The group $J_4 \times J_4$ is recognizable by spectrum

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Abstract The spectrum of a finite group is the set of its element orders. In this paper we prove that the direct product of two copies of the finite simple sporadic group $J_4$ is uniquely determined by its spectrum in the class of all finite groups.

Keywords: finite group, spectrum, recognizable group, non-simple group.

1 Introduction

Let $G$ be a finite group. Denote by $\omega(G)$ the spectrum of $G$, i.e. the set of all element orders of $G$. Recall that $G$ is recognizable by spectrum (or simply recognizable) if every finite group $H$ with $\omega(H) = \omega(G)$ is isomorphic to $G$. A finite group $L$ is isospectral to $G$ if $\omega(L) = \omega(G)$.

Denote by $\pi(G)$ the set of all prime divisors of the order of $G$. If $g \in G$, then denote $\pi(g) = \pi(\langle g \rangle)$. Let

$$\sigma(G) = \max\{|\pi(g)| \mid g \in G\}.$$

In 1994 W. Shi [17] proved that if a finite group $G$ has a non-trivial solvable normal subgroup, then there are infinitely many finite groups whose are isospectral to $G$. Moreover, in 2012 V. D. Mazurov and W. Shi [16] proved that there are infinitely many finite groups isospectral to a finite group $G$ if and only if there is a finite group $L$ such that $L$ is isospectral to $G$ and the solvable radical of $L$ is non-trivial. Thus, the socle of a recognizable finite group is a direct product of nonabelian simple groups. At the moment, for many finite nonabelian simple groups and their automorphism groups, it was proved that they are recognizable (see, for example, [19]). In 1997 V. D. Mazurov [15, Theorem 2] proved that the direct product of two copies of the group $Sz(2^7)$ is recognizable by spectrum. In this paper we prove the following theorem.

Theorem. The direct product of two copies of the finite simple sporadic group $J_4$ is recognizable by spectrum.

Note that if the direct product of $k$ copies of a finite group $G$ is recognizable by spectrum, then for each $i \leq k$ the direct product of $i$ copies of $G$ is recognizable by spectrum. Thus, the following problems are of interest.

Problem 1. Let $G$ be a finite group which is recognizable by spectrum. What is the largest number $k = k(G)$ such that the direct product of $k$ copies of the group $G$ is still recognizable by spectrum?

Problem 2. Is it true that for each integer $k \geq 1$ there exists a finite simple group $G = G(k)$ such that the direct product of $k$ copies of $G$ is recognizable by spectrum?

In proving Theorem, we use the following assertion which is interesting in its own right.
Proposition 1. Let $G$ be a finite solvable group such that $\sigma(G) = 2$ and for any $p, q \in \pi(G)$ the following conditions hold:
1. $p$ does not divide $q - 1$;
2. $pq \in \omega(G)$.
Then $|\pi(G)| \leq 3$.

Remark. The evaluation of Proposition 1 is the best possible. Indeed, let $V_1$ and $H_1$ be the additive and the multiplicative groups of the field of order $3^{16}$, respectively, $V_2$ and $H_2$ be the additive and the multiplicative groups of the field of order $81$, respectively. Assume that $H_i$ acts on $V_j$ by the following rules. Take $x \in H_i$ and $y \in V_j$. If $i = j$, then $x(y) = xy$. If $i \neq j$, then $x(y) = y$. Consider the group $G = (V_1 \times V_2) \times (L_1 \times L_2)$, where $L_1$ is the subgroup of order $17$ of $H_1$ and $L_2$ is the subgroup of order $5$ of $H_2$. Then $\pi(G) = \{3, 5, 17\}$, $\sigma(G) = 2$, and for any $p, q \in \pi(G)$, $p$ does not divide $q - 1$ and $pq \in \omega(G)$.

2 Preliminaries

Our terminology and notation are mostly standard and could be found in [4, 20, 6].

In this paper by “group” we mean “a finite group” and by “graph” we mean “an undirected graph without loops and multiple edges”.

Let $\pi$ be a set of primes. Denote by $\pi'$ the set of the primes not in $\pi$. Given a natural $n$, denote by $\pi(n)$ the set of its prime divisors. A natural number $n$ with $\pi(n) \subseteq \pi$ is called a $\pi$-number.

Let $G$ be a group. Note that $\pi(G)$ is exactly $\pi(|G|)$. The spectrum of $G$ defines the Gruenberg–Kegel graph (or the prime graph) $GK(G)$ of $G$; in this graph the vertex set is $\pi(G)$, and different vertices $p$ and $q$ are adjacent in $GK(G)$ if and only if $pq$ is an element of order $17$ of $G$.

A subgroup $H$ of a group $G$ is called a Hall subgroup if the numbers $|H|$ and $|G : H|$ are coprime. A group $G$ with $\pi(G) \subseteq \pi$ is called a $\pi$-group. A subgroup $H$ of a group $G$ is called a $\pi$-Hall subgroup if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. Note that $H$ is a $\pi$-Hall subgroup of a group $G$ if and only if $H$ is a Hall $\pi$-subgroup of $G$. We say that a finite group $G$ has the property $E_\pi$ if $G$ contains a Hall $\pi$-subgroup. We say that a finite group $G$ has the property $C_\pi$ if $G$ has the property $E_\pi$ and any two Hall $\pi$-subgroups of $G$ are conjugate in $G$. We denote by $E_\pi$ ($C_\pi$, respectively) the class of all groups $G$ such that $G$ has the property $E_\pi$ ($C_\pi$, respectively).

Recall that $Soc(G)$ and $F(G)$ denote the socle (the subgroup generated by all the minimal non-trivial normal subgroups of $G$) and the Fitting subgroup (the largest nilpotent normal subgroup) of $G$, respectively.

For a prime $p$ and a $p$-group $G$, $\Omega_1(G)$ denotes the subgroup of $G$ generated by the set of all its elements of order $p$.

Recall that a group $H$ is a section of a group $G$ if there exist subgroups $L$ and $K$ of $G$ such that $L$ is normal in $K$ and $K/L \cong H$.

