PARTIAL REGULARITY FOR DEGENERATE PARABOLIC SYSTEMS WITH NON-STANDARD GROWTH AND DISCONTINUOUS COEFFICIENTS

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Abstract. This article studies the partial Hölder continuity of weak solutions to certain degenerate parabolic systems whose model is the differentiable parabolic $p(x,t)$-Laplacian system,

$$\partial_t u - \text{div} [\mu(z)(1 + |Du|^{p(z)/2} Du)] = 0, \quad p(z) \geq 2.$$ 

Here, the exponential function $p(z)$ satisfies a logarithmic continuity condition. We show that if $\mu(z)$ satisfies a certain VMO-type condition, then $u$ is locally Hölder continuous except for a measure zero set.

1. Introduction

The aim of this paper is to establish a partial regularity result for weak solutions to parabolic systems of the type

$$\partial_t u - \text{div} [\mu(z) A(z, Du)] = 0,$$

where the coefficient $\mu(z)$ is discontinuous and the vector field $A(z, Du)$ exhibits non-standard $p(x,t)$-growth conditions. The partial regularity theory for weak solutions of parabolic systems was first studied by Campanato [7]. Unlike the case of single equation, in general not every weak solution of parabolic system is everywhere Hölder continuous and we can hope for is a result ensuring Hölder continuity outside a Lebesgue measure zero set. For Hölder continuous coefficients, the partial regularity theory for parabolic systems with standard $p$-growth has been established by Duzzar, Mingione, Steffen [9] in the superquadratic case and Scheven [16] in the subquadratic case. In [9, 16] the authors proved that the gradient of the weak solution is partial Hölder continuous. Moreover, in [8, 10] it has been proved that weak solutions to parabolic systems are Hölder continuous except a Lebesgue measure zero set, provided that the coefficients are merely continuous. Recently, Mons [15] proved the partial regularity result for the VMO coefficients, which allows the coefficients to be discontinuous.

In recent years, there has been tremendous interest in developing regularity theory for the parabolic systems with non-standard $p(x,t)$-growth. However, limited work has been done in the partial regularity problem for this kind of parabolic systems. The first result was established by Acerbi, Mingione and Seregin [1]. In [11] the authors obtained a Hausdorff dimension estimate of the singular set for the parabolic systems related to a class of non-Newtonian fluids. Subsequently, Duaaar and Habermann [8] studied the partial regularity problem with Hölder continuous coefficients by using the $A$-caloric approximation method. Motivated by this work, we are interested in extending the main result in [15].

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to the variable exponent case. Our main result states that any weak solution to degenerate parabolic systems with non-standard growth and VMO coefficients is partially Hölder continuous for any Hölder exponents.

Our approach is in the spirit of [4, 5, 8, 15] which uses the $A$-caloric approximation method. However, the treatment of non-standard growth condition is considerably more delicate. The higher integrability estimate is a standard tool in the proof and the higher integrability exponents play an important role in determining the parameters. From the higher integrability result, we shall derive an $L^p \log^q L$ estimate which is standard ingredient in the partial regularity proof. Contrary to [8], gradients of the affine functions used in our proof may not be bounded and the method of intrinsic scaling developed by DiBenedetto, Kinnunen and Lewis [13] is necessary. In the context of non-standard growth condition, we have to work with a non-standard version of the intrinsic geometry and establish the Caccioppoli inequality and decay estimates on the scaled parabolic cylinders.

An outline of this paper is as follows. In Sect. 2, we provide some preliminary material and state the main result. We also give the characterization of the singular sets in this section. In Sect. 3, we derive an estimate for gradients of solutions in $L^p \log^q L$ space by using the higher integrability estimates. Sect. 4 is devoted to the study of Poincaré type inequality for the weak solution. Furthermore, we obtain an alternative characterization of the regular points from the Poincaré inequality. In Sect. 5 we establish the Caccioppoli-type estimate which is a reverse Poincaré inequality. In Sec. 6 we use the $A$-caloric approximation method to derive a decay estimate. Finally the proof of the partial regularity result is presented in Sect. 7 by using an iteration method.

2. Preliminary material

In the present section, we set up notations and give the statement of the main result. Throughout the paper, we assume that $\Omega$ is an open bounded domain in $\mathbb{R}^n$ with $n \geq 2$. We write $\{e_i\}_{i=1}^n$ for the standard basis of $\mathbb{R}^n$. For $T > 0$, let $\Omega_T = \Omega \times (-T, 0)$. Given a point $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $r > 0$, we set $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\Lambda_r(t_0) = (t_0 - r^2, t_0)$ and $\overline{Q}_r(z_0) = B_r(x_0) \times \Lambda_r(t_0)$. For $\lambda > 0$, we define the intrinsic parabolic cylinder $Q^{(\lambda)}_r(z_0)$ by $Q^{(\lambda)}_r(z_0) := B_r(x_0) \times \Lambda^{(\lambda)}_r(t_0)$ where $\Lambda^{(\lambda)}_r(t_0) := (t_0 - \lambda^2 r^2, t_0)$ and $p_0 = \rho(z_0)$. If the reference point $z_0$ is the origin, then we omit in our notation the point $z_0$ and write $B_r$, $\Lambda_r$ and $Q^{(1)}_r$ for $B_r(0)$, $\Lambda_r(0)$ and $Q^{(1)}_r(0)$. Specifically, if $\lambda = 1$, then we abbreviate $\Lambda_r = \Lambda^{(1)}_r$ and $Q_r = Q^{(1)}_r$. Given a function $f \in L^1(\Omega, \mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^{n+1}$ and $N \in \mathbb{N}$ we define

$$
(f)_{\Omega} = \int_{\Omega} |f| \, dz := \frac{1}{|\Omega|} \int_{\Omega} |f| \, dz.
$$

For $u \in L^2(Q^{(\lambda)}_r, \mathbb{R}^N)$ with $Q^{(\lambda)}_r \subset \Omega_T$, we denote by $l_{(\lambda), r} : \mathbb{R}^n \rightarrow \mathbb{R}^N$ the unique affine map minimizing the functional

$$
F[l] = \int_{Q^{(\lambda)}_{\omega}} |u - l|^2 \, dz
$$

among all affine maps $l = l(x_0) + DL \cdot (x - x_0)$ which are independent of $t$. In this work we are concerned with the quasilinear parabolic systems of the divergence form

$$
\partial_t u - \text{div} \left( \mu(z, w) A(z, Du) \right) = 0,
$$

where $\mu : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ is a coefficient and $u : \mathbb{R}^{n+1} \mapsto \mathbb{R}^N$ is an integrable map. We assume that $(z, w) \mapsto A(z, w)$ and $(z, w) \mapsto \partial_w \mu(z, w)$ are continuous in $\Omega_T \times \mathbb{R}^{N \times n}$ and satisfy the
following non-standard \( p(z) \)-growth and ellipticity conditions:

\[
\begin{align*}
\{ & (\partial_w A(z, w) \tilde{w}, \tilde{w}) \geq \sqrt{v} (1 + |w|^2)^{\frac{p(z)}{2}} |\tilde{w}|^2, \\
& |A(z, w)| + (1 + |w|^2)^{\frac{1}{2}} |\partial_w A(z, w)| \leq \sqrt{L} (1 + |w|^2)^{\frac{p(z)}{2}}, \tag{2.2}
\end{align*}
\]

for all \( z \in \Omega_T, w, \tilde{w} \in \mathbb{R}^{N \times n} \). Here, \( v \) and \( L \) are fixed structural parameters. In this paper, we only consider the degenerate case where \( p(z) \geq 2 \). For the exponent function \( p : \Omega_T \mapsto [2, +\infty) \), we assume that it is continuous with a modulus of continuity \( \omega_p : \Omega_T \mapsto [0, 1] \).

More precisely, we assume that for any \( z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in \Omega_T \),

\[
|p(z_1) - p(z_2)| \leq \omega_p(d_P(z_1, z_2)), \tag{2.3}
\]

where \( d_P(z_1, z_2) = \max \{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\} \). Moreover, we shall assume \( 2 \leq p(z) \leq \gamma_2 \) for a fixed constant \( \gamma_2 > 0 \). The modulus of continuity \( \omega_p \) is assumed to be a concave, non-decreasing function satisfying the following logarithmic continuity condition:

\[
\lim_{\varepsilon \to 0} \omega_p(\varepsilon) \log \left( \frac{1}{\varepsilon} \right) = 0. \tag{2.4}
\]

Concerning the coefficients \( \mu(z) \), we impose a certain VMO-type condition. More precisely, we assume that \( \sqrt{v} \leq \mu(z) \leq \sqrt{L} \) for all \( z \in \Omega_T \) and satisfy the following vanishing mean oscillation condition:

\[
\lim_{r \to 0} \nu(r) = 0, \tag{2.5}
\]

where

\[
\nu(r) = \sup \left\{ \int_{Q_{\rho,r}(z_0) \cap \Omega_T} |\mu(z) - (\mu)_{Q_{\rho,r}(z_0) \cap \Omega_T}| \, dz : \max\{\rho, \sqrt{r}\} \leq r, \quad z_0 \in \Omega_T \right\}
\]

and \( Q_{\rho,r}(z_0) = B_\rho(z_0) \times (t_0 - \sigma, t_0) \). Furthermore, we shall assume that the partial map \( z \mapsto A(z, w) \) satisfies the following continuity condition:

\[
|A(z_1, w) - A(z_2, w)| \leq L \omega_p(d_P(z_1, z_2)) \left[ (1 + |w|^{p(z_1)-1} + (1 + |w|^{p(z_2)-1}) \times [1 + \log(1 + |w|)] \right. \tag{2.6}
\]

for any \( z_1, z_2 \in \Omega_T \) and \( w \in \mathbb{R}^{N \times n} \). Finally, we shall assume that the partial map \( z \mapsto \partial_w A(z, w) \) is continuous in the sense that there exists a bounded, concave and non-decreasing modulus of continuity \( \omega_a(\cdot) \) such that \( \omega_a(0) = 0 \) and

\[
|\partial_w A(z_1, w_1) - \partial_w A(z_2, w_2)| \leq L \omega_a \left( \frac{|w_1 - w_2|}{1 + |w_1| + |w_2|} \right) (1 + |w_1| + |w_2|)^{p(z)-2} \tag{2.7}
\]

holds for any \( w_1, w_2 \in \mathbb{R}^{N \times n} \) and \( z \in \Omega_T \). Now we give the definition of a weak solution to the parabolic system (2.1).

**Definition 2.1.** A function \( u \in L^1(\Omega_T, \mathbb{R}^N) \) is called weak solution to the parabolic system (2.1) if and only if \( u \in C^0([-T, 0]; L^2(\Omega, \mathbb{R}^{Nn})), \|u\|^{p(z)}, \|D u\|^{p(z)} \in L^1(\Omega_T) \) and

\[
\int_{\Omega_T} [u \cdot \partial_t \varphi - \mu(z)(A(z, Du), D\varphi)] \, dz = 0 \tag{2.8}
\]

holds, whenever \( \varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N) \).

We are now in a position to state our main theorem which also present a characterizations of singular sets.
Theorem 2.2. Let $u$ be a weak solution of the parabolic system (2.1), where the assumptions (2.2)-2.7 are in force. Then there exists an open subset $\Omega_0 \subset \Omega_T$ with $|\Omega_T \setminus \Omega_0| = 0$ such that $u \in C^{2,\alpha,\gamma}(\Omega_0, \mathbb{R}^N)$, where $\alpha \in (0, 1)$. More precisely, we have that the singular set fulfills $\Omega_T \setminus \Omega_0 \subset \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ and $\Sigma_2$ are defined in the following way:

$$
\Sigma_1 := \left\{ z_0 \in \Omega_T : \liminf_{\varepsilon \to 0^+} \int_{Q_{\varepsilon}(z_0)} |Du - (Du)_{Q_{\varepsilon}(z_0)}| \, dz > 0 \right\},
$$

$$
\Sigma_2 := \left\{ z_0 \in \Omega_T : \limsup_{\varepsilon \to 0^+} (Du)_{Q_{\varepsilon}(z_0)} = +\infty \right\}.
$$

We conclude this section by pointing out that the Lebesgue measure $\mathcal{L}^n(\Sigma_1 \cup \Sigma_2) = 0$ by the Lebesgue differentiation theorem and this also implies that $|\Omega_T \setminus \Omega_0| = 0$.

3. Higher integrability

A key ingredient in the partial regularity proof is the higher integrability for the gradient of the solutions to the parabolic system (2.1). In this section, we collect some higher integrability results and obtain a logarithmic estimate for the gradients. Since $|Du|^{p(z)} \in L^1(\Omega_T)$, we assume in this paper that

$$
\int_{\Omega_T} (|Du| + 1)^{p(z)} \, dz \leq E, \quad (3.1)
$$

for some constant $E > 1$. Next, we recall the following higher integrability results for the degenerate case $p(z) \geq 2$, which were obtained from [3. Theorem 2.2].

Proposition 3.1. ([3. Theorem 2.2]) There exists $\varepsilon_0 = \varepsilon_0(n, N, \nu, \Lambda_2) \in (0, 1)$ such that the following holds: Whenever $u$ is a weak solution of the parabolic system (2.1), where the assumptions (2.2)-2.7 are in force, then there holds

$$
|Du|^{p(z(1+\varepsilon))} \in L^1_{\text{loc}}(\Omega_T). \quad (3.2)
$$

Moreover, if (3.1) holds, then there exists a radius $\varrho_c = \varrho_c(n, N, \nu, L, \gamma_2, E) > 0$ such that for any parabolic cylinder $Q_{2\varrho}(z_0) \subset \Omega_T$ with $\varrho \leq \varrho_c$ and $\varepsilon \in (0, \varepsilon_0)$ there holds

$$
\int_{Q_{\varepsilon}(z_0)} |Du|^{p_c(1+\varepsilon)} \, dz \leq \left\{ \int_{Q_{\varepsilon}(z_0)} |Du|^{p_c} \, dz \right\}^{1 + \frac{1}{p_0} + 1}, \quad (3.3)
$$

for a constant $c$ depending only upon $n, N, \nu, L, \gamma_2$ and $E$.

Our first goal is to improve the higher integrability (3.2) to a fixed scaled parabolic cylinder and derive some properties. Let $\varepsilon_0 \in (0, 1)$ be the exponent as in Proposition 3.1. Fix $z_0 \in \Omega_T$, $\varrho > 0$ and $\lambda \geq 1$, we now consider the oscillation of $p(z)$ on the parabolic cylinder $Q_{\varrho}^{(\lambda)}(z_0)$ and define

$$
p_1 = \inf_{Q_{\varrho}^{(\lambda)}(z_0)} p(z) \quad \text{and} \quad p_2 = \sup_{Q_{\varrho}^{(\lambda)}(z_0)} p(z). \quad (3.4)
$$

Since $p_0 = p(z_0) \geq 2$ and $\lambda \geq 1$, we infer from (2.2) and (3.4) that

$$
p_2 - p_1 \leq \max_{z_1, z_2 \in \partial Q_{\varrho}^{(\lambda)}(z_0)} \omega_{p}(d_{\varrho}(z_1, z_2)) \leq \omega_{p}^{(\varrho)}. \quad (3.5)
$$

This also implies that $p_2 - p_1 \leq 1$. Moreover, we conclude from (2.4) and (3.5) that

$$
\varrho^{-(p_2 - p_1)} \leq \varrho^{-\omega_{p}^{(\varrho)}} = e^{\omega_{p}^{(\varrho)} \log(\varrho)} \leq e. \quad (3.6)
$$
At this point, we choose \( \tilde{\varrho}_0 = \tilde{\varrho}_0(\varrho_0) > 0 \) small enough to have
\[
\tilde{\varrho}_0 \leq \varrho_* \quad \text{and} \quad \omega_p(16\tilde{\varrho}_0) \leq \varepsilon_0,
\]
where \( \varrho_* \) is the radius in Proposition 3.3. From (3.7), we deduce from (3.2) that for any \( \varrho < \tilde{\varrho}_0 \) there holds
\[
Du \in L^{p_0}(Q_{\varrho}^\alpha(z_0), \mathbb{R}^{Nn}).
\]  
Our task now is to establish a relationship between the exponential function and the scaling factor. To this end, we introduce the quantity
\[
\Phi^{l}(z_0, \varrho, l) = \int_{Q_{\varrho}^\alpha(z_0)} \left( \frac{|Du - Dl|}{1 + |Dl|} \right)^2 + \left( \frac{|Du - Dl|}{1 + |Dl|} \right)^{p_0} \, dz,
\]
where \( p_0 = p(z_0) \) and \( l \) is an affine function. Specifically, if \( \lambda = 1 \), then we simply write \( \Phi(z_0, \varrho, l) \) for \( \Phi^{l}(z_0, \varrho, l) \). The next lemma provides some useful estimates for the treatment of the non-standard growth.

**Lemma 3.2.** Let \( z_0 \in \Omega_T \) and \( \tilde{\varrho}_0 \) satisfies (3.7). Let \( p_1 \) and \( p_2 \) be the exponents defined in (3.4). Suppose that \( \lambda \geq 1 \) and \( \lambda \leq 1 + |Dl| \leq M \lambda \) for some \( M \geq 1 \). If \( \Phi^{l}(z_0, \varrho, l) \leq \frac{1}{16} \) and \( \varrho \leq \tilde{\varrho}_0 \), then we have the following properties:

1. \( \frac{1}{2} (1 + |Dl|) \leq 1 + |(Du)_{Q_{\varrho}^\alpha(z_0)}| \leq 3(1 + |Dl|) \),
2. \( \int_{Q_{\varrho}^\alpha(z_0)} |Du|^{p_0} \, dz \leq c(1 + |Dl|)^{p_0} \),
3. \( \lambda^{p_2 - p_1} \leq c \) and \( (1 + |Dl|)^{p_2 - p_1} \leq c \),
4. \( 1 + |Dl| \leq c \gamma^{-\frac{(\gamma + 2)}{\gamma}} \),

where the constant \( c \) depends only upon \( n, E, \gamma_2 \), and \( M \).

**Proof.** We first observe that the first two claims follows directly from the proof of Lemma 3.3 and the assumption \( \Phi^{l}(z_0, \varrho, l) \leq \frac{1}{16} \). For the proof of the third claim, we apply (3.1) and (3.6) to obtain
\[
\lambda^{p_2 - p_1} \leq (1 + |Dl|)^{p_2 - p_1} \leq 2 \left( 1 + \int_{Q_{\varrho}^\alpha(z_0)} |Du|^{p_0} \, dz \right)^{p_2 - p_1}
\]
\[
\leq c \left( 1 + \left( \int_{Q_{\varrho}^\alpha(z_0)} |Du|^{p_0} \, dz \right)^{\frac{1}{p_0}} \right)^{p_2 - p_1}
\]
\[
\leq c \frac{1}{\varrho^{n+2(k(p_2-p_1)+1)}} \lambda^{\frac{n-2}{n-1}(p_2-p_1)} (1 + E)^{p_2 - p_1} \leq c \lambda^{\frac{n-2}{n-1}(p_2-p_1)},
\]
where the constant \( c \) depending only upon \( n \) and \( E \). Since \( p_2 - p_1 \leq 1 \), we infer from (3.10) that \( \lambda^{p_2 - p_1} \leq c(n, E)^{\frac{p_1}{p_2 - p_1}} \leq c(n, E)^{\gamma_2} \). Furthermore, \( (1 + |Dl|)^{p_2 - p_1} \leq c(n, \gamma_2, E, M) \) follows by the assumption \( 1 + |Dl| \leq M \lambda \). Next, we consider the proof of the fourth claim. In view of the first claim, we use the Hölder’s inequality and (3.1) to obtain
\[
1 + |Dl| \leq 2 \left( 1 + \int_{Q_{\varrho}^\alpha(z_0)} |Du|^{p_0} \, dz \right)^{\frac{1}{p_0}} \leq 2 + \frac{2}{\lambda^{\frac{n-2}{n-1}}} \left( \int_{Q_{\varrho}^\alpha(z_0)} |Du|^{p_0} \, dz \right)^{\frac{1}{p_0}}
\]
\[
\leq c(E) \lambda^{\frac{n-2}{n-1}} \leq c(E) \left( \frac{1}{\varrho^{\frac{n-2}{n-1}}} \right)^{\frac{1}{p_0}} (1 + |Dl|)^{\frac{p_0}{n-1}},
\]
since \( p_0 \geq 2 \) and \( 1 \leq \alpha \leq 1 + |Dl| \). Since \( p_2 - p_1 \leq 1 \), we infer from (3.11) that
\[
1 + |Dl| \leq c \varrho^{-\frac{n-2}{n-1}(p_2-p_1)} \leq c \varrho^{-\gamma_2(n)(p_2-p_1)}
\]
and this proves the fourth claim. We have thus proved the lemma. \( \square \)
Our next destination is to refine the estimate (3.3) on the scaled parabolic cylinder \( Q_r^{(i)}(z_0) \). To this end, we establish the following version of the higher integrability estimate for non-uniformly parabolic cylinders.

