EIGENVALUES OF COLLAPSING DOMAINS AND DRIFT LAPLACIANS

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Abstract. By introducing a weight function to the Laplace operator, Bakry and Émery defined the “drift Laplacian” to study diffusion processes. Our first main result is that, given a Bakry-Émery manifold, there is a naturally associated family of graphs whose eigenvalues converge to the eigenvalues of the drift Laplacian as the graphs collapse to the manifold. Applications of this result include a new relationship between Dirichlet eigenvalues of domains in $\mathbb{R}^n$ and Neumann eigenvalues of domains in $\mathbb{R}^{n+1}$ and a new maximum principle. Using our main result and maximum principle, we are able to generalize all the results in Riemannian geometry based on gradient estimates to Bakry-Émery manifolds.

1. Introduction

Bakry-Émery geometry was introduced in [3] to study diffusion processes. For a Riemannian manifold $(M, g)$ and $\phi \in C^2(M)$, the Bakry-Émery manifold is a triple $(M, g, \phi)$, where the measure on $M$ is the weighted measure $e^{-\phi}dV_g$. If $\text{Ric}$ and $\Delta$ are, respectively, the Ricci curvature and Laplacian with respect to the Riemannian metric $g$, then the Bakry-Émery Ricci curvature is defined to be

$$\text{Ric}_\infty = \text{Ric} + \text{Hess}(\phi),$$

and the Bakry-Émery Laplacian is

$$\Delta_\phi = \Delta - \nabla \phi \cdot \nabla.$$

The operator can be extended as a self-adjoint operator with respect to the weighted measure $e^{-\phi}dV_g$; it is also known as a “drifting” or “drift” Laplacian.

Theorem 1. Let $(M, g, \phi)$ be a compact Bakry-Émery manifold. Let

$$M_\varepsilon := \{(x, y) \mid x \in M, \ 0 \leq y \leq \varepsilon e^{-\phi(x)}\} \subset M \times \mathbb{R}^+,$$

with $\phi \in C^2(M)$ and $e^{-\phi} \in C(M \cup \partial M)$. Let $\{\mu_k\}_{k=0}^\infty$ be the eigenvalues of the Bakry-Émery Laplacian on $M$. If $\partial M \neq \emptyset$, assume the Neumann boundary condition. Let $\mu_k(\varepsilon)$ be the Neumann eigenvalues of $M_\varepsilon$ for $\tilde{\Delta} := \Delta + \partial_y^2$. Then $\mu_k(\varepsilon) = \mu_k + O(\varepsilon^2)$ for $k \geq 0$.

A corollary of Theorem 1 gives a relationship between the Dirichlet eigenvalues in $\mathbb{R}^n$ and Neumann eigenvalues in $\mathbb{R}^{n+1}$.

Corollary 1. Let $M$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and let $\phi_1$ be the first Dirichlet eigenfunction of the Euclidean Laplacian on $M$. Define

$$M_\varepsilon := \{(x, y) \in \mathbb{R}^{n+1} \mid x \in M, 0 \leq y \leq \varepsilon \phi_1(x)^2\}.$$

\[1\text{In the notation of [14], this is the } \infty \text{ Bakry-Émery Ricci curvature.}\]
Let \( \{ \lambda_k \}_{k=1}^{\infty} \) be the Dirichlet eigenvalues of \( M \), and let \( \{ \mu_k(\varepsilon) \}_{k=0}^{\infty} \) be the Neumann eigenvalues of \( M_{\varepsilon} \). Then \( \lim_{\varepsilon \to 0} \mu_{k-1}(\varepsilon) = \lambda_k - \lambda_1 \), for all \( k \in \mathbb{N} \).

In the second part of the paper, we establish a new maximum principle which, together with Theorem 1 imply the following.

**Principle.** There is a one-one correspondence between the gradient estimate on a Riemannian manifold and on a Bakry-Émery manifold. More precisely, the eigenvalue estimate on the Bakry-Émery manifold \((M, g, \phi)\) is equivalent to that on the Riemannian manifold \((M_{\varepsilon}, g + d\phi^2)\) for \( \varepsilon \) small enough.

The method of gradient estimates in eigenvalue problems was first used by Li-Yau [11]. The papers [2, 4, 10, 18, 21–24] are the most influential to this work. The point of the above Principle is that one may apply all the proofs of gradient estimates directly to Bakry-Émery geometry without repeating the calculations.

The organization of the paper is as follows. In §2, we present the variational principles for the drift Laplacian which we use heavily in our proof. The proof of Theorem 1 and Corollary 1 comprise §3. We prove the new maximum principle and discuss its applications in §4; finally, §5 contains technical results on Schauder estimates which are of independent interest.

2. Variational principles

On a Riemannian manifold \((M, g)\) with boundary \( \partial M \), the Laplace operator can be written as

\[
\Delta = \frac{1}{\sqrt{\det(g)}} \sum_{i,j} \partial_i g^{ij} \sqrt{\det(g)} \partial_j,
\]

and in particular on \( \mathbb{R}^n \) with the Euclidean metric,

\[
\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.
\]

The Dirichlet (respectively, Neumann) eigenvalues of the Laplace operator are the real numbers \( \lambda \) for which there exists an eigenfunction

\[
u \in C^\infty(M) \text{ such that } -\Delta u = \lambda u \text{ and } u|_{\partial M} = 0, \text{ (respectively, } \frac{\partial u}{\partial n}|_{\partial M} = 0).\]

The eigenvalues of the drift Laplace operator are defined analogously.

We shall use \( \lambda \) to denote Dirichlet eigenvalues, \( \mu \) to denote Neumann eigenvalues, and index the Dirichlet eigenvalues by \( \mathbb{N} \) and the Neumann eigenvalues by \( 0 \cup \mathbb{N} \). The Dirichlet and Neumann\(^2\) eigenvalues, respectively, satisfy the following variational principles [5].

\[
\lambda_k = \inf_{\varphi \in C^1(M)} \left\{ \int_M |\nabla \varphi|^2 \quad \varphi|_{\partial M} = 0, \varphi \neq 0 = \int_M \varphi \phi_l, 0 \leq l < k \right\},
\]

\[
\mu_j = \inf_{\varphi \in C^1(M)} \left\{ \int_M |\nabla \varphi|^2 \quad \varphi \neq 0 = \int_M \varphi \phi_l, -1 \leq l < j \right\},
\]

\(^2\)Note that the Neumann boundary condition is automatically satisfied if no boundary condition is imposed in the variational principle.
for $k \geq 1$ and $j \geq 0$, where $\phi_j$ and $\varphi_l$ are, respectively, eigenfunctions for $\lambda_j$ and $\mu_l$ (assuming that $\phi_0 \equiv 0$ and $\varphi_{-1} \equiv 0$).

**Proposition 1.** Let $(M, g, \phi)$ be a Bakry-Émery manifold with boundary. Then, the Dirichlet and Neumann eigenvalues of the associated drift Laplacian satisfy the following variational principles.

$$
\lambda_k = \inf_{\varphi \in C^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2 e^{-\phi}}{\int_M \varphi^2 e^{-\phi}} \left| \varphi|_{\partial M} = 0, \varphi \not\equiv 0 = \int_M \varphi \phi_j e^{-\phi}, 0 \leq j < k, \right. \right\},
$$

$$
\mu_j = \inf_{\varphi \in C^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2 e^{-\phi}}{\int_M \varphi^2 e^{-\phi}} \left| \varphi \not\equiv 0 = \int_M \varphi \phi_l e^{-\phi}, -1 \leq l < j \right. \right\}
$$

for $k \geq 1$ and $j \geq 0$.

**Remark 1.** Let $\{\lambda_k\}_{k=1}^\infty$ be the Dirichlet eigenvalues and let the associated orthonormal basis of eigenfunctions be $\{\phi_k\}_{k=1}^\infty$. Setting the weight function $\phi = -2 \log \varphi_1$, the variational principle for $(M, g, \phi)$ is that for all $k \geq 1$,

$$
\lambda_k - \lambda_1 = \inf_{\varphi \in C^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2 \varphi_1^2}{\int_M \varphi^2 \varphi_1^2} \left| \varphi \not\equiv 0 = \int_M \varphi \phi_j \varphi_1, 0 \leq j < k \right. \right\}.
$$

When $k = 2$, and the domain $M \subset \mathbb{R}^n$, the following variational principle is Corollary 1.3 of [9] and is based on results of [6]. The following proposition is a useful tool.

