RADIUS OF INJECTIVITY FOR HARMONIC MAPPINGS WITH FIXED ANALYTIC PART

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Abstract. In this paper, we study non sense-preserving harmonic mappings \( f = h + g \) in \( \mathbb{D} \) when its analytic part \( h \) is convex and injective in \( \mathbb{D} \) and obtain radius of injectivity.

1. Introduction

A complex valued function \( f \) is said to be harmonic in a domain \( \Omega \subset \mathbb{C} \) if it satisfies \( f_{zz}(z) = 0 \) for all \( z \in \Omega \). If \( \Omega \) is simply connected then such functions can be represented as \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \Omega \). Furthermore, if \( g(0) = 0 \), then this representation is unique. Let \( \text{Har}(\mathbb{D}) \) denote the class of harmonic mappings \( f \) in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) with the normalization \( h(0) = h'(0) - 1 = 0 \) and \( g(0) = 0 \). Such mappings \( f \) are uniquely determined by the coefficients of power series

\[
\begin{align*}
   h(z) &= z + \sum_{n=2}^{\infty} a_n z^n, \\
   g(z) &= \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{D}).
\end{align*}
\]

Here \( h \) is analytic and \( g \) is co-analytic part of \( f \). The Jacobian \( J_f(z) \) of \( f = h + \overline{g} \in \text{Har}(\mathbb{D}) \) is \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 \). A function \( f \in \text{Har}(\mathbb{D}) \) is locally injective in \( \mathbb{D} \) if and only if the Jacobian \( J_f(z) \) is non-vanishing in \( \mathbb{D} \), and sense-preserving if \( J_f(z) > 0 \) in \( \mathbb{D} \) (see [8]). A harmonic mapping \( f \) is said to be close-to-convex if \( f(\mathbb{D}) \) is close-to-convex, i.e., the complement of \( f(\mathbb{D}) \) can be written as disjoint union of non-intersecting half lines. The study of harmonic mappings have attracted the attention of complex analysts after Clunie and Sheil-Small [5]. For recent results in harmonic mappings, we refer to [1, 3, 9, 13] and the references therein.

Let \( \text{Hol}(\mathbb{D}) \) denote the class of holomorphic functions \( f \) in \( \mathbb{D} \) that are normalized by \( f(0) = f'(0) - 1 = 0 \) and \( \mathcal{S} \) denote the subclass of \( \text{Hol}(\mathbb{D}) \) of injective holomorphic functions in \( \mathbb{D} \). Note that \( \text{Hol}(\mathbb{D}) \subset \text{Har}(\mathbb{D}) \). Let \( \mathcal{K} \) denote the class of analytic functions \( f \in \text{Hol}(\mathbb{D}) \) such that \( f(\mathbb{D}) \) is convex. It is well known that, convexity of analytic functions in \( \mathbb{D} \) is a hereditary property; that is, if \( f \) is convex in \( \mathbb{D} \), then \( f(\mathbb{D}_r) \) is convex for every \( r \) \((0 < r < 1)\), where \( \mathbb{D}_r = \{ z : |z| < r, \ 0 < r < 1 \} \). An analytic function \( \text{Hol}(\mathbb{D}) \) is said to be starlike function of order \( \alpha \) \((0 \leq \alpha < 1)\), if \( \Re(zf'(z)/f(z)) > \alpha \) \((z \in \mathbb{D})\). Let

2010 Mathematics Subject Classification. 30C45, 30C20, 31A05.

Key words and phrases. Harmonic mappings, Univalent Functions, Convex functions, Radius of injectivity.
\( \mathcal{B} \) denote the set of all analytic functions \( w \) in \( \mathbb{D} \) such that \(|w(z)| \leq 1\) in \( \mathbb{D} \). A function \( w \in \mathcal{B} \) satisfies the inequality
\[
|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D},
\] (see [10, p. 168]).

The analytic parts of harmonic mappings are significant in shaping their geometric properties. For example, if \( h \) is convex injective and \( f = h + g \in \mathcal{H}ar(\mathbb{D}) \) is sense-preserving, then \( f(\mathbb{D}) \) is close-to-convex [5]. In ([7, 12]) harmonic mappings \( f = h + g \in \mathcal{H}ar(\mathbb{D}) \) have been studied, where \(|g'(0)| = \alpha \in (0, 1)\), \( h \) is convex in one direction in \( \mathbb{D} \) and the dilatation \( w \) is given by \( w(z) = (z + \alpha)/(1 + \alpha z) \). In [2], Bshouty et al. proved the following result of \( f = h + g \in \mathcal{H}ar(\mathbb{D}) \) when \( h \) is convex in \( \mathbb{D} \).

