Worldsheet Dynamics of String Junctions

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Abstract

We analyze scattering of string modes at string junctions of type IIB string theory. In the infrared limit, certain orthogonal linear combinations of the fields on the different strings satisfy either Dirichlet or Neumann boundary conditions. We confirm that the worldsheet theory of a general string network has eight conserved supercharges in agreement with target space BPS considerations. As an application, we obtain the band spectrum of some simple string lattices.

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1 Introduction

As several authors have recently pointed out, triple junctions of type IIB \((p,q)\) strings \[1, 2\] preserve supersymmetry and can be used to build supersymmetric networks of junctions \[3, 4\]. Other applications include gauge symmetry enhancement to exceptional groups in type IIB string theory \[5\] and description of \(\nu = 1/4\) BPS states in \(N = 4\) supersymmetric Yang-Mills theories \[6\]. Natural questions arise concerning the behavior of excitations of such networks: What is the S-matrix describing the scattering from a single junction? Are there BPS excited states of networks laid out on compact tori and, if so, what can we say about their entropy?

In this paper, we will explore what can be learned from an extremely simple-minded view of the problem. The individual \((p,q)\) strings all support the standard massless multiplet of 8 bosons and 8 fermions (both left-moving and right-moving). The triple junction is a common boundary to the three string worldsheets at which the left-moving excitations on any one string can scatter into right-moving excitations on all of them. More complicated things can happen at high energies, but at energies well below the string scale the scattering should be linear (an “in” quantum of given energy on one string should scatter into a linear combination of “out” quanta on all the strings entering the junction).

As we will show, rather simple physical arguments suffice to determine the linearized junction S-matrix. The reflection and transmission amplitudes for modes on the individual strings that extend from the junction depend on the tensions and relative orientation of the strings. It turns out, however, that the S-matrix can always be diagonalized to give standard Dirichlet or Neumann boundary conditions for particular combinations of modes on the different strings. We carry out the analysis both for fluctuations that are transverse to the plane of the string junction and for ‘in-plane’ modes. We then generalize the discussion to include higher order junctions where \(n > 3\) co-planar strings meet.

The worldsheet theory of a general BPS network of \((p,q)\) strings is expected to have eight of the thirty-two supergenerators of type IIB string theory unbroken. We verify that the S-matrices at the various string junctions in the network indeed preserve precisely the right number of supercharges.

We close with a discussion of the excitation spectrum of a periodic string lattice. Sen has proposed that such lattices, viewed as string networks laid out on tori, can be used as novel building blocks for string compactification \[8\]. Carrying out this idea in practice will require some understanding of the basic dynamical properties of triple
string junctions. Initial steps in that direction have been taken by Rey and Yee [7]. Our line of approach differs from theirs and gives, we believe, a more detailed understanding of the dynamics.

2 Transverse Mode Scattering at a Junction

Let us consider a junction where three strings meet, as shown in Figure 1. We’ll generalize to higher order string junctions in Section 4. We take the strings to lie in the complex plane with the equilibrium position of the junction at \( z = 0 \) and the strings making angles \( \theta_i \) with the positive real axis. The three strings have NS-NS and R-R charges \( p_i \) and \( q_i \) respectively.

![FIGURE 1: A three-string junction.](image)

The configuration is stable and preserves \( \nu = 1/4 \) of the type IIb supersymmetry if the charges are conserved at the junction and each string is aligned with its charge vector \( p_i + q_i \tau \), where \( \tau \) is the usual axion-dilaton modulus of type IIb theory [3, 4]. Charge conservation and force balance are then expressed as

\[
0 = \sum_{i=1}^{3} q_i = \sum_{i=1}^{3} p_i = \sum_{i=1}^{3} e^{i\theta_i} t_i, \tag{1}
\]

where \( \theta_i \) is the argument of \( p_i + q_i \tau \):

\[
p_i + q_i \tau = |p_i + q_i \tau| e^{i\theta_i}, \tag{2}
\]
and $t_i$ is the scalar string tension,

$$
t_i = \frac{1}{\sqrt{\text{Im} \tau}} |p_i + q_i \tau|.
$$

(3)

We first consider fluctuations $\phi_i$ that are transverse to the plane of the three-string junction. Here the subscript $i = 1, 2, 3$ denotes the string on which the fluctuation is found. There are seven independent transverse fluctuations on each string. The scattering problem is diagonal in this flavor space and we have suppressed the corresponding indices on $\phi$. Since we are dealing with real massless fields, the general expression for a mode of frequency $\omega$ can be written

$$
\phi_i(x_i, t) = \text{Re}\{ (A_i e^{i \omega x_i} + B_i e^{-i \omega x_i}) e^{-i \omega t} \},
$$

(4)

where $A_i, B_i$ are complex mode amplitudes and $x_i > 0$ is the distance from the junction measured along the given string. The physical matching conditions at $x_i = 0$ are continuity, expressed as

$$
\phi_1(0) = \phi_2(0) = \phi_3(0),
$$

(5)

and ‘vertical’ (i.e. transverse to plane of junction) tension balance,

$$
\sum_{i=1}^{3} t_i \phi_i'(0) = 0.
$$

(6)

In these equations, we suppress the time argument: they are supposed to hold at all times. As is to be expected, the matching conditions treat the three strings in a completely symmetric fashion.

