Reconstructing the formalism of quantum mechanics
in the “contextual objectivity” point of view.

Philippe Grangier

Laboratoire Charles Fabry, Institut d’Optique Théorique et Appliquée, F-91403 Orsay, France

In a previous preprint we introduced a “contextual objectivity” formulation of quantum mechanics (QM). A central feature of this approach is to define the quantum state in physical rather than in mathematical terms, in such a way that it may be given an “objective reality”. Here we use some ideas about the system dimensionality, taken from, to propose a possible axiomatic approach to QM. In this approach the structure of QM appears as a direct consequence of the non-commutative character of the (classical geometrical) group of “knobs transformations”, that relate between themselves the different positions of the measurement apparatus.

I. INTRODUCTION

In a previous preprint we introduced and discussed a “physical” (as opposed to mathematical) definition of a quantum state that reads in the following way:

The quantum state of a physical system is defined by the values of a complete set of physical quantities, which can be predicted with certainty and measured repeatedly without perturbing in any way the system (the set of quantities is complete in the sense that the value of any other quantity which satisfies the same criteria is a function of the set values).

As discussed in detail in ref., and briefly recalled below, this definition is in full agreement with the usual formalism of QM (it was actually deduced from it). But here we would like to address another question: is possible to deduce the usual formalism of QM from this definition? In order to address this question, we will discuss a procedure that was introduced by Lucien Hardy in. Then we will introduce an alternative approach, that differs significantly from, since it is based upon geometry rather than upon probabilities.

II. A FEW WORDS ABOUT CONTEXTUAL OBJECTIVITY

The definition of the quantum state that is given above is clearly in agreement with the usual formalism of QM, as it can be seen when using the notion of “complete set of commuting observables” (CSO). A quantum state is specified by the ensemble of eigenvalues corresponding to a CSO, that can obviously be measured repeatedly without perturbing in any way the system. Actually, as said in the definition, any physical quantity that satisfies the definition can be expressed as a function of the CSO ones. This also appears in the sentence “measured repeatedly without perturbing in any way the system”: if the subset is not a CSO, a measurement will generally change the state. It should be clear also that unitary evolution from Schroedinger’s equation transforms a state which satisfies our definition into a similar state, associated with a different set of physical quantities, corresponding to new well-defined measurements (that may however be not easy to perform).

An obvious but fundamental point is that given a quantum state, not all possible physical quantities can be predicted with certainty, but only a subset of them. That the subset is the largest possible set of independent quantities is just the definition of a CSO. How to deal with physical observables which do not commute with those of the CSO is a crucial point, that is discussed in detail in. It is also argued in that within this framework, there is no need to add a “measurement postulate”: this postulate is already contained in the definition of the quantum state, i.e., the very possibility to define a quantum state and the measurement postulate are essentially identical. For more details about that, including how it relates to the EPR “paradox” or to the decoherence approach, we refer the reader to.

It is worth pointing out that our definition implies that some “objectivity” can be attached to the quantum state. This is because the quantum state is associated with a fully predictable course of events, that is independent of the observer. Actually, our definition assumes that the sentence “predicted with certainty and measured repeatedly” itself has a meaning, i.e., that we are able to make experiments that correspond to well-defined measurements, and that give predictable outcomes. This is why we call our point of view “contextual objectivity”: the quantum state does have an objective existence, but its definition is inferred from observations that are made at the macroscopic level. We point out that there is no need to refer to “observer’s consciousness” or anything like that: what we need is simply the usual classical world, as the place where the measurements are made, and where results can be recorded by any (conscious or unconscious) observer. How exactly the “quantum reality” is related with the “macroscopic reality” will be spelled out in the conclusion section of this paper.

1Throughout this paper “state” means “pure state”. Mixed states, when needed, will be called “statistical mixtures”. 

arXiv:quant-ph/011154v1 29 Nov 2001
III. EXCLUSIVE AND NON-EXCLUSIVE MODALITIES

We shall now forget about the QM formalism, and look how far we can go by using only our definition of a quantum state. What we have at hand is only pure states, that yield predictions with unit probability for a given set of measurements. This can be detailed further:

(i) for a given set of (orthogonal) measurements, the different pure states corresponding to different results are exclusive, and we will call the associated pure states “exclusive modalities”. We may assume that the number of exclusive modalities is a property of the system, that will be called the dimension, $N$.

