Output Feedback Control of the One-Phase Stefan Problem

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Abstract—In this paper, a backstepping observer and an output feedback control law are designed for the stabilization of the one-phase Stefan problem. The present result is an improvement of the recent full state feedback backstepping controller proposed in our previous contribution. The one-phase Stefan problem describes the time-evolution of a temperature profile in a liquid-solid material and its liquid-solid moving interface. This phase transition problem is mathematically formulated as a 1-D diffusion Partial Differential Equation (PDE) of the melting zone defined on a time-varying spatial domain described by an Ordinary Differential Equation (ODE). We propose a backstepping observer allowing to estimate the temperature profile along the melting zone based on the available measurement, namely, the solid phase length. The designed observer and the output feedback controller ensure the exponential stability of the estimation errors, the moving interface, and the $H_2$-norm of the distributed temperature while keeping physical constraints, which is shown with the restriction on the gain parameter of the observer and the setpoint.

I. INTRODUCTION

Liquid-solid phase transition appear in various kinds of science and engineering processes. Typical applications include sea-ice melting and freezing [1], continuous casting of steel [2], crystal-growth [3], and thermal energy storage system [4]. The physical description of these processes is that, a temperature profile in a liquid-solid material promotes the dynamics of a liquid-solid interface due to the phase transition induced by melting or solidification processes. A mathematical model of such a physical process is called the Stefan problem [5], which is formulated by a diffusion PDE defined on a time-varying spatial domain. The domain’s dynamics is described by an ODE actuated by the Neumann boundary value of the PDE state.

For control objectives, infinite-dimensional frameworks that lead to significant mathematical complexities in the process characterization have been developed for the stabilization of the temperature profile and the moving interface of a 1D Stefan problem. For instance, [2] proposed an enthalpy-based feedback to ensure asymptotical stability of the temperature profile and the moving boundary at a desired reference. In [6] a geometric control approach which enables to adjust the position of a liquid-solid interface at a desired setpoint, and the exponential stability of the $L_2$-norm of the distributed temperature is developed using a Lyapunov analysis. It is worth to mention that [6] considers a priori strictly positive boundary input and a moving boundary which is assumed to be a non-decreasing function of time.

In this paper, a backstepping observer [7], [8] and an output feedback control law are developed for the stabilization of the interface position and the temperature profile of the melting zone of the one-phase Stefan problem [2], [5], [6]. The present work is an improvement of state-feedback result which was proposed in our previous contribution [9]. While [10] designed an output feedback controller that ensures the exponential stability of an unstable parabolic PDE system coupled with a priori-known moving interface, the observer design for the Stefan problem is rarely addressed in the literature. We propose a backstepping transformation which enables to deal with the one-phase Stefan problem in which the dynamics of the moving boundary is not explicitly given due to its state-dependency. Such a transformation is exploited to design an estimator and a controller standing as an extension of the one proposed in [11], [12] for coupled linear PDE-ODEs systems defined on a fixed spatial domain. Our designed observer and the output feedback controller achieve the exponential stabilization of the estimation error, the temperature profile, and the moving interface to the desired references in the $H_2$-norm under the restrictions on the observer gain and the setpoint.

This paper is organized as follows: The one-phase Stefan problem is presented in Section II and a brief review of full-state feedback result [9] is stated in Section III. Section IV explains the observer design and the output feedback control problem with the statement of the main results. Section V introduces a backstepping transformation for moving boundary problems which allows to design the observer gains and the output feedback control law. The physical constraints of this problem are stated and guaranteed by restricting the setpoint and the gain parameter of the observer in Section VI. The Lyapunov stability of the closed-loop system is established in Section VII. Supportive numerical simulations are provided in Section VIII. The paper ends with final remarks and future directions discussed in Section IX.