Lemma 1 (See [13, Lemma 1] and [14, Lemma 1]). Let a Frobenius group $H = F \rtimes C$ with kernel $F$ and cyclic complement $C = \langle c \rangle$ of order $n$ acts on a vector space $V$ of non-zero characteristic $p$ coprime to $|F|$. Assume that $F \not\leq C_H(V)$. Then the correspondent semidirect product $V \rtimes C$ contains an element of order $pn$ and $\dim C_V(\langle c \rangle) > 0$. 

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Lemma 2 (See [19] Lemmas 3.3, 3.6). Let $s$ and $p$ be distinct primes, a group $H$ be a semidirect product of a normal $p$-subgroup $T$ and a cyclic subgroup $C = \langle g \rangle$ of order $s$, and let $[T,g] \neq 1$. Suppose that $H$ acts faithfully on a vector space $V$ of positive characteristic $t$ not equal to $p$.

If the minimal polynomial of $g$ on $V$ equals to $x^s - 1$, then $C_V(g)$ is non-trivial.
If the minimal polynomial of $g$ on $V$ does not equal $x^s - 1$, then

(i) $C_T(g) \neq 1$;
(ii) $T$ is nonabelian;
(iii) $p = 2$ and $s = 2^{2s} + 1$ is a Fermat prime.

Lemma 3 (See, for example, [1]). Let $G = F \rtimes H$ be a Frobenius group with kernel $F$ and complement $H$. Then the following statements hold.

1. The subgroup $F$ is the largest nilpotent normal subgroup of $G$, and $|H|$ divides $|F| - 1$.
2. Any subgroup of order $pq$ from $H$, where $p$ and $q$ are (not necessarily distinct) primes, is cyclic. In particular, any Sylow subgroup of $H$ is a cyclic group or a (generalized) quaternion group.
3. If the order of $H$ is even, then $H$ contains a unique involution.
4. If the group $H$ is non solvable, then it contains a normal subgroup $S \times Z$ of index 1 or 2, where $S \cong SL_2(5)$ and $(|S|, |Z|) = 1$.

Lemma 4 ([10] Lemma 1). Let $G$ be a finite group and $\pi$ be a set of primes. If $G \in E_\pi$, then $S \in E_\pi$ for every composition factor $S$ of $G$.

Lemma 5 (See [8] and [9]). Let $\pi$ be a set of primes such that $2 \notin \pi$. Then $E_\pi = C_\pi$.

Lemma 6. Let $H$ be a finite solvable group such that $\sigma(H) = 1$. Then $|\pi(H)| \leq 2$.

Proof. Follows directly from [12] Theorem 1.

Lemma 7. Let $H$ be a finite solvable group such that $\sigma(H) = 2$. Then $|\pi(H)| \leq 5$.

Proof. Follows directly from [24] Theorem 1.

Lemma 8 (See [11]). (1) $|J_4| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$;
2. $\omega(J_4)$ consists from all the divisors of numbers from the set $\{16, 23, 24, 28, 29, 30, 31, 35, 37, 40, 42, 43, 44, 46\}$;
3. $\omega(J_4 \times J_4) = \{x \mid x \text{ divides } \text{lcm}(a, b), \text{ where } a, b \in \omega(J_4)\}$.

Lemma 9. Let $H$ be a finite simple group. Assume that the following conditions hold:

(i) $\pi(H) \subseteq \pi(J_4)$;
(ii) $\omega(H) \cap \{9, 25\} = \emptyset$;
(iii) $|\pi(H) \cap \{11, 23, 29, 31, 37, 43\}| \geq 2$.

Then one of the following statements holds:

1. $H \cong L_2(23)$ and $\pi(H) \cap \{11, 23, 29, 31, 37, 43\} = \{11, 23\}$;
2. $H \cong M_{23}$ and $\pi(H) \cap \{11, 23, 29, 31, 37, 43\} = \{11, 23\}$;
3. $H \cong L_2(29)$ and $\pi(H) \cap \{11, 23, 29, 31, 37, 43\} = \{11, 23\}$;
4. $H \cong L_2(31)$ and $\pi(H) \cap \{11, 23, 29, 31, 37, 43\} = \{11, 31\}$;
5. $H \cong U_3(11)$ and $\pi(H) \cap \{11, 23, 29, 31, 37, 43\} = \{11, 37\}$;
6. $H \cong L_2(43)$ and $\pi(H) \cap \{11, 23, 29, 31, 37, 43\} = \{11, 43\}$;
7. $H \cong J_4$ and $\{11, 23, 29, 31, 37, 43\} \subseteq \pi(H)$.
Proof. In view of [21], if \( \pi(H) \subseteq \pi(J_4) \) and \( |\pi(H) \cap \{11, 23, 29, 31, 37, 43\}| \geq 2 \), then \( H \) is one of the following groups: \( L_2(23), M_{23}, L_2(32), U_3(11), L_2(43), U_7(2), L_2(43^2), S_4(43), J_4 \).

If \( H \subseteq \{Co_3, Co_2\} \), then \( 9 \in \omega(H) \) in view of [4].

If \( H \cong U_7(2) \), then \( 9 \in \omega(H) \) in view of [3, Corollary 3].

If \( H \cong L_2(43^2) \), then \( 25 \in \omega(H) \) in view of [3, Corollary 3].

If \( H \cong S_4(43) \), then \( 25 \in \omega(H) \) in view of [2, Corollary 2].

Lemma 10. Let \( H \) be a finite simple group. Assume that the following conditions hold:

(i) \( \pi(H) \subseteq \pi(J_4) \);

(ii) \( \omega(H) \cap \{9, 25\} = \emptyset \);

(iii) \( |\pi(H) \cap \{5, 23, 29, 37, 43\}| \geq 2 \).

