The Locating Chromatic Number of Book Graph

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Abstract

The locating chromatic number of a graph is defined as the length of the shortest path connecting a vertex to the partition of vertices to which it belongs in a vertex coloring. In this paper, we determine the locating chromatic number of a book graph. The book graph is a graph obtained from n copies of a cycle of order 4 sharing one common edge. We prove that the locating chromatic number of the book graph is 3 for odd n ≥ 3 and 4 for even n ≥ 4.

1. Introduction

All graphs considered in this paper are assumed to be simple, connected, and undirected. Let G = (V(G), E(G)) be a graph with vertex set V(G) and edge set E(G). By P_n and S_n, we denote a path on n vertices and a star on n + 1 vertices, respectively. The distance, d(u, v), between two vertices u and v is defined as the length of the shortest path connecting them in G.

For a graph G and a positive integer k, a coloring c: V(G) → {1, 2, . . . , k} with c(u) ≠ c(v) for every two adjacent vertices u and v is called a proper k-coloring of G. Let Π = {C_1, C_2, . . . , C_k} be a partition of vertices of G induced by the coloring c. We define the color code c_Π(v) of a vertex v ∈ V(G) as an ordered k-tuple that contains the distance between each partition to the vertex v. If distinct vertices have distinct color code, then c is called a locating k-coloring of G. The locating chromatic number of G is the smallest k such that G has a locating k-coloring. In this paper, we determine the locating chromatic number of book graph.

Theorem 1. Let c be a locating coloring in a connected graph G and N(v) be the set of vertices adjacent to v. If u and v are distinct vertices of G such that d(u, w) = d(v, w) for all w ∈ V(G) − {u, v}, then c(u) ≠ c(v). In particular, if u and v are nonadjacent vertices of G such that N(u) = N(v), then c(u) ≠ c(v).

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2. Result and Discussion

In this section, we determine the exact value for the locating chromatic number of book graphs. Let $B_n$ be a book graph with the vertex set $V(B_n) = \{u_i, v_i; i = 0, 1, \ldots, n\}$ and the edge set $E(B_n) = \{u_iv_i; i = 0, 1, \ldots, n\} \cup \{u_0u_i, v_0v_i; i = 1, 2, \ldots, n\}$.

Two observations below follow from definition of proper coloring.

Observation 1. Let $c$ be a locating coloring of the book graph $B_n$, $n \geq 1$. Then, $c(u_i) \neq c(v_i)$ for $i = 0, 1, \ldots, n$.

Observation 2. Let $c$ be a locating coloring of the book graph $B_n$, $n \geq 1$. Then, $c(u_0) \neq c(u_i)$ and $c(v_0) \neq c(v_i)$ for $i = 1, 2, \ldots, n$.

Lemma 1. Let $c$ be a locating coloring of the book graph $B_n$, $n \geq 1$. Then, $(c(u_i), c(v_i)) \neq (c(u_j), c(v_j))$ for $1 \leq i \neq j \leq n$.

Proof. Assume that $(c(u_i), c(v_i)) = (c(u_j), c(v_j))$ for some $i$ and $j$, $i \neq j$. This means that $c(u_i) = c(u_j)$ and $c(v_i) = c(v_j)$. We know that $d(u_i, x) = d(u_j, x)$ for every $x \in V(B_n) - \{u_i, u_j, v_i, v_j\}$. As $c(v_i) = c(v_j)$ and $d(u_j, v_i) = d(u_j, v_j) = 1$, regardless the color of $x$, we have $c(u_i) = c(u_j)$, a contradiction.

Lemma 2. Let $c$ be a locating $k$-coloring of the book graph $B_n$, for $k \geq 4$ and $n \geq 1$. Then,

$$n \leq 2 \left(\frac{k-2}{2}\right) + 2(k - 1) - 1. \quad (1)$$

Proof. Start coloring the graph by giving color $C_1$ and $C_2$ for the vertices in the middle, that is, $c(u_i) = C_1$ and $c(v_0) = C_2$. From Observation 2, we know that we should have $c(u_i) \neq C_1$ and $c(v_i) \neq C_2$ for each $i = 1, 2, \ldots, n$.

Case 1. Assume $\{c(u_a), c(v_a)\}$ does not contain either $C_1$ or $C_2$. Thus, the total color that we can choose is $\left(\frac{k-2}{2}\right) \times 2$ because every outer part of sheets requires 2 different colors from observation 1, which means there are $\left(\frac{k-2}{2}\right)$ possibilities. But, since $c(u_a)$ and $c(v_a)$ are not reversed without any consequences, the total that can be obtained is $\left(\frac{k-2}{2}\right) \times 2$.

Case 2. If $c(u_a) = 2$, the total possibility of the colors being used is $(k-1)$ because $c(v_a)$ can only be given with 1, 3, 4, \ldots, $k$, where only one color can be used by book graph $B_n$ for $n \geq 2$. The illustration is shown in Figure 1.

