Research Article

Weaving Frames in Hilbert $C^*$-Modules

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Received 29 April 2021; Accepted 3 August 2021; Published 21 August 2021

Academic Editor: Ding-Xuan Zhou

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In this paper, we investigate weaving frames in Hilbert $C^*$-modules. We show that the equivalence of woven and weakly woven frames is still true for modular frames under certain conditions. By using the analysis operators of frames and frame operators of canonical duals, we obtain several perturbation results for given weaving frames and different weaving frame pairs. When the $C^*$-algebra is nonunital, we derive a correspondence of adjointable operators which is bounded below woven families. Finally, we discuss the redundancy of weaving frames in Hilbert $C^*$-modules.

1. Introduction

Due to the useful applications in the characterization of function spaces, signal processing, and many other fields of applications, the theory of frames has developed rather rapidly in recent years. Various generalizations of frames have been developed. For example, $g$-frames [1], fusion frames [2], pseudoframes [3], and operator-valued frames [4]. Recently, Bemrose et al. [5] introduced a new concept of “weaving frames” in separable Hilbert spaces. This notion has potential applications in distributed signal processing and wireless sensor networks. Two frames $\{x_j\}_{j \in J}$ and $\{y_j\}_{j \in J}$ are said to be woven if there exist constants $0 < C \leq D$ such that, for every subset $\sigma \subset J$ ($J$ is a finite or countable index set), the family $\{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in \sigma^c}$ is a frame with frame bounds $(C, D)$. Over the years, various extensions of weaving frames have been investigated, see [6–10].

By allowing the inner product to take values in a $C^*$-algebra, Hilbert $C^*$-modules are natural generalizations of Hilbert spaces. Note that the theory of Hilbert $C^*$-modules is quite different from that of Hilbert spaces. For example, not all bounded linear operators on Hilbert $C^*$-modules are adjointable. Thus, there are many essential differences between Hilbert space frames and modular frames. The problems about modular frames are more complicated than those in Hilbert spaces. In this paper, we investigate the weaving properties of modular frames. We refer to [11–13] for more information on frames in Hilbert $C^*$-modules.

The paper is organized as follows. Section 2 contains the definitions and some basic results about frames in Hilbert $C^*$-modules. In Section 3, we introduce the weaving frames in Hilbert $C^*$-modules and show that the equivalence of woven and weakly woven frames is still effective for Hilbert $C^*$-modules under certain conditions. Sufficient conditions for perturbations of weaving frames are given in Section 4. In Section 5, we give some weaving results for the nonunital case, and we investigate the redundancy property of weaving frames in Section 6.

2. Preliminaries

We first recall some basics about Hilbert $C^*$-modules.

Definition 1. Let $\mathcal{A}$ be a $C^*$-algebra. A pre-Hilbert $\mathcal{A}$-module $\mathcal{H}$ is a linear space and an algebraic (left) $\mathcal{A}$-module with compatible scalar multiplication: $\lambda (ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in C$, $a \in \mathcal{A}$, and $x \in \mathcal{H}$, together with a map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ that possesses the following properties:

(i) $\langle x, x \rangle \geq 0$, for every $x \in \mathcal{H}$
For more information on Hilbert bounded. In the case \( M \), called full if linear\span{\( \langle \rangle \)} generates Hilbert (see \cite{13}). Proposition 1 which is easy to be applied.

\[ |\langle x, y \rangle| \leq \|x\| \|y\|, \quad \forall x, y \in H. \quad (3) \]

for all \( x \in \mathcal{H}. \)

For a unital \( C^* \)-algebra \( \mathcal{A} \), let \( \ell^2(\mathcal{A}) \) be a Hilbert \( \mathcal{A} \)-module defined by

\[ \ell^2(\mathcal{A}) = \bigg\{ \{a_j\}_{j \in J} \in \mathcal{A} : \sum_{j \in J} a_j^* a_j \text{ converges in norm} \bigg\}, \]

with the inner product \( \langle \{a_j\}_{j \in J}, \{b_j\}_{j \in J} \rangle = \sum_{j \in J} a_j^* b_j \). Note that \( \ell^2(\mathcal{A}) \) possesses a canonical basis \( \{e_j\}_{j \in J} \) where \( e_j \) takes value 1 at \( j \) and 0 everywhere else. For a given Bessel sequence \( X = \{x_j\}_{j \in J} \) in \( \mathcal{H} \), its analysis operator,

\[ U_X : \mathcal{H} \rightarrow \ell^2(\mathcal{A}), \]

is well defined and adjointable. The adjoint operator \( T_X \) of \( U_X \) fulfills \( T_X e_j = x_j \) for every \( j \) and is called the synthesis operator. By composing \( U_X \) and \( T_X \), we obtain the frame operator \( S_X \):

\[ S_X : \mathcal{H} \rightarrow \mathcal{H}, \]

\[ S_X x = T_X U_X x = \sum_{j \in J} \langle x, x_j \rangle x_j, \quad (6) \]

where \( S_X \) is invertible and each \( x \in \mathcal{H} \) can be represented as

\[ x = \sum_{j \in J} \langle x, S_{X}^{-1} x_j \rangle x_j, \quad (7) \]

where \( \{S_{X}^{-1} x_j\}_{j \in J} \) is called the canonical dual frame of \( \{x_j\}_{j \in J} \).

At the end of this section, we present some fundamental results which will be used in the following sections.

Proposition 2 (see \cite{17}). For an operator \( F : \mathcal{H} \rightarrow \mathcal{H}, \) the following conditions are equivalent:

(i) \( F \) is a positive element of \( \mathcal{L}(\mathcal{H}) \)

(ii) For every \( x \in \mathcal{H}, \) \( \langle Fx, x \rangle \) is positive in \( \mathcal{A} \)

Proposition 3 (see \cite{20}). Let \( F \in \mathcal{L}(\mathcal{H}). \) Then, for any \( x \in \mathcal{H}, \) we have \( \langle Fx, Fx \rangle \leq \|F\|^2 \langle x, x \rangle. \)

3. Weaving Frames in Hilbert \( C^* \)-Modules

In this section, we investigate the weaving frames in Hilbert \( C^* \)-modules. It can be observed from (7) that each frame for a Hilbert \( C^* \)-module \( \mathcal{H} \) generates \( \mathcal{H}. \) This means that a Hilbert \( C^* \)-module that admits frames is necessarily countably generated.

Throughout the rest of the paper, for ease of notation, let

\[ [m] = \{1, \ldots, m\}, \]

for a given natural number \( m. \) First, we give the definition of weaving frames in Hilbert \( C^* \)-modules.
A finite family of frames \( \{ x_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) in a Hilbert \( \mathcal{A} \)-module \( \mathcal{H} \) is said to be woven if there are universal constants \( C \) and \( D \) such that, for every partition \( \{ \sigma_1, \ldots, \sigma_m \} \) of \( \mathcal{J} \), the family \( \{ x_{ij} \}_{i=1, j \in \sigma_i} \) is a frame for \( \mathcal{H} \) with lower and upper frame bounds \( C \) and \( D \), respectively. In this case, we usually call \( \{ x_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) woven with universal bounds \( (C, D) \). Each family \( \{ x_{ij} \}_{i=1, j \in \sigma_i} \) is called a weaving.

It is known from Proposition 3.1 in [5] that every weaving for a woven family automatically has a universal upper frame bound. By the positivity of the summand in (2), we can also derive a universal Bessel bound for every weaving in a Hilbert \( C^* \)-module.

Proposition 4. Suppose that \( \{ x_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) are Bessel sequences for a Hilbert \( \mathcal{A} \)-module \( \mathcal{H} \) with Bessel bounds \( D_j \), respectively. Then, every weaving is a Bessel sequence with a Bessel bound \( \sum_{i=1}^m D_i \).

Proof. For every partition \( \{ \sigma_1, \ldots, \sigma_m \} \) of \( \mathcal{J} \) and any \( x \in \mathcal{H} \), we have

\[
\sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \leq \sum_{i=1}^m \sum_{j \in \mathcal{J}} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \leq \sum_{i=1}^m D_i \langle x, x \rangle,
\]

as required.

Consequently, we only need to check the lower frame bound when studying the property of woven frames. The following result implies that the woven property is preserved under some adjointable operators.

Proposition 5. Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( \{ x_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) be a woven family of frames for \( \mathcal{H} \) with universal bounds \( (C, D) \).

