Miura Map between Lattice KP and its Modification is Canonical

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Abstract

We consider the Miura map between the lattice KP hierarchy and the lattice modified KP hierarchy and prove that the map is canonical not only between the first Hamiltonian structures, but also between the second Hamiltonian structures.

1 Introduction

It is well-known that Miura map, a transformation between the KdV equation and MKdV equation, plays a central role in the development of soliton theory. Indeed, the celebrated Inverse Scattering Method for solving nonlinear equations starts with the Miura map [1]. This type of transformations turns out to exist in the context of other integrable equations (see [1]-[10],[13]-[14] and the references there).

Kupershmidt, in a recent paper [8], considered the canonical properties of Miura maps between KP and MKP hierarchies. He shown that, both in continuous and discrete cases, Miura transformations are canonical between the first Hamiltonian structures. For the ordinary or continuous KP and its modification, Shaw and Tu [13] generalized the results of Kupershmidt and proved that the very Miura map is also canonical between the second Hamiltonian structures.

We will consider the canonical property of the Miura map between the Lattice MKP (lMKP) and the Lattice KP (lKP) hierarchies. The lKP hierarchy is a bi-Hamiltonian system and two Hamiltonian structures were constructed by using the residue calculus in [7]. For the lMKP hierarchy, the first Hamiltonian structure was also found in [8]. A slight different version of the lMKP hierarchy was proposed by Oevel and he further obtained the bi-Hamiltonian description for this hierarchy by means of r-matrix approach [12]. By introducing a parameter, we unify Kupershmidt’s version of the lMKP hierarchy and Oevel’s version into a single system. Our main purpose of the paper is to prove that Kupershmidt’s Miura map is a canonical map not only between the first Hamiltonian structures of lKP and lMKP, but also between the second Hamiltonian structures.

The paper is organized as follows. In the next section, we introduce notations and recall the relevant formulae such as bi-Hamiltonian structures of the lKP and lMKP hierarchies. In section 3 and section 4, we show that Kupershmidt’s Miura map is a canonical transformation for the first Hamiltonian structures and the second Hamiltonian structures respectively. The last section is intended to summary and discussions.

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2 Background and Notations

To introduce the lKP and lMKP hierarchies, we consider the algebra of shift operators

\[ g = \{ u_N(n)T^N + u_{N-1}(n)T^{N-1} + \cdots + u_0(n) + u_{-1}(n)T^{-1} + \cdots \}, \]

where \( u_j \) are scalar functions of integer \( n \). The shift operator \( T \) is given by

\[(T f)(n) = f^{(1)}(n) := f(n + 1),\]

and for arbitrary integer \( k \), \((T^k f)(n) = f^{(k)}(n) = f(n + k)\).

For any operator \( \xi = \sum_j u_j T^j \in g \), the projections to various shift orders are denoted by

\[ \xi_j = u_j T^j, \quad \xi_{\geq k} = \sum_{j \geq k} u_j T^j, \quad \xi_{< k} = \sum_{j < k} u_j T^j, \]

\[ \xi_{> k} = \sum_{j > k} u_j T^j, \quad \xi_{\leq k} = \sum_{j \leq k} u_j T^j. \]

From the shift operator \( T \), we also have the difference operator

\[ \Delta = T - 1, \]

and its formal inverse

\[ \Delta^{-1} = \sum_{j \geq 1} T^{-j}. \]

Another important notation is so called the trace, which is defined as

\[ \text{tr}(\sum_i u_i T^i) = \sum_n u_0(n), \]

this permits us to identify \( g \) and its dual by the metric \( g^* : < u^*, u > = \text{tr}(u^* u) \). It can be shown that the metric is bi-invariant.