**Proposition 2.** For $k \geq 1$, let $\xi_0, \ldots, \xi_{k-1}$ be a nontrivial orthogonal set with respect to the weighted $L^2$ measure; that is

$$
\int_M \xi_i \xi_j e^{-\phi} = 0
$$

for $i \neq j$ and $\xi_i \not\equiv 0$. Then, we have for the Neumann eigenvalues

$$
\sum_{j=0}^k \mu_j \leq \sum_{j=0}^k \frac{\int_M |\nabla \xi_j|^2 e^{-\phi}}{\int_M |\xi_j|^2 \varphi_1^2}.
$$

The proof is well known and is omitted. □

We demonstrate that the difference between the $k^{th}$ and the first Dirichlet eigenvalues is the Neumann eigenvalue of a certain drift Laplacian. This result was known to Singer-Wong-Yau-Yau [18].

**Proposition 3.** For a bounded domain $M \subset \mathbb{R}^n$, let $\{\lambda_k\}_{k=1}^\infty$ be the Dirichlet eigenvalues of the Euclidean Laplacian with orthonormal basis of eigenfunctions $\{\phi_k\}_{k=1}^\infty$, and let $\{\mu_k\}_{k=0}^\infty$ be the Neumann eigenvalues of the drift Laplacian on $M$ with respect to the weight function $-2 \log \varphi_1$. Then, $\lambda_k - \lambda_1 = \mu_k - 1$ for all $k \in \mathbb{N}$.

**Proof.** This follows from the following formula (cf. [18])

$$
\Delta \left( \frac{\phi_k}{\varphi_1} \right) + 2 \nabla \log \varphi_1 \nabla \left( \frac{\phi_k}{\varphi_1} \right) = - (\lambda_k - \lambda_1) \left( \frac{\phi_k}{\varphi_1} \right).
$$

□
Finally, throughout this paper we will use the following notations: for a function \( f(t) \) and fixed \( k \geq 0 \),

\[ f(t) = O(t^k) \text{ as } t \to 0 \text{ if there exists } C, \delta > 0 \text{ such that } |f(t)| \leq Ct^k \text{ for all } |t| \leq \delta; \]

\[ f(t) = o(t^k) \text{ as } t \to 0 \text{ if } \lim_{t \to 0} \frac{f(t)}{t^k} = 0. \]

Also, throughout this paper, a constant \( C \) is independent of \( \varepsilon \), but may differ from line to line.

3. Eigenvalue convergence: A coarse estimate

In this section, we prove a coarse version of Theorem 1. Let \((M, g, \phi)\) be the compact Bakry-Émery manifold, with or without boundary, and let

\[ M_\varepsilon = \{(x, y) \mid x \in M, \quad 0 \leq y \leq \varepsilon f(x)\} \subset M \times \mathbb{R}^+, \quad f(x) := e^{-\phi(x)}. \]

Let \( \{\mu_k\}_{k=0}^\infty \) and \( \{\psi_k\}_{k=0}^\infty \) be respectively the eigenvalues and eigenfunctions for the drift Laplacian \( \Delta \) on \( M \) (if \( \partial M \neq \emptyset \), we endow it the Neumann boundary condition), and let \( \{\mu_k(\varepsilon)\}_{k=0}^\infty \) be the eigenvalues for \( \tilde{\Delta} = \Delta + \partial_y^2 \) on \( M_\varepsilon \) with corresponding orthogonal eigenfunctions \( \{\varphi_{j,\varepsilon}\}_{j=0}^\infty \). We assume the eigenfunctions are normalized so that

\[ \int_{M_\varepsilon} \varphi_{j,\varepsilon} \varphi_{k,\varepsilon} = \varepsilon \delta_j^k. \]

In particular, the volume of \( M_\varepsilon \) is \( \varepsilon \). This normalization depends only on \( f \) and \( M \).

We use \( \nabla \) and \( \Delta \) as the gradient and Laplace operators, respectively, of \( M \), and \( \tilde{\nabla} = (\nabla, \frac{\partial}{\partial y}) \) and \( \tilde{\Delta} \) as the gradient and Laplace operators, respectively, of \( M_\varepsilon \subset M \times \mathbb{R}^+ \).

We prove the theorem by induction. For \( k = 0 \), the statement of Theorem 1 is trivial. We shall prove the theorem for \( k \geq 1 \), assuming that for \( 1, \ldots, k-1 \), the theorem has been proven. By a theorem of Uhlenbeck [19], for generic manifold \( \mu_1, \ldots, \mu_k \) are simple; that is, all eigenspaces with respect to the eigenvalues \( \mu_1, \ldots, \mu_k \) are of multiplicity one. Since the eigenvalues are continuous with respect to continuous deformations of the domain, it is sufficient to prove the theorem under this additional assumption.

**Lemma 1.** Using the above notation, \( \mu_k(\varepsilon) \leq \mu_k + O(\varepsilon^2) \).

**Proof.** Considering \( \psi_k \) as functions on \( M_\varepsilon \), they are orthogonal with respect to the measure \( dV_gdy \). By Proposition 2 we have

\[ \sum_{j=0}^k \mu_j(\varepsilon) \leq \sum_{j=0}^k \mu_j. \]

By the inductive assumption, we have

\[ \mu_j \leq \mu_j(\varepsilon) + O(\varepsilon^2) \]

for all \( j < k \). The lemma follows from the above two inequalities. \( \square \)

For any \( 0 \leq r \leq \varepsilon \), and for \( 0 \leq i \leq k \), let

\[ b_i(x, r) := \varphi_{i, \varepsilon}(x, rf(x)) \quad \text{and} \quad A_k = \sum_{j=0}^k \int_{M_\varepsilon} \left( \frac{\partial \varphi_{j,\varepsilon}}{\partial y} \right)^2 (x, y). \]
Lemma 2. Using the above notations, we have

\[ \left| \int_M b_i(x,r)b_j(x,r)f(x) - \delta_i^j \right| \leq C\varepsilon A_k \quad \forall \ 0 \leq i, j \leq k \]

for all \( 0 \leq r \leq \varepsilon \).

Proof. \[ \] For any \( 0 \leq r \leq \varepsilon \), \( 0 \leq y \leq \varepsilon f(x) \), and \( 0 \leq i, j \leq k \),

\[ |b_i(x,r)b_j(x,r) - \varphi_i(x,y)\varphi_j(x,y)| \leq \int_0^{\varepsilon f(x)} |\partial_y (\varphi_i(x,y)\varphi_j(x,y))| \, dy \]

(3.3)

\[ \leq \int_0^{\varepsilon f(x)} \left( \left| \frac{\partial \varphi_i}{\partial y} \right| \cdot |\varphi_j| + \left| \frac{\partial \varphi_j}{\partial y} \right| \cdot |\varphi_i| \right) (x,y) \, dy. \]

Note that for any \( 0 \leq r \leq \varepsilon \),

\[ \varepsilon \int_M b_i(x,r)b_j(x,r)f(x) = \int_0^{\varepsilon f(x)} \int_M b_i(x,r)b_j(x,r) = \int_{M_x} b_i(x,r)b_j(x,r), \]

and

\[ \int_{M_x} \varphi_i(x,y)\varphi_j(x,y) = \varepsilon \delta_i^j. \]

Then

\[ \left| \varepsilon \int_M b_i(x,r)b_j(x,r)f(x) - \varepsilon \delta_i^j \right| = \left| \int_{M_x} (b_i(x,r)b_j(x,r) - \varphi_i(x,y)\varphi_j(x,y)) \right|, \]

which by (3.3),

\[ \leq \int_{M_x} \int_0^{\varepsilon f(x)} \left( \left| \frac{\partial \varphi_i}{\partial y} \right| \cdot |\varphi_j| + \left| \frac{\partial \varphi_j}{\partial y} \right| \cdot |\varphi_i| \right) (x,t), \]

(3.4)

\[ \leq \varepsilon \|f\|_\infty \int_{M_x} \left( \left| \frac{\partial \varphi_i}{\partial y} \right| \cdot |\varphi_j| + \left| \frac{\partial \varphi_j}{\partial y} \right| \cdot |\varphi_i| \right) \]

\[ \leq \varepsilon \|f\|_\infty \left( \sqrt{A_k} \cdot \|\varphi_i\|_{L^2(M_x)} + \sqrt{A_k} \cdot \|\varphi_i\|_{L^2(M_x)} \right). \]

Since \( \|\varphi_i\|_{L^2(M_x)} = \sqrt{\varepsilon} \),

\[ \left| \varepsilon \int_M b_i(x,r)b_j(x,r)f(x) - \varepsilon \delta_i^j \right| \leq C\varepsilon^{3/2} \sqrt{A_k}. \]

\[ \square \]

Corollary 2. Using the same notations as above, we have

\[ \left| \int_M b_i(x,r)b_j(x,r)f(x) - \delta_i^j \right| \leq C\varepsilon \quad \forall \ 0 \leq i, j \leq k, 0 \leq r \leq \varepsilon. \]

Proof. This follows from the fact that \( A_k \leq \sum_{j=1}^k \|\nabla \varphi_j\|_{L^2(M_x)}^2 = \sum_{j=1}^k \mu_j(\varepsilon) \leq C\varepsilon \), where the constant \( C \) depends only on \( k \) (not on \( \varepsilon \)).

\[ \square \]

For simplicity of notation, we drop the subscript \( \varepsilon \) from \( \varphi \).
Lemma 3. Under the same condition as in Theorem 1 and assuming that Theorem 1 is true for \( j < k \), we have
\[
\mu_k \leq \mu_k(\varepsilon) + C(\varepsilon^2 + \sqrt{\varepsilon A_k}).
\]
In particular, this estimate and Lemma 2 imply that \( \mu_k(\varepsilon) = \mu_k + O(\varepsilon) \) for all \( k \geq 0 \).

