NORMS ON CATEGORIES AND ANALOGS OF THE
SCHRÖDER-BERNSTEIN THEOREM

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ABSTRACT. We generalize the concept of a norm on a vector space to one of a
norm on a category. This provides a unified perspective on many specific mat-
ters in many different areas of mathematics like set theory, functional analysis,
measure theory, topology, and metric space theory. We will especially address
the two last areas in which the monotone-light factorization and, respectively,
the Gromov-Hausdorff distance will naturally appear.

In our formalization a Schröder-Bernstein property becomes an axiom of
a norm which constitutes interesting properties of the categories in question.
The proposed concept provides a convenient framework for metrizations.

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1. Introduction

Norms form a corner stone of any quantitative research and belong to the foundational notions of most areas of mathematics. The subject of this paper is the generalization of the concept of a norm on a vector space or, more generally, on an abelian group to one on a category.

Our attempt is preceded by many approaches using enriched monoidal categories going back to Lawvere [Law73] and an industry arising out of this [Gra07; Nee20]. Independently, Ghez, Lima, and Roberts [GLR85] suggested a notion of normed *-category. Recently, interleaving distance has gained some favour as a playing ground [Sco20; BSS18]. Another recent approach outside of enriched or monoidal category theory is by Kubiš [Kub17] and Perrone [Per23].

Our motivation in the present work is threefold, aspiring a framework

1. to find analogs of the Cantor-Schröder-Bernstein theorem (CSB theorem) from set theory in other categories,
2. for systematic and convenient metrization of families of equivalence classes of spaces, like the Gromov-Hausdorff space, moduli spaces, and representation spaces,
3. to prove general theorems in the developed categorical framework that give insights and concrete useful applications in many different areas of mathematics,
4. to work with categories with large classes of morphisms.

The guiding example of our approach is the following norm on the category of sets: To each function $f: X \to Y$ assign the non-negative extended real number

\begin{equation}
\|f\|_{\text{inj}} := \sup_{x \in X} \log(\# \{ y \mid f(x) = f(y) \})
\end{equation}

where $\#$ assigns to a set the numbers of its elements (a member of $\{0, 1, \ldots, \infty\}$) and $\sup a f(x) = \sup \{a \mid f(x) \mid x \in X\}$. Note that $f$ is injective if and only if $\|f\|_{\text{inj}} = 0$. This is to say that $\| \cdot \|_{\text{inj}}$ is measuring the deviation from being injective. Hence the idea for the generalization of the CSB theorem is that in a normed category $(C, \| \cdot \|)$ two objects $X$ and $Y$ are isomorphic as soon as there are morphisms $f: X \to Y$ and $g: Y \to X$ such that $\|X\| = \|Y\| = 0$.

Returning to our list of motivations, we mention with regard to the first one that the CSB theorem is a fundamental theorem in set theory stating that there is a bijection between two sets as soon as there are injective maps between the sets both ways. In conceptual terms it states that if $X$ can be embedded into $Y$ and vice-versa, then $X$ and $Y$ are isomorphic. A direct formalization of this conceptual idea in category theoretic terms would be the property that in a category $C$ two objects $X, Y \in C$ are isomorphic as soon as there are monomorphisms $X \to Y$ and $Y \to X$. But, unfortunately, most categories considered in practice do not have this property, cf. [Laa10]. Though, there are some notable exceptions, which include measure spaces [Sri98, § 3.3] and a noncommutative version thereof for von Neumann algebras [KR97, Proposition 6.2.4]. Efforts to generalize the CBS theorem in alternative set-ups have recently revived, including results for categories of universal algebras [Fre19] and in homotopy type theory and boolean $\infty$-topoi [Esc20b] including a formalization in Agda [Esc20a]. Of course, one can also weaken the property by replacing “monomorphism” by some stronger or related notion of morphism. In our approach it will be a morphism with vanishing norm.
As for the second motivation, the problems with doing metrizations in practise are often that they become very technical, involve arbitrary choices, and basic properties like the triangle inequality or completeness become hard to check. The theory of uniform structure can be seen as an attempt to abstract from these choices, but it lacks a measure of the size of entourages. In many examples we present it will turn out that the norms can be defined in terms of a capacity, a quantity measuring the size of a subobject of an object \( X \in C_0 \). It is natural to proceed by a category theoretical approach by looking at many examples: representatives of a point (i.e. equivalence classes of spaces) are objects of a category and morphisms are comparison maps.

As for the last motivation, note that when working in some area of mathematics using the language of category theory one often has to limit the class of morphisms under consideration. For instance the category of metric spaces is normally defined with morphisms to be non-expansive maps in order to guarantee nice properties like existence of limits.

1.1. Organization of the paper. As for notation the reader is invited to consult appendix \( A \). We start the paper by introducing the basic definitions of seminorm, norm, and constructions based thereon (cf. Section 2). A categorically minded reader may enjoy complementing this perspective by a 2-categorical point of view elaborated in appendix \( B \). Acquaintance with these definitions is facilitated in Section 3 by a plethora of examples. Thereafter we turn to the notation of capacities in Section 4 that will be the convenient framework for most seminorms under investigation in this paper. On this occasion we will also provide a first overview of seminorm central to our investigation in Subsection 4.3.

Section 5 addresses the category of topological spaces. We will introduce a seminorm \( \| \cdot \|_{\text{top}} = \| \cdot \|_{\text{comp}} + \| \cdot \|_{\text{dim}} \), where \( \| \cdot \|_{\text{comp}} \) measures the increase in the number of connected components when pulling back a subset of the codomain to a subset of the domain and \( \| \cdot \|_{\text{dim}} \) measures the increase in dimension under such a transition. In other terms these seminorms measure the deviation of a map from being monotone or light respectively. The classic monotone-light factorization theorem implies that, when restricting to compact metrizable spaces, \( \| \cdot \|_{\text{top}} \) is a norm.

The subject of Section 6 is the category of compact metric spaces \( \text{Met} \) endowed with all multi-valued set theoretic maps as morphisms. The notion of dilatation gives rise to the seminorm

\[
\| f \|_{\text{diam}} = \sup \{ d_M(x, x') - d_N(y, y') \mid x, x' \in M, y \in f[x], y' \in f[x'] \}.
\]

Generalizing a classical theorem of Fendelthal and Hurewicz we show that \( \| f \|_{\text{diam}} \) is a norm on \( \text{Met} \). Moreover the metric \( d_{\text{diam}}^+ \) induced by this norm is almost the Gromov-Hausdorff distance \( d_{\text{GH}} \); to be precise the identity map

\[
(\{ \text{isometry classes of compact metric spaces} \}, d_{\text{GH}}) \to (\{ \text{isometry classes of compact metric spaces} \}, d_{\text{diam}}^+)
\]

is 2-Lipschitz with Cauchy continuous inverse.

1.2. Future work. The next steps in our investigation outline a follows:

- generalize Lemma 22—which goes back to Freudenthal and Hurewicz—to categories with a capacity. An assumption will be that for every \( X \in C_0 \) there is a map \( N \to \text{Sub}(X) \) compatible with \( c: \text{Sub}(X) \to [0, \infty] \).
• find various applications thereof, e.g. to the category of metric measure spaces.
• starting with a normed category $\mathcal{C}$ define a normed completion of the category consisting ind-objects and norm-converging pro-ind-limits of morphisms: Before taking a limit we take the under category $T/\mathcal{C}$—in order to fix a base point. Then objects are defined by some directed index category $I$ and a morphism $\vec{X}: I \to T/\mathcal{C}$ satisfying the Cauchy condition (compare [Kub17, Def. 3.3]):

$$\forall \varepsilon > 0: \exists i_\varepsilon: \forall i, i' \geq i_\varepsilon: \|X_i \to X_{i'}\|, \varepsilon.$$ 

As a motivation why this Cauchy condition is impose consider, for instance, in the case of the Lipschitz norm (13) that the diagram $X_n = ([0, 1], \frac{1}{n}d_{[0, 1]}), X_{n \to m} = \text{id}_{[0, 1]}$ should be ruled out as an object in the completion since this would be only a pseudometric space leading to division by zero in (13).

Finally, the set of morphisms between $\vec{X}: I \to T/\mathcal{C}$ and $\vec{Y}: J \to T/\mathcal{C}$ is given by all $f_{ij} \in \mathcal{C}[\vec{X}, \vec{Y}] = \lim_{i \to j} \mathcal{C}[X_i, Y_j]$ such that $\limsup_{i \to j} \|f_{ij}\|_{\mathcal{R}} \leq \limsup_{i \to j} \|f_{ij}\| < \infty$. This condition corresponds to Kubis’s axiom (N3) (cf. remark 3).

• Define a norm on this category by means of a Choquet style integral: For a directed set $I = (I, \leq)$ and an order preserving function $F: I \to [0, 1]$, thought of as the distribution of a probability measure, set for $f \in \mathcal{C}[\vec{X}, \vec{Y}]$

$$\int f(i) d\hat{F} := \int 1 - F(\sup \{ i \mid f(i) \leq t \}) dt$$

where $f(i) := \inf \{ \|g\| \mid g \in \mathcal{C}[X_i, Y_j] \text{ with } \iota_{ij}(g) = \text{pr}_i f \}$

where $\iota_{ij}$ is the universal map $\mathcal{C}[X_i, Y_j] \to \text{colim}_{j \in I} \mathcal{C}[X_i, Y_j]$. In the example ($\text{MET}, \| \cdot \|_{\text{diam}}$) of metric spaces this corresponds to the pointed Gromov-Hausdorff distance.

• Generalize the notion of a normed category to 2-categories: Require the norm to be a 2-morphism and weaken (N3) by requiring only being isomorphic up to 2-morphisms. This generalization should for instance capture coarse structure.

1.3. Versions of this paper. Changes from version 2 to version 3:

• Lemma 11 added.
• Approach to compact metric spaces and definition of $\| \cdot \|_{\text{diam}}$ changed (formerly confusingly denoted by $\| \cdot \|_{\text{dil}}$).
• Subsection 3.3, example of automatons, added.

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2. Definitions

Definition 1. A seminorm on a category $\mathcal{C}$ is a function $\| \cdot \|: \mathcal{C} \to [0, \infty]$ such that

(N1) $\|\text{id}_X\| = 0$ for every object $X \in \mathcal{C}_0$, 

(MET, $\| \cdot \|_{\text{diam}}$) of metric spaces this corresponds to the pointed Gromov-Hausdorff distance.
The tuple \((C, \| \cdot \|)\) is called a \textit{seminormed category}. 

Note that in the literature a seminorm is often called a norm \cite{Law73,Gra07}. Moreover, note that we do not require the obvious strengthening of (N1), namely that the seminorm of every categorical isomorphism vanishes. An explanation how to view this as a generalization of a seminorm on a vector space is found in Subsection 3.5. An isomorphism \(f: X \rightarrow Y\) with inverse \(g: Y \rightarrow X\) is called a \textit{norm isomorphism} if \(\|f\| = \|g\| = 0\). By (N2) being norm isomorphic is an equivalence relation. Moreover any morphism with norm zero is called a \textit{modulator}. Often the category with objects \(C_0\) and all modulators of \(C_1\) as morphisms has good categorical properties. We will denoted it by \(\mathcal{M}(C, \| \cdot \|)\). Two seminormed categories are called \textit{isomorphic} if there is a norm preserving categorical isomorphism between them. A seminorm or norm, respectively, induces a seminorm or norm, resp., on the opposite category in the obvious way.

\textbf{Definition 2.} A seminorm is called a \textit{norm} if for all objects \(X, Y\) the following holds

\begin{enumerate}
\item[(N3)] if there are modulators \(f: X \rightarrow Y\) and \(g: Y \rightarrow X\), then \(X\) and \(Y\) are norm isomorphic; and
\item[(N4)] if for all \(\varepsilon > 0\) there is \(f: X \rightarrow Y\) with \(\|f\| \leq \varepsilon\), then there is a modulator \(f: X \rightarrow Y\).
\end{enumerate}

The way to view (N3) is that a form of CSB theorem holds. The moral idea is that \(\|f\| = 0\) is a property that is stronger than being monic and \(\| \cdot \|\) measures the deviation from this property.

