HOMFLY POLYNOMIALS, STABLE PAIRS AND MOTIVIC DONALDSON-THOMAS INVARIANTS

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Abstract. Hilbert scheme topological invariants of plane curve singularities are identified to framed threefold stable pair invariants. As a result, the conjecture of Oblomkov and Shende on HOMFLY polynomials of links of plane curve singularities is given a Calabi-Yau threefold interpretation. The motivic Donaldson-Thomas theory developed by M. Kontsevich and the third author then yields natural motivic invariants for algebraic knots. This construction is motivated by previous work of V. Shende, C. Vafa and the first author on the large $N$ duality derivation of the above conjecture.

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1. Introduction

The starting point of this work is a conjecture of Oblomkov and Shende [39] relating the HOMFLY polynomial of the link of a plane curve singularity to topological invariants of its Hilbert scheme of points. It was then explained in [10] that this conjecture has a natural physical interpretation in terms of large $N$ duality for conifold transitions. The conifold transition is a topology changing process from a smooth hypersurface

$$xz - yw = \mu, \quad \mu \neq 0,$$

in $\mathbb{C}^4$ to a small resolution the conifold singularity

$$xz - yw = 0,$$

which is isomorphic to the total space $Y$ of the rank two bundle $\mathcal{O}_Y(-1)^\oplus 2$ on $\mathbb{P}^1$. In this context, the construction of [10] assigns to an algebraic knot $K$ in $S^3$ a lagrangian cycle $M_K$ in $Y$ which intersects a singular plane curve $C^\circ$ contained in a fiber of $Y \to \mathbb{P}^1$ along a circle. Moreover, $C^\circ$ has a unique singular point at the intersection with the zero section, its link being isotopic to $K$. Then large $N$ duality leads to a conjectural relation between HOMFLY polynomials of algebraic knots and Gromov-Witten theory on $Y$ with lagrangian boundary conditions on $M_K$. This conjecture has been tested in [10] by explicit $A$-model computations for torus knots.

The relation between large $N$ duality and the conjecture of Oblomkov and Shende follows from the observation that Gromov-Witten theory is conjecturally equivalent to Donaldson-Thomas theory [29], and also stable pair theory [42]. In the absence of lagrangian boundary conditions, these relations have been proven for toric threefolds in [30], [31]. String duality arguments [11], [25], [28] predict that such a relation holds in the presence of lagrangian cycles as well. For the unknot, this statement is equivalent via [22] to certain Hodge integral identities on the moduli space of curves which have been proven in [10], [26], [27]. Similar physical arguments predict that an analogous correspondence should hold for all lagrangian cycles $M_K$ corresponding to algebraic knot. In particular, this is confirmed for torus knots by the $A$-model computations of [10], which are analogous to those in [22].

The main goal of the present paper is to study the stable pair side of this correspondence. Given a singular plane curve $C^\circ$ in a fiber of the projection $Y \to \mathbb{P}^1$, there is a natural moduli space of $C^\circ$-framed stable pairs on $Y$. These are pairs $\mathcal{O}_Y \to F$ on $Y$ where $F$ is topologically supported on the union of $C^\circ$ with the zero section $C_0 \subset Y$, and has multiplicity one along $C^\circ$. Then the main result is that such moduli spaces are related to the nested Hilbert schemes employed in [39], [38] by a variation of stability condition. For technical reasons, this is proven embedding of the affine curve $C^\circ$ in a suitable compact Calabi-Yau threefold $X$. In particular the embedding will factor through the natural projective completion $C \subset \mathbb{P}^2$ of $C^\circ$. Using previous results on stability conditions for perverse coherent sheaves [17], [40], the nested Hilbert schemes of [39], [38] are then geometrically related to moduli spaces of framed stable objects in a certain stability chamber.

Enumerative invariants for $C$-framed stable pairs are defined by integration of a certain constructible function $\nu$ on the moduli space of $C^\circ$-framed stable pairs. Since the Hilbert scheme invariants used in [39] are topological, one can simply take $\nu = 1$ obtaining the topological Euler numbers of the moduli spaces. Then a wallcrossing formula shows that the resulting invariants are then in agreement with
those of [39]. Alternative constructions may be carried out, using either Behrend constructible functions [2] as in [21] or motivic weight functions as in [23]. Motivated by previous connections between motivic and refined Donaldson-Thomas invariants [11, 23, 33, 35], the second approach will be considered in this paper. Assuming the foundational aspects of [23], it will be shown that the virtual motivic invariants of $C$-framed objects are in agreement with the refined conjecture formulated in [38] if certain technical conditions are met. Removing the technical conditions in question reduces to a comparison conjecture between motivic weights of stable pairs and sheaves (see Section 4.2) which is at the moment open.

Appearance of motivic Donaldson-Thomas invariants supports an old idea of S. Gukov and third author that there should exist a motivic knot invariants theory. In such theory skein relations should correspond to wall-crossing formulas for the motivic Donaldson-Thomas invariants introduced in [23] (and further developed in [24]). Knot invariants themselves should be derived from an appropriate 3-dimensional Calabi-Yau category.

The idea can be traced back to [17], where Khovanov-Rozansky theory was linked to the count of BPS states in topological string theory. It was further developed in [13] in the form of a conjecture about knot superpolynomial. After the work [23] of Kontsevich and third author it became clear that motivic Donaldson-Thomas invariants (DT-invariants for short) introduced in the loc. cit. provide the right mathematical foundation for the notion of (refined) BPS state. This was pointed out in [11] based on physics arguments, rigorous mathematical statements confirming this claim being first formulated and proved in [3]. Further results along these lines have been obtained in [34, 35]. The parameter $y$ which appears in knot invariants should correspond to the motive $L = [\mathbb{A}^1]$ of affine line in the theory of motivic DT-invariants. Then the question is: what is an appropriate 3-dimensional Calabi-Yau category? From the point of view of the large $N$ duality it is natural to expect that the 3-dimensional Calabi-Yau category should be somehow derived from the resolved conifold $Y$. Unfortunately it is difficult to make this idea mathematically precise since $Y$ is non-compact (as well as the lagrangian cycle $M_K$). One can see that the partition function for the unknot derived in [11] coincides with the motivic DT-series for the 3-dimensional Calabi-Yau category generated by one spherical object (both are given essentially by the quantum dilogarithm). But there was no general conjecture about the desired relationship. Although such a conjecture does not exist at present, the works [39] and [38] give a hope that it can be formulated soon. Our paper can be considered as another step in this direction.

A more detailed overview including technical details is presented at length below.

1.1. **The conjectures of Oblomkov, Rasmussen and Shende.** Let $C^o \subset \mathbb{C}^2$ be a reduced pure dimension one curve with one singular point $p \in C^o$. Let $H^o_p(C^o)$ be the punctual Hilbert scheme parameterizing length $n$ zero dimensional subschemes of $C$ with topological support at $p$. Let $m : H^o_p(C^o) \to \mathbb{Z}$ be the constructible function assigning to any subscheme $Z \subset C^o$ the minimal number of generators of the defining ideal $I_{Z,p} \subset O_{C^o,p}$ at $p$. For any scheme $X$ of finite type over $\mathbb{C}$, and any constructible function $\nu : X \to \mathbb{Z}$ let

$$\int_X \nu d\chi = \sum_{n \in \mathbb{Z}} n \chi(\nu^{-1}(n))$$
where $\chi$ denotes the topological Euler character. Then let
\begin{equation}
Z_{C^p}(q, a) = \sum_{n \geq 0} q^{2n} \int_{H^p(C^p)} (1 - a^2)^m d\chi.
\end{equation}

Let $K_{C^p}$ denote the link of the plane curve singularity at $p$ and $\mu$ the Milnor number of the singularity. Let $P_{K_{C^p}}(a, q)$ denote the HOMFLY polynomial of $K_{C^p}$. It satisfies the skein relation of the type:
\[ aP_L - a^{-1}P_{L'} = (q - q^{-1})P_{L''}. \]

As opposed to [39], the HOMFLY polynomial will be normalized such that it takes value
\[ \frac{a - a^{-1}}{q - q^{-1}} \]
for the unknot. Then the conjecture of Oblomkov and Shende [39] states that
\begin{equation}
P_{K_{C^p}}(a, q) = (a/q)^{\mu-1} Z_{C^p}(q, a),
\end{equation}

1.1.1. Refinement. The correspondence between knot polynomial invariants and Hilbert scheme invariants of curve singularities admits a refined generalization [38] due to Oblomkov, Rasmussen and Shende. Given an algebraic knot or link $K$, let $P_{K}^{ref}(q, a, y)$ denote the refined HOMFLY polynomial introduced in [13, 18]. This is the polynomial invariant called reduced superpolynomial in [13], which specializes to the HOMFLY polynomial at $y = -1$. In the previous notation consider the incidence cycle
\[ H^l_p(C^\mu) \subset H^l_p(C^\mu) \times H^{l+r}_p(C^\mu) \]
parameterizing pairs of ideals $(J, I)$ in the local structure ring $O_{C^p}$ satisfying the following condition
\[ m_pJ \subseteq I \subseteq J, \]
where $m_p \subseteq O_{C^p}$ is the maximal ideal of the singular point. Let $H^l_{p}^{[l,r]}(C^\mu)$ be equipped with the reduced induced subscheme structure and
\begin{equation}
Z_{C^p}^{ref}(q, a, y) = \sum_{l, r \geq 0} q^{2l} a^{2r} y^r P_y(H^{l,r}_p(C^\mu)),
\end{equation}
where $P_y$ denotes the virtual Poincaré polynomial (also known as Serre polynomial). Then Oblomkov, Rasmussen and Shende [38] conjecture the following relation
\begin{equation}
P_{K}^{ref}(q, a, y) = \left( \frac{a}{q} \right)^{\mu-1} Z_{C^p}^{ref}(q, a, y),
\end{equation}
where $\mu$ is the Milnor number of the singularity at $p$.

For future reference, note that the Hilbert scheme $H^l_p(C^\mu)$ of $l$ points on $C^\mu$, $l \geq 0$ has a stratification
\[ \cdots \subset H^l_{\geq m}(C^\mu, p) \subset \cdots \subset H^l_{\geq 1}(C^\mu, p) = H^l_p(C^\mu) \]
where $H^l_{\geq m}(C^\mu, p)$, $m \geq 1$ denotes the closed subset parameterizing ideal sheaves $I \subset O_{C^p}$ with at least $m$ generators at $p$. Let $S^l_{m}(C^\mu, p) = H^l_{\geq m}(C^\mu, p) \setminus H^l_{\geq m+1}(C^\mu, p)$ denote the locally closed strata. Then the natural projection morphism $\pi : H^l_{[l,k]}(C^\mu) \to H^l_p(C^\mu)$ is a smooth $Gr(k, m)$-fibration over the locally closed stratum $S^l_{m}(C^\mu, p)$, where $Gr(k, m)$ is the Grassmannian of $k$-dimensional quotients of $C^m$. In particular the fibers of $\pi$ are empty over strata with $m < k$. 


1.2. Framed stable pair invariants of the conifold. The resolved conifold $Y$ is a small crepant resolution of the nodal hypersurface $xz - yv = 0$ in $\mathbb{C}^4$. It can be easily identified with the total space of the rank two bundle $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ such that the exceptional cycle $C_0 \cong \mathbb{P}^1$ of the resolution is the zero section.

There is closed embedding $C^0 \hookrightarrow Y$ which factors through the natural embedding of $C$ in a fiber of the projection $Y \rightarrow \mathbb{P}^1$. Therefore the curve $C$ in the conjecture of Oblomkov and Shende is naturally identified with a vertical complete intersection $C$ such that the exceptional cycle $C_0$ is topologically supported on the union $C^0 \cup C_0$.

Note that the second condition is equivalent to the requirement that the scheme theoretic support $Z_F$ of $F$ have at most two irreducible components, $C^0$ and an additional component supported on $C_0$, which may be empty. The numerical invariants of a $C^0$-framed stable pair on $Y$ will be the generic multiplicity $r$ of $F$ along the zero section, and $l = \chi(\text{Coker}(s))$.

Let $\overline{Y} = \mathbb{P}(\mathcal{O}_Y(-1) \oplus \mathcal{O}_Y)$ be a projective completion of $Y$, and $C \subset \overline{Y}$ the resulting projective completion of $C^0$. Projective plane curve $C$ is contained in a fiber of the projection $\overline{Y} \rightarrow \mathbb{P}^1$. According to [22], there exists a fine projective moduli space $\mathcal{P}(\overline{Y}, r, n)$ of stable pairs $(G, v)$ on $\overline{Y}$, where $\text{ch}_2(G) = [C] + r[C_0]$, and $\chi(G) = n$. Then it can be easily proved that there exists a fine quasi-projective moduli space $\mathcal{P}(Y, C^0, r, l)$ of $C^0$-framed stable pairs on $Y$ with $l = n - \chi(\mathcal{O}_C)$. Moreover, $\mathcal{P}(Y, C^0, r, l)$ is the locally closed subscheme of $\mathcal{P}(\overline{Y}, r, n)$ determined by the conditions

- $\text{Ann}(G) \subset \mathcal{I}_C$
- The support of $\text{Coker}(v)$ is contained in the open part $Y \subset \overline{Y}$.

Let $\mathcal{P}^0(\overline{Y}, r, n)$ denote the open subspace of $\mathcal{P}(\overline{Y}, r, n)$ parameterizing pairs satisfying only the second condition above. As observed above, counting invariants $P^\nu(Y, C^0, r, l)$ can be defined by integration of a numerical constructible function or a motivic weight function as in [23]. Suppose $\nu$ is either an $\mathbb{Z}$-valued constructible function on $\mathcal{P}^0(\overline{Y}, r, n)$ or a motivic weight constructible function with values in a ring of motives. Let

$$Z^\nu(Y, C^0, u, T) = \sum_{l \in \mathbb{Z}_{\geq 0}} \sum_{r \geq 0} P^\nu(Y, C^0, r, l) u^r T^r.$$ 

Let also $P^\nu(Y, r, n)$ denote the counting invariants for stable pairs $(F, s)$ on $Y$ with $\text{ch}_2(Y) = r[C_0]$ and $n = \chi(F)$, and

$$Z^\nu(Y, u, T) := \sum_{n \in \mathbb{Z}} \sum_{r \geq 0} P^\nu(Y, r, n) u^n T^r.$$ 

Then large N duality [10] leads to a conjectural factorization formula

$$Z^\nu(Y, C^0, u, T) = Z^\nu(Y, u, T) Z^\nu(C^0, u, T).$$  

(1.5)
where $Z^v(C^\circ, u, T)$ should be closely related to the generating function $Z_{CS,p}(q,a)$ in equation (1.2) or its refined counterpart (1.3). This is a large $N$ dual reflection of the natural factorization of Wilson loop expectation values in large $N$ Chern-Simons theory,

$$\langle W_K(U) \rangle_{CS,N \to \infty} = P_K(q,a)Z_{CS}(q,a).$$

It will be explained next that identities of the form (1.5), follow naturally from a wallcrossing formulas analogously to [47, 46], once the above construction is embedded in a compact Calabi-Yau threefold. In particular Theorem 1.1 below proves an identity analogous to (1.6) in a compact model containing the projective completion $C \subset \mathbb{P}^2$ of $C^\circ$.

1.3. Embedding in a compact Calabi-Yau threefold. Theory of stable pairs of Pandharipande and Thomas deals with compact varieties. Since the resolved conifold $Y$ is non-compact we need to formulate the problem in an appropriate compactification. We start with some generalities. Let $X_0$ be a projective Calabi-Yau threefold with a single conifold singularity $q \in X_0$ lying on a Weil divisor $\Delta \simeq \mathbb{P}^2 \subset X_0$. Suppose that there exist two smooth crepant resolutions $X^\pm \to X_0$ related by a flop, in each case the exceptional locus being a rational $(-1, -1)$ curve $C^\pm_0 \subset X^\pm$. Let also $D^\pm \subset X^\pm$ denote the strict transforms of the Weil divisor $\Delta$. A local computation shows that in one case, say $X^+$, $D^+ \simeq \Delta$ intersects $C^+_0$ transversely at a point $p$. In the second case, $D^-$ is isomorphic to the one point blow-up of $\Delta$, the Hirzebruch surface $F_1 = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1))$, and $C^-_0 \subset D^-$ is the exceptional curve of the blow-up. It will be further assumed that the exceptional curves $C^\pm_0$ are complete intersections in $X^\pm$.

Although the considerations below are not particular to a specific model, an example will be provided next for concreteness. Let $X^-$ be a smooth elliptic fibration with a section over the Hirzebruch surface $F_1$. Let $D^- \subset X^-$ denote the image of the canonical section, and $C^-_0 \subset D^-$ the unique $(-1)$ curve on $D^-$. As shown in [36] using toric methods, there exists a morphism $X^- \to X_0$ contracting the curve $C^-_0$, where $X_0$ is a nodal Calabi-Yau threefold. Moreover there is a second smooth crepant resolution of $X^+ \to X_0$, which is a fibration with a section over $\mathbb{P}^2$. The exceptional locus is in this case a rational $(-1, -1)$ curve intersecting the image $D^+$ of the base transversely at a point $p$. More examples with a two or four conifold singularities where $D^\pm$ are toric surfaces have been studied in the context of large $N$ duality in [9].

In this context, let $T \subset X_0$ be a reduced irreducible plane curve contained in the Weil divisor $\mathbb{P}^2$ passing through the conifold point $q$. Suppose $T$ has a singularity at $q$ and is otherwise smooth. Let $C^\pm \subset X^\pm$ be the strict transforms of $T$ in $X^\pm$ respectively. Note that $C^+$ is a plane curve isomorphic to $\Gamma$ in $D^+ \simeq \mathbb{P}^2 \subset X^+$ while $C^- \subset D^-$ is an embedded blow-up of $T$ at $q$. Moreover $C^+$ intersects the exceptional curve $C^+_0 \subset X^+$ at the point $p$, which is the only singular point of $C^+$ under the current assumptions. The other strict transform $C^-$ also intersects the exceptional curve $C^-_0 \subset D^-$, but the intersection may consists of several closed points $p_1, \ldots, p_k$ and is not in general transverse. In the following the labels $\pm$ will be omitted whenever the discussion is valid equally well for both $X^+$ and $X^-$. By analogy with Section 1.2 a stable pair $(F, s)$ on $X$ will be called $C$-framed of type $(r, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$

- $F$ is topologically supported on the union $C \cup C_0$
Then there is a closed subscheme \( P(X, C, r, n) \subset P(X, \beta, n) \), with \( \beta = [C] + r[C_0] \) parameterizing \( C \)-framed stable pairs.

Enumerative invariants are defined as explained above equation (1.5) by integration with respect to an appropriate constructible function \( \nu \) on the ambient space \( P(X, \beta, n) \). For \( \nu = 1 \), the resulting invariants are topological Euler numbers of the moduli spaces \( P(X, C, r, n) \) and they will be denoted by \( P^{top}(X, C, r, n) \). Then the following holds.

**Theorem 1.1.**

\[
Z^{top}(X^+, C^+, q^2, -a^2) = Z^{top}(q^2, -a^2)(1 - q^2)^{\chi(C^+)} Z_{C^+}(q, a).
\]

The proof of Theorem 1.1 is based on wallcrossing for framed stable pair invariants, as explained in the next subsection.

Virtual motivic invariants are constructed by taking \( \nu \) to be the motivic weight function \[29\] Sect. 6.2 on the ambient space \( P(X, \beta, n) \). Refined invariants are then obtained taking the virtual Poincaré polynomial \( P_y \) of the virtual motivic invariants. String theory predicts that there should be an analogous formula relating the generating function function \( Z_{ref}^+(X, C, u, T, y) \) to the series \[1.3\] of refined Hilbert scheme invariants. It will be explained below that such a relation also follows from refined wallcrossing formulas. By analogy with \[29\], one can also define numerical invariants by taking the semiclassical limit \( y \rightarrow -1 \) of the refined invariants.

### 1.4. \( C \)-framed perverse coherent sheaves and stability.

Let \( D^b(X) \) be the bounded derived category of \( X \). Let \( A \subset D^b(X) \) be the heart of the perverse t-structure on \( D^b(X) \) determined by the torsion pair \((\text{Coh}_{\geq 2}(X), \text{Coh}_{\leq 1}(X))\). The objects of \( A \) are objects \( E \) of \( D^b(Y) \) such that the cohomology sheaves \( \mathcal{H}^i(E) \) are nontrivial only for \( i = -1, 0 \), \( \mathcal{H}^{-1}(E) \) has no torsion in codimension \( \geq 2 \), and \( \mathcal{H}^0(E) \) is torsion, of dimension \( \leq 1 \). Let \( \omega \) be a fixed Kähler class on \( X \).

