Unitary Operators Over Quantum Systems with Several Levels

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Abstract. High-dimensional quantum states generalise multi-valued logics. The analogous of Pauli transformations acting on these quantum states determine a subgroup structure in U(n), which acts over the maximally entangled Bell states. These properties are suitable to produce superdense coding communication protocols.

1. Introduction

Qubits are realised as points in the unit sphere of the 2-dimensional complex Hilbert space. When considering \textit{k}-truth values, or equivalently, quantum states with \textit{k} levels or \textit{k} observable values, any superposition of them is a point in the unit sphere of the \textit{k}-dimensional complex Hilbert space, in this sense it represents a higher-dimensional quantum state [1].

We analyse several generalisations of Pauli transformations and maximally entangled orthonormal bases in the corresponding Hilbert spaces and the generated subgroups structures generated in the corresponding symmetry groups U(\textit{k}) and SU(\textit{k}). The maximally entangled orthonormal bases are generalisations of Bell basis [2, 3]. Our goal is to synthesize some natural algebraic relations between the operator groups and maximally entangled bases. These notions play essential roles in the development of superdense coding and teleportation.

We will use the notation \([i, j]\) to denote the set of integers \([i, i + 1, \ldots, j - 1, j]\).

2. Pauli transformations and Bell basis

2.1. Case of two qubits

Let \(\mathbb{H}_1 = \mathbb{C}^2\) be the 2-dimensional complex Hilbert space and let \(\mathbb{H}_2 = \mathbb{H}_1 \otimes \mathbb{H}_1\) be its tensor power of exponent 2.

We consider the vectors displayed at Table 1 where \(e_{ij} = e_i \otimes e_j\) is the \(ij\)-th canonical vector of \(\mathbb{H}_2\).

The collection \((b_{ij})_{i,j\in[0,1]^2}\) is called the \textit{Bell basis} of \(\mathbb{H}_2\). As a mnemonic rule, we may write

\[
b_{ij} = \frac{1}{\sqrt{2}} \left( e_{j0} + (-1)^i e_{1-j,1} \right).
\]

Pauli matrices are the following

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
Consider now the 3-fold tensor product $H_2$. Case of three qubits where Bell vectors $R$ Pauli transformations in $U(2)$, it has 16 elements, and can be realized as the Cartesian product subgroup $C$ and they conform a basis of the vector space $S$ pairs).

Table 1. Bell basis in $H_2$.

| $b_{00}$ | $b_{01}$ | $b_{10}$ | $b_{11}$ |
|---------|---------|---------|---------|
| $\frac{1}{\sqrt{2}} (e_{00} + e_{11})$ | $\frac{1}{\sqrt{2}} (e_{10} + e_{01})$ | $\frac{1}{\sqrt{2}} (e_{00} - e_{11})$ | $\frac{1}{\sqrt{2}} (e_{10} - e_{01})$ |

and they conform a basis of the vector space $\mathbb{C}^{2\times 2}$:

$$\forall A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in \mathbb{C}^{2\times 2} \exists c_0, c_x, c_y, c_z \in \mathbb{C} : A = c_0 \sigma_0 + c_x \sigma_x + c_y \sigma_y + c_z \sigma_z$$

namely

$$(c_0, c_x, c_y, c_z) = \frac{1}{2} ((a_{00} + a_{11}), (a_{01} + a_{10}), (a_{01} - a_{10}), (a_{00} - a_{11})).$$

We number Pauli matrices as $\sigma_{00} = \sigma_0, \sigma_{01} = \sigma_x, \sigma_{10} = \sigma_y, \sigma_{11} = \sigma_z$. Then by letting $\tau_{ij} = (1_2 \otimes \sigma_{ij})$, each vector in the Bell basis can be transformed, up to constant multiples, into the others by the matrices $\tau_{ij}$. Namely, in Table 2 we put at each entry the matrix $T_{\epsilon \delta}$ such that $b_\epsilon = T_{\epsilon \delta} b_\delta$, with $\delta, \epsilon \in [0,1]^2$, or in order to make explicit this last relation: boldface greek letter indices correspond to bit pairs.

