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Dedicated to the late Professor Jean-Pierre Demailly.

Communicated by Irene Sabadini.

This article is part of the topical collection “Higher Dimensional Geometric Function Theory and Hypercomplex Analysis” edited by Irene Sabadini, Michael Shapiro and Daniele Struppa.

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1 Introduction

The pluripotential theory in several complex variables, especially the complex Monge-Ampère operator, has generated considerable interest over the past five decades, thanks to the many applications on complex analysis, complex differential geometry and number theory. By letting ourselves inspired by techniques go back to Demailly [9] and Ben Messaoud and El Mir [2], we extend profound results in pluripotential theory to the superformalism setting via the supercurrent theory originated by Lagerberg [16] and Berndtsson [3]. Regarding positive supercurrents, there are at least two kinds of examples of positive supercurrents. The first kind is in tropical geometry, which consists of supercurrents of the form $dd^c u$, where $u$ is a convex function. This is strongly related to the study of the tropical hypersurfaces. Indeed, it was stressed by Lagerberg [16] that there is a one to one correspondence between tropical hypersurface and closed positive $(1, 1)$-supercurrent with adequate hypothesis on the support. The second kind is in analytic geometry over fields equipped with a non-archimedean absolute value, which consists of the associated $(p, p)$-supercurrent attached to the $v_p$-analytification $Y_{v_p}^{an} \subset (\mathbb{P}_Q^N)_{v_p}^{an}$ of a purely dimensional effective cycle $Y \subset \mathbb{P}_Q^N$ defined over $\mathbb{Q}$ in the Berkovich analytic space $(\mathbb{P}_Q^N)_{v_p}^{an}$, where $v_p$ is the ultrametric place corresponding to a prime number $p$ (see [7] and [13]). Our first main result is to associate to a given positive closed supercurrent $T$, a local potential $U$ satisfying many interesting properties similar to the complex setting. Next, we investigate the weak convergence of the superHessian operator relatively to a smooth regularisations of $T$ and $U$ with respect to local uniform converging sequences of convex functions. Comparing with the complex context, this is a clear difference. Namely, it is well known that the analogous weak continuity for the complex Monge-Ampère operator requires a convergence monotonicity condition on the sequences of plurisubharmonic functions. Among other properties of $U$ and as in the paper of Ghiloufi, Zaway and Hawari [14] in the complex setting, we describe a relation between the Lelong numbers of $T$ and of $U$. The second important aim of this paper is the study of the definition and the continuity of the $m$-superHessian operator associated to an $m$-positive closed supercurrent $T$, for some classes of unbounded $m$-convex functions as well as when we consider a regularization sequence of $T$. Recall that we consider the concepts of $m$-positivity of superforms and supercurrents introduced by Elkhadra and Zahmoul [12] analogously with the complex Hessian theory. As an immediate consequence we introduce the $m$-generalized Lelong number of $T$ relatively to a given $m$-convex function. It is worth mentioning that the results obtained here are the counterparts of corresponding one in the complex Hessian pluripotential theory and they had a crucial role in
the development of complex geometry. We hope that our superformalism adaptation attracts some problems within the tropical geometry and the Riemannian geometry. Next, as a generalization of the work of Labutin [15] and the work of Trudinger and Wang [23], we obtained new outcomes on the weighted $m$-Hessian capacity for $m$-convex functions on a Borel subset of $\mathbb{R}^n$, that are real counterparts of recent results due to Nguyen [19] on generalized $m$-capacity for $m$-subharmonic functions. Let us describe more precisely the content of the paper. Besides the introduction, this paper has four sections. In Sect. 2, we recall some basic notations and definitions necessarily for the rest of this paper. The third section is where we introduce and investigate a variant of important related properties of the local potential $U$ of a positive closed current $T$ in the superformalism setting. Also, we investigate the weak convergence of the sequence of the superHessian operators $(U_j \wedge dd^#u_1^j \wedge ... \wedge dd^#u_k^j)_j$, where $U_j$ is a smooth regularization by convolution of $U$ and $(u_j^k)_j$ are sequences of locally uniformly converging convex functions. The same weak convergence for $T_j$ instead of $U_j$ was guaranted by means of the link between $T$ and its potential. Moreover, we establish a result concerning the relation between the Lelong number of $T$ and $U$. Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and denote by $\beta = \frac{1}{2}dd^{|x|^2}$ the Kähler form on $\mathbb{R}^n \times \mathbb{R}^n$. Strongly motivated by the work of Demailly [9] on the Monge-Ampère operator, we study in Sect. 4 the definition and the continuity of the $m$-superHessian operator $T \wedge \beta^{n-m} \wedge dd^#u_1 \wedge ... \wedge dd^#u_k$ for monotone decreasing sequences of $m$-convex functions bounded near $\partial \Omega \cap \text{Supp } T$. Here $T$ is a closed $m$-positive supercurrent of bidimension $(p, p)$ on $\Omega \times \mathbb{R}^n$ such that $T \wedge \beta^{n-m}$ is positive and $\text{Supp } T$ is the support of $T$. As an application, we introduce the $m$-generalized Lelong number of $T$ relatively to an $m$-convex weight $\phi$ as the measure of the asymptotic behaviour of the supercurrent $T \wedge \beta^{n-m} \wedge (dd^#\phi)^{m+p-n}$ near the $m$-polar set $\{\phi = -\infty\}$. In analogy with the complex case, we investigate the counterpart of the famous Demailly comparison theorem and the transformation of the Lelong number by a direct image of a projection. The remaining of Sect. 4 contains several topics in superformalism setting, including the integrability exponents of $m$-convex functions and $m$-generalized Lelong numbers in terms of capacity. In the last section, we introduce two classes of $m$-convex functions $C^0_m(\Omega)$ and $\mathcal{F}_m(\Omega)$. These classes turn out to the Cegrell classes of plurisubharmonic functions which are a basic tool in the study of interesting problems in the complex setting. Namely, the domain of definition of the Monge-Ampère operator. After establishing some elementary properties of such classes in the superformalism setting and in analogy with Nguyen [19], we introduce and investigate the weighted relative extremal function $R_{m,u}(E)$ as a generalization of the one given by Labutin [15]. We also define the weighted $m$-Hessian capacity $cap_{m,u}(E)$ for an $m$-convex weight $u$. Finally, we discover a link between $cap_{m,u}(E)$ and $R_{m,u}(E)$ for Borel compact subset $E$ as well as for $u \in \mathcal{F}_m(\Omega)$.

2 Preliminaries

In this part, we recall definitions and basic properties of the superforms and supercurrents as introduced by Lagerberg [16], Berndtsson [3] and Elkhadhra and Zahrnouli...
[12], that will be used throughout this paper. Let $V$ and $W$ to be two $n$-dimensional vector spaces over $\mathbb{R}$, so that $x = (x_1, ..., x_n)$ and $\xi = (\xi_1, ..., \xi_n)$ are the corresponding coordinates, and let $J : V \to W$ be an isomorphism such that $J(x) = \xi$, and denote its inverse by $J(\xi) = -x$ if $x \in V$ is the element for which $J(x) = \xi$. By setting $E = V \times W = \{(x, \xi); x \in V, \xi \in W\}$, we observe that the map $J$ can be extended over $E$ by setting $J(x, \xi) = (J(\xi), J(x))$, so that $J^2 = -id$. For $0 \leq p, q \leq n$, any smooth superform on $E$ of bidegree $(p, q)$ can be written as

$$\alpha = \sum_{K,L} \alpha_{KL}(x) dx_K \wedge d\xi_L,$$

where $K = (k_1, ..., k_p)$, $dx_K = dx_{k_1} \wedge ... \wedge dx_{k_p}$, $L = (l_1, ..., l_q)$, $d\xi_L = d\xi_{l_1} \wedge ... \wedge d\xi_{l_q}$ and each coefficient $\alpha_{KL}(x)$ is smooth and depends only on $x$. We denote by $\mathcal{E}^{p,q} := \mathcal{E}^{p,q}(E)$ the set of smooth superforms on $E$ of bidegree $(p, q)$. In particular, if $p = q$ we say that $\alpha$ is symmetric if and only if $\alpha_{KL} = \alpha_{LK}$, $\forall K, L$. It is clear that $J^*(dx_i) = d\xi_i$ and $J^*(d\xi_i) = -dx_i$. In order to simplify the notation, we denote the operator $J^*$ by $J$, which can be extended on $\mathcal{E}^{p,q}$ by a map denoted again by $J : \mathcal{E}^{p,q} \to \mathcal{E}^{q,p}$ and defined by

$$J(\alpha) = J \left( \sum_{K,L} \alpha_{KL}(x) dx_K \wedge d\xi_L \right) = (-1)^q \sum_{K,L} \alpha_{KL}(x) dx_K \wedge d\xi_K, \quad \forall \alpha \in \mathcal{E}^{p,q}.$$  

In particular, if $\alpha \in \mathcal{E}^{p,p}$, then $\alpha$ is symmetric if and only if $J(\alpha) = \alpha$. Now, let us consider the Kähler form on $\mathbb{R}^n \times \mathbb{R}^n$ to be

$$\beta := \sum_{i=1}^n dx_i \wedge d\xi_i \in \mathcal{E}^{1,1}.$$  

A simple computation gives $\beta^n = n!dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n$. Next, we introduce three notions of positivity on $\mathcal{E}^{p,p}$. A superform $\varphi \in \mathcal{E}^{n,n}$ is said to be positive ($\varphi \geq 0$) if $\varphi = f \beta^n$, where $f$ is a positive function. Let $\varphi \in \mathcal{E}^{p,p}$ be symmetric, we say that $\varphi$ is:

1. Weakly positive if $\varphi \wedge \alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-p} \wedge J(\alpha_{n-p}) \geq 0$, $\forall \alpha_1, ..., \alpha_{n-p} \in \mathcal{E}^{1,0}$.
2. Positive if $\sigma_{n-p} \varphi \wedge J(\alpha) \geq 0$, $\forall \alpha \in \mathcal{E}^{n-p,0}$; $\sigma_k = (-1)^{\frac{k(k-1)}{2}}$.
3. Strongly positive if $\varphi = \sum_{s=1}^N \lambda_s \alpha_{1,s} \wedge J(\alpha_{1,s}) \wedge ... \wedge \alpha_{p,s} \wedge J(\alpha_{p,s})$; $\lambda_s \geq 0$, $\alpha_{i,s} \in \mathcal{E}^{1,0}$.

For every $\alpha \in \mathcal{E}^{n,n}$, there exists a function $\alpha_0$ defined on $V$ such that

$$\alpha = \alpha_0 dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n.$$
When an orientation on $V$ is chosen and $\alpha_0$ is integrable, the superintegral of $\alpha$ is defined by

$$\int_E \alpha = \int_V \alpha_0 dx_1 \wedge ... \wedge dx_n.$$  

The operators $d$ and $d^#$ are of type $(1, 0)$ and $(0, 1)$ respectively and acting on $\mathcal{E}^{p,q}$ by the following expressions

$$d = \sum_{i=1}^{n} \partial_i dx_i \quad \text{and} \quad d^# = \sum_{j=1}^{n} \partial_j d\xi_j.$$  

It is easy to see that $d^2 = (d^#)^2 = 0$, $dd^# = -d^#d$ and $\beta = \frac{1}{2}dd^#|x|^2$. Moreover, in this situation we can present a Stokes’ formula as follow: Assume that $\Omega_1 \subset V$ is an open bounded set with smooth boundary and let $\alpha \in \mathcal{E}^{n-1,n}$. Then,

$$\int_{\Omega \times W} d\alpha = \int_{\partial \Omega \times W} \alpha.$$  

Denote by $\mathcal{D}^{p,q}$ the set of smooth and compactly supported superforms of bidegree $(p, q)$ on $E$, whose topology can be defined by means of the inductive limit. We introduce the space $\mathcal{D}_{p,q}$ of supercurrents of bidimension $(n-p, n-q)$ as the topological dual of $\mathcal{D}^{n-p,n-q}$. This means that a supercurrent $T$ of bidimension $(n-p, n-q)$ is nothing but a continuous linear map on $\mathcal{D}^{n-p,n-q}$. More precisely, $T$ is a superform of bidegree $(p, q)$ which has distributions coefficients depending only on $x$. That is

$$T = \sum_{|I|=p, |J|=q} T_{IJ} dx_I \wedge d\xi_J,$$

where $T_{IJ}$ are distributions defined uniquely on $x$. In particular, as with superforms if $p = q$ we say that the supercurrent $T$ is symmetric if and only if $T_{IJ} = T_{JI} \forall I, J$. For any $\alpha \in \mathcal{D}^{n-p,n-q}$, denote by $\langle T, \alpha \rangle$ the action of $T$ on $\alpha$. A supercurrent $T$ is said to be closed if $dT = 0$. Assume that $T$ is symmetric and of bidegree $(p, p)$. $T$ is said to be:

1. Weakly positive if $\langle T, \alpha \rangle \geq 0$ for any $\alpha \in \mathcal{D}^{n-p,n-p}$ strongly positive.
2. Positive if $\langle T, \sigma_{n-p} \alpha \wedge J(\alpha) \rangle \geq 0$ for any $\alpha \in \mathcal{D}^{n-p,0}$.
3. Strongly positive if $\langle T, \alpha \rangle \geq 0$ for any $\alpha \in \mathcal{D}^{n-p,n-p}$ weakly positive.
4. Convex if $dd^# T$ is weakly positive and concave if $-T$ is convex.

For $K \subset \mathbb{R}^n$ and $T$ a supercurrent of order zero, we define the mass measure of $T$ on $K$ by

$$\|T\|_K = \sum_{IJ} |T_{IJ}|(K),$$
where \( T_{IJ} \) are the coefficients of \( T \). Due to Lagerberg [16], the mass of a positive supercurrent \( T \) of bidimension \((p, p)\) on \( K \) is proportional to the positive measure \( T \wedge \beta^p(K) \). Now, we adapt to the superformalism context the following notions of \( m \)-positivity and \( m \)-convexity from Elkhadhra and Zahmoul [12] in the complex hessian setting:

1. A symmetric superform \( \alpha \) of bidegree \((1, 1)\) is said \( m \)-positive if at every point we have \( \alpha^j \wedge \beta^{n-j} \geq 0, \forall j = 1, \ldots, m \).
2. A symmetric supercurrent \( T \) of bidimension \((n-p, n-p)\) such that \( p \leq m \leq n \), is called \( m \)-positive if we have \( T \wedge \beta^{n-m} \wedge \alpha_1 \wedge \ldots \wedge \alpha_{m-p} \geq 0 \), for all \( m \)-positive superforms \( \alpha_1, \ldots, \alpha_{m-p} \) of bidegree \((1, 1)\).
3. A function \( u : V \to \mathbb{R} \cup \{-\infty\} \) is called \( m \)-convex if it is subharmonic and the supercurrent \( dd^#u \) is \( m \)-positive. Denote by \( C_m \) the set of \( m \)-convex functions.

Next, let us recall some basic facts about \( m \)-convex functions due to Wan and Wang [25]:

1. If \( u \) is of class \( C^2 \) then \( u \) is \( m \)-convex if and only if \( dd^#u \) is \( m \)-positive superform.
2. Convex functions = \( C_n \subset C_{n-1} \subset \cdots \subset C_1 = \) subharmonic functions.
3. If \( u \) is \( m \)-convex then the standard regularization \( u_j = u \ast \chi_j \) is smooth and \( m \)-convex. Moreover, \( (u_j) \) decreases pointwise to \( u \).
4. Let \( u, v \in C_m \) then \( \max(u, v) \in C_m \).
5. If \( \{u_\alpha\}_{\alpha} \subset C_m \), \( u = \sup_{\alpha} u_\alpha < +\infty \) and \( u \) is upper semi-continuous then \( u \) is \( m \)-convex.

For the sake of simplicity, in the rest of this paper we consider two copies of \( \mathbb{R}^n \), i.e. \( V = W = \mathbb{R}^n \) and we say form instead of superform and current instead of supercurrent.

### 3 Local Potential Associated with Positive Closed Supercurrent

#### 3.1 Definition and Properties

In complex analysis, a possible way of studying interesting properties of a given current is to consider the associated local potential. Namely, it was stressed by Ben Messaoud and El Mir [2] that the local potential associated to a positive closed current in \( \mathbb{C}^n \) is crucial in the study of the complex Monge-Ampère operator. By a strong analogy with the complex theory, we try to extend this notion to the superformalism setting as follows:

**Definition 1** Assume that \( T \) is a weakly positive closed current of measure coefficients and of bidimension \((p, p)\) on \( \mathbb{R}^n \times \mathbb{R}^n \) with \( 1 \leq p \leq n - 1 \). Let \( \Omega \subset \mathbb{R}^n \) and \( \eta \) be a smooth compactly supported function on \( \mathbb{R}^n \) such that \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( \partial \Omega \). The local potential \( U = U(\eta, T) = U(\Omega, T) \) associated to \( T \) is a weakly negative current of bidimension \((p + 1, p + 1)\) on \( \mathbb{R}^n \times \mathbb{R}^n \) defined by

\[
U(x, \xi) = -c_n \int_{(y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n} \eta(y)T(y, \zeta) \wedge \beta^{n-1}(x - y, \xi - \zeta) \left/ |x - y|^{n-2} \right.,
\]
where  \( c_n = \frac{1}{(n-2)Vol_n(B(0,1))} \).

Analogously to the complex setting, if  \( h(x) = -c_n |x|^{2-n} \), it is important to observe that the coefficients of  \( U \) are nothing but the convolutions of the measure coefficients of  \( \eta T \) by the function  \( h \) which is in  \( L_{loc}^{1+\frac{1}{n}} \). Therefore, the coefficients of  \( U \) are in  \( L_{loc}^{1+\frac{1}{n}} \). Moreover, we can easily deduce the positivity of the local potential by using the equality (3.2) in the proof of the following Lemma 1 and the fact that the push forward of a (weakly) positive current is (weakly) positive thanks to Lagerberg [16].

**Example 1** (1) Assume that  \( T \) is a weakly positive closed current of bidimension  \( (n-1, n-1) \). Thanks to Lagerberg [16], there exists a convex function  \( u \) such that  \( T = dd^nu \). A direct computation gives

\[
U(x, \xi) = -c_n \int_{(y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n} \eta(y) dd^nu(y, \zeta) \wedge \frac{\beta^n-1(y, \zeta)}{|x-y|^{n-2}}.
\]

It follows that for  \( x \in \Omega \), we have  \( U(x, \xi) = (\eta \Delta u) \ast h(x) = \Delta (\eta u) \ast h + v(x) \), where  \( v \) is smooth because it is a product of convolution of  \( h \) with measures involving derivations of  \( \eta \) which vanish on  \( \Omega \). Since  \( \Delta h = 0 \), we get  \( U(x, \xi) = \eta(x)u(x) + v(x) = u(x) + v(x) \) on  \( \Omega \).

(2) Let  \( M \) be a minimal, smooth and  \( p \)-dimensional submanifold of  \( \mathbb{R}^n \), and let  \([M]_s\) be the associated minimal current as introduced by Berndtsson [3]. This means that  \([M]_s \wedge \beta^{p-1} \) is a weakly positive closed current of bidimension  \( (1, 1) \) on  \( \mathbb{R}^n \times \mathbb{R}^n \). The local potential associated to the current  \([M]_s \wedge \beta^{p-1} \) is

\[
U(x, \xi) = -c_n \int_{(y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n} \eta(y) [M]_s \wedge \beta^{p-1}(y, \zeta) \wedge \frac{\beta^{n-1}(x - y, \xi - \zeta)}{|x-y|^{n-2}}.
\]

where  \( \sigma_M := [M]_s \wedge \beta^{p-1} \) is the volume form defined by Berndtsson [3].

