Rings of microdifferential operators for arithmetic $\mathcal{D}$-modules.

— Construction and an application to the characteristic varieties for curves

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Abstract

An aim of this paper is to develop a theory of microdifferential operators for arithmetic $\mathcal{D}$-modules. We first define the rings of microdifferential operators of arbitrary levels on arbitrary smooth formal schemes. A difficulty lies in the fact that there is no homomorphism between rings of microdifferential operators of different levels. To remedy this, we define the intermediate differential operators, and using these, we define the ring of microdifferential operators for $\mathcal{D}^!$. We conjecture that the characteristic variety of a $\mathcal{D}^!$-module is computed as the support of the microlocalization of a $\mathcal{D}^!$-module, and prove it in the curve case.

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Introduction

This paper is aimed to construct a theory of rings of microdifferential operators for arithmetic $\mathcal{D}$-modules. Let $X$ be a smooth variety over $\mathbb{C}$. Then the sheaf of rings of microdifferential operators denoted by $\mathcal{E}_X$ is defined on the cotangent bundle $T^*X$ of $X$. This ring is one of basic tools to study $\mathcal{D}$-modules microlocally, and it is used in various contexts. One of the most important and fundamental properties is

$$\operatorname{Char}(\mathcal{M}) = \operatorname{Supp}(\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{G}_X} \mathcal{M})$$

for a coherent $\mathcal{D}_X$-module $\mathcal{M}$, where $\pi : T^*X \to X$ is the projection. A goal of this study is to find an analogous equality in the theory of arithmetic $\mathcal{D}$-modules.

We should point out two attempts to construct rings of microdifferential operators. The first attempt was made by R. G. López in [Lo]. In there, he constructed the ring of microdifferential operators of \textit{finite order on curves}. However, the relation between his construction and the theory of arithmetic $\mathcal{D}$-modules was not clear as he pointed out in the last remark of \textit{ibid}. The second construction was carried out by A. Marmora in [Ma]. Our work can be seen as
a generalization of this work, and we will explain the relation with our construction in the following.

Now, let $R$ be a complete discrete valuation ring of mixed characteristic $(0, p)$. Let $\mathcal{X}$ be a smooth formal scheme over $\text{Spf}(R)$, and we denote the special fiber of $\mathcal{X}$ by $X$. For an integer $m \geq 0$, P. Berthelot defined the ring of differential operators of level $m$ denoted by $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$. He also defined the characteristic varieties for coherent $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules using almost the same way as we used to define the characteristic varieties for analytic $\mathcal{D}$-modules. It is natural to hope that there exists a theory of microdifferential operators and we can define the ring of microdifferential operators $\mathcal{E}^{(m)}_{\mathcal{X}, \mathbb{Q}}$ of level $m$ associated with $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$ satisfying an analog of \ref{eq:basic}. When $\mathcal{X}$ is a curve, this was done by Marmora in [Ma] as we mentioned above, in his study of Fourier transformation. He fixed a system of local coordinates, and constructed the ring of microdifferential operators using explicit descriptions as in [Bi, Chapter VIII], and proved that the construction does not depend on the choice of local coordinates. In this paper, we use a general technique of G. Laumon of formal microlocalization of certain filtered rings (cf. [Lau]) to define rings of microdifferential operators of level $m$. An advantage of this construction is that we do not need to choose coordinates in the construction. It follows also formally using the result of Laumon that for a coherent $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-module $\mathcal{M}$, we get

$$\text{Char}(\mathcal{M}) = \text{Supp}(\mathcal{E}^{(m)}_{\mathcal{X}, \mathbb{Q}} \otimes_{\pi^{-1}} \mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}} \pi^{-1} \mathcal{M})$$

in $T^{(m)*}X := \text{Spec}(\text{gr}(\mathcal{D}^{(m)}_{X}))$, where $\pi : T^{(m)*}X \to X$ is the projection.

Before explaining the construction of sheaves of microdifferential operators associated with $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$, let us review the theory of Berthelot, and see why we need to consider $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules. Berthelot proved that many fundamental theorems in the theory of analytic $\mathcal{D}$-modules hold also for $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules. For example, we can define fundamental functors such as push-forwards, pull-backs, and so on, and the coherence is preserved under push-forwards along proper morphisms. However, Kashiwara’s theorem, which states an equivalence between the category of coherent $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules which are supported on a smooth closed formal subscheme $\mathcal{Z}$ of $\mathcal{X}$ and the category of coherent $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules, does not hold. This failure makes it difficult to define a suitable subcategory of holonomic modules in the category of $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules. To remedy this, Berthelot took inductive limit on the levels to define the ring $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$ and proved Kashiwara’s theorem for coherent $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules. As in the analytic $\mathcal{D}$-module theory, we need to consider holonomic modules to deal with push-forwards along open immersions, and we need to define characteristic varieties to define holonomic modules. When a coherent $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-module possesses a Frobenius structure (i.e. an isomorphism $\mathcal{M} \xrightarrow{\sim} F^* \mathcal{M}$), Berthelot defined the characteristic variety. He reduced the definition to a finite level situation using a marvelous theorem of Frobenius descent, and proved Bernstein’s inequality by using Kashiwara’s theorem. However, if there is no Frobenius structure, the situation was complete mystery.

In this paper, we propose a new formalism which allows us at least conjecturally to interpret this characteristic varieties by means of microlocalizations, and use them to define the characteristic varieties for general coherent $\mathcal{D}^{(m)}_{\mathcal{X}, \mathbb{Q}}$-modules which may not carry Frobenius structures. We also prove the conjecture in the case of curves (cf. Theorem 7.2). Let us describe a more precise statement and difficulties to carry this out. One of the difficulties of defining microdifferential operators associated with $\mathcal{D}$ is that there does not exist a transition homomorphism (cf. 4.1)

$$\mathcal{E}^{(m)}_{\mathcal{X}, \mathbb{Q}} \to \mathcal{E}^{(m+1)}_{\mathcal{X}, \mathbb{Q}}$$
compatible with the transition homomorphism $\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)} \to \widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m+1)}$, and we are not able to define the ring of microdifferential operators corresponding to $\mathcal{D}_{X,\mathbb{Q}}^{(1)}$ in a naive way. Let $\pi : T^{*}X \to X$ be the projection. To remedy this, we define a $\pi^{-1}\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)}$-algebra $\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m')}$ for any integer $m' \geq m$ called the “intermediate ring of microdifferential operators of level $(m,m')$” so that there exist homomorphisms of $\pi^{-1}\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)}$-algebras

$$\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m'+1)} \to \widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m')}, \quad \widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m')} \to \widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m+1,m')},$$

and $\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m)} = \widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m)}$. We define

$$\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,\dagger)} := \lim_{\substack{\longrightarrow \atop m'}} \widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m')}.$$ 

On this level, we have a transition homomorphism $\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,\dagger)} \to \widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m+1,\dagger)}$ compatible with the transition homomorphism of the ring of differential operators of level $m$ and $m+1$. We define

$$\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{\dagger} := \lim_{\substack{\longrightarrow \atop m}} \widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,\dagger)}.$$

Unfortunately, we no longer have the equality

$$\text{Char}(\mathcal{M}) = \text{Supp}(\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,\dagger)} \otimes_{\pi^{-1}\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)}} \pi^{-1}\mathcal{M})$$

for coherent $\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)}$-module $\mathcal{M}$ in general (cf. [71]). However, we conjecture the following.

**Conjecture.** — Let $X$ be a smooth formal scheme of finite type over $R$, and $\mathcal{M}$ be a coherent $\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)}$-module. Then there exists $N > m$ such that for any $m' \geq N$,

$$\text{Char}(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m')} \otimes \widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)} \mathcal{M}) = \text{Supp}(\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{(m',\dagger)} \otimes_{\pi^{-1}\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)}} \pi^{-1}\mathcal{M}).$$

This conjecture implies that $\text{Car}(\mathcal{M}) = \text{Supp}(\widehat{\mathcal{E}}_{X,\mathbb{Q}}^{\dagger} \otimes \mathcal{M})$ for a coherent $F^{\dagger}\mathcal{D}_{X,\mathbb{Q}}$-module $\mathcal{M}$ where $\text{Car}$ denotes the characteristic variety defined by Berthelot. It is also worth noticing here that if this conjecture is true, the characteristic varieties for coherent $\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m)}$-modules stabilizes when we raise level $m$, and in particular we are able to define characteristic varieties for coherent $\mathcal{D}_{X,\mathbb{Q}}^{(1)}$-modules even without Frobenius structures. In the last part of this paper, we prove the following.

**Theorem 7.2** — When $X$ is a curve, the conjecture is true.

Finally, let us point out one of the most important applications of this theorem. In the celebrated paper of Lafforgue [81], he proved Langlands’ program for function field. This is a certain correspondences between $\ell$-adic Galois representations and cuspidal automorphic forms. A natural question is if there is any analogous correspondences for overconvergent $F$-isocrystals. We believe that there are similar correspondences, and we observe that the conjecture of Deligne [1, 12.10 (vi)] (or more precisely [Gr 4.13]) can be understood as a consequence of this unknown correspondences. In Lafforgue’s proof, he used the so called “product formula” for $\ell$-adic epsilon factors. The construction of the ring of microdifferential operators in this paper and Theorem 7.2 are crucial technical tools for the proof of the product formula for $p$-adic epsilon factors in [AM], which is expected to be used to establish the “Langlands’ correspondence” of overconvergent $F$-isocrystals. Especially, the result of this paper is used to establish the theory of $p$-adic local Fourier transform and the “principle of stationary phase”.
To conclude Introduction, let us see the structure of this paper. In §1, we review the theory of formal microlocalization of certain filtered rings, and prove some basic facts on the noetherian property of rings. Using these results, we define the naive ring of microdifferential operators $\hat{E}^{(m)}_{X,\mathbb{Q}}$, and prove some basic facts in the next section §2. Before proceeding to the definition of the intermediate rings of microdifferential operators, we study some properties of $\text{gr}(E^{(m)}_{X,\mathbb{Q}})$ in §3. These are used to study the intermediate rings, which are defined in §4. In §5, we prove the flatness of transition homomorphisms and related results. One of the most important properties of $E^{(m)}_{X,\mathbb{Q}}$ is that its section on a (strict) affine open subscheme is a Fréchet-Stein algebra. In §6, we prove a finiteness property of certain sheaves of modules, which may be useful to deal with sheaves on formal schemes. In the last section, we formulate the conjecture, and prove it in the case of curves.

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Conventions
In this paper, all rings are assumed to be associative with unity. Filtered groups are assumed to be exhaustive (cf. §1.1.1), and modules are left modules unless otherwise stated. In general, we use Raman fonts (e.g. $X$) for schemes, and script fonts (e.g. $\mathcal{X}$) for formal schemes.

1. Preliminaries on filtered rings
The aim of this section is to review the formal construction of the microlocalization of certain filtered rings due to O. Gabber and G. Laumon. To fix notation and terminology, we begin by reviewing well-known definitions and properties of filtered modules.

1.1. The reader can refer to [Bo III, §2] and [HO] for more details.

1.1.1. An increasing sequence $\{G_n\}_{n \in \mathbb{Z}}$ of subgroups of a group $G$ is called an increasing filtration on $G$. The filtration is said to be positive if $G_n = 0$ for all $n < 0$. We say that the filtration is separated if $\bigcap_n G_n = \{e\}$ where $e$ is the unit. If $G_n$ is a normal subgroup of $G$ for any $n$, the filtration defines a canonical topology, which makes $G$ a topological group (cf. [Bo III, §2.5]).

Let $A$ be a ring (not necessary commutative), and $\{A_i\}_{i \in \mathbb{Z}}$ be a filtration of the additive group $A$. We say that the couple $(A, \{A_i\}_{i \in \mathbb{Z}})$ is a filtered ring if $A_i \cdot A_j \subset A_{i+j}$, and $1 \in A_0$. We always assume that the filtration is exhaustive (i.e. $\bigcup_i A_i = A$). If there is no possible confusion, we abbreviate it by $(A, A_i)$. Let $M$ be an $A$-module, and $\{M_i\}_{i \in \mathbb{Z}}$ be a filtration of the additive group $M$ such that $A_i \cdot M_j \subset M_{i+j}$ for any $i, j \in \mathbb{Z}$. Then the couple $(M, \{M_i\}_{i \in \mathbb{Z}})$ is said to be a filtered $(A, A_i)$-module. We often denote $(M, \{M_i\}_{i \in \mathbb{Z}})$ by $(M, M_i)$ for short.

1.1.2. Let $A$ be a ring, and $I$ be a two-sided ideal. We put $A_n := I^{-n}$ for $n \leq 0$, and $A_n := A$ for $n > 0$. The couple $(A, \{A_n\}_{n \in \mathbb{Z}})$ is a filtered ring, and the filtration is called the $I$-adic filtration.
Let $(M, M_i)$ be a filtered $(A, A_i)$-module. We say that the filtration $\{M_i\}_{i \in \mathbb{Z}}$ of $M$ is good if there exist $m_1, \ldots, m_s \in M$ and $k_1, \ldots, k_s \in \mathbb{Z}$ such that $M_n = \sum_{i=1}^s A_{n-k_i} \cdot m_i$ for any $n$.

1.1.3. A filtered homomorphism $f : (A, A_i) \to (B, B_i)$ is a ring homomorphism $f : A \to B$ such that there exists an integer $n$ satisfying $f(A_i) \subset B_{i+n}$ for any integer $i$. Such a homomorphism is continuous with respect to the topology defined by the filtration on $A$ and $B$. The filtered homomorphism $f$ is said to be strict if $f(A_i) = f(A) \cap B_i$ for any $i \in \mathbb{Z}$.

1.1.4. For a filtered ring $(A, A_i)$, we put $\text{gr}_i(A) := A_i/A_{i-1}$, and $\text{gr}(A) := \bigoplus_i \text{gr}_i(A)$. The module $\text{gr}(A)$ is naturally a graded ring, and it is called the associated graded ring. We define the principal symbol map $\sigma : A \to \text{gr}(A)$ in the following way: let $x \in A$. If $x \in \bigcap_i A_i$, then we put $\sigma(x) = 0$. Otherwise there exists an integer $i$ such that $x \in A_i$ and $x \notin A_{i-1}$. We define $\sigma(x)$ to be the image of $x$ in $\text{gr}_i(A) \subset \text{gr}(A)$.

1.1.5. We introduce the completion of a filtered ring. Let $(A, A_i)$ be a filtered ring. We refer to [HO, Ch.I, §3] for the details. Let $A[\nu, \nu^{-1}]$ be the ring of Laurent polynomials with one variable $\nu$ over $A$, graded by the degree of $\nu$. Here, the element $\nu$ is in the center by definition. We define the graded subalgebra of $A[\nu, \nu^{-1}]$ denoted by $A_\bullet$, called the Rees ring of $(A, A_i)$ by the formula

$$A_\bullet := \bigoplus_{i \in \mathbb{Z}} A_i \cdot \nu^i.$$ 

For an integer $n \geq 1$, we define a graded ring $A_{\bullet, n} := A_\bullet / \nu^n A_\bullet$. Let $i \in \mathbb{Z}$. We denote by $A_{i,n}$ the $i$-th degree part of $A_{\bullet, n}$ (i.e. the image of $A_i \cdot \nu^i$). We get a projective system of graded rings

$$\to A_{\bullet, n+1} \to A_{\bullet, n} \to \cdots \to A_{\bullet, 1} \cong \text{gr}(A).$$

We define a module and a ring by

$$\hat{A}_i := \varprojlim_{n \to -\infty} A_{i,n} = \varprojlim_{n \to -\infty} A_i / A_{i+n}, \quad \hat{A} := \varprojlim_{i \to -\infty} \hat{A}_i.$$ 

The couple $(\hat{A}, \{\hat{A}_i\}_{i \in \mathbb{Z}})$ is a filtered ring, and is called the completion of $(A, A_i)$. We note that the canonical homomorphism $\text{gr}(A) \to \text{gr}(\hat{A})$ is an isomorphism by [HO, Ch.I, 4.2.2]. We say that the filtered ring $(A, A_i)$ is complete if the canonical homomorphism $A \to \hat{A}$ is an isomorphism. Complete filtered rings are separated (cf. [HO, Ch.I, 3.5]).

1.1.6. We say that a filtered ring $(A, A_i)$ is left (resp. right, two-sided) noetherian filtered if the Rees ring $A_\bullet$ is left (resp. right, two-sided) noetherian. If $A$ is a noetherian filtered ring, the associated graded ring $\text{gr}(A)$ is noetherian since $\text{gr}(A) \cong A_\bullet / \nu A_\bullet$. If $A$ is moreover complete, the converse is also true, and $A$ is noetherian filtered if and only if $\text{gr}(A)$ is a noetherian ring (cf. [HO, Ch.II, 1.2.3]). This shows that the completion of a noetherian filtered ring is noetherian filtered. Moreover, if $(A, A_i)$ is a noetherian filtered ring, the canonical homomorphism $A \to \hat{A}$ is flat by [HO, Ch.I, 1.2.1]. We say that a noetherian filtered ring is Zariskian if any good filtered module is separated. Any noetherian filtered complete ring is known to be Zariskian (cf. [HO, Ch.II, 2.2.1]).

1.1.7. Let $X$ be a topological space or, more generally, topos. The terminologies defined so far except for principal symbol in 1.1.4 and those defined in 1.1.6 can be defined also in the language of sheaves by replacing “ring” by “sheaf of rings on $X$” and so on. See [B] A.III.2 for more details.

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(1) However, in some definitions, we used other equivalent definitions to make the explanation shorter. The equivalence is proven in ibid.
1.2. Let us review the Ore condition (we refer to [Lam, 4. §10A] for more details). Let \( R \) be a ring, and \( S \) be a multiplicative system. The following conditions for \( S \) are called the right (resp. left) Ore condition:

1. For every \( a \in R \) and \( s \in S \), \( aS \cap sR \neq \emptyset \) (resp. \( Sa \cap Rs \neq \emptyset \)).

2. If \( sa = 0 \) (resp. \( as = 0 \)), then there exists \( s' \in S \) such that \( as' = 0 \) (resp. \( s'a = 0 \)).

Suppose \( S \) satisfies the right Ore condition. We denote by \( RS^{-1} \) the set \( R \times S / \sim \). Here \( \sim \) is the equivalence relation such that \( (a, s) \) and \( (b, t) \), with \( a, b \in R \) and \( s, t \in S \), are equivalent if and only if there exists a pair \( (c, c') \) in \( R \times R \) such that \( ac = bc' \) in \( R \) and \( sc = tc' \) in \( S \). We usually denote the couple \( (a, s) \) by \( as^{-1} \). We can show that there exists a ring structure on \( RS^{-1} \) which makes the canonical homomorphism \( j: R \to RS^{-1} \) sending \( a \) to \( a1^{-1} \) a ring homomorphism, and satisfies the following universal property: given a homomorphism \( \varphi: R \to B \) of rings such that \( \varphi(s) \) is invertible in \( B \) for any \( s \in S \), there exists a unique homomorphism \( \tilde{\varphi}: RS^{-1} \to B \) such that \( \tilde{\varphi} \circ j = \varphi \). The ring \( RS^{-1} \) is called the right ring of fractions. When \( S \) satisfies the left Ore condition, we may define the left ring of fractions in a similar way. If \( S \) satisfies both left and right Ore conditions (in other words two-sided Ore condition), left and right ring of fractions coincide by the universal property. Obviously, when \( R \) is commutative, any multiplicative system satisfies the Ore condition, and the ring of fractions coincides with the usual one.

1.3. Let \( (A, A_1) \) be a filtered ring which is complete and the associated graded ring \( \text{gr}(A) \) is commutative. Let \( S_1 \subset \text{gr}(A) \) be a homogeneous multiplicative set (i.e. a multiplicative set consisting of homogeneous elements). Let \( c_n: A_{\bullet, n} \to A_{\bullet, 1} \cong \text{gr}(A) \) be the canonical homomorphism. We put

\[
S_n := \{ x \in A_{\bullet, n} \mid c_n(x) \in S_1 \}.
\]

By [Lau, A.2.1], the multiplicative set \( S_n \) satisfies the two-sided Ore condition. We define a graded ring by

\[
A'_{\bullet, n} := S_n^{-1}A_{\bullet, n} \cong A_{\bullet, n}S_n^{-1}.
\]

This defines a projective system of graded rings \( \{ A'_{\bullet, n} \} \). Let us denote by \( A'_{i,n} \) the \( i \)-th degree part of \( A'_{\bullet, n} \). We define

\[
A'_i := \lim_{n \to -\infty} A'_{i,n}, \quad A' := \lim_{i \to \infty} A'_i.
\]

The filtered ring \( (A', A'_i) \) is complete. We denote this filtered ring by \( (A, A_i)_{S_1} \), and we call it the microlocalization of \( (A, A_i) \) with respect to \( S_1 \). If \( \text{gr}(A) \) is noetherian, then \( A'_i \) and \( A' \) are noetherian and the canonical homomorphism \( A \to A' \) is flat by [Lau, Corollaire A.2.3.4].

Example. — Let \( s \) be an element of \( A \) such that \( \sigma(s) \in S_1 \). By definition, \( s \) is invertible in \( (A, A_i)_{S_1} \). Given an element \( a \in A'_i \), for any integer \( k \leq i \), there exist \( a_k \in A_{i,k} \) and \( s_k \in S \cap A_{i,k} \) such that \( a = \sum_{k \leq i} a_k s_k^{-1} \). Here, we considered the topology defined by the filtration of \( A \) (cf. 1.1.1). Moreover, assume that \( \sigma(s) \in \text{gr}_n(A) \) and \( S_1 = \{ \sigma(s)^n \}_{n \geq 0} \). Then for any \( a' \in A'_i \), there exist an integer \( n_k \geq 0 \) and \( a'_k \in A_{i,k+n_k} \) such that \( a' = \sum_{k \leq i} a'_k s^{-n_k} \).

1.4. Let \( (A, A_i) \) be a complete filtered ring whose associated graded ring is commutative. The constructions in the previous subsection can be carried out in almost the same way also for filtered \( (A, A_i) \)-modules. For the details see [Lam] A.2. For example, for a filtered \( A \)-module \( (M, M_i) \) and a homogeneous multiplicative system \( S_1 \subset \text{gr}(A) \), we are able to define the microlocalization of \( (M, M_i) \) with respect to \( S_1 \) denoted by \( (M, M_i)_{S_1} \), which is complete.
1.5. Let us sheafify the results. Let \((A, \{A_i\}_{i \in \mathbb{Z}})\) be a positively filtered ring such that the associated graded ring \(\text{gr}(A)\) is commutative. Let \(R := \text{gr}(A)\) be the positively graded commutative ring. Note that \(A_0 = R_0\) is a commutative ring by assumption. We let \(X := \text{Spec}(R_0), \ V := \text{Spec}(R), \ P := \text{Proj}(R)\). Let \(s \colon X \to V\) be the morphism defined by the canonical projection \(R \to R_0\). We put \(\tilde{V} := V \setminus s(X)\). We have the following canonical commutative diagram (cf. [EGA II, 8.3]).

\[
\begin{array}{ccc}
V & \xrightarrow{q} & P \\
\downarrow{p} & & \downarrow{=} \\
\tilde{V} & & X
\end{array}
\]

We define a topological space \(V'\) in the following way: as a set, \(V' := V\). The topology of \(V'\) is generated by the basis of open sets

\[\{D(f) \mid f \in R \text{ and } f \text{ is homogeneous}\}\]

We denote by \(\epsilon : V \to V'\) the identity map as sets, which is continuous. We put \(\mathcal{O}_{V'} := \epsilon_*\mathcal{O}_V\). For \(n \in \mathbb{Z}\), we denote by \(\mathcal{O}_{V'}(n)\), the subsheaf of \(\mathcal{O}_{V'}\) consisting of the homogeneous sections of degree \(n\). We put \(\mathcal{O}_V(n) := e^{-1}(\mathcal{O}_{V'}(n))\), and \(\mathcal{O}_V(*) := \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_V(n)\). We note that \(\mathcal{O}_V(*) \cong e^{-1}e_*\mathcal{O}_{V'}\). We get \(\mathcal{O}_V(n) \otimes \mathcal{F} \cong q^{-1}\mathcal{O}_P(n)\) for any integer \(n\) by [Lau A.3.0.5]. By [Lau A.3.0.5], we also have

\[(1.5.1) \quad p_*\mathcal{O}_V(*) \cong s^{-1}\mathcal{O}_V(*) \cong \tilde{R} \]

where \(\sim\) denotes the associated quasi-coherent \(\mathcal{O}_X\)-module.

Now, for a topological space \(X\), let us denote by \(\mathcal{O}(X)\) the category of open sets of \(X\). The canonical functor \(e^{-1} : \mathcal{O}(V') \to \mathcal{O}(V)\) admits a left adjoint denoted by \(e_\ast\). Let us describe this functor. There is a natural action of \(G_{m, X}\) on \(V\), and we have an isomorphism \(\mu : G_{m, X} \times V \xrightarrow{\sim} G_{m, X} \times V\) such that \(\text{pr}_1 \circ \mu = \text{pr}_1\) and \((\text{pr}_2 \circ \mu)^* : R \to R[t, t^{-1}]\) sends \(f \in R_t\) to \(f \cdot t^i\). For \(U \in \mathcal{O}(V)\),

\[e^{-1}e_\ast(U) = (\text{pr}_2 \circ \mu \circ \text{pr}_2^{-1})(U) = \bigcup_{\lambda \in G_{m, X}} \lambda \cdot U\]

For the details, see [Lau A.3.0]. For a sheaf \(\mathcal{F}'\) on \(V'\), we have

\[(1.5.2) \quad (e^{-1}\mathcal{F}'(U) = \mathcal{F}'(e_\ast(U))\]

by [Lau A.3.0.2]. This is showing that if \(\mathcal{A}'\) is a coherent ring on \(V'\), \(e^{-1}(\mathcal{A}')\) is also a coherent ring as explained in [Lau A.3.1.8].

1.6. Let \((\mathcal{A}, \mathcal{A}_i)\) be the filtered quasi-coherent \(\mathcal{O}_X\)-algebra associated to \((A, A_i)\) on \(X\). Let \(f\) be a homogeneous element of \(\text{gr}(A)\), and we put \(S_1(f) := \{f^m\}_{m \geq 0} \subset \text{gr}(A)\). Let \(S_n(f)\) be the multiplicative set of \(A_{\bullet, n}\) using the notation in [1.3] and define \(A'_{\bullet, n}(f) := S_n(f)^{-1}A_{\bullet, n}\). We define a sheaf \(B'_{\bullet, n}\) on \(V'\) to be the sheaf associated to the presheaf

\[(1.6.1) \quad D(f) \mapsto A'_{\bullet, n}(f)\]

over the open basis of \(V'\) consisting of \(D(f)\) with a homogeneous element \(f\) in \(\text{gr}(A)\). By [Lau A.3.1.1], we know that

\[(1.6.2) \quad \Gamma(D(f), B'_{\bullet, n}) = A'_{\bullet, n}(f)\]

We define

\[B'_t := \lim_{n \to -\infty} B'_{t, n}, \quad B'_t := \lim_{i \to \infty} B'_i\]
Then
\[ \Gamma(D(f), (B', B'_i)) \sim (A, A_i)_{S(f)} \]
for a homogeneous element \( f \) of \( \text{gr}(A) \) by [Lau, (A.3.1.2)]. We define a filtered sheaf of rings
\[ (B, B_i) := \epsilon^{-1}(B', B'_i) \]
on \( V \). There is a canonical homomorphism of filtered rings on \( V \)
\[ \varphi: p^{-1}(A, A_i) \to (B, B_i). \]
The filtered ring \((B, B_i)\) is called the microlocalization of \((A, A_i)\). By [Lau, A.3.1.6], we have canonical isomorphisms of graded rings
(1.6.3) \[ \text{gr}_n(B) \cong \mathcal{O}_V(n), \quad \text{gr}(B) \cong \mathcal{O}_V(\ast). \]

1.7. Let \((M, M_i)\) be a filtered \((A, A_i)\)-module such that \( M_i = 0 \) for \( i \ll 0 \) (and \( \bigcup_{i \in \mathbb{Z}} M_i = M \)). Let \((\mathcal{M}, M_i)\) be the quasi-coherent \( \mathcal{O}_X \)-module associated to the filtered module \((M, M_i)\). This is a filtered \((A, A_i)\)-module.

Using exactly the same construction (cf. [Lau A.3.2]), we are able to define a \((B', B'_i)\)-module \((N', N'_i)\) on \( V' \) such that
\[ (M, M_i)_{S(f)}(f) \sim \Gamma(D(f), (N', N'_i)) \]
for a homogeneous element \( f \) of \( \text{gr}(A) \). We define a filtered \((B, B_i)\)-module by
\[ (N, N_i) := \epsilon^{-1}(N', N'_i). \]
There is a homomorphism
\[ \varphi_M: p^{-1}(\mathcal{M}, M_i) \to (N, N_i). \]
over \( \varphi \). Now we define
\[ \text{Char}((\mathcal{M}, M_i)) := \text{Supp}(\text{gr}(N)) \subset V. \]
This is called the characteristic variety of the filtered module \((\mathcal{M}, M_i)\).

1.8 Lemma. — Suppose \( \text{gr}(M) \) is finitely presented over \( \text{gr}(A) \). Then
\[ \text{Char}((\mathcal{M}, M_i)) = \text{Supp}(\mathcal{B} \otimes_{p^{-1}A} p^{-1}M). \]

Proof. Let \( U := D(f) \) with a homogeneous element \( f \) of \( \text{gr}(A) \). Since \( \Gamma(U, N) \) is complete, it is in particular separated with respect to the filtration. Thus the local sections \( \Gamma(U, N) \) is equal to 0 if and only if \( \Gamma(U, \text{gr}(N)) \cong \text{gr}(\Gamma(U, N)) = 0 \) where the first isomorphism follows from [Lau A.3.2]. Combining this with [Lau Proposition A.3.2.4 (ii)], the lemma follows.

Remark. — Assume \( \mathcal{M} \) is an \( A \)-module of finite type. Then there exists a good filtration \( \{M_i\}_{i \in \mathbb{Z}} \) (cf. [Bj A.III 2.15]) of \( \mathcal{M} \). Suppose \( \text{gr}(A) \) is noetherian. The above lemma implies that the characteristic variety does not depend on the choice of a good filtration. We call \( \text{Char}((\mathcal{M}, M_i)) \) the characteristic variety of \( \mathcal{M} \) and denote by \( \text{Char}(\mathcal{M}) \).

1.9 Remark. — These constructions are functorial and, in particular, we may globalize the definitions of microlocalizations on schemes not necessary affine (cf. [Lau A.3.3]).
1.10. Now, we will collect some basic facts on noetherian conditions.

**Definition.** — Let \( X \) be a topological space, \( \mathcal{A} \) be a sheaf of rings on \( X \), and \( \mathfrak{B} \) be an open basis of the topology.

(i) The ring \( \mathcal{A} \) is said to be **left noetherian with respect to** \( \mathfrak{B} \) if it satisfies the following conditions.

1. It is a left coherent ring (i.e. locally, any finitely generated left ideal of \( \mathcal{A} \) is finitely presented).
2. For any point \( x \in X \), the stalk \( \mathcal{A}_x \) is a left noetherian ring.
3. For any \( U \in \mathfrak{B} \), \( \Gamma(U, \mathcal{A}) \) is a left noetherian ring.

In the same way, we define a **right (resp. two-sided) noetherian ring with respect to** \( \mathfrak{B} \). When there is no possible confusion, we abbreviate two-sided noetherian sheaf of rings with respect to \( \mathfrak{B} \) by noetherian ring.

(ii) A filtered ring \( (\mathcal{A}, \mathcal{A}_i) \) is said to be **pointwise left (resp. right, two-sided) Zariskian** if the stalk \( \mathcal{A}_x \) is left (resp. right, two-sided) Zariskian for any \( x \in X \).

(iii) An \( \mathcal{A} \)-algebra \( \mathcal{B} \) is said to be **of finite type over** \( \mathcal{A} \) if for any \( x \in X \), there exists an open neighborhood \( U \) of \( x \) and a surjection \( \mathcal{A}[T_1, \ldots, T_n]|_U \rightarrow \mathcal{B}|_U \).

**Remark.** — This definition of noetherian ring is slightly different from that of [KK, Definition 1.1.1], who replaced 3 by Condition (c): for any open set \( U \) of \( X \), a sum of left coherent \( \mathcal{A}|_U \)-ideals are also coherent. In §6 we show that a stronger condition than Condition (c) holds for some of the noetherian rings defined in this paper.

