A Class of Delay Optimal Control Problems and Viscosity Solutions to Associated Hamilton-Jacobi-Bellman Equations

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Abstract

In this article, a class of optimal control problems of differential equations with delays are investigated for which the associated Hamilton-Jacobi-Bellman (HJB) equations are nonlinear partial differential equations with delays. This type of HJB equation has not been previously studied and is difficult to solve because the state equations do not possess smoothing properties. We introduce a slightly different notion of viscosity solutions and identify the value function of the optimal control problems as a unique viscosity solution to the associated HJB equations.

Key Words: Hamilton-Jacobi-Bellman equations; Viscosity solutions; Optimal control; Differential equations with delays; Existence and uniqueness

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1 Introduction

In this paper, we consider the following controlled differential equations with delays:

\[ \begin{cases} 
    dX^u(s) = F(s, X^u(s), (a, X_s^u)H, u(s))ds + b(s)X^u(s - \tau)ds, s \in [t, T], \\
    X_t^u = x \in \mathcal{D}, 
\end{cases} \tag{1.1} \]

where

\[ X_s^u(\theta) = X^u(s + \theta), \quad \theta \in [-\tau, 0], \quad (a, X_s^u)H = \int_{-\tau}^{0} (a(\theta), X_s^u(\theta))_{R^d}d\theta. \]

In the equations above, \( \mathcal{D} \) denotes the space of bounded, right continuous, \( R^d \)-valued functions on \( [-\tau, 0] \), and \( F : [0, T] \times R^d \times R \times U \to R^d \) is a given map, where \( U \) is a metric space in which the control \( u(\cdot) \) takes values. For any initial state \( x \in \mathcal{D} \) and control \( u(\cdot) \in U[t, T] := \{ u : [t, T] \to U | u(\cdot) \text{ is measurable} \} \), the corresponding trajectory \( X(\cdot) \) is a solution to (1.1). \( a \) and \( b \) are two given functions that satisfy suitable smoothness properties, and the coefficient \( F \) is assumed to satisfy a Lipschitz condition with respect to the appropriate norm. Thus, the solution to (1.1) is uniquely determined by the initial state and the control.
The control problem consists of minimizing a cost functional of the following form:

\[ J(t, x, u) = \int_t^T q(\sigma, X^u(\sigma), u(\sigma)) d\sigma + \phi(X^u(T)), \]

over all of the controls \( u(\cdot) \in U[t, T] \). Here, \( q \) and \( \phi \) are functions on \([0, T] \times \mathbb{R}^d \times U \) and \( \mathbb{R}^d \), respectively. We define the value function of the optimal control problem as follows:

\[ V(t, x) := \inf_{u \in U[t, T]} J(t, x, u), \quad t \in [0, T], \ x \in \mathcal{D}. \]  

(1.3)

We assume that \( q \) and \( \phi \) satisfy suitable conditions and consider the following Hamilton-Jacobi-Bellman (HJB) equations:

\[
\begin{cases}
\frac{\partial}{\partial t} V(t, x) + S(V)(t, x) + H(t, x, \nabla_x V(t, x)) = 0, & t \in [0, T], \ x \in \mathcal{D}, \\
V(T, x) = \phi(x(0)),
\end{cases}
\]

(1.4)

where

\[
H(t, x, p) = \inf_{u \in U} [(F(t, x(0), (a, x)H, u) + b(t)x(-\tau)]1_0(t), p) + q(t, x(0), u)], (t, x, p) \in [0, T] \times \mathcal{D} \times \mathcal{D}^*.
\]

Here \( 1_0 \) denotes the character function of \( \{0\} \). The definition of a weak infinitesimal generator \( S \) will be given in section 3.

The primary objective of this article is to develop the notion of a viscosity solution to the HJB equations given by (1.4). We then show the value function \( V \) defined in (1.3) is a unique viscosity solution to the HJB equations given in (1.4).

The type of problem above arises in many different fields of application, including engineering, economics and biology. These problems typically disrupt the optimum operation of a system in the form of a time lag in the response to a given input. References [1], [2], [4], and [17] present models with delays in economics; references [7] and [18] present deterministic advertising models with delay effects; references [5] and [6] present population models.

These optimal control problems for differential equations with delays have been thoroughly investigated in recent years (see [4], [5], [6], [7], [8], [17], [18], and [19]). However, to the best of our knowledge, none of these results are directly applicable to our case. In reference [8] and [19], the optimal control problem was embedded in a Hilbert space, and the viscosity solutions for the associated HJB equations were investigated. These results do not hold when \( b(\cdot) \neq 0 \) in the state equation. In [5], [6], [17], and [18], the term \( b(\cdot)X(\cdot - \tau) \) in the state equation is considered, the results obtained in these references only apply when the state equation is a linear differential equation with delays. Optimal control problems of a state equation with memory were investigated in [7]; in these problems, however, the control must satisfy a linear condition, which is not fulfilled if \( F \) is a genuinely nonlinear function.

It is well known that the optimal control problem given by (1.1) and (1.2) can be reformulated as an optimal control problem of the evolution equation in a Hilbert space (see, e.g. [8] and [19]). However, in this case, the initial value must have the following form: \( X^u_t = x \in H \) and \( X^u(t) = x^0 \in \mathbb{R}^d \). This form ensures that the value function is not a viscosity solution of the associated HJB equations because of the \( b(\cdot)X^u(\cdot - \tau) \) term in the state equations. To the aforementioned challenges, we study the associated HJB equations in an infinite dimensional space \( \mathcal{D} \).

Crandall-Lions [10] introduced the notion of viscosity solutions to the HJB equations in the early 1980’s and showed that the dynamic programming method could be applied to optimal control
problems. Since then, many papers have been published on the development of the theory of viscosity solutions (see, e.g., [3], [9], [11], [20], [21], and [23]). References for dealing with equation in an infinite dimensional Hilbert space include [12], [13], [14], [15], [16], and [22]. In references [12], [13], [14], [15], and [16], Crandall and Lions systematically introduced the basic theories for viscosity solutions. Then, Zhou and Yong [22] proved the existence and uniqueness of a viscosity solution to general unbounded first-order HJB equations in infinite dimensional Hilbert spaces.

To the best of our knowledge, the associated HJB equations (1.4) have not been previously studied. The primary difficulty in solving these equations is caused by the infinite-dimensionality of the space of variables and thus the non-compactness of the space. Hence, our problem does not fall into the framework used in references [12], [13], [14], [15], [16], and [22]. Thus, the standard proofs of the comparison theorem rely heavily on compactness arguments and are not applicable to our case.

To overcome this difficulty, we first prove a left maximization principle for the space $[0, T] \times \mathcal{D} \times [0, T] \times \mathcal{D}$ (see Lemma 4.1), i.e., variables exist that maximize functions defined. The proof of the comparison theorem involves maximizing the auxiliary function. The underlying principle is to use the left maximization principle to find a variable that maximizes the auxiliary function.

We next introduce a slightly different notion of a viscosity solution to the HJB equations given in (1.4). We use the left maximization principle to prove the uniqueness of a viscosity solution that corresponds to our new definition of a viscosity solution. At the same time, we show that the value function is a viscosity solution to the HJB equations.

Our results rely heavily on the construction of state equations. We hope to overcome this serious limitation of our approach in future work. However, our method is suitable for a large class of optimal control problems for differential equations with delays.

The paper is organized as follows. In the following section, we define our notation and review the background for differential equations with delays are studied. In section 3 we prove the dynamic programming principle (DPP) and Lemma 3.5 which are used in the following sections. In section 4, we define viscosity solutions and show that the value function $V$ defined by (1.3) is a viscosity solution to the HJB equations given in (1.4). Finally, the uniqueness of viscosity solutions to (1.4) is proved in section 5.

2 Preliminary work

Here, we define the notations that are used in this paper. We use the symbol $|\cdot|$ to denote the norm in a Banach space $F$, the norm symbol is subscripted when necessary. For the vectors $x, y \in R^d$, the scalar product is denoted by $(x, y)_{R^d}$ and the Euclidean norm $(x, x)^{1/2}_{R^d}$ is denoted by $|x|$. For $T > 0$ and $0 \leq t < T$, let $C([t, T], R^d)$ denote the space of continuous functions from $[t, T] \to R^d$, which is associated with the usual norm $|f|_{C} = \sup_{\theta \in [t, T]} |f(\theta)|$. Let $\tau > 0$ be fixed; then $H$ denotes the real, separable Hilbert space $L^2([-\tau, 0]; R^d)$ for scalar product $(\cdot, \cdot)_H$. Let $\mathcal{D}$ denote the set of bounded, right continuous, $R^d$-valued functions on $[-\tau, 0]$. We define a norm on $\mathcal{D}$ as follows:

$$|\omega|_{\mathcal{D}} = \sup_{\theta \in [-\tau, 0]} |\omega(\theta)|, \ \omega \in \mathcal{D}.$$ 

Then, $(\mathcal{D}, |\cdot|_{\mathcal{D}})$ is Banach space.
We define the $| \cdot |_B$-norm on $H$ as follows:

$$|x|_B^2 := \int_{-\tau}^{0} (Bx)^2(s)ds,$$

where

$$(Bx)(s) = \int_{s}^{0} x(\theta)d\theta, \quad s \in [-\tau, 0].$$

Let $0 \leq t \leq \bar{t} \leq T$, $0 \leq s \leq \bar{s} \leq T$, and $\omega, \bar{\omega}, \nu, \bar{\nu} \in D$ be given. We define $(t, \omega) \otimes (\bar{t}, \bar{\omega}) \in [0, T] \times D$ and $(t, \omega, \nu, \bar{\nu}) \otimes (\bar{t}, \bar{\omega}, \bar{\nu}) \in [0, T] \times D \times [0, T] \times D$ by

$$(t, \omega) \otimes (\bar{t}, \bar{\omega}) := (\bar{t}, \bar{\omega}), \quad (t, \omega, \nu, \bar{\nu}) \otimes (\bar{t}, \bar{\omega}, \bar{\nu}) := (\bar{t}, \bar{\omega}, \bar{\nu}),$$

where

$$\bar{\omega}(\theta) = \begin{cases} \bar{\omega}(\theta), & t - \bar{t} \leq \theta \leq 0, \\
\omega(\bar{t} - t + \theta), & -\tau \leq \theta < t - \bar{t}, \end{cases}$$

$$\bar{\nu}(\theta) = \begin{cases} \bar{\nu}(\theta), & t - \bar{t} \leq \theta \leq 0, \\
\nu(\bar{t} - t + \theta), & -\tau \leq \theta < t - \bar{t}. \end{cases}$$

We denote the boundary of a given open subset $Q \subset R^d$ by $\partial Q$ and $\bar{Q} = Q \cup \partial Q$. Let us define

$$D_Q := \{\omega \in D : \omega(\theta) \in Q, \quad \theta \in [-\tau, 0]\}$$

and

$$\bar{D}_Q := \{\omega \in D : \omega(\theta) \in \bar{Q}, \quad \theta \in [-\tau, 0]\}.$$

Let us consider the controlled state equations:

$$\begin{aligned}
\begin{cases}
    dX^u(s) = F(s, X^u(s), (a, X^u)_H, u(s))ds + b(s)X^u(s - \tau)ds, \quad s \in [t, T], \\
    X^u_t = x \in D,
\end{cases}
\end{aligned}$$

where

$$X^u_s \in D, \quad X^u_s(\theta) = X^u(s + \theta), \quad \theta \in [-\tau, 0].$$

Here, the control $u(\cdot)$ belongs to

$$U[t, T] := \{u(\cdot) : [t, T] \rightarrow U \mid u(\cdot) \text{ is measurable}\},$$

and where $U$ is a metric space. We make the following assumptions.

