Priority Promotion with Parysian Flair

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Abstract

We develop an algorithm that combines the advantages of priority promotion - one of the leading approaches to solving large parity games in practice - with the quasi-polynomial time guarantees offered by Parys’ algorithm. Hybridising these algorithms sounds both natural and difficult, as they both generalise the classic recursive algorithm in different ways that appear to be irreconcilable: while the promotion transcends the call structure, the guarantees change on each level. We show that an interface that respects both is not only effective, but also efficient.

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1 Introduction

Parity games have many applications in model checking [38, 18, 16, 1, 65, 39] and synthesis [65, 38, 62, 55, 50, 57, 58]. In particular, modal and alternating-time μ-calculus model checking [65, 1], synthesis [58, 50, 57] and satisfiability checking [65, 38, 62, 55] for reactive systems, module checking [39], and ATL* model checking [16, 1] can be reduced to solving parity games. This relevance of parity games led to a series of different approaches to solving them [46, 19, 45, 51, 67, 12, 66, 32, 33, 64, 47, 40, 9, 10, 35, 22, 7, 13, 34, 42, 23, 48, 44, 49, 15, 43, 59, 41].

The research falls into two categories: on the one hand to develop fast solvers; on the other hand to determine the complexity of parity games or to find algorithms with a good worst-case complexity. With its practical motivation, one of the leading algorithms most efficient for solving parity games is currently priority promotion techniques [3, 7, 6], a refinement of the classic recursive algorithm [46, 19, 66] that follows the iterated fixed-point structure induced by the parity condition. The complexity of solving parity games is still an open problem. Parity games are memoryless determined [18, 11], which implies that nondeterministic algorithms can determine winning regions and strategies for both players. Due to their symmetry, they are therefore in NPTIME ∩ CoNPTIME [18], and by reduction to payoff games [67], in UPTIME ∩ CoUPTIME [32]. While determining their membership in PTIME continues to be a major challenge, one of the most celebrated results in recent years has been the landmark result of Calude et al. [13], which established that parity games can be solved in quasi-polynomial time (QP). This was a major step from former deterministic algorithms, which were (at least) exponential in the number of priorities [46, 19, 67, 12, 66, 33, 10, 53, 54, 7] (nO(3)), or in the square-root of the number of game positions [45, 35, 10] (approximately nO(√n)). The breakthrough of Calude et al. [13] has triggered a new line of research into QP algorithms, including [34, 42, 23, 48, 44, 49, 15].
Algorithms that are good in practice do not tend to display their worst-case behaviour, except for in carefully designed hostile examples. This holds in particular for strategy improvement algorithms [45, 51, 64, 10, 53, 20, 56], which were considered candidates for tractable algorithms until they were shown to be exponential by Friedman’s delicate lower bound constructions [24, 29, 26] (with the notable exception of the symmetric approach from [56], for which no hard families are known). But while it is easier to design hard classes for recursive [25, 5] and priority-promotion algorithms [6], these classes are still not relevant in practice. However, with a host of QP algorithms at hand, an upgrade of priority promotion that offers QP lower bounds without undue compromise on efficiency will be an attractive challenge that combines the best of both worlds.

Interestingly, Parys’ algorithm [48] and variations thereof [44, 43], which, like priority promotion techniques, adjust the classic recursive algorithm [46, 19, 66], are relatively fast among the QP algorithms, where [48] has the edge on benchmarks, while [44] has the edge on theoretical guarantees. On first glance, this seems to invite synthesising one of these algorithms with priority promotion. On second glance, the prospect of this synthesis seems less promising. Priority promotion techniques [3, 7, 6] achieve their advancement over the previously leading recursive algorithms [46, 19, 66] by globally bypassing the call structure through temporally increasing the priority of a position. Parys’ approach, on the other hand, locally creates sets with guarantees with quickly falling strength along the recursive call structure, where subgames are split into areas that contain all 0-dominions with size up to a bound \( b_0 \) and all 1-dominions of size up to a bound \( b_1 \); one of these bounds is halved in each call until the guarantees are trivial. Prima facie, it seems clear that such guarantees are ill suited for a promotion across the call structure. We did, however, find that, when one shifts the view on the essence of a promotion from creating quasi-dominions to creating regions and promoting them to the lowest level where they are no longer dominions, this allows for a concurrent treatment of sets with bounded guarantees (the Parysian flair of our hybrid algorithm) and with unbounded guarantees (the Priority Promotion core of our algorithm). While the integration of these seemingly antagonistic concepts is intricate, it provides an efficient bridge between the behaviour and the data structure of [7] and [48]: the resulting algorithm guarantees a quasi-polynomial running time, and offers excellent practical behaviour on the benchmarks we have tested it against.

2 Preliminaries

A two-player turn-based arena is a tuple \( A = (P_{S_0}, P_{S_1}, M_v) \), with \( P_{S_0} \cap P_{S_1} = \emptyset \) and \( P_S \triangleq P_{S_0} \cup P_{S_1} \), such that \((P_S, M_v)\) is a finite directed graph without sinks. \( P_{S_0} \) (resp., \( P_{S_1} \)) is the set of positions of the Player (resp., the Opponent) and \( M_v \subseteq P_S \times P_S \) is a left-total relation describing all possible moves. A path in \( V \subseteq P_S \) is a finite or infinite sequence \( \pi \in \text{Pth}(V) \) of positions in \( V \) compatible with the move relation, i.e., \((\pi_i, \pi_{i+1}) \in M_v\), for all \( i \in [0, |\pi| - 1] \). A positional strategy for player \( \alpha \in \{0, 1\} \) on \( V \subseteq P_S \) is a function \( \sigma_\alpha \in \text{Str}_\alpha(V) \subseteq (V \cap P_{S_\alpha}) \rightarrow V \), mapping each \( \alpha \)-position \( v \) in \( V \) to position \( \sigma_\alpha(v) \) compatible with the move relation, i.e., \( (v, \sigma_\alpha(v)) \in M_v \). With \( \text{Str}_\alpha(V) \) we denote the set of all \( \alpha \)-strategies on \( V \). When talking about players, with \( \overline{\alpha} \) we will refer to the opponent player of \( \alpha \). A play in \( V \subseteq P_S \) from a position \( v \in V \) w.r.t. a pair of strategies \((\sigma_0, \sigma_1) \in \text{Str}_0(V) \times \text{Str}_1(V)\), called \(((\sigma_0, \sigma_1), v)\)-play, is a path \( \pi \in \text{Pth}(V) \) such that \( (\pi_0) = v \) and, for all \( i \in [0, |\pi| - 1] \), if \( (\pi_i) \in P_{S_0} \) then \( (\pi_{i+1}) = \sigma_0((\pi_i)) \) else \( (\pi_{i+1}) = \sigma_1((\pi_i)) \). The play function \( \text{play} : (\text{Str}_0(V) \times \text{Str}_1(V)) \times V \rightarrow \text{Pth}(V) \) returns, for each position \( v \in V \) and pair of strategies \((\sigma_0, \sigma_1) \in \text{Str}_0(V) \times \text{Str}_1(V)\), the maximal \(((\sigma_0, \sigma_1), v)\)-play \( \text{play}((\sigma_0, \sigma_1), v) \).
Given a partial function \( f : A \rightarrow B \), \( \text{dom}(f) \subseteq A \) and \( \text{rng}(f) \subseteq B \) we indicate the domain and the range of \( f \).

A parity game is a tuple \( \mathcal{G} = \langle A, P, \pi \rangle \in \mathcal{PG} \), where \( A \) is an arena, \( P \subseteq \mathbb{N} \) is a finite set of priorities, and \( \pi : P \rightarrow \mathbb{N} \) is a priority function assigning a priority to each position. The priority function can be naturally extended to games and paths as follows: \( \pi(v) \triangleq \max_{p \in P} \pi(v) \); for a path \( \pi \in \mathcal{P} \), we set \( \pi(v) \triangleq \max_{i \in [0,|\pi|]} \pi((\pi)_i) \), if \( \pi \) is finite, and \( \pi(v) \triangleq \lim_{i \in \mathbb{N}} \pi((\pi)_i) \), otherwise. A set of positions \( V \subseteq P \) is an \( \alpha \)-dominion, with \( \alpha \in \{0,1\} \), if there exists an \( \alpha \)-strategy \( \sigma_{\alpha} \in \text{Str}_{\alpha}(V) \) such that, for all \( \pi \)-strategies \( \sigma_{\pi} \in \text{Str}_{\pi}(V) \) and positions \( v \in V \), the induced play \( \pi = \text{play}(\sigma_{\alpha}, \sigma_{\pi}, v) \) is infinite and \( \pi(v) \equiv \alpha \). In other words, \( \sigma_{\alpha} \) only induces on \( V \) infinite plays whose maximal priority visited infinitely often has parity \( \alpha \). The winning region for player \( \alpha \in \{0,1\} \) in game \( \mathcal{G} \), denoted by \( W_{\alpha} \), is the greatest set of positions that is also a \( \alpha \)-dominion in \( \mathcal{G} \). Since parity games are memoryless determined [17], meaning that from each position one of the two players wins, the two winning regions of a game \( \mathcal{G} \) form a partition of its positions, i.e., \( W_{0} \cup W_{1} = \mathcal{P}_{\mathcal{G}} \).

By \( \mathcal{G} \setminus V \) we denote the maximal subgame of \( \mathcal{G} \) with set of positions \( P \mathcal{S} \) contained in \( P \mathcal{S} \setminus V \) and move relation \( M_{\mathcal{S}} \) equal to the restriction of \( M_{\mathcal{S}} \) to \( P \mathcal{S} \). The \( \alpha \)-predecessor of \( V \), in symbols \( \text{pre}^\alpha(V) \triangleq \{ v \in P \mathcal{S} \mid M_{\mathcal{S}}(v) \cap V \neq \emptyset \} \cup \{ v \in P \mathcal{S} \mid M_{\mathcal{S}}(v) \subseteq V \} \), collects the positions from which player \( \alpha \) can force the game to reach some position in \( V \) with a single move. The \( \alpha \)-attractor \( \text{atr}^\alpha(V) \) generalises the notion of \( \alpha \)-predecessor \( \text{pre}^\alpha(V) \) to an arbitrary number of moves. Thus, it corresponds to the least fix-point of that operator. When \( V = \text{atr}^\alpha(V) \), player \( \alpha \) cannot force any position outside \( V \) to enter this set that is, therefore, called \( \alpha \)-maximal. For such a \( V \), the set of positions of the subgame \( \mathcal{G} \setminus V \) is precisely \( P \mathcal{S} \setminus V \). When the computation of the attractor is restricted to a given set of positions \( X \), we will use the notation \( \text{atr}^\alpha(V, X) \) which corresponds to the least fix-point of \( \text{pre}^\alpha(V) \setminus X \). Finally, the set \( \text{esc}^\alpha(V) \triangleq \text{pre}^\alpha(P \mathcal{S} \setminus V) \setminus V \), called the \( \alpha \)-escape of \( V \), contains the positions in \( V \) from which \( \alpha \) can leave \( V \) in one move. Observe that all the operators and sets described above actually depend on the specific game \( \mathcal{G} \) they are applied to. In the rest of the paper, we shall only add \( \mathcal{G} \) as subscript of an operator, e.g. \( \text{atr}^\alpha_{\mathcal{G}}(V) \), when the game is not clear from the context.

3 A Hybrid Priority-Promotion Algorithm

We introduce the hybrid algorithm in three steps. In the first step (Section 3.1), we introduce a variation of classic Priority Promotion, which serves as the backbone of our hybrid algorithm in Section 3.3. We provide a recap of how Priority Promotion operates and an introduction to its data structure, which we later extend for our hybrid algorithm. In a nutshell, Priority Promotion accelerates the classic recursive algorithm, by allowing to merge dominions in subgames that span non-adjacent recursive calls, which is the essence of the promotion operations. In the following subsection (Section 3.2), we outline Parys’ algorithm, which does not seek to identify all dominions on a level, but merely all small dominions up to given bounds \( b_0 \) and \( b_1 \) for the dominions of Player 0 and Player 1, respectively. It truncates the size of the call tree by making all but one call with half the precision for one of the players. Here, we formulate the algorithm with a terminology analogous to Priority Promotion, and present it in a form similar to our hybrid algorithm.

The two concepts of Priority Promotion and truncated tree size through limited guarantees appear to be unlikely allies: not only does the presence of Parys’ sets with limited guarantees impede the promotion sets of dominions, any attempt to promote sets with bounded guarantees
are set to fail, when the bounds are larger (and thus the required guarantees stronger) along the call tree. In Section 3.3, we see that, when synthesising the algorithms carefully, sets with the ‘region guarantees’ from Priority Promotion and with ‘bounded guarantees’ from Parys’ approach can co-exist, so long as they are kept carefully apart and treated differently.

The resulting algorithm can identify dominions in many places, and these dominions can be promoted. This promotion can be to a set with ‘region guarantees’ at a higher level, but it can also be that the correct target is a set with ‘bounded guarantees’ (which works across levels because dominions have unbounded guarantees). The identification of the right set to promote to, instead, remains fairly similar to the way it is identified in classic Priority Promotion. While sets with bounded guarantees cannot be promoted along the data structure (which follows the call tree), they lose parts of their locality: positions can be promoted into them, and, crucially, they do not prevent promotions to higher levels. This way, we can keep the Priority Promotion part, which usually carries the main burden of solving the parity game and can play out its practical efficiency in full, while we also retain the quasi-polynomial complexity from Parys’ algorithm [48], bypassing the known hard cases for recursive algorithms. For practical considerations, it is still computationally attractive to grow the bounded sets more slowly: we found that some of the points where Parys’ algorithm applies a closure of sets with bounded guarantees are merely for the convenience of the proof. For efficiency, we have restricted the closure under attractor of these sets to the places where it is necessary for correctness.

3.1 The Priority-Promotion Approach

The priority-promotion approaches [7, 6] attack the problem of solving a parity game $\mathcal{G} \in \mathcal{PG}$ by iteratively computing, one at a time, a sequence of $\alpha$-dominions $D_0, D_1, \ldots \subseteq \mathcal{P}s$, for some player $\alpha \in \mathbb{B} \triangleq \{0, 1\}$. These, indeed, are portions of the two winning regions, $W_0$ and $W_1$, that need to be identified. The idea here is to start from a weaker notion, called quasi dominion, and then compose them until a dominion is obtained. The name of the approach comes precisely from the fact that this composition is computed by applying the following operation of promotion: given two quasi dominions $Q_1$ and $Q_2$ to which some priorities $p_1 < p_2$ of the same parity are assigned, $Q_1$ is combined with $Q_2$ by promoting the former to the priority of the latter.