1. If \( F \in \mathcal{L}(\mathcal{H}) \) is invertible, then the family \( \{ Fx_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) is woven with universal bounds \( (C\|F^{-1}\|^{-2}, D\|F\|^2) \).

2. If \( G \in \mathcal{L}(\mathcal{H}) \) is a co-isometry, then \( \{ Gx_{ij} \}_{i=1, j \in \mathcal{J}} \) is woven with universal bounds \( (C, D) \).

Proof

(1) For every partition \( \{ \sigma_1, \ldots, \sigma_m \} \) of \( \mathcal{J} \) and any \( x \in \mathcal{H} \),

\[
C\|F^*x, F^*x\| \leq \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, Fx_{ij} \rangle \langle Fx_{ij}, x \rangle \leq D\|F^*x, F^*x\|.
\]

It follows from Proposition 3 that

\[
\|F^{-1}\|^2 \langle x, x \rangle \leq \|F^*x, F^*x\| \leq \|F\|^2 \langle x, x \rangle.
\]

Hence, \( \{ Fx_{ij} \}_{i=1, j \in \sigma} \) is a frame with bounds \( (C\|F^{-1}\|^{-2}, D\|F\|^2) \).

(2) Since \( G \) is an co-isometry, we know \( \|G^*x\| = \|x\| \).

Then, by E. C. Lance's theorem [16], \( \langle G^*x, G^*x \rangle = \langle x, x \rangle \). Now, the conclusion follows from a similar discussion of (1).

We show that multiplying the frames in the woven family by individual elements still preserves the woven property.

Proposition 6. Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( \{ x_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) be a woven family of frames for \( \mathcal{H} \) with universal frame bounds \( (C, D) \). If there exist \( \alpha_{ij} \in \mathcal{A} \) and \( 0 < \alpha \leq \beta \leq \infty \) which satisfies \( \alpha \cdot 1_{\mathcal{J}} \leq \alpha_{ij} \leq \beta \cdot 1_{\mathcal{J}} \), then \( \{ \alpha_{ij} x_{ij} \}_{i=1, j \in \mathcal{J}} \) is also woven with universal bounds \( (\alpha C, \beta D) \).

Proof. Recall that \( a \leq b \) implies \( c^*ac \leq c^*bc \), for any \( a, b \in \mathcal{A} \) and \( c \in \mathcal{A} \).

On the one hand, we have for any partition \( \{ \sigma_1, \ldots, \sigma_m \} \) of \( \mathcal{J} \) and any \( x \in \mathcal{H} \),

\[
\sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, \alpha_{ij} x_{ij} \rangle \langle \alpha_{ij} x_{ij}, x \rangle = \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \alpha_{ij}^* \alpha_{ij} \langle x_{ij}, x \rangle \leq \beta \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \leq \beta D \langle x, x \rangle.
\]

(12)

On the other hand, we also have

\[
\sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, \alpha_{ij} x_{ij} \rangle \langle \alpha_{ij} x_{ij}, x \rangle = \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \alpha_{ij}^* \alpha_{ij} \langle x_{ij}, x \rangle \geq \alpha \sum_{i=1}^m \sum_{j \in \sigma_i} \langle x, x_{ij} \rangle \langle x_{ij}, x \rangle \geq \alpha C \langle x, x \rangle.
\]

(13)

This gives the desired result.

Now, we give a definition of weakly woven frames.

Definition 4. A finite family of frames \( \{ x_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) in a Hilbert \( \mathcal{A} \)-module \( \mathcal{H} \) is called weakly woven if, for every partition \( \{ \sigma_1, \ldots, \sigma_m \} \) of \( \mathcal{J} \), the family \( \{ x_{ij} \}_{i=1, j \in \sigma_i} \) is a frame for \( \mathcal{H} \).

Bemrose et al. presented a characterization of the equivalence of woven and weakly woven frames in [5]. At the end of this section, we consider whether this equivalence is still effective for Hilbert \( C^* \)-modules. As a preparation, we need the following proposition.

Proposition 7. Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( \{ x_{ij} \}_{j \in \mathcal{J}, i \in [m]} \) be a family of frames for \( \mathcal{H} \). Assume, for every partition \( \{ \Lambda_1 \}_{i=1}^m \) of \( \mathcal{J} \) and every \( \epsilon > 0 \), there exists a \( \Lambda_i \subset \Gamma_i \) \( (i \in [m]) \) partition \( \{ \Gamma_i \}_{i=1}^m \) of \( \mathcal{J} \) such that and the lower frame bound of
\{x_{ij}\}_{i=1}^m \text{ is less than } \varepsilon. \text{ Then, there exists a partition } \{\sigma_i\}_{i=1}^m \text{ of } J \text{ such that } \{x_{ij}\}_{i=1,j \in \sigma_i} \text{ is not a frame.}

\textbf{Proof.} Assume without loss of generality that } J = \mathbb{N}. \text{ Choose } \varepsilon = 1/2 \text{ and } \Lambda_i^\varepsilon = \emptyset \text{ for } i \in [m]. \text{ Then, by assumption, there exists a partition } \{\Gamma_i\}_{i=1}^m \text{ of } \mathbb{N} \text{ such that } \Lambda_i^\varepsilon \subset \Gamma_i^\varepsilon \text{ for all } i \in [m] \text{ and the lower frame bound of } \{x_{ij}\}_{i=1,j \in \sigma_i} \text{ is less than } 1/2. \text{ So, there exists a element } y_1 \text{ such that}

\sum_{i=1}^m \sum_{j \in \Gamma_i^\varepsilon} \langle x_{ij}, x_{ij}, y_1 \rangle < \frac{1}{2} \langle y_1, y_1 \rangle. \quad (14)

By the convergence of the series \(\sum_{i=1}^m \sum_{j \in \Gamma_i} \langle x_{ij}, x_{ij}, y_2 \rangle\) for each \(x \in \mathcal{H}\), we can find a \(k_1 \in \mathbb{N}\) such that \(\sum_{i=1}^m \sum_{j \in \Gamma_i} \langle x_{ij}, x_{ij}, y_1 \rangle < 1/2 \langle y_1, y_1 \rangle\) for all \(i \in [m]\) and \(\Lambda_i^\varepsilon = \Gamma_i \cap [k_1]\) and \(\varepsilon = 1/2^2\). Then, there exists a partition \(\{\Gamma_i\}_{i=1}^m\) of \(\mathbb{N}\) and an element \(y_2 \in \mathcal{H}\) such that \(\Lambda_i^\varepsilon \subset \Gamma_i^\varepsilon\) for any \(i \in [m]\) and

\sum_{i=1}^m \sum_{j \in \Gamma_i^\varepsilon} \langle x_{ij}, x_{ij}, y_2 \rangle < \frac{1}{2^2} \langle y_2, y_2 \rangle. \quad (15)

Since \(\sum_{i=1}^m \sum_{j \in \Gamma_i} \langle x_{ij}, y_1, x_{ij} \rangle \langle x_{ij}, y_2 \rangle\) converges, there exists a \(k_2 > k_1\) such that

\sum_{i=1}^m \sum_{j \geq k_2} \langle x_{ij}, x_{ij}, y_2 \rangle < \frac{1}{2^2} \langle y_2, y_2 \rangle. \quad (16)

Continuing in this way, for a partition \(\Lambda_i^{n+1} = \Gamma_i^{n+1} \cap [k_n]\) of \([k_n]\), we can find a partition \(\{\Gamma_i^{n+1}\}_{i=1}^m\) of \(\mathbb{N}\) and \(y_{n+1} \in \mathcal{H}\) such that \(\Lambda_i^{n+1} \subset \Gamma_i^{n+1}\) for every \(i \in [m]\) and

\sum_{i=1}^m \sum_{j \geq k_{n+1}} \langle x_{ij}, x_{ij}, y_{n+1} \rangle < \frac{1}{2^{n+1}} \langle y_{n+1}, y_{n+1} \rangle. \quad (17)

Similar to above, there exists a \(k_{n+1} > k_n\) such that

\sum_{i=1}^m \sum_{j \geq k_{n+1}} \langle x_{ij}, x_{ij}, y_{n+1} \rangle < \frac{1}{2^{n+1}} \langle y_{n+1}, y_{n+1} \rangle. \quad (18)

Now, set \(\sigma_i = \bigcup_{n=1}^{\infty} \Lambda_i^{n+1}\) for \(i \in [m]\) and observe that \(\{\sigma_i\}_{i=1}^m\) is a partition of \(\mathbb{N}\). It follows from construction along with (17) and (18) that

\sum_{i=1}^m \sum_{j \in \sigma_i} \langle x_{ij}, x_{ij}, y_{n+1} \rangle \langle x_{ij}, x_{ij}, y_{n+1} \rangle

so that a lower frame bound of \(\{x_{ij}\}_{ij \in \sigma_i}\) is zero. Hence, it is not a frame and we have the result. \qed

\textbf{Corollary 1.} Let \(\mathcal{H}\) be a full Hilbert \(\mathcal{A}\)-module over a unital \(C^*\)-algebra \(\mathcal{A}\) and \(\{\{x_{ij}\}_{ij \in \sigma_i}\}_{i=1}^m\) be a weakly woven family of frames for \(\mathcal{H}\). Then, there exist disjoint finite sets \(\Lambda_1, \ldots, \Lambda_m \subset J\) and a constant \(C > 0\) such that, for any partition \(\{\sigma_i\}_{i=1}^m\) of \(J\) with \(\Lambda_i \subset \sigma_i\) for \(i \in [m]\), the family \(\{x_{ij}\}_{ij \in \sigma_i}\) has a lower frame bound \(C\).

