COHOMOLOGY OF GROUP THEORETIC DEHN FILLINGS II

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Abstract. We study the cohomology of group theoretic Dehn fillings. Applying the Cohen-Lyndon property for sufficiently deep Dehn fillings of hyperbolically embedded subgroups $H \hookrightarrow G$, obtained by the second named author in [Sun20], we derive a spectral sequence that computes the cohomology of the corresponding Dehn filling quotients $\overline{G}$. As an application, we establish an isomorphism between the relative cohomology of the group pair $(G,H)$ and its sufficiently deep Dehn filling quotient pair $(\overline{G},\overline{H})$. This allows us to generalise the results of Fujiwara and Manning on simplicial volume of Dehn fillings of hyperbolic manifolds to Dehn fillings of Poincaré duality pairs.

We also generalize the results of Olshanskii [Ols95], Dahmani-Guirardel-Osin [DGO17] and Hull [Hul16] on SQ-universality and common quotients of acylindrically hyperbolic groups by adding cohomological finiteness conditions. We apply these results to obtain hyperbolic and acylindrically hyperbolic quotients with special properties.

1. Introduction

1.1. Dehn surgery in 3-manifolds. In the late 1970’s, Thurston dramatically changed the study of 3-manifolds by introducing his Geometrization Conjecture. As supporting evidence, Thurston proved that many non-Haken 3-manifolds satisfy the conjecture [Thu82], using the notion of a Dehn surgery, which is a two-step procedure of modifying a 3-manifold by first cutting off a solid torus and then gluing the torus back in a different way. Another motivation of Dehn surgery comes from the Lickorish-Wallace theorem, which states that every closed connected orientable 3-manifold can be constructed from the 3-sphere by using finitely many Dehn surgeries.

The second step of the surgery, called Dehn filling, starts with a 3-manifold $M$ with toral boundary and constructs a new manifold by gluing a solid torus to $M$ by identifying their boundaries. Topologically distinct ways of gluing a solid torus are parametrized by free homotopy classes of essential simple closed curves on $\partial M$ (the image of the meridian circle of the solid torus under the identification), called slopes. For a slope $s$, the new manifold constructed by the corresponding Dehn filling is denoted by $M_s$. A celebrated result of Thurston asserts that most Dehn fillings preserve hyperbolicity.

Theorem 1.1 (Thurston [Thu82]). Let $M$ be a compact orientable 3-manifold with boundary a torus, and with interior admitting a complete finite volume hyperbolic structure. Then for all but finitely many slopes $s$ on $\partial M$, $M_s$ admits a hyperbolic structure.

1.2. Group theoretic Dehn fillings. There is an analogous construction in group theory, called (group theoretic) Dehn filling, which can be formalized as follows. Given a group $G$
with a subgroup $H$ and a normal subgroup $N$ of $H$, the Dehn filling associated with the triple $(G, H, N)$ is the quotient $G/\langle \langle N \rangle \rangle$, where $\langle \langle N \rangle \rangle$ is the normal closure of $N$ in $G$.

The relation between these two versions of Dehn fillings can be seen as follows: under the assumptions of Theorem 1.1, the natural homomorphism $\pi_1(\partial M) \to \pi_1(M)$ is injective and thus $\pi_1(\partial M)$ can be thought of as a subgroup of $\pi_1(M)$. Let $G = \pi_1(M)$ and $H = \pi_1(\partial M)$. Every slope $s$ on $\partial M$ generates a normal subgroup $N_s \lhd H$. As $s$ is the image of the meridian circle of the solid torus, which bounds a disc, we have $\pi_1(M_s) = G/\langle \langle N_s \rangle \rangle$.

Dehn filling is a fundamental tool in group theory. It appears, for instance, in the solution of the Virtual Haken Conjecture [Ago13], the study of the Farrell-Jones Conjecture and the isomorphism problem of relatively hyperbolic groups [ACG18, DG18], and the construction of purely pseudo-Anosov normal subgroups of mapping class groups [DGO17]. Other applications of Dehn fillings can be found for example in [AGM16, GMS16].

Algebraic analogs of Theorem 1.1 can be proved for groups satisfying certain negative curvature conditions. The first result of this kind was for relatively hyperbolic groups by Osin [Osi07] and independently, by Groves-Manning [GM08]. Later, Dahmani-Guirardel-Osin [DGO17] introduced a generalization of relative hyperbolicity based on the notion of a hyperbolically embedded subgroup and proved a generalization of the main results of [Osi07, GM08]. We postpone the definition and motivation of hyperbolically embedded subgroups until Section 3.2 and only discuss several examples for the moment. The reader is referred to the survey [Osi18] for other examples. We use $H \hookrightarrow_h G$ to indicate that $H$ is a hyperbolically embedded subgroup of $G$.

**Example 1.2.** If a group $G$ is hyperbolic relative to its subgroup $H$, then $H \hookrightarrow_h G$ [DGO17, Proposition 2.4]. In particular, if $M$ is a compact orientable manifold with one boundary component and $M \setminus \partial M$ admits a complete finite volume hyperbolic structure, then $\pi_1(\partial M) \hookrightarrow_h \pi_1(M)$ [Bow12, Far98].

**Example 1.3.** Another typical example arises if a group $G$ acts on a Gromov hyperbolic space $S$ acylindrically by isometries and $g \in G$ is a loxodromic element. Then there exists a maximal virtually-cyclic subgroup $E(g) \leq G$ containing $g$ such that $E(g) \hookrightarrow_h G$ [DGO17, Corollary 2.9]. In particular, if $G$ is a word-hyperbolic group (resp. mapping class group of a finite type surface [DGO17, Theorem 2.19], outer automorphism group of a finite rank free group [DGO17, Theorem 2.20]) and $g$ is an infinite order (resp. a pseudo-Anosov, a fully irreducible) element, then $E(g) \hookrightarrow_h G$.

The following is a group theoretic analog of Thurston’s Theorem 1.1 due to Dahmani-Guirardel-Osin.

**Theorem 1.4** (Dahmani-Guirardel-Osin [DGO17]). Let $G$ be a group with a subgroup $H \hookrightarrow_h G$. Then there exists a finite set $F \subseteq H \setminus \{1\}$ such that if $N \lhd H$ and $N \cap F = \emptyset$, then the natural homomorphism $H/N \to G/\langle \langle N \rangle \rangle$ maps $H/N$ injectively onto a hyperbolically embedded subgroup of $G/\langle \langle N \rangle \rangle$.

In fact, Theorem 1.4 generalizes Theorem 1.1: let $M$ be as in Theorem 1.1. By Example 1.2, for all but finitely many slopes $s$ on $\partial M$, Theorem 1.4 implies that $\pi_1(M_s)$ is word-hyperbolic and [GM18, Corollary 1.11] implies that $\pi_1(M_s)$ is one-ended. The Geometrization Conjecture, proved by Perelman, then implies that $M_s$ admits a hyperbolic structure.
1.3. Motivation: cohomology of Dehn fillings. Theorem 1.1 asserts that $M_s$ is often hyperbolic and thus its universal cover is $\mathbb{H}^3$. It follows that the cohomology of $\pi_1(M_s)$ can be understood by studying the cohomology of $M_s$ and the action of $\pi_1(M_s)$ on $\mathbb{H}^3$. It is therefore natural to investigate the cohomology of $G/\langle\langle N\rangle\rangle$ in the more general setting of Theorem 1.4 where one may look for a similar geometric footing.

Question 1.5. For a group $G$ with a subgroup $H \hookrightarrow_h G$ and a normal subgroup $N \triangleleft H$, what can be said about the cohomology of $G/\langle\langle N\rangle\rangle$?

The main goal of this series of two papers is to address this question and to illustrate the implications of the results in this direction.

We should point out that even though we consider the general case of hyperbolically embedded subgroups $H \hookrightarrow_h G$, all of our results are new in the special case when $G$ is hyperbolic relative to $H$.

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2. Statements of main results

2.1. Cohomological properties of Dehn fillings. To simplify the statement, we introduce the following terminology and notation.

Definition 2.1. Let $G$ be a group and $H$ a subgroup of $G$. We say that a property $\mathcal{P}$ holds for sufficiently deep normal subgroups if there is a finite set $\mathcal{F} \subseteq H \setminus \{1\}$ such that $\mathcal{P}$ holds whenever $N$ is a normal subgroup of $H$ and $N \cap \mathcal{F} = \emptyset$.

Given a normal subgroup $N$ of $H$, let $\overline{G} = G/\langle\langle N\rangle\rangle$ and $\overline{H} = H/N$.

Theorem 1.4 can now be restated as: let $G$ be a group with a subgroup $H \hookrightarrow_h G$. Then for sufficiently deep $N \triangleleft H$, the natural homomorphism $\overline{H} \to \overline{G}$ maps $\overline{H}$ injectively onto a hyperbolically embedded subgroup of $\overline{G}$.

The following is a summary of our main results on cohomological properties of Dehn fillings.

Theorem A. Let $G$ be a group with a subgroup $H \hookrightarrow_h G$. Then the following hold for all sufficiently deep $N \triangleleft H$ and all $\overline{G}$-modules $A$.

(i) There is a spectral sequence

$$E_2^{p,q}(A) = \begin{cases} H^p(\overline{H}; H^q(N; A)) & \text{for } q > 0 \\ H^p(\overline{G}; A) & \text{for } q = 0 \end{cases}$$

where the action of $G$ on $A$ factors through $\overline{G}$. In particular, the action of $N$ on $A$ fixes $A$ pointwise.
(ii) (Algebraic Excision) For all \( n \geq 0 \), there is a natural isomorphism induced by the quotient maps \( G \to \overline{G} \) and \( H \to \overline{H} \),
\[
H^n(\overline{G}; \overline{H}; A) \cong H^n(G, H; A).
\]

The spectral sequence together with the algebraic excision have the following application.

**Corollary 2.2.** Let \( G \) be a group with a subgroup \( H \hookrightarrow G \). Then for all sufficiently deep \( N \lhd H \) and all \( \overline{G} \)-modules \( A \), we have

(i) For all \( n \geq \text{cd}(H) + 1 \),
\[
H^n(\overline{G}; A) \cong H^n(G; A) \oplus H^n(\overline{H}; A).
\]

(ii) \( \text{cd}(\overline{G}) \leq \max\{\text{cd}(G), \text{cd}(H) + 1, \text{cd}(\overline{H})\} \).

(iii) If \( G \) is of type \( FP_n \) for some \( n \in \mathbb{N}^+ \cup \{\infty\} \) (resp. \( FP \)), then \( \overline{G} \) is of type \( FP_n \) (resp. \( FP \)) if and only if \( \overline{H} \) is of type \( FP_n \) (resp. \( FP \)).

Here, \( \text{cd}(G) \) stands for the **cohomological dimension** of a group \( G \), and \( \mathbb{N}^+ \) stands for the set of positive integers. This notion, **property \( FP_n \)**, **property \( FP \)**, and **relative cohomology of group pairs** are reviewed in Section 3.1.

Analogous homological statements to Theorem A and Corollary 2.2 also hold. The spectral sequence in Theorem A (i) is a refinement of the classical Lyndon-Hochschild-Serre spectral sequence \([HS53, \text{Lyn}48]\) in the setting of Dehn fillings.

Let \( M \) and \( M_s \) be as in Theorem 1.1 and let \( G = \pi_1(M), H = \pi_1(\partial M), N = \langle s \rangle \). Then Theorem A (ii) is an immediate consequence of excision. Therefore, Theorem A (ii) can be thought of as an algebraic analog of excision.

Instead of proving Theorem A, we will prove more general results (see Section 4), which cover the case of a **hyperbolically embedded family of subgroups** and will be useful in the proof of Theorems B, C, and E below, and also cover the case of **weakly hyperbolically embedded subgroups** and can be applied to graph of groups (see Example 3.7 (e)).

Corollary 2.2 was recently used by Arenas \([\text{Are}22]\), who gave a variation of the Rips Construction that produces cubulated hyperbolic groups of cohomological dimension bounded above by the cohomological dimension of an associated compact special cube complex.

### 2.2. Poincaré duality and simplicial volume

In \([FM10]\), Fujiwara and Manning generalize Gromov-Thurston’s 2\( \pi \)-theorem on Dehn fillings of 3-manifolds to higher dimensional finite volume hyperbolic manifolds \( M^n \) with toral cusps. The resulting 2\( \pi \)-fillings are pseudo-manifolds and are manifolds if and only if all the filling cores have dimension exactly \( n - 2 \). They prove that every 2\( \pi \)-filling admits a complete locally CAT(0) metric. In \([FM11]\), they show that the simplicial volume of every 2\( \pi \)-filling is positive and bounded above by the relative simplicial volume \( ||M, \partial M|| \) of \( M \).

The following two theorems can be seen as natural generalizations and group theoretic analogs of the results of Fujiwara and Manning.

**Theorem B.** Let \( G \) be a group and \( \mathcal{H} = \{H_i\}_{i=1}^m \) a collection of subgroups such that \( \mathcal{H} \hookrightarrow G \). Suppose, for some integer \( 2 \leq n \), \( (G, \mathcal{H}) \) is a \( \text{PD}(n) \)-pair and that there are...
sufficiently deep \( \{ \mathbb{Z} \cong N_i \triangleleft H_i \}_{i=1}^m \), such that every \( \prod_i \) is a PD \((n-2)\)-group. Then \( G \) is a PD \((n)\)-group.

**Theorem C.** Let \( G \) be a group and \( \mathcal{H} = \{ H_i \}_{i=1}^m \) a collection of subgroups such that \( \mathcal{H} \hookrightarrow G \). Suppose, for some integer \( n \geq 2 \), \((G, \mathcal{H})\) is a PD \((n)\)-pair and for a sufficiently deep \( \{ N_i \triangleleft H_i \}_{i=1}^m \), \( \text{cd}(H_i) \leq n - 2 \) for each \( 1 \leq i \leq m \). Then, \( \text{cd}(G) = n \), \( H_n(G; \mathbb{Z}) = \mathbb{Z} \).

In addition,

(i) if the group \( H_i \) is amenable for each \( 1 \leq i \leq m \), then \( ||G|| \leq ||G, \mathcal{H}|| \), where \( || \cdot || \) denotes the simplicial volume;

(ii) if \( G \) is hyperbolic relative to \( \mathcal{H} \), then \( ||G|| > 0 \).

We should point out that conjecturally by Kropholler, amenable groups of finite cohomological dimension such as \( \prod_i \) are virtually solvable [Deg16, Question, p. 2]. Also, by Poincaré duality group or pair we will always mean orientable ones.

Theorem C can for example be applied when \( G \) is the fundamental group of a Riemannian manifold with a complete pinched negative sectional curvature and finite volume. We give one such application in Corollary 6.6 which generalises Theorem 1.5 of [FM11].

**Corollary 2.3.** Let \( M \) be a compact oriented \( n \)-manifold with nilmanifold boundary components such that the centre of the fundamental group of each boundary component is of rank at least 2. Suppose the interior of \( M \) admits a Riemannian manifold with a complete pinched negative sectional curvature and finite volume. If \( M_T \) is a sufficiently deep Dehn filling manifold of \( M \), then \( M_T \) is a closed oriented aspherical \( n \)-manifold with

\[
0 < ||M_T|| \leq ||M, \partial M||.
\]

### 2.3. Quotients of acylindrically hyperbolic groups

The notion of an acylindrically hyperbolic group was introduced by Osin [Osi16] as a generalization of non-elementary hyperbolic and non-elementary relatively hyperbolic groups. Examples of acylindrically hyperbolic groups can be found in many classes of groups that attracted group theorists for years, e.g., mapping class groups of surfaces [MM99, Bow08], outer automorphism groups of free groups [BF10], small cancellation groups [GS18], convergence groups [Sun19], the Cremona group (see [DGO17] and references therein; the main contribution towards showing the acylindrical hyperbolicity of the Cremona group is due to [CL13]), and tame automorphism groups of 3-dimensional affine spaces [LP19]. We refer to [Osi18] for details and other examples.

Every acylindrically hyperbolic group \( G \) contains hyperbolically embedded subgroups [DGO17, Theorem 6.14] and Dehn fillings can often be applied to construct useful quotients of \( G \). We use Theorem A to study homological properties of those quotients.

Recall that every acylindrically hyperbolic group \( G \) has a maximal finite normal subgroup denoted by \( K(G) \) [DGO17, Theorem 6.14].

**Theorem D.** Let \( G \) be an acylindrically hyperbolic group, and let \( C \) be any countable group. Then \( C \) embeds into a quotient \( \overline{G} \) of \( G/K(G) \) (in particular, \( \overline{G} \) is a quotient of \( G \)) such that

(i) \( \overline{G} \) is acylindrically hyperbolic;
(ii) if $C$ is finitely generated, then $C \rightarrow h \overline{G}$;

(iii) if $G$ and $C$ are torsion-free, then so is $\overline{G}$;

(iv) for all $n \geq 3$ and every $\overline{G}$-module $A$, we have

$$H^n(\overline{G}; A) \cong H^n(G/K(G); A) \oplus H^n(C; A),$$

where the action of $G/K(G)$ (resp. $C$) on $A$ is induced by the quotient map $G/K(G) \rightarrow \overline{G}$ (resp. the embedding $C \leftarrow \overline{G}$);

(v) $\text{cd}(\overline{G}) \leq \max\{\text{cd}(G), \text{cd}(C)\}$;

(vi) if $C$ is finitely generated and $G$ is of type $FP_n$ for some $n \in \mathbb{N}^+ \cup \{\infty\}$ (resp. $FP$), then $\overline{G}$ is of type $FP_n$ (resp. $FP$) if and only if $C$ is of type $FP_n$ (resp. $FP$).

As an application, we strengthen SQ-universality of hyperbolic groups given by Olshanskii [Ols95] and independently by Delzant [Del96] by adding cohomological conditions.

**Corollary 2.4.** Let $G$ be a non-elementary hyperbolic group and $C$ any hyperbolic group. Then there is a hyperbolic quotient $\overline{G}$ of $G/K(G)$ (in particular, $\overline{G}$ is a quotient of $G$), where $K(G)$ is the maximal finite normal subgroup of $G$, such that $C$ embeds into $\overline{G}$ and the following hold.

(i) For all $n \geq 3$ and every $\overline{G}$-module $A$, we have

$$H^n(\overline{G}; A) \cong H^n(G/K(G); A) \oplus H^n(C; A),$$

where the action of $G/K(G)$ (resp. $C$) on $A$ is induced by the quotient map $G/K(G) \rightarrow \overline{G}$ (resp. the embedding $C \leftarrow \overline{G}$).

(ii) $\text{cd}(\overline{G}) \leq \max\{\text{cd}(G), \text{cd}(C)\}$.

**Theorem E.** Let $G_1$ and $G_2$ be finitely generated acylindrically hyperbolic groups. Then there exists a common quotient $\overline{G}$ of $G_1/K(G_1)$ and $G_2/K(G_2)$ (in particular, $G$ is a common quotient of $G_1$ and $G_2$) such that

(i) $G$ is acylindrically hyperbolic;

(ii) for all $n \geq 3$ and every $G$-module $A$, we have

$$H^n(G; A) \cong H^n(G_1/K(G_1); A) \oplus H^n(G_2/K(G_2); A),$$

where the actions of $G_1/K(G_1)$ and $G_2/K(G_2)$ on $A$ factor through $G$;

(iii) $\text{cd}(G) \leq \max\{\text{cd}(G_1), \text{cd}(G_2)\}$;

(iv) if $G_1$ and $G_2$ are of type $FP_n$ for some $n \in \{2, 3, ..., \infty\}$ (resp. $FP$), then so is $G$.

Homological analogs of Theorem D (iv), (v) and Theorem E (ii), (iii) also hold (see Remarks 7.3 and 7.11). Theorem D (i), (ii) are proved in [DGO17, Theorem 2.33] and
Theorem E (i) is proved in [Hul16, Corollary 7.4]. The benefit of Theorems D and E is that they allow one to control the cohomology of the resulting acylically hyperbolic quotients. As we illustrate in Section 8, this facilitates the constructions of various acylically hyperbolic groups satisfying certain cohomological properties. We list some of them below.