Proof. Define inductively that \( \tilde{b}_0(x,r) = b_0(x,r) \),
\[
\tilde{b}_k(x,r) = b_k(x,r) + \sum_{j=0}^{k-1} c_{kj}(r)b_j(x,r),
\]
where for any \( k \geq 0 \), \( c_{kj}(r) \) are functions of \( r \) such that \( \tilde{b}_k \perp b_1, \ldots, b_{k-1} \) with respect to the measure \( fdV_g \). By Proposition 2, we have
\[
\sum_{j=0}^{k} \mu_j \leq \sum_{j=0}^{k} \frac{\int_M |\nabla \tilde{b}_j|^2 f}{\int_M \tilde{b}_j^2 f}.
\]
By Lemma 2 \( c_{kj}(r) \sim O(\sqrt{\varepsilon A_k}) \) uniformly for \( 0 < r \leq \varepsilon \). Thus by the definition of \( \tilde{b}_k \), using Lemma 2 again, we have
\[
\sum_{j=0}^{k} \mu_j \leq \sum_{j=0}^{k} \frac{\int_M |\nabla b_j|^2 f}{\int_M \tilde{b}_j^2 f} \leq (1 + C \varepsilon A_k) \sum_{j=0}^{k} \int_M |\nabla b_j|^2 f.
\]
Since the above inequality holds for all \( r \), integrating from 0 to \( \varepsilon \), we have
\[
\varepsilon \sum_{j=0}^{k} \mu_j \leq (1 + C \varepsilon A_k) \sum_{j=0}^{k} \int_0^{\varepsilon} \int_M |\nabla b_j|^2 f.
\]
We compute
\[
\nabla b_j(x,r) = (\nabla \varphi_j)(x,rf(x)) + r \frac{\partial \varphi_j}{\partial y}(x,rf(x)) \nabla f(x).
\]
Using the Cauchy inequality, we get
\[
|\nabla b_j(x,r)|^2 \leq (1 + \varepsilon^2 |\nabla f|^2) |\nabla \varphi_j,\varepsilon|^2.
\]
Therefore,
\[
\int_0^{\varepsilon} \int_M |\nabla b_j(x,r)|^2 f(x) \leq (1 + C \varepsilon^2) \int_M |\nabla \varphi_j,\varepsilon|^2 = (1 + C \varepsilon^2) \mu_j(\varepsilon) \varepsilon
\]
for \( j \leq k \).

The above estimate together with (3.7) show that
\[
\varepsilon \sum_{j=0}^{k} \mu_j \leq \varepsilon(1 + C \sqrt{\varepsilon A_k})(1 + C \varepsilon^2) \sum_{j=0}^{k} \mu_j(\varepsilon).
\]
Dividing by \( \varepsilon \) and letting \( \varepsilon \to 0 \), this estimate together with an induction argument completes the proof of the lemma. The precise estimate \( \mu_k(\varepsilon) = \mu_k + O(\varepsilon^2) \) to complete the proof of Theorem 1 will be demonstrated in the final section. \( \square \)

Proof of Corollary 1. The corollary follows immediately from Lemma 3 and Proposition 3. \( \square \)
4. A maximum principle

The Neumann eigenvalues are continuous functions with respect to the manifold \( M \). Therefore, to estimate the eigenvalues, we may use an exhaustion of \( M \),

\[
M^\delta = \{ x \in M \mid \text{dist}(x, \partial M) \geq \delta \}, \quad \delta > 0.
\]

On \( M^\delta \), \( f \) has a positive lower bound. Thus using the variational principle, we may, without loss of generality assume that \( f \) is not only positive but is a constant in a neighborhood of \( \partial M^\delta \). For the rest of the paper we make such an assumption.

The usual maximum principle for the gradient estimate is the following. Let \( H = \frac{1}{2} |\nabla \varphi|^2 + F(\varphi) \), where \( F \) is a smooth function of one variable, and let \( x_0 \) be an interior point of \( M \) at which \( H \) reaches its maximum. Then

\[
0 \geq |\nabla^2 \varphi|^2 + \nabla \varphi \nabla (\Delta \varphi) + \text{Ric}(\nabla \varphi, \nabla \varphi) + F'(\varphi) \Delta \varphi + F''(\varphi)|\nabla \varphi|^2.
\]

The above inequality is very useful for obtaining lower bounds on the first eigenvalue of a Laplace or Schrödinger operator; for more details, we refer to the book [17].

However, it is not appropriate to apply the above maximum principle directly to the manifold \( M_\varepsilon \) for the following reasons:

1. \( M_\varepsilon \) need not be convex, even if \( M \) is. As we know, if \( M \) is convex, the maximum of \( H \) must be reached in the interior of \( M \). In general, we don’t have such a property for \( M_\varepsilon \).
2. The natural Ricci curvature attached to the problem is \( \text{Ric}_\infty \), not the Ricci curvature of \( M_\varepsilon \), which is essentially \( \text{Ric} \).

Remark 2. The choice of \( F \) is highly technical. In the Li-Yau’s case [11], which is the simplest case, \( F(x) = \frac{1}{2} x^2 \). In Zhong-Yang’s case [24], \( F \) is (up to a constant)

\[
F(x) = 1 - x^2 + a \left( \frac{4}{\pi} \arcsin x + x \sqrt{1 - x^2} - 2x \right),
\]

where \( a \) is a positive constant. More sophisticated choices of \( F \) can be found in [12] and [13].

As in the previous sections, we assume \( M \) is a compact manifold with or without boundary. Let \( U \) be an open set of \( M \) and let \( (x_1, \cdots, x_n) \) be a local coordinates system on \( U \). Let \( \varphi_{k,\varepsilon} \) be the Neumann eigenfunctions of the eigenvalues \( \mu_k(\varepsilon) \) with the \( L^2 \) norm normalized to be \( \sqrt{\varepsilon} \). We let

\[
\psi(x) = \varphi_{k,\varepsilon}(x, 0), \quad x \in M.
\]

The technical heart of this paper is Theorem 5 which implies the following key results of this section.

Lemma 4. With the above notation, as \( \varepsilon \to 0 \),

\[
(4.1) \quad \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} - \nabla \log f(x) \nabla \psi = o(1),
\]

\[
(4.2) \quad (\nabla \psi, \nabla \frac{\partial \varphi_{k,\varepsilon}}{\partial y^2}) - (\nabla^2 \log f)(\nabla \psi, \nabla \psi) - \nabla^2 \psi(\nabla \psi, \nabla \log f) = o(1).
\]
Proof. Since $\varphi_{k,\varepsilon}$ satisfies the Neumann condition, we have
\begin{equation}
\frac{\partial \varphi_{k,\varepsilon}}{\partial y}(x, 0) = 0;
\end{equation}
(4.3)
\[
\frac{\partial \varphi_{k,\varepsilon}}{\partial y}(x, \varepsilon f(x)) - \varepsilon \nabla f(x) \nabla \varphi_{k,\varepsilon}(x, \varepsilon f(x)) = 0,
\]
for any $x \in M$. Applying the mean value theorem to the above equations, we have
\[
\varepsilon f(x) \left( \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, \xi(x)) - \nabla \log f(x) \nabla \varphi_{k,\varepsilon}(x, \varepsilon f(x)) \right) = 0,
\]
where $\xi(x) \in (0, \varepsilon f(x))$. Theorem 5 then implies (4.1).

