The Logarithmic Fib-Binomial Formula

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Abstract

Steven Romans Logarithmic Binomial Formula analogue has been found and is presented here also for the case of fibonomial coefficients - which recently have been given a combinatorial interpretation by the present author.

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1 Introduction

The aim of this note is to find out - as in [14, 15] of Steven Roman - the form of "Fib corresponding" Logarithmic Fib-Binomial Formula. In [14, 15] Steven Roman introduced The Logarithmic Binomial Formula :

$$\lambda_n^{(t)}(x + a) = \sum_{k \geq 0} \left[ \begin{array}{c} n \\ k \end{array} \right] \lambda_{n-k}^{(t)}(a)x^k; \quad t = 0, 1; \quad |x| < a; \quad n \in \mathbb{Z}$$

where

$$\left[ n \right] = \begin{cases} n & n \neq 0 \\ 1 & n = 0 \end{cases}$$
and the Roman factorial is given by

\[
[n]! = \begin{cases} 
  n! & n \geq 0 \\
  \frac{(-1)^{n+1}}{(-n-1)!} & n < 0 
\end{cases}
\]  

while hybrid binomial coefficients (Roman coefficients) read:

\[
\binom{n}{k} = \frac{[n]!}{[k]! [n-k]!}  
\]  

One may show that (Propositions 3.2, 4.1, 4.2, 4.3 in [11])

\[
\binom{n}{k} = \begin{cases} 
  \binom{n}{k} & n, k \geq 0 \\
  (-1)^k \binom{-n-1+k}{k} & k \geq 0 \geq n \\
  (-1)^{k+n} \binom{-k-1}{n-k} & 0 > n \geq k \\
  (-1)^{(n+k)} \left[ \Delta^n \frac{1}{x-k} \right]_{k=0} & k > n \geq 0 
\end{cases}
\]  

As seen from the above the hybrid binomial coefficients (Roman coefficients) are the intrinsic natural extension of binomial coefficients.

The Logarithmic Binomial Formula extends the notion of binomiality of polynomials as used in the Generalized Umbral Calculus (see Chapter 6 in [16] for functional formulation and see [8] for abundant references on Finite Operator Calculus of Rota formulation).
The great invention of Steven Roman - among others - relies on the fact that the real i.e. $R$-linear span $L$ of the basis functions (harmonic logarithms-see: Proposition 4.1 in [14])

$$L = \text{span} \left\{ \lambda_{n}^{(t)} \right\}_{n \in \mathbb{Z}, t=0,1}$$

allows the Fundamental Theorem of Calculus to hold on $L$ i.e. $D^{-1}D = DD^{-1} = id_L$. Here $D^{-1}$ depending on whether $t = 0$ or $t = 1$ acts as follows:

$$D^{-1} = \int_{0}^{x} \text{ on } \lambda_{n}^{(0)} \text{, } n \neq -1, \text{ } n \in \mathbb{Z} \text{ and gives } 0 \text{ for } n = -1$$

and

$$D^{-1} = \int_{1}^{x} \text{ on } \lambda_{n}^{(1)}; \text{ } n \in \mathbb{Z}$$

## 2 Fibonomial Coefficients

In [5] Fibonomial coefficients [12, 2, 3, 4] have been given a combinatorial interpretation as counting the number of finite "birth-self similar" subposets of an infinite poset. We shall use here the following notation: Fibonomial coefficients are defined as \( \left( \begin{array}{c} n \\ k \end{array} \right)_F = \frac{F_n}{F_k!F_{n-k}} \) or - usefully for our purpose here

\( \left( \begin{array}{c} n \\ k \end{array} \right)_F \equiv \frac{n_F!}{k_F!} \) where we make an analogy driven [8] identifications: \( n_F \neq 0,n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F1_F; \text{ } 0_F! = 1; \)

\( n_F = n_F(n-1)_F \ldots (n-k+1)_F \). This is the appropriate specification of notation from [8] for the purpose Fibonomial Finite Operator Calculus case investigation (see Example 2.1 in [9]).

Let us now introduce an infinite poset $P$ (for further details see: [8]) via its finite part subposet $P_m$ Hasse diagram to be continued ad infinitum in an obvious way as seen from the figure below. It looks like the Fibonacci tree with a specific "cobweb": see Figure 1. One sees that the $P_m$ is the subposet of $P$ consisting of points up to $m$-th level points

$$\bigcup_{s=1}^{m} \Phi_s; \text{ } \Phi_s \text{ is the set of elements of the } s \text{ - th level}$$

How many $P_m$'s rooted at the $k$-th level might be found?

We answer this question in the following sequence of observations right after Figure 1.
Observation 2.1. The number of maximal chains starting from the root (level $F_1$) to reach any point at the $n$-th level labeled by $F_n$ is equal to $n F!$.

