OPTIMAL BOUNDS FOR SELF-INTERSECTION LOCAL TIMES

GEORGE DELIGIANNIDIS AND SERGEY UTEV

Abstract. For a random walk \( S_n, n \geq 0 \) in \( \mathbb{Z}^d \), let \( l(n,x) \) be its local time at the site \( x \in \mathbb{Z}^d \). Define the \( \alpha \)-fold self intersection local time \( L_n(\alpha) := \sum_{x} l(n,x)^{\alpha} \), and let \( L_n(\alpha) \) be the corresponding quantity for \( d \)-dimensional simple random walk. Without imposing any moment conditions, we show that the variances of the local times \( \text{var}(L_n(\alpha)) \) of all genuinely \( d \)-dimensional random walk are bounded above by the corresponding characteristics of the simple symmetric random walk in \( \mathbb{Z}^d \), i.e. \( \text{var}(L_n(\alpha)) \leq C \text{var}(L_n(\alpha|, d)) \sim K_{d, \alpha} \nu_{d, \alpha}(n) \). In particular, variances of local times of all genuinely \( d \)-dimensional random walks, \( d \geq 4 \), are similar to the \( 4 \)-dimensional symmetric case \( \text{var}(L_n(\alpha)) = O(n) \). On the other hand, in dimensions \( d \leq 3 \) the resemblance to the simple random walk \( \lim_{n \to \infty} \text{var}(L_n(\alpha))/\nu_{d, \alpha}(n) > 0 \) implies that the jumps must have zero mean and finite second moment.

1. Introduction and main results

Let \( X, X_1, X_2, \ldots \) be independent, identically distributed, \( \mathbb{Z}^d \) valued random variables, and define the random walk \( S_0 := 0, S_n = \sum_{j=1}^n X_j \), for \( n \geq 1 \). Let \( l(n, x) = \sum_{j=1}^n I(S_j = x) \) be the local time of \((S_n)_n\) at the site \( x \in \mathbb{Z}^d \), and define for a positive integer \( \alpha \) the \( \alpha \)-fold self intersection local time

\[
L_n = L_n(\alpha) = \sum_{x \in \mathbb{Z}^d} l(n,x)^{\alpha} = \sum_{i_1, \ldots, i_\alpha = 0}^n I(S_{i_1} = \cdots = S_{i_\alpha}).
\]

Our method also applies to the more general case where the \( X_i \) are independent but not identically distributed. To distinguish between the two cases, we shall refer to random walk with independent identically distributed increments as the i.i.d. case. Following Spitzer [19], in the i.i.d. case, we call \( X_i \) and the random walk it generates genuinely \( d \)-dimensional if the support of the variable \( X_1 - X_2 \) linearly generates \( d \)-dimensional space. Finally let \( \Gamma = [0, 2\pi]^d \).

The quantity \( L_n(\alpha) \) has received considerable attention in the literature due to its relation to self-avoiding walks and random walks in random scenery. In particular let the random scenery \( \{\xi_x, x \in \mathbb{Z}^d\} \) be a collection of i.i.d. random variables, independent of the \( X_i \), and define the process \( Z_0 = 0, Z_n = \sum_{i=1}^n \xi_{S_i} \). Then \((Z_n)_n\) is commonly referred to as random walk in random scenery and was introduced in Kesten and Spitzer [13], where functional limit theorems were obtained for \( Z_{[nt]} \) under an appropriate normalization for the case \( d = 1 \). The case \( d = 2 \), with \( X_i \) centered with non-singular covariance matrix, was treated in [4] where it was shown that \( Z_{[nt]}/\sqrt{n \log n} \) converges weakly to Brownian motion. As is obvious from the identities \( Z_n = \sum_{x \in \mathbb{Z}^d} l(n,x)\xi_x \), \( \text{var}(Z_n) = \text{var}[L_n(2)]\text{var}(\xi_x) \), limit theorems for \( Z_n \) usually require asymptotics for the local times of the random walk \((S_n)_n\).

Such asymptotics are usually obtained from Fourier techniques applied to the characteristic function \( f(t) = \mathbb{E}[\exp(it \cdot X)] \) under the additional assumption of a Taylor expansion of the form \( f(t) = 1 - (\Sigma t, t) + o(|t|^2) \) where \( \Sigma \) is the positive definite covariance matrix \([4, 5, 6, 12, 20]\), which further requires that \( \mathbb{E}|X|^2 < \infty \) and \( \mathbb{E}X = 0 \). Similar restrictions are also required for the application of local limit theorems such as in \([14, 17]\).

In this paper, motivated by the results of Spitzer [19] for genuinely \( d \)-dimensional random walks and the approach of Becker and König [3] (see also Asselah [2] where non-integer \( \alpha \) is also treated) we shall study the asymptotic behavior of \( \text{var}(L_n(\alpha)) \) without imposing any moment.

2000 Mathematics Subject Classification. Primary 60G50, 60F05.

Key words and phrases. Self-intersection local time, random walk in random scenery.

1
assumptions on the random walk. The central idea behind our approach is to compare the self-intersection local times \( L_n(\alpha) \) of a general \( d \)-dimensional walk with those of its symmetrised version. In addition we will compare the self-intersection local times of a general \( d \)-dimensional random walk with those of the \( d \)-dimensional simple symmetric random walk, \( S_n^{d,\alpha} \) which we denote by \( L_n(\alpha|\epsilon, d) \). Recall that simple random walk in \( \mathbb{Z}^d \) is defined as \( S_n^{d,\alpha} := 0, S_n^{d,\alpha} := \sum_{j=1}^n X_j^{d,\alpha} \) for \( n \geq 1 \), where for \( k = 1, \ldots, d \) \( \mathbb{P}(X_j^{d,\alpha} = \pm e_k) = 1/2d \) and \( e_k \) is the \( k \)-th unit coordinate vector. It is well-known that with some positive constant \( K_{\alpha,d} \), \( \text{var}[L_n(\alpha|\epsilon, d)] \sim K_{\alpha,d}v_{d,a}(n) \) where

\[
\begin{align*}
v_{1,a}(n) &= n^{1+\alpha}, \quad v_{2,a}(n) = n^2 \log(n)^{2\alpha-4}, \quad v_{3,a}(n) = n \log(n) \quad \text{and} \quad v_{d,a}(n) = n, \quad d \geq 4.
\end{align*}
\]

Several other cases have been treated in the literature, using a variety of methods.

