Existence of solutions of a two-dimensional boundary value problem for a system of nonlinear equations arising in growing cell populations

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In the paper [A. Ben Amar, A. Jeribi, and B. Krichen, Fixed point theorems for block operator matrix and an application to a structured problem under boundary conditions of Rotenberg’s model type, to appear in Math. Slovaca. (2014)], the existence of solutions of the two-dimensional boundary value problem (1) and (2) was discussed in the product Banach space $L_p \times L_p$ for $p \in (1, \infty)$. Due to the lack of compactness on $L_1$ spaces, the analysis did not cover the case $p = 1$. The purpose of this work is to extend the results of Ben Amar et al. to the case $p = 1$ by establishing new variants of fixed-point theorems for a $2 \times 2$ operator matrix, involving weakly compact operators.

Keywords: operator matrix; fixed-point theory; weak compactness; growing cell populations

1. Introduction

In this paper, we are concerned with the existence of solutions for the following two-dimensional boundary value problem introduced in [4]:

$$\begin{pmatrix} \frac{-v}{\partial \mu} - \sigma_1(\mu, v, \cdot) & R_{12} \\ R_{21} & \frac{-v}{\partial \mu} - \sigma_2(\mu, v, \cdot) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

(1)

where $R_{ij}(\mu, v) = \int_a^b r_{ij}(\mu, v, v', \psi_j(\mu, v')) \, dv'$, $(i, j) \in \{(1, 2), (2, 1)\}$, $\mu \in [0, 1]$, $v, v' \in [a, b]$ with $0 \leq a < b < \infty$. The functions $\sigma_i(\cdot, \cdot, \cdot)$ and $r_{ij}(\cdot, \cdot, \cdot, \cdot)$ are nonlinear and $\lambda$ is a complex number.
The boundary conditions are given by

\[ \psi_{i|\Gamma_0} = K_i(\psi_{i|\Gamma_1}), \quad i = 1, 2, \] (2)

where \( \Gamma_0 = \{0\} \times [a, b] \) and \( \Gamma_1 = \{1\} \times [a, b] \). We denote by \( \psi_{i|\Gamma_0} \) (resp. \( \psi_{i|\Gamma_1} \)) the restriction of \( \psi_i \) to \( \Gamma_0 \) (resp. \( \Gamma_1 \)), while \( K_i \) are nonlinear operators from a suitable function space on \( \Gamma_1 \) to a similar one on \( \Gamma_0 \). The main point in Equation (1) of this model is the nonlinear dependence of the functions \( r_{ij}(\mu, v, v', \psi_j(\mu, v')) \) on \( \psi_j \). More specifically, we suppose that

\[ r_{ij}(\mu, v, v', \psi_j(\mu, v')) = k_{ij}(\mu, v, v')f(\mu, v', \psi_j(\mu, v')) \], \((i, j) \in \{(1, 2), (2, 1)\},\)

where \( f \) is a measurable function defined by

\[ f: [0, 1] \times [a, b] \times C \rightarrow C, \]

\[ (\mu, v, u) \rightarrow f(\mu, v, u), \]

with \( k_{ij}(\cdot, \cdot, \cdot) \) are measurable functions from \([0, 1] \times [a, b] \times C\) to \( C \).

Rotenberg [17] proposed the singular partial differential equation:

\[ v \frac{\partial \psi}{\partial \mu}(\mu, v) + \sigma(\mu, v)\psi(\mu, v) + \lambda \psi(\mu, v) - \int_{a}^{b} r(\mu, v', \psi(\mu, v')) dv' = 0, \] (3)

which models the evolution of a cell population. Each cell is distinguished by two parameters, the degree of maturity \( \mu \) and the velocity \( v \).

Latrach and Jeribi [14] examined the existence of Equation (3) supplemented with the boundary conditions

\[ \psi_{|\Gamma_0} = K(\psi_{|\Gamma_1}). \] (4)

Boulanouar [5] studied the one-dimensional cell proliferating model with linear boundary conditions (3) and (4), which generalize the known biological rules and proved that this model is governed by a strongly continuous semigroup.

Recently, Ben Amar et al. [4] proved an existence result for integrable solutions of the two-dimensional boundary problem (1) and (2) which were obtained in the Banach space \( L_p \times L_p \) for \( p \in (1, \infty) \). The analysis was carried out via topological arguments and uses the compactness results established in [14] for a one-dimensional transport equation and the Schauder and Krasnoselskii fixed-point theorems [13,18]. The purpose of this work is to continue this analysis in the Banach space \( L_1 \times L_1 \), due to the lack of compactness in \( L_1 \) spaces, by using the results concerning the weak compactness in [3]. Our strategy consists in establishing fixed-point theorems for a \( 2 \times 2 \) operator matrix on the general product Banach spaces which can be applied directly to solve our problem.

Note that the boundary value problem (1) and (2) may be transformed into the following fixed-point problem:

\[ \mathcal{L} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]

where

\[ \mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \] (5)

is a \( 2 \times 2 \) block operator matrix defined on the product Banach space \( \mathcal{X} \times \mathcal{Y} \).

The outline of the paper is as follows. In Section 2, we recall some definitions and give basic results for future use. In Section 3, we establish some fixed-point results for the operator matrix \( \mathcal{L} \) (5). In Section 4, we use the results of Section 3 to derive the existence of
solutions to problem (1) and (2) in the Banach space \( L_1 \times L_1 \). In Theorem 4.1, we consider the special case where each \( \sigma_i \) does not depend on the density of the population \( i \), that is, 
\[
\sigma_i(\mu, v, \psi_1(\mu, v)) = \sigma_i(\mu, v)\psi_1(\mu, v), \quad i = 1, 2.
\]
The general boundary value problem (1) and (2) (i.e. \( \sigma_i(\cdot, \cdot, \cdot) \) is a nonlinear function of \( \psi_1(\cdot, \cdot) \)) is discussed in Theorem 4.2.

2. Preliminaries

Throughout this section, \( \mathcal{X} \) denotes a Banach space. For any \( r > 0 \), \( B_r \) denotes the closed ball in \( \mathcal{X} \) centred at \( 0 \) with radius \( r \). Here \( \rightharpoonup \) denotes weak convergence and \( \rightarrow \) denotes strong convergence in \( \mathcal{X} \), respectively.

\( \Omega_\mathcal{X} \) is the collection of all nonempty bounded subsets of \( \mathcal{X} \) and \( \mathcal{K}_w \) is the subset of \( \Omega_\mathcal{X} \) consisting of all weakly compact subsets of \( \mathcal{X} \). Recall that the notion of the measure of weak noncompactness was introduced by De Blasi [11]; it is the map \( \omega : \Omega_\mathcal{X} \rightarrow [0, +\infty) \) defined in the following way:

\[
\omega(\mathcal{M}) = \inf \{ r > 0 : \text{there exists } K \in \mathcal{K}_w \text{ such that } \mathcal{M} \subseteq K + B_r \},
\]
for all \( \mathcal{M} \in \Omega_\mathcal{X} \). For convenience, we recall some basic properties of \( \omega(\cdot) \) needed below [2,11].