A string junction has an M-theory description in terms of a three-pronged wrapped membrane in $\mathbb{R}^9 \times T^2$ [8, 9]. Our matching conditions can easily be derived in that context. The continuity condition follows immediately from the fact that the junction is described by a single membrane. The tension balance condition requires more work, but it follows from a variational calculation, involving the Nambu-Goto action for the membrane, applied to a pants-like section of the membrane that includes the junction and connects to each of the extended strings. We will not go into the details here but simply consider the condition (6) to be physically well-motivated and proceed to work out the resulting scattering problem.

It is easy to solve the boundary conditions to find the $3 \times 3$ scattering matrix relating the “in” and “out” modes: $\vec{A} = S \cdot \vec{B}$, where

$$
S = -1 + \frac{2}{\sum_{i=1}^{3} t_i} \begin{pmatrix} t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.
$$

(7)
The matrix $S$ is real valued and therefore it acts independently on the real and imaginary parts of the complex mode amplitudes. It has eigenvalues $\pm 1$, with one $+1$ eigenvector and two $-1$ eigenvectors: $S \cdot (1, 1, 1) = (1, 1, 1)$ and $S \cdot \vec{x} = -\vec{x}$ if $\vec{t} \cdot \vec{x} = 0$, where $\vec{t} = (t_1, t_2, t_3)$ (there are two such vectors).

The action which describes the dynamics of the fields $\phi_i$ is

$$L = \sum_i t_i \int_0^\infty dx_i (\partial \phi_i)^2 . \quad (8)$$

If we supplement it with the constraint that the three fields be equal at the origin, it contains all the boundary conditions discussed above. For a quadratic action like this, we can eliminate the tensions from the action and the energy by rewriting them in terms of the rescaled fields $\hat{\phi}_i = \sqrt{t_i} \phi_i$. On the other hand, the tensions then appear in the boundary conditions in a slightly more complicated way:

$$\frac{\hat{\phi}_1(0)}{\sqrt{t_1}} = \frac{\hat{\phi}_2(0)}{\sqrt{t_2}} = \frac{\hat{\phi}_3(0)}{\sqrt{t_3}} , \quad \sum_{i=1}^3 \sqrt{t_i} \hat{\phi}_i'(0) = 0 . \quad (9)$$

The $S$-matrix for the rescaled fields is very simply related to the old one: $\hat{S} = \sqrt{t} S \sqrt{t}^{-1}$ where $t$ is a diagonal matrix whose entries are the scalar tensions. Recalling (7), we find

$$\hat{S} = -1 + 2 \vec{y} \otimes \vec{y} , \quad (10)$$

where $\vec{y}$ is the unit three-vector

$$\vec{y} = \frac{1}{\sqrt{\sum t_i}} (\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3}) . \quad (11)$$

The rescaled S-matrix is symmetric and squares to the identity. It has eigenvalues $\pm 1$ and its eigenvectors are orthogonal. It follows immediately that energy is conserved in the scattering process. In terms of rescaled fields, the energy carried by the incoming modes is $E_{in} = \sum_{i=1}^3 \omega^2 |\vec{B}_i|^2$, while the energy carried by the outgoing modes is the same thing with $\vec{B} \to \vec{A}$.

The eigenvalues $+1$ and $-1$ correspond to Neumann and Dirichlet boundary conditions respectively. The $+1$ eigenvector is $\vec{y}$ and any vector normal to $\vec{y}$ is a $-1$ eigenvector. Thus, we can “trivialize” the scattering problem by taking the appropriate linear combinations of the fields on the three different strings! Note that, according to (4), only the mode with $+1$ eigenvalue can correspond to a zero mode (a mode whose amplitude does not vanish in the $\omega \to 0$ limit). We easily see from (7) that this mode has $A_1 = A_2 = A_3$, which is to say that it amounts to a uniform translation of the junction transverse to its plane. This is precisely the zero mode we would expect.
To further check that these results correspond to expectations, consider a \((1,0)\) string (i.e. a fundamental string) attached to \((0,q)\) and \((-1,-q)\) strings at weak string coupling. In this case \(t_1 << t_2 \sim t_3\) and the +1 eigenvector is \(\vec{y} \sim (0,1/\sqrt{2},1/\sqrt{2})\). A fluctuation along the fundamental string \((1,0,0)\) is orthogonal to \(\vec{y}\). It therefore has eigenvalue \(-1\) and satisfies a Dirichlet boundary condition as one would expect for transverse fluctuations of a fundamental string attached to a D-string.

It is also easy to check that the transmission and reflection amplitudes of three-string junctions transform appropriately under the usual \(SL(2,Z)\) transformations of type IIb theory,

\[
\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d},
\]

\[
\begin{pmatrix} p \\ q \end{pmatrix} \rightarrow \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},
\]

where \(a,b,c,d\) are integers satisfying \(ad - bc = 1\). To prove this, one first shows that

\[
p' + q'\tau' = \frac{p + q\tau}{c\tau + d},
\]

and then observes that the S-matrix entries in (7) only depend on a ratio of first order expressions involving the different string tensions. It follows that the factors of \(\sqrt{|\text{Im}\tau'| |c\tau + d|}\) cancel and the form of the S-matrix is left invariant by the \(SL(2,Z)\) transformation (12). This does not appear to be a very restrictive test of the structure of the scattering matrix, but it is not without content.

