DUALITY PROBLEM FOR DISJOINTLY HOMOGENEOUS REARRANGEMENT INVARIANT SPACES

SERGEY V. ASTASHKIN

ABSTRACT. Let $1 \leq p < \infty$. A Banach lattice $E$ is said to be disjointly homogeneous (resp. $p$-disjointly homogeneous) if two arbitrary normalized disjoint sequences from $E$ contain equivalent in $E$ subsequences (resp. every normalized disjoint sequence contains a subsequence equivalent in $E$ to the unit vector basis of $l_p$). Answering a question raised in the paper [12], for each $1 < p < \infty$, we construct a reflexive $p$-disjointly homogeneous rearrangement invariant space on $[0,1]$ whose dual is not disjointly homogeneous. Employing methods from interpolation theory, we provide new examples of disjointly homogeneous rearrangement invariant spaces; in particular, we show that there is a Tsirelson type disjointly homogeneous rearrangement invariant space, which contains no subspace isomorphic to $l_p$, $1 \leq p < \infty$, or $c_0$.

1. Introduction

A Banach lattice $E$ is called disjointly homogeneous (shortly DH) if two arbitrary normalized disjoint sequences in $E$ contain equivalent subsequences. In particular, given $1 \leq p \leq \infty$, a Banach lattice $E$ is $p$-disjointly homogeneous (shortly $p$-DH) if each normalized disjoint sequence in $E$ has a subsequence equivalent to the unit vector basis of $l_p$ ($c_0$ when $p = \infty$). These notions were first introduced in [14] and proved to be very useful in studying the general problem of identifying Banach lattices $E$ such that the ideals of strictly singular and compact operators bounded in $E$ coincide [11] (see also survey [13] and references therein). Results obtained there can be treated as a continuation and development of a classical theorem of V. D. Milman [31] which states that every strictly singular operator in $L_p(\mu)$ has compact square. This is a motivation to find out how large is the class of DH Banach lattices. As is shown in the above cited papers, it contains $L_p(\mu)$-spaces, $1 \leq p \leq \infty$, Lorentz function spaces $L_{q,p}$ and $\Lambda(W,p)$, a certain class of Orlicz function spaces, Tsirelson space and some others.

The next step in the case of rearrangement invariant (in short, r.i.) function spaces was undertaken in the paper [4]. By the complex method of interpolation, it was proved there that for every $1 \leq p \leq \infty$ and any increasing concave function $\varphi$ on $[0,1]$, which is not equivalent to neither 1 nor $t$, there exists a $p$-DH r.i. space on $[0,1]$ with the fundamental function $\varphi$ (see definitions in the next section). Observe that there is the only r.i. space on $[0,1]$, $L_\infty$ (resp. $L_1$), having the fundamental function equivalent to 1 (resp. $t$). Moreover, in [4, Theorem 4] it is obtained the following sharp version of the classical Levy’s result [24] for sequences of pairwise disjoint functions. If $X_0$ and $X_1$ are r.i. spaces such that $X_0$ is strictly embedded into $X_1$, then every sequence $\{x_n\}_{n=1}^\infty$ of normalized pairwise disjoint functions from the real interpolation space $(X_0,X_1)_{\theta,p}$, $0 < \theta < 1$, $1 \leq p < \infty$,
\( \|x_n\|(x_0, x_1)_{\theta, p} = 1, \ n = 1, 2, \ldots, \) contains a subsequence equivalent to the unit vector basis of \( l_p. \)

Here, we continue the above direction of research considering a special case of the real method of interpolation, which was introduced and studied by Lions and Peetre [29] (see also [28] 2g]). While parameters of the functors \((\cdot, \cdot)_{\theta, p}, 0 < \theta < 1, 1 \leq p < \infty, \) are only weighted \( l_p \)-spaces, the interpolation spaces from [29] are generated by arbitrary Banach spaces having a normalized 1-unconditional basis \( \{e_n\} \). It turns out that in this case there is still a direct link between some properties of block bases of \( \{e_n\} \) and sequences of pairwise disjoint functions from the respective interpolation space, which allows to construct r.i. spaces with a certain prescribed lattice structure. This applies not only to the equivalence of subsequences but also to their complementability. As was shown in the paper [12], DH properties of Banach lattices are closely connected with the following concept. A Banach lattice \( E \) is called disjointly complemented (DC) if every disjoint sequence from \( E \) contains a subsequence complemented in \( E. \)

The above approach based on using tools from interpolation theory allows to construct new examples of \( p \)-DH and DH r.i. spaces. In particular, we show that there is a Tsirelson type DH and DC r.i. space, which contains no subspace isomorphic to \( l_p, 1 \leq p < \infty, \) or \( c_0 \) (Theorem 4). Moreover, we solve the following duality problem posed in [12]: Is there a reflexive \( p \)-DH r.i. space on \([0, 1]\) whose dual is not DH (see also Question 3 in the survey [13])? Answering this question, we construct such a space in Theorem 2. We believe that the approach developed here is interesting in its own and may be useful in solving other problems related to the study of lattice properties of r.i. spaces.

In the concluding part of the paper, we show that the existence of sequences of equimeasurable pairwise disjoint functions in the case of infinite measure leads to the essential difference of DH and DC properties of r.i. spaces on \((0, \infty)\) and \([0, 1]\). As was shown in [12] Theorem 5.2, for each \( 1 < p < \infty \) there is a \( p \)-DH Orlicz space on \((0, \infty)\) whose dual is not DH. In fact, by using known results on subspaces generated by translations in r.i. spaces due to Hernandez and Semenov [15], we are able to give a characterization of \( L_p \)-spaces via DH (resp. DC) property (Theorem 5). Namely, we show that if \( X \) is a reflexive r.i. space on \((0, \infty)\) such that \( X \neq L_p \) for every \( 1 < p < \infty \), then at least one of the spaces \( X \) and \( X^* \) is not DH.

2. Preliminaries

2.1. Banach lattices and rearrangement invariant spaces. Let \( I = [0, 1] \) or \([0, \infty)\). A Banach lattice \( E = (E, \|\cdot\|) \) on \( I \) is a Banach space of real–valued Lebesgue measurable functions (of equivalence classes) defined on \( I \), which satisfies the so–called ideal property: if \( F \) is a measurable function, \( |f| \leq |g| \) almost everywhere (a.e.) with respect to the Lebesgue measure on \( I \) and \( g \in E, \) then \( f \in E \) and \( \|f\|_E \leq \|g\|_E. \)

If \( E \) is a Banach lattice on \( I \), then the K"{o}the dual space (or associated space) \( E' \) consists of all real–valued measurable functions \( f \) such that

\[
\|f\|_{E'} := \sup_{g \in E, \|g\|_E \leq 1} \int_I |f(x)g(x)| \, dx < \infty.
\]

The K"{o}the dual \( E' \) is a Banach lattice. Moreover, \( E \subset E'' \) and the equality \( E = E'' \) holds isometrically if and only if the norm in \( E \) has the Fatou property, meaning that the conditions \( 0 \leq f_n \searrow f \) a.e. on \( I \) and \( \sup_{n \in \mathbb{N}} \|f_n\|_E < \infty \) imply that \( f \in E \) and \( \|f_n\|_E \searrow \|f\|_E. \) For a separable Banach lattice \( \hat{E} \) the K"{o}the dual \( E' \) and the (Banach) dual space \( E^* \) coincide. Moreover, a Banach lattice \( \hat{E} \) with the Fatou property is reflexive if and only if both \( E \) and its K"{o}the dual \( E' \) are separable.
Let $E$ be a Banach lattice on $I$. A function $f \in E$ is said to have an order continuous norm in $E$ if for any decreasing sequence of Lebesgue measurable sets $B_n \subset I$ with $m(\bigcap_{n=1}^{\infty} B_n) = 0$, where $m$ is the Lebesgue measure, we have $\|f\chi_{B_n}\|_E \to 0$ as $n \to \infty$.

