A problem of optimal switching and singular control with discretionary stopping in portfolio selection

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Abstract

In this paper we study the optimization problem of an economic agent who chooses a job and the time of retirement as well as consumption and portfolio of assets. The agent is constrained in the ability to borrow against future income. We transform the problem into a dual two-person zero-sum game, which involves a controller, who is a minimizer and chooses a non-increasing process, and a stopper, who is a maximizer and chooses a stopping time. We derive the Hamilton-Jacobi- Bellman quasi-variational inequality(HJBQV) of a max-min type arising from the game. We provide a solution to the HJBQV and verification that it is the value of the game. We establish a duality result which allows to derive the optimal strategies and value function of the primal problem from those of the dual problem.

Keywords: consumption, portfolio selection, job switch, early retirement, borrowing constraint, zero-sum game, Nash-equilibrium, Hamilton-Jacobi-Bellman quasi-variational inequality

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1 Introduction

In this paper we study the optimization problem of an economic agent who chooses a job and the time of retirement as well as consumption and portfolio of assets. There has been increasing interest in studying life-cycle patterns of consumption and asset allocation. An important realistic aspect in the study is to incorporate labor supply and human capital and their effects on consumption and risky investments. Job choice and retirement decisions are two important factors determining labor supply and human capital.

As the average life span and flexibility in the job market increase, there exist higher chances that people change their jobs, being motivated by the consideration of leisure and job satisfaction than that of higher salaries (Shim et al. (2018)). Thus, a job which promises more leisure time but with a lower salary can be a substitute for retirement. Taking such a job allows one to have an option to choose another job with a higher salary and less leisure time and thus to keep a positive value of human capital, while retirement is an irreversible decision and makes human capital equal to 0 permanently. We study the choice between the two substitutes, a job with more leisure time and lower salary and permanent retirement in this paper.

Another important aspect regarding human capital is the constraint which restricts people from borrowing against future income. Researchers have shown that the borrowing constraint has significant effects on consumption, investment, labor supply, and wealth accumulation (He and Pagés (1993), Domeij and Flodén (2006), Rendon (2006), Dybvig and Liu (2010)). We study the optimization problem of a borrowing constrained agent.

Mathematically, the problem is a combination of continuous control, discrete control and optimal stopping. The optimal retirement problem alone introduces complication, as retirement decision interacts with optimal consumption and portfolio choice (Yang and Koo (2018)). The borrowing constraint significantly complicates the problem, as it makes the financial market incomplete from the agent’s perspective (Dybvig and Liu (2011)).

We extend the previous work on optimal consumption and portfolio choice (Farhi and Panageas (2007), Dybvig and Liu (2010, 2011), and Lee et al. (2019)) by considering the job choice or retirement decision. Moreover, we employ a class of general utility functions. Farhi and Panageas (2007) and Dybvig and Liu (2010, 2011) consider the optimization problem with a constant relative risk aversion (CRRA) utility function. Thanks to the homogeneity of the CRRA utility function, the free boundaries associated with retirement and borrowing constraints are characterized by only one algebraic equation in their work. In contrast, those in our problem are expressed as solutions of two coupled algebraic equations, which makes the problem more complex and difficult.

There is a substantial work investigating the combined problem of the optimal retirement decision and the borrowing constraint (Farhi and Panageas (2007), Choi et al. (2007), Dybvig and Liu (2010, 2011), Lim and Shin (2011)). While these studies utilize the dual method, the dual problem is not specified in a clear and comprehensive manner, and often lack mathematical rigor for the verification of the dual problem and the duality theorem. To clarify the issue, we take a novel approach by transforming the problem into a two-person zero-sum game, which involves a controller, who is a minimizer and chooses a non-increasing process, and a stopper, who is a maximizer and chooses a stopping time.

Specifically, the transformation involves two critical steps. First, we transform the wealth dynamics into
a budget constraint in static form. We use the martingale approach developed by Karatzas et al. (1987) and Cox and Huang (1989). In order to incorporate the borrowing constraint we introduce a non-increasing shadow price process as in He and Pagés (1993) and El Karoui and Jeanblanc-Picqué (1998). Next we adopt the approach to change the order of optimization with in Karatzas and Wang (2000) and consider the problem of finding the optimal retirement time after selecting the optimal consumption in a dual problem. After the two steps are carried out, the problem becomes that involving the choice of an optimal retirement time which maximizes a dual objective function and that of a shadow price process which minimizes the dual objective function. Hence, the problem can be cast into a two-person zero-sum game; one player maximizes the objective by choosing the retirement time and the other player minimizes the objective by choosing the shadow price process. To our knowledge, this paper is the first that clarified the dual problem in the optimal retirement decision of a borrowing constrained agent.

We derive a Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQV) which is satisfied by the value function of the game. The HJBQV involves both minimum and maximum. We construct an explicit-form solution of the HJBQV and provide a verification that the solution is the value function of the game. Finally we establish a duality theorem which allows to derive the value function of the original problem from that of the game. By the duality theorem we can derive the optimal strategies of the primal problem from those of the dual problem.

The job choice is made by considering the trade-off between job satisfaction and salary. The agent chooses a job whenever the utility value from one job is greater than that from the other job, where the utility value includes that from the wage income.

The dual objective function has two components: (i) the present value of the convex conjugate of optimal consumption, and (ii) the present value of the difference between the utility values after and before retirement. We show that the retirement decision is made when the difference at the time is negative, i.e., the instantaneous utility after retirement is smaller than that before retirement. This is because retirement is an irreversible decision similar to exercise of a financial option; the exercise of an option occurs when it is in the money (Dixit and Pindyck (1994)). The characteristic of the problem as a game, however, implies that the decision is influenced by the presence of the borrowing constraint.

The literature on the two person zero-sum game of a controller and stopper includes Maitra and Sudderth (1996), Weerasinghe (2006), Karatzas and Zamfirescu (2008), Hamadéne (2006), Bayraktar and Young (2011), Hernández-Hernández et al. (2015), Hernández-Hernández and Yamazaki (2015), and references therein. In particular, the mathematical structure in our dual problem is similar to that in Hernández-Hernández et al. (2015) and Hernández-Hernández and Yamazaki (2015), as the two parties are a singular controller and a discretionary stopper. We add to the literature by considering a problem of optimal switching and singular control with discretionary stopping in portfolio selection.

The paper is organized as follows. We explain the model and the optimization problem in Section 2. We explain a dual problem by formulating a Lagrangian and the transformation into a two-person zero-sum game in Section 3. We derive a HJBQV and provide optimal strategies for the optimization problem in Section 4. We conclude in Section 5.
2 Model

We consider an agent whose objective is to maximize the following expected utility:

\[ U \equiv E \left[ \int_0^\tau e^{-\delta t} \left( u(\kappa_1 c_t) 1_{\{\zeta_t = B_1\}} + u(\kappa_2 c_t) 1_{\{\zeta_t = B_2\}} \right) dt + 1_{\{\tau < \infty\}} \int_\tau^\infty e^{-\delta t} u(c_t) dt \right], \quad (1) \]

Here \( \delta > 0 \) is the subjective discount rate, \( c_t \geq 0 \) is the rate of consumption at time \( t \), \( 1_A \) is the indicator function of set \( A \), \( \zeta_t \in \{ B_1, B_2 \} \) is the agent’s job at \( t \), \( \tau \) is the time of retirement, \( \kappa_i > 0 \) \( (i = 1, 2) \) is a constant, and \( u(\cdot) \) is the felicity function of consumption. For simplicity, we assume that there exist two different jobs available to the agent, that is, \( \zeta_t \) takes one of the two values, \( B_1 \) and \( B_2 \). Constant \( \kappa_i \) describes the agent’s satisfaction with job \( B_i \), a larger value implying higher satisfaction with the job.

The agent receives labor income at a rate \( \epsilon_i \) when she takes job \( B_i \). We assume that

\[ 0 < \epsilon_2 < \epsilon_1 \quad \text{and} \quad 0 < \kappa_1 < \kappa_2 < 1. \]

That is, job \( B_1 \) provides higher income but lower satisfaction than job \( B_2 \). Here, \( \kappa_i < 1 \) means that retirement provides higher satisfaction than working, and the marginal utility of consumption is greater after retirement than before retirement.

The agent has two options: the first is to switch jobs when working, and the second is to retire voluntarily. After retirement, the agent cannot go back to work, and thus retirement is an irreversible decision. In contrast, the job switching decision is freely reversible; the agent can switch from the current job to the other job at any time before retirement.

Financial Market: There exist two financial assets trading in the economy, a riskless asset and a risky asset, whose prices at \( t \) are denoted by \( S_{0,t} \) and \( S_{1,t} \), respectively. The asset prices satisfy the dynamics:

\[ dS_{0,t}/S_{0,t} = rd t \quad \text{and} \quad dS_{1,t}/S_{1,t} = \mu dt + \sigma dW_t, \]

where \( r > 0 \) is the risk-free rate, \( \mu > r \) is the drift of the risky asset price, \( \sigma \) is the volatility of the risky asset returns, and \( W_t \) is a standard Brownian motion on \( (\Omega, \mathcal{F}_\infty, P) \). We will denote the augmented filtration generated by the Brownian motion \( W_t \) by \( \mathcal{F} = \{ \mathcal{F}_t \}_{t \geq 0} \). Here, for simplicity of the model, we assume that the investment opportunity is constant, that is, the interest rate, the mean and standard deviation of the risky asset returns are constant. Under the constant investment opportunity assumption, the assumption of two assets is without loss of generality; by the two-fund separation theorem, the general case where there exist multiple risky assets with a constant covariance matrix of returns can be subsumed in our model by treating the market portfolio of risky assets in the general case as the single risky asset in our model (see Grossman and Laroque (1990)).

Wealth and Budget Constraint: We will now explain the budget constraint of the agent. Let us denote the agent’s investment in the risky asset at time \( t \) by \( \pi_t \) (in dollar amount). The agent’s wealth follows the dynamics:

\[ dX_t = [rX_t + (\mu - r)\pi_t - c_t + (\epsilon_1 1_{\{\zeta_t = B_1\}} + \epsilon_2 1_{\{\zeta_t = B_2\}}) 1_{\{t < \tau\}}] dt + \sigma \pi_t dW_t. \quad (2) \]
with \( X_0 = x \)

Let \( \theta := (\mu - r) / \sigma \), the risk premium on the return of the risky asset for one unit of standard deviation, or the Sharpe ratio. We define the stochastic discount factor, \( \mathcal{H}_t \):

\[
\mathcal{H}_t \equiv \exp \left( - \left\{ r + \frac{1}{2} \theta^2 \right\} t - \theta W_t \right).
\]

Since \( \epsilon_1 > \epsilon_2 \), the natural limit to the agent’s borrowing constraint is determined by the present value of the income stream:

\[
X_t \geq -\mathbb{E}_t \left[ \int_t^\infty \mathcal{H}_s \epsilon_1 ds \right] = -\frac{\epsilon_1}{r}, \quad \text{for all } t \geq 0,
\]

(3)

where \( \mathbb{E}_t[\cdot] = \mathbb{E}_t[\cdot | \mathcal{F}_t] \) is the expectation conditional on the filtration \( \mathcal{F}_t \).

We consider the **borrowing constraint** which restricts the agent from borrowing against future labor income:

\[
X_t \geq 0 \quad \text{for all } t > 0.
\]

(4)

Given \( X_0 = x \), we call a quadruple \((\delta, c, \pi, \tau)\) admissible if

(a) the job process \( \zeta := (\zeta_t)_{t=0}^\infty \) is \( \mathcal{F}_t \)-adapted and takes one of the two values, \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \),

(b) \( c_t \geq 0 \) and \( \pi_t \) are \( \mathcal{F}_t \)-progressively measurable processes satisfying the following integrability conditions:

\[
\int_0^t c_s ds < \infty \quad \text{a.s., and} \quad \int_0^t \pi_s^2 ds < \infty \quad \text{a.s.} \quad \forall \ t \geq 0,
\]

(c) \( \tau \) belongs to \( \mathcal{S} \) which is the set of all \( \mathcal{F} \)-stopping times taking values in \([0, \infty)\),

(d) the agent’s financial wealth \( X_t \) in (2) for all \( t \geq 0 \) satisfies

\[
X_t \geq 0.
\]

We denote the set of all admissible strategies by \( \mathcal{A}(x) \).

We make the following assumptions on the felicity function to guarantee the existence of a solution to the agent’s optimization problem:

**Assumption 2.1.** The felicity function \( u : [0, \infty) \to \mathbb{R} \) is strictly increasing, strictly concave and continuously differentiable, and \( \lim_{c \to +\infty} u'(c) = 0 \).

The strictly decreasing and continuous function \( u' : (0, \infty) \overset{\text{onto}}{\longrightarrow} (0, u'(0)) \) has a strictly decreasing, continuous inverse \( I : (0, u'(0)) \overset{\text{onto}}{\longrightarrow} (0, \infty) \). We extend \( I \) by setting \( I(y) = 0 \) for \( y \geq u'(0) \). Then, we have

\[
u'(I(y)) = \begin{cases} y, & 0 < y < u'(0), \\ u'(0), & y \geq u'(0), \end{cases}
\]

and \( I(u'(c)) = c \) for \( 0 < c < \infty \). Note that \( \lim_{y \to \infty} I(y) = 0 \).

**Assumption 2.2.** For \( y > 0 \),

\[
\int_0^y \eta^{-n_2} I(\eta)d\eta < \infty.
\]
where $n_1 > 0$ and $n_2 < 0$ are two roots of the quadratic equation:

$$\frac{\theta^2}{2} n^2 + \left( \delta - r - \frac{\theta^2}{2} \right) n - \delta = 0. \quad (5)$$

We now state the agent's optimization problem.

**Problem 1.** Given $X_0 = x > 0$, we consider the following agent’s utility maximization problem:

$$V(x) = \sup_{(c, \pi, \zeta, \tau) \in A(x)} \mathbb{E} \left[ \int_0^\tau e^{-\delta t} \left( u(\kappa_1 c_t) 1_{\{G_t = \mathcal{B}_1\}} + u(\kappa_2 c_t) 1_{\{G_t = \mathcal{B}_2\}} \right) dt + 1_{\{\tau < \infty\}} \int_\tau^\infty e^{-\delta t} u(c_t) dt \right]. \quad (6)$$

Our strategy of solving Problem 1 consists of the following steps.

(Step 1) By using the well-known method developed by Karatzas and Shreve (1998) and Cox and Huang (1989), we transform Problem 1 into a static problem.

(Step 2) Using the budget constraint in static form, we formulate a Lagrangian for Problem 1. By maximizing the Lagrangian, we have the candidates of optimal consumption and job. Putting these in the Lagrangian, we obtain the dual problem, which takes the form of a two-person zero-sum game of a singular stochastic controller and a discretionary stopper.

(Step 3) Utilizing the dynamic programming principle, we derive the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQV) arising from the game. We provide an explicit-form solution to the HJBQV and verification that it is the value of the game.

(Step 4) Finally, we establish the duality theorem and characterize the optimal strategies.

### 3 A Dual Optimization Problem

We use the martingale-dual approach (see Karatzas et al. (1987), Cox and Huang (1989)) which allows us to use the Lagrangian method to solve Problem 1. In order to apply the approach we need to transform the dynamic constraint (4) into a static form. He and Pagès (1993) and El Karoui and Jeanblanc-Picqué (1998) study a consumption and portfolio selection problem with constraint (4) by introducing a non-increasing process, which can be thought of as the integral of infinitesimal Lagrange multipliers, but they do not consider the job-switching options and irreversible retirement decision. Recently, Lee et al. (2019) investigate optimal job switching and consumption-investment problems under the borrowing constraint, but they also do not consider the irreversible retirement decision. Our problem described in Problem 1, however, involves the job switching, borrowing constraint, and irreversible retirement decision. To obtain a constraint of static form for our problem, we combine the results of Lemma 6.3 in Karatzas and Wang (2000), Proposition 2.2 in El Karoui and Jeanblanc-Picqué (1998), and Lemmas 1 and 2 in Lee et al. (2019). As a result, we derive the following proposition.