**Corollary 2.5.** Every torsion-free acylically hyperbolic group $G$ of type $FP_\infty$ has a torsion-free acylically hyperbolic quotient $\overline{G}$ of type $FP_\infty$ which contains the Thompson group $F$.

The existence of a pair $H < G$, where $G$ is hyperbolic and $H$ is of type $F$ but not hyperbolic, was a well-known open problem, raised in particular by Bestvina [Besb, Question 2.1], Brady [Bra99, Question 7.2], Bridson [Besa, Question 4.1], and Jankiewicz-Norin-Wise [JNW21, Section 7]. The first example of such a pair was constructed recently by Italiano-Martelli-Migliorini [IMM21, Corollary 2]. By Corollary 2.4, we obtain:

**Corollary 2.6.** Let $n \geq 5$ be an integer. Every non-elementary hyperbolic group $G$ with $cd(G) \leq n$ has a hyperbolic quotient $\overline{G}$ with $cd(\overline{G}) = n$ such that $\overline{G}$ contains the Italiano-Martelli-Migliorini group. In particular, there is a type $F$ non-hyperbolic subgroup $H < \overline{G}$.

The next two corollaries strengthen a result of [Hul16, Corollary 1.7] stating that every acylically hyperbolic group has an acylically hyperbolic quotient with Kazhdan’s Property (T).

**Corollary 2.7.** Every acylically hyperbolic group $G$ of type $FP_n$ for some $n \in \{2, 3, ..., \infty\}$ (resp. $FP$) has an acylically hyperbolic quotient $\overline{G}$ of type $FP_n$ (resp. $FP$) with Kazhdan’s Property (T) such that $cd(\overline{G}) \leq \max\{cd(G), 7\}$.

**Corollary 2.8.** Let $G$ be any acylically hyperbolic group of type $FP_\infty$. Then $G$ has a family of acylically hyperbolic quotients $\{G_k\}_{k=2}^\infty$ such that for each $k$, $G_k$ has Kazhdan’s Property (T), is of type $FP_{k-1}$ but not of type $FP_k$.

In particular, since mapping class groups of surfaces of finite type, outer automorphism groups of free groups of finite rank and most 3-manifold groups are acylically hyperbolic and of type $FP$, they all exhibit such quotients.

### 2.4. A few words on the proofs of the main results

The first step of the proof of Theorem A is to establish, under the assumptions of the theorem, the isomorphism

$$H^n(\langle N \rangle; A) \cong \text{ColInd}_{H} \overline{G} H^n(N; A)$$

for all $n > 0$. Here, $\text{ColInd}_{H}$ stands for the co-induction from $\mathbb{Z}H$ to $\mathbb{Z}\overline{G}$. The Lyndon-Hochschild-Serre spectral sequence associated to the triple $\langle G, \langle N \rangle, A \rangle$ takes the form

$$E_2^{p,q}(A) = H^p(\overline{G}; H^q(\langle N \rangle; A)) \Rightarrow H^{p+q}(G; A).$$

Shapiro’s lemma together with (1) yields Theorem A (i). In fact, we will establish a more precise result which involves a morphism between the Lyndon-Hochschild-Serre spectral sequences associated to $\langle G, \langle N \rangle, A \rangle$ and $\langle H, N, A \rangle$. Parts (ii) of Theorem A will be proved by an inspection of this morphism.

To prove Theorem B, we analyse the spectral sequence of Theorem A (i) and apply parts (ii) and (v). To show that $\overline{G}$ is a Poincaré duality group, we use Johnson-Wall
characterisation [JW72]; namely, a group $\Gamma$ is a Poincaré duality group of dimension $n$ if and only if $\Gamma$ is of type $FP$, $H^i(\Gamma, \mathbb{Z}) = 0$ for $i \neq n$ and $H^n(\Gamma, \mathbb{Z}) = \mathbb{Z}$.

The algebraic excision Theorem A (ii) is again key in proving Theorem C. Since all $\overline{\mathcal{H}}_i$ are amenable, the natural map in bounded cohomology $H^n_b(\overline{\mathcal{G}}, \mathbb{R}) \to H^n_b(\overline{\mathcal{G}}; \mathbb{R})$ is an isometric isomorphism. The duality pairing between bounded cohomology and ordinary homology leads to the inequality $\|\overline{\mathcal{G}}\| \leq \|\mathcal{G}, \mathcal{H}\|$. When $G$ is hyperbolic relative to $\mathcal{H}$, then a sufficiently deep Dehn filling quotient $\overline{\mathcal{G}}$ is hyperbolic relative to $\overline{\mathcal{H}}$ [Osi07, Theorem 1.1]. This allows us to show that $\|\overline{\mathcal{G}}\| > 0$.

The proof of Theorem D is a modification of the proof of [DGO17, Theorem 2.33]. Given any acylindrically hyperbolic group $G$, one can find a non-cyclic free group $F \hookrightarrow G_0 = G/K(G)$. For any countable group $C$ and any finite subset $\mathcal{F} \subseteq F \setminus \{1\}$, we will use small cancellation theory to construct a normal subgroup $N < F$ such that $N \cap \mathcal{F} = \emptyset$, $C$ embeds into $F/N$, and $F/N$ has the desired cohomological properties. Theorem 1.4 then implies that $C$ embeds into $G_0/\langle \langle N \rangle \rangle$ and Theorem A applied to $N < F' \hookrightarrow G_0$ yields the desired cohomological results.

The proof of [Hul16, Corollary 7.4] uses small cancellation theory instead of Dehn filling. In order to apply our main result, we carry out an alternative approach. Given finitely generated acylindrically hyperbolic groups $G_1$ and $G_2$, we construct subgroups $H_1, H_2 \lhd \tilde{G} = G'_1 * G'_2$, where $G'_1 = G_1/K(G_1)$ and $G'_2 = G_2/K(G_2)$, such that the family $\{H_1, H_2\}$ hyperbolically embeds into $\tilde{G}$. For any finite sets $\{F_i \subseteq H_i \setminus \{1\}\}_{i=1,2}$, we will use small cancellation theory to construct normal subgroups $\{N_i < H_i\}_{i=1,2}$ such that $N_i \cap F_i = \emptyset$ and $N_1$ (resp. $N_2$) identifies a finite set of generators of $G'_2$ (resp. $G'_1$) with certain elements of $G'_2$ (resp. $G'_1$). The quotient $G = \tilde{G}/\langle \langle N_1 \cup N_2 \rangle \rangle$ is thus a common quotient of $G'_1$ and $G'_2$. Theorem E is then proved by applying general versions of Theorems 1.4 and A. The main difficulty of this argument is the construction of $H_1$ and $H_2$, which is presented in Section 7.2, using a technical tool provided by [DGO17] (see also [Osi07]) called isolated components.

2.5. Organization of the paper. We will start with preliminaries in Section 3, recalling basic definitions of group cohomology, the notion of (weakly) hyperbolically embedded subgroups, isolated components, acylindrically hyperbolic groups, and the structural result of [Sun20] called the Cohen-Lyndon property. The proof of (the general version of) Theorem A is given in Section 4. Theorems B and C are proved in Sections 5 and 6, respectively. Theorems D and E are proved in Section 7 with applications given in Section 8.

3. Preliminaries

3.1. Cohomology of groups. Let $G$ be a group. Recall that the homological and cohomological dimension of $G$ can be defined by

$$
\text{hd}(G) = \sup\{n \in \mathbb{N} \mid H_n(G, A) \neq 0 \text{ for some } \mathbb{Z}G\text{-module } A\},
$$

$$
\text{cd}(G) = \sup\{n \in \mathbb{N} \mid H^n(G, A) \neq 0 \text{ for some } \mathbb{Z}G\text{-module } A\},
$$

respectively.
$G$ is of type $FP_n$ for some $n \in \mathbb{N}^+ \cup \{\infty\}$ if there is a projective resolution
\[ \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \]
over $\mathbb{Z}G$ such that $P_k$ are finitely generated $G$-modules for all $k \leq n$. $G$ is of type $FP$ if $\text{cd}(G) < \infty$ and $G$ is of type $FP_{\infty}$.

Property $FP_n$ can be characterized by the cohomology functor. The following will be useful in the proof of Theorem 4.7.

**Theorem 3.1** (Bieri [Bie81, Theorem 1.3], Brown [Bro75, Theorem 2]). For a group $G$, the following are equivalent.

(a) $G$ is of type $FP_n$ for some $n \in \mathbb{N}^+ \cup \{\infty\}$.

(b) For every $k \leq n$ and every direct system $\{A_i\}_{i \in I}$ of $G$-modules such that $\lim_{\rightarrow} A_i = 0$, we have $\lim_{\rightarrow} H^k(G; A_i) = 0$.

Given a family $\{H_\lambda\}_{\lambda \in \Lambda}$ of subgroups of $G$, [BE78] defined the relative (co)homology of the group pair $(G, \{H_\lambda\}_{\lambda \in \Lambda})$. We briefly recall the definition. Let $\Delta$ be the kernel of the augmentation $\bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda] \rightarrow \mathbb{Z}$ which sends every left $H_\lambda$-coset to 1. By definition,
\[ H_n(G, \{H_\lambda\}_{\lambda \in \Lambda}; A) = \text{Tor}_{n-1}^G(\Delta, A), \quad H^n(G, \{H_\lambda\}_{\lambda \in \Lambda}; A) = \text{Ext}_{n-1}^G(\Delta, A) \]
for any $G$-module $A$. The dimension shift in the above definition ensures a long exact sequence between the absolute and relative (co)homology (see [BE78, Proposition 1.1]).

**3.2. (Weakly) hyperbolically embedded subgroups.** The notion of (weakly) hyperbolically embedded subgroups was introduced by [DGO17], which is our main reference for Sections 3.2, 3.3, and 3.4. We first recall the definition and present some examples. The motivation will be discussed afterwards.

Let $G$ be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a family of subgroups of $G$, $X$ a subset of $G$ such that $G$ is generated by $X$ together with the union of all $H_\lambda$ (in which case $X$ is called a relative generating set of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$), and $\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda$. Consider the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$. Note that, for $\lambda \in \Lambda$ there is a natural embedding $\Gamma(H_\lambda, H_\lambda) \hookrightarrow \Gamma(G, X \sqcup \mathcal{H})$ whose image is the subgraph of $\Gamma(G, X \sqcup \mathcal{H})$ with vertices and edges labeled by elements of $H_\lambda$.

**Remark 3.2.** We do allow $X \cap H_\lambda \neq \emptyset$ and $H_\lambda \cap H_\mu \neq \{1\}$ for distinct $\lambda, \mu \in \Lambda$, in which case there will be multiple edges between some pairs of vertices of $\Gamma(G, X \sqcup \mathcal{H})$.

For $\lambda \in \Lambda$, an edge path in $\Gamma(G, X \sqcup \mathcal{H})$ between vertices of $\Gamma(H_\lambda, H_\lambda)$ is called $H_\lambda$-admissible if it does not contain any edge of $\Gamma(H_\lambda, H_\lambda)$. Note that an $H_\lambda$-admissible path is allowed to pass through vertices of $\Gamma(H_\lambda, H_\lambda)$.

For example, consider the simple case where $\{H_\lambda\}_{\lambda \in \Lambda} = \{H\}$ consists of only one subgroup $H \leq G$. The Cayley graph $\Gamma(G, X \sqcup H)$ is displayed in Figure 1. The blue path is admissible. The red path is an edge from 1 to $h$ labeled by $h \in H$, and thus is inadmissible. If $h$ happens to be an element of $X$, i.e., there exists $x \in X$ with $x = h$, and the red path were labeled by $x$ instead of $h$, then the red path would be admissible.

**Definition 3.3.** For every pair of elements $h, k \in H_\lambda$, let $\hat{d}_\lambda(h, k) \in [0, \infty]$ be the length of a shortest $H_\lambda$-admissible path connecting the vertices labeled by $h$ and $k$. If no such path
exists, set $d_{\lambda}(h, k) = \infty$. The laws of summation on $[0, \infty)$ extend naturally to $[0, \infty]$ and it is easy to verify that $\hat{d}_{\lambda} : H_{\lambda} \times H_{\lambda} \to [0, +\infty]$ defines a metric on $H_{\lambda}$, which is called the relative metric on $H_{\lambda}$ with respect to $X$.

**Definition 3.4.** We say that the family $\{H_{\lambda}\}_{\lambda \in \Lambda}$ weakly hyperbolically embeds into $(G, X)$ (denoted by $\{H_{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow_{wh} (G, X)$) if $G$ is generated by the set $X$ together with union of all $H_{\lambda}, \lambda \in \Lambda$, and the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ is a Gromov hyperbolic space.

If $\{H_{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow_{wh} (G, X)$ and for each $\lambda \in \Lambda$, the metric space $(H_{\lambda}, \hat{d}_{\lambda})$ is proper, i.e., every ball of finite radius contains only finitely many elements, then $\{H_{\lambda}\}_{\lambda \in \Lambda}$ hyperbolically embeds into $(G, X)$ (denoted by $\{H_{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$). If in addition, $X$ and $\Lambda$ are finite, then we say that $G$ is hyperbolic relative to $\{H_{\lambda}\}_{\lambda \in \Lambda}$.

Further, we say that the family $\{H_{\lambda}\}_{\lambda \in \Lambda}$ hyperbolically embeds into $G$, denoted by $\{H_{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow_h G$, if there exists some subset $X \subseteq G$ such that $\{H_{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$.

**Notation 3.5.** In case $\{H_{\lambda}\}_{\lambda \in \Lambda} = \{H\}$ is a singleton, we will drop the braces and write $H \hookrightarrow_{wh} (G, X)$ and $H \hookrightarrow_h G$.

We refer to Figure 1 for an illustration of the situation $H \hookrightarrow_h G$. The grey discs represent cosets of $H$ in $G$. The black edges are labeled by elements of $X$. The edges and discs appear in a tree-like pattern as $\Gamma(G, X \sqcup H)$ is Gromov hyperbolic.

![Figure 1. Illustration of $H \hookrightarrow_h G$](image)

**Remark 3.6.** Notice that the above definition of relative hyperbolicity is not commonly used in literature. One of the most commonly used definitions for relative hyperbolicity which we will use later is the following: a group $G$ is hyperbolic relative to a family $\{H_{\lambda}\}_{\lambda \in \Lambda}$ of its subgroups if $G$ has a finite relative presentation with respect to $\{H_{\lambda}\}_{\lambda \in \Lambda}$ with a linear relative isoperimetric function. The equivalence of these two definitions is proved in [DGO17, Remark 4.41 and Theorem 4.42].

**Example 3.7.**

(a) $H \hookrightarrow_{wh} (G, G)$ for every subgroup $H \leq G$. 
\( (a) \) illustrates a triviality of this notion, \( (b) \) and \( (c) \) exhibit two kinds of groups via Gromov hyperbolic spaces. Example of weak hyperbolic embedding is thus an attempt to study possibly non-word-hyperbolic any finite generating set the corresponding Cayley graph is Gromov hyperbolic. The notion of hyperbolic embedding, and a further refinement is given by the notion of an acylindrically hyperbolic group (see Section 3.4).

One might therefore refine the notion by imposing additional conditions on the relative metric, for example, requiring local finiteness of the relative metric and obtaining the notion of hyperbolic embedding. We note that Examples 3.7 \( (b) \) and \( (c) \) exhibit two kinds of trivialities of hyperbolic embedding, and a further refinement is given by the notion of an acylindrically hyperbolic group (see Section 3.4).

The next lemma tells us how to find hyperbolically embedded subgroups.

**Lemma 3.8** (Dahmani-Guirardel-Osin [DGO17, Lemma 4.21]). Suppose that \( \text{card} (\Lambda) < \infty \), \( G \) acts on a Gromov hyperbolic space \( (S, d) \) by isometries, and the following three conditions are satisfied, then \( \{ H_\lambda \}_{\lambda \in \Lambda} \hookrightarrow_h G \).

\( (C_1) \) Every \( H_\lambda \) acts on \( S \) properly.

\( (C_2) \) There exists \( s \in S \) such that for every \( \lambda \in \Lambda \), the \( H_\lambda \)-orbit \( H_\lambda (s) \) of \( s \) is quasi-convex in \( S \).

\( (C_3) \) For every \( \epsilon > 0 \) and some \( s \in S \), there exists \( R > 0 \) such that the following holds. Suppose that for some \( g \in G \) and \( \lambda, \mu \in \Lambda \), we have

\[ \text{diam} \left( H_\mu (s) \cap (gH_\lambda (s))^{+\epsilon} \right) \geq R, \]

then \( \lambda = \mu \) and \( g \in H_\lambda \), where \( (gH_\lambda (s))^{+\epsilon} \) denotes the \( \epsilon \)-neighborhood of \( gH_\lambda (s) \) in \( S \).

The following proposition, which roughly says that being a hyperbolically embedded subgroup is a transitive property, will be used later.

**Proposition 3.9** (Dahmani-Guirardel-Osin [DGO17, Proposition 4.35]). Let \( G \) be a group, let \( \{ H_\lambda \}_{\lambda \in \Lambda} \) be a finite family of subgroups of \( G \), let \( X \subseteq G \), and let \( Y_\lambda \subseteq H_\lambda \) for every \( \lambda \in \Lambda \). Suppose that \( \{ H_\lambda \}_{\lambda \in \Lambda} \hookrightarrow_h (G, X) \) and for every \( \lambda \in \Lambda \), there is a family of subgroups \( \{ K_{\lambda, \mu} \}_{\mu \in M_\lambda} \hookrightarrow_h (H_\lambda, Y_\lambda) \). Then

\[ \bigcup_{\lambda \in \Lambda} \{ K_{\lambda, \mu} \}_{\mu \in M_\lambda} \hookrightarrow_h \left( G, X \cup \left( \bigcup_{\lambda \in \Lambda} Y_\lambda \right) \right). \]
3.3. **Isolated components.** In the proof of Theorem E, we need to construct specific hyperbolically embedded subgroups. A tool to do this is the notion of an isolated component, which was introduced by [Osi07] for relatively hyperbolic groups and generalized to hyperbolically embedded subgroups by [DGO17]. In this section, we recall the definition and collect several results.

We start with conventions. Let $G$ be a group and $X$ a generating set of $G$. Consider the Cayley graph $\Gamma(G, X)$. If $p$ is a path in $\Gamma(G, X)$, then the label of $p$ is denoted by $\text{Lab}(p)$, the length of $p$ is denoted by $\ell_X(p)$, and the initial (resp. terminal) vertex of $p$ is denoted by $p^-$ (resp. $p^+$).

Now suppose that $\{H_\lambda\}_{\lambda \in \Lambda}$ is a family of subgroups of $G$. Let $\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda$, and let $X$ be a relative generating set of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$. For $\lambda \in \Lambda$, let $d_\lambda$ be the relative metric on $H_\lambda$ with respect to $X$. The following terminology goes back to [Osi06].

**Definition 3.10.** Let $p$ be a path in $\Gamma(G, X \sqcup \mathcal{H})$. For every $\lambda \in \Lambda$, an $H_\lambda$-subpath $q$ of $p$ is a nontrivial subpath of $p$ such that $\text{Lab}(q)$ is a word over the alphabet $H_\lambda$ (if $p$ is a cycle, we allow $q$ to be a subpath of some cyclic shift of $p$). An $H_\lambda$-subpath $q$ of $p$ is an $H_\lambda$-component if $q$ is not properly contained in any other $H_\lambda$-subpath. Two $H_\lambda$-components $q_1$ and $q_2$ of $p$ are connected if there exists a path $t$ in $\Gamma(G, X \sqcup \mathcal{H})$ such that $t$ connects a vertex of $q_1$ to a vertex of $q_2$, and that $\text{Lab}(t)$ is a letter from $H_\lambda$. An $H_\lambda$-component $q$ of $p$ is isolated if it is not connected to any other $H_\lambda$-component of $p$.

