The elastic contribution to the Burkhardt – Cottingham and Generalized Gerasimov–Drell–Hearn Sum Rules

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Abstract

The elastic contribution to the first moment of $g_2(x, Q^2)$ is analysed using a Drell-Yan-West type of relation and is shown to be negative. For a qualitative estimate the one-loop contributions to the polarized DIS sum rules in QED are studied. The behaviour of the generalized Gerasimov–Drell–Hearn sum rule is sensitive to infrared regularization. With a lower threshold for the gluon virtuality the relation of the generalized GDH sum rule to the Burkhardt–Cottingham sum rule is studied. We conclude that the elastic part has to be included for long range interactions but can be consistently discarded for short range interactions.

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The generalized Gerasimov-Drell-Hearn (GDH) sum rule [1, 2] is just being tested experimentally [3] and the available proton data are in good agreement with the predictions made in [4, 5], which used the relationship between GDH and Burkhardt-Cottingham (BC) sum rule. They also agree with a new estimate of the contributions from low-lying resonances [6]. We stress, that such a similarity is by no means surprising, since the dominant magnetic form factor of Δ(1232), being the main source of the rapid variation of GDH [7] is contributing entirely through the structure function \(g_2\) [5], which is the key ingredient to the approach of [4].

The starting point of this approach is the simultaneous analysis of GDH and BC sum rules, inspired by the paper of Schwinger [8]. To verify the latter, the check of BC and GDH sum rules in QED was performed almost 20 years ago [9] (although the BC sum rule was not mentioned in this paper). In the present article we complete their calculation and use the model QED case in order to make (qualitative) statements about the behaviour of the GDH sum rule for the proton. Also we try to clarify the role of the elastic contribution for \(x \to 1\).

The main problem with the (generalized) GDH sum rule is the following. Let us introduce the \(Q\)-dependent integral [10]

\[
I_1(Q^2) = \frac{2M^2}{Q^2} \int_0^1 g_1(x, Q^2) dx = \int_{Q^2/2M}^{\infty} \frac{d\nu}{\nu} G_1(\nu, Q^2),
\]

\[
I_2(Q^2) = \frac{2M^2}{Q^2} \int_0^1 g_2(x, Q^2) dx = M^2 \int_{Q^2/2M}^{\infty} d\nu G_2(\nu, Q^2).
\]

(1)

defined for all \(Q\). There are solid theoretical arguments to expect a strong \(Q^2\)-dependence of \(I_2\). It is the well-known Burkhardt-Cottingham sum rule [12], derived independently by Schwinger [8] with a rather different method. It states that

\[
I_2(Q^2) = \frac{1}{4} \mu G_M(Q^2)[\mu G_M(Q^2) - G_E(Q^2)],
\]

(2)

where \(\mu\) is the nucleon magnetic moment and the \(G\)’s are the familiar Sachs form factors which are dimensionless and normalised to unity at \(Q^2 = 0\). For large \(Q^2\) one can neglect the r.h.s. and gets

\[
\int_0^1 g_2(x) dx = 0.
\]

(3)

The latter equation is often called the BC sum rule and applies only up to corrections of twist higher than four.

One of the crucial points of the whole discussion is the treatment of the elastic contribution. Being of high twist it is not explicitly treated in standard OPE analyses. For the small \(Q^2\) values relevant here we follow the arguments of [4, 5] which show that for kinematic reasons if one requires a smooth interpolation to \(Q^2 = 0\) the elastic contribution at \(x = 1\) should not be included in the sum rule (4). One recovers then at \(Q^2 = 0\) the GDH sum rule:

\[
I_1(0) = -\frac{\mu_A^2}{4}.
\]

