Quantum Minkowski spaces

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One of the main problems of theoretical physics is to find a satisfactory theory which would generalize both the quantum field theory and the general theory of relativity. It is widely recognized that in such a theory the geometry of the space–time should drastically change at very small distances, comparable with the Planck’s length. One of possibilities is to replace the space–time by so called quantum space. In such an approach the role of the commutative algebra of functions on the space–time (generated in the simplest case by the coordinates) is played by some noncommutative algebra. At the present stage we are only able to test this idea in particular examples, which are important in physics. Many classical objects were already deformed in the above sense (cf. e.g. [19], [10], [17], [11], [3], [4], [20]). Here we deal with the most interesting case, namely that of Minkowski space $M$ endowed with the action of Poincaré group $P$. Examples of quantum Poincaré groups and their actions on quantum Minkowski spaces appeared e.g. in [7], [5], [21] (the case of quantum Poincaré groups and algebras extended by dilatations was considered e.g. in [9], [18], [12], [6]), see also [1], [2]. The aim is to find the classification of quantum Poincaré groups and quantum Minkowski spaces as well as mathematical and physical properties of those objects. We sketch the results of four papers [13], [14], [15], [16].

The (connected component of) vectorial Poincaré group

$$\tilde{P} = SO_0(1,3) \ltimes \mathbb{R}^4 = \{(M, a) : M \in SO_0(1,3), a \in \mathbb{R}^4\}$$

has the multiplication $(M, a) \cdot (M', a') = (MM', a + Ma')$. By the Poincaré group we mean spinorial Poincaré group (which is more important in quantum field theory then $\tilde{P}$)

$$P = SL(2, \mathbb{C}) \ltimes \mathbb{R}^4 = \{(g, a) : g \in SL(2, \mathbb{C}), a \in \mathbb{R}^4\}$$
with multiplication \((g, a) \cdot (g', a') = (gg', a + \lambda_g(a'))\) where the double covering \(SL(2, \mathbb{C}) \ni g \mapsto \lambda_g \in SO_0(1, 3)\) is the standard one. The group homomorphism \(\pi : P \ni (g, a) \mapsto (\lambda_g, a) \in \tilde{P}\) is also a double covering. In particular, \((-1, 0) \in P\) can be treated as rotation about \(2\pi\) which is trivial in \(\tilde{P}\) but nontrivial in \(P\) (it changes the sign of wave functions for fermions). Both \(P\) and \(\tilde{P}\) act on Minkowski space \(M = \mathbb{R}^4\) as follows \((g, a)x = (\lambda_g x + a, g)\), \(g \in SL(2, \mathbb{C}), a, x \in \mathbb{R}^4\).

Let us consider continuous functions \(w_{AB}, y_i\) on \(P\) defined by \(w_{AB}(g, a) = g_{AB}, y_i(g, a) = a_i\).

We introduce Hopf \(*\)-algebra \(Poly(P) = (\mathcal{B}, \Delta)\) of polynomials on the Poincaré group \(P\) as the \(*\)-algebra \(\mathcal{B}\) with identity \(I\), generated by \(w_{AB}\) and \(y_i\), \(A, B = 1, 2, i \in S = \{0, 1, 2, 3\}\) endowed with the comultiplication \(\Delta\) given by \(\Delta f(x, y) = f(x \cdot y), f \in \mathcal{B}, x, y \in P\) \((f^*(x) = \overline{f(x)})\). In particular,

\[
\Delta w_{CD} = w_{CF} \otimes w_{FD},
\]

\[
\Delta y_i = y_i \otimes I + \Lambda_{ij} \otimes y_j,
\]

\(y_i^* = y_i\), where

\[
\Lambda = V^{-1} (w \otimes \bar{w}) V,
\]

\[
V = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -i & 0 \\
0 & 1 & i & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}
\]