2.1. \textbf{Induced norms and distances.} Let \((C, \| \cdot \|)\) be a seminormed category. We define the \textit{left dual seminorm} as

\begin{equation}
\|f\|^{*L} := \sup_{f'} \left( \|f'\| - \|f' \circ f\| \right),
\end{equation}

where \(X' \xrightarrow{f'} X \xrightarrow{f} Y\), and the \textit{right dual seminorm} as

\begin{equation}
\|f\|^{*R} := \sup_{f''} \left( \|f''\| - \|f: f''\| \right)
\end{equation}

where \(X \xrightarrow{f} Y \xrightarrow{f''} Y'\). The seminorm \(\| \cdot \|\) is called \textbf{left reflexive} if \(\| \cdot \|^{*L+L} = \| \cdot \|\) and \textbf{right reflexive} if \(\| \cdot \|^{*R+R} = \| \cdot \|\). As opposed to the case of normed spaces, the dual in our case does not define an entirely new category but merely a new norm on the same category.

To check that the left dual and right dual seminorms are actually seminorms observe for (N1) that \(\|\text{id}\|_{*L} = \sup_{f'} \|f'\| - \|f' \circ \text{id}\| = 0 = \sup_{f'} \|f'\| - \|\text{id} \circ f'\| = \|\text{id}\|_{*R}\).

We show that (N2) holds for left duals and then apply that in the opposite category to show that it holds for right duals. To this end, observe that for any diagram
Remark. These arguments transfer to the right dual by the fact that the seminorm induced by the right dual on the opposite category coincides with the left dual of norm induced on the opposite category by the original norm.

3.1. Sets. On the category \text{SET} of sets we define for a function \( f : X \to Y \) the norm \( \| \cdot \|_{\text{set}} \) measuring the deviation of a function from being injective: we set

\[
\| f \|_{\text{set}} = \log \sup_{y \in Y} \# f^*(\{y\}).
\]

Note the foundational remark in the introduction. In many examples \( \text{sk}(\text{C}, \| \cdot \|) \) admits a set of representatives.
We check that \(\|\cdot\|_{\text{set}}\) is a norm. For the seminorm properties observe that \(\|\text{id}_X\|_{\text{set}} = 0\) for any set \(X\). Moreover the triangle inequality is satisfied as it holds trivially whenever \(\|f\|_{\text{set}} = \infty\) or \(\|g\|_{\text{set}} = \infty\) and otherwise—using (5)—

\[
\|f : g\|_{\text{set}} = \log\sup_{z \in Z} \#(f : g)^*\{z\} \\
= \log\sup_{z \in Z} \# \big\{ f^*(\{y\}) \mid y \in g^*\{z\} \big\} \\
\leq \log \left( \sup_{y \in Y} \# f^*(\{y\}) \cdot \sup_{z \in Z} \# g^*\{z\} \right) \\
= \|f\|_{\text{set}} + \|g\|_{\text{set}}.
\]

Hence \(\|\cdot\|_{\text{set}}\) is a seminorm. We continue with the norm properties: \(\|f\|_{\text{set}} = 0\) is to say that \(f\) is injective. Property (N3) is exactly the Schröder-Bernstein theorem. The property (N4) is trivial since the image of \(\|\cdot\|_{\text{set}}\) is discrete.

3.2. Simplicial complexes. We can use the same norm \(\|\cdot\|_{\text{set}}\), from the previous subsection, on simplicial complexes. Recall that an (unoriented) simplicial complex on a set \(V\) is a pair \((V,X)\) where \(X\) is a subset of \(\mathcal{P}(V)\) such that

1. each \(s \in X\) is finite and nonempty,
2. \(\{x\} \in X\) for each \(x \in V\), and
3. \(s' \in X\) for all \(s' \subseteq s \in X\) and \(s' \neq \emptyset\).

The elements of \(V\) are called vertices. A set \(s \in X\) is called simplex, and \(n\)-simplex if \(s\) has \(n\) elements. Further let \(X_n\) denote the set of all \(n\)-simplices in \(X\). A simplicial complex is called finite whenever it is finite as a set. A morphism of simplicial complexes is a function \(f: V \rightarrow W\) such that \((f)_s(X) \subseteq Y\). Let \(\text{SimpCplx}\) denote the category of simplicial complexes and morphisms of simplicial complexes. Let \(\text{FinSimpCplx}\) denote the full subcategory of finite simplicial complexes, i.e. simplicial complexes with a finite set of vertices (or equivalently a finite set of simplices).

**Proposition 4.** The seminormed category \((\text{FinSimpCplx}, \|\cdot\|_{\text{set}})\) is normed.

**Proof.** The axiom (N4) is trivial since the image of \(\|\cdot\|_{\text{set}}\) is discrete. For (N3) assume that \((V,X), (W,Y)\) are two simplicial complexes and \(f: V \rightarrow W, g: W \rightarrow V\) morphisms of simplicial complexes with \(\|f\|_{\text{set}} = \|g\|_{\text{set}} = 0\). Thus \(f\) and \(g\) are injective. Hence \(f_\ast\) and \(g_\ast\) are injective. This is to say that both functions map \(n\)-simplices to \(n\)-simplices for every \(n\). Hence both complexes have the same number of \(n\)-simplices for every \(n\). Since for each \(n\) the set of simplices \(X_n\) is finite and \(f_\ast\) is injective, the map \(f_\ast\) is actually bijective. Hence both simplicial complexes are norm isomorphic. \(\square\)

3.3. Automatons. By an automaton we understand a triple \(M = (S,A,\delta)\) where \(S\) and \(A\) are sets and \(\delta\) is a relation on \(S \times A \times S\). The set \(S\) is interpreted as states; \(A\) as action and \(\delta\) as a (non-deterministic) transition rule.

This gives naturally rise to the following normed category:

\[
M_0 := S ; \\
M_0[s, t] := \left\{ \text{words } a_1 \ldots a_n \text{ in } A \mid (s_{0}, a_{0}, s_{1}, a_{1}, \ldots, a_{n-1}, a_{n}, s_{n}) \in \delta \right\} ,
\]
note that $M[n,s]$ always contains the empty word by choice $s_0 = s$;

$$a_1 \ldots a_n ; b_1 \ldots b_m := a_1 \ldots a_n b_1 \ldots b_m;$$

$$\|a\|_M := \text{length}(a).$$

### 3.4. Cost functions and polarization

Let $c$ be a cost function on $M$, i.e. a map $c: \text{Conf}_2 M \to [0, \infty]$ where the configuration space $\text{Conf}_2(M)$ is defined as \{$(x, y) \in M \times M \mid x \neq y$\}. Cost functions form the corner stone of transportation theory [Vil08]. Remember that a square-free word is a word in which no pattern of the form $xx$ occurs. Define the normed category $(\underline{M}, \| \cdot \|_c)$ by

\[
\underline{M}_0 := M,
\]

\[
\underline{M}_1 := \{\text{square-free words over the alphabet } M\},
\]

\[
\underline{M}[x, y] := \{w \in \underline{M}_1 \mid w \text{ starts with } x \text{ and ends with } y\}
\]

\[
(x \xi_1 \ldots \xi_n y); (y \eta_1 \ldots \eta_m z) := (x \xi_1 \ldots \xi_n y \eta_1 \ldots \eta_m z)
\]

\[
\text{id}_x := (x),
\]

\[
\|(\xi_1 \ldots \xi_n)\|_c := c(\xi_1, \xi_2) + \ldots + c(\xi_{n-1}, \xi_n)
\]

for $(\xi_1 \ldots \xi_n): x \to y$. This obviously defines a seminorm since the triangle inequality (N2) is even an equality.

Moreover this construction induces a qmetric on $M$, namely $d_c := \| \cdot \|_c$. This is automatically a metric since any path from $x$ to $y$ can be transformed into a morphism from $y$ to $x$ of the same length by reversing. We have $d_c \leq c$ and equality holds if and only if the extension of $c$ to $M \times M$ by $0$ is a metric. In other words, $d_c$ is the largest pseudometric bounded by $c$.

Now assume that $M = (M, d_M)$ is a metric space. Then $(\underline{M}, \| \cdot \|_c)$ with $c(x, y) = d_M(x, y)$ is a normed category: for (N3) observe that a morphism $w = (\xi_1 \ldots \xi_n): x \to y$ has vanishing norm only if $|x \xi_1| = |\xi_1 \xi_2| = \ldots = |\xi_{n-1} \xi_n| = |\xi_n y| = 0$ and, hence, $x = \xi_1 = \ldots = \xi_n = y$. Hence $w$ is a word of length one. For (N4) observe that the norm of any morphism $x \to y$ is bounded from below by $|x y|$. All norm isomorphism classes are singletons.

The left and right duals of $\| \cdot \|_c$ vanish. Both cases are parallel. In case of the right dual—for instance—observe that any composition $f; f'$ has norm at least $\|f'\|$. Hence the norm $\| \cdot \|_c$ is only reflexive in the trivial case that $d_M$ is the vanishing distance.

### 3.5. Grothendieck norm for monoids

Let $M = (M, +, 0)$ be a (not necessarily commutative) monoid. Further let $\| \cdot \|_M$ be a seminorm on $M$, i.e. a map $M \to [0, \infty]$ such that $\|a + b\| \leq \|a\| + \|b\|$. In the spirit of the previous example we define the category $\underline{M}$ and the Grothendieck seminorm $\| \cdot \|$ on $\underline{M}$ by

\[
\underline{M}_0 := M
\]

\[
\underline{M}[a, b] := \{(f_+, f_-) \in M^\times 2 \mid f_+ + a = b + f_- \}
\]

\[
(f_+, f_-); (g_+, g_-) := (g_+ + f_+, g_- + f_-)
\]

for $(f_+, f_-): a \to b$ and $(g_+, g_-): b \to c$ (well-defined since $g_+ + f_+ + a = g_+ + b + f_- = c + g_- + f_-$)

\[
\text{id}_a := (0, 0) \quad \text{for all } a \in \underline{M}_0
\]

\[
\|(f_+, f_-)\| := \|f_+\|_M + \|f_-\|_M.
\]
This defines indeed a seminorm since \( \|(f_+, f_-); (g_+, g_-)\| = \|(g_+ + f_+, g_- + f_-)\| \leq \|g_+\|_M + \|f_+\|_M + \|g_-\|_M + \|f_-\|_M = \|(f_+, f_-)\| + \|(g_+, g_-)\| \).

If \( \| \cdot \|_M \) is a norm (i.e. \( \|a\|_M = 0 \) if and only if \( a = 0 \)) then \( (M, \| \cdot \|) \) is a normed category. For all \( (f_+, f_-) \in M \), we have that \( \|(f_+, f_-)\| = 0 \) implies \( f_+ = f_- = 0 \). Thus in this case \( \| \cdot \| \) fulfills (N3) on the category \( M \). For (N4) observe that \( \|f\| < \varepsilon \) implies that \( \|a-b\| = \|f_- - f_+\| \leq \|f_-\| + \|f_+\| < \varepsilon \). Thus is for \( a \) and \( b \) and every \( \varepsilon > 0 \) there is \( f : a \to b \) with \( \|f\| < \varepsilon \), then \( \|a-b\| = 0 = \varepsilon \). Since \( a \) and \( b \) we arbitrary, \( \| \cdot \| \) fulfills (N4). Thus \( \| \cdot \| \) is a norm.

**Proposition 5.** Assume that \( G \) is a group with a seminorm \( \| \cdot \|_G \). Then

(i) a normed category canonically isomorphic to \( (G', \| \cdot \|) \) is defined by

\[
G'_a := G, \\
G'[a, b] := G, \\
f \mapsto g := g - b + f, \quad \text{for } f : a \to b \text{ and } g : b \to c, \\
\|f\|' := \|f\|_G + \|-b + f + a\|_G \quad \text{for } f : b \to c;
\]

(ii) assuming that \( \|a\|_M = \| -a\|_M \), we have \( d_{a,b} \| (a, b) = \| -b + a\|_G \);

(iii) under the same assumption we have \( \| \cdot \|^{*L} = \| \cdot \|^{*R} = \| \cdot \| \).

