An Optimal Condition of Robust Low-rank Matrices Recovery

Jianwen Huang, Jianjun Wang, Feng Zhang, and Wendong Wang

Abstract—In this paper we investigate the reconstruction conditions of nuclear norm minimization for low-rank matrix recovery. We obtain sufficient conditions $\delta_r < t/(4 - t)$ with $0 < t < 4/3$ to guarantee the robust reconstruction ($z \neq 0$) or exact reconstruction ($z = 0$) of all rank $r$ matrices $X \in \mathbb{R}^{m \times n}$ from $b = A(X) + z$ via nuclear norm minimization. Furthermore, we not only show that when $t = 1$, the upper bound of $\delta_r < 1/3$ is the same as the result of Cai and Zhang [14], but also demonstrate that the gained upper bounds concerning the recovery error are better. Moreover, we prove that the restricted isometry property condition is sharp. Besides, the numerical experiments are conducted to reveal the nuclear norm minimization method is stable and robust for the recovery of low-rank matrix.

Index Terms—Low-rank matrix recovery, nuclear norm minimization, restricted isometry property condition, compressed sensing, convex optimization.

I. INTRODUCTION

SUPPOSE that $X \in \mathbb{R}^{m \times n}$ is an unknown low rank matrix, $A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^q$ is a known linear map, $b \in \mathbb{R}^q$ is a given observation and $z \in \mathbb{R}^q$ is measurement error. The rank minimization problem is defined as follows:

$$\min_X \text{rank}(X) \text{ s.t. } \|A(X) - b\|_2 \leq \epsilon, \tag{I.1}$$

where $b = A(X) + z$ and $\epsilon$ stands for the noise level. Since the problem (I.1) is NP-hard in general, Recht et al. [1] introduced a convex relaxation, which minimizes the nuclear norm (also known as the Schatten 1-norm or trace norm)

$$\min_X \|X\|_* \text{ s.t. } \|A(X) - b\|_2 \leq \epsilon, \tag{I.2}$$

where $\|X\|_* = \sum_{i=1}^{\min\{m, n\}} \sigma_i(X)$ and $\sigma_i(X)$ is the $i$-th largest singular values of matrix $X$. The problem (I.2) is convex, thus there are a large number of approaches which can be used to solve it. Some researchers have developed fast algorithms for solving it, see [2–10].

When $m = n$ and the matrix $X = \text{diag}(x)$ ($x \in \mathbb{R}^m$) is a diagonal matrix, the problems (I.1) and (I.2) degenerate to the $l_0$-minimization and $l_1$-minimization, respectively, which belong to the main optimization problems in compressed sensing (CS).

In order to study the relationship between the rank minimization problem and the nuclear norm minimization problem, Candès and Plan [11] extended the notion of restricted isometry constant proposed by Candès [12] to low-rank matrix recovery case. The concept is as follows:

**Definition I.1.** Let $A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^q$ be a linear map. For any integer $r$ ($1 \leq r \leq \min\{m, n\}$), the restricted isometry constant (RIC) of order $r$ is defined as the smallest positive number $\delta_r$ that satisfies

$$(1 - \delta_r)\|X\|_F^2 \leq \|A(X)\|_2^2 \leq (1 + \delta_r)\|X\|_F^2 \tag{I.3}$$

for all rank-$r$ matrices $X$ (i.e., the rank of $X$ is at most $r$), where $\|X\|_F^2 = \langle X, X \rangle = \text{Tr}(X^T X)$ is the Frobenius norm of $X$, which is also equal to the sum of the square of singular values and the inner product in $\mathbb{R}^{m \times n}$ as $\langle X, Y \rangle = \text{Tr}(X^T Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij}$ for matrices $X$ and $Y$ of the same dimension.

By the aforementioned definition, it is easy to see that if $r_1 \leq r_2$, then $\delta_{r_1} \leq \delta_{r_2}$.

Although it is not easy to examine the restricted isometry property for a given linear map, it is one of the central notions in low-rank matrix recovery. In fact, it has been shown [1] that Gaussian or sub-Gaussian random measurement map $A$ fulfills the restricted isometry property with high probability.

There exist many sufficient conditions based on RIP for the exact recovery (i.e., in the case of $z = 0$) of any rank-$r$ matrices through the formulation (I.2). These contain $\delta_{4r} < 1/10\| [1], \delta_{4r} < \sqrt{2} - 1 \| [11], \delta_{4r} < 0.558 \| [13], \delta_r < 1/3 \| [14], \text{ and } \delta_{2r} < \sqrt{2}/2 \| [15].$ For other related works, see, e.g., [16–24].

In special, Cai and Zhang [25] showed that for any given $t \geq 4/3$, $\delta_{tr} < \sqrt{(t - 1)/t}$ ensures the exact reconstruction for all matrices with rank no more than $r$ in the noise-free case via the constrained nuclear norm minimization (I.2). Furthermore, for any $\epsilon > 0$, $\delta_{tr} < \sqrt{(t - 1)/t + \epsilon}$ doesn’t suffice to make sure the exact recovery of all rank-$r$ matrices for large $r$. Besides, they showed that condition $\delta_{tr} < \sqrt{(t - 1)/t}$ suffices for robust reconstruction of nearly low-rank matrices in the noisy case.

Motivated by the aforementioned papers, we further discuss the upper bounds of $\delta_{tr}$ associated with some linear map $A$ as $0 < t < 4/3$. Sufficient conditions regarding $\delta_{tr}$ with $0 < t < 4/3$ are established to guarantee the robust reconstruction ($\epsilon \neq 0$) or ($\epsilon = 0$) of all rank-$r$ matrices $X \in \mathbb{R}^{m \times n}$ satisfying $b = A(X) + z$ with $\|z\|_2 \leq \epsilon$ and $\|A^*(z)\| \leq \epsilon$, respectively.

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Thereby, combined with [25], a complete description for sharp restricted isometry property (RIP) constants for all \( t > 0 \) is established to ensure the exact reconstruction of all matrices with rank no more than \( r \) via nuclear norm minimization.

The construction of this paper is as follows. In Section II, we will provide some fundamental lemmas that be employed. We present the main results and the proofs in Sections III and V, respectively. A series of numerical simulation experiments on low-rank matrix reconstruction are carried out in Section IV. Lastly, conclusion is drawn in in Section VI.

II. PRELIMINARIES

We begin by introducing basic notations. We also gather a few lemmas needed for the proofs of main results.

