INITIAL PARTICLE INSTABILITY IN MUON COLLISIONS

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Abstract

I consider the process $\mu^+\mu^- \rightarrow e\bar{\nu}W^+$ in the case when the effective mass of the $(e\bar{\nu})$ system than the muon mass. In this case the momentum transferred from the initial muon to the $e\bar{\nu}$ system (the virtual neutrino momentum) can be both time–like and space–like. Since the path of integration over $k^2$ goes through a pole at $k^2 = 0$, it gives a divergent cross section.

In the ideal case of large enough beams this divergence disappears if the finite width $\Gamma$ of the initial muon is taken into account. The obtained cross section corresponds to the flux of equivalent neutrino, which coincides with that of muon (with some energy distribution).

In practice, the effect of final size of the muon beam reduces this cross section very strong, and the effect is hardly observable.

Recently, muon collisions have been proposed as the next step for high–energy colliders (see e.g. [1]). This idea provides a problem:

To find (if possible) the point where the muon collisions differs substantially from the electron ones. My first impression was: I found this point; it relates to the muon instability, which gives really new option to consider muon collider as neutrino collider simultaneously. The subsequent studies shows that the discussed effect is small in practice, the basic problem has a negative solution for the muon collider project. Nevertheless, the discussed problem seems to be important for the particle theory. The first part of the discussion below reproduces the paper [2], the final result for the ”realistic” beams is given from ref. [3].

1 The problem

We discuss the effects of the muon instability for the process

$$\mu^- (p_1)\mu^+ (p_2) \rightarrow e (q_1)\bar{\nu} (q_2)W^+(p_3).$$

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where the momenta of the particles are shown in brackets. We use the following notation: $M$ is the $W$ boson mass, $m$ is the muon mass, $s \equiv 4E^2 = (p_1 + p_2)^2$; $x = M^2/s$; $q = q_1 + q_2$, $k = p_1 - q \equiv p_3 - p_2$ and we neglect the electron mass.

1. We study the specific kinematical region where the effective mass of the $e\bar{\nu}$ system is less than the mass of the muon, $q^2 < m^2$. In this case the transferred momentum $k$ can be time–like, the maximal value of $k^2$ is positive:

$$k^2 \leq t_{\text{max}} = \frac{x}{1-x} \left[ m^2(1-x) - q^2 \right] > 0.$$ (2)

With the increase of the total transverse momentum of the produced system, this momentum–transfer becomes space–like, $k^2 < 0$. Therefore the integration over this transverse momentum (at fixed $q$) goes through the point $k^2 = 0$.

The main contribution to the cross section in this region is due to the diagram of Fig. 1, where the neutrino $t$–channel exchange. It gives a factor $(k^2)^{-2}$ in the matrix element squared. The standard integration over $k^2$ results in a divergent cross section in this case! (This longstanding problem has got no satisfactory solution till now (see [4, 5, 6], for recent review see [7]).)

![Figure 1](image)

Figure 1: The diagram considered ($q^2 < \mu^2$)

This paradox originates from the instability of the muon, decaying into the $e\bar{\nu}\nu$ system: the point $k^2 = 0$, $q^2 < m^2$ is within the physical region for this decay.

2 Solution for the ideal case, large enough beams

First, we neglect the beam size effects. In this case, the above divergence is eliminated if one takes into account the fact that, because the muon is instable, the wave function differs from the standard plane wave. To obtain Lorentz covariant solution, we start from the muon rest frame

$$e^{-imt/\hbar} \Rightarrow e^{-i(m-i\Gamma/2)t/\hbar}. \quad (3)$$

In this frame the 4-momentum of $\mu^-$ is $\tilde{p}_1 t = (m - i\Gamma/2, 0, 0, 0)$. To obtain the energy of the produced $e\bar{\nu}$ system, $\tilde{q}^0$, in this frame, we use the simple kinematical relations $2p_1q = m^2 + q^2 - k^2$, $2p_1q \equiv 2mq^0$, which give $\tilde{q}^0 = (m^2 + q^2 - k^2)/2m$. \footnote{This way of deriving $k_{\text{new}}^2$ was proposed by V.G. Serbo.}
The new value of $k^2$ is obtained from the relation $k^2_{\text{new}} \equiv (p_1 - q)^2 = m^2 - i m \Gamma + q^2 + i \Gamma \tilde{q}^0$. Using the value of $\tilde{q}^0$ given previously, we obtain

\[ k^2 \Rightarrow k^2 - i \gamma ; \quad \gamma = \frac{m \Gamma (m^2 - q^2)}{2m^2}. \tag{4} \]

One can now calculate the cross section of the process in the standard way. The calculation is simplified since one can neglect the small quantities $\sim m^2/M^2$, $\Gamma/m$ in the result. We finally obtain:

\[ d\sigma = \frac{|M|^2 dk^2 dq^2}{4(4\pi^3 s(s - 4m^2))} \frac{d\Omega_q}{4\pi} \frac{d\varphi}{2\pi}, \]
\[ |M|^2 = \frac{(4\pi \alpha)^3}{M^2} \frac{(2q_1 p_1)(2k q_2)}{k^4 + \gamma^2}. \tag{5} \]

where $d\Omega_q^*$ is the solid angle element in the center of mass of the produced $e\bar{\nu}$ system.