Suppose that $q$ is an $H_\lambda$-component of a path $p \subseteq \Gamma(G, X \sqcup \mathcal{H})$. Then $q^-$ (resp. $q^+$) is labeled by an element $g \in G$ (resp $h \in G$) and we have $g^{-1}h \in H_\lambda$. In this case, let

$$\hat{\ell}_\lambda(q) = d_\lambda(1, g^{-1}h).$$

A nice property of isolated components is that in a geodesic polygon $p$, the total $\hat{\ell}$-length of isolated components of $p$ is bounded linearly by the number of sides of $p$. More precisely:

**Proposition 3.11** (Dahmani-Guirardel-Osin [DGO17, Proposition 4.14] (see also [Osi07, Proposition 3.2])). If $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{\text{wh}} (G, X)$, then there exists a number $D > 0$ satisfying the following property: Let $p$ be an $n$-gon in $\Gamma(G, X \sqcup \mathcal{H})$ with geodesic sides $p_1, ..., p_n$ and let $I$ be a subset of the set of sides of $p$ such that every side $p_i \in I$ is an isolated $H_\lambda$-component of $p$ for some $\lambda_i \in \Lambda$. Then

$$\sum_{p_i \in I} \hat{\ell}_{\lambda_i}(p_i) \leq Dn.$$

In fact, the above proposition is the reason why certain properties (for example, Theorems 3.20 and 3.22 below) hold for sufficiently deep Dehn fillings (see Definition 3.16).

The technical lemma below will be used in Section 7.2 along with Lemma 3.8 to construct hyperbolically embedded subgroups.

**Lemma 3.12** (Dahmani-Guirardel-Osin [DGO17, Lemma 4.21]). Suppose $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{\text{wh}} (G, X)$. Let $W$ be the set consisting of all words $w$ over $X \sqcup \mathcal{H}$ such that

(W1) $w$ contains no subwords of type $xy$, where $x, y \in X$;

(W2) if $w$ contains a letter $h \in H_\lambda$ for some $\lambda \in \Lambda$, then $d_\lambda(1, h) > 50D$, where $D$ is given by Proposition 3.11;
(W3) if $h_1xh_2$ (resp. $h_1h_2$) is a subword of $w$, where $x \in X, h_1 \in H_\lambda, h_2 \in H_\mu$, then either
\[ \lambda \neq \mu \]
or the element represented by $x$ in $G$ does not belong to $H_\lambda$ (resp. $\lambda \neq \mu$).

Then the following hold.

(a) Every path in the Cayley graph $\Gamma(G, X \sqcup H)$ labeled by a word from $W$ is a $(4,1)$-quasi-geodesic.

(b) If $p$ is a path in $\Gamma(G, X \sqcup H)$ labeled by a word from $W$, then for every $\lambda \in \Lambda$, every $H_\lambda$-component of $p$ is isolated.

(c) For every $\epsilon > 0$, there exists $R > 0$ satisfying the following condition. Let $p, q$ be two paths in $\Gamma(G, X \sqcup H)$ such that
\[ \ell_{X \sqcup H}(p) \geq R, \ \text{Lab}(p), \ \text{Lab}(q) \in W, \]
and $p, q$ are oriented $\epsilon$-close, i.e.,
\[ \max\{d_{X \sqcup H}(p^-, q^-), d_{X \sqcup H}(p^+, q^+)\} \leq \epsilon, \]
where $d_{X \sqcup H}$ is the combinatorial metric of $\Gamma(G, X \sqcup H)$. Then there exist five consecutive components of $p$ which are respectively connected to five consecutive components of $q$. In other words,
\[ p = ra_1x_1a_2x_2a_3x_3a_4x_4a_5s, \ \ q = tb_1y_1b_2y_2b_3y_3b_4y_4b_5u, \]
such that the following hold.

(i) $r$ (resp. $t$) is a subpath of $p$ (resp. $q$) whose label does not end with a letter from $H$.

(ii) $s$ (resp. $u$) is a subpath of $p$ (resp. $q$) whose label does not start with a letter from $H$.

(iii) For $i = 1, \ldots, 4$, $x_i$ and $y_i$ are either trivial subpaths or subpaths labeled by a word over $X$;

(iv) For $i = 1, \ldots, 5$, $a_i$ and $b_i$ are connected $H_\lambda$-components. Moreover, if $x_i$ or $y_i$ is the trivial path for some $i \in \{1, \ldots, 4\}$, then $\lambda_i \neq \lambda_{i+1}$.

**Remark 3.13.** Conclusion (b) of Lemma 3.12 is not stated in [DGO17, Lemma 4.21], but it is proved in the second paragraph of the proof of [DGO17, Lemma 4.21].

**3.4. Acylindrical hyperbolicity.** We notice that Examples 3.7 (b) and (c) are two occasions where having a hyperbolically embedded subgroup does not provide any information about the group $G$. We also notice that in those two cases, the hyperbolically embedded subgroup is either finite or improper. It is therefore natural to look at the groups $G$ with a proper infinite hyperbolically embedded subgroup. By [DGO17, Theorem 7.19] and [Osi16, Theorem 1.2], this is equivalent to saying that $G$ is acylindrically hyperbolic.

**Definition 3.14.** A group $G$ is acylindrically hyperbolic if $G$ admits a non-elementary acylindrical action on some Gromov hyperbolic space by isometries.

For the definition an acylindrical action, the reader is referred to [Osi18]. Intuitively, one can think of acylindricity as an analog of properness. An acylindrical action of a group $G$ is non-elementary if its orbits are unbounded and $G$ is not virtually-cyclic [Osi16, Theorem 1.1].
Techniques of hyperbolically embedded subgroups are mainly applied to acylindrically hyperbolic groups, because every acylindrically hyperbolic group contains infinitely many hyperbolically embedded virtually free subgroups.

**Theorem 3.15** (Dahmani-Guirardel-Osin [DGO17, Theorem 6.14]). Let \( G \) be an acylindrically hyperbolic group. Then \( G \) has a maximal finite normal subgroup, denoted by \( K(G) \). Moreover, for every \( n \in \mathbb{N}^+ \), there exists a free group \( F \) of rank \( n \) such that \( F \times K(G) \leq_h G \).

### 3.5. Sufficient deepness and Cohen-Lyndon triples

One consequence of Proposition 3.11 is that acylindrical hyperbolicity is preserved by Dehn fillings, provided that the Dehn fillings are sufficiently deep and done on hyperbolically embedded subgroups (see Theorem 3.20).

**Definition 3.16.** Let \( G \) be a group with a family of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \leq_h (G, X) \) for some subset \( X \subseteq G \). For every \( \lambda \in \Lambda \), let \( \delta_\lambda \) be the relative metric on \( H_\lambda \) with respect to \( X \).

A property \( \mathcal{P} \) holds for all sufficiently deep normal subgroups if there exists a constant \( C > 0 \) such that \( \mathcal{P} \) holds for every family of normal subgroups \( \{N_\lambda < H_\lambda\}_{\lambda \in \Lambda} \) with \( \delta_\lambda(1, n) > C \) for all \( n \in N_\lambda \setminus \{1\} \).

**Example 3.17.** If \( G = H_1 \ast_A H_2 \) is a free product with amalgamation, then \( \{H_1, H_2\} \leq_{wh} (G, \varnothing) \) by Example 3.7 (e). For normal subgroups \( \{N_i < H_i\}_{i = 1, 2} \), consider the property

\[
\mathcal{P}: \text{The quotient } G/\langle\langle N_1 \cup N_2 \rangle\rangle \text{ splits as an amalgamated free product, where } \langle\langle N_1 \cup N_2 \rangle\rangle \text{ denotes the normal closure of } N_1 \cup N_2 \text{ in } G.
\]

Then \( \mathcal{P} \) holds for all sufficiently deep normal subgroup \( \{N_i < H_i\}_{i = 1, 2} \), because \( \mathcal{P} \) holds whenever \( N_i \cap A = \{1\} \), which amounts to saying that \( \delta_i(n) > 1 \) for all \( n \in N_i \setminus \{1\} \), where \( \delta_i \) is the relative metric on \( H_i \) corresponding to the weak hyperbolic embedding \( \{H_1, H_2\} \leq_{wh} (G, \varnothing) \).

**Remark 3.18.** In Definition 3.16, if \( \{H_\lambda\}_{\lambda \in \Lambda} \leq_h (G, X) \), then the relative metrics \( \delta_\lambda \) are locally finite. Thus,

\[
\text{card} \left( \{h \in H_\lambda \mid \delta_\lambda(1, h) \leq C\} \right) < \infty
\]

for all \( C > 0 \). Therefore:

Let \( G \) be a group with a family of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \leq_h G \). Suppose that a property \( \mathcal{P} \) holds for all sufficiently deep normal subgroups \( \{N_\lambda < H_\lambda\}_{\lambda \in \Lambda} \). Then there exist finite sets \( \{F_\lambda \subseteq H_\lambda \setminus \{1\}\}_{\lambda \in \Lambda} \) such that \( \mathcal{P} \) holds whenever \( N_\lambda \cap F_\lambda = \varnothing \) for all \( \lambda \in \Lambda \).

To simplify statements, we use the following notation.

**Notation 3.19.** Let \( G \) be a group and \( S \) a subset of \( G \). Then \( \langle\langle S \rangle\rangle \) denotes the normal closure of \( S \) in \( G \). Suppose that \( \{H_\lambda\}_{\lambda \in \Lambda} \) is a family of subgroups of \( G \) and \( \{N_\lambda < H_\lambda\}_{\lambda \in \Lambda} \) is a family of normal subgroups. We call \( (G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda}) \) a group triple. We also let

\[
\mathcal{N} = \bigcup_{\lambda \in \Lambda} N_\lambda, \quad \mathcal{C} = G/\langle\langle \mathcal{N} \rangle\rangle, \quad \overline{H}_\lambda = H_\lambda/N_\lambda.
\]

**Theorem 3.20** (Dahmani-Guirardel-Osin [DGO17, Theorem 7.19], Osin [Osi16, Theorem 1.2]). Let \( G \) be a group with a family of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \leq_h G \). Then for all sufficiently deep normal subgroups \( \{N_\lambda < H_\lambda\}_{\lambda \in \Lambda} \), the natural homomorphism \( \overline{H}_\lambda \rightarrow \mathcal{C} \) is injective for
Let $\lambda \in \Lambda$ and we have $\{\overline{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow \overline{\mathcal{G}}$. Moreover, if for some $\lambda \in \Lambda$, $\text{card}(\overline{\lambda}) = \infty$ and $\overline{\lambda}$ is a proper subgroup of $\overline{\mathcal{G}}$, then $\overline{\mathcal{G}}$ is acylindrically hyperbolic.

The normal subgroup $\langle \mathcal{N} \rangle$ in the above theorem can be described more precisely: it has a particular free product structure.

**Definition 3.21.** A group triple $(G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})$ is called a Cohen-Lyndon triple if there exist left transversals $T_{\lambda}$ of $H_{\lambda}\langle \mathcal{N} \rangle$ in $G$ such that

$$\langle \mathcal{N} \rangle = \ast_{\lambda \in \Lambda, t \in T_{\lambda}} t N_{\lambda} t^{-1}.$$ 

The above free product structure was first proved by Cohen-Lyndon [CL63] for free groups, hence the name. The following theorem was partially proved by [DGO17, Theorem 7.19], which was later improved by [Sun20].

**Theorem 3.22** (Sun [Sun20, Theorem 5.1]). Let $G$ be a group with a family of subgroups $\{H_{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow (G, X)$ for some $X \subseteq G$. Then for all sufficiently deep normal subgroups $N_{\lambda} < H_{\lambda}$, $(G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})$ is a Cohen-Lyndon triple.

4. Cohomology of Dehn fillings

In this section, we prove Theorem A. To simplify the notation, we use $3.19$.

**Proposition 4.1.** Let $(G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})$ be a Cohen-Lyndon triple. Then for any $G$-module $A$ and $q > 1$ (also for any $\overline{\mathcal{G}}$-module $A$ and $q > 0$) there are isomorphisms

$$H_q(\langle \mathcal{N} \rangle; A) \cong \bigoplus_{\lambda \in \Lambda} \text{Ind}_{\overline{\lambda}}^{G} H_q(N_{\lambda}; A),$$

$$H^q(\langle \mathcal{N} \rangle; A) \cong \prod_{\lambda \in \Lambda} \text{CoInd}_{\overline{\lambda}}^{G} H^q(N_{\lambda}; A)$$

induced by the inclusions $N_{\lambda} \hookrightarrow \langle \mathcal{N} \rangle$.

We remark that an easy consequence of $(G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})$ being a Cohen-Lyndon triple is that the natural maps $\overline{\lambda} : \overline{H}_{\lambda} \to \overline{\mathcal{G}}$ are injective [Sun20, Lemma 6.4], and thus it makes sense to consider the (co)inductions $\text{Ind}_{\overline{\lambda}}^{\overline{\mathcal{G}}}$ and $\text{CoInd}_{\overline{\lambda}}^{\overline{\mathcal{G}}}$.

**Proof.** We will prove the homological version. The proof of the cohomological version will be analogous.

Let $\{T_{\lambda}\}_{\lambda \in \Lambda}$ be the family of left transversals associated to the Cohen-Lyndon triple $(G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})$, given by Definition 3.21. Then $\langle \mathcal{N} \rangle = \ast_{\lambda \in \Lambda, t \in T_{\lambda}} t N_{\lambda} t^{-1}$. Consider the Bass-Serre tree associated to this free product decomposition with vertex set $V$ and edge set $E$. We then get a short exact sequence of $\langle \mathcal{N} \rangle$-modules

$$0 \to \mathbb{Z}[E] \to \mathbb{Z}[V] \to \mathbb{Z} \to 0.$$ 

Let $F_i \to \mathbb{Z}$ be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Tensoring the above sequence by $(F_i \otimes A) \otimes_{\langle \mathcal{N} \rangle} -$,

we obtain a short exact sequence

$$(3) \quad 0 \to (F_i \otimes A) \otimes_{\langle \mathcal{N} \rangle} \mathbb{Z}[E] \to (F_i \otimes A) \otimes_{\langle \mathcal{N} \rangle} \mathbb{Z}[V] \to F_i \otimes_{\langle \mathcal{N} \rangle} A \to 0.$$
Note that there is an isomorphism of $\langle \mathcal{N} \rangle$-modules
\[
Z[V] \cong \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} Z[\langle \mathcal{N} \rangle/tN_\lambda t^{-1}].
\]

The short exact sequence (3) then transforms to a short exact sequence of chain complexes
\[
0 \to (F_i \otimes A) \otimes \langle \mathcal{N} \rangle Z[E] \to \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} F_i \otimes tN_\lambda t^{-1} A \overset{\theta}{\to} F_i \otimes \langle \mathcal{N} \rangle A \to 0.
\]

The long exact sequence corresponding to (4) yields
\[
\cdots \to H_q(\langle \mathcal{N} \rangle; Z[E] \otimes A) \to \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} H_q(tN_\lambda t^{-1}; A) \overset{\theta_*}{\to} H_q(\langle \mathcal{N} \rangle; A) \to \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} Z \otimes \langle \mathcal{N} \rangle (Z[E] \otimes A) \overset{L}{\to} Z \otimes tN_\lambda t^{-1} A \to Z \otimes \langle \mathcal{N} \rangle A \to 0.
\]

Since $Z[E]$ is a free $\langle \mathcal{N} \rangle$-module, $H_q(\langle \mathcal{N} \rangle; Z[E] \otimes A) = 0$ for $q > 0$ and $f$ is injective if $\langle \mathcal{N} \rangle$ acts trivially on $A$. This yields an isomorphism
\[
\theta_* : \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} H_q(tN_\lambda t^{-1}; A) \to H_q(\langle \mathcal{N} \rangle; A)
\]
for $q > 1$ and also for $q = 1$ if $\langle \mathcal{N} \rangle$ acts trivially on $A$. The action of $\mathcal{G}$ on $H_q(\langle \mathcal{N} \rangle; A)$ and the isomorphism $\theta_*$ endow $\bigoplus_{\lambda \in \Lambda, t \in T_\lambda} H_q(tN_\lambda t^{-1}; A)$ with a $\mathcal{G}$-action. Next, we will show that this action is a direct sum of permutation actions.

The $\mathcal{G}$-action on $H_q(\langle \mathcal{N} \rangle; A)$ comes from the $G$-action on $A$ and conjugation of $G$ on $\langle \mathcal{N} \rangle$ which is induced by the diagonal $G$-action on $F_i \otimes \langle \mathcal{N} \rangle A$. More explicitly, each $g \in G$ acts as
\[
\tau_g : F_i \otimes \langle \mathcal{N} \rangle A \to F_i \otimes \langle \mathcal{N} \rangle A, \quad [x, a] \mapsto [gx, ga].
\]

For each $\lambda \in \Lambda$ and $t \in T_\lambda$, the group $tH_\lambda t^{-1}$ acts on $F_i \otimes tN_\lambda t^{-1} A$ by
\[
[x, a] \mapsto [gx, ga], \quad \forall g \in tH_\lambda t^{-1}.
\]
This gives us a $tH_\lambda t^{-1}$-equivariant restriction of $\theta$
\[
\theta : F_i \otimes tN_\lambda t^{-1} A \to F_i \otimes \langle \mathcal{N} \rangle A,
\]
which in turn induces the $tH_\lambda t^{-1}$-equivariant inclusion
\[
\theta_* : H_q(tN_\lambda t^{-1}; A) \to H_q(\langle \mathcal{N} \rangle; A).
\]

Now, for each $\lambda \in \Lambda$ and $t \in T_\lambda$, the action of $t$ on $H_q(\langle \mathcal{N} \rangle; A)$ induces a map between two summands on the left-hand side of the isomorphism (5). To see this, note that the map defined by
\[
\sigma_t : F_i \otimes N_\lambda A \to F_i \otimes tN_\lambda t^{-1} A,
\]
\[
[x, a] \mapsto [tx, ta],
\]
satisfies $\theta \circ \sigma_t = \tau_t \circ \theta$. In homology, we then have $(\sigma_t)_* = \theta_*^{-1} \circ (\tau_t)_* \circ \theta_*$ as claimed.

We have shown that for each $\lambda \in \Lambda$, the $G$-action on $\bigoplus_{t \in T_\lambda} H_q(tN_\lambda t^{-1}; A)$ restricts to the $tH_\lambda t^{-1}$-action on the summand $H_q(tN_\lambda t^{-1}; A)$ and that it permutes $H_q(N_\lambda; A)$ and
\[ H_q(tN_\lambda t^{-1}; A). \] It is not difficult to show now, see for example [Bro94, Proposition 5.3], that

\[ \bigoplus_{t \in T_\lambda} H_q(tN_\lambda t^{-1}; A) \cong \text{Ind}_{\overline{H}_\lambda}^{\overline{H}_\lambda} H_q(N_\lambda; A). \]

Let \((G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})\) be a Cohen-Lyndon triple and \(A\) any \(G\)-module. There are Lyndon-Hochschild-Serre spectral sequences

\[ E^2_{p,q} = H_p(\overline{P}_\lambda; H_q(N_\lambda; A)) \Rightarrow H_{p+q}(H_\lambda; A), \]

\[ F^2_{p,q} = H_p(\overline{G}; H_q(\langle \mathcal{N} \rangle; A)) \Rightarrow H_{p+q}(G; A) \]

associated with the triples \((H_\lambda, N_\lambda, A)\) and \((G, \langle \mathcal{N} \rangle, A)\), respectively (see for example [Bro94, Chapter VII]). Let \(E_{p,q} = \bigoplus_{\lambda \in \Lambda} E^\lambda_{p,q}\). Then

\[ E^2_{p,q} = \bigoplus_{\lambda \in \Lambda} H_p(\overline{P}_\lambda; H_q(N_\lambda; A)) \Rightarrow \bigoplus_{\lambda \in \Lambda} H_{p+q}(H_\lambda; A). \]

The inclusions \(\overline{P}_\lambda \hookrightarrow \overline{G}\) and \(\overline{N}_\lambda \hookrightarrow \langle \mathcal{N} \rangle\) induce a morphism \(\phi : E_{p,q} \to F_{p,q}\). By Proposition 4.1 and Shapiro’s lemma, the restriction \(\phi : E^2_{p,q} \to F^2_{p,q}\) is an isomorphism for all \(q > 1\) and also for \(q = 1\) if \(\langle \mathcal{N} \rangle\) acts trivially on \(A\). In short:

**Theorem 4.2.** Let \((G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})\) be a Cohen-Lyndon triple. Then for every \(G\)-module \(A\), there is a natural morphism \(\phi : E_{p,q} \to F_{p,q}\) such that

\[ E^2_{p,q} = \bigoplus_{\lambda \in \Lambda} H_p(\overline{P}_\lambda; H_q(N_\lambda; A)) \Rightarrow \bigoplus_{\lambda \in \Lambda} H_{p+q}(H_\lambda; A), \]

\[ F^2_{p,q} = H_p(\overline{G}; H_q(\langle \mathcal{N} \rangle; A)) \Rightarrow H_{p+q}(G; A), \]

\(\phi\) is induced by the inclusions \(\overline{P}_\lambda \hookrightarrow \overline{G}\), \(\overline{N}_\lambda \hookrightarrow \langle \mathcal{N} \rangle\), and \(\phi\) restricts to an isomorphism \(\phi : E^2_{p,q} \cong F^2_{p,q}\) for all \(q > 1\) and also for \(q = 1\) if \(\langle \mathcal{N} \rangle\) acts trivially on \(A\). Moreover, the analogous statement holds for cohomology as well.