Table 2. Each entry $T_{\epsilon \delta}$ is such that $b_\epsilon = T_{\epsilon \delta} b_\delta$ (observe that the boldface greek letter indices refer to bit pairs).

| $b_\epsilon \backslash b_\delta$ | $b_{00}$ | $b_{01}$ | $b_{10}$ | $b_{11}$ |
|-----------------|---------|---------|---------|---------|
| $b_{00}$        | $\tau_{00}$ | $\tau_{01}$ | $-i \tau_{11}$ | $\tau_{10}$ |
| $b_{01}$        | $\tau_{01}$ | $\tau_{00}$ | $-i \tau_{11}$ | $\tau_{10}$ |
| $b_{10}$        | $\tau_{10}$ | $-i \tau_{11}$ | $\tau_{00}$ | $\tau_{01}$ |
| $b_{11}$        | $-i \tau_{11}$ | $\tau_{10}$ | $\tau_{01}$ | $\tau_{00}$ |

Let $R_4 = \langle i \rangle = \{1, i, -1, -i\}$ be the cyclic group generated by $i = \sqrt{-1}$ in the multiplicative subgroup $S_1$ consisting of the unit circle of $\mathbb{C}$. The Pauli group $P$ is the subgroup generated by Pauli transformations in $U(2)$, it has 16 elements, and can be realized as the Cartesian product $R_4 \times (\sigma_\epsilon)_{\epsilon \in [0,1]^2}$ with the composition determined by the well known relations:

$$\forall \epsilon \in \{01, 10, 11\} : \sigma_\epsilon^2 = \sigma_{00} \land -i \sigma_{01} \sigma_{10} \sigma_{11} = \sigma_{00}.$$

### 2.2. Case of three qubits

Consider now the 3-fold tensor product $H_3 = H_2 \otimes H_1$. An orthonormal basis consists of the Bell vectors

$$\forall i, j, k \in [0,1] : b_{ijk} = \frac{1}{\sqrt{2}} (e_{ijk} + (-1)^i e_{1-j,1-k,1}),$$

where $(e_{ijk} = e_i \otimes e_j \otimes e_k)$ is the canonical basis of $H_3$. The maps of the form $\sigma_{00} \otimes \sigma_\delta \otimes \sigma_\epsilon$, with $\delta, \epsilon \in \{0,1\}^2$, applied on the Bell vector $b_{000} \in H_3$ give $R_4$-multiples of the other elements in the Bell basis, as can be seen in Table 3.
Table 3. Maps of the form $\sigma_{00} \otimes \sigma_\delta \otimes \sigma_\varepsilon$ acting on $b_{000}$ in $H_3$ (Values of $(\sigma_{00} \otimes \sigma_\delta \otimes \sigma_\varepsilon) b_{000}$).

| $\sigma_\delta \setminus \sigma_\varepsilon$ | $\sigma_{00}$ | $\sigma_{01}$ | $\sigma_{10}$ | $\sigma_{11}$ |
|------------------------------------------|-------------|-------------|-------------|-------------|
| $\sigma_{00}$                           | $b_{000}$   | $b_{001}$   | $b_{100}$   | $i b_{101}$ |
| $\sigma_{01}$                           | $b_{010}$   | $b_{011}$   | $b_{110}$   | $i b_{111}$ |
| $\sigma_{10}$                           | $b_{100}$   | $b_{101}$   | $b_{000}$   | $i b_{001}$ |
| $\sigma_{11}$                           | $i b_{110}$ | $i b_{111}$ | $i b_{010}$ | $-b_{011}$  |

From there,

$$(\sigma_{00} \otimes \sigma_{0\delta_1} \otimes \sigma_{0\varepsilon_1}) b_{000} \in \mathcal{L}(b_{\eta_{01}, \eta_{12}}),$$

