NEW OBSERVATIONS ON COHOMOLOGY RINGS OF REEB SPACES OF EXPLICIT FOLD MAPS AND MANIFOLDS ADMITTING THESE MAPS

NAOKI KITAZAWA

Abstract. As a branch of algebraic and differential topology of manifolds, the theory of Morse functions and their higher dimensional versions or fold maps and its application to algebraic and differential topology of manifolds is fundamental, important and interesting.

This paper is on explicit construction of fold maps and homology groups and cohomology rings of their Reeb spaces: they are defined as the spaces of all the connected components of inverse images of the maps, and in suitable situations inherit some topological information such as homology groups and cohomology rings of the manifolds.

Explicit construction of the maps is a fundamental and difficult task even on manifolds which are not so complicated. The author has constructed explicit fold maps systematically and performed several calculations of homology groups and cohomology rings of the Reeb spaces. This paper concerns new observations on this task.

1. Introduction and fundamental notation and terminologies.

Fold maps are smooth maps regarded as higher dimensional versions of Morse functions and fundamental and important tools in investigating algebraic and differential topological properties of manifolds. The present paper concerns explicit construction of fold maps. Constructing explicit fold maps on explicit manifolds is fundamental and important related to the studies and it is also difficult.

1.1. Fold maps and their Reeb spaces.

1.1.1. Fold maps. First we review the definition of a fold map. Before this, we explain fundamental terminologies and notation related to singular points of smooth maps.

A singular point of a smooth map \( c \) is a point at which the rank of the differential of the map drops. A singular value of the map is a point realized as a value at a singular point. The set \( S(c) \) of all singular points is the singular set of the map. The singular value set is the image of the singular set. The regular value set of the map is the complement of the singular value set and a regular value is a point in the regular value set.

Definition 1. Let \( m > n \geq 1 \) be integers. A smooth map between an \( m \)-dimensional smooth manifold without boundary and an \( n \)-dimensional smooth manifold without
boundary is said to be a fold map if at each singular point \( p \), the form is

\[
(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)
\]

each singular point \( p \) of the map.

For a fold map, the following hold.

- For any singular point \( p \), the \( i(p) \) in Definition 1 is unique (\( i(p) \) is called the index of \( p \)).
- The set consisting of all singular points of a fixed index of the map is a smooth closed submanifold of dimension \( n-1 \) of the source manifold.
- The restriction to the singular set of the map is a smooth immersion of codimension 1.

A stable fold map is a fold map whose restriction to the singular set is a smooth immersion such that the crossings of the image is normal. For fundamental theory of fold maps and more general generic maps including stable maps, see [3] for example.

### 1.1.2. Reeb spaces

The Reeb space of a continuous map is defined as the space of all connected components of inverse images of the map.

**Definition 2.** Let \( X \) and \( Y \) be topological spaces. For \( p_1, p_2 \in X \) and for a continuous map \( c : X \to Y \), we define as \( p_1 \sim_c p_2 \) if and only if \( p_1 \) and \( p_2 \) are in a same connected component of \( c^{-1}(p) \) for some \( p \in Y \). Thus \( \sim_c \) is an equivalence relation. We denote the quotient space \( X/\sim_c \) by \( W_c \) and call \( W_c \) the Reeb space of \( c \).

We denote the induced quotient map from \( X \) into \( W_c \) by \( q_c \). We can define \( \bar{c} : W_c \to Y \) uniquely so that the relation \( c = \bar{c} \circ q_c \) holds.

**Proposition 1** ([19]). For stable fold maps, the Reeb spaces are polyhedra and the dimensions are equal to those of the target manifolds.

Reeb spaces are also fundamental and important tools in knowing the source manifolds of smooth maps whose codimensions are negative. For suitable classes of these maps, they inherit topological properties of the manifolds. For Reeb spaces, see also [14] for example.

We introduce a class of stable fold maps such that the Reeb spaces inherit much information of algebraic topological invariants of the manifolds in Proposition 2.

A simple fold map \( f \) is a fold map such that the restriction \( q_f|_{S(f)} \) is injective.

An almost-sphere is a homotopy sphere obtained by gluing two copies of a standard closed disc on the boundaries by a diffeomorphism.

On commutative rings, we call a so-called principal ideal domain a PID.

**Proposition 2** ([18] ([5] and [6])). Let \( m \) and \( n \) be integers satisfying \( m > n \geq 1 \). Let \( M \) be a smooth, closed, connected and orientable manifold of dimension \( m \) and \( N \) be an \( n \)-dimensional smooth manifold without boundary.

Then, for a simple fold map \( f : M \to N \) such that inverse images of regular values are always disjoint unions of almost-spheres and that indices of singular points are always 0 or 1, two induced homomorphisms \( q_{f*} : \pi_j(M) \to \pi_j(W_f) \), \( q_{f*} : H_j(M; R) \to H_j(W_f; R) \), and \( q_f^* : H^j(W_f; R) \to H^j(M; R) \) are isomorphisms for \( 0 \leq j \leq m - n - 1 \) and for any ring \( R \). Furthermore, let \( J \) be the set of all integers greater than or equal to 0 and smaller than or equal to \( m - n - 1 \) and
if $\oplus_{j \in J} H^j(W_f; R)$ is an algebra where the sums and the products are canonically induced from the cohomology ring $H^*(W_f; R)$, then $q_f$ induces an isomorphism between these algebras $\oplus_{j \in J} H^j(W_f; R)$ and $\oplus_{j \in J} H^j(M; R)$.

