ASYMPTOTIC BEHAVIOR OF FLOWS BY POWERS OF THE GAUSSIAN CURVATURE

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Abstract. We consider a one-parameter family of strictly convex hypersurfaces in \( \mathbb{R}^{n+1} \) moving with speed \(-K^\alpha \nu\), where \( \nu \) denotes the outward-pointing unit normal vector and \( \alpha \geq \frac{1}{n+2} \). For \( \alpha > \frac{1}{n+2} \), we show that the flow converges to a round sphere after rescaling. In the affine invariant case \( \alpha = \frac{1}{n+2} \), our arguments give an alternative proof of the fact that the flow converges to an ellipsoid after rescaling.

1. Introduction

Parabolic flows for hypersurfaces play an important role in differential geometry. One most fundamental example is the flow by mean curvature (see \[15\]). In this paper, we consider flows where the speed is given by some power of the Gaussian curvature. More precisely, given \( \alpha > 0 \), a one-parameter family of immersions \( F : M^n \times [0,T) \to \mathbb{R}^{n+1} \) is a solution of the \( \alpha \)-Gauss curvature flow, if for each \( t \in [0,T) \), \( F(M^n,t) = \Sigma_t \) is a complete convex hypersurface in \( \mathbb{R}^{n+1} \), and \( F(\cdot,t) \) satisfies

\[
\frac{\partial}{\partial t} F(p,t) = -K^\alpha(p,t) \nu(p,t).
\]

Here, \( K(p,t) \) and \( \nu(p,t) \) are the Gauss curvature and the outward pointing unit normal vector of \( \Sigma_t \) at the point \( F(p,t) \), respectively.

Theorem 1. Let \( \Sigma_t \) be a family of closed, strictly convex hypersurfaces in \( \mathbb{R}^{n+1} \) moving with speed \(-K^\alpha \nu\), where \( \alpha \geq \frac{1}{n+2} \). Then either the hypersurfaces \( \Sigma_t \) converge to a round sphere after rescaling, or we have \( \alpha = \frac{1}{n+2} \) and the hypersurfaces \( \Sigma_t \) converge to an ellipsoid after rescaling.

Tso \[21\] showed that the flow exists up to some maximal time, when the enclosed volume converges to 0. The convergence result for \( \alpha = \frac{1}{n+2} \) is due to Chow \[8\]. In \[9\], Chow obtained interesting entropy and Harnack inequalities for flows by powers of the Gaussian curvature (see also \[14\]). In the affine invariant case \( \alpha = \frac{1}{n+2} \), Andrews \[1\] showed that the flow converges to an ellipsoid. This result can alternatively be derived from Calabi’s classification of self-similar solutions \[6\]. In the special case of surfaces in \( \mathbb{R}^3 \) (\( n = 2 \)), Andrews \[1\] proved the convergence of the flow for \( \alpha = 1 \); this was extended to \( \alpha \in [\frac{1}{2},2] \) in \[14\]. Moreover, it is known that the flow converges to a self-similar solution for every \( \alpha \geq \frac{1}{n+2} \). This was proved by Andrews \[3\] for \( \alpha \in [\frac{1}{n+2},\frac{1}{n}] \), by Guan and Ni \[13\] for \( \alpha = 1 \), by Kim and
Lee [17] for $\alpha \in (\frac{1}{n}, 1]$, and by Andrews, Guan, and Ni [5] in the remaining case $\alpha \in (1, \infty)$.

Thus, the problem can be reduced to the classification of self-similar solutions. In the affine invariant case $\alpha = \frac{1}{n+2}$, the self-similar solutions were already classified by Calabi [6]. In the special case when $\alpha \geq 1$ and the hypersurfaces are invariant under antipodal reflection, it was shown in [5] that the only self-similar solutions are round spheres. Very recently, the case $\frac{1}{n} \leq \alpha < 1 + \frac{1}{n}$ was solved in [7] as part of K. Choi’s PhD thesis. In particular, this includes the case $\alpha = 1$ conjectured by Firey [12].

Finally, we note that there is a substantial literature on other fully nonlinear parabolic flows for hypersurfaces; see e.g. [10], [16], [19], [20]. There also are interesting connections between the Gauss curvature flow and the cross curvature flow introduced by Chow and Hamilton [11]. Indeed, if we have a one-parameter family of hypersurfaces in $\mathbb{R}^4$ evolving by Gauss curvature flow ($\alpha = 1$), then the induced metrics evolve by cross curvature flow. Thus, the cross curvature flow can be viewed as an analogue of Gauss curvature flow in the intrinsic setting.

We now give an outline of the proof of Theorem 1. In view of results in [3], [5], [13], and [17], it suffices to classify all the closed self-similar solutions to the flow. The self-similar solutions $\Sigma = F(M^n)$ satisfy

\((\ast_\alpha)\) \[ K^\alpha = \langle F, \nu \rangle. \]

To classify the solutions of \((\ast_\alpha)\), we distinguish two cases.

First, suppose that $\alpha \in \left[\frac{1}{n+2}, \frac{1}{2}\right]$. In this case, we consider the quantity

\[ Z = K^\alpha \text{tr}(b) - \frac{n\alpha - 1}{2\alpha} |F|^2, \]

where $b$ denotes the inverse of the second fundamental form. The motivation for the quantity $Z$ is that $Z$ is constant when $\alpha = \frac{1}{n+2}$ and $\Sigma$ is an ellipsoid. Indeed, if $\alpha = \frac{1}{n+2}$ and $\Sigma = \{ x \in \mathbb{R}^{n+1} : \langle Sx, x \rangle = 1 \}$ for some positive definite matrix $S$ with determinant 1, then $K^{\frac{1}{n+2}} = \langle F, \nu \rangle$ and

\[ Z = K^{\frac{1}{n+2}} \text{tr}(b) + |F|^2 = \text{tr}(S^{-1}). \]

Hence, in this case $Z$ is constant, and equals the sum of the squares of the semi-axes of the ellipsoid.

