Gradient Flow in Sparse Neural Networks and How Lottery Tickets Win

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Abstract

Sparse Neural Networks (NNs) can match the generalization of dense NNs using a fraction of the compute/storage for inference, and have the potential to enable efficient training. However, naively training unstructured sparse NNs from random initialization results in significantly worse generalization, with the notable exceptions of Lottery Tickets (LTs) and Dynamic Sparse Training (DST). Through our analysis of gradient flow during training we attempt to answer: (1) why training unstructured sparse networks from random initialization performs poorly and; (2) what makes LTs and DST the exceptions? We show that sparse NNs have poor gradient flow at initialization and demonstrate the importance of using sparsity-aware initialization. Furthermore, we find that DST methods significantly improve gradient flow during training over traditional sparse training methods. Finally, we show that LTs do not improve gradient flow, rather their success lies in re-learning the pruning solution they are derived from — however, this comes at the cost of learning novel solutions.

1 Introduction

Deep Neural Networks (DNNs) are the state-of-the-art method for solving problems in computer vision, speech recognition, and many other fields. While early research in deep learning focused on application to new problems, or pushing state-of-the-art performance with ever larger/more computationally expensive models, a broader focus has emerged towards their efficient real-world application. One such focus is on the observation that only a sparse subset of this dense connectivity is required for inference, as apparent in the success of pruning.

Pruning has a long history in Neural Network (NN) literature (Mozer and Smolensky 1989; Han et al. 2015), and remains the most popular approach for finding sparse NNs. Sparse NNs found by pruning algorithms (Han et al. 2015; Zhu and Gupta 2018; Molchanov, Ashukha, and Vetrov 2017; Louizos, Ullrich, and Welling 2017) (i.e. pruning solutions) can match dense NN generalization with much better efficiency at inference time. However, naively training an (unstructured) sparse NN from a random initialization (i.e. from scratch), typically leads to significantly worse generalization.

Two methods in particular have shown some success at addressing this problem — Lottery Tickets (LTs) and Dynamic Sparse Training (DST). However, we don’t know how to find Lottery Tickets (LTs) efficiently; while RigL (Evci et al. 2020), a recent DST method, requires 5× the training steps to match dense NN generalization. Only in understanding how these methods overcome the difficulty of sparse training can we improve upon them.

A significant breakthrough in training DNNs — addressing vanishing and exploding gradients — arose from understanding gradient flow both at initialization, and during training. In this work we investigate the role of gradient flow in the difficulty of training unstructured sparse NNs from random initializations and from LT initializations. Our experimental investigation results in the following insights:

1. Sparse NNs have poor gradient flow at initialization. In §3.1 §4.1 we show that the predominant method for initializing sparse NNs is incorrect in not considering heterogeneous connectivity. We believe we are the first to show that sparsity-aware initialization methods improve gradient flow and training.

2. Sparse NNs have poor gradient flow during training. In §3.2 §4.2 we observe that even in sparse NN architectures less sensitive to incorrect initialization, the gradient flow during training is poor. We show that DST methods achieving the best generalization have improved gradient flow, especially in early training.

3. Lottery Tickets don’t improve upon (1) or (2), instead they re-learn the pruning solution. In §3.3 §4.3 we show that a LT initialization resides within the same basin of attraction as the original pruning solution it is derived from, and a LT solution is highly similar to the pruning solution in function space.

2 Related Work

Pruning Pruning is used commonly in NN literature to obtain sparse networks (Mozer and Smolensky 1989; Han et al. 2015; Kusupati et al. 2020). While the majority of pruning algorithms focus on pruning after training, a subset focuses on pruning NNs before training (Lee, Ajanthan, and Torr 2019; Wang, Zhang, and Grosse 2020; Tanaka et al. 2020). Gradient Signal Preservation (GRaSP) (Wang, Zhang, and Grosse 2020) is particularly relevant to our study, since
their pruning criteria aims to preserve gradient flow, and they observe a positive correlation between gradient flow at initialization and final generalization. However, the recent work of Frankle et al. (2020b) suggests that the reported gains are due to sparsity distributions discovered rather than the particular sub-network identified. Another limitation of these algorithms is that they don’t scale to large-scale tasks like ResNet-50 training on ImageNet-2012.

Lottery Tickets Frankle and Carbin (2019) showed the existence of sparse sub-networks at initialization — known as Lottery Tickets — which can be trained to match the generalization of the corresponding dense DNN. The initial work of Frankle and Carbin (2019) inspired much follow-up work. Liu et al. (2019); Gale, Elsen, and Hooker (2019) observed that the initial formulation was not applicable to larger networks with higher learning rates. Frankle et al. (2019, 2020b) proposed late rewinding as a solution. Morcos et al. (2019); Sabatelli, Kestemont, and Geurts (2020) showed that LTs trained on large datasets transfer to smaller ones, but not vice versa. Zhou et al. (2019); Frankle, Schwab, and Morcos (2020); Ramanujan et al. (2019) focused on further understanding LTs, and finding sparse sub-networks at initialization. However, it is an open question whether finding such networks at initialization could be done more efficiently than with existing pruning algorithms.

Dynamic Sparse Training Most training algorithms work on pre-determined architectures and optimize parameters using fixed learning schedules. Dynamic Sparse Training (DST), on the other hand, aims to optimize the sparse NN connectivity jointly with model parameters. Mocanu et al. (2018); Mostafa and Wang (2019) propose replacing low-magnitude parameters with random connections and report improved generalization. Detmers and Zettlemoyer (2019) proposed using momentum values, whereas Evci et al. (2020) used gradient estimates directly to guide the selection of new connections, reporting results that are on par with pruning algorithms, and has been applied to vision transformers (Chen et al. 2021), language models (Dietrich et al. 2021), reinforcement learning (Sokar et al. 2021), training recurrent neural networks (Liu et al. 2021b) and fast ensembles (Liu et al. 2021a). In §4.2 we study these algorithms and try to understand the role of gradient flow in their success.

Random Initialization of Sparse NN In training sparse NN from scratch, the vast majority of pre-existing work on training sparse NN has used the common initialization methods (Glorot and Bengio 2010; He et al. 2015) derived for dense NNs, with only a few notable exceptions. Liu et al. (2019); Ramanujan et al. (2019); Gale, Elsen, and Hooker (2019) scaled the variance (fan-in/fan-out) of a sparse NN layer according to the layer’s sparsity, effectively using the standard initialization for a small dense layer with an equivalent number of weights as in the sparse model. Lee et al. (2020) measures the singular values of the input gradients and proposes to use a dense orthogonal initialization to ensure dynamical isometry and improve one-shot pruning performance before training. Similar to our work, Tessera, Hooker, and Rosman (2021); Lubana and Dick (2021) compare dense and sparse networks at initialization and during training using gradient flow, whereas Golubeva, Neyshabur, and Gur-Ari (2021) study the distance to the infinite-width kernel.

3 Analyzing Gradient Flow in Sparse NNs

A significant breakthrough in training very deep NNs arose in addressing the vanishing/exploding gradient problem, both at initialization, and during training. This problem was understood by analyzing the signal propagation within a DNN, and addressed in improved initialization methods (Glorot and Bengio 2010; He et al. 2015; Xiao et al. 2018) alongside normalization methods, such as Batch Normalization (BatchNorm) (Ioffe and Szegedy 2015). In our work, similar to Wang, Zhang, and Grossel (2020), we study these problems using gradient flow, \( \nabla \mathcal{L}(\theta)^\top \nabla \mathcal{L}(\theta) \) which is the first order approximation of the decrease in the loss expected after a gradient step. We observe poor gradient flow for the predominant sparse NN initialization strategy and propose a sparsity-aware generalization in §3.1. Then in §3.2 and §3.3 we summarize our analysis of DST methods and the LT hypothesis respectively.

