Riemann, Thorin, van Dantzig Pairs, Wald Couples and Hadamard Factorisation

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Abstract

Zeros of entire functions can be found using either the Fourier methods of Riemann-Pólya or the Generalised Gamma Convolution (GGC) methods of Thorin-Bondesson. This connection is based on a duality between the Hadamard-Weierstrass factorisation and van Dantzig Pairs-Wald couples of random variables. We demonstrate the methodology on particular functions including the Riemann $\zeta$ and $\xi$-functions, $L$-functions, Gamma function, Trigonometric and Hyperbolic functions.

Keywords: van Dantzig pairs, Wald couples, Thorin’s condition, Hadamard factorisation, Laguerre-Pólya class, Lévy, Thorin measure, Generalised Gamma Convolutions (GGC), Hyperbolically Completely Monotone (HCM), Riemann $\xi$-function, $\zeta$-function, $L$-functions, Bessel functions, Hyperbolic functions.

1 Introduction

An entire function has a Hadamard-Weierstrass factorization. This represents the function in terms of its zeros. A question of interest is how to characterise when an entire functions has only real zeros. Our motivation is to develop a methodology that can be applied to a question of Pólya (1926): what properties of kernels, $\Phi(t)$, are sufficient to secure the Fourier transform $f(s) = \int_{-\infty}^{\infty} e^{ist} \Phi(t) dt$ has only real zeros?

By constructing van Dantzig pairs-Wald couples, one implicitly finds the Hadamard factorization of an entire function and in turn one characterizes its zeros. To illustrate our methodology, we consider particular classes $\Gamma$ and special functions, trigonometric and hyperbolic functions, the Riemann $\xi$-function, Dedekind $\eta$-function and Ramanujan $\tau$-function and general class of $L$-functions.

The Laguerre-Pólya (LP) class consists of real entire functions or order at most two with only real zeros, given by $\rho_n$, such that $\sum_{n=1}^{\infty} |\rho_n|^{-2} < \infty$, with Hadamard product

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of the form
\[ f(s) = C s^m e^{bs+cs^2} \prod_{n=1}^{\infty} \left( 1 - \frac{s}{\rho_n} \right) e^{\frac{s}{\rho_n}}. \]

The Laplace transform of a Pólya frequency function necessarily is the reciprocal of an LP entire function, due to a result of Schoenberg (1948) which has since been studied by many others. The goal is to develop a duality between a Hadamard factorization and random variables that generate van Dantzig pairs and Wald couples.

The two analytic functions \([f, g]\) form a van Dantzig pair if \(f(s)\) and \(g(s) = 1/f(is)\) are both Fourier transforms. By construction \(g(s)f(is) = 1\). A related construction is a Wald couple of (infinitely divisible) random variables \((X, H)\) where \(H \geq 0\) is such that
\[
E(e^{sX})E(e^{-s^2H}) = 1.
\]

Our focus will be on a specific case where \(H\) is a Generalized Gamma Convolution (GGC) (Bondesson, 1992) and, in particular, falls into the sub-class generated by Thorin measures (Thorin, 1977). When available we explicitly calculate the corresponding van Dantzig pairs and Wald couples.

The rest of the paper is outlined as follows. Section 1.1 provides definitions and notation. Section 2 discusses the Fourier methods of Riemann-Pólya and the dual approach of Thorin-Bondesson that analyses the reciprocal function as a LT of a Generalised Gamma Convolutions (GGC). a duality between van Dantzig pairs, Wald couples of random variables and Hadamard-Weierstrass products of entire functions. Dmitrov and Rusev (2011) provide a review of finding real zeros of entire functions and the methods of Riemann and Pólya. Section 3 provides a number of examples of special functions including the Gamma hyperbolic classes, Bessel and MacDonal functions.

Section 4 discusses the Riemann \(\zeta\) and \(\xi\) (Polson, 2018), Ramanujan \(\tau\), Dedekind \(\eta\), and the general class of L-functions. This has implications for the Generalized Riemann Hypothesis (GRH) and the Birch-Swinnerton-Dyer (BSD) conjecture. Ostrovsky (2016) provides results for Barnes Beta distributions. Walker (1988) studies certain trigonometric functions. Grosswald (1964) calculates the Mellin transform for related Reisz-type functions. Barndorff-Nielsen et al (1982) discuss \(Z\)-distribution and provide a series representation of the density of a convolution mixture of exponentials related to Maslanka’s Pochammer representation of the reciprocal \(\xi\)-function. Finally, Section 5 concludes.

The following are key definitions and notation.

### 1.1 Definitions and Notation

The following list of definitions and notation is useful in the sequel.

**van Dantzig Pair.** The pair of analytic functions \([f, g]\) lie in the van Dantzig class if \(f(s)\) and \(g(s) = 1/f(is)\) are both characteristic functions. By construction,
\[
g(s)f(is) = 1.
\]

See van Dantzig (1958) and Lukacs (1968) for further discussion. Lukacs (1968) provides
many fundamental results and examples. A typical case, where \( f \) is even, is given by

\[
f(s) = \prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{\rho_k^2} \right) \quad \text{and} \quad g(s) = \prod_{k=1}^{\infty} \left( 1 + \frac{s^2}{\rho_k^2} \right)^{-1} = E(e^{isL})
\]

where \( L = \sum_{k=1}^{\infty} L_k / \rho_k \) and \( L_k \) is Laplace (a.k.a. double exponential). \( L \) is a EGGC random variable.

**Wald Couples.** A related condition is a Wald couple. The (infinitely divisible) random variables \((X, H)\) constitute a Wald couple if \( H \geq 0 \) is such that

\[
E(e^{sX})E(e^{-s^2H}) = 1.
\]

**Hadamard-Weierstrass Factorisation.** Suppose that \( f(s) \) is an entire function with zeros at \( \rho_1, \rho_2, \ldots \) and a zero of order \( m \) at \( s = 0 \). Then there is a sequence, \( n_j \), and an entire function, \( g(s) \), with \( E_{n_j}(s) = (1 - s) \exp (s + \frac{1}{2} s^2 + \ldots + \frac{1}{n} s^n) \), such that

\[
f(s) = s^m e^{g(s)} \prod_{j=1}^{\infty} E_{n_j} \left( \frac{s}{\rho_j} \right).
\]

Hadamard (1893) showed that if \(|f(s)| \leq C \exp(|z|^n)\), then the function \( g(s) \) in the Weierstrass product is polynomial of degree \( \leq \lfloor \alpha \rfloor \).

**Laguerre-Pólya Class.** A function, \( f \), lies in the LP class of entire functions if and only if all its roots, \( \rho_n \), are real, \( \sum_{n=1}^{\infty} |\rho_n|^{-2} < \infty \), and its Hadamard product takes the form

\[
f(s) = C s^m e^{bs+cs^2} \prod_{n=1}^{\infty} \left( 1 - \frac{s}{\rho_n} \right) e^{\frac{s}{\rho_n}}.
\]

Hence the LP class consists of real entire functions or order at most two with only real zeros. Lee and Yang (1953), Newman (1974) and Kac (1974) provide characterisations of spin-type variables. For examples, see Hirschman and Widder (1949), Cardon (2002, 2005), Csordas and Varga (1989), Dmitrov and Xu (2016), Baricz and Singh (2017).

**Pólya Frequency Function.** A PFF, denoted by \( \Lambda(x) \), is a non-negative measurable function, \( 0 < \int_{-\infty}^{\infty} \Lambda(x) < \infty \), such that for every two sets of increasing numbers \( x_1 < x_2 < \ldots x_n \quad y_1 < y_2 < \ldots < y_n \quad n = 1, 2, \ldots \)

the inequality \( \| \Lambda(x_i - y_i) \|_{1,n} \geq 0 \) holds. For \( n = 1, \Lambda(x) \geq 0 \) and with \( n = 2, \) this is equivalent to convexity of \( -\log \Lambda(x) \).

**LT of a PFF.** This is an entire function, \( \psi(s), \rho_n \) is real, \( 0 < \gamma + \sum_n |\rho_n|^2 < \infty \), and

\[
\int_{-\infty}^{\infty} e^{-xs} \Lambda(x) dx = \frac{1}{f(s)} \quad \text{where} \quad f(s) = C e^{-cs^2+bs} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{\rho_n} \right) e^{-s/\rho_n}.
\]
For example, the normal and one-sided exponential $\lambda(x) = e^{-x^2} \mathbb{I}(x \geq 0)$ are PFFs.

The shifted one-sided exponential has LT

$$\frac{1}{|\delta|} \int_{-\infty}^{\infty} e^{-sx} \lambda \left( \frac{x}{\delta} + 1 \right) dx = \frac{e^{\delta s}}{1 + \delta s}. $$

See Schoenberg (1948), Schoenberg and Whitney (1953) and Curry and Schoenberg (1966).

