Greedy Algorithms for Decision Trees with Hypotheses

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Abstract. We investigate at decision trees that incorporate both traditional queries based on one attribute and queries based on hypotheses about the values of all attributes. Such decision trees are similar to ones studied in exact learning, where membership and equivalence queries are allowed. We present greedy algorithms based on diverse uncertainty measures for construction of above decision trees and discuss results of computer experiments on various data sets from the UCI ML Repository and randomly generated Boolean functions. We also study the length and coverage of decision rules derived from the decision trees constructed by greedy algorithms.

Keywords: Decision tree · Hypothesis · Uncertainty measure · Greedy algorithm · Decision rule.

1 Introduction

Decision trees are well known as classifiers, as a tool for knowledge representation, and as algorithms. Conventional decision trees are studied, in particular, in rough set theory initiated by Pawlak and in test theory initiated by Chegis and Yablonskii. These trees use simple queries based on one attribute each. In contrast to these theories, exact learning initiated by Angluin studied not only membership queries that correspond to attributes from rough set theory and test theory but also the so-called equivalence queries.
In [4,5,6,7,8], we added the notion of a hypothesis to the model that has been considered in rough set theory as well in test theory. This model allows us to use an analog of equivalence queries and to consider different types of decision trees based on various combinations of attributes and hypotheses.

Experimental results discussed in [7] show that the optimal decision trees with hypotheses can have less complexity than the conventional decision trees and can be used as a tool for knowledge representation. However, dynamic programming algorithms for the optimization of decision trees considered in [7] are too complicated to be used in practice. Therefore in [4] we proposed an entropy-based greedy algorithm for the construction of different types of decision trees.

The present paper is a generalization of [4]. For an arbitrary uncertainty measure, we propose a greedy algorithm that, for given decision table and type of decision trees, constructs a decision tree of the considered type for this table.

The first goal of the present paper is to understand which uncertainty measures and types of decision trees should be chosen if we would like to minimize the depth or the number of realizable nodes in the constructed decision trees. To this end, we compare parameters of the decision trees of different types constructed for 10 decision tables from the UCI ML Repository [12] using five uncertainty measures. We do the same for 100 randomly generated Boolean functions with \( n = 3, \ldots, 6 \).

We also study the length and coverage of decision rules derived from the decision trees constructed by greedy algorithms. Previously in [8], we studied decision rules derived from optimal decision trees constructed by dynamic programming algorithms. The second goal of the paper is to understand which uncertainty measures and types of decision trees should be chosen if we would like to minimize the length or to maximize the coverage of the derived decision rules.

The main contributions of the paper are (i) the design of the greedy algorithms that can work with arbitrary uncertainty measures and different types of decision trees and (ii) the understanding (based on the experimental results) which uncertainty measures and which types of decision trees should be chosen if we would like to optimize the decision trees constructed by greedy algorithms or the decision rules derived from these trees.

The obtained experimental results for Boolean functions do not depend on the used uncertainty measures. We found the explanation of this interesting fact.

The rest of the paper is organized as follows. In Section 2 we consider main notions and in Section 3—greedy algorithms for the decision tree construction. Sections 4–6 contain results of computer experiments and their analysis, and Section 7—short conclusions.

2 Main Notions

Detailed definitions related to the decision tables, decision trees, and decision rules can be found in [4,8]. In this section, we restrict ourselves to the necessary short comments. We also add some new definitions compared to the papers [4,8].
Let $T$ be a decision table with $n$ conditional attributes $f_1, \ldots, f_n$ having values from the set $\omega = \{0, 1, 2, \ldots\}$ in which rows are pairwise different and each row is labeled with a decision from $\omega$. This table is called degenerate if it is empty or all rows of $T$ are labeled with the same decision. We denote $F(T) = \{f_1, \ldots, f_n\}$ and $D(T)$ the set of decisions attached to rows of $T$. For $f_i \in F(T)$, we denote by $E(T, f_i)$ the set of values of the attribute $f_i$ in the table $T$.

A subtable of the table $T$ is a decision table obtained from $T$ by removal of some rows. Let $S$ be a system of equations of the kind $f_i = \delta$ where $\delta \in E(T, f_i)$. By $TS$ we denote a subtable of the table $T$ containing only rows satisfying all equations from $S$.

We denote by $N(T)$ the number of rows in $T$ and, for any $t \in D(T)$, we denote by $N_t(T)$ the number of rows of $T$ labeled with the decision $t$. By $mcd(T)$ we denote a most common decision for $T$. If $T$ is empty, then $mcd(T) = 0$.

We denote by $T$ the set of all decision tables. An uncertainty measure is a function $U : T \to \mathbb{R}$ such that $U(T) \geq 0$ for any $T \in T$, and $U(T) = 0$ if and only if $T$ is a degenerate table. One can show (see book [1]) that the following functions (we assume that, for any empty table, the value of each of the considered functions is equal to 0) are uncertainty measures:

- Misclassification error $me(T) = N(T) - N_{mcd(T)}(T)$.
- Relative misclassification error $rme(T) = (N(T) - N_{mcd(T)}(T))/N(T)$.
- Entropy $ent(T) = -\sum_{t \in D(T)} (N_t(T)/N(T)) \log_2 (N_t(T)/N(T))$.
- Gini index $gini(T) = 1 - \sum_{t \in D(T)} (N_t(T)/N(T))^2$.
- Function $R$, where $R(T)$ is the number of unordered pairs of rows of $T$ labeled with different decisions (note that $R(T) = N(T)^2 gini(T)/2$).

For a given row of $T$, we should recognize the decision attached to this row. To this end, we can use decision trees based on two types of queries. We can ask about the value of an attribute $f_i$ on the given row. This query has the set of answers $A(f_i) = \{\{f_i = \delta\} : \delta \in E(T, f_i)\}$. We can formulate a hypothesis over $T$ in the form of $H = \{f_1 = \delta_1, \ldots, f_n = \delta_n\}$, where $\delta_1 \in E(T, f_1), \ldots, \delta_n \in E(T, f_n)$, and ask about this hypothesis. This query has the set of answers $A(H) = \{H, \{f_1 = \sigma_1\}, \ldots, \{f_n = \sigma_n\} : \sigma_1 \in E(T, f_1) \setminus \{\delta_1\}, \ldots, \sigma_n \in E(T, f_n) \setminus \{\delta_n\}\}$. The answer $H$ means that the hypothesis is true. Other answers are counterexamples. The hypothesis $H$ is called proper for $T$ if $(\delta_1, \ldots, \delta_n)$ is a row of the table $T$.

In this paper, we consider the following five types of decision trees:

1. Decision trees that use only attributes.
2. Decision trees that use only hypotheses.
3. Decision trees that use both attributes and hypotheses.
4. Decision trees that use only proper hypotheses.
5. Decision trees that use both attributes and proper hypotheses.

We consider the depth $h(\Gamma)$ of a decision tree $\Gamma$ as its time complexity, which is equal to the maximum number of queries in a path from the root to
a terminal node of the tree. As the space complexity of a decision tree $\Gamma$, we consider the number of its realizable relative to $T$ nodes $L(T, \Gamma)$. A node is called realizable relative to $T$ if, for a row of $T$ and some choice of counterexamples, the computation in the tree will pass through this node.

A complete path $\xi$ in $\Gamma$ is an arbitrary directed path from the root to a terminal node. Denote $T(\xi) = TS(\xi)$, where $S(\xi)$ is the union of systems of equations attached to edges of the path $\xi$.