For any matrix \( X \in \mathbb{R}^{m \times n} \), we assume w.o.l.g. that \( m \leq n \), and the singular value decomposition (SVD) of \( X \) is represented by

\[
X = U \text{diag}(\sigma(X)) V^T,
\]

where \( U \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{n \times n} \) are orthogonal matrices, and \( \sigma(X) = (\sigma_1(X), \ldots, \sigma_m(X))^\top \) indicates the vector of the singular values of \( X \). Assume that \( \sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_m(X) \). Consequently, the best \( r \)-rank approximation to the matrix \( X \) is

\[
X^{(r)} = U \begin{bmatrix}
\text{diag}(\sigma_r(X)) & 0 \\
0 & 0
\end{bmatrix} V^T,
\]

where \( \sigma_r(X) = (\sigma_1(X), \ldots, \sigma_r(X))^\top \).

For a linear map \( A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^q \), denote by its adjoint operator \( A^* : \mathbb{R}^q \rightarrow \mathbb{R}^{m \times n} \). Then, for all \( X \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^q \), \( \langle X, A^*(b) \rangle = \langle A(X), b \rangle \).

Without loss of generality, let \( X \) be the original matrix that we want to find and \( X^* \) be an optimal solution to the problem (I.2). Let \( Z = X - X^* \). Let SVD of \( U^\top ZV \in \mathbb{R}^{m \times m} \) be provided by

\[
U^\top ZV = U_0 \begin{bmatrix}
\text{diag}(\sigma(T^\top ZV)) & 0 \\
0 & 0
\end{bmatrix} V_0^\top
\]

where \( U_0, V_0 \in \mathbb{R}^{m \times m} \) are orthogonal matrices, \( \sigma_T(U^\top ZV) = (\sigma_1(U^\top ZV), \ldots, \sigma_r(U^\top ZV))^\top \), \( \sigma_{r+1}(U^\top ZV), \ldots, \sigma_m(U^\top ZV))^\top \), and we suppose that \( \sigma_1(U^\top ZV) \geq \cdots \geq \sigma_r(U^\top ZV) \geq \sigma_{r+1}(U^\top ZV) \geq \cdots \geq \sigma_m(U^\top ZV) \). Therefore, the matrix \( Z \) is decomposed as

\[
Z = Z^{(r)} + Z_c^{(r)},
\]

where

\[
Z^{(r)} = UU_0 \begin{bmatrix}
\text{diag}(\sigma_T(U^\top ZV)) & 0 \\
0 & 0
\end{bmatrix} V_0^\top V^T
\]

and

\[
Z_c^{(r)} = UU_0 \begin{bmatrix}
0 & 0 \\
0 & \text{diag}(\sigma_{r+1}(U^\top ZV))
\end{bmatrix} V_0^\top V^T.
\]

It is not hard to see that \( \sigma_r(U^\top ZV) = 0 \) and \( (X^{(r)})^\top Z_c^{(r)} = 0 \).

In order to show the main results, we need some elementary identities, which were given in [26] (see Lemma 1).

Lemma II.1. Give matrices \( \{V_i : i \in T\} \) in a matrix space \( \mathcal{V} \) with inner product \( \langle \cdot, \cdot \rangle \), where \( T \) denotes the index set with \( |T| = r \). Select all subsets \( T_i \subset T \) with \( |T_i| = k, i \in I \) and \( |I| = {r \choose k} \), then we get

\[
\sum_{i \in I} \sum_{p \in T_i} V_{p} = \left( \frac{r-1}{k-1} \right) \sum_{p \in T} V_{p} (k \geq 1),
\]

and

\[
\sum_{i \in I} \sum_{p \neq q \in T_i} \langle V_p, V_q \rangle = \left( \frac{r-2}{k-2} \right) \sum_{p \neq q \in T} \langle V_p, V_q \rangle (k \geq 2).
\]

Cai and Zhang developed a new elementary technique which states an elementary geometric fact: Any point in a polytope can be represented as a convex combination of sparse vectors (see Lemma 1.1 in [25]). It gives a crucial technical tool for the proof of our main results. It is also the special case \( p = 1 \) of Zhang and Li’s result (see Lemma 2.2 in [27]).

Lemma II.2. Let \( r \leq m \) be an integer, and \( \alpha \) be a positive real number. We can represent any vector \( x \) in the set

\[
V = \{ x \in \mathbb{R}^m : \|x\|_1 \leq r\alpha, \|x\|_{\infty} \leq \alpha \},
\]

as a convex combination of \( r \)-sparse vectors, i.e.,

\[
x = \sum_i \lambda_i u_i
\]

where \( \sum_i \lambda_i = 1 \) with \( \lambda_i \geq 0, \sup(u_i) \leq r, \sup(u_i) \subset \sup(x) \) and \( \sum_i \lambda_i \|u_i\|_2^2 \leq r \alpha^2 \).

Lemma II.3. (Lemma 2.3 in [1]) Let \( X, Y \) be the matrices of same dimensions. If \( XY^\top = 0 \) and \( X^\top Y = 0 \), then

\[
\|X + Y\|_* = \|X\|_* + \|Y\|_*.
\]

Lemma II.4. We have

\[
\|Z_c^{(r)}\|_* \leq \|Z^{(r)}\|_* + 2\|X - X^{(r)}\|_*.
\]

Proof. Since \( X^* \) is the optimal solution to the problem (I.2), we get

\[
\|X\|_* \geq \|X^*\|_* = \|X - Z\|_*.
\]

Applying the reverse inequality to (II.5), we get

\[
\|X - Z\|_* = \|X^{(r)} - Z_c^{(r)} + (X - X^{(r)} - Z^{(r)})\|_*
\]

\[
\geq \|X^{(r)} - Z_c^{(r)}\|_* - \|X - X^{(r)} - Z^{(r)}\|_*.
\]

By Lemma II.3 and the forward inequality, we get

\[
\|X^{(r)} + (-Z_c^{(r)})\|_* - \|X - X^{(r)} + (-Z^{(r)})\|_*
\]

\[
\geq \|X^{(r)}\|_* + \|Z_c^{(r)}\|_* - \|X - X^{(r)}\|_* - \|Z^{(r)}\|_*.
\]

Combining with (II.5), (II.6) and (II.7), we get

\[
\|Z_c^{(r)}\|_* \leq \|X\|_* - \|X^{(r)}\|_* + \|X - X^{(r)}\|_* + \|Z^{(r)}\|_*
\]

\[
\leq \|Z^{(r)}\|_* + 2\|X - X^{(r)}\|_*.
\]