Assume now that at $x$, the local coordinate system is normal. For the second statement, taking partial derivatives with respect to $x_j$ in the second equation of (4.3) gives
\[
\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y}(x, \varepsilon f(x)) + \varepsilon \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, \varepsilon f(x)) \frac{\partial f}{\partial x_j}
- \varepsilon \frac{\partial^2 f(x)}{\partial x_i \partial x_j}(x, \varepsilon f(x)) \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i}(x, \varepsilon f(x))
- \varepsilon^2 \frac{\partial f}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial y}(x, \varepsilon f(x)) \frac{\partial f}{\partial x_j} = 0.
\]
Since $\partial \varphi_{k,\varepsilon}/\partial y = 0$ on $\{y = 0\}$, we have
\[
\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y}(x, 0) = 0, \quad \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial x_i \partial y}(x, 0) = 0.
\]
The mean value theorem implies
\[
\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y}(x, \varepsilon f(x)) = \varepsilon f(x) \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial x_i \partial y}(x, \xi(x))
\]
for some $\xi(x) \in (0, \varepsilon f(x))$. Using Theorem 4 again, we have
\[
\varepsilon f(x) \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y}(x, 0) + \varepsilon \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, 0) \frac{\partial f}{\partial x_j}
- \varepsilon \frac{\partial^2 f(x)}{\partial x_i \partial x_j}(x, 0) \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i}(x, 0)
- \varepsilon^2 \frac{\partial f}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial y}(x, 0) \frac{\partial f}{\partial x_j} = o(\varepsilon f(x))
\]
as $\varepsilon \to 0$. Thus we have
\[
\frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial y^2}(x, 0) + \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, 0) \frac{\partial f}{\partial x_j}
- \frac{\partial^2 f(x)}{\partial x_i \partial x_j}(x, 0) \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i}(x, 0)
- \frac{\partial \log f(x)}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial x_j}(x, 0) = o(1).
\]
Using (4.1), we get
\[
\frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial y^2}(x, 0) + \nabla \log f(x) \nabla \psi(x) \frac{\partial f}{\partial x_j} \frac{\partial \log f(x)}{\partial x_j}
- \frac{1}{f(x)} \frac{\partial^2 f(x)}{\partial x_j \partial x_i}(x, 0) \frac{\partial \psi(x)}{\partial x_i}
- \frac{\partial \log f(x)}{\partial x_i} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x, 0) = o(1),
\]
which implies (4.2).
For our new maximum principle, we consider

\[ H(x, y) = \frac{1}{2} |\tilde{\nabla} \psi| + F(\psi), \]

where \( F \) is a smooth function of one variable. Assume that \((x_0, 0)\) is the point at which \( H \) reaches the maximum on \( \{y = 0\} \), where \( x_0 \) is in the interior of \( M \).

At \((x_0, 0)\), we have

\[ \nabla H(x_0, 0) = 0 \text{ and } \Delta H(x_0, 0) \leq 0. \]

The difficulty is that \( H \) satisfies an elliptic equation with respect to \( \tilde{\Delta} \), rather than \( \Delta \). To obtain the new maximum principle, we need to estimate the second derivative of \( H \) in the \( y \)-direction.

**Lemma 5.** At \((x_0, 0)\),

\[ \frac{\partial^2 H}{\partial y^2} = \nabla^2 \log f(\nabla \psi) + \left( \frac{\partial^2 \phi_{k, \e}}{\partial y^2} \right)^2 + o(1), \text{ as } \e \to 0. \]

**Proof.** Using the normal coordinates at \( x_0 \), we have

\[ \frac{\partial H}{\partial y} = \sum_{i=1}^{n} \frac{\partial \phi_{k, \e}}{\partial x_i} \frac{\partial^2 \phi_{k, \e}}{\partial x_i \partial y} + \sum_{i=1}^{n} \frac{\partial^2 \phi_{k, \e}}{\partial y^2} + F'(\phi_{k, \e}) \frac{\partial \phi_{k, \e}}{\partial y}, \]

and

\[ \frac{\partial^2 H}{\partial y^2} = \sum_{i=1}^{n} \left( \frac{\partial^2 \phi_{k, \e}}{\partial x_i \partial y} \right)^2 + \sum_{i=1}^{n} \frac{\partial \phi_{k, \e}}{\partial y} \frac{\partial^3 \phi_{k, \e}}{\partial x_i \partial y^2} + \left( \frac{\partial^2 \phi_{k, \e}}{\partial y^2} \right)^2 + F''(\phi_{k, \e}) \frac{\partial \phi_{k, \e}}{\partial y}. \]

Since \( \phi_{k, \e} \) satisfies the Neumann boundary condition, \( \frac{\partial \phi_{k, \e}}{\partial y} \) and \( \frac{\partial^2 \phi_{k, \e}}{\partial x_i \partial y} \) vanish on \( \{y = 0\} \). Thus we have

\[ \frac{\partial^2 H}{\partial y^2} = \sum_{i=1}^{n} \frac{\partial \phi_{k, \e}}{\partial x_i} \frac{\partial^3 \phi_{k, \e}}{\partial x_i \partial y^2} + \left( \frac{\partial^2 \phi_{k, \e}}{\partial y^2} \right)^2 + F'(\phi_{k, \e}) \frac{\partial \phi_{k, \e}}{\partial y}. \]

Using Lemma 4, we have

\[ \frac{\partial^2 H}{\partial y^2} = \nabla^2 \log f(\nabla \psi) + \nabla^2 \psi(\nabla \nabla f) + \left( \frac{\partial^2 \phi_{k, \e}}{\partial y^2} \right)^2 + F'(\psi) \frac{\partial \phi_{k, \e}}{\partial y^2} + o(1). \]

Since at \( x_0 \), \( \nabla H = 0 \), we have

\[ \sum_{j=1}^{n} \frac{\partial \psi}{\partial x_i \partial x_j} \frac{\partial \psi}{\partial x_j} + F'(\psi) \frac{\partial \psi}{\partial x_i} = 0, \text{ for each } 1 \leq i \leq n. \]

Using the above equality and Lemma 3, the second and fourth terms on the right side of the expression for \( \frac{\partial^2 H}{\partial y^2} \) cancel. \( \Box \)

**Theorem 2 (Maximum Principle).** With the above notations, we have at \((x_0, 0)\)

\[ o(1) \geq |\nabla^2 \psi|^2 + \nabla \psi(\nabla \phi_{k, \e}) + \text{Ric}_{\infty}(\nabla \psi, \nabla \psi) + F'(\phi) \tilde{\Delta} \phi_{k, \e} + F''(\psi) |\nabla \psi|^2. \]
Proof. By the Bochner formula, we have
\[
\tilde{\Delta} H = |\tilde{\nabla}^2 \varphi_{k,\varepsilon}|^2 + \text{Ric}(\tilde{\nabla} \varphi_{k,\varepsilon}, \tilde{\nabla} \varphi_{k,\varepsilon}) + \tilde{\nabla} \varphi_{k,\varepsilon} \tilde{\nabla}(\tilde{\Delta} \varphi_{k,\varepsilon}) \\
+ F'(\varphi_{k,\varepsilon}) \tilde{\Delta} \varphi_{k,\varepsilon} + F''(\varphi_{k,\varepsilon}) |\nabla \varphi_{k,\varepsilon}|^2.
\]
On \( \{y = 0\} \), we have
\[
|\tilde{\nabla}^2 \varphi_{k,\varepsilon}|^2 = |\nabla^2 \psi|^2 + \left( \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2,
\]
\[
\text{Ric}(\tilde{\nabla} \varphi_{k,\varepsilon}, \tilde{\nabla} \varphi_{k,\varepsilon}) = \text{Ric}(\nabla \psi, \nabla \psi),
\]
\[
\tilde{\Delta} \varphi_{k,\varepsilon} = \Delta \psi + \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2},
\]
Thus we have
\[
\tilde{\Delta} H = |\nabla^2 \psi|^2 + \left( \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2 + \text{Ric}(\nabla \psi, \nabla \psi) \\
+ \nabla \psi \nabla(\tilde{\Delta} \varphi_{k,\varepsilon}) + F'(\psi) \tilde{\Delta} \varphi_{k,\varepsilon} + F''(\psi) |\nabla \psi|^2.
\]
Using Lemma 5 noting that at \((x_0, 0)\)
\[
\tilde{\Delta} H = \Delta H + \frac{\partial^2 H}{\partial y^2} \leq \frac{\partial^2 H}{\partial y^2},
\]
completes the proof. \(\square\)

4.1. Applications. Our work not only has applications to Bakry-Émery geometry but also to Ricci solitons. We recall the main result of Futaki and Sano [7].

**Theorem 3** (Futaki-Sano). Let \(M^n\) be a compact smooth manifold of dimension at least 4. If \(g\) is a non-trivial gradient shrinking Ricci soliton on \(M\) (see definition 1.1 of [7]), then the diameter of \(M\) with respect to \(g\) is bounded below by \(\frac{10\pi}{13\sqrt{\gamma}}\), where \(\gamma\) is a constant determined by \(g\).

