Charge ordering and interlayer phase coherence in quantum Hall superlattices

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The possibility of the existence of states with a spontaneous interlayer phase coherence in multilayer electron systems in a high perpendicular to the layers magnetic field is investigated. It is shown that phase coherence can be established in such systems only within individual pairs of adjacent layers, while such coherence does not exist between layers of different pairs. The conditions for stability of the state with interlayer phase coherence against transition to a charge-ordered state are determined. It is shown that in the system with the number of layers \( N \leq 10 \) these conditions are satisfied at any value of the interlayer distance \( d \). For \( N > 10 \) there are two intervals of stability: at sufficiently large and at sufficiently small \( d \). For \( N \to \infty \) the stability interval in the region of small \( d \) vanishes.

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I. INTRODUCTION

In the last 10 years main attention in studying the quantum Hall effect and related phenomena was given to bilayer systems. One of the most interesting features of bilayer quantum Hall systems is the possibility of realizing in them a superfluid state of a gas of electron-hole pairs with spatially separated components. For the first time, the possibility of superfluidity of this type was predicted in Refs. 1 and 2 with reference to bilayer electron-hole systems in zero magnetic field (see also Refs. 3 and 4). In Ref. 5 (see also Refs. 6, 7, 8, 9 for more details) it was shown that strong magnetic field applied to a bilayer electron-hole system favors the formation of a superfluid state, since, unlike the situation considered in Refs. 1 and 2, in the latter case it is not necessary to satisfy the condition of nesting of the Fermi surfaces of the electrons and holes.

Owing to the electron-hole symmetry, a bilayer electron-electron system in a quantizing magnetic field with filling factors of the layers equal to \( \nu_1 = \nu \) and \( \nu_2 = 1 - \nu \) (\( \nu < 1 \)) is a practically complete analog of bilayer electron-hole systems, and a transition to a superfluid phase is possible in such a system as well\(^{10,11,12,13,14,15,16} \). Recently, interest in this question has risen significantly in connection with experiments\(^{17,18} \) in which an effect of the Josephson type was observed (a sharp peak of the differential tunneling conductance at gate voltage biased near zero). Furthermore, in a recent experiment\(^{19} \) on the interlayer drag in bilayer systems a strong suppression of the longitudinal drag current and an appearance of the Hall component of the drag current was observed. This effect takes place in the situation when the total filling factor \( \nu_{\text{tot}} = \nu_1 + \nu_2 \) becomes close to unity. The effects observed in Refs. \(^{17,18,19} \) can be explained on the assumption that a spontaneous interlayer phase coherence arises in the system and a transition of the system to a superfluid state occurs. A direct experimental proof of Bose-Einstein condensation of interwell excitons (photoexcited electron-hole pairs with spatially separated components) in double quantum wells in zero magnetic field was also obtained recently\(^{20} \). The proof\(^{20} \) was based on the measured temperature dependence of the luminescence spectra.

For further understanding of this phenomenon it is naturally to study the superfluid properties of multilayer electron systems. Multilayer systems are of particular interest for studying collective properties of electron-hole superfluids by optical and acoustic methods, since the integrated intensity of the interaction of such systems with external fields increases in proportion to the number of layers. The present study is devoted to investigation of the effect of interlayer phase coherence in quantum Hall superlattices.

The realization of multilayer quantum Hall systems was reported in Refs. \(^{21,22,23} \). The research\(^{21,22,23} \) was mainly concerned with analysis of the properties of chiral edge states in such systems. Previously, the question of interlayer phase coherence in multilayer quantum Hall systems was considered in Refs. \(^{24} \) and \(^{25} \). The authors of Ref. \(^{25} \) have shown that a state in which phase coherence arises between all layers can be realized in such systems (below, this is referred to as a state with global phase coherence). It has been shown in Ref. \(^{24} \) that in a multilayer quantum Hall system another state can arise (which we shall call a dimer state) in which the system separates into pairs of adjacent...
layers and the interlayer phase coherence is established only within each pair, while there is no such coherence between layers belonging to different pairs. This raises the question of reconciliation of the results of Refs. 24 and 25. In the present paper we show that the states of Refs. 24 and 25 correspond to two different solutions of the self-consistency equations for the order parameters describing the interlayer phase coherence. We find that the energy of the dimer state is less than the energy of the state found in Ref. 25 and, hence, it is the dimer state that should be considered as a candidate for the ground state of such a system.

