Fast scrambling dynamics in an all-to-all disordered quantum spin model

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We study the quantum thermalization and information scrambling dynamics of an experimentally realizable quantum spin model with homogeneous XX-type all-to-all interactions and random local potentials. We identify the thermalization-localization transition by changing the disorder strength, under a proper relative all-to-all interaction strength. The operator scrambling has no light-cone behavior and grows almost equally fast in both phases. In the thermal phase, we find that the scrambling dynamics exhibits fast scrambling without appealing to the semi-classical limit. The fast scrambling dynamics always exists at a fixed bare interaction strength regardless of the relative strength in the Hamiltonian. The model also shows faster or slower scrambling dynamics related to the bare interaction strength. We show that the seeming violation of the fast scrambling conjecture arises from the absence of a natural timescale in the infinite temperature ensemble. We suggest that one needs a proper timescale and the corresponding dimensionless time for fast scrambling dynamics. After introducing such a timescale, we give a general phase diagram of the fast scrambling dynamics in the thermal phase. We also briefly discuss the experimental realization of the model using superconducting qubit quantum simulators.

Introduction—Isolated out-of-equilibrium quantum many-body systems tend to become thermal and serve as their own thermal bath due to the interaction, known as quantum thermalization [1, 2]. Recently, quantum chaos and information scrambling [3–12] in thermalized systems have attracted great interest for its importance in understanding the non-equilibrium dynamics in the strongly interacting system and quantum gravity. Information scrambling describes how local information spreads to other degrees of freedom in the quantum chaotic system under the unitary evolution. It is fundamental for studying dynamics of black holes and quantum information processing.

A particular recent focus is the speed limit of information scrambling dubbed fast scrambling conjecture [6, 13], in which the scrambling time \( t_s \) for information spreading to the entire system satisfies

\[
    t_s \sim \log(N),
\]

where \( N \) is the system size. Black holes are known as the fast scrambler in nature that saturates the upper bound. Another celebrated model that exhibits fast scrambling is the Sachdev-Ye-Kitaev (SYK) model [14–17], which has been proved to be holographically dual to quantum gravity. On the other hand, the rapid development of highly controlled quantum simulators enable us experimentally study information scrambling [18–25]. Thus, it is interesting to find other experimental realizable quantum models that exhibit fast scrambling.

Generally, accessing fast scrambling in quantum many-body systems needs non-local interaction and chaotic dynamics [26–30]. Quantum systems with short-range or even power-law long-range interactions are prevented from fast scrambling for the existence of light-cone [31–38]. Recently, several models are proposed by using quasi-random all-to-all interaction [26], tree-like interaction [27] or the combination of all-to-all and local interactions [28, 29] to reach experimentally accessible fast scrambling. However, most of these models manifest fast scrambling only when considering the semi-classical limit and the direct signature Eq. (1) is absent for quantum spin models. It still remains challenging to find a quantum model that exhibits fast scrambling without appealing to semi-classical limits, besides the SYK model.

In this paper, we study an all-to-all quantum spin model with local quenched disorders that exhibits fast scrambling. A weak disorder strength can induce a thermal phase in the quantum many-body system, simultaneously the all-to-all interaction prevents the existence of light-cone that limits the information spreading. This model is easy to realize in quantum simulation experiments, meanwhile it hosts fast scrambling without taking the semi-classical limit.

