Efficient polarization entanglement purification based on parametric down-conversion sources with cross-Kerr nonlinearity

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We present a way for entanglement purification based on two parametric down-conversion (PDC) sources with cross-Kerr nonlinearities. It is comprised of two processes. The first one is a primary entanglement purification protocol for PDC sources with nondestructive quantum nondemolition (QND) detectors by transferring the spatial entanglement of photon pairs to their polarization. In this time, the QND detectors act as the role of controlled-not (CNot) gates. Also they can distinguish the photon number of the spatial modes, which provides a good way for the next process to purify the entanglement of the photon pairs kept more. In the second process for entanglement purification, new QND detectors are designed to act as the role of CNot gates. This protocol has the advantage of high yield and it requires neither CNot gates based on linear optical elements nor sophisticated single-photon detectors, which makes it more convenient in practical applications.

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I. INTRODUCTION

Quantum entanglement plays an important role in quantum information processing and transmission, such as quantum computation [1], quantum teleportation [2], quantum dense coding [3], quantum state sharing [4] and certain types of quantum cryptography [5, 6, 7, 8, 9, 10]. In order to complete these tasks efficiently, people need to share some maximally entangled states. In a practical transmission, the interaction between a quantum system and the innocent noise of quantum channel (such as optical fibers or a free space) will inevitably occur, which will degrade the entanglement of the quantum system or even make it in a mixed state. The imperity of the quantum system will make the outcome of the quantum computation anamorphic, the fidelity of quantum teleportation degraded, quantum dense coding failed and the key in quantum cryptography insecure. If the destructive effect of the noise is not very much, one can exploit entanglement concentration or entanglement purification to improve the entanglement of the quantum system first, and then achieve the goals of the applications above with maximally entangled state.

Entanglement concentration [11, 12, 13, 14] is used to increase the entanglement of some pure entangled pairs at the risk of that of some others. For the more general case of the quantum system transmitted through a noisy channel, it is in a mixed state and the process for reconstructing it in a maximally entangled state with an ensemble is termed as entanglement purification or distillation [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Generally, the implementation of entanglement purification schemes requires two or more controlled-not (CNot) gates which is not experimentally feasible with linear optical elements at present. In 1996, Bennett et al. [17] proposed an original entanglement purification scheme for purifying a Werner state [26] with two CNOT gates and single-photon measurements. Subsequently, Deutsch et al. [16] optimized this scheme for quantum privacy amplification with two CNOT operations and two special unitary transforms.

In 2001, Pan et al. [18] proposed an entanglement purification protocol with linear optical elements such as polarizing beam splitters (PBSs) and quarter wave plates (QWP’s). In their protocol [18], the two PBSs are used to complete the task of parity-check measurements of polarized photons with their spatial modes. We call it PBS protocol below. This protocol succeeds, provided that two ideal entangled sources are used. That is, both emit one and only one entangled photon pair synchronously at each time slot. As pointed out by Simon and Pan [27] in 2002, the currently available source of entangled photons, parametric down-conversion (PDC), is not an ideal entangled source. The feature of PDC seems to fail for the PBS protocol [18]. They then proposed a new entanglement purification protocol by exploiting spatial entanglement to purify polarization entanglement, which solves the problem above perfectly [27], and called it Simon-Pan protocol. However, in order to improve the fidelity of the entangled pairs kept more with the PBS protocol [18], the two parties should exploit quantum nondemolition (QND) measurement to determine whether there are photons after the PBS or not, which can not be ac-
complished only with PBS. Moreover, photon number detectors should be used to distinguish the two-photon cases from the cases with four photons in the same modes such as two photons in the upper modes of both Alice’s and Bob’s location. This task can not be accomplished simply with linear optical elements.

