R-matrix formulation of affine Yangian of \( \hat{\mathfrak{gl}}(1|1) \)

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Abstract

We study \( \mathcal{N} = 2 \) superconformal field theory and define the R-matrix which acts as an intertwining operator between different realizations of \( \mathcal{N} = 2 \) \( W \)-algebras of type \( A \). Using this R-matrix we define \( RLL \) algebra and relate it to current realization of affine Yangian of \( \hat{\mathfrak{gl}}(1|1) \).

1 Introduction

It has been known for a while that integrability in CFT can be studied using the affine Yangian symmetry. The basic example – the affine Yangian of \( \hat{\mathfrak{gl}}(1) \), or \( Y(\hat{\mathfrak{gl}}(1)) \) for shortness, has been introduced in R-matrix formulation by Maulik and Okounkov [1]. The R-matrix of [1] is closely related to the Liouville reflection operator [2], which intertwines two Miura representations for Virasoro algebra. The algebra under the same name has been introduced by Tsymbaliuk in [3] by explicit commutation relations (the so called current realization). In [4] it has been shown that both algebras differ by an infinite-dimensional center.

In [5, 6] the relation between \( Y(\hat{\mathfrak{gl}}(1)) \) and bosonic \( W_{1+\infty} \) algebra has been established. In particular, some of low lying generators of \( W_{1+\infty} \) algebra were expressed explicitly in terms of the affine Yangian generators. Later in [7, 8] Gaberdiel et al studied \( \mathcal{N} = 2 \) superconformal version of \( W_{1+\infty} \) algebra and defined, what they called "the supersymmetric affine Yangian algebra". The analysis in [7, 8] has been using the fact that \( \mathcal{N} = 2 \ W_{1+\infty} \) algebra contains two commuting bosonic \( W_{1+\infty} \) subalgebras. The corresponding Yangian algebra were built in the so called gluing basis: having two explicitly commuting \( Y(\hat{\mathfrak{gl}}(1)) \) algebras and additional fermionic generators, which transform as bi-fundamental representations (see also [9, 10] for later developments).

In [11] (see also [12, 13]) the current realization of \( Y(\hat{\mathfrak{gl}}(1|1)) \) has been given\(^1\). Moreover later in [13] it has been explicitly stated that \( Y(\hat{\mathfrak{gl}}(1|1)) \) defined in [11] (together with Serre relations, which appear later in [13]) should correspond to \( \mathcal{N} = 2 \ W_{1+\infty} \) algebra in the same way as \( Y(\hat{\mathfrak{gl}}(1)) \) corresponds to bosonic \( W_{1+\infty} \) algebra.

In this paper we relate \( \mathcal{N} = 2 \ W \)-algebras and \( Y(\hat{\mathfrak{gl}}(1|1)) \), but follow another approach similar to [4]. Namely, we define the R-matrix, which acts in the tensor product of two typical representations of \( \hat{\mathfrak{gl}}(1|1) \) current algebra, then define the \( RLL \) algebra and derive current relations of \( Y(\hat{\mathfrak{gl}}(1|1)) \) out of it. This paper is organized as follows. In section 2 we review \( \mathcal{N} = 2 \ W_n \)-algebras

\(^1\)We note that [11] aimed to construct various Yangian algebras from geometry and contains much more examples. The algebra \( Y(\hat{\mathfrak{gl}}(1|1)) \) corresponds to the resolved conifold geometry and the so called pyramid partitions.
of type $A$ and their free field representations. In section 3 we introduce $\widehat{\mathfrak{gl}}(1|1)$ algebra and its representations through $bc\beta\gamma$-system. In section 4 we define the $R-$matrix for $\widehat{\mathfrak{gl}}(1|1)$. In sections 5 and 6 we relate $RLL$ algebra and current realization of $Y(\widehat{\mathfrak{gl}}(1|1))$. Sections 4, 5 and 6 constitute main results of our paper. In section 7 we provide concluding remarks and directions for future studies.

## 2 \mathcal{N} = 2 superconformal $W$–algebras

The $\mathcal{N} = 2 \ W_{n+1}$-algebras, which play the key role in our study, correspond to $SU(n+1)/U(n)$ Kazama-Suzuki models [14]. These algebras contain one $\mathcal{N} = 2$ multiplet of holomorphic currents for each integers spin $s = 1, \ldots, n$. In particular, for $s = 1$ one has the currents $(J, G^+, G^-, T)$ of spins $(1, \frac{3}{2}, \frac{3}{2}, 2)$, which form $\mathcal{N} = 2$ Virasoro algebra (whose commutation relations we present in the form of OPE's)

$$
\begin{align*}
T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots, \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \ldots, \\
T(z)G^\pm(w) &= \frac{3G^\pm(w)}{2(z-w)^2} + \frac{\partial G^\pm(w)}{z-w} + \ldots, \\
J(z)J(w) &= \frac{c}{3(z-w)^2} + \ldots, \\
J(z)G^\pm(w) &= \pm \frac{G^\pm(w)}{z-w} + \ldots, \\
G^+(z)G^-(w) &= \frac{c}{3(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w) + \frac{1}{2} \partial J(w)}{z-w} + \ldots
\end{align*}
$$

with other OPE's regular. Commutation relations for higher spin currents become much more involved and are, in principle, not sufficiently illuminating and convenient to handle.

Instead, we will use free-field representation of $\mathcal{N} = 2 \ W_{n+1}$-algebras, which is obtained by Hamiltonian reduction of $\widehat{\mathfrak{sl}}(n+1|n)$ [15–17]. We work in $\mathcal{N} = 1$ formalism and introduce $2n$ real holomorphic superfields $\Phi(z, \theta) = (\Phi_1(z, \theta), \ldots, \Phi_{2n}(z, \theta))$

$$
\Phi_k(z, \theta) \overset{\text{def}}{=} \Phi_k(z) + \theta \chi_k(z),
$$

$$
\Phi_i(z)\Phi_j(w) = -\frac{\delta_{ij}}{2} \log(z-w) + \ldots, \quad \chi_i(z)\chi_j(w) = \frac{\delta_{ij}}{2} \frac{1}{z-w} + \ldots \tag{2}
$$

Let $\tilde{a}_r = (\alpha_{r,1}, \ldots, \alpha_{r,2n})$ with $r = 1, \ldots, 2n$ be the simple fermionic roots of $\widehat{\mathfrak{sl}}(n+1|n)$ with the Gram matrix

$$
(\tilde{a}_r \cdot \tilde{a}_s) = \begin{pmatrix}
0 & -1 & 0 & \cdots & \cdots \\
-1 & 0 & 1 & \cdots & \cdots \\
0 & 1 & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix} \tag{3}
$$

$\tilde{\omega}_r$ being the dual basis ($\tilde{\omega}_r \cdot \tilde{a}_s = \delta_{rs}$) and

$$
\tilde{h}_r \overset{\text{def}}{=} (-1)^r (\tilde{\omega}_r - \tilde{\omega}_{r-1}) \quad \text{for} \quad r = 1, \ldots, 2n + 1, \tag{4}
$$

2
where $\bar{\omega}_0 = \bar{\omega}_{2n+1} = 0$. Then $\mathcal{N} = 2$ $W_{n+1}$-algebra is defined by [15–17]

\[
\begin{align*}
(b^{-1}D - D(h_1 \cdot \Phi(z, \theta))) (b^{-1}D - D(h_2 \cdot \Phi(z, \theta))) \cdots (b^{-1}D - D(h_{2n+1} \cdot \Phi(z, \theta))) := \\
(b^{-1}D)^{2n+1} + W(z, \theta)(b^{-1}D)^{2n} + W(z, \theta)(b^{-1}D)^{2n-1} + W(z, \theta)(b^{-1}D)^{2n-2} + \ldots
\end{align*}
\]

(5)

where

\[
D = \frac{\partial}{\partial \theta} - \theta \partial.
\]

(6)

We have $W(z, \theta) = 0$,

\[
W(z, \theta) = J(z) + i \theta(G^+(z) - G^-(z)), \quad b W(z, \theta) = i G^- (z) + \theta \left( T(z) + \frac{1}{2} \partial J(z) \right)
\]

(7)

e
c.

d. The currents (7) satisfy commutation relations of $\mathcal{N} = 2$ Virasoro algebra (1) with the central charge

\[
c = 3n \left( 1 + \frac{n + 1}{b^2} \right).
\]

(8)

Higher currents in (5) provide its extension.

