Misconceptions About General Relativity in Theoretical Black Hole Astrophysics

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ABSTRACT
The fundamental role played by black holes in our study of microquasars, gamma ray bursts, and the outflows from active galactic nuclei requires an appreciation for, and at times some in-depth analysis of, curved spacetime. We highlight misconceptions surrounding the notion of coordinate transformation in general relativity as applied to metrics for rotating black holes that are beginning to increasingly appear in the literature. We emphasize that there is no coordinate transformation that can turn the metric of a rotating spacetime into that for a Schwarzschild spacetime, or more generally, that no coordinate transformation exists that can diagonalize the metric for a rotating spacetime. We caution against the notion of “local” coordinate transformation, which is often incorrectly associated with a global analysis of the spacetime.

Key words: relativity - rotating black holes

1 INTRODUCTION
Not surprisingly, the overwhelming observational evidence which has propelled black holes center stage in astrophysics over the past few decades, has had the consequence of making theoretical aspects of general relativity and the curved spacetime around black holes, indispensable tools. As a result, theoretical research in such areas can require a mastery of apparently disparate topics ranging from magnetohydrodynamics of turbulent flows to differential geometry, a daunting task that is not easily accomplished well. Given this state of affairs, it is perhaps not surprising that misconceptions arise. In particular, the spacetime of a rotating, isolated, black hole, in which commonly used coordinate systems suffer singularities at the horizon and complications due to off-diagonal terms, have been studied via the introduction of new coordinates that simplify calculations and avoid singularities. Because the metric tensor is usually cast in its one-form expression in terms of basis one-forms or differentials, differentials of coordinate transformations are needed to recast the metric in the new coordinate system. Often times, this differential approach in recasting the metric tensor is undertaken directly without care to ensure the validity of the actual coordinate transformation. This misconception has appeared in papers and books in the last several years, causing confusion and erroneous results. In this short paper, we describe this coordinate transformation misconception and ways to avoid it.

2 COORDINATE TRANSFORMATIONS FOR ROTATING SPACETIMES:THE BOYER-LINDQUIST CASE
In commonly used Boyer-Lindquist coordinates, the metric for a rotating black hole in the coordinate basis assumes its standard form (e.g. Poisson 2004),

\[
\begin{align*}
\text{d}S^2 &= -\left(1 - \frac{2Mr}{\rho^2}\right)\text{d}t^2 - \frac{4Mar}{\rho^2}\sin^2\theta\text{d}t\text{d}\phi \\
&\quad + \frac{\Sigma}{\rho^2}\sin^2\theta\text{d}\phi^2 + \frac{\rho^2}{\Delta}\text{d}r^2 + \frac{\rho^2}{\Sigma}\text{d}\theta^2,
\end{align*}
\]

(1)

where

\[
\rho^2 = r^2 + a^2\cos^2\theta,
\]

(2)

\[
\Delta = r^2 - 2Mr + a^2,
\]

(3)

and

\[
\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta,
\]

(4)

which can be recast in the following form, using the same coordinates,

\[
\begin{align*}
\text{d}S^2 &= -\left(\frac{\rho^2\Delta}{\Sigma}\right)\text{d}t^2 + \frac{\Sigma\sin^2\theta}{\rho^2}(\text{d}\phi - \omega\text{d}t)^2 \\
&\quad + \frac{\rho^2}{\Delta}\text{d}r^2 + \rho^2\text{d}\theta^2,
\end{align*}
\]

(5)

where

\[
\omega = \frac{2Mar}{\Sigma}.
\]

(6)
From the above form, it is easy to see that if one allows the differential of a new coordinate \( \phi' \) to have the form

\[
d\phi' = d\phi - \omega dt,
\]

the second term in the above metric tensor (colloquially referred to as the line element) fails to generate off-diagonal metric terms and one is tempted to claim that a global coordinate transformation from \( \phi \) to \( \phi' \) exists which accomplishes this (Krolik 1998). This is a misconception that arises from the differential approach. The coordinates \((t, r, \theta, \phi')\) are valid local coordinates (in the same manner as spacetime is locally Minkowskian), but they are not valid global coordinates. Let us illustrate this by going back to the actual coordinate transformation that is implicit in the one-form of equation (7), which is

\[
\phi' = \phi - \omega t. \tag{8}
\]