Cross-Kerr nonlinearity provides a good tool to construct nondestructive quantum nondemolition detectors "which have the potential available of being able to condition the evolution of our system but without necessarily destroying the single photons" [28]. QND with a cross-Kerr medium and a coherent state can be used for checking the purity of the polarization states of two photons [28], operating as a controlled-not (CNOT) gate [28], and analyzing the Bell states [30]. The Hamiltonian of a cross-Kerr nonlinear medium can be described by the form as follows:

\[ H_{QND} = \hbar \chi \hat{n}_a \hat{n}_c \]  

where \( \hat{n}_a \) (\( \hat{n}_c \)) denotes the number operator for mode a (c) and \( \hbar \chi \) is the coupling strength of the nonlinearity, which is decided by the property of material. For example, for a signal photon state \( |\varphi\rangle = a|0\rangle + b|1\rangle \) and a coherent state \( |\alpha\rangle \), the cross-Kerr interaction causes the combined system composed of a single photon and a coherent state to evolve as

\[ U_{ck} |\varphi\rangle |\alpha\rangle = e^{iH_{QND} t/\hbar} (a|0\rangle + b|1\rangle)|\alpha\rangle = |0\rangle |\alpha\rangle + \beta|1\rangle |\alpha e^{i\varphi}\rangle. \]  

We note that \( |0\rangle \) and \( |1\rangle \) are not the polarization of the photons, but the number of the photons. \( |\alpha\rangle \) is also called the Fock state which means the state contains \( n \) photons. Now one can see that the signal photon state is unaffected by the interaction, but the coherent state makes a phase shift of \( \varphi \). Here \( \theta = \chi t \) and \( t \) is the interaction time. The phase shift is directly proportional to the number of photons. This is the main principle of the cross-Kerr nonlinearity [28]. In 2005, Song et al. [31] presented a protocol for entanglement purification using cross-Kerr nonlinearity to complete parity check. It works for the original entanglement purification model proposed by Bennett et al. [17]. The biggest advantage of their protocol is that its successful probability can be nearly enhanced to a same order of magnitude for the case where Alice and Bob get a nonzero phase shift. First, we provide a primary entanglement purification protocol for PDC sources with QND detectors by transferring the spatial entanglement of photon pairs to their polarization. In this protocol, the QND detectors act as not only the role of CNot gates but also of that of photon number detectors, which provides a good way for the next process to purify the entanglement of the photon pairs more as they make the photon pairs equivalent to those coming from two ideal sources. In the second process for entanglement purification, new QND detectors are designed to act as the role of CNOT gates. This protocol has the advantage of high yield and it requires neither CNOT gates based on linear optics nor sophisticated single-photon detectors, which makes it more convenient in practical applications.

II. ENTANGLEMENT PURIFICATION BASED ON PDC SOURCES

A. The principle of primary entanglement purification based on bit-flipping errors with QND

The principle of our entanglement purification protocol is shown in Fig.1. The PDC sources can produce polarization and spatial entanglement naturally [27]. A pump pulse of ultraviolet light passes through a beta barium borate (BBO) crystal and produces correlated pairs of photons into the modes \( a_1 \) and \( b_1 \). Then it is reflected and traverses the crystal a second time, and produces correlated pairs of photons into the modes \( a_2 \) and \( b_2 \). The Hamiltonian can be approximately described as

\[ H_{PDC} = \gamma (|a_{1H}^+ b_{1V}^+ + a_{1V}^+ b_{1H}^+| + \text{H.c.}) + r e^{i\phi} (|a_{2H}^+ b_{2V}^+ + a_{2V}^+ b_{2H}^+| + \text{H.c.}), \]  

where \( H \) and \( V \) in subscripts present horizontal and vertical polarization, \( r \) denotes the relative probability of emission of photons into the lower modes compared to the upper modes, and \( \phi \) is the phase between these two possibilities [27]. The same as the Simon-Pan protocol [24], in a simple case we assume \( r = 1 \) and \( \phi = 0 \). So the single-pair state can be described by \( |a_{1H}^+ b_{1V}^+ + a_{1V}^+ b_{1H}^+ + a_{2H}^+ b_{2V}^+ + a_{2V}^+ b_{2H}^+|0\rangle \). It also can be written as \((|a_1\rangle |b_1\rangle + |a_2\rangle |b_2\rangle)(|H_a\rangle |H_b\rangle + |V_a\rangle |V_b\rangle)\). The four-photon state produced by this PDC source also can be written as \( |a_{1H}^+ b_{1V}^+ + a_{1V}^+ b_{1H}^+ + a_{2H}^+ b_{2V}^+ + a_{2V}^+ b_{2H}^+|^0\rangle \) and discussed in the same way.

