STABILITY OF TRAVELLING WAVES IN
STOCHASTIC BISTABLE REACTION-DIFFUSION
EQUATIONS

WILHELM STANNAT

Abstract. We prove stability of travelling waves for stochastic bistable reaction-diffusion equations with both additive and multiplicative noise, using a variational approach based on functional inequalities. Our analysis yields explicit estimates on the rate of stability that can be shown in special examples to be optimal.

1. Introduction

The purpose of this paper is to generalize the main results of [12] on the stability of travelling waves in Nagumo equation with multiplicative noise to general bistable reaction-diffusion equations with noise. To this end let us first consider the deterministic reaction-diffusion equation

\[ \partial_t v(t, x) = \nu v_{xx}(t, x) + bf(v(t, x)), \quad v(t, x) = v_0(x) \]

for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\). Here, \(f : \mathbb{R} \to \mathbb{R}\) is a continuously differentiable function satisfying

\[
\begin{align*}
  f(0) &= f(a) = f(1) = 0 \quad \text{for some } a \in (0, 1) \\
  f(x) &< 0 \quad \text{for } x \in (0, a), \ f(x) > 0 \text{ for } x \in (a, 1) \\
  f'(0) &< 0, \ f'(a) > 0, \ f'(1) < 0.
\end{align*}
\]

(A1)

Theorem 12 in [4] implies for \(\nu, b > 0\) the existence of a travelling wave connecting the stable fixed points 0 and 1 of the reaction term, i.e., a monotone increasing \(C^2\) function \(\hat{v}\) satisfying

\[ c\hat{v}_x = \nu \hat{v}_{xx} + bf(\hat{v}) \]

for some wavespeed \(c \in \mathbb{R}\) and boundary conditions \(\hat{v}(-\infty) = 0, \ \hat{v}(+\infty) = 1\). It follows that \(\hat{v}(t) := \hat{v}(\cdot + ct)\) and all its spatial translates \(\hat{v}(\cdot + x_0 + ct)\) are solutions of (1). A particular example is the Nagumo equation with \(f(v) = v(1-v)(v-a)\) where the travelling wave is explicitly given by \(\hat{v}(x) = \left(1 + e^{-\sqrt{\frac{b}{\nu^2}}x}\right)^{-1}\).
It is known that the wave speed $c$ and the integral $\int_0^1 f(v) \, dv \geq 0$ have the same sign and that in particular $c = 0$ if and only if $\int_0^1 f(v) \, dv = 0$. To simplify the presentation of our results we will therefore assume from now on that

\[(A2) \quad \int_0^1 f(v) \, dv \geq 0\]

hence that the wave speed $c$ is nonnegative.

So far the assumptions on the reaction term $f$ are classical. The existing results in the literature on the stability of the travelling wave can be divided up into results based on maximum principle and comparison techniques, see in particular [3] for a stability result w.r.t. initial conditions $v_0$ satisfying $0 \leq v_0 \leq 1$, $\lim\inf_{x \to -\infty} v_0(x) < a$ and $\lim\sup_{x \to \infty} v_0(x) > a$, and results w.r.t. $L^2$- or $H^1$-norms, based on spectral information on the linearization of (1) along the travelling wave $\hat{v}$ (see, e.g. [3, 10]). Whereas the first approach is not appropriate for stochastic perturbations, unless the noise terms would satisfy unnatural monotonicity conditions, the second approach can be in principle generalized to the stochastic case. However, in order to do this, the existing spectral information on the linearization of (1) has to be considerably refined. Abstract perturbation results on the spectral gap below the eigenvalue corresponding to the travelling wave cannot be easily generalized to the stochastic case. We will therefore use functional inequalities to derive Lyapunov stability of the travelling wave in the space $L^2(\mathbb{R})$. To be more precise, we will show in Theorem 1.5 under the following additional assumptions on the reaction term

\[(A3) \quad \exists v_* \in (a, 1) \text{ such that } f''(v) > 0 \text{ (resp. } < 0)\]

on $[0, v_*]$, (resp. $(v_*, 1]$)

saying that $f$ is strictly convex on $[0, v_*)$ and strictly concave on $(v_*, 1]$, that the $L^2$-norm is a Lyapunov function restricted to the orthogonal complement of $\hat{v}_x$. As a consequence of this phase-space stability, the stochastic case will become much easier to investigate. Our assumptions are satisfied in the case of the Nagumo equation (for all $a \in (0, 1)$) and do not require any estimates on the unknown wave speed $c$.

Our interest in the above reaction diffusion equation is motivated by the fact that (1) can be seen as a singular limit $\epsilon \downarrow 0$ of Fitz-Hugh Nagumo systems

\[
\begin{align*}
\partial_t v(t, x) &= \nu v_{xx}(t, x) + bf(v(t, x)) - w(t, x) + I \\
\partial_t w(t, x) &= \epsilon (v(t, x) - \gamma w(t, x)) \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}
\end{align*}
\]

when the adaptation variable $w$ is set constant to the value of the input current $I$ (see the monograph [1]). The Fitz-Hugh Nagumo system, a
mathematical idealization of the Hodgkin Huxley model, admits, under appropriate assumptions on the coefficients, pulse solutions that serve as a mathematical model for the action potential travelling along the nerve axon. By adding noise to this system, e.g. channel noise, the resulting dynamical system exhibits many interesting features like propagation failure of the pulse solution, backpropagation, annihilation and spontaneous pulse solutions. Recent computational studies can be found in [13, 14].

We are therefore interested in a rigorous mathematical analysis of stochastic reaction-diffusion systems with bistable reaction terms. With a view towards the above mentioned features of the noisy system, we are in particular interested to establish a multiscale analysis of the whole dynamics which requires in a first step a robust stability result of the travelling pulse solution. As already mentioned for the scalar-valued case, the existing stability results (e.g. [2, 6] for systems) cannot be carried over to the stochastic case. In order to reduce the mathematical difficulty of the problem, we therefore consider the scalar-valued case in the present paper as a starting point.

Before we proceed let us first draw a couple of conclusions on the travelling wave resulting from our assumptions.

**Lemma 1.1.** Assume that (A1) and (A2) hold. Then:

(i) $\hat{v}_x^2(x) \leq \frac{2b}{\nu} \int_{\hat{v}(x)}^{1} f(v) \, dv$ for all $x$. In particular,

$$\lim_{x \to +\infty} e^{-\alpha \hat{v}_x^2} = 0 \quad \text{for } \alpha \geq 0.$$ 

(ii) $e^{-2\hat{v}_x^2}$ is increasing (resp. decreasing) for $x \leq \hat{v}^{-1}(a)$ (resp. $x \geq \hat{v}^{-1}(a)$). In particular,

$$\lim_{x \to \pm \infty} e^{-\alpha \hat{v}_x^2} = 0 \quad \text{for } \alpha \in [0, 2[.$$ 

The proof of Lemma 1.1 is given in Section 4 below. The next Proposition summarizes the main conclusions implied by the additional assumption (A3).

**Proposition 1.2.** Assume that (A1) - (A3) hold. Then:

(i) \(\frac{L(\hat{v})}{\hat{v}_x}\) is strictly monotone increasing. In particular,

$$-\frac{d^2}{dx^2} \log \hat{v}_x = \frac{d}{dx} \frac{\hat{v}_{xx}}{\hat{v}_x} = \frac{b}{\nu} \frac{d f(\hat{v})}{d\hat{v}_x} \hat{v}_x > 0,$$

i.e., $\hat{v}_x$ is strictly log-concave (but not uniformly).

(ii) \(\gamma_- := \inf \frac{b f(\hat{v})}{\nu \hat{v}_x} = \frac{c}{2\nu} - \sqrt{\left(\frac{c}{2\nu}\right)^2 - \frac{b}{\nu} f'(0)}\)

\(\gamma_+ := \sup \frac{b f(\hat{v})}{\nu \hat{v}_x} = \frac{c}{2\nu} + \sqrt{\left(\frac{c}{2\nu}\right)^2 - \frac{b}{\nu} f'(1)}\).
\[
\int_{-\infty}^{0} e^{-20\frac{\nu}{c}x} (\hat{v}_x^2 + \hat{v}_{xx}^2) \, dx < \infty \quad \text{for all } \alpha \frac{c}{\nu} < \frac{c}{\nu} - \gamma_-
\]
\[
\int_{0}^{\infty} e^{-20\frac{\nu}{c}x} (\hat{v}_x^2 + \hat{v}_{xx}^2) \, dx < \infty \quad \text{for all } \alpha \frac{c}{\nu} > \frac{c}{\nu} - \gamma_+
\]

In particular,
\[
\int e^{-\frac{\nu}{c}x} (\hat{v}_x^2 + \hat{v}_{xx}^2) \, dx < \infty.
\]

The proof of Proposition 1.2 is given in Section 4 below.

The next theorem contains the essential functional inequality that is implied by (A3).

**Theorem 1.3.** Assume that (A1) - (A3) hold. Then there exists some \( \kappa > 0 \) such that
\[
-\frac{d^2}{dx^2} \log \hat{v}_x + \left( \frac{d}{dx} \log \hat{v}_x \right)^2 - \frac{c}{\nu^2} \frac{d}{dx} \log \hat{v}_x \geq \kappa.
\]

The proof of Theorem 1.3 is given in Section 4 below. We will assume from now on for all subsequent results that (A1) - (A3) hold.

**Example 1.4.** In the particular case of the Nagumo equation, i.e., \( f(v) = v(1-v)(v-a) \) for \( a \in (0, 1) \), the travelling wave is explicitly given as \( \hat{v}(x) = (1 + e^{-kx})^{-1} \) (resp. its spatial translates) with \( k = \sqrt{\frac{b}{2\nu}} \). The corresponding wave speed \( c \) can be calculated as \( c = \sqrt{2\nu b} \left( \frac{1}{2} - a \right) \). The logarithmic derivative \( \rho := \frac{d}{dx} \log \hat{v}_x = \frac{\hat{v}_{xx}}{\hat{v}_x} \) is given as \( \rho = \frac{c}{\nu} - \frac{b}{\nu} \hat{v} = \sqrt{\frac{2b}{\nu}} \left( \frac{1}{2} - \hat{v} \right) \). Thus
\[
-\rho' + \rho^2 - \frac{c}{\nu} \rho = b \left( (\hat{v} - a)^2 + a(1 - a) \right) \geq b \nu a(1 - a) > 0.
\]

With the functional inequality (2) of Theorem 1.3 we can now state the mentioned result on the Lyapunov stability of the linearization of (1) along the travelling wave \( \hat{v} \) in the deterministic case. To state our result precisely, let us introduce the Hilbert space \( H = L^2(\mathbb{R}) \) and the Sobolev space \( V = H^{1,2}(\mathbb{R}) \), defined as the closure of \( C_0^1(\mathbb{R}) \) w.r.t. the norm
\[
\|u\|_V^2 = \int_{\mathbb{R}} u^2 + u_x^2 \, dx
\]
in \( H \). Identifying \( H \) with its dual \( H' \) we obtain dense and continuous embeddings \( V \hookrightarrow H \equiv H' \hookrightarrow H' \). Note that w.r.t. this embedding the dualization between \( V' \) and \( V \) reduces for \( f \in H \) to the inner product in \( H \), i.e., \( \langle f, g \rangle_V = \langle f, g \rangle_H = \int f g \, dx \). The elementary estimate
\[ u^2(y) = 2 \int_{-\infty}^{y} u_x(x)u(x) \, dx \leq \int u_x^2 + u^2 \, dx \leq \|u\|_V^2 \text{ for } u \in C^1_c(\mathbb{R}) \] can be extended to the estimate \[ \|u\|_\infty \leq \|u\|_V \] for all \( u \in V \) that turns out to be crucial in the following.