Then one of the following statements holds:

1. \( H \cong M_{23} \) and \( \pi(H) \cap \{5, 23, 29, 37, 43\} = \{5, 23\} \);

2. \( H \cong M_{24} \) and \( \pi(H) \cap \{5, 23, 29, 37, 43\} = \{5, 23\} \);

3. \( H \cong L_2(29) \) and \( \pi(H) \cap \{5, 23, 29, 37, 43\} = \{5, 29\} \);

4. \( H \cong U_3(11) \) and \( \pi(H) \cap \{5, 23, 29, 37, 43\} = \{5, 37\} \);

5. \( H \cong J_4 \) and \( \{5, 23, 29, 37, 43\} \subset \pi(H) \).

Proof. In view of [21], if \( \pi(H) \subseteq \pi(J_4) \) and \( |\pi(H) \cap \{5, 23, 29, 37, 43\}| \geq 2 \), then \( H \) is one of the following groups: \( M_{23}, M_{24}, Co_3, Co_2, L_2(29), U_3(11), U_4(7), U_7(2), L_2(43^2), S_4(43), J_4 \).

As in the proof of Lemma 9 we exclude the following groups: \( Co_3, Co_2, U_7(2), L_2(43^2), S_4(43) \). Moreover, \( 25 \in \omega(U_4(7)) \) in view of [3, Corollary 3].

Lemma 11. Let \( G \) be a group and

\[
1 = G_n < G_{n-1} < \ldots < G_1 < G_0 = G
\]

be a normal series in \( G \). Let \( \pi = \{p_1, \ldots, p_n\} \) be a set of pairwise distinct primes such that \( p_k \in \pi(G_{i_k}/G_{i_k+1}) \) and \( i_k \neq i_l \) if \( k \neq l \). Then \( G \) contains a solvable subgroup \( H \) such that \( \pi(H) = \pi \).

Proof. Without loss of generality we can assume that \( m = n \) and \( p_i \in G_{i-1}/G_i \). Let \( T \) be a Sylow \( p_n \)-subgroup of \( G_{n-1} \). Using the Frattini argument we conclude that \( G = N_G(T)G_{n-1} \). Now \( N_G(T)/N_{G_{n-1}}(T) \cong G/G_{n-1} \) and in view of induction reasonings, \( N_G(T)/T \) contains a solvable subgroup \( H_1 \) such that \( \pi(H_1) = \{p_1, \ldots, p_{n-1}\} \). Thus, we conclude that \( N_G(T) \) contains a solvable subgroup \( H \) such that \( \pi(H) = \pi \).

Lemma 12. Let \( G \) be a group and

\[
1 = G_n < G_{n-1} < \ldots < G_1 < G_0 = G
\]

be a normal series in \( G \). Let \( \pi_1, \ldots, \pi_n \) be sets of odd primes such that \( \pi_k \subseteq \pi(G_{i_k}/G_{i_k+1}) \) and \( i_k \neq i_l \) if \( k \neq l \). Assume that \( G_{i_k}/G_{i_k+1} \in E_{\pi_k} \) for each \( k \). Then \( G \) contains a solvable subgroup \( H \) such that \( \pi(H) = \bigcup_{i=1}^{n} \pi_i \).
Proof. Without loss of generality we can assume that $\pi_i \subseteq G_{i-1}/G_i$.

Let $T$ be a Hall $\pi_n$-subgroup of $G_{n-1}$. In view of the Feit-Thompson theorem \[5\], $T$ is solvable. In view of Lemma \[5\] we have $G_{n-1} \in C_{\pi_n}$. Thus, using the Frattini argument we conclude that $G = N_G(T)G_{n-1}$. Now $N_G(T)/N_{G_{n-1}}(T) \cong G/G_{n-1}$. In view of induction reasonings, the group $N_G(T)/N_{G_{n-1}}(T)$ contains a solvable subgroup $H_1$ such that $\pi(H_1) = \cup_{i=1}^{n-1}\pi_i$.

If $N_{G_{n-1}}(T)/T$ is solvable, then we consider the complete preimage $H_2$ of $H_1$ in $N_G(T)$. Note that $H_2$ is solvable. In view of the Hall theorem \[6\] Theorem 6.4.1], $H_2$ contains a Hall $(\cup_{i=1}^{n-1}\pi_i)$-subgroup $H$. Note that in this case $\pi(H) = \cup_{i=1}^{n-1}\pi_i$.

Thus, we can assume that $N_{G_{n-1}}(T)/T$ is non-solvable. In view of the Feit-Thompson theorem \[5\], $|N_{G_{n-1}}(T)/T|$ is even. Put $R = N_G(T)/T$ and $A = N_{G_{n-1}}(T)/T$. Let $S$ a Sylow $2$-subgroup of $A$. Using the Frattini argument we conclude that $R = N_R(S)A$. Thus, $N_G(T)/N_{G_{n-1}}(T) = R/A \cong N_R(S)/N_A(S)$ and so, $N_R(S)/N_A(S)$ contains a solvable subgroup $H_2$ isomorphic to $H$. Note that in view of the Feit-Thompson theorem \[5\], $N_A(S)$ is solvable. Let $H_3$ be the complete preimage of $H_2$ in $N_R(S)$. Note that $H_3$ is solvable. Thus, in view of the Hall theorem \[6\] Theorem 6.4.1], $H_3$ contains a Hall $(\cup_{i=1}^{n-1}\pi_i)$-subgroup $H_4$ and $\pi(H_4) = \cup_{i=1}^{n-1}\pi_i$. Let $H$ be the complete preimage of $H_4$ in $N_G(T)$. Note that $H$ is solvable and $\pi(H) = \cup_{i=1}^{n-1}\pi_i$.

\[\square\]

Lemma 13 (See \[7\] Lemma 10)]. For a finite group $G$ take a coclique $\rho$ in $GK(G)$ with $|\rho| = 3$. Then the following claims hold:

(i) there exists a nonabelian composition factor $S$ of $G$ and a normal subgroup $K$ of $G$ such that $S \cong Inn(S) \leq G/K \leq Aut(S)$ and $|\pi(S) \cap \rho| \geq 2$.