A careful look at the literature reveals that the most difficult case in \( d = 2 \) is the near transient recurrent case, where \( \mathbb{P}(S_n) = 0 \sim C/n \), which corresponds to genuinely 2-dimensional symmetric recurrent random walks, which will be referred to as a critical case. Surprisingly enough, the variance of the self-intersection local times in the critical case is asymptotically the largest.

**Theorem 1.** Let \( X_i \) be i.i.d., genuinely \( d \)-dimensional. Then,

\[
\text{var}(L_n(\alpha)) \leq c_{n,X} \text{var}(L_n(\alpha|\epsilon, d)) \leq C_{n,X}v_{d,a}(n).
\]

The result was motivated by [19] and [3] (and improves related results of Becker and König for \( d = 3 \) and \( d = 4 \)). Several cases treated in [2, 4, 5, 8, 10, 7, 3, 17] can then be obtained as particular cases.

Moreover, we also show the surprising reverse, more exactly that the right asymptotic of \( \text{var}(L_n) \) implies that the jumps must have zero mean and finite second moment.

**Theorem 2.** Let \( X_i \) be i.i.d., genuinely \( d \)-dimensional and \( d = 1, 2, 3 \). If

\[
\liminf_{n \to \infty} \frac{\text{var}(L_n(\alpha))}{v_{d,a}(n)} > 0,
\]

then \( \mathbb{E}|X|^2 < \infty \) and \( \mathbb{E}X = 0 \).

As it follows from Theorem 3, given below, for \( d = 2, 3 \) and Theorem 5.2.3 in Chen [8] for \( d = 1 \), if \( \mathbb{E}X = 0 \) and \( 0 < \mathbb{E}|X|^2 < \infty \), then \( \liminf_n \text{var}(L_n(\alpha)) / v_{d,a}(n) > 0 \).

For general genuinely \( d \)-dimensional random walks with finite second moments and zero mean, the asymptotic behavior is similar to \( d \)-dimensional simple symmetric random walk, again the most complicated case being \( d = 2 \). Also, as it follows from our general bounds (see Proposition 4 and Corollary 7), the asymptotics for the genuinely \( d \)-dimensional random walk can be reproduced by those of the symmetric one-dimensional random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer [13]. The proofs are based on adapting the Tauberian approach developed in [10].

**Theorem 3.** Let \( d = 1, 2, 3 \), and suppose that for \( t \in [-\pi, \pi]^d \) we have

\[
f(t) = 1 - \gamma|t| + R(t), \quad \text{for} \quad d = 1, \quad \text{or} \quad f(t) = 1 - \langle \Sigma t, t \rangle + R(t), \quad \text{for} \quad d = 2, 3,
\]

where \( \Sigma \) is a non-singular covariance matrix and \( R(t) = o(|t|) \) for \( d = 1 \), and \( o(|t|^2) \) for \( d = 2, 3 \) as \( t \to 0 \). Then

\[
\text{var}(L_n(\alpha)) \sim \begin{cases}
\frac{\pi^2}{12}(\alpha^2)(\alpha-1)^2 - n^2 \log(n)^{2\alpha-4}, & \text{for} \quad d = 1, \\
\frac{2(\alpha)^2(\alpha-1)^2}{(\pi)^{2\alpha-2}} - n^2 \log(n)^{2\alpha-4}(\kappa + 1), & \text{for} \quad d = 2, \quad \text{and} \\
(\kappa_1 + \kappa_2)n \log n, & \text{for} \quad d = 3, \quad \alpha = 2,
\end{cases}
\]

where \( \kappa = \int_0^\infty \int_0^\infty dr ds \left[(1 + r)(1 + s)\sqrt{(1 + r + s)^2 - 4rs} \right]^{-1} - \pi^2/6 \) and \( \kappa_1, \kappa_2 \) are defined in (7) and (9) respectively.

Moreover, if \( L'(n, \alpha) \) is the self-intersection local time of another random walk whose characteristic function also satisfies (1) then \( L'(n, \alpha) = L(n, \alpha)(1 + o(1)) \).
The methods developed in this paper are used by the first author and K. Zemer in [11] to prove that the range of 1-stable random walk in \( \mathbb{Z} \) and simple random walk in \( \mathbb{Z}^2 \) has the Fölner property and therefore to compute the relative complexity of random walk in random scenery in the sense of Aaronson [1].

2. Proofs

2.1. General bounds. We first develop a technique to treat random walks with independent but not necessarily identically distributed increments.

**Proposition 4.** (General upper bound) Assume that \( X_i \) are independent \( \mathbb{Z}^d \)-valued random variables and let \( S_{a,v} := X_a + \ldots + X_{a+v} \). Suppose further that for all \( n \in \mathbb{N} \), and integers \( a, u, b, v \geq 0 \), with \( a + u \leq b \) and any \( x \in \mathbb{Z}^d \) we have

\[
\begin{align*}
(A) & \quad \mathbb{P}(S_{a,u} + S_{b,v} = x) \leq \phi(u + v), \\
(B) & \quad \mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,v} = 0) \leq \psi(u, v),
\end{align*}
\]

where \( \phi(u) \) is non-increasing, \( \psi(u, v) \) is non-increasing in \( u \) and is non-decreasing and sub-additive in \( v \) in the sense that \( \psi(u, v + w) \leq A_\psi[\psi(u, v) + \psi(v, w)] \), for some constant \( A_\psi \) independent of \( u, v \). Then, for some constant \( K = c A_\psi(1 + A_\psi)^{\alpha - 2} \) depending only on \( \alpha \)

\[
\text{var}(L_n(\alpha)) \leq Kn \left( \sum_{i=0}^{n-1} \phi(i) \right)^2 \sum_{i,j,k=0}^{n-1} \left[ \phi(j \vee i) \phi(k \vee i) + \phi(j) \psi(i + k, j) \right].
\]

**Proof of Proposition 4.** We first write out the variance as a sum

\[
\text{var} L_n(\alpha) = \langle \alpha \rangle^2 \sum_{k_1 \leq \ldots \leq k_{n-1} \leq \ldots \leq k_1} \left( \mathbb{P}[S_{k_1} = \ldots = S_{k_n}, S_{l_1} = \ldots = S_{l_n}] \right.
\]

\[
\left. - \mathbb{P}[S_{k_1} = \ldots = S_{k_n}] \mathbb{P}[S_{l_1} = \ldots = S_{l_n}] \right).
\]