The stable pair theory of \( X \) has been studied in \[17, 40\] employing a construction of limit (or weak) stability conditions on \( A \) which we review in Section 2.1. The main motivation for the study of limit stability conditions in the loc.cit. was to prove the rationality conjecture of Pandharipande and Thomas \[12\]. The main tool in the proof is the wallcrossing formalism of \[29, 91\] applied to a one-parameter family of stability conditions on \( A \) parameterized by a \( B \)-field, \( B = b\omega \in H^2(X) \). In fact, as was pointed out in \[29\], the wall-crossing formulas for the weak stability conditions is a special case of those considered in the loc.cit. as soon as one allows the central charge to take values in an ordered field. Weak stability conditions are easy to construct for the derived category of coherent sheaves \( D^b(X) \) on a Calabi-Yau manifold \( X \), differently from conventional Bridgeland stability conditions. More specifically, there is a slope function \( \mu_{(\omega, b)} \) on the Grothendieck group \( K_0(A) \) which defines a family of weak stability conditions on \( A \), as reviewed in Section 2.1. Moreover, the following results are proven in \[40\].

1. For fixed \((\beta, n)\) there is an algebraic moduli stack of finite type \( \mathcal{M}^{ss}_{(\beta, n)}(A, \beta, n) \) of \( \mu_{(\omega, b)} \)-semistable objects of \( A \) with \( \text{ch}(E) = (-1, 0, \beta, n) \).

2. For fixed \((\beta, n)\) there are finitely many critical parameters \( b_c \) such that strictly \( \mu_{(\omega, b)} \)-semistable objects exist. The moduli stacks \( \mathcal{M}^{ss}_{(\omega, b)}(A, \beta, n) \), \( \mathcal{M}^{ss}_{(\omega, b)}(A, \beta, n) \) are canonically isomorphic if there is no critical stability parameter in the interval \([b', b'']\). Moreover, if \( b \) is not critical all closed points of
\( M_{(\omega, b)}^* (A, \beta, n) \) are \( \mu_{(\omega, b)} \)-stable and their stabilizers are canonically isomorphic to \( \mathbb{C}^* \).

3. For fixed \( \omega, (\beta, n) \), there exists \( b_{-\infty} \) such that for any \( b < b_{-\infty} \) the moduli stack \( M_{(\omega, b)}^{ss} (A, \beta, n) \) is an \( \mathbb{C}^* \)-gerbe over the moduli space \( P(X, \beta, n) \) of stable pairs on \( X \).

A similar construction will be employed in the proof of Theorem 1.1. A full subcategory \( \mathcal{A}^C \) of \( \mathcal{A} \) consisting of \( C \)-framed perverse coherent sheaves \( \mathcal{A}^C \) is defined by conditions (C.1), (C.2) in Section 2.2. Then it is shown that the slope construction of weak stability conditions \( 46 \) and basic properties of slope limit semistable objects carry over to the \( C \)-framed category. In particular, one can construct a one parameter family of weak stability conditions parameterized by the \( B \)-field \( B = b \omega \in H^2 (X) \).

The moduli stacks of \( \mu_{(\omega, b)} \)-semistable objects \( E \) in \( \mathcal{A}^C \) with numerical invariants \( \text{ch}(E) = (-1, 0, [C] + r[C_0], n), r \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z} \), will be denoted by \( \mathcal{P}_{(\omega, b)} (X, C, r, n) \). Their properties are completely analogous (1) - (3) above. In particular they are algebraic stacks of finite type, and for fixed \( \omega \) and numerical invariants \( (r, n) \) strictly semistable objects exist only for finitely many critical values of \( b \). Moreover, there exists \( b_{-\infty} \in \mathbb{R}_{>0} \) such that for \( b < b_{-\infty} \) \( \mathcal{P}_{(\omega, b)} (X, C, r, n) \) is a \( \mathbb{C}^* \)-gerbe over the moduli space of \( C \)-framed stable pairs.

Let \( \text{Ob}(\mathcal{A}) \) be the stack of all objects of \( \mathcal{A} \), which is an algebraic stack locally finite type over \( \mathbb{C} \). For all \( b \in \mathbb{R} \) and all \( (r, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \) the natural forgetful morphism

\[
\mathcal{P}_{(\omega, b)} (X, C, r, n) \rightarrow \text{Ob}(\mathcal{A})
\]

determine a stack function in the motivic Hall algebra \( H(\mathcal{A}) \).

Counting invariants \( P_{(\omega, b)} (X, C, r, n) \) are again defined by integration with respect to a suitable constructible function \( \nu \) on the stack of all objects \( \text{Ob}(\mathcal{A}) \). Let

\[
(1.7) \quad Z_{(\omega, b)} (X, C; u, T) = \sum_{n \in \mathbb{Z}} \sum_{r \geq 0} P_{(\omega, b)} (X, C, r, n) u^n T^r
\]

denote the resulting generating series. When \( \nu \) is a motivic weight function, the invariants \( P_{(\omega, b)} (X, C, r, n) \) take values in a ring of motives, and refined invariants \( P_{(\omega, b)}^{ref} (X, C, r, n; y) \) are obtained by taking virtual Poincaré polynomials. For future reference note that counting invariants for objects \( E \) of \( \mathcal{A}^C \) with \( \text{ch}(E) = (0, 0, r[C_0], n) \) are defined analogously, and coincide with the counting invariants of the conifold 45, 57, 21 Ex 6.2, 34. Their generating function will be denoted by \( Z_{(\omega, b)} (X, C_0, u, T) \). Let us fix \( \omega \) and \( (r, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \). The stability parameter \( b > 0 \) will be called small if there are no critical stability parameters of type \( (r, n) \) in the interval \((0, b]\). The corresponding invariants will be denoted by \( P_{0+} (X, C, r, n) \), and their generating function, \( Z_{0+} (X, C, u, T) \). Moreover, for \( b << 0 \), the corresponding invariants \( P_{-\infty} (X, C, r, n) \) specialize to stable pair invariants. In the following the function \( \nu \) will be either the constant function \( \nu = 1 \) or the motivic weight function defined in 23 Sect. 6.2.

1.5. Factorization via wallcrossing. The first step in the proof of Theorem 1.1 is the derivation of a wallcrossing formula relating \( b << 0 \) invariants to small \( b > 0 \) invariants. Given the close analogy between \( \mu_{(\omega, b)} \)-semistable objects in \( \mathcal{A}^C \), and \( \mathcal{A} \) respectively, the wallcrossing computation for Euler characteristic invariants is identical to those carried out in 40 Sect 4.3 or the proof of 37 Thm 3.15.
For refined motivic invariants, one can employ similarly the refined wall-crossing formulas of [23] (see also [12, 5, 14] or the polynomial version conjectured in [7]). Alternatively, the wall-crossing formula of [23] can be employed in the semiclassical limit $y \to -1$ by analogy with [6, Sect. 4]. The details will not be repeated here because these computations have been carried out in minute detail in loc. cit. The final answer is a factorization formula of the form

$$Z_{\nu}^{-\infty}(X, C, u, T) = Z_{\nu}^0(X, C_0, u, T)Z_{0+}^\nu(X, C, u, T).$$

This is in agreement with the natural factorization of Wilson loop expectation values in Chern-Simons theory, as we explained below equation (1.5).

In conclusion, in order to find a connection between stable pair theory and the Hilbert scheme invariants of [38, 39] one has to analyze the moduli spaces of stable $C$-framed objects for small $b > 0$. We will explain that below.

1.6. Small $b$ chamber. Now take $X = X^+$ such that $C_0^+$ intersects $D^+$ transversely at $p$, and $p$ is a singular point of the plane curve $C_0^+ \subset D^+$. The upper label $+$ will be omitted keeping in mind that this is the only case considered in this section. Then the following result for topological Euler character invariants follows from Propositions 3.5, 3.10.

**Theorem 1.2.** There is an identity

$$Z^{0+}_{\nu}((C, X; q^2, -a^2)) = (1 - q^2)^{1 - \chi(C)} q^{2\chi(\mathcal{O}_C)} Z_{C,p}(q, a)$$

where $Z_{C,p}(q, a)$ is the series (1.1).

The proof of Theorem 1.2 relies on the construction in Section 3.2 of a moduli stack $\mathcal{Q}(X, C, r, n)$ of decorated sheaves on $X$ interpolating between the nested Hilbert schemes $H^{[l,r]}(C)$, $l = n - \chi(\mathcal{O}_C)$ and the moduli stacks $\mathcal{P}_{0+}(X, C, r, n)$. More precisely, Proposition 3.5 proves that $\mathcal{Q}(X, C, r, n)$ is a $\mathbb{C}^\times$ gerbe over a relative Quot scheme $Q^{[r,n]}(C)$ which is closely related to $H^{[r,n]}(C)$. At the same time Proposition 3.10 shows that $\mathcal{Q}(X, C, r, n)$ is equipped with a natural geometric bijection $f : \mathcal{Q}(X, C, r, n) \to \mathcal{P}_{0+}(X, C, r, n)$. This implies a relation of the form

$$[\mathcal{P}_{0+}(X, C, r, n)] = (L - 1)^{-1}[H^{[r,n]}(C)]$$

in the Grothendieck ring of algebraic stacks over $\mathbb{C}$. Then the proof of Theorem 1.2 reduces to a straightforward computation using the stratification of nested Hilbert schemes which we explained below equation (1.4).

The motivic Donaldson-Thomas theory of $C$-framed stable objects at small $b > 0$ is analyzed in Section 4. Agreement with (1.3) follows if the virtual motive of the moduli stack $\mathcal{P}_{0+}(X, C, r, n)$ is related to the Chow motive by the formula

$$[\mathcal{P}_{0+}(X, C, r, n)]^{vir} = L^{(e^2 - k^2 - n + 1)/2}[\mathcal{P}_{0+}(X, C, r, n)],$$

where $k$ is the degree of the curve $C$ in $\mathbb{P}^2$. This formula is proven in Section 4 for sufficiently high degree $n >> 0$, assuming the foundational aspects of the motivic Donaldson-Thomas theory of [23], as well as a specific choice of orientation. For arbitrary values of $n \in \mathbb{Z}$, the equation (1.10) reduces to a relation (1.3) between motivic weights of moduli stacks of pairs and sheaves for irreducible curve classes. This is a virtual motivic counterpart to [43, Thm. 4]. Motivated by this analogy, it is natural to conjecture that this relation equation holds for all $n \in \mathbb{Z}$ with a suitable choice of orientation data. Granting equation (1.10), it is checked in
Section 4.5 that the generating series of motivic invariants is related to the refined Hilbert series by a simple change of variables.

1.7. **Outlook and future directions.** This section records potential generalizations and extensions of the conjecture of Oblomkov and Shende motivated by the string theory construction of [10]. These are just possible future directions of study, not established mathematical results, or, in some cases, not even precise conjectures. Nevertheless, they are recorded here for the interested reader in the hope that they will lead to interesting developments at some point in the future.

1.7.1. **BPS states and nested Jacobians.** As observed in Remarks 3.4, 3.7, a second moduli space \( M_{\lfloor,\rfloor}(\mathbb{C}) \) naturally enters the picture, which can be identified with a moduli space of nested Jacobians. The closed points of \( M_{\lfloor,\rfloor}(\mathbb{C}) \) are pairs \((J, \psi)\) where \( J \) is a rank one torsion free sheaf on \( \mathbb{C} \) of degree \(-l\), and \( \psi : J \to \mathcal{O}_p^{\oplus r} \) a surjective morphism. According to Lemma 3.8 and Remark 3.9, allowing the curve \( C \) to vary in the linear system \( |kH| \) on \( D \) results in a smooth moduli space \( \mathcal{N}(D, k, r, n) \). Moreover, this moduli space is equipped with a natural determinant map

\[
h : M(D, k, r, n) \to |kH|
\]

to the linear system and \( M_{\lfloor,\rfloor}(\mathbb{C}) \) is the fiber of \( h \) at the point corresponding to \( C \).

Then physics arguments [16, 15] predict that the cohomology of \( M_{\lfloor,\rfloor}(\mathbb{C}) \) should admit a perverse sheaf decomposition

\[
H(M_{\lfloor,\rfloor}(\mathbb{C})) \simeq \oplus_p \text{Gr}^p H(M_{\lfloor,\rfloor}(\mathbb{C}))
\]
determined by an \( h \)-relative ample class. Moreover the dimensions of the perverse graded pieces, \( N_p = \dim(\text{Gr}^p H(M_{\lfloor,\rfloor}(\mathbb{C}))) \), should be independent of the polarization and \( n \), and the \( C \)-framed small \( b \) generating function \( Z_{0+}(X, C; u, T) \) should admit a Gopakumar-Vafa expansion

\[
Z_{0+}(X, C; u, T) = \sum_{r \geq 1} \sum_p N_p T^r u^p \left(1 - u\right)^2.
\]

Note that the \( r = 0 \) version of these conjectures is a rigorous mathematical result by work of [44, 32, 33]. The construction sketched above provides a possible generalization for \( r \geq 1 \) which deserves further study.

1.7.2. **A conjecture for colored HOMFLY polynomials.** Theorem 1.1 and the conjecture of Oblomkov and Shende imply that \( C^o \)-framed stable pairs on the conifold are related to the HOMFLY polynomial of the link of the singular point \( p \in C^o \). Large \( N \) duality arguments [10] lead to the following generalization.

Let \((x, y, z)\) be the affine local coordinates on \( Y \) such that the projection \( Y \to \mathbb{P}^1 \) is locally given by \((x, y, z) \to z \) and \( C^o \) is contained in the fiber \( z = 0 \). Hence \( C^o \) is a complete intersection of the form

\[
z = 0, \quad f(x, y) = 0
\]

where \( f \) is a degree \( k \geq 1 \) irreducible polynomial of two variables.

Let \( \mu \) be a Young diagram consisting of \( m_i \) columns of height \( h_i \in \mathbb{Z}_{\geq 1} \), where \( 1 \leq i \leq s \) and \( h_1 > h_2 > \cdots > h_s \).

Let \( C^o_\mu \) be the complete intersection on \( Y \) determined by the equations

\[
z^{h_i} f(x, y)^{m_1 + \cdots + m_{i-1}} = 0, \quad 1 \leq i \leq s,
\]

(1.11)
where by convention \( m_0 = 0 \). Note that \( C^0_\mu \) is a nonreduced irreducible subscheme of \( Y \) of pure dimension one.

In complete analogy with Sections 12 one can define \( C^0_\mu \)-framed stable pair invariants of \( Y \) employing the framing condition \( \text{Ann}(F) \subset \mathcal{I}_{C^0_\mu} \). Let \( P(Y, C^0_\mu; r, n) \) denote the counting invariants obtained by taking the quasiclassical limit of motivic Donaldson-Thomas invariants of the ambient space \( \mathcal{P}^{\text{stirc}}(Y, \beta, n) \), where \( \beta = [C_\mu] + r[C_0] \). Based on large \( N \) duality, the generating function

\[
Z(Y, C^0_\mu, q^2, a^2) = \sum_{n \in \mathbb{Z}} \sum_{r \geq 0} q^{2n} a^{2r} P(Y, C^0_\mu, r, n).
\]

is expected to be related to the \( \mu \)-colored HOMFLY polynomial \( P_{K, \mu}(q, a) \). More specifically, a relation of the form

\[
P_{K, \mu}(q, a) = a^\alpha q^\beta Z^B(Y, C^0_\mu, q^2, a^2) Z^B(Y, q^2, a^2).
\]

is expected to hold, for certain integral exponents \( \alpha, \beta \), possibly depending on \( \mu \).

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2. Framed stable pairs in the derived category

2.1. Review of slope limit stability. This section is a brief review of limit slope stability conditions on the derived category of a smooth projective Calabi-Yau threefold following [1, 17, 40].

Let \( D^b(X) \) be the bounded derived category of \( X \). Let \( \mathcal{A} \) be the heart of the \( t \)-structure determined by the torsion pair \((\text{Coh}_{\geq 2}(X), \text{Coh}_{\leq 1}(X))\). The objects of \( \mathcal{A} \) are objects \( E \) of \( D^b(Y) \) such that the cohomology sheaves \( H^i(E) \) are nontrivial only for \( i = -1, 0 \), \( H^{-1}(E) \) has no torsion in codimension \( \geq 2 \), and \( H^0(E) \) is torsion, of dimension \( \leq 1 \).

Let \( \omega \) be a Kähler class on \( X \) and \( B \in H^2(X) \), a real cohomology class i.e. a \( B \)-field. Let \( Z_{(\omega, B)} : K(X) \to \mathbb{C} \) be the central charge function

\[
Z_{(\omega, B)}(E) = -\int_X \text{ch}(E) e^{-(B+i\omega)} \sqrt{\text{Td}(X)}
\]

(more precisely, instead of the Todd class one should use the \( \gamma \)-class). For any \( m \in \mathbb{R}_{>0} \) let

\[
Z^f_{(\omega, B)}(E) = (\text{Re} Z_{(m\omega, B)}(E)) + i(\text{Im} Z_{(m\omega, B)}(E))
\]

where \( f^\dagger(m) \) denotes the leading monomial of a polynomial \( f(m) \). Then for \( m \gg 0 \) the following

\[
\mu_{(\omega, B)}(E) = -\frac{(\text{Re} e^{-i\pi/4} Z_{(m\omega, B)}(E))}{(\text{Im} e^{-i\pi/4} Z_{(m\omega, B)}(E))}
\]

is a well-defined map to the field of rational functions \( \mathbb{R}(m) \).
An object $E$ of $\mathcal{A}$ is said to be $\mu_{(\omega,B)}$-(semi)stable if any proper nonzero subobject $0 \subset F \subset E$ in $\mathcal{A}$ satisfies
\[
\mu_{(\omega,B)}(F) \leq \mu_{(\omega,B)}(E).
\]
Here rational functions $f, g \in \mathbb{R}(m)$ are ordered by
\[
f \geq g \iff f(m) \geq g(m) \quad \forall m >> 0.
\]
According to [40] the above slope stability gives rise to a weak stability condition on $\mathcal{A}$.

In order to study the properties of semistable objects of $\mathcal{A}$, it is helpful to consider the following full subcategories $(\mathcal{A}_1, \mathcal{A}_{1/2})$ of $\mathcal{A}$ (see [17,46]). The category $\mathcal{A}_1 \subset \mathcal{A}$ consists of objects $E$ such that $\mathcal{H}^{-1}(E)$ is torsion and $\mathcal{H}^0(E)$ is zero dimensional. By definition $\mathcal{A}_{1/2}$ is the subcategory of $\mathcal{A}$ consisting of objects $E$ such that $\text{Hom}_{\mathcal{A}}(A_1, E) = 0$ (i.e. it is right orthogonal to $\mathcal{A}_1$). Note that $\mathcal{H}^{-1}(E)$ is torsion-free for all objects $E$ of $\mathcal{A}_{1/2}$, and also $\text{Hom}(T, E) = 0$ for any zero-dimensional sheaf $T$. According to [17, Lemm. 2.16] the subcategories $(\mathcal{A}_1, \mathcal{A}_{1/2})$ define a torsion pair in $\mathcal{A}$. A morphism $E \to F$ of objects in $\mathcal{A}_i$, $i = 1, 1/2$ will be called a strict monomorphism/epimorphism if it is injective/surjective as a morphism in $\mathcal{A}$, and its cokernel/kernel belongs to $\mathcal{A}_i$.

In the following we consider objects $E$ of $\mathcal{A}$ with $\text{ch}_0(E) = -1$ and $\text{ch}_1(E) = 0$. The first observation following from [40, Lemm. 3.8] is that if such an object is $\mu_{(\omega,B)}$-semistable, then it must belong to $\mathcal{A}_{1/2}$. Moreover the following stability criterion holds [40, Prop. 3.13].

**Proposition 2.1.** An object $E$ of $\mathcal{A}_{1/2}$ with $\text{ch}_0(E) = -1$ and $\text{ch}_1(E) = 0$ is $\mu_{(\omega,B)}$-stable if and only if the following hold.

(i) For any strict epimorphism $E \twoheadrightarrow G$ in $\mathcal{A}_{1/2}$, with $G$ a pure dimension one sheaf on $X$
\[
\mu_{(\omega,B)}(G) > -\frac{3B\omega^2}{\omega^3}.
\]

(ii) For any strict monomorphism $F \hookrightarrow E$ in $\mathcal{A}_{1/2}$, with $F$ a pure dimension one sheaf on $X$,
\[
\mu_{(\omega,B)}(F) < \frac{3B\omega^2}{\omega^3}.
\]

Next let $\beta \in H_2(X)$ and $n \in \mathbb{Z}$. Suppose $B = b\omega, b \in \mathbb{R}$. Then the following results are proven in [40] for fixed $\omega$, $(\beta, n)$.

1. For any $b \in \mathbb{R}$, there is an algebraic moduli stack of finite type $\mathcal{M}^{ss}_{(\omega,B)}(\mathcal{A}, \beta, n)$ of $\mu_{(\omega,B)}$-semistable objects of $\mathcal{A}$ with $\text{ch}(E) = (-1, 0, \beta, n)$.