In fact, it is more convenient to work in $\mathfrak{gl}(n + 1|n)$ rather than in $\mathfrak{sl}(n + 1|n)$ terms. In order to do it, we extend our field as $\bar{\Phi}(z, \theta) = (\Phi_0(z, \theta), \Phi_1(z, \theta), \ldots, \Phi_{2n}(z, \theta))$. In this case we can choose simple roots as

\[
(\bar{\alpha}_{2r+1} \cdot \bar{\Phi}(z, \theta)) = i \Phi_{2r}(z, \theta) - \Phi_{2r+1}(z, \theta), \quad (\bar{\alpha}_{2r} \cdot \bar{\Phi}(z, \theta)) = \Phi_{2r-1}(z, \theta) - i \Phi_{2r}(z, \theta).
\]

(9)

In this case the map (5) takes very simple form

\[
(b^{-1}D - iD \Phi_0(z, \theta))(b^{-1}D - D \Phi_1(z, \theta)) \cdots (b^{-1}D - D \Phi_{2n}(z, \theta))
\]

(10)

We denote the corresponding algebra as $W_{\mathfrak{gl}(n+1|n)}$. Similarly $W_{\mathfrak{gl}(n|n)}$ corresponds to (10) with the factor $(b^{-1}D - iD \Phi_0(z, \theta))$ dropped.

It is important to note that both algebras $W_{\mathfrak{gl}(n+1|n)}$ and $W_{\mathfrak{gl}(n|n)}$ can be defined as commutants of certain set of screening operators [17]. For $W_{\mathfrak{gl}(n|n)}$ it will be convenient to use complex fields $(k = 1, \ldots, n)$

\[
X_k(z, \theta) \overset{\text{def}}{=} \Phi_{2k-1}(z, \theta) + i \Phi_{2k}(z, \theta) = X_k(z) + \theta \psi_k(z),
\]

\[
X_k^*(z, \theta) \overset{\text{def}}{=} \Phi_{2k-1}(z, \theta) - i \Phi_{2k}(z, \theta) = X_k^*(z) + \theta \psi_k^*(z),
\]

(11)

where $(X_k(z), X_k^*(z), \psi(z), \psi^*(z))$ have the following OPE’s

\[
X_i(z)X_j^*(w) = -\delta_{ij} \log(z + w) + \ldots, \quad \psi_i(z)\psi_j^*(w) = \frac{\delta_{ij}}{z - w} + \ldots
\]

(12)

In these terms the screening operators for $W_{\mathfrak{gl}(n|n)}$ algebra ($S_1, S_{1,2}, S_2, S_{2,3}, \ldots, S_{n-1,n}, S_n$) have the form [17]

\[
S_k = b^{-1} \oint e^{b(\bar{\alpha}_{2k-1} \cdot \bar{\Phi}(z, \theta))} dzd\theta = b^{-1} \oint e^{bX_k^*(z, \theta)} dzd\theta = \oint \psi_k^*(z) e^{bX_k^*(z)} dz,
\]

\[
S_{k,k+1} = 2b^{-1} \oint e^{b(\bar{\alpha}_{2k} \cdot \bar{\Phi}(z, \theta))} dzd\theta = 2b^{-1} \oint e^{b(X_k(z, \theta) - X_{k+1}(z, \theta))} dzd\theta = \oint e^{b(X_k(z) - X_{k+1}(z))}(\psi_k(z) - \psi_{k+1}(z) - \psi_k^*(z) - \psi_{k+1}^*(z)) dz.
\]

(13)

We note that all screening fields (13) are fermionic ones. There are also bosonic screening fields [17].
3 \( \mathfrak{gl}(1|1) \) current algebra and \( bc\beta\gamma \)-system

The commutation relations of \( \mathfrak{gl}(1|1) \) algebra [18] can be written in terms of OPE of two bosonic currents \( J^E(z), J^N(z) \) and two fermionic currents \( J^\pm(z) \):

\[
J^N(z)J^E(w) = \frac{\kappa}{(z-w)^2} + \ldots \\
J^+(z)J^-(w) = \frac{\kappa}{(z-w)^2} + \frac{J^E(w)}{(z-w)^2} + \ldots \\
J^\pm(z)J^N(w) = \mp \frac{J^\pm(w)}{z-w} + \ldots
\]

(14)

The parameter \( \kappa \) called the level can be set \( \kappa = 1 \) by the simple rescaling \( J^E(z) \rightarrow \kappa J^E(z) \) and \( J^+(z) \rightarrow \kappa J^+(z) \). It implies that \( \mathfrak{gl}(1|1) \) algebra admits free field representation [19] via the \( bc\beta\gamma \)-system. Namely, \( bc\beta\gamma \)-system is the system of complex fermions \( \alpha(z), \alpha^*(z) \) and complex bosons \( a(z), a^*(z) \) with OPE's (other OPE's are trivial)

\[
\alpha(z)\alpha^*(w) = \frac{1}{z-w} + \ldots, \quad a(z)a^*(w) = \frac{1}{z-w} + \ldots
\]

(15)

Then one can realize the currents \( J^E(z), J^N(z) \) and \( J^\pm(z) \) as bilinears (Wick normal ordering is assumed)

\[
E \overset{\text{def}}{=} \left( \begin{array}{cc}
\frac{J^E}{2} + J^N & J^+ \\
J^- & \frac{J^E}{2} - J^N
\end{array} \right) = \begin{pmatrix} \alpha\alpha & \alpha^*a \\ \alpha^*a & a\alpha \end{pmatrix}
\]

(16)

For our purposes it will be convenient to further realize \( (\alpha(z), \alpha^*(z), a(z), a^*(z)) \) by the complex superfield \( X(z, \theta) = X(z) + \theta \psi(z), X^*(z, \theta) = X^*(z) + \theta \psi^*(z), \) as the Wick ordered fields

\[
\begin{align*}
\alpha &= \psi e^{-\frac{X}{b}}, & a &= e^{\frac{X}{b}}, \\
\alpha^* &= \psi^* e^{\frac{X}{b}}, & a^* &= (\psi^* e^X - b \partial X^*) e^{\frac{X}{b}}.
\end{align*}
\]

(17)

Here \( b \) is an arbitrary parameter, which can be re-scaled out by the transformation \( X \rightarrow bX, X^* \rightarrow b^{-1}X^* \), but we will keep it as it is. It is important to note that the currents (17) can be equivalently defined by the following Wakimoto type screening operator\(^2\)

\[
S = b^{-1} \oint e^{bX(z, \theta)} d\theta d\theta = \oint \psi^*(z) e^{bX^*(z)} dz.
\]

(19)

We note also that (17) admit the following simple superfield expression

\[
a(z) - \frac{\theta}{b} \alpha(z) = e^{-\frac{X(z, \theta)}{b}}, \quad a^*(z) + \frac{\theta}{b} \alpha^*(z) = DX^*(z, \theta) e^{\frac{X(z, \theta)}{b}}.
\]

(20)

\(^2\)We remind that \( \mathfrak{gl}(2) \) algebra admits free field representation in terms of two bosonic fields \( \varphi_1(z), \varphi_2(z) \) and \( \beta\gamma \)-system realized by Wakimoto screening operator [20]

\[
S = \oint e^{b(\varphi_1(z) - \varphi_2(z))} \beta(z) dz,
\]

(18)

with \( \kappa = -2 - b^{-2} \). In \( \mathfrak{gl}(1|1) \) case \( (\varphi_1(z) - \varphi_2(z)) \) is replaced by \( X^*(z) \) and \( \beta(z) \) by \( \psi^*(z) \).
In terms of the fields \( (X, X^*, \psi, \psi^*) \) the representation of \( \hat{gl}(1|1) \) takes the form

\[
E = \begin{pmatrix} b^{-1} \partial X - \psi \psi^* & \psi^* \\ \partial \psi - b \psi \partial X^* & \psi^* - b^{-1} \partial X - b \partial X^* \end{pmatrix}
\]  

