Symmetry of singular solutions of degenerate quasilinear elliptic equations

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In ricordo di Fabio

Summary. - We prove radial symmetry of singular solutions to an overdetermined boundary value problem for a class of degenerate quasilinear elliptic equations.

1. Introduction

We consider solutions \( u \) to
\[
\text{div}(a(|\nabla u|)\nabla u) = 0, \quad \text{in } \Omega \setminus \{O\},
\]
which vanish on \( \partial \Omega \)
\[
u = 0, \quad \text{on } \partial \Omega,
\]
and have a positive singularity at the origin \( O \)
\[
limit_{x \to O} u(x) = M \in (0, +\infty].
\]
We prove that if \( u \) satisfies the overdetermined boundary condition
\[
\frac{\partial u}{\partial \nu} = -c, \quad \text{on } \partial \Omega,
\]

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with $c > 0$ constant, then $\Omega$ is a ball centered at $O$ and $u$ is radially symmetric.

To be more specific, we shall assume $\Omega$ to be a bounded connected open set in $\mathbb{R}^n$, $n \geq 2$, containing the origin $O$, and with $C^{2,\alpha}$-smooth boundary $\partial \Omega$. The nonlinearity $a$ is assumed to be a $C^1$ function from $(0, +\infty)$ to $(0, +\infty)$ and satisfying the degenerate ellipticity condition

$$0 < \lambda \leq 1 + \frac{sd'(s)}{a(s)} \leq \Lambda, \text{ for every } s > 0,$$

for some positive constants $\lambda$, $\Lambda$.

Such a class of quasilinear degenerate elliptic equations, which strictly contains the one of $p$-Laplacian type equations, was introduced by Lieberman [5] and independently, in the two dimensional case, by Alessandrini, Lupo and Rosset [3].

With such assumptions the main result of this note is the following.

**Theorem 1.1.** Let $u \in C^{1,\alpha}(\overline{\Omega} \setminus \{O\})$ be a weak solution to (1.1), satisfying the conditions (1.2), (1.3). If, in addition, $u$ satisfies (1.4) then $\Omega$ is a ball centered at the origin $O$ and $u$ is radially symmetric.

Let us observe that in view of the regularity results by Lieberman [5] it is reasonable to treat solutions in the $C^{1,\alpha}$-class.

Note also that for singular solutions satisfying (1.1)–(1.3) the limit $M$ in (1.3) may be finite or infinite depending on the nonlinearity $a$. This fact is particularly evident in the special case when $a(t) = t^{p-2}$, $p > 1$. One readily sees that when $p \leq n$ then we have $M = +\infty$, whereas for $p > n$ one must have $M < +\infty$. See in this respect Kichenassamy and Veron [4] and, for a detailed study of singular solutions in two variables we refer to Rosset [6].

Our proof is based on an adaptation of the well-known method of moving planes by Alexandrov and Serrin. The adaptation of the method to degenerate equations was initiated in [2]. Here we refer mostly to arguments introduced in [1], however, the presence of the singularity adds a little further difficulty, since it may appear, at a first glance, that the method of moving planes cannot be used after the moving plane has crossed the singularity. We shall show that
if this is the case for a certain direction $\xi$, then for the opposite direction $-\xi$ the problem of “hitting the singularity” cannot occur.

**Remark 1.2.** We take this opportunity to point out that an erratum is in order in [1]. In fact, it is improperly stated there that the nonlinearity $a$ may depend on $|\nabla u|$ and also on $u$. This is not correct, in fact one should assume $a = a(|\nabla u|)$ and with this proviso all the statements there are correct.

### 2. Proof of Theorem 1.1

We recall some basic properties of solutions to equation (1.1) that we shall use repeatedly in our arguments. Local solutions to the equation

$$\text{div}(a(|\nabla u|)\nabla u) = 0,$$  \hspace{1cm} (2.1)

are obtained as limits in $C^{1,\alpha}$ of solutions $u_{\epsilon}$ to regularized equations

$$\text{div}(a_{\epsilon}(|\nabla u_{\epsilon}|)\nabla u_{\epsilon}) = 0,$$  \hspace{1cm} (2.2)

where $a_{\epsilon}$ satisfies the same conditions as $a$ and in addition is $C^\infty$ and $a_{\epsilon} \geq \epsilon > 0$ everywhere. Consequently $u_{\epsilon}$ can also be seen as a strong solution to the non-divergence uniformly elliptic equation

$$\sum_{i,j} \left( \delta_{ij} + \frac{a_{\epsilon}'(|\nabla u_{\epsilon}|)}{|\nabla u_{\epsilon}|^2 a_{\epsilon}(|\nabla u_{\epsilon}|)} \frac{\partial u_{\epsilon}}{\partial x_i} \frac{\partial u_{\epsilon}}{\partial x_j} \right) \frac{\partial^2 u_{\epsilon}}{\partial x_i \partial x_j} = 0.$$  \hspace{1cm} (2.3)

Consequently solutions to (2.1) inherit some properties of strong solutions to uniformly elliptic equations.