After receiving the signals, the user Alice (Bob) lets them pass through QND detectors whose principle is shown in Fig.2. For a two-photon state without suffering from decoherence (including bit-flipping and phase-flipping) \( |a_{1H}^+ b_{1V}^+ + a_{1V}^+ b_{1H}^+ + a_{2H}^+ b_{2V}^+ + a_{2V}^+ b_{2H}^+|0\rangle \), the two parties Alice and Bob will get the same phase shifts on their coherent states as QND detectors evolve the
they retain the pair and perform no local unitary operations on their photons but link the photons with couplers shown in Fig.3. If both Alice and Bob get the phase shift $\theta'$, their photon pair in the state $(a^+_{1H}b^+_1 + a^+_{2H}b^+_2)|0\rangle$ will appear at the lower output modes $a_2b_2$ (upper modes $a_1b_1$) of the couplers. If a bit-flipping error takes place, i.e., the state of the pair becoming $(|a_1\rangle|b_1\rangle + |a_2\rangle|b_2\rangle)(|V\rangle|H_0\rangle + |H_0\rangle|V\rangle)$, Alice and Bob will get two different results with their homodyne measurements on their coherent states $|\alpha\rangle$ as QND detectors evolve the combined system to

\[
\begin{aligned}
&\rightarrow (a^+_{1H}b^+_1 + a^+_{2H}b^+_2)|0\rangle|\alpha e^{i\theta}\rangle_a|\alpha e^{i\theta'}\rangle_b + (a^+_{1V}b^+_1 + a^+_{2V}b^+_2)|0\rangle|\alpha e^{i\theta'}\rangle_a|\alpha e^{i\theta}\rangle_b.
\end{aligned}
\] (5)

One will get the result $\theta$ and the other $\theta'$. Therefore, by performing a bit-flipping operation $\sigma_x = |H\rangle\langle V| + |V\rangle\langle H|$, Alice and Bob can get rid of all bit-flip errors and obtain their uncorrupted pairs $(a^+_{1H}b^+_1 + a^+_{2H}b^+_2)|0\rangle$ by coupling the two spatial modes with their couplers.

Certainly, a phase-flipping error can not be directly purified in this way. However, as pointed out by others [15, 16, 17, 18], a phase-flipping error can be transformed into a bit-flipping error using a bilateral local operation. If a bit-flipping error purification has been successfully solved, phase-flipping errors also can be solved perfectly. In this way the two parties can purify a general mixed state. We only discuss the case with bit-flipping errors below.

For the four-photon state $(a^+_{1H}b^+_1 + a^+_{1V}b^+_1 + a^+_{2H}b^+_2 + a^+_{2V}b^+_2)|0\rangle$ which has the same order of magnitude of probability as the two-photon state $(a^+_{1H}b^+_1 + a^+_{1V}b^+_1 + a^+_{2H}b^+_2 + a^+_{2V}b^+_2)|0\rangle$, if it does not suffer from decoherence, the QND detectors evolve the combined system to

\[
\begin{aligned}
&\rightarrow (a^+_{1H}b^+_1 + a^+_{2H}b^+_2)^2|0\rangle|\alpha e^{i\theta}\rangle_a|\alpha e^{i\theta'}\rangle_b + (a^+_{1V}b^+_1 + a^+_{2V}b^+_2)^2|0\rangle|\alpha e^{i\theta'}\rangle_a|\alpha e^{i\theta}\rangle_b + 2(a^+_{1H}b^+_1 + a^+_{2V}b^+_2)(a^+_{1V}b^+_1 + a^+_{2H}b^+_2)|0\rangle|\alpha e^{i(\theta+\theta')}\rangle_a|\alpha e^{i(\theta+\theta')}\rangle_b,
\end{aligned}
\] (6)