II. DESCRIPTION OF THE PHYSICAL PROCESS

Consider a physical model which describes the melting or the solidification mechanism in a pure one-component material of length $L$ in one dimension. In order to describe the position at which phase transition from liquid to solid occurs (or equivalently, in the reverse direction) mathematically, we divide the domain $[0, L]$ into the two time-varying sub-domains, namely, $[0, s(t)]$ occupied by the liquid phase, and $[s(t), L]$ by the solid phase. A heat flux is entering the system at the boundary at $x = 0$ of the liquid phase, which
affects the dynamics of the liquid-solid interface. Assuming that the temperature in the liquid phase is not lower than the melting temperature of the material, we derive the following system described by:

- the diffusion equation of the temperature in the liquid phase which is written as

\[ T_l(x, t) = \alpha T_{xx}(x, t), \quad 0 \leq x \leq s(t), \quad \alpha := \frac{k}{\rho C_p}, \tag{1} \]

with the boundary conditions

\[ -kT_l(0, t) = q_c(t), \tag{2} \]
\[ T_l(s(t), t) = T_m, \tag{3} \]

and the initial values

\[ T_l(x, 0) = T_0(x), \quad s(0) = s_0, \tag{4} \]

where \( T_l(x, t), q_c(t), \rho, C_p \) and \( k \) are the distributed temperature of the liquid phase, manipulated heat flux, liquid density, the liquid heat capacity, and the liquid heat conductivity, respectively.

- the local energy balance at the liquid-solid interface \( x = s(t) \) which yields to the following ODE

\[ \dot{s}(t) = -\beta T_l(s(t), t), \quad \beta := \frac{k}{\rho \Delta H^*} \tag{5} \]

that describes the dynamics of moving boundary where \( \Delta H^* \) denotes the latent heat of fusion.

For the sake of brevity, we refer the readers to [5], where the Stefan condition in the case of a solidification process is derived.

**Remark 1:** As the moving interface \( s(t) \) is not explicitly given, the problem defined in (1)-(5) is a highly nonlinear problem.

**Remark 2:** Due to the so-called isothermal interface condition that prescribes the melting temperature \( T_m \) at the interface through (3), this form of the Stefan problem is a reasonable model only if the following conditions hold:

\[ T_l(x, t) \geq T_m, \quad \text{for all} \quad x \in [0, s(t)], \tag{6} \]
\[ \dot{s}(t) \geq 0, \quad \text{for all} \quad t \geq 0. \tag{7} \]

From Remark 2 it is plausible to assume \( s_0 > 0 \) and the existence of a positive constant \( H > 0 \) such that

\[ 0 \leq T_0(x) - T_m \leq H(s_0 - x). \tag{8} \]

We recall the following lemma that ensures the validity of the model (1)-(5).

**Lemma 1:** For any \( q_c(t) > 0 \) on the finite time interval \((0, \bar{t})\), \( T(x, t) > T_m, \forall x \in (0, s(t)) \) and \( \forall t \in (0, \bar{t}) \). And then \( \dot{s}(t) > 0, \forall t \in (0, \bar{t}) \).

The proof of Lemma 1 is provided by maximum principle and Hopf’s Lemma as shown in [5], [13].

### III. State Feedback Control

In this section, we recall the main result of the backstepping state-feedback control of the 1D Stefan problem [9]. The control objective is to drive the moving boundary \( s(t) \) to a reference setpoint \( s_r \) by manipulating the heat controller \( q_c(t) \). From a physical point of view, for a positive heat controller \( q_c(t) \), the irreversibility of the process restrict a priori the choice of the desired setpoint \( s_r \). The control objective can be achieved only if such constraints are satisfied. The following assumption is stated to satisfy the aforementioned physical constraints.