Suppose now that $\Sigma = F(M^n)$ is a solution of \((\ast_\alpha)\) for some $\alpha \in \left[\frac{1}{n+2}, \frac{1}{2}\right]$. We show that $Z$ satisfies an inequality of the form

\[ \alpha K^\alpha b^{ij} \nabla_i \nabla_j Z + (2\alpha - 1)b^{ij} \nabla_i K^\alpha \nabla_j Z \geq 0. \]

The strong maximum principle then implies that $Z$ is constant. By examining the case of equality, we are able to show that either $\nabla h = 0$ or $\alpha = \frac{1}{n+2}$ and the cubic form vanishes. This shows that either $\Sigma$ is a round sphere, or $\alpha = \frac{1}{n+2}$ and $\Sigma$ is an ellipsoid.
Finally, we consider the case \( \alpha \in (\frac{1}{2}, \infty) \). As in [7], we consider the quantity
\[
W = K^\alpha \lambda_1^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2.
\]
By applying the maximum principle, we can show that any point where \( W \) attains its maximum is umbilic. From this, we deduce that any maximum point of \( W \) is also a maximum point of \( Z \). Applying the strong maximum principle to \( Z \), we are able to show that \( Z \) and \( W \) are both constant. This implies that \( \Sigma \) is a round sphere.

2. Preliminaries

We first recall the notation:

- The metric is given by \( g_{ij} = \langle F_i, F_j \rangle \), where \( F_i := \nabla_i F \), and its inverse matrix \( g^{ij} \) of \( g_{ij} \), namely \( g^{ij} g_{jk} = \delta_i^k \). Also, we use the notation \( F_i = g^{ij} F_j \).
- We denote by \( H \) and \( h_{ij} \) the mean curvature and second fundamental form, respectively.
- For a strictly convex hypersurface, we denote by \( b_{ij} \) the inverse of the second fundamental form \( h_{ij} \), so that \( b_{ij} h_{jk} = \delta_i^k \). Moreover, \( \text{tr}(b) \) will denote the trace of \( b \), i.e. the reciprocal of the harmonic mean curvature.
- We denote by \( L \) the operator \( L = \alpha K^\alpha b^{ij} \nabla_i \nabla_j \).
- We denote by \( C_{ijk} \) the cubic form
\[
C_{ijk} = \frac{1}{2} K^{-\frac{1}{n+2}} \nabla_i K^{-\frac{1}{n+2}} + \frac{1}{2} h_{ij} \nabla K^{-\frac{1}{n+2}} + \frac{1}{2} h_{ij} \nabla K^{-\frac{1}{n+2}}.
\]

We next derive some basic equations.

**Proposition 2.** Given a strictly convex smooth solution \( F : M^n \to \mathbb{R}^{n+1} \) of \( (\ast) \), the following equations hold:

1. \( \nabla_i b^{jk} = -b^{ij} b^{km} \nabla_i h_{lm} \),
2. \( \mathcal{L} |F|^2 = 2\alpha K^\alpha b^{ij} (g_{ij} - h_{ij} K^\alpha) = 2\alpha K^\alpha \text{tr}(b) - 2n\alpha K^{2\alpha} \),
3. \( \nabla_i K^\alpha = h_{ij} \langle F, F_j \rangle \),
4. \( \mathcal{L} K^\alpha = \langle F, F_i \rangle \nabla_i K^\alpha + n\alpha K^\alpha - \alpha K^{2\alpha} H \),
5. \( \mathcal{L} h_{ij} = -K^{-\alpha} \nabla_i K^\alpha \nabla_j K^\alpha + \alpha K^\alpha b^{ps} b^{is} \nabla_i h_{rs} \nabla_j h_{pq} \langle F, F_k \rangle \nabla^k h_{ij} + h_{ij} + (n\alpha - 1) h_{ik} h_{kj} K^\alpha - \alpha K^\alpha H h_{ij} \),
6. \( \mathcal{L} b^{pq} = K^{-\alpha} b^{pr} b^{qs} \nabla_i K^\alpha \nabla_s K^\alpha + \alpha K^\alpha b^{ps} b^{iq} b^{km} \nabla_i h_{ik} \nabla_s h_{jm} + \langle F, F_i \rangle \nabla_i b^{pq} - b^{pq} - (n\alpha - 1) g^{pq} K^\alpha + \alpha K^\alpha H b^{pq} \).
Applying (5), we conclude that $h_{kl} \nabla_i b^{jk} = \nabla_i b^{jk} = 0$ gives $h_{kl} \nabla_i b^{jk} = -b^{jk} \nabla_i h_{kl}$. This directly implies (1). We next compute
\[
\nabla_i \nabla_j |F|^2 = 2 \langle \nabla_i F, \nabla_j F \rangle + 2 \langle \nabla_i \nabla_j F, F \rangle = 2 g_{ij} - 2 h_{ij} \langle F, \nu \rangle = 2 g_{ij} - 2 K^\alpha h_{ij},
\]
hence
\[
\mathcal{L} |F|^2 = 2 \alpha K^\alpha \text{tr}(b) - 2 n \alpha K^{2\alpha}.
\]
This proves (2).

