On the epistemic view of quantum states

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We investigate the strengths and limitations of the Spekkens toy model, which is a local hidden variable model that replicates many important properties of quantum dynamics. First, we present a set of five axioms that fully encapsulate Spekkens’ toy model. We then test whether these axioms can be extended to capture more quantum phenomena, by allowing operations on epistemic as well as ontic states. We discover that the resulting group of operations is isomorphic to the projective extended Clifford Group for two qubits. This larger group of operations results in a physically unreasonable model; consequently, we claim that a relaxed definition of valid operations in Spekkens’ toy model cannot produce an equivalence with the Clifford Group for two qubits. However, the new operations do serve as tests for correlation in a two toy bit model, analogous to the well known Horodecki criterion for the separability of quantum states.

I. INTRODUCTION

Spekkens introduced a toy theory that demonstrates how a local hidden variable model with a classical information-based restriction can capture a great deal of seemingly quantum phenomena, including non-commutativity of measurement, remote steering and teleportation [1]. Spekkens’ toy model (STM) builds upon other information-based models with similar aims [2, 3, 4, 5]. Although by no means a proposed axiomatization of quantum theory, STM aims to strengthen the view that the quantum state is a statistical distribution over a hidden variable space in which there exists a balance of knowledge and ignorance about the true state of the system.

In this paper, we axiomatize STM, and test it by relaxing its axioms. We claim that STM can be formalized into five axioms describing valid states, allowable transformations, measurement outcomes, and composition of systems. Arguing on empirical grounds, we relax the axiom regarding valid operations on toy bits to obtain larger groups of operations for one and two toy bits. We claim that these larger groups are isomorphic to the projective extended Clifford Group for one and two qubits respectively. However, these larger groups of operations contain elements that do not necessarily compose under the tensor product. That is to say, there exist operations that do not take valid states to valid states when composed under the tensor product, as one would demand of a physical model. These operations are analogous to positive maps in quantum theory. Just as positive (but not completely positive) maps can be used to test whether a quantum state is entangled or not [20], validity-preserving (but not completely validity-
preserving) maps can be used to test for correlations in the two toy bit STM. Finally, we claim that relaxing the transformations of STM to an epistemic perspective gives rise to physically unreasonable alternatives, and as such, no equivalence with the extended Clifford Group for two qubits can be established by relaxing STM’s operations.

The outline of the paper is as follows. In Section II, we present STM as a series of axioms and compare them to the axioms of quantum theory. We provide a brief review of the original model for an elementary toy system (a toy bit) and for two toy bits and provide a number of different ways of representing one and two toy bits. In Section III we propose a relaxation of the criterion for valid operations on elementary systems, identify the resulting groups of operations, and analyze both their mathematical and physical properties. We conclude with a discussion of our results in Section IV.

II. THE SPEKKENS TOY MODEL AND QUANTUM THEORY

In this section we present STM in its axiomatic basis and state the axioms of quantum mechanics for comparison. Using the axioms of STM we develop several ways of representing toy bits including a vector space, a tetrahedron, and a toy analogue of the Bloch sphere. We also develop two ways of representing two toy bits: a product space and a four-dimensional cube. We show how states, operations, and tensor products stem from the axioms of STM, and we draw parallels to the equivalent axioms and concepts in quantum theory.

STM is based on a simple classical principle called the knowledge balance principle:

If one has maximal knowledge, then for every system, at every time, the amount of knowledge one possesses about the ontic state of the system at that time must equal the amount of knowledge one lacks.

Spekkens realizes the knowledge balance principle using canonical sets of yes/no questions, which are minimal sets of questions that completely determine the actual state of a system. For any given system, at most half of a canonical set of questions can be answered. The state a system is actually in is called an ontic state, whereas the state of knowledge is called an epistemic state.

STM can be succinctly summarized using the following axioms:

STM 0: All systems obey the knowledge balance principle.

STM 1: A single toy bit is described by a single hidden variable that can be in 1 of 4 possible states, the ontic states.

The knowledge balance principle insists that the hidden variable is known to be in a subset of 2 or 4 of the ontic states—that subset is the epistemic state of the system.
STM 2: A valid reversible operation is a permutation of ontic states of the system that also permutes the epistemic states amongst themselves.

STM 3: A reproducible measurement is a partition of the ontic states into a set of disjoint epistemic states, with the outcome of a measurement being a specific epistemic state. The probability of a particular outcome is proportional to the number of ontic states that outcome has in common with the current epistemic state. Immediately after the process of measurement, the epistemic state of the system is updated to the outcome of the measurement.

STM 4: Elementary systems compose under the tensor product giving rise to composite systems; the knowledge balance principle applies to the composite system as well as to the parts.

To help make the comparison with quantum theory, the corresponding axioms of quantum mechanics are given below.

