Multi-particle and High-dimension Controlled Order Rearrangement Encryption Protocols

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Based on the controlled order rearrange encryption (CORE) for quantum key distribution using EPR pairs [Fu.G.Deng and G.L.Long Phys.Rev.A68 (2003) 042315], we propose the generalized controlled order rearrangement encryption (GCORE) protocols of \(N\) qubits and \(N\) qutrits, concretely display them in the cases using 3-qubit, 2-qutrit maximally entangled basis states. We further indicate that our protocols will become safer with the increase of number of particles and dimensions. Moreover, we carry out the security analysis using quantum covariant cloning machine for the protocol using qutrits. Although the applications of the generalized scheme need to be further studied, the GCORE has many distinct features such as great capacity and high efficiency.

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I. INTRODUCTION

Cryptography is an art of providing secure communication over insecure communication channels. Now, in the information community, the safety of transmission of secret information is getting more and more important. One essential theme of secure communication is to distribute secret keys between sender and receiver. Quantum cryptography (QC) is secure based on the fundamental principles of quantum mechanics rather than classical cryptography. An important application of QC is the quantum key distribution (QKD), which

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concerns the generation and distribution of secret key between two legitimate users. The security of key distribution is the most important part in the secret communication. QKD exploits quantum mechanics principles for the secret communication, which provides a secure way for transmitting the key. So far, there are many quantum secret key protocols such as BB84 protocol, Ekert91, B92, six-state protocol etc.[1-5], and some new quantum secret key protocols [6-12] are continually suggested.

The security of some QKD protocols in Refs.[1,2,3,6,10-14] are based on random choices of different measuring-base, so the randomness is usually a useful ingredient in QC. The security of other some QKD protocols in Refs.[7,8,15-17] lies on the nonlocality nature of quantum systems. Goldenberg-Vaidman scheme [7] first presents QKD protocol by two transmission lines. This protocol uses orthogonal states and has full efficiency, all the particles transmitted are used to generate secret keys. Then, Koashi-Imoto protocol [8] improves Goldenberg-Vaidman scheme by using an asymmetric interferometer to reduce the time delay. However, two factors lead that these schemes’ time delay can not be too short. Subsequently, F.G.Deng and G.L.Long propose a controlled order rearrange encryption (CORE) scheme [9] to overcome this drawback and realize a secure QKD. In the nonlocality based QKD protocols, orthogonal quantum states are used. Security is assured by not allowing an eavesdropper such as Eve to acquire both parts simultaneously.

Actually, the CORE technique is implemented not only suitable to use Einstein-Podolsky-Rosen (EPR) pairs, but also suitable to use other quantum information carriers (QICs) [9]. In recent years researchers have drawn their attentions to the QKD protocols that involve multilevel systems with two parties, or multiple parties with two-level systems. A pursuit motivation of multilevel QKD is that more information can be carried by each particle thereby the information flux is increased, and some multilevel protocols have been shown to have greater security against eavesdropping attacks than their qubit-based counterparts [16,18,19]. Thus, the use of multi-particle maximally entangled state can guarantee the security further and has higher efficiency in general.

In this paper, our main purpose is to generalize the CORE of QC to multi-particle and/or high dimension quantum systems. Our generalized protocols can be thought of having higher efficiency because the generalized protocols, which is called as the GCORE of QC here, exploit the facts that a possible eavesdropper with no access to the whole quantum system at the same time, cannot recover the whole information without being detected, and
the protocols employ a larger alphabet, a few-dimensional orthogonal basis of pure state. Consequently, we obtain the full efficiency from this point of view. The generalized protocols also have great capacity based on the reason that $M$ adopted $N$-qudit maximally entangled states can send $M \log_2 d^N$ bits of information in our schemes if we assume there are $N$ particles with each being $d$ dimension.

The paper is organized as follows. In Sec.II, we simply review the CORE protocol using EPR pairs provided by Fu.G.Deng and G.L.Long. Then we generalize the CORE protocol to $N$-qubit case, specially, we present the GCORE protocol using 3-qubit state and check its security by the correlated matrix method. In Sec.III, the GCORE protocol using $N$ qutrits is proposed, GCORE protocol using 2-qutrit is presented in detail. Moreover, we discuss the security of qutrit GCORE using the quantum covariant cloning machine. In Sec.IV, we present a uniform expression of multi-particle and/or high dimension situation. Advantages of GCORE are analogized and concluding remarks are given.

**II. GCORE USING $N$-QUBIT MAXIMALLY ENTANGLED BASIS STATES**

**A. Explanation of CORE protocol**

At the beginning, let us review briefly the meaning of CORE. Assume the keys are distributed between Alice and Bob. Before transmission, Alice rearranges the order of correlated particles and sends them to Bob. The aim of random rearrangement is to prevent the eavesdropper obtaining correlated particles simultaneously from different transmission channels as possible as they can, and we also need an evening process to make transmission in equal time intervals. Once Bob receives these particles, he restores the order of the particles and undoes Alice’s operations by synchronizing their measure devices using repeatedly a prior shared control key, so that he can make orthogonal basis measurement. The measurement outcome is exactly what Alice has prepared. The essence of CORE is use of a a control key as has been used in the modified BB84 scheme [6]. The noncloning nature ensures it viable.

The whole process of CORE protocol using EPR states [9] has been demonstrated clearly in Ref.[9]. In the following contexts, we generalize it to multi-particle and high-dimensional cases, hence the generalized protocol is denoted as GCORE.
B. GCORE protocol using GHZ-basis states

In the following, we firstly discuss concrete GCORE example using 3-qubit GHZ-basis states without loss of generalization.

(i) Alice generates a sequence of GHZ-basis states \((a_1, b_1, c_1), \ldots, (a_m, b_m, c_m)\) randomly, where \((a_i, b_i, c_i)\) denotes one GHZ-basis state \((1 \leq i \leq m, m\) is an integer) and every eight adjoining triplets are taken as one unit of QICs. Without loss of generality, we consider the first carrier unit \(\{(a_1, b_1, c_1), (a_2, b_2, c_2), \ldots, (a_8, b_8, c_8)\}\) which are randomly in eight GHZ-basis states that can be expressed as [20]:

\[
|\psi^\pm_j \rangle = \frac{1}{\sqrt{2}} (|j\rangle_{AB} |0\rangle_C \pm |3-j\rangle_{AB} |1\rangle_C), \tag{1}
\]

where \(j = j_1j_2\) denotes binary notations. In their explicit forms, eight GHZ-basis states reads:

\[
|\psi_0^+ \rangle = (|000\rangle + |111\rangle) / \sqrt{2} \\
|\psi_0^- \rangle = (|000\rangle - |111\rangle) / \sqrt{2} \\
|\psi_1^+ \rangle = (|010\rangle + |101\rangle) / \sqrt{2} \\
|\psi_1^- \rangle = (|010\rangle - |101\rangle) / \sqrt{2} \\
|\psi_2^+ \rangle = (|100\rangle + |011\rangle) / \sqrt{2} \\
|\psi_2^- \rangle = (|100\rangle - |011\rangle) / \sqrt{2} \\
|\psi_3^+ \rangle = (|110\rangle + |001\rangle) / \sqrt{2} \\
|\psi_3^- \rangle = (|110\rangle - |001\rangle) / \sqrt{2}) \tag{2}
\]

we indicate them by 000, 001, 010, 011, 100, 101, 110, and 111, respectively.

