TWO GENERATOR SUBGROUPS OF RIGHT-ANGLED ARTIN GROUPS ARE QUASI-ISOMETRICALLY EMBEDDED

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Abstract. We show that two-generator subgroups of right-angled Artin groups are quasi-isometrically embedded. This provides an alternate proof of a theorem of A. Baudisch: that these subgroups are free or free abelian. As a consequence, we may detect groups that do not quasi-isometrically embed in a RAAG, and thus are not the fundamental group of a special cube complex.

1. Introduction

Definition 1.1. Given a combinatorial graph $\Gamma$ with vertex set $V$ and edge set $E$, the right-angled Artin group $A_\Gamma$ is the group presented by the generating set $V$ and the relations $\{[v_i, v_j] \mid (v_i, v_j) \in E\}$, where $[v_i, v_j]$ denotes the commutator.

Right-angled Artin groups (RAAGs) constitute a spectrum between free abelian groups, given by complete graphs, and free groups, given by edgeless graphs. The simple definition belies that fact that complicated groups can exist as subgroups of right-angled Artin groups, including nearly all surface groups [CW04]. In addition to providing a wealth of interesting examples, RAAG subgroups have underpinned several recent results in group theory and topology. The recent proof of the virtual Haken conjecture [AGM12] relied on showing that every hyperbolic 3-manifold group had a finite index subgroup that embeds in a RAAG via a map defined in [HW08]. Hsu and Wise also used embeddings of subgroups to show that certain graph groups are linear [HW10].

The behavior of two-generator subgroups of RAAGs is much more circumscribed. A theorem of A. Baudisch completely describes their group structure.

Theorem 1.2. [Bau81, 1.3] Every two generator subgroup of $A_\Gamma$ is either free or free abelian.

This description of two-generator subgroups is interesting for at least two reasons. First, it passes to subgroups. If this description holds for a group $G$, then it holds for every subgroup $H < G$. Second, two generator subgroups of many other well-studied groups have this description or some approximation of it. In mapping class groups, it is false in general, but given two generators, one can take appropriate powers of each such that the group those powers generate is free or free abelian.

Two generator subgroups of pure braid groups do fit this description. This fact that was used by [Scr10] and [KK13] who recovered Baudish’s result by embedding any $A_\Gamma$ in some pure braid group.

Our main result concerns the metric structure of two-generator subgroups in a right-angled Artin group. We use the word metric to realize each group as a metric space. The desired metric property is the following.
Definition 1.3. A quasi-isometric embedding is a map $f : X \to Y$ between metric spaces such that there are constants $\lambda \geq 1$ and $\epsilon \geq 0$ for which
\[
\frac{1}{\lambda} d_X(p, q) - \epsilon \leq d_Y(f(p), f(q)) \leq \lambda d_X(p, q) + \epsilon
\]
for all $p, q \in X$.

The metric behavior of abelian subgroups is already well understood, via the Flat Torus Theorem (see in [BH99 II.7.1 and II.7.17]). Our main theorem resolves the nonabelian case.

Theorem 1.4. Let $F = \langle u, v \rangle$ be a free group. If $\phi : F \to A_\Gamma$ is a homomorphism, such that $\phi(u)$ and $\phi(v)$ do not commute, then $\phi$ is a quasi-isometric embedding.

We retrieve Baudisch’s original, algebraic description [1.2] as a corollary. Combining with the abelian case, we produce the following metric description.

Corollary 1.5. Every two generator subgroup of $A_\Gamma$ is a quasi-isometrically embedded free or free abelian group.

One consequence is that if an injective homomorphism $\psi : F_2 \to G$ is not quasi-isometric, then $G$ cannot quasi-isometrically embed in any RAAG. The map produced in [HW08] is a quasi-isometric embedding. Thus we can construct examples where the main theorem of [AGM12] (which finished the proof of the virtual Haken conjecture) applies, but passing to a proper subgroup is mandatory.

There is also a divergence between the algebraic and metric treatments in higher rank free groups. We produce a case where a free group injects into a RAAG, but the injection is not a quasi-isometric embedding.

Acknowledgements: I am grateful to Ruth Charney for her support and keen editorial insight. Thanks also to Danny Calegari for (indirectly) suggesting some consequences of the main theorem.

2. Preliminaries

2.1. CAT(0) spaces. A cube complex is a metric space $X$ that is the union of cubes $[-\frac{1}{2}, \frac{1}{2}]^n$ glued by isometries of their faces. Each cube is given the euclidean metric, and $X$ inherits a path metric. We will specifically be interested in cube complexes that are also CAT(0) spaces. Gromov gives a condition for a cube complex to be CAT(0) in [Gro87 4.2.C]. In a CAT(0) space, distances between points on a geodesic triangle $(a, b, c)$ are less than or equal to distances between corresponding points on a comparison triangle $(A, B, C)$ (of equal side lengths) in euclidean space.

This means, among other things, that any two points in $X$ are joined by a unique geodesic segment, and that geodesics vary continuously with the location of their endpoints [BH99 II.1.4].

Isometries of a CAT(0) space are categorized, but we will make use of only one type. Given a group $G$ acting cocompactly and freely by isometries on $X$, any non-identity $w \in G$ is a hyperbolic isometry. It has at least one axis [BH99 II.6.8(1)], a binate geodesic $L$ for which $wL = L$. Furthermore, since $w$ acts by isometries, any two axes $L_1$ and $L_2$ have finite Hausdorff distance.
2.2. Hyperplanes in a CAT(0) cube complex.

**Definition 2.1.** Given a metric \( n \)-cube: \([-\frac{1}{2}, \frac{1}{2}]^n\), we define the \( i \)th midcube to be a set of points that is 0 in the \( i \)th coordinate. A midcube naturally inherits the metric and cell structure of an \((n - 1)\)-cube.

In a cube complex \( X \) we define a relation on all midcubes in \( X \), \( C_1 \sim C_2 \) if and only if they share a vertex. There is a unique weakest equivalence relation generated by \( \sim \). This relates any midcubes that can be connected by a succession of vertices. The union of cubes in an equivalence class is a hyperplane. Figure 1 shows examples in a small complex.

We’ll say that the vertices that belong to edges crossing a hyperplane \( h \) are adjacent to \( h \).

![Figure 1. 3 hyperplanes in a cube complex. \( p \) is adjacent to \( h_1 \) and \( h_2 \) but not \( h_3 \).](image)

If \( X \) is CAT(0) then a hyperplane \( h \) contains at most one midcube of a given cube, and divides \( X \) into two connected components [Sag95, 4.10]. The cubes containing \( h \) form an \( h \times I \) neighborhood with a metric \( \sqrt{d_h^2 + d_I^2} \). This makes geodesics near \( h \) very easy to compute, and since geodesic segments are unique in \( X \) we obtain the following standard results:

1. If \( p, q \in h \), then the geodesic segment from \( p \) to \( q \) (in \( X \)) is also contained in \( h \). Specifically, \( h \) is simply connected.
2. A geodesic in \( X \) is either contained in \( h \) or intersects \( h \) at most once.
3. If \( p \notin h \) then any geodesic segment through \( p \) intersects \( h \) transversely.
4. If \( p \) and \( q \) are vertices of \( X \), then the geodesic from \( p \) to \( q \) meets \( h \) in zero or one points.