The unbounded linear operator \( \nu u_{xx} \) induces a continuous mapping \( A : V \to V' \), because for \( u \in C^1_c(\mathbb{R}) \)

\[ \langle Au, v \rangle_{V'} = \int \nu u_{xx} v \, dx = -\nu \int u_x v_x \, dx \leq \nu \|u\|_V \|v\|_V. \]

**Theorem 1.5.** Let \( u \in V \). Then

\[ \langle Au + b f'(\hat{v})u, u \rangle_V \leq -\kappa \|u\|_V^2 + C_s \langle u, \hat{v}_x \rangle^2 \]

where

\[ \kappa := \frac{\kappa}{\kappa + \left(\frac{c}{2\nu}\right)^2 q_1} \frac{\nu}{\kappa} \]

and

\[ C_s = \left( \kappa q_2 + \frac{\nu}{\kappa} \left(\frac{c}{2\nu}\right)^2 \left( \kappa + \left(\frac{c}{2\nu}\right)^2 \right) \right) \left( \int e^{-\frac{c}{2\nu} \hat{v}_x^2} \, dx \right)^2. \]

Here, \( \kappa \) is the lower bound obtained in Theorem 1.3 and \( q_1 \) and \( q_2 \) are defined in Lemma 5.4 below.

The proof of Theorem 1.5 is given in Section 5 below.

The previous Theorem states that the flow generated by the semilinear diffusion equation is contracting in the direction that is orthogonal to \( \hat{v}_x \) (and its spatial translates). To properly quantify this contraction we will need to model the equation (1) as an evolution equation in the appropriate function space.

**1.1. Realization of (1) as evolution equation.** In the next step we want to realize the reaction diffusion equation (1) as an evolution equation on a suitable function space. To this end we need to impose yet additional assumptions on the reaction term, but now concerning only its global behaviour at infinity and not affecting its behaviour on \([0, 1]\) hence also not the travelling wave \( \hat{v} \). We assume that the derivative \( f' \) of the reaction term is bounded from above

(B1) \[ \eta_1 := \sup_{x \in \mathbb{R}} f'(x) < \infty, \]

that there exists a finite positive constant \( L \) such that

(B2) \[ |f(x_1) - f(x_2)| \leq L|x_1 - x_2| \left(1 + x_1^2 + x_2^2\right) \quad \forall x_1, x_2 \in \mathbb{R}, \]

which is typically satisfied for polynomials of third degree with leading negative coefficient and that there exists \( \eta_2 \) such that

(B3) \[ |f(u + v) - f(v) - f'(v)u| \leq \eta_2 (1 + |u|)|u|^2 \quad \forall v \in [0, 1], u \in \mathbb{R}. \]
Since we are interested in the asymptotic stability of the travelling wave also w.r.t. stochastic perturbations, it is now natural to consider the following decomposition $v(t, x) = u(t, x) + \hat{v}(x)$ of the solution $v$ of (1), where $u$ now satisfies the following equation

$$u_t(t, x) = \nu u_{xx}(t, x) + b (f(u(t, x) + \hat{v}(x)) - f(\hat{v}(x))$$

on $\mathbb{R}_+ \times \mathbb{R}$ that can be analysed best in a variational framework.

2. The deterministic case

The nonlinear term

$$G(t, u) := f(u + \hat{v}(t)) - f(\hat{v}(t))$$

can be realized as a continuous mapping

$$G : [0, \infty) \times V \to V'$$

being Lipschitz w.r.t. second variable $u$ on bounded subsets of $V$. Indeed, condition (B2) on $f$ implies that

$$\langle G(t, u), w \rangle_{V'} = \int_{\mathbb{R}} G(t, u)w \, dx = \int_{\mathbb{R}} (f(u + \hat{v}(t)) - f(\hat{v}(t))) w \, dx$$

$$\leq L \int_{\mathbb{R}} |u| (2 + u^2) |w| \, dx \leq L \|u\|_H (3 + 2\|u\|_{H}^2) \|w\|_H$$

hence

$$\|G(t, u)\|_{V'} \leq L \|u\|_H (3 + 2\|u\|_{H}^2)$$

and similarly

$$\langle G(t, u_1) - G(t, u_2), w \rangle_{V'} = \int_{\mathbb{R}} (f(u_1 + \hat{v}(t)) - f(u_2 + \hat{v}(t))) w \, dx$$

$$\leq L \|u_1 - u_2\|_H (4 + 2\|u_1\|_{H}^2 + 2\|u_2\|_{H}^2) \|w\|_H$$

which implies

$$\|G(t, u_1) - G(t, u_2)\|_{V'} \leq 2L (2 + \|u_1\|_{V}^2 + \|u_2\|_{V}^2) \|u_1 - u_2\|_H.$$ 

The sum $Au + bG(t, u)$ of both operators now satisfies the global monotonicity condition

$$\langle Au_1 + bG(t, u_1) - Au_2 - bG(t, u_2), u_1 - u_2 \rangle$$

$$= \int_{\mathbb{R}} A(u_1 - u_2)(u_1 - u_2) \, dx + b \int_{\mathbb{R}} (G(t, u_1) - G(t, u_2))(u_1 - u_2) \, dx$$

$$= -\nu \int_{\mathbb{R}} (u_1 - u_2)^2 \, dx + b \int_{\mathbb{R}} (G(t, u_1) - G(t, u_2))(u_1 - u_2) \, dx$$

$$\leq b\eta_1 \|u_1 - u_2\|_{H}^2$$

using (B1) and similarly the coercivity condition

$$\langle Au + bG(t, u), u \rangle \leq -\nu \|u\|_{V}^2 + (\nu + b\eta_1) \|u\|_{H}^2$$
since \( f(s)s = (f(s) - f(0))(s - 0) \leq \eta_1 s^2 \) for all \( s \in \mathbb{R} \) using (B1).

Theorem 1.1 in [8] now implies for all initial conditions \( u_0 \in H \) and all finite times \( T \) existence and uniqueness of a variational solution \( u \in L^\infty([0,T]; H) \cap L^2([0,T]; V) \) satisfying the integral equation

\[
(9) \quad u(t) = u_0 + \int_0^t (Au(s) + b(f(u(s) + \hat{v}(s)) - f(\hat{v}(s)))) \, ds
\]

and we may extend the solution to the whole time axes \( \mathbb{R}_+ \).

The integral on the right hand side of (9) is well-defined as a Bochner integral in \( L^2([0,T]; V') \) using (5) which implies in particular that the mapping \( t \mapsto u(t), \mathbb{R}_+ \rightarrow V' \), is differentiable with differential

\[
(10) \quad \frac{du}{dt} = Au(t) + b(f(u(t) + \hat{v}(t)) - f(\hat{v}(t))) \in V',
\]

hence continuous.

We are now ready to state precisely our notion of stability we are going to prove in the following.

**Definition 2.1.** The travelling wave solution \( \hat{v} \) is called locally asymptotically stable w.r.t. the \( H \)-norm if there exists \( \delta > 0 \) such that for initial condition \( v_0 \) with \( v_0 - \hat{v} \in H \) and \( \|v_0 - \hat{v}\|_H \leq \delta \) the unique variational solution \( u(t, x) = v(t, x) - \hat{v}(x) \) of (3) satisfies

\[
\lim_{t \to \infty} \|v_0 - \hat{v}(\cdot + x_0)\|_H = 0
\]

for some (phase) \( x_0 \in \mathbb{R} \).

In order to apply Theorem 1.5 we need to control the tangential component \( \langle v(t) - \hat{v}(\cdot + x_0), v_x(\cdot + x_0) \rangle \) of the given solution \( v(t) = u(t) + \hat{v}(t) \) w.r.t. the appropriate phase-shift \( x_0 \), i.e., the phase-shift \( x_0 \) that minimizes the \( L^2 \)-distance between the solution \( v(t) \) and the orbit consisting of all phase-shifted travelling waves \( \hat{v}(\cdot + x_0) \). This can be achieved asymptotically by introducing dynamically by by introducing the following ordinary differential equation

\[
(11) \quad \dot{C}(t) = c + m \langle v(t) - \hat{v}(\cdot + C(t)), v_x(\cdot + C(t)) \rangle,
\]

\[
C(0) = 0
\]

for \( m \geq 0 \). To simplify notations, let

\[
\tilde{v}(t) := \hat{v}(\cdot + C(t))
\]

so that we can rewrite equation (11) as

\[
(12) \quad \dot{C}(t) = c + m \langle v(t) - \tilde{v}(t), \tilde{v}_x(t) \rangle,
\]

\[
C(0) = 0.
\]

The next Proposition first shows that (11) is well-posed.
Proposition 2.2. Let \( v = u + \hat{v}(t) \) be a solution of (10) with \( u \in L^\infty([0, T], H) \cap L^2([0, T]; V) \). Then
\[
B(t, C) = \langle v(t) - \hat{v}(\cdot + C), \hat{v}_x(\cdot + C) \rangle_H
\]
is continuous in \( (t, C) \in [0, T] \times \mathbb{R} \) and Lipschitz continuous w.r.t. \( C \) with Lipschitz constant independent of \( t \).

Proof. First note that
\[
B(t, C_1) - B(t, C_2) = \langle \hat{v}_x(\cdot + C_1) - \hat{v}_x(\cdot + C_2), u(t) \rangle_H
- \langle \hat{v}_x(\cdot + C_1), \hat{v}(\cdot + C_1) - \hat{v}(\cdot) \rangle_H
+ \langle \hat{v}_x(\cdot + C_2), \hat{v}(\cdot + C_2) - \hat{v}(\cdot) \rangle_H
\]
Using
\[
\hat{v}_x(x + C_1) - \hat{v}_x(x + C_2) = \int_{C_1}^{C_2} \hat{v}_{xx}(x + y) \, dy
\leq \int_{C_1}^{C_2} |\hat{v}_{xx}|(x + y) \, dy
\]
we conclude that the first term on the right hand side can be estimated from above by
\[
\|\hat{v}_x(\cdot + C_1) - \hat{v}_x(\cdot + C_2)\|_H \|u(t)\|_H
\leq \left( |C_1 - C_2| \int_{\mathbb{R}} \int_{C_1}^{C_2} \hat{v}_{xx}^2(x + y) \, dy \, dx \right)^{\frac{1}{2}} \|u(t)\|_H
= |C_1 - C_2| \|\hat{v}_{xx}\|_H^2 \|u(t)\|_H
\]
which implies that this term is Lipschitz continuous with Lipschitz constant independent of \( t \in [0, T] \).

The second and the third term can be rewritten as follows:
\[
\left| \langle \hat{v}_x(\cdot + C_1), \hat{v}(\cdot + C_1) - \hat{v}(\cdot) \rangle_H \right|
- \left| \langle \hat{v}_x(\cdot + C_2), \hat{v}(\cdot + C_2) - \hat{v}(\cdot) \rangle_H \right|
= \left| \langle \hat{v}_x, \hat{v}(\cdot - C_2) - \hat{v}(\cdot - C_1) \rangle_H \right|
\leq \|\hat{v}_x\|_H \left( \int_{\mathbb{R}} \left( \int_{C_1}^{C_2} \hat{v}_x(x + y) \, dy \right)^2 \, dx \right)^{\frac{1}{2}}
\leq \|\hat{v}_x\|_H |C_1 - C_2| \|\hat{v}_x\|_H
\]
so that also these two terms are Lipschitz continuous with Lipschitz constant independent of \( t \). \qed

In the following let
\[
\tilde{u}(t) := u(t) + \hat{v}(t) - \hat{v}(t) = v(t) - \hat{v}(t).
\]
Proposition 2.3. Let \( u = v - \hat{v}(t) \in L^\infty ([0, T]; H) \cap L^2 ([0, T]; V) \) be a solution of (10) and \( \tilde{u} \) be given by (13). Then \( \tilde{u} \in L^\infty ([0, T]; H) \cap L^2 ([0, T]; V) \) again and \( \tilde{u} \) satisfies the evolution equation

\[
\frac{d\tilde{u}}{dt}(t) = \nu \Delta \tilde{u}(t) + b \tilde{G}(t, \tilde{u}(t)) - (\tilde{C}(t) - c) \tilde{v}_x(t)
\]

with

\[
\tilde{G}(t, u) = f(u + \hat{v}(t)) - f(\hat{v}(t)) , \\
\tilde{\tilde{R}}(t, u) = \tilde{G}(t, u) - f'(\hat{v}(t)) u .
\]