(we sum over repeated indices, one has \(V_{CD,i} = (\sigma_i)_{CD}\) where \(\sigma_i\) are the Pauli matrices). Moreover, \(w\) is a representation, i.e. it is invertible (as \(2 \times 2\) matrix with elements in \(\mathcal{B}\)) and satisfies (4). Equivalences of representations are defined as usual (by means of invertible matrices with complex entries), e.g. \(\Lambda \simeq w \otimes \bar{w}\). We put \(y = (y_i)_{i \in S}\). One can also treat \(w_{CD}\) as continuous functions on the Lorentz group \(L = SL(2, \mathbb{C})\) \((w_{CD}(g) = g_{CD}, g \in L)\). We define Hopf \(*\)-algebra \(Poly(L) = (\mathcal{A}, \Delta)\) of polynomials on \(L\) as \(*\)-algebra with \(I\), generated by all \(w_{CD}\) endowed with \(\Delta\) obtained by restriction of \(\Delta\) for \(\mathcal{B}\) to \(\mathcal{A}\). Clearly \(w\) and \(\Lambda\) are representations of \(L\). It is easy to check that

1. \(\mathcal{B}\) is generated as algebra by \(\mathcal{A}\) and the elements \(y_i, i \in S\).
2. \( \mathcal{A} \) is a Hopf \(*\)-subalgebra of \( \mathcal{B} \).

3. \( \mathcal{P} = \begin{pmatrix} \Lambda & y \\ 0 & I \end{pmatrix} \) is a representation of \( \mathcal{B} \) where \( \Lambda \) is given by (3).

4. There exists \( i \in \mathcal{S} \) such that \( y_i \not\in \mathcal{A} \).

5. \( \Gamma \mathcal{A} \subset \Gamma \) where \( \Gamma = \mathcal{A}X + \mathcal{A}, \ X = \text{span}\{y_i : i \in \mathcal{S}\} \).

6. The left \( \mathcal{A} \)-module \( \mathcal{A} \cdot \text{span}\{y_iy_j, y_i, I : i, j \in \mathcal{S}\} \) has a free basis consisting of \(10 + 4 + 1\) elements.

(5. and 6. follow from the relations \( y_ia = ay_i, \ y_iy_j = y_jy_i, \ a \in \mathcal{A} \), and elementary computations, a free basis is given by \( \{y_iy_j, y_i, I : i \leq j, i, j \in \mathcal{S}\} \)).

According to [20], Poly(\( L \)) satisfies:

i. \( (\mathcal{A}, \Delta) \) is a Hopf \(*\)-algebra such that \( \mathcal{A} \) is generated (as \(*\)-algebra) by matrix elements of a two-dimensional representation \( w \)

ii. \( w \otimes w \simeq I \oplus w^1 \) where \( w^1 \) is a representation

iii. the representation \( w \otimes \bar{w} \simeq \bar{w} \otimes w \) is irreducible

iv. if \( \mathcal{A}', \Delta', w' \) satisfy i.–iii. and there exists a Hopf \(*\)-algebra epimorphism \( \rho : \mathcal{A}' \rightarrow \mathcal{A} \) such that \( \rho(w') = w \) then \( \rho \) is an isomorphism (the universality condition).

We define quantum Lorentz groups and quantum Poincaré groups as objects having the same properties as the classical Lorentz and Poincaré groups:

**Definition 1** We say [20] that \( H \) is a quantum Lorentz group if Poly(\( H \)) = \( (\mathcal{A}, \Delta) \) satisfies i.–iv. We say [13] that \( G \) is a quantum Poincaré group if Hopf \(*\)-algebra Poly(\( G \)) = \( (\mathcal{B}, \Delta) \) satisfies the conditions 1.–6. for some quantum Lorentz group \( H \) with Poly(\( H \)) = \( (\mathcal{A}, \Delta) \) and a representation \( w \) of \( H \).

**Remark.** The condition 5. follows from \( \mathcal{P} \otimes w \simeq w \otimes \mathcal{P}, \mathcal{P} \otimes \bar{w} \simeq \bar{w} \otimes \mathcal{P}, \) while 6. is suggested by the requirement \( W(\mathcal{P} \otimes \mathcal{P}) = (\mathcal{P} \otimes \mathcal{P})W \) for a “twist-like” matrix \( W \). Moreover, the condition 4. is superfluous (it follows from the condition 6.).