(1)

where $\mathcal{L}(b)$ is the one-dimensional space, or ray, spanned by vector $b$, and

$$\eta_0 = (1 - \delta_0)\varepsilon_0 + \delta_0(1 - \varepsilon_0), \quad \eta_1 = \delta_1, \quad \eta_2 = \varepsilon_1.$$

Let $CN : \{0, 1\}^2 \to \{0, 1\}$, $(\delta, \varepsilon) \mapsto CN(\delta, \varepsilon) = (1 - \delta)\varepsilon + \delta(1 - \varepsilon)$, be the “controlled-not” map, and define recursively,

$$CN_1(\delta_0, \delta_1) = CN(\delta_0, \delta_1),$$

$$CN_{n+1}(\delta_0, \delta_1, \ldots, \delta_{n+1}) = CN(\delta_0, CN_n(\delta_1, \ldots, \delta_{n+1})).$$

For each integer $n \geq 1$, $CN_n$ is a balanced map, and consequently

$$\text{card } CN_n^{-1}(0) = 2^n = \text{card } CN_n^{-1}(1).$$

One can also see that, for $n = 4$, in $H_4$:

$$(\sigma_{00} \otimes \sigma_{0\delta_1} \otimes \sigma_{0\varepsilon_1} \otimes \sigma_{0\zeta_1}) b_{0000} \in \mathcal{L}(b_{\eta_0, \eta_2, \eta_3}),$$

where

$$\eta_0 = CN_2(\delta_0, \varepsilon_0, \zeta_0), \quad \eta_1 = \delta_1, \quad \eta_2 = \varepsilon_1, \quad \eta_3 = \zeta_1.$$

2.3. Case of many qubits

In general, for any $n \geq 3$, in $H_n = H_{n-1} \otimes H_1$, for the Bell basis

$$\forall i_0, i_1, \ldots, i_{n-1} \in [0, 1]: \ b_{i_0 i_1 \ldots i_{n-1}} = \frac{1}{\sqrt{2}} \left( e_{i_1 \ldots i_{n-1} 0} + (-1)^{i_0} e_{1-i_1 \ldots 1-i_{n-1} 1} \right),$$

one has

$$\left( \sigma_{00} \otimes \bigotimes_{j=1}^{n-1} \sigma_{\varepsilon_0^{(j)} \varepsilon_1^{(j)}} \right) b_{0^n} \in \mathcal{L}(b_{\eta_0, \ldots, \eta_{n-1}}),$$

(2)

where

$$\eta_0 = CN_{n-2}(\varepsilon_0^{(1)}, \ldots, \varepsilon_0^{(n-1)}), \quad \eta_j = \varepsilon_1^{(j)} \quad \forall j = 1, \ldots, n - 1.$$

(3)

Thus, for $n \geq 4$, for each $\xi \in R_4$ and each $\eta_0, \eta_1, \ldots, \eta_{n-1} \in \{0, 1\}$, there are exactly

$$\frac{4^{n-1}}{4 \cdot 2^n} = \frac{2^{2n-2}}{2^{n+2}} = 2^{n-4}.$$
operators of the form \( (\sigma_0 \otimes \bigotimes_{j=1}^{n-1} \sigma^{(j)}_{\xi_0 \xi_1}) \) mapping \( b_0 \) into \( \xi b_{(0)\eta_1\cdots \eta_{n-1}} \), with \( \xi \in \mathbb{C}, |\xi| = 1 \).

If this last quregister is measured with respect to the Bell basis, then it is produced \( b_{(0)\eta_1\cdots \eta_{n-1}} \).

Consequently, after measurement on the Bell basis, each \( b_{(0)\eta_1\cdots \eta_{n-1}} \) is obtained in \( 2^{n-2} \) possible ways.

Just as an example of the above assertions, let us see Table 4 for the case \( n = 4 \). There are \( 4^{n-1} = 64 \) triplets \( (\xi_0, \xi_1, \eta_0, \eta_1) \) and there are \( 4 \cdot 2^n = 64 \) pairs \( (\xi, b_{(0)\eta_1\eta_2\eta_3}) \). Thus, each pair is obtained just once \( (2^{n-4} = 1) \), but each direction \( b_{(0)\eta_1\eta_2\eta_3} \) appears four times \( (2^{n-2} = 4) \). A quick glimpse in the indexes allows to check that relations (3) do hold indeed.