Let  \( \chi \) be a radial smooth function and with support included in the unit ball such that  \( \int \chi(x) dx = 1 \), and let  \( \chi_j(x) = j^n \chi(jx) \) for  \( j \in \mathbb{N}^* \). Let us set

\[
U_j(x, \xi) = \int_{(y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n} \eta(y) h \ast \chi_j(x - y) T(y, \zeta) \wedge \beta^{n-1}(x - y, \xi - \zeta).
\]

Since  \( h \) is subharmonic on  \( \mathbb{R}^n \), the sequence  \( (h \ast \chi_j)_j \) is smooth and monotone decreasing. It follows that  \( (U_j)_j \) is a sequence of forms of bidegree  \( (n-p-1, n-p-1) \)
on \( \mathbb{R}^n \times \mathbb{R}^n \) which decreases weakly to \( U \). Assume that

\[
U(x, \xi) = \sum_{|I|=|J|=n-p-1} U_{IJ}(x)dx_J \wedge d\xi_J
\]

is the canonical expression of \( U \). Then, it is not hard to see that

\[
U_{II}(x) = U \wedge (-1)^{(p+1)(n-p-1)}dx_I^c \wedge d\xi_I^c, \quad \text{where } I^c = \{1, \ldots, n\} \setminus I.
\]

Hence, in view of the above observation, \( U_{II} \) is a subharmonic function on \( \mathbb{R}^n \), since it coincides with the convolution of a positive measure by the function \( h \). Moreover, we have

**Proposition 1** Let \( u = \sum_{|I|=n-p-1} U_{II} \). Then, for all \( x \in \mathbb{R}^n \),

\[
u(x) = \frac{(n-p)(n-1)!}{p!} \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(y)h(x - y)T(y, \xi) \wedge \beta^p(y, \xi).
\]

**Proof** Since the desired equality is made up of two subharmonic functions, it suffices to show it almost everywhere. Clearly we have \( \beta(x - y, \xi - \zeta) = \beta(x, \xi) - \beta(x, \zeta) - \beta(y, \zeta) + \beta(y, \zeta) \). By the expression of the local potential, observe that the form \( \beta^{n-1}(x - y, \xi - \zeta) \) contribute only by its \((p, p)\)-component in \((y, \zeta)\) and \((n - p - 1, n - p - 1)\)-component in \((x, \xi)\), which is equals to

\[
\sum_{s=0}^N \frac{(n-1)!}{(n-p-1-s)!(p-s)!(s!)^2} \beta^{p-s}(y, \zeta) \wedge (\beta(x, \zeta) \wedge \beta(y, \xi))^s \wedge \beta^{n-p-1-s}(x, \xi),
\]

for \( N = \min(p, n-p-1) \). Since

\[
(\beta(x, \zeta) \wedge \beta(y, \xi))^s = \left( \sum_{i=1}^n dx_i \wedge d\xi_i \right) \wedge \left( \sum_{j=1}^n dy_j \wedge d\xi_j \right)^s = \left( \sum_{i,j=1}^n dx_i \wedge d\xi_i \wedge dy_j \wedge d\xi_j \right) = (-1)^s(s!)^2 \sum_{|I|=|J|=s} dx_I \wedge d\xi_J \wedge dy_J \wedge d\xi_J.
\]

Then,

\[
U(x, \xi) = \sum_{s=0}^N \frac{(n-1)!(-1)^s}{(n-p-1-s)!(p-s)!} \left( \sum_{|I|=|J|=s} B_{IJ}(x)\beta^{n-p-1-s}(x, \xi) \wedge dx_I \wedge d\xi_J \right),
\]

(3.1)
where $B_{IJ}(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(y) h(x-y) T(y, \zeta) \wedge \beta^{p-s}(y, \zeta) \wedge dy_I \wedge d\zeta_I$. Consequently, we get

$$U \wedge \beta^{p+1}(x, \xi) = \sum_{s=0}^{N} \frac{(n-1)!(n-s)!(-1)^s}{(n-p-1-s)!(p-s)!} \left( \sum_{|I|=s} B_{II}(x) \beta^{n-s}(x, \xi) \wedge dx_I \wedge d\xi_I \right).$$

Since $\beta^{n-s}(x, \xi) \wedge dx_I \wedge d\xi_I = \frac{(n-s)!}{s!} \beta^n(x, \xi)$ and $\sigma_s \sum_{|I|=s} dy_I \wedge d\xi_I = \frac{1}{s!} \beta^s(y, \zeta)$, then

$$U \wedge \beta^{p+1}(x, \xi) = \sum_{s=0}^{N} \frac{(n-1)!(n-s)!(-1)^s}{(n-p-1-s)!(p-s)!s!n!} \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(y) h(x-y) T(y, \zeta) \wedge \beta^p(y, \zeta) \beta^n(x, \xi).$$

To calculate $c(N) = \sum_{s=0}^{N} \frac{(n-1)!(n-s)!(-1)^s}{(n-p-1-s)!(p-s)!s!n!}$ and since the above argument is still holds when $T$ is only of order zero, let us take $T = \delta_0 \beta^{n-p}$. According to the previous equality, one has $U \wedge \beta^{p+1}(x, \xi) = n! c(N) \eta(0) h(x) \beta^n(x, \xi)$. Moreover, by the integral expression of $U$ and by changing variables $((y, \zeta), (x, \xi)) \rightarrow ((t, \mu), (x, \xi)) = ((x - y, \xi - \zeta), (x, \xi))$, we obtain

$$U(x, \xi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(x - t) h(t) \delta_x(t) \beta^{n-p}(x - t, \xi - \mu) \wedge \beta^{n-1}(t, \mu).$$

We need only the $(1, 1)$-component in $(t, \mu)$ and the $(n-p-1, n-p-1)$-component in $(x, \xi)$ of the form $\beta^{n-p}(x - t, \xi - \mu)$, which is equals to

$$w(x, \xi, t, \mu) = (n-p) \beta(t, \mu) \wedge \beta^{n-p-1}(x, \xi) - (n-p-1)(n-p) \beta^{n-p-2}(x, \xi)$$

\[ \wedge \sum_{i,j=1}^{n} dx_i \wedge d\xi_j \wedge dt_j \wedge d\mu_i. \]

Hence,

$$w(x, \xi, t, \mu) \wedge \beta^{n-1}(t, \mu) = \frac{(n-p)(p+1)}{n} \beta^{n-p-1}(x, \xi) \wedge \beta^n(t, \mu)$$
and $U \wedge \beta^{p+1}(x, \xi) = \frac{(n-p)(p+1)}{n!} n! \eta(0) h(x) \beta^n(x, \xi)$. It follows that $c(N) = \frac{(n-p)(p+1)}{n}$. Then, by going back to the canonical expression of $U$, we deduce that

$$u(x) \beta^n(x, \xi) = \frac{n!}{(p+1)!} U(x, \xi) \wedge \beta^{p+1}(x, \xi)$$

$$= \frac{(n-p)(n-1)!}{p!} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(y) h(x-y) T(y, \xi) \wedge \beta^n(y, \xi) \right) \beta^n(x, \xi).$$

**Proposition 2** Let $T$ be a positive closed current of bidimension $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$. If $x_0 \in \mathbb{R}^n$ and $K$ is a compact of $\mathbb{R}^n$, then:

1. There exists a constant $c = c(\eta, x_0, n, p) > 0$ such that $\|\eta T\|_{\mathbb{R}^n} \leq -cu(x_0)$.
2. There exists a constant $c_K = c(K, n, p) \geq 0$ such that $\|U\|_K \leq c_K \|\eta T\|_{\mathbb{R}^n}$.

**Proof** (1) There is nothing to prove when $u(x_0) = -\infty$. The function $g : x \mapsto -h(x_0 - x)$ is lower semi-continuous, then $g$ reaches its minimum on the compact subset $\text{Supp} \eta$. By Proposition 4.1 in [16], there exists $c_1 > 0$ such that

$$\|\eta T\|_{\mathbb{R}^n} \leq c_1 \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(y) T(y, \xi) \wedge \beta^n(y, \xi).$$

Then, by Proposition 1, it suffices to take $c = \frac{c_1 p!}{(n-p)(n-1)!} \min_{x \in \text{Supp} \eta} g(x)$.

(2) Since $U$ is a negative current of bidimension $(p + 1, p + 1)$ on $\mathbb{R}^n \times \mathbb{R}^n$. Then, by Proposition 4.1 in [16], there exists a constant $c = c(n, p) \geq 0$ such that

$$\|U\|_K \leq -c \int_{K \times \mathbb{R}^n} U(x, \xi) \wedge \beta^{p+1}(x, \xi).$$

By the integral formula of $U$ and Fubini’s theorem, the previous inequality becomes

$$\|U\|_K \leq \frac{c}{n!} \int_{(y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n} \eta(y) T(y, \xi) \wedge \beta^n(y, \xi) \left( \int_{x \in K} \frac{d\lambda}{|x-y|^{n-2}} \right),$$

where $d\lambda = dx_1 \wedge ... \wedge dx_n$. Let $r > 0$ such that $K \subset B(0, r)$. The invariance by the orthogonal group $O(\mathbb{R}^n)$ of the Lebesgue’s measure $d\sigma = \sum_{i=1}^n (-1)^{i-1} dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_n$ on the unit sphere of $\mathbb{R}^n$, gives

$$\int_{S(0, 1)} \frac{d\sigma(x)}{|x-y|^{n-2}} = \min \left(1, \frac{1}{|y|^{n-2}} \right) \leq 1.$$

Which conclude our proof. □

Following [16], the convolution of a given current $T(x, \xi) = \sum_{IJ} T_{IJ}(x) dx_I \wedge d\xi_J$ on $\mathbb{R}^n \times \mathbb{R}^n$ with $\chi_j$ is defined by

$$(T * \chi_j)(x, \xi) = \sum_{IJ} (T_{IJ} * \chi_j)(x) dx_I \wedge d\xi_J,$$
and \((T * \chi_j)_j\) converges weakly to \(T\), as \(j \to +\infty\). The following lemma is fundamental for what is to come, indeed it gives the relationship between the current \(T\) and its local potential \(U\). Moreover, it is the counterpart in the superformalism setting of Lemma 5 in [2].

**Lemma 1** (1) Let \(K(x, \xi) = \frac{h(x)}{n!} \beta^{n-1}(x, \xi)\) and \(\beta_n(x, \xi) = \frac{1}{n!} \beta^n(x, \xi) = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \wedge d\xi \). Then, we have \(dd^\# K(x, \xi) = \delta_0 \beta_n(x, \xi)\).

(2) \(dd^\# K_j(x, \xi) = \chi_j(x) \beta_n(x, \xi)\), where \(K_j(x, \xi) = (K * \chi_j)(x, \xi)\).

(3) \(U * \chi_j = U_j\).

(4) There is a smooth form \(R_j\) of bidegree \((n-p, n-p)\) on \(\mathbb{R}^n \times \mathbb{R}^n\), such that \(dd^\# R_j = (\eta T) * \chi_j + R_j\). Moreover, the sequence \((R_j)\) converges in \(\mathcal{C}^\infty_{n-p,n-p}\) to a form \(R\) of bidegree \((n-p, n-p)\) satisfying the equality \(dd^\# U = T + R\) on \(\Omega \times \mathbb{R}^n\).

**Proof** (1) For all \((x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n\), we have

\[
dd^\# K(x, \xi) = \frac{1}{n!} dd^\# h(x) \wedge \beta^{n-1}(x, \xi) = \frac{1}{n!} \Delta(h(x)) \beta^n(x, \xi) = \frac{1}{n!} \delta_0 \beta^n(x, \xi).
\]

(2) For all \((x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n\), we have

\[
dd^\# K_j(x, \xi) = dd^\# (K * \chi_j)(x, \xi) = (dd^\# K)(x, \xi) \beta_n(x, \xi) = \chi_j(x) \beta_n(x, \xi).
\]

(3) The equation (3.1), implies that

\[
(U * \chi_j)(x, \xi) = \sum_{s=0}^{N} \frac{(n-1)!(-1)^s}{(n-p-1-s)!(p-s)!} \left( \sum_{|I|=|J|=s} (B_{IJ} * \chi_j)(x) \beta^{n-p-1-s}(x, \xi) \wedge dx_I \wedge d\xi_J \right).
\]

Now, since the coefficients of \(T\) are measures and depending only on \(y\), it is clear that if we put \(S_{IJ}(y, \xi) = \eta(y) T(y, \xi) \wedge \beta^{p-s}(y, \xi) \wedge dy_j \wedge d\xi_I\), then we get

\[
(B_{IJ} * \chi_j)(x) = ((S_{IJ} * h) * \chi_j)(x, \xi) = (S_{IJ} * (h * \chi_j))(x, \xi)
\]

\[
= \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(y)(h * \chi_j)(x-y) T(y, \xi) \wedge \beta^{p-s}(y, \xi) \wedge dy_j \wedge d\xi_I,
\]

which completes the proof of the statement.

(4) We denote by \(p_1\) and \(p_2\) the projections of \((\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)\) on \(\mathbb{R}^n \times \mathbb{R}^n\) such that \(p_1((x, \xi), (y, \zeta)) = (x, \xi)\) and \(p_2((x, \xi), (y, \zeta)) = (y, \zeta)\). Let \(\tau((x, \xi), (y, \zeta)) = (x - y, \xi - \zeta)\), when we integrate on the fibers of \(p_2\), we get

\[
U = p_2 * [p_1^*(\eta T) \wedge \tau^*(K)] \quad \text{and} \quad U_j = p_2 * [p_1^*(\eta T) \wedge \tau^*(K_j)]. \quad (3.2)
\]
Since $p_{2*}$ and $p_1^*$ are commute with $d$ and $d^\#$ (see [16]), we obtain

\[ dd^\#U_j = p_{2*}[p_1^*(\eta T) \wedge \tau^*(\chi_j \beta_n)] + R_j \] and

\[ dd^\#U = p_{2*}[p_1^*(\eta T) \wedge \tau^*(\delta_0 \beta_n)] + R, \]

where

\[ R_j = p_{2*}[p_1^*(dd^\# \eta \wedge T) \wedge \tau^*(K_j)] - p_{2*}[p_1^*(d^\# \eta \wedge T) \wedge \tau^*(dK_j)] \]
\[ + p_{2*}[p_1^*(d \eta \wedge T) \wedge \tau^*(d^\# K_j)] \]

and

\[ R = p_{2*}[p_1^*(dd^\# \eta \wedge T) \wedge \tau^*(K)] - p_{2*}[p_1^*(d^\# \eta \wedge T) \wedge \tau^*(dK)] \]
\[ + p_{2*}[p_1^*(d \eta \wedge T) \wedge \tau^*(d^\# K)] \]

are two forms of bidegree $(n - p, n - p)$ on $\mathbb{R}^n \times \mathbb{R}^n$. By writing each term of $R$ in the integral form as in Definition 1 and using the fact that $\eta \equiv 1$ on $\overline{\Omega}$, it is obvious that $R$ is smooth on $\Omega \times \mathbb{R}^n$. If $T(y, \xi) = \sum_{|I| = |J| = n- p} T_{IJ}(y) dy_I \wedge d\xi_J$, $f_j(x, \xi) = p_{2*}[p_1^*(\eta T) \wedge \tau^*(\chi_j \beta_n)]$ and $g(x, \xi) = p_{2*}[p_1^*(\eta T) \wedge \tau^*(\delta_0 \beta_n)]$, then

\[ f_j(x, \xi) = \frac{1}{n!} \sum_{|I| = |J| = n- p} \int (y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \eta(y) \chi_j(x - y) T_{IJ}(y) dy_I \wedge d\xi_J \wedge \beta^n(x - y, \xi - \zeta) \]
\[ g(x, \xi) = \frac{1}{n!} \sum_{|I| = |J| = n- p} \int (y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \eta(y) \delta_0(x - y) T_{IJ}(y) dy_I \wedge d\xi_J \wedge \beta^n(x - y, \xi - \zeta). \]

Since $\beta^n(x - y, \xi - \zeta) = \bigwedge_{j=1}^n (dx_I \wedge d\xi_J - dx_J \wedge d\xi_I - dy_J \wedge d\xi_J + dy_J \wedge d\xi_J)$, then by induction on $q = n - p$, the $(n, n)$-component in $(y, \xi)$ and the $(n - p, n - p)$-component in $(x, \xi)$ of the form $dy_I \wedge d\xi_J \wedge \beta^n(x - y, \xi - \zeta)$ is equals to

\[ \frac{n!}{(n - p)!p!} dy_I \wedge d\xi_J \wedge \beta^{n-p}(x, \xi) \wedge \beta^p(y, \xi) = dx_I \wedge d\xi_J \wedge \beta^n(y, \xi) \]
\[ = n! dx_I \wedge d\xi_J \wedge \beta_n(y, \xi). \]

Thus,

\[ f_j(x, \xi) = \sum_{|I| = |J| = n- p} dx_I \wedge d\xi_J \left( \int (y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \chi_j(x - y) \eta(y) T_{IJ}(y) \beta_n(y, \xi) \right) \]
\[ = \sum_{|I| = |J| = n- p} (\eta T_J \ast \chi_j)(x) dx_I \wedge d\xi_J = (\eta T \ast \chi_j)(x, \xi) \]
and

\[ g(x, \xi) = \sum_{|I|=|J|=n-p} dx_I \wedge d\xi_J \left( \int_{(y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n} \delta_0(x-y) \eta(y) T_{IJ}(y) \beta_n (y, \zeta) \right) \]

\[ = \sum_{|I|=|J|=n-p} (\eta T_{IJ} \ast \delta_0)(x) dx_I \wedge d\xi_J = \eta T(x, \xi). \]

Hence, \(\mathcal{d}^\# U_j = (\eta T) \ast \chi_j + R_j\) and \(dd^\# U = \eta T + R\) on \(\mathbb{R}^n \times \mathbb{R}^n\). In particular, we obtain \(dd^\# U = T + R\) on \(\Omega \times \mathbb{R}^n\).

In the following theorem, we use the local potential as a tool to extend Theorem 6.1 in [12] for the case where \(T\) is weakly positive and \(T \wedge \beta^{n-p}\) is convex.

**Theorem 1** Let \(\Omega\) be an open subset of \(\mathbb{R}^n\) and \(T\) be a weakly positive convex current of bidimension \((p, p)\) on \(\{\Omega \setminus K\} \times \mathbb{R}^n\) with locally finite mass near \(K\), where \(K\) is a compact subset of \(\mathbb{R}^n\) with sigma-finite \((p-2)\)-dimensional Hausdorff measure. Then, there exists a positive measure \(S\) such that

\[ dd^\# T \wedge \beta^{p-1} = dd^\# \tilde{T} \wedge \beta^{p-1} + S, \]

where \(\tilde{T}\) and \(dd^\# \tilde{T} \wedge \beta^{p-1}\) are the extensions by \(0\) of \(T\) and \(dd^\# T \wedge \beta^{p-1}\) across \(K \times \mathbb{R}^n\).

**Proof** We prove firstly that \(dd^\# T \wedge \beta^{p-1}\) is of locally finite mass near \(K \times \mathbb{R}^n\). Consider \(U\) as the local potential associated to \(dd^\# T\). Since the problem is local, we can assume that \(\Omega = \{\psi < 0\}\) is a strictly convex domain on \(\mathbb{R}^n\), where \(\psi\) is a smooth strictly convex exhaustion function on \(\Omega\). As in Lemma 1, we have \(dd^\# U = dd^\# T + R\), where \(R\) is with uniformly bounded coefficients on \(\Omega \times \mathbb{R}^n\), then there exists \(A > 0\) such that \(R \geq -A(dd^\# \psi)^{n-p+1}\). Setting \(W = U - T + A\psi(dd^\# \psi)^{n-p}\), then it is clear that \(W\) is a weakly negative convex current of bidimension \((p, p)\) on \(\Omega \setminus K\) \(\times \mathbb{R}^n\). Since \(W\) has a locally finite mass near \(K \times \mathbb{R}^n\), then by applying Theorem 6.1 in [12] for the current \(-W\), there exists a positive measure \(S_1\) such that

\[ dd^\# W \wedge \beta^{p-1} + S_1 = dd^\# \tilde{W} \wedge \beta^{p-1}. \quad (3.3) \]

On the other hand, we have

\[ dd^\# \tilde{W} \wedge \beta^{p-1} = dd^\# U \wedge \beta^{p-1} - dd^\# \tilde{T} \wedge \beta^{p-1} + A(dd^\# \psi)^{n-p+1} \wedge \beta^{p-1} \]

and

\[ dd^\# \tilde{W} \wedge \beta^{p-1} = dd^\# U \wedge \beta^{p-1} - dd^\# T \wedge \beta^{p-1} + A(dd^\# \psi)^{n-p+1} \wedge \beta^{p-1}. \]
Next, by applying once again Theorem 6.1 in [12] on the current \(-U - A\psi(dd^# \psi)^{n-p}\) which is weakly positive and concave on \([\Omega \setminus K] \times \mathbb{R}^n\), there exists a positive measure \(S_2\) such that

\[
dd^# U \wedge \beta^{p-1} + A(dd^# \psi)^{n-p+1} \wedge \beta^{p-1} + S_2 = \dd^# U \wedge \beta^{p-1} + A(dd^# \psi)^{n-p+1} \wedge \beta^{p-1}.
\]

Consequently, in view of (3.3), the proof is completed by considering \(S = S_1 - S_2\) which is a positive measure due to the proof of the Theorem 6.1 in [12].