We further investigate the morphism \(\phi\) of the above theorem. Let \(P_1 \to \mathbb{Z}\) be a free resolution of \(\mathbb{Z}\) over \(\mathbb{Z}\overline{G}\) and \(S_i \to \mathbb{Z}\) a free resolution of \(\mathbb{Z}\) over \(\mathbb{Z}\overline{G}\). The spectral sequence \(E_{p,q}\) is induced by the double complex \(C_{p,q} = \bigoplus_{\lambda \in \Lambda} (P_p \otimes S_q) \otimes_{H_\lambda} A\) and \(F_{p,q}\) is induced by \(D_{p,q} = (P_p \otimes S_q) \otimes_{\overline{G}} A\). The surjections

\[ (P_p \otimes S_q) \otimes_{H_\lambda} A \to (P_p \otimes S_q) \otimes_{\overline{G}} A \]

induce a surjection \(C_{p,q} \to D_{p,q}\), which in turn induces the morphism \(\phi\). Let \(R_{p,q}\) be the kernel of the surjection \(C_{p,q} \to D_{p,q}\) and \(E_{p,q}(R)\) the spectral sequence associated with \(R_{p,q}\). It turns out that \(E_{p,q}(R) \Rightarrow H_{p+q+1}(G, \{H_\lambda\}_{\lambda \in \Lambda}; A)\) and we have the following commutative diagram of long exact sequences.

**Proposition 4.3.** Let \((G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})\) be a Cohen-Lyndon triple and \(A\) a \(\overline{G}\)-module. Then there is a commutative diagram of exact sequences

\[
\begin{array}{cccccccc}
\cdots & \to & \bigoplus_{\lambda \in \Lambda} H_t(H_\lambda; A) & \xrightarrow{\tau_*} & H_t(G; A) & \to & H_t(G, \{H_\lambda\}_{\lambda \in \Lambda}; A) & \xrightarrow{\oplus_{\lambda \in \Lambda} H_{t-1}(H_\lambda; A)} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \bigoplus_{\lambda \in \Lambda} H_t(\overline{P}_\lambda; A) & \xrightarrow{\tau_*} & H_t(\overline{G}; A) & \to & H_t(G, \{H_\lambda\}_{\lambda \in \Lambda}; A) & \xrightarrow{\oplus_{\lambda \in \Lambda} H_{t-1}(\overline{P}_\lambda; A)} & \cdots
\end{array}
\]
where $\iota_*$ and $\tau_*$ are induced by the inclusions $H_\lambda \hookrightarrow G$ and $\overline{\mu}_\lambda \hookrightarrow \overline{G}$, respectively. Moreover, the cohomological analog of the above statement holds as well.

**Proof.** We will only prove the homological version since the cohomological version is similar. First, we compute the limit of the spectral sequence $E_{p,q}$. 

**Claim 4.3.1.** $E_{p,q}(R) \Rightarrow H_{p+q+1}(G, \{H_\lambda\}_{\lambda \in \Lambda}; A)$.

**Proof of the claim.** Let $\Delta$ be the kernel of the augmentation $\oplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda] \to \mathbb{Z}$. Then we have a short exact sequence

$$0 \to P_p \otimes S_q \otimes \Delta \to P_p \otimes S_q \otimes \left( \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda] \right) \to P_p \otimes S_q \to 0.$$  

Note that

$$(P_p \otimes S_q \otimes \mathbb{Z}[G/H_\lambda]) \otimes_G A \cong (P_p \otimes S_q) \otimes_{H_\lambda} A.$$  

Thus, tensoring (6) by $- \otimes_G A$ yields

$$0 \to (P_p \otimes S_q \otimes \Delta) \otimes_G A \to C_{p,q} \to D_{p,q} \to 0.$$  

Let $T_n = \bigoplus_{p+q=n} P_p \otimes S_q$. Notice that $T_n \otimes \Delta \to \Delta$ is a free resolution of $\Delta$ over $\mathbb{Z}G$. Therefore, to prove the claim, it suffices to show that

$$(P_p \otimes S_q \otimes \Delta) \otimes_G A \cong R_{p,q},$$  

which will be established once we show that (7) is exact. It is easy to check that

$$0 \to A \otimes \Delta \to A \otimes \left( \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda] \right) \to A \to 0$$

is exact. Since $P_p$ is free abelian and $S_q$ is a free $G$-module, we have an exact sequence

$$0 \to (P_p \otimes \Delta \otimes A) \otimes_G S_q \to \left[ P_p \otimes A \otimes \left( \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda] \right) \right] \otimes_G S_q \to (P_p \otimes A) \otimes_G S_q \to 0,$$

which, by basic tensor identities, transforms to the desired one. $\square$

Let us return to the proof of the proposition. We have a short exact sequence

$$0 \to R_{p,q} \to C_{p,q} \to D_{p,q} \to 0,$$

whose vertical homology gives the following long exact sequence of the $E^1$-terms of the associated spectral sequences

$$\cdots \to H_q(R_{p,*}) \to H_q(C_{p,*}) \to H_q(D_{p,*}) \to H_{q-1}(R_{p,*}) \to \cdots$$

By Proposition 4.1 and Shapiro’s lemma,

$$H_q(D_{p,*}) \cong P_p \otimes \bigoplus_{\lambda \in \Lambda} \text{Ind}_{H_\lambda}^{\overline{G}} H_q(N_\lambda; A) \cong H_q(C_{p,*}).$$
This shows that $E^1_{p,q}(R) = H_q(R_{p,*}) = 0$ for all $q > 0$ and hence $E^2_{p,0}(R) = H_{p+1}(G, \{H_\lambda\}_{\lambda \in \Lambda}; A)$ for all $p$. Also, since $\mathbb{Z} \otimes \langle \lambda \rangle \otimes_{\langle \lambda \rangle} A = A$, we have

$$E^1_{p,0}(C) = H_0(C_{p,*}) = \bigoplus_{\lambda \in \Lambda} P_\lambda \otimes_{\overline{\Pi}_\lambda} A,$$

$$E^1_{p,0}(D) = H_0(D_{p,*}) = P_p \otimes_G A.$$

So, we get a short exact sequence

$$(8) \quad 0 \to E^1_{p,0}(R) \to \bigoplus_{\lambda \in \Lambda} P_\lambda \otimes_{\overline{\Pi}_\lambda} A \to P_p \otimes_G A \to 0,$$

whose long exact sequence is the bottom row of the desired diagram.

Similar to the proof of the claim, one can show that there is a short exact sequence

$$(9) \quad 0 \to (S_q \otimes \Delta) \otimes_G A \to \bigoplus_{\lambda \in \Lambda} S_q \otimes_{\Pi_\lambda} A \to S_q \otimes_G A \to 0.$$  

There is a $G$-module homomorphism from the resolution $S_q \to \mathbb{Z}$ to $P_p \to \mathbb{Z}$, which gives rise to a map from (9) to (8), which in turn yields the desired commutative diagram of the associated long exact sequences.$\square$

**Remark 4.4.** Similar to the proof of Proposition 4.3, one can show that there is a short exact sequence

$$(10) \quad 0 \to (P_p \otimes \overline{\Delta}) \otimes_G A \to \bigoplus_{\lambda \in \Lambda} P_p \otimes_{\overline{\Pi}_\lambda} A \to P_p \otimes_G A \to 0,$$

where $P_p$ and $A$ are as in the proof of Proposition 4.3, and $\overline{\Delta}$ is the kernel of the augmentation $\bigoplus_{\lambda \in \Lambda} \mathbb{Z} \otimes [\mathbb{G}/\overline{\Pi}_\lambda] \to \mathbb{Z}$. The natural map from (8) to (10) gives rise to a commutative diagram of the corresponding long exact sequences, which together with the five lemma yields an isomorphism

$$H_*([\overline{G}, \overline{\Pi}_\lambda]_{\lambda \in \Lambda}; A) \cong H_*(G, \{H_\lambda\}_{\lambda \in \Lambda}; A)$$

(of course, under the assumption that $(G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})$ is a Cohen-Lyndon triple and $A$ is a $\overline{G}$-module). Moreover, The analogous isomorphism for cohomology holds as well.

An easy consequence of Proposition 4.3 is a direct sum decomposition of (co)homology.

**Corollary 4.5.** Let $(G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})$ be a Cohen-Lyndon triple and $A$ a $\overline{G}$-module.

(a) If for some $p \in \mathbb{N}$, $\bigoplus_{\lambda \in \Lambda} H_p(H_\lambda; A) = 0$ and the natural map $\bigoplus_{\lambda \in \Lambda} H_{p-1}(H_\lambda; A) \to H_{p-1}(G; A)$ is injective, then

$$H_p([\overline{G}, \overline{\Pi}_\lambda]_{\lambda \in \Lambda}; A) \cong H_p(G; A) \oplus \left( \bigoplus_{\lambda \in \Lambda} H_p(\overline{\Pi}_\lambda; A) \right).$$

(b) If for some $p \in \mathbb{N}$, $\prod_{\lambda \in \Lambda} H^p(H_\lambda; A) = 0$ and the natural map $H^{p-1}(G; A) \to \prod_{\lambda \in \Lambda} H^{p-1}(H_\lambda; A)$ is surjective, then

$$H^p([\overline{G}, \overline{\Pi}_\lambda]_{\lambda \in \Lambda}; A) \cong H^p(G; A) \oplus \left( \prod_{\lambda \in \Lambda} H^p(\overline{\Pi}_\lambda; A) \right).$$
Proof. We only prove the cohomological version (b) and point out that the proof of (a) is analogous. To shorten the notation, we denote \( H^*(-; A) \) by \( H^*(-) \), \( \prod_{\lambda \in \Lambda} H^*(H\lambda; A) \) by \( H^*(H) \), \( \prod_{\lambda \in \Lambda} H^*(\overline{H\lambda}; A) \) by \( H^*(\overline{H}) \), and \( H^*(G, \{H\lambda\}_{\lambda \in \Lambda}; A) \) by \( H^*(G, H) \). Use Proposition 4.3 and consider the commutative diagrams of exact sequences

\[
\begin{array}{c}
H^{p-1}(\overline{G}) \xrightarrow{\phi^{p-1}} H^{p-1}(H) \xrightarrow{\lambda^{p-1}} H^p(G, H) \xrightarrow{\xi^p} H^p(\overline{G}) \\
\downarrow \psi^{p-1} \downarrow \downarrow \downarrow \downarrow \downarrow \\
H^{p-1}(G) \xrightarrow{\phi^p} H^{p-1}(H) \xrightarrow{\lambda^p} H^p(G, H) \xrightarrow{\xi^{p+1}} H^p(\overline{G}) \\
\downarrow \psi^p \downarrow \downarrow \downarrow \downarrow \\
H^p(G) \xrightarrow{\phi^p} H^p(H) \xrightarrow{\lambda^p} H^{p+1}(G, H) \xrightarrow{\xi^{p+1}} H^{p+1}(\overline{G})
\end{array}
\]

where the horizontal maps come from the exact sequences for the pairs \((G, H), (\overline{G}, H)\) and \(\psi^*, \theta^*\) are natural maps induced by the surjections \( G \twoheadrightarrow \overline{G}, H\lambda \twoheadrightarrow \overline{H\lambda}, \) respectively.

The hypothesis imply that \( \eta^p : H^p(G, H) \to H^p(G) \) is an isomorphism which in turn shows that the map \( \xi^p \) is injective. Since \( H^p(H) = 0 \), \( \eta^{p+1} \) is injective. So \( \xi^{p+1} \) is also injective. This shows \( \lambda^p = 0 \) and hence \( \phi^p \) is onto. Since \( (\eta^p)^{-1} \circ \phi^p \) is a section for \( \xi^p \), the result follows. \( \square \)

An immediate corollary to the above is an estimate of the (co)homological dimension. To short the notation, if \((G, \{H\lambda\}_{\lambda \in \Lambda}, \{N\lambda\}_{\lambda \in \Lambda})\) is a Cohen-Lyndon triple, let

\[
\text{hd}(H) = \sup_{\lambda \in \Lambda} \text{hd}(H\lambda), \quad \text{hd}(\overline{H}) = \sup_{\lambda \in \Lambda} \text{hd}(\overline{H\lambda}),
\]

\[
\text{cd}(H) = \sup_{\lambda \in \Lambda} \text{cd}(H\lambda), \quad \text{cd}(\overline{H}) = \sup_{\lambda \in \Lambda} \text{cd}(\overline{H\lambda}).
\]

Corollary 4.6. Let \((G, \{H\lambda\}_{\lambda \in \Lambda}, \{N\lambda\}_{\lambda \in \Lambda})\) be a Cohen-Lyndon triple. Then

\[
\text{hd}(\overline{G}) \leq \max\{\text{hd}(G), \text{hd}(H) + 1, \text{hd}(\overline{H})\}, \quad \text{cd}(\overline{G}) \leq \max\{\text{cd}(G), \text{cd}(H) + 1, \text{cd}(\overline{H})\}.
\]

By Theorem 3.1, the commutativity of colimits of coefficients with the cohomology functor can be used to characterize property \( FP_n \), which is our next goal.

Theorem 4.7. Let \( G \) be a group with a finite family of subgroups \( \{H_i\}_{i=1}^m \hookrightarrow_n G \). Suppose that \( G \) is of type \( FP_n \) for some \( n \in \mathbb{N}^+ \cup \{\infty\} \). Then for sufficiently deep \( \{N_i \triangleleft H_i\}_{i=1}^m \), \( \overline{G} \) is of type \( FP_n \) if and only if all \( \overline{H}_i \) are of type \( FP_n \).

Combining the above with Corollary 4.6, we obtain the following.

Corollary 4.8. Let \( G \) be a group with a finite family of subgroups \( \{H_i\}_{i=1}^m \hookrightarrow_h G \). Suppose that \( G \) is of type \( FP \). Then for sufficiently deep \( \{N_i \triangleleft H_i\}_{i=1}^m \), \( \overline{G} \) is of type \( FP \) if and only if all \( \overline{H}_i \) are of type \( FP \).

Proof of Theorem 4.7. Suppose that \( \overline{G} \) is of type \( FP_n \). By Theorem 3.20, we may assume that \( \{\overline{H}_i\}_{i=1}^m \hookrightarrow_h \overline{G} \). Then \( \overline{H}_i \) are of type \( FP_n \) as \( \overline{G} \) is \([\text{DGO}17, \text{Remark 4.26 and Corollary 4.32}]\).
Conversely, suppose that all $\varpi_i$ are of type $FP_n$. To shorten notations, we write $H^*(G, \mathcal{H}; -)$ for $H^*(G, \{H_i\}_{i=1}^m; -)$, $H^*(\mathcal{H}; -)$ for $\prod_{i=1}^m H^*(H_i; -)$, and $H^*(\mathcal{H}; -)$ for $\prod_{i=1}^m H^*(\mathcal{H}_i; -)$. Let $\{A_j\}_{j \in J}$ be a direct system of $\mathcal{G}$-modules such that $\lim_{j \to \infty} A_j = 0$. By Proposition 4.3, we have a commutative diagram of exact sequences for each $j \in J$:

$$
\cdots \to H^k(G, \mathcal{H}; A_j) \to H^k(\mathcal{G}; A_j) \to H^k(\mathcal{H}; A_j) \to H^{k+1}(G, \mathcal{H}; A_j) \to \cdots
$$

For $k \leq n$, after taking direct limits, we have $\lim_{j \to \infty} H^k(G; A_j) = 0$ by Theorem 3.1. By [DG017, Remark 4.26 and Corollary 4.32], we also have $\lim_{j \to \infty} H^k(\mathcal{H}; A_j) = 0$, and thus $\lim_{j \to \infty} H^k(G, \mathcal{H}; A_j) = 0$, which implies that $\lim_{j \to \infty} H^k(\mathcal{G}; A_j) = \lim_{j \to \infty} H^k(\mathcal{H}; A_j) = 0$ except for $k = n$. Since $\lim_{j \to \infty} H^n(\mathcal{H}; A_j) = 0$, we have $\lim_{j \to \infty} H^n(\mathcal{G}; A_j) = 0$ as well, and thus $\mathcal{G}$ is of type $FP_n$, again by Theorem 3.1. 

We emphasize that the finiteness of $\Lambda$ is needed in the above proof to guarantee that $\lim_{j \to \infty} H^k(\mathcal{H}; A_j) = \lim_{j \to \infty} H^k(\mathcal{H}; A_j) = 0$ for $k \leq n$.

Proof of Theorem A. By Theorem 3.22, $(G, H, N)$ is a Cohen-Lyndon triple for sufficiently deep $N \vartriangleleft H$. Items (ii) follows directly from Propositions 4.3 and Remark 4.4.

Theorem 4.2 provides us a spectral sequences

$$E_2^{p,q} \Rightarrow H^{p+q}(G; A)$$

and isomorphisms

$$E_2^{p,q} \cong H^p(\mathcal{H}; H^q(N; A))$$

for $q > 0$. Theorem A (i) then follows by observing that $H^q(\mathcal{G}; H^0(\langle N \rangle; A)) \cong H^p(\mathcal{G}; A)$. Similarly, one can prove the homological version. 

Corollary 2.2 follows directly from Proposition 4.5, Corollary 4.6, Theorem 4.7, and Corollary 4.8.

We collect the results of this section and state the full version of Theorem A.

Theorem 4.9. Let $G$ be a group with a family of hyperbolically embedded subgroups $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h G$. Then the following hold for all sufficiently deep $N_\lambda \vartriangleleft H_\lambda, \lambda \in \Lambda$ and all $\mathcal{G}$-modules $A$, where $\mathcal{G} = G/\langle \cup_{\lambda \in \Lambda} N_\lambda \rangle$.

(i) There is a spectral sequence

$$E_2^{p,q}(A) = \begin{cases} 
\prod_{\lambda \in \Lambda} H^p(\mathcal{H}_\lambda; H^q(N_\lambda; A)) & \text{for } q > 0 \\
H^0(\mathcal{G}; A) & \text{for } q = 0
\end{cases} \Rightarrow H^{p+q}(G; A),$$

where $\mathcal{H}_\lambda = H_{\lambda/N_\lambda}$ for all $\lambda$ and the action of $G$ on $A$ factors through $\mathcal{G}$. In particular, the action of $\langle \cup_{\lambda \in \Lambda} N_\lambda \rangle$ on $A$ fixes $A$ pointwise.
Example 4.10. Let $G$ be a free group with basis $\{x,y\}$ and let $H = \langle h \rangle \leq G$ where $h = xyx^{-1}y^{-1}$. Then $H \hookrightarrow h G$ by Example 1.3 and $\text{cd}(H) + 1 = 2$. Let $N = \langle h^k \rangle \lhd H$. Note that we can pick $k$ large enough so that $N$ avoids any given finite subset of $H \setminus \{1\}$. By [Lyn50, Theorem 11.1], $H^2(G;\mathbb{Z}) \cong \mathbb{Z}$, and it is well-known that $H^2(G;\mathbb{Z}) = 0$ and $H^2(H;\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$. Thus, $H^2(G;\mathbb{Z}) \neq H^2(G;\mathbb{Z}) \oplus H^2(H;\mathbb{Z})$. Similarly, $\text{hd}(H) + 1 = 2$ and one can show that $H^2(G;\mathbb{Z}) \neq H^2(G;\mathbb{Z}) \oplus H^2(H;\mathbb{Z})$.