Observation 2.2. The number of maximal chains starting from the level labeled by $F_k$ to reach any point at the $n$-th level labeled by $F_n$ is equal to $n^m_{F}$, $(n = k + m)$.

Observation 2.3. Let $n = k + m$. The number of subposets $P_m$ rooted at the level labeled by $F_k$ and ending at the $n$-th level labeled by $F_n$ is equal to

$$\binom{n}{m}_F = \binom{n}{k}_F = \frac{n^k_F}{k_F!}.$$

3 The Logarithmic Fibonomial Case

We shall now adopt the $\ast_{\psi}$ product formalism [9] (see also Appendix in [6] and [7]) to the Fibonomial case with $\exp_F \{x\} = \sum_{k=0}^{\infty} x^k / k_F!$ defining the $F$-exponential series.

3.1 $F$ product

Let $n > 0$ and let $\partial_F$ be a linear operator acting on formal series and defined accordingly by $\partial_F x^n = n_F x^{n-1}; \ n \geq 0, \ \partial_F x^0 = 0$.

We shall call the $F$-multiplication the new $F$ product of functions or formal series specified below.
Notation 3.1. \( x^* F x^n = \hat{x}_F(x^n) = \frac{(n+1)}{(n+1)!} x^{n+1}; \) \( n \geq 0 \) hence \( x^* F 1 = x \) and \( x^* F \alpha = \alpha x \) \( \forall x, \alpha \in \mathbb{R}, f(x) * F x^n = f(\hat{x}_F) x^n. \)

For \( k \neq n \), \( x^n * F x^k \neq x^k * F x^n \) as well as \( x^n * F x^k \neq x^{n+k} \) - in general.

**Definition 3.1.** With Notation 3.1 adopted define the \( * F \) powers of \( x \) according to

\[
x^{n \hat{F}} \equiv x * F x^{(n-1) \hat{F}} = \hat{x} (x^{(n-1) \hat{F}}) = x * F x * F \ldots * F x = \frac{n!}{n!} x^n; \quad n \geq 0.
\]

Note that \( x^{n \hat{F}} * F x^k \hat{F} = \frac{n!}{n!} x^{(n+k) \hat{F}} \neq x^k \hat{F} * F x^n \hat{F} = \frac{k!}{k!} x^{(n+k) \hat{F}} \) for \( k \neq n \) and \( x^0 \hat{F} = 1. \)

This noncommutative \( F \)-product \( * F \) is devised so as to ensure the observations below.

**Observation 3.1.**

(a) \( \partial_F x^{n \hat{F}} = n x^{(n-1) \hat{F}}; \quad n \geq 0; \)
(b) \( \exp_F[\alpha x] \equiv \exp\{\alpha \hat{x}_F\} 1; \)
(c) \( \exp[\alpha x] * F \{\exp_F \{\beta \hat{x}_F\} 1\} = \exp_F \{[\alpha + \beta] \hat{x}_F\} 1; \)
(d) \( \partial_F (x^k * F x^{n \hat{F}}) = (D x^k) * F x^{n \hat{F}} + x^k * F (\partial_F x^n \hat{F}); \)
(e) Leibniz rule \( \partial_F (f * F g) = (D f) * F g + f * F (\partial_F g); f, g \)- formal series;
(f) \( f(\hat{x}_F) g(\hat{x}_F) 1 = f(x) * F g; \quad \hat{g}(x) = g(\hat{x}_F) 1. \)

### 3.2 \( F \)-Integration

Let : \( \partial_0 x^n = x^{n-1} \). The linear operator \( \partial_0 \) is identical with divided difference operator. Let \( \hat{Q} f(x) = f(qx) \). Recall that to the Jackson \( \partial_q \) derivative \( \hat{Q} \) there corresponds the \( q \)-integration which is a right inverse operation to ”\( q \)-difference-ization”. Namely \( \hat{Q} \)

\[
F(z) := \left( \int_q \varphi \right) (z) := (1 - q)z \sum_{k=0}^{\infty} \varphi(q^k z) q^k
\]  

(6)
\[ F(z) \equiv \left( \int_q \varphi \right)(z) = (1-q)z \left( \sum_{k=0}^{\infty} q^k \hat{Q}^k \varphi \right)(z) = \left( (1-q)\hat{z} \frac{1}{1-q\hat{Q}} \right)(z) \]

where \((\hat{z}\varphi)(z) = z\varphi(z)\).

Of course
\[ \partial_q \circ \int_q = id \]

as
\[ \frac{1-q\hat{Q}}{1-q} \partial_0 \left( (1-q)\hat{z} \frac{1}{1-qQ} \right) = id \]

Naturally (9) might serve to define a right inverse to Jackson's "q-difference-ization" \((\partial_q \varphi)(x) = \frac{1-q\hat{Q}}{1-q} \partial_0 \varphi(x)\) and consequently the "q-integration " as represented by (6) and (7). As it is well known the definite q-integral is an numerical approximation of the definite integral obtained in the \(q \rightarrow 1\) limit.