The lKP hierarchy is defined by the following Lax operator

\[ L = T + \sum_{i=0}^{\infty} A_i T^{-i}, \quad (1) \]

and the flow equations are constructed as

\[ L_{tn} = [(L^n)_{\geq 0}, L]. \quad (2) \]

The lKP hierarchy (2) is a bi-Hamiltonian system. Its two Hamiltonian structures are constructed by means of the residue calculus in [7]. Recently, Oevel proposed a \( r \)-matrix setting for these Hamiltonian structures. The two Hamiltonian structures are given by the following Poisson tensors

\[ P_1(\nabla H) = [\nabla H, L]_{\leq 0}, \]

\[ P_2(\nabla H) = (L\nabla H)_{\geq 1}L - L(\nabla HL)_{\geq 1} + \frac{1}{2}[(L\nabla H + \nabla HL)_0, L] + \frac{1}{2}[\rho(\nabla H, L)_0, L], \quad (4) \]

where \( \rho \) is a skew-symmetric linear map on the algebra \( g_0 \) given explicitly by

\[ \rho = \frac{T + 1}{T - 1}, \quad (5) \]
As for the lMKP hierarchy, we consider the following Lax operator

\[ \alpha T \]

where \( \alpha \) and \( \alpha \) satisfy the conditions of Suris’s theorem (or the theorem 1 of Oevel [12]), therefore they also solves the modified Yang-Baxter equation only and only if \( \alpha = 0 \) or \( \alpha = 1 \). As we mentioned above, these are exactly the two cases studied by Kupershmidt and by Oevel. In the following, our parameter \( \alpha \) will take the value either one or zero. The above r–matrix leads to the first Hamiltonian structure for the lMKP hierarchy.

To get the second Poisson tensor, one may use Suris’s construction [15] by considering the following linear operators

\[
\begin{align*}
\mathcal{A}_1(\xi) &= \xi_{\geq 1} - \xi_{< 0} - 2\alpha(\xi\Delta^{-1})_0 - \rho(\xi_0) + 2\alpha\Delta^{-1}\xi_0, \\
\mathcal{A}_2(\xi) &= \xi_{\geq 1} - \xi_{< 0} + \rho(\xi_0), \\
\mathcal{S}(\xi) &= \rho(\xi_0) - \xi_0 - 2\alpha\Delta^{-1}\xi_0, \\
\mathcal{S}^1(\xi) &= -\rho(\xi_0) - \xi_0 - 2\alpha(\xi\Delta^{-1})_0.
\end{align*}
\]

When \( \alpha = 1 \), the above operators are those presented by Oevel and are lead to the second Poisson tensor for this case. It can be proved that in the case \( \alpha = 0 \), these operators are satisfied the conditions of Suris’s theorem (or the theorem 1 of Oevel [12]), therefore they also lead to a Poisson tensor, this time is for Kupershmidt’s case. Unifying both Kupershmidt’s case and Oevel’s case, we have the following two Poisson tensors

\[
\begin{align*}
\tilde{\mathcal{P}}_1(\nabla H) &= \left[ (\nabla H)_{\geq 1}, \mathcal{L} \right] - [\nabla H, \mathcal{L}]_{\leq 0} - \alpha([\nabla H\Delta^{-1}]_0, \mathcal{L}) - \alpha\Delta^{-1}[\nabla H, \mathcal{L}]_0, \\
\tilde{\mathcal{P}}_2(\nabla H) &= (\mathcal{L}\nabla H)_{\geq 1} - \mathcal{L}([\nabla H]_{\geq 1}) + \frac{1}{2} [\mathcal{L}, \nabla H]_0 \mathcal{L} + \frac{1}{2} \mathcal{L} [\mathcal{L}, \nabla H]_0 + \\
&+ \alpha\Delta^{-1}[\mathcal{L}, \nabla H]_0 \mathcal{L} + \alpha [\mathcal{L}, (\mathcal{L}\nabla H\Delta^{-1})_0] + \frac{1}{2} \rho([\nabla H, \mathcal{L}]_0, \mathcal{L}),
\end{align*}
\]