This result is proven by using Ling’s gradient estimates [12] to demonstrate a lower bound for the first non-zero eigenvalue of a certain Bakry-Émery Laplacian. Our **Principle** shows that one may directly apply Ling’s estimates to the Bakry-Émery Laplacian to obtain the result. It is reasonable to expect that one may similarly express elliptic geometric equations, like the Ricci soliton equation, in terms of a Bakry-Émery Laplacian and exploit the eigenvalue estimates from Riemannian geometry together with our **Principle** to produce interesting results.

Another application arises from the so-called fundamental gap: the difference between the first two Dirichlet eigenvalues of a domain in \(\mathbb{R}^n\). Andrews and Clutterbuck [2] recently demonstrated an optimal lower bound of \(3\pi^2/d^2\) for the fundamental gap of any convex domain in \(\mathbb{R}^n\) with diameter \(d\). By Proposition 3, the fundamental gap can be interpreted as the first Neumann eigenvalue on certain Bakry-Émery manifold, and in particular, techniques of [1], [2] together with our work imply the following.

**Theorem 4.** Let \(\Omega \subset \mathbb{R}^n\) be a convex domain with piecewise smooth boundary and diameter \(d\). Let \(f \in C^2(\Omega)\). If \(f\) satisfies
\[
(\nabla \log f(y) - \nabla \log f(x)) \cdot \frac{y - x}{|y - x|} \geq \frac{4\pi}{d} \tan \left( \frac{\pi |y - x|}{d} \right) \quad \forall \ x \neq y \text{ in } \Omega,
\]
then the first non-trivial Neumann eigenvalue of the Bakry-Émery Laplacian with respect to the weight function \( \phi = -\log(f^2) \) is bounded below by \( 3\pi^2/d^2 \). Moreover, the first Neumann eigenfunction for the Euclidean Laplacian on

\[
\Omega_\varepsilon := \{(x,y) \mid x \in \Omega, 0 \leq y \leq \varepsilon f^2(x)\} \subset \mathbb{R}^{n+1}
\]

is bounded below by \( 3\pi^2/d^2 - C\varepsilon^2 \), where \( C \) is a fixed constant that depends only on \( n \) and \( \Omega \).

\[\square\]

5. The approximation of eigenfunctions

It is not hard to write down the eigenfunctions formally. Let \( \phi \) be a Neumann eigenfunction of \( M_\varepsilon \) with eigenvalue \( \lambda \). Write

\[
\varphi = \sum_{k=0}^{\infty} y^k \varphi_k,
\]

where \( \varphi_k \) are functions on \( M \). Then we have (formally)

\[
\Delta \varphi_k + \lambda \varphi_k + (k+1)(k+2)\varphi_{k+2} = 0
\]

for all \( k \geq 0 \). Since \( \partial \varphi/\partial y = 0 \) on \( \{y = 0\} \), we have \( \varphi_1 = 0 \) and hence \( \varphi_{2k+1} = 0 \) for all \( k \). Let

\[
H \varphi = -\Delta \varphi - \lambda \varphi.
\]

Then

\[
\varphi_{2k+2} = \frac{H \varphi_{2k}}{(2k+1)(2k+2)} = \frac{H^{k+1} \varphi_0}{(2k+2)!}
\]

Formally, we have

\[
\varphi = \sum_{k=0}^{\infty} \frac{y^{2k} H^k}{(2k)!} \varphi_0 = \cosh(y\sqrt{H}) \varphi_0.
\]

The differential equation for \( \varphi_0 \) follows from the Neumann boundary condition

\[
\sqrt{H} \sinh \left( \varepsilon f(x) \sqrt{H} \right) \varphi_0 - \varepsilon \nabla f \cdot \nabla \left( \cosh(y\sqrt{H}) \varphi_0 \right) \bigg|_{y=\varepsilon f(x)} = 0.
\]

We are not able to prove the full regularity of the above equation at this moment. But a partial solution, namely, a good approximation to the eigenfunctions, is enough for our application. Very roughly speaking, in this section, we prove

\[
\varphi = \varphi_0 + y^2 \varphi_2 + O(\varepsilon^3).
\]

To state our results precisely, we recall the global Schauder estimates [8, Theorem 6.6, Theorem 6.30] and the interpolation inequalities.

We let

\[
B_1 = \{(x,y) \mid y = 0\};
\]

\[
B_{11} = \{(x,y) \mid y = \varepsilon f(x)\};
\]

\[
B_{111} = \{(x,y) \mid x \in \partial M\}.
\]

Then \( B_1 \cup B_{11} \cup B_{111} = \partial M_\varepsilon \).

Let

\[
u_1 = u\mid_{\partial M_\varepsilon} \text{ and } u_2 = \frac{\partial u}{\partial n}\mid_{\partial M_\varepsilon}
\]

on the smooth part of \( \partial M_\varepsilon \).
Define the weighted Hölder norm by
\[ ||u||_{C^{k,a}_0} = \varepsilon^{k+\alpha}||u||_{C^{k,a}} + \cdots + \varepsilon^{\alpha}||u||_{C^\alpha} + ||u||_{C^0}, \]
and \[ ||u||_{C^{k,a}_y} = ||u||_{C^{k,a}_y(\partial M)}, \] where \[ |\cdot|_{C^{k,a}} \] are the standard notations defined in [8]. Using these weighted norms, the constants in the Schauder estimates on \( M_{\varepsilon} \) are independent of \( \varepsilon \). Let 0 < \( \alpha < 1 \). Let \( L \) be a second order uniform elliptic operator with \( C^\alpha \)-bounded coefficients. Then we have the following version of global Schauder estimates on \( M_{\varepsilon} \)
\[ ||u||_{C^{2,\alpha}} \leq C(||u||_{C^0} + \varepsilon^2||Lu||_{C^{2,\alpha}} + \varepsilon||u_2||_{C^{2,\alpha}}), \tag{5.1} \]
and
\[ ||u||_{C^{2,\alpha}} \leq C(||u||_{C^0} + \varepsilon^2||Lu||_{C^{2,\alpha}} + \varepsilon||u_2||_{C^{2,\alpha}(B_{11})} + ||u_1||_{C^{2,\alpha}(B_{1,11})}). \tag{5.2} \]

The Sobolev inequality on \( M_{\varepsilon} \) is
\[ \left( \int_{M_{\varepsilon}} |u|^2 \right)^{\frac{1}{2}} \leq C \varepsilon^{-\frac{n}{4}} \left( \int_{M_{\varepsilon}} |\nabla u|^2 + \int_{M_{\varepsilon}} |u|^2 \right). \tag{5.3} \]
Define the Hölder norm in the \( y \)-direction to be
\[ [u]_{C^\alpha_y} = \max_{x \in M} \sup_{0 \leq y_1, y_2 \leq f(x)} \frac{|u(x, y_1) - u(x, y_2)|}{|y_1 - y_2|^\alpha}, \]
Then we have the following.

**Theorem 5.** For \( M_{\varepsilon} \) defined by (5.1) such that \( f(x) = e^{-\phi(x)} \) is constant in a neighborhood of \( \partial M \), the Neumann eigenfunctions \( \varphi_{k,\varepsilon} \) of \( M_{\varepsilon} \) satisfy
\[ \nabla \varphi_{k,\varepsilon} = O(1), \]
\[ \left\| \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right\|_{C^\alpha_y} + \left\| \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y} \right\|_{C^\alpha_y} = O(1), \quad 1 \leq j \leq n \]
\[ \left\| \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_i \partial x_j \partial y} \right\|_{C^\alpha_y} + \left\| \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_i \partial x_j \partial y} \right\|_{C^\alpha_y} = O(1), \quad 1 \leq i, j \leq n \]
for any 0 < \( \alpha < 1 \), where \( (x_1, \cdots, x_n) \) is any local coordinate system of \( M \).