(ii) Alice sends the three parts out in equal time intervals to Bob through three channels. Before these GHZ-basis states enter into the insecure transmission channel, their orders are rearranged by the GCORE system. Here are eight choices of GCORE operations, corresponding relations are the following:

\[
E_0 \leftrightarrow 000, \quad E_1 \leftrightarrow 001, \quad E_2 \leftrightarrow 010, \quad E_3 \leftrightarrow 011 \\
E_4 \leftrightarrow 100, \quad E_5 \leftrightarrow 101, \quad E_6 \leftrightarrow 110, \quad E_7 \leftrightarrow 111 
\]

and the GCORE is done for eight GHZ-basis states. Let us use permutation group notation to express them as following
\[ E_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = (1) (2) (3) (4) (5) (6) (7) (8) \]
\[ E_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \]
\[ E_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 7 \\ 6 & 8 \end{pmatrix} \]
\[ E_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 5 & 8 \\ 6 & 7 \end{pmatrix} \]
\[ E_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix} \]
\[ E_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 2 & 7 \\ 3 & 6 \\ 4 & 5 \end{pmatrix} \]
\[ E_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 2 & 8 \\ 3 & 5 \\ 4 & 6 \end{pmatrix} \]
\[ E_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 8 \\ 4 & 7 \end{pmatrix} \]

we also show this protocol using Fig. 1.

Three quantum channels in this GCORE protocol are denoted upper, middle, and lower channel. The upper QIC parts are transmitted according to their temporal orders. A control key is used to rearrange the order of middle and lower QIC parts. For instance, the value of control key is 000, the operation \( E_0 \) is applied. In Fig. 2 there are seven switches, the order of eight GHZ-basis states is unchanged with switch 1, 2, 3, 4, 5, 6, 7 in position (up, up, up, up, up, up, down). When the value of control key is 001, the operation \( E_1 \) is performed, and it is done by putting the seven switches into position(down, up, up, up, up, up, down), (up, up, up, up, up, down,up), (down, up, up, up, up, up, down), (up, up, up, up, up, down,up), (down, up, up, up, up, down,up), (up, up, up, up, up, down,up) for eight particles, respectively. In fact, five switches are enough. When the operation \( E_1 \) is performed, it is done by putting the five switches into position(down, down, up, up, down), (up, down, up, up, down, up), (down, down, up, up, down, up), (up, down, up, up, down, up), (down, down, up, up, down, up), (up, down, up, up, down, up) for eight particles, respectively. The
FIG. 1: Example of GCORE using GHZ-basis states. There are eight different GCORE operations.

The effect of using seven switches is the same as that of using five switches. Similar combination can be written explicitly for operations $E_2, E_3, E_4, E_5, E_6, E_7$.

Five switches

Seven switches

FIG. 2: Devices to perform GCORE operations, the loop represents a time delay of a fixed interval.

(iii) Bob undoes the effect of order rearrangement. At Bob’s site, he just exchanges upper, middle, and lower parts of Alice’s GCORE apparatus and the GCORE operations performed by Alice will be undone. (iv) Bob measures these carrier units to obtain the key. After these
particles are dearranged. Bob uses the GHZ-basis measurement to read out the information determinatively, which is exactly the same as Alice prepared one because the measurement here is orthogonal basis measurement and obviously the eight GHZ-basis states are mutually orthogonal.

Remark: To prevent Eve from stealing, we need an evening process to ensure the same time interval between different batches of QICs travel. Now, we need three transmission lines to ensure the application of current proposed scheme because 3-qubit GCORE uses GHZ-basis state, and each particle transmitted through a quantum transmission line in equal time interval. It is obviously different from the case using two-transmission lines in Refs.[7,8]. Detailed analysis will be presented in subsection C below. In addition, the control keys can be used to control the GCORE operation of a group of units to reduce resources. For example, instead of using 001 controls GCORE operation of one unit of QICs (eight GHZ-basis states), we can use 001 to control more units of QICs consecutively, say 4 units or 32 GHZ-basis states.

C. Security of GCORE using GHZ-basis states

Let us look at the security of GCORE using 3-qubit GHZ-basis states. Eve has only 1/8 chance to guess the right GCORE operation for the eight GHZ-basis states. If she uses a wrong GCORE operation, the three particles measured by her will be anticorrelated. Firstly we assume that A particle from the first GHZ-basis state, B particle from the second GHZ-basis state and C particle from the third GHZ-basis state are mistreated by Eve as a GHZ-basis state, then the density operator will be

$$\rho_{A_1B_2C_3} = \tilde{\rho}_{A_1} \otimes \tilde{\rho}_{B_2} \otimes \tilde{\rho}_{C_3} = \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \frac{1}{8} I_{8 \times 8} \quad (3)$$

where $\tilde{\rho}_{A_1} = \text{Tr}_{B_1C_1} (\rho_{A_1B_1C_1})$, $\tilde{\rho}_{B_2} = \text{Tr}_{A_2C_2} (\rho_{A_2B_2C_2})$, $\tilde{\rho}_{C_3} = \text{Tr}_{A_3B_3} (\rho_{A_3B_3C_3})$. When $\rho_{A_1B_2C_3}$ is measured in the GHZ-basis state, the result can be any one of eight GHZ-basis states with 12.5% probability each. Thus Eve will introduce 66.99% error rate in the results. Then we assume A particle from the first GHZ-basis state, B and C particles from the second
GHZ-basis state are mistreated by Eve as a GHZ-basis state, the density operator will be

\[
\rho_{A_1B_2C_3} = \tilde{\rho}_{A_1} \otimes \tilde{\rho}_{B_2C_3} = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}
\]

(E4)

