Gravitational Waves with Orbital Angular Momentum

Pratysuava Baral, Anarya Ray, and Ratna Koley

Department of Physics, Presidency University, Kolkata 700073, India.

Parthasarathi Majumdar

Ramkrishna Mission Vivekananda Educational and Research Institute, Belur Math 711202
and Indian Association for the Cultivation of Science, Kolkata 700032, India

Compact orbiting binaries like the black hole binary system observed in GW150914 carry large amount of orbital angular momentum. The post-ringdown compact object formed after merger of such a binary configuration has a small spin, and this results in a large orbital angular momentum excess. One significant possibility is that the gravitational waves generated by the system carry away this excess orbital angular momentum. An estimate of this excess is made within a primarily Newtonian gravity framework. Arguing that plane gravitational waves cannot possibly carry any orbital angular momentum, a case is made in this paper for gravitational wave beams carrying orbital angular momentum, akin to optical beams. Restricting to certain specific beam-configurations, we predict that such beams may produce a shear strain, in addition to the longitudinal strains measured at aLIGO for GW150914 and GW170817. Current constraints on post-ringdown spins, derived within the plane-wave approximation of gravitational waves, therefore stand to improve. The minimal modification that might be needed on a laser-interferometer detector (like aLIGO or VIRGO) to detect such shear strains is also briefly discussed.

INTRODUCTION

Gravitational waves (GWs) detected by the Advanced Laser Interferometer Gravitational Wave Observatory (aLIGO) have established the existence of inspiralling compact object binaries. Within general relativity (GR), such systems radiate gravitational waves, carrying energy and angular momenta (AM), while spiraling into each other. The amount of AM carried, as viewed by an observer at infinity (assuming the spacetime to be asymptotically flat) can be estimated by the difference in angular momentum of the initial and final stages of a merger. GW150914 confirmed the merger of two black holes separated by a radius of 210 km and of masses around $36M_\odot$ and $29M_\odot$, forming a resultant Kerr black hole of mass $\sim 62M_\odot$ and a low spin parameter of $\sim 0.67$. Although much is unknown about the formation of two such compact objects at such a low distance, for simplicity of our estimation we assume a radius of separation of 2000 km ($10R_s$) to be a valid stage in the evolution of the binary system, as a lower radius would make the system too relativistic for a Newtonian analysis. Such a radius gives rise to a Keplerian frequency of $200$ Hz and thus an orbital angular momentum (OAM) $\sim M m^2 \omega \sim 10^8 \frac{GM\omega}{c^2}$. The spin angular momentum (SAM) of the compact objects in binary, though negligible in this stage only increases the number. The total AM of the post ringdown object (assuming to be a Kerr Black-Hole) as measured by aLIGO, is less by two orders of magnitude. This implies that the excess AM must be carried away by GWs. A similar estimate about a high rate of AM radiation has been also given recently by Bialynicki-Birula et al confirming GWs should carry AM.

By measuring spacetime fluctuations as a function of time only, measurements at aLIGO have successfully constrained the masses and spins of the initial and final compact objects. From these estimated parameters it is possible to infer about the rate at which AM is radiated by the system. This has presumably been included already in the design of templates employed in the aLIGO experiment, and is not the focus in this paper. Here, our aim is to investigate possible evidence of such orbital AM excesses in laser-interferometric detection of GWs, either on earth or in space.

The motivation of directly measuring angular momentum from GWs is plenty. A direct measurement of AM would provide us with an estimate of its rate of loss from the inspiralling binary. This, in turn, might allow us to impose additional constraints on the parameters over and above those obtained by cross-correlation with various templates. This would further enable us to settle many controversies relating to various mergers like GW170817 (NS-NS merger) which are expected to be routine in the near future. Using this, we may also expect to put constraints on the exotic alternative compact objects like fuzz balls, gravastars, wormholes, boson stars and so on. Comparing how well the estimated angular momentum loss of the system compares to the angular momentum carried by GW’s as detected by a faraway observer, additional restrictions on the validity of GR in the linearized regime may perhaps be ascertained. Lastly, the third-generation run of the aLIGO and VIRGO is expected to detect certain gravitational lensing events. An additional probe of AM is expected to give us additional knowledge of the medium through which it passes. This may vastly improve our understanding of interactions of GWs with matter as
it passes through astrophysical objects such as stars or galactic clusters. However for the time-being, we focus on a direct independent study of angular momentum carried by GWs.