The proof of the Proposition is an immediate consequence of (10) and (11) (resp. (12)). (B3) implies for the remainder \( \tilde{\tilde{R}} \) the following estimate

\[
\langle \tilde{\tilde{R}}(t, u), u \rangle \leq \eta_2 \int (1 + |u|)|u|^3 \, dx \leq \eta_2 \left( \|u\|_\infty + \|u\|_H^2 \right) \|u\|_V^2
\]

We are now ready to state our main result in the deterministic case:

Theorem 2.4. Recall the definition of \( \kappa_* \) and \( C_* \) in Theorem 1.2. Let \( m \geq C_* \). If the initial condition \( v_0 = u_0 + \hat{v} \) is close to \( \hat{v} \) in the sense that

\[
\|v_0\|_H < \left( \delta \frac{\kappa_*}{2b\eta_2} \right) \land 1
\]

for some \( \delta < 1 \) and \( v(t) = u(t) + \hat{v}(t) \), where \( u(t) \) is the unique solution of (10), then

\[
\|v(t) - \hat{v} (\cdot + C(t))\|_H \leq e^{-(1-\delta)\kappa_* t} \|v_0 - \hat{v}\|_H .
\]

Proof. Let \( \tilde{u}(t) := v(t) - \hat{v}(t) \) be as in (13). Then Proposition 2.3 and equation (15) imply that

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_H^2 = \langle \nu \Delta \tilde{u}(t) + b f'(\hat{v}(t)) \tilde{u}(t), \tilde{u}(t) \rangle + b \langle \tilde{\tilde{R}}(t, \tilde{u}(t)), \tilde{u}(t) \rangle
\]

\[
- m \langle \tilde{v}_x(t), \tilde{u}(t) \rangle^2
\]

\[
\leq \langle \nu \Delta \tilde{u}(t) + b f'(\hat{v}(t)) \tilde{u}(t), \tilde{u}(t) \rangle + b \eta_2 \left( \|\tilde{u}(t)\|_H + \|\tilde{u}(t)\|_H^2 \right) \|\tilde{u}(t)\|_V^2 - m \langle \tilde{v}_x(t), \tilde{u}(t) \rangle^2 .
\]

Using translation invariance of \( \nu \Delta \) and \( \int u_x^2 \, dx \), Theorem 1.5 yields the estimate

\[
\langle \nu \Delta \tilde{u}(t) + b f'(\hat{v}(t)) \tilde{u}(t), \tilde{u}(t) \rangle \leq -\kappa_* \|\tilde{u}(t)\|_V^2 + C_* \langle \tilde{u}(t), \tilde{v}_x(t) \rangle^2 .
\]

Inserting (17) into (16) yields that

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_H^2 \leq -\kappa_* \|\tilde{u}(t)\|_V^2 + b\eta_2 \left( \|\tilde{u}(t)\|_H + \|\tilde{u}(t)\|_H^2 \right) \|\tilde{u}(t)\|_V^2 .
\]
In the next step we define the stopping time
\[ T := \inf \left\{ t \geq 0 \mid \| \hat{u}(t) \|_H \geq \left( \frac{\kappa_*}{2b_1} \right) \wedge 1 \right\} \]
with the usual convention \( \inf \emptyset = \infty \). Continuity of \( t \mapsto \| \hat{u}(t) \|_H \)
implies that \( T > 0 \) since \( \| u_0 \|_H < \left( \frac{\delta \kappa_*}{2b_1} \right) \wedge 1 \). For \( t < T \) note that
\[
\frac{1}{2} \frac{d}{dt} \| \hat{u}(t) \|_H^2 \leq -(1 - \delta)\kappa_* \| \hat{u}(t) \|_V \leq -(1 - \delta)\kappa_* \| \hat{u}(t) \|_H^2
\]
which implies that
\[
\| \hat{u}(t) \|_H^2 \leq e^{-2(1-\delta)\kappa_* t} \| u_0 \|_H^2
\]
for \( t < T \). Suppose now that \( T < \infty \). Then continuity of \( t \mapsto \| \hat{u}(t) \|_H \)
implies on the one hand that \( \| \hat{u}(T) \|_H = \left( \frac{\delta \kappa_*}{2b_1} \right) \wedge 1 \) and on the other
hand, using the last inequality,
\[
\| \hat{u}(T) \|_H = \lim_{t \uparrow T} \| \hat{u}(t) \|_H \leq e^{-(1-\delta)\kappa_* T} \| u_0 \|_H < \left( \frac{\delta \kappa_*}{2b_1} \right) \wedge 1
\]
which is a contradiction. Consequently, \( T = \infty \) and thus
\[
\| \hat{u}(t) \|_H \leq e^{-(1-\delta)\kappa_* t} \| u_0 \|_H \quad \forall t \geq 0
\]
which implies the assertion. \( \square \)

3. THE REACTION-DIFFUSION EQUATION WITH NOISE

In this section we will generalize the stability result for the reaction-diffusion equation (11) to the stochastic case. To this end we consider the following equation
\[
\frac{dv(t)}{dt} = \left[ \nu \partial_{xx}^2 v(t) + bf(v(t)) \right] dt + \Sigma_0(v(t)) dW(t)
\]
where \( W = (W(t))_{t \geq 0} \) is a cylindrical Wiener process with values in some separable real Hilbert space \( U \) defined on some underlying filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \geq 0}, P) \) and
\[
\Sigma_0 : \hat{v} + H \mapsto L_2(U, H)
\]
is a measurable map with values in the linear space of all Hilbert-Schmidt operators from \( U \) to \( H \) such that there exists some constant \( L_{\Sigma_0} \) with
\[
\| \Sigma_0(\hat{v} + u_1) - \Sigma_0(\hat{v} + u_2) \|_{L_2(U, H)} \leq L_{\Sigma_0} \| u_1 - u_2 \|_H \quad \forall u_1, u_2 \in H.
\]
For the theory of cylindrical Wiener processes see [11]. To simplify presentation of the results we also assume the following translation invariance
\[
\| \Sigma_0(\hat{v}(-C)) \|_{L_2(U, H)} = \| \Sigma_0(\hat{v} + (\hat{v}(\cdot - C) - \hat{v})) \|_{L_2(U, H)} \forall C \in \mathbb{R}.
\]

A typical example covered by the assumptions is
\[
dv(t) = \left[ \nu \partial_{xx}^2 v(t) + bf(v(t)) \right] dt + \sigma(v(t)) dW^Q(t)
\]
where \( \sigma : \mathbb{R} \to \mathbb{R} \) is Lipschitz, \( \sigma(0) = \sigma(1) = 0 \), \( W^Q \) is a \( Q \)-Wiener process with covariance operator \( Q \) for which its square-root \( \sqrt{Q} \) admits a kernel \( k\sqrt{Q}(x, y) \in L^2(\mathbb{R}^2) \) satisfying

\[
\sup_{x \in \mathbb{R}} \int k^2 \sqrt{Q}(x, y) \, dy < \infty
\]

(see [12]).

Similar to the deterministic case we can give the equation a rigorous formulation as a stochastic evolution equation with values in the Hilbert space \( H = L^2(\mathbb{R}) \) by decomposing \( v(t) = u(t) + \hat{v}(t) \) w.r.t. the travelling wave to obtain the following stochastic evolution equation

\[
(21) \quad du(t) = [\nu \Delta u(t) + bG(t, u(t))] \, dt + \Sigma(t, u(t)) \, dW(t)
\]

where the nonlinear term \( G \) is as in (11) and

\[
(22) \quad \Sigma(t, u) h := \Sigma_0 (\hat{v}(t) + u) h, \quad u \in H, h \in U,
\]

is a continuous mapping

\[
\Sigma(\cdot, \cdot) : [0, \infty) \times H \to L_2(U, H).
\]

The assumptions \((19)\) and \((20)\) on the dispersion operator imply

\[
(23) \quad \| \Sigma(t, u_1) - \Sigma(t, u_2) \|_{L_2(U, H)} \leq L_{\Sigma_0} \| u_1 - u_2 \|_H
\]

and

\[
(24) \quad \| \Sigma(t, u) \|_{L_2(U, H)} \leq \| \Sigma_0(\hat{v}) \|_{L_2(U, H)} + L_{\Sigma_0} \| u \|_H.
\]

We now consider the equation \((18)\) w.r.t. the same triple \( V \hookrightarrow H \equiv H' \hookrightarrow V' \) as in the deterministic case. Due to the properties \((5), (6), (7)\) and \((8)\), we can deduce from Theorem 1.1. in [8] for all finite \( T \) and all (deterministic) initial conditions \( u_0 \in H \) the existence and uniqueness of a solution \((u(t))_{t \in [0, T]}\) of \((18)\) satisfying the moment estimate

\[
E \left( \sup_{t \in [0, T]} \| u(t) \|_{H}^2 + \int_0^T \| u(t) \|_{V}^2 \, dt \right) < \infty.
\]

In particular, for any \( m \in \mathbb{R} \), we can apply Proposition 2.2 to a typical trajectory \( u(\cdot)(\omega) \) to obtain a unique solution \( C(\cdot)(\omega) \) of the ordinary differential equation (12). It is also clear that the resulting stochastic process \((C(t))_{t \geq 0}\) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted, since \((u(t))_{t \geq 0}\) is.

In the next step let us consider the stochastic process

\[
\tilde{u}(t) = u(t) + \hat{v}(t) - \hat{v}(\cdot + C(t)) = v(t) - \hat{v}(t)
\]

which is \((\mathcal{F}_t)_{t \geq 0}\) adapted too and satisfies the stochastic evolution equation

\[
d\tilde{u}(t) = \left[ \nu \Delta \tilde{u}(t) + b\tilde{G}(t, \tilde{u}(t)) - \hat{C}(t) - c \right] \, dt + \hat{\Sigma}(t, \tilde{u}(t)) \, dW(t),
\]

where the nonlinear term \( \tilde{G} \) is as in (4) and

\[
\tau(\cdot, \cdot) : [0, \infty) \times H \to L_2(U, H).
\]
where
\[ \tilde{G}(t, u) = f(u + \tilde{v}(t)) - f(\tilde{v}(t)), \tilde{\Sigma}(t, u) = \Sigma(t, u + \tilde{v}(t)), \]
and the moment estimates
\[ E \left( \sup_{t \in [0, T]} \| \tilde{u}(t) \|^2_H + \int_0^T \| \tilde{u}(t) \|^2_V dt \right) < \infty. \]

Theorem 4.2.5 in [11] now implies that the real-valued stochastic process \( \| \tilde{u} \|^2_H(t) \) is a continuous local semimartingale so that we have in particular the following time-dependent Ito-formula
\[
\varphi(t, \| \tilde{u}(t) \|^2_H) = \int_0^t \varphi_x(s, \| \tilde{u}(s) \|^2_H) + 2 \varphi_x(s, \| \tilde{u}(s) \|^2_H) \langle \nu \Delta \tilde{u}(s) \\
+ b \tilde{G}(s, \tilde{u}(s)) - \dot{\tilde{C}}(s) \tilde{w}(s), \tilde{u}(s) \rangle \\
+ \varphi_x(s, \| \tilde{u}(s) \|^2_H) \| \tilde{\Sigma}(s, \tilde{u}(s)) \|_{L_2(H)} \| \tilde{\Sigma}(s, \tilde{u}(s)) \|_{L_2(H)} ds \\
+ \int_0^t \varphi_x(s, \| \tilde{u}(s) \|^2_H) d\tilde{M}_s
\]
for any \( \varphi \in C^{1,2}([0, T] \times \mathbb{R}_+) \). Here, \( \tilde{\Sigma}(s, u) \) denotes the adjoint operator of \( \Sigma(s, u) \).

**Theorem 3.1.** Recall the definition of \( \kappa_* \) and \( C_* \) in Theorem [13] and assume that \( L^2_{\Sigma_0} \leq \frac{\kappa_*}{4} \). Let \( v_0 = u_0 + \tilde{v} \) and \( v(t) = u(t) + \tilde{v}(t) \), where \( u(t) \) is the unique solution of the stochastic evolution equation \( \langle 18 \rangle \) and \( \tilde{u}(t) = u(t) + \tilde{v}(t) - \tilde{v}(t) \). Then
\[
P(T < \infty) \leq \frac{1}{C_*^2} \left( \| \tilde{u}(0) \|^2_H + \frac{4}{\kappa_*} \| \Sigma_0(\tilde{v}) \|^2_{L_2(U, H)} \right)
\]
where \( T \) denotes the first exit time
\[
(26) \quad T := \inf \{ t \geq 0 \mid \| \tilde{u}(t) \|^2_H > c_* \}, \quad c_* = \left( \frac{\kappa_*}{4b\eta_2} \right) \wedge 1,
\]
with the usual convention \( \inf \emptyset = \infty \).