It was shown in Ref. 25 that conditions for the existence of the state with interlayer phase coherence in multilayer systems are more restrictive than for bilayer systems. Multilayer quantum Hall systems, unlike the bilayer ones, demonstrate a tendency toward formation of a charge-ordered state (where the filling factors of adjacent layers become equal to 0 and 1). In Ref. 25 the stability conditions for the state with interlayer phase coherence were analyzed under the assumption that the phase coherence has a global character. Since this assumption is not confirmed, the stability conditions will have to be reexamined. We find that in the limit of an infinite number of layers (the case considered in Ref. 25) the dimer state is stable against transition to the charge-ordered state already at interlayer distances \( d > d_c \approx 1.45\ell_B \) (\( \ell_B \) is the magnetic length), which is less than the critical value \( d_c \approx 1.7\ell_B \) obtained in Ref. 25.

In connection with the fact that the properties of systems with number of layers \( N > 2 \) and \( N \rightarrow \infty \) are substantially different, and in real physical systems the number of layers is always finite, an important question arises as to how the stability conditions depend on \( N \). In this paper this question is studied with reference to the system with even \( N \). It is found that for \( N \leq 10 \) the state with interlayer phase coherence is stable for arbitrary \( d \). For \( N > 10 \) stability takes place only for \( d < d_{c1} \) or for \( d > d_{c2} \), where \( d_{c1} \) decreases and \( d_{c2} \) increases with increasing \( N \) while remaining in the interval \( 0 < d_{c1} < d_{c2} < d_c \).

II. INTERLAYER PHASE COHERENCE IN BILAYER SYSTEM WITH TOTAL FILLING FACTOR \( \nu_{tot} = 1 \)

Prior studying the multilayer system, we describe the approach used in this paper with reference to the simplest case \( N = 2 \) and recall the bilayer situation. Let us consider a double-layer electron system subjected by a high magnetic field applied perpendicular to the layers. We imply the filling factor \( \nu_{tot} = \nu_1 + \nu_2 = 1 \). Tunneling between layers is assumed to be so weak to be neglected in the Hamiltonian. If the cyclotron frequency is much higher than all the other characteristic energies of the problem, the system can be treated in the lowest Landau level approximation, in which the influence of the upper unfilled Landau levels on the dynamics of the system is neglected.

The Hamiltonian of this system has the form

\[
H = \frac{1}{2} \sum_{n,n'=1,2} \int d^2r_n d^2r_{n'} \Psi_n^+(r_n) \Psi_{n'}^+(r_{n'}) V_{nn'}(|r_n - r_{n'}|) \Psi_{n'}(r_{n'}) \Psi_n(r_n) - \sum_{n=1,2} \int d^2r_n \mu_n \Psi_n^+(r_n) \Psi_n(r_n) + H_{BG},
\]

(1)

where \( \Psi_n \) are the Fermi field operators,

\[
V_{nn'}(r) = \frac{e^2}{\varepsilon [r^2 + (n - n')^2d^2]^{1/2}}
\]

(2)

is the Coulomb potential, \( \varepsilon \), the dielectric constant, and \( \mu_n \), the chemical potential in layer \( n \). The term \( H_{BG} \) in (1) describes the interaction of electrons with positively charged impurities in doping layers. For simplicity we assume that the \( z \) coordinates of the electron layers and doping layers coincide. If one takes into account the difference of the \( z \) coordinates of the electron and doping layers which takes place in the real physical situation it results in an appearance of additional constant term in the energy that does not influence the effects studied in this paper.

For further analysis it is convenient to transform to the operators of the Fourier components of the electron density \( \rho_n(q) \), in terms of which the Hamiltonian (1) can be written in the form

\[
H = \frac{1}{2\nu} \sum_{n,n'=1,2} \sum_q \left\{ V_{nn'}(q) \left[ \rho_n(q) \rho_{n'}(-q) - \delta_{nn'} \exp(-\frac{q^2\ell_B^2}{2})\rho_n(0) \right] \right\} - \sum_{n=1,2} \mu_n \rho_n(0) + H_{BG},
\]

(3)
where $S$ is the area of the layer. The operator $\rho_n(q)$ expressed in terms of the creation and annihilation operators for electrons in the lowest Landau level ($a^+(X), a(X)$) ($X$ is the guiding center of the orbit) has the form:

$$\rho_n(q) = \sum_X a^+_n(X + qy_2B/2)a_n(X - qy_2B/2) \exp(iq_xX - q^2\ell_B^2/4),$$

(4)

In Eq. (4) the quantity

$$V_{mn'}(q) = \frac{2\pi e^2}{\varepsilon q} \exp(-qd|n - n'|)$$

(5)

is the Fourier component of the Coulomb potential.

