BOUNDED MONOTONE ENUMERATIONS AND APPROXIMATION OF PARTIAL FUNCTIONS

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Abstract. Recently, Chong, Slaman, and Yang showed that the stable Ramsey’s theorem ($\text{SRT}_2^2$) does not imply $\Sigma^0_2$-induction ($I\Sigma^0_2$), see [1]. In their proof, so called, bounded monotone enumerations and the principle $\text{BME}_*$, which states that each tree enumerated by such an enumeration is finite, play a crucial role. They show that $\text{BME}_*$ is strictly weaker than $I\Sigma^0_2$ and incomparable to the $\Sigma^0_2$-bounded collection principle ($B\Sigma^0_2$).

In this note, we relate $\text{BME}_*$ to a principle by Hájek on the approximation of partial functions. This principle is also a consequence of $I\Sigma^0_2$ and incomparable to $B\Sigma^0_2$. Also, we show that both principles prove the totality of the Ackermann-Péters function.

Recently, Chong, Slaman, and Yang showed that the stable Ramsey’s theorem ($\text{SRT}_2^2$) does not imply $\Sigma^0_2$-induction ($I\Sigma^0_2$) and that it is strictly weaker than Ramsey’s theorem for pairs ($\text{RT}_2^2$), see [1]. Their proof uses, so called, bounded monotone enumerations (defined below) and relies on the principle that trees enumerated by them are finite. This principle will be called $\text{BME}_*$.

It is not known whether $\text{BME}_*$ is a consequence of $\text{SRT}_2^2$ or $\text{RT}_2^2$. However, the fact that this principle shows up in the previously mentioned proof gives some indications that this might be the case, see [1, Question 6.3].

In this note, we investigate the strength of $\text{BME}_*$ and relate it to a principle of Hájek on the approximation of partial functions, and we show that the statement that the Ackermann-Péters function is total follows from $\text{BME}_*$. With this we connect the question whether $\text{BME}_*$ follows from $\text{RT}_2^2$ to the open question whether $\text{RT}_2^2$ implies the totality of the Ackermann function, see [1, Question 2].

We will now define bounded monotone enumerations and $\text{BME}_*$. Let $E$ be a function enumerating a tree. That is a function $E : \mathbb{N} \rightarrow P_{fin}(\mathbb{N}^{<\mathbb{N}})$, such that $E[s] \subseteq E[s + 1]$. We will refer to the parameter $s$ of $E$ as stage of the enumeration and use $E$ also to refer to the tree enumerated by $E$, i.e.

$$\{ \tau \in \mathbb{N}^{<\mathbb{N}} \mid \exists s \exists \sigma \in E[s] (\tau \prec \sigma) \}.$$  

Definition 1 ([1]). We say that $E$ is a monotone enumeration if the following holds.

1. The empty sequence is enumerated at the first stage.
2. At each stage only finitely many sequences are enumerated by $E$.
3. If $\tau$ is enumerated by $E$ at stage $s$ and $\tau_0$ is the longest initial segment enumerated by $E$ at a prior stage. Then
(a) no extension of \( \tau_0 \) has been enumerated by \( E \) before the stage \( s \) and
(b) all sequences enumerated at stage \( s \) are extensions of \( \tau_0 \).

Let \( E \) be a monotone enumeration. For an element \( \tau \) enumerated by \( E \) at stage \( s \) we call the maximal initial segments \( (\tau_i) \) of \( \tau \) enumerated at a stage prior to \( s \) the stage-by-stage sequence of \( \tau \).

We say that a monotone enumeration \( E \) is bounded by \( b \) if for each \( \tau \) in \( E \) the length of its stage-by-stage sequence is bounded by \( b \).

**Definition 2.** \( \text{BME}_* \) is the statement that a tree enumerated by a bounded monotone enumeration is finite.

In the following we will use \( P^- \), that is Peano Arithmetic without the induction axiom, as base system.