(21)

In these notes we consider the so called typical (or generic) representations of \( \hat{gl}(1|1) \). One distinguishes between NS and R modules. We consider NS module for simplicity. We define the mode expansion of the currents on the cylinder \( z = e^{-ix}: x \sim x + 2\pi \)

\[
\partial X(x) = \sum_{k \in \mathbb{Z}} X_k e^{-ikx}, \quad \partial X^*(x) = \sum_{k \in \mathbb{Z}} X_k^* e^{-ikx},
\]

\[
\psi(x) = i^{\frac{1}{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r e^{-irx}, \quad \psi^*(x) = i^{\frac{1}{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi^*_r e^{-irx}.
\]

(22)

with commutation relations (other commutation relations are trivial)

\[
[X_m, X_n^*] = m \delta_{m-n}, \quad \{\psi_r, \psi^*_s\} = \delta_{r-s}.
\]

(23)

Then the Fock module \( \mathcal{F}_u \), where \( u = (u, u^*) \), is generated from the vacuum state \( |u\rangle \)

\[
X_n|u\rangle = X_n^*|u\rangle = \psi_r|u\rangle = \psi^*_r|u\rangle = 0 \quad \text{for} \quad n > 0, r > 0,
\]

\[
X_0|u\rangle = -iu|u\rangle, \quad X_0^*|u\rangle = -iu^*|u\rangle,
\]

(24)

by the action of the creation operators

\[
\mathcal{F}_u = \text{Span} \left( \psi^*_{-s}, \psi_{-s} X^*_{-\lambda} X_{-\lambda} |u\rangle \right), \quad \text{where} \quad X_{-\lambda} = X_{-\lambda_1} X_{-\lambda_2} \ldots \text{etc.}
\]

(25)

where \( \lambda \) and \( \lambda^* \) are partitions

\[
\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots\}, \quad \lambda^* = \{\lambda_1^* \geq \lambda_2^* \geq \ldots\},
\]

(26)

and \( s \) and \( s^* \) are strict partitions (fermionic partitions)

\[
s = \{s_1 > s_2 > \ldots\}, \quad s^* = \{s_1^* > s_2^* > \ldots\},
\]

(27)

The character of \( \mathcal{F}_u \) has the form

\[
\chi(q, t) = \text{Tr} \left( q^{L_0} e^{c_0} \right)_{\mathcal{F}_u} = \prod_{k=1}^{\infty} \frac{\left(1 + t q^{k-\frac{1}{2}} \right) \left(1 + t^{-1} q^{k-\frac{1}{2}} \right)}{(1-q^k)^2},
\]

(28)

where

\[
L_0 = \sum_{k>0} (X_{-k} X_k^* + X_{-k}^* X_k) + \sum_{s>0} s \left( \psi_{-s} \psi^*_s + \psi^*_{-s} \psi_s \right), \quad c_0 = \sum_{s>0} (\psi_{-s} \psi^*_s - \psi^*_{-s} \psi_s).
\]

(29)

### 4 R-matrix

In order to define the R-matrix for \( \hat{gl}(1|1) \), we take two copies of \( \hat{gl}(1|1) \) algebra defined by two screening fields

\[
S_1 = \int \psi_1^*(z) e^{hx_1(z)} dz \quad \text{and} \quad S_2 = \int \psi_2^*(z) e^{hx_2(z)} dz
\]

(30)
and define the additional screening field
\[ \mathcal{S}_{1, 2} \overset{\text{def}}{=} \int (\psi_1(z) - \psi_2(z)) e^{\frac{1}{2} (X_1(z) - X_2(z))} dz, \] (31)
which commutes with the diagonal \( \mathfrak{gl}(1|1) \) current \( E_1 + E_2 \).

The commutant of this additional screening is actually larger and can be associated with the \( W(\mathfrak{gl}(2|2)) \) algebra introduced in section 2. We note that the screening operator \( \mathcal{S}_{1, 2} \) given by (31) differs from \( S_{1, 2} \) given by (13) by the replacement in the prefactor
\[ \psi_1(z) - \psi_2(z) \rightarrow \psi_1(z) - \psi_2(z) - \psi_1^*(z) - \psi_2^*(z). \] (32)

However two algebras defined by \( (S_1, S_{1, 2}, S_2) \) and by \( (S_1, \mathcal{S}_{1, 2}, S_2) \) are actually isomorphic and are related by marginal deformations. It can be seen as follows. Consider the Lax operator for \( (S_1, S_{1, 2}, S_2) \)
\[ L = (b^{-1} D - D \Phi_1)(b^{-1} D - i D \Phi_2) = (b^{-1} D)^2 - DX^* b^{-1} D + \frac{1}{2} DX^* DX + \frac{1}{2b} D^2 (X^* - X), \] (33)
and perform the formal scale transformation
\[ \psi \rightarrow \lambda^{-\frac{1}{2}} \psi, \quad \psi^* \rightarrow \lambda^{\frac{1}{2}} \psi^*, \quad \theta \rightarrow \lambda \theta, \quad \frac{\partial}{\partial \theta} \rightarrow \lambda^{-\frac{1}{2}} \frac{\partial}{\partial \theta}, \] (34)
which does not change commutation relations. Then one has
\[ L = L^{(0)} + \lambda L^{(1)}, \] (35)
where (we use \( D^2 = -\partial \))
\[ L^{(0)} = -b^{-2} \partial + \frac{1}{2} (b^{-1} \partial X - \psi \psi^* - b^{-1} \partial X^*) + \frac{1}{2} (\partial \psi - b \psi \partial X^*) \theta - b^{-1} \psi^* \frac{\partial}{\partial \theta} + b^{-1} \partial X^* \theta \frac{\partial}{\partial \theta}, \] (36)
and
\[ L^{(1)} = \left( b^{-1} \psi^* \partial + \frac{1}{2} \psi^* \partial X - \frac{1}{2b} \partial \psi^* \right) \theta. \] (37)

It is clear that if \( L_1 \cdot L_2 \) "commutes" with the screening charges \( (S_1, S_{1, 2}, S_2) \) then \( L_1^{(0)} \cdot L_2^{(0)} \) "commutes" with \( (S_1, \mathcal{S}_{1, 2}, S_2) \).

Having defined the Lax operator \( L^{(0)} \), we define the R–matrix as an intertwining operator
\[ R_{ij} L^{(0)}_i = L^{(0)}_j R_{ij}, \] (39)
where \( L^{(0)}_j \overset{\text{def}}{=} L^{(0)}(X_i, X_i^*, \psi_i, \psi_i^*) \). We note that (39) automatically implies the Yang-Baxter equation
\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \] (40)

\(^3\)The current \( A(z) \) is said to "commute" with the screening field \( S = \int V(\xi) d\xi \) if
\[ \oint_{C_z} V(\xi) A(z) d\xi = 0. \] (38)
The $W_{\text{gl}(2)}$ algebra defined by $\mathcal{L}^{(0)}_1\mathcal{L}^{(0)}_2$ is obtained from the one defined by $\mathcal{L}_1\mathcal{L}_2$ in the limit $\lambda \to 0$. It consists of eight currents

\[
\Psi_{12}^a \equiv \psi_1^a + \psi_2^a, \quad U_{12} \equiv \partial X_1^a + \partial X_2^a, \quad J_{12} \equiv \psi_1\psi_2^*-1/b(\partial X_1 + \partial X_2),
\]

\[
G_{12} \equiv i\left(\psi_1\partial X_1^a + \psi_2\partial X_2^a - \frac{1}{b}(\partial \psi_1 + \partial \psi_2)\right),
\]

\[
G_{12}^* \equiv i\left(\psi_1^*(\partial X_1 - \partial X_2^*) + \psi_2^*(\partial X_2 + \partial X_1^*) - \frac{1}{b}(\partial \psi_1^* - \partial \psi_2^*)\right),
\]

\[
T_{12} \equiv -\partial X_1\partial X_1^* - \partial X_2\partial X_2^* + \frac{1}{b}(\partial^2 X_1^2 - \partial^2 X_2^2) + \partial \psi_1\psi_2^* + \partial \psi_2\psi_1^* - \frac{1}{2}\partial J_{12},
\]