**Lemma 2.2.** If \( L \) is a bi-infinite geodesic in a CAT(0) cube complex that properly intersects a hyperplane \( h \), then the distances from points on \( L \) to \( h \) is unbounded, traveling in either direction along \( L \).

**Proof.** Suppose \( L \cap h = p \). Take \( q \in L \) at distance 1 from \( p \). Now take \( r \in L \) distance \( t > 1 \) from \( p \) along the same direction as \( q \). Let \( r' \) be a point on \( h \) of minimum distance to \( r \).

Take the comparison triangle \((P, R, R')\) and let \( Q' \) be the point on \( PP' \) with \( d(P, Q') = \frac{1}{t}d(P, R') \). The corresponding point \( q' \) lies in \( h \) along the geodesic from \( p \) to \( q \).
p to r'. Then we have

\[ d(q, h) \leq d(q, q') \leq d(Q, Q') = \frac{1}{t} d(R, R') = \frac{1}{t} d(r, h). \]

Since \( d(q, h) \) is a constant, we can make \( r \) arbitrarily far from \( h \) by increasing \( t \).

\[ \square \]

**Figure 2.** The comparison triangle for Lemma 2.2

Here is an immediate consequence.

**Corollary 2.3.** If \( L_1 \) and \( L_2 \) are axes of \( w \), then \( L_1 \) properly intersects \( h \) if and only if \( L_2 \) does.

Furthermore \( h \) has the structure of a cube complex, given by the midcubes it comprises. If \( X \) satisfies Gromov’s condition, it is straightforward to check that so does \( h \). If \( w \) is a hyperbolic isometry of \( X \) fixing \( h \), then \( w \) has an axis in \( h \) which must also be an axis in \( X \).

2.3. **The Salvetti complex.** Given a RAAG \( A_\Gamma \), we take a cube \([-\frac{1}{2}, \frac{1}{2}]^\#V \) with the euclidean metric and with opposite faces identified to make an \( n \)-torus. The faces of this torus form a complex of metric cubes, specifically, each edge is an interval associated to an element of \( V \). We consider the subcomplex of only those cubes whose edges all commute pairwise in \( A_\Gamma \).

This subcomplex has fundamental group \( A_\Gamma \). Its universal cover is the *Salvetti complex*, which we will denote \( X_\Gamma \). Since each cube of \( X_\Gamma \) has a metric, \( X \) inherits a natural path metric. With this metric, \( X_\Gamma \) satisfies Gromov’s condition for a CAT(0) cube complex [CD95, 3.1.1].

2.4. **Quasi-isometries.**

**Definition 2.4.** A *quasi-isometry* is a quasi-isometric embedding such that all of \( Y \) lies within a fixed distance of the image \( f(X) \). If such a map exists, then \( X \) and \( Y \) are *quasi-isometric*.

Quasi-isometric is an equivalence relation. In some sense it is the natural equivalence relation to study groups as metric spaces. If two finite generating sets of a group \( G \) give word metrics \( d_1 \) and \( d_2 \), then the identity is a quasi-isometry between \((G, d_1)\) and \((G, d_2)\). Additionally the Svarc-Milnor Lemma [Mil68, BH99, I.8.19] states that a group is quasi-isometric to a geodesic metric space on which it acts properly, cocompactly and by isometries.

Thus \( A_\Gamma \) and \( X_\Gamma \) are quasi-isometric. They are also both quasi-isometric to \( X_\Gamma^{(1)} \), the 1-skeleton of \( X_\Gamma \).
3. $X^{(1)}$ Geodesics and Hyperplanes

In this section $A$ will denote a RAAG and $X$ its Salvetti complex.

Most of our arguments exploit our ability to relate the combinatorics of $A$ to the geometry of $X$. We can, in fact, see the group directly in $X$ by tracing our paths not in $X$ but in its 1-skeleton.

We’ll call paths in $X$ that stay in $X^{(1)}$ and don’t change direction mid-edge edge paths. Edge paths between vertices are nothing more than words in the generators of $A$. The path follows the edge corresponding to each letter in turn.

Unless $A$ is free, $X^{(1)}$ is not CAT(0) and does not have unique geodesics. We’ll use $[p,q]$ to denote some choice of minimal length path between $p$ and $q$ in $X^{(1)}$. We’ll call this an $X^{(1)}$ geodesic segment. Given an edge path from $p$ to $q$ encoded by $w \in A$, finding an $X^{(1)}$ geodesic $[p,q]$ is equivalent to finding a shortest form for $w$. The distance from 1 to $w$ in the word metric, denoted $|w|$ is also the length of $[p,q]$.

Here are some quick results connecting $X^{(1)}$ geodesics segments to hyperplanes.

Observe that an edge path is transverse to the hyperplanes of $X$. Furthermore, two vertices of $X$ lie on the same side of a hyperplane $h$ if and only if every edge path between them crosses $h$ an even number of times.

**Lemma 3.1.** An $X^{(1)}$-geodesic segment crosses no hyperplane more than once.

**Proof.** Suppose $[p,q]$ does cross some hyperplane twice. Consider an innermost pair of crossings. Let $h$ be the hyperplane they cross. The path between the crossings does not meet any hyperplane $h'$ disjoint to $h$, otherwise it would have to recross $h'$ in order to get back to $h$. Thus the path stays adjacent to $h$, in an $h \times I$ neighborhood. This means $[p,q]$ can be shortened by removing two crossings of $h$, so it is not a geodesic segment. □

**Corollary 3.2.** An $X^{(1)}$ geodesic segment is any segment that crosses no hyperplane more than once. Given two points, the $X$ geodesic segment and $X^{(1)}$ geodesic segments between them all cross the same set of hyperplanes.

The following corollary is an application of the triangle inequality.

**Corollary 3.3.** Given $w \in A$ not equal to 1 and a vertex $p \in X$ such that $d(p,wp)$ is minimal, the set $\xi = \bigcup_{i \in \mathbb{Z}} w^i[p,wp]$ is an axis of $w$, that is a $w$-invariant bi-infinite $X^{(1)}$ geodesic.

Note that $\xi$ depends both on the choice of $p$ and $[p,wp]$, so it is far from unique. However, $\xi$ does remain within bounded distance of any (non-$X^{(1)}$) axis $L$ of $w$, since it is the $w$ orbit of a (bounded) segment.

The last lemmas rely on the specific construction of $X$, rather than general CAT(0) hyperplane facts. By inspection, the hyperplanes of $X/A$ meet a single edge each. Thus we can assign to each hyperplane in $X$ a type, according to which hyperplane in $X/A$ lifts to it. Moreover, since the hyperplanes of $X/A$ do not self-intersect, distinct hyperplanes of the same type in $X$ do not intersect. Finally, note that the action of $A$ preserves type, as well as the orientation of the edges that cross each hyperplane.
Lemma 3.4 (Slope Lemma). Suppose \(w \in A\) has axis \(\xi\). Suppose \(p\) is a point on \(\xi\), and \(q\) is a point on a hyperplane \(h\). \(h\) crosses \(\xi\) beyond \(wp\) (that is, on the ray \(\xi = \bigcup_{i=1}^{\infty} w^i[p, wp]\)) if and only if \(h\) crosses \([wp, wq]\). Note figure 3.