2. There are finitely many critical parameters $b_c$ such that strictly $(\omega, B_c)$-semistable objects exist. The moduli stacks $\mathcal{M}^{ss}_{(\omega,B')}((\mathcal{A}, \beta, n)$, $\mathcal{M}^{ss}_{(\omega,B'')}((\mathcal{A}, \beta, n)$ are isomorphic if there is no critical stability parameter in the interval $[b', b'']$. Moreover, if $b$ is not critical all closed points of $\mathcal{M}^{ss}_{(\omega,B)}((\mathcal{A}, \beta, n)$ are $\mu_{(\omega,B)}$-stable and have $\mathbb{C}^\times$ stabilizers.

3. There exists $b_{-\infty}$ such that for any $b < b_{-\infty}$ the moduli stack $\mathcal{M}^{ss}_{(\omega,B)}((\mathcal{A}, \beta, n)$ is a $\mathbb{C}^\times$-gerbe over the the moduli space $\mathcal{P}(Y, \beta, n)$ of stable pairs on $Y$ constructed in [12].

4. One can define counting invariants and wallcrossing formulas using either the formalism of Joyce and Song or the one of Kontsevich and Soibelman. In particular
there is a Hall algebra of motivic stack functions associated to the abelian category $\mathcal{A}$. The corresponding wallcrossing formulas are in agreement with those of Kontsevich and Soibelman [23].

### 2.2. A C-framed subcategory

In this section $X$ will be one of the two small resolutions of a nodal Calabi-Yau threefold $X_0$, as explained in Section 1.4. In particular, it will be assumed that all conditions listed there are satisfied. Therefore there is only one conifold point lying on a Weil divisor $\Delta \simeq \mathbb{P}^2 \subset X_0$. The exceptional locus consists of a rational $(-1, -1)$ curve $C_0 \subset X$, which is furthermore a complete intersection on $X$. Let $B = b\omega$, $b \in \mathbb{R}$, where $\omega$ is a fixed Kähler class on $X$ as above. Without loss of generality, it will be assumed from now on that $\omega$ is normalized such that $\int_{C_0} \omega = 1$.

Let $C \subset X$ be a framing plane curve contained in the strict transform $D \subset X$ of the Weil divisor. All conditions specified in Section 1.4 are assumed to hold. In particular $C$ will meet the exceptional curve $C_0 \subset X$. If $X = X^+$ the scheme theoretic intersection is the closed point $p$ where $C_0$ intersects $D$ transversely. If $X = X^-$, the set theoretic intersection is the finite set of closed points, while the scheme theoretic intersection will be in general more complicated. For simplicity the labels $\pm$ will be omitted in the remaining part of this subsection since both cases can be treated uniformly.

Consider the full subcategory $\mathcal{A}^C$ of $\mathcal{A}$ consisting of objects $E$ satisfying the conditions

(C.1) $\mathcal{H}^{-1}(E)$ is a subsheaf of the defining ideal $\mathcal{I}_C$. In particular, if $\mathcal{H}^{-1}(E)$ is not trivial, it must be the ideal sheaf of a proper closed subscheme $Z_E \subset X$.

(C.2) The structure sheaf $\mathcal{O}_{Z_E}$ and the cohomology sheaf $\mathcal{H}^0(E)$ are topologically supported on the union $C \cup C_0$. Moreover, the quotient $\mathcal{H}^0(E)/Q$ is topologically supported to $C_0$, where $Q \subset \mathcal{H}^0(E)$ is the maximal dimension zero subsheaf.

**Lemma 2.2.** Consider an exact sequence

\[(2.1)\quad 0 \to F \to E \to G \to 0\]

in $\mathcal{A}$ where $\text{ch}_0(E) \in \{0, -1\}$. Then the following statements hold

(i) If $F, G$ belong to $\mathcal{A}^C$ and then $E$ belongs to $\mathcal{A}^C$.

(ii) If $F, E$ belong to $\mathcal{A}^C$ then $G$ belongs to $\mathcal{A}^C$.

(iii) If $E, G$ belong to $\mathcal{A}^C$ then $F$ belongs to $\mathcal{A}^C$.

**Proof.** The above statements are obvious if $\text{ch}_0(E) = 0$ since then $\text{ch}_0(F) = \text{ch}_0(G) = 0$ and $\text{ch}_0(E) = -1$, which implies $\text{ch}_0(F) = 0$, $\text{ch}_0(G) = -1$ or $\text{ch}_0(F) = -1$, $\text{ch}_0(G) = 0$. Then all the above statements follow easily from the long exact sequence

\[(2.2)\quad 0 \to \mathcal{H}^{-1}(F) \to \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(G) \to \mathcal{H}^0(F) \to \mathcal{H}^0(E) \to \mathcal{H}^0(G) \to 0\]

except case (ii), $\text{ch}_0(G) = -1$, $\text{ch}_0(F) = 0$, which requires more work. In this case, $\mathcal{H}^{-1}(F) = 0$ and $\mathcal{H}^{-1}(G)$ is a rank one sheaf on $X$ which admits torsion at most in codimension one. Let $T \subset \mathcal{H}^{-1}(G)$ be the maximal torsion submodule and let $I$ denote the image of $\mathcal{H}^{-1}(G)$ in $\mathcal{H}^0(F)$. By assumption, $I$ is a torsion sheaf with topological support on $C \cup C_0$. Then the kernel of the induced morphism $T \to I$ must be a torsion subsheaf of $\mathcal{H}^{-1}(E)$, which leads to a contradiction unless $T = 0$. 


Therefore \( \mathcal{H}^{-1}(G) \) must be a rank one torsion free sheaf with trivial determinant i.e. it must be isomorphic to the ideal sheaf of a closed subscheme \( Z_G \) on \( X \).

Moreover, there is an inclusion \( \mathcal{H}^{-1}(E) \hookrightarrow \mathcal{H}^{-1}(G) \) which implies that \( Z_G \) is a closed subscheme of \( Z_E \) and a simple application of the snake lemma yields an isomorphism

\[
K = \mathcal{H}^{-1}(G) / \mathcal{H}^{-1}(E) \cong \ker(\mathcal{O}_{Z_E} \to \mathcal{O}_{Z_G})
\]

Since \( K \subset \mathcal{H}^0(F) \) and both \( \mathcal{O}_{Z_E} \) and \( \mathcal{H}^0(F) \) are topologically supported on \( C \cup C_0 \), it follows that \( \mathcal{O}_{Z_G} \) satisfies the same condition. Moreover, in the exact sequence \( \mathcal{H}^0(F), \mathcal{H}^0(E) \) satisfy condition (C.2), which implies that \( \mathcal{H}^0(G) \) also satisfies (C.2). Finally note that \( K \subset \mathcal{H}^0(F) \) is topologically supported on a union of \( C_0 \) and a finite set of closed points lying on \( C \). Therefore \( \text{Hom}_X(K, \mathcal{O}_C) = 0 \) since \( \mathcal{O}_C \) is pure dimension one by assumption. This implies that the canonical projection \( \mathcal{O}_{Z_E} \to \mathcal{O}_C \) factors through \( \mathcal{O}_{Z_E} \to \mathcal{O}_{Z_G} \). Hence \( \mathcal{H}^{-1}(G) \) is a subsheaf of \( \mathcal{I}_C \).

\[
\square
\]

Limit slope stability for objects of \( \mathcal{A}^C \) will be defined by analogy with [47, 40]. An object \( E \) of \( \mathcal{A}^C \) will be (semi)stable if

\[
\mu_{(\omega, B)}(F) (\leq) \mu_{(\omega, B)}(E)
\]

for any proper nontrivial subobject \( 0 \subset F \subset E \) in \( \mathcal{A}^C \). Since the Kähler class \( \omega \) will be fixed, and \( B = b\omega \) with \( b \in \mathbb{R} \), the slope \( \mu_{(\omega, B)} \) will be denoted by \( \mu_{(\omega, b)} \).

Moreover, \( (\omega, B) \)-limit slope (semi)stable objects of \( \mathcal{A}^C \) will be called simply \( \mu_{(\omega, b)} \)- (semi)stable when the meaning is clear from the context.

Let \( \mathcal{A}_i^C \) be the full subcategories of \( \mathcal{A}^C \) consisting of objects belonging to \( \mathcal{A}_i \), \( i = 1, 1/2 \). Given the definition of \( \mathcal{A}^C \), it follows that \( \mathcal{A}_i^C \) is the subcategory of zero dimensional subsheaves with topological support on \( C \cup C_0 \). Let \( E \) be an object of \( \mathcal{A}^C \). Since the pair \( (\mathcal{A}_1, \mathcal{A}_{1/2}) \) is a torsion pair in \( \mathcal{A} \) (see [47]), there is an exact sequence

\[
0 \to E_1 \to E \to E_{1/2} \to 0
\]

in \( \mathcal{A} \) with \( E_i \) in \( \mathcal{A}_i \), \( i = 1, 1/2 \). Then the following holds

**Lemma 2.3.** Let \( E \) be an object of \( \mathcal{A}^C \). Then \( E_i \) belongs to \( \mathcal{A}_i^C \), \( i = 1, 1/2 \).

**Proof.** Consider again the exact sequence

\[
0 \to \mathcal{H}^{-1}(E_1) \to \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(E_{1/2}) \to \mathcal{H}^0(E_1) \to \mathcal{H}^0(E) \to \mathcal{H}^0(E_{1/2}) \to 0.
\]

By definition, \( \mathcal{H}^{-1}(E_1) \) must be a torsion sheaf of dimension two, hence it must be trivial since \( \mathcal{H}^{-1}(E) \) is torsion free. Therefore \( E_1 \cong \mathcal{H}^0(E_1) \) must be a zero dimensional sheaf. Let \( I \subset \mathcal{H}^0(E) \) denote its image in \( \mathcal{H}^0(E) \) and \( K = \ker(\mathcal{H}^0(E_1) \to I) \).

Then there is an exact sequence of sheaves

\[
0 \to \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(E_{1/2}) \to K \to 0.
\]

Note that both \( I \) and \( K \) are zero dimensional sheaves and \( I \) is topologically supported on \( C \cup C_0 \). Suppose there exists a subsheaf \( K' \subset K \) with support disjoint from \( C, C_0 \). Since \( \mathcal{H}^{-1}(E) = \mathcal{I}_{Z_E} \), and \( Z_E \) is topologically supported on \( C \cup C_0 \), it follows that

\[
\text{Ext}_X^1(K', \mathcal{H}^{-1}(E)) \cong \text{Ext}_X^1(K', \mathcal{O}_X).
\]

However [20, Prop. 1.1.6] shows that \( \text{Ext}_X^1(K', \mathcal{O}_X) = 0 \) since \( K' \) is zero dimensional. Therefore, using the local to global spectral sequence, \( \text{Ext}_X^1(K', \mathcal{H}^{-1}(E)) = 0 \).
Lemma 2.4. A consequence of Lemma 2.2 is that properties of limit slope semistable objects in \( \mathcal{A} \) proven in \([17, 16]\) also hold in \( \mathcal{A}^C \). More specifically, strict monomorphisms and epimorphisms of objects in \( \mathcal{A}^C \), \( i = 1, 1/2 \) may be defined again by requiring that the cokernel, respectively kernel belong to \( \mathcal{A}^C \). Then, by analogy with \([17, 2.27], [46, 3.8]\), it follows again that any \( \mu_{(\omega, b)} \) (semi)stable object of \( \mathcal{A}^C \) with \( \text{ch}_0(E) = -1 \), must belong to \( \mathcal{A}^C_{1/2} \). Moreover the stability criterion in Proposition 2.1 holds for objects of \( \mathcal{A}^C_{1/2} \) provided that \( F \rightarrow E, E \rightarrow G \) are strict monomorphisms, respectively epimorphisms in \( \mathcal{A}^C_{1/2} \).

Some more specific properties of limit slope semistable objects in \( \mathcal{A}^C \) for \( X = X^+ \) are recorded below.

2.3. Properties of \( C \)-framed limit slope stable objects. In this subsection \( X = X^+ \) and \( C = C^+ \). The other case has to be studied separately. First note that any nontrivial object \( E \) of \( \mathcal{A}^C \) with \( \text{ch}_0(E) = 0 \) must be a sheaf with topological support on \( C \cup C_0 \) and \( \text{ch}_2(E) = r[C_0] \), \( r \geq 0 \). Moreover, if \( r \geq 1 \),

\[
\mu_{(\omega, b)}(E) = \frac{\chi(E)}{r} - b
\]

Therefore \( \mu_{(\omega, b)} \) -stability for such objects reduces to \( \omega \)-slope stability for dimension one sheaves on \( X \). For completeness recall that a sheaf \( E \) as above with \( r \geq 1 \) is \( \omega \)-slope (semi)stable if

\[
\langle \omega, \text{ch}_2(E) \rangle \chi(E') (\leq) \langle \omega, \text{ch}_2(E') \rangle \chi(E)
\]

for any nontrivial proper subsheaf \( 0 \subset E' \subset E \). Since in the present case \( \text{ch}_2(E) = r[C_0], \text{ch}_2(E') = r'[C_0] \) for some \( r, r' \in \mathbb{Z}_{\geq 0} \), and \( \omega \) is normalized such that \( \int_{C_0} \omega = 1 \), this condition reduces to

\[
r\chi(E') (\leq) r'\chi(E).
\]

Under the current assumptions the exceptional curve \( C_0 \) is a complete intersection determined by two two sections \( s_i \in H^0(X, L_i), i = 1, 2, \) where \( L_i = L_{i1} \) are line bundles on \( X \) such that \( L_i|_{C_0} \simeq \mathcal{O}_{C_0}(-1), i = 1, 2 \). Then note the following lemma.

Lemma 2.4. Let \( F \) be an \( \omega \)-slope semistable sheaf supported on \( C_0 \) with \( \text{ch}_2(F) = rC_0 \), \( r \geq 1 \). Then \( F \) is the extension by zero of a semistable locally free sheaf on \( C_0 \).

Proof. Suppose \( F \) is as in Lemma 2.4 and suppose the morphism

\[
F \xrightarrow{s_i \otimes 1_F} F \otimes_X L_i
\]

is nonzero for some \( i = 1, 2 \). Let \( G \subset F \otimes_X L_i \) denote its image and \( K \subset F \) its kernel. Both \( G, K \) are pure of dimension one with \( \text{ch}_2(G) = r_G[C_0] \), \( \text{ch}_2(K) = r_K[C_0] \),
\[ r_G, r_K \geq 1, r_G + r_K = r. \]
Then
\[ \frac{\chi(G)}{r_G} \geq \frac{\chi(F)}{r}, \quad \text{and} \quad \frac{\chi(K)}{r_K} \leq \frac{\chi(F)}{r}. \]
Moreover, it is straightforward to show that \( F \otimes_X L_i \) must be slope semistable as well and \( \chi(F \otimes_X L_i) = \chi(F) - r. \) Therefore
\[ \frac{\chi(G)}{r_G} \leq \frac{\chi(F)}{r} - 1, \]
which leads to a contradiction. In conclusion \( s_i \otimes 1_F = 0 \) for both \( i = 1, 2. \)

Another simple class of objects of \( \mathcal{A}^C \) are stable pairs \( E = (\mathcal{O}_X - s \to F) \) with \( F \) a pure dimension one sheaf supported on \( C_0 \) and \( \text{Coker}(s) \) zero dimensional. The following result will be useful later.

**Lemma 2.5.** Suppose \( E = (\mathcal{O}_X - s \to F) \) is a stable pair on \( X \) with \( F \) a pure dimension one sheaf with topological support on \( C_0 \) and \( \text{ch}_2(F) = r|C_0|, r \geq 1. \) Then \( \chi(F) \geq r. \)

**Proof.** Let \( 0 = F_0 \subset F_1 \subset \cdots \subset F_h = F \) be the Harder-Narasimhan filtration of \( F \) with respect to \( \omega \)-slope stability. The successive quotients \( F_i/F_{i-1}, 1 \leq i \leq h \) are \( \omega \)-slope semistable sheaves on \( Y \) with topological support on \( C_0. \) Hence according to Lemma 2.4 each of them is isomorphic to the extension by zero of a locally free semistable \( \mathcal{O}_{C_0} \)-module \( E_i, 1 \leq i \leq h. \) Moreover, by the general properties of Harder-Narasimhan filtrations
\[ r_G(F_i) < r_G(F), \quad \mu_\omega(F_i) > \mu_\omega(F) \]
for any \( 1 \leq i \leq h - 1. \) Then \( \text{Im}(s) \) is not a subsheaf of \( F_i \) for any \( 1 \leq i \leq h - 1. \) Otherwise the sub-object \((C, s, F_i)\) would violate \((\omega, \delta)\)-semistability. Therefore the composition
\[ \mathcal{O}_Y - s \to F \to F/F_{h-1} \]
must be nontrivial. Therefore \( E_h \) must have a nontrivial section on \( C_0. \) Since \( C_0 \simeq \mathbb{P}^1 \) and \( E_h \) is semistable, it must be of the form \( E_h \simeq \mathcal{O}_{C_0}(d)_{\mathbb{P}^1} \) for some \( d \in \mathbb{Z}. \) Moreover \( d \geq 0 \) in order for \( E_h \) to have a nontrivial section. This implies
\[ \mu_\omega(F/F_{h-1}) \geq 1, \]
and \( \mu_\omega(F) \geq 1. \)

In order to derive similar structure results for more general \( \mu(\omega, \delta) \)-semistable objects, it will be helpful to note the following technical result.

**Lemma 2.6.** Let \( F_C \) be a pure dimension one sheaf on \( X \) with scheme theoretic support on \( C \) and \( F_0 \) a pure dimension one sheaf on \( X \) with topological support on \( C_0. \) Recall that the curve \( C, \) hence also the sheaf \( F_C, \) is scheme theoretically supported on a divisor \( D \simeq \mathbb{P}^2 \) in \( X \) which intersects \( C_0 \) transversely at one point. Then there are isomorphisms
\[ \varphi_k : \text{Ext}^k_X(F_C, F_0) \to \text{Ext}^{k-1}_D(F_C, \mathcal{O}_D \otimes_X F_0) \]
for all \( k \in \mathbb{Z}, \) where \( \text{Ext}^k_D \) are global extension groups of \( \mathcal{O}_D \)-modules. Moreover, suppose
\[ 0 \to F_0 \to F \to F_C \to 0 \]
is an extension of $\mathcal{O}_X$-modules corresponding to an extension class $e \in \text{Ext}^1_X(F_C, F_0)$ and let $F_C' \subset F_C$ be a subsheaf of $F_C$. Then $e$ is in the kernel of the natural map

$$\text{Ext}^1_X(F_C, F_0) \to \text{Ext}^1_X(F_C', F_0)$$

if and only if $F_C' \subset \text{Ker}(\varphi_1(e))$.

**Proof.** The adjunction formula the canonical embedding $i : D \hookrightarrow X$ yields a quasi-isomorphism

$$\text{RHom}_X(F_C, F_0) \simeq \text{RHom}_D(F_C, i^! F_0)$$

(2.7)

where

$$i^! F_0 = Li^* F_0 \otimes \omega_D[-1].$$

Note that the cohomology sheaves of the complex $Li^* F_0$ are isomorphic to the local tor sheaves

$$\mathcal{H}^k(Li^* F_0) \simeq \mathcal{T}\text{or}^X_k(F_0, \mathcal{O}_D)$$

for all $k \in \mathbb{Z}$. Moreover local tor is symmetric in its arguments, that is

$$\mathcal{T}\text{or}^X_k(F_0, \mathcal{O}_D) \simeq \mathcal{T}\text{or}^X_k(\mathcal{O}_D, F_0).$$

Since $F_0$ is pure of dimension one, using the canonical locally free resolution

$$\mathcal{O}_X(-D) \xrightarrow{c_D} \mathcal{O}_X$$

of $\mathcal{O}_D$, it follows that $\mathcal{T}\text{or}^X_k(\mathcal{O}_D, F_0) = 0$ for all $k \neq 0$. Therefore the complex $Li^* F_0$ is quasi-isomorphic to the sheaf $\mathcal{T}\text{or}^X_0(\mathcal{O}_D, F_0) \simeq \mathcal{O}_D \otimes_X F_0$. Then (2.7) yields isomorphisms of the form (2.8).

The second statement follows from the functoriality of the adjunction formula.

For future reference note the following corollary of Lemma 2.6.

**Corollary 2.7.** Under the same conditions as in Lemma 2.6, suppose $F_0 = V \otimes \mathcal{O}_{C_0}(-1)$ with $V$ a finite dimensional vector space and let $e \in \text{Ext}^1_X(F_C, F_0)$ be an extension class. Let $\psi = \varphi_1(e) \in \text{Hom}_D(F_C, V \otimes \mathcal{O}_p)$ be the corresponding morphism of $\mathcal{O}_D$-modules, where $\mathcal{O}_p$ is the structure sheaf of the transverse intersection point $\{p\} = D \cap C_0$. Then the following conditions are equivalent

(a) Given any nontrivial quotient $q : V \to V'$ the class $e$ is not in the kernel of the natural map

$$q_* : \text{Ext}^1_X(F_C, V \otimes \mathcal{O}_{C_0}(-1)) \to \text{Ext}^1_X(F_C, V' \otimes \mathcal{O}_{C_0}(-1)).$$

(b) The morphism $\psi : F_C \to V \otimes \mathcal{O}_p$ is surjective.