Let $\Gamma$ be a decision tree for $T$, $\xi$ be a complete path in $\Gamma$ such that $T(\xi)$ is a nonempty table, and the terminal node of the path $\xi$ be labeled with the decision $d$. We now define a system of equations $S'(\xi)$. If there are no working nodes in $\xi$, then $S'(\xi)$ is the empty system. Let us assume now that $\xi$ contains at least one working node. We now transform systems of equations attached to edges leaving working nodes of $\xi$. If an edge is labeled with an equation system containing exactly one equation, then we will not change this system. Let an edge $e$ leaving a working node $v$ be labeled with an equation system containing more than one equation. Then $v$ is labeled with a hypothesis $H$ and $e$ is labeled with the equation system $H$. Note that if such a node exists, then it is the last working node in the complete path $\xi$. In this case, we remove from the equation system $H$ attached to $e$ all equations, which follow from the union of equation systems attached to edges of the path from the root to the node $v$. Then $S'(\xi)$ is the union of new equation systems attached to the edges of the path $\xi$. Note that the removed equations are redundant: $T(\xi) = TS'(\xi)$.

We correspond to the complete path $\xi$ the decision rule

$$
\bigwedge_{f_i=\delta \in S'(\xi)} (f_i = \delta) \to d.
$$

We denote this rule by $rule(\xi)$. The number of equations in the equation system $S'(\xi)$ is called the length of the rule $rule(\xi)$ and is denoted $l(rule(\xi))$. The number of rows in the subtable $T(\xi)$ is called the coverage of the rule $rule(\xi)$ and is denoted $c(rule(\xi))$.

Denote $\Xi(T, \Gamma)$ the set of complete paths $\xi$ in $\Gamma$ such that the table $T(\xi)$ is nonempty and $Rows(T)$ the set of rows of the decision table $T$. For a row $r \in Rows(T)$, we denote by $l(r, T, \Gamma)$ the minimum length of a rule $rule(\xi)$ such that $\xi \in \Xi(T, \Gamma)$ and $r$ is a row of the subtable $T(\xi)$, and we denote by $c(r, T, \Gamma)$ the maximum coverage of a rule $rule(\xi)$ such that $\xi \in \Xi(T, \Gamma)$ and $r$ is a row of the subtable $T(\xi)$.

We will use the following notation:

$$
\begin{align*}
    l(T, \Gamma) &= \frac{\sum_{r \in Rows(T)} l(r, T, \Gamma)}{|Rows(T)|}, \\
    c(T, \Gamma) &= \frac{\sum_{r \in Rows(T)} c(r, T, \Gamma)}{|Rows(T)|}.
\end{align*}
$$
3 Greedy Algorithms

Let $U$ be an uncertainty measure, $T$ be a nondegenerate decision table with $n$ conditional attributes $f_1, \ldots, f_n$, and $\Theta$ be a nondegenerate subtable of the table $T$. We now define the impurity of a query for the table $\Theta$ and uncertainty measure $U$. The impurity of the query based on an attribute $f_i \in F(T)$ (impurity of query $f_i$) is equal to $I_U(f_i, \Theta) = \max\{U(\Theta S) : S \in A(f_i)\}$. The impurity of the query based on a hypothesis $H$ (impurity of query $H$) is equal to $I_U(H, \Theta) = \max\{U(\Theta S) : S \in A(H)\}$.

An attribute $f_i$ is called admissible for $\Theta$ if it is not constant in $\Theta$. A hypothesis $\{f_1 = \delta_1, \ldots, f_n = \delta_n\}$ over $T$ is called admissible for $\Theta$ if it satisfies the following condition. For $i = 1, \ldots, n$, if $f_i$ is constant in $\Theta$, then $\delta_i$ is the only value of $f_i$ in $\Theta$.

We can find by simple search among all attributes an admissible for $\Theta$ attribute $f_i$ with the minimum impurity $I_U(f_i, \Theta)$. We can also find by simple search among all proper hypotheses an admissible for $\Theta$ proper hypothesis $H$ with the minimum impurity $I_U(H, \Theta)$. It is not necessary to consider all hypotheses to find an admissible for $\Theta$ hypothesis with the minimum impurity. For $i = 1, \ldots, n$, we denote by $\delta_i$ a number from $E(T, f_i)$ such that

$$U(\Theta \{f_i = \delta_i\}) = \max\{U(\Theta \{f_i = \sigma\}) : \sigma \in E(T, f_i)\}.$$  

Then the hypothesis $H = \{f_1 = \delta_1, \ldots, f_n = \delta_n\}$ is admissible for $\Theta$ and has the minimum impurity $I_U(H, \Theta)$ among all admissible for $\Theta$ hypotheses.

We now describe a greedy algorithm $A_U$ based on the uncertainty measure $U$ that, for a given nonempty decision table $T$ and $k \in \{1, \ldots, 5\}$, constructs a decision tree of type $k$ for the table $T$. This algorithm is a generalization of the algorithm considered in [4].

**Algorithm $A_U$.**

*Input*: A nonempty decision table $T$ and a number $k \in \{1, \ldots, 5\}$.

*Output*: A decision tree of type $k$ for the table $T$.

1. Construct a tree $G$ consisting of a single node labeled with $T$.
2. If no node of the tree $G$ is labeled with a table, then the algorithm ends and returns the tree $G$.
3. Choose a node $v$ in $G$, which is labeled with a subtable $\Theta$ of the table $T$.
4. If $\Theta$ is degenerate, then instead of $\Theta$, we label the node $v$ with 0 if $\Theta$ is empty and with the decision attached to each row of $\Theta$ if $\Theta$ is nonempty.
5. If $\Theta$ is nondegenerate, then depending on $k$ we choose an admissible for $\Theta$ query $X$ (either attribute or hypothesis) in the following way:
   a. If $k = 1$, then we find an admissible for $\Theta$ attribute $X \in F(T)$ with the minimum impurity $I_U(X, \Theta)$.
   b. If $k = 2$, then we find an admissible for $\Theta$ hypothesis $X$ over $T$ with the minimum impurity $I_U(X, \Theta)$.
   c. If $k = 3$, then we find an admissible for $\Theta$ attribute $Y \in F(T)$ with the minimum impurity $I_U(Y, \Theta)$ and an admissible for $\Theta$ hypothesis $Z$ over $T$ with the minimum impurity $I_U(Z, \Theta)$. Between $Y$ and $Z$, we choose a query $X$ with the minimum impurity $I_U(X, \Theta)$. 
(d) If \( k = 4 \), then we find an admissible for \( \Theta \) proper hypothesis \( X \) over \( T \) with the minimum impurity \( I_U(X, \Theta) \).

(e) If \( k = 5 \), then we find an admissible for \( \Theta \) attribute \( Y \in F(T) \) with the minimum impurity \( I_U(Y, \Theta) \) and an admissible for \( \Theta \) proper hypothesis \( Z \) over \( T \) with the minimum impurity \( I_U(Z, \Theta) \). Between \( Y \) and \( Z \), we choose a query \( X \) with the minimum impurity \( I_U(X, \Theta) \).

6. Instead of \( \Theta \), we label the node \( v \) with the query \( X \). For each answer \( S \in A(X) \), we add to the tree \( G \) a node \( v(S) \) and an edge \( e(S) \) connecting \( v \) and \( v(S) \). We label the node \( v(S) \) with the subtable \( \Theta S \) and label the edge \( e(S) \) with the answer \( S \). Proceed to step 2.