3 In-Plane Scattering at a Junction

One can also consider fluctuations in the plane of the three-string junction. This case is somewhat more complicated to deal with than the out-of-plane fluctuations because an in-plane fluctuation that is transverse with respect to one of the strings induces motion at the junction that is both transverse and longitudinal with respect to the other two strings. The longitudinal fluctuations are unphysical and will eventually be eliminated, but the boundary conditions are easier to describe if they are left in temporarily. As an aside, we note that in the worldvolume gauge theory approach to this problem, longitudinal displacements are accounted for by the worldvolume gauge field \(A_\mu(x,t)\). The fact that each string terminates at the junction means that the gauge field cannot be completely eliminated by a choice of gauge. For each string, there remains a degree
of freedom corresponding to the longitudinal position of the end of the string and when several strings are coupled at a junction, this degree of freedom plays an important role in the scattering dynamics. The treatment of in-plane scattering given below is not directly derived from the coupled worldvolume gauge theory approach but gives, we believe, the same result a more formal treatment would give.

We again take the strings to lie in the complex plane with the equilibrium position of the junction at $z = 0$ and $x_i > 0$ measuring the distance from the junction along a given string. The in-plane transverse and longitudinal fluctuations on each string can be conveniently combined into a single complex-valued field $z(x,t) = \chi(x,t) + i\varphi(x,t)$ where $\chi$ and $\varphi$ are the longitudinal and transverse fluctuations, respectively. Consider first a string extended along the positive real axis. Its endpoint is at

$$z(x = 0) = \chi(0) + i\varphi(0),$$

where $\chi$ is the longitudinal fluctuation and $\varphi$ is transverse. To rotate to a given $(p,q)$ string one simply multiplies by the appropriate phase,

$$z_i = e^{i\theta_i}(\chi_i + i\varphi_i).$$

Continuity at the junction gives two complex equations:

$$z_1(0) = z_2(0) = z_3(0).$$

This amounts to four real valued equations, of which three can be used to eliminate the unphysical longitudinal fluctuations $\chi_i$, leaving behind a single linear condition on the transverse fluctuations, which reads:

$$0 = \varphi_1(0) \sin \theta_{23} + \varphi_2(0) \sin \theta_{31} + \varphi_3(0) \sin \theta_{12},$$

where $\theta_{ij} \equiv \theta_i - \theta_j$.

The remaining two boundary conditions come from tension balance at the junction. When the junction is perturbed the strings will come into it at angles that differ somewhat from the equilibrium angles, and the condition for tension balance becomes

$$0 = \sum_{i=1}^{3} e^{i(\theta_i + \delta\theta_i)}t_i.$$  

For small fluctuations the angles $\delta\theta_i$ will be small and to leading order they only depend on the transverse fluctuations. In fact, they are simply given by the slope of the transverse fluctuation field at the string endpoints:

$$\delta\theta_i \approx \varphi_i'(0).$$
Expanding the tension balance condition (18) to first order in fluctuations gives the linear relation

\[ 0 = \sum_{i=1}^{3} e^{i\eta t_i} \phi'_i(0). \]  

(20)

This is a complex equation and the real and imaginary parts give the remaining two boundary conditions that we need in order to determine the S-matrix relating the in and out parts of the transverse fluctuation fields \( \phi_i \).

After some straightforward algebra one finds the following expression:

\[ S = 1 - \frac{2}{D} \begin{pmatrix}
    t_2 t_3 \sin^2 \theta_{23} & t_2 t_3 \sin \theta_{23} \sin \theta_{31} & t_2 t_3 \sin \theta_{12} \sin \theta_{23} \\
    t_1 t_3 \sin \theta_{23} \sin \theta_{31} & t_1 t_3 \sin^2 \theta_{31} & t_1 t_3 \sin \theta_{31} \sin \theta_{12} \\
    t_1 t_2 \sin \theta_{12} \sin \theta_{23} & t_1 t_2 \sin \theta_{31} \sin \theta_{12} & t_1 t_2 \sin^2 \theta_{12}
\end{pmatrix}, \]

(21)

where \( D = t_1 t_2 \sin^2 \theta_{12} + t_2 t_3 \sin^2 \theta_{23} + t_3 t_1 \sin^2 \theta_{31} \). It is easy to see that this matrix has two +1 eigenvectors spanning the space orthogonal to \((\sin \theta_{23}, \sin \theta_{31}, \sin \theta_{12})\) (i.e. satisfying (17)) and one -1 eigenvector proportional to \((t_2 t_3 \sin \theta_{23}, t_1 t_3 \sin \theta_{31}, t_1 t_2 \sin \theta_{12})\). Once again, the zero modes of the system must belong to the +1 eigenvalues. Simple trigonometry shows that the defining condition for these eigenvectors is satisfied by any bodily translation of the string junction in its plane. These are precisely the two zero modes we would expect to find.