Now, given \( \eta_0, \eta_1, \ldots, \eta_{n-1} \in \{0, 1\} \), from relations (3), it is direct to find the corresponding operator satisfying (2), and for any other \( \theta_0, \theta_1, \ldots, \theta_{n-1} \in \{0, 1\} \) through the operation in Pauli group it is also direct to find \( (\xi_0, \xi_1) \), such that

\[
\left( \sigma_0 \otimes \bigotimes_{j=1}^{n-1} \sigma^{(j)}_{\xi_0 \xi_1} \right) b_{(0)\eta_1\cdots \eta_{n-1}} \in \mathcal{L} \left( b_{\theta_0\theta_1\cdots \theta_{n-1}} \right).
\]

3. Several levels quantum states

Let \( k \geq 2 \) be a positive integer, which counts the number of truth values, or quantum levels. Let \( \rho_k = e^{i2\pi k} \) be the primitive \( k \)-th root of unity. We denote by \( \mathbb{C}^k \) the \( k \)-dimensional complex Hilbert space, and, again, by \( e_0, \ldots, e_{k-1} \) the vectors in its canonical basis.

3.1. Case of two quantum states

We will refer to the elements of the unit sphere in \( \mathbb{C}^k \) as registers, namely, as “concatenations” of quantum states. For any \( m, n \in [0, k-1] \) let

\[
b_{mn} = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} \rho_k^{mj} e_j \otimes e_{(j+n) \ mod \ k}.
\]

Then \( B_{k2} = (b_{mn})_{m,n \in [0,k-1]} \) is an orthonormal basis of \( \mathbb{C}^k \), called Bell basis.

For any \( m, n \in [0, k-1] \) let \( U_{mn} = [u_{mn \mu \nu}]_{\mu,\nu \in [0,k-1]} \) be the matrix with entries in the unit circle of \( \mathbb{C} \) such that

\[
u_{mn \mu \nu} = \rho_k^{mn(\nu-n)} \delta_{\mu, (\nu+n) \ mod \ k},
\]

where, as usual, \( \delta_{ij} \) is Kroenecker’s delta. Then,

\[
U_{mn} e_j = \sum_{\mu=0}^{k-1} \rho_k^{mj(\nu-n)} \delta_{\mu, (j+n) \ mod \ k} e_{\mu} = \rho_k^{mj} e_{(j+n) \ mod \ k}
\]
Table 4. For \( n = 4 \), each “signed” Bell vector appears just once, but each direction appears 4 times.