**Theorem 2** Let \( G \) be a quantum Poincaré group, Poly(\( G \)) = \( (\mathcal{B}, \Delta) \). Then \( \mathcal{A} \) is linearly generated by matrix elements of irreducible representations of
\( G \), so \( A \) is uniquely determined. Moreover, we can choose \( w \) in such a way that \( A \) is the universal \( \ast \)-algebra generated by \( w_{AB}, A, B = 1, 2 \), satisfying

\[
(w \otimes w)E = E,
\]

\[
E'(w \otimes w) = E',
\]

\[
X(w \otimes \bar{w}) = (\bar{w} \otimes w)X,
\]

where the triples \( E \in M_{4 \times 1}(C), E' \in M_{1 \times 4}(C), X \in M_{4 \times 4}(C) \) are listed in Theorem 1.4 of [13]. We can (and will) choose \( y_i \) in such a way that \( y_i^* = y_i \).

In particular, it turns out that only quantum Lorentz groups with the parameter \( q = \pm 1 \) are admissible. Nevertheless, there are many families of admissible quantum Lorentz groups (numbered by some other parameters). In the following we assume that \( G \) is a quantum Poincaré group, \( \text{Poly}(G) = (B, \Delta) \) and \( w, y \) are as in Theorem 2. Using the general theory of inhomogeneous quantum groups [12], we find the full system of commutation relations for \( B \):

**Theorem 3** \( B \) is the universal \( \ast \)-algebra with \( I \), generated by \( A \) and \( y_i \) with relations

\[
y \otimes v = G_v(y \otimes y) + H_v(y - \Lambda \otimes v)H_v,
\]

\[
(R - 1)(y \otimes y - Zy + T - (\Lambda \otimes \Lambda)T) = 0,
\]

\[
y_i^* = y_i, \quad i \in S,
\]

for any \( v \in \text{Rep } G \) (the set of all representations of \( G \)), where \( (G_v)_{iC,Dj} = f_{ij}(v_{CD}), (H_v)_{iC,D} = \eta_i(v_{CD}), R = G_\Lambda, Z = H_\Lambda \) and \( T_{ij} \in C, f_{ij}, \eta_i \in A' \) (i, j \( \in S \)) satisfy the conditions of Theorem 1.5 of [13]. \( \Delta \) is given by (4), (5).

Moreover, \( (B, \Delta) \) gives a quantum Poincaré group if and only if a system of linear and quadratic equations listed in the proof of Theorem 1.6 of [13] is fulfilled.

It turns out that there are two choices for \( f \) determined by a number \( s = \pm 1 \) - the calculations are made for each \( s \) separately and the results are given in terms of \( H_{EFCD} = V_{EF_i}\eta_i(w_{CD}) \) and \( T_{EFCD} = V_{EF_i}V_{CD,j}T_{ij} \).

We solve [13] the above system of equations for almost all quantum Lorentz groups (except two cases, including the classical Lorentz group for which a large class of solutions is known). Moreover, we single out unisomorphic
objects. The classification is presented in Theorem 1.6 of [13]. We also identify few examples which were known earlier (cf. [7], [5], [21]). We prove that \( B \) has exactly the same “size” as in the undeformed case. Namely,

\[
B^N = \mathcal{A} \cdot \text{span}\{y_{i_1} \cdot \ldots \cdot y_{i_n} : i_1, \ldots, i_n \in S, \quad n = 0, 1, \ldots, N\}
\]

is a free left \( \mathcal{A} \)-module and

\[
dim_{\mathcal{A}} B^N = \sum_{n=0}^{N} d_n
\]

where \( d_n \) is the number of classical monomials of \( n \)th degree in 4 variables.