From here we calculate the differential or one-form of \( \phi' \) with respect to the coordinates \( r, \theta, \phi \) and \( t \), via

\[
d\phi' = \frac{\partial \phi'}{\partial r} dr + \frac{\partial \phi'}{\partial \theta} d\theta + \frac{\partial \phi'}{\partial \phi} d\phi + \frac{\partial \phi'}{\partial t} dt. \tag{9}
\]

Therefore,

\[
d\phi' = -d\omega tdr + \frac{\partial \omega}{\partial \theta} d\theta + \frac{\partial \omega}{\partial \phi} d\phi - \omega dt. \tag{10}
\]

If \( \omega \) is not a function of \( t \), we have

\[
d\phi' = -d\omega tdr + \frac{\partial \omega}{\partial \theta} d\theta + d\phi - \omega dt, \tag{11}
\]

and only when \( \omega \) is neither a function of \( r \) nor \( t \) does the differential reduce to

\[
d\phi' = d\phi - \omega dt. \tag{12}
\]

This means that if \( \omega \) has a value at some \( r, t, \theta \), such that the metric is diagonal there, it will be diagonal only at that point. Therefore, no global analysis of the spacetime can be carried out with a diagonal metric. The coordinate singularity appears in the one-form of \( \phi' \) (10), which is valid at one value of \( r \) and \( \theta \). It is true that along the worldline of an observer for which \( \frac{dt}{\rho} = \omega \), no \( d\phi' \) term exists. However, this is a local statement that refers to the values of \( r \) and \( \theta \) followed by this particular observer in spacetime. With Krolik (1998) is in the statement “the two metrics agree” when referring to the coordinate transformation between the Kerr metric in Boyer-Lindquist coordinates and the metric in “locally non-rotating frame coordinates”. It would be better to emphasize the “agreement” between the metric in Boyer-Lindquist coordinates and an infinite number of different “locally non-rotating frame coordinate” metrics, one for each of the different values of the coordinates \( r \) and \( \theta \).

Alternatively, one could take a more rigorous and possibly more direct approach by considering the metric as a \((0,2)\) tensor resulting from the tensor product of one-forms as in

\[
g = \sum_{i,j} g_{ij}(dx^i \otimes dx^j). \tag{13}
\]

Accordingly, \( d\phi' \) is the one-form basis in terms of unprimed coordinates that enters the metric via equation (13), which would have the following expression

\[
g = -\left( \frac{\rho^2 \Delta}{\Sigma} \right)(dt \otimes dt) + \frac{\rho^2 \Sigma}{\rho^2} \omega \sin^2 \theta (d\phi' \otimes d\phi') + \frac{\rho^2}{\rho^2} (dr \otimes dr) + \rho^2 (d\theta \otimes d\theta). \tag{14}
\]

However, in order to cast \( g \) in the form above, there must be a one-form basis \( d\phi' \) that is independent of \( r \) and \( \theta \) to take expression (5) into expression (14), but, by virtue of the fact that \( \omega = \omega(r, \theta) \) via expression (5), no such form exists.

Despite the absence of a diagonal metric, it is still possible to avoid the complications due to off diagonal metric terms by employing the methods of local frames or the tetrad formalism. This simplifies calculations by projecting tensors onto a local orthonormal basis of four linearly independent vector fields, the frames of locally non-rotating observers (Bardeen et al 1973; Chandrasekhar 1992), as done, for example, in Shafee et al (2008), via the use of the stress tensor component in the orthonormal basis of the comoving fluid. Even within the tetrad formalism, however, the Kerr metric in Boyer-Lindquist coordinates remains of the form (1) or (5) and is in general non-diagonal in any coordinates. In the next section we briefly illustrate this by showing how the introduction of one set of coordinates (which avoids the coordinate singularity at the horizon) inevitably produces off-diagonal terms in the spatial 3-metric.