**Generalised Gamma Convolutions (GGC).** Bondesson (1992) defines the GGC class of probability distributions on $[0, \infty)$ whose Laplace transform (LT) takes the form,

$$E(e^{-sH}) = \exp \left( -as + \int_{(0,\infty)} \log \left( \frac{z}{z + s} \right) U \, dz \right) $$

for $s > 0$ with (left-extremity) $a \geq 0$ and $U(dz)$ a non-negative measure on $(0, \infty)$ (with finite mass on any compact set of $(0, \infty)$) such that $\int_{(0,1)} |\log t| U(dz) < \infty$ and $\int_{(0,\infty)} z^{-1} U(dz) < \infty$.

The sigma-finite measure $U$ on $(0, \infty)$ is chosen so that the exponent

$$\phi(s) = \int_{(0,\infty)} \log(1 + s/z) U(dz) = \int_0^\infty \int_0^\infty (1 - e^{-sz}) t^{-1} e^{-tz} U(dz) \, dz < \infty $$

and $U$ is often referred to as the Thorin measure, which can have infinite mass. The corresponding Lévy measure is $t^{-1} \int_{(0,\infty)} e^{-tz} U(dz)$.

**Extended Generalised Gamma Convolutions (EGGC).** A symmetric extended GGC probability distribution (symEGGC) on $(-\infty, \infty)$ has mgf defined at least for $Re \ s = 0$ of the form

$$E(e^{s\tilde{H}}) = \exp \left( \frac{1}{2}cs^2 + \int_{-\infty}^{\infty} \left\{ \log \left( \frac{z}{z - s} \right) - \frac{s^2}{1 + z^2} \right\} U(dz) \right) $$

where $U(dz)$ is a non-negative Thorin measure on $\mathbb{R} \backslash \{0\}$ and $c$ is nonnegative.

The class of mean-zero symEGGC random variables is equivalent a Gaussian scale mixture (GSM), $\tilde{H} = \sqrt{2H}Z$ with $Z \sim \mathcal{N}(0,1)$ where the mixing measure, $H \sim$ GGC. The FT of an EGGC random variable can be replaced with an equivalent GGC condition,

$$E(e^{is\tilde{H}}) = E(e^{is\sqrt{2H}Z}) = E(e^{-s^2H}) .$$

See Roynette, Vallois and Yor (2009) for classes of GGC distributions.

As noted by Kent (1982) and Bondesson (1981) the class of ID distributions with completely monotone Lévy densities coincides with the class of generalized convolutions of mixtures of exponentials, which we denote by $\sum_{\rho>0} \text{Exp}(\rho)$.

**Bernstein Function.** The mapping $s \mapsto \exp(-\psi(s))$ is completely monotone iff $\psi(s)$ is a
Bernstein function where $\nu(du)$ is a Lévy measure if

$$\psi(s) = a + bs + \int_0^\infty (1 - e^{-st})\nu(dt), \quad \forall s > 0.$$ 

Thorin Measure. The Thorin class of GGC distributions is characterised by the property that its Laplace transform (LT) can be expressed in terms of a Bernstein function and Lévy measure, $\nu(dt)$, with $\int_0^\infty \min(1,t)\nu(dt) < \infty$ and

$$E(e^{-sx}) = \exp \left( -bs - \int_0^\infty (1 - e^{-st})\nu(dt) \right)$$

$$\nu(dt) = \frac{g(t)}{t} dt \text{ where } g(t) \text{ is CM.}$$

Then, write $g(t) = \int_0^\infty e^{-ty}U(dy)$ where $U(dy)$ is the Thorin measure.

Hyperbolically Completely Monotone (HCM). Bondesson (1992) defines a HCM function as one where $h(w) = f(u/v)f(uv)$ with $w = v + v^{-1}$ is a completely monotone Bernstein function. The LT of a GGC is an HCM function. Jedidi and Simon (2013) and Bosch (2014) provide further examples.

Mean-Zero Ferromagnetic. A random variable $Y$ is mean-zero ferromagnetic if there exists a sequence of random variables $Y^{(n)} = \sum_{i=1}^n a_i X_i$ where $X_i$ are spin variables that weakly converge to $Y$. Lee-Yang-Newman shows that the LT of a mean-zero ferromagnetic random variable can be expressed as

$$E(e^{sY}) = e^{cs^2} \prod_{\rho} \left( 1 + \frac{s^2}{\rho} \right).$$

Mixtures of Exponentials. The following result is extremely insightful and provides a probabilistic interpretation of the Mittag-Liffler partial fraction expansion.

Let $X = \sum_{j=1}^\infty X_j$ where $X_j \sim \text{Exp}(\lambda_j)$. Then $X$ has LT given by

$$E(e^{-sX}) = \prod_{j=1}^\infty \frac{1}{1 + s/\lambda_j}.$$ 

Let $\phi(s) = E(e^{sX})$. If $\lambda_j$ are all unequal, this can be given a partial fraction expansion

$$E(e^{-sX}) = \sum_{j=1}^\infty \frac{\lambda_j}{\lambda_j + s} \prod_{i \neq j, i=1}^\infty \frac{\lambda_i}{\lambda_i - \lambda_j}.$$
If \( |\text{Res}(\phi, \lambda_k)| = O(e^{\lambda_k}) \) \( \forall \epsilon \) as \( k \to \infty \), we can find the density (Kent, 1982) as

\[
p(u) = -\sum_{k=1}^{\infty} \text{Res}(\phi, \lambda_k)e^{-\lambda_k u}, \quad u > 0
\]

where \( \text{Res}(\phi, \lambda_k)^{-1} = d(1/\phi)/ds|_{s=\lambda_k} \) is the residue of \( \phi(s) \) at \( s = \lambda_k \).

2 Riemann-Pólya vs Thorin-Bondesson

The goal is to find the zeros of the entire function \( f_\alpha(s) := f(\alpha + s)/f(\alpha) \), \( \alpha > 0 \) via its Hadamard product. In particular, we consider the special case \( f_\alpha(s) = f_\alpha(-s) \) is even.

Thorin’s approach is to represent the reciprocal, \( f(\alpha)/f(\alpha + \sqrt{s}) = E(e^{-sH}), \ s > 0 \) as the Laplace transform of a GGC random variable, \( H \). This is known as Thorin’s condition (Bondesson, 1992). A key point is that this is only a real condition, namely \( s > 0 \). This is because analytical properties of the LT of a GGC allow us to extend the identity to the cut plane \( \mathbb{C} \setminus (-\infty, 0) \) and hence deduce that \( f \) has only real zeros.

Given a Wald couple \((X, H)\) then, if \( H \geq 0 \), we have

\[
E(e^{sX})E(e^{-s^2H}) = 1.
\]

Let \( H = B_H \) where \( B_t \) is a standard Brownian motion. Then, equivalently

\[
E(e^{sX})E(e^{is\bar{H}}) = 1.
\]

Given \( f_\alpha \), we construct pairs \((X, H)\) such that \( H \sim GGC \) and

\[
\frac{f(\alpha + s)}{f(\alpha)} = E(e^{sX}) \quad \text{and} \quad \frac{f(\alpha)}{f(\alpha + \sqrt{s})} = E(e^{-sH})
\]

The latter identity is related to Thorin’s condition.

The Riemann-Pólya framework directly tries to show that the Fourier transform

\[
\frac{f(\alpha + is)}{f(\alpha)} = E(e^{isX})
\]

is non-zero. This is true if \( X \) is infinitely divisible, then \( f \) has only real zeros. Pólya (1926) develops methods for finding real zeros of mixtures of such functions.

The Thorin-Bondesson is to find \( H \sim GGC \) such that, for the real condition,

\[
\frac{f(\alpha)}{f(\alpha + \sqrt{s})} = E(e^{-sH}), \ s > 0.
\]
2.1 van Dantzig Pairs-Wald Couples to Hadamard Products

Let \( f_\alpha(s) := f(\alpha + is)/f(\alpha) \) be an entire function of order one. Suppose that its zeros are given by \( \rho \) and the Hadamard product for \( f_\alpha(s) \) takes the form

\[
\frac{f_\alpha(s)}{f_\alpha(s+is)} = e^{-bf_s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} = \exp\left(-bf_s + \sum_{\rho} \frac{s}{\rho} + \log \left(1 - \frac{s}{\rho}\right)\right).
\]

Inverting yields \( f(\alpha)/f(\alpha + is) = \exp\left(bf_s - \sum_{\rho} s/\rho + \log \left(1/(1 - s/\rho)\right)\right) \).

