AN EXAMPLE OF LIFTINGS WITH DIFFERENT HODGE NUMBERS

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Abstract. In this paper, we exhibit an example of a smooth proper variety in positive characteristic possessing two liftings with different Hodge numbers.

1. Introduction

Does a smooth proper variety in positive characteristic know the Hodge number of its liftings? In this paper, we construct an example showing that the answer is no in general. There are some obstructions to make such an example. For instance, such an example must be of dimension at least 3 (see Proposition 4.4). The examples we constructed here are 3-folds in characteristic at least 3, see Section 3 and Subsection 4.1.

2. Notations

Throughout this paper, unless specified otherwise, let $p \geq 5$ be a prime, let $R = \mathbb{Z}_p[\zeta_p]$ where $\zeta_p$ is a $p$-th root of unity. Let $E/\text{Spec}(R)$ be an ordinary elliptic curve possessing a $p$-torsion $P \in E(R)[p]$ which does not specialize to identity element. Fix such an auxiliary elliptic curve along with this $p$-torsion point. Denote the uniformizer $\zeta_p - 1 \in R$ by $\pi$. Denote the fraction field of $R$ by $K$, the residue field by $\kappa$. We will use $\mathcal{O}$ to denote an unspecified mixed characteristic discrete valuation ring, which will only show up in Proposition 4.4.

We will use curly letters to denote integral objects over $\text{Spec}(R)$, use the corresponding straight letter to denote its generic fibre and use subscript $(\cdot)_0$ to denote its special fibre, i.e., reduction mod $\pi$. For example, we will denote the generic fibre of $E$ by $E$ and the special fibre by $E_0$. To simplify the notations, whenever no confusion seems to arise, we will not denote the base over which we make the fibre product.

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1There are such pairs over $\mathbb{Z}_p$. Indeed, the Honda–Tate theory tells us the polynomial $x^2 - x + p$ corresponds to an ordinary elliptic curve $E_0$ over $\mathbb{F}_p$ with $p$ rational points (c.f. [Int74, THÉORÈME 1.1]). In particular, we see that $E_0(\mathbb{F}_p) \cong \mathbb{Z}/p$. Now the Serre–Tate theory (c.f. [Kat81, Chapter 2]) tells us $E$, the canonical lift of $E_0$ over $\mathbb{Z}_p$, satisfies $E[p] \cong \mathbb{Z}/p \times \text{Spec}(\mathbb{Z}_p)$. Hence we see that all the rational points of $E_0$ are liftable over $\mathbb{Z}_p$. 

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3. L’exemple

Let $C$ be the proper smooth hyper-elliptic curve over $\text{Spec}(\mathbb{R})$ defined by

$$v^2 = \sum_{i=0}^{p-1} \binom{\frac{p}{2}}{i} u^{p-i}$$

One checks easily that this curve has genus $\frac{p-1}{2}$ and $C_0$, its reduction mod $\pi$, is the hyper-elliptic curve defined by

$$v^2 = u^p - u.$$

After inverting $\pi$ and making the substitution

$$x = (\zeta_p - 1)u + 1$$
$$y = v,$$

we see that $C$, the generic fibre of $C$, is the hyper-elliptic curve defined by

$$(\zeta_p - 1)^p y^2 = x^p - 1.$$

There is an $\mathbb{R}$-linear $\mathbb{Z}/p = \langle \sigma \rangle$-action on $C$ given by

$$\sigma(u) = \zeta_p \cdot u + 1$$
$$\sigma(v) = v.$$

One checks that in the generic fibre, using $xy$-coordinate, this action becomes $\sigma(x) = \zeta_p \cdot x$ and $\sigma(y) = y$. In the special fibre, this action becomes $\sigma(u) = u + 1$ and $\sigma(v) = v$.