**Proof.** Suppose an extension class $e$ satisfies condition (a) and $\psi$ is not surjective. Then the image of $\psi$ is $V'' \otimes \mathcal{O}_p$ where $V'' \subset V$ is a proper subspace of $V$. Let $V \simeq V' \oplus V''$ be a direct sum decomposition, and $q : V \to V'$ the natural projection. Then the second part of Lemma 2.6 implies that such that $e$ lies in the kernel of the map $q_*$, leading to a contradiction. The proof of the converse statement is analogous.

Now let $E$ be an object of $\mathcal{A}^C$ with $\text{ch}_0(E) = -1$ and let $F$ be a torsion sheaf on $X$ of dimension at most one. Then there is an exact sequence

$$0 \to \text{Ext}^1_X(F, \mathcal{H}^{-1}_E) \to \text{Hom}_{D^+(X)}(F, E) \to \text{Ext}^0_X(F, \mathcal{H}(E)) \to \text{Ext}^1_X(F, \mathcal{H}^{-1}(E)) \to \cdots$$

(2.8)
Moreover, since $H^{-1}(E) = \mathcal{I}_Z$ is the ideal sheaf of a dimension one subscheme, there is also an exact sequence

\[(2.9) \quad 0 \to \text{Ext}^1_X(F, \mathcal{O}_Z) \to \text{Ext}^1_X(F, H^{-1}(E)) \to \text{Ext}^2_X(F, \mathcal{O}_X) \to \cdots\]

**Lemma 2.8.** Suppose $E$ is an object of $A^{1/2}$ with $\text{ch}_0(E) = -1$, $\text{ch}_2(E) = [C] + r[C_0]$, $r \geq 0$. Then the following hold

(i) $Z_E$ is a pure dimension one subscheme of $X$.

(ii) There is a commutative diagram of morphisms of $\mathcal{O}_X$-modules

\[(2.10) \quad \begin{array}{cccccc}
0 & & 0 \\
\downarrow & & \downarrow & & \\
K_C & \xrightarrow{1} & K_C & \xrightarrow{f} & \mathcal{O}_C & \xrightarrow{\phi} 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{O}_Z & \to & \mathcal{O}_C & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{O}_Z & \xrightarrow{1} & \mathcal{O}_D \otimes_X K_0 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & \\
\end{array}\]

where $Z_0 \subset X$ is a pure dimension one closed subscheme of $X$ with topological support on $C_0$.

(iii) $\chi(K_0) \geq 0$.

**Proof.** Purity of $\mathcal{O}_Z$ follows from the observation that any nontrivial morphism $F \to \mathcal{O}_Z$ with $F$ zero dimensional would yield via the exact sequences $(2.8), (2.9)$ a nontrivial morphism $F \to E$ in $A^C$. This contradicts the assumption that $E$ belongs to $A^{1/2}$.

Next, the given conditions on the Chern classes of $E$ imply that

\[(2.11) \quad \text{ch}_2(H^0(E)) = r_0[C_0], \quad \text{ch}_2(H^{-1}(E)) = -[C] - r_1[C_0]
\]

with $r_0, r_1 \geq 0$, $r_0 + r_1 = r$. Moreover there is an exact sequence of $\mathcal{O}_X$-modules

\[(2.12) \quad 0 \to K_0 \to \mathcal{O}_Z \to \mathcal{O}_C \to 0\]

where $K_0$ is a pure dimension one sheaf with topological support on $C_0$. According to Lemma 2.6, there is an isomorphism

$$\varphi_1 : \text{Ext}^1_X(\mathcal{O}_C, K_0) \simeq \text{Hom}_X(\mathcal{O}_C, \mathcal{O}_D \otimes_X K_0).$$

identifying the extension class $e \in \text{Ext}^1_X(\mathcal{O}_C, K_0)$ determined by $(2.12)$ with a morphism $\phi \in \text{Hom}_X(\mathcal{O}_C, \mathcal{O}_D \otimes_X K_0)$. Let $K_C = \text{Ker}(\phi)$. Then Lemma 2.6 also implies that the restriction of the extension class $e$ to $K_C \subset \mathcal{O}_C$ is trivial. Therefore
there is a commutative diagram

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
K_C & K_C \\
\downarrow & \downarrow \\
K_0 & \mathcal{O}_{Z_E} \\
\downarrow & \downarrow \\
G & \mathcal{O}_D \otimes_X K_0 \\
\downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

with exact rows and columns. Obviously, \( G \) is the structure sheaf of a closed subscheme \( Z_0 \subset X \). The support conditions on \( \mathcal{O}_{Z_E} \) and equations (2.11) imply that \( G \) is topologically supported on \( C_0 \) and \( \text{ch}^2(G) = r - 1 \). In order to prove that \( G \) is pure, suppose \( T \subset G \) is the maximal 0 dimensional subsheaf and let \( G' = G/T \). Then \( G' \) is pure dimension one with \( \text{ch}^2(G') = r - 1 \). Let \( K_C' \) be the kernel of the resulting epimorphism \( \mathcal{O}_{Z_E} \to G' \). \( K_C' \) is a pure dimension one subsheaf of \( \mathcal{O}_{Z_E} \) with support on \( C \), \( K_C \subseteq K_C' \) and \( \text{ch}^2(K_C') = \text{ch}^2(K_C) = [C] \). Then the restriction \( f_{|K_C'} : K_C' \to \mathcal{O}_C \) must be injective. Therefore the restriction

\[
0 \to K_0 \to F \to K_C' \to 0
\]

of the extension (2.12) to \( K_C' \subset \mathcal{O}_C \) must be trivial. This implies that \( K_C' \) is contained in the kernel of \( \phi \), which is \( K_C \). Therefore \( K_C' = K_C \), which implies \( G' = G \), and \( T = 0 \). In conclusion \( G \) is of pure dimension one.

The third statement of Lemma 2.8 follows from the observation that the canonical surjective morphism \( \mathcal{O}_X \to \mathcal{O}_{Z_0} \) determines a stable pair on \( X \) with support on \( C_0 \). According to Lemma 2.36 this implies that

\[
\chi(\mathcal{O}_{Z_0}) \geq r - 1.
\]

However, the Riemann-Roch theorem implies that

\[
\chi(\mathcal{O}_D \otimes_X K_0) = r - 1
\]

since \( \text{ch}_2(K_0) = r - 1[\text{C}_0] \) and \( C_0 \cdot D = 1 \). Therefore

\[
\chi(K_0) = \chi(\mathcal{O}_{Z_0}) - r - 1 \geq 0.
\]

\[\square\]

Next note that there is an injective morphism

\[
\mathcal{O}_{Z_E} \hookrightarrow \mathcal{H}^{-1}(E)[1]
\]

in \( \mathcal{A} \) corresponding to the canonical extension

(2.13) \[
0 \to \mathcal{H}^{-1}(E) \to \mathcal{O}_X \to \mathcal{O}_{Z_E} \to 0.
\]

Therefore the canonical inclusion \( K_0 \subset \mathcal{O}_{Z_E} \) is a subobject of \( \mathcal{H}^{-1}(E)[1] \subset E \).
Lemma 2.9. Suppose $E$ is an object of $\mathcal{A}^{C}_{1/2}$ with $\text{ch}_0(E) = -1$, $\text{ch}_2(E) = [C] + r[C_0]$, $r \geq 0$. Then there is an exact sequence

$$0 \rightarrow K_0 \rightarrow E \rightarrow G \rightarrow 0$$

in $\mathcal{A}^C$ where $\mathcal{H}^{-1}(G) \simeq \mathcal{I}_C$ and $\mathcal{H}^0(G) \simeq \mathcal{H}^0(E)$.

Proof. Since $K_0$ is pure of dimension one, it belongs to $\mathcal{A}^{C}_{1/2}$. According to Lemma [2.2], $G = E/K_0$ belongs to $\mathcal{A}^{C}$. Moreover, note that the morphism $K_0 \rightarrow E \rightarrow \mathcal{H}^0(E)$ is trivial since $K_0$ is a subobject of $\mathcal{H}^{-1}(E)[1]$. Then the long exact cohomology sequence of (2.14) yields exact sequences of $\mathcal{O}_X$-modules

$$0 \rightarrow \mathcal{H}^{-1}(E) \rightarrow \mathcal{H}^{-1}(G) \rightarrow K_0 \rightarrow 0$$

$$0 \rightarrow \mathcal{H}^0(E) \rightarrow \mathcal{H}^0(G) \rightarrow 0.$$

The first exact sequence is the restriction of (2.14) to $K_0 \subset \mathcal{O}_E$. Then a simple application of the snake lemma shows that $\mathcal{H}^{-1}(G) \simeq \mathcal{I}_C$. \qed

Finally note the following observation.

Lemma 2.10. Let $E$ be an object of $\mathcal{A}^{C}_{1/2}$ with $\text{ch}(E) = (-1, 0, [C], n)$, $n \in \mathbb{Z}$ such that $\mathcal{H}^0(E)$ is a zero dimensional sheaf. Then $E$ is isomorphic to a stable pair $F \rightarrow C = (\mathcal{O}_X \xrightarrow{\omega} \mathcal{F}_C)$ with $\mathcal{F}_C$ a pure dimension one sheaf with scheme theoretic support on $C$. Moreover $E$ is $\mu(\omega, b)$-stable for any $b \in \mathbb{R}$.

Proof. The first part follows from [17, Lemm. 4.5]. In particular there is an exact sequence

$$0 \rightarrow F_C \rightarrow E \rightarrow \mathcal{O}_X[1] \rightarrow 0$$

in $\mathcal{A}$. For the second, note that there are no strict epimorphisms $E \rightarrow G$ in $\mathcal{A}^{C}_{1/2}$ with $G$ pure dimension one since $\text{Hom}_{\mathcal{A}}(E, G) \subset \text{Hom}_{\mathcal{X}}(\mathcal{H}^0(E), G)$ and $\mathcal{H}^0(E)$ is zero dimensional. Furthermore, for any pure dimension one sheaf $G$ with support on $C_0$, there is an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(E, G) \rightarrow \text{Ext}_X^1(\mathcal{O}_X, G) \rightarrow \cdots$$

since $\text{Hom}_{\mathcal{X}}(G, F_C) = 0$. Serre duality yields isomorphism $\text{Ext}_X^1(\mathcal{O}_X, G) \simeq H^2(G)^\vee$, hence there are no strict monomorphisms $G \hookrightarrow E$ in $\mathcal{A}^{C}_{1/2}$. \qed

3. Stable pairs at small $b$

The goal of this section is to analyze the structure of the moduli stacks $\mathcal{P}_{(\omega, b)}(X, C, r, n)$ for $b > 0$ sufficiently close to 0, in particular to prove Theorem [1.2]. Here again $X = X^+$ and $C = C^+$ and one fixes a Kähler class $\omega$ such that $\int_{C^b} \omega = 1$. For the present purposes it suffices to consider only generic values of $b$, in which case strictly semistable objects with numerical invariants $(r, n)$ do not exist. The main technical result in this section is the stability criterion obtained in Proposition [3.3] below. The proof is fairly long and complicated, and will be carried out in several stages in Section [3.1]. Sections [3.2] and [3.3] summarize several applications of this result, explaining the connection between moduli stacks of $C$-framed perverse coherent sheaves and nested Hilbert schemes.
3.1. A stability criterion. Fix $\text{ch}(E) = (-1, 0, [C] + r[C_0], n)$, $r \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$ and Kähler class $\omega$ on $X$ such that $\int_X \omega = 1$.

**Lemma 3.1.** For fixed Kähler class $\omega$, and fixed $(r, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ the following holds for any stability parameter $b > 0$ such that

\[ b < \frac{1}{2r} \]

if $r > 0$.

Any $\mu_{(\omega, b)}$-stable object $E$ of $\mathcal{A}^C$ with $\text{ch}(E) = (-1, 0, [C] + r[C_0], n)$ fits in an exact sequence

\[ 0 \to P_C \to E \to G \to 0 \]

in $\mathcal{A}^C$ such that

(i) $P_C = (\mathcal{O}_X \to F_C)$ is a stable pair on $X$ with $F_C$ scheme theoretically supported on $C$.

(ii) $G$ is a pure dimension one sheaf on $X$ with topological support on $C_0$ and $\text{ch}_2(G) = r[C_0]$. Moreover its Harder-Narasimhan filtration

\[ 0 = G_0 \subset G_1 \subset \cdots \subset G_h = G \]

with respect to $\omega$-slope stability satisfies

\[ G_j/G_{j-1} \cong \mathcal{O}_{C_0}(a_j)^{\oplus s_j}, \]

where $a_j \in \mathbb{Z}_{\geq 1}$ for $j = 1, \ldots, h$, and $a_1 > a_2 > \cdots > a_h$.

**Proof.** It will be first proven that for $b > 0$ sufficiently close to 0, the stability criterion in Proposition 2.1 implies that $\mathcal{H}^{-1}(E) = I_C$ for any $\mu_{(\omega, b)}$-stable object $E$ of $\mathcal{A}^C$ with $\text{ch}(E) = (-1, 0, [C] + r[C_0], n)$.

Suppose $E$ is $\mu_{(\omega, b)}$-stable for some $b > 0$. This implies that $E$ is an element of $\mathcal{A}^C_{1/2}$ satisfying the stability criterion in Proposition 2.1 with respect to strict morphisms in $\mathcal{A}^C_{1/2}$. According to Lemmas 2.3, 2.8 there is an injective morphism $\kappa : K_0 \to E$ where $K_0$ is pure of dimension one and $\chi(K_0) \geq 0$. Suppose $K_0$ is nontrivial, and $\text{ch}_2(K_0) = r^{-1}[C_0]$ for some $r^{-1} > 0$. Lemma 2.3 implies that the cokernel $G = \text{Coker}(\kappa)$ fits in an exact sequence

\[ 0 \to G_1 \to G \to G_{1/2} \to 0 \]

where $G_i \in \mathcal{A}^C_i$, $i = 1, 1/2$. In particular $G_1$ is a zero dimensional sheaf. Then the snake lemma implies that the kernel $K$ of the projection $E \to G_{1/2}$ is a one dimensional sheaf on $X$ which fits in an exact sequence

\[ 0 \to K_0 \to K \to G_1 \to 0 \]

Moreover, $K$ must be pure of dimension one since $E$ belongs to $\mathcal{A}^C_{1/2}$. Then the stability criterion in Proposition 2.1 implies that

\[ \frac{\chi(K)}{r^{-1}} < -2b \]

if $r^{-1} > 0$. Since $b > 0$, this implies that $\chi(K) < 0$, hence also $\chi(K_0) < 0$ since $G_1$ is a zero dimensional sheaf. This contradicts Lemma 2.3 (iii) unless $K_0$ is trivial.

In conclusion, $\mathcal{H}^{-1}(E) = I_C$ for any $b > 0$.

This implies in particular that $\text{ch}_2(\mathcal{H}^0(E)) = r[C_0]$ for $b > 0$. Therefore, if $r = 0$, $\mathcal{H}^0(E)$ must be a zero dimensional sheaf with topological support on $C$. Since $E$
This implies that there is an extension as well. Therefore there is an isomorphism by Serre duality and the structure of the Harder-Narasimhan filtration of Note also that \(A\) in stability criterion in Proposition 2.1 implies that \(r > 0\), hence the epimorphism \(E \rightarrow G\). Then \(F\) belongs to \(\mathcal{A}_{C_{ij}}\) since \(E\) does, hence the epimorphism \(E \rightarrow G\) is strict. Moreover \(\mathcal{H}^{-1}(F) \simeq \mathcal{H}^{-1}(E)\). Then the stability criterion in Proposition 2.1 implies that

\[
\frac{\chi(G)}{r_G} > -2b.
\]

Note that \(-2b > -1/r\) if the bound \(3.1\) holds. Since \(0 < r_G \leq r\) this implies that \(\chi(G) > -\frac{r_G}{r} \geq -1\), hence \(\chi(G) \geq 0\), since \(\chi(G) \in \mathbb{Z}\).

Now consider the exact sequence

\[
0 \rightarrow Q \rightarrow \mathcal{H}^0(E) \rightarrow G \rightarrow 0
\]

where \(Q \subset \mathcal{H}^0(E)\) is the maximal zero dimensional subsheaf of \(\mathcal{H}^0(E)\) and \(G\) is pure of dimension one supported on \(C_0\). The previous argument implies that any pure dimension one quotient \(G \rightarrow G'\) must have \(\chi(G') \geq 0\) if \(3.1\) holds. In particular, using Lemma 2.4 \(G\) has a Harder-Narasimhan filtration of the form \(5.3\).

Let \(E' = \text{Ker}(E \rightarrow G)\). Obviously, \(E'\) belongs to \(\mathcal{A}_{C_{ij}}\) and there is an exact sequence

\[
(3.4) \quad 0 \rightarrow \mathcal{I}_C[1] \rightarrow E' \rightarrow Q \rightarrow 0
\]

in \(\mathcal{A}^C\). Then \(4.5\) Lemm. 4.5] implies that \(E'\) is isomorphic to the stable pair \(P_C = (\mathcal{O}_X \xrightarrow{s} F_C)\) in \(\mathcal{A}^C\), where \(s\) is determined by the natural projection \(\mathcal{O}_X \rightarrow \mathcal{O}_C\). □

Lemma 3.2. Under the same conditions as in Lemma 3.1 suppose the bound \(3.1\) is satisfied. Then for any \(\mu_{\omega,b}\)-stable object \(E\) of \(\mathcal{A}^C\) with \(\text{ch}(E) = (-1, 0, [C] + r[C_0], n)\), with \(r > 0\), the quotient \(G\) in 5.2 must be of the form \(G \simeq \mathcal{O}_{C_0}(-1)^{\oplus r}\).

Proof. Using the notation in Lemma 3.1 note the exact sequence

\[
(3.5) \quad 0 \rightarrow F_C \rightarrow P_C \rightarrow \mathcal{O}_X[1] \rightarrow 0
\]

in \(\mathcal{A}\), which yields a long exact sequence exact sequence

\[
\cdots \rightarrow \text{Ext}^1_X(G, \mathcal{O}_X) \rightarrow \text{Ext}^1_X(G, F_C) \rightarrow \text{Ext}^1_A(G, P_C) \rightarrow \text{Ext}^2_X(G, \mathcal{O}_X) \cdots
\]

Note also that

\[
\text{Ext}^1_X(G, \mathcal{O}_X) \simeq H^2(X, G)^\vee = 0
\]

by Serre duality and the structure of the Harder-Narasimhan filtration of \(G\) implies

\[
\text{Ext}^2(G, \mathcal{O}_X) \simeq H^1(X, G)^\vee = 0
\]

as well. Therefore there is an isomorphism

\[
(3.6) \quad \text{Ext}^1_X(G, F_C) \simeq \text{Ext}^1_A(G, P_C).
\]

This implies that there is an extension

\[
(3.7) \quad 0 \rightarrow F_C \rightarrow F \rightarrow G \rightarrow 0
\]
determined by the extension class of \( E \) up to isomorphism, which fits in a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & F_C \\
\downarrow & & \downarrow \\
0 & \rightarrow & E \\
\end{array}
\begin{array}{ccc}
F & \rightarrow & G \\
\downarrow & & \downarrow 1 \\
E & \rightarrow & G \\
\end{array}
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\end{array}
\]

in \( \mathcal{A} \). In particular the middle vertical morphism is in injective in \( \mathcal{A} \), and

\[
E/F \cong P_C/F_C \cong \mathcal{O}_X[1].
\]

Since \( E \) belongs to \( \mathcal{A}_{1/2}^C \), \( F \) has to be a pure dimension one sheaf.

Let \( F_C' \) be the quotient \( F \otimes_X \mathcal{O}_C/T \) where \( T \subset F \otimes_X \mathcal{O}_C \) is the maximal zero dimensional submodule. Then there is an exact sequence

\[
0 \rightarrow G' \rightarrow F \rightarrow F_C' \rightarrow 0
\]

where \( G' \) is pure dimension one with topological support of \( C_0 \) and \( F_C' \) pure dimension one with support on \( C \). Moreover, \( \text{ch}_2(G') = \text{ch}_2(G) = r[C_0] \) and \( \text{ch}_2(F_C') = \text{ch}_2(F_C) = [C] \). In particular \( F_C' \) is scheme theoretically supported on \( C \). Obviously \( G' \subset F \subset E \) in \( \mathcal{A}^C \), and there is an exact sequence

\[
0 \rightarrow F_C' \rightarrow E/G' \rightarrow \mathcal{O}_X[1] \rightarrow 0
\]

in \( \mathcal{A} \). This implies that \( E/G' \) belongs to \( \mathcal{A}_{1/2} \) since both \( F_C', \mathcal{O}_X[1] \) do. As \( E/G' \) also belongs to \( \mathcal{A}^C \) according to Lemma 2.2, it follows that \( E/G' \) belongs to \( \mathcal{A}_{1/2}^C \) i.e. the inclusion \( G' \subset E \) is strict.