(ii) from our definition of a quantum state, there are other sets of measurements that are not predicted with certainty if the state of the system is one of the $N$ exclusive modalities. It is thus natural to consider all possible pure states, that constitute “non-exclusive modalities” (using the usual terminology for clarity, they are eigenvectors of other observables that do not commute with the ones in the initial set, but the QM formalism is not required at that stage).

The existence of non-exclusive modalities is a specific quantum feature: in classical physics, it should be possible to “give more details” about the state (*e.g.* by increasing the number of measurements), so that the “fully defined” states are exclusive, but this is not possible in QM. This is contained in our definition of the quantum state, and corresponds physically both to the existence of Heisenberg inequalities, and to the fact that a pure quantum state cannot be “completed” \[3\]. This specifically quantum difference between exclusive and non-exclusive modalities will be very important in the following.

One may thus ask the question: is it possible to reconstruct the QM formalism by appropriately combining exclusive and non-exclusive modalities? It should be clear that if one restricts to exclusive modalities, one will obtain something very close to classical probability theory over a discrete set of $N$ exclusive events, but there will be no room for interference effects, and thus we will miss QM. Lucien Hardy proposed in \[2\] an approach based on using $K$ probabilities defined on non-exclusive modalities, in such a way that they define an arbitrary (pure or mixed) quantum state. This approach is discussed in some details in the Appendix, and as a conclusion of this discussion we argue that the required axioms on $K$ are difficult to justify. In the following we will thus present another approach, that does not require the axioms on $K$ (H2 and H4b, see Appendix). On the other hand, we essentially keep H1 (statistical interpretation of probabilities), H3 and H4a (definition and properties of $N$), and H5 (continuity). As a central difference with Hardy’s approach, we will consider this continuity axiom from a geometrical point of view. This will lead us to the conclusion that QM is not only a probability theory, but has also to deal with geometry as an essential ingredient.

IV. RECONSTRUCTING QUANTUM MECHANICS

A. Restating the question

The question that we want to address is: since we know that there are $N$ exclusive modalities associated to each given CSCO, how to connect between themselves all the non-exclusive modalities corresponding to all possible CSCO? We should emphasize that the notion of Hilbert space is not yet there: beyond the axioms themselves, what we have is simply our definition of a quantum state, as the values of a set of physical quantities, that can be predicted with certainty and measured repeatedly.

By definition, changing the CSCO results from changing the measurement apparatus at the macroscopic level, that is, “turning the knobs”. A typical example is changing the orientations of a Stern-Gerlach magnet. These transformations have the mathematical structure of a continuous group: the combination of several transformations is associative and gives a new transformation, there is a neutral element (doing nothing), and each transformation has an inverse. Generally this group is not commutative: for instance, the three-dimensional rotations associated with the orientations of a Stern-Gerlach magnet do not commute. For a given position of the knob settings (given CSCO), there is a given set of $N$ exclusive modalities, that we can denote $\{b_i\}$. By turning the knobs, one obtains $N$ other exclusive modalities, that we can denote $\{b'_j\}$. The question is then how to relate the $\{b_i\}$ and the $\{b'_j\}$, that is, to define the transformation that allows one to go from a set of states of the other one. The reasoning presented below is not a demonstration, but a justification of a “reasonable” way to establish this relation.

B. The probability formula

Since the $\{b_i\}$ and $\{b'_j\}$ are by definition non-exclusive modalities, one has first to introduce the probabilities of finding the particular state $b'_j$ (after “turning the knobs”), when one starts in state $b_i$ (before “turning the knobs”). Since there are $N^2$ such probabilities, one can arrange them in a matrix $\Pi = (p_{i,j})$, corresponding to all the probabilities connecting the two sets $\{b_i\}$ and $\{b'_j\}$. Due to normalization conditions, it is easy to check that the number of independent numbers in $\Pi$ is equal to $(N - 1)^2$ (these numbers have to fulfill appropriate inequalities for consistency). In order to manipulate the $\Pi$ matrix, it is convenient to introduce the orthogonal $(N \times N)$ projectors $P_i$, with $P_i P_j = \delta_{ij}$. A useful operation is then to extract the particular probability $p_{i,j}$ from the matrix $\Pi$. It is easy to check that one may write:

$$p_{i,j} = \text{Trace}(P_i \Sigma P_j \Sigma^t)$$  \hspace{1cm} (1)
where $\Sigma = (\sqrt{P_{ij}})$ is the matrix formed by the square roots of the probabilities, and $^t$ denotes the transpose operation. Since we are looking for a state transformation, and since the $\{P_i\}$ define a set of orthogonal projectors corresponding to the initial set of states, it is very tempting to consider that the relations $P'_j = \Sigma P_j \Sigma^t$ define the set of projectors corresponding to the transformed set of states $\{b'_i\}$. However, though the $P'_j$ are indeed projectors (because the diagonal terms of $^{t}\Sigma \Sigma$ are all equal to 1), they are not orthogonal projectors (because the off-diagonal terms of $^{t}\Sigma \Sigma$ are not zero). The solution to that problem is obvious: the real-numbers matrix $\Sigma$ has to be replaced by the matrix $\tilde{\Sigma} = (e^{i\phi_{ij}} \sqrt{P_{ij}})$, where the phase factors $e^{i\phi_{ij}}$ are chosen so that $\tilde{\Sigma}$ is unitary (the normalization conditions on the $p_{i,j}$ warrants that this is possible; note that complex numbers are required: $\Sigma$ cannot be simply an orthogonal matrix). The equation for picking up a particular probability becomes:

$$p_{i,j} = \text{Trace}(P_i \tilde{\Sigma} P_j \tilde{\Sigma}^t)$$

(2)

that is the expected formula. As a consequence, what we actually need is a set of $N \times N$ unitary matrices, each one being associated to an element of the group of "knobs transformation", that will be denoted as $G_K$. For general consistency of the approach, we can conclude that this set of matrices gives a representation of the group of knobs transformations; this is fully consistent with the well known Wigner theorem [3]. We note that we never used the definition of a state as a "ray in an Hilbert space". The state keeps its initial definition as a set of measurement results, but what we get is a way to connect between themselves the probabilities of the "non-exclusive modalities" associated to all possible pure states. This connection is simply made by eq. (3), where the $\tilde{\Sigma}$ are unitary matrices that form a representation of the group of "knobs transformations" $G_K$.

It is noticeable that if $G_K$ is commutative, then its $N \times N$ representations are diagonal matrices with complex numbers of modulus unity of the diagonal, and zeros everywhere else. These matrices commute with $P_j$, and thus $p_{i,j} = \delta_{i,j}$. Therefore, there is actually no need for non-exclusive modalities, the initial $N$ exclusive ones are enough. This provides a straightforward way to recover classical probability theory, and the mathematical structure of QM thus appears as a direct consequence of the non-commutative character of $G_K$.

### V. CONCLUSION

As a summary, we sketched a possible way to reconstruct quantum mechanics in the framework of the "contextual objectivity" point of view. Like in [4], we use an axiom on the system’s dimension, that is related to a well-defined number of "exclusive modalities" for a given quantum system. Then we use geometry to connect the different "exclusive modalities" corresponding to different settings of the measurement apparatus. Our reconstruction axioms can thus be written as:

- **Axiom 1**: The quantum state of a physical system is defined as the values of a complete set of physical quantities, that can be predicted with certainty and measured repeatedly without perturbing in any way the system. In the axioms below a quantum state will be called a "modality".

- **Axiom 2**: For a given "knob settings" of the measurement apparatus, there exist $N$ distinguishable states $\{b_i\}$, that are called "exclusive modalities". The value of $N$, called the dimension, is a characteristic property of a given quantum system [6].

- **Axiom 3**: Various knob settings are related between themselves by (classical geometrical) transformations $g$ that have the structure of a continuous group $G_K$.

- **Axiom 4 (Theorem?)**: If the system is known to be in the state $b_i$ from the set $\{b_i\}$, the probability that it is found in state $b'_j$ from the set $\{b'_j\}$ corresponding to another knob settings obtained by the knob transformation $g$ is:

$$p_{i,j} = \text{Trace}(P_i \tilde{\Sigma} P_j \tilde{\Sigma}^t)$$

(3)

where the $P_i$ are orthogonal projectors, and $\tilde{\Sigma}$ is a unitary matrix corresponding to $g$, within the $N \times N$ matrix representation of the group $G_K$.