\[
\Theta_{12} \equiv -\frac{1}{8b^2}((\partial X_1 - \partial X_1^*)^2 + (\partial X_2 - \partial X_2^*)^2) - \frac{1}{2b^3}(\partial^2 X_2 - \partial^2 X_2^*) + \frac{1}{4b}(\partial X_1 - \partial X_1^*)\psi_1\psi_2^* + \\
+ \frac{1}{4b}(\partial X_2 - \partial X_2^*)\psi_2\psi_1^* + \frac{1}{2b}\partial X_2^*\psi_1\psi_2 + \frac{1}{8}(\psi_1^*\partial \psi_1 + \psi_1\partial \psi_1^* + \psi_2^*\partial \psi_2 + \psi_2\partial \psi_2^*) + \\
+ \frac{1}{2b^2}(\partial^3 \psi_2^* + \partial \psi_2\psi_2^* - \partial \psi_2\psi_2^*)
\]

\[
H_{12} \equiv \frac{1}{4}\psi_1\psi_1^*\psi_2\psi_2^* + \frac{1}{4}\psi_2\psi_2^*\psi_1\partial X_1^* - \\
- \frac{1}{4b}\left(\left(\partial X_1 + \partial X_1^*\right)\partial X_2\psi_2 + \left(\partial X_2 - \partial X_2^*\right)\partial X_1^*\psi_1 + \psi_1\partial \psi_1\partial \psi_2 + \psi_2\partial \psi_2\partial \psi_1\right) + \\
+ \frac{1}{4b^2}\left(\left(\partial X_1 + \partial X_1^* + 2\partial X_2\right)\partial \psi_2 + \left(\partial X_2 - \partial X_2^*\right)\partial \psi_1 + 2\partial \psi_2\partial X_2^*\right) - \frac{1}{2b^3}\partial^2 \psi_2,
\]

where all the densities are Wick ordered. We note that $(J_{12}, G_{12}, G_{12}^*, T_{12})$ form $\mathcal{N} = 2$ Virasoro algebra with the central charge

\[
c = 6.
\]

At the same time the currents $(\Psi_{12}^a, U_{12}, J_{12}, G_{12})$ correspond to the diagonal current $E_1 + E_2$.

Working in a particular representation, the relations (39) can be used to define the matrix of $R_{12}$. We consider the tensor product of two NS representations (25): $\mathcal{F}_u \otimes \mathcal{F}_v$ with the vacuum state

\[
|\emptyset\rangle \equiv |u\rangle \otimes |v\rangle.
\]

Since the normalization of $R_{12}$ is not fixed from (39), it can be chosen at will. We fix it by $R_{12}|\emptyset\rangle = |\emptyset\rangle$. Then using (39) one can find the action of $R_{12}$ on any state in $\mathcal{F}_u \otimes \mathcal{F}_v$. In particular, using two equations

\[
R_{12}(\Psi_{12}^a)_{-\frac{1}{2}}|\emptyset\rangle = (\Psi_{21}^a)_{-\frac{1}{2}}|\emptyset\rangle, \quad R_{12}(G_{12}^*)_{-\frac{1}{2}}|\emptyset\rangle = (G_{21}^*)_{-\frac{1}{2}}|\emptyset\rangle,
\]

we find that

\[
R_{12}(\psi_1^a)_{-\frac{1}{2}}|\emptyset\rangle = \frac{(u-v)-(u^*-v^*)}{(u-v)-(u^*+v^*)+b^{-1}(\psi_1^a)_{-\frac{1}{2}}|\emptyset\rangle} - \\
\frac{2u^*-b^{-1}}{(u-v)-(u^*+v^*)+b^{-1}(\psi_2^a)_{-\frac{1}{2}}|\emptyset\rangle},
\]
Indeed, from the relation
\[(\psi_1^s, \psi_2^s)_{\frac{1}{2}} | \varnothing \rangle = \frac{2v^* - b^{-1}}{(u - v) - (u^* + v^*) + b^{-1} (\psi_1^s)_{\frac{1}{2}} | \varnothing \rangle + (u - v) + (u^* - v^*)}{(u - v) - (u^* + v^*) + b^{-1} (\psi_2^s)_{\frac{1}{2}} | \varnothing \rangle} \]
Similarly, using
\[\mathcal{R}_{12}(G_{12})_{\frac{1}{2}} | \varnothing \rangle = (G_{21})_{\frac{1}{2}} | \varnothing \rangle, \quad \mathcal{R}_{12}(H_{12})_{\frac{1}{2}} | \varnothing \rangle = (H_{21})_{\frac{1}{2}} | \varnothing \rangle, \]
we find that
\[\mathcal{R}_{12}(\psi_1^s)_{\frac{1}{2}} | \varnothing \rangle = \frac{(u - v) + (u^* - v^*)}{(u - v) + (u^* + v^*) + b^{-1} (\psi_1^s)_{\frac{1}{2}} | \varnothing \rangle + 2v^* + b^{-1}}{(u - v) + (u^* + v^*) + b^{-1} (\psi_2^s)_{\frac{1}{2}} | \varnothing \rangle}, \]
and
\[\mathcal{R}_{12}(\psi_2^s)_{\frac{1}{2}} | \varnothing \rangle = \frac{2u^* + b^{-1}}{(u - v) + (u^* + v^*) + b^{-1} (\psi_1^s)_{\frac{1}{2}} | \varnothing \rangle + (u - v) - (u^* - v^*)}{(u - v) + (u^* + v^*) + b^{-1} (\psi_2^s)_{\frac{1}{2}} | \varnothing \rangle}. \]
Proceeding in this way, one can compute the matrix of \(\mathcal{R}_{12}\) at any given order.

In general, it is clear that the matrix elements of \(\mathcal{R}_{12}\) are some rational functions of three parameters
\[u - v, \quad u^* \quad \text{and} \quad v^*. \]
Indeed, from the relation \([\mathcal{R}_{12}, U_{12}] = 0\) it follows that \(\mathcal{R}_{12}\) does not depend on \(u + v\). Generic expression for \(\mathcal{R}_{12}\) is unlikely can be obtained in a closed form. However, some particular cases and expansions might be useful. In particular, following \([4, 21]\), one can develop the expansion of \(\mathcal{R}_{12}(u, v)\) at large \(u - v\).

5 \textit{RLL algebra}

In this section we study \textit{RLL} algebra provided by the \(R\)--matrix defined in section 4
\[\mathcal{R}_{12}(u, v)\mathcal{L}_1(u)\mathcal{L}_2(v) = \mathcal{L}_2(v)\mathcal{L}_1(u)\mathcal{R}_{12}(u, v), \tag{53}\]
where as usual \(\mathcal{L}_1(u)\) and \(\mathcal{L}_2(v)\) are treated as matrices acting in \(\widehat{gl}(1|1)\) modules \(\mathcal{F}_u\) and \(\mathcal{F}_v\) correspondingly (see (25)) those entries are operators acting in some unspecified space (called quantum space). Then (53) is equivalent to the infinite set of quadratic relations between the matrix elements
\[\langle l | \mathcal{L}(u) | k \rangle, \tag{54}\]
where \(|k\) and \(|l\) are arbitrary states from \(\mathcal{F}_u\).
However, for various reasons it is more convenient to use the so called current realization of the same algebra. The isomorphism between the $R$–matrix and current realizations is well understood in the case of classical algebras (see e.g. [22]). For affine algebras it has been studied in [4] for $\tilde{g}l(1)$ and in [21] for $\tilde{g}l(2)$ cases.

We will proceed exactly as in [4, 21]. We note however, that there is a conceptual difference compared to $\tilde{g}l(1)$ and $\tilde{g}l(2)$ cases. Namely, the R-matrix constructed in section 4 is rather unusual since unlike most other known R-matrices it cannot be written as a function of the difference of two spectral parameters. The well known example of non-difference R-matrix has been found by Shastry [23] and depends on two spectral parameters individually rather than on their difference. Our R-matrix depends on three combinations of four spectral parameters. Nevertheless, we will show that the relations (53) imply that some of the currents are functions of only one spectral parameter, while the others are expressed in terms of them.