The same holds for any proper saturated subsheaf \( G'' \subset G' \) (that is a subsheaf such that \( G'/G'' \) is pure dimension one.) For any such sheaf there is an exact sequence

\[
0 \rightarrow G'/G'' \rightarrow E/G'' \rightarrow E/G' \rightarrow 0
\]

in \( \mathcal{A} \) which implies that \( E/G'' \) belongs to \( \mathcal{A}_{1/2}^C \). Then the stability criterion implies that

\[
\frac{\chi(G'')}{r_{G''}} < -2b < 0
\]

for any saturated subsheaf \( G'' \subset G' \), where \( \text{ch}_2(G'') = r_{G''}[C_0] \), \( 0 < r_{G''} \leq r \).

Next recall that according to Lemma 2.6 there is an isomorphism

\[
\varphi_1 : \text{Ext}^1_X(F_C', G') \cong \text{Hom}_D(F_C', \mathcal{O}_D \otimes_X G').
\]

Since \( G' \) is pure dimension one supported on \( C_0 \), there is an exact sequence

\[
0 \rightarrow G'(-D) \rightarrow G' \rightarrow \mathcal{O}_D \otimes_X G' \rightarrow 0
\]

This implies via the Riemann-Roch theorem that

\[
\chi(\mathcal{O}_D \otimes_X G') = r.
\]

Now let \( \phi : F_C' \rightarrow \text{Ext}^1_D(\mathcal{O}_D, G') \) be the morphism corresponding to the extension class of \( (3.9) \) under the isomorphism \((3.11)\). The exact sequences \((3.7), (3.9)\) imply
that there is an injective morphism $F_C \hookrightarrow F'_C$ such that the following diagram commutes

$$
\begin{array}{ccc}
F_C & \xrightarrow{1} & F_C \\
\downarrow & & \downarrow \\
F_C & \xrightarrow{1} & F_C
\end{array}
$$

since both $F_C, F'_C$ are pure supported on $C$ and $\text{ch}_2(F_C) = \text{ch}_2(F'_C) = [C]$. This implies that the restriction of the extension (3.9) to $F_C \subset F'_C$ is split i.e. $F_C \subset \text{Ker}(\phi)$. In conclusion the quotient $F'_C/F_C$ is a subsheaf of $\mathcal{O}_D \otimes_X G'$ and there is a commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & 0 & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & F_C & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & G' & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & F'_C & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & F'_C/F_C & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & 0 & \rightarrow \\
\end{array}
$$

with exact rows and columns. The bottom row of this diagram yields

$$
(3.12) \quad \chi(G') = \chi(G) - \chi(F'_C/F_C) \geq \chi(G) - \chi(\mathcal{O}_D \otimes_X G') = \chi(G) - r.
$$

In order to conclude the proof, let

$$
0 = G'_0 \subset G'_1 \subset \cdots \subset G'_{h'} = G
$$

be the Harder-Narasimhan filtration of $G'$ with respect to $\omega$-slope stability. Each nontrivial quotient $G'_j/G'_{j-1}$, $j = 1, \ldots, h'$ must be isomorphic to a sheaf of the form $\mathcal{O}_{C_0}(a'_j)^{s'_j}$, $s'_j \geq 1$, such that $a'_1 > a'_2 > \cdots > a'_{h'}$. Inequality (3.10) implies that $a'_1 \leq -2$, therefore $a'_j \leq -2$ for all $j = 1, \ldots, h'$. This implies

$$
\chi(G') \leq -r,
$$

hence inequality (3.12) yields

$$
\chi(G) \leq 0.
$$

Taking into account the constraints on the Harder-Narasimhan filtration of $G$ in Lemma 3.1 it follows that

$$
G \simeq \mathcal{O}_{C_0}(-1)^{\oplus r}.
$$

\[ \square \]

**Proposition 3.3.** For fixed Kähler class $\omega$, and fixed $(r, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ the following holds for any $b > 0$ satisfying the bound (3.1).

An object $E$ of $\mathcal{A}^C$ is $\mu_{(\omega, b)}$-stable if and only if there is an exact sequence of the form

$$
(3.13) \quad 0 \rightarrow P_C \rightarrow E \rightarrow \mathcal{O}_{C_0}(-1)^{\oplus r} \rightarrow 0
$$
in $\mathcal{A}^C$ such that:

(i) $P_C = (\mathcal{O}_X \to F_C)$ is a stable pair on $X$ with $F_C$ scheme theoretically supported on $C$.

(ii) There is no linear subspace $0 \subset V' \subset \mathbb{C}^r$ such that the restriction

$$0 \to P_C \to E' \to V' \otimes \mathcal{O}_{C_0}(-1) \to 0$$

of the extension (5.13) to $V' \otimes \mathcal{O}_{C_0}(-1)$ is trivial. Extensions (5.13) satisfying this property will be called nondegenerate.

Proof. ($\Rightarrow$) First note that if $r = 0$ Lemma 3.1 shows that $E$ must be isomorphic to a stable pair $P_C$. Furthermore, any such stable pair is stable for all $b > 0$ according to Lemma 2.10.

Suppose $r > 0$. The existence of an extension of the form (5.13) follows from Lemmas 3.1 and 3.2. Nondegeneracy follows easily noting that if the restriction of the extension (5.13) to some subsheaf $G' = V' \otimes \mathcal{O}_{C_0}(-1) \subset \mathcal{O}_{C_0}(-1)^{\oplus r}$ is trivial, then there is an epimorphism $G' \to E$. Moreover, there is an exact sequence

$$0 \to P_C \to E/G' \to V'' \otimes \mathcal{O}_{C_0}(-1) \to 0$$

in $\mathcal{A}$ where $V'' \simeq \mathbb{C}^r/V'$. This implies that that $E/G'$ belongs to $\mathcal{A}^{C}_{1/2}$ i.e. the epimorphism $G' \to E$ is strict. Then $G' \subset E$ violates the stability criterion for $b > 0$.

($\Leftarrow$) Conversely, suppose an object $E$ of $\mathcal{A}^C$ fits in an extension of the form (3.2) satisfying the nondegeneracy condition. Then it follows easily that $E$ belongs to $\mathcal{A}^C_{1/2}$. One has to check the stability criterion in Proposition 2.1 with respect to strict monomorphisms and epimorphisms in $\mathcal{A}^C$. Note that property (C.2) implies that all pure dimension one sheaves in $\mathcal{A}^C$ must be topologically supported on $C_0$. Consider first strict epimorphisms $E \to G$, with $G$ a pure dimension one sheaf supported on $C_0$. It is straightforward to check that $\text{Hom}_\mathcal{A}(P_C, G) = 0$ as in the proof of Lemma 2.10. Therefore the exact sequence (5.13) yields an isomorphism

$$\text{Hom}_\mathcal{A}(E, G) \simeq \text{Hom}_X(\mathcal{O}_{C_0}(-1)^{\oplus r}, G).$$

Since $\mathcal{O}_{C_0}(-1)^{\oplus r}$ is $\omega$-slope semistable, any quotient $G$ must satisfy

$$\chi(G) \geq 0 > -2b$$

for any $b > 0$.

Next suppose $F \hookrightarrow E$ is a strict monomorphism in $\mathcal{A}^C$ with $F$ a nontrivial pure dimension one sheaf on $X$ supported on $C_0$. Let $F'$ denote the image of $F$ in $\mathcal{O}_{C_0}(-1)^{\oplus r}$ and $F''$ the kernel of $F \to F'$. Then $F''$ must be a subobject of $P_C$ in $\mathcal{A}^C$, hence it must be trivial, as shown in the proof of Lemma 2.10. Therefore $F = F'$ must be a subsheaf of $\mathcal{O}_{C_0}(-1)^{\oplus r}$, which is $\omega$-slope semistable with $\chi(\mathcal{O}_{C_0}(-1)) = 0$. This implies $\chi(F) \leq 0$. Since the bound (3.1) yields,

$$-\frac{1}{r} < -2b < 0,$$

$F$ destabilizes $E$ only if $\chi(F) = 0$, which implies that $F \simeq V' \otimes \mathcal{O}_{C_0}(-1)$ for some linear subspace $V' \subseteq \mathbb{C}^r$. However this contradicts the nondegeneracy assumption. 

□
3.2. Moduli spaces of decorated sheaves. Consider the moduli problem for data \((V,L,F,s,f)\) where \(V,L\) are vector spaces of dimension \(r,1, r \geq 1\), respectively, \(F\) is a coherent sheaf on \(X\), and

\[
s : L \otimes \mathcal{O}_X \to F, \quad f : F \to V \otimes \mathcal{O}_{C^0}(-1)
\]

are morphisms of coherent sheaves satisfying the following conditions:

(a) \(F\) is pure of dimension one with \(\text{ch}_2(F) = |C| + r|C_0|\), \(\chi(F) = n\).

(b) \(f : F \to V \otimes \mathcal{O}_{C^0}(-1)\) is surjective and \(\text{Ker}(f)\) is scheme theoretically supported on \(C\).

(c) \(s : L \otimes \mathcal{O}_X \to F\) is a nonzero morphism.

(d) The extension

\[
0 \to \text{Ker}(f) \to F \to V \otimes \mathcal{O}_{C^0}(-1) \to 0
\]

satisfies the nondegeneracy condition of Proposition 3.3. That is there is no proper nontrivial subspace \(0 \subset V' \subset V\) such that the restriction of the above extension to \(V' \otimes \mathcal{O}_{C^0}(-1)\) is trivial.

Two collections \((V,L,F,s,f)\) and \((V',L',F',s',f')\) are isomorphic if there exist linear isomorphisms \(V \sim \nrightarrow V'\), \(L \sim \nrightarrow L'\) and an isomorphism of sheaves \(F \sim \nrightarrow F'\) satisfying the obvious compatibility conditions with the data \((s,f)\), \((s',f')\) satisfied. Then it is straightforward to prove the automorphism group of any collection \((V,L,F,s,f)\) is isomorphic to \(\mathbb{C}^\times\).

Let \(T\) be a scheme over \(\mathbb{C}\), \(X_T = X \times T\) and \(\pi_T : X_T \to T\) denote the canonical projection. For any closed point \(t \in T\), let \(X_t = X \times \{t\}\) denote the fiber of \(\pi_T\) over \(t\). Let also \(C_{0T} \subset X_T\), \(C_T \subset X_T\) denote the closed subschemes \(C_0 \times T \subset X \times T\), \(C \times T \subset X \times T\) respectively, and \(\mathcal{O}_{C_{0T}}(d)\) denote the pull-back of the sheaf \(\mathcal{O}_{C_0}(d)\) to \(X_T\), for any \(d \in \mathbb{Z}\). Similar notation will be employed for each closed fiber \(X_t\), \(t \in T\).

A flat family of data \((V,L,F,s,f)\) on \(X\) parameterized by \(T\) is a collection \((V_T,L_T,F_T,s_T,f_T)\) where

- \(V_T, L_T\) are locally free \(\mathcal{O}_T\)-modules and \(F_T\) is a coherent \(\mathcal{O}_{X_T}\)-module flat over \(T\).
- \(s_T : V_T \to F_T\) and \(f_T : F_T \to \pi_T^*L_T \otimes \mathcal{O}_{C_{0T}}\) are morphisms of \(\mathcal{O}_{X_T}\)-modules.
- The restriction of the data \((V_T,L_T,F_T,s_T,f_T)\) to any fiber \(X_t\), with \(t \in T\), a closed point is a collection satisfying conditions (a)-(d) above.

For any \((r,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}\) let \(Q(X,C,r,n)\) denote the resulting moduli stack of data \((V,L,F,s,f)\) on \(X\) satisfying conditions (a)-(d). Let \(Q(X,C,r,n)\) be the rigidification of \(Q(X,C,r,n)\) obtained by fixing isomorphisms \(L \simeq \mathbb{C}\) and \(V \simeq \mathbb{C}'\). Then the closed points of \(Q(X,C,r,n)\) have trivial stabilizers and \(Q(X,C,r,n)\) is a \(\mathbb{C}^\times\)-gerbe over \(Q(X,C,r,n)\).

The moduli stacks \(Q(X,C,r,n)\) will be used as an interpolating tool between the nested Hilbert schemes \(\mathcal{H}^n_{[b]}(C)\) introduced in Section 1.1.1 and stable \(C\)-framed perverse coherent sheaves at small \(b > 0\).

Remark 3.4. For future reference, let \(\mathcal{M}(X,C,r,n)\) denote the moduli stack of data \((V,F,s,F)\) satisfying conditions (a), (b), (d) above. Obviously, there is a natural morphism \(\pi : Q(X,C,r,n) \to \mathcal{M}(X,C,r,n)\) forgetting the data \((L,s)\). It is straightforward to check that the stabilizers of all closed points of \(\mathcal{M}(X,C,r,n)\) are isomorphic to \(\mathbb{C}^\times\).
3.3. **Relation to nested Hilbert schemes.** Suppose \( C \subset \mathbb{P}^2 \) is a reduced irreducible divisor with one singular point \( p \), otherwise smooth. For any \( l \in \mathbb{Z}_{\geq 0} \) let \( \mathcal{H}^l(C) \) denote the Hilbert scheme of length \( l \) zero dimensional subschemes of \( C \). Let \( \mathcal{H}^{[l,k]}(C) \subset \mathcal{H}^l(C) \times \mathcal{H}^{l+k}(C) \) denote the cycle consisting of pairs of ideal sheaves \((J,I)\) such that
\[
m_p J \subseteq I \subseteq J,
\]
where \( m_p \subset \mathcal{O}_{C,p} \) is the maximal ideal in the local ring at \( p \).

The main observation is that the nested Hilbert schemes \( \mathcal{H}_p^{[l,r]}(C) \), equipped with an appropriate scheme structure, are isomorphic to relative Quot schemes over \( \mathcal{H}^l(C) \). Let \( J \) denote the universal ideal sheaf on \( H^l(C) \times C \) and \( J_p \) its restriction to the closed subscheme \( H^l(C) \times \{p\} \). Let \( Q^{[l,r]}(C) \) the relative Quot-scheme parametrizing rank \( r \) locally free quotients of \( J_p \) over \( \mathcal{H}^l(C) \). Standard results on Quot-schemes show that \( Q^{[l,r]}(C) \) is a quasi-projective scheme over \( H^l(C) \). Note that a closed point of \( Q^{[l,r]}(C) \) over a closed point \( |J| \in H^l(C) \) is a pair \((V,\xi)\) where \( V \) is a \( r \)-dimensional vector space over \( \mathbb{C} \) and \( \xi : J \otimes \mathcal{O}_p \rightarrow V \) is a surjective map of complex vector spaces. In particular the fiber of \( Q^{[l,r]}(C) \) is empty if \( J \) has less than \( r \) generators at \( p \). Let
\[
I = \ker(J \rightarrow J \otimes \mathcal{O}_{C,p} \xrightarrow{\xi} V).
\]
Then it is straightforward to check that \((J,I)\) is a pair of ideal sheaves on \( C \) satisfying conditions (3.14) at \( p \). Note that the resulting scheme structure on \( H^{[l,r]}(C) \) may be different from the reduced induced scheme structure.

The main result of this section is:

**Proposition 3.5.** For any \((r,n) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}, n \geq \chi(\mathcal{O}_C)\), there is an isomorphism
\[
q : Q(X,C,r,n) \sim Q^{[l,r]}(C)
\]
on \( H^l(C) \), where \( l = n - \chi(\mathcal{O}_C) \).

The first step in the proof of Proposition 3.5 is the observation that the moduli stack \( Q(X,C,r,n) \) admits a dual formulation which makes the connection with the Hilbert scheme of \( C \) manifest. Let \( J \subset \mathcal{O}_C \) be the ideal sheaf of a zero dimensional subscheme of \( C \) and consider an exact sequence of \( \mathcal{O}_X \)-modules
\[
0 \rightarrow V \otimes \mathcal{O}_{C_0}(-1) \rightarrow F \rightarrow J \rightarrow 0,
\]
with \( V \) a finite dimensional vector space. The extension (3.16) is called nondegenerate if for any nontrivial quotient \( V \rightarrow V' \), the corresponding extension class \( e \) is not in the kernel of the natural map
\[
\text{Ext}^1_X(J,V \otimes \mathcal{O}_{C_0}(-1)) \rightarrow \text{Ext}^1_X(J,V' \otimes \mathcal{O}_{C_0}(-1)).
\]
Now let \( \pi : Q^*(X,C,r,l) \rightarrow H^l(C) \) be a moduli stack over \( H^l(C) \) defined as follows. For any scheme \( \tau : T \rightarrow H^l(C) \) let \( J_T \) be the flat family of ideal sheaves on \( C \) obtained by pull-back. The objects of \( Q^*(X,C,r,l) \) are collections \((V_T,F_T,f_T,g_T)\) where \( V_T \) is a locally free \( \mathcal{O}_T \)-module, \( F_T \) is a flat family of pure dimension one sheaves on \( X \), and \( g_T : \pi_T^*V_T \otimes \mathcal{O}_{C_0}(-1) \rightarrow F_T, h_T : F_T \rightarrow J_T \) are morphisms of \( \mathcal{O}_{X_T} \)-modules such that

\((a^*)\) For any closed point \( t \in T \) there is an exact sequence of \( \mathcal{O}_{X_t} \)-modules
\[
0 \rightarrow V_t \otimes \mathcal{O}_{C_{0}(t)}(-1) \xrightarrow{g_t} F_t \xrightarrow{h_t} J_t \rightarrow 0
\]
(b∗) The extension \( \textbf{[3.17]} \) is nondegenerate. Isomorphisms are defined naturally. Then the following holds.

**Lemma 3.6.** For any \((r, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}, n \geq \chi(O_{C})\), there is an isomorphism \( j : Q(X, C, r, n) \simto Q^{*}(X, C, r, l), l = n - \chi(O_{C}) \).

**Proof.** Given any collection \((V, L, F, s, f)\) satisfying conditions (a)-(d), let \( G = \text{Ker}(f) \). Note that the section \( s : O_{X} \to F \) must factor through \( s_{C} : O_{C} \to G \) since \( H^{0}(O_{C_{a}}(-1)) = 0 \) and \( G \) is scheme theoretically supported on \( C \). According to \([\text{33}] \) Prop. B8], the moduli space of pairs \((G, s_{C})\) is isomorphic to the Hilbert scheme \( \mathcal{H}(C) \). The isomorphism is obtained by taking the derived dual \( G^{\vee} = R\text{Hom}_{C}(G, O_{C}) \), which is an ideal sheaf on \( C \).

The isomorphism \( j \) will be first constructed on closed points. Note that taking derived duals on \( X \) one obtains an exact sequence

\[
0 \to V^{\vee} \otimes O_{C_{a}}(-1) \to Ext^{2}(F, O_{X}) \to Ext^{2}(G, O_{X}) \to 0.
\]

The duality theorem for the closed embedding \( \iota : C \hookrightarrow X \) yields an isomorphism

\[
R\iota_{*}R\text{Hom}_{C}(G, \omega_{C})[-2] \simeq R\text{Hom}_{X}(R\iota_{*}G, O_{X}).
\]

Note also that \( \omega_{C} \simeq O_{X}((k - 3)H) \), where \( k \in \mathbb{Z}_{> 0} \) is the degree of \( C \subset D \). Therefore there is an isomorphism of \( O_{X} \)-modules

\[
\iota_{*}G^{\vee} \simeq Ext^{2}_{X}(G, O_{X}) \otimes O_{X}((3 - k)H),
\]

where \( G^{\vee} \) denotes the derived dual on \( C \). Moreover it is straightforward to check that the extension

\[
0 \to G \to F \to \frac{f}{j}V \otimes O_{C_{a}}(-1) \to 0
\]

is nondegenerate if and only if the dual \( \textbf{[3.18]} \) is nondegenerate.

In conclusion the functor \( j \) has been constructed on closed points. The construction in families is analogous, using \([\text{33}] \) Prop. B8].

In order to conclude the proof of Proposition \( \textbf{3.5} \), recall that according to Lemma \( \textbf{2.6} \) there is an isomorphism

\[
\varphi_{1} : Ext^{1}_{X}(J, V \otimes O_{C_{a}}(-1)) \simto \text{Hom}_{D}(J, V \otimes O_{p}).
\]

Moreover Corollary \( \textbf{2.7} \) shows that for given a morphism \( \psi : J \to V \otimes O_{p} \), the extension

\[
0 \to V \otimes O_{C_{a}}(-1) \to F_{\varphi_{1}^{-1}(\psi)} \to J \to 0
\]

is nondegenerate if and only if \( \psi \) is surjective. Now note that there is an isomorphism

\[
\text{Hom}_{C}(J, V \otimes O_{p}) \simeq \text{Hom}_{C}(J \otimes C O_{p}, V \otimes O_{p}), \quad \psi \mapsto \tilde{\psi},
\]

such that \( \tilde{\psi} \) is surjective if and only if \( \psi \) is surjective. Then Proposition \( \textbf{3.5} \) follows from Lemma \( \textbf{3.6} \) by a straightforward comparison of flat families.