**Theorem 3** ([1, Propositions 3.5, 3.6]).

1. \( I\Sigma_0 \vdash \text{BME}_* \)
2. \( B\Sigma_0 \not\vdash \text{BME}_* \)

We will relate \( \text{BME}_* \) with the following principle introduced by Hájek in Hájek, Paris [2], see also [3, Chap. I.2.(b)], about approximating iterations of partial functions (represented as formulas).

A formula \( \phi(x, y) \) represents a total function if \( \forall x \exists! y \phi(x, y) \), it represents a partial function if for all \( x \) there is at most one \( y \) satisfying \( \phi(x, y) \). We will denote these statements by \( \text{TFUN}(\phi) \) resp. \( \text{PFUN}(\phi) \).

We say that \( s \) is an approximation to the iteration of such a function, if \( s \) is a finite sequence such that

\[
\forall i < \text{lth}(s) - 1 \forall x, y \ ((x \leq (s)_i \land \phi(x, y)) \rightarrow y \leq (s)_{i+1})
\]

We will denote this statement by \( \text{Approx}_\phi(s) \). The statement that all finite approximations of the iterations of a total resp. partial function given by \( \phi \) is then given by the following.

\[
(T\phi) : \quad \text{TFUN}(\phi) \rightarrow \forall z \exists s \text{Approx}_\phi(s) \land \text{lth}(s) = z
\]
\[
(P\phi) : \quad \text{PFUN}(\phi) \rightarrow \forall z \exists s \text{Approx}_\phi(s) \land \text{lth}(s) = z
\]

(These definitions are made relative to \( I\Sigma_1^0 \).) For a class of formulas \( \mathcal{K} \), the sets \( \{ T\phi \mid \phi \in \mathcal{K} \} \cup I\Sigma_1 \), \( \{ P\phi \mid \phi \in \mathcal{K} \} \cup I\Sigma_1 \) will be denoted by \( T\mathcal{K} \) resp. \( P\mathcal{K} \).

The following theorem collects the known facts about \( T, P \).

**Theorem 4** ([2, 3, Chap. I.2.(b)]).

1. \( T\Sigma_{n+1} \leftrightarrow T\Pi_n, \ P\Sigma_{n+1} \leftrightarrow P\Pi_n, \ P\Sigma_0 \leftrightarrow P\Sigma_1 \).
2. \( T\Sigma_{n+1} \leftrightarrow I\Sigma_{n+1} \).
3. \( I\Sigma_{n+1} \rightarrow P\Sigma_{n+1} \rightarrow I\Sigma_n \). Here all implications are strict.
4. \( P\Sigma_{n+1} \) is incomparable with \( B\Sigma_{n+2} \).
5. \( P\Sigma_{n+1} + B\Sigma_{n+2} \) is strictly weaker than \( I\Sigma_{n+2} \).

1. \( \text{BME}_* \) and \( P\Sigma_1^0 \)

**Theorem 5.** \( P\Sigma_1^0 \vdash \text{BME}_* \).
Proof. Let $E$ be a monotone enumeration. Assume that $E$ is bounded by $b$. Define the partial functions

$$F^i(\tau) := [\text{first stage } s \text{ such that extensions of } \tau \text{ are enumerated in } E]$$

and

$$F(\tau) := E[F^i(\tau)] \setminus E[F^i(\tau) - 1].$$

The partial function $F(\tau)$ yields the set of all extensions of $\tau$ that are newly enumerated at the first stage where extensions of $\tau$ enter into $E$. Since $E$ is a monotone enumeration, this is all (and only) the direct extensions of $\tau$.

The graph of $F'$ can be defined by the following $\Sigma^0_i$-formula

$$\phi'(\tau, s) := \exists \exists \tau' \in E[s] \setminus E[s - 1] \ (\tau \prec \tau') \land \forall \tau' \in E[s - 1] \ (\tau \not\prec \tau').$$

The partial function $F$ can then be defined by the $\Sigma^0_i$-formula

$$\phi(\tau, x) := \exists s \ (\phi'(\tau, s) \land x = [E[s] \setminus E[s - 1]]).$$

We make the assumption that for each code of a finite set $x$ we have that $y \in x$ implies $y \leq x$. (This is for instance the case for the usual coding based on Cantor pairing.)

