TEST ELEMENTS, RETRACTS AND AUTOMORPHIC ORBITS

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Abstract. Let $A_2$ be a free associative or polynomial algebra of rank two over a field $K$ of characteristic zero. Based on the degree estimate of Makar-Limanov and J.-T.Yu, we prove: 1) An element $p \in A_2$ is a test element if $p$ does not belong to any proper retract of $A_2$; 2) Every endomorphism preserving the automorphic orbit of a nonconstant element of $A_2$ is an automorphism.

1. Introduction and main results

In the sequel, $K$ always denotes a field of of characteristic zero. Automorphisms (endomorphisms) always mean $K$-automorphisms ($K$-endomorphisms).

Let $A_n$ be a free associative or polynomial algebra of rank $n$ over $K$. An element $p \in A_n$ is called a test element if every endomorphism of $A_n$ fixing $p$ is an automorphism. A subalgebra $R$ of $A_n$ is called a retract if there is an idempotent endomorphism $\pi (\pi^2 = \pi)$ of $A_n$ (called a retraction or a projection) such that $\pi(A_n) = R$. Test elements and retracts of groups and other algebras are defined in a similar way. Test elements and retracts of algebras and groups have recently been studied in [3, 5, 6, 7, 12, 18, 20, 21, 22, 23, 24, 29, 30, 32, 33].

A test element does not belong to any proper retract for any algebra or group as the corresponding non-injective idempotent endomorphism is not an automorphism. The converse is proved by Turner [34] for free groups, by Mikhailov and Zolotykh [24] and by Mikhailov and J.-T. Yu [21, 22] for free Lie algebras and free Lie superalgebras respectively.
and by Mikhalev, Umirbaev and J.-T. Yu [19] for free nonassociative algebras. See also Mikhalev, Shpilrain and J.-T. Yu [18].

In view of the above, we may raise the following

**Conjecture 1.** If an element \( p \in A_n \) does not belong to any proper retract of \( A_n \), then \( p \) is a test element.

Recently, V. Shpilrain and J.-T. Yu [33] proved Conjecture 1 for \( \mathbb{C}[x, y] \). A key lemma in their proof is the degree estimate of Shestakov and Umirbaev [26], which plays a crucial role in the recent celebrated solution of the Nagata conjecture [27, 28] and the Strong Nagata conjecture [35].

More recently, Makar-Limanov and J.-T. Yu [16] developed a new combinatorial method based on the Lemma on radicals and obtained a sharp degree estimate for the ‘free’ case, namely, for a free associative algebra or a polynomial algebra over a field of characteristic zero. It has found applications for automorphisms and coordinates of polynomial and free associative algebras. See S.-J. Gong and J.-T. Yu [9].

Now we consider another related problem. In an algebra or a group, certainly an automorphism preserves the automorphic orbit of an element \( p \). The converse is proved by Shpilrain [31] and Ivanov [10] for free groups of rank two, by D. Lee [14] for free groups of any rank, by Mikhalev and J.-T. Yu [22] for free Lie algebras and by Mikhalev, Umirbaev and J.-T. Yu [19] for free non-associative algebras, by van den Essen and Shpilrain [7] for \( A_2 \) when \( p \) is a coordinate, by Jelonek [11] for polynomial algebras over \( \mathbb{C} \) when \( p \) is a coordinate. For the related linear coordinate preserving problem, see, for instance, S.-J. Gong and J.-T. Yu [8]. See also the book [18].

In view of the above, we may raise the following

**Conjecture 2.** Let \( p \in A_n - K \). Then any endomorphism of \( A_n \) preserving the automorphic orbit of \( p \) must be an automorphism.

Conjecture 2 has recently been settled affirmatively by J.-T. Yu [36] for \( A_2 = \mathbb{C}[x, y] \) based on Shpilrain and J.-T.Yu’s characterization of test elements of \( \mathbb{C}[x, y] \) in [33] and the main result in Drensky and J.-T.Yu [6].
In this paper, based on the recent degree estimate of Makar-Limanov and J.-T.Yu [16], the main ideals and techniques in Drensky and J.-T.Yu [6], Shpilrain and J.-T. Yu [32, 33], and J.-T.Yu [36], we prove both Conjecture 1 and Conjecture 2 for \( n = 2 \). Our main results are

**Theorem 1.1.** If an element \( p \in A_2 \) does not belong to any proper retract of \( A_2 \), then \( p \) is a test element of \( A_2 \).