For a given nonempty decision table \( T \) and number \( k \in \{1, \ldots, 5\} \), the algorithm \( \mathcal{A}_U \) constructs a decision tree \( \Gamma \) of type \( k \) for the table \( T \). We will use the following notation:

\[
\begin{align*}
\hat{h}_U^{(k)}(T) &= h(\Gamma), \\
L_U^{(k)}(T) &= L(T, \Gamma), \\
l_U^{(k)}(T) &= l(T, \Gamma), \\
c_U^{(k)}(T) &= c(T, \Gamma).
\end{align*}
\]

4 Results of Experiments with Decision Tables from [12]

We now consider results of experiments with decision tables described in Table 1.

| Decision table          | Number of rows | Number of attributes |
|-------------------------|----------------|----------------------|
| BALANCE-SCALE           | 625            | 5                    |
| BREAST-CANCER           | 266            | 10                   |
| CARS                    | 1728           | 7                    |
| HAYES-ROTH-DATA         | 69             | 5                    |
| LYMPHOGRAPHY            | 148            | 18                   |
| NURSERY                 | 12960          | 9                    |
| SOYBEAN-SMALL           | 47             | 36                   |
| SPECT-TEST              | 169            | 22                   |
| TIC-TAC-TOE             | 958            | 10                   |
| ZOO-DATA                | 59             | 17                   |
Using the algorithm $\mathcal{A}_U$ with five uncertainty measures, we construct for these tables decision trees of different types, evaluate complexity of these trees and study decision rules derived from them.

### 4.1 Results for Misclassification Error $me$

Results for decision trees can be found in Tables 2 and 3. In Table 2, we consider parameters $h_{me}^{(1)}(T), \ldots, h_{me}^{(5)}(T)$ (minimum values are in bold).

| Decision table $T$ | $h_{me}^{(1)}(T)$ | $h_{me}^{(2)}(T)$ | $h_{me}^{(3)}(T)$ | $h_{me}^{(4)}(T)$ | $h_{me}^{(5)}(T)$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| BALANCE-SCALE     | 4                 | 4                 | 4                 | 4                 | 4                 |
| BREAST-CANCER     | 7                 | 6                 | 5                 | 6                 | 5                 |
| CARS              | 6                 | 6                 | 6                 | 6                 | 6                 |
| HAYES-ROTH-DATA   | 4                 | 4                 | 4                 | 4                 | 4                 |
| LYMPHOGRAPHY      | 7                 | 5                 | 5                 | 7                 | 5                 |
| NURSERY           | 8                 | 8                 | 8                 | 8                 | 8                 |
| SOYBEAN-SMALL     | 2                 | 5                 | 2                 | 7                 | 2                 |
| SPECT-TEST        | 15                | 4                 | 4                 | 4                 | 4                 |
| TIC-TAC-TOE       | 7                 | 7                 | 7                 | 8                 | 7                 |
| ZOO-DATA          | 4                 | 4                 | 4                 | 6                 | 4                 |

Average: 6.4 5.3 4.9 6 4.9

In Table 3, we consider parameters $L_{me}^{(1)}(T), \ldots, L_{me}^{(5)}(T)$ (minimum values are in bold).

### 4.2 Results for Relative Misclassification Error $rme$

Results for decision trees can be found in Tables 6 and 7. In Table 6, we consider parameters $l_{rme}^{(1)}(T), \ldots, l_{rme}^{(5)}(T)$ (minimum values are in bold).

In Table 5, we consider parameters $c_{rme}^{(1)}(T), \ldots, c_{rme}^{(5)}(T)$ (maximum values are in bold).

In Table 7, we consider parameters $L_{rme}^{(1)}(T), \ldots, L_{rme}^{(5)}(T)$ (minimum values are in bold).

Results for decision rules can be found in Tables 8 and 9. In Table 8, we consider parameters $l_{rme}^{(1)}(T), \ldots, l_{rme}^{(5)}(T)$ (minimum values are in bold).

In Table 9, we consider parameters $c_{rme}^{(1)}(T), \ldots, c_{rme}^{(5)}(T)$ (maximum values are in bold).
Table 3. Results for \( m_e \) and \( L \)

| Decision table \( T \) | \( L^{(1)}_{m_e}(T) \) | \( L^{(2)}_{m_e}(T) \) | \( L^{(3)}_{m_e}(T) \) | \( L^{(4)}_{m_e}(T) \) | \( L^{(5)}_{m_e}(T) \) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| BALANCE-SCALE           | 556             | 5.234           | 3.694           | 5.234           | 3.694           |
| BREAST-CANCER           | 238             | 21.922          | 285             | 33.099          | 254             |
| CARS                    | 1,462           | 66.593          | 1,590           | 66.593          | 1,590           |
| HAYES-ROTH-DATA         | 65              | 353             | 88              | 348             | 88              |
| LYMPHOGRAPHY            | 92              | 49,780          | 126             | 165,481         | 126             |
| NURSERY                 | 4,623           | 13,487,465      | 5,473           | 13,487,465      | 5,473           |
| SOYBEAN-SMALL           | 23              | 5.375           | 23              | 44.601          | 23              |
| SPECT-TEST              | 91              | 3.266           | 168             | 6.229           | 168             |
| TIC-TAC-TOE             | 547             | 287,504         | 61,315          | 703,982         | 613             |
| ZOO-DATA                | 27              | 1,482           | 27              | 4,411           | 27              |
| **Average**             | **772.4**       | **1,392,897.4** | **7,278.9**     | **1,451,744.3** | **12,05.6**     |

Table 4. Results for \( m_e \) and \( l \)

| Decision table \( T \) | \( l^{(1)}_{m_e}(T) \) | \( l^{(2)}_{m_e}(T) \) | \( l^{(3)}_{m_e}(T) \) | \( l^{(4)}_{m_e}(T) \) | \( l^{(5)}_{m_e}(T) \) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| BALANCE-SCALE           | 3.64            | 3.20            | 3.24            | 3.20            | 3.24            |
| BREAST-CANCER           | 3.61            | 2.76            | 3.48            | 2.79            | 3.55            |
| CARS                    | 5.05            | 2.47            | 4.99            | 2.47            | 4.99            |
| HAYES-ROTH-DATA         | 2.87            | 2.26            | 2.86            | 2.25            | 2.86            |
| LYMPHOGRAPHY            | 3.48            | 1.99            | 3.53            | 2.14            | 3.53            |
| NURSERY                 | 4.82            | 3.34            | 4.69            | 3.34            | 4.69            |
| SOYBEAN-SMALL           | 1.89            | 1.00            | 1.89            | 1.57            | 1.89            |
| SPECT-TEST              | 5.44            | 2.27            | 4.81            | 2.05            | 4.81            |
| TIC-TAC-TOE             | 5.48            | 3.35            | 3.36            | 3.19            | 5.47            |
| ZOO-DATA                | 2.53            | 1.56            | 2.53            | 1.85            | 2.53            |
| **Average**             | **3.88**        | **2.42**        | **3.54**        | **2.48**        | **3.76**        |

4.3 Results for Entropy \( \text{ent} \)

For the completeness, we repeat experiments considered in [4] and related to entropy and decision trees.

Results for decision trees can be found in Tables 10 and 11. In Table 10, we consider parameters \( h^{(1)}_{\text{ent}}(T), \ldots, h^{(5)}_{\text{ent}}(T) \) (minimum values are in bold).

In Table 11, we consider parameters \( L^{(1)}_{\text{ent}}(T), \ldots, L^{(5)}_{\text{ent}}(T) \) (minimum values are in bold).

Results for decision rules can be found in Tables 12 and 13. In Table 12, we consider parameters \( l^{(1)}_{\text{ent}}(T), \ldots, l^{(5)}_{\text{ent}}(T) \) (minimum values are in bold).