Define the functions
\[ \eta_k := -\nabla \phi \nabla \psi_k, \]
on \( M \), and let
\[ U_k := \psi_k + \frac{1}{2}y^2 \eta_k \]
be functions on \( M_{\varepsilon} \). By our definition of \( f(x) = e^{-\phi(x)} \), \( \eta_k \) are smooth up to the boundary. Note that since \( \psi_k \) is a Bakry-Émery eigenfunction, it satisfies
\[ \Delta \psi_k + \eta_k = -\mu_k \psi_k. \tag{5.4} \]
Since \( \psi_k \) and \( \eta_k \) are independent of \( y \),
\[ \Delta U_k = -\mu_k \psi_k + \frac{1}{2}y^2 \Delta \eta_k = -\mu_k U_k + \frac{1}{2}y^2 (\Delta \eta_k + \mu_k \eta_k). \tag{5.5} \]
We compute directly,
\[ \frac{\partial U_k}{\partial n} \bigg|_{\partial M_{\varepsilon}} = \begin{cases} 0 & \text{on } B_I \cup B_{11}; \\ -\frac{\varepsilon^3 f^2 \nabla f \nabla \eta_k}{2(1 + \varepsilon^2 |\nabla f|^2)^{3/2}} & \text{on } B_{11}. \end{cases} \tag{5.6} \]
Define

\[ w_{k,\varepsilon} = \alpha_{k,k} \varphi_{k,\varepsilon} + U_k + \sum_{j=0}^{k-1} \alpha_{k,j} \varphi_{j,\varepsilon}, \]

where \( \alpha_{k,j} \) are defined such that \( w_{k,\varepsilon} \perp \varphi_{0,\varepsilon}, \ldots, \varphi_{k,\varepsilon} \) in \( L^2(M_\varepsilon) \). The following inequality will be used repeatedly in the rest of the paper: let \( \varphi \) be a function on \( M_\varepsilon \). Then for any \( p > 0 \), we have

\[ \int_{B_{1/\varepsilon}} |\varphi|^p \leq C \left( \frac{1}{\varepsilon} \int_{M_\varepsilon} |\varphi|^p + p \int_{M_\varepsilon} |\varphi|^{p-1} \right). \]

To prove (5.7), we observe that for any \( 0 \leq y \leq \varepsilon f(x) \), we have

\[ |\varphi|^p(x, \varepsilon f(x)) \leq |\varphi|^p(x, y) + p \int_0^{\varepsilon f(x)} |\varphi|^{p-1} \sqrt{\nabla \varphi} dy. \]

Integrating over \( M_\varepsilon \) to both sides of the above equation and using the Cauchy-Schwarz inequality implies (5.7).

**Lemma 6.** For all \( j < k \), \( \alpha_{k,j} = O(\varepsilon^2) \), and \( \alpha_{k,k} = O(1) \).

**Proof.** Integrating by parts, (5.5) and (5.6) give

\[ \mu_j(\varepsilon) \int_{M_\varepsilon} \varphi_{j,\varepsilon} U_k = -\int_{\partial M_\varepsilon} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} - \int_{M_\varepsilon} \varphi_{j,\varepsilon} \Delta U_k = \int_{\partial M_\varepsilon} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} - \int_{M_\varepsilon} \varphi_{j,\varepsilon} \Delta U_k \]

Thus by (5.6) again, we have

\[ (\mu_j(\varepsilon) - \mu_k) \int_{M_\varepsilon} \varphi_{j,\varepsilon} U_k = -\int_{\partial M_\varepsilon} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} - \int_{M_\varepsilon} \varphi_{j,\varepsilon} \Delta U_k. \]

We clearly have

\[ \left| \int_{M_\varepsilon} \varphi_{j,\varepsilon} y^2 (\Delta \eta_k + \mu_k \eta_k) \right| \leq C \sqrt{\int_{M_\varepsilon} y^4} \cdot \sqrt{\int_{M_\varepsilon} |\varphi_{j,\varepsilon}|^2} = O(\varepsilon^3). \]

Using (5.7) for \( p = 1 \), we have

\[ \left| \int_{B_{1/\varepsilon}} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} \right| \leq C \varepsilon. \]

By the generic assumption of the manifold \( M \) and Lemma 3, \( \mu_j(\varepsilon) = \mu_j + O(\varepsilon) < \mu_k \) for all \( j < k \) for \( \varepsilon \) sufficiently small. Thus, dividing by \( (\mu_j(\varepsilon) - \mu_k) \) in (5.8) gives

\[ \int_{M_\varepsilon} \varphi_{j,\varepsilon} U_k = O(\varepsilon^3) \Rightarrow \alpha_{k,j} = O(\varepsilon^2). \]

That \( \alpha_{k,k} \) is bounded follows from its definition. \( \square \)

A straightforward computation gives

\[ F_1 := \tilde{\Delta} w_{k,\varepsilon} + \mu_k(\varepsilon) w_{k,\varepsilon} = (\mu_k(\varepsilon) - \mu_k) U_k + \frac{1}{2} y^2 (\Delta \eta_k + \mu_k \eta_k) + \sum_{j=0}^{k-1} \alpha_{k,j} (\mu_k(\varepsilon) - \mu_j) \varphi_{j,\varepsilon}. \]
with the boundary conditions
\begin{equation}
\frac{\partial w_{k,\varepsilon}}{\partial n} = \begin{cases} 
0 & \text{on } B_I \cup B_{III} \\
O(\varepsilon^3) & \text{on } B_{II} 
\end{cases}
\end{equation}

Let
\[ r_{k,\varepsilon} = \frac{\partial w_{k,\varepsilon}}{\partial y}. \]
Then
\[ F_2 := \tilde{\Delta} r_{k,\varepsilon} + \mu_k(\varepsilon) r_{k,\varepsilon} \]
with the boundary conditions
\begin{equation}
\begin{cases}
  r_{k,\varepsilon} = 0 & \text{on } B_I \\
  r_{k,\varepsilon} = \varepsilon \nabla f(x) \nabla w_{k,\varepsilon}(x,\varepsilon f(x)) + O(\varepsilon^3) & \text{on } B_{II} \\
  \frac{\partial r_{k,\varepsilon}}{\partial n} = 0 & \text{on } B_{III}
\end{cases}
\end{equation}

Inductively, we assume that the Theorem 5 is true for \( j \leq k-1 \). Then by Lemma 3 and Lemma 6, we have
\begin{equation}
||F_1||_{C_\alpha} = O(B_k) \text{ and } ||F_2||_{C_\alpha} = O(\varepsilon),
\end{equation}
where
\[ B_k = \varepsilon^2 + \sqrt{\varepsilon} A_k. \]

**Lemma 7.** With the above notations, we have
\[ ||w_{k,\varepsilon}||_{L^2(M_{\varepsilon})} = O(\varepsilon^{1/2} B_k) \text{ and } ||\nabla w_{k,\varepsilon}||_{L^2(M_{\varepsilon})} = O(\varepsilon^{1/2} B_k). \]

**Proof.** Multiplying both sides of (5.11) by \( w_{k,\varepsilon} \) and integrating by parts, using (5.15), we get
\begin{equation}
||\nabla w_{k,\varepsilon}||_{L^2(M_{\varepsilon})} \leq C B_k \cdot ||w_{k,\varepsilon}||_{L^1(M_{\varepsilon})} + \int_{B_{III}} w_{k,\varepsilon} \frac{\partial w_{k,\varepsilon}}{\partial n}.
\end{equation}

By (5.17), we have
\[ \left| \int_{B_{III}} w_{k,\varepsilon} \frac{\partial w_{k,\varepsilon}}{\partial n} \right| \leq \int_{B_{III}} w_{k,\varepsilon} \frac{\partial U_k}{\partial n} \leq C \varepsilon^{5/2} (||w_{k,\varepsilon}||_{L^2(M_{\varepsilon})} + ||\nabla w_{k,\varepsilon}||_{L^2(M_{\varepsilon})}). \]

Thus we have
\[ ||\nabla w||_{L^2(M_{\varepsilon})} \leq C \varepsilon^{5/2} (||w_{k,\varepsilon}||_{L^2(M_{\varepsilon})} + ||\nabla w_{k,\varepsilon}||_{L^2(M_{\varepsilon})}). \]

By the Poincaré inequality, we have
\[ \mu_{k+1}(\varepsilon) \int_{M_{\varepsilon}} |w_{k,\varepsilon}|^2 \leq \int_{M_{\varepsilon}} |
abla w_{k,\varepsilon}|^2. \]
The lemma is proved since by our “generic” assumption, there is a gap between \( \mu_{k+1}(\varepsilon) \) and \( \mu_{k}(\varepsilon) \) that is independent of \( \varepsilon \). \qed
Lemma 8. Let $w_k$ be a solution to \( \nabla^2 w_k + \mu_k w_k = 0 \) in $B_k \times (0,1)$ with $w_k = 0$ on $\partial B_k \times (0,1)$. Then

\[
\int_{B_k} |w_k|^p \leq C_k \int_{B_k} |\nabla w_k|^2 + C_k \int_{B_k} |w_k|^{p-1} |\nabla w_k|^2.
\]

Proof. We prove the lemma using Moser iteration. By \( (5.6) \) and Theorem \( 2 \), we have

\[
\int_{B_k} |w_k|^p \leq C_k \int_{B_k} |\nabla w_k|^2 + C_k \int_{B_k} |w_k|^{p-1} |\nabla w_k|^2.
\]