3.1 The Initialization Problem in Sparse NNs

Here we analyze the gradient flow at initialization for random sparse NNs, motivating the derivation of a more general initialization for NN with heterogeneous connectivity, such as in unstructured sparse NNs.

In practice, without a method such as BatchNorm (Ioffe and Szegedy 2015), using the correct initialization can be the difference between being able to train a DNN, or not — as observed for VGG16 in our results (§4.1, Table 1). The initializations proposed by Glorot and Bengio (2010), He et al. (2015) ensure that the output distribution of every neuron in a layer is zero-mean and of unit variance by sampling initial weights from a Gaussian distribution with a variance based on the number of incoming/outgoing connections for all the neurons in a dense layer, as illustrated in Fig. 1a, which is assumed to be identical for all neurons in the layer.

In an unstructured sparse NN however, the number of incoming/outgoing connections is not identical for all neurons in a layer, as illustrated in Fig. 1b. In Appendix B.2 we derive the initialization for this more general case. Here we will focus only on explaining the generalized He et al. (2015) initialization for forward propagation, which we used in our experiments. In Appendix B.3 we explain in full the generalized Glorot and Bengio (2010), He et al. (2015) initialization, in the forward, backward and average use cases.

We propose to initialize every weight \( w_{ij}^{[\ell]} \in W_{n^{[\ell]} \times n^{[\ell-1]}} \) in a sparse layer \( \ell \) with \( n^{[\ell]} \) neurons, and connectivity mask \([m_{ij}^{[\ell]}] = M^{[\ell]} \in [0, 1]^{n^{[\ell]} \times n^{[\ell-1]}} \) with,

\[
    w_{ij}^{[\ell]} \sim \mathcal{N} \left( 0, \frac{2}{\text{fan-in}_{i}^{[\ell]}}, \right),
\]

where \( \text{fan-in}_{i}^{[\ell]} = \sum_{j=1}^{n^{[\ell-1]}} m_{ij}^{[\ell]} \) is the number of incoming connections for neuron \( i \) in layer \( \ell \).

*We omit learning rate for simplicity and ensure different methods have same learning rate schedules when compared.
In the special case of a dense layer where $m_{ij}^\ell = 1, \forall i, j$, Eq. (1) reduces to the initialization proposed by He et al. (2015) since fan-in$^{\ell} = n^{\ell-1}, \forall i$. Similarly, the initialization proposed by Liu et al. (2019) is another special case where it is assumed fan-in$^{\ell} = \mathbb{E}_i \{\text{fan-in}^i\}, i.e. all neurons have the same number of incoming connections in a layer which is often not true with unstructured sparsity. Using the dense initialization in a sparse DNN causes signal to vanish, as empirically observed in Fig. 1C, whereas sparsity-aware initialization techniques (ours and Liu et al. (2019)) keep the variance of the signal constant.

### 3.2 Dynamic Sparse Training

While initialization is important for the first training step, the gradient flow during the early stages of training is not well addressed by initialization alone, rather it has been addressed in dense DNNs by normalization methods [Ioffe and Szegedy 2015]. Our findings show that even with BatchNorm however, the gradient flow during the training of unstructured sparse NNs is poor.

Recently, a promising new approach to training sparse NNs has emerged — Dynamic Sparse Training (DST) — that learns connectivity adaptively during training, showing significant improvements over baseline methods that use fixed masks. These methods perform periodic updates on the sparse connectivity of each layer: commonly replacing least magnitude connections with new connections selected using various criteria. We consider two of these methods and measure their effect on gradient flow during training: Sparse Evolutionary Training (SET) (Mocanu et al. 2018), which chooses new connections randomly and Rigged Lottery (RigL) (Evci et al. 2019), which chooses the connections with high gradient magnitude. Understanding why and how these methods achieve better results can help us in improving upon them.

### 3.3 Lottery Ticket Hypothesis

A recent approach for training unstructured sparse NNs while achieving similar generalization to the original dense solution is the Lottery Ticket Hypothesis (LTH) (Frankle and Carbin 2019). Notably, rather than training a pruned NN structure from random initialization, the LTH uses the dense initialization from which the pruning solution was derived.

**Definition [Lottery Ticket Hypothesis]:** Given a NN architecture $f$ with parameters $\theta$ and an optimization function $O^N(f, \theta) = \theta^N$, which gives the optimized parameters of $f$ after $N$ training steps, there exists a sparse sub-network characterized by the binary mask $M$ such that for some iteration $K$, $O^K(f, \theta_k^K * M)$ performs as well as $O^N(f, \theta) * M$, whereas the model trained from another random initialization $\theta_k$, using the same mask $O^K(f, \theta_k * M)$, typically does not.

Initial results of Frankle and Carbin (2019) showed the LTH held for $K = 0$, but later results (Liu et al. 2019) showed a larger $K$ is necessary for larger datasets and NN architectures, i.e. $N \gg K \geq 0$.

We measure the gradient flow of LTs and observe poor gradient flow overall. Despite this, LTs enjoy significantly faster convergence compared to regular NN training, which motivates our investigation of the success of LTs beyond gradient flow. LTs require the connectivity mask as found by the pruning solution along with parameter values from the early part of the dense training. Given the importance of the early phase of training (Frankle, Schwab, and Morcos 2020; Lewkowycz et al. 2020), it is natural to ask about the difference between LTs and the solution they are derived from (i.e. pruning solutions). Answering this question can help us understand if the success of LTs is primarily due to its relation to the solution, or if we can identify generalizable characteristics that help with sparse NNs training.

### 4 Experiments

Here we show empirically that (1) sparsity-aware initialization improves gradient flow at initialization for all methods, and achieves higher generalization for networks without BatchNorm, (2) the mask updates of DST methods increase gradient flow and create new negative eigenvalues in the Hessian; which we believe to be the main factor for improved

*See Frankle et al. (2019) for details. * indicates element-wise multiplication, respecting the mask.
Table 1: Results of Trained Sparse/Dense Models from Different Initializations. The initialization proposed in Eq. (1) (Ours) and [19] improve generalization consistently over masked dense (Original) except for in ResNet50. Note that VGG16 trained without a sparsity-aware initialization fails to converge in some instances. Baseline corresponds to the original dense architecture, whereas Small Dense corresponds to a smaller dense model with approx. the same parameter count as the sparse models.

| MNIST               | ImageNet-2012 | ImageNet-2012 |
|---------------------|---------------|---------------|
|                      | VGG16 (80%) sparse | ResNet50 (80%) sparse |
| Baseline            | 69.25 ± 0.13  | 76.75 ± 0.12  |
| Lottery             | 10.0 ± 0.01   | 75.75 ± 0.12†|
| Small Dense         | 61.75 ± 0.09  | 71.95 ± 0.24  |
| Dense               | 70.58 ± 0.18  | 70.72 ± 0.16  |
| Sparse (Liu)        | 62.52 ± 0.05  | 72.93 ± 0.27  |
| Sparse (Ours)       | 62.52 ± 0.10  | 72.77 ± 0.27  |
| Dense               | 74.41 ± 0.05  | 74.38 ± 0.10  |
| Sparse (Liu)        | 74.38 ± 0.06  | 74.38 ± 0.01  |
| Sparse (Ours)       | 74.38 ± 0.01  | 74.38 ± 0.01  |

* With late-rewinding (i.e. $K = 5000$).

Figure 2: Gradient Flow of Sparse Models during Training. Gradient flow during training averaged over multiple runs, ‘+’ indicates training runs with our proposed sparse initialization and Small Dense corresponds to training of a dense network with same number of parameters as the sparse networks. Lottery ticket runs for ResNet-50 include late-rewinding.

generalization, (3) lottery tickets have poor gradient flow, however they achieve good performance by effectively re-learning the pruning solution, meaning they do not address the problem of training sparse NNs in general. Our experiments include the following settings: LeNet5 on MNIST, VGG16 on ImageNet-2012 and ResNet-50 on ImageNet-2012. Experimental details can be found in Appendix A.