3.2 SuperHessian Operator

In this subsection, we will focus on the convergence of the sequence of operators \(U_j \wedge dd^# v_1 \wedge ... \wedge dd^# v_q\), where \(U_j = U \ast \chi_j\) is a smooth regularization of the local potential \(U\) and \((v_1^j, ..., v_q^j)\) are sequences of convex functions which converge locally uniformly respectively towards \(v_1, ..., v_q\). Strongly motivated by techniques goes back to Ben Messaoud and El Mir [2] in the complex analysis, we extend firstly to the superformalism context the following convergence result:

**Theorem A** Let \(1 \leq p \leq n\), \(S\) is a current of bidimension \((p, p)\) on \(\Omega \times \mathbb{R}^n\) and \(v_1, ..., v_q\) are convex functions on \(\Omega\) for \(1 \leq q \leq p\). If we assume that there exists \((S_j)_j\) a sequence of smooth forms of bidegree \((n-p, n-p)\) on \(\Omega \times \mathbb{R}^n\) such that \(S_j\) is negative, \(dd^# S_j\) is positive and \((S_j)_j\) decreases weakly to \(S\), and if we take \(v_1^j, ..., v_q^j\) to be sequences of convex functions which converge locally uniformly respectively to \(v_1, ..., v_q\). Then, we have:

1. \((S_j \wedge dd^# v_1 \wedge ... \wedge dd^# v_q)_j\) converges weakly on \(\Omega\), where \(j \to +\infty\). Denote its limit by \(S \wedge dd^# v_1 \wedge ... \wedge dd^# v_q\).
2. If we also assume that \(v_i^j \geq v_i, \forall i = 1, ..., q,\) then:
   i) \(S \wedge dd^# v_1^j \wedge ... \wedge dd^# v_q^j \to S \wedge dd^# v_1 \wedge ... \wedge dd^# v_q\) weakly on \(\Omega\).
   ii) \(S_j \wedge dd^# v_1^i \wedge ... \wedge dd^# v_q^i \to S \wedge dd^# v_1 \wedge ... \wedge dd^# v_q\) weakly on \(\Omega\).
   iii) \(dd^#(S \wedge dd^# v_1 \wedge ... \wedge dd^# v_q) = dd^# S \wedge dd^# v_1 \wedge ... \wedge dd^# v_q\).

**Example 2** The current \(S = \frac{-\beta^{p+1}}{|x|^p}\) satisfies the hypothesis of Theorem A. Indeed, if we consider the sequence \(S_j = \frac{-\beta^{p+1}}{(|x|^2 + \frac{1}{j})^{p+1}}\), \(0\leq 0\), a simple computation gives

\[
dd^# S_j = p \left( \frac{\beta^{p+2}}{|x|^2 + \frac{1}{j}} \right) - \frac{p + 2 d|x|^2 \wedge dd^# |x|^2 \wedge \beta^{p+1}}{4} \left( |x|^2 + \frac{1}{j} \right)^{\frac{p+1}{2}} \left( |x|^2 + \frac{1}{j} \right)^{\frac{p+1}{2}} \geq 0.
\]
The proof of Theorem A needs some tools such as the Chern-Levine-Nirenberg type inequality and two lemmas in the superformalism setting with proofs almost identical to the complex case.

**Proposition 3** (Chern-Levine-Nirenberg type inequality) Let $T$ be a positive closed current of bidimension $(p, p)$ on $\Omega \times \mathbb{R}^n$ and let $v_1, \ldots, v_q$ are convex functions on $\Omega$ for $1 \leq q \leq p$. Then, for every compact $K$ and $L$ with $K \subset \mathring{L} \Subset \Omega$, there exists a constant $C_{K, L} \geq 0$ such that

$$
\|T \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_q\|_K \leq C_{K, L} \|v_1\|_{L^\infty(L)} \ldots \|v_q\|_{L^\infty(L)} \|T\|_L.
$$

**Lemma 2** Let $\varphi$ be a symmetric (closed) form of bidegree $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$. Then, there exist two positive (closed) forms $\varphi_1$ and $\varphi_2$ of bidegree $(p, p)$ such that $\varphi = \varphi_1 - \varphi_2$.

**Lemma 3** Let $\Omega_0 \Subset \Omega$ and $1 \leq q \leq n$. Let $\varphi$ be a form of bidegree $(n - q, n - q)$ on $\Omega_0 \times \mathbb{R}^n$ such that $\varphi$ is smooth on $\overline{\Omega_0} \times \mathbb{R}^n$ and $dd^c \varphi$ is positive, and let $v_1, \ldots, v_q$ and $w_1, \ldots, w_q$ are convex functions on a neighborhood of $\overline{\Omega_0}$, such that:

i) $\forall i = 1, \ldots, q$, $w_i \geq v_i$ on $\Omega_0$,  
ii) $\forall i = 1, \ldots, q$, $w_i = v_i$ on a neighborhood of $\partial \Omega_0$.

Then,

$$
\int_{\Omega_0 \times \mathbb{R}^n} \varphi \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_q \geq \int_{\Omega_0 \times \mathbb{R}^n} \varphi \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_q.
$$

**Proof of Theorem A** Since the problem is locally, then we can work on a strictly convex subset $\Omega_0 \Subset \Omega$ such that $\Omega_0 = \{ \psi < 0 \}$. Thanks to Theorem B in [4], we chose $\gamma$ to be a smooth strictly convex exhaustion function on $\overline{\Omega_0}$ such that $dd^c (\gamma - |x|^2) \geq 0$. For $i = 1, \ldots, q$, the function $v_i$ is convex on $\overline{\Omega_0}$, then it is locally bounded and we can assume that $-M \leq v_i \leq -1$, where $M > 0$.

1) Let $\varphi$ be a symmetric form of bidegree $(p - q, p - q)$ on $\Omega_0 \times \mathbb{R}^n$, we need to prove that the sequence $((S_j \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_q, \varphi))_j$ converges. Thanks to Lemma 2, it suffices to suppose that $\varphi$ is positive. So that $((S_j \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_q, \varphi))_j$ is decreasing and it remains to show that it is bounded from below. Let $K \Subset \Omega_0$ such that $\text{Supp} \varphi \subset K$, and let $c > 0$ such that $\varphi \leq c \|dd^c \psi\|^{p-q}$. Since $S_j \leq 0$, then

$$
\int_{\Omega_0 \times \mathbb{R}^n} S_j \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge \varphi \geq c \int_{K \times \mathbb{R}^n} S_j \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge (dd^c \psi)^{p-q}.
$$

Let $\delta > 0$ small enough such that $K \subset \Omega_\delta = \{ \psi < -\delta \}$, and let $w_i = \max \left( \frac{M}{\delta} \psi, v_i \right)$, $\forall i = 1, \ldots, q$. So, $w_i$ is convex in a neighborhood of $\overline{\Omega_0}$, $w_i = v_i$ on
\[\Omega_\delta \text{ and } w_i = \frac{M}{\delta} \psi \text{ on } \Omega_0 \setminus \Omega_\frac{\delta}{M}. \text{ Thus,} \]
\[
\int_{K \times \mathbb{R}^n} S_j \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q \wedge (dd^#\psi)^{p-q} \\
\geq \int_{\Omega_\frac{\delta}{M} \times \mathbb{R}^n} S_j \wedge dd^#w_1 \wedge \ldots \wedge dd^#w_q \wedge (dd^#\psi)^{p-q} \\
\geq \int_{\Omega_\frac{\delta}{M} \times \mathbb{R}^n} S_j \wedge dd^#w_1 \wedge \ldots \wedge dd^#w_q \wedge (dd^#\psi)^{p-q}.
\]

Let \(\chi\) be a positive and compactly supported function on \(\Omega_\frac{\delta}{M}\) such that \(\chi = -\psi\) on \(\Omega_\frac{\delta}{M}\), then
\[
\int_{\Omega_\frac{\delta}{M} \times \mathbb{R}^n} S_j \wedge dd^#w_1 \wedge \ldots \wedge dd^#w_q \wedge (dd^#\psi)^{p-q} \\
= -\int_{\Omega_\frac{\delta}{M} \times \mathbb{R}^n} S_j \wedge dd^#w_1 \wedge \ldots \wedge dd^#w_q \wedge (dd^#\psi)^{p-q-1} \wedge dd^#\chi \\
= -\int_{\Omega_\frac{\delta}{M} \times \mathbb{R}^n} \chi dd^#S_j \wedge dd^#w_1 \wedge \ldots \wedge dd^#w_q \wedge (dd^#\psi)^{p-q-1} \\
+ \frac{M^q}{\delta^q} \int_{(\Omega_\frac{\delta}{M} \setminus \Omega_\frac{\delta}{M}) \times \mathbb{R}^n} S_j \wedge (dd^#\psi)^{p-1} \wedge dd^#\chi.
\]

By using Proposition 3, we have
\[
\left| \int_{\Omega_\frac{\delta}{M} \times \mathbb{R}^n} S_j \wedge dd^#w_1 \wedge \ldots \wedge dd^#w_q \wedge (dd^#\psi)^{p-q} \right| \\
\leq M^q C_{K,\Omega_0} \left( \|dd^#S_j\|_{\Omega_\frac{\delta}{M}} + \|S_j\|_{\Omega_\frac{\delta}{M}} \right).
\]

Now, let \(\chi'\) be a smooth and compactly supported function on \(\Omega_0\) such that \(0 \leq \chi \leq 1\) and \(\chi \equiv 1\) on \(\Omega_\frac{\delta}{M}\). Since \(dd^#S_j\) is positive, Proposition 4.1 in [16] and an integration by parts, yield
\[
\|dd^#S_j\|_{\Omega_\frac{\delta}{M}} \leq C \int_{\Omega_\frac{\delta}{M} \times \mathbb{R}^n} dd^#S_j \wedge \beta^{n-p-1} \\
\leq C \int_{\Omega_0 \times \mathbb{R}^n} \chi' dd^#S_j \wedge \beta^{n-p-1} \\
\leq C \int_{\Omega_0 \times \mathbb{R}^n} S_j \wedge dd^#\chi' \wedge \beta^{n-p-1} \leq C_{\Omega_0} \|S_j\|_{\Omega_0}.
\]
Thus, since \( 0 \geq S_j \geq S \), we have
\[
\left| \int_{K \times \mathbb{R}^n} S_j \wedge d\# v_1 \wedge \ldots \wedge d\# v_q \wedge (d\# \psi)^{p-q} \right| \leq M^q C_{K, \Omega_0} \| S \|_{\Omega_0}. \tag{3.4}
\]
and (1) will follows.

(2) i) Firstly, we assume that \( q < p \). Let \( \Theta \) be a limit of the sequence \((S \wedge d\# v_1^j \wedge \ldots \wedge d\# v_q^j)_j\), which is locally bounded in mass thanks to (3.4). To prove that \( \Theta = S \wedge d\# v_1 \wedge \ldots \wedge d\# v_q \), we take a sub-sequence of \((S \wedge d\# v_1^j \wedge \ldots \wedge d\# v_q^j)_j\) which converges weakly to \( \Theta \). Let \( k > l \geq 1 \), then
\[
S_k \wedge d\# v_1^j \wedge \ldots \wedge d\# v_q^j \leq S_l \wedge d\# v_1^j \wedge \ldots \wedge d\# v_q^j.
\]
When we let \( k \to +\infty \), \( j \to +\infty \) and \( l \to +\infty \) in this order, the first result (1) yields
\[
\Theta \leq S \wedge d\# v_1 \wedge \ldots \wedge d\# v_q. \tag{3.5}
\]
Since every \((v_i^j)_j\) converges locally uniformly to \( v_i \), we can assume, without loss of generality, that \( \forall i = 1, \ldots, q \) and \( \forall j \in \mathbb{N}^* \), \(-M < v_i^j < -1\) on \( \overline{\Omega_0} \). Let \( K \in \Omega_0 \) and let \( \delta \) small enough such that \( K \subseteq \Omega_\delta \). Let us set \( w_i^j = \max \left( \frac{M}{\delta} \psi, v_i^j \right) \) and \( w_i = \max \left( \frac{M}{\delta} \psi, v_i \right), \forall i = 1, \ldots, q \). So, \( w_i^j \) is convex in a neighborhood of \( \overline{\Omega_0} \), \( w_i^j = v_i^j \) on \( \Omega_\delta \) and \( w_i^j = \frac{M}{\delta} \psi \) on \( \Omega_0 \setminus \Omega_\delta \). Let \( \Theta' \) be another limit of \((S \wedge d\# v_1^j \wedge \ldots \wedge d\# v_q^j)_j\) on a neighborhood of \( \overline{\Omega_0} \times \mathbb{R}^n \), as in (3.5) we have
\[
\Theta' \leq S \wedge d\# w_1 \wedge \ldots \wedge d\# w_q. \tag{3.6}
\]
Lemma 3 implies
\[
\int_{\Omega_0 \times \mathbb{R}^n} S_l \wedge d\# w_1^j \wedge \ldots \wedge d\# w_q^j \wedge (d\# \psi)^{p-q} \\
\geq \int_{\Omega_0 \times \mathbb{R}^n} S_l \wedge d\# w_1 \wedge \ldots \wedge d\# w_q \wedge (d\# \psi)^{p-q},
\]
and thanks to (1), we get
\[
\int_{\Omega_0 \times \mathbb{R}^n} S \wedge d\# w_1^j \wedge \ldots \wedge d\# w_q^j \wedge (d\# \psi)^{p-q} \\
\geq \int_{\Omega_0 \times \mathbb{R}^n} S \wedge d\# w_1 \wedge \ldots \wedge d\# w_q \wedge (d\# \psi)^{p-q}. \tag{3.7}
\]
Now, we replace the sequence of positive measures \((S \wedge d\# w_1^j \wedge \ldots \wedge d\# w_q^j \wedge (d\# \psi)^{p-q})_j\) with a sub-sequence \( \eta_j \) which converges weakly to \( \eta = \Theta' \wedge (d\# \psi)^{p-q} \)
on $\Omega_0 \times \mathbb{R}^n$. Since $\lim \sup_{j \to +\infty} \eta_j(\Omega_0) \leq \eta(\Omega_0)$, then (3.7) yields
\[
\int_{\Omega_0 \times \mathbb{R}^n} \Theta' \wedge (dd^\# \psi)^{p-q} \geq \int_{\Omega_0 \times \mathbb{R}^n} S \wedge dd^\# w_1 \wedge ... \wedge dd^\# w_q \wedge (dd^\# \psi)^{p-q}.
\]
Thus, by (3.6), we have
\[
\int_{\Omega_0 \times \mathbb{R}^n} \Theta' \wedge (dd^\# \psi)^{p-q} = \int_{\Omega_0 \times \mathbb{R}^n} S \wedge dd^\# w_1 \wedge ... \wedge dd^\# w_q \wedge (dd^\# \psi)^{p-q}.
\]
(3.8)
Let $\varphi$ be a symmetric form of bidegree $(p-q-1, p-q-1)$ on $\Omega_0 \times \mathbb{R}^n$, then according to Lemma 2, there exist two positive closed forms $\varphi_1$ and $\varphi_2$ of bidegree $(p-q, p-q)$ on $\Omega_0 \times \mathbb{R}^n$ such that $dd^\# \varphi = \varphi_1 - \varphi_2$. Thus, (3.8) remains valid for $\varphi_1$ and $\varphi_2$ instead of $(dd^\# \psi)^{p-q}$, and then it is valid for $dd^\# \varphi$ which implies that $dd^\# E = 0$. Since $E = S \wedge dd^\# w_1 \wedge ... \wedge dd^\# w_q - \Theta'$ is positive and is compactly supported, then Proposition 4.2 in [12] yields $E = 0$, and therefore $(S \wedge dd^\# w_1^j \wedge ... \wedge dd^\# w_q^j)_j$ converges weakly to $S \wedge dd^\# w_1 \wedge ... \wedge dd^\# w_q$. Since $w_1^j = v_1^j$ and $w_i = v_i$ on $K$, then $(S \wedge dd^\# v_1^j \wedge ... \wedge dd^\# v_q^j)_j$ converges weakly to $S \wedge dd^\# v_1 \wedge ... \wedge dd^\# v_q$ on $K \times \mathbb{R}^n$, which concludes (2)i). Secondly, for $q = p$, let $\pi$ be the projection of $\Omega \times \mathbb{R}$ on $\Omega$. Then, the currents $\pi^\ast(S)$ and $\pi^\ast(S_j)$ are of bidimension $(p+1, p+1)$ and verifies the hypothesis of Theorem A. Since $p+1 > q$, then we can use the first step by setting $\tilde{v}_i = v_i \circ \pi$ and $\tilde{v}_i^j = v_i^j \circ \pi$, for all $i = 1, ..., q$, and our result will be concluded by Fubini’s theorem.

ii) Let $v_i^{j, k} = v_i^j \ast \chi_k$ be the standard regularization of $v_i^j$. Since $dd^\# v_1^{j, k} \wedge ... \wedge dd^\# v_q^{j, k}$ is positive and the sequence $(S_j)_j$ is decreasing, we have
\[
S_j \wedge dd^\# v_1^{j, k} \wedge ... \wedge dd^\# v_q^{j, k} \leq S_i \wedge dd^\# v_1^{j, k} \wedge ... \wedge dd^\# v_q^{j, k}, \, \forall j > l.
\]
Furthermore, if $k \to +\infty$, then
\[
S_j \wedge dd^\# v_1^j \wedge ... \wedge dd^\# v_q^j \leq S_i \wedge dd^\# v_1^j \wedge ... \wedge dd^\# v_q^j, \, \forall j > l.
\]
Since the sequence $(S_j \wedge dd^\# v_1^j \wedge ... \wedge dd^\# v_q^j)_j$ is locally bounded in mass, then we can extract a weakly convergent sub-sequence and denote by $\Theta$ his limit. Then,
\[
\Theta \leq S_l \wedge dd^\# v_1 \wedge ... \wedge dd^\# v_q.
\]
By (1), when $l \to +\infty$, we get
\[
\Theta \leq S \wedge dd^\# v_1 \wedge ... \wedge dd^\# v_q.
\]
Next, since $S_j \geq S$, we have
\[
S_j \wedge dd^\# v_1^{j, k} \wedge ... \wedge dd^\# v_q^{j, k} \geq S \wedge dd^\# v_1^{j, k} \wedge ... \wedge dd^\# v_q^{j, k}.
\]
By (2)i), when $k \to +\infty$ and $j \to +\infty$ respectively, we get

$$\Theta \geq S \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q.$$ 

iii) Here, all we need to do is to replace $v_i$ with the standard regularization $v_i \ast \chi_k$, and just use (2)i) and the continuity of the operator $dd^#$.

As an immediate consequence of Theorem A, we get our main theorem of this section which is the counterpart of a result due to [2] in the complex setting:

**Theorem B** Let $1 \leq p < n$ and let $v_1, \ldots, v_q$ are convex functions on $\Omega$ for $1 \leq q \leq p + 1$. Let $U$ be the local potential associated to $T$ a positive closed current of bidimension $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$ and $U_j = U \ast \chi_j$. If we take $v_1^j, \ldots, v_q^j$ to be sequences of convex functions which converge locally uniformly respectively to $v_1, \ldots, v_q$. Then, we have:

1. $(U_j \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q)_j$ converges weakly on $\Omega$, where $j \to +\infty$. Denote its limit by $U \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q$.
2. If we also assume that $v_i^j \geq v_i, \forall i = 1, \ldots, q$, then:
   i) $U_j \wedge dd^#v_1^j \wedge \ldots \wedge dd^#v_q^j \to U \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q$ weakly on $\Omega$.
   ii) $U_j \wedge dd^#v_1^j \wedge \ldots \wedge dd^#v_q^j \to U \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q$ weakly on $\Omega$.
   iii) $dd^#(U \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q) = dd^#U \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q$.

**Proof** Since the problem is locally, then we can work on a strictly convex subset $\Omega_0 \subseteq \Omega$ such that $\Omega_0 = \{\psi < 0\}$, where $\psi$ is a smooth strictly convex exhaustion function on $\overline{\Omega}_0$. As in Lemma 1, $R_j$ is with uniformly bounded coefficients on $\Omega_0 \times \mathbb{R}^n$, then there exists $A > 0$ such that $dd^#(U_j + A\psi(dd^#\psi)^{n-p-1})$ is positive. Finally, the proof is completed by using [16] and applying Theorem A with $S_j = U_j + A\psi(dd^#\psi)^{n-p-1}$. 

As an application and in view of Lemma 1, we obtain:

**Corollary 1** Let $T$ be a positive closed current of bidimension $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$, $T_j = T \ast \chi_j$ and $v_0, \ldots, v_q$ are convex functions on $\Omega$ for $1 \leq q \leq p$. If we take $v_0^j, \ldots, v_q^j$ to be sequences of convex functions which converge locally uniformly respectively to $v_0, \ldots, v_q$ such that $v_i^j \geq v_i, \forall i = 1, \ldots, q$. Then, we have:

1. $T_j \wedge dd^#v_1^j \wedge \ldots \wedge dd^#v_q^j \to T \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q$ weakly on $\Omega$.
2. $v_0^jT_j \wedge dd^#v_1^j \wedge \ldots \wedge dd^#v_q^j \to v_0T \wedge dd^#v_1 \wedge \ldots \wedge dd^#v_q$ weakly on $\Omega$.

Notice that Corollary 1 is the corresponding result of the one given by [2] in complex analysis for plurisubharmonic functions and closed positive currents.