An important role is played by the manner in which the two sequences are interleaved, since for example if \( k_1 \leq l_1 \) and we arrange the two sequences in an ordered sequence of combined length \( 2\alpha \) which we denote as \( (p_1, \ldots, p_{2\alpha}) \); we also define \( (\epsilon_1, \ldots, \epsilon_{2\alpha}) \) where \( \epsilon_i = 0 \) if \( p_i \) came from \( k := \{k_1, \ldots, k_n\} \), and \( \epsilon_i = 1 \) if \( p_i \) came from \( l := \{l_1, \ldots, l_n\} \). Finally we define two new sequences \( m_0, m_1, \ldots, m_{2\alpha-1} \), and \( \delta_1, \ldots, \delta_{2\alpha-1} \), where \( m_0 := p_1, m_i = p_{i+1} - p_i \) and \( \delta_i = \epsilon_{i+1} - \epsilon_i \), for \( i = 1, \ldots, 2\alpha - 1 \). Notice that since we assume that \( k_1 \leq l_1 \), we have \( p_1 = k_1 \) and \( \epsilon_1 = 0 \). Let \( v(\delta) := \sum_{i=1}^{2\alpha-1} |\delta| \), denote the interlacement index. The terms with \( v = 1 \) vanish, while the terms with \( v = 2 \) will be considered separately.

We first consider the sum \( I_n \) of the terms with \( v \geq 3 \) for which we drop the negative part and sum over the free index \( m_0 = k_1 \) to obtain the bound

\[
I_n \leq c(\alpha)n \sum_{m_1, \ldots, m_{2\alpha-1}} \sup_{x \in \mathbb{Z}^d} \mathbb{P}(S_{w,m_t} = \delta_t x),
\]

where \( c(\alpha) \) denotes generic constants depending only on \( \alpha \), which may change from line to line. Of these \( 2\alpha - 1 \delta \)'s, exactly \( u := 2\alpha - 1 - v \) are equal to 0, and therefore

\[
I_n \leq c(\alpha)n \left( \sum_{i=0}^{n-1} \phi(i) \right)^u \sum_{j_1, \ldots, j_v=0}^{n} \prod_{x = 1}^{v} \sup_{x \in \mathbb{Z}^d} \mathbb{P}(S_{w_t,j_t} = \delta_t x).
\]

Notice that if \( S^{(1)}, \ldots, S^{(v)} \) denote independent random walks then, assuming without loss of generality that \( j_1 \leq \cdots \leq j_v \), we have that

\[
\sum_{x} \prod_{t=1}^{v} \mathbb{P}(S_{w_t,j_t} = \delta_t x) \leq \left( \prod_{t=2}^{v-1} \max_{x} \mathbb{P}(S^{(t)}_{j_t} = x) \right) \mathbb{P}(S^{(1)} = \delta_v S^{(v)}).
\]
Writing $G_n := \sum_{i=0}^{n} \phi(i)$, since $\phi$ is non-increasing we have that

$$\Delta_{n,v} := \sum_{0 \leq j_1 \leq \cdots \leq j_v \leq n} \prod_{l=2}^{v} \phi(j_l \vee j_{l-1}) = G_n \Delta_{n,v-1},$$

and repeating this procedure, for $v \geq 3$ we have that $\Delta_{n,v} \leq \Delta_{n,3}G_n^{v-3}$. Combining the two bounds and summing over $v = 3, \ldots, 2\alpha - 1$, we have the upper bound

$$\sum_{v=3}^{2\alpha-1} c(\alpha) nG_n^{2\alpha-1-v} \Delta_{n,v} \leq c(\alpha) nG_n^{2\alpha-1-v+3} \Delta_{n,3} = c(\alpha) nG_n^{2\alpha-4} \Delta_{n,3}.$$

Next we consider the sum $J_n$ over the terms with $v = 2$, which occurs when for some $j$, the indices $l_1, \ldots, l_\alpha$ all lie in $[k_j, k_{j+1}]$. Then it is easy to see that this sum $J_n$ is bounded above by

$$J_n \leq c(\alpha) n \sup_{w_0, \ldots, w_{2\alpha-1}=0} \sum_{m_0, \ldots, m_{2\alpha-2}=0}^{2\alpha-2} \prod_{l=1}^{\alpha-1} \mathbb{P}(S_{w_l, m_l} = 0)$$

$$\leq c(\alpha) nG_n^{2\alpha-2} \sup_{w_0, \ldots, w_{\alpha}} \sum_{m_0, \ldots, m_{\alpha}=0}^{\alpha-1} \prod_{l=1}^{\alpha-1} \mathbb{P}(S_{w_l, m_l} = 0)$$

$$\leq c(\alpha) nG_n^{2\alpha-2} A_\psi (1 + A_\psi)^{\alpha-2} \left( \sum_{m_2, \ldots, m_{\alpha-1}}^{\alpha-1} \prod_{l=2}^{\alpha-1} \phi(m_l) \right) \times \sum_{m_0, m_1, m_\alpha}^{\alpha-1} \phi(m_1) \psi(m_0 + m_\alpha, m_1)$$

$$\leq c(\alpha) A_\psi (1 + A_\psi)^{\alpha-2} nG_n^{2\alpha-4} \sum_{i,j,k=0}^{\alpha-1} \phi(j) \psi(i + k, j). \quad \square$$

The following corollary provides explicit bounds in the cases that are usually considered in the literature.

**Corollary 5.** Assume that the conditions of Proposition 4 are satisfied with $\phi(m) = Tm^{-r}$ and $\psi(m, k) = Tm^{-r-1}(k \wedge m)$. Then,

$$\text{var}(L_n(\alpha)) \leq c(\alpha) n^{2\alpha-2} \times \begin{cases} n^2 \log(n)^{2\alpha-4}, & \text{if } r = 1, \\ n^{1-2r}, & \text{if } 1 < r < 3/2, \\ n \log(n), & \text{if } r = 3/2, \text{ and} \\ n, & \text{if } r > 3/2. \end{cases}$$

Several relevant results treated so far in [3, 4, 7, 20, 8, 10, 14, 17] are not only obtained as a special case but also extended to the case of independent but not necessarily identically distributed variables, for example by applying the local limit theorem, as it is conducted in [14].