QM 1: Any isolated physical system corresponds to a complex vector space with an inner product, a Hilbert space. A system is completely described by a ray in Hilbert space.

QM 2: Evolution of a closed system is described by a unitary transformation through the Schrödinger equation

\[ \dot{\psi} = i\hbar \frac{\partial \psi}{\partial t} \] (1)

whereas \( \hat{H} \) is a Hermitian operator.

QM 3: Measurement is described by a collection, \( \{ M_m \} \), of measurement operators. These are operators acting on the state space of the system being measured. The index \( m \) refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is \( \psi \) immediately before the measurement then the probability that result \( m \) occurs is given by

\[ p(m) = \langle \psi | \hat{M}_m^\dagger \hat{M}_m | \psi \rangle, \] (2)

and the state after measurement is given by

\[ \frac{\hat{M}_m | \psi \rangle}{\sqrt{\langle \psi | \hat{M}_m^\dagger \hat{M}_m | \psi \rangle}}. \] (3)

Measurement operators satisfy \( \sum_m \hat{M}_m^\dagger \hat{M}_m = I \).

QM 4: The state space of a composite system is the tensor product of the state space of the component systems.

The simplest system that can exist is a single toy bit system: there are two yes/no questions in a canonical set, yielding four ontic states, which we label \( o_1, o_2, o_3, \) and \( o_4 \). A pair of ontic states forms the answer to one of the two
questions in a canonical set. The knowledge balance principle restricts us to knowing the answer to at most one of two questions, resulting in a pure epistemic state. The six pure states are shown pictorially in Fig. 1. (In Spekkens’ original notation, the state \( e_{ij} \) was denoted \( i \lor j \).)

By way of example, the questions “Is the ontic state in \( \{ o_1, o_2 \} \)?” and “Is the ontic state in \( \{ o_1, o_3 \} \)?” form one particular canonical set. The epistemic state \( e_{12} = o_1 + o_2 \) corresponds to the situation in which the first question can be answered, and it is in the affirmative. The model also includes a single mixed epistemic state, namely \( e_{1234} = o_1 + o_2 + o_3 + o_4 \), corresponding to knowing absolutely nothing about the system.

At this point we introduce the linear representation for the toy model which will be convenient for describing operations later. Let \( \{ o_1, o_2, o_3, o_4 \} \) be a basis for a real vector space, and express the epistemic states in that basis. Each pure epistemic state is then a vector with exactly two 1’s and two 0’s; for example,

\[
e_{12} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.
\]

Note that epistemic states that are disjoint (that is, have no ontic states in common) are orthogonal as vectors in \( \mathbb{R}^4 \).

Now that states in the toy model are defined, we turn our attention to transformations between states. STM 2 states that valid operations are permutations of ontic states. The group of permutations of four objects is denoted \( S_4 \), and permutations are usually summarized using cyclic notation (see [1, p. 7] for details). By way of example, the permutation \((123)(4)\) maps \( o_1 \) to \( o_2 \), \( o_2 \) to \( o_3 \), \( o_3 \) to \( o_1 \), and \( o_4 \) to \( o_4 \). In terms of epistemic states, \((123)(4)\) maps \( e_{12} \) to \( e_{23} \). In the linear representation, each transformation in \( S_4 \) is a \( 4 \times 4 \) permutation matrix that acts on the left of the epistemic state vectors. For example,

\[
(123)(4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

We call this the regular representation of \( S_4 \), and we will call this description of STM the linear model.

Since the group of operations on a single toy bit is such a well-studied group, there are other classical systems of states and transformations that may be readily identified with the single toy bit. One such system uses a regular
tetrahedron. In this geometric representation, the vertices of the tetrahedron represent the ontic states of the system, whereas pure epistemic states are represented by edges (see Fig. 2). The action of a transformation in $S_4$, then, is a symmetry operation on the tetrahedron. For example, the transformation $(123)(4)$ permutes vertices $o_1$, $o_2$, and $o_3$ of the tetrahedron by rotating counter-clockwise by $2\pi/3$ about the axes which passes through the center of the tetrahedron and vertex $o_4$. Since $S_4$ is the entire group of permutations of $\{o_1, o_2, o_3, o_4\}$, it is also the complete group of symmetry operations for the regular tetrahedron. Notice that $A_4$, the alternating group (or group of even permutations), corresponds to the group of rotations, whereas odd permutations correspond to reflections and roto-reflections.

