CATEGORIES AS MODELS
ON A SUITABLE ALGEBRAIC THEORY

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Abstract. We explain how categories, and groupoids, can be seen as models for a Lawvere \( \mathcal{G}_r \)-theory, where \( \mathcal{G}_r \) is the category of graphs, and show that for Lawvere \( \mathcal{G}_r \)-theories finitely presentable models are finitely presentable objects.

1. Introduction

Lawvere theories were introduced by Bill Lawvere in his doctoral thesis \[L64\] in 1963 as a categorical formulation of universal algebra. The correspondence between Lawvere theories and finitary monads on \( \mathcal{S}et \) is one of the deepest relationships in category theory. In \[P99\] Lawvere theories were generalized to enriched Lawvere theories, substituting \( \mathcal{S}et \) with an arbitrary base category \( \mathcal{V} \) satisfying axioms that make \( \mathcal{V} \) an appropriate base category for enrichment in the sense of \[K82\], and a correspondence between \( \mathcal{V} \)-enriched Lawvere theories and \( \mathcal{V} \)-enriched monads on \( \mathcal{V} \) was achieved. A further step was taken in \[NP09\] and \[LP11\] with the notion of Lawvere \( \mathcal{A} \)-theories: first a category \( \mathcal{V} \) in which to enrich and then a base \( \mathcal{V} \)-category \( \mathcal{A} \) were chosen. The correspondence above was extended to one between Lawvere \( \mathcal{A} \)-theories and finitary \( \mathcal{V} \)-enriched monads on the \( \mathcal{V} \)-category \( \mathcal{A} \). This allowed to view as models for Lawvere \( \mathcal{A} \)-theories structures for which this interpretation was not possible with \( \mathcal{A} = \mathcal{V} \).

In this paper we first show, as an application of what explained above, that categories and groupoids can be seen as models for certain Lawvere \( \mathcal{G}_r \)-theories, where \( \mathcal{A} = \mathcal{G}_r \) is the category of graphs and \( \mathcal{V} = \mathcal{S}et \).

Another property of Lawvere theories on \( \mathcal{S}et \) is that a model \( M \) for a given theory is finitely presentable exactly when \( \text{Mod}(M, -) : \text{Mod} \to \mathcal{S}et \) preserves filtered colimits, where \( \text{Mod} \) denotes the category of models for the given theory. This provides an equivalence between an extrinsic (the former) and an intrinsic (the latter) characterization

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of finitely presentability. We show that this still holds for categories, seen, as said, as models for a Lawvere \( G_r \)-theory, where the fact that \( A = G_r \) is decisive. We do not know if this equivalence holds for generic Lawvere \( A \)-theories and at the moment we have not counterexamples.

The paper is organized as follows: in the second chapter we remind the notion of graph and resume their basic properties; in the third we remember Lawvere \( A \)-theories, for a locally finitely presentable \( \mathcal{V} \)-category \( A \), where \( \mathcal{V} \) is a locally finitely presentable symmetric monoidal closed category, and their \( \mathcal{V} \)-category of models, particularly we show how categories and groupoids can be seen each one as models for a suitable Lawvere \( G_r \)-theory, where \( G_r \) denotes the category of graphs; finally, in the fourth, we show that finitely presentable categories are just finitely presentable models, establishing an equivalence between an intrinsic and extrinsic characterization.

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2. Graphs

We introduce here the notion of graph, explaining some of their properties, and the category of graphs and graphs morphisms.

**Definition 2.1.** A (directed) graph \( G \) consists of

1. a class \( G_0 \), whose elements are called vertices (or 0-cells);
2. for each pair \( (A, B) \in G_0 \times G_0 \) a set \( G(A, B) \), whose elements are called the arrows (or 1-cells or edges) from \( A \) to \( B \).

Equivalently, we can assign a graph \( G \) by giving a class \( G_0 \) of vertices and a class \( G_1 \) of arrows, together with two maps of classes \( s, t: G_1 \to G_0 \), called source and target, such that the arrows with given source and target form a set.

**Definition 2.2.** A morphism of graphs \( \alpha : G \to H \) between two graphs \( G \) and \( H \) consists of

1. a map \( \alpha_0 : G_0 \to H_0 \)
2. for each \( (A, B) \in G \times G \) a map \( \alpha_{A,B} : G(A, B) \to H(\alpha A, \alpha B) \)

Equivalently, a morphism of graphs \( \alpha \) is assigned by giving maps \( \alpha_0 : G_0 \to H_0 \) and \( \alpha_1 : G_1 \to H_1 \) commuting with \( s \) and \( t \).