**Example.** — Let \( X \) be a noetherian scheme. Let \( \mathfrak{B} \) be the open basis consisting of open affine subschemes of \( X \). Then \( \mathcal{O}_X \) is a noetherian ring with respect to \( \mathfrak{B} \). More generally, let \( \mathfrak{X} \) be a locally noetherian adic formal scheme (cf. [EGA I, 10.4.2]). Then \( \mathcal{O}_{\mathfrak{X}} \) is noetherian with respect to \( \mathfrak{B} \) as well by [EGA I, 10.1.6].

1.11. The following lemma is a generalization of [Be1, 3.3.6] to filtered rings.

**Lemma.** — Let \( (\mathcal{A}, \{\mathcal{A}_i\}_{i \in \mathbb{Z}}) \) be a filtered ring on a topological space \( X \). Let \( \mathfrak{B} \) be an open basis of the topological space \( X \). Suppose that the following conditions hold.

1. For any \( U \in \mathfrak{B} \), the filtered ring \( (\Gamma(U, \mathcal{A}), \Gamma(U, \mathcal{A}_i)) \) is complete.
2. The graded rings \( \text{gr}(\mathcal{A}) \) is left noetherian with respect to \( \mathfrak{B} \).
3. For \( V, U \in \mathfrak{B} \) such that \( V \subseteq U \), the restriction homomorphism \( \Gamma(U, \text{gr}(\mathcal{A})) \rightarrow \Gamma(V, \text{gr}(\mathcal{A})) \) is right flat.
4. For any \( U \in \mathfrak{B} \), the canonical homomorphism

\[
\text{gr}(\Gamma(U, \mathcal{A})) \rightarrow \Gamma(U, \text{gr}(\mathcal{A}))
\]

is an isomorphism.

Then, for any \( x \in X \), the canonical homomorphism

\[
(1.11.1) \quad \mathcal{A}_x \rightarrow \mathcal{A}_x^\wedge
\]

is right faithfully flat, where \( ^\wedge \) denotes the completion with respect to the filtration on \( \mathcal{A}_x \). Moreover, \((\mathcal{A}, \mathcal{A}_i)\) is pointwise left Zariskian, and \( \mathcal{A} \) is left noetherian with respect to \( \mathfrak{B} \). The statement is also valid if we replace left (resp. right) by right (resp. left).
Proof. We only deal with the left case, and modules are always assumed to be left modules. Let \( x \in X \), and take \( U \in \mathcal{B} \) such that \( x \in U \). Let us see that the restriction homomorphism

\[
(1.11.2) \quad r : \Gamma(U, A) \rightarrow A_x^\wedge
\]

is flat. Indeed, consider the following commutative diagram

\[
\begin{array}{ccc}
gr(\Gamma(U, A)) & \sim & \Gamma(U, gr(A)) \\
\downarrow \quad & & \downarrow \\
gr(A_x^\wedge) & \sim & gr(A_x) \\
\sim & & \sim \\
gr(A_x^\wedge) & \sim & gr(A)_x
\end{array}
\]

where the vertical homomorphisms are the restriction homomorphism of \( A \) and \( gr(A) \). The upper horizontal homomorphism is an isomorphism by condition 4. The right vertical homomorphism is flat by condition 3, and thus \( gr(r) \) is flat as well. Since the filtered rings \( \Gamma(U, A) \) and \( A_x^\wedge \) are complete and their associated graded rings are noetherian by conditions 2 and 4, these filtered rings are in fact noetherian filtered (cf. 1.1.6). Since the source and the target of \( r \) are noetherian filtered complete rings, \( r \) is flat by the flatness of \( gr(r) \) and [HO, Ch.II, 1.2.1]. By taking the inductive limit over \( U \), (1.11.1) is flat.

We say that an \( A_x \)-module \( M \) is monogenerated of finitely presented if there exists a surjection \( A_x \rightarrow M \) such that the kernel is a finitely generated ideal of \( A_x \). By [Be1, 3.3.5], to see that (1.11.1) is faithful, it suffices to show that for any monogenerated of finitely presented \( A_x \)-module \( M \) such that \( A_x \otimes M_U = 0 \), we get \( M = 0 \). Since we are assuming \( M \) to be a monogenerated of finitely presented \( A_x \)-module, there exist \( U \in \mathcal{B} \) and a monogenerated of finitely presented \( \Gamma(U, A) \)-module \( M_U \) such that \( A_x \otimes M_U \sim M \). We fix a surjection \( \phi : A_U : = \Gamma(U, A) \rightarrow M_U \).

This induces a good filtration on \( M_U \). We define an \( A_U \)-ideal \( K \) by the following short exact sequence

\[
0 \rightarrow K \rightarrow A_U \phi \rightarrow M_U \rightarrow 0.
\]

We endow these two modules with the tensor filtrations (cf. [HO, p.57]). Consider the following diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & K \\
\sim & & \sim \\
gr(A_x^\wedge) \otimes gr(K) & \rightarrow & gr(A_x^\wedge) \\
\downarrow \beta & & \downarrow \\
gr(A_x^\wedge) & \sim & gr(A_x^\wedge) \otimes gr(M_U) \\
\sim & & \sim \\
gr(A_x^\wedge) \otimes K & \rightarrow & gr(A_x^\wedge) \otimes A_U
\end{array}
\]

where the vertical homomorphisms are canonical ones. The right vertical homomorphism is an isomorphism by [ibid., Ch.I, 6.14]. Since the upper row is exact and

\[
gr(A_x^\wedge) \otimes gr(K) \rightarrow gr(A_x^\wedge) \otimes K
\]
is surjective by \cite[p.58]{ibid.}, \( \alpha \) is injective. This implies that the homomorphism (1.11.4) is strict by \cite[Ch.I, 4.2.4 (2)]{ibid.}, and \( \alpha \) is an isomorphism. Thus, by diagram chasing, \( \beta \) is also an isomorphism, and

\[
\text{gr}(A_x) \otimes \text{gr}(M_U) \cong \text{gr}(A_x^\wedge) \otimes \text{gr}(M_U) = 0.
\]

Since \( \text{gr}(A_x) \cong (\text{gr}(A))_x \), this shows that there exists \( V \in \mathcal{B} \) such that \( x \in V \), \( V \subset U \), and

\[
\Gamma(V, \text{gr}(A)) \otimes \Gamma(U, \text{gr}(A)) \text{gr}(M_U) = 0.
\]

Let \( M_V := \Gamma(V, A) \otimes \Gamma(U, A) M_U \), and equip it with the tensor filtration. Since the filtration on \( M_U \) is good, the filtration on \( M_V \) is also good by \cite[Ch.I, 6.14]{ibid.}. Since the canonical homomorphism

\[
\Gamma(V, \text{gr}(A)) \otimes \Gamma(U, \text{gr}(A)) \text{gr}(M_U) \cong \text{gr}(\Gamma(V, A)) \otimes \text{gr}(\Gamma(U, A)) \text{gr}(M_U) \to \text{gr}(M_V)
\]

is surjective where the first isomorphism is by condition 4, we have \( \text{gr}(M_V) = 0 \). Since \( \Gamma(V, A) \) is complete and the filtration on \( M_V \) is good, we obtain that \( M_V = 0 \) (cf. 1.1.6). Since \( M \cong A_x \otimes_{A_V} M_V = 0 \), the fully faithfulness follows.

Since \( \text{gr}(A_x) \) is noetherian, \( A_x \) is Zariskian by \cite[Ch.II, 2.1.2 (4)]{ibid.}, and in particular \( A_x \) is noetherian. By \cite[3.1.1]{Be1}, the ring \( A \) is coherent. Thus we obtain that the sheaf of rings \( A \) is noetherian.

\[\Box\]

1.12 Example. — We use the notation of subsection 1.5. We further assume that \( \text{gr}(A) \) is a noetherian ring. By \cite[II, 2.1.5]{EGA}, the ring \( A_0 \) is also noetherian, and \( \text{gr}(A) \) is of finite type over \( A_0 \). Then by the same argument as \cite[A.3.1.8]{Lau}, the rings \( O_V(0) \) and \( O_V(*) \) are noetherian. Moreover, \( O_V(n) \) is a coherent \( O_V(0) \)-module on \( V \) for any integer \( n \), since \( O_P(n) \) is a coherent \( O_P \cong O_P(0) \)-module, and \( O_X(*) \) is of finite type over \( O_X(0) \).

Applying Lemma 1.11 to the microlocalization \( \mathcal{B} \) on \( \hat{V} \), \( \mathcal{B} \) is a noetherian ring and pointwise Zariskian on \( \hat{V} \) by (1.6.3). Let us show that \( \mathcal{B} \) is in fact noetherian and pointwise Zariskian on \( V \). It is pointwise Zariskian by (1.5.1). To see Definition 1.10 (i)-2, we apply (1.5.1), and for \cite[(i)-3, we apply (1.5.2)]. It remains to show that \( \mathcal{B} \) is coherent. For this, use \cite[3.1.1]{Be1}, and we get what we wanted. Moreover, \( \varphi \) in (1.6) is flat by \cite[A.3.1.7]{Lau}.

1.13 Lemma. — Let \( (A, \{A_i\}_{i \in \mathbb{Z}}) \) be a filtered ring such that \( A_0 \) is noetherian filtered, \( \bigoplus_{i \geq 0} \text{gr}_i(A) \) is noetherian. Then \( A \) is noetherian filtered.

Proof. By \cite[II, 2.1.5, 2.1.6]{EGA}, \( \text{gr}_i(A) \) is finitely generated over \( A_0 \) for any \( i \in \mathbb{Z} \). Then the statement is nothing but \cite[Proposition 1.1.5]{KK} applying in the case where the topological space is just a point.

\[\Box\]

1.14 Lemma. — Let \( (A, A_i) \) be a pointwise Zariskian filtered ring on a topological space \( X \). Let \( (M_i, M_i) \) be a good filtered \( (A, A_i) \)-module. Then the filtration \( \{M_i\} \) is separated (i.e. \( \varprojlim_i M_i = 0 \)).

Proof. Since \( \varprojlim_i M_i \to M \), we get the following commutative diagram for any \( x \in X \).

\[
\begin{array}{ccc}
(\varprojlim_i M_i)_x & \longrightarrow & M_x \\
\downarrow & & \\
\varprojlim_i M_{i,x} & \\
\end{array}
\]

Since \( A \) is pointwise Zariskian, \( \varprojlim_i M_{i,x} = 0 \), and thus, \( \varprojlim_M = 0 \).

\[\Box\]
2. Microdifferential sheaves

We apply the results of the previous section to the theory of arithmetic \(D\)-modules, and define the rings of naive microdifferential operators of finite level.

2.1. Let \(S\) be a scheme over \(\mathbb{Z}_p\) (which may not be locally of finite type). Let \(X\) be a smooth scheme over \(S\), and let \(m\) be a non-negative integer. Then we may consider the sheaf of differential operators of level \(m\) denoted by \(\mathcal{D}_X^{(m)}\) on \(X\). For the details on this sheaf, we can refer to [Be1, Be2, Be]. For \(i \in \mathbb{Z}\), let \(\mathcal{D}_{X,i}^{(m)}\) be the sub-\(\mathcal{O}_X\)-module consisting of operators whose orders are less than or equal to \(i\) in \(\mathcal{D}_X^{(m)}\) (cf. [Be1 2.2.1]). By definition, \(\mathcal{D}_{X,i}^{(m)} = 0\) for \(i < 0\). Then \(\{\mathcal{D}_{X,i}^{(m)}\}_{i \in \mathbb{Z}}\) is an increasing filtration of \(\mathcal{D}_X^{(m)}\), which we call the filtration by order. By [Be1 2.2.4], the ring \(\text{gr}(\mathcal{D}_X^{(m)})\) is commutative. Let

\[
T^{(m)}X := \text{Spec}(\text{gr}(\mathcal{D}_X^{(m)})), \quad P^{(m)}X := \text{Proj}(\text{gr}(\mathcal{D}_X^{(m)})�
\]

We call these the pseudo cotangent bundles of level \(m\). When \(m = 0\), we denote \(T^{(m)}X\) and \(P^{(m)}X\) by \(T^*X\) and \(P^*X\) respectively, which are nothing but the usual cotangent bundles of \(X\). Let \(T^{(m)}X := T^{(m)}X \setminus s(X)\) where \(s : X \to T^{(m)}X\) denotes the zero section. Then there exist the canonical morphisms (cf. 1.5) as follows:

\[
\begin{array}{ccc}
T^{(m)}X & \xrightarrow{\pi_m} & T^{(m)}X \\
& \downarrow q & \downarrow \\
P^{(m)}X & \xrightarrow{} & P^{(m)}X
\end{array}
\]

Recall the notation \(\mathcal{O}_{T^{(m)}X}(n)\) for \(n \in \mathbb{Z}\) of the subsection 1.3 which is a subsheaf of \(\mathcal{O}_{T^{(m)}X}(\ast)\) consisting of homogeneous elements of degree \(n\). There is a canonical isomorphism \(q^{-1}\mathcal{O}_{P^{(m)}X}(n) \cong \mathcal{O}_{T^{(m)}X}(n)\) on \(T^*X\) for any integer \(n\) (cf. 1.5). We remind that \(\mathcal{O}_{T^{(m)}X}(\ast)\) does not coincide with \(\mathcal{O}_{T^{(m)}X}(\ast)\). The following lemma is immediate from Example 1.12.

**Lemma.** — The rings \(\mathcal{O}_{T^{(m)}X}(0), \mathcal{O}_{T^{(m)}X}(\ast)\) are noetherian, and \(\mathcal{O}_{\tilde{T}^{(m)}X}(n)\) is a coherent \(\mathcal{O}_{\tilde{T}^{(m)}X}(0)\)-module for any integer \(n\). Moreover, \(\mathcal{O}_{\tilde{T}^{(m)}X}(\ast)\) is an \(\mathcal{O}_{\tilde{T}^{(m)}X}(0)\)-algebra of finite type.

2.2. We can consider the microlocalization of \((\mathcal{D}_X^{(m)}, \mathcal{D}_{X,i}^{(m)})\) denoted by \((\mathcal{E}_X^{(m)}, \mathcal{E}_{X,i}^{(m)})\) using the technique of subsection 1.5. This is a filtered ring on \(T^{(m)}X\). Then there exists a canonical homomorphism of filtered rings

\[
\varphi_m : \pi_m^{-1}(\mathcal{D}_X^{(m)}, \mathcal{D}_{X,i}^{(m)}) \to (\mathcal{E}_X^{(m)}, \mathcal{E}_{X,i}^{(m)}).
\]

By 1.6.3, we have canonical isomorphisms

\[
(2.2.1) \quad \text{gr}_n(\mathcal{E}_X^{(m)}) \cong \mathcal{O}_{T^{(m)}X}(n), \quad \text{gr}(\mathcal{E}_X^{(m)}) \cong \mathcal{O}_{T^{(m)}X}(\ast).
\]

Since \(\text{gr}(\mathcal{D}_X^{(m)})\) is a noetherian ring by [Be1 5.2.3], \(\mathcal{E}_X^{(m)}\) is a pointwise Zariskian and noetherian, and moreover \(\varphi_m\) is flat by Example 1.12. Moreover, since the canonical homomorphism \(\pi_1^{-1}\mathcal{O}_{T^{(m)}X}(\ast) \to \mathcal{O}_{T^{(m)}X}(\ast)\) is injective, \(\text{gr}(\varphi_m)\) is injective as well, and thus \(\varphi_m\) is strictly injective by [HO, Ch.I, 4.2.4 (2)].

**2.3 Lemma.** — We preserve the notation. We further assume that \(S\) and \(X\) are affine, and \(S = \text{Spec}(A)\). Let \(S' := \text{Spec}(B)\) be an affine scheme finite over \(S\). We put \(X' := X \times_S S'\), and
we have the base change isomorphism $T^{(m)*}X' \cong T^{(m)*}X \times_S S'$ (cf. [Be1 2.2.2]). Let $f$ be a homogeneous section of $\Gamma(T^{(m)*}X, \mathcal{O}_{T^{(m)*}X})$, and $f'$ be the image in $\Gamma(T^{(m)*}X', \mathcal{O}_{T^{(m)*}X'})$. We put $U := D(f)$ and $U' := D(f')$. Then there exists a canonical isomorphism of filtered rings

$$\Gamma(U, (\mathcal{E}_X^{(m)}, \mathcal{E}_{X,i}^{(m)})) \otimes_A B \cong \Gamma(U', (\mathcal{E}_{X'}^{(m)}, \mathcal{E}_{X',i}^{(m)})).$$

Proof. By [Be1 2.2.2], there exists an isomorphism

$$(2.3.1) \quad \Gamma(X, (\mathcal{D}_X^{(m)}, \mathcal{D}_{X,i}^{(m)})) \otimes_A B \cong \Gamma(X', (\mathcal{D}_{X'}^{(m)}, \mathcal{D}_{X',i}^{(m)})).$$

We denote $\Gamma(U, (\mathcal{E}_X^{(m)}, \mathcal{E}_{X,i}^{(m)}))$ by $(E_X, E_{X,i})$, and $\Gamma(X', (\mathcal{D}_{X'}^{(m)}, \mathcal{D}_{X',i}^{(m)}))$ by $(D_{X'}, D_{X',i})$. The isomorphism (2.3.1) induces a homomorphism of filtered rings $(D_{X'}, D_{X',i}) \to (E_X, E_{X,i}) \otimes_A B$. Since $B$ is finite over $A$, $(E_X, E_{X,i}) \otimes_A B$ is a complete filtered ring by [HO] Ch.II, 1.2.10 (5)]. By the universality [Lau Proposition A.2.3.3], $(E_X, E_{X,i}) \otimes_A B$ is the microlocalization of $(D_{X'}, D_{X',i})$, and the lemma follows. ■

2.4. Now, we will pass to the limit. Let $R$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ whose residue field is denoted by $k$. We denote the field of fractions by $K$, and let $\pi$ be a uniformizer of $R$, and for a non-negative integer $i$, we put $R_i := R/(\pi^{i+1})$. From now on, we use these notation freely without referring to this subsection.

Let $\mathcal{X}$ be a smooth formal scheme over $R$. We denote by $X_i$ the reduction of $\mathcal{X}$ over $R_i$, and we sometimes denote $X_0$ simply by $X$. We define $T^{(m)*}\mathcal{X}$ and $P^{(m)*}\mathcal{X}$ by the limit of $T^{(m)*}X_i$ and $P^{(m)*}X_i$ over $i$ respectively. Let $\epsilon: \mathcal{O}(V) \to \mathcal{O}(V')$ be the functor in 1.5 where $V = T^{(m)*}\mathcal{X}$. Let $\mathcal{B}$ be the open basis of $V'$ consisting of $D(f)$ in $T^*\mathcal{X}$ over an affine open subscheme $\mathcal{U}$ of $\mathcal{X}$ where $f$ is a homogeneous element of $\Gamma(T^{(m)*}\mathcal{U}, \mathcal{O}_{T^{(m)*}\mathcal{X}})$. We define an open basis $\mathcal{B}$ of $V$ to be the set consisting of $U \in \mathcal{O}(V)$ such that $\epsilon(U) \in \mathcal{B}'$.

Lemma. — The rings $\mathcal{O}_{T^{(m)*}\mathcal{X}}(0)$ and $\mathcal{O}_{T^{(m)*}\mathcal{X}}(*)$ are noetherian with respect to $\mathcal{B}$, and $\mathcal{O}_{T^{(m)*}\mathcal{X}}(n)$ is a coherent $\mathcal{O}_{T^{(m)*}\mathcal{X}}(0)$-module for any integer $n$. Moreover, $\mathcal{O}_{T^{(m)*}\mathcal{X}}(*)$ is an $\mathcal{O}_{T^{(m)*}\mathcal{X}}(0)$-algebra of finite type.

Proof. The proof is the same as Example 1.12 so we only sketch the proof here. We put $\mathcal{Y} := T^{(m)*}\mathcal{X}$ and $\mathcal{Y}' := T^{(m)*}\mathcal{X}'$. To see that $\mathcal{O}_{\mathcal{Y}}(n)$ is a coherent $\mathcal{O}_{\mathcal{Y}}(0)$-module, it suffices to point out that $\mathcal{O}_{P^{(m)*}\mathcal{Y}}(n)$ is a coherent $\mathcal{O}_{P^{(m)*}\mathcal{Y}}(0)$-module. The proof that $\mathcal{O}_{\mathcal{Y}}(*)$ is of finite type is the same. It remains to show that $\mathcal{O}_{\mathcal{Y}}(0)$ is noetherian. The only thing we need to check is the coherence of $\mathcal{O}_{\mathcal{Y}}(0)$ and $\mathcal{O}_{\mathcal{Y}}(*)$ around the zero section, and for this, apply [Be1 3.1.1] as in Example 1.12. ■

We define a sheaf of rings on the topological space $T^{(m)*}\mathcal{X} \cong T^{(m)*}X$ ($\cong$ denotes a canonical homeomorphism of topological spaces) by

$$\mathcal{E}_{\mathcal{X}}^{(m)} := \lim_{i} \mathcal{E}_{X_i}^{(m)}.$$

For $j \in \mathbb{Z}$, we also define

$$\mathcal{E}_{\mathcal{X},j}^{(m)} := \lim_{i} \mathcal{E}_{X_i,j}^{(m)}.$$

We remark that the “filtration” $\mathcal{E}_{\mathcal{X},j}^{(m)}$ of $\mathcal{E}_{\mathcal{X}}^{(m)}$ is not exhaustive. We define a submodule (which is in fact a ring) by Lemma 2.5 (iii) below by

$$\mathcal{E}_{\mathcal{X}}^{(m)} := \lim_{j \to \infty} \mathcal{E}_{\mathcal{X},j}^{(m)} \subset \mathcal{E}_{\mathcal{X}}^{(m)}.$$
There is a canonical homomorphism of rings on $T(m)^* \mathcal{X}$

\[(2.4.1) \quad \hat{\varphi}_m : \pi_m^{-1} \mathcal{D}^{(m)}_X \to \hat{\mathcal{E}}^{(m)}_X.\]

This homomorphism is injective by the injectivity of $\varphi$ in 2.2. Since $\varphi_m(\pi_m^{-1} \mathcal{D}^{(m)}_X) \subset \mathcal{E}^{(m)}_X$, the isomorphism \[\hat{\varphi}_m_{|\pi_m^{-1} \mathcal{D}^{(m)}_X} \text{ induces a homomorphism of modules } \pi_m^{-1} \mathcal{D}^{(m)}_X \to \mathcal{E}^{(m)}_X.\] We abuse the notation to denote this homomorphism by $\hat{\varphi}_m$. We see from the following Lemma 2.5 (iii) that this homomorphism is in fact a homomorphism of rings.

2.5 Lemma. — Let $\mathcal{X}$ be a smooth formal scheme over $R$. Let $\mathcal{U}$ be an open formal subscheme of $T(m)^* \mathcal{X}$ belonging to $\mathfrak{B}$. Let $i$ be a non-negative integer, and we denote $\mathcal{U} \otimes R_i$ by $U_i$.

(i) The ring $\Gamma(\mathcal{U}, \mathcal{E}^{(m)}_X)$ is $\pi$-adically complete and flat over $R$, and for any integer $j$,

\[\Gamma(U_i, \mathcal{E}^{(m)}_{X,j}) \cong \Gamma(\mathcal{U}, \mathcal{E}^{(m)}_X) \otimes_R \mathcal{R}_i, \quad \Gamma(U_i, \mathcal{E}^{(m)}_{X_i}) \cong \Gamma(\mathcal{U}, \mathcal{E}^{(m)}_X) \otimes_R \mathcal{R}_i.\]

In particular, the canonical homomorphisms $\mathcal{E}^{(m)}_X \otimes R_i \to \mathcal{E}^{(m)}_{X_i}$ and $\mathcal{E}^{(m)}_X \otimes R_i \to \mathcal{E}^{(m)}_{X_i}$ are isomorphisms.

(ii) Let $j$ be an integer, and $k$ be a positive integer. Let $\mathcal{E}$ be one of $\mathcal{E}^{(m)}_{X,j+k}$, $\mathcal{E}^{(m)}_{X,j}$, $\mathcal{E}^{(m)}_X$. We have

\[\Gamma(\mathcal{U}, \mathcal{E}/\mathcal{E}^{(m)}_X) \cong \Gamma(\mathcal{U}, \mathcal{E}/\mathcal{E}^{(m)}_{X,j}), \quad \mathcal{E}^{(m)}_{X,j+k}/\mathcal{E}^{(m)}_{X,j} \cong \lim_{i} \mathcal{E}^{(m)}_{X_i,j+k}/\mathcal{E}^{(m)}_{X_i,j}.\]

(iii) Let $k$ and $j$ be integers. Then $\mathcal{E}^{(m)}_{X,j} \cdot \mathcal{E}^{(m)}_{X,k} \subset \mathcal{E}^{(m)}_{X,j+k}$ in $\mathcal{E}^{(m)}_X$, and in particular, $\mathcal{E}^{(m)}_X \cdot \mathcal{E}^{(m)}_{X,j} \subset \mathcal{E}^{(m)}_{X,j}$ is a filtered ring. Moreover, the $\pi$-adic completion of $\mathcal{E}^{(m)}_X$ is isomorphic to $\mathcal{E}^{(m)}_X$.

(iv) We have the following isomorphisms

\[\mathcal{E}^{(m)}_{X_i} \cong \lim_{j \to -\infty} \mathcal{E}^{(m)}_{X_i}/\mathcal{E}^{(m)}_{X_i,j}, \quad \mathcal{E}^{(m)}_X \cong \lim_{j \to -\infty} \mathcal{E}^{(m)}_X/\mathcal{E}^{(m)}_{X,j}, \quad \mathcal{E}^{(m)}_{X,j+k} \cong \lim_{j \to -\infty} \mathcal{E}^{(m)}_{X,j+k}/\mathcal{E}^{(m)}_{X,j}.\]

In particular the filtered ring $\mathcal{E}^{(m)}_X$ is complete with respect to the filtration by order.

Proof. For a projective system $\{\mathcal{F}_i\}_{i \geq 0}$ on a topological space $T$ and for an open subset $U$ of $T$,

\[(2.5.1) \quad \Gamma(U, \lim_i \mathcal{F}_i) \cong \lim_i \Gamma(U, \mathcal{F}_i)\]

by [EGA 01, 3.2.6]. For an inductive system $\{\mathcal{F}_i\}_{i \geq 0}$ on a noetherian topological space $T$ and for an open subset $U$ of $T$,

\[(2.5.2) \quad \Gamma(U, \lim_i \mathcal{F}_i) \cong \lim_i \Gamma(U, \mathcal{F}_i)\]

by [H1, Exercise II.1.11] (or [SGA1, Exp. VI, §5] for a thorough treatment). Since $\mathcal{U}$ is an open subset of an affine smooth formal scheme $\mathfrak{e}(\mathcal{U})$, $\mathfrak{U}$ is a noetherian space.

By the commutation (2.5.1) and the definition of $\mathcal{E}^{(m)}_X$ in 2.4, $\Gamma(\mathcal{U}, \mathcal{E}^{(m)}_X) \cong \lim_i \Gamma(U_i, \mathcal{E}^{(m)}_{X_i})$. Since $\Gamma(U_i, \mathcal{E}^{(m)}_{X_i})$ is flat over $\Gamma(U_i, \mathcal{E}^{(m)}_{X_i})$ (cf. 2.2), the ring $\Gamma(U_i, \mathcal{E}^{(m)}_{X_i})$ is flat over $R_i$. Thus we get (i) by Lemma 2.3 and the following Lemma 2.6.

Let us prove (ii). Let $k$ be a positive integer or $\infty$. When $k = \infty$, we put $\mathcal{E}^{(m)}_{X,i,j+k} = \mathcal{E}^{(m)}_{X,i,j}$ and $\mathcal{E}^{(m)}_{X,i,j+k} = \mathcal{E}^{(m)}_{X,i,j}$. By (1.6.2),

\[\Gamma(U_i, \mathcal{E}^{(m)}_{X,i,j+k}/\mathcal{E}^{(m)}_{X,i,j}) \cong \Gamma(U_i, \mathcal{E}^{(m)}_{X,i,j+k}/\mathcal{E}^{(m)}_{X,i,j}).\]
Since the projective system \( \{ \Gamma(U_i, \mathcal{E}_{X_i,j}^{(m)}) \}_{i \geq 0} \) satisfies the Mittag-Leffler condition by (i), the sequence
\[
0 \rightarrow \lim_{i} \Gamma(U_i, \mathcal{E}_{X_i,j}^{(m)}) \rightarrow \lim_{i} \Gamma(U_i, \mathcal{E}_{X_i,j+k}^{(m)}) \rightarrow \lim_{i} \Gamma(U_i, \mathcal{E}_{X_i,j+k}/\mathcal{E}_{X_i,j}^{(m)}) \rightarrow 0
\]
is exact. Considering (2.5.1), this shows that
\[
(2.5.3) \quad \Gamma(\mathcal{U}, \lim_{i} \mathcal{E}_{X_i,j+k}^{(m)}/\mathcal{E}_{X_i,j}^{(m)}) \cong \Gamma(\mathcal{U}, \mathcal{E}_{X_i,j+k}/\mathcal{E}_{X_i,j}^{(m)}).
\]
Since \( \mathcal{B} \) is a basis of the topology, the canonical homomorphism \( \mathcal{E}_{X_i,j+k}/\mathcal{E}_{X_i,j}^{(m)} \rightarrow \lim_{i} \mathcal{E}_{X_i,j+k}/\mathcal{E}_{X_i,j}^{(m)} \) is an isomorphism, and the second equality of (ii) follows. The first equality of (ii) except for the \( \mathcal{E}_{X} \) case follows by using (2.5.3) again. For \( \mathcal{E}_{X} = \mathcal{E}_{X}^{(m)} \) case, use (2.5.2) and \( \mathcal{E}_{X} = \mathcal{E}_{X}^{(m)} \) case.

The first claim of (iii) follows since \( \mathcal{E}_{X_i}^{(m)} \) is a filtered ring. By (i), \( \mathcal{E}_{X}^{(m)} \) is the \( \pi \)-adic completion of \( \mathcal{E}_{X}^{(m)} \).

Let us prove (iv). To see the first two isomorphisms, we use the commutativity (2.5.1), and the fact that two projective limits commute. We will show the last isomorphism. For an open affine subscheme \( \mathcal{U} \) in \( \mathcal{B} \), consider the following exact sequence
\[
0 \rightarrow \Gamma(\mathcal{U}, \mathcal{E}_{X,k}^{(m)}/\mathcal{E}_{X,j}^{(m)}) \rightarrow \Gamma(\mathcal{U}, \mathcal{E}_{X}^{(m)}/\mathcal{E}_{X,j}^{(m)}) \rightarrow \Gamma(\mathcal{U}, \mathcal{E}_{X}^{(m)}/\mathcal{E}_{X,k}^{(m)}) \rightarrow 0
\]
for integers \( k \geq j \). The last surjection is deduced by using (ii). Since the projective system \( \{ \Gamma(\mathcal{U}, \mathcal{E}_{X}^{(m)}/\mathcal{E}_{X,j}^{(m)}) \}_{j \geq 0} \) satisfies the Mittag-Leffler condition by (ii), the following sequence is exact:
\[
0 \rightarrow \lim_{j} \mathcal{E}_{X,k}^{(m)}/\mathcal{E}_{X,j}^{(m)} \rightarrow \lim_{j} \mathcal{E}_{X,j+k}^{(m)}/\mathcal{E}_{X,j}^{(m)} \rightarrow \lim_{j} \mathcal{E}_{X,j+k}^{(m)}/\mathcal{E}_{X,k}^{(m)} \rightarrow 0
\]
where \( j \rightarrow -\infty \) and \( k \rightarrow \infty \). Thus the lemma is proven.

2.6 Lemma. — Let \( \{ E_i \}_{i \geq 0} \) be a projective system of \( R \)-modules such that for each \( i \), \( E_i \) is a flat \( R \)-module. Assume that the homomorphism \( E_{i+1} \otimes R_i \rightarrow E_i \) induced by the transition homomorphism is an isomorphism for any non-negative integer \( i \). Let \( E := \lim_{i} E_i \). Then the canonical homomorphism
\[
E \otimes R_j \rightarrow E_j
\]
is an isomorphism for any non-negative integer \( j \). Moreover, \( E \) is \( \pi \)-adically complete and flat over \( R \).

