Existence and Hölder regularity of infinitely many solutions to a $p$-Kirchhoff type problem involving a singular and a superlinear nonlinearity without the Ambrosetti-Rabinowitz (AR) condition

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Abstract
We carry out an investigation of the existence of infinitely many solutions to a fractional $p$-Kirchhoff type problem with a singularity and a superlinear nonlinearity with a homogeneous Dirichlet boundary condition. Further the solution(s) will be proved to be bounded and a weak comparison principle has also been proved. A ‘$C^1$ versus $W^{s,p}_0$’ analysis has also been discussed.

Keywords: singularity, non-Ambrosetti-Rabinowitz condition, Cerami condition, multiplicity, symmetric Mountain-Pass theorem.

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1. Introduction

Off late, the problems involving a nonlocal and fractional operators have become hugely popular area of investigation owing to its manifold applications, viz. stratified materials, population dynamics, continuum mechanics, water waves, minimal surface problems etc. Interested readers may refer to [2, 5, 9, 16, 18, 23, 39] and the references therein.

The problem addressed in this article is as follows.

\[
\left( a + b \int_{\mathbb{R}^N} |u(x) - u(y)|^p K(x-y) dx dy \right) \mathcal{L}_p u - \lambda g(x) u^{p-1} = \mu h(x) u^{-\gamma} + f(x,u), \text{ in } \Omega \\
u > 0, \text{ in } \Omega \\
u = 0, \text{ in } \mathbb{R}^N \setminus \Omega \tag{1.1}
\]
where $\lambda, \mu > 0$, $f, g \geq 0$ are functions defined and bounded over $\Omega$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipshitz boundary $\partial \Omega$, $a, b > 0$, $0 < \gamma < 1$, $1 < p < \infty$, $p_s < N$, $s \in (0, 1)$. The function $f$ is a carathéodory function and the operator $\mathcal{L}_p^s$ is defined as

$$
\mathcal{L}_p^s u = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x-y)dy
$$

for all $x \in \mathbb{R}^N$ where $B_\epsilon = \{y : |y-x| < \epsilon\}$. The function $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ is measurable with the following properties ($P$):

1. $\rho K \in L^1(\mathbb{R}^N)$ where $\rho(x) = \min\{|x|^p, 1\}$
2. There exists $\delta > 0$ such that $K(x) \geq \delta|x|^{-N-ps}$ for all $x \in \mathbb{R}^N$
3. $K(x) = K(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

One can retrieve the fractional $p$-Laplacian operator if the ‘kernel’ $K$ is chosen to be $K(x) = |x-y|^{-N-ps}$. The discussion in Section 4.1 uses following condition.

$$(P') : \delta_1|x|^{-N-ps} \geq K(x) \geq \delta_2|x|^{-N-ps}$$

for all $x \in \mathbb{R}^N$ where $\delta_1, \delta_2 > 0$. In general, it is a practice to denote the Kirchhoff function as $\mathcal{M}$. In the current case $\mathcal{M}(t) = a + bt$. Therefore when $\mathcal{M}(t) \equiv 1$, $p = 2$, $\lambda = 0$, $g(x) = 1$ a.e. in $\Omega$, we reduce to the problem in (1.1) to

$$(-\Delta)^s u = \mu u^{-\gamma} + f(x, u), \text{ in } \Omega$$
$$u > 0, \text{ in } \Omega$$
$$u = 0, \text{ in } \mathbb{R}^N \setminus \Omega.$$  

(1.2)

For further details on the problem in (1.2) one may refer to [39]. The authors have used a variational technique to guarantee the existence of multiple solutions. Further results on existence of multiple solutions can be found in [4, 41]. In most of these studies, the authors obtained two distinct weak solutions. For the existence of infinite number of solutions but with a sublinear growth function can be found in [14].

Meanwhile, we direct the readers to a variety of forms for the function $\mathcal{M}$ [47, 10, 27, 17, 21, 19, 31, 43, 44]. With the advent of these references and with the help of fountain theorem, the authors in [20] have proved the existence of infinitely many solutions for a fractional $p$-Kirchhoff problem. In [33], the authors showed the existence and multiplicity of solutions to a degenerate fractional $p$-Kirchhoff problem. Recently Ghosh [14] has proved the existence of infinitely many solutions to a system of fractional Laplacian Kirchhoff type problem with a sublinear growth. Motivated from the work due to Ren et al [45] we will show the existence of the existence of infinitely many solutions to a $p$-Kirchhoff type problem with a superlinear growth without the AR condition. It will
also be proved that the solution (if exists) is in $L^\infty(\Omega)$. A weak comparison principle has also been proved. A little bit of history about the AR condition - this condition was first introduced by Ambrosetti and Rabinowitz (refer [1]) in 1973. Thereafter this condition formed a formidable tool in the analysis of elliptic PDEs, especially to prove the boundedness of the Palais-Smale (PS) sequences for the associated energy functional to the problem. To our knowledge there is no evidence in the literature that considered a $p$-Kirchhoff type problem with a singular nonlinearity. Therefore the problem considered and the results obtained here are new.

2. A simple physical motivation

This section is devoted to a physical motivation to the problem considered in this article. The explanation is physically heuristic but nevertheless gives a strong mathematical motivation to take-up this problem. We will confine ourselves to the one dimensional case of the model of an elastic string of finite length fixed at both the ends. The vertical displacement of the string will be represented by $u : [-1, 1] \times (0, \infty) \to \mathbb{R}$. Then, mathematically, the end point constraints can be expressed as

$$u(0, t) = u(2, t) = 0$$

for all $t \geq 0$. In order to identify this finite string with an infinite string one can consider

$$u(x, t) = 0$$

for all $x \in \mathbb{R} \setminus [0, 2]$, $t \geq 0$. Thus, the acceleration $\frac{\partial^2 u}{\partial t^2}$ of the vertical displacement $u$ of the vibrating string must be balanced (thanks to Newton's laws) by the elastic force of the string and by the external force field $f$. Therefore we have

$$\frac{\partial^2 u(x)}{\partial t^2} = m \frac{\partial^2 u(x)}{\partial x^2} + f(x), \text{ for all } x \in [0, 2], \ t \geq 0.$$

When the steady case is considered, we have

$$m \frac{\partial^2 u(x)}{\partial x^2} = f(x), \text{ in } [0, 2].$$

We quote here G.F. Carrier [12] which says that - it is well known that the classical linearized analysis of the vibrating string can lead to results which are reasonably accurate only when the minimum (rest position) tension and the displacements are of such magnitude that the relative change in tension during the motion is small. Taking this into account one can suppose that the tension due to small deformation is linear in form, then we have the following expression.

$$\mathcal{M}(l) = m_0 + 2Cl$$
where \( l \) is the increment in the length of the string with respect to its mean position, i.e.

\[
l = \int_0^2 \sqrt{1 + \left( \frac{\partial u}{\partial x} \right)^2} \, dx - 2,
\]

\( C > 0 \) is a constant of proportionality. Thus for small deformations we have

\[
\sqrt{1 + \left( \frac{\partial u}{\partial x} \right)^2} = 1 + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2.
\]

Hence,

\[
l = \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2.
\]

Therefore, the problem now boils down to a Kirchhoff type problem

\[
\left( m_0 + C \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 \right) \frac{\partial^2 u(x)}{\partial x^2} = f(x), \text{ for all } x \in [0, 2], \, t \geq 0
\]

\[
u(x) = 0, \text{ in } \mathbb{R} \setminus [0, 2]. \tag{2.1}
\]

In other words it models the vibration of a string.

\[3.\text{ Technical preliminaries and functional analytic set up}\]

We begin by giving the conditions of the function \( f \) which is is assumed to have a superlinear growth. Note that there are functions which are superlinear yet not satisfying the AR condition. Before that we quickly will recall the AR condition.

\[(AR) : \text{there exists constants } r > 0, \theta > \eta > 1 \text{ such that } 0 < \theta F(x, t) \leq tf(x, t) \text{ for any } x \in \Omega, t \in \mathbb{R} \text{ and } |t| \geq r. \tag{3.1}\]

Here \( F(x, t) = \int_0^t f(x, t) dt \). For example, \( f(x, t) = t^{-1} \sin(t) \). From (3.1) we have that \( F(x, t) \geq c_1 |t|^{\theta} - c_2 \) for any \((x, t) \in \Omega \times \mathbb{R}\), where \( \theta > \eta \) for \( c_1, c_2 > 0 \) constants. Another form is given by

\[
\lim_{|t| \to \infty} \frac{F(x, t)}{|t|^\theta} = \infty \text{ uniformly for } x \in \Omega. \tag{3.2}\]

Under the condition (3.2) the function \( f \) is superlinear at infinity. Observe that the example cited above satisfies (3.2) but not \( F(x, t) \geq c_1 |t|^{\theta} - c_2 \). Some important works
that has proved the existence of infinitely many solutions but without the AR condition can be found in [34]. we now give the assumptions made on the function \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \).

\((f_1)\) \( \exists C > 0 \) and \( q \in (p, p^*_p) \) such that \( |f(x, t)| \leq C(1 + |t|^{q-1}) \)

\((f_2)\) \( f(x, -t) = f(x, t) \forall (x, t) \in \Omega \times \mathbb{R} \)

\((f_3)\) \( \lim_{|t| \rightarrow \infty} \frac{F(x, t)}{|t|^{2p}} = \infty \) uniformly for all \( x \in \Omega \)

\((f_4)\) \( \lim_{|t| \rightarrow 0} \frac{f(x, t)}{|t|^{p-1}} = 0 \) uniformly for all \( x \in \Omega \)

\((f_5)\) \( \exists \tau > 0 \) such that \( t \mapsto \frac{f(x, t)}{t^{2p-1}} \) is decreasing if \( t \leq -\tau < 0 \) and increasing if \( t \geq \tau > 0 \forall x \in \Omega \)

\((f_6)\) \( \exists \sigma \geq 1 \) and \( T \in L^1(\Omega) \) satifying \( T(x) \geq 0 \) such that \( \mathcal{S}(x, t) \leq \sigma \mathcal{S}(x, t) + T(x) \forall x \in \Omega \) and \( 0 \leq |s| \leq |t| \), where \( \mathcal{S}(x, t) = \frac{1}{2p} tf(x, t) - F(x, t) \).