**Hypothesis 2.1.**

(i) The mapping $F : [0, T] \times R^d \times R \times U \rightarrow R^d$ is measurable and a constant $L > 0$ exists such that, for every $t, s \in [0, T], x, y \in R^d \times R, u \in U$,

$$|F(t, x, u)| \leq L(1 + |x|) \quad \text{and} \quad |F(t, x, u) - F(s, y, u)| \leq L(|s - t| + |x - y|).$$

(ii) $a(\cdot) \in W^{1,2}([-\tau, 0]; R^d)$ with $a(-\tau) = 0$ and $b(\cdot) \in W^{1,2}([0, T]; R)^d$, and a constant $L > 0$ exists such that, for every $t, s \in [0, T]$,

$$|b(s) - b(t)| \leq L|s - t|.$$
A function \( X^u : [t, T] \to R^d \) is a solution to equation (2.1) if the function satisfies the following condition:

\[
X^u(s) = x(0) + \int_t^s F(\sigma, X^u(\sigma), (a, X^u_\sigma)_H, u(\sigma))d\sigma + \int_t^s b(\sigma)X^u(\sigma - \tau)d\sigma, \quad s \in [t, T],
\]

where \( X^u_t = x \in D, X^u(s) = x(s-t), \ t - \tau \leq s < t \). To emphasize the dependence of the solution on the initial data, we denote the solution by \( X^u(s, t, x) \).

**Theorem 2.2.** Let us assume that Hypothesis 2.1 holds. Then, a unique function \( X \in C([t,T];R^d) \) exists that is a solution to (2.1). Moreover,

\[
\sup_{s \in [t,T]} |X^u(s, t, x)| \leq C_1 \left( 1 + |x(0)| + \sup_{t \in \mathbb{R}} \left| \int_{-\tau}^t x(\theta)d\theta \right| + |x|_B \right) \leq C_2 (1 + |x(0)| + |x|_H),
\]

where the constants \( C_1 \) and \( C_2 \) depend only on \( L, T, \tau \ a(\cdot) \) and \( b(\cdot) \).

**Proof.** For every initial value \( x \in D \), we define the mapping \( \Phi \) from \( C([t,T];R^d) \) to itself as

\[
\Phi(X^u)(s) = x(0) + \int_t^s F(\sigma, X^u(\sigma), (a, X^u_\sigma)_H, u(\sigma))d\sigma + \int_t^s b(\sigma)X^u(\sigma - \tau)d\sigma, \quad s \in [t, T],
\]

where \( X^u(s) = x(s-t) \) if \( s < t \). We first show that \( \Phi(X^u) \) is continuous with respect to the time \( s \). To this end, for every \( t \leq s_1 \leq s_2 \leq T \), there is a constant \( C > 0 \) that satisfies the following condition:

\[
|\Phi(X^u)(s_1) - \Phi(X^u)(s_2)| \leq L \int_{s_1}^{s_2} (1 + |X^u(\sigma)| + |(a, X^u_\sigma)_H|)d\sigma + \int_{s_1}^{s_2} |b(\sigma)||X^u(\sigma - \tau)|d\sigma
\]

\[
\leq C(1 + \sup_{s \in [t,T]} |X^u(\sigma)| + |x|_D) |s_2 - s_1|.
\]

We next show that it is a contraction, under an equivalent norm. We define the norm \( \| X^u \| = \sup_{s \in [t,T]} e^{-\beta s} |X^u(s)| \), where \( \beta > 0 \) will be chosen later. This norm is equivalent to the original norm on the space \( C([t,T];R^d) \). Then, the definition of the mapping yields

\[
\| \Phi(X^u) \| = \sup_{s \in [t,T]} |e^{-\beta s} \Phi(X^u)(s)|
\]

\[
\leq |x(0)| + \sup_{s \in [t,T]} e^{-\beta s} \left[ \int_t^s \left| F(\sigma, X^u(\sigma), (a, X^u_\sigma)_H, u(\sigma)) \right|d\sigma + \int_t^s |b(\sigma)X^u(\sigma - \tau)|d\sigma \right]
\]

\[
\leq |x(0)| + (T - t) \left[ L + \left( \sup_{s \in [0, T]} |b(s)| + |b|_{W^{1,2}(T-t)^{1/2}} \right) \sup_{t \in \mathbb{R}} \left| \int_{-\tau}^t x(\theta)d\theta \right| + L \sup_{s \in [0, T]} |a|_{W^{1,2}} \right]
\]

\[
+ \frac{1}{\beta} \left( L + \sup_{s \in [0, T]} |b(s)| + |b|_{W^{1,2}(T-t)^{1/2}} + 2L \tau \sup_{s \in [0, T]} |a|_{W^{1,2}} \right) \| X^u \|
\]

(2.4)

This result shows that \( \Phi \) is a well-defined mapping on \( C([t,T];R^d) \). If \( X^u, X^u_1 \) are functions belonging to this space, similar sequences of inequalities show that

\[
\| \Phi(X^u) - \Phi(X^u_1) \| \leq \frac{1}{\beta} \left( L + \sup_{s \in [0, T]} |b(s)| + |b|_{W^{1,2}(T-t)^{1/2}} + 2L \tau \sup_{s \in [0, T]} |a|_{W^{1,2}} \right) \| X^u - X^u_1 \|
\]

(2.5)

Therefore, for a sufficiently large \( \beta \), the mapping \( \Phi \) is a contraction. In addition, (2.4) can be used to obtain (2.3). This result completes the proof. \( \square \)

**Remark 2.3.** (i) The theorem above show that the solution \( X^u(\cdot) \) to equation (2.1) is continuous with respect to the time \( s \in [t, T] \) even if the initial value \( x \) belongs to \( D \).
(ii) Theorem 2.2 also holds true when the initial state $X^u_t = x \in D$ is replaced by $X^u_t = x \in H$ and $X^u(t) = x^0 \in \mathbb{R}^d$.

Let us now consider some continuities of the solution $X^u(\cdot)$ to equation (2.1), these properties will be used in the proof of Theorem 3.2.

**Theorem 2.4.** Let us assume that Hypothesis 2.1 holds. Then, constants $C_3, C_4 > 0$ exist that depend only on $L, T, \tau a(\cdot)$ and $b(\cdot)$, such that, for every $t_1, t_2 \in [0, T]$, and $x_1, x_2 \in D$,

\[
\sup_{u \in U([t_1, t_2], T]} \sup_{s \in [t_1, t_2]} |X^u(s, t_1, x_1) - X^u(s, t_2, x_2)| \leq C_3(1 + |x_1(0)| + |x_2(0)|_H + |x_1| + |x_2|_H) \times \left| |x_1(0) - x_2(0)| + |t_2 - t_1| + \sup_{t \in [0, T]} \int_0^t |x_1(\theta) - x_2(\theta)| d\theta \right|
\]

\[
\leq C_4(|x_1(0) - x_2(0)| + |x_1| - |x_2|_H).
\] (2.6)

**Proof.** For any $t_1, t_2 \in [0, T]$ and $x_1, x_2 \in D$, we assume that $t_1 \leq t_2 < t_1 + \tau$. Let $X^{u,i}(s)$ denote $X^u(s, t, x_i)$ for $s \in [t_i, T]$, where $i = 1, 2$. Thus, we obtain the following results:

\[
\sup_{s \in [t_1, t_2]} |X^{u,1}(s) - X^{u,2}(s)| \leq |x_1(0) - x_2(0)| + L(1 + |a|_H \tau) \sup_{s \in [t_1, T]} |X^{1,u}(s)| + |a|_H |x_1|_H |t_2 - t_1|
\]

\[
+ \sup_{s \in [0, T]} |b(s)| \left| \int_{t_2 - \tau}^{t_2} x_1(\theta) d\theta \right| + (1 + \tau^2 |a|_{W^{1,2}}) \int_{t_2}^t \sup_{s \in [t_2, \sigma]} |X^{u,1}(s) - X^{u,2}(s)| d\sigma
\]

\[
+ L |a|_{W^{1,2}} \tau^\frac{1}{2} |T - t_2| \sup_{t \in [0, t_2 + \tau - t_1]} \left| \int_0^t x_1(\theta) - x_2(\theta + t_1 - t_2) d\theta \right|
\]

\[
+ L |a|_{W^{1,2}} \tau^\frac{1}{2} |T - t_2| (t_2 - t_1)^{\frac{1}{2}} |t_2 - t_1|^2 \sup_{s \in [t_1, T]} |X^{u,1}(s)| + |x_2|
\]

\[
+ \sup_{s \in [0, T]} |b(s)| \left[ \int_{t_2 - \tau}^{t_2} x_1(\theta) - x_2(\theta + t_1 - t_2) d\theta \right] + (t_2 - t_1) \sup_{s \in [t_1, T]} |X^{u,1}(s)|
\]

Using the Gronwall-Bellman inequality, we obtain the following result, for a constant $C > 0$,

\[
\sup_{s \in [t_1, t_2]} |X^{u,1}(s) - X^{u,2}(s)| \leq C(1 + |x_1(0)| + |x_2(0)| + |x_1|_H + |x_2|_H)(|x_1(0) - x_2(0)| + |t_2 - t_1| + |t_2 - t_1|^{\frac{1}{2}})
\]

\[
+ 2C |x_1| |t_2 - t_1|^{\frac{1}{2}} + C \sup_{t \in [0, T]} \left| \int_0^t x_1(\theta) - x_2(\theta) d\theta \right|.
\]

Applying the supremum i.e., $\sup_{u \in U([t_1, t_2], T]}$, to both sides of the previous inequality, we obtain (2.6). We can show that (2.7) holds using a similar (even simpler) procedure. □

## 3 A DPP for optimal control problems

In this section, we consider the controlled state equations:

\[
X^u(s, t, x) = x(0) + \int_t^s F(\sigma, X^u(\sigma, t, x), (a, X^u_{\sigma}(t, x)), u(\sigma)) d\sigma + \int_t^s b(\sigma) X^u(\sigma - \tau, t, x) d\sigma, \quad s \in [t, T],
\] (3.1)
where $X_t^u = x \in \mathcal{D}$, and the cost function

$$J(t, x, u) = \int_t^T q(\sigma, X^u(\sigma, t, x), u(\sigma))d\sigma + \phi(X^u(T, t, x)) \tag{3.2}$$

Our purpose is to minimize the function $J$ over all controls $u \in \mathcal{U}[t, T]$. We define the function $V : [0, T] \times \mathcal{D} \to \mathbb{R}$ by the following:

$$V(t, x) := \inf_{u \in \mathcal{U}[t, T]} J(t, x, u). \tag{3.3}$$

The function $V$ is called the *value function* of optimal control problem (3.1) and (3.2). The goal of this paper is to characterize this value function.