Energy is conserved as a matter of course in the scattering process for the same reasons as before. The same logic as for the out-of-plane fluctuations leads us to consider rescaled fields \( \hat{\phi}_i = \sqrt{t_i} \phi_i \) and the rescaled S-matrix, \( \hat{S} = \sqrt{t} S \sqrt{t}^{-1} \). Carrying out the rescaling on (21), we get

\[ \hat{S} = 1 - 2 \vec{z} \otimes \vec{z}, \]

(22)

where \( \vec{z} \) is the unit three-vector

\[ \vec{z} = \frac{1}{\sqrt{D}}(\sqrt{t_2 t_3} \sin \theta_{23}, \sqrt{t_1 t_3} \sin \theta_{31}, \sqrt{t_1 t_2} \sin \theta_{12}) \].

(23)

This is again an orthogonal matrix which squares to the identity and its eigenvectors are orthogonal.

We can again perform a simple check by considering a fundamental string attached to a D-string, such that \( t_1 << t_2 \sim t_2 \), and \( \vec{z} \sim (0, 1/\sqrt{2}, 1/\sqrt{2}) \). A fluctuation on the fundamental string \((1, 0, 0)\) is orthogonal to \( \vec{z} \) and has eigenvalue +1. It thus satisfies a Neumann boundary condition as expected.

The argument for \( SL(2, Z) \) covariance of the scattering amplitudes proceeds in much the same way as for the out-of-plane fluctuations. The only new twist is that the S-matrix now depends on the angles between the strings as well as their scalar tensions. It
follows from (13) that the orientation angle of a \((p, q)\) string transforms under \(SL(2, \mathbb{Z})\) as \(\theta'_i = \theta_i - \text{arg}(c\tau + d)\). Since the \(S\)-matrix only involves differences of angles and is homogeneous in powers of the scalar tension, it manifestly maintains its form under \(SL(2, \mathbb{Z})\) transformations. Again, the test does not seem very restrictive, but it is not entirely trivial that it is met.

4 Higher Order String Junctions

Our treatment of scattering at a three-string junction can be generalized to planar \(n\)-string junctions, with \(n \geq 4\). An \(n\)-string junction preserves \(\nu = 1/4\) supersymmetry when the strings that emerge from it all lie in a single two-dimensional plane and the charge conservation and tension balance conditions are satisfied,

\[
0 = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} e^{i\theta_i} t_i .
\] (24)

As far as we can tell, such objects should exist in type II superstring theory. Consider, for example, the IR limit of a general string network formed out of a collection of three-string junctions with \(n\) external strings attached. For wavelengths large compared to the size of the network the physics will be that of an \(n\)-string junction with appropriate matching conditions on fluctuations.

Let us first consider out-of-plane modes \(\phi_i(t, x_i) = e^{-i\omega t} \tilde{\phi}_i(x_i)\), where \(i = 1, \ldots, n\) labels the strings that attach to the junction. Continuity at \(x_i = 0\) gives rise to \(n - 1\) equations,

\[
\tilde{\phi}_1(0) = \tilde{\phi}_2(0) = \ldots = \tilde{\phi}_n(0) ,
\] (25)

and vertical tension balance adds the equation

\[
\sum_{i=1}^{n} t_i \tilde{\phi}_i'(0) .
\] (26)

The resulting \(S\)-matrix relating in- and out-modes generalizes the answer for a three-string junction (7) in a straightforward way,

\[
S = -1 + \frac{2}{\sum_{i=1}^{n} t_i} \begin{pmatrix} t_1 & \cdots & t_n \\ \vdots & \ddots & \vdots \\ t_1 & \cdots & t_n \end{pmatrix} .
\] (27)

The eigenvalues of this \(S\)-matrix are \((1, -1, \ldots, -1)\) for any \(t_i\). There is a single linear combination of incoming modes that satisfies a Neumann condition at the junction and
all orthogonal combinations satisfy Dirichlet conditions. The Neumann mode is the one where all the strings have equal excitation, i.e. $B_i = b$ for all $i = 1, \ldots, n$ and some constant $b$. In the $\omega \to 0$ limit this mode describes a translational zero mode that uniformly moves the string junction in a direction perpendicular to its plane. There are a total of seven such zero modes for a junction embedded in $9 + 1$ dimensional spacetime.

For in-plane scattering at an $n$-string junction we can generalize the discussion of the previous section in the obvious way. Continuity at the junction gives $n - 1$ complex equations,

$$z_1(0) = z_2(0) = \ldots = z_n(0). \quad (28)$$

We use $n$ real valued equations to eliminate the unphysical longitudinal fluctuations $\chi_i$, leaving behind $n - 2$ linear conditions on the transverse fluctuations,

$$0 = \varphi_{i-1}(0) \sin \theta_{i,i+1} + \varphi_i(0) \sin \theta_{i+1,i-1} + \varphi_{i+1}(0) \sin \theta_{i-1,i}, \quad (29)$$

where $\theta_{i,j} \equiv \theta_i - \theta_j$.