Also when $X_t$ is in the domain of attraction of the one-dimensional symmetric Cauchy law ([9, 10]), or in the case of strongly aperiodic planar random walk with second moments ([4, 7, 20, 14, 17]), it is well known that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m, k) = Tm^{-2}(k \wedge m)$.

However, we can do better for symmetrized variables and show that condition (A) implies (B), which together with the comparison technique motivate the following results.
Proposition 6 (Bound via comparison with symmetrised). Let $X_i$ be independent, $d$-dimensional random variables and $f_i(t) := \mathbb{E}\exp(itX_i)$, and assume that there exists a non-negative measurable function $f(t)$, $0 \leq f(t) \leq 1$ and positive non-increasing sequence $\phi(m)$ such that

$$
|1 - f_i(t)| \leq T f(t), \quad |f_i(\pm t)| \leq f(t), \quad \text{and} \quad \int_{\Gamma} f(t)^m dt \leq \phi(m),
$$

for all $i, m \geq 0$, and $t \in \Gamma$. Then, for some constant $K = c(\alpha, d, T)$

$$
\text{var}(L_n(\alpha)) \leq Kn \left( \sum_{i=0}^{n-1} \phi([i/2]) \right)^{2\alpha-4} \sum_{j=0}^{n} \sum_{k=j}^{2n} \phi([j/2]) \phi([k/2]) =: \Delta_n(\alpha, \phi).
$$

Proof of Proposition 6. Using the notation of Proposition 4, for positive integers $a, u, b, v$, with $a + u \leq b$, $c_\gamma = \pm 1$ and any $x \in \mathbb{Z}^d$

$$
\mathbb{P}(S_{a,u} + (\epsilon, S_{b,v}) = x) \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f_j(\epsilon, t)| dt \leq \frac{1}{(2\pi)^d} \int_{\Gamma} f(t)^{u+v} dt \leq \frac{1}{(2\pi)^d} \phi(u + v)
$$

To find $\psi(u, v)$, notice that since $f(t) \geq 0$,

$$
\phi(m) \geq \int_{\Gamma} f(t)^m [1 - f(t)^m] dt = \int_{\Gamma} f(t)^{m+j} dt \geq m \int_{\Gamma} f(t)^m (1 - f(t)) dt =: Q(2m)
$$

whence $Q(m) \leq \phi(m/2)/m$, where $[\cdot]$ denotes integer part. Therefore,

$$
\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,v} = 0) \leq CT \int_{\Gamma} f(t)^m (1 - f(t)) dt \leq CT \phi([u/2])/u,
$$

and it easily follows that (B) is satisfied with $\psi(u, v) := \phi([u/2]) \min(u, v)/u$. Thus all conditions of Proposition 4 are satisfied and the result follows from direct application of (4).

The following Corollary, allows for the case where $\phi(m)$ is regularly varying.

Corollary 7. Assume that the conditions of Proposition 6 are satisfied with $\phi(m) = h(m)m^{-r}$, $r \geq 1$, where $h(x)$ is a slowly varying at $x \to \infty$. Then,

$$
\text{var}(L_n(\alpha)) \leq K \Delta_n(\alpha, \phi) \leq c_a T^{2\alpha-2} \begin{cases} 
\sum_{k=1}^{n} k^{2(n-k)} k^{2\alpha-4}, & \text{for } r = 1, \\
(n^4 - 2r^2) h^2(n), & \text{for } 1 < r < 3/2, \\
n \sum_{k=1}^{n} k h(k)^2/k, & \text{for } r = 3/2, \text{ and } \\
n, & \text{for } r > 3/2.
\end{cases}
$$

Again, the cited relevant results treated so far are not only obtained as a special case but also extended to dependent variables such as a random walk on a hidden Markov chain. In addition, following Kesten and Spitzer [13] we can mimic the behaviour of genuinely $d$-dimensional random walk by constructing a one dimensional symmetric random walk with characteristic function $f(t) = 1 - c|t|^{1/r} + o(|t|^{1/r})$ with $r = 2/d$ for $d = 2, 3$ and $r = 1/2$ for $d \geq 4$.

The following example of genuinely $2$-dimensional recurrent walk with infinite variance was motivated by Spitzer [19, pp. 87].

Example 8. Let $S_n = \sum_{i=1}^{n} X_i$ be a random walk in $\mathbb{Z}^2$, such that $\mathbb{P}(|X| = k) = c/(k^3 \log(k)^\gamma)$, for $k \geq 4$ and $\gamma \in [0, 1]$. Then we have $\text{var}(L_n(\alpha)) \leq cn^2 \max\{\log n, \log \log n\}^{2\alpha-4} \log n^{-2(1-\gamma)}$, for $n \geq 10$. Under these assumptions we have $\mathbb{P}(S_n = 0) \leq c/n \log(n)^{1-\gamma}$, which is in the critical range, where the random walk is recurrent, without second moment. To show it, we notice that by lengthy straightforward calculation the characteristic function of $X$ satisfies (4) with

$$
\phi(n) = c \frac{n}{n \log(e \vee n)^{1-\gamma}}, \quad f(t) = \exp[-A|t|^2 h(|t|^2)], \quad \text{where} \quad h(r) := [1 + \log(1/r)]^{1-\gamma},
$$

and the sequence $\phi(m)$ is identified via Fourier inversion, polar coordinates and a Laplace argument

$$
\int_{\Gamma} f(t)^m dt \leq c \int_0^1 \exp [-nr (1 + \log(1/r))^{1-\gamma}] + O(e^{-n}) \leq \frac{c}{n \log(e \vee n)^{1-\gamma}} =: \phi(n).
$$
2.2. Bounds for identically distributed variables.