**Remark 3.7.** Note that Proposition \( \textbf{3.5} \) implies that the stack \( Q(X, C, r, n) \) is a \( \mathbb{C}^{x} \)-gerbe over the Quot schemes \( \text{Q}^{[l, r]}(C) \). A similar result holds for the moduli stacks \( M(X, C, r, n) \) of decorated objects satisfying conditions (a), (b), (d) introduced in Remark \( \textbf{2.4} \). Let \( M^{[l, r]}(C) \) be the moduli stack of pairs \((J, \psi)\) where \( J \) is a length \( l \) ideal sheaf on \( C \) and \( \psi : J \to O_{p}^{\text{dr}} \) a surjective morphism. Two such pairs are isomorphic if there is an isomorphism \( \xi : J \to J' \) such that \( \psi' \circ \xi =
ψ. By analogy with the moduli spaces \( Q^{[l,r]}(C) \), \( M^{[l,r]}(C) \) are naturally identified with relative Quot schemes over the compactified Jacobian of \( C \) of degree \( l = n - \chi(O_C) \). By analogy with Proposition 3.7 the stacks \( \mathcal{M}(X,C,r,n) \) are \( \mathbb{C}^\times \)-gerbes over the moduli spaces \( M^{[l,r]}(C) \). Moreover there is an obvious forgetful morphism \( \pi : Q^{[l,r]}(C) \to M^{[l,r]}(C) \) determined by the natural morphism from the Hilbert scheme to the compactified Jacobian.

Let \( \mathcal{N}(D,k,r,n) \) be the moduli stack of pairs \((J,\psi)\) where \( J \) is a rank one torsion free sheaf on a degree \( k \) reduced irreducible divisor on \( D \) and \( \psi : J \to O_p^{\oplus r} \) a surjective morphism. Obviously there is a natural projection \( \mathcal{N}(D,k,r,n) \to \mathcal{U} \) to an open subset of the linear system \( |kH| \) on \( D \). \( M^{[l,r]}(C) \) is the fiber of this projection over the point \([C] \in \mathcal{U}\). Since any \( \mathcal{O}_D \)-module \( J \) as above is automatically slope and Gieseker stable on \( D \), one can easily check that such a pair \((J,\psi)\) is \( \delta \)-stable in the sense of [19] for sufficiently small \( \delta > 0 \). Then the results of [19] imply that \( \mathcal{N}(D,k,r,n) \) is a quasi-projective moduli scheme.

**Lemma 3.8.** If \( r < 3k \), \( \mathcal{N}(D,k,r,n) \) is smooth.

**Proof.** According to [19], the deformation theory of a pair \((J,\psi)\) is determined by the extension groups \( \text{Ext}_D^k(J,C(\psi)[-1]) \), \( k = 1, 2 \), where \( C(\psi) \) is the cone of \( \psi \). In order to prove smoothness it suffices to show that \( \text{Ext}_D^1(J,C(\psi)[-1]) = 0 \). Let \( T = O_p^{\oplus r} \). The long exact sequence of the exact triangle \( C(\psi)[-1] \to J \to T \) reads in part

\[
\cdots \to \text{Ext}_D^1(J,J) \to \text{Ext}_D^1(J,T) \to \text{Ext}_D^2(J,C(\psi)[-1]) \to \text{Ext}_D^2(J,J) \to \cdots
\]

Since \( J \) is a stable \( \mathcal{O}_D \)-module and \( D \simeq \mathbb{P}^2 \) is Fano, \( \text{Ext}^2(J,J) = 0 \). Therefore it suffices to prove that the natural map

\[
\psi_* : \text{Ext}_D^1(J,J) \to \text{Ext}_D^1(J,T)
\]

is surjective. Let \( I = \text{Ker}(\psi) \). Then there is an exact sequence

\[
\cdots \to \text{Ext}_D^1(J,J) \to \text{Ext}_D^1(J,I) \to \text{Ext}_D^2(J,I) \to \cdots
\]

Using Serre duality on \( D \), \( \text{Ext}_D^2(J,I) \simeq \text{Ext}_D^0(I,J \oplus D \omega_D)^\vee \). Since \( I, J \) are both slope stable on \( D \) with \( \text{ch}_1(I) = \text{ch}_1(J) = kH \), this group is trivial if \( \chi(I) > \chi(J \oplus D \omega_D) \). However

\[
\chi(I) = \chi(J) - \chi(T) = \chi(J) - r, \quad \chi(J \oplus D \omega_D) = \chi(J) - 3k.
\]

Therefore the conclusion follows if \( r < 3k \).

\[\square\]

**Remark 3.9.** Note that

\[
\text{Hom}_C(J,O_p^{\oplus r}) \simeq \text{Hom}(J \otimes_C O_p, C^r).
\]

Therefore the existence of a surjective morphism \( \psi : J \to O_p^{\oplus r} \) requires \( r \) to be smaller than the minimal number of generators of \( J \) at \( p \), \( m(J) = \dim(J \otimes_D O_p) \). However this number is bounded above by the degree \( k \) of \( C \), therefore the condition \( r < 3k \) is always satisfied.

\[\text{We thank Vivek Shende for pointing out this bound.}\]
3.4. Relation to small $b$ moduli spaces. Let $\mathcal{P}_{0+}(X, C, r, n)$ denote the moduli stack of $\mu_{(\omega, b)}$-slope stable objects of $\mathcal{A}^C$, where $b$ satisfies the bound (3.1). By analogy with [47, 46], $\mathcal{P}_{0+}(X, C, r, n)$ is an algebraic stack of finite type over $\mathbb{C}$, and all stabilizers of closed points are isomorphic to $\mathbb{C}^\times$. Recall that an object of $\mathcal{P}_{0+}(X, C, r, n)$ is a perfect complex $E_T$ on $X_T$ such that its restriction $\operatorname{Li}_t^*E_T$ is a $\mu_{(\omega, b)}$-slope stable object of the category $\mathcal{A}^C$ associated to the fiber $X_t$ for any closed point $\iota_t : \{t\} \to T$. In this subsection $b > 0$ will be a small stability parameter of type $(r, n)$ satisfying the bound (3.1).

Any flat family $(V_T, L_T, F_T, s_T, f_T)$ over $T$, determines a complex

$$E_T = (\pi^*_T L_T \xrightarrow{s_T} F_T)$$

on $X_T$. Since $F_T$ is flat over $T$, and $X_T$ is smooth projective over $T$, $E_T$ is perfect. Moreover, the derived restriction of $E_T$ to any closed fiber $X_t$ is simply obtained by restricting the terms of $E_T$ to $X_t$. It follows that the complex $\operatorname{Li}_t^*E_T$ satisfies the conditions of Proposition 3.3. Therefore this construction defines a morphism of stacks

$$f : \mathcal{Q}(X, C, r, n) \to \mathcal{P}_{0+}(X, C, r, n).$$

**Proposition 3.10.** The morphism $f$ is geometrically bijective i.e. it yields an equivalence

$$f(\mathbb{C}) : \mathcal{Q}(X, C, r, n)(\mathbb{C}) \sim \mathcal{P}_{0+}(X, C, r, n)(\mathbb{C}).$$

of groupoids of $\mathbb{C}$-valued points.

**Proof.** Proposition 3.3 implies that any object of $\mathcal{P}_{0+}(X, C, r, n)(\mathbb{C})$ is quasi-isomorphic to an object in the image of $f(\mathbb{C})$. One has to prove that if two data $(V, L, F, s, f)$ and $(V', L', F', s', f')$ are mapped to quasi-isomorphic complexes $E, E'$ then they must be isomorphic. This can be proven by analogy with [42, Prop. 1.21]. Given an object $E$ of $\mathcal{A}^C$ satisfying the conditions of Proposition 3.3 there is an exact triangle

$$\mathcal{O}_X \xrightarrow{s} F \to E$$

in $D^b(X)$ where $F$ is a nondegenerate extension

$$(3.19) \quad 0 \to G \to F \to V \otimes \mathcal{O}_{C_0}(-1) \to 0.$$ 

This yields a long exact sequence

$$\cdots \to \operatorname{Hom}(F, \mathcal{O}_X) \to \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Hom}(E, \mathcal{O}_X[1]) \to \operatorname{Hom}(F, \mathcal{O}_X[1]) \to \cdots$$

The first term is obviously trivial since $F$ is torsion and Serre duality implies that

$$\operatorname{Hom}(F, \mathcal{O}_X[1]) \simeq H^2(F)^\vee = 0$$

since $F$ is supported in dimension at most one. Therefore

$$\operatorname{Hom}(E, \mathcal{O}_X[1]) \simeq \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathbb{C},$$

which implies that there is a unique morphism $E \to \mathcal{O}_X[1]$ up to multiplication by nonzero complex numbers. Then $F$ is quasi-isomorphic to the mapping cone of the morphism $E \to \mathcal{O}_X[1]$, and the section $s$ is recovered from the induced map $\mathcal{O}_X \to F$ as in [42, Prop. 1.21]. In order to finish the proof, note that given two extensions $F, F'$ of the form (3.19), an isomorphism of sheaves $F \sim \to F'$ induces isomorphisms $G \sim \to G'$, respectively $V \sim \to V'$ using the snake lemma. Therefore the data $(V, L, F, s, f)$ can be recovered up to isomorphism from the complex $E$. 

$\square$
4. Motivic invariants at small $b$

Composing the morphism $f : Q(X,C,r,n) \to P_{0+}(X,C,r,n)$ with the natural morphism $p : P_{0+}(X,C,r,n) \to Ob(A)$ one obtains a stack function
$$q : Q(X,C,r,n) \to Ob(A)$$
of the motivic Hall algebra $H(A)$. The motivic Donaldson-Thomas theory of [23] assigns to any stack function an invariant with values in a certain ring of motives, as reviewed below. The goal of this section is to compare the resulting motivic invariants with the refined Hilbert scheme invariants introduced in Section 1.1.1. It will be shown that complete agreement follows from a conjectural comparison formula between the motivic weights of moduli stacks of stable pairs and sheaves. This is a natural motivic counterpart of previous results for numerical invariants [43], which will be proven here only for sheaves of sufficiently high degree. The general case is an open conjecture. The required elements in the construction of motivic Donaldson-Thomas invariants after [23] will be assumed without proof. More specifically, it will be assumed that the derived category $D^b(X)$ is endowed with orientation data as defined in [23] and the integral identity conjectured in [23, Conj. 4, Sect. 4.4] holds, such that there is a well defined motivic integration map as in [23 Thm. 8, Sect 6.3]. Moreover, explicit computations will be carried out in sections 4.3, 4.4 by reduction to a triangulated subcategory of quiver representations. In that context it will be assumed that the orientation data on the ambient category $D^b(X)$ agrees with orientation data on the derived category of quiver representations constructed in [24, 8].

4.1. Review of motivic Donaldson-Thomas invariants.

Recall that if $X$ is a compact complex Calabi-Yau 3-fold then the derived category of coherent sheaves $D^b(X)$ carries a structure of 3-dimensional Calabi-Yau category (3CY category for short), see [23] for details. In particular we endow it with a cyclic $A_\infty$-structure, for example by fixing a Calabi-Yau metric on $X$. Then according to [23, Sect 3] there is a formal potential function $W_E$ on the vector space $\text{Hom}^1(E,E)$ for any object $E$ of $D^b(X)$. Replacing the category by its minimal model we can treat $W_E$ as a formal function on $\text{Ext}^1(E,E)$. Moreover, explicit computations will be carried out in sections 4.3, 4.4 by reduction to a triangulated subcategory of quiver representations. In that context it will be assumed that the orientation data on the ambient category $D^b(X)$ agrees with orientation data on the derived category of quiver representations constructed in [24, 8].
Furthermore, suppose that the category \( D^b(X) \) is endowed with orientation data and a polarization such that the construction of motivic Donaldson-Thomas series in \([23]\) Sect 6] applies. In particular, to each object \( E \) of \( \mathcal{A} \) one assigns a motivic weight

\[
(4.1) \quad w_E = \mathbb{L}^{(E,E)_{\leq 1}/2}(1 - MF_0(W_E))L^{-\text{rk}(Q_E)/2}.
\]

Following the conventions of \([23]\), given any two objects \( E_1, E_2 \), set

\[
(E_1, E_2)_j = \dim(\text{Ext}^j(E_1, E_2)), \quad (E_1, E_2)_{\leq j} = \sum_{i \leq j} (-1)^i \dim(\text{Ext}^i(E_1, E_2))
\]

for any \( j \in \mathbb{Z} \).

Since \( \text{Ob}(\mathcal{A}) \subset \text{Ob}(D^b(X)) \) we can treat constructible families over \( \text{Ob}(\mathcal{A}) \) as constructible families over \( \text{Ob}(D^b(X)) \). This gives a homomorphism at the level of stack functions and motivic Hall algebras (since \( \mathcal{A} \) is a heart of t-structure there are no negative \( \text{Ext}^i \) between its objects). As a result, we can apply the formalism of \([23]\) to the category of perverse coherent sheaves. The motivic invariant for a stack function \([X \to \text{Ob}(\mathcal{A})]\) is defined by integration of motivic weights, which is defined in \([23]\) Sect 4.4. The result is encoded in the morphism \( \Phi \) constructed in \([23]\) Thm. 8, Sect 6.3] from the motivic Hall algebra \( H(\mathcal{A}) \) to the quantum torus. Let \( \Gamma \) denotes the intersection of the image of the Chern character \( ch : K_0(D^b(X)) \to H^{ev}(X, \mathbb{Q}) \) with \( H^{ev}(X, \mathbb{Z}) \) (instead of \( \Gamma \) one can take the quotient of \( K_0(\mathcal{A}) \) by the subgroup generated by the numerical equivalence). In particular the lattice \( \Gamma \) is equipped with a natural nondegenerate antisymmetric pairing \( \langle , \rangle \). The quantum torus is the associative algebra \( \mathcal{R} \) over an appropriate motivic ring described in \([23]\) spanned by the symbols \( \hat{e}_\gamma, \gamma \in \Gamma \) over the ring of motivic weights, where

\[
\hat{e}_{\gamma_1} \hat{e}_{\gamma_2} = \mathbb{L}^{\langle \gamma_1, \gamma_2 \rangle/2} \hat{e}_{\gamma_1 + \gamma_2}.
\]

Here \( \mathbb{L} \) denotes the motive of the affine line.

Then the integration map \( \Phi : H(\mathcal{A}) \to \mathcal{R} \) assigns to a stack function \([Y \to \text{Ob}(\mathcal{A})]\) which factors through the substack \( \text{Ob}_\gamma(\mathcal{A}) \), the element

\[
\int_Y w_{\pi(y)} \hat{e}_\gamma.
\]

4.2. Motivic weights at small \( b \). The next goal is to evaluate the integration map \( \Phi \) on the stack function

\[
q = p \circ f : \mathcal{Q}(X, C, r, n) \to \text{Ob}(\mathcal{A})
\]

determined by Proposition 3.10. As observed in Remark 3.4 there is a natural forgetful morphism \( \pi : \mathcal{Q}(X, C, r, n) \to \mathcal{M}(X, C, r, n) \) to the stack of nondegenerate extensions \( (V,F,f) \). The fiber of \( \pi \) over a closed point \( (V,F,f) \) is isomorphic to the projective space \( \mathbb{P}H^0(F) \). Note that there is also a natural obvious morphism \( m : \mathcal{M}(X, C, r, n) \to \text{Ob}(\mathcal{A}) \) sending the sheaf \( F \) to itself. Then the integration of motivic weights may be carried out in two stages, first along the fibers of \( \pi \), and then then on \( \mathcal{M}(X, C, r, n) \). The first step will be considered below, while the second one will be postponed for Section 4.3.

Note that there is a one-to-one correspondence between nonzero sections in \( H^0(F) \) and nontrivial extensions

\[
0 \to F \to E \to \mathcal{O}_X[1] \to 0
\]
Therefore the Ext for all \((4.3)\). Moreover, note that for all \(k\), if 
\[\text{Condition } (c)\] is satisfied because the automorphism group of a nondegenerate extension \(F\) is \(\mathbb{C}^*\). Condition \((b)\) is satisfied since 
\[\text{Ext}^0(F, O_X[1]) \simeq \text{Ext}^2(O_X, F) \simeq H^2(F) = 0.\]
Condition \((c)\) is also satisfied since 
\[\text{Ext}^{-1}(F, O_X[1]) = \text{Ext}^0(F, O_X) = 0.\]

Moreover, note that 
\[\text{Ext}^k(O_X, O_{\mathcal{C}_0}(-1)) \simeq H^k(O_{\mathcal{C}_0}(-1)) = 0\]
for all \(k \in \mathbb{Z}\), hence 
\[\text{Ext}^k(O_X, F) \simeq \text{Ext}^k(O_X, G),\]
for all \(k \in \mathbb{Z}\). Using Serre duality, 
\[\text{Ext}^1(G, O_X[1]) \simeq \text{Ext}^1(O_X, G) \simeq H^1(G) = 0.\]
Therefore the Ext quiver of the collection of objects \(\{E_1, E_2\}\) is of the form

\[\begin{align*}
E_1 & \rightarrow G \\
E_2 & \rightarrow V \otimes O_{\mathcal{C}_0}(-1) \\
E_0 & \rightarrow W_{E_0} = p^*W_F.
\end{align*}\]
where \( n = \dim(\text{Ext}^1(E_2, E_1)) \approx H^0(G) \), and \( d = \dim(\text{Ext}^1(E_1, E_1)) \). Note that there are no left directed arrows, hence all polynomial invariants of any quiver representation are determined by paths of the form \( b_1b_2 \cdots b_j \). Then [23 Thm. 9, Sect. 8] implies that \( W_{E_0} = p^*W_F \).

Since \( E_2 = \mathcal{O}_X[1] \), it follows that Lemma [1.1] yields
\[
w_{E_0} = \mathbb{L}[\sum_{(E_0, E_1) \leq 1}(E_1, E_1) \leq 1/\mathbb{L}] w_F
\]
when the conditions of Lemma [1.3] are satisfied. If this is the case, equation (4.2) yields
\[
\int_{\alpha \in \text{Ext}^1(E_2, E_1)} w_{E_0} - w_{E_0} = \mathbb{L}^{(1-n)/2}(\mathbb{L}^n - 1) w_F,
\]
where \( n = h^0(F) = \chi(F) \). In particular this holds for all sheaves \( F \) for sufficiently large \( n \). By analogy with [43 Thm. 4] it is natural to conjecture that the following holds for general \( n \in \mathbb{Z} \)
\[
\int_{\alpha \in \text{Ext}^1(E_2, E_1)} w_{E_0} - w_{E_0} = \mathbb{L}^{(1-n)/2}(\mathbb{L}^{h^0(F)} - 1) w_F.
\]

Using local toric models, the motivic weights \( w_F \) will be represented below as of motivic Milnor fibers of polynomial Chern-Simons functions.

4.3. Local toric models. The upper \( \pm \) labels making the distinction between the two small resolutions of the nodal threefold \( X_0 \) will be restored in this section. A straightforward local computation shows that the infinitesimal neighborhood of the union \( D^+ \cup C_0^+ \) equipped with the reduced scheme structure is isomorphic to the infinitesimal neighborhood of an identical configuration in a toric Calabi-Yau threefold. This is in fact easier to see starting with with the small crepant resolution \( X^- \), which is a smooth elliptic fibration with canonical section over the Hirzebruch surface \( \mathbb{F}_1 \). The exceptional curve \( C_0^- \) is contained in the section \( D^- \), which is identified with \( \mathbb{F}_1 \). Then the infinitesimal neighborhood of \( D^- \) in \( X^- \) is isomorphic to the total space \( Z^- \) of the canonical bundle \( K_{\mathbb{F}_1} \). Moreover \( D^- \) is identified with the zero section and \( C_0^- \) is identified with the unique \( (-1) \)-curve on \( Z^- \). Then one can construct a second smooth toric Calabi-Yau threefold \( Z^+ \) related to \( Z^- \) by a toric flop along the curve \( C_0^+ \) as shown in detail below. This threefold contains a compact divisor \( D^+ \simeq \mathbb{P}^2 \) and an exceptional \((-1, -1)\) curve \( C_0^+ \) intersecting \( D^+ \) transversely at a point \( p \).