To derive the equation (3), we differentiate (1):
\[
\nabla_i K^\alpha = h_{ik} \langle F, F^k \rangle.
\]
If we differentiate this equation again, we obtain
\[
\nabla_i \nabla_j K^\alpha = \langle F, F^k \rangle \nabla_i h_{jk} + h_{ij} - h_{ik} h_{j}^k \langle F, \nu \rangle = \langle F, F^k \rangle \nabla_k h_{ij} + h_{ij} - K^\alpha h_{ik} h_{j}^k;
\]
hence
\[
\mathcal{L} K^\alpha = \langle F, F^k \rangle \nabla_k K^\alpha + n \alpha K^\alpha - \alpha K^{2\alpha} H.
\]
On the other hand, using (1) we compute
\[
\nabla_i \nabla_j K^\alpha = \nabla_i (\alpha K^\alpha b^{pq} \nabla_j h_{pq}) = \alpha K^\alpha b^{pq} \nabla_i \nabla_j h_{pq} + \alpha^2 K^\alpha b^{rs} b^{pq} \nabla_i h_{rs} \nabla_j h_{pq} - \alpha K^\alpha b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq}.
\]
Using the commutator identity
\[
\nabla_i \nabla_j h_{pq} = \nabla_i \nabla_p h_{jq} = \nabla_p \nabla_i h_{jq} + R_{ipjm} h_{q}^m + R_{ipqm} h_{j}^m = \nabla_p \nabla_q h_{ij} + (h_{ij} h_{pm} - h_{im} h_{jp}) h_{q}^m + (h_{iq} h_{pm} - h_{im} h_{pq}) h_{j}^m,
\]
we deduce that
\[
\alpha K^\alpha b^{pq} \nabla_i \nabla_j h_{pq} = \alpha K^\alpha b^{pq} \nabla_p \nabla_q h_{ij} + \alpha K^\alpha H h_{ij} - \alpha K^\alpha h_{im} h_{j}^m = \mathcal{L} h_{ij} + \alpha K^\alpha H h_{ij} - \alpha K^\alpha h_{im} h_{j}^m.
\]
Combining the equations above yields
\[
\mathcal{L} h_{ij} = -\alpha^2 K^\alpha b^{rs} b^{pq} \nabla_i h_{rs} \nabla_j h_{pq} + \alpha K^\alpha b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq} + \langle F, F^k \rangle \nabla_k h_{ij} + h_{ij} + (n \alpha - 1) h_{ik} h_{j}^k - K^\alpha h_{ij} - \alpha K^\alpha H h_{ij}.
\]
This completes the proof of (3).

Finally, using (1), we obtain
\[
\mathcal{L} b^{pq} = \alpha K^\alpha b^{ij} \nabla_i \nabla_j (-b^{pq} b^{rs} \nabla_j h_{rs}) = 2 \alpha K^\alpha b^{ij} b^{pq} b^{rs} \nabla_i h_{km} \nabla_j h_{rs} - b^{pq} b^{rs} \mathcal{L} h_{rs}.
\]
Applying (3), we conclude that
\[
\mathcal{L} b^{pq} = \alpha^2 \alpha K^\alpha b^{pq} b^{ij} b^{km} \nabla_i h_{ij} \nabla_s h_{km} + \alpha K^\alpha b^{pr} b^{qs} b^{ij} b^{km} \nabla_i h_{ik} \nabla_s h_{jm} + \langle F, F^k \rangle \nabla_k h^{pq} - b^{pq} - (n \alpha - 1) g^{pq} K^\alpha + \alpha K^\alpha H b^{pq}.
\]
Since $\nabla K^\alpha = \alpha K^\alpha b^{ij} \nabla h_{ij}$, the identity (4) follows.
3. Classification of self-similar solutions: The case $\alpha \in [\frac{1}{n+2}, \frac{1}{2}]$

**Lemma 3.** Assume $\alpha \in [\frac{1}{n+2}, \frac{1}{2}]$, $\lambda_1, \ldots, \lambda_n$ are positive real numbers, and $\sigma_1, \ldots, \sigma_n$ are real numbers. Suppose that $I(n, \alpha, \lambda_1, \ldots, \lambda_n, \sigma_1, \ldots, \sigma_n)$ is a real-valued function defined by

$$
I = \sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i=1}^{n-1} \lambda_i \lambda_i^{-1} \sigma_i^2
- 4\alpha \lambda_n \left[ \sum_{i=1}^{n} \lambda_i^{-1} \sigma_i + \left( (n\alpha - 1)\lambda_n^{-1} - \alpha \sum_{i=1}^{n} \lambda_i^{-1} \right) \sum_{i=1}^{n} \sigma_i \right] \left( \sum_{i=1}^{n} \sigma_i \right)
- 2\alpha^2 \lambda_n \left( \sum_{i=1}^{n} \lambda_i^{-1} \right) \left( \sum_{i=1}^{n} \sigma_i \right)^2 + \left[ 2n\alpha^2 + (n-1)\alpha - 1 \right] \left( \sum_{i=1}^{n} \sigma_i \right)^2.
$$

Then

$$
I(n, \alpha, \lambda_1, \ldots, \lambda_n, \sigma_1, \ldots, \sigma_n) \geq 0.
$$

Moreover, if equality holds, then we either have $\sigma_1 = \ldots = \sigma_n = 0$ or we have $\alpha = \frac{1}{n+2}$ and $\sigma_1 = \ldots = \sigma_{n-1} = \frac{1}{3} \sigma_n$.

**Proof.** If $\sum_{i=1}^{n} \sigma_i = 0$, the assertion is trivial. Hence, it suffices to consider the case $\sum_{i=1}^{n} \sigma_i \neq 0$. By scaling, we may assume $\sum_{i=1}^{n} \sigma_i = 1$. Let us define

$$
\tau_i = \begin{cases} 
\sigma_i - \alpha & \text{for } i = 1, \ldots, n-1 \\
\sigma_n - 1 + (n-1)\alpha & \text{for } i = n.
\end{cases}
$$