Let $\mathcal{N}$ be the set of all non-negative, non-increasing, right-continuous processes with left limits (RCLL) and starting at 1.

**Proposition 3.1.**
(a) For any given \( \tau \in S \), the triplet \((c, \zeta, \tau)\) of the consumption, job, and retirement strategies such that \((c, \pi, \zeta, \tau) \in A(x)\) with a portfolio \( \pi \), satisfies the following budget constraint:

\[
\sup_{S \in \mathcal{A}} E \left[ \int_0^\tau H_tD_t (c_t - (\epsilon_11_{\{t=\tau\}} + \epsilon_21_{\{t>\tau\}})) \ dt + 1_{\{t<\infty\}} \mathcal{H}_tX_tD_t \right] \leq x. \tag{7}
\]

Budget constraint (7) is equivalent to

\[
\sup_\xi E \left[ \int_0^\tau H_t (c_t - (\epsilon_11_{\{t=\tau\}} + \epsilon_21_{\{t>\tau\}})) \ dt + 1_{\{\xi<\infty\}} \mathcal{H}_tX_t \right] \leq x, \tag{8}
\]

where \( S_\tau \) is the set of all stopping times such that \( 0 \leq \xi \leq \tau \).

(b) For any \((c, \zeta, \tau)\) satisfying budget constraint (8), then there exists a portfolio process \( \pi \) such that \((c, \pi, \zeta, \tau) \in A(x)\). Moreover, for \( t \in [0, \tau) \):

\[
\mathcal{H}_tX_t = E \left[ \int_t^\tau H_s (c_s - (\epsilon_11_{\{s=\tau\}} + \epsilon_21_{\{s>\tau\}})) \ ds + 1_{\{t<\infty\}} \mathcal{H}_tX_t \right]. \tag{9}
\]

3.1 Dual Value Function

In this subsection we study the dual optimization problem. We first explain the agent’s optimization problem after retirement. Assume \( t \geq \tau \), where \( t \) is a fixed constant denoting the current time. We define the agent’s optimization problem after retirement \( t \geq \tau \):

\[
V_R(X_t) := \sup_{(c, \pi) \in \mathcal{A}_R(X_t)} E_t \left[ \int_t^\infty e^{-\delta(s-t)}u(c_s)ds \right], \tag{10}
\]

where \( \mathcal{A}_R(X_t) \) is the set of all admissible consumption and portfolio strategies for given \( X_t > 0 \) satisfying the conditions: (i) for any \( T > 0 \), \( \int_t^T c_s \ ds < \infty \ a.s. \) and \( \int_t^T \pi_s^2 \ ds < \infty \ a.s. \), (ii) for \( s \geq t \), \( X_s \geq 0 \) with \( dX_s = [rX_s + (\mu - r)\pi_s - c_s]ds + \sigma \pi_s dB_s \).

By the well-known results in Section 3.9 in Karatzas and Shreve (1998), the following duality relationship holds:

\[
V_R(X_t) = \inf_{\mathcal{Y}_t} (J_R(\mathcal{Y}_t) + \mathcal{Y}_tX_t), \tag{11}
\]

where \( \mathcal{Y}_t = ye^{\delta t} \mathcal{H}_t \) and the dual value function \( J_R(y) \) after retirement is defined by

\[
J_R(y) := E \left[ \int_0^\infty e^{-\delta s} u(\mathcal{Y}_s)ds \right] \quad \text{with} \quad \tilde{u}(y) = \sup_{c \geq 0} (u(c) - yc). \tag{12}
\]

In particular,

\[
V_R(X_t) = E \left[ \int_t^\infty e^{-\delta(s-t)}u(c_s^R)\ ds \right], \tag{13}
\]

where \( c_s^R = I(\mathcal{Y}_s^R) \), \( \mathcal{Y}_s^R := \mathcal{Y}_t^Re^{\delta(s-t)} \mathcal{H}_s / \mathcal{H}_t \) and \( \mathcal{Y}_t^R \) is a unique solution of

\[
X_t = -J_R(\mathcal{Y}_t). \tag{14}
\]

The explicit form of \( J_R(y) \) is given by

\[
J_R(y) = \frac{2}{\theta^2(n_1 - n_2)} \left[ y^{n_2} \int_0^y \eta^{-n_2-1} \tilde{u}(\eta) \ d\eta + y^{n_1} \int_y^\infty \eta^{-n_1-1} \tilde{u}(\eta) \ d\eta \right].
\]
In Appendix A, we provide useful properties of $J_R(y)$.

We formulate the Lagrangian $\mathcal{L}$ of Problem 1 using constraint (7):

$$\mathcal{L} \equiv \sup_{\{\zeta, \zeta_t\}} \left\{ \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(\kappa_1 c_t 1_{\zeta_t = \mathfrak{B}_1}) + u(\kappa_2 c_t 1_{\zeta_t = \mathfrak{B}_2}) \right) dt + 1_{\{\zeta \leq \infty\}} \int_\tau e^{-\delta t} u(c_t) dt \right] \right\} + y \left( x - \mathbb{E} \left[ \int_0^\tau \mathcal{H}_t \mathcal{D}_t \left( \epsilon_t - \left( \epsilon_1 1_{\zeta_t = \mathfrak{B}_1} + \epsilon_2 1_{\zeta_t = \mathfrak{B}_2} \right) \right) dt + 1_{\{\zeta \leq \infty\}} \mathcal{H}_\tau \mathcal{X}_\tau \mathcal{D}_\tau - \right] \right) \right) \} \right) \leq \sup_{\{\zeta, \zeta_t\}} \left\{ \mathbb{E} \left[ \int_0^\tau e^{-\delta t} \left( \left( \bar{u} \left( \frac{Y_t D_t}{\kappa_1} \right) + \epsilon_1 Y_t D_t \right) 1_{\zeta_t = \mathfrak{B}_1} + \left( \bar{u} \left( \frac{Y_t D_t}{\kappa_2} \right) + \epsilon_2 Y_t D_t \right) 1_{\zeta_t = \mathfrak{B}_2} \right) \right] dt + 1_{\{\zeta \leq \infty\}} e^{-\delta t} J_R(Y_t \mathcal{D}_t - \right) \right) + y x \right) \right) ,$$

where $y > 0$ is the Lagrangian multiplier associated with constraint (7) and $\mathcal{D}_t \in \mathcal{N} \mathcal{T}$.

Since $\bar{u}(y) = \sup_{c \geq 0} (u(c) - yc) = u(I(y)) - y I(y)$, we deduce that the candidate of optimal consumption $\hat{c}(Y_t D_t)$ for $y > 0$ is given by

$$\hat{c}(z^D_t) := \begin{cases} \frac{1}{\kappa_1} \left( \frac{Z^D_t}{\kappa_1} \right) & \text{for } \zeta_t = \mathfrak{B}_1, \\ \frac{1}{\kappa_2} \left( \frac{Z^D_t}{\kappa_2} \right) & \text{for } \zeta_t = \mathfrak{B}_2, \end{cases}$$

where $Z^D_t := Y_t D_t$.

**Remark 3.1.** Note that the dynamics of $Z^D_t$ is given by

$$\frac{dZ^D_t}{Z^D_t} = (\delta - r) dt - \theta dt + \frac{dD_t}{D_t} \quad \text{with} \quad Z^D_0 = y. \quad (17)$$

Let us define $f(y)$ by

$$f(y) := \frac{1}{y} \left( \bar{u} \left( \frac{y}{\kappa_1} \right) - \bar{u} \left( \frac{y}{\kappa_2} \right) + y(\epsilon_1 - \epsilon_2) \right).$$

Quantity $\bar{u} \left( \frac{y}{\kappa_1} \right) - \bar{u} \left( \frac{y}{\kappa_2} \right) + y(\epsilon_1 - \epsilon_2)$ in the definition of $f(y)$ can be regarded as the difference between the utility value from job $\mathfrak{B}_1$ and that from $\mathfrak{B}_2$ for a given $y$. It compares utility values of consumption and those of income, for the latter of which we use the Lagrange multiplier $y$, the marginal value of wealth, to transform the monetary value to the utility value.

**Lemma 3.1.** $f(y)$ is strictly increasing in $y > 0$ and there exists a unique $z_S > 0$ such that

$$f(z_S) = 0.$$

**Proof.** Since

$$f'(y) = -\frac{1}{y^2} \left( \bar{u} \left( \frac{y}{\kappa_1} \right) - \bar{u} \left( \frac{y}{\kappa_2} \right) \right) + \frac{1}{y} \left( -\frac{1}{\kappa_1} \frac{y}{\kappa_1} + \frac{1}{\kappa_2} \frac{y}{\kappa_2} \right)$$

$$\quad = -\frac{1}{y^2} \left( u \left( \frac{y}{\kappa_1} \right) - u \left( \frac{y}{\kappa_2} \right) \right) > 0,$$
\( f(y) \) is strictly increasing in \( y > 0 \).

Moreover, the mean value theorem implies that there exists \( \tilde{y} \in (y/\kappa_2, y/\kappa_1) \) such that
\[
\ddot{u}
\left( \frac{y}{\kappa_1} \right) - \ddot{u}
\left( \frac{y}{\kappa_2} \right) = -I(\tilde{y}) \left( \frac{y}{\kappa_1} - \frac{y}{\kappa_2} \right).
\]
(18)

Since \( \lim_{y \to 0} I(y) = \infty \), \( \lim_{y \to \infty} I(y) = 0 \), and \( I\left( \frac{y}{\kappa_1} \right) < I(\tilde{y}) < I\left( \frac{y}{\kappa_2} \right) \), we deduce that
\[
\lim_{y \to \infty} f(y) = (\epsilon_1 - \epsilon_2) > 0 \quad \text{and} \quad \lim_{y \to 0} f(y) = -\infty.
\]
(19)

Thus, the intermediate value theorem implies that there exists a unique \( z_S > 0 \) such that
\[
f(z_S) = 0.
\]

For any job process \( \zeta \), Lemma 3.1 implies that
\[
\left( \ddot{u}
\left( \frac{Z^D_t}{\kappa_1} \right) + \epsilon_1 Z^D_t \right) 1_{\{\zeta_1 = \mathcal{B}_1\}} + \left( \ddot{u}
\left( \frac{Z^D_t}{\kappa_2} \right) + \epsilon_2 Z^D_t \right) 1_{\{\zeta_1 = \mathcal{B}_2\}}
\leq \left( \ddot{u}
\left( \frac{1}{\kappa_1} \right) + \epsilon_1 Z^D_t \right) 1_{\{Z^P_t \geq z_S\}} + \left( \ddot{u}
\left( \frac{1}{\kappa_2} \right) + \epsilon_2 Z^D_t \right) 1_{\{Z^P_t < z_S\}}.
\]
Hence, the candidate of optimal job state process \( \hat{\zeta}(Z^D_t) \) is given by
\[
\hat{\zeta}(Z^D_t) := \begin{cases} \mathcal{B}_1 & \text{for } Z^P_t \geq z_S, \\ \mathcal{B}_2 & \text{for } Z^P_t < z_S. \end{cases}
\]
(20)

That is, job \( \mathcal{B}_j \) is chosen if the utility value from it is greater than that from job \( \mathcal{B}_i \) for \( i \neq j \).

It follows from (15) that
\[
\mathcal{L} \leq \mathbb{E}
\left[ \int_0^\tau e^{-\delta t} \left\{ \left( \ddot{u}
\left( \frac{Z^D_t}{\kappa_1} \right) + \epsilon_1 Z^D_t \right) 1_{\{Z^P_t > z_S\}} + \left( \ddot{u}
\left( \frac{Z^D_t}{\kappa_2} \right) + \epsilon_2 Z^D_t \right) 1_{\{Z^P_t \leq z_S\}} \right\} dt \right.
+ \left. 1_{\{\tau < \infty\}} e^{-\delta \tau} J_R(Z^D_{\tau^-}) \right] + \rho x.
\]
(21)

Minimizing over \( y > 0 \) and \( D_i \) in (21) yields
\[
\mathcal{L} \leq \inf_{y > 0, D_i \in N_I} \left\{ J_0(y; D, \tau) + \rho x \right\},
\]
where
\[
J_0(y; D, \tau) := \mathbb{E}
\left[ \int_0^\tau e^{-\delta t} \left\{ \left( \ddot{u}
\left( \frac{Z^D_t}{\kappa_1} \right) + \epsilon_1 Z^D_t \right) 1_{\{Z^P_t > z_S\}} + \left( \ddot{u}
\left( \frac{Z^D_t}{\kappa_2} \right) + \epsilon_2 Z^D_t \right) 1_{\{Z^P_t \leq z_S\}} \right\} dt \right.
+ \left. 1_{\{\tau < \infty\}} e^{-\delta \tau} J_R(Z^D_{\tau^-}) \right].
\]

Therefore, we deduce that
\[
V(x) \leq \sup_{\tau \in S} \inf_{y > 0, D_i \in N_I} \left\{ J_0(y; D, \tau) + \rho x \right\} \leq \inf_{y > 0} \left( \sup_{\tau \in S} \inf_{D_i \in N_I} J_0(y; D, \tau) + \rho x \right) \leq \inf_{y > 0} \left( \inf_{D_i \in N_I} \sup_{\tau \in S} J_0(y; D, \tau) + \rho x \right),
\]
(22)
3.2 A Two-person Zero-sum Game

We will now consider a two-person zero-sum game arising from (22). The game involves a controller, who is a minimizer and chooses a process $D \in \mathcal{N}I$, and a stopper, who is maximizer and chooses a stopping time $\tau \in \mathcal{S}$. The two agents share the same performance criterion, which is given by

$$J_0(z; D, \tau) = \mathbb{E} \left[ \int_0^\tau e^{-\delta t} \left( \left( \frac{Z_D}{\kappa_1} \right) + \epsilon_1 Z_D^D \right) \mathbf{1}_{\{Z_D^D > z\}} + \left( \frac{Z_D}{\kappa_2} \right) + \epsilon_2 Z_D^D \right) \mathbf{1}_{\{Z_D^D \leq z\}} dt \right]$$

(23)

with $Z_D^D = z$ (we recall $Z_D^D = \gamma_D D_t$).

Next define the lower value $\underline{J}$ and the upper value $\overline{J}$ as

$$\underline{J}(z) := \sup_{\tau \in \mathcal{S}} \inf_{D \in \mathcal{N}I} J_0(z; D, \tau)$$

and

$$\overline{J}(z) := \inf_{D \in \mathcal{N}I} \sup_{\tau \in \mathcal{S}} J_0(z; D, \tau).$$

If $\underline{J}(z) = \overline{J}(z)$, then the game is said to have a value and we denote the common value $\underline{J}(z) = \overline{J}(z)$ by $J(z)$.

A Nash equilibrium, or equivalently a saddle point $(\hat{D}, \hat{\tau}) \in \mathcal{N}I \times \mathcal{S}$, such that $\hat{D}$ is the best response given $\hat{\tau}$ while $\hat{\tau}$ is simultaneously the best response given $\hat{D}$, i.e.,

$$J_0(z; \hat{D}, \tau) \leq J_0(z; \hat{D}, \hat{\tau}) \leq J_0(z; D, \hat{\tau})$$

(24)

for any $D \in \mathcal{N}I$, $\tau \in \mathcal{S}$. It is clear that if there exists a saddle point for the game, then the game has a value, i.e.,

$$J(z) = J_0(z; \hat{D}, \hat{\tau}).$$

(25)

We can now define the dual problem as finding a saddle point of the game described above.