**Assumption 1:** The setpoint \( s_r \) is chosen to satisfy the following inequality

\[ s_r > s_0 + \frac{\beta}{\alpha} \int_0^{s_0} (T_0(x) - T_m)dx. \tag{9} \]

Assumption 1 is necessary to achieve the control objective due to the energy conservation law given by

\[ \frac{d}{dt} \left( \frac{1}{\alpha} \int_0^{s(t)} (T(x, t) - T_m)dx + \frac{1}{\beta} (s(t) - s_r) \right) = \frac{q_c(t)}{k}. \tag{10} \]

The left hand side of (10) denotes the growth of internal energy. With a positive heat control \( q_c(t) > 0 \), the internal energy for a given setpoint must be greater than the initial internal energy, which leads to the condition (9).

Suppose that both \( T(x, t) \) and \( s(t) \) are measured \( \forall x \in [0, s(t)] \) and \( \forall t \geq 0 \). Then, the following theorem holds:

**Theorem 1:** Consider a closed-loop system consisting of the plant (1)-(5) and the control law

\[ q_c(t) = -c k \left( \frac{1}{\alpha} \int_0^{s(t)} (T(x, t) - T_m)dx + \frac{1}{\beta} (s(t) - s_r) \right). \tag{11} \]

where \( c > 0 \) is an arbitrary controller gain. Assume that the initial values \( (T_0(x), s_0) \) are compatible with the control law and satisfies (8). Then, for any reference setpoint \( s_r \) satisfying (9), the closed-loop system is exponentially stable in the sense of the norm

\[ ||T - T_m||_{L_2}^2 + (s(t) - s_r)^2. \tag{12} \]

**Proof:** The control law (11) was derived using the following backstepping transformation

\[ u(x, t) = T(x, t) - T_m, \quad X(t) = s(t) - s_r, \tag{13} \]
\[ w(x, t) = u(x, t) - \frac{c}{\alpha} \int_x^{s(t)} (x - y)u(y, t)dy + \frac{c}{\beta} (s(t) - x)X(t), \tag{14} \]

introduced in [9] with the aim to transform the system (1)-(5) into a target system. Noting that \( q_c(t) > 0 \) is required by Remark 2 and Lemma 1 to remain the model validity, the
overshoot beyond the reference $s_r$ is prohibited to achieve the control objective $s(t) \rightarrow s_r$ due to its irreversible process (7), which means $s(t) < s_r$ is required to be satisfied for $\forall t > 0$. These two conditions

$$q_c(t) > 0, \quad s(t) < s_r, \quad \forall t > 0, \quad (15)$$

namely the "physical constraints", are satisfied under the setpoint restriction (9). With the help of (7) and (15), the target system was shown to be exponentially stable. The detailed proof of Theorem 1 is established in [9].

IV. CONTROL AND ESTIMATION PROBLEM STATEMENT AND MAIN RESULTS

A. Problem Statement

In [9], the authors considered the full-state feedback control problem, in which the controller implementation requires available measurements of the temperature profile $T(x,t)$ along the domain $[0, s(t)]$ and the moving interface position $s(t)$. Under these conditions, the practical relevance of the proposed solution is relatively limited. In the present work, we extended the full-state feedback results considering moving interface position $s(t)$ as the only available measurement.

B. Observer Design

Suppose that the interface position is obtained as the only available measurement $Y(t) = s(t)$. Then, denoting the estimates of the temperature $\hat{T}(x,t)$, the following theorem holds:

**Theorem 2**: Consider the following closed-loop system of the observer

$$\dot{\hat{T}}_t(x,t) = \alpha \hat{T}_{xx}(x,t) - P_1(x,s(t)) \left( \frac{\dot{Y}(t)}{\beta} + \hat{T}_x(s(t),t) \right), \quad (16)$$

$$-k\hat{T}_x(0,t) = q_c(t), \quad (17)$$

$$\hat{T}(s(t),t) = T_m, \quad (18)$$

where $x \in [0, s(t)]$, and the observer gain $P_1(x,s(t))$ is

$$P_1(x,s(t)) = -\lambda s(t) \frac{I_1 \left( \sqrt{\frac{\Delta}{\alpha} (s(t)^2 - x^2)} \right)}{\sqrt{\frac{\Delta}{\alpha} (s(t)^2 - x^2)}} \quad (19)$$

with an observer gain $\lambda > 0$. Assume that the two physical constraints (15) are satisfied. Then, for all $\lambda > 0$, the observer error system is exponentially stable in the sense of the norm

$$||T - \hat{T}||^2_{H_1}. \quad (20)$$

C. Output Feedback Control

The design of the output feedback controller is achieved using the reconstruction of the state through the exponentially convergent observer defined in Theorem 2 and based on the only available measurement $Y(t)$. We propose the following theorem:

**Theorem 3**: Consider the closed-loop system (11)-(15) with the measurement $Y(t) = s(t)$ and the observer output feedback control law

$$q_c(t) = -ck \left( \frac{1}{\alpha} \int_0^t \hat{T}(x,t) - T_m \, dx \right) + \frac{1}{\beta} (Y(t) - s_r). \quad (21)$$

Assume that the initial values $(\hat{T}_0(x), s_0)$ are compatible with the control law and the initial plant states $(T_0(x), s_0)$ satisfy (5). Additionally, assume that the upper bound of the initial temperature is known, i.e. the Lipschitz constant $H$ in (8) is known. Then, by setting an initial temperature estimation $\hat{T}_0(x)$, gain parameter of the observer $\lambda$, and the setpoint $s_r$ to satisfy

$$\hat{T}_0(x) = T_m + \hat{H}(s_0 - x), \quad (22)$$

$$\lambda < \frac{4\alpha}{s_0} \left( 1 - \frac{H}{\hat{H}} \right), \quad (23)$$

$$s_r > s_0 + \frac{\beta s_0^2}{2\alpha} \hat{H}, \quad (24)$$

with a choice of a parameter $\hat{H} > H$, the closed-loop system is exponentially stable in the sense of the norm

$$||T - \hat{T}||^2_{H_1} + ||T - T_m||^2_{H_1} + (s(t) - s_r)^2. \quad (25)$$

The design procedure is presented in Section V and VI, and the proofs of Theorem 2 and Theorem 3 are provided in Section VII.

V. BACKSTEPPING TRANSFORMATION FOR MOVING BOUNDARY FORMULATION

We recall that the reference error of liquid temperature and the moving interface are denoted as $u(x,t) = T(x,t) - T_m$ and $X(t) = s(t) - s_r$, respectively (see Section III). Defining the controller and the available measurement as $U(t) = -q_c(t)/k$ and $Y(t) = s(t)$, respectively, the coupled system (11)-(15) are written as

$$u(x,t) = \alpha u_{xx}(x,t), \quad 0 \leq x \leq s(t) \quad (26)$$

$$u_x(0,t) = U(t), \quad u(s(t),t) = 0, \quad (27)$$

$$\dot{X}(t) = -\beta u_x(s(t),t). \quad (28)$$

For the reference error system, namely, the $u$-system (26)-(28), we consider the following observer:

$$\dot{\tilde{u}}_t(x,t) = \alpha \tilde{u}_{xx}(x,t) + P_1(x,s(t)) \left( \frac{1}{\beta} \tilde{Y}(t) - \tilde{u}_x(s(t),t) \right), \quad (29)$$

$$\tilde{u}(s(t),t) = 0, \quad (30)$$

$$\tilde{u}_x(0,t) = U(t), \quad (31)$$

where $P_1(x,s(t))$ is the observer gain that needs to be determined. Defining error variable of $u$-system as $\tilde{u}(x,t) = u(x,t) - \tilde{u}(x,t)$ and combining (26)-(28) with (29)-(31), the $\tilde{u}$-system is written as

$$\tilde{u}_t(x,t) = \alpha \tilde{u}_{xx}(x,t) - P_1(x,s(t)) \tilde{u}_x(s(t),t), \quad (32)$$

$$\tilde{u}(s(t),t) = 0, \quad \tilde{u}_x(0,t) = 0. \quad (33)$$
A. Observer Target System