The proof of the lemma is completed.
Select positive integers $a$ and $b$ satisfying $a + b = tr$ and $b \leq a \leq r$. We use $T_i$, $S_j$ to represent all possible index set contained in $\{1, 2, \cdots, r\}$ (i.e., $T_i, S_j \subset \{1, \cdots, r\}$) and $|T_i| = a$, $|S_j| = b$, where $i \in A$ and $j \in B$ with $|A| = \binom{r}{a}$ and $|B| = \binom{r}{b}$. Define

$$Z_{T_i}^{(r)} = UU_0 \begin{bmatrix} \text{diag} \left( \sigma_{T_i}(U^T Z V) \right) & 0 \\ 0 & 0 \end{bmatrix} V_0^T V^T,$$

and

$$Z_{S_j}^{(r)} = UU_0 \begin{bmatrix} \text{diag} \left( \sigma_{S_j}(U^T Z V) \right) & 0 \\ 0 & 0 \end{bmatrix} V_0^T V^T.$$

Here $\sigma_{T_i}(U^T Z V)$ (or $\sigma_{S_j}(U^T Z V)$) denotes the vector that equals to $\sigma_{T_i}(U^T Z V)$ on $T_i$ ($S_j$), and zero elsewhere.

**Lemma II.5.** We have

$$Z_c^{(r)} = \sum_k \mu_k U_k, \quad Z_c^{(r)} = \sum_k \nu_k V_k, \quad Z_c^{(r)} = \sum_k \tau_k W_k,$$

where $\sum_k \mu_k = \sum_k \nu_k = \sum_k \tau_k = 1$ with $\nu_k, \mu_k, \tau_k \geq 0$, $U_k, V_k, W_k$ are b-rank, a-rank and (t-1)r-rank ($t > 1$) with

$$\sum_k \mu_k \|U_k\|_F^2 \leq \frac{r^2}{b} \alpha^2,$$

$$\sum_k \nu_k \|V_k\|_F^2 \leq \frac{r^2}{a} \alpha^2,$$

and

$$\sum_k \tau_k \|W_k\|_F^2 \leq \frac{r^2}{t-1} \alpha^2.$$  

**Proof.** Set

$$\alpha = \frac{\|Z^{(r)}\|_* + 2 \|X - X^{(r)}\|_*}{r}.$$

By Lemma II.4, then

$$\|Z_c^{(r)}\|_* \leq r \alpha.$$

By the definition of $Z_c^{(r)}$, we get

$$\|\sigma_{T_i}(U^T Z V)\|_1 \leq \frac{r}{a} \alpha.$$  

By the decomposition of $Z$, we get

$$\|\sigma_{T_i}(U^T Z V)\|_\infty \leq \frac{\|\sigma_{T_i}(U^T Z V)\|_1}{r} \leq \frac{\|Z^{(r)}\|_* + 2 \|X - X^{(r)}\|_*}{r} \leq \alpha \leq \frac{r}{b} \alpha.$$  

Combining with Lemma II.2, (II.11) and (II.12), $\sigma_{T_i}(U^T Z V)$ is decomposed into the convex combination of $b$- sparse vectors, i.e., $\sigma_{T_i}(U^T Z V) = \sum_k \mu_k u_k$ with

$$\sum_k \mu_k \|u_k\|_2^2 \leq \frac{r^2}{b} \alpha^2.$$

Define

$$U_k = UU_0 \begin{bmatrix} 0 & 0 \\ 0 & \text{diag}(u_k) \end{bmatrix} V_0^T V^T.$$

It is easy to see that $U_k$ is $b$-rank. Therefore, $Z_c^{(r)}$ is decomposed as $Z_c^{(r)} = \sum_k \mu_k U_k$ with

$$\sum_k \mu_k \|U_k\|_F^2 = \sum_k \mu_k \|u_k\|_2^2 \leq \frac{r^2}{b} \alpha^2.$$

Likewise, $Z_c^{(r)}$ can also be denoted by

$$Z_c^{(r)} = \sum_k \nu_k V_k, \quad Z_c^{(r)} = \sum_k \tau_k W_k,$$

where $V_k$ is a-rank, $W_k$ is $(t-1)r$-rank ($t > 1$) with

$$\sum_k \nu_k \|V_k\|_F^2 \leq \frac{r^2}{a} \alpha^2,$$

and

$$\sum_k \tau_k \|V_k\|_F^2 \leq \frac{r^2}{t-1} \alpha^2.$$  

One can easily check that $\left\langle Z_{T_i}^{(r)}, U_k \right\rangle = 0$, $\left\langle Z_{S_j}^{(r)}, V_k \right\rangle = 0$ and $\left\langle Z^{(r)}, W_k \right\rangle = 0$. 

**Lemma II.6.** We have that for $0 < t < 1$,

$$\frac{\rho(t)}{(t)_a} \sum_{T_i \cap S_j = \emptyset} \left[ \|A \left( Z_{T_i}^{(r)} + Z_{S_j}^{(r)} \right) \|_2^2 - \left( (t-1)A \left( Z^{(r)} - W_k \right) \right) \|_2^2 \right] = -2t^2(2-t)ab \left\langle A Z^{(r)}, AZ \right\rangle + t \Delta_{a,b},$$

and for $1 \leq t < 4/3,$

$$\rho_a(t) \sum_k \tau_k \left[ \|A \left( Z^{(r)} + (t-1)W_k \right) \|_2^2 - \left( (t-1)A \left( Z^{(r)} - W_k \right) \right) \|_2^2 \right] = -2t^3(ab - (t-1)r^2) \left\langle A Z^{(r)}, AZ \right\rangle + (4-3t) \Delta_{a,b},$$

