Repeated Extraction of Scrambled Quantum Data: Sustainability of the Hayden-Preskill Type Protocols

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We introduce and study the problem of scrambler hacking, which is the procedure of quantum-information extraction from and installation on a quantum scrambler given only partial access. This problem necessarily emerges from a central topic in contemporary physics — information recovery from systems undergoing scrambling dynamics, such as the Hayden–Preskill protocol in black hole studies — because one must replace quantum data with another when extracting it due to the no-cloning theorem. For large scramblers, we supply analytical formulas for the optimal hacking fidelity, a quantitative measure of the effectiveness of scrambler hacking with limited access. In the two-user scenario where Bob attempts to hack Alice’s data, we find that the optimal fidelity converges to \( \frac{64}{9\pi^2} \approx 0.72 \) with increasing Bob’s hacking space relative to Alice’s user space. We apply our results to the black hole information problem and show that the limited hacking fidelity implies the reflectivity decay of a black hole as an information mirror, which questions the solvability of the black hole information paradox through the Hayden-Preskill type protocol.

In many-body quantum systems, due to rapid and complex interaction between subsystems, initially localized quantum information dissipates quickly and spreads throughout the whole system. This delocalization of quantum information is called quantum scrambling, and retracting scrambled quantum information is one of the most important central topics of contemporary physics [1–3]. From the perspective of the decoupling approach [4] to quantum information, perfect recovery is equivalent to implementing a quantum channel without leaking information to a particular subsystem, which is also studied in the context of catalysis of quantum randomness, quantum secret sharing and quantum masking [5–9].

One of the most important examples of recovery of quantum information from quantum scrambler is the Hayden–Preskill protocol for recovering quantum information from the Hawking radiation of old black holes [10–14]. The facts that black hole evaporates by emitting Hawking radiation which is predicted to be semi-classical thermal radiation and that no quantum information can be destroyed because of the unitarity of time evolution in quantum mechanics lead us to the famous black hole information paradox. The Hayden–Preskill protocol proposes a resolution for this paradox. The protocol is still under active research as its optimal decoding map is not completely understood and requires creative construction [15].

Although the Hayden–Preskill protocol was proposed for extraction of quantum data from black holes, it can be applied to any quantum scrambler that allows attachment and detachment of subsystems. The setting of the protocol is as follows. Suppose that Alice inputs a piece of quantum data into a part of the input register of a multipartite unitary operator (‘scrambler’) with a publicly known architecture. Bob acquires an access to a part of input/output ports of the network, and Bob attempts to extract as much data from Alice as possible.

The Hayden–Preskill protocol says that, if Bob inputs a highly entangled state and the multipartite unitary operator has strong scrambling property, then by collecting a little more than \( k \) qubits of output of the unitary operator one can successfully extract \( k \) qubits of quantum data prepared by Alice [11]. Old black holes are believed to satisfy the aforementioned conditions, thus, old black holes ‘act like information mirrors’. This is the reason why the Hayden–Preskill protocol could possibly resolve the black hole information paradox.

For the Hayden–Preskill protocol to be a true solution to the black hole information paradox, however, old black holes must function as information mirrors consistently. In other words, one should be able to recover one qubit after another without limit, not just a few initial qubits. However, the protocol seemingly consumes entanglement in the process, thus one can...
naturally question if this protocol can be repeated for the extraction of the next qubits.

Moreover, by the no-cloning theorem [16], Bob cannot simply copy out Alice’s quantum data, but has to replace it with another. For the case of many-body systems such as black holes, the installed quantum data will be fed into the next round of scrambling as a part of input. Therefore, one might wonder if installing a certain piece of quantum data into quantum scramblers can enhance the performance of the data extraction for the next round, so that the sustainability of quantum data extraction from quantum scramblers is better.

Significance of quantum data installation in the studies on the Hayden–Preskill type protocols has been generally overlooked. For example, in the proposal of efficient decoding scheme for the Hayden–Preskill protocol by Yoshida and Kitaev [15], the remnant entangle state after decoding is only referred to as “arbitrary state” (in Figure 1 or Ref. [15]) and no attention was given to its role in data extraction. In this work, we focus on this unnoticed, yet important problem of installing quantum data into the target system and how well this installation can be implemented at the same time with data extraction.

**Hayden–Preskill protocol.**— The original Hayden–Preskill protocol can be stated as follows. An old black hole (denoted as $B$) is believed to be maximally entangled with all the Hawking radiation it has emitted (denoted as $B'$) so far. Now let $A$ be an infalling object. Under the assumption that black holes are undergoing rapid scrambling interaction [1], we can say that systems $AB$ undergo a bipartite unitary operator $U : AB \rightarrow KL$. After this interaction, a part of output systems, $L$, whose size is slightly larger than $A$ is emitted from the black hole. By applying a suitable recovery map $R$ on $B'L$, one can recover the quantum state of $A$ almost perfectly [11]. One might interpret that quantum information is not destroyed in the process, so the black hole information paradox is resolved, at least from the perspective of the outer observer.

However, after this data extraction, the remaining black hole interior $K$ can interact with another infalling system $A_1$ and subsequently emit Hawking radiation $L_1$. What happens to the quantum information of $A_1$? Although systems $BB'$ were initially maximally entangled, the quantum correlation between the black hole and the data extractor may be degraded over time as the data extraction continues. The sheer fact that the size of $L$ must be larger than that of $A$ for accurate data recovery alone suggests that entanglement is consumed in the process. Is the Hayden–Preskill protocol sustainable after many rounds of data extraction?

The discussion so far can be applied to any quantum scrambler $U$ that behaves similarly. In the Hayden–Preskill setting, the initial maximal entanglement between a black hole and the systems outside is built through the natural Hawking radiation, but for general quantum scramblers, one may try to optimize data extraction by establishing a certain entangled state instead of the maximally entangled state. Also, as discussed before, one must install a certain quantum state in the scrambler after data extraction. Would it be possible?

**Quantum-scrambler hacking.**— Let us consider a general scrambler instead of a black hole. The quantum scrambler, accessed by only two users, Alice and Bob, is described as a $d_Ad_B$-dimensional unitary operator $U$ on systems $AB$. After the interaction, the joint system is decomposed into $K$ and $L$, possessed by Alice and Bob, respectively. Note that it implies that $d_A d_B = d_K d_L$.

Alice inputs her secret quantum state in $A$, and Bob tries to extract as much quantum information stored in $A$ as possible and replace it with an arbitrary quantum state of his choice. This is done by feeding a ‘probe’ quantum state that is going to interact with $A$ and applying a recovery map to the output state to simulate the SWAP operation [17].

