Solution space of 2+1 gravity on $\mathbb{R} \times T^2$
in Witten’s connection formulation

Jorma Louko and Donald M. Marolf
Department of Physics, Syracuse University, Syracuse, New York 13244–1130, USA
(Revised version, October 1993)

Abstract

We investigate the space $\mathcal{M}$ of classical solutions to Witten’s formulation of 2+1 gravity on the manifold $\mathbb{R} \times T^2$. $\mathcal{M}$ is connected, unlike the spaces of classical solutions in the cases where $T^2$ is replaced by a higher genus surface. Although $\mathcal{M}$ is neither Hausdorff nor a manifold, removing from $\mathcal{M}$ a set of measure zero yields a manifold which is naturally viewed as the cotangent bundle over a non-Hausdorff base space $\mathcal{B}$. We discuss the relation of the various parts of $\mathcal{M}$ to spacetime metrics, and various possibilities of quantizing $\mathcal{M}$. There exist quantizations in which the exponentials of certain momentum operators, when operating on states whose support is entirely on the part of $\mathcal{B}$ corresponding to conventional spacetime metrics, give states whose support is entirely outside this part of $\mathcal{B}$. Similar results hold when the gauge group $\text{SO}(2,1)$ is replaced by $\text{SU}(1,1)$.

Pacs: 04.60.+n, 04.20.Jb
I. INTRODUCTION

The observation that vacuum Einstein gravity in 2+1 spacetime dimensions has no local dynamical degrees of freedom \[1,2\] has created interest in 2+1 gravity as an arena where quantum gravity can be investigated without many of the technical complications that are present in 3+1 spacetime dimensions. Of particular interest for the 3+1 theory is to understand the relation between the various 2+1 quantum theories that have been constructed in the metric, connection, and loop formulations \[3–13\]. For a recent review, see Ref. \[14\].

In this paper we shall be interested in Witten’s connection formulation \[4\] of 2+1 gravity and its relation to spacetime metrics on spacetimes of the form \(\mathbb{R} \times \Sigma\) where \(\Sigma\) is a closed orientable surface of genus \(g\). The case \(g > 1\) is well understood: the space of classical solutions to Witten’s theory contains several disconnected components, one of which is the cotangent bundle over the Teichmüller space of \(\Sigma\), and this component is isomorphic to the solution space of the conventional metric formulation \[4,15–18\]. The case \(g = 0\), corresponding to \(\Sigma = S^2\), is also well understood: it is trivial, in the sense that Witten’s theory possesses only one classical solution and the conventional metric formulation possesses no solutions \[17,18\]. For the remaining case \(g = 1\), corresponding to \(\Sigma = T^2\), the conventional metric formulation is well understood \[17,19\] and partial discussions of Witten’s formulation have been given in Refs. \[6,7,9,20\]. However, a complete analysis of the space \(\mathcal{M}\) of classical solutions to Witten’s formulation for \(g = 1\) appears never to have been given. The purpose of this paper is to provide such an analysis, and to explore the resulting possibilities for quantizing the theory.

One motivation for looking at this question, besides its intrinsic interest, is that the case \(g = 1\) has been much invoked as a testing ground for calculations which for \(g > 1\) would require considerably more sophistication. An example is the analysis of the relation of Witten’s connection quantization to metric quantization in Refs. \[9–11,21\]. Another example is the analysis of the Rovelli-Smolin loop transform in Refs. \[6,7,20\]. It is clearly important to understand to what extent such analyses represent features that are present also in the case \(g > 1\).

Conventional metric formulation of 2+1 gravity on \(\mathbb{R} \times T^2\) assumes that the three-metric is nondegenerate and that there exists a foliation in which the induced metrics on the tori are spacelike. Such spacetimes are known to correspond to roughly speaking half of the space \(\mathcal{M}\) of classical solutions in Witten’s formulation \[17,18\]. We shall however see that all points in \(\mathcal{M}\), except a set of measure zero, do correspond to nondegenerate spacetime metrics on \(\mathbb{R} \times T^2\). The nondegenerate spacetimes arising from those points of \(\mathcal{M}\) that do not correspond to the conventional metric formulation possess no slicing in which the induced metrics on the tori would be spacelike.

The main difference between \(g = 1\) and the higher genus case is that \(\mathcal{M}\) is connected, even though it is neither Hausdorff nor a manifold. This raises various possibilities for quantizing \(\mathcal{M}\). We shall see that removing from \(\mathcal{M}\) a suitable subset of measure zero yields a manifold which can be naturally viewed as the cotangent bundle over a non-Hausdorff base space \(\mathcal{B}\). Quantization of this cotangent bundle yields a quantum theory which contains all the individual quantum theories previously presented in Refs. \[6,7,9\]; these smaller theories each correspond to quantizing roughly speaking half of \(\mathcal{M}\). Further, the larger quantum theory contains operators which induce transitions between the theories of Refs. \[6,7,9\].
Such operators can in particular take states whose support is entirely on the part of $\mathcal{B}$ which corresponds to conventional spacetime metrics and map them to states whose support is entirely outside this part of $\mathcal{B}$.

On a par with Witten’s theory, we shall consider its modification in which the gauge group $\text{SO}_0(2, 1)$ (the connected component of $\text{SO}(2, 1)$) is replaced by its double cover $\text{SU}(1, 1) \simeq \text{SL}(2, \mathbb{R})$. Although the technical details differ, the overall qualitative picture is highly similar, and the space of classical solutions in the modified theory will be just a four-fold cover of $\mathcal{M}$. One reason for studying the modified theory is that this theory is closer to the work done with the 3+1 loop transform in Ashtekar’s variables [22,23]. Also, the work done with the 2+1 loop transform in Refs. [6,7] is perhaps more appropriately understood in terms of the modified theory, and the detailed study of the 2+1 loop transform in Ref. [20] was explicitly carried out within the modified theory.

The rest of the paper is as follows. In section II we briefly outline Witten’s formulation (and its SU(1, 1) modification) of 2+1 gravity on a spacetime with topology $\mathbb{R} \times \Sigma$. In section III we specialize to the case $\Sigma = T^2$ and give a detailed analysis of the space of classical solutions. The relation of the connection formulation to spacetime metrics is examined in section IV, and various avenues for quantization are explored in section V. Section VI contains a brief summary and discussion.

II. CONNECTION DYNAMICS

We consider a formulation of 2+1 gravity [4,24] in which the fundamental variables of the theory are the co-triad $\tilde{e}_{aI}$ and the connection $\tilde{A}^I_a$ defined on a three-dimensional manifold $M$. The connection takes values in the Lie algebra of $\text{SO}(2, 1)$, and the co-triad takes values in the dual of this Lie algebra. The internal indices $I, J, \ldots$ thus take values in $\{0, 1, 2\}$, and they are raised and lowered by the internal Minkowski metric, $\eta_{IJ} = \text{diag}(-1, 1, 1)$. The abstract indices $a, b, \ldots$ are spacetime indices on $M$. We shall regard $\tilde{A}^I_a$ either as an $\text{SO}(2, 1)$ connection or as an $\text{SU}(1, 1)$ connection. The differences between the two cases will become clear in the later sections.

The dynamics of the system is derived from the action

$$S \left( \tilde{e}_{aI}, \tilde{A}^I_a \right) = \frac{1}{2} \int_M d^3 x \tilde{\eta}^{abc} \tilde{e}_{aI} \tilde{F}^I_{bc},$$  \hspace{1cm} (2.1)

where $\tilde{\eta}^{abc}$ is the Levi-Civita density on $M$ and $\tilde{F}^I_{bc}$ is the curvature of the connection,

$$\tilde{F}^I_{bc} = 2 \partial_{[b} \tilde{A}^I_{c]} + \epsilon^I_{JK} \tilde{A}^J_a \tilde{A}^K_c.$$  \hspace{1cm} (2.2)

The structure constants $\epsilon^I_{JK}$ are obtained from the totally antisymmetric symbol $\epsilon_{IJK}$ by raising the index with the Minkowski metric. Our convention is $\epsilon_{012} = 1$. For later convenience, we have followed Ref. [24] and taken the action (2.1) to differ from that adopted in Refs. [4,11] by a factor of $\frac{1}{2}$. The action may need to be supplemented with boundary terms depending on what is held fixed in the variational principle. However, we shall not need to discuss these boundary terms here.

The classical equations of motion are
\( \bar{F}_{ab} = 0 \) \hspace{1cm} (2.3a)
\( \bar{D}_{[a} \bar{e}_{b]} = 0 \) \hspace{1cm} (2.3b)

where \( \bar{D}_a \) is the gauge covariant derivative determined by \( \bar{A}^I_a \),
\[
\bar{D}_a v_K = \partial_a v_K - \epsilon^{IJK} \bar{A}^J_a v_I .
\] \hspace{1cm} (2.4)

Note that \( \bar{D}_a \) acts only on the internal indices. Equations (2.3) therefore say that the connection \( \bar{A}^I_a \) is flat and the co-triad is compatible with the connection. If the co-triad is nondegenerate, the metric \( g_{ab} = \bar{e}_a \bar{e}_b \) has signature \((- , +, +)\), and the equations of motion imply that this metric is flat, i.e., it satisfies the vacuum Einstein equations. However, the theory remains well-defined also for degenerate co-triads.