Finally we introduce the analogous representation for \(\partial_F\) difference-ization

\[ \partial_F = \hat{n}_F \partial_0; \quad \hat{n}_F x^{n-1} = n_F x^{n-1}; \quad n \geq 1. \]

Then
\[ \int_F x^n = \left( \hat{x} \frac{1}{\hat{n}_F} \right) x^n = \frac{1}{(n+1)_F} x^{n+1}; \quad n \geq 0 \]

and of course
\[ \partial_F \circ \int_F = id. \]

Naturally \((\int_F \equiv \int d_F)\)

\[ \partial_F \int_a^x f(t)d_F t = f(x). \]

The formula of "per partes" \(F\)-integration is easily obtainable from Observation (3.1) and it reads:

\[ \int_a^x (f \ast_F \partial_F g)(t)d_F t = [(f \ast_F g)(t)]_a^x - \int_a^x (D_f \ast_F g)(t)d_F t. \]

Now in order to have \(\partial_F^{-1}\) an \(F\)-analogue of \(D^{-1}\) as in [13, 14] thus causing the fundamental Theorem of Calculus to hold for \(\partial_F\)-difference-ization and \(\int_F\)-integration on some linear space \(L_F\) being the linear span of
"F-harmonic logarithms" - we shall proceed exactly as Steven Roman in [14 15].

4 The Logarithmic Fib-Binomial Formula

As in [14 15] of Roman - we have also The Logarithmic Fib-Binomial Formula (see: Propositions 4.1, 4.2 below):

\[
\phi_n^{(t)}(x_F a) \equiv \exp \left\{ a \partial_F \phi_n^{(t)} \right\} (x) = \sum_{k \geq 0} \left[ \begin{array}{c} n \\ k \end{array} \right]_F \phi_{n-k}^{(t)}(a) x^k \quad t = 0, 1; \ |x| < a; \ n \in \mathbb{Z}
\]

where ( more on "\(+ F\)" see [8, 9])

\[
[n_F] = \begin{cases} 
   n_F & n \neq 0 \\
   1 & n = 0 
\end{cases}
\]

and the Roman Fib-factorial is given by

\[
[n_F]! = \begin{cases} 
   n_F! & n \geq 0 \\
   \frac{(-1)^{n+1}}{(-n-1)!} & n < 0 
\end{cases}
\]

while Fib-hybrid binomial coefficients or Roman Fib-coefficients (see: [10 11]) read:

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_F = \frac{[n]!}{[k]![n-k]!} 
\]

One observes (as in Propositions 3.2, 4.1, 4.2, 4.3 in [11]) that:

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_F = \begin{cases} 
   \left( \begin{array}{c} n \\ k \end{array} \right)_F & n, k \geq 0 \\
   (-1)^k \left( \begin{array}{c} n-1+k \\ n \end{array} \right)_F & k \geq 0 > n \\
   (-1)^{k+n} \left( \begin{array}{c} n-k-1 \\ n-k \end{array} \right)_F & 0 > n \geq k \\
   (-1)^{n+k} \left[ \Delta F_{\frac{n}{2-k}} \right]_{k=0} & k > n \geq 0 
\end{cases}
\]

\[
\left[ \begin{array}{c} 0 \\ k \end{array} \right]_F = \left[ \begin{array}{c} 0 \\ -k \end{array} \right]_F = \frac{(-1)^{k+1}}{k_F}
\]
\[
\begin{align*}
[n]_F &= \begin{bmatrix} n \end{bmatrix}_F, \quad [n]_F \cdot [j]_F = \begin{bmatrix} n \end{bmatrix}_F \cdot \begin{bmatrix} n-j \end{bmatrix}_F \\
[k]_F &= \begin{bmatrix} n-k \end{bmatrix}_F, \quad [j]_F \cdot [k]_F = \begin{bmatrix} n-j \end{bmatrix}_F \cdot \begin{bmatrix} n-j-k \end{bmatrix}_F
\end{align*}
\]

(17)

\[
\begin{align*}
[n]_F &= \begin{bmatrix} n-1 \end{bmatrix}_F + \begin{bmatrix} n-1 \end{bmatrix}_F \\
[k]_F &= \begin{bmatrix} n-k-1 \end{bmatrix}_F + \begin{bmatrix} n-k \end{bmatrix}_F
\end{align*}
\]

(18)

where (see: pp.333-334 in [8])

\[
\Delta_F = \exp_F \{ \partial_F \} - id.
\]

Fib-Roman coefficients (as seen from the above) are then also natural "relative" of binomial coefficients among the family of ψ - binomial ones [8] (consult also Example 2.1 in [9]).