We make the following assumptions.

**Hypothesis 3.1.**

(i) The mappings $q : [0, T] \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ are measurable and there exists a constant $L > 0$, such that, for every $t \in [0, T], x \in \mathbb{R}^d, u \in \mathcal{U}$,

$$|q(t, x, u)| + |\phi(x)| \leq L(1 + |x|).$$

(ii) There exist a constant $L > 0$ and a local modulus of continuously $\rho$ such that, for every $t, s \in [0, T], x, y, \in \mathbb{R}^d, u \in \mathcal{U}$,

$$|q(t, x, u) - q(s, y, u)| + |\phi(x) - \phi(y)| \leq L|x - y| + \rho(|s - t|, |x| \vee |y|).$$

Our first result is the local boundedness and two kinds of continuities of the value function.

**Theorem 3.2.** Suppose that Hypothesis 2.1 and Hypothesis 3.1 hold true. Then, there exists a constant $C_5 > 0$ such that, for every $t, s \in [0, T], x, y \in \mathcal{D}$,

$$|V(t, x)| \leq C_5(1 + |x(0)| + |x|_H), \tag{3.4}$$

$$|V(t, x) - V(t, y)| \leq C_5(|x(0) - y(0)| + |x - y|_H), \tag{3.5}$$

and

$$|V(t, x) - V(s, y)| \leq C_5(1 + |x(0)| + |y(0)| + |x|_H + |y|_H) \times \left( |x(0) - y(0)| + |s - t|^{\frac{\gamma}{2}} + \sup_{t \in [-\tau, 0]} \left| \int_t^0 x(\theta) - y(\theta)d\theta \right| \right). \tag{3.6}$$

**Proof.** We let $0 \leq t \leq s \leq T, x, y \in \mathcal{D}$, by Hypothesis 3.1 (ii), (2.3) and (2.6), for any $u \in \mathcal{U}[t, T]$, we have

$$|J(t, x, u) - J(s, y, u)|$$

$$\leq (T + 1)L\sup_{\sigma \in [s, T]} |X^u(\sigma, t, x) - X^u(\sigma, s, y)| + L\int_t^s (1 + |X^u(\sigma, t, x)|)d\sigma$$

$$\leq L(1 + C_2(1 + |x(0)| + |x|_H))(s - t) + (T + 1)LC_3(1 + |x(0)| + |y(0)| + |x|_H + |y|_H) \times \left( |x(0) - y(0)| + |s - t|^{\frac{\gamma}{2}} + \sup_{t \in [-\tau, 0]} \left| \int_t^0 x(\theta) - y(\theta)d\theta \right| \right).$$

Thus, taking the infimum in $u \in \mathcal{U}[t, T]$, we obtain (3.6). By the similar procedure, we can show (3.4) and (3.5) hold true. The theorem is proved. $\Box$

We note that $V(t, x)$ is not necessarily Lipschitz continuous in $t$.

Secondly, we present the following result, which is called the dynamic programming principle (DPP).
Theorem 3.3. Assume the Hypothesis 2.1 and Hypothesis 3.1 hold true. Then, for every \((t, x) \in [0, T) \times \mathcal{D}\) and \(s \in [t, T]\), we have that

\[
V(t, x) = \inf_{u \in \mathcal{U}[t, T]} \left[ \int_t^s q(\sigma, X^u(\sigma, t, x), u(\sigma))d\sigma + V(s, X^u_s(t, x)) \right].
\]  

(3.7)

Proof. First of all, for any \(u \in \mathcal{U}[s, T], s \in [t, T]\) and any \(u \in \mathcal{U}[t, s]\), by putting them concatenatively, we get \(u \in \mathcal{U}[t, T]\). Let us denote the right-hand side of (3.7) by \(\mathcal{V}(t, x)\). By (3.3), we have

\[
V(t, x) \leq J(t, x, u) = \int_t^s q(\sigma, X^u(\sigma, t, x), u(\sigma))d\sigma + J(s, X^u_s(t, x), u), \ u(\cdot) \in \mathcal{U}[t, T].
\]

Thus, taking the infimum over \(u(\cdot) \in \mathcal{U}[s, T]\), we obtain

\[
V(t, x) \leq \int_t^s q(\sigma, X^u(\sigma, t, x), u(\sigma))d\sigma + V(s, X^u_s(t, x)).
\]

Consequently,

\[
V(t, x) \leq \mathcal{V}(t, x).
\]

On the other hand, for any \(\varepsilon > 0\), there exists a \(u^\varepsilon \in \mathcal{U}[t, T]\), such that

\[
V(t, x) + \varepsilon \geq J(t, x, u^\varepsilon) \geq \int_t^s q(\sigma, X^{u^\varepsilon}(\sigma, t, x), u^\varepsilon(\sigma))d\sigma + J(s, X^{u^\varepsilon}_s(t, x), u^\varepsilon) \geq \int_t^s q(\sigma, X^{u^\varepsilon}(\sigma, t, x), u^\varepsilon(\sigma))d\sigma + V(s, X^{u^\varepsilon}_s(t, x)) \geq \mathcal{V}(t, x).
\]

Hence, (3.7) follows. \(\square\)

Our next goal is to derive the so-called Hamilton – Jacobi – Bellman equation for the value function \(V\). To begin with, let us introduce the operator \(\mathcal{S}\). For a Borel measurable function \(f : \mathcal{D} \to \mathbb{R}\), we define

\[
\mathcal{S}(f)(x) = \lim_{h \to 0^+} \frac{1}{h} [f(\hat{x}_h) - f(x)], \quad x \in \mathcal{D},
\]

where \(\hat{x} : [-\tau, T] \to \mathbb{R}^d\) is an extension of \(x\) defined by

\[
\hat{x}(s) = \begin{cases} x(s), & s \in [-\tau, 0), \\ x(0), & s \geq 0, \end{cases}
\]

and \(\hat{x}_s\) is defined by

\[
\hat{x}_s(\theta) = \hat{x}(s + \theta), \quad \theta \in [-\tau, 0] .
\]

We denote by \(\hat{D}(\mathcal{S})\) the domain of the operator \(\mathcal{S}\), be the set of \(f : \mathcal{D} \to \mathbb{R}\) such that the above limit exists for all \(x \in \mathcal{D}\). Define \(D(\mathcal{S})\) as the set of all functions \(g : [0, T] \times \mathcal{D} \to \mathbb{R}\) such that \(g(t, \cdot) \in \hat{D}(\mathcal{S})\) for all \(t \in [0, T]\). For simplicity, we define

\[
\Phi = \{ \varphi \in C^1([0, T] \times \mathcal{D}) \cap D(\mathcal{S}) \mid \exists \varphi_0 \in C^1([0, T] \times \mathbb{R}^d \times H), \\
\text{such that } \varphi(t, x) = \varphi_0(t, x(0), x), \ \forall (t, x) \in [0, T] \times \mathcal{D} \}.
\]

Theorem 3.4. Let \(V\) denote the value function defined by (3.3), if the function \(V(t, x) \in \Phi\). Then, \(V(t, x)\) satisfies the following HJB equation:

\[
\begin{cases}
\frac{\partial}{\partial t} V(t, x) + \mathcal{S}(V)(t, x) + H(t, x, \nabla_x V(t, x)) = 0, & t \in [0, T], \ x \in \mathcal{D}, \\
V(T, x) = \phi(x(0)), & \end{cases}
\]

(3.8)
where

\[ H(t, x, p) = \inf_{u \in U} \left[ ([F(t, x(0), (a, x)_H, u) + b(t)x(-\tau)]1_0(t), p) + g(t, x(0), u) \right], \quad (t, x, p) \in [0, T] \times \mathcal{D} \times \mathcal{D}^* \]

Here the function \( 1_0 : [-\tau, 0] \to R \) is the characteristic function of \( \{0\} \).

In order to prove this theorem we need the following lemma.

**Lemma 3.5.** Suppose that Hypothesis 2.1 holds. If \( g \in \Phi \), then, for each \( (t, x) \in [0, T] \times \mathcal{D} \), the following convergence holds uniformly in \( u(\cdot) \in \mathcal{U}[t, T] \):

\[
\lim_{\epsilon \to 0^+} \frac{g(t + \epsilon, X_{t+\epsilon}^u) - g(t, x)}{\epsilon} - g(t, x) - S(g)(t, x) - \langle \nabla_x g(t, x), \nabla W(t, x, \epsilon) 1_0(t) \rangle = 0, \quad (3.9)
\]

where we let \( \nabla W(t, x, \epsilon) \) denote \( \frac{1}{\epsilon} \int_t^{t+\epsilon} F(t, x(0), (a, x)_H, u(\sigma)) d\sigma + b(t)x(-\tau) \).