**Proof.** Similar to the proof of Theorem [13] we have the following inequality
\[ \langle \nu \Delta \tilde{u}(t) + b \tilde{G}(t, \tilde{u}(t)) - (\dot{\tilde{C}}(t) - c) \tilde{v}_x(t), \tilde{u}(t) \rangle \]
\[ \leq -\kappa_* \| \tilde{u}(t) \|^2_V + b \eta_2 \left( \| \tilde{u}(t) \|^2_H + \| \tilde{u}(t) \|^2_H \right) \| \tilde{u}(t) \|^2_V. \]
In particular,
\[ \langle \nu \Delta \tilde{u}(t) + b \tilde{G}(t, \tilde{u}(t)) - (\dot{\tilde{C}}(t) - c) \tilde{v}_x(t), \tilde{u}(t) \rangle \leq -\frac{\kappa_*}{2} \| \tilde{u}(t) \|^2_V \]
for \( t \leq T \), where \( T \) is as in \( (26) \). [24] and [20] imply
\[ \| \tilde{\Sigma}(\tilde{u}(t)) \|^2_{L_2(H)} \leq 2 \left( L^2_{\Sigma_0} \| \tilde{u}(t) \|^2_H + \| \Sigma_0(\tilde{v}) \|^2_{L_2(U, H)} \right) \]
4.1 Proof of Lemma 1.1 and Proposition 1.2. 

Proof. (of Lemma 1.1) Let \( \hat{v} \) be the solution of (1.2). Then, for all \( x \) in \( \mathbb{R}^d \), we have
\[
\frac{d}{dt} \hat{v}(t) = -\Delta \hat{v}(t) + \hat{b}(t) \hat{v}(t) - \hat{c}(t) \hat{\varphi}(t, \hat{v}(t)) = -\Delta \hat{v}(t) + \hat{b}(t) \hat{v}(t) - \hat{c}(t) \hat{\varphi}(t, \hat{v}(t)),
\]
and therefore
\[
2\nu \Delta \hat{v}(t) + b \hat{c}(t, \hat{v}(t)) - \hat{c}(t) \hat{\varphi}(t, \hat{v}(t), \hat{v}(t)) + \| \hat{\Sigma}(t, \hat{v}(t)) \|^2_{L^2(H)} \leq -\frac{\kappa_\nu}{2} \| \hat{v}(t) \|^2_V + 2 \| \Sigma_0(\hat{v}) \|^2_{L^2(U,H)}. 
\]

Applying Ito’s formula \( \text{(25)} \) to \( e^{\frac{2\nu}{\kappa_\nu} t}x \), then yields for \( t < T \) that
\[
e^{\frac{2\nu}{\kappa_\nu} t} \| \hat{v}(t) \|^2_H \leq \| \hat{v}(0) \|^2_H + \frac{4}{\kappa_\nu} \left( e^{\frac{2\nu}{\kappa_\nu} t} - 1 \right) \| \Sigma_0(\hat{v}) \|^2_{L^2(U,H)} + \int_0^t e^{\frac{2\nu}{\kappa_\nu} s} d\tilde{M}_s.
\]
Taking expectations we obtain
\[
E \left( \| \hat{v}(t \wedge T) \|^2_H \right) \leq \| \hat{v}(0) \|^2_H + \frac{4}{\kappa_\nu} \| \Sigma_0(\hat{v}) \|^2_{L^2(U,H)}
\]
and thus in the limit \( t \uparrow \infty \)
\[
c^2 T < \infty = E \left( \| \hat{v}(T) \|^2_H \right) \leq \lim_{t \to \infty} E \left( \| \hat{v}(t \wedge T) \|^2_H \right) \leq \| \hat{v}(0) \|^2_H + \frac{4}{\kappa_\nu} \| \Sigma_0(\hat{v}) \|^2_{L^2(U,H)}
\]
which implies the assertion. 

4. PROOF OF LEMMA 1.1, PROPOSITION 1.2 AND THEOREM 1.3

4.1 Proof of Lemma 1.1 and Proposition 1.2.

Proof. (of Lemma 1.1) For the proof of (i) note that \( \hat{v}_x \geq 0 \) and \( \int_{-\infty}^\infty \hat{v}_x dx = \lim_{x \to \infty} \hat{v}(x) - \hat{v}(-x) = 1 \). In particular, \( \hat{v}_x \in L^1(\mathbb{R}) \) which implies that \( \lim_{n \to \infty} \hat{v}_x(x_n) = 0 \) for some sequence \( x_n \uparrow \infty \). It follows for all \( x \) that
\[
\hat{v}_x^2(x) = \hat{v}_x^2(x_n) - 2 \int_x^{x_n} \hat{v}_x \hat{v}_x dx
\]
\[
= \hat{v}_x^2(x_n) - 2 \nu \int_x^{x_n} \hat{v}_x^2 dx + 2 \nu \int_x^{x_n} f(\hat{v}) \hat{v}_x dx
\]
\[
\leq \hat{v}_x^2(x_n) + 2 \nu \int_{\hat{v}(x)}^{\hat{v}(x_n)} f(v) dv \quad \forall n.
\]
Consequently,
\[
\hat{v}_x^2(x) \leq \lim_{n \to \infty} \hat{v}_x^2(x_n) + 2 \nu \int_{\hat{v}(x)}^1 f(v) dv = \frac{2b}{\nu} \int_{\hat{v}(x)}^1 f(v) dv.
\]
In particular,
\[
\lim_{x \to \infty} \hat{v}_x^2(x) \leq \limsup_{x \to \infty} \frac{2b}{\nu} \int_{\hat{v}(x)}^1 f(v) dv = 0
\]
and thus also \( \lim_{x \to \infty} e^{-\alpha \frac{2\nu}{\kappa_\nu} \hat{v}_x^2(x)} = 0 \) for all \( \alpha \geq 0 \).
For the proof of (ii) note that for all $\alpha \in \mathbb{R}$

$$\frac{d}{dx}(e^{-\alpha \hat{\nu} x} \hat{\nu}_x^2) = \left(-\alpha \frac{c}{\nu} \hat{\nu}_x + 2 \hat{\nu}_{xx}\right) e^{-\alpha \hat{\nu} x} \hat{\nu}_x$$

(27)

$$= (2 - \alpha) \frac{c}{\nu} e^{-\alpha \hat{\nu} x} \hat{\nu}_x^2 - \frac{b}{\nu} e^{-\alpha \hat{\nu} x} f(\hat{\nu}) \hat{\nu}_x.$$  

Taking $\alpha = 2$ we conclude in particular that $\frac{d}{dx}(e^{-\alpha \hat{\nu} x} \hat{\nu}_x^2) \geq 0$ (resp. $\leq 0$) for $x \leq v^{-1}(a)$ (resp. $x \geq v^{-1}(a)$), since $\nu_x \geq 0$ and $f(\hat{\nu}(x)) \leq 0$ (resp. $\geq 0$) for $x \leq v^{-1}(a)$ (resp. $x \geq v^{-1}(a)$). Consequently, for $c \geq 0$,

$$\lim_{x \to -\infty} e^{-2\hat{\nu} x} \hat{\nu}_x^2(x) \leq \lim_{x \to -\infty} \sup e^{(2-\alpha)\hat{\nu} x} \gamma = 0.$$  

and thus for $\alpha < 2$

$$\lim_{x \to -\infty} e^{-\alpha \hat{\nu} x} \hat{\nu}_x^2(x) \leq \lim_{x \to -\infty} \sup e^{(2-\alpha)\hat{\nu} x} \gamma = 0.$$  

Similarly in the case $c \leq 0$

$$\lim_{x \to \infty} e^{-2\hat{\nu} x} \hat{\nu}_x^2(x) = \inf_{x \geq v^{-1}(a)} e^{-2\hat{\nu} x} \hat{\nu}_x^2(x) =: \gamma < \infty$$

and thus for $\alpha < 2$

$$\lim_{x \to \infty} e^{-\alpha \hat{\nu} x} \hat{\nu}_x^2(x) \leq \lim_{x \to \infty} \sup e^{(2-\alpha)\hat{\nu} x} \gamma = 0.$$  

Combining with (i) we obtain the assertion. \hfill \Box

Let us now turn to the proof of Proposition 1.2. Let $x_\ast = \hat{\nu}^{-1}(v_\ast)$ and $w(x) := e^{-\hat{\nu} x} \hat{\nu}_x(x)$. Then

$$w_{xx} = \left(\left(\frac{c}{2\nu}\right)^2 - \frac{b}{\nu} f'(\hat{\nu})\right) w,$$

since differentiating $c\hat{\nu}_x = \nu \hat{\nu}_{xx} + bf(\hat{\nu})$ implies $c\hat{\nu}_{xx} = \nu \hat{\nu}_{xxx} + bf'(\hat{\nu})\hat{\nu}_x$.

**Proof of Proposition 1.2 (i)** Note that

$$\frac{d}{dx} \left(w_x^2 + \left(\frac{b}{\nu} f'(\hat{\nu}) - \left(\frac{c}{2\nu}\right)^2\right) w^2\right) = \frac{b}{\nu} f''(\hat{\nu}) \hat{\nu}_x w^2$$

is strictly increasing (resp. decreasing) for $x < x_\ast$ (resp. $x > x_\ast$). According to Lemma 1.1

$$\lim_{|x| \to \infty} \left(w_x^2 + \left(\frac{b}{\nu} f'(\hat{\nu}) - \left(\frac{c}{2\nu}\right)^2\right) w^2\right) = 0$$

so that

$$w_x^2 + \left(\frac{b}{\nu} f'(\hat{\nu}) - \left(\frac{c}{2\nu}\right)^2\right) w^2 \geq 0 \quad \forall x.$$  

Using $w_x = \left(\frac{c}{2\nu} - \frac{b f(\hat{\nu})}{\nu \hat{\nu}_x}\right) w$, we conclude that

$$\left(\frac{c}{2\nu} - \frac{b f(\hat{\nu})}{\nu \hat{\nu}_x}\right)^2 + \frac{b}{\nu} f'(\hat{\nu}) - \left(\frac{c}{2\nu}\right)^2 > 0.$$
or equivalently

\[
(28) \quad \frac{b}{\nu} f'(\hat{v}) - \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \left( \frac{c}{\nu} - \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \right) > 0.
\]

In particular,

\[
\frac{b}{\nu} \frac{d f(\hat{v})}{dx} \frac{\hat{v}_x}{\hat{v}} = \frac{b}{\nu} \frac{f'(\hat{v})}{\hat{v}_x} - \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \hat{v}_x > 0
\]

so that \( \frac{f(h)}{v_x} \) is strictly increasing which implies that \( \hat{v}_x \) is log-concave, because

\[
-d^2 \frac{dx^2}{d \log \hat{v}_x} = -d \frac{\hat{v}_x}{d \hat{v}_x} = -d \left( \frac{c}{\nu} - \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \right) > 0.
\]

For the proof of part (ii) of Proposition \([1,2]\) we will first need the following

**Lemma 4.1.** Let \( K_+ := \frac{1 - \hat{v}(x)}{v_x(x)} \) and \( K_- := \frac{\hat{v}(x)}{v_x(x)} \). Then

- (i) \( \frac{1 - \hat{v}(x)}{v_x(x)} \leq K_+ \) for \( x \geq x_0 \),
- (ii) \( \frac{\hat{v}(x)}{v_x(x)} \leq K_- \) for \( x \leq x_0 \).