For further analysis we use the mean field approximation. In this approximation the Hamiltonian takes the form

$$H_{MF} = \sum_X \{\epsilon_1a^+_1(X)a_1(X) + \epsilon_2a^+_2(X)a_2(X) - [J\Delta a^+_1(X)a_2(X) + h.c.]\}.$$  

(6)

The energies $\epsilon_n$ in (6) are

$$\epsilon_n = V_n - I\nu_n - \mu_n,$$

(7)

where $V_n$ are the Fourier components of the Coulomb interaction at $q = 0$ screened by positively charged doping impurities. For the bilayer system these quantities are equal to

$$V_1 = \frac{1}{2\pi \ell_B^2} \lim_{q \to 0} \left[V_{11}(q) \left(\nu_1 - \frac{1}{2}\right) + V_{12}(q) \left(\nu_2 - \frac{1}{2}\right)\right] = W\tilde{\nu},$$

$$V_2 = \frac{1}{2\pi \ell_B^2} \lim_{q \to 0} \left[V_{22}(q) \left(\nu_2 - \frac{1}{2}\right) + V_{21}(q) \left(\nu_1 - \frac{1}{2}\right)\right] = -W\tilde{\nu}.$$  

(8)

The parameter $\tilde{\nu} = (\nu_1 - \nu_2)/2$ describes the value of the imbalance of the filling factors of the layers. In Eq. (6) $\Delta$ is the order parameter, defined as $\Delta = \langle a^+_1(X)a_1(X) \rangle$. In general case the order parameter $\Delta = |\Delta|e^{i\varphi}$ is complex valued. We restrict consideration to the case where its modulus $|\Delta|$ and phase $\varphi$ are independent of $X$. In deriving Eq. (6) we have also taken into account that the averages $\langle a^+_n(X)a_n(X) \rangle$ are equal to $\nu_n$. In Eqs. (6), (7) and (8) we have used the following energy parameters: the parameter

$$W = \frac{e^2d}{\varepsilon \ell_B^2}$$

(9)

that describes the energy of the direct Coulomb interaction between layers, and the parameters

$$I = \frac{1}{S} \sum_q V_{11}(q) \exp(-q^2\ell_B^2/2) = \sqrt{\frac{\pi e^2}{2 \varepsilon \ell_B}},$$

(10)

$$J = \frac{1}{S} \sum_q V_{12}(q) \exp(-q^2\ell_B^2/2) = \sqrt{\frac{\pi e^2}{2 \varepsilon \ell_B}} \exp\left(\frac{d^2}{2\ell_B}\right) \text{erfc}\left(\frac{d}{\ell_B\sqrt{2}}\right)$$

(11)

that describe the energy of the intralayer and interlayer exchange interactions.

In a state in which the order parameter $\Delta$ is nonzero, the motion of electrons in one layer is correlated with the motion of holes (empty single-particle states on the lowest Landau level) in the other layer. This state can be treated as a gas of electron-hole pairs, which are composite bosons. The Bose-Einstein condensate of such pairs (a true condensate at $T = 0$ or a quasi-condensate with a fluctuating phase at $T \neq 0$) will demonstrate superfluid properties. The state with nonzero superflow is the state in which nondissipative electrical current in one layer is accompanied by equal and oppositely directed nondissipative electrical current in the other layer.

For finding the self-consistency conditions for the order parameter $\Delta$ and chemical potentials $\mu_1$ and $\mu_2$ we subject the operators $a_1$ and $a_2$ to a $u - v$ transformation of the form

$$a_1 = u\alpha + v^*\beta^+, \quad a_2 = u^*\beta^+ - v\alpha.$$  

(12)
The operators \( \alpha \) and \( \beta \) will satisfy the Fermi commutation relations if \( |u|^2 + |v|^2 = 1 \). This allows to seek the coefficients of the \( u - v \) transformation in the form \( u = \cos(\Theta/2) \) and \( v = \sin(\Theta/2) e^{i\chi} \). From the requirement of vanishing the terms nondiagonal in \( \alpha \) and \( \beta \) in the transformed Hamiltonian, we find that

\[
\sin \Theta = \frac{J|\Delta|}{\sqrt{\epsilon^2 + (J|\Delta|)^2}}, \quad \cos \Theta = \frac{\hat{\epsilon}}{\sqrt{\epsilon^2 + (J|\Delta|)^2}},
\]

(13)

and that \( \chi \) coincides with the phase \( \varphi \) of the order parameter. Here we introduce the notation \( \hat{\epsilon} = (\epsilon_1 - \epsilon_2)/2 \).