**Important special case:** \( f_\alpha(s) = f_\alpha(-s) \) is even

Here \( b_f = 0 \) and the zeros are \( \pm \rho \). Then replacing \( s \to is \), gives

\[
\frac{f(\alpha)}{f(\alpha + s)} = \prod_{\rho > 0} \frac{\rho^2}{\rho^2 + s^2}.
\]

Then, using Frullani’s identity, \( \log(z/z + s^2) = \int_0^\infty (1 - e^{-s^2 t}) e^{-tz} dt/t \), write

\[
\frac{f(\alpha)}{f(\alpha + s)} = \prod_{\rho > 0} \frac{\rho^2}{\rho^2 + s^2} = \exp\left\{ \int_0^\infty \log \left(\frac{z}{z + s^2}\right) U(dz) \right\}
\]

\[
= \exp\left( - \int_0^\infty (1 - e^{-s^2 t}) \frac{g(t)}{t} dt \right)
\]

\[
= E(\exp(-s^2 H)) \text{ where } H \overset{D}{=} \sum_{\rho > 0} \text{Exp}(\rho^2)
\]

where \( H \) is GGC and \( U(dz) \) is the Thorin measure, with \( \delta \) a Dirac measure.

\[
g(t) = \int_0^\infty e^{-tz} U(dz) \text{ is CM and } U(dz) = \sum_{\tau > 0} \delta_{\tau^2}(dz).
\]

An important consequence is that the zeros of \( f \) can be determined from the \( \text{Exp}(\rho^2) \) components in the convolution structure of \( H \).

2.2 Thorin’s condition

The following lemma relates a Wald-couple \((X, H)\) to a representation of an entire function and its reciprocal. In turn, this can be used to find its Hadamard product from the properties of the Thorin measure of \( H \).

**Lemma 1.** Let \((X, H)\) be a Wald couple. Suppose that the function of interest, \( f \), can be expressed as the LT

\[
\frac{f(\alpha + s)}{f(\alpha)} = E(e^{sX}).
\]
Then, the reciprocal-$f$ function is the Laplace transform (LT) of $H$, namely

$$\frac{f(\alpha)}{f(\alpha + \sqrt{s})} = E(e^{-sH}) \text{ for } s > 0.$$ 

This follows from the Wald-couple condition, $E(e^{sX})E(e^{-s^2H}) = 1$.

Equivalently, $f(\alpha)/f(\alpha + \sqrt{s})$ is an HCM function and, with $\hat{H} = B_H$,

$$\frac{f(\alpha)}{f(\alpha + is)} = \int_0^\infty e^{-s^2\Lambda(x)}dx$$

where $\Lambda(x)$ is a Pólya frequency function.

Thorin’s condition (Bondesson, 1992, p.124) is a real condition, $s > 0$,

$$\frac{f(\alpha)}{f(\alpha + \sqrt{s})} = E(\exp(-sH)) \text{ for } s > 0.$$ 

Whilst simple, the power of checking Thorin’s condition is that the LT of GGC is analytic in $C \setminus (-\infty, 0)$. Then, by analytic continuation, this equality must hold for all values of $s$ in the cut plane, namely $C \setminus (-\infty, 0)$.

Hence, the denominator, $f(\alpha + \sqrt{s})$ cannot have any zeros there, and $f(\alpha + s)$ has no zeros for $Re(s) > 0$. Then $f_\alpha(s)$ has no zeroes for $Re(s) > \alpha$ and, as $f_\alpha(s) = f_\alpha(-s)$, no zeroes for $Re(s) < \alpha$ either.

### 2.3 Finding the Thorin Measure

The following lemma identifies the Thorin measure from a Hadamard product representation and vice-versa

**Lemma 2.** (Polson, 2018). Suppose that the function $f$ (taking $b_f = 0$ w.l.o.g.) satisfies

$$\log \left( \frac{f(\alpha + s)}{f(\alpha)} \right) - s \frac{f'(\alpha)}{f(\alpha)} = \int_0^\infty (e^{-sx} - 1 + sx)e^{-\alpha x} \mu(dx)$$

for some arbitrary $\mu(dx)$. This can then be re-expressed as

$$\log \left( \frac{f(\alpha)}{f(\alpha + s)} \right) + s \frac{f'(\alpha)}{f(\alpha)} = - \int_0^\infty (1 - e^{\frac{1}{2}s^2t}) \frac{\nu_\alpha(t)}{t} dt$$

where $\nu_\alpha(t)$ is the completely monotone function

$$\nu_\alpha(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2z^2} \left( \left( \int_0^\infty 2\sin^2(x\sqrt{z}/2)e^{-\alpha x} \mu(dx) \right) \frac{dz}{\sqrt{\pi z}}. \right.$$
This follows from the identities, valid for \( s > 0 \) and \( x > 0 \),

\[
e^{-sx} + sx - 1 = \int_0^\infty (1 - e^{-\frac{x^2}{2t}})(1 - e^{-\frac{x^2}{2t}}) \frac{x}{\sqrt{2\pi t^3}} dt
\]

\[
\frac{1 - e^{-x^2/2t}}{\sqrt{t}} = \int_0^\infty e^{-tz} \frac{2\sin^2(x\sqrt{z/2})}{\sqrt{\pi z}} dz
\]

In particular,

1. When \( f(s) = \Gamma(1 + s) \) then \( \mu^\Gamma(dx) = dx/(e^x - 1) \).
2. When \( f(s) = \zeta_p(s) \) then \( \mu^\zeta(dx) = \sum_{k \geq 1} (\log p)^k \delta_{k \log p}(dx) \).

We now explicitly calculate the Thorin measure and Hadamard factorizations for particular special functions.

### 2.4 Ramanujan’s Master Theorem (RMT)

A key tool for analytical continuation is via Ramanujan’s Master Theorem (RMT), see Hardy and Amdeberhan et al, (2012). Given an analytic function of the form

\[
f(x) = \sum_{k=1}^\infty \frac{\phi(k)}{k!} (-x)^k.
\]

Then, under growth conditions (Hardy-Ramanujan) on \( \phi \), for \( s \in \mathbb{C} \),

\[
\int_0^\infty x^{s-1} f(x) dx = \Gamma(s) \phi(-s).
\]

For example, if \( f(s) \) is defined for \( \Re(s) > 1 \) we can use RMT to analytically extend \( f(s) \) to \( 0 < s < 1 \) by calculating the coefficients in the expansion terms of a GGC as

\[
\frac{f(\alpha + 1)}{f(\alpha + \sqrt{1 + k})} = E[e^{-kH_{\alpha+1}}].
\]

Then RMT applies, under growth conditions, to give

\[
\frac{f(\alpha + 1)}{f(\alpha + \sqrt{1 - s})} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \left\{ \sum_{k=0}^\infty \frac{(-x)^k}{k!} \frac{f(\alpha + 1)}{f(\alpha + \sqrt{1 + k})} \right\} dx
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \left\{ \sum_{k=0}^\infty \frac{(-x)^k}{k!} E[e^{-kH_{\alpha+1}}] \right\} dx
\]

\[
= E[e^{sH_{\alpha+1}}].
\]
Replacing $s \rightarrow 1 - s$,

$$\frac{f(\alpha + 1)}{f(\alpha + \sqrt{s})} = E[e^{(1-s)H_{\alpha+1}}] = \frac{f(\alpha + 1)}{f(\alpha)} E[e^{-sH_{\alpha}^*}]$$  \hfill (2)

$$\frac{f(\alpha)}{f(\alpha + \sqrt{s})} = E[e^{-sH_{\alpha}^*}]$$  \hfill (3)

We need to check that $H_{\alpha}^*$ is still a GGC. Now the GGC property is preserved under exponential tilting and composition by Theorem 3.3.1 in Bonessson, [1, p.41]. Moreover, $\int_0^\infty e^x f_H(x) dx < \infty$. Hence $H_{\alpha}^*$ is GGC and Thorin’s condition holds for $f_\alpha(s)$.

### 3 Applications: $\Gamma$, Bessel and MacDonald functions

**Example: Gamma.** The log-gamma distribution, $X_{\alpha,\lambda} = -\lambda \log \gamma_{\alpha}$, has the following representation

$$E\left(e^{-sX_{\alpha,\lambda}}\right) = \exp\left(b_{\alpha,\lambda}s - \int_0^\infty (e^{-st} + st - 1) \frac{\nu_{\alpha,\lambda}(t)}{t} dt\right)$$

where $\nu_{\alpha,\lambda}(t) = e^{-at/\lambda}/(e^{-t/\lambda} - 1)$ is completely monotone (CM), see Pitman and Yor (2003) and Grigelionis (2008).