(4)

where \(\mu_A\) is the nucleon anomalous magnetic moment in nuclear magnetons. While \(I_1(0)\) is always negative, its value at large \(Q^2\) is determined by the integral \(\int_0^1 g_1(x, Q^2) dx\) and is thus positive for the proton. This illustrates the existence of strong scaling violations for \(I_1\) in the region \(0 < Q^2 < 1 \text{ GeV}^2\). Its origin can be elucidated somewhat
using a modified Drell-Yan-West relation \[11\]. As for the unpolarized case one can relate the elastic and quasielastic part of the first moment of the spin-dependent structure functions to formfactors (at large $Q^2$) according to

$$\int_{1-\text{cons}/Q^2}^1 g_2(x, Q^2)dx = -\frac{Q^2}{2} F_2(Q^2) \left( F_1(Q^2) + \frac{F_2(Q^2)}{2M} \right)$$  \hfill (5)

where the right hand side is just the elastic contribution. In the language of OPE it corresponds to contributions from cat-ear diagrams. The constant $\text{cons}$ is left free. In QCD it would be proportional to the duality interval ($\sim \nu M$). If one makes the usual ansatz for the form of $g_2(x) \sim (1 - x)^n$ for $x \to 1$ and uses the fact that $F_2(Q^2) \sim (1/Q^2)^3$ and $F_1(Q^2) \sim (1/Q^2)^2$ equation (3) gives $n = 3$. This prediction could be tested by planned SLAC and CEBAF experiments. It is of the same nature as the usual quark-counting rule predictions.

The role of the elastic contribution is in principle quite similar for polarized and unpolarized structure functions the only difference is that because the leading contributions are zero the cat-ear contribution is dominant for the first moment of $g_2(x)$. It leads to the non-trivial prediction of the negative sign.

It is possible to decompose $I_1$ into the contributions from the two form factors $I_{1+2}$ and $I_2$:

$$I_1 = I_{1+2} - I_2,$$  \hfill (6)

where

$$I_{1+2}(Q^2) = \frac{2M^2}{Q^2} \int_0^1 g_{1+2}(x, Q^2)dx$$  \hfill (7)

$$g_{1+2} = g_1 + g_2.$$  \hfill (8)

Note that this decomposition corresponds to extracting coefficient functions in front of two independent tensor combinations in the spin–dependent (antisymmetric) part of the hadron tensor:

$$W^a_{\mu\nu} \sim g_{1+2} \epsilon_{\mu\nu\rho\sigma} q^\rho s^\sigma - g_2 \epsilon_{\mu\nu\rho\sigma} q^\rho p^\sigma.$$  \hfill (9)

For $Q^2 \to 0$ one finds

$$I_2(0) = \frac{\mu_A^2 + \mu_A e}{4},$$  \hfill (10)

e being the nucleon charge in elementary units. To reproduce the GDH value one should have

$$I_{1+2}(0) = \frac{\mu_A e}{4}.$$  \hfill (11)

Note that $I_{1+2}$ does not differ from $I_1$ for large $Q^2$ because the BC sum rule holds there and that it is positive for the proton. A smooth interpolation for $I_{1+2}(Q^2)$ between large $Q^2$ and $Q^2 = 0$ can be found in [4].

To better understand the issue of the GDH sum rule it seems reasonable to investigate its generalized version in a simple theory, such as a perturbative gauge field model.

An implication for the BC sum rule comes from the check of sum rules (7) and (10) in QED performed immediately after Schwinger’s paper [9]. This pioneering paper is
hardly known in the spin community, probably, due to two main reasons. First, it uses the Schwinger sources theory, which is actually unproblematic in this case, because the result are the same in the diagrammatic approach, as we shall show below. Second, they use unconventional definitions for spin structure functions (also first introduced by Schwinger, e.g. $H_4$ stands for $g_2$).

Note that it is better to speak here about "perturbative QCD", just because the emission of a real photon by the Bethe-Heitler process (not taken into account in [9]) is of the same order as its emission by the "internal" quark. To exclude the former, one may change this photon to a gluon. Since we consider only the first order in $\alpha$, the result is the same apart from a trivial color factor. The BC sum rule in such an approximation is obviously valid, if the elastic contribution is included into the integrand [9].

