Quantum Information Approach to Rotating Bose-Einstein Condensate

Zhao Liu, Hongli Guo, Shu Chen, and Heng Fan

Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
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We investigate the 2D weakly interacting rotating Bose-Einstein condensate by the tools of quantum information theory. The critical exponents of the ground state fidelity susceptibility and the correlation length of the system are obtained for the sudden change of the ground state when the first vortex is formed. This sudden change can also be indicated by the ground state entanglement. We also find the single-particle entanglement can be an indicator of the angular momentums for some real ground states. The single-particle entanglement of fractional quantum Hall states such as Laughlin state and Pfaffian state is also studied.

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I. INTRODUCTION

Since the Bose-Einstein condensate (BEC) was observed in trapped alkali-metal atoms [1, 2, 3], the response of those systems to rotation has attracted considerable attention. Unlike classical systems, the BEC can only gain angular momentum by forming quantized vortices when it is stirred. When rotation frequency is small enough, no motion of the system can be observed, while above some critical frequencies, vortices are formed. These vortices are signatures of superfluidity [4]. The formation of the first vortex is perhaps especially interesting. Before the formation of the first vortex, the ground state of BEC does not rotate while after the formation the ground state is a single-vortex state, in which all particles rotate around their mass center. Therefore a macroscopic symmetry breaking must happen. However, this sudden change of the ground state has not been studied quantitatively. Some important physical quantities such as critical exponents are still unknown.

Besides the sudden change of the ground state, many authors focused on the the ground state energy and the ansatz ground state wave function when the system gains a fixed angular momentum through rotation in the weak interaction limit [5, 6, 7, 8, 9, 10, 11, 12, 13]. Generally speaking, the theoretical methods they used mainly include Gross-Pitaevskii (GP) mean field theory and exact diagonalization (ED). In GP mean field theory, the ground state many body wave function is simply expressed as a product of $N$ single-particle states, namely a non-entangled state. However, sometimes the mean field theory cannot give a good description to the system. As pointed out in Ref. [14], when the first vortex is formed, the ground state of the rotating BEC will change from a product state to a strongly-correlated entangled state, leading to the invalidity of the description of the mean field theory. Furthermore, when the number of vortices $N_v$ is large, the mean field theory predicts the ground state of the system is a vortex lattice phase [13]. This prediction is correct only when the filling fraction $\nu \equiv N/N_v \gtrsim \nu_c$ with $\nu_c \sim 6$ [12], where $N$ is the particle number. When $\nu \lesssim \nu_c$, the ground state is a strongly-correlated vortex liquid phase. Therefore it is the entanglement between particles that makes GP mean field theory invalid. So studying entanglement in rotating BEC will be very important to help us understand the properties of this system, as what has been done in other condensed matter systems [16]. However, to our knowledge, this aspect has not yet been investigated.

In this Article, we use the method of ED in the weakly interacting regime to remain the entanglement property of rotating BEC. Several tools of quantum information theory are used to investigate this system. First we use the ground state fidelity and fidelity susceptibility, which can precisely locate the critical point of a possibly unknown quantum phase transition [17, 18, 19, 20, 21, 22, 23], to study the ground state sudden change when the first vortex is formed. By using finite-size scaling analysis for even $N$, we obtain the critical exponents of both fidelity susceptibility and correlation length. We find the ground state single-particle entanglement can also indicate this sudden change of the ground state. Then we use the von Neumann entropy to calculate the single-particle entanglement of the ground states of subspaces of fixed $z$-component angular momentum $L_z$ of the system. Interestingly, we find that this single-particle entanglement can indicate some $L_z=0$ of the real ground states, namely those stable states. Finally, we study the relation between single-particle entanglement and $N$ for some special subspace ground states. We find the single-particle entanglement of both bosonic Pfaffian state and bosonic Laughlin state diverges logarithmical with $N$, showing a strongly-correlated characteristics of vortex liquid phase, while the single-particle entanglement of single-vortex state, namely the ground state in the subspace $L_z = Nh$, decays with $N$.