Problem 2 (Game). Find a Nash equilibrium $(\hat{D}, \hat{\tau}) \in \mathcal{N}I \times \mathcal{S}$ such that

$$J_0(z; \hat{D}, \tau) \leq J_0(z; \hat{D}, \hat{\tau}) \leq J_0(z; D, \hat{\tau}) \quad \text{for any } D \in \mathcal{N}I, \tau \in \mathcal{S}.$$

(26)

If a Nash equilibrium exists, the dual value function $J(y)$ is defined as

$$J(z) = \sup_{\tau \in \mathcal{S}} \inf_{D \in \mathcal{N}I} J_0(z; D, \tau) = \inf_{D \in \mathcal{N}I} \sup_{\tau \in \mathcal{S}} J_0(z; D, \tau) = J_0(z; \hat{D}, \hat{\tau})$$

(27)

with $J_0 = z$.

Remark 3.2. We can get the following weak duality

$$V(x) = \sup_{(c, \pi, \delta, \tau) \in \mathcal{A}(x)} \mathbb{E} \left[ \int_0^\tau e^{-\delta t} \left( u\kappa_1 c_t \mathbf{1}_{\{\zeta_t = \mathcal{B}_1\}} + u\kappa_2 c_t \mathbf{1}_{\{\zeta_t = \mathcal{B}_2\}} \right) dt + \mathbf{1}_{\{\tau < \infty\}} \int_\tau^\infty e^{-\delta t} u(c_t) dt \right]$$

(28)

$$\leq \inf_{y > 0} (J(y) + y x).$$

We will show in Theorem 4.2 that the maximized value is indeed equal to the right-hand side of the last inequality in (28) with infimum being replaced by minimum, i.e.,

$$V(x) = \min_{y > 0} (J(y) + y x).$$

(29)
Remark 3.3. Farhi and Panageas (2007) or Lim and Shin (2011)\(^1\) state that the sup\(_{\tau \in \mathcal{S}}\) and inf\(_{D \in \mathcal{N}I}\) in (27) can be interchanged. However, they do not provide a theoretical proof of the fact. We provide a rigorous proof by considering the two person zero-sum game in Problem 2.

4 Optimal Strategies

In this section we derive optimal strategies for the main optimization problem (Problem 1) by obtaining the value of the two-person zero-sum game in Problem 2.

4.1 Heuristic derivation of Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQV) for the dual value function \(J\)

In this subsection, we derive HJBQV for \(J(z)\), the value function of the game, by relying on heuristic and intuitive arguments. We will derive a concrete solution to the HJBQV and provide the verification that the solution is equal to the value function of the game in later subsections.

If a Nash-equilibrium \((\hat{D}, \hat{\tau}) \in \mathcal{N}I \times \mathcal{S}\) of the game (23) exists, then the lower value \(\underline{J}\) and the upper value \(\overline{J}\) can be written as

\[
\underline{J}(z) = \sup_{\tau \in \mathcal{S}} \mathbb{E} \left[ \int_0^\tau e^{-\delta t} \left\{ \left( \frac{\tilde{u} \left( \frac{Z^D_t}{\kappa_1} \right) + \epsilon_1 Z^D_t}{\kappa_1} \right) 1_{\{z > z_S\}} + \left( \frac{\hat{u} \left( \frac{Z^D_t}{\kappa_2} \right) + \epsilon_2 Z^D_t}{\kappa_2} \right) 1_{\{z \leq z_S\}} \right\} dt \right] + 1_{\{\tau < \infty\}} e^{-\delta \tau} J_R(Z^D_\tau \hat{\tau}) \right],
\]

and

\[
\overline{J}(z) = \inf_{D \in \mathcal{N}I} \mathbb{E} \left[ \int_0^\tau e^{-\delta t} \left\{ \left( \frac{\tilde{u} \left( \frac{Z^D_t}{\kappa_1} \right) + \epsilon_1 Z^D_t}{\kappa_1} \right) 1_{\{z > z_S\}} + \left( \frac{\hat{u} \left( \frac{Z^D_t}{\kappa_2} \right) + \epsilon_2 Z^D_t}{\kappa_2} \right) 1_{\{z \leq z_S\}} \right\} dt \right] + 1_{\{\tau < \infty\}} e^{-\delta \tau} J_R(Z^D_\tau \hat{\tau}) \right].
\]

Considering dynamic programming principle, we expect that \(\underline{J}\) satisfies the following HJB equation in the region \(\{z > 0 \mid \underline{J}'(z) < 0\}\):

\[
\max \left\{ \mathcal{L} Q(z) + \left( \frac{\tilde{u}}{\kappa_1} \right) + \epsilon_1 z \right\} 1_{\{z > z_S\}} + \left( \frac{\hat{u}}{\kappa_2} \right) + \epsilon_2 z \right\} 1_{\{z \leq z_S\}}, J_R(z) - Q(z) \right\} = 0, \quad (30)
\]

where the differential operator \(\mathcal{L}\) is given by

\[
\mathcal{L} = \frac{\theta^2}{2} z^2 \frac{d^2}{dz^2} + (\delta - r) z \frac{d}{dz} - \delta. \quad (31)
\]

In the region \(\{z > 0 \mid \overline{J}(z) > J_R(z)\}\), we also expect that \(\overline{J}\) satisfies the following HJB equation with a gradient constraint

\[
\min \left\{ \mathcal{L} Q(z) + \left( \frac{\tilde{u}}{\kappa_1} \right) + \epsilon_1 z \right\} 1_{\{z > z_S\}} + \left( \frac{\hat{u}}{\kappa_2} \right) + \epsilon_2 z \right\} 1_{\{z \leq z_S\}}, -Q'(z) \right\} = 0. \quad (32)
\]

\(^1\)See the equation (50) in extended appendix of Farhi and Panageas (2007) or the equation (A10) in Lim and Shin (2011).
Since \( J(z) = \bar{J}(z) = \underline{J}(z) \) when a *Nash-equilibrium* exists, in view of (30) and (32), we expect that the dual value function \( J(z) \) is a solution \( \mathcal{Q}(z) \) to the following max-min type of Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQV) arising from Problem 2: for \( z > 0 \)

\[
\max \left\{ \min \left\{ \mathcal{L}\mathcal{Q}(\zeta) + \left[ \frac{\zeta}{\kappa_1} + \epsilon_1 \zeta \right] \mathbf{1}_{\{\zeta > z_s\}} + \left[ \frac{\zeta}{\kappa_2} + \epsilon_2 \zeta \right] \mathbf{1}_{\{\zeta \leq z_s\}}, -\mathcal{Q}'(\zeta) \right\}, J_R(\zeta) - \mathcal{Q}(\zeta) \right\} = 0. \tag{33}
\]

### 4.2 Solution to HJBQV (33)

To find a solution to HJBQV (33), we will employ a *guess-and-verify* approach. We will provide formal results in the next section.

From the perspective of the stopper in (33), the state space \( \mathbb{R}_+ \) splits into two regions:

\[
\text{RR} := \{ z > 0 \mid \mathcal{Q}(z) = J_R(z) \} \quad \text{(retirement region)},
\]
\[
\text{WR} := \{ z > 0 \mid \mathcal{Q}(z) > J_R(z) \} \quad \text{(working region)}.
\]

Similarly, from the perspective of the controller, the state space \( \mathbb{R}_+ \) can be decomposed into two regions:

\[
\text{IR} := \{ z > 0 \mid \mathcal{Q}'(z) < 0 \} \quad \text{(in-action region)},
\]
\[
\text{AR} := \{ z > 0 \mid \mathcal{Q}'(z) = 0 \} \quad \text{(adjustment region)}.
\]

In the retirement region \( \text{RR} \), \( \mathcal{Q}(z) = J_R(z) \) satisfies

\[
\min \left\{ \mathcal{L}J_R(\zeta) + \left[ \frac{\zeta}{\kappa_1} + \epsilon_1 \zeta \right] \mathbf{1}_{\{\zeta > z_s\}} + \left[ \frac{\zeta}{\kappa_2} + \epsilon_2 \zeta \right] \mathbf{1}_{\{\zeta \leq z_s\}}, -\frac{dJ_R(\zeta)}{dz} \right\} \leq 0.
\]

It follows from (104) that

\[
J'_R(y) = \frac{2}{\theta^2(n_1 - n_2)} \left[ y^{n_2 - 1} \int_0^y \eta^{-n_2} I(\eta) d\eta + y^{n_1 - 1} \int_y^\infty \eta^{-n_1} I(\eta) d\eta \right] < 0.
\]

Thus, we have

\[
\mathcal{L}J_R(\zeta) + \left[ \frac{\zeta}{\kappa_1} + \epsilon_1 \zeta \right] \mathbf{1}_{\{\zeta > z_s\}} + \left[ \frac{\zeta}{\kappa_2} + \epsilon_2 \zeta \right] \mathbf{1}_{\{\zeta \leq z_s\}} \leq 0 \quad \text{for } z \in \text{RR}.
\]

Since \( \mathcal{L}J_R(z) + \tilde{u}(z) = 0 \), we deduce that

\[
h(z) \leq 0 \quad \text{for } z \in \text{RR},
\]

where

\[
h(z) := \left( \tilde{u} \left( \frac{z}{\kappa_1} \right) - \tilde{u}(z) + \epsilon_1 z \right) \mathbf{1}_{\{z > z_s\}} + \left( \tilde{u} \left( \frac{z}{\kappa_2} \right) - \tilde{u}(z) + \epsilon_2 z \right) \mathbf{1}_{\{z \leq z_s\}}.
\]

Function \( h(z) \) gives the difference in the utility value after retirement and that before retirement for a given \( z \). Note that \( z \) is used to transform the monetary value of income \( \epsilon_i \ (i = 1, 2) \) to the utility value.

**Lemma 4.1.**

(a) \( h(z)/z \) is strictly increasing in \( z > 0 \).
(b) There exists a unique \( \hat{z} > 0 \) such that \( h(\hat{z}) = 0 \).

(c) \( h(z) < 0 \) for \( z \in (0, \hat{z}) \), and \( h(z) > 0 \) for \( z \in (\hat{z}, \infty) \).

Proof. The proof is almost identical with that of Lemma 3.1 and we omit its detail. \( \square \)

**Remark 4.1.** Lemma 4.1 implies that \( z \leq \hat{z} \) for \( z \in \mathbb{R} \). Moreover, the inequality \( z_S > \hat{z} \) holds if and only if \( h(z_S) > 0 \).

Suppose that there exist two boundaries \( z_R \in (0, \hat{z}) \) and \( z_B \in (z_R, \infty) \) such that the agent chooses the option to retire if \( Z_t \leq z_R \) and the agent’s wealth is zero if \( Z_t \geq z_B \). That is,

\[
\begin{align*}
Q(z) &= J_R(z) \quad \text{if} \quad z \leq z_R, \\
Q'(z) &= 0 \quad \text{if} \quad z \geq z_B,
\end{align*}
\]

and it follows from the smooth-pasting(or super-contact) condition (see Dumas (1989)) that

\[
Q(z_R) = J_R(z_R), \quad Q'(z_R) = J_R'(z_R), \quad Q'(z_B) = 0, \quad \text{and} \quad Q''(z_B) = 0.
\] (35)

**Remark 4.2.** We will show later that a Nash-equilibrium for the game in (23) is given by a pair of barrier strategies \((D_t^{z}, \tau_{z_t})\) for \( z_R < z_B \), where we define

\[
D_t^{z} := \min \left\{ 1, \inf_{0 \leq s \leq t} \frac{z_B}{Y_s} \right\} \quad \text{for} \quad t \geq 0, \quad \tau_{z_t} := \inf \{ t \geq 0 \mid Z_t^{D_t^{z}} < z_R \}
\]

That is,

\[
Q(z) = J(z) = J_0(z; D_t^{z}, \tau_{z_t}).
\] (36)

In the region \( \mathbb{W}R \cap \mathbb{R} \), \( Q(z) \) satisfies

\[
\mathcal{L}Q(z) + h(z) + \bar{u}(z) = 0.
\] (37)

A general solution to the equation (37) can be written as the sum of a general solution to the homogeneous equation and a particular solution:

\[
Q(z) = E_1 z^{n_1} + E_2 z^{n_2} + \frac{2}{\theta^2(n_1 - n_2)} \left[ z^{n_2} \int_0^z \eta^{-n_2-1}(h(\eta) + \bar{u}(\eta))d\eta + z^{n_1} \int_z^\infty \eta^{-n_1-1}(h(\eta) + \bar{u}(\eta))d\eta \right]
\]

\[
= E_1 z^{n_1} + E_2 z^{n_2} + J_R(z) + \Psi_h(z),
\]

where

\[
\Psi_h(z) := \frac{2}{\theta^2(n_1 - n_2)} \left[ z^{n_2} \int_0^z \eta^{-n_2-1}h(\eta)d\eta + z^{n_1} \int_z^\infty \eta^{-n_1-1}h(\eta)d\eta \right].
\]

Since \( Q(z_R) = J_R(z_R) \) and \( Q'(z_R) = J_R'(z_R) \), we have

\[
E_1 z_R^{n_1} + E_2 z_R^{n_2} + J_R(z_R) + \Psi_h(z_R) = J_R(z_R),
\]

\[
n_1 E_1 z_R^{n_1-1} + n_2 E_2 z_R^{n_2-1} + J_R'(z_R) + \Psi'_h(z_R) = J_R'(z_R),
\]

which implies that

\[
E_1 = -\frac{2}{\theta^2(n_1 - n_2)} \int_{z_R}^\infty \eta^{-n_1-1}h(\eta)d\eta \quad \text{and} \quad E_2 = -\frac{2}{\theta^2(n_1 - n_2)} \int_0^{z_R} \eta^{-n_2-1}h(\eta)d\eta.
\] (38)
From (101), we can easily deduce that
\[
\int_{0}^{z} \eta^{-n_2-1}|h(\eta)|d\eta + \int_{z}^{\infty} \eta^{-n_1-1}|h(\eta)|d\eta < \infty.
\]
Proposition A.1 implies that
\[
\lim_{y \to 0} \inf z^{-n_2}|h(z)| = \lim_{z \to \infty} z^{-n_1}|h(z)| = 0.
\]
Since
\[
E_{1} = \frac{2}{\theta^{2}(n_1 - n_2)} \int_{z_{R}}^{\infty} \eta^{-n_1}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta
\]
and
\[
E_{2} = \frac{2}{n_2 \theta^{2}(n_1 - n_2)} \int_{0}^{z_{R}} \eta^{-n_2}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta.
\]
which implies that
\[
E_{1} = \frac{2}{n_1 \theta^{2}(n_1 - n_2)} \int_{z_{R}}^{\infty} \eta^{-n_1}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta
\]
and
\[
E_{2} = \frac{2}{n_2 \theta^{2}(n_1 - n_2)} \int_{0}^{z_{R}} \eta^{-n_2}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta.
\]
From (38) and (39), we deduce that \(z_{B} \) and \(z_{R} \) satisfy the coupled algebraic equations:
\[
\phi_{1}(z_{R}, z_{B}) = 0, \quad \phi_{2}(z_{R}, z_{B}) = 0,
\]
where
\[
\phi_{1}(z_{1}, z_{2}) := \int_{z_{1}}^{z_{2}} \eta^{-n_1}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta - n_1 \int_{z_{1}}^{\infty} \eta^{-n_1-1}h(\eta)d\eta,
\]
\[
\phi_{2}(z_{1}, z_{2}) := \int_{0}^{z_{2}} \eta^{-n_2}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta - n_2 \int_{0}^{z_{1}} \eta^{-n_2-1}h(\eta)d\eta.
\]
In summary, we set \(Q(z)\), a candidate solution to the HJBQV (33), as follows:
\[
Q(z) = \begin{cases} 
J_{R}(z) & \text{for } z \leq z_{R}, \\
E_{1}z^{n_1} + E_{2}z^{n_2} + \phi_{h}(z) + J_{R}(z) & \text{for } z_{R} \leq z \leq z_{B}, \\
E_{1}z^{n_1} + E_{2}z^{n_2} + \phi_{h}(z_{B}) + J_{R}(z_{B}) & \text{for } z_{B} \leq z.
\end{cases}
\]
From (38) and (39), we deduce that for \(z_{R} < z < z_{B}\)
\[
Q(z) = \frac{2}{\theta^{2}(n_1 - n_2)} \left[ z^{n_2} \int_{z_{R}}^{z} \eta^{-n_2-1}h(\eta)d\eta + z^{n_1} \int_{z}^{z_{R}} \eta^{-n_1-1}h(\eta)d\eta \right] + J_{R}(z)
\]
and
\[
Q'(z) = \frac{2}{\theta^{2}(n_1 - n_2)} \left[ z^{n_2-1} \int_{z_{R}}^{z} \eta^{-n_2}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta + z^{n_1-1} \int_{z}^{z_{R}} \eta^{-n_1}(\tilde{c}(\eta) - \tilde{c}(\eta))d\eta \right].
\]
Remark 4.3. According to (40), the candidate solution has two components: (i) $J_R(z)$, which is the present value of the convex conjugate of consumption and equal to the dual value function after retirement, and (ii) the difference $Q(z) - J_R(z)$ which can be regarded as the option value of retirement. In the two-person zero-sum game, the stopper attempts to maximize the option value, while the controller tries to minimize it.