Theorem 1.1 was proved by Shpilrain and J.-T.Yu [33] for \( A_2 = \mathbb{C}[x, y] \).

**Theorem 1.2.** If an endomorphism \( \phi \) of \( A_2 \) preserves the automorphic orbit of a nonconstant element \( p \in A_2 \), then \( \phi \) is an automorphism of \( A_2 \).

Theorem 1.2 was proved by J.-T.Yu [36] for \( A_2 = \mathbb{C}[x, y] \).

Crucial to the proofs of the above two theorems are the following two results, which have their own interests.

**Theorem 1.3.** Let \( p \in A_2 \) have outer rank two. Then any injective endomorphism \( \phi \) of \( A_2 \) is an automorphism if \( \phi(p) = p \).

Theorem 1.3 may be viewed as an analogue of a result in Turner [34] for free groups. It was proved for \( A_2 = \mathbb{C}[x, y] \) in J.-T.Yu [36] based on a result in Shpilrain and J.-T.Yu [33].

**Theorem 1.4.** An element \( p(x, y) \in A_2 \) belongs to a proper retract of \( A_2 \) if \( p(x, y) \) is fixed by a noninjective endomorphism \( \phi \) of \( A_2 \). Moreover, in this case there exists a positive integer \( m \) such that \( \phi^m \) is a retraction of \( A_2 \).

Theorem 1.4 was proved for \( A_2 = \mathbb{C}[x, y] \) in Drensky and J.-T.Yu [6].

2. **Proofs**

The following two lemmas are Theorem 1.1 and Proposition 1.2 in Makar-Limanov and J.-T.Yu [16].

**Lemma 2.1.** Let \( A_n = K\langle x_1, \ldots, x_n \rangle \) be a free associative algebra over a field \( K \) of characteristic zero, \( f, g \in A \) be algebraically independent, \( f^+ \) and \( g^+ \) are algebraically independent, or \( f^+ \) and \( g^+ \) are algebraically dependent and neither \( \deg(f) | \deg(g) \) nor \( \deg(g) | \deg(f) \), \( p \in K\langle x, y \rangle \). Then

\[
\deg(p(f, g)) \geq \frac{\deg(f, g)}{\deg(fg)} w_{\deg(f), \deg(g)}(p).
\]
Here \( \text{deg} \) is the total degree, \( w_{\text{deg}(f),\text{deg}(g)}(p) \) is the weighted degree of \( p \) when the weight of the first variable is \( \text{deg}(f) \) and the weight of the second variable is \( \text{deg}(g) \), \( f^+ \) and \( g^+ \) are the highest homogeneous components of \( f \) and \( g \) respectively, and \([f,g] = fg - gf\) is the commutator of \( f \) and \( g \).

**Lemma 2.2.** Let \( A_n = K[x_1, \cdots, x_n] \) be a polynomial algebra over a field \( K \) of characteristic zero, \( f, g \in A \) be algebraically independent, \( p \in K[x, y] \). Then

\[
\text{deg}(p(f, g)) \geq w_{\text{deg}(f),\text{deg}(g)}(p)[1 - \frac{(\text{deg}(f),\text{deg}(g))(\text{deg}(fg) - \text{deg}(J(f,g)) - 2)}{\text{deg}(f) \text{deg}(g)}].
\]

Here \( \text{deg} \) is the total degree, \( w_{\text{deg}(f),\text{deg}(g)}(p) \) is the weighted degree of \( p \) when the weight of the first variable is \( \text{deg}(f) \) and the weight of the second variable is \( \text{deg}(g) \), \( (\text{deg}(f),\text{deg}(g)) \) is the greatest common divisor of \( \text{deg}(f) \) and \( \text{deg}(g) \), \( \text{deg}(J(f,g)) \) is the largest degree of non-zero Jacobian determinants of \( f \) and \( g \) with respect to two of \( x_1, \cdots, x_n \).