4.1 Gradient Flow at Initialization

In this section, we measure the gradient flow over the course of the training (Fig. 2) and evaluate the performance of generalized He initialization method (Table 1), and that proposed by [19], over the commonly used dense initialization in sparse NN. Additional gradient flow plots for the remaining methods are shared in Appendix C. Sparse NNs initialized using the initialization distribution of a dense model (Scratch in Fig. 2) start in a flat region where gradient flow is very small and thus initial progress is limited. Learning starts after 1000 iterations for LeNet5 and 5000 for VGG-16, however, generalization is sub-optimal. [19] reported their proposed initialization has no empirical effect as compared to the masked dense initialization. In contrast, our results show their method to be largely as effective as our proposed initialization, despite the [19] initialization being incorrect in the general case (see §3). This indicates that the assumption of a neuron having roughly uniform connectivity is sufficient for the ranges of sparsity considered, possibly due to a law-of-large-numbers-like averaging effect. We expect this effect to disappear with higher sparsity and our initialization to be more important. Results in Table 2 for a 98% sparse LeNet5 demonstrate this, in which our sparsity-aware initialization provides 40% improvement on test accuracy. Our initialization remedy the vanishing gradient problem at initialization (Scratch+ in Fig. 2) and result in better generalization for all methods. For instance, improved initialization results in an 11% improvement in Top-1 accuracy for VGG16 (62.52 vs 51.81). While initialization is extremely important for NNs without BatchNorm and skip connections, its effect on modern architectures, such

1Implementation of our sparse initialization, Hessian calculation and code for reproducing our experiments can be found at [https://github.com/google-research/rigl/tree/master/rigl/rigl_tf2](https://github.com/google-research/rigl/tree/master/rigl/rigl_tf2). Additionally we provide videos that shows the evolution of Hessian during training under different algorithms in the supplementary material.

2Models with BatchNorm and skip connections are less affected by initialization, and this is likely why the authors did not observe this effect.
verify our hypothesis, we measure the change in gradient variance, which suggests studying the gradient norm in the flow and Hessian spectrum whenever the sparse connectivity, which in turn results in better performance. To bring better performance. This is reflected in Table 1: Scratch, SET and RigL obtains 63%, 63% and 81% respectively; all initializations seem to have no effect on final generalization. Some initial improvement in gradient flow, sparsity-aware initialization methods demonstrate the same behaviour. As Resnet-50, is limited (Evci et al. 2019; Zhang, Dauphin, and Ma 2019; Frankle et al. 2020b). We confirm these observations in our ResNet-50 experiments in which, despite some initial improvement in gradient flow, sparsity-aware initializations seem to have no effect on final generalization. Additionally, we observe significant increase in gradient norm after each learning rate drop (due to increased gradient variance), which suggests studying the gradient norm in the latter part of the training might not be helpful.

The LT hypothesis holds for MNIST and ResNet50 (when K=5000) but not for VGG16. We observe poor gradient flow for LTs at initialization similar to Scratch. After around 2000 steps gradients become non zero for Scratch, while gradient flow for LT experiments stay constant. Our sparse initialization improves gradient flow at initialization and we observe a significant difference in gradient flow during early training between RigL and Scratch+. DST methods alone can help resolve gradient flow problem at initialization and thus bring better performance. This is reflected in Table 1: Scratch, SET and RigL obtains 63%, 63% and 81% respectively; all starting from the bad initialization (Dense).

4.2 Gradient Flow During Sparse Training

Our hypothesis for Fig. 2 is that the DST methods improve gradient flow through the updates they make on the sparse connectivity, which in turn results in better performance. To verify our hypothesis, we measure the change in gradient flow and Hessian spectrum whenever the sparse connectivity is updated. We also run the inverted baseline for RigL (RigL Inverted), in which the growing criteria is reversed and connections with the smallest gradients are activated as in Frankle et al. 2020b.

DST methods such as RigL replace low saliency connections during training. Assuming the pruned connections indeed have a low impact on the loss, we might expect to see increased gradient norm after new connections are activated, especially in the case of RigL, which picks new connections with high magnitude gradients. In Fig. 4c we confirm that RigL updates increase the norm of the gradient significantly, especially in the first half of training, whereas SET, which picks new connections randomly, seems to be less effective at this. Using the inverted RigL criteria doesn’t improve the gradient flow, as expected, and without this RigL’s performance degrades (73.83 ± 0.12 for ResNet-50 and 92.71 ± 7.67 for LeNet5). These results suggest that improving gradient flow early in training — as RigL does — might be the key for training sparse networks. Various recent works that improve DST methods support this hypothesis: longer update intervals (Liu et al. 2021c) and adding parallel dense components (Price and Tanner 2021) or activations (Curci, Mocanu, and Pechenizkiy 2021) that improve gradient flow brings better results. Additional plots (Appendix C) using different initialization methods demonstrate the same behaviour.

When the gradient is zero, or uninformative due to the error term of the approximation, analyzing the Hessian could provide additional insights (Sagun et al. 2017; Ghorbani, Kriishnan, and Xiao 2019; Papyan 2019). In Appendix E we show the Hessian spectrum before and after sparse connectivity updates. After RigL updates we observe more negative eigenvalues with significantly larger magnitudes as compared to SET. Large negative eigenvalues can further accelerate optimization if corresponding eigenvectors are aligned with the gradient. We leave verifying this hypothesis as a future work.

4.3 Why Lottery Tickets are Successful

We found that LTs do not improve gradient flow, either at initialization, or early in training, as shown in Fig. 2. This may be surprising given the apparent success of LTs, however the
questions posed in §3.3 present an alternative hypothesis for the ease of training from a LT initialization: Can the success of LTs be due to their relationship to well-performing pruning solutions? Here we present results showing that indeed (1) LTs initializations are consistently closer to the pruning solution than a random initialization, (2) trained LTs (i.e. LT solutions) consistently end up in the same basin as the pruning solution and (3), LT solutions are highly similar to pruning solutions under various function similarity measures. Our resulting understanding of LTs in the context of the pruning solution and the loss landscape is illustrated in Fig. 4.

**Experimental Setup** To investigate the relationship between the pruned and LT solutions we perform experiments on two models/datasets: a 95% sparse LeNet\textsuperscript{\(\dagger\)} architecture (LeCun et al. 1989) trained on MNIST (where the original LT formulation works, i.e. \(K = 0\)), and an 80% sparse ResNet-50 (Wu, Zhong, and Liu 2018) on ImageNet-2012 (Russakovsky et al. 2015) (where \(K = 0\) doesn’t work (Frankle et al. 2019), for which we use values from \(K = 2000\) (\(\approx 6^{th}\) epoch). In both cases, we find a LT initialization by pruning each layer of a dense NN separately using magnitude-based iterative pruning (Zhu and Gupta 2018). Further details of our experiments are in Appendix A.

**Lottery Tickets Are Close to the Pruning Solution** We train 5 different models using different seeds from both scratch (random) and LT initializations, the results of which are in Figs. 5a and 5b. These networks share the same pruning mask and therefore lie in the same solution space. We visualize distances between initial and final points of these experiments in Figs. 5a and 5b using 2D Multi-dimensional Scaling (MDS) (Kruskal 1964) embeddings. LeNet5/MNIST: In Fig. 5a, we provide the average L2 distance to the pruning solution at initialization (d\textsubscript{init}), and after training (d\textsubscript{final}). We observe that LT initializations start significantly closer to the pruning solution on average (d\textsubscript{init} = 13.61 v.s. 17.46). After training, LTs end up more than 3× closer to the pruning solution compared to scratch. ResNet-50/ImageNet-2012: We observe similar results for Resnet-50/ImageNet-2012. LTs, again, start closer to the pruning solution, and solutions are 5× closer (d\textsubscript{final} = 39.35 v.s. 215.98). These results explain non-random initial loss values for LT initializations (Zhou et al. 2019) and their inability to find good solutions if repelled by pruning solutions (Maene, Li, and Moens 2021). LTs are biased towards the pruning solution they are derived from, but are they in the same basin? Can it be the case that LTs learn significantly different solutions each time in function space despite being linearly connected to the pruning solution?