We now take \( \mathcal{M} \) to be \( \mathbb{R} \times \Sigma \), where \( \Sigma \) is a closed orientable two-manifold. The 2+1 decomposition of the action takes the form \[ S = \int dt \int_{\Sigma} d^2x \left[ \tilde{E}^I_j \left( \partial_t A^I_j \right) + \bar{A}^I_i \left( \bar{D}_j \tilde{E}^I_j \right) + \frac{1}{2} \bar{e}_I F^I_{ij} \tilde{\eta}^{ij} \right] . \] \hspace{1cm} (2.5)

The abstract indices \( i, j, \ldots \) live on \( \Sigma \), and \( t \) is the coordinate on \( \mathbb{R} \). The SO(2, 1) or SU(1, 1) connection \( A^I_j \) is the pull-back of \( \bar{A}^I_a \) to \( \Sigma \), \( F^I_{ij} \) is its curvature given by
\[
F^I_{ij} = 2 \partial_i [A^I_j] + \epsilon^{IJK} A^J_i A^K_j \ ,
\] \hspace{1cm} (2.6)
and \( \tilde{\eta}^{ij} \) is the Levi-Civita density on \( \Sigma \). The vector density \( \tilde{E}^I_j \) is given by \( \tilde{E}^I_j = \tilde{\eta}^{ij} e_i t \), where \( e_i t \) is the pull-back of \( \bar{e}_a t \) to \( \Sigma \). \( \bar{D}_j \) is the gauge-covariant derivative on \( \Sigma \) determined by \( A^I_j \),
\[
\bar{D}_j v_K = \partial_j v_K - \epsilon^{IJK} A^I_j v_I \ .
\] \hspace{1cm} (2.7)

Note that the decomposition involves no assumptions about the rank of the metric \( g_{ab} = \bar{e}_a \bar{e}_b \), or about the sign of the quantity \( \bar{e}_I \bar{e}^I = g_{ab} (\partial/\partial t)^a (\partial/\partial t)^b \). This means that the induced metric on \( \Sigma \), \( e_i \bar{e}^I_j \), can (locally) have any of the signatures \((+, +),(−, +),(0, +),(−, 0),(0, 0)\). The canonical pair is now \( \left( A^I_j, \tilde{E}^I_j \right) \), with the Poisson brackets
\[
\left\{ A^I_i (x), \tilde{E}^J_j (x') \right\} = \delta^I_i \delta^J_j \delta^2 (x, x') \ ,
\] \hspace{1cm} (2.8)
where \( x \) denotes the points on \( \Sigma \). \( \bar{e}_I \) and \( \bar{A}^I_a \) act as Lagrange multipliers enforcing the constraints
\[
F^I_{ij} = 0 \hspace{1cm} (2.9a)
\]
\[
\bar{D}_j \tilde{E}^I_j = 0 \ .
\] \hspace{1cm} (2.9b)

The first of these constraints generates analogues of the Hamiltonian gauge transformations of the metric formulation of 2+1 gravity and their extensions to degenerate co-triads. The second constraint is analogous to the Gauss law constraint in electrodynamics. It generates internal SO(2, 1) or SU(1, 1) gauge transformations, depending on whether \( \bar{A}^I_a \) is an SO(2, 1) connection or an SU(1, 1) connection.
When $\vec{A}_a^I$ is an SO(2,1) connection, the pair $(A_1^I, e_{Ij})$ defines on $\Sigma$ the ISO(2,1) connection $e_1^I P_I + A_1^I K_I$, where $P_I$ and $K_I$ are the generators of translations and Lorentz transformations in 2+1 dimensional Minkowski space, satisfying the commutator algebra

\[
[K_I, K_J] = \epsilon_{IJK} K_K \\
[K_I, P_J] = \epsilon_{IJK} P_K \\
[P_I, P_J] = 0 .
\] (2.10)

Witten [4] observed that the constraints (2.9) make the curvature of this ISO(2,1) connection vanish. Further, these constraints generate local ISO$_0(2,1)$ gauge transformations, where ISO$_0(2,1)$ is the connected component of ISO(2,1). Taking the connection to be in the trivial principal bundle ISO$_0(2,1) \times \Sigma$, this means that the space of classical solutions is the moduli space of flat ISO$_0(2,1)$ connections on $\Sigma$, modulo ISO$_0(2,1)$ gauge transformations. When $\vec{A}_a^I$ is an SU(1,1) connection the situation is analogous, and the space of classical solutions is the moduli space of flat ISU(1,1) connections on $\Sigma$, modulo ISU(1,1) gauge transformations.

A flat connection on $\Sigma$ is completely determined by its holonomies around closed non-contractible loops. The space of classical solutions is therefore the space of group homomorphisms from the fundamental group of $\Sigma$ to $G$, where $G$ is respectively ISO$_0(2,1)$ or ISU(1,1), modulo conjugation by $G$. In the case $\Sigma = S^2$ the fundamental group is trivial, and the space of classical solutions consists of only one point. In all cases where $\Sigma$ is a surface of genus two or higher the spaces of classical solutions are nontrivial and can be described in a unified manner [4,15,16]. In the remaining case, $\Sigma = T^2$, the space of classical solutions is nontrivial but qualitatively different from those of the higher genus surfaces. Our aim is to analyze this remaining case in detail.

**III. SPACE OF CLASSICAL SOLUTIONS FOR $\Sigma = T^2$**

In this section we shall describe the space of classical solutions for $\Sigma = T^2$. As the fundamental group of the torus is the Abelian group $\mathbb{Z} \times \mathbb{Z}$, we see from the end of the previous section that points in the space of classical solutions are just equivalence classes of pairs of commuting elements of $G$ under $G$ conjugation. We shall devote separate subsections to the cases $G = \text{ISO}_0(2,1)$ and $G = \text{ISU}(1,1)$.

**A. $G = \text{ISO}_0(2,1)$**

We first consider the case where $\vec{A}_a^I$ is an SO(2,1) connection. Then $G$ is ISO$_0(2,1)$, that is, the group of proper Poincare transformations in 2+1 dimensional Minkowski space $M^{2+1}$. We denote the space of classical solutions by $\mathcal{M}$.

Recall that ISO$_0(2,1)$ can be defined as the group of pairs $(R, w)$, where $R$ is an SO$_0(2,1)$ matrix and $w$ is a column vector. The group multiplication law is

\[(R_2, w_2) \cdot (R_1, w_1) = (R_2 R_1, R_2 w_1 + w_2) .\] (3.1)
When points in $M^{2+1}$ are represented by column vectors as $v = (T, X, Y)^T$ and the entries are the usual Minkowski coordinates associated with the line element $ds^2 = -dT^2 + dX^2 + dY^2$, the action of a group element $(R, w)$ on $M^{2+1}$ is

$$(R, w) : v \mapsto Rv + w .$$  \hspace{1cm} (3.2)$$

We now wish to obtain a unique parametrization of pairs of commuting elements of ISO(0,2) modulo ISO(0,2) conjugation. For this purpose, let $(R_\alpha, w_\alpha)$, $\alpha = 1, 2$, be commuting. Clearly then $R_1R_2 = R_2R_1$. From the SU(1,1) parametrization of SO(0,2) given in Refs. [25,26], it is straightforward to show that every matrix $R \in SO(0,2)$ can be written in the form $\exp(v^IK_I)$, where $v^Iv_I \geq -\pi^2$ and $K_I$ are the $3 \times 3$ matrices spanning the adjoint representation of the Lie algebra of SO(2,1), $(K_J)^I_K = \epsilon^I_{JK}$. The only redundancy in this parametrization of SO(0,2) in terms of the Lorentz vector $v^I$ is that when $v^Iv_I = -\pi^2$, $v^I$ and $-v^I$ give the same element of SO(0,2). Writing now $R_\alpha = \exp(v^I_\alpha K_I)$ in this fashion, one sees that the two Lorentz vectors $v^I_\alpha$ must be proportional to each other. One has then four qualitatively different cases, depending on whether the two Lorentz vectors are spacelike, timelike, null but nonvanishing, or vanishing.

*Case (i).* Suppose first that the vectors $v^I_\alpha$ are spacelike, or that one of them is spacelike and the other vanishes. This means that $R_\alpha$ are boosts with at least one having a nonvanishing boost parameter. The holonomies $(R_\alpha, w_\alpha)$ can be conjugated to the form

$$R_\alpha = \exp(\lambda_\alpha K_2) = \begin{pmatrix} \cosh \lambda_\alpha & \sinh \lambda_\alpha & 0 \\ \sinh \lambda_\alpha & \cosh \lambda_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_\alpha = \begin{pmatrix} 0 \\ 0 \\ a_\alpha \end{pmatrix} ,$$  \hspace{1cm} (3.3)$$

where the four parameters $\lambda_\alpha$ and $a_\alpha$ can take arbitrary values except that $\lambda_\alpha$ are by assumption not both equal to zero. The only remaining redundancy in this parametrization is that conjugation by the rotation $R = \exp(\pi K_0)$ reverses the signs of all the four parameters. Thus, a unique parametrization is obtained from (3.3) through the identification $(\lambda_\alpha, a_\alpha) \sim (-\lambda_\alpha, -a_\alpha)$. We denote this part of $\mathcal{M}$ by $\mathcal{M}_s$.

*Case (ii).* Suppose then that the vectors $v^I_\alpha$ are timelike, or that one of them is timelike and the other vanishes. This means that $R_\alpha$ are rotations, with at least one differing from the identity rotation. The holonomies can be conjugated to the form

$$R_\alpha = \exp(\tilde{\lambda}_\alpha K_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \tilde{\lambda}_\alpha & -\sin \tilde{\lambda}_\alpha \\ 0 & \sin \tilde{\lambda}_\alpha & \cos \tilde{\lambda}_\alpha \end{pmatrix}, \quad w_\alpha = \begin{pmatrix} -\tilde{a}_\alpha \\ 0 \\ 0 \end{pmatrix} ,$$  \hspace{1cm} (3.4)$$

where $\tilde{\lambda}_\alpha$ and $\tilde{a}_\alpha$ are four parameters, with $|\tilde{\lambda}_\alpha| \leq \pi$ and $\tilde{\lambda}_\alpha$ not both equal to zero. This parametrization becomes unique when we interpret $\tilde{\lambda}_\alpha$ as angular parameters, identified according to $(\tilde{\lambda}_1, \tilde{\lambda}_2) \sim (\tilde{\lambda}_1 + 2\pi m, \tilde{\lambda}_2 + 2\pi n)$ for $m, n$ in $\mathbb{Z}$. We denote this part of $\mathcal{M}$ by $\mathcal{M}_t$.

*Case (iii).* Suppose then that the vectors $v^I_\alpha$ are null, with at least one of them nonvanishing. This means that $R_\alpha$ are null rotations, with at least one differing from the identity rotation. The holonomies can now be conjugated to the form
Here $\theta$ is identified with period $2\pi$, and $p_\theta$ and $p_r$ are arbitrary. It can be verified that this parametrization is unique; the reason for the nomenclature will become clear shortly. We denote this part of $\mathcal{M}$ by $\mathcal{M}_0$.