where \( \rho \) is the one defined by (3) and \( \nabla H \) is parametrized as

\[
\nabla H = T^{-1} \frac{\delta H}{\delta q} + \frac{\delta H}{\delta a_0} + T \frac{\delta H}{\delta a_1} + \cdots
\]
In the remaining part of this section, we introduce the Miura map between lKP hierarchy and lMKP hierarchy following Kupershmidt. With the aid of a new field \( w \), we introduce a map between lKP and lMKP hierarchies via the conjugacy

\[
L = e^w \mathcal{L} e^{-w} = e^w q T e^{-w} + \sum_{i=0}^{\infty} e^w a_i T^{-i} e^{-w},
\]

comparing the coefficients of different power of the shift operator of two sides leads to a transformation

\[
q = e^{w(1) - w}, \quad A_i = a_i e^{w(-i)}, \quad (i \geq 0).
\]

Let us introduce the new notations

\[
R_i = R_i(q) := \prod_{s=0}^{i} q^{(-s)}/q, \quad (i \geq 0).
\]

By eliminating the intermediate variable \( w \), we reach the Miura map between the two sets of variables

\[
M : \quad A_0 = a_0, \quad A_i = R_i a_i \quad (i > 0).
\]

this is the Miura map constructed in [8].

Now we prove that if \( \mathcal{L} \) solves the mlKP hierarchy, \( L = e^w \mathcal{L} e^{-w} \) solves the lKP hierarchy. From \( q = e^{w(1) - w} \), we obtain \( q_t = q(T - 1)w_t \). On the other hand, the time evolution of \( q \) can be read from the mlKP hierarchy, that is \( q_t = q(T - 1) \left( (L^n)_0 + \alpha (L^n \Delta^{-1})_0 \right) \), so \( w_t = (L^n)_0 + \alpha (L^n \Delta^{-1})_0 \). Now

\[
L_t = [w_t, L] + e^w [(L^n)_{\geq 1} - \alpha (L^n \Delta^{-1})_0, L] e^{-w}
\]

\[
= [w_t, L] + [e^w (L^n)_{\geq 1} e^{-w}, L] - \alpha [(L^n \Delta^{-1})_0, L] = [(L^n)_{\geq 1}, L],
\]

where we used \((L^n)_0 = (L^n)_0 \) and \( e^w (L^n)_{\geq 1} = (L^n)_{\geq 1} e^w \). Thus the Miura map (12) indeed converts the lMKP hierarchy into the lKP hierarchy.

### 3 Canonical Properties for First Hamiltonian Structures

In this section, we prove that Miura map is canonical between the first Hamiltonian structures. First we calculate the Hamiltonian matrices from the Poisson tensors (8) and (9). By substituting (8) into \( P_1 \) and (11) into \( P_2 \), it is straightforward to get

\[
B_{ij}^{lKP} = (B_{ij}), B_{ij} = T^j A_{i+j} - A_{i+j} T^{-i}, \quad (i, j \geq 0),
\]

and

\[
B_{ij}^{lMKP} = q a_i \begin{pmatrix}
0 & 0 & 0 \\
q(T - 1) & \alpha q(T - 1) T^j & 0 \\
(1 - T^{-1})q & 0 & B_{ij}^{(lMKP)}
\end{pmatrix},
\]

where

\[
B_{ij}^{(lMKP)} = T^j a_{i+j} - a_{i+j} T^{-i} + \alpha (a_i T^{j-i} - T^{j-i} a_j + T^{-i} a_j - a_i T^j).
\]

The Jacobian matrix of the Miura map (12) is easily calculated as

\[
J = A_n \begin{pmatrix}
a_n D_n & R_n \delta^n_0 & R_n \delta^n_i
\end{pmatrix},
\]
where $\delta^i_j$ is the standard Kronecker symbol and $D_n$ is the abbreviated notation for the Fréchet derivative given by

$$D_n := D(R_n) = R_n \frac{1 - T^{-n}}{T-1} q^{-1}, \quad D^i_j = q^{-1} \frac{1 - T^m}{T-1} R_n. \quad (17)$$