Eve will introduce 76.56% error rate in these results. In both situations, Alice and Bob can detect Eve easily by checking a sufficiently large subset of results randomly chosen. Surely, Eve can perform a generalized Bell inequality measurement on the particles, but it is useless for decrypting the control key. Let us choose \(\vec{a}(a_x, a_y, a_z), \vec{b}(b_x, b_y, b_z)\), as the directions of Alice’s and Bob’s measurements, at the same time, \(\vec{c}(c_x, c_y, c_z)\) is also Bob’s measurement direction. Then the correlation operator can be written as following:

\[
\hat{E} = (\hat{\sigma} \cdot \vec{a}) \otimes (\hat{\sigma} \cdot \vec{b}) \otimes (\hat{\sigma} \cdot \vec{c})
\]

(E5)

where \(\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), \(\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\), \(\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\). The expectation values

\[
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_\psi = \langle \psi \mid (\hat{\sigma} \cdot \vec{a}) \otimes (\hat{\sigma} \cdot \vec{b}) \otimes (\hat{\sigma} \cdot \vec{c}) \mid \psi \rangle
\]

are different for the different GHZ-basis states. They are

\[
\begin{align*}
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_0^+} &= (a_x - ia_y)(b_x - ib_y)(c_x - ic_y) + (a_x + ia_y)(b_x + ib_y)(c_x + ic_y) \\
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_0^-} &= -(a_x - ia_y)(b_x - ib_y)(c_x - ic_y) - (a_x + ia_y)(b_x + ib_y)(c_x + ic_y) \\
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_1^+} &= (a_x - ia_y)(b_x + ib_y)(c_x - ic_y) + (a_x + ia_y)(b_x - ib_y)(c_x + ic_y) \\
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_1^-} &= -(a_x - ia_y)(b_x + ib_y)(c_x - ic_y) - (a_x + ia_y)(b_x - ib_y)(c_x + ic_y) \\
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_2^+} &= (a_x + ia_y)(b_x - ib_y)(c_x - ic_y) + (a_x - ia_y)(b_x + ib_y)(c_x + ic_y) \\
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_2^-} &= -(a_x + ia_y)(b_x - ib_y)(c_x - ic_y) - (a_x - ia_y)(b_x + ib_y)(c_x + ic_y) \\
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_3^+} &= (a_x + ia_y)(b_x + ib_y)(c_x - ic_y) + (a_x - ia_y)(b_x - ib_y)(c_x + ic_y) \\
\left\langle E(\vec{a}, \vec{b}, \vec{c}) \right\rangle_{\psi_3^-} &= -(a_x + ia_y)(b_x + ib_y)(c_x - ic_y) - (a_x - ia_y)(b_x - ib_y)(c_x + ic_y)
\end{align*}
\]

(6)

Note their coefficients are 1/2.

For the product states \(|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\), the expected values are:

\[
a_x b_x c_x, \quad -a_x b_x c_x, \quad -a_x b_x c_x, \quad a_x b_x c_x, \quad -a_x b_x c_x, \quad a_x b_x c_x, \quad -a_x b_x c_x
\]

(7)
respectively. If Eve takes general Bell inequality measurements on the three uncorrelated particles, she will get 0 for a large number of measurements when the particles are randomly distributed among the eight GHZ-basis states. If Eve does take three correlated particles, she will also get 0 when eight GHZ-basis states are taken with the equal probability. So Eve gets nothing about the control key except for guessing it randomly. Because the control key can be repeatedly used, the probability that Eve guesses the right control key is \( \left( \frac{1}{2} \right)^{3N_k} \), where \( 3N_k \) is the number of bits in the control key. When \( N_k = 100 \), the probability is \( \left( \frac{1}{8} \right)^{100} \), which is practically zero. Naturally, GCORE protocol is suitable to N-qubit setting scenario, too. N-qubit maximally entangled basis states are defined as following \[20\]:

\[
|\psi_j^\pm\rangle = \frac{1}{\sqrt{2}} \left( |j\rangle |0\rangle \pm |2^{N-1} - j - 1\rangle |1\rangle \right)
\]  
(8)

where \( j = j_1j_2 \cdots j_{N-1} \) denotes binary notations. Then there are \( 2^N \) different control keys, \( 2^N \) operations corresponding to \( E_0, E_1, \cdots E_{2^N-1} \), and we need \( N \) quantum channels with \( \frac{2^N}{2} + 1 = 2^{N-1} + 1 \) switches each. The eavesdropper Eve only guesses the right \( N \)-GHZ-basis states with probability \( \frac{1}{2^N} \), as the density operation is \( \rho_{AB \cdots N} = \frac{1}{2^N} I_{2^N \times 2^N} \).

### III. GCORE USING N-QUTRIT MAXIMALLY ENTANGLED BASIS STATES

One of the motivations of considering a high dimensional system for QKD is to increase the information per particle. Another context where using higher-dimensional space might be advantageous is the key growing. However, the practical limitations might be more severe in realistic high-dimension cryptosystems, in particular the influence of the detector’s quantum efficiency and dark count rate \[21,22\]. This has been discussed in the related Ref.[23]. Here, we start to consider the qutrit quantum system.

#### A. GCORE protocol using 2-qutrit general Bell-basis states

Here, let’s consider the simplest scenario, two particles, each particle has three levels, i.e. a 2-qutrit system. On the whole, concrete four processes are similar to analysis in Sec.II.B. The recapitulation is presented in the following. As we know, the general Bell-basis states can be written as \[24\]:

\[
|\psi_{nm}\rangle = \sum_j e^{2\pi ij/3} |j\rangle \otimes |j + m \; \text{mod} \; 3\rangle / \sqrt{3}
\]  
(9)
where \( n, m, j = 0, 1, 2 \), the explicit expressions are then

\[
|\psi_{00}\rangle = (|00\rangle + |11\rangle + |22\rangle) / \sqrt{3}
\]
\[
|\psi_{10}\rangle = (|00\rangle + e^{2i\pi/3} |11\rangle + e^{4i\pi/3} |22\rangle) / \sqrt{3}
\]
\[
|\psi_{20}\rangle = (|00\rangle + e^{4i\pi/3} |11\rangle + e^{2i\pi/3} |22\rangle) / \sqrt{3}
\]
\[
|\psi_{01}\rangle = (|01\rangle + |12\rangle + |20\rangle) / \sqrt{3}
\]
\[
|\psi_{11}\rangle = (|01\rangle + e^{2i\pi/3} |12\rangle + e^{4i\pi/3} |20\rangle) / \sqrt{3}
\]
\[
|\psi_{21}\rangle = (|01\rangle + e^{4i\pi/3} |12\rangle + e^{2i\pi/3} |20\rangle) / \sqrt{3}
\]
\[
|\psi_{02}\rangle = (|02\rangle + |10\rangle + |21\rangle) / \sqrt{3}
\]
\[
|\psi_{12}\rangle = (|02\rangle + e^{2i\pi/3} |10\rangle + e^{4i\pi/3} |21\rangle) / \sqrt{3}
\]
\[
|\psi_{22}\rangle = (|02\rangle + e^{4i\pi/3} |10\rangle + e^{2i\pi/3} |21\rangle) / \sqrt{3}
\]  