**Proposition 2.3.** Small graphs and morphisms of graphs form a category, which we denote by \( G_r \).
Another useful characterization of graphs is that of presheaves over a suitable category. Let $\mathcal{S}et$ be the category of sets and $\mathcal{D}$ is the subcategory of $\mathcal{S}et$, whose objects are the sets $\bar{0} := \{0\}$ and $\bar{1} := \{0, 1\}$, and whose non-trivial morphisms are the obvious inclusions $i_0, i_1 : \{0\} \to \{0, 1\}$ respectively.

**Proposition 2.4.** $\mathcal{G}r$ is isomorphic to $\mathcal{S}et^{\mathcal{D}^{\text{op}}}$.

**Proof.** Given a graph $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, s, t)$ we define a presheaf $\Phi$ on $\mathcal{D}$ by setting $\Phi(\bar{0}) = \mathcal{G}_0$, $\Phi(\bar{1}) = \mathcal{G}_1$, $\Phi(i_0) = s$, $\Phi(i_1) = t$; conversely, the same definitions assign to a given presheaf $\Phi$ a graph $\mathcal{G}$. Given a morphism $\alpha : \mathcal{G} \to \mathcal{H}$, clearly from the equality above, it defines a morphism between presheaves $\Phi$ and $\Psi$ defined by $\mathcal{G}$ and $\mathcal{H}$ respectively, and the converse holds too. □

As examples we compute the graphs associated to the representable functors $h_{\bar{0}}(\bar{0}) = \text{Hom}_\mathcal{D}(\bar{0}, \bar{0})$ and $h_{\bar{1}}(\bar{0}) = \text{Hom}_\mathcal{D}(\bar{0}, \bar{1})$ in $\mathcal{S}et^{\mathcal{D}^{\text{op}}}$.

**Example 2.5.** From the definition of $\mathcal{D}$, we have that $h_{\bar{0}}(\bar{0}) = \{\text{id}_0\}$ and $h_{\bar{1}}(\bar{0}) = \emptyset$, so that $h_{\bar{0}}$ is the graph with one vertex and no arrows;

$$\bullet_{\bar{0}} \xrightarrow{\text{id}_0} \bullet_{\bar{1}}.$$

Instead $h_{\bar{1}}(\bar{0}) = \{i_0, i_1\}$ and $h_{\bar{1}}(\bar{1}) = \{\text{id}_1\}$, so that $h_{\bar{1}}$ is a graph with two vertexes and one arrow $\text{id}_1$ from $i_0$ to $i_1$;

$$\bullet_{i_0} \xrightarrow{\text{id}_1} \bullet_{i_1}.$$

**Corollary 2.6.** $\mathcal{G}r$ is locally finitely presentable.

**Proof.** It follows from the fact that $\mathcal{G}r$ is a category of presheaves by proposition 2.4. □

In particular, $\mathcal{G}r$ is complete and cocomplete such that limits and colimits can be computed pointwisely, or, equivalently, according to definition 2.1 cellwisely.

The following proposition establishes a relation between the category $\mathcal{C}at$ of small categories and the category $\mathcal{G}r$ of graphs:

**Proposition 2.7.** As a functor between $\mathcal{S}et$-categories, the forgetful functor $U : \mathcal{C}at \to \mathcal{G}r$ has a left adjoint $F$.

**Proof.** See [Bo94]. □

**Remark 2.8.** $\mathcal{G}r$ is a symmetric monoidal closed category. $\mathcal{G}r$ and $\mathcal{C}at$ are enriched over $\mathcal{G}r$, however proposition 2.7 does not extend to $\mathcal{G}r$-adjunction.
3. Lawvere $\mathcal{A}$-theories

As explained in remark 2.8 we will be concerned with Lawvere $\mathcal{A}$-theories when $\mathcal{A} = \mathcal{G}r$ and $\mathcal{V} = \mathcal{S}et$, however, following [NP09], we introduce them in generality. Suppose that $\mathcal{V}$ is locally finitely presentable as a symmetric monoidal closed category and that $\mathcal{A}$ is a locally finitely presentable $\mathcal{V}$-category. Denote by $\mathcal{A}_{fp}$ a skeleton of the full sub-$\mathcal{V}$-category of $\mathcal{A}$ given by finitely presentable objects of $\mathcal{A}$. Let $i : \mathcal{A}_{fp} \to \mathcal{A}$ be the inclusion $\mathcal{V}$-functor and $\tilde{i}$ the following composition:

$$
\mathcal{A} \overset{Y}{\longrightarrow} \mathcal{V} \overset{[\mathcal{V}]^{\mathcal{V}}}{{}^\mathcal{V}} \overset{[\mathcal{A}_{fp}^{\mathcal{V}}]}{{}\longrightarrow} \mathcal{A}^{\mathcal{V}}
$$

where $Y$ is the enriched Yoneda embedding. As to $\mathcal{G}r$, note that finitely presentable objects are just finite graphs; we will denote $\mathcal{G}r_{fp}$ simply by $\mathcal{G}r_{f}$.