Case (iv). The remaining case is when $v_\alpha$ both vanish, so that $R_\alpha$ are identity Lorentz transformations and $(R_\alpha, w_\alpha)$ are purely translational. We denote this part of $\mathcal{M}$ by $\mathcal{M}_0$. There are several subcases depending on the spacelike/timelike/null character of $w_\alpha$, and the plane or line which they span. For example, when $w_\alpha$ span a spacelike plane, the holonomies can be uniquely conjugated to the form

$$w_1 = \frac{1}{2}p_\theta \cos \theta \begin{pmatrix} 1 + \frac{1}{2} \cos^2 \theta & \sin \theta & -\frac{1}{2} \sin^2 \theta \\ \sin \theta & 1 & -\sin \theta \\ \frac{1}{2} \sin^2 \theta & \sin \theta & 1 - \frac{1}{2} \sin^2 \theta \end{pmatrix} \frac{1}{2}p_r \sin \theta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$w_2 = \frac{1}{2}p_\theta \sin \theta \begin{pmatrix} 1 + \frac{1}{3} \sin^2 \theta & \sin \theta & -\frac{1}{3} \sin^2 \theta \\ \sin \theta & 1 & -\sin \theta \\ \frac{2}{3} \sin^2 \theta & \sin \theta & 1 - \frac{2}{3} \sin^2 \theta \end{pmatrix} + \frac{1}{2}p_r \cos \theta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$ 

$$R_1 = \exp (\cos \theta (K_0 + K_2)) = \begin{pmatrix} 1 + \frac{1}{2} \cos^2 \theta & \cos \theta & -\frac{1}{2} \cos^2 \theta \\ \cos \theta & 1 & -\cos \theta \\ \frac{1}{2} \cos^2 \theta & \cos \theta & 1 - \frac{1}{2} \cos^2 \theta \end{pmatrix},$$

$$w_1 = \frac{1}{2}p_\theta \cos \theta \begin{pmatrix} 1 + \frac{1}{2} \cos^2 \theta \\ \cos \theta \\ -1 + \frac{1}{3} \cos^2 \theta \end{pmatrix} - \frac{1}{2}p_r \sin \theta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$R_2 = \exp (\sin \theta (K_0 + K_2)) = \begin{pmatrix} 1 + \frac{1}{2} \sin^2 \theta & \sin \theta & -\frac{1}{2} \sin^2 \theta \\ \sin \theta & 1 & -\sin \theta \\ \frac{1}{2} \sin^2 \theta & \sin \theta & 1 - \frac{1}{2} \sin^2 \theta \end{pmatrix},$$

$$w_2 = \frac{1}{2}p_\theta \sin \theta \begin{pmatrix} 1 + \frac{1}{3} \sin^2 \theta \\ \sin \theta \\ -1 + \frac{1}{3} \sin^2 \theta \end{pmatrix} + \frac{1}{2}p_r \cos \theta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

where $r_\alpha > 0$ and $0 < \phi < \pi$. The global structure of $\mathcal{M}_0$ is fairly complicated, and we shall not need an explicit parametrization for all the different subcases.

We have thus shown that $\mathcal{M}$ is the disjoint union of $\mathcal{M}_\alpha$, $\mathcal{M}_t$, $\mathcal{M}_n$, and $\mathcal{M}_0$. The topology on $\mathcal{M}$ is induced from that of the manifold $\text{ISO}_0(2,1) \times \text{ISO}_0(2,1) \simeq T^2 \times \mathbb{R}^{10}$, from which it is clear that $\mathcal{M}$ is connected. We shall now examine how close $\mathcal{M}$ comes to being a four-dimensional manifold, and what structure $\mathcal{M}$ inherits from the action (3.1). The set $\mathcal{M}_\alpha$ is open in $\mathcal{M}$, and it is a four-dimensional manifold with topology $S^1 \times \mathbb{R}^3$. A pair $(\bar{A}_\alpha, \bar{e}_{aI})$ with the holonomies (3.3) is given by

$$\bar{A}^2 = \lambda_\alpha dx^\alpha,$$

$$\bar{e}_2 = a_\alpha dx^\alpha,$$

with all other components vanishing. Here, and from now on, $(x^1, x^2)$ will denote a pair of angular coordinates on $\Sigma$, periodic with the identifications $(x^1, x^2) \sim (x^1 + m, x^2 + n)$ for $m, n$ in $\mathbb{Z}$; the Greek index in Eqs. (3.7) and henceforth should therefore be understood not as abstract index notation but as referring to this particular coordinate system. From the discussion of the holonomies above it is clear that the fields (3.7) define an almost unique
gauge fixing, the only degeneracy being that the transformation $(\lambda_\alpha, a_\alpha) \mapsto (-\lambda_\alpha, -a_\alpha)$ gives a gauge-equivalent pair. Pulling the action (2.5) back to these spatially homogeneous fields, the induced Poisson brackets on the parameters are found to be

$$\{\lambda_1, a_2\} = -\{\lambda_2, a_1\} = 1 \quad .$$

(3.8)

$\mathcal{M}_s$ has therefore the structure of a phase space, and it is naturally viewed as the cotangent bundle over the punctured plane. The symplectic form is

$$\Omega_s = -\epsilon^{\alpha\beta} d\lambda_\alpha \wedge d\lambda_\beta \quad ,$$

(3.9)

where $\epsilon^{\alpha\beta}$ is the antisymmetric symbol with the convention $\epsilon^{12} = 1$.

The set $\mathcal{M}_t$ is open in $\mathcal{M}$, and it is a four-dimensional manifold with the topology of a punctured torus times $\mathbf{R}^2$. A pair $(\vec{A}_a^I, \vec{e}_aI)$ with the holonomies (3.4) is given by

$$\vec{A}^0 = \vec{\lambda}_\alpha dx^\alpha$$
$$\vec{e}_0 = \vec{a}_\alpha dx^\alpha \quad .$$

(3.10)

It is clear that this pair defines an almost unique gauge fixing, the only degeneracy being that adding multiples of $2\pi$ to $\vec{\lambda}_\alpha$ gives gauge equivalent pairs. It can be verified as above that $\mathcal{M}_t$ inherits the symplectic structure

$$\Omega_t = -\epsilon^{\alpha\beta} d\vec{a}_\alpha \wedge d\vec{\lambda}_\beta \quad ,$$

(3.11)

and $\mathcal{M}_t$ can therefore be naturally viewed as the cotangent bundle over the punctured torus.

The set $\mathcal{M}_n$ is a three-dimensional manifold with topology $S^1 \times \mathbf{R}^2$. It is not open in $\mathcal{M}$, and the neighborhoods of every point in $\mathcal{M}_n$ intersect both $\mathcal{M}_s$ and $\mathcal{M}_t$. A pair $(\vec{A}_a^I, \vec{e}_aI)$ with the holonomies (3.5) is given by

$$\vec{A}^0 = \vec{A}^2 = du$$
$$\vec{e}_0 = -\frac{1}{2} p_r dv - \frac{1}{2} p_\theta du$$
$$\vec{e}_2 = \frac{1}{2} p_r dv - \frac{1}{2} p_\theta du \quad ,$$

(3.12)

where we have introduced the notation

$$du = \cos \theta \, dx^1 + \sin \theta \, dx^2$$
$$dv = -\sin \theta \, dx^1 + \cos \theta \, dx^2 \quad .$$

(3.13)

This pair clearly defines a unique gauge fixing.

Finally, the set $\mathcal{M}_0$ is neither open in $\mathcal{M}$ nor a manifold. It however contains several subsets that are three-dimensional manifolds, such as the set (3.6) homeomorphic to $\mathbf{R}^3$. A pair $(\vec{A}_a^I, \vec{e}_aI)$ with the appropriate holonomies is obtained by setting $\vec{A}_a^0 = 0$ and taking $\vec{e}_j^I dx^j$ equal to the components of the column vector valued one-form $w_\alpha dx^\alpha$. Note that all neighborhoods of the point $R_\alpha = 1, w_\alpha = 0$ intersect both $\mathcal{M}_s$, $\mathcal{M}_t$, and $\mathcal{M}_n$.

The emerging picture of $\mathcal{M}$ is thus that of two four-dimensional manifolds, $\mathcal{M}_s$ and $\mathcal{M}_t$, glued together by the three-dimensional manifold $\mathcal{M}_n$ and by the set $\mathcal{M}_0$ which is somewhere close to being a three-dimensional manifold. $\mathcal{M}$ itself is not a manifold. One way to see this
is to consider the set $\overline{\mathcal{M}}_t$ (the closure of $\mathcal{M}_t$). It is easily seen that the holonomies in $\overline{\mathcal{M}}_t$ are given by Eqs. (3.4) with arbitrary values of $\tilde{\lambda}_\alpha$ and $\tilde{a}_\alpha$. $\overline{\mathcal{M}}_t$ is clearly a manifold. It has the symplectic structure given by the expression (3.11), and it can be naturally viewed as the cotangent bundle over the torus. Suppose now that $\mathcal{M}$ were a manifold; if so, it would have to be four-dimensional. Consider a point $x \in \overline{\mathcal{M}}_t \cap \mathcal{M}_0$. Let $U \simeq \mathbb{R}^4$ be a neighborhood of $x$ in $\overline{\mathcal{M}}_t$. Then $U \cap \overline{\mathcal{M}}_t$ is open in $\overline{\mathcal{M}}_t$. As $\overline{\mathcal{M}}_t$ is a four-manifold, there exists a set $V \subset U \cap \overline{\mathcal{M}}_t$ such that $x \in V$ and $V \simeq \mathbb{R}^4$. Then, $V \subset U$ implies that $V$ is open as a subset of $U$; hence $V$ is open in $\overline{\mathcal{M}}_t$. This however is a contradiction, since any neighborhood of $x$ in $\overline{\mathcal{M}}_t$ contains points that are not in $\overline{\mathcal{M}}_t$. Therefore $\overline{\mathcal{M}}_t$ cannot be a manifold.

Note incidentally that the set $\overline{\mathcal{M}}_s$ (closure of $\mathcal{M}_s$), where the holonomies are given by Eqs. (3.3) with arbitrary values of $\lambda_\alpha$ and $a_\alpha$, is not a manifold. The reason is that the parametrization of $\overline{\mathcal{M}}_s$ by Eqs. (3.3) is unique only after the identification $(\lambda_\alpha, a_\alpha) \sim (-\lambda_\alpha, -a_\alpha)$, which has a fixed point at $\lambda_\alpha = a_\alpha = 0$. This makes $\overline{\mathcal{M}}_s$ an orbifold [27], or a conifold [28].