(10)

It is clear that these states are orthogonal. They can be presented by 00, 01, 02, 10, 11, 12, 20, 21, 22, respectively. It can be shown that single-body operators \( U_{ij} (i, j = 0, 1, 2) \) will transform \( |\psi_{00}\rangle \) into the corresponding other eight states. The expressions of these operators are:

\[
U_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; U_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2i\pi/3} & 0 \\ 0 & 0 & e^{4i\pi/3} \end{pmatrix}; U_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{4i\pi/3} & 0 \\ 0 & 0 & e^{2i\pi/3} \end{pmatrix}
\]

\[
U_{01} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; U_{11} = \begin{pmatrix} 0 & 0 & e^{4i\pi/3} \\ 1 & 0 & 0 \\ 0 & e^{2i\pi/3} & 0 \end{pmatrix}; U_{21} = \begin{pmatrix} 0 & 0 & e^{2i\pi/3} \\ 1 & 0 & 0 \\ 0 & e^{4i\pi/3} & 0 \end{pmatrix}
\]

\[
U_{02} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; U_{12} = \begin{pmatrix} 0 & e^{2i\pi/3} & 0 \\ 0 & 0 & e^{4i\pi/3} \\ 1 & 0 & 0 \end{pmatrix}; U_{22} = \begin{pmatrix} 0 & e^{4i\pi/3} & 0 \\ 0 & e^{2i\pi/3} & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]  

(11)

The GCORE operations using qutrit states are similar to the cases in Sec.II. However, there are nine choices of GCORE operations, corresponding relations are the following

\[
E_0 \leftrightarrow 00, \quad E_1 \leftrightarrow 01, \quad E_2 \leftrightarrow 02
\]
\[
E_3 \leftrightarrow 10, \quad E_4 \leftrightarrow 11, \quad E_5 \leftrightarrow 12
\]
\[
E_6 \leftrightarrow 20, \quad E_7 \leftrightarrow 21, \quad E_8 \leftrightarrow 22
\]
and the GCORE is done for every nine general Bell-basis states. These operations are denoted by the denotation of permutation group.

\[
E_0 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{pmatrix} =
(1 \ 1) (2 \ 2) (3 \ 3) (4 \ 4) (5 \ 5) (6 \ 6) (7 \ 7) (8 \ 8) (9 \ 9)
\]

\[
E_1 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2
\end{pmatrix} =
(1 \ 2) (2 \ 3) (3 \ 4) (4 \ 5) (5 \ 6) (6 \ 7) (7 \ 8) (8 \ 9) (9 \ 1)
\]

\[
E_2 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2
\end{pmatrix} =
(1 \ 3) (2 \ 4) (3 \ 5) (4 \ 6) (5 \ 7) (6 \ 8) (7 \ 9) (8 \ 1) (9 \ 2)
\]

\[
E_3 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3
\end{pmatrix} =
(1 \ 4) (2 \ 5) (3 \ 6) (4 \ 7) (5 \ 8) (6 \ 9) (7 \ 1) (8 \ 2) (9 \ 3)
\]

\[
E_4 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4
\end{pmatrix} =
(1 \ 5) (2 \ 6) (3 \ 7) (4 \ 8) (5 \ 9) (6 \ 1) (7 \ 2) (8 \ 3) (9 \ 4)
\]

\[
E_5 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5
\end{pmatrix} =
(1 \ 6) (2 \ 7) (3 \ 8) (4 \ 9) (5 \ 1) (6 \ 2) (7 \ 3) (8 \ 4) (9 \ 5)
\]

\[
E_6 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6
\end{pmatrix} =
(1 \ 7) (2 \ 8) (3 \ 9) (4 \ 1) (5 \ 2) (6 \ 3) (7 \ 4) (8 \ 5) (9 \ 6)
\]

\[
E_7 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix} =
(1 \ 8) (2 \ 9) (3 \ 1) (4 \ 2) (5 \ 3) (6 \ 4) (7 \ 5) (8 \ 6) (9 \ 7)
\]
\[ E_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = (1 9)(2 1)(3 2)(4 3)(5 4)(6 5)(7 6)(8 7)(9 8) \]

the permutation has been shown clearly in Fig. 3. Fig. 4 gives the main device to perform GCORE operation by a specific instance.

![Diagram](image) FIG. 3: Example of GCORE using 2-qutrit Bell-basis states; There are nine different GCORE operations

![Diagram](image) FIG. 4: Devices to perform GCORE operations, the loop represents a time delay of a fixed interval.

According to Fig. 4, the upper QIC parts are transmitted according to their temporal order. A control key is used to rearrange the order of the lower QIC parts. For instance, the value of control key is 00, the operation \( E_0 \) is applied. In Fig. 4 there are five switches, the order of nine general Bell-basis states is unchanged with switch 1,2,3,4 and 5 in position (up, up, down, up, up ). When control key is 01, \( E_1 \) is performed, and it is done by putting the nine switches into position (down, down, up, up, down ), (up, down, up, up, up ), (up, down,
up, up, up ), (up, down, up, up, up ), (up, down, up, up, up ), (up, down, up, up, up ),
for the nine particles, respectively. Similar combination can be written explicitly for
$E_2, E_3, E_4, E_5, E_6, E_7, E_8$.

Now we can consider the cases of multi-particle and/or high-dimension quantum systems.
Firstly the method is generalized to high dimension quantum system ($d > 3$) of two particles.

