LAURENT POLYNOMIAL SOLUTIONS OF THE BOUNDARY QUANTUM KNIZHNIK–ZAMOLODCHIKOV EQUATION

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Abstract. We construct Laurent polynomial solutions of the boundary quantum Knizhnik–Zamolodchikov equation for $U_q(\widehat{\mathfrak{sl}}_2)$ on the parabolic Kazhdan–Lusztig bases. They are characterized by non-symmetric Koornwinder polynomials with the specialized parameters. As a special case, we obtain the solution of the minimal degree.

1. Introduction

The quantum Knizhnik–Zamolodchikov (qKZ) equation is a system of first order $q$-difference equations [40], which is satisfied by the matrix coefficients of the products of the vertex operators in the representation theory of quantum affine algebras [18]. It was shown that the correlation functions for XXZ spin chain with quasi-periodic boundary conditions satisfy the qKZ equation [21, 22]. The qKZ equations reappeared in the Razumov–Stroganov correspondence [16, 35, 34]: the solution is the ground state of the XXZ spin chain at $q$ root of unity. The qKZ equations for arbitrary root system were obtained by Cherednik [5, 6]. We call them boundary (or reflection) qKZ equations. In the case of type A, one can construct a solution by using multidimensional integrals of hypergeometric type [31, 32, 42]. Recently, polynomial solutions were constructed by using the representation theory of the affine Hecke algebra. They are level one for $U_q(\widehat{\mathfrak{sl}}_n)$ [16, 17], level $-1/2$ for $U_q(\widehat{\mathfrak{sl}}_2)$ [25] and level $\frac{k+1}{2}$ for $U_q(\widehat{\mathfrak{sl}}_2)$ [26].

The integrability of a physical system with boundaries is ensured by the Yang–Baxter equation [1, 43] and the reflection (or boundary Yang–Baxter) equation [7, 39]. The boundary qKZ equations are classified into three classes depending on the choice of $K$-matrices (or boundary $R$-matrices). The first class is the cases where two $K$-matrices are diagonal. The boundary qKZ of this class is discussed in [12, 15, 19, 20]. The second class is the case where one $K$-matrix is diagonal and the other is not diagonal. This class is discussed in [12, 44]. The third class is the case where two $K$-matrices are not diagonal. In the case of $U_q(\widehat{\mathfrak{sl}}_2)$, the boundary qKZ equation of the third class has eight parameters: three Hecke parameters $q, q_0, q_N$, the shift parameter $s$, the Baxterization parameter $\zeta_0, \zeta_N$ and two more parameters $\kappa_0, \kappa_N$. The Laurent polynomial solutions in the case of $q_0 = q_N = \sqrt{-1}q^{1/2}$ is discussed in [4, 10]. In this paper, we consider the boundary qKZ equations of the second and the third classes for $U_q(\widehat{\mathfrak{sl}}_2)$. We call the boundary qKZ equation of the second (resp. third) class the one-boundary (resp. two-boundary) case.

We construct Laurent polynomial solutions of the boundary qKZ equations on the parabolic Kazhdan–Lusztig bases for the Hermitian symmetric pair $(B_N, A_{N-1})$ [2, 3, 38] (see also [27, 13]). The boundary qKZ equation can be regarded as compatibility conditions of the two representations: a representation by Kazhdan–Lusztig bases and a polynomial representation of the affine Hecke algebra. The affine Hecke algebra consists of $T_i, 0 \leq i \leq N$, and $Y_i, 1 \leq i \leq N$ (see Section 2). The Cherednik–Noumi [33] $Y$-operators commute with each other and generalize Dunkl operators.

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In the polynomial representation, the non-symmetric Macdonald–Koornwinder polynomials \([28]\) are the simultaneous eigenfunctions of the operators \(Y_i\) \([36]\). The (double) affine Hecke algebra of type \((C_N^+ , C_N)\) with \(N \geq 2\) has six parameters: \(q,q_0,q_N,s,\zeta_0\) and \(\zeta_N\). The appearance of the non-symmetric Koornwinder polynomial for generic parameters in the theory of the XXZ spin chain is discussed in \([41]\). The polynomial representations for specialized parameters are exposed in \([24]\). In this paper, we consider the specialization \(q^{2(k+1)}s^{2(r-1)} = 1\) where

\[
(1) \quad (k, r) = \begin{cases} 
(1, r+1), & \text{for two-boundary case}, \\
(2, 2r+1), & \text{for one-boundary case},
\end{cases}
\]

where \(r \in \mathbb{N}_+\). There, the polynomial representation of the double affine Hecke algebra can be non-\(Y\)-semisimple or reducible. In the case of \(q^{2(k+1)}s^{2(r-1)} = 1\), a polynomial representation considered in this paper is characterized by the so-called wheel conditions \([14,23]\). On the other hand, there exists a Kazhdan–Lusztig basis such that the basis is a simultaneous eigenfunction of the \(Y\)-operators. Thus we can identify the two representations by finding the non-symmetric Koornwinder polynomial associated with the Kazhdan–Lusztig basis. If the parameters satisfy

\[
q_N^2 = -q, \quad \omega_m^{\pm mJ/r} q^{\mp(N-1+4J/r)} (q_0 q_N)^{\pm 1} = \kappa_0 \kappa_N, \quad \text{for two-boundary case}
\]

where \(\omega_m\) is a primitive \(m\)-th root of unity, \(m = \text{GCD}(k+1,r-1)\) and \(J, r \in \mathbb{N}_+\), we have a Laurent polynomial solution characterized by a non-symmetric Koornwinder polynomial with the specialized parameters. As a special case, we obtain the solution of the minimal degree.

The plan of the paper is as follows. We briefly review the affine Hecke algebra and the two-boundary Temperley–Lieb algebra in Section 2. In Section 3, we summarize the representations of the one- and two-boundary Temperley–Lieb algebra on the standard bases and on the parabolic Kazhdan–Lusztig bases. We collect the definitions and properties of non-symmetric Koornwinder polynomials under the specialization (1). In section 4, we recall the boundary qKZ equations for one- and two-boundary cases. In Section 5, we consider the reduction of the boundary qKZ equation. Since the boundary qKZ equations contain many equivalent equations, we extract non-trivial equations from them. We also show the conditions which a solution of the boundary qKZ equation satisfies. In Section 6, we construct Laurent polynomial solutions of the boundary qKZ equation by combining the results of Section 3 and Section 5.

2. Algebras

2.1. Affine Hecke algebra. The affine root system of type \(C_N\) can be realized in \(\mathbb{R}^N \oplus \mathbb{R}\delta\) where \(\delta\) is a radical element. The affine simple roots are given by

\[
\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad (1 \leq i \leq N - 1), \quad \alpha_N = 2\epsilon_N, \quad \alpha_0 = \delta - 2\epsilon_1,
\]

where \(\epsilon_i\) is the standard orthonormal bases satisfying \(\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}\). We denote by \(\alpha_i^\vee := 2\alpha_i/\langle \alpha_i, \alpha_i \rangle\) simple co-roots.

Let \(W_0 := \langle s_1, \ldots, s_N \rangle\) and \(W := \langle s_0, \ldots, s_N \rangle\) be the finite and affine Weyl group of type \(C_N\) respectively. The faithful action of \(W\) on \(\mathbb{R}^N\) is given by

\[
s_0 \cdot (v_1, \ldots, v_N) = (-1 - v_1, v_2, \ldots, v_N),
\]

\[
s_i \cdot (v_1, \ldots, v_N) = (v_1, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \ldots, v_N), \quad 1 \leq i \leq N - 1,
\]

\[
s_N \cdot (v_1, \ldots, v_N) = (v_1, \ldots, v_{N-1}, -v_N),
\]
The \( s \)-dependent action of the affine Weyl group on \( z := (z_1, \ldots, z_N) \in (\mathbb{C}^*)^N \) is given by

\[
\begin{align*}
    s_0 z &= (s^2 z_1^{-1}, z_2, \ldots, z_N), \\
    s_i z &= (z_1, \ldots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \ldots, z_N), \quad 1 \leq i \leq N - 1, \\
    s_N z &= (z_1, \ldots, z_{N-1}, z_N^{-1}).
\end{align*}
\]

The affine Hecke algebra \( \mathcal{H}_N \) is the unital associative \( \mathbb{C}(q, q_0, q_N) \)-algebra generated by \( T_i \) (0 \leq i \leq N) and \( Y_j \) (1 \leq j \leq N) satisfying the relations:

\[
\begin{align*}
    (T_i + q)(T_i - q^{-1}) &= 0, \quad 1 \leq i \leq N - 1, \\
    (T_N + q_N)(T_N - q_N^{-1}) &= 0, \\
    (T_0 + q_0)(T_0 - q_0^{-1}) &= 0, \\
    T_0T_1T_0 &= T_1T_0T_1, \\
    T_N T_{N-1} T_N T_N &= T_N^{-1} T_N T_{N-1} T_N, \\
    T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq N - 2, \\
    T_i T_j &= T_j T_i, \quad |i - j| > 1, \\
    Y_i Y_j &= Y_j Y_i, \quad \forall i, j, \\
    T_i Y_j &= Y_j T_i, \quad (\alpha_i, \epsilon_j) = 0, \\
    T_i Y_{i+1} T_i &= Y_i, \quad 1 \leq i \leq N - 1, \\
    T_N^{-1} Y_N &= Y_N^{-1} T_N - (q_0 - q_0^{-1}).
\end{align*}
\]

By the Bernstein–Zelevinsky presentation of the affine Hecke algebra (see, e.g., [29]), we have the correspondence

\[
Y_i \mapsto T_i \ldots T_{N-1} T_N \ldots T_0^{-1} T_1^{-1} \ldots T_{i-1}^{-1}.
\]

We denote by \( \mathcal{H}_N^0 \) the Hecke algebra of type B generated by \( T_i, 1 \leq i \leq N \).

### 2.2. Two-boundary Temperley–Lieb algebra.

The \emph{two-boundary Temperley–Lieb algebra} [9, 11] is a unital associative algebra over \( \mathbb{C}(q, q_0, q_N, \kappa_0, \kappa_N) \) generated by \( e_i \), 0 \leq i \leq N, with the relations

\[
\begin{align*}
    e_i^2 &= -(q_i + q_i^{-1}) e_i, \quad 0 \leq i \leq N, \\
    e_i e_{i+1} &= a_{i,i+1} e_i, \quad 1 \leq i \leq N - 1, \\
    e_i e_j &= e_j e_i, \quad |i - j| > 1,
\end{align*}
\]

where \( q_1 = q_2 = \ldots = q_{N-1} := q \) and

\[
(3) \quad a_{i,j} := \begin{cases} \kappa_N (qq_N^{-1} + q^{-1} q_N), & (i, j) = (N - 1, N), \\ \kappa_0 (qq_0^{-1} + q^{-1} q_0), & (i, j) = (1, 0), \\ 1, & \text{otherwise.} \end{cases}
\]

The two-boundary Temperley–Lieb algebra is infinite dimensional. We consider the following two conditions to make the algebra finite dimensional:

\[
I_N J_N I_N = \alpha I_N, \quad J_N I_N J_N = \alpha J_N
\]
where

\[
\begin{align*}
I_{2n} &:= \prod_{i=0}^{n-1} e_{2i+1}, \quad I_{2n+1} := e_0 \prod_{i=1}^{n} e_{2i}, \\
J_{2n} &:= e_0 \prod_{i=1}^{n-1} e_{2i} \cdot e_N, \quad J_{2n+1} := \prod_{i=0}^{n-1} e_{2i+1} \cdot e_N,
\end{align*}
\]

\[
\alpha = \begin{cases} 
(k_N^{-1} + \kappa_0 q_0 q_N^{-1})(\kappa_0^{-1} + \kappa_N q_0^{-1} q_N), & \text{for } N \text{ odd}, \\
(k_N^{-1} q_0^{-1} q_N^{-1} q_0 - \kappa_0^{-1} q_N)(\kappa_N q_N - \kappa_0^{-1} q_0^{-1} q_N), & \text{for } N \text{ even}.
\end{cases}
\]

The subalgebra generated by \( e_1, \ldots, e_N \) is the one-boundary Temperley–Lieb algebra.