| \( \xi \) : \( \eta_0 \eta_1 \eta_2 \eta_3 \) | \( \varepsilon_0 \) | \( \varepsilon_1 \) | \( \varepsilon_2 \) | \( \varepsilon_3 \) |
|---------------------------------------------|-------------|-------------|-------------|-------------|
| \( 00 \) | \( 00 \) | \( 00 \) | \( 1 \) : \( 0000 \) |
| \( 00 \) | \( 00 \) | \( 01 \) | \( 1 \) : \( 0001 \) |
| \( 00 \) | \( 00 \) | \( 10 \) | \( 1 \) : \( 1000 \) |
| \( 00 \) | \( 01 \) | \( 00 \) | \( i \) : \( 1001 \) |
| \( 00 \) | \( 01 \) | \( 01 \) | \( 1 \) : \( 0010 \) |
| \( 00 \) | \( 01 \) | \( 10 \) | \( 1 \) : \( 1010 \) |
| \( 00 \) | \( 11 \) | \( 00 \) | \( i \) : \( 1011 \) |
| \( 00 \) | \( 10 \) | \( 00 \) | \( i \) : \( 0000 \) |
| \( 00 \) | \( 10 \) | \( 10 \) | \( 1 \) : \( 0000 \) |
| \( 01 \) | \( 00 \) | \( 00 \) | \( 1 \) : \( 1000 \) |
| \( 01 \) | \( 00 \) | \( 01 \) | \( 1 \) : \( 1001 \) |
| \( 01 \) | \( 00 \) | \( 10 \) | \( 1 \) : \( 0000 \) |
| \( 01 \) | \( 01 \) | \( 01 \) | \( 1 \) : \( 1010 \) |
| \( 01 \) | \( 01 \) | \( 10 \) | \( 1 \) : \( 0010 \) |
| \( 01 \) | \( 11 \) | \( 00 \) | \( i \) : \( 0011 \) |
| \( 01 \) | \( 10 \) | \( 00 \) | \( i \) : \( 0000 \) |
| \( 01 \) | \( 10 \) | \( 01 \) | \( 1 \) : \( 0001 \) |
| \( 01 \) | \( 10 \) | \( 10 \) | \( 1 \) : \( 1000 \) |
| \( 01 \) | \( 11 \) | \( 00 \) | \( i \) : \( 1001 \) |
| \( 01 \) | \( 11 \) | \( 01 \) | \( i \) : \( 0011 \) |
| \( 01 \) | \( 11 \) | \( 10 \) | \( i \) : \( 1010 \) |
| \( 01 \) | \( 11 \) | \( 11 \) | \( i \) : \( 1011 \) |
| \( 01 \) | \( 11 \) | \( 11 \) | \( -1 \) : \( 1011 \) |

| \( \xi \) : \( \eta_0 \eta_1 \eta_2 \eta_3 \) | \( \varepsilon_0 \) | \( \varepsilon_1 \) | \( \varepsilon_2 \) | \( \varepsilon_3 \) |
|---------------------------------------------|-------------|-------------|-------------|-------------|
| \( 01 \) | \( 00 \) | \( 00 \) | \( 1 \) : \( 0100 \) |
| \( 01 \) | \( 00 \) | \( 01 \) | \( 1 \) : \( 0101 \) |
| \( 01 \) | \( 00 \) | \( 10 \) | \( 1 \) : \( 1100 \) |
| \( 01 \) | \( 01 \) | \( 00 \) | \( i \) : \( 1101 \) |
| \( 01 \) | \( 01 \) | \( 01 \) | \( 1 \) : \( 0110 \) |
| \( 01 \) | \( 01 \) | \( 10 \) | \( 1 \) : \( 1110 \) |
| \( 01 \) | \( 11 \) | \( 00 \) | \( i \) : \( 1111 \) |
| \( 01 \) | \( 11 \) | \( 01 \) | \( i \) : \( 0110 \) |
| \( 01 \) | \( 11 \) | \( 10 \) | \( i \) : \( 1110 \) |
| \( 01 \) | \( 11 \) | \( 11 \) | \( i \) : \( 1111 \) |

| \( \xi \) : \( \eta_0 \eta_1 \eta_2 \eta_3 \) | \( \varepsilon_0 \) | \( \varepsilon_1 \) | \( \varepsilon_2 \) | \( \varepsilon_3 \) |
|---------------------------------------------|-------------|-------------|-------------|-------------|
| \( 10 \) | \( 00 \) | \( 00 \) | \( 1 \) : \( 1100 \) |
| \( 10 \) | \( 00 \) | \( 01 \) | \( 1 \) : \( 1101 \) |
| \( 10 \) | \( 00 \) | \( 10 \) | \( 1 \) : \( 0100 \) |
| \( 10 \) | \( 01 \) | \( 00 \) | \( i \) : \( 1101 \) |
| \( 10 \) | \( 01 \) | \( 01 \) | \( 1 \) : \( 0110 \) |
| \( 10 \) | \( 01 \) | \( 10 \) | \( 1 \) : \( 1110 \) |
| \( 10 \) | \( 11 \) | \( 00 \) | \( i \) : \( 0111 \) |
| \( 10 \) | \( 11 \) | \( 01 \) | \( i \) : \( 1111 \) |
| \( 10 \) | \( 11 \) | \( 10 \) | \( i \) : \( 0110 \) |
| \( 10 \) | \( 11 \) | \( 11 \) | \( i \) : \( 1110 \) |
| \( 10 \) | \( 11 \) | \( 11 \) | \( -1 \) : \( 1111 \) |
and, by denoting by $1_k$ the identity matrix of order $k \times k$,