5. Dehn fillings and duality

Let $G$ a group and $H = \{H_i\}_{i=1}^m$ a finite collection of subgroups. Following Bieri-Eckmann [BE78], we say that $(G, H)$ is a duality pair of dimension $n$, with dualizing module $C$, if for all $k \in \mathbb{Z}$ and all $G$-modules $A$, one has
\[
H^k(G; A) \cong H_{-k}(G, H; C \otimes A)
\]
given by the cap product by the fundamental class $c \in H_n(G, H; \mathbb{Z}G)$. In which case, it follows that $C \cong H^n(G, H; \mathbb{Z}G)$. If $C = \mathbb{Z}$ with trivial $G$-action, the pair is called an (orientable) Poincaré duality pair, in short, a PD($n$)-pair.

Let $(G, H)$ be a duality pair of dimension $n$ with dualizing module $C$ and let $\Delta$ be the kernel of the augmentation $\bigoplus_{i=1}^m \mathbb{Z}[G/H_i] \rightarrow \mathbb{Z}$. By [BE78, Theorem 4.2], $G$ is a duality group of dimension $n - 1$ with dualizing module $\Delta \otimes C$ and each $H_i$ is a duality group of dimension $n - 1$ with dualizing module $C$ (thinking as an $H_i$-module).

The following result generalises [Wan18, Corollary 1.5] which deals with the case where $(G, H)$ is a type $F_\infty$ relatively hyperbolic group pair.

Corollary 5.1. Let $G$ be a group and $H = \{H_i\}_{i=1}^m$ a collection of subgroups such that $H \hookrightarrow h G$. Suppose $(G, H)$ is a duality pair of dimension $n$, with dualizing module $C$. Then
for all sufficiently deep \( \{N_i \triangleleft H_i\}_{i=1}^m \) and \( k \in \mathbb{Z} \)

\[
H^k(G, \mathcal{H}; \mathbb{Z}G) \cong \begin{cases} 
\bigoplus_{i=1}^m \text{Ind}_H^{H_i} H_{n-k}(N_i; C) & \text{for } k \neq n, \\
C & \text{for } k = n.
\end{cases}
\]

**Proof.** By Theorem A(ii) (see also Remark 4.4) and duality, for \( k < n \) one has

\[
H^k(G, \mathcal{H}; \mathbb{Z}G) \cong H_{n-k}(G; C \otimes \mathbb{Z}G) \cong H_{n-k}(G; \text{Ind}_{\langle \langle N \rangle \rangle}^G \text{Res}_{\langle \langle N \rangle \rangle}^G C) \cong H_{n-k}(\langle \langle N \rangle \rangle; C),
\]

where the second isomorphism follows from [Bro94, Chapter III, Proposition 5.6]. The claim now follows from Theorem 3.22 and Proposition 4.1. \( \square \)

**Theorem 5.2.** Let \( G \) be a group and \( \mathcal{H} = \{H_i\}_{i=1}^m \) a collection of subgroups such that \( \mathcal{H} \hookrightarrow_h G \). Suppose, for some integer \( 0 < n \), \( (G, \mathcal{H}) \) is a PD(n)-pair and that there are sufficiently deep \( \{Z \cong N_i \triangleleft H_i\}_{i=1}^m \) such that each member of \( \overline{\mathcal{H}} = \{\overline{H_i}\}_{i=1}^m \) is a PD(n - 2)-group. Then \( \overline{G} \) is a PD(n)-group.

**Proof.** For each \( 1 \leq i \leq m \), \( H^k(\overline{H}_i; \mathbb{Z}\overline{G}) = 0 \) if \( k \neq n - 2 \) and by Corollary 5.1, \( H^k(G, \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) = 0 \) if \( k \neq n - 1, n \). The long exact sequence in cohomology for the pair \( (\overline{G}, \overline{\mathcal{H}}) \) shows that \( H^k(G; \mathbb{Z}\overline{G}) = 0 \) if \( k \neq n - 2, n - 1, n \) and for \( k = n \) it gives \( H^n(G, \mathcal{H}; \mathbb{Z}\overline{G}) \cong H^n(G, \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) \cong \mathbb{Z} \).

Next, we consider the spectral sequence of Theorem 4.2

\[
E_2^{p,q} = \begin{cases} 
\prod_{i=1}^m H^p(\overline{H}_i; H^q(N_i; \mathbb{Z}\overline{G})) & \text{for } q > 0 \\
H^p(\overline{G}; \mathbb{Z}\overline{G}) & \text{for } q = 0
\end{cases} \Rightarrow H^{p+q}(G; \mathbb{Z}\overline{G}).
\]

Since each \( N_i \cong \mathbb{Z} \), \( E_2^{p,q} = 0 \) for \( q > 1 \) and \( E_2^{0,1} \cong \prod_{i=1}^m H^p(\overline{H}_i; \mathbb{Z}\overline{G}) \). Since there are only two nontrivial rows on the \( E_2^{p,q} \)-page of the spectral sequence, we obtain a Gysin long exact sequence

\[
\cdots \rightarrow \prod_{i=1}^m H^{k-2}(\overline{H}_i; H^1(N_i; \mathbb{Z}\overline{G})) \overset{d_2}{\rightarrow} H^k(\overline{G}; \mathbb{Z}\overline{G}) \rightarrow H^k(G; \mathbb{Z}\overline{G}) \rightarrow \prod_{i=1}^m H^{k-1}(\overline{H}_i; H^1(N_i; \mathbb{Z}\overline{G})) \overset{d_2}{\rightarrow} \cdots
\]

For \( k = n - 2 \), (11) yields

\[
H^{n-2}(G; \mathbb{Z}\overline{G}) \cong H^{n-2}(G; \mathbb{Z}\overline{G}) \cong H_2(G; \mathcal{H}; \mathbb{Z}\overline{G}) \cong H_2(\overline{G}; \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) = 0.
\]

For \( k = n - 1 \), (11) and the morphism between the Lyndon-Hochschild-Serre spectral sequences associated to the inclusions \( (H_\lambda, N_\lambda) \hookrightarrow (G, \langle \langle N \rangle \rangle) \) give us
\[ 0 \rightarrow H^{n-1}(\mathcal{G}; \mathbb{Z}\mathcal{G}) \rightarrow H^{n-1}(G; \mathcal{Z}\mathcal{G}) \rightarrow \prod_{i=1}^m H^{n-2}(\mathcal{H}_i; H^1(N_i; \mathcal{Z}\mathcal{G})) \]
\[ \xrightarrow{r^{n-1}} \prod_{i=1}^m H^{n-1}(\mathcal{H}_i; \mathcal{Z}\mathcal{G}) \rightarrow \prod_{i=1}^m H^{n-2}(\mathcal{H}_i; H^1(N_i; \mathcal{Z}\mathcal{G})) \]

where the bottom isomorphism follows from the Lyndon-Hochschild-Serre spectral sequence applied to the extensions \( 1 \rightarrow N_i \rightarrow H_i \rightarrow \mathcal{H}_i \rightarrow 1 \). To show that \( H^{n-1}(\mathcal{G}; \mathcal{Z}\mathcal{G}) = 0 \) amounts to showing that \( r^{n-1} \) is injective. By duality, this is equivalent to showing that the connecting homomorphism

\[ \partial : H_1(G; \mathcal{H}; \mathcal{Z}\mathcal{G}) \rightarrow \bigoplus_{i=1}^m H_0(H_i; \mathcal{Z}\mathcal{G}) \]

is injective, which follows from the commutativity of the diagram

\[ H_1(G; \mathcal{H}; \mathcal{Z}\mathcal{G}) \xrightarrow{\partial} \bigoplus_{i=1}^m H_0(H_i; \mathcal{Z}\mathcal{G}) \]

\[ \xrightarrow{\cong} H_1(\mathcal{G}; \mathcal{H}; \mathcal{Z}\mathcal{G}) \xrightarrow{\cong} \bigoplus_{i=1}^m H_0(\mathcal{H}_i; \mathcal{Z}\mathcal{G}) \]

and the fact the kernel of \( \partial \) is \( H_1(\mathcal{G}, \mathcal{Z}\mathcal{G}) = 0 \). This finishes the proof of \( H^{n-1}(\mathcal{G}; \mathcal{Z}\mathcal{G}) = 0 \).

By [BE78, Theorem 6.2], \( G \) and each \( \mathcal{H}_i \) are of type \( FP \). So, by Corollary 4.8, \( \mathcal{G} \) is of type \( FP \). Again, by [BE78, Theorem 6.2], \( \mathcal{G} \) is a PD(\( n \))-group. \( \square \)

6. Simplicial volume of Dehn fillings

For detailed background on bounded cohomology and simplicial volume, we refer to [Gro82], [Iva85] and [Fri17].

Let \( G \) be a group. Consider the singular chain complex \( C_*(BG; \mathbb{R}) \) endowed with the \( \ell^1 \)-norm

\[ |c|_1 = \sum_{i=1}^k |a_i|, \quad \forall c = \sum_{i=1}^k a_i \sigma_i \in C_*(BG; \mathbb{R}). \]

The cochain complex of bounded cochains \( C_b^*(BG; \mathbb{R}) \) coincides with the normed dual complex of \( C_*(BG; \mathbb{R}) \). The norm on chains induces a \( \ell^1 \)-semi-norm \( | \cdot |_1 \) on \( H_*(G; \mathbb{R}) = H_*(C_*(BG; \mathbb{R})) \) and \( \ell^\infty \)-semi-norm \( | \cdot |_\infty \) on \( H^*_b(G; \mathbb{R}) = H^*(C^*_b(BG; \mathbb{R})) \). When \( \mathcal{H} = \{ H_i \}_{i=1}^m \) is a collection of subgroups of \( G \), \( H_*(G, \mathcal{H}; \mathbb{R}) \) and \( H^*_b(G, \mathcal{H}; \mathbb{R}) \) and their semi-norms are defined analogously [MY07, §9.2].

As before, when \( N_i < H_i \), we let \( \mathcal{N} = \bigcup_{i=1}^m N_i, \quad \mathcal{H} = \{ \mathcal{H}_i \}_{i=1}^m \text{ and } \mathcal{G} = G/\langle \mathcal{N} \rangle \).

**Lemma 6.1.** Let \( G \) be a group and \( \mathcal{H} = \{ H_i \}_{i=1}^m \) a collection of subgroups such that \( \mathcal{H} \hookrightarrow_h G \). Suppose, for some integer \( n \geq 2 \), \( (G, \mathcal{H}) \) is a PD(\( n \))-pair and for a sufficiently deep \( \{ N_i < H_i \}_{i=1}^m \), \( \text{cd}(\mathcal{H}_i) \leq n - 2 \) for each \( 1 \leq i \leq m \). Then \( H_n(\mathcal{G}, \mathcal{H}; \mathbb{Z}) = \mathbb{Z} \).

**Proof.** The result follows from the long exact sequence in homology of the pair \( (\mathcal{G}, \mathcal{H}) \) and the isomorphism \( H_n(\mathcal{G}, \mathcal{H}; \mathbb{Z}) \cong H_n(G, \mathcal{H}; \mathbb{Z}) = \mathbb{Z} \). \( \square \)
**Definition 6.2.** Under the hypothesis of Lemma 6.1, we define the *simplicial volume* of \( \overline{G} \) by
\[
|G| = \inf \{ |c| \mid c \in C_n(BG; \mathbb{R}) \},
\]
where \( [G] \) is the image of the fundamental class under the change of coefficients map \( H_n(\mathbb{G}; \mathbb{Z}) \to H_n(\overline{G}; \mathbb{R}) \). Similarly, we define
\[
|G, \mathcal{H}| = \inf \{ |f| \mid f \in C_n(BG, \bigcap_{i=1}^m BH_i); \mathbb{R}) \},
\]
where \( [f] \) is a collection of subgroups such that \( \mathcal{H} \) is amenable for each \( 1 \leq i \leq m \). The simplicial volume of \( \overline{G}, \overline{\mathcal{H}} \) is defined analogously.

The following result is a generalisation of [FM11, Theorem 1.5].

**Theorem 6.3.** Let \( G \) be a group and \( \mathcal{H} = \{ H_i \}_{i=1}^m \) a collection of subgroups such that \( G \to_h \mathcal{H} \). Suppose, for some integer \( n \geq 2 \), \( (G, \mathcal{H}) \) is a PD\((n)\)-pair and for a sufficiently deep \( \{ N_i \leq H_i \}_{i=1}^m \), \( \text{cd}(G) = n \). Then, \( \text{cd}(\overline{G}) = n \). Let \( \text{cd}(\overline{G}) = n \), \( H_n(\overline{G}; \mathbb{Z}) = \mathbb{Z} \). In addition,

1. if the group \( \overline{H}_i \) is amenable for each \( 1 \leq i \leq m \), then \( |\overline{G}| \leq |G, \mathcal{H}| \);
2. if \( G \) is hyperbolic relative to \( \mathcal{H} \), then \( |\overline{G}| > 0 \).

**Proof.** The first two claims follow from Corollary 2.2 (ii) and Lemma 6.1.

For part (i), by Theorem A (ii), \( p_n : H_n(G, \mathcal{H}; \mathbb{Z}) \to H_n(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z}) \) induced by \( (G, \mathcal{H}) \to (\overline{G}, \overline{\mathcal{H}}) \) is an isomorphism. Since each \( \overline{H}_i \) is amenable, \( j^n : H^n_\infty(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \to H^n_\infty(\overline{G}; \mathbb{R}) \) is an isometric isomorphism [Fri17, Theorem 5.14]. By Duality Principle [Fri17, Lemma 6.1], we obtain
\[
|\overline{G}| = \sup \{ \langle j^n(\psi), [\overline{G}] \rangle \mid \psi \in H^n_\infty(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}), \|\psi\|_\infty \leq 1 \},
\]
where the last inequality follows from the functoriality of the \( \ell^1 \)-semi-norm [Löh11, Proposition 2.1].

For part (ii), since \( G \) is hyperbolic relative to \( \mathcal{H} \), then \( \overline{G} \) is hyperbolic relative \( \overline{\mathcal{H}} \) [Osi07, Theorem 1.1]. Consider the commutative diagram of long exact sequences
\[
\begin{array}{cccccc}
\prod_{i=1}^m H^{n-1}_0(\overline{H}_i; \mathbb{R}) & \to & H^n_0(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) & \to & H^n_0(\overline{G}; \mathbb{R}) & \to & \prod_{i=1}^m H^n_0(\overline{H}_i; \mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{i=1}^m H^n(\overline{H}_i; \mathbb{R}) & \to & H^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) & \to & H^n(\overline{G}; \mathbb{R}) & \to & \prod_{i=1}^m H^n(\overline{H}_i; \mathbb{R})
\end{array}
\]
Since \( \text{cd}(\overline{H}_i) \leq n - 2 \), the map \( H^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \to H^n(\overline{G}; \mathbb{R}) \) is onto. Since \( \overline{G} \) is hyperbolic relative to \( \overline{\mathcal{H}} \), the comparison map \( H^n_\infty(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \to H^n(\overline{G}; \mathbb{R}) \) is onto [Fra18, Theorem 1.1]. It follows that the comparison map \( c : H^n_\infty(\overline{G}; \mathbb{R}) \to H^n(\overline{G}; \mathbb{R}) \) is also onto which shows that \( |\overline{G}| > 0 \). \qed

\[\text{COHOMOLOGY OF GROUP THEORETIC DEHN FILLINGS II 25}\]
6.1. Dehn fillings of pinched negatively curved manifolds. In this section, we illustrate how the results of the previous sections apply in a geometric setting. First, we need a lemma.

**Lemma 6.4.** Let $G$ be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of finitely generated torsion-free nilpotent subgroups with $rk(Z(H_i)) \geq 2$, for all $i$. Then there are infinitely many collections of subgroups $\{N_i\}_{i=1}^m$ such that $Z \cong N_i \leq Z(H_i)$, $\overline{\mathcal{H}}_i$ is torsion-free and $j_k : H_k(\overline{G};\mathbb{Z}) \rightarrow H_k(\overline{G},\overline{\mathcal{H}};\mathbb{Z})$ is an isomorphism for all $k > \max\{h(H_i) \mid 1 \leq i \leq m\}$ where $h(H_i)$ is the Hirsh length of $H_i$.

**Proof.** For each $i$, the center $Z(H_i)$ contains a factor $\langle x_i, y_i \rangle \cong \mathbb{Z}^2$. Let $N_i^{s,t} = \langle x_i^s, y_i^t \rangle < H_i$ where $s, t \in \mathbb{Z}$ are co-prime. Then the quotient $\overline{\mathcal{H}}_i = H_i/N_i^{s,t}$ is a torsion-free nilpotent group. The long exact sequence in homology for the pair $(\overline{G}, \overline{\mathcal{H}})$ establishes the isomorphism $j_k : H_k(\overline{G};\mathbb{Z}) \rightarrow H_k(\overline{G},\overline{\mathcal{H}};\mathbb{Z})$ for all $k > \max\{h(\overline{\mathcal{H}}_i) \mid 1 \leq i \leq m\} + 1 = \max\{h(\overline{\mathcal{H}}_i) \mid 1 \leq i \leq m\} + 1 = \max\{h(H_i) \mid 1 \leq i \leq m\}$.

The collections $\{N_i^{s,t}\}_{i=1}^m$ enumerated by the co-prime pairs $(s, t)$ have the required properties. \hfill $\square$

**Definition 6.5.** Let $M$ be a Riemannian $n$-manifold with a complete pinched negative sectional curvature and finite volume. All the cuspidal ends of $M$ are almost-flat manifolds and hence by the results of Gromov and Ruh are infra-nil [Gro78, Ruh82]. In particular, they are finitely covered by nilmanifolds. Let us assume here that all the cuspidal ends $\{L_i\}_{i=1}^m$ of $M$ are nilmanifolds such that $rk(Z(\pi_1(L_i))) \geq 2$. Let $\overline{M}$ be the natural compactification of $M$ with boundary components $\{L_i\}_{i=1}^m$. Let $G = \pi_1(\overline{M})$ and $H_i = \pi_1(L_i)$. Then $G$ is hyperbolic relative to the fundamental groups of the cuspidal ends and in particular $\{H_i\}_{i=1}^m \to G$ [Bow12, Far98]. By Theorem A (ii) and Lemma 6.4, there are infinitely many collections of sufficiently deep normal subgroups $\{N_i\}_{i=1}^m$ satisfying the conclusions of both such that $j_n : H_n(\overline{G};\mathbb{Z}) \xrightarrow{\cong} H_n(\overline{G},\overline{\mathcal{H}};\mathbb{Z})$. Let $\{N_i\}_{i=1}^m$ be one such collection. By [Osi07, Theorem 1.1], we can also assume that $\{N_i\}_{i=1}^m$ are sufficiently deep so that $\overline{G}$ is hyperbolic relative to $\overline{\mathcal{H}}$. Let $T_i \hookrightarrow L_i \xrightarrow{\pi_1} \mathcal{B}$ be the circle bundle with a nilmanifold base corresponding to the short exact sequence $N_i \to H_i \to \overline{\mathcal{H}}_i$ and denote by $C(L_i, T_i)$ the mapping cylinder of $\pi_i$ for each $i$ (see Figure 2 for an illustration). Note that $C(L_i, T_i)$ is manifold with boundary $L_i$. We define $M_T = \overline{M} \cup \bigcup_{\phi_1 \cup \cdots \cup \phi_m} (C(L_1, T_1) \cup \cdots \cup C(L_m, T_m)),$

where each $\phi_i$ is the canonical identification of $\partial C(L_i, T_i)$ with $L_i$ and call it a sufficiently deep Dehn filling of $\overline{M}$.

Given a closed oriented $n$-manifold $M$ possibly with boundary $\partial M$, the simplicial volume of $(M, \partial M)$ is defined as $||M, \partial M|| = ||(M, \partial M)||_1$
where \([M, \partial M] \in H_n(M, \partial M; \mathbb{R})\) is the image of the fundamental class under the change of coefficients map \(H_n(M, \partial M; \mathbb{Z}) \to H_n(M, \partial M; \mathbb{R})\). The following result is again a generalisation of \([FM11, \text{Theorem 1.5}]\).