Proof. For non-negative integers \( a \geq b \), we denote by \( \phi_{b,a} : E_a \rightarrow E_b \) the transition homomorphism. Consider the short exact sequence:
\[
0 \rightarrow R_i \xrightarrow{\pi^{i+1}} R_{i+j+1} \rightarrow R_j \rightarrow 0.
\]
Now, we take \( E_i \otimes_{R_{i+j+1}} \), and we have the following diagram.

\[
(2.6.1) \quad \begin{array}{ccccccccc}
0 & \rightarrow & E_{i+j+1} & \otimes_{R_i} & R_{i+j+1} & \xrightarrow{\pi^{i+1}} & E_{i+j+1} & \otimes_{R_{i+j+1}} & R_{i+j+1} & \rightarrow & 0 \\
& & \sim & \phi_{i,j+i+1} & \sim & \phi_{i,j+i+1} & \sim & \phi_{i,j+i+1} & \sim & \phi_{i,j+i+1} \\
& & \psi_i & \rightarrow & E_{i+j+1} & \xrightarrow{\phi_{i,j+i+1}} & E_{j} & \rightarrow & 0 \\
\end{array}
\]
Here the tensor product is taken over \( R_{i+j+1} \), and the dotted arrow is defined so that the diagram is commutative. By the flatness of \( E_{i+j+1} \) over \( R_{i+j+1} \), the upper horizontal sequence is exact. We note that \( \psi_i \) does not depend on \( j \), and the following diagram is commutative for any non-negative integers \( i \) and \( k \).

\[
\begin{array}{ccc}
E_{i+k} & \xrightarrow{\psi_{i+k}} & E_{i+j+k+1} \\
\phi_{i,i+k} & & \phi_{i+j+1,i+j+k+1} \\
E_i & \xrightarrow{\psi_i} & E_{i+j+1}
\end{array}
\]

Thus the lower exact sequence of \( (2.6.1) \) induces an exact sequence of projective systems with respect to the index \( i \in \mathbb{N} \). Since the projective system \( \{E_i\}_{i \in \mathbb{Z}} \) satisfies the Mittag-Leffler condition, the sequence

\[
0 \to E \xrightarrow{\psi} E \to E_j \to 0
\]

where \( \psi := \lim_{i \to \infty} \psi_i \), is exact. By definition, \( \psi \) is the homomorphism of multiplication by \( \pi^{i+1} \).

Thus the lemma follows. ■

2.7 Lemma. — Let \( \hat{\mathcal{X}} := \mathcal{X}^{(m)} \). Let \( \mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i \) be a graded \( \mathcal{O}_{\hat{\mathcal{X}}}(0) \)-algebra of finite type on \( \hat{\mathcal{X}} \) such that \( \mathcal{A}_i \) is a coherent \( \mathcal{O}_{\hat{\mathcal{X}}}(0) \)-module for any \( i \in \mathbb{Z} \). Then for any \( V \subset U \) in \( \mathcal{B} \), the restriction homomorphism \( \Gamma(U, \mathcal{A}) \to \Gamma(V, \mathcal{A}) \) is flat, and \( \mathcal{A} \) is noetherian with respect to \( \mathcal{B} \).

Proof. We put \( \mathcal{O} := \mathcal{O}_{\hat{\mathcal{X}}(0)} \). Let us check the conditions of Definition [1.10] (i). The condition 2 of \textit{ibid.} follows since \( \mathcal{A} \) is of finite type over \( \mathcal{O} \). Let \( U \) be an open subset of \( \hat{\mathcal{X}} \) in \( \mathcal{B} \) such that there exists a surjection \( \phi: \mathcal{O}[T_1, \ldots, T_n]|_U \to \mathcal{A}|_U \). We claim that the homomorphism

\[
\Gamma(U, \mathcal{O}[T_1, \ldots, T_n]) \to \Gamma(U, \mathcal{A})
\]

is surjective. Indeed, since \( \mathcal{A}_i \) is a coherent \( \mathcal{O} \)-module for any \( i \), \( \text{Ker}(\phi) \) is an inductive limit of coherent \( \mathcal{O}|_U \)-modules. Since \( U \) is noetherian and separated, we may use \cite{SGA3, Exp. VI \S5} on \( U \), and thus \( H^1(U, \text{Ker}(\phi)) = 0 \), which implies the claim. Thus the condition 3 is fulfilled. It remains to show that \( \mathcal{A} \) is a coherent ring. For this, it suffices to check the conditions of [Be1, 3.1.1]. For \( V \subset U \) in \( \mathcal{B} \), we have the restriction isomorphism \( \Gamma(V, \mathcal{O}) \cong \Gamma(U, \mathcal{O}) \to \Gamma(V, \mathcal{A}_i) \) for any \( i \) since \( \mathcal{A}_i \) is a coherent \( \mathcal{O} \)-module. This induces an isomorphism

\[
\Gamma(V, \mathcal{O}) \cong \Gamma(U, \mathcal{A}_i) \cong \Gamma(V, \mathcal{A}_i).
\]

Since the restriction homomorphism \( \Gamma(U, \mathcal{A}) \to \Gamma(V, \mathcal{A}) \) is flat, this isomorphism is showing that \( \Gamma(U, \mathcal{A}) \to \Gamma(V, \mathcal{A}) \) is flat as well. Thus the claim follows. ■

2.8 Proposition. — Let \( \mathcal{X} \) be a smooth formal scheme over \( R \).

(i) The rings \( \mathcal{E}_{\mathcal{X}}^{(m)}, \mathcal{E}_{\mathcal{X}}^{(m)}, \mathcal{E}_{\mathcal{X},0}^{(m)} \) are noetherian with respect to \( \mathcal{B} \).

(ii) The homomorphism \( \mathcal{E}_m \) of \( (2.4.1) \) is flat.

(iii) Let \( \mathcal{E} \) be either \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) or \( \mathcal{E}_{\mathcal{X}}^{(m)} \) or \( \mathcal{E}_{\mathcal{X}}^{(m)} \). For any open subsets \( \mathcal{U} \supset \mathcal{V} \) in \( \mathcal{B} \), the restriction homomorphism

\[
\Gamma(\mathcal{U}, \mathcal{E}) \to \Gamma(\mathcal{V}, \mathcal{E})
\]

is flat.

Proof. Let us prove (i). First, we will show the claim for \( \mathcal{E}_{\mathcal{X}}^{(m)} \) and \( \mathcal{E}_{\mathcal{X},0}^{(m)} \). Let us check the conditions of Lemma [1.11] for \( \mathcal{E}_{\mathcal{X}}^{(m)} \) (resp. \( \mathcal{E}_{\mathcal{X},0}^{(m)} \)) on \( \mathcal{V} := \mathcal{T}^{(m)} \mathcal{X} \). The conditions 1 and 4 hold
by Lemma 2.5. By Lemma 2.5 (ii), \( \text{gr}(\mathcal{E}_{\mathcal{X}}^{(m)}) \cong \mathcal{O}_{\mathcal{Y}}(s) \) as graded rings. By Lemma 2.4 this implies that \( \text{gr}_i(\mathcal{E}_{\mathcal{X}}^{(m)}) \) is a coherent \( \mathcal{O}_{\mathcal{Y}}(0) \)-module on \( \mathcal{Y} \) for any \( i \in \mathbb{Z} \), and \( \text{gr}(\mathcal{E}_{\mathcal{X}}^{(m)}) \) (resp. \( \text{gr}(\mathcal{E}_{\mathcal{X},0}^{(m)}) \)) is an \( \mathcal{O}_{\mathcal{Y}}(0) \)-module of finite type on \( \mathcal{Y} \). Thus by Lemma 2.7 the conditions 2 and 3 are fulfilled. This implies that \( \mathcal{E}_{\mathcal{X}}^{(m)} \) and \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) are noetherian with respect to \( \mathfrak{B} \) on \( \mathcal{Y} \). It remains to check that the rings are noetherian around the zero section. For this, we only need to see the coherence by using (1.5.2). Using [HO, Ch.II, 1.2.1], \( \varphi_m|_{\mathcal{Y}} \) is flat. Thus, we conclude the proof by using [Be1, 3.1.1].

For \( \mathcal{E}_{\mathcal{X}}^{(m)} \), let us endow with the \( \pi \)-adic filtration \( \{ \pi^{-i}\mathcal{E}_{\mathcal{X}}^{(m)} \}_{i \leq 0} \) (cf. 1.1.2). Since \( \mathcal{E}_{\mathcal{X}}^{(m)} \) is \( \pi \)-torsion free by Lemma 2.5 (i), the homomorphism \( \mathcal{E}_{\mathcal{X},0}^{(m)}[T] \to \text{gr}(\mathcal{E}_{\mathcal{X}}^{(m)}) \) sending \( T \) to \( \pi \in \text{gr}_1(\mathcal{E}_{\mathcal{X}}^{(m)}) \) is an isomorphism. It is straightforward to check the conditions of Lemma 1.11 We remind that the \( \pi \)-adic filtrations can also be used to show that \( \mathcal{E}_{\mathcal{X}}^{(m)} \) is noetherian.

To prove (ii) and (iii), it suffices to apply (i) and [HO, Ch.II, 1.2.1].

Remark. — By the proof, we can moreover say that \( \mathcal{E}_{\mathcal{X}}^{(m)} \) and \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) are pointwise Zariskian with respect to the filtration by order on \( \mathcal{T}^{(m)*}\mathcal{X} \), and \( \mathcal{E}_{\mathcal{X}}^{(m)} \) and \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) are pointwise Zariskian with respect to the \( \pi \)-adic filtration on \( \mathcal{T}^{(m)*}\mathcal{X} \).

2.9. Now, we define
\[
\hat{\mathcal{E}}_{\mathcal{X}}^{(m)} := \mathcal{E}_{\mathcal{X}}^{(m)} \otimes \mathbb{Q}, \quad \mathcal{E}_{\mathcal{X},0}^{(m)} := \mathcal{E}_{\mathcal{X}}^{(m)} \otimes \mathbb{Q}.
\]
The homomorphism \( \varphi_m|_{\mathcal{Y}} \) induces a canonical injective homomorphism
\[
\hat{\varphi}_m \otimes \mathbb{Q} : \pi_m^{-1}\hat{\mathcal{E}}_{\mathcal{Y},0}^{(m)} \to \hat{\mathcal{E}}_{\mathcal{X},0}^{(m)}.
\]
If there is no risk of confusion, we sometimes denote \( \hat{\varphi}_m \otimes \mathbb{Q} \) abusively by \( \hat{\varphi}_m \). We call the sheaves \( \mathcal{E}_{\mathcal{X},0}^{(m)} \), \( \mathcal{E}_{\mathcal{X}}^{(m)} \), \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) the rings of \( (\text{naive}) \) microdifferential operators of level \( m \). Proposition 2.8 implies the following.

Corollary. — The rings \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) and \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) are noetherian with respect to \( \mathfrak{B} \). Moreover, \( \varphi_m \otimes \mathbb{Q} \) and the restriction homomorphism \( \Gamma(\mathcal{U}, \mathcal{E}) \to \Gamma(\mathcal{Y}, \mathcal{E}) \) are flat for \( \mathcal{U} \supset \mathcal{Y} \) in \( \mathfrak{B} \), where \( \mathcal{E} \) is either \( \mathcal{E}_{\mathcal{X},0}^{(m)} \) or \( \mathcal{E}_{\mathcal{X},0}^{(m)} \).

2.10. Let us describe sections of rings of microdifferential operators explicitly. We use the notation of 2.4. Suppose in addition that \( \mathcal{X} \) is affine, and possesses a system of local coordinates \( \{x_1, \ldots, x_d\} \). Let \( \{\partial_1, \ldots, \partial_d\} \) be the corresponding differential operators. Let \( k \) be a positive integer. We have a differential operator \( \partial_i^{(k)} \) for any \( 1 \leq i \leq d \) in \( \mathcal{X}_i^{(m)} \) for any integer \( l \geq 0 \) or in \( \mathcal{E}_{\mathcal{Y},0}^{(m)} \) (cf. [Be1, 2.2.3]). Write \( k = pmq + r \) with \( 0 \leq r < pm \). Recall that there is a relation (cf. [Be1, (2.2.3.1)])
\[
k! \partial_i^{(k)} = q! \partial_i^k.
\]
Now, these operators define elements in \( \text{gr}(\mathcal{X}_i^{(m)}) \) by taking the principal symbol (cf. 1.1.4). We denote \( \mathcal{O}(\partial_i^{(k)}) \) by \( \Sigma_i^{(k)} \) in \( \text{gr}_k(\mathcal{X}_i^{(m)}) \subset \text{gr}(\mathcal{X}_i^{(m)}) \). From now on, we use the multi-index notation. For example, for \( \underline{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d \), we denote \( \Sigma^{\underline{k}} := \Sigma_1^{k_1} \cdots \Sigma_d^{k_d} \), \( \partial^{(k)} := \partial_1^{(k_1)} \cdots \partial_d^{(k_d)} \), and \( |\underline{k}| := k_1 + \cdots + k_d \).

Let \( \Theta \in \Gamma(T^*\mathcal{X}, \mathcal{O}_{T^*\mathcal{X}}) \).
be a homogeneous section of degree $n$. We consider $\pi^{-1}_m \mathcal{O}_\mathcal{X}$ as a subring of $\mathcal{O}_{T(m)\times \mathcal{X}}$. This section induces a section $\Theta^{(m)} \in \Gamma(T^{(m)}\times \mathcal{X}, \mathcal{O}_{T^{(m)}\times \mathcal{X}})$ for any integer $m \geq 0$ in the following way: we may write
\begin{equation}
\Theta = \sum_{|k|=n} a_k \xi^k
\end{equation}
where $k \in \mathbb{N}^d$ and $a_k \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ in a unique way. We put
\[ \Theta^{(m)} := \sum_{|k|=n} a_k^{(m)} \xi^{k(p^{(m)})}, \]
where $\xi^{k(p^{(m)})} := (\xi \cdot (p^{(m)}))^{k}$ with $p^{(m)} := (p^m, \ldots, p^m)$. The homogeneous element $\Theta$ induces also elements in $\mathcal{D}^{(m)}(\mathcal{X})$ or $\mathcal{D}^{(m)}(X_l)$. We put
\[ \Theta_l^{(m)} := \sum_{|k|=n} a_k^{(m)} \xi^{k(p^{(m)})} \quad \Theta_r^{(m)} := \sum_{|k|=n} a_k^{(m)} \xi^{k(p^{(m)})} \]
where the subscripts $l$ and $r$ stand for left and right respectively. Since $\mathcal{D}^{(m)}(\mathcal{X}, \mathcal{Q}) \cong \mathcal{D}^{(m')}(\mathcal{X})$ for any non-negative integer $m'$, we sometimes consider these operators as sections of $\mathcal{D}^{(m')}(\mathcal{X}, \mathcal{Q})$.

Let $\mathcal{U}$ be the open affine subset of $T^{(m)} \times \mathcal{X}$ defined by $\Theta^{(m)}$. We claim that the operator $\Theta^{(m)}$ (resp. $\Theta^{(m)}_*$) is invertible in $\Gamma(\mathcal{U}, \mathcal{O}^{(m)}_{\mathcal{X}})$. Indeed, the inverse of $\Theta^{(m)}_*$ in $\mathcal{O}_{X_l}$ has degree $-np^m$ for any $l$. Since the inverse of $\Theta^{(m)}_*$ is unique in $\mathcal{O}(X_l)$, these elements induces an element of $\mathcal{D}(\mathcal{X}, \mathcal{Q}) \cong \mathcal{D}(\mathcal{X}, \mathcal{Q})$.

Let $(b_{\xi}^{(m)})$ and $(b_{\xi}^{(m)})$ be sequences in $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ for $k \in \mathbb{N}^d$ and $i \in \mathbb{N}$ such that the following hold: for each integer $N$, let $\beta_{N,i} := \sup_{|k|=np^m+N} |b_{\xi}^{(m)}|$, where $| \cdot |$ denotes the spectral norm (cf. [Bel, 2.4.2]) on $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Then
\begin{equation}
\lim_{i \to \infty} \beta_{N,i} = 0, \quad \lim_{N \to +\infty} \sup_i \beta_{N,i} = 0.
\end{equation}
We assume that the same conditions hold for $b_{\xi}^{(m)}$. In the sequel, we consider the ring $\pi^{-1}_m \mathcal{O}_{X_l}$ (resp. $\pi^{-1}_m \mathcal{O}_{\mathcal{X}}$) as a subring of $\mathcal{O}_{X_l}$ (resp. $\mathcal{O}_{\mathcal{X}}$) for any $\varphi_m$ of $2.2$ (resp. $\varphi_m$ of $2.4.1$). The sums
\begin{equation}
\sum_{N \in \mathbb{Z}} \sum_{|k|=np^m=N} b_{\xi}^{(m)} \xi^{k(p^{(m)})} (\Theta_l^{(m)})^{-i}, \quad \sum_{N \in \mathbb{Z}} \sum_{|k|=np^m=N} (\Theta_r^{(m)})^{-i} \xi^{k(p^{(m)})} b_{\xi}^{(m)}
\end{equation}
converge in $\Gamma(U_l, \mathcal{O}_{X_l})$ for any $l$. We note that $N$ is the order of $\Theta_l^{(m)}(\Theta_l^{(m)})^{-i}$ or $\Theta_r^{(m)}(\Theta_r^{(m)})^{-i} \xi^{k(p^{(m)})}$, and $\sum_{|k|=np^m=N} \ldots$ are finite sums by the first condition of $2.10.2$, and $N \gg 0$ by the second condition. Since these elements form projective systems over $l$, we have two elements in $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$. Even though the sums of $2.10.3$ do not converge in $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$ with respect to the $\pi$-adic topology in general\footnote{However, we are able to define a reasonable weaker topology on $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$ such that these sums converge. In the curve case, see [AM, 1.2.2].}, we abusively denote by $2.10.3$ for these two operators in $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$.

**Lemma.** — For any element $P \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$, there exist sequences $(b_{\xi}^{(m)})$ and $(b_{\xi}^{(m)})$ for $k \in \mathbb{N}^d$ and $i \in \mathbb{N}$ satisfying $2.10.2$ such that $P$ can be written as $2.10.3$. Moreover, if $P \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{X}})$ for some integer $j$, we can take $b_{\xi}^{(m)} = b_{\xi}^{(m)} = 0$ for $|k|-np^m > j$.

The first one is called a left presentation, and the second a right presentation. We remark here that presentations are not unique.
Proof. We only see the left presentation case. Since \( \Gamma(\mathcal{W}, \mathcal{E}_{X,j}^{(m)}) \) and \( \Gamma(\mathcal{W}, \hat{\mathcal{E}}_{X,j}^{(m)}) \) are flat over \( R \) and \( \pi \)-adically complete by Lemma 2.5(i), it suffices to show that any element of \( \Gamma(\mathcal{W}, \mathcal{E}_{X,j}^{(m)}) \) can be written as

\[
\sum_{N \in \mathbb{Z}} \sum_{\|k\| = \text{inp}^m = N} c_{k,i} \partial^{(k)}(m) (\hat{\Theta}_i^{(m)})^{-i},
\]

with \( c_{k,i} \in \Gamma(X_0, \mathcal{O}_{X_0}) \) such that the following holds: for each integer \( N \), \( c_{k,i} = 0 \) for almost all couples \( (k, i) \in \mathbb{N}^d \times \mathbb{N} \) such that \( \|k\| - \text{inp}^m = N \), and \( c_{k,i} = 0 \) for any \( \|k\| - \text{inp}^m > j \). This follows from Example 1.3. \( \blacksquare \)

2.11. Let \( \bullet \in \{t, r\} \). We used \( \hat{\Theta}_i^{(m)} \) to describe elements of \( \hat{\mathcal{E}}_{X,j}^{(m)} \). We may also use a variant of \( \hat{\Theta}_i^{(m+j)} \) for \( j \geq 0 \) to describe them. Suppose \( \Theta \) is written as (2.10.1). Then we put

\[
\Theta^{(m+m+j)} := \sum_{\|k\| = n} a_{k}^{m+j} \xi^{(p)^m}_i (\partial^{(k)}(m)_i), \quad \hat{\Theta}_i^{(m+m+j)} := \sum_{\|k\| = n} a_{k}^{m+j} \partial^{(k)}(m)_i \xi^{(p)^m}_i, \quad \Theta^{(m,m+j)} := \sum_{\|k\| = n} \partial^{(k)}(m)_i a_{k}^{m+j}. \]

Lemma. — Let \( m' \geq m \) be an integer, and we put \( j := m' - m \).

(i) Let \( r_{m,m'} := (p^{m'}!) \cdot (p^{m'})^{-p'} \in \mathbb{Z}_{p} \). Then

\[
\Theta^{(m,m')} \cdot r_{m,m'} = \Theta^{(m')} \cdot r_{m,m'}^n \cdot \Theta^{(m')}. \]

(ii) The operator \( \hat{\Theta}_i^{(m,m')} \) is invertible in \( \Gamma(\mathcal{W}, \mathcal{E}_{X,j}^{(m)}) \).

Proof. We know that for any \( 1 \leq i \leq d \),

\[
\xi^{(p)^m}_i = r_{m,m'} \cdot \xi^{(p)^{m'}}_i. \]

Since \( r_{m,m'} \) does not depend on \( i \), we get (i) by definition. For the proof of (ii), just copy the proof of the invertibility of \( \hat{\Theta}_i^{(m)} \) in 2.4.10. \( \blacksquare \)

Let \( m' \geq m \) be an integer. We claim that any \( S \in \Gamma(\mathcal{W}, \mathcal{E}_{X,j}^{(m)}) \) can also be written as

\[
\sum_{N \in \mathbb{Z}} \sum_{\|k\| = \text{inp}^{m'} = N} c_{k,i} \partial^{(k)}(m) (\hat{\Theta}_i^{(m,m')})^{-i}, \quad \sum_{N \in \mathbb{Z}} \sum_{\|k\| = \text{inp}^{m'} = N} (\Theta^{(m,m')})^i \partial^{(k)}(m) c_{k,i} \]

with sequences \( \{c_{k,i}\}, \{c_{k,i}^{(m')}\} \in \Gamma(\mathcal{W}, \mathcal{O}_X) \) for \( k \in \mathbb{N}^d \) and \( i \in \mathbb{N} \), such that the following holds: for each integer \( N \), let \( \gamma_{N,i} := \sup_{\|k\| = \text{inp}^{m'} = N} \|c_{k,i}\| \). Then

\[
\lim_{i \to \infty} \beta_{N,i} = 0, \quad \lim_{N \to +\infty} \sup_{i} \{\gamma_{N,i}\} = 0.
\]

The verification is left to the reader.

2.12. We will see a relation between the characteristic varieties and the supports of microlocalizations. Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_{\mathcal{W}, \mathcal{Q}}^{(m)} \)-module. The characteristic variety of level \( m \) denoted by \( \text{Char}^{(m)}(\mathcal{M}) \) is defined in [Bel 5.2.4], where the notation \( \text{Car}^{(m)} \) is used. Let us briefly recall the definition. First, we take a \( p \)-torsion free coherent \( \mathcal{D}_{\mathcal{W}, \mathcal{Q}}^{(m)} \)-module \( \mathcal{M}' \) such that \( \mathcal{M}' \otimes \mathbb{Q} \cong \mathcal{M} \) using [Bel 3.4.5]. Then, \( \mathcal{M}'/\pi \) is a coherent \( \mathcal{D}_{\mathcal{W}, \mathcal{Q}}^{(m)} \)-module. We take the characteristic variety of filtered module (cf. Remark 1.7), which is defined as a closed subvariety of \( T^{(m)*}X \). This is
called the characteristic variety of $\mathcal{M}$. We can check that this does not depend on the choice of $\mathcal{M}'$ as written in *ibid.* If it is unlikely to cause any confusion, we sometimes abbreviate $\text{Char}^{(m)}$ as Char.

Let us define another subvariety of $T^{(m)*}X$ defined by $\mathcal{M}$. Consider the following coherent $\hat{E}_{\mathcal{X},\mathbb{Q}}$-module

$$\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}(\mathcal{M}) := \hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)} \otimes_{\pi_m^{-1}\hat{E}_{\mathcal{X},\mathbb{Q}}} \pi_m^{-1}\mathcal{M},$$

which is called the microlocalization of $\mathcal{M}$. Note here that since $\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}(\mathcal{M})$ is an $\hat{E}_{\mathcal{X},\mathbb{Q}}$-module of finite type, the support $\text{Supp}(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}(\mathcal{M})) \subset T^{(m)*}\mathcal{X}$ is closed by [EGA] 01, 5.2.2 (3).

**2.13 Proposition.** — Let $\mathcal{M}$ be a coherent $\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}$-module. Then, we have the following equality of closed subsets of $T^{(m)*}\mathcal{X}$:

$$\text{Char}^{(m)}(\mathcal{M}) = \text{Supp}(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}(\mathcal{M})).$$

**Proof.** Take a coherent $\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}$-module $\mathcal{M}'$ flat over $R$ and $\mathcal{M}' \otimes \mathbb{Q} \cong \mathcal{M}$. By definition,

$$\text{Char}^{(m)}(\mathcal{M}) = \text{Char}(\mathcal{M}' \otimes k).$$

Let us calculate the support of the microlocalization. Since $\hat{E}_{\mathcal{X}}^{(m)}$ is pointwise Zariskian with respect to the $p$-adic filtration by Remark 2.8, the $p$-adic filtration on $\hat{E}_{\mathcal{X}}^{(m)} \otimes \pi_m^{-1}\mathcal{M}'$ is separated by Lemma 1.14, and thus,

$$\text{Supp}(\hat{E}_{\mathcal{X}}^{(m)} \otimes \pi_m^{-1}\mathcal{M}') = \text{Supp}(\hat{E}_{\mathcal{X}}^{(m)} \otimes \pi_m^{-1}\mathcal{M'}).$$

Moreover, since $\hat{E}_{\mathcal{X}}^{(m)}$ is flat over $\pi_m^{-1}\hat{E}_{\mathcal{X}}$, $\hat{E}_{\mathcal{X}} \otimes \mathcal{M}'$ is $p$-torsion free. Thus,

$$\text{Supp}(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}(\mathcal{M})) = \text{Supp}(\hat{E}_{\mathcal{X}}^{(m)} \otimes \pi_m^{-1}\mathcal{M}).$$

Using Lemma 1.14, we get the result. ■

**2.14 Remark.** — P. Berthelot pointed out to the author another method to define $\hat{E}_{\mathcal{X}}^{(m)}$. Let $\mathcal{X}$ be a smooth affine formal scheme over $R$. Let $\mathcal{O}$ be a homogeneous section of $\Gamma(T^*\mathcal{X}, \mathcal{O}_{T^*}\mathcal{X})$. For each $i \geq 0$, there exists an integer $m' \geq m$ such that $\overline{\Theta}^{(m,m')}_i$ is contained in the center of $\mathcal{O}^{(m)}_{X_i}$. Note that in this case, $\overline{\Theta}^{(m,m)}_i = \overline{\Theta}^{(m,m')}_i$. Let $A$ be a ring, and $S$ be a multiplicative system of $A$ consisting of elements in the center of $A$. We can construct the ring of fractions $S^{-1}A$ as the commutative case. (The details are left to the reader.) Using this, we define

$$\Gamma(D(\Theta^{(m)}), \mathcal{L}\mathcal{D}_{\mathcal{X}_i}^{(m)}) := S^{-1}_{\Theta^{(m,m')}_i} \Gamma(D(\Theta^{(m)}), \pi^{-1}\mathcal{D}_{\mathcal{X}_i}^{(m)}),$$

where $S_{\Theta^{(m,m')}}$ denotes the multiplicative system generated by $\overline{\Theta}^{(m,m')}_i$. We see easily that this does not depend on the choice of $m'$ and defines a sheaf. By taking the completion with respect to the filtration by order, we get $\hat{E}_{\mathcal{X}_i}^{(m)}$. By definition, the sheaf $\mathcal{L}\mathcal{D}_{\mathcal{X}_i}^{(m)}$ is a noetherian ring.

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(3) In *ibid.*, only commutative case is treated, but the same argument can be used also for non-commutative case.
3. Pseudo cotangent bundles and pseudo-polynomials

3.1. Recall the notation of 2.3. Let \( A \) be a commutative \( R \)-algebra, and \( m \) be a non-negative integer, \( d \) be a positive integer. We define

\[
A[\xi_1, \ldots, \xi_d](m) := A[\xi_j^{(p^i)}(m) \mid j = 1, \ldots, d, i = 0, \ldots, m]/I_m
\]

where \( \xi_j^{(p^i)}(m) \) is an indeterminant for any \( i \) and \( j \), and \( I_m \) is the ideal generated by the relations

\[
(\xi_j^{(p^i)}(m))^p = \frac{(p^{i+1})!}{(p^i)!} \xi_j^{(p^{i+1})(m)}
\]

for \( 1 \leq j \leq d \) and \( 0 \leq i < m \). We note that \((p^{i+1})!:(p^i)!p \in \mathbb{Z}_p\). We call this the ring of pseudo-polynomials over \( A \). We denote by \( A\{\xi_1, \ldots, \xi_d\}(m) \) the \( p \)-adic completion of \( A[\xi_1, \ldots, \xi_d](m) \). We call this the pseudo-Tate algebra over \( A \). We note that for an \( R \)-algebra \( A \),

\[
(3.1.1) \quad A \otimes_R R[\xi_1, \ldots, \xi_d](m) \cong A[\xi_1, \ldots, \xi_d](m).
\]

Lemma. — Let \( A \) be a commutative \( R \)-algebra. For any non-negative integers \( m' \geq m \), there exists a unique isomorphism of rings

\[
A[\xi_1, \ldots, \xi_d](m) \otimes \mathbb{Q} \xrightarrow{\sim} A[\xi_1, \ldots, \xi_d](m') \otimes \mathbb{Q}
\]

sending \( \xi_i^{(1)(m)} \) to \( \xi_i^{(1)(m')} \) for \( 1 \leq i \leq d \).

Proof. By (3.1.1), it suffices to show that \( K[\xi_1, \ldots, \xi_d](m) \cong K[\xi_1, \ldots, \xi_d](m') \). By the definition of \( I_m \), we have \( K[\xi_1, \ldots, \xi_d](m) \cong K[\xi_1^{(1)}(m'), \ldots, \xi_d^{(1)}(m')] \). Thus the lemma follows. ■

3.2 Lemma. — Let \( \mathcal{X} = \text{Spf}(A) \) be an affine smooth formal scheme over \( R \) possessing a system of local coordinates \( \{x_1, \ldots, x_d\} \) on \( \mathcal{X} \). Let \( A_i := A \otimes_R R_i \). Then there exists a ring isomorphism

\[
A_i[\xi_1, \ldots, \xi_d](m) \cong \Gamma(X_i, \text{gr}(\mathcal{D}_X^{(m)}))
\]

sending \( \xi_k \) to \( \sigma(\partial_k) \), where \( \sigma \) denotes the principal symbol (cf. 1.1.4), for \( 1 \leq k \leq d \).

Proof. The surjectivity follows from [Be1, 2.2.5]. To see the injectivity, we note that the set \( \{\partial(\mathcal{D}^{(m)})\} \) where \( k \in \mathbb{N}^d \) forms a basis of \( \mathcal{D}_X^{(m)} \) as an \( \mathcal{O}_{X_i} \)-module, and use [ibid., 2.2.4]. ■

3.3. Let \( X \) be a smooth scheme over \( k \), and \( m \geq 0 \) be an integer. Let \( X^{(m)} := X \otimes_k F_k^m \otimes k \)

where \( F_k^m : k \to k \) is the \( m \)-th absolute Frobenius homomorphism (i.e. the homomorphism sending \( x \) to \( x^{p^m} \)). By [Be1, 5.2.2], we have a canonical isomorphism

\[
(T^{(m)*}X)_{\text{red}} \cong X \times_{X^{(m)}} T^*X^{(m)}
\]

where \( \text{red} \) denotes the underlying reduced scheme. The scheme \( T^*X^{(m)} \) is deduced from \( T^*X \) by the base change \( X^{(m)} \to X \). This induces the canonical morphism of schemes (which may not be a morphism over \( k \))

\[
(T^{(m)*}X)_{\text{red}} \to T^*X
\]

such that the underlying continuous map is a homeomorphism of topological spaces. Since the topological space of \( T^{(m)*}X \) is homeomorphic to that of \( T^{(m)*}\mathcal{X} \), we also get a canonical homeomorphism \( T^{(m)*}\mathcal{X} \cong T^*\mathcal{X} \). Consider the situation as in 2.10. The affine open subset of \( T^*\mathcal{X} \) defined by \( \Theta \) and that of \( T^{(m)*}\mathcal{X} \) defined by \( G^{(m)} \) are homeomorphic under this canonical
homeomorphism. From now on, we identify the spaces \( T^* \mathcal{X} \), \( T^{(m)*} \mathcal{X} \), \( T^* X \), and \( T^{(m)*} X \) using these homeomorphisms. In particular, we consider \( \mathcal{E}_{\mathcal{X}}^{(m)} \) etc. as sheaves on \( T^* \mathcal{X} \) or \( T^* X \). We denote the projection \( \pi : T^* \mathcal{X} \to \mathcal{X} \). The notation \( \pi^m \) is the same as the uniformizer of \( R \), but we do not think there will be any confusion. This identification also induces the identification of topological spaces

\[
P^* \mathcal{X} \approx P^* X \approx P^{*(m)} X \approx P^{*(m)} \mathcal{X}.
\]

### 3.4 Lemma. — Let \( \mathcal{X} \) be an affine smooth formal scheme over \( R \) of dimension \( d \). We use the notation and the identifications in \textsection 3.3. Let \( \bullet \in \{l, r \} \). We take non-negative integers \( m' \geq m \).