Note that the $d$-dimension Bell-basis states in a symmetric channel [8, 21, 23] are expressed
as
$$|\psi_{nm}\rangle = \sum_j e^{2\pi ijn/d} |j\rangle \otimes |j + m \mod d\rangle /\sqrt{d}$$
(12)

where $n, m, j = 0, 1, \cdots d - 1$. The unitary operator is
$$U_{nm} = \sum_j e^{2\pi ijn/d} |j + m \mod d\rangle \langle j|$$
(13)
which can transfer $d$-dimension Bell-basis state
$$|\psi_{00}\rangle = \sum_j |j\rangle \otimes |j\rangle /\sqrt{d}$$
(14)
to other $d$-dimension Bell-basis state $|\psi_{nm}\rangle$, i.e. $U_{nm} |\psi_{00}\rangle = |\psi_{nm}\rangle$. So we can use the
same method like 2-qutrit GCORE to analyze this problem completely. Thus, we have
presented the GCORE of two-particle high dimensional generalization, we will give multi-
particle situation next. At first, we consider a less complicated three particle quantum
system. For 3-qutrit quantum system, its generalized maximally entangled basis states are :
$$|\psi_{nm}^k\rangle = \sum_j e^{2\pi ijk/3} |j\rangle \otimes |j + n \mod 3\rangle \otimes |j + m \mod 3\rangle /\sqrt{3}$$
(15)

where $n, m, k = 0, 1, 2$, the explicit expressions are then
$$|\psi_{00}^0\rangle = (|000\rangle + |111\rangle + |222\rangle) /\sqrt{3}$$
$$|\psi_{01}^0\rangle = (|001\rangle + |112\rangle + |220\rangle) /\sqrt{3}$$
$$|\psi_{02}^0\rangle = (|002\rangle + |110\rangle + |221\rangle) /\sqrt{3}$$
$$\cdots$$
$$|\psi_{22}^2\rangle = (|022\rangle + e^{4\pi i/3} |100\rangle + e^{2\pi i/3} |222\rangle) /\sqrt{3}$$
(16)
There are 27 corresponding GCORE operations, denoted by:

\[
E_0 \leftrightarrow 000, \quad E_1 \leftrightarrow 001, \quad E_2 \leftrightarrow 002, \quad E_3 \leftrightarrow 100, \quad E_4 \leftrightarrow 101, \quad E_5 \leftrightarrow 102, \quad E_6 \leftrightarrow 200, \\
E_7 \leftrightarrow 201, \quad E_8 \leftrightarrow 202, \quad E_9 \leftrightarrow 010, \quad E_{11} \leftrightarrow 011, \quad E_{12} \leftrightarrow 012, \quad E_{13} \leftrightarrow 110, \quad E_{14} \leftrightarrow 111, \\
E_{15} \leftrightarrow 112, \quad E_{16} \leftrightarrow 210, \quad E_{17} \leftrightarrow 211, \quad E_{18} \leftrightarrow 212, \quad E_{19} \leftrightarrow 020, \quad E_{20} \leftrightarrow 021, \quad E_{21} \leftrightarrow 022, \\
E_{22} \leftrightarrow 120, \quad E_{23} \leftrightarrow 121, \quad E_{24} \leftrightarrow 122, \quad E_{25} \leftrightarrow 220, \quad E_{26} \leftrightarrow 221, \quad E_{27} \leftrightarrow 222
\]

(17)

Due to the complication of GCORE operations, more resources are needed, and the analysis of security also becomes more complicated. But the maximal advantage is the swell of security. And the probability that Eve guesses the right control key is near 0. The corresponding fig. 5 is given below.

![Diagram](image)

**FIG. 5:** (a). Example of GCORE using 3-qutrit maximally entangled basis states; (b). Devices to perform GCORE operations, the loop represents a time delay of a fixed interval.

Generally, a uniform expression of \(N\)-qutrit maximally entangled basis state can be expressed as the following form

\[
|\psi_{i_1,i_2,\cdots,i_{N-1}}^N\rangle = \sum_j e^{2\pi ijN/3} |j\rangle \otimes |j + i_1 \text{ mod } 3\rangle \otimes |j + i_2 \text{ mod } 3\rangle \otimes \cdots \otimes |j + i_{N-1} \text{ mod } 3\rangle / \sqrt{3}
\]

(18)

where \(i_1,i_2,\cdots,i_N = 0,1,2\). Similar analysis can be given, but there is a little difference. In short, there are \(3^N\) different control keys, \(3^N\) operations corresponding to \(E_0,E_1,\cdots,E_{3^N-1}\),
and we need \( N \) quantum channels with \( 3^{N-1} + 1 \) switches each. The eavesdropper only guesses the right general Bell-basis state with probability \( \frac{1}{3^N} \), as the density operation is 
\[
\rho_{AB\cdots N} = \frac{1}{3^N} I_{3^N \times 3^N}.
\]

B. Security of GCORE using 2-qutrit general Bell-basis states

Now, let us look at the security of above GCORE protocol using 2-qutrit states. Eve has only 11.1% chance to guess the right GCORE operation for nine general Bell-basis states. If she uses a wrong GCORE operation, the two particles she measured will be anticorrelated. Assume that A particle from the general Bell-basis state, B particle from the second general Bell-basis state are mistreated by Eve as a general Bell-basis state, then the density operator will be
\[
\rho_{A_1B_2} = \tilde{\rho}_{A_1} \otimes \tilde{\rho}_{B_2} = \left( \begin{array}{ccc} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{array} \right) \otimes \left( \begin{array}{ccc} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{array} \right) = \frac{1}{9} I_{9 \times 9} \tag{19}
\]
where \( \tilde{\rho}_{A_1} = \text{Tr}_{B_1}(\rho_{A_1B_1}) \), \( \tilde{\rho}_{B_2} = \text{Tr}_{A_2}(\rho_{A_2B_2}) \).

The result indicates that any one of the nine general Bell-basis states appears with 11.1% probability each. Thus Eve will introduce 79.01% error rates in the results. Alice and Bob can detect Eve easily by checking a sufficiently large subset of results randomly chosen. Surely, Eve can take the generalized Bell inequality measurement on the particles, but it is useless for decrypting the control key. There are eight (Hermitian) generators of SU(3), i.e. eight Gell-Mann matrices, which are defined by
\[
\begin{align*}
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \quad \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Let us choose directions \( \vec{M} \) and \( \vec{N} \) as the directions of measurement of Alice and Bob respectively, these measurements satisfy the orthogonal relations. The correlation operator
can be written as:

\[ \hat{E} = \hat{S} \cdot \hat{M} \otimes \hat{S} \cdot \hat{N} \]  \hspace{1cm} (20)

The expectation values \( \langle E(\vec{M}, \vec{N}) \rangle_{\psi} \) are not equal for different general Bell-basis states.

