Neutrino mixing, flavor states and dark energy

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We shortly summarize the quantum field theory formalism for the neutrino mixing and report on recent results showing that the vacuum condensate induced by neutrino mixing can be interpreted as a dark energy component of the Universe.

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The study of neutrino mixing\(^{[1,2]}\) in the quantum field theory (QFT) framework\(^{[3-11]}\) has led to the discovery of the unitary inequivalence between the massive neutrino vacuum and the flavor vacuum. This has shed new light on the proper definition of neutrino flavor states and has brought to a deeper understanding of the non-perturbative nature of the particle mixing phenomenon. Pontecorvo oscillation formulae have been recognized to be the quantum mechanical approximation of the exact QFT oscillation formulas and the vacuum energy due to the neutrino-antineutrino pair condensate has been computed\(^{[12]}\), which can be interpreted as dynamically evolving dark energy\(^{[13]}\).

Here we summarize these results by considering two flavor neutrino mixing. Extension to three flavors has also been done\(^{[3-13]}\). The Pontecorvo mixing transformations for Dirac neutrino fields are

\[
\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \\
\nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta ,
\]

where \(\theta\) is the mixing angle, \(\nu_e\) and \(\nu_\mu\) are the fields with definite flavors and \(\nu_1\) and \(\nu_2\) are those with definite masses \(m_1 \neq m_2\):

\[
\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{k,v} \left[ u_{k,i}^T \alpha_{k,i}^r(t) + v_{k,i}^r \beta_{k,i}^r(t) \right] e^{i k \cdot x} ,
\]

with \(i = 1, 2\), \(r = 1, 2\), \(\alpha_{k,i}^r(t) = \alpha_{k,i}^r \ e^{-i \omega_{k,i} x} , \beta_{k,i}^r(t) = \beta_{k,i}^r \ e^{i \omega_{k,i} x} , \omega_{k,i} = \sqrt{k^2 + m_i^2}\). The operators \(\alpha_{k,i}^r\) and \(\beta_{k,i}^r\) annihilate the vacuum state \(|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2\).

The mixing transformation Eqs.\(^{(1)}\), \(^{(2)}\) can be written as\(^{(4)}\):

\[
\nu_e^\circ(x) = G_{\theta}^{-1}(t) \nu^\circ_1(x) G_\theta(t) \\
\nu_\mu^\circ(x) = G_{\theta}^{-1}(t) \nu^\circ_2(x) G_\theta(t)
\]

where \(G_\theta(t)\) is the mixing generator given by

\[
G_\theta(t) = \exp \left[ \theta \int d^3 x \left( \nu^r_1(x) \nu_2(x) - \nu^r_2(x) \nu_1(x) \right) \right] .
\]

At finite volume, \(G_\theta(t)\) is a unitary operator, \(G_\theta^{-1}(t) = G_{-\theta}(t) = G_\theta^\dagger(t)\), preserving the canonical anticommutation relations; it maps the Hilbert space \(\mathcal{H}_{1,2}\) for \(\nu_1\) and \(\nu_2\) fields to the Hilbert space \(\mathcal{H}_{e,\mu}\) for the flavor fields. In particular, \(|0(t)\rangle_{e,\mu} = G_{\theta}(t) |0\rangle_{1,2}\), where \(|0\rangle_{e,\mu} \in \mathcal{H}_{e,\mu}\) is the flavor vacuum. The crucial point is that in the infinite volume limit \(\mathcal{H}_{1,2}\) turns out to be unitarily inequivalent to \(\mathcal{H}_{e,\mu}\)\(^{(4)}\). A feature which marks the essential difference with the quantum mechanical formalism\(^{(3)}\).