**Proof.** (i) Consider the function \( h := \frac{1 - \hat{v}}{v_x} \). Clearly, \( \hat{h} = -1 - \frac{\hat{v}_x}{v_x}h \) is negative, hence \( h \) decreasing, in a neighborhood of \( x_0 \). Since \( \hat{v}_x \) is log-concave it follows that \( -\frac{\hat{v}_x}{v_x} \) is increasing on \([x_0, \infty)\). We may assume in the following that there exists some \( x_+ > x_0 \) with

\[
\frac{\hat{v}_x}{v_x}(x_+) = \frac{\hat{v}_x}{v_x}(x_+).
\]

In fact, if this is not the case, then \( \hat{h} \leq 0 \) for all \( x \geq x_0 \), hence \( h \) decreasing on \([x_0, \infty)\) which already implies the assertion.

So let us assume that \( h \) is decreasing on \([x_0, x_+]\) only. In particular, \( \frac{1 - \hat{v}(x)}{v_x(x_+)} \leq K_+ \). For \( x \geq x_+ \) it follows that \( -\frac{\hat{v}_x}{v_x}(x) \leq -\frac{\hat{v}_x}{v_x}(x_+) = \frac{\hat{v}_x}{v_x}(x_+)\), hence \( d\frac{d}{dx} (e^{-\frac{\hat{v}_x}{v_x}(x_+)} \hat{v}_x) \leq 0 \), and consequently,

\[
1 - \hat{v}(x) = \int_x^\infty \hat{v}_x(s) ds = \int_x^\infty e^{\frac{\hat{v}_x(s)}{v_x(x_+)}} \left( e^{-\frac{\hat{v}_x(s)}{v_x(x_+)}} \hat{v}_x(s) \right) ds \\
\leq \int_x^\infty e^{\frac{\hat{v}_x(s)}{v_x(x_+)}} ds \left( e^{-\frac{\hat{v}_x(s)}{v_x(x_+)} \hat{v}_x(s)} \right) \\
= \frac{\hat{v}}{1 - \hat{v}_x} (x+) \hat{v}_x(x) \leq K_+ \hat{v}_x(x).
\]

(ii) is shown similar. \( \square \)
Proof of Proposition 1.2 (ii) Since \( f(0) = 0 \) it follows that \( \lim_{v \to 0} \frac{|f(v)|}{v} < \infty \) and thus
\[
\limsup_{x \to -\infty} \frac{|f(\hat{v})|}{\hat{v}_x}(x) = \limsup_{x \to -\infty} \frac{|f(\hat{v})|}{\hat{v}} \frac{\hat{v}_x}{\hat{v}}(x) < \infty
\]
due to the previous Lemma 4.1. Similarly, \( f(1) = 0 \) implies that \( \lim_{v \to 1} \frac{f(v)}{1-v} < \infty \) and thus
\[
\limsup_{x \to \infty} \frac{f(\hat{v})}{\hat{v}_x}(x) = \limsup_{x \to \infty} \frac{f(\hat{v})}{1-\hat{v}} \frac{1-\hat{v}_x}{\hat{v}_x}(x) < \infty.
\]
To compute \( \gamma_- \) note that \( \frac{b f(\hat{v})}{\nu \hat{v}_x} \) is increasing in \( x \), hence \( \gamma_- = \lim_{x \to -\infty} \frac{b f(\hat{v})}{\nu \hat{v}_x}(x) = \inf_{x \in \mathbb{R}} \frac{b f(\hat{v})}{\nu \hat{v}_x}(x) \) exists, must be strictly negative and is finite. Applying l’Hospital’s rule we obtain that
\[
\gamma_- = \lim_{x \to -\infty} \frac{b f(\hat{v})}{\nu \hat{v}_x}(x) = \lim_{x \to -\infty} \frac{b f'(\hat{v})}{\nu} \frac{\hat{v}_x}{\hat{v}}(x) = \frac{b f'(0)}{\nu} \frac{1}{\nu - \gamma_-}
\]
or equivalently, \( \gamma_- (\frac{c}{\nu} - \gamma_-) = \frac{b}{\nu} f'(0) \). Since \( \gamma_- < 0 \) we obtain the assertion. \( \gamma_+ \) can be computed similarly.

Proof of Proposition 1.2 (iii) The previous part implies for the logarithmic derivative of \( \hat{v}_x \) that
\[
\lim_{x \to -\infty} \frac{\hat{v}_{xx}}{\hat{v}_x} = \frac{c}{\nu} - \gamma_- > \frac{c}{\nu}
\]
and
\[
\lim_{x \to \infty} \frac{\hat{v}_{xx}}{\hat{v}_x} = \frac{c}{\nu} - \gamma_+ < \frac{c}{\nu}
\]
so that for every \( \alpha \) satisfying \( \frac{c}{\nu} < \frac{c}{\nu} - \gamma_- \) (resp. \( \frac{c}{\nu} > \frac{c}{\nu} - \gamma_+ \)) it follows that \( e^{-\alpha \hat{v}_x \hat{v}_x^2} \) is increasing for small \( x \) (resp. decreasing for large \( x \)). Hence \( \int_{-\infty}^{0} e^{-\alpha \hat{v}_x \hat{v}_x^2} \, dx < \infty \) (resp. \( \int_{0}^{\infty} e^{-\alpha \hat{v}_x \hat{v}_x^2} \, dx < \infty \)) in both cases.

We can also now estimate
\[
\int_{-\infty}^{0} e^{-\alpha \hat{v}_x \hat{v}_x^2} \, dx \leq \sup_{x \in \mathbb{R}} \frac{\hat{v}_{xx}}{\hat{v}_x} \int_{-\infty}^{0} e^{-\alpha \hat{v}_x \hat{v}_x^2} \, dx < \infty
\]
for \( \frac{c}{\nu} < \frac{c}{\nu} - \gamma_- \) and
\[
\int_{0}^{\infty} e^{-\alpha \hat{v}_x \hat{v}_x^2} \, dx \leq \sup_{x \in \mathbb{R}} \frac{\hat{v}_{xx}}{\hat{v}_x} \int_{0}^{\infty} e^{-\alpha \hat{v}_x \hat{v}_x^2} \, dx < \infty
\]
for \( \frac{c}{\nu} > \frac{c}{\nu} - \gamma_+ \), since
\[
\sup_{x \in \mathbb{R}} \frac{\hat{v}_{xx}}{\hat{v}_x} \leq \frac{|c|}{\nu} + \sup_{x \in \mathbb{R}} \frac{|f(\hat{v})|}{\hat{v}_x} < \infty
\]
again due to the previous part (ii).
4.2. Proof of Theorem 1.3. Inequality (2) is equivalent to

\begin{equation}
\frac{b}{\nu} f'(\hat{v}) + 2 \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \left( \frac{b f(\hat{v})}{\nu} - \frac{c}{\nu} \right) \geq \kappa.
\end{equation}

Since

\begin{equation*}
\frac{b}{\nu} f'(\hat{v}) + 2 \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \left( \frac{b f(\hat{v})}{\nu} - \frac{c}{\nu} \right) > \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \left( \frac{b f(\hat{v})}{\nu} - \frac{c}{\nu} \right)
\end{equation*}

and \( \lim_{x \to \pm\infty} \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \left( \frac{b f(\hat{v})}{\nu} - \frac{c}{\nu} \right) > 0 \), it remains to prove that

\begin{equation}
g_2 := \frac{1}{2} f'(\hat{v})\hat{v}_x^2 - f(\hat{v})\hat{v}_{xx} > 0
\end{equation}

for \( x \) with \( 0 \leq \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x}(x) \leq \frac{c}{\nu} \) in order to be able to find \( \kappa > 0 \) satisfying (29). In the particular case \( c = 0 \) this is obvious.

We therefore assume from now on that \( c > 0 \). Since \( \frac{f(\hat{v})}{\hat{v}_x} \) is strictly increasing, it follows that for all \( \alpha \in ] \inf \frac{b f(\hat{v})}{\hat{v}_x}, \sup \frac{b f(\hat{v})}{\hat{v}_x} [ \) there exists a unique \( x_\alpha \in \mathbb{R} \) with

\[
\frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x}(x_\alpha) = \alpha \frac{c}{\nu}.
\]

In particular, \( \hat{v}(x_0) = a \) and \( \hat{v}(x_1) \) is the unique root of \( \hat{v}_{xx} \), that is, \( x_1 \) is the location of the maximum of \( \hat{v}_x \) and \( x_0 \leq x_1 \) and both, \( f(\hat{v}), f'(\hat{v}) \geq 0 \) on \([x_0, x_1]\).

We will subdivide the proof of (30) into the three cases \( x \in [x_0, x_{0,5} \land x_*] \), \( x \in [x_{0,5} \lor x_*, x_1] \) and \( x \in [x_{0,5} \land x_*, x_{0,5} \lor x_*] \).

Lemma 4.2. \( g_2(x) > 0 \) for \( x \in [x_0, x_{0,5} \land x_*] \).

Proof. We may suppose that \( x_* \geq x_0 \), because otherwise, the interval is empty. Let

\[
\bar{x} := \inf\{x \geq x_0 \mid g_2(x) = 0\}.
\]

We will show that \( \bar{x} > x_{0,5} \land x_* \). Since \( g_2(x_0) = \frac{1}{2} f'(a)\hat{v}_x^2(x_0) > 0 \) we certainly have that \( \bar{x} > x_0 \). Suppose now that \( \bar{x} \leq x_{0,5} \land x_* \). Then for all \( m \in \mathbb{N} \)

\[
\frac{d}{dx} f(\hat{v})\hat{v}_{xx}^{\hat{v}_x^m} = f'(\hat{v})\hat{v}_{xx}^{\hat{v}_x^{1+m}} + f(\hat{v}) \left( \hat{v}_{xxx} + m \frac{\hat{v}_x^2}{\hat{v}_x} \right) \hat{v}_x^m
\]

\[
= \frac{c}{\nu} f(\hat{v})\hat{v}_x^{\hat{v}_x^m} + f'(\hat{v}) \left( \frac{c}{\nu} - 2 \frac{b}{\nu} \frac{f(\hat{v})}{\hat{v}_x} \right) \hat{v}_x^{\hat{v}_x^{m+2}}
\]