**Definition 3.1.** A Lawvere $\mathcal{A}$-theory is a small $\mathcal{V}$-category $\mathcal{L}$ together with an identity-on-objects strict finite $\mathcal{V}$-limit-preserving $\mathcal{V}$-functor $J : \mathcal{A}_{fp}^{\mathcal{V}} \to \mathcal{L}$.

**Definition 3.2.** Given a Lawvere $\mathcal{A}$-theory $(\mathcal{L}, J)$, its $\mathcal{V}$-category of models is defined by the following pull-back in the $\mathcal{V}$ – $\mathcal{C}at$ of locally small $\mathcal{V}$-categories:

$$
\begin{array}{ccc}
\text{Mod}(\mathcal{L}) & \xrightarrow{P_{\mathcal{L}}} & [\mathcal{L}, \mathcal{V}] \\
\downarrow U_{\mathcal{L}} & & \downarrow [J, \mathcal{V}] \\
\mathcal{A} & \xrightarrow{i} & [\mathcal{A}_{fp}^{\mathcal{V}}, \mathcal{V}] \\
\end{array}
$$

We quote the following result from [NP09]:

**Proposition 3.3.** $U_{\mathcal{L}}$ is finitary monadic, particularly it has a left $\mathcal{V}$-adjoint $F_{\mathcal{L}}$

For simplicity, when the theory $\mathcal{L}$ is fixed, we will use the notation $U$ and $F$ for the forgetful functor and its left adjoint.

As said, we want to show that categories can be seen as models for an $\mathcal{A}$-Lawvere theory with $\mathcal{V} = \mathcal{S}et$ and $\mathcal{A} = \mathcal{G}r$.

Let $\tilde{\mathcal{D}}$ be the following graph which is isomorphic to the graph corresponding to the representable functor $h_0$ in $\mathcal{S}et^{\mathcal{P}^{op}}$:

$$
\tilde{\mathcal{D}} := \bullet_a
$$
and \( \vec{1} \) the following graph which is isomorphic to the graph corresponding to the representable functor \( h_{\vec{1}} \) in \( \mathcal{S}et^{D^{op}} \)

\[
\vec{1} := \bullet_a \longrightarrow \bullet_b .
\]

By abuse of notations, \( s \) and \( t \) denote the two morphisms of graphs from \( \vec{0} \) to \( \vec{1} \), mapping the only vertex of \( \vec{0} \) to \( a \) and \( b \) respectively

\[
\begin{array}{cccc}
\bullet_a & \stackrel{t}{\longrightarrow} & \bullet_b \\
\downarrow & & & \downarrow \\
\bullet_a & \stackrel{s}{\longrightarrow} & \bullet_b
\end{array}
\]

Note that the graph \( \vec{2} \), defined as the graph with three vertexes \( a, b \) and \( c \) and two arrows from \( a \) to \( b \) and from \( b \) to \( c \)

\[
\vec{2} := \bullet_a \longrightarrow \bullet_b \longrightarrow \bullet_c
\]

is the push-out of \( s \) and \( t \) in \( \mathcal{G}r \)

\[
\begin{array}{cccc}
\vec{0} & \stackrel{t}{\longrightarrow} & \vec{1} \\
\downarrow & & & \downarrow \\
\vec{1} & \stackrel{s}{\longrightarrow} & \vec{0} \\
\end{array}
\]

, i.e., \( \vec{2} \cong \vec{1} +_0 \vec{1} \). In a similar way, the graph

\[
\vec{3} := \bullet_a \longrightarrow \bullet_b \longrightarrow \bullet_c \longrightarrow \bullet_d
\]

is isomorphic to \( \vec{1} +_0 \vec{1} +_0 \vec{1} \) in \( \mathcal{G}r \).