4.3 Verification I: Dual Problem

Proposition 4.1. The coupled algebraic equations $\phi_1(z_R, z_B) = 0$ and $\phi_2(z_R, z_B) = 0$ have a unique pair $(z_R, z_B)$ such that $0 < z_R < \tilde{z}$ and $z_B > z_R$.

Proof. Since $\hat{c}(\eta) - \tilde{c}(\eta)$ is strictly increasing in $\eta > 0$ and $\lim_{\eta \to 0} \hat{c}(\eta) - \tilde{c}(\eta) = -\infty$, $\lim_{\eta \to \infty} \hat{c}(\eta) - \tilde{c}(\eta) = \epsilon_1$, there exists a unique $\tilde{z} > 0$ such that

$$\hat{c}(\tilde{z}) - \tilde{c}(\tilde{z}) = 0.$$ 

From

$$\int_0^{z_B} \eta^{-n_2}(\hat{c}(\eta) - \tilde{c}(\eta))d\eta = n_2 \int_0^{z_B} \eta^{-n_2 - 1}h(\eta)d\eta,$$

we deduce that

$$z_B > \tilde{z} \text{ for } 0 < z_R \leq \tilde{z}.$$

Note that for $0 < z_R \leq \tilde{z}$, $z_R < z_B$,

$$\phi_2(z_R, z_B) = \int_0^{z_R} \eta^{-n_2}(\hat{c}(\eta) - \tilde{c}(\eta))d\eta - n_2 \int_0^{z_R} \eta^{-n_2 - 1}h(\eta)d\eta$$

$$= \int_0^{z_R} \eta^{-n_2}(\hat{c}(\eta) - \tilde{c}(\eta))d\eta + z_R^{-n_2}h(z_R) - \int_0^{z_R} \eta^{-n_2}(\hat{c}(\eta) - \tilde{c}(\eta) + I(\eta))d\eta$$

$$= z_R^{-n_2}h(z_R) - \int_0^{z_R} \eta^{-n_2}I(\eta)d\eta < 0,$$

where we have used integration by parts in the second equality.

Since $\lim_{\eta \to \infty}(\hat{c}(\eta) - \tilde{c}(\eta)) = \epsilon_1$, there exists a sufficiently large $M > 0$ such that

$$\hat{c}(\eta) - \tilde{c}(\eta) \geq \frac{\epsilon_1}{2} \text{ for } \eta \geq M.$$

Note that

$$\int_M^{\infty} \eta^{-n_2}(\hat{c}(\eta) - \tilde{c}(\eta))d\eta \geq \frac{\epsilon_1}{2} \int_M^{\infty} \eta^{-n_2}d\eta = \infty.$$

It follows that

$$\lim_{z_R \to \infty} \phi_2(z_R, z_B) = \infty. \quad (43)$$

Since

$$\frac{\partial \phi_2}{\partial z_R}(z_R, z_B) = z_B^{-n_2}(\hat{c}(z_B) - \tilde{c}(z_B)) > 0 \text{ for } 0 < z_R \leq \tilde{z},$$

the intermediate value theorem implies that for given $0 < z_R \leq \tilde{z}$ there exists a unique $\partial(z_R) > z_R$ such that

$$\phi_2(z_R, \partial(z_R)) = 0.$$

Note that $0 < \tilde{z} < \partial(\tilde{z})$. 

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Thus, we have
\begin{align*}
\phi_1(\hat{z}, \vartheta(\hat{z})) &= \int_0^\infty (\hat{c}(\eta) - \hat{c}(\eta)) d\eta - n_1 \int_{\hat{z}}^\infty \eta^{-n_1-1} h(\eta) d\eta \\
&= \int_0^\infty \eta^{-n_1} (\hat{c}(\eta) - \hat{c}(\eta) + I(\eta)) d\eta - \int_0^\infty \eta^{-n_1} I(\eta) d\eta - n_1 \int_{\hat{z}}^\infty \eta^{-n_1-1} h(\eta) d\eta \\
&= - \vartheta(\hat{z})^{-n_1} h(\vartheta(\hat{z})) - n_1 \int_{\hat{z}}^\infty \eta^{-n_1-1} h(\eta) d\eta - \int_0^\infty \eta^{-n_1} I(\eta) d\eta - n_1 \int_{\hat{z}}^\infty \eta^{-n_1-1} h(\eta) d\eta \\
&= - \vartheta(\hat{z})^{-n_1} h(\vartheta(\hat{z})) - n_1 \int_{\hat{z}}^\infty \eta^{-n_1-1} h(\eta) d\eta - 0 \int_{\hat{z}}^\infty \eta^{-n_1} I(\eta) d\eta < 0,
\end{align*}
where we have used integration by parts in the third equality and $0 = h(\hat{z}) < h(\vartheta(\hat{z}))$.

Since
\[ \int_0^z \eta^{-n_2} (\hat{c}(\eta) - \hat{c}(\eta)) d\eta < 0, \quad \lim_{z \to 0^+} \int_0^z \eta^{-n_2} (\hat{c}(\eta) - \hat{c}(\eta)) d\eta = \infty, \]
and
\[ \frac{d}{dz} \left( \int_0^z \eta^{-n_2} (\hat{c}(\eta) - \hat{c}(\eta)) d\eta \right) = z^{-n_2} (\hat{c}(z) - \hat{c}(z)) > 0 \quad \text{for } z > \hat{z}, \]
we deduce that there exists a unique $\hat{z} > \hat{z}$ such that
\[ \int_0^{\hat{z}} \eta^{-n_2} (\hat{c}(\eta) - \hat{c}(\eta)) d\eta = 0. \]

Letting $z \to 0^+$,
\[ 0 = \lim_{z \to 0^+} n_2 \int_0^z \eta^{-n_2-1} h(\eta) d\eta = \lim_{z \to 0^+} \int_{\vartheta(z)}^{\hat{z}} \eta^{-n_2} (\hat{c}(\eta) - \hat{c}(\eta)) d\eta. \]
That is, $\vartheta(0^+) = \lim_{z \to 0^+} \vartheta(z) = \hat{z}$.

Hence, we have
\[ \lim_{z \to 0^+} \phi_1(z, \vartheta(z)) = \int_\hat{z}^\infty \eta^{-n_1}(\hat{c}(\eta) - \hat{c}(\eta)) d\eta - n_1 \lim_{z \to 0^+} \int_{\hat{z}}^\infty \eta^{-n_1-1} h(\eta) d\eta \]
\[ > -n_1 \lim_{z \to 0^+} \int_{\hat{z}}^\infty \eta^{-n_1-1} h(\eta) d\eta. \]

For a sufficiently small $0 < \nu < \hat{z}$, it follows from Lemma 4.1 that
\[ \lim_{z \to 0^+} \int_{\hat{z}}^\nu \eta^{-n_1-1} h(\eta) d\eta = \lim_{z \to 0^+} \int_{\hat{z}}^\nu \eta^{-n_1} \frac{h(\eta)}{\eta} d\eta < \frac{h(\nu)}{\nu} \lim_{z \to 0^+} \int_{\hat{z}}^\nu \eta^{-n_1} d\eta = -\infty. \]
Thus,
\[ \lim_{z \to 0^+} \phi_1(z, \vartheta(z)) = \infty. \quad (45) \]

Note that
\[ 0 = \frac{\partial}{\partial z_R} (\phi_2(z_R, \vartheta(z_R))) = \frac{\partial \phi_2}{\partial z_1} + \frac{\partial \phi_2}{\partial z_2} \frac{d \vartheta(z_R)}{dz_R}. \]

Since
\[ \frac{\partial \phi_2}{\partial z_1}(z_R, z_B) = -n_2z_R^{-n_2-1} h(z_R) \quad \text{and} \quad \frac{\partial \phi_2}{\partial z_2}(z_R, z_B) = z_B^{-n_2} (\hat{c}(z_B) - \hat{c}(z_B)), \]
we have
\[ \frac{d \vartheta(z_R)}{dz_R} = \frac{n_2z_R^{-n_2-1} h(z_R)}{z_B^{-n_2} (\hat{c}(z_B) - \hat{c}(z_B))}. \]
For $0 < z_R < \hat{z}$, we deduce that
\[
\frac{d}{dz_R} \phi_1(z_R, \vartheta(z_R)) = \frac{\partial \phi}{\partial z_1}(z_R, \vartheta(z_R)) + \frac{\partial \phi}{\partial z_2}(z_R, \vartheta(z_R)) \frac{d\vartheta(z_R)}{dz_R} = 0.
\] (46)

From (44), (45), and (46), we conclude that there exists a unique $0 < z_R < \hat{z}$ such that
\[
\phi_1(z_R, \vartheta(z_R)) = 0.
\]

Thus, there exists a unique pair $(z_R, z_B)$ such that $0 < z_R < \hat{z}, z_B > z_R$, and
\[
\phi_1(z_R, z_B) = \phi_2(z_R, z_B) = 0.
\]

The fact $z_R < \hat{z}$ in Proposition 4.1 implies that optimal retirement decision is made only when $h(z) < 0$, i.e., the instantaneous utility value after retirement is strictly smaller than that before retirement. This is consistent with the observation that it is optimal to exercise an option only when it is in the money (Dixit and Pindyck (1994)).

**Proposition 4.2.**

(a) $Q(z)$ given in (40) satisfies the HJBQV (33). Moreover, $Q(z)$ is a continuously differentiable function for $z > 0$ and twice continuously differentiable function for $z \in (0, \infty) \setminus \{z_R\}$.

(b) The four regions $WR, RR, AR, and IR$ can be rewritten as follows:

\[
RR = \{z > 0 \mid z \leq z_R\}, \quad WR = \{z > 0 \mid z > z_R\},
\]
\[
AR = \{z > 0 \mid z_B \leq z\}, \quad IR = \{z > 0 \mid 0 < z < z_B\}.
\]

(c) $Q(z)$ is strictly convex in $z \in (0, z_B)$.

(d)
\[
\lim_{z \to z_B^-} Q'(z) = 0 \quad \text{and} \quad \lim_{z \to 0^+} Q'(z) = -\infty.
\]

**Proof.** (a) By construction of $Q(z)$ in (40), we can easily confirm that $Q(z)$ is a continuously differentiable function for $z > 0$ and twice continuously differentiable function for $z \in (0, \infty) \setminus \{z_R\}$.

We will prove that $Q(z)$ in (40) satisfies the HJBQV (33).

(i) the case $z \in (0, z_R]$.

Since $Q(z) = J_R(z)$ for $z \in (0, z_R]$, we have
\[
-\frac{dQ}{dz} = -J'_R(z) > 0.
\]

It follows from $\mathcal{L}J_R + \bar{u}(z) = 0$ that
\[
\mathcal{L}Q(z) + \bar{u}(z) + h(z) = \mathcal{L}J_R + \bar{u}(z) + h(z) = h(z) < 0 \quad \text{for} \quad z \in (0, z_R].
\]
Thus, we deduce that
\[
\min \left\{ \mathcal{L}Q(z) + \hat{u}(z) + h(z), -\frac{dQ}{dz}(z) \right\} < 0.
\]
That is,
\[
\max \left\{ \min \left\{ \mathcal{L}Q(z) + \hat{u}(z) + h(z), -\frac{dQ}{dz}(z) \right\}, J_R(z) - Q(z) \right\} = 0.
\]
(ii) the case \( z \in (z_R, z_B) \).

By construction of \( Q(z) \),
\[
\mathcal{L}Q(z) + \hat{u}(z) + h(z) = 0.
\]
From (41), we deduce that for \( z_R < z < z_B \),
\ [
Q(z) - J_R(z) = \frac{2}{\theta^2(n_1 - n_2)} \left[ z^{n_2} \int_{z_R}^{z} \eta^{-n_2-1} h(\eta) d\eta + z^{n_1} \int_{z_R}^{z} \eta^{-n_1-1} h(\eta) d\eta \right]
\]
\[
= \frac{2}{\theta^2(n_1 - n_2)} \left[ z^{n_2} \int_{z_R}^{z} \eta^{-n_2-1} h(\eta) d\eta - z^{n_1} \int_{z_R}^{z} \eta^{-n_1-1} h(\eta) d\eta \right].
\]
This leads to
\[
(Q - J_R)'(z) = \frac{2}{\theta^2(n_1 - n_2)} \left[ n_2 z^{n_2-1} \int_{z_R}^{z} \eta^{-n_2-1} h(\eta) d\eta - n_1 z^{n_1-1} \int_{z_R}^{z} \eta^{-n_1-1} h(\eta) d\eta \right].
\]
Since \( h(z) < 0 \) for \( 0 < z < \hat{z} \) and \( h(z) > 0 \) for \( z > \hat{z} \), \((Q - J_R)'(z)\) is increasing in \( z \in (z_R, \hat{z}) \) and decreasing in \( z \in (\hat{z}, z_B) \).

It follows from \( Q(z_R) = J_R(z_R) \) and \( Q'(z_B) = 0 \) that
\[
(Q - J_R)(z_R) = 0
\]
and
\[
(Q - J_R)'(z_B) = -J_R'(z_B) > 0.
\]
Hence, we conclude that
\[
(Q - J_R)'(z) > 0 \quad \text{for} \quad z \in (z_R, z_B).
\]
It follows from \( Q(z_R) = J_R(z_R) \) that
\[
Q(z) > J_R(z) \quad \text{for} \quad z \in (z_R, z_B). \quad (47)
\]
From (42),
\[
J'(z) = \frac{2}{\theta^2(n_1 - n_2)} \left[ z^{n_2-1} \int_{z_R}^{z} \eta^{-n_2} (\hat{c}(\eta) - \hat{\epsilon}(\eta)) d\eta + z^{n_1} \int_{z_R}^{z} \eta^{-n_1} (\hat{c}(\eta) - \hat{\epsilon}(\eta)) d\eta \right]
\]
for \( z_R < z < z_B \).

Hence, we have
\[
J''(z) = \frac{2}{\theta^2(n_1 - n_2)} \left[ (1 - n_2) z^{n_2-2} \int_{z}^{z_B} \eta^{-n_2} (\hat{c}(\eta) - \hat{\epsilon}(\eta)) d\eta + (1 - n_1) z^{n_1-2} \int_{z}^{z_B} \eta^{-n_1} (\hat{c}(\eta) - \hat{\epsilon}(\eta)) d\eta \right]
\]
for \( z_R < z < z_B \).
Since $\hat{c}(\eta) - \check{c}(\eta) > 0$ for $\eta > \bar{z}$ and $\hat{c}(\eta) - \check{c}(\eta) < 0$ for $\eta < \bar{z}$, $J''(z)$ is increasing in $z \in (z_R, \bar{z})$ and decreasing in $z \in (\bar{z}, z_B)$.