![Figure 3. An illustration of the Slope Lemma](image)

Proof. If \(h\) crosses \(\xi\) then \(wh \neq h\). If the crossing is beyond \(wp\) then \(h\) must cross \([wp, wq]\) to avoid crossing \(wh\).

On the other hand, if \(h\) crosses \([wp, wq]\), we have that \(w^{-1}h\) crosses \([p, q]\). Since \(wh \neq h\), \(w^{-1}h\) must also cross either \(\xi\) or \([wp, wq]\). Continuing by induction, we either get that some \(w^{-k}h\) crosses \(\xi\) (thus producing the desired conclusion) or that infinitely many hyperplanes cross \([p, q]\), an absurdity.

If \(h_1\) and \(h_2\) are two hyperplanes whose projections in \(X/A\) intersect transversely we write \(h_1 \perp h_2\). If their projections are disjoint or identical we write \(h_1 \parallel h_2\). No hyperplane of \(X/A\) contains two midcubes of the same cube, so if \(h_1 \parallel h_2\), then \(h_1\) and \(h_2\) are either identical or disjoint. However not all pairs \(h_1 \perp h_2\) intersect.

If two hyperplanes \(h_1 \perp h_2\) are adjacent to the same vertex, then they do in fact intersect in a square that meets that vertex. Also, only one edge of each type and direction meets a given vertex. These facts are immediate from our construction of \(X\). They are also, however, the remaining two conditions for \(X/A\) to be a special cube complex (originally termed A-special in [HW08 3.2]). The following lemma is true in all special cube complexes.

Lemma 3.5. Let \(p\) be a vertex and \(h\) a hyperplane. The vertex \(p\) is adjacent to \(h\) if and only if every geodesic segment \([q, p]\) from any \(q \in h\) crosses only hyperplanes \(h_i \perp h\).

Proof. Let \(p_i\) be the \(i^{th}\) vertex in \([q, p]\), and \(h_i\) be the hyperplane crossing \([p_i, p_{i+1}]\). Suppose \(h_i \perp h\) for all \(i\). We’ll show \(p\) is adjacent to \(h\) by induction. The first vertex, \(p_1\), is adjacent by definition. Now suppose \(p_i\), is adjacent to \(h\), via an edge we’ll denote \(e\). The edges \([p_i, p_{i+1}]\) and \(e\) span a square, \(h\) is a midcube of the square, and \(p_{i+1}\) is still adjacent to \(h\).

On the other hand if some \(h_i \parallel h\) then \(h_i\) separates \(h\) from \([p_{i+1}, p]\), since \([q, p]\) crosses no hyperplane twice. Thus any path from \(h\) to \(p\) must cross \(h_i\), and \(p\) is not adjacent to \(h\).
Any element of $A$ that fixes $h$ also preserved the set of adjacent vertices on each side. Given a point $p$ adjacent to $h$, the stabilizer of $h$ in $A$ sends $p$ along the edges that cross $h_i \perp h$. Thus it is conjugate to the subgroup generated by those vertices of $\Gamma$ adjacent to the vertex representing the edges crossing $h$. This characterization allows us to conclude that if $w^ih = h$ then $wh = h$. This permits a useful variant of the Slope Lemma.

**Lemma 3.6 (Parallel Axis Lemma).** Let $w \in A$ have axis $\xi$ as above, and $q \in \xi$. Suppose that some hyperplane $h$ does not meet $\xi$ and $r$ is some point in $A$ such that $[p,r]$ meets $h$. Then either $h \cap w^n[p,r] = \emptyset$ for all $n \neq 0$ or $wh = h$.

**Proof.** Let $q = h \cap [p,r]$. Suppose that for some $n \neq 0$, $h$ meets $w^n[p,r]$ at $q'$. If $d(p,q) < d(w^np,q')$, then the Slope Lemma implies that $h$ meets $\xi$ beyond $p$ (on the ray not containing $w^np$. Similarly, if $d(p,q) > d(w^nqp,q')$, then $h$ meets $\xi$ beyond $w^np$. Since $h$ doesn’t meet $\xi$ at all by hypothesis. We conclude that $d(p,q) = d(w^np,q')$ and $w^nq = q'$ and thus $w^nh = h$. However, as noted above, if $w^n$ is in the stabilizer of $h$ then so is $w$. □

4. A Standard Form for $X^{(1)}$ Geodesics

Suppose $A$, a RAAG acts on $X$, its Salvetti complex. $X/A$ has a single vertex. If we choose a base vertex, which we’ll call 1, then every other vertex can be written as $w \cdot 1$ for some $w \in A$. We’ll refer to this vertex as $w$.

Given an element $w \in A$ we’ll choose and label a few useful objects, also illustrated in Figure 4:

1. A vertex $a_w$ on an axis of $w$ that is minimal distance from 1 among all vertices that lie on axes of $w$
2. A geodesic $w$ from $a$ to $wa$
3. A geodesic $s_w$ from 1 to $a$.
4. The concatenation $\overset{\rightarrow}{W} = \bigcup_{i \in \mathbb{Z}} w^i w$, which if $w \neq 1$ is an axis of $w$.

![Figure 4](image-url)

**Figure 4.** The decomposition of $[1,w]$, along with its translates, contain an axis of $w$.

Notice that if 1 lies on an axis of $w$, then $a_w = 1$. Also, our geodesics are in $X^{(1)}$, so these choices are not unique. Henceforth, we’ll omit the subscripts when there is no ambiguity.

**Lemma 4.1 (Standard Form).** Given an element $w \in A$ and a choice of $w$, $a$ and $s$ as above,

1. No hyperplane meets $\overset{\rightarrow}{W} = \overset{\rightarrow}{W} \cup \bigcup_{i \in \mathbb{Z}} w^i w^s$ more than once.
2. The concatenation of $s$, $w$ and $ws$ is a geodesic from 1 to $w$. 
Notation. Given a set of hyperplanes $\mathcal{H}$ and an edge path $\gamma$, let $\mathcal{H}_{\gamma}$ denote the set of hyperplanes in $\mathcal{H}$ that meet $\gamma$ an odd number of times. This is equivalent to saying $\mathcal{H}_{\gamma}$ is the set of $h \in \mathcal{H}$ that separate the endpoints of $\gamma$, if it has them. Note $\mathcal{H}_{\gamma'} = \mathcal{H}_{\gamma}$ for any $\gamma'$ with the same endpoints. If $\alpha, \beta, \gamma$ is a triangle, then $\mathcal{H}_{\gamma} = \mathcal{H}_{\alpha} \Delta \mathcal{H}_{\beta}$ (the symmetric difference).

Proof of Lemma 4.1. (1) Suppose some $h$ meets $W$ more than once. No hyperplane meets an axis twice. After perhaps translating by $w$, we can take $h$ to meet $s$. Furthermore, either $h$ is adjacent to $a$ or it is separated by some $h' \parallel h$ by Lemma 3.5. This $h'$ separates $a$ from $h \cap W$ and thus meets $W$ both on $s$ and at some other point. Thus we can take $h$ adjacent to $a$, letting $p = h \cap s$ and letting $q$ be a different point of $h \cap W$. By Lemma 3.5, $[wa, wp]$ only meets hyperplanes $h_i \perp wh$, but $h' \parallel wh$. Thus by the Slope Lemma [3.4], the point $q$ cannot lie on $\overrightarrow{w}$ beyond $wa$. Similarly $q$ cannot lie on $\overrightarrow{w}$ beyond $w^{-1}a$. This leaves the following possibilities for $h$:

- $h$ meets $\overrightarrow{w}$ between $w^{-1}a$ and $wa$.
- $h$ does not meet $\overrightarrow{w}$ at all. Thus $q \in w^n s$ for some $n \neq 0$ and by the Parallel Axis Lemma [3.6], $wh = h$.