**Lemma 2.1** Let \( \mathcal{M}_1, \mathcal{M}_2 \) be two elements of \( \Omega_\mathcal{X} \). Then, the following conditions are satisfied:

1. \( \mathcal{M}_1 \subseteq \mathcal{M}_2 \) implies \( \omega(\mathcal{M}_1) \leq \omega(\mathcal{M}_2) \).
2. \( \omega(\mathcal{M}_1) = 0 \) if and only if \( \overline{\mathcal{M}_1}^w \in \mathcal{K}_w \), that is, \( \overline{\mathcal{M}_1}^w \) is the weak closure of \( \mathcal{M}_1 \).
3. \( \omega(\overline{\mathcal{M}_1}^w) = \omega(\mathcal{M}_1) \).
4. \( \omega(\mathcal{M}_1 \cup \mathcal{M}_2) = \max\{\omega(\mathcal{M}_1), \omega(\mathcal{M}_2)\} \).
5. \( \omega(\lambda \mathcal{M}_1) = |\lambda|\omega(\mathcal{M}_1) \) for all \( \lambda \in \mathbb{R} \).
6. \( \omega(\text{co}(\mathcal{M}_1)) = \omega(\mathcal{M}_1) \), that is, \( \text{co}(\mathcal{M}_1) \) is the convex hull of \( \mathcal{M}_1 \).
7. \( \omega(\mathcal{M}_1 + \mathcal{M}_2) \leq \omega(\mathcal{M}_1) + \omega(\mathcal{M}_2) \).
8. if \( (\mathcal{M}_n)_{n \geq 1} \) is a decreasing sequence of nonempty bounded and weakly closed subsets of \( \mathcal{X} \) with \( \lim_{n \rightarrow \infty} \omega(\mathcal{M}_n) = 0 \), then \( \mathcal{M}_\infty := \cap_{n=1}^{\infty} \mathcal{M}_n \) is nonempty and \( \omega(\mathcal{M}_\infty) = 0 \), that is, \( \mathcal{M}_\infty \) is relatively weakly compact.

**Definition 2.1** A map \( A : \mathcal{M} \subseteq \mathcal{X} \rightarrow \mathcal{X} \) is said to be weakly compact, if \( A\mathcal{M} \) is relatively weakly compact for every bounded subset \( \mathcal{M} \subseteq \mathcal{X} \).

**Definition 2.2** A map \( A : \mathcal{M} \subseteq \mathcal{X} \rightarrow \mathcal{X} \) is said to be \( \omega \)-contractive (or \( \omega \)-contraction) if it maps bounded sets into bounded sets, and there exists some \( \alpha \in [0, 1) \) such that \( \omega(A\mathcal{N}) \leq \alpha \omega(\mathcal{N}) \) for all bounded subsets \( \mathcal{N} \subseteq \mathcal{M} \).

**Definition 2.3** A map \( A : \mathcal{X} \rightarrow \mathcal{X} \) is said to be weakly--strongly sequentially continuous if for every sequence \( (x_n)_{n \in \mathbb{N}}, x_n \rightarrow x \) implies \( Ax_n \rightarrow Ax \).

Let \( A \) be a nonlinear operator from \( \mathcal{X} \) into itself. Following Latrach et al. [15], we introduce the following conditions:

\[
(\mathcal{A}1) \begin{cases} 
\text{If } (x_n)_{n \in \mathbb{N}} \text{ is a weakly convergent sequence in } \mathcal{X}, \text{ then } \\
(Ax_n)_{n \in \mathbb{N}} \text{ has a strongly convergent subsequence in } \mathcal{X}.
\end{cases}
\]

\[
(\mathcal{A}2) \begin{cases} 
\text{If } (x_n)_{n \in \mathbb{N}} \text{ is a weakly convergent sequence in } \mathcal{X}, \text{ then } \\
(Ax_n)_{n \in \mathbb{N}} \text{ has a weakly convergent subsequence in } \mathcal{X}.
\end{cases}
\]

Regarding these two conditions, Latrach et al. [15, Remark 2.1] noted the following.
Remark 2.1
(a) Operators satisfying \((A1)\) or \((A2)\) are not necessarily weakly continuous.
(b) Every \(\omega\)-contractive map satisfies \((A2)\).
(c) A map \(A\) satisfies \((A2)\) if and only if it maps relatively weakly compact sets into relatively weakly compact ones (use the Eberlein–Šmulian theorem [12, p. 430]).
(d) A map \(A\) satisfies \((A1)\) if and only if it maps relatively weakly compact sets into relatively compact ones.
(e) The condition \((A2)\) holds true for every bounded linear operator.

Moreover, note that \((A1)\) is weaker than the weakly–strongly sequentially continuity of the operator \(A\) [1]. Now, we shall recall the following well-known results in [15].

**Theorem 2.1** Let \(M\) be a nonempty closed convex subset of a Banach space \(X\). Assume that \(A : M \to M\) is a continuous map which verifies \((A1)\). If \(AM\) is relatively weakly compact, then there exists \(x \in M\) such that \(Ax = x\).

**Theorem 2.2** Let \(M\) be a nonempty bounded closed convex subset of a Banach space \(X\). Assume that \(A : M \to M\) is a continuous map satisfying \((A1)\). If \(A\) is \(\omega\)-contractive, then there exists \(x \in M\) such that \(Ax = x\).

**Remark 2.2** Assume that a mapping \(A : X \to X\) is a contraction and satisfies \((A2)\), then \(A\) is \(\omega\)-contractive.

**Theorem 2.3** Let \(M\) be a nonempty closed bounded convex subset of a Banach space \(X\). Suppose that \(A : M \to X\) and \(B : X \to X\) such that

(i) \(A\) is continuous, \(AM\) is relatively weakly compact and \(A\) satisfies \((A1)\),
(ii) \(B\) is a contraction satisfying \((A2)\),
(iii) \(AM + BM \subseteq M\).

Then, there is \(x \in M\) such that \(Ax + Bx = x\).

3. Fixed-point theory

Let \(M_1\) and \(M_2\) be closed convex nonempty subsets of two Banach spaces \(X\) and \(Y\). We consider the \(2 \times 2\) block operator matrix \(L(5)\) defined on the Banach space \(X \times Y\), that is, the nonlinear operator \(A\) maps \(M_1\) into \(X\), \(B\) from \(M_2\) into \(X\), \(C\) from \(M_1\) into \(Y\) and \(D\) from \(M_2\) into \(Y\).

Our aim is to develop a general matrix fixed-point theory which allows to treat the biological application described in the introduction. In the following, we discuss the existence of fixed points for the block operator matrix \(L(5)\) by imposing some conditions on the entries, which are in general nonlinear operators. This discussion is based on the invertibility or not of the diagonal terms of \(I - L\).

First case: \(I - A\) and \(I - D\) are invertible.

Assume that

\((J1)\) the operator \(I - A\) is invertible and \((I - A)^{-1}BM_2 \subseteq M_1\);
\((J2)\) \((I - A)^{-1}B\) is a operator continuous satisfying \((A1)\) and \((I - A)^{-1}BM_2\) is relatively weakly compact;
\((J3)\) \(C\) is a operator continuous satisfying \((A2)\);
(J4) the operator $I - D$ is invertible and their inverse $(I - D)^{-1}$ is continuous on $(I - D)\mathcal{M}_2$ and satisfies (A$_2$);

(J5) $(I - D)^{-1}C(I - A)^{-1}BM_2 \subseteq \mathcal{M}_2$.