where

$$\rho_a(t) = (a + b)^2 - 2ab(4 - t),$$

and

$$\Delta_{a,b} = \frac{r - b}{a(t)} \sum_{i \in A, k} \mu_k \left[ \|A \left( Z_{T_i}^{(r)} + bU_k \right) \|_2^2 - b^2 \left\langle A \left( Z_{T_i}^{(r)} - \frac{a}{r} U_k \right) \right\rangle \|_2^2 \right] + \frac{r - a}{b(t)} \sum_{j \in B, k} \nu_k \left[ \|A \left( Z_{S_j}^{(r)} + \frac{a}{r} V_k \right) \|_2^2 - a^2 \left\langle A \left( Z_{S_j}^{(r)} - \frac{b}{r} V_k \right) \right\rangle \|_2^2 \right].$$
Proof. The proof takes advantage of the ideas from [14], [26]. By Lemma I.1, we get
\[
\Delta_{a,b} = (a^2 - b^2) \left[ \frac{r - b}{a(\zeta)} \sum_{i \in A} \|AZ^{(r)}_{T_i}\|_2^2 - \frac{r - a}{b(\zeta)} \sum_{j \in B} \|AZ^{(r)}_{S_j}\|_2^2 \right] \\
+ 2(a^2b + ab^2) \times \\
\left( \frac{r - b}{a(\zeta)} \sum_{i \in A} AZ^{(r)}_{T_i} + \frac{r - a}{b(\zeta)} \sum_{j \in B} AZ^{(r)}_{S_j}, AZ_c^{(r)} \right) \\
= (a^2 - b^2) \left( \frac{r - b}{a(\zeta)} \zeta^\frac{-1}{2} \|AZ^{(r)}\|_2^2 \right) \\
- \frac{r - a}{b(\zeta)} \zeta^\frac{-1}{2} \|AZ^{(r)}\|_2^2 \\
+ 2ab(a + b) \times \\
\left( \frac{r - b}{a(\zeta)} \zeta^\frac{-1}{2} AZ^{(r)} + \frac{r - a}{b(\zeta)} \zeta^\frac{-1}{2} AZ^{(r)}, AZ^c \right) \\
= (a^2 - b^2) \frac{a - b}{r} \|AZ^{(r)}\|_2^2 \\
+ 2ab \frac{2r - a - b}{r} \left( AZ^{(r)}, AZ_c^{(r)} \right) \\
= t\rho_{a,b}(t) \|AZ^{(r)}\|_2^2 + 2ab(2 - t) \left( AZ^{(r)}, AZ \right) .
\] (II.16)
where the first equality follows from Lemma II.5, i.e., $Z^{(r)}_i$ has the convex decomposition, and in the second equality, we used the identity (II.1).

As $0 < t < 1$, by Lemma 2 in [26], we get
\[
LHS = \rho_{a,b}(t) \left( \frac{a + b}{r} \right)^2 \|AZ^{(r)}\|_2^2 \\
= \rho_{a,b}(t) \|AZ^{(r)}\|_2^2.
\] (II.17)
Substituting (II.16) to the right hand side of (II.13), we get
\[
RHS = t \left( \rho_{a,b}(t) \|AZ^{(r)}\|_2^2 + 2ab(2 - t) \left( AZ^{(r)}, AZ \right) \right) \\
- 2t^2(2 - t)ab \left( AZ^{(r)}, AZ \right) \\
= LHS.
\]
Accordingly, the identity (II.13) holds.
As $1 \leq t < 4/3$, we get
\[
LHS = \rho_{a,b}(t) \left\{ [1 - (t - 1)^2] \|AZ^{(r)}\|_2^2 \\
+ 2(t - 1)t \left( AZ^{(r)}, \sum_k \tau_k AW_k \right) \right\} \\
= \rho_{a,b}(t) \left\{ [1 - (t - 1)^2] \|AZ^{(r)}\|_2^2 \\
+ 2(t - 1)t \left( AZ^{(r)}, AZ^{(r)}_c \right) \right\} \\
= \rho_{a,b}(t) \left\{ (4t - 3t^2) \|AZ^{(r)}\|_2^2 \\
+ 2(t - 1)t \left( AZ^{(r)}, AZ \right) \right\}.
\]
We have
\[
RHS = (4 - 3t) \left( \rho_{a,b}(t) \|AZ^{(r)}\|_2^2 \\
+ 2ab(2 - t) \left( AZ^{(r)}, AZ \right) \right) \\
- 2t^3(ab - (t - 1)^2) \left( AZ^{(r)}, AZ \right) \\
= (4t - 3t^2)\rho_{a,b}(t) \|AZ^{(r)}\|_2^2 + 2t \left( ab(2 - t)(4 - 3t) \\
- t^2(ab - (t - 1)r^2) \right) \left( AZ^{(r)}, AZ \right) \\
=LHS.
\]
Therefore, the identity (II.14) holds.

Lemma II.7. It holds that
\[
\{ [(a + b)^2 - 4ab]t - [(a + b)^2 - 2ab](2 - t)\delta_{tr} \} \|Z^{(r)}\|_F^2 \\
- 2abr\delta_{tr}(2 - t) \leq \Delta_{a,b}.
\] (II.18)

Proof. Note that the ranks of matrices $U_k$, $Z_{S_j}^{(r)}$ are no more than $b$, the ranks of matrices $V_k$, $Z_{T_i}^{(r)}$ are at most $a$ and $a + b = tr$. By the $tr$-order restricted isometry property, we get
\[
\Delta_{a,b} \geq \frac{r - b}{a(\zeta)} \sum_{i \in A, k} \mu_k \left[ a^2(1 - \delta_{tr}) \|Z^{(r)}_{T_i}\|_F^2 \\
- b^2(1 + \delta_{tr}) \|Z^{(r)}_{T_i}\|_F^2 \right] \\
+ \frac{r - a}{b(\zeta)} \sum_{j \in B, k} \nu_k \left[ b^2(1 - \delta_{tr}) \|Z^{(r)}_{S_j}\|_F^2 + \frac{b}{r} \|V_k\|_F^2 \right] \\
- a^2(1 + \delta_{tr}) \|Z^{(r)}_{S_j}\|_F^2 \leq \Delta_{a,b}.
\]
Since the inner product of $Z^{(r)}_{T_i}$ ($Z^{(r)}_{S_j}$) and $U_k$ ($V_k$) equals to zero, by some elementary calculation, we get
\[
\Delta_{a,b} \geq (a^2 - b^2) \left[ \frac{r - b}{a(\zeta)} \sum_{i \in A} \|Z^{(r)}_{T_i}\|_F^2 - \frac{r - a}{b(\zeta)} \sum_{j \in B} \|Z^{(r)}_{S_j}\|_F^2 \right] \\
- (a^2 + b^2)\delta_{tr} \left[ \frac{r - b}{a(\zeta)} \sum_{i \in A} \|Z^{(r)}_{T_i}\|_F^2 \\
+ \frac{r - a}{b(\zeta)} \sum_{j \in B} \|Z^{(r)}_{S_j}\|_F^2 \right] \\
- \frac{2ab)(r - a)\delta_{tr}}{r^2} \sum_k \mu_k \|U_k\|_F^2 \\
- \frac{2a^2b(r - a)\delta_{tr}}{r^2} \sum_k \nu_k \|V_k\|_F^2.
\] (II.19)
By Lemma I.1, we get
\[
\sum_{i \in A} \|Z^{(r)}_{T_i}\|_F^2 = \zeta^\frac{-1}{2} \|Z^{(r)}\|_F^2,
\] (II.20)
and
\[
\sum_{j \in B} \|Z^{(r)}_{S_j}\|_F^2 = \zeta^\frac{-1}{2} \|Z^{(r)}\|_F^2.
\] (II.21)
Substituting (II.20) and (II.21) into (II.19) and combining with inequalities (II.8) and (II.9), we get
\[
\Delta_{a,b} \geq \frac{(a - b)(a + b)}{r} \|Z^{(r)}\|_F^2 - (a^2 + b^2)\delta_{tr}(2 - t) \|Z^{(r)}\|_F^2
\]
Lemma II.8. (Lemma 4.1 in [14]) For all linear maps $A: \mathbb{R}^{m \times n} \to \mathbb{R}^r$ and $r \geq 2$, $s \geq 2$, we have
$$\delta_{sr} \leq (2s-1)\delta_r,$$
where $\delta_{sr}$ is defined in (II.10). \hfill (II.22)