We denote by \( l : P \times M \rightarrow M \) the action of Poincaré group on Minkowski space, \( C = \text{Poly}(M) \) denotes the unital algebra generated by coordinates \( x_i \) (\( i \in S \)) of the Minkowski space \( M = \mathbb{R}^4 \). The coaction \( \Psi : C \rightarrow B \otimes C \) and \( \ast \) in \( C \) are given by \( (\Psi f)(x, y) = f(l(x, y)), \ f^\ast(y) = f(y), \ x \in P, \ y \in M \). One gets

\[
\Psi x_i = \Lambda_{ij} \otimes x_j + y_i \otimes I. \tag{4}
\]

We define a quantum Minkowski space as object having the same properties as the classical Minkowski space:

**Definition 4** We say that \( (C, \Psi) \) describes a quantum Minkowski space associated with a quantum Poincaré group \( G \), \( \text{Poly}(G) = (B, \Delta) \), if \( C \) is a unital \( \ast \)-algebra generated by \( x_i \), \( i \in S \), \( \Psi : C \rightarrow B \otimes C \) is a unital \( \ast \)-homomorphism, \( \Psi f)(x, y) = f(l(x, y)), \ f^\ast(y) = f(y), \ x \in P, \ y \in M \). Moreover, \( \Psi \) is given by (4).

**Theorem 5** Let \( G \) be a quantum Poincaré group with \( w, y \) as in Theorem 3. Then there exists a unique (up to a \( \ast \)-isomorphism) pair \( (C, \Psi) \) describing associated Minkowski space:

- \( C \) is the universal unital \( \ast \)-algebra generated by \( x_i \), \( i = 0, 1, 2, 3 \), satisfying \( x_i^\ast = x_i \) and

\[
(R - 1)(x \otimes x - Z x + T) = 0,
\]

and \( \Psi \) is given by (4). Moreover,

\[
dim C^N = \sum_{n=0}^{N} d_n,
\]

where \( C^N = \text{span}\{x_{i_1} \cdot \ldots \cdot x_{i_n} : i_1, \ldots, i_n \in S, \quad n = 0, 1, \ldots, N\} \).
The next our goal is to find the differential structure on quantum Minkowski spaces. It turns out \cite{14} that there exists a unique 4-dimensional covariant first order differential calculus on a quantum Minkowski space provided $\tilde{F} = 0$

\[
\tilde{F} = [(R - 1) \otimes 1] \{(1 \otimes Z)Z - (Z \otimes 1)Z + T \otimes 1 - (1 \otimes R)(R \otimes 1)(1 \otimes T)\}
\]

(otherwise there is no such a calculus). This condition singles out a large class of quantum Minkowski spaces \cite{14}. From now on we assume that this condition is fulfilled. In particular, there exists a $\mathcal{C}$-bimodule $\Gamma^\Lambda_1$ (of differential forms of the first order) and a linear mapping $d: \mathcal{C} \rightarrow \Gamma^\Lambda_1$ such that $d(ab) = a(db) + (da)b$, $a, b \in \mathcal{C}$, and $dx_i, i \in \mathcal{S}$, form a basis of $\Gamma^\Lambda_1$ (as right $\mathcal{C}$-module). We prove

\[
x_i dx_j = R_{ij,kd} dx_k x_l + Z_{ij,k} dx_k, \ i, j \in \mathcal{S}.
\]

This calculus prolongates to a unique exterior algebra of differential forms, with the same properties as in the undeformed case. In particular it possesses * such that $(dx_i)^* = dx_i$.

The partial derivatives $\partial_i : \mathcal{C} \rightarrow \mathcal{C}$ are uniquely defined by

\[
da = dx_i \partial_i (a), \ a \in \mathcal{S}.
\]

They can be also obtained as

\[
\partial_i = (Y_i \otimes \text{id})\Psi
\]

where $Y_i \in \mathcal{A}'$ are introduced in the proof of Proposition 3.1.2 of \cite{14}.

The metric tensor $g = (g_{ij})_{i,j \in \mathcal{S}}$ (its entries are called in \cite{14} by $g^{ij}$) is defined by the equations $(\Lambda \otimes \Lambda)g = g$ (or $\Lambda g \Lambda^T = g$) and $g_{ij} = g_{ji}$. After the choice of a real factor we fix it as

\[
g = -2q^{1/2}(V^{-1} \otimes V^{-1})(1 \otimes X \otimes 1)(E \otimes \tau E),
\]

where $\tau$ is always a standard twist. Then the Laplacian is defined by $\square = g_{ij} \partial_j \partial_i$. It commutes with the partial derivatives and therefore the momenta $P_l = i \partial_l$ are well defined in the spaces of solutions of the Klein–Gordon equation $(\square + m^2) \varphi = 0$. One proves that the momenta $P_k = g_{kl} P_l$.