The toric presentation of both \( Z^\pm \) is of the form
\[
\begin{array}{cccc}
x_1 & x_2 & x_3 & x_4 & x_5 \\
1 & 0 & 1 & 1 & -3 \\
0 & 1 & 0 & 1 & -2.
\end{array}
\]
The disallowed locus is \( \{x_1 = x_3 = 0\} \cup \{x_2 = x_4 = 0\} \) for \( Z^- \) and \( \{x_1 = x_3 = x_4\} \cup \{x_2 = x_4 = 0\} \) for \( Z^+ \). The toric fans \( \nabla^\pm \) of \( Z^\pm \) are generated by the vectors
\[
v_1 = (1, 0, 1), \quad v_2 = (1, 1, 1), \quad v_3 = (0, 1, 1), \quad v_4 = (-1, -1, 1), \quad v_5 = (0, 0, 1)
\]
in \( \mathbb{R}^3 \). In each case the fan is a cone over a two dimensional polytope embedded in the plane \( z = 1 \) in \( \mathbb{R}^3 \). The toric flop relating \( Z^- \) and \( Z^+ \) corresponds to a change of triangulation of the two dimensional polytopes, as shown in Fig. [4].
The canonical toric divisors $x_i = 0$ are denoted by $D^\pm_i$, $i = 1, \ldots, 5$. They are in one-to-one correspondence with the rays of the toric fans as shown in Fig. 1. Note that $D^\pm_1 = D^\pm_5$ are the only compact divisors on $Z^\pm$.

The derived categories of $Z^\pm$ are equivalent and are generated by line bundles. A collection of line bundles generating $D^b(Z^-)$ is obtained by pulling back an exceptional collection on the Hirzebruch surface $F_1$ of the form

$$\mathcal{O}_{F_1}, \quad \mathcal{O}_{F_1}(C^-_0), \quad \mathcal{O}_{F_1}(H), \quad \mathcal{O}_{F_1}(2H).$$

Here $C^-_0$ denotes the exceptional curve on $F_1$ and $H$ the hyperplane class. Note that the resulting line bundles on $Z^-$ are isomorphic to the toric line bundles

$$\mathcal{O}_{Z^-}, \quad \mathcal{O}_{Z^-}(D^-_2), \quad \mathcal{O}_{Z^-}(D^-_4), \quad \mathcal{O}_{Z^-}(2D^-_4).$$

The direct sum $\mathcal{T}^-$ of all above line bundles is a tilting object, and the derived category of $Z^-$ is equivalent to the derived category of modules over the algebra $R\text{End}_{Z^-}(\mathcal{T}^-)^{\text{op}}$. The equivalence is given by the derived functor $R\text{Hom}_{Z^-}(\mathcal{T}^-, \bullet)$. As a result the derived category of $Z^-$ is equivalent to the 3CY category which is a Calabi-Yau category associated with the abelian category $(Q, W)^{-\text{mod}}$ of finite-dimensional representations of the following quiver $Q$

\begin{align}
\begin{array}{ccccc}
& & a_2 & & \\
& & \downarrow & & \\
& & a_3 & & \\
& & \downarrow & & \\
& & a_1 & & \\
\end{array} & \begin{array}{ccccc}
& & b_2 & & \\
& & \downarrow & & \\
& & b_3 & & \\
& & \downarrow & & \\
& & b_1 & & \\
\end{array} & \begin{array}{ccc}
& & c \\
& & \downarrow \\
& & r \\
& & \downarrow \\
& & s_1 \\
& & \downarrow \\
& & s_2 \\
\end{array}
\end{align}

with potential

\begin{align}
W = r(b_1a_2 - b_2a_1) + s_1(cb_1a_3 - b_3a_1) + s_2(cb_2a_3 - b_3a_2).
\end{align}

Recall that this category can be described as the category of finite-dimensional representations of the Jacobi algebra $CQ/(\partial W)$, the quotient of the path algebra of $Q$ by the ideal generated by cyclic derivatives of $W$. 

![Figure 1. Local toric models related by a flop. The polytope on the left is the $z = 1$ section of the toric fan of the local $F_1$ model. The polytope on the right is a similar section of the toric fan of the local $\mathbb{P}^2 \cup \mathbb{P}^1$ model. The two models are related by a toric flop corresponding to the obvious change of triangulation.](image)
For future reference note that the line bundles
\[ \mathcal{O}_{D^-}, \quad \mathcal{O}_{D^-}(D_2^+), \quad \mathcal{O}_{D^-}(D_3^-), \quad \mathcal{O}_{D^-}(2D_4^-) \]
form an exceptional collection \( \mathcal{T}_{D^-} \) on the Hirzebruch surface \( D^- \simeq \mathbb{F}_1 \). The functor \( R\text{Hom}(\mathcal{T}_{D^-}, \bullet) \) yields an equivalence of the derived category \( D^b(D^-) \) to the derived category of the abelian category \((Q_0, S) - \text{mod}\) of the finite-dimensional representations of following quiver \( Q_0 \)

\[
\begin{array}{c}
1 \\
1 \\
3 \\
1 \\
0
\end{array}
\]

with relations

\[
S : \quad b_1a_2 - b_2a_1, \quad cb_1a_3 - b_3a_4, \quad cb_2a_3 - b_3a_2.
\]

The abelian category \((Q_0, S) - \text{mod}\) has homological dimension 2, and there is an obvious injective fully faithful exact functor of abelian categories

\[
\iota : (Q_0, S) - \text{mod} \longrightarrow (Q, W) - \text{mod}.
\]

For simplicity, extension groups in the two categories will be denoted by \( \text{Ext}^\bullet_{(Q_0, S)}, \text{Ext}^\bullet_{(Q, W)} \) respectively. It will be useful to note that the following relations hold:

\[
\begin{align*}
\text{Ext}^0_{(Q, W)}(\iota \rho_1, \iota \rho_2) &\simeq \text{Ext}^0_{(Q_0, S)}(\rho_1, \rho_2) \\
\text{Ext}^k_{(Q, W)}(\iota \rho_1, \iota \rho_2) &\simeq \text{Ext}^k_{(Q_0, S)}(\rho_1, \rho_2) \oplus \text{Ext}^{3-k}_{(Q_0, S)}(\rho_2, \rho_1)^\vee, \quad k = 1, 2.
\end{align*}
\]

Using the results of [14], the direct sum \( \mathcal{T}^+ \) of the following collection of line bundles

\[
\mathcal{L}_1 = \mathcal{O}_{Z^+}(2D_4^-), \quad \mathcal{L}_2 = \mathcal{O}_{Z^+}(D_2^+), \quad \mathcal{L}_3 = \mathcal{O}_{Z^+}(D_3^+), \quad \mathcal{L}_4 = \mathcal{O}_{Z^+}
\]
is a tilting object in the derived category of \( Z^+ \). Therefore it yields a similar equivalence of \( D^b(Z^+) \) to the derived category of the same quiver with potential.

The next step is to compute the image of dimension one sheaves on \( Z^+ \) via the tilting functor. First note the following result which follows from [18, Lemm 9.1].

**Lemma 4.2.** Let \( G \) be a rank one torsion free sheaf on a degree \( k \in \mathbb{Z}_{>0} \) reduced irreducible divisor on \( D^+ \simeq \mathbb{F}^2 \) with \( H^0(G) = 0 \). Then the complex \( R\text{Hom}(\mathcal{T}^+, G)[1] \) is quasi-isomorphic to a quiver representation \( \rho_G \) of dimension vector

\[
v_G = (2k - \chi(G), k - \chi(G), -\chi(G), -\chi(G)).
\]

which belongs to the subcategory of \((Q_0, S)\)-modules. Moreover

\[
\text{Ext}^2_{(Q_0, S)}(G, G) = 0,
\]
and \( \rho_G(c) \) is an isomorphism if \( \chi(G) \neq 0 \).

**Proof.** Note that the open subset \( U = \{x_2 \neq 0\} \subset Z^+ \) is isomorphic to the total space of the normal bundle \( N_{D^+/Z^+} \simeq \omega_{\mathbb{F}^2} \). This follows observing that \( U \) is isomorphic to a toric variety determined by the toric data

\[
\begin{pmatrix}
x'_1 & x'_3 & x'_4 & x'_5 \\
1 & 1 & 1 & -3
\end{pmatrix}
\]
where
\[ x'_1 = x_1, \quad x'_2 = x_3, \quad x'_4 = x_2^{-1}x_4, \quad x'_5 = x_2^{-2}x_5 \]
and the disallowed locus is \( \{ x'_1 = x'_3 = x'_4 = 0 \} \).

Denote the open immersion \( U \hookrightarrow \mathbb{P}^2 \) by \( j \) and the close immersions of \( D^+ \) into \( \mathbb{P}^2 \) and \( U \) by \( i \) and \( i' \) respectively. Clearly, \( i = j \circ i' \). Denote the tilting bundle on \( \mathbb{P}^2 \) by \( T^+ \). Given a sheaf on \( \mathbb{P}^2 \) of the form \( i_* G \), there is an isomorphism
\[
\text{RHom}_{\mathcal{O}_Z}(T, i_* G) \simeq \text{RHom}_{\mathcal{O}_{\mathbb{P}^2}}(j^* T, i'_* G).
\]

By adjunction, this is further isomorphic to \( \text{RHom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}(\mathcal{I}(1)) \oplus \mathcal{O}(\mathcal{I}(2)), G) \). By the derived Morita equivalence, this induces an equivalence between \( D^h(\mathbb{P}^2) \) and the derived category of the abelian category \( \mathcal{A} \) consisting of representations \( \rho \) of the directed quiver \( Q_0 \) with dimension vectors \( (v_1, v_2, v_3, v_4) \) and \( \rho(c) \) an invertible linear map.

Since \( \mathcal{A} \) is a fully faithful subcategory of \( (Q_0, S) - \text{mod} \), we have
\[
\text{Ext}^2_{(Q_0, S)}(\rho_G, \rho_G) = \text{Ext}^2_{\mathcal{A}}(\rho_G, \rho_G) = \text{Ext}^2_{\mathcal{O}_{\mathbb{P}^2}}(G, G) = 0
\]
when \( G \) is stable.

The next goal is to compute the image of nondegenerate extensions
\[
0 \to G \to F \to V \otimes \mathcal{O}_{C^+_0}(-1) \to 0
\]
via the tilting functor. In order to obtain a single quiver representation as a opposed to a complex thereof, \( F \) must be twisted by a suitable line bundle \( L \) prior to tilting. There are several possible results depending on the choice of \( L \). The one recorded below turns out to be most effective for the computation of motivic weights.

As shown in the proof of Lemma 3.6 taking derived duals on \( X \) sends a nondegenerate extension as above to an extension of the form
\[
0 \to V' \otimes \mathcal{O}_{C^+_0}(-1) \to \text{Ext}^2_{\mathcal{O}_{Z^+}}(F, \mathcal{O}_{Z^+}) \to J \otimes \mathcal{O}_C \omega_C \to 0
\]
where \( J = \text{RHom}_C(G, \mathcal{O}_C) \) is an ideal sheaf on \( C \). The dualizing sheaf of \( C \) is \( \omega_C \simeq \mathcal{O}_Z((k-3)D^+_4)|_C \). Let \( W = V' \). The dual extension is also subject to a nondegeneracy condition. Namely the corresponding extension class \( e \in \text{Ext}^1(J \otimes \mathcal{O}_C \omega_C, W \otimes \mathcal{O}_{C^+_0}(-1)) \) is not in the kernel of the map
\[
\text{Ext}^1(J \otimes \mathcal{O}_C \omega_C, W \otimes \mathcal{O}_{C^+_0}(-1)) \to \text{Ext}^1(J \otimes \mathcal{O}_C \omega_C, W' \otimes \mathcal{O}_{C^+_0}(-1))
\]
for any nontrivial quotient \( W \to W' \). The tilting functor will be applied to the twist \( F' = \mathcal{E}xt^2_{\mathcal{O}_{Z^+}}(F, \mathcal{O}_{Z^+}) \otimes \mathcal{O}_{Z^+}((2-k)D^+_4) \) which fits in an extension
\[
0 \to W \otimes \mathcal{O}_{C^+_0}(-1) \to F' \to J(-D^+_4) \to 0.
\]

Then the following holds:

**Lemma 4.3.** Consider a nondegenerate extension
\[
0 \to W \otimes \mathcal{O}_{C^+_0}(-1) \to F' \to J' \to 0
\]
where \( J' = J(-D^+_4) \) for an ideal sheaf \( J \) on a degree \( k \in \mathbb{Z}_{>0} \) reduced irreducible divisor \( C^+ \) on \( D^+ \simeq \mathbb{P}^2 \). Then \( \text{RHom}_{\mathcal{O}_{Z^+}}(T^+, F')[1] \) is quasi-isomorphic to a quiver representation \( \rho_F \) which fits in an extension
\[
0 \to W \otimes \rho_3 \to \rho_F \to \rho_{J'} \to 0,
\]
where \( J' = J(-D^+_4) \) for an ideal sheaf \( J \) on a degree \( k \in \mathbb{Z}_{>0} \) reduced irreducible divisor \( C^+ \) on \( D^+ \simeq \mathbb{P}^2 \). Then \( \text{RHom}_{\mathcal{O}_{Z^+}}(T^+, F')[1] \) is quasi-isomorphic to a quiver representation \( \rho_F \) which fits in an extension
\[
0 \to W \otimes \rho_3 \to \rho_F \to \rho_{J'} \to 0.
\]
In addition, \( \rho_F \) belongs to the subcategory of \( (Q_0, S) \)-modules, and
\[
\text{Ext}^2_{(Q_0, S)}(\rho_F, \rho_F) = 0.
\]

Proof. Observe that \( O_{C^0_0}(-1) \) is mapped to the simple module \( \rho_3[-1] \) corresponding to the third vertex of the quiver \( Q \). According to Lemma 4.2, the twisted derived dual \( J' \) of \( G \) will be mapped to a representation \( \rho_{J'}[-1] \) of \( Q_0 \) since \( H^0(J') = 0 \). Moreover the linear map \( \rho_{J'}(c) \) is invertible. Then we claim
\[
\text{Ext}^k_{(Q_0, S)}(\rho_3, \rho_3) = 0
\]
for all \( k \in \mathbb{Z} \). Suppose \( \rho_{J'} \) has dimension vector \( d_1, \ldots, d_4 \), recall that \( \rho_{J'} \) corresponds to a Maurer-Cartan element \( x \) of the \( L_\infty \) algebra \( \text{Ext}^*_{(Q_0, S)}(\oplus \rho_i \otimes V_i, \oplus \rho_i \otimes V_i) \), where dimension of \( V_i \) equals \( d_i \). The extension space \( \text{Ext}^0_{(Q_0, S)}(\rho_3, \oplus \rho_i \otimes V_i) \) is an \( L_\infty \) module over \( \text{Ext}^*_{(Q_0, S)}(\oplus \rho_i \otimes V_i, \oplus \rho_i \otimes V_i) \). The MC element \( x \) defines a differential \( \delta^x \) on \( \text{Ext}^0_{(Q_0, S)}(\rho_3, \oplus \rho_i \otimes V_i) \) such that the cohomology groups compute \( \text{Ext}^*_{(Q_0, S)}(\rho_3, \rho_{J'}) \). The complex \( \text{Ext}^*_{(Q_0, S)}(\rho_3, \oplus \rho_i \otimes V_i) \) has the form
\[
0 \longrightarrow \text{Hom}(\mathbb{C}, V_3) \overset{\delta^x}{\longrightarrow} \text{Hom}(\mathbb{C}, V_4) \longrightarrow 0
\]
Since the linear map \( \rho_{J'}(c) \) is invertible, this complex is acyclic. For future reference, note that a similar argument proves that
\[
\text{Ext}^k_{(Q_0, S)}(\rho_3, \rho_3) = 0
\]
for all \( k \in \mathbb{Z} \setminus \{0\} \).

According to relations (4.9) the extension group \( \text{Ext}^1_{(Q_0, S)}(\rho_{J'}, \rho_3) \) decomposes into \( \text{Ext}^1_{(Q_0, S)}(\rho_{J'}, \rho_3) \oplus \text{Ext}^2_{(Q_0, S)}(\rho_3, \rho_{J'})^* \). Since we have just proved the second summand vanishes, it follows any extension of the form (4.11) must be mapped by tilting to a representation \( \rho_F \) of \( (Q_0, S) \).

Since \( \rho_F \) is an extension of \( \rho_{J'} \) by \( W \otimes \rho_3 \), the extension group \( \text{Ext}^2_{(Q_0, S)}(\rho_F, \rho_F) \) is computed by the complex \( \text{Ext}^*(\rho_{J'} \oplus \rho_3, \rho_{J'} \oplus W \otimes \rho_3) \) with the differential \( \delta^x \) where \( x \) is the MC element corresponding to the extension class in \( \text{Ext}^0_{(Q_0, S)}(\rho_{J'}, W \otimes \rho_3) \). The vanishing results (4.14), (4.15) imply that \( \text{Ext}^2_{(Q_0, S)}(\rho_F, \rho_F) \) is isomorphic to the cokernel of the map
\[
\text{Ext}^1_{(Q_0, S)}(\rho_{J'}, \rho_3) \oplus \text{Ext}^1_{(Q_0, S)}(\rho_{J'}, W \otimes \rho_3) \overset{\delta^x}{\longrightarrow} \text{Ext}^2_{(Q_0, S)}(\rho_{J'}, W \otimes \rho_3)
\]
Because \( x \in \text{Ext}^1_{(Q_0, S)}(\rho_{J'}, W \otimes \rho_3) \), the above morphism simplifies to
\[
\text{Ext}^1_{(Q_0, S)}(\rho_{J'}, \rho_{J'}) \overset{\delta^x}{\longrightarrow} \text{Ext}^2_{(Q_0, S)}(\rho_{J'}, W \otimes \rho_3)
\]
Vanishing of \( \text{Ext}^2_{(Q_0, S)}(\rho_F, \rho_F) \) is equivalent with the above morphism being surjective. Furthermore, relations (4.9) and the vanishing results (4.14) imply that
\[
\text{Ext}^1_{(Q_0, S)}(\rho_{J'}, \rho_{J'}) = \text{Ext}^1_{(Q_0, S)}(\rho_{J'}, 1),
\]
\[
\text{Ext}^2_{(Q_0, S)}(\rho_{J'}, W \otimes \rho_3) = \text{Ext}^2_{(Q_0, S)}(\rho_{J'}, W \otimes \rho_3)
\]
Then derived equivalence with \( D^b(Z^+) \) maps the morphism (4.16) to the connecting morphism
\[
\text{Ext}^1_{Z^+}(J', J') \overset{\delta}{\longrightarrow} \text{Ext}^2_{Z^+}(J', W \otimes O_{C^0_0}(-1))
\]
In order to show that $\delta$ is a surjection recall that according to Lemma 2.6 there are isomorphisms

$$\varphi_k : \text{Ext}^k_{Z^+}(J', W \otimes O_{C^+_0}(-1)) \rightarrow \text{Ext}^{k-1}_{D^+(J, W \otimes O_p)}.$$ 

Moreover Corollary 2.7 shows that an extension $e \in \text{Ext}^1_{Z^+}(J', W \otimes O_{C^+}(-1))$ is nondegenerate if and only if the corresponding morphism $\varphi_1(e)$ is surjective. In particular this holds for the extension class $e^{\pm}$ corresponding to the Maurer-Cartan element $x$. Let $\psi = \varphi_1(e^{\pm})$ and $\psi_* : \text{Ext}^1_{D^+}(J', J') \rightarrow \text{Ext}^1_{Z^+}(J', W \otimes O_p)$ the natural induced morphism of extensions. Clearly the following diagram commutes.

$$
\begin{array}{ccc}
\text{Ext}^1_{Z^+}(J', J') & \xrightarrow{\delta} & \text{Ext}^2_{Z^+}(J', W \otimes O_{C^+_0}(-1)) \\
\downarrow & & \downarrow \\
\text{Ext}^1_{D^+}(J', J') & \xrightarrow{\psi} & \text{Ext}^1_{D^+}(J', W \otimes O_p)
\end{array}
$$

Since $\psi$ is surjective, surjectivity of $\psi_*$ follows from Lemma 3.8 and Remark 3.9.

4.4. Motivic weights in local model. Next it will be shown that Lemma 4.3 yields a presentation of the motivic weights $w_F$ as motivic Milnor fibers of polynomial functions. Note that the quiver $Q$ in (4.10) is the Ext$^1$ quiver associated to a pair of spherical objects $S_i$, $i = 1, \ldots, 4$ in the derived category $D^b(Z^+)$. Moreover the objects $S_i$, $i = 1, \ldots, 4$ generate the subcategory of complexes with topological support on $D^+ \cup C^+_0$. The images of this objects via the tilting functor generate the subcategory of complexes of quiver representations with nilpotent cohomology. In particular the representation $\rho_F$ corresponding to a sheaf $F$ as in Lemma 4.3 is obtained by successive extensions of the $S_i$, $i = 1, \ldots, 4$.

For a dimension vector $v = (v_i)_{1 \leq i \leq 4}$, let $A(v)$ denote the affine space parameterizing all representations of the quiver $Q$ without relations. Note that there is an obvious direct sum decomposition

$$A(v) = A^*(v) \oplus A^!(v)$$

where $A^*(v)$, $A^!(v)$ denote the linear subspaces associated to the right directed, and left directed arrows respectively in diagram (4.10). There is also a natural $GL(v)$ action on $A(v)$.