1) Direct transformation: In this section, we introduce a moving boundary backstepping transformation and observer gains motivated by a fixed boundary backstepping transformation [11], [12] as
\[ \tilde{u}(x, t) = \tilde{w}(x, t) + \int_{0}^{\xi(t)} P(x, y) \tilde{w}(y, t) dy, \]
which transforms the \( \tilde{u} \)-system in (32)-(33) into the following exponentially stable target system
\[ \begin{align*}
\tilde{w}_x(x, t) &= -\lambda \tilde{w}(x, t), \\
\tilde{w}(s, t) &= 0, \quad \tilde{w}_x(0, t) = 0.
\end{align*} \tag{35} \tag{36} \]
Taking the derivative of (34) with respect to \( t \) and \( x \) along the solution of (35)-(36) respectively, the solution of the gain kernel and the observer gain are given by
\[ P(x, y) = \frac{\alpha}{\lambda y} J_1 \left( \sqrt{\frac{\lambda}{\alpha}(y^2 - x^2)} \right), \]
\[ P_1(x, s(t)) = -\alpha P(x, s(t)), \] \tag{37} \tag{38}
where \( J_1(x) \) is a modified Bessel function of the first kind. In order to show the negativity of estimation error, we recall a following lemma. The importance of such a property is stated in Section VI.

Lemma 2: Suppose that \( \tilde{w}(0, t) < 0 \). Then, the solution of (35)-(36) satisfies
\[ \tilde{w}(x, t) < 0, \quad \forall x \in (0, s(t)), \quad \forall t > 0. \] \tag{39}
The proof of Lemma 2 is given by maximum principle [13].

2) Inverse transformation: By the same procedure, one can derive the inverse transformation from \( \tilde{w} \)-system in (35)-(36) to \( \tilde{u} \)-system in (32)-(33) as
\[ \begin{align*}
\tilde{w}(x, t) &= \tilde{u}(x, t) - \int_{0}^{\xi(t)} Q(x, y) \tilde{u}(y, t) dy, \\
Q(x, y) &= \frac{\lambda}{\alpha y} J_1 \left( \sqrt{\frac{\lambda}{\alpha}(y^2 - x^2)} \right).
\end{align*} \tag{40} \tag{41} \]
where \( J_1(x) \) is a Bessel function of the first kind.

B. Output Feedback Control

By equivalence, the transformation of the variables (\( \tilde{u}, X \)) into (\( \tilde{w}, X \)) leads to the gain kernel functions defined by the state-feedback backstepping transformation [14] given by
\[ \begin{align*}
\tilde{w}(x, t) &= \tilde{u}(x, t) - \frac{c}{\lambda} \int_{0}^{\xi(t)} (x - y) \tilde{u}(y, t) dy \\
&\quad + \frac{c}{\beta}(s(t) - x) X(t),
\end{align*} \tag{42} \]
with an associated target system given by
\[ \begin{align*}
\hat{w}_x(x, t) &= -\lambda \hat{w}(x, t) + \frac{c}{\beta} \hat{s}(t) X(t) \\
+ f(x, s(t)) \hat{w}(s, t), \\
\hat{w}(s, t) &= 0, \quad \hat{w}_x(0, t) = 0,
\end{align*} \tag{43} \tag{44} \]
\[ \begin{align*}
\hat{X}(t) &= -c X(t) - \beta \hat{w}_x(s(t), t) - \beta \hat{w}(s, t),
\end{align*} \tag{45} \]
where
\[ f(x, s(t)) = P_1(x, s(t)) - \frac{c}{\alpha} \int_{0}^{\xi(t)} (x - y) P_1(y, s(t)) dy - c(s(t) - x). \] \tag{46}
Evaluating the spatial derivative of (42) at \( t = 0 \), we derive the output feedback controller as
\[ q_c(t) = -ck \left( \frac{1}{\alpha} \int_{0}^{\xi(t)} \tilde{u}(x, t) dx + \frac{1}{\beta} X(t) \right). \] \tag{47}
By the same procedure as [9], one can derive an inverse transformation written as
\[ \tilde{u}(x, t) = \tilde{w}(x, t) + \frac{\beta}{\alpha} \int_{0}^{\xi(t)} \psi(x - y) \tilde{w}(y, t) dy \\
+ \psi(s(t)) X(t), \]
\[ \psi(x) = \frac{c}{\beta} \sqrt{\frac{\alpha}{c}} \sin \left( \sqrt{\frac{c}{\alpha}} x \right). \] \tag{48} \tag{49}