In general,

\[
\vec{n} := \bullet_{a_0} \rightarrow \bullet_{a_1} \longrightarrow \bullet_{a_n} \cong \vec{1} +_0 \cdots +_0 \vec{1}
\]

We may consider that above graphs and morphisms are in \( \mathcal{G}r_f \) and above finite colimits are those in \( \mathcal{G}r_f \) since \( i : \mathcal{G}r_f \rightarrow \mathcal{G}r \) preserves finite colimits.

Note that for any graph \( G \)

\[
\mathcal{G}r(\vec{0}, G) \cong G_0, \mathcal{G}r(\vec{1}, G) \cong G_1, \mathcal{G}r(\vec{n}, G) \cong G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1
\]

In particular, we have the following cartesian (pullback) diagram \( \mathcal{G}r(\vec{2}, G) \) in \( \mathcal{S}et \) corresponding to the pushout diagram \( \vec{1} +_0 \vec{1} \) in
\[ G_1 \times_{G_0} G_1 \xrightarrow{t'} G_1 \]
\[ s' \downarrow s \]
\[ G_1 \xrightarrow{t} G_0. \]

Denote the obvious inclusions in \( \mathcal{G} \) by
\[ l_j : \overrightarrow{1} \rightarrow \overrightarrow{3}, j = 1, 2, 3 \]
\[ l_{jk} : \overrightarrow{2} \rightarrow \overrightarrow{3}, (j, k) = (1, 2), (2, 3). \]

We define now the Lawvere theory we are interested in.

**Definition 3.4.** \( \mathcal{L}_\mathcal{E} \) is the Lawvere \( \mathcal{G} \)-theory having the following presentation:

- **Generators:** \( m : \overrightarrow{2} \rightarrow \overrightarrow{1}, e : \overrightarrow{0} \rightarrow \overrightarrow{1} \)
- **Axioms (relations):**
  \[
  \begin{align*}
  \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{3} & \xrightarrow{\psi} \overrightarrow{2} \\
  \overrightarrow{1} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{0} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} \\
  \overrightarrow{0} & \xrightarrow{e} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{e} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{\delta} \overrightarrow{2} & \xrightarrow{\rho} \overrightarrow{1} \\
  & \xrightarrow{id} \overrightarrow{0} & \xrightarrow{id} \overrightarrow{0} & \xrightarrow{id} \overrightarrow{1} & \xrightarrow{id} \overrightarrow{1} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{1} & \xrightarrow{t^{\text{op}}} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \\
  \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{2} & \xrightarrow{m} \overrightarrow{1} & \overrightarrow{0} & \xrightarrow{s^{\text{op}}} \overrightarrow{0} \end{align*} \]

where \( \psi, \phi, \delta, \rho \) are the unique morphisms in \( \mathcal{L}_\mathcal{E} \) making the following diagrams in \( \mathcal{L}_\mathcal{E} \) commute.
Note that such unique morphisms \( \psi, \phi, \delta, \rho \) exist in \( \mathbb{L}_C \), since the bottom diagrams are cartesian in \( \mathbb{L}_C \) and the outer diagrams commute (by axioms).

The next theorem says that categories are the models for this theory.

**Theorem 3.5.** The category \( \text{Mod}(\mathbb{L}_C) \) of \( \mathbb{L}_C \)-models is equivalent to the category \( \mathcal{C} \).

**Proof.** From definition 3.2 we have that for any model \( M \) there exists a graph \( G \in \mathfrak{Gr} \) such that \( M \circ J = \mathfrak{Gr}(i-, G) \).

The first two diagrams yield the following commutative diagrams in \( \mathcal{S}et \)

\[
\begin{array}{ccc}
G_1 \times_{G_0} G_1^{M(m)} & \rightarrow & G_1 \\
\downarrow s' & & \downarrow s \\
G_1 & \rightarrow & G_0
\end{array}
\quad \quad \quad \quad
\begin{array}{ccc}
G_1 \times_{G_0} G_1^{M(m)} & \rightarrow & G_1 \\
\downarrow t' & & \downarrow t \\
G_1 & \rightarrow & G_0
\end{array}
\]

which says that when applying ”the composition” \( M(m) \) to a pair of arrows \( (f, g) \) such that \( t(f) = s(g) \), we get an arrow \( g \circ f := M(m)(f, g) \) such that \( s(g \circ f) = s(f), t(g \circ f) = t(g) \).

Apply \( M \) to the commutative diagram which was used to define \( \psi \), we have the commutative diagram
where $p_{12}, p_3$ are the obvious projections. Indeed, $M(\psi)$ is the obvious projection $(M(m), id)$, since the bottom diagram is cartesian in $\mathfrak{Set}$ and the outer diagram commutes (by the second axiom). By analogous consideration, we have that $M(\phi) = (id, M(m))$.

