Optimal Tracking Portfolio with A Ratcheting Capital Benchmark

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Abstract

This paper studies the finite horizon portfolio management by optimally tracking a ratcheting capital benchmark process. To formulate such an optimal tracking problem, we envision that the fund manager can dynamically inject capital into the portfolio account such that the total capital dominates the nondecreasing benchmark floor process at each intermediate time. The control problem is to minimize the cost of the accumulative capital injection. We first transform the original problem with floor constraints into an unconstrained control problem, however, under a running maximum cost. By identifying a controlled state process with reflection, we next transform the problem further into an equivalent auxiliary problem, which leads to a nonlinear Hamilton-Jacobi-Bellman (HJB) with a Neumann boundary condition. By employing the dual transform, the probabilistic representation approach and some stochastic flow arguments, the existence of the unique classical solution to the dual HJB is established. The verification theorem is carefully proved, which gives the complete characterization of the primal value function and the feedback optimal portfolio.

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1 Introduction

Portfolio allocation with benchmark performance has been an active research topic in recent years, see some related work in Browne (2000), Gaivoronski et al. (2005), Yao et al. (2006), Strub and Baumann (2018) and many others. The target benchmark is either a prescribed capital process or a fixed portfolio in the financial market, and the goal is to choose the portfolio in a passive way to dynamically track the return or the value of the benchmark process. In practice, both professional and individual investors may measure their portfolio performance using different benchmarks, such as S&P500 index, Goldman Sachs commodity index, special liability, inflation and exchange rates. Some dominating mathematical problems in the existing studies are to minimize the difference between the controlled portfolio and the benchmark as either a linear quadratic control problem using the mean-variance analysis or a utility maximization problem at the terminal time. In the present paper, we aim to enrich the research on optimal tracking by formulating a different tracking procedure and examining the associated control problem. Taking a fund portfolio management for instance, we assume that the fund manager can dynamically inject capital into the portfolio account such that the total capital stays above the benchmark process as an American type floor constraint at each intermediate time. The control problem combines the regular portfolio control and the singular capital injection control and the optimality is attained when the cost from the accumulative

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capital injection is minimized. In particular, we are interested in the case when the benchmark process in nondecreasing, representing a ratcheting capital that the fund manager wants to closely follow.

On the other hand, another well known optimal tracking problem in the literature is the monotone follower problem, see for instance Karatzas and Shreve (1984) and Bayraktar and Egami (2008), in which one needs to choose a monotone process as a singular control to closely track a given diffusion process such as a Brownian motion with drift. Our problem formulation focuses on the opposite side as we look for a regular control such that the controlled diffusion process can closely track a given monotone process. It is mathematically new and appealing to study this type of tracking problem and characterize the value function and the optimal portfolio.

Our optimal tracking problem is also motivated by some stochastic control problems with a minimum guaranteed floor constraint, which is usually defined as a utility maximization problem such that the controlled wealth process dominates an exogenous deterministic or stochastic process at the terminal time or at each intermediate time. See some related work among El Karoui et al. (2005), El Karoui and Meziou (2006), Bouchard et al. (2010), Di Giacinto et al. (2011), Sekine (2012), Di Giacinto et al. (2014) and Chow et al. (2020), in which European type floor constraint or dynamic American type floor constraint have been investigated in various market models. In the aforementioned research, some typical techniques to handle the floor constraint are to introduce the option based portfolio or the insured portfolio allocation such that the floor constraint is guaranteed. Contrary to these work, in the first step, we instead reformulate the optimal tracking problem with dynamic floor constraints into a constraint-free stochastic control problem under a running maximum cost criterion, see Lemma 2.2.

Stochastic control with a running maximum cost or a running maximum process is itself an interesting topic and attracts a lot of attention in the past decades, see Barron and Ishii (1989), Barron (1993), Barles et al. (1994), Bokanowski et al. (2015) and Kröner et al. (2018) for various finite horizon and infinite horizon control problems. The viscosity solution approach is the main tool in the aforementioned work. However, the present paper has a special payoff function and underlying state processes such that we are able to prove the existence of a classical solution to the HJB equation for our control problem (2.6). See also some control problems on infinite horizon optimal consumption in Guasoni et al. (2020) and Deng et al. (2020), in which the utility function involves the running maximum of the control and the value function can be obtained explicitly. However, as opposed to Guasoni et al. (2020) and Deng et al. (2020), the running maximum of a controlled diffusion process appears in our finite horizon control problem. We therefore choose to introduce an auxiliary state process with reflection similar to Weerasinghe and Zhu (2016) to reformulate the control problem again, which leads to a nonlinear HJB equation with a Neumann boundary condition. By using a heuristic dual transform, we can employ a novel probabilistic representation method and some stochastic flow arguments to establish the existence and uniqueness of the classical solution to the dual PDE first. Based on the fact that our primal value function is not strictly concave, the inverse dual transform needs to be carefully carried out on a restricted domain. We eventually extend the full characterization of the primal value function on the whole domain using some delicate continuity and convergence analysis and also the previous probabilistic representation results. Moreover, an explicit threshold for the initial wealth can be derived, beyond which the ratcheting benchmark process is dynamically superhedgable by the portfolio process and no capital needs to be injected to catch up with the nondecreasing benchmark floor. The feedback optimal portfolio in risky assets across different regions can also be carefully addressed in the verification theorem for the primal control problem.

The rest of the paper is organized as follows. Section 2 introduces the model of the optimal tracking portfolio with a ratcheting capital benchmark. To overcome the floor constraint, we consider an equivalent formulation with a running maximum cost. This problem is further transformed in Section 3 into an auxiliary one, in which a state process with reflection is identified and the nonlinear HJB equation with a Neumann boundary condition is derived. Section 4 studies the existence of a classical solution to the HJB equation using the dual transform and some tailor-made probabilistic representation techniques. The feedback optimal portfolio of the original optimal tracking problem and the proof of the verification theorem are given in Section 5. Section 6 collects the proofs of some main results in previous sections.
2 Market Model and Problem Formulation

Under the filtered probability space \((Ω, F, P)\), in which \(F = (F_t)_{t ∈ [0,T]}\) satisfies the usual conditions, the process \((W^1, \ldots, W^d)\) is a \(d\)-dimensional Brownian motion adapted to \(F\). Let \(T ∈ \mathbb{R}_+ := (0, ∞)\) be the finite terminal horizon. The financial market consists of \(d\) risky assets and the price processes are described by, for \(t ∈ [0, T]\),

\[
\frac{dS^i_t}{S^i_t} = \mu_i dt + \sum_{j=1}^d \sigma_{ij} dW^j_t, \quad i = 1, \ldots, d,
\]

with constant drift \(\mu_i ∈ \mathbb{R}\) and constant volatility \(\sigma_{ij} ∈ \mathbb{R}\) for \(i, j = 1, \ldots, d\). To simplify the presentation, we will only focus on zero interest rate \(r = 0\). In view of our control problem and underlying processes, the case \(r > 0\) can be easily transformed into an equivalent control problem with \(r = 0\) by considering the controlled state process discounted by \(e^{-rt}\) in the auxiliary problem (3.5).

For \(t ∈ [0, T]\), let us denote \(θ^t\) the amount of wealth (as an \(F\)-adapted process) that the fund manager allocates in asset \(S^i\) \(= (S^i_t)_{t ∈ [0,T]}\) at time \(t\). The self-financing wealth process under the control \(θ = (θ^1_t, \ldots, θ^d_t)_{t ∈ [0,T]}\) is given by

\[
V^θ_t = v + \int_0^t θ^T_s µds + \int_0^t θ^T_s σdW_s, \quad t ∈ [0, T],
\]

with the initial wealth \(V^θ_0 = v ≥ 0\), the return vector \(µ = (µ_1, \ldots, µ_d)^T\) and the volatility matrix \(σ = (σ_{ij})_{d × d}\) that is assumed to be invertible (the invertible matrix is denoted by \(σ^{-1}\)).

We are interested in the present paper a passive portfolio selection by a fund manager to optimally track an exogenous ratcheting capital benchmark. In particular, the benchmark process is defined as a nondecreasing process \(A = (A_t)_{t ∈ [0,T]}\) taking the absolutely continuous form that

\[
A_t := a + \int_0^t f(s, Z_s)ds, \quad t ∈ [0, T],
\]

where \(a ≥ 0\) represents the initial benchmark that the fund account needs to track at time \(t = 0\). The function \(f(\cdot, \cdot)\), representing the benchmark growth rate, is required to satisfy the condition:

\((A_f)\): the function \(f : [0, T] × \mathbb{R} → \mathbb{R}_+\) is continuous and for \(t ∈ [0, T]\), \(f(t, \cdot) ∈ C^2(\mathbb{R})\) with bounded first and second order derivatives.

The stochastic factor process \(Z = (Z_t)_{t ∈ [0,T]}\) appearing in (2.2) satisfies the SDE:

\[
dZ_t = μ_Z(Z_t)dt + σ_Z(Z_t)dW^γ_t, \quad t ∈ [0, T],
\]

with the initial value \(Z_0 = z ∈ \mathbb{R}\) and \(W^γ = (W^γ_t)_{t ∈ [0,T]}\) is a linear combination of the \(d\)-dimensional Brownian motion \((W^1, \ldots, W^d)\) with weights \(γ = (γ_1, \ldots, γ_d)^T ∈ [-1, 1]^d\), which itself is a Brownian motion. We enforce the condition on coefficients \(μ_Z(\cdot)\) and \(σ_Z(\cdot)\) that:

\((A_Z)\): the coefficients \(μ_Z : \mathbb{R} → \mathbb{R}\) and \(σ_Z : \mathbb{R} → \mathbb{R}\) belong to \(C^2(\mathbb{R})\) with bounded first and second order derivatives.

If \(Z\) is an OU process or a geometric Brownian motion, the assumption \((A_Z)\) clearly holds. In real life applications, the stochastic factor process \(Z\) can be understood as the random inflation rate process affecting the benchmark growth dynamically. The fund manager is required to choose the portfolio in a way such that the fund capital is sufficiently competitive with respect to the growing inflation-driven
capital benchmark. On the other hand, by its definition in (2.3), the stochastic factor process Z can also depend on one or some risky asset price dynamics, which allows us to accommodate the possible scenario when the growing capital benchmark is influenced by some risky asset prices.

Given the nondecreasing benchmark process A, unlike the conventional portfolio tracking problem formulated as a linear quadratic control or a utility maximization problem, we aim to choose the portfolio control with another singular capital injection control such that its total capital outperforms the benchmark A_t as a floor constraint at each intermediate time t. This variant of the optimal tracking formulation combines some mathematical features from both the conventional tracking problem and the stochastic control problem with a minimum guarantee constraint, which is both theoretically and practically interesting and new from the literature.

To be precise, we assume that the fund manager can inject capital carefully to the fund account from time to time whenever it is necessary such that the total capital dynamically dominates the benchmark floor process A. That is, the goal of the fund manager is to optimally track the process A by choosing the regular control θ as the dynamic portfolio in risky assets and the singular control \( C = (C_t)_{t \in [0,T]} \) as the accumulative capital injection such that \( C_t + V_t^\theta \geq A_t \) at each intermediate time \( t \in [0,T] \). The optimal tracking problem is defined to minimize the expected cost of the discounted accumulative capital injection subject to a American-type floor constraint that

\[
u(a, v, z) := \inf_{C, \theta} \mathbb{E} \left[ C_0 + \int_0^T e^{-\rho t} dC_t \right] \quad \text{subject to} \quad A_t \leq C_t + V_t^\theta \quad \text{at each} \quad t \in [0, T],
\]

where the constant \( \rho \geq 0 \) is the discount rate and \( C_0 = (a - v)^+ \) is the initial injected capital to match with the initial benchmark.

**Remark 2.1.** For a large initial wealth \( v \gg a \) and some special choices of \( f(t, z) \) and \( (Z_t)_{t \in [0,T]} \), it is possible that the benchmark process \( A_t \) is dynamically superhedgeable by a portfolio in risky assets at each time \( t \in [0,T] \). That is, there exists a portfolio \( \theta^* \) such that \( V_t^{\theta^*} \geq A_t \), for any \( t \in [0,T] \). Then \( C_t^* \equiv 0 \) for any \( t \in [0,T] \) is an admissible capital injection control and \((0, \theta^*)\) is an optimal control for the problem (2.4) and the value function \( u(a, v, z) \equiv 0 \). We will characterize the region for \( v \) explicitly in Remark 5.3 such that there is no need to inject capital for the problem (2.4).

To make the problem in (2.4) more tractable, our first step is to reformulate the problem (2.4) with dynamic American-type constraints using the observation that for a fixed control \( \theta \), the optimal C is always the smallest adapted right-continuous and nondecreasing process that dominates \( A - V^\theta \). Let \( \mathcal{U} \) be the set of regular \( \mathbb{F} \)-adapted control processes \( \theta = (\theta_t)_{t \in [0,T]} \) such that (2.1) is well-defined. Then, the following lemma states that this corresponds to its running maximum process and its proof is given in Section 6.