**Remark 3.1.** The condition \((f_6)\) was assumed by Jeanjean [24]. When \( \sigma = 1 \) one can see that the conditions \((f_5)\) and \((f_6)\) are equivalent. In general, there are functions (for example \( f(x, t) = 2p|t|^{2p-2t} \ln(1 + t^{2p}) + p \sin(t) \)) that satisfy \((f_6)\) but not \((f_5)\).

We assign \( Q = \mathbb{R}^{2N} \setminus C(\Omega) \times C(\Omega) \subset \mathbb{R}^{2N} \) where \( C(\Omega) = \mathbb{R}^N \setminus \Omega \). The space \( X \) will denote the space of Lebesgue measurable functions from \( \mathbb{R}^N \) to \( \mathbb{R} \) such that its restriction to \( \Omega \) of any function \( u \) in \( X \) belongs to \( L^p(\Omega) \) and

\[
\int_Q |u(x) - u(y)|^p K(x-y) dxdy < \infty.
\]

The space \( X \) is equipped with the norm

\[
\|u\|_X = \|u\|_{L^p(\Omega)} + \left( \int_Q |u(x) - u(y)|^p K(x-y) dxdy \right)^{\frac{1}{p}}.
\]

We define the subspace \( X_0 \) of \( X \) as

\[
X_0 = \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}
\]

equipped with the norm

\[
\|u\| = \left( \int_Q |u(x) - u(y)|^p K(x-y) dxdy \right)^{\frac{1}{p}}.
\]

The space \( X_0 \) is a Banach and a reflexive space (refer Lemma 2.4 of [27] and Theorem 1.2 of [37]). We will denote the usual fractional Sobolev space by \( W^{s,p}(\Omega) \) equipped with norm

\[
\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left( \int_{\Omega \times \Omega} |u(x) - u(y)|^p K(x-y) dxdy \right)^{\frac{1}{p}}.
\]
Note that the norms $\| \cdot \|_X$ and $\| \cdot \|_{W^{s,p}}$ are not equivalent when $K(x) = |x|^{-N-ps}$ as $\Omega \times \Omega$ is strictly contained in $Q$. This makes the space $X_0$ different from the usual fractional Sobolev space. Thus the fractional Sobolev space is insufficient for dealing with our problem from the variational method point of view.

We now define the definition of a solution to (1.1) in a weaker sense.

**Definition 3.2.** A function $u \in X_0$ is a weak solution to (1.1) if $hu^{-\gamma} \phi \in L^1(\Omega)$ and

$$
(a + b\|u\|^p) \int_Q |u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))K(x - y)dxdy
$$

$$
-\lambda \int_\Omega g(x)|u|^{p-2}u\phi dx - \mu \int_\Omega h(x)u^{-\gamma}\phi dx - \int_\Omega F(x, u)dxdy = 0
$$

for all $\phi \in X_0$.

**Remark 3.3.** Henceforth, we will mean a weak solution whenever we use the term solution.

**Remark 3.4.** Throughout the article any constant will be denoted by alphabets $C$ or $K$.

With these developments, we are now in a position to state our main result(s).

**Theorem 3.5.** Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ be a function as above and let conditions $(f_1) - (f_5)$ hold. Then, for any $\lambda \in \mathbb{R}$ and for small enough $\mu \in \mathbb{R}$ the problem in (1.1) has infinitely many solutions in $X_0$ with unbounded energy.

**Theorem 3.6.** Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ be a function as above and let conditions $(f_1) - (f_4)$ hold. If condition $(f_6)$ is considered instead of $(f_5)$ then also the conclusion of the Theorem 3.5 holds.

**Theorem 3.7.** Let $u_0 \in C^1(\overline{\Omega})$ which satisfies

$$
u_0 \geq Kd(x, \partial \Omega)^{\eta} \text{ for some } \eta > 0 \tag{3.3}$$

be a local minimizer of $I$ (defined later in this section) in $C^1(\overline{\Omega})$ topology; that is, there exists $\epsilon > 0$ such that, $u \in C^1(\overline{\Omega})$, $\|u - u_0\|_{C^1(\overline{\Omega})} < \epsilon$ implies $I(u) \leq I(u_0)$. Then, $u_0$ is a local minimum of $I$ in $W^{s,p}_0(\Omega)$ as well.

We quickly recall that

$$
\mathcal{L}^*_{p}u = \lambda|u|^{p-2}u \text{ in } \Omega \\
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \tag{3.4}
$$

This has a divergent sequence of positive eigen values

$$
0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots
$$
which has eigen values say \((e_n)_{n \in \mathbb{N}}\). Refer to Proposition 9 of [40] where it has been shown that this sequence provides an orthonormal basis in \(L^p(\Omega)\) and an orthogonal basis in \(X_0\).

We first define

\[
I(u) = A(u) - B(u) - C(u)
\]

where

\[
A(u) = \frac{a}{p} \|u\|^p + b \|u\|^2_{X_0}
\]

\[
B(u) = \frac{\lambda}{p} \int_\Omega g(x)|u|^p dx
\]

\[
C(u) = \frac{\mu}{1 - \gamma} \int_\Omega h(x)u^{1-\gamma} dx + \int_\Omega F(x,u) dx.
\]

As said earlier \(F(x,t) = \int_0^t f(x,t) dx\). From here onwards we will denote \(\| \cdot \|_{L^p(\Omega)}\) as \(\| \cdot \|_p\). To our dissatisfaction, the functional \(I\) is not a \(C^1\) functional. However we redefine this functional \(I\) as follows. Define

\[
\bar{f}(x,t) = \begin{cases}
\mu h(x)t^{-\gamma} + f(x,t), & \text{if } t > u_\mu \\
\mu h(x)u_\mu^{-\gamma} + f(x,u_\mu), & \text{if } t \leq u_\mu
\end{cases}
\]

where \(f(x,t) = \frac{\mu g(x)}{t^\gamma} + f(x,t)\) and \(u_\mu\) is a solution to

\[
\left( a + b \int_{\mathbb{R}^N} |u(x) - u(y)|^p K(x-y) dxdy \right) \mathcal{L}_p u - \lambda g(x)u^{p-1} = f(x,u), \text{ in } \Omega
\]

\[
u > 0, \text{ in } \Omega
\]

\[
u = 0, \text{ in } \mathbb{R}^N \setminus \Omega
\]

(3.7)

whose existence can be guaranteed from [45]. Let \(\overline{F}(x,s) = \int_0^s \bar{f}(x,s) ds\). Now the functional

\[
\overline{I}(u) = A(u) - B(u) - \overline{C}(u)
\]

where \(\overline{C}(u) = \int_\Omega \overline{F}(x,u) dx\). The way the functional has been defined, it is easy to see that the critical points of the functional in (3.8) are also the critical points of the functional (3.5). Most importantly, the functional \(\overline{I}\) which is defined in (3.8) is \(C^1\) which allows us to use the variational methods. Further

\[
\langle \overline{I}(u), \phi \rangle = (a + b\|u\|^p) \int_{\mathcal{Q}} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))K(x-y) dxdy
\]

\[-\lambda \int_\Omega g(x)|u|^{p-2}u\phi dx - \int_\Omega h(x)\overline{f}(x,u)\phi dx
\]

(3.9)
for all $\phi \in X_0$.
The following lemma will be used in this work.

**Lemma 3.8.** (Refer Lemma 2.4 [13]) Let the kernel $K$ be as above. We then have the following.

1. For any $r \in [1, p_\ast)$, the embedding $X_0 \hookrightarrow L^r(\Omega)$ is compact when $\Omega$ is bounded with smooth enough boundary.

2. For all $r \in [1, p_\ast]$, the embedding $X_0 \hookrightarrow L^r(\Omega)$ is continuous.

Following is the definition of Cerami condition (Definition 2.1 [8]).

**Definition 3.9.** [Cerami condition] Let $I$ be a $C^1(X_0, \mathbb{R})$ functional. $I$ is said to satisfy the $(Ce)_c$ at level $c \in \mathbb{R}$, if any sequence $(u_n) \subset X_0$ for which $I(u_n) \to c$ in $X_0$, $I'(u_n) \to 0$ in $X_0'$, the dual space of $X_0$, as $n \to \infty$, then there exists a strongly convergent subsequence of $(u_{n_k})$ of $(u_n)$ in $X_0$.

**Remark 3.10.** Henceforth, a subsequence of any sequence, say $(v_n)$, will also be denoted by $(v_n)$.

We now give the symmetric mountain pass theorem [11].

**Theorem 3.11.** (Symmetric mountain pass theorem) Let $X$ be an infinite dimensional Banach space. $Y$ is a finite dimensional Banach space and $X = Y \oplus Z$. For any $c > 0$ if $I \in C^1(X, \mathbb{R})$ satisfies $(Ce)_c$ and

1. $I$ is even and $I(0) = 0$ for all $u \in X$
2. There exists $r > 0$ such that $I(u) \geq R$ for all $u \in B_r(0) = \{u \in X : \|u\|_X \leq r\}$
3. For any finite dimensional subspace $X \subset X$, there exists $r_0 = r(X) > 0$ such that $I(u) \leq 0$ on $X \setminus B_{r_0}(0_X)$, where $0_X$ is the null vector in $X$.

then there exists an unbounded sequence of critical values of $I$ characterized by a minimax argument.