As pointed out by Spekkens, another way of viewing the single toy bit is using a toy analogue of the Bloch sphere. In the toy Bloch sphere, epistemic states are identified with particular quantum states on the traditional Bloch sphere and are embedded in $S^2$ accordingly. In particular, $e_{13}$, $e_{23}$, and $e_{12}$ are identified with $|+\rangle$, $|i\rangle$, and $|0\rangle$ and are embedded on the positive $x$, $y$, and $z$ axes respectively (see Fig. 3). States that are orthogonal in the linear model are embedded as antipodal points on the toy Bloch sphere, just as orthogonal quantum states are embedded antipodally on the quantum Bloch sphere. Distance on the toy Bloch sphere corresponds to overlap between states: two epistemic states have an angle of $\pi/2$ between them if and only if they have exactly one ontic state in common.

On the quantum Bloch sphere, single qubit transformations are represented by rotations in the group $SO(3)$, and they may be characterized using Euler rotations. More precisely, if $R_x(\theta)$ denotes a rotation about the $x$-axis by $\theta$, then any $T \in SO(3)$ may be written in the form

$$T = R_x(\theta)R_z(\phi)R_x(\psi), \quad 0 \leq \theta \leq \pi, \quad -\pi < \phi, \psi \leq \pi.$$  (5)
For example, the rotation by $2\pi/3$ about the $x+y+z$ axis may be written as $R_{x+y+z}(2\pi/3) = R_x(\pi)R_z(\pi/2)R_x(\pi/2)$ (see Fig. 4).

FIG. 4: The element $R_{x+y+z}(2\pi/3)$ expressed as a series of Euler rotations.

On the toy Bloch sphere, in contrast, transformations are elements of $O(3)$, not all of which are rotations. For example, the permutation $(12)(3)(4)$ is not a rotation of the toy Bloch sphere but a reflection through the plane perpendicular to the $x-y$ axis (see Fig. 5). Thus, there are operations in the single toy bit model that have no quantum analogue. (We will see shortly that such toy operations correspond to anti-unitary quantum operations.)

FIG. 5: The element $(12)(3)(4)$ acts as a reflection on the toy sphere.

The toy operations that do correspond to rotations on the Bloch sphere are precisely the operations in $A_4$, the group of even permutations. In terms of the linear model, these are the transformations of $S_4$ with determinant 1. Toy operations not in $A_4$ may be expressed as a rotation composed with a single reflection. When $T$ is a rotation on the toy Bloch sphere, its Euler rotations $R_x(\theta)R_z(\phi)R_x(-\psi)$ satisfy $\theta \in \{0, \pi/2, \pi\}$ and $\phi, \psi \in \{-\pi/2, 0, \pi/2, \pi\}$. For example, the permutation $(123)(4)$ corresponds to the rotation $R_{x+y+z}(2\pi/3)$ seen in Fig. 4.

STM 3 addresses the problem of measurement in the toy theory. For a single toy bit, a measurement is any one question from a canonical set; thus there are a total of six measurements that may be performed. After a measurement is performed and a result is obtained, the observer has acquired new information about the system and updates his state of knowledge to the result of the measurement. This ensures that a repeat of the question produces the same outcome. Note that the outcome of a measurement is governed by the ontic state of the system and not the measurement itself. The question “Is the ontic state in $\{o_m, o_n\}$?” can be represented by a vector $r_{mn} = o_m + o_n$. The probability of
getting “yes” as the outcome is then

\[ p_{mn} = \frac{r_{mn}^T e_{ij}}{2}, \]

(6)

where \( e_{ij} \) is the current epistemic state of the system. After this outcome, the epistemic state is updated to be \( e_{mn} \). The vectors \( r_{mn} \) and probabilities \( p_{mn} \) are analogous to the measurement operators and outcome probabilities in QM 3.

STM 4 concerns the composition of one or more toy bits. For the case of two toy bits there are four questions in a canonical set, two per bit, giving rise to 16 ontic states, which we denote \( o_{ij}, \ i, j = 1 \ldots 4 \). In the linear model this is simply the tensor product of the 4-dimensional vector space with itself, and the ontic state \( o_{ij} \) is understood to be \( o_i \otimes o_j \). The types of epistemic states arising in this case are of three types; maximal, non maximal, and zero knowledge, corresponding to knowing the answers to two, one, or zero questions respectively. It suffices for our purposes to consider only states of maximal knowledge (pure states). These, in Spekkens’ representation, are of two types (see Fig. 6), called uncorrelated and correlated states respectively. An uncorrelated state is the tensor product of two pure single toy bit states. If each of the single toy bits satisfy the knowledge balance principle, then their composition will also satisfy the knowledge balance principle for the composite system. A correlated state is one in which nothing is known about the ontic state of each elementary system, but everything is known about the classical correlations between the ontic states of the two elementary toy systems. If the two single bit systems in Fig. 6(b) are labelled A and B, then nothing is known about the true state of either A or B, but we know that if A is in the state \( o_i \), then B is also in the state \( o_i \).