In this paper we examine the analytic structure of GWs in the linearized regime, starting from the gauge fixed linearized vacuum Einstein equation. We further demand that the wave-like solutions carry orbital angular momentum. This naturally enables us to go beyond plane waves and discuss GW beams. We show en passant that plain waves cannot carry OAM, implying that recourse to GW beams is mandatory. Then we choose a particular set of linearly independent beams which form a basis for GWs with orbital AM (OAM). A brief discussion is presented on what effects these beams would have on spacetime. We also give a schematic outline, how these beams carrying OAM may be detected and the contribution of the beam to the overall signal measured in a generic Laser-interferometer GW detector.

**GRAVITATIONAL WAVE BEAMS**

We employ here the linearized *tetrad* formalism for discussing GWs, for two reasons : the ease to discuss fermionic interactions in astrophysically relevant quantum field theoretic analyses, and to understand better the transition from local Lorentz invariance to global Lorentz symmetry under linearization - a phenomenon which remains slightly obscure within the metric formalism. However, the prescription to change to metric computations is included, for the ease of the readers.

For the purpose of linearization, the spacetime tetrad components $e^a_\mu$ are decomposed as $e^a_\mu = \tilde{e}^a_\mu + \varepsilon^a_\mu$, where $\tilde{e}^a_\mu$ is the background Minkowski spacetime tetrad and $\varepsilon^a_\mu$ is the linear fluctuation. Linearized gravity in the harmonic gauge using this perturbed tetrad can be expressed as, $\Box \varepsilon^a_\mu + \varepsilon^a_\mu = 0$ where Greek indices specify spacetime labels and early Latin indices are tangent space labels. The late Latin indices are reserved for three dimensional space. Therefore, the metric and fluctuations turns out to be

$$\eta_{\mu \nu} = \eta_{ab} \tilde{e}^a_\mu \tilde{e}^b_\nu$$ (1)

$$h_{\mu \nu} = \tilde{e}^a_\mu \varepsilon_{a \nu} + \tilde{e}^a_\nu \varepsilon_{a \mu} + O(\varepsilon^2)$$ (2)

Eq. (2) explicitly states how to transform from the metric fluctuations to the tetrad fluctuations and vice-versa.

The gauge fixed linearized tetrad equation admits a wave like solution given by,

$$\varepsilon_{a \mu} = \partial_{a \mu}(x^\sigma) \exp(ik_\chi x^\chi) + \partial^*_{a \mu}(x^\sigma) \exp(-ik_\chi x^\chi)$$ (3)

with the * representing complex conjugation. This solution imposed on linearized tetrad equation gives

$$\left( \partial^\nu \partial_\nu + 2ik_\nu \partial^\nu \right) \varepsilon_{a \mu}(x) = 0 .$$ (4)

The Lagrange density and the energy-momentum tensor for linearized gravity \[18\] are

$$\mathcal{L} = -\frac{c^4}{32\pi G} (\partial^\mu \varepsilon_{a \sigma} \partial_{\mu} \varepsilon^{a \sigma} + \tilde{\varepsilon}^a_\sigma \tilde{\varepsilon}^b_\rho \partial^\mu \varepsilon_{a \sigma} \partial^r \varepsilon_{b \rho} \varepsilon_{r b})$$ (5)

$$T^{\mu \nu} = \frac{c^4}{16\pi G} (\partial^\rho \varepsilon_{a \sigma} \partial^{\nu} \varepsilon^{a \sigma} + \tilde{\varepsilon}^a_\sigma \tilde{\varepsilon}^b_\rho \partial^\mu \varepsilon_{a \sigma} \partial^r \varepsilon_{b \rho} \varepsilon_{r b})$$ (6)

Since we are dealing with small fluctuations around a Minkowski spacetime tetrad in the linearized region, the system is globally Lorentz-symmetric. The conserved Noether charge density corresponding to this symmetry can be expressed as,

$$\frac{8\pi G}{c^3} M_{\mu \sigma} = \dot{\varepsilon}_{a \mu} [x_{a \mu} (\partial_{a \sigma} \varepsilon^{a \mu} + \tilde{\varepsilon}^a_\rho \tilde{\varepsilon}^b_\sigma \partial_{a \rho} \varepsilon_{b \sigma})]$$ (7)

where, dot(.) represents time derivative, and the square brackets denote anti-symmetry. Integrating this charge density over all space gives the infinitesimal Lorentz generators.