The following characterization of a proper retract of \( A_2 \) was obtained by Shpilrain and J.-T. Yu [32] based on a result of Costa [3].

**Lemma 2.3.** Let \( R \) be a proper retract of \( A_2 \). Then \( R = K[r] \) for some \( r \in A_2 \). Moreover, there exists an automorphism \( \alpha \) of \( A_2 \) such that \( \alpha(r) = x + w(x, y) \), where \( w(x, y) \) belongs to the ideal of \( A_2 \) generated by \( y \).

**Lemma 2.4.** Let \( p \in A_2 \) with outer rank 2 and \( f, g \in A_n \). Then \( w_{\text{deg}(f),\text{deg}(g)}(p) \geq \text{deg}(f) + \text{deg}(g) \). If every monomial of \( p \) contains both \( x \) and \( y \) and \( \text{deg}(p) > 2 \), then \( w_{\text{deg}(f),\text{deg}(g)}(p) > \text{deg}(f) + \text{deg}(g) \).

**Proof.** 1) If \( p \) contains a monomial containing both \( x \) and \( y \), where \( i \neq 0, j \neq 0 \), \( w_{\text{deg}(f),\text{deg}(g)}(p) \geq i(\text{deg}(f)) + j(\text{deg}(g)) \geq \text{deg}(f) + \text{deg}(g) \).

If every monomial of \( p \) contains both \( x \) and \( y \) and \( \text{deg}(p) > 2 \), then the second inequality becomes strict.

2) Otherwise \( p \) must contain monomials \( x^i \) and \( y^j \) where \( i \geq 2, j \geq 2 \). Then \( w_{\text{deg}(f),\text{deg}(g)}(p) \geq 2 \max\{\text{deg}(f), \text{deg}(g)\} \geq \text{deg}(f) + \text{deg}(g) \). \qed

**Lemma 2.5.** Let \( A_n = K\langle x_1, \cdots, x_n \rangle \) be a free associative algebra over an arbitrary field \( K \) of zero characteristic, \( f, g \in A_2 \) be algebraically independent, \( p \in K\langle x, y \rangle \) have outer rank two. Then

\[
\text{deg}(p(f, g)) \geq \text{deg}[f, g].
\]
If every monomial of $p$ contains both $x$ and $y$ and $\deg(p) > 2$, then
\[
\deg(p(f, g)) > \deg(f, g).
\]

Proof. Let 1) If $f^+$ and $g^+$ are algebraically independent; or $f^+$, $g^+$ are algebraically dependent, but $\deg(f) \nmid \deg(g)$ and $\deg(g) \nmid \deg(f)$. Then by Lemma 2.1 and Lemma 2.4, $\deg(p(f, g)) \geq \deg(f, g)$. If, in addition, every monomial of $p$ contains both $x$ and $y$ and $\deg(p) > 2$, then by Lemma 2.1 and Lemma 2.4, $\deg(p(f, g)) > \deg(f, g)$.

2) Otherwise there exists an automorphism $\alpha$, which is the composition of a sequence of elementary automorphisms, such that $\alpha(f) = \tilde{f}$, $\alpha(g) = \tilde{g}$, $p = \alpha^{-1}(p)$ satisfying the condition in 1). Then $\deg(p(f, g)) = \deg(\tilde{f}, \tilde{g}) \geq \deg(f, g)$.

Lemma 2.6. Let $A_n = K[x_1, \ldots, x_n]$ be a polynomial algebra over an arbitrary field $K$ of zero characteristic, $f, g \in A_n$ be algebraically independent, $p \in K[x, y]$ has outer rank two. Then
\[
\deg(p(f, g)) \geq \deg(J(f, g)) + 2.
\]

Proof. We may assume $\deg(f) = m, \deg(g) = n$. As $p$ has outer rank 2, by Lemma 2.4 then $p$ contains a monomial with both $x$ and $y$, or contains monomials $x^i$ and $y^j$ where $i \geq 2, j \geq 2$.