We first study the thermalization-localization transition between the thermal and many-body localization (MBL) phases [39–51] to locate the thermal phase and find that the transition is well defined under a proper relative all-to-all interaction strength. Then, we study the information scrambling dynamics by using the out-
spectra structure, so that the local field term $h$ model Eq. (2) hosts a global normalized by $1/d$ dynamics in the thermal phase, we can multiply a factor non-equilibrium phases. To study the fast scrambling ˜ the exponent ˜ small $N$ to the total number of sites ˜ induces different fast scrambling dynamics with respect $\tilde{N}$ fields $\tilde{h}$ reduces to the exactly solvable Lipkin-Meshkov-Glick model [53, 54]. The disordered field is introduced to enforce the model non-integrable. Here the coupling strength $J$ is normalized by a factor $1/N^\alpha$ with respect to the total number of sites $N$. It is clear that with small $\tilde{\alpha}$, $\tilde{H}$ is dominated by the coupling $J$. Thus the exponent $\tilde{\alpha}$ is crucial to realize possible different non-equilibrium phases. To study the fast scrambling dynamics in the thermal phase, we can multiply a factor $1/N^{\alpha-\tilde{\alpha}}$ to the Hamiltonian without changing the energy spectra structure, so that the local field term $\tilde{h}_i$ and $J$ are normalized by $1/N^{\alpha-\tilde{\alpha}}$ and $1/N^\alpha$ respectively. This will induce different fast scrambling dynamics with respect to $\alpha$. To be clear, in the rest of the paper we denote $\alpha$ as the bare all-to-all strength and $\tilde{\alpha}$ as the relative one. The model Eq. (2) hosts a global $Z_2$ symmetry, $\tilde{\sigma}^y \to -\tilde{\sigma}^y$. In
where \( W \) is the disorder strength consistent with that determined by the crossing of \( O(N^2/2) \) with \( W = 0.03 \) and \( W = 0.56 \), respectively. (c) Early time behavior of \( C(t) \) in a log-log plot. The disorder average is done over 50 samples. (d) Long time behavior of \( C(t) \) in the thermal and MBL phases.

at the transition point \( W_c = 0.14(2) \) for different system sizes. Away from the transition point, it approaches the theoretical value 0.38 or 0.53 in the localized and thermal phases respectively.

Furthermore, we perform a finite-scaling analysis of \( S(N^2/2) \) in Fig. 2 (b). The entanglement entropy satisfies volume and area law in the thermal and localized phases respectively, as expected (see inset of Fig. 2 (b)). We then perform a data collapse according to the scaling form \( S(N^2/2) = N f((W - W_c) N^{1/\nu}) \), where \( \nu \) is the critical exponent. The curves of different sizes accurately collapse onto each other with \( W_c = 0.14(2) \) and \( \nu = 0.89(2) \). The critical point obtained in this way is consistent with that determined by the crossing of \( \langle r \rangle \), demonstrating that the critical properties obtained here are accurate and reliable. These results provide the first numerical evidence supporting the existence of thermal-localization transition in all-to-all models.

Scrambling dynamics in the thermal and MBL phases—After identifying the thermal and localized phases, we continue to study the information scrambling in both phases. The most important quantity is the squared commutator of two local operators \( \hat{W}_i \) and \( \hat{V}_j \) at infinite temperature

\[
C(r, t) = \frac{1}{2} \langle [\hat{W}_0(t), \hat{V}_r][\hat{W}_0(t), \hat{V}_r] \rangle,
\]

where \( \hat{O}_r(t) = e^{i\hat{H}t} \hat{O}_r e^{-i\hat{H}t} \) denotes a Heisenberg operator. The commutator \( C(r, t) \) is closely related to the OTOC \( F(r, t) = \langle \hat{W}_0(t) \hat{V}_r(0) \hat{W}_0(t) \hat{V}_r(0) \rangle \) according to \( C(r, t) = (1 - \text{Re}[F(r, t)]) \). It provides a clear insight to the scrambling dynamics. That is, a local operator under time evolution grows non-local with time and becomes non-commuting with operators at other sites. It is expected that \( C(r, t) \) grows exponentially before saturation for fast scramblers [6], analogous to the classical butterfly effect. In the following we focus on \( C(r, t) \) at the transition point \( \alpha = 0.38 \) or 0.53 in the localized and thermal phases.

It has been demonstrated that in the thermal phase the commutator \( C(r, t) \) spreads ballistically with a broadened linear light cone in local systems [61]. With disorder, in the localized phase, \( C(r, t) \) exhibits a logarithmic linear light cone [62–64] and grows in a power-law way for a long time [65–67]. However, with non-local all-to-all interactions, \( C(r, t) \) of our model presents unique time evolution behaviors as shown in Fig. 3 (a-b). It grows quickly to \( O(1) \) simultaneously for all distances no matter in the thermal or localized phases. This is because there are no local interactions in our model and all sites are exactly equal to each other after disorder averaging. It is more interesting that even in the localized phase the early-time growth of the commutator is not arrested in our model. This is more clearly shown in Fig. 3(c), where the mean commutator \( \langle C(t) \rangle = \frac{1}{N} \sum \langle C(r, t) \rangle \) grows almost equally fast to saturation for both phases in an approximate power-law form. The difference between the two phases lies in the late time behavior, where \( C(t) \) is almost invariant in the thermal phase while periodically oscillates to long times in the MBL phase, see Fig. 3(d). The late-time oscillation of \( C(t) \) indicates the breakdown of the chaotic dynamics in the MBL phase.