**Theorem 3.1** Under assumptions (J1)–(J5) the block matrix operator (5) has a fixed point in $\mathcal{M}_1 \times \mathcal{M}_2$.

**Proof** Let $\Gamma$ be the operator defined by $\Gamma := (I - D)^{-1}C(I - A)^{-1}B : \mathcal{M}_2 \to \mathcal{M}_2$.

In order to prove the theorem, we have to check that

1. $\Gamma \mathcal{M}_2$ is relatively weakly compact.

For this, let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\Gamma \mathcal{M}_2$, there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}_2$ such that $y_n = \Gamma x_n$ for all $n \in \mathbb{N}$, because $((I - A)^{-1}Bx_n)_{n \in \mathbb{N}} \subseteq (I - A)^{-1}BM_2$ and $(I - A)^{-1}BM_2$ is relatively weakly compact then, by Eberlein–Šmulian theorem [12], $((I - A)^{-1}Bx_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence. On the other hand, using the fact that $C$ and $(I - D)^{-1}$ verify (A$_2$), the sequence $(y_n)_{n \in \mathbb{N}}$ has also a weak converging subsequence, it follows that $\Gamma \mathcal{M}_2$ is relatively weakly compact.

2. $\Gamma$ satisfies the condition (A$_1$).

Let $(x_n)_{n \in \mathbb{N}}$ be a weakly convergent sequence of $\mathcal{M}_2$, since $(I - A)^{-1}B$ satisfies (A$_1$) and $(I - A)^{-1}Bx_n)_{n \in \mathbb{N}}$ has a strongly convergent subsequence. By the continuity of the operator $C$ and $(I - D)^{-1}$, $(\Gamma x_n)_{n \in \mathbb{N}}$ has also a strongly convergent subsequence, that is, $\Gamma$ satisfies (A$_1$).

Clearly, $\Gamma$ is continuous; consequently, $\Gamma$ satisfies the hypotheses of Theorem 2.1 as we claimed, and there exists $y_0 \in \mathcal{M}_2$ such that

$$\Gamma y_0 = y_0.$$

Let $x_0 := (I - A)^{-1}By_0$, hence $L\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. \hfill \blacksquare

In the other cases, we will assume furthermore that $\mathcal{M}_1$ and $\mathcal{M}_2$ are bounded.

**Second case:** $I - A$ or $I - D$ is invertible.

We shall treat only the case of invertibility of $I - A$, the other case is similar just simply exchanging the roles of $A$ and $D$ and $B$ and $C$.

Assume that

(K1) the operator $I - A$ is invertible and $(I - A)^{-1}BM_2 \subseteq \mathcal{M}_1$;

(K2) $S := C(I - A)^{-1}B$ is a contraction satisfying (A2) with constant $k$;

(K3) $D$ is a continuous operator satisfying (A1) and $\omega$-$\alpha$-contractive for some $\alpha \in [0, 1 - k)$;

(K4) $S\mathcal{M}_2 + D\mathcal{M}_2 \subseteq \mathcal{M}_2$.

**Theorem 3.2** Under assumptions (K1)–(K4), the block matrix operator (5) has a fixed point in $\mathcal{M}_1 \times \mathcal{M}_2$.

**Proof** Since $S$ is a contraction with a constant $k \in (0, 1)$, the mapping $I - S$ is a homeomorphism from $\mathcal{M}_2$ into $(I - S)\mathcal{M}_2$ [18]. Let $y'$ be fixed in $\mathcal{M}_2$, the map which assigns to each $y \in \mathcal{M}_2$ the value $Sy + Dy'$ defines a contraction from $\mathcal{M}_2$ into $\mathcal{M}_2$. Therefore, by the Banach fixed-point theorem, the equation $y = Sy + Dy'$ has a unique solution $y = (I - S)^{-1}Dy'$ in $\mathcal{M}_2$. Therefore,

$$(I - S)^{-1}D\mathcal{M}_2 \subseteq \mathcal{M}_2.$$ 

Next, we will prove that $T := (I - S)^{-1}D$ satisfies the conditions of Theorem 2.2. It is clear that $T$ is continuous and satisfies (A$_1$). Now, we check that $T$ is $\omega$-$\beta$-contractive for some $\beta \in [0, 1)$.

To do so, let $\mathcal{N}$ be a subset of $\mathcal{M}_2$. Using the following equality:

$$(I - S)^{-1}D = D + S(I - S)^{-1}D,$$
we infer that
\[ \omega(TN) = \omega(DN + STN). \]
The properties of \( \omega(\cdot) \) in Lemma 2.1 and the assumptions on \( S \) and \( D \) imply that
\[ \omega(TN) \leq \omega(DN) + \omega(STN) \leq \alpha \omega(N) + k \omega(TN), \]
and therefore,
\[ \omega(TN) \leq \frac{\alpha}{1 - k} \omega(N). \]
This inequality means that \( T \) is \( \omega \)-\( \beta \)-contractive with \( \beta := \alpha/(1 - k) \).
Consequently, \( T \) satisfies the hypotheses of Theorem 2.2 as we claimed, and hence, such operator has a fixed point in \( M_2 \), so the operator matrix (5) has at least a fixed point in \( M_1 \times M_2 \).

Third case: Neither \( I - A \) nor \( I - D \) is invertible.
Here, we discuss the existence of fixed points for the following perturbed block operator matrix by imposing some conditions on the entries:
\[
\tilde{L} = \begin{pmatrix} A_1 & B \\ C & D_1 \end{pmatrix} + \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.
\] (6)
Assume that the nonlinear operators \( A_1 \) and \( P_1 \) maps \( M_1 \) into \( \mathcal{X} \), \( B \) from \( M_2 \) into \( \mathcal{X} \), \( C \) from \( M_1 \) into \( \mathcal{Y} \) and \( D_1 \) and \( P_2 \) from \( M_2 \) into \( \mathcal{Y} \). Suppose that Equation (6) fulfills the following assumptions:

\( (F1) \) The operator \( I - A_1 \) (resp. \( I - D_1 \)) is invertible from \( M_1 \) into \( \mathcal{X} \) (resp. from \( M_2 \) into \( \mathcal{Y} \)).
\( (F2) \) \( (I - A_1)^{-1}B \) and \( (I - D_1)^{-1}C \) are continuous, weakly compact maps and verify (\( A_1 \)).
\( (F3) \) \( (I - A_1)^{-1}P_1 \) and \( (I - D_1)^{-1}P_2 \) are contraction maps and verify (\( A_2 \)).
\( (F4) \) \( (I - A_1)^{-1}P_1 M_1 + (I - A_1)^{-1}B M_2 \subseteq M_1 \) and \( (I - D_1)^{-1}C M_1 + (I - D_1)^{-1}P_2 M_2 \subseteq M_2 \).

**Theorem 3.3** Under assumptions (\( F1 \))–(\( F4 \)), the block matrix operator (6) has a fixed point in \( M_1 \times M_2 \).