**Proposition 9** (General upper bound for i.i.d.). Let \( X_i \) be i.i.d. \( \mathbb{Z}^d \)-valued random variables, and suppose that for all \( n \in \mathbb{N} \), positive integers \( a, u, b, v \), with \( a + u \leq b \), and any \( x \in \mathbb{Z}^d \)

\[
\mathbb{P}(S_{a,u} + S_{b,v} = x) \leq \phi(u + v),
\]

where \( \phi(m) \) is a non-increasing sequence. Then, for some constant \( K = c(\alpha) \)

\[
\text{var}(L_n(\alpha)) \leq Kn\left(\sum_{i=0}^{n-1} \phi(i)\right)^{2a-4} \sum_{j=0}^{n} j \phi(j) \sum_{k=j}^{\lfloor an \rfloor + 1} \phi([k/\alpha]) .
\]

**Proof of Proposition 9.** By inspecting the proof of Proposition 6, we notice that only need to bound the \( J_n \) term. Consider a typical ordering

\[
0 \leq i_1 \leq \cdots \leq i_k \leq j_1 \leq \cdots \leq j_a \leq i_{k+1} \leq \cdots \leq i_n \leq n.
\]

Let us change variables to \((m_0, \ldots, m_{2\alpha})\) such that \( m_0 + \cdots + m_{2\alpha} = n \). Then the contribution from this case to \( J_n \) is

\[
\sum_{m_0, \ldots, m_{2\alpha}} \prod_{1 \leq j \leq 2\alpha-1, j \neq k, k+\alpha} \mathbb{P}(S_{m_j} = 0) \left[ \mathbb{P}(S_{m_k + m_{k+\alpha}} = 0) - \mathbb{P}(S_{m_k + \cdots + m_{k+\alpha}} = 0) \right].
\]

For \( j \neq \alpha, k + \alpha \) keep \( m_j \) fixed and sum over \( m = m_k + m_{k+\alpha} \), from 0 to \( M \) which depends on \( n \), and the \( m_j \) for \( j \neq k, k + \alpha \). Then for given \( m_{k+1}, \ldots, m_{k+\alpha-1} \), the term in the sum is

\[
\sum_{m=0}^{M} (m + 1) [\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)],
\]

where \( q := m_{k+1} + \cdots + m_{k+\alpha-1} \). Then since \( M \leq n - q \), it is an easy exercise to show that this sum is bounded above by

\[
\sum_{m=0}^{M} (m + 1) [\mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0)]
\]

\[
\leq \sum_{m=0}^{n-q} (m + 1) \mathbb{P}(S_m = 0) + q \mathbb{I}(n - q \geq q) \sum_{m=q}^{n-q} \mathbb{P}(S_m = 0)
\]

\[
\leq \sum_{m=0}^{(\alpha m^*) \wedge n} (m + 1) \mathbb{P}(S_m = 0) + \alpha m^* \sum_{m=m^*}^{n} \mathbb{P}(S_m = 0)
\]

where \( m^* := \max\{m_{k+1}, \ldots, m_{k+\alpha-1}\} \). The result follows by summing over all indices apart from \( m^* \) and changing the order of summation. \( \square \)

2.3. Proofs of main results.

**Proof of Theorem 1.** We apply a comparison argument found to be useful in many areas (e.g. Pruss and Montgomery-Smith [18], and Lefevre and Utev [16]), more exactly, we bound \( \text{var}(L_n) \) by the corresponding characteristic for the symmetrised random walk.

Following Spitzer’s argument we notice that with \( f(t) = \mathbb{E}[\exp(it \cdot X_1)] \)

\[
\mathbb{P}(S_{a,u} + cS_{b,v} = x) \leq c \int_{\Gamma} |f(t)|^u |f(-t)|^v dt = c \int_{\Gamma} \left[ |f(t)|^2 \right]^{u/2} \left[ |f(-t)|^2 \right]^{v/2} dt.
\]

Since \( |f(t)|^2 \) is a characteristic function of \( d \)-dimensional symmetric integer variable, for some positive \( \lambda \), \( 1 - |f(t)|^2 \geq \lambda |t|^2 \), and hence,

\[
\mathbb{P}(S_{a,u} + cS_{b,v} = x) \leq c \int_{\Gamma} \exp \left[ - \frac{\lambda(u + v)}{2} |t|^2 \right] dt \leq c(u + v)^{-d/2}
\]

and the proof follows from Proposition 9 applied with \( \phi(m) = m^{-d/2} \). \( \square \)

The proof of Theorem 2 will be based on the following Lemma.
Lemma 10. Assume $X$ is genuinely $d$-dimensional and $\mathbb{E}|X|^2 = \infty$. Then there exists a monotone slowly varying function $h_n \to 0$ as $n \to \infty$ such that
\[
\sup_{x \in \mathbb{R}^d} \mathbb{P}(S_n = x) \leq c_d \int_0^{1/2} \mathbb{E} |e^{i t \cdot X}|^n dt \leq h_n n^{-d/2}.
\]

Proof of lemma 10. Without loss of generality assume that $X$ is symmetrized. Let $\sigma_{e,L} := \mathbb{E}[(e \cdot X)^2 I(|X| \leq L)]$. Following Spitzer, since $X$ is genuinely $d$-dimensional, we may assume that there exist positive constants $c$ and $W$ such that for any unit vector $|e| = 1$, $\sigma_{e,W} \geq c$ and $1 - f(t) \geq c|t|^2$. Let $\lambda_t$ be the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$, and $\mu_d$ the Lebesgue-Haar measure on $S^{d-1} := \{e \in [-\pi, \pi]^d : |e| = 1\}$. Notice that since $\mathbb{E}|X|^2 = \infty$, for any $K$ we have $\mu_d\{e : \sigma_{e,\infty} < K\} = 0$.