Furthermore, if $R$ is a PID and the relation $m = 2n$ holds, then the rank of $H_n(M; R)$ is twice the rank of $H_n(W_f; R)$. In addition, if $H_{n-1}(W_f; R)$, which is isomorphic to $H_{n-1}(M; R)$, is free, then they are also free.

1.2. Explicit fold maps and their Reeb spaces. It is fundamental and important to construct explicit fold maps. However, even on manifolds which are not so complicated, it is difficult. We present known examples here.

For a topological space $X$, an $X$-bundle is a bundle whose fiber is $X$. For a smooth manifold $X$, a smooth $X$-bundle is an $X$-bundle whose structure group is (a subgroup of) the diffeomorphism group.

Example 1. (1) A stable special generic map is a specific version of simple fold maps in Proposition 2 and defined as a stable fold map such that the index of each singular point is 0. Canonical projections of unit spheres are most simplest examples. According to [15], [16], [17] and [20], homotopy spheres not diffeomorphic to standard spheres do not admit special generic maps into Euclidean spaces if the dimensions of the target Euclidean spaces are sufficiently high and smaller than the dimensions of the source manifolds.

Furthermore, the maximal degree $j = m - n - 1$ in Proposition 2 can be replaced by $j = m - n$ in cases of special generic maps.

Last, the Reeb space of a (stable) special generic map from a closed and connected manifold of dimension $m$ into $\mathbb{R}^n$ satisfying the relation $m > n \geq 1$ is an $n$-dimensional compact manifold we can immerse into $\mathbb{R}^n$. The image is the image of the immersion of the $n$-dimensional manifold.

This is a fundamental fact explained in [15] etc.

(2) ([4], [5] and [7]) Let $m > n \geq 1$ be integers. We can construct a stable fold map on a manifold represented as a connected sum of total spaces of smooth $S^{m-n}$-bundles over $S^n$ into $\mathbb{R}^n$ satisfying the following.

(a) The singular value set is embedded concentric spheres and $\bigcup_{k=1}^l \{||x|| = k \mid x \in \mathbb{R}^n\}$ for an integer $l > 0$.

(b) Inverse images of regular values are disjoint unions of standard spheres.

(c) In $\mathbb{R}^n$, the number of connected components of an inverse image increases as we go into the origin $0 \in \mathbb{R}^n$ of the target Euclidean space.

The Reeb space is simple homotopy equivalent to a bouquet of $l - 1$ copies of a sphere of dimension $n$ or a point if $l = 1$.

(3) In [8] and [9], stable fold maps such that the restrictions to the singular sets are embedding satisfying the assumption of Proposition 2 have been constructed. They have been obtained by finite iterations of surgery operations called bubbling operations starting from fundamental fold maps. More precisely, starting from stable special generic maps etc., by changing maps and manifolds by bearing new connected components of singular sets one after another, we obtain desired maps.

1.3. Construction of explicit fold maps by surgery operations called bubbling operations and the organization of this paper. In this paper, we present further studies on construction in Example 1 (3). We use bubbling operations. We try new methods and obtain new families of explicit stable fold maps.
The organization of the paper is as the following. In the next section, we introduce bubbling operations first introduced in [4]. The last section is devoted to main results. We present construction of new families of explicit fold maps and investigate the cohomology rings of the Reeb spaces. Proposition 2 is a key tool in knowing the cohomology rings of the manifolds from Reeb spaces in suitable cases.

Throughout this paper, \( M \) is a smooth closed manifold of dimension \( m \), \( N \) is a smooth manifold of dimension \( n \) without boundary, the relation \( m > n \geq 1 \) holds and \( f : M \to N \) is a smooth map unless otherwise stated. As this, we assume manifolds and maps to be smooth and of class \( C^\infty \) unless otherwise stated. In addition, the structure groups of bundles such that the fibers are manifolds are assumed to be (subgroups of) diffeomorphism groups. A linear bundle is a smooth bundle whose fiber is a \( k \)-dimensional unit disc or the \( k \)-dimensional unit sphere in \( \mathbb{R}^{k+1} \) and whose structure groups are subgroups of the \( k \)-dimensional orthogonal group \( O(k) \) and the \((k+1)\)-dimensional one \( O(k+1) \) acting canonically, respectively.

The author is a member of and supported by the project Grant-in-Aid for Scientific Research (S) (17H06128 Principal Investigator: Osamu Saeki) "Innovative research of geometric topology and singularities of differentiable mappings" (https://kaken.nii.ac.jp/en/grant/KAKENHI-PROJECT-17H06128/).

2. Bubbling operations and fold maps such that inverse images of regular values are disjoint unions of spheres.

2.1. Definitions and fundamental properties of Reeb spaces and bubbling operations. We introduce bubbling operations, first introduced in [8], referring to the article. We revise some terminologies etc. from the original definition.