This phenomenon can be demonstrated at lower levels. Let us define the following currents

\[
\begin{align*}
  h(u) &\equiv \langle \emptyset | L(u) | \emptyset \rangle, \\
  e(u) &\equiv h^{-1}(u)\langle \emptyset | L(u) \psi_{-\frac{1}{2}} | \emptyset \rangle, \\
  f(u) &\equiv \langle \emptyset | \psi_{\frac{1}{2}} L(u) | \emptyset \rangle h^{-1}(u), \\
  e^*(u) &\equiv \frac{1}{2u^* - b^{-1}} h^{-1}(u) \langle \emptyset | L(u) \psi_{\frac{1}{2}}^* | \emptyset \rangle, \\
  f^*(u) &\equiv \frac{1}{2u^* + b^{-1}} \langle \emptyset | \psi_{\frac{1}{2}}^* L(u) | \emptyset \rangle h^{-1}(u). 
\end{align*}
\]

(55)

Taking vacuum average of $RLL = LLR$ relation, one finds

\[
[h(u), h(v)] = 0. 
\]

(56)

Similarly, using (47), (48), (50) and (51), we find

\[
((u + u^*) - (v - v^*) + b^{-1})h(u)e(v) = ((u - u^*) - (v - v^*))e(v)h(u) + (2u^* + b^{-1})h(u)e(u), 
\]

(57)

\[
((u - u^*) - (v + v^*) + b^{-1})h(u)e^*(v) = ((u + u^*) - (v + v^*))e^*(v)h(u) - \\
- (2u^* - b^{-1})h(u)e^*(u), 
\]

(58)

\[
((u - u^*) - (v + v^*) + b^{-1})f(v)h(u) = ((u + u^*) - (v + v^*))h(u)f(v) - (2u^* - b^{-1})f(u)h(u), 
\]

(59)

and

\[
((u + u^*) - (v - v^*) + b^{-1})f^*(v)h(u) = ((u - u^*) - (v - v^*))h(u)f^*(v) + \\
+ (2u^* + b^{-1})f^*(u)h(u). 
\]

(60)

The terms shown in blue are called local terms, since they depend only on $u$, but not on $v$.

Consider for example the relation (57), set $v - v^* = u - u^*$ and multiply it by $h^{-1}(u)$ from the left. Then the first term in the r.h.s. of (57) vanishes and one obtains

\[
e(v + v^*, u - u^*) = e(u + u^*, u - u^*) \implies e = e(u - u^*). 
\]

(61)

Similarly one finds

\[
f = f(u + u^*), \quad e^* = e^*(u + u^*), \quad \text{and} \quad f^* = f^*(u - u^*). 
\]

(62)
The relations similar to (61)–(62) persist at higher levels as well. Let us consider the basic currents at level 1

\[ a(u) \overset{\text{def}}{=} h^{-1}(u) \langle \emptyset | \mathcal{L}(u) a_{-1} | \emptyset \rangle, \quad a^+(u) \overset{\text{def}}{=} \frac{1}{2u^*-b^{-1}} h^{-1}(u) \langle \emptyset | \mathcal{L}(u) a^*_+ | \emptyset \rangle, \]

\[ k(u) \overset{\text{def}}{=} \frac{1}{2u^*-b^{-1}} h^{-1}(u) \langle \emptyset | \mathcal{L}(u) \psi_{-\frac{1}{2}} | \emptyset \rangle, \]

and their linear combinations

\[ \rho(u) \overset{\text{def}}{=} a(u) - (2u^*-b^{-1}) a^+(u) \quad \text{and} \quad \rho^+(u) \overset{\text{def}}{=} a(u) + (2u^*-2b^{-1}) a^+(u) + 2i k(u). \quad (64) \]

Using commutation relations (88) and (89) presented in the appendix, one finds

\[ \rho = \rho(u - u^*), \quad \rho^+ = \rho^+(u + u^*) \quad (65) \]

Also the currents at level \(-1\)

\[ \bar{a}(u) \overset{\text{def}}{=} \langle \emptyset | a_1 \mathcal{L}(u) | \emptyset \rangle h^{-1}(u), \quad \bar{a}^+(u) \overset{\text{def}}{=} \frac{1}{2u^* + b^{-1}} \langle \emptyset | a_1^* \mathcal{L}(u) | \emptyset \rangle h^{-1}(u), \]

\[ \bar{k}(u) \overset{\text{def}}{=} \frac{1}{2u^* + b^{-1}} \langle \emptyset | \bar{\psi}_{-\frac{1}{2}} | \emptyset \rangle h^{-1}(u) \]

and

\[ \mu(u) \overset{\text{def}}{=} \bar{a}(u) + (2u^* + b^{-1}) \bar{a}^+(u) \quad \text{and} \quad \mu^+(u) \overset{\text{def}}{=} \bar{a}(u) - (2u^* - 2b^{-1}) \bar{a}^+(u) - 2i \bar{k}(u), \quad (67) \]

satisfy similar relations with \(h(u)\), which imply

\[ \mu = \mu(u + u^*), \quad \mu^+ = \mu^+(u - u^*). \quad (68) \]

In addition to (65) and (68) there are composite currents, which also depend on one parameter instead of two. Namely, two level 1 currents

\[ \sigma(u) \overset{\text{def}}{=} (2u^* - b^{-1}) (e^+(u) e(u) + k(u)) : \quad \sigma = \sigma(u - u^*), \]

\[ \sigma^+(u) \overset{\text{def}}{=} (2u^* + b^{-1}) (e(u) e^+(u) - k(u)) - 2i a^+(u) : \quad \sigma^+ = \sigma^+(u + u^*), \quad (69) \]

and two level \(-1\) currents

\[ \nu(u) \overset{\text{def}}{=} (2u^* + b^{-1}) (f(u) f^+(u) - \bar{k}(u)) : \quad \nu = \nu(u + u^*), \]

\[ \nu^+(u) \overset{\text{def}}{=} (2u^* - b^{-1}) (f^+(u) f(u) + \bar{k}(u)) - 2i \bar{a}^+(u) : \quad \nu^+ = \nu^+(u - u^*). \quad (70) \]

The derivation of (69) and (70) is straightforward, but tedious task and is based on commutation relation with \(h(u)\).

We will also need the following currents

\[ \psi(u) \overset{\text{def}}{=} \langle \emptyset | \bar{\psi}_{-\frac{1}{2}} \mathcal{L}(u) \psi_{-\frac{1}{2}} | \emptyset \rangle h^{-1}(u) - \langle \emptyset | \bar{\psi}_{-\frac{1}{2}} \mathcal{L}(u) | \emptyset \rangle h^{-1}(u) \langle \emptyset | \mathcal{L}(u) \psi_{-\frac{1}{2}} | \emptyset \rangle h^{-1}(u), \]

\[ (71) \]

and

\[ \psi^+(u) \overset{\text{def}}{=} \langle \emptyset | \bar{\psi}_{-\frac{1}{2}} \mathcal{L}(u) | \emptyset \rangle h^{-1}(u) \langle \emptyset | \mathcal{L}(u) \psi_{-\frac{1}{2}} | \emptyset \rangle h^{-1}(u) - \langle \emptyset | \bar{\psi}_{-\frac{1}{2}} \mathcal{L}(u) \psi_{-\frac{1}{2}} | \emptyset \rangle h^{-1}(u), \]

\[ (72) \]

which can be shown to satisfy

\[ \psi(u) = \psi(u - u^*) \quad \text{and} \quad \psi^+(u) = \psi(u + u^*). \quad (73) \]
6 Current relations

We define the parameters
\[ h_1 \equiv \frac{b}{2} + b^{-1}, \quad h_2 \equiv \frac{b}{2} - b^{-1} \] (74)
and the currents
\[ e_1(u) \equiv e(u), \quad e_2(u) \equiv e^*\left( u + \frac{b}{2} \right), \quad f_1(u) \equiv f^*(u) \quad \text{and} \quad f_2(u) \equiv f \left( u + \frac{b}{2} \right), \] (75)
\[ \psi_1(u) \equiv \psi(u), \quad \psi_2(u) \equiv \psi^*(u) \] (76)