**Proof** Since the problem is locally, then we can work on a strictly convex subset $\Omega_0 \subseteq \Omega$ such that $\Omega_0 = \{\psi < 0\}$, where $\psi$ is a smooth strictly convex exhaustion function on $\overline{\Omega}_0$. 

(1) Let $U$ be the local potential associated to $T$ on $\Omega_0 \times \mathbb{R}^n$ and $U_j = U * \chi_j$. By using Theorem B, Lemma 1 and the continuity of the operator $dd^\#$, we obtain

$$T_j \wedge dd^\# v_1^j \wedge \ldots \wedge dd^\# v_q^j + R_j \wedge dd^\# v_1^j \wedge \ldots \wedge dd^\# v_q^j \longrightarrow dd^\# U \wedge dd^\# v_1 \wedge \ldots \wedge dd^\# v_q.$$ 

As $R$ is with uniformly bounded coefficients on $\Omega_0 \times \mathbb{R}^n$, then by [16], we have

$$R \wedge dd^\# v_1^j \wedge \ldots \wedge dd^\# v_q^j \longrightarrow R \wedge dd^\# v_1 \wedge \ldots \wedge dd^\# v_q.$$ 

Then,

$$T_j \wedge dd^\# v_1^j \wedge \ldots \wedge dd^\# v_q^j + (R_j - R) \wedge dd^\# v_1^j \wedge \ldots \wedge dd^\# v_q^j \longrightarrow T \wedge dd^\# v_1 \wedge \ldots \wedge dd^\# v_q.$$ 

Since $(R_j - R)_j$ converges uniformly to 0 on $\Omega_0 \times \mathbb{R}^n$, we get

$$T_j \wedge dd^\# v_1^j \wedge \ldots \wedge dd^\# v_q^j \longrightarrow T \wedge dd^\# v_1 \wedge \ldots \wedge dd^\# v_q.$$ 

(2) For $q < p$, let $\Theta$ be a weak limit of $(v_0^j T_j \wedge dd^\# v_1^j \wedge \ldots \wedge dd^\# v_q^j)_j$. Thus, by regularizing $v_0$ and $v_q^j$, a simple computation gives $\Theta \leq v_0 T \wedge dd^\# v_1 \wedge \ldots \wedge dd^\# v_q$. Moreover, in a similar way as in the proof of Theorem A (2)i), the current $E = v_0 T \wedge dd^\# v_1 \wedge \ldots \wedge dd^\# v_q - \Theta$ of bidimension $(p-q, p-q)$ on $\Omega_0 \times \mathbb{R}^n$ is compactly supported and positive. Hence, according to previous argument, we have $dd^\# E = 0$, and thus $E = 0$. In case $q = p$, we place ourselves in $\Omega \times \mathbb{R}$ and we proceed as the previous case, then we conclude our proof thanks to Fubini’s theorem. \hfill \Box

**Remark 1** It should be noted that Corollary 1 can be obtained directly without passing by Theorem A and for a general sequence of positive closed currents $(T_j)_j$ converging weakly to $T$. In fact, it suffices to adapt Proposition 3.2 in [12] to our situation. In particular, when $T_j = T$ we recover a result goes back to Lagerberg [16].

### 3.3 Lelong Number of a Local Potential

Let us start by defining the Euclidean sphere and ball of centre $a$ and radius $r$ respectively by

$$S(a, r) = \{ x \in \Omega; \ |x - a| = r \} \quad \text{and} \quad B(a, r) = \{ x \in \Omega; \ |x - a| < r \}.$$ 

Next, for every weakly positive current $T$ of bidimension $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$, we define the Lelong number of $T$ at $a$ (when it exists) by

$$\nu_T(a) = \lim_{r \to 0} \nu_T(a, r), \quad \text{where} \quad \nu_T(a, r) = \frac{1}{r^p} \int_{B(a, r) \times \mathbb{R}^n} T \wedge \beta^n.$$ 

If $T$ is assumed to be closed, it was proved by [16] that $\nu_T(a)$ exists for any $a$. In analogue with the work of Ghiloufi, Zaway and Hawari [14], the next result describe
a relation between the Lelong number of a positive closed current and the Lelong number of his local potential.

**Theorem 2** Let $T$ be a weakly positive closed current of bidimension $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$ with measure coefficients and $1 \leq p \leq n - 1$. Let $\Omega \subset \mathbb{R}^n$ and $\eta$ be a smooth and compactly supported function on $\mathbb{R}^n$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\overline{\Omega}$. If we assume that the Lelong number of $T$ vanishes at every point of $\Omega$, then the Lelong number of the local potential $U = U(\eta, T)$ at every point of $\Omega$ exists and is equal to zero.

If $p = n - 1$, Theorem 2 is obvious. Indeed, $T = \frac{1}{p} \partial^p u$ for a convex function and by Example 1, we can take $U = u$ (modulo a smooth function), so we have the equality between the Lelong numbers of $T$ and $U$.

**Proof** Let $x_0 \in \Omega$ and $0 < r < d(x_0, \partial \Omega)$. The calculus in the proof of Proposition 1, yields

$$-\nu_U(x_0, r) = \frac{c_n}{r^{p+1}} \int_{(x, \xi) \in B(x_0, r) \times \mathbb{R}^n} \eta(y) T(y, \xi) \wedge \beta^{p-1}(x, \xi) \wedge \beta^p (y, \xi) \int_{x \in B(x_0, r)} \frac{1}{|x - y|^{n-2}} d\lambda,$$

where $d\lambda = dx_1 \wedge ... \wedge dx_n$. Let $d\sigma = \sum_{i=1}^n (-1)^{i-1} dx_1 \wedge ... \wedge dx_{i-1} \wedge dx_{i+1} \wedge ... \wedge dx_n$ and $S = S(0, 1)$, then we have

$$\int_{x \in B(x_0, r)} \frac{1}{|x - y|^{n-2}} d\lambda = \int_{x \in B(x_0, r)} \frac{1}{|x - x_0 - (y - x_0)|^{n-2}} d\lambda,$$

$$= \int_0^r \int_{S} \frac{1}{t^{n-1}} |t u - (y - x_0)|^{n-2} dt d\sigma(u),$$

$$= \int_0^r t \left( \int_{S} \frac{1}{|u - \frac{y - x_0}{t}|^{n-2}} dt \right),$$

$$= \int_0^r t \min \left( 1, \left( \frac{t}{|y - x_0|} \right)^{n-2} \right) dt.$$

Assume first that $T$ is smooth, then for $r < r_0 < d(x_0, \partial \Omega)$ and

$$\sigma_T(x_0, t) = \int_{(y, \xi) \in B(x_0, r) \times \mathbb{R}^n} T(y, \xi) \wedge \beta^p (y, \xi), \forall t > 0,$$
we get

$$\frac{1}{n!cn} v_U(x_0, r) = \frac{1}{r^{p+1}} \int_{\{(y, \zeta) \in [y - x_0, |r|] \times \mathbb{R}^n \}} \left( \frac{2}{2} - \frac{n-2}{2} |y - x_0|^2 \right) T(y, \zeta) \wedge \beta^p(y, \zeta)$$

$$+ \frac{1}{r^{p+1}} \int_{\{(y, \zeta) \in [y - x_0, |r|] \times \mathbb{R}^n \}} \eta(y) \left( \frac{r^n}{n!|y - x_0|^{n-2}} \right) T(y, \zeta) \wedge \beta^p(y, \zeta)$$

$$\leq \frac{r}{2} v_T(x_0, r) + \frac{p^{n-p-1}}{n} \int_{\{(y, \zeta) \in [y - x_0, |r|] \times \mathbb{R}^n \}} \eta(y) \left( \frac{r^n}{n!|y - x_0|^{n-2}} \right) T(y, \zeta) \wedge \beta^p(y, \zeta)$$

$$+ \frac{p^{n-p-1}}{n} \int_{\{(y, \zeta) \in [y - x_0, |r|] \times \mathbb{R}^n \}} \frac{1}{|y - x_0|^{n-2}} T(y, \zeta) \wedge \beta^p(y, \zeta)$$

$$= \frac{r}{2} v_T(x_0, r) + \frac{p^{n-p-1}}{n} \int_{\{(y, \zeta) \in [y - x_0, |r|] \times \mathbb{R}^n \}} \eta(y) \left( \frac{r^n}{n!|y - x_0|^{n-2}} \right) T(y, \zeta) \wedge \beta^p(y, \zeta)$$

$$+ \frac{p^{n-p-1}}{n} \int_{r}^{r_0} \frac{1}{t^{n-2}} dT(x_0, t)$$

$$= \frac{r}{2} v_T(x_0, r) + \frac{p^{n-p-1}}{n} \int_{\{(y, \zeta) \in [y - x_0, |r|] \times \mathbb{R}^n \}} \eta(y) \left( \frac{r^n}{n!|y - x_0|^{n-2}} \right) T(y, \zeta) \wedge \beta^p(y, \zeta)$$

$$+ \frac{p^{n-p-1}}{n} \frac{v_T(x_0, r)}{r_0^{p-n-2}} + \frac{v_T(x_0, r)}{r_0^{p-n-2}} \int_{r}^{r_0} \frac{v_T(x_0, t)}{t^{n-2}} dt$$

$$\leq \frac{r}{2} v_T(x_0, r) + \frac{p^{n-p-1}}{n} \int_{\{(y, \zeta) \in [y - x_0, |r|] \times \mathbb{R}^n \}} \eta(y) \left( \frac{r^n}{n!|y - x_0|^{n-2}} \right) T(y, \zeta) \wedge \beta^p(y, \zeta)$$

$$+ \frac{p^{n-p-1}}{n} \frac{v_T(x_0, r)}{r_0^{p-n-2}} v_T(x_0, r_0) + \frac{v_T(x_0, r)}{r_0^{p-n-2}} \int_{r}^{r_0} v_T(x_0, t) dt.$$

Since $v_T(x_0) = 0$, the last inequality leads to $v_U(x_0) = 0$. In the case where $T$ is not smooth, it suffices to use the above discussion for a smooth regularization of $T$ and passing to the limit to obtain our result. \hfill \Box

4 Continuity of $m$-superHessian Operator and $m$-generalized Lelong Number

4.1 $m$-SuperHessian Operator

Let $\Omega$ be a bounded and strictly convex domain of $\mathbb{R}^n$. According to Theorem B in [4], there exists a smooth convex exhaustion function $\psi$ on $\Omega$ such that $dd^c(\psi - |x|^2) \geq 0$. Motivated by the work of Demailly [9] on the complex Monge-Ampère
operator for unbounded plurisubharmonic functions, we are going to define the m-superHessian operator $T \wedge \beta^{n-m} \wedge dd^#u_1 \wedge ... \wedge dd^#u_k$, where $T$ is a closed $m$-positive current of bidimension $(p, p)$ on $\Omega \times \mathbb{R}^n$ such that $T \wedge \beta^{n-m}$ is positive and the functions $u_1, ..., u_k$ are m-convex and bounded near $\partial \Omega \cap \text{Supp} T$. For this aim, we need the following version of the Chern-Levine-Nirenberg type inequality in the m-superformalism setting.

**Proposition 4** (Chern-Levine-Nirenberg type inequality) Assume that $T$ is a closed $m$-positive current of bidimension $(p, p)$ such that $n - p \leq m \leq n$ and $T \wedge \beta^{n-m}$ is positive, and let $u_1, ..., u_k, k \leq m + p - n$ be locally bounded $m$-convex functions and $v$ is an $m$-convex function locally integrable with respect to $T \wedge \beta^p$. Then, for every compact $K$ and open subset $L$ with $K \subset L \subset \Omega$, there exists a constant $C_{K,L} \geq 0$ such that:

1. $\| T \wedge \beta^{n-m} \wedge dd^#u_1 \wedge ... \wedge dd^#u_k \|_K \leq C_{K,L} \| u_1 \|_{L^\infty(L)} ... \| u_k \|_{L^\infty(L)} \| T \wedge \beta^{n-m} \|_L.$
2. $\| vT \wedge \beta^{n-m} \wedge dd^#u_1 \wedge ... \wedge dd^#u_k \|_K \leq C_{K,L} \| u_1 \|_{L^\infty(L)} ... \| u_k \|_{L^\infty(L)} \int_{L \times \mathbb{R}^n} |v| |T \wedge \beta^p|.$

Note that for the border case $m = n$, (1) is reduced to Proposition 3. Moreover, in the case of the trivial current $T = 1$, the above inequalities are weaker than the one obtained by [23] Lemma 2.1 and by [22] Theorem 3.1 when $u_1 = ... = u_k = u$.

**Proof** (1) Using induction, it suffices to prove that

$$\| T \wedge \beta^{n-m} \wedge dd^#u_1 \wedge ... \wedge dd^#u_k \|_K \leq C_{K,L} \| u_1 \|_{L^\infty(L)} \| T \wedge \beta^{n-m} \wedge dd^#u_2 \wedge ... \wedge dd^#u_k \|_L.$$ 

Observe that $T \wedge \beta^{n-m} \wedge dd^#u_1 \wedge ... \wedge dd^#u_k$ is positive because $T \wedge \beta^{n-m}$ is positive. Thus, we derive the proof of (1) by following the lines of proof of the inequality (3.3) in [9] or Proposition 1.4 in [18] and by using Proposition 4.1 in [16] combined with an integration by parts.

(2) Using once again Proposition 4.1 in [16], inequality (1) and the same technics as in the proof of Proposition 3.11 in [9] or Proposition 1.4 in [18], the inequality (2) follows. \qed

**Theorem 3** Assume that $u$ is $m$-convex and bounded near $\partial \Omega \cap \text{Supp} T$, where $T$ is a closed $m$-positive current of bidimension $(p, p)$ on $\Omega \times \mathbb{R}^n$ such that $m + p \geq n + 1$ and $T \wedge \beta^{n-m}$ is positive. Then, the current $uT \wedge \beta^{n-m}$ has locally finite mass on $\Omega \times \mathbb{R}^n$.

Theorem 3 fails when $m + p = n$ as shown in the example: $\Omega$ is the unit ball of $\mathbb{R}^n$, $u(x) = -\frac{1}{(\frac{n}{m} - 2)|x|^2}$ if $m < \frac{n}{2}$ and $T = (dd^#u)^m$.

**Proof** We argue as in [9]. Since $\Omega$ is strictly convex, shrinking it a bit, we may assume that $\Omega = \{ \psi < 0 \}$, where $\psi$ is a smooth strictly convex exhaustion function on $\overline{\Omega}$ such that $dd^#\psi \geq \beta$ and $\psi = 0, d\psi \neq 0$ on $\partial \Omega$. We keep the same notations as in the proof of Proposition 4.1 in [9]: $\varepsilon, \Omega_\delta, L(u), A, M, w$, the max procedure $u_s$ and
we replace the Monge–Ampère operator $T \wedge (dd^c \psi)^p$ from the complex theory by $T \wedge \beta^{n-m} \wedge (dd^\#)^{p+m-n}$ in the superformalism setting, we get
\[
\int_{\Omega \times \mathbb{R}^n} u T \wedge \beta^{n-m} \wedge (dd^\#)^{p+m-n} \geq -(M + \|\psi\|_{L^\infty(\Omega)}/M/\delta) \int_{\Omega \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^\#)^{p+m-n}.
\]
Since $T \wedge \beta^{n-m}$ is positive, Proposition 3.2 in [16] implies that the last integral is finite. Therefore, $uT \wedge \beta^{n-m}$ has locally finite mass.

Lemma 4 Let $(u_k)$ be a decreasing sequence of $m$-convex functions converging pointwise to $u$ on $\Omega$ and $(v_k)$ is a sequence of positive measure converging weakly to $v$ on $\Omega$. Then every weak limit $\mu$ of $(u_kv_k)$ satisfies $\mu \leq uv$.

The proof of Lemma 4 is matching the one given by Demailly in [9]. Assume that $u_1, \ldots, u_k$ are $m$-convex on $\Omega$ such that each $u_j$ is bounded near $\partial \Omega \cap \text{Supp } T$, where $T$ is a closed $m$-positive current of bidimension $(p, p)$ on $\Omega \times \mathbb{R}^n$ such that $m + p \geq n + 1$ and $T \wedge \beta^{n-m}$ is positive. As an immediate consequence of Theorem 3 and similarly to the complex Hessian theory made by Dhouib and Elkhadhra [10], we define by induction the following positive current:
\[
T \wedge \beta^{n-m} \wedge dd^\# u_1 \wedge \ldots \wedge dd^\# u_k = dd^\#(u_1 T \wedge \beta^{n-m} \wedge dd^\# u_2 \wedge \ldots \wedge dd^\# u_k).
\]

In fact, it suffices to explain the case $k = 1$. Setting $T \wedge \beta^{n-m} \wedge dd^\# u_1 = dd^\#(u_1 T \wedge \beta^{n-m})$, we get a positive current. To see the positivity we consider a sequence $(u_1^j)_j$ of smooth and $m$-convex functions that decreases to $u_1$ and observe by the Lebesgue’s bounded convergence theorem that the positive sequence of currents $T \wedge \beta^{n-m} \wedge dd^\# u_1^j$ converges weakly to $T \wedge \beta^{n-m} \wedge dd^\# u_1$. Next, according to Lagerberg [16], we point out that there is a significant difference between the complex theory and the context of supercurrent. Indeed weakly positive current do not satisfies a trace measure inequality as obtained by Demailly [9] in the complex setting. Hence, the continuity of the $m$-superHessian operator relatively to a given $m$-positive closed current $T$ and under decreasing sequence of $m$-convex functions requires the positivity of the current $T \wedge \beta^{n-m}$. More precisely, we obtain the following main superformalism version:

Theorem C Let $\Omega$ be a bounded and strictly convex domain of $\mathbb{R}^n$. Assume that $u_0, \ldots, u_k$ are $m$-convex on $\Omega$ such that each $u_s$ is bounded near $\partial \Omega \cap \text{Supp } T$, where $T$ is a closed $m$-positive current of bidimension $(p, p)$ on $\Omega \times \mathbb{R}^n$ such that $1 \leq k \leq m + p - n$ and $T \wedge \beta^{n-m}$ is positive. Then, if $u_0^j, \ldots, u_k^j$ are decreasing sequences of $m$-convex functions which converging pointwise and respectively to $u_0, \ldots, u_k$, in the sense of currents, we have:

1. $u_0^j T \wedge \beta^{n-m} \wedge dd^\# u_1^j \wedge \ldots \wedge dd^\# u_k^j \longrightarrow u_0 T \wedge \beta^{n-m} \wedge dd^\# u_1 \wedge \ldots \wedge dd^\# u_k$, for $k \neq m + p - n$.
2. $T \wedge \beta^{n-m} \wedge dd^\# u_1^j \wedge \ldots \wedge dd^\# u_k^j \longrightarrow T \wedge \beta^{n-m} \wedge dd^\# u_1 \wedge \ldots \wedge dd^\# u_k$. 
Notice that Theorem C is the corresponding result in the superformalism setting of Theorem 2 in [10] for the complex hessian theory. On the other hand, it should be noted that when \( T = 1 \) and \( k = m \) the second statement of Theorem C was proved by [23] under the weaker hypothesis: for \( s = 1, \ldots, k \), the sequence \((u^i_s)\) converges weakly to \( u_s \). Comparing with Theorem 2.6 in [23], we see that the first statement of Theorem C was proved for the particular case \( T = 1 \) and \( k = m \) but with the hypothesis \( u_0 \) is locally bounded instead of each \( u_s \) is bounded near the boundary. Moreover, statement (1) fails when \( k = m + p - n \) as shown by the following example: \( T = 1 \) and \( u_0(x) = \ldots = u_k(x) = -\frac{1}{(\frac{n}{m} - 2)|x|^{\frac{n}{m} - 2}} \) if \( m < \frac{n}{2} \).