**Proof.** For claim (i) we will show that the isomorphism and its inverse are given by the reparameterization functors \( F \) and \( F \) that are the identity on the objects set \( G \) and on morphisms given by the assignments

\[
F_1 : (f_+, f_-) \mapsto f_+ \quad \text{and} \quad F_1 : f \mapsto (f, -b + f + a)
\]

Both maps are inverse to each other as \( f_- = -b + (f_+ + a) = F_1(f_+). \) Hence we only have to check compatibility of \( F \) and \( ' : F(f_+, f_-) \mapsto F(g_+, g_-) = g_+ + f_+ = F((f_+ + f_-, g_+, g_-)) \) for a \( (f_+ + f_-, g_+, g_-) \). This concludes the proof that \( G' \)

defines a category isomorphic to \( G \). The equality of the norms \( \| \cdot \| \) and \( \| \cdot \|' \) follows directly from the identity \( f_- = -b + (f_+ + a) \).

For claim (ii) observe that for every two objects \( a, b \in G \) we have \( d_{a,b} \| (a, b) \leq \|0\| + \|-b + 0 + a\| = \|-b + a\| \) and \( \|f\| + \|-b + f + a\| \geq \|-f - b + f + a\| = \| -b + a\| \) for all \( f, g \in G \). Hence \( d_{a,b} \| (a, b) = \|-b + a\|_G \).

For claim (iii) observe for the left dual that for \( f' = (f'_+, f'_-) \) we have \( \|f'|| - \|f'\| = \|f'_+; f; (-f)\| \leq \|f'_+; f\| + \|-f\| = \|f\|\) and \( \|-f\| = \|f\| \).

Hence \( \| \cdot \|^{*L} = \| \cdot \| \). By the parallel argument \( \| \cdot \|^{*R} = \| \cdot \| \).

Claim (iii) of Proposition 5 is the motivation to consider a seminorm on a category a generalization of a seminorm on a vector space where the underlying abelian group takes the role of \( G \).

**Example 6** (Word metric on a group). As an example of a norm on a monoid take generating set \( \{ g_i \}_{i \in I} \) of a group \( G = (G, +, \cdot, 0) \). Then the free monoid \( M \) generated by \( \bigcup_{i \in I} \{ g_i, -g_i \} \) has the evaluation function \( ev : M \to G \). On \( M \) there is a norm given on \( m \in M \) by the minimal word length of some word \( w \) over the alphabet \( M \) such that \( ev(w) = m \). Word metrics are a fundamental tool in geometric group theory [Löh17].
3.6. Operators and normed vector spaces. Let $\text{NVECT}_\mathbb{R}$ denote the category of normed vector spaces over the reals with linear maps as morphisms. Define the norm of a linear map $A: V \to W$ as

$$\|A\|_{\text{op}} := \log \sup_{v \in V} \frac{\|Av\|_W}{\|v\|_V}.$$ 

The pair $(\text{NVECT}_\mathbb{R}, \|\cdot\|)$ is a seminormed category. Being norm isomorphic is equivalent to being isometric as linear normed spaces.

It is not a normed category and especially does not satisfy the CSB axiom (N3). For an example consider the vector space $C^0([0, 1])$ of real valued continuous functions on the unit interval $[0, 1]$ vanishing at 0 and 1 and endow this space with the supremum norm $\|\cdot\|_c$. Define the spaces

$$V := C^0([0, 1])$$
$$W := C^0([0, 1]) \oplus \{ f \in C^0([0, 1]) \mid f \text{ smooth} \}$$

where $W$ is endowed with the norm $\|(f, g)\|_c := \|f\| \vee \|g\|$. The space $W$ is not complete, but $V$ is, so these spaces are not norm isomorphic. But we can find expansions in both directions:

$$f: V \to W, \quad f \mapsto (f, 0)$$
$$g: W \to V, \quad (f, g) \mapsto \begin{cases} f(2x) & x \leq 1/2 \\ g(2x) & x \geq 1/2 \end{cases}.$$ 

It’s even not possible to ensure (N3) by restricting to the category of Banach space, i.e. complete normed vector spaces. Corresponding examples are more complicated but one was found in a celebrated result by Gowers [ Gow96]. On the other hand the fully faithful subcategory $\text{Hilb} \subseteq \text{NVECT}_\mathbb{R}$ of $\text{NVECT}_\mathbb{R}$ consisting of Banach spaces that admit an inner product (i.e. admit the structure of a Hilbert space). Recall that the Hilbert dimension of a Hilbert space is defined as the cardinality of a basis. A basis is by definition a maximal orthonormal set $E \subseteq V$, i.e. $\langle e, e' \rangle = 0$ for all $e, e' \in E$ with $e \neq e'$ and $\|e\| = 1$ for all $e \in E$. Especially, $E$ is linearly independent. Hilbert spaces are norm isomorphic if and only if they have the same Hilbert dimension [Con94, Theorem 5.4].

If there is an expansive operator $A: V \to W$ than the dimension of $W$ is not smaller than the dimension of $V$: if $E$ is a maximal orthonormal set in $V$ then $A(E)$ is still linearly independent due to linearity and injectivity of $A$. Since $W$ is a Hilbert space there is a decomposition $W \simeq A(V) \oplus W'$. By the Gram-Schmidt process we can find a basis for $A(V)$. Extending this basis to a basis of $W$ be see that the dimension of $W$ is not smaller than the dimension of $V$. Thus if we have expansive operators in both direction, $V$ and $W$ are of the same dimension and hence there is a norm isomorphism.

The left dual of $\|\cdot\|_{\text{op}}$ is the re-scaled operator norm $\log \sup_{v \in V} \frac{\|Av\|_W}{\|v\|_V}$ since for any $v$ with $\|Av\|_c \neq 0$ one can consider the re-scaled embedding of the one-dimensional subspace $f': \mathbb{R} v \to V, v' \mapsto \frac{1}{\|Av\|} v'$. Repeating this argument shows that $\|\cdot\|$ is left reflexive.
4. Norms from capacities

For concrete categories \( C \) most seminorms arise from a function on the subobjects of objects in \( C_0 \)—or an extension of this concept—valued in the extended real numbers, called precapacity. We will define precapacities as a function on subobjects of objects in \( C \), the category to be given a seminorm. Given an object \( X \) in a category \( C \) the slice category \( C/X \) is defined as the category with morphisms \( B \in C[Y,X] \) (for any \( Y \in C_0 \)) as objects and commuting diagrams

\[
\begin{array}{ccc}
\text{source } B & \xrightarrow{\varphi} & \text{source } C \\
\downarrow \quad \quad B \quad \quad \quad \downarrow \quad \quad \quad X \quad \quad \quad \quad C \\
\end{array}
\]

as morphisms. Composition is defined by composition of morphisms \( \varphi; \psi \):

\[
\begin{array}{ccc}
\text{source } B & \xrightarrow{\varphi} & \text{source } C & \xrightarrow{\psi} & \text{source } D \\
\downarrow \quad \quad B \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad C \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \quad D \\
\end{array}
\]

We repeat the standard notion of subobjects from category theory [MM94, p. 11; Joh02, A.1.3]. A subobject of an object \( X \) in a category \( C \) is an equivalence class of monomorphisms to \( X \), where equivalent means isomorphic in \( C/X \). We denote the set of such equivalence classes by \( \text{Sub}_0(X) \). Note that the composition of two monomorphisms is a monomorphism.

Two objects \( B, C \in (C/X)_0 \) are isomorphic if there are morphisms \( \varphi \in C/X[B,C] \) and \( \psi \in C/X[C,B] \) such that \( \varphi \) and \( \psi \) are monomorphisms in \( C \): assume that there are morphisms \( \varphi \in C/X[B,C] \) and \( \psi \in C/X[C,B] \) such that \( \varphi \) and \( \psi \) are monomorphisms in \( C \). Then, in \( C \), we have \( C = \text{id}; C \) and \( C = \varphi; \psi; C \). Hence \( \text{id} = \varphi; \psi \). By the parallel argument we get \( \text{id} = \psi; \varphi \). This shows that \( \varphi, \psi \) are isomorphisms in \( C \). Further the property of admitting a monomorphism from \( B \) to \( C \) gives a partial order on \( \text{Sub}_0(X) \), written \( \subseteq \):

1. antisymmetry was just proven above,
2. transitivity means that the composition of monomorphisms is a monomorphism, and
3. reflexivity means that the identity is a monomorphism.

We will denote by \( \text{Sub}(X) \) the set \( \text{Sub}_0(X) \) endowed with the partial order \( \subseteq \). From category theory, the pullback\(^2\) of a monomorphism \( C \) along a morphism \( f \) is a monomorphism again; it is denoted by \( f^*C \):

\[
\begin{array}{ccc}
\text{source}(f^*C) & \xrightarrow{f^*C} & \text{source } C \\
\downarrow f^*C \quad \quad \quad \quad \quad \quad \quad \downarrow C \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

**Definition 7.** A **concrete category** \((C,F)\) is a category \( C \) together a faithful functor \( C \to \text{SET} \). The functor \( F \) is called **forgetful functor**.

\(^2\)We denote pullback diagrams by \( \xrightarrow{\quad} \).
**Definition 8.** By a concrete category with generalized subobjects \((C, F; S, F_S, GS)\), which by abuse of notation we will often denote by \((C; GS)\), we understand a concrete category \((C, F)\) additionally endowed with an extension of the concrete category \((C, F)\), meaning a commutative triangle of functors

\[
\begin{array}{ccc}
SC & \xrightarrow{S} & C \\
\uparrow F_S & \downarrow F & \downarrow \text{SET} \\
\end{array}
\]

and a selection function \(GS\) that assigns to each object \(X \in C\) a family of subobjects in \(\text{Sub}(SX)\), called (generalized) subobjects, such that

1. For each \(X \in C\) the order preserving induced functor
   \[|GS|(X): GS(X) \to \text{Sub}(F(X)), \quad C \mapsto (F_S)^*(C)\]
   is well-defined, i.e. \(F_S(C)\) is a monomorphism again, and full, i.e. \(|B| \subseteq |C|\) implies \(B \subseteq C\) where

   \[|C| := (F_S C)(\text{source } C) \subseteq F(X)\]

   for any \(C \in GS(X)\) with \(X \in C\), (note that (7) is independent of the representative in \(\text{SET}\)).

2. If \(f: X \to Y\) and \(C \in GS(Y)\), then there is a \(B \in GS(X)\) with \(|B| = (f^*)^*(|C|)\), that is maximal in \(GS(Y)\) with this property. This generalized subobject is called the preimage of \(C\) under \(f\) and written \(B = f^*(C)\).

Often we will encounter the case \(SC = C\) and \(S = \text{id}_C\). By abuse of notation we will often write \(B \in GS(X)\) for a representative of \(B\). Note that a concrete category with generalized subobjects has some similarity to a Grothendieck topology: think of a representative of an element of \(GS(X)\) as singleton set. Then one may compare the assignment \(GS\) to a Grothendieck topology, that assigns to each object a family of sieves.

An example of a concrete category with subobjects is given by

\[(\text{Top}, F; \text{id}_\text{Top}, F, C)\]

where \(C(X)\) is the collection of equivalence classes of homeomorphisms onto closed subspaces of \(X\) and \(F\) is the canonical forgetful functor \(\text{Top} \to \text{SET}\); the property of having enough subobjects follows immediately from the fact that preimages of closed sets under continuous maps are closed.

**Definition 9.** A precapacity \(w\) on a concrete category with subobjects \((C; GS)\) is a function

\[c: \bigsqcup_{X \in \{S\}C} GS(X) \to [-\infty, \infty]\]

and it is called a capacity if it is monotone or antimonotone, i.e. for any two subobjects \(A, A' \in GS(X)\) with \(A \subseteq A'\) we have that \(c(A) \leq c(A')\) or \(c(A') \leq c(A)\), resp.