Similar to the case with the two-photon state, Alice and Bob will get the same phase shifts with their homodyne measurements on their coherent states. That is, they both get $2\theta$, $2\theta'$, or $\theta + \theta'$ which corresponds to the four-photon state $(a^+_{1H}b^+_1 + a^+_{1V}b^+_1 + a^+_{2H}b^+_2 + a^+_{2V}b^+_2)|0\rangle$, $(a^+_{1H}b^+_1 + a^+_{1V}b^+_1 + a^+_{2H}b^+_2 + a^+_{2V}b^+_2)|0\rangle$, respectively. The state $(a^+_{1H}b^+_1 + a^+_{2V}b^+_2)(a^+_{1V}b^+_1 + a^+_{2H}b^+_2)|0\rangle$ represents the case that one pair appears at the upper modes and the other at the lower modes after the couplers, and both in the desired state $(a^+_{1H}b^+_1 + a^+_{1V}b^+_1)|0\rangle$. The state $(a^+_{1H}b^+_1 + a^+_{2V}b^+_2)(a^+_{1V}b^+_1 + a^+_{2H}b^+_2)|0\rangle$ denotes that the two pairs both appear at the lower (upper) modes after the couplers. That is, the QND detectors can pick up the state wanted from others with the spatial entanglement resource.

If a bit-flipping error takes place on one of the two photon pairs in the four-photon state, i.e., the state of the two photon pairs becoming $(a^+_{1H}b^+_1 + a^+_{1V}b^+_1 + a^+_{2H}b^+_2 + a^+_{2V}b^+_2)|0\rangle$...
Alice and Bob should discard the instances for which one photon pairs appear at the same mode simultaneously.

QND gates. The protocol in Ref. [31] with QND detectors of photon pairs purified is only half of that with CNOT gates. The fidelity of the photon pairs kept becomes

\[
F_1 = \frac{p_1 + \frac{1}{2} p_2 F_0^2}{p_1 + \frac{1}{2} p_2 [F_0^2 + (1 - F_0)^2]}.
\]

III. ENTANGLEMENT PURIFICATION BASED ON IDEAL SOURCES

After the primary entanglement purification based on PDC sources in Sec. III the photon pairs kept are equivalent to those coming from two ideal sources as the QND1 can distinguish the two-photon states from the four-photon states. Moreover, it shows there are useful photon pairs or not clearly for the two users. In this time, Alice and Bob can exploit the entanglement purification protocols with CNOT gates such as those in Refs. [16, 17] or the PBS protocol proposed by Pan et al. [18] to improve the fidelity of the photon pairs more. Here, a CNOT gate with single photons is far beyond what is experimentally feasible. The PBS protocol requires sophisticated single-photon detectors and its yield of photon pairs purified is only half of that with CNOT gates. The protocol in Ref. [31] with QND detectors designed by Nemoto and Munro [28] can also be used to purify less entangled pairs with X quadrature measurements in a nearly deterministic way as the two users Alice and Bob should ensure that the states \( |\alpha e^{i\theta}\rangle \) cannot be distinguished.

In this section, we will present a different entanglement purification protocol for ideal sources in a completely deterministic way without CNOT gates and sophisticated single-photon detectors. It has the same yield of photon pairs purified as those [16, 17] with CNOT gates, double that of the PBS protocol [18].
cross-Kerr nonlinearities provide the same phase shift with the phase shift $\theta$ with a probability of $\frac{1}{2}$.

Each QND measurement with the diagonal basis is composed of two same cross-Kerr nonlinearities that in Ref. [18], described as follows:

$|\xi\rangle_2=|\alpha\rangle_a|\beta\rangle_b$.

The two pairs can be seen projected to the state $|\Phi^+\rangle_{ab}$ by performing a bit-flipping operation $\sigma_z=|\Phi^+\rangle\langle\Phi^+| - |\Phi^-\rangle\langle\Phi^-|$ on their first photons $a_1$ and $b_1$. With the same way as in Ref. [18] Alice and Bob can make the photon pair in the state $|\Phi^+\rangle_{ab}$. In detail, Alice and Bob first take a measurement with the diagonal basis on their second photons $a_2$ and $b_2$. When they both get the results $|+\rangle$ (or $|-\rangle$), the photon pairs $a_1b_1$ are projected to the state $|\Phi^\pm\rangle_{ab}$. When one gets the result $|+\rangle$ (or the other gets $-\rangle$), they can obtain the state $|\Phi^\pm\rangle_{ab}$ by performing the phase-flipping operation $\sigma_x=|\Phi^\pm\rangle\langle\Phi^\pm| - |\Phi^\mp\rangle\langle\Phi^\mp|$ on the photon $a_1$.