Similarly to a dominion, a quasi dominion is a set of positions over which one of the two players, called the leading player, has a strategy defined on that set, whose induced plays, if infinite, are winning for that player. As opposed to dominions, however, some of these plays may be finite, since the opponent may have the possibility to escape from those positions towards a different part of the game, hoping for a better outcome.

**Definition 1 (Quasi Dominion).** A set of positions $Q \subseteq \mathcal{P}s$ is a quasi $\alpha$-dominion, for some player $\alpha \in \{0, 1\}$, if there exists an $\alpha$-strategy $\sigma_\alpha \in \text{Str}^{\alpha}(Q)$, called $\alpha$-witness for $Q$, such that, for all $\pi$-strategies $\sigma_\pi \in \text{Str}^{\pi}(Q)$ and positions $v \in Q$, the induced play $\pi = \text{play}((\sigma_0, \sigma_1), v)$, satisfies $\text{pr}(\pi) \equiv_2 \alpha$, if infinite.

The usefulness of the above concept, in addition to the property of being suitably composable, resides in the fact that quasi dominions are closed under inclusion. Thus, when a closed subset of a quasi dominion is found, a dominion is identified. Notice that, differently from the definition given in [4], no constraint is given on the finite plays, since the type of quasi dominions of [1] are not closed under inclusion. However, such a constraint is implicitly stated in Definition 5 below.
Theorem 2 (Induced Dominion). Let $\alpha \in \{0, 1\}$ be a player, $Q \subseteq Ps$ a quasi $\alpha$-dominion, $\sigma \in \text{Str}_\alpha(Q)$ one of its $\alpha$-witnesses, and $D \subseteq Q$ a subset such that $\sigma(v) \in D$, if $v \in Ps_\alpha$, and $Me(v) \subseteq D$, otherwise, for all positions $v \in D$. Then, $D$ is an $\alpha$-dominion.

Intuitively, the solution algorithms following this approach carry on the search for a dominion by exploring a finite strict partial order $(St, s_I, \prec)$, whose elements, called states, record information about the quasi dominions computed up to a certain point. In the initial state $s_I$, the quasi dominions are initialised to the sets of positions with the same priority. At each step, a new quasi $\alpha$-dominion $Q$, for some player $\alpha \in B$, is extracted from the current state $s$ and used to compute a successor state w.r.t. the order $\prec$, if $Q$ is open, i.e., if it is not an $\alpha$-dominion. If, on the other hand, it is closed, the search is over and $Q$ is added to the portion of the winning region $W_{\alpha}$ computed so far.

We start by describing a new priority-promotion algorithm that instantiates the above partial order and serves as a basis for the hybrid approach presented later in this section. To do so, we first need to introduce few technical notions, all of which refer to some fixed parity game $\mathcal{G} \in PG$. By $Pr_{\bot} \triangleq Pr \cup \{\bot\}$ and $Pr_{\top} \triangleq Pr \cup \{\top_0, \top_1\}$ we denote the set of priorities in $\mathcal{G}$ extended with the bottom symbol $\bot$ and two top symbols $\top_0$ and $\top_1$, one for each player. The standard ordering $\prec$ on $Pr$ is extended to these additional elements in the natural way: $\bot$ is the smallest element, while both $\top_0$ and $\top_1$ are strictly greater than every other priority; we do not assume any specific order between the two maximal elements, though, we consider $\top_0$ even and $\top_1$ odd.

The first step in the formalisation of the notion of state requires the concept of promotion function, which represents the backbone of the algorithm, being the data structure to which the promotion operation is applied. Intuitively, it is a partial function from positions to priorities that over-approximates the priority function of the game.

Definition 3 (Promotion Function). A promotion function $r \in Pm \triangleq Ps \rightarrow Pr_{\top}$ is a partial function such that $r(v) \geq pr(v)$, for every position $v \in \text{dom}(r)$.

In the following, we adopt the same notation as in [7]. Given a promotion function $r \in Pm$ and a priority $p \in Pr$, we denote with $r^{<p}$, for $\sim \in \{<, \leq, \equiv_2, \geq, >\}$, the function obtained by restricting the domain of $r$ to those positions $v \in \text{dom}(r)$ whose priority $r(v)$ satisfies the relation $r(v) \sim p$, i.e., $r^{<p} \triangleq r | \{v \in \text{dom}(r) \mid r(v) \sim p\}$, where $|$ is the standard operation of domain restriction. We may also use Boolean combinations of the above restrictions, as in $r^{(\equiv_2 \alpha) \wedge (\geq p)}$. By $H^{p} \triangleq \text{dom}(r^{(\equiv_2 \alpha) \wedge (\geq p)})$ we denote the set of positions in $r$ with a priority congruent to $\alpha \in B$ and with $H^{p} \wedge p \triangleq \text{dom}(r^{(\equiv_2 \alpha) \wedge (\geq p)})$ its subset with priorities greater than or equal to $p$.

A state encodes information about the quasi dominions computed up to a certain point of the computation. To this end, we require all positions in a promotion function $r$ with priority of parity $\alpha \in B$, i.e., the set $H^{p}_\alpha$, to form a quasi $\alpha$-dominion. Moreover, the idea is to store all $\alpha$-dominions already identified by associating them with the corresponding maximal priority $\top_\alpha$.

Definition 4 (Quasi-Domination Function). A quasi-domination function $r \in Qs \subseteq Pm$ is a promotion function satisfying the following conditions, for every $\alpha \in B$: 1) the set $H^{p}_\alpha$ is a quasi $\alpha$-dominion; 2) the set $r^{-1}(\top_\alpha)$ is an $\alpha$-dominion.

An important property of dominions is that the extension of an $\alpha$-dominion $Q$ by means of its $\alpha$-attractor is still an $\alpha$-dominion. This property, however, is not enjoyed by arbitrary quasi dominions. Indeed, there may even be cases where the $\alpha$-attractor of a quasi $\alpha$-dominion is a $\pi$-dominion. Moreover, to efficiently verify whether a quasi dominion is actually a
dominion, an explicit representation of one of its witnesses is usually required. To overcome these complications, we consider a subclass of quasi dominions that meets the following requirements: 1) the set \( \text{esc}(Q, \sigma) \triangleq \{ v \in Ps_\sigma \cap Q \mid \sigma(v) \not\in Q \} \) of \( \alpha \)-positions, which leave a quasi \( \alpha \)-dominion \( Q \subseteq Ps \) by following one of its \( \alpha \)-witnesses \( \sigma \in \text{Str}^\alpha(Q) \), is a subset of the \( \overline{\pi} \)-escape positions \( \overline{\text{esc}}^\overline{\pi}(Q) \); 2) all these \( \overline{\pi} \)-escape positions have priorities congruent to \( \alpha \) and greater than the ones of the positions that can be attracted by \( \alpha \) to \( Q \). The first requirement ensures that, to verify whether \( Q \) is an \( \alpha \)-dominion, it suffices to check for the emptiness of \( \overline{\text{esc}}^\overline{\pi}(Q) \). The second one, instead, can be exploited to regain closure under extension by \( \alpha \)-attractor.

\textbf{Definition 5 (Region Function).} A region function \( r \in \text{Rg} \subseteq Q_8 \) is a quasi-dominion function satisfying the following conditions, for every \( \alpha \in B \): 1) there exists an \( \alpha \)-witness \( \sigma_\alpha \in \text{Str}^\alpha(H^\alpha_\sigma) \) for \( H^\alpha_\sigma \) such that \( \text{esc}(H^\alpha_\sigma, \sigma_\alpha) \subseteq \overline{\text{esc}}^\overline{\pi}(H^\alpha_\sigma) \), for all \( p \in \text{rng}(r) \), with \( p \equiv_2 \alpha \); 2) \( p \leq \text{pr}(v) \equiv_2 \alpha \), for all \( p \in \text{rng}(r) \), with \( p \equiv_2 \alpha \), and \( v \in \overline{\text{esc}}^\overline{\pi}(H^\alpha_\sigma) \).

Notice that, every set \( H^\alpha_\sigma, p \), with \( p \in \text{rng}(r) \), is a quasi \( \alpha \)-dominion, being a subset of the quasi \( \alpha \)-dominion \( H^\alpha_\sigma \). Also, it is immediate to see that the priority function \( \text{pr} \) of a given parity game \( \mathcal{G} \) is always a region function. Indeed, it is trivially a promotion function. Moreover, the positions with a priority of parity \( \alpha \), i.e., \( H^\alpha_\sigma \), form a quasi \( \alpha \)-dominion with \( \alpha \)-witness any strategy that always chooses to remain inside the set, if allowed by the move relation. Thus, it is a quasi-dominion function as well. Finally, since \( H^\alpha_\sigma, p \) cannot contain positions of parity \( \overline{\pi} \) and thanks to the way the \( \alpha \)-witness is chosen, it is clear that \( \text{pr} \) also satisfies the conditions of Definition 5.

At this point, we have the technical tools to introduce the search space that instantiates the finite strict partial order described in the intuitive explanation of the approach. In particular, to account for the current status of the search of a dominion in a game \( \mathcal{G} \), we define a state \( s \) as a pair, comprising a region function \( r \) and a priority \( p \), with the idea that 1) all quasi \( \alpha \)-dominions computed so far are contained in \( H^\alpha_\sigma, q \), for some \( q > p \), 2) the current quasi dominion to focus on is contained in \( r \) at priority \( p \), and 3) all positions with priorities smaller than or equal to \( p \) correspond to the portion of the game that has still to be processed. The initial state is composed of the priority function \( \text{pr} \) of the game and its maximal priority \( \text{pr}(\mathcal{G}) \). Finally, we assume that a state \( s_1 \) is lower than another state \( s_2 \) w.r.t. the partial order relation \( \prec \), if the set of unprocessed positions in \( s_1 \) is a subset of those in \( s_2 \).

\textbf{Definition 6 (Search Space).} A search space is a tuple \( \mathcal{S} \triangleq (\text{St}, s_1, \prec) \), whose three components are defined as follows:

1. \( \text{St} \subseteq \text{Rg} \times \text{Pr} \upharpoonright \) is the set of all pairs \( s \triangleq (r, p) \), called states, where \( \text{dom}(r) = Ps \); for every state \( s \in \text{St} \), we set (i) \( \alpha_s \triangleq p \) mod 2, (ii) \( H^\alpha_\sigma \triangleq H^\alpha_\sigma \), (iii) \( H^\alpha_\sigma, q \triangleq H^\alpha_\sigma, q \), (iv) \( R_s \triangleq r^{-1}(p) \), and (v) \( L_s \triangleq \text{dom}(r^{[p \leq \alpha]} \); 2. \( s_1 \triangleq(\text{pr}, \text{pr}(\mathcal{G})) \) is the initial state; 3. \( s_1 \prec s_2 \) if either \( L_{s_1} \subseteq L_{s_2} \) or \( L_{s_1} = L_{s_2} \) and \( p_{s_1} < p_{s_2} \).

Given a state \( s \in \text{St} \), we refer to \( L_s \) as the local area, i.e., the set of unprocessed positions yet to be analysed. This also includes the quasi \( \alpha_s \)-dominion \( R_s \), called region, on which the next step of the search will focus. The two quasi dominions \( H^\alpha_\sigma \) and \( H^\alpha_\sigma \) partition the entire set of positions in the game, while \( H^\alpha_\sigma, q \) and \( H^\alpha_\sigma, q \) represent the portions of these quasi dominions to which the region function has assigned a priority at least equal to \( q \in \text{Pr} \). Notice that the pseudo-priority \( \bot \) is used to indicate the situation where all positions have been processed, which corresponds to an empty local area.
To exemplify the above notions, consider the game depicted in Figure 1, where circled shaped positions belong to Player 0 and square shaped ones to Player 1. Clearly, g and h are won by Player 0, while the rest of the game is won by the opponent. At the state \( s = (r, 3) \), where \( r = \{a\mapsto 0; c\mapsto 1; e\mapsto 3; d, f, h\mapsto 6; b\mapsto 7; g\mapsto \top\} \), the local area \( L_s \) contains the positions \( a, c, g, h \). Of these only \( e \) is part of the current region \( R_e \). The quasi 0-dominion \( H_0^e \) contains the positions \( a, d, f, h, g \), while the quasi 1-dominion \( H_1^e \) takes the remaining ones, namely b, c, and e. Position g forms a 0-dominion on its own, represented in the picture by the solid closed line.

Apart from this position, all the other ones are contained in open quasi dominions, indicated, instead, by the dashed closed lines. For example, the set \( r^{-1}(6) = \{d, f, h\} \) is a quasi 0-dominion, since, if Player 1 decides to remain inside, the adversary wins the play. However, Player 1 also has the choice to escape from position \( e \) moving to \( f \), i.e., \( \text{esc}^1(r^{-1}(6)) = \{f\} \). Similarly, \( \text{esc}^0(r^{-1}(7)) = \{b\} \). Finally, notice that \( H_0^e = \{d, f, g, h\} \) and \( H_1^e = \{b\} \). During the exploration of the search space, a priority-promotion algorithm typically traverses several types of states, some of which enjoy important properties that need to be explicitly identified, as they are exploited during the search for a dominion. Given a player \( \alpha \in B \), we say that a state \( s \in St \) is \( \alpha \)-maximal, if the quasi \( \alpha \)-dominion \( H_\alpha^s \setminus L_s \) is \( \alpha \)-maximal w.r.t. \( L_s \), i.e., the \( \alpha \)-attractor \( \text{at}^\alpha(H_\alpha^s \setminus L_s, L_s) \) to \( H_\alpha^s \setminus L_s \) of positions from the local area \( L_s \) is empty. If \( s \) is \( \alpha \)-maximal w.r.t. both players \( \alpha \in B \), we simply say that it is \( \alpha \)-maximal and denote by \( St_\alpha \subseteq St \) the corresponding subset of states and by \( \partial_\alpha \triangleq \partial \setminus \text{dom}(s^{>\partial_\alpha}) \) the induced subgame. A maximal state \( s \) is \( \alpha \)-maximal, if the current region \( R_s \) is \( \alpha \)-maximal w.r.t. \( L_s \). By \( St_\alpha \subseteq St_\beta \) we denote the set of strongly maximal states. Recall that region \( R_s \) of a state \( s \) is contained in the quasi \( \alpha_s \)-dominion \( H_{\alpha_s}^s \setminus P_s \). We say that \( s \) is open if the opponent \( \pi_s \) can escape from \( H_{\alpha_s}^s \setminus P_s \) starting from \( R_s \) using a single move, i.e., if \( R_s \cap \text{esc}^\pi_s(H_{\alpha_s}^s \setminus P_s) \neq \emptyset \). In this case, the opponent may escape from \( R_s \) by either moving to the remaining portion of local area \( L_s \setminus R_s \) or to the quasi \( \pi_s \)-dominion \( H_{\pi_s}^s \setminus P_s \). The state is said to be closed, otherwise. For technical convenience, a state with an empty region is always considered open. Finally, a closed state \( s \) is promotable, if it \( \pi_s \)-maximal and \( R_s \) is \( \alpha_s \)-maximal w.r.t. \( L_s \). By \( St_\pi \subseteq St \) we denote the set of promotable states.