Every bounded sequence in a Hilbert space has a weakly convergent subsequence. However, this is usually not true for bounded sequences in general Hilbert \(C^*\)-modules. Thus, the computations will be quite different than given in Proposition 4.5 in [5] and more conditions will be needed when investigating the analogous result.

\begin{proposition}
Let \(\mathcal{H}\) be a full Hilbert \(\mathcal{A}\)-module over a unital \(C^*\)-algebra \(\mathcal{A}\) and \(X = \{x_j\}_{j \in J}\) and \(Y = \{y_j\}_{j \in J}\) be frames for \(\mathcal{H}\). Then, the following are equivalent:

(i) \(X\) and \(Y\) are woven

(ii) \(X\) and \(Y\) are weakly woven

\end{proposition}

\textbf{Proof.} \((i) \Rightarrow (ii)\) is clear. For the proof of \((ii) \Rightarrow (i)\), we see from Proposition 4 that only a universal lower bound for \(X\) and \(Y\) needs to be shown. By Corollary 1, there exist disjoint finite sets \(\Lambda_0, \Gamma_0 \subset J\) and a constant \(C > 0\) such that, for any subset \(\sigma_0\) of \(\mathbb{N}\) with \(\Lambda_0 \subset \sigma_0\) and \(\Gamma_0 \subset \sigma_0\) and every \(x \in \mathcal{H}\),

\[\left\| \sum_{j \in \Lambda_0} \langle x, x_j \rangle \langle x_j, x \rangle + \sum_{j \in \Gamma_0} \langle x, y_j \rangle \langle y_j, x \rangle \right\| \geq C \|x\|^2. \quad (20)\]
Since permuting both frames simultaneously does not affect weaving, we permute \( X \) and \( Y \) such that \( \Lambda_{\alpha} \cup \Gamma_{\alpha} = [m] \).

If we can prove that, for any partition \( \{ \Lambda_{\alpha}, \Gamma_{\alpha} \} \) of \([m]\), there exist \( C_{\alpha} > 0 \) such that, for every subset \( \sigma_{\alpha} \) of \([n] \) with \( \Lambda_{\alpha} \subset \sigma_{\alpha} \) and \( \Gamma_{\alpha} \subset \sigma_{\alpha}^{c} \), the family \( \{ x_{j} \}_{j \in \sigma_{\alpha}} \cup \{ y_{j} \}_{j \in \sigma_{\alpha}^{c}} \) has a lower frame bound \( C_{\alpha} \); then, the frames \( X \) and \( Y \) are woven with a universal lower bound \( C_{\alpha} = \min \{ C_{\alpha} : \alpha \} \), and we will obtain the desired result. Now, assume the above result is not true, and we will show that this yields a contradiction.

First, by hypothesis, we can find a partition \( \{ \Lambda_{1}, \Gamma_{1} \} \) of \([m]\) such that, for any \( \varepsilon > 0 \), there exists a subset \( \sigma_{1} \) of \([n]\) with \( \Lambda_{1} \subset \sigma_{1} \) and \( \Gamma_{1} \subset \sigma_{1}^{c} \) such that the family \( \{ x_{j} \}_{j \in \sigma_{1}} \cup \{ y_{j} \}_{j \in \sigma_{1}^{c}} \) has a lower frame bound less than \( \varepsilon \). Then, for all \( n \in \mathbb{N} \), there exist subsets \( \sigma_{n} \subset \mathbb{N} \) with \( \Lambda_{1} \subset \sigma_{n} \) and \( \Gamma_{1} \subset \sigma_{n}^{c} \) and \( y_{n} \in \mathbb{C} \) with \( \| y_{n} \| = 1 \) such that

\[
\sum_{j \in \sigma_{n}} \langle h - h_{n}, x_{j} \rangle \langle x_{j}, h - h_{n} \rangle + \sum_{j \in \sigma_{n}^{c}} \langle h - h_{n}, y_{j} \rangle \langle y_{j}, h - h_{n} \rangle < \frac{1}{n}, \tag{21}
\]

and the sets \( \sigma_{n} \) satisfy the following properties:

(1) For every \( k = 1, 2, \ldots \), either \( m + k \in \sigma_{n} \), for all \( n \geq k \), or \( m + k \in \sigma_{n}^{c} \), for all \( n \geq k \).

(2) There exist a subset \( \sigma \) of \([n] \) with \( \Lambda_{1} \subset \sigma \) and \( \Gamma_{1} \subset \sigma^{c} \) such that \( m + k \in \sigma \) implies that \( m + k \in \sigma_{n} \), for all \( n \geq k \), or if \( m + k \in \sigma^{c} \), then \( m + k \in \sigma_{n}^{c} \), for all \( n \geq k \).

Since \( \mathcal{H} \) is a Hilbert \( C^{*} \)-module over a finite-dimensional \( C^{*} \)-algebra \( A \), we see from Proposition 2.1 in [21] that, for the norm-bounded sequence \( \{ h_{n} \}_{n=1}^{\infty} \), there exists a subsequence \( \{ h_{n} \}_{n=1}^{\infty} \) of \( \{ h_{n} \}_{n=1}^{\infty} \) and \( h \in \mathcal{H} \) such that

\[
\langle y, h_{n} \rangle - \langle y, h \rangle \longrightarrow 0 \quad (n \longrightarrow \infty)
\]

for any \( y \in \mathcal{H} \). \tag{22}

We reindex \( h_{n} \longrightarrow h_{i} \) and \( \sigma_{n} \longrightarrow \sigma_{i} \) and notice that (21), (1), and (2) are still satisfied in this construction. Now, fix \( k \in \mathbb{N} \) so that \( k > 2/C \), where \( C \) is the constant in (20). The fact that \( \mathcal{H} \) is full (Theorem 2.5 in [22]) and that finite-dimensional \( C^{*} \)-algebras are unital imply that \( \{ h_{n} \}_{n=1}^{\infty} \) converges in norm to \( h \) on finite-dimensional subspaces of \( \mathcal{H} \). Hence, we can find an \( N_{k} \in \mathbb{N} \) such that, for all \( n \geq N_{k} > k \),

\[
\sum_{j \in \sigma_{n}} \langle h - h_{n}, x_{j} \rangle \langle x_{j}, h - h_{n} \rangle + \sum_{j \in \sigma_{n}^{c}} \langle h - h_{n}, y_{j} \rangle \langle y_{j}, h - h_{n} \rangle < \frac{1}{2k}, \tag{23}
\]

Denote \( \delta = \Lambda_{0} \cup (\sigma \cap [m]^{c}) \) and \( \delta_{n} = \Lambda_{0} \cup (\sigma_{n} \cap [m]^{c}) \). It follows from (20) that

\[
\sum_{j \in \delta_{n}} \langle h_{n}, x_{j} \rangle \langle x_{j}, h_{n} \rangle + \sum_{j \in \delta_{n}^{c}} \langle h_{n}, y_{j} \rangle \langle y_{j}, h_{n} \rangle \geq C. \tag{24}
\]
\[ \sum_{j \in \delta_n} \langle h_n, x_j \rangle \langle x_j, h_n \rangle + \sum_{j \in \delta_n} \langle h_n, y_j \rangle \langle y_j, h_n \rangle - \sum_{j \in \delta_n \cap \{m+k\}} \langle h_n, x_j \rangle \langle x_j, h_n \rangle \]

\[ + \sum_{j \in \delta_n \cap \{m+k\}} \langle h_n, y_j \rangle \langle y_j, h_n \rangle \]

\[ - \sum_{j \in \{m+k\}} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \{m+k\}} \langle h, y_j \rangle \langle y_j, h \rangle \]

\[ \geq \frac{1}{2} \cdot \frac{1}{C} - \frac{1}{2} \cdot \frac{1}{n} - \frac{1}{2k} \]