We need to calculate the matrix operator $JB^1_{lMKP} J^i$, but first it is easy to find that

$$JB^1_{lMKP} = A_0 \begin{pmatrix} q & a_0 \ & \ & a_{j>0} \end{pmatrix} A_{i>0} \begin{pmatrix} (1 - T^{-1})q & 0 \ \alpha R_i T^{-i} (1 - T^{-1})q & a_i D_i q(T - 1) \ \alpha a_i D_i q(T - 1) T^j + R_i B^1_{ij}^{lMKP} \end{pmatrix},$$

now the entries of the first row of the $JB^1_{lMKP} J^i$ are seen as

$$(JB^1_{lMKP} J^i)^{0,m} = (1 - T^{-1})q D^i_m a_m = -(1 - T^m) R_m a_m = (T^m - 1) A_m,$$

which coincide with the $(B^1_{lKP})^{0,m}$. It is noticed that we have used the second formula of (17).

Therefore, for the first row and the first column, two matrix operators $B^1_{lKP}$ and $JB^1_{lMKP} J^i$ are just the same as expected. We turn our attention to other entries of matrices. We find that

$$(JB^1_{lMKP} J^i)^{m,n} = \alpha R_m T^{-m} (1 - T^{-1})q D^i_m a_n + \alpha a_m D_m q(T - 1) T^m R_n + \alpha R_m (a_m T^{-m} - T^{-m} a_n + T^{-m} a_n - a_m T^m) R_n + \alpha R_m (a_m T^{-m} - T^{-m} a_n + T^{-m} a_n - a_m T^m) R_n + R_m (T^m a_m + n - a_n + m T^{-m}) R_n = R_m (T^m a_m + n - a_n + m T^{-m}) R_n,$$

now we use the formula in [8]

$$R_n T^m R_n = T_m R_n + m,$$

and obtain the desired the results $(JB^1_{lMKP} J^i)^{m,n} = (B^1_{lKP})_{m,n}$. Thus, Miura map is indeed canonical.

### 4 Canonical Property for Second Hamiltonian Structures

We now show that the Miura map $B^1_{lKP}$ is also canonical between the second Hamiltonian structure of the IKP hierarchy and the second Hamiltonian structure of the IMKP hierarchy. As in last section, we first calculate the Hamiltonian matrix operators from the Poisson tensors [4] and [10]. The calculation in the present case is a bit cumbersome although it is straightforward. For the IKP hierarchy we have

$$\begin{align*}
  A_{k,t} &= \sum_{\ell=0}^{\infty} (B^1_{lKP})_{k\ell} \delta H \delta A_{\ell}, \quad k \geq 0, \quad t \equiv t_n, \quad H \equiv H_n = \frac{1}{n} \text{tr}(L^n), \\
  (B^1_{lKP})_{k\ell} &= \sum_{j=1}^{\ell+1} (A_{\ell-j} T^j A_{k+j} - A_{k+j} T^{\ell-k-j} A_{\ell-j}) + A_k (1 - T^{-k}) (1 + T + \cdots + T^\ell) A_t, \quad A_{-1} \equiv 1, \quad k \geq 0, \ell \geq 0.
\end{align*}$$
For the mlKP hierarchy, we have

\[ B_{2}^{\text{MLKP}} = \begin{pmatrix} q & a_{m \geq 0} \\ q(T - T^{-1})q & \alpha q(T - 1) \sum_{i=1}^{m+1} a_{m-i}T^{i} + \\ \alpha \sum_{j=1}^{k+1} T^{-j}a_{k-j}(1 - T^{-1})q + & B_{km}^{\text{MLKP}}, \\ a_{k}(T^{-k} - T^{-1})q & \end{pmatrix} \]

with

\[ B_{km}^{\text{MLKP}} = \sum_{i=1}^{m+1} (a_{m-i}T^{i}a_{k+i} - a_{k+i}T^{m-k-i}a_{m-i}) + a_{k}\frac{(1 - T^{-k+1})(1 - T^{m})}{1 - T}a_{m} + \]

\[ \alpha a_{k}(T^{-k} - 1) \sum_{i=1}^{m+1} a_{m-i}T^{i} + \alpha \sum_{j=1}^{k+1} T^{-j}a_{k-j}(1 - T^{m})a_{m}, \quad a_{-1} \equiv q. \]