![Figure 1](image.png)

The main function sol of the new priority-promotion-based approach, called \textit{recursive priority promotion} (RPP, for short), is reported in Algorithm 1. The auxiliary function \textsf{NextPr} and the two procedures \textsf{Maximise} and \textsf{Promote} are, instead, reported in Appendix D. The function sol assumes the input state \( s \) to be maximal, i.e., \( s \in St_\alpha \). At Line 1 it checks if there are still unprocessed positions in the game, namely if the priority of the current state is different from \( \perp \). If this is the case, Line 2 maximises the region of the current state, namely \( R_s \triangleq r^{-1}(p) \), by computing its \( \alpha_s \)-attractor, so that the resulting set is \( \alpha_s \)-maximal and, therefore, \( s \) becomes strongly maximal, i.e., \( s \in St_\alpha \).

For convenience, we abbreviate the update of some component in a state \( s \), say component \( R_s \) for instance, simply as \( R_s \Leftarrow \text{exp} \), for some expression \text{exp}. Therefore, the instruction
at Line 2 updates the state $s$ by replacing the original region $R_s$ with $\text{atr}^\alpha_p(R_s)$ within the state. If the resulting state $s$ is closed, it is also promotable, \(i.e., s \in \text{St}_p\), being maximal by hypothesis, and, therefore, a promotion can be applied at Line 8, by means of a call to procedure \text{Promote}. If, instead, $s$ is open, which means that the opponent can escape from $R_s$ moving outside the quasi $\alpha_s$-dominion $H^\alpha_s$, the algorithm proceeds to analyse the part of the game still unprocessed. To do this, we first compute the next state by means of \text{NextPr}(s)$, which simply identifies the next priority to consider, namely the maximal priority of the unprocessed positions. The resulting state is then given as input to the recursive call at Line 4. Once the recursive call completes, the state is updated with the new region function returned by the call. The new state $s$ is such that the local area $L_s$ coincides with $R_s$, since the recursive call ends after analysing all the previously unprocessed positions. As a consequence, either the opponent cannot escape from $R_s$ anymore or it can only move to its own quasi dominion $H^\pi_s$. Line 5 checks which one of the two possibilities occurs. In the first case, the new state $s$ is closed, hence $\pi_s$-maximal. Moreover, since $L_s = R_s$, the region $R_s$ cannot attract any other positions and is, therefore, $\alpha_s$-maximal. This means that $s$ is promotable, \(i.e., s \in \text{St}_p\), and Line 7 promotes the region within the quasi $\alpha_s$-dominion. If, on the other hand, $s$ is still open, then the opponent can escape to $H^\pi_n$ from some positions in $R_s$. This means that the current state is not $\pi_s$-maximal and Line 6 fixes this by calling the procedure \text{Maximise}. The aim of this function is to reestablish maximality of the quasi dominions $H^\alpha_n$ and $H^\pi_n$ associated with the state $s$. This is done by attracting positions from the current region $R_s = L_s$. The surviving positions in $R_s$, if any, need not form a quasi $\alpha_s$-dominion anymore and are set by \text{Maximise} to their original priority according to the priority function $pr$ of the game. In any case, when the computation reaches Line 9, whether coming from Line 6, Line 7 or Line 8, the state $s$ is maximal, \(i.e., s \in \text{St}_n\), and a second, and final, recursive call is performed on $s$ to process the remaining positions in $L_s$, if any. We refer to Appendix D for the detailed description of the auxiliary function \text{NextPr} and the two procedures \text{Maximise} and \text{Promote}.

### 3.2 Parys’ Algorithm

In order to obtain a quasi-polynomial time priority-promotion-based solution procedure, we entangle the algorithm of the previous subsection with Parys’ idea \cite{48} to suitably truncate the recursion tree. Naturally, cutting some of the recursive calls may prevent us from deciding the winner for some of the positions with certainty. These intermediate results are thus \textit{undetermined} (we use a function $u$, read ‘undetermined’, to refer to these results). Parys’ contribution was to design the truncation in a way that offers bounded guarantees, namely that the undetermined sets contain all small dominions of one player and do not intersect with small dominions of the other. The \textit{bounds} up to which these limited guarantees hold are shed quickly in the call tree: most of the calls are made with half precision (meaning that one of the bounds is halved) and only one is made with full precision, meaning that both bounds are kept. A first step in the integration of Parys’ approach with Priority Promotion is to formulate it in the same terms and to introduce the notation needed when hybridising the approaches. To this end, we assume $U_s \triangleq u_s^{-1}(p_s)$ is the set of

---

**Algorithm 1** RPP Solver

```
function sol(s; St_p): Rg
    if p_s ≠ ⊥ then
        R_s ← atr^\alpha_p(R_s)
        if s is open then
            r_s ← sol(NextPr(s))
            if s is open then
                { Maximise(s)
                } else
                    { Promote(s)
                    } else
                        { Promote(s)
                        }
        return r_s
```
undetermined positions at a certain stage $s$ of the search. We require that $U_s$ satisfies the following property: it must contain all the $\pi$-dominions of size no greater than a given bound $b_s$ and it cannot intersect any $\alpha$-dominion of size no greater than a second given bound $b_s$. 

Just as pure Priority Promotion does not use undetermined positions, Parys’ approach does not use regions (beyond the attractor of the nodes with highest priority). Consequently, little happens to the region functions in our representation of Parys’ algorithm: positions that are added to $U_s$ are removed from the region function, and when $U_s$ is destroyed, they are added (with their native priorities) back to $r_s$. 

We only outline the principles here together with slight generalisations of the standard lemmas employed in each step, required later for hybrid approach. The main function sol and the halved solver hsol are reported in Algorithm 2 and Algorithm 10. The procedure Maximise and the auxiliary functions NextPr, Half, and Und are, instead, reported in Appendix D. In hsol, the attractor of the positions with maximal priority is removed, and the recursive call of sol with half precision adds all small dominions $\leq \lfloor \frac{b_s}{2} \rfloor$ (but no part of any small region ($\leq b_s$)) of player $\alpha$ of the remaining subgame to $U_s$. Lemma 8 provides that, if such an $\pi_s$ dominion was contained in the game before removing $R_s$ then a sub-dominion of it still remains after $R_s$ is removed. 

**Lemma 7.** If a dominion $D$ for Player $\alpha$ in a game $\mathcal{G}$ does not intersect with a set $S$ of nodes, then it does not intersect with $\operatorname{atr}_S^\pi(D)$ either.

For $\alpha \equiv 2 p$, a $p$-Region is a quasi $\alpha$-dominion, such that (1) all nodes have priority $\leq p$, and (2) all escape positions have priority $p$.

**Lemma 8.** If the highest priority in a non-empty dominion $D$ for player $\alpha$ is $p$ and it intersects with a $q$-region $R$ for $q \neq 2 \alpha$ and $q \geq p$, then $D$ contains a non-empty sub-dominion that does not intersect with $\operatorname{atr}_S^\pi(D)$.

These dominions are then closed under attractor by a call of Maximise, until $s$ does not change, and thus until no $\pi_s$ dominion $\leq \lfloor \frac{b_s}{2} \rfloor$ is left. This guarantee is used in sol to ensure that a dominion $D$ of player $\pi_s$ is in $U_s$ at the end of the function: all $\pi_s$ dominions returned in the recursive call of line 5 are $> \lfloor \frac{b_s}{2} \rfloor$. Closing them under attractor through a call of Maximise (line 6) leaves, by Lemma 7, a (possibly empty) $\pi_s$ dominion of size $\leq \lfloor \frac{b_s}{2} \rfloor$, when hsol is called for the second time, and thus fully included in $U_s$, by means of Und, that simply updates the data structures $r_s$ and $u_s$. After Parys’ Solver is run (with full precision) on a game with maximal priority $p_{\max}$ identified by the function NextPr, we have that $u^{-1}(p_{\max} + 1)$ contains the winning region of player $p_{\max} \mod 2$.

### 3.3 A Hybrid Algorithm

In our hybrid approach, we have to synthesise the use of regions and the use of undetermined sets. To formalise this intuition, a state for the hybrid algorithm needs to embed quite a bit
of additional information \(w.r.t\) a simple state of RPP. It obviously contains both a region function \(r\), tracking the quasi dominions already analysed, and the current priority \(p\). It also features an additional promotion function \(u\), used to maintain the set of undetermined positions, which only satisfy the relative guarantees mentioned above, the priority of the caller used in the update of this function, and two numbers \(b_0\) and \(b_1\), representing the two bounds \(w.r.t\) which the guarantees are expressed. The initial state, from which the search starts, contains, as is the case in the exponential algorithm, the priority function \(pr\) and the maximal priority \(pr(\emptyset)\). In addition, the two bounds are both set to the number of positions in the game, while the accessory function \(u\) is empty. For technical convenience, we set the caller priority to \(\top\). Finally, the ordering between the states is again defined in terms of the sets of unprocessed positions in the two states.

**Definition 9 (Hybrid Search Space).** A hybrid search space is a tuple \(S \triangleq (St, s_1, \prec)\), whose three components are defined as follows:

1. \(St \subseteq (R_k \times Pr_L) \times (Pm \times Pr_T \times N \times N)\) is the set of all tuples \(s \triangleq ((r, p), (u, c, b_0, b_1))\), \(r\) being the initial state, where:
   a) \(\text{dom}(r) \cap \text{dom}(u) = \emptyset\), \(\text{dom}(r) \cup \text{dom}(u) = Ps\), \(\text{dom}(r \cup u)^{(p)} = \emptyset\), \(p < c\);
   b) \(u^{-1}(\top_0) = \text{dom}(u)^{(p)} = \emptyset\) and if \(c \neq \top_0\) then \(u^{-1}(\top_0) = \emptyset\);
   c) \(\text{dom}(u) \cap \text{esc}(H^\alpha_0,q) = \emptyset\), with \(H^\alpha_0,q \equiv H^\alpha_0,q \cup H^\alpha,q\), for all \(\alpha \in \mathbb{B}\) and \(q \geq p\);

   for every state \(s \in St\), we set (i) \(\alpha_s \triangleq p\) mod 2, (ii) \(H_s^0 \equiv H^0_s \cup H^\top_0\), (iii) \(U_s \equiv u^{-1}(p)\), (iv) \(R_s \equiv r^{-1}(p)\), and (v) \(L_s \equiv \text{dom}(r^{(p)})\);

2. \(s_1 \triangleq ((pr, pr(\emptyset)), (\emptyset, \top_0, |\emptyset|, |\emptyset|))\) is the initial state;
3. \(s_1 \prec s_2\) if either \(L_{s_1} \subset L_{s_2}\) or \(L_{s_1} = L_{s_2}\) and \(p_{s_1} < p_{s_2}\).

Intuitively, Item 1a ensures that the set of positions in the game is partitioned into two categories: (i) those contained in the region function \(r\), which are considered determined, in the sense that they belong to known quasi dominions; and (ii) those contained in the promotion function \(u\), which are undetermined, since they form sets that only satisfy the bounded guarantees. Obviously the priority of the caller has to be higher than the current one and no position can be associated with a priority between those two. Moreover, since positions assigned to the two top pseudo-priorities must be determined, as they form dominions, \(u\) cannot refer to those two values, except for the outermost call, when \(c = \top_0\). In this case, indeed, any undetermined position is necessarily won by player 1, i.e., \(u^{-1}(\top_0) \subseteq W_1\). Moreover, all positions with priorities lower than the current one are still unprocessed, therefore they cannot be determined. Both these requirements are expressed by Item 1b. Finally, we need to ensure that a player cannot immediately escape from a set of undetermined positions, as specified in Item 1c. This property is crucial to maintain, after the update of the promotion function \(u\), the implicit invariant stating that, if a strongly-maximal state is closed, then it is promotable.

Figure 2 reports a graphical representation of the structure of a hybrid state. Most of the concepts and notation introduced for the states of RPP have a similar meaning and play a similar role for hybrid states. In particular, given a hybrid state \(s \in St\), the set \(L_s\) identifies the local area, i.e., the set of positions yet to analyse, while \(R_s\) is the quasi \(\alpha_s\)-dominion, called region, included in \(L_s\), which the algorithm is currently focusing on. Moreover, the two sets \(H_0^0\) and \(H^2_0\) partition the game, while \(H^\alpha_0,q\) and \(H^\alpha,q\) represent the portions of these sets having a priority, assigned either in \(r\) or in \(u\), at least equal to \(q \in Pr\). As opposed to the previous notion of state, however, \(H_0^0\) and \(H^2_0\) are not necessarily quasi dominions, since they may include undetermined positions, namely those contained in \(H^\alpha_0,q\) and \(H^\alpha,q\). Only
the two subsets $H^0_\alpha$ and $H^1_\alpha$, as well as their relativised versions $H^0_{\alpha,q}$ and $H^1_{\alpha,q}$, are known to be quasi dominions.

Given a player $\alpha \in \mathbb{B}$, we say that a hybrid state $s \in St$ is $\alpha$-maximal, if the quasi $\alpha$-dominion $H^0_\alpha \setminus L_s$ is $\alpha$-maximal w.r.t. $L_s$. If $s$ is $\alpha$-maximal w.r.t. both players $\alpha \in \mathbb{B}$, we say that it is maximal. We denote with $St_\alpha \subseteq St$ the set of maximal hybrid states and with $O_\alpha \triangleq \alpha \setminus dom\left(\left(\alpha^{p_r}\right) \cup u_s\right)$ the induced subgame over the local area $L_s$. A maximal hybrid state $s$ is strongly maximal, if the current region function $r_s$ is $\alpha$-maximal w.r.t. $L_s$ and the quasi $\alpha$-dominion $H^0_{\alpha} \setminus (L_s \cup U_s)$ is $\alpha$-maximal w.r.t. $L_s \cup U_s$.

By $St_\delta \subseteq St$ we denote the set of strongly maximal hybrid states. Again, we say that $s$ is open if $R_s \cap esc^\alpha(H^0_{\alpha,p_r}) \neq \emptyset$, and we say that it is closed, otherwise. For technical convenience, we always consider a hybrid state with an empty region open. Finally, a closed hybrid state $s$ is promotable, if it is $\alpha$-maximal and $R_s$ is $\alpha$-maximal w.r.t. $L_s$. By $St_\alpha \subseteq St$ we denote the set of promotable hybrid states.