II. MODEL

In rotating reference frame, the Hamiltonian of a 2D rotating $N$-boson system with rotation frequency $\Omega$ trapped in a harmonic oscillator potential is $H = \sum_{i=1}^N \mathcal{H}_{0,i} + \mathcal{U}$, where $\mathcal{H}_{0,i} = -\frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2}m\omega^2 r_i^2 - \Omega L_{z,i}$ is the single-particle Hamiltonian and $\mathcal{U} \propto \sum_{i<j}^N \delta(r_i - r_j)$
When \( \Omega \) varies from 0 to \( \omega \), we can diagonalize the Hamiltonian (1) to find its subenergy. It’s obvious that in our system, the single-particle lowest Landau level (LLL) wave function of \( H_{\Omega,i} \). If the interaction is weak enough (this means in Eq. (1) \( N U_0 \ll \hbar \omega \), throughout the calculation we make \( N U_0 = \hbar \omega / 2 \) to keep the validity of the LLL approximation), the dynamics of the system is restricted in the LLL, from which other Landau levels are separated by a large energy gap \( 2 \hbar \omega \), so that we can use \( \varphi_l \) to do the second quantization of the Hamiltonian, leading to

\[
H_L = (L + N)\hbar \omega - L \hbar \Omega + \sum_{i,j,k,l} U_{i,j,k,l} a_i^\dagger a_j a_k a_l \tag{1}
\]

with \( U_{i,j,k,l} = \frac{1}{\sqrt{L}} \sum_{l=0}^{L-1} \varphi_{i+j+k+l} \). \( a_i^\dagger \) (\( a_i \)) creates (annihilates) a particle in the state \( \varphi_i \). The basis \( \mathfrak{B}_L \) of the Hilbert subspace of our system with fixed \( L_z = L \hbar \) in Fock representation is \( |N_0 N_1 \ldots N_L \rangle \) with the constraints that \( \sum_{i=0}^{L} N_i = N \) and \( \sum_{i=1}^{L}(iN_i) = L \). Under this basis, we can diagonalize the Hamiltonian (1) to find its subspace ground state \( |\Psi_{0,L} \rangle \) and the ground state energy \( E_{0,L}(\Omega) \).

Now let’s consider the total Hamiltonian \( \mathcal{H} = \bigoplus_L H_L \). When \( \Omega \) varies from 0 to \( \omega \), the one among all \( |\Psi_{0,L} \rangle \) with the lowest \( E_{0,L}(\Omega) \) is the real ground state, namely the stable state \( |\Psi_0 \rangle \) of \( \mathcal{H} \). The \( L_z \) of \( |\Psi_0 \rangle \), denoted by \( L_{z,0} \), forms a series of sharp steps from 0 to \( N(N-1) \) (in \( \hbar \) unit from now on) \( \mathfrak{B} \). In the first step at \( \Omega = 0.75 \omega \), \( L_{z,0} \) varies from 0 to \( N \), corresponding to the formation of the first vortex, where the ground state changes suddenly from a non-rotating state to a single-vortex state.

III. GROUND STATE FIDELITY AND FIDELITY SUSCEPTIBILITY

At the beginning we consider a general Hamiltonian \( \mathcal{H}(\lambda) = \mathcal{H}_0 + \lambda \mathcal{H}_v \), where \( \lambda \) is a parameter that can be changed, then the ground state fidelity is defined as \( F = |\langle \Psi_0(\lambda + \delta \lambda) | \Psi_0(\lambda) \rangle| \). The ground state fidelity susceptibility can be calculated from the formula \( \chi(\lambda) = \lim_{\delta \lambda \to 0} \frac{-2 \ln F}{\delta \lambda^2} \sum_{n \neq 0} \frac{|\langle \Psi_n(\lambda) | H_b | \Psi_0(\lambda) \rangle|^2}{(E_n(\lambda) - E_0(\lambda))^2} \),

where \( |\Psi_0(\lambda) \rangle \) (\( |\Psi_n(\lambda) \rangle \)) is the ground (excited) state of \( \mathcal{H}(\lambda) \) and \( E_0(\lambda) \) (\( E_n(\lambda) \)) is the ground (excited) state energy. It’s obvious that in our system, \( \lambda = \Omega \) and \( \mathcal{H}_v = L_z \).