3. Representations

3.1. Two-boundary Temperley–Lieb algebra. Denote \( V \cong \mathbb{C}^2 \) be a \( \mathbb{C} \)-vector space with an ordered basis \( (v_+, v_-) \). We consider the lexicographic order of basis in \( V^\otimes m \). For example, the ordered bases in \( V^\otimes 2 \) are \( (v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-) \). Let \( \epsilon \in \{+,-\}^N \) be a binary string of length \( N \). We abbreviate by \( v_\epsilon \) a basis \( v_{\epsilon_1} \otimes \ldots \otimes v_{\epsilon_N} \).

The two-boundary Temperley–Lieb algebra has a natural representation in \( \text{End}_\mathbb{C}(V^\otimes N) \) (see, e.g., [9] and references therein). This representation has two parameters \( \kappa_0 \) and \( \kappa_N \). The matrix representation of the generators are

\[
e_i = \begin{cases} 
\epsilon \otimes \ldots \otimes 1 \otimes \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -q^{-1} & 1 & 0 \\
0 & 1 & -q & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \otimes 1 \otimes \ldots \otimes 1, & 1 \leq i \leq N-1,
\end{cases}
\]

\[
e_N = \begin{cases} 
\epsilon \otimes \ldots \otimes 1 \otimes \begin{pmatrix}
-q_0^{-1} & \kappa_0^{-1} \\
\kappa_N & -q_N
\end{pmatrix},
\end{cases}
\]

\[
e_0 = \begin{cases} 
\begin{pmatrix}
-q_0 & \kappa_0^{-1} \\
\kappa_0 & -q_0^{-1}
\end{pmatrix} \otimes 1 \otimes \ldots \otimes 1.
\end{cases}
\]

We define the \( R \)-matrix acting on \( V \otimes V \) by

\[
R_i(z) := \frac{q z - q^{-1}}{q - q^{-1} z} 1 + \frac{z - 1}{q - q^{-1} z} e_i.
\]

The \( R \)-matrix satisfies the unitarity condition \( R(z) R(1/z) = 1 \) and the Yang–Baxter equation [1, 43]:

\[
R_i(z) R_{i+1}(z w) R_i(w) = R_{i+1}(w) R_i(z w) R_{i+1}(z).
\]

Similarly, we define the \( K \)-matrices (or boundary \( R \)-matrices) as

\[
K_N(z) := \frac{(z + q_N z_N)(z - q_N z_N^{-1})(1 + q_N z_N z)(1 - q_N z_N^{-1} z)}{(1 + q_N z_N z)(1 - q_N z_N^{-1} z)} 1 + \frac{q_N(z^2 - 1)}{(1 + q_N z_N z)(1 - q_N z_N^{-1} z)} e_N,
\]

\[
K_0(z) := \frac{(z^{-1} + q_0 z_0)(z^{-1} - q_0 z_0^{-1})(1 + q_0 z_0 z^{-1})(1 - q_0 z_0^{-1} z^{-1})}{(1 + q_0 z_0 z^{-1})(1 - q_0 z_0^{-1} z^{-1})} 1 + \frac{q_0(z^2 - 1)}{(1 + q_0 z_0 z^{-1})(1 - q_0 z_0^{-1} z^{-1})} e_0.
\]
where $\zeta_0$ and $\zeta_N$ are free parameters which appear by the Baxterization. These $K$-matrices satisfy the unitarity equation $K_0(z)K_0(1/z) = 1 = K_N(z)K_N(1/z)$ and the boundary Yang–Baxter (or reflection) equations [7, 39]:

$$K_N(w)\bar{R}_{N-1}(1/(zw))K_N(z)\bar{R}_{N-1}(w/z) = \bar{R}_{N-1}(w/z)K_N(z)\bar{R}_{N-1}(1/(wz))K_N(w),$$

$$K_0(z)\bar{R}_{1}(zw)K_0(w)\bar{R}_{1}(w/z) = \bar{R}_{1}(w/z)K_0(w)\bar{R}_{1}(zw)K_0(z).$$

3.2. Kazhdan–Lusztig bases. We consider the representation of the affine Hecke algebra in $V^\otimes N$. The two-boundary Temperley–Lieb algebra can be regarded as the affine Hecke algebra with quotient relations through $T_i \mapsto e_i + q_i^{-1}$. We will define three types of (parabolic) Kazhdan–Lusztig bases of $\mathcal{H}_N^0$ following [13, 27, 30, 38]. We call these bases type BI, BII and BIII respectively. The Hecke parameters satisfy $q_N = q^M$, $M \in \mathbb{Z}_{\geq 1}$, for type BI and $q_N$ and $q$ are algebraically independent for type BII and BIII. We consider the abelian groups $A' := \{q^i\kappa_N^j|i \in \mathbb{Z}, j \in \mathbb{Z}_{\geq 0}\}$ for type BI and $A^{II} := \{q^i q_N^j\kappa_N^k|i, j \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}\} =: A^{III}$ for type BII and BIII. The lexicographic order of $A^X$ ($X=I, II, III$) is defined by $A_X = A^{I}_+ \cup \{\kappa_N^i|i \in \mathbb{Z}_{\geq 0}\} \cup A_X^-$ where

$$A^I_+ := \{q^i\kappa_N^j|i > 0, j \geq 0\},$$

$$A^{II}_+ := \{q^i q_N^j\kappa_N^k|i > 0, j \in \mathbb{Z}, k \geq 0\} \cup \{q^i q_N^j\kappa_N^k|i > 0, j \geq 0\},$$

$$A^{III}_+ := \{q^i q_N^j\kappa_N^k|i \in \mathbb{Z}, j > 0, k \geq 0\} \cup \{q^i q_N^j\kappa_N^k|i > 0, j \geq 0\}.$$

We define the involutive ring automorphism of $\mathcal{H}_N^0$, known as the bar involution, via $\overline{T_i} = T_i^{-1}$, $\overline{q_i} = q_i^{-1}$ for $1 \leq i \leq N$. The action of $\mathcal{H}_N^0$ on $V^\otimes N$ can be realized by $T_i \mapsto e_i + q_i^{-1}$ for $0 \leq i \leq N$. On the module $V^\otimes N$, we define $\overline{\kappa_N} = \kappa_N$ and $\overline{b_+} = b_+$ where $b_+ = (\ldots +)$.

The (parabolic) Kazhdan–Lusztig bases $C^X_\epsilon$, $\epsilon \in \{\pm\}^N$, $X = I, II$ or III, satisfy

1. $C^X_\epsilon = C^X_{\epsilon'}$
2. $C^X_\epsilon = v_+ + \sum_{\epsilon' < \epsilon} \mathbb{Z}(A^X)v_{\epsilon'}$ for $X = I, II$ or III.

If we set $\kappa_N = 1$, we have standard parabolic Kazhdan–Lusztig bases studied in [38].

We briefly review a graphical presentation of Kazhdan–Lusztig bases following [38]. Let $b$ be a binary string $b \in \{\pm\}^N$. We place an up arrow (resp. a down arrow) from left to right according to $b_i = +$ (resp. $b_i = -$). First we make a pair between a down arrow and an up arrow which are next to each other and in this order. We connect this pair of arrows via an arc. Repeat this procedure until all the up arrows are to the left to down arrows. We have three cases according to the type of the lexicographic order of $A^X$.

**Type BI.** We put an integer $p$, $1 \leq p \leq M$, on the $(M+1-p)$-th down arrow from right. For remaining down arrows, we make a pair of adjacent down arrows from right to left. Then we connect the pair via a dashed arc.

**Type BII.** We put a vertical line with a mark $e$ (resp. $o$) on the $2i$-th (resp. $(2i-1)$-th) down arrow from right.

**Type BIII.** We put a vertical line with a circled integer $i$ on the $i$-th down arrow from right to left.
A partial diagram corresponds to a vector in $V^\otimes N$ as follows:

\[
\begin{align*}
\bigcup & = v_{-+} - q^{-1}v_{++}, \\
\bigcap' & = v_{--} - \kappa_Nq^{-1}v_{++}, \\
& \quad = v_- - \kappa_Nq^{-p}v_+, \\
& \quad = v_- - \kappa_Nq_N^{-1}v_+, \\
& \quad = v_- + \kappa_Nq^{-1}q_Nv_+, \\
& \quad = v_- - \kappa_Nq^{p-1}q_N^{-1}v_+
\end{align*}
\]

and an unpaired up (resp. down) arrow corresponds to $v_+$ (resp. $v_-$). Since a diagram for $b \in \{\pm\}^N$ can be regarded as a tensor product of the building blocks, we obtain a vector in $V^\otimes N$ by tensoring the above expressions.

**Example 3.1.** Let $\epsilon = (- - + +)$.

\[
\begin{align*}
C^I_\epsilon &= \bigcup_{N} \bigcup_{1} = v_{-++} - q^{-1}v_{++-} - \kappa_Nq^{-1}v_{-++} + \kappa_Nq^{-2}v_{++-}, \\
C^II_\epsilon &= \bigcup_{e} \bigcup_{o} = v_{-++} - q^{-1}v_{++-} - \kappa_Nq_N^{-1}v_{-++} + \kappa_Nq^{-1}q_Nv_{++-} + \kappa_Nq^{-2}q_Nv_{++-}, \\
C^III_\epsilon &= \bigcup_{2} \bigcup_{1} = v_{-++} - q^{-1}v_{++-} - \kappa_Nq_N^{-1}v_{-++} + \kappa_Nq^{-1}q_Nv_{++-} + \kappa_Nq^{-2}q_Nv_{++-} + \kappa_Nq_N^{-2}v_{++-}.
\end{align*}
\]

The actions of the two-boundary Temperley–Lieb algebra in the case of $\kappa_0, \kappa_N = 1$ are summarized in [37, Section 3]. Due to the existence of $\kappa_0$ and $\kappa_N$, a slight modification is necessary. For example, the action of $e_i$, $1 \leq i \leq N - 1$, on the Kazhdan–Lusztig basis of type BII is given by

\[
\begin{align*}
e_i\left(\begin{array}{c}
\otimes \\
\end{array}\right) & = k_N(qq_N^{-1} + q^{-1}q_N) \bigcup, \\
e_i\left(\begin{array}{c}
\otimes \\
\end{array}\right) & = -k_N(q_N + q_N^{-1}) \bigcup, \\
e_N\left(\begin{array}{c}
\bigcup \\
\end{array}\right) & = k_N^{-1} \begin{array}{c}
\otimes \\
\end{array}.
\end{align*}
\]

### 3.3. One-boundary Temperley–Lieb algebra

The space $V^\otimes N$ is reducible as a representation of the one-boundary Temperley–Lieb algebra. Let $B_N$ be a set of binary strings $b = (b_1 \ldots b_N) \in$
$\{\pm\}^N$ satisfying $\sum_{j=1}^j b_j \leq 0$ for all $1 \leq j \leq N$. Let $L_N$ be a vector space spanned by $\{C^I_\epsilon | \epsilon \in B_N\}$. Then, the space $L_N$ is irreducible as a representation of the one-boundary Temperley–Lieb algebra.

The dimension of $L_N$ is $\dim(L_N) = \binom{N}{N/2}$.

In the case of Type BII, a basis $C^I_\epsilon \in L_N$ is studied as a link pattern with a boundary in literatures (see, e.g., [8, 44]).