In Table 13, we consider parameters \( c^{(1)}_{\text{ent}}(T), \ldots, c^{(5)}_{\text{ent}}(T) \) (maximum values are in bold).
4.4 Results for Gini Index \( gini \)

Results for decision trees can be found in Tables 14 and 15. In Table 14, we consider parameters \( h_{gini}^{(1)}(T), \ldots, h_{gini}^{(5)}(T) \) (minimum values are in bold).

In Table 15, we consider parameters \( L_{gini}^{(1)}(T), \ldots, L_{gini}^{(5)}(T) \) (minimum values are in bold).

Results for decision rules can be found in Tables 16 and 17. In Table 16, we consider parameters \( l_{gini}^{(1)}(T), \ldots, l_{gini}^{(5)}(T) \) (minimum values are in bold).

In Table 17, we consider parameters \( c_{gini}^{(1)}(T), \ldots, c_{gini}^{(5)}(T) \) (maximum values are in bold).

| Decision table \( T \) | \( c_{me}^{(1)}(T) \) | \( c_{me}^{(2)}(T) \) | \( c_{me}^{(3)}(T) \) | \( c_{me}^{(4)}(T) \) | \( c_{me}^{(5)}(T) \) |
|------------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| Balance-scale          | 2.44                | 4.21                | 4.05                | 4.21                | 4.05                |
| Breast-Cancer          | 4.72                | 8.38                | 5.16                | 8.84                | 4.97                |
| Cars                   | 5.74                | 332.63              | 5.97                | 332.63              | 5.97                |
| Hayes-Roth-Data        | 3.58                | 6.19                | 3.62                | 6.19                | 3.62                |
| Lymphography           | 4.14                | 18.19               | 4.51                | 18.70               | 4.51                |
| Nursery                | 300.54              | 1,523.16            | 304.06              | 1,523.16            | 304.06              |
| Soybean-Small          | 3.47                | 12.32               | 3.47                | 9.32                | 3.47                |
| Spect-Test             | 8.75                | 57.98               | 18.69               | 57.87               | 18.69               |
| Tic-Tac-Toe            | 6.31                | 49.52               | 50.26               | 56.26               | 6.70                |
| Zoo-Data               | 6.83                | 10.59               | 6.83                | 10.80               | 6.83                |
| Average                | 34.65               | 202.32              | 40.66               | 202.80              | 36.29               |

| Decision table \( T \) | \( h_{me}^{(1)}(T) \) | \( h_{me}^{(2)}(T) \) | \( h_{me}^{(3)}(T) \) | \( h_{me}^{(4)}(T) \) | \( h_{me}^{(5)}(T) \) |
|------------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| Balance-scale          | 4                   | 4                   | 4                   | 4                   | 4                   |
| Breast-Cancer          | 9                   | 9                   | 9                   | 8                   | 9                   |
| Cars                   | 6                   | 6                   | 6                   | 6                   | 6                   |
| Hayes-Roth-Data        | 4                   | 4                   | 4                   | 4                   | 4                   |
| Lymphography           | 10                  | 11                  | 10                  | 12                  | 10                  |
| Nursery                | 8                   | 8                   | 8                   | 8                   | 8                   |
| Soybean-Small          | 2                   | 6                   | 2                   | 8                   | 2                   |
| Spect-Test             | 20                  | 5                   | 5                   | 14                  | 11                  |
| Tic-Tac-Toe            | 7                   | 8                   | 7                   | 8                   | 7                   |
| Zoo-Data               | 8                   | 7                   | 6                   | 10                  | 6                   |
| Average                | 7.8                 | 6.8                 | 6.1                 | 8.2                 | 6.7                 |
| Decision table T | $L_r^{(1)}(T)$ | $L_r^{(2)}(T)$ | $L_r^{(3)}(T)$ | $L_r^{(4)}(T)$ | $L_r^{(5)}(T)$ |
|------------------|----------------|----------------|----------------|----------------|----------------|
| BALANCE-SCALE    | 556            | 5,234          | 3,694          | 5,234          | 3,694          |
| BREAST-CANCER    | 255            | 446,170        | 304            | 103,642        | 266            |
| CARS             | 1,592          | 66,593         | 1,831          | 66,593         | 1,831          |
| HAYES-ROTH-DATA  | 73             | 428            | 72             | 378            | 72             |
| LYMPHOGRAPHY     | 116            | 2,475,650      | 143            | 1,952,599      | 143            |
| NURSERY          | 4,493          | 13,487,465     | 13,667         | 13,487,465     | 13,667         |
| SOYBEAN-SMALL    | 7              | 11,403         | 7              | 113,855        | 7              |
| SPECT-TEST       | 123            | 6,983          | 5,495          | 398,926        | 1,116          |
| TIC-TAC-TOE      | 648            | 864,578        | 200,847        | 946,858        | 940            |
| ZOO-DATA         | 35             | 6,536          | 37             | 30,889         | 37             |
| Average          | 789.8          | 1,737,104      | 22,609.7       | 1,710,643.9    | 2,177.3        |

| Decision table T | $l_r^{(1)}(T)$ | $l_r^{(2)}(T)$ | $l_r^{(3)}(T)$ | $l_r^{(4)}(T)$ | $l_r^{(5)}(T)$ |
|------------------|----------------|----------------|----------------|----------------|----------------|
| BALANCE-SCALE    | 3.64           | **3.20**       | 3.24           | **3.20**       | 3.24           |
| BREAST-CANCER    | 6.11           | **2.68**       | 6.01           | 2.73           | 6.06           |
| CARS             | 5.30           | **2.47**       | 5.18           | **2.47**       | 5.18           |
| HAYES-ROTH-DATA  | 3.22           | **2.20**       | 3.20           | 2.22           | 3.20           |
| LYMPHOGRAPHY     | 6.97           | **2.01**       | 6.80           | 2.12           | 6.80           |
| NURSERY          | 4.72           | **3.34**       | 4.33           | **3.34**       | 4.33           |
| SOYBEAN-SMALL    | 1.34           | **1.00**       | 1.34           | 1.53           | 1.34           |
| SPECT-TEST       | 8.37           | **2.28**       | 2.18           | **1.79**       | 2.21           |
| TIC-TAC-TOE      | 5.77           | 3.52           | 3.70           | **3.19**       | 5.63           |
| ZOO-DATA         | 4.90           | **1.66**       | 4.61           | 1.83           | 4.61           |
| Average          | 5.03           | 2.43           | 4.06           | 2.44           | 4.26           |

### 4.5 Results for Uncertainty Measure $R$

Results for decision trees can be found in Tables 18 and 19. In Table 18, we consider parameters $h_R^{(1)}(T), \ldots, h_R^{(5)}(T)$ (minimum values are in bold).

In Table 19, we consider parameters $L_R^{(1)}(T), \ldots, L_R^{(5)}(T)$ (minimum values are in bold).

Results for decision rules can be found in Tables 20 and 21. In Table 20, we consider parameters $l_R^{(1)}(T), \ldots, l_R^{(5)}(T)$ (minimum values are in bold).