4. **Auxilliary and main result**

We begin this section by proving a few auxiliary lemmas.

**Lemma 4.1.** Let $(f_1)$ hold. Then any bounded sequence $(u_n)$ in $X_0$ which satisfies $(1 + \|u_n\|)I'(u_n) \to 0$ as $n \to \infty$ possesses a strongly convergent subsequence in $X_0$. 
Proof. Let \( (u_n) \) be a bounded sequence in \( X_0 \). Since \( X_0 \) is reflexive we have
\[
\begin{align*}
&u_n \rightharpoonup u \text{ in } X_0, \\
&u_n \to u \text{ in } L^r(\Omega), 1 \leq r < p^*_s, \\
&u_n \to u \text{ a.e. in } \Omega.
\end{align*}
\] (4.1)

All we need to prove is that \( u_n \to u \) strongly in \( X_0 \). By the Hölder’s inequality we have
\[
0 \leq \int_\Omega |f(x, u_n)|(u_n - u)dx \leq \int_\Omega C(1 + |u_n|^{q-1})(u_n - u)dx
\]
\[
\leq C(|\Omega|^{\frac{q-1}{q}} + \|u_n\|^{q-1}_q\|u_n - u\|_q).
\] (4.2)

By (4.1) we obtain
\[
\lim_{n \to \infty} \int_\Omega |f(x, u_n)|(u_n - u)dx = 0.
\]

Further we have from the Hölder’s inequality that
\[
\int_\Omega |u_n|^{p-2}u_n(u_n - u)dx \leq \|u_n\|^{p-1}_p\|u_n - u\|_p.
\]

Therefore again from (4.1) we have
\[
\lim_{n \to \infty} \int_\Omega |u_n|^{p-2}u_n(u_n - u)dx = 0.
\]

Note that
\[
\lim_{n \to \infty} |u_n^{1-\gamma} - uu_n^{-\gamma}| = \begin{cases} 
\lim_{n \to \infty} |u_n^{1-\gamma} - uu_n^{-\gamma}|, & \text{if } u_n > u^\mu, \\
\lim_{n \to \infty} |uu_n^{-\gamma}u_n - uu_n^{-\gamma}|, & \text{if } u_n \leq u^\mu.
\end{cases}
\]

This
\[
\lim_{n \to \infty} \int_\Omega u_n^{-\gamma}(u_n - u)dx = 0
\]
in either cases. We now define a linear functional
\[
J_v(u) = \int_{\mathbb{R}^N} |v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))K(x - y)dxdy.
\]

The Hölder inequality yields
\[
|J_v(u)| \leq \|v\|^{p-1}_p\|u\|_{X_0}
\]
thereby implying that \( J \) is a continuous linear functional on \( X_0 \). Therefore
\[
\lim_{n \to \infty} J_v(u_n - u) = 0.
\]
From the discussion so far in the proof of this theorem and since \( u_n \to u \) in \( X_0 \), we conclude that \( \langle \tilde{T}(u_n), u_n - u \rangle \to 0 \) as \( n \to \infty \). Also \( (1 + \|u_n\|_{X_0}) \tilde{T}(u_n) \to 0 \) in \( X'_0 \).

Therefore, we have

\[
o(1) = \langle \tilde{T}(u_n), u_n - u \rangle - \lambda \int g(x)|u_n|^{p-2}u_n(u_n - u)dx \\
- \mu \int h(x)u_n^{-\gamma}(u_n - u)dx - \int f(x, u_n)(u_n - u)dx
\]

\[
= (a + b\|u_n\|^p)J_{u_n}(u_n - u) + o(1) \quad \text{as} \quad n \to \infty.
\]

Hence, by the boundedness of \( (u_n) \) in \( X_0 \) and \( \lim_{n \to \infty} J_{u}(u_n - u) = 0 \) we have

\[
\lim_{n \to \infty} (J_{u_n}(u_n - u) - J_{u}(u_n - u)) = 0.
\]

Recall the Simon inequalities which is as follows.

\[
|U - V|^p \leq C_p(|U|^{p-2}U - |V|^{p-2}V)(U - V), \quad p \geq 2
\]

\[
|U - V|^p \leq C'_p(|U|^{p-2}U - |V|^{p-2}V)^{\frac{p}{2}}(|U|^p + |V|^p)^{\frac{2-p}{2}}, \quad 1 < p < 2
\]

for all \( U, V \in \mathbb{R}^N \), \( C_p, C'_p > 0 \) are constants.

When \( p \geq 2 \), we have

\[
\|u_n - u\|^p \leq C_p \int_{\mathbb{R}^{2N}} [|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y))] \\
\times [(u_n(x) - u_n(y)) - (u(x) - u(y))]K(x - y)dxdy
\]

\[
= C_p[J_{u_n}(u_n - u) - J_{u}(u_n - u)] \to 0 \quad \text{as} \quad n \to \infty.
\]

(4.6)

When \( 1 < p < 2 \)

\[
\|u_n - u\|^p \leq C'_p[J_{u_n}(u_n - u) - J_{u}(u_n - u)]^\frac{2}{p}(\|u_n\|^p + \|u\|^p)^{\frac{2-p}{p}}
\]

\[
\leq C'_p[J_{u_n}(u_n - u) - J_{u}(u_n - u)]^\frac{2}{p}(\|u_n\|^{\frac{2-p}{2}} + \|u\|^{\frac{2-p}{2}}) \to 0 \quad \text{as} \quad n \to \infty.
\]

(4.7)

Thus \( u_n \to u \) strongly in \( X_0 \) as \( n \to \infty \).

\[
\square
\]

**Lemma 4.2.** Let \((f_1), (f_3), (f_5)\) hold. Then the functional \( \tilde{T} \) satisfies the \((Ce)_c\) condition.

**Proof.** Let \((f_5)\) hold. From the Remark 3.1 there exists \( C_1 > 0 \) such that

\[
\mathcal{G}(x, s) \leq \mathcal{G}(x, t) + C,
\]

(4.8)
for all \( x \in \Omega \) and \( 0 \leq |s| \leq |t| \). Refer (3.3) for the definition of \( G(., .) \). Let \((u_n)\) be a Cerami sequence in \( X_0 \). Thus

\[
\bar{T}(u_n) \to c \text{ in } X_0 \\
(1 + \|u_n\|)|\bar{T}'(u_n)| \to 0 \text{ in } X'_0
\]
as \( n \to \infty \). All we need to show is that the sequence \((u_n)\) is bounded in \( X_0 \). The conclusion will follow from the Lemma 4.1.

Suppose not, i.e. then upto a subsequence at least we have \( \|u_n\|_{X_0} \to \infty \). By the second condition of (4.9) we have \( |\bar{T}'(u_n)| \to 0 \) as \( n \to \infty \). Hence

\[
\|u_n\| \left\langle |\bar{T}(u_n)|, \frac{u_n}{\|u_n\|} \right\rangle \to 0
\]
as \( n \to \infty \). We define \( \xi_k = \frac{u_n}{\|u_n\|} \) so that \( \|\xi_n\| = 1 \). Thus \((\xi_n)\) is a bounded sequence and hence

\[
\xi_n \to \xi \text{ in } L^p(\Omega)
\]
and up to a subsequence

\[
\xi_n \to \xi \text{ a.e. in } \Omega.
\]

Further, from Lemma A.1 of [28] there exists a function \( \alpha(x) \) such that

\[
|\xi_n(x)| \leq \alpha(x) \text{ in } \mathbb{R}^N.
\]

This lead to the consideration of two cases, viz. \( \xi = 0 \) and \( \xi \neq 0 \). Since \( \bar{T} \) is a \( C^1 \) functional, therefore \( \bar{T}(\alpha_n u_n) = \max_{\alpha \in [0,1]} \bar{T}(\alpha u_n) \).

Let \( \xi = 0 \) and define \( h_T = \left( \frac{4pT}{b} \right)^{\frac{1}{2p}} \) such that \( h_T \|u_n\| \in (0, 1) \) for \( T \in \mathbb{N} \) and sufficiently large \( n \). Since \( \xi = 0 \) and \( \xi_n \to \xi \) a.e. in \( \Omega \) we have

\[
\int_\Omega |h(T)\xi_n|^p \, dx \to 0. \tag{4.9}
\]

By the continuity of \( F \) we obtain

\[
F(x, h_T \xi_n(x)) \to F(x, h_T \xi(x)) = F(x, 0) \text{ in } \Omega \text{ as } n \to \infty. \tag{4.10}
\]