Finally, we will show that \( \{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in \sigma} \) is not a frame for \( \mathcal{H} \). Similar to the above, for the set \( \sigma \) given in (2), \( \sigma \cap \{m+k\} = \sigma \cap \{m+k\} \). Thus, using (21) and (23),

\[ \sum_{j \in \sigma \cap \{m+k\}} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \sigma \cap \{m+k\}} \langle h, y_j \rangle \langle y_j, h \rangle \]

\[ \leq 2 \sum_{j \in \sigma \cap \{m+k\}} \langle h_{N_k}, x_j \rangle \langle x_j, h_{N_k} \rangle + \sum_{j \in \sigma \cap \{m+k\}} \langle h_{N_k}, y_j \rangle \langle y_j, h_{N_k} \rangle \]

\[ + 2 \sum_{j \in \sigma \cap \{m+k\}} \langle h - h_{N_k}, x_j \rangle \langle x_j, h - h_{N_k} \rangle + \sum_{j \in \sigma \cap \{m+k\}} \langle h - h_{N_k}, y_j \rangle \langle y_j, h - h_{N_k} \rangle \]

\[ \leq 2 \frac{1}{N_k} + 2 \cdot \frac{1}{2k} \]
Letting \( k \to \infty \) in the above formula,
\[
\left\| \sum_{j \in \mathcal{J}} \langle h, x_j \rangle \langle x_j, h \rangle + \sum_{j \in \mathcal{J}_o} \langle h, y_j \rangle \langle y_j, h \rangle \right\| = 0,
\] (27)

implying that \( \{ x_j \}_{j \in \mathcal{J}} \cup \{ y_j \}_{j \in \mathcal{J}_o} \) is not a frame. Thus, a contradiction is met, concluding the proof. \( \square \)

\section{4. Perturbations of Weaving Frames}

Let \( \mathcal{H} \) be a finitely or countably generated Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( X = \{ x_j \}_{j \in \mathcal{J}} \) be a Bessel sequence for \( \mathcal{H} \) with the analysis operator \( U_X \). To characterize weaving frames, we need to define the analysis operator relevant to the partitions. For any given subset \( \sigma \) of \( \mathcal{J} \), denoted by \( P_{\sigma} \), the orthogonal projection is onto span\{\( e_j \)\}_{\sigma} where \( \{ e_j \}_{j \in \mathcal{J}} \) is the standard orthonormal basis of \( \ell^2 (\mathcal{A}) \). Then, the analysis operator relevant to \( \sigma \) is defined by
\[
U^\sigma_X : \mathcal{H} \to \ell^2 (\mathcal{A}),
U^\sigma_X x = P_\sigma U_X x = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle e_j.
\] (28)

Similarly, the synthesis operator \( T^\sigma_X \) relevant to \( \sigma \) is given by
\[
T^\sigma_X : \ell^2 (\mathcal{A}) \to \mathcal{H},
T^\sigma_X \{ e_j \}_{j \in \mathcal{J}} = T_X P_\sigma \{ e_j \}_{j \in \mathcal{J}} = \sum_{j \in \mathcal{J}} c_j x_j.
\] (29)

By composing \( U^\sigma_X \) and \( T^\sigma_X \), we obtain the \textit{frame operator} \( S^\sigma_X \) relevant to \( \sigma \):
\[
S^\sigma_X : \mathcal{H} \to \mathcal{H},
S^\sigma_X x = T^\sigma_X U^\sigma_X x = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle x_j.
\] (30)

Let \( X = \{ x_j \}_{j \in \mathcal{J}} \) and \( Y = \{ y_j \}_{j \in \mathcal{J}_o} \) be frames for \( \mathcal{H} \). We compute, for any \( \sigma \subseteq \mathcal{J} \) and any \( x \in \mathcal{H} \),
\[
\left\| U^\sigma_X x + U^\sigma_Y x \right\|^2 = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle \langle x_j, x \rangle + \sum_{j \in \mathcal{J}_o} \langle x, y_j \rangle \langle y_j, x \rangle.
\] (31)

Then, by Proposition 1, we know \( X \) and \( Y \) are woven with universal bounds \( (C_i, D_i) \) if and only if, for any \( \sigma \subseteq \mathcal{J} \),
\[
\sqrt{C} \| x \| \leq \left\| U^\sigma_X x + U^\sigma_Y x \right\| \leq \sqrt{D} \| x \|.
\] (32)

We begin investigating the perturbations of weaving frames.

\textbf{Theorem 2.} Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( X = \{ x_j \}_{j \in \mathcal{J}} \) and \( Y = \{ y_j \}_{j \in \mathcal{J}_o} \) be frames for \( \mathcal{H} \) with frame bounds \( (C_1, D_1) \) and \( (C_2, D_2) \), respectively. Assume that there exist constants \( \lambda_1, \lambda_2, \mu \geq 0 \) such that
\[
\lambda_1 \sqrt{C_1} + \lambda_2 \sqrt{D_2} + \mu < \sqrt{C_1},
\] (33)

and for any \( x \in H \),
\[
\left\| U_X x - U_Y x \right\| \leq \lambda_1 \left\| U_X x \right\| + \lambda_2 \left\| U_Y x \right\| + \mu \| x \|.
\] (34)

where \( U_X \) and \( U_Y \) denote the analysis operators for \( X \) and \( Y \), respectively. Then, \( X \) and \( Y \) are woven frames with universal bounds \( (\sqrt{C_1} - \lambda_1 \sqrt{C_1} - \lambda_2 \sqrt{D_2} - \mu)^2 \) and min \( \{ \sqrt{(D_1 + \lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2} + 2 \mu)^2}, D_1 + D_2 \} \).

\textbf{Proof.} For any given \( \sigma \subseteq \mathcal{J} \), we denote \( \{ \eta_j \}_{j \in \mathcal{J}} = \{ x_j \}_{j \in \mathcal{J}_o} \cup \{ y_j \}_{j \in \mathcal{J}_o} \) and define an operator \( U_\eta : H \to \ell^2 (\mathcal{A}) \) by
\[
U_\eta x = U^\sigma_X x + U^\sigma_Y x = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle e_j + \sum_{j \in \mathcal{J}_o} \langle x, y_j \rangle e_j.
\] (35)

It is clear that \( U_\eta \) is the analysis operator for \( \{ \eta_j \}_{j \in \mathcal{J}} \). Hence, for any \( x \in \mathcal{H} \),
\[
\left\| U_X x - U_\eta x \right\| = \left\| U_X x - U^\sigma_X x - U^\sigma_Y x \right\|
\]
\[
= \sum_{j \in \mathcal{J}} \langle x, x_j \rangle e_j - \sum_{j \in \mathcal{J}_o} \langle x, x_j \rangle e_j - \sum_{j \in \mathcal{J}_o} \langle x, y_j \rangle e_j
\]
\[
= \sum_{j \in \mathcal{J}} \langle x, x_j - y_j \rangle e_j
\]
\[
\leq \sum_{j \in \mathcal{J}} \left\| \langle x, x_j - y_j \rangle \langle x_j - y_j, x \rangle \right\|^{1/2}
\]
\[
= \left\| U_X x - U_Y x \right\|.
\] (36)

On the one hand, we see from (34) that
\[
\left\| U_X x \right\| - \left\| U_\eta x \right\| \leq \left\| U_X x - U_\eta x \right\| \leq \lambda_1 \left\| U_X x \right\| + \lambda_2 \left\| U_Y x \right\| + \mu \| x \|.
\] (37)

Hence,
\[
\left\| U_\eta x \right\| \geq (1 - \lambda_1) \left\| U_X x \right\| - \lambda_2 \left\| U_Y x \right\| - \mu \| x \|
\]
\[
\geq (1 - \lambda_1) \sqrt{C_1} \| x \| - \lambda_2 \sqrt{D_2} \| x \| - \mu \| x \|
\]
\[
= (\sqrt{C_1} - \lambda_1 \sqrt{C_1} - \lambda_2 \sqrt{D_2} - \mu) \| x \|.
\]