Then we have for each stage-by-stage enumeration $(\tau_i)$ that $\tau_{i + 1} \leq F(\tau_i)$. As consequence each element in any $b$-bounded state-by-stage enumeration is bounded by $\max_{i \leq b} \{F^i()\}$. Now by $P \Sigma^0_i$ we can bound this value and obtain that $E$ is finite.

Together with Theorem 4 this theorem reproves Theorem 3.

We now separate $P \Sigma^0_i$ form $\text{BME}_i$.

Theorem 6. The theory $P \Sigma_1$ can be written entirely in $\Pi^1_3$ sentences. This is optimal, i.e., it cannot be axiomatized by $\Sigma_3$ sentences over $P^-$. Since $\text{BME}_i$ is of the form a finiteness statement implies another finiteness statement, (i.e., of the form $\Sigma_2 \rightarrow \Sigma_2$), it can be prenexed to either a $\Sigma_3$ or $\Pi_3$ statement and, thus, is $\Delta_3$. Therefore we obtain immediately.

Corollary 7. $\text{BME}_i$ does not imply $P \Sigma^0_1$ over $P^-$. Proof of Theorem 7. To see that the theory $P \Sigma_1$ can be axiomatized with $\Pi^1_3$ sentences one just has to write out the axioms, see Theorem IV.1.27 of [3]. To show that $\Sigma_3$ sentences are not sufficient to axiomatize $P \Sigma^0_1$, we assume that this is the case and alter the construction of the first-order model $H^1(M)$ of $\neg P \Sigma_1$ defined in IV.3.2 from [3] to obtain a contradiction. Let $M \models I \Sigma_3$ and be non-standard. (For the definition of the models below $I \Sigma_1$ suffice but latter we will need $I \Sigma_3$ and its consequence $I \Sigma_2$, $P \Sigma_1$.) Define

$$I^1(M) := \{ a \in M \ | \ a \text{ is majorized by a } \Sigma_1\text{-definable element of } M \},$$

and let $I^1(M, X)$ be defined like $I^1(M)$ but with $\Sigma_1$-definable replaced by $\Sigma_1$-definable in $X$. Further, define $H^1(M, X) := \bigcup_n H^1_n(M, X)$ with $H^1_0(M, X) := I^1(M, X)$ and $H^1_{n+1}(M, X) := I^1(M, H^1_n(M, X))$. (In [3] one considers the model $H^1(M) := H^1(M, \emptyset)$.)

Now assume that $P \Sigma_1$ can be written with sentence of the form

$$(1) \quad \exists x \forall y \exists z \phi(x, y, z).$$
Since $M \models P\Sigma_1$ the models also satisfies these sentences, and since $M \models I\Sigma_2$ there is a minimal $x$ satisfying each of these sentence. Let $X \subseteq M$ be the set of these $x$. By Lemma IV.1.37 of [3] and the fact that $M \models I\Sigma_3$ the set $X$ is not cofinal in $M$.

Now consider the model $H^1(M,X)$. Obviously $X \subseteq H^1(M,X)$. By IV.1.43 of [3] all $\Pi_2$ sentences satisfied by $M$ are also satisfied by $H^1(M,X)$. Combining these shows that $H^1(M,X)$ satisfies the sentences (2) and thus $H^1(M,X) \models P\Sigma_1$.

Replacing $\Sigma_1$-definable by $\Sigma_1$-definable from $X$ and using that $X$ is not cofinal in $M$ in the proof Lemma IV.1.48 and Corollary IV.1.49 one obtains that $H^1(M,X) \not\models P\Sigma_1$ and with this the desired contradiction. □

Remark 8. The above proof actually shows that $P\Sigma_k$ for $k > 0$ cannot be axiomatized by $\Sigma_{k+2}$ sentences.

2. BME$_*$ and the Ackermann function

Theorem 9. $P\Sigma^0_1$ proves the totality of the Ackermann-Péter function.