**Lemma 2.2.** For each fixed regular control \( \theta \), the optimal singular control \( C^* \) satisfies

\[
C_t^* = 0 \lor \sup_{s \in [0,t]} (A_s - V_s^\theta), \quad t \in [0,T].
\]

The problem (2.4) with the American-type floor constraints \( A_t \leq C_t + V_t^\theta \) for all \( t \in [0,T] \), admits the equivalent formulation as an unconstrained control problem under a running maximum cost that

\[
u(a, v, z) = (a - v)^+ + \inf_{\theta \in \mathcal{U}} \mathbb{E} \left[ \int_0^T e^{-\rho t} d \left( 0 \lor \sup_{s \in [0,t]} (A_s - V_s^\theta) \right) \right].
\]

**Remark 2.3.** To handle the running maximum term in the objective function, one can choose the monotone running maximum process as a controlled state process as in Barles et al. (1994), Kröner et al. (2018) and et al. to derive the HJB equation with a free boundary condition. Or one can choose the distance between the underlying process and its running maximum as a state process with reflection as in Weerasinghe and Zhu.
(2016) and derive the HJB equation with a Neumann boundary condition. In the present paper, we plan to tackle the control problem following the second method that allows us to prove the existence of a classical solution using the probabilistic representation approach. Our problem differs substantially from the one in Weerasinghe and Zhu (2016) because we consider the control in both the drift and volatility of the state process together with a stochastic factor process affecting the benchmark capital, which renders our value function multi-dimensional. Motivated by inventory control, Weerasinghe and Zhu (2016) only studied the control in the drift term in a simple model without stochastic factors.

3 Auxiliary Control Problem and HJB Equation

In this section, we choose to introduce a new controlled state process to replace the current controlled process $V^\theta = (V^\theta_t)_{t \in [0,T]}$ given by (2.1). Let us first define the difference process $D_t := A_t - V^\theta_t + v - a$ with the initial value $D_0 = 0$. For any $x \geq 0$, we then consider its running maximum process $L = (L_t)_{t \in [0,T]}$ defined by

$$L_t := x \vee \sup_{s \in [0,t]} D_s \geq 0, \quad t \in [0,T], \quad (3.1)$$

with the initial value $L_0 = x \geq 0$.

One can easily see that $(a - v)^+ - u(a, v, z)$ with $u(a, v, z)$ given in (2.6) is equivalent to the auxiliary control problem

$$\sup_{\theta \in \mathcal{U}} \mathbb{E} \left[ -\int_0^T e^{-\rho s} dL_s \right], \quad (3.2)$$

when we set the initial level $L_0 = x = (v - a)^+$. We can start to introduce our new controlled state process $X = (X_t)_{t \in [0,T]}$ for the problem (3.2), which is defined as the reflected process $X_t := L_t - D_t$ for $t \in [0,T]$ that satisfies the SDE, for $t \in [0,T]$,

$$X_t = -\int_0^t f(s, Z_s)ds + \int_0^t \theta_s^T \mu ds + \int_0^t \theta_s^T \sigma dW_s + L_t, \quad (3.3)$$

with the initial value $X_0 = x \geq 0$. In particular, the running maximum process $L_t$ increases if and only if $X_t = 0$, i.e., $L_t = D_t$. In view of “the Skorokhod problem”, it satisfies the representation that

$$L_t = x \vee \int_0^t 1_{\{X_s = 0\}} dL_s, \quad t \in [0,T].$$

We shall change the notation from $L_t$ to $L^X_t$ from this point onwards to emphasize its dependence on the new state process $X$ given in (3.3). Moreover, the stochastic factor process $Z = (Z_t)_{t \in [0,T]}$ defined in (2.3) is chosen as another state process.

To simplify the future presentation, we shall denote the variable domain $\mathcal{D}_T := [0, T] \times \mathbb{R} \times [0, \infty)$. Let us denote $\mathcal{U}$ the set of admissible controls taking the feedback form as $\theta_s = \theta(s, Z_s, X_s)$ for $s \in [t, T]$, where $\theta : \mathcal{D}_T \to \mathbb{R}^n$ is a measurable function such that the following reflected SDE has a weak solution:

$$X_t = -\int_0^t f(s, Z_s)ds + \int_0^t \theta(s, Z_s, X_s)^T \mu ds + \int_0^t \theta(s, Z_s, X_s)^T \sigma dW_s + L^X_t, \quad (3.4)$$

with $X_0 = x \geq 0$, where $L^X_t = x \vee \int_0^t 1_{\{X_s = 0\}} dL^X_s$ is a continuous, nonnegative and nondecreasing process, which increases only when the state process $X_t$ hits the level 0. For $(t, z, x) \in \mathcal{D}_T$, the dynamic version of the auxiliary stochastic control problem (3.2) is given by

$$v(t, z, x) := \sup_{\theta \in \mathcal{U}} \mathbb{E}_{t, z, x} \left[ -\int_t^T e^{-\rho s} dL^X_s \right], \quad (3.5)$$
where \( \mathbb{E}_{t,z,x}[\cdot] := \mathbb{E}[\cdot | Z_t = z, X_t = x] \) denotes the conditional expectation and the underlying state processes \((Z_t)_{t \in [0,T]} \) and \((X_t)_{t \in [0,T]} \) are given in (2.3) and (3.4) respectively.

It is important to note the equivalence that \( v(0, z, (v-a)^+) = (a-v)^+ - u(a, v, z), \) i.e. we have

\[
u(a, v, z) = \begin{cases} 
  a - v - v(0, z, 0), & \text{if } a \geq v, \\
  -v(0, z, v - a), & \text{if } a < v,
\end{cases}
\]

where \( u(a, v, z) \) is the value function of the original optimal tracking problem defined by (2.4) and \( a \) and \( v \) represent the initial benchmark level and initial wealth respectively. Starting from this section, we mainly focus on the auxiliary control problem (3.5) and seek to obtain its optimal portfolio in a feedback form.

We first have the next result on some properties of the value function \( v \) on \( D_T \) defined in (3.5). The proof is standard following the solution representation of “the Skorokhod problem” and hence is omitted.

**Lemma 3.1.** For \((t, z, x) \in D_T \)

\[
\text{the value function } v(t, z, x) \text{ defined by (3.5) is nondecreasing in } x \geq 0.
\]

Moreover, for all \((t, z) \in [0, T] \times \mathbb{R} \), we have \( |v(t, z, x_1) - v(t, z, x_2)| \leq e^{-\rho t} |x_1 - x_2| \) for all \( x_1, x_2 \geq 0 \).

**Remark 3.2.** For \((t, z) \in [0, T] \times \mathbb{R} \), if \( x \rightarrow v(t, z, x) \) is \( C^1([0, \infty)) \), Lemma 3.1 implies that the following range for the partial derivative \( v_x(t, z, x) \) needs to be considered: \( 0 \leq v_x(t, z, x) \leq e^{-\rho t} \leq 1 \) for all \((t, z, x) \in D_T \). Hereafter, we shall use \( v_x, v_t, v_{xx}, v_{tx} \) and \( v_{zz} \) to denote the (first, second order or mixed) partial derivatives of the value function \( v \) with respect to its arguments, if exists.

By some heuristic arguments of dynamic programming, we can show that \( v \) satisfies the following HJB equation:

\[
\begin{cases}
  v_t + \sup_{\theta \in \mathbb{R}^n} \left[ v_x \theta^T \mu + \frac{v_{xx}}{2} \theta^T \sigma \sigma^T \theta + v_{xz} \sigma_Z(z) \theta^T \sigma \gamma \right] \\
  + v_x \mu_Z(z) + \frac{\sigma^2_Z(z)}{2} - f(t, z) \rho v,
\end{cases}
\]

\[v(T, z, x) = 0, \quad \forall \ (z, x) \in \mathbb{R} \times [0, \infty);\]

\[v_x(t, z, 0) = 1, \quad \forall \ (z, t) \in [0, T] \times \mathbb{R},\]

in which the Neumann boundary condition \( v_x(t, z, 0) = 1 \) stems from the martingale optimality condition because the process \( L^X_t \) increases whenever the process \( X_t \) visits the value 0. Suppose \( v_{xx} < 0 \) on \([0, T] \times \mathbb{R} \times \mathbb{R}_+\), the feedback optimal control determined by (3.6) is obtained by

\[
\theta^*(t, z, x) = - (\sigma \sigma^T)^{-1} \frac{v_x(t, z, x) \mu + v_{xx}(t, z, x) \sigma_Z(z) \sigma \gamma}{v_{xx}(t, z, x)}, \quad (t, z, x) \in D_T.
\]

Plugging (3.7) into the HJB equation (3.6), we have for \((t, z, x) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+ \) that

\[
v_t - \rho v - \alpha \frac{v_x^2}{v_{xx}} + \frac{\sigma_Z^2(z)}{2} \left( v_{zz} - \frac{v_x^2}{v_{xx}} \right) - \phi(z) \frac{v_x v_{xz}}{v_{xx}} + \mu_Z(z) v_x - f(t, z) v_x = 0,
\]

where the coefficients are given by

\[
\alpha := \frac{1}{2} \mu^T (\sigma \sigma^T)^{-1} \mu, \quad \phi(z) := \sigma_Z(z) \mu^T (\sigma \sigma^T)^{-1} \sigma, \quad z \in \mathbb{R}.
\]

Note that the HJB equation (3.6) is fully nonlinear. To study the existence of a classical solution to (3.6), we will first apply the heuristic dual transform to linearize the original HJB equation (3.6) and establish the existence and uniqueness of a classical solution to the dual PDE using the probabilistic representation approach and stochastic flow analysis in the next section.
4 Dual Transform and Probabilistic Representation

To reduce the challenge from the nonlinear PDE, we choose to employ the Legendre-Fenchel dual transform to the primal HJB equation \((4.6)\). We start by assuming that the value function \(v\) satisfies \(v \in C^{1.2}(\mathbb{R} \times [0, T] \times \mathbb{R})\) and \(v_{xx} < 0\) on \([0, T] \times \mathbb{R} \times \mathbb{R}_+\), which will be verified later. For \((t, z, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+\), let us consider the dual transform function given by

\[
\hat{v}(t, z, y) := \sup_{x > 0} \{v(t, z, x) - xy\} \quad \text{and} \quad x^*(t, z, y) := v_z(t, z, \cdot)^{-1}(y),
\]

where \(y \mapsto v_z(t, z, \cdot)^{-1}(y)\) denotes the inverse function of \(x \mapsto v_z(t, z, x)\), and \(x^* = x^*(t, z, y)\) in \((4.1)\) satisfies the equation:

\[
v_x(t, z, x^*) = y, \quad (t, z) \in [0, T] \times \mathbb{R}.
\]

(4.2)

On the other hand, in view of Lemma 3.1 and Remark 3.2, the variable \(y\) in fact only takes values in the set \((0, 1)\). It then follows by \((4.1)\) that, for all \((t, z, y) \in [0, T] \times \mathbb{R} \times (0, 1),\)

\[
\hat{v}(t, z, y) = v(t, z, x^*) - x^*y.
\]

(4.3)

Taking the derivative w.r.t. \(y\) on both sides of \((4.3)\), we deduce that

\[
\hat{v}_y(t, z, y) = v_x(t, z, x^*)x^*_y - x^*_y y - z^* = yx^*_y - x^*_y y - z^* = -z^*.
\]

(4.4)

By taking the derivative w.r.t. \(y\) on both sides of \((4.2)\), we also have \(v_{xx}(t, z, x^*)x^*_y = 1\) and hence \(x^*_y = \frac{1}{v_{xx}(t, z, x^*)}\). Because of \((4.4)\), we can obtain that

\[
\hat{v}_{yy}(t, z, y) = -x^*_y = -\frac{1}{v_{xx}(t, z, x^*)}, \quad x^*_z = -\frac{v_{xz}(t, z, x^*)}{v_{xx}(t, z, x^*)}.
\]

(4.5)

It follows by \((4.2)\) and \((4.3)\) that

\[
\hat{v}_t(t, z, y) = v_t(t, z, x^*), \quad \hat{v}_z(t, z, y) = v_z(t, z, x^*), \quad \hat{v}_{zz}(t, z, y) = v_{zz}(t, z, x^*) - \frac{v_{xx}(t, z, x^*)^2}{v_{xx}(t, z, x^*)^2}.
\]

(4.6)

Moreover, by the second equality in \((4.5)\) and \((4.6)\), we further have that

\[
\hat{v}_{yz}(t, z, y) = v_{xz}(t, z, x^*)x^*_y = \frac{v_{xz}(t, z, x^*)}{v_{xx}(t, z, x^*)}.
\]

(4.7)

By virtue of \((3.8)\) and \((4.3)\), we obtain that

\[
v_t(t, z, x^*) - \rho v(t, z, x^*) - \alpha x^*_y v_x(t, z, x^*)^2 - \frac{\sigma^2_z(z)}{2} v_{xx}(t, z, x^*)^2 - \phi(z) v_{xx}(t, z, x^*) v_{zz}(t, z, x^*) v_{xx}(t, z, x^*)
\]

\[
+ \mu_Z(z) v_z(t, z, x^*) + \frac{\sigma^2_z(z)}{2} v_{zz}(t, z, x^*) - f(t, z) v_x(t, z, x^*) = 0.
\]

(4.8)

Plugging \((4.2)\), \((4.5)\), \((4.6)\) and \((4.7)\) into \((4.8)\), we can derive that, for \((t, z, y) \in [0, T] \times \mathbb{R} \times (0, 1),\)

\[
\hat{v}_t(t, z, y) - \rho \hat{v}(t, z, y) + \rho y \hat{v}_y(t, z, y) + \alpha y^2 \hat{v}_{yy}(t, z, y) + \mu_Z(z) \hat{v}_z(t, z, y) + \frac{\sigma^2_z(z)}{2} \hat{v}_{zz}(t, z, y)
\]

\[
- \phi(z) y \hat{v}_{yz}(t, z, y) - f(t, z) y = 0.
\]

(4.9)

We next derive the terminal condition and the boundary condition of the linear PDE \((4.9)\). By the terminal condition \(v(T, z, x) = 0\) of the HJB equation \((3.6)\), it follows that

\[
\hat{v}(T, z, y) = \sup_{x > 0} \{v(T, z, x) - xy\} = \sup_{x > 0} \{-xy\} = 0, \quad (z, y) \in \mathbb{R} \times (0, 1).
\]