Thus, the matrix operator \( JB_{2}^{\text{MLKP}} \) reads as

\[ JB_{2}^{\text{MLKP}} = \begin{pmatrix} q & a_{m \geq 0} \\ qT^{m+1}a_{m+1} - a_{m+1}T^{-1}q + & \alpha T^{-1}q(1 - T^{m})a_{m} \\ a_{k}D_{k}q(T - T^{-1})q + & a_{k}D_{k}(q(T - T^{-m})a_{m} + \\ a_{k}R_{k}(T^{-k} - T^{-1})q + & \alpha R_{k}q(T - 1) \sum_{i=1}^{m+1} a_{m-i}T^{i}) + R_{k}B_{km}^{\text{MLKP}}, \end{pmatrix} \]

With all these formulae in hand, we find that the entries of the first row of \( JB_{2}^{\text{MLKP}}J^{\dagger} \) are

\[ (JB_{2}^{\text{MLKP}}J^{\dagger})_{0,n} = (a_{0} + \alpha T^{-1}q)(1 - T^{-1})qD_{n}^{\dagger}a_{n} + \]

\[ \left(qT^{n+1}a_{n+1} - a_{n+1}T^{-1}q + \alpha T^{-1}q(1 - T^{n})a_{n}\right)R_{n} \]

\[ = -(a_{0} + \alpha T^{-1}q)(1 - T^{n})A_{n} + qT^{n+1}a_{n+1}R_{n} - \]

\[ a_{n+1}T^{-1}qR_{n} + \alpha T^{-1}q(1 - T_{n})A_{n} \]

\[ = -a_{0}(1 - T^{n})A_{n} + qT^{n+1}a_{n+1}R_{n} - a_{n+1}T^{-1}qR_{n} = \]

\[ = A_{0}(T^{n} - 1)A_{n} + T^{n+1}A_{n+1} - A_{n+1}T^{-1} = (B_{2}^{\text{MLKP}})_{0,n}, \]

where we used

\[ q^{(-n-1)}R_{n} = R_{n+1}, \quad q^{-1}R_{n}^{(-1)} = R_{n+1}, \]

which hold identically. For the remaining entries, we have,

\[ (JB_{2}^{\text{MLKP}}J^{\dagger})_{mn} = \]

\[ a_{m}D_{m}q(T - T^{-1})qD_{n}^{\dagger}a_{n} + R_{m}a_{m}(T^{-m} - T^{-1})qD_{n}^{\dagger}a_{n} + \]

\[ a_{m}D_{m}q(T - T^{n})a_{n} + R_{m}a_{m}\frac{(1 - T^{-m+1})(1 - T^{n})}{1 - T}a_{n}R_{n} + \]

\[ \alpha R_{m} \sum_{j=1}^{m+1} T^{-j}a_{m-j}\left((1 - T^{-1})qD_{n}^{\dagger}a_{n} + (1 - T^{n})a_{n}R_{n}\right) + \]

\[ (18) \]

\[ (19) \]

\[ (20) \]
\[ \alpha \left( a_m D_m q(T - 1) + R_m a_m (T^m - 1) \right) \sum_{j=1}^{n+1} a_{n-j} T^j R_n + \]
\[ R_m \left( \sum_{t=1}^{n+1} (a_{n-t} T^i a_{m+i} - a_{m+i} T^{m-i} a_{n-i}) \right) R_n, \]

so we need to prove that above expression is \((B_2^{lKP})_{mn}\).

It is easy to see that \(\alpha = 0\) in terms of \(D_m\).