### 3.4. The non-symmetric Koornwinder polynomials.

Let $K := C(s, q, q_N, q_0, \zeta_N, \zeta_0)$ and $P_N := K[z_1^\pm 1, \ldots, z_N^\pm 1]$ be the ring of $N$-variable Laurent polynomials. Following [33], we define linear operators $\hat{T}_0, \ldots, \hat{T}_N$ on $P_N$ by

$$
\hat{T}_0 := -q_0 - q_0^{-1} \frac{(1 - sq_0\zeta_0^{-1}z_1^{-1})(1 + sq_0\zeta_0^{-1})}{1 - s^2z_1^{-2}}(s_0 - 1)
$$

$$
\hat{T}_i := -q - q^{-1} \frac{1 - q^2z_i z_i^{-1}}{1 - z_i z_i^{-1}}(s_i - 1), \quad 1 \leq i \leq N - 1,
$$

$$
\hat{T}_N := -q_N - q_N^{-1} \frac{(1 + q_N\zeta_N z_N)(1 - q_N\zeta_N^{-1}z_N)}{1 - z_N^2}(s_N - 1).
$$

The map $T_i \mapsto \hat{T}_i, Y_i \mapsto \hat{Y}_i$ gives the dominant representation of the (double) affine Hecke algebra.

Fix an element $\lambda \in Z^N$. Let $\lambda^+$ be a unique dominant element in $W_0\lambda$ and $w_\lambda^+$ be the shortest element in $W_0$ such that $w_\lambda^+\lambda^+ = \lambda$. We define $\rho(\lambda) := w_\lambda^+\rho$ where $\rho = (N - 1, N - 2, \ldots, 0)$ and $\sigma(\lambda) = (\text{sign}(\lambda_1), \ldots, \text{sign}(\lambda_N))$. We define the dominance order $\geq$ and a partial order $\preceq$ on $Z^N$ as follows. We denote by $\lambda \geq \mu$ if $\lambda - \mu \in \sum_{i=1}^{N} Z_{\geq 0}\alpha_i^\vee$ and by $\lambda \preceq \mu$ if $\lambda^+ > \mu^+$, or $\lambda^+ = \mu^+$ and $\lambda \geq \mu$.

Set $\lambda \in Z^N$ and $z^\lambda := z_1^{\lambda_1} \cdots z_N^{\lambda_N}$. The non-symmetric Macdonald–Koornwinder polynomial [36] $E_\lambda(z; s^2, q^2)$ with parameters $s^2$ and $q^2$ is a Laurent polynomial satisfying

$$
\hat{Y}_i E_\lambda = y(\lambda)_i E_\lambda,
$$

$$
E_\lambda = z^\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} z^\mu, \quad c_{\lambda\mu} \in K,
$$

where

$$
y(\lambda)_i := s^{2\lambda_i} q^{2\rho(\lambda)_i} (q_0 q_N)^{\sigma(\lambda)_i}.
$$

In this paper, we are interested in a specialization of parameters [23, 24]:

$$
(5) \quad s^{2(r'-1)}q^{2(k+1)} = 1,
$$

where $N \geq k + 1 \geq 0$ and $r' - 1 \geq 1$. More precisely, we consider

$$
s^{2(r'-1)/m}q^{2(k+1)/m} = \omega_m,
$$

where $m = \text{GCD}(k + 1, r - 1)$ and $\omega_m$ is a primitive $m$-th root of unity. Recall that we have the one- and two-boundary Temperley–Lieb algebras as quotient algebras of the affine Hecke algebra of type C. We consider the following specialization:

$$(k, r') = (1, r + 1) \text{ for two-boundary case,}$$

$$(k, r') = (2, 2r + 1) \text{ for one-boundary case,}$$

where $r \geq N_+$. 

For a positive integer \( J \in \mathbb{N}_+ \), we define \( \nu^{J,\pm}_i := (\nu^{J,\pm}_{1,i}, \ldots, \nu^{J,\pm}_{N,i}) \in \mathbb{Z}^N \) and \( \xi^0, \xi^1 \in \mathbb{Z}^N \) by

\[
\nu^{J,\pm}_i := \begin{cases} 
J + r(N - i), & \text{for } 1 \leq i \leq N \\
-J - r(i - 1), & \text{for } 1 \leq i \leq N 
\end{cases}
\]

\[
\xi^0_i := \begin{cases} 
2([(N + 1)/2] - i)r, & 1 \leq i \leq [(N + 1)/2], \\
(2N - 2i + 1)r, & [(N + 1)/2] + 1 \leq i \leq N,
\end{cases}
\]

\[
\xi^1_i := \begin{cases} 
(N - 2i)r, & 1 \leq i \leq (N - 1)/2, \\
(2N - 2i)r, & (N + 1)/2 \leq i \leq N
\end{cases}
\]

Note that the dominant element \( \xi^+ := (\xi^+_1, \ldots, \xi^+_N) \in \mathbb{Z}_{\geq 0} \) associated with \( \xi^0 \) and \( \xi^1 \) is \( \xi^+_i = (N - i)r \).

Recall the definition of admissibility in the case of \( s^{2(r'-1)}q^{2(k+1)} = 1 \):

**Definition 3.2** (Definition 4.3 in [24]). Fix \( (k, r') \) and \( \lambda \in \mathbb{Z}^N \). A pair \( (i, j) \) is said to be a \( (k + 1, r' - 1) \)-neighbourhood if \( (i, j) \) satisfies the following three conditions:

1. \( |\rho(\lambda)_i - \rho(\lambda)_j| = k \),
2. \( |\lambda_i - \lambda_j| \leq r' - 1 \),
3. if \( |\lambda_i - \lambda_j| = r' - 1 \), then the sign \( \sigma(\lambda) \) satisfies one of the following three conditions:
   a. \( (\sigma(\lambda)_i, \sigma(\lambda)_j) = (+, +) \) and \( i > j \),
   b. \( (\sigma(\lambda)_i, \sigma(\lambda)_j) = (-, -) \) and \( i < j \),
   c. \( (\sigma(\lambda)_i, \sigma(\lambda)_j) = (-, +) \).

We call \( \lambda \) admissible if there is no neighbourhood pairs in \( \lambda \).

**Definition 3.3.** Take two elements \( \lambda_1, \lambda_2 \in \mathbb{Z}^N \). The element \( \lambda_2 \) is said to be connected to \( \lambda_1 \) if and only if there exist a sequence of admissible elements \( \lambda^{(0)} = \lambda_1, \lambda^{(1)}, \ldots, \lambda^{(l)} = \lambda_2 \) and a sequence of integers \( i_1, \ldots, i_l \) such that \( \lambda^{(j)} = s_{i_j} \lambda^{(j-1)} \) with \( s_{i_j} \in W_0 \).

Suppose that an element \( \mu \) is admissible.

**Definition 3.4.** We denote by \( I^{(k, r')}(\mu) \) a vector space spanned by

\[
\{ E_\lambda | \lambda \in W_0 \mu^+, \lambda \text{ is admissible and connected to } \mu \}
\]

at \( s^{2(r'-1)}q^{2(k+1)} = 1 \).

**Theorem 3.5** (Theorem 4.6 in [24]). The space \( I^{(k, r')}(\mu) \) is a representation of the affine Hecke algebra.

The space spanned by \( \{ E_\lambda | \lambda \text{ is admissible} \} \) is an irreducible representation of the double affine Hecke algebra [24, Theorem 4.6]. In general, the space \( I^{(k, r')}(\mu) \) may be reducible as a representation of the affine Hecke algebra. However, one can construct an irreducible representation from \( I^{(k, r')}(\mu) \).

### 3.4.1. Two-boundary case.

**Proposition 3.6.** The dimension of the space \( I^{(1, r+1)}(\nu^{J,+}) \) is \( 2^N \).

**Proof.** Since \( I := I^{(1, r+1)}(\nu^{J,+}) \) is spanned by admissible elements such that \( \nu \in W_0 \nu^{J,+} \), the dimension of \( I \) is equal to the number of admissible elements. We prove Proposition by induction on \( N \). When \( N = 2 \), admissible elements are \( (J + r, J), (J + r, -J), (-J, J + r) \) and \( (-J, -J - r) \), which implies \( \dim(I) = 4 \). We assume that Proposition holds true up to \( N - 1 \) and \( \dim(I) = 2^{N-1} \).
Let \( \nu \in W_0 \nu^{I, +} \). Suppose \( \nu_1 \) is neither \( J + (N - 1)r \) nor \( -J \). When \( \text{sign}(\nu_1) = + \), there exists \( i, 2 \leq i \leq N \) such that \( |\nu_i| = |\nu_1| + r \). From Definition 3.2, the pair \((1, i)\) is a neighbourhood. Similarly, when \( \text{sign}(\nu_1) = - \), there exists \( i \) such that \( |\nu_i| = |\nu_1| - r \). Then, the pair \((1, i)\) is a neighbourhood. Thus if \( \nu \) is admissible, then \( \nu_1 = J + (N - 1)r \) or \( \nu_1 = -J \).

Let \( \tilde{\nu} \) be an admissible element of length \( N - 1 \) in \( W_0 \nu^{I, +} \). Suppose \( \nu_1 = J + (N - 1)r \). There exists \( i \) such that \( |\nu_i| = J + (N - 2)r \). From Definition 3.2, the pair \((1, i)\) is not a neighbourhood. Set \( \nu_1 = J + (N - 1)r \) and \( \nu_i = \tilde{\nu}_{i-1} \) for \( 2 \leq i \leq N \). Since all the pairs \((j, k)\) with \( 2 \leq j < k \leq N \) are not neighbourhood, \( \nu \) is admissible. Suppose \( \nu_1 = -J \). The pair \((1, i)\), \( 2 \leq i \leq N \), is not a neighbourhood. Set \( \nu_1 = -J \) and \( \nu_i = \text{sign}(\tilde{\nu}_{i-1})(|\tilde{\nu}_{i-1}| + r) \) for \( 2 \leq i \leq N \). Since the pairs \((i, j)\) with \( 2 \leq i < j \leq N \) are not neighbourhood, \( \nu \) is admissible. From the construction of admissible elements, the dimension of \( I_{(1, r + 1)}(\nu^{I, +}) \) is given by \( 2^N \).

**Lemma 3.7.** Set \( \nu^\pm := \nu^{I, \pm} \). At \( s^{2r}q^4 = 1 \), we have
\[
(T_i - q^{-1})E_{\nu^\pm} = 0, \quad 1 \leq i \leq N - 1.
\]

**Proof.** The (anti-)dominant element \( \nu^\pm \) is admissible and \( s_i \cdot \nu^\pm 1 \leq i \leq N - 1 \) is not admissible. There exist linear operators (called intertwiner) \( \phi_i, 0 \leq i \leq N \) which send \( E_\lambda \) to \( E_{s_i \lambda} \), i.e., \( E_{s_i \lambda} = \phi_i E_\lambda \) (see Definition 2.4 in [24]). Especially, we have
\[
\phi_i := T_i + \frac{q^{-1} - q}{Y_{i+1}/Y_i - 1}.
\]

From Lemma 4.7 in [24], we have \( (\phi_i E_{\nu^\pm})|_{s^r q^4 = 1} = 0 \) for \( 1 \leq i \leq N - 1 \). At \( s^{2r}q^4 = 1 \), the ratio of eigenvalues of \( \hat{Y} \) is \( y(\nu^\pm)_{i+1}/y(\nu^\pm)_i = q^2 \). Thus the action of \( \phi_i \) on \( E_{\nu^\pm} \) is equal to the action of \( T_i - q^{-1} \).

We consider a graph whose vertices are labelled by an admissible elements in \( \mathbb{Z}^N \). We connect two vertices labelled by \( \lambda \) and \( \mu \) if and only if \( \mu = s_i \lambda \) with \( s_i \in W_0 \). We put the integer \( i \) on the edge connecting vertices labelled by \( \lambda \) and \( \mu \). When an element \( \lambda \) is admissible, we call this graph \( \Gamma(\lambda) \).

**Example 3.8.** Set \( N = 3 \) and take \( \nu^{1, +} \). We have eight admissible elements and the graph \( \Gamma(\nu^{1, +}) \) is as follows:
\[
(3, 2, 1) \xrightarrow{3} (3, 2, -1) \xrightarrow{2} (3, -1, 2) \xrightarrow{1} (1, -1, 3, -2) \xrightarrow{2} (-1, -2, 3) \xrightarrow{3} (-1, -2, -3)
\]

3.4.2. One-boundary case.

**Proposition 3.9.** The dimension of the space \( I_{(2, 2r + 1)}(\xi^0) \) is \( 2^{[N/2]} \left( \begin{array}{c} N \\ [N/2] \end{array} \right) \).