1) Let $f^+$ and $g^+$ be algebraically independent.
   a) If there exists a monomial in $p$ containing both $x$ and $y$, then $\deg(p(f, g)) \geq \deg(f) + \deg(g) \geq \deg(J(f, g)) + 2$;
   b) Otherwise $p$ must have a monomial of $x^i$ where $i \geq 2$, and another monomial $y^j$ where $j \geq 2$, then $\deg(p(f, g)) \geq 2 \max\{m, n\} \geq \deg(f) + \deg(g) \geq \deg(J(f, g)) + 2$;

2) Let $f^+, g^+$ be algebraically dependent, and $m \nmid n$ and $n \nmid m$.
   c) If $w_{\deg(f), \deg(g)}(p) < \lcm(m, n)$, then in $p(f, g)$, $f^+$ and $g^+$ cannot cancel out, hence similar to the case 1 a), $\deg(p(f, g)) \geq \deg(f) + \deg(g) \geq \deg(J(f, g)) + 2$.
   d) Otherwise $w_{\deg(f), \deg(g)}(p) \geq \lcm(m, n) = mn/(m, n)$. We also have $mn = (m, n)\lcm(m, n) \geq (m, n)(m + n)$. Hence $\deg(p(f, g)) \geq \deg(J(f, g)) + 2$ by Lemma 2.2.

3) Let $f^+, g^+$ be algebraically dependent, but $m \mid n$ or $n \mid m$. Then by same process in the Proof 2) of Lemma 2.4, we may reduce to the above case 1) or case 2).

Lemma 2.7. Let $\phi = (f, g)$ be an injective endomorphism of $K\langle x, y \rangle$ but not an automorphism. Then $\deg([\phi^k(x), \phi^k(y)]) \geq k + 2$ for $k \geq 0$. 

Proof. We use induction. \(\deg[\phi^0(x), \phi^0(y)] = \deg[x, y] = 0 + 2\). Assuming \(\deg[\phi^{k-1}(x), \phi^{k-1}(y)] \geq (k-1)+2\). Define \(p(x, y) := [f(x, y), g(x, y)]\). As \(\phi = (f, g)\) is injective, every monomial of \(p(x, y)\) contains both \(x\) and \(y\). Since \(\phi = (f, g)\) is not an automorphism, by the well-known result of Dicks (see, Dicks \[4\], or Cohn \[2\]), \(\deg[p(x, y)] > \deg(x) + \deg(y) = 2\). Applying Lemma 2.5. \(\deg(p(u, v)) > \deg[u, v]\) for \(u = \phi^{k-1}(x), v = \phi^{k-1}(y)\), hence
\[
\deg[\phi^k(x), \phi^k(y)] = \deg(p(\phi^{k-1}(x), \phi^{k-1}(y))) > \deg[\phi^{k-1}(x), \phi^{k-1}(y)] \geq (k-1)+2 = k+1.
\]
Therefore, \(\deg[\phi^k(x), \phi^k(y)] \geq (k+1)+1 = k+2\). \(\square\)

Lemma 2.8. Let \(\phi = (f, g)\) be an injective endomorphism of \(K[x, y]\) but not an automorphism and there exists an element \(p \in K[x, y]\) fixed by \(\phi\). Then \(\deg(J(\phi^k(x), \phi^k(y))) \geq k\) for \(k \geq 0\).

Proof. As \(\phi\) fixes \(p\), \(\phi\) is not an automorphism, by a result of Kraft \[13\] (see also Shpilrain and J.-T. Yu \[32\]), \(\deg(J(\phi(x), \phi(y))) = \deg(J(f, g)) \geq 1\). By the chain rule for the Jacobian,
\[
\deg(J(\phi^k(x), \phi^k(y))) = \deg(J(f, g))(\phi^{k-1}(x), \phi^{k-1}(y))(J(\phi^{k-1}(x), \phi^{k-1}(y))) \\
\geq \deg(J(\phi^{k-1}(x), \phi^{k-1}(y))) + 1.
\]
The proof is concluded by induction. \(\square\)

Lemma 2.9. Let \(\phi = (f, g)\) be an injective endomorphism of \(A_2\) but not an automorphism. Then any element \(p \in A_2\) with outer rank 2 cannot be fixed by \(\phi\).