Then

$$
\sum_{i=1}^{n} \tau_i = \sum_{i=1}^{n} \sigma_i - 1 = 0
$$

and

$$
\sum_{i=1}^{n} \lambda_i^{-1} \tau_i = \sum_{i=1}^{n} \lambda_i^{-1} \sigma_i + (n\alpha - 1)\lambda_n^{-1} - \alpha \sum_{i=1}^{n} \lambda_i^{-1}.
$$
Therefore, the function $I$ satisfies

\begin{align*}
I &= \sum_{i=1}^{n-1} (\tau_i + \alpha)^2 + (\tau_n + 1 - (n - 1)\alpha)^2 + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} (\tau_i + \alpha)^2 \\
&\quad - 4\alpha \sum_{i=1}^{n} \lambda_n \lambda_i^{-1} \tau_i - 2\alpha^2 \sum_{i=1}^{n} \lambda_n \lambda_i^{-1} + 2n\alpha^2 + (n - 1)\alpha - 1 \\
&= \sum_{i=1}^{n} \tau_i^2 + 2\alpha \sum_{i=1}^{n-1} \tau_i + 2(1 - (n - 1)\alpha)\tau_n + (n - 1)\alpha^2 + (1 - (n - 1)\alpha)^2 \\
&\quad + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} (\tau_i^2 + 2\alpha \tau_i + \alpha^2) - 4\alpha \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \tau_i - 4\alpha \tau_n \\
&\quad - 2\alpha^2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} - 2\alpha^2 + 2n\alpha^2 + (n - 1)\alpha - 1 \\
&= \sum_{i=1}^{n} \tau_i^2 + 2\alpha \sum_{i=1}^{n-1} \tau_i + 2(1 - (n + 1)\alpha)\tau_n + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \tau_i^2 \\
&\quad + (n - 1)\alpha((n + 2)\alpha - 1).
\end{align*}

Using the identity $\sum_{i=1}^{n} \tau_i = 0$, we obtain

\begin{align*}
I &= \sum_{i=1}^{n} \tau_i^2 + 2(1 - (n + 2)\alpha)\tau_n + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \tau_i^2 + (n - 1)\alpha((n + 2)\alpha - 1).
\end{align*}

Moreover, the identity $\sum_{i=1}^{n} \tau_i = 0$ yields

\begin{align*}
\sum_{i=1}^{n} \tau_i^2 &= \sum_{i=1}^{n-1} \left(\tau_i + \frac{1}{n-1} \tau_n\right)^2 - \frac{2}{n-1} \tau_n \sum_{i=1}^{n-1} \tau_i + \frac{n-2}{n-1} \tau_n^2 \\
&= \sum_{i=1}^{n-1} \left(\tau_i + \frac{1}{n-1} \tau_n\right)^2 + \frac{n}{n-1} \tau_n^2.
\end{align*}

Thus,

\begin{align*}
I &= \frac{n}{n-1} \left(\tau_n + \frac{n-1}{n} (1 - (n + 2)\alpha)\right)^2 + \sum_{i=1}^{n-1} \left(\tau_i + \frac{1}{n-1} \tau_n\right)^2 \\
&\quad + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \tau_i^2 + \frac{n-1}{n} (1 - 2\alpha)((n + 2)\alpha - 1).
\end{align*}

The right hand side is clearly nonnegative. Moreover, if equality holds, then $\tau_1 = \ldots = \tau_n = 0$, and $\alpha = \frac{1}{n+2}$. This proves the lemma.
Theorem 4. Assume $\alpha \in \left[\frac{1}{n+2}, \frac{1}{2}\right]$ and $\Sigma$ is a strictly convex closed smooth solution of \((r, \alpha)\). Then, either $\Sigma$ is a round sphere, or $\alpha = \frac{1}{n+2}$ and $\Sigma$ is an ellipsoid.

Proof. Taking the trace in equation \((3)\) gives
\[
\mathcal{L} \operatorname{tr}(b) = K^{-\alpha} b^{pr} b^p r_i K^\alpha \nabla_s K^\alpha + \alpha K^\alpha b^{pr} b^p r_i b^km \nabla_r h_{ik} \nabla_s h_{jm} + \langle F, F_i \rangle \nabla^i \operatorname{tr}(b) - \operatorname{tr}(b) - n(\alpha - 1)K^\alpha + \alpha K^\alpha H \operatorname{tr}(b).
\]
Using equation \((2)\), we obtain
\[
\mathcal{L} (K^\alpha \operatorname{tr}(b)) = K^\alpha \mathcal{L} \operatorname{tr}(b) + \mathcal{L} K^\alpha \operatorname{tr}(b) + 2\alpha K^\alpha b^{ij} \nabla_i K^\alpha \nabla_j \operatorname{tr}(b)
\]
\[
\begin{align*}
&= b^{pr} b^p r_i K^\alpha \nabla_s K^\alpha + \alpha K^{2\alpha} b^{pr} b^p r_i b^km \nabla_r h_{ik} \nabla_s h_{jm} + \langle F, F_i \rangle \nabla^i (K^\alpha \operatorname{tr}(b)) + (n\alpha - 1)K^\alpha \operatorname{tr}(b) - n(\alpha - 1)K^{2\alpha} \\
&\quad + 2\alpha K^\alpha b^{ij} \nabla_i K^\alpha \nabla_j \operatorname{tr}(b).
\end{align*}
\]
Using \((2)\), it follows that the function
\[
Z = K^\alpha \operatorname{tr}(b) - \frac{n\alpha - 1}{2\alpha} |F|^2
\]
satisfies
\[
\mathcal{L} Z = b^{pr} b^p r_i K^\alpha \nabla_s K^\alpha + \alpha K^{2\alpha} b^{pr} b^p r_i b^km \nabla_r h_{ik} \nabla_s h_{jm} + \langle F, F_i \rangle \nabla^i (K^\alpha \operatorname{tr}(b)) + 2\alpha K^\alpha b^{ij} \nabla_i K^\alpha \nabla_j \operatorname{tr}(b).
\]
Differentiating \((3)\) yields
\[
\frac{1}{2} \nabla^i |F|^2 = (F, F_i) = b^{ij} h_{jk} \langle F, F^k \rangle = b^{ij} \nabla_j \langle F, v \rangle = b^{ij} \nabla_j K^\alpha.
\]
Moreover, the derivatives of $Z$ satisfy
\[
\nabla_i Z = K^\alpha \nabla_i \operatorname{tr}(b) + \operatorname{tr}(b) \nabla_i K^\alpha - \frac{n\alpha - 1}{\alpha} b^{ij}_i \nabla_j K^\alpha.
\]
This gives
\[
\langle F, F_i \rangle \nabla^i (K^\alpha \operatorname{tr}(b)) = b^{ij} \nabla_i K^\alpha \nabla_j Z + \frac{n\alpha - 1}{\alpha} b^{ik} b^{ij}_i \nabla_j K^\alpha \nabla_j K^\alpha
\]
and
\[
2\alpha K^\alpha b^{ij} \nabla_i K^\alpha \nabla_j \operatorname{tr}(b) = 2ab^{ij} \nabla_i K^\alpha \nabla_j Z - (2\alpha) b^{ij} \operatorname{tr}(b) - 2(\alpha - 1) b^{ik} b^{ij}_i \nabla_j K^\alpha \nabla_j K^\alpha.
\]
Substituting these identities into the equation for $\mathcal{L} Z$ gives
\[
(7) \quad \mathcal{L} Z + (2\alpha - 1) b^{ij}_i \nabla_j K^\alpha \nabla_j Z
\]
\[
= 4\alpha b^{ij} \nabla_i K^\alpha \nabla_j Z + \alpha K^{2\alpha} b^{pr} b^p r_i b^km \nabla_r h_{ik} \nabla_s h_{jm} + (2\alpha - 1) b^{ik} b^{ij}_i \nabla_j K^\alpha \nabla_j K^\alpha.
\]
Let us fix an arbitrary point \( p \). We can choose an orthonormal frame so that \( h_{ij}(p) = \lambda_i \delta_{ij} \). With this understood, we have