(ii) If $\rho'$ is a coclique in $GK(G)$ with $|\rho'| \geq 3$ and $|\pi(S) \cap \rho'| \geq 1$, then $|G|/|S|$ is divisible by at most one element of $\rho'$. In particular, $|\pi(S) \cap \rho'| \geq |\rho'| - 1$ and $S$ is a unique composition factor of $G$ with $|\pi(S) \cap \rho'| \geq 2$.

Lemma 14 (See \[22\] Lemma 10)]. Let $V$ be a normal elementary abelian subgroup of a group $G$. Put $H = G/V$ and denote by $G_1 = V \rtimes H$ the natural semidirect product. Then $\omega(G_1) \subseteq \omega(G)$.

Lemma 15. If $G$ is an extension of an elementary abelian group $V$ with the group $H \cong J_4 \times J_4$, then $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

Proof. We can assume that $\pi(V) \subseteq \pi(J_4) = \{2, 3, 5, 7, 11, 23, 29, 31, 37, 43\}$. In view of Lemma \[14\] we can assume that $G = V \rtimes H$, where $H \leq G$ and $H = H_1 \times H_2$ for $H_1 \cong H_2 \cong J_4$.

Assume that $V$ is a $2$-group. In view of \[20\], $H_2$ contains a subgroup isomorphic to $U_3(11)$. So, in view of \[23\] Lemma 5], the subgroup $V \rtimes H_2$ contains an element of order $2 \cdot 37$. Therefore there is an element $z \in H_2$ such that $|z| = 37$ and $C_V\langle z \rangle$ is non-trivial. Let $V_1 = C_V\langle z \rangle$. Note that $C_G\langle z \rangle$ contains a subgroup $V_1 \rtimes H_1$. In view of \[20\], $H_1$ contains a Frobenius group $\langle x \rangle \rtimes \langle y \rangle$, where $|x| = 29$ and $|y| = 28$. In view of Lemma \[11\] the group $V_1 \rtimes H_1$ contains either an element of order $2 \cdot 29$ or an element of order $2 \cdot 28$. So, $G$ contains either an element of order $2 \cdot 29 \cdot 37$ or an element of order $2 \cdot 29 \cdot 37$. Thus, $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

Assume that $V$ is a $3$-group. In view of \[20\], $H_2$ contains a Frobenius group $\langle z \rangle \rtimes \langle t \rangle$, where $|z| = 37$ and $|t| = 3$. If $9 \in \omega(V \rtimes H_2)$, then $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$. Suppose that
9 \not\in \omega(V \rtimes H_2)$. In view of Lemma 1, $V \leq C_G(\langle z \rangle)$. So, $V \rtimes H_1 \leq C_G(\langle z \rangle)$. In view of (20), $H_1$ contains a Frobenius group $\langle x \rangle \rtimes \langle y \rangle$, where $|x| = 29$ and $|y| = 28$. In view of Lemma 11, the group $V \rtimes H_1$ contains either an element of order $3 \cdot 29$ or an element of order $3 \cdot 28$. So, $G$ contains either an element of order $3 \cdot 28 \cdot 37$ or an element of order $3 \cdot 29 \cdot 37$. Thus, $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

Assume that $V$ is a 5-group. In view of (20), $H_2$ contains a Frobenius group $\langle z \rangle \rtimes \langle t \rangle \leq L_2(11) \leq M_{22} \leq H_2$, where $|z| = 11$ and $|t| = 5$. If $25 \in \omega(V \rtimes H_2)$, then $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$. Suppose that $25 \not\in \omega(V \rtimes H_2)$. In view of Lemma 1, $V \leq C_G(\langle z \rangle)$. So, $V \rtimes H_1 \leq C_G(\langle z \rangle)$. In view of (20), $H_1$ contains a Frobenius group $\langle x \rangle \rtimes \langle y \rangle \leq L_5(2) \leq H_1$, where $|x| = 31$ and $|y| = 5$. In view of Lemma 1, the group $V \rtimes H_1$ contains either an element of order $25$ or an element of order $5 \cdot 31$. So, $G$ contains either an element of order $25$ or an element of order $5 \cdot 11 \cdot 31$. Thus, $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

Assume that $V$ is an 11-group. In view of (20), $H_2$ contains a Frobenius group $\langle z \rangle \rtimes \langle t \rangle$, where $|z| = 43$ and $|t| = 7$, and $H_1$ contains a Frobenius group $\langle x \rangle \rtimes \langle y \rangle$, where $|x| = 29$ and $|y| = 7$. Similar as above, we conclude that either $49 \in \omega(G)$ or $7 \cdot 29 \cdot 43 \in \omega(G)$. Thus, $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

Assume that $V$ is a 23-group. In view of (20), $H_2$ contains a Frobenius group $\langle z \rangle \rtimes \langle t \rangle$, where $|z| = 11$ and $|t| = 5$, and $H_1$ contains a Frobenius group $\langle x \rangle \rtimes \langle y \rangle$, where $|x| = 29$ and $|y| = 28$. Similar as above, we conclude that $\omega(G)$ contains an element of one of the following orders: $5 \cdot 23 \cdot 28$, $5 \cdot 23 \cdot 29$, $11 \cdot 23 \cdot 28$, $11 \cdot 23 \cdot 29$. Thus, $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

Assume that $V$ is a 29-group. In view of (20), $H_2$ contains a Frobenius group $\langle z \rangle \rtimes \langle t \rangle$, where $|z| = 23$ and $|t| = 11$, and $H_1$ contains a Frobenius group $\langle x \rangle \rtimes \langle y \rangle$, where $|x| = 29$ and $|y| = 7$. Similar as above, we conclude that $\omega(G)$ contains an element of one of the following orders: $7 \cdot 11 \cdot 29$, $7 \cdot 23 \cdot 29$, $11 \cdot 29 \cdot 43$, $23 \cdot 29 \cdot 43$. Thus, $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