Thus, the third diagram yields the commutative diagram

\[
\begin{array}{ccc}
G_1 \times G_0 & G_1 \times G_0 & G_1 \\
(id, M(m)) & (M(m), id) & M(m) \\
G_1 \times G_0 & G_1 & G_1
\end{array}
\]

which expresses the associativity of the composition $M(m)$, i.e., $h \circ (g \circ f) = (h \circ g) \circ f$ for any triple $(f, g, h)$ of arrows with $t(f) = s(g), s(h) = t(g)$.

The 4-th, 5-th diagrams yield the commutative diagrams

\[
\begin{array}{ccc}
G_1 & G_1 & G_1 \\
\downarrow & \downarrow & \downarrow \\
G_0 & G_0 & G_0
\end{array}
\]

which say that "the unit map" $M(e)$ assigns an arrow $id_a := M(e)(a) \in G_1$ with $S(id_a) = a = t(id_a)$ to each vertex $a \in G_0$.

Similar arguments for showing $M(\psi) = (M(m), id)$ show that

\[
M(\delta) = (M(e), id) : G_1 \cong G_0 \times G_0 G_1 \to G_1 \times G_0 G_1
\]

and

\[
M(\rho) = (id, M(e)) : G_1 \cong G_1 \times G_0 G_0 \to G_1 \times G_0 G_1
\]

Thus, the last diagram yields the commutative diagram

\[
\begin{array}{ccc}
G_1 \cong G_0 \times G_0 & G_1 \times G_0 & G_1 \\
(id, M(e), id) & (id, M(e), id) & (id, M(e), id) \\
G_1 & G_1 & G_1
\end{array}
\]

which says that $f \circ id_a = f$ for any $(a, f) \in G_0 \times G_1$ with $s(f) = a$ and $g = id_b \circ g$ for any $(g, b) \in G_1 \times G_0$ with $t(g) = b$.

All of these say that $(G, M(m), M(e))$ is a category.

For the converse, given a category $\mathcal{C}$, define the functor $M : \mathcal{L}_\mathcal{C} \to \mathfrak{Set}$ by the following:

$M(G) = \mathfrak{Set}(G, U(\mathcal{C}))$ for $G \in \text{ob}(\mathcal{L}_\mathcal{C}) = \text{ob}(\mathfrak{Set}^{op}_f)$,
$M(\alpha) = \mathfrak{G}(\alpha, U(\mathcal{C}))$ for morphisms $\alpha$ in $\mathfrak{G}_f$,
$M(m) : U(\mathcal{C})_1 \times_{U(\mathcal{C})_0} U(\mathcal{C})_1 \to U(\mathcal{C})_1, (f, g) \mapsto g \circ f,$
$M(e) : U(\mathcal{C})_0 \to U(\mathcal{C})_1, a \mapsto id_a.$

Then, all diagrams commute obviously. Finally, one can easily check that two constructions are mutually inverse.

\[\square\]

Remark 3.6. For the Lawvere theory $\mathcal{C}$ we have defined, the functors $U\mathcal{C}$ and $F\mathcal{C}$ coincide with forgetful functor and free construction of proposition 2.7.

In a similar way we can show that groupoids are models for a Lawvere $\mathfrak{G}$-theory.

Definition 3.7. $L_G$ is the Lawvere $\mathfrak{G}$-theory having the following presentation:

- generators: $m : 2 \to 1$, $e : 0 \to 1$, $t : 1 \to 1$

- axioms (relations): all those appearing in definition 3.4 plus

\[\begin{array}{c}
\begin{array}{c}
1 \quad 1 \\
\downarrow \quad \downarrow \\
0 \quad 0
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
1 \quad 1 \\
\downarrow \quad \downarrow \\
0 \quad 0
\end{array}
\end{array}\]

where $\xi$ and $\zeta$ are the unique morphisms in $L_G$ making the following diagrams in $L_G$ commute

\[\begin{array}{c}
\begin{array}{c}
1 \quad 1 \\
\downarrow \quad \downarrow \\
0 \quad 0
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
1 \quad 1 \\
\downarrow \quad \downarrow \\
0 \quad 0
\end{array}
\end{array}\]

Note that such unique morphisms $\xi, \zeta$ exist in $L_G$, since the bottom diagrams are cartesian in $L_G$ and the outer diagrams commute.