The main functions of the hybrid priority-promotion algorithm (HPP, for short) reported in Algorithm 4 combines the recursive priority-promotion technique of Algorithm 1 and the recursion-tree truncation idea of Algorithm 2 in a single approach. Again, the auxiliary functions and procedures are reported in Appendix D. As for the RPP, the main function $sol$ assumes the input state $s$ to be maximal, i.e., $s \in St_\alpha$. Line 1 checks whether (i) there are no unprocessed positions in the game or (ii) one of the two bounds on the guarantees over the undetermined positions has reached threshold zero. If one of these conditions is satisfied, the current region function $r_s$ and the promotion function $u_s$ are returned unmodified at Line 2, as no further progress can be achieved in the current recursive call. Otherwise, similarly, to Parys’ approach, the search for a dominion is split into three phases: (i) a first search with halved precision made by calling the auxiliary mutually-recursive procedure $hsol$ (Line 3); (ii) a second search with full precision via a recursive call to $sol$ itself (Lines 4 to 8); (iii) a final search by means of $hsol$, again with halved precision, conditioned to the actual progress obtained during the previous phase (Line 9). Once these three phases terminate, the information about the undetermined positions contained in the local area $L_s$ or in the undetermined set $U_s$ is suitably updated by function $Und$ at Line 10.

To discuss the guarantees and their effects in more detail, let us fix a small dominion $D$, with $|D| \leq b_{2,\alpha}$. The call to $hsol$ at Line 3 modifies in-place the maximal state $s$ given as input into a strongly-maximal one such that $dom(r^{(\leq p_r)})$ does no longer contain any tiny dominions of player $\pi_s$ of size $\leq \frac{b_{2,\alpha}}{2}$. Moreover, $D' = D \cap dom(r^{(\leq p_r)})$ is a dominion in $dom(r^{(\leq p_r)})$, while $D \setminus D'$ has been processed and added to $dom(u^{(\geq p_r)} \cup dom(r^{(> p_r)}))$.

After that, at Line 4, the obtained state is locally recorded in order to determine, later on, whether the second phase achieves any progress. The algorithm then proceeds to analyse the remaining part of the game still unprocessed. To do so, the next state computed by $NextPr(s)$ is given as input to the recursive call at Line 5. Once the call completes, the state is updated with the two new functions returned by the call. At this point, the guarantee that all $\pi_s$ dominions in $dom(r^{(\leq p_r)})$ are larger than $\frac{b_{2,\alpha}}{2}$ entails that, if $D'$ is not empty (and thus $D$ completely processed), then a non-empty sub-dominion $D''$ of $D'$ is part of the call. As $dom(r^{(\leq p_r)})$ contains no tiny $\pi_s$ dominion, $|D''| > \frac{b_{2,\alpha}}{2}$, and $D'' \setminus D' \leq \frac{b_{2,\alpha}}{2}$.

Depending on whether the state is closed or not, either a promotion or a maximisation operation is performed (Lines 6 to 8), to ensure that the new state is maximal. If the middle phase has made some progress in the search, a last call to $hsol$ at Line 9 is performed, which
again modifies in-place the current state into a strongly-maximal one. As $D'' \setminus D' \leq \lfloor \frac{1}{2} \rfloor$, it is processed in $\text{hsol}$. If no progress occurred, instead, the current state is equal to the one previously returned by the first call to $\text{hsol}$ and, thus, strongly-maximal. It also entails that $D'$ was empty, and $D$ therefore processed completely in the first call of $\text{hsol}$. In both cases, the state is fed to the function $\text{Und}$, after which the current call terminates.

The procedure $\text{hsol}$ simply executes the main body of the RPP algorithm by making mutually-recursive calls to the $\text{sol}$ function (Line 5) with halved precision, until no progress on the search for a dominion can be made. As for the RPP, the auxiliary function $\text{NextPr}$ identifies the next priority to consider, and the promotion and maximisation procedures, $\text{Promote}$ and $\text{Maximise}$, generalise the corresponding ones associated with RPP. Hence, $\text{Promote}$ applies a promotion while $\text{Maximise}$ makes a state $\pi$-maximal. Similarly to Parys’ algorithm, the $\text{Half}$ function halves the bound of the opponent player $\pi_s$, and, finally, the $\text{Und}$ function, updates or reset the 'undetermined' set of positions.

At this point, by defining the winning regions of the players as $W_1 = r^{-1}(T_1) \cup u^{-1}(T_0)$ and $W_0 = P_s \setminus W_1$, i.e., $\text{sol}(\emptyset) \equiv (W_0, W_1)$, where $(r, u) \equiv \text{sol}(s_f)$, we obtain a sound and complete solution algorithm, whose time-complexity is quasi-polynomial, as we shall show in the next section.

Algorithm 4  HPP Solver  

\begin{verbatim}
function sol(s: St): Rg × Pm
1 if $p_s = \perp \lor b_{b_s} = 0 \lor b_{1s} = 0$ then
2 return $(r_s, u_s)$
3 else
4 $\text{hsol}(s)$
5 $\hat{s} \leftarrow s$
6 $(r_s, u_s) \leftarrow \text{sol}(\text{NextPr}(s))$
7 if $s$ is open then
8 $\text{Maximise}(s)$
9 else
10 $\text{Promote}(s)$
11 if $s < \hat{s}$ then $\text{hsol}(s)$
12 return $\text{Und}(s)$
\end{verbatim}

Algorithm 5  Half-Solver

\begin{verbatim}
procedure $\text{hsol}(s: St)$
1 repeat
2 $\text{R}_s \leftarrow \text{atr}_{\hat{\pi}_s}(\text{R}_s)$
3 $\hat{s} \leftarrow s$
4 if $s$ is open then
5 $(r_s, u_s) \leftarrow \text{sol}(\text{Half}(s))$
6 if $s$ is open then
7 $\text{Maximise}(s)$
8 else
9 $\text{Promote}(s)$
10 else
11 $\text{Promote}(s)$
12 until $s \neq \hat{s}$
\end{verbatim}

4  Correctness and Complexity

We now discuss how we can entangle the concepts of Priority Promotion—the transfer of information across the call structure, which makes it so efficient in practice—with the concept of relative guarantees that provides favourable complexity guarantees to Parys’ algorithm. Before turning to the principle guarantees provided by the algorithm, we note that the two algorithms from the previous sections, Parys’ algorithm and the selected variation of Priority Promotion, can be viewed as variations of our hybrid algorithm. This is particularly easy to see for the exponential Priority Promotion algorithm from Section 3.1: when we set the bounds to infinity—or to $2^c$, where $c$ is the number of different priorities of the game—then the algorithm never runs out of bounds. In this case, the function $u$ is never used, and the algorithm behaves exactly as Algorithm 2. The connection to Parys’ algorithm is slightly looser, but essentially it replaces $\text{Maximise}$ by closing only $U_s$ under attractor, and skips the promotions (through calling $\text{Promote}$) altogether. Note that these changes would not impact
the partial correctness argument, while the remaining parts of the algorithm alone are strong enough to guarantee progress.

As the algorithm is a hybrid one, its correctness proof has both local and global aspects. The global guarantees are that the regions stored in \( r(\geq p_r) \) and the bounded dominions stored in \( u(\geq p_u) \) retain their properties in all function calls. These properties are not entirely local, and to conveniently reason about the effect of updates, we use \( \text{B}_s = \text{H}^0_s \setminus \text{R}_s = \text{H}^{p_r}_s \cup (\text{H}^{p_r}_s \setminus \text{R}_s) \) for the states that are bad for Player \( \alpha_s \) in that they contain the states in \( u^{-1}_s(q) \) for all \( q \geq p_s \) with \( q \equiv \alpha_s \) and all states in \( r^{-1}_s(q) \) for all \( q > p_s \) with \( q \neq \alpha_s \), and \( \text{G}_s = \text{H}^0_s \setminus \text{R}_s = \text{H}^{p_r}_s \cup (\text{H}^{p_r}_s \setminus \text{R}_s) \) for the states that are good for Player \( \alpha_s \) in that they contain the states in \( u^{-1}_s(q) \) for all \( q > p_s \) with \( q \neq \alpha_s \) and all states in \( r^{-1}_s(q) \) for all \( q > p_s \) with \( q \equiv \alpha_s \).

We also introduce additional data for each function, namely a set \( \text{P}_s \), which stores the local area \( \text{L}_s \) at the beginning of the call of \( \text{sol} \), which is then available also in the \( \text{hsol} \) at the level where they are called. This additional set \( \text{P}_s \) of initial positions is relevant, as the guarantees of finding small dominions is formulated relative to this initial set, and not relative to the local area \( \text{L}_s \) at the end of \( \text{sol} \). The correctness proof falls into lemmas that refer to the guarantees maintained by the auxiliary functions, and an inductive proof of the main theorem. While the proofs for the auxiliary functions and of the main result will be reported in the extended version, the inductive proof of the correctness is outlined below.

**Theorem 10.** Let \( s \in \text{St}_H \) be a maximal hybrid state for a parity game \( \mathcal{G} \), where \( c_s = \min\{\text{dom}(\text{P}_s) \cup \top_0\} \) and \( b_{b_s}, b_{s_1} \geq 1 \). Assume \( \text{sol} \) is called on \( s \) and let \( s' \equiv ((\hat{r}, \hat{p}'), (\hat{u}, \hat{c}', b_{b_s}, b_{s_1})) \) be the hybrid state, for \( (\hat{r}, \hat{u}) \equiv \text{sol}(s) \), \( p' \equiv c_s \), and some \( c' > c_s \). The following holds:

- if \( c_s \equiv \alpha_s \) then \( \text{B}_s = \text{B}_{s'} \), hence, \( \text{B}_{s'} \) preserves all global guarantees of \( \text{B}_s \).
- if \( c_s \neq \alpha_s \) then:
  - \( \text{G}_s \) contains all small dominions of player \( \alpha_s \) of size \( \leq b_{\alpha_s} \) in \( \text{P}_s \) and intersects with no small \( \pi_s \) dominions of size \( \leq b_{\pi_s} \) in \( \text{P}_s \); and
  - \( \text{B}_{s'} = \text{G}_s \).

Moreover, if \( \text{hsol} \) is called on \( s \in \text{St}_H \), then it modifies \( s \) into a strongly-maximal hybrid state with the following property: \( \text{L}_s \) does not contain a small dominion of player \( \pi_s \) of size \( \leq \lfloor \frac{m}{2}\rfloor \), and \( \text{R}_s \) and \( \text{B}_s \) are closed under attractor in \( \text{P}_s \).

**Proof sketch.** We prove this theorem by induction, using the lemmas from the previous section. Establishing the base case for \( \text{sol} \) and maximal priority \( \bot \), \( \text{hsol} \) and maximal priority \( 0 \), and \( \text{sol} \) and maximal priority \( 0 \) is straight forward.

The induction step for \( \text{hsol} \) then establishes that, after return, \( \text{L}_s \) contains no tiny \( \pi_s \) dominion of size \( \leq \lfloor \frac{m}{2}\rfloor \), as they would otherwise (using Lemma 8) be found in the last recursive call of \( \text{sol} \). We also have that \( \text{R}_s \) is closed under attractor because, since closed states lead to promotion, the resulting state \( s \) must be open, so that \( \text{Maximise} \) provides closure of \( \text{R}_s \) under attractor.

Using this, the induction step for \( \text{sol} \) has mainly to show that an initial small dominion \( \text{D} \) (\( |\text{D}| \leq b_{\pi_s} \)) of player \( \pi_s \) is entirely in \( \text{B}_s \) when \( \text{Und} \) is called. Let \( \text{D}' \) be the intersection of \( \text{D} \) with \( \text{L}_s \). Obviously, \( \text{D}' \) is a dominion of player \( \pi_s \) (Lemma 7). If \( \text{D}' \) is empty, we are done. Otherwise, \( \text{D}' \) must have a non-empty sub-dominion \( \text{D}'' \) that does not intersect with \( \text{R}_s \) (Lemma 8), and these are added, by inductive hypothesis, to \( \text{B}_s \) by the full precision call (Line 5). In addition, \( |\text{D}''| > \lfloor \frac{m}{2}\rfloor \) by the return guarantees of \( \text{hsol} \). The rest of the
dominion included in $D' \setminus D''$, is then $\leq \left\lfloor \frac{p_s}{2^l} \right\rfloor$ and, by the guarantees of $hsol$, is added to $B_s$ by the second call of $hsol$ (note that we have $s' \prec s$).

With the global guarantee that $B_s$ does not intersect with small dominions of size $\leq b_{\alpha_s}$, of player $\alpha_s$, we have added all small dominions of the players $\pi_s$ and $\alpha_s$ to $B_s$ and $G_{s} \cup L_s$, respectively and, depending on the priority, we can insert either $U_s$ or $L_s$ into $u^{-1}(c_s)$. ◀

Correctness is then the special case that we call $sol$ with $c_s = \top_0$ and full precision.

Corollary 11. When $sol$ is called with the initial state $s_I$, i.e. with $c_s = \top_0$ and full precision $b_{\alpha} = |\mathcal{S}|$, for all $\alpha \in \mathcal{B}$, then, after $sol$ returns, $r^{-1}(\mathcal{T}_1) \cup u^{-1}(\mathcal{T}_0)$ is the winning region of player $1$. ◀

This is because $r^{-1}(\mathcal{T}_1)$ and $r^{-1}(\mathcal{T}_0)$ contain dominions of the respective players and are closed under attractor by our global guarantees. It is also clear that $u^{-1}(\mathcal{T}_0)$ can only be filled in the final call of $Und$, such that $u^\top \cup r^\top$ contains all dominions (as the bound does not exclude any) of Player 1, but does not intersect with any dominion (as the bound again does not include any) dominion of Player 0. Note that the winning region of Player 0 is simply the complement of the winning region of Player 1. It is interesting to note an algorithmic difference between the parts of the winning regions in the dominions $r^{-1}(\mathcal{T}_0)$ and $r^{-1}(\mathcal{T}_1)$, and the rest of the the winning regions of both players: $r^{-1}(\mathcal{T}_0)$ and $r^{-1}(\mathcal{T}_1)$, are computed constructively, and winning strategies are simply the contribution attractor strategies / arbitrary exits from states with dominating priority (on their subgame). This is not the case for the remainder of the winning regions, as their calculation is not constructive.