**Proof** Using assumption (\( F1 \)), the following equation
\[
\begin{pmatrix} A_1 + P_1 & B \\ C & D_1 + P_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}
\]
may be transformed into
\[
Z_1 \begin{pmatrix} x \\ y \end{pmatrix} + Z_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},
\]
where
\[
Z_1 = \begin{pmatrix} (I - A_1)^{-1}P_1 & 0 \\ 0 & (I - D_1)^{-1}P_2 \end{pmatrix}
\]
and
\[
Z_2 = \begin{pmatrix} 0 & (I - A_1)^{-1}B \\ (I - D_1)^{-1}C & 0 \end{pmatrix}.
\]
Obviously, the operator matrix \( Z_2 \) is continuous. Now, we check that \( Z_2 \) is a weakly compact operator and satisfies \( A_1 \).
To see this, let \( ((x_n, y_n))_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{M}_1 \times \mathcal{M}_2 \); since \((I - A_1)^{-1}B \) is weakly compact, the sequence \(((I - A_1)^{-1}B y_n)_{n \in \mathbb{N}} \) has a weakly convergent subsequence, say \((I - A_1)^{-1}B y_{n_k} \) for \( k \in \mathbb{N} \). On the other hand, the sequence \((I - D_1)^{-1}C x_{n_k} \) has a weak converging subsequence \(((I - D_1)^{-1}C x_{n_k})_{k \in \mathbb{N}} \); hence, \( Z_2 \) is a weakly compact operator. Also, we show that the operator matrix \( Z_2 \) verifies \((A_1)\) and from \((F_3)\) the operator matrix \( Z_1 \) is a contraction that satisfies \((A_2)\).

It follows with \((F_4)\) and Theorem 2.3 that the operator matrix \((6)\) has at least a fixed point in \( \mathcal{M}_1 \times \mathcal{M}_2 \).

4. Application to transport equations

The aim of this section is to apply Theorems 3.1 and 3.3 to discuss existence results for the two-dimensional boundary value problem \((1)\) and \((2)\) in the Banach space \( L_1 \times L_1 \). To do so, let us first make precise the functional setting of the problem. Let \( X := L_1([0, 1] \times [a, b]; d\mu dv) \), where \( 0 \leq a < b \leq \infty \). We denote by \( X^0 \) and \( X^1 \) the following boundary spaces

\[
X^0 := L_1([0] \times [a, b]; v dv) \\
X^1 := L_1([1] \times [a, b]; v dv)
\]

endowed with their natural norms. Let \( \mathcal{W} \) be the space defined by

\[
\mathcal{W} = \left\{ \psi \in X \text{ such that } v \frac{\partial \psi}{\partial \mu} \in X \right\}.
\]

It is well known [7,8,10] that any \( \psi \) in \( \mathcal{W} \) has traces on the spatial boundary \( \{0\} \) and \( \{1\} \) which belong, respectively, to the spaces \( X^0 \) and \( X^1 \).

We define the free streaming operator \( S_{K_i}, i = 1, 2 \), by

\[
S_{K_i} : D(S_{K_i}) \subseteq X \longrightarrow X, \\
\psi_i \longrightarrow S_{K_i} \psi_i(\mu, v) = -v \frac{\partial \psi_i}{\partial \mu}(\mu, v) - \sigma_i(\mu, v) \psi_i(\mu, v), \\
D(S_{K_i}) = \{ \psi_i \in \mathcal{W} \text{ such that } \psi_i^0 = K_i(\psi_i^1) \},
\]

where \( \sigma_i(\cdot, \cdot) \in L^\infty([0, 1] \times [a, b]) \), \( \psi_i^0 = \psi_i|_{\Gamma_0}, \psi_i^1 = \psi_i|_{\Gamma_1} \) and \( K_i, i = 1, 2 \), are the following nonlinear boundary operators

\[
K_i : X^1 \longrightarrow X^0, \\
u \longrightarrow K_i u,
\]

satisfying the following conditions:

\((H1)\) There exists \( \alpha_i > 0 \) such that

\[
\|K_i \varphi_1 - K_i \varphi_2\| \leq \alpha_i \|\varphi_1 - \varphi_2\| \text{ for all } \varphi_1, \varphi_2 \in X^1, \quad i = 1, 2.
\]

\((H2)\) \( K_2 \) is a weakly compact operator on \( X^1 \).
As an immediate consequences of (H1) we have the continuity of the operator \( K_i \) from \( X^1 \) into \( X^0 \) and
\[
\|K_i \varphi\| \leq \alpha_i \|\varphi\| + \|K_i(0)\| \quad \text{for all } \varphi \in X^1.
\]
Let us consider the equation
\[
(\lambda - S_{K_i}) \psi_i = g.
\]
Our objective is to determine a solution \( \psi_i \in \mathcal{D}(S_{K_i}) \) where \( g \) is given in \( X \) and \( \lambda \in \mathbb{C} \). Let \( \sigma \) be real defined by
\[
\sigma := \text{ess- inf}\{\sigma_i(\mu, v), \ (\mu, v) \in [0, 1] \times [a, b], \ i = 1, 2\}.
\]
For \( \text{Re} \, \lambda > -\sigma \), the solution is formally given by
\[
\psi_i(\mu, v) = \psi_i(0, v) e^{-(1/v) \int_0^\mu (\lambda + \sigma_i(\mu', v)) \, d\mu'} + \frac{1}{v} \int_0^\mu e^{-(1/v) \int_0^\tau (\lambda + \sigma_i(\tau, v)) \, d\tau} g(\mu', v) \, d\mu'.
\]
Accordingly, for \( \mu = 1 \), we get
\[
\psi_i(1, v) = \psi_i(0, v) e^{-(1/v) \int_0^1 (\lambda + \sigma_i(1, v)) \, d\mu'} + \frac{1}{v} \int_0^1 e^{-(1/v) \int_0^\tau (\lambda + \sigma_i(\tau, v)) \, d\tau} g(\mu', v) \, d\mu'. \tag{7}
\]
Let the following operators:
\[
P_{i, \lambda} : X^0 \rightarrow X^1, \quad u \mapsto (P_{i, \lambda} u)(1, v) := u(0, v) e^{-(1/v) \int_0^1 (\lambda + \sigma_i(1, v)) \, d\mu'},
\]
\[
Q_{i, \lambda} : X^0 \rightarrow X, \quad u \mapsto (Q_{i, \lambda} u)(\mu, v) := u(0, v) e^{-(1/v) \int_0^\mu (\lambda + \sigma_i(\mu', v)) \, d\mu'},
\]
\[
\Pi_{i, \lambda} : X \rightarrow X^1, \quad u \mapsto (\Pi_{i, \lambda} u)(1, v) := \frac{1}{v} \int_0^1 e^{-(1/v) \int_0^\tau (\lambda + \sigma_i(\tau, v)) \, d\tau} u(\mu', v) \, d\mu',
\]
and finally
\[
R_{i, \lambda} : X \rightarrow X, \quad u \mapsto (R_{i, \lambda} u)(\mu, v) := \frac{1}{v} \int_0^\mu e^{-(1/v) \int_0^\tau (\lambda + \sigma_i(\tau, v)) \, d\tau} u(\mu', v) \, d\mu'.
\]
Clearly, for \( \lambda \) satisfying \( \text{Re} \, \lambda > -\sigma \), the operators \( P_{i, \lambda}, Q_{i, \lambda}, \Pi_{i, \lambda} \) and \( R_{i, \lambda} \), \( i = 1, 2 \), are bounded. It is not difficult to check that
\[
\|P_{i, \lambda}\| \leq e^{-(1/b)(\text{Re} \, \lambda + \sigma)} \tag{8}
\]
and
\[
\|Q_{i, \lambda}\| \leq (\text{Re} \, \lambda + \sigma)^{-1}. \tag{9}
\]
Moreover, simple calculations show that
\[
\|\Pi_{i, \lambda}\| \leq 1 \tag{10}
\]
and
\[
\|R_{i, \lambda}\| \leq (\text{Re} \, \lambda + \sigma)^{-1}. \tag{11}
\]
Thus, Equation (7) may be written abstractly as