In practise, capacities are often non-negative. Each precapacity \(c\) gives rise to an assignment

\[\|f\|_c := \sup \left\{ c(f^* B) - c(B) \mid B \in GS(Y), c(B) < \infty \right\}\]
where \(-\infty - (-\infty) := 0\). For a concrete category with enough subobjects it is called the **seminorm induced by** \(w\). The seminorm properties are checked immediately by

\[
\|\text{id}_X\|_c = \sup_B^0 c(\text{id}_X(B)) - c(B) = \sup_B^0 c(B) - c(B) = 0
\]

and for any diagram \(X \xrightarrow{f} Y \xrightarrow{g} Z\ in \mathcal{C}\)

\[
\|f : g\|_c = \sup_B^0 (c(f : g)^{\ast}B - c(B))
\]

\[
= \sup_B^0 (c(f : g)^{\ast}B - c(g^*B) + c(g^*B) - c(B))
\]

\[
\leq \sup_B^0 (c(f : g)^{\ast}B - c(g^*B)) + \sup_B^0 (c(g^*B) - c(B))
\]

\[
= \|f\|_c + \|g\|_c.
\]

**Example 10.** As an example we will explain how to formulate the seminormed category \((\text{NVect}_R, \|\cdot\|_\text{op})\) from Subsection 3.6 in the framework of capacities explained above. Set

\[
(\text{SNVect}_R)_0 := \{ (F, V) \mid F \subseteq V, V \in \text{NVect}_R \}
\]

\[
\text{SNVect}_R[(F, V), (G, W)] := \{ f \in S_{\text{Set}}(F, G) \mid \exists A \in (\text{NVect}_R)_1: A|_F = f \}
\]

\[
S : \text{SVect}_R \to \text{SNVect}_R
\]

\[
S(V) := (V, V)
\]

\[
F_S(F, V) := F
\]

\[
G_S(V) := \{ (F, V) \xrightarrow{(v, w) \mapsto (v, w)} (V, V) \mid F \subseteq V \}
\]

\[
c(F, V) := \sup_{v \in F} \log\|v\|_V
\]

where \(V = (V, \|v\|_V)\). This actually defines a category with subobject as both properties 1 and 2 are obvious. Moreover for any \(A : V \to W\)

\[
\|A\|_c = \sup_{G \subseteq W, \|G\|_c < \infty}^0 c(f^*G, V) - \log\|G\|_c
\]

\[
= \sup_{G \subseteq W, \|G\|_c < \infty}^0 \left( \sup_{v \in f^*G} \log\|v\|_V \right) - \left( \sup_{w \in G} \log\|w\|_W \right)
\]

Thus we have \(\|A\|_c \geq \sup_{w \in W}^0 (\sup_{v \in V, f(v)=w} \|v\|_V) - \|w\|_W\) and also \(\|A\|_c \leq \sup_{G \subseteq W, \|G\|_c < \infty}^0 (\log\|v\|_V) - \log\|f(v)\|_W\). Hence \(\|A\|_c = \|A\|_{\text{op}}\).

**4.1. Capacities by images.** A natural question is, when one can compute capacities given an "image" function \(g : \text{GS}(X) \to \text{GS}(Y)\). The leading example in this regard are the subobjects given by the power set and the corresponding adjunctions \(f, f^1 : \mathcal{P}(X) \to \mathcal{P}(Y)\) characterized by the adjunctions \(f^\ast \dashv f^\ast \dashv f^!\) (appendix A.1). As the collections of subobjects are posets, these adjunctions are actually Galois connections.

**Lemma 11.** Let \(X\) and \(Y\) be two objects in a category with generalized subobjects. Assume either that

(1) \(c\) is monotone and there is a direct image \(f^\ast : \text{GS}(X) \to \text{GS}(Y)\), i.e. a Galois connection \(f^\ast \dashv f^\ast\), or
Then
\[ \| \cdot \|_c = \lambda f. \sup \{ c(A) - c(f_* A) \mid A \in \text{GS}(X), c(f_* A) < \infty \}, \text{ or} \]
\[ \| \cdot \|_c = \lambda f. \sup \{ c(A) - c(f_1 A) \mid A \in \text{GS}(X), c(f_1 A) < \infty \}, \text{ resp.} \]

**Proof.** Both claims are implied by the following estimate, where \( g \) equals \( f_* \) or \( f_1 \):
\[ \| f \|_c = \sup \{ c(f^* B) - c(B) \mid B \in \text{GS}(Y), c(B) < \infty \} \]
\[ \geq \sup \{ c(f^* B) - c(B) \mid B = g(A), A \in \text{GS}(X), c(B) < \infty \} \]
\[ = \sup \{ c(f^* g A) - c(g A) \mid A \in \text{GS}(X), c(g A) < \infty \} \]
we have \( c(f^* g A) \geq c(A) \) due to—in the first case—\( f^* f_* A \supset A \) and monotonicity of \( c \) and—in the second case—\( f^* f_1 A \supset A \) and anti-monotonicity
\[ \geq \sup \{ c(A) - c(g A) \mid A \in \text{GS}(X), c(g A) < \infty \} \]
\[ \geq \sup \{ c(A) - c(g A) \mid A = f^*(B), B \in \text{GS}(Y), c(g A) < \infty \} \]
\[ = \sup \{ c(f^* B) - c(g f^* B) \mid B \in \text{GS}(Y), c(g f^* B) < \infty \} \]
we have \( c(g f^* B) \leq c(B) \) due to—in the first case—\( f_* f^* B \subset B \) and monotonicity of \( c \) and—in the second case—\( f_1 f^* B \supset B \) and anti-monotonicity
\[ \geq \sup \{ c(f^* B) - c(g f^* B) \mid B \in \text{GS}(Y), c(B) < \infty \} \]
and due to the same fact also
\[ \geq \sup \{ c(f^* B) - c(B) \mid B \in \text{GS}(Y), c(B) < \infty \} \]
\[ = \| f \|_c \]

\[ \Box \]

4.2. **Dual seminorms from capacities.** Let \( c \) be a precapacity. In view of (9) one is of course tempted to look at the quantity

\[ (10) \quad \| f \|_{-c} := \sup \{ c(B) - c f^* B \mid B \in \text{GS}(X), c(B) < \infty \}. \]

It turns out to be related by the inequalities

\[ (11a) \quad \| f \|_{-c}^R \leq \| f \|_{-c} \]

as is easily checked:
\[ \| f \|_{-c}^R = \sup_{f'} \| f' \|_c - \| f : f' \|_c \]
\[ = \sup_{f'} \sup_B c f'^* B - c B - \sup_B c (f : f')^* B - c B \]
\[ \leq \sup_{f', B} c f'^* B - c B - (c (f : f')^* B - c B) \]
\[ \leq \sup_{f', B} c f'^* B - c f^* B \]
\[ = \| f \|_{-c}. \]

For the **left** and **right biduals** we have that

\[ (12) \quad \| \cdot \|_{-L^*}, \| \cdot \|_{-R^*} \leq \| \cdot \|. \]
we check this in the case of the left bidual by estimating for any morphism $f$

$$\|f\|^{L_1L} = \sup_{f'} \|f'\|^{L_1} - \|f' \cdot f\|^{L_1}$$

$$= \sup_{f'} \sup_{f''} (\|f'\| - \|f''\|) = \sup_{f'} (\|f''\| - \|f'' \cdot f'\|)$$

$$\leq \sup_{f', f''} (\|f''\| - \|f'' \cdot f'\| - \langle f''\| = \sup_{f', f''} \|f''\| - \|f'' \cdot f'\|$$

$$\leq \|f\|$$

using triangle inequality in the last step; the case of the right bidual is checked by a parallel argument.

4.3. Some crucial seminorms. In this preprint and future work we will study the following examples of capacities and develop simplified characterizations for the induced norms in the following cases. Note the conventions $\log 0 = -\infty$ and $|\infty| = \infty$.

1. topological dimension $C \mapsto |\log(1 + \dim C)|$ for $SO(X) = P(X)$ in Section 5 giving rise to a seminorm measuring deviation of a map from being light.

2. logarithmic number of connected components

$$SO(X) = \mathcal{C}(\mathcal{X}) \quad \text{and} \quad \|\log \# I\|: \mathcal{X} \mapsto |\log(\#(I \mathcal{X}))|$$

for where $\mathcal{X}$ the set of connected components of $\mathcal{X}$. In Section 5 this gives rise to a seminorm measuring deviation of a map from being monotone.

3. diameter $diam M$ of a metric space $M$ for $SO(M) = P(M)$ giving rise to a norm, that quasi-metrizes Gromov-Hausdorff convergence.

4. negative logarithmic diameter $-\log diam M = -c_{\text{Lip}}$ giving rise to the Lipschitz norm $\|\cdot\|_{\text{Lip}}$, that quasi-metrizes Lipschitz convergence of metric spaces. For a map $f: M \rightarrow N$ it may be expressed as

$$\|f\|_{\text{Lip}} := \|f\| - c_{\text{Lip}}$$

$$= \sup \left\{ \log(diam B) - \log(diam f^*B) \mid B \subseteq N, \log diam B > -\infty \right\}$$

$$= \sup \left\{ \log \frac{\text{diam } B}{\text{diam } f^*B} \mid B \subseteq N, \text{diam } B > 0 \right\}.$$  \hspace{1cm} (13)

Note that $\text{diam } B > 0$ implies $\text{diam } f^*B > 0$ for metric spaces.

5. Topological spaces

The subject of this section are two seminorms both measuring the increase of complexity when passing from a subset of the domain to its preimage. In the first case complexity is measured by topological dimension in the second one by number of connected components. These norms measure the deviation from being light and monotone respectively. The well-known monotone-light factorization implies that in the case of compact spaces the sum of both norms, which we call the topological norm $\|\cdot\|_{\text{top}}$, is a norm. Actually, the monotone-light factorization is a strengthening of the norm property (N3). The idea of the monotone-light factorization goes back to Eilenberg [Eil34] and Whyburn [Why34] independently. For a historical overview about the monotone-light factorization and its variations consult [Lor97].
where we used the convention  \( \log(0) = -\infty \). For any topological space \( X \), we call a map \( f: X \to Y \) **light** if the fiber \( f^{-1}(y) \) is totally disconnected for every \( y \in Y \). Further we define the **dimension seminorm** using the precapacity \( \sup_{A \in P(Y)} \dim(A) \) by

\[
\|f\|_{\dim} := \|f\|_{\log(1 + \dim)} = \sup_{A \in P(Y), \dim(A) < \infty} \|\log(1 + \dim f^*A)\| - \|\log(1 + \dim A)\|
\]

where we used 2 of Definition 8 in the last step.

To further simplify this expression we use the Hurewicz formula which states

\[
\|f\|_{\dim} = \sup_{A \in P(Y), \dim f^*A < \infty} \|\log(1 + \dim A)\| - \|\log(1 + f_* \dim A)\|.
\]

Another consequence of (14) is that

\[
\|f\|_{\dim} = \sup_{y \in Y} \|\log(1 + \dim f^*\{y\})\|
\]

for a map \( f \) from a \( T_1 \)-space \( X \) to a metrizable space \( Y \). Indeed, for any singleton \( \{y\} \) we have \( \|\log(1 + \dim \{y\})\| = 0 \) and hence \( \|f\|_{\dim} \geq \sup_{y \in Y} \|\log(1 + \dim f^*\{y\})\| \).

But also for any \( C \subseteq Y \) we can apply Hurewicz’s formula (14) to \( f\mid_{f^{-1}C} f^*C \to C \) getting \( \dim(f^*C) \leq \dim(C) + \sup_{y \in C} \dim(f^*\{y\}) \leq \dim(C) + \sup_{y \in Y} \dim(f^*\{y\}) \). Hence by sublinearity and monotonicity of \( \log \) we have \( \log(1 + \dim f^*C) \leq \log(1 + \dim C) + \sup_{y \in Y} \log(1 + \dim f^*\{y\}) \). Since \( C \) was arbitrary this proves the lemma.

Actually, (15b) allows one to rephrase Hurewicz’s formula (14) as

\[
\dim X \leq \dim Y + \exp\|\|_{\dim}
\]

since surjectivity of \( f \) implies that \( \dim(f^*\{y\}) \leq 0 \).

5.2. **Component seminorm.** Following Whyburn [Why50] and Carboni et al. [Car+97] we call a map \( f: X \to Y \) **monotone** if the preimage of every singleton \( \{y\} \subset Y \) is nonempty and connected. Note that this property implies surjectivity since the empty set consists of zero connected components. For any topological space \( X \) let \( \mathcal{I}(X) = (I(X), \tau_{\mathcal{I}(X)}) \) denote the collection of connected components of \( X \) endowed with the quotient topology. Define the **component seminorm** as

\[
\|f\|_{\comp} := \|f\|_{\log \# I} = \sup_{\substack{C \subseteq Y \text{ closed,} \\ \log \#(I C) < \infty}} |\log(\#(f^*C)) - \log(\#(I C))|
\]

where we used the convention \( \log(0) = -\infty \). Note that we can express the number of connected components \( \#I X \) of a nonempty space \( X \) as \( \exp X \to \{\ast\} \) where
Let $\mathcal{X} \to \{\ast\}$ be the canonical map to the singleton space. This norm relates to the dimension of fibers by the obvious inequality

$$\|f\|_{\text{comp}} \geq \sup_{p \in \mathcal{Y}} \log(|\{f^*\{p\}\}|) =: \text{mon}(f).$$

**Lemma 13.** Let $f: \mathcal{X} \to \mathcal{Y}$ be a continuous function. Then

(i) We have $\|f\|_{\text{comp}} = \sup\left\{ \log\left(\#(f^*C)\right) \mid \emptyset \neq C \subseteq \mathcal{Y}, \#C = 1 \right\}$.