5 Discussion

As a hybrid between Priority Promotion and Parys’ approach, the algorithm retains the quasi-polynomial bound of Parys and the practical efficiency of Priority Promotion [3, 7, 6]. Our argument is exactly the same as Parys’ (Section 5 of [48]): use 2 parameters, $c$ for the number of priorities, and $l = 2\lfloor \log_2(n) \rfloor + 1$, where $n$ is the number of positions. We estimate the number of times $sol$ is called, excluding the trivial calls that return immediately (in line 2), because we run out of priorities or bounds, by $R(h, l)$. If $h = 0$, then we have run out of priorities ($p_s = \bot$), and $R(h, l) = 0$. If $l = 0$, then we have run out of bounds ($b_{\alpha} = 0$ or $b_{\alpha} = 0$ or $b_{\alpha} = 0$, with the other value being 1). As argued in Section 5 of [48], we can estimate $R(h, l) \leq n^l \left( \frac{h+1}{l} \right) - 1$. As the cost of all operations is linear in the size of the game, and as $l$ is logarithmic in the number of positions, this provides a quasi-polynomial running time.

Evaluation and Discussion. We have combined two generalisations of the classic recursive algorithm: the quasi-polynomial recursion scheme of Parys, which relies on the local spread of imperfect guarantees, and a Priority Promotion scheme, which relies on identifying and realising the global potential of perfect guarantees. That these improvements can be synthesised bodes well, as it promises to perfectly join the advantages of both schemes, and the first experimental data collected suggests that the algorithm lives up to this promise.

The practical effectiveness of the solution algorithms presented here, namely RPP and HPP, has been assessed by means of an extensive experimentation on both concrete and synthetic benchmarks. The algorithms have been incorporated in Oink [60], a tool written in C++ that collects implementations of several parity game solvers proposed in the literature, including the known quasi-polynomial ones. We shall compare solution times against the quasi-polynomial solvers SSPM [34], QPT [21], and the improved version of Parys’ algorithm Par [44], as well as the original version of the best exponential solver classes, namely the recursive algorithm ZLK from [66] and the original priority promotion PP [7], whose
superiority in practical contexts is widely acknowledged\(^1\) (see e.g., \([52, 60]\)).

The results give a simple argument for why—and when—to use this algorithm\(^2\). The first question is why one should use a QP algorithm. The answer to that question is quite simple: it is not hard to produce pathological cases for exponential time solvers. For complex games, it is well known that even the most efficient solvers in practice, \(i.e.,\) ZLK \([25, 61, 8]\) and PP \([4, 61, 8]\), take exponential time, while HPP has a quasi-polynomial worst case complexity. To show this behaviour, we have evaluated these three solvers and the improved version of Parys’ algorithm Par, which is a quasi polynomial version of ZLK, on the worst case family developed for the approaches based on quasi-dominions \([4]\). The results are reported in Table 1. Clearly the complex infrastructure required by the HPP can pay off in terms of running time, while Par does not outperform ZLK on these small examples. We guess it will eventually, but the game size has to grow significantly for this to show.

The reason for why most QP algorithms should not be used in practice is given with Keiran’s family of (for current solvers) simple benchmarks \([36]\) in Appendix E: our algorithm takes twice as long as the leading algorithm (PP) on this benchmark as a whole, the difference is between 0.17 and 0.36 second on average; the classic quasi-polynomial algorithms (QPT and SSPM) already time out; and Par fares similar to our algorithm. This is because these benchmarks are quite simple, and neither our algorithm nor Par runs out of bound. We also included the results for the non-bounded recursive version RPP.

We then tried to assess scalability \(w.r.t.\) the number of positions and priorities, so as to evaluate how sensitive the solvers are to variations of those two parameters. To this end, we

\[^1\text{Variations of Zielonka’s algorithm as well as of Priority Promotion approaches [3, 7, 6] (including Tangle Learning [59] and Justification [41] based approaches), who share the same basic data structure and promotion principles—are available. Their performance on benchmarks does not vary significantly. We therefore went with the original Parity Promotion approach to compare the principle performance.}\]

\[^2\text{Experiments were carried out on a 64-bit 3.9 GHz Intel\textsuperscript{®} quad-core machine, with i5-6600K processor and 16GB of RAM, running Ubuntu 18.04 with Linux kernel version 4.15.0. Oink was compiled with gcc 7.4.}\]

![Figure 3](image-url) Results on random games with a linear (left) and fixed (right) number of priorities.

| Index | Number of positions | Time | Iterations | Time | Iterations | Time | Iterations |
|-------|---------------------|------|------------|------|------------|------|------------|
| 10    | 88                  | 0.60 | 11267      | 0.00 | 1295       | 0.00 | 69         |
| 15    | 170                 | 0.18 | 524294     | 0.00 | 10175      | 0.00 | 177        |
| 20    | 278                 | 7.70 | 22020104   | 0.03 | 133631     | 0.00 | 339        |
| 25    | 410                 | -    | -          | 0.17 | 796671     | 0.00 | 879        |
| 30    | 568                 | -    | -          | 1.99 | 8863743    | 0.00 | 1217       |
| 35    | 750                 | -    | -          | 11.25| 47120383   | 0.01 | 2125       |
| 40    | 958                 | -    | -          | -    | -          | 0.02 | 2952       |

\(Table 1\) Solution times in seconds on the worst case family \([4]\).
set up two types of synthetic benchmarks. The first kind of benchmarks keeps the number of priorities fixed and only increases the size of the underlying graph, while the second one maintains a linear relation between positions and priorities. Here we drop both SSPM and QPT, since they could not solve any of these benchmarks.

Figure 3 reports the solution times of the quasi polynomial solvers on 10 clusters, each composed of 100 randomly generated games, of increasing size varying from $10^4$ to $10^5$. Each point corresponds to the total time to solve all the games in the cluster\(^3\). On the left-hand side the number of priorities grows linearly with the positions, i.e., equal to $\frac{n}{4}$, with $n$ the number of positions. On the right-hand side, instead, all the games have 500 priorities. In both cases, the timeout was set to 25 seconds. In these experiments, HPP and Par are tested together with the exponential solvers. HPP definitely scales very well w.r.t. the number of priorities, as opposed to Par, which is very sensitive to this parameter and starts hitting the timeout already on the smallest instances. HPP, indeed, behaves very much like the exponential solvers, none of which seems to be particularly sensitive to this parameter in practice, despite requiring time exponential in the number of priorities in the worst case.

What seems to emerge from the experimental analysis is that HPP behaves quite nicely in practice, competing with the leading exponential solution algorithms: the algorithmic overhead that guarantees its quasi-polynomial upper bound does not seem to impact the performance in any meaningful way, unlike what happens for all the other quasi-polynomial solvers, which do not scale with the number of positions and/or the number of priorities. This bodes well for the applicability of the hybrid approach in more challenging practical contexts, such as deciding temporal logic properties or solving reactive synthesis problems, where the number of priorities is typically higher.

Thus, HPP should be used when the game is hard; for this, it should have some structure (as opposed to being essentially random) as well as a high number of priorities, as the used advanced data structure only kicks in when the number of priorities is higher than $\log(n)$.

The excellent performance of the basic interplay between the two parts of the data structure invites exploring the limits of this approach. In future work, we will refine the interplay between these parts in our algorithm, e.g. by extracting the guarantees on the bounds provided upon return instead of the guarantees required when a call is made.

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\(^3\) The instances were generated by issuing the following OINK command for left-side games: rngame n n/4 2 10; right-side games: rngame n 500 2 10
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Lemma 7. If a dominion $D$ for Player $\alpha$ in a game $\mathcal{G}$ does not intersect with a set $S$ of nodes, then it does not intersect with $\text{atr}_{\mathcal{G}}(S)$ either.

Proof. From the definition of a dominion, Player $\alpha$ has a strategy that, from within $D$, only agrees with plays that stay within $D$, contradicting any node in $D$ being within the attractor of $S$.

Lemma 8. If the highest priority in a non-empty dominion $D$ for player $\alpha$ is $p$ and it intersects with a $q$-region $R$ for $q \not\equiv 2^k \alpha$ and $q \geq p$, then $D$ contains a non-empty sub-dominion that does not intersect with $\text{atr}_{\mathcal{G}}(R)$.

Proof. Let us fix a strategy, which witnesses that $D$ is a dominion for Player $\alpha$ in $\mathcal{G}$. We now assume for contradiction that player $\alpha$, for this strategy, player $\alpha$ can attract the player from any point in $D$ to $R$. Then, by playing this attractor strategy outside of $R$ and a witnessing strategy for $R$ being a $q$-region in $R$. We note that a play also stays in $D$ due to the witness strategy of player $\alpha$. An ensuing play is a lasso path, which either eventually stays in $R$ (in which case player $\alpha$ wins due to her witness strategy), or it infinitely often enters and leaves $R$, in which case it passes escape positions of $R$ infinitely often. We would then have that $q$ is the highest priority that occurs infinitely often, such that player $\alpha$ wins. This contradicts that $D$ is a dominion. Thus, there is a well defined non-empty area $A$ in $D$ from which player $\alpha$ cannot attract to $R$. $A$ is a dominion, as player $\alpha$ can win on $A$ with the same witness strategy. As $A$ clearly does not intersect with $R$, it does not intersect with $\text{atr}_{\mathcal{G}}(R)$ either By Lemma 7.

Having found such a dominion it can be closed under attractor and, after removing this attractor, what remains from a dominion is still a dominion, which allows for stepwise collecting dominions.

Lemma 12. Let $D$ be a dominion for Player $\alpha$ in a game $\mathcal{G}$. Then, for $A = \text{atr}_{\mathcal{G}}(D)$, then $A$ is a dominion for Player $\alpha$ in $\mathcal{G} \setminus A$.

Proof. It suffices to use a witness strategy for $D$ being a dominion in $D$, and an attractor strategy to $D$ in the remainder of $A$.

Lemma 13. Let $D$ be a dominion for Player $\alpha$ in a game $\mathcal{G}$. Then, for all sets $S$, if $A = \text{atr}_{\mathcal{G}}(S)$, then $D \setminus A$ is a dominion for Player $\alpha$ in $\mathcal{G} \setminus A$.

Proof. The same strategy that witnesses $D$ being a dominion for Player $\alpha$ in $\mathcal{G}$ witnesses $D \setminus A$ being a dominion for Player $\alpha$ in $\mathcal{G} \setminus A$.

Together, this implies that $\text{hsol}$ returns in a situation, where there are no small dominions left outside of $U_s$. Broadly speaking, this means that, if the full precision call of $\text{sol}$ in $\text{sol}$ cuts a chunk $> \lfloor \frac{b_{\alpha \cdot s}}{2} \rfloor$ from any remaining dominion $\leq b_{\alpha \cdot s}$, leaving a remainder of that dominion of size $\leq \lfloor \frac{b_{\alpha \cdot s}}{2} \rfloor$, which is then found by the second call of $\text{hsol}$. After Parys’ Solver is run (with full precision) for a parity game with maximal priority $p_{\text{max}}$, $u^{-1}(p_{\text{max}} + 1)$ contains the winning region of player $p_{\text{max}} \mod 2$. 
B Properties of the auxiliary functions for Section 4

Lemma 14. Given a strongly-maximal hybrid state $s \in St_\alpha$, the functions $\text{NextPr}(s)$ and $\text{Half}(s)$ return a maximal hybrid state $\hat{s} \in St_\alpha$ such that $\hat{s} \prec s$.

Proof. Let $s \mathrel{\triangleq} ((r, p), (u, c, b_o, b_1))$ be a strongly-maximal hybrid state. For function $\text{NextPr}$ let be $\hat{s} = \text{NextPr}(s)$ the result obtained by computing the function $\text{NextPr}$ on $s$. Due to Line 2 of Algorithm $\text{NextPr}$, $\hat{s} = ((r, q), (u, p_s, b_{o_s}, b_{1_s}))$ where $q = \max(\operatorname{rng}(s_{\beta_p}^{(s_{\beta_p} < p_s)}))$ by Line 1. Consequently, trivially holds Condition 1a of Definition 9 and that $r_s$ is a region function, $u_s$ is a promotion function, $p_s$ is a priority, and $b_{o_s}$ for $\alpha \in B$ are two integers. Moreover, clearly $q \in \downarrow, p_s$ and, therefore, also $q$ is a priority and $u_{\downarrow}^{-1}(\downarrow) = \operatorname{dom}(u_{\downarrow}^{\leq q}) = \emptyset$ as required by Condition 1b of the same definition. In addition, since $H^\beta_s q = H^\beta_s p^\alpha \cup r^{-1}(q)$ and $H^\beta_s q = H^\beta_s p^\alpha$, with $\beta = q \bmod 2$, it also follows Condition 1c. Finally, to prove that $\hat{s}$ is maximal it suffice to observe that $H^\beta_s = H^\alpha_\beta \cup R_s$ and $H^\beta_s = H^\beta_s \hat{s}$. Consequently, since $s$ is strongly-maximal it holds that $R_s$ is $\alpha$-maximal w.r.t. $L_s$ and, therefore, $H^\beta_s = H^\alpha_\beta$-maximal as well. To conclude, if $R_s = \emptyset$ trivially we have that $L_s \subseteq L_s$, otherwise we have that $L_s = L_s$ and $p_{s^*} = q < p_s$. In both case we can conclude that $\hat{s} \prec s$.

We can now focus on $\text{Half}$ function. Let us consider $\hat{s}$ as the argument of function $\text{NextPr}$ at Line 2 of Algorithm $\text{Half}$. Due to Line 1, we have that $\hat{s} = ((r, p), (u, c, b_o, b_1))$, where $b_o = \frac{1}{\frac{1}{\alpha}}$ and $b_1 = \frac{1}{\frac{1}{\alpha}}$. It is easy to observe, by the fact that $s$ is a strongly-maximal hybrid state and that both $b_0$ and $b_1$ are integers, that $\hat{s}$ is a strongly-maximal hybrid state as well. Hence, $\text{Half}$ returns $\text{NextPr}(\hat{s})$ that in turn returns a new maximal hybrid state $s' \in St_\alpha$ such that $s' \prec \hat{s}$.

Lemma 15. Given a hybrid state $s \in St$, the procedure $\text{Maximise}(s)$ modifies in-place $s$ into a maximal hybrid state $\hat{s} \in St_\alpha$ such that $\hat{s} \prec s$.