\footnote{in this case we relax our rule of writing all indices in the subscript position}
and Laplacian are hermitian and have good transformation properties. The
commutation relations among partial derivatives and with the coordinates
are as follows:

\[ \partial_i \partial_k = R_{ij,kl} \partial_j \partial_l, \]
\[ \partial_i x_k = \delta_{ki} + (R_{kl,in} x_n + Z_{kl,i}) \partial_l. \]

Let us now pass to the particles of spin 1/2 [16]. First we define the space
of bispinors as \( \mathbb{C}^4 \) endowed with a representation \( G \simeq \mathbb{C} \otimes \mathbb{C} \) of a quantum
Poincaré group. We choose \( G = \mathbb{C} \oplus \mathbb{C} \) where \( \mathbb{C} \oplus \mathbb{C} \) where
\( \mathbb{C} = \mathbb{C}^T \mathbb{C} \). We are
going to find the gamma matrices \( \gamma_i \in M_{4 \times 4}(\mathbb{C}), i \in S \). At the moment they
are not determined yet. The Dirac operator has form \( \partial / \) \( \gamma_i \). It acts
on the bispinor functions \( \phi \in \tilde{C} \equiv \mathbb{C}^4 \otimes \mathbb{C} \) (in a more advanced approach
we should consider square integrable functions \( \phi \)). They can be written as
\( \phi = \varepsilon_a \otimes \phi_a \) where \( \varepsilon_a, a = 1, 2, 3, 4, \) form the standard basis of \( \mathbb{C}^4 \). We define
the action \( \tilde{\Psi} : \tilde{C} \rightarrow \mathcal{B} \otimes \tilde{C} \) of quantum Poincaré group on \( \tilde{C} \) by

\[ \tilde{\Psi}(\varepsilon_a \otimes \phi_a) = G_{al} \phi_a(1) \otimes \varepsilon_l \otimes \phi_a(2) \]

where \( \Psi(\phi) = \phi_a(1) \otimes \phi_a(2) \) (Sweedler’s notation, exception of the summation
convention). Then the classical condition of invariance of the Dirac operator
is generalized to

\[ \tilde{\Psi}(\phi) = (\text{id} \otimes \phi)[\tilde{\Psi}(\phi)], \quad \phi \in \tilde{C}. \]

According to Theorem III.1 of [16], the above condition is equivalent to

\[ \gamma_i = \begin{pmatrix} 0 & bA_i \\ a\sigma_i & 0 \end{pmatrix}, \quad i \in S, \]

where

\[ A_i = q^{-1/2} E^T (\sigma_i \circ D) E, \]

\[(\sigma_i \circ D)_{KL} = (\sigma_i)_{AB} D_{AB,KL}, \]
\[ D = \tau X^{-1} \tau, \quad a, b \in \mathbb{C} \] (\( E \) is regarded here as
\( 2 \times 2 \) matrix). Thus we have found the form of the Dirac operator up to two
constants. But Theorem III.2 of [16] says that the following are equivalent:

1. \( \partial^2 = \Box \).
2. \( \gamma_i \gamma_j + R_{ji,k} \gamma_k \gamma_l = 2g_{ji} \mathbf{1}, \quad i, j \in S. \)
3. \( ab = 1. \)
Moreover, the remaining freedom in the choice of $a$ results in a trivial scaling of the undotted spinor and we can set $a = b = 1$. Thus we have obtained

$$\gamma_i = \begin{pmatrix} 0 & A_i \\ \sigma_i & 0 \end{pmatrix}$$

where $A_i$ are given by (2). Now the form of the Dirac equation $(i\partial\!\!\!/ + m)\varphi = 0$ is determined.