(ii) The map $f$ is monotone if and only if $\text{mon}(f) = 0$.

(iii) If $f$ is closed and monotone then $\|f\|_{\text{comp}} = 0$.

(iv) Assume that $\mathcal{X}$ is compact and that $\mathcal{Y}$ is Hausdorff. Then $f$ is monotone, if and only if $\|f\|_{\text{comp}} = 0$.

**Proof.** For claim (i) observe

$$\|f\|_{\text{comp}} = \sup_{0 < \#(IC) < \infty} \log\left(\sum_{C' \subseteq IC} \#(f^*C')\right) - \log\left(\#(IC)\right)$$

$$\leq \sup_{0 < \#(IC) < \infty} \log\left(\sum_{C' \subseteq IC} \#(f^*C')\right) - \log\left(\#(IC)\right)$$

$$= \sup_{0 < \#(IC) < \infty} \left\{ \log\left(\sum_{C' \subseteq IC} \#(f^*C')\right) \mid \#C = 1 \right\}$$

$$\leq \sup_{0 < \#(IC) < \infty} \left\{ \log\left(\#(f^*C')\right) \mid \#C = 1 \right\}$$

Claim (ii) clearly follows.

Claim (iii) follows from the fact that $\log(|\{f^*\{p\}\}|) = 0$ if and only if $f^*\{p\}$ is a singleton.

For claim (iv) assume that $f$ is monotone and closed. Note that the restriction $f|_{f^\ast C}: f^\ast C \to C$ is closed as well: take any relatively closed $A \subseteq f^\ast C$. Any point $x$ in the closure of $f_\ast(A)$ relative to $C$ must be in the image $f_\ast(A^Y)$ of the closure of $C$ in $X$, but then already $y \in A$ for any $y \in f^x$. Thus $f_\ast(A) = f_\ast(A^Y)$.

Let $C$ be an arbitrary closed connected subset of $Y$. The preimage $f^\ast C$ must not be empty because otherwise monotonicity of $f$ would be violated. Since $C$ is closed, so is $f^\ast C$. Assume that $f^\ast C$ is a disjoint union of two sets $K$ and $L$ that are clopen in the relative topology on $f^\ast C$. For any $y \in C$ the preimage $f^\ast\{y\}$ is connected in the relative topology. Hence either $f^\ast\{y\} \subseteq K$ or $f^\ast\{y\} \subseteq L$. Since this holds for all $y \in C$, the set $C$ is actually the disjoint union of $f_\ast K$ and $f_\ast L$.

Due to our observation on the closedness of $f|_{f^\ast C}: f^\ast C \to C$ both $f_\ast K$ and $f_\ast L$
are open in the relative topology. Thus by connectedness of \( C \) either \( f^\ast K = \emptyset \) or \( f^\ast L = \emptyset \), a contradiction. Consequently, \( f^\ast C \) is connected. Hence \( \| f \|_{\text{comp}} = 0 \).

In claim (iv) the direction monotone“\( \ldots \Rightarrow \| f \|_{\text{comp}} = 0 \)“ follows from claim (iii) and the fact that a continuous function from a compact space to a Hausdorff space is closed. The other direction is implied by (16) and claim (ii). □

Remark 14. In claim (iv) the assumption that \( \tau_\mathcal{Y} \) is Hausdorff is necessary for the theorem that monotonicity of \( f \) implies \( \| f \|_{\text{comp}} = 0 \): For a counterexample let \( \mathcal{X} \) be the discrete space on a two element set and \( \mathcal{Y} \) the Sierpiński space, i.e. \( \mathcal{Y} = \{0,1\} \) and \( \tau_\mathcal{Y} = \{\emptyset,\{1\},\mathcal{Y}\} \). Being finite \( \mathcal{X} \) is compact. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a bijection. Since the map \( f \) is bijective, it is monotone. On the other hand \( \mathcal{Y} \) is connected but its preimage consists of two connected components.

Theorem 15. A map \( f : \mathcal{X} \to \mathcal{Y} \) between totally disconnected compact Hausdorff spaces having \( \| f \|_{\text{comp}} = 0 \) is a homeomorphism.

Proof. Assume that \( \| f \|_{\text{comp}} = 0 \). Then this map is surjective. Since every fiber is totally disconnected, \( \| f \|_{\text{comp}} = 0 \) implies that each fiber is a singleton. Hence \( f \) is bijective. As for compact Hausdorff spaces the notions of closed and compact subsets coincide, and compact subsets are mapped to compact subsets under continuous function, the inverse of \( f \) is continuous. Thus \( f \) is a homeomorphism. □

5.3. The topological norm. Define the topological norm as

\[
\| f \|_{\text{top}} := \| f \|_{\text{comp}} + \| f \|_{\text{dim}}
\]

Proposition 16. Let \( \mathcal{X} \) be a compact \( T_4 \) space and \( \mathcal{Y} \) be metrizable. If \( \| f \|_{\text{top}} = 0 \) for a continuous function \( \mathcal{X} \to \mathcal{Y} \), then \( f \) is monotone and light.

Proof. Assume that \( \| f \|_{\text{top}} = 0 \). Then \( \| f \|_{\text{comp}} = \| f \|_{\text{dim}} = 0 \). The fact \( \| f \|_{\text{comp}} = 0 \) implies that \( f \) is monotone because points in \( \mathcal{Y} \) are closed. The other fact \( \| f \|_{\text{dim}} = 0 \) implies that \( f \) is light by claim (iv) of Lemma 13. □

Theorem 17. Let \( \mathcal{X} \) be a compact \( T_4 \) space and \( \mathcal{Y} \) be metrizable. If there is map with \( \| f \|_{\text{top}} = 0 \), then \( f \) is a homeomorphism. Especially, the category of compact metrizable spaces is a normed category with respect to \( \| \cdot \|_{\text{top}} \).

Proof. By Proposition 16 the map \( f \) is monotone and light. Since the identity on a topological space is monotone and light as well, we have two factorizations of \( f \) in a monotone and a light map:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{id} & \mathcal{X} \\
\downarrow f & \downarrow \varphi & \downarrow \text{id} \\
\mathcal{Y} & \xrightarrow{id} & \mathcal{Y}
\end{array}
\]

By the classical uniqueness of the monotone-light factorization [Car+97, 2.8, 7.3] there is a homeomorphism \( \varphi : \mathcal{X} \to \mathcal{Y} \). □
6. Metric spaces

Let $\mathbf{Met}$ denote the category of compact metric spaces with multi-valued functions among them as morphisms, i.e. functions $f: M \to \mathcal{P}(N) \setminus \{\emptyset\}$, $x \mapsto f(x)$. We will always write $f[x]$ instead of $f(x)$ for multi-valued functions to avoid any confusion with normal functions. Objects of $\mathbf{Met}$ we denote by curly letters, e.g. $M = (M, d_M)$, and the metric is abbreviated by $|\, . \, | = d_M(\, . \, , \, . \, )$ if no confusion can arise.

Actually, our arguments in this section extend to densely defined multi-valued functions, i.e. functions $f: M \to \mathcal{P}(N)$ such that points with $f[x] \neq \emptyset$ are dense. This is done by transforming such a function to a morphism in $\mathbf{Met}$ by the closure

$$\tilde{f}[x] := \left\{ y \mid y = \lim_{n \to \infty} y_n \text{ for } y_n \in f[x_n] \text{ with } x_n \xrightarrow{n \to \infty} x \right\}.$$ 

Let’s start with a naïve approach by considering subsets $A \subseteq M$ as subobjects and considering the perhaps most natural quantity as a capacity, the diameter

$$\text{diam} = \lambda A. \sup_{x,y \in A} |x y|.$$ 

This induces the seminorm

$$\|f\|_{\text{diam}} = \sup_{A \subseteq N} [\text{diam}(f^* A) - \text{diam}(A)]$$

for a morphism $f: M = (M, d_M) \to N = (N, d_N)$. Unfortunately, as we want to include multi-valued functions to our discussion, the adjunction $f^* \dashv f^*$ is no longer valid for the power sets.

But there is a solution to this problem: The idea is to choose the power set of a power set as set of subobjects. Moreover,—since a metric is a map on pairs of points—it’s suitable to consider the Cartesian product $M \times M$ in lieu of $M$. This is done by making $\mathbf{Met}$ into a concrete category with generalized subobject (cf. Definition 8) as follows:

$$\begin{align*}
S & \downarrow F_S \\
\mathbf{Met} & \xrightarrow{F} \mathbf{Set} \\
S & \downarrow F \\
\mathbf{SMet} & \xrightarrow{S} \mathbf{Set} \\
\mathbb{S} & \downarrow F_S \\
\mathbf{SMet} & \xrightarrow{S} \mathbf{Set} \\
\mathbb{S} & \downarrow F \\
\mathbf{GS(M)} & \xrightarrow{S} \mathbf{Set} \\
(18a) & \quad F(M) = \mathcal{P}(M \times M) \\
(18b) & \quad F(f) = \lambda P\{ (y, y') \mid y \in f[x], y' \in f[x'], (x, x') \in P \}, \\
(18c) & \quad \mathbf{SMet} = \mathbf{Set} \\
(18d) & \quad S = F \\
(18e) & \quad \mathbb{S} = F(M)
\end{align*}$$

for $M \in \mathbf{Met}_0$ and $f \in \mathbf{Met}_1$. Note that by this definition subobjects are simply normal subsets (i.e. subobjects in set) of the power set. Hence the pullback of subobjects is just the preimage, i.e.

$$(Sf)^*(B) = \{ P \subseteq \mathcal{P}(M \times M) \mid (Ff)(P) \in B \}.$$ 

On the subobjects of a $S(M)$ for $M = (M, d_M) \in \mathbf{Met}_0$ we define the capacity

$$c_{\text{diam}} = \lambda A. \sup_{p \in A} \inf_{p \in P} d_M(p)$$

for $A \subseteq M$. 


which is actually a capacity, i.e. monotone, being defined by a supremum. Any subset \( A \subseteq M \) naturally corresponds to \( \mathcal{P}(A \times A) \). Observe

\[
\text{(20)} \quad \text{diam}(A) = c_{\text{diam}}(\mathcal{P}(A \times A)).
\]

By definition and the fact that for any compact space there is a maximum value (i.e. the diameter of the space) attained for the distances of two point sets we have

\[
\text{(21a)} \quad \|f\|_{\text{diam}} = \sup \{ c_{\text{diam}}((Sf)^* (B)) - c_{\text{diam}}(B) \mid B \subseteq \mathcal{P}(N \times N) \}.
\]

As with \( \mathbf{SMET} = \mathbf{SET} \) we have the usual adjunction—i.e. Galois connection in this case—\( f_* \dashv f^* \). Thus we can apply Lemma 11 and again finiteness of the diameter obtaining

\[
\text{(21b)} \quad \|f\|_{\text{diam}} = \sup \{ c_{\text{diam}}(A) - c_{\text{diam}}((Sf)_*(A)) \mid A \subseteq \mathcal{P}(M \times M) \}
\]

\[
= \sup \{ c_{\text{diam}}(\{P\}) - c_{\text{diam}}((Sf)*(\{P\})) \mid \{P\} \subseteq \mathcal{P}(M \times M) \}
\]

where "\( \geq \)" is obvious and "\( \leq \)" holds since \( c_{\text{diam}}(A) = \sup_{P \in A} c_{\text{diam}}(\{P\}) \).

\[
= \sup \left\{ \inf_{p \in P} d_M(p) - \inf_{q \in (Sf)(P)} d_M(q) \mid P \subseteq \mathcal{P}(M \times M), q \in (Sf)(P) \right\}
\]

\[
= \sup \{ d_M(p) - d_M(q) \mid p \in \mathcal{P}(M \times M), q \in (Sf)(P) \}
\]

\[
= \sup \{ d_M(x, x') - d_N(y, y') \mid x, x' \in M, y, y' \in f[x'] \}.
\]

For \( r > 0 \) let \( M_r = \{ (x, y), d_r \} \) be the two point metric space such that \( d_r(x, y) = r \). Next we calculate

**Lemma 18.** The seminorm \( \| \cdot \|_{\text{diam}} \) has the left dual

\[
\text{(22)} \quad \|f\|_{\text{diam}}^* = \sup \{ |y y'| - |x x'| \mid x, x' \in M, y \in f[x], y' \in f[x'] \}
\]

and is left reflexive.