$$\psi_i^1 = P_{i,\lambda} \psi_i^0 + \Pi_{i,\lambda} g.$$  

On the other hand, $\psi_i$ must satisfy the boundary condition (2); thus, we obtain

$$\psi_i^1 = P_{i,\lambda} K_i \psi_i^1 + \Pi_{i,\lambda} g. \tag{12}$$

Observe that the operator $P_{i,\lambda} K_i$ in Equation (12) is defined from $X^1$ into $X^1$.

Let $\phi_1, \phi_2 \in X^1$; from $(H1)$ and estimate (8), we have

$$\|P_{i,\lambda} K_i \phi_1 - P_{i,\lambda} K_i \phi_2\| \leq \alpha_i e^{-(\text{Re}\lambda + \sigma)/b} \|\phi_1 - \phi_2\|. \tag{13}$$

Consider now the equation

$$u = P_{i,\lambda} K_i u + \varphi, \quad \varphi \in X^1, \tag{14}$$

where $u$ is the unknown function and define the operator $A_{i,\lambda,\varphi}$ on $X^1$ by

$$A_{i,\lambda,\varphi} : X^1 \longrightarrow X^1,$$

$$u \longrightarrow (A_{i,\lambda,\varphi} u)(1, v) := P_{i,\lambda} K_i u + \varphi.$$

It follows from estimate (13) that

$$\|A_{i,\lambda,\varphi} \phi_1 - A_{i,\lambda,\varphi} \phi_2\| = \|P_{i,\lambda} K_i \phi_1 - P_{i,\lambda} K_i \phi_2\| \leq \alpha_i e^{-(\text{Re}\lambda + \sigma)/b} \|\phi_1 - \phi_2\|.$$  

Consequently, for $\text{Re}\lambda > -\sigma + b \log(\alpha_i)$, the operator $A_{i,\lambda,\varphi}$ is a contraction mapping and therefore Equation (14) has a unique solution

$$u_{i,\lambda,\varphi} = u_i.$$

Let $W_{i,\lambda}$ the nonlinear operator defined by

$$W_{i,\lambda} \varphi = u_i, \tag{15}$$

where $u_i$ is the solution of Equation (14). Arguing as the proof of Lemma 2.1 and Proposition 2.1 in [14], we have the following result.

**Lemma 4.1** Assume that $(H1)$ holds. Then,

(1) for every $\lambda$ satisfying $\text{Re}\lambda > -\sigma + b \log(\alpha_i)$, $i = 1, 2$, the operator $W_{i,\lambda}$ is continuous and maps bounded sets into bounded ones and satisfies the following estimate

$$\|W_{i,\lambda} \varphi_1 - W_{i,\lambda} \varphi_2\| \leq (1 - \alpha_i e^{-(\text{Re}\lambda + \sigma)/b})^{-1} \|\varphi_1 - \varphi_2\| \quad \text{for all } \varphi_1, \varphi_2 \in X^1.$$  

(2) If $\text{Re}\lambda > \max(-\sigma, -\sigma + b \log(\alpha_i))$, then the operator $(\lambda - S_{K_i})$ is invertible and $(\lambda - S_{K_i})^{-1}$ is given by

$$(\lambda - S_{K_i})^{-1} = Q_{i,\lambda} K_i W_{i,\lambda} \Pi_{i,\lambda} + R_{i,\lambda}.$$  

Moreover, $(\lambda - S_{K_i})^{-1}$ is continuous on $X$ and maps bounded sets into bounded ones.
In what follows and for our subsequent analysis, we need the following hypothesis:

\[(H_3) \quad r_{ij}(\mu, v, v', \psi_j(\mu, v')) = k_{ij}(\mu, v, v')f(\mu, v', \psi_j(\mu, v')) \quad (i, j) \in \{(1, 2), (2, 1)\},\]

where \(f\) is a measurable function defined by

\[f : [0, 1] \times [a, b] \times \mathbb{C} \rightarrow \mathbb{C}, \quad (\mu, v, u) \rightarrow f(\mu, v, u),\]

with \(k_{ij}(\cdot, \cdot, \cdot), (i, j) \in \{(1, 2), (2, 1)\}\), is a measurable function from \([0, 1] \times [a, b] \times \mathbb{C}\) to \(\mathbb{C}\) which defines a bounded linear operator \(B_{ij}\) by

\[B_{ij} : \mathcal{X} \rightarrow \mathcal{X}, \quad \psi_j \mapsto \int_a^b k_{ij}(\mu, v, v')\psi_j(\mu, v') \, dv'. \quad (16)\]

**Definition 4.1 [16]** Let \(B_{ij}, (i, j) \in \{(1, 2), (2, 1)\}\), be the operator defined by Equation (16). Then, \(B_{ij}\) is said to be a regular operator if \(\{k_{ij}(\mu, \cdot, \cdot) \text{ such that } (\mu, v') \in [0, 1] \times [a, b]\}\) is a relatively weakly compact subset of \(L^1([a, b]; \, d\mu)\).

**Lemma 4.2 [3]** If \(B_{ij}, (i, j) \in \{(1, 2), (2, 1)\}\), is a regular operator then \((\lambda - S_K)^{-1}B_{ij}\) is weakly compact on \(\mathcal{X}\), for \(\Re \lambda > \max(-\sigma, -\sigma + b \log(\alpha_i))\).

Let \(D\) be a subset of \(\mathbb{R}^n\). Recall that a function \(g : D \times \mathbb{C} \rightarrow \mathbb{C}\) is said to satisfy the Carathéodory conditions on \(D \times \mathbb{C}\) if

\[t \rightarrow g(t, x) \text{ is measurable on } D \text{ for all } x \in \mathbb{C}, \quad x \rightarrow g(t, x) \text{ is continuous on } \mathbb{C} \text{ for almost all } t \in D.\]

Observe that if \(f\) is a Carathéodory function, then we can define the operator \(N_g\) on the set of functions \(\psi : D \rightarrow \mathbb{C}\) by \((N_g\psi)(y) := g(y, \psi(y))\) for every \(y \in D\). The operator \(N_g\) is called the Nemytskii operator generated by \(g\).