Using Eqs. (12) and (13), we obtain the following self-consistency equations:

\[
\nu_1 + \nu_2 = 1 = \langle a_1^+ a_1 \rangle + \langle a_2^+ a_2 \rangle = 1 + N_F(\mathcal{E}_\alpha) - N_F(\mathcal{E}_\beta),
\]

(14)

\[
2\tilde{\nu} = \langle a_1^+ a_1 \rangle - \langle a_2^+ a_2 \rangle = -\frac{\hat{\epsilon}}{\sqrt{\epsilon^2 + (J|\Delta|)^2}}[1 - N_F(\mathcal{E}_\alpha) - N_F(\mathcal{E}_\beta)],
\]

(15)

\[
\Delta = \langle a_2^+ a_1 \rangle = \frac{J\Delta}{2\sqrt{\epsilon^2 + (J|\Delta|)^2}}[1 - N_F(\mathcal{E}_\alpha) - N_F(\mathcal{E}_\beta)],
\]

(16)

where

\[
\mathcal{E}_{\alpha(\beta)} = \sqrt{\epsilon^2 + (J|\Delta|)^2} \pm \frac{\epsilon_1 + \epsilon_2}{2} - \frac{J\Delta}{2}\sqrt{\epsilon^2 + (J|\Delta|)^2} + \frac{J\Delta}{2}\sqrt{\epsilon^2 + (J|\Delta|)^2} - \{\mathcal{E}_{\alpha(\beta)}\}^2
\]

(17)

are the energies of elementary excitations and \( N_F(\mathcal{E}) = [\exp(\mathcal{E}/T) + 1]^{-1} \) is a the Fermi distribution function. It follows from Eq. 11 that \( \mathcal{E}_\alpha = \mathcal{E}_\beta \), which leads to the condition \( \epsilon_1 + \epsilon_2 = 0 \). From Eqs. (15) and (16) one can easily find the dependence of the modulus of the order parameter on the value of the imbalance and temperature. Below we specify the case of zero temperatures, when this dependence has the form

\[
|\Delta(\tilde{\nu})| = \sqrt{\frac{1}{4} - \tilde{\nu}^2}.
\]

(18)

The differential of the free energy of the system at \( \nu_1 + \nu_2 = 1 \) and \( T = 0 \) is equal to

\[
dF = \mu_1 d\nu_1 + \mu_2 d\nu_2 = 2\tilde{\mu} d\tilde{\nu},
\]

(19)

where \( \tilde{\mu} = (\mu_1 - \mu_2)/2 \). Here and below we give the energy per electron. If the analytical form of the function \( \tilde{\mu}(\tilde{\nu}) \) is known, the energy of the system as a function of the imbalance value is easily found:

\[
F(\tilde{\nu}) = F(-1/2) + 2 \int_{-1/2}^{\tilde{\nu}} d\tilde{\nu}' \tilde{\mu}(\tilde{\nu}'),
\]

(20)

where \( F(-1/2) \) is the energy of the system for \( \tilde{\nu} = -1/2 \), i.e., for \( \nu_1 = 0 \) and \( \nu_2 = 1 \). At such filling factors the energy of the system is the sum of the energies of the direct Coulomb interaction between layers and the intralayer exchange interaction in layer 2, i.e., \( F(-1/2) = W/4 - I/2 \). Using Eq. 14 and taking into account the relation \( \hat{\epsilon} = -J\tilde{\nu} \) that follows from Eqs. (15) and (16), we find the following expression for \( \tilde{\mu} \):

\[
\tilde{\mu}(\tilde{\nu}) = (W - I + J)\tilde{\nu}.
\]

Accordingly,

\[
F(\tilde{\nu}) = -\frac{I + J}{4} + (W - I + J)\tilde{\nu}^2.
\]

(21)

It is useful to present expression (21) in the form

\[
F = W\tilde{\nu}^2 - \frac{I}{2} [\nu_1^2 + (1 - \nu_1)^2] - J|\Delta|^2
\]

(22)

from which the physical meaning of each of the terms becomes obvious. The first term is the energy of the direct Coulomb interaction at filling factor imbalance \( \tilde{\nu} \). The second term is the sum of the energies of the intralayer exchange interaction in layers 1 and 2. The third term is the energy connected with the interlayer phase coherence.
Since energy \( \tilde{H} \) is independent of the phase \( \varphi \) of the order parameter, the interlayer phase coherence is spontaneous one. The minimum energy of a state with interlayer phase coherence is reached for the modulus and phase of the order parameter independent of the coordinate. A state with nonzero superfluid current that corresponds to nonzero gradients of the phase is higher in energy, but in the case where the gradient of the phase is less than certain critical value (which depends on the ratio \( d/\ell_B \)), such a state remains metastable, because the system cannot emit normal excitations, i.e., the Landau criterion of superfluidity is satisfied.

To conclude this Section we turn our attention to an important observation that follows from formula (21). Using Eqs. (19)-(21), we see that the coefficient of the \( \tilde{W}^2 \) term on the right-hand side of (21) is positive for arbitrary \( d \neq 0 \). This plays a fundamental role for the existence of the state with interlayer phase coherence in bilayer systems. Indeed, if the sign of the expression \( W - I + J \) were negative, then the system would go to a state with a maximum imbalance of the filling factors, \( \tilde{\nu} = \pm 1/2 \), for which \( |\Delta| = 0 \) (see Eq. (18)) and, consequently, interlayer phase coherence is absent. Skipping ahead, we note that in multilayer systems, in contrast, such a mechanism of destroying the phase coherence can indeed be realized in a certain range of \( d \). This question is analyzed in details in Sec. IV.