Thus, exploiting symmetry, we can take $h$ to cross $w$ or we can stipulate $h = wh$ (but not both, since no hyperplane meets $\overrightarrow{w}$ twice). In either case, we reach the same contradiction as follows:

Let $a'$ be the vertex adjacent to $a$ that shares the edge through $h$. Let $\mathcal{H}$ be the set of all hyperplanes of $X$. Then we have $\mathcal{H}_{[1,a']} = \mathcal{H}_s - \{h\}$. We also have $\mathcal{H}_{[a',wa']} = \{h\} \Delta \mathcal{H}_w \Delta \{wh\}$, which in either case means that $d(a', wa') = d(a, wa)$. We conclude $a'$ lies on an axis and is closer to 1 than $a$, violating our construction of $a$.

(2) Since the concatenation of $s$, $w$ and $ws$ crosses every hyperplane between 1 and $w$ once, but none of them twice, it must be a geodesic segment. \hfill \square

Remark. The use of Lemma 3.5 is a convenience here. One can prove this lemma with only the assumption that $X$ is CAT(0), and hyperplanes of $X/A$ don’t self-intersect.

Lemma 4.2 (Separating Lemma). Let $w \neq 1$ be an element of $A$ and suppose that a hyperplane $h$ is disjoint from one (and hence every) axis of $w$. Suppose that $wh \neq h$. Then there is a hyperplane $h_1 \parallel h$ that separates $h$ from $wh$, and does not intersect $w^i h$ for any $i \in \mathbb{Z}$.

Proof. Let $[p, a]$ be an edge path from $h$ to an axis of $w$, minimal in length among all such paths for all axes of $w$. $p$ is not a vertex, so we’ll name the vertices of the edge it lies on. Let $q$ be the one that lies in $[p, a]$. Set the other vertex as the basepoint 1. Then $[1, w \cdot 1]$ decomposes into a geodesics $s$, $w$ and $ws$, with $s = [1, a] \supset [q, a]$. By the Standard Form Lemma (1.1),

- $h \cap [1, w] = \{p\}$
- $wh \cap [1, w] = \{wp\}$
- $w^i h \cap [1, w] = \emptyset$ for $i \neq 0, 1$.

We conclude that $[q, wq]$ meets no $w^i h$.

Suppose that $[q, wq]$ meets no hyperplanes $h_1 \parallel h$. Then by Lemma 3.5, $wq$ is adjacent to both $h$ and $wh$. But $w$ preserves orientation, and only one edge of
each type and direction meets the vertex $wq$. This implies that $wh = h$, which violates our hypothesis. We conclude that $[q,wq]$ does not cross a hyperplane $h_1 \parallel h$, which must therefore separate $h$ from $wh$. For all $i$, the hyperplane $w^i h \parallel h_1$, but $w^i h \neq h_1$ since it doesn’t meet $[q,wq]$. Thus $w^i h$ and $h_1$ are disjoint. □

Repeated application of this lemma gives a corollary.

**Corollary 4.3.** As in the lemma let $w \neq 1$ be an element of $A$ and suppose that a hyperplane $h_0$ is disjoint from one (and hence every) axis of $w$. Suppose that $wh_0 \neq h_0$. Then there is a hyperplane $h_N$ meeting $\overrightarrow{w}$ that separates $h_0$ from $wh_0$, and does not intersect $w^ih_0$ for any $i \in \mathbb{Z}$.

**Proof.** We’ll argue by induction. Given an $h_n$ that satisfies the hypotheses of Lemma 4.2 we produce $h_{n+1}$. If $h_{n+1}$ crosses $\overrightarrow{w}$, then we are done. If not, then it crosses $s$ or $ws$, but not both. This means that $h_{n+1}$ separates either $h_n$ or $wh_n$ from $\overrightarrow{w}$. Furthermore, $wh_n \neq h_n$, and we can apply the Lemma again. Since $s$ crosses finitely many hyperplanes, repeated application of the lemma will eventually produce some $h_N$ that crosses $\overrightarrow{w}$. Each hyperplane $w^ih_0$ is separated from $\overrightarrow{w}$ and $h_N$ by a sequence hyperplanes $w^ih_n$. □

5. **Essential hyperplanes**

A 2 generator subgroup of $A$ is the image of a homomorphism $\phi : F \to A$, where $F$ is a free group on two generators. Since $A$ acts on $X$, so does $F$. We’ll produce the following tree $T$ and extend the map $\phi : F \to A \subset X$ to this tree.

First we assume that $F = \langle u, v \rangle$ and we choose a basepoint 1 on some axis of $u$. We then choose $u$, $a_v$, $s_v$, $v$ as defined in the previous section (note that $a_u = 1$, so we don’t bother with $s_u$), and immediately drop the subscripts.

$T$ is a trivalent tree with vertices labeled $w$ and $wa$ for all $w \in F$. We label the edges as well as follows:

1. $w$ is adjacent to $wu$ via edge $wu$
2. $w$ is adjacent to $wa$ via edge $ws$
3. $wa$ is adjacent to $wva$ via edge $wv$

![Figure 5. A diagram of $T$ near the vertex 1](image)
We extend the map $\phi$ to $T$ by mapping each vertex to its eponymous vertex in $X$ and each edge to its eponymous geodesic in some equivariant way (say, constant speed). This has the effect that for some $q \in T$ with $\phi(q) \in h$, we have $\phi(wq) \in h$ if and only if $wh = h$.

**Definition 5.1.** We say a hyperplane $h$ is essential if $\phi^{-1}(h)$ is a single point.

The advantage of essential hyperplanes to a metric argument should be clear. If a reduced word $w \in F$, taken as a geodesic in $T$ crosses such a hyperplane once, then it will never cross it again. Thus our strategy is to produce sufficiently many of these to justify the main theorem. Our tool for proving that a hyperplane is essential is the following definition and lemma.

**Definition 5.2.** Suppose $h$ is a hyperplane and $p \in \phi^{-1}(h)$. We denote $T_p$ to be the closure of the connected component of $T - \{wp \mid w \in F - \{1\}\}$ containing $p$.

**Lemma 5.3.** If $\phi^{-1}(h) \cap T_p = \{p\}$, then $h$ is essential.

**Proof.** Suppose there is some $q \neq p$ in $\phi^{-1}(h)$. Let $\gamma$ be the geodesic in $T$ from $p$ to $q$. Then $\gamma$ intersects finitely many hyperplanes. The intersections of $\phi(\gamma)$ with each $wh$ for $w \in F$ occur in pairs, since edge paths are transverse to hyperplanes. Pick a pair that is innermost among those that include a point $p'$ of the form $wp$. Then the second point $q'$ of the pair must not be in $T_{p'}$ by hypothesis. But then $\gamma$ crosses some $w'p'$ in the boundary of $T_{p'}$ between $p'$ and $q'$, contradicting our assumption that $p', q'$ was innermost. □

We’re now ready to prove the main theorem, via a supporting proposition. Recall the following notation: Given a set of hyperplanes $\mathcal{H}$, and a geodesic $\gamma$, $\mathcal{H}_\gamma$ denotes the set of hyperplanes of $\mathcal{H}$ that intersect $\gamma$.