In the next step we find (formal) solutions of Klein–Gordon and Dirac equations in two important cases (Section 4 of [14]). In one of them we use (as a tool during calculations) an additional algebra $\mathcal{F}$ and its representations. Specific calculations are made in [16] (including the form of metric tensor, gamma matrices, representations of $\mathcal{F}$ and the solutions of Klein–Gordon and Dirac equations). For spin 0 particles the momenta $P_j = \partial_j$. For spin 1/2 particles we set [16] the momenta as

$$\tilde{P}_j = i\tilde{\partial}_j, \quad (6)$$

$$\tilde{\partial}_j = (Y_j \otimes \text{id})\tilde{\psi} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}} \quad (7)$$

(motivation: (6)–(7) remain true if we omit tildas everywhere). We get four objects $\tilde{P}_k = g_{kj}\tilde{P}_j$ which have good transformation properties, commute with the Dirac operator and are hermitian w.r.t. (indefinite) inner product in the space of bispinors. However, in many cases their spectral properties are not satisfactory. The problem of further improvement in this matter remains open. We also study certain expressions like the deformed Lagrangian. They transform themselves in a similar way as in the standard theory.

It turns out (cf. Theorem 1.13 of [13]) that there exist invertible matrices $W$ such that $W(P \otimes P) = (P \otimes P)W$ and the Yang–Baxter equation

$$(W \otimes \mathbf{1})(\mathbf{1} \otimes W)(W \otimes \mathbf{1}) = (\mathbf{1} \otimes W)(W \otimes \mathbf{1})(\mathbf{1} \otimes W)$$

is satisfied. Namely, up to a constant they are given by the unit matrix and

$$R_Q = \begin{pmatrix} R & Z & -R \cdot Z & (R - \mathbf{1} \otimes \mathbf{2})T + b \cdot g \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{C}$. 

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In [13] we show the existence of \( R \in (B \otimes B)' \) such that \((B, \Delta, R)\) is a coquasitriangular (CQT) Hopf algebra. In other words, for any \( v, z \in \text{Rep} \ G \) we define
\[
R^{vz} \in \text{Lin}(C^{	ext{dim } v} \otimes C^{	ext{dim } z}, C^{	ext{dim } z} \otimes C^{	ext{dim } v})
\]
by
\[
(R^{vz})_{ij,kl} = R(v_{jk} \otimes z_{il}), \quad j, k = 1, \ldots, \text{dim } v, \quad i, l = 1, \ldots, \text{dim } z,
\]
and prove
\[
R^{1v} = R^{v1} = 1,
\]
\[
R^{v_1 \otimes v_2, z} = (R^{v_1 z} \otimes 1)(1 \otimes R^{v_2 z}),
\]
\[
R^{v, z_1 \otimes z_2} = (1 \otimes R^{v z_2})(R^{v z_1} \otimes 1),
\]
\[
(z \otimes v)R^{vz} = R^{vz}(v \otimes z),
\]
for all \( v, v_1, v_2, z, z_1, z_2 \in \text{Rep} \ G \) (1 = (I) is the trivial representation).

The classification of all CQT Hopf algebra structures (for all quantum Poincaré groups) is done in Theorem 3 of [15]. In particular, \( R^{PP} = R_Q \),
\[
R^{vP} = \begin{pmatrix} G_v, & H_v \\ 0, & 1 \end{pmatrix},
\]
\[
R^{vw} = (R^{vP})^{-1}, \quad R^{ww} = kL, \quad R^{w\bar{w}} = k\bar{X}, \quad R^{\bar{w}w} = qkX^{-1}, \quad R^{\bar{w}\bar{w}} = k\tau L\tau,
\]
for all representations \( v \) of the corresponding quantum Lorentz group \( H \) (these data determine \( R \) uniquely), where \( L = sq^{1/2}(1 + qEE') \), \( k = \pm 1 \) (two possible \( R \) for each \( b \in C \)).

We have to do with CQT Hopf *-algebra iff \( \overline{R(x^\ast \otimes y^\ast)} = R(x \otimes y) \), \( x, y \in B \), iff \( q = 1 \) and \( b \in R \). We have cotriangular (CT) Hopf algebra iff \( (R^{vz})^{-1} = R^{zv} \) for all \( v, z \in \text{Rep} \ G \) iff \( q = 1 \) and \( b = 0 \) (so then it is also a CT Hopf *-algebra).