However, if this task can be done with error $\epsilon$ (measured by the average infidelity of pure state inputs), then the unitary operator $U$ itself should be close to the SWAP operator (up to local operations) with error $\epsilon$ and vice versa. It is thus impossible to substitute quantum data through a non-SWAP operator (See Appendix A).

The next optimal strategy for Bob is to build as much correlation as possible with the target system $A$ while extracting quantum information out of it (See Fig. 1). This is because building correlations allows the extraction of quantum information from the next (undecided) computation step provided Bob has access to a part of the current computation output. In general, Bob possesses a reference system $B'$ ($d_B = d_{B'}$) and prepares an entangled input probe state $|\phi\rangle_{BB'}$. Bob’s goals are, therefore, to extract the input information stored in $A$, and install an output
maximally entangled state in $AB$. The latter can be interpreted as preparation of extraction of quantum data of future quantum computation on $A$ [18, 19], because having a maximally entangled state with the target system yields the maximum side information.

Our protocol is set in the following situation.

(i) A bipartite unitary operator $U : AB \to KL$ is randomly chosen, and Bob is informed about it.

(ii) Alice prepares a quantum state in $A$ that Bob wants to extract. Bob prepares a quantum state $|\phi\rangle_{BB'}$ (‘probe state’) with an ancillary system $B'$. For a given $U$, which is equivalent to inverting a possibly non-unitary operator $U$ for $\chi$, the output system decomposes into $A''$, where $A''$ should be prepared to achieve the largest hacking fidelity. A natural candidate would be $\chi = I_B / \sqrt{d_B}$, which corresponds to a maximally entangled probe state. Equation (3) then immediately yields the fidelity $p_{\text{hack}}$ of $U$.

Since the fidelity never decreases under a partial trace, $p_{\text{hack}}$ serves as a lower bound for both fidelities of the extracted quantum data (systems $A'B'$) and implemented entangled state (systems $AB$). We can parametrize any bipartite entangled pure state $|\phi\rangle_{BB'}$ with an operator $\chi$ acting on system $B'$ such that $|\phi\rangle_{BB'} = \sum_i |i\rangle_B \otimes |\chi i\rangle_{B'}$ with $||\chi||_2 = 1$. Here $||X||_p := (\text{Tr} |X|^p)^{1/p}$, where $|X| := \sqrt{X^\dagger X}$, is the Schatten-$p$ norm. With these, Eq. (1) is simplified to

$$p_{\text{hack}}^{(R,\chi)} = \left| \text{Tr}[R(I_L \otimes \chi)U^{o}]\right|^2 / (d_A^2 d_K).$$

Here, the (generally non-unitary) map $U^o : AA' \to BB'$ is represented by a matrix, understood as a tensor, formed by cyclically rotating the indices of $U$ clockwise by one position—$U^{oij}_{kl} := U^{ki}_{lj}$. Each pair $(R,\chi)$ constitutes a hacking strategy for Bob. For a given $\chi$, which is identical to fixing Bob’s probe state, the optimal unitary recovery $R$ is the one that gives the polar decomposition $(I_L \otimes \chi)U^o = R^\dagger (I_L \otimes \chi)U^o$. This leads to

$$p_{\text{hack}} = \max_R p_{\text{hack}}^{(R,\chi)} = ||(I_L \otimes \chi)U^o||_2^2 / (d_A^2 d_K),$$

which is equivalent to inverting a possibly non-unitary operator with a unitary one [28].

This leaves us the problem of finding an optimal $\chi$ that achieves the largest hacking fidelity. A natural candidate would be $\chi = I_B / \sqrt{d_B}$, which corresponds to a maximally entangled probe state. Equation (3) then immediately yields the fidelity $p_{\text{hack}}^{\text{MB}} = ||U^o||_2^2 / (d_A^2 d_B d_K)$, which is a unitarity measure of

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**Figure 1:** Schematic diagram of scrambler hacking of a unitary process $U$ (quantum scrambler or quantum computer) in the two-user scenario. Ideally, Bob, the hacker, would desire to extract Alice’s information and plant a part of a maximally entangled state for the next quantum computation.
\[ \|U^o\|_1 \] is the maximal inner product of \( U^o \) and an isometry. With this strategy, perfect hacking \( (p_{\text{hack}}^{\text{ME}} = p_{\text{hack}}^{\text{PG}} = p_{\text{hack}}^\text{opt} = 1) \) is only possible when \( U^o \) is proportional to an isometry—\( U^o U^o = (d_B/d_B^o) I_{AA'} \). Unitary operators \( U \) with such a property are known to be \textit{dual-unitary} in the studies of quantum lattice models [29–31], and we give a new operational meaning to them as \textit{completely hackable} unitary operators. On the contrary, \( p_{\text{hack}}^{\text{ME}} \) reaches its minimum \( 1/d_A^2 \) when \( U^o \) is rank-1, which happens if \( U = I_A \otimes I_B \), for instance. This value serves as a lower bound of the optimal hacking fidelity and shows that the degree of unitarity of \( U^o \) directly affects the performance of scrambler hacking. (See FIG. 2.)

Physical intuition may lead to the putatively obvious conclusion that a maximally entangled probe state is optimal for scrambler hacking. As it turns out, this is, however, not the case in general. For example, for a qubit-quantum controlled unitary operator given as \( U_c = I_A \otimes |0\rangle\langle 0|_B + X_A \otimes (I_B - |0\rangle\langle 0|_B) \), with the Pauli \( X \) operator acting on \( A \), \( p_{\text{hack}}^{\text{ME}} \) is smaller than the \( p_{\text{hack}}^{\text{opt}} \) with \( \chi = (|0\rangle\langle 0|_B + |1\rangle\langle 1|_B) \sqrt{2} \).