From \((f_1)\) and \(|\xi_n(x)| \leq \alpha(x) \in \mathbb{R}^N \) in combination with the Hölder inequality we get

\[
|F(x, h_T \xi_n)| \leq C|h_T \alpha(x)| + \frac{C}{q} |h_T \alpha(x)|^q \in L^1(\Omega) \tag{4.11}
\]
for any \( n, T \in \mathbb{N} \). Therefore from the Lebesgue dominated convergence theorem we get

\[
F(., h_T \xi_n(.,)) \to F(., h_T \xi(.,)) \text{ in } L^1(\Omega) \text{ as } n \to \infty. \tag{4.12}
\]
for any $T \in \mathbb{N}$. Thus

$$\int_{\Omega} F(x, h_T \xi_n(x)) dx \to 0 \text{ as } n \to \infty.$$  

We further have

$$T(\alpha_n u_n) \geq T\left(\frac{h_T}{\|u_n\|} u_n\right)$$

$$= T(h_T \xi_n)$$

$$\geq \frac{a}{p}\|h_T \xi_n\|^p + \frac{b}{2p}\|h_T \xi_n\|^{2p} - \frac{\lambda \|f\|_{\infty}}{p} \int_{\Omega} |h_T \xi_n|^p dx$$

$$- \frac{\mu \|g\|_{\infty}}{1 - \gamma} \int_{\Omega} |h_T \xi_n|^{1 - \gamma} dx - \int_{\Omega} F(x, h_T \xi_n) dx$$

$$\geq \frac{b}{2p}\|h_T \xi_n\|^{2p} = 2T. \quad (4.13)$$

Therefore

$$T(\alpha_n u_n) \to \infty \quad (4.14)$$

as $n \to \infty$. We now show that

$$\lim_{n \to \infty} \sup T(\alpha_n u_n) \leq \beta$$

for some $\beta > 0$. Observe that $\int_{\Omega} |u_n|^p dx \leq C$. Since the boundary $\partial \Omega$ is Lipshitz continuous and $\frac{1}{\|u_n\|} \to 0$ as $n \to \infty$ we have that

$$\left| \frac{\lambda}{2p} \int_{\Omega} g(x) |u_n|^p dx \right| \leq C(\lambda, p) \|g\|_{\infty}$$

$$\left| \mu \left(\frac{1}{1 - \gamma} - \frac{1}{2p}\right) \int_{\Omega} h(x)(\alpha_n u_n)^{1 - \gamma} dx \right| \leq C(\mu, \gamma, p) \|h\|_{\infty}. \quad (4.15)$$

Since $\frac{d}{d\alpha}|_{\alpha=\alpha_n} T(\alpha u_n) = 0$ for all $n$ we have

$$\langle T'(\alpha_n u_n), \alpha_n u_n \rangle = \alpha_n \frac{d}{d\alpha}|_{\alpha=\alpha_n} T(\alpha u_n) = 0. \quad (4.16)$$
We further have
\[
\mathcal{T}(\alpha_n u_n) = \mathcal{T}(\alpha_n u_n) - \frac{1}{2p} \langle \mathcal{T} (\alpha_n u_n), \alpha_n u_n \rangle \\
= \frac{a}{2p} \|\alpha_n u_n\|^p - \frac{\lambda}{2p} \int \frac{g(x)|\alpha_n u_n|^p}{\int_\Omega} dx - \mu \left( \frac{1}{1 - \gamma} - \frac{1}{2p} \right) \int_\Omega h(x)(\alpha_n u_n)^{1-\gamma} dx \\
- \int_\Omega F(x, \alpha_n u_n) dx + \frac{1}{2p} \int_\Omega f(x, \alpha_n u_n) \alpha_n u_n dx
\]
\[
\leq \frac{a}{2p} \|\alpha_n u_n\|^p + C(\lambda, p) + C(\mu, \gamma, p) + \int_\Omega \mathcal{G}(x, \alpha_n u_n) dx
\]
\[
\leq \frac{a}{2p} \|\alpha_n u_n\|^p + C(\lambda, p) + C(\mu, \gamma, p) + \int_\Omega \mathcal{G}(x, u_n) dx + C|\Omega|
\]
\[
= \frac{a}{2p} \|\alpha_n u_n\|^p + C(\lambda, p) + C(\mu, \gamma, p) + \int_\Omega \mathcal{G}(x, u_n) dx + C|\Omega|
\]
\[
- \frac{\lambda}{2p} \int_\Omega f(x)|\alpha_n u_n|^p dx + \frac{\lambda}{2p} \int_\Omega f(x)|\alpha_n u_n|^p dx
\]
\[
- \mu \left( \frac{1}{1 - \gamma} - \frac{1}{2p} \right) \int_\Omega g(x)(\alpha_n u_n)^{1-\gamma} dx + \mu \left( \frac{1}{1 - \gamma} - \frac{1}{2p} \right) \int_\Omega g(x)(\alpha_n u_n)^{1-\gamma} dx
\]
\[
+ C|\Omega|
\]
\[
\leq \mathcal{T}(\alpha_n u_n) - \frac{1}{2p} \langle \mathcal{T} (\alpha_n u_n), \alpha_n u_n \rangle + 2C(\lambda, p) + 2C(\mu, \gamma, p) + C|\Omega|
\]
\[
= c + o(1) + 2C(\lambda, p) + 2C(\mu, \gamma, p) + C|\Omega| < \infty \text{ as } n \rightarrow \infty.
\]
(4.17)

where $|\Omega|$ is the lebesgue measure of $\Omega$. This is a contradiction to (4.14). Thus $(u_n)$ is bounded $X_0$.

Let $\xi_n \neq 0$. Define
\[
A = \{x \in \Omega : \xi(x) \neq 0\}.
\]

Therefore e have
\[
|u_n(x)| = |\xi_n(x)||u_n| \rightarrow \infty \text{ in } A \text{ as } n \rightarrow \infty.
\]
(4.18)

Further from $(f_3)$
\[
\frac{G(x, u_n(x))}{\|u_n\|^{2p}} = \frac{G(x, u_n(x)) |u_n(x)|^{2p}}{|u_n(x)|^{2p} \|u_n\|^{2p}}
\]
\[
= \frac{G(x, u_n(x))}{|u_n(x)|^{2p}} |\xi_n|^{2p} \rightarrow \infty \text{ in } A \text{ as } n \rightarrow \infty.
\]
(4.19)

By the Fatou’s lemma
\[
\int_\Omega \frac{G(x, u_n(x))}{\|u_n\|^{2p}} dx \rightarrow \infty.
\]
(4.20)
Let us now analyse over $\Omega \setminus A$. Thus from $(f_3)$ again we have

$$\lim_{|t| \to \infty} F(x, t) = \infty$$

for $x \in \Omega$. Therefore for arbitrary $M > 0$ there exists $t'$ such that

$$F(x, t) \geq M \text{ whenever } |t| \geq t', x \in \Omega. \quad (4.21)$$

Hence

$$F(x, t) \geq \min \left\{ M, \min_{(x, t) \in \Omega \times [-t', t']} \{ F(x, t) \} \right\} = M' \text{ say.} \quad (4.22)$$

So we get

$$\lim_{|t| \to \infty} \int_{\Omega \setminus A} F(x, u_n(x)) \frac{\| u_n \|^{2p}}{2p} dx \geq 0. \quad (4.23)$$

Also

$$0 \leq \frac{\mu}{1 - \gamma} \int_{\Omega} h(x) \frac{u_n^{1-\gamma}}{\| u_n \|^{2p}} dx \leq \frac{\mu \| h \|_{\infty}}{1 - \gamma} \| u_n \|^{1-\gamma-2p} = o(1). \quad (4.24)$$

By the variational characterization of $\lambda_j$, the $j$-th eigen value of $\Omega_p^*$, i.e.

$$\lambda_j = \min_{u \in X_0 \setminus \{0\}} \left\{ \int_{\mathbb{R} \times \mathbb{R}^2} |u(x) - u(y)|^p K(x - y) dx dy \right\} \int_{\Omega} f(x)|u(x)|^pdx$$

where $g > 0$ and bounded in $\Omega$. Now since

$$o(1) = \frac{\mathcal{I}(u_n)}{\| u_n \|^{2p}} = \frac{a}{p \| u_n \|^p} + \frac{b}{2p} - \frac{\lambda}{p} \int_{\Omega} f(x) \frac{|u_n(x)|^p}{\| u_n \|^{2p}} dx - \frac{\mu}{1 - \gamma} \int_{\Omega} g(x) \frac{u_n^{1-\gamma}}{\| u_n \|^{2p}} dx - \int_{\Omega \setminus A} F(x, u_n) \frac{\| u_n \|^{2p}}{2p} dx$$

$$\leq o(1) + \frac{b}{2p} + \frac{\| g \|_{\infty} \lambda}{p \lambda_j} \frac{1}{\| u_n \|^p} - \frac{\mu}{1 - \gamma} \int_{\Omega} h(x) \frac{u_n^{1-\gamma}}{\| u_n \|^{2p}} dx - \int_{\Omega \setminus A} F(x, u_n) \frac{\| u_n \|^{2p}}{2p} dx$$

$$\leq - \infty \quad (4.25)$$

where the last step is due to $(4.20)$, $(4.23)$, $(4.24)$. This is absolutely a contradiction!. Thus the sequence $(u_n)$ is bounded in $X_0$ and hence by the Lemma 4.1 we conclude that $(u_n)$ possesses a strongly convergent subsequence in $X_0.$
We now prove the results stated in the Theorems 3.5 and 3.6. For this let us develop some prerequisites. It is well known that the space $X_0$ is a Banach space and we have that

$$X_0 = \bigoplus_{i \geq 1} X_i$$

where $X_i = \text{span}\{e_j\}_{j \geq i}$. Define

$$Y_m = \bigoplus_{1 \leq j \leq m} X_j$$
$$Z_m = \bigoplus_{j \geq m} X_j.$$

Clearly $Y_m$ is a finite dimensional sub space of $X_0$.

**Theorem 4.3.** Let $\kappa \in [1, p^*_s)$. We have

$$\zeta(\kappa) = \sup\{\|u\|_{L^\kappa}\|u\|: u \in Z_m, \|u\| = 1\} \to 0$$

as $m \to \infty$.