Proof. Let $s \mathrel{\triangleq} ((r, p), (u, c, b_o, b_1))$ be an hybrid state and $\hat{s} \mathrel{\triangleq} ((\hat{r}, \hat{p}), (\hat{u}, \hat{c}, \hat{b_o}, \hat{b_1})) = \text{Maximise}(s)$ the modified state obtained by computing the function $\text{Maximise}$ on $s$. It is easy to see that $\hat{p} = p$, $\hat{c} = c$, and $\hat{b}_o = b_o$ for $\alpha \in B$. Moreover, due to Line 3 of Algorithm $\text{Maximise}$, it holds that $X \subseteq R_s \cup L_s \subseteq \operatorname{dom}(p_{\alpha}^{(\leq p)})$ while, due to Line 4, we have that $q \geq p$. As a consequence, by Lines 6 and 8, $\hat{r}$ is a a promotion function. Furthermore, from the observation that in case $q = \top$, with $\beta \in B$, it necessarily holds that $\alpha \equiv_2 \beta$, we can also conclude that, by Lines 7 and 9, $\hat{u}$ is a promotion function as well. To prove that $\hat{r}$ is a region function, let us consider again the case in which $q = \top$ with $\beta \in B$. Now, since $\alpha \equiv_2 \beta$, the set $X$ corresponds to the $\beta$-attractor of $R_s \cup L_s$ by Line 3. Once $X$ is merged to $r^{-1}(\top)$, by Line 6, it is obvious that $r^{-1}(\top)$ is still a $\beta$-dominion due to the fact that the $\beta$-attractor of a $\beta$-dominion is a $\beta$-dominion as well. We can now consider $q < \top$ for any $\beta \in B$. Let us first focus on the iteration of Line 2 in which $\alpha = \alpha_s$. Two cases may arises: $q \equiv_2 \alpha$ or $q \not\equiv_2 \alpha$. In the first case, due to Lines 5-6, we have that $H^\beta_r = H^\alpha_r \cup X$ which is clearly a quasi $\alpha$-dominion since $X$ corresponds to the $\alpha$-attractor of $R_s \cup L_s$, while $H^\beta_r = H^\beta_s$ is a quasi $\beta$-dominion function. Hence, $\hat{r}$ is a quasi dominion function. Finally, clearly $X$ does not belongs to $\text{esc}(H^\beta_r)$, since Player $\alpha$ has a strategy to attract every position $v \in X$ to $r^{-1}(q)$, form which the opponent can only escape through positions having priority $q' \geq q$ and congruent to the parity of $q$. Summing up, the resulting $\hat{r}$ is a region function and due to the fact that the priority function $p_{\alpha}$ is always a region function, we can conclude that also after Line 10 $\hat{r}$ is a region function. On the other hand, if $q \not\equiv_2 \alpha$, we can easily conclude that $\hat{r}$ is a region function due to Lines 5, 8, and 10. When, instead, the loop at Line 2
selects a parity $\alpha \neq \alpha_s$ and the priority selected at Line 4 has the same parity as $\alpha$, the
proof that $\tilde{r}$ is a region function is equivalent to the one provided above for the case in which
$\varphi \equiv \alpha$ and $\alpha = \alpha_s$. Similarly, when $\varphi \equiv \alpha$ the proof corresponds to the one provided
for the case in which $\varphi \equiv \alpha$ and $\alpha = \alpha_s$. To prove Condition 1a of Definition 9 it suffice
to observe that whenever $X$ is merged to $r$ (resp. $u$) due to Lines 6-7 and 8-9, while the operation at Line 10 does not change $\text{dom}(r)$, since $L_s \triangleq \text{dom}(r^{(\leq p)})$.
Condition 1b, instead, follows form the fact that that $\varphi \geq p$ and $\varphi = \top_\beta$ with $\beta \in \mathcal{B}$ only if
$\alpha \equiv \beta$. Therefore, it holds that $\tilde{u}^{-1}(\top_1) = u^{-1}(\top_1)$, and $\text{dom}(\tilde{u}^{(\leq p)}) = \text{dom}(u^{(\leq p)})$ which are all empty. It remains to prove Condition 1c. Since $s$ is a hybrid state, it holds that $U_s$ and every $v \in H^s_\alpha$ cannot be attracted by $R_s$ and more in general by $L_s$, due to Condition 1c. Due to the same condition, it also holds that $\text{dom}(\tilde{u}) \cap \text{esc}(H^s_\beta) = \emptyset$ for every $q' > p$. In
addition, by Lines 5 and 9, the domain of $u$ only increase when the priority selected at Line 4
has parity $\overline{\alpha}$. However, the positions of $X$ merged in this case are not part of $\text{esc}(H^s_\beta)$ since
$X$ has been computed as the $\overline{s}$-attractor to $H^s_\beta \setminus L_s$ and, therefore, Player $s$ has a strategy
to reach $H^s_\beta \setminus L_s$ from $X$. The maximality of $\tilde{s}$ is a trivial consequence of Line 3, together
with the observation that the operation at Line 10 cannot affect the maximality of the state.
Finally, since $\text{dom}(\tilde{u}) \supseteq \text{dom}(u)$ and $\text{dom}(\tilde{r}) \supseteq \text{dom}(r)$, while $\tilde{p} = p$, we can also conclude that
$\tilde{s} < s$. □

Lemma 16. Given a promotable hybrid state $s \in St_p$, the procedure $\text{Promote}(s)$ modifies
in-place $s$ into a maximal hybrid state $\tilde{s} \in St_p$ such that $\tilde{s} < s$.

Proof. Let $s \triangleq (r,p), (u,c,b_0,b_1)$ be a promotable hybrid state and $\tilde{s} \triangleq (\tilde{r}, \tilde{p}), (\tilde{u}, \tilde{c}, \tilde{b}_0, \tilde{b}_1) =$
$\text{Promote}(s)$ the modified state obtained by computing the function $\text{Promote}$ on $s$. It is easy
to see that $\tilde{p} = p$, $\tilde{c} = c$, and $\tilde{b}_0 = b_0$ for $\alpha \in \mathcal{B}$. At Line 1 of Algorithm $\text{Promote}$ the two
best escape priorities $(p_0, p_u)$ from $R_s$ to $r$ and $u$ are computed by the function $\text{bep}$. Due to the
definition of $\text{bep}$ and the fact that $R_s$ is an $\alpha$-dominion in $\bar{\alpha}$, it follows that both $p_0$ and
$p_u$ are priorities higher than $p$. Moreover, by the same definition of $\text{bep}$ and the fact that $s$ is
promotable, it also holds that $p_\alpha \equiv p$, while $p_u \not\equiv p$ if $p_u < \top_\alpha$. Now, two cases may arises:
$p_0 \leq p_u$ or $p_u > p$. In the fist case, by Line 2 and 3, we have that $\tilde{u} = u$ and $\tilde{r} = \top[R_s \mapsto p_\alpha]$. Hence, $\tilde{u}$ is a promotion function and Condition 1b of Definition 9 is satisfied. Moreover,
since $R_s \subseteq r^{(\leq p)}$, we have that $\text{dom}(\tilde{r}) = \text{dom}(r)$ and, therefore, also the two requirements
of Condition 1a are guaranteed. Finally, since $p_\alpha \equiv p$, it follows that $H^\alpha_\beta = H^\beta_\beta$ for any
$\beta \in \mathcal{B}$. Hence, Condition 1c is satisfied. At this point, it remains to prove that $\tilde{r}$ is a region
function. Now, since $R_s \subseteq r^{(\leq p)}$ and $p_\alpha > p$, $\tilde{r}$ is clearly a promotion function. In addition, it
is also a quasi dominion function as a consequence of the fact that $H^\alpha_\beta = H^\beta_\beta$ for any $\beta \in \mathcal{B}$. Moreover,
being $\tilde{r}$ a region function, it holds that $\text{pr}(v) \geq p$ and $\text{pr}(v) \equiv \alpha$ for each position
$v$ in $\text{esc}(H^\alpha_{\varphi \equiv p})$. However, due to Line 1 we have that $\text{bep}(R_s, r_\alpha) = p_\alpha$, which implies that
$\text{pr}(v) \geq p$, and $\text{pr}(v) \equiv \alpha$ for each position $v$ in $\text{esc}(H^\alpha_{\varphi \equiv p})$. Hence, $\tilde{r}$ is a region function.
Finally, since $s$ is promotable, it follows that $H^\alpha_s = H^\beta_s \cup R_s$ is $\alpha$-maximal, while $H^\alpha_\overline{\alpha}$ is $\overline{s}$-
maximal and, therefore, $\tilde{s}$ is maximal. To conclude, being $s$ promotable, it necessarily
holds that $R_\alpha \not\subseteq\emptyset$, which implies that $L_\overline{\alpha} \subset L_\alpha$. Hence, $\tilde{s} < s$. Let us now consider the
case in which $p_\alpha > p_u$. Due to Line 2 and 4 of Algorithm $\text{Promote}$ we have that $\tilde{r} = r \setminus R_s$
and $\tilde{u} = u[R_s \mapsto p_u]$. Conditions 1a and 1b follow from the fact that $\top_\alpha > p_u > p$ and
$R_s \subseteq r^{(\leq p)}$, which also imply that $\tilde{u}$ is a promotion function. In addition, since $R_s \triangleq r^{-1}(p)$,
it holds that $H^\alpha_{\varphi \not\equiv p} = H^\beta_{\varphi \equiv p} \setminus R_s$, hence $\tilde{r}$ is a region function. To prove Condition 1c, instead,
let us observe that $R_s$ is an $\alpha$-dominion in $\bar{s}$ and, consequently, $\text{esc}(R_s) \subseteq H^\alpha_s$. Moreover,
we have that $p_u \not\equiv p$, hence, as a consequence of the fact that $\tilde{u} = u[R_s \mapsto p_u]$, it holds that
$R_s \subseteq H^\alpha_{\varphi \not\equiv p} \subseteq H^\beta_{\varphi \equiv p}$. In addition, it suffice to observe that $\text{dom}(\tilde{r}^{(\varphi \equiv p)}) = \text{dom}(r^{(\varphi \equiv p)})$ and
Given a strongly-maximal hybrid state $s \in St_{\slr}$, the function $Und(s)$ returns a pair $(\hat{r}, \hat{u}) \in R_{\slr} \times P_{\slr}$ of a region function $\hat{r}$ and a promotion function $\hat{u}$ such that: 1) $r_{\slr}^{(s,p)} \subseteq \hat{r}^{(s,p)}$; 2) $u_{\slr}^{(s,p)} \subseteq \hat{u}^{(s,p)}$; 3) $\hat{s} = ((\hat{r}, c_s), (\hat{u}, q, b_0, b_1))$ is a promotable hybrid state if it $s$ is closed and, an hybrid state otherwise, for all priority $q \in \PR$ such that $c_s < q \leq \min(\rgdom(r_s \cup u_s^{>c_s}))$, and $b_0, b_1 \in \N$.

Proof. Let be $s \triangleq ((r, p), (u, c, b_0, b_1))$ a strongly-maximal hybrid state and $(\hat{r}, \hat{u})$ the result obtained by computing the function $Und$ on $s$. Moreover, let $\hat{s} = ((\hat{r}, c_s), (\hat{u}, q, b_0, b_1))$ the composition of state $s$ and the pair $(\hat{r}, \hat{u})$. Two cases may arise: $c \equiv_2 \alpha$ or $c \equiv_2 \pi_s$. In the first case, by Lines 1-2 of Algorithm $Und$, we have that $\hat{r} = r$ and $\hat{u} = u[U_s \mapsto c]$, where $U \triangleq u^{-1}(p)$. By the fact that the priorities of the states are strictly decreasing we have that $c > p$. Moreover, it holds that $\dom(\hat{u}) = \dom(u)$ and, therefore, $\hat{r}$ is a region function, $\hat{u}$ is a promotion function, both Points 1) and 2) of the Lemma and both Conditions 1a and 1b of Definition 9 are satisfied for the composed state $\hat{s}$. To prove Condition 1c it suffice to observe that due to Line 3 and the fact that $c \equiv_2 \alpha$, it holds that $H^{(\hat{r})}_u = H^{(r)}_u$ for any $\beta \in \SLR$. In particular, we have that $H^{(\hat{r})}_{\beta, \alpha} = H^{(r)}_{\beta, \alpha}$ for all priority $q > c$ and $H^{(\hat{r})}_{\pi_s, \alpha} = H^{(r)}_{\pi_s, \alpha}$ for all priority $q > c$ and $H^{(\hat{r})}_{\Sigma, \alpha} = H^{(r)}_{\Sigma, \alpha} \cup U_s$. Finally, if $\hat{s}$ is closed, it necessarily holds that $\hat{s}$ is $\pi_s$-maximal and that $R_\alpha$ is $\alpha$-maximal as a direct consequence of the fact that $s$ is strongly-maximal and $r^{(s,p)} \land (\Sigma, c_s) = \emptyset$. Hence, $\hat{s}$ is also promotable. We can now focus on the case in which $c \equiv_2 \pi_s$. By Lines 2 and 4-5 of Algorithm $Und$ we have that $\hat{r} = r^{(\hat{s})}[v[U_s \mapsto \pr(v)]]$ and $\hat{u} = u^{(\hat{s})}[U_s \mapsto c]$. Now, it is easy to see that also in this case both Points 1) and 2) of the Lemma and both Conditions 1a and 1b are satisfied for the composed state $\hat{s}$, since $L \triangleq \dom(r^{(\hat{s})})$ and $U \triangleq u^{-1}(p)$ and, also holds that $\hat{u}$ is a promotion function and $\hat{r}$ is a region function due to the fact that the priority function $\pr$ is always a region function. Condition 1c, instead, follows form the fact that $s$ is a strongly-maximal hybrid state and, therefore, it holds that $esc^\alpha(L_s) \subseteq H^{\hat{r}}_\pi$. By the same Condition 1c, we have that $esc^\alpha(H^{(r)}_{\pi, \alpha} \setminus L_s)$ and, consequently, for each priority $q > p$ the set $u^{-1}(q)$ cannot reach $U_s$ that, due to Line 5, is reset to the values of the priority function $\pr$. The latter observation also implies that, if $\hat{s}$ is closed, $\hat{s}$ is also $\pi_s$-maximal. Indeed, when $\hat{s}$ is closed, it holds that player $\pi_s$ cannot attract $R_\pi$ and since $L^{(\hat{r})} = R^{(r)}_{(\hat{r})} \cup U_s$ we can conclude that $H^{(\hat{r})}_{\pi_s} \setminus L^{(\hat{r})}$ is $\pi_s$-maximal w.r.t. $L^{(\hat{r})}$. Finally, similarly the previous case, $R^{(r)}_\pi$ turns out to be $\alpha$-maximal as a direct consequence of the fact that $s$ is strongly-maximal and that $r^{(s,p)} \land (\Sigma, c_s) = \emptyset$. Hence, $\hat{s}$ is also promotable. $
$

Given a maximal hybrid state $s \in St_{\slr}$, the function $sol(s)$ terminates and returns a pair $(\hat{r}, \hat{u}) \in R_{\slr} \times P_{\slr}$ of a region function $\hat{r}$ and a promotion function $\hat{u}$ such that: 1) $r_{\slr}^{(s,p)} \subseteq \hat{r}^{(s,p)}$; 2) $u_{\slr}^{(s,p)} \subseteq \hat{u}^{(s,p)}$; 3) $\hat{s} = ((\hat{r}, c_s), (\hat{u}, q, b_0, b_1))$ is a promotable hybrid state if it $s$ is closed and, an hybrid state otherwise, for all priority $q \in \PR$ such that $c_s < q \leq \min(\rgdom(r_s \cup u_s^{>c_s}))$, and $b_0, b_1 \in \N$. Moreover, the procedure $hsol(s)$ modifies in-place $s$ into a strongly-maximal hybrid state $\hat{s} \in St_{\slr}$ such that $\hat{s} \leq s$.