In \(L_p\) spaces, \(1 \leq p < \infty\), the Nemytskii operator has been extensively investigated [9,10]. However, we recall the following result which states a basic fact for the theory of these operators on \(L_1\) spaces.

**Lemma 4.3 [9]** Assume that \(g\) satisfies the Carathéodory conditions. If the operator \(N_g\) acts from \(L_1\) into \(L_1\), then \(N_g\) is continuous and takes bounded sets into bounded sets.

We shall also assume that

\[(H_4) \quad f \text{ satisfies the Carathéodory conditions and } N_f \text{ acts from } \mathcal{X} \text{ into } \mathcal{X}.\]

We recall the following lemma established in [15] which will play a crucial role below.

**Lemma 4.4** If condition (H4) holds true, then \(N_f\) satisfies \((A2)\).

We are now ready to state our first existence result.
Theorem 4.1  Assume that \((\mathcal{H}1)-(\mathcal{H}4)\) hold. If \(B_{12}\) is a regular collision operator on \(\mathcal{X}\), then for each \(r > 0\) there is \(\lambda_r > 0\) such that for each \(\lambda\) satisfying \(\text{Re} (\lambda) > \lambda_r\), the problem

\[
\begin{pmatrix}
-v \frac{\partial}{\partial \mu} - \sigma_1 (\mu, v) I & R_{12} \\
R_{21} & -v \frac{\partial}{\partial \mu} - \sigma_2 (\mu, v) I
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= \lambda
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},
\]

(17)

has at least one solution in \(B_r \times B_r\).

Proof  Let \(\lambda\) be a complex number such that \(\text{Re} \lambda > \max (-\sigma, -\sigma + b \log (\alpha))\) with \(\alpha = \max (\alpha_1, \alpha_2)\). Then, according to Lemma 4.1, \(\lambda - S_{K_i}\) is invertible and therefore the problem (17) and (18) may be transformed into

\[
\mathcal{L}_{\lambda} \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}, \quad \psi_i^0 = K_i \psi_i^1, \quad i = 1, 2,
\]

where

\[
\mathcal{L}_{\lambda} = \begin{pmatrix}
S_{K_i} - (\lambda - 1) I & B_{12} N_f \\
B_{21} N_f & S_{K_2} - (\lambda - 1) I
\end{pmatrix},
\]

Claim 1  Let \(r > 0\). We first check that, for suitable \(\lambda, S_{\lambda} := (\lambda - S_{K_i})^{-1} B_{12} N_f\) leaves \(B_r\) invariant. Let \(\psi \in B_r\); therefore, from Lemma 4.1 and the estimates (8)–(11), we have

\[
\| S_{\lambda} \psi \| \leq \| Q_{1,3} K_1 W_{1,3} \Pi_{1,3} B_{12} N_f \psi + R_{1,3} B_{12} N_f \psi \|
\leq \left[ 1 + \frac{\alpha_1}{1 - \alpha_1 e^{-(\text{Re} \lambda + \sigma)/b}} \right] \| B_{12} \| \frac{M(r)}{\text{Re} \lambda + \sigma} + \frac{\alpha_1 \| W_{1,\lambda} (0) \| + \| K_1 (0) \|}{\text{Re} \lambda + \sigma},
\]

where \(M(r)\) is the upper-bound of \(N_f\) on \(B_r\). Let \(\varepsilon > \max (-\sigma, -\sigma + b \log (\alpha))\). For \(\text{Re} \lambda > \varepsilon\), we have

\[
(1 - \alpha_1 e^{-(\text{Re} \lambda + \sigma)/b})^{-1} \leq (1 - \alpha_1 e^{-(\varepsilon + \sigma)/b})^{-1}.
\]

Therefore,

\[
\| S_{\lambda} \psi \| \leq \left[ 1 + \frac{\alpha_1}{1 - \alpha_1 e^{-(\varepsilon + \sigma)/b}} \right] \| B_{12} \| \frac{M(r)}{\text{Re} \lambda + \sigma} + \frac{\alpha_1 \| W_{1,\lambda} (0) \| + \| K_1 (0) \|}{\text{Re} \lambda + \sigma}.
\]

Using Equation (15), we have

\[
P_{1,\lambda} K_1 W_{1,\lambda} (0) = W_{1,\lambda} (0).
\]

Let \(0 < \delta < 1/\alpha_1\), and from the estimate (8) there exists \(\lambda_1\) such that for any \(\lambda\) satisfying \(\text{Re} \lambda > \max (-\sigma, -\sigma + b \log (\alpha), \lambda_1)\), we have \(|| P_{1,\lambda} || \leq \delta\); then using \((\mathcal{H}1)\) we obtain

\[
\| W_{1,\lambda} (0) \| \leq \| P_{1,\lambda} \| || K_1 W_{1,\lambda} (0) ||
\leq \delta (\alpha_1 || W_{1,\lambda} (0) || + || K_1 (0) ||).
\]

It follows that

\[
\| W_{1,\lambda} (0) \| \leq \frac{\delta \| K_1 (0) \|}{1 - \delta \alpha_1}.
\]
Therefore,
\[
\|S_\lambda \psi\| \leq \left[ 1 + \frac{\alpha_1}{1 - \alpha_1 e^{-(\epsilon + \sigma)/b}} \right] \|B_{12}\| M(r) \frac{\|K_1(0)\|}{\text{Re} \lambda + \sigma} + \frac{(\alpha_1 \delta/(1 - \delta) + 1)\|K_1(0)\|}{\text{Re} \lambda + \sigma}
\]
where
\[
Q(t) = \left[ 1 + \frac{\alpha_1}{1 - \alpha_1 e^{-(\epsilon + \sigma)/b}} \right] \|B_{12}\| M(r) \frac{\|K_1(0)\|}{t + \sigma} + \frac{(\alpha_1 \delta/(1 - \delta) + 1)\|K_1(0)\|}{t + \sigma}.
\]

Clearly, \(Q(\cdot)\) is continuously strictly decreasing in \(t > 0\) and satisfies \(\lim_{t \to +\infty} Q(t) = 0\). Hence, there exists \(\lambda_2\) such that \(Q(\lambda_2) \leq r\). Obviously, if \(\text{Re} \lambda \geq \max(\lambda_1, \lambda_2)\), then \((\lambda - S_{\lambda_1})^{-1}B_{12}\mathcal{N}_f\) maps \(B_r\) into itself.

**Claim 2** It is immediate that the operator \(S_\lambda\) is continuous and weakly compact on \(X\). Now, we check that \(S_\lambda\) satisfies the condition \((A_1)\). For this, let \((\psi_n)_{n \in \mathbb{N}}\) be a weakly convergent sequence of \(X\). Using the fact \(\mathcal{N}_f\) satisfies \((A_2)\), \((\mathcal{N}_f \psi_n)_{n \in \mathbb{N}}\) has a weakly convergent subsequence, say \((\mathcal{N}_f \psi_n_k)_{k \in \mathbb{N}}\). Moreover, using Proposition 2.12 and Lemma 4.7 in [3], we have \((\lambda - S_{\lambda_1})^{-1}B_{12}\mathcal{N}_f\psi_n_k\) converges strongly in \(X\). Then, \(S_\lambda\) satisfies \((A_1)\).