**Lemma 1.1.** Let \( h \) be analytic and convex in \( \mathbb{D} \). Then every harmonic mapping \( f = h + g \) where \( g'(z) = w(z)h'(z); \ |w(z)| < 1 \) is close-to-convex in \( \mathbb{D} \).

Note that, the harmonic mapping in Lemma 1.1 is sense-preserving. In this article, we consider the case of Lemma 1.1 when harmonic mapping \( f = h + g \) is not necessarily sense-preserving in \( \mathbb{D} \) but satisfies \( g(z) = w(z)h(z) \ (w \in \mathcal{B}) \). We observe that such harmonic mappings are not necessarily sense-preserving and injective in \( \mathbb{D} \). For example, the harmonic mapping
\[
(1.3) \quad f_1(z) = \frac{z}{1 - z} - \frac{\bar{z}}{2}, \quad z \in \mathbb{D},
\]
is not sense-preserving in \( \mathbb{D} \) as \(|g'(-1/2)/h'(-1/2)| = 9/8 > 1 \) and not injective in \( \mathbb{D} \) (see Figure 1).

![Figure 1. The images of \( \mathbb{D} \) under \( f_1 \).](image-url)
Lemma 1.2. Suppose that \( f = h + \overline{g} \in \text{Har}(\mathbb{D}) \) is sense-preserving in \( \mathbb{D} \) such that \( h \) is injective in \( \mathbb{D} \). Then the radius of injectivity and close-to-convexity of \( f \) is \( 2 - \sqrt{3} \).

In this article, we consider the case of Lemma 1.2 when harmonic mapping \( f = h + \overline{g} \) is not necessarily sense-preserving in \( \mathbb{D} \) but satisfies \( g(z) = w(z)h(z) \) (\( w \in \mathcal{B} \)). We observe that such harmonic mappings are not necessarily sense-preserving and injective in \( \mathbb{D} \). For example, the harmonic mapping

\[
(1.4) \quad f_2(z) = \frac{z}{(1-z)^2} - \frac{z}{2(1-z)}, \quad z \in \mathbb{D},
\]

is not sense-preserving as \( |g'(-1/2)/h'(-1/2)| = 3/2 > 1 \) and not injective in \( \mathbb{D} \) (see Figure 2).

2. Main Results

To prove our results, we shall use the following Lemma. This Lemma was appeared in [4] for \( \mathbb{D} \), but observing it’s proof, we see that this result is valid for all subdisk \( \mathbb{D}_r \). The proof of this special case is so short that we include it here for completeness.

Lemma 2.1. Let \( f = h + \overline{g} \) be a sense-preserving harmonic mapping in \( \mathbb{D}_r \), \( 0 < r < 1 \) and \( h \) is injective convex in \( \mathbb{D} \). Then \( f \) is injective in \( \mathbb{D}_r \).

Proof. Let \( \Omega = h(\mathbb{D}_r) \), \( 0 < r < 1 \). Define \( \psi : \Omega \to \mathbb{C} \) by \( \psi(w) = g \circ h^{-1}(w) \). Then \( \psi \) is analytic in convex domain \( \Omega \) and \( \psi'(w) = g'(w)/h'(w) \), where \( w = \psi^{-1}(z) \) and \( |\psi'(w)| < 1 \). Now, let \( z_1, z_2 \in \mathbb{D}_r \), \( z_1 \neq z_2 \) such that \( f(z_1) = f(z_2) \), this is equivalent to \( h(z_1) - h(z_2) = -(g(z_1) - g(z_2)) \). Set \( w_1 = h(z_1) \) and \( w_2 = h(z_2) \) so that \( w_1 - w_2 = -(g(z_1) - g(z_2)) \). As \( h^{-1}(w_1) = z_1 \) and \( h^{-1}(w_2) = z_2 \), we have

\[
w_1 - w_2 = -\left( g(h^{-1}(w_1)) \right) - \left( g(h^{-1}(w_2)) \right) = \overline{\psi(w_2)} - \overline{\psi(w_1)}.
\]
Because $\psi$ is analytic on the convex domain $\Omega$, we have $w_1 - w_2 = \int_{[w_1, w_2]} \psi'(w) \, dw$, which is not possible as $|\psi'(w)| < 1$ in $\Omega$. Thus $f(z_1) \neq f(z_2)$. This shows the injectivity of $f$ in $\mathbb{D}_r$. \hfill \square

First we prove the following sharp result for harmonic mappings with injective and convex analytic part.

**Theorem 2.1.** Let $f = h + \overline{g} \in \text{Har}(\mathbb{D})$ such that $h$ is injective and convex in $D_{1/3}$ and $g(z) = w(z)h(z)$, where $w \in \mathcal{B}$. Then $f$ is sense-preserving and injective in $D_{1/3}$. The results is sharp.