To maximize \( p_{\text{hack}} \) in Eq. (2), recalling that \( \|\chi\|_2 = 1 \), we may invoke the Cauchy–Schwarz inequality, \( |\text{Tr}B[\chi^\dagger \text{Tr}B[U^o R]]| \leq \|\text{Tr}B[U^o R]\|_2 \). This bound is saturated when \( \chi = \text{Tr}B[U^o R]/\|\text{Tr}B[U^o R]\|_2 \). Hence, the true optimal hacking fidelity reads

\[ p_{\text{hack}}^\text{opt} = \max_R \|\text{Tr}B[U^o R]\|_2^2/d_A^2, \]

where the maximization is over all \( d_A^2 \times d_B^2 \) cosymmetry operator \( R \). By exploiting the polar decomposition once more, a natural choice of \( R \) is \( U^o R = |U^o| \) and yields the fidelity \( p_{\text{hack}}^\text{PG} = \|\text{Tr}B[U^o R]\|_2^2/d_A^2 \). As we shall soon demonstrate that this hacking strategy is near-optimal, we will call this the “pretty good” (PG) strategy. As this strategy also outperforms that using a maximally entangled probe state, the following inequalities hold:

\[ p_{\text{hack}}^{\text{ME}} \leq p_{\text{hack}}^{\text{PG}} \leq p_{\text{hack}}^\text{opt}. \]
increases, specifically—

peaked iteration of the corresponding extremal equa-

rical methods to acquire $\chi$ has no known analytical form, we propose two nu-

totically optimal for quantum-scrambler hacking—

that a maximally entangled probe state is asymp-

dom matrix theory [35–37], in the scenario where

Using properties of this measure and results from ran-

Interestingly, one can calculate an analytical form

for sufficiently large dimensions; more

specifically $d_B d_K \gg 1$. To do this, we observe

that a maximally entangled probe state is asympto-

tically optimal for quantum-scrambler hacking—

$\chi_{\text{opt}} \rightarrow I_B/\sqrt{d_B}$. This follows from the fact that the

restricted state of any high-dimensional pure state ap-

proaches the maximally mixed state [34], which then

implies that $p_{\text{hack}}^{\text{opt}} \rightarrow \|U^\rangle\|_2^2/(d_A^2 d_B d_K)$. We shall

consider $U$ as a random unitary operator distributed

according to the Haar measure of the unitary group.

Using properties of this measure and results from ran-

dom matrix theory [35–37], in the scenario where

only Alice and Bob are influenced by the action of a

generic quantum scrambler, we have the asymptotic

Haar-averaged formula for $\kappa \equiv \sqrt{d_B d_L}/(d_A d_K) =

d_B/d_K \geq 1$,

$$p_{\text{hack}}^{\text{opt}} \approx I_2^2 + (1 - I_1^2)/(d_A d_K),$$

$$I_\kappa = 2F_1 \left(2^{-1}, -2^{-1}; 2; \kappa^{-2}\right). \quad (6)$$

with $2F_1(\cdot, \cdot; \cdot)$ being the hypergeometric func-

tion [21]. In the specific circumstance where $U$ is the

interaction unitary operator for the Hayden–Preskill

scenario, we have $\kappa = 1$, so that $I_1 = 8/(3\pi)$ or

$p_{\text{hack}}^{\text{opt}} \approx 0.72$. If $\kappa < 1$, we instead have $p_{\text{hack}}^{\text{opt}} \approx

\kappa^{-2}\left(2^{-1}/d_A d_K\right)/(1 - I_1^2)$. The results indi-
cate that efforts in using optimal probe states for a
given scrambler $U$ do pay off with a much higher hack-
ing fidelity compared to all other random choices of Bob’s probe state.

The quantity $I_\kappa$ is an important indicator of the limiting perfor-

mance for hacking large quantum scramblers of a

dimension ratio $\kappa$. It also suggests that in the

two-user scenario, a larger Hilbert space of Bob rela-
tive to Alice’s results in a larger $p_{\text{hack}}^{\text{opt}}$. A single-

qubit ancilla ($\kappa = 2$) is enough to boost $p_{\text{hack}}^{\text{opt}}$ all

the way to $\approx 0.936$. We also find that the PG strategy

$\tilde{\chi} := \chi = \text{Tr}_L[U^\rangle\|/\|\text{Tr}_L[U^\rangle\|]$ is almost optimal

for any $d_A$ and $d_B$. In particular, when $d_A = d_B$, we

precisely get $\tilde{\chi} = \chi_{\text{opt}} = I_B/\sqrt{d_B}$ (see Ap-

pendix E).

Scrambler hacking and entanglement recycling.—The operator $U^\rangle$

appears in various

scenarios. In some, the input and output sys-

tems of the unitary operator $U$ need not match. Let $U : AB \to KL$ be a map of dimension $D = d_A d_B = d_K d_L$. By using a probe state ($\chi$) and recovery map $R$ on $LB'$ with matching dimensions, we get the modified hacking fidelity $p_{\text{hack}}^{(R,\chi)} = \|\text{Tr}[R(I_L \otimes \chi) U^\rangle]\|_2^2/(d_A^2 d_B d_K)$. One re-

markable case is where Bob receives continual emission of data packets from the target system.

An example of such a situation is data extraction

from the evaporation of an old black hole, where the

probe state is maximally entangled between the inner

degrees of freedom of the black hole and all Hawking

radiation emitted from the back hole up to that point.

The black hole degrees of freedom is typically much

larger than those of matter falling into it momentarily.

Let the former be $D_B = d_B = d_K$ and the latter be,

say, qudith: $d_M = d_A = d_L$. If Bob collects an ad-

ditional $d_M$-dimensional Hawking radiation, the res-

ulting optimal hacking fidelity is $p_{\text{hack}}^{BH} = \|U^\rangle\|_2^2/D^2$.

We remark that the dimension of the black hole

interior state remains the same since a qudit enters

the black hole and another exits it. Depending on the

assumptions made on the dynamics of black holes

(see [38] for the recent discussion on the effect of sym-

metry for information recovery), there may be an

Figure 3: Averaged quantum-scrambler hacking perfor-

mance (over 20 randomly-generated Haar unitary scram-

blers $U$) featuring the optimal strategy (Opt) via (4),

the PG strategy with $\tilde{\chi}$, and a random one (Rand) using

an arbitrarily-chosen probe state. Without loss of gen-

erality, we show graphs only for $\kappa \geq 1$. When $\kappa = 1$

(corresponding to the Hayden–Preskill scenario), PG is

almost the same as Opt in hacking performance. As $\kappa$

increases, $p_{\text{hack}} \to I_{\kappa > 1} \to 1$. All theoretical dashed

curves are computed with (6).
estimated value $1 - \epsilon_{\text{BH}}$ of $p_{\text{BH}}^{\text{hack}}$. For example, for Haar random $U$, $p_{\text{BH}}^{\text{hack}}$ tends to $(8/3\pi)^2 \approx 0.72$ for large $D$ [21]. This also serves as a lower bound for the fidelity between the posterior probe state and a maximally entangled state. If one uses a probe state whose maximal fidelity with a maximally entangled state is $f$ for the information extraction of the next qudit falling into the black hole, where optimal hacking fidelity is typically $p_{\text{BH}}^{\text{hack}}$ for large $D$, then the optimal hacking fidelity approaches to the product $fp_{\text{BH}}^{\text{hack}}$. However, for a given hacking fidelity $p_{\text{hack}}$ and the fidelities of the extracted quantum data ($f_{\text{ext}}$), and between the posterior probe state and a maximally entangled state ($f_{\text{post}}$), the following trade-off relation exists [21]:

$$f_{\text{ext}} + f_{\text{post}} \leq 1 + p_{\text{hack}}. \quad (7)$$

So one should choose between accurate data extraction and good entanglement recycling for imperfect hacking ($p_{\text{hack}} < 1$); giving up the former means inaccurate data extraction for the current round of hacking, and giving up the latter leads to a worse fidelity in the next round. We remark that the same argument can be applied to any large and generic quantum scrambler, not only to black holes, hence the same issue of entanglement recycling happens there, too. It suggests that if the target quantum scrambler does not allow perfect scrambler hacking, then the quality of the extracted data must be degraded over many rounds extraction.

**Discussion.**—We proposed a scrambler hacking task, which entails the extraction and replacement of quantum data through limited interaction with a quantum scrambler, and analyzed its performance in terms of the hacking fidelity. In finding good hacking strategies and calculating the optimal hacking fidelity for generic multipartite unitary scramblers, we give explicit operational meaning to $\pi/2$ tensor rotations of unitary operators. Moreover, we proved that finding an optimal decoder for this stronger task is equivalent to that for Hayden–Preskill-type protocols.

While Bob’s hacking fidelity on Alice saturates at a nonzero value in the two-network-user scenario, we find that with multiple users, naive attempts to hack any single user would generally result in exceedingly-low hacking fidelity. To improve the hacking success, it is necessary to perform program modifications to $U$, akin to hackers introducing malware to control classical computers. For quantum networks, quantum circuits would constitute such a program, but since an arbitrary quantum program cannot be encoded into a state owing to the no-programming theorem [39], Bob would need to supplement his quantum resources with additional classical attacks to improve the hacking fidelity.

As an interesting application, we considered an information-reflection model for black holes, and surveyed the sustainability of black hole mirroring. From our trade-off relation in (7) and analysis of black hole hacking, we conclude that the black hole in this model indeed functions as a mirror [11], but its “reflectivity” may be gradually degraded over time (See Ref. [40, 41] for different notions of reflectivity of quantum black holes). One could collect more Hawking radiation to increase the hacking fidelity, but that necessitates an entanglement reduction [11], thereby leading to yet another type of reflectivity degradation. This questions whether quantum information gets destroyed when it falls into an ‘older’ black hole that has already reflected a significant amount of quantum information that fell into it.

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### A Implausibility of arbitrary quantum-state installation

For any $d$-dimensional quantum channel $\Lambda$, the infidelity $1 - F_{\text{ent}}(\Lambda) = 1 - \langle \psi | (I \otimes \Lambda)(|\psi\rangle\langle\psi|) |\psi\rangle$ for maximally entangled input state $|\psi\rangle = \sum_{i=1}^{d} |ii\rangle$ and the average infidelity over Haar random pure input state $1 - F_{\text{pure}}(\Lambda) = 1 - \int d\phi \langle \phi | \Lambda(|\phi\rangle\langle\phi|) |\phi\rangle$ have the following linear dependence [20],

$$1 - F_{\text{ent}}(\Lambda) = \frac{d}{d+1} (1 - F_{\text{pure}}(\Lambda)),$$

we will use the infidelity $1 - F_{\text{ent}}(\Lambda)$ instead and have the equivalent result without losing generality.

Suppose, with the unitary operators $U$, $P$ and $R$, that the SWAP operator can be approximated with error $\epsilon$ according to

$$|0\rangle_B U P R |s\rangle_B \approx \epsilon |s\rangle_B.$$

Here, $\approx_\epsilon$ means that the two circuits are close to each other with error $\epsilon$ in the infidelity for maximally entangled input states. It means that $\Sigma \approx_\epsilon \Lambda$ is equivalent to $F(J_\Sigma, J_\Lambda) \geq 1 - \epsilon$ where $J_\mathcal{N}$ is the normalized Choi matrix for quantum channel $\mathcal{N}$. From the Uhlmann theorem [42], it follows that there exists a pure state $|s\rangle_B$ such that

$$|0\rangle_B U P R |0\rangle_B \approx_\epsilon \Sigma |0\rangle_B.$$

Since the fidelity never decreases under partial trace, it follows that

$$|0\rangle_B U |0\rangle_B \approx_\epsilon \Sigma |0\rangle_B,$$

where

$$C \Phi B = |0\rangle_B,$$

and

$$C \Psi = |s\rangle_B.$$

From the cyclicity of the fidelity for the maximally entangled input state, it follows that

$$A U B \approx_\epsilon A \Phi B.$$
It can be interpreted that unless the target scrambler $U$ itself is already close to a swapping operator followed by local operations, it is impossible to nearly perfectly substitute the quantum information out of the scrambler. Conversely, if $U$ is close to the $\text{SWAP}$ operator with error $\epsilon$ (if the dimensions of $A$ and $B$ do not match, then it can be $\text{SWAP} \oplus I$), then by choosing $P$ and $R$ also as $\text{SWAP}$ operators, one can achieve data substitution with error $\epsilon$.