Lemma II.9. It holds that for $0 < t < 1$,
$$\rho_{a,b}(t) \sum \tau_k \left[ \left\| A \left( Z^{(r)} + (t-1)W_k \right) \right\|_2^2 - \left\| (t-1)A \left( Z^{(r)} + W_k \right) \right\|_2^2 \right] \leq \rho_{a,b}(t) \left[ t(2-t) - (t^2 - 2t + 2)\delta_r \right] \left\| Z^{(r)} \right\|_F^2 - 2\alpha^2 \delta_r (t-1), \hfill (II.24)$$
and for $1 \leq t < 4/3$,
$$\rho_{a,b}(t) \sum_{\tau_k} \left[ \left\| A \left( Z^{(r)} + (t-1)W_k \right) \right\|_2^2 - \left\| (t-1)A \left( Z^{(r)} + W_k \right) \right\|_2^2 \right] \leq \rho_{a,b}(t) \left( t(2-t) - (t^2 - 2t + 2)\delta_r \right) \left\| Z^{(r)} \right\|_F^2 - 2\alpha^2 \delta_r (t-1),$$
where
$$\rho_{a,b}(t) = (a + b)^2 - 2ab(4-t).$$

Proof. We first consider the case of $0 < t < 1$. As $t_r$ equals to even, we can fix $a = b = t_r/2$. As $t_r$ equals to odd, we can set $a = b = 1 + (t_r - 1)/2$. For both cases, one can easily prove that $\rho_{a,b}(t) < 0$. Since $Z_{T_1}, Z_{S_1}$ are $r$-rank and $s$-rank, respectively, by utilizing $tr$-order RIP, we get
$$\rho_{a,b}(t) \sum_{\tau_k} \left[ \left\| A \left( Z^{(r)} + (t-1)W_k \right) \right\|_2^2 - \left\| (t-1)A \left( Z^{(r)} + W_k \right) \right\|_2^2 \right] \leq \rho_{a,b}(t) \left( t(2-t) - (t^2 - 2t + 2)\delta_r \right) \left\| Z^{(r)} \right\|_F^2 - 2\alpha^2 \delta_r (t-1),$$
where the first equality follows from the fact that $\langle Z^{(r)}, W_k \rangle = 0$, and for the last inequality, we used the inequality (II.10). \hfill \Box

As we described in the Introduction part, Cai and Zhang [14] established the sharp sufficient conditions to ensure the recovery of low-rank matrices via nuclear norm minimization. Their main results are stated as follows.

Theorem II.1. (Theorem 3.7 in [14]) Consider the affine rank minimization problem $b = AX + z$ with $\|z\|_2 \leq \epsilon$. Let $X_*$ be the minimizer of $\arg \min_{X} \|X\|_* : AX - z \in B$ with $B = \{z : \|z\|_2 \leq \eta\}$ for some $\eta \geq \epsilon$. If $\delta_r < 1/3$ with $r \geq 2$, then
$$\|X_* - X\|_F \leq \sqrt{2(1 + \delta_r)}(\epsilon + \eta) + 2\sqrt{2} \left\{ \sqrt{2} + 2\delta_r + \frac{(1 + 3\delta_r)(\epsilon + \eta)}{1 - 3\delta_r} \right\} \cdot \|X - X^{(r)}\|_*.$$  \hfill (II.25)

Theorem II.2. (Theorem 3.8 in [14]) Consider the affine rank minimization problem $b = AX + z$ with $\|A^*z\| \leq \epsilon$. Let $X_*$ be the minimizer of $\arg \min_{X} \|X\|_* : AX - z \in B$ with  \hfill (II.25)
B = \{z : \|A^*(z)\| \leq \eta\} for some \eta \geq \epsilon. If \delta_r < 1/3 with r \geq 2, then

\|X_r - X\|_F \leq \frac{\sqrt{2}\tau}{1 - 3\delta_r} (\epsilon + \eta) + 2\sqrt{\frac{2}{T} \left[ \frac{2}{3} + \frac{2\delta_r + \sqrt{(1 - 3\delta_r)\epsilon_r}}{1 - 3\delta_r} \right]} \cdot \|X - X^{(r)}\|_*.

### III. Main results

**Theorem III.1.** Consider rank minimization problem \(b = AX + z\) with \(\|z\|_2 \leq \epsilon\). If \(\delta_{tr} < t/(4 - t)\) with \(0 < t < 4/3\), then the solution \(X^*\) to the nuclear norm minimization problem (I.2) fulfills

\[ \|X - X^*\|_F \leq C_1 \epsilon + C_2 \|X - X^{(r)}\|_*, \]  

where

\[ C_1 = \frac{2\sqrt{2(1 + \delta_{tr})}\kappa}{4 - t - \delta_{tr}}, \]  

and

\[ C_2 = \frac{2\sqrt{2}}{\sqrt{T}} \left\{ \frac{1}{4} + \frac{2\delta_{tr} + \sqrt{\delta_{tr}(4 - t)\left( \frac{t}{4 - t} - \delta_{tr}\right)}}{\frac{t}{4 - t} - \delta_{tr}} \right\}, \]  

with

\[ \kappa = \max \left\{ \frac{t}{4 - t}, \frac{\sqrt{t}}{4 - t} \right\}. \]

Similarly, Consider rank minimization problem \(b = AX + z\) with \(z\) such that \(\|A^*(z)\| \leq \epsilon\). If \(\delta_{tr} < t/(4 - t)\) with \(0 < t < 4/3\), then the solution \(X^*\) to the nuclear norm minimization problem \(\min_X \|X\|_* \text{ s.t. } \|A^*(z)\| \leq \epsilon\) fulfills

\[ \|X - X^*\|_F \leq C_1 \epsilon + C_4 \|X - X^{(r)}\|_*, \]  

where

\[ C_3 = \frac{2\sqrt{2}\tau\kappa}{4 - t - \delta_{tr}}, \]  

and \(C_4 = C_2\).