Definition 3. For a stable fold map \( f : M \to N \), let \( R \) be a connected component of \( (W_f - q_f(S(f))) \cap f^{-1}(N - f(S(f))) \), which we may regard as an open manifold diffeomorphic to an open manifold \( f(R) \) in \( N \). Let \( S \) be a connected, orientable and compact submanifold with no boundary of \( R \) such that the normal bundle is orientable. Let \( N(S), N(S)_i \) and \( N(S)_o \) be small closed tubular neighborhoods of \( S \) in \( R \) such that the relations \( N(S)_i \subset \text{Int} N(S) \) and \( N(S) \subset \text{Int} N(S)_o \) hold. Let \( Q := q_f^{-1}(N(S)_o) \) be the connected manifold such that \( q_f|Q \) makes \( Q \) a bundle over \( N(S)_o \). Assume that we can construct a stable fold map \( f' \) of an \( m \)-dimensional closed manifold \( M' \) into \( \mathbb{R}^n \) satisfying the following.

1. \( M - \text{Int} Q \) is realized as a compact submanifold (with non-empty boundary) of \( M' \) of dimension \( m \) by considering a suitable smooth embedding \( e \) into \( M' \).
2. \( f|_{M - \text{Int} Q} = f' \circ e|_{M - \text{Int} Q} \) holds.
3. \( f'(S(f')) \) is the disjoint union of \( f(S(f)) \) and \( f(\partial N(S)) \).
4. \( (M' - e(M - Q)) \cap q_{f'}^{-1}(N(S)_i) \) is empty or \( q_{f'}|_{(M' - e(M - Q)) \cap q_{f'}^{-1}(N(S)_i)} \) makes \( (M' - e(M - Q)) \cap q_{f'}^{-1}(N(S)_i) \) a bundle over \( N(S)_i \).

From this, we can define a procedure of constructing \( f' \) from \( f \). We call it a normal bubbling operation to \( f \). Furthermore, \( S \) is called the generating manifold of the normal bubbling operation.

Furthermore, the following are defined.

1. \( f'|_{(M' - e(M - Q)) \cap q_{f'}^{-1}(N(S)_i)} \) makes \( (M' - e(M - Q)) \cap q_{f'}^{-1}(N(S)_i) \) the disjoint union of two bundles over \( N(S) \), then the procedure is called a normal \( M \)-bubbling operation to \( f \).
let \( N \) to bubbling operation and call the operation a are isomorphic as regular neighborhoods. By a similar way, we define a similar op-

operation, connected components of the fiber are also called inver se images 

an S-bubbling operation. In the case of an M-bubbling operation or a n S-bubbling 

the relations 

a bubble 

obtained by an M-bubbling operation to 

an inverse image 

born by a bubble if the surgery is not an M-bubbling operation or an S-bubbling operation. In the case of an M-bubbling operation or an S-bubbling 

operation, connected components of the fiber are also called inverse images born by a bubble.

In the definition above, let \( S \) be the bouquet of finitely many connected, ori-

entable and compact submanifolds without boundaries whose dimensions are smaller than \( n \) and let the normal bundles be orientable and admit sections. Furthermore, 

let \( N(S), N(S)_i \) and \( N(S)_j \) be small regular neighborhoods of \( S \) in \( P \) such that the relations \( N(S)_i \subset \text{Int}N(S) \) and \( N(S) \subset \text{Int}N(S)_j \) hold and that these three 

are isomorphic as regular neighborhoods. By a similar way, we define a similar operation and call the operation a bubbling operation and an \( M(S) \)-bubbling operation to \( f \). We call \( S \) the generating polyhedron of the bubbling operation.

Hereafter, we abuse notation in Definition 3.

Example 2. In Definition 3, if \( S \) is in an open disc of \( R \) or more generally, \( q_f|_Q \) makes \( Q \) a trivial bundle over \( N(S)_i \), then we can perform a (normal) bubbling operation to \( f \). We can replace the phrase ”bubbling” by ”M-bubbling” and ”S-bubbling”.

Furthermore, we can change the fiber of the original bundle so that the resulting new inverse image including no singular points is as the following.

(1) Any closed and connected manifold obtained by a handle attachment: more precisely consider the product of the manifold appearing as the fiber and the closed interval, attach a handle to the one connected component of the boundary of the cylinder and consider the resulting boundary.

(2) A disjoint union of arbitrary two manifolds whose connected sum is diffeo-
morphic to the original manifold appearing as the fiber.

Ideas for the operations are based on stuffs in [10], [11] and [12]. Especially, through [10] and [11], bubbling surgeries have been established: a bubbling surgery is the case where the generating polyhedron is a point.

The following is a key lemma implicitly used throughout the present paper and it follows immediately from the definition of an M-bubbling operation.

Lemma 1. Let \( f \) be a stable fold map. If an M-bubbling operation is performed to \( f \) and a new map \( f' \) is obtained, then \( W_f \) is a proper subset of \( W_{f'} \) such that for the map \( \tilde{f'}: W_{f'} \to N \), the restriction to \( W_f \) is \( \tilde{f}: W_f \to N \).

The following proposition is a fundamental and key tool in the present paper.

Proposition 3. Let \( f: M \to N \) be a stable fold map. Let \( f': M' \to N \) be a fold map obtained by an M-bubbling operation to \( f \). Let \( S \) be the generating polyhedron of the M-bubbling operation. Let \( k \) be a positive integer and \( S \) be represented as the bouquet of submanifolds \( S_j \) where \( j \) is an integer satisfying \( 1 \leq j \leq k \). Then, for any integer \( 0 \leq i < n \), we have

\[
H_i(W_{f'}; R) \cong H_i(W_f; R) \oplus \bigoplus_{j=1}^{k} (H_i-(n-\dim S_j)) (S_j; R))
\]
and we also have $H_n(W_f; R) \cong H_n(W_f; R) \oplus R$.