**Proof.** Recall that the dominant element \( \xi^+ \) is \((N - 1)r, (N - 2)r, \ldots, r, 0\) and this element is admissible and connected to \( \xi^0 \) and \( \xi^1 \). Suppose an element \( \xi \in \mathbb{Z}^N_{\geq 0} \) is admissible. From the definition of admissibility, \((N - i)r\) is left to \((N - i - 2)r\) in \( \xi \). The cardinality of \( \xi \)'s satisfying \( \xi \in \mathbb{Z}^N_{\geq 0} \) is \( (N \choose [N/2]) \). Take a sub-element \( \xi' \) of length \([N/2] \) satisfying \( \xi' \in W_0(2[N/2] - 1, \ldots, 3r, r) \). By a similar argument to Proposition 3.6, the cardinality of admissible \( \xi' \)'s is \( 2^{[N/2]} \). Thus the total dimension of \( I_{(2, 2r + 1)}(\xi^+) \) is \( 2^{[N/2]} \left( \begin{array}{c} N \\ [N/2] \end{array} \right) \). □
We denote by $I_+$ a vector space spanned by $\{E_\lambda | \lambda \in \mathbb{Z}^N_{\geq 0}\}$. We define

$$I^{(k,r')}_+ = I^{(k,r')} \cap I_+.$$  

Let $\xi \in W_0 \xi^+$ be an admissible element for $(k,r') = (2,2r+1)$ and $\xi_N = r$. We have intertwiners $\phi_i$, $0 \leq i \leq N$, which send a non-symmetric Koornwinder polynomial to another one (see e.g. [24, Definition 2.4]). We have $\phi_N E_\xi = c_{N,\xi} E_{s_N \xi}$. By a straightforward computation with Definition 2.4 in [24], we have $c_{N,\xi} = 0$ if $q_N^2 = -q$. The space $I^{(2,2r+1)}_+$ is closed under the action of $Y_i$, $1 \leq i \leq N$, and $T_i$, $0 \leq i \leq N$, at $q_N^2 = -q$. Thus, we have

**Proposition 3.10.** Suppose that $q_N^2 = -q$. The space $I^{(2,2r+1)}_+$ is an irreducible representation of the affine Hecke algebra. The dimension of $I^{(2,2r+1)}_+$ is $\binom{N}{(N/2)}$.

**Proposition 3.11.** At $s^{4r} q^6 = 1$ and $q_N^2 = -q$, we have

$$(\hat{T}_i - q^{-1}) E_{\xi^0} = 0, \quad i \neq [(N+1)/2],$$

$$(\hat{T}_N - q_N^{-1}) E_{\xi^0} = 0.$$  

**Proof.** From the definition of the admissibility, an element $\xi' := s_i \cdot \xi^0$, $1 \leq i \leq N-1$, $i \neq [(N+1)/2]$, is not admissible. Therefore, by a similar argument to the proof of Lemma 3.7, we have $(\hat{T}_i - q^{-1}) E_{\xi^0} = 0$.

Let $\xi' := s_N \cdot \xi^0$. Since the element $\xi'$ is admissible, we have $\phi_N E_{\xi^0} = c_{N,\xi^0} E_{\xi'}$ with the intertwiner $\phi_N$ and a rational function $c_{N,\xi^0}$ in $K$ (see [24, Proposition 2.5]). By a straightforward calculation, we have $c_{N,\xi^0} = 0$ at $q_N^2 = -q$. The intertwiner $\phi_N$ is written as

$$\phi_N = \hat{T}_N + \frac{(q_N^{-1} - q_N) + (q_0^{-1} - q_0) Y_N^{-1}}{Y_N^{-2} - 1}.$$  

Inserting $Y_N = q^{-1} q_N$ at $s^{4r} q^6 = 1$ and $q_N^2 = -q$, we obtain the action of $\phi_N$ is equal to the action of $\hat{T}_N - q_N^{-1}$ on $E_{\xi^0}$. This completes the proof. 

4. **Boundary quantum Knizhnik–Zamolodchikov equation**

We introduce boundary quantum Knizhnik-Zamolodchikov equations associated with the one- and two-boundary Temperley–Lieb algebras.

4.1. **Two-boundary case.** Let $z := (z_1, \ldots, z_N)$ and $\Psi(z)$ be a function taking values in $V \otimes N$, i.e., $\Psi(z) := \sum_b \Psi_0(z) C_b$ where $C_b$ is the Kazhdan–Lusztig bases. We define the scattering matrices $S_i, 1 \leq i \leq N-1$, by

$$(6) \quad S_i(z) := \hat{R}_i(z_i/(s^2 z_{i+1})) \hat{R}_{i+1}(z_i/(s^2 z_{i+2})) \ldots \hat{R}_{N-1}(z_i/(s^2 z_N)) K_N(s^2 / z_i)$$

$$\times \hat{R}_{N-1}(s^{-2} z_N) \ldots \hat{R}_i(s^{-2} z_1) K_0(s^{-1} z_i)$$

$$\times \hat{R}_1(z_i/z_1) \ldots \hat{R}_{i-1}(z_i/z_{i-1}).$$  

The boundary quantum Knizhnik–Zamolodchikov equation [5, 6, 18] is

$$(7) \quad S_i(z) \Psi(z) = \Psi(z_1, \ldots, z_i, s^{-2} z_i, z_{i+1}, \ldots, z_N), \quad 1 \leq i \leq N.$$
Suppose that the function $\Psi$ satisfies

\begin{align}
(8) & \quad \Psi(s_0z) = K_0(s^{-1}z_1)\Psi(z), \\
(9) & \quad \Psi(s_iz) = \tilde{R}_i(z_{i+1}/z_i)\Psi(z), \\
(10) & \quad \Psi(s_Nz) = K_N(z_N)\Psi(z).
\end{align}

Then, it is easy to show that $\Psi$ is a solution of the boundary qKZ equation. Hereafter, we solve the set of equations (8), (9) and (10) and call them boundary qKZ equation.

Define a linear operator $\hat{e}_i := \hat{T}_i - q^{-1}$. The boundary quantum Knizhnik–Zamolodchikov equation is rewritten as

\begin{align}
(11) & \quad e_i \Psi(z) = \hat{e}_i \Psi(z), \quad 0 \leq i \leq N.
\end{align}

Recall that $\Psi = \sum_b \Psi_b(z)C_b$. In Eqn.(11), the generator $e_i$ acts on a basis $C_b$ and the operator $\hat{e}_i$ acts on a function $\Psi_b(z)$. We call Eqn.(11) with $1 \leq i \leq N$ non-affine part of the boundary quantum Knizhnik–Zamolodchikov equation.

4.2. One-boundary case. Let $\Psi(z) := \sum_{b \in B_N} \Psi_b(z)C_b$ be a function taking values in $L_N$. We define scattering matrices $\tilde{S}_i(z)$, $1 \leq i \leq N$, by replacing $K_0(s^{-1}z_1)$ with the identity in Eqn.(6). The boundary quantum Knizhnik–Zamolodchikov equation is of the form Eqn.(7) by replacing $S_i(z)$ with $\tilde{S}_i(z)$. The factorized form of the boundary quantum Knizhnik–Zamolodchikov equation is Eqs.(9), (10) and $\Psi(z) = \Psi(s_0z)$.

5. Reduction of the boundary qKZ equation

5.1. Two-boundary case. Suppose that a binary string $\alpha \in \{+, -\}^N$ satisfies $\alpha_i \geq \alpha_{i+1}$ for some $i$. There is no $C_{\beta}$ such that the expansion of $e_i C_{\beta}$ contains the term $C_{\alpha}$. Therefore, we have $\hat{e}_i \Psi_{\alpha}(z) = 0$. Similarly, suppose that a binary string $\alpha$ satisfies $\alpha_N = +$. Then, there is no $C_{\beta}$ such that the expansion of $e_N C_{\beta}$ contains the term $C_{\alpha}$. Thus, we have $\hat{e}_N \Psi_{\alpha} = 0$. We have non-trivial equations for $\hat{e}_i \Psi_{\alpha}$ when $\alpha_i < \alpha_{i+1}$ for $1 \leq i \leq N - 1$ or $\alpha_N = -$ for $i = N$. Explicitly, non-trivial equations are written as

\begin{align}
(12) & \quad (\hat{e}_i + q + q^{-1})\Psi_{\alpha} = \Psi_{\alpha} + \sum_{\beta > \alpha} c_{\alpha\beta}^i \Psi_{\beta}, \quad 1 \leq i \leq N - 1,
\end{align}

where $(\alpha_i, \alpha_{i+1}) = (-, +)$, $\beta > \alpha$ is the lexicographic order, $\beta_i \geq \beta_{i+1}$ and $c_{\alpha\beta}^i \in \mathbb{K}$. Similarly, we also have

\begin{align}
(13) & \quad (\hat{e}_N + q_N + q_N^{-1})\Psi_{\alpha} = \Psi_{\alpha} + \sum_{\beta > \alpha} c_{\alpha\beta}^N \Psi_{\beta},
\end{align}

where $\alpha_N = -$, $\beta_N = +$ and $c_{\alpha\beta}^N \in \mathbb{K}$.

Set $b_j := (\ldots - + \ldots -)$ for $1 \leq i \leq N$ and $b_0 := (- \ldots -)$ and $\Psi_i := \Psi_{b_i}$.

Lemma 5.1. The non-affine part of quantum KZ equations $e_i \Psi(z) = \hat{e}_i \Psi(z)$, $1 \leq i \leq N$ is equivalent to the following set of equations: Eqs. (12), (13) and

\begin{align}
(14) & \quad \hat{e}_i \Psi_0 = 0, \quad 1 \leq i \leq N - 1.
\end{align}
Proof. We prove Proposition in the case of type BI. One can prove Proposition for other types by a similar argument. To prove Proposition, it is enough to show that one can obtain \( \hat{e}_i \Psi_\alpha = 0 \) with \( \alpha_i \geq \alpha_{i+1} \) for \( 1 \leq i \leq N-1 \) and with \( \alpha_N = + \) for \( i = N \) from Eqns.(12), (13) and (14).

We prove Proposition by induction in the reversed lexicographic order. Since \( e_N C_{---+} = \kappa^{-1} C_{---+} + \ldots \), we have

\[
\hat{e}_i \Psi_N = \hat{e}_i (\hat{e}_N + q_N + q_N^{-1}) \Psi_0 \\
= (\hat{e}_N + q_N + q_N^{-1}) \hat{e}_i \Psi_0 \\
= 0
\]

where \( 1 \leq i \leq N - 2 \) and we have used the commutation relation of \( \hat{e}_i \).

Suppose that Proposition holds true up to \( \beta > \alpha \). We have three cases for \( (\alpha_1, \alpha_{i+1}) \): 1) (+, −), 2)(−, −) and 3) (+, +). We also have a case for \( \alpha_N \): 4) \( \alpha_N = + \). We consider only the case 1, 2 and 4 since one can apply a similar argument to case 3.

**Case 1.** Let \( P_i \) and \( Q_i \) be the statements:

\( (P_i) \) The \( i \)-th and \( (i + 1) \)-th site are connected via a dashed arc,

\( (Q_i) \) The \( i \)-th arrow is the down arrow with a star.

