On the rate of convergence of \( L_p \) norms in the CLT for Poisson random sum

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Abstract. In the paper, we present the upper bound of \( L_p \) norm \( \Delta_{\lambda, p}^{1/\lambda} \) of the order \( \lambda^{-\delta/2} \) for all \( 1 \leq p \leq \infty \), in the central limit theorem for a standardized random sum \( (S_{N_{\lambda}} - \mathbb{E} S_{N_{\lambda}})/\sqrt{D S_{N_{\lambda}}}, \) where \( S_{N_{\lambda}} = X_1 + \cdots + X_{N_{\lambda}} \) is the random sum of independent identically distributed random variables \( X, X_1, X_2, \ldots \) with \( \beta_{2+\delta} = \mathbb{E}|X|^{2+\delta} < \infty \) where \( 0 < \delta \leq 1 \), \( N_{\lambda} \) is a random variable distributed by the Poisson distribution with the parameter \( \lambda > 0 \), and \( N_{\lambda} \) is independent of \( X_1, X_2, \ldots \).

Keywords: central limit theorem, \( L_p \) norms, Poisson random sum.

1. Introduction
Let \( X, X_1, X_2, \ldots \) be independent identically distributed random variables (r.v.’s) with \( \mu = \mathbb{E} X, \alpha_2 = \mathbb{E} X^2, \sigma^2 = \mathbb{D} X, \) and
\[
\beta_{2+\delta} = \mathbb{E}|X|^{2+\delta} < \infty, \quad 0 < \delta \leq 1.
\]
Let \( N_{\lambda} \) be a r.v. distributed by the Poisson distribution with the parameter \( \lambda > 0 \) (for short, \( N_{\lambda} \sim \mathcal{P}(\lambda) \)), i.e.,
\[
P\{N_{\lambda} = k\} = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots.
\]
Moreover, assume that the r.v. \( N_{\lambda} \) is independent of the \( X_1, X_2, \ldots \), and consider the so-called Poisson random sum
\[
S_{N_{\lambda}} = X_1 + \cdots + X_{N_{\lambda}} \left( \sum_{i=1}^{0} = 0 \right).
\]
Denote
\[
Z_{N_{\lambda}} = \frac{S_{N_{\lambda}} - \mathbb{E} S_{N_{\lambda}}}{\sqrt{\mathbb{D} S_{N_{\lambda}}}} = \frac{S_{N_{\lambda}} - \lambda \mu}{\sqrt{\lambda \alpha_2}}, \quad \chi^{2+\delta} = \frac{\beta_{2+\delta}}{(\mu^2 + \sigma^2)^{(2+\delta)/2}}.
\]

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\[ \Delta_\lambda(x) = \mathbb{P}(Z_{N_\lambda} < x) - \Phi(x), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du, \]
\[ \Delta_{\lambda,p} = \begin{cases} \left( \int_{-\infty}^{\infty} |\Delta_\lambda(x)|^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{x \in \mathbb{R}} |\Delta_\lambda(x)| & \text{if } p = \infty. \end{cases} \]

Here and in what follows \( \mathbb{R} \) is a real line.

We are interested in the rate of convergence of the \( L_p \) norm \( \Delta_{\lambda,p} \) for all \( 1 \leq p \leq \infty \).

First of all, note that there is a rich literature on normal approximations for random sums of independent sequences of r.v.’s, see, for example, papers [4,9], books [7,11], and the references therein. In most of these works the upper bounds of the uniform distance \( \Delta_{\lambda,\infty} \) are investigated. The obtained bounds contain, as a rule, several terms, outlined by the distributions of summands \( X_1, X_2, \ldots \) (not necessarily identically distributed r.v.’s), and the random numbers of summands \( N \) are also not necessarily Poissonian. These general bounds are provided with some loss of accuracy for concrete distribution of the random number of summands \( N \).

Recall that Robbins (1948) [16] has proved that, if a distribution of a r.v. \( N \) depends on the parameter \( \lambda \), then \( X_1, X_2, \ldots \) are independent identically distributed r.v.’s and \( N \) is independent of \( X_1, X_2, \ldots \), the following equality is valid
\[ \lim_{\lambda \to \infty} \mathbb{P} \left\{ \frac{N - \mathbb{E} N}{\sqrt{\mathbb{D} N}} < x \right\} = \Phi(x), \quad (3) \]

for a random sum \( S_N = X_1 + \cdots + X_N \), where \( X_1, X_2, \ldots \) are independent identically distributed r.v.’s and \( N \) is independent of \( X_1, X_2, \ldots \), the following equality is valid
\[ \lim_{\lambda \to \infty} \mathbb{P} \left\{ \frac{S_N - \mathbb{E} S_N}{\sqrt{\mathbb{D} S_N}} < x \right\} = \Phi(x). \quad (4) \]

In [20], we have obtained the upper estimates of \( L_p \) norms for all \( 1 \leq p \leq \infty \) of the order \( (\mathbb{D} N)^{-1/2} \) in the limit (3) as \( N \) is either a Poisson r.v. or gamma one.