There exists, nevertheless, an open subset of $\mathcal{M}$ that contains both $\mathcal{M}_s$ and $\mathcal{M}_t$ and is a manifold: it is $\mathcal{M} \setminus \mathcal{M}_0 = \mathcal{M}_t \cup \mathcal{M}_n \cup \mathcal{M}_s$. To show this, let us first perform on $\mathcal{M}_s$ the coordinate transformation

\[
\begin{align*}
\lambda_1 &= (2 \sinh 2r)^{1/2} \cos \theta \\
\lambda_2 &= (2 \sinh 2r)^{1/2} \sin \theta \\
a_1 &= -\frac{p_r (2 \sinh 2r)^{1/2} \sin \theta}{2 \cosh 2r} - \frac{p_\theta \cos \theta}{(2 \sinh 2r)^{1/2}} \\
a_2 &= \frac{p_r (2 \sinh 2r)^{1/2} \cos \theta}{2 \cosh 2r} - \frac{p_\theta \sin \theta}{(2 \sinh 2r)^{1/2}},
\end{align*}
\]  

(3.14)

where $r > 0$, $\theta$ is periodic with period $\pi$, and $p_r$ and $p_\theta$ take arbitrary values. The symplectic structure (3.9) is given by

\[
\Omega_s = dp_r \wedge dr + dp_\theta \wedge d\theta.
\]  

(3.15)

Next, we use the Gauss constraint (2.9b) to perform on the pair (3.7) a gauge transformation which amounts to a Lorentz boost with rapidity $\frac{1}{2} \ln(\coth r)$ in the internal (02) plane. Finally, we use the curvature constraint (2.9a) to perform a second gauge transformation, taking only the $I = 1$ component of the gauge transformation parameter to be nonvanishing and equal to $p_\theta/(2 \sinh 2r)$. The new pair is

\[
\begin{align*}
\bar{A}^0 &= e^{-r} du \\
\bar{A}^2 &= e^r du \\
\bar{e}_0 &= -\frac{p_r e^{-r} dv}{2 \cosh 2r} - \frac{p_\theta du}{2 \cosh r} \\
\bar{e}_2 &= \frac{p_r e^r dv}{2 \cosh 2r} - \frac{p_\theta du}{2 \cosh r},
\end{align*}
\]  

(3.16)

where $du$ and $dv$ are as in (3.13). This pair is well-defined for $-\infty < r < \infty$. For $r = 0$ it reduces to (3.12), provided that $\theta$ is regarded as periodic with period $2\pi$. For $r < 0$, the pair (3.16) is gauge-equivalent to the pair (3.10), via the coordinate transformation
\[ \lambda_1 = (-2 \sinh 2r)^{1/2} \cos \theta \]
\[ \lambda_2 = (-2 \sinh 2r)^{1/2} \sin \theta \]
\[ \tilde{a}_1 = \frac{pr (-2 \sinh 2r)^{1/2} \sin \theta}{2 \cosh 2r} - \frac{p_\theta \cos \theta}{(-2 \sinh 2r)^{1/2}} \]
\[ \tilde{a}_2 = -\frac{pr (-2 \sinh 2r)^{1/2} \cos \theta}{2 \cosh 2r} - \frac{p_\theta \sin \theta}{(-2 \sinh 2r)^{1/2}}, \]

again provided that \( \theta \) is understood periodic with period \( 2\pi \). The required gauge transformation is the inverse of the one used when going from (3.7) to (3.16). The periodic identifications of \( \tilde{\lambda}_\alpha \) induce a set of corresponding identifications for \((r, \theta, p_r, p_\theta)\) in the domain \( r < 0 \) via (3.17); however, from the first two lines in (3.17) it is seen that no quadruplet \((r, \theta, p_r, p_\theta)\) in the range \(-\frac{1}{2} \arcsinh(\pi^2/2) < r < 0\) is identified with another quadruplet in the same range.

Thus, the coordinates \((r, \theta, p_r, p_\theta)\) extend from \( \mathcal{M}_s \) through \( \mathcal{M}_n \) to \( \mathcal{M}_t \), with \( \theta \) being periodic with period \( \pi \) for \( r > 0 \) and with period \( 2\pi \) for \( r \leq 0 \). This shows that \( \mathcal{M} \setminus \mathcal{M}_0 \) is a manifold, although not a Hausdorff one, since points with \( r = 0 \) and \( \theta \) differing by \( \pi \) do not have disjoint neighborhoods. \( \mathcal{M} \setminus \mathcal{M}_0 \) can therefore be viewed as the cotangent bundle whose base space consists of the base space of \( \mathcal{M}_s \) (punctured plane) and the base space of \( \mathcal{M}_t \) (punctured torus) glued together at the punctures; the circle which provides the glue is the image of \( \mathcal{M}_n \) under the projection that maps the holonomies (3.5) to pure Lorentz transformations. The circle joins to the base space of \( \mathcal{M}_t \) in a one-to-one fashion, but the joining of the base space of \( \mathcal{M}_s \) to the circle is two-to-one.

**B. \( G = \text{ISU}(1, 1) \)**

We now turn to the case where \( \bar{A}^a_I \) is an \( \text{SU}(1, 1) \) connection. The group \( G \) is now \( \text{ISU}(1, 1) \), and we denote the space of classical solutions by \( \mathcal{N} \).

Recall that \( \text{SU}(1, 1) \) can be defined as the group of pairs \((U, H)\), where \( U \) is an \( \text{SU}(1, 1) \) matrix and \( H \) is a Hermitian \( 2 \times 2 \) matrix whose diagonal elements are equal. The group multiplication law is

\[
(U_2, H_2) \cdot (U_1, H_1) = (U_2 U_1, U_2 H_1 U_2^\dagger + H_2).
\]

Since \( \text{SU}(1, 1) \) is a double cover of \( \text{SO}_0(2, 1) \), it follows that \( \text{ISU}(1, 1) \) is a double cover of \( \text{ISO}_0(2, 1) \) \([23, 24]\). Representing points in \( \mathbb{M}^{2+1} \) by the matrices

\[
V = \begin{pmatrix} -T & Y + iX \\ Y - iX & -T \end{pmatrix},
\]

where \( T \), \( X \), and \( Y \) are the usual Minkowski coordinates, the \( \text{ISO}_0(2, 1) \) action of \( \text{ISU}(1, 1) \) on \( \mathbb{M}^{2+1} \) is given by

\[
(U, H) : \quad V \mapsto UVU^\dagger + H.
\]

We now wish to obtain a unique parametrization of pairs of commuting elements of \( \text{ISU}(1, 1) \) modulo \( \text{ISU}(1, 1) \) conjugation. The main difference from the previous subsection
arises from the fact that not every element of SU(1, 1) can be written as the exponential of an element in the Lie algebra. Recall [24] that a basis for the Lie algebra of SU(1, 1) is given by the three matrices

\[
\tau_0 = -\frac{1}{2}i\sigma_3, \quad \tau_1 = \frac{1}{2}\sigma_1, \quad \tau_2 = \frac{1}{2}\sigma_2, \quad (3.21)
\]

where \(\sigma_1, \sigma_2\) and \(\sigma_3\) are the Pauli matrices. An isomorphism with the Lie algebra of SO(2, 1) is given by \(\tau_I \mapsto k_I\), where the matrices \(k_I\) were introduced in the previous subsection.

From the parametrization of SU(1, 1) given in Refs. [25,26], it is easily verified that every element of SU(1, 1) can be written either as \(\exp(v^I\tau_I)\) or \(-\exp(v^I\tau_I)\). For an SU(1, 1) element for which the corresponding Lorentz transformation is a rotation, the former expression is sufficient (and still redundant); for an SU(1, 1) element for which the corresponding Lorentz transformation is a boost or a null rotation, there is a unique parametrization by exactly one of the two expressions. Using this, the analysis proceeds in analogy with that in the previous subsection. There are again four qualitatively different cases.

Case (i). When the Lorentz transformations corresponding to the SU(1, 1) matrices are boosts, with at least one having a nonvanishing boost parameter, the holonomies \((U_\alpha, H_\alpha)\) can be conjugated to the form

\[
U_\alpha = (-1)^{\eta_\alpha}\exp(\lambda_\alpha \tau_2) = (-1)^{\eta_\alpha}\begin{pmatrix}
\cosh(\lambda_\alpha/2) & -i\sinh(\lambda_\alpha/2) \\
i\sinh(\lambda_\alpha/2) & \cosh(\lambda_\alpha/2)
\end{pmatrix}, \quad H_\alpha = \begin{pmatrix}
0 & a_\alpha \\
a_\alpha & 0
\end{pmatrix}, \quad (3.22)
\]

where the four parameters \(\lambda_\alpha\) and \(a_\alpha\) can take arbitrary values except that \(\lambda_\alpha\) are not both equal to zero, and the parameters \(\eta_\alpha\) take the discrete values 0 and 1. A unique parametrization is obtained from (3.22) through the identification \((\lambda_\alpha, a_\alpha) \sim (-\lambda_\alpha, -a_\alpha)\). We denote these four parts of \(\mathcal{N}\) respectively by \(\mathcal{N}^{\eta_1\eta_2}\), where the upper indices correspond to the four possible combinations of \(\eta_\alpha\) in Eq. (3.22).

Case (ii). When the Lorentz transformations corresponding to the SU(1, 1) matrices are rotations, with at least one differing from the identity rotation, the holonomies \((U_\alpha, H_\alpha)\) can be conjugated to the form

\[
U_\alpha = \exp(\tilde{\lambda}_\alpha \tau_0) = \begin{pmatrix}
\exp(-i\tilde{\lambda}_\alpha/2) & 0 \\
0 & \exp(i\tilde{\lambda}_\alpha/2)
\end{pmatrix}, \quad H_\alpha = \begin{pmatrix}
\tilde{a}_\alpha & 0 \\
0 & \tilde{a}_\alpha
\end{pmatrix}, \quad (3.23)
\]

where \(\tilde{\lambda}_\alpha\) and \(\tilde{a}_\alpha\) are four parameters, arbitrary except that \((\tilde{\lambda}_1, \tilde{\lambda}_2) \neq (2\pi m, 2\pi n)\) for \(m, n\) in \(\mathbb{Z}\). This parametrization becomes unique when the parameters \(\tilde{\lambda}_\alpha\) are identified according to \((\tilde{\lambda}_1, \tilde{\lambda}_2) \sim (\tilde{\lambda}_1 + 4\pi m, \tilde{\lambda}_2 + 4\pi n)\) for \(m, n\) in \(\mathbb{Z}\). We denote this part of \(\mathcal{N}\) by \(\mathcal{N}_t\).