**Proof** The proof follows the same techniques used by Demailly [9] for plurisubharmonic functions. Observing that (1) \( \Rightarrow \) (2), it then suffices to prove (1). Since \( T \) is a closed \( m \)-positive current, (1) is obvious for \( k = 1 \) thanks to Theorem 3 and Lebesgue’s bounded convergence theorem. Since the sequence \((u^i_j)\), is decreasing and since \( u_i \) is bounded near \( \partial \Omega \cap \text{Supp } T \), the family \((u^i_j)_{j \in \mathbb{N}} \) is uniformly bounded near \( \partial \Omega \cap \text{Supp } T \). Without loss of generality, we can assume that \( \Omega = \{ \psi < 0 \} \), where \( \psi \) is a smooth strictly convex exhaustion function on \( \Omega \) such that \( dd^c \psi \geq \beta \) and \( \psi = 0 \), \( d\psi \neq 0 \) on \( \partial \Omega \). We set \( \Omega_\lambda = \{ \psi < -\lambda \} \), for all \( \lambda > 0 \). After addition of a constant we can assume that \( u^i_j \leq -1 \) on \( \Omega \). We fix \( \lambda \) so small such that \( u^i_j \) is bounded near \( \Omega \setminus \Omega_\lambda \) and we select a neighborhood \( L \) of \( \Omega \setminus \Omega_\lambda \cap \text{Supp } T \) such that \( u^i_j \) is bounded near \( L \). Assume that \( u^i_j \geq -M \) on \( L \) for some constant \( M > 0 \) and setting \( A = \frac{M}{\lambda} \). Then by considering the same max procedure sequences \( v^i_j \) (respectively \( v^i_j, v \)) as in the proof of Corollary 4.2 in [9] which are \( m \)-convex by Proposition 2.2 in [25], we may assume that all \( u^i_j \) (and similarly all \( v^i_j, v \)) coincide with \( A\psi \) on a fixed neighborhood of \( \partial \Omega \cap \text{Supp } T \). Assume that (1) has been proved for \( k \), then

\[
S^i_j := T \wedge \beta^{n-m} \wedge dd^# u^j_1 \wedge \ldots \wedge dd^# u^j_k 
\rightarrow S := T \wedge \beta^{n-m} \wedge dd^# u_1 \wedge \ldots \wedge dd^# u_k.
\]

Since \( T \wedge dd^# u^j_1 \wedge \ldots \wedge dd^# u^j_k \) is a closed \( m \)-positive current and \( S^i_j \) is positive, proof of Theorem 3 implies that the sequence \((u^i_j, S^i_j)\), has locally uniformly bounded mass, hence is relatively compact for the weak topology. In order to prove (1), we only have to show that every weak limit \( \Theta \) of \( u^i_j S^i_j \) is equal to \( u_0 S \). Let \((q, q)\) be the bimension of \( S \) and let \( \alpha \) be an arbitrary smooth compactly supported form of bidegree \((q, 0)\). Then, the positive measure \( S^i_j \wedge \sigma_q \alpha \wedge J(\alpha) \) converges weakly to \( S \wedge \sigma_q \alpha \wedge J(\alpha) \) and by Lemma 4, we can see that \( \Theta \wedge \sigma_q \alpha \wedge J(\alpha) \leq u_0 S \wedge \sigma_q \alpha \wedge J(\alpha) \), which means that \( u_0 S - \Theta \) is positive. Next, an integration by parts exactly as in the proof of Theorem 3.7 in [9], yields the following inequality

\[
\int_{\Omega \times \mathbb{R}^n} u_0 T \wedge \beta^{n-m} \wedge dd^# u_1 \wedge \ldots \wedge dd^# u_k \wedge (dd^# \psi)^q \leq \int_{\Omega \times \mathbb{R}^n} u_0^{j, x_0} T \wedge \beta^{n-m} \wedge dd^# u^j_1 \wedge \ldots \wedge dd^# u^j_k \wedge (dd^# \psi)^q.
\]
where \( u_s \leq u^j_s \leq u^j_{s\varepsilon}, \varepsilon > 0, s = 1, \ldots, k\). Then, we let \( \varepsilon_k \to 0, \ldots, \varepsilon_0 \to 0 \) in this order, we have weak convergence at each step and \( u^j_{0\varepsilon} = 0 \) on the boundary. Hence, the last integral converges and we get the inequality

\[
\int_{\Omega \times \mathbb{R}^n} u_0 T \wedge \beta^{n-m} \wedge dd^# u_1 \wedge \ldots \wedge dd^# u_k \wedge (dd^# \psi)^q \leq \liminf_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} u^j_0 T \wedge \beta^{n-m} \wedge dd^# u^j_1 \wedge \ldots \wedge dd^# u^j_k \wedge (dd^# \psi)^q.
\]

Therefore, \( (u_0 S - \Theta) \wedge (dd^# \psi)^q = 0 \) and since \( u_0 S - \Theta \) is positive, then thanks to Lagerberg [16] we obtain the equality \( u_0 S = \Theta \).

4.2 \( m \)-Generalized Lelong Numbers

Let us first define the pseudo-sphere and the pseudo-ball associated with an \( m \)-convex function \( \varphi \), respectively by

\[
S(r) = \{ x \in \Omega; \varphi(x) = r \} \quad \text{and} \quad B(r) = \{ x \in \Omega; \varphi(x) < r \}.
\]

Thanks to the argument before Theorem C, the \( m \)-superHessian operator \( T \wedge \beta^{n-m} \wedge (dd^# \varphi)^{m+p-n} \) is a well defined positive measure provided that \( T \) is a closed \( m \)-positive current of bidimension \( (p, p) \) on \( \Omega \times \mathbb{R}^n \) such that \( n - p \leq m \leq n \), \( T \wedge \beta^{n-m} \) is positive and \( \varphi \) is \( m \)-convex bounded near \( \partial \Omega \cap \text{Supp} \ T \). Hence, inspired by the work of Elkhadhra [11], we state the following definition:

**Definition 2** Assume that \( 1 \leq m \leq \frac{n}{2} \) and \( \varphi \) is semi-exhaustive on \( \text{Supp} \ T \), i.e. there exists \( R \) such that \( B(R) \cap \text{Supp} \ T \subseteq \Omega \), then \( S(-\infty) \cap \text{Supp} \ T \) is compact and therefore for \( r < R \), we set

\[
\nu^m_T(\varphi, r) = \int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^# \varphi)^{m+p-n} \quad \text{and} \quad \nu^m_T(\varphi) = \lim_{r \to -\infty} \nu^m_T(\varphi, r).
\]

The number \( \nu^m_T(\varphi) \) will be called the \( m \)-generalized Lelong number with respect to the weight \( \varphi \).

Observe that \( \nu^m_T(\varphi) \) measure the asymptotic behaviour of the current \( T \wedge \beta^{n-m} \wedge (dd^# \varphi)^{m+p-n} \) near the \( m \)-polar set \( \{ \varphi = -\infty \} \). Moreover, this notion is a real Hessian version of the definition of \( m \)-generalized Lelong number given by Elkhadhra [11] in the complex Hessian setting.

**Proposition 5** For any convex increasing function \( \chi : \mathbb{R} \to \mathbb{R} \), we have

\[
\int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^# \chi \circ \varphi)^{m+p-n} = (\chi'(r - 0))^{m+p-n} \nu^m_T(\varphi, r), \tag{4.1}
\]

where \( \chi'(r - 0) \) is the left derivative of \( \chi \) at \( r \).
Proof According to Trudinger and Wang [22], the composit function \( \chi \circ \varphi \) is \( m \)-convex. Now, by adapting the same techniques of Demailly [9], let \( \chi_\varepsilon \) be the convex function equal to \( \chi \) on \([r - \varepsilon, +\infty[\) and to a linear function of slope \( \chi'(r - \varepsilon - 0) \) on \([-\infty, r - \varepsilon]\). So, we get \( dd^\# (\chi_\varepsilon \circ \varphi) = \chi'(r - \varepsilon - 0) dd^\# \varphi \) on \( B(r - \varepsilon) \) and Stokes formula implies

\[
\int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^\# \chi \circ \varphi)^{m+p-n} = \int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^\# \chi_\varepsilon \circ \varphi)^{m+p-n} \\
= \int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^\# \chi_\varepsilon \circ \varphi)^{m+p-n} = \chi'(r - \varepsilon - 0)^{m+p-n} v^m_T(\varphi, r - \varepsilon).
\]

Similarly, taking \( \tilde{\chi}_\varepsilon \) equal to \( \chi \) on \([-\infty, r - \varepsilon]\) and linear on \([r - \varepsilon, r]\), we obtain

\[
\int_{B(r - \varepsilon) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^\# \chi \circ \varphi)^{m+p-n} \leq \int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^\# \tilde{\chi}_\varepsilon \circ \varphi)^{m+p-n} = \chi'(r - \varepsilon - 0)^{m+p-n} v^m_T(\varphi, r).
\]

The desired formula follows when \( \varepsilon \) tends to 0. \( \Box \)

Special cases. Assume that \( a \in \Omega \) and let \( \varphi_m(x) = -\frac{1}{(\frac{n}{2} - 2)|x-a|^{\frac{n}{2} - 2}} \) if \( m \neq \frac{n}{2} \) and \( \log |x-a| \) if \( m = \frac{n}{2} \). A straightforward computation gives

\[
dd^\# \varphi_m(x) = \begin{cases} 
|x-a|^{-\frac{n}{2}} \left( \beta - \frac{n}{m} |x-a| - 2 \frac{1}{2} d|x-a|^2 \wedge \frac{1}{2} d^\# |x-a|^2 \right) & \text{if } m \neq \frac{n}{2} \\
|x-a|^{-2} \left( \beta - 2|x-a| - 2 \frac{1}{2} d|x-a|^2 \wedge \frac{1}{2} d^\# |x-a|^2 \right) & \text{if } m = \frac{n}{2}.
\end{cases}
\]

By using the Binôme formula, for \( x \neq a \), a simple computation gives

\[
\forall s < m, \quad (dd^\# \varphi_m(x))^s \wedge \beta^{n-s} = \begin{cases} 
(1 - \frac{s}{m}) |x - a|^{-\frac{ns}{2}} \beta^n & \text{if } m \neq \frac{n}{2} \\
|x - a|^{-2s} (1 - \frac{2s}{n}) \beta^n & \text{if } m = \frac{n}{2}.
\end{cases}
\]

In particular, we see that \( \varphi_m \) is \( m \)-convex and we have the following two important cases:

1. For \( 1 \leq m < \frac{n}{2} \), put \( \mu = 1 - \frac{m}{2m} \), then by applying formula (4.1) for the convex increasing function \( \chi \) defined by \( \chi(x) = (-x)^{\frac{1}{2}} \) if \( x \leq R \) and \( \chi \) is linear on \( \{x \geq R\} \). Then, for \( r < R \), we obtain

\[
\int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge \left(dd^\# (-\varphi_m)^\frac{1}{n}\right)^{m+p-n} = \left(\frac{2m}{n-2m}\right)^{m+p-n} \left(-r\right)^{\frac{n(m+p-n)}{2m-n}} v^m_T(\varphi_m, r).
\]

For \( t > 0 \), let \( r = -\frac{1}{m-2} t^{2\mu} \), we have

\[
v^m_T(\varphi_m, r) = \frac{1}{t^{\frac{(m+p-n)}{m}}} \int_{\{ |x-a| < t \} \times \mathbb{R}^n} T \wedge \beta^p = \frac{p!}{t^{\frac{(m+p-n)}{m}}} \Theta_T(a, t),
\]

where \( \Theta_T(a, t) \) is the fundamental solution of the equation \( -\Delta u + \beta^p u = -t^p \).
here $\Theta_T(a, t)$ is the trace measure of $T$ on the euclidean ball $B(a, t)$ (see [16]). Now, for $m = \frac{n}{2}$, we apply formula (4.1) for the convex increasing function $\chi$ defined by $\chi(x) = \exp(2x)$. Then, for $r < R$, we obtain

$$\int_{B(r) \times \mathbb{R}^n} \left( dd^# \exp(2\varphi_m) \right)^{m+p-n} \wedge T \wedge \beta^{n-m} = 2^{m+p-n} \exp(2r(m + p - n)) \nu_T^m(\varphi_m, r).$$

For $t > 0$, let $r = \log t$, we have

$$v_T^m(\varphi_m, r) = v_T^m(\varphi_m, \log t) = \frac{1}{t^{2(p - \frac{m}{2})}} \int_{\{|x - a| < t\} \times \mathbb{R}^n} T \wedge \beta^p = \frac{p!}{t^{2(p - \frac{m}{2})}} \Theta_T(a, t).$$

Thus, the number

$$v_T^m(a) := \lim_{t \to 0^+} \frac{p!}{t^{n(m+p-n)}} \Theta_T(a, t)$$

exists and it coincides with the $m$-Lelong number of $T$ at $a$ as established by Elkhadhra and Zahmoul [12] in the case where $T$ is $m$-positive and $T \wedge \beta^{p-1}$ is convex.

(2) For $\frac{n}{2} < m \leq n$ and according to Trudinger and Wang [21], every $m$-convex function satisfies a local H"{o}lder estimate, and therefore its $m$-polar set is empty. However, in this case and by an adaptation of the proof of Proposition 4.1 in [12], if we assume that $\varphi$ is a positive $m$-convex function such that $\varphi^\mu$ is $m$-convex, then we have

$$\int_{B(r_1, r_2) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^# \varphi^\mu)^{m+p-n}$$

$$= \frac{\mu^m + p-n}{r_2^m (m+p-n)} \int_{B(r_2) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^# \varphi)^{m+p-n}$$

$$= \frac{\mu^m + p-n}{r_1^m (m+p-n)} \int_{B(r_1) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^# \varphi)^{m+p-n}.$$ 

In particular, we see that the map

$$r \mapsto v_T^m(\varphi, r) := \frac{\mu^m + p-n}{r^m (m+p-n)} \int_{B(r) \times \mathbb{R}^n} T \wedge \beta^{n-m} \wedge (dd^# \varphi)^{m+p-n}$$

is positive and increases, and therefore we can define the $m$-generalized Lelong number of $T$ relatively to the weight $\varphi$ by setting $v_T^m(\varphi) = \lim_{r \to 0} v_T^m(\varphi, r)$. In particular, for $m = n$,

$$v_T(\varphi) := \lim_{r \to 0} \frac{1}{2pr^2} \int_{B(r) \times \mathbb{R}^n} T \wedge (dd^# \varphi)^p$$

exists and it is the Lelong number of $T$ relative to the weight $\varphi$ as defined by [12]. Moreover, if we assume that $\varphi = |x - a|^2$ and $r = t^2$, then the $m$-superHessian
operator $T \wedge \beta^p$ is well defined without the condition $T \wedge \beta^{n-m}$ is positive. Hence, as an immediate consequence of the above discussion, the function

$$ t \longmapsto \frac{p!}{t^{n(m+p-n)m}} \Theta_T(a, t) $$

is positive and increasing on $[0, +\infty[$. Therefore, the $m$-Lelong number $v^m_T(a)$ of $T$ at $a$ exists. In particular, for $m = n$, we recover the Lelong number defined by Lagerberg [16].

Remark 2 (1) As a direct applications of the monotonicity statement and by an adaptation of the proof of Proposition 5.11 in [9], we see that for each $\delta > 0$, the set

$$ \mathcal{E}_\delta = \{ z \in \Omega : v^m_T(z) \geq \delta \} $$

is closed and has a locally finite $\mathcal{H}_{n(m+p-n)}$ Hausdorff measure in $\Omega$. Also, by the same lines of the proof of Proposition 4.2 in [12], we get that for each compact subset $K$ of $\mathbb{R}^n$, if $\mathcal{H}_{n(m+p-n)}(K \cap \text{Supp } T) = 0$, then $\|T\|_K = 0$. Finally, for $m = n - p$, it is clear that the $(n - p)$-generalized Lelong number of $T$ at the point $a$ is exactly the trace measure of $T$ on $[a]$.

(2) As a special case when $T = dd^\#u$ and $0 < m \leq \frac{n}{2}$, where $u$ is an $m$-convex function, we recover the definition of the generalized Lelong number of $u$ with respect to the weight $\phi$ given by Wan and Wang [25] (modulo a constant). According to Theorem 1.3 in [25], if $m < \frac{n}{2}$ and $(dd^\#\phi)^m \wedge \beta^{n-m} = 0$ on $\Omega \setminus \{\phi = -\infty\}$, then we have the relation

$$ v^m_T(\phi) = \lim_{r \to -\infty} \frac{\mu_{\phi, r}(u)}{r}, $$

where $\mu_{\phi, r}$ is the swept out Hessian measure for $(dd^\#\phi)^m \wedge \beta^{n-m}$ on the pseudosphere $S(r) = \{\phi = r\}$, i.e.

$$ \mu_{\phi, r}(u) = \int_{S(r) \times \mathbb{R}^n} u \left( (dd^\#(\max(\phi, r))^m \wedge \beta^{n-m} - (dd^\#\phi)^m \wedge \beta^{n-m}) \right). $$

(3) If $T$ is $m$-positive and $(m - 1)$-positive (for example when $T = (dd^\#u)^p$, $u$ is $m$-convex and bounded near $\partial \Omega$), then $v^{m-1}_T(a) = 0$, $\forall a \in \Omega$. In particular, if $T$ is a strongly positive closed current, then $T$ is $m$-positive for every $p \leq m \leq n$, and therefore $v^j_T(a) = 0$, $\forall j = p, \ldots, n - 1$ and $a \in \Omega$. Moreover, if $u$ is $m$-convex, then $v^j_{dd^\#u}(a) = 0$, $\forall j = 2, \ldots, m - 1$ and $a \in \Omega$.

(4) Assume that $\Omega = \mathbb{R}^n$ and $T = 1$. Since $p = n$, By Example 4.1 in [16] which gives $\Theta_T(a, t) = t^n \text{Vol}_n(\mathbb{B}(a, 1))$ and by the above computations, for all $r > 0$ and $1 \leq m \leq \frac{n}{2}$, we have

$$ \int_{\mathbb{B}(a, r) \times \mathbb{R}^n} (dd^\#\phi_m)^m \wedge \beta^{n-m} = n! \text{Vol}_n(\mathbb{B}(a, 1)). $$
In particular, we derive the following fundamental formula which is the corresponding of a well-known formula in the complex Hessian theory:

$$(dd^c \varphi_m)^m \wedge \beta^{n-m} = n! \text{Vol}_n(\mathbb{B}(a, 1)) \delta_a, \quad \forall 1 \leq m \leq \frac{n}{2}. $$

Next, we state an analogue of the first Demailly comparison theorem in our setting which is the counterpart of Theorem 1 in [11]. We omit the proof since it is almost identical to the one given by Demailly [9].

**Theorem 4** (Demailly’s comparison theorem) Assume that $T$ is a closed $m$-positive current of bidimension $(p, p)$ on $\Omega \times \mathbb{R}^n$ such that $T \wedge \beta^{n-m}$ is positive. Let $\varphi, \psi : \Omega \rightarrow [-\infty, +\infty]$ be two continuous $m$-convex functions. We assume that $\varphi, \psi$ are semi-exhaustive on $\text{Supp} T$ and that $l := \limsup \frac{\psi(x)}{\varphi(x)} < +\infty$ as $x \in \text{Supp} T$ and $\varphi(x) \rightarrow -\infty$.

Then, $\nu^m_T(\psi) \leq l^{m+p-n} \nu^m_T(\varphi)$, and the equality holds if $l = \lim \frac{\psi}{\varphi}$.

As a consequence of the comparison theorem, we see that the $m$-generalized Lelong number $\nu^m_T(\varphi)$ only depends on the asymptotic behavior of $\varphi$ near $(\varphi = -\infty) \cap \text{Supp} T$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be an affine function. Following Lagerberg [16], the function $f$ can be extended to unique affine application $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ such that $\tilde{f} \circ J = J \circ \tilde{f}$. If $T$ is a current of bidimension $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$ and $f$ is proper on $\text{Supp} T$, then the direct image of $T$ by $\tilde{f}$ is a current of bidimension $(p, p)$ on $\mathbb{R}^m \times \mathbb{R}^m$ noted $f_* T$ and defined by

$$\langle f_* T, \alpha \rangle = \langle T, f^* \alpha \rangle, \quad (4.2)$$

for every compactly supported form $\alpha$ of bidegree $(p, p)$ on $\mathbb{R}^m \times \mathbb{R}^m$. The definition makes sense, because $\text{Supp} T \cap f^{-1}(\text{Supp} \alpha)$ is compact. Moreover, thanks to Lagerberg [16], if $T$ is (weakly) positive, then $f_* T$ is also (weakly) positive. Our aim now is to find a relationship between the Lelong number of an $m$-positive current and the Lelong number of his direct image by a projection. Strongly inspired by the work of Demailly [9] on the same subject in the complex setting, we have proven the following result.

**Proposition 6** Let $\pi$ be the projection of $\mathbb{R}^n$ on $\mathbb{R}^{n-k}$ and $T$ be an $m$-positive closed current of bidimension $(p, p)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $T \wedge \beta^{n-m}$ is positive, $1 \leq k \leq \inf(m - 1, n - m)$ and $\pi_* T$ is $(m-k)$-positive closed on $\mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$. Let $\psi$ be an $m$-convex function on $\mathbb{R}^{n-k}$ which is semi-exhaustive on $\text{Supp} \pi_* T$, i.e. there exists $R$ such that $B(R) \cap \text{Supp} \pi_* T \subseteq \Omega$. Then, $\varphi = \psi \circ \pi$ is $m$-convex on $\mathbb{R}^n$ and semi-exhaustive on $\text{Supp} T$, and we have

$$\int_{\{\varphi < r\} \times \mathbb{R}^n} T \wedge (dd^c \varphi)^{m+p-n} \wedge \beta^{n-m}$$
\[ \int_{\{\psi < r\} \times \mathbb{R}^{n-k}} \pi_* T \wedge (dd^\# \psi)^{(m-k)+p-(n-k)} \wedge \beta^{(n-k)-(m-k)}, \]

for all \( r < R \), where \( \beta' = \frac{1}{2} dd^\# |\pi(x)|^2 \), \( \forall x \in \mathbb{R}^n \). In particular, we have \( \nu^m_{\pi_* T}(\psi) = \nu^m_{\pi_* T}(\psi \circ \pi) \).