In Table 21, we consider parameters $c_R^{(1)}(T), \ldots, c_R^{(5)}(T)$ (maximum values are in bold).
Table 9. Results for \( rme \) and \( c \)

| Decision table \( T \)                  | \( c^{(1)}_{rme}(T) \) | \( c^{(2)}_{rme}(T) \) | \( c^{(3)}_{rme}(T) \) | \( c^{(4)}_{rme}(T) \) | \( c^{(5)}_{rme}(T) \) |
|----------------------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| BALANCE-SCALE                          | 2.44                    | 4.21                    | 4.05                    | 4.21                    | 4.05                    |
| BREAST-CANCER                          | 2.18                    | 9.12                    | 2.68                    | 9.46                    | 2.37                    |
| CARS                                   | 3.95                    | \textbf{332.63}         | 5.40                    | \textbf{332.63}         | 5.40                    |
| HAYES-ROTH-DATA                        | 1.81                    | 6.23                    | 1.81                    | 6.22                    | 1.81                    |
| LYMPHOGRAPHY                           | 3.32                    | 19.76                   | 4.12                    | \textbf{21.33}          | 4.12                    |
| NURSERY                                | 1,451.06                | \textbf{1,523.16}       | 1,460.80                | \textbf{1,523.16}       | 1,460.80                |
| SOYBEAN-SMALL                          | 11.51                   | \textbf{12.32}          | 11.51                   | 11.11                   | 11.51                   |
| SPECT-TEST                             | 5.54                    | 57.98                   | 57.98                   | \textbf{58.01}          | 54.76                   |
| TIC-TAC-TOE                            | 5.31                    | 50.48                   | 34.99                   | \textbf{56.25}          | 6.27                    |
| ZOO-DATA                               | 7.10                    | 10.69                   | 7.71                    | \textbf{10.97}          | 7.71                    |
| **Average**                            | 149.42                  | 202.66                  | 159.10                  | 203.33                  | 155.88                  |

Table 10. Results for \( \text{ent} \) and \( h \)

| Decision table \( T \)                  | \( h^{(1)}_{\text{ent}}(T) \) | \( h^{(2)}_{\text{ent}}(T) \) | \( h^{(3)}_{\text{ent}}(T) \) | \( h^{(4)}_{\text{ent}}(T) \) | \( h^{(5)}_{\text{ent}}(T) \) |
|----------------------------------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| BALANCE-SCALE                          | 4                       | 4                       | 4                       | 4                       | 4                       |
| BREAST-CANCER                          | 9                       | 9                       | 9                       | 8                       | 9                       |
| CARS                                   | 6                       | 6                       | 6                       | 6                       | 6                       |
| HAYES-ROTH-DATA                        | 4                       | 4                       | 4                       | 4                       | 4                       |
| LYMPHOGRAPHY                           | 11                      | 11                      | 9                       | 13                      | 10                      |
| NURSERY                                | 8                       | 8                       | 8                       | 8                       | 8                       |
| SOYBEAN-SMALL                          | 2                       | 6                       | 2                       | 8                       | 2                       |
| SPECT-TEST                             | 20                      | 5                       | 5                       | 14                      | 11                      |
| TIC-TAC-TOE                            | 7                       | 8                       | 7                       | 8                       | 7                       |
| ZOO-DATA                               | 8                       | 6                       | 5                       | 8                       | 5                       |
| **Average**                            | 7.9                     | 6.7                     | 5.9                     | 8.1                     | 6.6                     |

5 Results of Experiments with Randomly Generated Boolean Functions

For \( n = 3, \ldots, 6 \), we randomly generate 100 Boolean functions with \( n \) variables. The table representation \( T_f \) of a Boolean function \( f(x_1, \ldots, x_n) \) is considered as a decision table. This table contains \( n \) columns and \( 2^n \) rows. Columns are labeled with variables (attributes) \( x_1, \ldots, x_n \). The set of rows coincides with \( \{0, 1\}^n \). Each row is labeled with the value of the function \( f \) on it. Decision trees for this decision table are interpreted as decision trees computing the function \( f \).

Using the algorithm \( A_U \) with five uncertainty measures, we construct for the generated Boolean functions decision trees of different types, evaluate complexity of these trees and study decision rules derived from them.
Table 11. Results for \textit{ent} and \( L \)

| Decision table T | \( L_{\text{ent}}^{(1)}(T) \) | \( L_{\text{ent}}^{(2)}(T) \) | \( L_{\text{ent}}^{(3)}(T) \) | \( L_{\text{ent}}^{(4)}(T) \) | \( L_{\text{ent}}^{(5)}(T) \) |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| BALANCE-SCALE    | 556             | 5.234           | 4.102           | 5.234           | 4.102           |
| BREAST-CANCER    | 255             | 446.170         | 304             | 103.642         | 266             |
| CARS             | 1,136           | 65.624          | 3.944           | 65.624          | 3.944           |
| HAYES-ROTH-DATA  | 73              | 421             | 72              | 367             | 72              |
| LYMPHOGRAPHY     | 123             | 6,653.366       | 162             | 8,515,841       | 153             |
| NURSERY          | 4,460           | 12,790.306      | 14,422          | 12,790.306      | 14,422          |
| SOYBEAN-SMALL    | 7               | 10,029          | 7               | 157,640         | 7               |
| SPECT-TEST       | 123             | 6,983           | 2.28            | 2.18            | 5.495           |
| TIC-TAC-TOE      | 648             | 864,578         | 200,847         | 946,858         | 940             |
| ZOO-DATA         | 33              | 2,134           | 35              | 13,310          | 35              |
| **Average**      | 741.4           | 2,084,484.5     | 22,939          | 2,299,774.8     | 2,505.7         |

Table 12. Results for \textit{ent} and \( l \)

| Decision table T | \( l_{\text{ent}}^{(1)}(T) \) | \( l_{\text{ent}}^{(2)}(T) \) | \( l_{\text{ent}}^{(3)}(T) \) | \( l_{\text{ent}}^{(4)}(T) \) | \( l_{\text{ent}}^{(5)}(T) \) |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| BALANCE-SCALE    | 3.64            | **3.20**        | 3.24            | **3.20**        | 3.24            |
| BREAST-CANCER    | 6.11            | **2.68**        | 6.01            | 2.73            | 6.06            |
| CARS             | 4.68            | **2.45**        | 4.17            | **2.45**        | 4.17            |
| HAYES-ROTH-DATA  | 3.22            | **2.17**        | 3.20            | 2.22            | 3.20            |
| LYMPHOGRAPHY     | 6.61            | **1.99**        | 6.32            | 2.11            | 6.37            |
| NURSERY          | 4.72            | **3.80**        | 4.29            | **3.80**        | 4.29            |
| SOYBEAN-SMALL    | 1.34            | **1.00**        | 1.34            | 1.47            | 1.34            |
| SPECT-TEST       | 8.37            | 2.28            | 2.18            | **1.79**        | 2.21            |
| TIC-TAC-TOE      | 5.77            | 3.52            | 3.70            | **3.19**        | 5.63            |
| ZOO-DATA         | 4.27            | **1.66**        | 3.98            | 1.97            | 3.98            |
| **Average**      | 4.87            | 2.48            | 3.84            | 2.49            | 4.05            |

Since each hypothesis over the decision table \( T_f \) is proper, for each uncertainty measure \( U \), \( h_U^{(2)}(T_f) = h_U^{(4)}(T_f) \), \( h_U^{(3)}(T_f) = h_U^{(5)}(T_f) \), \( L_U^{(2)}(T_f) = L_U^{(4)}(T_f) \), \( L_U^{(3)}(T_f) = L_U^{(5)}(T_f) \), \( L_U^{(2)}(T_f) = L_U^{(4)}(T_f) \), \( l_U^{(2)}(T_f) = l_U^{(4)}(T_f) \), \( l_U^{(3)}(T_f) = l_U^{(5)}(T_f) \), \( c_U^{(2)}(T_f) = c_U^{(4)}(T_f) \), and \( c_U^{(3)}(T_f) = c_U^{(5)}(T_f) \).