If $\vartheta_{a \mu}$ is not a function of the spatial coordinates $(x^1)$ on integration the first term reduces to terms like $\int (x^\alpha k^3 - x^3 k^\alpha) d^3x$ which is 0 as $d^3x$ is a rotationally invariant measure and $x^\alpha$ is a four vector component. This implies that a constant polarization, like plane waves (which definitely satisfy equation (4)) cannot carry or give information about orbital angular momentum. It follows that for GWs to carry OAM, their polarization tensors must themselves be tensor fields. One way to enable polarization fields in GWs is through GW beams, akin to optical beams.

**Laguerre-Gaussian (LG) Beams**

Let $z$ direction be the direction of propagation of the GW beam. Since any conceivable detector has to be placed far away from the sources, we can safely assume the beam to be paraxial ($k_z \approx k$) \[21\]. So equation (3) can be expressed as

$$\nabla^2 T^2 + \partial^2 \partial_z - \partial^i \partial_i + 2ik_z \partial^2 - 2ik_i \partial^i \partial_{a \mu}(x) = 0$$ (8)

where $\nabla^2 T^2$ is a two dimensional Laplace operator in the plane perpendicular to z. The paraxial approximation also guarantees that the change in polarization tensor in the direction of propagation is negligible in comparison to the wave vector $\left( \frac{\partial^2 \vartheta_{a \mu}}{\partial z^2} \right) \ll \left| k_z \frac{\partial \vartheta_{a \mu}}{\partial z} \right|$. So equation (8) reduces to, $\left( \nabla^2 T^2 + 2ik_z \partial^2 \right) \vartheta_{a \mu}(x) = 0$. We choose the transverse plane to be spanned by $(r, \phi)$ then $\nabla^2 T^2 = \frac{1}{r} \partial_r \partial_r + \frac{1}{r^2} \partial^2 \phi^2$. For simplicity, we choose to work with one particular component of $\vartheta_{a \mu}$ and for the time being we drop the spacetime and tangent space indices. A solution of the form
\[ \partial_{m\!p}(r, \phi, z) = \frac{A_{m\!p}}{w(z)} \left( \frac{\sqrt{2}r}{w(z)} \right)^{|m|} \exp \left[ \frac{-ikr^2z}{2(z^2 + z_R^2)} \right] L_p^{|m|} \left( \frac{2r^2}{w^2(z)} \right) \exp \left[ \frac{im\phi - i(2p + |m| + 1) \tan^{-1} \frac{z}{z_R}}{z_R} \right] \times \exp \left( \frac{-r^2}{w^2(z)} \right) \]

satisfies equation \( \Box \) in its paraxial form with \( m, p \) taking integer values referring to various modes. The radius of the beam \( w(z) \) is given by \( w(z) = w(0) \sqrt{1 + \left( \frac{z}{z_R} \right)^2} \) where \( z_R = \frac{w(0)}{k} \) and \( k = \frac{2}{\lambda} \). \( L_p^{|m|}(x) \) is the associated Laguerre polynomial while \( A_{m\!p} \) is a normalization constant. By definition of Laguerre polynomials, \( p \) has to be an integer. The single valuedness of the field under a rotation of \( \pi \) radians forces the azimuthally dependent phase factor to be quantized with \( m, p \) taking only integral values. The LG modes are orthonormal in both labels \((m & p) \) i.e. \( \int_0^{2\pi} \! d\phi \int_0^\infty \! rdr \partial_{m\!p}(r, \phi, z)[\partial_{m\!p}(r, \phi, z)]^* = \delta_{m\!m}\!\delta_{p\!p}. \) This guarantees that the LG modes form a complete orthonormal family which can be used as a basis for a beam with an arbitrary polarization distribution. These beams exhibit a symmetry manifest in the use of cylindrical coordinates. Any other choice of a complete set of beams a different symmetry (like, e.g., the Bessel beam) is equally valid; these beams can of course be expressed as a linear combination of LG modes.