Let us first consider the state $|\Phi^+\rangle_{a_1b_1} \cdot |\Phi^+\rangle_{a_2b_2}$.

$$|\Phi^+\rangle_{a_1b_1} \cdot |\Phi^+\rangle_{a_2b_2} = \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{a_1}|H\rangle_{b_1} + |\Phi^+\rangle_{a_1}|V\rangle_{b_1})$$

$$\oplus \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{a_2}|H\rangle_{b_2} + |\Phi^+\rangle_{a_2}|V\rangle_{b_2})$$

$$= \frac{1}{2}(|\Phi^+\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{b_2} + |\Phi^+\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{b_2} + |\Phi^+\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{b_2} + |\Phi^+\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{b_2}|.$$  (11)

QND$_2$ detectors evolve the combined system composed of four photons and two coherent states to

$$\rightarrow \frac{1}{2}\{(|\Phi^+\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{b_2}$$

$+ |\Phi^+\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{b_2} + |\Phi^+\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{b_2}|$$

$$+ |\Phi^\mp\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{b_2}|$$

$$|\alpha\rangle_a|\beta\rangle_b\}.$$  (12)

When both Alice and Bob get the phase shift $\pi$ with their homodyne measurements on their coherent states, the two photon pairs project to the state $|\Phi^\pm\rangle_{ab}$ and $|\Phi^\pm\rangle_{ab}$. When they both get the phase shift $0$ (2$\pi$ is just the phase shift 0 for the coherent states), they get the state $|\Phi^\pm\rangle_{ab}$ and $|\Phi^\pm\rangle_{ab}$ and they can obtain the state $|\Phi^\pm\rangle_{ab}$ by performing a bit-flipping operation $\sigma_x=|\Phi^\pm\rangle\langle\Phi^\pm| - |\Phi^\mp\rangle\langle\Phi^\mp|$ on their first photons $a_1$ and $b_1$. With the same way as in Ref. [18] Alice and Bob can make the photon pair in the state $|\Phi^+\rangle_{ab}$. In detail, Alice and Bob first take a measurement with the diagonal basis on their second photons $a_2$ and $b_2$. When they both get the results $|+\rangle$ (or $|-\rangle$), the photon pairs $a_1b_1$ are projected to the state $|\Phi^\pm\rangle_{ab}$. When one gets the result $|+\rangle$ (or the other gets $-\rangle$), they can obtain the state $|\Phi^\pm\rangle_{ab}$ by performing the phase-flipping operation $\sigma_x=|\Phi^\pm\rangle\langle\Phi^\pm| - |\Phi^\mp\rangle\langle\Phi^\mp|$ on the photon $a_1$.

For the cross-combinations $|\Phi^+\rangle_{a_1b_1} \cdot |\Phi^\mp\rangle_{a_2b_2}$ and $|\Phi^\mp\rangle_{a_1b_1} \cdot |\Phi^+\rangle_{a_2b_2}$, the QND$_2$ detectors will evolve the combined system to the state in which Alice and Bob can not get the same phase shift with their homodyne measurements on their coherent states. In detail, $|\Phi^+\rangle_{a_1b_1} \cdot |\Phi^\mp\rangle_{a_2b_2}$ will be evolved to

$$\rightarrow \frac{1}{2}\{(|\Phi^+\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{b_2}$$

$+ |\Phi^+\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{b_2}|$$

$$|\alpha\rangle_a|\beta\rangle_b\}.$$  (13)

and $|\Phi^\mp\rangle_{a_1b_1} \cdot |\Phi^+\rangle_{a_2b_2}$ will be evolved to

$$\rightarrow \frac{1}{2}\{(|\Phi^\mp\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{b_2}$$