B Rotation of matrix and fidelity bounds

We first show how the fidelity expression

$$p^{(R, \chi)}_{\text{hack}} := \frac{1}{\sqrt{d_A}} \langle \psi |_{AB} \langle \psi |_{A'B'} U_{BB'} R_{BB'} U_{AB} | \psi \rangle_{AA'} | \phi \rangle_{BB'} |^2,$$  \hspace{1cm} (15)

is simplified with the rotated matrix $U^\circ$. First, note that $p_{\text{hack}}$ is the fidelity between the pure states

$$\frac{1}{\sqrt{d_A}} \begin{array}{ccc}
A' \\
B'
\end{array} \begin{array}{c}
U \\
R \\
\chi
\end{array} \begin{array}{ccc}
A \\
B
\end{array}$$

and

$$\frac{1}{\sqrt{d_A}} \begin{array}{ccc}
A' \\
B'
\end{array}$$

Here, $\begin{array}{c} \chi \\
\sum_i |ii\rangle \end{array}$ represents an unnormalized maximally entangled state with the appropriate Schmidt number for the system it is defined on. Therefore, it can be expressed with a tensor network diagram.

$$p^{(R, \chi)}_{\text{hack}} = \frac{1}{d_A^3} U R \chi,$$  \hspace{1cm} (17)

with the equivalence of $|\phi\rangle_{BB'}$ and $\chi$:

$$|\phi\rangle_{BB'} = \chi.$$  \hspace{1cm} (18)

We remark that the time flows from left to right in the diagram, as opposed to the matrix multiplication order. The definition of $U^\circ$ can be expressed in a circuit diagram as follows:

$$U^\circ = U.$$  \hspace{1cm} (19)

By plugging this diagram into Eq. (17), we get

$$p^{(R, \chi)}_{\text{hack}} = \frac{1}{d_A^3} U^\circ R \chi.$$  \hspace{1cm} (20)
This is equivalent to the expression

\[
p_{\text{hack}}^{(R, \chi)} = \frac{1}{d_A^2} \left| \text{Tr} \left[ R(I_B \otimes \chi) U^o \right] \right|^2.
\]  

(21)

Since \( \sqrt{d_B} \| \text{Tr}_B \left[ U^o \right] \|_2 \geq \| \text{Tr}_B \left[ U^o \right] \|_1 = \| U^o \|_1 \), \( p_{\text{hack}}^{\text{PG}} = \| \text{Tr}_B \left[ U^o \right] \|_2 / d_A^2 \) is higher than \( p_{\text{ME}} = \| U^o \|_2^2 / (d_A^3 d_B) \). Also, since the PG strategy is a particular strategy, the fidelity of it is not larger than that of the optimal strategy, so we have \( p_{\text{hack}}^{\text{PG}} \leq p_{\text{hack}}^{\text{opt}} \). In summary, we have

\[
p_{\text{ME}} \leq p_{\text{hack}}^{\text{PG}} \leq p_{\text{hack}}^{\text{opt}}.
\]  

(22)

When \( d_A = d_B = d \), \( p_{\text{hack}}^{\text{opt}} \) is the maximal fidelity between a maximally entangled state with the Schmidt rank \( d^2 \) and a pure state of the form \( \Omega_\chi := d(U_{AB} \otimes \chi_{B'}) |\psi\rangle \langle \psi|_{A'BB'} (U_{AB} \otimes \chi_{B'})^\dagger \) with \( \| \chi \|_2 = 1 \). Let \( \chi_M \) be a \( \chi \) that achieves the maximum. Since the partial trace never decreases the fidelity, by tracing out systems other than \( B' \), we get \( F(I_{B'} / d, \| \chi_M \|^2) \geq p_{\text{hack}}^{\text{opt}} \). Let the recovery map that achieves the optimal fidelity be \( R_{\text{opt}} \) and let \( \Theta_V := V_{BB'} |\psi\rangle \langle \psi|_{A'BB'} V_{BB'}^\dagger \) for any bipartite unitary operator \( V_{BB'} \), so that \( p_{\text{hack}}^{\text{opt}} = F(\Omega_{\chi_M}, \Theta_{R_{\text{opt}}}) \). Since there is a freedom of local unitary operation to the choice of \( R_{\text{opt}} \) and \( \chi_M \), without loss of generality, we can assume that \( \chi_M \) is positive semi-definite so that \( \text{Tr} \chi_M = \text{Tr} \chi_M \).

From the following relation for arbitrary pure quantum states \(|\eta_1\rangle \) and \(|\eta_2\rangle\),

\[
\frac{1}{2} \| \eta_1 \langle \eta_1 | - | \eta_2 \rangle \|_2 = \sqrt{1 - | \langle \eta_1 | \eta_2 \rangle |^2},
\]  

(23)

we have \( \sqrt{1 - p_{\text{ME}}^{\text{opt}}} = \min_W \| \Theta_W - \Omega_{\chi_M} \|_1 / 2 \), where \( \Omega_{\chi_M} := \Omega_{I_{B'} / \sqrt{d} \chi_M} \). Therefore \( \sqrt{1 - p_{\text{ME}}^{\text{opt}}} \leq \| \Theta_{R_{\text{opt}}} - \Omega_{\chi_M} \|_1 / 2 \). Because of the triangular inequality, we have \( \| \Theta_{R_{\text{opt}}} - \Omega_{\chi_M} \|_1 \leq \| \Theta_{R_{\text{opt}}} - \Omega_{\chi_M} \|_1 + \| \Omega_{\chi_M} - \Omega_{\chi_M} \|_1 \). By Eq. (23), we have \( \| \Theta_{R_{\text{opt}}} - \Omega_{\chi_M} \|_1 / 2 = \sqrt{1 - p_{\text{ME}}^{\text{opt}}} \) and \( \| \Omega_{\chi_M} - \Omega_{\chi_M} \|_1 / 2 = \sqrt{1 - d^{-1} (\text{Tr} \chi_M)^2} = \sqrt{1 - F(I_{B'} / d, \| \chi_M \|^2) / 2} \leq \sqrt{1 - p_{\text{hack}}^{\text{opt}}} \). As a result, we have \( \sqrt{1 - p_{\text{ME}}^{\text{opt}}} \leq 2 \sqrt{1 - p_{\text{ME}}^{\text{opt}}} \) thus \( 1 - p_{\text{ME}}^{\text{opt}} \leq 4(1 - p_{\text{hack}}^{\text{opt}}) \).

The trade-off relation between the data extraction fidelity \( f_{\text{ext}} \) and the posterior probe state fidelity \( f_{\text{prob}} \)

\[
f_{\text{ext}} + f_{\text{prob}} \leq 1 + p_{\text{hack}}^{\text{opt}},
\]  

(24)

directly follows from the following result.