Fix a small positive $x$ such that $\sqrt{c/x} \geq 2W$, and for any $\epsilon > 0$ let $K = K(\epsilon) = e^{-d/2}$. Then there exists $L = L(\epsilon) > 0$ small enough so that $\mu_d\{e : \sigma_{e,L} < K\} \leq e^{-d/2}$. We partition $S^{d-1}$ in two sets
\[
A_{L,K} = \{e \in S^{d-1} : \sigma_{e,L} \geq K\} \quad \text{and} \quad \tilde{A}_{L,K} = \{e \in S^{d-1} : \sigma_{e,L} < K\},
\]
so that for any direction $e \in \tilde{A}_{L,K}$,
\[
\{z \in \mathbb{R} : 1 - f(ze) \leq x\} \subseteq \{z : cz^2 \leq x\} \subseteq \{z : |z| \leq \sqrt{x/c}\}.
\]
Hence, using $d$-dimensional spherical coordinates,
\[
\lambda_d\{(z,e) \in \mathbb{R} \times \tilde{A}_{L,K} : 1 - f(ze) \leq x\} \leq \mu_d\{\tilde{A}_{L,K}\}(x/c)^{d/2}(1/d) \leq c^{d/2}(x/c)^{d/2}(1/d).
\]
On the other hand, for any $t$,
\[
1 - f(t) = 2 \sum_{k \in \mathbb{Z}^d} \sin((t \cdot k)/2)^2 P(X = k) \geq (1/4)\mathbb{E}[(t \cdot X)^2 I(|t \cdot X| \leq 1/2)] = (|t|^2/4)\sigma_{t/|t|,1/2}(t).
\]
Now, assume that $\sqrt{c/x} \geq 2L$. Then for any direction $e \in A_{L,K}$, by choice of $x$ and since $\sigma_{e,L}$ is increasing in $L$, for $cz^2 \leq 1 - f(ze) \leq x$ or $|z| \leq \sqrt{x/c}$, it must be the case that
\[
x \geq 1 - f(ze) \geq (z^2/4)\sigma_{e,1/2} \geq (z^2/4)\sigma_{e,L} \geq (z^2/4)K
\]
implying that on the set $A_{L,K}$, it must be that $|z| \leq 2\sqrt{x/K}$. Changing to $d$-dimensional polar coordinates, we find that
\[
\lambda_d\{(z,e) \in \mathbb{R} \times A_{L,K} : 1 - f(ze) \leq x\} \leq \int_{A_{L,K}} \int_0^{2\pi} \int_0^{\sqrt{1/r}} r^{d-1} dr de \leq C_d x^{d/2} x^{d/2}.
\]
Overall, for $x \leq c/4L^2$, $\lambda_d\{t : 1 - f(t) \leq x\} \leq c_d x^{d/2}$, and hence $\{t \in \Gamma : 1 - f(t) \leq x\}$ has Lebesgue measure $O(x^{d/2})$.

Let $F(x)$ be the cumulative distribution function of $\log(1/f(t))$ on the probability space $\Gamma$ with normalised Lebesgue measure. Then $F$ is continuous at $x = 0$ and supported on $\mathbb{R}^+$. Moreover, as $0 < x \to 0$, $F(x) = o(x^{d/2})$. Therefore, for some positive sequence $\epsilon_n \to 0$
\[
\frac{1}{2\pi^2} \int \int_0^{\infty} e^{-nx} e^{-y} dF(x) dx = n \int_0^{\infty} e^{-nx} F(x) dx \leq n^{-d/2} \epsilon_n.
\]
It remains to show that there exists a positive monotone slowly varying function $\epsilon_n \leq h(n) \to 0$ as $n \to \infty$. Let $\delta_n = sup_{j \geq n} \epsilon_j$, $a_0 := 0$ and for $n \geq 1$ define $a_n$ recursively by $a_n = min(2a_{2n-1}, 1/\delta_n)$, for $2^{r-1} < n \leq 2^r$, so that $a_n \to \infty$ is monotone, $a_{2n} \leq 2a_{2n-1}$ implying that $a_{2n} \leq 4\delta_n$, and $1/\delta_n \geq \epsilon_n$. Finally, take $h_n := 1/\max(a_0, \log a_n)$.

Proof of Theorem 2. Assume that $\mathbb{E}|X|^2 = \infty$ and $d = 2$ or $d = 3$. Then, by Lemma 10 there exists a slowly varying function $h(n) \to 0$ as $n \to \infty$ such that $\int_1^{\infty} \mathbb{E}(|\exp(it \cdot X)|^n) dt \leq h_n n^{-d/2}$. Applying Corollary 7 with $r = 1$ and $r = 3/2$ we respectively find that
\[
\text{var}(L_n(\alpha)) \leq \begin{cases} Kn^2 \left( \sum_{k=1}^{n} h(k)/k \right)^{2r-4} = o(n^2 (\log n)^{2r-4}), & \text{for } d = 2, \\
Kn^2 \left( \sum_{k=1}^{n} h(k)^2/k \right) = o(n \ln n), & \text{for } d = 3.
\end{cases}
\]
Finally assume that $\mathbb{E}|X|^2 < \infty$ and $E[X] = \mu \neq 0$. Then $\mathbb{P}(S_n = 0) = \mathbb{P}(S'_n = -n\mu)$ whence it follows that $\mathbb{P}(S_n = 0) = o(n^{-d/2})$ (see for example [15, Theorem 2.3.10]). Then inspecting the proof of Proposition 4, one can readily obtain the desired bound for the $J_n$ term, while with slight modification the bound for the $I_n$ term also follows.

Note that for $d = 1$ the situation is much simpler since then $\text{var}(L_n(\alpha|\epsilon, d)) \sim C[\mathbb{E}L_n(\alpha|\epsilon, d)]^2$ and if $\mathbb{E}|X|^2 = \infty$ or $E[X] \neq 0$, $\mathbb{E}L_n(\alpha|\epsilon, d) = o(n^{(1+\alpha)/2})$.

\textbf{Proof of Theorem 3.} We first give the proof for the case $d = 1$. As in the proof of Proposition 4 we begin from expression (2), and define the sequences $p_i$, and $\delta_i$ for $i = 1, \ldots, 2\alpha - 1$, and the quantity $v(\delta) = \sum_{i=1}^{2\alpha-1} |\delta_i|$. Recall that $v(\delta)$ measures the interlacement of the two sequences $k_1, \ldots, k_\alpha$, and $l_1, \ldots, l_\alpha$. For example $v(\delta) = 1$ occurs when either $k_\alpha \leq l_1$, or $l_\alpha \leq k_1$, in which case the contribution vanishes by the Markov property. On the other hand $v(\delta) = 2$ when for example $l_1, \ldots, l_\alpha \in [k_i, k_{i+1}]$ for some $i$. Finally $v(\delta) = 3$ occurs when for example $k_1 \leq \cdots \leq k_r \leq l_1 \leq \cdots \leq l_s \leq k_{r+1} \leq \cdots \leq k_\alpha \leq l_{s+1} \leq \cdots \leq l_\alpha \leq n$.

From the proof of Proposition 4, and using the bound $\mathbb{P}(S_n = 0) = O(1/n)$, the terms of the sum are bounded above by $n^2 \log(n)^{2\alpha-1-v(\delta)}$, and thus the leading term appears when either $v(\delta) = 2, 3$, with other terms giving strictly lower order. We shall therefore analyze these two situations in detail in order to derive the exact asymptotic constants. When $v = 3$, the two terms in the difference individually give the correct order and shall be treated by the classical Tauberian theory. However for $v = 2$, the two terms only give the correct order when considered together. This however forbids the use of Karamata’s Tauberian theorem since the monotonicity restriction would require roughly that $X_i$ is symmetrized. Thus the complex Tauberian approach, as developed in [10], is required to justify the answer.