(i) For any integer \( k \), there exist integers \( a_k, b_k \geq 0 \) such that the following holds: let \( \Theta \in \Gamma(T^* \mathcal{X}, \mathcal{O}_{T^* \mathcal{X}}) \) be a homogeneous section of degree \( n \).

(a) The operator

\[
p_{\alpha k}^m \mathcal{O}^{(m')}_{\Theta(m')} - i,
\]

which is a priori contained in \( \Gamma(D(\Theta), \mathcal{E}_{\mathcal{X}}^{(m')}) \) by Lemma \ref{lemma2}(i), is contained in \( \Gamma(D(\Theta), \mathcal{E}_{\mathcal{X}}^{(m')}) \) for any \( |l| - \text{inp}^m \geq k \). If \( dp^{m'+1} < k \), we may take \( a_k = 0 \).

(b) The operator

\[
p_{\beta k}^m \mathcal{O}^{(m')}_{\Theta(m')} - i
\]

is in \( \Gamma(D(\Theta), \mathcal{E}_{\mathcal{X}}^{(m')}) \) for any \( |l| - \text{inp}^m \leq k \). If \( k < p^{m+1} \), we may take \( b_k = 0 \).

(ii) Let \( \Theta \in \Gamma(T^* \mathcal{X}, \mathcal{O}_{T^* \mathcal{X}}) \) be a homogeneous section. Take an integer \( m'' \) such that \( m \leq m'' \leq m' \). Suppose \( P = \alpha \cdot \mathcal{O}^{(m')}_{\Theta(m')} - i \) with \( \alpha \in R \) is contained in \( \Gamma(D(\Theta), \mathcal{E}_{\mathcal{X}}^{(m')}) \). Then it is also contained in \( \Gamma(D(\Theta), \mathcal{E}_{\mathcal{X}}^{(m''+1)}) \).

**Proof.** First, let us show (i). Since the proof for (b) is essentially the same, we concentrate on proving (a). We show the following.

**Claim.** — Let \( m' \geq 0 \) be an integer. For integers \( m, a, k \) such that \( m' \geq m \geq 0, m' - m \geq a \geq 0, \) there exists an integer \( \alpha_{k,m,a} \geq 0 \) such that, for any \( \Theta \) and \( |l| - \text{inp}^m \geq k \), \( p_{\alpha_{k,m,a}}^{m} \mathcal{O}^{(m')}_{\Theta(m')} - i \) is equal to \( \alpha \cdot \mathcal{O}^{(m+a)}_{\Theta(m+a,m')} - i \) with some \( \alpha \in \mathbb{Z}_p \). If \( k > dp^{m'+1} \), we can take \( \alpha_{k,m,a} = 0 \).

Once this claim is proven, the lemma follows by taking \( a = m' - m \).

**Proof of the claim.** Let \( b := m' - m - 1 \geq -1 \). We will show the claim using the induction on \( b \). When \( b = -1 \) or more generally \( a = 0 \), we can take \( \alpha_{k,m,a} = 0 \). Since we can take \( \alpha_{k,m,a} = \alpha_{k,m+1,a-1} + \alpha_{k,m,1} \), it suffices to show the existence of \( \alpha_{k,m,1} \) by the induction hypothesis.

There exists a number \( c \in \mathbb{Z}_p^* \) such that

\[
\Theta_{\bullet}^{(m,m')} = c p^{np^b} \cdot \Theta_{\bullet}^{(m+1,m')}
\]

For \( l' \in \mathbb{N}^d \), we put

\[
g(l') := \sum_{j=1}^{d} \left\lfloor \frac{l'_j}{p^{m+1}} \right\rfloor
\]

where \( \lfloor \alpha \rfloor \) denotes the maximum integer less than or equal to \( \alpha \). Then

\[
\mathcal{O}^{(m')}_{\Theta} = c' p^{g(l')} \mathcal{O}^{(m+1)}_{\Theta}
\]
with $c' \in \mathbb{Z}_p$. Since
\[ \frac{l_j}{p^{m+1}} - 1 < \left\lfloor \frac{l_j}{p^{m+1}} \right\rfloor \leq \frac{l_j}{p^{m+1}}, \]
we get inequalities
\[ \text{inp}^b + \frac{k}{p^{m+1}} - d \leq \frac{|l|}{p^{m+1}} - d < \sum_{j=1}^{d} \left\lfloor \frac{l_j}{p^{m+1}} \right\rfloor = g(\ell). \]
Thus,
\[ \partial^m(\ell) (\Theta^{(m,m')}_{\ast} - i) = c' c^{-i} p^{q(\ell) - \text{inp}^b} \partial^m(\ell) (\Theta^{(m+1,m')}_{\ast}) - i, \]
and we may take $\alpha_{k,m,1} = \max\{0, [d - kp^{-(m+1)} + 1]\}$. Thus, we conclude the proof of the claim. 

Let us prove (ii) on $D(\Theta)$. We get
\[ \widetilde{\Theta}^{(m,m')}_{\ast} = u (p^{m'-m})^n \cdot \Theta^{(m,m')} \]
where $n$ denotes the order of $\Theta$, and $u$ denotes a number in $\mathbb{Z}_p$. Thus, for $m \leq l \leq m'$,
\begin{equation}
(3.4.1) \quad \Theta^{(m,m')} = u' (p^{m'-m})^n \cdot \Theta^{(m,m')}_{l} = u_l p^{a_l} \cdot \Theta^{(m,m')}_{l} \tag{3.4.1}
\end{equation}
where $u'$ and $u_l$ denote numbers in $\mathbb{Z}_p$, and $a_l$ is equal to $n \cdot (p^{m'-l} + \cdots + p^{m'-m-1})$. We also get
\begin{equation}
(3.4.2) \quad \partial^m(\ell) = u' p^{b_l} \partial^m(\ell)_{l}, \quad b_l = \sum_{j=1}^{d} \sum_{i=m+1}^{l} [p^{-i} k_j]. \tag{3.4.2}
\end{equation}
Now, we will define two functions $f, g: [m, m'] \rightarrow \mathbb{R}$. The function $f$ is the continuous function such that it is affine on the interval $[l, l+1]$ for any integer $l$ in $[m, m']$, and

\[ f(l) := \text{ord}_p (\alpha) + b_l = \text{ord}_p (\alpha) + \sum_{j=1}^{d} \sum_{i=m+1}^{l} [p^{-i} k_j], \]

where $\text{ord}_p$ denotes the $p$-adic order normalized so that $\text{ord}_p (p) = 1$. The function $g$ is the continuous function such that it is affine on the interval $[l, l+1]$ for any integer $l$ in $[m, m']$, and

\[ g(l) := i \cdot a_l = m i \cdot (p^{m'-l} + p^{m'-l+1} + \cdots + p^{m'-m-1}). \]

Since the operator $\partial^m(\ell)$ is a section of $\hat{\delta}^{(m)}_{\ell}$, we have $g(m) \leq f(m)$. By (3.1.1) and (3.1.2), it suffices to show that if $g(m') \leq f(m')$, then $g(l) \leq f(l)$ for any integer $l$ in $[m, m']$. We put

\[ Df(l) := f(l) - f(l-1) = \sum_{j=1}^{d} [p^{-i} k_j], \quad Dg(l) := g(l) - g(l-1) = n i p^{m'-l}. \]

For any $a \in \mathbb{R}$, we have $p^{-1} \cdot [a] \geq [p^{-1} a]$. Indeed, $p^{-1} a \geq [p^{-1} a]$, and $a \geq p \cdot [p^{-1} a]$. Since $p \cdot [p^{-1} a]$ is an integer, we get what we want by the definition of $[.]$. This implies that $p^{-1} \cdot [p^{-1} k_j] \geq [p^{-1} k_j]$, and thus
\[ p^{-1} \cdot Df(l) \geq Df(l+1). \]
In turn, we have $p^{-1} \cdot Dg(l) = Dg(l+1)$. Suppose there exists an integer $l$ in $[m, m']$ such that $f(l) < g(l)$ and $f(a) \geq g(a)$ for any integer $a$ in $[m, l]$. This shows that $Df(l) < Dg(l)$. Thus, $Df(b) < Dg(b)$ for any $l \leq b$, which implies $f(m') < g(m')$. This contradicts the assumption, and we conclude that $g(l) \leq f(l)$ for any $m \leq l \leq m'$. 

\[ \square \]
3.5. Let $M$ and $M'$ be $p$-torsion free $R$-modules. A $p$-isogeny $\phi: M \to M'$ is an isomorphism

$$\phi_Q : M \otimes \mathbb{Q} \xrightarrow{\sim} M' \otimes \mathbb{Q}$$

such that there exist positive integers $n$ and $n'$ satisfying

$$p^n \cdot \phi_Q(M) \subset M' \subset p^{-n'} \cdot \phi_Q(M).$$

Here $\phi_Q$ is called the realization of the $p$-isogeny. We say that the $p$-isogeny is a homomorphism if we can take $n$ to be 0.

**Lemma.** — Let $M$ and $M'$ be $p$-torsion free $R$-modules, and let $\phi: M \to M'$ be a $p$-isogeny. Then this induces a canonical $p$-isogeny

$$\hat{\phi}: M^\wedge \to M'^\wedge.$$

where $^\wedge$ denotes the $p$-adic completion. If the given $p$-isogeny is a homomorphism, the induced $p$-isogeny is also a homomorphism.

**Proof.** Let $\phi_Q : M \otimes \mathbb{Q} \to M' \otimes \mathbb{Q}$ be the realization of the isogeny. By definition, there exists an integer $n$ such that $p^n \cdot \phi_Q$ induces a homomorphism $M \to M'$. We denote this homomorphism by $\phi_n : M \to M'$. Let $C := \text{Coker}(\phi_n)$. Since $\phi$ is a $p$-isogeny, $C$ is a $\pi'^n$-torsion module for some integer $n' \geq 0$. We have an exact sequence of projective systems

$$0 \to \{C\}_{i \geq n'}' \to \{M \otimes R_i\}_{i \geq n'} \to \{M' \otimes R_i\}_{i \geq n'} \to \{C\}_{i \geq n'} \to 0,$$

where $\{C\}_{i \geq n'}'$ denotes the projective system $\{\cdots \to C \xrightarrow{\pi} C \xrightarrow{\pi} C \to \cdots\}$, and $\{C\}_{i \geq n'}$ is the projective system $\{\cdots \to C \xrightarrow{\text{id}} C \xrightarrow{\text{id}} C \to \cdots\}$. Since any projective system appearing in the short exact sequence above satisfies the Mittag-Leffler condition, the exact sequence induces an exact sequence

$$0 \to \hat{M} \xrightarrow{\widehat{\phi_n}} \hat{M}' \to C \to 0$$

by taking the projective limit. Thus, we get a $p$-isogeny $\hat{\phi}_Q := p^{-n} \cdot \hat{\phi}_n : \hat{M} \otimes \mathbb{Q} \to \hat{M}' \otimes \mathbb{Q}$ as desired. By construction, the homomorphism $\hat{\phi}_Q$ does not depend on the choice of the integer $n$. \[\square\]

Let $\mathcal{M}, \mathcal{M}'$ be $p$-torsion free $R$-modules a topological space $X$. Then exactly in the same way, we can define $p$-isogeny $\phi: \mathcal{M} \to \mathcal{M}'$. Namely, it is a homomorphism of sheaves of modules $\phi_Q: \mathcal{M} \otimes \mathbb{Q} \to \mathcal{M}' \otimes \mathbb{Q}$ such that there exist positive integers $n$ and $n'$ satisfying $p^n \cdot \phi_Q(\mathcal{M}) \subset \mathcal{M}' \subset p^{-n'} \cdot \phi_Q(\mathcal{M})$. We say that the $p$-isogeny is a homomorphism if we can take $n$ to be 0.

3.6. Let $\mathcal{X} = \text{Spf}(A)$ be an affine smooth formal scheme over $R$, and assume that it possesses a system of local coordinates $\{x_1, \ldots, x_d\}$. We identify the ring of global sections of $\mathcal{O}_{\mathcal{T}(\mathcal{X})}$ with $A[\xi_1, \ldots, \xi_d]^{(m)}$ using Lemma 3.2. Let $\Theta$ be a homogeneous element of $A[\xi_1, \ldots, \xi_d]$ whose degree is strictly greater than 0. For a commutative graded ring $\Lambda$ and a homogeneous element $f \in \Lambda$, we denote the submodule of degree $n$ of the graded ring $\Lambda_f$ by $\Lambda_f(n)$. Then by construction of $\mathcal{O}_{\mathcal{T}(\mathcal{X})}$,

$$\Gamma(D_+(\Theta), \mathcal{O}_{\mathcal{T}(\mathcal{X})}(n)) \cong (A[\xi_1, \ldots, \xi_d]_{(\Theta(n))}(m))^{\wedge}$$
Lemma. — Let $b$ restrictions. Moreover, since $n < p$ which is a homomorphism for $p$ for any topological spaces $p$

Using Lemma 3.4 (i)-(b), this homomorphism defines a $p$-isogeny $O$-$isogeny$

Lemma 2.11 (i) and the isomorphism $A[\xi_1, \ldots, \xi_d]^{(m)} \otimes \mathbb{Q} \cong A[\xi_1, \ldots, \xi_d]^{(m')} \otimes \mathbb{Q}$ of Lemma 3.1 induces the following homomorphism.

$$A[\xi_1, \ldots, \xi_d]^{(m')} \to A[\xi_1, \ldots, \xi_d]^{(m)}$$

Using Lemma 3.4 (i)-(b), this homomorphism defines a $p$-isogeny

$$A[\xi_1, \ldots, \xi_d]^{(m')} \to A[\xi_1, \ldots, \xi_d]^{(m)}(n)$$

for any $n \in \mathbb{Z}$. For $n < p^{m+1}$, this $p$-isogeny is moreover a homomorphism by the same lemma. This defines a $p$-isogeny

$$(A[\xi_1, \ldots, \xi_d]^{(m')} \to A[\xi_1, \ldots, \xi_d]^{(m)}(n))^\wedge$$

by Lemma 3.5 Composing this with 3.6.1, we get a canonical $p$-isogeny

$$(A[\xi_1, \ldots, \xi_d]^{(m')} \to A[\xi_1, \ldots, \xi_d]^{(m)}(n))^\wedge$$

which is a homomorphism for $n < p^{m+1}$. By construction, this $p$-isogeny is compatible with restrictions. Moreover, since $b_n$ of Lemma 3.4 does not depend on $\Theta$, this induces a $p$-isogeny of sheaves. Summing up, we obtain the following lemma.

**Lemma.** — Let $m' \geq m$ be non-negative integers. For any $n \in \mathbb{Z}$, there exist canonical $p$-isogenies of sheaves of modules

$$O_{P^{(m')}} (n) \to O_{P^{(m')}}(n), \quad O_{T^{(m')}} (n) \to O_{T^{(m')}}(n)$$

on the topological spaces $P^X$ and $T^X$ respectively. These are homomorphisms for $n < p^{m+1}$.

**3.7 Lemma.** — By using the homomorphism of Lemma 3.6 $O_{P^{(m')}} (n)$ can be seen as an $O_{P^{(m')}} (n)$-algebra. Then $O_{P^{(m')}} (n)$ is a coherent $O_{P^{(m')}} (n)$-algebra.

**Proof.** Let $\Theta$ be a homogeneous element of $A[\xi_1, \ldots, \xi_d]$ whose degree is strictly greater than 0. First of all, let us show that the homomorphism of rings

$$A[\xi_1, \ldots, \xi_d]^{(m')} (0) \to A[\xi_1, \ldots, \xi_d]^{(m)} (0)$$

is finite. By construction of 3.7.1, it suffices to show the finiteness of the homomorphism

$$A[\xi_1, \ldots, \xi_d]^{(m')} (0) \to A[\xi_1, \ldots, \xi_d]^{(m)} (0).$$

Let

$$S := \left\{(k, k', i) \in \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N} \mid k_j < p^{m'} \text{ for any } j, |k'| < \text{ord}(\Theta), \text{ and } \frac{|k| + |k'| p^{m'}}{\text{ord}(\Theta)} \right\}.$$
The condition $|k| + |k'| p^{m'} = ip^{m'} \ord(\Theta)$ means that the order of
\[ \xi(k + k')_{(m',m')}(\Theta^{(m,m')})^{-i} \]
is equal to 0. Obviously, $\# S < \infty$. Let
\[ T := \{ k \in \mathbb{N}^d \mid k \in p^{m'} \mathbb{N}^d, |k| = ip^{m'} \ord(\Theta) \text{ for some integer } i \} . \]
The set $T$ is a submonoid of the commutative monoid $\mathbb{N}^d$. For any $k \in T$, there exists $u \in \mathbb{Z}^*_p$ such that
\[ (3.7.2) \quad \xi(k)_{(m)}(\Theta^{(m,m')})^{-i} = u \cdot \xi(k')_{(m')}(\Theta^{(m')})^{-i} . \]
Let
\[ U := \{ k \in \mathbb{N}^d \mid |k| = ip^{m'} \ord(\Theta) \text{ for some } i \} . \]
This is also a submonoid of $\mathbb{N}^d$. Let
\[ S' := \{ l \in \mathbb{N}^d \mid \text{there exists } (k, k', i) \in S \text{ such that } l = k + p^{m'} k' \} . \]
The monoid $T$ is a submonoid of $U$, and $S'$ is a finite subset of $U$. We claim that $U = T + S'$. Indeed, take $l \in U$. We can write $l = l + p^{m'} k'$ such that $l + k' \in \mathbb{N}^d$ and $i_j < p^{m'}$ for any $j$. Now, there exists $k'$ such that $|k'| < \ord(\Theta)$, $i'_j \geq k_j$ for any $j$, and
\[ |l'| - |k'| = \left| l' \cdot (\ord(\Theta))^{-1} \right| \cdot \ord(\Theta) \]
where $[\alpha]$ denotes the maximum integer less than or equal to $\alpha$. We put $k := l$. Then there exists an integer $i$ such that $|l| + p^{m'} |k'| = ip^{m'} \ord(\Theta)$. By construction $(k, k', i) \in S$, and $p^{m'} \cdot (l' - k') \in T$. Since $l = p^{m'} \cdot (l' - k') + (k + p^{m'} k')$, the claim follows. Considering (3.7.2), this implies that the homomorphism
\[ \bigoplus_{l \in S} \mathcal{A}[\xi_1, \ldots, \xi_d]_{(\Theta^{(m')})}(0) \rightarrow \mathcal{A}[\xi_1, \ldots, \xi_d]_{(\Theta^{(m,m')})}(0) \]
sending 1 sitting at the $(k, k', i) \in S$ component to $\xi(k + k')_{(m',m')}(\Theta^{(m,m')})^{-i}$ is surjective. Thus the homomorphism (3.7.1) is finite.

Let us see the coherence. Let $\Xi$ be another homogeneous element of $\mathcal{A}[\xi_1, \ldots, \xi_d]$ whose degree is strictly greater than 0. Let $\mathcal{W}$ be the affine open subset of $P^* \mathcal{X}$ defined by $\Theta$, and $\mathcal{W}'$ by that of $\Xi$. $\Theta$. It suffices to show that the canonical homomorphism
\[ (3.7.3) \quad \Gamma(\mathcal{W}, \mathcal{O}_{P^*(\mathcal{X})}) \otimes \Gamma(\mathcal{W}', \mathcal{O}_{P^*(\mathcal{X})}) \rightarrow \Gamma(\mathcal{W}', \mathcal{O}_{P^*(\mathcal{X})}) \]
is an isomorphism. By changing $\Theta$ and $\Xi$ to some powers of $\Theta$ and $\Xi$ respectively, we may assume that $\ord(\Xi) = \ord(\Theta)$. We put
\[ \mathcal{A} := \mathcal{A}[\xi_1, \ldots, \xi_d]_{(\Theta^{(m')})}(0), \quad \Psi := \Xi^{(m')}/\Theta^{(m')}, \]
\[ \mathcal{B} := \mathcal{A}[\xi_1, \ldots, \xi_d]_{(\Theta^{(m,m')})}(0), \quad \Phi := \Xi^{(m')}/\Theta^{(m)}, \quad \Phi' := \Xi^{(m,m')}/\Theta^{(m,m')} . \]
Let $\phi: \mathcal{A} \rightarrow \mathcal{B}$ be the canonical homomorphism. Firstly, $\Phi' = \phi(\Psi)$ in $\mathcal{B}$ by Lemma 2.11 (i). Secondly,
\[ (\mathcal{B}_{\phi})^\wedge \cong (\mathcal{B}_{\Psi})^\wedge \]
by the same reason as (3.6.1). Thirdly,
\[ \mathcal{B}_{\phi}(\Psi) \cong \mathcal{B} \otimes_{\mathcal{A}} \mathcal{A}_{\Psi} . \]
Combining these, $(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{A}_{\Psi})^\wedge \cong (\mathcal{B}_{\Psi})^\wedge$. This is saying that the $p$-adic completion of the left hand side of (3.7.3) is isomorphic to the right hand side. However, by the finiteness of (3.7.1), the left hand side of (3.7.3) is already $p$-adically complete by [EGA 01, 7.3.6], and as a result, (3.7.3) is an isomorphism. Thus we obtain the lemma. 

\[ \blacksquare \]
3.8. Let \( \mathcal{X} \) be an affine smooth formal scheme over \( R \) possessing a system of local coordinates. For any \( n \in \mathbb{Z} \), the module \( \mathcal{O}_{T^*(m), \mathcal{X}}(n) \) is an \( \mathcal{O}_{T^*(m), \mathcal{X}}(0) \)-module on \( T^* \mathcal{X} \), and by using Lemma 3.6, \( \mathcal{O}_{T^*(m), \mathcal{X}}(n) \) can be seen as an \( \mathcal{O}_{T^*(m'), \mathcal{X}}(0) \)-module.

**Corollary.** The \( \mathcal{O}_{T^*(m'), \mathcal{X}}(0) \)-module \( \mathcal{O}_{T^*(m), \mathcal{X}}(n) \) is coherent.

**Remark.** We will see in Lemma 4.2 that the corollary holds for any smooth formal scheme \( \mathcal{X} \) not necessary affine.

4. Intermediate microdifferential sheaves

4.1. In section 2 we defined the naive ring of microdifferential operators. However, we do not have any natural homomorphism \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \to \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m+1)} \). This can be seen from the following example. Suppose we had a homomorphism \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \to \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m+1)} \) compatible with the canonical homomorphism \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \to \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m+1)} \). Then, for any coherent \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \)-module \( \mathcal{M} \), we would get

\[
\text{Char}(\mathcal{M}) \supset \text{Char}(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m+1)} \otimes \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \mathcal{M})
\]

by Proposition 2.13. However, this does not hold by Lemma 4.1 below. To remedy this situation, we will consider the intermediate ring of microdifferential operators denoted by \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m, m')} \) for \( m' \geq m \), which is an “intersection” of \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \) and \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \). In this section, we will define these rings and prove some basic properties.

**Lemma.** Let \( \mathcal{X} := \hat{A}_R^1 \), \( X \) be the special fiber, \( x \) be the canonical coordinate, and \( \partial \) be the corresponding differential operator. We put \( \mathcal{M} = \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(0)} \otimes \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(0)} (\partial - x) \). Then,

1. \( \text{Char}^{(0)}(\mathcal{M}) = X \).
2. \( \text{Char}^{(1)}(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(1)} \otimes \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(0)} \mathcal{M}) \cap T^* X \neq \emptyset \).

**Proof.** Since \( \mathcal{M} \) is a coherent \( \mathcal{O}_{\mathcal{X}, \mathbb{Q}} \)-module, the first claim follows. Let us see the second claim. First, let us see that \( \mathcal{D}_{\mathcal{X}, \mathbb{Q}}^1 \otimes \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(0)} \mathcal{M} \neq 0 \). Let \( f_n \in K\{x\} \) (i.e. the Tate algebra), and \( \sum_{n \geq 0} f_n \partial^{[n]} \in \Gamma(\mathcal{X}, \mathcal{D}_{\mathcal{X}, \mathbb{Q}}^1) \). We get

\[
\sum_{n \geq 0} f_n \partial^{[n]} \cdot (\partial - x) = \sum_{n \geq 0} \left( f_n \partial^{[n]} \partial - x f_n \partial^{[n]} - f_n \partial^{[n-1]} \right) = \sum_{n \geq 0} \left( n f_n x - f_n - f_n x \right) \partial^{[n]}.
\]

Assume \( \sum_{n \geq 0} f_n \partial^{[n]} \cdot (\partial - x) = 1 \). Then there exist \( g_n, h_n \in K[x] \), \( \deg(g_n) < n - 1 \) and \( \deg(h_n) < n \), such that the equality

\[
(-1)^n f_n = (x^{n-1} + g_n) + (x^n + h_n) \cdot f_0
\]

should hold for any \( n > 0 \). However, there is no \( f_0 \in K\{x\} \) such that \( \sum_{n \geq 0} f_n \partial^{[n]} \in \Gamma(\mathcal{X}, \mathcal{D}_{\mathcal{X}, \mathbb{Q}}^1) \) (since \( |f_n| = \max\{1, |f_0|\} \) by the equality), and \( \mathcal{D}_{\mathcal{X}, \mathbb{Q}}^1 \otimes \mathcal{M} \neq 0 \).

Now, let \( e \) be the element of \( \Gamma(\mathcal{X}, \mathcal{M}) \) defined by \( 1 \in \Gamma(\mathcal{X}, \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(0)}) \). As an \( \mathcal{O}_{\mathcal{X}, \mathbb{Q}} \)-module, \( \mathcal{M} \) is free of rank 1. Since

\[
\partial^n \cdot e = (x^n + (\text{polynomial in } K[x] \text{ whose degree is less than } n)) \cdot e,
\]

by using Lemma 4.1 below.
we see that the $\widehat{D}^{(0)}_{X,\mathbb{Q}}$-module structure on $\mathcal{M}$ does not extend continuously to a $\widehat{D}^{(1)}_{X,\mathbb{Q}}$-module structure. This shows that the canonical homomorphism $\mathcal{M} \to \mathcal{M}^{(1)} := \widehat{D}^{(1)}_{X,\mathbb{Q}} \otimes \mathcal{M}$ is not an isomorphism.

Garnier showed in [Ga, 5.2.4] that for any coherent $\widehat{D}^{(0)}_{X,\mathbb{Q}}$-module $\mathcal{M}$, the characteristic variety \text{Char}^{(0)}(\mathcal{M}) satisfies the Bernstein inequality (i.e. the dimension of the characteristic variety is greater than or equal to 1 unless $\mathcal{M} = 0$). Using the relation of characteristic varieties of Frobenius descents (cf. [BeI, 5.2.4 (iii)], and for a similar result and its proof, see [Ab, 2.1.7]), the Bernstein inequality also holds for any coherent $\widehat{D}^{(m)}_{X,\mathbb{Q}}$-modules. Thus there are three possibilities for the characteristic variety $V$ of $\mathcal{M}^{(1)}$: either $0$ or $[X]$ or $V \cap \tilde{T}^*X \neq 0$. Since $\mathcal{M}^{(1)}$ is not 0, $V$ is not empty. If $V = [X]$, $\mathcal{M}^{(1)}$ would be a coherent $O_{X,\mathbb{Q}}$-module, and since $\mathcal{M}$ is a coherent $O_{X,\mathbb{Q}}$-module of rank 1, we would get that $\mathcal{M} \cong \mathcal{M}^{(1)}$, which is a contradiction. Thus the lemma follows.  

\textbf{Lemma.} — There exists a unique filtered strictly injective homomorphism of rings

\begin{equation}
\phi_{m',m}: E_{X,\mathbb{Q}}^{(m')} \to E_{X,\mathbb{Q}}^{(m)}
\end{equation}

such that the following diagram is commutative:

\begin{equation}
\begin{array}{ccc}
\pi^{-1}E_{X,\mathbb{Q}}^{(m')} & \xrightarrow{\phi_{m',m} \otimes \mathbb{Q}} & \pi^{-1}E_{X,\mathbb{Q}}^{(m)} \\
\varphi_{m'} & & \varphi_{m}
\end{array}
\end{equation}

where we refer to subsection 2.2.4 for $\varphi_m$. For $m'' \geq m' \geq m$, $\psi_{m,m'} \circ \psi_{m',m''} = \psi_{m,m''}$. By using (4.2.1), $\text{gr}_n(\psi_{m,m'})$ can be identified with the p-isogeny in Lemma 3.6 locally.

\textbf{Proof.} Once the existence and the uniqueness is proven, the compatibility $\psi_{m,m'} \circ \psi_{m',m''} = \psi_{m,m''}$ automatically holds by the compatibility of $\phi_{m',m}$. Let us see the uniqueness first. Since the problem is local, we may assume that $X$ possesses a system of local coordinates, and it suffices to show the claim for the ring of sections on $D(\Theta)$ where $\Theta \in \Gamma(\mathcal{X},\mathcal{O}_{X^\nu})$ is a homogeneous element. Suppose there are two homomorphisms $\psi, \psi'$ satisfying the condition. By the commutativity of the diagram, $\psi((\Theta_{t}^{(m')})^{-1}) = \psi'((\Theta_{t}^{(m')})^{-1})$. Since $\psi$ and $\psi'$ are filtered homomorphisms, these homomorphisms are continuous with respect to the topology defined by the filtrations (cf. 1.1.1). Let $E$ be the subring of $E^{(m')} := \Gamma(D(\Theta), E_{X,\mathbb{Q}}^{(m')})$ generated by $\Gamma(D(\Theta), E_{X,\mathbb{Q}}^{(m')})$ and $(\Theta_{t}^{(m')})^{-1}$. Then, $\psi|_E = \psi'|_E$. Since $\Gamma(D(\Theta), E_{X,\mathbb{Q}}^{(m')})$ is separated and $E$ is dense in $E^{(m')}$, we get $\psi = \psi'$, and the uniqueness follows.
Now, let us see the existence. Since the problem is local by the uniqueness, we may suppose that \( \mathcal{X} \) is affine. Let \( \Theta \) be a homogeneous element of \( \Gamma(\mathcal{X}, \mathcal{O}_{T^* \mathcal{X}}) \). It suffices to prove the claim on \( \mathcal{U} := D(\Theta) \). We denote \( \Gamma(\mathcal{U}, \mathcal{E}(m)_{\mathcal{X}, (\mathbb{Q})}) \) by \( E(m)_{\mathcal{X}, (\mathbb{Q})} \) and so on. Let \( (E(m'))' \) be the microlocalization of \( D(m') \) by using the multiplicative set of \( \text{gr}(D(m')) \) generated by \( \Theta(m') \in \text{gr}(D(m')) \) (cf. \[\text{3}\]), and let \( (E'_Q(m))' \) be the completion of \( E(m) \) with respect to the filtration by order. Note that the canonical homomorphism \( E(m)_{\mathcal{X}, (\mathbb{Q})} \rightarrow (E'_Q(m))' \) is injective, since the filtration is separated. Since \( \overline{\Theta}_{\mathcal{U}}(m,m) \) is invertible in \( (E'_Q(m))' \), \( \overline{\Theta}_{\mathcal{U}}(m,m) \) is also invertible by Lemma \[\text{2.11} \ (i) \]. Thus, by the universal property of the microlocalization \( \text{Lau} \ A.2.3.3 \), there exists a unique homomorphism \( \alpha: (E(m'))' \rightarrow (E(m))' \). Let \( (E(m'))'_n \) be the \( p \)-adic completion of \( (E(m'))' \). Then \( \beta_n: (E(m'))'_{n} \rightarrow E(m') \). Indeed, by Lemma \[\text{2.3} \]

\[
(E(m'))'_{n} \otimes R_{i} \cong \Gamma(X_i, \mathcal{E}(m')_{X_i,n}).
\]

Since \( E(m') \) is \( p \)-adically complete, the claim follows. For any \( n \), there exists an integer \( N \) such that the homomorphism \( p^N \cdot \alpha_n \) induces a homomorphism \( (E(m'))'_{n} \rightarrow E(m) \) by the concrete description Lemma \[\text{2.10} \] and Lemma \[\text{3.4} \] (i)-(b). Since \( E(m) \) is \( p \)-adically complete, this induces the homomorphism \( (p^N \cdot \alpha_n)^\wedge: (E(m'))'_{n} \rightarrow E(m) \). We define

\[
\hat{\alpha}_n := p^{-N} \cdot (p^N \cdot \alpha_n)^\wedge: (E(m'))'_{n} \rightarrow E(m) \otimes \mathbb{Q}.
\]

By construction, \( \{\hat{\alpha}_n\}_{n \in \mathbb{Z}} \) is compatible with each other, and defines

\[
\lim_n (\alpha_n \circ \beta_n^{-1}): E(m') \rightarrow E(m) \otimes \mathbb{Q},
\]

which is what we are looking for.