Assume that $V$ is a $p$-group, where $p \in \{31, 37, 43\}$. In view of (20), $H_2$ contains a Frobenius group $\langle z \rangle \rtimes \langle t \rangle$, where $|z| = 23$ and $|t| = 11$, and $H_1$ contains a Frobenius group $\langle x \rangle \rtimes \langle y \rangle$, where $|x| = 29$ and $|y| = 7$. Similar as above, we conclude that $\omega(G)$ contains an element of one of the following orders: $7 \cdot 11 \cdot p$, $7 \cdot 23 \cdot p$, $11 \cdot 29 \cdot p$, $23 \cdot 29 \cdot p$. Thus, $\omega(G) \setminus \omega(J_4 \times J_4) \neq \emptyset$.

\[ \square \]

3 Proof of Proposition 1

Let $G$ be a solvable group such that $\sigma(G) = 2$ and for any $p, q \in \pi(G)$ the following conditions hold:

1. $p$ does not divide $q - 1$;
2. $pq \in \omega(G)$.

In view of the Hall theorem [6, Theorem 6.4.1], it’s enough to prove that $|\pi(G)| \neq 4$. Suppose to the contradiction that $G$ is a group of the least order satisfying the conditions (1) and (2), and $|\pi(G)| = 4$.

In view of condition (1), any element of $\pi(G)$ is odd. Thus, $G$ does not contain a generalized quaternion group as its Sylow subgroup. Moreover, for any Sylow $p$-subgroup
$S$ of $G$, the subgroup $\Omega_1(S)$ is non-cyclic, in particular, $S$ is non-cyclic. Otherwise there is $g \in G$ such that $|g| = p$ and $|\pi(C_G(g)) \setminus \{p\}| = 3$. A contradiction to Lemma 3.

Let $H_1$ be a minimal normal subgroup of $G$. From solvability of $G$ it follows that $H_1$ is an elementary abelian $p_1$-group for some $p_1 \in \pi(G)$. Let $T_1$ be a Hall $(\pi(G) \setminus \{p_1\})$-subgroup of $G$. Sylow subgroups of $T_1$ are non-cyclic. It follows from Lemma 3 that $p_1t_1 \in \omega(H_1T_1)$ for each $t_1 \in \pi(T_1)$. Thus, $G = H_1T_1$ in view of minimality of $G$.

Let $H_2$ be a minimal normal subgroup of $T_1$. Then $H_2$ is an elementary abelian $p_2$-group for some $p_2 \in \pi(T_1)$. Let $T_2$ be a Hall $(\pi(T_1) \setminus \{p_2\})$-subgroup of $T_1$. Since $p_2 - 1$ is not divisible by primes from $\pi(T_2)$, we see that $H_2$ is non-cyclic. Otherwise, it follows from Lemma 3 that any element of prime order from $T_2$ centralizes $H_2$ and $p_3p_4 \in \pi(T_2)$. So, there exists $g \in C_{T_1}(H_2)$ such that $|\pi(g)| \geq 3$. Sylow subgroups of $T_1$ are non-cyclic, consequently, $p_2t_2 \in \omega(H_2T_2)$ for each $t_2 \in \pi(T_2)$. Since $H_2$ is non-cyclic, we have $p_1p_2 \in \omega(H_1H_2)$. Thus, $G = H_1H_2T_2$ in view of minimality of $G$.

Let $R = Soc(T_2)$. Suppose that $R$ is cyclic.

Suppose that there exists a non-trivial subgroup $L$ of $R$ such that $|L| = p_3 \in \pi(T_2)$ and $C_HH_2(L)$ is non-trivial. Then $L$ is a characteristic subgroup of $R$ and so, $L$ is normal in $T_2$. Since $p_3 - 1$ is not divisible by $p_4$, we see that $\Omega_1(H_4) < C_G(L)$, where $H_4$ is a Sylow $p_4$-subgroup of $T_2$ with $p_4 \neq p_3$. Note that $C_{H_1H_2}(L) \subseteq C_G(L)$. Since the subgroup $\Omega_1(H_4)$ is non-cyclic, in view of Lemma 3 there exists an element $g \in C_G(L)$ such that $|\pi(g)| \geq 3$, a contradiction. Thus, $L$ acts fixed-point free on $H_1H_2$. So, by the Thompson theorem, $H_1H_2$ is nilpotent.

Let $F = F(T_2)$. Suppose that $F$ is cyclic.

Note that $F \neq T_2$ since Sylow subgroups of $T_2$ are non-cyclic. Let $T < K \triangleleft T_2$ and $K/F$ is an elementary abelian $p_j$-group for some $p_j \in \pi(T_2)$. Since $K \neq F$, a Sylow $p_j$-subgroup of $K$ acts non-trivially on the Sylow $p_j$-subgroup of $F$, where $p_j \neq p$. Thus, an element whose order is a power of $p_j$ acts non-trivially on a cyclic $p_j$-subgroup, and $p_j$ does not divide $p_i - 1$, a contradiction to Lemma 3.

So, $F$ is non-cyclic. Consequently, there exists a non-cyclic Sylow $p_3$-subgroup $P_3$ of $F$, which is characteristic in $F$, and so, is normal in $T_2$. Consider the group $T_3 = P_3H_4$, where $H_4$ is a Sylow $p_4$-subgroup of $T_2$. Suppose that $\Omega_1(H_4) < C_{T_3}(P_3)$. Since $p_2p_4 \in \omega(G)$, there is $g \in \Omega_1(H_4)$ such that a subgroup $C_{T_1}(g) \cap H_2$ is non-trivial. Note that $\Omega_1(H_4)$ is non-cyclic. Consequently, in view of Lemma 3 there exists an element $t \in C_{T_1}(g)$ such that $|t| = p_2p_3p_4$. A contradiction. Thus, there is $h \in H_4$ acting non-trivially on $P_3$.