We quote, in particular, the Harnack inequality, that is, there exists $C = C(\lambda, \Lambda)$ such that: if $u$ solves (2.1) in $B_R(x_0)$ and $u \geq 0$ then

$$\max_{B_{R/2}(x_0)} u \leq C \min_{B_{R/2}(x_0)} u.$$  \hspace{1cm} (2.4)

A further consequence of the use of the regularized solutions is that solutions to (2.1) satisfy the comparison principle in the weak form. That is, if $v, u$ solve (2.1) in a domain $G$ and $v \leq u$ on $\partial G$ then $v \leq u$ also inside. In addition, if $|\nabla u| + |\nabla v| > 0$ in $\overline{G}$ then the strong version of the comparison principle holds, that is if $v \leq u$ on $\partial G$ then either $v \equiv u$ or $v < u$ in $G$. 


Let us also observe that, if \( u \) is a solution to (2.1), then for every constant \( C \), also \( C - u \) is a solution. Hence, by the Harnack inequality, one readily obtains that the solution to (1.1)–(1.3) satisfies
\[
0 < u(x) < M, \quad \text{for every } x \in \Omega \setminus \{O\}. \tag{2.5}
\]

Let us now introduce the moving plane apparatus. For any direction \( \xi \in \mathbb{R}^n, |\xi| = 1 \), and for any \( t \in \mathbb{R} \), we define the hyperplane
\[
\Pi_t^\xi = \{ x \in \mathbb{R}^n \mid x \cdot \xi = t \}.
\]
We denote by \( R_t^\xi \) the reflection in \( \Pi_t^\xi \), that is
\[
R_t^\xi x = 2(t - x \cdot \xi)\xi + x.
\]
We shall denote
\[
(R_t^\xi u)(x) = u(R_t^\xi x).
\]
If we agree to say that if \( x \cdot \xi < t \), \( x \) is on the left hand side of \( \Pi_t^\xi \), and conversely \( x \) is on the right hand side of \( \Pi_t^\xi \) if \( x \cdot \xi > t \), we denote by \( R_t^\xi \Omega \) the reflection of the part of \( \Omega \) which is on the left hand side of \( \Pi_t^\xi \), that is
\[
R_t^\xi \Omega = \{ x \in \mathbb{R}^n \mid x \cdot \xi > t, R_t^\xi x \in \Omega \}.
\]
Given \( \xi \), we fix \( \overline{t} \) such that \( R_{\overline{t}}^\xi \Omega = \emptyset \). Letting \( t > \overline{t} \) increase, we denote by \( t(\xi) \) the largest number such that
\[
R_t^\xi \Omega \subset \Omega, \quad \text{for every } t \in (\overline{t}, t(\xi)).
\]
As is well-known since Serrin [7], when \( t = t(\xi) \) one of the following two cases is satisfied
\[
\begin{align*}
\text{I) } & \partial(R_t^\xi \Omega) \text{ is tangent to } \partial \Omega \text{ at a point } P \notin \Pi_t^\xi, \\
\text{II) } & \partial(R_t^\xi \Omega) \text{ is tangent to } \partial \Omega \text{ at a point } P \in \Pi_t^\xi.
\end{align*}
\]
Let us consider the family of moving planes associated to the opposite direction \( -\xi \) and the corresponding reflections. One can easily verify that
\[
\Pi_t^\xi = \Pi_{-t}^{-\xi}, \quad \text{for every } t,
\]
and also
\[ R_t^\xi = R_{-t}^\xi, \quad \text{for every } t. \]

Now we observe that for every \( s < t(\xi) \) we also have \(-s > t(-\xi)\). In fact
\[ R_s^\xi \Omega \subsetneq \Omega \cap \{x \cdot \xi > s\} \]
and therefore, applying \( R_{-s}^{\xi} \) to both sides,
\[ \Omega \cap \{x \cdot \xi < s\} \not\subsetneq R_{-s}^{\xi} \Omega, \]
that is \( R_{-s}^{-\xi} \Omega \) is not contained in \( \Omega \) and hence \(-s > t(-\xi)\).

Hence, letting \( s \) increase to \( t(\xi) \), we obtain
\[ t(\xi) + t(-\xi) \leq 0 \quad (2.6) \]

Consequently either \( t(\xi) = t(-\xi) = 0 \) or one of the two numbers \( t(\xi), t(-\xi) \) is strictly negative.

If \( t(\xi) = t(-\xi) = 0 \) then, obviously, \( \Omega \) is symmetric in \( \Pi_0^\xi = \Pi_0^{-\xi} \).

Assume now \( t(\xi) < 0 \) (the other case \( t(-\xi) < 0 \) being equivalent).

We simplify our notation by posing
\[ \Pi = \Pi_{t(\xi)}^\xi, \quad R = R_{t(\xi)}^\xi, \quad G = R_{t(\xi)}^\xi \Omega, \quad v = R_{t(\xi)}^\xi u. \]

Since the origin \( O \) is on the right hand side of \( \Pi \), by (2.5), we have that there exists \( N, 0 < N < M \) such that the level set
\[ E = \{x \in \Omega \setminus \{O\} | u(x) \geq N\} \]
is strictly on the right hand side of \( \Pi \).