We have a canonical character $\mathbb{Z}/p \to K^\times$ given by

$$\chi: \langle \sigma \rangle \to K^\times$$
$$\sigma \mapsto \zeta_p.$$

**Proposition 3.1.** Using notations as above, we have

1. in the special fibre, the action of $\sigma$ and $\sigma^4$ are conjugate by an automorphism of $C_0$;
2. in the generic fibre, we have a decomposition

$$H^0(C, \Omega^1) = \bigoplus_{1 \leq i \leq \frac{p-1}{2}} \chi^i$$

as representations of $\langle \sigma \rangle$.

**Proof.** (1) Consider the automorphism $\tau: C_0 \to C_0$ given by

$$\tau(u) = 4u$$
$$\tau(v) = 2v.$$

One easily verifies that this preserves the equation $v^2 = u^p - u$ hence an automorphism of $C_0$, and that $\tau \circ \sigma \circ \tau^{-1} = \sigma^4$. This completes the proof of (1).

(2) Recall that \( \left\{ \frac{dx}{y}, \frac{x \, dx}{y}, \ldots, \frac{x^{p-1} \, dx}{y} \right\} \) form a basis of $H^0(C, \Omega^1)$ whenever $C$ is a genus $g$ hyper-elliptic curve given by $y^2 = f(x)$. One checks immediately that under this basis, $\sigma$ acts by the characters as in the Proposition. \( \Box \)

\[ ^2 \text{We leave it to the readers to verify that this indeed defines a smooth proper curve with the other affine piece given by } v^2 = \sum_{i=0}^{p-1} \binom{\frac{p}{2}}{i} u^{i+1}. \]
we find above correspond to is with respect to the subgroup $\mu$. Chaps have realized the curve $\{ \}$

**Construction 3.2.** Let $X := (C \times C \times E)/(\langle \sigma, \sigma, \tau \rangle)$ and let $Y := (C \times C \times E)/(\langle \sigma, \sigma^4, \tau \rangle)$. Here we mean the schematic quotient by the indicated diagonal action.

Then we have the following:

**Proposition 3.3.** Both $X$ and $Y$ are smooth proper over $\text{Spec}(R)$, and their special fibers are isomorphic as smooth proper $k$-varieties. Moreover we have $H^0(X, \Omega^3_X) = 0$ and $H^0(Y, \Omega^3_Y) \neq 0$.

**Proof.** The third component ensures that the action is fixed point free. Therefore the quotient is smooth and proper, and it satisfies the following base change of taking quotient:

$$X_0 \equiv (C_0 \times C_0 \times E_0)/(\langle \sigma, \sigma, \tau \rangle),$$

$$Y_0 \equiv (C_0 \times C_0 \times E_0)/(\langle \sigma, \sigma^4, \tau \rangle).$$

By [3.1](1), $\sigma$ and $\sigma^4$ are conjugate to each other by $\tau$ (with notations loc. cit.). We see that $(\text{id}, \tau, \text{id})$ induces an isomorphism between $X_0$ and $Y_0$.

In the generic fibre, we have that the global 3-forms of the quotient are identified as the invariant (regarding respective actions) global 3-forms of $C \times C \times E$. By Künneth formula and [3.1](2), we have the following decomposition

$$H^3,0(C \times C \times E) = \bigoplus_{1 \leq i \leq 3} \bigoplus_{1 \leq j \leq 3} \chi^i \otimes \chi^j \otimes \mathbb{L}$$

as $(\sigma, \sigma, \tau)$-representations. Therefore we see that $H^3,0(X) = 0$. To see that $H^3,0(Y, \Omega^3_Y) \neq 0$, we note that in the above decomposition $\frac{dz}{y_1} \wedge \frac{dz}{y_2} \wedge \omega$ is invariant under $(\sigma, \sigma^4, \tau)$, where $\omega$ is some translation invariant nonzero 1-form on $E$. Here we have used $p \geq 5$, so that $\frac{dz}{y_1}$ is a holomorphic global 1-form on $C$. Hence we get that $H^3,0(Y, \Omega^3_Y) \neq 0$.