By \( (5.7) \), we have

\[
\int_{B_k} |w_k|^p \leq \left( \frac{1}{\varepsilon} \int_{B_k} |w_k|^p + \frac{1}{\varepsilon} \int_{B_k} |w_k|^{p-1} \right)^{p-1} \int_{B_k} |w_k|^{p-1} |\nabla w_k|^2.
\]

It follows that

\[
\int_{B_k} |w_k|^p \leq \frac{1}{4} \int_{B_k} |w_k|^p + \varepsilon^{p-1} \int_{B_k} |w_k|^{p-1} + \varepsilon^{p+1} \int_{B_k} |w_k|^p.
\]

By the Young’s inequality, we have

\[
\varepsilon^2 |w_k|^p + \varepsilon^{p+1} |w_k|^{p-1} \leq |w_k|^p + \varepsilon^{2(p+1)}.
\]

Thus we have

\[
\int_{B_k} |w_k|^p \leq \frac{p}{4} \int_{B_k} |w_k|^p + \varepsilon^{p-1} \int_{B_k} |w_k|^{p-1} |\nabla w_k|^2 + \varepsilon^{2(p+1)} \int_{B_k} |w_k|^p.
\]

from which we have

\[
p \int_{B_k} |w_k|^p \leq C \left( \int_{B_k} |w_k|^p + \varepsilon^{2(p+1)} \right).
\]

Using the above inequality and the Sobolev inequality \( (5.3) \), we have

\[
\left( \int_{B_k} |w_k|^{p+1} \right)^{p+1} \leq C \varepsilon^{p+1} \left( \int_{B_k} |w_k|^p + \varepsilon^{2(p+1)} \right).
\]
We thus have
\[
\left( \int_{M_\varepsilon} |w_{k,\varepsilon}|^{(p+1)\frac{n+1}{n-1}} + \varepsilon^{2(p+1)\frac{n+1}{n-1}+1} \right)^{\frac{n-1}{n+1}} \leq Cp^2 \varepsilon^{-\frac{2}{p+1}} \left( \int_{M_\varepsilon} |w_{k,\varepsilon}|^{p+1} + \varepsilon^{2(p+1)+1} \right).
\]

Let
\[
a_k = 2 \left( \frac{n+1}{n-1} \right)^k
\]
for \(k \geq 0\). Let
\[
b_k = \left( \int_{M_\varepsilon} |w_{k,\varepsilon}|^{a_k} + \varepsilon^{2(a_k)+1} \right)^{1/a_k}.
\]
Then we have
\[
b_{k+1} \leq \left( C(a_k)^2 \varepsilon^{-\frac{2}{p+1}} \right)^{1/a_k} b_k.
\]
The standard iteration process shows that
\[
||w_{k,\varepsilon}||_{C^\alpha} \leq C\varepsilon^{-\frac{1}{p+1}} \sum \left[ \frac{1}{a_k} b_0 \right] \leq C\varepsilon^2.
\]

\[\square\]

\textbf{Lemma 9.} \(r_{k,\varepsilon} = O(\varepsilon^2)\).

\textit{Proof.} We could run Moser iteration again to get the estimate. However, the following proof using the maximum principle seems to be simpler. Using (5.1), (5.15), and the boundary conditions (5.14), we have
\[
||w_{k,\varepsilon}||_{C^\alpha} \leq C\varepsilon^{-\frac{1}{p+1}} \sum \left[ \frac{1}{a_k} b_0 \right] \leq C\varepsilon^2.
\]

\textbf{Lemma 10.} \(||r_{k,\varepsilon}||_{C^{2,\alpha}_\varepsilon} = O(\varepsilon^2), \text{ and } |
\hat{\nabla}^2 w_{k,\varepsilon}| = O(\varepsilon)\).

\textit{Proof.} For fixed \(y\), let \(w = w_{k,\varepsilon}\) and let \(h = w(x, yf(x))\). Then \(h\) satisfies the equation
\[
\Delta h + \mu_k(\varepsilon)h
\]
\[
= - (1 - y^2|\nabla f|^2) \frac{\partial r_{k,\varepsilon}}{\partial y} + 2y\langle \nabla f, \nabla r_{k,\varepsilon} \rangle + yr_{k,\varepsilon}\Delta f + F_1(x, yf(x)).
\]
Let \(\Omega \subset M\) such that on \(M \setminus \Omega\), \(f\) is identically equal to a positive constant \(\delta\). By the Schauder interior estimate, we have
\[
||h||_{C^{3,\alpha}(\Omega)} \leq C(\varepsilon^2 + ||r_{k,\varepsilon}||_{C^{2,\alpha}_\varepsilon}).
\]

Note that the above \(C^{2,\alpha}\)-norm is a function of \(y\), and this norm is unscaled. By Lemma 2 (5.13) and (5.14), the global Schauder estimate (5.2) gives
\[
||r_{k,\varepsilon}||_{C^{2,\alpha}_\varepsilon} \leq C(\varepsilon^2 + ||\varepsilon\nabla f \nabla h||_{C^{2,\alpha}_\varepsilon}) \leq C(\varepsilon^2 + ||h||_{C^{3,\alpha}_\varepsilon(\Omega)}).
\]
The relation between weighted and the usual Hölder norms is (up to a constant)
\[ \varepsilon^{k+\alpha} \|u\|_{C^{k,\alpha}} \leq \|u\|_{C^{k,\alpha}} \leq \varepsilon^{k+\alpha} \|u\|_{C^{k,\alpha}} + \|u\|_{C^{\alpha}}. \]

Thus we have
\[ \|r_{k,\varepsilon}\|_{C^{2,\alpha}}^{2} \leq C(\varepsilon^{2} + \varepsilon^{3+\alpha} \|h\|_{C^{3,\alpha}(\Omega)} + \|h\|_{C^{\alpha}(\Omega)}) \]
\[ \leq C(\varepsilon^{2} + \varepsilon^{3+\alpha}(\varepsilon^{2} + \|r_{k,\varepsilon}\|_{C^{2,\alpha}})) \leq C(\varepsilon^{2} + \varepsilon\|r_{k,\varepsilon}\|_{C^{2,\alpha}}). \]

Therefore \( \|r_{k,\varepsilon}\|_{C^{2,\alpha}} = O(\varepsilon^{2}) \), which also implies that
\[ \frac{\partial^{2}w_{k,\varepsilon}}{\partial y^{2}}, \frac{\partial^{2}w_{k,\varepsilon}}{\partial x_{j}\partial y} = O(\varepsilon), \]
where \((x_{1}, \ldots, x_{n})\) is any local coordinate system on \( M \). Using the global Schauder estimate on (5.17) again, we get
\[ \frac{\partial^{2}w_{k,\varepsilon}}{\partial x_{j}\partial x_{i}} = O(\varepsilon). \]

\[ \square \]

**Lemma 11.** \( |\nabla r_{k,\varepsilon}| = O(\varepsilon^{2}) \).

**Proof.** We need to prove that for any first order differential operator \( R \) on \( M \),
\[ R(r_{k,\varepsilon}) = O(\varepsilon^{2}) \]
uniformly for any \( 0 \leq y \leq \varepsilon \). By Lemma 10, this is equivalent to
\[ v = R(r_{k,\varepsilon}(x,yf(x))) = O(\varepsilon^{2}) \]
uniformly for any \( 0 \leq y \leq \varepsilon \).

We first assume that the vector field \( R \) is vertical to \( \partial M \). Then by Lemma 10, \( v = 0 \) on \( B_{I} \), and
\[ v = R(\varepsilon\nabla f(x)\nabla w_{k,\varepsilon}(x,\varepsilon f(x))) = O(\varepsilon^{2}) \] on \( B_{II} \).

On \( B_{III} \), since \( R \) is vertical to \( \partial M \), \( v = g\frac{\partial}{\partial n} \) on \( \partial M \) for some function \( g \). Thus by Lemma 10 again,
\[ v = g\frac{\partial}{\partial n}(r_{k,\varepsilon}(x,yf(x))) = O(\varepsilon^{2}). \]

By (5.13) and Lemma 10, we have
\[ \Delta R(r_{k,\varepsilon}) + \mu_{k}(\varepsilon)R(r_{k,\varepsilon}) = R(F_{2}) + [\Delta, R]r_{k,\varepsilon} = O(1). \]

Thus for \( C > 0 \) large enough, we have
\[ \Delta(R(r_{k,\varepsilon}) + Cy^{2}) > 0, \]
and by the maximum principle, \( R(r_{k,\varepsilon}) \leq O(\varepsilon^{2}) \). Like in the proof of Lemma 9 the other side of the inequality can be obtained by estimating \( R(r_{k,\varepsilon}) - Cy^{2} \).