Proof. Recall that the Ackermann-Péter function $A(m,n)$ is defined by the following equation.

$$A(m,n) := \begin{cases} n + 1 & \text{if } m = 0, \\
A(m - 1,1) & \text{if } m > 0 \text{ and } n = 0, \\
A(m - 1,A(m,n-1)) & \text{if } m > 0. 
\end{cases}$$

Let $\phi_A(m,n,k)$ be the $\Sigma^0_1$-formula describing the graph of $A$ and $\psi_A(m,n) \equiv \exists k \phi_A(m,n,k)$ be the $\Sigma^0_1$-formula which states that $A(m,n)$ is defined. It is clear that

(2) $\forall n \psi_A(0,n)$.

We claim that $I\Sigma^0_1$ proves

(3) $\forall m,n \ (\neg \psi_A(m,n) \rightarrow \exists n' \neg \psi_A(m-1,n'))$.

Indeed, suppose $\neg \psi_A(m,n)$ and in particular that $m > 0$. Then by $I\Sigma^0_1$ we can find a $k$ which is minimal with $\neg \psi_A(m,k)$. If $k = 0$ then by definition of $A$ we have $\neg \psi_A(m-1,1)$. If $k > 0$ then by minimality $A(m,k-1)$ is defined, thus $A(m-1,A(m,k-1))$ cannot be defined and therefore $\neg \psi_A(m-1,A(m,k-1))$.

$\Sigma^0_2$-induction applied to (3) would now immediately give that $\neg \psi_A(m,n)$ implies $\exists n' \neg \psi_A(0,n')$. ($\Sigma^0_2$-induction is required since $\exists n' \neg \psi_A(m,n')$ is $\Sigma^0_2$. Together with (2) this would yield the totality of $A$.

We will show how to use $P\Sigma^0_1$ to bound $n'$ occurring in (3). With this, $I\Sigma^0_1$ suffices to carry out this induction.

Let $\langle m,n \rangle$ denote that Cantor paring function and $(x)_0,(x)_1$ the unparing functions. Recall that $m,n < \langle m,n \rangle$. To cover both parameters of $A(m,n)$ we will use the following modification

$$A'(x) := A((x)_0,(x)_1),A((x)_0,(x)_1))$$

Let $\phi_A'(x,k)$ be the $\Sigma^0_1$-formula describing the graph of $A'$.

Suppose that $A(m,n)$ is not defined or in other words $\neg \psi_A(m,n)$. Let $c := \max(m,n)$. 


Now by $PΣ_1^0$ arbitrary long approximations to $A'$ exists. Since $A(0, n) = n + 1$, and assuming that $(0, 0) = 0$, which is the case for Cantor paring, we have for any approximation $s$ of $A'$

$$(s)_j \geq (j + 1, j + 1), \quad \text{for } j < \text{lth}(s).$$

Therefore, if $A(n, m)$ with $n, m < c$ is defined then $A(n, m) \leq (s)_c$ for any approximation $s$ to $A'$ of length $> c$.

Now as in the argument above, assume that $A(n, m)$ is not defined. Then we know that there is an $k < m$ such that $A(m, k - 1)$ is defined and $A(m - 1, A(m, k - 1))$ is not defined or $A(m - 1, 1)$ is not defined. In particular, for a long enough approximation $s$ of $A'$ we have

$$\exists n' < (s)_c \neg \phi_A(m - 1, n').$$

Since $m, n'$ are bound by $(s)_c$ one obtains by the same argument that

$$\exists n'' < (s)_{c+1} \neg \phi_A(m - 2, n'').$$

Iterating this argument gives then

$$\exists n^* < (s)_{c+m-1} \neg \phi_A(0, n^*)$$

and with this the desired contradiction to (2). This argument can be carried out in $PΣ_1^0$ since this iteration is—after building the approximation $s$ of sufficient ($= 2c$) length—provable in $PΣ_1^0$ which is a consequence of $PΣ_1^0$.