**Corollary 6.6.** Let \(\overline{M}\) be a compact oriented \(n\)-manifold with nilmanifold boundary components \(\{L_i\}_{i=1}^m\) as above. Suppose \(M_T\) is a sufficiently deep Dehn filling of \(\overline{M}\). Then, \(M_T\) is a closed oriented aspherical \(n\)-manifold with fundamental group \(G\) and 
\[
0 < ||M_T|| \leq ||\overline{M}, \partial \overline{M}||.
\]

**Proof.** We first show that \(M_T\) is aspherical with fundamental group \(G\).

Let \(\pi : \widetilde{M} \to M\) be the universal covering map and for each \(1 \leq i \leq m\), denote by \(L_i\) the universal cover of \(L_i\) which is a simply connected nilpotent Lie group containing \(H_i\) as a uniform lattice. The universal cover \(Y\) of \(\overline{M}\) is a subspace of \(\widetilde{M}\) with boundary a \(G\)-orbit of almost-flat totally geodesic submanifolds homeomorphic to \(L_i\) for each \(1 \leq i \leq m\). Let \(K = Y / \langle \langle \mathcal{N} \rangle \rangle\), which is a cover of \(\overline{M}\) with fundamental group \(\langle \langle \mathcal{N} \rangle \rangle\). The boundary of \(K\) is a disjoint union of \(\overline{G}\)-orbits of submanifolds homeomorphic to \(L_i = L_i / N_i\) for each \(1 \leq i \leq m\).

The circle bundle \(T_i \hookrightarrow L_i \overset{\pi_i}{\twoheadrightarrow} B_i\) is the quotient of the short exact sequence \(\mathbb{R} \hookrightarrow L_i \twoheadrightarrow B_i\) of simply connected nilpotent Lie groups by the action of \(H_i\). Hence, it lifts to the circle bundle \(T_i \hookrightarrow L_i' \overset{\pi_i}{\twoheadrightarrow} B_i\). Denote by \(C(L_i', T_i)\) the mapping cylinder of \(\pi_i\). Define 
\[
M' = \bigcup_{\psi_1 \cdots \psi_m} (\overline{G} \times \prod_{i=1}^m C(L_i', T_i) \sqcup \cdots \sqcup \overline{G} \times \prod_{i=m}^m C(L_i', T_i)),
\]
where each \(\psi_i\) is the canonical identification of \(\overline{G} \times \prod_{i=1}^m \partial C(L_i', T_i)\) with \(\overline{G} L_i\). The manifold \(M'\) is simply connected by construction. By Mayer-Vietoris homology sequence and the Cohen-Lyndon property of Theorem 3.22, it follows \(M'\) has also trivial homology and is therefore contractible. The group \(\overline{G}\) acts freely on \(M'\) and the quotient \(M'/\overline{G}\) is homeomorphic to \(M_T\).

Since \((\overline{M}, \partial \overline{M}) \approx (BG, \mathbb{Z} \mathbb{L}_1)^m BH_i\), we have \([\overline{M}, \partial \overline{M}] \in H_n(G, \mathcal{H}; \mathbb{R})\). Similarly, since \(M_T \approx B\overline{G}\), we have \([M_T] \in H_n(\overline{G}; \mathbb{R})\). Applying Theorem 6.3, we obtain 
\[
0 < ||M_T|| \leq ||\overline{M}, \partial \overline{M}||. \quad \Box
\]

**Remark 6.7.** In the geometric setting of Corollary 6.6, the isomorphism 
\[
p_n : H_n(G, \mathcal{H}; \mathbb{Z}) \xrightarrow{\cong} H_n(\overline{G}, \mathcal{H}; \mathbb{Z})
\]
used in the proof of Theorem 6.3 also follows from excising the interiors of the attached sub-
manifolds \( \{C(L_i, T_i)\}_{i=1}^m \) from \( M_T \) which gives an isomorphism between the (co)homologies
of \((M, \partial M)\) and \( (M_T, \{C(L_i, T_i)\}_{i=1}^m) \). So, one can think of the isomorphism in Theorem A
(ii) as a group theoretic analog of topological excision.

We should also remark that the right-hand-side inequality of Corollary 6.6 can be deduced
from Gromov’s Additivity Theorem [Fri17, Theorem 7.6].

7. Quotients of acylindrically hyperbolic groups

7.1. Cohomology and embedding theorems. We prove Theorem D in this subsection.
The reader is referred to Section 2.4 for an outline of the proof. In the sequel, we employ
the convention that if \( X \) is a set of alphabets and \( w \) is a word over \( X \), then \( \|w\| \) denotes
the length of \( w \). In certain cases, it might be possible to view \( w \) as a word over another alphabet \( Y \). In such a case, we will use \( \|w\|_X \) (resp. \( \|w\|_Y \)) to denote the number of letters of \( X \) (resp. \( Y \)) in \( w \). For two words \( w \) and \( v \), we write \( w \equiv v \) to indicate that there is a
letter-by-letter equality between \( w \) and \( v \).

**Lemma 7.1.** Let \( F_4 \) be a free group of rank 4, \( \mathcal{F} \subseteq F_4 \) a finite set, and \( C \) a countable group
with \( cd(C) \geq 2 \). Then there exists a quotient \( R \) of \( F_4 \) such that the following hold.

1. \( R \) can be decomposed as a free product \( R = \mathbb{Z} \ast R_0 \) with \( \text{card}(R_0) = \infty \). Moreover,
   \( C \) embeds into \( R_0 \).
2. The quotient map \( F_4 \rightarrow R \) is injective on \( \mathcal{F} \).
3. If \( C \) is torsion-free, then so is \( R \).
4. \( \text{hd}(R) \leq \max\{\text{hd}(C), 2\} \), \( \text{cd}(R) \leq \text{cd}(C) \).
5. For every \( n \geq 3 \) and every \( R \)-module \( A \), we have
   \[ H_n(R; A) \cong H_n(C; A), \quad H^n(R; A) \cong H^n(C; A), \]
   where the action of \( C \) on \( A \) is induced by the embedding \( C \hookrightarrow R \).
6. If \( C \) is finitely generated, then \( R_0 \) is hyperbolic relative to \( C \).
7. If \( C \) is of type FP\( n \) for some \( n \in \mathbb{N}^+ \cup \{\infty\} \), then so is \( R \).

**Remark 7.2.** Lemma 7.1 (1)–(6) are proved in [DGO17, Lemma 8.4]. We refine the method
therein so as to impose (co)homological conditions.

The proof of Lemma 7.1 relies on small cancellation theory, the reader is referred to
[LS01, Chapter V] for a treatment.

**Proof.** Let \( \{x, y, z, t\} \) be a free basis of \( F_4 \), let \( F_3 < F_4 \) be the subgroup generated by \( x, y, t \),
and let \( \{c_i\}_{i \in I} \) be a generating set of \( C \). There exist freely reduced words \( \{w_i\}_{i \in I}, \{v_i\}_{i \in I} \)
over the alphabet \( \{x, y\} \) such that

1. the words \( \{c_iw_i\}_{i \in I} \) satisfy the \( C'(1/2) \) small cancellation condition over the free
   product \( \langle x \rangle \ast \langle y \rangle \ast C \);
2. the words \( \{v_i\}_{i \in I} \) satisfy the \( C'(1/2) \) small cancellation condition over the alphabet
   \( \{x, y\} \);
3. the words \( \{tc_iw_it^{-1}v_i\}_{i \in I} \) satisfy the \( C'(1/6) \) small cancellation condition over the free
   product \( \langle x \rangle \ast \langle y \rangle \ast \langle t \rangle \ast C \).
Indeed, we can first construct words \( w_i \) satisfying condition (a), and then pick sufficiently long words \( v_i \) to ensure conditions (b) and (c).

Let \( N \) (resp. \( N_0 \)) be the normal subgroup of \( F_4 \ast C \) (resp. \( F_3 \ast C \)) generated by \( \{tc_iw_it^{-1}v_i\}_{i \in I} \), and let

\[
R_0 = (F_3 \ast C)/N_0, \; R = (F_4 \ast C)/N.
\]

For \( i \in I \), let \( \overline{t} \) (resp. \( \overline{u}, \overline{v}, \overline{w}, \overline{z} \)) be the image of \( t \) (resp. \( c_i, w_i, v_i, z \)) under the quotient map \( F_4 \ast C \to R \). Then we have

\[
R = \langle \overline{z} \rangle * R_0 = \mathbb{Z} * R_0.
\]

Note that \( \overline{t} \overline{c_iw_i} \overline{t}^{-1} \overline{v_i} = 1 \) and we can rewrite this equation as \( \overline{c_i} = \overline{t}^{-1} \overline{v_i}^k \overline{t} \overline{w_i}^{-1} \). Thus, \( R \) is generated by \( \overline{t}, \overline{u}, \overline{v}, \overline{w}, \overline{z}, i \in I \), and hence is a quotient of \( F_4 \).

Let

\[
\alpha : F_4 \to R
\]

be the corresponding quotient map, which is the restriction of the quotient map \( F_4 \ast C \to R \) to \( F_4 \). It follows from the Greendlinger’s lemma for free products \([LS01, \text{Chapter V Theorem 9.3}]\) that if \( |w_i|, |v_i|, i \in I \), are sufficiently large, then \( \alpha \) is injective on \( \mathcal{F} \) and thus statement (2) is guaranteed.

Let \( L = \langle x \rangle * \langle y \rangle * C \), let \( W \leq L \) be the subgroup generated by \( \{c_iw_i\}_{i \in I} \), and let \( V \leq L \) be the subgroup generated by \( \{v_i\}_{i \in I} \).

**Claim 7.2.1.** \( W \) (resp. \( V \)) is a free group with basis \( \{c_iw_i\}_{i \in I} \), (resp. \( \{v_i\}_{i \in I} \)). In particular, \( W \) and \( V \) are both of rank \( \text{card}(I) \).

**Proof of Claim 7.2.1.** We prove the claim for \( W \). The proof for \( V \) is similar. Let

\[
u = \prod_{k=1}^{\ell} (c_{i_k}w_{i_k})^{\epsilon_k}
\]

be a nonempty freely reduced word over the alphabet \( \{c_iw_i\}_{i \in I} \), where \( i_k \in I \) and \( \epsilon_k = \pm 1 \) for \( k = 1, \ldots, \ell \). Think of \( u \) as a word over the alphabet \( \langle x \rangle \cup \langle y \rangle \cup C \) and then reduce \( u \) to its normal form \( \overline{u} \) corresponding to the free product \( \langle x \rangle * \langle y \rangle * C \) (see \([LS01, \text{Chapter IV}]\) for the definition of normal forms). By condition (a) and that the relations \( c_iw_i \) do not involve inverses of the generators \( x, y \), for each factor \( (c_{i_k}w_{i_k})^{\epsilon_k} \) of \( u \), a non-empty subword of \( (c_{i_k}w_{i_k})^{\epsilon_k} \) survives in \( \overline{u} \). In particular, \( \overline{u} \) is nonempty and thus \( u \) does not represent 1 in \( L \).

Note that the relations \( \overline{t} \overline{c_iw_i} \overline{t}^{-1} \overline{v_i} = 1 \) can be rewritten as \( \overline{t} \overline{c_iw_i} \overline{t}^{-1} = \overline{v_i}^{-1} \). Thus, \( R_0 \) is the HNN-extension of \( L \) with associated subgroups \( W \) and \( V \). In particular, \( L \) embeds into \( R_0 \). As \( \text{card}(L) = \infty \), we have \( \text{card}(R_0) = \infty \). Since \( C \) embeds into \( L \), \( C \) embeds into \( R_0 \). Thus, statement (1) holds.

If \( C \) is torsion-free, then so is \( L \). Being an HNN-extension of \( L \), \( R_0 \) is also torsion-free, and thus so is \( R = \mathbb{Z} * R_0 \), which is statement (3).

By \([Bie75, \text{Theorem 3.1}]\), there is a long exact sequence for any \( R_0 \)-module \( A \),

\[
\cdots \to H^{n-1}(W; A) \to H^n(R_0; A) \to H^n(L; A) \to H^n(W; A) \to \cdots
\]
As $W$ is free, exact sequence (12) implies for $n \geq 3$,
\[ H^n(R_0; A) \cong H^n(L; A) \cong H^n(C; A). \]

As $R = \mathbb{Z} \ast R_0$, the cohomology part of statement (5) holds. Similarly, one can prove the homology part of statement (5). Statement (4) follows from statement (5) and $\text{cd}(C) \geq 2$.

If $C$ is finitely generated, then we can construct $R$ using a finite generating set of $C$. Then $R_0$ is the quotient of $F_3 \ast C$ by adding finitely many relations $tc_iw_it^{-1}v_i$, and thus has a finite relative presentation over $C$. The Greendlinger’s lemma for free products implies that the relative isoperimetric function of $R_0$ with respect to $C$ is linear. Thus, $R_0$ is hyperbolic relative to $C$ (see Remark 3.6), which is statement (6).

If $C$ is of type $FP_n$ for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then $C$ is finitely generated and we can construct $R$ using a finite generating set of $C$, that is, $\text{card}(I) < \infty$. Note that the rank of the free group $W$ is $\text{card}(I)$. Thus, $W$ is of type $FP_\infty$. Note also that $L$ is the free product of a finite rank free group with $C$ and thus is of type $FP_n$. Let $\{A_i\}_{i \in I}$ be a direct system of $R_0$-modules such that $\lim A_i = 0$. For every $k \leq n$, Theorem 3.1 implies
\[ \lim H^k(W; A_i) = \lim H^k(L; A_i) = 0, \]
where the actions of $W$ and $L$ on $A_i$ are induced by their embeddings into $R_0$. It follows from the five lemma and exact sequence (12) that $\lim H^k(R_0; A_i) = 0$. Therefore, $R_0$ is of type $FP_n$ by Theorem 3.1. As $R = \mathbb{Z} \ast R_0$, $R$ is of type $FP_n$, which is statement (7).

**Proof of Theorem D.** By Theorem 3.15, $G$ has a maximal finite normal subgroup $K(G)$. By [Hul16, Lemma 5.10], $G_0 = G/K(G)$ is acylindrically hyperbolic.

If $\text{cd}(C) = 0$, then $C = \{1\}$. Let $\overline{G} = G_0$. By Theorem 3.15, $C \hookrightarrow_h \overline{G}$. First consider statement (vi). As $\overline{G}$ and $G$ are quasi-isometric (note that the assumption of (vi) implies that $G$ and $\overline{G}$ are finitely generated), [Alo94, Corollary 9] implies (vi). Other conclusions of Theorem D hold trivially.

If $\text{cd}(C) = 1$, then by the Stallings-Swan theorem [Swa69, corollary to Theorem 1], $C$ is free. By Theorem 3.15, there exists a finitely generated non-cyclic free group $F \hookrightarrow_h G_0$. Let $\overline{G} = G_0$. It is well-known that the free group $C$ embeds into $F$. Thus, $C$ also embeds into $\overline{G}$. All conclusions except for (ii) hold trivially. If in addition, $C$ is finitely generated, then $C$ is a finite rank free group and we can let $F = C$. Thus, (ii) also holds.

Let us assume $\text{cd}(C) \geq 2$. By Theorem 3.15, there exists $X \subseteq G_0$ and a free subgroup $F_4 \subseteq G_0$ of rank 4 such that $F_4 \hookrightarrow_h (G_0, X)$. There exists a finite set $F \subseteq F_4 \setminus \{1\}$ such that if $N \lhd F_4$ satisfies $N \cap F = \emptyset$, then the conclusions of Theorems A and 3.20 and [DGO17, Theorem 7.15] hold.

By Lemma 7.1, $C$ embeds into an infinite quotient $R = \mathbb{Z} \ast R_0$ of $F_4$ such that the conclusions of Lemma 7.1 hold and the quotient map $F_4 \to R$ is injective on $F$. Let $N$ be the kernel of $F_4 \to R$. Then $N \cap F = \emptyset$. Let $\overline{G} = G/\langle\langle N\rangle\rangle$.

As $R = \mathbb{Z} \ast R_0$, $R_0$ is a proper subgroup of $R$ and in particular, $R_0$ is a proper subgroup of $G$. By Example 3.7 (d), $R_0 \hookrightarrow_h R$. Proposition 3.9 then implies that $R_0 \hookrightarrow_h G$. As $\text{card}(R_0) = \infty$, Theorem 3.20 then implies that $\overline{G}$ is acylindrically hyperbolic, that is, statement (i) holds. As $C$ embeds into $R_0$, $C$ also embeds into $\overline{G}$. 


If $C$ is finitely generated, then Lemma 7.1 implies that $R_0$ is hyperbolic relative to $C$, in particular, $C \to \pi_1 R_0$. As $R_0 \to \pi_1 \overline{G}$, we have $C \to \pi_1 \overline{G}$ by Proposition 3.9. Thus, statement (ii) holds.

Suppose that $G$ and $C$ are torsion-free and there is a non-trivial finite-order element $g \in \overline{G}$. Then $G_0 = G$. Denote the image of $X$ under the quotient map $G \to \overline{G}$ by $\overline{X}$. Then $R \to \pi_1 (\overline{G}, \overline{X})$ [DGO17, Theorem 7.15 (b)]. As $\overline{g}$ has finite order, it acts elliptically on the Cayley graph $\Gamma(\overline{G}, \overline{X} \cup R)$. By [DGO17, Theorem 7.15 (f)], there is an element $g \in G$ such that $g$ is mapped to $\overline{g}$ under the quotient map $G \to \overline{G}$ and $g$ acts elliptically on $\Gamma(G, X \cup F_1)$. As $\overline{g}$ has finite order, $g^n \in \langle N \rangle$ for some $n > 0$. Since $g^n$ is elliptic as $g$ is, there is some $h \in G$ such that $hg^n h^{-1} \in N \leq F_4$ [DGO17, Theorem 7.15 (d)]. By Lemma 7.1 (7), $R$ is torsion free and thus $hg^{-1} \not\in F_4$. But

$$|(hgh^{-1})_4 (hgh^{-1})^{-1} \cap F_4| \geq \langle hg^n h^{-1} \rangle = \infty,$$

which is in contradiction with the almost malnormality of $F_4$ in $G$ [DGO17, Proposition 2.10]. We have proved statement (iii).

Statement (iv) follows from Corollary 2.2 (i).

Consider statement (v). Corollary 2.2 (ii) implies that

$$\text{cd}(\overline{G}) \leq \max\{\text{cd}(G_0), \text{cd}(F_4) + 1, \text{cd}(R)\}.$$ 

If $K(G) \neq \{1\}$, then $G$ has torsion and thus $\text{cd}(G) = \infty$ [Bro94, Chapter VIII Corollary 2.5], in which case (v) is a void statement. Thus, let us assume $K(G) = \{1\}$ and thus $G_0 = G$. As $\text{cd}(R) \leq \text{cd}(C)$ and $\text{cd}(C) \geq 2$, we have

$$\text{cd}(\overline{G}) \leq \max\{\text{cd}(G), \text{cd}(F_3) + 1, \text{cd}(R)\} \leq \max\{\text{cd}(G), 2, \text{cd}(C)\} = \max\{\text{cd}(G), \text{cd}(C)\}.$$ 

If $G$ and $C$ are of type $FP_n$ for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then Lemma 7.1 implies that so is $R$. As $G$ and $G_0$ are quasi-isometric, $G_0$ is of type $FP_n$ [Alo94, Corollary 9]. As $F_4$ is of type $FP_\infty$, Corollary 2.2 (iii) implies that $\overline{G}$ is of type $FP_n$. Conversely, if $C$ is finitely generated and $G$ and $\overline{G}$ are of type $FP_n$ for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then $C \to \pi_1 \overline{G}$ by the previously proved statement (ii). Thus, $C$ is of type $FP_n$ [DGO17, Theorem 2.11]. The previously proved statement (v) implies that $\text{cd}(\overline{G}) < \infty$ if and only if $\max\{\text{cd}(G), \text{cd}(C)\} < \infty$, which further implies the statement about type $FP$. This finishes the proof of statement (vi). \qed

Remark 7.3. Analogous to the above proof, in the setting of Theorem D, we have $\text{hd}(\overline{G}) \leq \max\{\text{hd}(G), \text{hd}(C), 2\}$.