III. STATES WITH INTERLAYER PHASE COHERENCE IN MULTILAYER SYSTEM

Let us consider a system with the number of layers \( N \to \infty \) with an average filling factor per layer equal 1/2. The Hamiltonian of such a system can be written in the form (3) with the sum over two layers replaced by a sum over an infinite number of layers. In the mean field approximation, the Hamiltonian of the multilayer systems is reduced to

\[
H_{MF} = \sum_n \sum_{X} \left\{ \epsilon_n a_n^+ a_n - \frac{1}{2} \sum_{m \neq 0} [J_m \Delta(n,m) a_n^+ a_{n+m} + h.c.] \right\},
\]

where the energy parameters \( J_m \) are equal to

\[
J_m = \sqrt{\frac{\pi}{2\epsilon_B}} \exp \left( \frac{d^2 m^2}{2 \epsilon_B^2} \right) \text{erfc} \left( \frac{d|m|}{\epsilon_B \sqrt{2}} \right)
\]

(the parameter \( J \) introduced in the previous Section coincides with \( J_1 \)) and the order parameters are defined as

\[
\Delta(n,m) = \langle a_{n+m}^+ a_n \rangle
\]

In general case the quantities \( \Delta(n,m) \) can be nonzero for arbitrary \( m \) (not only for \( m = \pm 1 \)). This corresponds to the situation when a correlation arises between the electrons belonging not only to the adjacent layers but also to the layers arbitrarily far apart. Since the Coulomb potential is long-ranged, such correlations may give significant contribution to the energy, and it is important to take them into account for finding the ground state of the system.

It is natural to expect from physical reasons that if an imbalance of the filling factors arise in such a system it has a periodic character. Let us consider the situation when such a period does not exceed twice the distance between adjacent layers. As it shown below, the doubling of the period may happen even in the absence of imbalance of the filling factors. Latter possibility corresponds to the state with interlayer phase coherence of the dimer type.

Let the filling factors of the layers be equal to \( \nu_n = 1/2 + (-1)^n \tilde{\nu} \). In this case the quantity \( \tilde{\nu} \) can be treated as a charge-ordering parameter. Basing on the assumption of periodicity of the system, we seek the order parameter \( \Delta(n,m) \) in the form

\[
\Delta(n,m) = \Delta_1(m) + (-1)^n \Delta_2(m).
\]

In general case the quantities \( \Delta_{1(2)}(m) \) can be complex valued, i.e., they can contain phase factors \( e^{i\varphi_{1(2)}(m)} \). The phase factors can no longer be specified independently, since the self-consistency equation imposes certain conditions on the phase difference \( \varphi_{1(2)}(m) \). However, these conditions do not fix all of the phases, and a certain arbitrariness in the choice of the quantities \( \varphi_{1(2)}(m) \) remains. It can be shown that the energy of the system is independent of the phases of the order parameters if the latter satisfy the self-consistency equations (we stress that we are considering the case when the modulus and phase of the order parameter are independent of \( X \)). An arbitrariness in the choice of phases is just a reflection of the spontaneous character of the phase coherence. It allows us to be more specific and restrict consideration to the simplest case of all the phases are equal to zero and the quantities \( \Delta_1(m) \) and \( \Delta_2(m) \) are real valued.

Transforming in Eq. (23) to the Fourier components of the operators \( a_n \)

\[
a_n = \frac{1}{\sqrt{N}} \sum_q e^{i q_n} a_q,
\]

where \( q_n = n \) and \( n = 0, \pm 1, \pm 2, \ldots \).
we obtain the following expression for the Hamiltonian of the system:

\[
H_{MF} = \sum_{q_x} \sum_{q_z} \{ [\epsilon - \Delta_1(q_z)] a_{q_x}^* a_{q_z} + [\tilde{\epsilon} - \Delta_{2r}(q_z) - i\Delta_{21}(q_z)] a_{q_x + q_z}^* a_{q_z} \}. \tag{28}
\]

Here the following notations are introduced \( \epsilon = (\epsilon_{2k} + \epsilon_{2k-1})/2, \tilde{\epsilon} = (\epsilon_{2k} - \epsilon_{2k-1})/2 \). In view of the assumed periodicity in \( z \) the quantities \( \epsilon \) and \( \tilde{\epsilon} \) are independent of \( k \). The functions \( \Delta_1(q_z) \) and \( \Delta_{2r}(q_z) \) in (28) are defined as

\[
\Delta_1(q_z) = 2 \sum_{m=1}^{\infty} J_m \Delta_1(m) \cos(mq_z), \tag{29}
\]

\[
\Delta_{2r}(q_z) = 2 \sum_{m=1}^{\infty} J_{2m} \Delta_2(2m) \cos(2mq_z), \tag{30}
\]

\[
\Delta_{21}(q_z) = 2 \sum_{m=0}^{\infty} J_{2m+1} \Delta_2(2m + 1) \sin((2m + 1)q_z). \tag{31}
\]