**Theorem 10.** BME$_*$ proves the totality of the Ackermann-Péters function.

**Proof.** We proceed like in the proof of Theorem 9 and use BME$_*$ to bound the existential quantifier in (3).

Thus, suppose that $A(m, n)$ is not defined. We will build a monotone enumeration $E$ containing representations for $A(m, 0), \ldots, A(m, n - 1)$ and $A(m - 1, 1), \ldots, A(0, 1)$ and which is closed under the following rule.

If $A(i, j)$ is defined and represented in $E$

then also $A(i - 1, A(i, j))$ is represented in $E$.

We represent “$A(i, j)$” via the following association to elements of $\mathbb{N}^{<\mathbb{N}}$

$$\tau = \langle τ_0, \ldots, τ_k-1 \rangle \mapsto "A(m + 1 - k, τ_k-1)".$$ 

At stage 0 the empty sequence is enumerated into $E$. At the first stage we enumerate $(0), (1), \ldots, (n - 1)$ and $(n, 1), (n, 0, 1), \ldots (n, 0, \ldots, 0, 1)$ into $E$. This represents the starting condition from above. If at a stage $s$ we realize that an $A(i, j)$, represented in $E$ by $\tau$, is defined, we enumerate the sequences $\tau * (0), \ldots, \tau * \langle A(i, j) \rangle$ into $E$. Now $E$ satisfies all condition for a monotone enumeration except for $\Sigma_1^0$ from Definition 1. However, $E$ can be easily modified to satisfy also this condition. (For instance we may assume that for each $s$ we only realize for one pair $i, j$ that $A(i, j)$ is defined. Then we enumerate the extensions $\tau * (0), \ldots, \tau * \langle A(i, j) \rangle$ of $\tau$ into $E$ at stage $\langle s', s \rangle$ where $s'$ is the stage where $\tau$ was enumerated into $E$. This modification ensures that at each stage only extensions of one $\tau$ are enumerated into $E$.)

The monotone enumeration $E$ is obviously bounded by $m$. Thus by BME$_*$ the tree enumerated by $E$ is finite.
Solid branches are enumerated at the first stage, dashed branches at a later stage (if the corresponding value of $A(i,j)$ is defined). In the figure it is assumed that $A(m-1, A(m,k-1))$ is not defined.

**Figure 1.** Tree enumerated by $E$ in the proof of Theorem 10.

Now looking at the proof of (3) we see that if “$A(m,n)$” is represented in $E$ and undefined then there is an $n'$ which is either $n':=A(m,k-1)$ with $k \leq n$ or $n'=1$ such that also “$A(m-1, n')$” is represented in $E$ and undefined. Now by construction of $n'$ and the tree $E$ we have that $n'$ is contained in a $\tau \in E$. Since the tree enumerated by $E$ is finite we can find an $n^*$ which bounds all $n'$ occurring in iterated applications of (3). With this we can carry out the induction as in Theorem 9 using only $I\Sigma^0_1$. □

We summarize the situation in Figure 2.

**Figure 2.** Status of $BME_*$

Chong, Slaman, Yang actually used certain iterations of the principle $BME_*$ in [1] called $BME_k$ and $BME := \bigcup_k BME_k$ for the union of all these. Since these
principles are complicated to state we refer the reader to Definition 3.10 in [1] and just state that this principle for 1-iterations ($BME_1$) does follow from $BME_*$ and $WKL$.

**Theorem 11.**

$$WKL_0 + BME_* \vdash BME_1$$

In particular, $WKL_0 + P\Sigma^0_1 \vdash BME_1$.

**Proof.** We will use the notation of [1]. Let $(V_1, E_1)$ be a 1-iterated monotone enumeration. Then by $WKL$ there exists an infinite path $X$ through $V_1$. Now $E_1(X)$ is a monotone enumeration that is—by definition of 1-iterated monotone enumeration—infinite. Thus $BME_*$ yields that it is not bounded and, with this, $BME_1$. 

It remains open whether $BME_2$ (or $BME$) follows from $BME_*$, $P\Sigma^0_1$ or $I\Sigma^0_2$. 

**References**

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