Rigorous proofs with discussions on Mayer-Vietoris sequences, homology groups of product bundles etc. are presented in [8] and [9].

**Proof.** For $S_j$, we can take a small closed tubular neighborhood, regarded as the total space of a linear $D^{n-\dim S_j}$-bundle over $S_j$. By the definition of the operation, a small regular neighborhood $N(S)$ of $S$ is represented as a boundary connected sum of these closed tubular neighborhoods. $W_f'$ is obtained by attaching a manifold represented as a connected sum of total spaces of linear $S^{n-\dim S_j}$-bundles over $S_j$ (1 ≤ $j$ ≤ $k$) by considering $D^{n-\dim S_j}$ in the beginning as a hemisphere of $S^{n-\dim S_j}$ and identifying the subspace obtained by restricting the space to fibers $D^{m-\dim S_j}$ with the original regular neighborhood: this discussion is also related to Lemma 1. For the manifold represented as a connected sum of total spaces of linear $S^{n-\dim S_j}$-bundles over $S_j$ (1 ≤ $j$ ≤ $k$), the bundles are treated as product bundles in knowing the homology groups. In fact, they are orientable and admit sections, corresponding to the submanifolds $S_j$ and regarded as subbundles corresponding to $\{0\} \subset D^{n-\dim S_j} \subset S^{n-\dim S_j}$. Observing $W_f$ and $W_f'$ carefully, we have the result. □

The following has been first shown in [8]. We can show this by applying Proposition 3 one after another. As each generating polyhedron, we take a suitable bouquet of a finite number of standard spheres in open balls in $R$ in Definition 3.

**Proposition 4.** Let $R$ be a PID and $\{G_j\}_{j=0}^n$ be a family of free and finitely generated modules over $R$ so that $G_0$ is trivial and that $G_n$ is not zero. Then, by a finite iteration of normal M-bubbling operations starting from a map $f$, we obtain a fold map $f'$ such that $H_j(W_f; R)$ is isomorphic to $H_j(W_f; R) \oplus G_j$.

### 3. Main results

We will investigate homology groups and cohomology rings of the resulting Reeb spaces and present these results as main results.

#### 3.1. CPS manifolds

In [9], a CPS manifold and a CPS and GCPS graded commutative algebras are defined.

**Definition 4.** A manifold $S$ is said to be CPS if either of the following hold.

1. $S$ is a standard sphere whose dimension is positive.
2. $S$ is represented as a connected sum or a product of two CPS manifolds.

We can know the following by virtue of fundamental differential topological discussions and omit the proof.

**Proposition 5.** CPS manifolds can be embedded into one-dimensional higher Euclidean spaces.

**Definition 5.** A graded commutative algebra $A$ over a PID $R$ is said to be CPS if either of the following hold.

1. $A$ is isomorphic to the cohomology ring $H^*(S^k; R)$ ($k \geq 1$).
2. $A$ is represented as a tensor product of two CPS graded commutative algebras over $R$ or a graded commutative algebra obtained from two CPS graded commutative algebras $A_1$ and $A_2$ over $R$ satisfying the following.
are negative. In [15] and also in [12], connected sums of two smooth maps on closed manifolds of dimensions greater than 2 into the plane.

3.2. A connected sum of two smooth maps whose codimensions are negative. First we introduce a connected sum of two smooth maps whose codimensions are negative. In [15] and also in [12], connected sums are important in constructing new maps from given pairs of special generic maps into fixed Euclidean spaces and stable maps on closed manifolds of dimensions greater than 2 into the plane.

Let $\pi_{m+1,n} : \mathbb{R}^{m+1} \to \mathbb{R}^n$ defined by $\pi_{m+1,n}(\{x_1, \cdots, x_{m+1}\}) = (x_1, \cdots, x_n)$ be the canonical projection. Set $\mathbb{R}^{n+} := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$. The restriction of the map $\pi_{m+1,n}$ to the unit sphere $S^m$ is the canonical projection and we denote its restriction to the inverse image of $\mathbb{R}^{n+}$ by $\pi_{S^m, n}$: its source manifold is diffeomorphic to $D^m$ and the target is $\mathbb{R}^{n+}$.

Let $m > n \geq 1$ be integers, $M_i \ (i = 1, 2)$ be a closed and connected manifold of dimension $m$ and $f_1 : M_1 \to \mathbb{R}^n$ and $f_2 : M_2 \to \mathbb{R}^n$ be smooth maps. Let $P_i \ (i = 1, 2)$ be the closure of a region obtained by a hyperplane including the origin in $\mathbb{R}^n$ such that for the map $f_1|_{f_2^{-1}(P_i)} : f_2^{-1}(P_i) \to P_i$ there exist diffeomorphisms $\Phi$ and $\phi$ satisfying the relation

$$\phi \circ f_1|_{f_2^{-1}(P_i)} = \pi_{S^m, n} \circ \Phi.$$ 

Two maps $f_1|_{f_2^{-1}(P_i)}$ and $\pi_{S^m, n}$ are $C^\infty$ equivalent: see [3] for the terminology for example.