**Remark III.1.** As \(t = 1\), the upper bound \(\delta_{r} < 1/3\) is coincident with Theorems I.1 and II.2 of [14]. Furthermore, the upper bounds of error estimates \(\|X - X^*\|_F \div (\|X - X^*\|_F)\) are smaller than the results of [14]. In theory, the recovered precision is given by our results is higher than that of theirs.

**Corollary III.1.** Assume that \(X \in \mathbb{R}^{m \times n}\) is a r-rank matrix. Let \(b = AX\). If

\[ \delta_{tr} < t/(4 - t) \]  

for \(0 < t < 4/3\), then the solution \(X^*\) to the nuclear norm minimization problem (I.2) in the noiseless case (i.e., \(\epsilon = 0\)) reconstructs \(X\) exactly.

**Remark III.2.** As \(t = 1\), the upper bound \(\delta_{r} < 1/3\) is the same as Theorem 3.5 of [14].

The Gaussian noise situation is of special interest in statistics and image processing. Note that the Gaussian random variables are essentially bounded. The results given in Theorem III.1 regarding the bounded noise situation are immediately applied to the Gaussian noise situation, which employs the similar discussion as that in [28].

**Theorem III.2.** Assume that the low-rank recovery model \(b = AX + z\) with \(z \sim N_q(0, \sigma^2I)\). \(\delta_{tr} < t/(4 - t)\) for some \(0 < t < 4/3\). Let \(X^*\) represent the minimizer of \(\min_{X} \|X\|_* \text{ s.t. } \|z\|_2 \leq \sqrt{q} + 2\sigma \log q\) and let \(X^*\) be the minimizer of \(\min_{X} \|X\|_* \text{ s.t. } \|A^*(z)\| \leq 2\sigma \sqrt{\log n}\). We have with probability at least \(1 - 1/q\),

\[ \|X - X^*\|_F \leq \frac{2 \sqrt{(1 + \delta_{tr})\kappa}}{\tau - \delta_{tr}} \sigma \sqrt{q} + 2\sigma \sqrt{q \log q} + 2 \sqrt{\frac{2}{q} \left\{ \frac{1}{4} + \frac{2\delta_{tr} + \sqrt{\delta_{tr}(4 - t)\left( \frac{t}{4 - t} - \delta_{tr}\right)}}{\frac{t}{4 - t} - \delta_{tr}} \right\}} \|X - X^{(r)}\|_* \]  

and probability at least \(1 - 1/\sqrt{\pi \log n}\),

\[ \|X - X^*\|_F \leq \frac{4 \sqrt{(1 + \delta_{tr})\kappa}}{\tau - \delta_{tr}} \sigma \sqrt{\log n} + 2 \sqrt{\frac{2}{q} \left\{ \frac{1}{4} + \frac{2\delta_{tr} + \sqrt{\delta_{tr}(4 - t)\left( \frac{t}{4 - t} - \delta_{tr}\right)}}{\frac{t}{4 - t} - \delta_{tr}} \right\}} \|X - X^{(r)}\|_* \]  

where \(\kappa\) is defined in Theorem III.1.

**Theorem III.3.** Let \(1 \leq r \leq m/2\). There is a linear map \(A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^q\) with \(\delta_{tr} < t/(4 - t) + \epsilon\) with \(0 < t < 4/3\), \(\epsilon > 0\) such that for some \(r\)-rank matrices \(Y_1, Y_2 \in \mathbb{R}^{m \times n}\) with \(Y_1 \neq Y_2\), \(AY_1 = AY_2\). Hence, there don’t exist any approach to exactly reconstruct all \(r\)-rank matrices \(X\) based on \((A, z)\).

**Remark III.3.** Theorems III.1 and III.3 jointly indicate the condition \(\delta_{tr} < t/(4 - t)\) with \(0 < t < 4/3\) is sharp.

### IV. Numerical experiments

**A. Solution algorithm**

In this section, we carry out some numerical experiments to verify our theoretical results. In order to solve the nuclear norm minimization model (I.2), we will utilize alternating direction method of multipliers (abbreviated as ADMM), which is usually applied in sparse signal recovery and low-rank matrix reconstruction, see references [29] [30] [31]. The constrained optimization problem (I.2) could be converted into the unconstrained optimization problem as follows:

\[ \min_{\hat{X}} \|\hat{X}\|_* + \frac{\lambda}{2} \|\text{vec}(\hat{X}) - b\|_2^2, \]  

where \(\lambda\) is a regularization parameter, and vec(\(\hat{X}\)) denotes the vectorization of \(\hat{X}\). Then, we employ ADMM technique to solve the problem (IV.1). In particular, introducing a new
variable $V \in \mathbb{R}^{m \times n}$, the problem (IV.1) could be equivalently transformed into
\[
\min_{\hat{X}} \|V\|_* + \frac{\lambda}{2} \|\text{Vec}(\hat{X}) - b\|_2^2 \quad \text{s.t.} \quad \hat{X} = V.
\] (IV.2)

The augmented Lagrangian function is
\[
L(\hat{X}, V, Y) = \|V\|_* + \frac{\lambda}{2} \|\text{Vec}(\hat{X}) - b\|_2^2 + \langle Y, \hat{X} - V \rangle
+ \frac{\mu}{2} \|\hat{X} - V\|_F^2,
\] (IV.3)

where $Y \in \mathbb{R}^{m \times n}$ is the dual variable, and $\mu$ is the penalty parameter associating to augmented Lagrangian function. Then, applying ADMM to (IV.3), we could obtain the following iterations:
\[
\hat{X}^{k+1} = \arg \min_{\hat{X} \in \mathbb{R}^{m \times n}} \frac{\lambda}{2} \|\text{Vec}(\hat{X}) - b\|_2^2 + \frac{\mu}{2} \|\hat{X} - V^k\|_F^2 + \frac{\mu}{2} \|\hat{X} - V^k\|_F^2
\]
\[
V^{k+1} = \arg \min_{V \in \mathbb{R}^{m \times n}} \|V\|_* + \frac{\mu}{2} \|\hat{X} - V^k\|_F^2
\]
\[
Y^{k+1} = Y^k + \hat{X}^{k+1} - V^{k+1}.
\] (IV.4)