**Proof** If \( \alpha \) is an an \( m \)-positive form of bidegree \((1,1)\) on \( \mathbb{R}^{n-k} \), then \( \pi^* \alpha \) is also an \( m \)-positive form on \( \mathbb{R}^n \). Indeed, for \( s \leq m \),

\[(\pi^* \alpha)^s \wedge \beta^{n-s} = (\pi^* \alpha)^s \wedge (\beta' + \beta'')^{n-s} = (\pi^* \alpha)^s \wedge \beta^{m-k-s} \wedge \beta^m \geq 0.\]

Assume first that \( \psi \) is smooth. In order to prove that \( \varphi = \psi \circ \pi \) is \( m \)-convex, it suffices to prove that \( dd^\# \varphi \) is \( m \)-positive. Since \( dd^\# \psi \) is \( m \)-positive and since \( \pi^* \) commute with \( d \) and \( d^\# \) (see [16]), then \( dd^\# \varphi = dd^\# (\psi \circ \pi) = dd^\# (\pi^* \psi) = \pi^* (dd^\# \psi) \) is \( m \)-positive thanks to the above discussion. For \( r < R \), we have

\[ \int_{\{\psi < r\} \times \mathbb{R}^{n-k}} \pi_* T \wedge (dd^\# \psi)^{(m-k)+p-(n-k)} \wedge \beta^{(n-k)-(m-k)} \]

This follows almost immediately from the equality (4.2) when \( \psi \) is smooth and when we write \( \Pi_{\{\psi < r\}} \) as the limit of an increasing sequence of smooth functions. In general, if \( \psi \) is not necessarily smooth, we take a smooth regularization \( (\psi_j)_j \) of \( \psi \). Then, the equality is hold for \( \varphi_j \) and \( \psi_j \). Thanks to Theorem C, the associated superHessian operators converge to measures which can be considered with compact supports because our functions are semi-exhaustive. So, by passing to the limit it is not difficult to deduce the equality for \( \varphi \) and \( \psi \). The later statement of Proposition 6 is obtained by sending \( r \rightarrow -\infty \).

### 4.3 Applications

As a meaningful estimates and applications of the \( m \)-generalized Lelong numbers, we get:
4.3.1 Integrability Exponents of $m$-convex Functions

Assume that $u$ is an $m$-convex function on an open subset $\Omega$ of $\mathbb{R}^n, m < \frac{n}{2}$ and $K$ is a compact subset of $\Omega$. The integrability exponent of $u$ at $K$ is defined by

$$\iota_K(u) = \sup\{c > 0; |u|^c \in L^1(\mathcal{V}(K))\},$$

where $\mathcal{V}(K)$ is an open neighbourhood of $K$. For simplicity, if $K = \{x\}$, then we denote $\iota_{\{x\}}(u)$ by $\iota_x(u)$. According to Theorem 4.1 in [22] (see also [5]), for $m < \frac{n}{2}$, we have

$$\iota_x(u) \geq \frac{nm}{n - 2m}.$$

Then, similarly as in [1], $\iota_K(u) = \inf_{x \in K} \iota_x(u) \geq \frac{nm}{n - 2m}$. Assume that $u$ is an $m$-convex function which is negative in a neighbourhood of $K$ and let $\alpha$ be a real number such that $0 < \alpha < \iota_K(u)$. Then, we have

$$C(\alpha, u) = \int_K |u(x)|^\alpha d\lambda(x) < +\infty.$$

An adaptation of the proof of Lemma 5 in [1] to the real Hessian theory, yields for any $t < 0$ the following estimate:

$$\lambda(\{x \in K, u(x) \leq t\}) \leq \frac{C(\alpha, u)}{|t|^\alpha}.$$

In particular, if $\int_K |u(x)|^\alpha d\lambda(x)$ is bounded by an absolute constant $C = C(\alpha, K)$. Then $\lambda(\{x \in K, u(x) \leq t\})$ decreases at most like $C|t|^{-\alpha}$. Finally, in strong analogy with the complex Hessian theory (see [1]), we obtain a relationship between the $m$-Lelong number and the integrability exponents of a given $m$-convex function. More precisely, assume that $u$ is a negative $m$-convex function on an open subset $\Omega$ of $\mathbb{R}^n$ and assume that $0 < m < \frac{n}{2}$. Then, for any $a \in \Omega$ such that $\nu^m_{dd^c u}(a) > 0$, we have

$$\iota_a(u) = \frac{nm}{n - 2m}.$$

Notice that this result can be seen as the corresponding one of Theorem 5 in [1] from complex Hessian theory. On the other hand if $m = 1$, we have $\iota_a(u) = \frac{n}{n-2}$ for $u$ subharmonic on $\Omega$. In particular, when $\Omega$ is an open subset of $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, we recover the second statement of Remark 2 in [1]. In order to prove the preceding equality, we keep the notations in [1] and we follow the same line of the proof of Theorem 3 in the same paper, firstly we establish that the $m$-Lelong number of $u$ at $a$ can be obtained by the following limits:

$$v^m_{du}(a) := v^m_{dd^c u}(a) = \lim_{r \to 0^+} \frac{\lambda(u, a, r)}{\phi_m(r)} = \frac{1}{n} \left( n - \frac{n}{2m} + 1 \right) \lim_{r \to 0^+} \frac{\wedge(u, a, r)}{\phi_m(r)}.$$
where \( \lambda(u, a, r) = \frac{(n-1)!}{2\pi^n r^{n-1}} \int_{\mathbb{B}(a, r)} u(x) d\sigma(x) \), \( \wedge (u, a, r) = \frac{n!}{\pi^n r^n} \int_{\mathbb{B}(a, r)} u(x) d\lambda(x) \) and \( \phi_m(r) = -\frac{1}{(\frac{m}{m-2})^{m-2}} \). For simplicity, assume that \( a = 0 \). Hence, by a minor change of the proof of Theorem 5 in [1], if \( 0 < p < u_0(u) \), we derive the inequality
\[
\int_{\mathbb{B}(0, r)} (-u(x))^p d\lambda(x) \geq \frac{1}{2np} \frac{\pi^n}{(n-1)!} \left( \frac{v^m_u(0)}{(\frac{n}{2m} - 1)(n - \frac{n}{2m} + 1)} \right)^p \\
\times \int_0^r \left( 1 - \frac{\phi_m(2r)}{\phi_m(r)} \right)^p t^{n-1-p(\frac{m}{m-2})} dt.
\]
So, the last integral is finite which gives \( p < \frac{nm}{n-2m} \).

4.3.2 \( m \)-Generalized Lelong Numbers in Terms of Capacity

As another application, we are going to show that the \( m \)-generalized Lelong numbers of \( T \) with respect to a weight \( \phi \) can be expressed in terms of \( cap_m, T \) the associated capacity to \( T \) defined by Elkhadhra and Zahmoul [12] (see Remark 3.2). By keeping the notations of Definition 2 and similarly as in the complex Hessian theory (see [11]), for an \( m \)-positive current \( T \) of bidegree \( (p, p) \) on \( \Omega \) and \( r < R, t < R \) we get
\[
(R - t)^{m-p} cap_m, T(B(r + t - R), B(R)) \leq v^m_T(\phi, r) \\
\leq (R - r)^{m-p} cap_m, T(B(r), B(R)).
\]

Hence, in analog with [11], for the complex Hessian setting, we deduce that
\[
v^m_T(\phi) = \int_{\{|\phi| = \infty\} \times \mathbb{R}^n} (dd^c\phi)^{m-p} \wedge \beta^{n-m} \wedge T \\
= \lim_{r \to -\infty} (R - r)^{m-p} cap_m, T(B(r), B(R)).
\]
Let \( a \in \Omega \) and let \( R > 0 \) be such that \( \mathbb{B}(a, R) \cap \text{Supp } T \subseteq \Omega \). We have:

1. If \( m = \frac{n}{2} \), then by applying (4.3) for \( \phi(x) = |x - a| \), \( \log R \), \( \log t \) and \( \log r \), we immediately see that
\[
\left( \log \frac{R}{t} \right)^{m-p} cap_m, T\left( \mathbb{B}(a, \frac{t}{R}), \mathbb{B}(a, R) \right) \leq v^m_T(\phi, a, r) \\
\leq \left( \log \frac{R}{r} \right)^{m-p} cap_m, T\left( \mathbb{B}(a, r), \mathbb{B}(a, R) \right).
\]

Therefore, similarly as in [11], we deduce that the \( m \)-Lelong number of \( T \) at \( a \), can be expressed as
\[
v^m_T(\phi) = \lim_{r \to 0} v^m_T(a, r) = \lim_{r \to 0} \left( \log \frac{R}{r} \right)^{m-p} cap_m, T\left( \mathbb{B}(a, r), \mathbb{B}(a, R) \right).
\]
(2) If \( m < \frac{n}{2} \) and \( \mu = \frac{n}{m} - 2 \), then applying (4.3) for \( \varphi_m(x) = -|x - a|^{-\mu} \), 
\(-R^{-\mu}, -r^{-\mu}\), and 
\(-t^{-\mu}\), we obtain
\[
(t^{-\mu} - R^{-\mu})^{m-p} \cap m,T(\mathbb{B}(a, s), \mathbb{B}(a, R)) \leq v_T^m(a, r)
\]
\[
\leq (r^{-\mu} - R^{-\mu})^{m-p} \cap m,T(\mathbb{B}(a, r), \mathbb{B}(a, R)),
\]
where \( s = (r^{-\mu} + t^{-\mu} - R^{-\mu})^{-\frac{1}{p}} \). So, as in the preceding case we conclude that
the \( m\)-Lelong number of \( T \) at \( a \), can be expressed as
\[
v_T^m(a) = \lim_{r \to 0} v_T^m(a, r) = \lim_{r \to 0} (r^{-\mu} - R^{-\mu})^{m-p} \cap m,T(\mathbb{B}(a, r), \mathbb{B}(a, R)).
\]

As a direct consequence of the preceding discussion, we get:

(1) \( \cap m,\mathbb{B}(a, r), \mathbb{B}(a, R) = n!\text{Vol}_n(\mathbb{B}(a, 1)) \left( \log \frac{R}{r} \right)^{-\frac{n}{2}}. \)

(2) For \( m < \frac{n}{2} \), \( \cap m,\mathbb{B}(a, r), \mathbb{B}(a, R) = \frac{n!\text{Vol}_n(\mathbb{B}(a, 1))}{(2 - \frac{n}{m})^m} \frac{1}{(r^{-\mu} - R^{-\mu})^m}. \)

Notice that the later result was obtained by Labutin [15] (see Example 2), by using a formulas on the link between the Hessian measure of an \( m\)-convex function on an open set \( \Omega \) and the elementary symmetric function of the curvatures of \( \partial \Omega \).

5 Weighted Relative Extremal Function and Weighted \( m\)-Hessian Capacity

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) and denote by \( \mathcal{C}^{-m}(\Omega) \) the set of all negative \( m\)-convex functions on \( \Omega \). In this section we begin by introducing analogously to the complex Hessian setting (see Lu [17]) two classes of Cegrell-type \( \mathcal{E}_m^0(\Omega) \) and \( \mathcal{F}_m(\Omega) \). After investigating some properties of these classes and by using ideas of Nguyen [19] from pluripotential theory in \( \mathbb{C}^n \), we introduce the weighted relative extremal function as well as the weighted \( m\)-Hessian capacity in the real Hessian context. The aim is to generalize the notions of \( m\)-Hessian capacity and relative extremal function studied by Trudinger and Wang [23] and Labutin [15]. Now, in a similar way as in [17] and [6], we define:
\[
\mathcal{E}_m^0(\Omega) = \left\{ u \in \mathcal{C}^{-m}(\Omega) \cap L^\infty(\Omega); \lim_{x \to \partial \Omega} u(x) = 0, \int_{\Omega \times \mathbb{R}^n} (dd^au)^m \wedge \beta^{n-m} < +\infty \right\}.
\]
\[
\mathcal{F}_m(\Omega) = \left\{ u \in \mathcal{C}^{-m}(\Omega); \exists (u_j) \subset \mathcal{E}_m^0(\Omega), u_j \downarrow u, \sup_j \int_{\Omega \times \mathbb{R}^n} (dd^au_j)^m \wedge \beta^{n-m} < +\infty \right\}.
\]

Note that \( \mathcal{E}_m^0(\Omega) \) is a subclass of the one defined and used by Wan [24] for proper \( m\)-
convex functions in order to prove several estimates for the mixed \( m\)-Hessian operator.
According to [21] and [23] it was proved that for every bounded functions $g, h \in \mathcal{E}_m(\Omega)$, we have:

$$g = h \text{ on } \partial \Omega \text{ and } g \leq h \text{ in } \Omega \Rightarrow \int_{\Omega \times \mathbb{R}^n} (dd^# h)^m \wedge \beta^{n-m} \leq \int_{\Omega \times \mathbb{R}^n} (dd^# g)^m \wedge \beta^{n-m}. \quad (5.1)$$

For the discontinuous functions, the equality $g = h$ is understood in the sense of limit.

The following proposition was given by Cegrell [4] for negative plurisubharmonic functions and extended by Lu [17] for $m$-subharmonic functions.

**Proposition 7**

(1) $\mathcal{E}_0^0(\Omega) \subset \mathcal{F}_m(\Omega) \subset \mathcal{E}_m^-(\Omega)$.

(2) If $u \in \mathcal{F}_m(\Omega)$, then

$$\int_{\Omega \times \mathbb{R}^n} (dd^# u)^m \wedge \beta^{n-m} < +\infty.$$  

(3) The class $\mathcal{E}_0^0(\Omega)$ (as well as $\mathcal{F}_m(\Omega)$) is convex cone.

(4) If $\varphi \in \mathcal{E}_m^0(\Omega)$ (respectively in $\mathcal{F}_m(\Omega)$) and $\psi \in \mathcal{E}_m^-(\Omega)$, then $\max(\varphi, \psi) \in \mathcal{E}_m^0(\Omega)$ (respectively in $\mathcal{F}_m(\Omega)$).

**Proof** It is clear that (1) and (2) are immediate from the definitions and Theorem 2.4 in [23]. So, all we need is to prove (3) and (4). It is not hard to see that if $\varphi \in \mathcal{E}_0^0(\Omega)$ then $\alpha \varphi \in \mathcal{E}_0^0(\Omega)$, $\forall \alpha \in \mathbb{R}^+$. Now, if $\varphi, \psi \in \mathcal{E}_0^0(\Omega)$, then

$$\int_{\Omega \times \mathbb{R}^n} (dd^# (\varphi + \psi))^m \wedge \beta^{n-m} = \int_{\{\varphi < \psi\} \times \mathbb{R}^n} (dd^# (\varphi + \psi))^m \wedge \beta^{n-m} + \int_{\{\psi \leq \varphi\} \times \mathbb{R}^n} (dd^# (\varphi + \psi))^m \wedge \beta^{n-m}.$$  

We have $\{\varphi < \psi\} \subset \{2\varphi < \varphi + \psi\}$, then by (5.1), we obtain

$$\int_{\{\varphi < \psi\} \times \mathbb{R}^n} (dd^# (\varphi + \psi))^m \wedge \beta^{n-m} \leq \int_{\{2\varphi < \varphi + \psi\} \times \mathbb{R}^n} (dd^# (\varphi + \psi))^m \wedge \beta^{n-m} \leq \int_{\{2\varphi < \varphi + \psi\} \times \mathbb{R}^n} (dd^# (2\varphi))^m \wedge \beta^{n-m} = 2^m \int_{\{2\varphi < \varphi + \psi\} \times \mathbb{R}^n} (dd^# \varphi)^m \wedge \beta^{n-m} \leq 2^m \int_{\Omega \times \mathbb{R}^n} (dd^# \varphi)^m \wedge \beta^{n-m} < +\infty.$$  

Let $\lambda > 1$, then $\lambda \varphi < \psi$ and $\{\psi \leq \varphi\} \subset \{\lambda \varphi < \varphi\} \subset \{\lambda + 1\varphi < \varphi + \psi\}$. Hence, the implication (5.1) implies that

$$\int_{\{\psi \leq \varphi\} \times \mathbb{R}^n} (dd^# (\varphi + \psi))^m \wedge \beta^{n-m} \leq (\lambda + 1)^m \int_{\Omega \times \mathbb{R}^n} (dd^# \psi)^m \wedge \beta^{n-m} < +\infty.$$  

Thus,

$$\int_{\Omega \times \mathbb{R}^n} (dd^# (\varphi + \psi))^m \wedge \beta^{n-m} < +\infty,$$
and then $\varphi + \psi \in \mathcal{E}_m^0(\Omega)$. Finally, for $\varphi \in \mathcal{E}_m^0(\Omega)$, $\psi \in \mathcal{E}_m^-(\Omega)$ and $\lambda > 1$, the implication (5.1) yields

$$\int_{\Omega \times \mathbb{R}^n} (dd^# \max(\varphi, \psi))^m \wedge \beta^{n-m} = \int_{\{\lambda \varphi < \max(\varphi, \psi)\} \times \mathbb{R}^n} (dd^# \max(\varphi, \psi))^m \wedge \beta^{n-m} \leq \lambda^m \int_{\{\lambda \varphi < \max(\varphi, \psi)\} \times \mathbb{R}^n} (dd^# \varphi)^m \wedge \beta^{n-m} = \lambda^m \int_{\Omega \times \mathbb{R}^n} (dd^# \varphi)^m \wedge \beta^{n-m} < +\infty,$$

and then $\max(\varphi, \psi) \in \mathcal{E}_m^0(\Omega)$. For the other class $\mathcal{F}_m(\Omega)$, the properties (3) and (4) can be proved by repeating almost the same arguments as above. □

**Lemma 5** Let $u \in \mathcal{C}_m^-(\Omega) \cap \mathcal{C}(\Omega)$, $K \subset \Omega$ and

$$u_K = \sup\{v \in \mathcal{C}_m^-(\Omega); v \leq u \text{ on } K\}.$$ 

Then, $\operatorname{Supp}(dd^# \bar{u}_K)^m \wedge \beta^{n-m} \subset \bar{K}$, where $\bar{u}_K$ is the upper semi-continuous regularization of $u_K$.

**Proof** By Dini’s theorem,

$$u_K = \sup\{v \in \mathcal{C}_m^-(\Omega) \cap \mathcal{C}(\Omega); v \leq u \text{ on } K\}.$$ 

Hence, Choquet’s Lemma implies that there is an increasing sequence $(u_j)_j \subset \mathcal{C}_m^-(\Omega) \cap \mathcal{C}(\Omega)$ with $u_j \leq u$, $\forall j$ on $K$ and $\bar{u}_K = \sup_j(u_j)$. If $B$ is a ball in $\Omega \setminus K$, then by Theorem 1.1 in [20] (The Dirichlet problem) we can take $w_j$ to be the unique continuous $m$-convex negative function on $B$ with $w_j = u_j$ on $\partial B \subset \Omega$ and $(dd^# w_j)^m \wedge \beta^{n-m} = 0$ in $B$. Next, we set

$$\tilde{u}_j = \begin{cases} u_j & \text{on } \Omega \setminus B \\ w_j & \text{on } B. \end{cases}$$

Then, by Proposition 2.2 in [25], $\tilde{u}_j \in \mathcal{C}_m^-(\Omega)$. Also, by Theorem 3.1 in [21], $\tilde{u}_j \geq u_j$ and the sequence $(\tilde{u}_j)_j$ is increasing. Hence, $\bar{u}_K = \sup_j(\tilde{u}_j)$ and due to Theorem 2.4 in [23], the sequence $(dd^# \tilde{u}_j)^m \wedge \beta^{n-m}$ converges to $(dd^# \bar{u}_K)^m \wedge \beta^{n-m}$. Thus, since $(dd^# \tilde{u}_j)^m \wedge \beta^{n-m} = 0$ on $B$, then $(dd^# \bar{u}_K)^m \wedge \beta^{n-m} = 0$ on $B$. Furthermore, since $B$ is arbitrary on $\Omega \setminus K$, then $(dd^# \bar{u}_K)^m \wedge \beta^{n-m} = 0$ on $\Omega \setminus K$. □

**Proposition 8** Let $u \in \mathcal{C}_m^-(\Omega)$, then $u$ is locally in $\mathcal{F}_m(\Omega)$; i.e. for all $K \subset \Omega$, there is a function $u_K \in \mathcal{F}_m(\Omega)$ such that $u = u_K$ on $K$.