**Theorem 1.** For arbitrary bipartite quantum state \( \rho_{AB} \) and pure states \(|\psi\rangle_A \) and \(|\phi\rangle_B \), let \( F_A := \langle \psi | \rho_A | \psi \rangle \), \( F_B := \langle \phi | \rho_B | \phi \rangle \) and \( F_{AB} := \langle \psi |_A \langle \phi |_B \rho_{AB} | \psi \rangle_A | \phi \rangle_B \). Then the following inequality holds

\[
F_A + F_B \leq 1 + F_{AB}.
\]  

(25)

**Proof.** It is enough to realize that \( 1 - F_A - F_B + F_{AB} \) equals to \( \text{Tr} \left[ \rho_{AB} (\psi_A^\dagger \otimes \phi_B^\dagger) \right] \) which is always non-negative. Here, \( \psi_A^\dagger := 1_A - \psi_A \) is the projector onto the kernel of \( \psi_A \) and similarly for \( \phi_B^\dagger \). \( \square \)

Now, consider the black hole radiation problem of **Hacking as entanglement recycling** section. When the probe state is a general mixed bipartite state \( \Pi_{BB'} = \sum_{i} \rho_i \left| \phi_i \right\rangle \left\langle \phi_i \right|_{BB'} \) with \( |\phi_i\rangle_{BB'} = \sum_k (I_B \otimes \chi_i) |kk\rangle_{BB'} \), the hacking fidelity is given as

\[
p_{\text{hack}}^{(R, \Pi)} = \frac{1}{\sum_{i} \rho_i \left| \text{Tr} \left[ R(I_B \otimes \chi_i) U^o \right] \right|^2 / d_M^3 D_B^2}.
\]  

(26)

If the dimension of the Hilbert space of black hole state is large enough, then the PG strategy becomes nearly optimal thus, \( p_{\text{hack}}^{(R, \Pi)} \) reduces to \( \sum_{i} \rho_i \left| \text{Tr} \left[ \chi_i \text{Tr}_B \left[ U^o \right] \right] \right|^2 / d_M^3 \). Moreover, as \( D \to \infty \), \( \text{Tr}_B \left[ U^o \right] \) converges to
\[\|U^o\|_1 P_B/D_B(\text{See Sec. F}), \text{ so we have}\]

\[
\max_R p_{\text{hack}}^{(R, B)} \approx \sum_i p_i \frac{\|\text{Tr} \chi_i\|^2}{D_B D^2} \approx \sum_i p_i \frac{\|\text{Tr} \chi_i\|^2}{D_B} \approx p_{\text{hack}}^{\text{opt}} f'_{\text{prob}},
\]

where \(f'_{\text{prob}} = \sum_i p_i \|\text{Tr} \chi_i\|^2/D_B\) is the fidelity between \(\Pi_{BB'}\) and a maximally entangled state. Therefore the hacking fidelity is asymptotically the product of the optimal hacking fidelity and \(f'_{\text{prob}}\).

### C Duality with Hayden–Preskill protocols

Surprisingly, the seemingly harder problem of finding an optimal scrambler hacking strategy by Bob on Alice is equivalent to that of a Hayden–Preskill-type protocol of Alice on Bob. In this setting we assume that, instead of \(U\), its (computational-basis) transpose \((U^\top)^{\otimes 2}_o = U^{kl}\) is applied to systems \(AB\). Now, Alice wants to extract information from Bob’s system \(B\). Similar to scrambler hacking, to model such information extraction, we assume that a maximally entangled state \(|\psi\rangle_{BB'}\) is fed into \(U^\top\). Alice also chooses a maximally entangled state \(|\psi\rangle_{AA'}\) as a probe state. Systems \(AB\) interact with \(U^\top\) and Alice applies a \(d_B^2\)-dimensional unitary operator \(W\) on \(AA'\). Alice’s goal is to prepare a maximally entangled state on systems \(AB'\). The optimal fidelity between the actual and ideal states is \(p_{\text{HP}}^{\text{opt}} = \max_W \langle\psi_{AB}\rangle_{AB} \text{Tr}_{AB}[W \circ U(|\psi\rangle_{A'AB'B}\|^{\otimes 2}_{AB})] |\psi\rangle_{AB'}\rangle.\) Here, \(W(\rho) := W_{AA'} \rho W_{AA'}^\dagger \equiv W \rho W^\dagger\) and \(U(\rho) := U_{AB}\rho U_{AB}^\dagger = U \rho U^\top\). This expression can also be simplified in terms of the SWAP operator \(F\) to

\[
p_{\text{HP}}^{\text{opt}} = \max_W \|\text{Tr}_B[U^o W^\top F]\|_2^2/(d_A d_B^2).\]

It follows that the optimal hacking strategy of

\[
p_{\text{hack}}^{\text{opt}} = \max_R \|\text{Tr}_B[U^o R]\|_2^2/d_A^2,
\]

and the optimal strategy to \(p_{\text{HP}}\) are related by \(R = W^\top F\). Therefore, finding an optimal strategy for scrambler hacking is formally equivalent to finding an optimal strategy for the Hayden–Preskill protocol, in the sense that if one problem is solvable for an arbitrary \(U\), then so is the other. It follows that \(p_{\text{HP}}^{\text{opt}} = p_{\text{hack}}^{\text{opt}}/\kappa^2\) and \(p_{\text{HP}}^{\text{opt}} < 1\) when \(d_B > d_A\).

### D Numerical maximization of \(p_{\text{hack}}^{\text{opt}}\)

Given a scrambler described by \(U\), it is possible to derive an iterative numerical scheme to obtain the optimal probe state \((\chi_{\text{opt}})\) that achieves the optimal hacking fidelity \(p_{\text{hack}}^{\text{opt}}\). Rather than directly solving the numerical problem in (5) of the main text, we can instead start with \(f_X = \|\langle I_L \otimes \chi\rangle U^o\|_1\), which is the objective function involving the square root of the rightmost side in (4), and perform a variation with respect to \(\chi\). Furthermore, the constraint \(\|\chi\|_2 = 1\) invites the following parametrization \(\chi = Z/\|Z\|_2\), such that

\[
\delta \chi = \frac{\delta Z}{\|Z\|_2} - \frac{Z}{2\|Z\|_2^2} \text{Tr}[\delta ZZ^\dagger + Z \delta Z^\dagger].
\]

Upon denoting \(M = (I_L \otimes \chi) U^o\), we consequently have

\[
\delta f_X = \frac{1}{2} \text{Tr} \left[\text{Tr}_B[|M^\top|^{-1} M U^o |\delta ZZ^\dagger + Z \delta Z^\dagger|/\|Z\|_2^2] + \text{c.c.}\right]
\]

\[
= \frac{1}{2} \text{Tr} [M^\top |\delta ZZ^\dagger + Z \delta Z^\dagger|/\|Z\|_2^2],
\]

\(\text{(31)}\)
which leads to the operator gradient
\[
\frac{\delta f_{\chi}}{\delta Z} = \frac{1}{2\|Z\|_2} \left( \text{Tr}_B[|M\rangle\langle M|^{-1}MU^o] - \text{Tr}_B[|M\rangle\langle M| |Z\rangle\langle Z|] \right)
\]  
(32)

with respect to $Z^\dagger$. Setting it to zero would then give the extremal equation
\[
\chi = \frac{\text{Tr}_B[|M\rangle\langle M|^{-1}MU^o]}{\|\text{Tr}_B[|M\rangle\langle M|^{-1}MU^o]\|_2},
\]  
(33)

which may alternatively be gotten from reasoning with the Cauchy-Schwarz inequality. As $\|(I_L \otimes \chi)U^o\|_1$ is concave in $\rho_{B'} = \chi^\dagger\chi$, one can generally expect a convex solution set of $\rho_{B'}$'s that solve (33), all of which give the unique maximal fidelity $p_{\text{hack}}^{\text{opt}}$.