The Mellin-Weierstrass factorisation for the $\Gamma$-function is

$$\frac{\Gamma(a)}{\Gamma(a + s)} e^{s\psi(a)} = \prod_{k=0}^\infty \left(1 + \frac{s}{a + k}\right) e^{-s/a + k} = E\left(e^{-\frac{1}{2}s^2H_a^\Gamma}\right)$$

where $H_a^\Gamma = \sum_{k=0}^\infty (a + k)^{-2} H_{1,k}$.

Let $H_a \sim \frac{1}{2}a^2\gamma_2$ with density $(1/a^2\sqrt{2\pi}x^5) \exp(-1/(2a^2x))$. Let $\hat{H}_a = B_{H_a}$. Then

$$E\left(e^{is\hat{H}_a}\right) = E\left(e^{-\frac{1}{2}s^2H_a}\right) = \left(1 + \frac{|s|}{a}\right) e^{-\frac{|s|}{a}}.$$  

This distribution is a convolution of double exponentials with density $\frac{1}{2a} e^{-\frac{|s|}{a}}$.

Direct calculation (Kendall, 1961) finds the convolution

$$\frac{1}{4a} \left(1 + \frac{|s|}{a}\right) e^{-\frac{|s|}{a}} = \int_0^\infty \frac{1}{2a} e^{-\frac{|s-z|}{a}} \frac{1}{2a} e^{-\frac{|z|}{a}} dz.$$  

Barndorff-Nielsen, Sorenson and Kent (1982) provide direct calculation of the GGC distribution, $H$, in the Wald couple representation for a number of hyperbolic functions.

**Example: 1/Gamma.** The reciprocal gamma function is a scale mixture of normals (Hart-
man (1976, section 6) which follows from the CM representation
\[
\frac{e^{-\gamma\sqrt{s}}}{\Gamma(1 + \sqrt{s})} = \int_{0}^{\infty} e^{-st} P_{\gamma}(dt)
\]
where \(\gamma = -\psi(1)\) is Euler’s constant and \(P_{\gamma}(dt)\) is a finite measure.

This can be used with the Gamma-distribution results to find a Wald couple \((X, H)\) for \(e^{-\gamma s}/\Gamma(1 + s)\), see Roynette and Yor (2005).

**Example: Gumbel.** The Gumbel distribution, \(X = \log E\), with cdf \(\exp(-e^{-x})\) has LT
\[
E(e^{-sX}) = \Gamma(1 + s) = \exp\left(-\gamma s + \int_{0}^{\infty} (e^{-st} - 1 + st) \frac{dt}{t(e^t - 1)}\right)
\]
As \(g(t) = 1/(e^t - 1)\) is CM, we have \(X \sim EGGC\).

Hinds (1974) shows that if \(X_1\) has density \(2/\sqrt{\pi}\exp(x - e^{2x})\) and \(X_2 \overset{D}= -X_1\), then the c.f.s are
\[
f_1(s) = \frac{1}{\sqrt{2\pi}} \Gamma(\frac{1}{2} + is) \quad \text{and} \quad f_2(s) = \frac{1}{\sqrt{2\pi}} \Gamma(\frac{1}{2} - is)
\]
respectively. Neither of which falls into the van Dantzig class. However, \(X = X_1 + X_2\) has c.f. \(f_1(s)f_2(s) = 1/cosh(\frac{1}{2}\pi s)\) which lies in the van Dantzig class.

**Example: symBeta.** Hinds (1974), Amdeberhan et al (2012), Hasebe, Simon and Wang (2018) consider the symmetric Beta distribution (a.k.a power semi-circular distribution). This generalises the arc-sine, uniform and Wigner’s semi-circle distribution \((\nu = -\frac{1}{2}, \frac{1}{2}, 1, \text{respectively})\), where \(X_\nu\) has density
\[
p_\nu(x) = \frac{1}{B(\nu + \frac{1}{2}, \frac{3}{2})} (1 - x^2)^{\nu - \frac{1}{2}} \mathbb{1}_{-1 < x < 1}, \quad \nu > -\frac{1}{2}.
\]
The characteristic function is given by a Bessel function of the first kind
\[
f_\nu(s) = E(e^{isX_\nu}) = (s/2)^{-\nu} \Gamma(\nu + 1) J_\nu(s) = \prod_{n \geq 1} \left(1 - \frac{s^2}{j_{\nu,n}^2}\right)
\]
where \(j_{\nu,n}\) denotes the Bessel zeros, see Jurek (2010). For \(\nu \geq -\frac{1}{2}\), \(g_\nu(s) = 1/f_\nu(is)\) is also a characteristic function.

Specifically, if \(H_\nu := \sum_{n \geq 1} L_n/j_{\nu,n}\) where \(L_n\) are i.i.d. Laplace with density \(e^{-\frac{1}{2}|x|}\),
\[
E(e^{isH_\nu}) = \prod_{n \geq 1} \left(1 + \frac{s^2}{j_{\nu,n}^2}\right)^{-1} = \frac{1}{f(is)}.
\]
Hence we have a van Dantzig class.

Special cases include the pairs \(f_{-\frac{1}{2}}(s) = \cos s, g_{-\frac{1}{2}}(s) = 1/cosh(s)\) and \(f_{\frac{1}{2}}(s) = \sin s/s, g_{\frac{1}{2}}(s) = s/sinh(s)\) which we discuss in Section 4.
Example: Bessel functions (Pólya, 1926) Define the Bessel and Macdonald functions

\[ K_z(a) = \int_0^\infty e^{-\frac{a}{2}(t+1)} t^{z-1} dt \]
\[ \mathcal{G}(z, a) = \int_{-\infty}^\infty e^{-a(u^2+e^{-u})+zu} du \]

where \( K_z(a) = \int_{-\infty}^{\infty} \cosh(zu)e^{-\frac{a}{2}(e^u+e^{-u})} du \) and \( K_x(2x) = \mathcal{G}(z, x) \), see Biane (2009).

For the Macdonald function, let \( T_a \) denote the inverse Gaussian, \( T_a \), with density

\[ p_a(t) = \frac{ae^a^2}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2}(t+1)^{-1}} dt. \]

The Mellin transform is given by

\[ E(T^s) = \sqrt{\pi} a^{-1} K_{\frac{1}{2}+s}(a^2) = \sqrt{\pi} a^{-1} \mathcal{G}\left\{ (\sigma - \frac{1}{2}) + it, a^2 \right\} \]

Pólya showed \( K_{\sigma-\frac{1}{2}+it}(\mu) \) has zeros only on the line \( \sigma = \frac{1}{2} \) by using the identity

\[ \int_0^\infty t^{(\sigma-\frac{1}{2})-1}e^{-\frac{a}{2}(t+1)^{-1}} dt = \sqrt{2\pi a^{-2}} e^{-a^2} \mathcal{G}\left\{ (\sigma - \frac{1}{2}) + it, a^2 \right\}. \]

4 Applications: Trigonometric and Hyperbolic functions

Example: \( \cosh \) and \( \sinh \). The Hadamard products for hyperbolic functions are

\[ \cosh(s) = \prod_{i=1}^\infty \left( 1 + \frac{s^2}{(n-\frac{1}{2})^2\pi^2} \right) \quad \text{and} \quad \frac{\sinh s}{s} = \prod_{n=1}^\infty \left( 1 + \frac{s^2}{n^2\pi^2} \right). \]

The latter is known as Euler’s product formula. Following Section 2.1, this Hadamard product can be represented as a LT of a GGC density with a Thorin measure.

For \( \sinh(s)/s \), let \( E_n \) be a sequence of i.i.d exponentials and define

\[ S_1 \overset{D}{=} \frac{2}{\pi^2} \sum_{n=1}^\infty \frac{E_n}{n^2} \quad \text{with} \quad E(e^{sS_1}) = \frac{s}{\sinh s}. \]

Here we can write

\[ \log (as/\sinh(as)) = \int_0^\infty (e^{-\frac{1}{2}a^2t} - 1) \sum_{n=1}^\infty e^{-\pi^2n^2/a^2t} dt. \]

The Lévy and Thorn measures are given by

\[ \mu(dt) = 2 \sum_{n=1}^\infty e^{-\frac{a^2}{2\pi^2}n^2} dt \quad \text{and} \quad \nu(dt) = 2 \sum_{n=1}^\infty \delta_{\frac{1}{2}n^2/a^2}(dt). \]
Let $W_a = S_a + S'_a$ where $S'_a = S_a$ and
\[
\mathbb{E} \left( e^{-\frac{1}{2} s^2 S_a} \right) = \frac{as}{\sinh(as)}.
\]
The Mellin transform, $\mathbb{E} (W_a^s) = 2(2a^2/\pi)^s \xi(s)$ relates to $\xi$-function (Williams, 1990).