The remaining matching conditions come from tension balance,

$$0 = \sum_{i=1}^{n} e^{i \theta_i} t_i \varphi_i'(0). \quad (30)$$

This complex equation can be rewritten as the following two real-valued equations:

$$0 = \sum_{i=2}^{n} \varphi_i'(0) t_i \sin \theta_{1,i},$$

$$0 = \sum_{i=1}^{n-1} \varphi_i'(0) t_i \sin \theta_{i,\ell}. \quad (31)$$

The matching conditions (29) and (31) define a linear system of $n$ equations which determines the in-plane $S$-matrix of the $n$-string junction,

$$S = 1 - \frac{2}{D} T. \quad (32)$$

Here $D = \sum_{i<j} t_i t_j \sin^2 \theta_{ij}$, and the diagonal and off-diagonal elements of the $n \times n$ matrix $T$ are given by

$$T_{ii} = \sum_{k<l \atop k,l \neq i} t_k t_l \sin^2 \theta_{kl}, \quad \text{(no sum on $i$),}$$

$$T_{ij} = \sum_{k \neq i,j} t_j t_k \sin \theta_{ik} \sin \theta_{kj}, \quad (i \neq j). \quad (33)$$
The eigenvalues of the in-plane S-matrix are $(1, 1, -1, \ldots, -1)$, which is once again independent of the tension of individual strings at the junction. The $+1$, or Neumann, eigenvectors are those that are annihilated by the matrix $T$. Some fairly tedious trigonometry applied to (33) shows that there are two independent vectors, corresponding to bodily displacements of the junction within its own plane, that satisfy this Neumann condition. As before, the $\omega \to 0$ limit of the Neumann eigensolutions are zero modes. Taking longitudinal and transverse modes together, we have identified a total of nine translational zero modes, just what we expect for a solitonic object in $9 + 1$ dimensions.

5 Supersymmetry

In the previous sections, we have computed the low-energy limit of the S-matrix for massless bosonic excitations of a string junction. The strings support fermionic excitations as well, and we should be able to say something about their S-matrix. Sen [3] has given a supergravity argument that a network of $(p,q)$ strings in a plane leaves exactly eight of the thirty-two type-IIB supersymmetry generators unbroken and we should at least be able to reproduce this result.

The low energy degrees of freedom of a $(p,q)$ string fall into eight 1+1-dimensional $(1,1)$ supermultiplets (corresponding to the eight directions transverse to the string). In a string network, we have multiple string segments lying in a plane and coupled to each other through boundary conditions at their junctions. The supermultiplets corresponding to the seven displacements perpendicular to the plane are decoupled from each other and completely equivalent. The single supermultiplet corresponding to displacements in the plane of the network is decoupled from, and inequivalent to, the rest. The question is whether, in the presence of the junctions, each of these eight theories manages to have one surviving supersymmetry. If they do, that would give the expected total of eight supersymmetries.

Let us consider fluctuations in a particular one of the directions transverse to the plane of the network. We know that the low-energy dynamics of an individual $(p,q)$ string is described by a two-dimensional massless Majorana fermion $\psi$ plus a real scalar $\phi$. Because the fields are massless, they can be decomposed into left- and right-moving supermultiplets $(\phi_+, \psi_+)$ and $(\phi_-, \psi_-)$. The $\psi_\pm$ are now single-component anticommuting objects and the $(1,1)$ supersymmetries basically exchange the $\psi_\pm$ with the $\phi_\pm$. A junction imposes a boundary condition which couples together the left-
and right-movers on different strings. The question is whether this can be done in a supersymmetric fashion.

The answer is pretty trivially yes. The first step is to rewrite the boundary conditions (5,6) on transverse bosonic coordinates in terms of the left- and right-moving components defined by \( \phi(\tau, \sigma) = \phi_+(\tau + \sigma) + \phi_-(\tau - \sigma) \). The boundary condition at \( \sigma = 0 \) can be recast as a relation between fields of the two different chiralities:

\[
\phi_1^+ - \phi_2^+ = -\phi_1^- + \phi_2^- , \quad \phi_2^+ - \phi_3^+ = -\phi_2^- + \phi_3^- , \quad \sum_{i=1}^3 t_i \phi_i^+ = \sum_{i=1}^3 t_i \phi_i^- . \tag{34}
\]

Modulo a possible zero-mode subtlety, we can integrate the last of these to eliminate the derivative and recast the boundary condition as a simple linear map between the \( \phi_i^+ \) and \( \phi_i^- \). By passing to the hatted fields \( \hat{\phi}_i = \sqrt{t_i} \phi_i \), we can then diagonalize the boundary conditions by an orthogonal transformation:

\[
\begin{align*}
\sum_i l_i \hat{\phi}_i^+ &= -\sum_i l_i \hat{\phi}_i^- , & \vec{l} &= \frac{(\sqrt{t_2}, -\sqrt{t_1}, 0)}{\sqrt{t_1 + t_2}} , \\
\sum_i m_i \hat{\phi}_i^+ &= -\sum_i m_i \hat{\phi}_i^- , & \vec{m} &= \frac{(\sqrt{t_3 t_1}, \sqrt{t_3 t_2}, -(t_1 + t_2))}{\sqrt{t_1 + t_2} \sqrt{t_1 + t_2 + t_3}} , \\
\sum_i n_i \hat{\phi}_i^+ &= +\sum_i n_i \hat{\phi}_i^- , & \vec{n} &= \frac{(\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3})}{\sqrt{t_1 + t_2 + t_3}} , \\
\vec{l}^2 &= \vec{m}^2 = \vec{n}^2 = 1 , & \vec{l} \cdot \vec{m} = \vec{l} \cdot \vec{n} = \vec{n} \cdot \vec{m} = 0 . \tag{35}
\end{align*}
\]