The set of all functions in $E$ with order continuous norm is denoted by $E_0$. A Banach lattice $E$ on $I$ is separable if and only if $E_0 = E$.

A Banach lattice $X$ on $I$ is said to be a rearrangement invariant (in short, r.i.) (or symmetric) space if from the conditions: functions $x(t)$ and $y(t)$ are equimeasurable, i.e.,

$$m\{ t \in I : |x(t)| > \tau \} = m \{ t \in I : |y(t)| > \tau \} \quad \text{for all } \tau > 0,$$

and $y \in X$ it follows $x \in X$ and $\|x\|_X = \|y\|_X$.

In particular, every measurable on $I$ function $x(t)$ is equimeasurable with the non-increasing, right-continuous rearrangement of $|x(t)|$ given by

$$x^*(t) := \inf \{ \tau > 0 : m \{ s \in I : |x(s)| > \tau \} \leq t \}, \quad t > 0.$$

If $X$ is a r.i. space on $I$, then the Köthe dual $X'$ is also r.i. In what follows, as in [28], we suppose that every r.i. space is either separable or maximal, i.e., $X = X''$.

The fundamental function $\varphi_X$ of a r.i. space $X$ is defined as

$$\varphi_X(t) := \|\chi_{[0,t]}\|_X, \quad t > 0,$$

where $\chi_B$, throughout, will denote the characteristic function of a set $B$. The function $\varphi_X$ is quasi-concave, that is, it is nonnegative and increases, $\varphi_X(0) = 0$, and the function $\varphi_X(t)/t$ decreases.

For any r.i. space $X$ on $[0,1]$ we have $L_\infty[0,1] \subseteq X \subseteq L_1[0,1]$. In the case when $X \neq L_\infty[0,1]$ the space $X_0$ is r.i. and it coincides with the closure of $L_\infty$ in $X$ (the separable part of $X$). Next, we will repeatedly use the fact that the conditional expectation generated by a $\sigma$-algebra of measurable subsets of $[0,1]$ is a projection of norm 1 in every r.i. space on $[0,1]$ [28, Theorem 2.a.4].

An important example of r.i. spaces are the Orlicz spaces. Let $\Phi$ be an increasing convex function on $[0,\infty)$ such that $\Phi(0) = 0$. Denote by $L_\Phi(I)$ the Orlicz space on $I$ (see e.g. [22]) endowed with the Luxemburg–Nakano norm

$$\|f\|_{L_\Phi} = \inf \{ \lambda > 0 : \int_I \Phi(|f(x)|/\lambda) \, dx \leq 1 \}.$$ 

In particular, if $\Phi(u) = u^p$, $1 \leq p < \infty$, we obtain $L_p(I)$.

Similarly, one can define Banach lattices and r.i. sequence spaces (i.e., on $I = \mathbb{N}$ with the counting measure) and all the above notions.

For general properties of Banach lattices and r.i. spaces we refer to the books [21], [2], [28], [23], [7] and [5].

2.2. Interpolation spaces. Recall that a pair $(A_0, A_1)$ of Banach spaces is called a Banach couple if $A_0$ and $A_1$ are both linearly and continuously embedded in some Hausdorff linear topological vector space. In particular, arbitrary Banach lattices $E_0$ and $E_1$ on $I$ form a Banach couple because of every such a lattice is continuously embedded into the space of all measurable a.e. finite functions on $I$ equipped with the convergence in measure on the sets of finite measure.

A Banach space $A$ is called interpolation with respect to the couple $(A_0, A_1)$ if $A_0 \cap A_1 \subset A \subset A_0 + A_1$ and each linear operator $T : A_0 + A_1 \to A_0 + A_1$, which is bounded in $A_0$ and in $A_1$, is bounded in $A$. 

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For a Banach couple \((A_0, A_1)\) we can define the intersection \(A_0 \cap A_1\) and the sum \(A_0 + A_1\) as the Banach spaces with the natural norms: \(\|a\|_{A_0 \cap A_1} = \max \{\|a\|_{A_0}, \|a\|_{A_1}\}\) and \(\|a\|_{A_0 + A_1} = k(1, 1, a; A_0, A_1)\), where \(k(\alpha, \beta, a; A_0, A_1)\) is the Peetre \(K\)-functional, i.e.,

\[
k(\alpha, \beta, a; A_0, A_1) := \inf \{\alpha\|a\|_{A_0} + \beta\|a\|_{A_1} : a = a_0 + a_1, a_i \in A_i, i = 0, 1\}
\]

for any \(a \in A_0 + A_1\) and \(\alpha, \beta > 0\).

Let \(E\) be a Banach space with an 1-unconditional normalized basis \(\{e_n\}_{n=1}^{\infty}\) and let \((\alpha_n)_{n=1}^{\infty}\) and \((\beta_n)_{n=1}^{\infty}\) be two sequences of positive numbers such that

\[
\sum_{n=1}^{\infty} \min(\alpha_n, \beta_n) < \infty.
\]

Following [29] (see also [28, Definition 2.g.3]), for every Banach couple \((A_0, A_1)\) we define the Lions-Peetre space \(W_E^K(A_0, A_1, (\alpha_n), (\beta_n))\) as the set of all elements \(a \in A_0 + A_1\), for which the series

\[
\sum_{n=1}^{\infty} k(\alpha_n, \beta_n, a; A_0, A_1)e_n
\]

converges in \(E\), and we set

\[
\|a\|_{W_E^K} := \left\| \sum_{n=1}^{\infty} k(\alpha_n, \beta_n, a; A_0, A_1)e_n \right\|_E.
\]

It is known [28, Proposition 2.g.4] that \(W_E^K(A_0, A_1, E, (\alpha_n), (\beta_n))\) is an interpolation Banach space with respect to the couple \((A_0, A_1)\). In particular, the following continuous embeddings hold:

\[
A_0 \cap A_1 \subset W_E^K(A_0, A_1, E, (\alpha_n), (\beta_n)) \subset A_0 + A_1.
\]

We shall concern with the case when \(\alpha_n = m_n^{-1}\) and \(\beta_n = m_n, n = 1, 2, \ldots\), where \((m_n)_{n=1}^{\infty}\) is any fixed increasing sequence such that \(m_1 \geq 2\) and

\[
m_n^{-1} \sum_{i=1}^{n-1} m_i + m_n \sum_{i=n+1}^{\infty} m_i^{-1} < 2^{-n-1}, \quad n = 1, 2, \ldots
\]

(by convention, \(\sum_{i=1}^{0} = 0\)).

Let \(A\) be a Banach space, \(a_n \in A, n = 1, 2, \ldots\) We shall denote by \([a_n]\) the closed linear span of a sequence \(\{a_n\}\) in \(A\). This sequence will be called complemented if \([a_n]\) is a complemented subspace in \(A\). Moreover, if \(A^*\) is the dual space for \(A\), the value of a functional \(a^* \in A^*\) at an element \(a \in A\) will be denoted by \(\langle a, a^* \rangle\). In particular, if \(E^*\) is the Köthe dual to a Banach function lattice \(E\) on \(I\) (resp. Banach sequence lattice \(E\)), \(x(t) \in E, y(t) \in E^*\) (resp. \(x = (x_k)_{k=1}^{\infty} \in E, y = (y_k)_{k=1}^{\infty} \in E^*\)), we have \(\langle x, y \rangle = \int_I x(t)y(t) dt\) (resp. \(\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k\)). Finally, the notation \(F \preceq G\) will mean that there exist constants \(C > 0\) and \(c > 0\) not depending on the arguments of the expressions \(F\) and \(G\) such that \(c \cdot F \leq G \leq C \cdot F\).

3. Lattice properties of interpolation spaces

In what follows we assume that \(X_0\) and \(X_1\) are r.i. spaces, \(X_0 \subset X_1\) and this embedding is strict, that is, for each sequence \(\{x_n\}_{n=1}^{\infty} \subset X_0\) such that \(\sup_{n=1,2,\ldots} \|x_n\|_{X_0} < \infty\) and \(m(\text{supp } x_n) \rightarrow 0\) we have \(\|x_n\|_{X_1} \rightarrow 0\) as \(n \rightarrow \infty\).