\[
+ m f(\hat{v}) \hat{v}_x^{\hat{v}_x^{m-1}}
\]
which implies
\[ f(\hat{v}) \hat{v}_{xx} \hat{v}_x^m(x) = \int_{x_0}^x e^{\mathcal{F}(x-s)} f'(\hat{v}) \left( \frac{c}{\nu} - 2 \frac{b f(\hat{v})}{\nu \hat{v}_x} \right) \hat{v}_x^{m+2} ds \]
(31)
\[ + m \int_{x_0}^x e^{\mathcal{F}(x-s)} f(\hat{v}) \hat{v}_x^2 \hat{v}_x^{m-1} ds \]
\[ =: I + II , \text{ say.} \]
Now \( f^{(2)}(\hat{v}) \geq 0 \), hence \( f'(\hat{v}) \) increasing, and \( \frac{c}{\nu} - 2 \frac{b f(\hat{v})}{\nu \hat{v}_x} \geq 0 \) due to \( x \leq x_{0.5} \land x_* \), implies that
\[ I \leq f'(\hat{v})(\bar{x}) \int_{x_0}^x e^{\mathcal{F}(x-s)} \left( \frac{c}{\nu} - 2 \frac{b f(\hat{v})}{\nu \hat{v}_x} \right) \hat{v}_x^{m+2} ds \]
\[ = \frac{1}{2} f'(\hat{v})(\bar{x}) \left( \hat{v}_x^{m+2}(\bar{x}) - e^{\mathcal{F}(\bar{x}-x_0)} \hat{v}_x^{m+2}(x_0) - m \int_{x_0}^x e^{\mathcal{F}(x-s)} \hat{v}_{xx} \hat{v}_x^{m+1} ds \right) \]
\[ + \frac{1}{2} f'(\hat{v})(\bar{x}) \int_{x_0}^x e^{\mathcal{F}(x-s)} \left( \frac{c}{\nu} - \frac{b f(\hat{v})}{\nu \hat{v}_x} \right) \hat{v}_x^{m+2} ds \]
thereby using \( e^{\mathcal{F}(x-s)} \left( \frac{c}{\nu} \hat{v}_x - 2 \frac{b f(\hat{v})}{\nu \hat{v}_x} \right) \hat{v}_x = \frac{d}{ds} e^{\mathcal{F}(x-s)} \hat{v}_x^2 \). Inserting the last estimate into (31) and using \( g_2(s) \geq 0 \) for \( s \leq \bar{x} \), hence
\[ II \leq \frac{m}{2} \int_{x_0}^x e^{\mathcal{F}(x-s)} f'(\hat{v}) \hat{v}_{xx} \hat{v}_x^{m+1} ds , \]
we arrive at
\[ f(\hat{v}) \hat{v}_{xx} \hat{v}_x^m(\bar{x}) < \frac{1}{2} f'(\hat{v}) \hat{v}_x^{m+2}(\bar{x}) \]
\[ - \frac{m}{2} \int_{x_0}^x e^{\mathcal{F}(x-s)} (f'(\hat{v})(\bar{x}) - f'(\hat{v})(s)) \hat{v}_{xx} \hat{v}_x^{m+1} ds \]
\[ + \frac{1}{2} f'(\hat{v})(\bar{x}) \int_{x_0}^x e^{\mathcal{F}(x-s)} \left( \frac{c}{2\nu} - \frac{b f(\hat{v})}{\nu \hat{v}_x} \right) \hat{v}_x^{m+2} ds . \]
We can now choose \( m \) sufficiently large such that
\[ f'(\hat{v})(\bar{x}) \int_{x_0}^x e^{\mathcal{F}(x-s)} \left( \frac{c}{2\nu} - \frac{b f(\hat{v})}{\nu \hat{v}_x} \right) \hat{v}_x^{m+2} ds \]
\[ < m \int_{x_0}^x e^{\mathcal{F}(x-s)} (f'(\hat{v})(\bar{x}) - f'(\hat{v})(s)) \hat{v}_{xx} \hat{v}_x^{m+1} ds \]
since \( f'(\hat{v})(\bar{x}) - f'(\hat{v})(s) > 0 \) for \( s < \bar{x} \). It follows that \( f(\hat{v}) \hat{v}_{xx} \hat{v}_x^m(\bar{x}) < \frac{1}{2} f'(\hat{v}) \hat{v}_x^{m+2}(\bar{x}) \), which is a contradiction to the definition of \( \bar{x} \). It follows that \( \bar{x} > x_{0.5} \land x_* \) and thus \( g_2(x) > 0 \) on \( [x_0, x_{0.5} \land x_*] \).

We now turn to the second subinterval \([x_{0.5} \lor x_*, x_1]\) where \( f'(\hat{v}) \) decreases.

**Lemma 4.3.** \( g_2(x) > 0 \) for \( x \in [x_{0.5} \lor x_*, x_1] \).
Proof. We may assume that \( x_* \leq x_1 \). Otherwise the interval \([x_0 \vee x_*, x_1]\) is empty. Let

\[
\bar{x} := \sup\{x \in [x_0, x_*] \mid g_2(x) = 0\}.
\]

In this case we will show that \( \bar{x} < x_0 \vee x_* \). Since \( g_2(x_1) = \frac{1}{2} f'(\hat{v}) \hat{v}^2(x_1) > 0 \) we certainly have that \( \bar{x} < x_1 \). Suppose now that \( \bar{x} \geq x_0 \vee x_* \). Then for all \( m \in \mathbb{N} \) we have that

\[
\frac{d}{dx} f(\hat{v}) \hat{v} \hat{v}^m = f'(\hat{v}) \hat{v} \hat{v}^{1-m} + f(\hat{v}) \left( \hat{v} \hat{v} - m \frac{\hat{v}^2}{\hat{v}^2} \right) \hat{v}^{1-m}
\]

which implies

\[
f(\hat{v}) \hat{v} \hat{v}^{-m}(\bar{x}) = \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} f'(\hat{v}) \left( 2 \frac{b}{\nu} f(\hat{v}) - \frac{c}{\nu} \right) \hat{v}^{1-m} ds
\]

(32)

\[
\begin{aligned}
&= m \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} f(\hat{v}) \hat{v}^{2} \hat{v}^{-(m+1)} ds \\
&=: I + II, \quad \text{say.}
\end{aligned}
\]

Now \( f^{(2)}(\hat{v}) \leq 0 \), hence \( f'(\hat{v}) \) decreasing, and \( 2 \frac{b}{\nu} f(\hat{v}) - \frac{c}{\nu} \bar{v} \geq 0 \) due to \( x \geq x_0 \vee x_* \), implies that

\[
I \leq f'(\hat{v})(\bar{x}) \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} \left( 2 \frac{b}{\nu} f(\hat{v}) - \frac{c}{\nu} \right) \hat{v}^{2-m} ds
\]

\[
= \frac{1}{2} f'(\hat{v})(\bar{x}) \left( \hat{v}^{2-m}(\bar{x}) - e^{\hat{v}(x-x_1)} \hat{v}^{2-m}(x_1) \right) - m \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} \hat{v} \hat{v}^{1-m} ds
\]

\[
+ \frac{1}{2} f'(\hat{v})(\bar{x}) \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} \left( \frac{b f(\hat{v})}{\nu} \hat{v} - \frac{c}{2\nu} \right) \hat{v}^{2-m} ds
\]

thereby using \( e^{\hat{v}(x-s)} \left( 2 \frac{b}{\nu} f(\hat{v}) - \frac{c}{\nu} \hat{v} \right) \hat{v} = -\frac{d}{ds} e^{\hat{v}(x-s)} \hat{v}^2 \). Inserting the last estimate into (32) and using \( g_2(s) \geq 0 \) for \( s \geq \bar{x} \), hence

\[
II \leq m \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} f'(\hat{v}) \hat{v} \hat{v}^{1-m} ds,
\]

we arrive at

\[
f(\hat{v}) \hat{v} \hat{v}^{-m}(\bar{x}) < \frac{1}{2} f'(\hat{v}) \hat{v}^{2-m}(\bar{x})
\]

\[
- \frac{m}{2} \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} \left( f'(\hat{v})(\bar{x}) - f'(\hat{v})(s) \right) \hat{v} \hat{v}^{1-m} ds
\]

\[
+ \frac{1}{2} f'(\hat{v})(\bar{x}) \int_{\bar{x}}^{x_1} e^{\hat{v}(x-s)} \left( \frac{b f(\hat{v})}{\nu} \hat{v} - \frac{c}{2\nu} \right) \hat{v}^{2-m} ds.
\]
We can now choose $m$ sufficiently large such that
\[
f'(\hat{v})(\bar{x}) \int_{\bar{x}}^{x_{1}} e^{\nu (\bar{x}-s)} \left( \frac{b f(\hat{v})}{\nu \hat{v}_{x}} - \frac{c}{2\nu} \right) \hat{v}_{x}^{2-m} ds
< m \int_{\bar{x}}^{x_{1}} e^{\nu (\bar{x}-s)} (f'(\hat{v})(\bar{x}) - f'(\hat{v})(s)) \hat{v}_{xx} \hat{v}_{x}^{1-m} ds
\]
since $f'(\hat{v})(\bar{x}) - f'(\hat{v})(s) > 0$ for $s > \bar{x}$. It follows that $f(\hat{v})\hat{v}_{xx} \hat{v}_{x}^{-m}(\bar{x}) < \frac{1}{2} f'(\hat{v}) \hat{v}_{x}^{2-m}(\bar{x})$, which is a contradiction to the definition of $\bar{x}$. It follows that $\bar{x} < x_{0.5} \lor x_{*}$ and thus $g_{2}(x) > 0$ on $[x_{0.5} \lor x_{*}, x_{1} \lor x_{*}]$. \hfill \Box

Finally we consider the third subinterval $[x_{0.5} \land x_{*}, x_{0.5} \lor x_{*}]$.

**Lemma 4.4.** $g_{2}(x) > 0$ for $x \in [x_{0.5} \land x_{*}, x_{0.5} \lor x_{*}]$.

**Proof.** We consider the two cases $x_{*} \leq x_{0.5}$ and $x_{0.5} > x_{*}$ separately.

**Case 1:** $x_{*} \leq x_{0.5}$, hence $[x_{0.5} \land x_{*}, x_{0.5} \lor x_{*}] = [x_{*}, x_{0.5}]$.

In this case $f'(\hat{v})$ is decreasing and $\frac{f'(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}}$ increases, since
\[
\frac{d}{dx} \left( \frac{f'(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \right) = \left( \frac{c}{\nu} - \frac{2b f(\hat{v})}{\nu \hat{v}_{x}} \right) \frac{d}{dx} \frac{f'(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \geq 0.
\]
Hence
\[
g_{2}(x) = \hat{v}_{x}^{2}(x) \left( \frac{1}{2} f'(\hat{v}) - \frac{f(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \right)(x)
\geq \hat{v}_{x}^{2}(x) \left( \frac{1}{2} f'(\hat{v}) - \frac{f(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \right)(x_{0.5})
\geq \frac{\hat{v}_{x}^{2}(x)}{\hat{v}_{x}^{2}(x_{0.5})} g_{2}(x_{0.5}) > 0
\]
according to Lemma 4.3.

**Case 2:** $x_{0.5} < x_{*}$, hence $[x_{0.5} \land x_{*}, x_{0.5} \lor x_{*}] = [x_{0.5}, x_{*}]$.

In this case $f'(\hat{v})$ is increasing and $\frac{f'(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}}$ decreases, since
\[
\frac{d}{dx} \left( \frac{f'(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \right) = \left( \frac{c}{\nu} - \frac{2b f(\hat{v})}{\nu \hat{v}_{x}} \right) \frac{d}{dx} \frac{f'(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \leq 0.
\]
Hence
\[
g_{2}(x) = \hat{v}_{x}^{2}(x) \left( \frac{1}{2} f'(\hat{v}) - \frac{f(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \right)(x)
\geq \hat{v}_{x}^{2}(x) \left( \frac{1}{2} f'(\hat{v}) - \frac{f(\hat{v}) \hat{v}_{xx}}{\hat{v}_{x}^{2}} \right)(x_{0.5})
\geq \frac{\hat{v}_{x}^{2}(x)}{\hat{v}_{x}^{2}(x_{0.5})} g_{2}(x_{0.5}) > 0
\]
according to Lemma 4.2. \hfill \Box
Recall that the travelling wave satisfies the equation \( c\dot{v}_x = \nu \dot{v}_{xx} + bf(\dot{v}) \), hence \( c\dot{v}_{xx} = \nu \dot{v}_{xx} + bf'(\dot{v})\dot{v}_x \). Given a function \( u \in C^1_c(\mathbb{R}) \) and writing \( u = h\dot{v}_x \) it follows that

\[
\nu \Delta u + bf'(\dot{v})u = \nu h_{xx} \dot{v}_x + 2\nu \dot{v}_{xx} h + c\dot{v}_{xx} h
\]

which implies

\[
-\langle \nu \Delta u + bf'(\dot{v})u, u \rangle = -\int (\nu h_{xx} + 2\nu \frac{\dot{v}_{xx}}{\dot{v}_x} h) \dot{v}_x^2 dx - c \int h \dot{v}_{xx} h \dot{v}_x dx \\
= \nu \int h_x^2 \dot{v}_x^2 dx + c \int h_x h \dot{v}_x^2 dx \\
= \nu \int (he^{2\nu \dot{v}_x^2})^2 e^{-\nu \dot{v}_x^2} \dot{v}_x^2 dx - \nu \left( \frac{c}{2\nu} \right)^2 \int h^2 \dot{v}_x^2 dx \\
=: \mathcal{E}(h).
\]

In the following, consider the two functions \( h_0(x) = 1 \) and \( h_1(x) = e^{-\frac{\nu}{\nu^2}} \). Notice that \( \mathcal{E}(h_0) = 0 \) and \( \mathcal{E}(h_1) = \nu \left( \frac{c}{2\nu} \right)^2 \int e^{-\nu \dot{v}_x^2} \dot{v}_x^2 dx > 0 \). Consequently, the Schrödinger operator \( \nu \Delta u + bf'(\dot{v})u \) is not negative definite on the subspace \( \mathcal{N} := \text{span}\{\dot{v}_x, e^{-\frac{\nu}{\nu^2}} \dot{v}_x\} \). \( \dot{v}_x \) can be interpreted as the vector pointing in the tangential direction of the orbit of the travelling wave solutions, since \( \frac{d}{dt} \dot{v}(\cdot + ct) = c\dot{v}_x(\cdot + ct) \) and the second function \( h_1(x) = e^{-\frac{\nu}{\nu^2}} \) measures the infinitesimal variation of the linearization of \( \nu \Delta u + b(f(u + \dot{v}) - f(\dot{v})) \) w.r.t. time. Notice that in the case \( c = 0 \) of a stationary wave both functions coincide, since the linearization is independent of the time.