\[
\nabla_i K^\alpha = \alpha K^\alpha \sum_{j=1}^{n} \lambda_j^{-1} \nabla_i h_{jj}, \quad \nabla_i \text{tr}(b) = -\sum_{j=1}^{n} \lambda_j^{-2} \nabla_i h_{jj}.
\]

Let \( D \) denote the set of all triplets \((i,j,k)\) such that \( i,j,k \) are pairwise distinct. Then, by using (7) and (8), we have

\[
\alpha^{-1} K^{-2\alpha} \left( \mathcal{L} Z + (2\alpha - 1)b^{ijj} \nabla_i K^\alpha \nabla_j Z \right)
\]

\[
= \sum_{D} \lambda_i^{-2} \lambda_j^{-1} \lambda_k^{-1} (\nabla_i h_{jk})^2 + 4\alpha \sum_{k} \lambda_k^{-1} (\nabla_k \log K) (K^{-\alpha} \nabla_k Z)
\]

\[
+ \sum_{k} \sum_{i} \lambda_k^{-2} \lambda_i^{-2} (\nabla_k h_{ii})^2 + 2 \sum_{k} \sum_{i \neq k} \lambda_k^{-1} \lambda_i^{-3} (\nabla_k h_{ii})^2
\]

\[
+ \sum_{k} \lambda_k^{-1} \left[ -2\alpha^2 \text{tr}(b) + (2n\alpha^2 + (n - 1)\alpha - 1) \lambda_k^{-1} \right] (\nabla_k \log K)^2.
\]

We claim that for each \( k \) the following holds:

\[
\sum_{i} \lambda_i^{-1} \lambda_i^{-2} (\nabla_k h_{ii})^2 + 2 \sum_{i \neq k} \lambda_i^{-3} (\nabla_k h_{ii})^2 + 4\alpha (\nabla_k \log K) (K^{-\alpha} \nabla_k Z)
\]

\[
+ \left[ -2\alpha^2 \text{tr}(b) + (2n\alpha^2 + (n - 1)\alpha - 1) \lambda_k^{-1} \right] (\nabla_k \log K)^2 \geq 0.
\]

Notice that

\[
K^{-\alpha} \nabla_k Z = -\sum_{i} \lambda_i^{-2} \nabla_k h_{ii} + (\alpha \text{tr}(b) - (n\alpha - 1) \lambda_k^{-1}) \nabla_k \log K.
\]

After relabeling indices, we may assume \( k = n \). If we put \( \sigma_i := \lambda_i^{-1} \nabla_n h_{ii} \), then the assertion follows from the previous lemma. Thus, we conclude that \( \mathcal{L} Z + (2\alpha - 1)b^{ijj} \nabla_i K^\alpha \nabla_j Z \geq 0 \) at each point \( p \). Therefore, by the strong maximum principle, \( Z \) is a constant. Hence, the left hand side of (9) is zero. Therefore, \( \nabla_i h_{jk} = 0 \) if \( i,j,k \) are all distinct. Moreover, since we have equality in the lemma, we either have \( \lambda_i^{-1} \nabla_k h_{ii} = 0 \) for all \( i,k \), or we have \( \alpha = \frac{1}{n+2} \) and \( \lambda_i^{-1} \nabla_k h_{ii} = \frac{3}{n+2} \lambda_i^{-1} \nabla_k h_{kk} \) for \( i \neq k \).

In the first case, we conclude that \( \nabla_i h_{jk} = 0 \) for all \( i,j,k \) and thus \( \Sigma \) is a round sphere.

In the second case, we obtain \( \lambda_i^{-1} \nabla_k h_{ii} = \frac{1}{n+2} \nabla_k \log K \) for \( i \neq k \) and \( \lambda_k^{-1} \nabla_k h_{kk} = \frac{3}{n+2} \nabla_k \log K \). This gives

\[
C_{ijk} = \frac{1}{2} K^{-\frac{1}{n+2}} \nabla_k h_{ij}
\]

\[
+ \frac{1}{2} h_{jk} \nabla_i K^{-\frac{1}{n+2}} + \frac{1}{2} h_{ki} \nabla_j K^{-\frac{1}{n+2}} + \frac{1}{2} h_{ij} \nabla_k K^{-\frac{1}{n+2}} = 0
\]

for all \( i,j,k \). Since the cubic form \( C_{ijk} \) vanishes everywhere, the surface is an ellipsoid by the Berwald-Pick theorem (see e.g. [18], Theorem 4.5). This
proves the theorem.