**Proof.** Since \( g \in C^1([0, T] \times \mathcal{D}) \), by Taylor’s theorem we get that

\[
g(t + \epsilon, X_{t+\epsilon}^u) - g(t, x) = g(t + \epsilon, X_{t+\epsilon}^u) - g(t, X_{t+\epsilon}^u) + g(t, \hat{x}_{t+\epsilon}) - g(t, x) + \langle \nabla_x g(t, \hat{x}_{t+\epsilon}), X_{t+\epsilon}^u - \hat{x}_{t+\epsilon} \rangle + o(\|X_{t+\epsilon}^u - \hat{x}_{t+\epsilon}\|), \quad t \in [0, T - \epsilon]. \quad (3.10)
\]

Again by \( g \in C^1([0, T] \times \mathcal{D}) \), we have that

\[
\lim_{\epsilon \to 0^+} \sup_{u(\cdot) \in \mathcal{U}[t, T]} \left| \frac{1}{\epsilon} \left[ g(t + \epsilon, X_{t+\epsilon}^u) - g(t, X_{t+\epsilon}^u) \right] - \frac{\partial}{\partial \epsilon} g(t, x) \right| = 0. \quad (3.11)
\]

From \( g \in D(S) \), it follows that

\[
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} (g(t, \hat{x}_{t+\epsilon}) - g(t, x)) = S(g)(t, x). \quad (3.12)
\]

By the definitions of \( X_s^u \) and \( \hat{x}_s \), we have that, for every \( \epsilon \in [0, T - t] \),

\[
X_{t+\epsilon}^u(\theta) - \hat{x}_{t+\epsilon}(\theta) = \begin{cases} \int_t^{t+\epsilon+\theta} F(\sigma, X_{t+\epsilon}^u(\sigma), (a, X_{t+\epsilon}^u)_H, u(\sigma)) + b(\sigma)X_{t+\epsilon}^u(\sigma - \tau) d\sigma, & \epsilon + \theta \geq 0, \\ 0, & \epsilon + \theta < 0. \end{cases}
\]

Thus, the following convergence holds uniformly in \( u(\cdot) \in \mathcal{U}[t, T] \):

\[
\lim_{\epsilon \to 0^+} \left| \frac{1}{\epsilon} (X_{t+\epsilon}^u(t, x) - \hat{x}_{t+\epsilon})(\theta) \right| = \begin{cases} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_t^{t+\epsilon+\theta} F(\sigma, X_{t+\epsilon}^u(\sigma), (a, X_{t+\epsilon}^u)_H, u(\sigma)) + b(\sigma)X_{t+\epsilon}^u(\sigma - \tau) d\sigma, & \theta = 0, \\ 0, & -\tau \leq \theta < 0. \end{cases}
\]

By (2.3), we get that, there exists a constant \( C > 0 \) independent of \( u(\cdot) \) such that

\[
\sup_{\theta \in [-\tau, 0]} \left| \frac{1}{\epsilon} (X_{t+\epsilon}^u - \hat{x}_{t+\epsilon})(\theta) \right| \leq C.
\]

Thus, by the continuity of \( \nabla_x g_0(t, x, \cdot) \) and \( \nabla_x g_0(t, x^0, \cdot) \), we obtain

\[
\lim_{\epsilon \to 0^+} \sup_{u(\cdot) \in \mathcal{U}[t, T]} \left| \frac{1}{\epsilon} \langle \nabla_x g(t, \hat{x}_{t+\epsilon}), X_{t+\epsilon}^u - \hat{x}_{t+\epsilon} \rangle - \frac{1}{\epsilon} \langle \nabla_x g(t, x), X_{t+\epsilon}^u - \hat{x}_{t+\epsilon} \rangle \right| \leq C \lim_{\epsilon \to 0^+} \left| \nabla_x g_0(t, x(0), \hat{x}_{t+\epsilon}) + \nabla_x g_0(t, x(0), \hat{x}_{t+\epsilon}) - \nabla_{x^0} g_0(t, x(0), x) - \nabla_x g_0(t, x(0), x) \right| = 0.
\]
Therefore, by the above inequality, we obtain that

\[
\lim_{\epsilon \to 0^+} \sup_{u(\cdot) \in U[t,T]} \left| \frac{1}{\epsilon} (\nabla_x g(t, \hat{x}_{t+\epsilon}) - \hat{X}_{t+\epsilon}^u) - \langle \nabla_x g(t, x), \hat{W}(t, x, \epsilon) \rangle \right| = 0.
\]

Hence, dividing by \( \epsilon \) in (3.10) and sending \( \epsilon \to 0^+ \), and putting together the results of (3.11), (3.12) and (3.13), we finally obtain (3.9). \( \Box \)

From the above lemma, the following two lemmas hold true, which will be used in the proof of uniqueness result for viscosity solution.

**Lemma 3.6.** Suppose that Hypothesis 2.1 holds. If \( g(t, x) = g_0(t, |x|^2_H) \), \( (t, x) \in [0, T) \times \mathcal{D} \), where \( g_0 \in C^1([0, T] \times R) \). Then the following holds:

\[
g(t + \epsilon, X_{t+\epsilon}^u) - g(t, x) = g_t(t, x) + g_0'(t, |x|^2_H)(x^2(0) - x^2(-\tau)) + o(1),
\]

where \( o(1) \) is uniformly in \( u(\cdot) \in U[t,T] \).

**Proof.** By Lemma 3.5, we only need to show that

\[
\lim_{\epsilon \to 0^+} \sup_{u(\cdot) \in U[t,T]} \left| \frac{1}{\epsilon} \int_t^{t+\epsilon} \left\{ F(\sigma, X_u^u(\sigma), u(\sigma)) + b(\sigma) X_u^u - \hat{x}_{t+\epsilon} \right\} d\tau - \hat{W}(t, x, \epsilon) \right| = 0.
\]

By the similar procedure of Lemma 3.5, we get (3.15) and (3.16) hold true. Now, let us prove (3.17). From the definition of \( g \), it follows that

\[
\frac{1}{\epsilon} (g(t, \hat{x}_{t+\epsilon}) - g(t, x)) = \frac{1}{\epsilon} \left( g_0(t, |\hat{x}_{t+\epsilon}|^2_H) - g_0(t, |x|^2_H) \right)
\]

\[
= \frac{1}{\epsilon} \int_0^1 g_0'(t, |x|^2_H) s(|\hat{x}_{t+\epsilon}|^2_H - |x|^2_H) (|\hat{x}_{t+\epsilon}|^2_H - |x|^2_H) ds \nonumber
\]

\[
= \frac{1}{\epsilon} \int_{-\tau}^{-\tau+\epsilon} x^2(\theta) ds \int_0^1 g_0'(t, |x|^2_H) s(|\hat{x}_{t+\epsilon}|^2_H - |x|^2_H) ds \nonumber
\]

By the similar procedure of Lemma 3.5, we get (3.15) and (3.16) hold true. Now, let us prove (3.17). From the definition of \( g \), it follows that

\[
\frac{1}{\epsilon} (g(t, \hat{x}_{t+\epsilon}) - g(t, x)) = \frac{1}{\epsilon} \left( g_0(t, |\hat{x}_{t+\epsilon}|^2_H) - g_0(t, |x|^2_H) \right)
\]

\[
= \frac{1}{\epsilon} \int_0^1 g_0'(t, |x|^2_H) s(|\hat{x}_{t+\epsilon}|^2_H - |x|^2_H) (|\hat{x}_{t+\epsilon}|^2_H - |x|^2_H) ds \nonumber
\]

\[
= \frac{1}{\epsilon} \int_{-\tau}^{-\tau+\epsilon} x^2(\theta) ds \int_0^1 g_0'(t, |x|^2_H) s(|\hat{x}_{t+\epsilon}|^2_H - |x|^2_H) ds \nonumber
\]

Letting \( \epsilon \to 0 \), we obtain (3.17). \( \Box \)

**Lemma 3.7.** Suppose that Hypothesis 2.1 holds. If \( \psi(x) = \psi_0(|x - \hat{a}|^2_H) \), \( \hat{a}, x \in \mathcal{D} \), where \( \psi_0 \in C^1(R) \). Then the following holds:

\[
\frac{1}{\epsilon} \psi(X_{t+\epsilon}^u) - \psi(x) = 2 \psi_0'(|x - \hat{a}|^2_H) (B(x - \hat{a}), x(0)) 1_{[-\tau, 0]} - x) + o(1),
\]

where \( o(1) \) is uniformly in \( u(\cdot) \in U[t,T] \). Here the function \( 1_{[-\tau, 0]} \) is the character function of \([-\tau, 0]\).
Proof. By Lemma 3.5, we only need to show that
\[
\lim_{\epsilon \to 0^+} \sup_{\epsilon \in (0, \epsilon)} |\nabla_{\epsilon} \psi(\tilde{x} t + \epsilon)(X^u t + \epsilon - \tilde{x} t + \epsilon)| = 0, \tag{3.18}
\]
and
\[
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} (\psi(\tilde{x} t + \epsilon) - \psi(x)) = 2\psi_0((|x - \hat{a}|_B^2)(B(x - \hat{a}), x(0)1_{[-\tau, 0]} - x)_H. \tag{3.19}
\]
By the similar procedure of Lemma 3.5, we can obtain (3.18). Now let us show (3.19) hold true.

By the definition of \(\psi\), we have that, for some \(s \in (0, 1)\),
\[
\frac{1}{\epsilon}(\psi(\tilde{x} t + \epsilon) - \psi(x)) = \frac{1}{\epsilon}[\psi'(x + s(\tilde{x} t + \epsilon - x)) - \psi'(\tilde{x} t + \epsilon - x)]
\]
\[
= 2\epsilon \psi_0'(|(x + s(\tilde{x} t + \epsilon - x) - \hat{a})|_B^2)(B(x + s(\tilde{x} t + \epsilon - x) - \hat{a}), B(\tilde{x} t + \epsilon - x)). \tag{3.20}
\]
On the other hand, we have that
\[
\frac{1}{\epsilon} \langle B(x - \hat{a}), B(\tilde{x} t + \epsilon - x) \rangle = \frac{1}{\epsilon} \int_{-\tau}^0 \int_s^0 (x(\theta) - \hat{a}(\theta))d\theta \int_s^0 (\tilde{x} t + \epsilon(\theta) - x(\theta))d\theta ds
\]
\[
= \frac{1}{\epsilon} \int_{-\tau}^0 \int_s^0 (x(\theta) - \hat{a}(\theta))d\theta \left( - \int_s^{(s + \epsilon) \land 0} x(\theta)d\theta + ((-s) \land \epsilon)x(0) \right) ds
\]
\[
\to \int_{-\tau}^0 (x(0) - x(s)) \int_s^0 (x(\theta) - \hat{a}(\theta))d\theta ds = (B(x - \hat{a}), x(0)1_{[-\tau, 0]} - x)_H \quad \text{as } \epsilon \to 0,
\]
and
\[
\frac{1}{\epsilon} \langle B((\tilde{x} t + \epsilon) - x), B(\tilde{x} t + \epsilon - x) \rangle = \frac{1}{\epsilon} \int_{-\tau}^0 \left( \int_s^0 (\tilde{x} t + \epsilon(\theta) - x(\theta))d\theta \right)^2 ds
\]
\[
= \frac{1}{\epsilon} \int_{-\tau}^0 \left( - \int_s^{(s + \epsilon) \land 0} x(\theta)d\theta + ((-s) \land \epsilon)x(0) \right)^2 ds
\]
\[
\leq \frac{2}{\epsilon} \int_{-\tau}^0 \left( \epsilon^2 x^2(0) + \epsilon \int_s^{(s + \epsilon) \land 0} x^2(\theta)d\theta \right) ds \to 0 \quad \text{as } \epsilon \to 0.
\]

Letting \(\epsilon \to 0\) in (3.20), we get (3.19). \(\Box\)

Remark 3.8. We note that \(\hat{D}(S)\) and \(D(S)\) are not empty. In fact, by (3.17) and (3.19), we have \(g_0(|x|_D^2), g_0(|x - a|_D^2) \in \hat{D}(S)\), if \(g_0 \in C^1(R)\) and \(a \in D\). Moreover, \(g_0(|x|_D^2) + l(t)\) and \(g_0(|x - a|_D^2) + l(t)\) belong to \(D(S)\), if \(l \in C^1(R)\).