Case (iii) When the Lorentz transformations corresponding to the SU(1, 1) matrices are null rotations, with at least one differing from the identity rotation, the holonomies \((U_\alpha, H_\alpha)\) can be conjugated to the form
\[ U_1 = (-1)^{n_1} \exp(\cos \theta(\tau_0 + \tau_2)) = (-1)^{n_1} \begin{pmatrix} 1 - \frac{i}{2} \cos \theta & -\frac{i}{2} \cos \theta \\ \frac{i}{2} \cos \theta & 1 + \frac{i}{2} \cos \theta \end{pmatrix} \]

\[ H_1 = \frac{1}{2} p_\theta \cos \theta \begin{pmatrix} -1 - \frac{1}{3} \cos^2 \theta & -1 + \frac{1}{3} \cos^2 \theta + i \cos \theta \\ 1 + \frac{1}{3} \cos^2 \theta - i \cos \theta & -1 - \frac{1}{3} \cos^2 \theta \end{pmatrix} - \frac{1}{2} p_r \sin \theta \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \]

\[ U_2 = (-1)^{n_2} \exp(\sin \theta(\tau_0 + \tau_2)) = (-1)^{n_2} \begin{pmatrix} 1 - \frac{i}{2} \sin \theta & -\frac{i}{2} \sin \theta \\ \frac{i}{2} \sin \theta & 1 + \frac{i}{2} \sin \theta \end{pmatrix} \]

\[ H_2 = \frac{1}{2} p_\theta \sin \theta \begin{pmatrix} -1 - \frac{1}{3} \sin^2 \theta & -1 + \frac{1}{3} \sin^2 \theta + i \sin \theta \\ 1 + \frac{1}{3} \sin^2 \theta - i \sin \theta & -1 - \frac{1}{3} \sin^2 \theta \end{pmatrix} + \frac{1}{2} p_r \cos \theta \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \]

(3.24)

Here \( \theta \) is identified with period \( 2\pi \), and \( p_\theta \) and \( p_r \) are arbitrary. The discrete parameters \( \eta_\alpha \) take the values 0 and 1 as in case (i). It can be verified that this parametrization is unique. We denote these four parts of \( \mathcal{N} \) by \( \mathcal{N}_{0}^{m,n_2} \).

Case (iv). The remaining case is when \( U_\alpha = (-1)^{n_\alpha} \) and \( H_\alpha \) are arbitrary. We denote these four parts of \( \mathcal{N} \) by \( \mathcal{N}_{0}^{m,n_2} \). As in the previous subsection, there are several subcases depending on the spacelike/timelike/null character of the two Lorentz-vectors encoded in \( \lambda \) gauge fixing, the only degeneracy being that the transformation \( (\alpha, \beta) \mapsto (-\lambda_\alpha, -a_\alpha) \) gives a gauge-equivalent pair. The induced symplectic structure on each \( \mathcal{N}_{s}^{m,n_2} \) is given by Eq. (3.9). All the spaces \( \mathcal{N}_{s}^{m,n_2} \) are therefore isomorphic to each other and to \( \mathcal{M}_{s} \) as symplectic manifolds.

The four sets \( \mathcal{N}_{s}^{m,n_2} \) are each open in \( \mathcal{N} \), and each is a four-dimensional manifold with topology \( S^1 \times \mathbf{R}^3 \). A pair \( (\bar{A}_\alpha^I, \bar{e}_{a_I}) \) with the holonomies (3.22) is given by

\[
\bar{A}^0 = -2\pi \eta_\alpha dx^\alpha \\
\bar{A}^1 = -\sin(2\pi \eta_3 x^\beta) \lambda_\alpha dx^\alpha \\
\bar{A}^2 = \cos(2\pi \eta_3 x^\beta) \lambda_\alpha dx^\alpha \\
\bar{e}_1 = -\sin(2\pi \eta_3 x^\beta) a_\alpha dx^\alpha \\
\bar{e}_2 = \cos(2\pi \eta_3 x^\beta) a_\alpha dx^\alpha ,
\]

(3.25)

as can be verified by direct computation.\(^1\) The fields (3.25) clearly define an almost unique gauge fixing, the only degeneracy being that the transformation \( (\lambda_\alpha, a_\alpha) \mapsto (-\lambda_\alpha, -a_\alpha) \) gives a gauge-equivalent pair. The induced symplectic structure on each \( \mathcal{N}_{s}^{m,n_2} \) is given by Eq. (3.9). All the spaces \( \mathcal{N}_{s}^{m,n_2} \) are therefore isomorphic to each other and to \( \mathcal{M}_{s} \) as symplectic manifolds.

The set \( \mathcal{N}_{s} \) is open in \( \mathcal{N} \), and it is a four-dimensional manifold with the topology of a four times punctured torus times \( \mathbf{R}^2 \). A pair \( (\bar{A}_a^I, \bar{e}_{a_I}) \) with the holonomies (3.23) is

\(^1\)That the fields (3.25) are homogeneous only in \( \mathcal{N}_{s}^{00} \) is not coincidental. A direct exponentiation shows that any flat ISU(1, 1) connection with a homogeneous global cross section will have its holonomies either in \( \mathcal{N}_{s}^{00}, \mathcal{N}_{n}^{00}, \mathcal{N}_{0}^{00} \), or \( \mathcal{N}_{s} \).
given by Eqs. (3.10). This pair clearly defines an almost unique gauge fixing, the only degeneracy being that adding multiples of \(4\pi\) to \(\lambda_\alpha\) gives gauge equivalent pairs. The symplectic structure on \(\mathcal{N}_t\) is given by (1.11). As a symplectic manifold, \(\mathcal{N}_t\) is therefore a four-fold cover of \(\mathcal{M}_s\). It can be naturally viewed as the cotangent bundle over the four times punctured torus.

The sets \(\mathcal{N}_n^{m_n2}\) are three-dimensional manifolds with topology \(S^1 \times \mathbb{R}^2\). They are not open in \(\mathcal{N}\), and the neighborhoods of every point in \(\mathcal{N}_n^{m_n2}\) intersect both \(\mathcal{N}_s^{m_n2}\) (with the same values of the indices) and \(\mathcal{N}_t\). A pair \((\tilde{A}_a^I, \tilde{e}_at)\) with the holonomies (3.24) is given by

\[
\tilde{A}^0 = du - 2\pi (\eta_\alpha dx^K) \\
\tilde{A}^1 = -\sin(2\pi \eta_\beta x^K)du \\
\tilde{A}^2 = \cos(2\pi \eta_\beta x^K)du \\
\tilde{e}_0 = -\frac{1}{2} p_r dv - \frac{1}{2} p_\vartheta du \\
\tilde{e}_1 = -\frac{1}{2} \sin(2\pi \eta_\beta x^K) (p_r dv - p_\vartheta du) \\
\tilde{e}_2 = \frac{1}{2} \cos(2\pi \eta_\beta x^K) (p_r dv - p_\vartheta du)
\]  

(3.26)

This pair clearly defines a unique gauge fixing in each of the \(\mathcal{N}_n^{m_n2}\).

Finally, the sets \(\mathcal{N}_s^{m_n2}\) are neither open in \(\mathcal{N}\) nor manifolds. A pair \((\tilde{A}_a^I, \tilde{e}_at)\) with the appropriate holonomies is obtained by setting \(\tilde{A}^0 = -2\pi \eta_\alpha dx^K\) and taking \(\tilde{e}_j^K\) to be given by

\[
\exp(2\pi \eta_\beta x^K \tau_0) H_\alpha \exp(-2\pi \eta_\beta x^K \tau_0) dx^K = \left( \begin{array}{cc} -\tilde{e}_0^j & \tilde{e}_1^j + i \tilde{e}_2^j \\ \tilde{e}_1^j - i \tilde{e}_2^j & -\tilde{e}_0^j \end{array} \right) dx^j \right)
\]

(3.27)

Thus, \(\mathcal{N}\) consists of the four-manifolds \(\mathcal{N}_n^{m_n2}\) respectively glued to the four-manifold \(\mathcal{N}_s\) by the three-manifolds \(\mathcal{N}_n^{m_n2}\) and the sets \(\mathcal{N}_0^{m_n2}\). From the previous subsection it is clear that the gluing of \(\mathcal{N}_s^{m_0}\) to \(\mathcal{N}_t\) by \(\mathcal{N}_s^{m_0}\) and \(\mathcal{N}_0^{m_0}\) is identical to the gluing of \(\mathcal{M}_s\) to \(\mathcal{M}_t\) by \(\mathcal{M}_s\) and \(\mathcal{M}_0\), and it is straightforward to verify that the same holds also for the three other gluings. We can therefore view \(\mathcal{N}\) as a four-fold covering of \(\mathcal{M}\). This means in particular that the open subset \(\mathcal{N} \setminus (\cup_{m_n2} N_n^{m_n2}) = \mathcal{N}_t \cup (\cup_{m_n2} N_n^{m_n2}) \cup (\cup_{m_n2} N_s^{m_n2})\) is a manifold that contains \(\mathcal{N}_t\) and all four \(\mathcal{N}_s^{m_n2}\). This non-Hausdorff symplectic manifold is a four-fold cover of \(\mathcal{M} \setminus \mathcal{M}_0\), and it is naturally viewed as the cotangent bundle over a non-Hausdorff base space.

IV. SPACETIME METRICS

In this section we shall briefly discuss the relation of spacetime metrics to the classical solutions in Witten’s connection formulation. We shall first concentrate on the case where \(\tilde{A}_a^I\) is an SO(2, 1) connection. The case when \(\tilde{A}_a^I\) is an SU(1, 1) connection will be discussed at the end of the section.