Finally, let us see that \( \psi_{m,m'} \) is strictly injective. By construction, locally, \( \text{gr}_n(\psi_{m,m'}) \) coincides with the \( p \)-isogeny of Lemma \[\text{3.6} \] for any \( n \). This implies that the canonical homomorphism \( \text{gr}(\mathcal{E}(m')) \rightarrow \text{gr}(\mathcal{E}(m)) \) is injective. Since \( \mathcal{E}(m') \) is separated with respect to the filtration by order, we get the result by \[\text{H}, \text{O} \] Ch.I, 4.2.4 (5).

4.3. We preserve the notation. For non-negative integers \( m' \geq m \), we define

\[
\mathcal{E}(m',m):= \psi^{-1}_{m,m'}(\mathcal{E}(m')) \cap \mathcal{E}(m')
\]

where the intersection is taken in \( \mathcal{E}(m')_{\mathcal{X}, \mathbb{Q}} \). By definition, \( \mathcal{E}(m,m) = \mathcal{E}(m') \). We denote \( \mathcal{E}(m',m) \otimes R_i \) by \( \mathcal{E}(m',m)_{n} \).

Let \( \mathcal{U} \) be an open subset of \( T^* \mathcal{X} \). We denote \( \Gamma(\mathcal{U}, \mathcal{E}(m')_{\mathcal{X}, (\mathbb{Q})}) \) by \( E_{\mathcal{U}, (\mathbb{Q})} \). Then the left exactness of the functor \( \Gamma \) implies that

\[
\Gamma(\mathcal{U}, \mathcal{E}(m',m)) \cong \psi^{-1}_{m,m'}(E_{\mathcal{U}}(m')) \cap E_{\mathcal{U}}(m') \subset E_{\mathcal{U}}(m').
\]

Since \( \psi_{m,m'}(\mathcal{E}(m')) \) and \( \mathcal{E}(m) \) are sub-\( \pi^{-1} \mathcal{G}(m) \)-algebras of \( \mathcal{E}(m)_{\mathcal{X}, \mathbb{Q}} \), the ring \( \mathcal{E}(m',m) \) is also a \( \pi^{-1} \mathcal{G}(m) \)-algebra on \( T^* \mathcal{X} \). Moreover, by putting

\[
\mathcal{E}(m',m)_{n} := \psi^{-1}_{m,m'}(\mathcal{E}(m)_{n}) \cap \mathcal{E}(m')_{n}
\]

we may equip \( \mathcal{E}(m',m) \) with a filtration, and we consider \( \mathcal{E}(m',m) \) as a filtered ring. By Lemma \[\text{4.2} \] \( \psi_{m,m'} \) is a strict homomorphism, and the canonical homomorphisms of filtered rings

\[
(\mathcal{E}(m',m)_{n}, \mathcal{E}(m',m)_{n}) \rightarrow (\mathcal{E}(m)_{n}, \mathcal{E}(m)_{n}), \quad (\mathcal{E}(m',m), \mathcal{E}(m',m)_{n}) \rightarrow (\mathcal{E}(m), \mathcal{E}(m)_{n})
\]

we have
are also strict injective homomorphisms. By the explicit presentation Lemma 2.10 and Lemma 3.4 (i)-(b), \( \psi_{m,m'}(\mathcal{E}_{\mathcal{X},n}^{(m)}) \subset \mathcal{E}_{\mathcal{X},n}^{(m)} \) for \( n < p^{m+1} \), and in particular

\[
\mathcal{E}_{\mathcal{X},0}^{(m,m')} = \mathcal{E}_{\mathcal{X},0}^{(m')}
\]

**4.4 Example.** — Let \( \mathcal{X} \) be a smooth affine formal scheme over \( R \). Let \( \Theta \) be a homogeneous section of \( \mathcal{O}_{T^* \mathcal{X}} \), and we put \( \mathcal{U} := D(\Theta) \). For any \( m'' \geq m' \geq m \), the operator \( (\tilde{\Theta}_{\mathcal{X}}^{(m',m'')})^{-1} \) is contained in \( \Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}^{(m)}) \). Indeed, there exist a non-negative integer \( n \) and a unit \( u \) of \( R \) such that \( \tilde{\Theta}_{\mathcal{X}}^{(m',m'')} = u p^{-n} \cdot \tilde{\Theta}_{\mathcal{X}}^{(m,m'')} \), and thus

\[
(\tilde{\Theta}_{\mathcal{X}}^{(m',m'')})^{-1} = u^{-1} p^{n} \cdot (\tilde{\Theta}_{\mathcal{X}}^{(m,m'')})^{-1} \in \Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}^{(m)}).
\]

This is showing that for integers \( M' \geq M \geq m' \geq m \), the operator \( (\tilde{\Theta}_{\mathcal{X}}^{(M,M')})^{-1} \) is contained in \( \Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}^{(m,m')}) \).

In turn, for any integers \( m'' \geq m' \), the operator \( (\tilde{\Theta}_{\mathcal{X}}^{(m',m'')})^{-1} \) is contained in \( \Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}^{(m)}) \).

Thus, for non-negative integers \( M' \geq M \) and \( M' \geq m' \geq m \), the operator \( (\tilde{\Theta}_{\mathcal{X}}^{(M,M')})^{-1} \) is contained in \( \Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}^{(m,m')}) \).

**4.5.** Let \( \iota: \mathcal{E}_{\mathcal{X}}^{(m)} \rightarrow \mathcal{E}_{\mathcal{X}}^{(m)} \) be the canonical inclusion. Take an integer \( n \). Consider the following diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{E}_{\mathcal{X},n-1}^{(m,m')} & \rightarrow & \mathcal{E}_{\mathcal{X},n-1}^{(m')} & \oplus & \mathcal{E}_{\mathcal{X},n-1}^{(m)} & \rightarrow & (\mathcal{E}_{\mathcal{X},Q}^{(m)})_{n-1} \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{E}_{\mathcal{X},n}^{(m,m')} & \rightarrow & \mathcal{E}_{\mathcal{X},n}^{(m')} & \oplus & \mathcal{E}_{\mathcal{X},n}^{(m)} & \rightarrow & (\mathcal{E}_{\mathcal{X},Q}^{(m)})_{n} \\
\end{array}
\]

whose rows are exact sequences.

**Lemma.** — The following diagram is cartesian

\[
\begin{array}{cc}
\text{Im}((\psi_{m,m'})_{n-1}, \iota_{n-1}) & \rightarrow (\mathcal{E}_{\mathcal{X},Q}^{(m)})_{n-1} \\
\downarrow & \downarrow \\
\text{Im}((\psi_{m,m'})_{n}, \iota_{n}) & \rightarrow (\mathcal{E}_{\mathcal{X},Q}^{(m)})_{n} \\
\end{array}
\]

In other words, the following sequence is exact:

\[
0 \rightarrow \text{Im}((\psi_{m,m'})_{n-1}, \iota_{n-1}) \rightarrow (\mathcal{E}_{\mathcal{X},Q}^{(m)})_{n-1} \oplus \text{Im}((\psi_{m,m'})_{n}, \iota_{n}) \rightarrow (\mathcal{E}_{\mathcal{X},Q}^{(m)})_{n},
\]

where the last homomorphism is \( a + b \).

**Proof.** Since the statement is local, we may suppose that \( \mathcal{X} \) is affine and possesses a system of local coordinates. Moreover, it suffices to show that the diagram is cartesian for the modules of sections on \( \mathcal{U} := D(\Theta) \) where \( \Theta \) is a homogeneous section of \( \Gamma(T^* \mathcal{X}, \mathcal{O}_{T^* \mathcal{X}}) \). We denote \( \Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}^{(m,m')}) \) by \( E_{\mathcal{X}}^{(m,m')} \) and so on.

An operator \( P \) of \( \mathcal{E}_{\mathcal{X},Q}^{(m)} \) is said to be homogeneous of degree \( n \) if we can write \( P = \sum_{|k|=n} a_k \partial_{k}^{(m)} \) with \( a_k \in \mathcal{O}_{\mathcal{X},Q} \). Take \( S \in \text{Im}((\psi_{m,m'})_{n}, \iota_{n}) \). Using a left presentation, we can write

\[
S := \psi_{m,m}(\sum_{k \leq n} P_{k,i}(\Theta_{l}^{(m')})^{i}) + \sum_{k \leq n} Q_{k,i}(\Theta_{l}^{(m,m')})^{i}
\]
where the first summation is an element of \( E_{n}^{(m')} \) and the second summation is one of \( E_{n}^{(m)} \), and \( P_{k,i}, Q_{k,i} \) are operators homogeneous of degree \( k - ip'' \) \( \text{ord}(\Theta) \) in \( \mathcal{D}_{\mathcal{X},\mathcal{Q}}^{(m')} \) and \( \mathcal{D}_{\mathcal{Y},\mathcal{Q}}^{(m)} \) respectively. Suppose \( S \in (E_{Q}^{(m)})_{n-1} \). We need to show that this element is contained in \( \text{Im}(\psi_{m,m'}^{(n-1),m-1,\ell-1}) \). There exists a finite subset \( I \subset \mathbb{Z} \) such that

\[
\psi_{m,m'}(\sum_{i \in I} P_{n,i}(\tilde{\Theta}_{i}^{(m)})) \in E_{n}^{(m)}.
\]

This is in fact contained in \( E_{n}^{(m)} \cap \psi_{m,m'}(E_{n}^{(m)}) \). Then for \( N > 0 \), there exists \( a_{\underline{k}} \in \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \) for \( |\underline{k}| = n - np'' \) \( \text{ord}(\Theta) = M \) such that

\[
\sum_{i} P_{n,i}(\tilde{\Theta}_{i}^{(m')}) \in \sum_{|\underline{k}| = M} a_{\underline{k}}\tilde{\Theta}_{i}^{(m')}N + E_{n-1}^{(m')} + (\psi_{m,m'}^{(n)}(E_{n}^{(m)}) \cap E_{n}^{(m')})
\]

and \( a_{\underline{k}}\tilde{\Theta}_{i}^{(m')}N \notin \psi_{m,m'}^{(n)}(E_{n}^{(m)}) \) for any \( |\underline{k}| = M \) such that \( a_{\underline{k}} \neq 0 \). We denote \( \psi_{m,m'}^{(n)}(E_{n}^{(m)}) \) by \( (E_{n}^{(m)})_{m}' \) for short. By the same argument for \( E_{n}^{(m)} \) and increasing \( N \) if necessary, we may also suppose that there exists \( b_{\underline{k}} \in \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \) for \( |\underline{k}| = M \) such that

\[
\sum_{i} Q_{n,i}(\tilde{\Theta}_{i}^{(m',m)}) \in \sum_{|\underline{k}| = M} b_{\underline{k}}\tilde{\Theta}_{i}^{(m',m)}N + E_{n}^{(m)} + ((E_{n}^{(m')})_{m}' \cap E_{n}^{(m)})
\]

in \( E_{Q}^{(m)} \). However, since \( S \in (E_{Q}^{(m)})_{n-1} \),

\[
\sum_{|\underline{k}| = M} a_{\underline{k}}\tilde{\Theta}_{i}^{(m')}N + b_{\underline{k}}\tilde{\Theta}_{i}^{(m',m)}N \in (E_{Q}^{(m)})_{n-1} + ((E_{n}^{(m')})_{m}' \cap E_{n}^{(m)}).
\]

By the choice of \( a_{\underline{k}} \) and \( b_{\underline{k}} \), this is possible only when \( a_{\underline{k}} = b_{\underline{k}} = 0 \), and the lemma is proven. \( \blacksquare \)

**Corollary.** — We have

\[
\text{gr}(E_{\mathcal{X}}^{(m,m')}) = \text{gr}(\psi_{m,m'}^{-1}(\text{gr}(E_{\mathcal{X}}^{(m)}))) \cap \text{gr}(E_{\mathcal{X}}^{(m')}).
\]

where the intersection is taken in the ring

\[
\text{gr}(E_{\mathcal{X}}^{(m)}) \equiv \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_{T^{*} \mathcal{X},Q}(i) = \mathcal{O}_{T^{*} \mathcal{X},Q}(*).
\]

**4.6 Lemma.** — There exist strict injective homomorphisms of filtered \( \pi^{-1} \mathcal{D}_{\mathcal{X},\mathcal{Q}}^{(m-1)} \)-algebras \( E_{\mathcal{X}}^{(m-1,m')} \rightarrow E_{\mathcal{X}}^{(m,m')} \) and \( E_{\mathcal{X}}^{(m,m'+1)} \rightarrow E_{\mathcal{X}}^{(m,m')} \) for \( m' \geq m \) making the following diagram commutative.

\[
\begin{array}{ccc}
E_{\mathcal{X}}^{(m-1,m')} & \longrightarrow & E_{\mathcal{X}}^{(m,m'+1)} \\
\downarrow & & \downarrow \\
E_{\mathcal{X}}^{(m-1,m')} & \longrightarrow & E_{\mathcal{X}}^{(m,m')}
\end{array}
\]

**Proof.** We will show that the canonical strict injection \( E_{\mathcal{X}}^{(m-1,m')} \rightarrow E_{\mathcal{X},Q}^{(m-1)} \) factors through the composition \( E_{\mathcal{X}}^{(m,m')} \rightarrow E_{\mathcal{X}}^{(m)} \rightarrow E_{\mathcal{X},Q}^{(m-1)} \). We consider \( E_{\mathcal{X}}^{(m-1,m')} \) and \( E_{\mathcal{X}}^{(m,m')} \) as subrings of \( E_{\mathcal{X},Q}^{(m-1)} \) using these injections. We may assume \( \mathcal{X} \) is affine, and let \( \mathcal{U} := D(\Theta) \) where \( \Theta \) is a
homogeneous section of $\Gamma(T^*X, \mathcal{O}_{T^*X})$. It suffices to show that $E^{(m-1,m')} := \Gamma(\mathcal{U}, \mathcal{E}_{X}^{(m-1,m')})$ is contained in $\Gamma(\mathcal{U}, \mathcal{E}_{X}^{(m,m')}) =: E^{(m,m')}$. Take $P \in (E^{(m-1,m')}_\mathcal{U})_N$ for some integer $N$. We inductively define $P_i \in (E^{(m-1,m')}_\mathcal{U})_{N-i}$ and $Q_i \in (E^{(m,m')}_\mathcal{U})_{N-i}$ such that $P_{i+1} = P_i - Q_i$ for $i \geq 0$. Put $P_0 := P$. Assume $P_i$ is constructed. We can write

$$\sigma(P_i) = \sum_{|k|=N-i} a_k \xi(k)(\Theta_i^{(m-1,m')})^n$$

with $a_k \in \Gamma(\mathcal{U}, \mathcal{O}_{T^*X})$ and $n \in \mathbb{Z}$. By Corollary 4.3, this is contained in $\text{gr}(E^{(m-1)}_\mathcal{U})$ and $\text{gr}(E^{(m,m')}_\mathcal{U})$. Thus, Lemma 3.4 (ii) is showing that

$$Q_i := \sum_{|k|=N-i} a_k \xi(k)(\Theta_i^{(m-1,m')})^n$$

is in $E^{(m,m')}_\mathcal{U}$. By construction, $P_{i+1} = P_i - Q_i$ is contained in $(E^{(m-1,m')}_\mathcal{U})_{N-i-1}$. The filtered ring $E^{(m,m')}_\mathcal{U}$ is complete by [4.3.1] and Lemma 2.5 (iv). Thus, $P = \sum_{i \geq 0} Q_i \in E^{(m,m')}_\mathcal{U}$. The rest of the claim follows in the same way. \(\blacksquare\)

4.7 Lemma. — For any $n \in \mathbb{Z}$, $\text{gr}_n(\mathcal{E}_{X}^{(m,m')})$ is a coherent $\mathcal{O}_{T^{(m')}X}(0)$-module on $\check{T}^*X$. Moreover, on $T^*X, \bigoplus_{i \geq 0} \text{gr}_i(\mathcal{E}_{X}^{(m,m')})$ and $\text{gr}(\mathcal{E}_{X}^{(m,m')})$ are of finite type over $\mathcal{O}_{T^{(m')}X}(0)$, and they are noetherian.

Proof. The modules $\text{gr}_n(\mathcal{E}_{X}^{(m)})$ and $\text{gr}_n(\mathcal{E}_{X}^{(m')})$ are coherent $\mathcal{O}_{T^{(m')}X}(0)$-modules on $\check{T}^*X$ by Corollary 3.8. Since

$$\text{gr}_n(\mathcal{E}_{X}^{(m)}) \simeq \mathcal{O}_{T^{(m')}X}(n) = \sum_{i \geq 0} p^{-i} \mathcal{O}_{T^{(m')}X}(n),$$

$\text{gr}_n(\mathcal{E}_{X}^{(m')})$ is a pseudo-coherent $\mathcal{O}_{T^{(m')}X}(0)$-module (cf. [KK, Definition 1.1.6]). Since the intersection of two coherent modules in a pseudo-coherent module is coherent, $\text{gr}_n(\mathcal{E}_{X}^{(m,m')})$ is also a coherent $\mathcal{O}_{T^{(m')}X}(0)$-module for any $n \in \mathbb{Z}$ by Corollary 4.5.

Claim. — Let $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$ be a finitely generated $\mathcal{O} := \mathcal{O}_{T^{(m')}X}(0)$-module such that $\mathcal{A}_i$ are coherent $\mathcal{O}$-modules for all $i$. Let $\mathcal{B} := \bigoplus_{i \in \mathbb{Z}} \mathcal{B}_i$ be a sub-$\mathcal{O}$-algebra of $\mathcal{A}$ such that $\mathcal{B}_i \subset \mathcal{A}_i$ and $\mathcal{B}_i$ are coherent $\mathcal{O}$-modules for all $i$. Assume $\mathcal{A}$ is finite over $\mathcal{B}$. Then $\mathcal{B}$ is a noetherian ring and finitely generated over $\mathcal{O}$.

The proof is similar to that of Lemma 2.7 using [AtMa, 7.8]. For any integer $m'' \geq 0$, $\bigoplus_{i \geq 0} \text{gr}_i(\mathcal{E}_{X}^{(m,m')})$ and $\bigoplus_{i \geq 0} \text{gr}_i(\mathcal{E}_{X}^{(m,m')})$ are finitely generated $\mathcal{O}_{T^{(m')}X}(0)$-algebras by Lemma 2.4. $\mathcal{O}_{T^{(m')}X}(0)$ is a coherent $\mathcal{O}_{T^{(m')}X}(0)$-module on $\check{T}^*X$ by Corollary 3.8 and $\bigoplus_{i \geq 0} \text{gr}_i(\mathcal{E}_{X}^{(m)})$ is finite over $\bigoplus_{i \geq 0} \text{gr}_i(\mathcal{E}_{X}^{(m,m')})$ by Lemma 3.3 (i)-(a). Using the claim, this is showing that $\bigoplus_{i \geq 0} \text{gr}_i(\mathcal{E}_{X}^{(m,m')})$ and $\text{gr}(\mathcal{E}_{X}^{(m,m')})$ are finitely generated over $\mathcal{O}_{T^{(m')}X}(0)$. Thus, using the claim again, they are noetherian, and the lemma follows. \(\blacksquare\)

4.8 Proposition. — (i) The filtered ring $(\mathcal{E}_{X}^{(m,m')}, \mathcal{E}_{X}^{(m,m')}_{n})$ is complete.

(ii) The rings $\mathcal{E}_{X,0}^{(m,m')}$ and $\mathcal{E}_{X,n}^{(m,m')}$ are noetherian on $\check{T}^*X$.

(iii) For open subsets $\check{T}^*X \supset \mathcal{U} \supset \mathcal{V}$ in $\mathcal{B}$, the restriction homomorphism

$$\Gamma(\mathcal{U}, \mathcal{E}) \to \Gamma(\mathcal{V}, \mathcal{E})$$

is flat. Here $\mathcal{E}$ denotes $\mathcal{E}_{X,0}^{(m,m')}$ or $\mathcal{E}_{X}^{(m,m')}$. 

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Proof. We have already used (i) but we rewrite the statement because of the importance. This follows from the fact that \( E^{(m,n)} = E^{(m)} \), for any \( n < p^{m+1} \) by (4.3.1) and Lemma 2.5 (iv). We also get (i) through (iii) for \( E = E^{(m,n)} \) by Proposition 2.5.

Let us see (ii) for \( E^{(m,n)} \). Let us check the conditions of Lemma 1.11 for the filtered ring \( (E^{(m,n)}', E^{(m,n)}) \). The first condition is nothing but (i). The second and the third conditions follow from Lemma 4.7. The last condition follows from Corollary 4.5. Hence \( E^{(m,n)} \) is a noetherian ring. Thus, for any open subscheme \( Y \) in \( \mathfrak{B} \), \( \Gamma(Y, E^{(m,n)}) \) is noetherian, and (iii) holds by using [HO] Ch.II, 1.2.1.

Remark. — By the proof, we can moreover say that \( E^{(m,n)} \) is pointwise Zariskian with respect to the filtration by order on \( T^*\mathcal{X} \). This implies that \( E^{(m,n)} \) is also pointwise Zariskian with respect to the filtration by order for any integer \( i \geq 0 \) on \( T^*\mathcal{X} \).

4.9 Definition. — We say an open subscheme \( Y \) of \( T^*\mathcal{X} \) is strictly affine if there exists an affine open subscheme \( Y' \) of \( P^*\mathcal{X} \) such that \( Y = q^{-1}(Y') \) where \( q: T^*\mathcal{X} \to P^*\mathcal{X} \) is the canonical morphism.

4.10 Lemma. — Let \( \mathcal{X} \) be an affine smooth formal scheme. Let \( \mathcal{U} \) be a strictly affine open subscheme of \( T^*\mathcal{X} \), and let \( \mathfrak{U} \) be a finite \( \mathfrak{B} \)-covering (i.e. a covering consisting of subsets in \( \mathfrak{B} \)) of \( \mathcal{U} \). Let \( E \) be either \( E^{(m,n)} \) or \( E^{(m,n)}_Q \). Then

\[
\check{H}^i_{\text{aug}}(\mathfrak{U}, E) = 0
\]

for \( i \in \mathbb{Z} \). Here \( \check{H}^i_{\text{aug}} \) denotes the augmented Čech cohomology (cf. [BGR 8.1.3]). In particular,

\[
H^1(\mathcal{U}, E) = 0.
\]

Proof. Let \( V, W \in \mathfrak{B} \). Since \( \epsilon(V \cap W) = \epsilon(V) \cap \epsilon(W) \), we may assume that \( \mathfrak{U} \) is a finite strictly affine covering. Let us show that \( \check{H}^i_{\text{aug}}(\mathfrak{U}, E^{(m,n)}_{\mathcal{X}, k}) = 0 \) for any \( k \). Since \( E^{(m,n)}_{\mathcal{X}, k} \) is complete by Proposition 4.3, (i),

\[
E^{(m,n)}_{\mathcal{X}, k} \cong \lim_{n \to \infty} E^{(m,n)}_{\mathcal{X}, k} / E^{(m,n)}_{\mathcal{X}, -n}.
\]

By Lemma 4.7, \( E^{(m,n)}_{\mathcal{X}, k} / E^{(m,n)}_{\mathcal{X}, -n} \) is a coherent \( \mathcal{O}_{T^{(m,n)} \mathcal{X}}(0) \)-module. Thus,

\[
\check{H}^i_{\text{aug}}(\mathfrak{U}, E^{(m,n)}_{\mathcal{X}, k} / E^{(m,n)}_{\mathcal{X}, -n}) = 0
\]

for \( i \in \mathbb{Z} \). By the coherence, the projective system \( \left\{ \Gamma(\mathcal{V}, E^{(m,n)}_{\mathcal{X}, k} / E^{(m,n)}_{\mathcal{X}, -n}) \right\}_{n \geq 0} \) satisfies the Mittag-Leffler condition for any strictly affine open subset \( \mathcal{V} \). By (4.3.1), this shows that \( C^q_{\text{aug}}(\mathfrak{U}, E^{(m,n)}_{\mathcal{X}, k} / E^{(m,n)}_{\mathcal{X}, -n}) \) satisfies the Mittag-Leffler condition for any \( q \in \mathbb{Z} \). Thus

\[
\check{H}^i_{\text{aug}}(\mathfrak{U}, E^{(m,n)}_{\mathcal{X}, k}) \cong \check{H}^i_{\text{aug}}(\mathfrak{U}, \lim_{n \to \infty} E^{(m,n)}_{\mathcal{X}, k} / E^{(m,n)}_{\mathcal{X}, -n}) \cong \lim_{n \to \infty} \check{H}^i_{\text{aug}}(\mathfrak{U}, E^{(m,n)}_{\mathcal{X}, k} / E^{(m,n)}_{\mathcal{X}, -n}) = 0
\]

for \( i \in \mathbb{Z} \), and we get what we wanted. Now, since

\[
E^{(m,n)} \cong \lim_{k \to \infty} E^{(m,n)}_{\mathcal{X}, k},
\]

and thus

\[
\check{H}^i_{\text{aug}}(\mathfrak{U}, E^{(m,n)}_{\mathcal{X}, k}) = 0
\]
by using \eqref{2.5.2}. The claim for $\mathcal{E}^{(m,m')}_X$ follows immediately from the $\mathcal{E} = \mathcal{E}^{(m,m')}_{X'}$ case. Now, let us see $H^1(\mathcal{U},\mathcal{E}) = 0$. We know that $H^1(\mathcal{U},\mathcal{E}) \cong \varinjlim \hat{H}^1(\mathfrak{U},\mathcal{E})$ where $\mathfrak{U}$ runs over open coverings of $\mathcal{U}$. Given an open covering $\mathfrak{U}$ of $\mathcal{U}$, there exists a refinement $\mathfrak{U}'$ which is a $\mathfrak{B}$-covering. Thus the statement follows from the first claim. \[\blacksquare\]

**Remark.** — We may also prove that $H^i(\mathcal{U},\mathcal{E}) = 0$ for $i > 0$. This can be proven in the same way by using [EGA, 0III 13.3.1].

4.11. We define
\[
\mathcal{E}^{(m,m')}_{\hat{X}} := \lim_{m' \to \infty} \mathcal{E}^{(m,m')}_{X_i}, \quad \mathcal{E}^{(m,m')}_{\hat{X},Q} := \mathcal{E}^{(m,m')}_{\hat{X}} \otimes \mathbb{Q}.
\]
These are $\pi^{-1}\hat{\mathcal{E}}^{(m)}_{\hat{X}}$-algebras and the latter is moreover a $\pi^{-1}\hat{\mathcal{E}}^{(m)}_{\hat{X},Q}$-algebra. We call the rings $\mathcal{E}^{(m,m')}_{\hat{X}}$ and $\mathcal{E}^{(m,m')}_{\hat{X},Q}$ the intermediate rings of microdifferential operators (of level $(m,m')$). Let $\mathcal{U}$ be a strictly affine open subset of $\hat{T}^*\hat{\mathcal{X}}$. Applying Lemma 4.10 to the exact sequence
\[
0 \to \mathcal{E}^{(m,m')}_{\hat{X}} \xrightarrow{\pi^{i+1}} \mathcal{E}^{(m,m')}_{\hat{X},X_i} \to \mathcal{E}^{(m,m')}_{\hat{X},Q} \to 0,
\]
we get an isomorphism
\[
\Gamma(\mathcal{U},\mathcal{E}^{(m,m')}_{\hat{X},X_i}) \cong \Gamma(\mathcal{U},\mathcal{E}^{(m,m')}_{\hat{X},Q}) \otimes R_i,
\]
and by taking the projective limit over $i$,
\[
\Gamma(\mathcal{U},\mathcal{E}^{(m,m')}_{\hat{X}}) \cong \Gamma(\mathcal{U},\mathcal{E}^{(m,m')}_{\hat{X},Q})^\wedge
\]
where $^\wedge$ denotes the $p$-adic completion. By Lemma 2.6, $\Gamma(\mathcal{U},\mathcal{E}^{(m,m')}_{\hat{X},Q}) \otimes R_i \cong \Gamma(\mathcal{U},\mathcal{E}^{(m,m')}_{\hat{X},X_i})$, and thus $\mathcal{E}^{(m,m')}_{\hat{X},Q} \otimes R_i \cong \mathcal{E}^{(m,m')}_{\hat{X},X_i}$. We also define
\[
\mathcal{E}^{(m,\dagger)}_{\hat{X},Q} := \lim_{m' \to \infty} \mathcal{E}^{(m,m')}_{\hat{X},Q}.
\]
This is a ring on $T^*\hat{\mathcal{X}}$. Note that there exists a canonical homomorphism
\[
\mathcal{E}^{(m,\dagger)}_{\hat{X},Q} \to \mathcal{E}^{(m+1,\dagger)}_{\hat{X},Q}
\]
of rings by Lemma 4.6. We define
\[
\mathcal{E}^{\dagger}_{\hat{X},Q} := \lim_{m \to \infty} \mathcal{E}^{(m,\dagger)}_{\hat{X},Q}.
\]
For a quasi-compact open subscheme $\mathcal{U}$ of $\hat{T}^*\hat{\mathcal{X}}$, we get the following isomorphism since $\mathcal{U}$ is a noetherian space using \eqref{2.5.2}
\[
\Gamma(\mathcal{U},\mathcal{E}^{\dagger}_{\hat{X},Q}) \cong \lim_{m' \to \infty} \Gamma(\mathcal{U},\mathcal{E}^{(m',\dagger)}_{\hat{X},Q}).
\]
By Lemma 4.6, we have the following inclusion relations between rings of microdifferential operators.

\[ \pi^{-1} \hat{\mathcal{D}}^{(m)}_{\mathcal{X}, Q} \subset \pi^{-1} \hat{\mathcal{D}}^{(m+1)}_{\mathcal{X}, Q} \subset \pi^{-1} \hat{\mathcal{D}}^{(m+2)}_{\mathcal{X}, Q} \subset \cdots \subset \pi^{-1} \hat{\mathcal{D}}^{1}_{\mathcal{X}, Q} \]

\[ \mathcal{E}^{(m,\dagger)}_{\mathcal{X}, Q} \subset \mathcal{E}^{(m+1,\dagger)}_{\mathcal{X}, Q} \subset \mathcal{E}^{(m+2,\dagger)}_{\mathcal{X}, Q} \subset \cdots \subset \mathcal{E}^{\dagger}_{\mathcal{X}, Q} \]

\[ \vdots \]

\[ \mathcal{E}^{(m,m+2)}_{\mathcal{X}, Q} \subset \mathcal{E}^{(m+1,m+2)}_{\mathcal{X}, Q} \subset \mathcal{E}^{(m+2)}_{\mathcal{X}, Q} \]

\[ \vdots \]

\[ \mathcal{E}^{(m,m+1)}_{\mathcal{X}, Q} \subset \mathcal{E}^{(m+1)}_{\mathcal{X}, Q} \]

\[ \vdots \]

\[ \mathcal{E}^{(m)}_{\mathcal{X}, Q} \]

4.12 Proposition. — The rings \( \mathcal{E}^{(m,m')}_{\mathcal{X}} \), \( \hat{\mathcal{E}}^{(m,m')}_{\mathcal{X}} \) are noetherian on \( T^* \mathcal{X} \), and Proposition 4.8 (iii) and Lemma 4.10 are also valid if we take \( \mathcal{E} \) to be either \( \mathcal{E}^{(m,m')}_{\mathcal{X}} \) or \( \hat{\mathcal{E}}^{(m,m')}_{\mathcal{X}} \).