**Remark 3.4.** Those readers who are familiar with Deligne–Lusztig varieties perhaps have realized the curve $C_0 = \{ y^2 = x^p - x \}$ is nothing but the quotient of the Drinfeld’s curve $\{ y^{p+1} = x^{p+1} - x \}$ (c.f. [DL76] Ch. 2), where the quotient is with respect to the subgroup $\mu_{p+1} \subset \mu_{p}$ acts on $y$ by multiplication and fixes $x$ and $z$. Hence the curve $C_0$ bears the action of $\text{SL}_2(\mathbb{F}_p) \times \mathbb{Z}/2$ where the second factor is the hyper-elliptic structure of $C_0$. Under this identification, the $\sigma$ (resp. $\tau$) we find above correspond to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$) (possibly times the nontrivial involution depending on whether $2 \equiv 1$ or $-2 \equiv 1$ in $\mathbb{F}_p$).

Following the same spirit, we construct similar example in the case $p = 3$ (see Subsection 3.3). Concerning the case $p = 2$ one may ask the following:

**Question 3.5.** Can one lift a Deligne–Lusztig curve of Suzuki type (c.f. [GKT06]) in characteristic 2 equivariantly with respect to an element in the Suzuki group $2B_2$, so that similar example can be made out of?
4. COMPLEMENTS AND REMARKS

4.1. Case $p = 3$. Let us consider the case $p = 3$ in this subsection. Let $R = \mathbb{Z}_3[\omega, i]$ where $\omega$ is a 3-rd root of unity and $i^2 = -1$. Denote the uniformizer $\omega - 1 \in R$ by $\pi$. Let $\mathcal{C}$ be the proper smooth hyper-elliptic curve over $\text{Spec}(R)$ defined by

$$v^2 = (u^3 + (\omega^2 - 1)u^2 - \omega^2 u)^3 + (u^3 + (\omega^2 - 1)u^2 - \omega^2 u).$$

One checks easily that this curve has genus 4 and $\mathcal{C}_0$, its reduction mod $\pi$, is the hyper-elliptic curve defined by $v^2 = u^9 - u$. After inverting $\pi$ and making the substitution

$$x = (\omega - 1)u + 1$$

$$y = v,$$

we see that $C$, the generic fibre of $\mathcal{C}$, is the hyper-elliptic curve defined by

$$y^2 = \frac{1}{(\omega - 1)^9} \cdot (x^3 - 1)^3 + \frac{1}{(\omega - 1)^3} \cdot (x^3 - 1).$$

There is an $R$-linear $\mathbb{Z}/3$-action on $\mathcal{C}$ given by

$$\sigma(u) = \omega \cdot u + 1$$

$$\sigma(v) = v.$$

Similar to the Section 3 and use analogous notation as there, we state the following:

**Proposition 4.1.** Using notations as above, we have

1. in the special fibre, the action of $\sigma$ and $\sigma^2$ are conjugate by an automorphism of $\mathcal{C}_0$;
2. in the generic fibre, we have a decomposition

$$H^0(C, \Omega^1_C) = \chi^{\otimes 2} \oplus \chi^2 \oplus 1$$

as representations of $\langle \sigma \rangle$.

The proof is similar, notice that now the automorphism group of $\mathcal{C}_0$ is $\text{SL}_2(\mathbb{F}_9) \times \mathbb{Z}/2$ and $2 = -1 = i^2$ is a square in $\mathbb{F}_9$.

Possibly passing to an unramified extension of $R$, we may assume as before that there is an elliptic curve $\mathcal{E}$ over $R$ together with a nonzero 3-torsion point $P$. Then we make the following:

**Construction 4.2.** Let $\mathcal{X} := (C \times C \times \mathcal{E})/\langle (\sigma, \sigma, \tau_P) \rangle$ and let $\mathcal{Y} := (C \times C \times \mathcal{E})/\langle (\sigma, \sigma^2, \tau_P) \rangle$.

**Proposition 4.3.** Both of $\mathcal{X}$ and $\mathcal{Y}$ are smooth proper over $\text{Spec}(R)$ and we have $h^{3,0}(X) = 5$ and $h^{3,0}(Y) = 6$.

4.2. Final Remarks. The following Proposition shows that our example is sharp in terms of its dimension (the case of curve is trivial).

