Comparison of Taylor Galerkin and FTCS models for dam-break simulation

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Abstract. The aim of this paper is to compare the ability of two numerical models to simulate two-dimensional dam-break flows. Dam-break modelling is important in reducing the high-risk of dam failure by determining the water flow movement and its effect on buildings downstream. In this study, the Taylor Galerkin and Forward Time Central Space method is presented to simulate 2-D dam-break flow. The Hansen filter is applied to both methods to reduce numerical instability oscillations in the model. The filters were used for each time step at each computing point. The model is verified by comparing the result simulation with numerical results from previous studies. The result show that Taylor Galerkin numerical scheme with Hansen filter is successful to simulate the problem of 2-D dam-break. Also, the Taylor Galerkin model has better results than the FTCS model. Studies like this are needed for mitigation efforts in the event of dam failure.

1. Introduction
Several factors cause dam failures, such as geotechnical failures, excessive pore water pressures, natural disasters (e.g. earthquake, erosion), inadequate construction strength due to poor quality of materials or design [1]. Dam failures can affect the dam downstream. The impact caused by dam failures is shown in [2]. Dam-break modeling is significant in reducing the high-risk of dam failure. The main goal of the analysis is to determine the water flow movement.

The shallow water equations (SWEs) is usually used to simulate the dam-break flow case. It can be solved using various types of numerical methods, e.g. finite difference method [3,4], finite element method [5], and finite volume method [6-8]. The advantage of the finite difference method is its simplicity, while the finite element method has advantages in terms of efficiency [9] and flexibility for some complex geometries. In this study, Taylor Galerkin (finite element method) and Forward Time
Central Space (finite difference method) methods were used. The Hansen filter [10] is applied to both methods to reduce oscillation due to numerical instability and it has been successfully used in [1,3]. Thus, the purpose of this study is to determine and compare the ability of the Taylor Galerkin (TG) and Forward Time Central Space (FTCS) methods to simulate two-dimensional dam-break flow.

2. Research methodology

2.1. Governing equation

The 2-D shallow water equations can be obtained with an average depth of the Navier-Stokes equation. Excluding the effects of wind, Coriolis force, and momentum due to turbulence and viscosity, the 2-D shallow water equations can be written in vector form as follows:

$$\frac{\partial U}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = S$$  \hspace{1cm} (1)

where

$$U = \begin{bmatrix} h \\ uh \\ v h \end{bmatrix} \quad \begin{bmatrix} uh \\ u^2h + \frac{1}{2}gh^2 \\ uvh \end{bmatrix} \quad \begin{bmatrix} uh \\ uvh \\ v^2h + \frac{1}{2}gh^2 \end{bmatrix} \quad S = \begin{bmatrix} 0 \\ gh(So_x - Sf_x) \\ gh(So_y - Sf_y) \end{bmatrix}$$  \hspace{1cm} (2)

where $h$ is the water depth, $u$ and $v$ are the velocities of $x$- and $y$-directions respectively, $So_x$ and $So_y$ is bed slopes for $x$- and $y$-directions, $g$ is acceleration of gravity, $Sf_x$ and $Sf_y$ is friction slopes along the $x$- and $y$-directions. The friction slopes $Sf_x$ and $Sf_y$ can be evaluated according to Manning’s formula:

$$Sf_x = \frac{n^2u(u^2 + v^2)^{0.5}}{h^{3/2}} \quad Sf_y = \frac{n^2v(u^2 + v^2)^{0.5}}{h^{3/2}}$$  \hspace{1cm} (3)

where $n$ is the Manning’s roughness coefficient.