Additionally, without loss of generality, we shall assume that

\[
\|x\|_{X_1} \leq \|x\|_{X_0} \text{ for all } x \in X_0
\]
and

\[ \phi_{X_0}(1) = \phi_{X_1}(1) = 1. \]

In the next two propositions the spaces \( X_0, X_1 \) and \( E \) will be fixed and so, for brevity, we set \( W := W^K_E(X_0, X_1) \). In the proof of the first of them we make use of an idea from the proof of Proposition 3.b.4 in [27].

**Proposition 1.** There exists a sequence \( (A_n)_{n=1}^{\infty} \) of pairwise disjoint measurable subsets of \([0, 1]\) such that the sequence \( \left\{ \frac{\chi_{A_n}}{\|\chi_{A_n}\|_W} \right\}_{n=1}^{\infty} \) is equivalent to the basis \( \{e_n\}_{n=1}^{\infty} \) of the space \( E \).

**Proof.** For each \( n \in \mathbb{N} \) we define on \( X_1 \) the norm \( \| \cdot \|_n \) by

\[ \|x\|_n := k(x, m_n^{-1}, m_n). \]

We claim that for any measurable set \( A \subset [0, 1] \), with \( m(A) = t \), it holds

\[ \|\chi_A\|_n = \min\{m_n^{-1}\phi_{X_0}(t), m_n\phi_{X_1}(t)\}. \]

Since the inequality \( \leq \) is obvious, it suffices to prove the opposite one. Recall that the expectation operator corresponding to the partition \( \{A, [0, 1] \setminus A\} \) of \([0, 1] \), i.e., the operator

\[ S_A x(t) := \frac{1}{m(A)} \int_A x(s) \, ds \cdot \chi_A(t) + \frac{1}{m([0, 1] \setminus A)} \int_{[0,1] \setminus A} x(s) \, ds \cdot \chi_{[0,1] \setminus A}(t) \]

is bounded in every r.i. space with the norm 1 (see Section 2 or [28, Theorem 2.a.4]). Therefore, the value of norm \( \|\chi_A\|_n \) can be computed by using only decompositions of \( \chi_A \) of the form

\[ \chi_A = \alpha \chi_0 + (1 - \alpha) \chi_1, \quad 0 \leq \alpha \leq 1. \]

Thus,

\[ \|\chi_A\|_n = \inf_{0 \leq \alpha \leq 1} (\alpha m_n^{-1}\|\chi_A\|_X_0 + (1 - \alpha)m_n\|\chi_A\|_X_1) \geq \min\{m_n^{-1}\phi_{X_0}(t), m_n\phi_{X_1}(t)\}, \]

and therefore (5) follows.

Further, since the embedding \( X_0 \subset X_1 \) is strict, we have

\[ \lim_{t \to 0} \frac{\phi_{X_1}(t)}{\phi_{X_0}(t)} = 0. \]

Therefore, due to the continuity of the fundamental functions \( \phi_{X_0} \) and \( \phi_{X_1} \) and equation (1), for each \( n \in \mathbb{N} \) we can find \( t_n \in (0, 1) \) such that

\[ \frac{\phi_{X_1}(t_n)}{\phi_{X_0}(t_n)} = m_n^{-2}. \]

Observe that \( t_n \leq m_n^{-2} \). Indeed, assuming that \( t_n > m_n^{-2} \), by the quasi-concavity of the function \( \phi_{X_1}(t) \), we infer

\[ \phi_{X_1}(t_n) \geq t_n \phi_{X_1}(1) > m_n^{-2} \phi_{X_0}(1) \geq m_n^{-2} \phi_{X_0}(t_n), \]

which contradicts the choice of \( t_n \). Hence, from the inequality \( m_n \geq 2 \) and (2) it follows

\[ \sum_{n=1}^{\infty} t_n \leq \sum_{n=1}^{\infty} m_n^{-2} < 1, \]
and so we can fix pairwise disjoint measurable subsets $A_k \subset [0, 1]$, $m(A_k) = t_k$, $k = 1, 2, \ldots$, such that (see (5) and (2))

$$\sup_{k=1,2,\ldots} \|\chi_{A_n}\|_k = \|\chi_{A_n}\|_n$$

for all $n, k \in \mathbb{N}$.

Show that the norms $\|\chi_{A_n}\|_k$, $k \neq n$, are negligible comparatively with the norm $\|\chi_{A_n}\|_n$.

First, if $1 \leq k < n$, then by (3) we have

$$\|\chi_{A_n}\|_k \leq m_k \phi_X(t_n) = \frac{m_k}{m_n} m_n \phi_X(t_n) = \frac{m_k}{m_n} \|\chi_{A_n}\|_n.$$  

Similarly, in the case when $k > n$ we obtain

$$\|\chi_{A_n}\|_k \leq m_k^{-1} \phi_X(t_n) = \frac{m_n}{m_k} \|\chi_{A_n}\|_n.$$  

Combining these estimates with (2), we deduce

$$\sum_{k \neq n} \|\chi_{A_n}\|_k \leq \left( m_n^{-1} \sum_{k=1}^{n-1} m_k + m_n \sum_{k=n+1}^{\infty} m_k^{-1} \right) \|\chi_{A_n}\|_n \leq 2^{-n-1} \|\chi_{A_n}\|_n.$$  

Since the basis $\{e_n\}_{n=1}^{\infty}$ is normalized and 1-unconditional, the latter estimate with the definition of the norm in $W$ guarantee that for every $n = 1, 2, \ldots$

$$\|\chi_{A_n}\|_n \leq \|\chi_{A_n}\|_W \leq (1 + 2^{-n-1}) \|\chi_{A_n}\|_n.$$  

Setting $u_j := \frac{\chi_{A_j}}{\|\chi_{A_j}\|_n}$, $j = 1, 2, \ldots$, in view of (6), (7) and the left-hand side of (8) we obtain

$$\|u_j\|_n \leq \min \left( \frac{m_j}{m_n}, \frac{m_n}{m_j} \right), \quad j \neq n.$$  

This and inequality (2) imply

$$\sum_{j \neq n} \|u_j\|_n \leq m_n^{-1} \sum_{i=1}^{n-1} m_i + m_n \sum_{i=n+1}^{\infty} m_i^{-1} < 2^{-n-1},$$

which yields

$$\left\| \sum_{j=1}^{\infty} c_j u_j \right\|_n - |c_n| = \left\| \sum_{j=1}^{\infty} c_j u_j \right\|_n - \|c_n u_n\|_n \leq \sum_{j \neq n} |c_j| \|u_j\|_n \leq 2^{-n-1} \sup_{j=1,2,\ldots} |c_j|$$

for arbitrary $c_j \in \mathbb{R}$, $j = 1, 2, \ldots$ Hence,

$$\left\| \sum_{j=1}^{\infty} c_j u_j \right\|_W = \left\| \sum_{n=1}^{\infty} \left( \sum_{j=1}^{\infty} c_j u_j \right) e_n \right\|_E$$

$$\leq \left\| \sum_{n=1}^{\infty} \left( |c_n| + 2^{-n-1} \sup_{j=1,2,\ldots} |c_j| e_n \right) \right\|_E$$

$$\leq \left\| \sum_{n=1}^{\infty} c_n e_n \right\|_E + \sup_{j=1,2,\ldots} |c_j| \leq 2 \left\| \sum_{n=1}^{\infty} c_n e_n \right\|_E.$$

On the other hand, since

$$\left\| \sum_{j=1}^{\infty} c_j u_j \right\|_n \geq |c_n| \frac{\|\chi_{A_n}\|_n}{\|\chi_{A_n}\|_W},$$
then from (8) and the 1-unconditionality of \( \{e_n\}^\infty_{n=1} \) it follows
\[
\left\| \sum_{j=1}^{\infty} c_j u_j \right\|_W \geq \left\| \sum_{n=1}^{\infty} |c_n| \frac{x_{An}}{\|x_{An}\|_W} e_n \right\|_E \geq \frac{4}{5} \left\| \sum_{n=1}^{\infty} c_n e_n \right\|_E.
\]
This completes the proof. \( \square \)

Now, we prove in a sense opposite result. We shall need some more notation. Let \( B \subset \mathbb{N} \). For any \( f \in E \) we put \( P_B f := f \chi_B \). Since the basis \( \{e_n\}^\infty_{n=1} \) is 1-unconditional, \( P_B \) is a bounded projection in \( E \) with norm 1. Moreover, we set
\[
Sx := \sum_{k=1}^\infty \|x\|_k e_k, \ x \in W,
\]
where \( \| \cdot \|_k, k = 1, 2, \ldots \), are the norms introduced in the proof of Proposition 1. Clearly, \( Sx \in E \) for each \( x \in W \) and \( \|x\|_W = \|Sx\|_E \).