**Lemma 3.3.** Let \( z_0 \in \Omega_T \) and \( Q_r^{(i)}(z_0) \subset \Omega_r \). Let \( p_1 \) and \( p_2 \) be the exponents defined in (3.4). Assume that \( \Phi^{(i)}(z_0, q, l) \leq \frac{1}{10}, \lambda \geq 1 \) and \( \lambda \leq 1 + |Dl| \leq M \lambda \) for some \( M \geq 1 \). Then, there exists a constant \( \delta_0 = \delta_0(n, E, \gamma_2, \nu, L, M) > 0 \) such that the following holds: Whenever \( \delta \in (0, \delta_0] \), there exists a radius \( \hat{\delta}_0 = \hat{\delta}_0(n, E, \gamma_2, \nu, L, M, \delta) > 0 \) such that for any \( \varrho \leq \hat{\delta}_0 \) the following inequality holds

\[
\int_{Q_{r/2}^{(i)}(z_0)} |Du|^{p_1(1+\varrho)} \, dz \leq c \left[ \int_{Q_r^{(i)}(z_0)} |Du|^{p_2} \, dz \right]^{1+\beta(\varrho)} + 1, \tag{3.12}
\]

where the constant \( c \) depends only upon \( n, E, \gamma_2, \nu, L \) and \( M \). Here, the constant \( \beta(\varrho) \) is defined by

\[
\beta(\varrho) = \frac{\delta}{1 - \frac{\varrho - \beta_p}{\varrho_p}(1 + \frac{\delta_p}{\varrho})}. \tag{3.13}
\]

**Proof.** To start with, we first choose \( \varrho < \hat{\delta}_0 \) where the radius \( \hat{\delta}_0 \) satisfies (3.7). This choice of \( \varrho \) allows us to use Lemma 3.3. Let \( \frac{1}{2} < s < t < \varrho \) and set \( \hat{z} = (\hat{x}, \hat{t}) \in Q_r^{(i)}(z_0) \). Moreover, we set \( r = t - s \) and consider the uniform parabolic cylinder

\[
Q_r^{(i)}(\hat{z}) := B_r(\hat{x}) \times (\hat{t} - \lambda^2 - p_0 r^2, \hat{t}), \tag{3.14}
\]

where \( p_0 = p(z_0) \). By abuse of notation, we continue to write \( Q_r^{(i)}(\hat{z}) \) for \( \hat{Q}_r^{(i)}(\hat{z}) \). It can be easily seen that \( Q_r^{(i)}(\hat{z}) \subset \hat{Q}_r^{(i)}(z_0) \). We divide our proof in two steps.

**Step 1:** Obtaining a higher integrability type estimate on \( Q_r^{(i)}(\hat{z}) \), where \( \hat{z} \in Q_r^{(i)}(z_0) \).

Introducing the change of variables and the new functions

\[
\begin{align*}
\vec{u}(y, \tau) &= \lambda^{-1}u(\hat{x} + y, \hat{t} + \lambda^2 - p_0 \tau), \\
\vec{A}(y, \tau, w) &= \lambda^{-1}A(\hat{x} + y, \hat{t} + \lambda^2 - p_0 \tau, \lambda w), \\
\vec{\mu}(y, \tau) &= \mu(\hat{x} + y, \hat{t} + \lambda^2 - p_0 \tau), \\
\vec{p}(y, \tau) &= p(\hat{x} + y, \hat{t} + \lambda^2 - p_0 \tau),
\end{align*}
\]

we rewrite the parabolic system (3.1) in terms of the new variables \((y, \tau)\) and new functions \(\vec{u}\) as follows:

\[
\partial_t \vec{u} - \text{div}_y \vec{\mu}(y, \tau) \vec{A}(y, \tau, D_y \vec{u}) = 0, \quad \text{in} \quad Q_{2r}(0). \tag{3.16}
\]

Our task now is to establish a reverse Hölder inequality similar to (3.3) for the map \(\vec{u}\). To this end, we first observe that \(\vec{u} \in C([-4r^2, 0]; L^2(B_{2r}(0); \mathbb{R}^N))\) as well as \(|\vec{u}|^{\beta(p)}) \in L^1(Q_{2r}(0))\). Moreover, we check that

\[
\int_{Q_{2r}(0)} \left[ \vec{u} \cdot \partial_t \vec{\phi} - \vec{\mu}(y, \tau) (\vec{A}(y, \tau, D_y \vec{u}), D_y \vec{\phi}) \right] \, dy \, dr = 0
\]

holds, whenever \( \vec{\phi}(y, \tau) \in C^\infty_0(Q_{2r}(0), \mathbb{R}^N) \). It follows that \(\vec{u}\) is indeed a weak solution to (3.16). The next thing to do in the proof is to check that the conditions in (3.3) and Lemma 3.2(3) are fulfilled. Since \( \varrho < \hat{\delta}_0 \) and \( \lambda \geq 1 \), we use (3.1) and Lemma 3.2(3) to deduce that

\[
\int_{Q_{2r}(0)} |D_y \vec{u}|^{\beta(p)(\tau, \rho)} \, dy \, dr \leq \lambda^2 - p_0 \lambda^2 - 2 \int_{Q_{2r}(0)} |Du|^{p_2} \, dz \leq c(n, E) \frac{E}{\lambda^2} \leq c_1(n, E), \tag{3.17}
\]
which is analogue to (3.1). Next, we check the continuity condition for the new exponent function \( \hat{p} \). From (2.3) and (3.15), we see that
\[
|\hat{p}(y_1, \tau_1) - \hat{p}(y_2, \tau_2)| \leq \omega_p \left( \max \left\{ |y_1 - y_2|, A^{2-p_0} \sqrt{\tau_1 - \tau_2} \right\} \right)
\leq \omega_p (d \hat{p}(z, \tau_1), (y, \tau_2)),
\]
(3.18)
since \( p_0 \geq 2 \) and \( \lambda \geq 1 \). This shows that the exponent function \( \hat{p} \) satisfies (2.3). Moreover, we consider the structural conditions for the vector field \( \tilde{A}(y, \tau, w) \). From Lemma 3.2 (2), (2.2) and (3.15), we obtain
\[
|\tilde{A}(y, \tau, w)| \leq \sqrt{L} \lambda^{1-p_0}(1 + |w|^2)^{\frac{p_0-1}{2}}
\leq \sqrt{L} \lambda^{p(c^2-p_0)}(1 + |w|^2)^{\frac{p_0-1}{2}} \leq L_1(1 + |w|^2)^{\frac{p_0-1}{2}},
\]
(3.19)
since \( z = (\tilde{x} + \tilde{y} + A^{2-p_0} \tau) \) and \( p(z) = \hat{p}(y, \tau) \). Here, the constant \( L_1 \) depends only upon \( L, n, E, \gamma_2 \) and \( M \). This establishes the growth condition for \( \tilde{A} \). In view of (2.2) and (3.15), we apply Lemma 3.2 (2) to deduce that
\[
\langle \tilde{A}(y, \tau, w) \rangle = \lambda^{2-p_0} \langle \tilde{A}(\tilde{x} + y, \tilde{y} + A^{2-p_0} \tau, A w) \rangle \tilde{w}, \tilde{w} \rangle
\geq \sqrt{L} \lambda^{p(c^2-p_0)}|w|^{p(c^2-2)} |\tilde{w}|^2 \gtrsim v_1 |w|^{\tilde{p}(c, \tau)-2} |\tilde{w}|^2,
\]
(3.20)
where the constant \( v_1 \) depends only upon \( n, E, \gamma_2 \) and \( M \). It follows from (3.20) that
\[
\langle \tilde{A}(y, \tau, w) \rangle = \int_0^1 \langle \tilde{A}(y, \tau, w) \rangle \, ds + \langle \tilde{A}(0, \tau, w) \rangle
\geq v_1 |w|^{p(c)} \int_0^1 s^{p(c)-2} \, ds = L_1|w| \gtrsim c|w|^{\tilde{p}(c, \tau)} - c(L_1),
\]
(3.21)
where we used Young's inequality in the last step. This shows the ellipticity condition for \( \tilde{A} \). Furthermore, we infer from (3.15) that \( \sqrt{r} \leq \tilde{p} \leq \sqrt{L} \). At this point, we observe from (3.17)-(3.21) that the parabolic system (3.16) fulfills the desired structure assumptions and the results of [15, Theorem 2.2] apply to weak solutions of (3.16). More precisely, there exist a constant \( \delta_0 > 0 \) and a radius \( r_0 > 0 \), depending only upon \( n, E, \gamma_2, \nu, L \) and \( M \), such that for any \( \hat{r} \leq r_0 \) and \( \delta \in (0, \delta_0] \) there holds
\[
\int_{Q_{r_0}(0)} |D_{\hat{r}} \tilde{A}|^{p(1+\delta)} \, dy \, dr \leq c \left( \int_{Q_{r}(0)} |D_{\hat{r}} \tilde{A}|^{\hat{p}(\nu, \gamma_2)} \, dy \, dr \right)^{1+\frac{\hat{p}(\nu, \gamma_2)}{p}} + 1,
\]
(3.22)
where the constant \( c \) depends only upon \( n, E, \gamma_2, \nu, L \) and \( M \). Let \( \delta \in (0, \delta_0] \) be a fixed constant, we now choose \( \hat{q}_0 = \tilde{q}_0(\delta) > 0 \) small enough to have
\[
\hat{q}_0 \leq \min(\hat{q}_0, r_0) \quad \text{and} \quad \omega_p(\hat{q}_0) \leq \frac{\delta}{1 + \delta^2 \nu^2}.
\]
(3.23)
At this stage, we conclude from (3.23) that for any \( q \leq \hat{q}_0 \), there holds \( r = t - s \leq \frac{r_0}{2} < r_0 \) and \( p_2 < p_1(1 + \delta) \). Therefore, the reverse Hölder inequality (3.22) holds for \( \hat{r} = r \). This gives
\[
\int_{Q_{r_0}(0)} |D_{\hat{r}} \tilde{A}|^{p_1(1+\delta)} \, dy \, dr \leq c \left( \int_{Q_{r}(0)} |D_{\hat{r}} \tilde{A}|^{p_2} \, dy \, dr \right)^{1+\frac{p_2}{p_1}} + 1.
\]
(3.22)
Rescaling back to \( u \), we deduce
\[
\int_{Q_{r_0}(2)} |Du|^{p_1(1+\delta)} \, dz \leq c \lambda^{p_1(1+\delta)} \left( \int_{Q_{r_0}(2)} |Du|^{p_2} \, dz \right)^{1+\frac{p_2}{p_1}} + c \lambda^{p_1(1+\delta)}.
\]
From Lemma 3.2, we obtain \( \lambda \leq 1 + |Dl| \leq 2(1 + |(Du)_{Q_s^4(z_0)}|) \) and therefore
\[
\int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz \leq c \lambda^{\delta p_1 - \frac{1}{2} \rho_{p_1 p_2}} \left( \int_{Q_s^4(z_0)} |Du|^{p_1} \, dz \right)^{1 + \frac{1}{2} \rho_{p_1 p_2}} + c \, (1 + |(Du)_{Q_s^4(z_0)}|)^{p_1(1+\delta)}.
\]
In view of \( p_1 \leq p_2 \leq p_1(1+\delta) \), the interpolation inequality allows us to conclude that
\[
\int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz 
\leq c \left( \int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz \right)^{\frac{1}{2}} \lambda^{\frac{1}{2}(1+\delta)} \left( \int_{Q_s^4(z_0)} |Du|^{p_1} \, dz \right)^{\frac{1}{2}} + c \, |(Du)_{Q_s^4(z_0)}|^{p_1(1+\delta)} + c,
\]
since
\[
\frac{1}{p_2} = \frac{\Theta}{p_1(1+\delta)} + \frac{1 - \Theta}{p_1}, \quad \text{where} \quad \Theta = \frac{p_2 - p_1 \delta + 1}{p_2}.
\]
Furthermore, from (3.23), we see that
\[
\sigma := \frac{1}{\frac{\delta p_2}{\delta p_1}(1 + \frac{\delta p_2}{\delta p_1})} > 1.
\]
This enables us to use the Young’s inequality with \( \sigma \) and \( \sigma/(\sigma - 1) \) to obtain
\[
\int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz \leq \kappa \int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz 
+ \frac{c(\kappa)}{|Q_s^4(z_0)|^{\frac{\delta p_2}{\delta p_1}(1 + \frac{\delta p_2}{\delta p_1})}} \lambda^{\frac{1}{2}(1+\delta)} \left( \int_{Q_s^4(z_0)} |Du|^{p_1} \, dz \right)^{1 + \frac{1}{2} \rho_{p_1 p_2}} + c |Q_s^4(z_0)|^{p_1(1+\delta)} + c |Q_s^4(z_0)|, \]
where \( \kappa \leq 1 \) and
\[
\hat{\beta}(\delta) = \frac{\delta p_2}{1 - \frac{p_2 - p_1 \delta + 1}{p_2}}.
\]

Step 2: Proof of (3.12) by using a covering argument. To start with, we cover the set \( Q_s^4(z_0) \) by a finite number of the uniform parabolic cylinders \( \{Q_{s_1}^4(z_i)\}_{i=1}^M \) of the type (3.14) such that only a finite number of cylinders \( Q_{s_1}^4(z_i) \) intersect. Therefore, we find that for any \( \frac{1}{2} \delta < s < t < \frac{1}{2}\delta \) there holds
\[
\int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz \leq \kappa N \int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz 
+ c(\kappa, N) \lambda^{\frac{1}{2}(1+\delta)} \left( \int_{Q_s^4(z_0)} |Du|^{p_1} \, dz \right)^{1 + \frac{1}{2} \rho_{p_1 p_2}} + cN |Q_s^4(z_0)| |(Du)_{Q_s^4(z_0)}|^{p_1(1+\delta)} + c |Q_s^4(z_0)|.
\]
At this stage, we can use an iteration lemma (cf. [3] Lemma 2.2)) to re-absorb the first integral of the right-hand side into the left. This gives
\[
\int_{Q_s^4(z_0)} |Du|^{p_1(1+\delta)} \, dz \leq \lambda^{\frac{1}{2}(1+\delta)} \left( \int_{Q_s^4(z_0)} |Du|^{p_1} \, dz \right)^{1 + \frac{1}{2} \rho_{p_1 p_2}} + c(\kappa, N) \frac{|Q_s^4(z_0)|^{\hat{\beta}(\delta)}}{|Q_s^4(z_0)|^{\hat{\beta}(\delta)}} \left( \int_{Q_s^4(z_0)} |Du|^{p_1} \, dz \right)^{1 + \frac{1}{2} \rho_{p_1 p_2}} + cN |Q_s^4(z_0)| |(Du)_{Q_s^4(z_0)}|^{p_1(1+\delta)} + c |Q_s^4(z_0)|.
Therefore, dividing by $|Q^{(1)}_{\delta_0}(z_0)|$, we obtain

$$
\int_{Q^{(1)}_{\delta_0}(z_0)} |Du|^{p_1(1+\delta)} \, dz \leq c \lambda^{-\frac{1}{2}(1-\delta_1)(1-\delta_2)} \left( \int_{Q^{(1)}_{\delta_0}(z_0)} |Du|^{p_1} \, dz \right)^{1+\beta(\delta)} + c(1 + |Du|)^{p_1} \leq c(M, \gamma_2) \lambda^{p_1(1+\beta(\delta))}
$$

(3.24)

On the other hand, we infer from the Hölder’s inequality and Lemma 3 (2) that

$$
\int_{Q^{(1)}_{\delta_0}(z_0)} |Du|^{p_1} \, dz \leq \left( \int_{Q^{(1)}_{\delta_0}(z_0)} |Du|^{p_1} \, dz \right)^{\frac{p_1}{p_1(1+\delta)}} \leq c(1 + |Du|)^{p_1} \leq c(M, \gamma_2) \lambda^{p_1},
$$

(3.25)

since $1 + |Du| \leq M \lambda$. In view of $\lambda \leq 1 + |Du| \leq 2(1 + |Du|_{Q^{(1)}_{\delta_0}(z_0)})$, we conclude from (3.25) that

$$
\lambda^{-\frac{1}{2}(1-\delta_1)(1-\delta_2)} \left( \int_{Q^{(1)}_{\delta_0}(z_0)} |Du|^{p_1} \, dz \right)^{1+\beta(\delta)} \leq c(\delta, M, \gamma_2) \lambda^{-\frac{1}{2}(1-\delta_1)(1-\delta_2)} \lambda^{p_1(1+\beta(\delta))}
$$

(3.26)

$$
= c(\delta, M, \gamma_2) \lambda^{p_1(1+\beta(\delta))} \leq c \left( \int_{Q^{(1)}_{\delta_0}(z_0)} |Du|^{p_1} \, dz \right)^{1+\beta(\delta)} + 1.
$$

where $\beta(\delta)$ satisfies (3.13). Plugging (3.26) into (3.24) and taking into account $\delta \leq \beta(\delta)$, we obtain the desired estimate (3.12). The proof of the lemma is now complete. \( \square \)

**Remark 3.4.** From the proof of Lemma 3.3 we conclude that for any fixed $\delta \in (0, \delta_0]$ the radius $\delta_0$ can be chosen according to (3.23) in order to obtain the higher integrability estimate (3.12).

With the help of the preceding two lemmas, we can now prove the following extension of [11, Lemma 2.9] to the parabolic case.

**Lemma 3.5.** Let $\theta \in (0, \frac{1}{2}]$, $M \geq 1$ and $\gamma \geq 1$ be fixed. Suppose that $Q^{(1)}(z_0, \theta, l) \leq \frac{1}{10} \lambda \geq 1$ and $\lambda \leq 1 + |Du| \leq M \lambda$. Then, there exists a radius $\delta_0 = \delta_0(n, E, \gamma_2, \gamma, \nu, L, M) > 0$ such that for any $q \leq \delta_0$ and $q \in \{p_1, p_1 + 6\omega_p(2q)\}$, there holds

$$
\int_{Q^{(1)}_{\delta_0}(z_0)} (1 + |Du| + |Du|^q \log^*(e + |Du| + |Du|) \, dz \leq c \log^* \left( \frac{1}{Q} \right) (1 + |Du|)^{p_0}.
$$

(3.27)

where the constant $c$ depends only upon $n, E, \gamma_2, \gamma, \nu, L, M$ and $\theta$. Here, $p_0 = p(z_0)$ and $p_1$ is the minimum of $p(z)$ on $Q^{(1)}_{\delta_0}(z_0)$.

**Proof.** We prove the estimate (3.27) by decomposing the integral of the left-hand side into two parts. To this aim, we decompose $Q^{(1)}_{\delta_0}(z_0) = Q^+ \cup Q^-$, where

$$
Q^+ = \left\{ z \in Q^{(1)}_{\delta_0}(z_0) : |Du| \geq 1 + |Du| \right\} \quad \text{and} \quad Q^- = Q^{(1)}_{\delta_0}(z_0) \setminus Q^+.
$$
We first consider the estimate on the set $Q^-$. From Lemma 3.2 (4) and (3.6), we conclude that

$$I_- := \frac{1}{|Q^{(1)}_{0_0}(z_0)|} \int_{Q^-} (1 + |Du| + |Du|^q \log^\gamma(e + |Du| + |Du|) \, dz$$

$$\leq c(1 + |Du|^q \log^\gamma[2(1 + |Du|)]$$

$$\leq c(1 + |Du|^p_0 \left( q^{-\frac{(n+2)2}{4}} \right)^{\frac{q-p_0}{p_0}} \log^\gamma \left[ c_0 q^{-\frac{(n+2)2}{4}} \right]$$

$$\leq c \log^\gamma \left( \frac{1}{\theta} \right) (1 + |Du|)^{p_0},$$

since $q \in [p_1, p_1 + 6\omega_p(2\theta)]$. Moreover, we estimate the left-hand side of (3.27) on the set $Q^+$ and obtain a decomposition as follows:

$$I_+ := \frac{1}{|Q^{(1)}_{0_0}(z_0)|} \int_{Q^+} (1 + |Du| + |Du|^q \log^\gamma(e + |Du| + |Du|) \, dz$$

$$\leq c \frac{1}{|Q^{(1)}_{0_0}(z_0)|} \int_{Q^+_{0_0}(z_0)} |Du|^q \log^\gamma(e + |Du|) \, dz$$

$$\leq c \int_{Q^+_{0_0}(z_0)} |Du|^q \log^\gamma(e + \frac{|Du|^q}{|Q^+_{0_0}(z_0)|}) \, dz$$

$$+ c \log^\gamma(e + \frac{|Du|^q}{|Q^+_{0_0}(z_0)|}) \int_{Q^+_{0_0}(z_0)} |Du|^q \, dz =: I_{+1}^{(1)} + I_{+2}^{(1)},$$

with the obvious meaning of $I_{+1}^{(1)}$ and $I_{+2}^{(1)}$. Next, we consider the estimate for $I_{+1}^{(1)}$. To start with, we apply Proposition A.1 with $(m, p, \sigma, f)$ replaced by $(n + 1, q, \delta_0, |Du|)$, where $\delta_0$ is the exponent in Lemma 3.3. This gives

$$I_{+1}^{(1)} \leq c \left( \int_{Q^+_{0_0}(z_0)} |Du|^{p(1+\delta_0)} \, dz \right)^{\frac{1}{\gamma n}}$$

$$\leq c(\theta) \left( \int_{Q^+_{0_0}(z_0)} |Du|^{p_0(1+3\omega_p(2\theta)(1+\delta_0))} \, dz + 1 \right)^{\frac{1}{\gamma n}},$$

since $\theta \leq \frac{1}{2}$. At this point, let $r_0 = r_0(n, E, \gamma_2, \nu, L, M)$ be the radius considered in the proof of Lemma 3.3 and hence the radius $\varrho_0$ can be determined a priori only in terms of $\varepsilon_0$, $\delta_0$, $\varrho_0$ and $r_0$, such that

$$\varrho_0 \leq \min(\varrho_0, r_0) \quad \text{and} \quad \omega_p(2\varrho_0) \leq \min \left\{ \frac{\delta_0}{1 + 13\delta_0 \gamma_2}, \frac{\varepsilon_0}{3} \right\}. \quad (3.28)$$

From Remark 3.4, the conditions for Lemma 3.3 are satisfied with $\delta = \delta_0 + 3\omega_p(2\varrho)(1 + \delta_0)$. We now apply Lemma 3.3 to conclude that for any $\varrho \leq \varrho_0$, there holds

$$I_{+1}^{(1)} \leq c \left( \int_{Q^+_{0_0}(z_0)} |Du|^{p(1+\delta_0)} \, dz \right)^{\frac{1}{\gamma n}} + c,$$

where the constant $c$ depends only upon $n, E, \gamma_2, \gamma, \nu, L, M$ and $\theta$. According to Lemma 3.2 (2), we use the Hölder’s inequality and (3.13) to deduce

$$\left( \int_{Q^+_{0_0}(z_0)} |Du|^{p(1+\delta_0)} \, dz \right)^{\frac{1}{\gamma n}} \leq c(M)(1 + |Du|)^{c\omega_p(2\varrho)} \leq c \varrho^{-c_{\omega_p}(\varrho)} \leq c, \quad (3.29)$$
where the constant $c$ depends only upon $n$, $E$, $\gamma_2$ and $M$. Finally, we use (3.29), Hölder’s inequality and Lemma 3.2 (2) to obtain

$$I^1_+ \leq c \left( \int_{Q^{(0)}_\gamma} |Du|^p \, dz \right) + c \leq c(1 + |Dl|)^{p_0} \leq c(1 + |Dl|)^{p_0}. \tag{3.30}$$

To estimate $I^2_+$, we infer from (3.28) that the inequality $q \leq p_1 + 6\omega_p(2\rho) \leq p_1(1 + \epsilon_0)$ holds. In view of $p_0 \geq 2$ and $\lambda \leq 1 + |Dl|$, we conclude from (3.1), Proposition 3.1 and Lemma 3.2 (4) that

$$\left( |Du|^q \right)_{Q^{(0)}_\gamma} \leq c(\theta) \int_{Q^{(0)}_\gamma} |Du|^q \, dz \leq c|Dl|^{p_0-2} \left[ \theta^{-n-2} \left( \int_{Q^\rho \gamma} |Du|^p \, dz \right)^{1+\frac{1}{p_0}} + 1 \right] \leq c|Dl|^{p_0} \theta^{(2-\gamma_2)r_0^\gamma - n - 2}.$$ 

By the Hölder’s inequality, we adopt the same procedure as in the proof of (3.30). This yields the estimate

$$I^2_+ \leq c \log^\gamma \left( \frac{1}{\theta} \right) \int_{Q^{(0)}_\gamma} |Du|^q \, dz \leq c \log^\gamma \left( \frac{1}{\theta} \right) (1 + |Dl|)^{p_0},$$

for a constant $c$ depending only on the data $n$, $E$, $\gamma_2$, $\gamma$, $v$, $L$, $M$ and $\theta$. Which proves the lemma. Combining the estimates for $I_-$, $I^1_+$ and $I^2_+$, we obtain the desired estimate (3.27).