Whether or not Axiom 4 can be seen as a theorem (similar to Gleason’s theorem [1], but with slightly different hypothesis) remains an open question. It should be clear that the standard structure of QM can be obtained from the above axioms (in particular rewriting physical quantities as operators and states as rays is straightforward). We note again that in our approach there is no "measurement postulate", since it is already included in Axiom 1 (see detailed discussion in [3]).

An important feature of our approach is that the mathematical structure of QM is a direct consequence of the non-commutative character of the group of knobs transformations $G_K$. In some sense, QM appears as the result of accommodating the "contradictory" requirements that the exclusive modalities have a discrete structure, and that the knobs transformations have a continuous, but non-commutative group structure.

The view about the "classical vs quantum" dilemma that emerges from our approach is thus the following. A physical quantity is defined as an ensemble of possible measurements, that are connected between themselves by "geometrical" transformations that are in the "knobs transformations" group $G_K$. The "classical illusion" is to identify this physical quantity with the numbers given by the measurement, and to attribute "reality" to these...
numbers. EPR themselves realized that this definition of “reality” was too restrictive, and proposed instead their definition based upon predictability and reproducibility; this is just the idea that we use as our definition of a quantum state. But as soon as this is done, it appears that this “reality” cannot be attributed simultaneously to all physical quantities: this is simply incompatible with the structure of $\mathcal{G}$.

Thus what is “real” at the macroscopic level is the definition of the physical quantities (i.e. of the possible measurements related by the group $\mathcal{G}$), and what is “real” at quantum level (i.e. at the level of the measured system) is the quantum state. These two “realities” are fully compatible - they are actually the only ones that can connect the experimental definition of a physical quantity and the measurement results in a consistent way.

ACKNOWLEDGEMENTS

Stimulating discussions with Jean-Louis Basdevant, Franck Laloe and Anton Zeilinger are acknowledged, as well as interesting email exchanges with David Deutsch, Lucien Hardy and Chris Fuchs.

APPENDIX : HARDY’S APPROACH

A. The problem

The main idea of this approach is to define an arbitrary state of the system (that may be pure or not pure) by using a set of real numbers, each representing the probability that the system is in a given pure state, chosen among a fixed set of non-exclusive modalities. We note that it is not warranted that such a procedure should be successful: there is an infinite number of non-exclusive modalities, and no obvious reason that all of them can be represented by using probabilities of being in a finite number of modalities taken from all possible ones. Nevertheless, following Hardy, we will assume that this can be done, and we will denote as K the number of probabilities needed to determine an arbitrary state, defined over a set of K non-exclusive modalities. The number K should be related to N, that is the number of exclusive modalities (dimension of the system). Therefore the questions can be rewritten as:

- it is possible to find out the number K of non-exclusive modalities that are required to define an arbitrary (pure or non pure) state of the system, by using the probabilities of being in each of the modalities?
- is it possible to recover the usual structure of QM from these hypothesis?

The answers to these questions are yes, as it was shown by Hardy in, provided that some simple axioms are introduced about the structure of a probability theory. In particular, Hardy shows that for classical probability theory one has $K = N$, and for quantum probability theory one has $K = N^2$ [this is how many real numbers are required to define a non-normalized density matrix].

B. “Five reasonable axioms”

The probability axioms introduced by Hardy are the following (here they are slightly reworded and reordered, keeping the initial numbering):

- Axiom on probabilities (H1) : Relative frequencies (measured by taking the proportion of times a particular outcome is observed) converge to the same value, which we call the probability, for any case where a given measurement is performed on an ensemble of n systems prepared by some given preparation in the limit as n becomes large.

- Axioms on N (H3 and H4a) : The dimension N is a property of a given system, related to the number of exclusive modalities that can be defined for that system. A system whose state is constrained to have support on only M of a set of N possible exclusive modalities behaves like a system of dimension M (H3). A composite system consisting of subsystems A and B satisfies $N = N_A N_B$ (H4a).

- Axioms on K (H2 and H4b) : An arbitrary state of the system can be defined by specifying probabilities over a set of K non-exclusive modalities. K is determined by a function of N (i.e. $K = K(N)$), and for each given N, K takes the minimum value consistent with the other axioms (H2). A composite system consisting of subsystems A and B satisfies $K = K_A K_B$ (H4b).

- Axiom on pure states (modalities) : The above axioms are consistent with both classical and quantum probability theories. To get quantum theory one must add another axiom about “continuity”: any pure state can be transformed continuously and reversibly along a path through the pure states to any other pure state (H5).