In other words, $p_{\text{hack}}^{\text{opt}}$ is achieved when a solution $\chi = \chi^{\text{opt}}$ for Eq. (33) is obtained. While there are no known closed-form expressions for this solution, we can nevertheless find explicit analytical forms for certain special cases. The most immediate one happens to be the limiting case $d_B \to \infty$, whence we have $\chi_{\text{opt}} \to I_B/\sqrt{d_B}$ since in this limit, $\text{Tr}_B O \to I_B \text{Tr}_O/d_B$ for any bipartite operator $O$ of systems $BB'$. For finite $d_B$, we may still have an estimate for $\chi_{\text{opt}} \approx \tilde{\chi}$. A straightforward way to do this is to simply iterate the extremal equation (33) once by substituting $I_B/\sqrt{d_B}$ for $\chi$ on the right-hand side. This gives us $\tilde{\chi} = \text{Tr}_B[U^o]/\|\text{Tr}_B[U^o]\|_2$, which is in practice very close to $\chi_{\text{opt}}$.

In practice, iterating Eq. (33) usually results in good convergence to $\chi_{\text{opt}}$. For the pedantic, we may additionally adopt the steepest-ascent methodology and require that $\delta f_{\chi} = \text{Tr}[(\delta f_{\chi}/\delta Z)\delta Z] + \delta Z(\delta f_{\chi}/\delta Z) \geq 0$. This amounts to defining the increment $\delta Z := \epsilon \delta f_{\chi}/\delta Z$ for some small real $\epsilon > 0$ that functions as a fixed iteration step size. This allows us to state the iterative equations
\[
Z_{k+1} = \left(1 - \frac{\epsilon}{2} \frac{\text{Tr}[M^k]}{\|Z_k\|_2} \right) Z_k + \frac{\epsilon}{2} \text{Tr}[M^k U^o],
\]
\[
\chi_{k+1} = \frac{Z_{k+1}}{\|Z_{k+1}\|_2},
\]  
(34)

that can be used to converge $\chi_k$ to $\chi_{\text{opt}}$ starting with $Z_1 = I_B$, where a factor of $\|Z_k\|_2$ has been neglected for a suitably chosen magnitude of $\epsilon$. As $\delta f_{\chi} = 2\epsilon \text{Tr}[|\delta f_{\chi}/\delta Z|^2] > 0$ by construction, convergence is guaranteed as long as $\epsilon$ is sufficiently small. Operationally, one can afford to choose a reasonably large $\epsilon$ to increase the convergence rate.

### E Optimal hacking of two-qubit quantum scramblers

The case where $d_A = d_B = d_C = d_L = 2$ presents the unique situation in which one can confirm, indeed, that $\chi_{\text{opt}} = I_B/\sqrt{d_B}$. To this end, we proceed to construct the exact expression of $|U^o\rangle$. Since $UU^\dagger = I$, in terms of the product computational basis $\langle jk|U|lm\rangle = U^{jk}_{lm}$, the basic relation
\[
\sum_{l,m=0}^{1} U^{jk}_{lm} U^{j_2k_2}_{lm} = \delta_{j_1,j_2}\delta_{k_1,k_2}
\]  
(35)
shall be immensely useful in the subsequent discussion.

Using Eq. (35), we first note that the product
\[
U^o U^o = \sum_{l,m,m'} \left[ |0, m\rangle(U^{00}_{lm} U^{00}_{lm'} + U^{10}_{lm} U^{10}_{lm'}) |0, m'\rangle + |1, m\rangle(U^{01}_{lm} U^{01}_{lm'} + U^{11}_{lm} U^{11}_{lm'}) |1, m'\rangle + |0, m\rangle(U^{10}_{lm} U^{10}_{lm'} + U^{11}_{lm} U^{11}_{lm'}) |1, m'\rangle + |1, m\rangle(U^{10}_{lm} U^{01}_{lm'} + U^{11}_{lm} U^{10}_{lm'}) |0, m'\rangle \right]
\]  
\[
\equiv \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} B^\dagger & A^{-1}\text{Det}A \end{pmatrix} \]  
(36)
may be characterized, in the product computational basis, by only two $2 \times 2$ matrices in a highly specific manner, where $B$ is traceless. Such a structure is absent in higher dimensions. For convenience, we may rewrite

$$U^o U^{\dagger} \equiv 1 + \begin{pmatrix} a \cdot \sigma & b^* \cdot \sigma \\ b \cdot \sigma & -a \cdot \sigma \end{pmatrix}$$

(37)

in terms of dot products $(v \cdot w = v^T w)$ of the vectorial parameters $a$ and $b$ with the standard vector of Pauli operators $\sigma = (\sigma_x, \sigma_y, \sigma_z)^T$ to separate the matrix representation of $U^o U^{\dagger}$ into the identity and another $4 \times 4$ traceless matrix, where $a$ is real and $b$ complex.

With the identity $(a \cdot \sigma)(a' \cdot \sigma) = a \cdot a' + i a \times a' \cdot \sigma$, it is a straightforward matter to verify that $|U^o|$ has the same matrix-representation structure, where all its parameters satisfy the following conditions:

$$|U^o| \equiv c' 1 + \begin{pmatrix} a' \cdot \sigma & b'^* \cdot \sigma \\ b' \cdot \sigma & -a' \cdot \sigma \end{pmatrix},$$

$$1 = c'^2 + |a'|^2 + |\text{Re}\{b'\}|^2 + |\text{Im}\{b'\}|^2,$$

$$a = 2c'a' - 2\text{Re}\{b'\} \times \text{Im}\{b'\},$$

$$\text{Re}\{b\} = 2c'i\text{Re}\{b'\} - 2\text{Im}\{b'\} \times a',$$

$$\text{Im}\{b\} = 2c'i\text{Im}\{b'\} - 2a' \times \text{Re}\{b'\}.$$  

(38)

Here, $\text{Re}\{\cdot\}$ and $\text{Im}\{\cdot\}$ respectively denote the real and imaginary parts of the argument. Evidently, in this fortuitously easy yet general two-qubit scenario, we find that $\text{Tr}_B|U^o| = 2c'I_B$, such that $\chi = \text{Tr}_B|U^o|/\|\text{Tr}_B|U^o||_2 = I_B/\sqrt{d_B} = \chi_{\text{opt}}$.