4. Poincaré type inequality

The aim of this section is to derive some smallness conditions for weak solutions near the regular points. These smallness conditions will be the starting point for the proof of the main result in Sect. 7. In order to reduce the alternative characterization of the regular points, we will need a Poincaré type inequality for the weak solutions. To achieve this, we first prove the following gluing lemma, which concerns weighted mean values of weak solutions on different time slices..

**Lemma 4.1.** Let $z_0 \in \Omega_T$ and let $l$ be an affine function. Let $\rho_0 > 0$ be the radius in Lemma 3.5. Assume that $\rho < \rho_0$ and $\Phi(z_0, 2\rho, l) \leq \frac{1}{16}$. Then, for any $r$, $s \in \Lambda_\rho(z_0)$, there exists a constant $c$ depending only upon $n$, $E$, $\gamma_2$, $v$ and $L$ such that the inequality

$$\int_{B_r(x_0)} (u_i(\cdot, r) - u_i(\cdot, s)) \psi(x) \, dx \leq c(r - s)^{\frac{1}{p}} |D\psi|_{L^p} Q_{\rho_0}^{\frac{1}{p}} \left( \int_{Q_{\rho}^{(0)}} |Du - Dl|^{p_0} \, dz \right)^{\frac{1}{p_0}} + c(r - s)^{\frac{1}{\nu}} (1 + |Dl|)^{p_0-2} |D\psi|_{L^p} Q_{\rho_0}^{\frac{1}{p}} \left( \int_{Q_{\rho}^{(0)}} |Du - Dl|^{p_0} \, dz \right)^{\frac{1}{p_0}} + c\left( \omega_p(\rho) \log \left( \frac{1}{\theta} \right) + v(\rho) \right)^{\frac{1}{p_0}} (r - s)^{\frac{1}{\nu}} (1 + |Dl|)^{p_0-1} |D\psi|_{L^p} Q_{\rho_0}^{\frac{1}{p}} \tag{4.1}$$

holds, where $\psi \in C^0_0(B_\rho(x_0))$, $i \in \{1, \cdots, N\}$ and $p_0 = p(\rho_0)$.
Proof. Without loss of generality, we may assume that $z_0 = 0$. Let $i \in \{1, \cdots, N\}$ and $0 < h < \frac{1}{2}(r - s)$. In the weak formulation we choose the test function
\[
\varphi_h(x, t) = \zeta_h(t)\psi(x)\varepsilon_i,
\]
where $\psi \in C_0^\infty(B_\rho)$ and $\zeta_h$ is a Lipschitz function given by
\[
\zeta_h(t) = \begin{cases} 
0, & t \leq s, \\
\frac{1}{h}(t - s), & s < t < s + h, \\
\frac{1}{h}(t - r), & r - h < t < r, \\
0, & t \geq r,
\end{cases}
\]
where $-\rho^2 < s < r < 0$. For the vector field $A = (A^1, \cdots, A^N) \in \mathbb{R}^{N\times N}$, we deduce
\[
\int_{-\rho^2}^0 \int_{B_\rho} u_i \cdot \partial_t \zeta_h \psi \, dx \, dt = \int_{-\rho^2}^0 \int_{B_\rho} \zeta_h \mu(z)A^i(z, Du) \cdot D\psi \, dx \, dt.
\]
From the definition of $\zeta_h$, we pass to the limit $h \downarrow 0$ to infer that
\[
\int_{B_\rho} (u_i(\cdot, r) - u_i(\cdot, s))\psi(x) \, dx = \int_s^r \int_{B_\rho} \mu(z)A^i(z, Du) \cdot D\psi \, dx \, dt.
\]
We now proceed to estimate the integral with respect to space from the right-hand side of the above equation by
\[
\int_{B_\rho \times [r]} \mu(z)A^i(z, Du) \cdot D\psi \, dx
\]
\[
= \int_{B_\rho \times [r]} \mu(z)[A^i(z, Du) - A^i(z, Dl)] \cdot D\psi \, dx
\]
\[
+ \int_{B_\rho \times [r]} [\mu(z) - (\mu)_{Q_\rho}]A^i(z, Dl) \cdot D\psi \, dx
\]
\[
+ \int_{B_\rho \times [r]} (\mu)_{Q_\rho}[A^i(z, Dl) - A^i(0, Dl)] \cdot D\psi \, dx
\]
\[
=: L_1 + L_2 + L_3,
\]
with the obvious meaning of $L_1$, $L_2$ and $L_3$. At this point, we define $p_1 = \inf_{Q_\rho} p(z)$ and $p_2 = \sup_{Q_\rho} p(z)$. By the fundamental theorem of calculus, we infer from (2.2) that
\[
|L_1| \leq \sqrt{L} \left| \int_{B_\rho \times [r]} \int_0^1 \partial_s A^i(z, Dl + s(Du - Dl)) \cdot (Du - Dl) \, ds \cdot D\psi \, dx \right|
\]
\[
\leq \sqrt{L} \int_{B_\rho \times [r]} \left( 1 + |Dl| + |Du - Dl|^{p_{i-2}} |Du - Dl| |D\psi| \right) \, dx
\]
\[
\leq c \int_{B_\rho \times [r]} (1 + |Dl|^{p_{i-2}} |Du - Dl| |D\psi| \, dx
\]
\[
+ c \int_{B_\rho \times [r]} |Du - Dl|^{p_{i-1}} |D\psi| \, dx =: L_1^{(1)} + L_1^{(2)},
\]
since $p_2 \geq 2$. We note that the choice of $Q < \rho_0$ allows us to use Lemma (3.2) with $\lambda = 1$. Now we come to the estimate of $L_1^{(1)}$. By Lemma (3.2) (3) and the Hölder’s inequality, we
obtain
\[
\int_s^t L_1^{(1)} \, dt \leq c(1 + |Dl|^{p_0}) \int_s^t \int_{B_{x, t}} |Du - Dl|^{\frac{p_0}{m}} \, dx \, dz
\]
\[
\leq c(1 + |Dl|^{p_0}) \int_s^t \int_{Q_{t, x}} |Du - Dl|^{\frac{p_0}{m}} \, dx \, dz
\]
\[
\leq c(1 + |Dl|^{p_0}) \int_s^t \int_{Q_{t, x}} |Du - Dl|^{\frac{p_0}{m}} \, dx \, dz
\]
\[
\leq c(1 + |Dl|^{p_0}) \int_s^t \int_{Q_{t, x}} |Du - Dl|^{\frac{p_0}{m}} \, dx \, dz
\]
since \( p_0 \geq 2 \). To estimate \( L_1^{(2)} \), we use the H"older’s inequality to deduce
\[
L_1^{(2)} \leq c \|D\psi\|_{L^m} \left( \int_{B_{x, t}} |Du - Dl|^{\frac{p_0}{m}} \, dx \right)^{\frac{m-1}{m}}.
\]
In the case \( |Du - Dl| \geq 1 \), we use the fundamental theorem of calculus and \((3.5)\) to obtain
\[
|Du - Dl|^{\frac{m}{m-1}}(p_2 - 1) \log(|Du - Dl|) \leq 2 \omega_p(q) |Du - Dl|^{\frac{m}{m-1}} \log(1 + |Du - Dl|)
\]
\[
(4.3)
\]
since \( p_0 \geq 2 \). In the case \( |Du - Dl| < 1 \), we observe that \( |Du - Dl|^{\frac{m}{m-1}}(p_2 - 1) \leq |Du - Dl|^{p_0} \). Using this together with \((4.3)\), we conclude that
\[
L_1^{(2)} \leq c \|D\psi\|_{L^m} \left( \int_{B_{x, t}} |Du - Dl|^{\frac{p_0}{m}} \, dx \right)^{\frac{m-1}{m}}
\]
\[
+ c \|D\psi\|_{L^m} \left( \int_{B_{x, t}} (1 + |Du - Dl|) \, dx \right)^{\frac{m-1}{m}}
\]
Noting that \( p_0 \frac{m-1}{m} \leq p_1 + 2 \omega_p(q) \), we can apply Lemma\(3.5\) with \( \theta = \frac{1}{2}, \gamma = 1 \) and \( \lambda = 1 \). Therefore, we apply Lemma\(3.5\) and the H"older’s inequality to obtain
\[
\int_s^t L_1^{(2)} \, dt \leq c[r]^{\frac{m}{m-1}} \|D\psi\|_{L^m} \left( \int_{Q_{t, x}} (1 + |Du - Dl|^{p_0} \, dx \right)^{\frac{m-1}{m}}
\]
\[
+ c \|D\psi\|_{L^m} \left( \int_{Q_{t, x}} |Du - Dl|^{\frac{m}{m-1}} \log(1 + |Du - Dl|) \, dx \right)^{\frac{m-1}{m}}
\]
\[
\leq c[r]^{\frac{m}{m-1}} \|D\psi\|_{L^m} \left( \int_{Q_{t, x}} (1 + |Du - Dl|^{p_0} \, dx \right)^{\frac{m-1}{m}}
\]
\[
+ c \|D\psi\|_{L^m} \left( \int_{Q_{t, x}} |Du - Dl|^{\frac{m}{m-1}} \log \left( \frac{1}{q} \right) \right)^{\frac{m-1}{m}}
\]
Consequently, we infer that
\[
\int_s^t L_1 \, dt \leq c(1 + |Dl|^{p_0}) |r - s|^{\frac{m}{m-1}} \|D\psi\|_{L^m} \left( \int_{Q_{t, x}} (1 + |Du - Dl|^{p_0} \, dx \right)^{\frac{m-1}{m}}
\]
\[
+ c[r]^{\frac{m}{m-1}} \|D\psi\|_{L^m} \left( \int_{Q_{t, x}} |Du - Dl|^{\frac{m}{m-1}} \log \left( \frac{1}{q} \right) \right)^{\frac{m-1}{m}}
\]
\[
+ c \|D\psi\|_{L^m} \left( \int_{Q_{t, x}} (1 + |Du - Dl|^{p_0} \, dx \right)^{\frac{m-1}{m}}
\]
\[
(4.4)
\]
We now turn our attention to the estimate of $L_2$. Since $\sqrt{v} \leq \mu(z) \leq \sqrt{L}$, we conclude from Lemma 3.2 (3) and the growth condition (2.2) to obtain

\[
\int_s^w L_2 v(t) \, dt = \int_s^w \int_{B_r} |(u - (\mu)_{Q_0})A'(z, Dl) \cdot D\psi| \, dx \, dt \\
\leq \sqrt{L} (1 + |Dl|)^{p_0 - 1} \int_s^w \int_{B_r} |(u - (\mu)_{Q_0})D\psi| \, dx \, dt \\
\leq c(1 + |Dl|)^{p_0 - 1} (r - s)^{\frac{1}{p_0}} \|D\psi\|_{L^{p_0}} |Q_0|^{\frac{p_0 - 1}{p_0}}. \tag{4.5}
\]

Finally, we consider the estimate for $L_3$. Recalling that $\sqrt{v} \leq \mu(z) \leq \sqrt{L}$, we infer from Lemma 3.2 (3), (4) and the continuity condition (2.6) to obtain

\[
\int_s^w L_3 v(t) \, dt = \int_s^w \int_{B_r} (\mu)_{Q_0}[A'(z, Dl) - A'(0, Dl)] \cdot D\psi \, dx \, dt \\
\leq \sqrt{L} \int_s^w \int_{B_r} \omega_p(q) \left[ (1 + |Dl|)^{\frac{2}{p_0 - 1}} + (1 + |Dl|)^2 \right] |D\psi| \, dx \, dt \\
\times \left[ 1 + \log(1 + |Dl|^2) \right] \\
\leq c(1 + |Dl|)^{p_0 - 1} \omega_p(q) \log \left( \frac{1}{q} \right) (r - s)^{\frac{1}{p_0}} \|D\psi\|_{L^{p_0}} |Q_0|^{\frac{p_0 - 1}{p_0}}. \tag{4.6}
\]

Combining the estimates (4.4)-(4.6), we obtain the desired estimate (4.1). This completes the proof.

With the help of Lemma 4.1, we are now ready for the proof of Poincaré inequality for weak solutions to the parabolic system (2.1).

**Lemma 4.2.** Let $z_0 \in \Omega_T$ and let $l$ be an affine function. Let $q_0 > 0$ be the radius in Lemma 3.5. Assume that $q < q_0$ and $\Phi(z_0, 2q, l) \leq \frac{1}{10}$. Then, there exists a constant $c$ depending only on $n, E, \gamma, \nu$ and $L$ such that the inequality

\[
\int_{Q_{r_0}(z_0)} \frac{|u - (u)_{Q_{r_0}(z_0)} - Dl \cdot (x - x_0)|^2}{r_0} \, dz + \frac{|u - (u)_{Q_{r_0}(z_0)} - Dl \cdot (x - x_0)|^{p_0}}{r_0} \leq c(1 + |Dl|)^{p_0(p_0 - 2)} \left[ \int_{Q_{r_0}(z_0)} |D(u - (u)_{Q_{r_0}(z_0)} - Dl \cdot (x - x_0))|^{p_0} \, dz \right]^{\frac{1}{p_0}} \\
+ c(1 + |Dl|)^{p_0(p_0 - 2)} \left[ \int_{Q_{r_0}(z_0)} |D(u - (u)_{Q_{r_0}(z_0)} - Dl \cdot (x - x_0))|^{p_0} \, dz \right]^{\frac{1}{p_0}} \\
+ c(1 + |Dl|)^{p_0(p_0 - 1)} \left[ \omega_{p_0}(q) \log \left( \frac{1}{q} \right) + \left( \omega_p(q) \log \left( \frac{1}{q} \right) \right)^{\frac{p_0 - 1}{p_0}} \right] \tag{4.7}
\]

holds, where $p_0 = p(z_0)$.

**Proof.** Without loss of generality, we may assume that $z_0 = 0$. We choose a radial function $\psi \in C_c^\infty(B_1)$ with $\int_{B_1} \psi(x) \, dx = 1$. Let $\psi_q(x) = q^{-n} \psi(x/q)$. For $t \in (-q^2, 0)$, we define some different types of mean values

\[
\bar{u}_q(t) = \int_{B_q} u(x, t) \, dx, \quad \bar{u}^\psi(t) = \int_{B_q} u(x, t)\psi_q(x) \, dx \quad \text{and} \quad (u)^\psi = \int_{-q^2}^0 \bar{u}^\psi(t) \, dt.
\]
For any fixed $t \in (-q^2, 0)$, we infer from Lemma 4.1 that
\[
|\tilde{u}^\beta(t) - (u)^\beta| \leq \int_{-q^2}^{0} |\tilde{u}^\beta(t) - \tilde{u}^\beta(s)| \, ds \\
\leq c \sum_{i=1}^{N} \int_{-q^2}^{0} \left| \int_{R_i} (u_i(x, t) - u_i(x, s))\psi_p(x) \, dx \right| \, ds \\
\leq c q \left( \int_{Q_0} |Du - Dl|^{p_0} \, dz \right)^{\frac{1}{p_0}} + c q (1 + |Dl|)^{p_0-2} \left( \int_{Q_0} |Du - Dl|^{p_0} \, dz \right)^{\frac{1}{p_0-2}} \\
+ c q (1 + |Dl|)^{p_0-1} \left( v^{p_0-1} \frac{q}{p_0} (\omega_p(q) \log \frac{1}{Q}) \right)^{\frac{1}{p_0-1}}.
\]

On the other hand, we apply the Poincaré inequality retrieved from [14, Theorem 12.36] to infer that there exists a constant $c$ depending only upon $n$ such that
\[
\int_{B_{r_k}} |u(x, t) - \tilde{u}_p(t) - Dl \cdot x|^{p_0} \, dx \leq c q \int_{B_{r_k}} |Du - Dl|^{p_0} \, dx.
\]

We emphasize that the constant $c$ in (4.9) is independent of $p(z)$ and therefore the estimate (4.9) is suitable for our purpose. To this end, we use (4.9) slicewise to deduce
\[
|\tilde{u}^\beta(t) - \tilde{u}_p(t)|^{p_0} = \left| \int_{R_i} (u(x, t) - \tilde{u}_p(t) - Dl \cdot x)\psi_p(x) \, dx \right|^{p_0} \\
\leq c(n) q^{p_0} \int_{Q_0} |Du - Dl|^{p_0} \, dx,
\]

for any $t \in (-q^2, 0)$. Furthermore, the inequality (4.10) also implies that
\[
|(u)^\beta - u_0|^{p_0} \leq c(n) q^{p_0} \int_{Q_0} |Du - Dl|^{p_0} \, dz.
\]

Combining the estimates (4.8)-(4.11), we conclude from the triangle inequality that
\[
\int_{Q_0} \left| \frac{u - (u)_{Q_0} - Dl \cdot x}{Q} \right|^{p_0} \, dz \\
\leq c \int_{Q_0} \left| \frac{u - \tilde{u}_p(t) - Dl \cdot x}{Q} \right|^{p_0} \, dz + c q^{p_0} \sup_{-q^2 < t < 0} |\tilde{u}^\beta(t) - (u)^\beta|^{p_0} \\
+ c q^{p_0} \sup_{-q^2 < t < 0} |\tilde{u}^\beta(t) - \tilde{u}_p(t)|^{p_0} + c q^{p_0} |(u)^\beta - u_0|^{p_0} \\
\leq c \int_{Q_0} |Du - Dl|^{p_0} \, dx + c \left( \int_{Q_0} |Du - Dl|^{p_0} \, dz \right)^{p_0-1} \\
+ c(1 + |Dl|)^{p_0(p_0-2)} \int_{Q_0} |Du - Dl|^{p_0} \, dz \\
+ c(1 + |Dl|)^{p_0(p_0-1)} \left( v^{p_0-1}(q) + (\omega_p(q) \log \frac{1}{Q}) \right)^{p_0-1}.
\]

Therefore, we have proved the desired estimate (4.7) by the Hölder’s inequality. The proof of the lemma is now complete. □

Before giving the precise statement of the smallness conditions near the regular points, we introduce the excess functional for weak solutions. To this end, we let $z_0 \in \Omega_T$, $\varrho > 0$
and let $l$ be an affine function. Moreover, we assume that $1 \leq \lambda \leq 1 + |Dl|$ and the excess functional is defined by

$$
\Psi_l(z_0, \varrho, l) = \int_{Q^+_{n}(z_0)} \frac{|u - l|}{\varrho(1 + |Dl|)}^2 \, dz + \int_{Q^+_{n}(z_0)} \frac{|u - l|}{\varrho(1 + |Dl|)}^{p_0} \, dz,
$$

(4.12)

where $p_0 = p(\bar{z}_0)$. Specifically, if $\lambda = 1$, then we simply write $\Psi_l(z_0, \varrho, l)$ for $\Psi_l(z_0, \varrho, 0)$. Similarly, if $\lambda = 1$, then we simply write $l_{\varrho, \varrho}$ for $l_{\varrho, \varrho}$. We are now in a position to give a new characterization for the regular points in terms of the excess functional and the following proposition is our main result in this section.

**Proposition 4.3.** Let $\Sigma_1$ and $\Sigma_2$ be the sets defined in Theorem 2.2 and fix $\tilde{z}_0 \in \Omega_T \setminus (\Sigma_1 \cup \Sigma_2)$. Then there exist a universal constant $M_0 > 0$ and a constant $\delta = \delta(n, \nu, L, E, \gamma_2, M_0) > 0$ such that the following holds: Whenever $\varepsilon_* < \delta$ is a fixed constant, then there exists a radius $\varrho = \varrho(n, \nu, L, E, \gamma_2, M_0, \varepsilon_*) > 0$ such that for any $z_0 \in Q_{\varrho/8}(\tilde{z}_0)$ there holds

$$
\left\{
\begin{array}{ll}
1 + |Dl_{\varrho, \varrho}| \leq M_0, \\
\Psi_l(z_0, \varrho, l_{\varrho, \varrho}) \leq \varepsilon_* \\
\Phi(z_0, \varrho, l_{\varrho, \varrho}) \leq \frac{1}{10}, \\
v(\varrho) + \omega(\varrho) \log \left(\frac{1}{\varrho}\right) \leq \varepsilon_*
\end{array}
\right.
$$

(4.13)

and $\varrho < \varrho_0$ where $\varrho_0 > 0$ is the radius in Lemma 3.5.