Case 3. If $c(v_a) = 1$, the total colors can be used are $(k-1)$ because $c(u_a)$ can only be given by color 2, 3, 4, \ldots, $k$, where only two colors can be used on book graph $B_n$ for $n \geq 2$. The illustration is shown in Figure 2.

Since the coloring $c(v_a) = 1$ and $c(u_a) = 2$ have been used in Case 2, then the coloring is not reusable, and thus, the total coloring is $(k-1) - 1$. If uses $k$ colors, then $x(L(B_n))$ must satisfy the following

$$n \leq 2 \left(\frac{k-2}{2}\right) + (k-1) + (k-1) - 1,$$

$$n \leq 2 \left(\frac{k-2}{2}\right) + 2(k - 1) - 1. \quad (2)$$

Theorem 2. Locating chromatic number of book graph $B_n$ is

$$x_L(B_n) = \min \left\{k: n \leq 2 \left(\frac{k-2}{2}\right) + 2(k - 1) - 1\right\}, \quad (3)$$

for $k \geq 4$.

Proof. Firstly, we will prove the existence of $x_L(B_n)$. We can clearly see that $\left\{k: n \leq 2 \left(\frac{k-2}{2}\right) + 2(k - 1) - 1\right\}$ is the subset from a set of real numbers. So from the well-ordering principle, the set must have the smallest element, say $k^* := \min \left\{k: n \leq 2 \left(\frac{k-2}{2}\right) + 2(k - 1) - 1\right\}. \quad (4)$
that yields
\[ (k^* - 1) \geq \min \left\{ k : n \leq 2 \left( \frac{k - 2}{2} \right) + 2(k - 1) - 1 \right\} = k^*. \]
(5)

It was contradiction, which means it is impossible that the locating chromatic of book graph has a coloring for \((k^* - 1)\) colors. It should be
\[ \chi_L(B_n) = k^* = \min \left\{ k : n \leq \left( \frac{k - 2}{2} \right) + 2(k - 1) - 1 \right\}, \]
for \(k \geq 4\).
(6)

**Corollary 1.** Locating chromatic number of book graph \(B_n\) is
\[ \chi_L(B_n) = \left\lceil \sqrt{n - \frac{3}{4}} + \frac{3}{2} \right\rceil. \]
(7)

**Proof.** Let \(k\) be the color needed for locating chromatic of book graph \(B_n\), then
\[ n \leq \left( \frac{k - 2}{2} \right) + 2(k - 1) - 1, \]
\[ n \leq \frac{(k - 2)(k - 3)}{2} \times 2 + 2k - 2 - 1, \]
\[ \left( k^2 - 5k + 6 + 2k - 3 \right) \geq n, \]
\[ k^2 - 3k + 3 \geq n, \]
\[ \left( k - \frac{3}{2} \right)^2 - \frac{9}{4} + 3 \geq n, \]
\[ \left( k - \frac{3}{2} \right)^2 + \frac{3}{4} \geq n, \]
\[ \left( k - \frac{3}{2} \right)^2 \geq n - \frac{3}{4}, \]
\[ k \geq \sqrt{n - \frac{3}{4}} + \frac{3}{2}. \]
(8)

Then, the smallest possible \(k\) is
\[ k_{\min} = \left\lceil \sqrt{n - \frac{3}{4}} + \frac{3}{2} \right\rceil. \]
(9)

\[ \chi_L(B_n) = k^* = \min \left\{ k : n \leq \left( \frac{k - 2}{2} \right) + 2(k - 1) - 1 \right\}, \]
for \(k \geq 4\),
(10)
or
\[ k_{\min} = \left\lceil \sqrt{n - \frac{3}{4}} + \frac{3}{2} \right\rceil. \]
(11)

**Data Availability**
No data were used to support this study.

**Conflicts of Interest**
The authors declare that they have no conflicts of interest.

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**References**
[1] G. Chartrand, D. Erwin, M. Henning, P. Slater, and P. Zhang, “The locating-chromatic number of a graph,” *Bulletin of the ICA*, vol. 36, pp. 89–101, 2002.
[2] A. Behtoei and B. Omoomi, “On the locating chromatic number of the cartesian product of graphs,” *Ars Combinatoria*, vol. 126, pp. 221–235, 2012.
[3] D. Welyyanti, E. T. Baskoro, R. Simanjuntak, and S. Uttunggadewa, “The locating-chromatic number of disconnected graphs,” *Far East Journal of Mathematical Sciences*, vol. 94, no. 2, pp. 169–182, 2014.
[4] C. D. Rianti and Narwen, “Bilangan kromatik lokasi dari graf spinner,” *Jurnal Matematika UNAND*, vol. VII, no. 4, pp. 19–23, 2018.