In the experiment, the $r$-rank matrix $X \in \mathbb{R}^{m \times n}$ is generated by $X = P \ast Q$, where $P \in \mathbb{R}^{m \times r}$ and $Q \in \mathbb{R}^{r \times n}$. We produce the measurement matrix $A \in \mathbb{R}^{q \times mn}$ with its elements being i.i.d. zero mean and $1/q$ Gaussian random variables. In all experiments, we take $m = n = 50$, and $r = 0.2 \ast m$. On the premise that $A$ and $X$ are known, the measurement $b$ is produced by $b = \text{Vec}(X) + \epsilon \ast z$, where the entries of $z$ follow zero mean and 0.05 standard variation Gaussian distribution, and $\epsilon$ represents the noise level whose range of value is 0, 0.05 and 0.1. In all experiments, we report the average result over 50 independent tests.

### B. Algorithm convergence

Fig. 4.1 shows the result about algorithm convergence for solving the problem (I.2). It is observed that the relative neighboring iteration error $(r(k) = \|X^{k+1} - X^k\|_F/\|X^k\|_F)$ decreases with the increase of iteration times $k$. When the number of iterations $k$ exceeds 210, it tends to become less than $10^{-4}$.

---

### C. Comparison of error bounds

In Fig. 4.2(a) $\|X - X^*\|_F$ is plotted versus the rank $r$ for different noise level $\epsilon = 0$, 0.05, 0.1. In Fig. 4.2(b) the relevant theoretical error bound determined by (III.1) is plotted with $t = 1$ and $\delta_k = 0.05$. One can easily see that $\|X - X^*\|_F$ is lower than the theoretical error bound. Fig. 4.3(a) and (b) present $\|X - X^*\|_F$ and the corresponding theoretical error bound defined by (I.4).

---

### D. Results of different measurement matrices

Fig. 4.4(a) plots the relationship between the relative error $\|X - X^*\|_F/\|X\|_F$ and the rank $r$ for Gaussian measurement matrix. Fig. 4.4(b) plots the relation the relative error and the number of measurement $q$. It is easy to see that a decreasing rank $r$ or an increasing number of measurement $q$ leads to a better performance of the model (I.2). Furthermore, the smaller the noise level, the better the model reconstruction effect.

In Figs. 4.5 and 4.6, the relative errors are plotted respectively for Bernoulli measurement matrix and Partial Fourier measurement matrix. It is observed from Figs. 4.4, 4.5 and 4.6 that the reconstruction performance of the nuclear norm
minimization method (I.2) is the best when the measurement matrix is a partial Fourier matrix.

V. PROOFS OF MAIN RESULTS

With above preparation, we present the proof of main results.

Proof of Theorem III.1. By the definition of $\alpha$ and notice that the rank of $Z^{(r)}$ is at most $r$, we get

$$\alpha^2 = \frac{\|Z^{(r)}\|_F^2 + 4\|Z^{(r)}\|_F \|X - X^{(r)}\|_F + 4\|X - X^{(r)}\|_F^2}{r^2} \leq \frac{\|Z^{(r)}\|_F^2}{r^2} \frac{1 + \delta_r}{\delta_r} \frac{\|X\|_F^2}{\|X\|_F^2} + \frac{4\|X - X^{(r)}\|_F^2}{r^2},$$

(V.1)

where in the last step, we used the fact that for any $X \in \mathbb{R}^{m \times n}$ ($m \leq n$) and $p \in (0, 1)$,

$$m^{p-\frac{1}{2}} \|X\|_F \geq \|X\|_p$$

(V.2)

with $\|X\|_p = (\sum_i \sigma_i^p(X))^{1/p}$. Additionally, due to the feasibility of $X^*$, we get

$$\|AZ\|_2 = \|AX - AX^*\|_2 \leq \|AX - b\|_2 + \|AX^* - b\|_2 \leq 2\epsilon.$$  

(V.3)

In the situation of $0 < t < 1$, by Lemma II.8, we have

$$\langle AZ^{(r)}, AZ \rangle \leq \|AZ^{(r)}\|_2 \|AZ\|_2 \leq \sqrt{1 + \delta_r \|Z^{(r)}\|_F} \|AZ\|_2$$

$$= \sqrt{1 + \delta_r \|Z^{(r)}\|_F} \|Z^{(r)}\|_F \|AZ\|_2$$

$$\leq \sqrt{1 + \frac{2}{t} \|Z^{(r)}\|_F} \|AZ\|_2$$

$$\leq \sqrt{1 + \frac{\delta_r}{t} \|Z^{(r)}\|_F} \|AZ\|_2$$

(V.4)

where in the first inequality, we used Cauchy-Schwarz inequality, and the second inequality follows from RIP of $r$-order.

Plugging (V.3) to (V.4), it follows that

$$\langle AZ^{(r)}, AZ \rangle \leq 2\epsilon \sqrt{\frac{1 + \delta_r}{t} \|Z^{(r)}\|_F}. \quad \text{(V.5)}$$

Combining with equation (II.13) and inequalities (II.18), (II.23) and (V.5), we have

$$\rho_{a,b}(t)(t - (2 - t)\delta_r)\|Z^{(r)}\|_F^2$$

$$+ 4ab\epsilon^2(2 - t)\sqrt{\frac{1 + \delta_r}{t} \|Z^{(r)}\|_F} - t \left\{ [(a + b)^2 - 4ab]t \right\}$$
Applying inequality (V.1) to above equality, we get

\[- [(a + b)^2 - 2ab](2 - t)\delta_t] \| Z(r) \|_F^2 \]

\[- 2abr\delta_t\alpha^2(2 - t) \geq 0. \]

Applying inequality (V.1) to above equality, we get

\[2ab(t - 2) \left(4 - t \right) \left(\frac{t}{4 - t} - \delta_t \right) \| Z(r) \|_F^2 \]

\[- \left[2\sqrt{\epsilon(1 + \delta_t)}t + \frac{4\delta_t \| X - X(r) \|_F^2}{\sqrt{t}} \right] \| Z(r) \|_F^2 \]

\[- \frac{4\delta_t \| X - X(r) \|_2^2}{r} \geq 0. \quad (V.6) \]