In this paper, we have obtained the upper estimate of the \( L_p \) norm \( \Delta_{\lambda,p} \) for all \( 1 \leq p \leq \infty \) (see Theorem 1). The obtained constants are not the best possible, but that was not the main author’s aim.

The uniform estimate (5) of \( \Delta_{\lambda,\infty} \) in the case \( \delta = 1 \) was first presented in [15], and the proof independently was given in [3] and [1]. It has been proved independently in [14] and [10] that the constant \( C_{\infty}(1) \) in (5) in this particular case \( (p = \infty, \delta = 1) \) is the same as in the Berry–Esséen inequality for the sum of a non-random number of independent identically distributed summands \( (C_{\infty}(1) \leq 0.7655 [17]). \)

As in [20], to obtain the upper estimates of the norm \( \Delta_{\lambda,\infty} \) (uniform metric) and the \( L_1 \) norm \( \Delta_{\lambda,1} \), we have formed linear differential equations from the characteristic function of the standardized r.v. \( Z_{N_\lambda} = \frac{S_N - \lambda \mu}{\sqrt{\lambda \sigma^2}} \) by virtue of which we succeeded in getting proper estimates of differences: between the characteristic function and the normal one, and between their derivatives as well. The proof of the estimate of the \( L_p \) norm is elementary.
The papers, for example, [5] and [19], in which the rate of convergence of \( L_p \) norms in the central limit theorem is investigated for sums of independent random variables, are close to that of ours.

The obtained results are produced under the influence of Stein’s method and the papers of Stein [18] and Tikhomirov [21].

2. Main and auxiliary results

Now, we formulate the main result.

**Theorem 1.** Let \( X, X_1, X_2, \ldots \) be independent identically distributed r.v.’s with \( \mu = \mathbb{E}X, \alpha_2 = \mathbb{E}X^2, \sigma^2 = \text{D}X, \beta_{2+\delta} = \mathbb{E}|X|^{2+\delta} < \infty \), for some \( 0 < \delta \leq 1 \), and the r.v. \( N_\lambda \sim \mathcal{P}(\lambda) \) with the parameter \( \lambda > 0 \) be independent of \( X_1, X_2, \ldots \). Then for all \( 1 \leq p \leq \infty \)

\[
\Delta_{\lambda,p} \leq C_p(\delta) \frac{\sigma^{2+\delta}}{\lambda^{8/2}},
\]

where the constant \( C_p(\delta) \) depends only on \( \delta \) and \( p \). Moreover, \( C_\infty(1) = \frac{2}{\sqrt{\pi}} + \frac{8}{\pi} \frac{\sqrt{2}}{\sqrt{\pi}} < 2.41 \), and \( C_p(1) < 20.87 \) for all \( 1 \leq p \leq \infty \).

Recall that \( \mathbb{E}N_\lambda = \text{D}N_\lambda = \lambda \) for the r.v. \( N_\lambda \sim \mathcal{P}(\lambda) \).

Denote the characteristic function of the standardized r.v. \( Z_{N_\lambda} = \frac{S_{N_\lambda} - \lambda \mu}{\sqrt{\lambda \alpha_2}} \) by \( f_\lambda(t) = \mathbb{E}e^{itZ_{N_\lambda}} \), and the derivative of the characteristic function \( f_\lambda(t) \) with respect to \( t \) by \( f'_\lambda(t) \).

To prove Theorem 1, we use an auxiliary result, Lemma 1, on the behaviour of the functions \( f_\lambda(t) \) and \( f'_\lambda(t) \).

**Lemma 1.** Let the conditions of Theorem 1 be satisfied. Then the characteristic function \( f_\lambda(t) \) satisfies the following homogeneous linear differential equation for all \( t \in \mathbb{R} \):

\[
f'_\lambda(t) = i \frac{\lambda}{\alpha_2} \left( \mathbb{E}X e^{i t \sqrt{\sigma^2 X}} - \mu f_\lambda(t) \right).
\]