In the metric formulation of 2+1 gravity one starts with the assumption that the space-time metric is nondegenerate. In the Hamiltonian metric formulation on the manifold \(\Sigma \times \mathbb{R}\), where \(\Sigma\) is a closed orientable two-manifold, one conventionally makes further the assumption that the induced metric on \(\Sigma\) is positive definite. As mentioned in section [1], Witten’s connection formulation does not contain these assumptions, neither in its Lagrangian nor Hamiltonian form. An immediate consequence of this and the spacetime diffeomorphism...
invariance of the actions (2.1) and (2.5) is that local statements about the degeneracy and signature of the induced metric on $\Sigma$ become gauge-dependent: when $\bar{e}_{aI}$ is not identically zero, one can locally change the signature of the induced metric on $\Sigma$ by spacetime diffeomorphisms that distort the constant $t$ surfaces sufficiently wildly. But also local statements about the degeneracy of the three-metric in Witten’s theory are gauge dependent [4]. The geometrical reason is perhaps most easily understood from the observation that as the action (2.1) and the field equations (2.3) contain only (Lie algebra valued) forms but no vectors, the action and the classical solutions can be pulled back by more general maps from $\mathbb{R} \times \Sigma$ to itself than just diffeomorphisms [29]. For example, consider a smooth map from $\mathbb{R} \times \Sigma$ to itself that takes an open neighborhood $U$ to a single point. Given any solution to Witten’s theory, the pull-back of this solution gives a solution for which $\bar{A}^I_a$ and $\bar{e}^I_a$, and hence in particular the metric $g_{ab} = \bar{e}_{aI} \bar{e}^I_b$, vanish in $U$. If the map is homotopic to the identity map, the two solutions related in this fashion are clearly gauge equivalent. These considerations indicate that the relation of Witten’s connection formulation to spacetime metrics should be addressed in terms of globally defined questions.

(Interestingly, in the Hamiltonian metric formulation it is possible, and arguably even natural, to relax the condition that the spacetime metric is nondegenerate while still keeping the metric on $\Sigma$ positive definite. The geometrical reason for this is analogous [30]: the lapse-function can be regarded as a one-form with respect to time reparametrizations, and the Hamiltonian action does not involve its inverse. Hence, the Hamiltonian formulation allows one to slice nondegenerate spacetimes in a fashion which may locally stay still or even go backwards in the proper time. For a discussion of some implications of this, see Refs. [31,32].)

One well-defined question is to start from the spacetimes of the conventional Hamiltonian metric formulation (that is, spacetimes with nondegenerate three-metrics and positive definite induced metrics on $\Sigma$) and ask to which solutions in the connection formulation they correspond [18]. For $\Sigma = T^2$, these spacetimes fall into two classes [17,18]. In the first class the holonomies are in $\mathcal{M}_0$ and given by (3.6). A global nondegenerate co-triad is

\begin{align*}
\bar{e}_0 &= dt \\
\bar{e}_1 &= r_1 dx^1 + r_2 \cos \phi \, dx^2 \\
\bar{e}_2 &= r_2 \sin \phi \, dx^2 ,
\end{align*}

(4.1)

which is obtained from the degenerate co-triad mentioned in subsection III A via a gauge transformation generated by the $F$-constraint. This spacetime is just $M^{2+1}$ divided by the two purely translational ISO$_0(2, 1)$ holonomies (3.3). In the second class the holonomies are in $\mathcal{M}_s$ and given by (3.3) with $\epsilon^{\alpha\beta} a_\alpha \lambda_{\beta} \neq 0$. A global nondegenerate co-triad is

\begin{align*}
\bar{e}_0 &= -dt \\
\bar{e}_1 &= t\lambda_\alpha dx^\alpha \\
\bar{e}_2 &= a_\alpha dx^\alpha ,
\end{align*}

(4.2)

where $t > 0$. This co-triad is obtained from the degenerate co-triad (3.7) via a gauge transformation generated by the $F$-constraint. The resulting spacetime can be constructed from the region $T > |X|$ of $M^{2+1}$ by taking the quotient with respect to the two ISO$_0(2, 1)$ holonomies (3.3) [9].

From these results one expects that more general nondegenerate spacetime metrics corresponding to points in $\mathcal{M}$ can be found in a similar fashion, dividing $M^{2+1}$ or some subset
of it by the two ISO\(_0(2, 1)\) holonomies. We shall now demonstrate that this is true for all points in \(M\) except a set of measure zero. It appears plausible that the metrics we shall exhibit are, up to diffeomorphisms, the most general nondegenerate metrics on \(\mathbb{R} \times T^2\) that correspond to the connection formulation.

Let us first reconsider the points in \(M\) for which \(\epsilon^{\alpha\beta} a_\alpha \lambda_\beta \neq 0\). The holonomies (3.3) do not act properly discontinuously on all of \(M^{2+1}\), and dividing \(M^{2+1}\) by these holonomies gives a modest generalization of Misner space [33]. However, the holonomies do act freely and properly discontinuously above (or below) the null plane \(T = X\), and similarly above (or below) the null plane \(T = -X\). For concreteness, consider the domain \(T > -X\). It can be verified that the co-triad

\[
\bar{e}_0 = -\frac{1}{2}dt + \frac{1}{2}(t-1)\lambda_\alpha dx^\alpha \\
\bar{e}_1 = \frac{1}{2}dt - \frac{1}{2}(t+1)\lambda_\alpha dx^\alpha \\
\bar{e}_2 = a_\alpha dx^\alpha ,
\]

(4.3)

which is gauge-equivalent to that given in Eq. (3.7), gives the metric of the space obtained by taking the quotient of the domain \(T > -X\) with respect to the two holonomies (3.3). The subdomains \(T > |X|\) and \(X > |T|\) are covered respectively by \(t > 0\) and \(t < 0\), and the surface \(T = X\) is at \(t = 0\). The \(t = \text{constant}\) tori go from timelike for \(t < 0\) via null at \(t = 0\) to spacelike for \(t > 0\). The situation is highly similar to that of Misner space [33]. Note that the induced metric on the \(t = 0\) torus is just the one given by the degenerate co-triad (3.7). The existence of co-triads corresponding to (4.3) has been previously pointed out by Unruh and Newbury [34].

For \(M_0\), where the holonomies are purely translational, the situation is straightforward. Whenever the two translations are linearly independent, one easily finds a nondegenerate co-triad gauge equivalent to that mentioned in subsection [11A] such that the resulting spacetime is simply \(M^{2+1}\) divided by the two translations. The tori are spacelike only in the case (4.4).

Consider then \(M_n\). For \(p_r \neq 0 \neq p_\theta\), a nondegenerate co-triad gauge equivalent to that given in Eq. (3.12) is

\[
\bar{e}_0 = -\frac{1}{2}[(p_r dv - 2tdu) + p_\theta du] \\
\bar{e}_1 = dt \\
\bar{e}_2 = \frac{1}{2}[(p_r dv - 2tdu) - p_\theta du] ,
\]

(4.4)

where \(-\infty < t < \infty\). The spacetime corresponding to (4.4) is obtained by taking the quotient of \(M^{2+1}\) with respect to the holonomies (3.3) (the action is free and properly discontinuous), and the \(t = \text{constant}\) tori are timelike. For \(p_r \neq 0 = p_\theta\) the co-triad (4.4) becomes degenerate, but a gauge equivalent nondegenerate co-triad is

\[
\bar{e}_0 = -\frac{1}{2}(p_r dv + dt) \\
\bar{e}_1 = -tdu \\
\bar{e}_2 = \frac{1}{2}(p_r dv - dt) ,
\]

(4.5)

where \(t > 0\). The spacetime corresponding to (4.5) is obtained by taking the quotient of the region \(T > Y\) (or, equivalently, the region \(T < Y\)) of \(M^{2+1}\) with respect to the holonomies (3.3). The \(t = \text{constant}\) tori are now null. The action of the holonomies is free
and properly discontinuous in this region but not on the plane $T = Y$. The geometrical picture is perhaps most easily seen from the observation that the holonomies (3.3) are respectively the exponential maps of the vectors $\cos \theta \xi_1 + \sin \theta \xi_2$ and $-\sin \theta \xi_1 + \cos \theta \xi_2$, where $\xi_1$ and $\xi_2$ are the two commuting Killing vectors on $M^{2+1}$ given by

$$
\xi_1 = (T - Y)(\partial/\partial X) + X[(\partial/\partial T) + (\partial/\partial Y)] + \frac{1}{2}p_0[(\partial/\partial T) - (\partial/\partial Y)]
$$

$$
\xi_2 = \frac{1}{2}p_r[(\partial/\partial T) + (\partial/\partial Y)] .
$$

Note that $\xi_1$ reduces to a generator of a pure null rotation for $p_\theta = 0$.

Consider finally $\mathcal{M}_t$. When $\epsilon^{\alpha\beta}\tilde{\alpha}_\beta \neq 0$, a nondegenerate co-triad gauge equivalent to that given in Eq. (3.10) is

$$
\tilde{e}_0 = \tilde{a}_\alpha dx^\alpha \\
\tilde{e}_1 = -t\tilde{\lambda}_\alpha dx^\alpha \\
\tilde{e}_2 = dt
$$

(4.7)

where $t > 0$. The spacetime corresponding to (4.7) is obtained by removing the $T$ axis from $M^{2+1}$, going to the universal covering space, and taking the quotient with respect to the holonomies (3.4). The $t = \text{constant}$ tori are timelike. This procedure of obtaining a spacetime metric from the pair (3.10) has, however, the feature that the metric depends on the gauge choice made for the pair by more than just spacetime diffeomorphisms. If, instead of the pair (3.10), one starts from a gauge equivalent pair in which multiples of $2\pi$ have been added to $\tilde{\lambda}_\alpha$, one finds a nondegenerate co-triad which is obtained from (4.7) by adding the same multiples of $2\pi$ to $\tilde{\lambda}_\alpha$. For generic values of the parameters, the metric obtained from this new co-triad will not be diffeomorphic to that obtained from (4.7). This means that in $\mathcal{M}_t$ the relation of the connection description to spacetime metrics is considerably looser than in $\mathcal{M}_0$ and $\mathcal{M}_a$.

One attempt to make the relation between points in $\mathcal{M}_t$ and spacetime metrics more rigid might be to declare that the pair (3.10) should correspond to the spacetime obtained by removing the $T$ axis from $M^{2+1}$ and taking the quotient with respect to the holonomies (3.4), without going to the covering space. Such a spacetime could be seen as an attempt to characterize the equivalence class of the co-triads (4.7) under the gauge transformations which add multiples of $2\pi$ to $\tilde{\lambda}_\alpha$. Whereas this viewpoint appears self-consistent, it suffers from the drawback that for generic values of the parameters the action of the holonomies is not properly discontinuous, and the space obtained after taking the quotient is nowhere near a manifold. This quotient space would therefore not correspond to the notion of “spacetime” as usually understood.