Fast scrambling—Now we focus on the fast scrambling dynamics in the thermal phase. We have shown that for a small system the early growth of \( C(t) \) is in an approximate power-law form rather than exponential. However, the exponential growth is generally expected for the semi-classical system with a large system size \( N \) in the large spin-\( S \) limit. While it is not guaranteed for quantum models with limited system size and small local Hilbert space [68–70]. Nevertheless, we will show that there exists fast scrambling in our model.

To study the fast scrambling dynamics, one needs to resort to larger system sizes, which are generally inaccessible to the ED study. We utilize the recently developed tensor-network method [27, 61, 71] based on the matrix product operator representation of Heisenberg operators and the time-dependent variational principle [72, 73] to study the OTOC in the early time growth region. This approach has been shown to accurately capture the
We next explore the scrambling dynamics by changing the bare interaction $\alpha$ while keeping the system in the thermal phase. Note that we realize this by introducing a normalization factor $1/N^{0-\alpha}$ to the Hamiltonian as we discussed previously, which maintains the energy level structure and statistics. In this case, the scrambling dynamics for other $\alpha$ can be derived by directly rescaling the time by $t = t N^{\alpha-0.5}$. Thus the scrambling time for general $\alpha$ can be obtained as $\tilde{t}_s = t_s N^{\alpha-0.5} \sim N^{\alpha-0.5} \log N$, as shown in Fig. 4 (c). Specifically, it increases faster than $\log N$ when $\alpha > 0.5$ and slower when $\alpha < 0.5$. Moreover, we find that this is generic for other values of relative strength $\bar{\alpha}$, hence different energy level structures [60]. It is interesting that the fast scrambling can always exist at the bare all-to-all strength $\bar{\alpha} \approx 0.5$ separating the two regions with faster and slower scrambling dynamics.

**Timescale in the infinite temperature ensemble**—The faster scrambling region with $\alpha < 0.5$ seems violate the fast scrambling conjecture [14–17]. However, we will show that this is due to the lack of a natural timescale in the infinite temperature ensemble used in this work. By introducing a proper timescale $t_0$, the fast scrambling is justified in our model.

Actually, the faster and slower type scrambling dynamics can exist in quantum lattice models. For instance, the quantum lattice models on a star graph [76] can have a constant scrambling time, which can be seen as the fastest scrambling. On the other hand, for the slower scrambling region, our results are in agreement with the analytical result $t_s \gtrsim N^{\alpha-0.5}$ [77] that the fast scrambling is absent for $\alpha > 0.5$. The fast scrambling point $\alpha \approx 0.5$ is more interesting. We have shown this value is not affected by $\bar{\alpha}$ which determines the energy level structures. It is also the same as that in previously proposed minimal models [28]. This indicates that $\alpha \approx 0.5$ is general for the 2-local all-to-all quantum spin models to have a holographic type scrambling dynamics, similar to the normalization factor in the SYK model.

In the finite temperature ensemble, the inverse temperature $\beta$ serves as a natural timescale, so that $t/\beta$ is dimensionless and the fast scrambler conjecture can be stated as the Lyapunov exponent is upper bounded by $2\pi$. In the infinite temperature calculation, however, there is no such a natural scale and the time $t$ has the dimension of inverse energy. In analogy to the finite temperature ensemble, we can introduce a proper timescale and one should study the information scrambling via the dimensionless $t/t_0$. In our model this timescale can naturally be chosen as $t_0 \sim N^{\alpha-0.5}$. Note that the timescale $t_0$ must not be divergent for large $N$. It is only valid for the region $\alpha < 0.5$ where $t_0 \to 0$ for large $N$.
consisting with the infinite temperature $\beta = 0$. Thus, we concludes that for $\alpha \leq 0.5$ our model is a fast scrambler, in which the rescaled scrambling time $t_s/t_0 \sim \log N$, see Fig. 1.