\textbf{Case 1:} $v(\delta) = 3$. Assume that part of the sequence $l = \{l_1, \ldots, l_\alpha\}$ lies between $k_r$ and $k_{r+1}$, and the rest between $k_1$ and $k_{\alpha+1}$. Then using the change of variables

\[ i_1 = m_0, i_2 = m_0 + m_1, \cdots, i_r = m_0 + \cdots + m_{r-1} \]

\[ j_1 = m_0 + \cdots + m_r, j_2 = m_0 + \cdots + m_{r+1}, \cdots, j_s = m_0 + \cdots + m_{s-1}, i_\alpha = m_0 + \cdots + m_{\alpha-1}, \]

\[ j_{s+1} = m_0 + \cdots + m_{\alpha+s}, j_{s+2} = m_0 + \cdots + m_{\alpha+s+1}, \cdots, j_\alpha = m_{2\alpha} - n = m_0 + \cdots + m_{2\alpha}, \]

we rewrite the positive term in (2) as

\[
  a(n) = \sum \mathbb{P}[S(i_1) = \cdots = S(i_\alpha); S(j_1) = \cdots = S(j_\alpha)]
  = \sum_{m_0, \ldots, m_{2\alpha-1}} \left[ \prod_{j=1}^{2\alpha-1} \mathbb{P}(S_{m_j} = 0) \right] \times \mathbb{P}(S_{m_r} + S'_{m_{r+s}} + S''_{m_{s+s}} = 0).
\]

Notice that from [10] we have that $\sum_{n \geq 0} \lambda^n \mathbb{P}(S_n = 0) \sim \log (1/(1-\lambda))/\pi \gamma$. Let

\[ a(\lambda) = (1-\lambda)^{-3}[-\log(1-\lambda)]^{2\alpha-4}, \quad c_\gamma = (\pi \gamma)^{-2\alpha+4}. \]

Then, by direct calculations and Fourier inversion formula

\[
  \sum_{n \geq 0} \lambda^n a(n) = c_\gamma (1-\lambda) a(\lambda) \sum_{x \in \mathbb{Z}_2} \sum_{k_1, k_2, k_3 \geq 0} \lambda^{k_1+k_2+k_3} \mathbb{P}(S_{k_1} = x) \mathbb{P}(S_{k_2} = -x) \mathbb{P}(S_{k_3} = x)
  = c_\gamma (1-\lambda) a(\lambda) \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{dt ds}{(1-\lambda f(t))(1-\lambda f(t+s))} \approx c_\gamma (1-\lambda) a(\lambda) \frac{1}{(4\pi^2 \gamma^2)} \frac{1}{1-\lambda} \int_{\mathbb{R}^2} \frac{dx dy}{(1+|x|)(1+|y|)(1+|x+y|)} \sim (1/4\gamma^2)c_\gamma a(\lambda)
\]

Next we consider the negative term in (2)

\[
  b(n) := \sum_{m_0, \ldots, m_{2\alpha-1}} \mathbb{P}[S_{m_1} = \cdots = S_{m_{r-1}} = S_{m_r} + \cdots + S_{m_{r+s}} = S_{m_{r+s+1}} = \cdots = S_{m_{\alpha+s-1}} = 0]
\]
By direct calculations and (1),
\[
\sum_n \lambda^n b(n) = \left(1 - \lambda \right) \left(1 - \lambda \right)^{-2} \sum_{m_r, \ldots, m_{\alpha + s} = 0} \lambda^{m_r + \cdots + m_{\alpha + s}} \prod_{t=r+1, \ldots, s+1} \prod_{t \neq r+s} \mathbb{P}(S_m = 0) \mathbb{P}(S_{m_r} + \cdots + S_{m_{\alpha + s}} = 0) \mathbb{P}(S_{m_{r+s}} + \cdots + S_{m_{\alpha + s}} = 0),
\]
and using Fourier inversion and (1) the internal sum behaves as
\[
(2\pi)^{-\alpha-s+r} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (1 - \lambda \phi(x))^{-1} (1 - \lambda \phi(x) \phi(y))^{-1} (1 - \lambda \phi(y))^{-1} dx dy \times \left[ \prod_{j=r+1}^{r+s-1} \prod_{k=r+s+1}^{\infty} (1 - \lambda \phi(x) \phi(t_j))^{-1} (1 - \lambda \phi(y) \phi(t_k))^{-1} dt_j dt_k \right] dx dy \sim (\pi \gamma)^{-\alpha-s+r} (1 - \lambda)^{-1} \log \left(1 - \lambda^{-1} \right).\]

Then we have \(\sum_n \lambda^n b(n) \sim (\pi^2 / 6(\pi \gamma)^{2\alpha-2}) a(\lambda)\), whence the Tauberian theorem implies that \(a(n) - b(n) \sim n^2 \log(n)^{2\alpha-4} / 4\pi^2 \gamma^{2\alpha-2}\). Most importantly we see that the lengths and locations of the chains, \(r\) and \(s\), do not affect the asymptotic. Noting that if \(1 \leq r, s \leq \alpha - 1\), we can partition \(2\alpha = r + s + (\alpha - r) + (\alpha - s)\) in \((\alpha - 1)^2\) ways, and thus overall the total contribution from terms with \(v = 3\) is
\[
[(\alpha(\alpha - 1))^2 / 12\pi^{2\alpha-4} \gamma^{2\alpha-2}] n^2 \log(n)^{2\alpha-4}.
\]

**Case 2**: \(v(\delta) = 2\). The typical term \(c(n)\) was introduced in (6) in the proof of Proposition 9. Now we let \(\lambda \in \mathbb{C}\), with \(|\lambda| < 1\). By lengthy but direct calculations we can derive an expression of the form
\[
\sum_n \lambda^n c(n) = \frac{\alpha - 1}{(\gamma \pi)^{2\alpha-2}} a(\lambda) + o(a(\lambda)), \quad \lambda \to 1.
\]

The approach developed in [10] can then be used to bound the error terms and show that \(c(n) \sim [(\alpha - 1) / 2(\gamma \pi)^{2\alpha-2}] n^2 \log(n)^{2\alpha-4}\).

