)^2}{4r^2} \phi^2 \right) \, dx > 0
\]
for $\phi \in H^1_{0,rad}(B_0) \setminus \{0\}$. Here, $B_0$ is a ball with radius the first positive zero of $u(\cdot)$. This inequality indicates that the first eigenvalue of the problem
\[
\begin{aligned}
\Delta \phi + f'(u)\phi &= -\nu \phi, \\
\phi &\in H^1_{0,rad}(B_0)
\end{aligned}
\]
is strictly positive. The first eigenvalue of the eigenvalue problem associated to $(\bar{\mu}, U(\bar{R}))$ is also strictly positive. We have a contradiction, since $(\bar{\mu}, U(\bar{R}))$ is a turning point. The proof is complete. □

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