Note that for $z_R < z < z_B$,

$$\frac{\theta^2}{2} z^2 Q''(z) + (\delta - r) z Q'(z) - \delta Q(z) + \tilde{u}(z) + h(z) = 0.$$ 

Letting $z \to z_R^+$, we derive that

$$0 = \lim_{z \to z_R^+} \left( \frac{\theta^2}{2} z^2 Q''(z) + (\delta - r) z Q'(z) - \delta Q(z) + \tilde{u}(z) + h(z) \right)$$

$$= \frac{\theta^2}{2} z_R^2 \lim_{z \to z_R^+} Q''(z) - \frac{\theta^2}{2} z_R^2 \tilde{J}_R''(z) + h(z_R),$$

where we have used the fact that $Q(z)$ is continuously differentiable and $L J_R(z_R) + \tilde{u}(z_R) = 0$.

It follows from the strictly convexity of $J_R$ that

$$Q''(z_R^+) = J_R''(z_R) - \frac{2}{z_R^2} h(z_R) > 0.$$ 

Since $Q''(z_B) = 0$, we deduce that $Q''(z) > 0$ for $z_R < z < z_B$. That is, $Q(z)$ is strictly increasing in $z \in (z_R, z_B)$. It follows from $Q'(z_B) = 0$ that

$$Q'(z) > 0 \quad \text{for} \quad z \in (z_R, z_B).$$

Thus, $Q(z)$ satisfies

$$\max \left\{ \min \left\{ L Q_0(z) + \tilde{u}(z) + h(z), - \frac{dQ}{dz}(z) \right\}, J_R(z) - Q(z) \right\} = 0.$$

(iii) the case $z \in [z_B, \infty)$.

Since $Q'(z) = Q''(z) = 0$ for $z \geq z_B$, we deduce that

$$L Q(z) + \tilde{u}(z) + h(z) = L Q(z_B) + \tilde{u}(z_B) + h(z_B) - \delta Q(z) + \delta Q(z_B) + \tilde{u}(z) + h(z) - \tilde{u}(z_B) - h(z_B)$$

$$= - \delta Q(z) + \delta Q(z_B) + \tilde{u}(z) + h(z) - \tilde{u}(z_B) - h(z_B)$$

$$= \int_{z_B}^{z} (\tilde{u}'(\eta) + h'(\eta) - \delta Q'(\eta)) \, d\eta$$

$$= \int_{z_B}^{z} (\hat{c}(\eta) - \check{c}(\eta)) \, d\eta > 0.$$ 

It follows from $J_R^H(z) = - X(z) < 0$ that for $z \geq z_B$ \n
$$J_R(z) - Q(z) \leq J_R(z_B) - Q(z_B) < 0,$$

where we have used the fact that $Q(z) > J_R(z)$ for $z \in (z_R, z_B]$ (see (47)).

Thus, we deduce that

$$\max \left\{ \min \left\{ L Q_0(z) + \tilde{u}(z) + h(z), - \frac{dQ}{dz}(z) \right\}, J_R(z) - Q(z) \right\} = \max \left\{ 0, J_R(z) - Q(z) \right\} = 0.$$
By (i), (ii), and (iii), we conclude that for all $z > 0$ $Q(z)$ satisfies
\[
\max \left\{ \min \left\{ \mathcal{L}Q_0(z) + \bar{u}(z) + h(z), -\frac{dQ}{dz}(z) \right\}, J_R(z) - Q(z) \right\} = 0.
\]

(b) From (i), (ii), and (iii) in the proof of part (a), we can rewrite the four regions $WR$, $RR$, $AR$, and $IR$ as follows:
\[
RR = \{ z > 0 \mid z \leq z_R \}, \quad WR = \{ z > 0 \mid z > z_R \},
\]
\[
AR = \{ z > 0 \mid zB \leq z \}, \quad IR = \{ z > 0 \mid 0 < z < z_B \}.
\]

(c) Since $Q(z) = J_R(z)$ for $z \in (0, z_R)$, it follows from (99) and (104) in Appendix A that
\[
Q''(z) = J''_R(z) > 0.
\]
From (ii) in the proof of part (a),
\[
Q''(z) > 0 \quad \text{for} \quad z \in (z_R, z_B).
\]
Hence, $Q(z)$ is strictly convex in $z \in (0, z_R)$.

(d) By construction of $Q(z)$ in (40), it is clear that
\[
\lim_{z \to z_B^-} Q'(z) = 0.
\]
On the other hand, it follows from (97) that
\[
\lim_{z \to +0} Q'(z) = \lim_{z \to +0} J''_R(z) = -\lim_{z \to +0} X''_R(z) = -\infty.
\]

We are now ready to state the verification theorem for Problem 2.

**Theorem 4.1.** Let $z_R$ and $z_B$ be as in Proposition 4.1, and consider strategies $(D^{zn}, \tau_{zn})$ defined by
\[
D^{zn}_t = \min \left\{ 1, \inf_{0 \leq s \leq t} \frac{z_B}{Y_s} \right\} \quad \text{for} \quad t \geq 0, \quad \tau_{zn} = \inf \{ t \geq 0 \mid z^{zn}_t < z_R \} \tag{48}
\]
with $z^{zn}_t = Y_t D^{zn}_t$ for $t \geq 0$.
For $Q(z)$ in (40), we have

(a) $Q(z) \leq J_0(z; D, \tau_{zn})$ for any $D \in \mathcal{N} T$.
(b) $Q(z) \geq J_0(z; D^{zn}, \tau)$ for any $\tau \in \mathcal{S}$
(c) $Q(z) = J_0(z; D^{zn}, \tau_{zn})$

That is, the pair $(\tilde{D}, \tilde{\tau}) = (D^{zn}, \tau_{zn})$ is a Nash-equilibrium for Problem 2 and $Q(z) = J(z)$ is the value function of the game in (23).
Proof. (a) Define a process $G$ by

$$G_t^P = \int_0^t e^{-\delta s} \left\{ \left( \tilde{u} \left( \frac{Z^D_s}{k_1} \right) + \epsilon_1 Z^P_s \right) 1_{\{Z^P > zs\}} + \left( \tilde{u} \left( \frac{Z^D_s}{k_2} \right) + \epsilon_2 Z^P_s \right) 1_{\{Z^P \leq zs\}} \right\} ds + e^{-\delta t} Q(Z^P_t). \quad (49)$$

Since $Q$ is $C^1$ as well as $C^2$ outside a finite set $\{z_R\}$, we can still apply generalized Itô’s lemma (see Proposition 9 in Harrison (1985), Exercise 6.24 in Karatzas and Shreve (1991)).

It follows that

$$dG_t^P$$

$$=e^{-\delta t} \left\{ \tilde{u} \left( \frac{Z^D_t}{k_1} \right) + \epsilon_1 Z^P_t \right\} 1_{\{Z^P > zs\}} + \left( \tilde{u} \left( \frac{Z^D_t}{k_2} \right) + \epsilon_2 Z^P_t \right) 1_{\{Z^P \leq zs\}} \right\} dt + e^{-\delta t} dQ(Z^P_t) - \delta Q(Z^P_t) dt$$

$$=e^{-\delta t} \left\{ \tilde{u} \left( \frac{Z^D_t}{k_1} \right) + \epsilon_1 Z^P_t \right\} 1_{\{Z^P > zs\}} + \left( \tilde{u} \left( \frac{Z^D_t}{k_2} \right) + \epsilon_2 Z^P_t \right) 1_{\{Z^P \leq zs\}} \right\} dt - \delta Q(Z^P_t) dt + e^{-\delta t} Q'(Z^P_t) dZ^P_t$$

$$+ e^{-\delta t} \frac{1}{2} Q''(Z^P_t) (dZ^P_t)^2 + e^{-\delta t} Q'(Z^P_t) dD_t$$

$$=e^{-\delta t} \left\{ \left( \frac{\theta}{2} \left( Z^P_t \right)^2 \right) + \left( \delta - \nu \right) Z^P_t Q'(Z^P_t) - \delta Q(Z^P_t) \right\} + \left( \tilde{u} \left( \frac{Z^D_t}{k_1} \right) + \epsilon_1 Z^P_t \right) 1_{\{Z^P > zs\}}$$

$$\left( \tilde{u} \left( \frac{Z^D_t}{k_2} \right) + \epsilon_2 Z^P_t \right) 1_{\{Z^P \leq zs\}} \right\} dt + e^{-\delta t} Q'(Z^P_t) dD_t + e^{-\delta t} \left( Q(Z^P_t) - Q(Z^P_{t-}) \right) = e^{-\delta t} Q'(Z^P_t) dW_t,$$

where $\Delta D_t \equiv D_t - D_{t-}$ and $D^c$ is the continuous part of $D$.

Let us denote $\tau_n$ by $\tau_n = \inf\{ t \geq 0 \mid Z^P_t > n \} \wedge \tau_n$ for each $n \in \mathbb{N}$. Then, we have

$$G_{\tau_n}^P = Q(z) + \int_0^{\tau_n} e^{-\delta s} Q'(Z^P_s) dD^c_s + \sum_{i=0}^{\tau_n} e^{-\delta s} \left( Q(Z^P_s) - Q(Z^P_{s-}) \right) + \int_0^{\tau_n} e^{-\delta s} \left( -\theta \right) Z^P_s Q'(Z^P_s) dW_s$$

$$+ \int_0^{\tau_n} e^{-\delta s} \left\{ \tilde{u} \left( \frac{Z^D_s}{k_1} \right) + \epsilon_1 Z^P_s \right\} 1_{\{Z^P > zs\}} + \left( \tilde{u} \left( \frac{Z^D_s}{k_2} \right) + \epsilon_2 Z^P_s \right) 1_{\{Z^P \leq zs\}} \right\} ds.$$

Taking expectations, we have

$$\mathbb{E} \left[ \int_0^{\tau_n} e^{-\delta s} \left\{ \tilde{u} \left( \frac{Z^D_s}{k_1} \right) + \epsilon_1 Z^P_s \right\} 1_{\{Z^P > zs\}} + \left( \tilde{u} \left( \frac{Z^D_s}{k_2} \right) + \epsilon_2 Z^P_s \right) 1_{\{Z^P \leq zs\}} \right\} ds + e^{-\delta(\tau_n \wedge t)} Q(Z^P_{(\tau_n \wedge t)-}) \right]$$

$$=Q(z) + \mathbb{E}[\text{(A)}] + \mathbb{E}[\text{(B)}] + \mathbb{E}[\text{(C)}] + \mathbb{E}[\text{(D)}].$$

Let us denote $\mathcal{M}_t$ by

$$\mathcal{M}_t = \int_0^t e^{-\delta s} \left( -\theta \right) Z^P_s Q'(Z^P_s) dW_s \quad \text{for } t \geq 0. \quad (51)$$

Since $Q$ satisfies HJBQV \((33)\) and $\mathcal{M}_{\tau_n \wedge t}$ is a martingale, we have

$$\mathbb{E}[(\text{A})] \geq 0, \quad \mathbb{E}[(\text{C})] = 0, \quad \text{and} \quad \mathbb{E}[(\text{D})] \geq 0. \quad (52)$$

Note that

$$Q(Z^P_s) - Q(Z^P_{s-}) = \int_{Z^P_{s-} \Delta Z^P_s} Q_s(\nu) d\nu \geq 0,$$  \(53\)
where $\Delta Z^D = \mathcal{Y}_t \Delta D_s \leq 0$.

It follows that

$$E[(B)] \geq 0. \quad (54)$$

Hence, we deduce that

$$Q(z) \leq E \left[ \int_0^{\tau_n \wedge t} e^{-s} \left\{ \bar{u} \left( \frac{Z^D}{\kappa_1} \right) + \epsilon_1 Z^D_s \right\} 1_{\{Z^D_s > z\}} + \left( \bar{u} \left( \frac{Z^D}{\kappa_2} \right) + \epsilon_2 Z^D_s \right) 1_{\{Z^D_s \leq z\}} \right] ds$$

$$+ e^{-\delta (\tau_n \wedge t)} Q(Z^D_{(\tau_n \wedge t-)}) \right]. \quad (55)$$

Let us temporarily denote $\varphi(z)$ by

$$\varphi(z) := h(z) + \bar{u}(z) = \left( \bar{u} \left( \frac{z}{\kappa_1} \right) + \epsilon_1 z \right) 1_{\{z > \kappa_\}} + \left( \bar{u} \left( \frac{z}{\kappa_2} \right) + \epsilon_2 z \right) 1_{\{z \leq \kappa_\}}.$$

For $s \in [0, \tau_n \wedge t)$, we have

$$z_R \leq Z^D_s \leq Y_s.$$

This leads to

$$(\varphi(Z^D_s))_+ = \left( \bar{u} \left( \frac{Z^D}{\kappa_1} \right) + \epsilon_1 Z^D_s \right) 1_{\{Z^D_s > z\}} + \left( \bar{u} \left( \frac{Z^D}{\kappa_2} \right) + \epsilon_2 Z^D_s \right) 1_{\{Z^D_s \leq z\}}$$

$$\leq \left( \bar{u} \left( \frac{z_R}{\kappa_2} \right) + \epsilon_1 Y_s \right) + \left( \bar{u} \left( \frac{z_R}{\kappa_2} \right) + \epsilon_1 Y_s \right).$$

Thus, we deduce that

$$E \left[ \int_0^{\tau_n \wedge t} e^{-s} (\varphi(Z^D_s))_+ ds \right] \leq E \left[ \int_0^{\tau_R} e^{-s} \left\{ \bar{u} \left( \frac{z_R}{\kappa_2} \right) + \epsilon_1 Y_s \right\} ds \right] < \infty.$$  

The dominated convergence theorem implies that

$$\lim_{t \uparrow \infty} \lim_{n \uparrow \infty} E \left[ \int_0^{\tau_n \wedge t} e^{-s} (\varphi(Z^D_s))_+ ds \right] = E \left[ \int_0^{\tau_R} e^{-s} (\varphi(Z^D_s))_+ ds \right] < \infty. \quad (56)$$

Moreover, the monotone convergence theorem implies that

$$\lim_{t \uparrow \infty} \lim_{n \uparrow \infty} E \left[ \int_0^{\tau_n \wedge t} e^{-s} (\varphi(Z^D_s))_- ds \right] = E \left[ \int_0^{\tau_R} e^{-s} (\varphi(Z^D_s))_- ds \right] \quad (57)$$

It follows from (56) and (57) that

$$\lim_{t \uparrow \infty} \lim_{n \uparrow \infty} E \left[ \int_0^{\tau_n \wedge t} e^{-s} \left\{ \bar{u} \left( \frac{Z^D}{\kappa_1} \right) + \epsilon_1 Z^D_s \right\} 1_{\{Z^D_s > z\}} + \left( \bar{u} \left( \frac{Z^D}{\kappa_2} \right) + \epsilon_2 Z^D_s \right) 1_{\{Z^D_s \leq z\}} \right] ds$$

$$= E \left[ \int_0^{\tau_R} e^{-s} \left\{ \bar{u} \left( \frac{Z^D}{\kappa_1} \right) + \epsilon_1 Z^D_s \right\} 1_{\{Z^D_s > z\}} + \left( \bar{u} \left( \frac{Z^D}{\kappa_2} \right) + \epsilon_2 Z^D_s \right) 1_{\{Z^D_s \leq z\}} \right] ds \right]. \quad (58)$$

Since $Q'(z) \leq 0$, we have

$$e^{-\delta (\tau_n \wedge t)} \left( Q(Z^D_{(\tau_n \wedge t-)}) \right)_+ \leq (Q(z_R))_+ < \infty,$$

where we have used fact that $z_R \leq Z^D_{(\tau_n \wedge t-)}$ for all $t \geq 0.$