Using the representation [33] we will now first consider the gradient form \( \int h_x^2 \dot{v}_x^2 dx \), where \( w = e^{-\frac{\nu}{\nu^2}} \dot{v}_x \). The logarithmic derivative

\[
\theta(x) := \frac{w_x(x)}{w(x)} = \frac{c}{2\nu} - \frac{b f(\dot{v})}{\nu \dot{v}_x}(x) = \frac{\dot{v}_{xx}}{\dot{v}_x} - \frac{c}{2\nu}
\]

of \( w \) satisfies the inequality

\[
-\theta' + \theta^2 = -\frac{d}{dx} \frac{\dot{v}_{xx}}{\dot{v}_x} + \left( \frac{\dot{v}_{xx}}{\dot{v}_x} - \frac{c}{2\nu} \right)^2 \\
= -\frac{d}{dx} \frac{\dot{v}_{xx}}{\dot{v}_x} + \left( \frac{\dot{v}_{xx}}{\dot{v}_x} \right)^2 - \frac{c}{\nu} \frac{\dot{v}_{xx}}{\dot{v}_x} + \left( \frac{c}{2\nu} \right)^2 \geq \kappa + \left( \frac{c}{2\nu} \right)^2
\]

for some \( \kappa > 0 \) according to Theorem [13, Proposition 5.5] below now implies the weighted Hardy type inequality

\[
\int h^2 w^2 dx \leq \frac{1}{\kappa + \left( \frac{c}{2\nu} \right)^2} \int h_x^2 w^2 dx
\]

for any \( h \in C^1_c(\mathbb{R}) \) with \( h(x_{0.5}) = 0 \), where \( x_{0.5} \) is the unique root of \( \frac{b f(\dot{v})}{\nu \dot{v}_x}(x) = \frac{c}{2\nu} \) (recall that \( \frac{b f(\dot{v})}{\nu \dot{v}_x} \) is strictly monotone increasing). Clearly,
the last inequality implies the Poincare inequality

\[ \int h^2 w^2 \, dx \leq \frac{1}{\kappa + (\frac{c}{2\nu})^2} \int h_x^2 w^2 \, dx + Z^{-1} \left( \int h w^2 \, dx \right)^2 \]

for the normalizing constant \( Z = \int e^{-\frac{c}{\nu} \hat{\nu}^2} \, dx \) and for any \( h \in C^1_b(\mathbb{R}) \).

Unfortunately, this is not yet enough, since for \( u = he^{\hat{\nu} \hat{x}} \hat{\nu}_x \) we cannot control the tangential direction \( \int h w^2 \, dx = \int ue^{-\frac{c}{\nu} \hat{\nu}^2} \hat{\nu}_x \, dx \) but only the tangential direction \( \int he^{\hat{\nu} \hat{x}} w^2 \, dx = \int u \hat{\nu}_x \, dx \). This is done in the following

**Proposition 5.1.** For \( h \in C^1_b(\mathbb{R}) \) the following inequality holds:

\[ \int h^2 w^2 \, dx \leq \frac{1}{\kappa + \left( \frac{c}{2\nu} \right)^2} \int h_x^2 w^2 \, dx + C_5.1 \left( \int h e^{\hat{\nu} \hat{x}} w^2 \, dx \right)^2. \]

with

\[ C_5.1 = \frac{\kappa + \left( \frac{c}{2\nu} \right)^2}{\kappa} \left( \frac{\int e^{-\frac{c}{\nu} \hat{\nu}^2} \, dx}{\left( \int e^{-\frac{c}{\nu} \hat{\nu}^2} \hat{\nu}_x^2 \, dx \right)^2} \right). \]

The proof of Proposition 5.1 requires the following lemma.

**Lemma 5.2.** There exists a function \( g \in C^1(\mathbb{R}) \cap L^2(\mathbb{R}, w^2 \, dx) \), \( g \geq 0 \), satisfying the equation

\[ \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) g - \left( g_{xx} + \left( \frac{c}{\nu} - \frac{b f(\hat{\nu})}{\nu} \right) g_x \right) = \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) e^{\hat{\nu} \hat{x}}. \]

Moreover, \( |g_x(x)| \leq \frac{c}{\nu} g(x) \) for all \( x \in \mathbb{R} \) and we have the lower bound

\[ \int g^2 w^2 \, dx \geq \frac{\left( \int e^{-\frac{c}{\nu} \hat{\nu}^2} \, dx \right)^2}{\left( \int e^{-\frac{c}{\nu} \hat{\nu}^2} \hat{\nu}_x^2 \, dx \right)^2}. \]

**Proof.** Fix a 1D-Brownian motion \((W_t)_{t \geq 0}\) defined on some underlying probability space \((\Omega, \mathcal{A}, P)\). For all initial conditions \( x \in \mathbb{R} \) let \( X_t(x) \) be the unique strong solution of the stochastic differential equation

\[ dX_t(x) = \left( \frac{c}{\nu} - \frac{b f(\hat{\nu})}{\nu} \hat{\nu}_x \right) (X_t(x)) \, dt + dW_t, X_0(x) = x. \]

The family of solutions is a Markov process on \( \mathbb{R} \) having invariant measure \( w^2 \, dx \), i.e.,

\[ \int_\mathbb{R} E(h(X_t(h))) w^2 \, dx = \int_\mathbb{R} h w^2 \, dx, t \geq 0. \]

It follows that the associated semigroup of transition operators \( p_t h(x) := E(h(X_t(x))) \) induces a contraction semigroup of Markovian integral operators on \( L^p(\mathbb{R}, w^2 \, dx) \) for all \( p \in [1, \infty] \).

Theorem V.7.4 in [7] yields that the function

\[ g(x) := \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) \int_0^\infty e^{-\left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right)t} E\left( e^{\hat{\nu} \hat{x}}X_t(x) \right) \, dt \]
is twice continuously differentiable and solves equation (57). Since $e^{\frac{\xi}{\nu}} \in L^2(\mathbb{R}, w^2 \, dx)$ we also have that $g \in L^2(\mathbb{R}, w^2 \, dx)$.

We will show next the pointwise estimate of the derivative $g_x$. The solution $X_t(x)$ of the stochastic differential equation (58) is differentiable w.r.t. its initial condition $x$. Its differential $DX_t(x)$ is the solution of the linear linear differential equation

$$dDX_t(x) = -2 \frac{d}{dx} \nu \frac{b (\hat{v} x)}{\nu_x} (X_t(x)) \, DX_t(x) \, dt, \, DX_0(x) = 1,$$

with explicit solution

$$DX_t(x) = \exp \left( -2 \frac{b}{\nu} \int_0^t \frac{d}{dx} \nu \frac{b (\hat{v} x)}{\nu_x} (X_s(x)) \, ds \right) < 1$$

for all $t > 0$, since $\frac{d}{dx} \nu \frac{b (\hat{v} x)}{\nu_x} > 0$ according to Proposition 1.2. Consequently,

$$g_x(x) = \frac{c}{2\nu} \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) \int_0^\infty e^{-\left( \frac{\kappa}{\nu} + \left( \frac{\nu}{\kappa} \right)^2 \right) t} E \left( \frac{g_{\hat{v} X_t(x)}}{\nu_x} DX_t(x) \right) \, dt$$

$$= \frac{c}{2\nu} \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) \int_0^\infty e^{-\left( \frac{\kappa}{\nu} + \left( \frac{\nu}{\kappa} \right)^2 \right) t} E \left( \frac{g_{\hat{v} X_t(x)}}{\nu_x} - 2 b \int_0^t \frac{b (\hat{v} x)}{\nu_x} (X_s(x)) \, ds \right) \, dt$$

which implies that

$$|g_x(x)| < \frac{c}{2\nu} \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) \int_0^\infty e^{-\left( \frac{\kappa}{\nu} + \left( \frac{\nu}{\kappa} \right)^2 \right) t} E \left( \frac{g_{\hat{v} X_t(x)}}{\nu_x} \right) \, dt = \frac{c}{2\nu} g(x).$$

It remains to prove the lower bound. To this end note that invariance of the measure $w^2 \, dx$ implies

$$\int g^2 w^2 \, dx \geq \left( \int w^2 \, dx \right)^{-1} \left( \int gw^2 \, dx \right)^2$$

$$= \left( \int w^2 \, dx \right)^{-1} \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) \int_0^\infty e^{-\left( \frac{\kappa}{\nu} + \left( \frac{\nu}{\kappa} \right)^2 \right) t} \int p_t \left( \frac{g_{\hat{v} x}}{\nu_x} \right) w^2 \, dx \, dt \right)^2$$

$$= \left( \int w^2 \, dx \right)^{-1} \left( \kappa + \left( \frac{c}{2\nu} \right)^2 \right) \int_0^\infty e^{-\left( \frac{\kappa}{\nu} + \left( \frac{\nu}{\kappa} \right)^2 \right) t} dt \int \frac{g_{\hat{v} x} w^2}{\nu_x} \, dx \right)^2$$

$$= \left( \int w^2 \, dx \right)^{-1} \left( \int \frac{g_{\hat{v} x}}{\nu_x} w^2 \, dx \right)^2.$$

\[\square\]

**Proof.** (of Proposition 5.1). Let $\tilde{h} := h - \frac{h(x_1)}{g(x_1)} g$, hence $\tilde{h}(x_1) = 0$. Then Proposition 5.3 implies that

$$\int \tilde{h}^2 w^2 \, dx \leq \frac{1}{\kappa + \left( \frac{c}{2\nu} \right)^2} \int \tilde{h}^2_x w^2 \, dx$$
or equivalently,
\[\int h^2 w^2 \, dx \leq \frac{1}{\kappa + (\frac{c}{2\nu})^2} \int h^2 w^2 \, dx + T(h)\]

with the remainder
\[T(h) := \frac{1}{\kappa + (\frac{c}{2\nu})^2} \left( -2 \int h_x \frac{h(x_1)}{g(x_1)} g_x w^2 \, dx + \left( \frac{h(x_1)}{g(x_1)} \right)^2 \int g^2 w^2 \, dx \right) \]
\[+ 2 \int h \frac{h(x_1)}{g(x_1)} g w^2 \, dx \left( \frac{h(x_1)}{g(x_1)} \right)^2 \int g^2 \, dx\]

Using Lemma 5.2 we obtain that
\[T(h) = 2 \int \frac{h(x_1)}{g(x_1)} \int \left( g - \frac{1}{\kappa + (\frac{c}{2\nu})^2} \left( g_{xx} - \left( \frac{c}{\nu} - \frac{b}{\nu} \right) g_x \right) \right) h w^2 \, dx \]
\[+ \left( \frac{h(x_1)}{g(x_1)} \right)^2 \left( \frac{1}{\kappa + (\frac{c}{2\nu})^2} \int g^2 w^2 \, dx - \int g^2 w^2 \, dx \right)\]
\[\leq 2 \int e^{\frac{c}{2\nu} x} h w^2 \, dx - \left( \frac{h(x_1)}{g(x_1)} \right)^2 \frac{\kappa}{\kappa + (\frac{c}{2\nu})^2} \int g^2 w^2 \, dx.\]