The content of this is that the fields \( \vec{l} \cdot \hat{\phi} \) and \( \vec{m} \cdot \hat{\phi} \) satisfy Dirichlet boundary conditions while the field \( \vec{n} \cdot \hat{\phi} \) satisfies a Neumann boundary condition. Now we can impose equivalent boundary conditions on the fermions by replacing the \( \hat{\phi}^\pm \) in these equations by the single-component anticommuting \( \hat{\psi}^\pm \). Since the bulk supersymmetry transformation is implemented by swapping \( \phi^\pm \) with \( \psi^\pm \), this is the only possible supersymmetric boundary condition. Indeed, the allowed boundary conditions for Majorana fermions are just \( \psi^+ = \pm \psi^- \) and we are choosing to impose precisely such boundary conditions on orthogonal linear combinations of \( \hat{\psi}_i \).

Now let’s think about supersymmetry of a network of junctions. In a network, for each leg, \( i \), we have one \( N = 1 \) supercharge:

\[
Q_i = \int_0^{\bar{l}_i} d\sigma (\hat{\psi}_i^+ \hat{\phi}_i^+ + \hat{\psi}_i^- \hat{\phi}_i^- ) . 
\]

Possible non-conservation of \( Q_i \) comes from boundary terms:

\[
\frac{d}{dt} Q_i = (\hat{\psi}_i^+ \hat{\phi}_i^+ - \hat{\psi}_i^- \hat{\phi}_i^- )|_{t=0}^{t_i} .
\]

11
This would vanish if Neumann/Dirichlet boundary conditions $\hat{\phi}_i^+ = \pm \hat{\phi}_i^-$ and $\hat{\psi}_i^- = \pm \hat{\psi}_i^-$ were applied directly to the fields on a given leg. That is not the situation in a network, however. In the network supercharge $Q = \sum_i Q_i$, the boundary terms organize themselves into a sum of contributions from the various junctions:

$$\frac{d}{dt} Q_{\text{junc}} = \sum_{i=1}^3 (\hat{\psi}_i^+ \hat{\phi}_i^+ - \hat{\psi}_i^- \hat{\phi}_i^-)$$

where the sum is over the legs that meet at the junction in question (and the fields are evaluated at the junction). By orthogonality, we can rewrite this in terms of the fields which diagonalize the boundary conditions:

$$\frac{d}{dt} Q_{\text{junc}} = (\vec{l} \cdot \hat{\psi}_i^+ \hat{\phi}_i^+ - \vec{l} \cdot \hat{\psi}_i^- \hat{\phi}_i^-) + (\vec{m} \cdot \hat{\psi}_i^+ \hat{\phi}_i^+ - \vec{m} \cdot \hat{\psi}_i^- \hat{\phi}_i^-) + (\vec{n} \cdot \hat{\psi}_i^+ \hat{\phi}_i^+ - \vec{n} \cdot \hat{\psi}_i^- \hat{\phi}_i^-)$$

This is a sum of terms which individually vanish because the relevant fields satisfy supersymmetric Neumann or Dirichlet boundary conditions. Therefore the sum automatically vanishes. This argument works at any junction in the network, even if the tensions involved and the orthogonal projections are different at different junctions. Consequently, for each transverse direction, we have exactly one supercharge. Much the same argument goes through for the in-plane excitations as well and we conclude that the network has a total of eight supercharges, as expected.

## 6 Excitations on String Lattices

In this section we will discuss the dynamics of a web of junctions that forms a periodic lattice. We will consider the system obtained by placing two identical triple junctions on a 2-torus and connecting up strings of like charge. The setup, then, is three different strings, of three different lengths (more or less freely adjustable by adjusting the modulus of the torus) coupled together at two mirror image triple junctions, as shown in Figure 2. The static geometry of this setup is determined by the string tensions and windings. We would like to determine the low-lying excitations of this system in order to assess how its entropy, conformal invariance and supersymmetry properties would differ from those of the more familiar simply-wound D-string.
FIGURE 2: A string lattice with two junctions per unit cell. Opposite sides of the parallelogram are identified so that one of the junctions is represented four times in the diagram.

Our first exercise will be to construct the eigenvalue condition for out-of-plane bosonic excitations. The three string segments joining the two junctions have scalar tensions and lengths $t_i, l_i$ ($i = 1, 2, 3$). By varying the torus parameters, the lengths can be made pretty much arbitrary, so we will keep them general for now. We can regard the displacement field on each of the three strings as an independent real free massless field $\phi_i(x_i, t)$ living on its own line segment $0 < x_i < l_i$. The most general disturbance of frequency $\omega$ has been written down in (4), but, in order to study the effect of the second junction (at $x_i = l_i$), it is helpful to also write the same fields in terms of “conjugate” variables $y_i = l_i - x_i$:

$$
\phi_i(x_i, t) = \text{Re}\{ (A_i e^{i \omega x_i} + B_i e^{-i \omega x_i}) e^{-i \omega t} \} = \text{Re}\{ (B_i e^{-i \omega y_i} e^{i \omega y_i} e^{-i \omega y_i} A_i e^{i \omega y_i} e^{-i \omega y_i}) e^{-i \omega t} \} .
$$

The parameters $A_i, B_i$ and $\omega$ are constrained by the boundary conditions imposed by the S-matrix (7) at the two junctions:

$$
\vec{A} = S \cdot \vec{B} \quad \text{at } x_i = 0 ,
$$
$$
P \cdot \vec{B} = S \cdot \vec{P}^* \cdot \vec{A} \quad \text{at } y_i = 0 ,
$$

where $P = \text{diag}(e^{-i \omega l_1}, e^{-i \omega l_2}, e^{-i \omega l_3})$ is a phase matrix which accounts for the different propagation phases along the different length legs between the two vertices.