Proof. Let be $\hat{s} \triangleq sol(s)$ the result obtained by computing $sol$ on a maximal hybrid state $s \triangleq ((r, p), (u, c, b_0, b_1))$. The proof proceeds by induction, where in the base case we have that $p = \bot$ or $b_0 = b_1 = 0$ for any $\beta \in \SLR$. In this case, by Line 1 and 2 of Algorithm $4$ we have that $\hat{s} = ((r, c_s), (u, q, b_0, b_1))$, which trivially satisfies all the requirements. Therefore, let
us now consider an arbitrary recursive call of $\text{sol}$ where $p > \bot$ and $b_\beta > 0$ for any $\beta \in \mathbb{B}$. At to Line 3 of $\text{sol}$, the procedure $\text{hsol}$ is called on the state $s$. Consequently, at Line 1 of Algorithm 9, the set $R_s$ is maximised by computing its $\alpha_s$-attractor $a_\alpha^{\alpha_s}$. It is not hard to show that the new state $s$ is still a strongly-maximal hybrid greater or equal to the old one, $w.r.t.$ the ordering of Definition 9. Then, two cases may arises: $s$ is open or closed. In the first case, at Line 5 a recursive call of $\text{sol}$ is performed on the state generated by $\text{Half}$. Now, due to the fact that $s$ is a strongly maximal hybrid state and by Lemma 14, it follows that the resulting state of $\text{Half}(s)$ is a maximal hybrid state greater or equal to the input $s$, $w.r.t.$ the ordering of Definition 9. Hence, by external induction, we have that $s = ((\widehat{r}, p), (\widehat{u}, c, b_0, b_1))$ is a hybrid state and, by Point 1) and 2) it is also greater or equal to the previous one, $w.r.t.$ the ordering of Definition 9. At this point, $s$ can be open or closed. Let us first consider again the case in which it is open. By Lemma 15 and Line 7, the function $\text{Maximise}$ modifies $s$ into a maximal hybrid state that is greater or equal to the previous $s$, $w.r.t.$ the ordering of Definition 9. On the other hand, when $s$ is closed, by external induction it holds that $s$ is promotable and, by Lemma 16 the function $\text{Promote}$ modifies $s$ into a maximal hybrid state greater than the previous one the ordering of Definition 9. This last case also apply when $s$ is closed at Line 4. Now, since the modified state by $\text{Promote}(s)$ is always greater than $s$, $w.r.t.$ the ordering of Definition 9, we can conclude that function $\text{hsol}$ only ends when the function $\text{Maximise}$ modify the input hybrid state such that there is no progress $w.r.t.$ the ordering of Definition 9, otherwise $\text{hsol}$ starts a new iteration due to the Loop at Line 1. By the fact that the state space is finite and that it starts a new iteration only if there is a progress in the ordering it is easy fo prove that $\text{hsol}$ always terminates. Observe that, when $s' \neq s$ where $s'$ is the state modified by $\text{Maximise}(s)$, it holds that $s'$ is also strongly-maximal and, therefore, $\text{hsol}$ returns a strongly maximal hybrid state. At this point, we can go back to Line 3 of Algorithm 4 where the modified state $s$ is strongly maximal as observed above. Now, by Lemma 14 the input state of $\text{sol}$ at Line 5 is maximal. Hence, by external induction, the modified state composed of the results of Line 5 $s = ((\widehat{r}, p), (\widehat{u}, c, b_0, b_1))$ is a hybrid state and, by Point 1) and 2) it is also greater or equal to the previous $s$, $w.r.t.$ the ordering of Definition 9. Similarly to the proof for $\text{hsol}$, the state $s$ can be open or closed. In the first case, two more cases may arise, indeed, if $s$ is open, then by Lemma 15 and Line 7, the function $\text{Maximise}$ modifies $s$ into a maximal hybrid state that is greater or equal to the previous $s$, $w.r.t.$ the ordering of Definition 9. On the other hand, if $s$ is closed, then by external induction it holds that $s$ is promotable and, by Lemma 16 the function $\text{Promote}$ modifies $s$ into a maximal hybrid state greater than the previous one the ordering of Definition 9. Thus, at Line 11, is case $s \sim \widehat{s}$ where $\widehat{s}$ is the state stored before the recursive calls at Line 4, $\text{hsol}$ is called on a maximal state and, as proved above, the modified resulting state is strongly maximal. Otherwise, as observer in the proof for $\text{hsol}$, $s$ is already strongly-maximal. As a consequence, by Lemma 17 and Line 10, we can conclude that $\text{sol}$ returns a pair of region function and priority function $(\widehat{r}, \widehat{u})$ such that $\widehat{s} = ((\widehat{r}, c_s), (\widehat{u}, q, b_0, b_1))$ is a promotable hybrid state if $\widehat{s}$ is closed and, an hybrid state otherwise, for all priority $q \in \text{pr}$ such that $c_s < q \leq \min(\text{rg}(r_s \cup u_s^{>c_s}))$, and $b_0, b_1 \in \mathbb{N}$.

## C Proof of Theorem 10

**Theorem 10.** Let $s \in \text{St}_\mathcal{G}$ be a maximal hybrid state for a parity game $\mathcal{G}$, where $c_s = \min(\text{dom}(P_s) \cup \top_\mathcal{G})$ and $b_{\alpha_s}, b_s \geq 1$. Assume $\text{sol}$ is called on $s$ and let $s' \triangleq ((\widehat{r}, p'), (\widehat{u}, c', b_0, b_1))$ be the hybrid state, for $(\widehat{r}, \widehat{u}) \triangleq \text{sol}(s)$, $p' \triangleq c_s$, and some $c' > c_s$. The following holds:
if \( c_s \equiv 2 \alpha_s \) then \( B_s = B_s' \), hence, \( B_s' \) preserves all global guarantees of \( B_s \).

if \( c_s \not\equiv 2 \alpha_s \) then:

- \( G_s \) contains all small dominions of player \( \alpha_s \) of size \( \leq b_{\alpha_s} \), in \( P_s \) and intersects with no small \( \pi_s \) dominions of size \( \leq b_{\pi_s} \) in \( P_s \); and

- \( B_s' = G_s \).

Moreover, if \( \text{hsol} \) is called on \( s \in \text{St}_R \), then it modifies \( s \) into a strongly-maximal hybrid state with the following property: \( L_s \) does not contain a small dominion of player \( \pi_s \) of size \( \leq \left\lfloor \frac{b_{\pi_s}}{2} \right\rfloor \), and \( R_s \) and \( B_s \) are closed under attractor in \( P_s \).

We provide an inductive proof over the highest priority.

For the induction basis, we consider, in this order, (1) \( \text{sol} \) with highest priority \( \perp \) (i.e. with an empty game); (2) \( \text{hsol} \) being called with highest priority 0; and (3) \( \text{sol} \) being called with highest priority 0.

For empty games—(1), highest priority \( \perp \)—there is nothing to do (and \( \text{sol} \) does nothing).

For (2), games with maximal (and thus only) priority 0 in \( \text{hsol} \), all states are in \( R_s \), \( s \) is closed, and all states in \( R_s \) are promoted. The game is then empty, \( \text{sol} \) is called with an empty game, and \( \text{Maximise} \) is called on an empty set. All global guarantees are retained due to Lemma 16, case (1), and Lemma 15. As \( \text{dom}(r_s^0) = \text{dom}(u_s^0) = \emptyset \), the additional requirement for \( \text{hsol} \) holds.

For (3), games with maximal (and thus only) priority 0 in \( \text{sol} \), all nodes are promoted—as we have shown for case (2)—in the first call of \( \text{hsol} \) (line 3). \( s \) is then empty. The call of \( \text{sol} \) in line 5 therefore does nothing, and neither do the call of \( \text{Maximise} \) in line 7, or of \( \text{Und} \) in line 10 for an empty game.

Thus, the global guarantees are retained by (2), (1), and due to Lemmas 15 and 17.

We now implement the induction step, again first for \( \text{hsol} \), and then for \( \text{sol} \), using the results from \( \text{hsol} \).

For the induction step of \( \text{hsol} \), we observe that, in each iteration of the loop, line 2 closes \( R_s \) under attractor, thus creating the precondition for the call of \( \text{sol} \) (establishing the closure condition when \( R_s \) is removed from the induced subgame on which the subcalls work).

When executed, the call of \( \text{sol} \) then provides, by induction hypothesis, an increase of \( B_s' \), which now contains every small dominion \( D \) of player \( 1 - \alpha \) of size \( \leq \left\lfloor \frac{b_{1-\alpha_s}}{2} \right\rfloor \) in \( S = \text{dom}(r_s^0) \setminus R_s \) from before the call, while retaining all global guarantees. Note that no such set exists if \( \left\lfloor \frac{b_{1-\alpha_s}}{2} \right\rfloor = 0 \) holds; while this case is not covered in the induction hypothesis, \( \text{sol} \) does nothing (or: returns immediately without implementing any change) in this case; thus, all global guarantees are retained.

Thus, the global guarantees are retained in every step either by induction hypothesis (Lemmas 14), by the special case \( \left\lfloor \frac{b_{1-\alpha_s}}{2} \right\rfloor = 0 \), or by Lemmas 15 and Lemma 16.

Finally, to see that \( B_s \) is closed, we observe that it is closed after \( \text{Maximise} \) is run, because closedness of \( s \) always leads to a promotion, and thus to \( s < \tilde{s} \), \( s \) is open in the last iteration. Thus, the last operation executed in \( \text{hsol} \) is the call of \( \text{Maximise} \), which entails closedness of \( B_s \) (Lemma 15).

We have shown that \( B_s' \) now contains all small dominions \( D \) of player \( 1 - \alpha \) of size \( \leq \left\lfloor \frac{b_{1-\alpha_s}}{2} \right\rfloor \) in \( S = \text{dom}(r_s^{\leq p_s}) \setminus R_s \) before the last call. As \( s \) has not changed during the call, it also contains all small dominions in \( S = \text{dom}(r_s^{< p_s}) \setminus R_s \) after the call, and on the time of return. As \( R_s \) is a \( p_s \)-region, Lemma 8 provides that, for any non-empty dominion \( D' \) in \( \text{dom}(r_s^{\leq p_s}) \), there must be a non-empty sub-dominion \( D'' \subseteq D' \), which is also a dominion in \( S \). Thus, as \( S \) does not contain a dominion \( D \) of player \( 1 - \alpha_s \) with size \( \leq \left\lfloor \frac{b_{1-\alpha_s}}{2} \right\rfloor \), neither does \( \text{dom}(r_s^{\leq p_s}) \) when \( \text{hsol} \) returns.
We can now turn to the induction step for sol. We first check that the global guarantees are retained up to the point where Und is called. We then make the argument that sol will, for a dominion \( D \) of player \( 1 - \alpha \) in \( P_s \) of size \( |D| \leq b_{1 - \alpha} \), guarantee that \( D \subseteq B_s \) holds. We will then show that, under these conditions, Und will retain the global guarantees.

The check that the global guarantees are retained up to the point where Und is called is straightforward: we have seen that \( hsol \) (line 3) does, and also provides that \( R_s \) is closed under attractor before sol is called in line 5.

Maximise and Promote do retain the global guarantees by Lemmas 15 and 16, respectively. We now argue that, when Und is called, \( B_s \) contains all dominions of player \( 1 - \alpha_s \) in \( P_s \) of size \( \leq b_{1 - \alpha_s} \). Let \( D \) be such a dominion.

We first observe that \( G_s \), and its \( \alpha \)-attractor \( \mathsf{atr}^\alpha(G_s) \), cannot intersect with \( D \) at any point, by the global guarantees.

Thus, \( D \) is a dominion in \( P_s \setminus \mathsf{atr}^\alpha(G_s) \) at any point by Lemma 7.

After the first call of \( hsol \) (in line 3), we have established that \( \mathsf{dom}(r^{\leq p_s}) \) does not contain a dominion of player \( 1 - \alpha_s \) of size \( \leq \left\lfloor \frac{b_{1 - \alpha_s}}{2} \right\rfloor \).

In particular, if \( \mathsf{dom}(r^{\leq p_s}) \) does not intersect with \( D \), then \( D \) must be contained in \( B_s \), as \( P_s \subseteq \mathsf{dom}(r^{\leq p_s}) \cup B_s \cup G_s \).

Otherwise, \( D' = D \cap \mathsf{dom}(r^{\leq p_s}) \) must be a dominion in \( \mathsf{dom}(r^{\leq p_s}) \) (with the same winning strategy for player \( \alpha \) as in \( P_s \) (or \( P_s \setminus \mathsf{atr}^\alpha(G_s) \)) by Lemma 7.

By Lemma 8, \( D' \) has a sub-dominion \( D'' \subseteq D' \) which does not intersect with \( R_s \).

\(|D''| > \left\lfloor \frac{b_{1 - \alpha_s}}{2} \right\rfloor \) follows from the fact that \( \mathsf{dom}(r^{\leq p_s}) \) contains no smaller dominions. Thus, by induction hypothesis, after the call of sol from line 5 is returned, \( B_s \) also contains \( D'' \).

In this case, \( s < \hat{s} \) in line 9, such that \( hsol \) is called. In this case, also calling Maximise (by Lemma 15 or Promote (By Lemma 16) retain the global guarantees.

Let us assume for contradiction that, after the return from \( hsol, I = \mathsf{dom}(r^{\leq p_s}) \cap (D' \setminus D'') \) is not empty. Noting that \( D \supseteq D' \supseteq D'' \setminus D', |D| \leq b_{1 - \alpha_s} \), and \( |D''| > \left\lfloor \frac{b_{1 - \alpha_s}}{2} \right\rfloor \).

Moreover, we have again that \( I \) is contained in \( P_s \setminus \mathsf{atr}^\alpha(G_s) \) by our global guarantees, \( P_s \subseteq \mathsf{dom}(r^{\leq p_s}) \cup B_s \cup G_s \), and \( B_s \) is closed under \( 1 - \alpha_s \) attractor in \( P_s \).