The subsequent angular integration is trivial. The integration over $k^2$ gives $\pi/\gamma$ for any $q^2$, since the bounds of the integration region are much higher than $\gamma$. The integration over $q^2$ covers the region $q^2 < m^2(1 - x)$ \(\text{(5)}\). This procedure is very close to the calculation of the muon decay width. We insert this width instead of the corresponding combination of factors in eq. \(\text{(5)}\), and it compensates the factor $\Gamma$ in the denominator; the final result is then:

\[ \sigma = \frac{\pi^2 \alpha}{s \sin^2 \Theta_W} f(x) \equiv 20 x f(x) \text{ nb}; \tag{6} \]
\[ f(x) = 4 x (1 - x)(2 - x). \tag{7} \]

This equation solves the discussed problem in the ideal case: with the above prescription we obtain a finite cross section. It is the final result for the description of hadron collisions with fast decay of one of collided particles.

But the obtained quantities correspond to the integration over the whole space–time irrespective to the size of interaction region. The effective spatial scale of the considered phenomena is $c \tau$ where $\tau = \hbar/\Gamma$ is the muon time of life. In reality, the size of beam is much less, and only small fraction of $\nu$ interacts. This very scale regularize the cross section, the effect of muon instability manifests itself only in the existence of region under interest with the possible time–like momentum of exchanged neutrino, the imaginary part of the muon mass become irrelevant to the observed phenomenon.

### 3 The finite size effect. Basic equations

In the subsequent calculations we neglect the muon instability in its mass, since considered sizes of bunch are very small. This approach is justified by the finiteness of the observed result.

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\(^2\) In our approach, both the energy and 3–momentum of muon have imaginary parts in the lab. frame. The idea of ref. \(\text{[4]}\) looks like similar to that, discussed here. The ansatz proposed can be written as $\gamma = m \Gamma (1 - x)$. To obtain this value, one should assume 3–momentum of muon to be real in the lab. system, and calculate its energy with complex mass. This approach is evidently not Lorentz covariant.
The calculations are based on the method, developed in refs. 8 (see also 2). We should take into account, that the wave functions of muons in the initial beams are no plane waves but wave packets with some distribution over momenta (the effect of complex muon mass is hidden within this distribution):

\[ |p_i \rangle \rightarrow \int \frac{d^3 P_i}{(2\pi)^{3/2}} |P_i \rangle \quad (i = 1, 2). \]  

When calculate cross section, we summarize over final states. Therefore, we can use here arbitrary whole set of states, in particular, plane waves \(|q_i \rangle, |p_3 \rangle\).

Therefore, the matrix element squared \(|\mathcal{M}|^2\) is expressed via the standard matrix elements in the momentum representation as

\[
|\mathcal{M}|^2 = \int \frac{d^3 P_1 d^3 P'_1 d^3 P_2 d^3 P'_2}{(2\pi)^6} M(P_1, P_2; q_1, q_2, p_3) M^*(P'_1, P'_2; q_1, q_2, p_3) \delta(P_1 + P_2 - q_1 - q_2 - p_3) \delta(P_1 + P_2 - P'_1 - P'_2).
\]

We can write the identity:

\[
2\pi \delta(\sum P_i - \sum P'_i) = \delta(\sum \vec{P}_i - \sum \vec{P}'_i) \int dt \exp[it(\sum \epsilon_i - \sum \epsilon'_i)] \quad (\epsilon_i \equiv P_i^0).
\]

Then the phase averaging results in density matrices for the muons in the beams:

\[
\Phi(P_i)\Phi(P'_i) \exp[it(\epsilon_i - \epsilon'_i)] \Rightarrow \rho(\vec{P}_i, \vec{P}'_i, t).
\]

Next, it is useful to go to the mixed representation of the density matrix — Wigner function \(n(p, r, t)\):

\[
\rho(\vec{P}_i, \vec{P}'_i, t) \frac{d^3 P_i d^3 P'_i}{(2\pi)^3/2} = \int n(\vec{p}_i, \vec{r}_i, t) e^{2i\vec{p}_i\vec{r}_i} \frac{d^3 p_i d^3 l_i d^3 r_i}{(2\pi)^3/2} \left( p_i = \frac{P_i + P'_i}{2}, l_i = \frac{P_i - P'_i}{2} \right).
\]

In the quasi–classical limit, which realized for particles in beam, this Wigner function coincides with the density in the phase space. That is the point, in which known distributions of particles within beams enter into the result.

We see, that the identical final state is obtained from the different (in plane wave language) initial states of all initial particles and they give different values of transferred momenta for the diagram and the conjugated one. Therefore, when calculate probability \(P \propto |\mathcal{M}|^2\), we obtain instead of (2)

\[
P \propto \int n(\vec{p}_1, \vec{r}_1, t)n(\vec{p}_2, \vec{r}_2, t) \frac{e^{2i\vec{p}\vec{r}}}{(k - l)^2(k + l)^2} d^3 r d^3 l d^2 k_\perp.
\]

4 Final result for realistic beams

The above integration was performed in the explicit form in ref. 3. The qualitative explanation of result was given by G.L. Kotkin. The final result is written via the transversal size of the beam \(a\) and muon time of life \(\tau\) in the form like eq. (3):

\[
\sigma_{eff} = \frac{a}{c\tau} \frac{g(x)}{\sigma_0}; \sigma_0 = 20 \text{ nb};
\]

\[
g(x) = \frac{12}{5} x \sqrt{x(1 - x)} \left( 1 + \frac{22}{9} x - \frac{16}{9} x^2 \right).
\]
For $a = 3\mu m$, the maximal value of this "cross section" about 0.3 fb at $\sqrt{s} \approx 100$ GeV.

Therefore, for the realistic beams, the instability in the muon mass is invisible in the result, the effect of the finite beam size is dominant.

5 Conclusion

The main conclusions are:

- It is necessary to consider the distinction of the initial state of unstable particle from the plane wave, to eliminate the t-channel singularity discussed. Depending of the problem considered, the dominant regularizing effect is due to either complex mass of unstable particle or beam sizes. There is no theory now to consider intermediate case.

- The discussed effect has no practical meaning for the muon collider.

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