On the other hand, we also have
\[
\left\| U_\eta x \right\| - \left\| U_X x \right\| \leq \left\| U_X x - U_\eta x \right\| \leq \lambda_1 \left\| U_X x \right\| + \lambda_2 \left\| U_Y x \right\| + \mu \| x \|.
\] (38)

Therefore,
\[
\left\| U_\eta x \right\| \leq (\lambda_1 + 1) \left\| U_X x \right\| + \lambda_2 \left\| U_Y x \right\| + \mu \| x \|
\]
\[
\leq (\lambda_1 + 1) \sqrt{D_1} \| x \| + \lambda_2 \sqrt{D_2} \| x \| + \mu \| x \|
\]
\[
= (\sqrt{D_1} + \lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2} + 2 \mu) \| x \|,
\] (40)

as required. \( \square \)
By choosing \( \lambda_1 = \lambda_2 = 0 \) in Theorem 2, we see that \( X \) and \( Y \) are woven with universal bounds \( \sqrt{C_1^2 - \mu^2} \) and \( \min\left\{ \sqrt{D_1^2 + \mu^2}, D_1 + D_2 \right\} \), where \( \mu < \sqrt{C_1^2} \) and \( \|U_Xx - U_Yx\| \leq \mu \|x\| \). If the value for the universal bounds is irrelevant, we can get the following result.

**Corollary 2.** Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( X = \{x_j\}_{j \in \mathcal{J}} \) and \( Y = \{y_j\}_{j \in \mathcal{J}} \) be frames for \( \mathcal{H} \) with frame bounds \( (C_1, D_1) \) and \( (C_2, D_2) \), respectively. If, for any \( x \in H \),

\[
\|U_Xx - U_Yx\| < \sqrt{C_1^2 - \mu^2} \|x\| \quad \text{or} \quad \|U_Xx - U_Yx\| < \sqrt{C_2^2 - \mu^2} \|x\|, \tag{41}
\]

then \( X \) and \( Y \) are woven.

The property of woven frames is preserved under an adjointable invertible operator. However, applying two different operators to woven frames does not always preserve the weaving property. Using Corollary 2, we consider when the given frames are woven under different operators.

**Corollary 3.** Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( X = \{x_j\}_{j \in \mathcal{J}} \) and \( Y = \{y_j\}_{j \in \mathcal{J}} \) be frames for \( \mathcal{H} \) with frame bounds \( (C_1, D_1) \) and \( (C_2, D_2) \), respectively. Assume \( F_1, F_2 \in \mathcal{L}(\mathcal{H}) \) are invertible operators on \( \mathcal{H} \). Then, \( \{F_1x_j\}_{j \in \mathcal{J}} \) and \( \{F_2y_j\}_{j \in \mathcal{J}} \) are woven frames when \( \sqrt{D_1^2 + \mu^2} \|F_1\| + \sqrt{D_2^2 + \mu^2} \|F_2\| \sqrt{C_1^2 - \mu^2} \|x\| \leq \sqrt{C_1^2} \|x\| \quad \text{or} \quad \sqrt{D_1^2 + \mu^2} \|F_1\| + \sqrt{D_2^2 + \mu^2} \|F_2\| \sqrt{C_2^2 - \mu^2} \|y\| \leq \sqrt{C_2^2} \|y\|. \tag{42} \]

**Proof.** Note that \( \{F_1x_j\}_{j \in \mathcal{J}} \) (resp. \( \{F_2y_j\}_{j \in \mathcal{J}} \)) is a frame with lower frame bound \( C_1 \|F_1^{-1}\|^{-2} \) (resp. \( C_2 \|F_2^{-1}\|^{-2} \)) and analysis operator \( U_XF_1^* \) (resp. \( U_YF_2^* \)). Thus, by assumption,

\[
\|U_XF_1^*x - U_YF_2^*x\| \leq \|U_XF_1^*x - U_YF_2^*x\| + \|U_YF_2^*x\| \\
\leq \sqrt{D_1^2 \|F_1\| \|x\|} + \sqrt{D_2^2 \|F_2\| \|y\|} \tag{42} \]

\[
< \sqrt{C_1^2 \|F_1^{-1}\|^{-2} \|x\|}. \]

Using Corollary 2, we obtain the desired result. \( \square \)

Canonical dual frames play a fundamental role in the study of woven frames. Using Corollary 3, we now derive a perturbation result for canonical dual frames.

**Theorem 3.** Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( X = \{x_j\}_{j \in \mathcal{J}} \) and \( Y = \{y_j\}_{j \in \mathcal{J}} \) be frames for \( \mathcal{H} \) with frame bounds \( (C_1, D_1) \) and \( (C_2, D_2) \), respectively. If, for any \( x \in H \),

\[
\|U_Xx - U_Yx\| < \frac{C_1C_2}{C_2\sqrt{D_1 + D_1\sqrt{D_2}} + D_2\sqrt{D_1}} \|x\|. \tag{43}
\]

or

\[
\|U_Xx - U_Yx\| < \frac{C_1C_2}{C_1\sqrt{D_2} + D_1\sqrt{D_2} + D_2\sqrt{D_1}} \|x\|, \tag{44}
\]

then \( \{S_X^jx_j\}_{j \in \mathcal{J}} \) and \( \{S_Y^jy_j\}_{j \in \mathcal{J}} \) are woven.

**Proof.** Let \( \mu \) denote \( C_1C_2\sqrt{D_1 + D_1\sqrt{D_2}} + D_2\sqrt{D_1} \) and \( \mu \) we have

\[
\|U_Xx - U_Yx\| < \mu \|x\|. \tag{45}
\]

\[
\frac{\mu}{C_1} + \frac{\mu}{C_2} \left( \sqrt{D_1^2 + D_2^2} + D_2 \right) = \frac{1}{D_1}. \tag{46}
\]

Now, using (45), we get, for any \( x \in \mathcal{H} \),

\[
\|S_X^jx_j - S_Y^jy_j\| = \left\|S_Y^j(S_Y - S_X)S_X^jx_j\right\| < \frac{\mu}{C_1C_2} \left( \sqrt{D_1^2 + D_2^2} \right). \tag{47}
\]

Note that \( \{S_X^jx_j\}_{j \in \mathcal{J}} \) and \( \{S_Y^jy_j\}_{j \in \mathcal{J}} \) are frames with frame bounds \( \{(1/D_1, 1/C_2)\} \) and \( \{(1/D_2, 1/C_2)\} \), respectively. Hence, for any \( x \in \mathcal{H} \),

\[
\|S_X^jx_j - S_Y^jy_j\| = \left\|S_Y^j(S_Y - S_X)S_X^jx_j\right\| < \frac{\mu}{C_1C_2} \left( \sqrt{D_1^2 + D_2^2} \right). \tag{48}
\]

Now, denote \( \{S_X^jx_j\}_{j \in \mathcal{J}} \) and \( \{S_Y^jy_j\}_{j \in \mathcal{J}} \) by \( S_X^jX \) and \( S_Y^jY \), respectively. Putting (45), (46), and (48) together, we get, for any \( x \in H \),

\[
\|U_X^jx - U_Y^jy\| = \left\|\sum_{j \in \mathcal{J}} \langle x, S_X^jx_j \rangle e_j - \sum_{j \in \mathcal{J}} \langle x, S_Y^jy_j \rangle e_j\right\| \leq \left\|\sum_{j \in \mathcal{J}} \langle x, S_X^jx_j \rangle e_j - \sum_{j \in \mathcal{J}} \langle x, S_Y^jy_j \rangle e_j\right\| \tag{49}
\]

Now, the conclusion follows from Corollary 3.

Canonical dual frames can be used in investigating the perturbation results of weaving frames. \( \square \)

**Proposition 8.** Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( X = \{x_j\}_{j \in \mathcal{J}} \) and \( Y = \{y_j\}_{j \in \mathcal{J}} \) be frames for
\(H\) with frame bounds \((C_1, D_1)\) and \((C_2, D_2)\), respectively. If, for any \(x \in H\),
\[
\sum \|x_j - y_j\| \cdot \|S^{-1}_X x_j\| = \mu < 1,
\]
then \(\{x_j\}_{j \in J}\) and \(\{y_j\}_{j \in J}\) are woven frames with universal bounds \((C_1 (1 - \mu)^2, D_1 + D_2)\).