**Proposition 2.** Every sequence \( \{x_n\}^\infty_{n=1} \) of pairwise disjoint functions, \( \|x_n\|_W = 1 \), contains a subsequence \( \{u_i\}^\infty_{i=1} \), which is equivalent in \( W \) to some block basis of \( \{e_n\}^\infty_{n=1} \).

**Proof.** Let \( \delta_i > 0, i = 1, 2, \ldots \), and also \( \delta_0 := \sum_{i=1}^{\infty} \delta_i < 1 \). Moreover, suppose that functions \( x_k, k = 1, 2, \ldots \), are pairwise disjoint, \( \|x_k\|_W = 1 \) for each \( k \). Show that there are a subsequence \( \{u_i\}^\infty_{i=1} \) of \( \{x_k\}^\infty_{k=1} \) and sets \( B_i = \{n \in \mathbb{N} : l_i \leq n \leq m_i \}, i = 1, 2, \ldots \), where \( 1 = l_1 \leq m_1 < l_2 \leq m_2 < \ldots \), such that for \( f_i := P_{B_i} Su_i, i = 1, 2, \ldots \), we have
\[
\|Su_i - f_i\|_E < \delta_i, \ i = 1, 2, \ldots
\]
First, since \( \|x_n\|_W = 1 \) and the basis \( \{e_n\}^\infty_{n=1} \) is 1-unconditional, then \( \|x_n\|_k \leq 1 \), or equivalently,
\[
\inf \{m_k^{-1} \|y_n\|_X, m_k \|z_n\|_X : y_n + z_n = x_n \} \leq 1.
\]
Therefore, for each \( k = 1, 2, \ldots \) there are functions \( y^k_n \in X_0 \) and \( z^k_n \in X_1, n = 1, 2, \ldots \), such that \( y^k_n + z^k_n = x_n \), \( \|y^k_n\|_X \leq m_k \) and \( \|z^k_n\|_X \leq m^{-1}_k \). This observation and inequality (3) imply
\[
\|x_n\|_X \leq \|y^k_n\|_X + \|z^k_n\|_X \leq m^{-1}_k + \|y^k_n\|_X.
\]
Further, \( m(\supp y^k_n) = m(\supp x_n) \to 0 \) as \( n \to \infty \) and \( \|y^k_n\|_X \leq m_k, n = 1, 2, \ldots \). Hence, because of \( X_0 \) is strictly embedded into \( X_1 \), we infer that \( \|y^k_n\|_X \to 0 \) as \( n \to \infty \) for each \( k = 1, 2, \ldots \). Thus, for every \( k = 1, 2, \ldots \) there exists an increasing sequence of positive integers \( \{n_k\}^\infty_{k=1} \) such that \( \|y^k_{n_k}\|_X \leq m^{-1}_k \). Therefore, in view of (10), we have
\[
\|x_{n_k}\|_X \leq 2m^{-1}_k, \ k = 1, 2, \ldots
\]
Moreover, by the definition of the norm \( \| \cdot \|_j \),
\[
\|x_{n_k}\|_j \leq m_j \|x_{n_k}\|_X, \ j = 1, 2, \ldots
\]
Combining the latter inequalities, we deduce
\[
\|x_{n_k}\|_j \leq \frac{2m_j}{m_k}, \ k, j = 1, 2, \ldots
\]
and therefore from (2) it follows
\[
\left( \sum_{j=1}^{k-1} |x_{n_k}\|_j e_j \right) \left( \sum_{j=1}^{k-1} |x_{n_k}\|_j \right) \leq \frac{2}{m_k} \sum_{j=1}^{k-1} m_j \sum_{j=1}^{k-1} m_j \leq 2^{-k}, \ k = 1, 2, \ldots
\]
We set $l_1 := 1$ and $u_1 := x_1$. Since $u_1 \in W$, we can find a positive integer $m_1$ such that
\[ \left\| \sum_{j=m_1+1}^{\infty} \| u_1 \|_j e_j \right\|_E < \delta_1. \]
Hence, if $B_1 := \{ n \in \mathbb{N} : l_1 \leq n \leq m_1 \}$ and $f_1 := P_{B_1}Su_1$ we get
\[ \left\| Su_1 - f_1 \right\|_E < \delta_1. \]
Next, we choose $k_2 > m_1 + 1$ such that $2^{-k_2} < \delta_2/2$. Then, setting $l_2 := k_2 - 1$ and $u_2 := x_{n_{k_2}}$ we have $l_2 > m_1$ and, by (11),
\[ \left\| \sum_{j=1}^{l_2} \| u_2 \|_j e_j \right\|_E < \frac{\delta_2}{2}. \]
Moreover, for some $m_2 \geq l_2$
\[ \left\| \sum_{j=m_2+1}^{\infty} \| u_2 \|_j e_j \right\|_E < \frac{\delta_2}{2}. \]
Combining the latter inequalities, we conclude
\[ \left\| Su_2 - f_2 \right\|_E < \delta_2, \]
where $B_2 := \{ n \in \mathbb{N} : l_2 \leq n \leq m_2 \}$ and $f_2 := P_{B_2}Su_2$.

Proceeding in the same way, we get a subsequence $\{ u_i \}_{i=1}^{\infty}$ of $\{ x_k \}_{k=1}^{\infty}$ and sets $B_i = \{ n \in \mathbb{N} : l_i \leq n \leq m_i \}, i = 1, 2, \ldots$, where $1 = l_1 < m_1 < l_2 < m_2 < \ldots$ such that the elements $f_i := P_{B_i}Su_i, i = 1, 2, \ldots$, satisfy (9). Clearly, from (9) and the fact that $\| u_i \|_W = 1, i = 1, 2, \ldots$, it follows
\[ \| f_i \|_E \geq 1 - \delta_i, \quad i = 1, 2, \ldots \]
Prove that the constructed block basis $\{ f_i \}_{i=1}^{\infty}$ of the basis $\{ e_k \}_{k=1}^{\infty}$ is equivalent to the sequence $\{ u_i \}_{i=1}^{\infty} \subset W$.