Proof of Theorem 3.4. First of all, by the definition of \(V\), we have that \(V(T, x) = \phi(x(0))\). Next, we fix a \(u \in U\) and \(x \in D\), from (3.7), it follows that
\[
0 \leq \int_t^S q(\sigma, X^u(\sigma, t, x), u)d\sigma + V(s, X^u_s(t, x)) - V(t, x).
\]
By Lemma 3.5, the above inequality implies that
\[
0 \leq \lim_{s \to t^+} \frac{1}{s - t} \left[ \int_t^S q(\sigma, X^u(\sigma, t, x), u)d\sigma + V(s, X^u_s(t, x)) - V(t, x) \right]
\]
\[
= \frac{\partial}{\partial t} V(t, x) + S(V)(t, x) + \langle [F(t, x(0), (a, x)_H, u) + b(t)x(-\tau)] \rangle_0(t), \nabla_x V(t, x) \rangle + q(t, x(0), u).
\]
Thus, we have that
\[
0 \leq \frac{\partial}{\partial t} V(t, x) + S(V)(t, x) + H(t, x, \nabla_x V(t, x)). \tag{3.21}
\]
On the other hand, let $x \in \mathcal{D}$ be fixed. For any $\varepsilon > 0$ and $s \geq t$, by (3.7), there exists a $\bar{u} \in \mathcal{U}[t,T]$ such that
\[
\varepsilon(s-t) \geq \int_t^s q(\sigma, X_{\bar{u}}(\sigma, t, x), \bar{u}(\sigma))d\sigma + V(s, X_s(t, x)) - V(t, x)
\]
\[
= \frac{\partial}{\partial t}V(t, x)(s-t) + \mathcal{S}(V)(t, x)(s-t) + \int_t^s q(t, x(0), \bar{u}(\sigma))d\sigma
\]
\[
+ (\nabla_x V(t, x), \int_t^s [F(t, x(0), (a, x)_H, \bar{u}(\sigma)) + b(t)x(-\tau)]d\sigma)_{10}(t) + o(|s-t|)
\]
\[
\geq \frac{\partial}{\partial t}V(t, x)(s-t) + \mathcal{S}(V)(t, x)(s-t) + H(t, x, \nabla_x V(t, x))(s-t) + o(|s-t|).
\]
Then, dividing through by $s-t$ and letting $s-t \to 0$, we have that
\[
\varepsilon \geq \frac{\partial}{\partial t}V(t, x) + \mathcal{S}(V)(t, x) + H(t, x, \nabla_x V(t, x)).
\]
Combining with (3.21), we get the desired result. □

4 Viscosity solution of HJB equations: Existence theorem.

In this section, we are going to introduce the notion of viscosity solution. M. G. Crandall and P. L. Lions [12, 13, 14, 15, 16] systematically introduced the basic theories of viscosity solutions for HJB equations in infinite dimensions. The proof of the uniqueness is mainly based on the weak compactness of separable Hilbert space (see also [22]). We note that the HJB equation (3.8) is defined in $\mathcal{D}$, which doesn't have the weak compactness. Thus, we need to give a new notion of viscosity solution of (3.8). To begin with, let us introduce the following key lemma that will be used in the proof of the uniqueness of viscosity solutions.

Lemma 4.1. (Left maximization principle) Let $Q$ be a bounded open subset of $\mathbb{R}^d$ and let $v : [0,T] \times \mathcal{D} \times [0,T] \times \mathcal{D} \to \mathbb{R}$ be continuous, and there exists an integer $k > 0$ such that, for every $t, s \in [0,T], x, x_1, x_2, y, y_1, y_2 \in \mathcal{D},$
\[
|v(t, x, s, y)| \leq L(1 + |x|_H + |x(0)| + |y|_H + |y(0)|)^k,
\]
\[
|v(t, x_1, s, y_1) - v(t, x_2, s, y_2)| \leq L(|x_1 - x_2|_H + |x_1(0) - x_2(0)| + |y_1 - y_2|_H + |y_1(0) - y_2(0)|)^k. \tag{4.1}
\]
Then, for each $(t_0, x_0, s_0, y_0) \in [0,T] \times \mathcal{D}_Q \times [0,T] \times \mathcal{D}_Q$, there exists $(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \in [t_0, T] \times \mathcal{D}_Q \times [s_0, T] \times \mathcal{D}_Q$, such that $(\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (t_0, x_0, s_0, y_0) \otimes (\bar{t}, \bar{x}, \bar{s}, \bar{y}), u(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \geq u(t_0, x_0, s_0, y_0)$, and
\[
v(\bar{t}, \bar{x}, \bar{s}, \bar{y}) = \sup_{(t, x, s, y) \in [t_0, T] \times \mathcal{D}_Q \times [s_0, T] \times \mathcal{D}_Q} v((t, x, s, y) \otimes (t, x, s, y)). \tag{4.2}
\]

Proof. Without loss of generality, we can assume that $v(t_0, x_0, s_0, y_0) \geq v(t_0, x_0 + \varepsilon l_0(\cdot), s_0, y_0 + l_1(\cdot))$ for all $e, l \in \mathbb{R}^d$ such that $e + x_0(0), l + y_0(0) \in \bar{Q}$. We set $m_0 = v(t_0, x_0, s_0, y_0)$ and
\[
m_0 := \sup_{(t, x, s, y) \in [t_0, T] \times \mathcal{D}_Q \times [s_0, T] \times \mathcal{D}_Q} v((t_0, x_0, s_0, y_0) \otimes (t, x, s, y)) \geq m_0.
\]
If $m_0 = m_0$, then we can take $(\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (t_0, x_0, s_0, y_0)$ and finish the procedure. Otherwise there exists $(t_1, x_1, s_1, y_1) \in (t_0, T] \times \mathcal{D}_Q \times (s_0, T] \times \mathcal{D}_Q$, such that $(t_1, x_1, s_1, y_1) = (t_0, x_0, s_0, y_0) \otimes (t_1, x_1, s_1, y_1)$ and
\[
m_1 := v(t_1, x_1, s_1, y_1) \geq \frac{m_0 + \bar{m}_0}{2}.
\]
We set
\[ \overline{m}_1 := \sup_{(x,s,y) \in [t_1,T] \times \mathcal{D}_Q \times [s_1,T] \times \mathcal{D}_Q} v((t_1, x_1, s_1, y_1) \otimes (t, x, s, y)) \geq m_1. \]
If \( \overline{m}_1 = m_1 \), then we can take \((\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (t_1, x_1, s_1, y_1)\) and finish the procedure. Otherwise we can find, for \( i = 2, 3, \ldots \), \((t_i, x_i, s_i, y_i) \in (t_{i-1}, T] \times \mathcal{D}_Q \times (s_{i-1}, T] \times \mathcal{D}_Q \) such that \((t_i, x_i, s_i, y_i) = (t_{i-1}, x_{i-1}, s_{i-1}, y_{i-1} - 1) \otimes (t_i, x_i, s_i, y_i)\), \(v(t_i, x_i, s_i, y_i) \geq v(t_i, x_i + e1_0(\cdot), s_i, y_i + l1_0(\cdot))\), for all \( e, l \) such that \( e + x_i(0), l + y_i(0) \in \bar{Q} \) and
\[ m_i := u(t_i, x_i, s_i, y_i) \geq \frac{m_{i-1} + \overline{m}_{i-1}}{2}, \]
\[ \overline{m}_i := \sup_{(x,s,y) \in [t_i,T] \times \mathcal{D}_Q [s_i,T] \times \mathcal{D}_Q} u((t_i, x_i, s_i, y_i) \otimes (t, x, s, y)) \geq m_i, \]
and continue this procedure till the first time when \( \overline{m}_i = m_i \) and the finish the proof by setting \((\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (t_i, x_i, s_i, y_i)\). For the last case in which \( m_i > m_i \) for all \( i = 1, 2, \ldots \), we have \( t_i \uparrow \bar{t} \in [0, T], s_i \uparrow \bar{s} \in [0, T] \). Then we can find \( \bar{x}, \bar{y} \in \mathcal{D}_Q \) such that \((\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (t_i, x_i, s_i, y_i) \otimes (\bar{t}, \bar{x}, \bar{s}, \bar{y})\).
We can choose \( \bar{x}(0), \bar{y}(0) \in \bar{Q} \) such that \( u(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \geq u(\bar{t}, \bar{x} + x1_0(\cdot), \bar{s}, \bar{y} + y1_0(\cdot)) \), for all \( x, y \) such that \( x + \bar{x}(0), y + \bar{y}(0) \in \bar{Q} \).

Since
\[ \overline{m}_{i+1} - m_i \leq \frac{\overline{m}_i + m_i}{2} = \frac{m_i - m_i}{2}, \]
thus there exists \( \bar{m} \in (m_0, \overline{m}_0) \), such that \( \overline{m}_i \downarrow \bar{m} \) and \( m_i \uparrow \bar{m} \). By the definitions of \( \bar{x} \) and \( \bar{y} \), we get \( x_i(s) \rightarrow \bar{x}(s) \) and \( y_i(s) \rightarrow \bar{y}(s) \) for almost all \( s \in [-\tau, 0] \) and there exist two subsequences of \( x_i(0) \) and \( y_i(0) \) still denoted by themselves such that \( x_i(0) \rightarrow \bar{a} \in Q \) and \( y_i(0) \rightarrow \bar{b} \in Q \), respectively.
Thus, by (4.1) we get that
\[ \bar{m} = \lim_{i \rightarrow \infty} m_i = \lim_{i \rightarrow \infty} v(t_i, x_i, s_i, y_i) \leq v(\bar{t}, \bar{x}, \bar{s}, \bar{y}). \]
We can claim that (4.2) holds for this \((\bar{t}, \bar{x}, \bar{s}, \bar{y})\). Indeed, otherwise there exist \((t, x, s, y) \in (\bar{t}, T] \times \mathcal{D}_Q \times (\bar{s}, T] \times \mathcal{D}_Q \) and \( \delta > 0 \) with \((t, x, s, y) = (\bar{t}, \bar{x}, \bar{s}, \bar{y}) \otimes (t, x, s, y)\), such that
\[ v((\bar{t}, \bar{x}, \bar{s}, \bar{y}) \otimes (t, x, s, y)) \geq v((\bar{t}, \bar{x}, \bar{s}, \bar{y}) + \delta \geq \bar{m} + \delta, \]
then the following contradiction is induced:
\[ v((\bar{t}, \bar{x}, \bar{s}, \bar{y}) \otimes (t, x, s, y)) = v((t_i, x_i, s_i, y_i) \otimes (t, x, s, y)) \leq m_i \rightarrow \bar{m}. \]
The proof is completed. \( \square \)