The density of $W_a$ is determined via Laplace inversion (Ciesielski and Taylor, 1962)
\[
\mathbb{P} (W_a \in dx) = \sum_{n=1}^{\infty} \pi^2 (\pi^2 n^2/a^2 - 3) \frac{n^2}{a^2} e^{-\pi^2 n^2 x/a^2} dx.
\]

Jurek (2003) and Jurek and Yor (2010) provide calculations of van Dantzig pairs and Wald couples for hyperbolic functions.

The sum representation of $\sinh x$ is given by
\[
\frac{\pi}{\sinh(\pi s)} = \frac{2\pi}{e^{\pi s} - e^{-\pi s}} = \pi \sum_{m \in \mathbb{Z}} (-1)^m \frac{s^2 + m^2}{s^2 + m^2}.
\]

Finally, $X \sim U(-a, a)$ is mean-zero ferromagnetic (Kac, 1974, Williams, 1990) with
\[
E(e^{sX}) = \frac{\sinh(as)}{as} = \prod_{n=1}^{\infty} \left( 1 + \frac{a^2 s^2}{n^2 \pi^2} \right).
\]

Biane, Pitman and Yor (2001) and Devroye (2009) define the following random variables
\[
C_1 \overset{D}{=} \sum_{n=1}^{\infty} \frac{\Gamma_{1,n}}{(n-\frac{1}{2})^2} \text{ with } \mathbb{E} \left( e^{-\frac{1}{2} s^2 C_1} \right) = \frac{1}{\cosh(s)}.
\]

Hyperbolic powers follow from
\[
C_2 \overset{D}{=} \sum_{n=1}^{\infty} \frac{\Gamma_{2,n}}{(n-\frac{1}{2})^2} \text{ with } \mathbb{E} \left( e^{-\frac{1}{2} s^2 C_2} \right) = \frac{1}{\cosh^2(s)}.
\]

Let $X$ be a random variable with LT given by
\[
E \left( e^{-sX} \right) = \frac{1}{\cosh(\sqrt{s})} = \text{sech}(\sqrt{s}) = 4\pi \sum_{k \geq 0} \frac{(-1)^k (2k + 1)}{\pi^2 (2k + 1)^2 + 4s}.
\]

The LT of an $Exp(\lambda)$ random variable is given by $\lambda/(\lambda + s)$. Hence inverting term-by-term, we see that $X$ is a sum of exponentials known as a Pólya-Gamma distribution and a GGC.

The Weierstrass-Hadamard product formula for $\cosh x$ is given by
\[
\cosh(s) = \prod_{k=0}^{\infty} \left( 1 + \frac{4s^2}{(2k + 1)^2 \pi^2} \right)
\]
Taking logs and differentiating gives

\[
\frac{\tanh(s)}{s} = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2 \pi^2 + 4s^2}
\]

Hence, we can also find a representation of \( \tanh(s)/s \).

This class is related to Ramanujan’s class \( \prod_{k=0}^{\infty} \left( 1 + \frac{s^2}{(a+kd)^2} \right) \).

**Example: Z Distributions.** Barndorff-Nielsen, Kent and Sorensen (1983) define the class of random variables \( H_{\delta, \gamma} \) on \( (0, \infty) \) which have mgf

\[
E \left( e^{sH_{\delta, \gamma}} \right) = \prod_{k=0}^{\infty} \left( 1 - \frac{s^2}{(\delta + k)^2 - \gamma} \right)^{-1}, \ \delta > 0, \gamma < \frac{1}{2}\delta^2.
\]

Now \( H_{\delta, \gamma} \) lies in the class of infinite convolutions of exponentials also known as the class of Pólya distributions with support \( (0, \infty) \).

The hyperbolic secant distribution \( 1/\sqrt{2\pi} \cosh(\sqrt{\pi/2}x) \) is a special case of a Z-distribution: \( Z(1, 1, 0, 0) \). The density of a symmetric \( Z(\delta, \delta, 1, 0) \) random variable is given by

\[
\frac{1}{4^\delta B(\delta, \delta)} \frac{1}{\cosh(x/2)^{2\delta}}
\]

The symmetric Z-distributions are normal variance mixtures under \( H(\delta, 0) \) with

\[
E \left( e^{isH_{\delta, 0}} \right) = g(s) = \prod_{k=0}^{\infty} \left( 1 + \frac{s^2}{(\delta + k)^2} \right)^{-1} = \frac{\Gamma(\delta + is)\Gamma(\delta - is)}{\Gamma(\delta)^2}
\]

Here \( \lambda_k = \frac{1}{2}(\delta + k)^2 \). Hence the density is given by

\[
p(u) = -\sum_{k=1}^{\infty} \left( -\frac{2\delta}{k} \right) \frac{\delta + k}{B(\delta, \delta)} e^{-\frac{1}{2}(\delta+k)^2u}, \ u > 0
\]

where the residues alternate in sign.

**Example: Ostrovskii.** Define a general class of hyperbolic functions, denoted by \( f_{\delta}(\cosh t) \), by the conditions \( \frac{1}{2} < C < B, \delta > 0 \) and \( \sum_{k=1}^{\infty} a_k < \infty \), with

\[
f_{\delta}(z) = \frac{1}{z^2} - \frac{1}{Cz} + \delta g(z), \ \text{where} \ g(z) = \sum_{k=1}^{\infty} \frac{a_k}{z + h_k}.
\]

Ostrovskaia (1970) shows that the following functions are characteristic functions (c.f.), when \( \delta \) is chosen to be small enough, \( f_{\delta}(\cosh t) \) is a van Dantzig class, with

\[
\phi_{\delta}(t) = \frac{f_{\delta}(\cosh t)}{f_{\delta}(1)} \quad \text{and} \quad \frac{1}{\phi_{\delta}(it)} = \frac{f_{\delta}(1)}{f_{\delta}(\cos t)}
\]
To show the $\phi_\delta(t)$ is a c.f. it is enough to show that the following function is summable and non-negative on $0 \leq x < \infty$

$$h_\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix} \phi_\delta(t) dt$$

The following formulas prove useful:

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{\cosh t + b} dt = \frac{2\pi}{\sin \alpha} \frac{\sinh(\pi - \alpha)x}{\sinh \pi x}, \quad -1 < b < 1, 0 < \alpha < \pi, \cos \alpha = -b$$

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{\cosh t + b} dt = \frac{2\pi}{\sinh \alpha \sinh \pi x}, \quad b > 1, \alpha > 0, \cosh \alpha = -b$$

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{\cosh^2 t} dt = \frac{2\pi}{\sinh \pi x} x \cosh(\frac{\pi}{2} x)$$

which follow from the theory of residuals. Now write, with $\alpha_k > 0$, $\cosh \alpha_k = h_k$,

$$f_\delta(1) h_\delta(x) = \frac{1}{\sinh \pi x} \left\{ x \cosh(\frac{\pi}{2} x) - \frac{1}{C} \sinh(\frac{\pi}{2} x) + \delta \sum_{k=1}^{\infty} \frac{a_k}{\sinh \alpha_k} \right\}$$

Thus the function $h_\delta(x)$ is summable.

Since $\alpha_k \geq \beta > 0$, where $\cosh \beta = B$, and $|\sin \alpha_k x|/\sinh \alpha_k \leq \alpha_k x/\sinh \alpha_k \leq \sinh \beta$ for $x \geq 0$, we have

$$f_\delta(1) p_\delta(x) \geq \frac{1}{\sinh \pi x} \left\{ x \cosh(\frac{\pi}{2} x) - \frac{1}{C} \sinh(\frac{\pi}{2} x) - x \frac{\delta \beta}{\sinh \beta} \sum_{k=1}^{\infty} a_k \right\}$$

When $\delta$ is small enough, the r.h.s. is non-negative $\forall x \geq 0$ as $C > \pi/2$. As $f_\delta(1) > 1$, when $\sigma$ is small enough, $h_\delta(x)$ is non-negative. Therefore, $\phi_\delta(t)$ is a c.f. Ostrovskii (1970) shows that $f_\delta(1)/f_\delta(\cos \sigma)$ is also a characteristic function.