(4.10)
Note that \( x_y^* = \frac{1}{v_y(t,z,x)} < 0 \), and for each \((t,z) \in [0,T] \times \mathbb{R}\), the map \( y \mapsto x^*(t,z,y) := v_x(t,z,\cdot)^{-1}(y) \) is one to one. Moreover, by the Neumann boundary condition of the HJB equation (3.6), we deduce from (4.2) that \( v_x(t,z,0) = 1 \) and \( x^*(t,z,1) = 0 \). Therefore, in view of (4.4), for all \((t,z) \in [0,T] \times \mathbb{R}, \)
\[
\hat{v}_y(t,z,1) = -x^*(t,z,1) = 0. \tag{4.11}
\]

In summary, the HJB equation (3.6) can be transformed into the linear dual PDE of \( \hat{v} \) that
\[
\begin{aligned}
\hat{v}_t + \alpha y^2 \hat{v}_{yy} + \rho y \hat{v}_y - \phi(z)y \hat{v}_{yz} + \mu_Z(z) \hat{v}_z + \frac{\sigma^2_Z(z)}{2} \hat{v}_{zz} & \quad - f(t,z,y), \quad (t,z,y) \in [0,T] \times \mathbb{R} \times (0,1) ; \\
\hat{v}(T,z,y) & \quad = 0, \quad \forall (z,y) \in \mathbb{R} \times [0,1] ; \\
\hat{v}_y(t,z,1) & \quad = 0, \quad \forall (t,z) \in [0,T] \times \mathbb{R}.
\end{aligned} \tag{4.12}
\]

We next study the existence and uniqueness of a classical solution to the Neumann boundary problem (4.12) with the extra condition that \( \hat{v}_{yy} \geq 0 \) on \([0,T] \times \mathbb{R} \times (0,1)\) using the probabilistic representation approach. To this purpose, for \((t,z,u) \in \mathcal{D}_T\), let us define the function
\[
h(t,z,u) := -\mathbb{E} \left[ \int_t^T e^{-\rho s} f(s, M^{t,z}_s)e^{-R^{t,u}_s} \, ds \right], \tag{4.13}
\]
where the process \((M^{t,z}_s)_{s \in [t,T]}\) with \((t,z) \in [0,T] \times \mathbb{R}\) satisfies the SDE, for \(s \in [t,T], \)
\[
M^{t,z}_s = z + \int_t^s \mu_Z(M^{t,z}_r) \, dr + \int_t^s \sigma_Z(M^{t,z}_r) \, dB^1_r + \sqrt{1 - \varrho^2} \int_t^s \sigma_Z(M^{t,z}_r) \, dB^2_r. \tag{4.14}
\]
The processes \(B^1 = (B^1_t)_{t \in [0,T]}\) and \(B^2 = (B^2_t)_{t \in [0,T]}\) are two standard Brownian motions with a specific correlation coefficient
\[
\varrho := \frac{(\sigma^{-1}\mu)^T}{\sigma^{-1}\mu} \gamma. \tag{4.15}
\]
Moreover, the process \((R^{t,u}_s)_{s \in [t,T]}\) with \((t,u) \in [0,T] \times [0,\infty)\) is a reflected Brownian motion with drift defined by
\[
R^{t,u}_s := u + \sqrt{2\alpha} \int_t^s \, dB^1_r + \int_t^s (\alpha - \rho) \, dr + \int_t^s dL^R_r \geq 0, \quad s \in [t,T], \tag{4.16}
\]
where \(t \mapsto L^R_t\) is a continuous and nondecreasing process that increases only on \(\{ t \in [0,T] ; R^{t,u}_t = 0 \}\) with \(L^R_0 = 0\). By the solution representation of “the Skorokhod problem”, we obtain that, for \((s,u) \in [t,T] \times [0,\infty), \)
\[
L^R_s = 0 \vee \left\{ -u + \max_{r \in [t,s]} \left[ -\sqrt{2\alpha}(B^1_r - B^1_t) - (\alpha - \rho)(r-t) \right] \right\}, \tag{4.17}
\]
It follows from assumptions \((A_f)\) and \((A_Z)\) that, for all \((t,z,u) \in \mathcal{D}_T, \)
\[
|h(t,z,u)| = \mathbb{E} \left[ \int_t^T e^{-\rho s} f(s, M^{t,z}_s)e^{-R^{t,u}_s} \, ds \right] \leq C \mathbb{E} \left[ \int_t^T e^{-\rho s}(1 + |M^{t,z}_s|) \, ds \right] \tag{4.18}
\]
\[
\leq C(T-t) + C(T-t)E \left[ \sup_{s \in [t,T]} |M^t_s| \right] \leq C(T-t)(1 + |z|),
\]
(4.18)

for some constant \( C = C_f > 0 \). Hence, the function \( h \) given in (4.13) is well-defined. We next study the regularity of the function \( h \) defined in (4.13) in the next result, and its proof is reported in Section 6.

**Proposition 4.1.** Let assumptions \((A_f)\) and \((A_Z)\) hold. We have that \( h \in C^{1,2,2}(\mathcal{D}_T) \). Moreover, for \((t,z,u) \in \mathcal{D}_T\), we get
\[
h_u(t,z,u) = E \left[ \int_t^T e^{-\rho s} f(s, M^t_s z) e^{-R^t_s u} 1_{\left\{ \max_{s \in [t,\tau]} [-\sqrt{\alpha B^1_s} - (\alpha - \rho)s] \leq u \right\}} \, ds \right]
= E \left[ \int_{\tau^*_u \wedge T} e^{-\rho s} f(s, M^t_s z) e^{-R^t_s u} \, ds \right],
\]
(4.19)

where \( \tau^*_u := \inf \{ s \geq t; -\sqrt{2\alpha B^1_s} - (\alpha - \rho)s = u \} \) (we assume \( \inf \emptyset = +\infty \) by convention).

Building upon Proposition 4.1, we have the next important auxiliary result and provides its proof in Section 6.

**Theorem 4.2.** Suppose that \((A_f)\) and \((A_Z)\) hold. Then, the function \( h \) defined in (4.13) solves the Neumann boundary problem:
\[
\begin{cases}
    h_t + \alpha h_{uu} + (\alpha - \rho)h_u + \phi(z)h_{uz} + \mu_Z(z)h_z + \frac{\sigma_Z^2(z)}{2} h_{zz} = f(t, z) e^{-u - \rho t}, & (t, z, u) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+; \\
    h(T, z, u) = 0, & \forall (z, u) \in \mathbb{R} \times [0, \infty); \\
    h_u(t, z, 0) = 0, & \forall (t, z) \in [0, T) \times \mathbb{R}.
\end{cases}
\]
(4.20)

On the other hand, if a function \( h \) defined on \( \mathcal{D}_T \) with a polynomial growth is a classical solution of the Neumann boundary problem (4.20), then \( h \) has the representation (4.13).

**Remark 4.3.** Under assumptions \((A_f)\) and \((A_Z)\), it follows by (4.19), together with (6.6) in Section 6, that for all \((t, z, u) \in \mathcal{D}_T\),
\[
h_u(t, z, u) + h_{uu}(t, z, u) = E \left[ e^{-\rho \tau^*_u} f(t, M^t_{\tau^*_u} z) \Gamma(\tau^*_u) + 2 \int_t^{\tau^*_u} e^{-\rho s} f(s, M^t_s z) e^{-R^t_s u} \, ds \right],
\]
(4.21)

where the stopping time \( \tau^*_u \) is given in Proposition 4.1 and the function \( \Gamma(t) \) for \( t \in [0, T] \) is given by (6.5) in Section 6. Note that \( f > 0 \) by the assumption \((A_f)\). Then, we have from (4.21) that \( h_{uu} + h_u \geq 0 \), while \( ">" \) holds when \((t, z, u) \in [0, T) \times \mathbb{R} \times [0, \infty)\).

The well-posedness of the Neumann problem (4.12) is now given in the next result and its proof is postponed to Section 6.

**Corollary 4.4.** Let assumptions of Theorem 4.2 hold. The Neumann problem (4.12) admits a unique classical solution \( \hat{v} \) such that for \((t, z, y) \in [0, T] \times \mathbb{R} \times (0, 1], \)
\[
|\hat{v}(t, z, y)| \leq C(1 + |z|^p + |\ln |y||^p), \quad \text{for some } p > 1,
\]
(4.22)

and the function
\[
h(t, z, u) := e^{-\rho t} \hat{v}(t, z, e^{-u}), \quad (t, z, u) \in \mathcal{D}_T,
\]
(4.23)

has the probabilistic representation (4.13). Moreover, for each \((t, z) \in [0, T) \times \mathbb{R}, \) the solution \((0, 1] \ni y \mapsto \hat{v}(t, z, y)\) is strictly convex.
5 Optimal Portfolio and Verification Theorem

In the previous section, Corollary 4.4 gives the existence and uniqueness of the classical solution \( \hat{v}(t,z,y) \) for \((t,z,y) \in [0,T] \times \mathbb{R} \times (0,1)\) to the dual PDE (4.12). We next recover the classical solution \( v(t,z,x) \) of the primal HJB equation (3.8) via \( \hat{v}(t,z,y) \) and prove the verification theorem of our original stochastic control problem (3.5).

**Theorem 5.1** (Verification theorem). Let assumptions \((A_f)\) and \((A_Z)\) hold. We have that:

(i) The primal HJB equation (3.8) admits a solution \( v \in C^{1,2,2}([0,T] \times \mathbb{R} \times [0,\infty)) \cap C(\mathcal{D}_T) \). Moreover, for \((t,z,x) \in \mathcal{D}_T\), the solution \( v \) of HJB equation (3.8) can be written as:

\[
v(t,z,x) = \begin{cases} 
\inf_{y \in (0,1)} \{ \hat{v}(t,z,y) + xy \}, & \text{if } (t,z,x) \in \mathcal{O}_T \text{ or } x = 0, \\
0, & \text{if } (t,z,x) \in \mathcal{O}_T^f \cap \mathcal{D}_T,
\end{cases}
\]

where the region \( \mathcal{O}_T \) in (5.1) is given by

\[
\mathcal{O}_T := \{(t,z,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}_+; \ x \in (0,\xi(t,z))\},
\]

and the function \( \xi(t,z) \) with \((t,z) \in [0,T] \times \mathbb{R} \) is defined by

\[
\xi(t,z) := \mathbb{E} \left[ \int_t^T e^{-\rho(s-t)} f(s, M_s^t z) e^{\sqrt{2\rho}(B_t^1 - B_{s+})} \right] ds,
\]

and the process \((M_s^t z)_{s \in [0,t]}\) with \((t,z) \in [0,T] \times \mathbb{R} \) is the strong solution of SDE (4.14). Here, for \((t,z,y) \in [0,T] \times \mathbb{R} \times (0,1] \), the function \( \hat{v}(t,z,y) = e^{\rho t} h(t,z, -\ln y) \) solves the dual PDE (4.12) with Neumann boundary condition.

(ii) Let us define the following feedback control function as, for \((t,z,x) \in \mathcal{D}_T\),

\[
\theta^*(t,z,x) := \begin{cases} 
-(\sigma \sigma^\top)^{-1} v_z(t,z,x) + v_{zz}(t,z,x) \sigma Z(z) \sigma^\top, & \text{if } (t,z,x) \in \mathcal{O}_T \text{ or } x = 0, \\
-\mu(\sigma \sigma^\top)^{-1} \lim_{y \downarrow 0} y v_y(t,z,y), & \text{if } (t,z,x) \in \mathcal{O}_T^f \cap \mathcal{D}_T,
\end{cases}
\]

For the processes \((Z,X) = (Z_t, X_t)_{t \in [0,T]} \) given by (3.4), define \( \theta^*_t := \theta^*(t, Z_t, X_t) \) for \( t \in [0,T] \). Then \( \theta^* = (\theta^*_t)_{t \in [0,T]} \in \mathcal{U}_t \) is an optimal strategy. Moreover, for all \( \theta \in \mathcal{U}_t \), it holds that \( \hat{J}(\theta; t,z,x) \leq e^{-\rho t} v(t,z,x) \), where \((t,z,x) \in [0,T] \times \mathbb{R} \times [0,\infty)\).

**Remark 5.2.** We explain here the role of the function \( \xi(t,z) \) defined by (5.3) in Theorem 5.1. In fact, for \((t,z) \in [0,T] \times \mathbb{R} \) and \( x \geq \xi(t,z) \), it follows from Theorem 5.1-(i) that the value function \( v(t,z,x) = 0 \). Then, by Theorem 5.1-(ii), we have that, for the strategy \( \theta^* \in \mathcal{U}_t \) given by (5.4),

\[
\mathbb{E}_{t,z} \left[ -\int_t^T e^{-\rho s} dL^x_s \right] = 0,
\]

where the process \( L^x = (L^x_t)_{t \in [0,T]} \) is the reflected term of the process \( X^* = (X^*_t)_{t \in [0,T]} \) which is given in (3.4) with \( \theta \) replaced by \( \theta^* \). This implies from integration by parts that \( e^{-\rho T} L^x_T + \rho \int_0^T e^{-\rho s} L^x_s ds = x \),
\( \mathbb{P}\)-a.s. and hence \( L_{\mathbb{T}^\ast}^X = L_t^X = x \), \( \mathbb{P}\)-a.s. because \( \xi(t, z) > 0 \) for \( (t, z) \in [0, T) \times \mathbb{R} \). Therefore, with the strategy \( \theta^\ast \in \mathcal{U}_t \), the (nonnegative) process \( X_t^\ast \) is given by

\[
X_t^\ast = x - \int_0^t f(s, Z_s)ds + \int_0^t (\theta_s^\ast) \sigma dW_s.
\]

On the other hand, for \( 0 \leq x < \xi(t, z) \), we have that \( v_x(t, z, x) > 0 \) and hence \( \nu(t, x, z) < 0 \). This implies that, with this initial value \( x \), the reflected term \( L_t^X \) is strictly increasing in \( t \in [0, T] \).