The obtained experimental results for Boolean functions do not depend on the used uncertainty measures. At the end of this section, we will explain this interesting fact.

Results for decision trees can be found in Tables 22 and 23. In Table 22 we consider parameters \( h_U^{(1)}, \ldots, h_U^{(5)} \), \( U \in \{ \text{me, rme, ent, gini, R} \} \), in the format \( \min \text{Avg} \max \) (minimum values of \( \text{Avg} \) are in bold).

In Table 23 we consider parameters \( L_U^{(1)}, \ldots, L_U^{(5)} \), \( U \in \{ \text{me, rme, ent, gini, R} \} \), in the format \( \min \text{Avg} \max \) (minimum values of \( \text{Avg} \) are in bold).
Table 13. Results for $ent$ and $c$

| Decision table $T$ | $c^{(1)}_{ent}(T)$ | $c^{(2)}_{ent}(T)$ | $c^{(3)}_{ent}(T)$ | $c^{(4)}_{ent}(T)$ | $c^{(5)}_{ent}(T)$ |
|-------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| balance-scale     | 2.44                | 4.21                | 4.03                | 4.21                | 4.03                |
| breast-cancer     | 2.18                | 9.12                | 2.68                | 9.46                | 2.37                |
| cars              | 15.97               | **332.71**          | 37.12               | **332.71**          | 37.12               |
| hayes-roth-data   | 1.81                | **6.38**            | 1.81                | 6.26                | 1.81                |
| lymphography      | 3.51                | **21.54**           | 4.72                | 21.53               | 4.42                |
| nursery           | 1,451.07            | **1,516.04**        | 1,462.03            | **1,516.04**        | 1,462.03            |
| soybean-small     | 11.51               | **12.32**           | 11.51               | 11.66               | 11.51               |
| spect-test        | 5.54                | 57.98               | 57.98               | **58.01**           | 54.76               |
| tic-tac-toe       | 5.31                | 50.48               | 34.99               | **56.25**           | 6.27                |
| zoo-data          | 7.24                | **10.66**           | 7.85                | 10.36               | 7.85                |
| **Average**       | **150.66**          | **202.14**          | **162.47**          | **202.65**          | **159.22**          |

Table 14. Results for $gini$ and $h$

| Decision table $T$ | $h^{(1)}_{gini}(T)$ | $h^{(2)}_{gini}(T)$ | $h^{(3)}_{gini}(T)$ | $h^{(4)}_{gini}(T)$ | $h^{(5)}_{gini}(T)$ |
|-------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| balance-scale     | 4                   | 4                   | 4                   | 4                   | 4                   |
| breast-cancer     | 9                   | 9                   | 9                   | 8                   | 9                   |
| cars              | 6                   | 6                   | 6                   | 6                   | 6                   |
| hayes-roth-data   | 4                   | 4                   | 4                   | 4                   | 4                   |
| lymphography      | **10**              | 11                  | **10**              | 13                  | **10**              |
| nursery           | 8                   | 8                   | 8                   | 8                   | 8                   |
| soybean-small     | 2                   | 6                   | 2                   | 8                   | 2                   |
| spect-test        | 20                  | 5                   | 5                   | 14                  | 11                  |
| tic-tac-toe       | 7                   | 8                   | 7                   | 8                   | 7                   |
| zoo-data          | 8                   | 7                   | 8                   | 9                   | 7                   |
| **Average**       | **7.8**             | 6.8                 | **6.3**             | 8.2                 | 6.8                 |

Results for decision rules can be found in Tables 24 and 25. In Table 24 we consider parameters $l^{(1)}_U, \ldots, l^{(5)}_U, U \in \{me, rme, ent, gini, R\}$, in the format $\min Avg_{max}$ (minimum values of $Avg$ are in bold).

In Table 25 we consider parameters $c^{(1)}_U, \ldots, c^{(5)}_U, U \in \{me, rme, ent, gini, R\}$, in the format $\max Avg_{max}$ (maximum values of $Avg$ are in bold).

We now explain why the results of experiments with Boolean functions do not depend on the choice of uncertainty measures from the set $M = \{me, rme, ent, gini, R\}$.

Let $f(x_1, \ldots, x_n)$ be a nonconstant Boolean function, $T_f$ be the decision table with attributes $x_1, \ldots, x_n$ representing $f$, and $T$ be a nondegenerate subtable of $T_f$ such that $T = T_f \{x_j = \delta_1, \ldots, x_jk = \delta_k\}, k < n, 1 \leq j_1 < \cdots < j_k \leq n$, and $\delta_1, \ldots, \delta_n \in \{0, 1\}$. 
Table 15. Results for gini and L

| Decision table T   | \( L^{(1)}_{gini}(T) \) | \( L^{(2)}_{gini}(T) \) | \( L^{(3)}_{gini}(T) \) | \( L^{(4)}_{gini}(T) \) | \( L^{(5)}_{gini}(T) \) |
|-------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| BALANCE-SCALE     | 556                      | 5.234                    | 4.006                    | 5.234                    | 4.006                    |
| BREAST-CANCER     | 255                      | 446.170                  | 304                      | 103.642                  | 266                      |
| CARS              | 1,511                    | 65.579                   | 3.643                    | 65.579                   | 3.643                    |
| HAYES-ROTH-DATA   | 73                       | 415                      | 72                       | 363                      | 72                       |
| LYMPHOGRAPHY      | 115                      | 2,735,180                | 140                      | 2,232,312                | 115                      |
| NURSERY           | 4,284                    | 12,795,294               | 16,566                   | 12,795,294               | 16,566                   |
| SOYBEAN-SMALL     | 7                        | 10,029                   | 7                        | 148,891                  | 7                        |
| SPECT-TEST        | 648                      | 864,578                  | 200,847                  | 946,858                  | 940                      |
| TIC-TAC-TOE       | 39                       | 4,702                    | 57                       | 20,896                   | 41                       |
| AVERAGE           | 761.1                    | 1,693,416.4              | 23,113.7                 | 1,671,799.5              | 2,677.2                  |

Table 16. Results for gini and l

| Decision table T   | \( l^{(1)}_{gini}(T) \) | \( l^{(2)}_{gini}(T) \) | \( l^{(3)}_{gini}(T) \) | \( l^{(4)}_{gini}(T) \) | \( l^{(5)}_{gini}(T) \) |
|-------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| BALANCE-SCALE     | 3.64                     | 3.20                     | 3.20                     | 3.20                     | 3.23                     |
| BREAST-CANCER     | 6.11                     | 2.68                     | 6.01                     | 2.73                     | 6.06                     |
| CARS              | 5.25                     | 2.45                     | 4.70                     | 2.45                     | 4.70                     |
| HAYES-ROTH-DATA   | 3.22                     | 2.20                     | 3.20                     | 2.23                     | 3.20                     |
| LYMPHOGRAPHY      | 7.01                     | 1.99                     | 6.91                     | 2.14                     | 7.01                     |
| NURSERY           | 4.63                     | 3.76                     | 4.12                     | 3.76                     | 4.12                     |
| SOYBEAN-SMALL     | 1.34                     | 1.00                     | 1.34                     | 1.47                     | 1.34                     |
| SPECT-TEST        | 8.37                     | 2.28                     | 2.18                     | 1.79                     | 2.21                     |
| TIC-TAC-TOE       | 5.77                     | 3.52                     | 3.70                     | 3.19                     | 5.63                     |
| ZOO-DATA          | 5.10                     | 1.66                     | 4.71                     | 2.05                     | 4.81                     |
| AVERAGE           | 5.04                     | 2.47                     | 4.01                     | 2.50                     | 4.23                     |

A subtable \( \Theta \) of the table \( T \) is called 1-subtable of \( T \) if it can be represented in the form \( \Theta = T\{x_i = \delta \} \), where \( i \notin \{j_1, \ldots, j_k \} \) and \( \delta \in \{0, 1\} \). This subtable contains exactly \( 2^t \) rows, where \( t = n - k - 1 \), i.e., \( N(\Theta) = 2^t \). Denote \( m(\Theta) = \min\{N_0(\Theta), N_1(\Theta)\} \). Then

- \( me(\Theta) = m(\Theta) \),
- \( rme(\Theta) = m(\Theta)/2^t \),
- \( ent(\Theta) = -p \log_2 p - (1 - p) \log_2 (1 - p) \), where \( p = m(\Theta)/2^t \),
- \( gini(\Theta) = m(\Theta)(2^t - m(\Theta))/2^{2t} \),
- \( R(\Theta) = m(\Theta)(2^t - m(\Theta))/2^t \).