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{00}} = \frac{2}{3} \left( M_1 N_1 - M_2 N_2 + M_3 N_3 + M_4 N_4 - M_5 N_5 + M_6 N_6 - M_7 N_7 + M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{01}} = \frac{1}{3} \left( 2 M_4 N_1 + 2 M_5 N_2 - M_3 N_3 - \sqrt{3} M_8 N_3 + 2 M_6 N_4 + 2 M_7 N_5 + 2 M_1 N_6 - 2 M_2 N_7 + \sqrt{3} M_3 N_8 - M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{02}} = \frac{1}{3} \left( 2 M_6 N_1 - 2 M_7 N_2 - M_3 N_3 + \sqrt{3} M_8 N_3 + 2 M_1 N_4 + 2 M_2 N_5 + 2 M_4 N_6 + 2 M_5 N_7 - \sqrt{3} M_3 N_8 - M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{10}} = \frac{1}{3} \left( \frac{1}{3} M_1 N_1 - \frac{1}{3} M_2 N_2 + \frac{1}{3} M_3 N_3 + \frac{\sqrt{3} i}{6} M_8 N_3 - \frac{1}{3} M_4 N_4 + \frac{1}{3} M_5 N_5 + \frac{2}{3} M_6 N_6 - \frac{2}{3} M_7 N_7 + \frac{\sqrt{3} i}{6} M_3 N_8 - \frac{1}{3} M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{11}} = \frac{1}{3} \left( \frac{1}{6} M_4 N_4 - \frac{1}{3} M_5 N_5 - \frac{1}{3} M_3 N_3 - \frac{2}{3} M_6 N_6 + \frac{2}{3} M_7 N_7 + \frac{1}{3} M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{12}} = \frac{1}{3} \left( \frac{1}{6} M_4 N_4 - \frac{1}{3} M_5 N_5 - \frac{1}{3} M_3 N_3 - \frac{2}{3} M_6 N_6 + \frac{2}{3} M_7 N_7 + \frac{1}{3} M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{20}} = \frac{1}{3} \left( \frac{1}{6} M_4 N_4 + \frac{1}{3} M_3 N_3 + \frac{\sqrt{3} i}{6} M_8 N_3 - \frac{1}{3} M_5 N_5 + \frac{2}{3} M_6 N_6 - \frac{2}{3} M_7 N_7 + \frac{\sqrt{3} i}{6} M_3 N_8 - \frac{1}{3} M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{21}} = \frac{1}{3} \left( \frac{1}{6} M_4 N_4 - \frac{1}{3} M_3 N_3 + \frac{\sqrt{3} i}{6} M_8 N_3 - \frac{1}{3} M_5 N_5 + \frac{2}{3} M_6 N_6 - \frac{2}{3} M_7 N_7 + \frac{\sqrt{3} i}{6} M_3 N_8 - \frac{1}{3} M_8 N_8 \right)
\]

\[
\langle E(\vec{M}, \vec{N}) \rangle_{\psi_{22}} = \frac{1}{3} \left( \frac{1}{6} M_4 N_4 + \frac{1}{3} M_5 N_5 - \frac{1}{3} M_3 N_3 + \frac{\sqrt{3} i}{6} M_8 N_3 + \frac{2}{3} M_6 N_6 + \frac{2}{3} M_7 N_7 + \frac{1}{3} M_3 N_8 + \frac{1}{3} M_8 N_8 \right)
\]
For product states $|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle$, the expected values are

$$
(M_3 + M_8 \sqrt{3}) (N_3 + N_8 \sqrt{3}), \quad \left( M_3 + M_8 \sqrt{3} \right) \left( -N_3 + N_8 \frac{\sqrt{3}}{3} \right), \quad -\left( M_3 + M_8 \sqrt{3} \right) \frac{N_8}{\sqrt{3}}
$$

$$
\left( -M_3 + \frac{M_8}{\sqrt{3}} \right) \left( N_3 + \frac{N_8}{\sqrt{3}} \right), \quad \left( \frac{M_8}{\sqrt{3}} - M_3 \right) \left( \frac{N_8}{\sqrt{3}} - N_3 \right), \quad -\frac{2}{\sqrt{3}} \left( -M_3 + \frac{M_8}{\sqrt{3}} \right) N_8,
$$

$$
-\frac{2}{\sqrt{3}} \left( N_3 + \frac{N_8}{\sqrt{3}} \right) N_8, \quad -\frac{2}{\sqrt{3}} \left( -N_3 + \frac{N_8}{\sqrt{3}} \right) M_8, \quad \frac{4}{3} M_8 N_8
$$

respectively. Subsequently, it’s easy to give the similar analysis like CORE using EPR pairs.

The experimental realization about qutrit for quantum cryptography is important. Up to now, the experiment has achieved much progress to realize the production of general Bell-basis state [25,26]. For example, in order to produce the state

$$
|\psi_{00}\rangle = (|00\rangle + |11\rangle + |22\rangle) / \sqrt{3}
$$

one uses a unbiased six-port beam splitter [27] which is a device with the following property: if a photon enters any single input port (out of the three ports), there is equal probability that it leaves one of the three output ports to produce state. In fact, one can always construct a special six-port beam splitter with the distinguishing trait that the elements of its unitary transition matrix, $T$, are solely powers of the complex number, $\alpha = \exp(i\frac{2\pi}{3})$, namely, $T_{kl} = \frac{1}{\sqrt{3}} \alpha^{(k-1)(l-1)}$. It has been shown in Ref.[22] that any six-port beam splitter can be constructed from the above-mentioned one by adding appropriate phase shifters at its exit and input ports (and by a trivial relabeling of the output ports). The phase shifters in front of the input ports of beam splitter can be tunable and used to change the phase of the incoming photon.

C. Security analysis of qutrit GCORE using the quantum cloning machine

From now on, we will analyze the security of qutrit GCORE against individual attacks (where Eve monitors the qutrit separately). So far, a lot work about the analysis of security for BB84 or generalized BB84 protocol using cloning machine have been done [18,19,28]. These workers are significative. Fortunately, GCORE protocols are also propitious to analyze using these methods. For this case, we consider a fairly general class of eavesdropping attack based on (not necessarily universal) quantum cloning machine. It is known that such a cloning-based attack is the optimal eavesdropping strategy, that is, the best Eve can do is
to clone (imperfectly) Alice’s qubit and keep a copy while sending the original to Bob [18].
An appropriate measurement of the clone (and the ancilla system) after disclosure of the
basis enables Eve to gain the maximally possible information on Alice’s key bit.