**Proof.** From the definition of $(Z_m)$ we have that $Z_m \subset Z_{m+1}$ and thus $0 \leq \zeta_m \leq \zeta_{m+1}$. This implies that $\zeta_m \to 0$ as $m \to \infty$. Further by the definition of supremum for every $m$ there exists $u_m \in Z_m$ such that $\|u_m\|$ and $\|u_m\|_{L^\kappa} > \frac{\kappa}{2}$. By the reflexivity of $X_0$ the sequence $(u_m)$ is weakly convergent to, say, $u$. Obviously $u = 0$ as $\langle u_m, v \rangle \to 0$ for every $v \in Y_m$. By the embedding results from Lemma 3.8 we have that $u_m \to 0$ in $L^\kappa(\Omega)$ for any $\kappa \in [1, p^*_s)$. Thus we conclude that $\zeta = 0$. \hfill \qed

**Proof of Theorem 3.5.** Since $Y_m$s are all finite dimensional spaces, hence the norms $\| \cdot \|$ and $\| \cdot \|_{L^\kappa}$ are equivalent for $\kappa \in [1, p^*_s)$. Observe that $\mathcal{T}(u) = 0$. If $\frac{a}{\lambda} > \lambda_1$, then from the definition of $\lambda_j$ given in the previous theorem, there exists $\lambda_j$ such that $\frac{a}{\lambda} \in [\lambda_{j-1}, \lambda_j)$, $\frac{a}{\lambda} \in [\lambda_{k-1}, \lambda_k)$. Further, from $(f_1)$ and $(f_4)$, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$F(x, t) \leq \frac{\epsilon}{p} |t|^p + \frac{C_\epsilon}{q} |t|^q, \quad (4.26)$$

for any $(x, t) \in \Omega \times \mathbb{R}$. By the definition of $\zeta_m$ in Lemma 4.3, fix $\epsilon > 0$ and choose $m' \geq 1$ such that

$$\|u\|_p^p \leq \frac{a - \frac{\lambda}{\lambda_j}}{2\|f\|_{\infty}} \|u\|_p^p$$

$$\|u\|_q^q \leq \frac{q(a - \frac{\lambda}{\lambda_j})}{2pC_\epsilon} \|u\|_q^q \text{ for every } u \in Z_{m'}.$$
Choose $r < 1$, in Lemma 3.11. Since $q > p$, we have

$$
\bar{T}(u) = \frac{a}{p} \|u\|^p + \frac{b}{2p} \|u\|^{2p} - \frac{\lambda}{p} \int_{\Omega} f(x)|u(x)|^p dx - \frac{\mu}{1 - \gamma} \int_{\Omega} g(x)u^{1-\gamma} dx - \int_{\Omega} F(x,u) dx \\
\geq \frac{a}{p} \|u\|^p - \lambda \frac{1}{p\lambda_j} \|u\|^p - \frac{\mu}{1 - \gamma} \|u\|^{1-\gamma} - \frac{\lambda}{p} \|u\|^{2p} - \frac{\mu}{1 - \gamma} \|u\|^{1-\gamma} \\
\geq \frac{a - \lambda}{2p} \|u\|^p - \frac{\mu}{1 - \gamma} \|u\|^{1-\gamma} = R > 0
$$

(4.28)

for sufficiently small $\mu > 0$. Finally, from $(f_3)$, there exists $A > \frac{b}{2pC^2}\gamma$ (a possible choice of $C$, as we shall see later, is a Sobolev constant), $B > 0$ such that

$$
F(x,t) \geq A|t|^{2p}
$$

(4.29)

for any $x \in \Omega$ and $|t| > B$. From $(f_1)$ we have

$$
|F(x,t)| \leq C(1+B^{q-1})|t| \text{ for every } x \in \Omega \text{ and } |t| \leq B.
$$

(4.30)

Let $C' = C(1+B^{q-1}) > 0$. Then we get

$$
F(x,t) \geq A|t|^{2p} - C'|t|, \text{ for } (x,t) \in \Omega \times \mathbb{R}.
$$

(4.31)

By the equivalence of norm in $Y_m$ and (4.31), we have

$$
\bar{T}(u) = \frac{a}{p} \|u\|^p + \frac{b}{2p} \|u\|^{2p} - \frac{\lambda}{p} \int_{\Omega} f(x)|u(x)|^p dx - \frac{\mu}{1 - \gamma} \int_{\Omega} g(x)u^{1-\gamma} dx \\
- \int_{\Omega} F(x,u) dx \\
\leq \frac{a}{p} \|u\|^p + \frac{b}{2p} \|u\|^{2p} - \frac{\lambda}{p\lambda_j} \|u\|^p - \frac{\mu}{1 - \gamma} \|u\|^{1-\gamma} - \int_{\Omega} F(x,u) dx \\
\leq \frac{a}{p} \|u\|^p + \frac{b}{2p} \|u\|^{2p} - \frac{\lambda}{p\lambda_j} \|u\|^p - \frac{\mu}{1 - \gamma} \|u\|^{1-\gamma} - \frac{\lambda}{p} \|u\|^{2p} + C'|u|_1 \\
\leq \frac{a}{p} \|u\|^p + \left(\frac{b}{2p} - AC^2\gamma\right) \|u\|^{2p} + C_2C'|u|.
$$

(4.32)

Thus for a sufficiently large $r_0 = r(X)$, we have $\bar{T}(u) \leq 0$ whenever $\|u\| \geq r_0$. 

Proof of Theorem 3.6: Suppose now \((f_6)\) holds. The proof follows verbatim of the Theorem 3.5, except that we need to prove the inequality in (4.17). Thus we have

\[
\frac{1}{\sigma} T(\alpha_n u_n) = \frac{1}{\sigma} \left( T(\alpha_n u_n) - \frac{1}{2p} \langle T(\alpha_n u_n), \alpha_n u_n \rangle \right)
\]

\[
= \frac{1}{\sigma} \left[ \left( \frac{a}{2p} \right) \| \alpha_n u_n \|^p + \frac{\lambda}{2p} \int_{\Omega} g(x) |\alpha_n u_n|^p dx - \mu \left( \frac{1}{1 - \gamma} - \frac{1}{2p} \right) \int_{\Omega} h(x)(\alpha_n u_n)^{1-\gamma} dx \right.
\]

\[
- \int_{\Omega} F(x, \alpha_n u_n) dx + \frac{1}{2p} \int_{\Omega} f(x, \alpha_n u_n) \alpha_n u_n dx \right]
\]

\[
\leq \frac{1}{\sigma} \left[ \left( \frac{a}{2p} \right) \| \alpha_n u_n \|^p + \int_{\Omega} \mathcal{G}(x, \alpha_n u_n) dx \right] + C(\lambda, p)\| g \|_{\infty}
\]

\[
\leq \left( \frac{a}{2p} \right) \| \alpha_n u_n \|^p + \int_{\Omega} \mathcal{G}(x, \alpha_n u_n) dx + \frac{1}{\sigma} \int_{\Omega} T(x) dx + C(\lambda, p)\| g \|_{\infty}
\]

\[
\leq \left( \frac{a}{2p} \right) \| \alpha_n u_n \|^p + \int_{\Omega} \mathcal{G}(x, \alpha_n u_n) dx + \frac{1}{\sigma} \int_{\Omega} T(x) dx + C(\lambda, p)\| g \|_{\infty}
\]

\[
- \frac{\lambda}{2p} \int_{\Omega} |u_n|^p dx + \frac{\lambda}{2p} \int_{\Omega} |u_n|^p dx
\]

\[
\leq T_{u_n} - \frac{1}{2p} \langle T(\alpha_n u_n), \alpha_n u_n \rangle + 2C(\lambda, p)\| g \|_{\infty} + \frac{1}{\sigma} \int_{\Omega} T(x) dx
\]

\[
\leq c + o(1) + 2C(\lambda, p)\| g \|_{\infty} + \frac{1}{\sigma} \int_{\Omega} T(x) dx < \infty.
\]

(4.33)

Hence the proof.

We will now show that the solution \(u\) to (1.1) is in \(L^\infty(\Omega)\), i.e. bounded. Firstly, let us recall the following elementary inequality needed for the proof of the \(L^\infty\) estimate.

**Lemma 4.4.** (Lemma 5.1 in [13]) For all \(c, d \in \mathbb{R}, \rho \geq p, p \geq 1, k > 0\) we have

\[
\frac{p^p(\rho + 1 - p)}{\rho^p} (c|c|^\frac{\rho - p - 1}{k} - d|d|^\frac{\rho - p - 1}{k})^p 
\]

\[
\leq (c|c|^\rho - p - 1) - d|d|^\rho - p - 1) (c - d)^{p - 1}
\]

with the assumption that \(c \geq d\).

**Proof.** Let us define

\[
m(t) = \begin{cases} 
sgn(t)|t|^\frac{\rho - 1}{p}, & |t| < k \\
sgn(t)k^\frac{\rho - 1}{p}, & |t| \geq k.
\end{cases}
\]
Note that
\[ \int_d^c m(t) dt = \frac{p}{\rho} (c|c|_k^{p-1} d|d|_k^{p-1}). \]

Similarly,
\[ \int_d^c m(t)^p dt \leq \frac{1}{\rho + 1 - p} (c|c|_k^{p-1} - d|d|_k^{p-1}). \]

Using the Cauchy-Schwartz inequality we obtain
\[ \left( \int_d^c m(t) dt \right)^p \leq (c - d)^{p-1} \int_d^c h(t)^p dt. \]

Thus
\[ \frac{p^p}{\rho^p} (c|c|_k^{p-1} - d|d|_k^{p-1})^p \]
\[ = \left( \int_d^c m(t) dt \right)^p \]
\[ \leq (c - d)^{p-1} \int_d^c m(t)^p dt \]
\[ \leq \frac{(c - d)^{p-1}}{\rho + 1 - p} (c|c|_k^{p-1} - d|d|_k^{p-1}). \]

\[ \Box \]

**Theorem 4.5.** Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be as defined in (f_1), then for any weak solution \( u \in X_0 \) we have \( u \in L^\infty(\Omega) \).