It turns out that these equations are separately satisfied by the real and imaginary parts of the mode amplitudes. This is a consequence of the fact that $S$ is a real matrix and it allows us to focus on, say, real parts of $A_i$ and $B_i$ only. Furthermore, since the problem is linear, the overall scale of the fields must drop out leaving only five field
parameters to determine, plus one energy, for a total of six. With some uninspiring
algebra, the system can be boiled down to a single eigenvalue condition on the energy:

\[
0 = (\sum_{i=1}^{3} t_i^2) s_1 s_2 s_3 + 2t_1 t_2 s_3(1 - c_1 c_2) + 2t_1 t_3 s_2(1 - c_1 c_3) + 2t_2 t_3 s_1(1 - c_2 c_3),
\]

(39)

where \( s_i = \sin \omega l_i, c_i = \cos \omega l_i \). For any given eigenvalue \( \omega \) that satisfies this condition
there is a mode vector \( \vec{A} \) which is, in general, a linear superposition of the three
eigenvectors of the S-matrix at one of the vertices. This means, roughly speaking, that
the general mode satisfies neither D nor N boundary conditions. This has implications
for conformal invariance, as we will discuss.

It is hard to say anything general about the transcendental condition \((39)\) other
than that the number and spacing of eigenvalues is roughly what is expected. We can
get some insight by studying some special cases. First, let the three string lengths
be equal, in which case the eigenvalue condition reduces to \( \sin \omega l = 0 \), which gives
the energy levels of an NN or DD open string. This is just right, because the whole
problem can be trivially diagonalized, by taking orthogonal linear combinations of the
\( \phi_i \), into one NN and two DD open strings of length \( l \). It is also worth noting that
we always have one zero mode \( \omega = 0 \), no matter what the lengths and tensions are.
Another special case of interest is when we have one fundamental string and two very
heavy D-strings. Then the tensions are related by \( t_1 \ll t_2 \sim t_3 \), and in this limit the
eigenvalue condition becomes

\[
\sin \omega l_1 (1 - \cos \omega (l_2 + l_3)) = 0.
\]

(40)

This gives the spectrum of the DD open string on \( l_1 \) plus that of a closed string (left-
movers plus right-movers) on \( l_2 + l_3 \). This is precisely what you would expect from
the Polchinski boundary condition approach to the dynamics of fundamental strings
attached to D-strings.

A similar exercise gives us the equation for the spectrum of in-plane disturbances,

\[
0 = ((\tau_{12})^2 + (\tau_{23})^2 + (\tau_{31})^2) s_1 s_2 s_3 + 2\tau_{12} \tau_{31} s_1(1 - c_2 c_3) \\
+ 2\tau_{12} \tau_{23} s_2(1 - c_1 c_3) + 2\tau_{23} \tau_{31} s_3(1 - c_1 c_2),
\]

(41)

where \( \tau_{ij} = t_i t_j \sin^2(\theta_{ij}) \) and \( \theta_{ij} \) is the angle between the the \( ij \) string pair. This is
similar, but by no means identical, in structure to the out-of-plane spectral equation.
By taking appropriate limits, we see that it does the right thing for the following special
cases: (1) all lengths equal, and (2) the weak coupling limit of a fundamental string
attached to a D-string.
Let us discuss the issue of conformal invariance. The individual junctions impose conformal boundary conditions (D or N) on orthogonal linear combinations of fields. When two junctions are connected up in the most general fashion, however, one does not see towers of equally-spaced levels corresponding to the Verma modules of the two-dimensional conformal group. Only in very special cases, such as those discussed above, do we find standard conformal towers. At some level this is to be expected: strict conformal invariance is after all a property of tree level string theory while the whole point of the string junction construction is to include physics that is non-perturbative in $g_s$ (through the string tensions at least). On the other hand, we might hope to use this setup as a toy model for understanding how the dynamical principle of conformal invariance gets generalized beyond string tree level, but we have not seen how to exploit this possibility concretely.

Our formalism can also be used to study the band structure of the string lattice. When we determined the lattice spectrum above, we imposed periodic conditions on the fields across a unit cell of the lattice. An infinite lattice also supports modes with wavelengths that span several unit cells. A mode which comes back to itself after $n$ lattice spacings only has to be periodic up to a phase across a single lattice spacing. This phase can be any $n$-th root of unity, $e^{2\pi i m/n}$, and since $n$ can take arbitrarily large values on an infinite lattice the phase is in fact completely unrestricted.