To end this section we shall briefly indicate how the above picture is modified when $\tilde{A}_a^I$ is an SU(1, 1) connection. Recall that $\mathcal{N}$ can be naturally understood as a four-fold cover of $\mathcal{M}$. Consider a point $x \in \mathcal{M} \setminus \mathcal{M}_t$ that corresponds to nondegenerate spacetime metrics as above. Let $y$ be a point in $\mathcal{N} \setminus \mathcal{N}_t$ which is taken to $x$ by the covering map. Then the nondegenerate spacetime metrics obtained above from $x$ are also described by co-triads corresponding to $y$. For $y \in \mathcal{N}_0^{\eta_1\eta_2}$, $\mathcal{N}_a^{\eta_1\eta_2}$, or $\mathcal{N}_n^{\eta_1\eta_2}$, appropriate co-triads are obtained from those in Eqs. (4.7) by an internal rotation according to $\tilde{e}^I \mapsto \left[\exp(2\pi \eta_\beta x^\beta K_0)\right]^I_{\tilde{e}^J}$. In $\mathcal{N}_t$ the situation differs from that in $\mathcal{M}_t$ only in that the large gauge transformations act on
the co-triad (1.7) by adding to $\tilde{\lambda}$ multiples of $4\pi$ rather than $2\pi$. An attempt to construct a spacetime characterizing such a gauge-equivalence class of triads might now be to remove the $T$ axis from $M^{2+1}$, go to a double cover, and take the quotient of this double cover with respect to the transformations corresponding to the holonomies (3.23). Again, for generic values of the parameters this quotient space is nowhere near a manifold.

V. QUANTUM THEORIES FOR $\Sigma = T^2$

We shall now explore formulations of quantum gravity based on the solution spaces $\mathcal{M}$ and $\mathcal{N}$. Because these spaces have neither a preferred set of global coordinates nor the structure of a cotangent bundle over a manifold, this will require some creativity. We shall follow the technique of judiciously choosing the functions that are to become operators in the quantum theory, in combination with replacing $\mathcal{M}$ and $\mathcal{N}$ by their simpler subsets. Some of the resulting theories coincide with those presented in Refs. [6,7,9,20], whereas others are new.

Let us first recall the geometric quantization framework of Refs. [35,36] for quantizing the cotangent bundle $T^*B$ of a manifold $B$. For any smooth functions $f, g$ and vector fields $v, w$ on $B$, one defines the commutation relations

$$[f, g] = 0, \quad [v, f] = -i \mathcal{L}_v f, \quad [v, w] = -i \{v, w\}. \quad (5.1)$$

Here, $\{v, w\}$ stands for the Lie bracket of vector fields and $\mathcal{L}_v$ is the Lie derivative along $v$. Together with the involution operation $\star$ defined by complex conjugation, Eq. (5.1) defines a $\star$-algebra of functions and vector fields on $B$. One may now seek Hilbert space representations of (5.1) which implement the $\star$-relations as Hermiticity relations.

One representation acts on smooth square integrable half-densities on $B$. Functions act by multiplication and vector fields act by $-i$ times Lie differentiation. Note that this representation is well-defined even when $B$ is non-Hausdorff. With respect to the inner product $(\alpha, \beta) = \int_B \bar{\alpha}\beta$, $\star$ is the adjoint operation when acting on any function or any vector field, with suitable fall-off conditions imposed when $B$ is non-compact. Completing the space of smooth half-densities in the corresponding norm gives a representation of the $\star$-algebra on $L^2(B)$. Note that if $B$ is disconnected, this representation is topologically reducible to a direct sum of representations corresponding to the connected components of $B$. One can therefore without loss of generality assume $B$ to be connected.

We would now like to apply this framework to some subsets of $\mathcal{M}$ and $\mathcal{N}$ that were seen in section 11 to be cotangent bundles. For $\mathcal{M}$, two natural subsets are $\mathcal{M}_t$ and $\mathcal{M}_s$; as these subsets are disconnected, considering them separately is equivalent to considering the set $\mathcal{M}_t \cup \mathcal{M}_s$ which consists of all of $\mathcal{M}$ except a set of measure zero. A third natural subset is $\mathcal{M} \setminus \mathcal{M}_0$ which contains both $\mathcal{M}_s$ and $\mathcal{M}_t$. For $\mathcal{N}$, the corresponding natural subsets are $\mathcal{N}_t$, the four $\mathcal{N}^{\eta_1\eta_2}_s$, and $\mathcal{N} \setminus \mathcal{N}_0$, where $\mathcal{N}_0 = \bigcup_{\eta_1\eta_2} \mathcal{N}^{\eta_1\eta_2}_0$.

When $T^*B$ is $\mathcal{M}_t$ or $\mathcal{M}_s$, or respectively $\mathcal{N}_t$ or $\mathcal{N}^{\eta_1\eta_2}_s$, the above quantization is straightforward to implement. In particular, for $\mathcal{N}_t$ and $\mathcal{N}^{\eta_1\eta_2}_s$ the resulting quantum theories are equivalent to those given in Refs. [6,7]. One further sees that replacing $\mathcal{N}_t$ and $\mathcal{M}_t$ by their
is an orbifold with three singular points. The quotient $T^*B = M \setminus M_0$ or $T^*B = N \setminus N_0$. In these cases the non-Hausdorff nature of $B$ leads to an unusual feature. Let us write $B = B_s \cup B_n \cup B_t$, where $B_s$, $B_n$, and $B_t$ are the parts of the base space that correspond to $\mathcal{M}_s$, $\mathcal{M}_n$, and $\mathcal{M}_t$, or respectively to $\mathcal{N}_s = \bigcup_{n_{12}} N_s^{n_{12}}$, $\mathcal{N}_n = \bigcup_{n_{12}} N_n^{n_{12}}$, and $\mathcal{N}_t$. Then, exponentials of self-adjoint operators corresponding to complete vector fields which are not tangent to $B_n$ are not defined on all densities. In particular, consider the map $\alpha : T^*B \to T^*B$, defined on $T^*B_s$ as the identity map, on $T^*B_t$ by $(\tilde{\lambda}_a, \tilde{a}_a) \mapsto (-\tilde{\lambda}_a, -\tilde{a}_a)$, and on the fibers over $B_n$ by $(\theta, p_\theta, p_r) \mapsto (\theta + \pi, p_\theta, p_r)$, where $\tilde{\lambda}_a$, $\tilde{a}_a$, $\theta$, $p_r$, and $p_\theta$ were defined in section $\text{III}$.

Then, exponentiation of all operators corresponding to complete vector fields that are invariant under $\alpha$ is defined only on densities that are also invariant under $\alpha$.

If the usual type of quantum theory is to be recovered, this problem must be corrected. One solution is to promote to operators only those functions and vector fields that are invariant under $\alpha$. A topologically irreducible representation of the resulting algebra is given as above when the operators are taken to act on half-densities that are invariant under $\alpha$. Now, in the resulting quantum theory, any complete smooth vector field may be exponentiated to act on any half-density. Since $B$ can be interpreted as a configuration space, these vector fields have an interpretation as momentum operators. In particular, there exist momentum operators whose exponentials take half-densities with support in $B_s$ to half-densities whose support is in $B_t$, and vice versa. In this sense, then, this quantum theory includes operators which induce transitions between the theories of Refs. $[6,7,9]$. This is a result of the fact that $B$ is a connected manifold.

Another approach is to consider the quotient of $B$ and its cotangent bundle by the map $\alpha$. Consider first the case $T^*B = N \setminus N_0$. The action of $\alpha$ on $B$ is free and properly discontinuous, and the quotient $(T^*B)/\alpha$ is a cotangent bundle over the manifold $B/\alpha$. It is easily seen that $B/\alpha$ consists of a four times punctured sphere glued to four punctured planes. The gluings at the punctures are now one-to-one, and $B/\alpha$ is Hausdorff. A Hilbert space representation of our algebra on half-densities on $B/\alpha$ follows as above, and the pull-back from $B/\alpha$ to $B$ provides an isomorphism with the representation constructed above from functions, vector fields, and half-densities on $B$ that are invariant under $\alpha$.

Consider then the case $T^*B = M \setminus M_0$. The action of $\alpha$ on $B$ leaves fixed the three points in $B_t$ for which $(\tilde{\lambda}_1, \tilde{\lambda}_2) = (m\pi, n\pi)$, with $m, n$ in $\mathbb{Z}$ and not both even. Hence $B/\alpha$ is an orbifold with three singular points. The quotient $(T^*B)/\alpha$ is not a bundle since the “fibers” over the singular points are not $\mathbb{R}^2$, but $\mathbb{R}^2$ with opposite points identified. Now let $B'$ be the complement in $B$ of the three fixed points of $\alpha$. Then $\alpha$ acts freely and properly discontinuously on $B'$ and $T^*B'$, and $B'/\alpha$ consists of a punctured plane glued to a four times punctured sphere at one of the punctures. The gluing at the puncture is one-to-one, so $B'/\alpha$ is Hausdorff, and a representation of our algebra on half-densities follows as above. Note

\[ \text{The difference between } \mathcal{N}_t \text{ and } \mathcal{N}_t \text{ (} \mathcal{M}_t \text{ and } \mathcal{M}_t, \text{ respectively) becomes however more involved when one only promotes into operators the elements of the classical } T\text{-algebra of Refs. [18]. See Ref. [20].} \]
that complete vector fields on $B'/\alpha$ pull back to vector fields on $B$ that vanish rapidly near the three fixed points of $\alpha$. Since all smooth complete vector fields on $B$ that are invariant under $\alpha$ also vanish rapidly at these three points, a pull-back again provides an isomorphism with the representation constructed from functions, vector fields, and half-densities on $B$ that are invariant under $\alpha$.

In passing, we note that $\alpha$ is the map on $T^*B$ induced by inversion of the torus: $x^\beta \mapsto -x^\beta$. The above construction thus describes theories invariant under this limited class of large diffeomorphisms.

Alternatively, we could return to $T^*B = \mathcal{M} \setminus \mathcal{M}_0$ or $T^*B = \mathcal{N} \setminus \mathcal{N}_0$ and promote to operators only those vector fields that are tangent to $B_n$. A Hilbert space representation of this algebra as above is topologically reducible to a sum of two representations, one on densities with support on $B_t$ and one with support on $B_s$. (For $T^*B = \mathcal{N} \setminus \mathcal{N}_0$, the representation with support on $B_s$ is further topologically reducible to a sum of four representations corresponding to the four disconnected components of $B_s$.) This again leads to an analysis similar to that of Refs. [6,7], since the vector fields $T^1(\gamma)$ used in Refs. [6,7] are tangent to $B_n$.

However, because Refs. [6,7] promote to operators only the functions $T^0(\gamma)$ and vector fields $T^1(\gamma)$ belonging to the $T$-algebra, all of whom are invariant under $\alpha$, the representation obtained on $B_t$ is further reducible.