Since \(-T^{-1} = -(1 + T)(T^{-1} - 1)\), we obtain
\[
\left[ 1 \right] + \left[ 2 \right] = -A_m \left( 1 - \frac{T^{-m}}{T-1} \right) (1 + T) (1 - T^m) A_n + A_m (T^{-m} - T^{-1}) \frac{1 - T^n}{T - 1} A_n
\]
\[ + A_m \left( 1 - \frac{T^{-m}}{T-1} \right) (T - T^n) A_n + A_m \frac{(1 - T^{-m+1})(1 - T^n)}{1 - T} A_n.
\]

Thus to complete the proof, we need to show that
\[ R_m \left( \sum_{i=1}^{n+1} a_{n-i} T^i a_{m+i} - a_{m+i} T^{m-i} a_{n-i} \right) R_n = \sum_{j=1}^{n+1} (A_{n-j} T^j A_{m+j} - A_{m+j} T^{m-j} A_{n-j}), \]

this amounts to the identity
\[ R_m T^j R_n = R_{n-j} T^j R_{m+j}, \quad 1 \leq j \leq n + 1, \]

which can be seen as follows
\[
R_m T^j R_n = q^{-1} \cdots q^{-m} q^{-1+j} \cdots q^{-n+j}
\]
\[ = q^{-1} \cdots q^{-m} q^{-1+j} \cdots q q^{-1} \cdots q^{-n+j} = \]
\[ = q^{-1} \cdots q^{-m+j} q^{-1+j} \cdots q a^{-1} \cdots q^{-m} = \]
\[ = R_{n-j} T^j R_{m+j}. \]

Thus, we conclude that the Miura map is canonical in the sense of the second Hamiltonian structures.

5 Conclusions and Discussions

We have proved that the canonical property of Miura map holds between the lKP hierarchy and the lMKP hierarchy, that is, it maps the bi-Hamiltonian structures of the lMKP hierarchy to those of the lKP hierarchy. In [3], the lattice KP hierarchy is extended and it turns out that the extended lattice KP hierarchy is isomorphic to the lattice MKP hierarchy. Since we are dealing a slight generalized version of lMKP hierarchy here (8), we have a different extended lKP hierarchy.

Introducing a new field \(u\) and define the following invertible transformation
\[ u = q, \quad A_0 = a_0, \quad A_i = R_a a_i, \]

It is easy to see that the first Hamiltonian matrix operator for our extended lKP hierarchy reads
\[
B_1^{lKP} = \begin{pmatrix}
0 & u(T - 1) & \alpha u(T - 1) T^m R_m \\
(1 - T^{-1}) u & 0 & (T^m - 1) A_m \\
\alpha R_a T^{-n} (1 - T^{-1}) u & A_n (1 - T^{-n}) & T^m A_{n+m} - A_{n+m} T^{-n}
\end{pmatrix}, \quad (23)
\]
and the flow equations are given
\[ u_t = u(T - 1) \frac{\delta H}{\delta A_0} + \alpha \sum_{m=1}^{\infty} u(T - 1)T^m \frac{\delta H}{\delta A_m}, \]
\[ A_{i,t} = \alpha R_n T^{-n} (1 - T^{-1}) \frac{\delta H}{\delta u} + \sum_{j=0}^{\infty} (B_1^e_{1KP})_{ij} \frac{\delta H}{\delta A_j}, \quad H \equiv H_{n+1} = \frac{1}{n+1} \text{tr}(L^{n+1}), \]
where the Hamiltonian \( H \) is the seam as in the \( lKP \) case. We could have a second Hamiltonian structure for the extended \( lKP \) hierarchy, but it is in a rather complicated form. So we omit it.

To conclude the paper, we point out that it seems interesting to prove the canonical property of the Miura map on the level of the Poisson tensors since that will hopefully make the proof more concise. For the Gelfand-Dickey hierarchy, such proof was given by Dickey \([3]\) and for the continuous KP hierarchy and the constrained KP hierarchy, it is provided in \([13, 14]\) respectively.

**Acknowledgements**

It is a pleasure to thank Professor S.Y. Lou for helpful discussions. The part of work was done during the author's stay in the Abdus Salam International Centre for Theoretical Physics and he should like to thank the AS ICTP for hospitality. The work is supported in part by National Natural Science Foundation of China.

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