The potential (4.7) determines a $G(v)$-invariant quartic polynomial function $W$ on $A(v)$ such that quiver representations of dimension vector $v = (v_i)_{1 \leq i \leq 4}$ are in one-to-one correspondence with closed points in the critical locus $\text{Crit}(W)$.

Let $\rho_F \in A(v)$ be a closed point corresponding to a sheaf $F$ satisfying the conditions of Lemma 4.3. Let $W_{\rho_F}$ be the Taylor series expansion of $W$ at $\rho_F$. Since $\rho_F$ is an iterated extension of the spherical objects $S_i$, $i = 1, \ldots, 4$, the computation of $w_{\rho_F} = w_F$ will be carried out in close analogy with the proof of [23, Thm. 8, Sect. 6.3].

Suppose $E_1, E_2$ are any two objects in derived category of quiver representations with nilpotent cohomology. Let $E_0 = E_1 \oplus E_2$. Suppose moreover that the potential function $W_{E_0}$ on

$$\text{Ext}^1(E_0, E_0) = \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_1) \oplus \text{Ext}^1(E_1, E_2) \oplus \text{Ext}^1(E_2, E_2)$$
is minimal i.e. has no quadratic part. Let \( \alpha \in \text{Hom}(E_2[-1], E_1) \) be a nontrivial element, and let \( E_\alpha = \text{Cone}(\alpha) \). As in \textit{Step 3} in the proof of [23, Thm. 8, Sect. 6.3], let \( W_{(0,\alpha,0,0)} \) denote the Taylor expansion of \( W_{E_\alpha} \) at the point \((0,\alpha,0,0)\) in \( \text{Ext}^1(E_0, E_\alpha) \). Then \( W_{(0,\alpha,0,0)} \) is related by a formal change of variables to a direct sum of the form
\[
W_{E_\alpha}^{\text{min}} \oplus \tilde{Q}_{E_\alpha} \oplus \tilde{N}_{E_\alpha}
\]
where \( \tilde{Q}_{E_\alpha} \) is a nondegenerate quadratic form and \( \tilde{N}_{E_\alpha} \) the zero function on a linear subspace. This implies that there is an identity
\[
(1 - MF_0(W_{(0,\alpha,0,0)})) = (1 - MF_0(W_{E_\alpha}^{\text{min}}))(1 - MF_0(\tilde{Q}_{E_\alpha})).
\]
Note that \( \tilde{Q}_{E_\alpha} \) is not the same as the intrinsic quadratic form \( Q_{E_\alpha} \). In fact the discrepancy between these two forms leads to the need to introduce orientation data in order to obtain a well defined integration map.

Two identities for the quadratic form \( \tilde{Q}_{E_\alpha} \) follow from the proof of [23, Thm. 8, Sect 6.3]. First, the rank of \( \tilde{Q}_{E_\alpha} \) is expressed in terms of dimensions of \( \text{Ext} \) groups as follows
\[
\text{rk}(\tilde{Q}_{E_\alpha}) = (E_\alpha, E_\alpha)_{\leq 1} - (E_0, E_0)_{\leq 1}.
\]
Next, there is a cocycle identity for motivic Milnor fibers above [23, Def. 18, Sect 6.3] which reads
\[
\mathbb{L}^{-\text{rk}(\tilde{Q}_{E_\alpha})/2} (1 - MF_0(\tilde{Q}_{E_\alpha})) = \prod_{i=1}^{2} \mathbb{L}^{-\text{rk}(\tilde{Q}_{E_\alpha})/2} (1 - MF_0(Q_{E_i})).
\]
In the present case \( E_0 \) is a direct sum of simple objects
\[
E_0 = \bigoplus_{i=1}^{4} S_i^{v_F(i)}
\]
where \( v_F = (v_F(i))_{1 \leq i \leq 4} \) is the dimension vector of the extension \( \rho_F \) of \( \text{Lemma 4.2} \).
\[
v_H = ((N + 2)k - n, (N + 1)k - n, Nk - n + r, Nk - n).
\]
Then equation (4.17) yields
\[
1 - MF_0(W_{\rho_F}) = (1 - MF_0(W_{\rho_F}^{\text{min}}))(1 - MF_0(\tilde{Q}_{\rho_F})).
\]
where \( \tilde{Q}_{\rho_F} \) is a quadratic form which satisfies two identities analogous to (4.18), (4.19). Therefore the rank of \( \tilde{Q}_{\rho_F} \) is given by
\[
\text{rk}(\tilde{Q}_{\rho_F}) = (F, F)_{\leq 1} - (E_0, E_0)_{\leq 1}
\]
\[
= (F, F)_{\leq 1} + \text{dim}(A(v_F)) - \text{dim}(G(v_F)).
\]
Moreover there is a cocycle identity
\[
\mathbb{L}^{-\text{rk}(\tilde{Q}_{\rho_F})/2} (1 - MF_0(\tilde{Q}_{\rho_F})) = \mathbb{L}^{-\text{rk}(Q_F)/2} (1 - MF_0(Q_F))
\]
since \( Q_{S_i} = 0, i = 1, \ldots, 4 \) for the spherical objects. Equations (4.20), (4.21), (4.22) then yield the following expression
\[
w_F = \mathbb{L}^{(\text{dim}(G(v_F)) - \text{dim}(A(v_F)))/2} (1 - MF_0(W_{\rho_F})).
\]
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where \( W_{\rho F} \) is the polynomial function

\[
W_{\rho F} = W(\rho + \rho F)
\]

for any \( \rho \in \mathbb{A}(vF) \). Note that \( MF_{0}(W_{\rho F}) = MF_{\rho F}(W) \) by functoriality of motivic Milnor fibers.

In general explicit computations of pointwise Milnor fibers are difficult. The following Lemma shows that the computation is tractable on a certain subset of the critical locus of \( W \). Let \( MC_{0} = \text{Crit}(W) \cap \mathbb{A}^{r}(v) \) be the subscheme of critical points with trivial left directed arrows. The potential \( W : \mathbb{A}(v) \to \mathbb{C} \) is of the form

\[
W = \sum_{\kappa=1}^{K} y_{\kappa} P_{\kappa}
\]

where \( (y_{\kappa})_{1 \leq \kappa \leq K} \) are natural linear coordinates on \( \mathbb{A}^{l}(v) \) and \( P_{\kappa} : \mathbb{A}^{r}(v) \to \mathbb{C} \) are polynomial functions. Then \( MC_{0} \) is determined by

\[
y_{\kappa} = 0, \quad P_{\kappa} = 0, \quad \kappa = 1, \ldots, K.
\]

Let \( X_{0} = W^{-1}(0) \) denote the central fiber. Note that there is a commutative diagram

\[
\begin{array}{ccc}
\text{Crit}(W) & \xrightarrow{\iota} & X_{0} \\
\downarrow p^{cr} & & \downarrow p \\
MC_{0} & \xrightarrow{\iota} & \mathbb{A}^{r}(v)
\end{array}
\]

where \( p : X_{0} \to \mathbb{A}^{r}(v) \) is the restriction of the canonical projection \( \mathbb{A}(v) \to \mathbb{A}^{r}(v) \) and \( \iota \) is the zero section \( y_{\kappa} = 0, \kappa = 1, \ldots, K \). Note that the fibers of \( p, p^{cr} \) are linear subspaces of \( \mathbb{A}^{l}(v) \). Let \( MC_{0}^{sm} \) denote the smooth open locus of \( MC_{0} \).

**Lemma 4.4.** Let \( \rho \in MC_{0}^{sm} \). Then the motivic weight at \( \iota(\rho) \) is

\[
1 - MF_{\iota(\rho)}(W) = L^{\dim \mathbb{A}^{l}(v)}.
\]

**Proof.** Let \( U \subset \mathbb{A}(v) \) be the open subset where the Jacobian matrix of the polynomial functions \( (P_{\kappa}), \kappa = 1, \ldots, K \) has maximal rank. Then \( U \cap MC_{0} = MC_{0}^{sm} \). Let \( \mathcal{Y}_{0} \) be the restriction of the central fiber \( X_{0} \) to \( U \) and \( q : \mathcal{Y}_{0} \to \mathbb{A}^{r}(v) \) the restriction of \( p \). Note that the singular locus \( \mathcal{Y}_{0}^{\text{sing}} \subset \mathcal{Y}_{0} \) is determined by the equations

\[
y_{\kappa} = 0, \quad P_{\kappa} = 0, \quad \kappa = 1, \ldots, K.
\]

This follows observing that there is a factorization

\[
U \xrightarrow{P} \mathbb{A}^{l}(v) \times \mathbb{A}^{l}(v) \xrightarrow{Q} \mathbb{C}
\]

of \( W|_{U} : U \to \mathbb{C} \), where

\[
P(y_{\kappa}, x) = (y_{\kappa}, P_{\kappa}(x))
\]

for any \( x \in \mathbb{A}^{r}(v), (y_{\kappa}) \in \mathbb{A}^{l}(v) \) and

\[
Q(y_{\kappa}, z_{\kappa}) = \sum_{\kappa=1}^{K} y_{\kappa} z_{\kappa}.
\]

Since the Jacobian matrix of \( (P_{\kappa}) \) has maximal rank on \( U \), the map \( P \) is smooth. Moreover, the singular locus of the central fiber of \( Q \) is obviously \( y_{\kappa} = z_{\kappa} = 0 \) for all \( \kappa = 1, \ldots, K \). This implies the claim.
In conclusion, $\mathcal{Y}_0^{\text{sing}}$ coincides with the image $\iota(MC_{0}^{\text{sm}}) \subset \mathcal{Y}_0$. Note also that the fibers of $p$ over closed points $\rho \in MC_{0}^{\text{sm}}$ are isomorphic to $\mathbb{A}^1(v)$. Then a normal crossing resolution of $\mathcal{Y}_0$ can be obtained by a single embedded blow-up. Let $\sigma: \mathcal{U}' \to \mathcal{U}$ be the blow-up of $\mathcal{U}$ along the linear subspace

$$y_\kappa = 0, \quad \kappa = 1, \ldots, K.$$ 

The total transform $\sigma^{-1}(\mathcal{Y}_0')$ consists of the strict transform $\mathcal{Y}_0'$ and an exceptional divisor $D$ isomorphic to a $\mathbb{P}(\mathbb{A}^1(v))$-bundle over $\mathbb{A}^r(v)$. The strict transform $\mathcal{Y}_0'$ is smooth and intersects $D$ transversely along a divisor $D' \subset \mathcal{Y}_0'$, which is isomorphic to a $\mathbb{P}(\mathbb{A}^1(v))$-bundle over $MC_{0}^{\text{sm}}$. Moreover both $\mathcal{Y}_0'$ and $D$ multiplicity 1 in $\sigma^{-1}(\mathcal{Y}_0)$.

For any point $\rho \in MC_{0}^{\text{sm}}$, $\sigma^{-1}(\iota(\rho))$ intersects both $\mathcal{Y}_0'$ and $D$ along the fiber $D_\rho \subset D$, which is isomorphic to $\mathbb{P}(\mathbb{A}^1(v))$. Therefore, from the definition \[23\] Sect. 4, pp. 67

$$1 - MF_{\iota(\rho)}(\mathcal{W}) = 1 - (1 - L) [P(\mathbb{A}^1(v))] = L^{\dim(\mathbb{A}^1(v))}.$$

\[\square\]

4.5. **Comparison with refined Hilbert scheme invariants.** The motivic version of the refined Hilbert scheme series \[123\] is

$$Z_{C,p}^{\text{mot}}(q,a,y) = \sum_{l,r \geq 0} q^l a^{2r} L^{r^2/2} [H^{l,r}(C)],$$

where $[H^{l,r}(C)]$ is the motive of the nested Hilbert scheme introduced in Section \[123\]. By construction $[H^{l,r}(C)] = [Q^{l,r}(C)]$, where $Q^{l,r}(C)$ is the relative Quot scheme defined above Proposition \[123\]. Moreover, the stack $Q(X,C,r,n)$ is a $C^\times$ gerbe over the relative Quot scheme $Q^{l,r}(C)$, $l = n - \chi(O_C)$, according to Proposition \[123\]. As observed in Remark \[123\], the moduli stack $\mathcal{M}(X,C,r,n)$ is also a $C^\times$ gerbe over a coarse moduli scheme $M^{l,r}(C)$, and there is a natural forgetful morphism $\pi: Q^{l,r}(C) \to M^{l,r}(C)$. Note also that there is a natural stratification of $M^{l,r}(C)$ such that the restriction of $\pi$ to each stratum is a smooth projective bundle with fiber $\mathbb{B}^{h(F)}$. Since the motivic weights $w_F$ are invariant under isomorphisms, $F \simeq F'$, they descend to motivic weights $w_{F'}$ on the coarse moduli space $M^{l,r}(C)$.

Then using the conjectural identity \[123\] a stratification argument implies that the virtual motive of the stack function $f: Q(X,C,r,n) \hookrightarrow Ob(A)$ is given by

$$\frac{1}{L - 1} \Phi([f: Q(X,C,r,n) \to Ob(A)]) = \text{L}^{(1-n)/2} \int_{M^{l,r}(C)} [\mathbb{B}^{h(F)} - 1] w_{[F]}.$$

Applying Lemmas \[123\] one then obtains

$$\frac{1}{L - 1} \Phi([f: Q(X,C,r,n) \to Ob(A)]) = \text{L}^{(1-n)/2} \text{L}^{(\dim(G(v_F)) - \dim(\mathbb{A}(v_F))/2 + \dim(\mathbb{A}(v_F))/2)} \int_{M^{l,r}(C)} [\mathbb{B}^{h(F)} - 1]$$

Note that

$$(\dim(G(v_F)) - \dim(\mathbb{A}(v_F))/2 + \dim(\mathbb{A}(v_F))/2 = (n^2 - k^2)/2$$
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by a straightforward computation. Therefore the final formula is

\[
\frac{1}{L-1} \Phi([f: \mathcal{Q}(X,C,r,n) \to Ob(A)]) = \mathbb{L}^{(r^2-k^2+1-n)/2}[\mathcal{Q}^{[l,r]}(C)].
\]

Then the resulting generating function of C-framed virtual motivic invariants is

\[
Z^{\text{mot}}(X,C;u,a) = \mathbb{L}^{(1-k^2)/2} \sum_{r \geq 0} \sum_{l \geq 0} u^{2n} a^{2r} L^{(r^2-n)/2}[\mathcal{Q}^{[l,r]}(C)]
\]

\[
= \mathbb{L}^{(1-k^2-\chi(O_C))/2} u^{2\chi(O_C)} \sum_{r \geq 0} \sum_{l \geq 0} u^{2l} a^{2r} L^{(r^2-l)/2}[\mathcal{Q}^{[l,r]}(C)]
\]

Up to the overall factor, this agrees with equation (4.24) provided that the formal variables are related by \( q^2 = u^2 L^{-1/2} \).

REFERENCES

[1] A. Bayer. Polynomial Bridgeland stability conditions and the large volume limit. Geom. Topol., 13(4):2389–2425, 2009.
[2] K. Behrend. Donaldson-Thomas type invariants via microlocal geometry. Ann. of Math. (2), 170(3):1307–1338, 2009.
[3] K. Behrend, J. Bryan, and B. Szendroi. Motivic degree zero Donaldson-Thomas invariants. arxiv.org:0909.5088.
[4] M. Bender and S. Mozgovoy. Crepant resolutions and brane tilings II: Tilting bundles. arXiv:0909.2013.
[5] S. Cecotti and C. Vafa. BPS Wall Crossing and Topological Strings. hep-th/0910.2615.
[6] W.-y. Chuang, D.-E. Diaconescu, and G. Pan. Rank Two ADHM Invariants and Wallcrossing. Commun. Num. Theor. Phys., 4:417–461, 2010. 1002.0579.
[7] W.-y. Chuang, D.-E. Diaconescu, and G. Pan. Wallcrossing and Cohomology of The Moduli Space of Hitchin Pairs. Commun. Num. Theor. Phys., 5:1–56, 2011.
[8] B. Davison. Invariance of orientation data for ind-constructible Calabi-Yau A_\infty categories under derived equivalence. arXiv:1006.5475.
[9] D.-E. Diaconescu and B. Florea. Large N duality for compact Calabi-Yau threefolds. Adv. Theor. Math. Phys., 9(1):31–128, 2005.
[10] D.-E. Diaconescu, V. Shende, and C. Vafa. Large N duality, lagrangian cycles and algebraic knots. arXiv:1111.6533.
[11] T. Dimofte and S. Gukov. Refined, Motivic, and Quantum. Lett. Math. Phys., 91:1, 2010.
[12] T. Dimofte, S. Gukov, and Y. Soibelman. Quantum Wall Crossing in N=2 Gauge Theories. Lett. Math. Phys., 95:1–25, 2011. 0912.1346.
[13] N. M. Dunfield, S. Gukov, and J. Rasmussen. The superpolynomial for knot homologies. Experiment. Math., 15(2):129–159, 2006.
[14] D. Gaiotto, G. W. Moore, and A. Neitzke. Framed BPS States. arXiv:1006.0146.
[15] R. Gopakumar and C. Vafa. M theory and topological strings. I. hep-th/9809187.
[16] S. Gukov, A. S. Schwarz, and C. Vafa. Khovanov-Rozansky homology and topological strings. Lett. Math. Phys., 74:53–74, 2005.
[17] Z. Hua. Chern-Simons functions on toric Calabi-Yau threefolds and Donaldson-Thomas theory. arXiv:1103.1921.
[18] D. Huybrechts and M. Lehn. Framed modules and their moduli. Internat. J. Math., 6(2):297–324, 1995.
[19] D. Huybrechts and M. Lehn. The geometry of moduli spaces of sheaves. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.
[20] D. Joyce and Y. Song. A theory of generalized Donaldson-Thomas invariants. arxiv.org:0810.5645.
[21] S. Katz and C.-C. M. Liu. Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc. Adv. Theor. Math. Phys., 5(1):1–49, 2001.
[22] M. Kontsevich and Y. Soibelman. Stability structures, Donaldson-Thomas invariants and cluster transformations. arXiv.org:0811.2435.
[24] M. Kontsevich and Y. Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. *Commun.Num.Theor.Phys.*, 5:231–352, 2011.
[25] J. M. F. Labastida, M. Marino, and C. Vafa. Knots, links and branes at large N. *JHEP*, 11:007, 2000.
[26] C.-C. M. Liu, K. Liu, and J. Zhou. A proof of a conjecture of Mariño-Vafa on Hodge integrals. *J. Differential Geom.*, 65(2):289–340, 2003.
[27] C.-C. M. Liu, K. Liu, and J. Zhou. A formula of two-partition Hodge integrals. *J. Amer. Math. Soc.*, 20(1):149–184 (electronic), 2007.
[28] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten and Donaldson-Thomas theory. I. *Compos. Math.*, 142(5):1263–1285, 2006.
[29] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande. Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds. arXiv:0809.3976.
[30] D. Maulik, R. Pandharipande, and R. P. Thomas. Curves on K3 surfaces and modular forms. *J. Topol.*, 3(4):937–996, 2010. With an appendix by A. Pixton.
[31] D. Maulik and Z. Yun. Macdonald formula for curves with planar singularities. arXiv:1107.2175.
[32] L. Migliorini and V. Shende. A support theorem for Hilbert schemes of planar curves. arXiv:1107.2355.
[33] A. Morrison, S. Mozgovoy, K. Nagao, and B. Szendroï. Motivic Donaldson-Thomas invariants of the conifold and the refined topological vertex. arXiv:1107.5017.
[34] A. Morrison and K. Nagao. Motivic Donaldson-Thomas invariants of toric small crepant resolutions. arXiv:1110.5976.
[35] K. Nagao and H. Nakajima. Counting invariant of perverse coherent sheaves and its wall-crossing. arXiv.org:0809.2992, to appear in IMRN.
[36] A. Oblomkov, J. Rasmussen, and V. Shende. The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link. arXiv:1201.2115.
[37] A. Oblomkov and V. Shende. The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link. arXiv:1003.1568.
[38] H. Ooguri and C. Vafa. Knot invariants and topological strings. *Nucl. Phys.*, B577:419–438, 2000.
[39] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. *Invent. Math.*, 178(2):407–447, 2009.
[40] V. Shende. Hilbert schemes of points on a locally planar curve and the Severi strata of its versal deformation. arXiv:1009.0914.
[41] B. Szendröi. Non-commutative Donaldson-Thomas invariants and the conifold. *Geom. Topol.*, 12(2):1171–1202, 2008.
[42] Y. Toda. Generating functions of stable pair invariants via wall-crossings in derived categories. arXiv.org:0806.0062.
[43] Y. Toda. Limit stable objects on Calabi-Yau 3-folds. arXiv.org:0803.2356.