Theorem 3.8. The category $\text{Mod}(L_G)$ of $L_G$-models is equivalent to the category $\mathfrak{G}_{\text{gpd}}$ of groupoids.

Proof. Following the proof of theorem 3.5, we have that for any model $M$ there exists a graph $G \in \mathfrak{G}$ such that $M \circ J = \mathfrak{G}(i, G)$.

We refer to the proof of theorem 3.5 for what concerns those diagrams already appearing there.
The first and second diagrams in definition \[ \text{3.7} \] yield the following diagram in \( \text{Set} \)

\[
\begin{array}{ccc}
G_1 & \xrightarrow{M(\iota)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_0 & \xrightarrow{s^{\text{op}}} & G_0
\end{array}
\]

which say that the “inverse map” \( M(\iota) \) assigns to any arrow \( f \in G_1 \) an arrow \( f^{-1} := M(\iota)(f) \in G_1 \) such that \( s(f^{-1}) = t(f) \) and \( t(f^{-1}) = s(f) \).

Applying \( M \) to the commutative diagram defining \( \xi \) we obtain another commutative diagram

\[
\begin{array}{ccc}
G_1 & \xrightarrow{M(\iota)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{id} & G_1
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{M(\iota)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{s^{\text{op}}} & G_0
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{M(\iota)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{s^{\text{op}}} & G_0
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{M(\iota)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{s^{\text{op}}} & G_0
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{M(\iota)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{s^{\text{op}}} & G_0
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{M(\iota)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{s^{\text{op}}} & G_0
\end{array}
\]

M(\xi) is \( (M(m),id) \), since the bottom diagram is cartesian in \( \text{Set} \) and the outer diagram commutes (by the second axiom). By analogous considerations, we have that \( M(\zeta) = (id,M(\iota)) \).

Therefore the third diagram yields the commutative diagram

\[
\begin{array}{ccc}
G_1 & \xrightarrow{(M(\iota),id)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{id} & G_1
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{(M(\iota),id)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{id} & G_1
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{(M(\iota),id)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{id} & G_1
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{(M(\iota),id)} & G_1 \\
\downarrow{t^{\text{op}}} & & \downarrow{t^{\text{op}}} \\
G_1 \times G_0 & \xrightarrow{id} & G_1
\end{array}
\]

which says that \( f \circ f^{-1} = id_{t(f)} \) and \( f^{-1} \circ f = id_{s(f)} \).

These, together with what proved in theorem \[ \text{3.5} \] say that \( (G, M(m), M(e), M(\iota)) \) is a groupoid.

For the converse, as in the proof of theorem \[ \text{3.5} \] given a groupoid \( \mathcal{G} \), using the inclusion \( \mathfrak{Grpd} \subset \mathfrak{Cat} \) to apply the forgetful functor \( U \) to \( \mathcal{G} \), define the functor \( M : \mathcal{L}_\mathfrak{G} \to \mathfrak{Set} \) by the following:

\[ M(G) = \mathfrak{Gr}(G,U(\mathcal{G})) \text{ for } G \in \text{ob}(\mathcal{L}_\mathfrak{G}) = \text{ob}(\mathfrak{Gr}_f), \]

\[ M(\alpha) = \mathfrak{Gr}(\alpha,U(\mathcal{G})) \text{ for morphisms } \alpha \in \mathfrak{Gr}_f, \]

\[ M(m) : U(\mathcal{G})_1 \times_U(\mathcal{G})_0 \to U(\mathcal{G})_1, (f,g) \mapsto g \circ f, \]

\[ M(e) : U(\mathcal{G})_0 \to U(\mathcal{G})_1, a \mapsto id_a. \]

\[ M(\iota) : U(\mathcal{G})_1 \to U(\mathcal{G})_1, f \mapsto f^{-1}. \]
Then all diagrams commute. Finally, one can check that two constructions are mutually inverse. □

4. Finitely presentable categories and models

We want now to prove that finitely presentable objects are just finitely presentable models for a Lawvere \( \mathcal{G} \)-theory.

In this section, \( \mathcal{L} \) will denote a Lawvere \( \mathcal{G} \)-theory where \( \mathcal{G} \) is considered as a category, i.e., a \( \mathcal{S}et \)-category. Recall that an object \( C \) in a category \( \mathcal{C} \) is \textit{finitely presentable} if the representable functor

\[
\mathcal{C}(C, -) : \mathcal{C} \to \mathcal{S}et
\]

preserves filtered colimits.