**Proof.** We first observe from the definition of $\Sigma_1$ and $\Sigma_2$ that for any $\tilde{z}_0 \in \Omega_T \setminus (\Sigma_1 \cup \Sigma_2)$, there exist a constant $M_* > 1$ and a sequence $\{\varrho_i\}_{i=1}^\infty$ with $\varrho_i \downarrow 0$ such that

$$
\lim_{i \to \infty} \int_{Q_{\varrho_i}(\tilde{z}_0)} |Dl - (Du)_{\varrho_i}(z_0)| \, dz = 0 \quad \text{and} \quad \limsup_{i \to \infty} \left( |Du|^{p(\varrho_i)} \right)_{Q_{\varrho_i}(\tilde{z}_0)} \leq M_*.
$$

From this, we conclude that for any fixed $0 < \varepsilon_* < 1$, then there exists a radius $\varrho > 0$ such that for any $\varrho_j < \varrho$ there holds

$$
\int_{Q_{\varrho_j}(\tilde{z}_0)} |Du - (Du)_{\varrho_j}(z_0)| \, dz \leq 2^{-\varepsilon_*} \frac{\varepsilon_0^{\gamma + (\gamma - 1)}}{\varrho_j^{\gamma + (\gamma - 1)}} \quad \text{and} \quad \int_{Q_{\varrho_j}(\tilde{z}_0)} |Du|^{p(\varrho_j)} \, dz \leq 2 M_*,
$$

where $\delta = \frac{1}{2} \varepsilon_0$ and $\varepsilon_0$ is the exponent defined in Proposition 3.1. At this stage, let $\varrho = \varrho_j/8 < \varrho_j/8$ be a fixed radius which will be determined later. For any fixed point $z_0 \in Q_{\varrho/8}(\tilde{z}_0)$, we infer that

$$
\int_{Q_{\varrho}(z_0)} |Du - (Du)_{\varrho_j}(z_0)| \, dz \leq 2 \int_{Q_{\varrho_j}(z_0)} |Du - (Du)_{\varrho_j}(z_0)| \, dz \leq \varepsilon_* \frac{\varepsilon_0^{\gamma + (\gamma - 1)}}{\varrho_j^{\gamma + (\gamma - 1)}}
$$

(4.14)

and

$$
\int_{Q_{\varrho}(z_0)} |Du|^{p(\varrho_j)} \, dz \leq 2^{\varepsilon_*} \int_{Q_{\varrho_j}(z_0)} |Du|^{p(\varrho_j)} \, dz \leq 2^{\varepsilon_*} M_*.
$$

(4.15)

We first consider the proof of (4.13). To this end, we choose a radial function $\psi \in C_0^\infty(B_{\varrho_j}(z_0))$ with $|D\psi| \leq c \varrho_j^{-1}$. Moreover, we define $u^\theta(t) = \int_{B_{\varrho_j}} u(x, t) \psi(x) \, dx$ and apply the Poincaré inequality slicewise to obtain

$$
\int_{Q_{\varrho}(z_0)} \left| \frac{u - (u)_{Q_{\varrho}(z_0)}}{\varrho} \right| \, dz \leq c \int_{Q_{\varrho}(z_0)} |Du| \, dz + c \varrho^{-1} \sup_{t \in (t_0 - \varrho_j, t_0)} |\tilde{u}^\theta(t) - \tilde{u}^\theta(t)|.
$$
Once again, we define $p_1 = \inf_{Q_{t_0}(x_0)} p(z)$ and $p_2 = \sup_{Q_{t_0}(x_0)} p(z)$. We now follow the proof of [5, Lemma 5] to deduce that for a.e. $t, \tau \in (t_0 - \theta, t_0)$ there holds

$$|u^\phi(t) - u^\phi(\tau)| \leq \sqrt{t_0}||D\phi||_{L^\infty} \theta \int_{Q_{t_0}(x_0)} (1 + |Du|^p(z))^{-1} \, dz$$

$$\leq c \theta \int_{Q_{t_0}(x_0)} (1 + |Du|)^{p-1} \, dz$$

$$\leq c \theta \int_{Q_{t_0}(x_0)} (1 + |Du|)^p \, dz \leq c \theta M_\infty,$$

since $p_2 - p_1 \leq \omega_p(4\theta) \leq 1$. Consequently, we conclude that there exists a constant $\hat{c}$ depending only upon $n, N, v, L$ and $\gamma_2$ such that the inequality

$$\int_{Q_{t_0}(x_0)} \frac{|u - (u)_{Q_{t_0}(x_0)}|}{\theta} \, dz \leq \hat{c} + \hat{c}M_\infty + \hat{c} \int_{Q_{t_0}(x_0)} |Du|^{p_0} \, dz \leq 3\hat{c}M_\infty$$

holds. Recalling that $l_{0, \theta}$ is the unique affine function minimizing $l \to \int_{Q_{t_0}(x_0)} |u - h|^2 \, dz$, we apply [5, (2.1)] to deduce that

$$|Dl_{0, \theta}| = \frac{n + 2}{\theta^2} \int_{Q_{t_0}(x_0)} \left( u - (u)_{Q_{t_0}(x_0)} \right) \otimes (x - x_0) \, dz$$

$$\leq (n + 2) \int_{Q_{t_0}(x_0)} \frac{|u - (u)_{Q_{t_0}(x_0)}|}{\theta} \, dz \leq 3(n + 2)\hat{c}M_\infty.$$

This proves (4.13) with the choice $M_0 = (1 + 3(n + 2)\hat{c})M_\infty$. To prove (4.13), the aim is to determine the upper bound of the constant $\hat{c}_\infty < \hat{c}$ and the radius $\theta$. We first claim that for $\epsilon_0$ and $\theta$ sufficiently small there holds

$$\Phi(z_0, 2\theta, l) \leq \frac{1}{16},$$

(4.16)

where $l = (u)_{Q_{t_0}(x_0)} + (Du)_{Q_{t_0}(x_0)}(x - x_0)$. Initially, we set $\theta < \theta_0$, where $\theta_0 > 0$ is the radius in Lemma [5, 5]. Note that this choice can be justified by the limitation $\theta \downarrow 0$. For the proof of (4.16), we apply the interpolation inequality to deduce

$$\left( \int_{Q_{t_0}(x_0)} |Du - (Du)_{Q_{t_0}(x_0)}|^{p_0} \, dz \right)^{\frac{1}{p_0}} \leq \left( \int_{Q_{t_0}(x_0)} |Du - (Du)_{Q_{t_0}(x_0)}|^{p_0(1+\delta)} \, dz \right)^{\frac{1}{p_0(1+\delta)}} \left( \int_{Q_{t_0}(x_0)} |Du - (Du)_{Q_{t_0}(x_0)}| \, dz \right)^{1-\Theta}$$

(4.17)

where $p_0 = p(z_0)$, $\delta = \frac{1}{2}\epsilon_0$ and the factor $\Theta$ can be bounded above by a constant $\hat{\Theta}$ depending only upon $\epsilon_0$ and $\gamma_2$,

$$\Theta = \frac{(p_0 - 1)(1 + \delta)}{p_0(1 + \delta) - 1} \leq \frac{(\gamma_2 - 1)(1 + \delta)}{\gamma_2(1 + \delta) - 1} =: \hat{\Theta}.$$

Since $\theta < \theta_0$, we observe from (3.28) and (3.7) that $p_2 - p_1 \leq \epsilon_0 \leq \frac{1}{2}p_1\epsilon_0$. Using this together with (3.3) and (4.18) we obtain

$$\int_{Q_{t_0}(x_0)} |Du|^{p_0(1+\delta)} \, dz \leq \int_{Q_{t_0}(x_0)} (1 + |Du|)^{p_0(1+\epsilon_0)} \, dz$$

$$\leq c \left[ \left( \int_{Q_{t_0}(x_0)} |Du|^{p_0} \, dz \right)^{1+\epsilon_0/p_0} + 1 \right] \leq c(M_0),$$

(4.18)
Consequently, we infer from (4.13), (4.15), (4.17) and (4.18) that there exists a constant \( \hat{c}_1 \geq 1 \) depending only on \( n, N, \nu, L, \gamma_2 \) and \( M_0 \) such that the inequality
\[
\left( \int_{Q_{a}(x_0)} |Du - (Du)_{Q_{a}(x_0)}|^{p_0} \, dz \right)^{\frac{1}{p_0}} \leq c(\gamma_2, M_0)^{\hat{\theta}} \left( \int_{Q_{a}(x_0)} |Du - (Du)_{Q_{a}(x_0)}| \, dz \right)^{1 - \hat{\theta}} \leq \hat{c}_1 \varepsilon_*
\]
holds. This estimate together with the Hölder’s inequality ensures us that
\[
\Phi(z_0, 2\varrho, l) = \int_{Q_{a}(x_0)} \left( \frac{|Du - Dl|}{1 + |Dl|} \right)^2 \, dz 
\leq 2 \left( \int_{Q_{a}(x_0)} |Du - (Du)_{Q_{a}(x_0)}|^{p_0} \, dz \right)^{\frac{1}{p_0}} 
\leq 2\hat{c}_1^2 \varepsilon_*^2,
\]
since \( p_0 \geq 2 \). At this point, we choose \( \hat{\varepsilon} \leq \frac{1}{32\hat{c}_1^{p_0 - 2}} \), which proves the claim (4.10) for \( \varepsilon_* \leq \hat{\varepsilon} \) and \( \varrho < \varrho_0 \). In view of (4.16) and \( \varrho < \varrho_0 \), we can apply Lemma 4.2 with \( l = (u)_{Q_{a}(x_0)} + (Du)_{Q_{a}(x_0)}(x - x_0) \) to obtain
\[
\int_{Q_{a}(x_0)} \left| u - (u)_{Q_{a}(x_0)} - (Du)_{Q_{a}(x_0)}(x - x_0) \right|^2 \, dz + \int_{Q_{a}(x_0)} \left| u - (u)_{Q_{a}(x_0)} - (Du)_{Q_{a}(x_0)}(x - x_0) \right|^2 \, dz 
\leq c(1 + |(Du)_{Q_{a}(x_0)}|)^{p_0(p_0 - 1)} \left( \int_{Q_{a}(x_0)} |Du - (Du)_{Q_{a}(x_0)}|^{p_0} \, dz \right)^{\frac{1}{p_0}} 
\leq c(n, N, \gamma_2, M_0) \varepsilon_*^2 + c(\gamma_2, M_0) \left[ \log \left( \frac{1}{\varrho_0} \right) \right]^{\frac{2(n - 1)}{n}}.
\]
To proceed further, we apply the quasi-minimality of the affine function \( l_{z_0, \varrho} \) from [8, Lemma 2.2] and conclude that there exists a constant \( \hat{c}_2 \geq 1 \) depending only on \( n, N, \nu, L, \gamma_2 \) and \( M_0 \) such that the inequality
\[
\Psi(z_0, \varrho, l_{z_0, \varrho}) = \int_{Q_{a}(x_0)} \left| u - l_{z_0, \varrho} \right|^2 \, dz + \int_{Q_{a}(x_0)} \left| u - l_{z_0, \varrho} \right|^2 \, dz 
\leq \hat{c}_2 \varepsilon_*^2 + \hat{c}_2 \left[ \log \left( \frac{1}{\varrho_0} \right) \right]^{\frac{2(n - 1)}{n}}
\]
holds. At this stage, we choose \( \hat{\varepsilon} > 0 \) such that \( \hat{\varepsilon} < \frac{1}{2} \hat{c}_2^{n-1} \). Moreover, for a fixed \( \varepsilon_* < \hat{\varepsilon} \), we choose \( \varrho < \varrho_0 \) small enough to have
\[
\hat{c}_2 \left[ \log \left( \frac{1}{\varrho_0} \right) \right]^{\frac{2(n - 1)}{n}} < \frac{1}{2} \varepsilon_*,
\]
since \( \varrho = \varrho_j \) for some \( j \in \mathbb{N} \) and \( \varrho_j \downarrow 0 \). This establishes the desired inequality (4.13) and (4.14). Finally, we come to the proof of (4.15). Applying [15, Lemma 2.5] with \( \lambda = 1, A^{(1)}_{\varrho, \varrho} = Dl_{z_0, \varrho}, \xi = (u)_{Q_{a}(x_0)} \) and \( w = (Du)_{Q_{a}(x_0)} \), we obtain
\[
|(Du)_{Q_{a}(x_0)} - Dl_{z_0, \varrho}|^2 \leq \frac{n(n + 2)}{\varrho^2} \int_{Q_{a}(x_0)} |u - (u)_{Q_{a}(x_0)} - (Du)_{Q_{a}(x_0)}(x - x_0)|^2 \, dz \leq c\varepsilon_*,
\]
Therefore, we conclude that there exists a constant \( \hat{c}_3 \geq 1 \) depending only upon \( n, N, \nu, \gamma_2 \) and \( M_0 \) such that the inequality

\[
\Phi(x_0, q_1, l_{n, \nu}) = \int_{\Omega_{n, \nu}^0} \left( \frac{|Du - Dl_{n, \nu}|}{1 + |Dl_{n, \nu}|} \right)^2 + \left( \frac{|Du - Dl_{n, \nu}|}{1 + |Dl_{n, \nu}|} \right)^{p_0} dz \\
\leq c \left( \int_{\Omega_{n, \nu}^0} |Du - (Du)_{\Omega_{n, \nu}^0}|^{p_0} dz \right)^{\frac{1}{p_0}} + c(|Du)_{\Omega_{n, \nu}^0} - Dl_{n, \nu}|^{p_0} \\
\leq \hat{c}_3 \varepsilon_* \leq \hat{c}_3 \hat{\varepsilon}
\]

holds. At this point, we choose \( \hat{\varepsilon} > 0 \) small enough to have \( \hat{\varepsilon} < \frac{1}{\hat{c}_3} \hat{\varepsilon}^{-1} \). This proves (4.13). In conclusion, we have proved (5.1) with the choice

\[
\hat{\varepsilon} = \min \left\{ \frac{1}{32} \hat{c}_3^{-1}, \frac{1}{2} \hat{c}_3^{-1}, \frac{1}{16} \hat{c}_3^{-1} \right\},
\]

while for any fixed \( \varepsilon_* < \hat{\varepsilon} \), the radius \( \rho < \min\left(\frac{\hat{\varepsilon}}{\hat{c}_3}, q_0\right) \) can be determined via (4.19). The proof of the proposition is now complete. \( \square \)

5. Caccioppoli Inequality

The aim of this section is to establish Caccioppoli type estimate for the weak solution to the parabolic system (2.1). In the context of the problem for discontinuous coefficients, the energy estimate should be established on non-uniformly parabolic cylinders and the bounds should be independent of \( |Dl| \). The next lemma is our main result in this section.

**Lemma 5.1.** Let \( x_0 \in \Omega_T \) and let \( l \) be an affine function. Let \( q_0 > 0 \) be the radius in Lemma 5.5. Assume that \( q < q_0, \lambda \geq 1 \) and \( \lambda \leq 1 + |Dl| \leq M \lambda \) for some \( M \geq 1 \). Moreover, suppose that \( \Phi^{(l)}(x_0, q_1, l) \leq \frac{q_0}{10} \). Then, there exists a constant \( \varepsilon \) depending only upon \( n, N, \nu, \gamma_2, \nu, L \) and \( M \) such that the inequality

\[
\sup_{t_0 - \varepsilon < t < t_0} \int_{|x - x_0| < \varepsilon} \frac{|u(t, x) - \bar{u}(t)|^2}{\varepsilon^{p_0}} dx + \int_{\Omega_{n, \nu}^0} \left( 1 + |Dl| \right)^{p_0} |Du - Dl|^2 dz \\
\leq c \left( 1 + |Dl| \right)^{p_0} \left( \int_{\Omega_{n, \nu}^0} \frac{|u - l|}{\varepsilon (1 + |Dl|)} \right)^2 dz + \int_{\Omega_{n, \nu}^0} \left( \frac{|u - l|}{\varepsilon (1 + |Dl|)} \right)^{p_0} dz \quad (5.1) \\
+ c (1 + |Dl|)^{p_0} \left( \omega_\nu(q) \log \left( \frac{1}{\varepsilon} \right) \right)^2
\]

holds, where \( p_0 = p(\varepsilon) \).

**Proof.** There is no loss of generality in assuming \( x_0 = 0 \). Once again, we set \( p_1 = \inf_{x_0} p(z) \) and \( p_2 = \sup_{x_0} p(z) \). For fixed \( \frac{1}{2} \mu < s_1 < s_2 < \frac{1}{2} \mu \), we consider the concentric parabolic cylinders \( Q^{(s)}_{\nu} \subset Q^{(s_1)}_{\nu} \subset Q^{(s)}_{\nu} \subset Q^{(s_1)}_{\nu} \). We choose a cut-off function \( \phi = \phi(x) \in C^\infty_0(B_\mu) \) with \( \phi \equiv 1 \) on \( B_{s_1} \), \( 0 \leq \phi \leq 1 \), \( |D\phi| \leq c(s_2 - s_1)^{-1} \). Moreover, we define a Lipschitz function \( \xi \in W^{1, \infty}(A_{\nu}^{(s_1)} \setminus \{0, 1\}) \) via

\[
\xi(t) = \begin{cases} \\
0, & \text{on } (-\lambda^2 - p_0, \lambda^2 - p_0), \\
\frac{1}{\lambda^2 - p_0} (t + \lambda^2 - p_0), & \text{on } (-\lambda^2 - p_0, \lambda^2 - p_0), \\
1, & \text{on } (-\lambda^2 - p_0, \lambda^2 - p_0), \\
\frac{1}{\lambda^2 - p_0} (t + \varepsilon - t), & \text{on } (t, t + \varepsilon), \\
0, & \text{on } (t + \varepsilon, 0)
\end{cases}
\quad (5.2)
\]
for a fixed \( \tilde{t} \in \Lambda_{n}^{(3)} \) and \( \varepsilon \in (0, \| \tilde{t} \|) \). In the weak formulation (2.8) we choose the test function \( \varphi(x, t) = \phi \zeta_{\varepsilon}(u - \tilde{t}) \). We first observe that
\[
\int_{Q_{\varepsilon}^{(3)}} 1 \cdot \partial_{t} \varphi \, dz = 0 \quad \text{and} \quad \int_{Q_{\varepsilon}^{(3)}} (\mu)_{Q_{\varepsilon}^{(3)}} (A(0, Dl), D\varphi) \, dz = 0.
\]
This leads us to
\[
I := \int_{Q_{\varepsilon}^{(3)}} \phi(x) \zeta_{\varepsilon}(t) \mu(z) (A(z, Du) - A(z, Dl), Du - Dl) \, dz \\
= - \int_{Q_{\varepsilon}^{(3)}} \zeta_{\varepsilon}(t) \mu(z) (A(z, Du) - A(z, Dl), (u - l) \otimes D\phi) \, dz \\
- \int_{Q_{\varepsilon}^{(3)}} [\mu(z) - (\mu)_{Q_{\varepsilon}^{(3)}}] (A(z, Dl), D\varphi) \, dz \\
- \int_{Q_{\varepsilon}^{(3)}} (\mu)_{Q_{\varepsilon}^{(3)}} (A(z, Dl) - A(0, Dl), D\varphi) \, dz \\
+ \int_{Q_{\varepsilon}^{(3)}} (u - l) \cdot \partial_{t} \varphi \, dz =: II + III + IV + V,
\]
with the obvious meaning of \( I - V \). Now we are going to estimate the first terms \( I \). By the fundamental theorem of calculus, we infer from the ellipticity condition (2.2) that
\[
I = \int_{Q_{\varepsilon}^{(3)}} \phi(x) \zeta_{\varepsilon}(t) \mu(z) \int_{0}^{1} \langle \partial_{n} A(z, Dl + s(Du - Dl))(Du - Dl), Du - Dl \rangle \, ds \, dz \\
\geq \nu \int_{Q_{\varepsilon}^{(3)}} \phi \zeta_{\varepsilon} \int_{0}^{1} (1 + |Dl + s(Du - Dl)|)^{p(z)-2} |Du - Dl|^{2} \, dz \\
\geq c_{1}(\nu, \gamma_{2}) \int_{Q_{\varepsilon}^{(3)}} \phi \zeta_{\varepsilon} (1 + |Du|) |Du - Dl|^{2} \, dz.
\]
In the last line we have used [4, Lemma 2.4] to estimate the integral from below. Furthermore, we set \( \tilde{v} = c_{1}(\nu, \gamma_{2}) \) and decompose the right-hand side by \( I \geq \tilde{v}(I_{1} + I_{2}) \), where
\[
I_{1} := \int_{Q_{\varepsilon}^{(3)}} \phi \zeta_{\varepsilon} (1 + |Dl| + |Du|)^{p_{0}-2} |Du - Dl|^{2} \, dz
\]
and
\[
I_{2} := \int_{Q_{\varepsilon}^{(3)}} \phi \zeta_{\varepsilon} ((1 + |Dl| + |Du|)^{p(z)-2} - (1 + |Dl| + |Du|)^{p_{0}-2}) |Du - Dl|^{2} \, dz.
\]
We now come to the estimate of \( I_{2} \). First, we use the fundamental theorem of calculus and Young’s inequality to obtain for any \( \kappa \in (0, 1) \) that
\[
I_{2} \leq c_{\omega_{p}(\epsilon)} \int_{Q_{\lambda_{2}}^{(3)}} (1 + |Du| + |Dl|)^{p_{0}-1} \log(1 + |Du| + |Dl|) |Du - Dl| \, dz \\
\leq \kappa \int_{Q_{\lambda_{2}}^{(3)}} (1 + |Du| + |Dl|)^{p_{0}-2} |Du - Dl|^{2} \, dz \\
+ c(\kappa) \omega_{p_{0}}^{2}(\epsilon) \int_{Q_{\lambda_{2}}^{(3)}} (1 + |Du| + |Dl|)^{2p_{0}-p_{0}} \log^{2}(1 + |Du| + |Dl|) \, dz \\
=: I_{3} + I_{4},
\]
where
with the obvious meaning of $I_3$ and $I_4$. Since $2p_2 - p_0 \leq p_1 + 2\omega_p(q)$, we apply Lemma 3.5 with $\gamma = 2$ and $\theta = \frac{1}{2}$ to obtain

\[ I_4 \leq c\omega_p^2(q) \int_{Q_0^{2\bar{r}}(0)} (1 + |Du| + |Dl|)^{p_0} \log^2(1 + |Dl| + |Du|) \, dz \]

\[ \leq c|Q_0^{2\bar{r}}(0)| (1 + |Dl|)^{p_0} \left( \omega_p(q) \log \left( \frac{1}{\bar{Q}} \right) \right)^2. \]  