**Proof.** Let \( 1 \leq q < p_s^* \) and let \( u \) be a weak solution to the given problem and \( \alpha = \left( \frac{p^*_s}{p} \right)^{\frac{1}{p}} \). For every \( \rho \geq p(p - 1) \), \( k > 0 \), the mapping \( t \mapsto t|t|_k^{p-1} \) is Lipschitz in \( \mathbb{R} \). Therefore, \( u|u|_k^{p-1} \in X_0 \). In general for any \( t \) in \( \mathbb{R} \) and \( k > 0 \), we have defined \( t_k = \text{sgn}(t) \min\{|t|, k\} \). We apply the embedding results due to Theorem 3.8, the previous lemma 4.4, test with the test function \( u|u|_k^{p-1} \) and on using the growth condition of \( f \).
which is given in (f₁) to get
\[
\|u\|_{L^\infty}^{p-1} \leq C\|u\|_{L^p}^{p-1} \|u\|^p
\]
\[
\leq C\frac{p^p}{\rho + 1 - p} \langle u, u \rangle_k^{\rho - p}
\]
\[
\leq \rho^p C' \frac{1}{(a + b\|u\|_k^{p-1})} \left( \lambda \int_\Omega g(x)|u|^{p-1}u|u_k^{\rho-p}dx + \mu \int_\Omega h(x)u^{-\gamma}|u|^{\rho-p}
\right.
\]
\[
+ \int_\Omega |f(x, u)||u|^{\rho-p}dx
\]
\[
\leq C'\rho^p \int_\Omega (g(x)|u|^{p-1}u|u_k^{\rho-p} + h(x)|u|^{1-\gamma}|u_k^{\rho-p} + |u||u_k^{\rho-p} + |u|^q|u_k^{\rho-p}) dx
\]
\[
(4.34)
\]
for some \( C > 0 \) independent of \( \rho \geq p \) and \( k > 0 \). On applying the Fatou’s lemma as \( k \to \infty \) gives
\[
\|u\|_{\alpha p_\rho} \leq C'\rho^p \left\{ \int_\Omega (|u|^{\rho-\gamma} + |u|^{\rho+q-p}
\right.
\]
\[
+ |u|^{\rho-p-\gamma+1})dx \right\}^{1/p}.
\]
(4.35)
The idea is to try and develop an argument to guarantee that \( u \in L^{p_1}(\Omega) \) for all \( p_1 \geq 1 \). Therefore define a recursive sequence \( (\rho_n) \) by choosing \( \mu > \mu_0 \) and setting \( \rho_0 = \mu, \rho_{n+1} = \alpha^p\rho_n + p - q \). The proof follows verbatim of the proof of the Theorem 5.2 in [13] hereafter which guarantees that the solution is in \( L^\infty(\Omega) \).

\[\square\]

**Lemma 4.6 (Weak Comparison Principle).** Let \( u, v \in X_0 \). Suppose, \( (a + b\|v\|^{p})\mathcal{L}_p^sv - h(x)\frac{\mu}{v^\gamma} \geq (a + b\|u\|^{p})\mathcal{L}_p^su - h(x)\frac{\mu}{u^\gamma} \) weakly with \( v = u = 0 \) in \( \mathbb{R}^N \setminus \Omega \). Then \( v \geq u \) in \( \mathbb{R}^N \).

**Proof.** Since, \( (a + b\|v\|^{p})\mathcal{L}_p^sv - h(x)\frac{\mu}{v^\gamma} \geq (a + b\|u\|^{p})\mathcal{L}_p^su - h(x)\frac{\mu}{u^\gamma} \) weakly with \( u = v = 0 \) in \( \mathbb{R}^N \setminus \Omega \), we have
\[
\langle (a + b\|v\|^{p})\mathcal{L}_p^sv, \phi \rangle - \int_\Omega h(x)\frac{\mu\phi}{v^\gamma}dx \geq \langle (a + b\|u\|^{p})\mathcal{L}_p^su, \phi \rangle - \int_\Omega h(x)\frac{\mu\phi}{u^\gamma}dx
\]
\[
(4.36)
\]
\( \forall \phi \geq 0 \in X_0 \).

In particular choose \( \phi = (u - v)^+ \). To this choice, the inequality in (4.36) looks as follows.
\[
\langle (a + b\|v\|^{p})\mathcal{L}_p^sv - (a + b\|u\|^{p})\mathcal{L}_p^su, (u - v)^+ \rangle
\]
\[
- \int_\Omega \mu h(x)(u - v)^+ \left( \frac{1}{v^\gamma} - \frac{1}{u^\gamma} \right) dx \geq 0.
\]
(4.37)
Define \( m(t) = (a + bt^p) \geq a > 0 \) for \( t \geq 0 \) and

\[
M(t) = \int_0^t m(t)dt.
\]

By the Cauchy-Schwartz inequality we have

\[
|\langle u(x) - u(y), v(x) - v(y) \rangle| \leq |u(x) - u(y)||v(x) - v(y)|
\]

\[
\leq \frac{|u(x) - u(y)|^2 + |v(x) - v(y)|^2}{2}.
\]

(4.38)

Consider \( I_1 = \langle M'(u), u \rangle - \langle M'(u), v \rangle - \langle M'(v), u \rangle + \langle M'(v), v \rangle \) and let \( |u(x) - u(y)| \geq |v(x) - v(y)| \). Therefore using (4.38) we get

\[
I_1 = pm(|u|^p \left( \int_Q |u(x) - u(y)|^{p-2} \{ |u(x) - u(y)\} dx dy \right)
\]

\[
- (u(x) - u(y))(v(x) - v(y)) \right) dx dy
\]

\[
+ pm(|v|^p \left( \int_Q |v(x) - v(y)|^{p-2} \{ |v(x) - v(y)\} dx dy \right)
\]

\[
- (u(x) - u(y))(v(x) - v(y)) \right) dx dy
\]

\[
\geq \frac{p}{2} m(|u|^p \left( \int_Q |u(x) - u(y)|^{p-2} \{ |u(x) - u(y)\} dx dy \right)
\]

\[
- |v(x) - v(y)|^2 \right) dx dy
\]

\[
+ \frac{p}{2} m(|v|^p \left( \int_Q |v(x) - v(y)|^{p-2} \{ |v(x) - v(y)\} dx dy \right)
\]

\[
- |u(x) - u(y)|^2 \right) dx dy
\]

\[
\geq pm(|u|^p \left( \int_Q (|u(x) - u(y)|^{p-2} - |v(x) - v(y)|^{p-2})(|u(x) - u(y)|^2 - |v(x) - v(y)|^2) dx \right)
\]

\[
\geq pa \left( \int_Q (|u(x) - u(y)|^{p-2} - |v(x) - v(y)|^{p-2})(|u(x) - u(y)|^2 - |v(x) - v(y)|^2) dx \right).
\]

(4.39)

When \( |u(x) - u(y)| \leq |v(x) - v(y)| \), we interchange the roles of \( u, v \) to get

\[
I_1 \geq pa \left( \int_Q (|u(x) - u(y)|^{p-2} - |v(x) - v(y)|^{p-2})(|u(x) - u(y)|^2 - |v(x) - v(y)|^2) dx \right).
\]

(4.40)

Thus

\[
\langle M'(u) - M'(v), u - v \rangle = I_1 \geq 0.
\]

(4.41)
Thus $M'$ is a monotone operator. This monotonicity sufficient for our work. Coming back to (4.37), we have
\[
0 \geq -\langle (a + b\|u\|^p)\mathcal{L}_p^su - (a + b\|v\|^p)\mathcal{L}_p^sv, (u - v)^+ \rangle = \langle (a + b\|v\|^p)\mathcal{L}_p^sv - (a + b\|u\|^p)\mathcal{L}_p^su, (u - v)^+ \rangle \geq 0.
\]
(4.42)

Therefore, $|\{x : u(x) > v(x)\}| = 0$. Hence $u \geq v$ a.e. in $\Omega$. \hfill \Box

4.1 $C^1$ versus $W^{s,p}$ local minimizers of the energy

This section is devoted towards discussing ‘$C^1$ versus $W^{s,p}$’ analysis for a particular class of Kernel satisfying $(P')$. Let us begin with some well-known results and prove a few lemmas towards which a geometrical property of a general bounded domain $\Omega$ with $C^{1,1}$ boundary is stated and is as follows.

**Lemma 4.7** (Lemma 3.5, Iannizzotto [6]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. Then, there exists $\rho > 0$ such that for all $x_0 \in \partial\Omega$ there exists $x_1, x_2 \in \mathbb{R}^N$ on the normal line to $\partial\Omega$ at $x_0$, with the following properties

(i) $B_\rho(x_1) \subset \Omega, B_\rho(x_2) \subset \Omega^c$;

(ii) $\bar{B}_\rho(x_1) \cap \bar{B}_\rho(x_2) = \{x_0\}$;

(iii) $d(x) = |x - x_0|$ for all $x \in [x_0, x_1]$.

Using the Lemma 4.7, we generalize two of the results from Iannizzotto [6]. Before that, we set $\forall R > 0, x_0 \in \mathbb{R}^N$

\[
Q(u; x_0, R) = \|u\|_{L^{\infty}(B_R(x_0))} + \text{Tail}(u; x_0, R)
\]

\[
Q(u, R) = Q(u; 0, R)
\]

\[
d(x, \partial\Omega) = \inf_{y \in \Omega} \{d(x, y)\}. \tag{4.43}
\]

**Lemma 4.8.** For any $r > 0$, there exists $K > 0$ such that $|\mathcal{L}_p^su| \leq K$ in $B_r(x)$, where $u$ is a weak solution to the problem (1.1).