From the above lemma, we can now give the following definition of viscosity solution:

**Definition 4.2.** \( w \in C([0, T] \times \mathcal{D}) \) is called a viscosity subsolution (supersolution) of (3.8) if the terminal condition \( w(T, x) \leq \phi(x(0)) \) (resp. \( w(T, x) \geq \phi(x(0)) \)) is satisfied and for every bounded open subset \( Q \) of \( \mathbb{R}^d \) and \( \varphi \in \Phi \), whenever the function \( w - \varphi \) (resp. \( w + \varphi \)) satisfies
\[ (w - \varphi)(s, z) = \sup_{(t,x) \in [s,T] \times \mathcal{D}_Q} (w - \varphi)((s, z) \otimes (t, x)), \]
\[ (w + \varphi)(s, z) = \inf_{(t,x) \in [s,T] \times \mathcal{D}_Q} (w + \varphi)((s, z) \otimes (t, x)), \]
where \((s, z) \in [0, T] \times \mathcal{D}_Q \) and \( z(0) \in Q \), we have
\[ \varphi_1(s, z) + \mathcal{S}(\varphi)(s, z) + H(s, z, \nabla_x \varphi(s, z)) \geq 0, \]
Taking the minimum in \( w \in C([0,T] \times \mathcal{D}) \) is said to be a viscosity solution of (3.8) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 4.3.** (i) A viscosity solution of the HJB equation (3.8) is a classical solution if it furthermore lies in \( \Phi \).

(ii) In the classical uniqueness proof of viscosity solution to HJB equation in infinite dimensions, the weak compactness of separable Hilbert spaces is used (see [22]). In our case, the HJB equation is defined on space \( \mathcal{D} \), which doesn’t have weak compactness. For the sake of the uniqueness proof, our new notion of viscosity solution is enhanced. At the same time, our modification doesn’t lead to additional difficulty in the existence proof.

(iii) Assume that the coefficient \( F(t,x,y,u) = \mathcal{T}(t,x,u), (t,x,y,u) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times U \) and \( b = 0 \). Let function \( V(t,x) : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) be a viscosity solution to (3.8) as a functional \( V(t,x) : [0,T] \times \mathcal{D} \rightarrow \mathbb{R} \). Then \( V \) is also a classical viscosity solution as a function of time and state.

We conclude this section with the existence result on viscosity solution.

**Theorem 4.4.** Suppose that Hypothesis 2.1 and Hypothesis 3.1 hold true. Then the value function \( V(t,x) \) defined by (3.3) is a viscosity solution of (3.8).

**Proof.** First, for every bounded open subset \( Q \subset \mathbb{R}^d \) we let \( \varphi \in \Phi \) such that

\[
0 = (V - \varphi)(t,x) = \sup_{(s,y) \in [t,T] \times \mathcal{D}_Q} (V - \varphi)((t,x) \otimes (s,y)),
\]

where \( (t,x) \in [0,T] \times \mathcal{D}_Q \) and \( x(0) \in Q \). Then, for fixed \( u \in U \) and \( t \leq s < T \), by the dynamic programming principle (Theorem 3.3), we get that, for \( s \) small enough,

\[
\varphi(t,x) = V(t,x) \leq \int_t^s q(\sigma, X^u(\sigma), u) d\sigma + V(s, X^u_s) \leq \int_t^s q(\sigma, X^u(\sigma), u) d\sigma + \varphi(s, X^u_s).
\]

Thus,

\[
0 \leq \frac{1}{s-t} \int_t^s q(\sigma, X^u(\sigma), u) d\sigma + \frac{1}{s-t}[\varphi(s, X^u_s) - \varphi(t,x)].
\]

Now, applying Lemma 3.5, we show that

\[
0 \leq q(t,x(0),u) + \varphi_t(t,x) + \mathcal{S}(\varphi)(t,x) + \langle \nabla_x \varphi(t,x), [F(t,x(0), (a,x)_H, u) + b(t)x(-\tau)]1_0(t) \rangle.
\]

Taking the minimum in \( u \in U \), we get that \( V \) is a viscosity subsolution of (3.8).

Next, for every bounded open subset \( Q \subset \mathbb{R}^d \), we let \( \varphi \in \Phi \) such that

\[
0 = (V + \varphi)(t,x) = \inf_{(s,y) \in [t,T] \times \mathcal{D}_Q} (V + \varphi)((t,x) \otimes (s,y)),
\]

where \( (t,x) \in [0,T] \times \mathcal{D}_Q \) and \( x(0) \in Q \). For any \( \varepsilon > 0 \) and \( s > t \), by (3.7), one can find a control \( u^\varepsilon(\cdot) \equiv u^\varepsilon,s(\cdot) \in U[t,T] \) such that, for \( s \) small enough,

\[
\varepsilon(s-t) \geq \int_t^s q(\sigma, X^{u^\varepsilon}(\sigma), u^\varepsilon(\sigma)) d\sigma + V(s, X^u_s) - V(t,x).
\]
Then, by Lemma 3.5, we obtain that
\[ \varepsilon \geq \frac{1}{s-t} \int_t^s q(\sigma, X^{u^\varepsilon}(\sigma), u^\varepsilon(\sigma))d\sigma - \frac{\varphi(s, X^{u^\varepsilon}_s) - \varphi(t, x)}{s-t} \]
\[ \geq -\varphi_t(t, x) - S(\varphi)(t, x) + \frac{1}{s-t} \int_t^s q(t, x(0), u^\varepsilon(\sigma)) \]
\[-\langle \nabla_x \varphi(t, x), [F(t, x(0), (a, x)_H, u^\varepsilon(\sigma)) + b(t)x(\tau)\delta_0(t)] \rangle d\sigma + o(1) \]
\[ \geq -\varphi_t(t, x) - S(\varphi)(t, x) + \inf_{u \in U} [q(t, x(0), u) \]
\[-\langle \nabla_x \varphi(t, x), [F(t, x(0), (a, x)_H, u) + b(t)x(\tau)\delta_0(t)] \rangle] + o(1). \]

Letting \( s \downarrow t \) and \( \varepsilon \to 0 \) we show that
\[ 0 \geq -\varphi_t(t, x) - S(\varphi)(t, x) + \inf_{u \in U} [q(t, x(0), u) - \langle \nabla_x \varphi(t, x), [F(t, x(0), (a, x)_H, u) + b(t)x(\tau)\delta_0(t)] \rangle]. \]

Therefore, \( V \) is also a viscosity supsolution of (3.8). This completes the proof of Theorem 4.4. \( \Box \)

5 Viscosity solution of HJB equations: Uniqueness theorem.

This section is devoted to a proof of uniqueness of the viscosity solution to (3.8). This result, together with those in the previous section, will give a characterization for the value function of optimal control problem (3.1) and (3.2).

We are now state the main result of this section.

**Theorem 5.1.** Suppose that Hypothesis 2.1 and Hypothesis 3.1 hold true. Let \( W \) (resp. \( V \)) be a viscosity subsolution (resp. supsolution) of (3.8) and there exists a constant \( \Lambda > 0 \) such that, for \((t, x), (s, y) \in [0, T] \times D,\)
\[ |W(t, x)| \vee |V(t, x)| \leq \Lambda (1 + |x(0)| + |x|_H), \] (5.1)
and
\[ |W(t, x) - W(s, y)| \vee |V(t, x) - V(s, y)| \leq \Lambda (1 + |x(0)| + |x|_H + |y(0)| + |y|_H) \]
\[ \times \left( |x(0) - y(0)| + |s - t|^{\frac{1}{2}} + \sup_{t \in [0, T]} \left| \int_0^{\tau}(x(\theta) - y(\theta))d\theta \right| \right). \] (5.2)
Then \( W \leq V. \)

From this theorem, the viscosity solution to HJB equation (3.8) can characterize the value function \( V(t, x) \) of our optimal control problem (3.1) and (3.2) as following:

**Theorem 5.2.** Let Hypothesis 2.1 and Hypothesis 3.1 hold true. Then the value function \( V \) defined by (3.3) is the unique viscosity solution of (3.8).

**Proof.** By Theorem 4.4, we know that \( V \) is a viscosity solution of (3.8). Thus, our conclusion follows from Theorem 3.2 and Theorem 5.1. \( \Box \)

We are now in a position of showing the proof the Theorem 5.1. We first note that for \( \delta > 0 \), the function defined by \( \tilde{W} := W - \frac{\delta}{T} \) is a subsolution of
\[
\begin{cases}
\frac{\partial}{\partial t} \tilde{W}(t, x) + S(\tilde{W})(t, x) + H(t, x, \nabla_x \tilde{W}(t, x)) = \frac{\delta}{T}, & t \in [0, T], \ x \in D, \\
\tilde{W}(T, x) = \phi(x(0)).
\end{cases}
\]
Since \( W \leq V \) follows from \( \bar{W} \leq V \) in the limit \( \delta \downarrow 0 \), it suffices to prove the theorem under the additional assumption:

\[
\frac{\partial}{\partial t} W(t, x) + S(W)(t, x) + H(t, x, \nabla_x W(t, x)) \geq c, \quad c := \frac{\delta}{T^2}, \quad t \in [0, T], \quad x \in D.
\]

**Proof of Theorem 5.1.** The proof of this theorem is rather long. Thus, we split it into several steps.

**Step 1.** Definition of auxiliary functions and sets.

We only need to prove that \( W(t, x) \leq V(t, x) \) for all \( (t, x) \in [T - \bar{a}, T) \times D \). Here

\[
\bar{a} = \frac{1}{8(1 + L)^2C^2} \wedge \frac{T}{2}, \quad \bar{C} = 1 + \tau|a|_{W,1.2} + \sup_{s \in [0,T]} |b(s)|.
\]

Then repeat the same procedure for cases \( [T - i\bar{a}, T - (i - 1)\bar{a}] \). To this end we assume to the contrary that there exists \( (\bar{t}, \bar{x}) \in [T - \bar{a}, T) \times D \), such that \( 2\bar{m} := W(\bar{t}, \bar{x}) - V(\bar{t}, \bar{x}) > 0 \). Since Lipschitz continuous functions are dense in \( H \), by (5.2) there exist a Lipschitz continuous function \( \bar{y} \) and \( \bar{a} \in \mathbb{R}^d \) such that \( W(\bar{t}, \bar{x}) - V(\bar{t}, \bar{x}) > \bar{m} \), where \( \bar{x} = \bar{y} + \bar{a}l_0(\cdot) \).