Assume that the series $\sum_{i=1}^{\infty} c_i u_i$ converges in $W$. Since the sequence $\{ u_i \}_{i=1}^{\infty}$ consists of pairwise disjoint functions and the basis $\{ e_j \}_{j=1}^{\infty}$ is 1-unconditional, from the definition of the norm in $W$ it follows
\[
\left\| \sum_{i=1}^{\infty} c_i u_i \right\|_W = \left\| \sum_{i=1}^{\infty} c_i u_i \right\|_W = \left\| \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} |c_i| u_i \right)_j e_j \right\|_E \\
\geq \left\| \sum_{k=1}^{m_k} \sum_{j=1}^{\infty} |c_i| u_i \right\|_j e_j \left\|_E \geq \left\| \sum_{k=1}^{m_k} \sum_{j=1}^{l_k} |c_k| u_k \right\|_j e_j \right\|_E \\
= \left\| \sum_{k=1}^{\infty} |c_k| f_k \right\|_E = \left\| \sum_{k=1}^{\infty} c_k f_k \right\|_E.
\]
Conversely, assume that the series $\sum_{k=1}^{\infty} c_k f_k$ converges in $E$. Then, by (11) and (12), we have

$$\left\| \sum_{k=1}^{\infty} c_k u_k \right\|_W = \left\| \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} c_k u_k \right)_j e_j \right\|_E \leq \left\| \sum_{k=1}^{\infty} c_k \left( \sum_{j=1}^{\infty} \left\| u_k \right\|_j e_j \right) \right\|_E$$

$$\leq \left\| \sum_{k=1}^{\infty} c_k \sum_{j \in B_k} \left\| u_k \right\|_j e_j \right\|_E + \left\| \sum_{k=1}^{\infty} c_k \sum_{j \notin B_k} \left\| u_k \right\|_j e_j \right\|_E$$

$$\leq \max_{k=1,2,\ldots} \left\| c_k \sum_{k=1}^{\infty} \delta_k \right\|_E \leq \frac{\delta_0}{1 - \delta_0} \left\| \sum_{k=1}^{\infty} c_k f_k \right\|_E,$$

and the proof is completed. \qed

4. DH and DC properties of interpolation spaces and duality problem

We start with the following definitions.

**Definition 1.** [13] A Banach lattice $E$ is disjointly homogeneous (shortly DH) if two arbitrary normalized disjoint sequences from $E$ contain equivalent subsequences.

Given $1 \leq p \leq \infty$, a Banach lattice $E$ is called $p$-disjointly homogeneous (shortly $p$-DH) if each normalized disjoint sequence has a subsequence equivalent in $E$ to the unit vector basis of $l_p$ ($c_0$ when $p = \infty$).

**Definition 2.** [12] A Banach lattice $E$ is called disjointly complemented ($E \in DC$) if every disjoint sequence from $E$ has a subsequence whose span is complemented in $E$.

For examples and other information related to DH, $p$-DH and DC Banach lattices and r.i. spaces see [14, 11, 12, 13, 17, 4].

Results obtained in Section 3 allow to get a direct link between DH and DC properties of the Lions-Peetre interpolation spaces and their parameters. Next, we consider the space $E$ as a Banach lattice ordered by the basis $\{e_n\}_{n=1}^{\infty}$, i.e., if $x = \sum_{n=1}^{\infty} a_n e_n$ then $x \geq 0$ iff $a_n \geq 0$ for each $n = 1, 2, \ldots$.

**Theorem 1.** Let $1 \leq p \leq \infty$. The following conditions are equivalent:

(a) $E$ is a DH (resp. $p$-DH) lattice;

(b) every two normalized block bases of the basis $\{e_n\}_{n=1}^{\infty}$ contain equivalent in $E$ subsequences (resp. every normalized block basis of $\{e_n\}_{n=1}^{\infty}$ contains a subsequence equivalent in $E$ to the unit vector basis of $l_p$);

(c) $W := W^k_E(X_0, X_1)$ is a DH (resp. $p$-DH) space.

**Proof.** Obviously, (a) implies (b), while the implication (b) $\Rightarrow$ (c) is an immediate consequence of Proposition 2.

Now, suppose $W$ is a DH space. Let $\{M_i\}_{i=1}^{\infty}$ and $\{L_i\}_{i=1}^{\infty}$ be two sequences of pairwise disjoint non-empty subsets of $\mathbb{N}$, and let $u_i = \sum_{k \in M_i} a^i_k e_k$ and $v_i = \sum_{k \in L_i} b^i_k e_k$, where $a^i_k, b^i_k \in \mathbb{R}$, $\left\| u_i \right\|_E = \left\| v_i \right\|_E = 1$, $i = 1, 2, \ldots$. By Proposition 1 there is a sequence $(A_i)_{i=1}^{\infty}$ of pairwise disjoint measurable subsets of $[0, 1]$ such that the functions $x_i := \frac{\chi_{A_i}}{\left\| x_{A_i} \right\|_W}$, $i = 1, 2, \ldots$, form a sequence equivalent in $W$ to the basis $\{e_i\}_{i=1}^{\infty}$. Hence, the functions $y_i := \sum_{k \in M_i} a^i_k x_k$, $i = 1, 2, \ldots$ (resp. $z_i := \sum_{k \in L_i} b^i_k x_k$, $i = 1, 2, \ldots$), are pairwise disjoint, $\left\| y_i \right\|_W \asymp \left\| z_i \right\|_W \asymp 1$, $i = 1, 2, \ldots$ and the sequence $\{y_i\}$ (resp. $\{z_i\}$) is equivalent in $W$ to the sequence $\{u_i\}$ (resp. $\{v_i\}$). By the condition, the sequences $\{y_i\}$ and $\{z_i\}$ contain equivalent in $W$ subsequences. Clearly, the sequences $\{u_i\}$ and $\{v_i\}$ share the same property (in $E$) and, as a result, we get (a).
Since for the $p$-DH property the implication $(c) \Rightarrow (a)$ can be deduced in the same way, the proof is completed.

\[\square\]

The following result answers the question raised in the paper [12] (see also Question 3 in the survey [13]).

**Theorem 2.** For every $1 < p < \infty$ there exists a $p$-DH reflexive r.i. space $X_p$ on $[0,1]$ such that its dual space $X_p^*$ is not DH.

**Proof.** Let $X_0$ and $X_1$ be arbitrary r.i. spaces on $[0,1]$ such that $X_0$ is strictly embedded into $X_1$ (for example, we can put $X_0 = L_r[0,1]$, $X_1 = L_t[0,1]$ with $1 < r < q < \infty$). In [12, Theorem 6.7] (see also [13, Example 1]) it is shown that for every $1 < p < \infty$ there is a $p$-DH Banach lattice $E_p$ ordered by an unconditional basis $\{e_n\}_{n=1}^\infty$, $\|e_n\|_{E_p} = 1$, $n = 1, 2, \ldots$, such that the dual space $E_p^*$ is not DH. Clearly, without loss of generality, we may assume that the basis $\{e_n\}$ is 1-unconditional (cf. [27, p. 19]). Therefore, by Theorem 1, $X_p := W^p_{E_p}(X_0, X_1)$ is a $p$-DH r.i. space. Since neither $c_0$ nor $l_1$ is lattice embeddable in $X_p$, by Lozanovsky theorem (see [30] or [2, Theorem 4.71]), $X_p$ is reflexive.

As above, there exists a sequence $(A_k)_{k=1}^\infty$ of pairwise disjoint measurable subsets of $[0,1]$ such that the sequence $\{x_k\}_{k=1}^\infty$, where $x_k := \frac{\sum \chi_{A_k} \cdot b_k}{\|\sum \chi_{A_k} \cdot b_k\|}$, $k = 1, 2, \ldots$, is equivalent to the basis $\{e_k\}_{k=1}^\infty$. Clearly, the functions $y_k := \frac{\sum \chi_{A_k} \cdot b_k}{\sum \chi_{A_k} \cdot b_k}$, $k = 1, 2, \ldots$, form the biorthogonal system to $\{x_k\}_{k=1}^\infty$. Denoting by $e_k^*$, $k = 1, 2, \ldots$, elements of biorthogonal system to the basis $\{e_k\}_{k=1}^\infty$, we show that for all $b_k \in \mathbb{R}$ we have

\begin{equation}
C^{-1} \left\| \sum_{k=1}^{\infty} b_k e_k^* \right\|_{E_p^*} \leq \left\| \sum_{k=1}^{\infty} b_k y_k \right\|_{X_p^*} \leq C \left\| \sum_{k=1}^{\infty} b_k e_k^* \right\|_{E_p^*},
\end{equation}

where $C$ is the equivalence constant of $\{x_k\}_{k=1}^\infty$ and $\{e_k\}_{k=1}^\infty$.