The flavor annihilators, relative to the fields \(\nu_e(x)\) and \(\nu_\mu(x)\), are \(\alpha_{k,\nu}^r(t) \equiv G_\theta^{-1}(t) \alpha_{k,\nu}^r(t) G_\theta(t)\), and similar for \(\beta_{k,\nu}^r\), with \((\sigma, i) = (e, 1), (\mu, 2)\). In the reference frame such that \(k = (0, 0, |k|)\) they are explicitly given by:

\[
\alpha_{k,\nu}^r(t) = \cos \theta \alpha_{k,1}^r(t) + \sin \theta \left( |U_k| \alpha_{k,2}^r(t) + e^r |V_k| \beta_{k,2}^r \right)
\]

and similar for \(\alpha_{k,\nu}^r\), \(\beta_{k,\nu}^r\), \(\beta_{k,\nu}^r\), where \(|U_k| \equiv u_{k,i}^r v_{k,j}^r = \epsilon^r u_{k,i}^r v_{k,j}^r , i, j = 1, 2\), \(i \neq j\), \(|V_k| \equiv e^r u_{k,1}^r v_{k,2}^r = -e^r u_{k,2}^r v_{k,1}^r\). We have \(|U_k|^2 + |V_k|^2 = 1\). For the explicit expressions of \(U_k\) and \(V_k\) see Ref.\(^{(3)}\).
The number of condensed neutrinos in the flavor vacuum is given by

$$e_{\mu}(0)\sigma_{k,n}^{\dagger}(0)e_{\mu} = e_{\mu}(0)\beta_{k,n}^{\dagger}(0)e_{\mu} = \sin^2 \theta |V_{kk}|^2,$$

with $i = 1, 2$. Notice that $|V_{kk}|^2 = 0$ for $m_1 = m_2$, it has a maximum at $|k|^2 = m_1m_2$ and goes to zero for large momenta (i.e., for $|k|^2 \gg m_1m_2$) as $|V_{kk}|^2 \approx (\Delta m^2)_{\text{sol}}$. As we will see below, the mixing of neutrinos contributes to the dark energy exactly because of the non-zero value of $|V_{kk}|^2$.

Flavor neutrinos are produced in charged current weak interaction processes, like $W^+ \rightarrow e^+ + \nu_e$, as described in the Standard Model. At tree level, flavor charge is strictly conserved in the vertex; field mixing allows for the possibility of violation of lepton number via loop corrections. Such corrections are however extremely small and practically unobservable. In practice neutrinos are identified by the observation of the corresponding charged leptons, assuming flavor conservation in production (detection) vertices. In the presence of the neutrino mixing, however, one has to consider the fact that selecting to work with the Hilbert space of the mass eigenstates poses a mathematical problem, because of the existence of a separate Hilbert space (the one for the flavor eigenstates), as well as a physical problem since it amounts to work with the Pontecorvo states which are not eigenstates of the flavor charges, in contrast with the observed conservation of lepton number in the neutrino production (detection). Lepton numbers are indeed good quantum numbers, provided the neutrino production (detection) vertex can be localized within a region much smaller than the region where flavor oscillations take place, which is what happens in practice, since typically the spatial extension of the neutrino source (detector) is much smaller than the neutrino oscillation length.

Thus, flavor neutrino states have to be eigenstates of the neutrino flavor charges $Q_{\nu_{e,\mu}}$. These operators may be expressed in terms of the (conserved) charges for the neutrinos with definite masses $Q_{\nu_{1}}$ and $Q_{\nu_{2}}$ as follows:

$$Q_{\nu_{\sigma}}(t) = \cos^2 \theta Q_{\nu_{1}} + \sin^2 \theta Q_{\nu_{2}} + \sin \theta \cos \theta \int d^3x \left[ \nu_{1}^\dagger(x)\nu_{2}(x) + \nu_{2}^\dagger(x)\nu_{1}(x) \right],$$

$$Q_{\bar{\nu}_{\sigma}}(t) = \sin^2 \theta Q_{\nu_{1}} + \cos^2 \theta Q_{\nu_{2}} - \sin \theta \cos \theta \int d^3x \left[ \nu_{1}^\dagger(x)\nu_{2}(x) + \nu_{2}^\dagger(x)\nu_{1}(x) \right].$$