Proof. It suffices to show the proposition for \( \mathcal{E} = \hat{\mathcal{E}}^{(m,m')}_{\mathcal{X}} \). First, let us see Lemma 4.10 for this \( \mathcal{E} \). Since \( \mathcal{E}^{(m,m')}_{\mathcal{X}} \) is p-torsion free and (4.11.1),

\[ \hat{H}^i_{\text{aug}}(\mathcal{U}, \mathcal{E}^{(m,m')}_{X_i}) = 0 \]

by Lemma 4.10. The projective system \( \left\{ \Gamma(\mathcal{Y}, \mathcal{E}^{(m,m')}_{\mathcal{X}} \otimes R_i) \right\}_{i \geq 0} \) satisfies the Mittag-Leffler condition for any strictly affine open subscheme \( \mathcal{Y} \), and we conclude that

\[ \hat{H}^i_{\text{aug}}(\mathcal{U}, \hat{\mathcal{E}}^{(m,m')}_{\mathcal{X}}) \cong \hat{H}^i_{\text{aug}}(\mathcal{U}, \lim_{\to i} \mathcal{E}^{(m,m')}_{X_i}) \cong \lim_{\to i} \hat{H}^i_{\text{aug}}(\mathcal{U}, \mathcal{E}^{(m,m')}_{X_i}) = 0. \]

To see that \( \hat{\mathcal{E}}^{(m,m')}_{\mathcal{X}} \) is noetherian, we check the conditions of Lemma 4.11 for the \( \pi \)-adic filtration. Conditions 2 and 3 follow from the fact that \( \mathcal{E}^{(m,m')}_{\mathcal{X}} \) is noetherian by Proposition 4.8. The condition 4 follows for any strictly open formal scheme \( \mathcal{Y} \) from (4.11.1).

Proposition 4.8 (iii) for the \( \mathcal{E} \) follows directly from the \( \mathcal{E} = \mathcal{E}^{(m,m')}_{\mathcal{X}} \) case by using [Be1, 3.2.3 (vii)], and we finish the proof.

Remark. — By the proof we can moreover say that \( \hat{\mathcal{E}}^{(m,m')}_{\mathcal{X}} \) is pointwise Zariskian with respect to the \( \pi \)-adic filtration on \( T^* \mathcal{X} \). We expect that \( \hat{\mathcal{E}}^{(m,m')}_{\mathcal{X}, Q} \) is flat over \( \pi^{-1} \hat{D}^{(m)}_{\mathcal{X}, Q} \).

4.13. We will describe the sections of intermediate microdifferential rings in terms of given local coordinates. We will use the notation of 2.10. Let \( \Theta \in \Gamma(\mathcal{X}, \mathcal{O}_{T^* \mathcal{X}}) \) be a homogeneous section, and let \( \mathcal{U} := D(\Theta) \). Then we get the following:
Lemma. — We use the above notation. Let \( n \) be the order of \( \Theta \). For \( m' \geq m \), the canonical homomorphism \( \mathcal{E}_{/Q}^{(m,m')} \rightarrow \mathcal{E}_{/Q}^{(m)} \) is injective, and the canonical homomorphism \( \mathcal{E}_{/Q}^{(m,m')} / (\mathcal{E}_{/Q}^{(m,m')})_0 \rightarrow \mathcal{E}_{/Q}^{(m)}/(\mathcal{E}_{/Q}^{(m)})_0 \) is an isomorphism. Moreover,

\[
\text{Im} \left( \Gamma(\mathcal{M}, \mathcal{E}_{/Q}^{(m,m')}) \rightarrow \Gamma(\mathcal{M}, \mathcal{E}_{/Q}^{(m)}) \right) = \left\{ \sum_{|k|-\text{in} p^m < 0} a_{k,i} \left( \overline{\Theta}_{i}^{(m',m)} \right)^{-i} + \sum_{|k|-\text{in} p^m \geq 0} a_{k,i} \left( \overline{\Theta}_{i}^{(m,m')} \right)^{-i} \right\}.
\]

(*) Let \( k \in \mathbb{N}^d, i \geq 0, a_{k,i} \in K \). For an integer \( N \), put \( \alpha_{N,i} := \sup_{|k|=\text{in} p^m \geq N} |a_{k,i}| \). Then \( \lim_{i \to \infty} \alpha_{N,i} = 0 \) for any \( N \), and \( \lim_{N \to \infty} \sup_i \{ \alpha_{N,i} \} = 0 \), and there exists a real number \( C > 0 \) such that \( C > \sup_i \{ \alpha_{N,i} \} \) for any \( N < 0 \).

Proof. First, let us see the former claim. To see this, it suffices to show the injectivity for the ring of sections on \( \mathcal{M} \). In this proof, we denote \( \Gamma(\mathcal{M}, \mathcal{E}_{/Q}^{(m)}) \) by \( E^{(m)} \) and so on. Consider the following diagram whose rows are exact:

\[
\begin{array}{cccccccc}
0 & \rightarrow & E^{(m,m')} & \rightarrow & E^{(m,m')} & \rightarrow & E^{(m,m')}/E^{(m,m')} & \rightarrow & 0 \\
& & \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma & & \\
0 & \rightarrow & E^{(m)} & \rightarrow & E^{(m)} & \rightarrow & E^{(m)}/E^{(m)} & \rightarrow & 0
\end{array}
\]

Since the injective homomorphism \( E^{(m,m')} \rightarrow E^{(m)} \) is strict, \( \gamma \) is injective as well. Moreover, by Lemma 3.4 (i), there exists an integer \( a \) such that \( \text{Coker}(\gamma) \) is killed by \( p^a \). Since \( E^{(m,m')}/E^{(m,m')} \) is \( p \)-torsion free, this implies that \( \gamma \) is a homomorphism and \( p \)-isogeny.

Thus we get the following commutative diagram, whose rows are exact

\[
\begin{array}{cccccccc}
0 & \rightarrow & \tilde{E}^{(m,m')} & \rightarrow & \tilde{E}^{(m,m')} & \rightarrow & \tilde{E}^{(m,m')}/\tilde{E}^{(m,m')} & \rightarrow & 0 \\
& & \downarrow \tilde{\gamma} & & \downarrow \tilde{\gamma} & & \downarrow \tilde{\gamma} & & \\
0 & \rightarrow & \tilde{E}^{(m)} & \rightarrow & \tilde{E}^{(m)} & \rightarrow & \tilde{E}^{(m)}/\tilde{E}^{(m)} & \rightarrow & 0
\end{array}
\]

where \( \wedge \) denotes the \( p \)-adic completion. By Lemma 3.5, \( \tilde{\gamma} \) is also a \( p \)-isogeny, and in particular, it is an isomorphism after tensoring with \( \mathbb{Q} \). Since \( \tilde{E}^{(m,m')} \) is already complete with respect to the \( p \)-adic topology by Lemma 2.5 (i),

\[
\tilde{E}^{(m,m')} \cong \tilde{E}^{(m,m')}.
\]

By Lemma 4.2, the homomorphism \( E^{(m,m')}_0 \rightarrow E^{(m)}_0 \) is injective. Combining these, we get the first assertion.

Let us prove the second assertion. It is straightforward to see that the right hand side of (4.13.1) is contained in the left hand side. Let us see the opposite inclusion. Let \( S \) be the subset of \( E^{(m)} \) consisting of elements which can be written as

\[
\sum_{|k|-\text{in} p^m \geq 0} a_{k,i} \left( \overline{\Theta}_{i}^{(m,m')} \right)^{-i}
\]

such that for each \( i \), we get \( \lim_{i \to \infty} \alpha_{N,i} = 0 \) and \( \lim_{N \to \infty} \sup_i \{ \alpha_{N,i} \} = 0 \). This set surjects into \( \tilde{E}^{(m)}/E^{(m)}_0 \) by Remark 2.11. Thus, we get the second assertion by (4.13.2).
Remark. — We also get another description of the image of $\Gamma(\mathcal{U}, \hat{\mathcal{E}}(m,m'))$ if we use $\Theta^{(m',m'+k)}$ (resp. $\Theta^{(m,m'+k)}$) instead of $\Theta^{(m')}$ (resp. $\Theta^{(m,m')}$) for $k' \geq 0$. For the proof, we only need to change the last subsection of the proof in a suitable way.

4.14 Corollary. — Let $m' > m \geq 0$ be integers. Then the canonical injection $\hat{\mathcal{E}}(m+1,m'+1) \to \hat{\mathcal{E}}(m+1,m')$ induces the isomorphism:

$$\hat{\mathcal{E}}(m+1,m'+1)/\hat{\mathcal{E}}(m,m'+1) \cong \hat{\mathcal{E}}(m+1,m')/\hat{\mathcal{E}}(m,m').$$

Proof. We have the following diagram whose rows are exact.

\[
\begin{array}{cccccc}
0 & \to & \hat{\mathcal{E}}(m,m'+1)/(\hat{\mathcal{E}}(m,m'+1))_0 & \to & \hat{\mathcal{E}}(m+1,m'+1)/(\hat{\mathcal{E}}(m+1,m'+1))_0 & \to & \hat{\mathcal{E}}(m+1,m')/\hat{\mathcal{E}}(m,m'+1) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \hat{\mathcal{E}}(m,m')/(\hat{\mathcal{E}}(m,m'))_0 & \to & \hat{\mathcal{E}}(m+1,m')/(\hat{\mathcal{E}}(m+1,m'))_0 & \to & \hat{\mathcal{E}}(m+1,m')/\hat{\mathcal{E}}(m,m') & \to 0 \\
\end{array}
\]

The first two vertical homomorphisms are isomorphisms by Lemma 4.13. Thus the right vertical homomorphism is an isomorphism as well, and the corollary follows.

5. Flatness results

5.1. Let $\mathcal{X}$ be an affine smooth formal scheme, and $\mathcal{U}$ be a strictly affine open subscheme of $\mathcal{T}^* \mathcal{X}$. Let $\mathcal{E}$ be one of the rings $\hat{\mathcal{E}}(m,m'), \hat{\mathcal{E}}(m,m'), \hat{\mathcal{E}}(m,m'), \hat{\mathcal{E}}(m,m')$. Let $M$ be a finite $\Gamma(\mathcal{U}, \mathcal{E})$-module. We use the terminologies in [BGR 9.1, 9.2] freely. Let $\mathfrak{T}$ be the Grothendieck topology (in the sense of [BGR 9.1.1/1]) on $\mathcal{U}$ defined in the following way.

- A subset is said to be admissible open if it is strictly affine open subset of $\mathcal{U}$.
- A covering is called an admissible covering if it is open covering in the usual sense.

We define a presheaf $M^\Delta$ on $(\mathcal{U}, \mathfrak{T})$ by associating

$$\Gamma(\mathcal{V}, \mathcal{E}) \otimes_{\Gamma(\mathcal{U}, \mathcal{E})} M$$

with an open strictly affine subset $\mathcal{V}$.

5.2 Lemma. — We preserve the notation. For any finite $\mathfrak{B}$-covering $\mathfrak{U}$ of $\mathcal{U}$,

$$\check{H}^i_{\text{aug}}(\mathfrak{U}, M^\Delta) = 0$$

for $i \in \mathbb{Z}$.

Proof. We just copy the proof of [BGR 8.2.1/5] using Lemma 4.10 and Lemma 4.12.

5.3 Corollary. — We preserve the notation.

(i) For any finite $\Gamma(\mathcal{U}, \mathcal{E})$-module $M$, the presheaf $M^\Delta$ defines a sheaf on $(\mathcal{U}, \mathfrak{T})$, and the functor $\Delta$ is exact.

(ii) Let $\mathcal{U}$ be a strictly affine open subscheme of $\mathcal{T}^* \mathcal{X}$, and suppose there exists the following exact sequence of modules on $\mathcal{U}$:

$$\mathcal{E}^\oplus_a \overset{\phi}{\to} \mathcal{E}^\oplus_b \to \mathcal{M} \to 0.$$

Then we have a canonical isomorphism

$$\Gamma(\mathcal{U}, \mathcal{M})^\Delta \cong \mathcal{M}.$$
Proof. Let us see (i). We see directly from Lemma [5.2] that the presheaf $M^\Delta$ is a sheaf. The functor $\Delta$ is exact since the restriction homomorphism $\Gamma(\mathcal{V}, \mathcal{E}) \to \Gamma(\mathcal{W}, \mathcal{E})$ is flat where $\mathcal{W} \subset \mathcal{V} \subset \mathcal{U}$ are strictly affine by Proposition [4.8] and [4.12].

Let us show (ii). We put $M := \text{Coker}(\Gamma(\mathcal{W}, \phi))$. Let $E := \Gamma(\mathcal{W}, \mathcal{E})$. By the definition of $M$, we have the following exact sequence

$$E^{\otimes a} \xrightarrow{\Gamma(\mathcal{W}, \phi)} E^{\otimes b} \to M \to 0.$$  

Taking the exact functor $\Delta$, we have an isomorphism $\mathcal{M} \cong M^\Delta$. Taking the global sections, $\Gamma(\mathcal{W}, \mathcal{M}) \cong M$, and the claim follows.

Remark. — We preserve the notation in the corollary. We did not prove that any coherent $\mathcal{E}$-module on $\mathcal{U}$ can be written as $M^\Delta$ with a finite $\Gamma(\mathcal{W}, \mathcal{E})$-module $M$. We do not go into the problem further in this paper. We believe, however, that for any coherent $\mathcal{E}$-module $\mathcal{M}$ on a strict affine open subscheme $\mathcal{U}$, the canonical homomorphism $\Gamma(\mathcal{W}, \mathcal{M})^\Delta \to \mathcal{M}$ is an isomorphism.

Let us use the notation of [1.5]. Now, we consider the induced topology from $(T^* \mathcal{X})'$ on $\mathcal{U}$, and denote the topological space by $\mathcal{V}'$. We denote by $\epsilon : \mathcal{U} \to \mathcal{V}'$ the continuous map induced by the identity. The topology of $\mathcal{V}'$ is slightly finer (cf. [BGR] 9.1.2/1) than $\Sigma$. Thus by [BGR] 9.2.3/1, the sheaf $M^\Delta$ extends uniquely to a sheaf on $\mathcal{V}'$, denoted by $(M^\Delta)'$. Now, we get the sheaf $\epsilon^{-1}((M^\Delta)')$. We also denote this sheaf on the topological space $\mathcal{U}$ by $M^\Delta$. From now on, $M^\Delta$ indicates the sheaf on $\mathcal{U}$ unless otherwise stated.

5.4. We briefly recall the definition of Fréchet-Stein algebra. For more details, we refer to [ST]. A $K$-algebra $A$ together with a projective system of $K$-Banach algebras $\{A_i\}_{i \geq 0}$ and a homomorphism of projective systems $A \to \{A_i\}$ where $A$ denotes the constant projective system is called a Fréchet-Stein algebra if the following hold.

1. For any $i \geq 0$, the ring $A_i$ is noetherian.
2. The transition homomorphism $A_{i+1} \to A_i$ is flat and the image is dense.
3. The given homomorphism of projective systems induces an isomorphism of $K$-algebras $A \to \varprojlim_i A_i$.

5.5. Let $\mathcal{X}$ be an affine smooth formal scheme over $R$ possessing a system of local coordinates. Take a homogeneous element $\Theta$ in $\Gamma(T^* \mathcal{X}, \mathcal{O}_{T^* \mathcal{X}})$. Let $\mathcal{U} := D(\Theta)$. In this subsection, we denote $\Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}(m, \mathbb{Q}))$ by $E^{(m, \mathbb{Q})}$ and $\Gamma(\mathcal{U}, \mathcal{E}_{\mathcal{X}}(m, \mathbb{Q}'))$ by $\tilde{E}^{(m, \mathbb{Q}')}$. From now on, $E^{(m, \mathbb{Q})}$ and $\tilde{E}^{(m, \mathbb{Q}')}$ are equipped with the $\mathbb{Q}$-filtration by order.

Recall the canonical homomorphism

$$\rho_{m, n} : \tilde{E}^{(m, \mathbb{Q}')} \to \tilde{E}^{(m, \mathbb{Q})}.$$  

We define $E^{(m, m')} := \rho_{m, n}^{-1}(E^{(m, n)})$, and equip it with the filtration by order. Since $\tilde{E}^{(m, \mathbb{Q}')} \otimes \mathbb{Q} \cong E^{(m, \mathbb{Q})}$, we get

$$E^{(m, m')} \otimes \mathbb{Q} \cong \tilde{E}^{(m, \mathbb{Q}')}.$$  

Lemma. — There exists a subring $E' \subset E^{(m, \mathbb{Q})}$ such that the following holds. We equip $E'$ with the induced filtration from $E^{(m, \mathbb{Q})}$.

1. The ring $E'$ contains $\tilde{E}^{(m, \mathbb{Q}')} + 1$, and the inclusion $E' \subset E^{(m, \mathbb{Q})}$ is a $p$-isogeny.
2. The ring $\text{gr}(E')$ is finitely generated over $\text{gr}(E^{(m, \mathbb{Q}')} + 1)$.
3. The ring $E'$ and the Rees ring $(E'_0)_\bullet$ of $E'_0$ are left and right noetherian.

Proof. In this proof, we denote $\tilde{E}(m,m'+1)$ by $E$ for simplicity. By Example 1.4, we see that $(\tilde{\Theta}(m,m'+1))^{-1} \in \tilde{E}(m,m')$ and $(\tilde{\Theta}(m,m'+1))^{-1} \in \tilde{E}(m,m'+1)_Q$. This shows that $(\tilde{\Theta}(m,m'+1))^{-1} \in E^{[m,m']}$. We put $E'$ to be the subring of $E^{[m,m']}$ generated by $E$ and $\theta := (\tilde{\Theta}(m,m'+1))^{-1}$. By construction, the condition 2 holds.

Let us show that the inclusion $E' \hookrightarrow E^{[m,m']}$ is a $p$-isogeny. It suffices to see $E'_0 \hookrightarrow E^{[m,m']}_0$ is a $p$-isogeny. Let $a := a_{\text{ord}(\theta)}$ in Lemma 3.1. Let $\partial_{(m',m'+1)}(\tilde{\Theta}(m,m'+1))^{-i}$ be an operator in $E^{[m,m']}$ whose order is less than or equal to 0. Then there exists an integer $j > 0$ such that the order of $\partial_{(m',m'+1)}(\tilde{\Theta}(m,m'+1))^{-i+j}$ is strictly greater than $\text{ord}(\theta)$ and less than or equal to 0. By the choice of $a$, the operator $p^a \cdot \partial_{(m',m'+1)}(\tilde{\Theta}(m,m'+1))^{-i+j}$ is in $E$, and thus

$$p^a \cdot \partial_{(m',m'+1)}(\tilde{\Theta}(m,m'+1))^{-i} \in E'.$$

Take any $P$ in $E^{[m,m']}_0$. Take a left presentation (2.10.3) of level $m'+1$. For an integer $M'$, we put

$$P_{\leq M'} := \sum_{N \leq M'} \sum_{u_{\mathbb{B}}^{\mathbb{B}}} b_{k,i} \partial_{(m'+1)}(\tilde{\Theta}(m'+1))^{-i},$$

$$P_{> M'} := \sum_{N > M'} \sum_{u_{\mathbb{B}}^{\mathbb{B}}} b_{k,i} \partial_{(m'+1)}(\tilde{\Theta}(m'+1))^{-i}.$$ 

There exists an integer $b$ such that $p^b \cdot P \in E$. Since $(\tilde{\Theta}(m'+1))^{-1} \cdot \tilde{\Theta}(m',m'+1) \in pE$, the operator $(P_{\leq M} \cdot (\tilde{\Theta}(m',m'+1))^{b})$ where $b = \text{ord}(\theta)$ is contained in $E$. Thus,

$$p^a \cdot P = p^a \cdot P_{\geq M} + p^a \cdot P_{\leq M} \in E' + E \cdot \theta^b \subset E',$$

which implies that $p^a \cdot E^{[m,m']}_0 \subset E'_0$, and the claim follows.

We will show that this $E'$ is left noetherian. A proof that it is right noetherian is the same. We define a filtration $G_i$ for $i \geq 0$ on $E'$ in the following way: we put $G_0(E') := E$. For $i > 0$, we inductively define $G_{i+1}(E') := E + G_i(E') \cdot \theta$. Let $P \in E_l$ for some integer $l$. Since $p^a \theta = u \cdot (\tilde{\Theta}(m'+1))^{-1}$ where $u \in \mathbb{Z}^+$ and $n$ denotes the order of $\Theta$,

$$\left(p^n \theta\right)P = P(p^n \theta) + \sum_{k>1} P_k (p^n \theta)^k$$

with $P_k \in E_{k-1}m^{m'+1}p^{l-1}$, thus $\theta \cdot P \in P \cdot \theta + E_{l-1} \cdot \theta$. This implies that the filtration is a filtration of ring, and the filtration $G$ is exhaustive. Let us show that $\text{gr}^G(E')$ is noetherian.

We put $\overline{E} := E/p^a E$. Let $A := E \oplus \bigoplus_{k>0} \overline{E} \cdot T^k$ be a graded ring, whose graduation is defined by the degree of the indeterminant $T$, and the ring structure is defined by

$$T \cdot P = P \cdot T + \sum_k P_k (p^n \theta)^{k-1} \cdot T \quad \in P \cdot T + E_{l-1} \cdot T$$

where we used the notation in (5.5.1). We denote by $A_i$ the homogeneous part of degree $i$. Since $p^a \theta \in E$, there exist the surjection $\overline{E} \twoheadrightarrow \text{gr}^G(E')$ for $i \geq 1$ sending $1$ to $\theta^i$, which defines the surjection of rings

$$A \twoheadrightarrow \text{gr}^G(E').$$

It suffices to show that $A$ is noetherian. Let $P \in A$. Then we may write in a unique way $P = \sum P_i$ where $P_i \in A_i$. Let $k$ be the largest integer such that $P_k \neq 0$. For $Q \in \overline{E}$, we denote
by \( \sigma(Q) \) the principal symbol in \( \gr(E) \) where the filtration is taken with respect to the filtration by order, and for \( Q' \in E \), we denote by \( \sigma(Q') \) the principal symbol of the image of \( Q' \) in \( E \). We denote \( \sigma(P_k) \in \gr(E) \) by \( \Sigma(P) \). Let \( I \) be a left ideal of \( A \). We define

\[
S := \{ \Sigma(P) \mid P \in I \} \subset \gr(E).
\]

Since \( T \cdot P \in P \cdot T + E_{i-1} \cdot T \) for \( P \in E_i \in \gr^G(A) \), the set \( S \) is also closed under multiplication by homogeneous elements of \( \gr(E) \). Moreover, \( S \) is closed under addition of two homogeneous elements with the same degree. Let \( I_S \) in \( \gr(E) \) be the ideal generated by \( S \). By the above properties, we get that \( S = I_S \cap \bigcup_{i \geq 2} \gr_i(E) \). Since \( \gr(E) \) is noetherian, we can take homogeneous generators \( P_{i,1}, \ldots, P_{i,k} \) of \( I_S \) such that \( \Sigma(P_i) = P_i' \) and the degrees of \( P_i \) are the same \( \nu = 0 \) for all \( i \). For any \( P \in I \), the completeness of \( E \) with respect to the filtration by order implies that there exists \( R_i \in E \) such that \( P = \sum R_i P_i \) is degree less than \( d \), and thus contained in \( \bigoplus_{i < d} A_i \). Since \( \bigoplus_{i < d} A_i \) is finite over the noetherian ring \( E \), there exists \( Q_1, \ldots, Q_{k'} \) generating \( I \cap \bigoplus_{i < d} A_i \) over \( E \). By the construction, \( P_1, \ldots, P_k, Q_1, \ldots, Q_{k'} \) generate \( I \), and in particular, \( I \) is finitely generated.

Thus, the ring \( \gr^G(E') \) is noetherian. Since the filtration \( G \) is positive, this implies that \( E' \) is noetherian as well.

It remains to show that \( (E'_0)_\bullet \) is noetherian. Although the proof is slightly more complicated, the idea is essentially the same. We define a filtration \( F \) on \( \bigoplus_{j \leq 0} E'_{\nu j} \) in the same way: \( F_0 \) is equal to \( \bigoplus_{j \leq 0} E_{\nu j} \), and we inductively define \( F_{i+1} := F_i + F_i \cdot \theta \), namely \( F_i = \bigoplus_{j \leq 0} G_i(E')_{\nu j} \).

We define the induced filtration on \( (E'_0)_\bullet \) from \( \bigoplus_{j \leq 0} E'_{\nu j} \) also denoted by \( F \). It suffices to show that \( \gr^F((E'_0)_\bullet) \) is noetherian. Let \( (\gr^G(E'))_j \) denote the image of \( G_i(E') \cap E'_{\nu j} \) in \( \gr^G(E') \), and we put \( N := np^{m' + 1} = -\ord(\theta) > 0 \). Then, for \( i > 0 \), we get a surjection

\[
\bigoplus_{i \leq 0} E_{Ni} \otimes \bigoplus_{i \leq 0} E_{Ni-1} \cdot \nu^{-1} \oplus \cdots \to \gr^F_i((E'_0)_\bullet) \cong \bigoplus_{j \leq 0} (\gr^G_i(E'))_j \cdot \nu^j,
\]

sending \( P \cdot \nu^j \) with \( P \in E_{Ni+j} \) to \( (P \cdot \theta^i - \nu^i) \cdot \nu^j \). We define a double graded ring by

\[
A' := \left( E_0 \oplus E_{-1} \cdot \nu^{-1} \oplus \cdots \right) \bigoplus_{i > 0} \left( E_{Ni} \otimes E_{Ni-1} \cdot \nu^{-1} \oplus \cdots \right) T^i
\]

and define the ring structure in the same way as before. If we simply say degree, it means the degree of \( T \). We denote by \( A'_i \) the \( i \)-th homogeneous part. Since there exists a surjection \( A' \to \gr^F((E'_0)_\bullet) \), it suffices to show that \( A' \) is noetherian. Let \( \gr_{[\nu]}(E) := \bigoplus_{i \leq 0} \gr^G_i(E) \). We define a double graded commutative ring

\[
B := \bigoplus_{a,b \geq 0} \gr_{[Na-b]}(E) \mu^a \nu^{-b},
\]

whose ring structure is defined in the obvious way. We claim that this ring is noetherian. For this, it suffices to show that \( B \) is finitely generated over \( \gr_0(E) \). We know that \( \bigoplus_{i \geq 0} \gr^G_i(E) \) and \( \bigoplus_{i \leq 0} \gr^G_i(E) \) are finitely generated over \( \gr_0(E) \). Then the following claim leads us to the desired conclusion.

Claim. — Let \( C = \bigoplus_{i \leq 0} C_i \) be a graded commutative ring. Assume that \( C \) is noetherian and \( C_{\leq 0} := \bigoplus_{i \leq 0} C_i \) and \( C_{\geq 0} := \bigoplus_{i \geq 0} C_i \) are finitely generated over \( C_0 \). Let \( C_{[\nu]} := \bigoplus_{i \geq 1} C_i \). Then for any positive integer \( N \), the ring \( D_N := \bigoplus_{i \geq 0} C_{[Ni]} \nu^{Nj} \), where \( \nu \) is an indeterminant, is also finitely generated over \( C_0 \). Moreover, the ring \( \bigoplus_{k,j \geq 0} C_{[Nj-k]} \nu^j \mu^k \) is finitely generated over \( C_0 \) where \( \nu \) and \( \mu \) are indeterminants.
It suffices to show that $\bigoplus_{j \geq 0} \left( \bigoplus_{i \geq 0} C_i \right) \nu^j \subset D_1$ and $\bigoplus_{j \geq 0} \left( \bigoplus_{i \geq 0} C_i \right) \nu^j \subset D_1$ are finitely generated over $C_0$. Since the former one is isomorphic to $C_{\leq 0} [\nu]$, it is finitely generated. Let $\{ x_i \}_{i \in I}$ be a finite set of generators of $C_{\geq 0} \cong \bigoplus_{i \geq 0} C_i \nu^i \subset D_1$ over $C_0$. Then the latter one is generated by $\{ x_i \}_{i \in I}$ and $\nu$, and the claim follows. □

For $P \in A'$, we can write $P = \sum_i P_i$ with $P_i \in A'_i$ in a unique way. Let $s$ be the maximal integer such that $P_s \neq 0$. We denote $P_s$ in $A'_i$ by $\tau(P)$. Let $\tau(P) = \sum_{0 \leq i \leq K} P_i \nu^{-i}$ with $P_K \neq 0$. We define $\Sigma'(P) \in B$ to be $\sigma(P_K) \mu^s \nu^{-K}$ where $\sigma$ denotes the principal symbol with respect to the filtration by order of $\mathcal{E}$. Let $I$ be an ideal of $A'$, and we put $S' := \{ \Sigma'(P) \mid P \in I \} \subset B$. This set is closed under addition of two elements with the same degree, and multiplication by homogeneous element. Take a finite set $\{ Q_i \}$ in $I$ such that $\{ \Sigma'(Q_i) \}$ is a set of generators of the ideal $BS' \subset B$. It is straightforward to check that the set $\{ Q_i \}$ generates $I$.

5.6 Proposition. — (i) The ring $\hat{E}_Q^{[m, m']} := \hat{E}^{[m, m']} \otimes \mathbb{Q}$ is noetherian where $\hat{\cdot}$ indicates the $p$-adic completion.

(ii) The canonical homomorphism

$$\alpha_{m, m'}: \hat{E}_Q^{(m, m' + 1)} \to \hat{E}_Q^{[m, m']}$$

is flat.

(iii) The canonical homomorphism

$$\beta_{m, m'}: \hat{E}_Q^{[m, m']} \to \hat{E}_Q^{(m, m')}$$

is flat.

Proof. We use the notation of Lemma 5.5. Since $E'$ is noetherian, the canonical homomorphism $E' \to \mathcal{E}'$ is flat and $\mathcal{E}'$ is noetherian by [Be1 3.2.3]. Since $E'$ is $p$-isogeneous to $E^{[m, m']}$, they are also $p$-isogeneous even after taking $p$-adic completion by Lemma 3.5. Thus we get (i). Since $E^{[m, m']} \otimes \mathbb{Q} \cong \hat{E}_Q^{[m, m' + 1]}$, the flatness of $\alpha_{m, m'}$ follows, which is (ii).

Let us see (iii). We put $E_f' := \bigcup_n E_f^n$. By the condition 2 of Lemma 5.5 and Lemma 4.7, $\bigoplus_{i \geq 0} \text{gr}_1(E_f')$ is finitely generated over $\text{gr}_0(E_f')$. Since $E_f'$ is a noetherian filtered ring with respect to the filtration by order, $E_f'$ is also a noetherian filtered ring by Lemma 1.13. Let $E''$ be the completion of $E_f'$ with respect to the filtration. Then the canonical homomorphism $E_f' \to E''$ is flat and $E''$ is noetherian (cf. 1.1.3). Thus, by taking the $p$-adic completion, the canonical homomorphism $\hat{E}_f' \to \hat{E}''$ is flat by [Be1 3.2.3 (vii)] where $\hat{\cdot}$ denote the $p$-adic completion. It suffices to show that

$$\hat{E}_f' \otimes \mathbb{Q} \cong \hat{E}_Q^{[m, m']}, \quad \hat{E}'' \otimes \mathbb{Q} \cong \hat{E}_Q^{(m, m')}.$$  \hfill (5.6.1)

Let $E_f^{[m, m']} := \bigcup_n E_f^n$. Since $E_f' \subset E_f^{[m, m']}$ is a $p$-isogeny and the $p$-adic completion of $E_f^{[m, m']}$ is $\hat{E}_Q^{[m, m']}$, we get the first isomorphism. Let us show the second one. Note that the completion of $E_f^{[m, m']}$ with respect to the filtration by order is $E^{(m, m')}$. There exists an integer $n$ such that $p^n E_f^{[m, m']} \subset E_f' \subset E_f^{[m, m']}$.

Since these inclusions are strict homomorphisms, the inclusions are preserved even after taking the completion with respect to the filtration by order, and we get $p^n E^{(m, m')} \subset E'' \subset E^{(m, m')}$. In particular, the inclusion $E'' \subset E^{(m, m')}$ is a $p$-isogeny, which implies the second isomorphism of (5.6.1). Thus we finish the proof. □
5.7 Lemma. — Let \( m' \) be an integer strictly greater than \( m \) or \( \dagger \). For simplicity, we denote \( E_Q^{(m,1)} \) by \( \hat{E}_Q^{(m,1)} \). Let \( E_{(m')} := \iota(\hat{E}_Q^{(m,m')}) \cap \hat{E}^{(m)} \) where \( \iota : \hat{E}_Q^{(m,m')} \to \hat{E}_Q^{(m)} \) is the canonical injection. Then \( E_{(m')} \) is isomorphic to the ring of sections \( LD_{X_i}^{(m)} := \Gamma(\mathcal{Y}, \mathcal{L} \mathcal{D}_{X_i}^{(m)}) \) (cf. Remark 2.14).