We can glue the maps $f_1|_{f_2^{-1}(\mathbb{R}^n - \text{Int} P_i)} : f_2^{-1}(\mathbb{R}^n - \text{Int} P_i) \to \mathbb{R}^n - \text{Int} P_i \ (i = 1, 2)$ on the boundaries to obtain a new map regarded as a smooth map into $\mathbb{R}^n$ so that the resulting source manifold is regarded as a connected sum of the original source manifolds. The resulting map is called a connected sum of $f_1$ and $f_2$.

For arbitrary two stable fold maps on closed and connected manifolds of dimension $m$ into $\mathbb{R}^n$, we can obtain a connected sum of them.

3.3. Special generic maps into Euclidean spaces. The following is a fundamental fact on special generic maps stated in [15] etc..

Fact 1. Let $m > n \geq 1$ be integers.

(1) For a closed and connected manifold of dimension $m$, it admits a special generic map into $\mathbb{R}^n$ if and only if it admits a stable simple fold map as in Proposition 3 whose Reeb space is a compact and connected manifold of dimension $n$ we can immerse into $\mathbb{R}^n$.
(2) For a compact and connected manifold of dimension \( n \) we can immerse into \( \mathbb{R}^n \), there exists a closed and connected manifold of dimension \( m \) admitting a stable special generic map into \( \mathbb{R}^n \) whose Reeb space is the \( n \)-dimensional compact manifold. Furthermore, the special generic map satisfies the following.

(a) On the inverse image of the interior of the Reeb space, the map onto this interior gives a \( S^{m-n} \)-bundle.

(b) On the inverse image of a small collar neighborhood of the Reeb space, the composition of the map onto this neighborhood and the canonical projection onto the boundary gives a linear \( S^{m-n+1} \)-bundle.

Definition 6. In the latter half of Fact 1 above, we can construct the map so that the two bundles are trivial smooth bundles (which may not be trivial linear bundles) on a suitable \( m \)-dimensional manifold. In this case, if the restriction to the singular set is an embedding, then we say that the map has almost-trivial monodromies.

In Example 1 (1) special generic maps whose Reeb spaces are standard discs are presented, for example. They have almost-trivial monodromies. Including the examples, most of special generic maps in existing studies are essentially satisfying the following condition.

Definition 7. Let \( m > n \geq 1 \) integers. A special generic map from a closed and connected manifold of dimension \( m \) into \( \mathbb{R}^n \) is said to be a standard GCPS special generic map if the following hold.

1. The Reeb space is represented as a boundary connected sum of finitely many manifolds each of which is represented as a product of a CPS manifold and a standard closed disc.
2. The restriction map to the singular set is an embedding.

Example 3. (1) Let \( n = 2 \). A result in [15] shows that a stable special generic map into the plane on a manifold of dimension \( m \geq 3 \) is essentially regarded as a standard GCPS special generic map. In fact the Reeb space is obtained as a boundary connected sum of finite copies of \( S^1 \times D^1 \).

(2) Let \( n = 3, 4 \). Saeki’s results and Nishioka’s results obtained later ([13] and [15]) imply that a 5-dimensional closed and simply-connected manifold admits a special generic map into \( \mathbb{R}^n \) if and only if it is represented as a connected sum of the total spaces of \( S^3 \)-bundles over \( S^2 \). According to the results, such a manifold also admits a standard GCPS special generic map into \( \mathbb{R}^n \) whose Reeb space is represented as a boundary connected sum of finite copies of \( S^2 \times D^{n-2} \).

Definition 8. Let \( m > n \geq 1 \) be integers. Consider a stable fold map \( f : M \to N \) on a closed and connected manifold of dimension \( m \) into a manifold with no boundary of dimension \( n \) such that the restriction to the singular set is an embedding. Then we can see the following.

1. On the inverse image of each connected component of \( W_f - q_f(S(f)) \), the map onto this interior gives a trivial bundle.
2. On the inverse image of a small regular neighborhood of each connected component of \( q_f(S(f)) \), the composition of the map onto this neighborhood and the canonical projection onto \( q_f(S(f)) \) gives a smooth bundle.
If these bundles are smooth trivial bundles, then we say that the map has almost-trivial monodromies.

Example 4. Example 2 implies that we can construct stable fold maps having almost-trivial monodromies one after another by considering compact and orientable submanifolds with no boundaries in the Reeb spaces inverse images of which contain no singular points starting from a stable fold map having almost-trivial monodromies.

3.4. New results. We show new results on cohomology rings of Reeb spaces. In the situation where we can apply Proposition 2, we can obtain a result on those of the resulting manifolds. For example, starting from a stable standard special generic map and performing a finite iteration of S-bubbling operations starting from the map such that inverse images born by bubbles are almost-spheres, we can obtain various maps to which we can apply Proposition 2.

Theorem 1. Let $R$ be a PID. Let $f : M \to \mathbb{R}^n$ be a stable fold map. We consider a standard GCPS special generic map $f_0$ and consider a finite iteration of normal $M$-bubbling operations to the map to obtain a new map $f_1$. Next, we consider a connected sum of $f$ and the map $f_1$ just before.