According to STM 2, operations on two toy bits are permutations of ontic states that map epistemic states to epistemic states. These permutations are of two types: tensor products of permutations on the individual systems, and indecomposable permutations (see Fig. 7). Moreover, STM 4 suggests that if an operation is valid on a given system, then it should still be valid when an ancilla is added to that system. That is, if \( T \) is a valid operation on a single toy bit, then \( T \otimes I \) ought to be valid on two toy bits. It follows that valid operations should compose under the tensor product.

Finally, STM 3 implies that a measurement of the two toy bit space is a partition of ontic states into disjoint epistemic

![FIG. 6: (a) Uncorrelated and (b) correlated states in the toy model.](image)
FIG. 7: Operations on two toy bits: (a) a tensor product operation and (b) an indecomposable permutation.

states: each epistemic state consists of 4 or 8 ontic states. There are in total 105 partitions of the two toy bit space into epistemic states of size 4.

In the linear model, epistemic states, operations, and measurements extrapolate in the manner anticipated. A pure epistemic state is a \( \{0, 1\}\)-vector of length 16 containing exactly 4 ones, whereas an operation is a \(16 \times 16\) permutation matrix. The group of operations can be computationally verified to be of order 11520. Measurement is a row vector \( r_{ijkl} \in \{0, 1\}^{16} \) with the state after measurement updated according to the outcome obtained. In the linear model STM 4 is understood as the composition of valid states and operations under the tensor product.

Finally, a two toy bit system can be geometrically realized by the four-dimensional cube (see Fig. 8). This is a new representation for the two toy bit system that in some ways generalizes Spekkens’ tetrahedral description of the single toy bit. By mapping the ontic states \( o_1 \ldots o_4 \) of an elementary system to the vertices \((x, y),\ x, y \in \{-1, 1\}\) of a square, the four-dimensional cube is the result of the tensor product of two elementary systems. Every epistemic state is an affine plane containing four vertices, and the group of permutations of two toy bits is a subgroup of \( B_4[3, 3, 4] \), the symmetry group of the four-dimensional cube (for more details, see [7]).

FIG. 8: The four-dimensional hypercube representation for the space of two toy bits.

In this section we reviewed STM, identifying its axioms and drawing a correspondence with the axioms of quantum theory. In the next section, we investigate a relaxation of STM 2.
III. RELAXING THE SPEKKENS TOY MODEL.

In this section we relax STM 2, the axiom describing valid reversible operations. We obtain a new group of operations which contains a subgroup isomorphic to the projective Clifford Group for two qubits, a characteristic of quantum theory not captured by STM. However, the operations in these new group fail to compose under the tensor product, rendering the relaxation of STM 2 physically unreasonable. Nevertheless, we claim that operations that fail to compose under the tensor product can be used as tests for correlations in STM.

Recall that STM 2 describes valid operations on toy states. In particular, STM 2 requires that valid operations act on the ontic states in a reversible manner (ontic determinism). Now consider an empiricist living in a universe governed by the axioms of STM—a toy universe. Such an empiricist has access only to epistemic states. As a result an empiricist sees determinism only at the epistemic scale (epistemic determinism); the knowledge balance principle forbids exact knowledge of the ontic state of the system. For an empiricist, ontic determinism is too strict a condition. We thus propose the following amendment.

STM 2’: A valid reversible operation is a linear transformation that permutes the epistemic states of the system.

. The requirement that transformations be linear implies that as \( e_{1234} = e_{12} + e_{34} \), then \( T(e_{1234}) = T(e_{12}) + T(e_{34}) \) for any valid \( T \); in other words, mixtures of epistemic states are transformed into other mixtures. It follows that pairs of disjoint epistemic states are mapped to other pairs of disjoint states, and the amount of overlap between epistemic states is preserved. This linearity condition is essential if the toy theory is to emulate significant aspects of quantum theory. Investigations into a non-linear theory of quantum mechanics \([9, 10, 11]\) have been experimentally tested and found to be “measurably not different from the linear formalism” \([12]\). Furthermore it was shown by Peres that a non-linear quantum mechanical theory would violate the second law of thermodynamics \([13]\).

We let \( TG(1) \) denote the group of operations obtained by replacing STM 2 with STM 2’. In terms of the linear model, an operation is in \( TG(1) \) if it can be represented as a \( 4 \times 4 \) orthogonal matrix that maps epistemic states to epistemic states. This includes all the operations in \( S_4 \), but it also includes operations such as

\[
\sqrt{Z} = \frac{1}{2} \begin{pmatrix}
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{pmatrix}, \quad \tilde{H} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1
\end{pmatrix}.
\]

On the toy Bloch sphere, \( TG(1) \) is the subgroup of operations in \( O(3) \) that preserve the set of six pure epistemic states. On the toy Bloch sphere, Eq. (7), are the Euler rotations

\[
\sqrt{Z} = R_z \left( -\frac{\pi}{2} \right), \quad \tilde{H} = R_x \left( \frac{\pi}{2} \right) R_z \left( \frac{\pi}{2} \right) R_x \left( \frac{\pi}{2} \right),
\]

(8)
respectively. We have called these operations $\sqrt{Z}$ and $\tilde{H}$ because their action on the toy Bloch sphere resembles the quantum operations $\sqrt{Z}$ and $H$ respectively.