Now we assume that \( R \) is tangential on \( \partial M \). We have similar estimates on \( B_{I} \) and \( B_{II} \) as above. On \( B_{III} \), we note that
\[ \frac{\partial}{\partial n}(R(r_{k,\varepsilon}(x,yf(x))) = \frac{\partial}{\partial n}(r_{k,\varepsilon}(x,yf(x))) + O(\varepsilon^{2}), \]
where \( \tilde{R} = [\frac{\partial}{\partial n}, R] \). Let \( \xi \) be a function on \( M \) such that on \( \partial M, \xi = 0; \partial \xi/\partial n = 1 \); and on \( M, \varepsilon \Delta \xi \) bounded; \( \xi = O(\varepsilon) \). We construct such a function \( \xi \) as follows: let \( \rho(x) \) be a cut-off function on \( \mathbb{R} \) such that \( \rho \geq 0; \rho'(0) = 0 \) and \( \rho(x) = 0 \) for \( x \geq \varepsilon \);
where $\varepsilon|\rho_1|$ and $\varepsilon^2|\rho_1''|$ are bounded. Let $\xi = -d\rho(d)$, where $d$ is the distance function to the boundary $\partial M$. Let

$$\tilde{v} = R(r_{k,\varepsilon}(x, yf(x))) - C_1 \xi \max |\nabla r_{k,\varepsilon}| + C_2 y^2$$

for large $C_1, C_2$. By choosing $C_1$ large enough, from (5.21), we have

$$\frac{\partial \tilde{v}}{\partial n} < 0$$

on $B_{III}$. Fixing $C_1$, we choose $C_2$ large enough. Then by Lemma 9, we have $\tilde{v} \leq C(\varepsilon^2 + \varepsilon \max |\nabla r_{k,\varepsilon}|)$.

Thus we have

$$R(r_{k,\varepsilon}(x, yf(x))) \leq C(\varepsilon^2 + \varepsilon \max |\nabla r_{k,\varepsilon}|).$$

Since $R$ is arbitrary, this yields

$$\max |\nabla r_{k,\varepsilon}| \leq C(\varepsilon^2 + C_1 \delta \max |\nabla r_{k,\varepsilon}|).$$

\[\square\]

**Proof of Theorem 5.** The proof is similar to that of Lemma 10. On $B_1$, $R(r_{k,\varepsilon}) = 0$; on $B_{II}$, by (5.18), we have

$$||R(r_{k,\varepsilon})||_{C^{2,\alpha}(B_{II})} \leq C \varepsilon^2 + C \varepsilon^{3+\alpha}||h||_{C^{4,\alpha}(\Omega)}.$$  

On $B_{III}$, we have two cases. If $R$ is vertical to $\partial M$, by (5.19),

$$||R(r_{k,\varepsilon})||_{C^{2,\alpha}(B_{III})} \leq C(\varepsilon^2 + \varepsilon ||\nabla r_{k,\varepsilon}||_{C^{2,\alpha}}).$$

If $R$ is tangential to $\partial M$, by (5.21),

$$\left|\frac{\partial}{\partial n} R(r_{k,\varepsilon})\right|_{C^{1,\alpha}(B_{III})} \leq C \varepsilon^2 + ||\nabla r_{k,\varepsilon}||_{C^{1,\alpha}}.$$  

By the Schauder estimate and (5.22),

$$||R(r_{k,\varepsilon})||_{C^{2,\alpha}} \leq C(\varepsilon^2 + \varepsilon^{3+\alpha}||h||_{C^{4,\alpha}(\Omega)} + ||\nabla r_{k,\varepsilon}||_{C^{2,\alpha}}).$$

By the Schauder interior estimate,

$$||h||_{C^{4,\alpha}(\Omega)} \leq C(\varepsilon^2 + ||r_{k,\varepsilon}||_{C^{3,\alpha}}) \leq C(\varepsilon^2 + \varepsilon^{-2-\alpha}||\nabla r_{k,\varepsilon}||_{C^{2,\alpha}}).$$

Combining (5.22), (5.23), (5.24), we get

$$||\nabla r_{k,\varepsilon}||_{C^{2,\alpha}} \leq C(\varepsilon^2 + \varepsilon ||\nabla r_{k,\varepsilon}||_{C^{2,\alpha}} + ||\nabla r_{k,\varepsilon}||_{C^{1,\alpha}}),$$

which implies the theorem by the interpolation inequalities.  

\[\square\]

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References

[1] B. Andrews, Gradient and oscillation estimates and their applications in geometric PDE, AMS/IP Studies in Advanced Mathematics, 5th ICCM (2010).

[2] B. Andrews and J. Clutterbuck, Proof of the fundamental gap conjecture, 2010. preprint, arXiv 1006.1686.

[3] Dominique Bakry and Michel Émery, Diffusions hypercontractives, Séminaire de probabilités, XIX, 1983/84. Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206, DOI 10.1007/BFb0075847, (to appear in print) (French). MR889476 (88j:60131)

[4] Dominique Bakry and Zhongmin Qian, Some new results on eigenvectors via dimension, diameter, and Ricci curvature, Adv. Math. 155 (2000), no. 1, 98–153, DOI 10.1006/aima.2000.1932. MR1789850 (2002g:58048)

[5] Isaac Chavel, Eigenvalues in Riemannian geometry, Pure and Applied Mathematics, vol. 115, Academic Press Inc., Orlando, FL, 1984. Including a chapter by Burton Randol; With an appendix by Jozef Dodziuk. MR768584 (86g:58140)

[6] E. B. Davies and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, J. Funct. Anal. 59 (1984), no. 2, 335–395, DOI 10.1016/0022-1236(84)90076-4. MR766493 (86e:47054)

[7] Akito Futaki and Yuji Sano, Lower Diameter Bounds for Compact Shrinking Ricci Solitons. arXiv:1007.1759v1.

[8] David Gilbarg and Neil S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[9] W. Kirsch and B. Simon, Comparison theorems for the gap of Schrödinger operators, J. Funct. Anal. 75 (1987), no. 2, 396–410.

[10] Pawel Kröger, On the spectral gap for compact manifolds, J. Differential Geom. 36 (1992), no. 2, 315–330. MR1180385 (94g:58236)

[11] P. Li and S. T. Yau, Estimates of eigenvalues of a compact Riemannian manifold, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 205–239.

[12] Jun Ling, Estimates on the lower bound of the first gap, Comm. Anal. Geom. 16 (2008), no. 3, 539–563.

[13] Jun Ling and Zhiqin Lu, Bounds of eigenvalues on Riemannian manifolds, Trends in partial differential equations, Adv. Lect. Math. (ALM), vol. 10, Int. Press, Somerville, MA, 2010, pp. 241–264.

[14] John Lott, Some geometric properties of the Bakry-Émery-Ricci tensor, Comment. Math. Helv. 78 (2003), no. 4, 865–883, DOI 10.1007/s00014-003-0775-8. MR2016700 (2004i:53044)

[15] L. Ma and B. Liu, Li-Yau type eigenvalue estimates for drifting Laplacians. preprint.

[16] , Convex eigenfunction of a drifting Laplacian operator and the fundamental gap, Pacific J. Math. 240 (2009), no. 2, 343–361.

[17] R. Schoen and S.-T. Yau, An estimate of the gap of the first two eigenvalues in the Schrödinger operator, Ann. of Math. (2) 12 (1985), no. 2, 319–333.

[18] Karen Uhlenbeck, Generic properties of eigenfunctions, Amer. J. Math. 98 (1976), no. 4, 1059–1078.

[19] Guofang Wei and Will Wylie, Comparison geometry for the Bakry-Émery Ricci tensor, J. Differential Geom. 83 (2009), no. 2, 377–405. MR2577473 (2011a:53064)

[20] Shing-Tung Yau, Nonlinear analysis in geometry, Enseign. Math. (2) 33 (1987), no. 1-2, 109–158. MR896385 (88g:58003)

[21] , An estimate of the gap of the first two eigenvalues in the Schrödinger operator, Lectures on partial differential equations, New Stud. Adv. Math., vol. 2, Int. Press, Somerville, MA, 2003, pp. 223–235. MR2055851 (2005c:35219)

[22] , Gap of the first two eigenvalues of the Schrödinger operator with nonconvex potential, Mat. Contemp. 35 (2008), 267–285. MR2584188 (2010m:53097)

[23] J. Q. Zhong and H. C. Yang, On the estimate of the first eigenvalue of a compact Riemannian manifold, Sci. Sinica Ser. A 27 (1984), no. 12, 1265–1273.
[25] Qi Huang Yu and Jia Qing Zhong, *Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator*, Trans. Amer. Math. Soc. 294 (1986), no. 1, 341–349.

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