The OAM density can be expressed in a simple form which follows directly from equation \( \Box \),

\[
\tilde{L} = \int d^3x \left[ -\frac{l}{\omega} \frac{z}{r^2} |\partial_{m\!p}|^2 \hat{r} + \frac{r}{\epsilon} \left( \frac{\tilde{z}^2}{z^2 + z_R^2} - 1 \right) |\partial_{m\!p}|^2 \hat{z} \right]
+ \frac{l}{\omega} |\partial_{m\!p}|^2 \hat{z} \right]
\]

Such solutions also exist in the case of electromagnetic waves and have been extensively studied in refs. [19]-[22]. Bylinski-Birula et al. [23] have also discussed GW beams using an electromagnetic-gravitational correspondence within a spinorial formalism. In this work we take an approach that appears to be more convenient for phenomenological applications.

Integrating the OAM density over all space, we get the total angular momentum. As stated earlier our analysis is only valid in the weak field regime. So, the integration domain must be restricted to that regime. Here we should mention the caveat that unlike in laser optics, \( w(0) \) has no significance, as our analysis is not strictly valid at \( z = 0 \). If a model of sources is constructed, in which GWs are allowed to take away angular momentum, then in the limit \( z \rightarrow \infty \) we recover our linearized analysis. From such a model, one can extract which LG modes are present in the strong gravity regime, near the source. For a study of the effects on available laser-interferometer detectors, an analysis of the wave in the linearized regime suffices.

**EFFECT OF A PASSING GW BEAM ON SPACETIME : POSSIBLE DETECTION**

We first describe the effect on spacetime in the plane perpendicular to the direction of propagation. Let this transverse plane be spanned by coordinates \( x, y \). It is well known that in standard TT gauge all perturbation except \( h_+ = \epsilon_x e^{i \phi} + \epsilon_x e^{i \phi} \) and \( h_\times = \epsilon_x e^{i \phi} + \epsilon_x e^{i \phi} \) can be gauged away to 0. To start with let us consider a GW beam consisting of only the component \( h_+ \). The corresponding tetrad fluctuation \( \epsilon_{\mu\nu} = \epsilon e^{i \phi} \) contains an infinite number of various LG modes. The proper distance between four test particles localized at \( A(0,0), B(L,0), C(L, L) \) and \( D(0, L) \) would change due to the passing GW. Firstly let us consider motion along or parallel to \( x \)-axis. Therefore, \( dt^2 = dy^2 = dz^2 = 0. \)

\[
dx^2 = (1 + 2\epsilon_x e^{i \phi})dx^2
\Rightarrow \Delta l \approx \int_0^l \epsilon_x e^{i \phi} dx \approx f(y)
\]

where, \( f(y) \) is an arbitrary function of \( y \).

If the polarization tensor does not depend on \( y \)-coordinate as in the case of plane waves we have \( AB = CD \). The differential elongation along \( x \) for various values of \( y \) creates a shear in spacetime. The same is true for motion along \( y \). As the polarization is a function of the transverse spatial coordinates, the change in spacetime can be decomposed without loss of generality into a longitudinal as well as a shear strain. The existence of a shear strain over and above the longitudinal strain is the hallmark of GWs carrying OAM.

Note that the spatial part energy-momentum tensor or the stress tensor of GWs carrying OAM contains diagonal as well as off-diagonal terms. The diagonal terms correspond to longitudinal strain while off-diagonal terms give the shearing strain. For plane waves, one also gets off-diagonal terms, but they vanish upon integration over the entire area.

If a GW of constant polarization passes over a circular ring of particles (shown by bold green dots), they would change to an elliptical ring (shown by red dots) as shown in Fig.1. The presence of a LG mode would deviate the masses from their expected places (shown by blue crosses) due to the shear stress. The deviation from an ellipse for the lowest mode is plotted in Fig.2. The symmetry in the figure is precisely due to the symmetry in \( x \) and \( y \) coordinates for \( m = 0 \) mode. Since each \( h_+ \) and \( h_\times \) will contain an infinite number of LG modes, instead
we get a smear of infinitely many polarization states.