In the last inequality we have used the pointwise estimate \(|g_x(x)| \leq \frac{c}{2\nu} g(x)|. Using the lower bound \(\int g^2 w^2 \, dx \geq \frac{(\int e^{-\frac{c}{2\nu} x} \hat{v}^2 \, dx)^2}{\int e^{-\frac{c}{2\nu} x} \hat{v}^2 \, dx}\) obtained in the previous Lemma we conclude that
\[T(h) \leq \frac{\kappa + (\frac{c}{2\nu})^2}{\kappa} \left( \int g^2 w^2 \, dx \right)^{-1} \left( \int h e^{\frac{c}{2\nu} x} w^2 \, dx \right)^2\]
\[\leq C_{5.1} \left( \int h e^{\frac{c}{2\nu} x} w^2 \, dx \right)^2\]

with
\[C_{5.1} = \frac{\kappa + (\frac{c}{2\nu})^2}{\kappa} \int e^{-\frac{c}{2\nu} x} \hat{v}^2 \, dx \frac{\int e^{-\frac{c}{2\nu} x} \hat{v}^2 \, dx}{\left( \int e^{-\frac{c}{2\nu} x} \hat{v}^2 \, dx \right)^2}\]

which implies the assertion.

\[\square\]

Having Proposition 5.1 we can now state the following

**Proposition 5.3.** Let \(u \in C^1_c(\mathbb{R})\) and write \(u = hw\) for \(h \in C^1_c(\mathbb{R})\).

Then
\[\langle \nu \Delta u + bf'(\hat{v})u, u \rangle \leq -\nu \frac{\kappa}{\kappa + (\frac{c}{2\nu})^2} \int h^2 w^2 \, dx\]
\[+ \nu \left( \frac{c}{2\nu} \right)^2 C_{5.1} \left( \int u \hat{v}_x \, dx \right)^2.\]
Proof. First note that \( h \in C^1_c(\mathbb{R}) \), and thus equations (33) and Proposition 5.1 imply that
\[
\langle \nu \Delta u + bf'(\hat{v})u, u \rangle = -\nu \int h_x^2 w^2 dx + \nu \left( \frac{\nu}{2c} \right)^2 \int h^2 w^2 dx \leq -\nu \frac{\kappa}{\kappa + \left( \frac{c}{2\nu} \right)^2} \int h_x^2 w^2 dx + \nu \left( \frac{\nu}{2c} \right)^2 C_{5.1} \left( \int \tilde{h} e^{-\frac{\nu}{c} x} w^2 dx \right)^2.
\]

In the next step we will show that for \( u \in C^1_c(\mathbb{R}) \) and \( u = h\hat{v}_x \) its \( V \)-norm \( \| u \|_V \) can be controlled by \( \int (he^{\frac{\nu}{c} x})_x^2 w^2 dx \).

Lemma 5.4. Let \( u \in C^1_c(\mathbb{R}) \) and write \( u = hw \). Then
\[
\| u \|^2_V \leq q_1 \int h_x^2 w^2 dx + q_2 \langle u, \hat{v}_x \rangle^2
\]
where
\[
q_1 := \left( 1 + \frac{b\eta}{\nu} + 1 \right) \frac{1}{\kappa + \left( \frac{c}{2\nu} \right)^2}, \quad q_2 := \left( \frac{b\eta}{\nu} + 1 \right) C_{5.1},
\]
and
\[
\eta := \max_{v \in [0,1]} f'(v).
\]

Proof. Using (33) we have that
\[
\nu \int u_x^2 dx = -\langle \nu \Delta u + bf'(\hat{v})u, u \rangle + b\langle f'(\hat{v})u, u \rangle \leq \nu \int h_x^2 w^2 dx + bn\| u \|^2_H.
\]

Proposition 5.1 now implies
\[
\| u \|^2_V \leq \left( 1 + \frac{b\eta}{\nu} + 1 \right) \frac{1}{\kappa + \left( \frac{c}{2\nu} \right)^2} \int h_x^2 w^2 dx + \left( \frac{b\eta}{\nu} + 1 \right) C_{5.1} \langle u, \hat{v}_x \rangle^2,
\]
which implies the assertion. \( \square \)

Proof. (of Theorem 1.5) First let \( u \in C^1_c(\mathbb{R}) \). Then Proposition 5.3 implies the estimate
\[
\langle \nu \Delta u + bf'(\hat{v})u, u \rangle \leq -\nu \frac{\kappa}{\kappa + \left( \frac{c}{2\nu} \right)^2} \int h_x^2 w^2 dx + \nu \left( \frac{c}{2\nu} \right)^2 C_{5.1} \langle u, \hat{v}_x \rangle^2.
\]
Combining the last estimate with the previous Lemma 5.4, we obtain that
\[ \langle \nu \Delta u + b f'(\hat{v})u, u \rangle \leq -\frac{\kappa}{\kappa + \left(\frac{c}{2\nu}\right)^2 q_1} \|u\|_V^2 + \left(\frac{\kappa}{\kappa + \left(\frac{c}{2\nu}\right)^2 q_1} + \nu \left(\frac{c}{2\nu}\right)^2 C_{5.1}\right) \langle u, \hat{v} \rangle^2 \]
which implies Theorem 1.5 with
\[ \kappa_* = \frac{\kappa}{\kappa + \left(\frac{c}{2\nu}\right)^2 q_1} \]
and
\[ C_* = \left(\kappa_* q_2 + \nu \left(\frac{c}{2\nu}\right)^2 \left(\kappa + \left(\frac{c}{2\nu}\right)^2 \int e^{-\frac{c}{2\nu} \hat{v}^2} dx\right)\right). \]
\[ \square \]

It remains to prove the weighted Hardy type inequality (34) which is of independent interest.

**Proposition 5.5.** Let \( w \in C^2_b(\mathbb{R}) \) and \( \theta = \frac{w}{w} \). Suppose that
\[ \inf_{x \in \mathbb{R}} -\theta'(x) + \theta^2(x) \geq \kappa_0 > 0 \]
and that there exists \( \hat{x} \) such that \( \theta(\hat{x}) = 0 \). Then
\[ \int h^2 w^2 \, dx \leq \frac{1}{\kappa} \int h_\hat{x}^2 w^2 \, dx \]
for any \( h \in C^1_b(\mathbb{R}) \) with \( h(\hat{x}) = 0 \).

**Proof.** Define the function \( g(x) := (-\theta'(x) + \theta^2(x)) \exp \left(-\int_{\hat{x}}^x \theta(s) \, ds\right) \) and notice that
\[ \exp \left(-\int_{\hat{x}}^x \theta(s) \, ds\right) = \exp (-\log w(x) + \log w(\hat{x})) = \frac{w(\hat{x})}{w(x)} \]
and thus
\[ g(x) = (-\theta'(x) + \theta^2(x)) \frac{w(\hat{x})}{w(x)} \geq \kappa \frac{w(\hat{x})}{w(x)}. \]
Then for \( x \geq \hat{x} \) we have that
\[
(h(x) - h(\hat{x}))^2 = \left(\int_{\hat{x}}^x h_x(s) \, ds\right)^2 \leq \int_{\hat{x}}^x \frac{1}{g(s)} h_x^2(s) \, ds \int_{\hat{x}}^x g(s) \, ds
\]
\[ = \int_{\hat{x}}^x \frac{1}{g(s)} h_x^2(s) \, ds \left(-\theta(x) \exp \left(-\int_{\hat{x}}^x \theta(s) \, ds\right)\right) \]
\[ = \int_{\hat{x}}^x \frac{1}{g(s)} h_x^2(s) \, ds \left(-\frac{w_x(x)w(\hat{x})}{w(x)w(x)}\right) \]
\[ \leq \frac{1}{\kappa} \int_{\hat{x}}^x \frac{w(s)}{w(\hat{x})} h_x^2(s) \, ds \left(-\frac{w_x(x)w(\hat{x})}{w(x)w(x)}\right). \]
Integrating against $w^2 \, dx$ for $x \geq \hat{x}$ now yields the following estimate (39)

$$\int_{\hat{x}}^{\infty} (h - h(x))^2 \, w^2 \, dx \leq \frac{1}{\kappa} \int_{\hat{x}}^{\infty} \frac{w(s)}{w(\hat{x})} h_x^2(s) \int_{s}^{\infty} -w_x(x) w(\hat{x}) \, dx \, ds$$

$$= \frac{1}{\kappa} \int_{\hat{x}}^{\infty} h_x^2(s) w^2(s) \, ds .$$

Similarly, for $x \leq \hat{x}$ we have that

$$(h(\hat{x}) - h(x))^2 = \left( \int_{x}^{\hat{x}} h_x(s) \, ds \right)^2 \leq \int_{x}^{\hat{x}} \frac{1}{g(s)} h_x^2(s) \, ds \int_{x}^{\hat{x}} g(s) \, ds$$

$$= \int_{x}^{\hat{x}} \frac{1}{g(s)} h_x^2(s) \, ds \, \left( \theta(x) \exp \left( -\int_{x}^{\hat{x}} \theta(s) \, ds \right) \right)$$

$$= \int_{x}^{\hat{x}} \frac{1}{g(s)} h_x^2(s) \, ds \, \frac{w_x(x) \, w(\hat{x})}{w(x) \, w(\hat{x})} \leq \frac{1}{\kappa} \int_{x}^{\hat{x}} \frac{w(s)}{w(\hat{x})} \frac{w_x(x) \, w(\hat{x})}{w(x) \, w(\hat{x})} .$$

Integrating against $w^2 \, dx$ now for $x \leq \hat{x}$ yields

$$(40) \quad \int_{-\infty}^{\hat{x}} (h - h(\hat{x}))^2 \, w^2 \, dx$$

$$\leq \frac{1}{\kappa} \int_{-\infty}^{\hat{x}} \frac{w(s)}{w(\hat{x})} h_x^2(s) \int_{-\infty}^{\hat{x}} w_x(x) w(\hat{x}) \, dx \, ds$$

$$= \frac{1}{\kappa} \int_{-\infty}^{\hat{x}} h_x^2(s) w^2(s) \, ds .$$

The assertion now follows from estimates (39) and (40).

\textbf{Acknowledgement} This work is supported by the BMBF, FKZ 01GQ1001B.

\textbf{References}

[1] Ermentrout, G.B., Terman, D.H., Mathematical Foundations of Neuroscience, Springer, Berlin, 2010.
[2] Evans, J.W., Nerve axon equation III: Stability of the nerve impulse, Indiana Univ. Mat. J., Vol. 22, 577–594, 1972.
[3] Fife, P.C., McLeod, J.B., The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Ration. Mech. Anal., Vol. 65, 335–361, 1977.
[4] Hadeler, K.P., Rothe, F., Travelling Fronts in Nonlinear Diffusion Equations, J. Math. Biol., Vol. 2, 251–263, 1975.
[5] Henry, D., Geometric theory of semilinear parabolic equations, LNM Vol. 840, Springer-Verlag, Berlin, 1981.
[6] Jones, C.K.R.T., Stability of the traveling wave solution of the FitzHugh-Nagumo equations, Trans A.M.S., Vol. 286, 431–469, 1984.
[7] Krylov, N.V., Introduction to the Theory of Diffusion Processes, American Mathematical Society, Providence, RI, 1995.
[8] Liu, W., Röckner, M., SPDE in Hilbert space with locally monotone coefficients, J. Funct. Anal., Vol. 295, 2902–2922, 2010.
[9] Lord, G.J., Thümmler, V., Computing Stochastic Travelling Waves, SIAM Journal of Scientific Computation, Vol. 34, 24–43, 2012.
[10] Otto, F., Reznikoff, M.G., Slow motion of gradient flows, J. Differential Equations, Vol. 237, 372–420, 2007.
[11] Prevot, C., Röckner, M., A Concise course on Stochastic Partial Differential Equations, Lecture Notes in Mathematics, Vol. 1905, Springer, Berlin, 2007.
[12] Stannat, W., Stability of travelling waves in stochastic Nagumo equations, arXiv:1301.6378 2013.
[13] Tuckwell, H.C., Jost, J., Weak noise in neurons may powerfully inhibit the generation of repetitive spiking but not its propagation, PLoS Comput. Biol., Vol. 6, 13 pp, 2010.
[14] Tuckwell, H.C., Jost, J., The effect of various spatial distributions of weak noise on rhythmic spiking, J. Comput. Neurosci., Vol 30, 361–371, 2011.

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI 136, D-10623 BERLIN, AND, BERNSTEIN CENTER FOR COMPUTATIONAL NEUROSCIENCE, PHILIPPSTR. 13, D-10115 BERLIN, GERMANY

E-mail address: stannat@math.tu-berlin.de