Proof. If \(p \in A_2\) with outer rank two fixed by \(\phi\), then \(\deg(p(f, g)) = \deg(p(\phi^k(x), \phi^k(y))) \geq k + 2\) for all \(k \geq 0\). By Lemma 2.5 and Lemma 2.7 for noncommutative case; and by Lemma 2.6 and Lemma 2.8 for polynomial case. The contradiction completes the proof. \(\square\)

Proof of Theorem 1.3.

By Lemma 2.9 \(\square\)

Proof of Theorem 1.4.

The proof presented here is similar to the proof of the main Theorem in Drensky and J.-T. Yu \[6\].

Let \(p \in A_2 - \{0\}\) fixed by a noninjective endomorphism of \(A_2\). Then \(\phi(x)\) and \(\phi(y)\) are algebraically dependent over \(K\). Let us denote the image of \(\phi(A_2)\) by \(S = K[\phi(x), \phi(y)]\) (since \(\phi(x)\) and \(\phi(y)\) are algebraically dependent, \(\phi(x)\) and \(\phi(y)\) are in a polynomial algebra of
rank one over $K$ as a consequence of a result of Bergman [1] for noncommutative case and as a consequence of a result of Shestakov and Umirbaev [20] for polynomial case) and by $Q(S)$ the field of fractions of $S$. Therefore the transcendence degree of $Q(S)$ over $K$ is 1. Let $0 \neq q(x, y) \in (\text{Ker}(\phi)) \cap S$. Since $p(x, y)$ also belongs to $S$, the polynomials $p$ and $q$ are algebraically dependent and

$$h(p, q) = a_0(q)p^n + a_1(q)p^{n-1} + \ldots + a_{n-1}(q)p + a_n(q) = 0$$

for an irreducible polynomial $h(u, v) \in K[u, v]$ and $a_i(t) \in K[t]$, $i = 0, 1, \ldots, n$. Hence $\phi(h(p, q)) = h(\phi(p), \phi(q)) = h(p, 0)$,

$$a_0(0)p^n + a_1(0)p^{n-1} + \ldots + a_{n-1}(0)p + a_n(0) = 0.$$ 

Therefore $a_0(0) = a_1(0) = \ldots = a_n(0) = 0$. Now the polynomials $a_i(t)$ have no constant terms and $h(u, v)$ is divisible by $v$ which contradicts to the irreducibility of $h(u, v)$. Therefore $(\text{Ker}(\phi)) \cap S = 0$ and $\phi$ acts injectively on its image $S$. Hence we may extend the action of $\phi$ on $Q(S)$ (because $a_1/b_1 = a_2/b_2$ in $Q(S)$ is equivalent to $a_1b_2 = a_2b_1$ and hence $\phi(a_1/b_1) = \phi(a_1)/\phi(b_1) = \phi(a_2)/\phi(b_2) = \phi(a_2/b_2)$). By Lüroth’s theorem (See, for instance, Schinzel [25]), $Q(S) = K(w)$ for some $w \in Q(S)$. The automorphism $\phi$ fixes $p(x, y)$ and its extension $\bar{\phi}$ on $Q(S)$ fixes $K(p)$. Since $w$ is algebraic over $K(p)$, $Q(S)$ is a finite dimensional vector space over $K(p)$ and $\bar{\phi}$ is a $K(p)$-linear operator of $Q(S)$ with trivial kernel. Hence $\bar{\phi}$ is invertible on $Q(S)$ and we may consider $\bar{\phi}$ as an automorphism of the finite field extension $Q(S)$ over $K(p)$ which fixes $K(p)$. By Galois theory ($\bar{\phi}$ interchanges the roots of the minimal polynomial of $w$ over $K(p)$ and there are finite number of possibilities for $\bar{\phi}(w)$), $\bar{\phi}$ has finite order. Let $\bar{\phi}^m \neq 1$. Then $\phi^{m+1}(r) = \phi^m(\phi(r)) = \bar{\phi}^m(\phi(r)) = \phi(r)$ for every $r \in A_2$ and $(\phi^m)^2 = \phi^{m+1}\phi^{m-1} = \phi\phi^{m-1} = \phi^m$. Therefore $\pi = \phi^m$ is a retraction (idempotent endomorphism) of $A_2$ with a nontrivial kernel and $\pi(p) = p$. Hence $p(x, y)$ is in the image of $\pi$ which is a proper retract $\pi(A_2)$ of $A_2$. \(\square\)