Thus, \( I = (D' \setminus D'') \cap \mathsf{dom}(r^{\leq p_s}) \) must be a dominion in \( \mathsf{dom}(r^{\leq p_s}) \) (with the same winning strategy for player \( \alpha \) as in \( P_s \) (or \( P_s \setminus \mathsf{atr}^\alpha(G_s) \)) by Lemma 8. (Contradiction between the assumption that \( I \) is non-empty, and that \( \mathsf{dom}(r^{\leq p_s}) \) cannot contain such a small dominion of player \( 1 - \alpha_s \).)

We have now established that the global guarantees hold, while \( B_s \) contains all dominions of size \( \leq b_{1 - \alpha_s} \) of player \( 1 - \alpha_s \) in \( P_s \), when Und is called. This entails that \( G_s \cup L_s \) contains all dominions of size \( \leq b_{\alpha_s} \) of player \( \alpha \) in \( P_s \), when Und is called.

Und then moves the respective set, \( U_s \) or \( L_s \), to \( u_s^{-1}(c_s) \) of the caller priority, such that \( B_{s'} \) on that level is the current \( B_s \) or \( G_s \cup L_s \) when \( c_s \equiv p_s \) or \( c_s \neq p_s \), respectively.

Thus, due to Lemma 17, Und retains the global guarantees.

This completes the induction step.

**D** Appendix for Section 3.1

The auxiliary function NextPr generates a new maximal state \( \hat{s} = \text{NextPr}(s) \in \text{Str}_R \), starting from a strongly-maximal one \( s \in \text{Str}_R \). The state \( \hat{s} \) is obtained by changing the current priority \( p_s \) to the highest priority \( q \) of the positions in \( L_s \setminus R_s \). Observe that when no such position exists, namely when \( L_s = R_s \), the new priority coincides with \( \perp \).

Maximise enforces the maximality property on the state \( s \) received as input, so that, in the resulting state obtained by modifying \( s \) in-place, no position of the local area \( L_s \) can be
where a result, in the modified state no position of the local area generates a new state with halved precision for the player in the current priority. This function is also called by another auxiliary function, the recursive nature of the algorithm, whose base case ensures that no position is left unprocessed from the fact that a sound and complete solution algorithm for parity games. In particular, the soundness follows the only possibility for player to the opponent can use to escape. The procedure, then, promotes one move when escaping from the region \( R \). Enjoyed by the input state. It first computes the opponent best-escape priority at Line 5. To conclude, the procedure \( \text{Promote} \) requires a promotable state \( s \in \text{St}_p \) and applies a promotion operation to the region \( R_s \), while preserving any maximality property already enjoyed by the input state. It first computes the opponent best-escape priority \( q \) for the set \( R_s \) w.r.t. \( r_s \) (Line 1). Intuitively, this is the smallest priority the opponent can reach with one move when escaping from the region \( R_s \). Formally, it is defined as:

\[
\text{bep}_s^\top (R_s, r_s) \triangleq \min(\text{rng}(r_s) \mid \text{rng}(I)),
\]

where \( I \triangleq M_{\top} \cap (\text{esc}^\top (R_s) \times (\text{dom}(r_s) \setminus R_s)) \) contains all the moves leading outside \( R_s \) that the opponent can use to escape. The procedure, then, promotes \( R_s \) to \( q \), by assigning at Line 2 the priority \( q \) to all the positions of \( R_s \) in the region function \( r_s \). Observe that, thanks to the \( \pi_s \)-maximality of the input state, \( q \) is necessarily congruent to \( \alpha_s \). In particular, when the only possibility for player \( \pi_s \) to escape from \( R_s \) is to reach \( r_s^{-1}(\top_{\alpha}) \), the value of \( q \) is \( \top_{\alpha} \). In this case, we are promoting \( R_s \) from the status of quasi \( \alpha \)-dominion to that of \( \alpha \)-dominion. The correctness of this is ensured by Theorem 2.

At this point, by defining \( \sigma_1 (\alpha) \triangleq (r_{-1}(\top_0), r_{-1}(\top_1)) \), where \( r_0 \triangleq \text{sol}(s_f) \), we obtain a sound and complete solution algorithm for parity games. In particular, the soundness follows from the fact that RPP always traverses states having as invariant the property that \( r_{-1}(\top_0) \) and \( r_{-1}(\top_1) \) are dominions (see Item 2 of Definition 4). Completeness, instead, is due to the recursive nature of the algorithm, whose base case ensures that no position is left unprocessed at any given priority.

The auxiliary function \( \text{NextPr} \) of the Parys Solver generates a new state by decreasing the current priority. This function is also called by another auxiliary function, the \( \text{hsol} \), that generates a new state with halved precision for the player \( \alpha \) and decreased priority.

As for the RPP solver, \( \text{Maximise} \) enforces the maximality property on current state \( s \). As a result, in the modified state no position of the local area \( L_\alpha \) can be attracted by the quasi
Priority Promotion with Parysian Flair

**Algorithm 8** Parys Solver

```plaintext
function sol(s: St): 2^Ps
  1. if p_s = ⊥ ∨ b_0s = 0 ∨ b_1s = 0 then
  2. return (r_s, u_s)
  3. else
  4. hsol(s)
  5. ̂s ← s
  6. (r_s, u_s) ← sol(NextPr(s))
  7. Maximise(s)
  8. if s < ̂s then hsol(s)
  9. return Und(s)
```

**Algorithm 10** Half-Solver

```plaintext
procedure hsol(s: St)
  1. repeat
  2. R_s ← atr_{G_s}^r(R_s)
  3. ̂s ← s
  4. (r_s, u_s) ← sol(Half(s))
  5. Maximise(s)
  6. until s ≠ ̂s
```

**Auxiliary Functions / Procedures I**

```plaintext
function NextPr(s: St): St
  1. return ((r_s, p_s - 1), (u_s, p_s, b_0s, b_1s))

function Half(s: St): St
  1. (b_0s, b_1s) ← (⌈ b_0s / 1 + α_s ⌉, ⌈ b_1s / 2 - α_s ⌉)
  2. return NextPr(s)
```

**Auxiliary Functions / Procedures II**

```plaintext
procedure Maximise(s: St)
  1. X ← atr_{H_s}(H^s, L_s)
  2. r_s ← r_s \ X
  3. u_s ← u_s[X ↦ p_s]

function Und(s: St): Rg × Pm
  1. r_s ← r_s^{\varphi_{G_{p_s}}}[v ∈ U_s ↦ pr(v)]
  2. u_s ← u_s[L_s ↦ p_s + 1]
  3. return (r_s, u_s)
```

α-dominions H\(^0_s\). To this end, the procedure computes at Line 1 the \(\varpi\)-attractor \(\text{atr}_{\varpi}^r(H^s, L_s)\), collecting all the position of \(L_s\) that player \(\varpi\) can force to move into \(H^s\). The attracted set is then assigned to \(r\) to a position in the attracting set is extracted at Line 3. The attracted positions are then assigned to \(u\) with the current priority \(p_s\).

Finally, the Und function, receive in input a state where all small dominions of player \(\overline{\varpi}_s\) of size \(≤ 2mc\), from the time of the call are processed. Therefore, none of this small dominions intersect with \(L_s\), that is in turn moved to the previous priority level. The positions of \(U_s\) can be instead reset in \(r\) to their original priority.

The auxiliary function NextPr generates a new maximal state \(\hat{s} = \text{NextPr}(s) \in S_{\text{tr}}\), starting from the strongly-maximal one \(s \in S_{\text{tr}}\) in input. The state \(\hat{s}\) is obtained by first setting the caller priority to the current one \(p_s\) and, then, by computing the highest priority \(q\) among the unprocessed positions in \(L_s \setminus R_s\). The promotion and maximisation procedures, \text{Promote} and \text{Maximise}, generalise the corresponding ones associated with RPP. The only difference is that here we need to determine which one, between the region function \(r\) and the promotion function \(u\), has to receive the promoted region \(Rg_s\), in case of \text{Promote}, or the positions attracted from \(L_s \cup U_s\), in case of \text{Maximise}. As before, \text{Promote} asks for the input state to be promotable, i.e., \(s \in S_{\text{tr}}\), while \text{Maximise} does not require any specific property on it.
Auxiliary Functions / Procedures I

function NextPr(s: Stₙ): Stₙ
1  q ← max(rₙ \cup \{cₙ\})
2  return ((rₙ, q), (uₛ, pₛ, bₖₛ, b₁ₛ))

function Half(s: Stₙ): Stₙ
1  (bₖₛ, b₁ₛ) ← (\[\frac{bₖ}{1+αₛ}, \frac{b₁}{2-αₛ}\])
2  return NextPr(s)

function Und(s: Stₙ): Rg×Pm
1  if cₛ ≡ 2 αₛ then
2      uₛ ← uₛ \{Uₛ → cₛ\}
3  else
4      rₛ ← rₛ \{≥ cₛ\}[v ∈ Uₛ → pr(v)]
5      uₛ ← uₛ \{≥ cₛ\}[Lₛ → cₛ]
6  return (rₛ, uₛ)

Auxiliary Functions / Procedures II

procedure Promote(s: Stₙ)
1  \{(pᵣ, pᵰᵣ)← (bepπₙ (Rₛ, rₛ), bepπₙ (Rₛ, uₛ))\}
2  if pᵣ ≤ pᵰᵣ then
3      rₛ ← rₛ \{Rₛ → pᵣ\}
4  else
5      (rₛ, uₛ) ← (rₛ \{Rₛ \rightarrow pᵣ\}, uₛ)

procedure Maximise(s: Stₙ)
1  Z ← Rₛ
2  foreach α ∈ B do
3      X ← atrⁿ(Hₛ \{Uₛ \cup Lₛ\})
4      q ← min(rₙ \cup uₛ) \{Hₛ \cup Lₛ\})
5  if q ≡ 2 α then
6      rₛ ← rₛ \{X → q\}
7      uₛ ← uₛ \{X\}
8  else
9      rₛ ← rₛ \{X \rightarrow pr(v)\}
10     uₛ ← uₛ \{X \rightarrow q\}
11     if Z ≠ Rₛ then rₛ ← rₛ \{v ∈ Lₛ \rightarrow pr(v)\}

Similarly to Paray's algorithm, the Half function halves the bound of the opponent player πₛ, leaving the bound of player αₛ unchanged. Finally, the Und function, starts from a strongly maximal state \(s ∈ Stₙ\) where all small dominions of player πₛ of size ≤ \(bπₙ\) from the time of the call are processed. Thus, neither do any of the small dominions (≤ \(bπₙ\)) of player πₛ intersect with \(Uₛ\), nor do any of the small dominions (≤ \(bπₙ\)) of player πₛ intersect with \(Lₛ\). Depending on the parity of the calling priority, we can then return the respective set and, where the parity is different, reset the positions of \(Uₛ\) in \(r\) to their original priority.
Table 2  Solution times in seconds on Keiren’s benchmarks (1012 games).

### E  Experimental Evaluation - Keiren’s Benchmarks

In this set of benchmarks we consider were first proposed in [36] and comprises a number of concrete verification problems, ranging from model-checking, to equivalence-checking and decision problems for different temporal logics. They can be divided in the following four categories.

**Model-checking benchmarks.** The first group contains 313 games, with size up to $O(10^7)$ positions. It includes a number of different verification problems. A first set contains encodings of a variety of communication protocols from [37, 14, 30, 2]: the alternating bit protocol, the positive acknowledgement with retransmission protocol, the bounded retransmission protocol, and the sliding window protocols. The protocols are parameterised with the number of messages to send and, when applicable, the window size. The set also contains verification problems for the cache coherence protocol of [63] and the wait-free handshake register of [31], as well as the classic elevator and towers-of-Hanoi benchmarks from [27]. The verification tasks under analysis cover fairness, liveness and safety properties. A second set, instead, contains encodings of two-player board games, such as Clobber, Domineering, Hex, Othello, and Snake, all parameterised by their board size. Here, the existence of a winning strategy for the game is the property considered. The encoding into parity games results in games with very few priorities: up to 4 in some cases.

**Equivalence checking benchmarks.** This group contains 216 games encoding equivalence tests between processes. The verification problems test various forms of process equivalences, such as strong, weak and branching bisimulation, as well as branching simulation. Most of the processes are the ones already considered in the model-checking benchmarks. The encoding into parity games results in games with at most two priorities, hence the only relevant measure of difficulty is the size, again reaching $O(10^7)$ nodes for the bigger instances.

**Decision problem benchmarks.** The third group contains encodings of satisfiability and validity problems for formulae of various temporal logics: LTL, CTL, CTL*, PDL and the $\mu$-CALCULUS, and comprises 192 games. The maximal size of a benchmark is around $3 \cdot 10^6$ positions. The parity games encoding have been obtained with the tool MLSolver [28]. The situation here is more interesting, since these concrete problems feature a higher number of priority, up to 20 in few cases. Hence, unlike the previous two groups, these benchmarks allow us to stress a bit more the scalability of the solution algorithms w.r.t. the increase in priorities.

**PGSolver.** This group contains 291 synthetic benchmarks, corresponding to known families.
of hard cases for specific solvers and randomly generated ones. The sizes and number of priorities vary significantly, depending on the specific class of games.

Table 2 reports the results of the experiments for all the solvers considered in the analysis, divided by class of benchmarks\(^4\). For each solver, the total completion time, the average time per benchmark and the percentage of timed-out executions are given. We set a timeout of 10 seconds for all the benchmarks, except for the equivalence-check class, for which 40 seconds is used instead. As expected, the exponential solvers perform better on all the classes, with PP taking the lead most of the time. SSPM and QPT both perform quite poorly, between two and three orders of magnitude worse than the other solvers, and do not seem to scale beyond the simplest instances, as also evidenced by the high number of timeouts. Both Par and HPP, instead, perform relatively well in all the benchmarks, being able to solve all the instances without incurring in timeouts and maintaining a short distance from the exponential solvers performance-wise. Par has a slight edge over HPP on the model-checking and equivalence checking problems, both of which feature a very low number of priorities, though the time advantage on average is typically negligible. On the other hand, when the number of priorities increases, like in the decision problems, the situation reverses and HPP takes the lead over Par and practically matches the performance of the exponential solvers. This seems to suggest that HPP may scale better w.r.t. the number of priorities in the games. To further investigate this behaviour we decided to perform additional experiments, whose results are reported in the next subsection.

\(^4\) The benchmarks were run by issuing the following OINK commands: `oink –no-single –no-loops –no-wcwc GameName SolverName`; where solvers are: ZLK, NPP, SSPM, QPT, ZLKQ.