Diagonalizing the Hamiltonian (28) and calculating the averages of (26), we arrive to the self-consistency conditions. Here we present these equations for the case \( T = 0 \):

\[
\tilde{\nu} = -\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dq_z \frac{\tilde{\epsilon} - \Delta_{2r}(q_z)}{E(q_z)}, \tag{32}
\]

\[
\Delta_1(2m) = 0, \tag{33}
\]

\[
\Delta_1(2m + 1) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dq_z \frac{\cos((2m + 1)q_z) \Delta_1(q_z)}{E(q_z)}, \tag{34}
\]

\[
\Delta_2(2m) = -\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dq_z \frac{\cos(2mq_z) [\tilde{\epsilon} - \Delta_{2r}(q_z)]}{E(q_z)}, \tag{35}
\]

\[
\Delta_2(2m + 1) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dq_z \frac{\sin((2m + 1)q_z) \Delta_{21}(q_z)}{E(q_z)}. \tag{36}
\]

In equations (32) and (33-36) the spectrum of energies of elementary excitations \( E(q_z) \) is determined by the following relation:

\[
E(q_z) = \sqrt{[\tilde{\epsilon} - \Delta_{2r}(q_z)]^2 + \Delta_1^2(q_z) + \Delta_{21}^2(q_z)}. \tag{37}
\]

The state with global phase coherence considered in Ref. 25 corresponds to the solution of (32)-(36) with all the quantities \( \Delta_2(2m + 1) \) equal zero (in this case Eq. (36) is satisfied automatically) and \( \Delta_1(2m + 1) \neq 0 \) for arbitrarily large \( m \). The latter means that phase coherence is established between all layers.

It is easy to see that the self-consistency equations (32-36) are also satisfied by a solution of an essentially different type: \( \Delta_1(\pm 1) = \pm \Delta_2(\pm 1) \neq 0 \), but \( \Delta_1(m) = \Delta_2(m) = 0 \) at \( m \neq \pm 1 \). In this case \( \Delta(2k, 1) = \Delta(2k + 1, -1) \neq 0 \) (or \( \Delta(2k - 1, 1) = \Delta(2k, -1) \neq 0 \)), but all the remaining \( \Delta(n, m) \) are zero. In the state corresponding to such a solution, the system separates into pairs of layers, and the interlayer phase coherence is established only within each pair. We call this state the dimer state.

Writing the differential of the free energy of a multilayer system and integrating it over \( \tilde{\nu} \), we obtain an expression for the free energy of the multilayer system:

\[
F(\tilde{\nu}) = V_c(\tilde{\nu}) - I \left( \frac{1}{4} + \tilde{\nu}^2 \right) - 2 \sum_{m=1}^{\infty} J_m [\Delta_1^2(m) + \Delta_{21}^2(m)]. \tag{38}
\]
where the first term \( V_c(\tilde{\nu}) \) corresponds to the energy of the direct Coulomb interaction, which is independent of the concrete form of \( \Delta_{1(2)}(m) \).

Using the answer \( 55 \), let us compare the energies of the state with global phase coherence and of the dimer state for \( \tilde{\nu} = 0 \). At zero imbalance Eqs. \( 32 \)–\( 36 \) give the following expressions for the order parameters \( \Delta_{1,2}(m) \). For a state with global coherence we have

\[
\Delta_1(m) = \frac{\sin(\pi m/2)}{m\pi}, \quad \Delta_2(m) = 0. \tag{39}
\]

For the dimer state \( \Delta_1(m) = \pm \Delta_2(m) = \delta_{1,|m|}/4 \). Consequently, the energy difference of the dimer state and the state with global phase coherence satisfies the inequality

\[
F_d - F_g = -J_1/4 + 2\pi^2 \sum_{k=0}^{\infty} J_{2k+1} + 1/(2k+1)^2 \leq -J_1/4 \left( 1 - \frac{8}{\pi^2} \sum_{k=0}^{\infty} 1/(2k+1)^2 \right) = 0. \tag{40}
\]

In deriving Eq. \( 40 \) we take into account that for \( d \neq 0 \) the parameters \( J_m \) the smaller the large \( |m| \). For \( d = 0 \), when all \( J_m \) are identical, the energies of the two states under consideration are equal. A numerical analysis of the self-consistency equations shows that the inequality \( F_d < F_g \) holds for arbitrary values of \( \tilde{\nu} \) and \( d \neq 0 \). Since the energy of the state with global phase coherence is higher than the energy of the dimer state, this state but not the state considered in Ref. \( 45 \) should be considered as a possible candidate for the ground state of the system.