**Proof of Theorem 1.1.**

As $p \in A_2$ does not belong to any proper retract of $A_2$, by Theorem 1.4 any endomorphism $\phi$ of $A_2$ fixing $p$ must be injective. By Lemma 2.3, obviously $p$ must have outer rank two, otherwise $p$ would belong to a proper retract of $A_2$. By Theorem 1.3 $\phi$ is an automorphism. Hence $p$ is a test element of $A_2$. \(\square\)

**Proof of Theorem 1.2.**
The proof presented here is similar to the proof of the main result Theorem 1.4 in J.-T. Yu [36].

We may assume that \( \phi(p) = p \). By the definition of the test element, we may assume \( p \) is not a test element. By Theorem 1.4, we may assume \( p \) belongs to a proper retract \( K[r] \) of \( A_2 \). By a result in J.-T. Yu [36], we may assume \( p \) has outer rank 2. By Theorem 1.3, we may assume \( \phi \) is non-injective. Suppose that \( p = f(r) \), where \( f \in K[t] - K \), \( \deg(f) = m \). By Theorem 1.4, \( \pi = \phi^m \) is a retraction of \( A_2 \) onto \( K[r] \). As \( \phi \) preserves the automorphic orbit of \( p \), so does \( \pi = \phi^m \). Applying Lemma 2.3 (suppose \( \alpha(r) = x + w(x, y) \), where \( w(x, y) \notin K[y] \) belongs to the ideal of \( A_2 \) generated by \( y \), \( \alpha \) is some automorphism of \( A_2 \), replace \( r \) by \( \alpha(r) \), and \( \pi \) by \( \alpha \pi \alpha^{-1} \)), we have reduced our proof to the following

**Lemma 2.10.** Let \( r = x + w(x, y) \), where \( w(x, y) \) belongs to the ideal of \( A_2 \) generated by \( y \) and \( w(x, y) \notin K[y] \), \( \pi \) the retraction of \( A_2 \) onto \( K[r] \) defined by \( \pi(x) = x + w(x, y) \), \( \pi(y) = 0 \), \( f \in K[t] - K \). Then \( \pi \) does not preserve the automorphic orbit of \( f(r) \).

**Proof.** Suppose on the contrary, \( \pi \) preserves the automorphic orbit of \( f(r) \). Then for any automorphism \( \alpha \) of \( A_2 \), \( \pi \alpha(f(r)) = \beta(f(r)) \in K[r] \) for some automorphism \( \beta \) of \( A_2 \). Note that \( \pi \beta(f(r)) = \pi^2 \beta(f(r)) = \pi \alpha(f(r)) = \beta(f(r)) \). By Theorem 1.4, \( \pi^{\deg(f)} = \pi \) is the retraction of \( A_2 \) onto the retract \( K[\beta(r)] \) taking \( \beta(r) \) to \( \beta(r) \). By hypothesis, \( \pi \) is also a retraction of \( A_2 \) onto the retract \( K[r] \) taking \( r \) to \( r \). This forces that \( \beta(r) = cr + d \) for some \( c \in K^* \), \( d \in K \). We have concluded that for any automorphism \( \alpha \) of \( A_2 \), there exists some \( c \in K^* \), \( d \in K \), such that \( \pi \alpha(f(r)) = f(cr + d) \).

Now we proceed the proof in two cases.

1. **Noncommutative case:** \( A_2 = K\langle x, y \rangle \).