The order of $TG(1)$ is 48, as the next lemma shows.

**Lemma 1.** $TG(1)$ is the set of all permutations of $\{e_{13}, e_{24}, e_{23}, e_{14}, e_{12}, e_{34}\}$ such that pairs of antipodal states are mapped to pairs of antipodal states.

**Proof.** Since $TG(1)$ contains $S_4$ as a proper subgroup, $TG(1)$ has order at least 48. Moreover, every element of $TG(1)$ is a permutation of epistemic states mapping pairs of antipodal points to pairs of antipodal points. We prove the lemma by counting those permutations; as only 48 such operations exist, they must all be in $TG(1)$.

There are three pairs of antipodal states on the toy sphere, namely $\{e_{13}, e_{24}\}$, $\{e_{23}, e_{14}\}$, and $\{e_{12}, e_{34}\}$. Therefore a map that preserves pairs of antipodal points must permute these three pairs: there are $3! = 6$ such permutations. Once a pair is chosen, there are two ways to permute the states within a pair. Therefore, there are a total of $3! \cdot 2^3 = 48$ distinct permutations that map pairs of antipodal states to pairs of antipodal states. \[\square\]

By the argument in Lemma 1, $TG(1)$ may be formally identified with the semidirect product $(\mathbb{Z}_2)^3 \rtimes S_3$, where $g \in S_3$ acts on $(\mathbb{Z}_2)^3$ by

$$g : (x_1, x_2, x_3) \mapsto (x_g(1), x_g(2), x_g(3)), \quad (x_1, x_2, x_3) \in \mathbb{Z}_2^3. \tag{9}$$

An element of $S_3$ permutes the three pairs of antipodal states, whereas an element of $(\mathbb{Z}_2)^3$ determines whether or not to permute the states within each antipodal pair. The following result explains how Spekkens' original group of operations fits into $TG(1)$.

**Lemma 2.** $S_4$ is the subgroup of $(\mathbb{Z}_2)^3 \rtimes S_3$ consisting of elements $((x, y, z), g)$ such that $(x, y, z) \in \mathbb{Z}_2^3$ has Hamming weight of zero or two.

**Proof.** Label the antipodal pairs $\{e_{13}, e_{24}\}$, $\{e_{23}, e_{14}\}$, and $\{e_{12}, e_{34}\}$ with their Bloch sphere axes of $x$, $y$, and $z$. Now $S_4$ is generated by the elements $(12)(3)(4)$, $(23)(1)(4)$, and $(34)(1)(2)$, and by considering the action on the Bloch sphere, we see that these elements correspond to $((0, 0, 0), (z)(xy))$, $((0, 0, 0), (z)(y))$ and $((1, 1, 0), (z)(xy))$ in $(\mathbb{Z}_2)^3 \rtimes S_3$ respectively. Note that $((0, 0, 0), (z)(xy))$ and $((0, 0, 0), (zx)(y))$ generate all elements of the form $((0, 0, 0), g)$ with $g \in S_3$, so adding $((1, 1, 0), (z)(xy))$ generates all elements of the form $((x, y, z), g)$ whereas $(x, y, z)$ has Hamming weight zero or two. \[\square\]

$TG(1)$ exhibits a relationship with the operations in quantum mechanics acting on a single qubit restricted to the six states shown in Fig. 3. To describe the connection, we must first describe the extended Clifford Group.
Recall that the Pauli Group for a single qubit, denoted $\mathcal{P}(1)$, is the group of matrices generated by $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Clifford Group, denoted $\mathcal{C}(1)$, is the normalizer of the Pauli Group in $U(2)$, and is generated by the matrices (see [14])

$$H = \sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \sqrt{Z} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \{e^{i\theta}I \mid 0 \leq \theta < 2\pi\}.$$  \hspace{1cm} (10)

Since $U$ and $e^{i\theta}U$ are equivalent as quantum operations, we focus on the projective group of Clifford operations, namely $\mathcal{C}(1)/U(1) \cong \mathcal{C}(1)/(e^{i\theta}I)$. This is a finite group of 24 elements. For our purposes, the significance of the Clifford Group is that it is the largest group in $U(2)$ that acts invariantly on the set of the six quantum states $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle, |i\rangle, |-i\rangle\} \subset \mathbb{C}^2$ (with $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ considered equivalent).