We remark that the last term in the above expressions forbids the construction of eigenstates of the $Q_{\nu_{e,\mu}}(t)$, $\sigma = e, \mu$, in the Hilbert space $\mathcal{H}_{1,2}$ for the fields with definite masses. One can also show that the above flavor charge operators are diagonal in the ladder operators $\alpha_{\sigma}$, $\beta_{\sigma}$, for the neutrino flavor fields:

$$Q_{\nu_{\sigma}}(t) = \sum_{\tau} \int d^3k \left( \alpha_{k,\sigma}^\dagger \alpha_{k,\sigma}^\tau(t) - \beta_{-k,\sigma}^\dagger \beta_{-k,\sigma}^\tau(t) \right),$$

where $\sigma = e, \mu$ and $\vdots \ldots \vdots$ denotes normal ordering with respect to $|0\rangle_{e,\mu}$. This makes straightforward the definition (at reference time $t = 0$) of flavor neutrino and antineutrino states as:

$$|\nu_{k,\sigma}\rangle = \alpha_{k,\sigma}^\dagger |0\rangle_{e,\mu}; \quad |\bar{\nu}_{k,\sigma}\rangle = \beta_{k,\sigma}^\dagger |0\rangle_{e,\mu}, \quad \sigma = e, \mu,$$

which are thus by construction eigenstates of the operators in Eq. (11) at $t = 0$, as it should be in agreement with the observed conservation of flavor charge in the neutrino production (detection) vertices.

The oscillation formulas are obtained by taking expectation values of the above charges on the flavor neutrino states. Consider for example an initial electron neutrino. Working in the Heisenberg picture, we obtain

$$Q_{\nu_{e}}^k(t) = \langle \nu_{k,e}^\dagger | Q_{\nu_{e}}(t) | \nu_{k,e}^\rangle = \left| \left\{ \alpha_{k,\sigma}^\dagger(t), \alpha_{k,\sigma}^\tau(0) \right\} \right|^2 + \left| \left\{ \beta_{-k,\sigma}^\dagger(t), \alpha_{k,\sigma}^\tau(0) \right\} \right|^2.$$  (13)

Charge conservation is obviously ensured at any time: $Q_{\nu_{e}}^k(t) + Q_{\bar{\nu}_{e}}^k(t) = 1$ and $e_{\mu}(0)Q_{\nu_{e}}(t)|0_{e,\mu} = 0$. The oscillation (in time) formula for the flavor charge on an initially electron neutrino state, is thus:

$$Q_{\nu_{e}}(t) = 1 - \sin^2(2\theta) |U_k|^2 \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}t}{2} \right) - \sin^2(2\theta) |V_k|^2 \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}t}{2} \right),$$

(14)

with $|U_k|^2 + |V_k|^2 = 1$. The differences with respect to the usual formula for neutrino oscillations are in the energy dependence of the amplitudes and in the additional oscillating term. In the relativistic limit $|k| \gg m_1m_2$, we have $|U_k|^2 \rightarrow 1$ and $|V_k|^2 \rightarrow 0$ and the traditional (Pontecorvo) oscillation formula is recovered.
As already observed, contrarily to the QFT flavor states considered above, the quantum mechanical (Pontecorvo) flavor states \[1\]:

\[
|\nu_{k,\alpha}^{\nu}\rangle = \cos \theta |\nu_{k,1}^{\nu}\rangle + \sin \theta |\nu_{k,2}^{\nu}\rangle ,
\]

\[
|\nu_{k,\mu}^{\nu}\rangle = -\sin \theta |\nu_{k,1}^{\nu}\rangle + \cos \theta |\nu_{k,2}^{\nu}\rangle ,
\]

are not eigenstates of the flavor charges \[10\]. This can be explicitly seen by using Eqs. (9) and (10). One can estimate how much the flavor charge is violated in the usual quantum mechanical states by taking the expectation values of the flavor charges on the above Pontecorvo states. Considering for example an electron neutrino state, we obtain:

\[
p\langle \nu_{k,\alpha}^{\nu} | Q_{\nu} : | \nu_{k,\alpha}^{\nu}\rangle = \left[ \cos^4 \theta + \sin^4 \theta + 2 |U_{k}| \sin^2 \theta \cos^2 \theta < 1, \right.
\]