   Denote by \( C \) the commutator ideal of \( K\langle x, y \rangle \).

   a) If \( w(x, y) \in C \), then take \( \alpha \) to be the automorphism of \( K\langle x, y \rangle \) defined by \( \alpha(x) = y + x^2 \), \( \alpha(y) = x \). Direct calculation shows that \( \pi \alpha(f(r)) = f(r^2 + w(r^2, r)) = f(r^2) \neq f(cr + d) \), a contradiction.

   b) If \( w(x, y) \notin C \), then \( w^a(x, y) = yv(x, y) \) for some \( v(x, y) \in K[x, y] - \{0\} \). Here \( w^a(x, y) \in K[x, y] \) is the image of \( w(x, y) \) under the abelianization from \( K\langle x, y \rangle \) onto \( K[x, y] \). Let \( M \) be a positive integer greater than \( \deg(v(x, y)) \), it is easy to see that \( x^M - y \) does not
divide $v(x,y)$ in $K[x,y]$. Let $\alpha$ be the automorphism of $K\langle x, y \rangle$ defined by $\alpha(x) = x$, $\alpha(y) = y + x^M$. Then $\pi\alpha(f(r)) = f(r + w(r,r^M)) = f(r + r^M v(r,r^M))$. As $x^M - y$ does not divide $v(x,y)$, $v(r,r^M) \neq 0$. Therefore $\pi\alpha(f(r)) = f(r + r^M v(r,r^M)) \neq f(cr+d)$, a contradiction.

2. Polynomial case: $A_2 = K[x,y]$.

In this case we write $w(x,y) = yq(x,y)$ where $q(x,y) \notin K[y]$. Let $M$ be a positive integer greater than $\deg(q(x,y))$, it is easy to see that $x^M - y$ does not divide $q(x,y)$ in $K[x,y]$. Let $\alpha$ be the automorphism of $K[x,y]$ defined by $\alpha(x) = x$, $\alpha(y) = y + x^M$. Then easy calculation shows that $\pi\alpha(f(r)) = f(r + r^M q(r,r^M))$. As $x^M - y$ does not divide $q(x,y)$, $q(r,r^M) \neq 0$. Therefore $\pi\alpha(f(r)) = f(r + r^M q(r,r^M)) \neq f(cr+d)$. The contradiction completes the proof. \hfill $\Box$

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References

[1] G. Bergman, Centralizers in free associative algebras, Trans. Amer. Math. Soc. 137 (1969), 327-344.
[2] P. M. Cohn, Free Rings and Their Relations, 2nd Edition, London Mathematical Society Monograph, 19, Academic Press, Inc. London, 1985.
[3] D. Costa, Retracts of polynomial rings, J. Algebra 44 (1977), 492-502.
[4] W. Dicks, A commutator test for two elements to generate the free algebra of rank two, Bull. London Math. Soc. 14 (1982), 48-51.
[5] V. Drensky, J.-T. Yu, Test Polynomials for automorphisms of polynomial and free associative algebras, J. Algebra 207 (1998), 491-510.
[6] V. Drensky, J.-T. Yu, Retracts and test polynomials of polynomial algebras, C. R. Acad. Bulgaria Sci. 55 (7) (2002), 11-14.
[7] A. van den Essen, V. Shpilrain, Some combinatorial questions about polynomial mappings, J. Pure Appl. Algebra 119 (1997), 47-52.
[8] S.-J. Gong, J.-T. Yu, The linear coordinate preserving problem, Comm. Algebra, to appear.
[9] S.-J. Gong, J.-T. Yu, The preimage of a coordinate, Algebra Colloquium, to appear.
[10] S. Ivanov, On endomorphisms of free groups that preserve primitivity, Arch. Math. 72 (1999), 92-100.
[11] Z. Jelonek, A solution of the problem of van den Essen and Shpilrain, J. Pure Appl. Algebra 137 (1999), 49-55.
[12] Z. Jelonek, Test polynomials, J. Pure Appl. Algebra 147 (2000), 125-132.
[13] H. Kraft, *On a question of Yosef Stein*, in ‘Automorphisms of Affine Spaces (Curacao, 1994)’ Proceedings of the Conference on Invertible Polynomial Maps held in Curacao, July 4-8, 1994. Ed. A. van den Essen, Kluwer Acad. Publ., Dordrecht, 1995, 225-229.