We will say that an uncertainty measure \( U \) is monotone for \( T \) if, for any 1-subtables \( \Theta_1 \) and \( \Theta_2 \) of \( T \), \( U(\Theta_1) \leq U(\Theta_2) \) if and only if \( m(\Theta_1) \leq m(\Theta_2) \). We now show that each uncertainty measure from the set \( M \) is monotone for \( T \).
Table 17. Results for \textit{gini} and \textit{c}

| Decision table T | \(c_{\text{gini}}^{(1)}(T)\) | \(c_{\text{gini}}^{(2)}(T)\) | \(c_{\text{gini}}^{(3)}(T)\) | \(c_{\text{gini}}^{(4)}(T)\) | \(c_{\text{gini}}^{(5)}(T)\) |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| BALANCE-SCALE    | 2.44            | 4.21            | 4.08            | 4.21            | 4.08            |
| BREAST-CANCER    | 2.18            | 9.12            | 2.68            | 9.46            | 2.37            |
| CARS             | 4.41            | 332.70          | 26.27           | 332.70          | 26.27           |
| HAYES-ROTH-DATA  | 1.81            | 6.23            | 1.81            | 6.19            | 1.81            |
| LYMPHOGRAPHY     | 3.28            | 20.85           | 3.49            | 21.45           | 3.28            |
| NURSERY         | 1,454.14        | \(1,516.81\)    | 1,467.10        | \(1,516.81\)    | 1,467.10        |
| SOYBEAN-SMALL    | 11.51           | 12.32           | 11.51           | 11.66           | 11.51           |
| SPECT-TEST       | 5.54            | 57.98           | 57.98           | 58.01           | 54.76           |
| TIC-TAC-TOE      | 5.31            | 50.48           | 34.99           | \(56.25\)       | 6.27            |
| ZOO-DATA         | 5.85            | 10.66           | 7.37            | \(10.86\)       | 6.46            |
| **Average**      | 149.65          | 202.14          | 161.73          | 202.76          | 158.39          |

Table 18. Results for \(R\) and \(h\)

| Decision table T | \(h_{R}^{(1)}(T)\) | \(h_{R}^{(2)}(T)\) | \(h_{R}^{(3)}(T)\) | \(h_{R}^{(4)}(T)\) | \(h_{R}^{(5)}(T)\) |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| BALANCE-SCALE    | \(4\)           | \(4\)           | \(4\)           | \(4\)           | \(4\)           |
| BREAST-CANCER    | \(6\)           | \(6\)           | \(5\)           | \(6\)           | \(5\)           |
| CARS             | \(6\)           | \(6\)           | \(6\)           | \(6\)           | \(6\)           |
| HAYES-ROTH-DATA  | \(4\)           | \(4\)           | \(4\)           | \(4\)           | \(4\)           |
| LYMPHOGRAPHY     | \(5\)           | \(6\)           | \(5\)           | \(7\)           | \(5\)           |
| NURSERY         | \(8\)           | \(8\)           | \(7\)           | \(8\)           | \(7\)           |
| SOYBEAN-SMALL    | \(2\)           | \(5\)           | \(2\)           | \(7\)           | \(2\)           |
| SPECT-TEST       | \(9\)           | \(4\)           | \(5\)           | \(6\)           | \(5\)           |
| TIC-TAC-TOE      | \(7\)           | \(6\)           | \(6\)           | \(8\)           | \(7\)           |
| ZOO-DATA         | \(4\)           | \(5\)           | \(4\)           | \(6\)           | \(4\)           |
| **Average**      | 5.5             | 5.4             | 4.8             | 6.2             | 4.9             |

For \textit{me} and \textit{rme}, the considered statement is obvious.

Let us consider the function \(H(x) = -x \log_2 x - (1-x) \log_2 (1-x)\), where \(x\) is a real number and \(0 \leq x \leq 1\). It is well known that this function is increasing if \(0 \leq x \leq 0.5\). Using this fact, it is easy to show that \textit{ent} is a monotone uncertainty measure for \(T\).

We now consider the function \(r(x) = x(2^t - x)\), where \(x\) is an integer and \(0 \leq x \leq 2^t\). One can show that this function is increasing if \(0 \leq x \leq 2^t-1\). Using this fact, it is not difficult to show that \textit{gini} and \(R\) are monotone uncertainty measures for \(T\).

Let \(U \in M\) and let us assume that, for each pair \(\Theta_1, \Theta_2\) of 1-subtables for \(T\), we know if the inequality \(U(\Theta_1) \leq U(\Theta_2)\) holds or not. This information determines the sets of queries with the minimum impurity among (i) all admissible for \(T\) attributes, (ii) all admissible for \(T\) hypotheses, and (iii) all admissible for
Table 19. Results for $R$ and $L$  

| Decision table $T$ | $L_R^{(1)}(T)$ | $L_R^{(2)}(T)$ | $L_R^{(3)}(T)$ | $L_R^{(4)}(T)$ | $L_R^{(5)}(T)$ |
|-------------------|----------------|----------------|----------------|----------------|----------------|
| BALANCE-SCALE     | **556**        | 5.234          | 4.006          | 5234           | 4.006          |
| BREAST-CANCER     | 222            | 19.698         | 320            | 29.308         | 264            |
| CARS              | **1,405**      | 65.579         | 1.541          | 65.579         | 1.541          |
| HAYES-ROTH-DATA   | 55             | 338            | 65             | 353            | 65             |
| LYMPHOGRAPHY      | 90             | 47.278         | 110            | 141.322        | 110            |
| NURSERY           | 4.366          | 12,795,294     | 3,880          | 12,795,294     | 3,889          |
| SOYBEAN-SMALL     | 23             | 5.335          | 23             | 36.733         | 23             |
| SPECT-TEST        | 53             | 2.730          | 2.126          | 5.920          | 656            |
| TIC-TAC-TOE       | 579            | 164,663        | 74,567         | 645,542        | 734            |
| ZOO-DATA          | 25             | 1,542          | 76             | 3,826          | 72             |
| **Average**       | **737.4**      | 1,310,769.1    | 8,672.3        | 1,372,911.1    | 1,136          |