VI. Physical Constraints

The two physical constraints defined in [15] are required to guarantee the physical validity of the model (1)-(5) and achieve the control objective \( s(t) \to s_f \). In this section, we derived sufficient conditions to guarantee (15) which is satisfied if only if the controller (47) is always injecting positive heat without any interface overshoot beyond the setpoint. First, we state the following lemma.

Lemma 3: Assume that the upper bound of the initial temperature is known, i.e. the Lipschitz constant \( H \) in (8) is known. Suppose that we set the initial estimation \( \hat{u}(x, 0) \) and the gain parameter of the observer \( \lambda \) chosen to satisfy
\[ \hat{u}(x, 0) = \hat{H}(s_0 - x), \]
\[ \lambda < \frac{4\alpha}{s_0^2} \left( 1 - \frac{H}{\hat{H}} \right), \] \tag{50} \tag{51}
respectively with a choice of a parameter \( \hat{H} > H \). Then, the following properties hold:
\[ \tilde{u}(x, t) < 0, \quad \tilde{u}_x(s(t), t) > 0, \quad \forall x \in (0, s(t)), \quad \forall t > 0. \] \tag{52}

Proof: Lemma 2 showed that if \( \tilde{w}(x, 0) < 0 \), then \( \tilde{w}(x, t) < 0 \). In addition, by the direct transformation (8), \( \tilde{w}(x, t) < 0 \) leads to \( \hat{u}(x, t) < 0 \) due to the positivity of the solution to the gain kernel (37). Therefore, with the help of (40), we deduce that \( \hat{u}(x, t) < 0 \) if \( \hat{u}(x, 0) \) satisfies
\[ \hat{u}(x, 0) < \int_{0}^{s_0} Q(x, y) \tilde{u}(y, 0) dy, \quad \forall x \in (0, s_0). \] \tag{53}
Considering the bound of the solution (41) under the condition of (50), the sufficient condition for (55) to hold is given by (51) which restricts the gain parameter \( \lambda \). Thus, we have shown that conditions (50) and (51) lead to \( \hat{u}(x, t) < 0 \) for all \( x \in (0, s_0) \). In addition, by the boundary condition (33) and Hopf’s lemma, it leads to \( \tilde{u}_x(s(t), t) > 0 \).

Next, we show the physical constraints (15) are satisfied with a restriction on the setpoint using (52).
Proposition 1: Suppose the initial values \((\tilde{u}_0(x), s_0)\) satisfy (50) and the setpoint \(s_r\) is chosen to satisfy
\[
s_r > s_0 + \frac{\beta s_0^2}{2c_0} \Delta H. \tag{54}
\]
Then, the following physical constraints are satisfied
\[
q_c(t) > 0, \quad u(x,t) > 0, \quad \hat{s}(t) > 0, \tag{55}
\]
\[
s_0 < \hat{s}(t) < s_r. \tag{56}
\]

Proof: Taking the time derivative of (47) along with the solution (29)-(31), with the help of the observer gain (38), we derive the following differential equation:
\[
\dot{q}_c(t) = -cq_c(t) + \left(1 + \int_0^{s(t)} P(x,s(t))dx\right) \tilde{u}_x(s(t),t). \tag{57}
\]