Indeed, the spaces $E_p$ and $X_p$ are separable and so their duals coincide with the Köthe duals. Hence,

\[
\left\| \sum_{k=1}^{\infty} b_k y_k \right\|_{X_p^*} \geq \sup \left\{ \left\langle \sum_{k=1}^{\infty} b_k x_k, \sum_{k=1}^{\infty} a_k y_k \right\rangle : \left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{X_p} \leq 1 \right\}
\geq C^{-1} \sup \left\{ \sum_{k=1}^{\infty} a_k b_k : \left\| \sum_{k=1}^{\infty} a_k e_k \right\|_{E_p} \leq 1 \right\}
= C^{-1} \left\| \sum_{k=1}^{\infty} a_k e_k \right\|_{E_p^*},
\]

Conversely (see Section 2 or [28, Theorem 2.4]), the projection

\[
P x(t) := \sum_{k=1}^{\infty} \langle x, y_k \rangle x_k(t), \quad 0 \leq t \leq 1,
\]
is bounded in \( X_p \) with norm 1. Therefore,

\[
\| \sum_{k=1}^{\infty} b_k y_k \|_{X_p} = \sup \left\{ \langle x, \sum_{k=1}^{\infty} b_k y_k \rangle : \| x \|_{X_p} \leq 1 \right\}
\]

\[
\leq \sup \left\{ \langle Px, \sum_{k=1}^{\infty} b_k y_k \rangle : \| Px \|_{X_p} \leq 1 \right\}
\]

\[
\leq K \sup \left\{ \sum_{k=1}^{\infty} a_k b_k : \left\| \sum_{k=1}^{\infty} a_k e_k \right\|_{E_p} \leq 1 \right\}
\]

\[
= C \left\| \sum_{k=1}^{\infty} a_k e_k \right\|_{E_p},
\]

and (13) is proved. Since \( E^*_p \) is not a DH lattice and the functions \( y_k, k = 1, 2, \ldots, \) are pairwise disjoint, from (13) it follows that \( X^*_p \) is also not DH.

\[\square\]

**Theorem 3.** The following conditions are equivalent:

(a) \( E \) is a DC lattice;

(b) each block basis of the basis \( \{ e_n \}_{n=1}^{\infty} \) contains a complemented in \( E \) subsequence;

(c) \( W := W^K_E(X_0, X_1) \) is a DC space.

**Proof.** First, the implication (a) \( \Rightarrow \) (b) is obvious.

(b) \( \Rightarrow \) (c). Suppose \( \{ x_k \}_{k=1}^{\infty} \) is a sequence of pairwise disjoint functions, \( \| x_k \|_W = 1, k \in \mathbb{N} \). We need to show that it contains a subsequence spanning in \( W \) a complemented subspace.

Let \( \delta_k > 0, k = 1, 2, \ldots, \) be such that \( \sum_{k=1}^{\infty} \delta_k < 1 \) (they will be specified later). Thanks to Proposition 2 and its proof, we can assume that there are sets \( B_k = \{ n \in \mathbb{N} : l_k \leq n \leq m_k \}, k = 1, 2, \ldots, \) where \( 1 = l_1 \leq m_1 < l_2 \leq m_2 < \ldots, \) such that \( \{ x_k \}_{k=1}^{\infty} \) is equivalent to the sequence \( \{ f_k \}_{k=1}^{\infty}, f_k = P_{B_k} S x_k, \) i.e., for some constant \( C > 0 \) and all \( c_k \in \mathbb{R} \)

\[
C^{-1} \left\| \sum_{k=1}^{\infty} c_k f_k \right\|_E \leq \left\| \sum_{k=1}^{\infty} c_k x_k \right\|_W \leq C \left\| \sum_{k=1}^{\infty} c_k f_k \right\|_E.
\]

Moreover, as above, we have

\[
\| S x_k - f_k \|_E < \delta_k \quad \text{and} \quad \| f_k \|_E \geq 1 - \delta_k, \quad k = 1, 2, \ldots
\]

Now, by [2, Theorem 1.25], for every \( k \in \mathbb{N} \) we can find a linear operator \( L_k : W \rightarrow E \) such that \( L_k x_k = S x_k \) and \( |L_k x| \leq S x \) for all \( x \in W \). Define on \( W \) the linear operator \( L \) by

\[
Lx := \sum_{i=1}^{\infty} P_{B_i} L_i x.
\]

Since the sets \( B_i, i = 1, 2, \ldots, \) are pairwise disjoint, then

\[
|Lx| \leq \sup_{i \in \mathbb{N}} |L_i x| \leq S x, \quad x \in W;
\]

whence \( L \) is bounded from \( W \) into \( E \). Moreover, setting \( u_k := L x_k, k = 1, 2, \ldots, \) we have

\[
u_k = P_{B_k} S x_k + P_{\mathbb{N} \setminus B_k} L x_k = f_k + P_{\mathbb{N} \setminus B_k} L x_k.
\]

Therefore, from pointwise estimate (16) and the first inequality in (15) it follows

\[
\| u_k - f_k \|_E = \| P_{\mathbb{N} \setminus B_k} L x_k \|_E \leq \| P_{\mathbb{N} \setminus B_k} S x_k \|_E = \| S x_k - f_k \|_E < \delta_k, \quad k = 1, 2, \ldots
\]
Combining this together with the second inequality in \((15)\), we get

\[
\sum_{k=1}^{\infty} \frac{\|u_k - f_k\|_E}{\|f_k\|_E} \leq \sum_{k=1}^{\infty} \frac{\delta_k}{1 - \delta_k}.
\]

Since the sequence \(\{f_k\}_{k=1}^{\infty}\) is a block basis of \(\{e_k\}_{k=1}^{\infty}\), by hypothesis, it contains a subsequence \(\{f_{k_i}\}\) complemented in \(E\). Therefore, if \(\delta_k > 0\) are sufficiently small, then, by the principle of small perturbations \([27, \text{Proposition 1.a.9}]\), from \((17)\) it follows that the sequence \(\{u_{k_i}\}\) is equivalent to the sequence \(\{f_{k_i}\}\), i.e.,

\[
K^{-1} \left\| \sum_{i=1}^{\infty} c_i f_{k_i} \right\|_E \leq \left\| \sum_{i=1}^{\infty} c_i u_{k_i} \right\|_E \leq K \left\| \sum_{i=1}^{\infty} c_i f_{k_i} \right\|_E,
\]

and is also complemented in \(E\). Moreover, since \(E\) is a separable space, then there is a bounded projection \(Q : E \to E\) of the form:

\[
Q x := \sum_{i=1}^{\infty} \langle x, v_i \rangle u_{k_i},
\]

where \(v_i \in E', \langle u_{k_i}, v_i \rangle = 1\) and \(\langle u_{k_i}, v_j \rangle = 0\) if \(i \neq j\).

Let us show that the linear operator

\[
Tx := \sum_{i=1}^{\infty} \langle Lx, v_i \rangle x_{k_i}
\]

is a bounded projection in \(W\). In fact, applying successively inequalities \((14), (18)\) and \((16)\), for any \(x \in W\) we obtain

\[
\|Tx\|_W \leq C \left\| \sum_{i=1}^{\infty} \langle Lx, v_i \rangle f_{k_i} \right\|_E \leq CK \left\| \sum_{i=1}^{\infty} \langle Lx, v_i \rangle u_{k_i} \right\|_E \leq CK \left\| Q(Lx) \right\|_E \leq CK \left\| Q \right\| \|Lx\|_E \leq CK \left\| Q \right\| \|Sx\|_E = CK \left\| Q \right\| \|x\|_W.
\]

Moreover, for every \(i = 1, 2, \ldots\)

\[
T x_{k_i} = \sum_{i=1}^{\infty} \langle Lx_{k_i}, v_i \rangle x_{k_i} = \sum_{i=1}^{\infty} \langle u_{k_i}, v_i \rangle x_{k_i} = x_{k_i}.
\]

Therefore, \([x_{k_i}]\) as the image of the bounded projection \(T\) is a complemented subspace of \(W\), and thus the proof of the implication \((b) \Rightarrow (c)\) is complete.