(1) By this construction, we can obtain a map $f'$ satisfying the following.
   (a) $H^*(W_{f'}; R)$ is isomorphic to a graded algebra represented as the direct sum of $H^*(W_f; R)$ and a suitable graded algebra $A$, which we will explain in the following.
   (b) $A$ is, as a graded $R$-module, represented as a direct sum of a free finitely generated $R$-module obtained by forgetting the ring structure of a GCPS algebra $A_0$ and a finitely generated $R$-module.
   (c) $A_0$ is regarded as a subalgebra of $A$ canonically.

(2) Let $n \geq 3$. Let $S$ be a closed, connected and orientable manifold satisfying the following.
   (a) $2\dim S \leq n$.
   (b) There exists $c \in H_k(S; R)$ for $k \geq 0$.
      (i) There exists no integer $a > 1$ such that there exists a class $c'$ satisfying $c = ac'$
      (ii) $c$ is represented by a closed submanifold $F$ whose normal bundle is trivial.

Suppose also that $f_0$ has almost-trivial monodromies and the generating polyhedra for construction of $f_1$ are of dimension smaller than $n - 1$. Then we can obtain a map $f''$ satisfying the following by an $M$-operation to $f'$ before whose generating manifold is diffeomorphic to $S$.
   (a) The $R$-module $H_j(W_{f''}; R)$ is represented as a direct sum of $H_j(W_{f'}; R)$ and $H_j(-(n-\dim S))(S; R)$. $H^j(W_{f''}; R)$ is as an $R$-module represented as a direct sum of $H^j(W_{f'}; R)$ and $H^j(-(n-\dim S))(S; R)$.
   (b) $H^*(W_{f'}; R)$ is regarded as a subalgebra of $H^*(W_{f''}; R)$ where we regard $H^*(W_{f'}; R)$ as an algebra before,
   (c) The products of two elements in $H^*(W_{f''}; R)$ satisfy the following.
      (i) Products of elements in $H^*(W_{f'}; R)$ are same as the given ones.
      (ii) Products of elements in $\bigoplus_{j=n-\dim S} H^j(-(n-\dim S))(S; R)$ always vanish.
Proof. The former half is almost straightforward from the definitions of standard GCPS special generic map and a connected sum of stable fold maps and the statement and the discussion of the proof of Proposition 3. We need to explain about the third property. \(A_0\) is free and realized as the cohomology ring of the Reeb space of the standard GCPS special generic map \(f_0\). After the finite iteration of bubbling operations, the original Reeb space is regarded as a subspace of the new Reeb space \(W_{f_t}\) by Lemma 1 or as in the proof of Proposition 3. We define a new cocycle of the new Reeb space \(W_{f_t}\) corresponding to a cocycle of degree \(k > 0\) of the original Reeb space \(W_{f_0}\) so that the following hold: we consider a natural triangulation.

1. At \(k\)-chains in the newly attached spaces including no \(k\)-simplices in the original Reeb space (seen as a subspaces of the new Reeb space canonically), the values are 0.
2. At \(k\)-chains regarded as ones in the original Reeb space, the values are same as the values of the original cocycles at the same chains.

This canonically gives an isomorphism of \(A_0\) into \(A\), which is realized as the cohomology ring of \(W_{f_t}\).

We show the latter part. The generating polyhedron is chosen in the interior of the Reeb space of \(W_{f_0}\). The first property is shown by a discussion similar to the proof of Proposition 3 (also for the cohomology rings). The second property is shown in a way similar to the proof of the former part. For the third property, the first case is shown by a discussion similar to one in the proof of the former part. The relation \(n - \dim S \geq \frac{4}{3}\) holds by the assumption and this leads us to

\[(iii) \text{ Let the rank of } H_{\dim S - k}(W_{f_0}; R) \subset A_0 \text{ be } b \geq 0 \text{ and set a basis } \{e_j\}_{j=1}^b : \text{ note that } A_0 \text{ is regarded as } H^*(W_{f_0}; R) \subset H^*(W_{f_t}; R), \text{ free, and as a graded } R\text{-module, isomorphic to the homology group of } W_{f_0}. \text{ We can define the dual of } e_j \text{ and denote it by } e_j^* \in H^{\dim S - k}(W_{f_0}; R) (e_j^*)^*(e_j^*) = \delta_{j_1, j_2} \text{ where } \delta_{j_1, j_2} \text{ is } \text{the Kronecker's delta: as this some other duals will be defined through the present theorem). Let } \{a_j\}_{j=1}^b \text{ be a sequence of integers. We can define the dual } c^* \text{ such that } c^*(c) = 1 \text{ and that by considering } H_k(S; R) \text{ as a direct sum of the } R\text{-submodule generated by } c \text{ and an } R\text{-submodule } C, c^*(C) = \{0\}. \text{ We denote the Poincaré dual to } c^* \text{ in } S \text{ by } d(c). \text{ Fix a generator of } H^*(W_{f_t}; R) \text{ isomorphic to } R: \text{ it is regarded as an element of degree } n - \dim S \text{ in the whole cohomology ring } H^*(W_{f_t}; R). \text{ Then the following hold.}