To proceed, we note that in a gauge theory (for definiteness, we will refer to perturbative QCD), while both parts (elastic and inelastic) of the BC sum rule are infrared stable, the generalized version of the GDH sum rule is infrared divergent (logarithmically) at any nonzero $Q^2$. It is well–known, however, that the infrared divergencies coming from the elastic process cancel those in the inelastic one, implying that the quantity

$$\tilde{I}_1(Q^2) = I_1(Q^2) + I_1^{\text{elastic}}(Q^2)$$

is infrared finite. Unfortunately, it diverges as $1/Q^2$ at small $Q^2$ due to kinematic factors in $I_1^{\text{elastic}}(Q^2)$. At the same time, the Born contribution to $I_2$ is zero. The IR divergent part of the elastic contribution has the kinematic structure of the Born term, so it is also zero in the case of $I_2$ making the inelastic contribution to be IR finite.

The physical reason for this problem is that the possibility of emitting soft gluons contradicts the existence of a finite threshold, which is assumed in the original version of the GDH sum rule. So, the generalized GDH sum rule needs to be defined more carefully for gauge models. Possible solutions could be a finite mass of the gluon or suppression of soft gluons with virtualities less than some value $\lambda^2$, which can be interpreted as the threshold of detector sensitivity.

In this paper we will use the latter option for the regularization of IR singularities in $I_{1+2}$. The calculation with regularization by a finite gluon mass is more complicated, and the result clearly should be the same.

Explicit calculations gives the following expression:

$$I_i = \int_0^{1-\delta} dx g_i(x), \quad \delta = \frac{2m\lambda}{Q^2}$$

$$g_{1+2} = \frac{1}{2} \delta(1-x) + \frac{1}{2} \frac{\alpha_s}{4\pi} C_F \left\{ -\frac{5y^2 + 2y^2 x + 6yx - 11y^2 x^2 + 36yx^2 + 4y^2 x^3 - 34yx^3 + 32x^3}{(y + x - xy)(y + 4x^2)(1-x)} 
- 2 \frac{-2y^2 x - 3yx + y^2 x^2 - 10yx^2 - y^2 + xy - 16x^3}{(1-x)(y + 4x^2)\sqrt{y(y + 4x^2)}} \cdot \log D \right\} ,$$
\[ D = \frac{2x + y + \sqrt{y(y + 4x^2)}}{2x + y - \sqrt{y(y + 4x^2)}}, \quad y = \frac{Q^2}{m^2}. \]  

(15)

Below we reproduce also the full expression for \( g_2 \) obtained first in the paper [9] (see also the more recent calculations [14]):

\[
g_2 = \frac{\alpha_s}{2\pi^4} C_F \cdot \frac{y}{2} \left\{ \frac{2}{1 + y} \left[ \frac{y}{(y + x - xy)^2} - \frac{6x(2(3y + 2)x + 2y + 3)y}{(y + 4x^2)^2} \right. \right.
\]

\[
\left. \left. - \frac{1}{y + 4x} - \frac{(y + x - xy)(y + 4x^2)}{x(y^2 + 1)} \right] \right. \]

\[
+ \left( 1 + 2y - 3x \right) \frac{4x(y - 1) - 5y}{y + 4x^2} \frac{4x^2}{\sqrt{y(y + x - xy)}^3} \log D \right\}. \]

(16)

For QED \( m \) is the mass of the Dirac particle. It corresponds to some constituent quark mass in QCD.

Comparison of this expression for \( g_2 \) with the asymptotic (in the limit \( Q^2 \to \infty \)) formula obtained in [13] immediately shows that the term omitted in that paper is the first term in the expression above:

\[
\Delta g_2 = \frac{\alpha_s}{2\pi^4} C_F \cdot \frac{1}{y} \cdot \frac{1}{[(1 - x)(1 - 1/y) + 1/y]^2} \]

(17)

and naively suppressed at high \( Q^2 \) as \( m^2/Q^2 \). However, due to the (integrated) singularity in the denominator, it should be taken into account for moments and leads to the correct result for the BC sum rule. More precisely, this term is proportional to

\[
\lim_{\epsilon \to 0} \frac{\epsilon}{(\bar{x} + \epsilon)^2} = \delta_+(\bar{x}); \quad (\bar{x} = 1 - x, \quad \epsilon = 1/y), \]

(18)

which looks like the elastic contribution (cf Ref. [14]).