It is also convenient to define the currents labeled by pyramid partitions (see (64), (67), (69) and (70))
\[ e_{\phi}(u) \equiv \frac{i \rho(u)}{2h_2}, \quad e_\sigma(u) \equiv \sigma(u) - \frac{i \rho(u)}{2h_2}, \]
\[ e_{\phi}(u) \equiv \sigma^*\left( u + \frac{b}{2} \right) - \frac{i \rho^*(u + \frac{b}{2})}{2h_1}, \quad e_{\psi}(u) \equiv \frac{i \rho^*(u + \frac{b}{2})}{2h_1}, \] (78)
\[ f_{\phi}(u) \equiv \frac{i \mu(u + \frac{b}{2})}{2h_2}, \quad f_\nu(u) \equiv \nu\left( u + \frac{b}{2} \right) - \frac{i \mu(u + \frac{b}{2})}{2h_1}, \]
\[ f_{\phi}(u) \equiv \nu^*(u) - \frac{i \mu^*(u)}{2h_1}, \quad f_{\psi}(u) \equiv \frac{i \mu^*(u)}{2h_1}. \] (79)

One can derive the following commutation relations (see appendix A for details)
\[ \left[ \psi_1(u), \psi_2(v) \right] = 0, \quad \{ e_i(u), e_i(v) \} = 0, \quad \{ f_i(u), f_i(v) \} = 0, \] (80)
\[ \{ e_i(u), f_j(v) \} = \delta_{ij} \frac{\psi_i(u) \psi_j(v)}{u - v}, \] (81)
\[ (u - v - h_1)(u - v + h_1) \left( e_1(u)e_2(v) + \frac{e_\phi(v)}{u - v - h_1} + \frac{e_\psi(v)}{u - v + h_1} \right) = \]
\[ = -(v - u - h_2)(v - u + h_2) \left( e_2(v)e_1(u) + \frac{e_\phi(u)}{v - u - h_2} + \frac{e_\psi(u)}{v - u + h_2} \right) \] (82)
\[ (u - v - h_1)(u - v + h_1) \left( f_2(u)f_1(v) + \frac{f_{\phi}(u)}{v - u - h_1} + \frac{f_{\psi}(u)}{v - u + h_1} \right) = \]
\[ = -(v - u - h_2)(v - u + h_2) \left( f_1(v)f_2(u) + \frac{f_{\phi}(u)}{u - v - h_2} + \frac{f_{\psi}(u)}{u - v + h_2} \right) \] (83)

\*The analog of MacMahon for $Y(\mathfrak{gl}(1|1))$ is naturally associated with the space of pyramid partitions (see [24]). It induces natural notations for the currents. There are two pyramid partitions with one stone which correspond to $e_i(u)$ and $e_\phi(u)$
\[ e_i(u) = e_i(u) \quad \text{and} \quad e_\phi(u) = e_\phi(u). \] (77)
Also there are 4 pyramid partitions with 2 stones which correspond to the currents (78). Similar identification holds for $f$ currents as well.
First, we defined the $R$-matrix, which serves as an intertwining operator for certain $\mathcal{N}=2$ superconformal algebra. The remarkable property of this $R$-matrix is that it depends on 4 spectral

\[\psi_i(u)e_j(v) = \frac{(u-v-h_j)(u-v+h_j)}{(u-v-h_i)(u-v+h_i)}e_j(v)\psi_i(u) + \frac{h_i^2-h_j^2}{2h_i(u-v-h_i)}e_j(u-h_i)\psi_i(u) - \frac{h_i^2-h_j^2}{2h_i(u-v+h_i)}e_j(u+h_i)\psi_i(u)\] (84)

\[\psi_i(u)f_j(v) = \frac{(u-v-h_i)(u-v+h_i)}{(u-v-h_j)(u-v+h_j)}f_j(v)\psi_i(u) + \frac{h_j^2-h_i^2}{2h_j(u-v-h_j)}f_j(u-h_j)\psi_i(u) - \frac{h_j^2-h_i^2}{2h_j(u-v+h_j)}f_j(u+h_j)\psi_i(u)\] (85)

The relations (80)–(85) should be understood in terms of modes of basic currents expanded at $u=\infty$

\[e_k(u) = \frac{e_k^{(0)}}{u} + \frac{e_k^{(1)}}{u^2} + \ldots, \quad f_k(u) = \frac{f_k^{(0)}}{u} + \frac{f_k^{(1)}}{u^2} + \ldots, \quad \psi_k(u) = 1 + \frac{\psi_k^{(0)}}{u} + \frac{\psi_k^{(1)}}{u^2} + \ldots\] (86)

We note that local terms in (80)–(85) depend only on one spectral parameter and do not contribute to commutation relations for modes with $(i,j) > 1$. The relations with either $i \leq 1$ or $j \leq 1$ can be used to express higher currents in terms of basic ones. In particular, one finds from (82)

\[e_k(u) = 1 + \frac{e_k^{(0)}}{u} + \frac{e_k^{(1)}}{u^2} + \ldots, \quad f_k(u) = 1 + \frac{f_k^{(0)}}{u} + \frac{f_k^{(1)}}{u^2} + \ldots, \quad \psi_k(u) = 1 + \frac{\psi_k^{(0)}}{u} + \frac{\psi_k^{(1)}}{u^2} + \ldots\]

Similar relations hold for higher currents (79) as well.

We note that (80)–(85) coincide exactly with commutation relations for $Y(\widehat{\mathfrak{gl}}(1|1))$ listed in [11]. It is important to note that $RLL$-algebra contains more currents than $Y(\widehat{\mathfrak{gl}}(1|1))$. In particular, it has the current $h(u)$, which is not in $Y(\widehat{\mathfrak{gl}}(1|1))$. Moreover this current depends on two spectral parameters rather than one. The precise relation between two algebras, $RLL$ and $Y(\widehat{\mathfrak{gl}}(1|1))$, should be described by studying the center of the first along the lines of [4]. We postpone this study to future publication.

Also the algebra $Y(\widehat{\mathfrak{gl}}(1|1))$ contains Serre relations. Their derivation from $RLL$ algebra is straightforward, but tedious task, which we expect to return in future publication (see [4] for similar derivation for $Y(\widehat{\mathfrak{gl}}(1))$).

7 Conclusion

In this paper we derived current relations of $Y(\widehat{\mathfrak{gl}}(1|1))$ (80)–(85) starting from $RLL$ realization. First, we defined the $R$-matrix, which serves as an intertwining operator for certain $\mathcal{N}=2$ superconformal algebra. The remarkable property of this $R$-matrix is that it depends on 4 spectral
parameters instead of 2. However, we have shown that the relations of \( RLL \) algebra imply that some of the currents depend only on one spectral parameter. In particular, we defined the currents \( e_i(u) \), \( f_i(u) \) and \( \psi_i(u) \) which satisfy the relations (80)–(85).

There are several open questions, which we plan to address in future publications:

- The relations (80)–(85) should be supplemented by Serre relations to define the correct algebra. The derivation of these relations starting from \( RLL \) algebra is rather tedious. It is expected that Serre relations should be of quartic order of total charge 0 (see e.g. [13]). This requires studying of the \( R \)–matrix acting in 39-dimensional space (as can be easily seen from the character formula (28)).

- It is interesting to describe explicitly the embedding \( Y(\hat{\mathfrak{gl}}(1)) \oplus Y(\hat{\mathfrak{gl}}(1)) \in Y(\hat{\mathfrak{gl}}(1|1)) \), which is suggested by the results of [7,8]. Namely, one expects to construct two copies of \( Y(\hat{\mathfrak{gl}}(1)) \) currents \((e(u), f(u), \psi(u))\) and \((\hat{e}(u), \hat{f}(u), \hat{\psi}(u))\), as well as auxiliary fermionic currents \( x(u), \bar{x}(u), y(u) \) and \( \bar{y}(u) \) (see [7,8]).

- Using current realization of \( Y(\hat{\mathfrak{gl}}(1|1)) \) it is straightforward to derive Bethe ansatz equations following the lines of [4,21]. These Bethe ansatz equations will correspond in particular to quantum version of \( \mathcal{N} = 2 \) KdV hierarchy and its \( W_n \) generalizations. We plan to provide more details elsewhere.