Consequently, on \( \overline{G} \), \( v = R_{t(\xi)}^\xi u < N \). Now we observe that on \( \partial(G \setminus E) \) we have \( v \leq u \), in fact \( \partial(G \setminus E) \) can be decomposed as
\[ \partial(G \setminus E) = (\partial G \cap \Pi) \cup (\partial G \setminus (E \cup \Pi)) \cup (\partial E \cap G) \]
and we have
\[ \begin{align*}
v &= u, \quad \text{on } \partial G \cap \Pi, \\
v &= 0 \leq u, \quad \text{on } \partial G \setminus (E \cup \Pi), \\
v &< N \leq u, \quad \text{on } \partial E \cap G.
\end{align*} \]
Hence, by the weak comparison principle,

\[ v \leq u, \quad \text{in } G \setminus E, \]

and also, trivially,

\[ v < N \leq u, \quad \text{in } G \cap E. \]

Consequently

\[ v \leq u \quad \text{in } G. \quad (2.7) \]

From now on we rephrase arguments in [1] to prove that \( \Pi \) is a plane of symmetry for \( \Omega \).

Let \( U \) be an \( \epsilon \)-neighborhood of \( \partial \Omega \) in \( \Omega \), with \( \epsilon \) small enough to have \( |\nabla u| > 0 \) in \( U \) and \( O \notin U \). Let \( A \) be the connected component of \( (RU) \cap U \) such that \( P \in \partial A \). In view of the boundary conditions (1.2), (1.4), we have

\[ (u - v)(P) = 0, \quad \nabla(u - v)(P) = 0. \]

Moreover, when case II) occurs, by applying the arguments in [7] to equation (2.1) we also have that

\[ \frac{\partial^2}{\partial \eta^2}(u - v)(P) = 0, \]

for every direction \( \eta \). Since \( A \subset G \cap U \), \( u - v \) is a non-negative solution of a uniformly elliptic equation in \( A \). By using the Hopf lemma when case I) occurs and its variant due to Serrin [7, Lemma 2] when case II) occurs and by the strong comparison principle, it follows that

\[ u = v, \quad \text{in } A. \quad (2.8) \]

Now, let us prove that \( R(\partial \Omega) \subset \partial \Omega \), obtaining that \( \Pi \) is a plane of symmetry of \( \Omega \). Assume, by contradiction, that there exists a point \( Q \in R(\partial \Omega) \setminus \partial \Omega \). Let \( \gamma \) be an arc in \( R(\partial \Omega) \) joining \( Q \) with \( P \). One can find a subarc \( \gamma' \) in \( \gamma \cap (U \cup \partial \Omega) \) having as endpoints \( P \) and a point \( R \in U \). Since in any neighborhood of \( \gamma' \) one can find points of \( RU \cap U \), which can be joined to \( P \) through paths inside \( RU \cap U \), it follows that \( \gamma' \subset \partial A \). Therefore the set \( R(\partial \Omega) \cap U \cap \partial A \) is non-empty and on such a set \( v = 0 < u \), contradicting (2.8).
We have therefore proved that for any direction $\xi$ there exists a plane $\Pi$ orthogonal to $\xi$ which is a plane of symmetry for $\Omega$. Hence $\Omega$ is a ball $B_R(x_0)$. It remains to prove that $x_0 = O$ and that $u$ is radially symmetric.

Let $b = b(s)$ be the inverse function to $ta(t)$, let $K = R^{n-1}c a(c)$ and set

$$v(r) = \int_r^R b(K \rho^{1-n}) d\rho, \quad 0 < r \leq R.$$ 

One can easily verify that $v(|x - x_0|)$ solves

$$\begin{cases} 
\text{div}(a(|\nabla v|) \nabla v) = 0, & \text{in } B_R(x_0) \setminus \{x_0\}, \\
v = 0, & \text{on } \partial B_R(x_0), \\
\frac{\partial v}{\partial \nu} = -c, & \text{on } \partial B_R(x_0).
\end{cases}$$

Now, in $B_R(x_0) \setminus \{O, x_0\}$, $u(x)$ and $v(|x - x_0|)$ satisfy the same elliptic equation and have the same Cauchy data on $\partial B_R(x_0)$.

We recall that for such an equation the unique continuation property holds as long as one of the two solutions has non-vanishing gradient. It follows from a standard continuity argument that $u(x) = v(|x - x_0|)$ for every $x \neq O, x_0$.

Consequently $u$ and $v$ must have the same singular point, that is $x_0 = O$ and finally $u(x) = v(|x|)$. We conclude the proof observing that the singular value $M$ can be computed as follows

$$M = \int_0^R b \left( \frac{R}{\rho} \right)^{n-1} c a(c) \, d\rho.$$ 

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