**Proof.** For any \(\sigma \subset J\), we denote \(\{\eta_j\}_{j \in J} = \{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in J - \sigma}\) and define an operator \(L: H \rightarrow H\) by
\[
Lx = \sum_{j \in J} \langle x, S^{-1}_X x_j \rangle \eta_j.
\]

Note that \(\{S^{-1}_X x_j\}\) is a \((1/D, 1/C)\)-frame. For any \(x \in H\), we compute
\[
\|x - Lx\| = \sum_{j \in J} \langle x, S^{-1}_X x_j \rangle \langle x_j - \eta_j \rangle\leq \sum_{j \in J} \langle x, S^{-1}_X x_j \rangle \langle x_j - y_j \rangle\leq \sum_{j \in J} \|x_j - y_j\| \cdot \|S^{-1}_X x\| \cdot \|x\| = \mu \|x\|.
\]

Hence, (50) implies that \(L\) is invertible and
\[
\|L\| \leq 1 + \mu, \quad \|L^{-1}\| \leq \frac{1}{1 - \mu}.
\]

Every \(x \in H\) can be written as
\[
x = LL^{-1}x = \sum_{j \in J} \langle L^{-1}x, S^{-1}_X x_j \rangle \eta_j,
\]

implying that
\[
\|x\|^2 = \langle x, x \rangle = \sum_{j \in J} \langle L^{-1}x, S^{-1}_X x_j \rangle \langle \eta_j, x \rangle^2 \leq \sum_{j \in J} \langle L^{-1}x, S^{-1}_X x_j \rangle \langle S^{-1}_X x_j, L^{-1}x \rangle \leq \frac{1}{C_1 \|L^{-1}x\|^2} \sum_{j \in J} \|\eta_j\| \|\eta_j, x\| \leq \frac{1}{C_1 (1 - \mu)^2} \sum_{j \in J} \|\eta_j\| \|\eta_j, x\|. \tag{55}
\]

Hence, we have obtained the claimed lower frame bound:
\[
\sum_{j \in J} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \geq C_1 (1 - \mu)^2 \|x\|^2, \tag{56}
\]
as required. \(\square\)

We have known frames that small perturbations of each other are woven. In this section, we consider the sufficient conditions under which every perturbation of a given woven frame pair \(\{x_j\}_{j \in J}, \{y_j\}_{j \in J}\) is still woven.

**Theorem 4.** Let \(H\) be a Hilbert \(\mathcal{A}\)-module over a unital \(C^*\)-algebra \(\mathcal{A}\). Suppose \(X = \{x_j\}_{j \in J}\) and \(Y = \{y_j\}_{j \in J}\) are woven frames with universal bounds \((M, N)\) and \(W = \{w_j\}_{j \in J}\) and \(Z = \{z_j\}_{j \in J}\) are frames for \(H\). Assume that there exist \(\mu, \mu' \geq 0\) such that
\[
\lambda_1 \sqrt{M} + \lambda'_1 \sqrt{M} + \mu + \mu' < \sqrt{M}, \tag{57}
\]
and for any \(x \in H\), we have
\[
\|U_x - U_{w, x}\| \leq \lambda_1 \|U_x\| + \lambda_2 \|U_{w, x}\| + \mu \|x\|, \tag{58}
\]
\[
\|U_x - U_{z, x}\| \leq \lambda_1 \|U_x\| + \lambda_2 \|U_{z, x}\| + \mu' \|x\|. \tag{59}
\]

Then, \(W\) and \(Z\) are woven with lower universal bound \((\sqrt{M} - \lambda_1 \sqrt{M} - \lambda'_1 \sqrt{M} - \mu - \mu' + 1 + \lambda_2 + \lambda'_2)^2\).

**Proof.** For any given \(\sigma \subset J\), we denote \(\{\eta_j\}_{j \in J} = \{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in J - \sigma}\) and \(\{\xi_j\}_{j \in J} = \{w_j\}_{j \in \sigma} \cup \{z_j\}_{j \in J - \sigma}\).

Then, by assumption, we know \(\{\eta_j\}\) is a \((M, N)\)-frame.

It follows from \(U_{\eta, x} = P_{\sigma, U} \eta\) that \(\|U_{\eta, x}\| \leq \|U_{\eta, x}\|\). Hence, using (58) and (59),
\[
\|U_{x, x}\| \leq \|U_{x, x}\| \leq \|U_{x, x}\| = \|U_{x, x}\| \leq \|U_{x, x}\| + \lambda_1 \|U_{x, x}\| + \lambda_2 \|U_{x, x}\| + \mu \|x\| \leq \lambda_1 \|U_{x, x}\| + \lambda_2 \|U_{x, x}\| + \mu \|x\| + \lambda_1 \|U_{x, x}\| + \lambda_2 \|U_{x, x}\| + \mu \|x\| = (\lambda_1 + \lambda_2) \|U_{x, x}\| + (\lambda_2' + \lambda_2') \|U_{x, x}\| + (\mu + \mu') \|x\|. \tag{60}
\]

Therefore,
\[
\|U_{x, x}\| \geq \frac{\sqrt{M} - \lambda_1 \sqrt{M} - \lambda'_1 \sqrt{M} - \mu - \mu'}{1 + \lambda_2 + \lambda_2'} \|x\|, \tag{61}
\]
and we have the result. \(\square\)

**Theorem 5.** Let \(H\) be a Hilbert \(\mathcal{A}\)-module over a unital \(C^*\)-algebra \(\mathcal{A}\). Suppose \(X = \{x_j\}_{j \in J}\) and \(Y = \{y_j\}_{j \in J}\) are
woven frames with universal bounds \((M, N)\) and \(W = \{w_j\}_{j \in J}\) and \(Z = \{z_j\}_{j \in J}\) are frames for \(\mathcal{H}\). Assume that there exist \(\mu, \mu' \geq 0\) such that \(\mu + \mu' < 1\), and for any \(x \in \mathcal{H}\), \(\begin{align*}
\sum_{j \in J} \| x_j - w_j \| \cdot \| S^{-1}_x x_j \| &= \mu, \\
\sum_{j \in J} \| y_j - z_j \| \cdot \| S^{-1}_y y_j \| &= \mu'.
\end{align*}\) (62)

Then, \(W\) and \(Z\) are woven with lower universal bounds \(M (1 - \mu - \mu')^2\).

**Proof.** Since \(X\) and \(Y\) are woven with universal bounds \((M, N)\), we obtain that \(\{\eta_j\}_{j \in J} = \{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in \sigma}\) is a \((M, N)\)-frame for any \(\sigma \subseteq J\). Now, we denote \(\{\xi_j\}_{j \in J} = \{w_j\}_{j \in \sigma} \cup \{z_j\}_{j \in \sigma}\) and define an operator \(L: \mathcal{H} \rightarrow \mathcal{H}\) by
\[
Lx = \sum_{j \in J} \langle x, S^{-1}_x \eta_j \rangle \xi_j.
\] (63)

Then, for assumption, we have, for any \(x \in \mathcal{H}\),
\[
\| x - Lx \| = \left\| \sum_{j \in J} \langle x, S^{-1}_x \eta_j \rangle \eta_j - \xi_j \right\| \\
\leq \sum_{j \in J} \left\| \langle x, S^{-1}_x \eta_j \rangle \eta_j - \xi_j \right\| + \sum_{j \in J} \left\| \langle x, S^{-1}_x \eta_j \rangle (x_j - w_j) \right\| \\
\leq \sum_{j \in J} \| \eta_j - \xi_j \| \cdot \| S^{-1}_x \eta_j \| \cdot \| x \| + \sum_{j \in \sigma} \| x_j - w_j \| \cdot \| S^{-1}_x \eta_j \| \cdot \| x \| \\
\leq (\mu + \mu') \| x \|.
\] (64)

It follows that \(L\) is invertible and
\[
\| L^{-1} \| \leq \frac{1}{1 - \mu - \mu'}.
\] (65)

Now, for any \(x \in \mathcal{H}\),
\[
x = LL^{-1} x = \sum_{j \in J} \langle L^{-1} x, S^{-1}_x \eta_j \rangle \xi_j.
\] (66)