**Claim 3** Clearly \(B_{21}\mathcal{N}_f\) is continuous on \(X\). Now, we check that \(B_{21}\mathcal{N}_f\) satisfies the condition \((A_2)\). To do so, let \((\psi_n)_{n \in \mathbb{N}}\) be a weakly convergent sequence of \(X\). Using the fact \(\mathcal{N}_f\) satisfies the condition \((A_2)\), \((\mathcal{N}_f \psi_n)_{n \in \mathbb{N}}\) has a weakly convergent subsequence, say \((\mathcal{N}_f \psi_n_k)_{k \in \mathbb{N}}\). Moreover, the continuity of the linear operator \(B_{21}\) implies that it is weakly continuous on \(X\) [6], so \((B_{21}\mathcal{N}_f \psi_n_k)_{k \in \mathbb{N}}\) converge weakly in \(X\). Then, \(B_{21}\mathcal{N}_f\) satisfies \((A_2)\).

**Claim 4** Clearly from Lemma 4.1, we have that \((\lambda - S_{K_1})^{-1}\) exists and is continuous on \(X\). Now, we check that \((\lambda - S_{K_1})^{-1}\) satisfies the condition \((A_2)\). For this, let \((\psi_n)_{n \in \mathbb{N}}\) be a weakly convergent sequence of \(X\). Using the fact \((W_{2,2,1}(\mathbb{R}, \mathbb{R}))\) is a bounded sequence and \(K_2\) is a weakly compact operator on \(X^\prime\), \((K_2 W_{2,2,1}(\mathbb{R}, \mathbb{R}))\) has a weakly convergent subsequence \((K_2 W_{2,2,1}(\mathbb{R}, \mathbb{R}))\); moreover, using the continuity of the linear operators \(Q_{2,2}\) and \(R_{2,2}\), we have that \((\lambda - S_{K_2})^{-1}\psi_n_k\) converge weakly in \(X\). Then, the operator \((\lambda - S_{K_2})^{-1}\) satisfies \((A_2)\).

Arguing as the claim 1 for \(S_\lambda\psi\), there exists \(\lambda_r\) such that for \(\text{Re} \lambda \geq \lambda_r\), we have \(\Gamma \psi := (\lambda - S_{K_1})^{-1}B_{21}\mathcal{N}_f S_\lambda \psi \in B_r\). Finally, \(\Gamma\) has a fixed point in \(B_r\); equivalently the problem (17) and (18) has a solution in \(B_r \times B_r\).

Now, we discuss the existence of solutions for the more general nonlinear boundary problem (1) and (2). When dealing with this problem, some technical difficulties arise. Therefore, we need the following assumption:

**\((H5)\)** \(K_i \in \mathcal{L}(X^1, X^0)\) and for each \(r > 0\), the function \(\sigma_i(\cdot, \cdot, \cdot), i = 1, 2\), satisfies
\[
|\sigma_i(\mu, \nu, \psi_1) - \sigma_i(\mu, \nu, \psi_2)| \leq |\omega_i(\mu, \nu)| |\psi_1 - \psi_2| \quad \text{for all } \psi_1, \psi_2 \in B_r,
\]
where \(\mathcal{L}(X^1, X^0)\) denotes the set of all bounded linear operators from \(X^1\) into \(X^0\) and \(\omega_i(\cdot, \cdot) \in L^\infty([0, 1] \times [a, b])\) and \(N_{\omega_i}\) acts from \(X\) into \(X\).
Define the free streaming operator $\hat{S}_K$, $i = 1, 2$, by

$$
\hat{S}_K : \mathcal{D}(\hat{S}_K) \subseteq \mathcal{X} \rightarrow \mathcal{X},
$$

$$
\psi_i \rightarrow \hat{S}_K \psi_i(\mu, \nu) = -v \frac{\partial \psi_i}{\partial \mu}(\mu, \nu),
$$

where

$$
\mathcal{D}(\hat{S}_K) = \{\psi_i \in \mathcal{W} \text{ such that } \psi_i^0 = K_i(\psi_i^1)\}.
$$

**Theorem 4.2** Assume that ($\mathcal{H}3$–($\mathcal{H}$5) hold. If $B_{ij}$, $(i, j) \in \{(1, 2), (2, 1)\}$, are regular collision operators on $\mathcal{X}$, then for each $r > 0$, there is $\lambda_r > 0$ such that for each $\lambda$ satisfying $\text{Re}(\lambda) > \lambda_r$ the problem (1) and (2) has at least one solution in $B_r \times B_r$.

**Proof** Since $K_i$, $i = 1, 2$, is linear, the operator $\hat{S}_K$ is linear too. Using Lemma 4.1, {$(\lambda, \in \mathbb{C}$ such that $\text{Re}\lambda > \max(0, b \log \|K_1\|, b \log \|K_2\|) \subset \rho(\hat{S}_K)$}, where $\rho(\hat{S}_K)$ denotes the resolvent set of $\hat{S}_K$. Let $\lambda \in \mathbb{C}$ such that $\text{Re}\lambda > \max(0, b \log \|K_1\|, b \log \|K_2\|)$. Then, by linearity of the operator $(\lambda - \hat{S}_K)^{-1}$, the problem (1) and (2) written in the form

$$
\hat{L}_\lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_i^0 = K_i \psi_i^1, \quad i = 1, 2,
$$

where

$$
\hat{L}_\lambda = \begin{pmatrix} \hat{S}_K - (\lambda - 1)I + N_{-\sigma_1} & B_{12}N_f \\ B_{21}N_f & \hat{S}_K - (\lambda - 1)I + N_{-\sigma_2} \end{pmatrix}
$$

may be transformed into the form

$$
G_{1,\lambda} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + G_{2,\lambda} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_i^0 = K_i \psi_i^1, \quad i = 1, 2,
$$

where

$$
G_{1,\lambda} = \begin{pmatrix} (\lambda - \hat{S}_K)^{-1}N_{-\sigma_1} & 0 \\ 0 & (\lambda - \hat{S}_K)^{-1}N_{-\sigma_2} \end{pmatrix},
$$

and

$$
G_{2,\lambda} = \begin{pmatrix} 0 & (\lambda - \hat{S}_K)^{-1}B_{12}N_f \\ (\lambda - \hat{S}_K)^{-1}B_{21}N_f & 0 \end{pmatrix}.
$$

**Claim 1** Check that, for suitable $\lambda$, the operator $G_{1,\lambda}$ is a contraction mapping. Indeed, let $(\psi_1, \psi_2) \in \mathcal{X} \times \mathcal{X},$

$$
\|(\lambda - \hat{S}_K)^{-1}(N_{-\sigma_1}\psi_i - N_{-\sigma_2}\psi_i)\| \leq \|(\lambda - \hat{S}_K)^{-1}\|\|N_{-\sigma_1}\psi_i - N_{-\sigma_2}\psi_i\|, \quad i = 1, 2.
$$