Suppose that there exists an element $h_3 \in P_3$ such that $h_3$ centralizes a subgroup $H_j$ for some $j \in \{1, 2\}$. Since $H_1H_2 = H_1 \times H_2$, $p_1p_4 \in \omega(G)$, and $p_2p_4 \in \omega(G)$, there exists $t \in H_j$ such that $|\pi(C_G(t))| = 4$. In view of Lemma 3 there exists $t_1 \in C_G(t)$ such that $|\pi(t_1)| \geq 3$; a contradiction. Similar, if $h$ centralizes a subgroup $H_j$ for some $j \in \{1, 2\}$, then taking into account that $H_1H_2 = H_1 \times H_2$ and $P_3$ is non-cyclic, we conclude that there exists $t \in H_j$ such that $|\pi(C_G(t))| = 4$ and receive a contradiction. Thus, the group $P_3\langle h \rangle$ acts faithfully both on $H_1$ and on $H_3$, and $[P_3, h] \neq 1$.

Since $p \neq 2$, with using Lemma 2 we obtain that for each $j \in \{1, 2\}$ the minimal polynomial of $h$ over $H_j$ is equal to $x^{p_4} - 1$. Consequently, $C_{H_j}(h) \neq \{1\}$ for each $j \in \{1, 2\}$. Thus, the intersections $C_G(h) \cap H_1$ and $C_G(h) \cap H_2$ are non-trivial and so, $p_1p_2p_4 \in \omega(G)$. A contradiction.

We have that $R$ is non-cyclic.

Let $H_3$ be a non-cyclic Sylow subgroup of $R$ and $\{p_3\} = \pi(H_3)$. Consider the subgroup $H = H_1H_2H_3H_4$ of $G$, where $H_1$, $H_2$, and $H_3$ are non-cyclic elementary abelian groups, and
conclude that

Consequently,

Therefore Frobenius group and

Moreover, there exists

trivial. Consider subgroups

way as before, we receive that the group $\langle h, H \rangle$ is non-cyclic. A contradiction. Thus, there exists $h \in \Omega_1(H_4)$ such that $h$ acts on $H_3$ non-trivially, i.e. $[H_3, h] \neq 1$.

Suppose that there exists $l \in \Omega_1(H_4)$ such that $H_3 < C_G(l)$. Then taking into account that $H_3$ is non-cyclic we conclude that $l$ acts fixed point free on $H_1H_2$. Consequently, the subgroup $H_1H_2$ is nilpotent by the Thompson theorem \[18\]. Since $H_3$ is abelian, we have $H_3 = [h, H_3] \times C_{H_3}(h)$ in view of \[6\] Theorem 5.2.3. Moreover, $[h, H_3]\langle h \rangle$ is a Frobenius group, and $|[h, H_3]| \geq p_3^2$ in view of Lemma \[3\] since $p_4$ does not divide $p_3 - 1$. In a similar way as before, we receive that the group $[h, H_3]\langle h \rangle$ acts non-trivially on both $H_1$ and $H_2$. Therefore $C_G(h) \cap H_1$ is non-trivial and $C_G(h) \cap H_2$ is non-trivial in view of Lemma \[2\]. So, $p_1p_2p_3 \in \omega(G)$. A contradiction. Thus, any element from $\Omega_1(H_4)$ acts non-trivially on $H_3$.

Since $\Omega_1(H_4)$ is non-cyclic, there exists $m \in \Omega_1(H_4)$ such that $C_G(m) \cap H_3$ is non-trivial. Consider subgroups $H_i([H_3, m]\langle m \rangle)$, where $H_i \in \{H_1, H_2\}$. Note, $[m, H_3]\langle m \rangle$ is a Frobenius group and $|[m, H_3]| \geq p_3^2$, so, $[H_3, m]$ acts non-trivially on $H_i$. In view of Lemma \[2\] $C_G(m) \cap H_i$ is non-trivial. Thus, $|\pi(C_G(m))| = 4$. Consequently, in view of Lemma \[6\] there exists an element $u \in C_G(m)$ such that $|\pi(u)| \geq 3$. A contradiction.

\[\square\]

4 Proof of Theorem

Let $G$ be a finite group such that $\omega(G) = \omega(J_4 \times J_4)$. The spectrum of $G$ could be found in Lemma \[8\]

Put

$$\pi_1 = \{5, 11, 23, 29, 31, 37, 43\},$$

$$\pi_2 = \{7, 11, 23, 29, 31, 37, 43\},$$

and

$$\pi = \pi_1 \cup \{7\} = \pi_2 \cup \{5\} = \pi_1 \cup \pi_2.$$ 

Lemma 16. Let $H$ be a section of $G$ such that $\pi(H) \subseteq \pi_i$ for some $i \in \{1, 2\}$. Then $\sigma(H) \leq 2$.

Proof. Follows directly from Lemma \[8\] 

\[\square\]

Lemma 17. Let $p \in \pi$ and $P \in Syl_p(G)$. Then $P$ is non-cyclic.

Proof. Suppose to the contradiction that there exists $p \in \pi$ such that $P \in Syl_p(G)$ and $P$ is cyclic. Denote by $\theta$ a set $\pi_i$ such that $p \in \pi_i$. Let $C = C_G(P)/P$. We have $\pi(C) = \pi(G)\backslash\{p\}$, and any two primes from $\theta \backslash \{p\}$ are no-adjacent in $GK(C)$ in view of Lemma \[8\]

Suppose that $C$ is solvable. In view of the Hall theorem \[6\] Theorem 6.4.1], there exists a $\theta$-Hall subgroup $C_1$ of $C$. Note that $\sigma(C_1) = 1$ and $|\pi(C_1)| = 6$, a contradiction to Lemma \[6\].
Note that \(|\pi(C) \cap \theta| = 6\). In view of Lemma 13, there exists a nonabelian composition factor \(R\) of \(C\) such that \(5 \leq |\pi(R) \cap \pi| \leq 6\) and \(\pi(R) \subseteq \pi(G) \setminus \{p\} = \{2, 3, 5, 7, 11, 23, 29, 31, 37, 43\} \setminus \{p\}\). In view of [21], there is no a finite nonabelian simple group \(R\) satisfying these conditions. A contradiction.