**Proof** Let $u \in \mathcal{C}_m^-(\Omega)$. Thanks to Theorem 1.1 in [24], we can find $(u_j)_j \subset \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ such that $u_j$ decreases to $u$ on $\Omega$. Let $K \subset \Omega$ and

$$u_{j,K} = \sup\{v \in \mathcal{C}_m^-(\Omega); v \leq u_j \text{ on } K\}, \forall j.$$
Then, $u_{j,K} \in E_0^m(\Omega)$ and $\text{Supp}(dd^#u_{j,K})^m \wedge \beta^{n-m} \subset K, \ \forall j$. Moreover, $u_{j,K}$ decreases to $u$ on $K$, $u_{j,K}$ decreases on $\Omega$ and $u_{j,K} \geq u$ everywhere on $\Omega$. Thus, $\lim_{j \to +\infty} u_{j,K} = u_K \geq u$. Therefore, since $(dd^#u_{j,K})^m \wedge \beta^{n-m}$ is weakly convergent, it follows that

$$\sup_j \int_{\Omega \times \mathbb{R}^n} (dd^#u_{j,K})^m \wedge \beta^{n-m} < +\infty.$$ 

Hence, $u_K \in \mathcal{F}_m(\Omega)$ such that $u = u_K$ on $K$. □

**Definition 3** A set $E \subset \mathbb{R}^n$ is said to be $m$-polar, if for each point $a \in E$ there exists a neighborhood $V_a$ of $a$ and a function $u \in C^m(V_a)$ such that $u = -\infty$ on $E \cap V_a$.

According to Trindinger and Wang [23], we can define the $m$-Hessian capacity of a compact subset $K \subset \Omega$ by

$$\text{cap}_m(K, \Omega) = \sup \left\{ \int_{K \times \mathbb{R}^n} (dd^#u)^m \wedge \beta^{n-m}; \ u \in \mathcal{C}_m(\Omega), \ -1 \leq u \leq 0 \right\}.$$ 

Furthermore, for an arbitrary subset $E \subset \Omega$, we define

$$\text{cap}_m(E, \Omega) = \inf \left\{ \text{cap}_m(\omega, \Omega); \ \omega \text{ is open, } E \subset \omega \subset \Omega \right\},$$

where, $\text{cap}_m(\omega, \Omega) = \sup \{ \text{cap}_m(K, \Omega); \ K \text{ compact, } K \subset \omega \}$. It was proved by Labutin [15] that every Borel subset $E \subset \Omega$ is capacitable, that is

$$\text{cap}_m(E, \Omega) = \sup \{ \text{cap}_m(K, \Omega); \ K \text{ compact, } K \subset E \}.$$ 

The second main tool in potential theory is the relative extremal function defined as

$$R_m(E, \Omega)(x) := \sup \{ u(x); \ u \in \mathcal{C}_m^{-}(\Omega), \ u \leq -1 \text{ on } E \}, \ x \in \Omega \text{ and } E \Subset \Omega.$$ 

By using the same notations as in [15], we denote by $\overline{R}_m(E, \Omega)$ the upper semi-continuous regularisation of $R_m(E, \Omega)$, which is negative and $m$-convex in $\Omega$. Moreover, we have $\overline{R}_m(E, \Omega) = R_m(E, \Omega)$ almost everywhere in $\Omega$. The following proposition is due to Labutin [15] and can be seen as the corresponding result of the one obtained by Lu [17] in the complex Hessian setting:

**Proposition 9** (1) Let $R > 0$ and $\Omega = B(0, R)$. If $E \Subset B(R)$, then $E$ is $m$-polar if and only if $\text{cap}_m(E, B_R) = 0$ if and only if $\overline{R}_m(E, B(R)) = 0$.

(2) Let $E \Subset \mathbb{R}^n$ be $m$-polar. Then, there exists $u \in \mathcal{C}_m(\mathbb{R}^n)$ such that $u = -\infty$ on $E$.

Now, strongly inspired by Nguyen [19], we introduce the following definition of weighted relative extremal function:

**Definition 4** Let $E \subset \Omega$ and $u \in \mathcal{C}_m^{-}(\Omega)$. The weighted relative extremal function associated to $E$ and $u$ is defined by
\( R_{m,u}(E) = R_{m,u}(E, \Omega) := \sup \{ v \in C_m(\Omega) ; \ v \leq u \text{ outside an } m-\text{polar subset on } E \} \).

Denote by \( \overline{R}_{m,u}(E) \) the upper semi-continuous regularization of \( R_{m,u}(E) \). By using the second statement of Proposition 9, it is not hard to see that \( \overline{R}_{m,u}(E) = \overline{R}_{m,u}(E \setminus F) \), for all \( m \)-polar \( F \subset \Omega \). This means in particular that when \( u \) is identically \(-1\), we recover the relative extremal function as defined by Labutin [15]. A function \( u \in C_m(\Omega) \) is called \( m \)-maximal if for each \( v \in C_m(\Omega) \) such that \( v \leq u \) outside a compact subset of \( \Omega \) implies that \( v \leq u \) in \( \Omega \). In particular, by Theorem A.1 in [25], if \( u \in C(\Omega) \) then the function \( u \) is \( m \)-maximal if and only if \( (dd^c u)^m \wedge \beta^{n-m} = 0 \).

Thanks to Theorem 1.1 in [24] and the proof of Theorem A.1 in [25] it is not hard to see that each \( m \)-maximal bounded convex function can be approximated locally by a decreasing sequence of continuous \( m \)-maximal function. Hence, in view of Theorem 2.4 in [23], we obtain the following corresponding of a well-known result in the complex hessian setting.

**Theorem 5** (maximality criterion) A function \( u \in C_m(\Omega) \cap L_\infty(\Omega) \) is \( m \)-maximal if and only if \( (dd^c u)^m \wedge \beta^{n-m} = 0 \).

**Proposition 10** Let \( u \in C_m^-(\Omega) \) and \( E \subset \Omega \), then:

1. \( R_{m,u}(E) \in C_m^-(\Omega) \).
2. \( R_{m,u}(E) \) is \( m \)-maximal on \( \Omega \setminus \overline{E} \).
3. \( E \subset \Omega \Rightarrow R_{m,u}(E) \in \mathcal{F}_m(\Omega) \).
4. Let \((u_j)_j \subset C_m^-(\Omega)\) be a decreasing sequence converging to \( u \). Then, the sequence \((R_{m,u_j}(E))_j\) decreases to \( R_{m,u}(E) \).

**Proof** (1) It is not hard to see that \( R_{m,u}(E) = \overline{R}_{m,u}(E) \) outside an \( m \)-polar subset of \( \Omega \). Thus, by definition of \( R_{m,u}(E) \), we can see that \( R_{m,u}(E) = \overline{R}_{m,u}(E) \in C_m^-(\Omega) \).

(2) Let \( v \in C_m^-(\Omega \setminus \overline{E}) \) and \( v \leq R_{m,u}(E) \) outside a compact subset of \( \Omega \setminus \overline{E} \). Let us set

\[
\varphi = \begin{cases} 
\sup \{v, R_{m,u}(E)\} & \text{on } \Omega \setminus \overline{E} \\
R_{m,u}(E) & \text{on } \overline{E} 
\end{cases}
\]

Thus, by Proposition 2.2 in [25], \( \varphi \in C_m^-(\Omega) \) and \( \varphi \leq R_{m,u}(E) \) on \( \Omega \). Hence, \( \varphi = R_{m,u}(E) \) on \( \Omega \), which implies that \( v \leq R_{m,u}(E) \) on \( \Omega \setminus \overline{E} \).

(3) Due to Proposition 8, there exists a decreasing sequence \((u_j)_j \subset \mathcal{E}_m^0(\Omega)\) converging to \( u \) on a neighborhood of \( \overline{E} \) such that

\[
\sup_j \int_{\Omega \times \mathbb{R}^n} (dd^c u_j)^m \wedge \beta^{n-m} < +\infty.
\]

Then, \((R_{m,u_j}(E))_j \subset \mathcal{E}_m^0(\Omega)\) is a decreasing sequence such that for all \( j \), \( u_j \leq R_{m,u_j}(E) \) on \( \Omega \) and \( u_j = R_{m,u_j}(E) \) on \( E \) outside an \( m \)-polar set. Thus, \((R_{m,u_j}(E))_j\) converges to a function \( \varphi \in C_m^-(\Omega) \) such that \( R_{m,u}(E) \leq \varphi \) on \( \Omega \) and \( \varphi = R_{m,u}(E) \) on \( E \) outside an \( m \)-polar set. Hence, \( \varphi = R_{m,u}(E) \) on \( \Omega \). Moreover, by (5.1), we have

\[
\int_{\Omega \times \mathbb{R}^n} (dd^c R_{m,u_j}(E))^m \wedge \beta^{n-m} \leq \int_{\Omega \times \mathbb{R}^n} (dd^c u_j)^m \wedge \beta^{n-m}.
\]
Thus, the sequence \((R_{m,u_j}(E))_j\) decreases to \(R_{m,u}(E)\) and

\[
\sup_j \int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u_j}(E))^m \wedge \beta^{n-m} \leq \sup_j \int_{\Omega \times \mathbb{R}^n} (dd^# u_j)^m \wedge \beta^{n-m} < +\infty,
\]

which implies that \(R_{m,u}(E) \in \mathcal{F}_m(\Omega)\).

(4) Assume that \((R_{m,u_j}(E))_j\) decreases to \(\varphi \in \mathcal{C}_m^-(\Omega)\). We have \(R_{m,u}(E) \leq R_{m,u_j}(E)\) on \(\Omega\), then \(R_{m,u}(E) \leq \varphi\). Moreover, \(u_j = R_{m,u_j}(E)\) and \(u = R_{m,u}(E)\) on \(E\) outside an \(m\)-polar set, thus \(\varphi = R_{m,u}(E)\). \(\square\)

**Definition 5** For each Borel subset \(E \subset \Omega\) and \(u \in \mathcal{C}_m^-(\Omega)\), we define the weighted \(m\)-Hessian capacity associated to \(u\) of \(E\) by

\[
cap_{m,u}(E) = \sup \left\{ \int_K (dd^# v)^m \wedge \beta^{n-m}; \ K \subset E, \ v \in \mathcal{C}_m^-(\Omega) \cap L^\infty(\Omega) \text{ and } u \leq v \leq 0 \right\}.
\]

In particular, if \(u \equiv -1\) and \(E \Subset \Omega\), we recover the \(m\)-Hessian capacity investigated by Trindinger and Wang [23]. Moreover, thanks to Proposition 4, if \(E \Subset \Omega\) then the capacity \(\cap_{m,u}(E)\) is finite provided that \(u\) is locally bounded.

**Proposition 11** Assume that \(u_j, u \in \mathcal{C}_m^-(\Omega)\) such that \((u_j)_j\) is monotone decreasing to \(u\). Then, the sequence \((\cap_{m,u_j}(E))_j\) is monotone increasing to \(\cap_{m,u}(E)\), for all Borel subset \(E \subset \Omega\) such that \(\cap_{m,u_j}(E) < +\infty\).

**Proof** It is not hard to see that \((\cap_{m,u_j}(E))_j\) is an increasing sequence and \(\cap_{m,u_j}(E)\) increases to \(c \leq \cap_{m,u}(E)\) as \(j \to +\infty\). Let \(v \in \mathcal{C}_m^-(\Omega) \cap L^\infty(\Omega)\) such that \(u \leq v \leq 0\). By Theorem 4.3 in [15] and by using the subadditivity of \(\cap_{m,u}\), it is not difficult to realize that for each \(\varepsilon > 0\) there exists an open subset \(U \subset \Omega\) such that \(\cap_{m,u}(U, \Omega) < \varepsilon\) and \(u_j, u\) are continuous on \(\Omega \setminus U\), for each \(j\). Let us take an open subset \(\Omega_0\) such that \(E \subset \Omega_0 \Subset \Omega\). So, according to Dini’s theorem, \((u_j)_j\) uniformly converges to \(u\) on \(\Omega_0 \setminus U\). Thus, we can choose \(j_0\) such that \(u_{j_0} < (1 - \varepsilon)u \leq (1 - \varepsilon)v\) on \(\Omega_0 \setminus U\). Then,

\[
c \geq \cap_{m,u_{j_0}}(E) \geq \int_{E \times \mathbb{R}^n} (dd^# \max((1 - \varepsilon)v, u_{j_0}))^m \wedge \beta^{n-m}
\geq \int_{E \setminus U \times \mathbb{R}^n} (dd^# \max((1 - \varepsilon)v, u_{j_0}))^m \wedge \beta^{n-m}
= (1 - \varepsilon)^m \int_{E \setminus U \times \mathbb{R}^n} (dd^# v)^m \wedge \beta^{n-m}
\geq (1 - \varepsilon)^m \left( \int_{E \times \mathbb{R}^n} (dd^# v)^m \wedge \beta^{n-m} - \sup_U |v| \right)^m \cap_{m,u}(U, \Omega)
\geq (1 - \varepsilon)^m \int_{E \times \mathbb{R}^n} (dd^# v)^m \wedge \beta^{n-m} - (1 - \varepsilon)^m \varepsilon \left( \sup_U |v| \right)^m.
\]
Hence, by letting $\varepsilon \to 0$, we obtain

$$c \geq \int_{E \times \mathbb{R}^n} (dd^# v)^m \wedge \beta^{n-m}.$$

Thus, $c \geq \text{cap}_{m,u}(E)$, which leads to the conclusion that $c = \text{cap}_{m,u}(E)$. \hfill \Box

**Lemma 6** Let $u, v \in \mathcal{C}_m^-(\Omega) \cap L^\infty_{loc}(\Omega)$, then

$$(dd^# \max(u, v))^m \wedge \beta^{n-m} \geq \mathbb{I}_{[u \geq v]}(dd^# u)^m \wedge \beta^{n-m} + \mathbb{I}_{[v > u]}(dd^# v)^m \wedge \beta^{n-m}.$$  

**Proof** For all compact $K \subset \{u \geq v\}$, we have

$$\int_{K \times \mathbb{R}^n} (dd^# \max(u, v))^m \wedge \beta^{n-m} \geq \limsup_{\varepsilon \to 0} \int_{K \times \mathbb{R}^n} (dd^# (u + \varepsilon, v))^m \wedge \beta^{n-m} = \limsup_{\varepsilon \to 0} \int_{K \times \mathbb{R}^n} \mathbb{I}_{[u + \varepsilon \geq v]}(dd^# (u + \varepsilon, v))^m \wedge \beta^{n-m} = \limsup_{\varepsilon \to 0} \int_{K \times \mathbb{R}^n} (dd^# u)^m \wedge \beta^{n-m}.$$

The other term is then obtained by reversing the roles of $u$ and $v$. \hfill \Box

**Theorem 6** (Integration by parts) Let the functions $v, u_1, \ldots, u_m \in \mathcal{C}_m^-(\Omega)$ and denote by $T = dd^# u_2 \wedge \ldots \wedge dd^# u_m \wedge \beta^{n-m}$, then:

1. If $v, u_1 \in L^\infty_{loc}(\Omega)$ and $u = v$ outside a compact subset of $\Omega$, then

$$\int_{\Omega \times \mathbb{R}^n} vdd^# u_1 \wedge T = \int_{\Omega \times \mathbb{R}^n} u_1 dd^# v \wedge T.$$  

2. If $v, u_1 \in L^\infty_{loc}(\Omega)$, $v$ is an exhaustion function for $\Omega$ and $\int_{\Omega \times \mathbb{R}^n} dd^# v \wedge T < +\infty$, then

$$\int_{\Omega \times \mathbb{R}^n} vdd^# u_1 \wedge T \geq \int_{\Omega \times \mathbb{R}^n} u_1 dd^# v \wedge T.$$  

3. If $v \neq 0$, $\lim_{x \to y} v(x) = 0$, $\forall y \in \partial \Omega$ and $\int_{\Omega \times \mathbb{R}^n} u_1 dd^# v \wedge T > -\infty$, then

$$\int_{\Omega \times \mathbb{R}^n} vdd^# u_1 \wedge T \geq \int_{\Omega \times \mathbb{R}^n} u_1 dd^# v \wedge T,$$

and the equality will holds if we also assume that $\lim_{x \to y} u_1(x) = 0$, $\forall y \in \partial \Omega$.

Theorem 6 is the corresponding of Theorems 3.1 and 3.3 in [8] and Theorem 3.2 in [6].
Proof (1) We assume first that \( v, u_1, \ldots, u_m \) are defined in a neighborhood of \( \partial \Omega \), \( v \) and \( u_1 \) are smooth and \( v = u_1 \) in a neighborhood of \( \partial \Omega \). Let \( K \) be a compact subset of \( \Omega \) such that \( \{ x \in \Omega; u_1(x) \neq v(x) \} \subset K \). Let \( \chi \) be a smooth and compactly supported function such that \( 0 \leq \chi \leq 1 \) on \( \Omega \) and \( \chi \equiv 1 \) on \( K \). Then

\[
\int_{\Omega \times \mathbb{R}^n} \chi v dd^h u_1 \wedge T = \langle dd^h u_1 T, \chi v \rangle = \langle u_1 T, dd^h(\chi v) \rangle = \langle u_1 T, dd^h(\chi v) - \chi dd^h v \rangle + \langle u_1 T, \chi dd^h v \rangle.
\]

Note that \( dd^h(\chi v) - \chi dd^h v = v dd^h \chi + d \chi \wedge d^h v - d^h \chi \wedge dv \) is a smooth form whose support is contained in an open set of \( \{ u_1 = v \} \), thus

\[
\langle u_1 T, dd^h(\chi v) - \chi dd^h v \rangle = \langle v T, dd^h(\chi v) \rangle - \langle v T, \chi dd^h v \rangle = \langle dd^h v \wedge T, \chi v \rangle - \langle v T, \chi dd^h v \rangle = 0.
\]

Since \( v \) is smooth, we obtain

\[
\int_{\Omega \times \mathbb{R}^n} \chi v dd^h u_1 \wedge T = \int_{\Omega \times \mathbb{R}^n} \chi u_1 dd^h v \wedge T,
\]

and our result will follow by extending the support of \( \chi \) to be \( \Omega \). Let us consider now the general case. Let \( K \) be a compact subset of \( \Omega \) such that \( u_1 = v \) in \( \Omega \setminus K \) and choose \( \lambda > 0 \) such that \( K_{2\lambda} = \{ x; \text{dist}(x, K) \leq 2\lambda \} \) is contained in \( \Omega \). For \( \varepsilon > 0 \), let \( u_{1,\varepsilon} \) and \( v_{\varepsilon} \) are smooth regularizations of \( u \) and \( v \) respectively. Then, \( u_{1,\varepsilon} = v_{\varepsilon} \) outside \( K_{\lambda} \) if \( \varepsilon < \lambda \). Finally, we choose \( \delta_0 > 0 \) small enough such that that \( K_{2\lambda} \subset \Omega_{3\delta_0} = \{ x \in \Omega; \text{dist}(x, \partial \Omega) > \delta_0 \} \). For \( \delta < \delta_0 \), we have \( \max\{v(x); x \in \Omega_{\delta} \} < 0 \) and since \( v_\varepsilon \) is smooth and decreases to \( v \) as \( \varepsilon \to 0 \) it follows that \( v_\varepsilon < 0 \) on \( \Omega_{\delta} \), if \( \varepsilon \) is small enough. Let \( \chi \) be a smooth and compactly supported function such that \( 0 \leq \chi \leq 1 \) on \( \Omega \) and \( \chi \equiv 1 \) on \( K_{2\lambda} \). Then, by the first step, we have

\[
\int_{\Omega_{\delta} \times \mathbb{R}^n} \chi v_{\varepsilon} dd^h u_{1,\varepsilon} \wedge T = \int_{\Omega_{\delta} \times \mathbb{R}^n} \chi u_{1,\varepsilon} dd^h v_{\varepsilon} \wedge T,
\]

which implies, by letting \( \varepsilon \to 0 \) and using the weak convergence of \( v_{\varepsilon} dd^h u_{1,\varepsilon} \wedge T \) and \( u_{1,\varepsilon} dd^h v_{\varepsilon} \wedge T \) (see Theorem 2.6 in [23]), that

\[
\int_{\Omega_{\delta} \times \mathbb{R}^n} \chi vdd^h u_1 \wedge T = \int_{\Omega_{\delta} \times \mathbb{R}^n} \chi u_1 dd^h v \wedge T.
\]

Hence,

\[
\int_{\Omega_{\delta} \times \mathbb{R}^n} vdd^h u_1 \wedge T = \int_{\Omega_{\delta} \times \mathbb{R}^n} u_1 dd^h v \wedge T,
\]

and we conclude by sending \( \delta \) to \( 0 \).
(2) Let $\varepsilon > 0$, $k \in \mathbb{N}^*$ and $u_{1,k} = \max(u_1 - \varepsilon, kv)$. Then, $u_{1,k} \in \mathcal{C}_m^-(\Omega)$ and since $v$ is an exhaustion function for $\Omega$, we have $u_{1,k} = kv$ near $\partial\Omega$. Thus, by (1), we get

$$\int_{\Omega \times \mathbb{R}^n} v dd^# u_{1,k} \wedge T = \int_{\Omega \times \mathbb{R}^n} u_{1,k} dd^# v \wedge T.$$  

In $\Omega$, $u_{1,k}$ decreases to $u_1 - \varepsilon$ as $k \to +\infty$, and by the monotone convergence theorem, we have

$$\int_{\Omega \times \mathbb{R}^n} (u_1 - \varepsilon) dd^# v \wedge T = \lim_{k \to +\infty} \int_{\Omega \times \mathbb{R}^n} v dd^# u_{1,k} \wedge T.$$ 

Since $dd^# u_{1,k} \wedge T$ converges to $dd^# u_1 \wedge T$ as $k \to +\infty$, it follows that

$$\lim_{k \to +\infty} \int_{\Omega \times \mathbb{R}^n} \chi dd^# u_{1,k} \wedge T = \int_{\Omega \times \mathbb{R}^n} \chi dd^# u_1 \wedge T,$$

for every smooth and compactly supported function $\chi$ on $\Omega$ such that $0 \leq \chi \leq 1$ (see Theorem 2.6 in [23]). Note also that

$$\int_{\Omega \times \mathbb{R}^n} \chi dd^# u_{1,k} \wedge T \geq \int_{\Omega \times \mathbb{R}^n} dd^# u_{1,k} \wedge T.$$ 

Hence, we conclude that

$$\int_{\Omega \times \mathbb{R}^n} (u_1 - \varepsilon) dd^# v \wedge T \leq \int_{\Omega \times \mathbb{R}^n} \chi dd^# u_1 \wedge T.$$ 

Thus,

$$\int_{\Omega \times \mathbb{R}^n} (u_1 - \varepsilon) dd^# v \wedge T \leq \int_{\Omega \times \mathbb{R}^n} dd^# u_1 \wedge T,$$

and since the positive measure $dd^# v \wedge T$ is finite on $\Omega$, then the desired result will follows by letting $\varepsilon \to 0$.