**Lottery Tickets are in the Pruning Solution Basin** Investigating paths between different solutions is a popular tool for understanding how various points in parameter space relate to each other in the loss landscape (Goodfellow, Vinyals, and Saxe 2015; Garipov et al. 2018; Draxler et al. 2018; Evci et al. 2019; Fort, Hu, and Lakshminarayanan 2020; Frankle et al. 2020a). For example, Frankle et al. (2019) use linear interpolations to show that LTs always go to the same basin when trained in different data orders. In Figs. 5c and 5d, we look at the linear paths between pruning solution and 4 other points: LT initialization/solution and random (scratch) initialization/solution. Each experiment is repeated 5 times with different random seeds, and mean values are provided with 80% confidence intervals. In both experiments we observe that the linear path between LT initialization and the pruning solution decreases faster compared to the path that originates from scratch initialization. After training, the linear paths towards the pruning solution change drastically. The path from the scratch solution depicts a loss barrier; the scratch solution seems to be in a different basin than the pruning solution\textsuperscript{\(\dagger\)}.

In contrast, LTs are linearly connected to the pruning solution in both small and large-scale experiments indicating that LTs have the same basin of attraction as the pruning solutions they are derived from. While it seems likely, these results do not however explicitly show that the LT and pruning solutions have learned similar functions.

**Lottery Tickets Learn Similar Functions to the Pruning Solution** Fort, Hu, and Lakshminarayanan (2020) motivate deep ensembles by empirically showing that models starting from different random initializations typically learn different solutions, as compared to models trained from similar initializations, and thus improve performance. In Appendix F, we adopt the analysis of (Fort, Hu, and Lakshminarayanan 2020), but in comparing LT initializations and random initializations using fractional disagreement — the fraction of class predictions over which the LT and scratch models disagree with the pruning solution they were derived from, Kullback–Leibler

\footnote{Note: We use ReLU activation functions, unlike the original architecture (LeCun et al. 1989).}

\footnote{§3.3 We define a basin as a set of points, each of which is linearly connected to at least one other point in the set.}

\footnote{\(\dagger\)This is not always true, it is possible that non-linear low energy paths exist between two solutions (Draxler et al. 2018; Garipov et al. 2018), but searching for such paths is outside the scope of this work.}
Figure 5: MDS Embeddings/L2 Distances. (a), (d): 2D Multi-dimensional Scaling (MDS) embedding of sparse NNs with the same connectivity/mask; (b), (e): the average L2-distance between a pruning solution and other derived sparse networks; (c), (f): linear path between the pruning solution ($\alpha = 1.0$) and LT/scratch at both initialization, and solution (end of training). Top and bottom rows are for MNIST/LeNet5 and ImageNet-2012/ResNet-50 respectively.

Table 3: Ensemble & Prediction Disagreement. We compare the function similarity (Fort, Hu, and Lakshminarayanan 2020) with the original pruning solution and ensemble generalization over 5 sparse models, trained from random initializations and LTs. As a baseline, we also show results for 5 pruned models trained from different random initializations. See Appendix F for full results.

| Initialization | (Top-1) Test Acc. | Ensemble | Disagreement | Disagree. w/ Pruned |
|---------------|-------------------|----------|--------------|---------------------|
| LeNet5        |                   |          |              |                     |
| LT            | 98.52±0.02        | 98.58    | 0.0043±0.0006| 0.0089±0.0002       |
| Scratch       | 97.04±0.15        | 98.00    | 0.0316±0.0023| 0.0278±0.0020       |
| Scratch (Diff. Init.) | 97.19±0.33 | 98.43    | 0.0352±0.0037| 0.0278±0.0032       |
| Prune Restart | 98.60±0.01        | 98.63    | 0.0027±0.0003| 0.0077±0.0003       |
| Pruned Soln.  | 98.53             |          |              |                     |
| 5 Diff. Pruned| 98.30±0.23        | 99.07    | 0.0214±0.0023| 0.0197±0.0019*      |
| MNIST         |                   |          |              |                     |
| LT            | 75.73±0.08        | 76.27    | 0.0894±0.0009| 0.0941±0.0009       |
| Scratch       | 71.16±0.13        | 74.05    | 0.2039±0.0013| 0.2033±0.0012       |
| Prune Restart | 95.60             |          |              |                     |
| Pruned Soln.  | 97.65±0.13        | 77.80    | 0.1620±0.0008| 0.1623±0.0011*      |
| 5 Diff. Pruned|                  |          |              |                     |

* Here we compare 4 different pruned models with the pruning solution LT/scratch are derived from.
Divergence (KL), and Jensen–Shannon Divergence (JSD). The results in Table 4 suggest LTs models converge on a solution almost identical to the pruning solution and therefore an ensemble of LTs brings marginal improvements. Our results also show that having a fixed initialization alone can not explain the low disagreement observed for LT experiments as Scratch solutions obtain an average disagreement of 0.0316 despite using the same initialization, which is almost 10 times larger than that of the LT solutions (0.0043). As a result, LT ensembles show significantly less gains on MNIST and ImageNet-2012 (+0.06% and +0.54% respectively) compared to a random sparse initialization (+0.96% and +2.89%).

Implications: (a) Rewinding of LTs. Frankle et al. (2019) argued that LTs work when the training is stable, and thus converges to the same basin when trained with different data sampling orders. In Figure 3, we show that this basin is the same one found by pruning, and since the training converges to the same basin as before, we expect to see limited gains from rewinding if any. This is partially confirmed by Renda, Frankle, and Carbin (2020) which shows that restarting the learning rate schedule from the pruning solution performs better than rewinding the weights. (b) Transfer of LTs. Given the close relationship between LTs and pruning solutions, the observation that LTs trained on large datasets transfer to smaller ones, but not vice versa (Morcos et al. 2019; Sabatelli, Kestemont, and Geurts 2020) can be explained by a common observation in transfer learning: networks trained in large datasets transfer to smaller ones. (c) LT’s Robustness to Perturbations. Zhou et al. (2019), Frankle, Schwab, and Morcos (2020) found that certain perturbations, like only using the signs of weights at initialization, do not impact LT generalization, while others, like shuffling the weights, do. Our results bring further insights to these observations: As long as the perturbation is small enough such that a LT stays in the same basin of attraction, results will be as good as the pruning solution. (d) Success of LTs. While it is exciting to see widespread applicability of LTs in different domains (Brix, Bahar, and Ney 2020; Li et al. 2020; Venkatesh et al. 2020), the results presented in this paper suggest this success may be due to the underlying pruning algorithm (and transfer learning) rather than LT initializations themselves.

5 Conclusion

In this work we studied (1) why training unstructured sparse networks from random initialization performs poorly and; (2) what makes LTs and DST the exceptions? We identified that randomly initialized unstructured sparse NNs exhibit poor gradient flow when initialized naively and proposed an alternative initialization that scales the initial variance for each neuron separately. Furthermore we showed that modern sparse NN architectures are more sensitive to poor gradient flow during early training rather than initialization alone. We observed that this is somewhat addressed by state-of-the-art DST methods, such as RigL, which significantly improves gradient flow during early training over traditional sparse training methods. Finally, we show that LTs do not improve gradient flow at either initialization or during training, but rather their success lies in effectively re-learning the original pruning solution they are derived from. We showed that a LTs initialization resides within the same basin of attraction as the pruning solution and, furthermore, when trained the LT solution learns a highly similar solution to the pruning solution, limiting their ensemble performance. These findings suggest that LTs are fundamentally limited in their potential for improving the training of sparse NNs more generally.