In view of Proposition 11 we have \(G \not\in E_{\rho}\) for \(\rho = \{29, 31, 37, 43\}\). From Lemma 11 and the Sylow theorems it follows that there exists a composition factor \(S\) of \(G\) such that \(|\{29, 31, 37, 43\} \cap \pi(S)| \geq 2\) and \(\pi(S) \subseteq \pi(J_4)\). Thus, \(S \cong J_4\) in view of Lemma 9.

Let \(G_1 = G\) and \(C_1 = \text{Soc}(G)\). For \(i \geq 2\) we put \(G_i = G_{i-1}/C_{i-1}\) and \(C_i = \text{Soc}(G_i)\).

Let \(s\) be the minimal number such that \(C_s\) contains a compositional factor of \(G\) which is isomorphic to \(S\).

**Lemma 18.** We have \(11 \in \pi(|G|/|S|)\).

**Proof.** Assume that \(11 \not\in \pi(|G|/|S|)\). The group \(S\) contains two conjugacy classes of elements of order 11. Since \(11 \not\in \pi(|G|/|S|)\), \(G\) does not contain more than two conjugacy classes of elements of order 11.

If \(G\) contains the only conjugacy class of elements of order 11, then we receive a contradiction by the same way as in the proof of Lemma 17. Thus, we can assume that in \(G\) there are exactly two conjugacy classes of elements of order 11.

Let \(x, y \in G\) such that \(|x| = |y| = 11\) and \(x \not\in yG\). We have \(\pi \subseteq \pi(C_G(x)) \cup \pi(C_G(y))\). Let \(\pi(C_G(x)) \cap \pi_1 \geq \pi(C_G(y)) \cap \pi_1\). Put \(\theta = \pi(C_G(x)) \cap \pi_1 \setminus \{11\}\). We have \(|\theta| \geq 3\) and the vertices from \(\theta\) are pairwise non-adjacent in \(G\).

Let \(r\) be a number such that \(H\) is a section of \(C_r\).

Suppose that \(r \geq s\), and let \(\bar{x}\) be the image of \(x\) in \(G_s\). Let \(\bar{H} < G_s\) be a minimal preimage of \(H\) in \(G_s\) such that \(\bar{H} \leq C_{G_s}(\bar{x})\). In view of Lemma 8, we have \(C_s(\bar{x})\) is a \(\{2, 3, 11\}\)-group and so, in view of [11], \(C_s(\bar{x})\) is solvable. Thus, \(\bar{H} \leq S\). Note that \(S\) is characteristic in \(C_s\) and therefore is normal in \(G_s\). Now consider \(R = SH\), which is a preimage of \(H\) in \(G_s\), and note that \(R\) contains nonabelian composition factors isomorphic to \(S\) and \(H\). Consider the factor-group \(R/C_R(S)\), which is isomorphic to a subgroup of \(\text{Aut}(S) \cong J_4\), and note that the order of \(C_R(S)\) is coprime to 11. Thus, in view of the Jordan–Holder theorem, \(R/C_R(S) \cong J_4\) and the group \(C_R(S)\) contains a nonabelian composition factor which is isomorphic to \(H\). Therefore \(H\) is a composition factor of \(C_R(y)\). We get that \(|\theta| \geq 5\) and \(|\pi(H) \cap \pi| \geq 4\).

Suppose that \(r < s\). Let \(\bar{y}\) be the image of \(y\) in \(G_r\) and \(C_r = T_0 \times \ldots \times T_k\), where \(T_i\) are simple groups. It is easy to see that there is \(i\) such that \(H\) is a section of \(T_i\), and without loss of generality we can assume that \(i = 1\) and so, \(\pi(H) \subseteq \pi(T_1)\).

Note that \(\pi(T_1) \subseteq \pi(J_4)\), therefore in view of [21], we have \(11 \not\in \pi(\text{Out}(T_1))\). Moreover, \(11 \not\in \pi(T_1)\) and so, \(11 \not\in \pi(\text{Aut}(T_1))\). Therefore \(N_{\langle \bar{y} \rangle}(T_1) = C_{\langle \bar{y} \rangle}(T_1)\). If \(\bar{y} \not\in C_{G_r}(T_1)\), then consider

\[
K = \langle T_1^w \mid w \in \langle \bar{y} \rangle \rangle.
\]

It is easy to see that \(C_K(\bar{y}) \cong T_1\). Thus, in any case \(\pi(H) \subseteq \pi(C_{G_r}(\bar{y}))\) and so, \(\pi(H) \subseteq \pi(C_G(y))\). It follows that \(|\theta| \geq 5\) and \(|\pi(H) \cap \pi| \geq 4\).

Now it is easy to see that \(11 \not\in \pi(|H|)\). In view of [21], there is no a finite nonabelian simple group \(H\) satisfying these conditions. A contradiction.

Let us prove that there exists a composition factor \(T \neq S\), such that \(T \cong J_4\).
From Lemma \[17\] it follows that a Sylow $p$-subgroup is not cyclic for any $p \in \pi_1 \cup \pi_2$. Therefore $(\pi \setminus \{11\}) \subseteq \pi(|G|/|S|)$, and Lemma \[18\] implies that $11 \in \pi(|G|/|S|)$. Thus, $\pi \subseteq \pi(|G|/|S|)$.

In view of Lemmas \[11\,7\] and \[16\], there exists a composition factor $T_1$ of $G$ such that $T_1 \neq S$ and at least two primes from the set $\{11, 23, 29, 31, 37, 43\}$ divide $|T_1|$. In view of Theorem \[9\] $T$ is isomorphic to one of the following groups: $L_2(23), M_{23}, M_{24}, L_2(32), U_3(11), L_2(43), J_4$.