Using the above results, universal enveloping algebras for quantum Poincaré groups are introduced. Their commutation relations are investigated. Moreover, we classify C(Q)T Hopf (*-)-algebra structures for quantum Lorentz groups. We also show some general statements concerning coquasitriangularity. The results of [13] are used in Section 5 of [14] to define the Fock space for non-interacting particles of spin 0 on a quantum Minkowski space.
Then we take a $CT$ Hopf $^*$-algebra structure $\mathcal{R}$ as above and introduce the particles interchange operator $K : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ by

$$K(x \otimes y) = \mathcal{R}(y^{(1)} \otimes x^{(1)})(y^{(2)} \otimes x^{(2)}),$$

where $\Psi(x) = x^{(1)} \otimes x^{(2)}$, $\Psi(y) = y^{(1)} \otimes y^{(2)}$. It defines the action of the permutation group $\Pi_n$ in $\mathcal{C}^\otimes_n$ ($\Pi_n \ni \sigma \rightarrow \pi_\sigma$) which agrees with the action of the quantum Poincaré group $G$. Thus $G$ acts in the boson subspace $\mathcal{C}^\otimes_n$. If $W : \mathcal{C} \rightarrow \mathcal{C}$ is an operator related to a single particle then the corresponding $n$-particle operator is given by

$$W^{(n)} = \sum_{m=1}^{n} \pi_{(1,m)}(W \otimes 1^\otimes_{(n-1)}) \pi_{(1,m)} : \mathcal{C}^\otimes_n \rightarrow \mathcal{C}^\otimes_n$$

(the $m$-th term is the operator in $\mathcal{C}^\otimes_n$ corresponding to the $m$-th particle).

We can also define the Fock space $F = \bigoplus_{n=0}^{\infty} \mathcal{C}^\otimes_n$ and the operator $\bigoplus_{n=0}^{\infty} W^{(n)}$ acting in $F$.

For particles of mass $m$ we should consider $\ker(\Box + m^2)$ instead of $\mathcal{C}$ and a scalar product there (heuristically e.g. $W = P^k$, $k \in \mathbb{S}$, would be hermitian operators in such a space).

Results of [14] and [15] are proven also for general inhomogeneous quantum groups (satisfying certain conditions). References to the existing literature are given in [13], [14], [15], [16].

Concluding, quantum Minkowski spaces and quantum Poincaré groups have a lot of properties similar to that of the classical ones. It suggests a possibility of building more advanced physical models which use those objects. However, it would need further studies concerning deformed quantum field theory and interaction of particles. There is also another advantage of quantum Minkowski spaces: the fact that there are many possibilities in the choice of parameters somehow forces us to find the proofs which have good geometric meaning. In particular, the invariance of the Dirac operator turns out to be equivalent to the fact that some object built from the gamma matrices intertwines two specific representations of the quantum Poincaré group. Then the form of gamma matrices is easy to find (without using the Lie algebra at all).
References

[1] Aschieri, P. and Castellani, L., R-matrix formulation of the quantum inhomogeneous groups $ISO_{qr}(N)$ and $ISP_{qr}(N)$, Lett. Math. Phys. 36 (1996), 197–211; Bicovariant calculus on twisted $ISO(N)$, quantum Poincaré group and quantum Minkowski space, Int. J. Mod. Phys. A 11 (1996), 4513; Aschieri, P., Castellani, L. and Scarfone, A.M., Quantum orthogonal planes: $ISO_{qr}(N)$ and $SO_{qr}(N)$ – bicovariant calculi and differential geometry on quantum Minkowski space, \texttt{q-alg/9709032}.

[2] de Azcárraga, J.A. and Rodenas, F., Differential calculus on $q$-Minkowski space, An. Fisica (Monogr.) 2 (1995), 107–130; de Azcárraga, J.A., Kulish, P.P. and Rodenas, F., On the physical contents of $q$-deformed Minkowski spaces, Phys. Lett. B 351 (1995), 123; Twisted $h$-spacetimes and invariant equations, Z. Phys. C 76 (1997), 567–576.

[3] Carow–Watamura, U., Schlieker, M., Scholl, M. and Watamura, S., Tensor representation of the quantum group $SL_q(2,\mathbb{C})$ and quantum Minkowski space, Z. Phys. C – Particles and Fields 48 (1990), 159–165.