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A Experimental Details

A.1 Details of Experiments in Section 4.3

The training hyper-parameters used in §4.3 are shared in Table 4. All experiments in this section start with a pruning experiment, after which the sparsity masks found by pruning are used to perform LT experiments. We use iterative magnitude pruning [Zhu and Gupta 2018] in our experiments, which is a well studied and more efficient pruning method as compared to the one used by Frankle and Carbin [2019]. Our pruning algorithm performs iterative pruning without rewinding the weights between intermediate steps and requires significantly less iterations. We expect our results would be even more pronounced with additional rewinding steps.

We use SGD with momentum in all of our experiments. Scratch and Lottery experiments use the same hyper-parameters. Additional specific details of our experiments are shared below.

LeNet5 We prune all layers of LeNet5, so that they reach 95% final sparsity (i.e. 95% of the parameters are zeros). We choose this sparsity, since at this sparsity, we start observing stark differences between Lottery and Scratch in terms of performance. We set the weight decay to zero, similar to the MNIST experiments done in the original LT paper [Frankle and Carbin 2019] and do a grid search over learning-rates={0.1,0.2,0.05,0.02,0.01}. Loss values for the linear interpolation experiments are calculated using 500,000 images from the ImageNet-2012 training set.

ResNet50 We prune all layers of ResNet50, except the first layer, so that they reach 80% final sparsity. In this setting rewinding to the original initialization doesn’t work, hence we use values from 6th epoch. Loss values for the linear interpolation experiments are calculated using 500,000 images from the ImageNet-2012 training set.

A.2 Details of Experiments in Section 4.1 and 4.2

Training hyper-parameters used for these experiments are shared in Table 5.

MNIST In this setting, the hyper-parameters are almost same as in §4.3 except we enable weight decay as it brings better generalization. We do a grid search over weight-decays={0.001,0.0001,0.00005,0.00001,0.0005} and learning-rates={0.1,0.2,0.05,0.02,0.01} and pick the values with top test accuracy. We use the masks found by pruning experiments in all of our MNIST experiments in this section to isolate the effect of the initialization. We simplify the update schedule of Dynamic Sparse Training (DST) methods such that they decay with learning rate. This approach fits well, since the original decay function used in these experiments is the cosine decay which is the same as our learning rate schedule. We scale learning rate such that it matches the initial drop fraction provided. Mask update frequency and initial weights between intermediate steps and requires significantly less iterations. We expect our results would be even more pronounced with additional rewinding steps.

We use SGD with momentum in all of our experiments. Scratch and Lottery experiments use the same hyper-parameters. Additional specific details of our experiments are shared below.

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ResNet50 We prune all layers of ResNet50, except the first layer, so that they reach 80% final sparsity. In this setting rewinding to the original initialization doesn’t work, hence we use values from 6th epoch. Loss values for the linear interpolation experiments are calculated using 500,000 images from the ImageNet-2012 training set.

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We use SGD with momentum in all of our experiments. Scratch and Lottery experiments use the same hyper-parameters. Additional specific details of our experiments are shared below.

LeNet5 We prune all layers of LeNet5, so that they reach 95% final sparsity (i.e. 95% of the parameters are zeros). We choose this sparsity, since at this sparsity, we start observing stark differences between Lottery and Scratch in terms of performance. We set the weight decay to zero, similar to the MNIST experiments done in the original LT paper [Frankle and Carbin 2019] and do a grid search over learning-rates={0.1,0.2,0.05,0.02,0.01}. Loss values for the linear interpolation experiments are calculated using 500,000 images from the ImageNet-2012 training set.

ResNet50 We prune all layers of ResNet50, except the first layer, so that they reach 80% final sparsity. In this setting rewinding to the original initialization doesn’t work, hence we use values from 6th epoch. Loss values for the linear interpolation experiments are calculated using 500,000 images from the ImageNet-2012 training set.
Table 4: [4.3: Experiment Details/Hyperparameters]. Initial Learning Rate (LR), LR Schedule (Sched.), Batchsize (Batch.), Momentum (m), Weight Decay (WD), $t_{\text{start}}$, $t_{\text{end}}$ and $f$ are the pruning starting iteration, end iteration, and mask update frequency respectively.

| Dataset     | Model     | $t_{\text{total}}$ | Epochs | Batch. | LR   | Sched. | m  | WD     | Sparsity | Pruning   | $t_{\text{start}}$ | $t_{\text{end}}$ | $f$ |
|-------------|-----------|--------------------|--------|--------|------|--------|----|--------|-----------|-----------|-------------------|------------------|----|
| MNIST       | LeNet5    | 11719              | 30     | 128    | 0.05 | Cosine | 0  | 95%    | 3000      | 7000      | 100               |                  |     |
| ImageNet    | ResNet50  | 32000              | $\approx$102 | 4096   | 1.6  | Step  | 0.9| $1 \times 10^{-4}$ | 80% | 5000     | 8000      | 2000     |                  |                  |     |

* Step schedule has a linear warm-up in first 5 epochs and decreases the learning rate by a factor of 10 at epochs 30, 70 and 90.

Table 5: [4.4.1: Experiment Details/Hyperparameters]. Initial Learning Rate (LR), LR Schedule (Sched.), Batchsize (Batch.), Momentum (m), Weight Decay (WD), Initial Drop Fraction (Drop.), $t_{\text{end}}$ and $f$ are the pruning mask update frequency and end iteration respectively. $LeNet5+$ row corresponds the LeNet5 experiments with our sparse initialization, whereas $LeNet5$ is the regular masked initialization.

| Dataset     | Model     | $t_{\text{total}}$ | Epochs | Batch. | LR   | Sched. | m  | WD     | Sparsity | Pruning   | $t_{\text{end}}$ | Drop. | $f$ | $t_{\text{end}}$ |
|-------------|-----------|--------------------|--------|--------|------|--------|----|--------|-----------|-----------|-------------------|------|----|-----------------|
| MNIST       | LeNet5    | 11719              | 30     | 128    | 0.05 | Cosine | 0  | 5 \times 10^{-95%} | 0.3 | 500      | 11719     |                  |      |
| ImageNet    | ResNet50  | 32000              | $\approx$102 | 4096   | 1.6  | Step  | 0.9| $1 \times 10^{-80%}$ | 0.3 | 100      | 25000     |                  | 0.1 | 500|

* Step schedule has a linear warm-up in first 5 epochs and decreases the learning rate by a factor of 10 at epochs 30, 70 and 90.

Hessian calculation The Hessian is calculated on full training set using Hessian-vector products. We mask our network after each gradient call and calculate only non-zero rows. After calculating the full Hessian, we use numpy.eigh (van der Walt, Colbert, and Varoquaux 2011) to calculate eigenvalues of the Hessian.

ImageNet-2012 In this setting, hyper-parameters are almost the same as in §4.4 except for VGG16 architecture, where we use a smaller batch size and learning rate. For all DST methods, we use a cosine drop schedule (Dettmers and Zettlemoyer 2019) and hyper-parameters proposed by Evci et al. (2019). For VGG, we reduce the mask update frequency and the initial drop fraction, as we observe better performance after doing a grid search over {50, 100, 500} and {0, 0.1, 0.3, 0.5} respectively. We also use a non-uniform (ERK) sparsity distribution among layers as described in Evci et al. (2020), since we observed that it brings better performance.

B Glorot/He Initialization Generalized to Neural Networks with Heterogeneous Connectivity: Full Explanation/Derivation

Here we derive the full generalized initialization for both the forwards/backwards cases (i.e. fan-in/fan-out), refer to Fig. 6 for an illustration of how the connectivity for the fan-in/fan-out cases are determined for each neuron.