**Remark 5.3.** Recall the equivalence that \( u(a, v, z) = -v(0, z, v - a) \) when \( v > a \), where \( u(a, v, z) \) is the value function of the optimal original tracking problem (2.4). According to Remark 5.2 above, if the initial wealth \( v \) is sufficiently large such that \( v - a > \xi(0, z) \) for the given benchmark growth rate function \( f(\cdot, \cdot) \) and \( \mu_Z(\cdot), \sigma_Z(\cdot) \) in the definition of the stochastic factor process \( Z = (Z_t)_{t \in [0, T]} \), we can conclude that \( u(a, v, z) = 0 \) and the optimal singular control \( C_t^v \equiv 0 \) for \( t \in [0, T] \) and we always have that \( A_t \) is dynamically superhedgeable that \( A_t \leq V_t^\theta^\ast \) for \( t \in [0, T] \).

**Proof of Theorem 5.1.** We first focus on the proof of (i). For this purpose, recall the function \( \xi(t, z) \) with \( (t, z) \in [0, T) \times \mathbb{R} \) defined by (5.3). Then, by the assumption (\( A_f \)), we have \( \xi(t, z) > 0 \) for all \( (t, z) \in [0, T) \times \mathbb{R} \). Moreover, in view of Proposition 4.1 and Theorem 4.2, for \( (t, z) \in [0, T) \times \mathbb{R} \), \( \xi(t, z) = -\lim_{y \to 0} \hat{\nu}_y(t, z, y) \), and the derivative \( \hat{\nu}_y \) satisfies that, for \( (t, z) \in [0, T) \times \mathbb{R} \),

\[
\begin{align*}
\hat{\nu}_y(t, z, 1) &= -e^{pl}h_u(t, z, 0) = 0, \\
\hat{\nu}_y(t, z, y) &= y^{-1}e^{pl}h_u(T, z, -\ln y) = 0, \\
\hat{\nu}_{yy}(t, z, y) &= e^{pl+2u}(h_u(t, z, u) + h_{uu}(t, z, u)) \geq 0, \\
\lim_{y \to 0} \hat{\nu}_y(t, z, y) &= -\xi(t, z), \\
\lim_{y \to 0} \hat{\nu}_{yy}(t, z, y) &= -\xi(t, z).
\end{align*}
\]

According to the definition (5.2) and (5.3), the region \( \mathcal{O}_T \) has a boundary that is at least \( C^1 \). We next consider the original HJB equation (3.6), however, restricted to the domain \( (t, y, z) \in \mathcal{O}_T \) that

\[
\begin{align*}
\nu_t &+ \sup_{\theta \in \mathbb{R}^n} \left[ \nu_y \mu + \frac{\nu_{yy}}{2} \sigma \sigma^\top \theta + \nu_{xz} \sigma_Z(\xi) \theta^\top \sigma \right] \\
&+ \nu_z \mu_Z(z) + \nu_{xz} \frac{\sigma_Z^2(z)}{2} - f(t, z)v_x = \rho v, \quad (t, y, z) \in \mathcal{O}_T; \\
v_x(t, z, 0) &= 1, \quad \forall \ (t, z) \in [0, T) \times \mathbb{R}.
\end{align*}
\]

First of all, for \( (t, z, x) \in \mathcal{O}_T \), let us define \( y^* = y^*(t, z, x) \in (0, 1] \) that satisfies

\[
\hat{\nu}_y(t, z, y^*) = -x.
\]

Thanks to (5.6), we have that

\[
v(t, z, x) = \inf_{y \in (0, 1]} \{ \hat{\nu}(t, z, y) + xy \} = \hat{\nu}(t, z, y^*(t, z, x)) + xy^*(t, z, x), \quad (t, z, x) \in \mathcal{O}_T.
\]

Note that \( (0, 1] \ni y \to \hat{\nu}_y(t, z, y) \) is strictly increasing for fixed \( (t, z) \in [0, T) \times \mathbb{R} \), as well as \( \hat{\nu}_y(t, z, 1) = 0 \) and \( \lim_{y \to 0} \hat{\nu}_y(t, z, y) = -\xi(t, z) \), we have that \( x \to y^*(t, z, x) \) is decreasing, \( \lim_{x \to 0} y^*(t, z, x) = 1 \), and \( \lim_{x \to \xi(t, z)} y^*(t, z, x) = 0 \). It follows from the implicit function theorem that \( y^* \) is \( C^1 \) on \( \mathcal{O}_T \). Therefore \( v \).
in (5.13) is well defined, and it is $C^{1,2,2}$ on $O_T$. On the other hand, a direct calculation yields that, for $(t, z, x) \in O_T$,
\[
y^*(t, z, x) = v_x(t, z, x), \quad \hat{v}_t(t, z, y^*(t, z, x)) = v_t(t, z, x), \quad \hat{v}_z(t, z, y^*(t, z, x)) = v_z(t, z, x),
\]
\[
\hat{v}_{yy}(t, z, y^*(t, z, x)) = \frac{1}{v_{xx}(t, z, x)}, \quad \hat{v}_{zy}(t, z, y^*(t, z, x)) = \frac{v_{xz}(t, z, x)}{v_{xx}(t, z, x)},
\]
\[
\hat{v}_{zz}(t, z, y^*(t, z, x)) = \left(v_{zz} - \frac{v_z^2}{v_{xx}}\right)(t, z, x).
\]
(5.8)

Recall that $v_{xx}(t, z, x) < 0$ for $(t, z, x) \in O_T$. Plugging (5.8) into (4.12), we deduce that $v$ defined in (5.13) solves the dual PDE (5.5). We next study the behavior of $v$ on $O_T^\ast \cap [0, T) \times R \times R_+$. To this end, for $(t, z) \in [0, T) \times R$, let $(t_n, z_n, x_n) \in O_T$ for $n \geq 1$ be a sequence such that $(t_n, z_n, x_n) \to (t, z, \xi(t, z))$. We then claim that
\[
\lim_{n \to +\infty} y^*(t_n, z_n, x_n) = 0.
\]
(5.9)

Let us verify (5.9) by contradiction. Suppose that, up to a subsequence, there exists a constant $\delta > 0$ such that $\lim_{n \to +\infty} y^*(t_n, z_n, x_n) = \delta$. Then, by (5.6), it yields that
\[
\hat{v}_y(t, z, \delta) = \lim_{n \to +\infty} \hat{v}_y(t_n, z_n, y^*(t_n, z_n, x_n)) = - \lim_{n \to +\infty} x_n = -\xi(t, z),
\]
which contradicts the definition (5.3) of $\xi(t, z)$. Moreover, from (5.9), it follows that
\[
\lim_{n \to +\infty} v(t_n, z_n, x_n) = \lim_{n \to +\infty} \left\{ \hat{v}(t_n, z_n, y^*(t_n, z_n, x_n)) + x_n y^*(t_n, z_n, x_n) \right\} = 0,
\]
(5.10)

and it holds that
\[
\lim_{n \to +\infty} v_t(t_n, z_n, x_n) = \lim_{n \to +\infty} \hat{v}_t(t_n, z_n, y^*(t_n, z_n, x_n)) = 0,
\]
\[
\lim_{n \to +\infty} v_z(t_n, z_n, x_n) = \lim_{n \to +\infty} \hat{v}_z(t_n, z_n, y^*(t_n, z_n, x_n)) = 0,
\]
\[
\lim_{n \to +\infty} v_{xz}(t_n, z_n, x_n) = \lim_{n \to +\infty} y^*(t_n, z_n, x_n) = 0,
\]
(5.11)
\[
\lim_{n \to +\infty} v_{xx}(t_n, z_n, x_n) = - \lim_{n \to +\infty} \hat{v}_{yy}(t_n, z_n, y^*(t_n, z_n, x_n))^{-1} = 0,
\]
\[
\lim_{n \to +\infty} v_{xz}(t_n, z_n, x_n) = \lim_{n \to +\infty} \hat{v}_{yz}(t_n, z_n, y^*(t_n, z_n, x_n)) = 0,
\]
\[
\lim_{n \to +\infty} v_{zz}(t_n, z_n, x_n) = \lim_{n \to +\infty} \left( \hat{v}_{zz} - \frac{v_z^2}{v_{xx}} \right)(t_n, z_n, y^*(t_n, z_n, x_n)) = 0.
\]

Let us define $v(t, z, x) = 0$ for $(t, z, x) \in O_T^\ast \cap [0, T) \times R \times R_+$. By (5.10) and (5.11), we have that $v$ given by (5.13) and its partial derivatives up to order two are continuous on $\partial O_T \cap ([0, T) \times R \times R_+)$. This implies that $v$ is $C^{1,2,2}$ on $[0, T) \times R \times R_+$. Moreover, using (5.8) and (4.12) on $[0, T) \times R \times R_+$, we have that $v$ given by (5.1) solves the following HJB equation:
\[
\begin{cases}
 v_t + \sup_{\theta \in R^n} \left[ v_x \theta^\top \mu + \frac{v_{xx}}{2} \theta^\top \sigma \sigma^\top \theta + v_{xz} \sigma_z (z) \theta^\top \sigma \gamma \right] \\
 + v_z \mu_z (z) + v_{zz} \frac{\sigma_z^2 (z)}{2} - f(t, z) v_x = \rho v_x, \quad \forall (t, z, x) \in [0, T) \times R \times R_+;
\end{cases}
\]
(5.12)

$$v_x(t, z, 0) = 1, \quad \forall (t, z) \in [0, T) \times R.$$
On the other hand, note that \( v_x \geq 0 \) on \((t, z, x) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+ \), and \( v(t, z, x) \to 0 \) as \( x \to +\infty \). By (3.5), it is obvious to have \( v(t, z, 0) \leq 0 \). Therefore \( v(t, z, x) \leq 0 \) for \((t, z, x) \in [0, T) \times \mathbb{R} \times [0, \infty) \), and it follows from (4.18) that, there exists a constant \( C > 0 \) independent of \( T \) such that, for \((t, z, x) \in [0, T) \times \mathbb{R} \times [0, \infty) \),

\[
|v(t, z, x)| = -v(t, z, x) = \sup_{y \in (0, 1]} \{ -\hat{v}(t, z, y) \} \leq \sup_{y \in (0, 1]} \{ -\hat{v}(t, z, y) \} \\
= \sup_{y \in (0, 1]} \{ -e^{\eta y}(t, z, -\ln y) \} \leq e^{\eta C(T-t)}(1+|z|),
\]

where the function \( h(t, z, u) \) is given by (4.13).

We next prove the continuity of \( v \) on the boundary of \([0, T) \times \mathbb{R} \times [0, +\infty) \). Note that \( v(t, z, 0) = \hat{v}(t, z, 1) \). Consider \((t, z) \in [0, T) \times \mathbb{R} \) and \((t_n, z_n, x_n) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+ \) satisfying \((t_n, z_n, x_n) \to (t, z, 0) \) as \( n \to \infty \). By mimicking the proof to show (5.9), one can also attain that

\[
\lim_{n \to \infty} y^*(t_n, z_n, x_n) = 1.
\]

An application of L’Hospital’s rule gives that

\[
\lim_{x \downarrow 0} \frac{1}{x} (v(t, z, x) - v(t, z, 0)) = \lim_{x \downarrow 0} \frac{x}{1} \left( \hat{y}_y(t, z, y^*(t, z, x)) + xy^*(t, z, x) - \hat{v}(t, z, 1) \right) \\
= \lim_{x \downarrow 0} y^*(t, z, x) - \lim_{x \downarrow 0} \frac{\hat{v}_y(t, z, y^*(t, z, x)) - \hat{v}(t, z, 1)}{y^*(t, z, x) - 1} \times \lim_{x \downarrow 0} \frac{y^*(t, z, x) - 1}{x} \\
= 1 - \hat{v}_y(t, z, 1) \times \left( \lim_{x \downarrow 0} y^*_y(t, z, x) \right) = 1.
\]

Moreover, by noting \( \lim_{n \to \infty} v_x(t_n, z_n, x_n) = \lim_{n \to \infty} y^*(t_n, z_n, x_n) = 1 \), it holds that

\[
\lim_{n \to \infty} v_x(t_n, z_n, x_n) = v_x(t, z, 0).
\]

Similarly, we also have that

\[
\lim_{x \downarrow 0} \frac{1}{x} (v_x(t, z, x) - v_x(t, z, 0)) = \lim_{x \downarrow 0} \frac{x}{1} (y^*_y(t, z, x) - 1) = \lim_{x \downarrow 0} y^*_y(t, z, x) = -\hat{v}_{yy}(t, z, 1)^{-1},
\]

and \( \lim_{n \to \infty} v_{xx}(t_n, z_n, x_n) = -\lim_{n \to +\infty} \hat{v}_{yy}(t_n, z_n, y^*(t_n, z_n, x_n))^{-1} = -\hat{v}_{yy}(t, z, 1)^{-1} \). Therefore

\[
\lim_{n \to +\infty} v_{xx}(t_n, z_n, x_n) = v_{xx}(t, z, 0).
\]

In a similar fashion, the limits (5.15) and (5.16) also hold for \( v_z \), \( v_{xz} \), and \( v_{zz} \). Hence, we have that \( v \in C^{1,2,2}([0, T) \times \mathbb{R} \times [0, \infty)) \). On the other hand, for \((z, x) \in \mathbb{R} \times [0, +\infty) \), we define \( v(T, z, x) = 0 \), and consider \((t_n, z_n, x_n) \in [0, T) \times \mathbb{R} \times [0, +\infty) \) satisfying \((t_n, z_n, x_n) \to (T, z, x) \) as \( n \to +\infty \). In view of (5.13), we have \( \lim_{n \to +\infty} v(t_n, z_n, x_n) = 0 \), which yields that \( v \in C(D_T) \). By combining Eq. (5.12), we deduce that \( v \in C^{1,2,2}([0, T) \times \mathbb{R} \times [0, +\infty)) \cap C(D_T) \), and \( v \) satisfies that

\[
\begin{cases}
\begin{aligned}
&v_t + \sup_{\theta \in \mathbb{R}^n} \left[ v_{\theta} \theta \mu + \frac{v_{xx}}{2} \theta \sigma \theta + v_{xz} \sigma(z) \theta \gamma \right] \\
&+ v_{z} \mu(z) + v_{xz} \frac{\sigma^2(z)}{2} - f(t, z)v_x = \rho v, \quad \forall \ (t, z, x) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+;
\end{aligned}
\end{cases}
\]

\[
\begin{cases}
\begin{aligned}
&v_z(t, z, 0) = 1, \quad \forall \ (t, z) \in [0, T) \times \mathbb{R},
\end{aligned}
\end{cases}
\]

\[
\begin{cases}
\begin{aligned}
&v(T, z, x) = 0, \quad \forall \ (z, x) \in \mathbb{R} \times [0, +\infty),
\end{aligned}
\end{cases}
\]
and also the estimate (5.13) for \((t, z, x) \in [0, T] \times \mathbb{R} \times [0, +\infty)\).