5 Application: Riemann $\xi$ and $L$ functions

5.1 Riemann $\zeta$-function

Riemann (1859) defines the zeta function, $\zeta(s)$, as the analytic continuation of $\sum_{n=1}^{\infty} n^{-s}$ on $\Re(s) > 1$ and Euler’s product formula, for $\alpha > 1$, gives

$$\zeta(\alpha + s) = \prod_{p \text{ prime}} (1 - p^{-\alpha-s})^{-1} = \prod_{p \text{ prime}} \zeta_p(\alpha + s) \text{ where } \zeta_p(s) := p^s/(p^s - 1)$$
Now \( \zeta(\alpha) = \prod_p \zeta_p(\alpha) \), yields

\[
\log \frac{\zeta(\alpha + s)}{\zeta(\alpha)} = \sum \log \frac{1 - p^{-s}}{1 - p^{-\alpha}} = \sum_p \sum_{r=1}^{\infty} \frac{1}{p^{s+\alpha r}(e^{-sr\log p} - 1)}
\]

\[
= \int_0^{\infty} (e^{-sx} - 1)e^{-\alpha x} \frac{\mu(\alpha)(dx)}{x}
\]

where \( \mu(\alpha)(dx) = \sum_p \sum_{r=1}^{\infty} (\log p) \delta_r p^{-r-\alpha} \).

Hence Lemma 2 applies.

### 5.2 Riemann \( \xi \)-function

The \( \xi \)-function is defined by

\[
\xi(s) = \frac{s}{2}(s - 1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right) \zeta(s).
\]

The \( \xi \)-function is an entire function of order one and hence admits a Hadamard factorisation, see Titchmarsh (2.12.5).

Riemann showed \( \xi(s) \) is a Mellin transform. Pólya (1926) constructed a random variable \( X_{\frac{1}{2}} \), with symmetric density such that

\[
\zeta(\frac{1}{2} + is) = E(e^{isX_{\frac{1}{2}}}), \quad \forall s \in \mathbb{C}.
\]

The density depends on the Jacobi function with tails \( p(x) \sim 4\pi^2 e^{\frac{2}{x} - \pi e^{2x}} \) as \( x \to \infty \),

\[
p(x) = \frac{1}{\xi(\frac{1}{2})} \sum_{n=1}^{\infty} p_n(x) \quad \text{and} \quad p_n(x) := 2n^2\pi(2\pi n^2 e^{-2x} - 3)e^{-\frac{2}{x} + \pi n e^{-2x}}.
\]

Khintchine showed infinitely divisibility of \( X_\sigma \) for \( \sigma > 1 \), where

\[
\frac{\xi(\sigma + is)}{\xi(\sigma)} = E(e^{isX_{\sigma}})
\]

This is not infinitely divisible for \( \frac{1}{2} < \sigma < 1 \), as it has a signed Lévy measure

\[
\frac{\xi(\sigma + is)}{\xi(\sigma)} = \exp \left( -2 \int_0^{\infty} (e^{-isx} - 1) \sum_{\tau} \frac{\cos(\tau x)}{xe^{(\sigma - \frac{1}{2})x}} dx \right).
\]

Polson (2018) constructs GGC representations, \( H_\alpha^\xi \) for \( \alpha > 1 \), such that

\[
\frac{\xi(\alpha)}{\xi(\alpha + \sqrt{s})} e^{-b_\alpha \sqrt{s}} = E(\exp(-sH_\alpha^\xi)), \quad s > 0
\]

where \( b_\alpha = -\frac{\xi'}{\xi}(\alpha) + \frac{1}{\alpha - 1} \).
Moreover, Thorin’s condition holds for the reciprocal \( \xi \)-function
\[
\frac{\xi(\frac{1}{2})}{\xi(\frac{1}{2} + \sqrt{s})} = E(\exp(-sH^c_\xi)), \ s > 0.
\]

Chaudry and Qadir (2012) provide growth conditions for the reciprocal zeta function, \( 1/\zeta(s) \) and reciprocal gamma \( 1/\Gamma(s) \) function thus allowing RMT to provide an analytical continuation of the \( \xi \)-function to the strip \( 0 < s < 1 \).

Hayman-Grosswald (1966) provides the growth condition for \( \xi \) for RMT to hold true.

5.3 L-functions

Consider a Dirichlet L-series, defined for \( \text{Re}(s) > \sigma \), given by
\[
L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.
\]

Here \( \chi \) is a completely multiplicative Dirichlet character. A Dirichlet L-series then admits an Euler product
\[
L_\chi(s) = \prod_p (1 - \chi(p)p^{-s})^{-1} \text{ for } \text{Re}(s) > \sigma.
\]

Define the regularized L-function by
\[
\Lambda(s, \chi) := \left(\pi/k\right)^{-\frac{1}{2}(s+\epsilon)}\Gamma(\frac{1}{2}(s + \epsilon))L_\chi(s)
\]
where \( \tau(\chi) \) is the Gauss sum, with \( \tau(\chi) = \sum_{n=1}^{k} \chi(n)e^{2\pi in/k} \) and \( \epsilon = 0 \) if \( \chi(-1) = 1 \) and \( \epsilon = 1 \) if \( \chi(-1) = -1 \). This construction leads to a functional equation, and a symmetry property, similar to the \( \xi \)-function, given by
\[
\Lambda(s, \chi) = (-1)^\epsilon \tau(\chi)\Lambda(1 - s, \bar{\chi}).
\]

Kuznetsov (2017) provides Lévy calculations for Dirichlet L-series. First, write
\[
L_\chi(s) = \prod_p (1 - \chi(p)p^{-s})^{-1} = \exp \left( - \sum_p \log(1 - \chi(p)p^{-s}) \right)
\]
\[
= \exp \left( \sum_p \sum_{n \geq 1} \frac{\chi(p)^k}{kp^{ks}} \right) = \exp \left( \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{\log(n)n^\sigma} \right).
\]

Then define the finite measure, \( \mu_L(dx) \), with mass \( \mu_L((0, \infty)) = \log L_\chi(\sigma) \), by
\[
\mu_L(dx) = \sum_{n \geq 2} \frac{\Lambda(n)\chi(n)}{\log(n)n^\sigma} \delta_{\log n}(dx).
\]
Dirichlet L-series Hadamard products follow from Section 3.4. Using Lemma 2, we can define the associated Thorin measure $\nu_L(dt)$ for the reciprocal $\Lambda_L$ function, namely

$$\nu_L(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-tz} \left( \int_0^\infty 2\sin^2(x\sqrt{z/2})e^{-\alpha x} \mu_L(dx) \right) \frac{dz}{\sqrt{\pi z}}.$$ 

Hence, there exists a corresponding GGC random variable that allows you to represent the reciprocal of $\Lambda(s, \chi)$ as the Laplace transform of $H_L$.

There is also a Poisson process interpretation. Specifically, let $(N_t)_{t \geq 0}$ be a Poisson process with rate $\lambda$ and $E(N_t) = \lambda t$. Any Bernstein function $\kappa + \delta s + \int_0^\infty (1 - e^{-sx}) \mu(dx)$ has an associated subordinator, $X_t$. This generates powers of $L$-series.

Define the subordinator $X_t = \delta t + \sum_{j=0}^{N_t} V_j$ where $V_j$ are realizations from $c_L^{-1} \mu_L(dx)$. Then the LT, with $\phi_X(s) = \int_0^\infty (1 - e^{-sx}) \mu(dx)$, satisfies

$$E(\exp(-sX_t)) = \exp(-t\phi_X(s)) = \left( \frac{L_X(s + \sigma)}{L_X(\sigma)} \right)^t.$$ 

Given $X_t$, let $Z_t = t/c - X_t$ with first passage time $Y_x := \inf \{t > 0 : Z_t > x\}$.

Then $Y_x$ is a subordinator such that for $w, x > 0$, $E(\exp(-wY_x)) = \exp(-x\phi_Y(w))$ where $z/c - \phi_X(z) = w \iff z = \phi_Y(w)$ for $w > 0$. Kendall’s identity then provides number theoretic results. See Kendall (1961).

### 5.4 Dedekind $\eta$-function

The Dedekind $\eta$-function is given by

$$\eta(ix) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{where} \quad q = e^{-2\pi x}.$$ 

Euler’s identity yields

$$\eta(ix) = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \cos\left(\frac{\pi}{6} (2n + 1)\right) q^{\frac{1}{48} (2n+1)^2}.$$ 

Glasser (2009) finds the LT of powers of $\eta$ as follows. Calculating, with $q = e^{-2\pi x}$,

$$\int_0^1 q^{y-1} \eta(ix) dx = \frac{2\pi}{\sqrt{3}} \sum_{n=0}^{\infty} \cos\left(\frac{\pi}{6} (2n + 1)\right) \int_0^\infty e^{-2\pi (y + \frac{1}{24} (2n+1)^2)x} dx = \pi \sqrt{\frac{2}{y}} \sinh\frac{\sqrt{8y/3}}{\sqrt{6y}}.$$ 

Using the identity (Appendix A)

$$\sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi}{6} (2n + 1)\right)}{(2n + 1)^2 + (2z)^2} = \frac{\pi}{8z} \frac{\sinh(2\pi z/3)}{\cosh(\pi z)}.$$
Letting $s = 3\pi y$, gives the LT of $\eta$ as

$$
\int_0^\infty e^{-sx} \eta(ix) dx = \sqrt{\frac{\pi}{s}} \sinh \frac{2\pi s/3}{3} \cosh \frac{\sqrt{3\pi s}}{s}.
$$

Jacobi’s triple product identity yields

$$
\eta^3(ix) = \sum_{n=0}^\infty (-1)^n (2n+1)q^{\frac{1}{2}(2n+1)^2} \text{ where } q = e^{-2\pi x}.
$$

This yields LT of $\eta^3$ as

$$
\int_0^\infty e^{-sx} \eta^3(ix) dx = \text{sech}(\sqrt{\frac{\pi s}{12}}) = 4\pi \sum_{k \geq 0} \frac{(-1)^k (2k+1)}{\pi^2 (2k+1)^2 + 4\pi s}.
$$

The Dirichlet beta function, $\beta(s) = \sum_{n\geq 1} (-1)^n / (2n+1)^s$ arises as the Mellin transform of $\text{sech}(\sqrt{\frac{2\pi}{3}}x)$.