(A) \text{ For } e_j^* \text{ and a fixed generator of } H^{(n - \dim S) - (n - \dim S)}(S; R), \text{ isomorphic to } R, \text{ which is regarded as an element of degree } n - \dim S \text{ in the whole cohomology ring } H^*(W_{f_t}; R), \text{ the product is } a_j \text{ times the dual of } d(c) \in H^{(n - \dim S) + (\dim S - k) - (n - \dim S)}(S; R), \text{ regarded as an element in } H^{\dim S - k}(S; R) \text{ and of degree } n - k \text{ in the whole cohomology ring.}

(B) \text{ For } e_j^* \text{ and the dual of } c \in H_k(S; R), \text{ regarded as an element of degree } n - \dim S + k \text{ in the whole cohomology ring, the product is } a_j \text{ times a fixed generator of } H^{n - (n - \dim S)}(S; R), \text{ isomorphic to } R, \text{ which is regarded as an element of degree } n \text{ in the whole cohomology ring.}
the second case: The product of two elements of degree $\frac{n}{2}$ is zero since it is, in the proof of Proposition 3, represented as product of two cycles represented by fibers of the $S^{n-\dim S_j}$-bundle over $S_j$: the number of the bundles is just one in this case. For the third case, first we choose a $(\dim S - k)$-dimensional standard sphere $S_0$ whose homology class is regarded as $\Sigma_{j=1}^b a_j e_j$ sufficiently close to the boundary of the original Reeb space $W_{f_0}$ of the special generic map. We can do this since the dimensions of generating polyhedra for the construction of $f_1$ are assumed to be smaller than $n - 1 < n$ and since two relations $n \geq 3$ and $2 \dim S \leq n$ are assumed; for example, as additionally needed discussions, in the case where $(n, \dim S) = (4, 2)$, we need [1], [21] etc.

The assumptions that $c$ in the manifold $S$ is represented by a closed submanifold $F$ whose normal bundle is trivial and that $\dim S \leq \frac{n}{2}$ enable us to obtain an embedding into a small open tubular neighborhood of $S_0$ so that the composition of the embedding and the projection to $S_0$ is a smooth map and that for the resulting map there exists the inverse image of a regular value regarded as $F \subset S$. Now we can demonstrate calculations. For $e_j$ and a fixed generator of $H^{(n-\dim S)-(n-\dim S_j)}(S; R)$, isomorphic to $R$, which is regarded as an element of degree $n - \dim S$ in the whole cohomology ring $H^*(W_{f_0}; R)$, the product can be calculated by considering the element of degree $n - \dim S$ as the dual of the homology class represented by a fiber of the $S^{n-\dim S_j}$-bundle over $S_j$ in the proof of Proposition 3. Note also that $d(c)$ is regarded as $\Sigma_{j=1}^b a_j e_j$ when we consider $S$ embedded in the tubular neighborhood of $S_0$. We can calculate the value at the tensor product of a cycle representing $\Sigma_{j=1}^b a_j e_j$ and the cycle obtained canonically from the fiber of the bundle before as $a_j$. This completes the first calculation. We can calculate the product similarly for the second case: we can calculate the value at the tensor product of a cycle representing $\Sigma_{j=1}^b a_j e_j$, the cycle obtained canonically from the fiber of the bundle, and $c$.

Example 5. In [9], we have studied maps obtained by finite iterations of M-bubbling operations whose generating polyhedra are bouquets of spheres starting from standard GCPS special generic maps from $m$-dimensional manifolds into $\mathbb{R}^n$ satisfying $m > n$. In these cases, by fundamental properties, if we consider two cohomology classes $c_1$ and $c_2$ such that the sum of the degrees is smaller than $n$ and that the class $c_i$ is not represented as $a_i c_i'$ for any integer $a_i > 1$ and any cocycle $c_i'$, then the product also satisfies this condition unless it vanishes. In Theorem 2, if we take these generating polyhedra not represented as bouquets of spheres, then this does not always hold.

For example, we can consider a suitable case where $n = 6$, the image of the given standard GCPS special generic map is $S^2 \times D^4$ and the generating manifold of a normal M-bubbling operation to obtain a new map $f$ is diffeomorphic to $S^2 \times S^1$.

Note that for the resulting Reeb space $W_f$, $H_j(W_f; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ for $j = 0, 2, 3, 4, 5, 6$ and zero for $j \neq 2, 3, 4, 5, 6$. It is also simply-connected.

We also present a result which may be fundamental and useful in constructing new stable fold maps. It resembles the last theorem (Theorem 9) of [8].

Theorem 2. Let $f$ be a stable fold map from a closed and connected manifold of dimension $m$ into $\mathbb{R}^n$ satisfying the relation $m > n \geq 1$. Let $R$ be a PID. For any integer $0 \leq j \leq n$ and a finitely generated module $G_j$ over $R$ satisfying that $G_0$ is trivial and that $G_n$ is not zero. Let a standard GCPS special generic map $f_0$
be given. Assume that by performing a finite iteration of S-bubbling operations starting from this special generic map whose generating polyhedra are in the interior of the Reeb space of this original special generic map disjointly and by considering a connected sum of \( f \) and the resulting map \( f_1 \) before, we can obtain a fold map \( f' \) such that the module \( H_j(W_{f'}; R) \) is isomorphic to the module \( H_j(W_f; R) \oplus G_j \); note that \( G_n \) is free by virtue of Proposition 3 and let \( H \) be a non-trivial submodule of \( G_n \) (and free). Assume also that the inverse images born by bubbles in the finite iteration of the operations to get \( f_1 \) from \( f_0 \) are all diffeomorphic to a given homotopy sphere.