$+ |\Phi^\mp\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{b_2} + |\Phi^\mp\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{b_2}|$$

$|\alpha\rangle_a|\beta\rangle_b\}.$$  (14)
When Alice gets the phase shift 0 and Bob gets π, their two photon pairs a₁b₁ and a₂b₂ are in the state $(|H⟩_{a₁}|H⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂} + |V⟩_{a₁}|V⟩_{b₁}|H⟩_{a₂}|V⟩_{b₂})$ or $(|V⟩_{a₁}|H⟩_{b₁}|H⟩_{a₂}|V⟩_{b₂} + |H⟩_{a₁}|V⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂})$ with the same probability. In this time, Alice and Bob can not determine in which pair takes place a bit-flipping error. For improving the fidelity of the photon pairs kept, Alice and Bob should discard both these photon pairs, the same as that in the protocol with CNOT gates [16, 17]. When Alice gets the phase shift π and Bob gets 0, they should also discard their two photon pairs.

For the state $|Ψ^+⟩_{a₁b₁} - |Ψ^+⟩_{a₂b₂}$, QND2 detectors evolve the combined system to

$$\rightarrow \frac{1}{2}\{(|V⟩_{a₁}|H⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂}
+ |H⟩_{a₁}|V⟩_{b₁}|H⟩_{a₂}|V⟩_{b₂}
+ |H⟩_{a₁}|V⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂}
+ |H⟩_{a₁}|V⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂})\{αα\}_a \{αα\}_b \}.$$  

When Alice and Bob both get the phase shift π, their two photon pairs are in the state $(|V⟩_{a₁}|H⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂} + |H⟩_{a₁}|V⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂})$. After Alice and Bob perform a measurement with the diagonal basis on their second photons a₂ and b₂, the first photon pair a₁b₁ projects to the state $|Ψ^+⟩_{a₂b₂}$ when they both obtain the outcome $|⟩$ (or $|⟩$); otherwise Alice and Bob will make the pair a₁b₁ in this state by performing a phase-flipping operation $σ_z$. When Alice and Bob both get the phase shift π, their two photon pairs are in the state $(|V⟩_{a₁}|H⟩_{b₁}|H⟩_{a₂}|V⟩_{b₂} + |H⟩_{a₁}|V⟩_{b₁}|V⟩_{a₂}|H⟩_{b₂})$. With the same operations as those in the case where both photon pairs do not contain errors, Alice and Bob will make their first photon pair in the state $|Ψ^+⟩_{a₂b₂}$. In other words, Alice and Bob can not distinguish the two cases that contain no errors in their two photon pairs or that both contain a bit-flipping error. They keep those photon pairs for improving their fidelity in the next round.

By postselection according to the phase shifts of the coherent states, Alice and Bob only keep the first photon pair in the instances where they get the same phase shifts. After this purification process, the new fidelity of the photon pairs kept becomes

$$F' = \frac{F^2}{F^2 + (1 - F)^2}.$$  

We get the same fidelity as in the PBS protocol [18], but the yield is double that in the PBS protocol as Alice and Bob will keep a photon pair when they get the same phase shift, no matter what it is. In PBS protocol, Alice and Bob only keep the instances that each mode has one and only one photon, which makes its yield half those with CNOT gates [16, 17]. Moreover, Alice and Bob use the homodyne measurements on their coherent states to replace the sophisticated single-photon detectors in PBS protocol [18]. This new entanglement purification protocol can be used to improve the fidelity of photon pairs more by iteration.

### IV. DISCUSSION AND SUMMARY

In the primary entanglement purification protocol, Alice and Bob can also use the QND, whose principle is shown in Fig.6, to purify the photon pairs produced by two PDC sources if they can control accurately the overlap time of the photons coming from the upper mode and the lower mode. In essence, the two parties exploit the cross-Kerr nonlinearities, instead of the sophisticated single-photon detectors in the Simon-Pan protocol [27], to complete the task of distinguishing the photon numbers from their modes, without destroying the photons in this time.