**Proof.** As convexity is hereditary property, $h$ is injective and convex in $D_{1/3}$. Thus, in view of Lemma 2.1, it is sufficient to prove that $f$ is sense-preserving in $D_{1/3}$. A convex function is starlike of order $1/2$ [6, Theorem 2.3.2]. Hence

$$\frac{zh'(z)}{h(z)} = \frac{1}{1 - w(z)},$$

where $w \in \mathcal{B}$. This gives $|h(z)| \leq |z|(1 + |z|)|h'(z)|$ ($z \in \mathbb{D}$). Using this inequality and (1.2), we have

$$|g'(z)| \leq |w(z)||h'(z)| + |h(z)||w'(z)|$$

$$= \left( |w(z)| + \frac{|z|(1 - |w(z)|^2)}{1 - |z|} \right)|h'(z)|.$$

If $|z| < \frac{1}{3}$, then

$$|w(z)| + \frac{|z|(1 - |w(z)|^2)}{1 - |z|} < \frac{1}{2}(2|w(z)| + 1 - |w(z)|^2)$$

$$\leq \frac{1}{2}(2|w(z)| + 2(1 - |w(z)|)) = 1.$$

Therefore $|g'| < |h'|$, hence $f$ is sense-preserving in $\mathbb{D}_{1/3}$. To show the sharpness, let

$$h(z) = \frac{z}{1 + z} \text{ and } w(z) = \frac{z + \zeta}{1 + \zeta z}$$

where $\zeta \in [-1, 1]$. We deduce that $g'(r) = U(r, \zeta)h'(r)$, where

$$U(r, \zeta) = \frac{r + \zeta}{1 + r\zeta} + \frac{r(1 + r)(1 - \zeta^2)}{(1 + r\zeta)^2}.$$

Note that $U(r, 1) = 1$ and

$$\left. \frac{\partial}{\partial \zeta} U(r, \zeta) \right|_{\zeta=1} = \frac{1 - 3r}{1 + r}.$$
Choose $r$ such that $r \in [1/3, 1)$, then $\frac{\partial}{\partial r} U(r, \zeta) \bigg|_{\zeta=1} \leq 0$. Hence $U(r, 1 - \varepsilon) > 1$ for each $\varepsilon > 0$. This gives $g'(r) > h'(r) > 0$, for each $r \in [1/3, 1)$. Thus $f$ is not sense-preserving in $|z| < r$ if $r > 1/3$. This complete the proof.

If dilatation $w$ has the form $w(z) = e^{i\theta} z^n$ ($\theta \in \mathbb{R}, n \geq 1$) and $w(z) = c$ ($c \in \mathbb{C}, |c| < 1$), then we have

\textbf{Corollary 2.1.} Let $f = h + \overline{g} \in \text{Har}(\mathbb{D})$ such that $h$ is injective and convex in $\mathbb{D}$ and $g(z) = e^{i\theta} z^n h(z)$ ($\theta \in \mathbb{R}, n \geq 1$), then $f$ is injective in $\mathbb{D}_{r_{n,1}}$, where $r_{n,1}$ is the unique root of $n r^{n+1} + (n+1) r^n - 1 = 0$ in the interval $(0, 1)$. The constant $r_{n,1}$ cannot be improved. The constant $r_{n,2}$ cannot be improved.

| $n$   | 1    | 2    | 3    | 4    | 5    |
|-------|------|------|------|------|------|
| $r_{n,1}$ | $\approx 0.414$ | 0.5  | $\approx 0.5604$ | $\approx 0.6058$ | $\approx 0.6415$ |

\textit{Proof.} From the hypothesis $g(z) = e^{i\theta} z^n h(z)$, we obtain

\[ |g'(z)| = \left| n e^{i\theta} z^{n-1} \frac{h(z)}{h'(z)} + e^{i\theta} z^n \right| \leq (n|z|^{n+1} + (n+1)|z|^n)|h'(z)|. \]

Hence, $|g'(z)| < |h'(z)|$ if $n|z|^{n+1} + (n+1)|z|^n \leq 1$. Thus, $f$ is sense-preserving in $\mathbb{D}_{r_{n,1}}$, where $r_{n,1}$ is the unique root of $n r^{n+1} + (n+1) r^n - 1 = 0$ in the interval $(0, 1)$.

\textbf{Corollary 2.2.} Let $f = h + \overline{g} \in \text{Har}(\mathbb{D})$ such that $h$ is injective and convex in $\mathbb{D}$ and $g(z) = c h(z)$ ($c \in \mathbb{C}, |c| < 1$), then $f$ is sense-preserving and close-to-convex in $\mathbb{D}$.

\textit{Proof.} Note that $f$ is sense-preserving in $\mathbb{D}$, hence in view of Lemma 1.1, $f$ is close-to-convex in $\mathbb{D}$.