For a binary string \( \epsilon \in \{\pm\}^N \), we define \( \theta(R; \epsilon) = \Psi_\epsilon \) if the diagram for \( \epsilon \) satisfies the statement \( R \), and \( \theta(R; \epsilon) = 0 \) otherwise. Eqn.(12) is explicitly written as follows. When \( (\alpha_1, \alpha_2) = (+, -) \) (the first and second sites are underlined), we have

\[
\Psi_{+++...} = (\hat{e}_1 + [2]) \Psi_{++...} - \kappa_N^2 \theta(P_2; --- - ...) \\
\Psi_{++...} = (\hat{e}_1 + [2]) \Psi_{++...} - \Psi_{++...} - \kappa_N \theta(Q_2; --- - ...).
\]

When \( (\alpha_{N-1}, \alpha_N) = (+, -) \) (the \( N-1 \)-th and \( N \)-th sites are underlined), we have

\[
\Psi_{...++} = (\hat{e}_{N-1} + [2]) \Psi_{...++} - \kappa_N \Psi_{...-}.
\]

In general, if \( (\alpha_i, \alpha_{i+1}) = (+, -) \) (the \( i \)-th and \( (i + 1) \)-th sites are underlined) we have

\[
\Psi_{+++...} = (\hat{e}_i + [2]) \Psi_{++...} - \Psi_{++...} - \kappa_N \theta(P_i; ... + --- - ...), \\
\Psi_{++...} = (\hat{e}_i + [2]) \Psi_{++...} - \Psi_{++...} - \Psi_{+++...} - \kappa_N \theta(Q_i; ... + --- - ...), \\
\Psi_{...++} = (\hat{e}_i + [2]) \Psi_{...++} - \kappa_N \theta(P_i; ... + --- - ...), \\
\Psi_{...-} = (\hat{e}_i + [2]) \Psi_{...-} - \Psi_{---} - \kappa_N \theta(Q_i; ... + --- - ...).
\]

From Eqns.(15) to (22), we have \( \hat{e}_i \Psi_\alpha = 0 \) by \( \hat{e}_i^2 = -[2] \hat{e}_i \) and the induction assumption \( \hat{e}_i \Psi_\beta = 0 \) for \( \beta_i \geq \beta_{i+1} \).

**Case 2.** Let \( h \) be the largest integer such that \( \alpha_h = + \) and \( \alpha_j = - \) for \( h + 1 \leq j \leq i - 1 \). There are three cases for \( h \): a) there is no such \( h \), b) \( h = 1 \) and c) \( h \geq 2 \).

Case 2-a. Since the binary string \( \alpha \) satisfies \( \alpha_j = - \) with \( 1 \leq j \leq i + 1 \), we have

\[
\Psi_\alpha = \sum_{k \geq 0} \sum_{i_1, i_2, \ldots, i_k} c_{i_1, i_2, \ldots, i_k} \hat{e}_{i_1} \cdots \hat{e}_{i_k} \Psi_0
\]

where \( c_{i_1,\ldots,i_k} \in \mathbb{K} \) and \( i_t \geq i + 2 \). Since \([\hat{e}_{i_t}, \hat{e}_{i_t}] = 0 \) and \( \hat{e}_i \Psi_0 = 0 \), we have \( \hat{e}_i \Psi_\alpha = 0 \).

Case 2-b. We have three subcases for \( i \): i) \( i = 2 \), ii) \( i = 3 \), and iii) \( i \geq 4 \). We consider the case i) and ii) since one can apply a similar argument to the case iii).
Case 2-b-i. From Eqn.(16), (21) and (22), we obtain

$$\hat{e}_2 \Psi_{\alpha} = \hat{e}_2(\hat{e}_1 + [2])(\hat{e}_2 + [2])\Psi_{\alpha} - \hat{e}_2 \Psi_{\alpha},$$

$$= (\hat{e}_2 \hat{e}_1 \hat{e}_2 - \hat{e}_2)\Psi_{\alpha},$$

$$= (\hat{e}_1 \hat{e}_2 \hat{e}_1 - \hat{e}_1)\Psi_{\alpha},$$

$$= 0,$$

where we have used the induction assumption and the braid relation for $\hat{e}_i$.

Case 2-b-ii. We have

$$\hat{e}_3 \Psi_{\alpha} = \hat{e}_3(\hat{e}_1 + [2])\Psi_{\alpha} - \kappa_N \hat{e}_3\theta(\hat{Q}_2; - - -) - \hat{e}_3 \Psi_{\alpha},$$

where we have used the induction assumption and the commutation relations for $\hat{e}_i$.

Case 2-c. We have two cases for $\alpha_{h-1}$: i) $\alpha_{h-1} = +$, and ii) $\alpha_{h-1} = -$. We consider the case i) only since one can apply a similar argument to the case ii.

Case 2-c-i. We have three cases for $i$: $i = h+1$, $i = h+2$ and $i \geq h+3$. We consider the case $i = h+1$ only since we can apply a similar argument to the other cases.

$$\hat{e}_i \Psi_{\alpha} = \hat{e}_i(\hat{e}_{i-1} + [2])\Psi_{\alpha} - \hat{e}_i \Psi_{\alpha},$$

$$= \hat{e}_i(\hat{e}_{i-1} + [2])(\hat{e}_1 + [2])\Psi_{\alpha} - \hat{e}_i \Psi_{\alpha},$$

$$= 0,$$

where we have used the induction assumption and the braid relation for $\hat{e}_i$.

Case 4. Since we have $\hat{e}_N \Psi_{\alpha} = \hat{e}_N(\hat{e}_N + q_N + q_N^{-1})\Psi_{\alpha} = 0$, Proposition holds true for $b_N$. We assume that Proposition is true up to $\beta$ with $\beta > \alpha$. Since $\alpha_N = +$, from Eqn.(13), we have

$$\hat{e}_N \Psi_{\beta} = \hat{e}_N(\hat{e}_N + q_N + q_N^{-1})\Psi_{\beta} = 0,$$

where we have used the induction assumption $\hat{e}_N \Psi_{\beta} = 0$.

We consider a graph whose vertices are labelled by a binary string in $\{\pm\}^N$. We connect two vertices labelled by $\alpha$ and $\beta$ if and only if $\alpha = s_1 \beta$, $s_i \in W_0$ and $\alpha < \beta$ in the lexicographic order. Further, we put the integer $i$ on the edge connecting the vertices $\alpha$ and $\beta$. We denote this graph by $\Gamma'$.

**Proposition 5.2.** Let $D_N$ be the number of edges in $\Gamma'$. Then, $D_N$ satisfies the following recurrence relation:

$$D_N = 2D_{N-1} + 2^{N-2}$$

with $D_2 = 3$.

**Proof.** From the construction of $\Gamma'$, the number of edges in $\Gamma'$ is equal to the number of non-trivial equations (12) and (13). Let $d_\alpha$ be the number of $i$, $1 \leq i \leq N$, such that $(\alpha_i, \alpha_{i+1}) = (-, +)$ or $\alpha_N = -$. Then we have $D_N = \sum_\alpha d_\alpha$. The number of $\alpha$ with $\alpha_N = -$ is $2^{N-1}$. When $\alpha_N = -$, the partial sum $\sum_{i, \alpha_N = -} d_\alpha$ is equal to $2^{N-1} + (D_{N-1} - 2^{N-2})$. When $\alpha_N = +$, the partial sum $\sum_{i, \alpha_N = +} d_\alpha$ is equal to $D_{N-1}$. Therefore, we have

$$D_N = 2^{N-1} + D_{N-1} + (D_{N-1} - 2^{N-2})$$

$$= 2D_{N-1} + 2^{N-2}.$$
By a direct computation, we have $D_2 = 3$. This completes the proof.

By a straightforward computation, one can show that

**Corollary 5.3.** We have $D_N \geq 2^N - 1$.

We define a map $\varphi_\pm$ from an admissible element $\nu$ in $W_0^{\nu^{J,+}}$ to a binary string $\alpha$ of length $N$:

$$\varphi_\pm : \nu_i \mapsto \alpha_i = \text{sign}(\nu_i), \quad 1 \leq i \leq N.$$  

The inverse $\varphi_{\pm}^{-1}$ is obtained by the following algorithm:

$$\varphi_+ : \alpha_i \mapsto \nu_i = \begin{cases} \max(S_1), & \text{for } \alpha_i = +, \\ -\min(S_1), & \text{for } \alpha_i = -, \end{cases}$$

$$\varphi_- : \alpha_i \mapsto \nu_i = \begin{cases} -\min(S_1), & \text{for } \alpha_i = +, \\ \max(S_1), & \text{for } \alpha_i = -, \end{cases}$$

$$S_{i+1} = S_i \setminus |\nu_i|$$

with $S_1 := \{J, J + r, \ldots, J + (N - 1)r\}$. From the explicit construction, $\varphi_\pm$ is a bijection.

**Proposition 5.4.** We have $\Gamma(\nu^{J,+}) = \Gamma'$ as a graph.

**Proof.** The number of vertices in $\Gamma(\nu^{J,+})$ and $\Gamma'$ is $2^N$. We have a bijection $\varphi_\pm$ from vertices in $\Gamma(\nu^{J,+})$ to vertices in $\Gamma'$. To prove $\Gamma(\nu^{J,+}) = \Gamma'$, it is enough to show that vertices $\mu$ and $\nu$ are connected by an edge with an integer $i$ in $\Gamma(\nu^{J,+})$ if and only if vertices $\alpha = \varphi_\pm(\mu)$ and $\beta = \varphi_\pm(\nu)$ are connected by an edge with the integer $i$ in $\Gamma'$. We prove Proposition for $\varphi_+$ since one can apply the same argument to $\varphi_-$. We first show that if the vertex $\alpha = \varphi_+(\mu)$ does not have an edge with integer $i$ in $\Gamma'$, then $s_i \mu$ is not admissible. We have $\alpha_i = \alpha_{i+1}$. From the explicit construction of $\varphi_+^{-1}$, we have $\mu_{i+1} = \mu_i - 1$ for $\alpha_i = +$ and $\mu_{i+1} = \mu_i + 1$ for $\alpha_i = -$. From Definition 3.2, the pair $(i, i + 1)$ in $s_i \mu$ is a neighbourhood. Thus $s_i \mu$ is not admissible.

Below, we will show that if $\Gamma'$ has an edge with the integer $i$, then $\Gamma(\nu^{J,+})$ has the corresponding edge. Suppose that $\alpha = s_i \beta$ with $\alpha < \beta$. Then, in $\Gamma'$, the two vertices $\alpha$ and $\beta$ are connected and the edge has the integer $i$. We have two cases for $i$: 1) $1 \leq i \leq N - 1$, and 2) $i = N$.

Case 1. Let $\nu = \varphi_+(\beta)$. Since $(\beta_i, \beta_{i+1}) = (\gamma, \gamma + 1)$, we have $|\nu_{i+1}| - |\nu_i| \geq 2$ for $1 \leq i \leq N - 2$. In this case, $s_i \nu$ is admissible since $s_i \nu$ does not have a neighbourhood. Suppose that $i = N - 1$. We have $(\rho_{N-1}, \nu_N) = (-p, p + 1)$ for some $p \in \mathbb{Z}_{>0}$. Let $\mu = s_i \nu$. We have $|\rho(\mu)_{N-1}| - |\rho(\mu)_N| = 1$, $|\mu_{N-1}| - |\mu_N| = 1$ and $(\sigma(\mu)_{N-1}, \sigma(\mu)_N) = (+, -)$. From the definition of neighbourhood (see Definition 3.2), $\mu$ is admissible. Thus we have an edge with the integer $i$.

Case 2. Let $\nu = \varphi_+(\beta)$ and $\mu = s_i \nu$. Since $\beta_N = -$, we have $\nu_N = -p$ with some $p \in \mathbb{Z}_{>0}$. From the explicit construction of $\varphi_+$, there exists no integer $i$, $1 \leq i \leq N - 1$, such that $v_i = -(p + 1)$ or $v_i = (p - 1)$. We have two cases for $v_i$ for some $1 \leq i \leq N - 1$: a) $v_i = p + 1$ and b) $v_i = -(p - 1)$. We consider only case a since we can apply the essentially same argument to case b.

Case 2-a. We have $|\rho(\mu)_i| - |\rho(\mu)_{N}| = 1$, $|\mu_i| - |\mu_N| = 1$ and $(\sigma(\mu)_i, \sigma(\mu)_N) = (+, +)$. From Definition 3.2, $(i, N)$ is not a neighbourhood. Thus $\mu$ is admissible. In $\Gamma(\nu^{J,+})$, vertices $\mu$ and $\nu$ are connected by an edge with the integer $i$. This completes the proof.