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Thus, Fatou’s lemma implies that
\[
\limsup_{n \to \infty} \mathbb{E} \left[ e^{-\delta(t_{\tau_n} \wedge t)} Q(Z^{D_{\tau_n} \wedge t}_{\tau_n} \wedge t) \right] \leq \mathbb{E} \left[ e^{-\delta(t_{\tau_n} \wedge t)} Q(Z^{D_{(\tau_n \wedge t) \wedge t}}_{\tau_n}) \right].
\] (59)

Note that
\[
e^{-\delta(t_{\tau_n} \wedge t)} Q(Z^{D_{(\tau_n \wedge t) \wedge t}}_{\tau_n}) = e^{-\delta t} Q(Z^{D}_{\tau_n}) 1_{(t < \tau_n)} + e^{-\delta \tau_n} J_R(Z^{D}_{\tau_n} \wedge t) 1_{(t \geq \tau_n)}.
\] (60)

Since \( Q(Z^{D}_{\tau_n}) \leq Q(z_R) \) for \( t \in [0, \tau_{z_R}] \) and \( J_R(Z^{D}_{\tau_n} \wedge t) \leq J_R(z_R) \), it follows from Fatou’s lemma that
\[
\limsup_{t \to \infty} \mathbb{E} \left[ e^{-\delta(t_{\tau_n} \wedge t)} Q(Z^{D_{(\tau_n \wedge t) \wedge t}}_{\tau_n}) \right] 
\leq \limsup_{t \to \infty} \mathbb{E} \left[ e^{-\delta t} Q(Z^{D}_{\tau_n}) 1_{(t < \tau_n)} \right] + \limsup_{t \to \infty} \mathbb{E} \left[ e^{-\delta \tau_n} J_R(Z^{D}_{\tau_n}) 1_{(t \geq \tau_n)} \right]
\leq \limsup_{t \to \infty} \mathbb{E} \left[ e^{-\delta \tau_n} J_R(Z^{D}_{\tau_n}) 1_{(t \geq \tau_n)} \right] 
\leq \mathbb{E} \left[ e^{-\delta \tau_n} J_R(Z^{D}_{\tau_n}) 1_{(t \geq \tau_n)} \right].
\] (61)

By (55), (58), (59), and (61), we conclude that
\[
Q(z) \leq \mathcal{J}_0(z; \mathcal{D}, \tau_{z_R})
\] (62)
for any \( \mathcal{D} \in \mathcal{N} \).

(b) Let \( D^{\tau_n} \) be as in the statement of theorem. For each \( n \in \mathbb{N} \) define the stopping time \( \tau_n \) as
\[
\tau_n := \inf \left\{ t > 0 \mid Z^{D^{\tau_n}}_{t} < \frac{1}{n} \right\} \wedge \tau
\]
for an arbitrary stopping time \( \tau \in \mathcal{S} \).

From the definition of \( D^{\tau_n} \), it is easy to see that \( D^{\tau_n} \) is continuous,
\[
\int_0^t e^{-\delta s} Q'(Z^{D^{\tau_n}}_s) dD^{\tau_n}_s = 0 \quad \text{for} \quad t \geq 0,
\] (63)
and
\[
\frac{\theta^2}{2} (Z^{D^{\tau_n}}_t)^2 Q''(Z^{D^{\tau_n}}_t) + (\delta - r) Z^{D^{\tau_n}}_t Q'(Z^{D^{\tau_n}}_t) - \delta Q(Z^{D^{\tau_n}}_t) + \bar{u}(Z^{D^{\tau_n}}_t) + h(Z^{D^{\tau_n}}_t) = 0
\]
for all \( t \in [0, \tau_{z_R}] \),
\[
\frac{\theta^2}{2} (Z^{D^{\tau_n}}_t)^2 Q''(Z^{D^{\tau_n}}_t) + (\delta - r) Z^{D^{\tau_n}}_t Q'(Z^{D^{\tau_n}}_t) - \delta Q(Z^{D^{\tau_n}}_t) + \bar{u}(Z^{D^{\tau_n}}_t) + h(Z^{D^{\tau_n}}_t) \leq 0
\]
for all \( t \in (\tau_{z_R}, \infty) \).

This leads to
\[
\mathbb{E}[(A)] = \mathbb{E}[(B)] = \mathbb{E}[(C)] = 0 \quad \text{and} \quad \mathbb{E}[(D)] \leq 0
\] (65)
for \( \mathcal{D} = D^{\tau_n} \) in (50).
By utilizing the above localizing argument, it follows from (65) that

\[
Q(z) \geq \mathbb{E} \left[ \int_0^{\tau_n \wedge t} e^{-\delta s} \left\{ \left( \tilde{u} \left( \frac{Z_{s}^{D^n}}{\kappa_1} \right) + \epsilon_1 Z_{s}^{D^n} \right) 1_{\{Z_{s}^{D^n} > z_B\}} + \left( \tilde{u} \left( \frac{Z_{s}^{D^n}}{\kappa_2} \right) + \epsilon_2 Z_{s}^{D^n} \right) 1_{\{Z_{s}^{D^n} \leq z_B\}} \right\} ds \right]
\]

\[
+ e^{-\delta (\tau_n \wedge t)} Q(Z_{\tau_n \wedge t}^{D^n})\right].
\]

Note that \(Z_{t}^{D^n} \leq z_B\) for all \(t \geq 0\). Thus, similarly to the derivation above in the part (a), we have

\[
\lim_{t \uparrow \infty} \liminf_{n \uparrow \infty} \mathbb{E} \left[ \int_0^{\tau_n \wedge t} e^{-\delta s} \left\{ \left( \tilde{u} \left( \frac{Z_{s}^{D^n}}{\kappa_1} \right) + \epsilon_1 Z_{s}^{D^n} \right) 1_{\{Z_{s}^{D^n} > z_B\}} + \left( \tilde{u} \left( \frac{Z_{s}^{D^n}}{\kappa_2} \right) + \epsilon_2 Z_{s}^{D^n} \right) 1_{\{Z_{s}^{D^n} \leq z_B\}} \right\} ds \right]
\]

\[
- e^{e^{-\delta (\tau_n \wedge t)} Q(Z_{\tau_n \wedge t}^{D^n})} \right].
\]

Since \(Q(Z_{\tau_n \wedge t}^{D^n}) \geq Q(z_B)\) for all \(t \geq 0\), Fatou's lemma implies that

\[
\lim_{t \uparrow \infty} \liminf_{n \uparrow \infty} \mathbb{E} \left[ e^{-\delta (\tau_n \wedge t)} Q(Z_{\tau_n \wedge t}^{D^n}) \right] \geq \mathbb{E} \left[ e^{-\delta \tau} Q(Z_{\tau}^{D^n}) 1_{\{\tau < \infty\}} \right] + \liminf_{t \uparrow \infty} \mathbb{E} \left[ e^{-\delta t} Q(Z_{t}^{D^n}) \right]
\]

\[
\geq \mathbb{E} \left[ e^{-\delta \tau} Q(Z_{\tau}^{D^n}) 1_{\{\tau < \infty\}} \right].
\]

Since \(Q(z) \geq J_R(z)\) for all \(z > 0\) in (33), we have

\[
\lim_{t \uparrow \infty} \liminf_{n \uparrow \infty} \mathbb{E} \left[ e^{-\delta (\tau_n \wedge t)} Q(Z_{\tau_n \wedge t}^{D^n}) \right] \geq \mathbb{E} \left[ e^{-\delta \tau} J_R(Z_{\tau}^{D^n}) 1_{\{\tau < \infty\}} \right].
\]

(68)

It follows from (66), (67), and (68) that

\[
Q(z) \geq J_0(z; D^{z_B}, \tau)
\]

for any \(\tau \in \mathcal{S}\).

(c) For \(D = D^{z_B}\) and \(\tau_n = \tau_R\),

\[
z_R \leq Z_{t}^{D^{z_B}} \leq z_B \quad t \in [0, \tau_R].
\]

(69)

By using the arguments in parts (a) and (b) with (69), it is easy to confirm that

\[
\mathbb{E}[\mathbb{A}] = \mathbb{E}[\mathbb{B}] = \mathbb{E}[\mathbb{C}] = \mathbb{E}[\mathbb{D}] = 0 \quad \text{in (50)}.
\]

That is,

\[
Q(z) = \mathbb{E} \left[ \int_0^{\tau_R \wedge t} e^{-\delta s} \left\{ \left( \tilde{u} \left( \frac{Z_{s}^{D}}{\kappa_1} \right) + \epsilon_1 Z_{s}^{D} \right) 1_{\{Z_{s}^{D} > z_B, \tau\}} + \left( \tilde{u} \left( \frac{Z_{s}^{D}}{\kappa_2} \right) + \epsilon_2 Z_{s}^{D} \right) 1_{\{Z_{s}^{D} \leq z_B, \tau\}} \right\} ds \right]
\]

\[
+ e^{-\delta (\tau_R \wedge \tau)} Q(Z_{\tau_R \wedge \tau}^{D})\right].
\]

From (69), we can easily get

\[
\mathbb{E} \left[ \int_0^{\tau_R} e^{-\delta s} \left\{ \left( \tilde{u} \left( \frac{Z_{s}^{D}}{\kappa_1} \right) + \epsilon_1 Z_{s}^{D} \right) 1_{\{Z_{s}^{D} > z_B, \tau\}} + \left( \tilde{u} \left( \frac{Z_{s}^{D}}{\kappa_2} \right) + \epsilon_2 Z_{s}^{D} \right) 1_{\{Z_{s}^{D} \leq z_B, \tau\}} \right\} ds \right] < \infty
\]
and

\[
E \left[ e^{-\delta \tau_n} \left| J_R(\mathcal{Z}_{\tau_n}^n) \right| \right] < \infty. 
\]

The dominated convergence theorem implies that

\[
Q(z) = \lim_{\epsilon \to 0} \left[ \int_0^{\tau_{\epsilon n}} e^{-\delta s} \left\{ \left( \tilde{u} \left( \frac{Z_s^D}{\kappa_1} \right) + \epsilon_z Z_s^D \right) 1_{\{Z_s^D > z\}} + \left( \tilde{u} \left( \frac{Z_s^D}{\kappa_2} \right) + \epsilon_z Z_s^D \right) 1_{\{Z_s^D \leq z\}} \right\} ds + e^{-\delta(\tau_{\epsilon n} \wedge \epsilon)} Q(Z_{\tau_{\epsilon n} \wedge \epsilon}^D) \right] 
\]

\[
= \mathbb{E} \left[ \int_0^{\tau_{\epsilon n}} e^{-\delta s} \left\{ \left( \tilde{u} \left( \frac{Z_s^D}{\kappa_1} \right) + \epsilon_z Z_s^D \right) 1_{\{Z_s^D > z\}} + \left( \tilde{u} \left( \frac{Z_s^D}{\kappa_2} \right) + \epsilon_z Z_s^D \right) 1_{\{Z_s^D \leq z\}} \right\} ds + 1_{\{\tau_{\epsilon n} \leq \epsilon\}} e^{-\delta \tau_n} J_R(\mathcal{Z}_{\tau_n}^D) \right].
\]

\[
\square
\]

### 4.4 Verification II: Duality Theorem

Since the pair \((D^n, \tau_n)\) is a Nash-equilibrium of Problem 2 (Theorem 4.1), the dual value function \(J(z)\) is given by

\[
J(y) = J_0(y; D^n, \tau_n) = Q(y).
\]

Hence, we can write the dual value function \(J(y)\) in the explicit form:

\[
J(y) = \begin{cases} 
J_R(y) & \text{for } y \leq z_R, \\
E_1 y_n^1 + E_2 y_n^2 + \phi_h(y) + J_R(y) & \text{for } z_R \leq y \leq z_B, \\
E_1 y_n^1 + E_2 y_n^2 + \phi_h(z_B) + J_R(z_B) & \text{for } z_B \leq y,
\end{cases}
\]

where

\[
E_1 = -\frac{2}{\theta^2(n_1 - n_2)} \int_{z_R}^{\infty} \eta^{-n_1-1} h(\eta) d\eta \quad \text{and} \quad E_2 = -\frac{2}{\theta^2(n_1 - n_2)} \int_{z_R}^{\infty} \eta^{-n_2-1} h(\eta) d\eta.
\]

#### Lemma 4.2

For given \(y > 0\) and the Nash-equilibrium \((D^n, \tau_n)\) of Problem 2, the following relationship holds:

\[
-J'(y) = \mathbb{E} \left[ \int_0^{\tau_R} \mathcal{H}_t \left( \tilde{c}(Z_t^{D, n}) - \tilde{e}(Z_t^{D, n}) \right) dt + 1_{\{\tau_{\epsilon n} \leq \epsilon\}} \mathcal{H}_{\tau_R} \mathcal{X}_R(Z_{\tau_R}^{D, n}) \right]
\]

\[
= \mathbb{E} \left[ \int_0^{\tau_R} \mathcal{H}_t D_t^{D, n} \left( \tilde{c}(Z_t^{D, n}) - \tilde{e}(Z_t^{D, n}) \right) dt + 1_{\{\tau_{\epsilon n} \leq \epsilon\}} \mathcal{H}_{\tau_R} D_{\tau_R}^{D, n} \mathcal{X}_R(Z_{\tau_R}^{D, n}) \right].
\]

Recall that \(\mathcal{X}_R(y) = -J_R'(y)\) (see Appendix A),

\[
\tilde{c}(z) = \frac{1}{\kappa_1} \left( \frac{z}{\kappa_1} \right) 1_{\{z > z_B\}} + \frac{1}{\kappa_2} \left( \frac{z}{\kappa_2} \right) 1_{\{z \leq z_B\}}, \quad \text{and} \quad \tilde{e}(z) = \epsilon_1 1_{\{z > z_B\}} + \epsilon_2 1_{\{z \leq z_B\}}.
\]

**Proof.** First, we will show that

\[
-J'(y) = \mathbb{E} \left[ \int_0^{\tau_R} \mathcal{H}_t D_t^{D, n} \left( \tilde{c}(Z_t^{D, n}) - \tilde{e}(Z_t^{D, n}) \right) dt + 1_{\{\tau_{\epsilon n} \leq \epsilon\}} \mathcal{H}_{\tau_R} D_{\tau_R}^{D, n} \mathcal{X}_R(Z_{\tau_R}^{D, n}) \right].
\]
Let us temporarily denote \( Q_1(y) \) by 
\[
Q_1(y) = -yJ'(y).
\]

Define a process \( \mathcal{N} \) by 
\[
\mathcal{N}_t = \int_0^t e^{-\delta s} \left( Z_s^{D^n} \dot{c}(Z_s^{D^n}) - Z_s^{D^n} \dot{c}(Z_s^{D^n}) \right) ds + e^{-\delta t} Q_1(Z_t^{D^n}). \tag{72}
\]

Since \( X_R(y) = -J_R'(y) \) is \( C^2 \), it is easy to confirm that \( Q_1 \) is \( C^2 \) outside \( \{Z_R, z_B\} \). Thus, we can apply generalized Itô’s lemma to \( Q_1(Z_t^{D^n}) \). Thus, we have 
\[
d\mathcal{N}_t = e^{-\delta t} \left( Z_t^{D^n} \dot{c}(Z_t^{D^n}) - Z_t^{D^n} \dot{c}(Z_t^{D^n}) \right) dt + e^{-\delta t} dQ_1(Z_t^{D^n}) - \delta e^{-\delta t} Q_1(Z_t^{D^n}) dt 
\]
\[
= e^{-\delta t} \left( \mathcal{L}Q_1(Z_t^{D^n}) + Z_t^{D^n} \dot{c}(Z_t^{D^n}) - Z_t^{D^n} \dot{c}(Z_t^{D^n}) \right) dt + e^{-\delta t} (-\theta) Z_t^{D^n} Q_1'(Z_t^{D^n}) dW_t + Q_1(Z_t^{D^n}) dD_t^{z_B}.
\]

From this, 
\[
\mathcal{N}_t^{\tau_{z_R} \wedge t} - Q_1(z) + \int_0^{\tau_{z_R} \wedge t} \left( \mathcal{L}Q_1(Z_s^{D^n}) + Z_s^{D^n} \dot{c}(Z_s^{D^n}) - Z_s^{D^n} \dot{c}(Z_s^{D^n}) \right) ds + \mathcal{M}_t^{\tau_{z_R} \wedge t} + \int_0^{\tau_{z_R} \wedge t} Q_1(Z_s^{D^n}) dD_t^{z_B}, \tag{73}
\]
where 
\[
\mathcal{M}_t := \int_0^t e^{-\delta s} (-\theta) Z_s^{D^n} Q_1'(Z_s^{D^n}) dW_s
\]

By construction of \( J(y) \) and the definition of \( D^{z_B} \), we deduce that 
\[
\int_0^{\tau_{z_R} \wedge t} Q_1'(Z_s^{D^n}) dD_t^{z_B} = - \int_0^{\tau_{z_R} \wedge t} Z_s^{D^n} J''(Z_s^{D^n}) dD_t^{z_B} = 0.
\]

Since \( \mathcal{L}J(Z_s^{D^n}) + h(Z_s^{D^n}) + \bar{u}(Z_s^{D^n}) = 0 \) for all \( t \in [0, \tau_{z_R}) \), we have 
\[
\mathcal{L}Q_1(Z_s^{D^n}) + Z_s^{D^n} \dot{c}(Z_s^{D^n}) - Z_s^{D^n} \dot{c}(Z_s^{D^n}) = 0
\]

for all \( t \in [0, \tau_{z_R}) \).