On experimental realization—Our model is directly achievable in state-of-the-art quantum simulation platforms, such as superconducting qubit quantum simulators [78–80]. With a cluster of qubits coupled to a single cavity resonator, the all-to-all XX-type interaction can be conveniently generated and accurately controlled. The on-site random transverse fields are also realizable by applying microwaves driving on each qubit with adjustable amplitudes and frequencies. The scrambling dynamics may be probed by statistical correlation of random measurements [23, 24] or considering the fidelity OTOC [22] that is easier to measure in experiments.

Summary and outlook—In summary, we have studied the thermalization-localization transition and scrambling dynamics in an all-to-all model with local quenched disorder. We characterize the transition under a proper relative all-to-all interaction strength $\tilde{\alpha}$ and show that the thermal and localized phases can be distinguished by OTOC at later time. Utilizing state-of-the-art tensor network simulation, we confirm the existence of fast scrambling in our model without appealing to the semiclassical limit. We argue that it is necessary to introduce a timescale $t_0$ and consider the dimensionless timescale $t/t_0$ to study the fast scrambling for general quantum models at infinite temperature. With this timescale, we give a phase diagram for the fast scrambling dynamics in quantum models with homogeneous all-to-all interactions.

Further works may include more detailed study of the MBL transition in all-to-all models using various scaling functions proposed recently [81, 82] and other non-equilibrium dynamical properties in the MBL phase. For the fast scrambling in the thermal phase, it is also interesting to study the scrambling dynamics at a finite temperature. The lack of exponential growth in finite-size spin-$1/2$ systems may be further addressed by considering the models without conservation laws such as Floquet systems [67] with all-to-all interactions.

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Supplementary material

RESULTS FOR OTHER RELATIVE INTERACTION STRENGTH \( \tilde{\alpha} \)

In the main text, we show that the thermalization-localization transition emerges when taking a relative all-to-all strength \( \tilde{\alpha} = 2 \). Here, we show the results for other \( \tilde{\alpha} \). Fig. 5 (a) presents the level statics with \( \tilde{\alpha} = 0.5 \). We observe that the intersecting points \( W_i \) of consecutive curves have a strong shift and diverge to infinite. This indicates the absence of the MBL phase in the thermodynamic limit. Furthermore, by studying the shift of intersecting points under different \( \tilde{\alpha} \), we show that \( \tilde{\alpha} \sim 2.0 \) is crucial for a well-defined thermalization-localization transition. Specifically, in Fig. 5 (b), the shift of the cross points versus system size indicates that the transition point evolves to infinity (zero) for \( \tilde{\alpha} < 2.0 \) (\( \tilde{\alpha} > 2.0 \)).

Figure 5. (a) Adjacent radio gap as a function of disorder strength, under the relative interaction strength \( \tilde{\alpha} = 0.5 \). The dashed lines mark out the region where the intersecting points shift with system size. (b) The intersecting points \( W_i \) with different \( \tilde{\alpha} \). These \( W_i \) converge to a finite value as increasing \( N \) under \( \tilde{\alpha} \sim 2 \).

Figure 6. (a) Fast scrambling dynamics with various relative interaction strengths \( \tilde{\alpha} \) and bare strength \( \alpha = 0.5 \). The disorder strengths are chosen as \( W = 2, 0.55, 0.15 \) for \( \tilde{\alpha} = 0.5, 1.0, 1.5 \), respectively. (b) Scrambling time versus the system size \( N \) for different \( \alpha \). The values of \( \alpha \) are \( 0.1-1.0 \), which are the same as that in Fig. 4. The dashed lines are the linear fit for \( \alpha = 0.5 \), indicating a logarithmic growth of scrambling time. Different \( t_s-N \) curves are rescaled by \( 12^{\alpha=0.5} \) for better presentation.
We also explore the fast scrambling dynamics at different $\tilde{\alpha}$. The choices of parameter $W$ ensure the system is in the thermal phase. In Fig. 6, we show that the scrambling dynamics is not significantly affected by this shift and the fast scrambling always occur at $\alpha \sim 0.5$. The scrambling dynamics is faster when $\alpha < 0.5$ and slower when $\alpha > 0.5$. Thus, in our all-to-all models, the fast scrambling dynamics is governed by the bare interaction strength $\alpha$ and is universal for all thermal phases.