It is interesting to investigate this term in the opposite limit of very small \( Q^2 \):

\[
\frac{y}{[x(1 - y) + y]^2} \to \frac{\epsilon}{(x + \epsilon)^2} \sim \delta_+(x); \quad (\epsilon = y). \]

(19)

It can be easily found that its contribution to the BC sum rule at \( Q^2 = 0 \) is \( 2I_2(0) \), i.e. without it one would obtain the correct absolute value but the wrong sign.

The numerical results for the generalized GDH sum rule in the lowest order are shown for different values of the IR cut-off parameter in Fig.1. It can be seen that smaller values of \( \delta \) lead to higher IR peaks closer to the abscissa. In the opposite regime of large \( \delta \) (which effectively is expected in QCD) it shows a rather smooth behavior, which is compatible with what was predicted for the generalized sum rule \( I_{1+2}(Q^2) \) for the proton in [4]

In conclusion, we presented here the investigation of the generalized Gerasimov-Drell-Hearn sum rule in the framework of perturbation theory. This simple example allows one to distinguish between the two ways of writing this sum rule, namely, keeping and omitting the elastic contribution.
The first way (which is the only meaningful one in long-range theories like QED, because both elastic and inelastic terms are IR divergent) leads to the smooth interpolation between low and high $Q^2$. This observation supports the suggestion of X. Ji \cite{15} to consider such a quantity for the interpolation between high and low $Q^2$ for the real proton. However, such an approach has nothing to do with the original GDH value for real photons, which is changed completely by the infinitely growing elastic background.

At the same time, in a short-range theory with a mass gap (finite threshold), like QED with IR cutoff (or real QCD, where it is implied by the confinement property), the interpolation between inelastic contribution at non-zero $Q^2$ and the GDH value at $Q^2 = 0$ is possible. The form factor $I_{1+2}$ is rather smooth. Our simple model is thus supporting the hypothesis about the dominant role of the $g_2(x, Q^2)$ structure function in the $Q^2$ dependence of the GDH sum rule. However, further investigations in the framework of, say, chiral models and/or QCD sum rules are highly desirable. It would be especially interesting to obtain a direct quantitative estimate for the relevant cat-ear contributions, e.g. from lattice gauge calculations.

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References

[1] S.B. Gerasimov, Yad. Fiz. 2, 598(1965) [Sov. J. Nucl Phys. 2, 430(1966)].
[2] S.D. Drell and A.C. Hearn, Phys. Rev. Lett. 16, 908(1966).
[3] E143Collaboration, K. Abe et al., Phys. Rev. Lett. 78, 815 (1997).
[4] J. Soffer and O. Teryaev, Phys. Rev. Lett. 70, 3373(1993).
[5] J. Soffer and O. Teryaev, Phys. Rev. 51, 25(1995).
[6] V. D. Burkert and B. L. Ioffe, Phys. Lett. B296, 223(1992);
[7] B.L. Ioffe, V.A. Khoze, L.N. Lipatov, Hard Processes, North-Holland, 1984.
[8] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. 72, 1559(1975).
[9] Wu-Yang Tsai, L. DeRaad and K.A. Milton, Phys. Rev. D11, 3537(1975).
[10] M. Anselmino, B.L. Ioffe and E. Leader, Yad. Fiz. 49 (1989) 214.
[11] S.D. Drell and T.M. Yan, Phys. Rev. Lett. 24 (1970)181
G.B. West Phys. Rev. Lett. 24 (1970) 1206
[12] H. Burkhardt and W.N. Cottingham, Ann. Phys. (N.Y.) 16, 543(1970).
[13] R. Mertig, W. L. van Neerven, Z. Phys. C60, 489 (1993).

[14] G. Altarelli et al., Phys. Lett. B334, 187 (1994).

[15] Xiangdong Ji, Phys. Lett. B309, 187 (1993).
4I_{1+2}/\mu_A as a function of \( y = Q^2/m^2 \) for different values of the threshold \( \lambda \): a) \( \lambda = 0.1 \) (solid), b) \( \lambda = 0.3 \) (dashed), c) \( \lambda = 1.0 \) (dotted); \( e = 1 \).
$\frac{4I_{1+2}}{\mu_A}$