**Proposition 5.4.** If $\phi(F)$ is not abelian, then we can choose a basepoint $1$ and generators $u, v$ of $F$ such that for the associated $T$, there exists an essential hyperplane $h$ in $X$ meeting $[1, v]$.

**Proof.** (1): Pick a generating set $\{u_{\text{orig}}, v_{\text{orig}}\}$ of $F$. Pick a hyperplane that crosses an axis of the commutator $c_{\text{orig}} = u_{\text{orig}}v_{\text{orig}}u_{\text{orig}}^{-1}v_{\text{orig}}^{-1}$. Let $\mathcal{H}$ be the set of all hyperplanes of that type.

Now pick a new generating set $u, v$ and a basepoint $1$ such that the following triple of numbers is minimal (lexicographically) among all such choices:

(1) \((\#\mathcal{H}_u, \#\mathcal{H}_{[1, v]}, \#\mathcal{H}_v)\).
We claim $\mathcal{H}_{[1,v]}$ contains an essential hyperplane. There are two cases which, unfortunately, admit almost no overlap in their discussion.

**Case 1:** Suppose $\mathcal{H}_u$ is nonempty. We will argue that each of $\mathcal{H}_u, \mathcal{H}_v$ contains an essential hyperplane.

Since the hyperplanes of $\mathcal{H}$ are pairwise disjoint, we can order the $\mathcal{H}_u$ by distance from 1, and let $h_1, h_2$ be the middle ones (if $\#\mathcal{H}_u$ is odd, $h_1 = h_2$). We claim that one of $h_1, h_2$ does not meet the nearby $v$-paths, seen also in Figure (7a):

$$[1, v], [1, v^{-1}], [u, uv], [u, uv^{-1}].$$

(2)

$$[1, v], [1, v^{-1}], [u, uv], [u, uv^{-1}].$$

(A) The four $v$ paths avoided by one of $h_1$ or $h_2$

(b) The hyperplanes between 1, $u$ and $uv^{-1}$, assuming $h_1$ meets $[u, uv^{-1}]

**Figure 7**

Similarly if we know that $\mathcal{H}_v$ is nonempty, otherwise we could switch $u$ and $v$ and take basepoint $a$ to reduce the triple. Take $h'_1$ and $h'_2$ to be the middle hyperplanes of $\mathcal{H}_v$. Then these are also the middle hyperplanes of $\mathcal{H}_{[1,v]}$, since $\#\mathcal{H}_v = \#\mathcal{H}_{uv}$. We make an analogous claim that one of $h'_1$ and $h'_2$ meets none of the nearby $u$-paths. The following three facts establish our claims about the $h_i$ and $h'_i$.

(1) Neither $h_i$ meets either of the “far” $v$-paths. For instance, if $h_1$ met $[u, uv^{-1}]$, then $[1, uv^{-1}]$ would meet fewer hyperplanes than $[u, uv^{-1}]$, as seen in Figure 7b. Defining $v_{\text{new}} = uv^{-1}$ would give $\#\mathcal{H}_{[1,v_{\text{new}}]} < \#\mathcal{H}_{[u, uv^{-1}]} = \#\mathcal{H}_{[1,v]}$. Thus $\{u, v_{\text{new}}\}$ would produce a smaller triple than $\{u, v\}$, violating our minimality assumption. Similarly, neither $h'_i$ meets either of the far $u$-paths.

In the case that $h_1 = h_2$, all four $v$-paths are “far,” so the claim is proven.

(2) The $h_i$ do not each meet the near $v$-paths on the same side. For instance, suppose $h_1$ met $[1, v]$. We claim that $h_2$ does not meet $[u, uv]$. If it did, then $h_2 = uh_1$ as in Figure 8a but no hyperplane meets $u$ twice. Similarly, since $h'_1$ and $h'_2$ meet the axis of $v$, we have $h'_1$ and $h'_2$ do not both meet the near $u$-paths on the same side.
(3) The $h_i$ do not each meet the near $v$-paths on the opposite sides. For instance, suppose $h_1$ met $[1, v]$ and $h_2$ met $[u, uv^{-1}]$.

Choose a point $e \in u$ between $h_1$ and $h_2$. Notice $H_{[e,u]}$ consists of $h_1$ and all the hyperplanes of $H$ that separate it from 1, while $H_{[e,u]}$ consists of $h_2$ and all the hyperplanes of $H$ that separate it from $u$.

By part (1), $h_1$ does not meet the far $v$-paths $[u, uv^{-1}]$. Also by part (1) no hyperplane of $H_{[1,u]}$ meets both $[1, u]$ and $[v, vu^{-1}]$ (one of these $u$-paths would be “far”). Since $h_1 \in H_{[1,e]}$, it meets $[u, uv^{-1}]$ (see Figure 8b).

Thus if we set $v_{new} = uv^{-1}$ we have that $h_1$ meets all of $[1, u], [1, v_{new}]$ and $[1, v_{new}]$. So $H_{[e,u]}$ is a subset of $H_{u}, H_{[1,v_{new}]}$, and $H_{[1,v_{new}]}$. Applying $v_{new}$ to the last inclusion, $H_{[v_{new}, v_{new}]} \subset H_{[1,v_{new}]}$. Let’s see what this implies.

\[
\begin{align*}
H_{[e,u]} &= H_{[1,e]} \Delta H_{u} \Delta H_{[u,e]} = (H_{u} - H_{[1,e]}) \cup \Delta H_{[u,e]} \\
\#H_{[e,u]} &= \#H_{u} \\
H_{[e,v_{new}]} &= H_{[1,e]} \Delta H_{[1,v_{new}]} \Delta H_{[v_{new}, v_{new}]} \\
&= H_{[1,v_{new}]} - (H_{[1,e]} \Delta H_{[v_{new}, v_{new}]}) \subset H_{[1,v_{new}]} \\
\#H_{[e,v_{new}]} &< \#H_{[1,v_{new}]}
\end{align*}
\]

Our hypothesis on $h_2$ gives $H_{[e,u]} = H_{u} \cap H_{[u,u,v]}$, so $H_{[1,v_{new}]} = H_{u} \Delta H_{[u,u,v]} = H_{[1,e]} \cup (H_{[u,u,v]} - H_{[e,u]})$. So $\#H_{[1,v_{new}]} = \#H_{[1,v]}$.

This means that replacing $v$ by $v_{new}$ and shifting the basepoint to $e$ would strictly reduce the triple, violating our assumptions about its minimality under $u, v$ and 1. The $v$ case is similar.

![Figure 8](image_url)

(A) Meeting the same side paths (b) The points $e$ and $v_{new} = uv^{-1}$ means $h_1$ and $h_2$ don’t cross $\overrightarrow{uv}$. When $h_1$ and $h_2$ meet opposite side paths

This proves our claim about $h_1$ and $h_2$ in $H_u$. Call one that meets none of the nearby $v$-paths $h_u$. Similarly call one of $h_1^i, h_2^i$ that meets none of the nearby $u$-paths $h_u$. Note that by the Slope Lemma, $h_u$ meets no translate $[v^n, v^{n+1}]_{n \in \mathbb{Z}}$.