(5.5)

At this stage, we choose $\kappa = \frac{1}{12}$ in (5.4). It follows from (5.4) and (5.5) that for any $\bar{r} \in (-\lambda^{2-p_2}s_1^2, 0)$ there holds

\[ \lim_{\varepsilon \downarrow 0} I \geq \bar{\bar{v}} \int_{-\bar{r}^{2-p_2}s_1^2}^0 \int_{B_{s_1}} (1 + |Dl| + |Du|)^{p_0-2} |Du - Dl|^2 \, dxdr \]

\[ - \frac{\bar{\bar{v}}}{12} \int_{Q_0^{2\bar{r}}(0)} (1 + |Dl| + |Du|)^{p_0-2} |Du - Dl|^2 \, dz \]

\[ - cQ_0^{2\bar{r}}(0) (1 + |Dl|)^{p_0} \left( \omega_p(q) \log \left( \frac{1}{\bar{Q}} \right) \right)^2. \]

(5.6)

To estimate $II$, we use the mean value theorem from calculus, growth condition (2.2) and the Young's inequality to obtain for any $\kappa \in (0, 1)$ that

\[ |II| = \left| \int_{Q_0^{2\bar{r}}(0)} \zeta(\bar{r}) \mu(z)(A(z, Du) - A(z, Dl), (u - l) \otimes D\phi) \, dz \right| \]

\[ \leq L \int_{Q_0^{2\bar{r}}(0)} \left| \int_0^{1} \partial_s A(z, Dl + s(Du - Dl)) \cdot (Du - Dl) \, ds \right| \left| \frac{u - l}{s_2 - s_1} \right| \, dz \]

\[ \leq c \int_{Q_0^{2\bar{r}}(0)} (1 + |Dl| + |Du|)^{\frac{p_1}{2}} |Du - Dl| \left| \frac{u - l}{s_2 - s_1} \right| \, dz \]

\[ \leq \kappa \int_{Q_0^{2\bar{r}}(0)} (1 + |Dl| + |Du|)^{p_0-2} |Du - Dl|^2 \, dz \]

\[ + c(\kappa) \int_{Q_0^{2\bar{r}}(0)} (1 + |Dl| + |Du|)^{p_0-2} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz \]

\[ =: II_1 + II_2, \]

with the obvious meaning of $II_1$ and $II_2$. In order to estimate $II_1$, we set $\kappa = \bar{\bar{v}}/12$ and decompose

\[ II_1 = \frac{\bar{\bar{v}}}{12} \int_{Q_0^{2\bar{r}}(0)} (1 + |Dl| + |Du|)^{p_0-2} |Du - Dl|^2 \, dz \]

\[ + \frac{\bar{\bar{v}}}{12} \int_{Q_0^{2\bar{r}}(0)} (1 + |Dl| + |Du|)^{2p_1-p_0-2} - (1 + |Dl| + |Du|)^{p_0-2} \cdot |Du - Dl|^2 \, dz. \]
It suffices to treat the second term on the right-hand side. To this end, we use the mean value theorem from calculus and Young’s inequality to obtain
\[
\frac{\bar{V}}{12} \int_{Q_{\varepsilon_2}^{(k)}} \left[ (1 + |Dl| + |Du|)^2p_1 - p_0 - 2 - (1 + |Dl| + |Du|)^{p_0 - 2} \right] |Du - Dl|^2 \, dz
\leq c \omega_p(q) \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl| + |Du|)^2 \log(1 + |Dl| + |Du|) \, dz
\leq c \int_{Q_{\varepsilon_2}^{(k)}} \omega_p(q)(1 + |Dl| + |Du|)^2 p_1^{-\frac{1}{2}} \log(1 + |Dl| + |Du|)
\times (1 + |Dl| + |Du|)^{\frac{p_0}{2}-1} |Du - Dl| \, dz
\leq c \omega_p^2(q) \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl| + |Du|)^{p_0 - 2} \log^2(1 + |Dl| + |Du|) |Du - Dl| \, dz
+ \frac{\bar{V}}{12} \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz.
\]
Since \(4p_2 - 3p_0 \leq p_1 + 4\omega_p(q)\), we apply Lemma 5.3 with \(\gamma = 2\) and \(\theta = \frac{1}{\bar{V}}\) to deduce
\[
\frac{\bar{V}}{12} \int_{Q_{\varepsilon_2}^{(k)}} \left[ (1 + |Dl| + |Du|)^2p_2 - p_0 - 2 - (1 + |Dl| + |Du|)^{p_0 - 2} \right] |Du - Dl|^2 \, dz
\leq \frac{\bar{V}}{12} \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz + c|Q_{\varepsilon_2}^{(k)}|(1 + |Dl|)^{p_0}(\omega_p(q) \log \left( \frac{1}{Q} \right))^2,
\]
and this leads us to
\[
H_1 \leq \frac{\bar{V}}{6} \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz + c|Q_{\varepsilon_2}^{(k)}|(1 + |Dl|)^{p_0}(\omega_p(q) \log \left( \frac{1}{Q} \right))^2.
\]
We now turn our attention to the estimate of \(H_2\). Taking into account that \(p_0 \geq 2\), we use the Young’s inequality with exponents \(\frac{p_0}{2}\) and \(\frac{p_0}{p_0 - 2}\) to obtain
\[
H_2 \leq c \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl|)^{p_0 - 2} \left| \frac{u - l}{s_2 - s_1} \right| |Du - Dl|^{p_0 - 2} \, dz
\leq c(1 + |Dl|)^{p_0 - 2} \int_{Q_{\varepsilon_2}^{(k)}} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz + c \int_{Q_{\varepsilon_2}^{(k)}} \left| \frac{u - l}{s_2 - s_1} \right|^{p_0} \, dz
+ \frac{\bar{V}}{12} \int_{Q_{\varepsilon_2}^{(k)}} |Du - Dl|^{p_0} \, dz
\leq c(1 + |Dl|)^{p_0 - 2} \int_{Q_{\varepsilon_2}^{(k)}} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz + c \int_{Q_{\varepsilon_2}^{(k)}} \left| \frac{u - l}{s_2 - s_1} \right|^{p_0} \, dz
+ \frac{\bar{V}}{12} \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz.
\]
Therefore, combining the estimates for \(H_1\) and \(H_2\), we conclude that
\[
|I| \leq c(1 + |Dl|)^{p_0 - 2} \int_{Q_{\varepsilon_2}^{(k)}} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz + c \int_{Q_{\varepsilon_2}^{(k)}} \left| \frac{u - l}{s_2 - s_1} \right|^{p_0} \, dz
+ \frac{\bar{V}}{4} \int_{Q_{\varepsilon_2}^{(k)}} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz
+ c|Q_{\varepsilon_2}^{(k)}|(1 + |Dl|)^{p_0}(\omega_p(q) \log \left( \frac{1}{Q} \right))^2. \tag{5.7}
\]
Now we come to the estimate of $III$. We first note that the smallness condition $\phi < \varrho_0$ allows us to use Lemma 3.2. To this end, we use the growth condition (2.2) and Lemma 3.2(3) to obtain

$$|III| = \left| \int_{Q^0_0} [\mu(z) - (\mu)_{Q^0_0}] (A(z, Dl), D\varphi) \, dz \right|$$

$$\leq c(1 + |Dl|)^{p_0-1} \int_{Q^0_0} |\mu(z) - (\mu)_{Q^0_0}| |D\varphi| \, dz$$

$$\leq c(1 + |Dl|)^{p_0-1} \int_{Q^0_0} |\mu(z) - (\mu)_{Q^0_0}| |Du - Dl| \, dz$$

$$+ c(1 + |Dl|)^{p_0-1} \int_{Q^0_0} |\mu(z) - (\mu)_{Q^0_0}| \left| \frac{u - l}{s_2 - s_1} \right| \, dz$$

$$=: III_1 + III_2,$$

with the obvious meaning of $III_1$ and $III_2$. Since $p_0 \geq 2$, we apply the Young’s inequality to obtain

$$III_1 \leq c(1 + |Dl|)^{\frac{p_0}{2}} \int_{Q^0_0} (1 + |Dl| + |Du|)^{\frac{p_0}{2}-1} |\mu(z) - (\mu)_{Q^0_0}| |Du - Dl| \, dz$$

$$\leq \frac{p}{12} \int_{Q^0_0} (1 + |Dl| + |Du|)^{p_0-2} |Du - Dl|^2 \, dz + c(1 + |Dl|)^p \int_{Q^0_0} |\mu(z) - (\mu)_{Q^0_0}|^2 \, dz$$

$$\leq \frac{p}{12} \int_{Q^0_0} (1 + |Dl| + |Du|)^{p_0-2} |Du - Dl|^2 \, dz + c(1 + |Dl|)^p \varrho^4 |v(\varrho)|,$$

where we used $\sqrt{\varrho} \leq \mu(z) \leq \sqrt{L}$ in the last step. To estimate $III_2$, we use the Hölder’s inequality to deduce

$$III_2 = c \int_{Q^0_0} \left[ (1 + |Dl|)^{\frac{p_0}{2}} |\mu(z) - (\mu)_{Q^0_0}| \right] \times \left[ (1 + |Dl|)^{\frac{p_0}{2}-1} \left| \frac{u - l}{s_2 - s_1} \right| \right] \, dz$$

$$\leq c(1 + |Dl|)^p |Q^0_0| |v(\varrho)| + (1 + |Dl|)^{p_0-2} \int_{Q^0_0} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz.$$

Consequently, we conclude from the estimates for $III_1$ and $III_2$ that

$$|III| \leq \frac{p}{12} \int_{Q^0_0} (1 + |Dl| + |Du|)^{p_0-2} |Du - Dl|^2 \, dz$$

$$+ c(1 + |Dl|)^p |Q^0_0| |v(\varrho)| + (1 + |Dl|)^{p_0-2} \int_{Q^0_0} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz. \quad (5.8)$$
Next, we consider the estimate for $IV$. To this end, we apply the continuity condition (2.6), Lemma(2.3), (3), (4) and take into account that $\sqrt{\nu} \leq \mu(\zeta) \leq \sqrt{L}$. This yields

$$
|IV| = |\int_{Q_{\nu}^{(n)}} (\mu)_{\nu}^{(n)}(A(\zeta, Dl) - A(0, Dl), D\varphi) \, dz|
$$

$$
\leq c \sqrt{L} \omega_p(\nu) \int_{Q_{\nu}^{(n)}} (1 + |Dl|)^{p_0 - 1} (1 + \log(1 + |Dl|)) |D\varphi| \, dz
$$

$$
\leq c \omega_p(\nu) \log \left( \frac{1}{\nu} \right) (1 + |Dl|)^{p_0 - 1} \int_{Q_{\nu}^{(n)}} |Du - Dl| \, dz
$$

$$
+ c \omega_p(\nu) \log \left( \frac{1}{\nu} \right) (1 + |Dl|)^{p_0 - 1} \int_{Q_{\nu}^{(n)}} \frac{|u - l|}{s_2 - s_1} \, dz
$$

$$
=: IV_1 + IV_2,
$$

with the obvious meaning of $IV_1$ and $IV_2$. To estimate $IV_1$, we use the Young’s inequality to deduce

$$
IV_1 \leq c \int_{Q_{\nu}^{(n)}} \omega_p(\nu) \log \left( \frac{1}{\nu} \right) (1 + |Dl|)^{\frac{2}{p_0}} \cdot \left( (1 + |Dl| + |Du|)^{\frac{2}{p_0} - 1} |Du - Dl| \right) \, dz
$$

$$
\leq \frac{c}{12} \int_{Q_{\nu}^{(n)}} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz + c |Q_{\nu}^{(n)}| (1 + |Dl|)^{p_0} \left( \omega_p(\nu) \log \left( \frac{1}{\nu} \right) \right)^2,
$$

since $p_0 \geq 2$. Similarly, in order to estimate $IV_2$, we use the Young’s inequality again and deduce

$$
IV_2 \leq c \int_{Q_{\nu}^{(n)}} (1 + |Dl|)^{p_0 - 2} \left( \frac{u - l}{s_2 - s_1} \right)^2 \, dz + c |Q_{\nu}^{(n)}| (1 + |Dl|)^{p_0} \left( \omega_p(\nu) \log \left( \frac{1}{\nu} \right) \right)^2.
$$

Consequently, we infer that

$$
IV \leq \frac{c}{12} \int_{Q_{\nu}^{(n)}} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz
$$

$$
+ c |Q_{\nu}^{(n)}| (1 + |Dl|)^{p_0} \left( \omega_p(\nu) \log \left( \frac{1}{\nu} \right) \right)^2 + c \int_{Q_{\nu}^{(n)}} (1 + |Dl|)^{p_0 - 2} \left( \frac{u - l}{s_2 - s_1} \right)^2 \, dz.
$$

Finally, we address the estimate of $V$. In the following we shall proceed formally by assuming that the time derivative exists, since the arguments can be made rigorous by the use of the Steklov averages. Therefore, we find that

$$
V = \int_{Q_{\nu}^{(n)}} \left| u - l \right|^2 \phi \partial_z \xi \, dz + \frac{1}{2} \int_{Q_{\nu}^{(n)}} \phi \xi \partial_z (\left| u - l \right|^2) \, dz
$$

$$
= \frac{1}{2} \int_{Q_{\nu}^{(n)}} \left| u - l \right|^2 \phi \partial_z \xi \, dz
$$

$$
\leq \frac{1}{2} |\nu|^{p_0 - 2} \int_{Q_{\nu}^{(n)}} \int_{B_{\nu/2}} \left| \frac{u - l}{s_2 - s_1} \right|^2 \phi \, dx \, dt - \frac{1}{2} \int_{Q_{\nu}^{(n)}} \int_{B_{\nu/2}} \left| u - l \right|^2 \phi \, dx \, dt.
$$

Recalling that $\lambda \leq 1 + |Dl|$, then we conclude that for any time level $\bar{t} \in (-\lambda^2 - p_0 (\nu/4)^2, 0)$ there holds

$$
\lim_{\nu \to 0} V \leq \frac{1}{2} (1 + |Dl|)^{p_0 - 2} \int_{Q_{\nu/2}^{(n)}} \left| \frac{u - \bar{l}}{s_2 - s_1} \right|^2 \, dz + \frac{1}{2} \int_{B_{\nu/4}} \left| u(\cdot, \bar{t}) - \bar{l} \right|^2 \, dx,
$$

(5.10)
since $p_0 \geq 2$. Combining (5.6), (5.7), (5.8), (5.9) and (5.10), we conclude that for any fixed $\frac{1}{\nu} < s_1 < s_2 < \frac{1}{\nu}$ and any time level $\tilde{t} \in (-\lambda^2 - p_0(\theta)/4)^2, 0)$ there holds
\[
\int_{B_{\delta}^n} |u(\cdot, \tilde{t}) - \overline{l}|^2 \, dx + \bar{\nu} \int_{\rho - \lambda^2 - p_0(\theta)/4} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dx
\leq \frac{\bar{\nu}}{2} \int_{Q_{\delta}^n} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz
\]
\[
+ c(1 + |Dl|)^{p_0 - 2} \int_{Q_{\delta}^n} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz + c \int_{Q_{\delta}^n} \left| \frac{u - l}{s_2 - s_1} \right|^{p_0} \, dz
\]
\[
+ c |Q_{\delta}^n| (1 + |Dl|)^{p_0} \left( v(\theta) + (\omega_p(\theta) \log \left( \frac{1}{\nu} \right))^2 \right).
\]
(5.11)

Taking into account that $\tilde{t} \in (-\lambda^2 - p_0(\theta)/4)^2, 0)$ is arbitrary, we first pass to the limit $\tilde{t} \uparrow 0$ in (5.11) and then take the supremum for $\tilde{t} \in (-\lambda^2 - p_0(\theta)/4)^2, 0)$ in (5.11) again. This gives
\[
\sup_{\tilde{t} \in (-\lambda^2 - p_0(\theta)/4)^2, 0)} \int_{B_{\delta}^n} |u(\cdot, \tilde{t}) - \overline{l}|^2 \, dx + \bar{\nu} \int_{\rho - \lambda^2 - p_0(\theta)/4} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dx
\]
\[
\leq \frac{\bar{\nu}}{2} \int_{Q_{\delta}^n} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dz
\]
\[
+ c(1 + |Dl|)^{p_0 - 2} \int_{Q_{\delta}^n} \left| \frac{u - l}{s_2 - s_1} \right|^2 \, dz + c \int_{Q_{\delta}^n} \left| \frac{u - l}{s_2 - s_1} \right|^{p_0} \, dz
\]
\[
+ c |Q_{\delta}^n| (1 + |Dl|)^{p_0} \left( v(\theta) + (\omega_p(\theta) \log \left( \frac{1}{\nu} \right))^2 \right).
\]

At this point, we can use an iteration lemma (cf. [8] Lemma 2.2) to re-absorb the first integral of the right-hand side into the left. This leads us to
\[
\sup_{\tilde{t} \in (-\lambda^2 - p_0(\theta)/4)^2, 0)} \int_{B_{\delta}^n} |u(\cdot, \tilde{t}) - \overline{l}|^2 \, dx + \bar{\nu} \int_{\rho - \lambda^2 - p_0(\theta)/4} (1 + |Dl| + |Du|)^{p_0 - 2} |Du - Dl|^2 \, dx
\]
\[
\leq c(1 + |Dl|)^{p_0 - 2} \int_{Q_{\delta}^n} \left| \frac{u - l}{\nu} \right|^2 \, dz + c \int_{Q_{\delta}^n} \left| \frac{u - l}{\nu} \right|^{p_0} \, dz
\]
\[
+ c |Q_{\delta}^n| (1 + |Dl|)^{p_0} \left( v(\theta) + (\omega_p(\theta) \log \left( \frac{1}{\nu} \right))^2 \right),
\]
which proves the desired estimate (5.11). Dividing by $|Q_{\delta}^n|$, it is now obvious that the lemma holds.

6. Decay estimate

This section is devoted to the study of a decay estimate which plays a crucial role in the partial regularity proof. We first show that the weak solution minus an affine function is approximatively A-caloric. Before giving the precise statement of this result we introduce the concept of the hybrid excess functional. For $z_0 \in \Omega^R, \theta \in (0, 1)$, $\lambda \geq 1$ and an affine function $l : \mathbb{R}^N \to \mathbb{R}^N$, we define the hybrid excess functional $\Psi_J(z_0, \theta, l)$ by
\[
\Psi_J(z_0, \theta, l) = \Psi_J(z_0, \theta, l) + v(\theta) + \omega_p(\theta) \log \left( \frac{1}{\theta} \right),
\]
(6.1)
where $\Psi_J(z_0, \theta, l)$ is defined in (4.12). We are now in a position to state the following linearization result.
Lemma 6.1. Let $z_0 \in \Omega_T$ and let $l$ be an affine function. Let $\varrho_0 > 0$ be the radius in Lemma 3.5. Assume that $\varrho < \varrho_0$, $\lambda \geq 1$ and $\lambda \leq 1 + |Dl| \leq M\lambda$ for some $M \geq 1$. Moreover, suppose that $\Phi^{(4)}(z_0, \varrho, l) \leq \frac{1}{16}$. Then, there exists a constant $C$ depending only upon $n$, $N$, $E$, $\gamma_2$, $\nu$, $L$ and $M$ such that for any $\varphi \in C^0_0(Q^4_0(z_0))$ there holds

$$\left| \int_{Q^4_0(z_0)} (u - l) \cdot \partial_i \varphi - \langle (\mu Q^4_0(z_0)) A(z_0, Dl) \cdot (Du - Dl), D\varphi \rangle \, dz \right| \leq c(1 + |Dl|)^{p_0 - 1}\left[ \Psi^*(z_0, \varrho, l) + \omega^*(\sqrt{\Psi^*_A(z_0, \varrho, l)}) \right] \sup_{Q^4_0(z_0)} |D\varphi|,$$

(6.2)

where $p_0 = p(z_0)$.