Then starting from the given map \( f \) and the standard GCPS special generic map \( f_0 \) above similarly, we can obtain a fold map \( f'' \) by a finite iteration of S-bubbling operations starting from this special generic map whose generating polyhedra are in the interior of the Reeb space of this original special generic map and after that by considering a connected sum of \( f \) and the new resulting map. Furthermore, \( f'' \) satisfies the following.

1. The module \( H_j(W_{f''}; R) \) is isomorphic to the module \( H_j(W_f; R) \oplus G_j \) for \( 0 \leq j \leq n - 1 \).
2. The module \( H_n(W_{f''}; R) \) is isomorphic to the module \( H_n(W_f; R) \oplus H \).
3. The product in the ring \( H^*(W_{f''}; R) \) is defined by composing the product in the ring \( H^*(W_f; R) \) and a natural projection to \( H^*(W_{f''}; R) \). This is regarded as a subalgebra of \( H^*(W_f; R) \) in a canonical way and for \( 0 \leq j \leq n \), only in the case \( j = n \), \( H^j(W_{f''}; R) \) is not isomorphic to \( H^j(W_f; R) \): they are regarded as \( H^n(W_f; R) \oplus H \subset H^n(W_f; R) \oplus G_n \), respectively.

**Proof.** The main ingredient of the proof is how we take the generating polyhedra. If the difference between the ranks of \( G_n \) and \( H \) is \( l > 0 \), then we can choose \( l + 1 \) of the original generating polyhedra in the interior of the Reeb space of the original standard GCPS special generic map \( f_0 \) and consider the bouquet without changing the isotopy classes of the original generating polyhedra in the Reeb space. We consider a new finite iteration of bubbling operations under this situation starting from the given special generic map and consider a connected sum of \( f \) and the resulting map obtained from the special generic map. Owing to some discussions as in the proofs of Proposition 3 and Theorem 1, we see that the resulting stable fold map is a desired one.

**References**

[1] P. M. Akhmet’ev, *On an isotopic and a discrete realization of mappings of an n-dimensional sphere in Euclidean space*, Mat. Sb. 187 (1996), 3–34.

[2] D. Barden, *Simply Connected Five-Manifolds*, Ann. of Math. (3) 82 (1965), 365–385.

[3] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics (14), Springer-Verlag (1974).

[4] N. Kitazawa, *On round fold maps* (in Japanese), RIMS Kokyuroku Bessatsu B38 (2013), 45–59.

[5] N. Kitazawa, *On manifolds admitting fold maps with singular value sets of concentric spheres*, Doctoral Dissertation, Tokyo Institute of Technology (2014).

[6] N. Kitazawa, *Fold maps with singular value sets of concentric spheres*, Hokkaido Mathematical Journal Vol.43, No.3 (2014), 327–359.

[7] N. Kitazawa, *Round fold maps and the topologies and the differentiable structures of manifolds admitting explicit ones*, submitted to a refereed journal, arXiv:1304.0618 (the title has changed).
[8] N. Kitazawa, Constructing fold maps by surgery operations and homological information of their Reeb spaces, submitted to a refereed journal, arxiv:1508.05630 (the title has been changed).
[9] N. Kitazawa, Notes on fold maps obtained by surgery operations and algebraic information of their Reeb spaces, arxiv:1811.04080.
[10] M. Kobayashi, Stable mappings with trivial monodromies and application to inactive log-transformations, RIMS Kokyuroku. 815 (1992), 47–53.
[11] M. Kobayashi, Bubbling surgery on a smooth map, preprint.
[12] M. Kobayashi and O. Saeki, Simplifying stable mappings into the plane from a global viewpoint, Trans. Amer. Math. Soc. 348 (1996), 2607–2636.
[13] M. Nishioka, Special generic maps of 5-dimensional manifolds, Revue Roumaine de Mathématiques Pures et Appliquées, Volume LX No.4 (2015), 507–517.
[14] G. Reeb, Sur les points singuliers d’une forme de Pfaff complètement intégrable ou d’une fonction numérique, -C. R. A. S. Paris 222 (1946), 847–849.
[15] O. Saeki, Topology of special generic maps of manifolds into Euclidean spaces, Topology Appl. 49 (1993), 265–293.
[16] O. Saeki, Topology of special generic maps into \( \mathbb{R}^3 \), Workshop on Real and Complex Singularities (Sao Carlos, 1992), Mat. Contemp. 5 (1993), 161–186.
[17] O. Saeki and K. Sakuma, On special generic maps into \( \mathbb{R}^3 \), Pacific J. Math. 184 (1998), 175–193.
[18] O. Saeki and K. Suzuoka, Generic smooth maps with sphere fibers J. Math. Soc. Japan Volume 57, Number 3 (2005), 881–902.
[19] M. Shioya, Thom’s conjecture on triangulations of maps, Topology 39 (2000), 383–399.
[20] D. J. Wrazidlo, Standard special generic maps of homotopy spheres into Euclidean spaces, Topology Appl. 234 (2018), 348–358, arxiv:1707.08646.
[21] M. Yamamoto, Lifting a generic map of a surface into the plane to an embedding into 4-space, Illinois Journal of Mathematics 51 (2007), 705–721.