**Proof.** The claim (22) follows from the estimate

\[
\|f\|_{\text{diam}}^* = \sup_f \{ \|f'\|_{\text{diam}} - \|f' ; f\|_{\text{diam}} \}
\]

\[
\overset{\text{(21c)}}{=} \sup_f \left( \sup_{z, z' \in M'} (|z z'| - |x x'|) - \sup_{y, y' \in f[x']} (|y y'| - |y y'|) \right)
\]

\[
\leq \sup_f \sup_{z, z' \in M', x \in f[z], x' \in f'[z']} ((|z z'| - |x x'|) - (|z z'| - |y y'|))
\]

\[
\leq \sup_f \sup_{z, z' \in M', x \in f[z], x' \in f'[z'], y \in f[x], y' \in f[x']} (|y y'| - |x x'|)
\]

\[
\leq \sup_{x, x' \in M, y, y' \in f[x']} (|y y'| - |x x'|)
\]

\[
= \sup_{x, x' \in M, y \in f[x], y' \in f[x']} (r - |y y'|) - (r - |x x'|)
\]

\[
= \sup \{ (r - |y y'|) - (r - |x x'|) \mid r > 0, x, x' \in M \text{ with } |x x'| = r, y \in f[x], y' \in f[x'] \}
\]

\[
= \sup_f \{ \|f'\|_{\text{diam}} - \|f' ; f\|_{\text{diam}} \mid f' : M_r \to \mathcal{M} \text{ with } r > 0 \}
\]

\[
\leq \sup_f \{ \|f'\|_{\text{diam}} - \|f' ; f\|_{\text{diam}} \}.
\]
Left reflexivity follows from (12), i.e. \( \|f\|_{\text{diam}}^L \leq \|f\|_{-\text{diam}} \), and the estimate
\[
\|f\|_{\text{diam}}^L = \sup_{f'} (\|f\|_{\text{diam}}^L - \|f'\|_{\text{diam}}^L)
\geq \sup_{f': M \to M} \left( \|f'\|_{\text{diam}}^L - \|f\|_{\text{diam}}^L \right)
\geq \sup_{x,y \in M} \left\{ \|f(x) - f(y)\| - \|x - y\| \right\}
\geq \sup_{x,y \in M} \left( \|f^* \{f(x), f(y)\}\| - \|x - y\| \right)
\geq \sup_{A \subseteq N} \left( \|f^* A\| - \|x - y\| \right)
\]

Lemma 19. We have that \( \|f\|_{\text{diam}}^L \geq \|f\|_{-\text{diam}} \).

Proof. Expressing \( \|f\|_{\text{diam}}^L \) by Lemma 18 we get the desired estimate:
\[
\|f\|_{\text{diam}}^L = \sup_{x,y \in M} (\|f(x) f(y)\| - |x y|)
\geq \sup_{x,y \in M} - \text{diam}\{x, y\} - (\text{diam}\{f(x), f(y)\})
\geq \sup_{x,y \in M} - \text{diam}(f^* \{f(x), f(y)\}) - \text{diam}\{x, y\} - (\text{diam}\{f(x), f(y)\})
\geq \sup_{A \subseteq N} - \text{diam}(f^* A) - (\text{diam} A)
\]

Example 20. The reverse inequality \( \|f\|_{\text{diam}}^L \leq \|f\|_{-\text{diam}} \) does not hold as is seen from the example of the space \( \{0, 1, 2\} \subset \mathbb{R} \) with the induced distance and the map \( 0 \mapsto 0; 1, 2 \mapsto 2 \) to \( \mathbb{R} \): we have \( \|f\|_{\text{diam}}^L \geq |0 2| - |0 1| = 1 \), but the preimage of any set containing 0 and 1 has always diameter 2.

Note that a function \( f \) with \( \|f\|_{\text{diam}}^L \) bounded is single-valued. Moreover let \( T \) be the one point metric space. Observe that
\[
\text{diam} M = \|M \to T\|_{\text{diam}}.
\]

Further set \( d_{\text{diam}}(\mathcal{M}, \mathcal{N}) := \|\| \text{diam}\{\mathcal{M}, \mathcal{N}\} = \inf\{ \|f\|_{\text{diam}} \mid f: \mathcal{M} \to \mathcal{N} \} \). We recall the well-known notion of Gromov-Hausdorff distance [Pet16, § 11.1.1; Gro99, § 3A] employing the following shorthand notations for any \( A \subseteq M \)
\[
A^r := \{ x \in M \mid |x A| < r \} \quad \text{for } r > 0 \quad \text{and}
A^r := \bigcap_{r > r} A^r \quad \text{for } r \geq 0.
\]

A subset \( X \subseteq M \) is said to be \( l \)-dense in \( M \) if \( X^l = M \). Let \( A, B \subseteq M \) be subsets of a metric space \( M \). The Hausdorff distance between \( A \) and \( B \) is given by
\[
d_H(A, B) := \inf\{ r \in [0, \infty] \mid A \subseteq B^r \quad \text{and} \quad B \subseteq A^r \}.
\]

Let \( M, N \) be metric spaces. Their Gromov-Hausdorff distance is
\[
d_{\text{GH}}(\mathcal{M}, \mathcal{N}) := \inf\{ d_H(f_* M, g_* N) \mid f: \mathcal{M} \to \mathcal{N} \}
\]
where \( \mathcal{L} \) ranges over all metric spaces and \( f, g \) are metric embeddings. Recall that a function is Cauchy continuous if it preserves Cauchy sequences.
Theorem 21. The identity map on $sk_0(M, \|_{\text{diam}})$ with the Gromov-Hausdorff metric $d_{GH}$ on the domain and the distance $d_{\text{diam}}^+$ on the codomain is 2-Lipschitz with Cauchy continuous inverse.

The next lemma is a quantitative version of [FH36], cf. also [BM15]. For its proof we require the terminology of packings, which can be elegantly be introduced by what we just developed:

For any metric space $M \in \text{Met}$ a collection $P := \{p_1, \ldots, p_n\} \subseteq M$ is called a packing of $M$. Further let the configurations of a packing be $Conf P := \{(p,q) \in X \times X \mid p \neq q\}$, the set of all pairs $(x,x')$ of distinct points $x$ and $x'$ from $X$. Further, we can assign a map $\text{pack}_M \colon N \rightarrow \text{GS}(M)$

$$\text{pack}_M = \lambda N. \{\text{Conf}(P) \mid P \subseteq M, \#P = n\}. $$

We can compose this map with the capacity $c_{\text{diam}}$

$$N \xrightarrow{\text{pack}_M} \text{GS}(M) \xrightarrow{c_{\text{diam}}} [0, \infty]$$

For $l > 0$ an $l$-packing of $M$ is a packing $P$ that $c_{\text{diam}} \text{Conf}(P) > l$. We define the metric $l$-packing number of $M$ by

$$\#_l^\text{pack}(M) := \sup_{\text{pack}_M} (c_{\text{diam}}^*(l, \infty))$$

$$= \sup \{n \mid \exists l\text{-packing } (p_1, \ldots, p_n) \text{ of } M\}. $$

This definition is extended to non-positive $l$ by $\#_l^\text{pack}(M) = \infty$ (also for the terminal space). As $M$ is compact, $N := \#_l^\text{pack}(M)$ is finite. A collection $P := (p_1, \ldots, p_N)$ is called a maximal $l$-packing of $M$.

Further define the total distances for a finite $P \subseteq M$ and of $M$ itself by

$$\|P\|_{\text{tot}} := \sum_{(p,q) \in \text{Conf } P} |p,q|$$

$$\|M\|_{\text{tot}} := \sup \{\|P\|_{\text{tot}} \mid \text{P is an } l\text{-packing of } M\}. $$

The integer valued function

$$l \mapsto \#_l^\text{pack}(M)$$

is continuous from the left and monotonically decreasing in $l$. Finally, for spaces $M$ and $N$ with it is easy to see that

$$\#_l^\text{pack}(N) \leq \#_{l-d_{\text{diam}}(N,M)}^\text{pack}(M),$$

indeed, given an $l$-packing $P \subseteq N$ with $\|P\|_{\text{tot}} = \|N\|_{\text{tot}}$ the set $f_*(P)$ is still an $l-d_{\text{diam}}(N,M)$-packing, provided that $l-d_{\text{diam}}(N,M) > 0$, or the right hand side if $\infty$, otherwise.

Lemma 22. Let $M, M'$ be compact metric spaces. For all $L, l$ with $l > L \geq 0$ it holds for sufficiently small $\delta > 0$ that for every $h \colon M' \rightarrow M$ with $\|h\|_{\text{diam}} < \delta$ and $d_{\text{diam}}(M',M) \leq L$ we have that

(i) $h_*(M')$ is $l$-dense, and

(ii) $\|h\|_{\text{diam}}^* \leq 4L + C\delta$ where $C = C(l - L, M)$.

\footnote{Note that the result of the latter is implied by the former using the closure (17).}
Proof. For the first claim, let \( h : \mathcal{M} \to \mathcal{M} \) be a map between compact metric spaces. By monotonicity and continuity from the left of \( l \mapsto \#_l^{\mathrm{pack}}(\mathcal{M}) \) we can find some small \( \varepsilon > 0 \) such that for all \( \delta \in (0, \varepsilon] \), we have \( \#_{l-\delta}^{\mathrm{pack}}(\mathcal{M}) = \#_l^{\mathrm{pack}}(\mathcal{M}) \). For any \( l \)-packing \( P = (p_1, \ldots, p_N) \) with \( N := \#_1^{\mathrm{pack}}(\mathcal{M}) \) and \( h \) with \( \|h\|_{\mathrm{diam}} < \delta \) \( \leq \varepsilon \) the collection \( h_*(P) \) is an \( (l-\delta) \)-packing. Since \( \#_{l-\delta}^{\mathrm{pack}}(\mathcal{M}) = \#_l^{\mathrm{pack}}(\mathcal{M}) = \#_{l-\varepsilon}^{\mathrm{pack}}(\mathcal{M}) \) this implies that \( h_*(P) \) is even a maximal \( (l-\delta) \)-packing by (27). Hence \( h_*(\mathcal{M}) \) is \( l \)-dense; actually even \( \{h_*(P)\}^\delta = \mathcal{M} \). Thus claim (i) holds.

For claim (ii), i.e. \( \|h\|_L^{\mathrm{diam}} \leq 4l + C\delta \), assume further that \( d_{\mathrm{diam}}(\mathcal{M}', \mathcal{M}) \leq L \). Observe that it is possible to find an \( l \)-packing \( P \) in \( h^* \mathcal{M}' \) such that

\[
\|P\|_{\mathrm{tot}} > \|\mathcal{M}'\|_{\mathrm{tot}} - \left( \#_1^{\mathrm{pack}}(\mathcal{M}') \right)^2 \delta.
\]

Thus by (27)

\[
\|P\|_{\mathrm{tot}} > \|\mathcal{M}'\|_{\mathrm{tot}} - \left( \#_{l-L}^{\mathrm{pack}}(\mathcal{M'}) \right)^2 \delta.
\]

We still assume \( \delta \leq \varepsilon \). Further observe that

\[
\frac{\|h(P)\|_{\mathrm{tot}}}{\|\mathcal{M}'\|_{\mathrm{tot}}} = \frac{\|\mathcal{M}'\|_{\mathrm{tot}}}{\|\mathcal{M}'\|_{\mathrm{tot}}} \leq \frac{\|\mathcal{M}'\|_{\mathrm{tot}}}{\|\mathcal{M}'\|_{\mathrm{tot}}} < \|P\|_{\mathrm{tot}} + \left( \#_1^{\mathrm{pack}}(\mathcal{M}') \right)^2 \delta \leq \|P\|_{\mathrm{tot}} + \left( \#_{l-L}^{\mathrm{pack}}(\mathcal{M}) \right)^2 \delta.
\]

and hence,

\[
\sum_{(p,q) \in \text{Conf } h(P)} |pq| \leq \sum_{(p,q) \in \text{Conf } P} |pq| + \left( \#_{l-L}^{\mathrm{pack}}(\mathcal{M}) \right)^2 \delta.
\]

From \( |h(p) h(q)| \geq |p q| - \delta \) and a summand-wise comparison we get that for all \( p, q \in P \) (using the notation \( \tilde{p} = h(p), \tilde{q} = h(q) \)),

\[
|\tilde{p} \tilde{q}| \leq |pq| + \left( \#_1^{\mathrm{pack}}(\mathcal{M}') - 1 \right)^2 \delta + \left( \#_{l-L}^{\mathrm{pack}}(\mathcal{M}) \right)^2 \delta
\]

\[
\leq |pq| + \left( \#_1^{\mathrm{pack}}(\mathcal{M}') - 1 \right)^2 \delta + \left( \#_{l-L}^{\mathrm{pack}}(\mathcal{M}) \right)^2 \delta
\]

\[
\leq |pq| + C' \delta
\]

where the parameter \( C' \) depends upon \( \mathcal{M} \) and \( l - L \).