**Proof.** By definition,

\[
|\mathcal{L}_p^su(x)| \leq C \int_{B_r(x)} |u(x) - u(y)|^{p-1}K(x - y)dy
\]

\[
\leq C \int_{B_r(x)} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+(p-1)s}} \frac{dy}{|x - y|^{p_0 s - (p-1)s}}dy. \tag{4.44}
\]

Converting this into polar coordinates and applying the Hölder’s inequality we obtain $|\mathcal{L}_p^su(x)| \leq K$ in $B_r(x)$. It follows from this that $|(a + b\|u\|^p)\mathcal{L}_p^su(x)| \leq K'$ in $B_r(x)$. \hfill \Box
Lemma 4.9. There exists $0 < \alpha \leq s$ such that any weak solution $u$ to the problem (1.1) we have $[u/d^s]_{C^0(\overline{\Omega})} \leq K$.

Proof. Note that,

$$
\text{Tail}(u/d^s; x, R_0)^{p-1} = R_0^{ps} \int_{B_{2R_0}(x_0) \setminus B_{R_0}(x_0)} \frac{|u(y)|^{p-1}}{d(y)^{s(p-1)}|x - y|^{N + sp}} dy
$$

$$
\leq CR_0^{ps} \left( \int_{B_{2R_0}(x_0) \setminus B_{R_0}(x_0)} \frac{\|u/d^s\|_{L^\infty(B_{2R_0}(x_0))}^{p-1}}{|x - y|^{N + ps}} dy \right)
$$

$$
+ \int_{B_{2R_0}(x_0) \setminus B_{R_0}(x_0)} \frac{|u(y)|^{p-1}}{d(y)^{s(p-1)}|x - y|^{N + ps}} dy
$$

$$
\leq C \left( \|u/d^s\|_{L^\infty(B_{2R_0}(x_0))}^{p-1} \right)
$$

$$
+ R_0^{ps} \int_{B_{2R_0}(x_0) \setminus B_{R_0}(x_0)} \frac{|u(y)|^{p-1}}{d(y)^{s(p-1)}|x - y|^{N + ps}} dy
$$

$$
= CQ(u/d^s; x_0, 2R_0).
$$

(4.45)

The reader may refer [6] for a definition of Tail. Thus we obtained $Q(u/d^s; x_0, R_0) \leq CQ(u/d^s; x_0, 2R_0)$, which implies the following Hölder seminorm estimate.

$$
[u/d^s]_{C^\alpha(B_{R_0}(x_0))} \leq C[(KR_0^{ps})^{1/(p-1)} + Q(u/d^s; x_0, 2R_0)]R_0^{-\alpha}.
$$

(4.46)

Here, $K$ is the bound of $\mathbb{L}^p u$ in $B_{2R_0}(x_0)$. Let $\alpha \in (0, s]$ and $\Omega' \subseteq \Omega$. Then we have through the compactness of $\Omega'$ and the estimate (4.46) that $\|u/d^s\|_{C^0(\overline{\Omega'})} \leq C$.

Let $\Pi : V \to \partial \Omega$ be a metric projection map defined as $\Pi(x) = \text{Argmin}_{y \in \partial \Omega} \{|x - y|\}$ where $V = \{x \in \overline{\Omega} : d(x, \partial \Omega) \leq \rho\}$. By (4.46) we have

$$
[u/d^s]_{C^\alpha(B_{r/2}(x))} \leq C[(Kr^{ps})^{1/(p-1)} + \|u/d^s\|_{L^\infty(B_{r}(x))} + \text{Tail}(u/d^s; x, r)]r^{-\alpha}.
$$

(4.47)

The first term in (4.47) is trivially controlled since $\alpha \leq s \leq \frac{sp}{p-1}$. The other terms are controlled uniformly due to the compactness of the set $V$.

Remark 4.10. From [7] we can say that $u \in C^1(\overline{\Omega})$. Further, it can be proved that for any $r > 0$, there exists $K > 0$ such that $|\mathbb{L}^p u| \leq K$ in $B_r(x)$, where $u$ is a weak solution to the problem (1.1). The proof follows the Lemma 4.8.
Lemma 4.11. There exists $0 < \alpha \leq s$ such that for any weak solution $u$ of the problem (1.1) we have $[Du]_{C^\alpha(\overline{\Omega})} \leq C$.

Proof. Observe that

$$
\text{Tail}(Du; x, R_0)^{p-1} = R_0^{ps} \int_{B_{R_0}(x)^c} \frac{|Du(y)|^{p-1}}{|x-y|^{N+sp}} dy 
\leq CR_0^{ps} \left( \int_{B_{2R_0}(x_0)\setminus B_{R_0}(x_0)} \frac{\|Du\|_{L^\infty(B_{2R_0}(x_0))}^{p-1}}{|x-y|^{N+ps}} dy \right) + \int_{B_{2R_0}(x_0)^c} \frac{|Du(y)|^{p-1}}{|x-y|^{N+ps}} dy
\leq C \left( \|Du\|_{L^\infty(B_{2R_0}(x_0))}^{p-1} + R_0^{ps} \int_{B_{2R_0}(x_0)^c} \frac{|Du(y)|^{p-1}}{|x-y|^{N+ps}} dy \right)
= CQ(Du; x_0, 2R_0).
$$

(4.48)

Therefore we obtained $Q(Du; x_0, R_0) \leq CQ(Du; x_0, 2R_0)$ which implies the following Hölder seminorm estimate.

$$
[Du]_{C^\alpha(B_{R_0}(x_0))} \leq C[(KR_0^{ps})^{1/(p-1)} + Q(Du; x_0, 2R_0)]R_0^{-\alpha}.
$$

(4.49)

Here $K$ is the bound of $|\mathcal{L}_p Du|$ in $B_{2R_0}(x_0)$. Let $\alpha \in (0, s]$ and $\Omega' \Subset \Omega$. Then we have through the compactness of $\Omega'$ and the the estimate (4.49) that $\|Du\|_{C^\alpha(\overline{\Omega'})} \leq C$.

Let $\Pi : V \to \partial \Omega$ be a metric projection map defined as $\Pi(x) = \text{Argmin}_{y \in \partial \Omega} \{|x-y|\}$ where $V = \{x \in \overline{\Omega} : d(x, \partial \Omega) \leq \rho\}$. By (4.49) we have

$$
[Du]_{C^\alpha(B_{r/2}(x))} \leq C[(K_{r^{ps}})^{1/(p-1)} + \|Du\|_{L^\infty(B_r(x))} + \text{Tail}(Du; x, r)]r^{-\alpha}.
$$

(4.50)

We now try to control the growth of the terms on the right hand side of (4.50). The first term is trivially controlled since $\alpha \leq s \leq \frac{sp}{p-1}$. The other terms are controlled uniformly due to the compactness of the set $V$.

We will now prove the Theorem 3.7. The main tool to prove this result requires an application of the lagrange multiplier rule which is given in the form of a theorem from [22].

Theorem 4.12. Let $I$ and $J$ be real $C^1$ functionals on a real Banach space say $X$. If $z_0 \in X$ satisfies the following problem:

minimizing $I(z)$ under the constraint $K(z) = 0$.

Then there exists $\mu \in \mathbb{R}$ such that $I'(z_0) = \mu J'(z_0)$.
For a more generalized version of the result, one may refer to [46] and the references therein.

**Proof of Theorem 3.7** Let $\Omega' \Subset \Omega$. We will only consider the subcritical case i.e. when $q < p^*_s - 1$. We prove by contradiction, i.e. suppose $u_0$ is not a local minimizer. Let $r \in (q, p^*_s - 1)$ and define

$$J(w) = \frac{1}{r + 1} \int_{\Omega'} |w - u_0|^r + 1 \, dx, \ (w \in W^{s,p}(\Omega')).$$

(4.51)

**Case i:** Let $J(v_\epsilon) < \epsilon$.

Define $S_\epsilon = \{ v \in W^{s,p}_0(\Omega) : 0 \leq J(v) \leq \epsilon \}$. Consider the problem $I_\epsilon = \inf_{v \in S_\epsilon} \{ I(v) \}$. The infimum exists since the set $S_\epsilon$ is bounded and the functional $I$ is $C^1$. Furthermore, $I$ is also weakly lower semicontinuous and $S_\epsilon$ is closed, convex. Thus $I_\epsilon$ is actually attained, at say $v_\epsilon \in S_\epsilon$, and $I_\epsilon = \tilde{I}(v_\epsilon) < \tilde{I}(u_0)$.

**Claim:** We now show that $\exists \eta > 0$ such that $v_\epsilon \geq \eta \phi_1$.

**Proof:** Define $v_\eta = (\eta \phi_1 - v_\epsilon)^+$. We prove the claim by contradiction, i.e. $\forall \eta > 0$ let $|\Omega_\eta| = |\text{supp}\{(\eta \phi_1 - v_\epsilon)^+\}| > 0$. For $0 < t < 1$, define $\xi(t) = \tilde{I}(v_\epsilon + tv_\eta)$. Thus

$$\xi'(t) = \langle \tilde{I}'(v_\epsilon + tv_\eta), v_\eta \rangle$$

$$= \langle (a + b\|u\|^p) \mathcal{L}_p^s(v_\epsilon + tv_\eta) - \lambda g(x)(v_\epsilon + tv_\eta)^{p-1} - \mu h(x)(v_\epsilon + tv_\eta)^{-\gamma} - f(x, v_\epsilon + tv_\eta), v_\eta \rangle.$$  

(4.52)

Similarly,

$$\xi'(1) = \langle \tilde{I}'(v_\epsilon + v_\eta), v_\eta \rangle$$

$$= \langle \tilde{I}'(\eta \phi_1), v_\eta \rangle$$

$$= \langle (a + b\|\eta \phi_1\|^p) \mathcal{L}_p^s(\eta \phi_1) - \lambda g(x)(\eta \phi_1)^{p-1} - \mu h(x)(\eta \phi_1)^{-\gamma} - f(x, \eta \phi_1), v_\eta \rangle < 0.$$  