C. Discussion

The axioms H1, H3 and H4a about probabilities and about N are easy to accept, since they deal with the definition of a probability theory over an ensemble of N exclusive modalities. In particular, the arguments about subspaces and composite systems are natural once it is admitted that the number of exclusive modalities is a characteristic property of the system. Similarly, the “quantum” axiom H5 is acceptable given the initial definition of a pure state as a modality, since an “infinitesimal” change to the measurement set-up is expected to change continuously the resulting pure states.
On the other hand, the axioms on \( K \) (H2 and H4b) are not obvious: they are trivial consequences of the previous ones if \( K = N \) (that is the classical probability situation), but far from trivial if \( K \neq N \). Can our approach be helpful? Accepting the (non-obvious) step that \( K \) is a function of \( N \) only, we interpret the equality \( K = K_A K_B \) as meaning that the set of non-exclusive modalities (i.e. the \( K \)'s) is treated just like the set of exclusive ones (i.e. the \( N \)'s) by the probability theory. This is consistent with considering the \( K \) states as “real”, though they are not exclusive of each other, but still does not provide a straightforward justification for these axioms.

The content of the axioms may be better understood by looking at their implications. An immediate consequence of H4a and H4b is that \( K(N^2) = K(N)^2 \). By adding some arguments to show that \( K \) must be a polynomial in \( N \), this implies that \( K(N) = N^r \), where \( r \) is an integer. In a previous version (v1) of [2] the value of \( r \) was claimed to be calculated, but in the present version (v3) it is simply shown that \( r = 2 \), i.e. \( K(N) = N^2 \), is the smallest value compatible with H5, and corresponds thus to quantum theory. If H5 is dropped, then \( K = N \) is acceptable and classical probability theory is obtained. After proving that \( K = N^2 \), Ref. [2] shows how to reconstruct a “Liouville space” (rather than Hilbert space) formulation of QM, including the trace formula and the general structure of the time evolution.

In our opinion, an important problem in the approach of [2] is that the axioms on \( K \) (H2 and H4b) are difficult to justify. In particular, it seems unlikely that (quoting [3]) “a 19th century theorist may have developed quantum theory without access to the empirical data that later became available to his 20th century descendants”. Without a good knowledge about QM, there is little chance to reach the crucial idea of “non-exclusive modalities” (non-orthogonal pure states as irreducible features of the reality), that looks crucial for the whole reasoning. Though our contextual objectivity approach may help to clarify physically the “K vs N” issue, it does not provide a firm basis to justify all the axioms of Ref. [2]. As we have shown above, the “missing axioms” can advantageously be replaced by geometrical considerations.

---

[1] Philippe Grangier, “Contextual objectivity : a realistic interpretation of quantum mechanics”, arXiv: quant-ph/0012122
[2] Lucien Hardy, “Quantum Theory From Five Reasonable Axioms”, arXiv: quant-ph/0101012
[3] A. Einstein, B. Podolsky and N. Rosen, “Can quantum mechanical description of reality be considered complete”, Phys. Rev. 47, 777 (1935)
[4] C. Cohen-Tannoudji, B. Diu and F. Laloe, “Mécanique Quantique”, Hermann, 1977

---

[5] The group of geometrical displacements is a very important example, where \( N \) is actually infinite. But the same considerations can be translated to that case, see e.g. Franck Laloe, “Les symétries en mécanique quantique”, Cours de DEA, Ecole Normale Supérieure (1980).

[6] The value of \( N \) may be finite or infinite, the important point is that there exist also a (much larger) number of “non-exclusive modalities” (associated with all possible knob settings), that cannot be reduced to exclusive modalities. For other discussions of extensions to infinite \( N \), see also e.g. [2] and [3].

[7] C. A. Fuchs, “Quantum Foundations in the Light of Quantum Information”, arXiv: quant-ph/0106166. It should be clear that we do not share all positions expressed in this reference, since our goal is rather to deal with “Quantum Information in the Light of Quantum Foundations”.

[8] Among open questions, we did not consider time evolution. One should also spell out in more detail the known connection between the physical quantities and the infinitesimal generators of \( G_K \). Another interesting question is the role of “projective” representations, that are connected to gauge theories for systems that involve charged particles and electromagnetic fields.