4. Classification of self-similar solutions: The case $\alpha \in (\frac{1}{2}, \infty)$

We now turn to the case $\alpha \in (\frac{1}{2}, \infty)$. In the following, we denote by $
abla \lambda$ for $1 \leq \lambda \leq n$ the eigenvalues of the second fundamental form, arranged in increasing order. Each eigenvalue defines a Lipschitz continuous function on $M$.

Lemma 5. Suppose that $\varphi$ is a smooth function such that $\lambda_1 \geq \varphi$ everywhere and $\lambda_1 = \varphi$ at $\bar{p}$. Let $\mu$ denote the multiplicity of the smallest curvature eigenvalue at $\bar{p}$, so that $\lambda_1 = \ldots = \lambda_\mu < \lambda_{\mu+1} \leq \ldots \leq \lambda_n$. Then, at $\bar{p}$,
\[
\n \nabla_i h_{kl} = \nabla_i \varphi \delta_{kl} \text{ for } 1 \leq k, l \leq \mu.
\]
Moreover,
\[
\n abla_i \nabla_i \varphi \leq \nabla_i \nabla_i h_{11} - 2 \sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} (\nabla_i h_{11})^2.
\]

at $\bar{p}$.

Proof. Fix an index $i$, and let $\gamma(s)$ be the geodesic satisfying $\gamma(0) = \bar{p}$ and $\gamma'(0) = e_i$. Moreover, let $v(s)$ be a vector field along $\gamma$ such that $v(0) \in \text{span}\{e_1, \ldots, e_\mu\}$ and $v'(0) \in \text{span}\{e_{\mu+1}, \ldots, e_n\}$. Then the function $s \mapsto h(v(s), v(s)) - \varphi(\gamma(s)) |v(s)|^2$ has a local minimum at $s = 0$. This gives
\[
0 = \frac{d}{ds} \left( h(v(s), v(s)) - \varphi(\gamma(s)) |v(s)|^2 \right) |_{s=0}
\]
\[
= \nabla_i h(v(0), v(0)) + 2 h(v(0), v'(0)) - \nabla_i \varphi |v(0)|^2 - 2 \langle v(0), v'(0) \rangle
\]
\[
= \nabla_i h(v(0), v(0)) - \nabla_i \varphi |v(0)|^2.
\]
Since $v(0) \in \text{span}\{e_1, \ldots, e_\mu\}$ is arbitrary, it follows that $\nabla_i h_{kl} = \nabla_i \varphi \delta_{kl}$ for $1 \leq k, l \leq \mu$ at the point $\bar{p}$.

We next consider the second derivative. To that end, we choose $v(0) = e_1$, $v'(0) = -\sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} \nabla_i h_{11} e_l$, and $v''(0) = 0$. Since the function $s \mapsto h(v(s), v(s)) - \varphi(\gamma(s)) |v(s)|^2$ has a local minimum at $s = 0$, we obtain
\[
0 \leq \frac{d^2}{ds^2} \left( h(v(s), v(s)) - \varphi(\gamma(s)) |v(s)|^2 \right) |_{s=0}
\]
\[
= \nabla_i \nabla_i h(v(0), v(0)) + 4 \nabla_i h(v(0), v'(0)) + 2 h(v'(0), v'(0))
\]
\[
- \nabla_i \nabla_i \varphi |v(0)|^2 - 4 \nabla_i \varphi \langle v(0), v'(0) \rangle - 2 \varphi |v'(0)|^2
\]
\[
= \nabla_i \nabla_i h_{11} - 4 \sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} (\nabla_i h_{11})^2 + 2 \sum_{l > \mu} \lambda_l (\lambda_l - \lambda_1)^{-2} (\nabla_i h_{11})^2
\]
\[
- \nabla_i \nabla_i \varphi - 2 \sum_{l > \mu} \lambda_1 (\lambda_l - \lambda_1)^{-2} (\nabla_i h_{11})^2
\]
\[
= \nabla_i \nabla_i h_{11} - 2 \sum_{l > \mu} (\lambda_l - \lambda_1)^{-2} (\nabla_i h_{11})^2 - \nabla_i \nabla_i \varphi.
\]

This proves the assertion.
Theorem 6. Assume $\alpha \in (\frac{1}{2}, \infty)$ and $\Sigma$ is a strictly convex closed smooth solution of $\{\varphi\}. Then $\Sigma$ is a round sphere.

Proof. Let us consider the function

$$W = K^{\alpha} \lambda_1^{\lambda} - \frac{n\alpha - 1}{2n\alpha} |F|^2.$$  

Let us consider an arbitrary point $\bar{p}$ where $W$ attains its maximum. As above, we denote by $\mu$ the multiplicity of the smallest eigenvalue of the second fundamental form. Let us define a smooth function $\varphi$ such that

$$W(\bar{p}) = K^{\alpha} \varphi^{\lambda} - \frac{n\alpha - 1}{2n\alpha} |F|^2.$$  

Since $W$ attains its maximum at $\bar{p}$, we have $\lambda_1 \geq \varphi$ everywhere and $\lambda_1 = \varphi$ at $\bar{p}$. Therefore, we may apply the previous lemma. Hence, at the point $\bar{p}$, we have $\nabla_i h_{kl} = \nabla_i \varphi \delta_{kl}$ for $1 \leq k, l \leq \mu$. Moreover,

$$\nabla_k \nabla_k \varphi \leq \nabla_k \nabla_k h_{11} - 2 \sum_{l > \mu} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{11})^2$$  

at $\bar{p}$. We multiply both sides by $\alpha K^{\alpha} \lambda_k^{-1}$ and sum over $k$. This gives

$$\mathcal{L} \varphi \leq \mathcal{L} h_{11} - 2 \alpha K^{\alpha} \sum_k \sum_{l > \mu} \lambda_k^{-1} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{11})^2.$$  