We use a general class of cloning transformations which is defined in Refs.[18,19], the
resulting joint state of the two clones (noted A and B) and of the cloning machine (noted
C) is

$$|\psi\rangle \rightarrow \sum_{m,n=0}^{N-1} a_{m,n} U_{m,n} |\psi\rangle_A |B_{m,-n}\rangle_{B,C}$$

$$= \sum_{m,n=0}^{N-1} b_{m,n} U_{m,n} |\psi\rangle_B |B_{m,-n}\rangle_{A,C}$$

(23)

where

$$U_{m,n} = \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |k+m\rangle \langle k|$$

(24)

$U_{m,n}$ forms a group of qudit error operators, generalizing the Pauli matrices for qubit: $m$
labels the shift errors (extending the bit flip $\sigma_x$), while $n$ labels the phase errors (extending
the phase flip $\sigma_z$). And

$$|B_{m,n}\rangle = N^{-\frac{1}{2}} \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |k\rangle |k+m\rangle$$

(25)

with $0 \leq m, n \leq N - 1$. Equation $|B_{m,n}\rangle$ defines the $N^2$ generalized Bell states for a pair of
$N$-dimensional systems. The final states of clone A, B are

$$\rho_A = \sum_{m,n=0}^{N-1} p_{m,n} |\psi_{m,n}\rangle \langle \psi_{m,n}| = \sum_{m,n=0}^{N-1} p_{m,n} U_{m,n} |\psi\rangle \langle \psi| U_{m,n}^\dagger$$

$$\rho_B = \sum_{m,n=0}^{N-1} q_{m,n} |\psi_{m,n}\rangle \langle \psi_{m,n}| = \sum_{m,n=0}^{N-1} q_{m,n} U_{m,n} |\psi\rangle \langle \psi| U_{m,n}^\dagger$$

(26)

In addition, the weight functions of the two clones are related by

$$p_{m,n} = |a_{m,n}|^2, \quad q_{m,n} = |b_{m,n}|^2$$

(27)

where $a_{m,n}, b_{m,n}$ are two (complex) amplitude functions that are dual under a Fourier trans-
form:

$$b_{m,n} = \frac{1}{N} \sum_{x,y=0}^{N-1} e^{2\pi i (nx-my)/N} a_{m,n}$$

(28)
Assume that Eve clones the qutrit state that is sent to Bob. Then Eve will measure her clone in the same basis as Bob and her ancilla in the conjugate basis. For deriving Eve’s information, we need first to rewrite the cloning transformation of these bases. If Alice sends any state $|k\rangle$ in the computational basis, the phase errors clearly do not play any role in the mixture $\rho_B$, so the fidelity can be expressed as:

$$F = \langle k | \rho_B | k \rangle = \sum_{n=0}^{N-1} |a_{0,n}|^2$$  \hspace{1cm} (29)

In the rest of this subsection, we will use this general characterization of cloning in order to investigate the state-dependent cloning of qutrit. Alice sends the input state $|\psi\rangle$ belonging to a 3-dimensional space. For the cloner to copy equally well the states of computational bases, we choose the amplitude $a_{m,n}$ characterizing the cloner, which must be of the form

$$
(a_{m,n}) = \begin{pmatrix}
v & x & x \\
y & y & y \\
z & z & z
\end{pmatrix}
$$  \hspace{1cm} (30)

such a cloner is phase covariant, which means it acts identically on each state of the computational base.

The fidelity of the first clone (the one that is sent to Bob) when copying a state $|\psi\rangle$ can be written, in general, as

$$F_A = \langle \psi | \rho_A | \psi \rangle = \sum_{m,n=0}^{N-1} |a_{m,n}|^2 |\langle \psi | \psi_{m,n} \rangle|^2 = \sum_{m,n=0}^{N-1} |\langle \psi | U_{m,n} | \psi_{m,n} \rangle|^2$$  \hspace{1cm} (31)

That is $F_A = v^2 + y^2 + z^2$. The Disturbances $D_{A1}$ and $D_{A2}$ of the first clone are:

$$D_{A1} = D_{A2} = x^2 + y^2 + z^2$$  \hspace{1cm} (32)

By view of equation

$$b_{m,n} = \frac{1}{N} \sum_{x,y=0}^{N-1} e^{2\pi i(nx-my)/N} a_{m,n}$$  \hspace{1cm} (33)

we can obtain that, for the second clone, which is the maximum when $y = z$, and the fidelity is given by

$$F_B = \frac{(v^2 + 2x^2 + 12y^2 + 8xy + 4vy)}{3}$$  \hspace{1cm} (34)

Again, we get the same disturbances (minimal when $y = z$) given by

$$D_{B1} = D_{B2} = \frac{(v^2 + 2x^2 + 3y^2 - 4xy - 2vy)}{3}$$  \hspace{1cm} (35)
For simplicity, it’s natural to consider the following amplitude matrix \[19\]

\[
(a_{m,n}) = \begin{pmatrix}
v & x & x \\
x & x & x \\
x & x & x
\end{pmatrix}
\] (36)

where \(v, x\) are real parameters that satisfy the normalization condition \(v^2 + 8x^2 = 1\). It’s easy to check that this cloner’s results in the same fidelity and same disturbance for any qutrit state:

\[F = v^2 + 2x^2 \quad \text{and} \quad D_1 = D_2 = 3x^2\] (37)

Of course we have the relation: \(F + D_1 + D_2 = 1\). We can easily know that the symmetric universal qutrit cloner is characterized by a fidelity of \(3/4\). Now, it is simple to analyze its security against an incoherent attack. Bob’s fidelity is \(F = v^2 + 2x^2\) and the corresponding mutual information between Alice and Bob (if the latter measures his clone in the good basis) \[18\] is given by

\[I_{AB} = \log_2 3 + F \log_2 F + (1 - F) \log_2 \frac{1 - F}{2}\] (38)

since two possible errors are equiprobable. The cloning fidelity for Eve is given by

\[F_E = \frac{(v + 8x)^2 + 2(v - x)^2}{9}\] (39)

Maximizing Eve’s fidelity using the normalization relation \(v^2 + 8x^2 = 1\) yields the optimal cloner

\[x = \sqrt{\frac{F (1 - F)}{2}}, \quad v = F\] (40)

The corresponding optimal fidelity for Eve is

\[F_E = \frac{F}{3} + \frac{2}{3} (1 - F) + \frac{2}{3} \sqrt{2F (1 - F)}\] (41)

Let us see how Eve can maximize her information on Alice’s state. If Alice sends the state \(|k\rangle (k = 0, 1, 2)\), then it is clear that Eve can obtain Bob’s error simply by performing a practical Bell measurement (measuring only the \(m\) index) on BC. In order to infer Alice’s state, Eve must distinguish between three states (\(|0\rangle, |1\rangle, |2\rangle\)) with a same scalar product \(\frac{3F - 1}{2}\) for all pairs of states, regardless of the measured value of \(m\). Consequently, Eve’s information \[18\] is

\[I_{AE} = \log_2 3 + F_E \log_2 F_E + (1 - F_E) \log_2 \frac{1 - F_E}{2}\] (42)
As a result, Bob’s and Eve’s information curves intersect exactly where the fidelities coincide. That is, at \( F = F_E = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{3}} \right) \).