IV. COMPARISON OF THE ENERGIES OF THE DIMER AND CHARGE-ORDERED STATES

Let us now consider the question of the stability of the dimer state in a multilayer system with even number of layers against a transition to the charge-ordered state. We should specify the explicit form of the energy of the direct Coulomb interaction \( V_c(\tilde{\nu}) \). It can be easily done by using the fact that this energy is equal to the energy of the electric field induced inside the superlattice due to deviation of the filling factors of individual layers from the average value. Since the system on the whole remains electrically neutral, the electric field outside the superlattice is equal to zero.

For instance, for the filling factors of the layers equal to \( \nu_{2k-1} = 1/2 - \tilde{\nu} \) and \( \nu_{2k} = 1/2 + \tilde{\nu} \) (\( k = 1, 2, \ldots, N/2 \)), the electric field is nonzero between layers 1 and 2, 3 and 4, 5 and 6, etc., while between layers 2 and 3, 4 and 5, etc. it equals to zero. Then, the electric field energy per electron, as in the bilayer system, would be equal to \( V_c = W\tilde{\nu}^2 \).

However, in contrast to the bilayer system, in the multilayer system substantial benefit in the energy of the direct Coulomb interaction can be achieved by changing the filling factors of only the two outer layers. Indeed, if the filling factors of the outer layers take the values \( \nu_1 = 1/2 - \tilde{\nu}/2 \) and \( \nu_N = 1/2 + \tilde{\nu}/2 \), then the electric field is induced between all the layers, and its absolute value would be smaller by a factor of two than in the previous case (the direction of the field would change sign on crossing a layer). Consequently, in the last case the energy \( V_c \) (energy per electron) is in two times smaller than in the first case (with an accuracy up to \( 1/N \)). In such a state the total energy is

\[
F = -I + J/4 + \left( \frac{W}{2} - I + J \right) \tilde{\nu}^2. \tag{41}
\]

In deriving Eq. \( 41 \) we take into account that in the dimer state the order parameter \( \Delta = 2\Delta_{1(2)} \) has the same dependence on \( \tilde{\nu} \) as in the bilayer system (Eq. \( 15 \)). In contrast to Eq. \( 21 \), the sign of the coefficient of the \( \tilde{\nu}^2 \) term in Eq. \( 41 \) depends on \( d \): for \( d < 1.45\ell_B \) this coefficient is negative, while at larger \( d \) it is positive. Consequently, for \( d < 1.45\ell_B \) and \( N \rightarrow \infty \) the minimum energy corresponds to the state with the maximum charge ordering, in which all the \( \Delta(n,m) \) go to zero and phase coherence is absent. In the opposite case \( d > 1.45\ell_B \) the minimum of energy \( 41 \) is reached for \( \tilde{\nu} = 0 \) and charge ordering becomes energetically unfavorable.

The above analysis is easily generalized to the case of finite even \( N \). Let us consider the configuration where the inner layers are in a completely or partially charge-ordered state, and for these layers the imbalance of the filling factors is equal to \( \tilde{\nu}_n \). The imbalance of the filling factors of the outer layers \( \tilde{\nu}_{ex} = (\nu_N - \nu_1)/2 \) can be different from \( \tilde{\nu}_n \). We imply that phase coherence is established only between the inner layers and has the dimer character. The energy (per electron) of such a state is equal to

\[
F(\tilde{\nu}_{ex}, \tilde{\nu}_n) = W \left( \tilde{\nu}_{ex}^2 + \frac{N-2}{N} (\tilde{\nu}_n - \tilde{\nu}_{ex})^2 \right) - I \left( \frac{N-2}{N} \left( \frac{1}{4} + \tilde{\nu}_n^2 \right) + \frac{2}{N} \left( \frac{1}{4} + \tilde{\nu}_{ex}^2 \right) \right) - J \frac{N-2}{N} \left( \frac{1}{4} - \tilde{\nu}_n^2 \right). \tag{42}
\]

The minimum of \( F(\tilde{\nu}_{ex}, \tilde{\nu}_n) \) should be found in the domain specified by inequalities \( |\tilde{\nu}_n| \leq 1/2 \) and \( |\tilde{\nu}_{ex}| \leq 1/2 \). The function \( F(\tilde{\nu}_{ex}, \tilde{\nu}_n) \) has only one extremum (or saddle point) at \( \tilde{\nu}_{ex} = \tilde{\nu}_n = 0 \). Therefore, its minimum is reached
either at the point of the extremum or at the edge of the domain of allowed values of $\tilde{\nu}_{\text{in}}, \tilde{\nu}_{\text{ex}}$. Thus, for determining the stability region of the dimer phase it is sufficient to find the minimum value of expression (42) at the edge of the domain and compare it to the energy of the dimer phase in the absence of charge ordering, $F_d = -(I + J)/4$.