From Proposition 5.2, we have $D_N$ non-trivial equations. Note that if $\alpha = s_i \beta$ with $\alpha < \beta$, we have a non-trivial equation (12) or (13). An edge of the graph $\Gamma'$ encodes these non-trivial equations. Since the graph $\Gamma'$ is connected, for any $v \in \{\pm\}^N$, there exists a sequence of vertices.
\( \mathbf{v} := (v_0, v_1, \ldots, v_l) \) and a sequence of integers \( \mathbf{i} := (i_1, \ldots, i_l) \) such that \( v_{j-1} > v_j \) and the vertices \( v_{j-1} \) and \( v_j \) are connected by an edge with an integer \( i_j \). We call the doublet \((\mathbf{v}, \mathbf{i})\) a path from \( v_0 \) to \( v_l \).

Suppose that \( p = (\mathbf{v}, \mathbf{i}) \) with \( \mathbf{v} = (v_0, v_1, v_2) \) and \( \mathbf{i} = (p, q) \) is a path from \( v_0 \) to \( v_2 \) with \( |p - q| > 1 \). Since \( v_2 = s_p s_q v_0 = s_p s_q v_0 \), we have another path \( p' = (\mathbf{v}', \mathbf{i}') \) with \( \mathbf{v}' = (v_0, v_1', v_2) \) and \( \mathbf{i}' = (q, p) \).

Proposition 5.5. The non-affine part of boundary qKZ equation is equivalent to the following set of 2\( N \) + 2 equations:

1. \( \hat{e}_i \Psi_0 = 0, 1 \leq i \leq N - 1 \).
2. Given \( v < b_0 \), fix a path \( p_v := (\mathbf{v}, \mathbf{i}) \) with \( \mathbf{i} = (i_1, \ldots, i_l) \). Then,
   \[ e_{i_1} \cdots e_{i_l} \Psi(z) = \hat{e}_{i_l} \cdots \hat{e}_{i_1} \Psi(z). \]

Let \( W^p \) be the following statement:

\( (W^p) \) The \( i \)-th arrow is the down arrow with the integer \( p \).

For a binary string \( \epsilon \in \{\pm\}^N \), we define \( \theta(W^p; \epsilon) := \Psi_\epsilon \) if the diagram for \( \epsilon \) satisfies the statement \( W^p \), and \( \theta(W^p; \epsilon) := 0 \) otherwise. Set \( \Psi_i := \Psi_i - q^{-1} \Psi_{i+1} \) for \( 1 \leq i \leq N - 1 \) and \( \tilde{\Psi}_N := \Psi_N \).

Lemma 5.6. For type BI, we have

\[
\begin{align*}
\hat{T}_0 \Psi_0 &= \kappa_0 \tilde{\Psi}_1 - \kappa_0 \kappa_N q^{-p} \theta(W^p_1; b_0), \\
\hat{T}_i \tilde{\Psi}_i &= \tilde{\Psi}_{i+1} - \kappa_N q^{-1} \theta(W^1_{i+1}; b_0), &1 \leq i \leq N - 1, \\
\hat{T}_N(\Psi_N - \kappa_N q^{-1} \Psi_0) &= \kappa_N \Psi_0.
\end{align*}
\]

Proof. We show Eqn. (24) since one can apply a similar argument to other cases. We have \( e_0(C_{b_i}) = \kappa_0 C_{b_0} + \ldots \) and \( e_0(C_{b_j}) = -\kappa_0 q^{-1} C_{b_0} + \ldots \). When the diagram for \( b_0 \) satisfies the statement \( W^p_1 \), we have \( e_0 C_{b_0} = -\kappa_0 \kappa_N q^{-p} C_{b_0} + \ldots \) This implies Eqn. (24). \( \square \)

Lemma 5.7. For type BII, we have

\[
\begin{align*}
\hat{T}_0 \Psi_0 &= \kappa_0 \tilde{\Psi}_1 + \begin{cases} 
\kappa_N \kappa_0 q^{-1} q_N \Psi_0, &\text{for } N \text{ even}, \\
-\kappa_N \kappa_0 q^{-1} \Psi_0, &\text{for } N \text{ odd},
\end{cases} \\
\hat{T}_i \tilde{\Psi}_i &= \tilde{\Psi}_{i+1} + \begin{cases} 
-\kappa_N q^{-1} (q^{-1} q_N + q q_N^{-1}) \Psi_0, &\text{for } i \equiv N + 1(\text{mod } 2), \\
\kappa_N q^{-1} (q_N + q_N^{-1}) \Psi_0, &\text{for } i \equiv N(\text{mod } 2),
\end{cases} \\
\hat{T}_N(\tilde{\Psi}_N - \kappa_N q^{-1} \Psi_0) &= \kappa_N \Psi_0.
\end{align*}
\]

Lemma 5.8. For type BIII, we have

\[
\begin{align*}
\hat{T}_0 \Psi_0 &= \kappa_0 (\tilde{\Psi}_1 - \kappa_N q^{-1} q_N^{-1} \Psi_0), \\
\hat{T}_i \tilde{\Psi}_i &= \tilde{\Psi}_{i+1}, &1 \leq i \leq N - 1, \\
\hat{T}_N(\tilde{\Psi}_N - \kappa_N q^{-1} \Psi_0) &= \kappa_N \Psi_0.
\end{align*}
\]
Proposition 5.9. We have
\[ \hat{Y}_i \Psi_0 = \kappa_0 \kappa_N q^{-N+2i-1} \Psi_0. \]

Proof. Since there is no basis \( C_b \) such that the expansion of \( e_i(C_b) \) contains the basis \( C_{b_0} \), we have \( \hat{e}_i \Psi_0 = 0 \) for \( 1 \leq i \leq N - 1 \). Thus we have \( \hat{T}^{N-1}_i \Psi_0 = q^{i+1} \Psi_0 \). From the correspondence (2), we have
\[
\begin{align*}
\hat{Y}_i \Psi_0 &= q^{i-1} \hat{T}_i \ldots \hat{T}_{N-1} \hat{T}_N \ldots \hat{T}_0 \Psi_0 \\
&= \kappa_0 \kappa_N q^{-N+2i-1} \Psi_0
\end{align*}
\]
where we have successively used Lemma 5.6, 5.7 or 5.8. \( \square \)

5.2. One-boundary case. Let \( b_0, b_r \in \{\pm\}^N \) be binary strings as
\[
\begin{align*}
b_0 := (\overbrace{- \ldots -}^{N/2} \overbrace{+ \ldots +}^{N/2}), & \text{ for } N \text{ even,} \\
b_0 := (\overbrace{- \ldots -}^{(N+1)/2} \overbrace{+ \ldots +}^{(N-1)/2}), & \text{ for } N \text{ odd.}
\end{align*}
\]
We abbreviate \( \Psi_{b_0} \) and \( \Psi_{b_r} \) as \( \Psi_0 \) and \( \Psi_r \).

We show that one can obtain a component of \( \Psi(z) \) from the generating vector \( \Psi_0 \) through the boundary qKZ equation.

Given two binary strings \( b_1 := (b_1, \ldots, b_{1,N}) \in \{\pm\}^N \) and \( b_2 := (b_{2,1}, \ldots, b_{2,N}) \in \{\pm\}^N \), we denote by \( b_1 \prec b_2 \) if and only if \( \sum_{j=1}^N b_{1,j} \leq \sum_{j=1}^N b_{2,j} \) for \( 1 \leq j \leq N \). Set \( b_- := \{-\}^N \) and abbreviate \( \Psi_{b_-} \) as \( \Psi_- \). The binary string \( b_- \) satisfies \( b_- < b \) for \( b \in B_N \).

We define \( B_N^{(i)} := \{b \in B_N | \sum_{j=1}^N b_j = -i\} \) for \( 0 \leq i \leq N \). Let \( L_N^{(i)} = \{C_b | b \in B_N^{(i)}\} \). The dimension of the space \( L_N^{(i)} \) is zero if \( N - i \equiv 1 \pmod{2} \) and one if \( i = N \).

Lemma 5.10. Given the component \( \Psi_- \), a component \( \Psi_{b} \) with \( b \in B_N \) is written in terms of \( \Psi_- \) through the boundary quantum Knizhnik–Zamolodchikov equation.

Proof. We prove Lemma by induction. Let \( b_i = (b_{1,i}, \ldots, b_{i,N}) \in B_N^{(i-1)} \), \( 1 \leq i \leq N - 1 \) be a binary string such that \( (b_{i,i}, b_{i,i+1}) = (-,+) \) and \( b_{i,j} = - \) for \( j \neq i, i+1 \). Then, we have \( b_- \prec b_{N-1} \prec \cdots \prec b_1 \). We have \( e_i(C_{b_{N-1}}) = \kappa_1 C_{b_1} \) and there exists no \( b \neq b_{N-1} \) such that \( C_{b} \) appears in the expansion of \( e_N(C_{b}) \). The boundary qKZ equation for \( b_- \)-component is written as
\[
\Psi_{b_{N-1}} = \kappa_N (\hat{T}_N + qN) \Psi_{b_-}.
\]
Similarly, we have \( e_i(C_{b_{N-1}}) = e_i(C_{b_{i+1}}) = C_{b_i} \) and \( e_i(C_{b_{-}}) = \alpha C_{b_i} \) where \( \alpha \) is \( \kappa_N (q q^{-1} + q^{-1} q) \) (resp. \( -\kappa_N (q q^{-1} + q^{-1} q) \)) for \( N - i \) odd (resp. even). The boundary qKZ equation is written as
\[
\Psi_{b_{i-1}} = (\hat{T}_i + q) \Psi_{b_i} - \Psi_{b_{i+1}} - \alpha \Psi_-,
\]
for \( 1 \leq i \leq N - 1 \) and \( \Psi_{b_N} = 0 \). From Eqns. (27) and (28), the component \( \Psi_{b_i}, 1 \leq i \leq N - 1 \) is written in terms of \( \Psi_{b_-} \).

Fix a binary string \( b \in B_N^{(p)} \) with \( p \neq N \). We assume that Lemma holds true for all \( b' \preceq b \). Since \( b \preceq b_- \), there exists \( i \) such that \( (b_i, b_{i+1}) = (-,+) \). Recall a diagram for the Kazhdan–Lusztig basis indexed by the binary string \( b \). The diagram has a little arc \( a \) connecting the \( i \)-th and the \((i+1)\)-th sites. We have two cases for \( a \): 1) there exists a larger arc outside of \( a \), and 2) there exists no larger arc outside of \( a \).
Case 1. Let $b'$ be a partial string $b' := (b_{i-1}b_{i+1}b_{i+2})$. We have four cases for $b'$. When $b' = (+ - ++)$, set a partial string $b_1 := (---+)$ or $(---+)$. When $b' = (+ - --)$, set a partial string $b_1 := (---+)$. When $b' = (- + +-)$, set a partial string $b_1 := (---+)$. When $b' = (- + --)$, set a partial string $b_1 := (---+)$. Note that there exists at most one partial string $b_1$ such that $b' \prec b_1$. In the first case, the boundary qKZ equation for $b$-component is written as

$$\Psi_{++-} = (\hat{T}_i + q)\Psi_b - \Psi_{---}.$$ 

Since the component $\Psi_{+++}$ is written in terms of $\Psi_-$ from the induction assumption, $\Psi_{++-}$ is also written in terms of $\Psi_-$. One can prove Lemma for other cases by a similar argument.

Case 2. Let $b'$ be a partial binary string $b' = (b_{i-1}b_{i+1}b_{i+2})$. We have two cases: i) $b' = (+ - -)$ and ii) $b' = (- - +)$. 

In the case of i), we assume that there exists o- or e-unpaired down arrow at the $j$-th ($j \leq i - 1$) site. Let $b_1 = (++, (--))$ and other binary strings are smaller than $b_1$. There is no $C_{b_0}$ appears in the expansion of $\Psi_{b_0}$ for $i \neq [(N+1)/2]$. The $b_0$-component of $\Psi$ satisfies Eqns.(29) and (30).