Moreover, it follows from \( z_R \leq Z_t^{D^n} \leq z_B \) for \( t \in [0, \tau_{z_R}) \) that \( \mathcal{M}_t^{\tau_{z_R} \wedge t} \) is a martingale.

By taking expectation to the both sides of (73), we derive that 
\[
Q_1(z) = E \left[ \int_0^{\tau_{z_R} \wedge t} e^{-\delta s} \left( Z_s^{D^n} \dot{c}(Z_s^{D^n}) - Z_s^{D^n} \dot{c}(Z_s^{D^n}) \right) ds + e^{-\delta (\tau_{z_R} \wedge t)} Q_1(Z_{\tau_{z_R} \wedge t}^{D^n}) \right]. \tag{74}
\]

From \( z_R \leq Z_t^{D^n} \leq z_B \) for \( t \in [0, \tau_{z_R}) \), it is easy to show that 
\[
E \left[ \int_0^{\tau_{z_R}} e^{-\delta t} \left( Z_t^{D^n} \dot{c}(Z_t^{D^n}) - Z_t^{D^n} \dot{c}(Z_t^{D^n}) \right) dt \right] < \infty \quad \text{and} \quad E \left[ e^{-\delta \tau_{z_R}} |Q_1(Z_{\tau_{z_R}}^{D^n})| \right] < \infty.
\]

Since \( Q_1(y) = -yJ_R'(y) = yX_R(y) \) for \( y \in (0, z_R] \), the dominated convergence theorem implies that 
\[
Q_1(z) = E \left[ \int_0^{\tau_{z_R}} e^{-\delta t} \left( Z_t^{D^n} \dot{c}(Z_t^{D^n}) - Z_t^{D^n} \dot{c}(Z_t^{D^n}) \right) dt + 1_{\{\tau_{z_R} < \infty\}} e^{-\delta \tau_{z_R}} Z_{\tau_{z_R}}^{D^n} X_R(Z_{\tau_{z_R}}^{D^n}) \right]. \tag{75}
\]

Since \( Z_t^{D^n} = ye^{\delta t} H_t D_t^{z_B} \), we have
\[-J'(y) = \mathbb{E} \left[ \int_0^T \mathcal{H}_t \left( \hat{c}(Z_t^{D^n}) - \hat{c}(Z_t^{D^n}) \right) dt + 1_{\{\tau_{z_R} < \infty\}} \mathcal{H}_{\tau_{z_R}} D_{\tau_{z_R}}^{z_R} X_R(Z_{\tau_{z_R}}^{D^n}) \right].\]

Next, we will show that
\[-J'(y) = \mathbb{E} \left[ \int_0^T \mathcal{H}_t \left( \hat{c}(Z_t^{D^n}) - \hat{c}(Z_t^{D^n}) \right) dt + 1_{\{\tau_{z_R} < \infty\}} \mathcal{H}_{\tau_{z_R}} X_R(Z_{\tau_{z_R}}^{D^n}) \right].\]

For any fixed \(T > 0\), let us define an equivalent martingale measure \(Q\) by
\[
\frac{dQ}{dp} = e^{-\frac{4}{3}T - \theta W_T}. \tag{76}
\]

The Girsanov theorem implies that \(W^Q_t = W_t + \theta t\) is a standard Brownian motion under the measure \(Q\).

Note that
\[
\frac{dZ_t^{D^n}}{Z_t^{D^n}} = (\delta - r + \theta^2) dt - \theta dW^Q_t + \frac{dD_t^{z_R}}{D_t^{z_R}} \tag{77}
\]
under the measure \(Q\).

Then, similarly to the derivation above, we have
\[
-J'(y) = \mathbb{E} \left[ \int_0^{\tau_{z_R} \wedge T} e^{-rt} \left( \hat{c}(Z_t^{D^n}) - \hat{c}(Z_t^{D^n}) \right) dt + e^{-r(\tau_{z_R} \wedge T)} \mathcal{H}_{\tau_{z_R} \wedge T} R(Z_{\tau_{z_R} \wedge T}^{D^n}) \right] \tag{78}
\]
\[
= \mathbb{E} \left[ \int_0^{\tau_{z_R} \wedge T} \mathcal{H}_t \left( \hat{c}(Z_t^{D^n}) - \hat{c}(Z_t^{D^n}) \right) dt + \mathcal{H}_{\tau_{z_R} \wedge T} R(Z_{\tau_{z_R} \wedge T}^{D^n}) \right].
\]

By the dominated convergence theorem, we can obtain that
\[-J'(y) = \mathbb{E} \left[ \int_0^T \mathcal{H}_t \left( \hat{c}(Z_t^{D^n}) - \hat{c}(Z_t^{D^n}) \right) dt + 1_{\{\tau_{z_R} < \infty\}} \mathcal{H}_{\tau_{z_R}} X_R(Z_{\tau_{z_R}}^{D^n}) \right]. \tag{79} \]

\[\square\]

**Theorem 4.2.** Let \(x > 0\) be given.

(a) There exists a unique \(y^* \in (0, z_B)\) such that
\[x = -J'(y^*).\]

(b) Consider the Nash-equilibrium \((D^*, \tau^*)\) of \(J(y^*)\) by
\[D_t^* = \min \left\{ 1, \inf_{0 \leq s \leq t} \frac{Z_s}{\mathcal{Y}_s^*} \right\} \quad t \geq 0 \tag{80}\]
and
\[\tau^* = \inf \{ t \geq 0 \mid Z_t^* < z_R \}, \tag{81}\]

where \(Z_t^* = \mathcal{Y}_t^* D_t^*\) with \(\mathcal{Y}_t^* = y^* e^{\delta t} \mathcal{H}_t\).

Let \(c^*, \zeta^*,\) and \(X_{\tau^*}\) be the consumption, job state process, and wealth at time \(\tau^*\) given by
\[c_t^* = \begin{cases} \hat{c}(Z_t^*) & \text{for } t \in [0, \tau^*), \\ I \left( \mathcal{Y}_t^{R,t^*} \right) & \text{for } t \geq \tau^*, \end{cases} \quad \zeta^* = \hat{\zeta}(Z_t^*) = \begin{cases} \mathbb{B}_1 & \text{for } Z_t^* > z_S, \\ \mathbb{B}_2 & \text{for } Z_t^* \leq z_S, \end{cases}\]

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and
\[ X_{t^*} = -J_R(Z_{t^*}) = \mathcal{X}_R(Z_{t^*}), \]
respectively. Here, \( y^{H}_t = Z_{t^*} e^{\delta(t-\tau^*)} \mathcal{H}_t / \mathcal{H}_{t^*} \). Then, there exists a portfolio \( \pi^* \) such that the strategy \((\pi^*, \pi^*, \xi^*, \tau^*) \in \mathcal{A}(x)\).

(c) \( V(x) \) and \( J(y) \) satisfy the duality relationship:
\[ V(x) = \inf_{y > 0} (J(y) + yx) \quad \text{and} \quad J(y) = \sup_{x > 0} (V(x) - yx). \] (82)

Moreover, \((\pi^*, \pi^*, \xi^*, \tau^*) \) is optimal.

Proof. (a) Since \( J(y) = Q(y) \), it follows from (c) and (d) in Proposition 4.2 that (i) \( J(y) \) is strictly convex in \( y \in (0, z_B) \), (ii) \( \lim_{y \to z_B^-} J'(y) = 0 \) and \( \lim_{y \to 0^+} J'(y) = -\infty \). Thus, for given \( x > 0 \), there exists a unique \( y^* > 0 \) such that
\[ x = -J'(y^*). \]

(b) For \( y^* \in (0, z_B) \) which is a unique solution to \( x = -J'(y^*), \) let \( \mathcal{Y}_t^*, \mathcal{D}_t^*, \mathcal{Z}_t^* \), and \( \tau^* \) be as in the statement of the theorem.

Then, it follows from Lemma 4.2 that
\[ x = \mathbb{E} \left[ \int_0^{\tau^*} \mathcal{H}_t(c_t - \epsilon(\mathcal{Z}_t^*)) dt + 1_{\{\tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{X}_R(\mathcal{Z}_{\tau^*}) \right] \] (83)
\[ = \mathbb{E} \left[ \int_0^{\tau^*} \mathcal{D}_t^* \mathcal{H}_t(c_t - \epsilon(\mathcal{Z}_t^*)) dt + 1_{\{\tau^* < \infty\}} \mathcal{D}_{\tau^*}^* \mathcal{H}_{\tau^*} \mathcal{X}_R(\mathcal{Z}_{\tau^*}) \right] \] (84)

For given \( \epsilon > 0 \), define a process \( \mathcal{D}_t^\epsilon \) by
\[ \mathcal{D}_t^\epsilon := \frac{y^\epsilon \mathcal{D}_t^\epsilon + \mathbb{1}_{(0,\xi)}}{y^\epsilon + \epsilon} \quad \text{for} \quad t \geq 0, \] (85)
where \( \xi \) is an arbitrary stopping time belongs to \( \mathcal{S}_{\tau^*} = \{ \xi \in \mathcal{S} \mid 0 \leq \xi \leq \tau^* \} \).

Since \( \mathcal{D}_0^\epsilon = 1 \), it is clear that \( \mathcal{D}_t^\epsilon \in \mathcal{N}_t \).

Let us define \( y^\epsilon, \mathcal{Y}_t^\epsilon, \text{ and } \mathcal{Z}_t^\epsilon \) by
\[ y^\epsilon = y^* + \epsilon, \quad \mathcal{Y}_t^\epsilon = y^\epsilon e^{\delta t} \mathcal{H}_t, \quad \text{and} \quad \mathcal{Z}_t^\epsilon = \mathcal{Y}_t^\epsilon \mathcal{D}_t^\epsilon, \]
respectively.

Note that
\[ \mathcal{Z}_t^\epsilon = \begin{cases} \mathcal{Z}_t^\epsilon + \epsilon e^{\delta t} \mathcal{H}_t & \text{for} \quad 0 \leq t < \xi, \\ \mathcal{Z}_t^\epsilon & \text{for} \quad t \geq \xi. \end{cases} \]

Let \((\hat{D}_t^\epsilon, \hat{\tau}^\epsilon)\) be the Nash-equilibrium of \( J(y^\epsilon) \), i.e.,
\[ J(y^\epsilon) = J_0(y^\epsilon; \hat{D}_t^\epsilon, \hat{\tau}^\epsilon), \] (86)
where
\[ \hat{D}_t^\epsilon = \min \left\{ 1, \inf_{0 \leq s \leq t} \frac{z_B}{y_t^\epsilon} \right\} \quad \text{and} \quad \hat{\tau}^\epsilon = \inf\{ t \geq 0 \mid \mathcal{Y}_t^\epsilon \hat{D}_t^\epsilon < z_B \}. \]
By the definition of Nash-equilibrium,
\[ J_0(y^*) + y^*x = J_0(y^*; \bar{D}^*, \bar{\tau}^*) + y^*x \leq J_0(y^*; D^*, \tau^*) + y^*x. \]

Since \( y^* \) is a unique minimizer of \( J(y) + yx \), we deduce that
\[ J_0(y^*; D^*, \tau^*) + y^*x = J(y^*) + y^*x \leq J_0(y^*; D^*, \tau^*) + y^*x. \] (87)

Note that
\[ J_0(y^*; D^*, \tau^*) + y^*x \leq J_0(y^*; D^*, \tau^*) + y^*x, \] (88)
where we have used the fact that \((D^*, \tau^*)\) is the Nash-equilibrium of \( J(y^*) \).

Define a process \( N^\epsilon \) by
\[
N^\epsilon = E \left[ \int_0^{\hat{\tau}^\epsilon} e^{-\delta t} \left\{ \left( \tilde{u} \left( \frac{Z^*_t}{\kappa_1} \right) + \epsilon_1 \right) 1\{Z^*_t \geq \tau\} + \left( \tilde{u} \left( \frac{Z^*_t}{\kappa_2} \right) + \epsilon_2 \right) 1\{Z^*_t < \tau\} \right\} dt + 1_{(\tau^* < \infty)} J_R(Z^*_\tau^\epsilon) \right] + y^*x. \] (89)

By Lemma 3.1,
\[
\left( \tilde{u} \left( \frac{Z^*_t}{\kappa_1} \right) + \epsilon_1 \right) 1\{Z^*_t \geq \tau\} + \left( \tilde{u} \left( \frac{Z^*_t}{\kappa_2} \right) + \epsilon_2 \right) 1\{Z^*_t < \tau\} \geq \left( \tilde{u} \left( \frac{Z^*_t}{\kappa_1} \right) + \epsilon_1 \right) 1\{Z^*_t \geq \tau\} + \left( \tilde{u} \left( \frac{Z^*_t}{\kappa_2} \right) + \epsilon_2 \right) 1\{Z^*_t < \tau\}. \]

This implies that
\[ N^\epsilon \leq J_0(y^*; D^*, \tau^*) + y^*x. \] (90)

From (87), (88), and (90), we deduce that
\[ N^\epsilon \leq J_0(y^*; D^*, \tau^*) + y^*x. \] (91)

Hence,
\[
0 \leq E \left[ \int_0^{\hat{\tau}^\epsilon} e^{-\delta t} \left\{ \left( \tilde{u} \left( \frac{Z^*_t}{\kappa_1} \right) - \frac{\tilde{u}(Z^*_t)}{\epsilon} + \epsilon_1 e^{\delta t} H_t \right) 1\{Z^*_t \geq \tau\} + \left( \tilde{u} \left( \frac{Z^*_t}{\kappa_2} \right) - \frac{\tilde{u}(Z^*_t)}{\epsilon} + \epsilon_2 e^{\delta t} H_t \right) 1\{Z^*_t < \tau\} \right\} dt + 1_{(\tau^* \leq \epsilon)} \left( \frac{J_R(Z^*_\tau^\epsilon) - J_R(Z^*_\tau^\epsilon)}{\epsilon} \right) \right] + x. \] (92)

By applying the dominated convergence theorem, it is easy to confirm that
\[
\lim_{\epsilon \downarrow 0} E \left[ \int_0^{\hat{\tau}^\epsilon} H_t \left( \epsilon_1 1\{Z^*_t \geq \tau\} + \epsilon_2 1\{Z^*_t < \tau\} \right) dt \right] = E \left[ \int_0^{\hat{\tau}^\epsilon} H_t \left( \epsilon_1 1\{Z^*_t \geq \tau\} + \epsilon_2 1\{Z^*_t < \tau\} \right) dt \right] \] (93)

\[ = E \left[ \int_0^{\hat{\tau}^\epsilon} H_t \delta(Z^*_t) dt \right]. \]

Note that
\[ \tilde{u} \left( \frac{Z^*_t}{\kappa_1} \right) - \tilde{u} \left( \frac{Z^*_t}{\kappa_1} \right) \leq 0, \quad \tilde{u} \left( \frac{Z^*_t}{\kappa_2} \right) - \tilde{u} \left( \frac{Z^*_t}{\kappa_2} \right) \leq 0, \quad \text{and} \quad J_R(Z^*_\tau^\epsilon) - J_R(Z^*_\tau^\epsilon) \leq 0. \]
Thus, it follows from Fatou’s lemma and (93) in (92) that
\[
0 \leq -\mathbb{E} \left[ \int_0^\xi \mathcal{H}_t(c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\xi = \tau^*, \tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{X}_{R}(Z_{\tau^*}^*) \right] + x,
\]
where we have used the fact that \( \lim_{\varepsilon \to 0} \hat{\tau}_\varepsilon = \tau^* \) and \( 0 \leq \xi \leq \tau^* \).