Proof. By definition, the canonical homomorphism \( E_{(m')} : k \to E_{X_0}^{(m)} := \Gamma(\mathcal{Y}, \mathcal{E}_{X_0}^{(m)}) \) is injective. Note that \( LD_{X_0}^{(m)} \subset E_{X_0}^{(m)} \). Assume for a moment that \( m' \) is an integer. The ring \( LD_{X_0}^{(m)} \) is generated by \( D_{X_0}^{(m)} \) and \( (\Theta_i^{(m,m')})_1 \). Since \( (\Theta_i^{(m,m')})_1 \equiv 0 \mod p\hat{E}^{(m)} \), we get

\[
E_{(m')} \otimes k \subset LD_{X_0}^{(m)}.
\]

Since \( E_{(\dagger)} \subset E_{(m')} \), \( (5.7, \text{1}) \) holds also for \( m' = \dagger \). Now, \( m' \) can be \( \dagger \), and we will see the opposite inclusion. For simplicity we denote \( E_{(m')} \) by \( E \). It suffices to see that \( (\Theta_i^{(m,m+1)})_1 \in E \otimes k \). Let

\[
\Xi := \lim_{k \to \infty} (\Theta_i^{(m,m+k+1)})_1 \cdot \left( (\Theta_i^{(m,m+k)})_1 \cdot (\Theta_i^{(m,m+1)})_1 \right)
\]

The limit exists in \( E_{Q}^{(m,m')} \) for any \( m' \geq m \) using the \( p \)-adic topology. Indeed, for any \( i \), there exists an integer \( N \geq m' \) such that \( \Theta_i^{(m,N)} \) and \( (\Theta_i^{(m,N)})_1 \) are in the center of \( E_{X_i}^{(m,m')} \). Let \( r \in \mathbb{Z}_p \) such that \( r \cdot (\Theta_i^{(m,N)})_1 = (\Theta_i^{(m,N)})_1 \). Then

\[
r \cdot (\Theta_i^{(m,m+k+1)})_1 \cdot \left( (\Theta_i^{(m,m+k)})_1 \cdot (\Theta_i^{(m,m+1)})_1 \right) = (\Theta_i^{(m,m+k+1)})_1 \cdot \left( (\Theta_i^{(m,m+k)})_1 \cdot (\Theta_i^{(m,m+1)})_1 \right)
\]

in \( E_{X_i}^{(m,m')} \), and the claim follows. Thus \( \Xi \) is defined in \( E_{Q}^{(m,\dagger)} \), and \( \Xi \in E \). Since the image of \( \Xi \) in \( E_{X_0}^{(m)} \) is \( (\Theta_i^{(m,m+1)})_1 \), we get the lemma for \( i = 0 \). For the general case, since there exists an inclusion

\[
E \otimes R_i \to LD_{X_i}^{(m)}
\]

and both sides are flat over \( R_i \), we may reduce to the \( i = 0 \) case. \[ \blacksquare \]

5.8 Corollary. — The image of the homomorphism

\[
E_{Q}^{(m,\dagger)} \to \hat{E}_Q^{(m,m')}
\]

is dense with respect to the \( p \)-adic topology on \( \hat{E}_Q^{(m,m')} \) for any \( m' \geq m \).

Proof. Lemma 5.7 tells us that the image of the homomorphism \( E_{Q}^{(m',\dagger)} \to \hat{E}_Q^{(m',m')} \) is dense for any \( m' \geq m \). This shows that the image of the homomorphism \( (E_{Q}^{(m,\dagger)})_0 \to (\hat{E}_Q^{(m',m')})_0 \) is also dense. Thus the image of the composition

\[
(E_{Q}^{(m,\dagger)})_0 \to (\hat{E}_Q^{(m',\dagger)})_0 \to (\hat{E}_Q^{(m',m')})_0
\]

is dense.
is dense as well. By using the fact that \( (\hat{E}_Q^{[m,m']}\) \( \sim (\hat{E}_Q^{[m',m']})_0 \), this implies that the image of the homomorphism \( (E_Q^{(m,t)})_0 \to (\hat{E}_Q^{[m,m']})_0 \) is dense. Since

\[
E_Q^{(m,t)}/(E_Q^{[m',m']})_0 \to \hat{E}_Q^{[m,m']}/(\hat{E}_Q^{[m,m']})_0,
\]

we conclude the proof.

**5.9 Theorem.** — (i) The ring \( E_Q^{(m,t)} \) is a Fréchet-Stein algebra with respect to the projective system \( \{\hat{E}_Q^{[m,m']}\}_{m' \geq m} \).

(ii) For a finitely presented \( E_Q^{(m,t)} \)-module \( M \), we have

\[
R^i \lim_{m'} (\hat{E}_Q^{(m,m')} \otimes E_Q^{(m,t)}) M \sim \begin{cases} M & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}
\]

**Proof.** (i) follows by combining the previous results. To see (ii), the projective systems \( \{\hat{E}_Q^{[m,m']} \otimes M\}_{m' \geq m} \) and \( \{\hat{E}_Q^{[m,m']} \otimes M\}_{m' \geq m} \) are cofinal in the projective system

\[
\cdots \to \hat{E}_Q^{[m,m'+1]} \otimes M \to \hat{E}_Q^{[m,m'+1]} \otimes M \to \hat{E}_Q^{[m,m']} \otimes M \to \hat{E}_Q^{[m,m']} \otimes M \to \cdots,
\]

and these three projective systems have the same \( R^i \lim_{m'} \). Thus, [ST, Corollary 3.1] leads us to (ii).

**5.10 Corollary.** — Let \( \mathcal{V} \) be a strictly affine open subscheme of \( T^* \mathcal{X} \), and \( M \) be a finitely presented \( \Gamma(\mathcal{V}, \mathcal{E}_Q^{(m,t)}) \)-module. We define the presheaf \( M^\Delta \) in the same way as subsection 5.1. Then Lemma 4.10, Lemma 5.2, Corollary 5.3 are also valid for \( \mathcal{E} = \mathcal{E}_Q^{(m,t)}, \mathcal{E}_Q^{(m,t)} \), and \( M \).

**Proof.** Let us see the claim for \( \mathcal{E} = \mathcal{E}_Q^{(m,t)} \). We note that even if we prove Lemma 4.10 for \( \mathcal{E}_Q^{(m,t)} \), the other claims do not follow from this using the argument that we used to deduce them, since the ring of sections of \( \mathcal{E}_Q^{(m,t)} \) may not be noetherian, and we may not be able to take a finite free resolution of \( M \), which is used in the proof of Lemma 5.2.

For any strictly open subscheme \( \mathcal{V} \subset \mathcal{V} \),

\[
R^i \lim_{m'} (\Gamma(\mathcal{V}, \hat{E}_Q^{(m,m')}) \otimes M) = 0
\]

for \( i > 0 \) by Theorem 5.9 (ii). Let us denote by \( \hat{E}_Q^{(m,m')} \otimes M \) the coherent \( \hat{E}_Q^{(m,m')} \)-module associated with \( M \). By Lemma 5.2, this is showing that the sequence

\[
\cdots \to \lim_{m'} C_{\text{aug}}^q (\mathcal{U}, \hat{E}_Q^{(m,m')} \otimes M) \to \lim_{m'} C_{\text{aug}}^{q+1} (\mathcal{U}, \hat{E}_Q^{(m,m')} \otimes M) \to \cdots
\]

is exact. Since

\[
\lim_{m'} C_{\text{aug}}^q (\mathcal{U}, \hat{E}_Q^{(m,m')} \otimes M) \cong C_{\text{aug}}^q (\mathcal{U}, \hat{E}_Q^{(m,m')} \otimes M)
\]

by Theorem 5.9 (ii), Lemma 4.10 and Lemma 5.2 for this \( \mathcal{E} \) follows. Let us show Corollary 5.3.

For a strictly affine open subscheme \( \mathcal{V} \), \( \Gamma(\mathcal{V}, \mathcal{E}_Q^{(m,t)}) \otimes M \) is finitely presented, and in particular, coadmissible. Thus,

\[
\Gamma(\mathcal{V}, M^\Delta) \cong \lim_{k} \Gamma(\mathcal{V}, \hat{E}_Q^{(m,m+k)}) \otimes \Gamma(\mathcal{V}, M^\Delta).
\]

Since we know that the presheaf associating \( \Gamma(\mathcal{V}, \hat{E}_Q^{(m,m+k)}) \otimes M \) with \( \mathcal{V} \) is a sheaf, the result follows from 2.5.4. For the claim on \( \mathcal{E}_Q^{(m,t)} \), we only note that the functor \( \lim \) is exact. ■
of local coordinates. It suffices to see that $(\mathcal{E}, E)$ is flat. By Lemma 3.4 (i), we have $\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$ is noetherian, and thus the homomorphism $\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$. Thus, the claim follows.

5.12.  Now, we will argue the flatness of $\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$.

Lemma. — The canonical homomorphism of rings

$$\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$$

is injective and flat for $m' > m$.

Proof. Since the verification is local, we may assume that $\mathcal{X}$ is affine and possesses a system of local coordinates. It suffices to see that $\mathcal{E}_{\mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{Q}}(m, m')$ is flat and injective. For the flatness, the proof is essentially the same as [Be1, 3.5.3], and we only sketch the proof. Let $E'$ be the subring of $\hat{\mathcal{E}}_{\mathcal{Q}}(m, m')$ generated by $\hat{\mathcal{E}}(m, m')$ and $\mathcal{E}(m, m')$. We may check that $E' = \hat{\mathcal{E}}(m, m') + \mathcal{E}(m, m')$. We have $E' \otimes \mathcal{Q} = \hat{\mathcal{E}}_{\mathcal{Q}}(m, m')$, and $\hat{\mathcal{E}}(m, m')$ is the $p$-adic completion of $E'$. Let $E''$ be the subring of $E'$ generated over $\hat{\mathcal{E}}(m, m')$ by $\{\partial_{i}^{(p^{m+1})/(m+1)}\}$. In the same way as to prove $D'$ is noetherian in [ibid.], $E''$ is noetherian, and thus the homomorphism $E'' \twoheadrightarrow (E'')^{\wedge}$ is flat. By Lemma 4.1 (i), $E''$ is $p$-isogenous to $E'$. Thus, the claim follows.

To see the injectivity, it suffices to show that $E'$ is $p$-adically separated. It suffices to show that for any $P \in E'$, there exists an integer $n > 0$ such that $p^{-n}P \not\in E'$. Let $P = Q + R$ where $Q \in \hat{\mathcal{E}}(m, m')$ and $R \in \mathcal{E}(m, m')$. Recall that $\hat{\mathcal{E}}(m, m') \subset \mathcal{E}(m, m')$ by 4.6. For $n > 0$, we get that $p^{-n}Q \not\in \hat{\mathcal{E}}(m, m')$, and $p^{-n}R \not\in \mathcal{E}(m, m')$. We will show that $p^{-n}P \not\in E'$. Suppose the contrary: $p^{-n}Q + p^{-n}R \in E'$. There exists $Q' \in \hat{\mathcal{E}}(m, m') \cap p^{-n} \hat{\mathcal{E}}(m, m')$ such that $p^{-n}Q - Q' \in \hat{\mathcal{E}}(m, m')$. This implies that $Q + p^{-n}R \in E' \cap \hat{\mathcal{E}}(m, m')$, thus $Q + p^{-n}R \in \mathcal{E}(m, m')$. However, since $p^{n}Q' \in \mathcal{E}(m, m') \subset \mathcal{E}(m, m')$, we get that $R \in p^{n} \mathcal{E}(m, m')$, which contradicts the hypothesis $p^{-n}R \not\in \mathcal{E}(m, m')$.

Corollary. — Let $m' > m > 0$ be integers. Then the canonical injection $\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$ induces the isomorphism:

$$\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \cong \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$$

Proof. The proof is the same as Corollary 4.14.

5.13.  We sum up the results we got in this section as the following theorem.

Theorem. — We get the following:

1. The canonical homomorphism $\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$ is injective and flat.

2. The canonical homomorphism $\mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m') \twoheadrightarrow \mathcal{E}_{\mathcal{X}, \mathcal{Q}}(m, m')$ is injective and flat.
3. Let $\mathcal{M}$ be a finitely presented $\hat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,\tilde{t})}$-module. Then we get that

$$\mathcal{M} \xrightarrow{\sim} \lim_{\leftarrow m'} \hat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m')} \otimes \hat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,\tilde{t})} \mathcal{M}.$$ 

4. The canonical homomorphism $\hat{\mathcal{E}}_{X,\mathbb{Q}}^{(m,m')} \to \hat{\mathcal{E}}_{X,\mathbb{Q}}^{(m+1,m')}\) is injective and flat.

Proof. The statement 2 and 4 are direct consequences of what we have proven. To see 1, it suffices to apply [ST, Remark 3.2]. Let us prove 3. Since $\mathcal{M}$ is finitely presented, there exists a strictly affine open subscheme $\mathcal{U}$ such that there exists a presentation

$$(\mathcal{E}_{\mathcal{U},\mathbb{Q}}^{(m,\tilde{t})})^a \to (\mathcal{E}_{\mathcal{U},\mathbb{Q}}^{(m,\tilde{t})})^b \to \mathcal{M} \to 0.$$ 

Then we apply Corollary 5.10.

6. On finiteness of sheaves of rings

In this section, we will introduce a finiteness property for modules on certain topological spaces: we prove some stationary type theorem. This finiteness is especially useful when we consider modules on formal schemes.

6.1. First, let us introduce conditions on topological spaces and on sheaves.

A ringed space $(X, \mathcal{O}_X)$ is said to satisfy the condition (FT) if the following two conditions hold.

1. The topological space $X$ is sober (i.e. any irreducible closed subset has a unique generic point, see [SGA4, Exp. IV, 4.2.1]) and noetherian (cf. [EGA, 0I, §2.2]).

2. The structure sheaf $\mathcal{O}_X$ is a noetherian ring (with respect to an open basis $\mathcal{B}$) (cf. Definition 1.10).

Let $(X, \mathcal{O}_X)$ be a ringed space satisfying (FT), and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Let $\mathfrak{Z} := \{Z_i\}_{i \in I}$ be a finite family of irreducible closed subsets. The module $\mathcal{M}$ is said to satisfy the condition (SH) with respect to $\mathfrak{Z}$ if the following holds.

For any section $s \in \Gamma(U, \mathcal{M})$ on any open subset $U$, there exists a subset $I' \subset I$ such that

$$\text{Supp}(s) = \bigcup_{i \in I'} Z_i \cap U.$$ 

We simply say that $\mathcal{M}$ satisfies the condition (SH) if there exists a finite family $\mathfrak{Z}$ such that $\mathcal{M}$ satisfies (SH) with respect to $\mathfrak{Z}$.

6.2 Lemma. — Let $(X, \mathcal{O}_X)$ be a ringed space satisfying (FT). Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module satisfying (SH) with respect to $\mathfrak{Z} = \{Z_i\}_{i \in I}$. Then for any sub-$\mathcal{O}_X$-module $\mathcal{K}$ of $\mathcal{M}$, there exists an open subset $Z'_i$ of $Z_i$ for each $i \in I$ such that

$$\text{Supp}(\mathcal{K}) = \bigcup_{i \in I} Z'_i.$$ 

(4) In this paper, we do not use the uniqueness of generic points, and this assumption is a little stronger than what is really needed.
Proof. Let $U$ be an open subset of $X$, and take $0 \neq s \in \Gamma(U, \mathcal{K})$. Let $\varphi_U: \Gamma(U, \mathcal{K}) \hookrightarrow \Gamma(U, \mathcal{M})$ be the inclusion. Since $\varphi$ is injective, $\text{Supp}(s) = \text{Supp}(\varphi_U(s))$. There exists a subset $I_s \subset I$ such that

$$\text{Supp}(s) = \text{Supp}(\varphi_U(s)) = \bigcup_{i \in I_s} Z_i \cap U.$$ 

by (SH) of $\mathcal{M}$. Note that this is an open subset of $\bigcup_{i \in I_s} Z_i$. Now, we get

$$\text{Supp}(\mathcal{K}) = \bigcup_{s \in \mathfrak{S}} \text{Supp}(s) = \bigcup_{i \in I} \left( \bigcup_{s \in \mathfrak{S}_i} \text{Supp}(s) \cap Z_i \right),$$

where $\mathfrak{S} := \bigcup_{U \subset X} \Gamma(U, \mathcal{K})$, and $\mathfrak{S}_i$ is the subset of $\mathfrak{S}$ consisting of the elements $s$ such that $i \in I_s$. Since $\text{Supp}(s) \cap Z_i$ is open in $Z_i$, the set $\bigcup_{s \in \mathfrak{S}_i} \text{Supp}(s) \cap Z_i$ is also open in $Z_i$, and since $I$ is a finite set, we conclude the proof. $\blacksquare$

6.3 Proposition. — Let $(X, \mathcal{O}_X)$ be a ringed space satisfying (FT). Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module, and assume that for any open subset $U \subset X$, (SH) holds for any coherent subquotient of $\mathcal{M}|_U$. Now, let

$$\mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_3 \subset \cdots \subset \mathcal{M}$$

be an ascending chain of sub-$\mathcal{O}_X$-modules (not necessarily coherent) of $\mathcal{M}$. Then the sequence is stationary.

Proof. Let $n \in \mathbb{N}$ and $Z$ be a closed subset of $X$. We say that the sequence is stationary for $(n, Z)$ if

$$\mathcal{K}_n|_{X \setminus Z} = \mathcal{K}_i|_{X \setminus Z}$$

for any $i \geq n$. We will show that if the sequence is stationary for $(n, Z)$ with $Z \neq \emptyset$, then there exists an integer $n'$ and $Z' \subset Z$ such that the sequence is stationary for $(n', Z')$. Once this is proven, (i) is proven since $X$ is a noetherian space.

We will show the claim. By Lemma 6.2, there exists an integer $a$ such that $\overline{\text{Supp}(\mathcal{K}_i)} = \overline{\text{Supp}(\mathcal{K}_a)}$ for any $i \geq a$. We may suppose that

$$Z \subset \overline{\text{Supp}(\mathcal{K}_a)}.$$ 

Take a generic point $\eta$ of $Z$. Since $\mathcal{O}_{X, \eta}$ is noetherian, there exists $n' \geq \max\{a, n\}$ such that

$$\mathcal{K}_{i, \eta} = \mathcal{K}_{n', \eta}$$

for any $i \geq n'$. Fix a set of generators $\{f_1, \ldots, f_\alpha\}$ of $\mathcal{K}_{n', \eta}$. There exists an open neighborhood $U$ of $\eta$ such that $\{f_1, \ldots, f_\alpha\}$ can be lifted on $U$ and $U \cap Z$ is irreducible. We fix a set of liftings $\{\tilde{f}_1, \ldots, \tilde{f}_\alpha\}$ in $\Gamma(U, \mathcal{K}_{n'})$. Let $S \subset \Gamma(U, \mathcal{M})$ be the submodule generated by $\{\tilde{f}_1, \ldots, \tilde{f}_\alpha\}$. These elements generate a coherent subsheaf $S$ of $\mathcal{M}|_U$ since $\mathcal{O}_X$ is a coherent ring. Now, let $\mathcal{M}'_U := \mathcal{M}|_U/S$ be a coherent $\mathcal{O}_X|_U$-module, and define $\mathcal{K}'_i$ to be the image of $\mathcal{K}_i|_U$ in $\mathcal{M}'_U$. We know that

$$\text{Supp}(\mathcal{K}_i) \cap U \supset \text{Supp}(\mathcal{K}'_i).$$

By construction, $\eta \notin \text{Supp}(\mathcal{K}'_i)$ for any $i \geq n'$. By assumption, $\mathcal{M}'_U$ also satisfies (SH). Let $\mathfrak{M} := \{W_j\}_{j \in J}$ be a finite family of irreducible closed subset of $U$ such that $\mathcal{M}'_U$ satisfies (SH) with respect to $\mathfrak{M}$. Let $J'$ be the subset of $J$ such that $\eta \notin W_j$, and we put $W' := \bigcup_{j \in J'} W_j$. We let

$$Z' := (Z \cap W') \cup (Z \setminus U).$$
Since $\eta \notin Z'$, we get $Z' \subset Z$. By Lemma [6.2] there exists an open subset $W'_j$ of $W_j$ for each $j \in J$ such that
\[ \text{Supp}(\mathcal{K}'_i) = \bigcup_{j \in J} W'_j. \]
We claim that $W'_j \cap Z = \emptyset$ for any $j \notin J'$. Indeed, $j \notin J'$ implies that $\eta \notin W_j$ and $Z \cap U \subset W_j$. If $W'_j \cap Z \neq \emptyset$, we would get that $\eta \in W'_j$ since $Z \cap U$ is irreducible closed and $W'_j$ is open in $W_j$. This contradicts with $\eta \notin \text{Supp}(\mathcal{K}'_i)$. Thus,
\[ (6.3.1) \quad \text{Supp}(\mathcal{K}'_i) \cap (Z \setminus W') \cap U = \emptyset. \]
The sequence is stationary for $(n', Z')$: it suffices to see that for any
\[ z \in Z \setminus Z' = (Z \setminus W') \cap U, \]
$K_{i,z} = K_{n',z}$. However, we get that $\mathcal{K}'_{i,z} = 0$ for any $i \geq n'$ by (6.3.1). Thus, $\mathcal{K}_{i,z} = S_z$ by the definition of $\mathcal{K}'_i$, which concludes the proof. 

6.4. We will show that coherent modules on some noetherian rings we have defined in this paper satisfy (SH). For this, we prepare some lemmas. In the following, let $(X, \mathcal{O}_X)$ be a ringed space satisfying (FT).

**Lemma.** — We preserve the notation. The condition (SH) is closed under extensions. Namely, suppose there exists an exact sequence of coherent $\mathcal{O}_X$-modules
\[ 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0 \]
such that $\mathcal{F}'$ and $\mathcal{F}''$ satisfy the condition (SH). Then $\mathcal{F}$ also satisfies the condition (SH).

**Proof.** Assume $\mathcal{F}'$ (resp. $\mathcal{F}''$) satisfies (SH) with respect to $\mathfrak{I} = \{Z_i\}_{i \in I}$ (resp. $\mathfrak{M} = \{W_j\}_{j \in J}$). Then we will see that $\mathcal{F}''$ satisfies (SH) with respect to $\mathfrak{I} \cup \mathfrak{M}$. Let $U$ be an open subset of $X$, and take $s \in \Gamma(U, \mathcal{F})$. Let $\mathfrak{F}$ be the image in $\Gamma(U, \mathcal{F}'')$. Then there exists a subset $J' \subset J$ such that
\[ \text{Supp}(\mathfrak{F}) = \bigcup_{j \in J'} W_j \cap U. \]
Now, let $V := U \setminus \text{Supp}(\mathfrak{F})$. Then $s_V \in \Gamma(V, \mathcal{F}')$ by the right exactness of the global section functor, and there exists a subset $I' \subset I$ such that
\[ \text{Supp}(s_V) = \bigcup_{i \in I'} Z_i \cap V. \]
This shows that
\[ \text{Supp}(s) = \left( \bigcup_{i \in I'} Z_i \cup \bigcup_{j \in J'} W_j \right) \cap U. \]

6.5 **Lemma.** — Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. The module $\mathcal{M}$ satisfies the condition (SH) if and only if there exists a covering $\{U_i\}_{i \in I}$ of $X$ such that $\mathcal{M}|_{U_i}$ satisfies the condition on $U_i$ for any $i$.

**Proof.** We only need the proof for the “if” part. Since $X$ is quasi-compact, we may assume that the covering is finite. By assumption, for each $i \in I$, there exists a family $\{Z_j\}_{j \in J_i}$ of closed subsets of $U_i$ such that $\mathcal{M}|_{U_i}$ satisfies (SH) with respect to this family. The module $\mathcal{M}$ satisfies (SH) with respect to the family $\bigcup_{i \in I} \{Z_j\}_{j \in J_i}$. Since the verification is straightforward, we leave the details to the reader. 

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6.6 Lemma. — Let \((F, F_i)\) \((i \in \mathbb{Z})\) be a separated filtered sheaf \(i.e., \lim_{i \to -\infty} F_i = 0\). Suppose that \(\text{gr}(F)\) satisfies (SH). Then \(F\) also satisfies (SH).

Proof. Assume that \(\text{gr}(F)\) satisfies (SH) with respect to \(\mathcal{F} = \{Z_i\}_{i \in I}\). Let \(U\) be an open subset of \(X\), and take a non-zero \(s \in \Gamma(U, F)\). There exists an integer \(i_0\) such that \(s \in \Gamma(U, F_{i_0})\) and \(s \not\in \Gamma(U, F_{i_0-1})\) since the filtration is separated. (We remind here that we are always assuming filtrations to be exhaustive as in Conventions.)

Let \(\sigma = \sigma(s) \in \Gamma(U, \text{gr}_{i_0}(F)) \subset \Gamma(U, \text{gr}(F))\) be the principal symbol of \(s\). Then there exists \(J_0 \subset I\) such that

\[
\text{Supp}(\sigma(s)) = \bigcup_{j \in J_0} Z_j \cap U.
\]

For \(k \geq 0\), we inductively define an open subset \(U_k\), and a subset \(J_k\) of \(J\) in the following way. We put \(U_0 := U\). Now, let \(U_{k+1} := U_k \setminus \text{Supp}(\sigma(s|_{U_k}))\). Then there exists \(J_{k+1}^\prime\) such that

\[
\text{Supp}(\sigma(s|_{U_{k+1}})) = \bigcup_{j \in J_{k+1}^\prime} Z_j \cap U_{k+1}.
\]

We define \(J_{k+1} := J_k \cup J_{k+1}^\prime\). Obviously, \(J_0 \subset J_1 \subset \cdots \subset I\). Since \(I\) is a finite set, this sequence is stationary. Let \(J := \bigcup_i J_i \subset I\). Then

\[
\text{Supp}(s) = \bigcup_{j \in J} Z_j \cap U,
\]

and \(F\) satisfies (SH) with respect to \(\mathcal{F}\) as well. \(\blacksquare\)

6.7 Definition. — Let \((X, \mathcal{O}_X)\) be a ringed space satisfying (FT). We say that this ringed space (or \(\mathcal{O}_X\)) is strictly noetherian if the condition (SH) is satisfied for any coherent \(\mathcal{O}_X|_U\)-modules for any open subset \(U\) of \(X\).

6.8 Corollary. — Let \((X, \mathcal{E})\) be a ringed space satisfying (FT), and let \((\mathcal{E}, \mathcal{E}_i)\) be a filtration on \(\mathcal{E}\). Suppose that the filtration is pointwise Zariskian (cf. Definition 1.10). If \(\text{gr}(\mathcal{E})\) is strictly noetherian, then so is \(\mathcal{E}\).

Proof. Let \(M\) be a coherent \(\mathcal{E}\)-module. Since the verification is local by Lemma 6.5, we may suppose that there exists a good filtration \((M, M_i)\). By Lemma 1.14, the filtration is separated. Now by Lemma 6.6, the corollary follows. \(\blacksquare\)

6.9 Lemma. — Let \(A\) be a strictly noetherian sheaf on a topological space \(X\). Then \(A[T]\) is strictly noetherian as well.

Proof. It is easy to see that \(A[T]\) is a noetherian ring since for \(U \in \mathfrak{B}\), \(\Gamma(U, A[T]) \cong \Gamma(U, A)[T]\). Let \(M\) be a coherent \(A[T]\)-module. Since the verification is local by Lemma 6.5, we may assume that there exists integers \(a, b \geq 0\) and a presentation

\[
A[T]^\oplus_a \xrightarrow{\psi} A[T]^\oplus_b \xrightarrow{\psi} M \to 0.
\]

Let \(A_n := \bigoplus_{i \leq n} A \cdot T^i\) and \(K'_n := \psi(A_n)\). Let \(K_{m,n} := K'_m \cap A_n^\oplus b\) in \(A[T]^\oplus b\), which is a coherent sub-\(A\)-module of \(A_n^\oplus b \subset A[T]^\oplus b\). Since \(A\) is strictly noetherian, \(K_n := \bigcup_{m \geq 0} K_{m,n}\) is a coherent \(A\)-module. We define a coherent \(A\)-module \(M_n\) by \(A_n^\oplus b / K_n \subset M\). By construction, \(\bigcup_n M_n = M\), and \((M, \{M_n\}_{n \in \mathbb{Z}})\) is a filtered \((A[T], \{A_n\}_{n \in \mathbb{Z}})\)-module.  

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It suffices to show that \( \text{gr}(\mathcal{M}) \) satisfies (SH) by Lemma 6.6. We note that \( \text{gr}_i(\mathcal{M}) \) is a coherent \( \mathcal{A} \)-module for any \( i \). By construction, the homomorphism \( T : \text{gr}_i(\mathcal{M}) \to \text{gr}_{i+1}(\mathcal{M}) \) is surjective for any \( i \), and we have the following sequence of surjections.

\[
\mathcal{M}_0 = \text{gr}_0(\mathcal{M}) \to \cdots \to \text{gr}_i(\mathcal{M}) \to \text{gr}_{i+1}(\mathcal{M}) \to \cdots
\]

Since \( \mathcal{M}_0 \) is coherent and \( \mathcal{A} \) is strictly noetherian, this sequence is stationary, and there exists an integer \( N \) such that \( T^i : \text{gr}_N(\mathcal{M}) \to \text{gr}_{N+i}(\mathcal{M}) \) is an isomorphism for any \( i \geq 0 \). Since \( \bigoplus_{0 \leq i \leq N} \text{gr}_i(\mathcal{M}) \) is a coherent \( \mathcal{A} \)-module, it satisfies (SH) with respect to a family \( \mathcal{F} \). Then \( \text{gr}(\mathcal{M}) \) satisfies (SH) with respect to \( \mathcal{F} \).

\[\text{(FT)}\]

6.10 Corollary. — Let \( X \) be a topological space satisfying (FT), and \( \mathcal{A} \) be a strictly noetherian \( R \)-module on \( X \). Then \( \mathcal{A} \otimes \mathbb{Q} \) is strictly noetherian as well.

Proof. Let \( \mathcal{A}_n := \ker(\mathcal{A} \to \mathcal{A}) \). Since \( \mathcal{A} \) is strictly noetherian, the sequence \( \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A} \) is stationary. Let \( \mathcal{A}_\infty := \lim_{\to n} \mathcal{A}_n \). Since \( \mathcal{A}/\mathcal{A}_\infty \) is a coherent \( \mathcal{A} \)-algebra, it is strictly noetherian, and we may assume that \( \mathcal{A} \) is a flat \( R \)-module in the corollary. Let \( F_i(\mathcal{A} \otimes \mathbb{Q}) := \pi^{-i} \mathcal{A} \) for \( i \geq 0 \) and \( F_i(\mathcal{A} \otimes \mathbb{Q}) = 0 \) for \( i < 0 \). Then it suffices to show that \( \text{gr}^F(\mathcal{A} \otimes \mathbb{Q}) \) is strictly noetherian by Corollary 6.8. Since \( \text{gr}^F(\mathcal{A} \otimes \mathbb{Q}) \) is a coherent \( \mathcal{A}[T] \)-algebra where the action of \( T \) is the multiplication by \( \pi^{-1} \in \text{gr}_1(\mathcal{A} \otimes \mathbb{Q}) \), it is reduced to showing that \( \mathcal{A}[T] \) is strictly noetherian, which follows from Lemma 6.9.

6.11 Lemma. — Let \( X \) be a noetherian scheme. Then \( \mathcal{O}_X \) is strictly noetherian.

Proof. The condition (FT) on \((X, \mathcal{O}_X)\) is a fundamental property of noetherian schemes. Let \( \mathcal{M} \) be a coherent \( \mathcal{O}_X \)-module, and let us check (SH) for this \( \mathcal{M} \). Since the statement is local by Lemma 6.5, we may suppose that \( X = \text{Spec}(\mathcal{A}) \) for a noetherian ring \( \mathcal{A} \). There exists a finite decreasing filtration \( \{\mathcal{M}_i\}_{0 \leq i \leq n} \) on \( \mathcal{M} \) such that \( \mathcal{M}_0 = \mathcal{M}, \mathcal{M}_n = 0 \), the quotient \( \mathcal{M}_i/\mathcal{M}_{i+1} \) is irredundant for any \( 0 \leq i < n \), and

\[
\text{Ass}(\mathcal{M}_i/\mathcal{M}_{i+1}) \subset \text{Ass}(\mathcal{M}).
\]

by [EGA IV 3.2.8]. By Lemma 6.4, it suffices to show the lemma for irredundant modules, but in this case, it follows by definition.