First consider $N$ even. Given a binary string $b_i := (\ldots + i + \ldots)$, $1 \leq i \leq N/2 - 1$, we abbreviate $\Psi_i := \Psi_{b_i}$. For a binary string $b_{N/2} := (\ldots + i + \ldots)$, we abbreviate $\Psi_{N/2} := \Psi_{b_{N/2}}$. Set $\hat{\Psi}_i := \Psi_i - q^{+1}\Psi_{i-1}$ for $0 \leq i \leq N/2 - 1$ with $\Psi_{-1} = 0$.

**Lemma 5.11.** We have

\begin{align*}
(29) & \quad (\hat{T}_i - q^{-1})\Psi_0 = 0, \quad 1 \leq i \leq N - 1, i \neq [(N+1)/2], \\
(30) & \quad (\hat{T}_N - q^{-1})\Psi_0 = 0.
\end{align*}

**Proof.** There is no $b \in \mathcal{B}_N$ such that $C_{b_0}$ appears in the expansion of $e_i C_b$ for $i \neq [(N+1)/2]$. The $b_0$-component of $\Psi$ satisfies Eqns.(29) and (30).

**Lemma 5.12.** We have

\begin{align*}
\hat{T}_{N/2+i}\hat{\Psi}_i^+ &= \hat{\Psi}_{i+1}^+, \quad \text{for } 0 \leq i \leq N/2 - 2, \\
\hat{T}_{N-1}\hat{\Psi}_{N/2-1}^+ &= -q\Psi_{N/2-1} + \alpha\Psi_{N/2}, \\
\hat{T}_N(-q\Psi_{N/2-1} + \alpha\Psi_{N/2}) &= q_N(q^{-1}\Psi_{N/2-1} - \alpha\Psi_{N/2}), \\
\hat{T}_{N-1}(q^{-1}\Psi_{N/2-1} - \alpha\Psi_{N/2}) &= -\hat{\Psi}_{N/2-1}, \\
\hat{T}_{N/2+i}\hat{\Psi}_{i+1}^- &= \hat{\Psi}_i^-, \quad \text{for } 0 \leq i \leq N/2 - 2,
\end{align*}

where $\alpha = \kappa_N(q^{-1} + q^{-1}q_N)$. 

Proof. Since we have $e_{N/2+i}(C_{b_{i+1}}) = C_{b_i}$, $e_{N/2+i}(C_{b_{i-1}}) = C_{b_i}$ and $e_{N/2+i}(C_{b_i}) = -(q + q^{-1})C_{b_i}$, we have

$$
\hat{T}_{N/2+i} \Psi_i^+ = q^{-1} \Psi_i - \Psi_{i-1} - (q + q^{-1}) \Psi_i + \Psi_{i-1} + \Psi_{i+1} = \hat{\Psi}_{i+1}^+,
$$

for $0 \leq i \leq N/2 - 2$. Other equations can be proven in a similar way.

Secondly, we consider $N$ odd. Given a binary string $b_i := (\ldots - + \ldots + + \ldots)$, $0 \leq i \leq (N-3)/2$, we abbreviate $\Psi_i := \Psi_{b_i}$. Let $b_{(N-1)/2}$ be a binary string $b_{(N-1)/2} := (\ldots - + \ldots + \ldots)$ and $b_{(N+1)/2} := (\ldots - + \ldots + \ldots)$. We abbreviate $\Psi_{(N+1)/2} := \Psi_{b_{(N+1)/2}}$. We set $\hat{\Psi}_i^+ := \Psi_{i - q^{\pm 1} \Psi_{i-1}}$ for $0 \leq i \leq (N-3)/2$ and $\Psi_{-1} = 0$.

Proposition 5.13. We have

$$
\hat{T}_{N+1/2+i} \hat{\Psi}_i = \hat{\Psi}_{i+1}, \quad \text{for } 0 \leq i \leq (N-5)/2,
$$

$$
\hat{T}_{N-1} \hat{\Psi}_{(N-3)/2} = -q \Psi_{(N-3)/2} + \alpha \Psi_{(N-1)/2},
$$

$$
\hat{T}_N (-q \Psi_{(N-3)/2} + \alpha \Psi_{(N-1)/2}) = q_N (q^{-1} \Psi_{(N-3)/2} - \alpha \Psi_{(N-1)/2}),
$$

$$
\hat{T}_{N-1} (q^{-1} \Psi_{(N-3)/2} - \alpha \Psi_{(N-1)/2}) = -\hat{\Psi}_{(N-3)/2},
$$

$$
\hat{T}_{N+1/2+i} \hat{\Psi}_{i+1} = \hat{\Psi}_i, \quad \text{for } 0 \leq i \leq (N-3)/2
$$

where $\alpha = \kappa_N (q q_N^{-1} + q^{-1} q_N)$.

We omit a proof of Proposition 5.13 since one can apply a similar argument in a proof of Proposition 5.12 to this case.

Proposition 5.14. We have

$$
\hat{Y}_i \Psi_0 = \begin{cases} 
q^{-(N+1-2i)} q_0 q_N \Psi_0, & 1 \leq i \leq (N+1)/2, \\
-q^{2i} q^{-1} q_0 q_N^{-1} \Psi_0, & (N+3)/2 \leq i \leq N,
\end{cases}
$$

$$
\hat{Y}_i \Psi_{(N+1)/2} = \begin{cases} 
q^{-(N-1-2i)} q_0 q_N^{-1} \Psi_{(N+1)/2}, & 1 \leq i \leq (N-1)/2, \\
q^{-2i} (N-i) q_0 q_N \Psi_{(N+1)/2}, & (N+1)/2 \leq i \leq N,
\end{cases}
$$

$$
\hat{Y}_i \Psi_0 = \begin{cases} 
q^{-(N-2i)} q_0 q_N \Psi_0, & 1 \leq i \leq N/2, \\
-q^{2i} q^{-1} q_0 q_N^{-1} \Psi_0, & N/2 \leq i \leq N.
\end{cases}
$$

Proof. By a straightforward calculation, one can verify Proposition holds true for $N = 2, 3$. Consider $N$ even first. We have $\hat{T}_0 \Psi_0 = q^{-1} \Psi_0$ for $i \neq N/2$. Since $\Psi(z) = \Psi(s_0 z)$, we have $\hat{T}_0 \Psi(z) = -q_0 \Psi(z)$. From the correspondence (2) and Lemma 5.12, we obtain the desired expression. One can prove Proposition for $N$ odd in a similar way.

6. Laurent polynomial solutions of QKZ equation

6.1. Two-boundary case.
Theorem 6.1. The non-symmetric Koornwinder polynomial $E_{\nu,+,\pm}$, $J \in \mathbb{Z}_{\geq 1}$, with the specialization $(5)$ with $(k,r') = (1,r+1)$ yields a solution of the boundary $qKZ$ equation. Especially, $\Psi_0 = E_{\nu,+,\pm}$. The parameters have a constraint
\begin{equation}
\hat{\nu}_{m}^{\pm mJ/r} q^{\pm(N-1+4J/r)} (q_0 q_N)^{\pm 1} = \kappa_0 \kappa_N.
\end{equation}

Proof. From Proposition 5.9, $\Psi_0$ is the simultaneous eigenfunction of the operator $\hat{Y}_i$. Thus, $\Psi_0$ is a non-symmetric Koornwinder polynomial with the constraints (14) in Lemma 5.1. From Lemma 3.7, these conditions are satisfied by taking $\Psi_0 = E_{\nu,+,\pm}$. At $s^{2r} q^4 = 1$, the $\hat{Y}_i$-eigenvalues $y_i$ of $E_{\nu,+,\pm}$ is written in terms of $q,q_0$ and $q_N$ explicitly, i.e.,
\begin{equation}
y_i = q^{-2(N-1)} s^{2J} q_0 q_N \text{ for } \nu^{J,+} \text{ and } y_i = q^{2(i-1)} s^{-2J} q_0^{-1} q_N^{-1} \text{ for } \nu^{J,-}.
\end{equation}
Together with Lemma 5.9, we obtain the constraint (31).

Given $v,v < b_0$, fix a path $p_v := (v,i)$ with $i = (i_1,\ldots,i_j)$. Form Proposition 5.5, $\Psi_v$ is written as
\begin{equation}
\Psi_v = \hat{e}_1 \cdots \hat{e}_j \Psi_0 + \sum_{w > v} c_{v,w} \Psi_w
\end{equation}
with $c_{v,w} \in \mathbb{K}$. From Proposition 5.4 and the definition of $\Gamma(\nu^{J,+})$, the function $\Psi_v$ is characterized by the non-symmetric Koornwinder polynomial $E_{\nu}$ with $\nu := \varphi_{\pm}(v)$, that is
\begin{equation}
\Psi_v \propto E_{\nu} + \sum_{\lambda} c_{\lambda \mu} E_{\lambda},
\end{equation}
where $\lambda \preceq \mu$ (resp. $\lambda \succeq \mu$) for $\varphi_+$ (resp. $\varphi_-$). It remains to show that $c_0 \Psi = \hat{e}_0 \Psi$. From Theorem 3.5, Proposition 3.6 and Eqn.(32), $\Psi_v,v \in \{\pm\}^N$, form the bases of the space $I(\nu^{J,+})$. Therefore, $c_0 \Psi_v$ is uniquely written in terms of $\Psi_w$, that is,
\begin{equation}
c_0 \Psi_v = \sum_{w} g_{vw} \Psi_w,
\end{equation}
where $g_{vw} \in \mathbb{K}$ at $s^{2r} q^4 = 1$. This $g_{vw}$ is nothing but the matrix representation of $c_0$. This implies that the functions $\Psi_v$ satisfy the boundary $qKZ$ equations. This completes the proof. \hfill \Box

We define the action of $\tau_i$, $1 \leq i \leq N$, on $z = (z_1,\ldots,z_N) \in (\mathbb{C}^*)^N$ by $\tau_i : z_i \mapsto z_i^{-1}$. From Eqns.(8) and (9), we have
\begin{equation}
\Psi(\tau_i z) = \hat{R}_i(1/(z_i z_{i+1})) \cdots \hat{R}_{N-1}(1/(z_i z_N)) K_N(z_i) \hat{R}_{N-1}(z_N/z_i) \cdots \hat{R}_i(1/(z_{i+1}/z_i)) \Psi(z)
\end{equation}
for $1 \leq i \leq N$. Let $b_+ := (+ \ldots +)$ and $\Psi_+ := \Psi_{b_+}$. There is no $C_v$ such that the expansion $e_i C_v$ contains the term $C_{b_+}$. Thus Eqn.(33) for $\Psi_+$ is equal to
\begin{equation}
\left(\frac{1 + q_N \zeta_N}{1 + q_N \zeta_N z_i}\right) \prod_{j=i+1}^N \frac{(q z_i^{-1} z_j^{-1} - q^{-1})(q z_i^{-1} z_j - q)}{(q - q^{-1} z_i^{-1} z_j^{-1})(q - q^{-1} z_i z_j)} \Psi_+(z) = \Psi(\tau_i z).
\end{equation}
Since we look for the Laurent polynomial solution, the function $\Psi_+(z)$ vanishes at $z_j = q^2 z_i$, $z_j = q^{-2} z_i^{-1}$, $z_i = -q_N^{-1} \zeta_N^{-1}$ and $z_i = q^{-1} \zeta_N$ for $1 \leq i < j \leq N$. Since $(\hat{T}_i - q^{-1}) \Psi_+(z) = 0$ for $1 \leq i \leq N$, $\Psi_+(z)$ is written as
\begin{equation}
\Psi_+ = C \prod_{i=1}^N z_i^{(N-2)} \prod_{1 \leq i < j \leq N} (q z_i - q^{-1} z_j)(q z_i z_j - q^{-1}) \prod_{i=1}^N (1 + q_N^{-1} \zeta_N^{-1} z_i^{-1})(1 - q_N^{-1} \zeta_N z_i^{-1}) \Psi_+(z)
\end{equation}
where $C$ is a constant term and $\tilde{\Psi}_+(z)$ satisfies $\tilde{\Psi}_+(z) = \tilde{\Psi}_+(\tau_i z) = \tilde{\Psi}_+(\tau_N z)$ for $1 \leq i < N$. A candidate for a solution of the minimal degree is $\tilde{\Psi}_+ = 1$. The dominant term in $\Psi_+$ is $\prod_{i=1}^N z_i^{-N-i+1}$. Therefore, the solution with the minimal degree corresponds to $(r,J) = (1,1)$. 