By (5), we have

$$\mathcal{L} h_{ij} = -K^{-\alpha} \nabla_i K^{\alpha} \nabla_j K^{\alpha} + \alpha K^{\alpha} h^{rs} h_{ij} \nabla_r \nabla_s h_{pq}$$

$$+ \langle F, F \rangle \nabla_k h_{ij} + h_{ij} + (n\alpha - 1) h_{ik} h_{jk} K^{\alpha} - \alpha K^{\alpha} H h_{ij}.$$  

Thus,

$$\mathcal{L} \varphi \leq -2 \alpha K^{\alpha} \sum_k \sum_{l > \mu} \lambda_k^{-1} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{11})^2$$

$$- K^{-\alpha} (\nabla_1 K^{\alpha})^2 + \alpha K^{\alpha} \sum_{k,l} \lambda_k^{-1} \lambda_l^{-1} (\nabla_k h_{11})^2$$

$$+ \sum_k \langle F, F \rangle \nabla_k \lambda_1 + \lambda_1 + (n\alpha - 1) \lambda_1^2 K^{\alpha} - \alpha K^{\alpha} H \lambda_1.$$  

Using the estimate

$$- 2 \alpha K^{\alpha} \sum_k \sum_{l > \mu} \lambda_k^{-1} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{11})^2 + \alpha K^{\alpha} \sum_{k,l} \lambda_k^{-1} \lambda_l^{-1} (\nabla_k h_{11})^2$$

$$= - \alpha K^{\alpha} \sum_k \sum_{l > \mu} \lambda_k^{-1} [2(\lambda_l - \lambda_1)^{-1} - \lambda_l^{-1}] (\nabla_k h_{11})^2 + \alpha K^{\alpha} \sum_{k,l} \lambda_k^{-1} \lambda_l^{-1} (\nabla_k h_{11})^2$$

$$\leq - \alpha K^{\alpha} \sum_{l > \mu} \lambda_l^{-1} [2(\lambda_l - \lambda_1)^{-1} - \lambda_l^{-1}] (\nabla_1 h_{11})^2 + \alpha K^{\alpha} \sum_{k} \lambda_k^{-1} \lambda_l^{-1} (\nabla_k h_{11})^2$$

$$= -2 \alpha K^{\alpha} \sum_{k > \mu} \lambda_k^{-1} [(\lambda_k - \lambda_1)^{-1} - \lambda_k^{-1}] (\nabla_k h_{11})^2 + \alpha K^{\alpha} \lambda_1^{-2} (\nabla_1 h_{11})^2,$$
we obtain

\[\mathcal{L}\varphi \leq -2\alpha K^{\alpha\lambda_1^{-1}} \sum_{k>\mu} [(\lambda_k - \lambda_1)^{-1} - \lambda_k^{-1}] (\nabla_k h_{11})^2
+ \alpha K^{\alpha\lambda_1^{-2}}(\nabla_1 h_{11})^2 - K^{-\alpha}(\nabla_1 K^{\alpha})^2
+ \sum_k \langle F, F_k \rangle \nabla_k \lambda_1 + \lambda_1 + (n\alpha - 1)\lambda_1^2 K^{\alpha} - \alpha K^{\alpha} H \lambda_1.\]

Since \(\nabla_k \varphi = \nabla_k h_{11}\), it follows that

\[\mathcal{L}(\varphi^{-1}) \geq 2\alpha K^{\alpha\lambda_1^{-3}} \sum_k \lambda_k^{-1}(\nabla_k h_{11})^2
+ 2\alpha K^{\alpha\lambda_1^{-3}} \sum_{k>\mu} [(\lambda_k - \lambda_1)^{-1} - \lambda_k^{-1}] (\nabla_k h_{11})^2
- \alpha K^{\alpha\lambda_1^{-4}}(\nabla_1 h_{11})^2 + K^{-\alpha}\lambda_1^{-2}(\nabla_1 K^{\alpha})^2
+ \sum_k \langle F, F_k \rangle \nabla_k (\lambda_1^{-1} - \lambda_1^{-1} - (n\alpha - 1)K^{\alpha} + \alpha K^{\alpha} H \lambda_1^{-1}
= 2\alpha K^{\alpha\lambda_1^{-3}} \sum_k [(\lambda_k - \lambda_1)^{-1} - \lambda_k^{-1} - (n\alpha - 1)K^{\alpha} + \alpha K^{\alpha} H \lambda_1^{-1}.
\]

This gives

\[\mathcal{L}(K^{\alpha\varphi^{-1}}) = K^{\alpha} \mathcal{L}(\varphi^{-1}) + \varphi^{-1} \mathcal{L}(K^{\alpha}) + 2\alpha K^{\alpha} \sum_k \lambda_k^{-1} \nabla_k K^{\alpha} \nabla_k (\varphi^{-1})
\geq 2\alpha \sum_k \lambda_k^{-1} \nabla_k K^{\alpha} \nabla_k (K^{\alpha}\varphi^{-1}) - 2\alpha \lambda_1^{-1} \sum_k \lambda_k^{-1} (\nabla_k K^{\alpha})^2
+ 2\alpha K^{2\alpha\lambda_1^{-3}} \sum_{k>\mu} [(\lambda_k - \lambda_1)^{-1} - \lambda_k^{-1} (\nabla_k h_{11})^2
+ \alpha K^{2\alpha\lambda_1^{-4}}(\nabla_1 h_{11})^2 + \lambda_1^{-2}(\nabla_1 K^{\alpha})^2
+ \sum_k \langle F, F_k \rangle \nabla_k (K^{\alpha}\varphi^{-1}) + (n\alpha - 1)K^{\alpha}\lambda_1^{-1} - (n\alpha - 1)K^{2\alpha}.\]
By assumption, the function $K^\alpha \varphi^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2$ is constant. Consequently, 
\[ \nabla_k (K^\alpha \varphi^{-1}) = \frac{n\alpha - 1}{2n\alpha} \nabla_k |F|^2, \]
and
\[ 0 = \mathcal{L}(K^\alpha \varphi^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2) \]
\[ \geq \frac{n\alpha - 1}{n} \sum_k \lambda_k^{-1} \nabla_k K^\alpha \nabla_k |F|^2 - 2\alpha \lambda_1^{-1} \sum_k \lambda_k^{-1} (\nabla_k K^\alpha)^2 \]
\[ + 2\alpha K^2 \lambda_1^{-3} \sum_{k>\mu} (\lambda_k - \lambda_1)^{-1} (\nabla_k h_{11})^2 \]
\[ + \alpha K^2 \lambda_1^{-4} (\nabla_1 h_{11})^2 + \lambda_1^{-2} (\nabla_1 K^\alpha)^2 \]
\[ + \frac{n\alpha - 1}{2n\alpha} \sum_k \langle F, F_k \rangle \nabla_k |F|^2 + (n\alpha - 1) K^\alpha (\lambda_1^{-1} - \frac{1}{n} \sum_{k=1}^n \lambda_k^{-1}). \]