An anti-linear map on $\mathbb{C}^2$ is a transformation $T$ that satisfies the following condition for all $u, v \in \mathbb{C}^2$ and $\alpha \in \mathbb{C}$:

$$T(\alpha u + v) = \bar{\alpha}T(u) + T(v).$$ \hspace{1cm} (11)

Every anti-linear map may be written as a linear map composed with the complex conjugation operation, namely

$$\text{conj} : \alpha |0\rangle + \beta |1\rangle \mapsto \bar{\alpha} |0\rangle + \bar{\beta} |1\rangle.$$ \hspace{1cm} (12)

An anti-unitary map is an anti-linear map that may be written as a unitary map composed with conjugation. The unitary maps $U(2)$ and their anti-unitary counterparts together form a group, which we denote $EU(2)$. Finally, the extended Clifford Group $\mathcal{EC}(1)$ is the normalizer of the Pauli Group in $EU(2)$. Working projectively, $\mathcal{EC}(1)/U(1)$ is a finite group of 48 elements, generated by $\sqrt{Z}(e^{i\theta}I)$, $H(e^{i\theta}I)$, and $\text{conj}(e^{i\theta}I)$. For more details about the extended Clifford Group, see for example [15].

The following proposition demonstrates the relationship between $TG(1)$ and $\mathcal{EC}(1)/U(1)$.

**Proposition 1.** The toy group $TG(1)$ is isomorphic to the projective extended Clifford Group $\mathcal{EC}(1)/U(1)$.

**Proof.** By Lemma 1, $TG(1)$ consists of all possible ways of permuting $\{e_{13}, e_{24}, e_{23}, e_{14}, e_{12}, e_{34}\}$ such that antipodal points are mapped to antipodal points. Now consider the quantum analogues of these states, namely $|+\rangle, |-\rangle, |i\rangle, |-i\rangle, |0\rangle$, and $|1\rangle$ respectively. For each $T(e^{i\theta}I)$ in $\mathcal{EC}(1)/U(1)$, $T$ is a normalizer of the Pauli Group, so $T(e^{i\theta}I)$ acts invariantly on the six quantum states as a set. Since $T$ is also unitary or anti-unitary, it preserves distance on the Bloch sphere and therefore maps antipodal points to antipodal points. By the argument in Lemma 1, there are only 48 such operations, and it is easy to verify that no two elements of $\mathcal{EC}(1)/U(1)$ act identically. It follows that $\mathcal{EC}(1)/U(1)$ and $TG(1)$ are isomorphic, as both are the group of operations on six points of the Bloch sphere that map pairs of antipodal points to pairs of antipodal points. \hfill $\square$

We now look at the composition of two elementary systems. In the linear model of two toy bits, every valid operation is an orthogonal matrix. As STM 2′ requires that valid operations map epistemic states to epistemic states reversibly,
FIG. 9: (a) $\overline{\text{SWAP}}$, (b) $\overline{P_1}$, (c) $\overline{P_2}$ and (d) $\overline{P_3}$: four operations on two toy bits.

it can be shown that operations such as $I \otimes \tilde{H}P$, with $P \in S_4$, fail to map correlated states to valid epistemic states and therefore are not valid operations. On the other hand, operations such as $P \tilde{H} \otimes Q \tilde{H}$, with $P, Q \in S_4$, are valid under STM $2'$. Let $TG(2)$ denote the group of valid operations for two toy bits. The order of $TG(2)$ can be verified computationally to be 23040, and Spekkens’ group of operations is a subgroup of $TG(2)$.

We discover that $TG(2)$ is very simply related to the extended Clifford Group for two qubits, $\mathcal{EC}(2)$. Let $\mathcal{P}(2)$ be the Pauli Group for two qubits; then the extended Clifford Group for two qubits, $\mathcal{EC}(2)$, is the group of all unitary and anti-unitary operators $U$ such that

$$U\mathcal{P}(2)U^\dagger = \mathcal{P}(2).$$

It is generated by

$$\sqrt{Z} \otimes I, I \otimes \sqrt{Z}, H \otimes I, I \otimes H, \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

the conjugation operation, and unitary multiples of the identity matrix. Working projectively, it can be shown that $\mathcal{EC}(2)/U(1)$ is a group of order 23040 (see [15]). The two-qubit Clifford Group $\mathcal{C}(2)$ is a subgroup of $\mathcal{EC}(2)$, and $\mathcal{C}(2)$ is the largest group in $U(4)$ that acts invariantly on a set of sixty states; this is the same size as the set of epistemic states for two toy bits. The following isomorphism was verified using the computation program GAP [21].