From the positivity of the solution (37) and the Neumann boundary value (52) by Lemma 3, it leads to the following differential inequality
\[
\dot{q}_c(t) \geq -cq_c(t). \tag{58}
\]
Hence, if the initial values satisfy \(q_c(0) > 0\), equivalently (54) by (47) and (50), we get
\[
q_c(t) > 0, \quad \forall t > 0. \tag{59}
\]
By Lemma 1 we derive the two other conditions stated in (55). Then, with the relation (52) given in Lemma 3 and the positivity of \(u(x,t)\) derived in (55), the following inequality is established
\[
\dot{u}(x,t) > 0, \quad \forall x \in (0,s(t)), \quad \forall t > 0. \tag{60}
\]
Finally, substituting the inequalities (59) and (60) into (47), we arrive at
\[
X(t) < 0, \quad \forall t > 0 \tag{61}
\]
which guarantees the second physical constraint (56). In the next section, we assume that the restrictions (50), (51) and (54) are satisfied. We use the physical constraints (55) and (56) to show the Lyapunov stability.

VII. LYAPUNOV STABILITY

In this section, we establish the exponential stability of the origin in the closed-loop system in \(H_1\)-norm based on the Lyapunov stability analysis for PDEs [7] of the associated target system (35)-(36) and (43)-(45).

A. Stability Analysis of \(\tilde{w}\)-System

Let \(\tilde{V}_1\) be a functional such that
\[
\tilde{V}_1 = \frac{1}{2} \|\tilde{w}\|_{H_1}^2. \tag{62}
\]

Taking the derivative of (62) along the solution of the target system (35)-(36), we obtain
\[
\dot{\tilde{V}}_1 = -\alpha \|\tilde{w}_x\|_{H_1}^2 - \lambda \|\tilde{w}\|_{H_1}^2 - \frac{\hat{s}(t)}{2} \tilde{w}_x(s(t),t)^2. \tag{63}
\]
With the help of (55) and (56), and applying Pointcare’s inequality, a differential inequality in \(\tilde{V}_1\) is obtained as
\[
\dot{\tilde{V}}_1 \leq -\left(\frac{\alpha}{4s_0^2} + \lambda\right) \tilde{V}_1. \tag{64}
\]
Hence, the origin of the target \(\tilde{w}\)-system is exponentially stable. Since the transformation (34) is invertible as in (40), the exponential stability of \(\tilde{w}\)-system at the origin deduces the exponential stability of \(\hat{w}\)-system at the origin with the help of (56), which completes the proof of Theorem 2.

B. Stability Analysis of the Closed-Loop System

Define the functional \(V_{tot}\) as
\[
V_{tot} = \frac{1}{2} \|\tilde{w}\|_{H_1}^2 + \frac{p}{2} X(t)^2 + d\tilde{V}_1, \tag{65}
\]
where \(d\) is chosen to be large enough and \(p\) is defined as
\[
p = \frac{cc_0}{16s^2}. \tag{66}
\]
Taking the derivative of (62) along the solution of the target system (43)-(45), and applying Young’s, Cauchy-Schwarz, Poincare’s, Agmon’s inequality, with the help of (55) and (56), the following holds:
\[
\dot{V}_{tot} \leq -bV_{tot} + a\tilde{s}(t)V_{tot}, \tag{67}
\]
where,
\[
a = \max\left\{s^2_1 \frac{16csr}{\alpha}, \frac{16csr}{\alpha} \right\}, \quad b = \min\left\{\frac{\alpha}{8s^2}, c, 2\lambda \right\}. \tag{68}
\]
Define the Lyapunov function \(V\) such that
\[
V = V_{tot} e^{-as(t)}. \tag{69}
\]
Taking the time derivative of (69) and applying (67), we get
\[
\dot{V} = (\dot{V}_{tot} - a\tilde{s}(t)V_{tot}) e^{-as(t)} \leq -bV \tag{70}
\]
From the equation above, we deduce the exponential decay of \(V\) as \(V(t) \leq V(0)e^{-bt}\). Then, using (56), we arrive at
\[
V_{tot} \leq e^{a\tilde{s}(t)}V_{tot}(0)e^{-bt}. \tag{71}
\]
Hence, the origin of the \((\hat{w}, X, \tilde{w})\)-system is exponentially stable. Since the transformation (34) and (42) are invertible as described in (40) and (48), the exponential stability of \((\hat{w}, X, \tilde{w})\)-system at the origin guarantees the exponential stability of \((\hat{u}, \hat{X}, \hat{u})\)-system at the origin, which completes the proof of Theorem 3.