To conclude the argument for claim (ii), let \( x, y \in \mathcal{M}' \). Set \( \tilde{x} = h(x) \) and \( \tilde{y} = h(y) \). We derive an estimate for \( \tilde{x} \tilde{p}, \tilde{y} \tilde{q} \) such that \( |\tilde{x} \tilde{p}|, |\tilde{y} \tilde{q}| < l \). Observe that \( |x p|, |y q| \leq l + \delta \), so we have

\[
|pq| \leq |x p| + |x y| + |y q| \leq |x y| + 2(l + \delta).
\]

Now we can apply (28)

\[
|x y| \geq |pq| - 2(l + \delta) \geq |\tilde{x} \tilde{p}| - C' \delta - 2(l + \delta).
\]

Finally, we obtain by setting \( C := C' + 2 \)

\[
|\tilde{x} \tilde{y}| \leq |\tilde{x} \tilde{p}| + |\tilde{p} \tilde{q}| + |\tilde{q} \tilde{y}| \leq l + (|x y| + C' \delta + 2(l + \delta)) + l = 4l + C\delta.
\]

This theorem in particular implies that in the category \( \mathbf{MET}(\mathcal{M}, \|\cdot\|_{\mathrm{diam}}) \) every endomorphism is an isomorphism. Such categories are called EI-categories and have been studied for several decades [Die87].

Example 23 (counterexample). With regard to a global estimate on the density of \( h(M) \) in claim (i) in Lemma 22 we give a counterexample that shows that the claim does not hold for \( \|\cdot\|_{-\mathrm{diam}} \). Consider the metric on \( M := \{0, 1, \ldots, n\} \) determined by

\[
|i j| := \begin{cases} j - 1 & \text{if } j \geq 2 \\ 1 & \text{if } j = 1 \end{cases}
\]
for \( i < j \) and the map \( h : \mathcal{M} \to \mathcal{M} \) defined by
\[
hiber
\]
Indeed \( \|h\|_{\text{diam}} = 0 \) but \( n \notin h(M) \) and, hence, \( B(n, n - 1) \subseteq (h_* M)^C \).

Proof of Theorem 22. Set \( sk_0 := sk_0(\mathcal{M}, \|\cdot\|_{\text{diam}}) \). First, we prove the 2-Lipschitz property of id: \( (sk_0, d_{GH}) \to (sk_0, d_{\text{diam}}) \). Set \( l := d_{GH}(\mathcal{M}, \mathcal{N}) \). For every \( \varepsilon > 0 \) we have embeddings \( \mathcal{M} \overset{f}{\to} \mathcal{L} \overset{g}{\to} \mathcal{N} \) such that \( f_* M \subseteq (g_* N)^{1+\varepsilon} \) and \( g_* N \subseteq (f_* M)^{1+\varepsilon} \). Set \( h(x) := B[x, l + \varepsilon] \cap N \), where the ball and the intersection are in \( \mathcal{L} \). Observe that
\[
\|h\|_{\text{diam}} = \sup_{x,y} |x - y| \geq \sup_{x,y} |f(x) - f(y)| \geq \sup_{x,y} |x - y| + |f(x) - f(y)| \geq 2(l + \varepsilon).
\]
Since \( \varepsilon > 0 \) can be chosen arbitrarily small, \( d_{\text{diam}}(\mathcal{M}, \mathcal{N}) \leq 2l \). The analog argument with \( \mathcal{M} \) and \( \mathcal{N} \) interchanged gives \( d_{\text{diam}}(\mathcal{M}, \mathcal{N}) \leq 2l \). This implies
\[
d_{\text{diam}}^+(\mathcal{M}, \mathcal{N}) \leq 2l.
\]
To show that id: \( (sk_0, d_{\text{diam}}^+) \to (sk_0, d_{GH}) \) is Cauchy continuous it suffices to show that for any Cauchy sequence \( \mathcal{M}_n \) with respect to \( d_{\text{diam}}^+ \) the following holds: for all \( N \in \mathbb{N}, L > 0 \) we have that if \( \forall n > N: d_{\text{diam}}^+(\mathcal{M}_n, \mathcal{M}_n) < L/2, \) then \( \exists M \geq N: \forall n, m \geq M: d_{GH}(\mathcal{M}_n, \mathcal{M}_m) \leq 5L \).

Take such \( N \) and \( L > 0 \) so that for all \( n > N \) we have \( d_{\text{diam}}^+(\mathcal{M}_n, \mathcal{M}_n) < L/2 \). We know that \( d_{\text{diam}}(\mathcal{M}_n, \mathcal{M}_n) < L \) for all \( n > N \). Let \( C = C(\mathcal{M}_N, \mathcal{L}) \) be the parameter from Lemma 22. Choose \( M \geq N \) so large that for all \( n, m > M \) there are maps \( \mathcal{M}_n \overset{f_{nm}}{\to} \mathcal{M}_m \overset{g_{nm}}{\to} \mathcal{M}_m \) such that \( \|f_{nm} ; g_{nm}\|_{\text{diam}} < L/C \otimes L \) and \( \|f_{nm}\|_{\text{diam}}, \|g_{nm}\|_{\text{diam}} < L \). Set \( h_{nm} := f_{nm} ; g_{nm} \). Hence by Lemma 22 for sufficiently large \( n \) we have that \( h_{nm}(\mathcal{M}_n) \) is \( L \)-dense in \( \mathcal{M}_n \) and \( \|h_{nm}\|_{\text{diam}}^+ \leq 5L \).

Thus \( g_{nm}(\mathcal{M}_m) \) is \( L \)-dense in \( \mathcal{M}_m \). Therefore \( f_{nm}(\mathcal{M}_m) \) must be \( 2L \)-dense in \( \mathcal{M}_m \).

On \( \mathcal{M}_n \cup \mathcal{M}_m \) consider the symmetric function determined by the assignment
\[
d_{nm}(x, y) := \begin{cases} 
|x y|_n & \text{if } x, y \in \mathcal{M}_n \\
|x y|_m & \text{if } x, y \in \mathcal{M}_m \\
3L + \inf_{x \in \mathcal{M}_n} |x h_{nm}(x')| + |f_{nm}(x') y|_n & \text{if } x \in \mathcal{M}_n, y \in \mathcal{M}_m.
\end{cases}
\]
Obviously, \( d_{nm} \) distinguishes points. Moreover it fulfills the triangle inequality: for convenience set \( f := f_{nm}, g := g_{nm}, \) and \( h := h_{nm} \). Take three points \( x, y, z \in \mathcal{M}_n \cup \mathcal{M}_m \).

The cases \( x, y, z \in \mathcal{M}_n \) and \( x, y, z \in \mathcal{M}_m \) are obvious.

In case \( x, y \in \mathcal{M}_n \) but \( z \in \mathcal{M}_m \) observe that
\[
\inf_{x' \in \mathcal{M}_n} |x h(x')| + |f(x') z|_m \leq \inf_{x' \in \mathcal{M}_n} |x h(x')| + |f(x') z|_n + |f(x') z|_m \leq d_{nm}(x, z)
\]
and, thus, \( d_{nm}(x, z) \leq d_{nm}(x, y) + d_{nm}(y, z) \). The case \( x \in \mathcal{M} \) and \( y, z \in \mathcal{M}_n \) is parallel.

In the case \( x, z \in \mathcal{M}_n \) and \( y \in \mathcal{M} \) observe
\[
d_{nm}(x, y) + d_{nm}(y, z)
\]
\[
= 6L + \inf \{ |x h(y')| + |f(x') y|_n + |z h(z')| + |f(z') y|_n | x', z' \in \mathcal{M} \}
\]
\[
\geq 6L + \inf \{ |x h(x')| + |f(x') f(z')|_n + |z h(z')| | x', z' \in \mathcal{M} \}
\]
\[
\geq 5L + \inf \{ |x h(x')| + |x' z'| + |z h(z')| | x', z' \in \mathcal{M} \}
\]
\[
\geq 5L + \inf \{ |x h(x')| + |h(x') h(z')| - \|h\|_{\text{diam}}^+ + |z h(z')| | x', z' \in \mathcal{M} \}
\]
\[
\geq \inf \{ |x h(x')| + |h(x') h(z')| + |z h(z')| | x', z' \in \mathcal{M} \}
\]
\[
\geq |x z|.
\]
In the remaining case \( x, z \in M_n \) and \( y \in M \) we get
\[
\begin{align*}
 d_{nm}(x, y) + d_{nm}(y, z) \\
= 6L + \inf\{ |h(y')| + |f(y') x|_n + |y h(y'')| + |f(y'')| z|_n | y', y'' \in M \} \\
\geq 6L + \inf\{ |h(y') h(y'')| + |f(y') x|_n + |f(y'')| z|_n | y', y'' \in M \} \\
\geq 5L + \inf\{ |f(y') f(y'')| + |f(y') x|_n + |f(y'')| z|_n | y', y'' \in M \} \\
\geq L + |x z|.
\end{align*}
\]

Within \((M_n \cup M_m, d_{nm})\) we have \( M_n^{2L} \supseteq M_n \) since \( f_{nm}(M_n) \) is \( 2L \)-dense in \( M_m \).
By the same fact, \( M_n^{2L} \supseteq ((f_{nm})_M)^{2L} \supseteq M_m \). Hence \( d_{GH}(M_n, M_m) \leq 5L \). □

The fact that the Gromov-Hausdorff space is complete [Pet16, § 11.1.1] implies:

**Corollary 24.** The space \((\text{sk}_0(\text{MET}, \| \cdot \|_{\text{diam}}), d^+_\text{diam})\) is a complete metric space.

**Corollary 25.** The category \((\text{MET}, \| \cdot \|_{\text{diam}})\) is normed.

**Proof.** For the first property, \((\text{N3})\), note that given two maps \( f : M \to N \) and \( g : N \to M \) with \( \| f \|_{\text{diam}} = \| g \|_{\text{diam}} = 0 \) their compositions \( f \circ g \) and \( g \circ f \) have vanishing dilatation norm as well. Lemma [22] implies in this case that \( f ; g \) and \( g ; f \) have a dense image and are contractions. Hence both maps are also isometries in this set up. Thus by set theoretic arguments \( f \) and \( g \) are bijections. Since \( f \) and \( g \) are both expansions they also have to be contractions: assume for instance in the case of \( f \) that the distance between two points \( x, y \in M \) is expanded, i.e. \( |x y| < |f(x) f(y)| \), then also \( |x y| < |g(f(x)) g(f(y))| \) in contradiction to \( \| f ; g \|_{\text{diam}} = 0 \).