We next move on to the proof of (ii). We first show the continuity of \(\theta^*(t, z, x)\) on \((t, z, x) \in D_T\), which verifies the admissibility of \(\theta_t^* = \theta^*(t, Z_t, X_t)\) for \(t \in [0, T]\) (i.e., \(\theta^* \in \mathcal{U}_t\)). Let us define \(y^*(t, z, 0) = 1\). Thanks to (5.14), \(y^*\) is continuous at \((t, z, 0)\). For \((t, z, x) \in D_T\), we rewrite (5.4) by

\[
\theta^*(t, z, x) = -\mu(\sigma \sigma^\top)^{-1}y^*(t, z, x)\dot{v}_{yy}(t, z, y^*(t, z, x)) + \sigma Z(z)\sigma\gamma \dot{v}_{yz}(t, z, y^*(t, z, x)).
\]  

(5.18)

It is easy to see that \(\theta^*(t, z, x)\) is continuous for \((t, z, x) \in \mathcal{O}_T \cup \{(t, z, 0); (t, z) \in [0, T) \times \mathbb{R}\}\. Therefore, it remains to show that

\[
\lim_{n \rightarrow +\infty} \theta^*(t_n, z_n, x_n) = \theta^*(t, z, x),
\]

where \(x = \xi(t, z)\), and \((t_n, z_n, x_n) \in \mathcal{O}_T\), \(\lim_{n \rightarrow +\infty} (t_n, z_n, x_n) = (t, z, x)\). By virtue of (5.18), we have that

\[
\theta^*(t_n, z_n, x_n) = -\mu(\sigma \sigma^\top)^{-1}y^*(t_n, z_n, x_n)\dot{v}_{yy}(t_n, z_n, y^*(t_n, z_n, x_n)) + (\sigma \sigma^\top)^{-1}\sigma Z(z)\sigma\gamma \dot{v}_{yz}(t_n, z_n, y^*(t_n, z_n, x_n)).
\]

(5.20)

Note that \(x^*(t_n, z_n, x_n) \rightarrow 0\) as \(n \rightarrow +\infty\). By sending \(n\) to \(+\infty\) on both sides of (5.20), we deduce that

\[
\lim_{n \rightarrow \infty} \theta^*(t_n, z_n, x_n) = -\mu(\sigma \sigma^\top)^{-1}\lim_{n \rightarrow \infty} x_n \dot{v}_{yy}(t, z, y) + (\sigma \sigma^\top)^{-1}\sigma Z(z)\sigma\gamma \lim_{n \rightarrow \infty} \dot{v}_{yz}(t, z, y) = \theta^*(t, z, x).
\]

(5.21)

Following the same argument, we can establish the convergence (5.19) for any \((t, z, x) \in \partial \mathcal{O}_T \cap (\{0, T\} \times \mathbb{R} \times [0, +\infty))\) and \((t_n, z_n, x_n) \in \mathcal{O}_T\). Hence \(\theta^*(t, y, z)\) is continuous for \((t, z, x) \in [0, T) \times \mathbb{R} \times [0, +\infty)\). Moreover, one can see from (4.23), (5.4) and (5.18) that there exists constant \(C > 0\) such that

\[
|\theta^*(t, z, x)| \leq C(1 + |z|), \quad \forall (t, z, x) \in D_T.
\]

With the continuity of \(\theta^*\) on \(D_T\) and the estimate (5.21), we can apply Theorem 2.2, Theorem 2.4 and Remark 2.1 in chapter 4 of Ikeda and Watanabe (1992) to conclude that the SDE below admits a weak solution: for \(t \in [0, T]\):

\[
\begin{aligned}
X_t &= -\int_0^t f(s, Z_s)ds + \int_0^t \theta^*(s, Z_s, \Phi(\dot{X})_s) \sigma ds + \int_0^t \theta^*(s, Z_s, \Phi(\dot{X})_s) \sigma dW_s, \\
\sigma Z_t &= \mu Z(Z_t)dt + \sigma Z(Z_t) dW_t, \\
\end{aligned}
\]

(5.22)

where the mapping \(\Phi : C([0, T]; \mathbb{R}) \rightarrow C([0, T]; \mathbb{R})\) satisfies that, for all \(\varphi \in C([0, T]; \mathbb{R})\),

(i) \(\Phi(\varphi)_t = \varphi_t + \eta_t\) for \(t \in [0, T]\), and \(\Phi(\varphi)_0 = \varphi_0\).

(ii) \(\Phi(\varphi)_t \geq 0\) for \(t \in [0, T]\).

(iii) \(t \rightarrow \eta_t\) is continuous, nonnegative and nondecreasing, and \(\eta_t = x \vee \int_0^t 1_{\{\varphi(s) = 0\}}ds\) for \(t \in [0, T]\). Define \(X^* := \Phi(\dot{X})\) and \(L^* := \Phi(\dot{X}) - \dot{X}\). Then \((X^*, L^*, W)\) solves the following reflected SDE:

\[
X^*_t = -\int_0^t f(s, Z_s)ds + \int_0^t \theta^*(s, Z_s, X^*_s) \sigma ds + \int_0^t \theta^*(s, Z_s, X^*_s) \sigma dW_s + L^*_t, \quad t \in [0, T],
\]

where \(L^*\) satisfies (iii). Therefore, we have shown that \(\theta^* \in \mathcal{U}_t\) is admissible.

Let us fix any \((t, z, x) \in [0, T) \times \mathbb{R} \times [0, +\infty)\), and \(\theta \in \mathcal{U}_t\). For any \(n > T^{-1}\), we define that

\[
\tau^*_n := \left(T - \frac{1}{n}\right) \wedge \inf\{s \geq t : |Z_s| + |X_s| > n\}.
\]

(5.23)
It holds that $\tau_n^t \uparrow T$ as $n \to \infty$, $\mathbb{P}\text{-a.s.}$ By applying Itô’s formula, we arrive at
\begin{equation}
\mathbb{E}_{t,z,x} \left[ -\int_t^{\tau_n^t} e^{-\rho s} dL_s^X + e^{-\rho \tau_n^t} v(\tau_n^t, Z_{\tau_n^t}, X_{\tau_n^t}) - e^{-\rho t} v(t, Z_t, X_t) \right] = \mathbb{E}_{t,z,x} \left[ \int_t^{\tau_n^t} e^{-\rho s} (v(s, Z_s, X_s) - 1) dL_s^X \right],
\end{equation}
where, for $\theta \in \mathbb{R}^n$, the operator $\mathcal{L}_\theta^t$ acted on $C^2(\mathbb{R} \times [0, \infty))$ is defined by
\begin{align*}
\mathcal{L}_\theta^t \varphi(z, x) := & \varphi_x(z, x) \theta^\top \mu + \frac{\varphi_{xx}(z, x)}{2} \theta^\top \sigma \sigma^\top \theta + \varphi_{xz}(z, x) \sigma^\top \varphi_{zz}(z, z) \sigma + \varphi(z, x) \mu_Z(z) \\
& + \varphi_{zz}(z, x) \frac{\sigma^2(z)}{2} - f(t, z, x) - \rho \varphi(z, x),
\end{align*}
for all $\varphi \in C^2(\mathbb{R} \times [0, \infty))$. The boundary condition in (5.17), together with (3.4), yields that
\begin{equation}
\mathbb{E}_{t,z,x} \left[ \int_t^{\tau_n^t} e^{-\rho s} (v(s, Z_s, X_s) - 1) dL_s^X \right] = \mathbb{E}_{t,z,x} \left[ \int_t^{\tau_n^t} e^{-\rho s} (v(s, Z_s, X_s) - 1) 1_{\{X_s = 0\}} dL_s^X \right] = 0.
\end{equation}
On the other hand, the HJB equation (5.17) satisfied by $v$ also gives that, for all $t \in [0, T]$, $\mathbb{P}\text{-a.s.}$
\begin{equation}
(v_t + \mathcal{L}_\theta^t v) (t, Z_t, X_t) \leq 0,
\end{equation}
where the equality holds in (5.25) if $\theta = \theta^*$. Hence, we arrive from (5.25) at
\begin{equation}
\mathbb{E}_{t,z,x} \left[ -\int_t^{\tau_n^t} e^{-\rho s} dL_s^X \right] \leq v(t, z, x) - \mathbb{E}_{t,z,x} \left[ e^{-\rho \tau_n^t} v(\tau_n^t, Z_{\tau_n^t}, X_{\tau_n^t}) \right].
\end{equation}

By applying the estimate (5.13), we have $|v(\tau_n^t, Z_{\tau_n^t}, X_{\tau_n^t})| \leq e^{\rho \tau_n^t} C(T - \tau_n^t) (1 + \sup_{s \in [t,T]} |Z_s|^p)$, $\mathbb{P}\text{-a.s.}$
By sending $n \to +\infty$ and noting that $\tau_n^t \uparrow T$, $\mathbb{P}\text{-a.s.}$, the dominated convergence theorem yields that
\begin{equation}
\lim_{n \to +\infty} \mathbb{E}_{t,z,x} \left[ e^{-\rho \tau_n^t} v(\tau_n^t, Z_{\tau_n^t}, X_{\tau_n^t}) \right] = 0.
\end{equation}
Therefore, as $n$ tends to $+\infty$ in (5.26), we have that, for all $\theta \in \mathcal{U}_t$,
\begin{equation}
\hat{J}(\theta; t, z, x) = \mathbb{E}_{t,z,x} \left[ -\int_t^T e^{-\rho s} dL_s^X \right] \leq v(t, z, x), \quad (t, z, x) \in \mathcal{D}_T,
\end{equation}
where the equality in (5.28) holds for $\theta = \theta^*$. This verifies that $\theta^* \in \mathcal{U}_t$ is an optimal strategy.

\section{Proofs of Main Results}
This section collects the proofs of some important results in the main body of the paper.

\textit{Proof of Lemma 2.2.} It is clear that, for $(a, v, z) \in [0, \infty) \times \mathbb{R}$,
\begin{equation}
u(a, v, z) = \inf_{\theta} \inf_C \mathbb{E} \left[ C_0 + \int_0^T e^{-\rho t} dC_t \right],
\end{equation}
and using integration by parts, we have $C_0 + \int_0^T e^{-\rho t} dC_t = e^{-\rho T} C_T + \rho \int_0^T e^{-\rho t} C_t dt$. For each fixed $	heta = (\theta_t)_{t \in [0,T]}$, we need to choose the optimal singular control $C = (C_t)_{t \in [0,T]}$ to minimize

$$\inf_C F(C), \quad \text{where } F(C) := \mathbb{E}\left[ e^{-\rho T} C_T + \rho \int_0^T e^{-\rho t} C_t dt \right],$$

subjecting to $C_t \geq A_t - V_t^\theta$ at each $t \in [0,T]$. Note that the cost functional $F(C)$ is strictly increasing in $C$. That is, if $C^1 \leq C^2$ and $C^1 \neq C^2$, then we have $F(C^1) < F(C^2)$. Therefore, the optimal choice of the control $C$ is the minimal nonnegative and nondecreasing process $C_t$ such that $C_t \geq A_t - V_t^\theta$ for $t \in [0,T]$. We claim the minimal process is the nondecreasing envelope $C_t^* := 0 \lor \sup_{s \leq t} (A_s - V_s^\theta)$. To wit, $C_t^*$ is nonnegative and satisfies the dynamic floor constraint. Let $\tilde{C}$ be another nonnegative and nondecreasing process satisfying $\tilde{C}_t \geq A_t - V_t^\theta$, $t \in [0,T]$. Suppose that $\tilde{C} \leq C^*$ and $\tilde{C} \neq C^*$. That is, there exists a $t \in [0,T]$ and a set $O$ with $\mathbb{P}(O) > 0$, such that $C_t(\omega) < C_t(\omega)$ for $\omega \in O$. By definition, we have

$$A_t(\omega) - V_t^\theta(\omega) \leq \tilde{C}_t(\omega) < C_t(\omega) = \sup_{s \leq t} (A_s - V_s^\theta(\omega)).$$