Table 20. Results for $R$ and $l$  

| Decision table $T$ | $l_R^{(1)}(T)$ | $l_R^{(2)}(T)$ | $l_R^{(3)}(T)$ | $l_R^{(4)}(T)$ | $l_R^{(5)}(T)$ |
|-------------------|----------------|----------------|----------------|----------------|----------------|
| BALANCE-SCALE     | 3.64           | **3.20**       | 3.23           | 3.20           | 3.23           |
| BREAST-CANCER     | 3.36           | 2.80           | 3.25           | **2.79**       | 3.33           |
| CARS              | 4.91           | **2.45**       | 4.82           | **2.45**       | 4.82           |
| HAYES-ROTH-DATA   | 2.68           | 2.26           | 2.55           | **2.23**       | 2.55           |
| LYMPHOGRAPHY      | 2.92           | **2.01**       | 3.02           | 2.14           | 3.02           |
| NURSERY           | 5.34           | **3.76**       | 5.29           | **3.76**       | 5.29           |
| SOYBEAN-SMALL     | 1.89           | **1.00**       | 1.89           | 1.68           | 1.89           |
| SPECT-TEST        | 3.93           | 2.24           | 1.88           | 2.09           | **1.84**       |
| TIC-TAC-TOE       | 5.16           | 3.44           | **3.43**       | 3.44           | 5.16           |
| ZOO-DATA          | 2.41           | **1.78**       | 2.53           | 2.17           | 2.53           |
| **Average**       | 3.62           | 2.49           | 3.19           | 2.60           | 3.37           |

$T$ attributes and hypotheses. Since all uncertainty measures from the set $M$ are monotone for $T$, the sets of admissible for $T$ queries with the minimum impurity do not depend on the choice of uncertainty measures from $M$.

We will not consider details of the software implementation for the greedy algorithms. However, the above reasoning allows us to understand independence of the experimental results for Boolean functions from the chosen uncertainty measure.

6 Analysis of Experimental Results

First, we evaluate results obtained for the decision tables from [12] based on average values of the parameters $h_{U}^{(k)}$, $L_{U}^{(k)}$, $l_{U}^{(k)}$, and $c_{U}^{(k)}$, where $U \in \{me, rme, ent, gini, R\}$ and $k = 1, \ldots, 5$. 


Table 21. Results for R and c

| Decision table T | \( c_R^{(1)}(T) \) | \( c_R^{(2)}(T) \) | \( c_R^{(3)}(T) \) | \( c_R^{(4)}(T) \) | \( c_R^{(5)}(T) \) |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| BALANCE-Scale    | 2.44            | 4.21            | 4.08            | 4.21            | 4.08            |
| BREAST-Cancer    | 4.62            | 7.56            | 4.74            | 8.14            | 4.81            |
| CARS             | 7.64            | 332.70          | 8.00            | 332.70          | 8.00            |
| HAYES-ROTH-Data  | 3.90            | 6.19            | 3.93            | 6.19            | 3.93            |
| LYMPHOGRAPHY     | 4.53            | 14.91           | 4.72            | 18.11           | 4.72            |
| NURSERY          | 36.52           | 1,516.81        | 36.77           | 1,516.81        | 36.77           |
| SOYBEAN-SMALL    | 3.47            | 12.32           | 3.47            | 10.04           | 3.47            |
| SPECT-TEST       | 20.46           | 55.73           | 55.73           | 56.72           | 56.75           |
| TIC-TAC-TOE      | 12.86           | 25.19           | 25.77           | 34.12           | 34.12           |
| ZOO-DATA         | 6.86            | 10.54           | 6.49            | 10.73           | 6.49            |
| **Average**      | **10.33**       | **198.62**      | **15.37**       | **199.78**      | **14.21**       |

Table 22. Results for \( U \in \{ me, rme, ent, gini, R \} \) and h

| Number of variables \( n \) | \( h_U^{(1)} \) | \( h_U^{(2)} \) | \( h_U^{(3)} \) | \( h_U^{(4)} \) | \( h_U^{(5)} \) |
|-------------------------------|----------------|----------------|----------------|----------------|----------------|
| 3                             | 0.2943         | 0.2023         | 1.8645         | 0.2023         | 1.8630         |
| 4                             | 4.0042         | 4.0054         | 2.9730         | 4.0054         | 2.9730         |
| 5                             | 5.0055         | 4.1114         | 3.9935         | 4.1114         | 3.9934         |
| 6                             | 6.0066         | 5.0996         | 5.0055         | 5.0996         | 5.0055         |

Table 23. Results for \( U \in \{ me, rme, ent, gini, R \} \) and L

| Number of variables \( n \) | \( L_U^{(1)} \) | \( L_U^{(2)} \) | \( L_U^{(3)} \) | \( L_U^{(4)} \) | \( L_U^{(5)} \) |
|-------------------------------|----------------|----------------|----------------|----------------|----------------|
| 3                             | 1.6015         | 1.9632         | 1.9615         | 1.9632         | 1.9615         |
| 4                             | 15.2102        | 14.44750       | 11.27695       | 14.44750       | 11.27695       |
| 5                             | 51.4254        | 125.21805292   | 25.7019176     | 125.21805292   | 25.7019176     |
| 6                             | 69.8630101     | 649.1171031538 | 75.29299867    | 649.1171031538 | 75.29299867    |

Table 24. Results for \( U \in \{ me, rme, ent, gini, R \} \) and l
Table 25. Results for $U \in \{\text{me, rme, ent, gini, } R\}$ and $c$

| Number of variables $n$ | $c_U^{(1)}$ | $c_U^{(2)}$ | $c_U^{(3)}$ | $c_U^{(4)}$ | $c_U^{(5)}$ |
|-------------------------|-------------|-------------|-------------|-------------|-------------|
| 3                       | 1.001.85    | 2.183.63    | 1.002.083.63| 1.252.183.63| 1.002.083.63|
| 4                       | 1.131.85    | 2.546.44    | 1.502.266.44| 1.312.546.44| 1.502.266.44|
| 5                       | 1.381.86    | 2.694.22    | 1.502.304.84| 1.502.694.22| 1.502.304.84|
| 6                       | 1.471.85    | 2.864.17    | 1.672.604.16| 2.002.864.17| 1.672.604.16|

To minimize the depth $h$ of the constructed decision trees, we should choose decision trees of type 3. The two best uncertainty measures are $R$ and $\text{me}$. To minimize the number of realizable nodes $L$ in the constructed decision trees, we should choose decision trees of type 1. The two best uncertainty measures are $R$ and $\text{ent}$. To minimize the length $l$ of decision rules derived from the constructed decision trees, we should choose decision trees of type 2. The two best uncertainty measures are $\text{me}$ and $\text{rme}$. To maximize the coverage $c$ of decision rules derived from the constructed decision trees, we should choose decision trees of type 4. The two best uncertainty measures are $\text{rme}$ and $\text{me}$.

The results for randomly generated Boolean functions are consistent with the results for decision tables from [12]. For the minimization of $h$, we should use decision trees of types 3 and 5, for the minimization of $L$ – trees of type 1, for the minimization of $l$ – trees of types 2 and 4, and for the maximization of $c$ – also trees of types 2 and 4. Note that the results obtained for Boolean functions do not depend on the choice of an uncertainty measure from the set $\{\text{me, rme, ent, gini, R}\}$.

7 Conclusions

In this paper, we studied decision trees with hypotheses. We designed greedy algorithms based on arbitrary uncertainty measures for the construction of such decision trees and made experiments with five uncertainty measures.

Using the results of experiments, we determined which types of decision trees and uncertainty measures should be chosen if we would like (i) to minimize the depth of decision trees, (ii) to minimize the number of realizable nodes in decision trees, (iii) to minimize the length of decision rules derived from the constructed decision trees, and (iv) to maximize the coverage of decision rules derived from the constructed decision trees.

Acknowledgments

Research reported in this publication was supported by King Abdullah University of Science and Technology (KAUST).
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