For the system with Hamiltonian \( \mathcal{H} = \bigoplus_L H_L \), an energy level crossing of the ground state exists at \( \Omega = 0.75 \omega \) therefore the ground state fidelity shows a simple drop at \( \Omega = 0.75 \omega \). One should notice that Eq. (2) is valid only when there is no degeneracy for the ground state of the system. So to study the ground state fidelity susceptibility, we have to eliminate this energy level crossing first. We adopt the method used in Ref. [14] to add the stirring potential \( V \propto \sum_{i=1}^{N}(x_i^2 - y_i^2) \) to the Hamiltonian \( \mathcal{H} \) to generate an energy gap \( \Omega = 0.75 \omega \). Then we diagonalize \( \mathcal{H} \) which has been added by \( V \) in the basis \( \mathfrak{B} = \bigcup_{L=0}^{N+2} \mathfrak{B}_L \) to calculate \( \chi(\Omega) \). \( \chi \) must be small enough to guarantee that we can still use \( \varphi_l \), the single-particle LLL wave function of \( H_{\Omega,l} \) in the absence of \( V \) to do the second quantization. \( V \) can be expressed as

\[ V_0 \sum_l \left( \sqrt{l(l-1)}a_i^\dagger a_i - \sqrt{l(l+1)}a_i a_i^\dagger \right), \]

where \( V_0 \ll \hbar \omega \) (\( V_0 = 0.003 \hbar \omega \) in this section). We find that after adding \( V \), the ground state degeneracy for odd \( N \) is not broken completely, meaning the ground state fidelity still shows a drop. But for even \( N \), the ground state is non-degenerate so the ground state fidelity is a smooth curve (FIG.1 and FIG.2). We focus on even \( N \) in the following.

We calculate \( \chi(\Omega) \) for even \( N \). We find that \( \chi(\Omega) \) shows a singularity at \( \Omega \approx 0.7525 \omega \), where a sudden change of the ground state is indicated (FIG.3(a)). When studying the sudden change of the ground state, it’s necessary to consider the problem of scaling. First we will study how does the fidelity susceptibility scale with \( N \) when \( \Omega \) is at and far from \( \Omega_{max} \) where the peak locates. The height of the peak of \( \chi(\Omega) \) at \( \Omega = \Omega_{max} \) diverges with \( N \) and scales like \( \chi(\Omega_{max}) \propto N^d \) with \( d \approx 2.9862 \) (FIG.3(b)). When \( \Omega \) is far from (either below or above) \( \Omega_{max} \), \( \chi(\Omega) \propto N^d \) with \( d = 1 \) (FIG.3(c)). In the thermodynamic limit, \( \Omega_{max} \rightarrow \Omega_c \). We are interested in the critical exponent \( \mu \) of the correlation length when \( \Omega \) approaches \( \Omega_c \). It’s known that the rescaled fi-
and its first order derivative will be defined and discussed in detail in the next section, the ground state single-particle entanglement when Ω approaches Ω_{\text{max}}. Near Ω_{\text{max}}, χ(Ω) must behave like χ(Ω) \propto 1/|Ω - Ω_{\text{max}}|^\alpha, where α = (d_c - d)/\mu. We can obtain α \approx 1.35 \approx 1.4713. So we can calculate the single-particle reduced density operator can be expressed in the form \rho_1 = \sum_{i=0}^{L_z} \xi_i |φ_i⟩⟨φ_i| and S_1 = -\sum_{i=0}^{L_z} \xi_i \ln \xi_i. Once L_z is fixed, according to Eq.(1), the ground state single-particle entanglement is uniquely determined by the operator \sum_{i,j,k,l} U_{i,j,k,l} a_{i}^\dagger a_{j}^\dagger a_{k} a_{l}. Therefore S_1 is irrelevant with U_0.