MPO METHODS AND NUMERICAL CONVERGENCE

In this section, we describe the tensor network simulation method and the numerical convergence of the results. The observable $C(r, t)$ is calculated by the following relation

$$C(r, t) = 1 - \text{Re} F(r, t) \tag{4}$$

, where $F(r, t)$ is defined as

$$F(r, t) = \langle \hat{\sigma}_0^z(t) \hat{\sigma}_r^z \hat{\sigma}_0^z(t) \hat{\sigma}_r^z \rangle \tag{5}$$

Our method for calculating Eq. (5) is by directly evolving the operator using the Heisenberg equation. The time evolved operators are represented as matrix product operator (MPO) with maximum bond-dimension $\chi$. Then we map the MPO to matrix product state (MPS) by vectorizing each local operator and employ the standard TDVP algorithm to evolve the MPS in time. The effective Hamiltonian for this MPS is constructed as the super-operator $H \otimes I - I \otimes H^*$, corresponding to the Heisenberg equation. After obtaining the time evolved MPS, we split it back to MPO and calculate $F(r, t)$.

![Figure 7](image.png)

Figure 7. Numerical convergence for the tensor network simulation with bond dimension $\chi$ and disorder realization number $N_r$, where the system size is $N = 48$.

In numerical simulations, we find that $\chi = 100$ is enough to obtain the dynamics up to $N = 48$ sites with truncation errors $10^{-5}$. In fact, this level of truncation errors can hold for system sizes from 24 to 48 and for the later evolution time when the commutators are saturated. We present the results obtained using $\chi = 100$ and $\chi = 120$ in Fig. 7 for system size $N = 48$, which show a good numerical convergence. We have also checked that the disorder-averaged number $N_r = 24$ already gives converged results. Using more disorder realizations, we obtain the same results.

SEMICLASSICAL DYNAMICS

As additional results for much larger system sizes, we study the chaotic dynamics in the semi-classical limit. That is, when the spin length $S$ tends to infinite, the spin operator $\hat{\mathbf{S}}$ can be replaced by the classical angular momentum
vector $\mathbf{S}$. Correspondingly, the commutator in the Heisenberg equation is replaced by the Poisson bracket. For large $S$ models, the results of thermal-localization transition derived for the spin-1/2 case are no longer applicable. We thus choose a value of $W$ that renders the semi-classical dynamics chaotic and easy to numerically integrate. Following Ref. [26, 70], we calculate the semi-classical averaged sensitivity

$$C_{cl}(r,t) = \frac{1}{S^2} \left( \left( \frac{dS_z^2(t)}{d\phi} \right)^2 \right),$$

where $\phi$ is an initial small rotation of spin 0 about the $z$ axis. $C_{cl}(r,t)$ can be seen as the semi-classical version of $C(r,t)$. The average in Eq. (6) is done for an initial ensemble that each spin randomly lies in the $x-y$ plane, for each disorder realization. The growth of the disorder averaged $C_{cl}(t)$ is shown in Fig. 8 (a), where an additional average is done over sites $j > 1$ since we have confirmed that all these $C_{cl}(j,t)$ behave identically similar to the quantum case. We can clearly see three stages of the growth. Following the first stage of rapid power-law growth, there exists an exponential growth of $C_{cl}(t)$ corresponding to the Lyapunov region. At last stage the growth of averaged sensitivity slows down and tends to saturate. We extract the scrambling time with the saturated value $C_{cl}(t) = 1$ in the second stage, up to the system size $N = 200$. It is shown in Fig. 8 (b) that the semi-classical dynamics also presents a fast scrambling.

![Figure 8. Semiclassical numerics for fast scrambling dynamics with $\alpha = \tilde{\alpha} = 0.5$ and $W = 1$. (a) The exponential growth of averaged sensitivity $C_{cl}(t)$ at intermediate-time region. (b) Fast scrambling in the semi-classical limit, with system size up to $N = 200$.](image-url)