$$(1_k \otimes U_{mn}) b_{00} = (1_k \otimes U_{mn}) \left( \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} e_j \otimes e_j \right)$$

$$= \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} (1_k \otimes U_{mn}) (e_j \otimes e_j)$$

$$= \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} e_j \otimes U_{mn} e_j$$

$$= \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} \rho_k^{m_j} e_j \otimes e_{(j+n) \mod k}$$

$$= b_{mn}.$$

Let $C_k = [\delta_{\mu,(\nu+1) \mod k}]_{\mu,\nu \in [0,k-1]}$ be the matrix representing the linear map given as the “rotation” of the canonical basis $e_\nu \mapsto e_{(\nu+1) \mod k}$. Then, from relation (4) we may see that

$$\forall m,n : U_{mn} = C^m_k U_{m0} = C^m_k U_{10} = C^m_k \left( \text{diag} [\rho_k^j]_{j \in [0,k-1]} \right)^m.$$ 

Besides,

$$U_{10} C_k = \rho_k C_k U_{10},$$

Consequently,

$$U_{10} C^p_k = \rho_k^p C^p_k U_{10} \quad \text{and} \quad U_{10}^q C^p_k = \rho_k^q C^p_k U_{10}^q,$$

which implies

$$\forall m,n,p,q : U_{mn} U_{pq} = \rho_k^{nq} U_{(m+p) \mod k,(n+q) \mod k}.$$

The relation (5) determines a subgroup structure, in $U(k)$, generated by the set $\mathcal{U}_k = \{ U_{mn} \}_{mn \in [0,k-1]^2}$, with unit $U_{00} = 1_k$ (see 4). This last class of maps is indeed a subset of $SU(k)$ when $k$ is odd. Thus,

$$(1_k \otimes U_{mn}) b_{pq} = (1_k \otimes U_{mn}) \circ (1_k \otimes U_{pq}) b_{00}$$

$$= (1_k \otimes (U_{mn} U_{pq})) b_{00}$$

$$= \rho_k^{nq} (1_k \otimes U_{(m+p) \mod k,(n+q) \mod k}) b_{00}$$

$$= \rho_k^{nq} b_{(m+p) \mod k,(n+q) \mod k},$$

and $\forall m,n,p,q$ :

$$\left[ \rho_k^{-(m-p)(n-q) \mod k} (1_k \otimes U_{(m-p) \mod k,(n-q) \mod k}) \right] b_{pq} = b_{mn}. $$

3.2. Case of $k$ quantum states

Let $k \geq 2$ be a positive integer, and let $\mathbb{H}^{(k)}_k = \left( \mathbb{H}_1^{(k)} \right)^\otimes_k$ be the $k$-fold tensor power of $\mathbb{H}_1^{(k)} = \mathbb{C}^k$. We will refer again to the elements of the unit sphere in $\mathbb{H}^{(k)}_k$ as registers, namely, as “concatenations” of quantum states. For any $n_0, n_1, \ldots, n_{k-1} \in [0, k-1]$ let

$$b^{(k)}_{n_0 n_1 \ldots n_{k-1}} = \frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} \rho_k^{j n_0} e_j \otimes \bigotimes_{\ell=1}^{k-1} e_{(j+n_\ell) \mod k}.$$
Then $B_{kk} = \left(b_{n_0n_1\ldots n_{k-1}}^{(k)}\right)_{n_0, n_1, \ldots, n_{k-1} \in \{0, k-1\}}$ is an orthonormal basis of $\mathbb{H}_k^{(k)}$. As in eq. (6), we have

$$\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right) b_{n_0n_1\ldots n_{k-1}}^{(k)} = \sum_{\ell=1}^{k-1} q_{\ell} n_{\ell} b_{n_0n_1\ldots n_{k-1}}^{(k)},$$

where

$$m_0 = \left(\sum_{\ell=1}^{k-1} p_{\ell} + n_0\right) \mod k;$$

$$m_1 = (q_1 + n_1) \mod k, \ldots, m_{k-1} = (q_{k-1} + n_{k-1}) \mod k,$$

which gives an equivalent form of eq. (7) to recover a map $\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right)$ transforming a given register in the Bell basis into another one in the same basis.