For the second property, \((\text{N4})\), take \( f_n : M \to N \) with \( \| f_n \|_{\text{diam}} = 0 \). We do a diagonal argument. Take a dense sequence \( x_1, \ldots \in M \). For each \( i = 1, \ldots \) choose a sequence \( x_{ij} \) such that \( x_{ij} \to x_i \) as \( j \to \infty \) and \( x_{ij} \in f_j^* N \) for each \( j \). We choose a sequence \( j_1(1), j_1(2), \ldots \) such that for some \( y_{jn} = f_{j_1(n)}(x_{jn}) \) the sequence \( y_n \) converges as \( n \to \infty \) (using compactness of \( N \)). We proceed by choosing a further sequence \( j_2(1), j_2(2), \ldots \) such that some \( y_{2n} = f_{j_2(n)}(x_{2n}) \) converge as \( n \to \infty \). We continue this procedure with sequences \( j_3(\cdot), j_4(\cdot), \ldots \). We define the densely defined function \( f : M \to N \) by \( f(x_i) = \lim_{n \to \infty} y_{jn} \). By construction \( \| f \|_{\text{diam}} = 0 \). Thus \( \| f \|_{\text{diam}} = 0 \) as follows from \((21c)\) and an easy limit argument. □

**Appendix A. Notation**

In this appendix we will describe much notation used throughout this work. Other notation can be found in our references or is defined along the way in the body of this article.

**A.1. Set theory.** Per common practice, we will typically use the notation of Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC), but in many cases, especially when we need to work with proper classes, we will actually use the relatively consistent extension referred to generally as GBN (Gödel-Bernaise-von Neumann) Class-Set Theory with the Axiom of Choice. Foundationally, we could use instead category theoretic foundations, but that seems to us to be merely a matter of taste. Accordingly, we leave it to the readers to adjust our recipes and seasonings for the dishes we describe to their preferences.

Given a set \( X \), we denote by \( \mathcal{P}(X) \) the power set of \( X \), i.e. the set of all subsets of \( X \). We will find it convenient to have a systematic notation for image and
preimage mappings on the power set of a set, but in fact, for more variants than merely the simplest pair of such, whence, we denote by $f_*(A)$, or just $f_*$, the set $\{ f(x) \mid x \in A \}$, where $A$ is a subset of the domain of $f$, denote the preimage of any set $B$ under $f$ by $f^*(B)$ or $f^*B$ and by $f_!$ or $f_!(A)$ we denote the small image of $A$, $f_!(A) := \{ y \mid f(x) = y \implies x \in A \}$ (note that $f_!(A)$ always contains all points in $Y$ that are not hit by $f$). One should be careful that the adjunctions for $\mathcal{P}(X)$, $\mathcal{P}(Y)$ are reversed compared to the adjunctions for a geometric morphism in algebraic geometry. This is because in algebraic geometry one would not study $\mathcal{P}(X)$ or $\mathcal{P}(Y)$ but sheaves thereon. Thus our adjunctions are

$$ f_* \dashv f^* \dashv f_! $$

Note that both $f_!$ or $f_*$ are completely determined by this property, as—dwelling in the set-up of posets—these adjunctions are actually Galois connections.

A.2. Orders. We denote partial orders by $\mathcal{X} = (X, \leq), \mathcal{Y} = (Y, \subseteq)$, and $\mathcal{Z} = (Z, \subseteq)$. Let $\mathcal{L} = (L, \leq)$ be a complete lattice, i.e. a poset that admits all suprema (and consequently, all infima). We define $\sup : L \times \mathcal{P}(L) \to L$ by

$$ \sup(m, M) := \sup(M \cap \{ m' \mid m \leq m' \}) \cup \{ m \}, $$

and then for any $m \in L$, set

$$ \sup^m M := \sup(m, M) $$

for any subset $M$ of $L$. Moreover, if $I$ is any (indexing) set, then

$$ \sup^a \mathcal{P} := \sup^a \{ f(i) \mid i \in \mathcal{P} \}, $$

for any subset $\mathcal{P}$ of $I$ and function $f$ on $I$. If not specified otherwise, $\sup$ is understood as the supremum function on the extended real numbers $[-\infty, \infty]$.

A.3. Categories. We list the (2-)-categorical notation used in this article:

- special (bi-)categories: $\mathbf{Top}$, $\mathbf{Met}$, etc.;
- variables for (bi-)categories: $\mathbf{C}, \mathbf{D}, \mathbf{E}$;
- variables for objects: $X, Y, M, \ldots$;
- variables for morphisms $f, g, h$;
- $f : \text{source}(f) \to \text{target}(f)$;
- variables for 2-morphisms: $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$, etc.;
- $\mathbf{C}_0/\mathbf{C}_2$ for the collection of objects/morphisms/2-morphisms
- $\mathbf{C}[X,Y]$ set of morphisms from $X$ to $Y$ in category $\mathbf{C}$, written $[X,Y]$ if category is specified by the context;
- composition $f : g = g \circ f$ of morphisms;
- vertical composition $\alpha \circ^v \beta = \beta \circ^v \alpha$ of 2-morphisms;
- horizontal composition $\alpha \circ^h \beta = \beta \circ^h \alpha$ of 2-morphisms;
- we write $\circ = \circ_{\mathbf{C}} = \circ_{\mathbf{C}}$, $\circ^v = \circ_{\mathbf{C}}$, $\circ^h = \circ_{\mathbf{C}}$, $\circ = \circ_{\mathbf{C}}$ to specify the category.

APPENDIX B. 2-CATEGORICAL VIEWPOINT ON SEMINORMS

Let $\mathbf{Cat}$ denote the category of small categories. Note that this category has products. Recall that a bifunctor from $\mathbf{C}$ and $\mathbf{D}$ to $\mathbf{E}$ is a functor from a product category $\mathbf{C} \times \mathbf{D}$ to $\mathbf{E}$.
B.1. **Strict 2-categories.** A strict 2-category $\mathcal{C}$ is a category enriched in $\text{CAT}$ meaning that $\mathcal{C}$ consists of

- a class $\mathcal{C}_0$ of objects;
- for each $X, Y \in \mathcal{C}_0$ a category $\mathcal{C}[X, Y] \in \text{CAT}$. Morphisms in $\mathcal{C}[X, Y]$ are called 2-morphisms and depicted by $\Rightarrow$. Composition of such morphisms is called vertical composition and denoted by $;$. For all objects $X, Y, Z \in \mathcal{C}_0$ there is a bifunctor $\triangleright: \mathcal{C}[X, Y] \times \mathcal{C}[Y, Z] \to \mathcal{C}[X, Z]$, $(f, g) \mapsto F^g_{\triangleright} f$, $(f, g) \mapsto \alpha; F^g_{\triangleright} f = \beta$. B.1.

Lax functors. B.2.

there are weaker notions of a 2-category, especially bicategories.

- for each object $X$ an identity element $\text{id}_X \in \mathcal{C}[X, X]$.

such that the following axioms are satisfied

- for object $X, Y, Z, X'$ an associativity law, the equality of functors

$$\triangleright_{XXY'} \circ \mathcal{C}_{XY} (\text{id}_{\mathcal{C}[X,Y]} \times (\triangleright_{X'YZ})) = \triangleright_{XZX'} \circ \mathcal{C}_{XY} ((\triangleright_{XYZ}) \times \text{id}_{\mathcal{C}[Z,X']})$$

(which for morphisms $X \overset{f}{\to} Y \overset{g}{\to} X' \overset{h}{\to} X''$ states that $f \triangleright_{XXY'} (g \triangleright_{X'YZ}) = (f \triangleright_{XYZ}) \circ (g \triangleright_{X'Y''Z})$).

- for each $X \overset{f}{\to} Y \in \mathcal{C}_0$ the identity law

$$f = \text{id}_X \triangleright_{XXY} f = f \triangleright_{XXY} \text{id}_Y .$$

Note that strict 2-categories are too restrictive for many applications. Therefore there are weaker notions of a 2-category, especially bicategories.

B.2. **Lax functors.** In the case of strict 2-categories a lax functor $F: \mathcal{C} \to \mathcal{D}$ can be defined as an assignment of

- (IF1) each $X \in \mathcal{C}_0$ to an object $F_X \in \mathcal{D}_0$;
- (IF2) each $\mathcal{C}[X, Y]$ to a functor $F_{XY}: \mathcal{C}[X, Y] \to \mathcal{C}[F_X, F_Y]$;
- (IF3) (lax preservation of identity) each $X \in \mathcal{C}_0$ to an invertible 2-morphism $\text{id}_{F_X}: \text{id}_{F_X} \Rightarrow F_X (\text{id}_X)$ in $\mathcal{D}_0$;
- (IF4) (lax preservation of composition) each $X, Y, Z \in \mathcal{C}_0$ to a natural transformation $F_{XY} \circ F_{YZ}$ from the bifunctor $(f, g) \mapsto F_{XY} (f \triangleright_{X'YZ} g)$, $\mathcal{C}[X, Y] \times \mathcal{C}[Y, Z] \to \mathcal{C}[F_X, F_Z]$ to the bifunctor $(f, g) \mapsto F_{F_{XY}} (f \triangleright_{X'Y''Z} g)$; such that

- (IF5) for each $X, Y \in \mathcal{C}_0$ and $f \in \mathcal{C}[X, Y]$ an identity law

$$F_{X} (\text{id}_X \triangleright_{XXY} f) \circ \text{id}_{F_{XY} (f)} = \text{id}_{F_{XY} (f)}$$

$$\text{id}_{F_{XY} (f)} \circ F_{X} (\text{id}_X \triangleright_{XXY} f) = \text{id}_{F_{XY} (f)}$$

- (IF6) for each diagram $X \overset{f}{\to} Y \overset{g}{\to} X' \overset{h}{\to} X''$ an associativity law

$$(F_{XY} (f \triangleright_{X'YZ} g)) \circ \text{id}_{F_{XZ} (h)} \circ F_{XZ} (f \triangleright_{X'Y''Z} g) = \text{id}_{F_{XY} (f \triangleright_{X'Y''Z} g)} \circ F_{XZ} (f \triangleright_{X'Y''Z} g)$$

Note that for more general bicategories the last two properties become more complicated. If the units $F_{\text{id}_X}$ are identities, i.e. $F_{\text{id}_X} = \text{id}_{F_{\text{id}_X}}$, the lax functor is called normal.
B.3. Seminorms as lax functors. Remember that every set $S$ can be regarded as a category by interpreting $S$ as the set of objects and allowing only trivial morphisms, $C[x, y] = \emptyset$ for $x \neq y$ and $C[x, x] = \{\text{id}_x\}$. In the same manner any (ordinary) category $C$ can be regarded as a strict 2-category by defining $C[X, Y]_0 = C[X, Y]$ and $C[f, g] = \emptyset$ for distinct $f, g \in C[X, Y]$ or $C[f, f] = \{\text{id}_f\}$. Another example is given by a $(\ast, [0, \infty], +, \geq)$ where
\[
\begin{align*}
(30a) & \quad (\ast, [0, \infty], +, \geq)_0 := \{\ast\} \\
(30b) & \quad (\ast, [0, \infty], +, \geq)[\ast, \ast] := ([0, \infty], \geq) \quad \text{with } \ast' = \geq \\
(30c) & \quad r, s \ast := r + s.
\end{align*}
\]
One immediately checks that $(30c)$ is functorial from the fact that $r \geq r'$ and $s \geq s'$ implies $r + s \geq r' + s'$.

**Proposition 26.** A seminorm $\| \cdot \| : C \to [0, \infty]$ on a (1-)category $C$ forms a lax functor $\| \cdot \| : C \to (\ast, [0, \infty], +, \geq)$ by the assignments
\[
\begin{align*}
(31a) & \quad F_X := \ast \quad \text{for all } X \in C_0 \\
(31b) & \quad F_{00}(f) := \|f\| \quad \text{for all } f \in C_i \\
(31c) & \quad F_{00}(f)(\text{id}_f) := ([\|f\| \geq \|f\|]) \quad \text{for all } f \in C_i \\
(31d) & \quad F_{00}(f)(0 \geq 0) := 0 \geq 0 \quad \text{for all } X \in C_0 \\
(31e) & \quad F_{000}(f, g) := ([\|f\| + \|g\| \geq \|f \cdot g\|]) \quad \text{for all } X \xrightarrow{f} Y \xrightarrow{g} Z.
\end{align*}
\]

**Proof.** First observe that $F$ is well-defined: the function defined by $(31c)$ is in fact total since the only 2-morphisms in $C$ are identity morphisms. In $(31e)$ the image exists by triangle inequality.

The identity laws hold automatically since the 2-homsets $(\ast, [0, \infty], +, \geq)[\ast, \ast][r, s]$ are at most singletons for all $r, s \in [0, \infty]$. The same applies to the associativity law.

Note that any lax functor to $(\ast, [0, \infty], +, \geq)$ is automatically normal.

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