6.2. One-boundary case.

**Theorem 6.2.** The non-symmetric Koornwinder polynomial $E_{\xi^+}$ with the specialization (5) with $(k, r') = (2, 2r + 1)$ yields a solution of the one-boundary qKZ equation. Especially, $\Psi_0 = E_{\xi^0}$. The parameters have a constraint $q_N^2 = -q$.

**Proof.** From Proposition 5.14, $\Psi_0$ is the simultaneous eigenfunction of the operator $Y_i$, $1 \leq i \leq N$. Hence, $\Psi_0$ is a non-symmetric Koornwinder polynomial satisfying Lemma 5.11. The non-symmetric Koornwinder polynomial $E_{\xi^0}$ satisfies the above criteria under the specialization $s^{4r}q^6 = 1$ and $q_N^2 = -q$. Since $E_{\xi^0} \in I^{(2,2r+1)}_+$ and $q_N^2 = -q$, one has a polynomial representation of the affine Hecke algebra with the dimension $\left(\binom{N}{[(N+1)/2]}\right)$ in $I^{(2,2r+1)}_+$ (see Proposition 3.10). The component $\Psi_-$ is written as

$$\Psi_- = \sum_\xi c_\xi E_\xi$$

where $c_\xi \in \mathbb{K}$ is an indeterminate and the sum is taken over all admissible $\xi$ with $E_\xi \in I^{(2,2r+1)}_+$. From Lemma 5.10, any component $\Psi_b$, $b \in B_N$ is written in terms of $\Psi_-$. The component $\Psi_-$ satisfies the following vanishing conditions:

$$\Psi_-|_{z_j = q^2 z_i} = 0, \quad 1 \leq i < j \leq N.$$  

From $\Psi_0 \propto E_{\xi^0}$ and the vanishing conditions, the indeterminates $c_\xi$ are determined except normalization. By a similar argument to Theorem 6.1, one can show that $\hat{T}_0 \Psi(z) = -q_0 \Psi(z)$, which implies that $\Psi(z) = \Psi(s_0z)$ under the specializations $s^{4r}q^6 = 1$ and $q_N^2 = -q$. Therefore, we have a Laurent polynomial solution of the boundary qKZ equation, which is characterized by $\xi^+$. This completes the proof. \hfill \Box

**Corollary 6.3.** We have $\Psi_{(N+1)/2} \propto E_{\xi^1}$.

**Proof.** From Proposition 5.14, $\Psi_{(N+1)/2}$ is a simultaneous eigenfunction of the operators $\hat{Y}_i$, $1 \leq i \leq N$. The non-symmetric Koornwinder polynomial $E_{\xi^1}$ has the same eigenvalues as $\Psi_{(N+1)/2}$ under the specializations $s^{4r}q^6 = 1$ and $q_N^2 = -q$. Further, the actions of $\hat{T}_i$ on $\Psi_{(N+1)/2}$ and $E_{\xi^1}$ are equivalent to each other. Thus we have $\Psi_{(N+1)/2} \propto E_{\xi^1}$. \hfill \Box

Since $(\hat{T}_i - q^{-1}) \Psi_- = 0$ for $1 \leq i \leq N - 1$, we have

$$\Psi_- = C \prod_{i=1}^N z_i^{-(N-1)} \prod_{1 \leq i < j \leq N} (qz_i - q^{-1}z_j)(qz_i z_j - q^{-1}) \Psi_-(z)$$

where $C$ is a constant term and $\Psi_-$ satisfies $\Psi_-(z) = \Psi_-(\tau z)$ for $1 \leq i \leq N$. The candidate for the solution with the minimal degree is $\Psi_- = 1$. In this case, the dominant term in $\Psi_-$ is given by $\prod_{i=1}^N z_i^{N-i}$. The solution corresponding to $(k, r') = (2, 3)$ is the one with the minimal degree and studied in [12, 44].

**REFERENCES**

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.
[2] B. D. Boe, *Kazhdan–Lusztig polynomials for Hermitian symmetric spaces*, Trans. Amer. Math. Soc. 309 (1988), 279–294.
[3] F. Brenti, *Parabolic Kazhdan–Lusztig polynomials for Hermitian symmetric spaces*, Trans. Amer. Math. Soc. 361 (2009), 1703–1729.
[4] L. Cantini, qKZ equations and ground state of the O(1) loop model with open boundary conditions, preprint (2009), arXiv:0903.5050.

[5] I. Cherednik, A unification of Knizhnik–Zamolodchikov and Dunkl operators via affine hecke algebras, Invent. Math. 106 (1991), 411–431.

[6] ______, Quantum Knizhnik–Zamolodchikov equations and affine root systems, Comm. Math. Phys. 150 (1992), 109–136.

[7] I. V. Cherednik, Factorized particles on the half-line and root systems, Theor. Math. Phys. 61 (1984), 977–983.

[8] J. de Gier, Loops, matchings and alternating-sign matrices, Discr. Math. 298 (2005), no. 1-3, 365–388, arXiv:math/0211285.

[9] J. de Gier and A. Nichols, The two-boundary Temperley–Lieb algebra, J. Algebra 321 (2009), 1132–1167, arXiv:math/0703338.

[10] J. de Gier, A. Ponsaing, and K. Shigechi, Exact finite size groundstate of the O(n=1) loop model with open boundaries, J. Stat. Mech. 0904 (2009), P04010, arXiv:0901.2961.

[11] J. de Gier and P. Pyatov, Bethe Ansatz for the Temperley–Lieb loop model with open boundaries, J. Stat. Mech. 0403 (2004), P03002, arXiv:hep-th/0312235.

[12] ______, Factorised solutions of Temperley–lieb qKZ equations on a segment, Adv. Theor. Math. Phys. 14 (2010), 795–877, arXiv:0710.5362.

[13] V. Deodhar, On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials, J. Algebra 111 (1987), no. 2, 483–506.

[14] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Symmetric polynomials vanishing on the shifted diagonals and macdonald polynomials, Int. Math. Res. Not. 2003 (2003), no. 18, 1015–1034, arXiv:math/0209042.

[15] P. Di Francesco, Boundary qKZ equation and generalized Razumov–Stroganov sum rules for open IRF models, J. Stat. Mech. (2005), P11003, arXiv:math-ph/0509011.

[16] P. Di Francesco and P. Zinn-Justin, Around the Razumov–Stroganov conjecture: proof of a multi-parameter sum rule, Elect. J. Comb. 12 (2005), R6, arXiv:math-ph/0410061.

[17] ______, Quantum Knizhnik–Zamolodchikov equation, generalized Razumov–Stroganov sum rules and extended Joseph polynomials, J. Phys. A: Math. Gen. 38 (2005), L815–L822, arXiv:math-ph/0508059.

[18] I. Frenkel and N. Reshetikhin, Quantum affine algebras and holonomic difference equations, Comm. Math. Phys. 146 (1992), 1–60.

[19] M. Jimbo, R. Kedem, T. Kojima, H. Konno, and T. Miwa, XXZ chain with a boundary, Nucl. Phys. B 441 (1995), no. 3, 437–470, arXiv:hep-th/9411112.

[20] M. Jimbo, R. Kedem, H. Konno, T. Miwa, and R. Weston, Difference equations in spin chains with a boundary, Nucl. Phys. B 448 (1995), no. 3, 429–456, arXiv:hep-th/9502060.

[21] M. Jimbo and T. Miwa, Algebraic analysis of solvable lattice models, CBMS Regional Conference Series in Mathematics, vol. 85, Amer. Math. Soc., Providence, RI, 1995.

[22] ______, Quantum KZ equation with $|q|=1$ and correlation functions of the XXZ model in the gapless regime, J. Phys. A: Math. Gen. 29 (1996), 2923–2958, arXiv:hep-th/9601135.

[23] M. Kasatani, Zeros of Symmetric Laurent Polynomials of type (BC)$_n$ and Koornwinder–Macdonald Polynomials Specialized at $q^k+q^{-k}=1$, Compos. Math. 141 (2005), no. 6, 1589–1601, arXiv:math/0312327.

[24] ______, The polynomial representation of the double affine Hecke algebra of type ($C'_n,C_n$) for specialized parameters, preprint (2008), arXiv:0807.2714.

[25] M. Kasatani and V. Pasquier, On Polynomials Interpolating Between the Stationary State of a $O(n)$ Model and a Q.H.E. Ground State, Comm. Math. Phys. 276 (2007), 397–435, arXiv:cond-mat/0608160.

[26] M. Kasatani and Y. Takeyama, The quantum Knizhnik–Zamolodchikov equation and non-symmetric Macdonald polynomials, Funkcialaj Ekvacioj 50 (2007), 491–509, arXiv:math/0608773.

[27] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), no. 2, 165–184.

[28] T. Koornwinder, Askey–Wilson polynomials for root systems of type BC, Contemp. Math. 138 (1992), 189–204.

[29] G. Lusztig, Affine Hecke algebras and their graded version, J. Amer. Math. Soc. 2 (1989), 599–635.

[30] ______, Hecke Algebra with Unequal Parameters, CRM monograph series, vol. 18, American Mathematical Society, 2003.

[31] K. Mimachi, A solution to quantum Knizhnik–Zamolodchikov equations and its application to eigenvalue problems of the Macdonald type, Duke Math. J. 85 (1996), no. 3, 635–658.

[32] T. Miwa, Y. Takeyama, and V. Tarasov, Determinant formula for solutions of the quantum Knizhnik–Zamolodchikov equation associated with $U_q(sl_n)$ at $|q|=1$, Publ. Res. Inst. Math. Sci. 35 (1999), no. 6, 871–892, arXiv:math/9905137.
[33] M. Noumi, *Macdonald–Koornwinder polynomials and affine Hecke rings*, Surikaisekikenkyusho Kokyuroku 919 (1995), 44–55.

[34] A. Razumov, Yu. Stroganov, and P. Zinn-Justin, *Polynomial solutions of qKZ equation and ground state of XXZ spin chain at $\Delta = -1/2$*, J. Phys. A: Math. Theor. 40 (2007), no. 39, 11827–11847, arXiv:0704.3542.

[35] A. V. Razumov and Yu. G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop model*, Theor. Math. Phys. 138 (2004), 333–337, [russian: Teor. Mat. Fiz. 138 (2004) 395–400], arXiv:math/0104216.

[36] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. of Math. 150 (1999), no. 1, 267–282, arXiv:q-alg/9710032.

[37] K. Shigechi, *A positive integral property of the ground state of the two-boundary Temperley–Lieb Hamiltonian*, preprint (2014), arXiv:1412.7617.

[38] , *Kazhdan–Lusztig polynomials for the Hermitian symmetric pair $(B_N, A_{N-1})*$, preprint (2014), arXiv:1412.6740.

[39] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen. 21 (1988), no. 10, 2375–2389.

[40] F. Smirnov, *A general formula for soliton form factors in the quantum sine-Gordon model*, J. Phys. A 19 (1986), no. 10, L575–L578.

[41] J. Stokman and B. Vlaar, *Koornwinder Polynomials and the XXZ Spin Chain*, preprint (2013), To appear in J. Approx. Theory, arXiv:1310.5545.

[42] A. N. Varchenko and V. O. Tarasov, *Jackson integral representations for solutions of the quantized Knizhnik–Zamolodchikov equation*, Algebra i Analiz 6 (1994), no. 2, 90–137, engish translation in St. Petersburg Math. J. 6 (1995), no. 2, 275–313, arXiv:hep-th/9311040.

[43] C. N. Yang, *Some Exact Results for the Many-Body Problem in one Dimension with Repulsive Delta-Function Interaction*, Phys. Rev. Lett. 19 (1967), 1312–1315.

[44] P. Zinn-Justin, *Loop model with mixed boundary conditions, qKZ equation and Alternating Sign Matrices*, J. Stat. Mech. (2007), P01007, arXiv:math-ph/0610067.

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