Finally taking into account the fact that the \(l_1, \ldots, l_\alpha\) can be in any of the \(\alpha - 1\) intervals \([k_i, k_{i+1}]\), for \(i = 1, \ldots, \alpha - 1\), the result follows the overall contribution of terms with \(v(\delta) = 2\) is
\[
\frac{(\alpha - 1)^2}{2(\gamma \pi)^{2\alpha-2}} n^2 \log(n)^{2\alpha-4}.
\]

The case for \(d = 2\) is very similar, so we move on to the case \(d = 3\).

**Case** \(d = 3, \alpha = 2\). Using the same notation as before, we have three terms to consider \(a(n), b(n), \) and \(c(n)\). We first consider \(c(n)\). Letting \(K := \epsilon / \sqrt{1 - \lambda} \) and using the usual power series construction and spherical coordinates
\[
\sum_n \lambda^n c(n) = (1 - \lambda)^{-2} (2\pi)^{-6} \int_{[0,1] \times [0,1]} \frac{\lambda f(y)(1 - f(x))}{(1 - \lambda f(x))(1 - \lambda f(y))(1 - \lambda f(x)f(y))} dx dy \sim 2(2\pi)^{-4} |\Sigma|^{-1} (1 - \lambda)^{-2} \int_0^K \int_0^K r^4 s^2 dr ds \sim 2(2\pi)^{-4} |\Sigma|^{-1} \frac{\pi}{2} (1 - \lambda)^{-2} \log \left(\frac{1}{1 - \lambda}\right) =: \kappa_1 (1 - \lambda)^{-2} \log \left(\frac{1}{1 - \lambda}\right),
\]
and thus \(c(n) \sim \kappa_1 n \log n\), where \(\kappa_1 > 0\), where the answer can be justified following [10].

The term \(a(n) - b(n)\) is trickier to compute. As usual we consider the power series
\[
\sum_{n \geq 0} \lambda^n (a(n) - b(n)) = (1 - \lambda)^{-2} (2\pi)^{-6} \int_{B(\epsilon)} \frac{dr dy}{(1 - \lambda f(x))(1 - \lambda f(y))(1 - \lambda f(x+y))}.
\]
(10) G. Deligiannidis and K. Zemer. Relative complexity of random walks in random scenery in the absence of a

Let $A \in [-1, 1]$ be the cosine of the angle between $x$ and $y$, which in spherical coordinates is

Then as $0 < \lambda < 1$, using the expansion (1)

The other integral is slightly easier

and thus overall we must have that

whence it follows that $\text{var}(L_n(2)) \sim (\kappa_1 + \kappa_2)n \log n$.

To prove the last claim, let $S_n' = X_1' + \cdots + X_n'$ be another random walk, independent of $S_n$, such that its characteristic function $f'(t) = E[\exp(itX_1')]$ also satisfies the expansion (1). Then using [10, Lemma 3.1] one can adapt the proof of [10, Theorem 2.1] to show that $L_n'(\alpha) = L_n(\alpha) + o(L_n(\alpha))$. \hfill $\Box$

References

[1] J. Aaronson. Relative complexity of random walks in random scenery. Ann. Probab. 40, 2012, no. 6, pp. 2469-2482.

[2] A. Asselah. Shape transition under excess self-intersections for transient random walk. Annales de l’institut Henri Poincare (B) Probability and Statistics, 46, no. 1, pp. 1250-278, 2010.

[3] M. Becker and W. König. Moments and distribution of the local times of a transient random walk on $\mathbb{Z}^d$. J. Theoret. Probab., 22 (2): 365–374, 2009.

[4] E. Bolthausen. A central limit theorem for two-dimensional random walks in random scenery. Ann. Probab., 17 (1): 108–115, 1989.

[5] A. N. Borodin. A limit theorem for sums of independent random variables defined on a recurrent random walk. Dokl. Akad. Nauk SSSR, 246 (4): 786–787, 1979.

[6] D. C. Brydges and G. Slade. The diffusive phase of a model of self-interacting walks. Probab. Theory Related Fields, 103 (3): 285–315, 1995.

[7] F. Castell, N. Guillotin-Plantard, and F. Péne. Limit theorems for one and two-dimensional random walks in random scenery. Annales de l’institut Henri Poincare (B) Probability and Statistics, 2012.

[8] X. Chen. Random walk intersections, volume 157 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2010. Large deviations and related topics.

[9] G. Deligiannidis and S. Utev. An asymptotic variance of the self-intersections of random walks, 2010, arXiv:1004.4845.

[10] G. Deligiannidis and S. Utev. Computation of the asymptotics of the variance of the number of self-intersections of stable random walks using Wiener-Darboux theory. Sib. Math. J, 52, 2011.

[11] G. Deligiannidis and K. Zemer. Relative complexity of random walks in random scenery in the absence of a weak invariance principle for the local times. preprint, 2015.
[12] Jürgen Gärtner and Rongfeng Sun. A quenched limit theorem for the local time of random walks on $\mathbb{Z}^2$. *Stochastic Process. Appl.*, 119 (4): 1198–1215, 2009.

[13] H. Kesten and F. Spitzer. A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. verw. Gebiete*, 50: 5–25, 1979.

[14] G.F. Lawler. *Intersections of Random Walks*. Birkhauser, 1991.

[15] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2010.

[16] C. Lefèvre and S. Utev. Exact norms of a Stein-type operator and associated stochastic orderings. *Probab. Theory Related Fields*, 127 (3): 353–366, 2003.

[17] T. M. Lewis. A law of the iterated logarithm for random walk in random scenery with deterministic normalizers. *J. Theoret. Probab.*, 6 (2): 209–230, 1993.

[18] S. J. Montgomery-Smith and A. R. Pruss. A comparison inequality for sums of independent random variables. *J. Math. Anal. Appl.*, 254 (1): 35–42, 2001.

[19] F. Spitzer. *Principles of Random Walk*. Springer, 1976.

[20] J. Černý. Moments and distribution of the local time of a two-dimensional random walk. *Stochastic Process. Appl.*, 117 (2): 262–270, 2007.

**Department of Statistics, University of Oxford, Oxford OX1 3TG, UK**

*E-mail address*: deligian@stats.ox.ac.uk

**Department of Mathematics, University of Leicester, LE1 7RH, UK**

*E-mail address*: su3501e.ac.uk