Assume that $T_1$ is isomorphic to one of the groups $L_2(23), M_{23}, M_{24}$. In view of \[41\], $T_1$ contains a subgroup isomorphic to $23:11$ which is a Hall $\{11, 23\}$-subgroup of $T_1$, therefore the corresponding chief factor of $G$ containing $T_1$ belongs to $E_{(11,23)}$. Thus, in view of Lemmas \[12\,7\] and \[16\] we conclude that there exists a composition factor $T \neq S$, such that at least two primes from the set $\{29, 31, 37, 43\}$ divide $|T|$. In view of Lemma \[9\] we conclude that $T \cong J_4$.

Assume that $T_1$ is isomorphic to the group $U_3(11)$. In view of \[41\], $T_1 \in E_{(5,11)},$ therefore the corresponding chief factor of $G$ containing $T_1$ belongs to $E_{(5,11)}$. Thus, in view of Lemmas \[12\,7\] and \[16\] we conclude that there exists a composition factor $T \neq S$, such that at least two primes from the set $\{23, 29, 31, 43\}$ divide $|T|$. In view of Lemma \[9\] we conclude that $T \cong J_4$.

Assume that $T_1$ is isomorphic to the group $L_2(43)$. In view of \[20\], $T_1 \in E_{(7,43)},$ therefore the corresponding chief factor of $G$ containing $T_1$ belongs to $E_{(7,43)}$. Thus, in view of Lemmas \[12\,7\] and \[16\] we conclude that there exists a composition factor $T \neq S$, such that at least two primes from the set $\{23, 29, 31, 43\}$ divide $|T|$. In view of Lemma \[9\] we conclude that $T \cong J_4$.

Assume that $T_1$ is isomorphic to the group $L_2(32)$. Note that $\pi(T_1) \cap \{5, 7, 23, 29, 31, 37, 43\} = \{31\}$ in view of \[41\]. Thus, in view of Lemmas \[11\,7\] and \[16\] we conclude that there exists a composition factor $T_2 \neq S$, such that at least two primes from the set $\{5, 23, 29, 31, 37, 43\}$ divide $|T_2|$. In view of Lemma \[10\] we conclude that $T_2$ is isomorphic to one of the following groups: $M_{23}, M_{24}, L_2(29), U_3(11), J_4$. The cases when $T_2$ is isomorphic to $M_{23}, M_{24},$ or $U_3(11)$ were considered above. If $T_2 \cong L_2(29)$, then in view of \[41\], $T_2 \in E_{(7,29)}.$ Therefore the corresponding chief factor of $G$ containing $T_2$ belongs to $E_{(7,29)}$. Thus, in view of Lemmas \[12\,7\] and \[16\] we conclude that there exists a composition factor $T \neq S$, such that at least two primes from the set $\{23, 31, 37, 43\}$ divide $|T|$. In view of Lemma \[9\] we conclude that $T \cong J_4$.

Assume that $T \leq C_t$, where $t$ is the minimal number such that $C_t$ contains a compositional factor which is isomorphic to $J_4$ and distinct from $S$. Without loss of generality we can assume that $t \geq s$.

Assume that $t = s$. Note that $C_s$ contains no more than two distinct compositional factors of $G$ whose are isomorphic to $J_4$. Thus, $S \times T$ is a characteristic subgroup of $C_s$ and so, $G$ has a chief factor isomorphic to $S \times T$.

Now assume that $t > s$. In this case $S$ is a characteristic subgroup of $C_s$ and $S$ is normal in $G_s$. We have $G_s = N_{G_s}(S), C_G(S)$ is a normal subgroup in $G_s$, and $G_s/C_G(S)$ is isomorphic to a subgroup of $\text{Aut}(S) \cong J_4$. Thus, in view of the Jordan–Holder theorem, $T$ is a composition factor of $C_G(S)$. Moreover, $SC_G(S) = S \times C_G(S)$ is a normal subgroup of $G_s$.

Hence, in any case there exists a normal subgroup $H$ of $G$ such that $\overline{G} = G/H$ has a normal subgroup $\overline{A}$ isomorphic to $J_4 \times J_4$. 

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Let us prove that $\overline{G} \cong J_4 \times J_4$. It is easy to see that $C_{\overline{G}}(A)$ is trivial. Otherwise if a prime $p_1$ divides $|C_{\overline{G}}(A)|$, then the group $\overline{G}$ contains an element of order $p_1 p_2 p_3$, where $p_2$ and $p_3$ are primes from the set $\{29, 31, 37, 43\}$, and $|\{p_1, p_2, p_3\}| = 3$. A contradiction. Note that $\text{Aut}(A) \cong J_4 \rtimes C_2 \cong (J_4 \times J_4).2$. Thus, $\overline{G}$ is isomorphic to either $J_4 \times J_4$ or $J_4 \rtimes C_2$. Now it is easy to see that $32 \in \omega(J_4 \rtimes C_2)$, and in view of Lemma 8, we have $\overline{G} \cong J_4 \times J_4$.

Assume that $H$ is non-trivial. If $H$ is solvable, then there exists a normal subgroup $H_1$ of $G$ such that $H_1 \leq H$ and $H/H_1$ is elementary abelian. It is easy to see that

$$\omega(J_4 \times J_4) = \omega(G/H) \subseteq \omega(G/H_1) \subseteq \omega(G) = \omega(J_4 \times J_4).$$

Thus, we obtain a contradiction to Lemma 15.

Assume that $H$ is non-solvable. In view of the Feit-Thompson theorem [5], $|H|$ is even. Let $S$ be a Sylow 2-subgroup of $H$. Using the Frattini argument we conclude that $G = N_G(S)H$ and so, $N_G(S)/N_H(S) \cong G/H \cong J_4 \times J_4$. Note that $N_H(S)$ is a non-trivial solvable subgroup of $N_G(S)$. Moreover,

$$\omega(J_4 \times J_4) \subseteq \omega(N_G(S)) \subseteq \omega(G) = \omega(J_4 \times J_4).$$

So, we receive a contradiction as above. Thus, $G \cong J_4 \times J_4$. 

5 Acknowledgements

The first author is supported by Russian Foundation for Basic Research (project 18-31-20011).

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