VIII. SIMULATION RESULTS

As in [9], the simulation is performed considering a strip of zinc whose physical properties are given in Table 1. The setpoint and the initial values are chosen as \(s_r = 0.35\ m, s_0 = 0.01\ m, T_0(x) - T_m = H(s_0 - x)\) with \(H = 100\ \text{Km}^{-1}\), and \(\lambda\) and \(s_r\) described in (22), (23) are satisfied, which are conditions for Theorem 2 and Theorem 3 to remain valid.
The interface $s(t)$ converges to the setpoint $s_r$, keeping increasing and without the overshoot, i.e. $s(t) > 0, s_0 < s(t) < s_r$.

(b) The output feedback controller $q_c(t)$ remains positive, i.e. $q_c(t) > 0$.

(c) The estimation error of the boundary temperature $\tilde{T}(0, t)$ converges to zero with keeping the negativity, i.e. $\tilde{T}(0, t) < 0$.

Fig. 2: The simulation of the closed-loop system (1) - (5) and the estimator (21) - (25) with the output feedback control law (21).

The dynamics of the moving interface $s(t)$, the output feedback controller $q_c(t)$, and the estimation error of the boundary temperature $\tilde{T}(0,t)$ are depicted in Fig. 2 (a) - (c), respectively. Fig. 2 (a) shows that the interface $s(t)$ converges to the setpoint $s_r$ with $\dot{s}(t) > 0$ and $s_0 < s(t) < s_r$, for $\forall t > 0$, which are guaranteed in Proposition 1. Fig. 2 (b) shows that the output feedback controller remains positive, which is a physical constraint for the model to be valid as stated in Lemma 3 and ensured in Proposition 1. The positivity of the controller results from the negativity of the distributed estimation error $\tilde{T}(x,t)$ as shown in Lemma 3 and Proposition 1. Fig. 2 (c) shows that the estimation error of boundary temperature $\tilde{T}(0,t)$ converges to zero and remains negative. Therefore, the numerical results are consistent with the theoretical results stated in Lemma 3 and Proposition 1.

**TABLE I: Physical properties of zinc**

| Description             | Symbol | Value         |
|-------------------------|--------|---------------|
| Density                 | $\rho$ | 6570 kg·m$^{-3}$ |
| Latent heat of fusion   | $\Delta H^*$ | 111.961 J·kg$^{-1}$ |
| Heat Capacity           | $C_p$  | 389.5687 J·kg$^{-1}$·K$^{-1}$ |
| Thermal conductivity    | $k$    | 116 W·m$^{-1}$·K$^{-1}$ |

**IX. CONCLUSIONS AND FUTURE WORKS**

In this paper we designed an observer and a boundary output feedback controller for the one-phase Stefan problem via the backstepping transformation. The proposed controller achieves the exponential stability of the closed-loop system using only a measurement of the moving interface and ensures that the physical constraints are satisfied under the restriction on the setpoint and the gain parameter of the designed observer assuming the upper bound of the initial temperature to be known. The main contribution of this paper is that, this is the first result which shows the convergence of the estimation error and output feedback control of the one-phase Stefan problem theoretically. Although the Stefan problem has been a well known model since 200 years ago related with phase transitions which appear in various nature and engineering processes, its control and estimation related problems have not been investigated in details. The estimation of the sea-ice melting problem in Arctic region is being considered as a future work.

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