**Definition 4.1.** A model \( M \in \text{Mod}(\mathcal{L}) \) is finitely presentable when there exist \( G \) and \( H \) in \( \mathcal{G} \) such that \( M \) is the coequalizer

\[
\begin{array}{ccc}
F(H) & \xrightarrow{\alpha} & F(G) \\
\beta & \xrightarrow{\text{q}} & M
\end{array}
\]

We call this a finite presentation of \( M \).

**Proposition 4.2.** \( \text{Mod}(\mathcal{L}) \) is a reflective subcategory of \( [\mathcal{L}, \mathcal{S}et] \).

*Proof.* See [LR11]. □

This implies in particular that \( \text{Mod}(\mathcal{L}) \) is complete and cocomplete.

**Lemma 4.3.** \( \mathcal{L}(G, -) = F(iG) \) for \( G \in \mathcal{G} \).

*Proof.* Our statement says that for a model \( M \)

\[
\text{Mod}(\mathcal{L})(\mathcal{L}(G, -), M) = \mathcal{G}(iG, U(M))
\]

but this follows from proposition 4.1 of [NP09]. □

**Proposition 4.4.** Free models on finite graphs form a dense family of generators of \( \text{Mod}(\mathcal{L}) \).

*Proof.* By proposition 4.2 \( \text{Mod}(\mathcal{L}) \) is a reflective subcategory of \( [\mathcal{L}, \mathcal{S}et] \); in \( [\mathcal{L}, \mathcal{S}et] \) every model \( M \) is the colimit of representable functors \( \mathcal{L}(JG, -) \) for \( G \) finite; these, on the other hand, are in \( \text{Mod}(\mathcal{L}) \) as, by lemma 4.3 \( \mathcal{L}(G, -) = F(iG) \) for \( G \in \mathcal{G} \); so the colimit \( M \) exists in \( \text{Mod}(\mathcal{L}) \). □

**Proposition 4.5.** If \( M \) is a finitely presentable model, then it admits a presentation (a coequalizer as in definition 4.1) such that the \( q \), as graph morphism, admits a section \( s \), that is, \( q \circ s = \text{id}_M \) in \( \mathcal{G} \)

\[
\begin{array}{ccc}
F(H) & \xrightarrow{\alpha} & F(G) \\
\beta & \xrightarrow{\text{q}} & M
\end{array}
\]
Proof. Let $M$ be a finitely presentable model and take a presentation of it

$$F(H') \xrightarrow{\alpha'} F(G') \xrightarrow{q'} M.$$ 

Consider the following adjunctions of $\alpha', \beta'$

$$H' \xrightarrow{\alpha''} UF(G').$$

Let $R_0$ be the smallest equivalence relation containing $<\alpha'(v), \beta'(v)>$, for $v \in |H'|$, and, since $|UF(G')| = |G'|$, let $r : G' \to G'/R_0$ be the quotient morphism. Applying $F$ we get a morphism $F(r) : F(G') \to F(G'/R_0)$. Note that $F(r)$ is an epimorphism, because $r$ is and $F$ is left-adjoint to $U$. We can now define a morphism $\bar{q} : M \to F(G'/R_0)$:

$$F(G'/R_0) \xrightarrow{\bar{q}} M$$

We have that $\bar{q} \circ F(r) \circ \alpha = q \circ \alpha = q \circ \beta = \bar{q} \circ F(r) \circ \beta$ and we want to show that

$$F(H) \xrightarrow{F(r) \circ \alpha} F(G/R_0) \xrightarrow{\bar{q}} M$$

is a coequalizer. It remains to prove the universal property. So let $(N, p)$ such that $p \circ F(r) \circ \alpha = p \circ F(r) \circ \beta$. By universality we have a unique morphism $t : M \to N$ such that $p \circ F(r) = t \circ q'$. Since $q' = \bar{q} \circ F(r)$ we have that $p \circ F(r) = t \circ \bar{q} \circ F(r)$, and, since $F(r)$ is an epimorphism, we get that $p = t \circ \bar{q}$. Observe now that, since $F(G/R_0)$ and $M$ are graphs with same vertexes, there exists a section $s : M \to F(G/R_0)$ to $\bar{q}$. Note finally that $H'$ is finite by assumption and $F(G/R_0)$ is finite since $G$ is and $R_0$ just identifies some vertexes.