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A Current commutation relations

Here we provide some details on derivation of commutation relations of \( Y(\hat{\mathfrak{gl}}(1|1)) \).
\textbf{\( h\rho \) and \( h\rho^* \) commutation relations.} Using \( RLL \) relations on level 1 and charge 0 one finds:

\[
R_{\varphi}(v) = \frac{((u-v *) - (v - v *)) ((u + u *) - (v - v *) - b + b^{-1})}{((u - u *) - (v - v *) - b + 2b^{-1}) ((u + u *) - (v - v *) + b + b^{-1})} \rho(v) h(u) + \frac{2(u^* + \frac{1}{2b})}{(u + u *) - (v - v *) - b + b^{-1}} h(u) \rho(v) - \frac{2((u - u *) - (v - v *)) (u^* - \frac{1}{2b})}{((u - u *) - (v - v *) - b + 2b^{-1}) ((u + u *) - (v - v *) + b + b^{-1})} h(u) \rho^*(u) + \frac{2((u - u *) - (v - v *)) (u^* + \frac{1}{2b})}{((u - u *) - (v - v *) - b + 2b^{-1}) ((u + u *) - (v - v *) + b + b^{-1})} h(u) a^*(u) + \frac{4i((u - u *) - (v - v *)) (u^* - \frac{1}{2b})}{((u - u *) - (v - v *) - b + 2b^{-1}) ((u + u *) - (v - v *) + b + b^{-1})} e(v) h(u) e^*(u) \quad (88)
\]

and

\[
R_{\varphi}^*(v) = \frac{((u + u *) - (v + v *)) ((u - u *) - (v + v *) + b + b^{-1})}{((u - u *) - (v + v *) + b^{-1}) ((u + u *) - (v + v *) + b + 2b^{-1})} \rho^*(v) h(u) - \frac{2(u^* - \frac{1}{2b})}{(u - u *) - (v + v *) - b + b^{-1}} h(u) \rho^*(u) + \frac{2((u + u *) - (v + v *)) (u^* + \frac{1}{2b})}{((u - u *) - (v + v *) + b^{-1}) ((u + u *) - (v + v *) + b + 2b^{-1})} h(u) (a(u) + a^*(u)) - \frac{4i((u + u *) - (v + v *)) (u^* + \frac{1}{2b})}{((u - u *) - (v + v *) + b^{-1}) ((u + u *) - (v + v *) + b + 2b^{-1})} e^*(v) h(u) e(u) \quad (89)
\]

\textbf{ee commutation relation.} We introduce the following notation

\[
L_{\varphi \varphi}(u) \overset{\text{def}}{=} \langle \varnothing \mid L(u) \psi_{\frac{1}{2}} \mid \varnothing \rangle, \quad L_{\varphi \varphi}^*(u) \overset{\text{def}}{=} \langle \varnothing \mid L^*(u) \psi_{-\frac{1}{2}} \mid \varnothing \rangle, \quad L_{\psi \varphi}(u) \overset{\text{def}}{=} \langle \varnothing \mid \psi_{\frac{1}{2}} L(u) \mid \varnothing \rangle, \quad L_{\psi \varphi}^*(u) \overset{\text{def}}{=} \langle \varnothing \mid \psi_{-\frac{1}{2}} L^*(u) \mid \varnothing \rangle, \quad L_{\psi \psi}(u) \overset{\text{def}}{=} \langle \varnothing \mid \psi_{\frac{1}{2}} L(u) \psi_{\frac{1}{2}} \mid \varnothing \rangle. \quad (90)
\]

Then the equation

\[
\langle \varnothing \mid R_{12} (u, v) L_{1}(u) L_{2}(v) (\psi_{1})_{\frac{1}{2}} (\psi_{2})_{-\frac{1}{2}} \mid \varnothing \rangle = \langle \varnothing \mid L_{2}(v) L_{1}(u) R_{12} (u, v) (\psi_{1})_{-\frac{1}{2}} (\psi_{2})_{\frac{1}{2}} \mid \varnothing \rangle \quad (91)
\]

can be rewritten in the explicit form

\[
L_{\varphi \psi}(u) L_{\varphi \psi}^*(v) = \frac{(u - u *) - (v + v *) - b^{-1}}{(u + u *) - (v - v *) + b^{-1}} L_{\varphi \psi}^*(v) L_{\varphi \psi}(u), \quad (92)
\]

where we considered the action of \( R \)-matrix on \( R_{12} (u, v) (\psi_{1})_{\frac{1}{2}} (\psi_{2})_{-\frac{1}{2}} \mid \varnothing \). It is important to note the presence of the "minus" sign in this equation due to taking into account the fermions.

Consider also the equation

\[
\langle \varnothing \mid R_{12} (u, v) L_{1}(u) L_{2}(v) (\psi_{1})_{\frac{1}{2}} \mid \varnothing \rangle = \langle \varnothing \mid L_{2}(v) L_{1}(u) R_{12} (u, v) (\psi_{1})_{\frac{1}{2}} \mid \varnothing \rangle, \quad (93)
\]
which takes the form (compare to (50))

\[
L_{\emptyset \psi}(u)h(v) = \frac{(u + u^*) - (v + v^*)}{(u + u^*) - (v - v^*) + b^{-1}}h(v)L_{\emptyset \psi}(u) + \frac{2v^* + b^{-1}}{(u + u^*) - (v - v^*) + b^{-1}}L_{\emptyset \psi}(v)h(u). \tag{94}
\]

Then multiplying (92) by \(h^{-1}(u)h^{-1}(v)\) on the left and using (94), we get the anticommutation relation

\[
\{e(u), e(v)\} = \frac{2u^* + b^{-1}}{(u + u^*) - (v - v^*) + b^{-1}}e^2(u) - \frac{2v^* + b^{-1}}{(u - u^*) - (v + v^*) - b^{-1}}e^2(v). \tag{95}
\]

The relations (95) should not have any poles. It implies that \(e^2(u) = 0\) and \(\{e(u), e(v)\}\).

\textbf{ee* commutation relation.} We consider the equation

\[
\langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(\psi_1)_{-\frac{1}{2}}(\psi_2)_{-\frac{1}{2}}| \emptyset \rangle = \langle \emptyset | L_2(v)L_1(u)R_{12}(u, v)(\psi_1)_{-\frac{1}{2}}(\psi_2)_{-\frac{1}{2}}| \emptyset \rangle. \tag{96}
\]

In principle, one can add to this formula an arbitrary linear combination of RLL-relations at level 1 with charge 0:

\[
\langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(\psi_1)_{-\frac{1}{2}}(\psi_2)_{-\frac{1}{2}}| \emptyset \rangle + \alpha_1 \langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(a_1)_{-\frac{1}{2}}| \emptyset \rangle + \alpha_2 \langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(a_2)_{-\frac{1}{2}}| \emptyset \rangle + \ldots \tag{97}
\]

but we leave only those terms which, after multiplying by \(h^{-1}(u)h^{-1}(v)\) from the left, turn out to be local:

\[
\langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(\psi_1)_{-\frac{1}{2}}(\psi_2)_{-\frac{1}{2}}| \emptyset \rangle + \alpha \langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(a_1)_{-\frac{1}{2}}| \emptyset \rangle + \beta \langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(a_2)_{-\frac{1}{2}}| \emptyset \rangle + \gamma \langle \emptyset | R_{12}(u, v)L_1(u)L_2(v)(\psi_2)_{-\frac{1}{2}}(\psi_2^*)_{-\frac{1}{2}}| \emptyset \rangle. \tag{98}
\]

The coefficients \(\alpha, \beta, \gamma\) are found from the condition of cancellation of the terms

\[
\langle \emptyset | L_2(v)L_1(u)(a_1)_{-\frac{1}{2}}| \emptyset \rangle, \quad \langle \emptyset | L_2(v)L_1(u)(a_2)_{-\frac{1}{2}}| \emptyset \rangle, \quad \langle \emptyset | L_2(v)L_1(u)(\psi_2)_{-\frac{1}{2}}(\psi_2^*)_{-\frac{1}{2}}| \emptyset \rangle, \tag{99}
\]

in the right side of the RLL-relation (98). Their explicit expressions are

\[
\alpha = \frac{i(2v^* - b^{-1})}{((u - u^*) - (v - v^*)((\Delta + h_1 + \frac{b}{2}))}, \quad \beta = \frac{-2i((u - u^*) - v + \frac{1}{2b})}{((u - u^*) - (v - v^*)((\Delta + h_1 + \frac{b}{2}))}, \tag{100}
\]

\[
\gamma = \frac{2}{((u - u^*) - (v - v^*)((\Delta + h_1 + \frac{b}{2}))} - \frac{2v^* + b^{-1}}{(u - u^*) - (v - v^*)}. \tag{101}
\]