2.2. Taylor Galerkin method

In this study, the governing equation is solved by using Taylor Galerkin formulation that has been developed first by Donea [11]. This formulation is Taylor Galerkin variant which has been known as Two-step Taylor Galerkin. The governing equation is discretized towards time by using second order of Taylor series as follow:

$$U^{n+1} = U^n + \Delta t \left( \frac{\partial U}{\partial t} \right)^n + \frac{\Delta t^2}{2} \left( \frac{\partial^2 U}{\partial t^2} \right)^n$$  \hspace{1cm} (4)

Rearranging equation (1), so the derivative of time as the derivative of space as follow:

$$\frac{\partial U}{\partial t} = S - \frac{\partial F_i}{\partial x_i}$$  \hspace{1cm} (5)

And for the second derivation is:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial S}{\partial t} - \frac{\partial}{\partial x_i} \left( \frac{\partial F_i}{\partial x_i} \right) = \frac{\partial S}{\partial U} \frac{\partial U}{\partial t} - \frac{\partial}{\partial x_i} \left( \frac{\partial F_i}{\partial U} \frac{\partial U}{\partial t} \right) = G \frac{\partial U}{\partial t} - \frac{\partial}{\partial x_i} \left( A_i \frac{\partial U}{\partial t} \right)$$  \hspace{1cm} (6)

By substituting equation (5) to (6), the new equation can be obtained as follow:

$$\frac{\partial^2 U}{\partial t^2} = G \left( S - \frac{\partial F_i}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left[ A_i \left( S - \frac{\partial F_i}{\partial x_i} \right) \right]$$  \hspace{1cm} (7)
Thus the time discretization is:

\[ U^{n+1} = U^n + \Delta t \left( S \frac{\partial F_i}{\partial x_i} \right)^n + \frac{\Delta t^2}{2} \left\{ G \left( S - \frac{\partial F_i}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left[ A_i \left( S - \frac{\partial F_i}{\partial x_i} \right) \right] \right\}^n \] (8)

In space discretization, several functions are used here for an approximation as below:

\[ U = U^i N_i \; ; \; F^j N_i \; ; \; S = S^i N_i \] (9)

where \( N_i \) is linear shape function for i node, and

\[ G = G^e P_e \; ; \; A_j = A^e_j P_e \] (10)

where \( P_e \) is constant shape function for e element. If weight residual process is applied to equation (8) in \( N_i \) function, equation can be obtained as follow:

\[ \int_{\Omega} \Delta U N_i d\Omega = \Delta t \int_{\Omega} \left( S - \frac{\partial F_i}{\partial x_i} \right)^n N_i d\Omega + \frac{\Delta t^2}{2} \int_{\Omega} \left\{ G \left( S - \frac{\partial F_i}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left[ A_i \left( S - \frac{\partial F_i}{\partial x_i} \right) \right] \right\}^n N_i d\Omega \] (11)

where \( \Omega \) is space domain and consiste mass matrix (M) as follow:

\[ M_{ij} = \int_{\Omega} N_i N_j d\Omega \] (12)

Gauss Theorem is applied to equation (11) in order to obtain weak form, then multypl both sides of equation (11) by \( N_j \) and using equation (12), the equation is transformed into the below

\[ M \Delta U = \Delta t \left\{ \int_{\Omega} \left( S - \frac{\partial F_k}{\partial x_k} \right)^n N_i N_j d\Omega \right\} + \frac{\Delta t^2}{2} \int_{\Omega} \left\{ G \left( S - \frac{\partial F_k}{\partial x_k} \right)^n N_i \right\} + \left\{ \int_{\Gamma} A_k \left( S - \frac{\partial F_k}{\partial x_k} \right)^n \partial N_i / \partial x_k \right\} \]

\[ N_j d\Omega - \int_{\Gamma} \left\{ \int_{\Gamma} A_k \left( S - \frac{\partial F_i}{\partial x_i} \right)^n \left[ N_i N_j d\Gamma \right] \right\} \]

where \( n_k \) is normal vector component with respect to boundary \( \Gamma \). The equation above is able to be solved by two step algorithm likely in [12] then the following equation can be obtained:

\[ M \Delta U = \Delta t \left\{ \int_{\Omega} \left( R_{si} N_i + \left( R_{s}^{n+1/2} - \bar{R}_s^n \right) \right) N_j \right\} d\Omega + \left\{ \int_{\Omega} \left[ F_{x}^{n+1/2} - \bar{F}_x^n + F_{xi} N_i \right] \frac{\partial N_j}{\partial x} \right\} + \left\{ \int_{\Gamma} \left[ -F_{xi} n_x N_i - \left( F_{x}^{n+1/2} - \bar{F}_x^n \right) n_x \right] N_j \right\} \]