In the situation of \(1 \leq t < 4/3\), due to the monotonicity of RIC \(\delta_t\), it implies that

\[\langle AZ^{(r)}, A\rangle \leq \sqrt{1 + \delta_t} \| Z(r) \|_F \| AZ \|_2 \]

\[\leq \sqrt{1 + \delta_t} \| Z(r) \|_F \| AZ \|_2 \]

\[\leq 2\epsilon \sqrt{1 + \delta_t} \| Z(r) \|_F. \quad (V.7) \]

It is easy to check that

\[ab \geq \left(\frac{tr}{2}\right)^2 - \frac{1}{4} = \frac{(2 - t)^2}{4} - \frac{1}{4} - (1 - t)r^2 \]

\[> -(1 - t)r^2. \quad (V.8) \]

Due to inequality (V.1), by fundamental calculation, we get

\[2t(1 - t)^r - ab \left(4 - t \right) \left(\frac{t}{4 - t} - \delta_t \right) \| Z(r) \|_F^2 \]

\[- \left[2\sqrt{\epsilon(1 + \delta_t)}t + \frac{4\delta_t \| X - X_{(r)} \|_F^2}{\sqrt{t}} \right] \| Z(r) \|_F^2 \]

\[- \frac{4\delta_t \| X - X_{(r)} \|_2^2}{r} \geq 0. \quad (V.10) \]
Then, a combination of (V.11) and (V.12) implies that
\[
\|X - X^{(r)}\|_F \leq \frac{1}{2(4 - t)\frac{1}{4-t} - \delta_{tr}} \left[ 4\delta_{tr} \|X - X^{(r)}\|_F \right] + 2\sqrt{\frac{\delta_{tr}}{4-t}} \left[ \frac{1}{4} + \frac{2\delta_{tr} + \sqrt{(4-t)(\frac{4-t}{4-t} - \delta_{tr})}}{4-t} \right] X - X^{(r)}\|_F.
\]
In the situation of the error bound \(\|A^*(z)\| \leq \epsilon\), set \(Z = X - X^{\circ}\). It holds that
\[
\|A^*AZ\| = \|A^*(AX - b) - A^*(AX^{\circ} - b)\| \leq \|A^*(AX - b)\| + \|A^*(AX^{\circ} - b)\| \leq 2\epsilon.
\]
Moreover,
\[
\langle AZ^{(r)}, AZ \rangle = \langle Z^{(r)}, A^*AZ \rangle \leq \|Z^{(r)}\|_F \cdot 2\epsilon \leq 2\epsilon \sqrt{\|Z^{(r)}\|_F^2}.
\]
The rest of steps are similar with the situation of the error bound \(\|z\|_2 \leq \epsilon\). The proof of Theorem III.1 is completed.

**Proof of Theorem III.3.** Let \(E = \text{diag}(x) \in \mathbb{R}^{m \times m}\) with
\[
x = \frac{1}{\sqrt{2r}} (1, \cdots, 1, 0, \cdots, 0).
\]
Define \(A : \mathbb{R}^{m \times m} \rightarrow \mathbb{R}^r\) as
\[
AX = \frac{2}{\sqrt{4 - t}} (\sigma(X) - \langle \sigma(X), \sigma(E) \rangle \sigma(E))
\]
Applying the Cauchy-Schwarz inequality, for all \([tr]-\text{rank}\) matrices \(X\), we get
\[
\|\sigma(X), \sigma(E)\| \leq \|\sigma(X)\|_2 \|\sigma(E)\|_F 1_{\sup(\sigma(X))} \leq \sqrt{\frac{\|X\|_F^2}{2r}}
\]
and
\[
\|AX\|_2^2 = \frac{4}{4-t} \times (\sigma(X) - \langle \sigma(X), \sigma(E) \rangle \sigma(E), \sigma(X) - \langle \sigma(X), \sigma(E) \rangle \sigma(E)) \leq \frac{4}{4-t} \left[ \|X\|_F^2 - \|\langle \sigma(X), \sigma(E) \rangle\|_2^2 \right].
\]
Therefore,
\[
\|AX\|_2^2 \leq \left( 1 + \frac{t}{4-t} + \epsilon \right) \|X\|_F^2.
\]
For \(r > 1/\epsilon\), we get
\[
\|AX\|_2^2 \geq \frac{4}{4-t} \left( 1 - \frac{[tr]}{2r} \right) \|X\|_F^2 \geq \frac{4}{4-t} \left( 1 - \frac{tr}{2r} - \frac{1}{2r} \right) \|X\|_F^2 \geq \frac{4}{4-t} \left( 1 - \frac{tr}{2r} - \frac{\epsilon}{2} \right) \|X\|_F^2.
\]


\[
\geq \left(1 - \frac{t}{4t - t} - \varepsilon\right) \|X\|_F^2.
\]

Accordingly, by Definition 1.1, we obtain \(\delta_{tr} = \delta_{tr}\) \(\frac{t}{4t - t} + \varepsilon\). Suppose \(Y_1 = \text{diag}(y_1), Y_2 = \text{diag}(y_2) \in \mathbb{R}^{m \times m}\) with

\[
y_1 = (1, \cdots, 1, 0, \cdots, 0)
\]

and

\[
y_2 = (0, \cdots, 0, -1, \cdots, -1, 0, \cdots, 0).
\]

It is easy to verify that \(Y_1\) and \(Y_2\) are both matrices of rank \(r\) such that \(Y_1 - Y_2 \in \mathcal{N}(A)\), i.e., \(AY_1 = AY_2\). Consequently, it is not possible to reconstruct both \(Y_1\) and \(Y_2\) based on \((z, A)\).

VI. Conclusion

In this paper, we establish sufficient conditions which ensure the stable recovery or exactly recovery of any \(r\)-rank matrix satisfying a given linear system of equality constraints via solving a convex optimization problem, i.e., nuclear norm minimization. When the parameter \(t\) is equal to 1, the bound of RIC \(\delta_r\) coincide with the result of [14]. Meanwhile, the derived upper bounds regarding the reconstruction error are better than those of [14]. Besides, the restricted isometry property condition is proved sharp. And integrated with the main results of [25], i.e., the case of \(t > 4/3\), for sharp RIP conditions for all \(t > 0\), we present an intact characterization that can guarantee the exact recovery of all \(r\)-rank matrices by way of nuclear norm minimization. Furthermore, the numerical experiments demonstrate the performance of nuclear norm minimization method.

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