We’ll use Lemma 5.3 to show that $h_u$ is essential. Consider $p = h_u \cap u$. Then to see that $T_p \cap h_u = \{p\}$ we check:

1. $\overrightarrow{u} \cap h_u = \{p\}$
2. $h_u$ doesn’t meet the four nearby $v$-paths: $[1, v], [1, v^{-1}], [u, uv], [u, uv^{-1}]$ by our choice of $h_u$. 


(3) $h_u$ does not meet any point that lies across any of the hyperplanes: $h_v, uh_v, v^{-1}h_v$, or $uv^{-1}h_v$ (see Figure 9), since none of these cross $h_u$.

This verifies that $T_p \cap h_u = \{p\}$ and by Lemma 5.3, $h_u$ is essential. By identical reasoning, so is $h_v$. As it lies in $H_{[1,v]}$, $h_v$ is the hyperplane we sought.

**Case 2:** Suppose $H_u$ is empty. Recall that $H$ is the set of all hyperplanes of some type that crosses the axis of $c_{orig}$, the commutator of the original generators of $F$. However, one can check that $c = uvuv^{-1}$ is equal to $gc_{orig}g^{-1}$ for some $g \in A$ (the reader can check this on his or her favorite generating set of $Aut(F)$). Thus $H$ also contains a hyperplane that crosses any axis of $c$. Consider a standard decomposition of $[1,c]$ into $c_s c$ and $c c_s$. Since $H$ is $c$-invariant, there is some $h \in H$ that meets $c^{-1} c$.

Let $\xi$ be the geodesic from 1 to $c^{-1}$ in $T$. It maps to a piecewise geodesic path in $X$, which is a concatenation of two translates of $v$ and two translates of $u$. By the Standard Form Lemma (4.1), $\phi^{-1}(h)$ meets $\xi$ at exactly one point. Since $H_u$ is empty, this point lies on either $[1,v]$ or $[uv, uv^{-1}]$, as in Figure 10. Assume, without loss of generality, that it meets $[1,v]$ at $p$.

We produce two facts about the stabilizer of $h$. Since $vuv^{-1}p \notin h$, we know $vuv^{-1}h \neq h$. We also claim that $uh \neq h$. If we suppose otherwise, then $h$ meets $c \xi$ at $up$. Since $h$ still cannot cross $[1,c]$ or any translate of $u$, it must meet $c \xi$ a second time, and that meeting must be some $q \in [uv^{-1},c]$. Since $h$ doesn’t meet $uv^{-1}c$, the Parallel Axis Lemma says that $q = cp$. But if $ch = h$, then $h$ cannot
meet \( \overrightarrow{c} \) at all, violating our choice of \( h \). Thus \( uh \neq h \) as claimed. We’ll use the fact that neither \( uh \) nor \( vw^{-1}h \) is equal to \( h \) in the argument that follows.

We’ll now argue that \( h \) is essential using Lemma 5.3. To do this we will consider the intersection of \( h \) with the following 5 sets (illustrated in Figure 11). These sets cover \( T_p \) regardless of whether \( p \) lies on \( s \), \( v \) or \( vs \). Computing these intersections will show that \( h \cap T_p \{p\} \).

1. \( V = \gamma \cup \bigcup_{n \in \mathbb{Z}} v^n s \). \( h \) meets \( V \) only at \( p \), by Lemma 4.1.
2. \( u \cup v \). The intersection of \( h \) with these axes is empty, because no \( F \)-translate of \( h \) meets \( u \).
3. \( \bigcup_{n \neq 0} [u^n, u^n v] \). Since \( h \) doesn’t meet \( u \), the Parallel Axis Lemma (3.6) states that if \( h \) meets one of these edges then \( uh = h \). We’ve shown that this isn’t the case, so \( h \) does not intersect this set.
4. \( \bigcup_{n \neq 0} [vu^n v^{-1}, vu^n] \). Note as in the previous step that \( h \) doesn’t meet \( \overrightarrow{u} \) and \( v^{-1}h \neq h \). Thus \( h \) does not intersect this set.
5. Finally, in the case \( p \in v \), the region \( T_p \) overlaps the paths \( [u^n v^{-1}, u^n] \) and \( [vu^n, vu^n v] \). We’ll use the following \( v \)-straightening argument to show that \( h \) doesn’t meet \( [u^n v^{-1}, u^n] \), noting that a similar one exists for \( [vu^n, vu^n v] \):

Suppose \( h \) meets \( [u^n v^{-1}, u^n] \). Since \( H_u \) is empty, \( [1, v] \) meets the same hyperplanes as \( [1, vu^{-n}] \). So if we set \( v_{new} = vu^{-n} \), then \( H_{[1, v_{new}]} = H_{[1, v]} \). We may then produce a \( T_{new} \) with \( v_{new} \) and \( s_{new} \). But \( h \) and every hyperplane of \( H \) between 1 and \( h \) (including those of \( H_s \)), meet both \( [1, v_{new}] \) and \( [1, v_{new}^{-1}] \). Thus \( h \cup H_s \subset H_{s_{new}} \), which means \( \#H_{s_{new}} < \#H_v \), violating the minimality of the triple \( [1, v, s] \).

Figure 11. \( T_p \) is divided into 5 regions to verify that \( h \cap T_p = \{p\} \). This is the \( p \in v \) case.

This establishes that \( \phi^{-1}(h) \cap T_p = \{p\} \). By Lemma 5.3 \( h \) is essential. \( \square \)

The following corollary may be of interest, but we won’t use it here.

**Corollary 5.5.** For \( F \rightarrow A \) a homomorphism and any hyperplane \( h \), there is a choice of generators \( u, v \in F \) and a basepoint 1 such that either

1. \( h \) does not intersect \( T \).
(2) There exists \( w \in F \) such that \( wh \) is essential at some \( p \in [1, v] \).
(3) There exists \( w \in F \) such that \( wh \) meets \([1, v]\) and either \( uh = h \) or \( vuv^{-1}h = h \) (possibly both).

**Proof.** Apply the methods of the previous proposition to \( \mathcal{H} = Fh \). Assuming the conditions of case 1 leads to essential hyperplanes \( h_u \in \mathcal{H}_u \) and \( h_v \in \mathcal{H}_v \). Since \( h_u = wh_u \) for some \( w \in F \), we have that \( h_v \) meets \( w^{-1}u \). This contradicts the claim that it is essential. In case 2 we showed that every hyperplane of \( \mathcal{H}_{[1,v]} \) is essential unless \( uh = h \) or \( vuv^{-1}h = h \). So either \( \mathcal{H}_{[1,v]} \) is empty (along with \( \mathcal{H}_u \) by assumption) or some \( w\)-translate of \( h \) satisfies (2) or (3).

Returning to the full statement of our proposition, we are ready to prove the main theorem: that \( \phi \) is a quasi-isometric embedding.