First, let \( \varepsilon > 0 \) be a small number such that

\[
W(\bar{t}, \bar{x}) - V(\bar{t}, \bar{x}) - 2\varepsilon \frac{\mu T - \bar{t}}{\mu T} (|\bar{x}|^2_H + |\bar{x}(0)|^2) > \frac{\bar{m}}{2},
\]

and

\[
\frac{9L^2\varepsilon}{8\mu T(1 + L)^2} \leq \frac{c}{2}, \quad (5.3)
\]

where

\[
\mu = 1 + \frac{1}{4T(1 + L)^2C^2}.
\]

Next, for every \( \alpha > 0 \) we define for any \( (t, x, s, y) \in [0, T] \times D \times [0, T] \times D \),

\[
\Psi(t, x, s, y) = W(t, x) - V(s, y) - \frac{\alpha}{2}d(t, s, x, y) - \varepsilon \frac{\mu T - t}{\mu T} (|x|^2_H + |x(0)|^2) - \varepsilon \frac{\mu T - s}{\mu T} (|y|^2_H + |y(0)|^2),
\]

where

\[
d(t, s, x, y) = |x(0) - y(0)|^2 + |x - y|^2_H + |s - t|^2.
\]

Finally, for every \( M > 0 \) satisfying \( \bar{x} \in \mathcal{D}_{Q^M} \), we define

\[
M_\alpha := \sup_{t, s \geq \bar{t}; x, y \in \mathcal{D}_{Q^M}} \Psi((\bar{t}, \bar{x}, \bar{t}, \bar{x}) \otimes (t, x, s, y)),
\]

and

\[
M_\alpha \geq M := \sup_{t \geq \bar{t}; x \in \mathcal{D}_{Q^M}} \Psi((\bar{t}, \bar{x}, \bar{t}, \bar{x}) \otimes (t, x, t, x)) \geq \frac{\bar{m}}{2},
\]

where

\[
Q^M := \{(x_1, x_2, \ldots, x_d)| |x_1|^2 + |x_2|^2 + \cdots + |x_d|^2 < M^2\}.
\]

**Step 2.** Properties of \( \Psi(t, x, s, y) \).

By the definition of \( M_\alpha \), we can fix \( (\bar{t}, \bar{x}), (\bar{s}, \bar{y}) \in [\bar{t}, T] \times \mathcal{D}_{Q^M} \) satisfying

\[
(\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (\bar{t}, \bar{x}, \bar{t}, \bar{x}) \otimes (\bar{t}, \bar{x}, \bar{s}, \bar{y}), \quad \Psi((\bar{t}, \bar{x}, \bar{t}, \bar{x}) \otimes (\bar{t}, \bar{x}, \bar{s}, \bar{y})) \leq \Psi((\bar{t}, \bar{x}, \bar{t}, \bar{x}) \otimes (t, x, t, x)) \quad \text{and} \quad \Psi((\bar{t}, \bar{x}, \bar{s}, \bar{y}) + \frac{T}{2} > M_\alpha.
\]
Now we can apply Lemma 4.1 to find \((\hat{t}, \hat{x}), (\hat{s}, \hat{y}) \in [T - \bar{a}, T] \times D_{Q^2}\) satisfying \((\hat{t}, \hat{x}, \hat{s}, \hat{y}) = (\bar{t}, \bar{x}, \bar{s}, \bar{y}) \otimes (\hat{t}, \hat{x}, \hat{s}, \hat{y})\) with \(\Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \geq \Psi(\bar{t}, \bar{x}, \bar{s}, \bar{y})\) such that

\[
\Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi((\hat{t}, \hat{x}, \hat{s}, \hat{y}) \otimes (t, x, s, y)), \quad t \geq \hat{t}, s \geq \hat{s}, x, y \in D_{Q^2}.
\]

We should note that the point \((\hat{t}, \hat{x}, \hat{s}, \hat{y})\) depends on \(\bar{t}, \bar{x}, \bar{s}, \bar{y}, \alpha, M\).

**Step 3.** For fixed \(M\), there exists a subsequence of \(\alpha\) still denoted by itself such that

\[
\frac{\alpha}{2} d(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \leq \frac{1}{\alpha} + |W(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y})| + |V(\hat{t}, \hat{x}) - V(\hat{s}, \hat{y})| \to 0 \text{ as } \alpha \to +\infty, \tag{5.4}
\]

and

\[
\alpha |b(\hat{t})\hat{x}(\tau) - b(\hat{s})\hat{y}(\tau)|^2 \to 0 \text{ as } \alpha \to +\infty. \tag{5.5}
\]

Let us show the above. We can check that

\[
\frac{\alpha}{2} d(\hat{t}, \hat{x}, \hat{s}, \hat{y}) + \frac{\alpha}{\mu T} \left( |\hat{x}|^2_H + |\hat{x}(0)|^2 \right) + \frac{\alpha}{\mu T} \left( |\hat{y}|^2_H + |\hat{y}(0)|^2 \right)
\]

\[
\leq \frac{1}{\alpha} + W(\hat{t}, \hat{x}) - V(\hat{s}, \hat{y}) - M_\alpha \leq \frac{1}{\alpha} + W(\hat{t}, \hat{x}) - V(\hat{s}, \hat{y}) - M_*
\]

\[
\leq \frac{1}{\alpha} + C - M_*, \tag{5.6}
\]

where \(C := 2\Lambda(1 + M + \tau^2 M)\). We also have that

\[
2M_* \leq \frac{2}{\alpha} + W(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y}) + W(\hat{s}, \hat{y}) - V(\hat{s}, \hat{y}) + W(\hat{t}, \hat{x}) - V(\hat{t}, \hat{x}) - V(\hat{s}, \hat{y})
\]

\[
- \alpha d(\hat{t}, \hat{x}, \hat{s}, \hat{y}) - 2 \frac{\alpha}{\mu T} \left( |\hat{x}|^2_H + |\hat{x}(0)|^2 \right) - 2 \frac{\alpha}{\mu T} \left( |\hat{y}|^2_H + |\hat{y}(0)|^2 \right)
\]

\[
\leq \frac{2}{\alpha} + |W(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y})| + |V(\hat{t}, \hat{x}) - V(\hat{s}, \hat{y})| + 2M_* - \alpha d(\hat{t}, \hat{x}, \hat{s}, \hat{y}).
\]

Thus

\[
\frac{\alpha}{2} d(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \leq \frac{1}{\alpha} + |W(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y})| + |V(\hat{t}, \hat{x}) - V(\hat{s}, \hat{y})|. \tag{5.7}
\]

By the definition of \(d\) and (5.6), we get that \(|\hat{x} - \hat{y}|_B \to 0\) as \(\alpha \to +\infty\). We note that \(|\hat{x}|_D \leq \hat{y}|_D \leq M\). Then, from the definition of \(B\), it follows that there exists a subsequence of \(\alpha\) still denoted by itself, such that

\[
\sup_{t \in [-\tau, 0]} \left| \int_0^t \hat{x}(\theta) - \hat{y}(\theta)d\theta \right| \to 0 \text{ as } \alpha \to +\infty.
\]

Combining (5.2) and (5.7) we see that (5.4) holds. On the other hand, by \(\hat{x} = \hat{y} + \hat{a}1_0(\cdot),\) \(0 < \hat{a} \leq \frac{\tau}{2},\)

\((\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (\hat{t}, \hat{x}, \hat{s}, \hat{y}) \otimes (\bar{t}, \bar{x}, \bar{s}, \bar{y})\) and \((\bar{t}, \bar{x}, \bar{s}, \bar{y}) = (\hat{t}, \hat{x}, \hat{s}, \hat{y}) \otimes (\bar{t}, \bar{x}, \bar{s}, \bar{y})\), we have that

\[
\hat{x}(\tau) \to \hat{y}(\hat{t} - \tau \to \bar{t}), \quad \hat{y}(\tau) \to \hat{y}(\hat{t} - \tau \to \bar{t}).
\]

Since \(b\) and \(\hat{y}\) are Lipschitz continuous, there exists a constant \(N > 0\) such that

\[
\alpha |b(\hat{t})\hat{x}(\tau) - b(\hat{s})\hat{y}(\tau)|^2 = \alpha |b(\hat{t})\hat{y}(\hat{t} - \tau \to \bar{t}) - b(\hat{s})\hat{y}(\hat{t} - \tau \to \bar{t})|^2 \leq N \alpha |\hat{t} - \hat{s}|^2.
\]

Then, (5.5) follows from (5.6).

**Step 4.** There exist \(N, M > 0\) such that \(\hat{t}, \hat{s} \in [\bar{t}, T]\) and \(\hat{x}(0), \hat{y}(0) \in Q^M\) for all \(\alpha \geq N\).

First, we note that, for any \(\alpha > 0\), there exists a \(M > 0\) be large enough such that

\[
\Psi(\hat{t}, \hat{x}, \hat{t}, \hat{x}) = W(\hat{t}, \hat{x}) - V(\hat{t}, \hat{x}) - 2 \frac{\alpha}{\mu T} \frac{\hat{t}}{(|\hat{x}|^2_H + |\hat{x}(0)|^2)}
\]
\[
W(t, x) - V(s, y) - \varepsilon \frac{\mu T - t}{\mu T} M^2 - \varepsilon \frac{\mu T - s}{\mu T} M^2 \\
\geq \Psi(t, x, s, y),
\]
where \((t, x, s, y) \in [0, T] \times \mathcal{D} \times [0, T] \times \mathcal{D}
\text{ and } |x(0)| = |y(0)| = M
\text{. Therefore, for this } M > 0, \text{ we have that } \hat{x}(0), \hat{y}(0) \in Q^M
\text{ for every } \alpha > 0.

Next, by (5.4), we can let \(N > 0\) be a large number such that
\[
\frac{\alpha}{2} d(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \leq \frac{1}{\alpha} + |W(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y})| + |V(\hat{t}, \hat{x}) - V(\hat{s}, \hat{y})| \leq \frac{1}{4}(\bar{m} \land c),
\]
for all \(\alpha \geq N\). Moreover, we have \(\hat{t}, \hat{s} \in [\hat{t}, T]\) for all \(\alpha \geq N\).
Indeed, if say \(\hat{s} = T\), then we will deduce the following contradiction:
\[
\frac{\bar{m}}{2} \leq M_\alpha \leq \frac{1}{\alpha} + \Psi(\hat{t}, \hat{s}, \hat{x}, \hat{y})
\leq \frac{1}{\alpha} + W(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y}) + W(\hat{s}, \hat{y}) - V(\hat{s}, \hat{y})
\leq \frac{1}{\alpha} + |W(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y})| \leq \frac{\bar{m}}{4}.
\]

Step 5. Completion of the proof.