\((c) \Rightarrow (a)\). Again, by Proposition \(12\) there is a sequence \(\{A_i\}_{i=1}^{\infty}\) of pairwise disjoint measurable subsets of \([0, 1]\) such that the sequence \(\left\{ \frac{x_{A_i}}{\|x_{A_i}\|_W} \right\}_{i=1}^{\infty}\) is equivalent in \(W\) to the basis \(\{e_i\}_{i=1}^{\infty}\) of \(E\). Setting \(y_i := \frac{x_{A_i}}{\|x_{A_i}\|_W}\), we see that the linear operator \(H : E \to W\) defined by \(H(e_i) := y_i, i = 1, 2, \ldots\), is an isomorphic embedding of \(E\) into \(W\) with the image \([y_i]\).

Let \(M_i, i = 1, 2, \ldots\), be a sequence of pairwise disjoint subsets of \(\mathbb{N}\), \(M_i \neq \emptyset, i = 1, 2, \ldots\) and let \(w_i = \sum_{k \in M_i} a_k^i e_k\), where \(a_k^i \in \mathbb{R}, k \in M_i, i = 1, 2, \ldots\) It is sufficient to find a subsequence \(\{w_{i_j}\} \subset \{w_i\}\), which is complemented in \(E\).

Clearly, the functions \(g_i := \sum_{k \in M_i} a_k^i y_k, i = 1, 2, \ldots\), are pairwise disjoint and the sequence \(\{g_i\}\) is equivalent in \(W\) to the sequence \(\{w_i\}\). By hypothesis, there is a subsequence \(\{g_{i_j}\} \subset \{g_i\}\), which is complemented in \(W\). Let \(P\) be a bounded projection in \(W\) whose image is \([g_{i_j}]\). Now, the operator \(Q := H^{-1}PH\) is bounded in \(E\) and it is
easy to check that \( Q \) is a projection whose image coincides with \([w_{ij}]\). Thus, \([w_{ij}]\) is a complemented subspace in \( E \), and the theorem is proved. \( \square \)

Let \( 1 < p < \infty \). It is well known that there are Orlicz sequence spaces, which contain a block basis equivalent to the unit vector basis of \( l_p \) but do not have any complemented subspace isomorphic to \( l_p \) \([27]\) Examples 4.c.6 and 4.c.7]. (see also \([25]\) and \([26]\)). Combining this together with the implication \((c) \Rightarrow (a)\) of Theorem \(3\) we can construct r.i. spaces without the DC property.

**Corollary 1.** There exists an Orlicz sequence space \( l_N \) such that the space \( W^K_{l_N}(X_0, X_1) \) is not DC.

Taking for \( E \) a DH (resp. DC) Banach lattice ordered by a normalized 1-unconditional basis, according to Theorem \(1\) (resp. Theorem \(3\)), we get a DH (resp. DC) r.i. space \( W^K_E(X_0, X_1) \) whenever r.i. spaces \( X_0 \) and \( X_1 \) are such that \( X_0 \subset X_1 \) and this embedding is strict. In particular, we have

**Corollary 2.** Let \( 1 \leq p \leq \infty \). Then, \( W^K_{l_p}(X_0, X_1) \) (\( W^K_0(X_0, X_1) \) if \( p = \infty \)) is a \( p \)-DH and DC r.i. space.

Another interesting class of parameters enjoying DH and DC properties is formed by separable Banach sequence spaces having the so-called (RSP) and (LSP) properties.

A given sequence \( x = (x(k))_{k=1}^\infty \) of reals, the support of \( x \) is the set \( \text{supp } x := \{ k \in \mathbb{N} : x(k) \neq 0 \} \). If \( A \subset \mathbb{N} \) and \( B \subset \mathbb{N} \), then the inequality \( A < B \) means that \( a < b \) for arbitrary \( a \in A, b \in B \). Let \( \{x_n\}_{n=1}^m \) and \( \{y_n\}_{n=1}^m \) be two families of sequences. The pair \( (x_n, y_n)_{n=1}^m \) is interlaced if supports of the sequences \( x_n, y_n, n = 1, 2, \ldots, m \), are finite and also

\[
\text{supp } x_n \subset \text{supp } y_n \ (1 \leq n \leq m), \quad \text{supp } y_n \subset \text{supp } x_{n+1} \ (1 \leq n \leq m - 1).
\]

We say that a Banach sequence space \( E \) has the right-shift property (RSP) if there exists \( C_{RS} > 0 \) such that for any interlaced pair \( (x_n, y_n)_{n=1}^m \) with \( \|y_n\|_E \leq \|x_n\|_E = 1, \ 1 \leq n \leq m \), and for all \( a_n \in \mathbb{R} \) we have

\[
\left\| \sum_{n=1}^m a_n y_n \right\|_E \leq C_{RS} \left\| \sum_{n=1}^m a_n x_n \right\|_E.
\]

Analogously, \( E \) has the left-shift property (LSP) if for some \( C_{LS} > 0 \), any interlaced pair \( (x_n, y_n)_{n=1}^m \), \( \|x_n\|_E \leq \|y_n\|_E = 1 \ (1 \leq n \leq m) \) and all \( a_n \in \mathbb{R} \)

\[
\left\| \sum_{n=1}^m a_n x_n \right\|_E \leq C_{LS} \left\| \sum_{n=1}^m a_n y_n \right\|_E.
\]

**Theorem 4.** (i) If the space \( E \) has at least one of the properties (RSP) or (LSP), then \( W := W^K_E(X_0, X_1) \) is a DH space.

(ii) If \( E \) has both properties (RSP) and (LSP), then \( W \) is a DC space.

**Proof.** (i) Assume that the space \( E \) possesses the (RSP)-property. According to Theorem \(1\), it is sufficient to check that every two normalized block bases \( \{f_k\}_{k=1}^\infty \) and \( \{g_k\}_{k=1}^\infty \) of \( \{e_n\}_{n=1}^\infty \) contain subsequences equivalent in \( E \). Since supports of the elements \( f_k \) and \( g_k \) are finite, we can find subsequences \( \{f'_i\}_{i=1}^\infty \subset \{f_k\} \) and \( \{g'_i\}_{i=1}^\infty \subset \{g_k\} \) such that the pair \( \{f'_i, g'_i\}_{i=1}^m \) is interlaced for each \( m = 1, 2, \ldots \). Therefore, since \( \|f'_i\|_E = \|g'_i\|_E = 1, \ i \in \mathbb{N} \), then applying the (RSP)-property of \( E \), for all \( a_i \in \mathbb{R} \) we have

\[
\left\| \sum_{i=1}^m a_i f'_i \right\|_E \leq C_{RS} \left\| \sum_{i=1}^m a_i g'_i \right\|_E.
\]
Hence, if the series $\sum_{i=1}^{\infty} a_i g'_i$ converges in $E$, we get

$$\left\| \sum_{i=1}^{\infty} a_i f'_i \right\|_E \leq C_{RS} \left\| \sum_{i=1}^{\infty} a_i g'_i \right\|_E.$$  

Similarly, there are further subsequences $\{f''_j\}_{j=1}^{\infty} \subseteq \{f'_j\}_{j=1}^{\infty}$ and $\{g''_j\}_{j=1}^{\infty} \subseteq \{g'_j\}_{j=1}^{\infty}$ such that the pair $(g''_j, f''_j)_{i=1}^m$ is interlaced for each $m = 1, 2, \ldots$. Hence, in the same way as above, we infer

$$\left\| \sum_{i=1}^{\infty} a_i g''_i \right\|_E \leq C_{RS} \left\| \sum_{i=1}^{\infty} a_i f''_i \right\|_E.$$  

Thus, $\{f''_j\}_{j=1}^{\infty}$ and $\{g''_j\}_{j=1}^{\infty}$ are equivalent in $E$. In the case when $E$ has (LSP), the proof follows by the same lines.