We calculate the ground state single-particle entanglement S_1 with fixed L_z = 1, 2, ..., N(N - 1) + 1. We find that S_1 has a tendency to grow with L_z but the whole curve shows a behavior of slight oscillation. To see this oscillation clearer, in the following we subtract a second order polynomial in L_z from S_1(L_z). For example, for N = 8 we subtract -0.0008L_z + 0.0817L_z^2 + 0.5957N from S_1(L_z). In FIG.5, we find there exist a series of local minima at L_z = L_m for S_1(L_z). Interestingly, most of these L_m are either the real ground state angular momentum L_{z,0} or the candidate of L_{z,0} predicted by the composite fermion theory. Therefore the calculation of S_1 of the subspace ground state |Ψ_{0,L_z}\rangle may give a way of electing some L_z s of real ground state |Ψ_0\rangle (TABLE I).

Now we focus on the S_1 of three subspace ground states with special L_z s. The first one is L_z = N, which belongs to L_{z,0} for any N. When L_z = N, the ground state wave function was conjectured as ψ_N = \prod_{i=1}^{N} [(z_i - z_c)e^{-|z_i|^2/2}], where z_c = \frac{1}{N} \sum_{i=1}^{N} z_i is the position of the center of mass. The single-particle reduced density matrix to accuracy O(1/N^2) is [12]

\[ ρ_1(z, z') = \int \left( \prod_{i=2}^{N} dz_i \right) ψ_N(z, z_2, ..., z_N) ψ_N^\dagger(z', z_2, ..., z_N) \]

= \frac{1}{N} ϕ_N(z)ϕ_N^\dagger(z') + \left( 1 - \frac{2}{N} \right) ϕ_1(z)ϕ_1^\dagger(z') + \frac{1}{N} ϕ_2(z)ϕ_2^\dagger(z').

The single-particle entanglement can be obtained as S_1^{L_z=N} \approx -2/N \ln(1/N) - \left( 1 - 2/N \right) \ln\left( 1 - 2/N \right). FIG.6(a) shows the S_1^{L_z=N} decays with the growth of
TABLE I: In this table, we list $L_m$, where $S_1(L_z)$ shows a local minimum; $L_{z,0}$, the real ground state angular momentum, and $L_{CF}$, the candidate of $L_{z,0}$ predicted by the composite fermion theory \[2\] for $N = 5, 6, 7, 8$.

| $N$ | $L_m$ | $L_{z,0}$ | $L_{CF}$ |
|-----|-------|-----------|----------|
| 5   | 8,10,12,15,20 | 8,10,12,15,20 | 8,10,12,15,20 |
| 6   | 6,10,12,15,18,20,24,30 | 6,10,12,15,20,24,30 | 6,10,12,15,18,20,24,30 |
| 7   | 7,12,14,18,20,22,24,27,30,35,42 | 7,12,15,18,24,30,35,42 | 7,12,15,18,20,22,24,27,30,35,42 |
| 8   | 8,14,16,18,21,24,28,30,32,35,38,42,48,56 | 8,12,14,18,24,30,35,42,56 | 8,14,16,18,21,24,28,30,32,35,38,42,48,56 |

FIG. 5: (Color online) The single-particle entanglement $S_1$ of the ground state of the subspaces of fixed $L_z$. A second order polynomial of $L_z$ has been subtracted from $S_1(L_z)$ to make the oscillation clearer. The red arrows point to the positions of local minima.