7.2. Constructing hyperbolically embedded subgroups. We will prove Theorem E in the next subsection, where the following technical result will be needed. The reader is referred to Section 2.4 for an outline of the proof.

Proposition 7.4. Suppose that $G$ is a group, $\Lambda$ is a finite set and $\{k_\lambda\}_{\lambda \in \Lambda}$ is a set of positive integers. Let $a_{\lambda,i} \in G$ for $\lambda \in \Lambda$ and $i = 1, 2, \cdots, k_\lambda$. Also suppose that there is a family of subgroups $\{F_\lambda\}_{\lambda \in \Lambda} \to_h G$ such that

(a) each $F_\lambda$ is free of rank $2k_\lambda$ with basis $\{f_{\lambda,i}; g_{\lambda,i}\}_{i=1}^{k_\lambda}$; and
(b) $a_{\lambda,i} \not\in F_\lambda \setminus \{1\}$ for all $\lambda \in \Lambda$ and $i = 1, 2, \cdots, k_\lambda$. 

Then for sufficiently large $n \in \mathbb{N}^+$, the set

$$\{f_{\lambda,i}^{n}a_{\lambda,i}^{n}g_{\lambda,i}^{n}\}_{i=1}^{k_{\lambda}} \subseteq G$$

freely generates a subgroup $H_{\lambda} \leq G$ and

$$(13) \quad \{H_{\lambda}\}_{\lambda \in \Lambda} \hookrightarrow G.$$

The rest of this subsection is devoted to the proof of the above proposition. We first find a “sufficiently large” $n \in \mathbb{N}^+$. As $F_{\lambda} = *_{i=1}^{k_{\lambda}}(\langle f_{\lambda,i} \rangle * \langle g_{\lambda,i} \rangle)$, we have $\{\langle f_{\lambda,i} \rangle, \langle g_{\lambda,i} \rangle\}_{i=1}^{k_{\lambda}} \hookrightarrow F_{\lambda}$ by Example 3.7. It follows from Proposition 3.9 that there exists a set $X \subseteq G$ such that

$$(14) \quad \{\langle f_{\lambda,i} \rangle, \langle g_{\lambda,i} \rangle\}_{\lambda \in \Lambda, i \in \{1, \ldots, k_{\lambda}\}} \hookrightarrow (G, X).$$

By [DGO17, Corollary 4.27], we may assume that $a_{\lambda,i} \in X$ for all $\lambda \in \Lambda$ and $i = 1, 2, \ldots, k_{\lambda}$, as $\Lambda$ and $k_{\lambda}$ are finite.

For $\lambda \in \Lambda$ and $i \in \{1, \ldots, k_{\lambda}\}$, let

$$\hat{d}_{\lambda,i,f} : \langle f_{\lambda,i} \rangle \times \langle f_{\lambda,i} \rangle \to [0, +\infty], \quad \hat{d}_{\lambda,i,g} : \langle g_{\lambda,i} \rangle \times \langle g_{\lambda,i} \rangle \to [0, +\infty]$$

be the relative metrics corresponding to (14). The metrics $\hat{d}_{\lambda,i,f}$ and $\hat{d}_{\lambda,i,g}$ are locally finite.

As $\text{card}(\Lambda), k_{\lambda} < \infty$, for sufficiently large $n$, we will have

$$\hat{d}_{\lambda,i,f}(1, f_{\lambda,i}^{n})^{\lambda}, \hat{d}_{\lambda,i,g}(1, g_{\lambda,i}^{n})^{\lambda} > 50D$$

for all $\lambda$ and $i$, where $D > 0$ is given by Lemma 3.11. We fix one such $n$ and let $H_{\lambda} \leq G$ be as in Proposition 7.4. For simplicity, denote the set $\{f_{\lambda,i}^{n}a_{\lambda,i}^{n}g_{\lambda,i}^{n}\}_{i=1}^{k_{\lambda}}$ by $U_{\lambda}$.

Notice that $F_{\lambda} \cap F_{\mu} = \{1\}$ whenever $\lambda \neq \mu$ [DGO17, Proposition 4.33]. The following are easy consequences of this fact.

$$(15) \quad \langle f_{\lambda,i} \rangle \cap \langle g_{\mu,j} \rangle = \{1\} \quad \text{for all } \lambda, \mu \in \Lambda,$$

$$(16) \quad \langle f_{\lambda,i} \rangle \cap \langle f_{\mu,j} \rangle = \langle g_{\lambda,i} \rangle \cap \langle g_{\mu,j} \rangle = \{1\} \quad \text{for all } \lambda, \mu \in \Lambda \text{ with } \lambda \neq \mu,$$

$$(17) \quad \langle f_{\lambda,i} \rangle \cap \langle f_{\lambda,j} \rangle = \langle g_{\lambda,i} \rangle \cap \langle g_{\lambda,j} \rangle = \{1\} \quad \text{for all } \lambda \in \Lambda \text{ and } i, j \in \{1, \ldots, k_{\lambda}\} \text{ with } i \neq j.$$

In particular, $\{\langle f_{\lambda,i} \rangle, \langle g_{\lambda,i} \rangle\}_{\lambda \in \Lambda, i \in \{1, \ldots, k_{\lambda}\}}$ is a distinct family of hyperbolically embedded subgroups of $G$.

For simplicity, let

$$K_{\lambda} = \left( \bigcup_{i=1}^{k_{\lambda}} \langle f_{\lambda,i} \rangle \right) \cup \left( \bigcup_{i=1}^{k_{\lambda}} \langle g_{\lambda,i} \rangle \right), \quad K = \bigsqcup_{\lambda \in \Lambda} K_{\lambda}.$$

Remark 7.5. For every $\lambda \in \Lambda$ and every freely reduced word $w$ over $U_{\lambda}$, we can think of $w$ as a word over the alphabet $X \sqcup K$, i.e., regard every $f_{\lambda,i}^{n}$ (resp. $g_{\lambda,i}^{n}$) as a letter from $\langle f_{\lambda,i} \rangle$ (resp. $\langle g_{\lambda,i} \rangle$) and regard every $a_{\lambda,i}$ as a letter from $X$. In this sense, $w$ satisfies the conditions (W1), (W2), and (W3) of Lemma 3.12.

Lemma 7.6. For all $\lambda \in \Lambda$, $H_{\lambda}$ is free with basis $U_{\lambda}$.

Proof. We need to show that for every $\lambda \in \Lambda$, every nonempty freely reduced word over $U_{\lambda}$ does not represent 1 in $G$. Suppose that

$$w = \prod_{k=1}^{\ell} (f_{\lambda,i}^{n}a_{\lambda,i}^{n}g_{\lambda,i}^{n})^{e_{k}}$$
is a freely reduced word over $U_\lambda$ for some $\lambda \in \Lambda$ such that $w$ represents 1 in $G$, where $\epsilon_k = \pm 1$ for $k = 1, \ldots, \ell$. As Remark 7.5, we think of $w$ as a word over $X \sqcup K$. Then $w$ labels a $3\ell$-gon $p$ in $\Gamma(G, X \sqcup K)$ with geodesic sides. Notice that $p$ has $2\ell$ components. By Lemma 3.12, each of these components is isolated. Proposition 3.11 then implies

$$2n \cdot 50D \leq 3nD,$$

which is absurd. Therefore, such a word $w$ does not exist. \hfill $\Box$

Consider the action $G \to \Gamma(G, X \sqcup K)$. Let $d_{X \sqcup K}$ be the combinatorial metric of $\Gamma(G, X \sqcup K)$. We verify that, with respect to this action, the family $\{H_\lambda\}_{\lambda \in \Lambda}$ satisfies the conditions $(C_1), (C_2), \text{ and } (C_3)$ of Lemma 3.8, which then implies $\{H_\lambda\}_{\lambda \in \Lambda} \to h G$. The verification is divided into the following Lemmas 7.7, 7.8, and 7.9.

**Lemma 7.7.** For every $\lambda \in \Lambda$, the action $H_\lambda \to \Gamma(G, X \sqcup K)$ is proper.

**Proof.** Fix $\lambda \in \Lambda$. It suffices to prove that for every $R > 0$, there are only finitely many $h \in H_\lambda$ such that $d_{X \sqcup K}(1, h) \leq R$. Let $h \in H_\lambda$ such that $d_{X \sqcup K}(1, h) \leq R$ and let $w$ be a freely reduced word over $U_\lambda$ representing $h$ in $G$. As in Remark 7.5, think of $w$ as a word over $X \sqcup K$. By Lemma 3.12, $w$ labels a $(4, 1)$-quasi-geodesic in $\Gamma(G, X \sqcup K)$. Thus,

$$\|w\|_{U_\lambda} = \|w\|_{X \sqcup K} \leq \frac{4R + 1}{3}.$$

There are only finitely many words $w$ satisfying the above inequality. It follows that the number of $h \in H_\lambda$ such that $d_{X \sqcup K}(1, h) \leq R$ is finite. \hfill $\Box$

For every $\lambda \in \Lambda$, we identify $H_\lambda$ with the subset of $\Gamma(G, X \sqcup K)$ labeled by elements of $H_\lambda$. Equivalently, we identify $H_\lambda$ with the $H_\lambda$-orbit of the identity vertex of $\Gamma(G, X \sqcup K)$.

**Lemma 7.8.** For every $\lambda \in \Lambda$, the orbit $H_\lambda$ is quasi-convex in $\Gamma(G, X \sqcup K)$.

**Proof.** Fix $\lambda \in \Lambda$. Let $h \in H_\lambda$ and let $\gamma$ be a geodesic in $\Gamma(G, X \sqcup K)$ from the vertex 1 to the vertex $h$. As $\Gamma(G, X \sqcup K)$ is a Gromov hyperbolic space, there exists $R > 0$ such that if $\alpha$ and $\beta$ are $(4, 1)$-quasi-geodesics with the same endpoint, then $d_{Hau}(\alpha, \beta) \leq R$, where $d_{Hau}$ is the Hausdorff metric with respect to $d_{X \sqcup K}$.

Let $w$ be a freely reduced word over $U_\lambda$ representing $h$ in $G$. As in Remark 7.5, think of $w$ as a word over $X \sqcup K$. By Lemma 3.12, $w$ labels a $(4, 1)$-quasi-geodesic $\alpha$ in $\Gamma(G, X \sqcup K)$. Note that $\alpha$ lies in the 2-neighborhood of the orbit $H_\lambda$, and $\gamma$ lies in the $R$-neighborhood of $\alpha$. Thus, $\gamma$ lies in the $(R + 2)$-neighborhood of $H_\lambda$. \hfill $\Box$

For $\lambda, \mu \in \Lambda$, the orbits $H_\lambda$ and $H_\mu$ are subsets of $\Gamma(G, X \sqcup K)$. Thus, it makes sense to talk about the diameter of $H_\mu \cap (gH_\lambda)^{+\epsilon}$ in $\Gamma(X \sqcup K)$, which is denoted by $\mathrm{diam} \left( H_\mu \cap (gH_\lambda)^{+\epsilon} \right)$.

**Lemma 7.9.** For every $\epsilon > 0$, there exists $R > 0$ such that the following holds. Suppose that for some $g \in G$ and $\lambda, \mu \in \Lambda$, we have

$$\mathrm{diam} \left( H_\mu \cap (gH_\lambda)^{+\epsilon} \right) \geq R.$$

Then $\lambda = \mu$ and $g \in H_\lambda$. 

Figure 3. Some possible configurations of the two pairs of adjacent components of $p$.

Proof. Fix $\epsilon > 0$ and let $R$ be the constant given by Lemma 3.8. Suppose that $\text{diam} (H_\mu \cap (gH_\lambda)^+ \epsilon) \geq R$ for some $g \in G$ and $\lambda, \mu \in \Lambda$. Then there exist vertices $v_1, v_2 \in H_\mu$ and $v_3, v_4 \in gH_\lambda$ such that $d(v_1, v_2) \geq R$, $d(v_1, v_3), d(v_2, v_4) \leq \epsilon$.

Let $p$ (resp. $q$) be a path from $v_1$ (resp. $v_3$) to $v_2$ (resp. $v_4$) such that $\text{Lab}(p)$ (resp. $\text{Lab}(q)$) is a freely reduced word over $U_\mu$ (resp. $U_\lambda$). Then $\ell(p) \geq R$ and $p, q$ are oriented $\epsilon$-close. By Lemma 3.12, there exist five consecutive components of $p$ which are connected to five consecutive components of $q$. In particular, there exist two pairs of adjacent components of $p$ which are connected to four consecutive components of $q$. Some of possible configurations of these two pairs of adjacent components are shown by Figure 3, where each horizontal line represents one possible configuration, the red and blue segments represent the two pairs of adjacent components of $p$, and the corresponding labels are written on top of the subpaths.

Below, we assume, without loss of generality, that these two pairs of adjacent components are of the form $f_{\mu, i}^{-n} g_{\mu, j}^{-n}, f_{\mu, j}^{-n} g_{\mu, r}^{-n}.

Other possible configurations can be analyzed similarly. We distinguish two cases.

Case 1. The first pair of adjacent components of $p$ are respectively connected to a pair of adjacent components of $q$.

Figure 4. Case 1
Case 1 is displayed by Figure 4, where the red (resp. blue) dashed line represents a path with label in $\langle f_{\mu,i}^n \rangle$ (resp. $\langle g_{\mu,j}^n \rangle$) connecting the corresponding red (resp. blue) components. Equations (15), (16), and (17) imply that $\lambda = \mu$ and the red (resp. blue) component of $q$ is labeled by $f_{\mu,i}^{-n}$ (resp. $g_{\mu,j}^{-n}$). As the red and blue dashed lines form a loop, another consequence of (15) is that both of these dashed lines are labeled by 1.

Let $p_1$ (resp. $q_1$) be the subpath of $p$ (resp. $q$) labeled by $uf_{\mu,i}^{-n}$ (resp. $v f_{\mu,i}^{-n}$). Then $p_1^+ = q_1^+$. By the structure of $U_{\mu}$, Lab($p_1$) and Lab($q_1$) are words over $U_{\mu}$ and thus represent elements in $H_{\mu}$. Therefore, $p_1^+ \in H_{\mu} \cap gH_{\mu}$. It follows that $g \in H_{\mu}$.

Case 2. The first pair of adjacent components of $p$ are respectively connected to two consecutive, but not adjacent, components of $q$.

\[ \text{Figure 5. Case 2} \]

Case 2 is displayed by Figure 5. Once again, Equations (15), (16), and (17) imply $\lambda = \mu$. The structures of $U_{\mu}$ imply $i = j = r$. The red (resp. blue) dashed line on the left is labeled by an element in $\langle f_{\mu,i}^n \rangle$ (resp. $\langle g_{\mu,j}^n \rangle$). As these dashed lines and the yellow segment labeled by $a_{\mu,i}$ form a loop, assumption (b) of Proposition 7.4 implies that both of these dashed lines are labeled by 1. Similarly, the red and blue dashed lines on the right are both labeled by 1.

Therefore, the word $g_{\mu,i}^{-2n} f_{\mu,i}^{2n} a_{\mu,i}$ labels a loop in $\Gamma(G, X \sqcup K)$ and thus represents 1 in $G$, which is in contradiction with assumption (b) of Proposition 7.4. Hence, Case 2 is in fact impossible.

\[ \text{Proof of Proposition 7.4.} \] The first assertion follows from Lemma 7.6 and formula (13) follows Lemmas 3.8, 7.7, 7.8, and 7.9.

7.3. Common quotients of acylindrically hyperbolic groups. In this subsection, we prove Theorem E. Given finitely generated acylindrically hyperbolic groups $G_1$ and $G_2$, we construct a common quotient $G$ of $G_1' = G_1/K(G_1)$ and $G_2' = G_2/K(G_2)$ satisfying the conclusions of that theorem, where $K(G_1)$ (resp. $K(G_2)$) is the maximal finite normal subgroup of $G_1$ (resp. $G_2$).

The idea is to consider $\tilde{G} = G_1' * G_2'$ and pick a finite generating set $A$ (resp. $B$) of $G_1'$ (resp. $G_2'$). The quotient $G$ is constructed by adding particular relations (which will be done by Dehn filling) to $\tilde{G}$ which identify elements of $A$ (resp. $B$) with certain elements of $G_2'$ (resp. $G_1'$).

There exists a finite generating set $A = \{a_1, ..., a_k\}$ (resp. $B = \{b_1, ..., b_k\}$) of $G_1'$ (resp. $G_2'$) for some $k \in \mathbb{N}^+$. To simplify the argument, let $a_{k+1} = a_{k+2} = b_{k+1} = b_{k+2} = 1$. 


To perform Dehn filling on \(\tilde{G}\), the first step is to find hyperbolically embedded subgroups. By [Hul16, Lemma 5.10], \(G'_1\) and \(G'_2\) are acylindrically hyperbolic with \(K(G'_1) = K(G'_2) = \{1\}\). Thus, Theorem 3.15 implies that there exist free groups 
\[
F_1 \hookrightarrow_h G'_1, \quad F_2 \hookrightarrow_h G'_2,
\]
each of which has rank \(2k + 4\). By Example 3.7, we have \(\{G'_1, G'_2\} \hookrightarrow_h \tilde{G}\). Thus, Proposition 3.9 implies 
\[
\{F_1, F_2\} \hookrightarrow_h \tilde{G}.
\]

In fact, \(F_1, F_2\) are not quite the subgroups that we want, and we will apply Proposition 7.4 to construct other hyperbolically embedded subgroups from \(A, B, F_1, F_2\). Note that for 
\[
i = 1, 2, \ldots, k + 2,
\]
\[
a_i \notin F_2 \setminus \{1\}, \quad b_i \notin F_1 \setminus \{1\}.
\]
Let \(\{f_{1,i}g_{1,i}^{k+2}\}_{i=1}^{k+2}\) (resp. \(\{f_{2,i}g_{2,i}^{k+2}\}_{i=1}^{k+2}\)) be a basis of the free group \(F_1\) (resp. \(F_2\)). Then Proposition 7.4 implies the following.

**Lemma 7.10.** For sufficiently large \(\ell \in \mathbb{N}^+\), \(\{f_{1,i}^{\ell}b_if_{1,i}^{\ell}\}_{i=1}^{k+2}\) (resp. \(\{f_{2,i}^{\ell}a_if_{2,i}^{\ell}\}_{i=1}^{k+2}\)) freely generates a subgroup \(H_1 \leq \tilde{G}\) (resp. \(H_2 \leq \tilde{G}\)) and 
\[
\{H_1, H_2\} \hookrightarrow_h \tilde{G}.
\]

**Proof of Theorem E.** As \(G'_1\) and \(G'_2\) are acylindrically hyperbolic, we have \(|G'_1| = |G'_2| = \infty\) and thus \(cd(G'_1), cd(G'_2) \geq 1\). Suppose \(cd(G'_1) = cd(G'_2) = 1\). Then \(G'_1\) and \(G'_2\) are free by the Stallings-Swan theorem [Swa69, corollary to Theorem 1]. Without loss of generality, we may assume that the rank of \(G'_1\) is greater than or equal to the rank of \(G'_2\). It follows that \(G'_2\) is a quotient of \(G'_1\). Let \(G = G'_2\). Statements (i), (ii), and (iii) follow trivially. Statement (iv) also holds because if \(G_2\) is of type \(FP_n\) for some \(n \in \{2, 3, \ldots, \infty\}\), then \(G'_2\) is a finite rank free group and thus of type \(FP_\infty\).

Thus, let us assume \(\max\{cd(G'_1), cd(G'_2)\} \geq 2\). Fix a sufficiently large \(\ell \in \mathbb{N}^+\) and let \(H_1\) and \(H_2\) be the subgroups given by Lemma 7.10. By Remark 3.18, Lemma 7.10, and Theorems 3.20, 3.22, there exist finite sets \(F_1 \subseteq H_1 \setminus \{1\}\) and \(F_2 \subseteq H_2 \setminus \{1\}\) such that if \(N_1 < H_1, N_2 < H_2\) and \(N_1 \cap F_1 = N_2 \cap F_2 = \emptyset\), then the following hold.