Therefore,
\[
\| x \|^2 = \langle x, x \rangle = \sum_{j \in J} \left\| \langle L^{-1} x, S^{-1}_x \eta_j \rangle \xi_j \right\|^2 \\
\leq \sum_{j \in J} \left\| \langle L^{-1} x, S^{-1}_x \eta_j \rangle \xi_j \right\|^2 = \sum_{j \in J} \langle x, \xi_j \rangle \langle \xi_j, x \rangle \\
\leq \frac{1}{M} \| L^{-1} x \|^2 \cdot \| x \| \| \xi_j \|.
\] (67)

Putting (65) and (66) together, we have
\[
\sum_{j \in J} \langle x, \xi_j \rangle \langle \xi_j, x \rangle \geq M (1 - \mu - \mu')^2 \| x \|^2,
\] (68)
as required. \(\square\)

### 5. Weaving Frames in Hilbert \(C^*\)-Modules over Nonunital \(C^*\)-Algebras

The frames for a Hilbert space \(\mathcal{H}\) are precisely the families \(\{T e_j\}_{j \in J}\) where \(T: \ell^2 (J) \rightarrow \mathcal{H}\) is a bounded surjection and \(\{e_j\}_{j \in J}\) is the canonical basis of \(\ell^2 (J)\). This correspondence is still true for modular frames when the \(C^*\)-algebra \(\mathcal{A}\) is unital. Assume \(\{x_j\}_{j \in J}\) is a frame for a Hilbert \(C^*\)-module over a unital \(C^*\)-algebra with analysis operator \(U\). We can derive from Corollary 2 that any adjointable operators belonging to a small neighborhood of \(U\) determine a frame which is woven with \(\{x_j\}_{j \in J}\).

However, for the nonunital case, the situation is more complicated. Let \(\mathcal{A}\) be an arbitrary \(C^*\)-algebra and \(\mathcal{H}\) be a Hilbert \(\mathcal{A}\)-module. An adjointable surjection from \(\ell^2 (\mathcal{A})\) to \(\mathcal{H}\) may not be the analysis operator for any frame. Furthermore, there exist operator whose any neighborhood can contain an operator that is not the analysis operator for any frame. Thus, we need a new notation to obtain analogues of the unital case.

Arambasic et al. introduced the concept of the outer frame in [23] to investigate the bijective correspondence of all surjections from \(\ell^2 (\mathcal{A})\) to \(\mathcal{H}\) and all both frames or outer frames for \(\mathcal{H}\). Let \(M (\mathcal{H})\) be the multiplier module of \(\mathcal{H}\). Note that \(M (\mathcal{H})\) is a Hilbert \(M (\mathcal{A})\)-module which contains \(\mathcal{H}\) as an ideal submodule. Recall that a sequence \(\{x_j\}_{j \in J}\) in \(M (\mathcal{H})\) is called an outer frame for \(\mathcal{H}\) if \(x_j \in M (\mathcal{H})\) for at least one \(i \in J\), and (2) holds for every \(x \in \mathcal{H}\) and suitable positive constants \((C, D)\). Let \(X = \{x_j\}_{j \in J}\) and \(Y = \{y_j\}_{j \in J}\) be frames or outer frames for \(\mathcal{H}\), and we say \(X\) and \(Y\) are woven if there are universal constants \((C, D)\) such that, for any subset \(\sigma \subseteq J\), \(\{x_j\}_{j \in \sigma} \cup \{y_j\}_{j \in \sigma}\) is a frame or an outer frame for \(\mathcal{H}\) with bounds \((C, D)\) (Theorem 3.19 in [23]) shows that, for any surjection \(T \in \mathcal{L} (\ell^2 (\mathcal{A}), \mathcal{H})\), there exists a unique frame or an outer frame \(\{x_j\}_{j \in J}\) whose synthesis operator coincides with \(T\). Now, we give the key result of this section.

**Theorem 6.** Let \(\mathcal{H}\) be a Hilbert \(\mathcal{A}\)-module over a \(C^*\)-algebra \(\mathcal{A}\) and \(U, V \in \mathcal{L} (\ell^2 (\mathcal{A}), \mathcal{H})\). Suppose \(U\) is bounded below and
\[
\| U - V \| < \max \left\{ \| (U^* U)^{-1/2} \|^{-1}, \| (V^* V)^{-1/2} \|^{-1} \right\}.
\] (69)

Then, \(U\) and \(V\) are analysis operators for a woven family of frames or outer frames.

**Proof.** Without loss of generality, we assume \(\| U - V \| < \| (U^* U)^{-1/2} \|^{-1}\). Since \(U\) is bounded below,
Write \( \mu := \|U - V\|. \) Then, by assumption,

\[
\|Vx\| \geq \|Ux - Vx\| \geq \left( \left\| (U^*U)^{-1/2} \right\|^\frac{1}{2} \right) \|x\|, \quad \forall x \in \mathcal{H},
\]

(71)

implying that \( V \) is bounded below. Hence, \( U \) and \( V \) are analysis operators of frames or outer frames denoted by \( \{x_j\}_{j \in \mathcal{J}} \) and \( \{y_j\}_{j \in \mathcal{J}'}, \) respectively.

For any \( \sigma \subset J \), we denote \( \{j\}_{j \in \mathcal{J}} \cup \{y_j\}_{j \in \mathcal{J}_\sigma} \) and define an operator \( U_\sigma : \mathcal{H} \to \ell^2 (\mathcal{A}) \) by

\[
U_\sigma x = \sum_{j \in \mathcal{J}_\sigma} (x, y_j) e_j + \sum_{j \in \mathcal{J}_\sigma} (x, y_j) e_j.
\]

(72)

Then, for any \( x \in \mathcal{H}, \)

\[
\left\| U_\sigma x - Ux \right\| = \sum_{j \in \mathcal{J}_\sigma} (x, y_j) e_j \leq \|Ux - Vx\|.
\]

(73)

It follows that

\[
\left\| U_\sigma x \right\| \geq \|Ux\| - \left( (U^*U)^{-1/2} \right) \|x\|.
\]

(74)

Thus, \( \{j\}_{j \in \mathcal{J}} \) is a frame or an outer frame for \( \mathcal{H} \). We can prove the result similarly using \( \|U - V\| < \left\| (U^*U)^{-1/2} \right\|^{-1}, \) and we have the result. \( \square \)

When \( \mathcal{A} \) is unital, \( M(\mathcal{H}) = \mathcal{H}. \) There are no outer frames for \( \mathcal{H}, \) and clearly, Theorem 6 holds for ordinary frames in the Hilbert \( \mathcal{A} \)-module.

6. Redundancy of Weaving Frames

In this section, we study the redundancy of frames in Hilbert \( C^* \)-modules. Recall that a frame \( \{x_j\}_{j \in \mathcal{J}} \) for a Hilbert \( \mathcal{A} \)-module \( \mathcal{H} \) is said to be a (standard) Riesz basis for \( \mathcal{H} \) if it satisfies the following:

(i) \( x_j \neq 0, \) for all \( j \)

(ii) If an \( \mathcal{A} \)-linear combination \( \sum_{j \in \mathcal{S}} c_j x_j \) with coefficients \( \{c_j : j \in \mathcal{S} \} \subset \mathcal{A} \) and \( \mathcal{S} \subset \mathcal{J} \) is equal to zero, then every summand \( c_j x_j \) is zero.

It is shown in [5] that a frame which is woven with a Riesz basis in a Hilbert space must be a Riesz basis and [24] shows that woven frames have the same excess. However, the redundancy property of modular frames is quite different from that of frames in Hilbert spaces. As was pointed out in Section 2, every countably generated Hilbert \( C^* \)-module admits a (standard) frame. However, there exist countably generated Hilbert \( C^* \)-modules that contain no orthonormal basis, even a Riesz basis, cf. [25].

We first state the following characterization for Riesz bases in Hilbert \( \mathcal{A} \)-modules.

**Proposition 9** (see [12]). Suppose that \( \{x_j\}_{j \in \mathcal{J}} \) is a frame of \( \mathcal{H} \); then, \( \{x_j\}_{j \in \mathcal{J}} \) is a Riesz basis if and only if

(i) \( x_j \neq 0 \) for all \( j \in J \)

(ii) \( \sum_{j \in \mathcal{J}} c_j x_j = 0 \) for some sequence \( \{c_j\}_{j \in \mathcal{J}} \in \ell^2 (\mathcal{A}), \) then \( c_j x_j = 0, \) for all \( j \)

Now, we derive the following result.