B.1 Generalized Glorot/He Initialization: Backwards, Forwards and Average Cases

For every weight $w_{ij}^{[\ell]} \in W^{n^{[\ell]} \times n^{[\ell-1]}}$ in a layer $\ell$ with $n^{[\ell]}$ neurons, connecting neuron $i$ in layer $\ell$ to neuron $j$ in layer $(\ell - 1)$ with $n^{[\ell-1]}$ neurons, and weight mask $[m_{ij}^{[\ell]}] = M^{\ell} \in [0, 1]^{n^{[\ell]} \times n^{[\ell-1]}}$,

\begin{equation}
\text{Glorot and Bengio (2010)}: \quad w_{ij}^{[\ell]} \sim \mathcal{N} \left(0, \frac{1}{n_{\ell}} \right) \\
\text{He et al. (2015)}: \quad w_{ij}^{[\ell]} \sim \mathcal{N} \left(0, \frac{2}{\ell+1} \right)
\end{equation}

\begin{equation}
\begin{aligned}
\text{forward} & \quad \left[ \begin{array}{c}
\text{fan-in}^{[\ell]} \\
\text{fan-out}^{[\ell]}
\end{array} \right]_i \\
\text{backward} & \quad \left[ \begin{array}{c}
\text{fan-out}^{[\ell]} \\
\text{fan-in}^{[\ell]} + \text{fan-out}^{[\ell]} / 2
\end{array} \right]_j \\
\text{average} & \quad \left[ \begin{array}{c}
\text{fan-in}^{[\ell]} \\
\text{fan-out}^{[\ell]}
\end{array} \right]_i + \text{fan-out}^{[\ell]} / 2
\end{aligned}
\end{equation}

where,

\begin{equation}
\begin{aligned}
\text{fan-in}^{[\ell]}_i & = \sum_{j=1}^{n^{[\ell-1]}} m_{ij}^{[\ell]} \\
\text{fan-out}^{[\ell]}_j & = \sum_{i=1}^{n^{[\ell]}} m_{ij}^{[\ell]}
\end{aligned}
\end{equation}

are the number of incoming and outgoing connections respectively. In the special case of a dense layer where $m_{ij}^{[\ell]} = 1$, $\forall i, j$, Eq. (1) reduces to the initializations proposed by Glorot and Bengio (2010) and He et al. (2015) since $\text{fan-in}^{[\ell]}_i = n^{[\ell-1]}$, $\forall i$, and $\text{fan-out}^{[\ell]}_j = n^{[\ell]}$, $\forall j$.

B.2 Derivation: Fixed Mask, Forward Propagation

Given a sparse NN, where the output of a neuron $a_i$ is given by, $a_i^{[\ell]} = f \left( z_i^{[\ell]} \right)$, where $z_i^{[\ell]} = \sum_j m_{ij}^{[\ell]} w_{ij}^{[\ell]} a_j^{[\ell-1]}$,
Figure 6: Glorot/He Initialization for a Sparse NN. [Glorot and Bengio 2010; He et al. 2015] restrict the outputs of all neurons to be zero-mean and of unit variance. All neurons in a dense NN layer (a) have the same fan-in/fan-out, whereas in a sparse NN (b) the fan-in/fan-out can differ for every neuron, potentially requiring sampling from a different distribution for every neuron. The fan-in matrix contains the values used in Eq. (1) for each neuron.

where \( m_{ij}^{[\ell]} \) and \( w_{ij}^{[\ell]} \) are the mask and weights respectively for layer \( \ell \), and \( a_{j}^{[\ell-1]} \) the output of the previous layer. Assume the mask \( M^{[\ell]} = 1^{n^{[\ell]} \times n^{[\ell-1]}} \) is constant, where \( 1^{n^{[\ell]} \times n^{[\ell-1]}} \) is an indicator matrix.

As in [Glorot and Bengio 2010] we want to ensure \( \text{Var}(a_{i}^{[\ell-1]}) = \text{Var}(a_{i}^{[\ell-1]}), \) and mean \( \langle a_{i}^{[\ell-1]} \rangle = 0 \). Assume that \( f(x) \approx x \) for \( x \) close to 0, e.g. in the case of \( f(x) = \tanh(x) \), and that \( w_{ij}^{[\ell]} \) and \( a_{j}^{[\ell-1]} \) are independent,

\[
\text{Var}(a_{i}^{[\ell-1]}) \approx \text{Var}(z_{i}^{[\ell]}) \\
= \text{Var} \left( \sum_{j=1}^{n^{[\ell-1]}} m_{ij}^{[\ell]} w_{ij}^{[\ell]} a_{j}^{[\ell-1]} \right) \\
= \sum_{j=1}^{n^{[\ell-1]}} \text{Var} \left( m_{ij}^{[\ell]} w_{ij}^{[\ell]} a_{j}^{[\ell-1]} \right) \\
\text{(independent sum) (4) \hspace{1cm} (5) \hspace{1cm} (6)} \\
= \sum_{j=1}^{n^{[\ell-1]}} \left( m_{ij}^{[\ell]} \right)^{2} \text{Var} \left( w_{ij}^{[\ell]} a_{j}^{[\ell-1]} \right) \\
\therefore: m_{ij}^{[\ell]} \text{ is constant, } \text{Var}(cX) = c^{2} \text{Var}(X) \hspace{1cm} (7) \\
= \sum_{j=1}^{n^{[\ell-1]}} m_{ij}^{[\ell]} \text{Var} \left( w_{ij}^{[\ell]} a_{j}^{[\ell-1]} \right) \\
\therefore: m_{ij}^{[\ell]} \in [0,1], \left( m_{ij}^{[\ell]} \right)^{2} = m_{ij}^{[\ell]} \hspace{1cm} (8) \\
= \sum_{j=1}^{n^{[\ell-1]}} m_{ij}^{[\ell]} \text{Var}(w_{ij}^{[\ell]}) \text{Var}(a_{j}^{[\ell-1]}) \hspace{1cm} (9) \hspace{1cm} (10) \\
\text{(independent product) (11) \hspace{1cm} (12) \hspace{1cm} (13) \hspace{1cm} (14) \hspace{1cm} (15) \hspace{1cm} (16)}
\]

Therefore, in order to have the output of each neuron \( a_{i}^{[\ell]} \) in layer \( \ell \) to have unit variance, and mean 0, we need to sample the weights for each neuron from the normal distribution,

\[
[w_{ij}^{[\ell]}] \sim \mathcal{N} \left( 0, \frac{1}{\text{fan-in}_{i}^{[\ell]}}, \right) \hspace{1cm} (17)
\]

where \( s_{i}^{\ell} \) is the sparsity of weights of the neuron with output \( a_{i} \). For the ReLU activation function, following the derivation in [He et al. 2015],

\[
[w_{ij}^{[\ell]}] \sim \mathcal{N} \left( 0, \frac{2}{\text{fan-in}_{i}^{[\ell]}}, \right) \hspace{1cm} (18)
\]

B.3 Fixed Mask: Backward Pass

Given a sparse NN, where the output of a neuron \( a_{i} \) is given by, \( a_{i}^{[\ell]} = f \left( z_{i}^{[\ell]} \right) \), where \( z_{i}^{[\ell]} = \sum_{j} m_{ij}^{[\ell]} w_{ij}^{[\ell]} a_{j}^{[\ell-1]} \),
where \( m_{ij}^{[\ell]} \in M^{[\ell]} \) and \( w_{ij}^{[\ell]} \in W^{[\ell]} \) are the mask and weights respectively for layer \( \ell \), and \( a_{ij}^{[\ell-1]} \) the output of the previous layer. Assume the mask \( M^{[\ell]} \in \mathbb{1}_{n^{[\ell]} \times n^{[\ell-1]}} \) is constant, where \( \mathbb{1}_{n^{[\ell]} \times n^{[\ell-1]}} \) is an indicator matrix, and let \( L(\theta = \{ W^{[\ell]}, \ell = 0 \ldots N \}) \) be the loss we are optimizing.