Proof. Without loss of generality, we may prove the lemma in the case that $z_0 = 0$ and $\sup_{Q^4_0} |D\varphi| = 1$. For simplicity of notation, we write $\Phi^{(4)}(0, \varrho, l)$, $\Psi^*_A(0, \varrho, l)$ and $\Psi^*_A(0, \varrho, l)$ for $\Phi^{(4)}(0, \varrho, l)$, $\Psi^*_A(0, \varrho, l)$ and $\Psi^*_A(0, \varrho, l)$, respectively. To start with, we first observe that

$$\int_{Q^4_0(z_0)} I \cdot \partial_i \varphi \, dz = 0$$

and

$$\int_{Q^4_0(z_0)} (\mu Q^4_0(z_0)) \langle A(0, Dl), D\varphi \rangle \, dz = 0,$$

since $\varphi \in C^0_0(Q^4_0(z_0))$. From the identities above and weak formulation (2.8), we have the decomposition

$$\int_{Q^4_0(z_0)} (u - l) \cdot \partial_i \varphi - \langle (\mu Q^4_0(z_0)) A(0, Dl) \cdot (Du - Dl), D\varphi \rangle \, dz = I + II + III,$$

where

$$I = \int_{Q^4_0} \langle (\mu - (\mu Q^4_0(z_0))) A(z, Du), D\varphi \rangle \, dz,$$

$$II = \int_{Q^4_0} \langle (\mu Q^4_0(z_0)) A(z, Du) - A(0, Du), D\varphi \rangle \, dz$$

and

$$III = \int_{Q^4_0} \langle (\mu Q^4_0(z_0)) A(0, Du) - A(0, Dl), D\varphi \rangle \, dz$$

$$- \int_{Q^4_0} \langle (\mu Q^4_0(z_0)) \partial_u A(0, Dl) \cdot (Du - Dl), D\varphi \rangle \, dz.$$
with the obvious meaning of $I_1$ and $I_2$. To estimate $I_1$, we use the definition of $v(q)$ to deduce $I_1 \leq c(1 + |Dl|)^{p_0-1}v(q)$. To estimate $I_2$, we decompose

$$
I_2 = c\frac{1}{|Q_{\frac{3}{4}}|} \int_{Q_{\frac{3}{4}}} |(\mu - (\mu_{Q_{\frac{3}{4}}})|Du - Dl|^{p(z)-1}x_{|Du - Dl| \leq 1}\,dz
$$

$$
+ c\frac{1}{|Q_{\frac{3}{4}}|} \int_{Q_{\frac{3}{4}}} |(\mu - (\mu_{Q_{\frac{3}{4}}})|Du - Dl|^{p(z)-1}x_{|Du - Dl| > 1}\,dz
$$

$$
\leq cv(q) + I_3,
$$

where

$$
I_3 = \int_{Q_{\frac{3}{4}}} |(\mu - (\mu_{Q_{\frac{3}{4}}})|Du - Dl|^{p(z)-1}x_{|Du - Dl| > 1}\,dz.
$$

To proceed further, we apply the Hölder’s inequality with exponents $p_0$ and $\frac{p_0}{p_0-1}$ to conclude that

$$
I_3 \leq c\left(\int_{Q_{\frac{3}{4}}} |(\mu - (\mu_{Q_{\frac{3}{4}}})|Du - Dl|^{p(z)-1}x_{|Du - Dl| > 1}\,dz\right)^{p_0-1}\frac{p_0}{p_0-1}
$$

$$
\times \left(\int_{Q_{\frac{3}{4}}} |(\mu - (\mu_{Q_{\frac{3}{4}}})|^{p_0}x_{|Du - Dl| > 1}\,dz\right)^{\frac{1}{p_0}},
$$

(6.3)

since $\sqrt[p_0]{\mu(z)} \leq \sqrt[p_0]{L}$. Moreover, for any $z \in \{z \in Q_{\frac{3}{4}} : |Du - Dl| > 1\}$, we use the mean value theorem from calculus to find that

$$
|Du - Dl|^{\frac{p_0}{p_0-1}} - |Du - Dl|^p_0 \leq \frac{p_0}{p_0-1}(|Du - Dl|^{\frac{p_0}{p_0-1}} - |Du - Dl|p_0) \leq 2\omega_p(|Du - Dl|^p_0 + 2\omega_p(|Du - Dl|p_0) \log |Du - Dl|,
$$

since $p_0 \geq 2$ and therefore $\frac{p_0}{p_0-1}(p_2 - 1) \leq p_1 + 2\omega_p(|Du - Dl|p_0$). Plugging this into (6.3), we apply Lemma 3.5 with $\gamma = 1$ and $\theta = \frac{1}{4}$ to deduce

$$
I_3 \leq cv(q)\frac{p_0}{p_0-1} \left(\int_{Q_{\frac{3}{4}}} |(\mu - (\mu_{Q_{\frac{3}{4}}})|Du - Dl|^{p(z)-1}\,dz\right)^{p_0-1}\frac{p_0}{p_0-1}
$$

$$
+ cv(q)^\frac{1}{p_0} \left(\int_{Q_{\frac{3}{4}}} \omega_p(q)(1 + |Du| + |Dl|p_0(1+2\omega_p(q))) \log(1 + |Du| + |Dl|) dz\right)^{p_0-1}\frac{p_0}{p_0-1}
$$

$$
\leq c(1 + |Dl|^{p_0-1}v(q)^{\frac{1}{p_0}} + \omega_p(q)^{\frac{1}{p_0}}(1 + |Dl|^{p_0-1}v(q) \log(1 + |Dl|^{p_0-1}))\,dz.
$$

From the Caccioppoli’s inequality (5.1), we see that $\Phi^{\mathcal{C}}(\frac{2}{4}) \leq c\Psi_0^+(q)$. We now use the Young’s inequality with exponents $p_0$ and $\frac{p_0}{p_0-1}$ to obtain

$$
I_3 \leq c\left(v(q) + \omega_p(q) \log\left(\frac{1}{q}\right) + \Psi_0^+(q)(1 + |Dl|^{p_0-1} \leq c\Psi_0^+(q)(1 + |Dl|^{p_0-1}),
$$

where we used (6.1) in the last step. The estimates above yield that

$$
I \leq c\Psi_0^+(q)(1 + |Dl|)^{p_0-1}.
$$

(6.4)
Next, we consider the estimate for $II$. To this end, we use the continuity condition (2.6) to obtain

$$ II \lesssim \int_{Q_{c}^{1/4}} |(\mu_{Q_{c}^{1/4}})(A(z, Du) - A(0, Du))| d\phi(z) $$

$$ \leq L \int_{Q_{c}^{1/4}} \omega_{p}(r)(1 + |Du|)^{p^{\theta-1}} + (1 + |Du|)^{p^{\theta-1}} \left(1 + \log(1 + |Du|)\right) d\phi(z) $$

$$ \leq L \int_{Q_{c}^{1/4}} \omega_{p}(r)(1 + |Du|)^{p^{\theta-1}} \log(e + |Du|) d\phi(z) $$

since $\sqrt{L} \leq \mu(z) \leq \sqrt{L}$. Noting that $\frac{p}{p^{\theta-1}} \geq 1$, we use the Hölder’s inequality and Lemma 3.5 with $\gamma = \frac{p}{p-1}$ and $\beta = \frac{1}{2}$ to obtain

$$ II \leq L \omega_{p}(r) \left( \int_{Q_{c}^{1/4}} (1 + |Du|)^{p^{\theta-1}} (e + |Du|) d\phi(z) \right)^{\frac{1}{p^{\theta-1}}} $$

$$ \leq c \omega_{p}(r) \left( \int_{Q_{c}^{1/4}} (1 + |Du|)^{p^{\theta-1}} d\phi(z) \right)^{\frac{1}{p^{\theta-1}}} $$

$$ \leq c(1 + |Du|)^{p^{\theta-1}} \omega_{p}(r) \left( \int_{Q_{c}^{1/4}} (1 + |Du|)^{p^{\theta-1}} d\phi(z) \right)^{\frac{1}{p^{\theta-1}}} $$

where we used Lemma 3.2 (3) in the last step. From the estimates above, we conclude that

$$ II \leq c\Psi_{A}(r)(1 + |Du|)^{p^{\theta-1}}. \quad (6.5) $$

Finally, we come to the estimate of $III$. Recalling that $\sqrt{L} \leq \mu(z) \leq \sqrt{L}$, we use the continuity condition (2.7) to deduce

$$ |III| \leq \sqrt{L} \int_{Q_{c}^{1/4}} \left| \int_{0}^{1} \left[ \partial_{n}A(0, Du + s(Du - Du)) - \partial_{n}A(0, Du) \right] : (Du - Du) d\phi(z) \right| d\phi(z) $$

$$ \leq c \int_{Q_{c}^{1/4}} \int_{0}^{1} \omega_{n} \left( \frac{s|Du - Du|}{1 + |Du + s(Du - Du)| + |Du|} \right) (1 + |Du| + |Du - Du|)^{p^{\theta-2}} d|Du - Du| d\phi(z) $$

Noting that $s \in (0, 1)$, we conclude from the arguments in [15, page 1808] that

$$ 1 + |Du + s(Du - Du)| + |Du| \geq \frac{3}{2}(1 + |Du| + |Du - Du|), $$

and this leads us to

$$ |III| \leq c(1 + |Du|)^{p^{\theta-2}} \int_{Q_{c}^{1/4}} \omega_{n} \left( \frac{|Du - Du|}{1 + |Du|} + \frac{|Du - Du|}{1 + |Du|} \right) d\phi(z) $$

Recalling that the Caccioppoli’s inequality (5.1) implies $\Phi^{(1)}(\frac{3}{2}) \leq c\Psi_{A}(r)$, we now proceed along the lines of the arguments in [15, page 1808-1809] to conclude that

$$ |III| \leq c(1 + |Du|)^{p^{\theta-1}} \left( \frac{|Du - Du|}{1 + |Du|} + \frac{|Du - Du|}{1 + |Du|} \right) \Psi_{A}(r) $$

Combining (6.4), (6.5) and (6.6), we obtain the desired estimate (6.2). The proof of the lemma is now complete. $\square$

With the help of Lemma 6.1 we can now establish a decay estimate in terms of the hybrid excess functional and the following proposition is our main result in this section.
Proposition 6.2. Let \( Q_0^{(0)}(z_0) \subset \Omega_T \) be a scaled parabolic cylinder on which the intrinsic coupling \( \lambda = 1 + |D\Psi|^{(1)}_{\xi} \) holds and let \( \varrho_0 > 0 \) be the radius in Lemma 3.5. Assume that \( \Psi^{(1)}(z_0, \varrho, t) \leq \frac{1}{16} \) and \( \varrho < \varrho_0 \). For any fixed \( \theta < 2^{-\frac{1}{2}}\gamma_\nu^{-6} \), there exists a constant \( \epsilon_1 = \epsilon_1(n, N, \nu, \gamma_\nu, E, M, \theta) \) such that if smallness conditions

\[
\Psi^{(1)}_A(z_0, \varrho, t_0^{(1)}_{z_0, \varrho}) \leq \epsilon_1 \quad \text{and} \quad \Phi^{(1)}_A(z_0, \varrho, t_0^{(1)}_{z_0, \varrho}) \leq \frac{1}{16} \quad (6.7)
\]

are satisfied, then there exists a new scaling factor \( \lambda_1 \in [\frac{1}{2} \lambda, 2M \lambda] \) such that \( 1 + |D\Psi^{(1)}|_\lambda = \lambda_1 \) and the following decay estimate holds:

\[
\Psi_\lambda(z_0, \varrho, t^{(1)}_{z_0, \varrho}) \leq c_\nu \Phi^{(1)}_A(z_0, \varrho, t_0^{(1)}_{z_0, \varrho}), \quad (6.8)
\]

where the constant \( c_\nu \) depends only upon \( n, N, \nu, \gamma_\nu, E \) and \( M \).

**Proof.** Without loss of generality, we may take \( z_0 = 0 \). Once again, we set \( p_0 = p(z_0) \), \( p_1 = \inf_{Q_0^{(0)}(z_0)} p(z) \) and \( p_2 = \sup_{Q_0^{(0)}(z_0)} p(z) \). For simplicity, we abbreviate

\[
\Psi^{(1)}_A(q) := \Psi^{(1)}_A(z_0, \varrho, t^{(1)}_{z_0, \varrho}) \quad \text{and} \quad \Psi^{(1)}_\lambda(q) := \Psi^{(1)}(z_0, \varrho, t^{(1)}_{z_0, \varrho}).
\]

We first observe that the assumption \( \theta < 2^{-\frac{1}{2}}\gamma_\nu^{-6} \) guarantees \( Q^{(1)}_{\varrho_0} \subset Q^{(1)}_{\varrho} \) for any fixed \( \lambda_1 \in [\frac{1}{2} \lambda, 2M \lambda] \). In order to prove (6.8), we only need to consider the case \( \Psi^{(1)}_A(q) > 0 \) due to the quasi-minimality of the affine map \( t^{(1)}_{z_0, \varrho} \). This enables us to define an auxiliary map \( \nu \) via

\[
\nu(x, t) = \frac{u(x, \lambda^{2-p_0} t) - t^{(1)}_\varrho}{c_1(1 + |D\Psi^{(1)}|)} \Psi^{(1)}_\lambda(q)
\]

for \((x, t) \in Q_\varrho := Q^{(1)}_\varrho \) and \( c_1 \geq 1 \) is to be determined later. Moreover, we set \( \nu = \sqrt{\Psi^{(1)}_\lambda(q)} \). Initially, we choose \( \epsilon_1 < 1 \) and this ensures that \( \gamma < 1 \). By a change of variable, we observe from \( p_0 \geq 2 \) that

\[
\int_{Q^{(1)}_\varrho} \gamma^{p_0-2} \frac{|v|^{p_0}}{q} \, dz \leq \frac{1}{c_1^2 \gamma^2} \int_{Q^{(1)}_\varrho} \frac{|u - t^{(1)}_\varrho|^{p_0}}{q(1 + |D\Psi^{(1)}|)} \, dz \leq \frac{c}{c_1^2 \Psi^{(1)}_\lambda(q)} \Psi^{(1)}_\lambda(q) \leq \frac{c}{c_1^2},
\]

where the constant \( c \) depending only upon \( n \) and \( \gamma_2 \). The next thing to do in the proof is to verify that the map \( \nu \) is approximatively \( A \)-caloric in the sense of (5.2). From the Caccioppoli’s inequality (5.1), we deduce

\[
\sup_{x \in \Lambda_{\nu}^{(1)}} \int_{B_{\nu^{(1)}}} \frac{|v(x, t)|^2}{q} \, dx \leq \frac{\lambda^{2-p_0}}{c_1^2 \gamma^2 (1 + |D\Psi^{(1)}|)^2} \sup_{x \in \Lambda_{\nu}^{(1)}} \int_{B_{\nu^{(1)}}} \frac{|u(x, t) - t^{(1)}_\varrho|^2}{q(1 + |D\Psi^{(1)}|)^2} \, dx \leq \frac{c_{Cacc} (1 + |D\Psi^{(1)}|)^{p_0-2}}{c_1^2 \gamma^2} \Psi^{(1)}_\lambda(q) \leq \frac{c_{Cacc} M, \gamma_2}{c_1^2},
\]

where \( c_{Cacc} \) is the constant in the inequality (5.1). Next, we use the Caccioppoli’s inequality (5.1) again to obtain

\[
\int_{Q^{(1)}_\varrho} |D\nu|^2 + \gamma^{p_0-2} |D\nu|^{p_0} \, dz \leq \frac{1}{c_1^2 \gamma^2} \Phi^{(1)}_A(0, \frac{\varrho}{4}, t^{(1)}_\varrho) \leq \frac{c_{Cacc}}{c_1^2 \Psi^{(1)}_\lambda(q)} \Psi^{(1)}_\lambda(q) = \frac{c_{Cacc}}{c_1^2}. \]

Therefore, we conclude from the above estimates that there exists a constant \( \tilde{c} \) depending only upon \( n, N, \nu, \gamma_\nu, E \) and \( M \) such that

\[
\sup_{-\frac{1}{16} \nu^{(1)} \subset 0} \int_{B_{\nu^{(1)}}} \frac{v}{q/4} \, dx + \int_{Q^{(1)}_\varrho} |D\nu|^2 \, dz + \int_{Q^{(1)}_\varrho} \gamma^{p_0-2} \left( \frac{|v|^{p_0}}{q^2} + |Dv|^{p_0} \right) \, dz \leq \frac{\tilde{c}}{c_1} \leq 1,
\]
provided that the constant $c_1$ can be chosen so large that $c_1 \geq \sqrt{R}$. Furthermore, let us now introduce the vector field $A \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ with constant coefficients
\[ A := (\mu)_{\epsilon,4}^{l^2-\nu_0} \partial_n A(0, Df^{(4)}). \]

Another step in the proof is to check that $A$ satisfies ellipticity and growth conditions similar to (B.1). We first apply the ellipticity condition (2.2), $\sqrt{V} \leq \mu(z) \leq \sqrt{E}$, $p_0 \geq 2$ and $\lambda \leq 1 + |Df^{(4)}|$ to deduce that for any $\bar{w} \in \mathbb{R}^{N \times n}$ there holds
\[ (\bar{w}, \bar{w}) = (\mu)_{\epsilon,4}^{l^2-\nu_0} (\partial_n A(0, Df^{(4)}) \bar{w}, \bar{w}) \geq \nu \lambda^{l^2-\nu_0} (1 + |Df^{(4)}|)^{p_0-2} |\bar{w}|^2 \geq \nu |\bar{w}|^2. \]

Moreover, for any $w, \bar{w} \in \mathbb{R}^{N \times n}$, we use the growth condition (2.2), $\sqrt{V} \leq \mu(z) \leq \sqrt{E}$, $p_0 \geq 2$ and $1 + |Df^{(4)}| \leq M \lambda$ to find that there exists a constant $\hat{c}$ depending only upon $L$, $\gamma_2$ and $M$ such that the following inequality holds:
\[ [(\bar{w}, \bar{w})] = (\mu)_{\epsilon,4}^{l^2-\nu_0} (\partial_n A(0, Df^{(4)}) \bar{w}, \bar{w}) \leq L \lambda^{l^2-\nu_0} (1 + |Df^{(4)}|)^{p_0-2} |\bar{w}|^2 \leq \hat{c} |w||\bar{w}|. \]

To proceed further, we define a rescaled test function $\hat{\varphi} \in C^\infty_0(Q^{(4)}_{\epsilon/4}, \mathbb{R}^N)$ by $\hat{\varphi}(x, \lambda^{l^2-\nu_0}t) = \varphi(x, t)$. By the chain rule, we conclude from Lemma [6.1] with $\ell = t^{(4)}_{\epsilon/4}$ that there exists a constant $\hat{c}$ depending only upon $n, N, \nu, L, \gamma_2, E$ and $M$ such that the inequality
\[ \int_{Q^{(4)}_{\epsilon/4}} v \cdot \partial_n \varphi - \langle A Dv, D\varphi \rangle \, dx \]
\[ = \frac{1}{c_1 \gamma (1 + |Df^{(4)}|)^{p_0-2}} \int_{Q^{(4)}_{\epsilon/4}} (\mu - \epsilon) \partial_n \hat{\varphi} - (\mu)_{\epsilon,4}^{l^2-\nu_0} (\partial_n A(0, Df^{(4)}) \cdot (Du - Df^{(4)}), D\hat{\varphi}) \, dx \]
\[ \leq \frac{c (1 + |Df^{(4)}|)^{p_0-2}}{c_1 \gamma^{l^2-\nu_0}} \left[ \Psi^*_4(\varphi) + \omega^*_4 \left( \sqrt{\Psi^*_4(\varphi)} \right) \right] \sup_{Q^{(4)}_{\epsilon/4}} |D\hat{\varphi}| \]
\[ \leq \hat{c} \left[ \sqrt{\Psi^*_4(\varphi)} + \omega^*_4 \left( \sqrt{\Psi^*_4(\varphi)} \right) \right] \sup_{Q^{(4)}_{\epsilon/4}} |D\hat{\varphi}| \leq \hat{c} \left[ \sqrt{V} + \omega^*_4 \left( \sqrt{V} \right) \right] \sup_{Q^{(4)}_{\epsilon/4}} |D\varphi| \]
holds. Our task now is to apply Lemma [B.2] to find an $A$-caloric map $f$ which approximate the map $\nu$ in the sense of (B.3). To this end, we fix $\epsilon \in (0, 1)$ which will be determined in the course of the proof. Moreover, let $\delta_0 = \delta_0(n, N, \gamma_2, \nu, \epsilon, \hat{c})$ be the constant determined by Lemma [B.2] with $A = \hat{c}$ and $A = \epsilon$. We now impose a condition that $\epsilon_1 \leq \hat{\epsilon}$ where $\hat{\epsilon} > 0$ satisfies
\[ \hat{\epsilon} \left[ \sqrt{\hat{\epsilon}} + \omega^*_4 \left( \sqrt{\hat{\epsilon}} \right) \right] \leq \delta_0. \] (6.9)

At this stage, we see that the hypotheses of Lemma [B.2] are fulfilled and we can apply Lemma [B.2] with $w = \nu$. Therefore, we conclude the existence of an $A$-caloric map $f \in C^\infty_0(Q^{(4)}_{\epsilon/8}, \mathbb{R}^N)$ satisfying
\[ \int_{Q^{(4)}_{\epsilon/8}} \frac{f}{2} + |Df|^2 \, dz + \int_{Q^{(4)}_{\epsilon/8}} \gamma^{p_0-2} \left( \frac{f}{2} \right)^{p_0} + |Df|^{p_0} \, dz \leq c(n, \gamma_2, \nu, L) \]
and
\[ \int_{Q^{(4)}_{\epsilon/8}} \frac{v - f}{2} \, dz + \gamma^{p_0-2} \left( \frac{v - f}{2} \right)^{p_0} \, dz \leq \epsilon. \]