**Theorem 7.** Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra \( \mathcal{A} \) and \( X = \{x_j\}_{j \in \mathcal{J}} \) and \( Y = \{y_j\}_{j \in \mathcal{J}'} \) be woven frames for \( \mathcal{H}. \) If \( X \) is a Riesz basis with \( \text{ran} \theta_X = \ell^2 (\mathcal{A}), \) then \( Y \) is a Riesz basis with \( \text{ran} \theta_Y = \ell^2 (\mathcal{A}). \)

**Proof.** Let \( \{e_j\}_{j \in \mathcal{J}} \) be the canonical basis of \( \ell^2 (\mathcal{A}). \) Note that every modular frame can be written as \( \{Te_j\}_{j \in \mathcal{J}} \) for a surjective adjointable operator \( T. \) Since \( \text{ran} \theta_X = \ell^2 (\mathcal{A}), \) the synthesis operator \( T_X \) is invertible. Then, by Proposition 5, we can assume that \( X = \{e_j\}_{j \in \mathcal{J}} \) and \( Y = \{Te_j\}_{j \in \mathcal{J}}, \) where \( T \) is a surjective operator in \( \mathcal{L}(\ell^2 (\mathcal{A})). \)

By way of contradiction, assume that \( \theta_Y \neq \ell^2 (\mathcal{A}). \) This means that \( \ker T_Y \neq \{0\}. \) Then, there exists \( \{c_j\}_{j \in \mathcal{J}} \) with \( c_i \neq 0 \) for some \( i, \) such that \( \sum_{j \neq i} c_j T e_j = 0. \) This implies that \( T e_i = \sum_{j \neq i} (-c_j/c_i) T e_j, \) and thus, \( T e_i \in \overline{\text{span}} \{Te_j\}_{j \neq i}. \) Without loss of generality it may be assumed that \( T e_i \in \overline{\text{span}} \{Te_j\}_{j \neq i}. \) For any \( \varepsilon > 0, \) there exists \( n \in \mathbb{N} \) such that \( d(T e_i, \overline{\text{span}} \{T e_j\}_{j \neq i}) < \varepsilon. \) Now, choose \( x \in \text{span} \{e_j\}_{j \in \mathcal{J}} \cap (\overline{\text{span}} \{Te_j\}_{j \neq i}) \) with \( \|x\| = 1. \) Then, for any \( y \in \text{span} \{Te_j\}_{j \neq i}, \) we compute

\[
\sum_{j=1}^n \langle x, e_j \rangle^2 + \sum_{j=1}^\infty \langle x, Te_j \rangle^2 = \|x, Te_1 \|^2 = \|x, T e_1 - y \|^2 \\
\leq \|T e_1 - y\|^2 < \varepsilon.
\]

(75)
So, \( \{e_j\}_{j=1}^n \cup \{T e_j\}_{j=n+1}^{+\infty} \) is not a frame. Then, we have \( \text{ran} \theta_\ell = \ell^2(\mathcal{A}) \). Hence, \( T_Y \{c_j\}_{j \in J} = 0 \) can imply \( c_j = 0 \) for all \( j \). Thus, \( c_j x_j = 0 \) for all \( j \), and it follows from Proposition 9 that \( X \) is a Riesz basis.

Theorem 4.15 in [23] proved that a frame or an outer frame for a full Hilbert \( \mathcal{A} \)-module has a unique dual if and only if the analysis operator is surjective. Combined with Theorem 7, we have the following.

Corollary 4. Let \( \mathcal{H} \) be a full Hilbert \( \mathcal{A} \)-module and \( X = \{x_j\}_{j \in J} \) and \( Y = \{y_j\}_{j \in J} \) be a woven family of frames or outer frames. If \( X \) has a unique dual, then \( Y \) has a unique dual.

Now, we derive a sufficient condition for the situation that a frame which is woven to a Riesz basis is a Riesz basis itself.

Proposition 10. Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( \text{C}^* \)-algebra \( \mathcal{A} \) and \( X = \{x_j\}_{j \in J} \) and \( Y = \{y_j\}_{j \in J} \) be frames for \( \mathcal{H} \) with frame bounds \( (C_1, D_1) \) and \( (C_2, D_2) \), respectively. Assume that \( X \) is a Riesz basis, and there exist constants \( \lambda_1 \), \( \lambda_2 \geq 0 \) with

\[
\lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2} < \min \{\sqrt{C_1}, \sqrt{C_2}\},
\]

such that, for any \( \{c_j\}_{j \in J} \in \ell^2(\mathcal{A}) \),

\[
\|T_X \{c_j\}_{j \in J} - T_Y \{c_j\}_{j \in J}\| \leq \lambda_1 \|T_X \{c_j\}_{j \in J}\| + \lambda_2 \|T_Y \{c_j\}_{j \in J}\|. \tag{77}
\]

where \( T_X \) and \( T_Y \) are the synthesis operators for \( X \) and \( Y \), respectively. Then, \( X \) and \( Y \) are woven frames and \( Y \) is a Riesz basis.

Proof. Define an operator \( L \) and a sequence \( \{\eta_j\}_{j \in J} \) in the same way as the proof of Theorem 9. Using (77), on the sequence \( \{c_j\}_{j \in J}^* = \{\langle x, S_X^{-1} y_j \rangle\}_{j \in J} \), we obtain that

\[
\|x - Lx\| = \left\|\sum_{j \in J} \langle x, S_X^{-1} y_j \rangle (x_j - \eta_j)\right\|
\]

\[
= \left\|\sum_{j \in J} c_j (x_j - y_j)\right\|
\]

\[
\leq \lambda_1 \|T_X \{c_j\}_{j \in J}\| + \lambda_2 \|T_Y \{c_j\}_{j \in J}\| \tag{78}
\]

\[
\leq \lambda_1 \|T_X\| \cdot \|\{c_j\}_{j \in J}\| + \lambda_2 \|T_Y\| \cdot \|\{c_j\}_{j \in J}\|
\]

\[
\leq \left(\lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2}\right) \left\|\{c_j\}_{j \in J}\right\|
\]

\[
\leq \lambda_1 \sqrt{D_1} + \lambda_2 \sqrt{D_2} \sqrt{\|x\|}.
\]

Hence, (76) implies that \( L \) is invertible and \( \|L^{-1}\| \leq \sqrt{C_1}/\sqrt{C_1} - \lambda_1 \sqrt{D_1} - \lambda_2 \sqrt{D_2} \). Every \( x \in \mathcal{H} \) can be written as

\[
x = LL^{-1} x = \sum_{j \in J} \langle L^{-1} x, S_X^{-1} x_j \rangle \eta_j,
\]

implying that

\[
\|x\|^2 = \|\langle x, x \rangle\|^2
\]

\[
= \sum_{j \in J} \langle L^{-1} x, S_X^{-1} x_j \rangle \langle \eta_j, x \rangle
\]

\[
\leq \sum_{j \in J} \langle L^{-1} x, S_X^{-1} x_j \rangle \langle S_X^{-1} x_j, L^{-1} x \rangle \cdot \sum_{j \in J} \langle x, \eta_j \rangle \langle \eta_j, x \rangle
\]

\[
\leq \frac{1}{C_1} \|L^{-1} x\|^2 \cdot \sum_{j \in J} \langle x, \eta_j \rangle \langle \eta_j, x \rangle.
\]

Therefore,

\[
\sum_{j \in J} \langle x, \eta_j \rangle \langle \eta_j, x \rangle \geq (\sqrt{C_1} - \lambda_1 \sqrt{D_1} - \lambda_2 \sqrt{D_2})^2 \|x\|^2,
\]

and thus, we get \( X \) and \( Y \) are woven.

Now, let us prove \( Y \) is a Riesz basis. Assume \( y_j = 0 \) for some \( j \). It follows from (76) that \( \lambda_1, \lambda_2 < 1 \) and \( \|x_j\| \leq \lambda_1 \|x_j\| \). So, \( x_j = 0 \), and it is a contradiction. Hence, \( y_j \neq 0 \), for each \( j \in J \). Now, let \( \{c_j\}_{j \in J} \in \text{ker} T_X \). Using (77), we obtain that \( \|T_Y \{c_j\}_{j \in J}\| \leq \lambda_2 \|T_Y \{c_j\}_{j \in J}\| \), which implies that \( \{c_j\}_{j \in J} \in \text{ker} T_Y \). So, \( \text{ran} \theta_X = \text{ran} \theta_Y \), and it follows from Theorem 3.1 in [12] that \( Y \) is a Riesz basis.

\[\]
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