### 3 COORDINATE TRANSFORMATIONS FOR ROTATING SPACETIMES: THE KERR-SCHILDT CASE

In this section we perform a coordinate transformation that avoids the coordinate singularity at the black hole horizon, emphasizing the role of its global nature in the failure of this transformation to produce a diagonal metric. The coordinate singularity appears in the term

\[
g_{tr} = \frac{\rho^2}{\Delta}, \tag{15}
\]

whose denominator goes to zero at the horizon. To perform a singularity-removing coordinate transformation requires transformation of both time and azimuthal angle in the form

\[
dt' = dt + \frac{2Mr}{\Delta} dr \tag{16}
\]

and

\[
d\phi' = d\phi + \frac{a}{\Sigma} dr. \tag{17}
\]

Given the fact that each second term is integrable in \( r \), this constitutes a global coordinate transformation. Integrating \( t' \), and \( \phi' \) gives the actual coordinate transformations as

\[
t' = t + \frac{M}{\sqrt{M^2 - a^2}^2} \left[ (M + \sqrt{M^2 - a^2}) \ln \frac{r}{M + \sqrt{M^2 - a^2}} - 1 \right] - (M - \sqrt{M^2 - a^2}) \ln \frac{r}{M + \sqrt{M^2 - a^2}} - 1 \right] \tag{18}
\]

and

\[
\phi' = \phi + \frac{\alpha}{2\sqrt{M^2 - a^2}} \ln \left[ \frac{r - (M + \sqrt{M^2 - a^2})}{r - (M - \sqrt{M^2 - a^2})} \right]. \tag{19}
\]

In coordinates \((t', r, \theta, \phi')\) the metric now becomes

\[
dS^2 = \left( 1 - \frac{2Mr}{\rho^2} \right) dt' dr' - 2\omega \sin^2 \theta (1 + \frac{2Mr}{\rho^2}) dr'd\phi' + \frac{\Delta}{\rho^2} (d\phi')^2 + \left( 1 + \frac{2Mr}{\rho^2} \right) dr^2 + \rho^2 d\theta^2. \tag{20}
\]

With the above coordinate transformations (18)/(19), the spatial part of the metric no longer suffers a coordinate singularity at the
black hole horizon where $\Delta = 0$. However, their is a price to pay, as the spatial 3-metric is no longer diagonal due to a new metric term $g_{r\phi'}$. Any attempt to produce horizon-penetrating coordinates that avoid the coordinate singularity at the horizon, while also leaving the spatial 3-metric diagonal, requires that the integrand in $t'$ be non-integrable in $r$. Also, in general, producing a diagonal metric requires the integrand in $\phi'$ be non-integrable in $r$ as well.

4 DISCUSSION AND CONCLUSIONS

The property of rotating spacetimes discussed above can strongly affect how codes that perform numerical simulations of accretion onto black holes are constructed. For example, Koide (2003) devised a clever method of performing numerical MHD simulations in a stationary Kerr metric using standard methods of curvilinear coordinates, if the spatial portion of the 4-metric were diagonal. This produced excellent results outside the horizon, but, as explained above, possessed a coordinate singularity at the horizon. Hence, matter could not flow naturally through the horizon in the simulation. In such cases, simulations could last for only a few tens of black hole light crossing times. On the other hand, McKinney (2006), for example, employed the correct non-diagonal Kerr-Schild coordinates in his simulation code, which allowed one to follow the flow of plasma well into the horizon without numerical problems. Such simulations can be run out to at least $\sim 10^4$ black hole light crossing times. However, obtaining this capability in one’s simulation code requires using more involved techniques to handle the inevitable off-diagonal 3-metric terms.

In this short letter, we highlight the increased appearance in the astrophysical literature of the misuse of a metric of the form (14) when working with asymptotically flat, stationary and axisymmetric spacetimes to address the global spacetime nature. We suggest that avoiding such problems can be accomplished by either starting with the full coordinate transformation before determining its differential form, or, by simply ensuring that the partial derivatives in the differential are taken with respect to the full set of coordinates.

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