**Proposition 2.** $TG(2)$ is isomorphic to $\mathcal{EC}(2)/U(1)$, the two qubit extended Clifford Group modulo phases.

We give one such isomorphism explicitly. Let $\overline{\text{SWAP}}$ denote the toy operation that swaps rows and columns of ontic states, and let $\overline{P_1}$ and $\overline{P_2}$ be as shown in Fig. 9. For convenience, we use the generating set $\{\text{conj}, \text{CNOT}, H \otimes I, H \otimes H, \sqrt{Z} \otimes \sqrt{Z}\}$ for $\mathcal{EC}(2)$. Then the following map, extended to the entire group, is an isomorphism from $\mathcal{EC}(2)/U(1)$
to $T\Gamma G(2)$:

$$
\text{conj} \langle e^{i\theta} I \rangle \mapsto \frac{1}{4} \left( \begin{array}{cccc}
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1
\end{array} \right) \otimes^2,
$$

$$
\text{CNOT} \langle e^{i\theta} I \rangle \mapsto \text{SWAP} \cdot \frac{1}{4} \left( \begin{array}{cccc}
-1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1
\end{array} \right),
$$

$$
(H \otimes I) \langle e^{i\theta} I \rangle \mapsto \frac{1}{4} \left( \begin{array}{cccc}
1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{array} \right) \otimes \left( \begin{array}{cccc}
1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{array} \right),
$$

$$
(H \otimes H) \langle e^{i\theta} I \rangle \mapsto \tilde{P}_1.
$$

$$
(\sqrt{Z} \otimes \sqrt{Z}) \langle e^{i\theta} I \rangle \mapsto \tilde{P}_2.
$$

A similar GAP computation shows that Spekkens’ group of operations for two toy bits is not isomorphic to $C(2)/U(1)$, despite the fact that both groups have 11520 elements. One way to verify that the two groups are not isomorphic is the following: while the projective Clifford group contains no maximal subgroups of order 720, Spekkens’ group does. One such maximal subgroup is generated by the operations $(12) \otimes (23), I \otimes (12)$, and $\tilde{P}_3$ (also shown in Fig. 9).

As $T\Gamma G(2)$ is isomorphic to the extended Clifford group—which contains the Clifford Group as a proper subgroup—the relaxation of STM 2 to STM 2′ results in a group of operations that is isomorphic to the Clifford Group of two qubits. We emphasize that this equivalence is a direct consequence of applying empiricism to STM.

Unfortunately, the relaxation of STM 2 to STM 2′ gives rise to a physically unreasonable state of affairs. For a physical model, we expect that if an operation is valid for a given system, then it should also be valid when we attach an ancilla to that system; the operations of $T\Gamma G(2)$ violate this condition. Consider the operation $\tilde{H} \otimes I$: under STM 2′, both $\tilde{H}$ and $I$ are valid operations on an elementary system, yet $\tilde{H} \otimes I$ is not a valid operation on the composite system, as it fails to map the correlated state shown in Fig. 6(b) to a valid epistemic state. In fact, the subgroups of $T\Gamma G(1)$ and $T\Gamma G(2)$ that preserve valid epistemic states when an ancilla is added are simply Spekkens’ original groups of operations for one and two toy bits respectively.

However, just as positive maps serve as tests for entanglement in quantum theory, validity-preserving maps serve as tests of correlation in the toy theory, as we now explain.
Formally, let $\mathcal{A}_i$ denote the set of operators acting on the Hilbert space $\mathcal{H}_i$. Then a linear map $\Delta : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is positive if it maps positive operators in $\mathcal{A}_1$ to positive operators in $\mathcal{A}_2$: in other words, $\rho \geq 0$ implies $\Delta \rho \geq 0$. On the other hand $\Delta$ is completely positive if the map

$$\Delta \otimes I : \mathcal{A}_1 \otimes \mathcal{A}_3 \rightarrow \mathcal{A}_2 \otimes \mathcal{A}_3$$

is positive for every identity map $I : \mathcal{A}_3 \rightarrow \mathcal{A}_3$. In other words, a completely positive map takes valid density operators to valid density operators even if an ancilla is attached to the system. Also recall that an operator $\rho \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is separable if it can be written in the form

$$\varrho = \sum_{i=1}^{n} p_i \varrho_i \otimes \tilde{\varrho}_i,$$

(15)

for $\varrho_i \in \mathcal{A}_1$, $\tilde{\varrho}_i \in \mathcal{A}_2$, and some probability distribution $\{p_i\}$. A well known result in quantum information is that positive maps can distinguish whether or not a state is separable (Theorem 2 [20, p. 5]):

**Theorem 1.** Let $\varrho$ act on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\varrho$ is separable if and only if for any positive map $\Delta : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, the operator $(\Delta \otimes I)\varrho$ is positive.