A simple calculation using the estimates (8)–(11) leads to

$$
\|(\lambda - \hat{S}_K)^{-1}\| \leq \frac{1}{\text{Re}\lambda} \left[ 1 + \frac{\gamma}{1 - \gamma e^{-\text{Re}\lambda/b}} \right], \quad i = 1, 2, \quad (19)
$$

where $\gamma = \max(\|K_1\|, \|K_2\|)$. Moreover, taking into account the assumption on $\sigma_i(\cdot, \cdot, \cdot)$, we get

$$
\|N_{-\sigma_1}\psi_i - N_{-\sigma_2}\psi_i\| \leq \|\omega\|_\infty \|\psi_i - \psi_i\|, \quad i = 1, 2,
$$

where \( \|\omega\|_\infty = \max(\|\omega_1\|_\infty, \|\omega_2\|_\infty) \). Using the estimate (19) we have
\[
\left\| G_{1,\lambda} (\psi_2) - G_{1,\lambda} (\psi_2) \right\| = \left\| (\lambda - \hat{S}_{K_1})^{-1}N_{\sigma_1}\psi_1 \right\| 
\leq \frac{\|\omega\|_\infty}{\text{Re} \lambda} \left[ 1 + \frac{\gamma}{1 - \gamma e^{-\text{Re} \lambda/b}} \right] \left\| (\psi_1 - \psi_2) \right\|. 
\]
\[
\leq E(\text{Re} \lambda) \left\| (\psi_1 - \psi_2) \right\|.
\]
Note that \( E \) is a continuous strictly decreasing function in \( t > 0 \) and \( \lim_{t \to \infty} E(t) = 0 \).

Therefore, there exists \( \lambda_1 \in \max(0, b \log \|K_1\|, b \log \|K_2\|) \) such that \( E(\lambda_1) < 1 \). Hence, for \( \text{Re} \lambda \geq \lambda_1 \), \( G_{1,\lambda} \) is a contraction mapping.

Claim 2 Using Lemma 4.2 and arguing as in the proof of Theorem 4.1, we show that \( G_{1,\lambda} \) satisfies (\( \mathcal{A}_2 \)) and \( G_{2,\lambda} \) is continuous, weakly compact on \( \mathcal{X} \times \mathcal{X} \) and satisfies (\( \mathcal{A}_1 \)).

Claim 3 Let \( r > 0 \) and \( \varphi_1, \psi_2 \in B_r \). According to estimation (19), we obtain
\[
\| (\lambda - \hat{S}_{K_1})^{-1}N_{\sigma_1}\psi_1 + (\lambda - \hat{S}_{K_1})^{-1}B_{12}\psi_2 \| \leq \frac{\|B_{12}\|M(r) + M'(r)}{\text{Re} \lambda} \left[ 1 + \frac{\|K_1\|}{1 - \|K_1\| e^{-\text{Re} \lambda/b}} \right]
\leq T(\text{Re} \lambda),
\]
where \( T(\cdot) \) has the same properties as \( E(\cdot) \), and \( M(r) \) and \( M'(r) \) are the upper-bounds of \( \mathcal{N}_r \) and \( \mathcal{N}_r \) on \( B_r \). Arguing as above, we show that there exists \( \lambda_2 \) such that, for all \( \lambda \) such that \( \text{Re} \lambda \geq \lambda_2 \), we have \( (\lambda - \hat{S}_{K_1})^{-1}N_{\sigma_1}\psi_r + (\lambda - \hat{S}_{K_1})^{-1}B_{12}\psi_r \subseteq B_r \).

By similar reasoning, we prove that there exists \( \lambda_3 \), such that \( \text{Re} \lambda \geq \lambda_3 \), and we have \( (\lambda - \hat{S}_{K_1})^{-1}N_{\sigma_2}\psi_r + (\lambda - \hat{S}_{K_1})^{-1}B_{21}\psi_r \subseteq B_r \).

Finally, if \( \lambda_\ast = \max(\lambda_1, \lambda_2, \lambda_3) \), then for all \( \lambda \) satisfying \( \text{Re} \lambda \geq \lambda_\ast \), the operators \( G_{1,\lambda} \) and \( G_{2,\lambda} \) satisfy the conditions of Theorem 3.3. Consequently, the problem (1) and (2) has a solution in \( B_r \times B_r \) for all \( \lambda \) such that \( \text{Re} \lambda \geq \lambda_\ast \).

\begin{thebibliography}{99}

[1] R.P. Agarwal, D. O'Regan, and X. Liu, A Leray-Schauder alternative for weakly–strongly sequentially continuous weakly compact maps, Fixed Point Theory Appl. 1 (2005), pp. 1–10.

[2] J. Appell and E. De Pascale, Su alcuni parametri connessi con la misura di non compattezza di Haussdorff in spazi di funzioni misurabili, Boll. Unione Mat. Ital. Sez. B (6) 3 (1984), pp. 497–515.

[3] A. Ben Amar, A. Jeribi, and M. Mnif, Some fixed point theorems and application to biological model, Numer. Funct. Anal. Optim. 29 (2008), pp. 1–23.

[4] A. Ben Amar, A. Jeribi, and B. Krichen, Fixed point theorems for block operator matrix and an application to a structured problem under boundary conditions of Rotenberg’s model type, to appear in Math. Slovaca. (2014).

[5] M. Bouloumié, Transport equations in cell population dynamics I, Electron. Differ. Equ. 144 (2010), pp. 1–20.

[6] H. Brezis, Analyse Fonctionnelle, Théorie et Applications, Masson, Paris, 1983.

[7] M. Cessenat, Théorèmes de trace \( L_p \) pour des espaces de fonctions de la neutronique, C. R. Acad. Sci. Paris Série I 299 (1984), pp. 831–834.

[8] M. Cessenat, Théorèmes de trace pour des espaces de fonctions de la neutronique, C. R. Acad. Sci. Paris Série I 300 (1985), pp. 89–92.

[9] S.N. Chow and J.K. Hale, Methods of Bifurcations Theory, Grundlehren der Mathematischen Wissenschaften 251, Springer-Verlag, New York, Berlin, 1982.

[10] R. Dautray and J.L. Lions, Analyse mathématiques et calcul numérique pour les sciences et les techniques, Masson, Paris, 1988.

[11] E.S. De Blasi, On a property of the unit sphere in Banach spaces, Bull. Math. Soc. Sci. Math. Roumanie 21 (1997), pp. 259–262.

[12] N. Dunford and J.T. Schwartz, Linear Operators, Part I, General Theory, Interscience, New York, 1988.

\end{thebibliography}
[13] M.A. Krasnosel’skii, *Some problems of nonlinear analysis*, Am. Math. Soc. Trans. Ser. 2 10(2) (1958), pp. 345–409.

[14] K. Latrach and A. Jeribi, *A nonlinear boundary value problem arising in growing cell populations*, Nonlinear Anal. T.M.A. 36 (1999), pp. 843–862.

[15] K. Latrach, M.A. Taoudi, and A. Zeghal, *Some fixed point theorems of the Schauder and the Krasnosel’skii type and application to nonlinear transport equations*, Differ. Equ. 221 (2006), pp. 256–271.

[16] B. Lods, *On linear kinetic equations involving unbounded cross-sections*, Math. Methods Appl. Sci. 27 (2004), pp. 1049–1075.

[17] M. Rotenberg, *Transport theory for growing cell populations*, J. Theor. Biol. 103 (1983), pp. 181–199.

[18] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1980.