From above all, for the fixed \(N, M > 0\) in step 4, we find \((\hat{t}, \hat{x}), (\hat{s}, \hat{y}) \in [\hat{t}, T] \times \mathcal{D}_{Q^M} \text{ satisfying } \hat{x}(0), \hat{y}(0) \in Q^M \text{ for all } \alpha \geq N \text{ and } (\tilde{t}, \tilde{x}, \tilde{s}, \tilde{y}) = (\hat{t}, \hat{x}, \hat{s}, \hat{y}) \otimes (\hat{t}, \hat{x}, \hat{s}, \hat{y}) \text{ with } \Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \text{ such that}
\[
\Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi((\hat{t}, \hat{x}, \hat{s}, \hat{y}) \odot (t, x, s, y)), \ t \geq \hat{t}, \ s \geq \hat{s}, x, y \in \mathcal{D}_{Q^M}. \quad (5.8)
\]
Then
\[
\Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi((\hat{t}, \hat{x}) \odot (t, x), \hat{s}, \hat{y}), \ t \geq \hat{t}, x \in \mathcal{D}_{Q^M}.
\]
Thus, by the definition of the viscosity subsolution, we get that
\[
\alpha(\hat{t} - \hat{s}) - \varepsilon \frac{\mu T - \hat{t}}{\mu T} (|\hat{x}(0)|^2 + |\hat{y}(0)|^2) + \alpha \frac{1}{2} (B(\hat{y} - \hat{\hat{x}}), \hat{x}(0)1_{[-\tau, 0]} - \hat{x})_H + \varepsilon \frac{\mu T - \hat{t}}{\mu T} (|\hat{x}(0)|^2 - |\hat{x}(-\tau)|^2)
\]
\[
+ H(\hat{t}, \hat{x}, [\alpha(\hat{x}(0) - \hat{y}(0))] + 2\varepsilon \frac{\mu T - \hat{t}}{\mu T} \hat{x}(0)1_0(t)) \geq c, \quad (5.9)
\]
Also, by (5.8), we have
\[
\Psi(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi((\hat{t}, \hat{x}, (s, y) \odot (s, y)), \ s \geq \hat{s}, y \in \mathcal{D}_{Q^M}.
\]
Thus, we obtain
\[
\alpha(\hat{t} - \hat{s}) + \frac{\varepsilon}{\mu T} (|\hat{y}|^2_H + |\hat{y}(0)|^2) - \alpha \frac{1}{2} (B(\hat{y} - \hat{x}), \hat{y}(0)1_{[-\tau, 0]} - \hat{y})_H - \varepsilon \frac{\mu T - \hat{s}}{\mu T} (|\hat{y}(0)|^2 - |\hat{y}(-\tau)|^2)
\]
\[
+ H(\hat{s}, \hat{y}, [-\alpha(\hat{y}(0) - \hat{x}(0))] - 2\varepsilon \frac{\mu T - \hat{s}}{\mu T} \hat{y}(0)1_0(t)) \leq 0. \quad (5.10)
\]
Combining (5.9) and (5.10), we obtain
\[
c + \frac{\varepsilon}{\mu T} (|\hat{x}|^2_H + |\hat{x}(0)|^2 + |\hat{y}|^2_H + |\hat{y}(0)|^2) + \varepsilon \frac{\mu T - \hat{t}}{\mu T} |\hat{x}(-\tau)|^2 + \varepsilon \frac{\mu T - \hat{s}}{\mu T} |\hat{y}(-\tau)|^2
\]
\[
\leq \frac{\alpha}{2} (B(\hat{x} - \hat{y}, \hat{x}(0)1_{[-\tau, 0]} - \hat{x} - \hat{y}(0)1_{[-\tau, 0]} + \hat{y}) + \varepsilon \frac{\mu T - \hat{t}}{\mu T} |\hat{x}(0)|^2 + \varepsilon \frac{\mu T - \hat{s}}{\mu T} |\hat{y}(0)|^2
\]
\[ + \dot{H}(\hat{t}, \hat{x}, [\alpha(\hat{x}(0) - \hat{y}(0))] + 2\varepsilon \frac{\mu T - \hat{t}}{\mu T} \hat{x}(0)) \]
\[ - \dot{H}(\hat{s}, \hat{y}, [\alpha(\hat{x}(0) - \hat{y}(0))] - 2\varepsilon \frac{\mu T - \hat{s}}{\mu T} \hat{y}(0)) \] \tag{5.11}

On the other hand, by simple calculation we obtain
\[
|H(\hat{t}, \hat{x}, [\alpha(\hat{x}(0) - \hat{y}(0))] + 2\varepsilon \frac{\mu T - \hat{t}}{\mu T} \hat{x}(0))| \leq |\sup_{\hat{u} \in U} (J_1 + J_2)|, \tag{5.12}
\]

where
\[
J_1 = \langle F(\hat{t}, \hat{x}(0), (a, \hat{x})_H, \nu) + b(\hat{t})\hat{x}(t) - \alpha(\hat{x}(0) - \hat{y}(0)) + 2\varepsilon \frac{\mu T - \hat{t}}{\mu T} \hat{x}(0) \rangle
\]
\[ - \langle F(\hat{s}, \hat{y}(0), (a, \hat{x})_H, \nu) + b(\hat{s})\hat{y}(t) - \alpha(\hat{x}(0) - \hat{y}(0)) - 2\varepsilon \frac{\mu T - \hat{s}}{\mu T} \hat{y}(0) \rangle \]
\[ \leq \alpha L(|\hat{x}(0) - \hat{y}(0)|^2 + |\hat{x}(0) - \hat{y}(0)||a|_{W^{1,1}}|\hat{x} - \hat{y}|_B + |b(\hat{t})\hat{x}(t) - b(\hat{s})\hat{y}(t)| + |\hat{t} - \hat{s}|) \]
\[ + 2\varepsilon L \left( \frac{\mu T - \hat{t}}{\mu T} \right)||\hat{x}(0)|(1 + \sup_{s \in [0,T]} |b(s)||\hat{x}(t) - \hat{y}(t)|) + |\hat{x}(0)| + |a|_{W^{1,1}}|\hat{x}|_B \]
\[ + 2\varepsilon L \left( \frac{\mu T - \hat{s}}{\mu T} \right)||\hat{y}(0)|(1 + \sup_{s \in [0,T]} |b(s)||\hat{y}(t) - \hat{y}(t)| + |\hat{y}(0)| + |a|_{W^{1,1}}|\hat{y}|_B); \tag{5.13}\]

and
\[
J_2 = q(\hat{t}, \hat{x}(0), \nu) - q(\hat{s}, \hat{y}(0), \nu) \leq L|\hat{x}(0) - \hat{y}(0)| + \rho(|\hat{t} - \hat{s}|, M). \tag{5.14}\]

Combining (5.11)-(5.14), we get
\[
c \leq - \frac{\varepsilon}{\mu T}(|\hat{x}_v|^2 + |\hat{x}(0)|^2 + |\hat{y}_v|^2 + |\hat{y}(0)|^2) - \varepsilon \frac{\mu T - \hat{t}}{\mu T} |\hat{x}(t) - \hat{y}(t)|^2 - \varepsilon \frac{\mu T - \hat{s}}{\mu T} |\hat{y}(t) - \hat{y}(t)|^2
\]
\[ + \frac{\alpha}{4} |(\hat{x} - \hat{y}|^2 + |\hat{x}(0) - \hat{y}(0)|^2) + \varepsilon \frac{\mu T - \hat{t}}{\mu T} |\hat{x}(0)|^2 + \varepsilon \frac{\mu T - \hat{s}}{\mu T} |\hat{y}(0)|^2
\]
\[ + \alpha L(3|\hat{x}(0) - \hat{y}(0)|^2 + |a|_{W^{1,1}}|\hat{x} - \hat{y}|_B + |b(\hat{t})\hat{x}(t) - b(\hat{s})\hat{y}(t)|^2 + |\hat{t} - \hat{s}|^2)
\]
\[ + 2\varepsilon L \left( \frac{\mu T - \hat{t}}{\mu T} \right)||\hat{x}(0)|(1 + \sup_{s \in [0,T]} |b(s)||\hat{x}(t) - \hat{y}(t)|) + |\hat{x}(0)| + |a|_{W^{1,1}}|\hat{x}|_B \]
\[ + 2\varepsilon L \left( \frac{\mu T - \hat{s}}{\mu T} \right)||\hat{y}(0)|(1 + \sup_{s \in [0,T]} |b(s)||\hat{y}(t) - \hat{y}(t)| + |\hat{y}(0)| + |a|_{W^{1,1}}|\hat{y}|_B
\]
\[ + L|\hat{x}(0) - \hat{y}(0)| + \rho(|\hat{t} - \hat{s}|, M). \]

Recalling \( \bar{a} = \frac{1}{s(1+L)^2} \) and \( \bar{\nu} = 1 + \frac{1}{4(1+L)^2}\bar{c}^2 \) and \( \hat{t}, \hat{s} \in [T - \bar{a}, T) \), we show that
\[
c \leq \frac{\varepsilon}{4\mu T(1 + L)^2} \left( \frac{1}{C} |\hat{x}(t) - \hat{y}(t)|^2 + (2(1 + L)|\hat{x} - \hat{y}(0)||\hat{x}(0)|^2 + (\hat{x}(0) - \frac{3L}{2})^2 \right)
\]
\[ - \frac{\varepsilon}{4\mu T(1 + L)^2} \left( \frac{1}{C} |\hat{y}(t) - \hat{y}(0)|^2 + (2(1 + L)|\hat{y} - \hat{y}(0)||\hat{y}(0)|^2 + (\hat{y}(0) - \frac{3L}{2})^2 \right)
\]
\[ + \frac{9L^2 \varepsilon}{8\mu T(1 + L)^2} + \alpha(3L + \tau)|\hat{x}(0) - \hat{y}(0)|^2 + |b(\hat{t})\hat{x}(t) - b(\hat{s})\hat{y}(t)|^2 + |\hat{t} - \hat{s}|^2
\]
\[ + \alpha(1 + \bar{L}C^2)|\hat{x} - \hat{y}|_B + L|\hat{x}(0) - \hat{y}(0)| + \rho(|\hat{t} - \hat{s}|, M)
\]
\[ \leq \frac{9L^2 \varepsilon}{8\mu T(1 + L)^2} + \alpha(3L + \tau)|\hat{x}(0) - \hat{y}(0)|^2 + |b(\hat{t})\hat{x}(t) - b(\hat{s})\hat{y}(t)|^2 + |\hat{t} - \hat{s}|^2
\]
\[ + \alpha(1 + \bar{L}C^2)|\hat{x} - \hat{y}|_B + L|\hat{x}(0) - \hat{y}(0)| + \rho(|\hat{t} - \hat{s}|, M). \]
Letting $\alpha \to +\infty$, it follows from (5.3) that

$$c \leq \frac{c}{2},$$

which induces a contradiction. The proof is completed. ☐

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