Theorem 1 says is that maps that are positive but not not completely positive serve as tests for detecting whether or not a density matrix is separable. An analogous statement can be made for validity preserving maps and correlated states in a two toy bit system.

Define a transformation $\Delta$ in STM to be validity-preserving if it maps all valid epistemic states to valid epistemic states in a toy system; all operations in $TG(1)$ and $TG(2)$ are validity-preserving. Define $\Delta$ to be completely validity-preserving if $\Delta \otimes I$ is validity-preserving for every $I$, where $I$ is the identity transformation on some ancilla toy system. For example, $\tilde{H} \in TG(1)$ is validity-preserving but not completely validity-preserving. Finally, a two toy bit state is perfectly correlated if for any acquisition of knowledge about one of the systems, the description of the other system is refined. The perfectly correlated two toy bit states are precisely the correlated pure states: no mixed states are perfectly correlated.

**Theorem 2.** Let $\sigma$ be a two toy bit epistemic state (pure or mixed). Then $\sigma$ is perfectly correlated if and only if there exists a one toy bit validity-preserving operation $\Delta$ such that $(\Delta \otimes I)\sigma$ is an invalid two toy bit state.

**Proof.** First suppose $\sigma$ is a pure state. If $\sigma$ is uncorrelated, then it has the form $e_{ab} \otimes e_{cd}$, and for any $\Delta \in TG(1)$, the state

$$(\Delta \otimes I)(e_{ab} \otimes e_{cd}) = (\Delta e_{ab}) \otimes e_{cd}$$

is a valid two toy bit state. On the other hand, if $\sigma$ is correlated, then it has the form $(I \otimes P)\sigma_0$, where $\sigma_0$ is the correlated state shown in Fig. 6(b) and $P \in S_4$ is some permutation of the second toy bit system. In this case, the
is an invalid state, as we have already seen that $(\tilde{H} \otimes I)\sigma_0$ is invalid.

Next suppose $\sigma$ is a mixed state. Then either $\sigma$ is uncorrelated, in which case it has the form $e_{ab} \otimes e_{1234}, e_{1234} \otimes e_{ab}$, or $e_{1234} \otimes e_{1234}$, or it is correlated, in which case it has the form $(e_{ab} \otimes e_{cd} + e_{mn} \otimes e_{pq})$, with $\{a, b\}$ disjoint from $\{m, n\}$ and $\{c, d\}$ disjoint from $\{p, q\}$. Any of these mixed states may be written as a sum of pure uncorrelated states. Since pure uncorrelated states remain valid under $\Delta \otimes I$ for any validity preserving $\Delta$, it follows that $(\Delta \otimes I)\sigma$ is also a valid state whenever $\sigma$ is a mixed state. Thus, invalidity of a state under a local validity-preserving map is a necessary and sufficient condition for a bipartite epistemic state (pure or mixed) to have perfect correlation.

In this section we introduced a possible relaxation of STM. Motivated by empiricism, we argued for the relaxation of STM 2, from ontic to epistemic determinism. We showed that this relaxation gives rise to a group of operations that is equivalent to the projective extended Clifford Group for one and two qubits. However, the operations of $TG(1)$ and $TG(2)$ are physically unreasonable as they do not represent completely validity-preserving maps. They do, however, serve as tests for correlations in the toy model. In the next section we discuss these results further.

IV. DISCUSSION

In this paper we formulated STM in an axiomatic framework and considered a possible relaxation—STM 2’—in its assumptions. The motivation for proposing STM 2’ is the empirical fact that in a toy universe, an observer is restricted to knowledge of epistemic states. We discovered that replacing STM 2 with STM 2’ gave rise to a group of operations that exhibit an isomorphism with the projective extended Clifford Group of operations (and consequently the projective Clifford group of operations) in quantum mechanics. This characteristic is not present in STM; while $S_4$ is isomorphic to $C(1)/U(1)$, the group of operations for two toy bits in STM is not isomorphic to $C(2)/U(1)$. However, due to the fact that operations arising from STM 2’ do not compose under the tensor product—they are not completely validity-preserving—the proposed relaxation does not give rise to a physically reasonable model.

Despite this failure, the group of operations generated by STM 2’ gives rise to a very useful tool; namely, the Horodecki criterion for separability in the toy model. The same operations that render the toy model physically unreasonable serve as tools for detecting correlations in the toy model. We believe that the investigation into possible relaxations of the axioms of STM gives rise demonstrates the power as well as the limitations of STM. Most significantly, we discover that no physically reasonable toy model can arise from relaxing STM 2 to an epistemic perspective; this robustness is an indication of the model’s power. On the other hand, we conclude that there is at least one characteristic of quantum theory that the STM cannot capture, an equivalence with the Clifford Group of operations.
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