**Proof of Theorem 1.4.** For a word \( w \) of length \( n \), we know that
\[
|\phi(w)| \leq n \max\{|\phi(u)|, |\phi(v)|\}.
\]
It remains to find a lower bound that is linear in \( n \). We’ll produce a pair of essential hyperplanes, meeting \( u \) and \( v \). We’ll consider their orbits \( Fh_u \) and \( Fh_v \), and count how many of these hyperplanes cross \([1, w]\). Note that in case 1 of Proposition 5.4 we already produced an adequate pair of essential hyperplanes, but the methods here only rely on the assumption that some essential \( h \) crosses \([1, v]\).

Take the choice of \( u, v \) and a basepoint from Proposition 5.4. Consider all possible generators \( v_{\text{new}} = u^{n_1}v^n \), and the \( T_{\text{new}} \) generated by \( \langle u, v_{\text{new}} \rangle \), but keeping the same basepoint. Let \( h \) be the hyperplane of \( X \) that lies closest to \( \vec{u} \) among all essential (with respect to \( T_{\text{new}} \) ) hyperplanes meeting \([1, v_{\text{new}}]\). Such hyperplanes exist by Proposition 5.4 and a closest one exists because \( \vec{u} \) is the \( u \) orbit of the compact set \( u \).

The Separating Lemma 4.2 part (1) states that there is a hyperplane \( h' \parallel h \) between \( h \) and \( uh \). Since any \( F \)-translate \( h \) meets \( T_p \) only once, we know \( h' \notin Fh \) so it is disjoint from all \( F \)-translates of \( h \). As a result, \( h' \cap T \) is contained in \( T_p \). In fact, it is contained in the component of \( T_p - \{p\} \) that contains \( uh \).

We claim \( h' \) crosses \( \vec{u} \). Suppose it does not. Then \( h' \) meets either \([1, p]\) or \([u, up] \) but not both. Without loss of generality, suppose it meets \([1, p]\). By the Parallel Axis Lemma, since \( h \) doesn’t meet \( \vec{u} \), it does not meet any \([u^n, u^n p] \) for \( n \neq 0 \). The only other possible intersection is on some \([u^n v^{-1}, u^n] \). If \( h' \) doesn’t meet any such edge, then it is essential and closer to \( \vec{u} \) than \( h \). This contradicts our choice of \( h \). If \( h' \) does meet this edge, then set \( v_{\text{new}} = vu^{-n} \), and one may check that \( T_{\text{new}} \cap \phi^{-1}(h') \) is a single point on \( s_{\text{new}} \). Thus \( h' \) would be essential for this \( v_{\text{new}} \) and closer to \( \vec{u} \) than \( h \). This also contradicts our choice of \( h \).

So \( h' \) meets some \( u^n u \). Now we claim that for some \( v_{\text{new}} \), both \( h \) and \( h' \) are essential on \( T_{\text{new}} \). That \( h \) is essential is immediate. If \( v_{\text{new}} = u^{n_2}hu^{n_2} \), then \([1, v_{\text{new}}]\) meets \( u^{n_1}h \) and no other \( F \)-translates. Thus \( u^{n_1}h \) is essential on \([1, v_{\text{new}}]\).

As noted above, \( h' \cap T \) is limited to one component of \( T_p - \{p\} \). If \( h' \) meets any \( u \)-translate of \([1, p]\) let \( u^{k_1} [1, p] \) be the one farthest from \( u^n u \). Repeated applications of the Slope Lemma show that only finitely many translates of \([1, p]\) meet \( h' \). They also show that \( h' \) meets every translate of \([1, p]\) between \( u^{k_1} [1, p] \) and \( u^m u \). We define \( u^{k_2} [1, v^{-1} p] \) similarly to produce the farthest translate meeting \( h' \), and apply the Slope Lemma similarly. Then we can choose \( v_{\text{new}} = u^{\sigma(m, k_1)v_{\text{new}}^{-\sigma(m, k_2)}} \) where
\[ \sigma(m, k) = \begin{cases} k - m - 1 & \text{if } k \leq m \\ k - m & \text{if } k > m \\ 0 & \text{if } k \text{ is not defined} \end{cases} \]

One can check that \( h' \) is essential, meeting \( T_{\text{new}} \) only once, on \( u^m u \).

\[ \begin{array}{c}
u^2 u
\end{array} \]

Figure 12. An example of a non-essential \( h' \) meeting \( u^2 u \). The set \( h' \cap T \) is limited to one component of \( T_i - \{p\} \). In this case \( k_1 = 1 \) and \( k_2 = 3 \).

Now let \( h_u = u^{-m} h' \). As noted above \( u^{\sigma(m,k_1)} h \) is essential. It lies on either \( v_{\text{new}}, s_{\text{new}} \) or \( v_{\text{new}} s_{\text{new}} \). In the first case, we’ll call it \( h_u \).

In the second two cases, Corollary [1.3] produces a \( h_v \) meeting \( v_{\text{new}} \) that doesn’t intersect \( u^{\sigma(m,k_1)} h \) or any \( u^{\sigma(m,k_1)} v_{\text{new}} h \). This restriction, along with the Standard Form Lemma [4.1], means that \( h_v \) is essential.

Either way we now have essential hyperplanes \( h_u \) and \( h_v \). The group \( F \) acts freely on the orbits \( Fh_u \) and \( Fh_v \), which consist entirely of essential hyperplanes. Any geodesic in \( T_{\text{new}} \) which crosses the preimage \( \phi^{-1}(h) \) of an essential hyperplane \( h \) will never cross that preimage again. Since we have one such hyperplane in \( Fh_u \cup Fh_v \) for each translate of \( u \) and \( v_{\text{new}} \), a reduced word \( w \) of length \( n \) in \( \{u, v_{\text{new}}\} \) will cross exactly \( n \) hyperplanes of \( Fh_u \cup Fh_v \).

We have been flexible with our choice of generators. Given a different set, say \( \{u_{\text{orig}}, v_{\text{orig}}\} \), we might have to compose with some automorphism of \( F \). But automorphisms are quasi-isometries, and quasi-isometric embeddings are closed under composition.

\[ \begin{array}{c}
u^2 u
\end{array} \]

Remark. Our procedure is nearly algorithmic. While some steps assume that a certain quantity is minimized, we also showed explicitly how to reduce it, should the properties we required be absent. Computationally one needs to be able to check whether some \( wh \) is equal to \( h \) and whether some \([p, q]\) crosses \( h \). One also must be able to produce the hyperplanes of the Separating Lemma. Neither of these tools is more difficult than placing an element of \( A_F \) in a standard form. Given those computational tools, the initial description of \( \phi \), and the steps of the proof, one can produce \( 1, u, v, \) and the essential \( h_u, h_v \).

The non-abelian case of Baudisch’s Theorem is recovered by the following corollary. The results listed in the introduction follow quickly.

**Corollary 5.6.** If \( \phi(F) \) is not abelian, then \( \phi \) is an injection.

**Proof.** By the previous theorem, given any choice of generators (and hence word metric), \( \phi \) is a quasi isometric embedding. There is a \( \lambda \geq 1 \) and \( \epsilon \geq 0 \) such that for all \( w \in F \),
Suppose $w \in F$ is not the identity. It translates along its axis in $T$ by some distance $d$. For sufficiently large $n$, we have $|w^n| \geq nd > \lambda \epsilon$. But then we have

$$0 = \frac{1}{\lambda} \lambda \epsilon - \epsilon < \frac{1}{\lambda} |w^n| - \epsilon \leq |\phi(w^n)|.$$ 

Since $\phi(w^n)$ is not the identity, neither is $\phi(w)$.