(4.53)

for sufficiently small $\eta > 0$. Moreover,

$$-\xi'(1) + \xi'(t) = \langle (a + b\|u\|^p) \mathcal{L}_p^s(v_\epsilon + tv_\eta) - (a + b\|u\|^p) \mathcal{L}_p^s(v_\epsilon + v_\eta)$$

$$+ \lambda g(x)((v_\epsilon + v_\eta)^{p-1} - (v_\epsilon + tv_\eta)^{p-1})$$

$$+ \mu h(x)((v_\epsilon + v_\eta)^{-\gamma} - (v_\epsilon + tv_\eta)^{-\gamma})$$

$$+ (f(x, v_\epsilon + v_\eta) - f(x, v_\epsilon + tv_\eta), v_\eta \rangle.$$  

(4.54)

since $\lambda g(x) t^{p-1} + \mu h(x) t^{-\gamma} + f(\cdot, t)$ is a uniformly nonincreasing function with respect to $x \in \Omega$ for sufficiently small $t > 0$. From the monotonicity of $(a + b\|u\|^p) \mathcal{L}_p^s$ (refer Theorem 4.6) we have that, for sufficiently small $\eta > 0$, $0 \leq \xi'(1) - \xi'(t)$. From the Taylor series expansion and the fact that $J(v_\epsilon) < \epsilon$, $\exists 0 < \theta < 1$ such that

$$0 \leq \tilde{I}(v_\epsilon + v_\eta) - \tilde{I}(v_\epsilon)$$

$$= \langle \tilde{I}'(v_\epsilon + \theta v_\eta), v_\eta \rangle$$

$$= \xi'(\theta).$$  

(4.55)
Thus for \( t = \theta \) we get \( \xi'(\theta) \geq 0 \) which is a contradiction to \( \xi'(\theta) \leq \xi'(1) < 0 \) as obtained above. Thus \( v_\epsilon \geq \eta \phi_1 \) for some \( \eta > 0 \).

In fact, from the Lemmas 4.9 and 4.11 we have \( \sup_{\epsilon \in [0,1]} \{ ||v_\epsilon||_{C^1,\alpha(\Omega)} \} \leq C \). By the compact embedding \( C^{1,\alpha}(\bar{\Omega}) \hookrightarrow C^{1,\kappa}(\Omega) \), for any \( \kappa < \alpha \), we have \( v_\epsilon \to u_0 \) which contradicts the assumption made.

**Case ii:** \( K(v_\epsilon) = \epsilon \).

Let \( \eta = (\eta \phi_1 - v_\epsilon)^+ \) and \( \xi(t) = \bar{T}(v_\epsilon + tv_0) \). Then by arguments as in Case i, we have that \( \xi \) is decreasing. This implies that \( \bar{T}(v_\epsilon) > \bar{T}(v_\epsilon + tv_0) \). Since the functionals \( \bar{T}, J \) are \( C^1 \), hence in this case from the Lagrange multiplier rule (refer Theorem 4.12) there exists \( \mu_\epsilon \in \mathbb{R} \) such that \( \bar{I}'(v_\epsilon) = \mu_\epsilon J'(v_\epsilon) \). We will first show that \( \mu_\epsilon \leq 0 \). Suppose \( \mu_\epsilon > 0 \), then \( \exists \phi \in X_0 \) such that

\[
\langle \bar{I}'(v_\epsilon), \phi \rangle < 0 \text{ and } \langle J'(v_\epsilon), \phi \rangle < 0.
\]

Then for small \( t > 0 \) we have

\[
\bar{I}(v_\epsilon + t\phi) < \bar{I}_\lambda(v_\epsilon) \\
J(v_\epsilon + t\phi) < J(v_\epsilon) = \epsilon
\]

which is a contradiction to \( v_\epsilon \) being a minimizer of \( \bar{I}_\lambda \) in \( S_\epsilon \).

We now consider the following two cases.

**Case a:** \( \mu_\epsilon \in (-l, 0) \) where \( l > -\infty \).

Now consider the sequence of problems

\[
(P_\epsilon) : \ (a + b\|u\|^p) \mathcal{L}_p^s u = \lambda g(x) u^{p-1} + \mu h(x) u^{-\gamma} + f(x, u) + \mu_\epsilon |u - u_0|^{r-1} (u - u_0)
\]

(4.56)

Observe that \( u_0 \) is a weak solution to \( (P_\epsilon) \). From the weak comparison principle (Theorem 4.6) we have \( v_\epsilon \geq \eta \phi_1 \) for some \( \eta > 0 \) small enough, independent of \( \epsilon \) since \( \eta \phi_1 \) is a strict subsolution to \( (P_\epsilon) \). Further, since \(-l \leq \mu_\epsilon \leq 0\), there exists \( M, c \) such that

\[
(a + b\|u\|^p) \mathcal{L}_p^s (v_\epsilon - 1)^+ \leq M + c((v_\epsilon - 1)^+)^r.
\]

(4.57)

Using the Moser iteration technique as in Theorem 4.5 we obtain \( \|v_\epsilon\|_{\infty} \leq C' \). Therefore \( \exists L > 0 \) such that \( \eta \phi_1 \leq v_\epsilon \leq L \phi_1 \). By using the arguments previously used, we end up getting \( \|v_\epsilon\|_{C^\alpha(\Omega)} \leq C' \). The conclusion follows as in the previous case of \( J(v_\epsilon) < \epsilon \).

**Case b:** \( \inf_{\mu_\epsilon} = -\infty \)

Let us assume \( \mu_\epsilon \leq -1 \). As above, we can similarly obtain \( v_\epsilon \geq \eta \phi_1 \) for \( \eta > 0 \) small enough and independent of \( \epsilon \). Further, there exists a constant \( M > 0 \) such that

\[
\lambda g(x) t^{p-1} + \mu h(x) t^{-\gamma} + f(x, t) + \tau |t - u_0(x)|^{r-1} (t - u_0(x)) < 0, \forall (\tau, x, t) \in (-\infty, -1] \times \Omega \times (M, \infty).
\]
From the weak comparison principle on \((a + b\|u\|^p)\mathcal{L}_p^s\), we get \(v_\epsilon \leq M\) for \(\epsilon > 0\) sufficiently small. Since \(u_0\) is a local \(C^1\) - minimizer, \(u_0\) is a weak solution to (4.1) and hence

\[
\langle (a + b\|u_0\|^p)\mathcal{L}_p^s u_0, \phi \rangle = \lambda \int_\Omega g(x)u_0^{p-1}\phi dx + \mu \int_\Omega h(x)u_0^{-\gamma}\phi dx + \int_\Omega f(x, u_0)\phi dx
\]

\(\forall \phi \in C_c^\infty(\Omega)\). Also, \(u_0\) satisfies

\[
\langle (a + b\|u_0\|^p)\mathcal{L}_p^s u_0, w \rangle = \lambda \int_\Omega g(x)u_0^{p-1}w + \mu \int_\Omega h(x)u_0^{-\gamma}wdx + \int_\Omega f(x, u_0)wdx.
\]

Similarly,

\[
\langle (a + b\|v_\epsilon\|^p)\mathcal{L}_p^s v_\epsilon, w \rangle = \lambda \int_\Omega g(x)v_\epsilon^{p-1}w + \mu \int_\Omega h(x)v_\epsilon^{-\gamma}wdx + \int_\Omega f(x, v_\epsilon)wdx.
\]

On subtracting (4.59) from (4.60) and testing with \(w = |v_\epsilon - u_0|^{\beta-1}(v_\epsilon - u_0)\), where \(\beta \geq 1\), we obtain

\[
0 \leq \beta \langle (a + b\|u_\epsilon\|^p)\mathcal{L}_p^s v_\epsilon - (a + b\|u_0\|^p)\mathcal{L}_p^s u_0, |v_\epsilon - u_0|^{\beta-1}(v_\epsilon - u_0) \rangle
\]

\[-\lambda \int_\Omega g(x)(v_\epsilon^{p-1} - u_0^{p-1})|v_\epsilon - u_0|^{\beta-1}(v_\epsilon - u_0)dx
\]

\[-\mu \int_\Omega h(x)(v_\epsilon^{-\gamma} - u_0^{-\gamma})|v_\epsilon - u_0|^{\beta-1}(v_\epsilon - u_0)dx
\]

\[= \int_\Omega (f(x, v_\epsilon) - f(x, u_0))|v_\epsilon - u_0|^{\beta-1}(v_\epsilon - u_0)dx
\]

\[+ \mu_\epsilon \int_\Omega |v_\epsilon - u_0|^{\beta+r}dx.
\]

By the Hölder’s inequality and the bounds of \(v_\epsilon, u_0\) we obtain

\[-\mu_\epsilon \|v_\epsilon - u_0\|^{r}_{\beta+r} \leq C|\Omega|^{-\frac{r}{\beta+r}}.
\]

Here \(C\) is independent of \(\epsilon\) and \(\beta\). On passing the limit \(\beta \to \infty\) we get

\[-\mu_\epsilon \|v_\epsilon - u_0\|_\infty \leq C.
\]

Working on similar lines we end up getting \(v_\epsilon\) is bounded in \(C^\alpha(\Omega)\) independent of \(\epsilon\) and the conclusion follows.

**Conclusions**

Existence of infinitely many solutions to the problem in (1.1) has been shown. In addition, a weak comparison result has been proved. It has also been shown that the solutions are in \(L^\infty(\Omega)\). Further it has also been proved that the \(C^1\) minimizers are the \(W_0^{s,p}\) minimizers as well. Some future scope of work on this line would be to prove that the \(C^1\) minimizers are the \(W_0^{s,p}\) minimizers as well for a general Kernel.
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Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed
in this study.

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