### 5.5 Ramanujan $\tau$-function

de Bruin (1950) and Walker (1988) consider zeros of trigonometric functions. Let $\tau(n)$ be defined via its generating function

$$
g(y) = \sum_{n=1}^\infty \tau(n)y^n = y \prod_{k=1}^\infty (1 - y^k)^{24}.
$$

Ramanujan defined his $\tau$-function, $L_\tau(s)$, by the series, where

$$
L_\tau(s) = \sum_{n=1}^\infty \tau(n)n^{-s}.
$$

This satisfies

$$
\sum_{n=1}^\infty \tau(n)n^{-s} = \prod_p (1 - \tau(p)p^{-s} + p^{1-2s})^{-1}
$$

where the series and product are absolutely convergent for $Re(s) > \frac{13}{2}$. Lemma 2 also applies. $L_\tau(s)$ can be continued to the whole plane by constructing $\xi_\tau(s)$ and $\Xi_\tau(s)$ as

$$
\xi_\tau(s) = (2\pi)^{-s}\Gamma(s)L_\tau(s) \text{ and } \Xi_\tau(s) = \xi(6 + is).
$$

The generating function satisfies a functional identity

$$
x^6 g(e^{-2\pi x}) = \left(g(e^{-2\pi x})g(e^{-2\pi /x})\right)^{\frac{3}{2}}.
$$
Then we can write this as a Fourier transform

\[ \Xi_t(s) = (2\pi)^{-s-6} \xi_t(s+6) \Gamma(s+6) = \int_0^\infty x^{s+5} g(e^{-2\pi x}) dx. \]

Using the functional identity, this implies \( \Phi_t(t) = \Phi_t(-t) \), and

\[ \Xi_t(s) = \int_{-\infty}^\infty e^{ist} \Phi_t(t) dt = \int_{-\infty}^\infty e^{ist} e^{-2\pi t} \prod_{k=1}^\infty (1 - e^{-2\pi k t})^{12} (1 - e^{-2\pi k t^{-1}})^{12} dt. \]

Alternatively, we can write \( F_t(t) = e^{6t} e^{-2\pi t^2} \prod_{n=1}^\infty (1 - e^{-2\pi t^2})^{24} \) and \( \Xi_t(t) = \int_{-\infty}^\infty e^{ist} F_t(t) dt. \)

The Weierstrass product form can be calculated using \( \sigma_{-1}(n) = \sum_{d|n} d^{-1} \) as

\[ \prod_{n=1}^\infty (1 - e^{-2\pi n x}) = \exp \left( -\sum_{n=1}^\infty \sigma_{-1}(n) e^{-2\pi n x} \right). \]

See Conrey and Ghosh (1994) for further discussion.

### 5.6 Birch-Swinnerton-Dyer (BSD) conjecture

Solvability of equations is directly related to the L-function of the elliptic curve, \( E \), that maps out the solution. Using the modularity of \( E \), one can associate with it the L-function, \( L_E(s) := \sum_{n=1}^\infty a(n)n^{-s}, s > \frac{3}{2} \) where, if \( n \) is prime, the coefficients \( a(n) \) are related to the number of solutions in integers modulo \( n \) of the equation defining the curve. Modularity allows us to extend \( L \) to \( s = 1 \) and write \( L_E(1) = S\Omega \) where \( \Omega > 0 \).

The BSD conjecture simply states that an elliptic curve has infinitely many rational points if and only if \( S = 0 \), otherwise \( S = 1 \). One can define a regularised L-function for \( E \), denoted by \( \Lambda_E(s) \), which satisfies a functional equation \( \Lambda_E(s) = \pm \Lambda_E(2-s) \), via an entire continuation to \( \mathbb{C} \), namely

\[ \Lambda_E(s) := \left( \frac{\sqrt{N_E}}{2\pi} \right)^{-s} \Gamma(s) L_E(s) \]

where \( N_E = \prod_p p^{f_p} \) is the conductor which can be computed by Tate’s algorithm. This too has a Thorin representation.

Applying Lemma 2, to the reciprocal \( \Lambda_E \)-function, allows us to write each of these terms in Thorin form, Hence, there exists a GGC random variable, denoted by \( H_E \), associated with \( E \) that calculates \( 1/\Lambda_E(s) \) and \( S \) can be calculated via the LT of \( H_E \).
6 Discussion

The Hadamard-Weierstrass factorisation of an entire function is related to van Dantzig pairs and Wald couples of random variables. In particular, we focus on entire functions from the Laguerre-Polya class and show how the Wald couple is related to generalized gamma convolutions (GGCs).

Our methodology characterizes the zeros of entire functions by essentially calculating the convolution representation of the underlying GGC Thorin measure. Applications to a variety of special functions are provided including the general class of L-functions and Riemann’s $\zeta$- and $\Xi$-functions, Ramanujan’s $\tau$-function and Dedekind’s $\eta$-function and $\Gamma$ and Hyperbolic functions.

7 References

Amdeberhan, T., O. espinosa, I. Gonzalez, M. Harrison, V. H. Moll and A. Straub (2012). Ramanujan’s Master Theorem. *The Ramanujan Journal*, 29, 103-120.

Amdeberhan, T., V. H. Moll and C. Vignat (2012). A Probabilistic Interpretation of a sequence related to Narayan polynomials. arXiv1202: 1203.

Baricz, A. and S. Singh (2017). Zeros of some Special Entire functions. arXiv: 1702.00626.

Barndorff-Nielsen, O., Kent, J., & Sørensen, M. (1982). Normal variance-mean mixtures and $z$ distributions. *International Statistical Review*, 145-159.

Biane, P. (2009). Matrix-Valued Brownian Motion and a paper by Pólya. Séminaire de Probabilités, XLII, 171-185.

Biane, P., Pitman, J., & Yor, M. (2001). Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bulletin of the American Mathematical Society*, 38(4), 435-465.

Bondesson, L. (1981). Classes of infinitely divisible distributions and densities. *Z. Wahrsch. Verw. Gebiete*, 57, 39-71.

Bondesson, L. (1992). *Generalised Gamma Convolutions and Related Classes of Distributions and Densities*. Springer-Verlag, New York.

Bosch, P. (2014). HCM property and the half-Cauchy distribution. arXiv:1402.1059.

Bruijn de, N. G. (1950). The roots of trigonometric integrals. *Duke Math. J.*, 17, 197-226.

Cardon, D. A. (2002). Convolution operators and zeros of entire functions. *Proc. Amer. Math. Soc.*, 130(6), 1725-1734.

Cardon, D. A. (2005). Fourier Transforms having real zeros. *Proc. Amer. Math. Soc.*, 133(5), 1349-1356.

Chaudhry, M. A. and A, Qadir (2012). Extension of Hardy’s class for Ramanujan’s interpolation formula and master theorem with applications. *J. Inequalities and Applications*, 52, 1-13.
Ciesielski, Z., & Taylor, S. J. (1962). First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. American Mathematical Society*, 103(3), 434-450.

Conrey, J. B. and A. Ghosh (1994). Turán Inequalities and Zeros of Dirichlet Series associated with certain Cusp Forms. *Trans. Amer. Math. Soc.*, 342(1), 407-419.

Csordas, G. and R. S. Varga (1989). Integral Transforms and Laguerre-Pólya class. *Complex Variables*, 12, 211-230.

Curry, H. B. and I. J. Schoenberg (1966). On Pólya Frequency functions: IV. *J. of Analyse Mathématique*, 17(1), 71-107.

Devroye, L. (2009). On exact simulation algorithms for some distributions related to Jacobi theta functions. *Statistics & Probability Letters*, 79(21), 2251-2259.