We have established the existence of the $A$-caloric map, and now we further study its decay estimate. To start with, we set $\theta = 2^{\frac{1}{2}} \gamma + 2 \hat{\theta}$. It follows from $\theta < 2^{-\frac{1}{2}} \gamma - \theta$ that $\theta < \frac{1}{2}$. We now
apply (6.10) and [8, Lemma 2.7] for $s = 2$ or $s = p_0$ to deduce that there exists a constant $c_{pa}$ depending only upon $n, N, \nu, L$ and $\gamma_2$ such that the inequality
\[
\gamma r_2 \left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |f - (f)_{Q_{\theta/8}} - (Df)_{Q_{\theta/8}} \cdot x|^s \, dz 
\leq c_{pa} \gamma r_2 \left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |f - (f)_{Q_{\theta/8}} - (Df)_{Q_{\theta/8}} \cdot x|^s \, dz 
\leq c_{pa} \theta \left( \gamma r_2 \right)^{-s} \int_{Q_{\theta/8}} |f - (f)_{Q_{\theta/8}} - (Df)_{Q_{\theta/8}} \cdot x|^s \, dz + \gamma r_2 \int_{Q_{\theta/8}} |Df|^s \, dz \leq c_{pa} \theta \left( \gamma r_2 \right)^{-s}
\]
holds for either $s = 2$ or $s = p_0$. Furthermore, we conclude from the above estimate and (6.11) that for $s = 2$ or $s = p_0$ there holds
\[
\gamma r_2 \left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |v - (f)_{Q_{\theta/8}} - (Df)_{Q_{\theta/8}} \cdot x|^s \, dz 
\leq 2^{-1} \gamma r_2 \left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |v - f|^s \, dz 
+ 2^{-1} \gamma r_2 \left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |f - (f)_{Q_{\theta/8}} - (Df)_{Q_{\theta/8}} \cdot x|^s \, dz 
\leq c \theta^{-s} \left( \gamma r_2 \right)^{-s} \int_{Q_{\theta/8}} |v - f|^s \, dz + c \theta^s \leq c \theta^{-s} \gamma r_2^{-s} + c \theta^s.
\]
At this stage, we choose $\varepsilon = \theta^{-s} \gamma r_2^{-s}$ and therefore $\varepsilon$ depends only upon $\theta$ and $\gamma_2$. The choice of $\varepsilon$ determines the value of $\delta_0$ in dependence of $n, N, \gamma_2, \nu, L$ and $\theta$. Moreover, the constant $\tilde{c}$ is fixed via (6.9). We now proceed to estimate (6.12) and obtain
\[
\left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |v - (f)_{Q_{\theta/8}} - (Df)_{Q_{\theta/8}} \cdot x|^s \, dz \leq c \theta^{-s} \gamma r_2^{-s}.
\]
Recalling the definition of the map $v$, we rescale back to $u$ in (6.13) and this yields that for $s = 2$ or $s = p_0$ there holds
\[
\left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |u - k^{(2)}(1 + |Dk^{(2)}|)u - (f)_{Q_{\theta/8}} - (Df)_{Q_{\theta/8}} \cdot x|^s \, dz 
\leq c(1 + |Dk^{(2)}|)^s \gamma r_2^{-s} \gamma r_2^{-s} \leq c \lambda^s \Psi_4(M).
\]
A careful examination of the proof of [8, Lemma 2.8] ensures the existence of a constant $\hat{c}_1 \geq 1$ depending only upon $n, N, \nu, L, \gamma_2$ and $M$ such that the inequality
\[
\left( \frac{\theta}{8} \right)^{-s} \int_{Q_{\theta/8}} |u - k^{(2)}| \, dz \leq \hat{c}_1 \lambda^s \Psi_4(M)
\]
holds for either $s = 2$ or $s = p_0$. Our task now is to find a scaling factor $\lambda \in \left[ \frac{1}{4} \lambda, 2M \lambda \right]$ such that $1 + |Dk^{(2)}| = \lambda_1$. For $\lambda \in \left[ \frac{1}{4} \lambda, 2M \lambda \right]$, we have $(\theta q)^2 \lambda^{2 - p_0} \leq \left( \frac{1}{8} \theta q \right)^2 \lambda^{2 - p_0}$, since $p_0 \geq 2$ and $\theta = 2^{1/2 + 1} \theta$. This implies that $Q_{\theta/8}^{(2)} \subset Q_{\theta/8}^{(1)} \subset Q_{\theta/8}^{(2)}$. Therefore, we apply [13, Lemma 2.5] with $z_0 = 0, \lambda = \lambda_1, A^{(2)}_{\xi, \tilde{c}} = Dk^{(2)}_{\theta/8}, \xi = k^{(2)}_{\theta/8}(0)$ and $w = Dk^{(2)}_{\theta/8}$ to conclude that there exists a constant $\hat{c}_1$ depending only upon $n, M$ and $\gamma_2$ such that the following
Lemma 7.1. Let $z_0 \in \Omega_T$ and $M \geq 1$. Moreover, let $\varrho_0 > 0$ be the radius in Lemma 6.15 and let $c$, be the constant in (6.8). Assume that $0 < \theta < \min\{2^{-\gamma_z - 6}, (3c_z)^{-1}\}$. Then, there

\begin{equation}
\left| D_{\varrho_0} f^{(\lambda_1)} - D_{\varrho} f^{(\lambda)} \right|^2 \leq \frac{n(n + 2)}{(\theta \varrho)^2} \int_{Q_{\varrho_0}^{(\lambda_1)}} |u - f^{(\lambda)}|^2 \, dz \\
\leq n(n + 2) \theta^{-2(n+4)} \left( \frac{\lambda}{\lambda_1} \right)^{2-p_0} \int_{Q_{\varrho_0}^{(\lambda)}} |u - f^{(\lambda)}|^2 \, dz \\
\leq \tilde{c}_1 \theta^{-(n+4)} \lambda^{2} \Psi^\prime (\varrho) \leq \tilde{c}_1 \theta^{-(n+4)} \lambda^{2} \varepsilon_1,
\end{equation}

since $1 + |D_{\varrho_0} f^{(\lambda_1)}| \leq M\lambda$. At this point, we assume that $\varepsilon_1 \leq \frac{1}{4} \tilde{c}_1^{-1} \theta^{n+4}$ and this yields $|D_{\varrho_0} f^{(\lambda_1)} - D_{\varrho} f^{(\lambda)}| \leq \frac{1}{4} \lambda$. By the triangle inequality, we observe from $\lambda \leq 1 + |D_{\varrho_0} f^{(\lambda_1)}| \leq M\lambda$ that for any $\lambda_1 \in \left[ \frac{1}{2}, 2M\lambda \right]$ there holds

\begin{equation}
1 + |D_{\varrho_0} f^{(\lambda_1)}| \leq 2M\lambda \quad \text{and} \quad 1 + |D_{\varrho_0} f^{(\lambda_1)}| \geq \frac{1}{2} \lambda.
\end{equation}

To proceed further, we now define the function $f : \left[ \frac{1}{2}, 2M\lambda \right] \rightarrow \mathbb{R}$ by $f(\lambda^*) = 1 + |D_{\varrho_0} f^{(\lambda_1)}| - \lambda^*$. From (6.9), we find that

\begin{equation}
D_{\varrho_0} f^{(\lambda_1)} = \frac{n + 2}{(\theta \varrho)^2} \int_{Q_{\varrho_0}^{(\lambda_1)}} u \otimes x \, dz
\end{equation}

and this implies that $f(\lambda^*)$ is continuous. From (6.15), we see that $f(\frac{1}{2} \lambda) \geq 0$ and $f(2M\lambda) \leq 0$. By continuity of the function $f(\lambda^*)$, there exists a scaling factor $\lambda_1 \in \left[ \frac{1}{2}, 2M\lambda \right]$ such that $f(\lambda_1) = 0$ which leads us to $1 + |D_{\varrho_0} f^{(\lambda_1)}| = \lambda_1$. Recalling that $Q_{\varrho_0}^{(\lambda_1)} \subset Q_{\varrho_0/8}^{(\lambda_1)}$, then we conclude from (6.14) and the proof of (6.15) that there exists a constant $\tilde{c}_2$ depending only upon $n$ and $\gamma_2$ such that the inequality

\begin{equation}
\int_{Q_{\varrho_0}^{(\lambda_1)}} \left| \frac{u - f^{(\lambda_1)}}{\theta \varrho} \right|^s \, dz \leq \tilde{c}_2 \int_{Q_{\varrho_0}^{(\lambda_1)}} \left| \frac{u - f^{(\lambda_1)}}{\theta \varrho/8} \right|^s \, dz
\end{equation}

\begin{equation}
\leq \tilde{c}_2 \left( \frac{\theta}{\theta \varrho} \right)^{n+2+s} \left( \frac{\lambda}{\lambda_1} \right)^{2-p_0} \int_{Q_{\varrho_0/8}^{(\lambda_1)}} \left| u - f^{(\lambda_1)} \right|^s \, dz
\end{equation}

\begin{equation}
\leq 2^{(\frac{1}{2} n+1)(n+2+s)} \tilde{c}_2 \lambda^{s} \tilde{c} \theta^s \Psi^\prime (\varrho) \leq 2^{5n^2 \tilde{c}_2 \lambda^{s} \tilde{c}} \theta^{2s} \Psi^\prime (\varrho) \leq 2^{5n^2 \tilde{c}_2 \lambda^{s} \tilde{c}} \theta^{2s} \Psi^\prime (\varrho)
\end{equation}

holds for either $s = 2$ or $s = p_0$. Therefore, the desired inequality (6.8) follows with the choice $c_s = 2^{5n^2 \tilde{c}_2 \lambda^{s} \tilde{c}}$, provided that the smallness condition $\Psi^\prime (\varrho) \leq \varepsilon_1$ holds with $\varepsilon_1 = \min\{1, \tilde{c}, \frac{1}{4} \theta^{n+4}\}$. The proof of the proposition is now complete. \hfill \Box

7. Proof of partial regularity

In this section, we give the proof of Theorem 2.2. Our proof starts from the estimate (4.13) in Proposition 4.3 and we will show that the weak solution to the parabolic system (2.1) is Hölder continuous in a neighborhood of any regular point. The strategy is standard and we will make use of an integral characterization of parabolic Hölder space due to Campanato and DaPrato (cf. (15, Theorem 2.10)). Before proving Theorem 2.2 we need the following iteration lemma.

Lemma 7.1. Let $z_0 \in \Omega_T$ and $M \geq 1$. Moreover, let $\varrho_0 > 0$ be the radius in Lemma 6.15 and let $c$, be the constant in (6.8). Assume that $0 < \theta < \min\{2^{-\gamma_z - 6}, (3c_z)^{-1}\}$. Then,
exist a constant $\varepsilon_2 = \varepsilon_2(n, N, L, \nu, \gamma_2, E, M, \theta) > 0$ such that the following holds: If there exists a radius $\varrho < \varrho_0$ such that the following inequalities
\[
\begin{aligned}
1 + |D\varphi| &\leq M,
\Psi(z_0, \varrho, l_{0} \varrho) \leq \varepsilon_2, \\
\Phi(z_0, \varrho, l_{0} \varrho) &\leq \frac{1}{16}, \\
v(\varrho) + \omega_\varrho(\varrho) \log \left(\frac{1}{\varrho}\right) &\leq \varepsilon_2 
\end{aligned}
\] (7.1)

hold, then there exists a sequence $\{\lambda_j\}_{j \in \mathbb{Z}}$ such that for all $j \in \mathbb{N}$ there holds
\[
\begin{aligned}
1 &\leq \lambda_j \leq (2M)^j, \\
\lambda_j &\leq 1 + |D\varphi_{\lambda_j}| \leq M\lambda_j, \\
\Psi_{\lambda_j}(z_0, \varrho, l_{\lambda_j} \varrho) &\leq \varepsilon_2, \\
\Phi(\lambda_j z_0, \varrho, l_{\lambda_j} \varrho) &\leq \frac{1}{16}.
\end{aligned}
\] (Ij)

Proof. We omit for simplification the reference point $z_0$ in our notation and we will prove the lemma by induction. We fix a constant $\delta_1 < \varepsilon_1$ where $\varepsilon_1$ is the constant from Proposition 6.2 and the constant $\delta_1$ will be determined in the course of the proof. Now, we set $\varepsilon_2 := \frac{\delta_1}{4\delta_1}$. Initially, the inequalities in (I0) hold true from (7.1).

It only suffices to prove that the implication (Ij) $\Rightarrow$ (Ij+1) holds true for any $j \geq 0$. For simplicity, we may take $l_j = l_{\lambda_j}^{(1)}$ for $j = 0, 1, \ldots$. From (7.1), (Ij), and (6.1), we conclude that for any $\varrho < \varrho_0$ there holds
\[
\Psi_{\lambda_j}(\theta^j \varrho, l_j) \leq \Psi_{\lambda_j}(\theta^j \varrho, l_j) + \nu(\varrho) + \omega_\varrho(\varrho) \log \left(\frac{1}{\varrho}\right) \leq 2\varepsilon_2 < \varepsilon_1. 
\] (7.2)

Keeping in mind that $\varrho < \varrho_0$ we deduce from (7.2) and (Ij) that the hypotheses of Proposition 6.2 with $(\varrho, \lambda)$ replaced by $(\theta^j \varrho, \lambda_j)$ are fulfilled. Therefore, we conclude that there exists a scaling factor $\lambda_{j+1} \in \left[\frac{1}{2}, 2M\lambda_j\right]$ such that $\lambda_{j+1} = 1 + |D\varphi_{\lambda_{j+1}}|$, hence that $\lambda_{j+1} \geq 1$, and finally that $\lambda_{j+1} \leq 2M\lambda_j \leq (2M)^{j+1}$. This implies that (Ij+1) and (Ij+2) hold true. Since $\theta < (3c_\nu)^{-\frac{1}{2}}$, we infer from (6.8) and (7.2) that
\[
\Psi_{\lambda_{j+1}}(\theta^{j+1} \varrho, l_{j+1}) \leq c_\nu^2 \Psi_{\lambda_j}(\theta^j \varrho, l_j) \leq 3c_\nu^2 \varepsilon_2 \leq \varepsilon_2.
\]

This proves that (Ij+3) holds true. The next thing to do in the proof is to verify that (Ij+4) holds true. Since $\theta < 2^{-3-\gamma}$ and $\lambda_{j+1} \in \left[\frac{1}{2}, 2M\lambda_j\right]$, we have $\sqrt{\theta} < \frac{1}{4}$ and $Q_{\theta^{j+1} \varrho}^{(1)} \subset Q_{\theta \varrho}^{(1)}$. Moreover, we apply the Caccioppoli inequality (5.1) with $(\varrho, \lambda, l_j)$ replaced by $(\theta^j \varrho, \lambda_j, l_j)$ to deduce that there exists a constant $\tilde{c}_1$ depending only upon $n, N, \nu, L, \gamma_2, E$ and $M$ such that the following inequality holds
\[
\Phi_{\lambda_{j+1}}(\theta^{j+1} \varrho, l_{j+1}) \leq \int_{Q_{\theta^{j+1} \varrho}^{(1)}} \left(\frac{|Du - DL|}{1 + |D\varphi|}\right)^2 + \left(\frac{|Du - DL|}{1 + |D\varphi|}\right)^{p_0} \Phi_{\lambda_{j+1}}(\theta^{j+1} \varrho, l_{j+1}) \right) \, dz
\leq \theta^{-n+2}(2M)^{\gamma_2-2} \Phi_{\lambda_{j+1}}(\theta^{j+1} \varrho, l_{j+1})
\leq c_{\text{Cacci}} \theta^{-n+2}(2M)^{\gamma_2-2} \Psi_{\lambda_j}(\theta^j \varrho, l_j) \leq \tilde{c}_1 \theta^{-n+2} \delta_1,
\] (7.3)
where we used \((7.2)\) in the last step. On the other hand, we apply \([15, \text{Lemma 2.5}]\) with \((\varrho A, \ldots)\) replaced by \((\theta^{1}|q, J_{j+1}, Dl_{j+1}, l_{j}(x_0), Dl_{j})\) to infer that there exists a constant \(c_2 \geq 1\) depending only upon \(n, \gamma_2\) and \(M\) such that the following inequality holds

\[
|Dl_{j+1} - Dl_j|^2 \leq \frac{n(n+2)}{(\theta^{1}|q)^2} \int_{Q_{\delta(\varrho)|q}} |u - l_j|^2 \, dz
\]

\[
\leq \frac{n(n+2)|\lambda_j|^2}{(\theta^{1}|q)^2} \int_{Q_{\delta(\varrho)|q}} |u - l_j|^2 \, dz \cdot (1 + |Dl_j|^2)
\]

\[
\leq n(n+2)(2M)^{\gamma_2 - 2}\theta^{n-4}(1 + |Dl_j|)^2 \Psi(\theta^{1}|q, l_j)
\]

\[
\leq c_2\theta^{n-4}(1 + |Dl_j|)^2 \delta_1.
\]

Moreover, the inequality \((7.4)\) also implies that

\[
|Dl_{j+1} - Dl_j|^{p_0} \leq c_2^\frac{2}{p_0} \theta^{-\left(n+\frac{2}{2}\right)}(1 + |Dl_j|)^{p_0} \delta_1,
\]

since \(p_0 \geq 2\) and \(\delta_1 < 1\). Recalling that \(\lambda_{j+1} \in \left[\frac{1}{2}\lambda_j, 2M\lambda_j\right]\) and \(\lambda_{j+1} = 1 + |Dl_{j+1}|\), then we obtain \(1 + |Dl_j| \geq \lambda_j \geq \frac{1}{2M}\lambda_{j+1} = \frac{1}{2M}(1 + |Dl_j|)\) and \(1 + |Dl_{j+1}| = \lambda_{j+1} \geq \frac{1}{2}\lambda_j \geq \frac{1}{2M}(1 + |Dl_j|)\). At this stage, we apply \((7.3), (7.5)\) to conclude with

\[
\Phi^{(j+1)}(z_0, \theta^{1}|q, l_{j+1})
\]

\[
\leq 2\gamma_1^{-1}(2M)^{\gamma_2}\Phi^{(j+1)}(z_0, \theta^{1}|q, l_j) + 2\gamma_1^{-1}\frac{|Dl_j - Dl_{j+1}|^{p_0}}{(1 + |Dl_j|)^{p_0} + 2|Dl_j - Dl_{j+1}|^2} + 2\gamma_1^{-1}\frac{|Dl_j - Dl_{j+1}|^{p_0}}{(1 + |Dl_j|)^{p_0} + 2|Dl_j - Dl_{j+1}|^2}
\]

\[
\leq 2\gamma_1^{-1}(2M)^{\gamma_2}\left(\Phi^{(j)}(z_0, \theta^{1}|q, l_j) + \frac{|Dl_j - Dl_{j+1}|^{p_0}}{(1 + |Dl_j|)^{p_0} + 2|Dl_j - Dl_{j+1}|^2} + \frac{|Dl_j - Dl_{j+1}|^{p_0}}{(1 + |Dl_j|)^{p_0} + 2|Dl_j - Dl_{j+1}|^2}\right)
\]

\[
\leq 2\gamma_1^{-1}(2M)^{\gamma_2}(2c_2^{\frac{2}{p_0}} + \delta_1)\theta^{-\left(n+\frac{4}{2}\right)}\delta_1.
\]

Finally, we choose

\[
\delta_1 = \min\left\{\frac{1}{2}\delta_1, \frac{1}{16}\left[2\gamma_1^{-1}(2M)^{\gamma_2}(2c_2^{\frac{2}{p_0}} + \delta_1)\right]^{-\frac{1}{4n+\frac{4}{2}}}\right\}
\]

and therefore the constant \(c_2 = \frac{1}{4}\delta_1\) depends only upon \(n, N, \nu, L, \gamma_2, E, M\) and \(\theta\). Moreover, we have established the inequality \((I_{j+1})\) for this choice of \(\delta_1\). The proof of the lemma is now complete. \(\Box\)

**Proof of Theorem 2.2** We fix a point \(z_0 \in \Omega \setminus (\Sigma_1 \cup \Sigma_2)\) and let \(\alpha \in (0, 1)\) be a fixed constant. From now on, we show that \(u\) is Hölder continuous with Hölder exponent \(\alpha\) on a neighborhood of \(z_0\). From Proposition 3.3 we can find a constant \(M > 0\) and a constant \(\delta = \delta(n, N, \nu, L, E, \gamma_2, M) > 0\) such that the following is true: For any \(\varepsilon < \delta\) there exists a radius \(\varrho = \varrho(n, N, \nu, L, E, \gamma_2, M, \varepsilon) < \varrho_0\) such that for any \(z_0 \in Q_{\varrho/8}(z_0)\) there holds

\[
\begin{align*}
1 + |Dl_{l_{z_0}}| & \leq M, \\
\Psi(z_0, \varrho, l_{z_0}, \varrho) & \leq \varepsilon, \\
\Phi(z_0, \varrho, l_{z_0}, \varrho) & \leq \frac{1}{M}, \\
v(\varrho) + \omega_p(\varrho) \log \left(\frac{1}{\varrho}\right) & \leq \varepsilon.
\end{align*}
\]

At this point, we fix

\[
\theta = \frac{1}{2} \min \left\{2^{-\gamma_2 - 6}, (3c_2)^{-\frac{1}{2}}, (2M)^{-\frac{4\alpha^{n+\frac{4}{2}}}{4n+\frac{4}{2}}}\right\}
\]

(7.8)
where \( c \) is the constant in (6.8). This also fixes \( \varepsilon_2 \) via (7.6). Moreover, we fix \( \varepsilon = \min(\delta, \varepsilon_2) \) where \( \varepsilon_2 \) is the constant in Lemma 7.1. Note that this particular choice of \( \varepsilon \) determines \( \varrho = \varrho(\varepsilon) \) from Proposition 4.3 and therefore (7.7) holds for such \( \varepsilon \) and \( \varrho \). Since \( \varepsilon \leq \varepsilon_2 \), we see that (7.1) holds true and the hypotheses of Lemma 7.1 are fulfilled. Our task now is to show that \( u \in C^{1,\gamma}_{\text{loc}}(Q_{\varepsilon/8}(30), \mathbb{R}^N) \) by using the iteration scheme from Lemma 7.1. To this end, we use the Campanato characterization for the parabolic Hölder space.