Moreover, for all \( |t| \leq C_0(\delta) \frac{\sigma^{\delta}}{\lambda^{4+6\delta}} = T \),

\[
|f_\lambda(t) - e^{-t^2/2}| \leq \frac{2^{1-\delta} \sigma^{2+\delta}}{(1+\delta)(2+\delta) \lambda^{8/2}} |t|^{2+\delta} e^{-t^2/4},
\]

\[
|f'_\lambda(t) - (e^{-t^2/2})'| \leq \frac{2^{1-\delta} \sigma^{2+\delta}}{1+\delta \lambda^{8/2}} |t|^{1+\delta} e^{-t^2/2} + \frac{2^{1-\delta} \sigma^{2+\delta}}{(1+\delta)(2+\delta) \lambda^{8/2}} |t|^{3+\delta} + \frac{2^{2(1-\delta)} \sigma^{2+\delta}}{(1+\delta)^2(2+\delta) \lambda^{8/2}} |t|^{3+2\delta} e^{-t^2/4}.
\]
where \( C_0(\delta) = \left( \frac{(1+\delta)(\sqrt{2+\delta})}{2+\delta} \right)^{1/\delta} \).

**Proof.** It is easy to see that \( \mathbb{E}S_{N_\lambda} = \lambda \mu \), \( \mathbb{D}S_{N_\lambda} = \lambda (\mu^2 + \sigma^2) = \lambda \sigma_2 \), and

\[
f_\lambda(t) = \exp \left\{ \lambda \left( \mathbb{E} \exp \left[ i \frac{t}{\sqrt{\lambda \sigma_2}} X \right] - 1 - i \frac{t}{\sqrt{\lambda \sigma_2}} \mu \right) \right\}.
\]

To make sure on the correctness of (6), it suffices to take the derivatives with respect to \( t \) on both sides of expression (9).

Denote by \( \theta_1 \) and \( \theta_2 \) complex functions such that \( |\theta_1| \leq 1 \).

**Estimation of** \( |f_\lambda(t) - e^{-t^2/2}| \). Since for all \( t \in \mathbb{R}, 0 < \delta \leq 1 \), and \( i = \sqrt{-1} \),

\[
|e^{ix} - 1 - ix| \leq \frac{2^{1-\delta}}{1+\delta} |x|^{1+\delta},
\]

we derive from (6) that, for all \( t \in \mathbb{R} \),

\[
f'_\lambda(t) = \left( -t + \theta_1 \frac{2^{1-\delta}}{1+\delta} \frac{x^{2+\delta}}{\lambda^{\delta/2}} |t|^{2+\delta} \right) f_\lambda(t).
\]

After solving the linear differential equation (10), we get that the solution of Eq. (10) can be written in the form

\[
f_\lambda(t) = \exp \left\{ -\frac{t^2}{2} + \theta_2 \frac{2^{1-\delta}}{(1+\delta)(2+\delta)} \frac{x^{2+\delta}}{\lambda^{\delta/2}} |t|^{2+\delta} \right\}, \quad t \in \mathbb{R}.
\]

To estimate the difference \( |f_\lambda(t) - e^{-t^2/2}| \), we use the well-known fact that \( |e^z - 1| \leq |z| |e^z| \) for all complex numbers \( z \). We obtain that for all \( |t| \leq C_0(\delta) \frac{\sqrt{2}}{\sqrt{\lambda^{\delta/2}}} \)

\[
|f_\lambda(t) - e^{-t^2/2}| \leq \frac{2^{1-\delta}}{(1+\delta)(2+\delta)} \frac{x^{2+\delta}}{\lambda^{\delta/2}} |t|^{2+\delta} e^{-t^2/4},
\]

i.e., (7) is proved.

Substituting (7) into (10), we get (8).

Lemma 1 is proved.

3. **Proof of Theorem 1**

**Estimation of** \( \Delta_{\lambda, \infty} \). To estimate the uniform metric \( \Delta_{\lambda, \infty} \), we use the smoothing inequality of Esséen [12, p. 297] with \( T = C_0(\delta) \frac{\sqrt{T}}{\sqrt{\lambda^{\delta/2}}} > 0 \), inequality (7), and obtain

\[
\Delta_{\lambda, \infty} \leq \frac{2}{\pi} \int_0^T \left| \frac{f_\lambda(t) - e^{-t^2/2}}{t} \right| \, dt + \frac{24}{\pi \sqrt{2\pi}} \frac{1}{T} \leq C(\delta) \left( \frac{x^{2+\delta}}{\lambda^{\delta/2}} + \left( \frac{x^{2+\delta}}{\lambda^{\delta/2}} \right)^{1/\delta} \right).
\]