Taken to the extreme, selective promotion of functions to operators becomes a new approach by itself. Let us consider only those smooth functions on $\mathcal{M}$ or $\mathcal{N}$ that vanish on $\mathcal{M}_0$ or $\mathcal{N}_0$. Examples are again $T^0(\gamma) - 1$ and $T^1(\gamma)$. Such functions are essentially functions on $T^*B = \mathcal{M} \setminus \mathcal{M}_0$ or $T^*B = \mathcal{N} \setminus \mathcal{N}_0$ and may again be interpreted as built from functions and vector fields on $B$. The further restriction to functions that vanish on $B_n$ and vector fields tangent to $B_n$ again leads to the analysis of Ref. [7].

VI. CONCLUSIONS AND DISCUSSION

In this paper we have investigated Witten's connection formulation of 2+1 gravity, and its SU(1,1) modification, on the manifold $\mathbf{R} \times T^2$. The space of classical solutions was shown to be connected, unlike in the case where the torus is replaced by a higher genus surface. Also, we saw that all points of this space except a set of measure zero correspond to nondegenerate spacetime metrics on $\mathbf{R} \times T^2$, even though not all of these spacetimes admit slicings with spacelike tori.

Removing from the space of classical solutions a set of measure zero, we obtained a manifold which is naturally viewed as the cotangent bundle over a non-Hausdorff base space. This allowed us to explore various possibilities for quantizing the theory. In addition to recovering the quantum theories previously presented in Refs. [6,7], we exhibited a larger quantum theory which contains the theories of Refs. [6,7] as its parts. In particular, the larger theory contains operators that induce transitions between the smaller theories.

While all our quantum theories incorporated invariance under the connected component of the diffeomorphism group of the torus, the only disconnected component considered was the one containing inversions of the torus. If one attempts to build connection quantum theories that are invariant under all diffeomorphisms, including the disconnected ones, one encounters the fact that the full diffeomorphism group does not act even remotely properly.
discontinuously on our base spaces. For example, consider the action of the large diffeomorphisms on $\mathcal{M}_s$, given by

$$
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
a_1 \\
a_2
\end{pmatrix} \mapsto
\begin{pmatrix}
M & 0 \\
0 & M
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
a_1 \\
a_2
\end{pmatrix},
$$

(6.1)

where $M \in \text{SL}(2, \mathbb{Z})$. Although the action (6.1) is properly discontinuous and free in the domains $\epsilon^{\alpha\beta}a_\alpha \lambda_\beta > 0$ and $\epsilon^{\alpha\beta}a_\alpha \lambda_\beta < 0$, it is not properly discontinuous on the surface $\epsilon^{\alpha\beta}a_\alpha \lambda_\beta = 0$. In particular, the projection of the action (6.1) to the base space $a_\alpha = 0$ is not properly discontinuous. This makes the formulation of connection quantum theories invariant under the large diffeomorphisms a highly subtle question.

In contrast, in the conventional metric theory the large diffeomorphisms do act properly discontinuously (although not freely) on the configuration space, and a quantum theory invariant under the large diffeomorphisms is readily constructed. This difference between the metric theory and the connection theory is related to the fact that the metric theory of Ref. [9] contains classically only the region $\epsilon^{\alpha\beta}a_\alpha \lambda_\beta > 0$ (or, equivalently, the region $\epsilon^{\alpha\beta}a_\alpha \lambda_\beta < 0$) of $\mathcal{M}_s$.

It should be noted that these subtleties with the large diffeomorphisms have no counterpart with the higher genus surfaces. There, the geometrodynamically relevant connected component of the classical solution space to Witten’s theory is the cotangent bundle over the Teichmüller space of $\Sigma$, and the action of the large diffeomorphisms on the Teichmüller space is properly discontinuous, although not free.

It is straightforward to generalize our results to modifications of Witten’s theory where the gauge group of the connection $\tilde{A}_I^I$ is taken to be the $n$-fold covering group of $\text{SO}_0(2, 1)$, $n > 2$. The space of classical solutions can be understood simply as the $n \times n$ cover of $\mathcal{M}$, containing $n \times n$ counterparts of $\mathcal{M}_s$, $\mathcal{M}_n$ and $\mathcal{M}_0$ attached to an $n \times n$ cover of $\mathcal{M}_t$. Similarly, if the gauge group is taken to be the universal covering group $\tilde{\text{SO}}_0(2, 1)$ of $\text{SO}_0(2, 1)$, the space of classical solutions will consist of infinitely many counterparts of $\mathcal{M}_s$, $\mathcal{M}_n$ and $\mathcal{M}_0$ attached to the universal covering space of $\mathcal{M}_t$. The discussion of the classical spacetime metrics for these theories is an obvious generalization of that given in section [V]. (Note, however, that for $\text{SO}_0(2, 1)$ there are no gauge transformations in the connection formulation that would add multiples of $2\pi$ to the parameters $\tilde{\lambda}_\alpha$ in the co-triad (4.7).) Also, quantization of these theories is an obvious generalization of that discussed in section [V].

In conclusion, we have seen that Witten’s formulation of 2+1 gravity on $\mathbb{R} \times T^2$ differs qualitatively from this theory on manifolds where the torus is replaced by a higher genus surface, both at the classical and quantum levels. While the torus case admits quantum theories that are reasonably close to the geometrodynamically relevant quantum theories of the higher genus cases, it also admits quantum theories that have no apparent analogue for

---

3In our understanding, a subtlety in the approach of Refs. [10,11] arises from the domain of integration in Eq. (3.3) of Ref. [10]. We thank Steve Carlip for discussions on this point.
higher genus. Incorporating invariance under large diffeomorphisms in a quantum theory in the torus case remains, however, a subtle question.

Note added. After the completion of this work we became aware of Ref. [37], in which the moduli space of flat $\text{SU}(1,1)$ connections on the torus is described in the context of $d = 3, N = 2$ supergravity. This moduli space is the subspace of our $\mathcal{N}$ for which the $\text{ISU}(1,1)$ holonomies are in $\text{SU}(1,1)$.

ACKNOWLEDGMENTS

We would like to thank Abhay Ashtekar, Steve Carlip, Alan Rendall, Bill Unruh, and Madhavan Varadarajan for helpful discussions. This work was supported in part by the NSF grants PHY90-05790 and PHY90-16733, and by research funds provided by Syracuse University.
REFERENCES

∗ Electronic address: louko@convex.csd.uwm.edu. Present address: Department of Physics, University of Wisconsin-Milwaukee, P.O. Box 413, Milwaukee, WI 53201, USA.
† Electronic address: marolf@hbar.phys.psu.edu. Present address: Department of Physics, The Pennsylvania State University, University Park, PA 16802, USA.

[1] Deser S, Jackiw R and ’t Hooft, G 1984 Ann. Phys. (N.Y) 152 220
[2] Deser S and Jackiw R 1984 Ann. Phys. (N.Y) 153 405
[3] Martinec E 1984 Phys. Rev. D 30 1198
[4] Witten E 1988 Nucl. Phys. B311 46
[5] Bengtsson I 1989 Phys. Lett. 220B 51
[6] Ashtekar A, Husain V, Rovelli C, Samuel J and Smolin L 1989 Class. Quantum Grav. 6 L185
[7] Ashtekar A 1991 Lectures on Non-Perturbative Canonical Gravity (World Scientific, Singapore) Chapter 17
[8] Hosoya A and Nakao K 1990 Prog. Theor. Phys. 84 739
[9] Carlip S 1990 Phys. Rev. D 42 2647
[10] Carlip S 1992 Phys. Rev. D 45 3584 [Erratum (1993) Phys. Rev. D 47 1729]
[11] Carlip S 1993 Phys. Rev. D 47 4520
[12] Manojlović N and Miković A 1992 Nucl. Phys. B385 571
[13] Barbero J F and Varadarajan M 1993 “The Phase Space of 2+1 Dimensional Gravity in the Ashtekar Formulation”, Syracuse Preprint SU-GP-93/64, gr-qc/9307006
[14] Carlip S 1993 “Six ways to quantize (2+1) dimensional gravity,” Preprint UCD-93-15, NSF-ITP-93-63, gr-qc/9305020, in Proceedings of the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics (World Scientific, Singapore) (to be published)
[15] Goldman W M 1984 Adv. Math. 54 200
[16] Goldman W M 1985 in Geometry and Topology (Lecture Notes in Mathematics, Vol. 1167), eds. Alexander J and Harer J (Springer, Berlin)
[17] Moncrief V 1989 J. Math. Phys. 30 2907
[18] Mess G 1990 “Lorentz spacetimes of constant curvature,” Institutes des Hautes Etudes Scientifiques Preprint IHES/M/90/28
[19] Hosoya A and Nakao K 1990 Class. Quantum Grav. 7 163
[20] Marolf D M 1993 “Loop Representations for 2+1 Gravity on a Torus,” Class. Quantum Grav. (in press)
[21] Anderson A 1993 Phys. Rev. D 47 4458
[22] Rovelli C and Smolin L 1990 Nucl. Phys. B331 80
[23] Ref. [1] Chapter 16
[24] Romano J D 1993 Gen. Rel. Grav. 25 759
[25] Bargmann V 1947 Ann. Math. 38 568
[26] Carmeli M 1977 Group Theory and General Relativity (McGraw-Hill, New York) Chapter 3
[27] Dixon L, Harvey J, Vafa C and Witten E 1986 Nucl. Phys. B274 285
[28] Schleich K and Witt D M 1993 Nucl. Phys. B402 411
[29] Horowitz G T 1991 Class. Quantum Grav. 8 587
[30] Teitelboim C (1982) Phys. Rev. D 25 3159
[31] Teitelboim C 1982 in Quantum structure of spacetime eds. Duff M J and Isham C J (Cambridge University Press, Cambridge)
[32] Halliwell J J and Hartle J B 1991 Phys. Rev. D 43 1170
[33] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-time (Cambridge University Press, Cambridge)
[34] Unruh W G and Newbury P 1993 Phys. Rev. D 48 2686
[35] Woodhouse N M J 1980 Geometric Quantization (Clarendon Press, Oxford)
[36] Ref. [7] Chapter 10 and Appendix C
[37] de Wit B, Matschull H J and Nicolai H 1993 “Physical States in $d = 3, N = 2$ Supergravity” Preprint DESY 93-125, THU-93/19, gr-qc/9309006