**Proposition 4.4.** Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth proper schemes over $\text{Spec}(\mathcal{O})$ of relative dimension 2. Suppose $\mathcal{X}_0 \cong \mathcal{Y}_0$, then $h^{i,j}(X) = h^{i,j}(Y)$ for all $i,j$. 
Proof. Since for surfaces we have \( \frac{1}{2}b_1 = h^{0,1} = h^{1,0} = h^{0,3} = h^{3,0} \), by smooth proper base change we know that these numbers only depend on the special fibre. Therefore the Hodge numbers of \( X \) and \( Y \) agree except for the degree 2 part. Now the fact that the Euler characteristic of a flat coherent sheaf stays constant in a family shows that the degree 2 Hodge numbers of \( X \) and \( Y \) also agree. \( \square \)

In order to make such an example, dimension is certainly not the only constraint.

**Proposition 4.5.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be smooth proper schemes over \( \text{Spec}(\mathcal{O}) \) with \( \mathcal{X}_0 \cong \mathcal{Y}_0 \). Suppose the Hodge-to-de Rham spectral sequence for \( \mathcal{X}_0 \) degenerates at \( E_1 \)-page and \( H^r_{\text{cris}}(\mathcal{X}_0/W(k)) \) is torsion-free for all \( r \). Then \( h^{i,j}(X) = h^{i,j}(Y) \) for all \( i,j \).

**Proof.** The crystalline cohomology being torsion-free implies that \( h^r_{\text{cris}}(X) = h^r_{\text{cris}}(Y) \). In the generic fibre, by Hodge theory, we have \( \sum_{i+j=r} h^{i,j}(X) = h^r_{\text{cris}}(X) \). In the special fibre, by the degeneration of Hodge-to-de Rham spectral sequence, we have \( \sum_{i+j=r} h^{i,j}(\mathcal{X}_0) = h^r_{\text{cris}}(\mathcal{X}_0) \). These three equalities along with upper semi-continuity of \( h^{i,j} \) imply \( h^{i,j}(\mathcal{X}_0) = h^{i,j}(X) \). Then same argument implies \( h^{i,j}(\mathcal{X}_0) = h^{i,j}(Y) \). Hence we see that the Hodge numbers of \( X \) and \( Y \) are the same. \( \square \)

**Remark 4.6.** Using the fact that \( H_1(C;\mathbb{Z}) \) as a \( \mathbb{Z}/p \)-module is the augmentation ideal in \( \mathbb{Z}/[\mathbb{Z}/p] \), one can show that \( h^1_{\text{cris}}(\mathcal{X}_0) = 4 \) and \( h^1_{\text{cris}}(X) = 2 \), which implies that \( \dim_{\mathbb{Z}}[H^2_{\text{cris}}(\mathcal{X}_0/W(k))[p] = 2. \)

A more detailed study shows that the length of torsions in the crystalline cohomology groups of our examples stay bounded for all primes \( p \), however the discrepancy between \( h^{3,0}(X) \) and \( h^{3,0}(Y) \) grows linearly in \( p \).

We conclude this paper by observing that the examples we found are over ramified base with absolute ramification index \( p-1 \) and asking:

**Question 4.7.** Is there a pair of smooth proper schemes \( \mathcal{X} \) and \( \mathcal{Y} \) over \( \text{Spec}(W(k)) \), such that

1. \( \mathcal{X}_0 \cong \mathcal{Y}_0 \) and;
2. \( h^{i,j}(X) \neq h^{i,j}(Y) \) for some \( i,j \)?

Note that by [DI87] Corollaire 2.4] the Hodge-to-de Rham spectral sequence for any smooth proper \( \mathcal{X}_0 \) degenerates at \( E_1 \)-page, provided that \( \dim(\mathcal{X}_0) < p \) and \( \mathcal{X}_0 \) lifts to \( W_2(k) \). In particular, the example asked for in Question 4.7, if it exists and is of small dimension (say, 3-fold), must have torsion in \( H^*_{\text{cris}}(\mathcal{X}_0/W(k)) \) by Proposition 4.5.

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