![FIG. 6: Schematic diagram showing the principle of simple nondestructive quantum nondemolition detectors (QND)](image)

For a two-photon state without suffering from decoherence $(a^+_{1H}b^+_{1V} + a^+_{1V}b^+_{1H} + a^+_{2H}b^+_{2V} + a^+_{2V}b^+_{2H})|0⟩$, the two parties Alice and Bob will get the same phase shift on their coherent states as QND detectors evolve the combined system to

$$\rightarrow (a^+_{1H}b^+_{1V} + a^+_{1V}b^+_{1H} + a^+_{2H}b^+_{2V} + a^+_{2V}b^+_{2H})|0⟩|αα⟩_a |αα⟩_b \}.$$  

where $θ \neq θ' \neq 2π$. If Alice and Bob get the same results with an X homodyne measurement $(θ$ or $θ')$, they get a photon pair in the state $(a^+_{1H}b^+_{1V} + a^+_{2V}b^+_{2H})|0⟩$. The homodyne measurement provides not only the information about the polarization state of the photon pair but also their spatial modes. If a bit-flipping error takes place, i.e., the state of the pair becoming $|H⟩_{a₁}|V⟩_{b₁} + |V⟩_{a₁}|H⟩_{b₁}$, Alice and Bob will get two different results with their homodyne measurements on their coherent states $|α⟩$ as QND detectors evolve the combined system to

$$\rightarrow (a^+_{1V}b^+_{1H} + a^+_{1H}b^+_{1V} + a^+_{2V}b^+_{2H} + a^+_{2H}b^+_{2V})|0⟩|αα⟩_a |αα⟩_b \}.$$  

One will get the result $θ$ and the other $θ'$. By performing a bit-flipping operation $σ_x = [H|V⟩ + |V⟩H]|0⟩$ on one photon such as the photon controlled by Alice, Alice
and Bob can get rid of all bit-flip errors and obtain their uncorrupted pair \( (a_1^+ b_1^- + a_1^- b_1^+) |0\). 

For the four-photon state \( (a_1^+ b_1^- + a_1^- b_1^+ + a_2^+ b_2^- + a_2^- b_2^+)^2 |0\), the QND detectors evolve the combined system to 
\[
\begin{align*}
(a_1^+ b_1^- + a_1^- b_1^+ + a_2^+ b_2^- + a_2^- b_2^+)^2 |0\rangle &\rightarrow |\alpha \alpha \rangle + |\alpha^\ast \alpha^\ast \rangle + 2 |\alpha \alpha^\ast \rangle + 2 |\alpha^\ast \alpha \rangle,
\end{align*}
\]
This measurement can not be accomplished in a deterministic way, just in a nearly deterministic way. That is, Alice and Bob can not obtain the state \(|\alpha \alpha \rangle |\alpha \alpha \rangle + |\alpha^\ast \alpha^\ast \rangle + 2 |\alpha \alpha^\ast \rangle + 2 |\alpha^\ast \alpha \rangle\) perfectly, which is different from that with QND2 in Sec. III.

In summary, we propose a different purification scheme based on two PDC sources with cross-Kerr nonlinearities. The task of entanglement purification can be completed with two steps in this scheme. First, we provides a primary entanglement purification protocol for PDC sources with QND detectors by transferring the spatial entanglement of photon pairs to their polarization. In this protocol, the QND detectors act as not only the role of CNOT gates but also that of photon number detectors, which provides a good way for the next process to purify the entanglement of the photon pairs more as they make the photon pairs equivalent to those coming from two ideal sources. Compared with the Simon-Pan protocol for PDC sources, this protocol does not require sophisticated single-photon detectors and can distinguish the number of the photons coming from the four modes. This advantage makes the two parties have the ability to complete the entanglement purification perfectly. In the second process for entanglement purification, new QND detectors are designed to act as the role of CNOT gates. This protocol does not require CNOT gates based on linear optical elements, but possesses the same yield of photon pairs purified as the protocols with CNOT gates, double that of the PBS protocol. As a perfect CNOT gate is far beyond what is experimentally feasible with linear optical elements, this protocol may be an optimal one.

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