\[
p\langle \nu_{k,\alpha}^{\nu} | Q_{\nu} : | \nu_{k,\alpha}^{\nu}\rangle = 2 (1 - |U_{k}|) \sin^2 \theta \cos^2 \theta > 0,
\]

for any \( \theta \neq 0, m_1 \neq m_2, k \neq 0 \) and where : \( \cdots \) : denotes normal ordering with respect to the vacuum state \( |0\rangle_{1,2} \). In the relativistic limit, \( |U_{k}| \rightarrow 1 \) and the Pontecorvo states are a good approximation for the exact charge eigenstates Eq. (12).

We now turn our attention to the vacuum energy due to neutrino condensate. We calculate the contribution \( \rho_{\text{vac}}^{\text{mix}} \) of the neutrino mixing to the vacuum energy density in a Minkowski space-time, but the computation can be extended to curved space-times. At the present epoch it is a good approximation of that in FRW space-time, since the characteristic oscillation length of the neutrino is much smaller than the universe curvature radius. Computations carried on in a curved background give a time dependent dark energy, leading, however, to the same final result we obtain in the flat space-time.

The Lorentz invariance of the vacuum requires that the vacuum energy-momentum tensor density \( T_{\mu\nu}^{\text{vac}} \) is equal to zero. The \((0,0)\) component of \( T_{\mu\nu} \) for the fields \( \nu_1 \) and \( \nu_2 \) is: \( T_{00} = : \bar{\Psi}_{m}(x) \gamma_0 \partial_0 \Psi_{m}(x) : \), where \( \Psi_{m} = (\nu_1, \nu_2)^T \). The \((0,0)\) component of the energy-momentum tensor \( T_{00} = \int d^3x T_{00}(x) \) is then given by: \( T_{00}^{(i)} = \sum_r \int d^3k \omega_{k,i} \left( \alpha_{k,i}^\dagger \alpha_{k,i} + \beta_{k,i}^\dagger \beta_{k,i} \right) \), with \( i = 1,2 \). We note that \( T_{00}^{(i)} \) is time independent.

In the early universe epochs, when the Lorentz invariance of the vacuum condensate is broken, \( \rho_{\text{vac}}^{\text{mix}} \) presents also space-time dependent condensate contributions [13]. Then \( \rho_{\text{vac}}^{\text{mix}} \) and the contribution of the neutrino mixing to the vacuum pressure \( p_{\text{vac}}^{\text{mix}} \) are given respectively by:

\[
\rho_{\text{vac}}^{\text{mix}} = - \frac{1}{V} \sum_i : T_{00}^{(i)} : |0\rangle_{e,\mu} ,
\]

\[
p_{\text{vac}}^{\text{mix}} = - \frac{1}{V} \sum_i T_{ij}^{(i)} |0\rangle_{e,\mu} ,
\]

where \( T_{ij}^{(i)} = \sum_r \int d^3k \frac{\bar{\Psi}_{k,i} \Psi_{k,i}}{(2\pi)^3} \left( \alpha_{k,i}^\dagger \alpha_{k,i} + \beta_{k,i}^\dagger \beta_{k,i} \right) \) (no summation on the index \( j \) is intended). We then obtain

\[
\rho_{\text{vac}}^{\text{mix}} = \sum_{i,r} \int \frac{d^3k}{(2\pi)^3} \omega_{k,i} \left( e_{,\mu} \langle 0 | \alpha_{k,i}^\dagger \alpha_{k,i} |0\rangle_{e,\mu} + e_{,\mu} \langle 0 | \beta_{k,i}^\dagger \beta_{k,i} |0\rangle_{e,\mu} \right) ,
\]

and, by introducing the cut-off \( K \),

\[
\rho_{\text{vac}}^{\text{mix}} = \frac{2}{\pi} \sin^2 \theta \int_0^K dk k^2 (\omega_{k,1} + \omega_{k,2}) |V_k|^2 .
\]

We also obtain

\[
p_{\text{vac}}^{\text{mix}} = \frac{2}{3 \pi} \sin^2 \theta \int_0^K dk k^4 \left[ \frac{1}{\omega_{k,1}} + \frac{1}{\omega_{k,2}} \right] |V_k|^2 .
\]

Eqs. (22) and (23) show that the adiabatic index is \( w = p_{\text{vac}}^{\text{mix}} / \rho_{\text{vac}}^{\text{mix}} \approx 1/3 \) when the cut-off is chosen to be \( K \gg m_1, m_2 \).