[14] D. Lee, *Endomorphisms of free groups that preserve automorphic orbits*, J. Algebra 248 (2002), 230-236.

[15] L. Makar-Limanov, *On automorphisms of free algebra with two generators*, Funk. Analiz. Prilozh. 4 (1970), no. 3, 107-108. English translation: Funct. Anal. Appl. 4 (1970), 262-264.

[16] L. Makar-Limanov, J. -T. Yu, *Degree estimate for two-generated subalgebras*, J. Euro. Math. Soc. 10 (2008), 533-541.

[17] A. A. Mikhalev, V. Shpilrain, J. -T. Yu, *On endomorphisms of free algebras*, Algebra Colloq. 6 (1999), 241-248.

[18] A. A. Mikhalev, V. Shpilrain, J. -T. Yu, Combinatorial Methods: Free Groups, Polynomials, and Free Algebras, CMS Books in Mathematics, Springer, New York, 2004.

[19] A. A. Mikhalev, U. U. Umirbaev, J. -T. Yu, *Automorphic orbits of elements of free non-associative algebras*, J. Algebra 243 (2001), 198-223.

[20] A. A. Mikhalev, U. U. Umirbaev, J. -T. Yu, *Generic, almost primitive and test elements of free Lie algebras*, Proc. Amer. Math. Soc. 130 (2002), 1303–1310.

[21] A. A. Mikhalev, J. -T. Yu, *Test elements and retracts of free Lie algebras*, Commun. Algebra 25 (1997), 3283-3289.

[22] A. A. Mikhalev, J. -T. Yu, *Test elements, retracts and automorphic orbits of free algebras*, Intern. J. Algebra Comput. 8 (1998), 295-310.

[23] A. A. Mikhalev, J. -T. Yu, *Primitive, almost primitive, test, and ∆-primitive elements of free algebras with the Nielsen-Schreier property*, J. Algebra 228 (2000), 603–623.

[24] A. A. Mikhalev, A. A. Zolotykh, *Test elements for monomorphisms of free Lie algebras and Lie superalgebras*, Commun. Algebra 23 (1995), 4995-5001.

[25] A. Schinzel, *Polynomials with Special Regard to Reducibility*, Encyclopedia of Mathematics, Cambridge University Press, Cambridge, 2000.

[26] I. P. Shestakov, U. U. Umirbaev, *Poisson brackets and two-generated subalgebras of rings of polynomials*, J. Amer. Math. Soc. 17 (2004), 181-196.

[27] I. P. Shestakov, U. U. Umirbaev, *The tame and the wild automorphisms of polynomial rings in three variables*, J. Amer. Math. Soc. 17 (2004), 197-227.

[28] I. P. Shestakov, U. U. Umirbaev, *The Nagata Automorphism is Wild*, Proc. Nat. Acad. Sci. 100 (2003), No. 22, 12561-12563.

[29] V. Shpilrain, *Recognizing automorphisms of the free groups*, Arch. Math. 62 (1994), 385–392.

[30] V. Shpilrain, *Test elements for endomorphisms of free groups and algebras*, Israel. J. Math. 92 (1995), 307-316.

[31] V. Shpilrain, *Generalized primitive elements of a free group*, Arch. Math. 71 (1998), 270-278.

[32] V. Shpilrain, J. -T. Yu, *Polynomial retracts and the Jacobian conjecture*, Trans. Amer. Math. Soc. 352 (2000), 477-484.

[33] V. Shpilrain, J. -T. Yu, *Test polynomials, retracts, and the Jacobian conjecture*, in Affine Algebraic Geometry, Contemp. Math. 369 (2005), 253-259, Amer. Math. Soc. Series, Providence, RI.

[34] E. Turner, *Test words for automorphisms of free groups*, Bull. London Math. Soc. 28 (1996), 255-263.

[35] U. U. Umirbaev, J. -T. Yu, *The Strong Nagata Conjecture*, Proc. Nat. Acad. Sci. 101 (2004), No. 13, 4352-4355.
[36] J. -T. Yu, Automorphic orbit problem for polynomial algebras, J. Algebra 319 (2008), 966-970.

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