For each fixed $\omega \in O$, let $t^* < t$ be the time such that $A_{t^*}(\omega) - V_{t^*}^\theta(\omega) = \sup_{s \leq t} (A_s - V_s^\theta(\omega))$. It follows that $\tilde{C}_t(\omega) < A_{t^*}(\omega) - V_{t^*}^\theta(\omega) \leq \tilde{C}_{t^*}(\omega)$. We obtain a contradiction that the process $\tilde{C}$ is nondecreasing. Therefore, the original problem can be written as

$$u(a, v, z) = C_0^* + \inf_{\theta} \mathbb{E}\left[ \int_0^T e^{-\rho t} dC_t^* \right] = (a - v)^+ + \inf_{\theta} \mathbb{E}\left[ \int_0^T e^{-\rho t} d(0 \lor \sup_{s \leq t} (A_s - V_s^\theta)) \right],$$

which completes the proof. \hfill \square

**Proof of Proposition 4.1.** We first derive the representation of the partial derivative $h_u$ of the function $h$ w.r.t. the variable $u$. Let $(t, z) \in [0, T] \times \mathbb{R}$ be fixed. For any $u_2 > u_1 \geq 0$, it follows from (4.13) that

$$\frac{h(t, z, u_2) - h(t, z, u_1)}{u_2 - u_1} = -\int_t^T \mathbb{E}\left[ e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s^{u_2} - e^{-R_s^{u_1}}} \right] ds.$$

A direct calculation yields that, for $s \in [t, T]$,

$$\lim_{u_2 \downarrow u_1} \frac{e^{-R_s^{u_2} - e^{-R_s^{u_1}}} - e^{-R_s^{u_2} - e^{-R_s^{u_1}}}}{u_2 - u_1} = \begin{cases} -e^{-R_s^{u_1} - \rho s}, & \max_{r \in [t,s]} [-\sqrt{2\alpha}B^1_r - (\alpha - \rho)r] \leq u_1, \\ 0, & \max_{r \in [t,s]} [-\sqrt{2\alpha}B^1_r - (\alpha - \rho)r] > u_1. \end{cases}$$

As $\sup_{(s,u_1,u_2) \in [t,T] \times [0,\infty)^2} \left| \frac{e^{-R_s^{u_2} - e^{-R_s^{u_1}}}}{u_2 - u_1} \right| \leq 1$, the dominated convergence theorem yields that

$$\lim_{u_2 \downarrow u_1} \frac{h(t, z, u_2) - h(t, z, u_1)}{u_2 - u_1} = \mathbb{E}\left[ \int_t^T e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s^{u_1}} 1 \{ \max_{r \in [t,s]} [-\sqrt{2\alpha}B^1_r - (\alpha - \rho)r] \leq u_1 \} ds \right]$$

$$= \mathbb{E}\left[ \int_{t \wedge T}^{\tau_{u_1}^{t,z}} e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s^{u_1}} ds \right],$$

(6.1)

where $\tau_{u_1}^{t,z} := \inf \{ s \geq t; -\sqrt{2\alpha}B^1_s - (\alpha - \rho)s = u_1 \}$. On the other hand, for the case $u_1 > u_2 \geq 0$, similar to the computation of (6.1), we have that $\lim_{u_2 \uparrow u_1} \frac{h(t, z, u_2) - h(t, z, u_1)}{u_2 - u_1} = \lim_{u_2 \downarrow u_1} \frac{h(t, z, u_2) - h(t, z, u_1)}{u_2 - u_1}$. Therefore the representation (4.19) holds.
We next derive the representation of $h_{u_0}$. In fact, let $(t, z) \in [0, T] \times \mathbb{R}$ be fixed. For any $u_0, u_n \geq 0$ and $u_n \to u_0$ as $n \to \infty$, we have that, for $n \geq 1$,

$$
\Delta_n = \frac{h_u(t, z, u_n) - h_u(t, z, u_0)}{u_n - u_0} = \mathbb{E} \left[ \frac{1}{u_n - u_0} \int_{\tau_0}^{\tau_n} e^{-\rho s} f(s, M_s t, z) e^{-R_s u_n} ds \right]
+ \mathbb{E} \left[ \frac{1}{u_n - u_0} \int_{t}^{\tau_0} e^{-\rho s} f(s, M_s t, z) \left( e^{-R_s u_n} - e^{-R_s u_0} \right) ds \right]
+ \mathbb{E} \left[ \frac{1}{u_n - u_0} \int_{\tau_0}^{\tau_n} e^{-\rho s} f(s, M_s t, z) \left( e^{-R_s u_n} - e^{-R_s u_0} \right) ds \right] := \Delta_n^{(1)} + \Delta_n^{(2)} + \Delta_n^{(3)}. \quad (6.2)
$$

where $\tau_0 := \tau_{u_0} \wedge T$ and $\tau_n := \tau_{u_n} \wedge T$. In order to deal with $\Delta_n^{(1)}$, we first focus on the case where $u_n \downarrow u_0$ as $n \to \infty$. To this end, we introduce $\hat{\Delta}_n^{(1)} := \mathbb{E} \left[ \frac{\tau_n - \tau_0}{u_n - u_0} e^{-\rho \tau_0} f(\tau_0, M_{\tau_0} t, z) e^{-R_{\tau_0} u_0} \right]$. For any $m > 0$, it follows from the assumption (A$_f$) that

$$
|\Delta_n^{(1)} - \hat{\Delta}_n^{(1)}| \leq \mathbb{E} \left[ \frac{\tau_n - \tau_0}{u_n - u_0} \xi_n \right] \leq m \mathbb{E} [\xi_n] + C \left\{ 1 + \mathbb{E} \left[ \max_{s \in [t, T]} |M_s t, z|^2 \right] \right\} \mathbb{P} \left( \frac{\tau_n - \tau_0}{u_n - u_0} > m \right), \quad (6.3)
$$

where for $n \geq 1$,

$$
\xi_n := \max_{s \in [\tau_0, \tau_n]} \left| e^{-\rho s} f(s, M_s t, z) e^{-R_s u_n} - e^{-\rho \tau_0} f(\tau_0, M_{\tau_0} t, z) e^{-R_{\tau_0} u_0} \right|.
$$

Note that $\xi_n \leq C(1 + \sup_{s \in [t, T]} |M_s t, z|)$ for all $n \geq 1$ by using the assumption (A$_f$). For any $m > 0$, it follows that $\xi_n \downarrow 0$, $\mathbb{P}$-a.s. as $n \to +\infty$. Therefore, we have that $\xi_n \downarrow 0$, as $n \to \infty$, $\mathbb{P}$-a.s., and hence $\mathbb{E}[\xi_n] \to 0$ as $n \to \infty$. On the other hand, by setting $\mu := \alpha - \rho$, we have that

$$
\mathbb{P} \left( \frac{\tau_n - \tau_0}{u_n - u_0} > m \right) \leq \int_{m(u_n - u_0)}^{+\infty} \frac{u_n - u_0}{\sqrt{4\alpha \pi t^3}} e^{-\frac{(u_n - u_0 - \mu t)^2}{4\alpha \pi t}} dt \leq \sqrt{u_n - u_0} \int_{m}^{+\infty} \frac{1}{\sqrt{4\alpha \pi s^3}} ds.
$$

Letting $n \to +\infty$ in (6.3), we arrive at

$$
\lim_{n \to +\infty} |\Delta_n^{(1)} - \hat{\Delta}_n^{(1)}| = 0. \quad (6.4)
$$

Moreover, using the strong Markov property of Brownian motion with drift, it follows that

$$
\mathbb{E} \left[ \frac{\tau_n - \tau_0}{u_n - u_0} \mathbb{I}_{\tau_n > T} \right] = \int_0^{T - \tau_0} \frac{1}{\sqrt{4\alpha \pi s}} e^{-\frac{(u_n - u_0 - \mu t)^2}{4\alpha \pi t}} ds + \left( T - \tau_0 \right) \int_{T - \tau_0}^{+\infty} \frac{1}{\sqrt{4\alpha \pi s^3}} e^{-\frac{(u_n - u_0 - \mu s)^2}{4\alpha \pi s}} ds.
$$

Therefore, for $n \geq 1$,

$$
\hat{\Delta}_n^{(1)} = \mathbb{E} \left[ e^{-\rho \tau_0} f(\tau_0, M_{\tau_0} t, z) e^{-R_{\tau_0} u_0} \int_0^{T - \tau_0} \frac{1}{\sqrt{4\alpha \pi s}} e^{-\frac{(u_n - u_0 - \mu t)^2}{4\alpha \pi t}} ds \right]
+ \mathbb{E} \left[ (T - \tau_0) e^{-\rho \tau_0} f(\tau_0, M_{\tau_0} t, z) e^{-R_{\tau_0} u_0} \int_{T - \tau_0}^{+\infty} \frac{1}{\sqrt{4\alpha \pi s^3}} e^{-\frac{(u_n - u_0 - \mu s)^2}{4\alpha \pi s}} ds \right].
$$

This yields that

$$
\lim_{n \to +\infty} \Delta_n^{(1)} = \lim_{n \to +\infty} \hat{\Delta}_n^{(1)} = \mathbb{E} \left[ e^{-\rho \tau_0} f(\tau_0, M_{\tau_0} t, z) e^{-R_{\tau_0} u_0} \Gamma(\tau_0) \right],
$$

where, for $t \in [0, T]$,

$$
\Gamma(t) := \int_0^{T - t} \frac{1}{\sqrt{4\alpha \pi s}} e^{-\frac{s^2}{4\alpha \pi}} ds + (T - t) \int_{T - t}^{+\infty} \frac{1}{\sqrt{4\alpha \pi s^3}} e^{-\frac{s^2}{4\alpha \pi}} ds. \quad (6.5)
$$
Finally, similar to (A2019), we can show by the assumption (A) that

\[ \Delta_n^{(3)} = \lim_{n \to +\infty} \Delta_n^{(3)} = \lim_{n \to +\infty} \mathbb{E} \left[ \frac{1}{\eta_n - \rho_n} \int_{t_0}^{\tau_n} e^{-\rho t_s f(s, M_{s}^{t,z})} \left( e^{R_{s}^{t,u,n}} - e^{R_{s}^{t,u,0}} \right) ds \right] \]

where, for \( t \in [0, T] \),

\[ \hat{\xi}_n := \max_{s \in [t_0, \tau_n]} e^{-\rho t_s f(s, M_{s}^{t,z})} e^{R_{s}^{t,u,n}} - e^{R_{s}^{t,u,0}}. \]

By the assumption (A), we have \( \mathbb{E}[\hat{\xi}_n] \to 0 \) as \( n \to \infty \). Hence \( \lim_{n \to +\infty} |\Delta_n^{(3)}| = 0 \). Putting all the pieces together, we can derive from the decomposition (6.2) and \( R_{t_0}^{t,u} = 0 \) that

\[ h_{uu}(t, z, u_0) = \mathbb{E} \left[ e^{-\rho \tau_0 f(\tau_0, M_{\tau_0}^{t,z})} \Gamma(\tau_0) \right] + \mathbb{E} \left[ \int_{t_0}^{\tau_0} e^{-\rho t_s f(s, M_{s}^{t,z})} e^{R_{s}^{t,u,0}} ds \right]. \] (6.6)

We next derive the representations of \( h_y, h_{yy} \) and \( h_{uy} \). Let \( (t, u) \in [0, T] \times [0, \infty) \). In view of the assumption (A), Theorem 3.3.2 in Kunita (2019) yields that, for \( s \in [t, T] \), the family \( (M_{s}^{t,z})_{s \in \mathbb{R}} \) admits a modification which is continuously differentiable w.r.t. \( z \). Moreover, \( \partial_z M_{s}^{t,z} \) is continuous in \( z \) and satisfies the following SDE for \( s \in [t, T] \) that

\[ \partial_z M_{s}^{t,z} = 1 + \int_{t}^{s} \mu_{Z}(M_{r}^{t,z}) \partial_{r} M_{r}^{t,z} dr + \varrho \int_{t}^{s} \sigma'_{Z}(M_{r}^{t,z}) \partial_{r} M_{r}^{t,z} dB_{r}^{1} + \sqrt{1 - \varrho^2} \int_{t}^{s} \sigma'_{Z}(M_{r}^{t,z}) \partial_{r} M_{r}^{t,z} dB_{r}^{2}, \] (6.7)

and for any \( p \geq 2 \), the following moment estimate holds that

\[ \sup_{z \in \mathbb{R}} \mathbb{E} \left[ \max_{s \in [t, T]} |\partial_z M_{s}^{t,z}|^p \right] < +\infty. \] (6.8)

In terms of (4.13), for distinct \( z, \hat{z} \in \mathbb{R} \) and some constant \( C > 0 \), we have that

\[ \frac{h(t, z, u) - h(t, \hat{z}, u)}{z - \hat{z}} = -\mathbb{E} \left[ \int_{t}^{T} e^{-\rho s - R_{s}^{t,u} f(s, M_{s}^{t,z})} - f(s, M_{s}^{t,z}) | \frac{z - \hat{z}}{z - \hat{z}} ds \right]. \] (6.9)

Finally, similar to (4.19), we also have that

\[ \lim_{n \to +\infty} \Delta_n^{(2)} = \lim_{n \to +\infty} \mathbb{E} \left[ \frac{1}{u_n - u_0} \int_{t_0}^{\tau_n} e^{-\rho t_s f(s, M_{s}^{t,z})} \left( e^{-R_{s}^{t,u,n}} - e^{-R_{s}^{t,u,0}} \right) ds \right] = \mathbb{E} \left[ \int_{t_0}^{\tau_n} e^{-\rho t_s f(s, M_{s}^{t,z})} e^{-R_{s}^{t,u,0}} ds \right]. \]
By the assumption \((A_f)\), for \(s \in [t, T]\), the next results hold \(\mathbb{P}\)-a.s. that
\[
\begin{cases}
f(s, M^{t,z}_s) - f(s, M^{t,z}_s) \xrightarrow{\tilde{z} \to z} f'(s, M^{t,z}_s) \frac{\partial_z M^{t,z}_s}{z - \tilde{z}}, \\
f(s, M^{t,z}_s) - f(s, M^{t,z}_s) \leq C \left| \frac{M^{t,z}_s - M^{t,z}_s}{z - \tilde{z}} \right|.
\end{cases}
\]