More precisely, we shall show that

\[
\sup_{z_0 \in Q_{\varepsilon/8}(30)} \sup_{r > 0} \frac{1}{|Q_r(z_0)|^{1+\gamma}} \int_{Q_r(z_0)} |u - (u)_{Q_r(z_0)}|^2 \, dz < +\infty.
\]

By a localization argument, our problem reduces to show that there exists a constant \( c = c(n, N, \nu, L, E, \gamma_2, \alpha, \varrho, M) > 0 \) such that for any \( z_0 \in Q_{\varepsilon/8}(30) \) and \( r \in (0, \varepsilon] \) there holds

\[
\int_{Q_r(z_0)} |u - (u)_{Q_r(z_0)}|^2 \, dz \leq cr^{n+2+2\alpha}.
\]

(7.9)

The proof of (7.9) follows in a similar manner as the arguments in [15, page 1817-1819] and we just sketch the proof. Once again, we set \( l_j = l_{j,0}^{j,3} \) for \( j = 0, 1, \ldots \). We firstly show that

\[
\int_{Q_{\varepsilon/4}(z_0)} |u - l_j(x_0)|^2 \, dz \leq (2M)^{2(j+1)}(\theta')^2
\]

(7.10)

holds for any \( j \in \mathbb{N} \). According to (7.10), it follows that

\[
\int_{Q_{\varepsilon/4}(z_0)} |u - l_j(x_0)|^2 \, dz = \int_{Q_{\varepsilon/4}(z_0)} |u - l_j + Dl_j \cdot (x - x_0)|^2 \, dz
\]

\[
\leq 2 \int_{Q_{\varepsilon/4}(z_0)} |u - l_j|^2 \, dz + 2|Dl_j|^2 \int_{Q_{\varepsilon/4}(z_0)} |x - x_0|^2 \, dz
\]

\[
\leq 2(\theta')^2(1 + |Dl_j|^2)\Psi_j(z_0, \nu, l_j) + 2(\theta')^2|Dl_j|^2
\]

\[
\leq 4(\theta')^2(1 + |Dl_j|^2) \leq (2M)^{2(j+1)}(\theta')^2,
\]

which proves (7.10). Therefore, we conclude from (7.10) and \( l_j(x_0) = (u)_{Q_{\varepsilon/4}(z_0)} \) that for any \( j \geq 0 \) there holds

\[
\int_{Q_{\varepsilon/4}(z_0)} |u - (u)_{Q_{\varepsilon/4}(z_0)}|^2 \, dz \leq (2M)^{2(j+1)}(\theta')^2|Q_{\varepsilon/4}(z_0)| = \alpha_n(2M)^{2(j+1)}(\theta')^{n+1}l_j^{n-\nu},
\]

where \( \alpha_n \) is the volume of the unit ball in \( \mathbb{R}^n \). Next, we set \( \hat{\theta} = (2M)^{2+\nu} \theta \) and this implies \( Q_{\hat{\theta}/4}(z_0) \subset Q_{\nu/4}(z_0) \). Consequently, we infer from (7.8) that

\[
\int_{Q_{\varepsilon/4}(z_0)} |u - (u)_{Q_{\varepsilon/4}(z_0)}|^2 \, dz \leq 2 \int_{Q_{\varepsilon/4}(z_0)} |u - (u)_{Q_{\varepsilon/4}(z_0)}|^2 \, dz \leq 2 \int_{Q_{\varepsilon/4}(z_0)} |u - (u)_{Q_{\varepsilon/4}(z_0)}|^2 \, dz
\]

\[
\leq 2\alpha_n(2M)^2\hat{\theta}^{n+1}l_j^{n-2\alpha}(2M)^{4(2+2\alpha)}\hat{\theta}^{2-2\alpha}\left(2(2M)^{2(j+1)}(\theta')^2\right)^{2-2\alpha}
\]

\[
\leq 2\alpha_n(2M)^2\hat{\theta}^{n+1}l_j^{n-2\alpha}(2M)^{4(2+2\alpha)}\hat{\theta}^{2-2\alpha}.
\]
Finally, for any fixed \( r \in (0, \varrho] \), there exists an integer \( j \in \mathbb{N} \) such that \( \hat{\theta}^{j+1} \varrho < r \leq \hat{\theta}^{j} \varrho \). Therefore, we conclude from the above estimate that

\[
\int_{Q_{r}(z_{0})} |u - (u)_{Q_{r}(z_{0})}|^2 \, dz \leq 2 \int_{Q_{r}(z_{0})} |u - (u)_{Q_{r}(z_{0})}|^2 \, dz \leq 2 \int_{Q_{r}(z_{0})} |u - (u)_{Q_{r}(z_{0})}|^2 \, dz \leq 4 \alpha_{s} (2M)^{2} \varepsilon^{\alpha (n+2+2\alpha)} \leq 4 \alpha_{s} (2M)^{2} \varepsilon^{\alpha (n+2+2\alpha)} \leq c(n, \gamma, M, \alpha, \theta) r^{\alpha (n+2+2\alpha)},
\]

which proves the desired estimate (7.9) for any fixed \( \alpha \in (0, 1) \). This finishes the proof of Theorem 2.2. \( \square \)

**Remark 7.2.** We finally remark that the method in this paper could be improved to address the problem for the full range \( p(z) > \frac{2n}{n+2} \). In the case that \( \frac{2n}{n+2} < p(z_{0}) < 2 \) for a fixed point \( z_{0} \in \Omega_{r} \), we could consider the scaled parabolic cylinder in the following form:

\[
Q_{r}^{(A)}(z_{0}) := B_{\frac{z_{0}}{\varrho}}(z_{0}) \times \Lambda_{r}(t_{0})
\]

where \( p_{0} = p(z_{0}), B_{\frac{z_{0}}{\varrho}}(z_{0}) = \{ x \in \mathbb{R}^n : |x - z_{0}| \leq \frac{z_{0}}{\varrho} \} \) and \( \Lambda_{r}(t_{0}) = (t_{0} - r^{2}, t_{0}) \). We leave the problem of the singular range \( \frac{2n}{n+2} < p(z) < 2 \) for future study.

**Appendix A. Estimate in \( L^p \log^\gamma L \) space**

For the sake of completeness we state in Proposition A.1 an embedding result \( L^{p+\sigma} \hookrightarrow L^p \log^\gamma L \) from [12, Lemma 8.6] which we used in the proof of Lemma 3.5. For completeness sake we also include the proof of this result here.

**Proposition A.1.** Let \( m \geq 2, \gamma > 1 \) and \( \Omega \subset \mathbb{R}^m \) be a measurable set. Let \( \gamma_{2} \geq 1 \) and \( 1 \leq \gamma \leq \gamma_{2} \). Assume that \( p \geq 1 \) and \( f \in L^{p} \log^\gamma L(\Omega) \). For any \( \sigma \in (0, \gamma) \), there exists a constant \( c \) depending only upon \( \gamma \) and \( \sigma \) such that

\[
\left( \int_{\Omega} |f|^{p} \log^\gamma \left( e + \frac{|f|}{\int_{\Omega} |f|^{p} \, dx} \right) \, dx \right)^{\frac{1}{p}} \leq c \left( \int_{\Omega} |f|^{p+\sigma} \, dx \right)^{\frac{1}{p+\sigma}}. \tag{A.1}
\]

**Proof.** Our proof is due to Iwaniec and Verde [12, Lemma 8.6]. We define the left-hand side of (A.1) by

\[
|f|_{L^p \log^\gamma L} := \left( \int_{\Omega} |f|^{p} \log^\gamma \left( e + \frac{|f|}{\int_{\Omega} |f|^{p} \, dx} \right) \, dx \right)^{\frac{1}{p}}
\]

and introduce the Luxemburg norm

\[
\|f\|_{L^p \log^\gamma L} = \inf \left\{ k > 0 : \int_{\Omega} |f|^{p} \log^\gamma \left( e + \frac{|f|}{k} \right) \, dx \leq k^p \right\}. \tag{A.2}
\]

For a fixed \( \alpha \in (0, 1) \), it is easy to verify that the inequality \( \log(1 + x) \leq \frac{1}{\alpha} x^\alpha \) holds for every \( x \geq 0 \). This gives \( \log^\gamma (1 + x) \leq \left( \frac{\gamma}{\alpha} \right)^{\gamma} x^\gamma \). It follows from (A.2) that

\[
\|f\|_{L^p \log^\gamma L} \leq c \left( \int_{\Omega} |f|^{p+\sigma} \, dx \right)^{\frac{1}{p+\sigma}}.
\]
Lemma B.2. \( \delta \) is a map satisfying \( \{f\}_{L^p} \leq c\{f\}_{L^p} \). To this end, we set \( K = \{f\}_{L^p} \) and deduce

\[
[f]_{L^p} \leq c_\gamma \int_{\Omega} |f|^p \log^\gamma \left( e + \frac{|f|}{K} \right) \, dx + c_\gamma \int_{\Omega} |f|^p \log^\gamma \left( e + \frac{K}{\left( \int_{\Omega} |f|^p \, dx \right)^{\gamma}} \right) \, dx
\]

\[
= cK^p + c \log^\gamma \left( e + \frac{K}{\left( \int_{\Omega} |f|^p \, dx \right)^{\gamma}} \right) \int_{\Omega} |f|^p \, dx.
\]

(A.3)

In the case \( \int_{\Omega} |f|^p \, dx \geq K^p \), we infer from (A.3) that

\[
[f]_{L^p} \leq c(\gamma, \sigma)(\int_{\Omega} |f|^p \, dx)^{\frac{1}{\gamma}}
\]

and (A.1) follows by the Hölder’s inequality. In the case \( \int_{\Omega} |f|^p \, dx < K^p \), we observe that the estimate \( \log^\gamma (e + x) \leq c(\gamma) x \) holds for any \( x \geq 1 \). It follows from (A.3) that \( [f]_{L^p} \leq c(\gamma) K^p \). We have thus proved the proposition. \( \Box \)

Appendix B. A-caloric approximation

In this section, we present a version of the A-caloric approximation that is compatible with the intrinsic geometry of non-standard growth. Our proof follows in a similar manner as the proofs of [8, Lemma 3.2] and [8, Lemma 2.8], and we just sketch the proof. We follow the terminology used in [9, Chapter 3] and introduce the definition of A-caloric map.

Definition B.1. Let \( \lambda, \Lambda > 0 \) be fixed constants and \( A : \mathbb{R}^{n \times N} \to \mathbb{R}^{n \times N} \) be a bilinear form which satisfies

\[
\lambda|\tilde{u}|^2 \leq \langle A\tilde{u}, \tilde{u} \rangle, \quad |\langle Aw, \tilde{w} \rangle| \leq \Lambda|w||\tilde{w}|, \quad w, \tilde{w} \in \mathbb{R}^{N \times n}.
\]

A map \( f \in L^2(t_0 - \varrho^2, t_0; W^{1,2}(B_\varrho(x_0)), \mathbb{R}^N) \) is called A-caloric in the parabolic cylinder \( Q_\varrho(z_0) \) if it satisfies

\[
\int_{Q_\varrho(z_0)} \left[ f \cdot \partial_t \varphi - \langle ADf, D\varphi \rangle \right] \, dz = 0, \quad \text{whenever } \varphi \in C_0^\infty(Q_\varrho(z_0); \mathbb{R}^N).
\]

We are now in a position to state our main result of this section.

Lemma B.2. Let \( \varepsilon > 0 \) and \( 0 < \lambda \leq \Lambda \) be fixed. Then, for any \( p \in [2, \gamma] \) there exists \( \delta_0 = \delta_0(n, N, \gamma_2, \lambda, L, \varepsilon) \in (0, 1] \) with the following property: Whenever \( \gamma \in (0, 1] \) and \( A \) is a bilinear form on \( \mathbb{R}^{N \times n} \) satisfying (B.1) and whenever \( w \in L^p(t_0 - \varrho^2, t_0; W^{1,2}(B_\varrho(x_0)), \mathbb{R}^N) \) is a map satisfying

\[
\sup_{t_0 - \varrho^2 < s < t_0} \int_{B_\varrho(s)} \frac{|w|^2}{\varrho^2} \, dx + \int_{Q_\varrho(z_0)} |Dw|^2 \, dz + \int_{Q_\varrho(z_0)} \gamma^{p-2} \left( \frac{|w|^p}{\varrho^2} + |Dw|^p \right) \, dz \leq 1
\]

which is approximatively A-caloric in the sense that

\[
\left| \int_{Q_\varrho(z_0)} w \cdot \partial_t \varphi - \langle ADw, D\varphi \rangle \, dz \right| \leq \delta \sup_{Q_\varrho(z_0)} |D\varphi|
\]

(B.2)

for every \( \varphi \in C_0^\infty(Q_\varrho(z_0), \mathbb{R}^N) \), where \( \delta \leq \delta_0(n, N, \gamma_2, \lambda, \Lambda, \varepsilon) \). Then there exists a map \( f \in C^\infty(Q_\varrho(z_0), \mathbb{R}^N) \) satisfying

\[
\int_{Q_\varrho(z_0)} \left[ f \right]^2 \, dz + \int_{Q_\varrho(z_0)} \gamma^{\gamma-2} \left( \frac{|f|^p}{\varrho^2} + |Df|^p \right) \, dz \leq c(n, \gamma_2, \lambda, \Lambda)
\]
At this point, we choose $q$. In the remainder of the proof, we will construct a sequence of $w_k$ with a constant $\tilde{\kappa}$ bilinear forms $A$ and $B$, the map $\tilde{\phi}$ is approximately $A_k$-caloric in the sense that

$$
\left| \int_{Q_{1/2}(y)} w_k \cdot \partial_\nu \phi - \langle A_k D w_k, D \phi \rangle \, dz \right| \leq \frac{1}{k} \sup_{Q_1} |D \phi|
$$

for all $\phi \in C_0^\infty(Q_1, \mathbb{R}^N)$ and satisfies

$$
\sup_{-1 < r < 0} \int_{B_1} |w_k(r, t)|^2 \, dt + \int_{Q_1} |D w_k|^2 \, dz + \int_{Q_{1/2}} \gamma_k^{p_2-2} (|w_k|^{p_2} + |D w_k|^{p_2}) \, dz \leq 1,
$$

while for all $A_k$-caloric maps $f \in C^\infty(Q_{1/2}, \mathbb{R}^N)$ satisfying

$$
\int_{Q_{1/2}} 4|f|^2 + |D f|^2 \, dz + \int_{Q_{1/2}} \gamma_k^{p_2-2} (2^{p_2}|f|^{p_2} + |D f|^{p_2}) \, dz \leq c_*,
$$

with a constant $c_* = c_*(n, N, \gamma_2, \lambda, \Lambda)$, there holds

$$
\int_{Q_{1/2}} 4|w_k - f|^2 \, dz + 2^{p_2} \gamma_k^{p_2-2} |w_k - f|^{p_2} \, dz \geq \varepsilon.
$$

In the remainder of the proof, we will construct a sequence of $A_k$-caloric maps $w_k$ such that (B.5) holds for a constant $c_*>0$, but (B.6) fails to hold, which leads to a contradiction.

Step 2: We discuss uniform bounds and weak convergence for $w_k$. To this end, we set $\tilde{w}_k \equiv \gamma_k^{(p_2-2)/p_2} w_k$ and it follows from (B.4) that

$$
\int_{Q_1} |\tilde{w}_k|^{p_2} + |D \tilde{w}_k|^{p_2} \, dz \leq 1.
$$

At this point, we choose $q \geq 2$ with $q \leq p_k \leq q(1 + \frac{2}{n})$ holds for all $k \geq 1$. By the Hölder’s inequality, we have

$$
\int_{Q_1} |\tilde{w}_k|^q + |D \tilde{w}_k|^q \, dz \leq 1.
$$

(B.7)

In view of $p_k \in [2, \gamma_2]$ and $\gamma_k \in (0, 1]$, we see that $\gamma_k^{\frac{2}{p_k}} \leq 1$ for all $k \in \mathbb{N}$. In the case $p_k \to 2$ and $\gamma_k \to 0$, the limitation of $\gamma_k^{\frac{2}{p_k}}$ may not exists, but the boundedness of $\gamma_k^{\frac{2}{p_k}} \leq 1$ ensures the existence of a not relabeled-subsequence $\{\gamma_k^{\frac{2}{p_k}}\}_{k \in \mathbb{N}}$ such that $\gamma_k^{\frac{2}{p_k}} \to \mu$ for a constant $\mu \in [0, 1]$. Moreover, from (B.4) and (B.7), we infer the existence of maps $w \in L^2(-1, 0; W^{1,2}(B_1, \mathbb{R}^N))$, $\tilde{w} \in L^q(-1, 0; W^{1,q}(B_1, \mathbb{R}^N))$, a bilinear form $A$ and constants...
\( (\gamma, p, \mu) \) such that

\[
\begin{align*}
  w_k \to w & \quad \text{weakly in } L^2(Q_1, \mathbb{R}^N), \\
  Dw_k \to Dw & \quad \text{weakly in } L^2(Q_1, \mathbb{R}^{Nn}), \\
  \tilde{w}_k \to \tilde{w} & \quad \text{weakly in } L^p(Q_1, \mathbb{R}^n), \\
  D\tilde{w}_k \to D\tilde{w} & \quad \text{weakly in } L^q(Q_1, \mathbb{R}^{Nn}), \\
  A_k \to A & \quad \text{as bilinear forms in } \mathbb{R}^{N\times n}, \\
  \gamma_k \to \gamma & \quad \text{in } [0, 1], \\
  p_k \to p & \quad \text{in } [2, \gamma_2], \\
  \gamma_k^{\frac{\mu}{\gamma}} \to \mu & \quad \text{in } [0, 1].
\end{align*}
\]

We observe from (B.8) \( _1 \) and (B.8) \( _8 \) that \( \tilde{w}_k \to \mu w \) weakly in \( L^2(-1, 0; W^{1,2}(B_1, \mathbb{R}^N)) \). On the other hand, (B.8) \( _3 \) and (B.8) \( _4 \) imply \( \tilde{w}_k \to \tilde{w} \) weakly in \( L^q(-1, 0; W^{1,q}(B_1, \mathbb{R}^N)) \). It is clear that \( \tilde{w} = \mu w \). By lower semicontinuity, we infer from (B.4) and (B.7) that

\[
\int_{Q_1} |w|^2 + |Dw|^2 + |\tilde{w}|^q + |D\tilde{w}|^q \, dz \leq 1. \tag{B.9}
\]

Furthermore, we follow the arguments in \cite{9}, page 28] to infer that the limit map \( w \) is an \( A \)-caloric map. More precisely, we have

\[
\int_{Q_1} [w \cdot \partial \varphi - \langle ADw, D\varphi \rangle] \, dz = 0,
\]

for any \( \varphi \in C^\infty_0(Q_1, \mathbb{R}^N) \). This also implies that \( w \in C^\infty(Q_1, \mathbb{R}^N) \).

Step 3: We aim to establish the strong convergence of \( w_k \) by using compactness method. To start with, we follow the arguments in \cite{9}, page 28-29] to deduce that for any \( h \in (0, 1) \) there holds

\[
\int_{-h}^h \|w_k(\cdot, t) - w_k(\cdot, t + h)\|^2_{W^{1,2}(Q_1, \mathbb{R}^N)} \, dt \leq c(n, L, \ell)(h + \frac{1}{k^2}),
\]

where \( l > \frac{\gamma-2}{\gamma} \) is fixed. This inequality together with a compactness result (cf. \cite{9}, Theorem 2.5]) implies that \( w_k \to w \) strongly in \( L^2(Q_1, \mathbb{R}^N) \). Moreover, since

\[
\|\tilde{w}_k - \tilde{w}\|_{L^2(Q_1, \mathbb{R}^N)} \leq \|w_k - w\|_{L^2(Q_1, \mathbb{R}^N)} + \|w\|_{L^2(Q_1, \mathbb{R}^N)}(\gamma_k^{\frac{\mu}{\gamma}} - \mu),
\]

we have \( \tilde{w}_k \to \tilde{w} \) strongly in \( L^2(Q_1, \mathbb{R}^N) \). Next, we claim that

\[
\tilde{w}_k \to \tilde{w} \quad \text{strongly in } L^\sigma(Q_{1/2}, \mathbb{R}^N), \tag{B.10}
\]

for some \( \sigma > q \). To prove the claim (B.10), we choose \( \sigma > 0 \) such that \( q < p < \sigma < q(1 + \frac{1}{n}) \) and set \( U_k = \tilde{w}_k - \tilde{w} \). By the Gagliardo-Nirenberg inequality (cf. \cite{8}, Lemma 2.5]), we deduce

\[
\int_{Q_{1/2}} |U_k|^\sigma \, dz \leq c \int_{-\sigma}^0 \left( \int_{B_{1/2}} |U_k|^q + |D\tilde{w}|^q \, dx \right)^{\frac{\sigma}{q}} \left( \int_{B_{1/2}} |\tilde{w}|^2 \, dx \right)^{\frac{\sigma(q-\gamma)}{q}} \, dr.
\]
Recalling that $\tilde{w}$ and therefore (B.10) holds true.

This together with (B.13) for $k$ sufficiently large. For $p < \sigma' < q(1 + \frac{2}{n})$, the estimate (B.12) yields that

$$\gamma_k^{\frac{p}{n-2}} w_k \rightarrow \tilde{w} \quad \text{strongly in } L^\sigma(Q_{1/2}, \mathbb{R}^N).$$

Moreover, we conclude from (B.10) and (B.13) that

$$\int_{Q_{1/2}} |w_k - v_k|^{p_n} \, dz \leq c \left( \int_{Q_{1/2}} \tilde{w}_k^{\frac{p}{n-2}} \, dz \right)^{\frac{p}{n}} + c \left( \int_{Q_{1/2}} \tilde{w} - \tilde{w}_k^{\frac{p}{n-2}} v_k \, dz \right)^{\frac{p}{n}} \rightarrow 0.$$

This together with (B.13) yields

$$\lim_{k \to \infty} \int_{Q_{1/2}} 4|w_k - v_k|^2 + 2|Dw_k|^2 \, dz = 0,$$

since $w_k \to w$ strongly in $L^2(Q_1, \mathbb{R}^N)$. Finally, recalling that $\gamma_k^{\frac{p}{n-2}} \leq 1$, we use (B.12) to deduce that

$$\lim_{k \to \infty} \int_{Q_{1/2}} 4|w_k|^2 + |Dw_k|^2 + \gamma_k^{\frac{p}{n-2}}(2^{p_1}|v_k|^{p_1} + |Dv_k|^{p_1}) \, dz \leq \hat{c}(n, N, \gamma, \lambda, \Lambda).$$
Therefore, the limit \( c^* \geq \tilde{c}(n, N, \gamma_2, \lambda, \Lambda) \) is contrary to (B.6). This leads to a contradiction for \( k \) sufficiently large. We have thus proved the lemma. \( \square \)

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