$N$. This is because when $N \rightarrow \infty$, all particles condensate to the single-particle state $\varphi_1$. We know that perfect condensate has zero particle entanglement \[24\].

Next we study other two cases which have close relation with quantum Hall effect \[23, 26, 27\]. One is when $L_z = \frac{1}{2}N(N-2)$ for even $N$ and $L_z = \frac{1}{2}(N-1)^2$ for odd $N$. This subspace ground state has a high overlap with the bosonic Pfaffian state of $\nu = 1$ \[9\]:

$$\psi_{PF} = \prod_{i<j}^{N} \left( z_i - z_j \right) \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i=1}^{N} e^{-|z_i|^2/2}.$$ Our numerical result is $S_1 \approx \ln(1.037N-1.1369)$. The other is when $L_z = N(N-1)$, which belongs to $L_{z,0}$ for any $N$. This subspace ground state is bosonic Laughlin state of $\nu = 1/2$:

$$\psi_{Laugh} = \prod_{i<j}^{N} (z_i - z_j)^2 \prod_{i=1}^{N} e^{-|z_i|^2/2}.$$ Our numerical result is $S_1 \approx \ln(1.941N-1.081)$. We can find the single-particle entanglement analytically for some quantum Hall effect states in spherical geometry obtaining $S_1^{Laugh} = \ln(2N-1)$ for $\nu = 1/2$ and $S_1^{Pf} = \ln(N-1)$ for $\nu = 1$ \[28\]. Now we are dealing with a system in disk geometry so our results are not exactly the same with this. However, the difference is not very large (FIG.6(b)).

It’s known that when $N$ and $N_c$ are both large and our system has a fixed $L_z$, if $\nu = N/N_c = N^2/(2L_z) \geq \nu_c$ with $\nu_c \sim 6$, the ground state is a vortex lattice state, otherwise the ground state is a vortex liquid state. In the vortex liquid regime for $\nu = 1/2$ and $\nu = 1$, the results above show that in the thermodynamic limit $S_1$ of the ground state is logarithmical divergent with $N$. But what’s the relation between $S_1$ of the ground state and $S_1$ for some $\nu$ in the vortex lattice regime? Considering this regime the mean field theory describes our system well, we conjecture that when $N \rightarrow \infty$, $S_1$ will not diverge with $N$ for some $\nu$ in the vortex lattice regime.

We hope the ground state sudden change when the first vortex is formed and the entanglement in it can be investigated by experiments. To achieve this goal, an energy gap between the ground state and excited states needs to be generated by stirring potential (see FIG.1). Considering this energy gap decays with the particle number, the number of particles in experiments should be restricted. On the other hand, for most experiments of rotating BEC realized in the laboratory, the filling factor $\nu \sim 10^3$, well inside the vortex lattice phase. Experimentalists have elaborate techniques to observe vortex lattice, while how to detect the vortex liquid state is still a challenge. However, some experimental methods have been proposed \[29\]. We hope the strongly-correlated characteristics of bosonic Laughlin and Pfaffian state as we show here by their logarithmical divergent single-particle-entanglement can be verified in experiments in the future.
V. SUMMARY

In summary, we investigate a 2D rotating BEC by tools of quantum information theory. The critical exponents of ground state fidelity susceptibility and the correlation length are obtained for the ground state sudden change when the first vortex is formed. We find the single-particle entanglement $S_1$ of the ground state can be used to detect this sudden change. We also find a novel property that $S_1$ can indicate some angular momentum $L_z$ of the real ground states, namely those stable states. At last, $S_1$ of the ground states in some special subspaces of fixed $L_z$ are calculated to show the strongly-correlated property of vortex liquid phase. Thus some basic properties underlying in rotating BEC are clarified and viewed from the point of quantum information. On the other hand, the formation of the first vortex in rotating BEC may provide a signature for the macroscopic entanglement [30].

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