However, at present case there are $(k^3)^{k-1} = k^{2k-2}$ maps of the form $\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right)$ and there are $k^k$ registers in the Bell basis. One can see that for any two Bell registers $b_{n_0n_1\ldots n_{k-1}}^{(k)}$ and $b_{n_0m_1\ldots m_{k-1}}^{(k)}$ there are exactly

$$\frac{k^{2k-2}}{k^k} = k^{k-2}$$

maps transforming the first register into the second one.

Indeed, for any index sequence $n_0n_1\ldots n_{k-1} \in \{0, k-1\}^{(k)}$ let for any two maps $\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right)$ and $\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right)$:

$$\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right) b_{n_0n_1\ldots n_{k-1}}^{(k)} \sim_{n_0n_1\ldots n_{k-1}} \left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right) \iff$$

$$\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right) b_{n_0n_1\ldots n_{k-1}}^{(k)} = \left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right) b_{n_0n_1\ldots n_{k-1}}^{(k)}.$$

Then, according to [8],

$$\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right) \sim_{n_0n_1\ldots n_{k-1}} \left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right) \iff$$

$$\left(\sum_{\ell=1}^{k-1} p_{\ell} + n_0\right) \mod k = \left(\sum_{\ell=1}^{k-1} p_{\ell} + n_0\right) \mod k \land$$

$$\forall \kappa \in [1, k-1]: \ (q_\kappa + n_\kappa) \mod k = (q_\kappa + n_\kappa) \mod k.$$

Hence, the index of the equivalence relation $\sim_{n_0n_1\ldots n_{k-1}}$ is $k^k$, the cardinality of the Bell basis. Thus, by fixing canonical representatives in each equivalence class, for instance, the first map $\left(1_k \otimes \bigotimes_{\ell=1}^{k-1} U_{p_{\ell}q_{\ell}}\right)$ according to the lexical ordering of the index sequence $p_1q_1\ldots p_{k-1}q_{k-1}$, superdense coding can be performed univocally.

4. Discussion

Superdense coding was originally presented in the literature in the 2-dimensional case. Modeled with $k$ quantum levels, superdense coding allows the communication of $2\log_2 k$ classical bits among two parts with the transmission of just one $k$-level quantum state.
Departing from the protocol presented in [4], we generalise it to many participants, in a purely algebraic way: a central part, Alice, is receiving independent pieces of information, from several correspondent participants, and we remark direct dependencies in order to recover the unitary operations performed by each participant when codifying his information pieces. Alice is able to receive $k \log_2 k$ classical bits by receiving $(k - 1)$ $k$-level quantum states, each from each correspondent. Although the generalisation of the proposed multipart superdense protocol enables univocity in the decoding process for the case of $k = 2$, the univocity is lost for $k > 2$. However an equivalence relation is well determined among the generalised operators, thus, by fixing a notion of canonical representatives, superdense can be fixed as well.

In [5] a nonoblivious encoding is proposed in order to produce superdense coding using two-dimensional qubits. In [6] a superdense coding is proposed by mixing GHZ-quantum states and Bell states. In [7, 8] a mixture of the so called $W$-states and Bell states is used to produce a form of superdense coding. Instead, in our approach we consider just direct generalisations of Bell states.

5. Conclusions

We have analised the subgroup structure by the operators analogous of Pauli operators for $k$-quantum levels states. The main interest in this study consists in the algebraic analysis of the involved operator compositions.

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