Now we prove the following sharp result for harmonic mappings with injective analytic part.

\textbf{Theorem 2.2.} Let $f = h + \overline{g} \in \text{Har}(\mathbb{D})$ such that $h$ is injective in $\mathbb{D}$ and $g(z) = w(z) h(z)$, where $w \in \mathcal{B}$. Then $f$ is sense-preserving and injective in $\mathbb{D}_{2 - \sqrt{3}}$. The result is sharp.

\textit{Proof.} It is well known that the radius of convexity for the class $\mathcal{S}$ is $2 - \sqrt{3}$ (see [6, Theorem 2.2.22]). Thus, in view of Lemma 2.1, it is sufficient to prove that $f$ is sense-preserving in $\mathbb{D}_{2 - \sqrt{3}}$. For $h \in \mathcal{S}$, we have

\[ |h(z)| \leq \frac{|z|(1 + |z|)}{1 - |z|} |h'(z)|, \quad z \in \mathbb{D}, \]
(see [6, Theorem 1.1.6]). Using this inequality and (1.2), we have
\begin{equation}
|g'(z)| \leq \left( |w(z)| + \frac{|z|(1 - |w(z)|^2)}{(1 - |z|)^2} \right) |h'(z)|.
\end{equation}

If $|z|^2 - 4|z| + 1 > 0$, then
$$|w(z)| + \frac{|z|(1 - |w(z)|^2)}{(1 - |z|)^2} < 1,$$

hence $|g'| < |h'|$ in $D$. Therefore, $f$ is sense-preserving in a disk $D_r$, where $r$ is unique root of $r^2 - 4r + 1 = 0$ in the interval $(0, 1)$. This shows that $f$ is sense-preserving in $D_{2-\sqrt{3}}$.

To show the sharpness, let $h(z) = \frac{z}{(1 + z)^2}$ and $w(z) = \frac{z + \zeta}{1 + \zeta z}$, where $\zeta \in [-1, 1]$. A computation gives $g'(r) = V(r, \zeta)h'(r)$, where
$$V(r, \zeta) = \frac{r + \zeta}{1 + r\zeta} + \frac{1 - \zeta^2}{(1 + r\zeta)^2} \frac{r(1 + r)}{1 - r}.$$  

Note that $V(r, 1) = 1$ and
$$\left. \frac{\partial}{\partial \zeta} V(r, \zeta) \right|_{\zeta=1} = \frac{1 - 4r + r^2}{1 - r^2}.$$

Choose $r$ such that $r \in [2 - \sqrt{3}, 1)$, then $\left. \frac{\partial}{\partial \zeta} V(r, \zeta) \right|_{\zeta=1} < 0$. Therefore $V(r, 1 - \epsilon) > 1$ for $\epsilon > 0$. This shows that $g'(r) > h'(r) > 0$ for $r \in (2 - \sqrt{3}, 1)$. Thus $f$ is not injective in $|z| < r$ if $r > 2 - \sqrt{3}$. This complete the proof. \hfill \Box

If dilatation $w$ has the form $w(z) = e^{i\theta} z^n$ ($\theta \in \mathbb{R}, n \geq 1$), then we have

**Corollary 2.3.** Let $f = h + \overline{g} \in \mathcal{H}ar(\mathbb{D})$ such that $h$ is injective in $\mathbb{D}$ and $g(z) = e^{i\theta} z^n h(z)$ ($\theta \in \mathbb{R}, n \geq 1$), then $f$ is injective in $D_{r_{n,2}}$, where $r_{n,2}$ is the unique root of $(n - 1)r^{n+1} + (n + 1)r^n + r - 1 = 0$ in the interval $(0, 1)$. The constant $r_{n,2}$ cannot be improved.

| $n$ | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|
| $r_{n,2}$ | $\approx 0.333$ | $\approx 0.414$ | $\approx 0.4738$ | $\approx 0.5201$ | $\approx 0.5574$ |

**Proof.** We have
\begin{align*}
|g'(z)| &= \left| n e^{\theta} z^{n-1} \frac{h(z)}{h'(z)} + e^{i\theta} z^n \right| \\
&\leq \left( n |z|^{n-1} \frac{|z|(1 + |z|)}{1 - |z|} + |z|^n \right) |h'(z)|.
\end{align*}
Hence, $|g'(z)| < |h'(z)|$ if $(n - 1)|z|^{n+1} + (n + 1)|z|^n + |z| - 1 \geq 0$. Thus, $f$ is sense-preserving in $D_{r_{n,2}}$, where $r_{n,2}$ is the unique root of $(n - 1)r^{n+1} + (n + 1)r^n + r - 1 = 0$ in the interval $(0, 1)$. □

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