We obtain that for $WN < 2I$ the minimum of (42) at the edge of the domain is reached for $\tilde{\nu}_{\text{in}} = \tilde{\nu}_{\text{ex}} = \pm 1/2$. Since $F(\pm 1/2, \pm 1/2) = W/4 - I/2 > F_d$ for such parameters the dimer state is stable against the transition to the charge ordered state.

For $WN > 2I$, the minimum at the edge domain is reached at the points

$$\tilde{\nu}_{\text{ex}} = \pm \frac{1}{4} \frac{N - 2}{N - 1 - WN}, \quad \tilde{\nu}_{\text{in}} = \pm \frac{1}{2},$$

(43)

where the value of the energy is equal to

$$\frac{I}{2N} - \frac{I - J}{4} + \frac{W(WN - 2I)(N - 2)}{2N(WN + I - W)},$$

(44)

In particular, it follows from (43) that for $N \to \infty$ the minimum of energy at the edge of the domain corresponds just to the state mentioned above ($\tilde{\nu}_{\text{ex}} = \tilde{\nu}_{\text{in}} = 2/1/2$).

The value of expression (44) can be larger or smaller than $F_d$. Therefore, there is a region of instability of the dimer phase in the space of parameters $(d, N)$. It is determined by the system of inequalities

$$\frac{I}{2N} - \frac{I - J}{4} + \frac{W(WN - 2I)(N - 2)}{2N(WN + I - W)} < 0$$

(45)

We analyze the system of inequalities numerically and find that for $N \leq 10$ the conditions (45) are not satisfied for any $d$. Consequently, in a multilayer system with even $N \leq 10$ the dimer state is stable against transition to the charge-ordered phase at any values of $d$. For $N > 10$ the inequalities (45) are satisfied in certain interval of $d$: $d_{c1} < d < d_{c2}$. At such $N$ and $d$ there will be complete charge ordering of the inner layers and no phase coherence between those layers. The dependence of $d_{c1}$ and $d_{c2}$ on $N$ is shown in Fig. 1. Since $\lim_{N \to \infty} d_{c1}(N) = 0$, for $N \to \infty$ the domain of values $d$ for which the dimer state can exist is restricted by the condition $d > 1.45\ell_B$. However, at large but finite $N$ there exists a region of small $d$ ($d \leq \ell_B/2\pi/N$) in which the formation of the dimer state is possible.

To conclude this section we discuss shortly the role of tunneling. It was shown in Ref. 27 that the interlayer tunneling unfavours the charge ordering in multilayer systems. Based on a comparison of the energy of the system for $\tilde{\nu} = 0$ and $\tilde{\nu} = \pm 1/2$ the authors of that paper arrived at the conclusion that in the case when the tunneling parameter $t$ exceeds $0.05e^2/\varepsilon\ell_B$ the charge ordering becomes energetically unfavorable for any values of the parameter $d/\ell_B$.

In the presence of tunneling between layers the phase coherence is no longer spontaneous, since tunneling leads to fixing of the phase of the order parameter. In that case the superfluid state with a constant phase gradient cannot be realized, but superfluidity of a "soliton" type can exist.26,27 But, according to the results of Ref. 14 already for $t \geq 0.01e^2/\varepsilon\ell_B$ the Landau criterium is not fulfilled at any superfluid velocities. Therefore, tunneling plays more of a
negative than a positive role for realization of the superfluid state in multilayer systems. On the contrary, the use of multilayer systems with parameters lying in the stability region of the dimer state (Fig. 1) and with a small value of the tunneling is more promising.

V. CONCLUSION

We have shown that a state with spontaneous interlayer phase coherence can be realized in multilayer quantum Hall systems in a certain range of parameters. The phase coherence in such systems has a two-dimensional rather than a three-dimensional character; namely, the system separates into pairs of adjacent layers, and coherence is established only within each pair, and coherence is absent between electrons belonging to different pairs. If the number of layers \( N \leq 10 \), then for an arbitrary distance between layers the state with spontaneous interlayer phase coherence is stable against transition to a charge-ordered phase. If the number of layers \( N > 10 \), the state with spontaneous phase coherence is stable only for sufficiently small or sufficiently large distances between layers. In the intermediate region of interlayer distances in a system with more than 10 layers a transition occurs to a charge-ordered state in which interlayer phase coherence is absent.

Since the phase coherence has a two-dimensional character, a multilayer quantum Hall system can be considered as a solid-state analog of ultracold atomic Bose gases in optical superlattices (see, e.g., Ref. 28). Among new effects that can expect in such systems we would mention the nondissipative interlayer drag of superfluid flow, in analogy with similar effect predicted for bilayer Bose systems.29,30,31

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