The Logarithmic Fib-Binomial Formula extends the notion of binomiality of polynomials as used in the Generalized Umbral Calculus (see Chapter 6 in [16] for functional formulation and see [8] for abundant references on Finite Operator Calculus of Rota formulation)- to sequences of functions - (compare with [1]).

Here the importance of the great invention of Steven Roman - among others - relies on the fact that the \( R \)-linear span \( L_F \) of now basis Fib-harmonic logarithms functions

\[
\{ \phi_{n}^{(t)} \}_{n \in \mathbb{Z}, t=0,1}, \quad L_F = \text{span} \left\{ \phi_{n}^{(t)} \right\}_{n \in \mathbb{Z}, t=0,1}
\]

allows the Fundamental Theorem of Calculus to hold also on \( L_F \), i.e. \( \partial_F^{-1} \partial_F = id_{L_F} \) for \( \partial_F \)- difference-ization and \( \int_F \) - integration acting on a linear space \( L_F \) being the linear span of "Fib-harmonic logarithms". Here anti-difference-ization operator \( \partial_F^{-1} \)- depending on whether \( t = 0 \) or \( t = 1 \) - acts as follows on Fib-harmonic logarithm functions:

\[
\partial_F^{-1} = \int_0^x d_F \text{ on } \phi_n^{(0)} \quad n \neq -1, \quad n \in \mathbb{Z} \text{ and gives } 0 \text{ for } n = -1
\]

\[
\text{and } \quad \partial_F^{-1} = \int_1^x d_F \text{ on } \phi_n^{(1)}; \quad n \in \mathbb{Z}
\]

Let us define these Fib-harmonic logarithms

\[
\{ \phi_{n}^{(t)} \}_{n \in \mathbb{Z}, t=0,1}
\]
-(see Proposition 2.2 in [14]) - as solutions of Fib-harmonic \( t \)-binomiality conditions. Thus Fib-harmonic logarithm functions are unique solutions of Fib-harmonic \( t \)-binomiality conditions; \( t = 0, 1 \) [19] - (compare with [1] and relaxation Lemma 2.12 therein):

1) \( \phi_0^{(0)}(x) = 1, \quad 2) \phi_n^{(0)}(0) = 0, \quad n \ni \mathbb{Z}\{0\}, \)

3) \( \partial_F \phi_n^{(0)} = [n_F] \phi_{n-1}^{(0)}, \quad n \ni \mathbb{Z} \)

\[ (19) \]

1) \( \phi_0^{(1)}(x) = \ln x, \quad 2) \phi_n^{(1)}(x) \text{ has no constant term}, \quad n \ni \mathbb{Z}, \)

3) \( \partial_F \phi_n^{(1)} = [n_F] \phi_{n-1}^{(1)}, \quad n \ni \mathbb{Z} \)

The Fib-harmonic \( t \)-binomiality conditions; \( t = 0, 1 \) [19] yield [14] what follows:

**Proposition 4.1.**

\[
\phi_n^{(0)}(x) = \begin{cases} 
  x^n & n \geq 0 \\
  0 & n < 0
\end{cases}, \quad \phi_n^{(1)}(x) = \begin{cases} 
  x^n(\ln x - f_n) & n \geq 0 \\
  x^n & n < 0
\end{cases},
\]

\[ f_0 = 0, \quad f_n = 1 + \frac{1}{2F} + \frac{1}{3F} + \ldots + \frac{1}{nF}, \quad n \ni \mathbb{N} \]

We shall call \( f_n, \quad n \ni \mathbb{N} \) the \textit{Fib-harmonic numbers} \( f_0 = 0 \), (see: [13]).

**Proposition 4.2.** The linear anti-difference-ization unique operator

\( \partial_F^{-1} : L_F \rightarrow L_F; \partial_F^{-1} \partial_F = id_{L_F} \) is given by

\[
\partial_F^{-1} \phi_n^{(0)} = \begin{cases} 
  \frac{1}{[n+1]_F} \phi_{n+1}^{(0)} & n \neq -1 \\
  0 & n = -1
\end{cases}, \quad \partial_F^{-1} \phi_n^{(1)} = \frac{1}{[n+1]_F} \phi_n^{(1)}, \quad n \ni \mathbb{Z}.
\]

**REMARK.** Instead of Roman Fib-coefficients and Roman Fib-factorial one may - (replace \( F \) by \( \psi \))- start to consider Roman \( \psi \)-coefficients, \( \psi \)-harmonic logarithms etc. However these seemingly might lack any "reasonable" combinatorial interpretation.
As the generally useful reading - also for this purpose one recommends here:

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