Here and below, we use the following notation

\[
\Delta = (u - u^*) - (v + v^*), \quad h_1 = \frac{b}{2} + b^{-1}, \quad h_2 = \frac{b}{2} - b^{-1}. \tag{101}
\]
Then, taking into account the action of the $R$-matrix on the states of level 1 with charge 0, we get

\[
\mathcal{L}_{\Psi^*}(u)\mathcal{L}_{\Phi\psi'}(v) + \alpha h(u)\mathcal{L}_{\Phi\alpha}(v) + \beta h(u)\mathcal{L}_{\Phi\alpha}(v) + \gamma \mathcal{L}_{\Phi\kappa}(v) = \\
= \frac{((u + u^*) - (v + v^*))(\Delta + h_2 + \frac{b}{2})}{((u - u^*) - (v - v^*)+(\Delta + h_1 + \frac{b}{2})}L_{\Phi\psi'}(v)L_{\Phi\psi}(u) - \\
- \frac{2v^* - b^{-1}(\Delta + h_2 + \frac{b}{2})}{((u - u^*) - (v - v^*)+(\Delta + h_1 + \frac{b}{2})}h(v)\mathcal{L}_{\Phi\kappa}(u) + \\
+ \frac{i(2v^* - b^{-1})}{((u - u^*) - (v - v^*)+(\Delta + h_1 + \frac{b}{2})}h(v)(\mathcal{L}_{\Phi\alpha}(u) - \mathcal{L}_{\Phi\alpha}(u)). \tag{102}
\]

Multiplying this expression by $h^{-1}(u)h^{-1}(v)$ on the left and using commutation relations $h^{-1}(v)L_{\Phi\psi}(u)$ and $h^{-1}(u)L_{\Phi\psi'}(v)$, one obtains

\[
e(u)e^*(v) - \frac{i\rho^*(v)}{(\Delta + h_1 + \frac{b}{2})(\Delta - h_1 + \frac{b}{2})} + \frac{(2v^* + b^{-1})(e(v)e^*(v) - k(v)) - 2ia^*(v)}{(\Delta - h_1 + \frac{b}{2})} = \\
= -\frac{1}{(\Delta + h_1 + \frac{b}{2})(\Delta - h_1 + \frac{b}{2})}e^*(v)e(u) - \frac{i\rho(u)}{(\Delta + h_1 + \frac{b}{2})(\Delta - h_1 + \frac{b}{2})} - \\
- \frac{(\Delta + h_2 + \frac{b}{2})}{(\Delta + h_1 + \frac{b}{2})(\Delta - h_1 + \frac{b}{2})}2u^* - b^{-1}(e^*(u)e(u) + k(u)), \tag{103}
\]

We see that, this commutation relation corresponds to (82).

**$e^*f$ commutation relation.** Let us consider the equation

\[
\langle \emptyset | (\psi_2)_{1/2} R_{12}(u, v) \mathcal{L}_1(u) \mathcal{L}_2(v)(\psi_1^*)_{-1/2} | \emptyset \rangle = \\
= \langle \emptyset | (\psi_2)_{1/2} \mathcal{L}_2(v) \mathcal{L}_1(u) R_{12}(u, v)(\psi_1^*)_{-1/2} | \emptyset \rangle. \tag{104}
\]

Then from the definition of the $R$-matrix one finds

\[
- \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}}\mathcal{L}_{\Phi\psi^*}(u)h(v) - \frac{(u + u^*) - (v + v^*)}{(u - u^*) - (v + v^*) + b^{-1}}\mathcal{L}_{\Phi\psi^*}(u)\mathcal{L}_{\Phi\psi}(v) = \\
= \frac{(u - u^*) - (v - v^*)}{(u - u^*) - (v + v^*) + b^{-1}}\mathcal{L}_{\Phi\psi}(v)\mathcal{L}_{\Phi\psi^*}(u) - \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}}\mathcal{L}_{\Phi\psi^*}(v)h(u). \tag{105}
\]

Let us also consider two auxiliary equations

\[
\langle \emptyset | R_{12}(u, v) \mathcal{L}_1(u) \mathcal{L}_2(v)(\psi_1^*)_{-1/2} | \emptyset \rangle = \langle \emptyset | \mathcal{L}_2(v) \mathcal{L}_1(u) R_{12}(u, v)(\psi_1^*)_{-1/2} | \emptyset \rangle, \tag{106}
\]
\[
\langle \emptyset | (\psi_2)_{1/2} R_{12}(u, v) \mathcal{L}_1(u) \mathcal{L}_2(v) | \emptyset \rangle = \langle \emptyset | (\psi_2)_{1/2} \mathcal{L}_2(v) \mathcal{L}_1(u) R_{12}(u, v) | \emptyset \rangle, \tag{107}
\]

16
which can be rewritten explicitly as

\[
\mathcal{L}_{\psi^*}(u)\mathcal{h}(v) = \frac{(u - u^*) - (v - v^*)}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{h}(v)\mathcal{L}_{\psi^*}(u) - \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{L}_{\psi^*}(v)\mathcal{h}(u),
\]

and

\[
\mathcal{L}_{\psi^*}(v)\mathcal{h}(u) = \frac{(u + u^*) - (v + v^*)}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{h}(u)\mathcal{L}_{\psi^*}(v) - \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{L}_{\psi^*}(u)\mathcal{h}(v),
\]

Multiplying (105) by \(h^{-1}(u)h^{-1}(v)\) from the right and using these relations, one finds

\[
- \mathcal{L}_{\psi^*}(u)h^{-1}(u)\mathcal{L}_{\psi^*}(v)h^{-1}(v) - \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{L}_{\psi^*}(u)h^{-1}(u) - \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{L}_{\psi^*}(u)h^{-1}(u) = \]

\[
= \mathcal{L}_{\psi^*}(v)h^{-1}(v)\mathcal{L}_{\psi^*}(u)h^{-1}(u) - \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{L}_{\psi^*}(v)h^{-1}(v) + \frac{2u^* - b^{-1}}{(u - u^*) - (v + v^*) + b^{-1}} \mathcal{L}_{\psi^*}(v)h^{-1}(v)\mathcal{L}_{\psi^*}(v)h^{-1}(v). \tag{110}
\]

Equation (58) implies

\[
e^*(u - u^* + b^{-1}) = \frac{1}{2u^* - b^{-1}} \mathcal{L}_{\psi^*}(u)h^{-1}(u),
\]

from which one can obtain

\[
\{e^*(u - u^* + b^{-1}), f(v)\} = \frac{\psi^*(u, u^* + b^{-1}) - \psi^*(v)}{(u - u^*) - (v + v^*) + b^{-1}}, \tag{112}
\]

where (compare to (72))

\[
\psi^*(u + u^*) \overset{\text{def}}{=} \mathcal{L}_{\psi^*}(u)h^{-1}(u)\mathcal{L}_{\psi^*}(u)h^{-1}(u) - \mathcal{L}_{\psi^*}(u)h^{-1}(u). \tag{113}
\]

So, this formula corresponds to (81). Similarly, one can get the commutation relation for \(e(u)\) and \(f^*(v)\):

\[
\{e(u + u^* + b^{-1}), f^*(v)\} = \frac{\psi(u, u^* + b^{-1}) - \psi(v)}{(u - u^*) - (v - v^*)} \tag{114}
\]

with

\[
\psi(u - u^*) \overset{\text{def}}{=} \mathcal{L}_{\psi^*}(u)h^{-1}(u) - \mathcal{L}_{\psi^*}(u)h^{-1}(u)\mathcal{L}_{\psi^*}(u)h^{-1}(u). \tag{115}
\]

The remaining relations from (80)–(85) can be derived in a similar way.
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