(ii). If the space $E$ has both the properties (RSP) and (LSP), then from [20, Lemma 2.6 and subsequent remark] it follows that each block basis of $\{e_n\}_{n=1}^{\infty}$ is complemented in $E$. Therefore, applying Theorem 3, we arrive at the desired result. \hfill $\square$

It is well known that Tsirelson’s space $T$ possesses the properties (RSP) and (LSP) (see [9] and [10]). Since $T$ does not contain subspaces isomorphic to $l_p$, $1 \leq p < \infty$ or $c_0$, from Proposition 2 and Theorem 3 we obtain

**Corollary 3.** $W^K_T(X_0, X_1)$ is a DH and DC r.i. space, in which no sequence of pairwise disjoint functions is equivalent to the unit vector basis of $l_p$, $1 \leq p < \infty$ or $c_0$. In particular, $W^K_T(X_0, X_1)$ is not a p-DH space for each $1 \leq p \leq \infty$.

A more special choice of the spaces $X_0$ and $X_1$ allows to construct a DH and DC r.i. space, which is even not isomorphic to any $p$-DH space (cf. [33]).

**Corollary 4.** Let $X_0$ and $X_1$ be r.i. spaces such that $X_0 \subset X_1$ and this embedding is strict. Moreover, suppose that $X_1 \not\supset (ExpL^2)_0$, where $(ExpL^2)_0$ is the separable part of the Orlicz space $ExpL^2$ generated by the function $e^{u^2} - 1$. Then, $W^K_T(X_0, X_1)$ is a DH and DC r.i. space, which contains no subspace isomorphic to $l_p$, $1 \leq p < \infty$, or $c_0$. In particular, the space $W^K_T(X_0, X_1)$ is not isomorphic to a $p$-DH Banach lattice for every $p \in [1, \infty]$.

**Proof.** Put $W := W^K_T(X_0, X_1)$. At first, we observe that the norms of $W$ and $L_1$ are not equivalent on any infinite dimensional subspace of $W$. Indeed, since $W \subset X_1$, then from the condition it follows that $W \not\supset (ExpL^2)_0$. Therefore, assuming the contrary, by [3, Theorem 2], we can find a sequence $\{x_k\}$ of pairwise disjoint functions from $W$ such that the norms of $W$ and $L_1$ are equivalent on the subspace $[x_k]$. Since $W$ is a r.i. space, $W \neq L_1$, this is a contradiction [32] (see also [3, Corollary 3]).

Now, let $Y$ be an arbitrary infinite dimensional subspace of $W$. According to the preceding observation, by Kadec-Pelczynski alternative [19], $Y$ contains a sequence, which is equivalent in $W$ to some sequence of pairwise disjoint functions. Clearly (see Corollary 3), the latter sequence cannot be equivalent to the unit vector basis of $l_p$, $1 \leq p < \infty$, or $c_0$. Hence, $Y$ is not isomorphic to $l_p$ for any $1 \leq p < \infty$ or $c_0$. As an immediate consequence, we deduce that $W$ is not isomorphic to a $p$-DH Banach lattice for each $p \in [1, \infty]$.

$\square$

5. A CHARACTERIZATION OF $L_p$-SPACES ON $(0, \infty)$ VIA DH PROPERTIES

In this concluding part of the paper, we turn to the case of r.i. spaces on $(0, \infty)$. Let $1 < p < \infty$. In Theorem 5.2 of the paper [12] one can find an example of a $p$-DH Orlicz
space $L_M(0, \infty)$ whose dual is not DH. In fact, known results on subspaces generated by translations in r.i. spaces due to Hernandez and Semenov [15] easily imply the following characterization of $L_p$-spaces.

**Theorem 5.** Let $X$ be a reflexive r.i. space on $(0,\infty)$. The following conditions are equivalent:

(a) $X$ and $X^*$ are DH;
(b) $X$ and $X^*$ are DC;
(c) $X = L_p(0, \infty)$ for some $1 < p < \infty$.

**Proof.** Since $X$ is reflexive, the implication $(a) \Rightarrow (b)$ is an immediate consequence of [12, Proposition 4.10].

$(b) \Rightarrow (c)$. Let $a \in X$, $a \neq 0$, supp $a \subset [0,1]$. Define the sequence of translations of the function $a$: $\tau_n a(t) = a(t - n + 1)$ if $t \in [n - 1, n)$ and $\tau_n a(t) = 0$ if $t \not\in [n - 1, n)$, $n = 1,2,\ldots$ We prove that the sequence $\{\tau_n a\}$ is complemented in $X$.

By hypothesis, there is subsequence $\{\tau_{n_k} a\} \subset \{\tau_n a\}$, which is complemented in $X$. Since $X$ is separable, the dual space $E^*$ coincides with the Köthe dual. Hence, there is a sequence $\{b_k\} \subset E^*$ such that supp $b_k \subset [n_k - 1, n_k]$, $\int_{n_k - 1}^{n_k} b_k \tau_{n_k} a \, ds = 1$ and the projection

$$P x(t) := \sum_{k=1}^{\infty} \int_{n_k - 1}^{n_k} b_k(s)x(s) \, ds \cdot \tau_{n_k} a(t)$$

is bounded in $X$. Let us define the isometric embedding $R : X \rightarrow X$ by

$$Rx(s) := \sum_{k=1}^{\infty} x(s + k - n_k)\chi_{[n_k - 1, n_k]}(s), \ s > 0.$$ 

Moreover, we put $c_k(s) := b_k(s - k + n_k)$, $k = 1, 2,\ldots$, and

$$Q x(t) := \sum_{k=1}^{\infty} \int_{k-1}^{k} c_k(s)x(s) \, ds \cdot \tau_k a(t).$$

Clearly, supp $c_k \subset [k-1,k]$, $k = 1,2,\ldots$, and

$$Q x(t) = \sum_{k=1}^{\infty} \int_{n_k - 1}^{n_k} b_k(s)R x(s) \, ds \cdot \tau_{n_k} a(t).$$

Therefore, the functions $Q x$ and $PR x$ are equimeasurable and so

$$\|Q x\|_X = \|PR x\|_X \leq \|P\|\|Rx\|_X = \|P\|\|x\|_X.$$ 

We have also $R \tau_k a = \tau_{n_k} a$ for all $k = 1, 2,\ldots$, whence $Q \tau_k a = \tau_{n_k} a, k = 1, 2,\ldots$ Thus, $Q$ is a bounded projection whose image coincides with the subspace $[\tau_k a]$, and our claim is proved.

Similarly, we can prove that the sequence $\{\tau_k a^*\}$ is complemented in $X^*$ for each $a^* \in X^*$, $a^* \neq 0$, supp $a^* \subset [0,1]$. Combining this together with Theorem 5.4 from the paper [15] (see also [16, Theorem 7]), we arrive to $(c)$.

Since the implication $(c) \Rightarrow (a)$ is obvious, the proof is completed. 

**Corollary 5.** If $X$ is a reflexive r.i. space on $(0,\infty)$ such that $X \neq L_p$ for every $1 < p < \infty$, then at least one of the spaces $X$ and $X^*$ is not DH.

**Remark 1.** An inspection of the proof of Theorem 3 shows that DH (resp. DC) properties of spaces $X$ and $X^*$ are used not to the full extent of their power. Indeed, it is sufficient that one can select "nice" subsequences only from positive integer-valued translations of
functions with support in $[0, 1]$. In this regard, it is worth to note that just the existence of such a class of sequences of equimeasurable pairwise disjoint functions in the case of infinite measure is a root of the difference of DH and DC properties of r.i. spaces on $(0, \infty)$ and $[0, 1]$.

**Remark 2.** For reflexive Banach lattices ordered by subsymmetric normalized bases there is an analogue of Theorem [2] (see [12] Proposition 6.9).

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(Sergey V. Astashkin) Department of Mathematics, Samara National Research University, Moskovskoye shosse 34, 443086, Samara, Russia

E-mail address: astash56@mail.ru