(3) Assume that $v, u_1 \in \mathcal{C}(\overline{\Omega})$ and $v = u_1 = 0$ on $\partial\Omega$. Then, by (2), we have

$$-\infty < \int_{\Omega \times \mathbb{R}^n} u_1 dd^# v \wedge T$$

$$\leq \int_{\{v < -\varepsilon\} \times \mathbb{R}^n} u_1 dd^# v \wedge T \leq \int_{\{v < -\varepsilon\} \times \mathbb{R}^n} (v + \varepsilon) dd^# u_1 \wedge T.$$ 

Thus, if we denote by $1_{\varepsilon}$ the characteristic function of $\{v < -\varepsilon\}$, then $(v + \varepsilon) 1_{\varepsilon}$ decreases to $v$ when $\varepsilon \to 0$. Hence,

$$\int_{\Omega \times \mathbb{R}^n} u_1 dd^# v \wedge T \leq \int_{\Omega \times \mathbb{R}^n} dd^# u_1 \wedge T.$$
In the general case, we use Theorem 1.1 in [24] and choose \(v_j, u_{1,j} \in C^0_m(\Omega) \cap C(\Omega)\) such that \(v_j\) and \(u_{1,j}\) decrease respectively to \(v\) and \(u_1\) as \(j \to +\infty\). Then, by Theorem 2.6 in [23], we have

\[
\int_{\Omega \times \mathbb{R}^n} u_{1,j} d\# v \wedge T \leq \int_{\Omega \times \mathbb{R}^n} u_{1,k} d\# v \wedge T \\
\leq \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} u_{1,k} d\# v_j \wedge T \\
= \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} v_j d\# u_{1,k} \wedge T = \int_{\Omega \times \mathbb{R}^n} v d\# u_{1,k} \wedge T.
\]

Since \(v\) is negative and upper semi-continuous and \(d\# u_{1,k} \wedge T\) is weakly convergent to \(d\# u_1 \wedge T\), then for all \(\varepsilon > 0\), we get

\[
\int_{\Omega \times \mathbb{R}^n} u_{1,j} d\# v \wedge T \leq \lim_{k \to +\infty} \int_{\Omega \times \mathbb{R}^n} v d\# u_{1,k} \wedge T \\
\leq \lim_{k \to +\infty} \int_{\{v < -\varepsilon\} \times \mathbb{R}^n} v d\# u_{1,k} \wedge T \\
\leq \int_{\{v < -\varepsilon\} \times \mathbb{R}^n} v d\# u_1 \wedge T.
\]

Hence, by letting \(\varepsilon \to 0\), we have

\[
\int_{\Omega \times \mathbb{R}^n} u_{1,j} d\# v \wedge T \leq \int_{\Omega \times \mathbb{R}^n} v d\# u_1 \wedge T,
\]

which concludes our proof. \(\Box\)

**Theorem 7** Let \(u_1, ..., u_m \in F_m(\Omega)\) and \(u_1^j, ..., u_m^j\) are sequences of functions in \(C^0_m(\Omega)\) which decrease respectively to \(u_1, ..., u_m\). Then, for all \(v \in C^0_m(\Omega)\), we have

\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} v d\# u_1^j \wedge ... \wedge d\# u_m^j \wedge \beta^{n-m} = \int_{\Omega \times \mathbb{R}^n} v d\# u_1 \wedge ... \wedge d\# u_m \wedge \beta^{n-m}.
\]

Theorem 7 is the corresponding of Proposition 5.1 in [6].

**Proof** Assume first that

\[
\sup_{k,j} \int_{\Omega \times \mathbb{R}^n} (\beta^{n-m})^m \wedge \beta^{n-m} < +\infty. \quad (5.2)
\]

Then, according to Corollary 1.1 in [24], it is clear that

\[
\sup_j \int_{\Omega \times \mathbb{R}^n} d\# u_1^j \wedge ... \wedge d\# u_m^j \wedge \beta^{n-m} < +\infty. \quad (5.2)
\]
Thanks to Theorem 2.4 in [23], the sequence \( dd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m} \) is weakly convergent to \( dd^\# u_1 \wedge ... \wedge dd^\# u_m \wedge \beta^{n-m} \). Hence, we have

\[
+\infty \geq \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} dd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m} \geq \int_{\Omega \times \mathbb{R}^n} dd^\# u_1 \wedge ... \wedge dd^\# u_m \wedge \beta^{n-m}.
\]

If \( v \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\overline{\Omega}) \), then by Theorem 6, we have that

\[
\int_{\Omega \times \mathbb{R}^n} vdd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m} \text{ decreases.}
\]

Moreover, we have

\[
\int_{\Omega \times \mathbb{R}^n} vdd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m} \geq (\inf_v \sup_j) \int_{\Omega \times \mathbb{R}^n} dd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m} > -\infty.
\]

Consequently,

\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} vdd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m} \text{ exists.}
\]

Now, if \((w_1^j)_j, ..., (w_m^j)_j\) are another sequences which decrease respectively to \(u_1, ..., u_m\), then by using again Theorem 6, we get

\[
\int_{\Omega \times \mathbb{R}^n} vdd^\# w_1^j \wedge dd^\# w_2^j \wedge ... \wedge dd^\# w_m^j \wedge \beta^{n-m}
\]

\[
= \int_{\Omega \times \mathbb{R}^n} w_1^j dd^\# v \wedge dd^\# w_2^j \wedge ... \wedge dd^\# w_m^j \wedge \beta^{n-m}
\]

\[
\geq \int_{\Omega \times \mathbb{R}^n} u_1 dd^\# v \wedge dd^\# w_2^j \wedge ... \wedge dd^\# w_m^j \wedge \beta^{n-m}
\]

\[
= \lim_{j_1 \to +\infty} \int_{\Omega \times \mathbb{R}^n} u_1^{j_1} dd^\# v \wedge dd^\# w_2^j \wedge ... \wedge dd^\# w_m^j \wedge \beta^{n-m}
\]

\[
= \lim_{j_1 \to +\infty} \int_{\Omega \times \mathbb{R}^n} w_2^j dd^\# v \wedge dd^\# u_1^{j_1} \wedge dd^\# w_3^j \wedge ... \wedge dd^\# w_m^j \wedge \beta^{n-m}
\]

\[
\geq ... \geq \lim_{j_1 \to +\infty} ... \lim_{j_m \to +\infty} \int_{\Omega \times \mathbb{R}^n} vdd^\# u_1^{j_1} \wedge ... \wedge dd^\# u_m^{j_m} \wedge \beta^{n-m}
\]

\[
\geq \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} vdd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m}.
\]

Thus, \( \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} vdd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m} \) exists and minorized by

\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} vdd^\# u_1^j \wedge ... \wedge dd^\# u_m^j \wedge \beta^{n-m}.
\]
But this is a symmetric situation, then we conclude that the limits are equal, and then
\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} v dd^# u_1^j \wedge \ldots \wedge dd^# u_m^j \wedge \beta^{n-m} = \int_{\Omega \times \mathbb{R}^n} v dd^# u_1 \wedge \ldots \wedge dd^# u_m \wedge \beta^{n-m}.
\]

Let us now remove the restriction (5.2). For this aim, we consider \((h_1^j)_j, \ldots, (h_m^j)_j\) are sequences in \(E_0^m(\Omega)\) which decrease respectively to \(u_1, \ldots, u_m\) and such that
\[
\sup_{k,j} \int_{\Omega \times \mathbb{R}^n} (dd^# h_k^j)^m \wedge \beta^{n-m} < +\infty.
\]

By setting \(g_k^j = \max(h_k^j, u_k^j)\) and using the implication (5.1), we get
\[
\sup_{k,j} \int_{\Omega \times \mathbb{R}^n} (dd^# g_k^j)^m \wedge \beta^{n-m} < +\infty.
\]

Therefore, by the above arguments of integration by parts, it is not hard to see that
\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} v dd^# u_1^j \wedge \ldots \wedge dd^# u_m^j \wedge \beta^{n-m} \leq \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} v dd^# g_{1,j} \wedge \ldots \wedge dd^# g_{m,j} \wedge \beta^{n-m} = \int_{\Omega \times \mathbb{R}^n} v dd^# u_1 \wedge \ldots \wedge dd^# u_m \wedge \beta^{n-m} \leq \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} v dd^# u_1^j \wedge \ldots \wedge dd^# u_m^j \wedge \beta^{n-m}.
\]

The last inequality because the sequence \(v dd^# u_1^j \wedge \ldots \wedge dd^# u_m^j \wedge \beta^{n-m}\) converge weakly to \(v dd^# u_1 \wedge \ldots \wedge dd^# u_m \wedge \beta^{n-m}\). Thus,
\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} v dd^# u_1^j \wedge \ldots \wedge dd^# u_m^j \wedge \beta^{n-m} = \int_{\Omega \times \mathbb{R}^n} v dd^# u_1 \wedge \ldots \wedge dd^# u_m \wedge \beta^{n-m}.
\]

Assume now that \(v \in \mathcal{C}_m^-(\Omega)\) and
\[
- \int_{\Omega \times \mathbb{R}^n} v dd^# u_1 \wedge \ldots \wedge dd^# u_m \wedge \beta^{n-m} < +\infty.
\]

For each \(j\), we choose \(v_j \in \mathcal{C}_m^0(\Omega) \cap \mathcal{C}(\Omega)\) decreasing to \(v\), \(q_j\) and \(s_j\) such that
\[
- \int_{\Omega \times \mathbb{R}^n} v dd^# u_1 \wedge \ldots \wedge dd^# u_m \wedge \beta^{n-m}
\]
\[
\begin{align*}
&\leq \frac{1}{j} - \int_{\Omega \times \mathbb{R}^n} v_j \dd^j u_1 \wedge \ldots \wedge \dd^j u_m \wedge \beta^{n-m} \\
&\leq \frac{2}{j} - \int_{\Omega \times \mathbb{R}^n} v_j \dd^j u_1^q \wedge \ldots \wedge \dd^j u_m^q \wedge \beta^{n-m} \\
&\leq \frac{2}{j} - \int_{\Omega \times \mathbb{R}^n} v \dd^j u_1^q \wedge \ldots \wedge \dd^j u_m^q \wedge \beta^{n-m} \\
&\leq \frac{4}{j} - \int_{\Omega \times \mathbb{R}^n} v_s \dd^j u_1^q \wedge \ldots \wedge \dd^j u_m^q \wedge \beta^{n-m} \\
&\leq \frac{4}{j} - \int_{\Omega \times \mathbb{R}^n} v \dd^j u_1 \wedge \ldots \wedge \dd^j u_m \wedge \beta^{n-m}.
\end{align*}
\]

Letting \( j \to +\infty \), we get
\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} \dd^j u_1^j \wedge \ldots \wedge \dd^j u_m^j \wedge \beta^{n-m} = \int_{\Omega \times \mathbb{R}^n} \dd^j u_1 \wedge \ldots \wedge \dd^j u_m \wedge \beta^{n-m}.
\]

Also, if \( \int_{\Omega \times \mathbb{R}^n} \dd^j u_1 \wedge \ldots \wedge \dd^j u_m \wedge \beta^{n-m} = -\infty \), then
\[
\lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} \dd^j u_1^j \wedge \ldots \wedge \dd^j u_m^j \wedge \beta^{n-m} = -\infty,
\]
and our proof is completed. \( \square \)

Finally, we state our main result in this section, which is the corresponding of Theorem 4.6 in [19] in the setting of \( m \)-convex functions:

**Theorem D** Let \( u \in C_m^-(\Omega) \), then
\[
cap_{m,u}(E) = \int_{E \times \mathbb{R}^n} (\dd R_{m,u}(E))^m \wedge \beta^{n-m},
\]
for all Borel compact subset \( E \) of \( \Omega \).

We must highlight an important point here, which is that Theorem D is a generalization of Lemma 4.6 in [15].

**Proof** First, assume that \( u \in \mathcal{E}_m^0(\Omega) \). Since \( u \leq R_{m,u}(E) \leq 0 \) and in view of implication (5.1), we have \( R_{m,u}(E) \in \mathcal{E}_m^0(\Omega) \). It follows by the definition of \( \cap_{m,u} \) that
\[
\cap_{m,u}(E) \geq \int_{E \times \mathbb{R}^n} (\dd R_{m,u}(E))^m \wedge \beta^{n-m}.
\]

Conversely, let \( v \in \mathcal{E}_m^-(\Omega) \cap L^\infty(\Omega) \) such that \( u \leq v \leq 0 \). Since \( u \in \mathcal{E}_m^0(\Omega) \), then \( v \in C_m(\Omega) \). Moreover, \( u = R_{m,u}(E) \) outside an \( m \)-polar subset on \( E \), so that...
\( R_{m,u}(E) \leq v \) outside an \( m \)-polar subset on \( E \). Since functions from \( \mathcal{E}_m^0(\Omega) \) put no mass on \( m \)-polar sets, then it follows from Lemma 6, the implication (5.1) and Theorem 5 that

\[
\int_{E \times \mathbb{R}^n} (dd^# v)^m \wedge \beta^{n-m} \leq \int_{E \times \mathbb{R}^n} (dd^# \max(R_{m,u}(E), v))^m \wedge \beta^{n-m} \\
\leq \int_{\Omega \times \mathbb{R}^n} (dd^# \max(R_{m,u}(E), v))^m \wedge \beta^{n-m} \\
\leq \int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m} \\
= \int_{E \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m}.
\]

Then,

\[
\text{cap}_{m,u}(E) = \int_{E \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m}, \quad \forall u \in \mathcal{E}_m^0(\Omega).
\]

In the general case, if \( u \in \mathcal{C}_m^{-}(\Omega) \) there exists a decreasing sequence \( (u_j)_j \subset \mathcal{E}_m^0(\Omega) \) such that \( u_j \) decreases to \( u \) as \( j \to +\infty \). By the first step, we have

\[
\text{cap}_{m,u_j}(E) = \int_{E \times \mathbb{R}^n} (dd^# R_{m,u_j}(E))^m \wedge \beta^{n-m}, \quad \forall j. \tag{5.3}
\]

On the one hand, Proposition 10 (4) implies that \( R_{m,u_j}(E) \) decreases to \( R_{m,u}(E) \) and thanks to Theorem 7, we have

\[
\lim_{j \to +\infty} \int_{E \times \mathbb{R}^n} (dd^# R_{m,u_j}(E))^m \wedge \beta^{n-m} = \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u_j}(E))^m \wedge \beta^{n-m} \\
= \int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m} \\
= \int_{E \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m} < +\infty.
\]

On the other hand, Proposition 11 implies that \( \text{cap}_{m,u_j}(E) \) increases to \( \text{cap}_{m,u}(E) \). Thus, we obtain the desired result by letting \( j \to +\infty \) in (5.3).

\[\blacksquare\]

**Corollary 2** Let \( u \in \mathcal{C}_m^{-}(\Omega) \), then:

1. If \( E \) is a Borel compact subset of \( \Omega \), then \( \text{cap}_{m,u}(E) < +\infty \).
2. If \( u \in \mathcal{F}_m(\Omega) \), then

\[
\int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m} \leq \text{cap}_{m,u}(E) \leq \int_{\Omega \times \mathbb{R}^n} (dd^# u)^m \wedge \beta^{n-m},
\]

for all open subset \( E \) of \( \Omega \). In particularly, if \( E = \Omega \), then

\[
\text{cap}_{m,u}(\Omega) = \int_{\Omega \times \mathbb{R}^n} (dd^# u)^m \wedge \beta^{n-m}.
\]
Proof (1) Assume that \( E \) is a Borel compact subset of \( \Omega \). In spite of Proposition 10, we have \( R_{m,u}(E) \in \mathcal{F}_m(\Omega) \). Thus, Theorem D and the second statement of Proposition 7 imply that
\[
cap_{m,u}(E) = \int_{E \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m} < +\infty.
\]

(2) Assume that \( u \in \mathcal{E}^0_m(\Omega) \) and let \( v \in \mathcal{C}_m(\Omega) \) such that \( u \leq v \leq 0 \). In view of Proposition 7 we have \( \max(u, v) = v \in \mathcal{E}^0_m(\Omega) \), and therefore by (5.1) we obtain
\[
\int_K \times \mathbb{R}^n (dd^# v)^m \wedge \beta^{n-m} \leq \int_{\Omega \times \mathbb{R}^n} (dd^# v)^m \wedge \beta^{n-m} \leq \int_{\Omega \times \mathbb{R}^n} (dd^# u)^m \wedge \beta^{n-m},
\]
for every compact \( K \subset E \). By taking the supremum over all \( v \) and over all \( K \), we get
\[
cap_{m,u}(E) \leq \int_{\Omega \times \mathbb{R}^n} (dd^# u)^m \wedge \beta^{n-m}.
\]

In order to prove the other inequality, let \((K_j)_j\) be an exhaustive sequence of compact subsets of \( E \) (\( K_j \subset K_{j+1} \) and \( \bigcup_j K_j = E \)). We claim that \( R_{m,u}(K_j) \) decreases to \( R_{m,u}(E) \). Indeed, it is clear that \( R_{m,u}(K_j) \) decreases to \( w \geq R_{m,u}(E) \), for some \( w \in \mathcal{C}_m(\Omega) \). To show that \( R_{m,u}(E) \leq w \), we see by definition of the weighted capacity that for each \( j \), there exists an \( m \)-polar subset \( F_j \subset K_j \) such that \( R_{m,u}(K_j) = u \) on \( K_j \setminus F_j \). Hence, for \( F = \bigcup_j F_j \), it is clear that \( w = u \) on \( E \setminus F \) and therefore \( w \leq R_{m,u}(E) \). Thus, by Theorem 7 and Theorem D, we get
\[
\int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u}(E))^m \wedge \beta^{n-m} = \lim_{j \to +\infty} \int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u}(K_j))^m \wedge \beta^{n-m} = \lim_{j \to +\infty} \cap_{m,u}(K_j) \leq \cap_{m,u}(E).
\]

Now, consider \((u_j)_j \subset \mathcal{E}^0_m(\Omega)\) such that \( u_j \) decreases to \( u \) and
\[
\sup_j \int_{\Omega \times \mathbb{R}^n} (dd^# u_j)^m \wedge \beta^{n-m} < +\infty.
\]

By the preceding argument, for each \( j \), we have
\[
\int_{\Omega \times \mathbb{R}^n} (dd^# R_{m,u_j}(E))^m \wedge \beta^{n-m} \leq \cap_{m,u_j}(E) \leq \int_{\Omega \times \mathbb{R}^n} (dd^# u_j)^m \wedge \beta^{n-m}.
\]

In order to complete the proof, it suffices to apply Proposition 11 and Theorem 7, by taking into account the fact that \( R_{m,u_j}(E) \in \mathcal{E}^0_m(\Omega) \) decreases to \( R_{m,u}(E) \in \mathcal{F}_m(\Omega) \).

\( \Box \)
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