![Graph of mutual information vs fidelity](image)

**FIG. 6:** Relation between fidelity and mutual information in which real curve represents mutual information between Alice and Eve; the dashed curve represents mutual information between Alice and Bob.

Due to a theorem given by Csiszar and Korner [21], which provides a lower bound on the secret key rate. Concretely, it is sufficient that \( I_{AB} > I_{AC} \) in order to establish a secret key with a nonzero rate, if the one-way communication on the classical channel is used, this is actually a necessary condition. Consequently, the GCORE protocols cease to generate secret key bits precisely at the point where Eve’s information attains Bob’s information.

We compute the disturbance \( D_{\text{qutrit}} = 1 - F = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{3}} \right) \) (or error rate) at which \( I_{AB} = I_{AE} \) (or \( F = F_E \)), that is, above which Alice and Bob can not distill a secret key any more by use of one–way privacy amplification protocol. While the disturbance for the protocol using qubit is \( D_{\text{qubit}} = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{2}} \right) \), we can see \( D_{\text{qubit}} < D_{\text{qutrit}} \) easily. Thus we say that disturbance increases with the dimension, suggesting mutual information between Alice and Eve of qutrit cryptosystem is getting smaller than that of qubit cryptosystem under the same condition. In other words, Eve obtains less information in the qutrit scheme. Our analysis thus confirms a seemingly general property that qutrit scheme for QKD to be more robust against eavesdropping than the corresponding qubit scheme.

**IV. ANALYSIS AND CONCLUSION**

For \( m \) particles and/or higher dimension quantum systems, we can provide the uniform expression of maximally entangled basis states. Assume the number of particle is \( n \), each
dimension is $d$, the maximally entangled basis states are

$$|\psi_{i_1,i_2,\ldots,i_n}^n\rangle = \sum_j e^{2\pi i j n/d} |j\rangle \otimes |j + i_1 \text{ mod } d\rangle \otimes |j + i_2 \text{ mod } d\rangle \otimes \cdots |j + i_n \text{ mod } d\rangle / \sqrt{d}$$

(43)

Similar analysis can be used, even if the dimension is not limited to 2, 3. If Eve wants to measure the states without disturbing the system only if they are eigenstates of the measuring operator, otherwise, she will produce errors at most of time. Meantime, Eve can only guess the control key randomly, she has no means to decipher the control key. In a word, the security of GCORE operation becomes better than ever. Like other QKD protocols using orthogonal states, one distinct feature of our scheme is its high efficiency. The information-theoretic efficiency defined in Ref.[16] is:

$$\eta = \frac{b_s}{q_t + b_t}$$

(44)

where $b_s$ is the number of secret bits received by Bob, $q_t$ is the number of qubits used, and $b_t$ is the number of classical bits exchanged between Alice and Bob during the QKD process. The efficiency of any protocol for QKD, defined as the number of secret (i.e. allowing eavesdropping detection) bits per transmitted bit plus qubit, satisfies $\eta \leq 1$. The protocol presented here becomes 100%, because $b_s = d^N \log_2 d^N$, $q_t = d^N \log_2 d^N$, $b_t = 0$. In this way, we can calculate out that the efficiency of BB84 is 25%, similarly the EPR protocol is 50%. To the best of our knowledge, only two protocols reach the limit value of $\eta = 1$, one protocol by Cabello (high capacity Cabello protocol, HCCP) [16] and one by Long and Liu (high capacity Long Liu protocol HCLLP) [17]. Both protocols exploit the fact that a possible eavesdropper with no access to the whole quantum system at the same time, cannot recover the whole information without being detected, and both employ a larger alphabet, a few-dimensional orthogonal basis of pure state. The GCORE has the same characters, so we can also obtain the full efficiency from this point of view.

Another feature of the scheme is its high capacity since the four possible states of the EPR pairs carry two bits of information($\log_2 4 = 2$), eight possible states of GHZ-basis states carry three bits of information($\log_2 2^3 = 3$). Similarly, the nine possible states of the 2-qutrit general Bell-basis states carry $\log_2 9$ bits of information, the 27 possible states of the 3-qutrit general maximally entangled basis states carry $\log_2 3^3$ bits of information, so we can think the possible states of the $N$-qudit maximally entangled basis state carry
log₂ \(d^N\) bits of information. In short, \(M\) adopted \(N\)-qudit maximally entangled state can send \(M \log₂ d^N\) bits of information in our GCORE scheme. On average, per particle of GCORE protocol carries \((\log₂ d^N) / Nd^N\) bits of information. Whereas in the EPR scheme (BB84) each adopted EPR pair (particles) carried only one bit of information, that is, 0.5 bit information per particle carries. But if we use the control key to control the GCORE operation of a group of units. We can save a large amount of resources. From this sense, we think the proposed scheme is better.

In QKD, our scheme is just one-to-one protocol, there are other protocols using different ways to distribute secret keys [1-3,6,7-17]. As we know, Townsend’s protocol [30] is a one-to-any protocol, where Alice acts as a single controller to establish and update a distinct secret key with each network user. An any-to-any protocol has been proposed to allow any two users to establish a secret key over an optical network by Phoenix et al.[31]. The present scheme can be generalized to distribute secret keys to multiple legitimate users. It is different from Townsend and Phoenix’s protocol in that the secret keys are common to all legitimate users. The procedure is given in the following. We demonstrate it using EPR pair for simplicity, after Alice has sent the keys to Bob, Bob can create an EPR pair sequence that carries the raw keys. Then he sends this EPR pair sequence to another legitimate user, Clare, using the same procedure and device as before. The key protocols common to Alice, Bob and Clare are those Bell-basis measurement results that are not chosen to check eavesdropping. In this way, the protocol can be generalized to a multiparty common key distribution protocol. Note that all of the GCORE protocols have a final step, i.e. error correction and privacy amplification [30], we shall not discuss these points, which are the same as in all cryptographic protocols, except that we have to use qutrits (qudits) instead of bits, and therefore parity checks becomes triality checks, that is sums of modulo3 (\(d\)).

In summary, we extend the idea of CORE to \(N\)-qubit, \(N\)-qudit quantum systems, propose the detailed protocols and give the corresponding security analysis of 3-qubit, 2-qudit maximally entangled states, finally, we obtain the GCORE using the general expression of multi-particle and high dimension maximally entangled basis state by using repeatedly a prior shared control key in this paper. The generalized version has great capacity and high efficiency. In addition, the control key can be used to control the GCORE operation of a group of units, so it greatly simplifies the experimental realization and enables quantum key distribution in a more efficient way.
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