**Proposition 4.6.** The finitely presentable models form a dense family of generators in $\text{mod}(\mathcal{L})$, stable under finite colimits, and every model is a filtered colimit of finitely presentable ones.
Proof. The proof with parallel that proposition 3.8.12 in [Bo94]. Let $\mathcal{F}$ be the full subcategory of finitely presentable models. For a model $M$ consider the overcategory $\mathcal{F}/M$ and the forgetful functor $\phi: \mathcal{F}/M \to Mod(\mathcal{L})$. Following [Bo94] and using proposition 4.5, we have colimit $\phi = (M, s_{(F,f)})$, where $s_{(F,f)} = f: \phi((F,f)) = F \to M$.

That the colimit above is cofiltered, that is, that $F/M$ is cofiltered, follows from the fact that $F$ is stable in $Mod(\mathcal{L})$ under finite colimits. Let us prove this. Following [Bo94], we soon have that $F$ is stable under finite coproducts. It is stable also under coequalizers. The proof is again similar to that in [Bo94], however we need to apply proposition 4.4. Suppose $P$ and $Q$ are finitely presentable, let $u, v: P \to Q$ be two morphism, and let $(R, r)$ be the coequalizer: we want to prove that $R$ is also finitely presentable. Since $P$ and $Q$ are finitely presentable we can consider the diagram

\[
\begin{array}{ccc}
F(H) & & F(K) \\
\downarrow a & & \downarrow d \\
F(G) & \xrightarrow{x} & F(J) \\
\downarrow p & & \downarrow s \\
P & \xrightarrow{u} & Q & \xrightarrow{r} & R
\end{array}
\]

the existence of the lifts $x$ and $y$ of respectively $u$ and $v$ is a consequence of proposition 4.4 since we can choose a presentation of $Q$ admitting a section $s: Q \to F(J)$ of $q$. The proof follows now as in citeB, showing that $R$ admits indeed a presentation

\[
F(G \amalg K) \xrightarrow{x \amalg c} F(J) \xrightarrow{r \circ q} R
\]

Lemma 4.7. Free models on finite graphs are finitely presentable models.

Proof. Let $F(G)$ be a free model with $G$ finite and consider a cofiltered colimit $X = \text{colim} X_i$, then by adjointness

\[
Mod(\mathcal{L})(F(G), \text{colim} X_i) = \mathcal{G}r(G, U(\text{colim} X_i))
\]

since $U$, being finitary monadic (see proposition 3.3) preserves filtered colimits, we have

\[
\mathcal{G}r(G, U(\text{colim} X_i)) = \text{colim} \mathcal{G}r(G, U(X_i))
\]
finally, since \( G \) is finitely presentable
\[
\text{colim}_c \mathcal{S}(G, U(X_i)) = \text{colim} \text{Mod}(\mathcal{L})(F(G), X_i)
\]
thus free finitely presentable models are finitely presentable objects. \( \Box \)

Before enouncing the main result, the following one is expected, having started our construction with finitely presentable categories:

**Proposition 4.8.** \( \text{Mod}(\mathcal{L}) \) is locally finitely presentable.

**Proof.** \( \text{Mod}(\mathcal{L}) \) is cocomplete by proposition 4.2. Free generators are finitely presentable by lemma 4.7 and by proposition 4.5 form a dense, thus strong, family of generators. \( \Box \)

We conclude with the main result:

**Theorem 4.9.** Finitely presentable models correspond to finitely presentable categories.

**Proof.** Let \( M \) a finitely presentable model and take a presentation
\[
F(H) \longrightarrow F(G) \longrightarrow M
\]
since \( F(H) \) and \( F(G) \) are finite presentable objects, and since these are stable under finite colimits, it follows that \( M \) is a finitely presentable object.

For the converse, suppose that for \( M \in \text{Mod}(\mathcal{L}) \) we have an isomorphism
\[
\text{Mod}(\mathcal{L})(M, \text{colim} X_i) \cong \text{colim} \text{Mod}(\mathcal{L})(M, X_i)
\]
for any filtered colimit \( X = \text{colim} X_i \). By proposition 4.6, \( M \) is a filtered colimit of finitely presentable ones: \( (M, s_{(F,f)}) = \text{colim} \phi(F,f) \); so, substituting, we obtain
\[
\text{Mod}(\mathcal{L})(M, M) \cong \text{colim} \text{Mod}(\mathcal{L})(M, \phi(F,f))
\]
Let \( f : M \to F \) be the morphism corresponding to the identity on \( M \): together with \( s_{(F,f)} \) expresses \( M \) as a retract of \( P \) and so \( M \) as a coequalizer of \( (id_F, f \circ s_{(F,f)}) : F \to F \). By proposition 4.6, \( M \) is finitely presentable. \( \Box \)

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