Proof of Theorem 1.2. Let $H$ be a two-generator subgroup of $A_\Gamma$. RAAGs are torsion free [Bau81, 2.3], so if $H$ is abelian, it is free abelian. If $H$ is not abelian, then apply the previous lemma.

Proof of Corollary 1.5. As above, suppose first $H$ is abelian. Free abelian subgroups of isometries of $X$ are quasi-isometrically embedded in $X$ (and hence in $A_\Gamma$) by the Flat Torus Theorem (presented in [BH99 II.7.1 and II.7.17]). If $H$ is not abelian then Theorem 1.4 applies.

6. Counterexamples in other settings

6.1. A three-generator subgroup. The result of Baudisch cannot be extended to subgroups of more generators. He produces a counterexample himself [Bau81, 6.2], which we repeat here. Let $\Gamma$ be the following graph:

```
  a
 / \  \
 b  x  y
 / \  \
 c  z
```

Write $F_3 = \langle u, v, w \rangle$ and let $\phi : F_3 \to A_\Gamma$ be defined by

$$\phi(u) = ax \quad \phi(v) = by \quad \phi(w) = cz$$

Notice that no two of $\phi(u), \phi(v), \phi(w)$ commute, yet Baudisch shows that the image of $\phi$ is not a free group, meaning that $\phi$ is not an injection. He goes further and shows that the image of $\phi$ is not even a RAAG.

6.2. A badly embedded $n$-generator free subgroup. However, even assuming that $F_n$ injects into $A_\Gamma$ is insufficient to guarantee a quasi-isometric embedding. Consider the group $G = F_2 \times \psi Z$ where $F_2 = \langle a, b \rangle$ and $Z = \langle t \rangle$ and $\psi(t)(a) = ab$ while $\psi(t)(b) = bab$. We can present this group as

$$G = \langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = bab \rangle$$

Now let $F_2 = \langle u, v \rangle$ and $\phi : F_2 \to G$ be the injection given by $\phi(u) = a$ and $\phi(v) = b$.

Consider the element $t^n at^{-n}$, which has length at most $2n + 1$ in $G$. We can equate this to a word in $a, b$ by taking $a$ and applying $\psi(t)$ $n$ times. Each application of $\psi(t)$ at least doubles the number of $a$’s and $b$’s. No cancellation is ever possible, since we only produce positive powers of $a$ and $b$. The result is that $t^n at^{-n}$ is equal to a word of length at least $2^n$ in $a$ and $b$. Thus there is an infinite sequence of elements $w_i \in F_2$ such that $|\phi(w_i)| \leq 2 \log_2 |w_i| + 1$.

However $G$ is the fundamental group of a hyperbolic 3 manifold, specifically, a punctured torus bundle over a circle. By a result [AGM12], $G$ has a finite index
subgroup $H$ that embeds quasi-isometrically into a RAAG. The group $\phi^{-1}H$ is a finite index subgroup of $F_2$ and thus finitely generated [Sch27], [LS77, 3.9]. Call it $F_n$. The inclusions of $H$ into $G$ and $F_n$ into $F_2$ are quasi-isometries (this is a standard exercise).

$$\begin{align*}
F_n & \xrightarrow{\phi} H \xrightarrow{\alpha} A\Gamma \\
i_F & \downarrow \\
F_2 & \xrightarrow{\phi} G \\
u, v & \longmapsto a, b
\end{align*}$$

We can produce an infinite sequence of elements $w_i' \in F_n$ such that each lies within bounded distance of the $w_i \in F_2$ defined above. Following through the quasi-isometries, we may compute that $|\alpha \circ \phi(w_i')|$ is less than some linear function of $\log_2 |w_i'|$ for all $i$. We conclude that $\alpha \circ \phi$ is an embedding of a free group in a RAAG that is not quasi-isometric.

6.3. Application to 3-manifolds. The proof of the virtual Haken conjecture involves three major ingredients. The first is a proof that every hyperbolic 3-manifold group acts on a $CAT(0)$ cube complex geometrically. The third is that if a 3-manifold group virtually embeds in a RAAG, then the 3-manifold is virtually Haken. It is the intermediate step that concerns us here.

To get such an embedding, the proof uses machinery of Haglund and Wise in [HW08]. They call a cube complex “special” if its hyperplanes satisfy certain properties. Their definition is constructed so that if $Y$ is a special cube complex, then $\tilde{Y}$ isometrically embeds in a Salvetti complex of some RAAG. Thus $\pi_1(Y)$ quasi-isometrically embeds in that RAAG. The main theorem of [AGM12] states that any quotient of a $CAT(0)$ cube complex by a geometric action has a finite cover which is special. In the case of a hyperbolic 3-manifold group, this guarantees the existence of an embedding into a RAAG. The proof, however, does not construct the cover.

Earlier work by Hsu and Wise show that many hyperbolic free by cyclic groups act on a $CAT(0)$ cube complex $Y$. They also conclude (independently) that the quotient cube complex has a special cover, again without an explicit construction [HW10, 15.2 and 15.3]. The class of groups they study includes the fundamental groups of cusped hyperbolic 3-manifolds.

In the construction above, the group $G = F_2 \times \mathbb{Z}$ is the fundamental group of a cusped hyperbolic 3-manifold $M$, and acts on a cube complex $Y$. We still have the map $\phi : F_2 \to G$ above. If $Y/G$ were special, then there would be a $(\lambda, \epsilon)$ quasi-isometric embedding $\alpha : G \to A\Gamma$. But then the sequence $w_i \in F_2$ has the property that $|\alpha \circ \phi(w_i)| \leq \lambda \log_2 |w_i| + 1 + \epsilon$. Thus $G$ does not quasi-isometrically embed in any $A\Gamma$ and $Y/G$ is not a special cube complex.

We can do a little better.

**Proposition 6.1.** Given a cube complex $Y$ with $\pi_1(Y) = F_2 \rtimes \psi \mathbb{Z}$, no double cover of $Y$ admits a special cubulation.

**Proof.** Let $H$ be an index 2 subgroup of $F_2 \rtimes \psi \mathbb{Z}$. Then $H$ is normal, and $a$ and $tat^{-1} = ab$ are either both in $H$ or both not in $H$. So $a^{-1}ab = b \in H$. Also,
\[tbt^{-1} = bab \in H, \text{ so } a \in H. \] Therefore for \( \phi : F_2 \to F_2 \rtimes \psi \mathbb{Z} \) as defined above, \( \text{Im}(\phi) \subset H \). Finally, \( t^2 \in H \). Thus we have a sequence of \( w_i \in F_2 \) such that \( \phi(w_i) = t^{2i}a^{-2i} \) and \( |\phi(w_i)| \leq 2\log_2 |w_i| + 1 \). Thus \( H \) does not quasi-isometrically embed in any RAAG by Theorem 1.4.

**Question.** How many sheets must a cover of \( Y \) have in order to admit a special cubulation? Does this number change for different hyperbolic groups of the form \( F_2 \rtimes \mathbb{Z} \)?

**Question.** Can we find fundamental groups of non-positively curved cube complexes such that no subgroup of a given index is special? Can we find these in the class of 3-manifold groups?

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