It is straightforward to allow for this phase freedom in our equations. With the lattice strings labelled as in Figure 2, this is achieved by a suitable modification of the matching conditions at one of the two string junctions:

$$
\vec{A} = S \cdot \vec{B} \quad \text{at } x_i = 0,
$$
$$
P \cdot \alpha \cdot \vec{B} = S \cdot P^* \cdot \alpha \cdot \vec{A} \quad \text{at } y_i = 0, \tag{42}
$$

where the matrix $\alpha = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1)$, with $0 \leq \alpha_1, \alpha_2 \leq 2\pi$, contains the phase information along the two independent lattice vectors. When general phases are inserted, the real and imaginary parts of the mode amplitudes no longer decouple in the matching conditions but at the end of the day the system of equations still reduces to a single real equation for the mode energy. For out-of-plane fluctuations the spectral equation (39) generalizes to

$$
0 = \sum_{i=1}^{3} t_i^2 s_1 s_2 s_3 + 2t_1 t_2 s_3 (\cos (\alpha_1 - \alpha_2) - c_1 c_2) \\
+ 2t_1 t_3 s_2 (\cos \alpha_1 - c_1 c_3) + 2t_2 t_3 s_1 (\cos \alpha_2 - c_2 c_3). \tag{43}
$$

At generic parameter values it is hard to extract useful information from this equation,
but the qualitative behavior is that allowed values of $\omega$ fall into continuous bands as the phase angles $\alpha_1$ and $\alpha_2$ range from 0 to $2\pi$. Let us illustrate this for the special case when the three strings in the unit cell all have the same length, $l_1 = l_2 = l_3 \equiv l$, so that $c_1 = c_2 = c_3 = \cos \omega l$ and $s_1 = s_2 = s_3 = \sin \omega l$. With this simplification the spectral equation (43) can be solved explicitly:

$$\sin \omega l = \pm \frac{2\sqrt{t_1 t_2 \sin^2\left(\frac{\alpha_1 - \alpha_2}{2}\right) + t_1 t_3 \sin^2\left(\frac{\alpha_1}{2}\right) + t_2 t_3 \sin^2\left(\frac{\alpha_2}{2}\right)}}{t_1 + t_2 + t_3}.$$  \hspace{1cm} (44)

The absolute value of the right hand side is always less than or equal to one so that the allowed values of $\omega$ lie in continuous bands centered around $\omega = n \pi$ for $n \in \mathbb{Z}$. The bands will overlap with each other if and only if the right hand side of (44) equals $\pm 1$ for some value of $\alpha_1$ and $\alpha_2$. This can only happen if the scalar tension $t_i$ of one of the strings equals the sum of the other two. In this case the strings are all parallel and the string lattice is degenerate. On a non-degenerate lattice the out-of-plane excitations will always have finite band gaps. If one of the strings is a fundamental string the band gap will be finite but very narrow at weak string coupling.

For generic values of the string lengths the detailed analysis of the band spectrum becomes more complicated but the qualitative features are unchanged. The same method can be used to derive the appropriate generalization of the in-plane spectral equation (41) and obtain from it the band spectrum of in-plane modes.

Rey and Yee [7] have also studied aspects of propagation on string lattices. They claim to find an ‘evanescent bound state’ for which we see no evidence. It may be that their state is an approximate manifestation of the opening of the band gap which we find to be a generic lattice feature, but we have not tracked down the precise correspondence.

7 Discussion

We have explored the infrared dynamics of string junctions and string networks. Only the most basic features of the relativistic string entered into our considerations, so our results are presumably quite reliable at low energy. If anything, it is surprising how much structure the system has, given how little goes into it in the way of dynamical information.

One important issue is the energy scale beyond which the simple viewpoint adopted in the present paper becomes inadequate. Clearly, when mode energies approach the string scale we expect particle production in the worldsheet theory at a string junction,
in which case the S-matrix can no longer be determined by its action on one-particle states alone. At weak string coupling, $g_s << 1$, there is a lower energy scale where new physics enters. The worldsheet theory of strings carrying $q$ units of R-R charge is an $N = 8$ supersymmetric $U(q)$ gauge theory and the $(p,q)$ string corresponds to a vacuum in this theory with $p$ quarks in the fundamental representation of $U(q)$ placed at infinity [10]. In this vacuum, massless excitations carry $U(1)$ charge but are $SU(q)$ singlets, as is required by the $SL(2,\mathbb{Z})$ duality of type IIB string theory. The mass gap for the non-abelian degrees of freedom is $\Delta m \sim g_s/\sqrt{\alpha'}$, and when our mode energies approach this scale the worldsheet dynamics will become non-trivial. For an isolated string junction, our results will be valid for modes with wavelengths longer than $\sqrt{\alpha'}/g_s$, and similarly, our spectral equations for string lattices will hold provided the lattice spacing is sufficiently large, $l > \sqrt{\alpha'}/g_s$. It would be interesting to identify the leading effects of the non-abelian worldsheet structure as the relevant energy scale is approached from below.

Our results lay the groundwork for attacking at least two more questions of potential interest. The first has to do with the nature of the conformal invariance that survives in a stringy system beyond the string tree level. The second has to do with the properties of black hole analog states constructed by wrapping brane networks, rather than the usual D-branes, about compact tori. Do we find a new class of extremal black holes and their near-extremal relatives, or are we seeing old friends in new clothes? We hope to examine these questions in future investigations.

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