Using Lemma 6.9 we have the following corollary.

Corollary. — Let \( X \) be a noetherian scheme and \( \mathcal{A} \) be a quasi-coherent \( \mathcal{O}_X \)-algebra of finite type. Then \( \mathcal{A} \) is strictly noetherian.

6.12 Lemma. — Let \((Y, \mathcal{A})\) be a ringed space satisfying (FT), and assume that \( \mathcal{A} \) is strictly noetherian. Let \( X \) be a sober and noetherian topological space, and \( f : X \to Y \) be an open continuous map of topological spaces. Then the ringed space \((X, f^{-1} \mathcal{A})\) is strictly noetherian.

Proof. Let \( \mathcal{N} \) be a coherent \( \mathcal{A} \)-module satisfying (SH) with respect to \( \mathcal{F} \). Then \( f^{-1} \mathcal{N} \) satisfies (SH) with respect to \( f^{-1} \mathcal{F} \). Thus, it suffices to show that for any coherent \( f^{-1} \mathcal{A} \)-module \( \mathcal{M} \), there exists a coherent \( \mathcal{A} \)-module \( \mathcal{N} \) such that \( \mathcal{M} \cong f^{-1}(\mathcal{N}) \). Now, the functor \( f_* \) is exact since \( f \) is open. Thus, for a coherent \( f^{-1} \mathcal{A} \)-module \( \mathcal{M} \), the canonical homomorphism \( f^{-1} f_* \mathcal{M} \to \mathcal{M} \) is an isomorphism, and \( f_* \mathcal{M}|_{f^{-1}(\mathcal{A})} \) is a coherent \( \mathcal{A}|_{f^{-1}(\mathcal{A})} \)-module.

6.13 Theorem. — Let \( \mathcal{X} \) be a smooth formal scheme of finite type over \( \text{Spf}(R) \). Then \( \mathcal{O}_{X_i}, \mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}, Q}, \mathcal{D}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}, Q}, \mathcal{G}_{\mathcal{X}}, \mathcal{G}_{\mathcal{X}, Q}, \mathcal{G}_{\mathcal{X}, Q}^{(m)}, \mathcal{G}_{\mathcal{X}, Q}^{(m)}, \mathcal{G}_{\mathcal{X}, Q}^{(m)}, \mathcal{G}_{\mathcal{X}, Q}^{(m)} \) are strictly noetherian sheaves on \( \mathcal{X} \). Moreover, \( \mathcal{E}_{X_i}, \mathcal{E}_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}, Q}, \mathcal{E}_{\mathcal{X}, Q}^{(m)}, \mathcal{E}_{\mathcal{X}, Q}^{(m)}, \mathcal{E}_{\mathcal{X}, Q}^{(m)} \) are strictly noetherian on \( \mathcal{T}^* \mathcal{X} \).

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Proof. Note first that $\mathcal{X}$ and $\tilde{T}^*\mathcal{X}$ are noetherian spaces. The ring $\mathcal{O}_{X_i}$ is strictly noetherian by Lemma 6.11. To see that $\mathcal{O}_{\mathcal{X}}$ is strictly noetherian, we consider the $\pi$-adic filtration. Since $\mathcal{O}_{\mathcal{X}}$ is Zariskian with respect to the $\pi$-adic filtration by [Be1, 3.3.6] and [HO, Ch.II, 2.2 (4)] (or we can use Lemma 6.11), it suffices to show that $\text{gr}_\pi(\mathcal{O}_\mathcal{X}) \cong \mathcal{O}_{X_0}[\pi]$ is strictly noetherian by Corollary 6.8 where $\text{gr}_\pi$ denotes the $\text{gr}$ with respect to the $\pi$-adic filtration and $T$ denotes the class of $\pi$. This follows from Lemma 6.9. For $\mathcal{O}_{\mathcal{X},\mathbb{Q}}$ use Corollary 6.10.

Let $X$ be either $\mathcal{X}$ or $X_i$ for some $i \geq 0$. Let us see that $\mathcal{D}_X^{(m)}$ is strictly noetherian. We consider the filtration by order. Since the filtration is positive, it suffices to show that $\text{gr}(\mathcal{D}_X^{(m)})$ is strictly noetherian by Corollary 6.8. Since $\text{gr}(\mathcal{D}_X^{(m)})$ is of finite type over $\text{gr}_0(\mathcal{D}_X^{(m)})$, and $\text{gr}_i(\mathcal{D}_X^{(m)})$ is coherent $\text{gr}_0(\mathcal{D}_X^{(m)})$-module for any $i$, $\text{gr}(\mathcal{D}_X^{(m)})$ can be seen as a coherent $\mathcal{O}_X[T_1, \ldots, T_n]$-algebra for some $n$, and the claim follows by using Corollary 6.11.

Let us prove that $\mathcal{D}_\mathcal{X}^{(m)}$ is strictly noetherian. Consider the $\pi$-adic filtration. Then $\mathcal{D}_\mathcal{X}^{(m)}$ is Zariskian filtered by [Be1, 3.3.6] and [HO, Ch.II, 2.2 (4)]. By Corollary 6.8 it suffices to show that $\text{gr}_\pi(\mathcal{D}_\mathcal{X}^{(m)}) \cong (\mathcal{D}_{X_0})[\pi]$ is strictly noetherian where $\text{gr}_\pi$ denotes the $\text{gr}$ with respect to the $\pi$-adic filtration and $T$ denotes the class of $\pi$. Since $\mathcal{D}_{X_0}$ is strictly noetherian by the argument above, $(\mathcal{D}_{X_0})[\pi]$ is strictly noetherian by Lemma 6.9, and thus $\mathcal{D}_\mathcal{X}^{(m)}$ is strictly noetherian.

Let $X$ be either $\mathcal{X}$ or $X_i$ for some $i$. Let us see that $\mathcal{E} := \mathcal{D}_X^{(m,m')} = \mathcal{E}_X^{(m,m')}$ is strictly noetherian on $\tilde{T}^*\mathcal{X}$. Consider the filtration by order. It suffices to show that $\text{gr}(\mathcal{E})$ is strictly noetherian by Corollary 6.8 and Remark 4.8. Let $q : \tilde{T}^*X \to P^*X$ be the canonical surjection. Then, it suffices to show that $q_*(\text{gr}(\mathcal{E}))$ is strictly noetherian by 1.5.2. By Lemma 4.7 we see that $q_*(\text{gr}(\mathcal{E}))$ is of finite type over $\mathcal{O}_{P^{(m,m')}X(0)}$-algebra and $q_*(\text{gr}_i(\mathcal{E}))$ is a coherent $\mathcal{O}_{P^{(m,m')}X(0)}$-module for any $i$, and we get the claim by using Lemma 6.9.

For $\mathcal{D}^{(m,m')}_\mathcal{X}$ and $\mathcal{D}^{(m,m')}_{\mathcal{X},\mathbb{Q}}$ the verifications are the same as those of $\mathcal{D}_\mathcal{X}^{(m)}$ and $\mathcal{D}_\mathcal{X}^{(m)}$, we leave the details to the reader. 

7. Application: Stability theorem for curves

In this section, we will focus on the relation between the support of the microlocalization and the characteristic variety. We formulate a conjecture on the relation, and prove the conjecture in the curve case.

7.1. Recall the setting 2.1 and let $\mathcal{X}$ be a smooth formal scheme of finite type over $R$. One might expect that, for a coherent $\mathcal{D}_{\mathcal{X},\mathbb{Q}}$-module $\mathcal{M}$,

$$\text{Char}^{(m)}(\mathcal{M}) = \text{Supp}(\mathcal{E}_\mathcal{X,\mathbb{Q}}^{(m,1)} \otimes_{\pi^{-1}\mathcal{D}_{\mathcal{X},\mathbb{Q}}} \pi^{-1}\mathcal{M}).$$

For the definition of the characteristic varieties, see 2.12. However, this does not hold in general. Indeed, suppose this were true. Then since $\text{Supp}(\mathcal{E}_\mathcal{X,\mathbb{Q}}^{(m,1)} \otimes \mathcal{M}) \supset \text{Supp}(\mathcal{E}_\mathcal{X,\mathbb{Q}}^{(m+1,1)} \otimes \mathcal{M})$, we would get $\text{Char}^{(m)}(\mathcal{M}) \supset \text{Char}^{(m+1)}(\mathcal{D}_{\mathcal{X},\mathbb{Q}}^{(m+1)} \otimes \mathcal{M})$. However, this does not hold by Lemma 4.1. Considering these, we conjecture the following.

Conjecture. — Let $\mathcal{X}$ be a smooth formal scheme of finite type over $R$, and $\mathcal{M}$ be a coherent $\mathcal{D}_{\mathcal{X},\mathbb{Q}}$-module. Then there exists an integer $N \geq m$ such that for any $m' \geq N$,

$$\text{Char}^{(m')}(\mathcal{D}_{\mathcal{X},\mathbb{Q}} \otimes \mathcal{D}_{\mathcal{X},\mathbb{Q}} \mathcal{M}) = \text{Supp}(\mathcal{E}_\mathcal{X,\mathbb{Q}}^{(m',1)} \otimes_{\pi^{-1}\mathcal{D}_{\mathcal{X},\mathbb{Q}}} \pi^{-1}\mathcal{M}).$$

We prove this conjecture in the case where $\mathcal{X}$ is a formal curve (i.e. dimension 1 connected smooth formal scheme of finite type over $R$). Namely,
7.2 Theorem. — Let $\mathcal{X}$ be a smooth formal curve over $R$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{(m)}$-module. Then there exists an integer $N \geq m$ such that we have

$$\text{Char}^{(m')} (\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{(m')} \otimes \mathcal{M}) = \text{Supp}(\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{\dagger} \otimes \mathcal{M})$$

for $m' \geq N$.

This theorem is proven in the last part of this section.

7.3 Remark. — The conjecture and Theorem 7.2 may seem to be different since we used $\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m, i)}$ in the conjecture and $\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{\dagger}$ in the theorem. However, these are equivalent. We use the notation in the conjecture. Since there exists a homomorphism $\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', i)} \rightarrow \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m'+1, i)}$ and the topological space $T^* \mathcal{X}$ is noetherian, there exists an integer $a$ such that for any $m' \geq a$,

$$\text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{\dagger} \otimes \mathcal{M}) \subset \text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', i)} \otimes \mathcal{M}) = \text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(a, i)} \otimes \mathcal{M}).$$

We remind that these supports are closed by [EGA IV 0, 5.2.2]. We will show that this inclusion is in fact an equality. Since the problem is local, we may assume that $\mathcal{X}$ is affine, and take global generators $m_1, \ldots, m_n \in \Gamma(\mathcal{X}, \mathcal{M})$ of $\mathcal{M}$ over $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{(m)}$. Suppose that the inclusion is not an equality, and take a point $x \in \text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(a, i)} \otimes \mathcal{M})$ which is not contained in $\text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{\dagger} \otimes \mathcal{M})$. This means that $(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{\dagger} \otimes \mathcal{M})_x = 0$. Now, we know that

$$(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{\dagger} \otimes \mathcal{M})_x \cong \lim_{m' \rightarrow \infty} (\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', i)} \otimes \mathcal{M})_x$$

by [Go II.1.11]. Thus there exists an integer $m' \geq a$ such that the images of $m_1, \ldots, m_n$ in $(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', i)} \otimes \mathcal{M})_x$ are 0. Since the latter module is generated by these elements over $(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', i)})_x$, we would have $(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', i)} \otimes \mathcal{M})_x = 0$, which contradicts with the assumption. Summing up, we obtain

$$\text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{\dagger} \otimes \mathcal{M}) = \bigcap_{m' \geq m} \text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', i)} \otimes \mathcal{M}).$$

7.4. Before we start proving the theorem, let us see some consequences of the theorem.

Definition. — Let $\mathcal{X}$ be a smooth formal curve over $R$, and let $\mathcal{M}$ be a coherent $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{\dagger}$ module. We define

$$\text{Char}(\mathcal{M}) := \text{Supp}(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{\dagger} \otimes \mathcal{M}).$$

7.5 Corollary. — Suppose that $k$ is perfect and there exists a lifting $\sigma : R \xrightarrow{\cong} R$ of the absolute Frobenius automorphism of $k$, and fix one. Let $\mathcal{X}$ be a smooth formal curve, and let $\mathcal{M}$ be a coherent $F^* \mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{\dagger}$-module. Then

$$\text{Car}(\mathcal{M}) = \text{Char}(\mathcal{M}),$$

where $\text{Car}$ denotes the characteristic variety defined by Berthelot (cf. [Be1 5.2.7]).

Proof. Since this follows immediately from the definition of Car, we recall the definition briefly. For a large enough integer $m$, we can take the Frobenius descent of level $m$ denoted by $\mathcal{M}^{(m)}$ by [Be2 4.5.4], which is a coherent $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{(m)}$-module with an isomorphism $\mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{(m+1)} \otimes \mathcal{M}^{(m)} \xrightarrow{\cong} F^* \mathcal{M}^{(m)}$. The characteristic variety of $\mathcal{M}$ is by definition $\text{Char}^{(m)}(\mathcal{M}^{(m)})$. A property of Frobenius descents tells us that $\mathcal{M}^{(m+k)} := \mathcal{D}_{\mathcal{X}, \mathbb{Q}}^{(m+k)} \otimes \mathcal{M}^{(m)} \cong F^{k*} \mathcal{M}^{(m)}$ and $\text{Char}^{(m)}(\mathcal{M}^{(m)}) = \text{Char}^{(m+k)}(\mathcal{M}^{(m+k)})$ by [ibid., 5.2.4 (iii)]. Thus the corollary follows by applying Theorem 7.2 to $\mathcal{M}^{(m)}$.
7.6. Let us prove the theorem. To prove the theorem, we will show the following proposition first.

Proposition. — Let \( \mathcal{X} \) be a smooth formal curve over \( R \), and \( \mathcal{M} \) be a coherent \( \hat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \)-module. Suppose moreover that there exists an integer \( N' \geq m \) such that

\[
\text{Char}^{(m')}(\hat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \otimes \mathcal{M}) = \text{Char}^{(N')}(\hat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(N')} \otimes \mathcal{M})
\]

for any \( m' \geq N' \). Then the conclusion of Theorem 7.2 holds.

The proof will be given in 7.7. For an interval \( I \subset \mathbb{R} \), we will denote \( I \cap \mathbb{Z} \) by \( I_{\mathbb{Z}} \) in the following.

7.7 Lemma. — Let \( \mathcal{X} \) be a smooth formal scheme. (We do not need to assume that \( \mathcal{X} \) is a curve in this lemma.) Let \( \mathcal{M} \) be a \( \hat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \)-module and \( x \in T^* \mathcal{X} \). Suppose there exist integers \( b \geq a \geq m \) such that

\[
(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \otimes \mathcal{M})_x = 0
\]

for any integer \( b \geq m' \geq a \). Then the canonical homomorphism

\[
(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, \bullet)} \otimes \mathcal{M})_x \to (\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m', \bullet)} \otimes \mathcal{M})_x
\]

is split surjective for any integers \( b \geq m' \geq a, m' \geq l \geq m \), and \( \bullet \in \{ \dag, [m', \infty] \} \). Here \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, \dag)} \) means \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, \dag)} \) by abuse of language. Moreover, we get

\[
\text{Ker}(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, \bullet)} \otimes \mathcal{M})_x \to (\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m', \bullet)} \otimes \mathcal{M})_x \cong \text{Tor}_{1, \mathbb{Q}}(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m')} / \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, \bullet)} \otimes \mathcal{M})_x
\]

by Corollary 5.11. We denote this quotient by \( \mathcal{D} \). We get the following diagram whose rows are exact:

\[
\begin{array}{ccc}
\text{Tor}_1(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m', \bullet)} \otimes \mathcal{M}) & \xrightarrow{\beta} & \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, \bullet)} \otimes \mathcal{M} \\
\text{Tor}_1(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \otimes \mathcal{M}) & \xrightarrow{\alpha} & \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, m')} \otimes \mathcal{M} \\
\end{array}
\]

where the \( \otimes \) and Tor\(_1\) are taken over \( \hat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m)} \), and we omit the pull-back of sheaves \( \pi^{-1} \) since it is obvious where to put them. Now, since \( \text{Tor}_1(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \otimes \mathcal{M}) = 0 \) by the flatness of \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \) over \( \hat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \) and \( \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \) over \( \hat{\mathcal{D}}_{\mathcal{X}, \mathbb{Q}}^{(m')} \) (cf. Proposition 2.8 (ii) and [Be1 3.5.3]), the homomorphism \( \alpha \) is injective. Moreover, since \( (\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}} \otimes \mathcal{M})_x = 0 \) by the hypothesis, the homomorphism \( \alpha_x \) is an isomorphism, and \( (\mathcal{D} \otimes \mathcal{M})_x = 0 \). Since \( \alpha \) is injective, \( \beta \) is injective as well. Moreover, the homomorphism (7.7.1) is split surjective and

\[
\text{Ker}(\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, \bullet)} \otimes \mathcal{M} \to \hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(m', \bullet)} \otimes \mathcal{M})_x \cong \text{Tor}_1(\mathcal{D} \otimes \mathcal{M})_x \xrightarrow{\alpha_x} (\hat{\mathcal{E}}_{\mathcal{X}, \mathbb{Q}}^{(l, m')} \otimes \mathcal{M})_x
\]

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Let us calculate $\text{Tor}_1(\mathcal{D}, \mathcal{M})_x$. We only treat the case where $l < a$, and since the proof is similar, the other case is left to the reader. To calculate this, it suffices to show

$$\left(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')} \otimes \mathcal{M}\right)_x \cong \left(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,a)} \otimes \mathcal{M}\right)_x \oplus \bigoplus_{i=a}^{m'-1} \left(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(i+1)} \otimes \mathcal{M}\right)_x.$$  

We use the induction on $k := m' - a$. For $k = 0$, the claim is redundant. Suppose that the statement holds to be true for $k = k_0 - 1 \geq 0$. Then it suffices to show the isomorphism for $m' = a + k_0$

$$(7.7.2) \quad \left(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')} \otimes \mathcal{M}\right)_x \cong \left(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')-1} \otimes \mathcal{M}\right)_x \oplus \left(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m'-1)} \otimes \mathcal{M}\right)_x.$$  

Indeed, we just apply the induction hypothesis to $\left(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')-1} \otimes \mathcal{M}\right)_x$ to get the conclusion. Let us show (7.7.2). Note that $m' - 1 \geq a$. This isomorphism can be shown using exactly the same method as before using the following diagram instead:

$$\begin{array}{cccccccc}
\text{Tor}_1(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')-1}, \mathcal{M}) & \longrightarrow & \text{Tor}_1(\mathcal{D}', \mathcal{M}) & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')} \otimes \mathcal{M} & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')-1} \otimes \mathcal{M} & \longrightarrow & \mathcal{D}' \otimes \mathcal{M} & \longrightarrow & 0 \\
\downarrow & & \sim & & \downarrow & & \sim & & \downarrow & & \sim
\text{Tor}_1(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')-1}, \mathcal{M}) & \longrightarrow & \text{Tor}_1(\mathcal{D}', \mathcal{M}) & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m'-1)}, \mathcal{M} & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m'-1)} \otimes \mathcal{M} & \longrightarrow & \mathcal{D}' \otimes \mathcal{M} & \longrightarrow & 0
\end{array}$$

where

$$\mathcal{D}' = \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')-1} / \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(l,m')} \cong \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')-1} / \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')-1}$$

using Corollary 6.12. \hspace{1cm} ■

7.8 Lemma. — Let $\mathcal{X}$ be a curve. Then the canonical homomorphism

$$\pi^{-1}\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m+k)} / \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m)} \to \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m+k)} / \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m)}$$

is an isomorphism.

Proof. It suffices to prove that the canonical homomorphism of sheaf of abelian groups

$$\pi^{-1}\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m)} \to \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m)} / (\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m)})^{-1}$$

is an isomorphism. To see this, it suffices to see that

$$\pi^{-1}\mathcal{E}_{\mathcal{X}_i}^{(m)} \to \mathcal{E}_{\mathcal{X}_i}^{(m)} / (\mathcal{E}_{\mathcal{X}_i}^{(m)})^{-1}$$

is an isomorphism, whose verification is straightforward. \hspace{1cm} ■

7.9. Proof of Proposition 7.6 Consider the following diagram of sheaves on $T^* \mathcal{X}$ for any integer $m' \geq m$.

$$\begin{array}{cccccccc}
\text{Tor}_1(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')}, \mathcal{M}) & \longrightarrow & \text{Tor}_1(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')}, \mathcal{M}) & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')} \otimes \mathcal{M} & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')} \otimes \mathcal{M} \\
\downarrow & & \sim & & \downarrow & & \sim & & \downarrow & & \sim
\text{Tor}_1(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')}, \mathcal{M}) & \longrightarrow & \text{Tor}_1(\mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m')}, \mathcal{M}) & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', m')} \otimes \mathcal{M} & \longrightarrow & \mathcal{E}_{\mathcal{X}, \mathbb{Q}}^{(m', m')} \otimes \mathcal{M}
\end{array}$$

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Here $\otimes$ and Tor are taken over $\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}$, and we omit the pull-back of sheaves $\pi^{-1}$. Since $\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m')}$ and $\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}$ are flat over $\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}$, the left vertical arrow of the diagram is just $0 \to 0$. Thus the homomorphism $\alpha$ is injective. Consider the following commutative diagram

$$
\begin{array}{c}
\text{Tor}_1(E_{\mathcal{X},\mathbb{Q}}^{(m')}/E_{\mathcal{X},\mathbb{Q}}^{(m)}, \mathcal{M}) \xrightarrow{\beta} E_{\mathcal{X},\mathbb{Q}}^{(m)} \otimes \mathcal{M} \\
\downarrow \sim \\
\text{Tor}_1(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m')}/\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}, \mathcal{M}) \xrightarrow{\alpha} \hat{E}_{\mathcal{X},\mathbb{Q}}^{(m')} \otimes \mathcal{M}
\end{array}
$$

where the isomorphism is by Lemma 5.11. Since $\alpha$ is injective, $\beta$ is also injective. This implies that

$$\mathcal{H}_{m'} := \text{Ker}(\varphi_{m'}: E_{\mathcal{X},\mathbb{Q}}^{(m')} \otimes \pi^{-1} \mathcal{M} \to E_{\mathcal{X},\mathbb{Q}}^{(m')}/\pi^{-1} \mathcal{M}) \cong \text{Tor}_1(E_{\mathcal{X},\mathbb{Q}}^{(m')}/E_{\mathcal{X},\mathbb{Q}}^{(m)}, \mathcal{M}) \cong \text{Tor}_1(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m')}/\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}, \mathcal{M}) \to \pi^{-1} \mathcal{M}.$$

Since $\pi^{-1} \hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)}$ is strictly noetherian by Lemma 6.12 and Theorem 6.13, there exists an integer $N$ such that $\mathcal{H}_k = \mathcal{H}_N$ for $k \geq N$. So far we have not used the assumption on the characteristic varieties.

By changing $m$ if necessarily, we may assume that $m = N$. Now, by this assumption,

$$Z := \text{Supp}(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m)} \otimes \mathcal{M}) = \text{Supp}(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m')} \otimes \mathcal{M})$$

for any $m' \geq m$. Take $x \in \mathcal{T}^* \mathcal{X} \setminus Z$. Then Lemma 7.7 is showing that $\varphi_{m',x}$ is split surjective. Thus, this is showing that the homomorphism

$$\psi_{m',x}: (E_{\mathcal{X},\mathbb{Q}}^{(N,1)} \otimes \mathcal{M})_x \to (E_{\mathcal{X},\mathbb{Q}}^{(m',1)} \otimes \mathcal{M})_x$$

is also surjective for any $m' \geq N$. Since the kernel is isomorphic to $(\mathcal{H}_{m'}/\mathcal{H}_N)_x$, the homomorphism $\psi_{m',x}$ is an isomorphism by the choice of $N$. Thus using Lemma 7.7 again,

$$(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m,m'')} \otimes \mathcal{M})_x = 0$$

for any integer $m' \geq N$ and $m'' \geq m'$.

Let $\mathcal{U}$ be the complement of $Z$. Let $\mathcal{V} \subset \mathcal{U}$ be a strictly affine open subscheme. By the above observation, we get that

$$\Gamma(\mathcal{V}, \hat{E}_{\mathcal{X},\mathbb{Q}}^{(m,m'')} \otimes \mathcal{M}) = 0$$

Since $\Gamma(\mathcal{V}, E_{\mathcal{X},\mathbb{Q}}^{(m',1)})$ is a Fréchet-Stein algebra, we get that

$$\Gamma(\mathcal{V}, E_{\mathcal{X},\mathbb{Q}}^{(m',1)} \otimes \mathcal{M}) = 0.$$

Thus the proposition follows.

7.10. Proof of Theorem 7.2 We use the notation in the proof of Proposition 7.6. There exists an integer $M$ such that $\mathcal{H}_k = \mathcal{H}_M$ for $k \geq M$. Note that to see the existence of this $M$, we did not use the assumption of Proposition 7.6. Let $x \in \mathcal{T}^* \mathcal{X}$. Suppose there is an integer $m' > M$ such that $(\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m')} \otimes \mathcal{M})_x = 0$. Then by using Lemma 7.7,

$$0 = (\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m')} \otimes \mathcal{M})_x \cong (\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m,m'')} \otimes \mathcal{M})_x / \mathcal{H}_{m',x} = (\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m,m'')} \otimes \mathcal{M})_x / \mathcal{H}_{m''} \cap (\hat{E}_{\mathcal{X},\mathbb{Q}}^{(m',m'')} \otimes \mathcal{M})_x.$$
for an integer $m' \geq m'' \geq M$. However, since the last inclusion is dense, we have $(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m'',m')} \otimes \mathcal{M})_x = 0$. This implies that $(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m')} \otimes \mathcal{M})_x = 0$ for $m' \geq m'' \geq M$. Thus,

$$\operatorname{Supp}(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m')} \otimes \mathcal{M}) \subset \operatorname{Supp}(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m'+1)} \otimes \mathcal{M})$$

for $m' \geq M$.

Now suppose there exists $M' \geq M$ such that $\operatorname{Supp}(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(M')} \otimes \mathcal{M}) = T^*\mathcal{X}$. Then there is nothing to prove. So we may suppose that the dimension of $\operatorname{Supp}(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m')} \otimes \mathcal{M})$ is equal to 1 for any $m' \geq M$.

In this case, for any $m' \geq M$, there exists an open formal subscheme $\mathcal{U}_{m'}$ of $\mathcal{X}$ such that

$$\operatorname{Char}(\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m')} \otimes \mathcal{M}|_{\mathcal{U}_{m'}}) \cap T^*\mathcal{U}_{m'} = \mathcal{U}_{m'},$$

and $\mathcal{U}_{m'} \supset \mathcal{U}_{m'+1}$. Let $\mathcal{M}^{(m')} := \widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(m')} \otimes \mathcal{M}$. The module $\mathcal{M}^{(m')}|_{\mathcal{U}_{m'}}$ is a coherent $\mathcal{O}_{\mathcal{U}_{m'},\mathbb{Q}}$-module. Let $r_{m'}$ be the projective rank as an $\mathcal{O}_{\mathcal{U}_{m'},\mathbb{Q}}$-module. Then we know that $r_{m'} \geq r_{m'+1}$ for any $m' \geq M$. There exists an integer $N \geq M$ such that $r_N = r_{m'}$ for any $m' \geq N$. By the choice of $N$, the canonical homomorphism

$$\Gamma(\mathcal{U}_{m'},\mathcal{M}^{(N)}) \to \Gamma(\mathcal{U}_{m'},\mathcal{M}^{(m')})$$

is an isomorphism for $m' \geq N$. Indeed, since both sides are finite over $\Gamma(\mathcal{U}_{m'},\mathcal{O}_{\mathcal{X},\mathbb{Q}})$, and the image is dense, the homomorphism is surjective. However, since the rank of both sides are the same by assumption, the homomorphism is an isomorphism. Now, the proof of [Og, 2.16] is saying that for a smooth curve $\mathcal{Y}$ and a $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m)}$-module $\mathcal{M}$ which is coherent as an $\mathcal{O}_{\mathcal{Y},\mathbb{Q}}$-module, if there exists an open formal subscheme $\mathcal{V}$ such that $\mathcal{M}|_{\mathcal{V}}$ is a $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m+1)}$-module, then $\mathcal{M}$ is a $\widehat{\mathcal{D}}_{\mathcal{Y},\mathbb{Q}}^{(m)}$-module. Thus $\mathcal{M}^{(N)}|_{\mathcal{V}_N}$ is already a $\widehat{\mathcal{D}}_{\mathcal{V}_N,\mathbb{Q}}^{(m')}$. This implies that the condition of Proposition [7,6] holds, and we obtain the theorem.

References

[Ab] Abe, T.: Comparison between Swan conductors and characteristic cycles, Compos. Math. 146, p.638–682 (2010).

[AM] Abe, T., Marmora, A.: Product formula for $p$-adic epsilon factors, available at http://arxiv.org/abs/1104.1563

[AtMa] Atiyah, M. F., MacDonald, I. G.: introduction to commutative algebra, Addison-Wesley Publishing (1969).

[Be1] Berthelot, P.: $\mathcal{D}$-modules arithmétiques. I. Opérateurs différentiels de niveau fini, Ann. Sci. École Norm. Sup. 29, p.185–272 (1996).

[Be2] Berthelot, P.: $\mathcal{D}$-modules arithmétiques. II. Descente par Frobenius, Mém. Soc. Math. Fr. 81 (2000).

[Be] Berthelot, P.: Introduction à la théorie arithmétique des $\mathcal{D}$-modules, Astérisque 279, p.1–80 (2002).

[BGR] Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis, Grundlehren der math. Wissenschaften 261, Springer (1984).

[Bj] Björk, J. E.: Analytic $\mathcal{D}$-modules and applications, Mathematics and its applications 247, Kluwer (1993).

[Bo] Bourbaki, N.: Algèbre Commutative, Hermann.

[CM] Christol, G., Mebkhout, Z.: Équations différentielles $p$-adiques et coefficients $p$-adiques sur les courbes, Astérisque 279, p.125–183 (2002).

[Cr] Crew, R.: F-isocrystals and their monodromy groups, Ann. Sci. École Norm. Sup. 25, p.429–464 (1992).

[De] Deligne, P.: La conjecture de Weil II, Publ. Math. IHES 52, p.137–252 (1980).

[EGA] Grothendieck, A.: Eléments de Géométrie Algébrique, Publ. Math. IHES

[Ga] Garnier, L.: Théorèmes de division sur $\widehat{\mathcal{D}}_{X,\mathbb{Q}}^{(0)}$ et applications, Bull. Soc. Math. Fr. 123, p.547–589 (1995).

[Go] Godement, R.: Topologie algébrique et théorie des faisceaux, Hermann.
Hartshorne, R.: *Algebraic geometry*, GTM 52, Springer.

Huishi, L., Oystaeyen, F.: *Zariskian filtrations*, K-Monographs in math. 2, Kluwer (1996).

Kashiwara, M., Kawai, T.: *On holonomic system of microdifferential equations. III*, Publ. RIMS 17, p.813–979 (1981).

Lafforgue, L.: *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. 147, p.1–241 (2002).

Lam, T. Y.: *Lectures on Modules and Rings*, GTM 189, Springer.

Laumon, G.: *Transformations canoniques et spécialisation pour les $\mathcal{D}$-modules filtrés*, Asterisque 130, p.56–129 (1985).

López, R. G.: *Microlocalization and stationary phase*, Asian J. Math. 8, p.747–768 (2004).

Marmora, A.: *Microdifferential arithmétiques sur une courbe*, in preparation.

Ogus, A.: *F-isocrystals and de Rham cohomology II*, Duke Math. J. 51, no. 4, p.765–850 (1984).

Grothendieck, A., et al.: *Théorie des topos et cohomologie étale des schémas (SGA 4).*

Schneider, P., Teitelbaum, J.: *Algebras of p-adic distributions and admissible representations*, Invent. Math. 153, p.145–196 (2003).

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