Let \( \frac{x^{2+\delta}}{\lambda^{\delta/2}} \leq 1 \). Then \( \left( \frac{x^{2+\delta}}{\lambda^{\delta/2}} \right)^{1/\delta} \leq \frac{x^{2+\delta}}{\lambda^{\delta/2}} \) and \( \Delta_{\lambda, \infty} \leq C(\delta) \frac{x^{2+\delta}}{\lambda^{\delta/2}} \). In the case \( \frac{x^{2+\delta}}{\lambda^{\delta/2}} > 1 \), the estimate of \( \Delta_{\lambda, \infty} \) is trivial since \( \Delta_{\lambda, \infty} \leq 0.5416 [2, \text{p. 103}] \).
Estimation of $\Delta_{1,1}$. To estimate the $L_1$ norm $\Delta_{1,1}$, we use the estimates of Lemma 1 and the following inequality with $T \geq 1$ ([8, p. 25] and [13, p. 395]):

\[
\int_{-\infty}^{\infty} \left| F_{Z_{\alpha}}(x) - \Phi(x) \right| dx 
\leq 3 \left( \int_0^T \left| \frac{f_\lambda(t) - e^{-t^2/2}}{t} \right|^2 dt \right)^{1/2} + \sqrt{2} \left( \int_0^T \left| \frac{d}{dt} \left( \frac{f_\lambda(t) - e^{-t^2/2}}{t} \right) \right|^2 dt \right)^{1/2} + \frac{8\pi}{T}
\]

\[
\leq 3I_1 + 2(I_2 + I_3) + \frac{8\pi}{T},
\]

where

\[
I_1 = \int_0^T \left| \frac{f_\lambda(t) - e^{-t^2/2}}{t} \right|^2 dt, \quad I_2 = \int_0^T \left| \frac{f_\lambda'(t) - (e^{-t^2/2})'}{t} \right|^2 dt,
\]

\[
I_3 = \int_0^T \left| \frac{f_\lambda(t) - e^{-t^2/2}}{t^2} \right|^2 dt.
\]

Using inequalities (7) and (8), we estimate the quantities $I_1$, $I_2$, and $I_3$ from (11) with $T = C_0(\delta)\frac{\sqrt{x}}{\varpi(x+\delta/\varpi)} > 1$, and obtain

\[
I_1 \leq C(\delta) \frac{x^{2+\delta}}{\lambda^{\delta/2}}, \quad I_2 \leq C(\delta) \left( \frac{x^{2+\delta}}{\lambda^{\delta/2}} + \left( \frac{x^{2+\delta}}{\lambda^{\delta/2}} \right)^2 \right), \quad I_3 \leq C(\delta) \frac{x^{2+\delta}}{\lambda^{\delta/2}}.
\]

To estimate $I_2$, we use the elementary inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$. Substituting these estimates into (11), we have that, for $T = C_0(\delta)\frac{\sqrt{x}}{\varpi(x+\delta/\varpi)} > 1$ and $\frac{x^{2+\delta}}{\lambda^{\delta/2}} \leq 1$,

\[
\Delta_{1,1} \leq 3I_1 + 2(I_2 + I_3) + \frac{8\pi}{T} \leq C(\delta) \frac{x^{2+\delta}}{\lambda^{\delta/2}}.
\]

In the case $\frac{x^{2+\delta}}{\lambda^{\delta/2}} \geq 1$ (in the case $C_0(\delta)\frac{\sqrt{x}}{\varpi(x+\delta/\varpi)} \leq 1$ as well), the estimate of $\Delta_{1,1}$ is trivial since $\Delta_{1,p} \leq \sqrt{T}$ for all $1 \leq p \leq \infty$ (for $\Delta_{1,1} \leq \sqrt{T}$, see [6, p. 528]).

The estimate (5) now follows from the estimates of $\Delta_{1,\infty}$ and $\Delta_{1,1}$, since

\[
\Delta_{1,p} \leq \Delta_{1,\infty}^{(p-1)/p} \Delta_{1,1}^{1/p}
\]

for all $1 \leq p < \infty$.

Theorem 1 is proved.
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REZIUMĖ

J. Sunklodas. _Apie normos $L_p$ konvergavimo greitį CRT Puasono atsitiktinei sumaui_.

Gautas $L_p$ normos $\Delta_{1,p}$ viršutinis ivertis $\lambda^{-1/2}$ su visais $1 \leq p < \infty$ centrinėje ribinėje teoremoje 
standartizuota suma $(S_n - \mathbb{E}S_n)/\sqrt{\mathbb{V}S_n}$, kur $S_n = X_1 + \cdots + X_n$ yra suma nepriklausomų vieno-
dai pasiskirstusių atsitiktinių dydžių $X_1, X_2, \ldots$ su $\mathbb{V}X_i = \mathbb{E}|X|^{2+\delta} < \infty$ nekuriam $0 < \delta \leq 1$. $N_\lambda$ yra 
Puasono atsitiktinis dydis su parametru $\lambda > 0$ ir $N_\lambda$ nepriklausu nuo atsitiktinių dydžių $X_1, X_2, \ldots$.

Raktiniai žodžiai: centrinė ribinė teorema, $L_p$ norma, Puasono atsitiktinė suma.