We have from (6.8) that, for any \(p \geq 2\), \(\sup_{\tilde{z} \neq z} \mathbb{E} \left[ \left| \frac{M^{t,z}_s - M^{t,z}_s}{z - \tilde{z}} \right|^p \right] < +\infty\). This implies that \(\frac{M^{t,z}_s - M^{t,z}_s}{z - \tilde{z}}\) is uniformly integrable. Therefore, in view of (6.9), we arrive at
\[
h_z(t, z, u) = -\lim_{\tilde{z} \to z} \mathbb{E} \left[ \int_t^T e^{-\rho s - R^1 u} \frac{f(s, M^{t,z}_s) - f(s, M^{t,z}_s)}{z - \tilde{z}} ds \right] = -\mathbb{E} \left[ \int_t^T e^{-\rho s - R^1 u} f'(s, M^{t,z}_s) \frac{\partial_z M^{t,z}_s}{z - \tilde{z}} ds \right],
\]
where \(f'(t, z)\) denotes the partial derivative of \(f\) w.r.t. \(z\). Similarly, we can obtain that
\[
h_{zu}(t, z, u) = \mathbb{E} \left[ \int_t^T e^{-\rho s - R^1 u} f'(s, M^{t,z}_s) \frac{\partial_z M^{t,z}_s}{z - \tilde{z}} ds \right].
\]

We next derive the expression of \(h_{yy}\). To this end, we need the dynamics of \(\partial_z^2 M^{t,z}_s\) for \(s \in [t, T]\). Following a similar argument of Theorem 3.4.2 of Kunita (2019), we can deduce that
\[
\partial_z^2 M^{t,z}_s = \int_t^s \{ \mu''_Z(M^{t,z}_r) |\partial_z M^{t,z}_r|^2 + \mu'_Z(M^{t,z}_r) \partial_z^2 M^{t,z}_r \} dr + \vartheta \int_t^s \{ \sigma''_Z(M^{t,z}_r) |\partial_z M^{t,z}_r|^2 + \sigma'_Z(M^{t,z}_r) \partial_z^2 M^{t,z}_r \} dB_r + \sqrt{1 - \vartheta^2} \int_t^s \{ \sigma''_Z(M^{t,z}_r) |\partial_z M^{t,z}_r|^2 + \sigma'_Z(M^{t,z}_r) \partial_z^2 M^{t,z}_r \} dB_r,
\]
and for all \(p \geq 1\), it holds that
\[
\sup_{z \in \mathbb{R}} \mathbb{E} \left[ \max_{s \in [t, T]} |\partial_z^2 M^{t,z}_s|^{2p} \right] < +\infty.
\]

The chain rule with the assumption \((A_f)\) yields that, \(\mathbb{P}\)-a.s.
\[
\frac{f'(s, M^{t,z}_s) \partial_z M^{t,z}_s - f'(s, M^{t,z}_s) \partial_z M^{t,z}_s}{z - \tilde{z}} \xrightarrow{\tilde{z} \to z} \frac{f''(s, M^{t,z}_s) |\partial_z M^{t,z}_s|^2}{z - \tilde{z}} + \frac{f'(s, M^{t,z}_s) \partial_z^2 M^{t,z}_s}{z - \tilde{z}},
\]
and there exists a constant \(C > 0\) such that, for all \(\tilde{z} \neq z\),
\[
\left| \frac{f'(s, M^{t,z}_s) \partial_z M^{t,z}_s - f'(s, M^{t,z}_s) \partial_z M^{t,z}_s}{z - \tilde{z}} \right| \leq C \left| \frac{M^{t,z}_s - M^{t,z}_s}{z - \tilde{z}} \right| \partial_z M^{t,z}_s + C \left| \partial_z M^{t,z}_s - \partial_z M^{t,z}_s \right|.
\]

Let \(p \geq 1\) and define \(I(s, z, \tilde{z}) := \mathbb{E} \left[ \max_{r \in [t, s]} |\partial_z M^{t,z}_r - \partial_z M^{t,z}_r|^{2p} \right]\) for \((s, z, \tilde{z}) \in [t, T] \times \mathbb{R}^2\). It follows from (6.7) and the assumption \((A_Z)\) that, for some \(C > 0\),
\[
I(s, z, \tilde{z}) \leq C \int_t^s \left\{ I(r, z, \tilde{z}) + \sup_{u \in \mathbb{R}} \mathbb{E} \left[ \max_{s \in [t, T]} |\partial_z M^{t,u}_s|^{2p} \right] \mathbb{E} \left[ \left| M^{t,z}_r - M^{t,z}_r \right|^{2p} \right] \right\} dr.
\]

The Gronwall’s lemma yields that, for all $s \in [t, T]$,

$$
\sup_{\tilde{z} \neq z} \mathbb{E} \left[ \sup_{r \in [t, s]} \left| \frac{\partial_{\tilde{z}} M_r^{t, z} - \partial_{	ilde{z}} M_t^{t, z}}{z - \tilde{z}} \right|^{2p} \right] < +\infty,
$$

and hence the left hand side of (6.12) with $\tilde{z} \neq z$ and $s \in [t, T]$ is uniformly integrable. Then

$$
\begin{align*}
\dot{h}_{zz}(t, z, u) &= -\lim_{\tilde{z} \to z} \frac{d}{dt} \mathbb{E} \left[ \int_t^T e^{-\rho s-R_s^{t,u}} \left( f'(s, M_s^{t, z}) \partial_{z} M_s^{t, z} - f'(s, M_s^{t, \tilde{z}}) \partial_{z} M_s^{t, \tilde{z}} \right) ds \right] \\
&= -\mathbb{E} \left[ \int_t^T e^{-\rho s-R_s^{t,u}} \left( f''(s, M_s^{t, z}) \left| \partial_{z} M_s^{t, z} \right|^2 + f'(s, M_s^{t, z}) \partial_{zz} M_s^{t, z} \right) ds \right], \tag{6.13}
\end{align*}
$$

where $f''(t, z)$ denotes the second-order partial derivative of $f$ w.r.t. $z$.

Next, we derive the representation of $h_t$. Let us consider the solutions $M_t^{t, u} = (M_s^{t, u})_{s \in [t, T]}$ of SDE (4.14) with parameters $(t, z) \in [0, T] \times \mathbb{R}$ and $(\hat{t}, \hat{z}) \in [0, T] \times \mathbb{R}$ respectively. Moreover, for $r \geq 0$, we introduce $\mathcal{F}^i_r := \mathcal{F}_{t+r}$, $B_{r}^{1,i} := B_{r+t}^1 - B_1^1$, and $B_{r}^{2,i} := B_{r+t}^2 - B_1^2$ and define $\mathcal{F}^i_r$, $B_{r}^{i,t}$, $i = 1, 2$ for $r \geq 0$ in a similar way. It is not difficult to check that

$$
(M_t^{t, u}, B_{r}^{1,i}, B_{r}^{2,i})_{r \geq 0} \overset{d}{=} (M_t^{t, u}, B_{r}^{1,i}, B_{r}^{2,i})_{r \geq 0}. \tag{6.14}
$$

It holds that, for any $\delta \in [0, T-t]$, 

$$
\begin{align*}
\dot{h}(t+\delta, z, u) &= -\mathbb{E} \left[ \int_{t+\delta}^T e^{-\rho s-R_s^{t,u}} ds \right] = -\mathbb{E} \left[ \int_{t}^{T-\delta} e^{-\rho s-R_s^{t,u}} ds \right].
\end{align*}
$$

It follows from the dominated convergence theorem that

$$
\lim_{\delta \downarrow 0} \frac{1}{\delta} \left( h(t+\delta, z, u) - h(t, z, u) \right) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{E} \left[ \int_{T-\delta}^{T} e^{-\rho s-R_s^{t,u}} ds \right] = \mathbb{E} \left[ e^{-\rho T} f(T, M_T^{t, u}) e^{-R_T^{t,u}} \right].
$$

Similarly, for $t \in (0, T]$, we have that

$$
\lim_{\delta \downarrow 0} \frac{1}{\delta} \left( h(t, u, z) - h(t-\delta, z, u) \right) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{E} \left[ \int_{T-\delta}^{T} e^{-\rho s-R_s^{t,u}} ds \right] = \mathbb{E} \left[ e^{-\rho T} f(T, M_T^{t, u}) e^{-R_T^{t,u}} \right].
$$

Therefore, we conclude that, for $(t, z, u) \in D_T$,

$$
\dot{h}_t(t, z, u) = \mathbb{E} \left[ e^{-\rho T} f(T, M_T^{t, u}) e^{-R_T^{t,u}} \right]. \tag{6.15}
$$

At last, we verify the continuity of $h_t$, $h_{uu}$, $h_{yu}$ and $h_{yy}$ in $(t, y, u)$ using expressions (6.15), (6.6), (6.11) and (6.13). In fact, by Theorem 3.4.3 of Kunita (2019), we have that $M^{t, z}_r$, $\partial_z M^{t, z}_r$ and $\partial_{zz} M^{t, z}_r$ for $s \in [t, T]$ admit the respective modifications which are continuous in $(t, z, u)$, P-a.s.. Moreover, by (4.16) and (4.17), $R^{t,u}_s$ is also continuous in $(t, u, s)$, P-a.s.. Then, we follow from the dominated convergence theorem that $h_t$, $h_{zu}$ and $h_{zz}$ are continuous in $(t, z, u)$. For the continuity of $h_{uu}$, in view of (6.14), for any $\epsilon \in \mathbb{R}$ satisfying $\tau^\epsilon_{0} := \tau^\epsilon_{u0} \wedge (T - \epsilon) \in [t, T]$, 

$$
\begin{align*}
\dot{h}_{uu}(t+\epsilon, z, u_0) &= \mathbb{E} \left[ e^{-\rho \tau_{0}^{\epsilon}} f(\tau_{0}^{\epsilon}, M_{\tau_{0}^{\epsilon}}^{t, z}) f(\tau_{0}^{\epsilon}, M_{\tau_{0}^{\epsilon}}^{t, z}) \right] + \mathbb{E} \left[ \int_{t}^{\tau_{0}^{\epsilon}} e^{-\rho s} f(s, M_s^{t, z}) e^{-R_s^{t,u_0}} ds \right]. \tag{6.16}
\end{align*}
$$

Note that $\tau_{0}^{\epsilon}$ is continuous in $(u_0, \epsilon)$, P-a.s.. The continuity of $h_{uu}$ in $(t, z, u)$ follows from (6.16) and the dominated convergence theorem, which completes the whole proof. □
Proof of Theorem 4.2. For \((t, u) \in [0, T] \times [0, \infty)\), we define \(B_{s,t}^{t,u} := u + \sqrt{2\alpha}(B_s^1 - B_t^2) + (\alpha - \rho)(s - t)\) for \(s \in [t, T]\) and \(M^{t,z}_s = \langle M_s^z \rangle_{s \in [t, T]}\) for \((t, z) \in [0, T] \times \mathbb{R}\) is the unique (strong) solution of SDE (4.14) under the assumption \((A_Z)\). By Remark 4.17 of Chapter 5 in Karatzas and Shreve (1991), the assumption \((A_Z)\) guarantees that the time-homogeneous martingale problem on \((M^{t,z}_s, B_{s,t}^{t,u})_{s \in [t, T]}\) is well posed. By applying Theorem 5.4.20 in Karatzas and Shreve (1991), \((M^{t,z}_s, B_{s,t}^{t,u})\) is a strong Markov process with \((t, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+\). For \(\varepsilon \in (0, u)\), let us define that

\[
\tau_{t}^\varepsilon := \inf \{ s \geq t; |B_{s,t}^{t,u} - u| \geq \varepsilon \text{ or } |M^{t,z}_s - z| \geq \varepsilon \} \land T. \tag{6.17}
\]

Because the paths of \((M^{t,z}_s, B_{s,t}^{t,u})\) are continuous, we have \(\tau_{t}^\varepsilon > t\), \(\mathbb{P}\)-a.s.. For any \(\hat{t} \in [t, T]\), it holds that

\[
B_{\hat{t},t}^{t,u} = R_{\hat{t} \land \tau_t^\varepsilon}^{t,u}. \tag{6.18}
\]

In fact, if \(\tau_{t}^\varepsilon \leq \hat{t}\), the following two cases may happen:

(i) \(|B_{\tau_{t}^\varepsilon}^{t,u} - u| < \varepsilon\): this yields that \(0 < -\varepsilon + u < B_{\tau_{t}^\varepsilon}^{t,u} < \varepsilon + u\), and hence (6.18) holds.

(ii) \(|B_{\tau_{t}^\varepsilon}^{t,u} - u| \geq \varepsilon\): this implies that \(B_{\tau_{t}^\varepsilon}^{t,u} = u + \varepsilon > 0\) or \(B_{\tau_{t}^\varepsilon}^{t,u} = u - \varepsilon > 0\), and hence (6.18) holds.