The values of \( \rho_{\text{vac}}^{\text{mix}} \) and \( p_{\text{vac}}^{\text{mix}} \) which we obtain are time-independent since we are taking into account the Minkowski metric. In a curved space-time, time-dependence has to be taken into account, but the essence of the result is the same.
At the present epoch, the breaking of the Lorentz invariance is negligible and then the contribution $\rho_{\Lambda}^{\text{mix}}$ of the neutrino mixing to the vacuum energy density comes from space-time independent condensate contributions (i.e. the contributions carrying a non-vanishing $\partial_\mu \sim k_\mu = (\omega_k, k_j)$ are missing). Thus, the energy-momentum density tensor of the vacuum condensate is

$$\rho_{\mu\nu} = \eta_{\mu\nu} \sum_i m_i \int \frac{d^3 x}{(2\pi)^3} \langle 0 : \bar{\nu}_i(x) \nu_i(x) : 0 \rangle \delta_{\mu\nu} + \eta_{\mu\nu} \rho_{\Lambda}^{\text{mix}}. \quad (24)$$

Since $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and, in a homogeneous and isotropic universe, $T_{\mu\nu} = \text{diag}(\rho, p, p, p)$, then the state equation is $\rho_{\Lambda}^{\text{mix}} = \rho_{\Lambda}^{\text{mix}} \pm \frac{\rho_{\Lambda}^{\text{mix}}}{m_\nu}$. This means that $\rho_{\Lambda}^{\text{mix}}$ contributes today to the dynamics of the universe by a cosmological constant behavior. We obtain

$$\rho_{\Lambda}^{\text{mix}} = \frac{2}{\pi} \sin^2 \theta \int_0^K dk \frac{k^2}{\omega_{k,1} + \omega_{k,2}} |V_k|^2. \quad (25)$$

Notice that $\rho_{\Lambda}^{\text{mix}}$ would be zero for $|V_k|^2 = 0$ for any $|k|$, as it is in the usual (Pontecorvo) approximation.

One can show \[13\] that the integral in Eq. \[25\] diverges in $K$ as $m_\nu^4 \log (2K/m_\nu^2)$. However, the divergence in $K$ is smoothed by the factor $m_\nu^4$. For neutrino masses of order of $10^{-3}eV$ and $\Delta m_{12}^2 \approx 10^{-5}eV^2$ we have $\rho_{\Lambda}^{\text{mix}} \approx 2.9 \times 10^{-47}GeV^4$, which is the observed dark energy value, even for a value of the cut-off of order of the Planck scale $K = 10^{19}GeV$ (see Fig. \[2\]). One can also see that $\frac{d\rho_{\Lambda}^{\text{mix}}(K)}{dK} \propto \frac{1}{K} \to 0$ for large $K$.

In conclusion, the vacuum condensate generated by neutrino mixing can be interpreted as an evolving dark energy that, at present epoch, behaves as the cosmological constant.

When the three generation mixing is considered we obtain a value for $\rho_{\Lambda}^{\text{mix}}$ which differs of about 5 order of magnitude from the observed dark energy value. It has been proposed in the literature \[14\] that heaviest neutrinos should not contribute to the vacuum energy density. The present result might suggest that this is indeed the case. However, it also points to the need of a further theoretical refinement of the present approach, which is in our future research plans. At present state of art, among the merits of the complete QFT treatment of neutrino mixing above summarized there is the remarkable improvement in the computed order of magnitude of the dark energy (to be compared with the 123 order of magnitude of disagreement in usual approaches) and the fact that in the present approach there is no need of postulating (till now) unobserved exotic fields.

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