As in (Glorot and Bengio 2010), from the backward-propagation standpoint, we want to ensure \( \operatorname{Var}(\partial L / \partial z_i^{[\ell]}) = \operatorname{Var}(\partial L / \partial z_i^{[\ell-1]}) \), and mean(\( \partial L / \partial z_i^{[\ell]} \)) = 0. Assume that \( f'(0) = 1 \),

\[
\operatorname{Var}(\partial L / \partial z_j^{[\ell]}) \approx \operatorname{Var}(\partial L / \partial a_j^{[\ell-1]}) \tag{19}
\]

\[
= \operatorname{Var}\left( \sum_{i=1}^{n^{[\ell]}} m_{ij}^{[\ell]} w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}} \right) \tag{20}
\]

\[
= \sum_{i=1}^{n^{[\ell]}} \operatorname{Var}(m_{ij}^{[\ell]} w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) . \tag{21}
\]

\[
= \sum_{i=1}^{n^{[\ell]}} \left( m_{ij}^{[\ell]} \right)^2 \operatorname{Var}(w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) . \tag{22}
\]

\[
\therefore \quad m_{ij}^{[\ell]} \text{ is constant, } \operatorname{Var}(cX) = c^2 \operatorname{Var}(X) \nonumber 
\]

\[
= \sum_{i=1}^{n^{[\ell]}} m_{ij}^{[\ell]} \operatorname{Var}(w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) . \tag{23}
\]

\[
\therefore \quad m_{ij}^{[\ell]} \in [0,1], \left( m_{ij}^{[\ell]} \right)^2 = m_{ij}^{[\ell]} .
\]

\[
= \sum_{i=1}^{n^{[\ell]}} m_{ij}^{[\ell]} \operatorname{Var}(w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) \operatorname{Var}(\frac{\partial L}{\partial z_i^{[\ell]}}) . \tag{24}
\]

\[
\text{(independent product)}
\]

Assume \( \operatorname{Var}(w_{ij}^{[m]}) = \operatorname{Var}(w_{ij}^{[n]}), \forall n, m \), i.e. the variance of all weights for a given neuron are the same, and \( \operatorname{Var}(\partial L / \partial z_i^{[\ell]}) = \operatorname{Var}(\partial L / \partial z_i^{[\ell-1]}), \) i.e. the variance of the output gradients of each neuron at layer \( \ell \) are the same. Then we can simplify Eq. (25).

\[
\operatorname{Var}(\partial L / \partial z_j^{[\ell]}) = \sum_{i=1}^{n^{[\ell]}} m_{ij}^{[\ell]} \operatorname{Var}(w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) \tag{25}
\]

\[
= \operatorname{Var}(w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) \sum_{i=1}^{n^{[\ell]}} m_{ij}^{[\ell]} . \tag{26}
\]

Let neuron \( i \)'s number of non-masked weights be denoted \( \text{fan-out}_i^{[\ell]} \), where \( \text{fan-out}_i^{[\ell]} = \sum_{i=1}^{n^{[\ell]}} m_{ij}^{[\ell]} \), then

\[
\text{Var}(\partial L / \partial z_j^{[\ell]}) = \text{fan-out}_j^{[\ell]} \operatorname{Var}(w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) \tag{27}
\]

Recall, \( \operatorname{Var}(\partial L / \partial z_j^{[\ell]}) = \operatorname{Var}(\partial L / \partial z_i^{[\ell]}) \)

\[
\Rightarrow \operatorname{Var}(w_{ij}^{[\ell]} \frac{\partial L}{\partial z_i^{[\ell]}}) = \frac{1}{\text{fan-out}_j^{[\ell]}}, \tag{28}
\]

Therefore, in order to have the output of each neuron \( a_i^{[\ell]} \) in layer \( \ell \) to have unit variance, and mean 0, we need to sample the weights for each neuron from the normal distribution,

\[
[w_{ij}^{[\ell]}] \sim \mathcal{N}\left(0, \frac{1}{\text{fan-out}_i^{[\ell]}}\right), \tag{29}
\]

where \( s_i^{[\ell]} \) is the sparsity of weights of the neuron with output \( a_i^{[\ell]} \). For the ReLU activation function, following the derivation in (He et al. 2015),

\[
[w_{ij}^{[\ell]}] \sim \mathcal{N}\left(0, \frac{2}{\text{fan-out}_i^{[\ell]}}\right) . \tag{30}
\]

C Additional Gradient Flow Plots

Here we share additional gradient flow figures for method(initialization) combinations presented in 3 and 2.

In Fig. 7b, we show the gradient flow for DST methods and scratch training. Using RigL helps improve gradient flow with both initialization; helping learning to start earlier than regular Scratch training. SET seem to have limited effect on the gradient flow.

In Fig. 7b, we share gradient flow when the scaled initialization of (Liu et al. 2019) is used. Similar to the proposed initialization, we observe improved gradient flow for all cases. Different than our initialization however, RigL doesn’t improve gradient flow in this setting; highlighting an interesting future research direction on the relationship between initialization and the DST methods.

In Fig. 7b, we share gradient flow when He initialization is used instead of Glorot initialization. We observe that Scratch training starts learning faster in this case. Gradient flow seems to be similar for other sparse initialization methods.

In Fig. 8 we share gradient flow improvements after DST updates on connectivity for different initialization methods. ResNet-50 curves match the results in Fig. 3. LeNet5 curves however seem to be adversely affected by poor initialization at the beginning of the training. We start observing improvements with RigL when the learning starts (around the 400th iteration).

D Fully Connected Neural Network

Experiments on MNIST

In this section we repeat our experiments from §4.1 using different sparse initialization methods, and analyzing gradient flow, for a standard 2-layer fully-connected NN with 2 hidden layers of size 300 and 100 units. We use the same grid used
Figure 7: **Gradient Flow of Sparse LeNet-5s during Training.** Gradient flow during training averaged over multiple runs, ‘+’ indicates training runs with our proposed sparse initialization. ‘+Liu’ indicates initialization proposed by (Liu et al. 2019). ‘He’ suffix refers to He initialization where ‘scale=2’ and ‘fanin’ options are used for the variance scaling initialization.

Figure 8: **Effect of Mask Updates in Dynamic Sparse Training.** Effect of mask updates on the gradient norm. We measure the gradient norm before and after the mask updates and plot the $\Delta$. ‘+’ indicates proposed initialization and used in MNIST experiments.
in LeNet5 experiments for hyper-parameter selection. Best results were obtained with a learning rate of 0.2, a weight decay coefficient of 0.0001 and an mask update frequency of 500 (used in DST methods). The rest of the hyperparameters remained unchanged from the LeNet5 experiments.

The results of training with various initialization methods is shown in Table 6. Although the results are not as drastic as with LeNet5, we see that here too sparsity aware initialization (the proposed initialization, and that of Liu et al. 2019) shows a significant improvement in the test accuracy of Scratch, and RigL or our proposed initialization, although not quite reaching lottery or RigL accuracy. Finally, we see no significant effect on SET training, with none of the initialization variants having a significant increase over any of the others, although the Liu et al. 2019 initialization does marginally better.

The gradient flow of this model is shown in Fig. 9. While we see moderate improvements to Scratch gradient flow early on in training with our proposed initialization (3), RigL shows significantly higher gradient flow throughout training, in particular after mask updates (5), mirroring the results of LeNet5. The interpolation graphs in (c) only differ slightly from that of LeNet5, again showing that our results for LeNet5 broadly hold for the fully-connected model.