Dmitrov, D.K. and Y. Xu (2016). Wronskians of Fourier and Laplace Transforms. *arXiv: 1606.05011*.

Dmitrov, D.K. and P. K. Rusev (2011). Zeros of entire Fourier Transforms. *East Journal of Approximations*, 17(1), 1-108.

Glasser, M. L. (2009). Some Integrals of the Dedekind Eta-function. *J. Math. Anal. Appl.*, 254(2), 490-493.

Grigelionis, B. (2003). On the self-decomposability of Euler’s gamma function. *Lithuanian Mathematical Journal*, 43(3), 295-305.

Grosswald, E. (1964). *Sur la fonction de Riesz*. Séminaire Delange-Pisot-Poitou. Théorie des nombres.

Grosswald, E. (1966). Generalization of a formula of Hayman and its application to the study of Riemann’s Zeta function. *Illinois J. Math*, 10(1), 9.23.

Hadamard, J. (1893). Étude sure les Propriétés des Fonctions Entireres et en Particulier d’une Fonction Considerée par Riemann. *J. Math. Pures Appl.*, 171-215.

Hartman, P. (1976). Completely monotone families of solutions of n-th order linear differential equations and infinitely divisible distributions. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 3(2), 267-287.

Hasebe, T., Simon, T. and M. Wang (2018). Some properties of the free Stable distributions. *Technical Report*.

Hinds, W. E. (1974). Moments of complex random variables related to a certain class of characteristic functions. *Sankhya*, 36(2), 219-222.

Hirschman, I.J. and D.V. Widder (1949). The inversion of a general class of convolution transforms. *Trans. Amer. Math. Soc.*, 66, 135-201.

Jedidi, W. and T. Simon (2013). Further examples of GGC and HCM densities. *Bernoulli*, 19(5A), 1818-1838.

Jurek, Z.J. (2003). Generalized Lévy Stochastic Areas and Self-Decomposability. *Stat. and Probab. Letters*, 64, 213-222.
Jurek, Z. J. and M. Yor (2010). Self-decomposable Laws associated with Hyperbolic functions. arXiv:1009.3542.

Kac, M. (1974). Comments on Pólya’s “Bemerkung Über die Integraldarstellung der Riemannschen ξ-Funktion”. Collected works of G. Pólya.

Kendall, D. G. (1961). Some problems in the Theory of Comets: II. Proc. IV Berk. Symp., 3, 121-147.

Kent, J. T. (1982). The Spectral Decomposition of a Diffusion Hitting Time. Ann. Prob., 10, 207-219.

Kuznetsov, A. (2017). On Dirichlet series and functional equations. arXiv1703.08827

Lukacs, E. (1968). Contributions to a problem of D. van Dantzig. Theory of Probability & its Applications, 13(1), 116-127.

Newman, C. M. (1974). Gaussian correlation inequalities for ferromagnets. Z. Wahrsch. Verw. Gebiete, 33, 75-93.

Ostrovskii, I. V. (1970). A certain class of characteristic functions. Trudy Matematicheskogo Instituta imeni VA Steklova, 111, 195-207.

Ostrovsky, D. (2016). On Barnes beta distributions, Selberg integral and Riemann Xi. Forum Mathematicum, 28 (1), 1-23.

Pitman, J. and M. Yor (2003). Infinitely divisible laws associated with hyperbolic functions. Canadian J. of Math., 55(2), 292-330.

Pólya, G. (1926). Bemerkung Über die Integraldarstellung der Riemannschen ξ-Funktion. Acta Math, 48, 305-317.

Polson, N. G. (2018). On Hilbert’s 8th problem. arXiv: 1708.02653.

Riemann, B. (1859). Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Berliner Akademie.

Roynette, B., Vallois, P., and M. Yor (2009). A family of Generalized Gamma Convoluted variables. Probability and Mathematical Statistics, 29(2), 181-204.

Roynette, B. and M. Yor (2005). Infinitely Divisible Wald’s Couples: Examples linked with the Euler gamma and the Riemann zeta functions. Annales de l’institut Fourier, 55 (4), 1219-1283.

Schoenberg, I. J. (1948). On variation-diminishing integral operators of the convolution type. Proc. Natl. Acad. Sci., 34(4), 164-169.

Schoenberg, I. J. (1951). On Polya frequency functions: I. Journal d’Analyse Mathematique, 1(1), 331-374.

Schoenberg, I. J. and A. Whitney (1953). On Polya frequency functions: III. Trans. Amer. Math. Soc., 74, 246-259.

Thorin, O. (1977). On the infinite divisibility of the log-normal distribution. Scand. Actuar. J., 3, 121-148.
Titchmarsh, E.C. (1974). *The Theory of the Riemann Zeta-function*. Oxford University Press.

van Dantzig, D. (1958). Prize questions. *Nieuw Archiel vaar Wiskunde*, 3(6), 28.

Walker, P. L. (1988). On the zeros of certain trigonometric integrals. *Journal of Mathematical Analysis and Applications*, 130(2), 439-450.

Williams, D. (1990). Brownian Motion and the Riemann-Zeta function. *Disorder in Physical Systems*, 361-372.

8 Appendix A

8.1 Dedekind Eta

First, we show the following identity

\[
\sum_{n=0}^{\infty} \frac{\cos\left[\frac{\pi}{6}(2n+1)\right]}{(2n+1)^2 + (2z)^2} = \frac{\pi}{8z} \frac{\sinh(2\pi z/3)}{\cosh(\pi z)}.
\]

Let \( f(x) = \cosh \alpha x \) with Fourier series, on \([-\pi, \pi]\), given by

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx
\]

where \( b_n = 0, \forall n \) by symmetry. Direct calculation yields

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} \cosh \alpha x \cos nx \, dx = \frac{2\alpha \sinh \pi \alpha}{\pi} \frac{\pi \alpha}{\alpha^2 + n^2} \cos \pi n
\]

Hence,

\[
\cosh \alpha x = \frac{\sinh \pi \alpha}{\pi \alpha} + \frac{2\alpha \sinh \pi \alpha}{\pi} \sum_{n=1}^{\infty} \frac{\cos \pi n \cos nx}{\alpha^2 + n^2}
\]

Evaluating at \( x = \pi - y \) yields

\[
\cosh \alpha y \cosh \pi \alpha - \sinh \alpha y \sinh \pi \alpha = \frac{\sinh \pi \alpha}{\pi \alpha} + \frac{2\alpha \sinh \pi \alpha}{\pi} \sum_{n=1}^{\infty} \frac{\cos ny}{\alpha^2 + n^2}
\]

Rearranging, for \( x \in [0, 2\pi] \), gives

\[
\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\alpha \cos nx}{\alpha^2 + n^2} = \cosh \alpha x \coth \pi \alpha - \sinh \alpha x - \frac{1}{\pi \alpha}.
\]

Define

\[
S(\alpha, x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\alpha \cos nx}{\alpha^2 + n^2}.
\]
Split into odd and even terms

\[
S(\alpha, x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\alpha \cos(2n)x}{\alpha^2 + (2n)^2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\alpha \cos(2n+1)x}{\alpha^2 + (2n+1)^2}
\]

\[
= \frac{1}{2} S\left(\frac{\alpha}{2}, 2x\right) + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\alpha \cos(2n+1)x}{\alpha^2 + (2n+1)^2}
\]

Therefore, for \( x \in [0, \pi] \),

\[
\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\alpha \cos(2n+1)x}{\alpha^2 + (2n+1)^2} = S(\alpha, x) - \frac{1}{2} S\left(\frac{\alpha}{2}, 2x\right)
\]

\[
= \cosh \alpha x \left( \coth \pi \alpha - \frac{1}{2} \coth \frac{\pi}{2} \alpha \right) - \frac{1}{2} \sinh \alpha x
\]

\[
= \frac{1}{2} \sinh \frac{\alpha}{2} \alpha \left( \pi - 2x \right) \cosh \frac{\alpha}{2} \pi.
\]

Hence, for \( x \in [0, \pi] \),

\[
\sum_{n=0}^{\infty} \frac{\cos \left[ \left(2n+1\right)x \right]}{\alpha^2 + (2n+1)^2} = \frac{\pi}{4\alpha} \frac{\sinh \alpha (\pi - 2x)/2}{\cosh \alpha \pi/2}.
\]

Let \( x = \pi/6 \) and \( \alpha = 2z \) to obtain

\[
\sum_{n=0}^{\infty} \frac{\cos \left[ \frac{\pi}{6} \left(2n+1\right) \right]}{(2n+1)^2 + (2z)^2} = \frac{\pi}{8z} \frac{\sinh (2\pi z)}{\cosh (\pi z)}.
\]

Setting \( z = \sqrt{b}y \) yields the desired result.