F Asymptotic formulas for $p_{\text{hack}}^{\text{opt}}$

For the problem of quantum-information extraction involving unitary scrambling dynamics described by a $d_A d_B$-dimensional unitary operator $U$, we may consider a general a very general setting where the bipartite output dimensions of $U$ are respectively $d_K$ and $d_L$ for systems $A$ and $B$, such that clearly $d_A d_B = d_K d_L$ owing to unitarity. In the asymptotic limit $d_A, d_B \to \infty$, according to the discussions in Sec. D, the corresponding optimal scrambler hacking fidelity takes the form $p_{\text{hack}}^{\text{opt}} \to ||U^o||^2_1/(d_A^2 d_B d_K)$. The analytical form of its average value then necessitates the calculation of the average term $||U^o||^2_1$ over all random $U$’s distributed according to some specific distribution measure, which we fix to be the Haar measure. We emphasize that since $U^o$ is represented by a $d_B d_L \times d_A d_K$ matrix that is obtained from just a sequence of index swapping operations, such a rectangular matrix still retains the statistical properties of a Haar unitary matrix elements, namely $U^o_{lm} = 0$ and $||U^o||^2_{lm} = 1/(d_A d_B)$ [35]. If we additionally suppose that $\kappa \equiv \sqrt{d_B d_L/(d_A d_K)} = d_B/d_K \geq 1$, then in the dimensional asymptotic limit, the random Haar unitary ensemble has elements that are so weakly correlated that they are approximately independently and identically distributed [43, 44]. Each eigenvalue $(\sigma_j)$ of the positive operator $\kappa^{-1}U^o U^{\dagger o}$ shall then independently follow the Marčenko–Pastur distribution [36]:

$$\sigma_j \sim \frac{1}{2\pi} \frac{\sqrt{\lambda_+ - x}(x - \lambda_-)}{\lambda x}, \quad \lambda_\pm = (1 \pm \sqrt{\lambda})^2,$$  

(39)

which is specified by the distribution’s characteristic variable $\lambda = \kappa^{-2}$. With these,

$$||U^o||^2_1 = \kappa \left( \sum_{j=1}^{d_A d_K} \sigma_j + \sum_{j \neq k} \sigma_j \sqrt{\sigma_k} \right)$$

$$= \kappa \left[ d_A d_K + (d_A d_K - 1)d_A d_K \mathcal{I}_\kappa^2 \right],$$  

(40)

where $\mathcal{I}_\kappa$ now translates to an average with respect to the distribution in (39), and we have used the fact that $\sigma_j = 1$. The quantity $\mathcal{I}_\kappa = x^{1/2}$ refers to the half-moment of this distribution. For completeness, we evaluate
The $m$th moment:

\[
\bar{x}^m = \int_{\lambda}^{\lambda+} \frac{dx}{2\pi\lambda} x^{m-1} \sqrt{(\lambda_+ - x)(x - \lambda_-)}
\]

\[
= \frac{2}{\pi} (1 + \lambda)^{m-1} \int_{-1}^{1} dt \left( 1 + \frac{2\sqrt{\lambda}}{1 + \lambda} t \right)^{m-1} \sqrt{1 - t^2}
\]

\[
= \left( 1 + \frac{1}{\kappa^2} \right)^{m-1} \frac{2F_1}{2} \left( 1 - \frac{m}{2}, 1 - \frac{m}{2}; \frac{2}{\kappa}; \left( \frac{2}{1 + 1/\kappa^2} \right)^2 \right). \tag{41}
\]

The variable substitution $x = (\lambda_+ + \lambda_-)/2 + (\lambda_+ - \lambda_-) t/2$ has been introduced after the second equality in (41). We emphasize that the last equality in (41) is valid for any real $m$ so long as the previous $t$ integral converges. Hence, $\bar{x}^m$ is proportional to a hypergeometric function $2F_1(\cdot, \cdot; \cdot; \cdot)$. Upon using the identity [37]

\[
2F_1(2a, 2a + 1 - \gamma; \gamma; z) = \frac{2F_1(a, a + \frac{1}{2}; \gamma; \frac{4z}{(1 + z)^2})}{(1 + z)^{2a}}, \tag{42}
\]

we get $\bar{x}^m = 2F_1(1 - m, -m; 2; \kappa^{-2})$. That $\pi = 1$ follows immediately from a direct evaluation of the hypergeometric function. Thereafter, the substitution $m = 1/2$ nab us the final answer $I_\kappa = 2F_1(2^{-1} - 2^{-1}; 2; \kappa^{-2})$, so that

\[
\bar{p}_{\text{hack}}^{\text{opt}} \approx I_\kappa^2 + \frac{1}{d_\lambda d_\kappa} (1 - I_\kappa^2) \quad \text{for } \kappa \geq 1. \tag{43}
\]

Moreover, we may simplify this expression further by considering a moderately large $\kappa$, for which the hypergeometric function has the simple second-order approximation $I_\kappa \approx 1 - 1/(8\kappa^2)$. This simplification works amazingly well even for $\kappa = 1 - I_1 = 8/(3\pi) \approx 0.875$—such that one might as well use this approximation for any $\kappa$.

Now, if $\kappa < 1$, one can go through a similar line of argument and arrive at $\|U^\kappa\|_F^2 = \kappa^{-1} \left[ d_B d_L + (d_B d_L - 1) d_B d_L I_1^{2/\kappa} \right]$, in which case, we get

\[
\bar{p}_{\text{hack}}^{\text{opt}} \approx \kappa^2 I_1^{2/\kappa} + \frac{1}{d_\lambda d_\kappa} (1 - I_1^{2/\kappa}) \quad \text{for } \kappa < 1, \tag{44}
\]

which tells us that the asymptotic optimal scrambler hacking fidelity is going to be smaller than that when $\kappa \geq 1$.

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