Therefore, we get
\[
\mathbb{E} \left[ \int_0^\xi \mathcal{H}_t(c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\xi = \tau^*, \tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{X}_{R}(Z_{\tau^*}^*) \right] \leq x. \tag{94}
\]
Since inequality (94) holds for any stopping time \( \xi \in \mathcal{S}_{\tau^*} \),
\[
\sup_{\xi \in \mathcal{S}_{\tau^*}} \mathbb{E} \left[ \int_0^\xi \mathcal{H}_t(c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\xi = \tau^*, \tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{X}_{R}(Z_{\tau^*}^*) \right] \leq x. \tag{95}
\]
By Proposition 3.1 (b), there exists a portfolio \( \pi^*_x \) such that
\[
(c^*, \pi^*, \zeta^*, \tau^*) \in \mathcal{A}(x)
\]
with \( X_{\tau^*} = \mathcal{X}_R(Z_{\tau^*}^*) \).

(c) By Lemma 4.2, we have
\[
x = -J'(y^*) = \mathbb{E} \left[ \int_0^{\tau^*} \mathcal{H}_t(c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{X}_{R}(Z_{\tau^*}^*) \right]
\]
\[
= \mathbb{E} \left[ \int_0^{\tau^*} \mathcal{H}_t \mathcal{D}_{t}^*(c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{D}_{\tau^*}^* \mathcal{X}_{R}(Z_{\tau^*}^*) \right].
\]
Note that
\[
x \leq \sup_{\xi \in \mathcal{S}_{\tau^*}} \mathbb{E} \left[ \int_0^\xi \mathcal{H}_t(c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\xi = \tau^*, \tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{X}_{R}(Z_{\tau^*}^*) \right] \leq x.
\]
That is,
\[
x = \sup_{\xi \in \mathcal{S}_{\tau^*}} \mathbb{E} \left[ \int_0^\xi \mathcal{H}_t(c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\xi = \tau^*, \tau^* < \infty\}} \mathcal{H}_{\tau^*} \mathcal{X}_{R}(Z_{\tau^*}^*) \right].
\]
On the other hand,
\[
xy^* = -y^* J'(y^*) = \mathbb{E} \left[ \int_0^{\tau^*} y^* \mathcal{H}_t \mathcal{D}_{t}^* (c_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\tau^* < \infty\}} y^* \mathcal{H}_{\tau^*} \mathcal{D}_{\tau^*}^* \mathcal{X}_{R}(Z_{\tau^*}^*) \right]
\]
\[
= \mathbb{E} \left[ \int_0^{\tau^*} e^{-\delta t} Z_t^* (\hat{c}_t^* - \hat{\iota}(Z_t^*)) dt + 1_{\{\tau^* < \infty\}} e^{-\delta \tau^*} Z_{\tau^*}^* \mathcal{X}_{R}(Z_{\tau^*}^*) \right]
\]
\[
= \mathbb{E} \left[ \int_0^{\tau^*} e^{-\delta t} \left( u(\kappa_1 c_t^*) 1_{\{\zeta_t^* = \theta_1\}} + u(\kappa_2 c_t^*) 1_{\{\zeta_t^* = \theta_2\}} \right) dt + 1_{\{\tau^* < \infty\}} \int_{\tau^*}^{\infty} e^{-\delta u} u(c_t^*) dt \right] - J(y^*). \]
Since \( J(y^*) + y^* x \geq \inf_{y > 0} (J(y) + y x) \), the weak duality in (28) implies that
\[
E \left[ \int_0^\tau e^{-\delta t} \left( u(\kappa_1 c_t^*) 1_{\{\zeta^*_t = B_1\}} + u(\kappa_2 c_t^*) 1_{\{\zeta^*_t = B_2\}} \right) dt + 1_{\{\tau^* < \infty\}} \int_{\tau^*}^\infty e^{-\delta t} u(c_t^*) dt \right] \geq \sup_{(c, \pi, \delta, \tau) \in \mathcal{A}(x)} E \left[ \int_0^\tau e^{-\delta t} \left( u(\kappa_1 c_t) 1_{\{\zeta_t = B_1\}} + u(\kappa_2 c_t) 1_{\{\zeta_t = B_2\}} \right) dt + 1_{\{\tau < \infty\}} \int_{\tau}^\infty e^{-\delta t} u(c_t) dt \right].
\]
Therefore, it follows from \((c^*, \pi^*, \zeta^*, \tau^*) \in \mathcal{A}(x)\) that
\[
V(x) = E \left[ \int_0^\tau e^{-\delta t} \left( u(\kappa_1 c_t^*) 1_{\{\zeta^*_t = B_1\}} + u(\kappa_2 c_t^*) 1_{\{\zeta^*_t = B_2\}} \right) dt + 1_{\{\tau^* < \infty\}} \int_{\tau^*}^\infty e^{-\delta t} u(c_t^*) dt \right].
\]
That is,
\[
V(x) = \inf_{y > 0} (J(y) + y x), \quad J(y) = \sup_{x > 0} (V(x) - y x)
\]
and \((c^*, \pi^*, \zeta^*, \tau^*) \in \mathcal{A}(x)\) is optimal. \(\square\)

**Remark 4.4.** The agent stays at \(B_2\) or switches from \(B_1\) to \(B_2\), a job with higher satisfaction before retirement if \(z_R < z_S < z_B\). If \(z_S \leq z_R\) or \(z_S \geq z_B\) the agent stays always at \(B_1\) or \(B_2\), respectively, before retirement.

## 5 Conclusion

We have studied the optimization problem of the job choice, retirement, consumption and portfolio selection of a borrowing constrained agent. We have derived a solution in concrete form by transforming the problem into a dual two-person zero-sum game. In this paper, we assume a constant investment opportunity, and consideration of a general market environment would be an interesting topic for future research.

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A Properties of $J_R(y)$

Proposition 4.1 in Knudsen et al. (1998) is useful to get the analytic properties of $J_R(y)$ as well as the closed form solution of $J_R(y)$. Thus, we provide a representation of the proposition in our notation:

**Proposition A.1** (Proposition 4.1 in Knudsen et al. (1998)). Let $g(y)$ be an arbitrary measurable function defined on $(0, \infty)$. Then the following conditions are equivalent:

(i) for every $y > 0$

$$E \left[ \int_0^\infty e^{-\delta t} |g(Y_t)| dt \right] < \infty,$$

(ii) for every $y > 0$

$$\int_0^y \eta^{-n_2-1} |g(\eta)| d\eta + \int_y^\infty \eta^{-n_1-1} |g(\eta)| d\eta < \infty.$$

Let us denote $\Xi_g(y)$ by

$$\Xi_g(y) = E \left[ \int_0^\infty e^{-\delta t} g(Y_t) dt \right].$$

Under the condition (i) or (ii), the following statements are true:

(a) $\lim_{y \downarrow 0} y^{-n_2}|g(y)| = \lim_{y \uparrow \infty} y^{-n_1}|g(y)| = 0$,

(b) $\Xi_g$ has a following form:

$$\Xi_g(y) = \frac{2}{\theta^2(n_1-n_2)} \left[ y^{n_2} \int_0^y \eta^{-n_2-1} g(\eta) d\eta + y^{n_1} \int_y^\infty \eta^{-n_1-1} g(\eta) d\eta \right],$$

(c) $\Xi_g$ is twice differentiable and

$$\frac{\theta^2}{2} y^2 \Xi_g''(y) + (\delta - r)y \Xi_g'(y) - \delta \Xi_g(y) + g(y) = 0,$$
(d) there exists a positive constant $C$ such that

\[ |\Xi_\eta'(y)| \leq C(y^{n_1-1} + y^{n_2-1}) \quad \text{for all } y > 0, \]

(e) \( \lim_{t \to \infty} e^{-\delta t} E[|\Xi_\eta(Y_t)|] = 0. \)

From Assumption 2.2,

\[ \E \left[ \int_0^\infty \mathcal{H}_t I(Y_t) dt \right] = \frac{1}{y} \E \left[ \int_0^\infty e^{-\delta t} \mathcal{Y}_t I(Y_t) dt \right] < \infty. \]

By Proposition A.1, we have

\[ \int_0^y \eta^{-n_2} I(\eta) d\eta + \int_y^\infty \eta^{-n_1} I(\eta) d\eta < \infty. \] (96)

Let us denote \( \mathcal{X}_R(y) \) by

\[ \mathcal{X}_R(y) = \E \left[ \int_0^\infty \mathcal{H}_t I(Y_t) dt \right] = \frac{1}{y} \E \left[ \int_0^\infty e^{-\delta t} \mathcal{Y}_t I(Y_t) dt \right]. \]

Since

\[ \mathcal{X}_R(y) = \E \left[ \int_0^\infty \mathcal{H}_t I(y e^{\delta t} I_t) dt \right], \]

the monotone convergence theorem and the dominated convergence theorem imply that

\[ \lim_{y \downarrow 0} \mathcal{X}_R(y) = \E \left[ \int_0^\infty \lim_{y \downarrow 0} \mathcal{H}_t I(y e^{\delta t} I_t) dt \right] = \infty \] (97)

and

\[ \lim_{y \uparrow \infty} \mathcal{X}_R(y) = \E \left[ \int_0^\infty \lim_{y \uparrow \infty} \mathcal{H}_t I(y e^{\delta t} I_t) dt \right] = 0. \] (98)

By Proposition A.1, we have

\[ \mathcal{X}_R(y) = \frac{2}{\theta^2 (n_1 - n_2)} \left[ y^{n_2-1} \int_0^y \eta^{-n_2} I(\eta) d\eta + y^{n_1-1} \int_y^{\infty} \eta^{-n_1} I(\eta) d\eta \right]. \]

and

\[ \liminf_{y \downarrow 0} y^{-n_2+1} I(y) = \liminf_{y \uparrow \infty} y^{-n_1+1} I(y) = 0. \]

By utilizing the integration by parts for Riemann-Stieltjes integral, we have

\[ \mathcal{X}_R'(y) = \frac{2}{\theta^2 (n_1 - n_2)} \left[ (n_2 - 1) y^{n_2-2} \int_0^y \eta^{-n_2} I(\eta) d\eta + (n_1 - 1) y^{n_1-2} \int_y^{\infty} \eta^{-n_1} I(\eta) d\eta \right] \]

\[ = \frac{2}{\theta^2 (n_1 - n_2)} \left[ y^{n_2-2} \liminf_{\eta \downarrow 0} (\eta^{1-n_2} I(\eta)) - y^{n_1-2} \liminf_{\eta \uparrow \infty} (\eta^{1-n_1} I(\eta)) + y^{n_2-2} \int_0^y \eta^{1-n_2} I(\eta) d\eta \right] \]

\[ + y^{n_1-2} \int_y^{\infty} \eta^{1-n_1} I(\eta) d\eta. \]

Since \( I(\eta) \) is strictly decreasing in \( 0 < \eta \leq u'(0) \), we deduce that \( \mathcal{X}_R(y) \) is strictly decreasing in \( y > 0 \), i.e.,

\[ \mathcal{X}_R'(y) < 0. \] (99)
Note that for any $y > 0$
\[ u(I(y)) - yI(y) = \tilde{u}(y) \text{ and } \tilde{u}'(y) = -I(y), \]
and thus
\[ u(I(\eta)) = u(I(y)) - yI(\eta) + \eta I(\eta) + \int_0^\eta I(\eta)d\eta. \]

Since $E \left[ \int_0^\infty e^{-\delta t} \mathcal{Y}_t I(\mathcal{Y}_t)dt \right] < \infty$, it follows from Proposition A.1 that
\begin{align*}
\int_0^\eta \eta^{-n_2-1}|u(I(\eta))|d\eta + \int_0^\infty \eta^{-n_1-1}|u(I(\eta))|d\eta \\
&\leq -\frac{1}{n_2} \eta^{-n_2}|u(I(y)) - yI(y)| + \int_0^y \eta^{-n_2}I(\eta)d\eta + \int_0^\eta \int_0^\eta \eta^{-n_2-1}I(\eta)d\eta d\eta \\
&\quad + \frac{1}{n_1} \eta^{-n_1}|u(I(y)) - yI(y)| + \int_y^\infty \eta^{-n_1}I(\eta)d\eta + \int_y^\infty \int_y^\infty \eta^{-n_1-1}I(\eta)d\eta d\eta
\end{align*}
(100)

where we have used Fubini’s theorem in last equality.

From (96) and (100),
\[ \int_0^\eta \eta^{-n_2-1}|\tilde{u}(\eta)|d\eta + \int_0^\infty \eta^{-n_1-1}|\tilde{u}(\eta)|d\eta < \infty. \]
(101)

By Proposition A.1, we deduce that
\[ J_R(y) = \frac{2}{\theta^2(n_1 - n_2)} \left[ \eta^{n_2} \int_0^y \eta^{-n_2-1} \tilde{u}(\eta)d\eta + \eta^{n_1} \int_y^\infty \eta^{-n_1-1} \tilde{u}(\eta)d\eta \right] \]
(102)

and
\[ \frac{\theta^2}{2} \eta^2 J_R''(y) + (\delta - r) y J_R'(y) - \delta J_R + \tilde{u}(y) = 0. \]
(103)

Moreover, it follows from (101) that
\[ \liminf_{y \downarrow 0} y^{-n_2}|\tilde{u}(y)| = \liminf_{y \uparrow \infty} y^{-n_1}|\tilde{u}(y)| = 0. \]

Therefore, the integration by parts implies that
\begin{align*}
J_R'(y) &= \frac{2}{\theta^2(n_1 - n_2)} \left[ n_2y^{n_2-1} \int_0^y \eta^{-n_2-1} \tilde{u}(\eta)d\eta + n_1 \eta^{n_1-1} \int_y^\infty \eta^{-n_1-1} \tilde{u}(\eta)d\eta \right] \\
&= -\frac{2}{\theta^2(n_1 - n_2)} \left[ n_2y^{n_2-1} \int_0^y \eta^{-n_2}I(\eta)d\eta + n_1 \eta^{n_1-1} \int_y^\infty \eta^{-n_1}I(\eta)d\eta \right] = -\mathcal{X}_R(y).
\end{align*}
(104)

From (99), we have $J_R''(y) = -\mathcal{X}_R'(y) > 0$. That is, $J_R(y)$ is strictly convex in $y > 0$. 

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