\[ M \Delta U = \Delta t \left\{ \int_{\Omega} \left( R_{si} N_i + \left( R_{s}^{n+1/2} - \bar{R}_s^n \right) \right) N_j \right\} d\Omega + \left\{ \int_{\Omega} \left[ F_{y}^{n+1/2} - \bar{F}_y^n + F_{yi} n_i \right] \frac{\partial N_j}{\partial y} \right\} + \left\{ \int_{\Gamma} \left[ -F_{yi} n_y N_i - \left( F_{y}^{n+1/2} - \bar{F}_y^n \right) n_y \right] N_j \right\} \]

2.3. Forward Time Central Space (FTCS) method

According to this scheme, the time derivative is approximated by a first order forward difference. Meanwhile, the space derivative is approached by a central difference. Therefore, the discrete form of SWEs are as follows:
2.4. Numerical filter

The modeling developed in this study is carried out by adding a numerical filter (Hansen filter) which aims to reduce oscillations that occur due to shock waves and to improve the numerical stability of the model. The filters are used in each time step at each computing point. Water depth and velocity values are filtered or updated in each iteration using the following equation:

$$\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} + \frac{F_{x i+1,j}^n - F_{x i-1,j}^n}{2\Delta x} + \frac{F_{y i+1,j}^n - F_{y i-1,j}^n}{2\Delta y} = S$$

(15)

The correction factor (CF) used is 0.99, with P is according to the parameters filtered.

3. Results and discussion

The performance of numerical schemes developed is then evaluated for 2-D partial dam break modeling. In this test case the computational domain is 200 m wide and 200 m long. The grid in this simulation is 5 m x 5 m. The element used in the Taylor Galerkin model is a linear triangular element. This case has been analyzed by various authors [13-18]. This case is simulated to know the ability of numerical model in solving the discontinuity of initial condition. The location of the dam is in the middle of the domain. The upstream water level (hu) is 10 m and in the downstream (hd) is 5 m; 0.1 m; 0 m (dry). The bottom is assumed to be frictionless (n = 0) with no slope. All boundaries are considered as reflective. Minimum water depth (hmin) is set as boundary condition for the dry condition case in the downstream (hd = 0 m). If water depth is smaller than hmin, water depth value is automatically equal to hmin and the velocity becomes zero. Since there is no analytic solution for this case, thus the model is verified by comparing the result simulation with numerical results from previous studies.

Figure 1 (a and b) show the water surface elevation with TG Hansen at time t = 7.2 s after dam break down instantaneously for downstream water level is 5 m and 0.1 m. Figure 1(c) shows the simulation over dry bed condition. Figure 2 shows simulation result using FTCS Hansen. It shows that developed model gives similar result with previous studies results. Furthermore, Taylor Galerkin model has better result than FTCS model especially when hd = 0 m. Due to the failure of the dam, water is released through the breach and forms waves that propagate forward. Based on Taylor Galerkin model, flood wave propagates faster as the downstream water depth is smaller.
Figure 1. Water surface elevation with TG Hansen (a) $hd = 5 \, m$; (b) $hd = 0.1 \, m$; (c) $hd = 0 \, m$, $h_{min} = 10^{-5} \, m$.

Figure 2. Water surface elevation with FTCS Hansen (a) $hd = 5 \, m$; (b) $hd = 0.1 \, m$; (c) $hd = 0 \, m$, $h_{min} = 0.04 \, m$. 
4. Conclusion
In this study, Taylor Galerkin and FTCS numerical scheme have been developed to solve the 2-D shallow water equations for computing dam-break flow. Taylor Galerkin scheme is based on Taylor series in time discretization and Galerkin scheme in space discretization. Whereas for the FTCS scheme, the time derivative is approximated by a first order forward difference and the space derivative is approached by a central difference. To reduce oscillation due to numerical instability in the model, the Hansen filter is applied. Then, the performance of the developed model in simulating 2-D shallow water waves is tested for 2D partial dam-break. Based on the numerical results, the Taylor Galerkin numerical scheme with the Hansen filter can successfully simulate the 2-D dam break problem and is better than the FTCS method.

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