1. \(\{H_1/N_1, H_2/N_2\} \hookrightarrow_h \tilde{G}/\langle\langle N_1 \cup N_2\rangle\rangle\).
2. \((\tilde{G}, \{H_1, H_2\}, \{N_1, N_2\})\) is a Cohen-Lyndon triple and thus Theorems 4.5 and 4.7 and Corollary 4.6 can be applied to it.

Let \(\{u_i\}_{i=1}^{k}\) (resp. \(\{v_i\}_{i=1}^{k}\)) be freely reduced words over the alphabet \(\{f_{1,k+1}^{\ell}b_{i,k+1}f_{1,k+2}^{\ell}g_{1,k+2}^{\ell}, f_{2,k+1}^{\ell}a_{i,k+1}f_{2,k+2}^{\ell}g_{2,k+2}^{\ell}\}\) (resp. \(\{f_{2,k+1}^{\ell}a_{i,k+1}g_{2,k+1}^{\ell}, f_{2,k+2}^{\ell}a_{i,k+2}g_{2,k+2}^{\ell}\}\)) satisfying the \(C'(1/6)\) small cancellation condition, and let \(N_1\) (resp. \(N_2\)) be the normal subgroup of \(H_1\) (resp. \(H_2\)) generated by \(\{f_{1,i}^{\ell}b_iu_i\}_{i=1}^{k}\) (resp. \(\{f_{2,i}^{\ell}a_ig_{2,i}^{k}\}_{i=1}^{k}\)).

By Lemma 7.10, \(H_1\) and \(H_2\) are freely generated by \(\{f_{1,i}^{\ell}b_ig_{1,i}^{\ell}\}_{i=1}^{k+2}\) and \(\{f_{2,i}^{\ell}a_ig_{2,i}^{k}\}_{i=1}^{k+2}\), respectively. Thus, \(H_1/N_1\) and \(H_2/N_2\) can be presented as 
\[
H_1/N_1 = \langle f_{1,i}^{\ell}b_ig_{1,i}^{\ell} (i = 1, \ldots, k + 2) | f_{1,j}^{\ell}b_jg_{1,j}^{\ell}u_j (j = 1, \ldots, k) \rangle
\]
\[
= \langle f_{1,k+1}^{\ell}b_{i,k+1}f_{1,k+2}^{\ell}g_{1,k+2}^{\ell}, f_{1,k+2}^{\ell}a_{i,k+2}g_{1,k+2}^{\ell} \rangle,
\]
\[
H_2/N_2 = \langle f_{2,i}^{\ell}a_ig_{2,i}^{k} (i = 1, \ldots, k + 2) | f_{2,j}^{\ell}a_jg_{2,j}^{k}v_j (j = 1, \ldots, k) \rangle
\]
3.20
E
4.7
and
4.5
E
D
LS01
LS01

is trivial. So let us assume that $K$ implies that $p(20) \text{card} \{x\}$. Remark 7.11.

Analogous to the above proof, in the setting of Theorem (iv) holds. □

Let

$G = \tilde{G}/\langle \langle N_1 \cup N_2 \rangle \rangle$.

As $a_{k+1} = a_{k+2} = b_{k+1} = b_{k+2} = 1$, $G$ is a common quotient of $G'_1$ and $G'_2$. In particular, $G$ is a common quotient of $G_1$ and $G_2$.

If $H_1/N_1 = G$, then $G$ is a non-cyclic free group and thus is acylindrically hyperbolic. If $H_1/N_1$ is a proper subgroup of $G$, then equation (20), statement (1), and Theorem 3.20 imply that $G$ is acylindrically hyperbolic. Statement (i) is proved.

Consider statement (ii). For every $n \geq 3$ and every $G$-module $A$, we have

$H^n(G; A)$

$\cong H^n(\tilde{G}; A) \oplus H^n(H_1/N_1; A) \oplus H^n(H_2/N_2; A)$ by Theorem 4.5

$\cong H^n(\tilde{G}; A)$ as $H_1/N_1$ and $H_2/N_2$ are free groups

$\cong H^n(G'_1; A) \oplus H^n(G'_2; A)$ as $\tilde{G} = G'_1 * G'_2$, which is (ii).

Consider statement (iii). If either $K(G_1)$ or $K(G_2)$ is not the trivial group $\{1\}$, then (iii) is trivial. So let us assume that $K(G_1) = K(G_2) = \{1\}$. Then Corollary 4.6 implies

$\text{cd}(G) \leq \text{max}\{\text{cd}(\tilde{G}), \text{cd}(H_1/N_1), \text{cd}(H_2/N_2), \text{cd}(H_1) + 1, \text{cd}(H_2) + 1\}$

$= \text{max}\{\text{cd}(\tilde{G}), 1, 2\}$ as $H_1, H_2, H_1/N_1$, and $H_2/N_2$ are free groups

$= \text{max}\{\text{cd}(G'_1), \text{cd}(G'_2)\}$ as $\text{cd}(G'_1), \text{cd}(G'_2) \geq 2$

$= \text{max}\{\text{cd}(G_1), \text{cd}(G_2)\}$.

Finally, suppose that $G_1$ and $G_2$ are of type $FP_n$ for some $n \in \{2, 3, \ldots, \infty\}$. Then so are $G'_1$ and $G'_2$ [Alo94, Corollary 9]. As $H_1/N_1$ and $H_2/N_2$ are free groups of finite rank, they are of type $FP_\infty$. Therefore, Theorem 4.7 implies that $G$ is of type $FP_n$ and thus statement (iv) holds.

Remark 7.11. Analogous to the above proof, in the setting of Theorem E, we have $\text{hd}(G) \leq \text{max}\{\text{hd}(G_1), \text{hd}(G_2), 2\}$.

8. Some applications

In this section, we give some applications of Theorems D and E.
8.1. Infinite dimension torsion-free $FP_\infty$ groups. If a group $G$ has $\text{cd}(G) < \infty$, then $G$ is necessarily torsion-free. Of course, the converse is not true, e.g., $\text{cd}(\mathbb{Z}^2) = \infty$. Observe however that $\mathbb{Z}^2$ is not of type $FP_\infty$. In fact, Bieri asked if every torsion-free group of type $FP_\infty$ is also of type $FP$ or if there is an $FP_\infty$ group with a non-finitely generated free abelian subgroup [Wal79, Problem F11]. In [BG84], Brown and Geoghegan settled both questions, by showing that Thompson’s group $F$ is of type $FP_\infty$.

For every torsion-free acylindrically hyperbolic group $G$, Theorem D embeds $F$ into an acylindrically hyperbolic quotient $\overline{G}$ of $G$, which is a torsion-free $FP_\infty$ group of infinite cohomological dimension. We thus have the following.

**Corollary 8.1.** Every torsion-free acylindrically hyperbolic group $G$ of type $FP_\infty$ has a torsion-free acylindrically hyperbolic quotient $\overline{G}$ of type $FP_\infty$ which contains the Thompson group $F$. In particular, $\text{cd}(\overline{G}) = \infty$.

8.2. Quotients of hyperbolic groups. SQ-universality of non-elementary hyperbolic groups was proved by Olshanskii [Ols95] and independently by Delzant [Del96]. In Theorem D, if $G$ and $C$ are word-hyperbolic, a hyperbolic quotient $\overline{G}$ of $G$ can be constructed so that the conclusions hold. The next corollary is thus a strengthening of the aforementioned result.

**Corollary 8.2.** Let $G$ be a non-elementary hyperbolic group and $C$ any hyperbolic group. Then there is a hyperbolic quotient $\overline{G}$ of $G/K(G)$ (in particular, $\overline{G}$ is a quotient of $G$), where $K(G)$ is the maximal finite normal subgroup of $G$, such that $C$ embeds into $\overline{G}$ and the following hold.

1. For all $n \geq 3$ and every $G$-module $A$, we have

$$H^n(\overline{G}; A) \cong H^n(G/K(G); A) \oplus H^n(C; A),$$

where the action of $G/K(G)$ (resp. $C$) on $A$ is induced by the quotient map $G/K(G) \to \overline{G}$ (resp. the embedding $C \hookrightarrow \overline{G}$).

2. $\text{cd}(\overline{G}) \leq \max\{\text{cd}(G), \text{cd}(C)\}$.

**Proof.** By passing to $G/K(G)$, we may assume that $K(G) = \{1\}$. By [Osi21, Corollary 4.21], there is a free subgroup $F_2 < G$ of rank 2 such that $G$ is hyperbolic relative to $F_2$. By [Jit02], four random elements of $F_2$ freely generate a free subgroup $F_4 < F_2$ that is malnormal in $F_2$. It is easy to see that $F_4$ is also quasi-convex in $F_2$, and thus [Bow12, Theorem 7.11] implies that $F_2$ is hyperbolic relative to $F_4$. Thus, $G$ is hyperbolic relative to $F_4$ [Osi06]. By Lemma 7.1, $C$ embeds into a quotient $R$ of $F_4$ such that $R$ is hyperbolic relative to $C$ and the quotient map $F_4 \to R$ is injective on any given finite set $F \subseteq F_4 \setminus \{1\}$. Since $C$ is hyperbolic, so is $R$ [Osi06, Corollary 2.41]. Let $\overline{G}$ be the quotient of $G$ by the Dehn filling $F_4 \to R$. By avoiding a suitable finite set $F$, we may assume that $\overline{G}$ is hyperbolic relative to $R$ [Osi07, Theorem 1.1]. Therefore, $\overline{G}$ is hyperbolic [Osi06, Corollary 2.41]. Items (i) and (ii) are immediate consequences of Theorem D.

Recently in [IMM21, Corollary 2], Italiano-Martelli-Migliorini settled a long-standing open problem by constructing the first example of hyperbolic group which contains a
subgroup of type $F$ that is not hyperbolic. Their group is the fundamental group of 5-dimensional hyperbolic pseudo-manifold that fibers over the circle. The fundamental group of the fiber is of type $F$ but not hyperbolic.

**Corollary 8.3.** Let $n \geq 5$ be an integer. Every non-elementary hyperbolic group $G$ with $\text{cd}(G) \leq n$ has a hyperbolic quotient $\overline{G}$ with $\text{cd}(\overline{G}) = n$ such that $\overline{G}$ contains the Italiano-Martelli-Migliorini group. In particular, there is a type $F$ non-hyperbolic subgroup $H < \overline{G}$.

*Proof.* Let $C_1$ be the group constructed by Italiano-Martelli-Migliorini in [IMM21, Corollary 2]. Then $\text{cd}(C_1) = 5$. Let $C_2$ be a hyperbolic group with $\text{cd}(C_2) = n$, and let $C = C_1 \ast C_2$. Since $\text{cd}(G) < \infty$, we have $K(G) = \{1\}$. Corollary 8.2 thus yields a hyperbolic quotient $\overline{G}$ such that $C$ embeds into $\overline{G}$ and $\text{cd}(\overline{G}) \leq \max\{\text{cd}(G), \text{cd}(C)\} = \max\{\text{cd}(G), \text{cd}(C_1), \text{cd}(C_2)\} = n$. Since $C$ embeds into $\overline{G}$, we also have $\text{cd}(\overline{G}) \geq \text{cd}(C) \geq \text{cd}(C_2) = n$, and thus $\text{cd}(\overline{G}) = n$. □

### 8.3. Property (T) quotients

A group $G$ has Kazhdan’s property (T) if every affine isometric action of $G$ on a Hilbert space has a global fixed point. The next result strengthens [Hul16, Corollary 1.7].

**Corollary 8.4.** Every acylindrically hyperbolic group $G$ of type $FP_n$ for some $n \in \{2, 3, \ldots, \infty\}$ (resp. $FP$) has an acylindrically hyperbolic quotient $\overline{G}$ of type $FP_n$ (resp. $FP$) with Kazhdan’s Property (T) such that $\text{cd}(\overline{G}) \leq \max\{\text{cd}(G), 7\}$.

*Proof.* By Kazhdan’s theorem, the Lie group $Sp(2,1)$ contains a acylindrically hyperbolic non-uniform arithmetic lattice $\Gamma$ with Property (T). By [ADMP17, Proposition 4.1], $\text{vcd}(\Gamma) = 7$. Since $\Gamma$ is a lattice in $Sp(2,1)$, it is of type $F$ and in particular of type $FP$ [Ash84]. Passing to a torsion-free finite index subgroup, we can assume that $\Gamma$ is torsion-free. The result now follows from Theorem E applied to $G$ and $\Gamma$. □

The above result can for example be applied to mapping class groups of surfaces of finite type, outer automorphism groups of free groups of finite rank and (non-virtually polycyclic) fundamental groups of compact orientable 3-manifolds which all exhibit nice cohomological finiteness conditions.

### 8.4. Acylindrically hyperbolic quotients distinguishable by cohomology

Let $G$ be any acylindrically hyperbolic group. For each $k \geq 3$, Theorem D implies that one can embed $\mathbb{Z}^k$ into a quotient $G_k$ of $G$ such that $H^n(G_k; \mathbb{Z}) \cong H^n(G; \mathbb{Z}) \oplus H^n(\mathbb{Z}^k; \mathbb{Z})$ for all $n \geq 3$. Suppose $\text{cd}(G) < \infty$. Then $H^*(G_k; \mathbb{Z}) \neq H^*(G_\ell; \mathbb{Z})$ for $k, \ell > \text{cd}(G)$. Suppose that $G$ is of type $FP_\infty$ instead. Then $H^k(G; \mathbb{Z})$ is finitely generated, and thus $H^k(G_k; \mathbb{Z}) \neq H^k(G_\ell; \mathbb{Z})$ for $k \neq \ell$. This establishes the following.

**Corollary 8.5.** Let $G$ be any acylindrically hyperbolic group. Then there exists an infinite family $\{G_k\}_{k=3}^\infty$ of acylindrically hyperbolic quotients of $G$ such that $H^n(G_k; \mathbb{Z}) \cong H^n(G; \mathbb{Z}) \oplus H^n(\mathbb{Z}^k; \mathbb{Z})$ for all $n, k \geq 3$. In particular, if $\text{cd}(G) < \infty$ or $G$ is of type $FP_\infty$, then each pair of elements of $\{G_k\}_{k=1}^\infty$ have non-isomorphic integral cohomology for $l = \text{cd}(G) + 1$ and $l = 3$, respectively.

If instead of embedding $\mathbb{Z}^k$, we embed the family of Houghton’s groups or Abel’s groups, then we will obtain quotients that are separated by homological finiteness properties.
Corollary 8.6. Let $G$ be any acylindrically hyperbolic group of type $FP_\infty$. Then $G$ has a family of acylindrically hyperbolic quotients $\{G_k\}_{k=2}^\infty$ such that for each $k$, $G_k$ has Kazhdan’s Property (T), is of type $FP_{k-1}$ but not of type $FP_k$.

We remark that being of type $FP_n$ is a quasi-isometric invariant [Alo94, Corollary 9]. Therefore, the quotients $\{G_k\}_{k\geq 1}$ in Corollary 8.6 are pairwise non-quasi-isometric.

Proof. By Corollary 8.4, we can pass to a quotient $\overline{G}$ of $G$ that is acylindrically hyperbolic and of type $FP_\infty$ and has property (T). There exists a family of groups $\{H_k\}_{k=2}^\infty$ such that for all $k$, $H_k$ is of type $FP_{k-1}$ but not of type $FP_k$. For example, one can let $\{H_k\}_{k=2}^\infty$ be the family of Houghton’s groups or Abel’s groups [Bro87, Theorems 5.1 and 6.1]. Theorem D then embeds each $H_k$ to a quotient $G_k$ of $\overline{G}$ with the desired properties. $\square$

8.5. First Betti number and a question of Osin. Motivated by the Virtual Positive First Betti Number Conjecture which has now been settled by Agol [Ago13], Osin posed the following question.

Question 8.7 (Osin, [Osi08, Problem 2.3]). Let $G$ be a Poincaré duality group of dimension 3 such that $G$ is hyperbolic relative to its subgroup $H$. If $G$ has positive virtual first Betti number, is it true that for most $N \triangleleft H$, $G/\langle\langle N\rangle\rangle$ also has positive virtual first Betti number?

Question 8.7 is only about virtual Betti numbers. For actual Betti numbers there are counterexamples. For example, let $S$ be an orientable surface of genus two. Take a suitable pseudo-Anosov homeomorphism $\phi$ on $S$ such that the mapping torus $T_\phi$ has $b_1(T_\phi) = 1$. Then $\pi_1(T_\phi)$ is hyperbolic and is a semi-direct product $\pi_1(T_\phi) = \pi_1(S) \rtimes \langle \tau \rangle$. $\pi_1(T_\phi)$ is hyperbolic relative to $\langle \tau \rangle$ [Bow12], and for every $n \geq 1$, $b_1(\pi_1(T_\phi)/\langle \tau^n \rangle) = 0$ since the image of $\tau$ generates $H_1(\pi_1(T_\phi); \mathbb{Q})$. Using this example, we prove the following.

Corollary 8.8. Let $G$ be any acylindrically hyperbolic group. Then $G$ has an acylindrically hyperbolic quotient $\overline{G}$ such that $b_1(\overline{G}) = 0$, $\text{cd}(\overline{G}) \leq \max\{\text{cd}(G), 3\}$, and $H^n(\overline{G}; A) \cong H^n(G; A)$ for all $n \geq 4$ and any $\overline{G}$-module $A$.

Proof. The idea is to construct an acylindrically hyperbolic group $G'$ such that $b_1(G') = 0$ and $\text{cd}(G') \leq 3$. Once such a $G'$ has been constructed, Theorem E will produce a common quotient $\overline{G}$ of $G$ and $G'$ with the desired properties.

So it remains to construct $G'$. Let $T_\phi, t$ be as above. Take eight elements $a_1, a_2, \ldots, a_8$ that generate $\pi_1(S)$ as a group. Then $a_1, a_2, \ldots, a_8, t$ generate $\pi_1(T_\phi)$ as a group. Let $X$ be the set consisting of $a_1, a_2, \ldots, a_8, t$ and their inverses. Randomly generate two words $w_1, w_2$ over $X$ of length $n$ and identify $w_1, w_2$ with the elements of $\pi_1(T_\phi)$ they represent.

By [MS19, Theorem 1], as $n \to \infty$, with probability 1 we have that $\langle w_1, w_2 \rangle \leq \pi_1(T_\phi)$ is free of rank 2 and $\langle w_1, w_2 \rangle \triangleleft \pi_1(T_\phi)$. Moreover, as $n \to \infty$, with positive probability we have that both $w_1$ and $w_2$ have non-zero $t$-exponents. Thus, there exist $w_1, w_2 \in \pi_1(T_\phi)$ such that

(a) $\langle w_1, w_2 \rangle \leq \pi_1(T_\phi)$ is free of rank 2,
(b) $\langle w_1, w_2 \rangle \triangleleft \pi_1(T_\phi)$, and
(c) $w_1$ and $w_2$ have non-zero $t$-exponents.
Let \( w \) be a word over \( w_1, w_2 \) such that \( w \) is not a proper power, satisfies the \( C'(1/6) \) small cancellation condition, and \( w \) has positive \( t \)-exponent. Let \( N \) be the normal subgroup of \( \langle w_1, w_2 \rangle \) generated by \( w \). By [Lyn50, Theorem 11.1], \( \langle w_1, w_2 \rangle / N \) has cd\( (\langle w_1, w_2 \rangle / N) \leq 2 \). Let \( \langle N \rangle \) be the normal closure of \( N \) in \( \pi_1(T_\phi) \). By making \( w \) sufficiently long, we can guarantee that \( N \) avoids any given finite subset of \( \langle w_1, w_2 \rangle \setminus \{1\} \), and therefore guarantee that \( \pi_1(T_\phi)/\langle N \rangle \) is acylindrically hyperbolic and cd\( (\pi_1(T_\phi)/\langle N \rangle) \leq 3 \) by Theorems 3.20 and A. Since \( w \) has positive \( t \)-exponent, we have \( b_1(\pi_1(T_\phi)/\langle N \rangle) = 0 \). So it suffices to let \( G' = \pi_1(T_\phi)/\langle N \rangle \).

\[\square\]

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