E Hessian Spectrum of LeNet5

Given a loss function $L$ and parameters $\theta$, we can write the first order Taylor approximation of the change in loss $\Delta L = L(\theta^{t+1}) - L(\theta^t)$ after a single training step with the learning rate $\epsilon > 0$ as:

$$\Delta L \approx -\epsilon \nabla L(\theta)^T \nabla L(\theta), \quad (31)$$

Note that as long as the error is small, gradient descent is guaranteed to decrease the loss by an amount proportional to $\nabla L(\theta)^T \nabla L(\theta)$, which we refer as the gradient flow. In practice large learning rates are used, and the first order approximation might not be accurate. Instead we can look at the second order approximation of $\Delta L$:

$$\Delta L \approx -\alpha \nabla L(\theta)^T \nabla L(\theta) + \frac{\alpha^2}{2} \nabla L(\theta)^T H(\theta) \nabla L(\theta), \quad (32)$$

where $H(\theta)$ is the Hessian of the loss function. The eigenvalue spectrum of Hessian can help us understand the local landscape (Sagun et al. 2017), and help us identify optimization difficulties (Ghorbani, Krishnan, and Xiao 2019). For example, if and when the gradient is aligned with large magnitude eigenvalues, the second term of Eq. (32) can have a significant effect on the optimization of $L$. If the gradient is aligned with large positive eigenvalues, it can prevent gradient descent from decreasing the loss and harm the optimization. Similarly, if it is aligned with negative eigenvalues it can help to accelerate optimization.

We show the Hessian spectrum before and after the topology updates in Fig. 10. After RigL updates we observe new negative eigenvalues with significantly larger magnitudes. We also see larger positive eigenvalues, which disappear after few iterations. In comparison, the effect of SET updates on the Hessian spectrum seems limited.

We also evaluate the Hessian spectrum of LeNet5 during the training. In Fig. 11b we observe similar shapes for each method on the positive side of the spectrum, however, on the negative side dense models seem to have more mass. We plot the magnitude of the largest negative eigenvalue to characterize this behaviour in Fig. 11c. We observe a significant difference between sparse and dense models and observe that sparse networks trained with RigL have larger negative eigenvalues.

F Comparing Function Similarity

Table 7 gives a full list of comparison metrics of the predictions on the test set for LeNet5 on MNIST and ResNet50 on ImageNet-2012, in particular here we also compare the output probability distributions using relevant metrics. Fort, Hu, and Lakshminarayanan (2020) motivate deep ensembles by empirically showing that models starting from different random initializations typically learn different solutions, as compared to models trained from similar initializations. Here we adopt the analysis of Fort, Hu, and Lakshminarayanan (2020), but in comparing LT initializations and random initializations using fractional disagreement — the fraction of class predictions over which the LT and scratch solutions disagree with the pruning solution they were derived from. In Table 7 we show the mean fractional disagreement over all pairs of models. We run two versions of scratch training: (1) Scratch (Diff. Init.) different weight initialization and different data order (2) Scratch same weight initialization and different data order for 5 different seeds the experiments are ran. Finally, we restart training starting from the pruning solution (Prune Restart) using, again, 5 different data orders.

The results presented in Table 7 suggest that all 5 LT models converge on a solution almost identical to the pruning solution. Interestingly, the 5 LT models are even more similar to each other (Disagree. column) than the pruning solution, possibly because they share an initialization and training is stable (Frankle et al. 2019). The disagreement of Prune Restart solutions with the original pruning solution matches the disagreement of lottery solutions; showing the extent of similarity between LT and pruning solutions.

Our results show that having a fixed initialization alone cannot explain the low disagreement observed for LT experi-

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**We share videos of these transitions in supplementary material.
Figure 9: **Sparse 300-100 MLP experiments.** Gradient flow during training averaged over multiple runs, ‘+’ indicates training runs with our proposed sparse initialization.

Figure 10: Hessian spectrum before and after mask updates: (left) SET (right) RigL. Similar to [Ghorbani, Krishnan, and Xiao 2019](#), we estimate the spectral density of Hessian using Gaussian kernels.
Figure 11: MNIST Hessian spectrum experiments.
Table 7: Ensemble/Prediction Disagreement. In order to show the function similarity of LTs to the pruning solution, we follow the analysis of [Fort, Hu, and Lakshminarayanan 2020], and compare the function similarity and ensemble generalization over 5 sparse models trained using random initializations and LTs with the original pruning solution they are derived from. The fractional disagreement is the pairwise disagreement of class predictions over the test set, as compared within the group of sparse models, and as compared to the pruned model whose mask they were derived from. KL and JSD compare the prediction distributions over all the test samples.

| Init. Method | Test Acc. (Top-1) | Ensemble | Pairwise Disagree. | Disagree. w/ Pruned | Pairwise KL | KL w/ Pruned | 5-model JSD | JSD w/ Pruned |
|--------------|------------------|----------|-------------------|---------------------|-------------|--------------|-------------|--------------|
| Pruned Soln.*| 98.53            | –        | –                 | –                   | –           | –            | –           | –            |
| LT           | 98.52 ±0.02      | 98.58    | 0.0043±0.0006     | 0.0089±0.0002       | 0.0030±0.0004| 0.0039±0.0028| 0.0140 ±0.0004| 0.0033±0.0001|
| Scratch      | 97.04 ±0.15      | 98.00    | 0.0316±0.0023     | 0.0278±0.0020       | 0.0917±0.0126| 0.0506±0.0238| 0.1506 ±0.0130| 0.0159±0.0009|
| Scratch (Diff. Init.) | 97.19 ±0.33 | 98.43    | 0.0352±0.0037     | 0.0278±0.0032       | 0.1358±0.0202| 0.0650±0.0301| 0.1468 ±0.0295| 0.0157±0.0021|
| Prune Restart| 98.60 ±0.01      | 98.63    | 0.0027±0.0003     | 0.0077±0.0003       | 0.0014±0.0001| 0.0019±0.0014| 0.0105 ±0.0001| 0.0027±0.0000|
| Pruned w/ Diff. Init.** | 98.30 ±0.23 | 99.07    | 0.0214±0.0023     | 0.0197±0.0019*** | 0.0741±0.0107| 0.0438±0.0241| 0.0752 ±0.0133| 0.0103±0.0010|

| Init. Method      | Test Acc. (Top-1) | Ensemble | Pairwise Disagree. | Disagree. w/ Pruned | Pairwise KL | KL w/ Pruned | 5-model JSD | JSD w/ Pruned |
|-------------------|------------------|----------|-------------------|---------------------|-------------|--------------|-------------|--------------|
| Pruned Soln.*     | 0.7554           | –        | –                 | –                   | –           | –            | –           | –            |
| LT                | 75.730±0.008     | 76.27    | 0.0894±0.0008     | 0.0941±0.0009       | 3325.0±26.4 | 3605.0±12.4 | 787.20±5.24 | 853.70±3.55  |
| Scratch           | 71.16 ±0.13      | 74.05    | 0.2039±0.0013     | 0.2033±0.0012       | 15.787 ±105 | 20.442 ±178 | 3316.0 ±13.2 | 3847.0 ±26.0 |
| Pruned w/ Diff. Init.** | 75.65 ±0.13 | 77.78    | 0.1620±0.0008     | 0.1623±0.0011***    | 13.158 ±143 | 13.087±55.9 | 2760.0 ±16.8 | 2755.00±4.96 |

* This is the pruning solution that the LT and scratch models are derived from.
** 5 pruning solutions found with different random initialization, one of which is the pruning solution above.
*** Here we compare 4 different pruned models with the pruning solution the LT/Scratch are derived from.
ments as Scratch solutions obtain an average disagreement of 0.0316 despite using the same initialization, which is almost 10 times more than the LT solutions (0.0043). Finding different LT initialization is costly, however using a different initialization in Scratch (Diff. Init.) training is free as the initializations are random. Using different initializations we can obtain more diverse solutions and thus achieve higher ensemble accuracy. As suggested by the analysis of Fort, Hu and Lakshminarayanan (2020), ensembles of different solutions are more robust, and generalize better, than ensembles of similar solutions. An ensemble of 5 LT models with low disagreement doesn’t significantly improve generalization as compared to an ensemble of 5 different pruning solutions with similar individual test accuracy. We further demonstrate these results by comparing the output probability distributions using the KL, and JSD.