If \(\hat{t} < \tau_{t}^\varepsilon\), then \(|B_{\hat{t}}^{t,u} - u| < \varepsilon\) and \(|M_{\hat{t}}^{t,z} - z| < \varepsilon\). This results in that \(0 < -\varepsilon + u < B_{\hat{t}}^{t,u} < \varepsilon + u\), and hence (6.18) holds. It follows from (6.18) and the strong Markov property that

\[
-\mathbb{E} \left[ \int_{\hat{t} \land \tau_t^\varepsilon}^{T} f(t, M_s^{t,z}) e^{-R_s^{t,u} - \rho s} ds \mid \mathcal{F}_{\hat{t} \land \tau_t^\varepsilon} \right] = h \left( \hat{t} \land \tau_t^\varepsilon, M_{\hat{t} \land \tau_t^\varepsilon}^{t,z}, B_{\hat{t} \land \tau_t^\varepsilon}^{t,u} \right),
\]

where, thanks to (4.13), the function \(h\) is given by

\[
-\mathbb{E} \left[ \int_{\hat{t} \land \tau_t^\varepsilon}^{T} e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s} ds \right].
\]

Therefore, for \((t, z, u) \in \mathcal{D}_T\), it holds that

\[
h(t, z, u) = \mathbb{E} \left[ h \left( \hat{t} \land \tau_t^\varepsilon, M_{\hat{t} \land \tau_t^\varepsilon}^{t,z}, B_{\hat{t} \land \tau_t^\varepsilon}^{t,u} \right) - \int_{\hat{t} \land \tau_t^\varepsilon}^{T} e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s} ds \right]. \tag{6.19}
\]

By Proposition 4.1 and Itô’s formula, we have that

\[
\frac{1}{t - \hat{t}} \mathbb{E} \left[ \int_{\hat{t}}^{\hat{t} \land \tau_t^\varepsilon} e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s} ds \right] = \frac{1}{t - \hat{t}} \mathbb{E} \left[ h \left( \hat{t} \land \tau_t^\varepsilon, M_{\hat{t} \land \tau_t^\varepsilon}^{t,z}, B_{\hat{t} \land \tau_t^\varepsilon}^{t,u} \right) - h(t, z, u) \right]
\]

\[
= \frac{1}{t - \hat{t}} \mathbb{E} \left[ \int_{\hat{t}}^{\hat{t} \land \tau_t^\varepsilon} (h_t + L h) \left( s, M_s^{t,z}, B_{s,t}^{t,u} \right) ds \right] + \frac{1}{t - \hat{t}} \mathbb{E} \left[ \int_{\hat{t}}^{\hat{t} \land \tau_t^\varepsilon} h_u \left( s, M_{s,t}^{t,z}, B_{s,t}^{t,u} \right) dR_s \right], \tag{6.20}
\]

where the operator \(L\) acted on \(C^2(\mathbb{R} \times [0, \infty))\) is defined for \(g \in C^2(\mathbb{R} \times [0, \infty))\) that

\[
\mathcal{L} g := \alpha g_{uu} + (\alpha - \rho) g_u + \phi(y) g_{uz} + \mu Z(z) g_z + \frac{\sigma_z^2(z)}{2} g_{zz}. \tag{6.21}
\]

By (6.17), the assumption \((A_f)\) implies that \((e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s})_{s \in [t, \hat{t} \land \tau_t^\varepsilon]}\) is bounded. The bounded convergence theorem yields that

\[
\lim_{\hat{t} \downarrow t} \frac{1}{t - \hat{t}} \mathbb{E} \left[ \int_{\hat{t}}^{\hat{t} \land \tau_t^\varepsilon} e^{-\rho s} f(s, M_s^{t,z}) e^{-R_s} ds \right] = f(t, z) e^{-u - \rho t}.
\]

\[\text{---}

\(1\) The definition of well-posedness of a time-homogeneous martingale problem can be found in Definition 4.15 of Chapter 5 in Karatzas and Shreve (1991), page 320.

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Similarly, we have that
\[
\lim_{\tilde{t} \to t} \frac{1}{1-t} \mathbb{E} \left[ \int_{\tilde{t}}^{\tilde{t}+\tau_{\tilde{t}}^n} (h_t + \mathcal{L}h) \left( s, M_s^{t,z}, B_s^{t,u} \right) ds \right] = (h_t + \mathcal{L}h) (t, z, u).
\]

Note that \( R_s^{t,u} > 0 \) on \( s \in [t, \tilde{t} + \tau_{\tilde{t}}^n] \) for all \((t, u) \in [0, T] \times [0, \infty)\). We then have that
\[
\frac{1}{1-t} \mathbb{E} \left[ \int_{\tilde{t}}^{\tilde{t}+\tau_{\tilde{t}}^n} h_u \left( s, M_s^{t,z}, B_s^{t,u} \right) dL_s^R \right] = 0.
\]

By applying (6.20), we obtain that \( (h_t + \mathcal{L}h)(t, z, u) = f(t, z)e^{-u-\rho t} \) on \((t, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+\).

We next verify that the function \( h \) in (4.13) satisfies the boundary conditions of the Neumann problem (4.20). By the representation form (4.13), it is easy to see that \( h(T, z, u) = 0 \) for all \((z, u) \in \mathbb{R} \times [0, \infty)\).

It remains to show the validity of homogeneous Neumann boundary condition. In fact, for \( s \in [t, T] \), we have that, for any positive sequence \((u_n)_{n \geq 1}\) satisfying \( u_n \downarrow 0 \) as \( n \to \infty \),
\[
\bigcup_{n \geq 1} A^n_s := \bigcup_{n \geq 1} \left\{ \max_{r \in [t, s]} \left\{ -\sqrt{2\alpha} B^1_r - (\alpha - \rho) r \right\} > u_n \right\} \in \mathcal{F}_s, \quad \text{and} \quad \mathbb{P} \left( \bigcup_{n \geq 1} A^n_s \right) = 1. \tag{6.22}
\]
In view of (4.19) in Proposition 4.1 and the assumption \((A_2)\), it follows from the dominated convergence theorem that
\[
h_u(t, z, 0) = \lim_{n \to \infty} \int_t^T \mathbb{E} \left[ f(s, M_s^{t,z})e^{-R_s^{t,u}-\rho s}1_{\{A^n_s\}} \right] ds = 0, \quad (t, z) \in [0, T] \times \mathbb{R}. \tag{6.23}
\]
That is, the Neumann boundary condition in (4.20) holds.

We next assume that the Neumann problem (4.20) admits a classical solution \( h \) with a polynomial growth. For \( n \in \mathbb{N} \) and \( t \in [0, T] \), we define \( \tau_{\tilde{t}}^n := \inf \{ s \geq t; |M_s^{t,z}| \geq n \text{ or } |R_s^{t,u}| \geq n \} \wedge T \). Ró’s formula gives that, for \((t, z, u) \in \mathcal{D}_T\),
\[
\mathbb{E} \left[ h \left( \tau_{\tilde{t}}^n, M_{\tau_{\tilde{t}}^n}^{t,z}, R_{\tau_{\tilde{t}}^n}^{t,u} \right) \right] = h(t, z, u) + \mathbb{E} \left[ \int_t^{\tau_{\tilde{t}}^n} (h_t + \mathcal{L}h) \left( r, M_r^{t,z}, R_r^{t,u} \right) dr \right] \tag{6.24}
\]
\[
+ \mathbb{E} \left[ \int_t^{\tau_{\tilde{t}}^n} h_u(r, M_r^{t,z}, R_r^{t,u})1_{\{R_r^{t,u}=0\}} dL_r^R \right] = h(t, z, u) + \mathbb{E} \left[ \int_t^{\tau_{\tilde{t}}^n} f(r, M_r^{t,z})e^{-R_r^{t,u}-\rho r} dr \right].
\]
Moreover, the polynomial growth of \( h \) implies the existence of a constant \( C = C_T > 0 \) such that, for some \( p \geq 1 \), \( |h(t, z, u)| \leq C \left\{ 1 + \max_{r \in [t, T]} |M_r^{t,z}|^p + \max_{r \in [t, T]} |R_r^{t,u}|^p \right\} \). Note that \( \lim_{n \to \infty} h(\tau_{\tilde{t}}^n, M_{\tau_{\tilde{t}}^n}^{t,z}, R_{\tau_{\tilde{t}}^n}^{t,u}) = h(T, M_T^{t,z}, R_T^{t,u}) = 0 \) in view of (4.20). Letting \( n \to \infty \) on the both sides of (6.24), by dominated convergence theorem and monotone convergence theorem, we obtain the representation (4.13) for the solution \( h \), which completes the proof.

**Proof of Corollary 4.4.** We first show the existence of a classical solution to the Neumann problem (4.12). By Theorem 4.2, the function \( h \) defined by (4.13) solves the Neumann problem (4.20). It readily follows that for \((t, z, y) \in [0, T] \times \mathbb{R} \times (0, 1)\), \( \hat{v}(t, z, y) := e^\rho t h(t, z, -\ln y) \) solves the Neumann problem (4.12).

The existence of a classical solution to the Neumann problem (4.12) then follows by Proposition 4.1. For the uniqueness, let \( \hat{v}(i) \) for \( i = 1, 2 \) be two classical solutions of the Neumann problem (4.12) such that \( h^{(i)}(t, z, u) := e^{-\rho t} \hat{v}^{(i)}(t, z, e^{-u}) \) for \((t, z, u) \in \mathcal{D}_T\) satisfies the polynomial growth for \( i = 1, 2 \). Theorem 4.2 implies that both \( h^{(1)} \) and \( h^{(2)} \) admit the probabilistic representation (4.13), and hence \( h^1 = h^2 \) on \( \mathcal{D}_T \).

Therefore \( \hat{v}^{(1)}(t, z, y) = e^\rho t h^{(1)}(t, z, -\ln y) = e^\rho t h^{(2)}(t, z, -\ln y) = \hat{v}^{(2)}(t, z, y) \) for \((t, z, y) \in [0, T] \times \mathbb{R} \times (0, 1)\). Moreover, the strict convexity of \((0, 1) \ni y \to \hat{v}(t, z, y)\) for fixed \((t, z) \in [0, T] \times \mathbb{R}\) follows from Remark 4.3. The corollary is proved as desired. \(\square\)
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References

G. Barles, C. Daher and M. Romano (1994): Optimal control on the $L^\infty$ norm of a diffusion process. *SIAM J. Contr. Optim.* 32(3), 612-634.

E.N. Barron and H. Ishii (1989): The bellman equation for minimizing the maximum cost. *Nonlinear Anal.: Theor. Meth. Appl.* 13(9), 1067-1090.

E.N. Barron (1993): The bellman equation for control of the running max of a diffusion and applications to look-back options. *Appl. Anal.* 48, 205-222.

E. Bayraktar and M. Egami (2008): An analysis of monotone follower problems for diffusion processes. *Math. Oper. Res.* 33(2), 336-350.

O. Bokanowski, A. Picarelli and H. Zidani (2015): Dynamic programming and error estimates for stochastic control problems with maximum cost. *Appl. Math. Optim.* 71, 125-163.

B. Bouchard, R. Elie and C. Imbert (2010): Optimal control under stochastic target constraints. *SIAM J. Contr. Optim.* 48(5), 3501-3531.

S. Browne (2000): Risk-constrained dynamic active portfolio management *Manage. Sci.* 46(9), 1188-1199.

Y. Chow, X. Yu and C. Zhou (2020): On dynamic programming principle for stochastic control under expectation constraints. *J. Optim. Theor. App.* 185(3), 803-818.

S. Deng, X. Li, H. Pham and X. Yu (2020): Optimal consumption with reference to past spending maximum. *Preprint, arXiv:2006.07223.*

A. Gaivoronski, S. Krylov and N. Wijst (2005): Optimal portfolio selection and dynamic benchmark tracking. *Euro. J. Oper. Res.* 163, 115-131.

M. Di Giacinto, S. Federico and F. Gozzi (2011): Pension funds with a minimum guarantee: a stochastic control approach. *Finance Stoch.* 15, 297-342.

M. Di Giacinto, S. Federico, F. Gozzi and E. Vigna (2014): Income drawdown option with minimum guarantee. *Euro. J. Oper. Res.* 234, 610-624.

P. Guasoni, G. Huberman and D. Ren (2020): Shortfall aversion. *Math. Financ.* forthcoming.

M. Harrison (1985): *Brownian Motion and Stochastic Flow Systems.* John Wiley and Son, New York.

N. Ikeda and S. Watanabe (1992): *Stochastic Differential Equations and Diffusion Processes.* North-Holland Mathematical Library, North Holland.

I. Karatzas, J. Lehoczky, S.E. Shreve and G. Xu (1991): Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Contr. Optim.* 29(3), 702-730.

I. Karatzas and S.E. Shreve (1984): Connections between optimal stopping and singular stochastic control: i. monotone follower problems. *SIAM J. Contr. Optim.* 22(6), 856-877.

I. Karatzas and S.E. Shreve (1991): *Brownian Motion and Stochastic Calculus, 2nd Ed.* Springer-Verlag, New York.

N. El Karoui, M. Jeanblanc and V. Lacoste (2005): Optimal portfolio manangement with American capital guarantee. *J. Econ. Dyn. Contr.* 29, 449-468.
N. El Karoui and A. Meziou (2006): Constrained optimization with respect to stochastic dominance: application to portfolio insurance. *Math. Financ.* 16(1), 103-117.

A. Kröner, A. Picarelli and H. Zidani (2018): Infinite horizon stochastic optimal control problems with running maximum cost. *SIAM J. Contr. Optim.* 56(5), 3296-3319.

H. Kunita (2019): *Stochastic Flows and Jump-Diffusions.* Springer-Verlag, New York.

J. Sekine (2012): Long-term optimal portfolios with floor. *Finance Stoch.* 16, 369-401.

O. Strub and P. Baumann (2018): Optimal construction and rebalancing of index-tracking portfolios. *Euro. J. Oper. Res.* 264, 370-387.

A. Weerasinghe and C. Zhu (2016): Optimal inventory control with path-dependent cost criteria. *Stoch. Process. Appl.* 126, 1585-1621.

D. Yao, S. Zhang and X. Zhou (2006): Tracking a financial benchmark using a few assets. *Oper. Res.* 54(2), 232-246.