\[
\delta \phi(L) = \frac{2\pi(L_0 - \tau_y)}{L_{\text{light}}} = f(L) \approx \alpha + \beta L + \gamma L^2
\]

where \(\alpha, \beta\) and \(\gamma\) are constants. For plane waves \(\gamma = 0\). Figure (3) shows how \(\delta \phi(L)\) depends on \(L\) when \(m = p = 0\). It clearly deviates from the linear nature shown by the blue dots. The value of 4 km is chosen for our numerical analysis. It has no special significance and any other choice is equally valid. The ratio does not depend on the value of \(w(0)\) (in the range \(10^{-2} - 10^{-8}\)) in this case. For \(m = 1\), the nature of the plot remains similar to that of figure (3), although it is now sensitive to \(w(0)\). The strains definitely depend on the value of \(w(0)\) (as well as \(L\)) as shown in figure (3). As is clear from the graph, typical strain values predicted for a binary black hole merger source as reported for GW150914, is about \(\sim 10^{-21}\) which is very much detectable by aLIGO type detectors. If \(m = 1\) we have a \((1/2)\) factor suppression of the GW signal, leading to shear strain values that are far smaller than current sensitivities.

\[c_{dx} = dx \sqrt{1 + 2\varepsilon^2 \varepsilon_{ax}|y=0} \approx dx(1 + \varepsilon^2 \varepsilon_{ax}|y=0)\]

FIG. 3. The variation of \(\frac{\delta \phi(L)}{\delta \phi(4\text{km})}\) with interferometer arm length has been shown for \(m = 0, p = 0\).

**Detection Scheme**

Thus, it can be seen that the unlike the case for a plane wave where \(\delta \phi\) is directly proportional to the wave...
amplitude and the arm length $L$, if the incident wave carries angular momentum and hence a beam of polarization, the phase difference will vary as a nonlinear and complicated function of arm-length.

If it is possible to vary the arm length, we can measure $\delta \phi$ for different values of $L$, and by comparing the data obtained with the functional dependence shown in figures (3), (4) and using sophisticated statistical techniques and proper source modeling, it is possible to get values of $w(0)$, the normalization of various modes, and constraints on luminosity distance, frequency and other required parameters. Knowing them would enable us to directly measure the total angular momentum carried by GWs.

![Graph](image)

**FIG. 4.** Strain has been plotted against $w(0)$, for $L = 4$ km and $m = p = 0$.

### CONCLUSIONS

As electromagnetic beams carry orbital angular momentum, we have shown in this paper that there is sufficient reason to expect the same for GWs. However, the main point of difference is that unlike lasers we cannot make any form of ‘gaser’ (a gravitational laser!!) and thus are dependent on nature to produce GW beams that carry OAM. Luckily it turns out that the simplest detectable GW emitting system, should radiate angular momentum. Unlike man-made lasers which usually have 1 particular mode, a GW will be a collection of various modes. Although any particular mode of a specific class of beams, can be expressed as a linear superposition of various modes of another class of beams, we have chosen LG modes as they are perhaps the simplest and most elegant solution, suffering no problem of divergence at asymptotic regions.

In this paper, in addition to showing the necessity of considering GW beams in place of plane waves in order to explain the excess of OAM via GWs emitted by the merger of inspiralling compact binaries, we have presented an account of the effects these GW beams will have on spacetime in general and on laser interferometer detectors. Further, perhaps for the first time, we have proposed a schematic way of measuring the amplitudes of various modes present in the GW beams, those that can be achieved by incorporating minimal changes in extant interferometers. Since the orbital angular momentum of gravitational waves can be directly calculated from these amplitudes, we have thus, again for the first time, proposed a schematic method of direct measurement of angular momentum carried by GWs.

We are primarily interested in the lowest order mode, because the first higher mode for GW150914 like sources will have non-unique values of strains dependent on the normalization factor $w(0)$ and primarily because we have a $1/z^m$-suppression. This might make the interference signal too weak to detect. Having said that though, the expressions are dependent on various non-linear parameters and a particular set of $w(0)$; tweaking the frequency and distance it may be possible to produce shear strains for higher modes, which are more realistic.

### ACKNOWLEDGEMENTS

The authors would like to thank Soumendra Kishore Roy and Sk. Sajid of Presidency University, Kolkata for valuable discussions and inputs.

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