[4] Carow–Watamura, U., Schlieker, M. and Watamura, S., $SO_q(N)$ covariant differential calculus on quantum space and quantum deformation of Schrödinger equation, Z. Phys. C – Particles and Fields 49 (1991), 439–446.

[5] Chaichian, M. and Demichev, A.P., Quantum Poincaré group, Phys. Lett. B304 (1993), 220–224.

[6] Dobrev, V.K., Canonical $q$-deformations of noncompact Lie (super-) algebras, J. Phys. A: Math. Gen. 26(1993), 1317–1334.

[7] Lukierski, J., Nowicki, A. and Ruegg, H., New quantum Poincaré algebra and $\kappa$–deformed field theory, Phys. Lett. B293 (1992), 344–352; Zakrzewski, S., Quantum Poincaré group related to the $\kappa$-Poincaré algebra, J. Phys. A: Math. Gen. 27 (1994), 2075–2082.

[8] Majid, S., Braided momentum in the $q$-Poincaré group, J. Math. Phys. 34 (1993), 2045–2058.
[9] Ogievetsky, O., Schmidke, W.B., Wess, J. and Zumino, B., \textit{q-Deformed Poincaré algebra}, \textit{Commun. Math. Phys.} \textbf{150} (1992), 495–518.

[10] Podleś, P., \textit{Quantum spheres}, \textit{Lett. Math. Phys.} \textbf{14} (1987), 193–202; The classification of differential structures on quantum 2-spheres, \textit{Commun. Math. Phys.} \textbf{150} (1992), 167–179; Quantization enforces interaction. Quantum mechanics of two particles on a quantum sphere, \textit{Int. J. Mod. Phys. A, 7}, Suppl. 1B (1992), 805–812.

[11] Podleś, P. and Woronowicz, S.L., \textit{Quantum deformation of Lorentz group}, \textit{Commun. Math. Phys.} \textbf{130} (1990), 381–431.

[12] Podleś, P. and Woronowicz, S.L., \textit{On the structure of inhomogeneous quantum groups}, \texttt{hep-th/9412058}, \textit{Commun. Math. Phys.} \textbf{185} (1997), 325–358.

[13] Podleś, P. and Woronowicz, S.L., \textit{On the classification of quantum Poincaré groups}, \textit{Commun. Math. Phys.} \textbf{178} (1996), 61–82.

[14] Podleś, P., \textit{Solutions of Klein–Gordon and Dirac equations on quantum Minkowski spaces}, \textit{Commun. Math. Phys.} \textbf{181} (1996), 569–585.

[15] Podleś, P., \textit{Quasitriangularity and enveloping algebras for inhomogeneous quantum groups}, \textit{J. Math. Phys.} \textbf{37} (1996), 4724–4737.

[16] Podleś, P., \textit{The Dirac operator and gamma matrices for quantum Minkowski spaces}, \textit{J. Math. Phys.} \textbf{38} (1997), 4474–4491.

[17] Reshetikhin, N. Yu., Takhtadzyan, L. A. and Faddeev, L. D., Quantization of Lie groups and Lie algebras, \textit{Leningrad Math. J.} \textbf{1:1} (1990), 193–225. Russian original: \textit{Algebra i analiz} \textbf{1:1} (1989), 178–206.

[18] Schlicker, M., Weich, W. and Weixler, R., \textit{Inhomogeneous quantum groups}, \textit{Z. Phys. C. – Particles and Fields} \textbf{53} (1992), 79–82.

[19] Woronowicz, S.L., \textit{Twisted SU(2) group. An example of a non-commutative differential calculus}, \textit{Publ. RIMS, Kyoto University} \textbf{23} (1987), 117–181; \textit{Compact matrix pseudogroups}, \textit{Commun. Math. Phys.} \textbf{111} (1987), 613–665.
[20] Woronowicz, S.L. and Zakrzewski, S., Quantum deformations of the Lorentz group. The Hopf $\ast$-algebra level, *Comp. Math.* 90 (1994), 211–243.

[21] Zakrzewski, S., Geometric quantization of Poisson groups – diagonal and soft deformations, *Contemp. Math.* 179 (1994), 271–285.