Recall that
\[ \frac{1}{2} \nabla_k |F|^2 = \langle F, F_k \rangle = \lambda_k^{-1} \nabla_k K^\alpha. \]

Moreover, using the identity $\nabla_k \varphi = \nabla_k h_{11}$, we obtain
\[ 0 = \nabla_k (K^\alpha \varphi^{-1}) - \frac{n\alpha - 1}{2n\alpha} \nabla_k |F|^2 \]
\[ = (\lambda_1^{-1} - \frac{n\alpha - 1}{n\alpha} \lambda_k^{-1}) \nabla_k K^\alpha - K^\alpha \lambda_1^{-2} \nabla_k h_{11} \]
at $\bar{p}$. Note that, if $2 \leq l \leq \mu$, then $\nabla_l h_{11}$ for all $k$. Putting $k = 1$ gives $\nabla_l h_{11} = 0$ and $\nabla_l K = 0$ for $2 \leq l \leq \mu$. Putting these facts together, we obtain
\[ 0 \geq \frac{2(n\alpha - 1)}{n} \sum_k \lambda_k^{-2} (\nabla_k K^\alpha)^2 - 2\alpha \lambda_1^{-1} \sum_k \lambda_k^{-1} (\nabla_k K^\alpha)^2 \]
\[ + 2\alpha \lambda_1 \sum_{k>\mu} (\lambda_k - \lambda_1)^{-1} (\lambda_1^{-1} - \frac{n\alpha - 1}{n\alpha} \lambda_k^{-1})^2 (\nabla_k K^\alpha)^2 \]
\[ + \frac{1}{n^2\alpha} \lambda_1^{-2} (\nabla_1 K^\alpha)^2 + \lambda_1^{-2} (\nabla_1 K^\alpha)^2 \]
\[ + \frac{n\alpha - 1}{n\alpha} \sum_k \lambda_k^{-2} (\nabla_k K^\alpha)^2 + (n\alpha - 1) K^\alpha (\lambda_1^{-1} - \frac{1}{n} \sum_{k=1}^n \lambda_k^{-1}). \]

Using the identities
\[ \frac{2(n\alpha - 1)}{n} \lambda_k^{-2} - 2\alpha \lambda_1^{-1} \lambda_k^{-1} \]
\[ + 2\alpha \lambda_1 (\lambda_k - \lambda_1)^{-1} (\lambda_1^{-1} - \frac{n\alpha - 1}{n\alpha} \lambda_k^{-1})^2 \]
\[ + \frac{n\alpha - 1}{n\alpha} \lambda_k^{-2} \]
\[ = \left( \frac{n\alpha - 1}{n\alpha} \right) \lambda_1^{-2} \]
\[ + \frac{2}{n} \left( \lambda_1 (\lambda_k - \lambda_1)^{-1} \right) \lambda_k^{-2} \]
and
\[ \frac{2(n\alpha - 1)}{n} \lambda_1^{-2} - 2\alpha \lambda_1^{-2} + \frac{1}{n^2\alpha} \lambda_1^{-2} + \lambda_1^{-2} + \frac{n\alpha - 1}{n\alpha} \lambda_1^{-2} = \frac{n - 1}{n^2\alpha} (2n\alpha - 1) \lambda_1^{-2}, \]
the previous inequality can be rewritten as follows:
\[ 0 \geq \sum_{k>\mu} \left( \frac{n\alpha - 1}{n\alpha} + \frac{2}{n} + \frac{2}{n^2\alpha} \lambda_1 (\lambda_k - \lambda_1)^{-1} \right) \lambda_k^{-2} (\nabla_k K^{\alpha})^2 \]
\[ + \frac{n - 1}{n^2\alpha} (2n\alpha - 1) \lambda_1^{-2} (\nabla_1 K^{\alpha})^2 + (n\alpha - 1) K^{\alpha} (\lambda_1^{-1} - \frac{1}{n} \sum_{k=1}^{n} \lambda_k^{-1}). \]
Since \( \alpha > \frac{1}{n} \), it follows that \( \bar{p} \) is an umbilic point. Since \( \bar{p} \) is an umbilic point and \( \alpha > \frac{1}{2} \), there exists a neighborhood \( U \) of \( \bar{p} \) with the property that
\[ \alpha^{-1} K^{-2\alpha} \left( L Z - (2\alpha + 1) b^{ij} \nabla_i K^{\alpha} \nabla_j Z \right) \]
\[ = \sum_D \lambda_i^{-2} \lambda_j^{-1} \lambda_k^{-1} (\nabla_i h_{jk})^2 \]
\[ + \sum_k \sum_i \lambda_k^{-3} (\nabla_k h_{ii})^2 + 2 \sum_k \sum_{i \neq k} \lambda_k^{-1} \lambda_i^{-3} (\nabla_k h_{ii})^2 \]
\[ + \sum_k \lambda_k^{-1} \left[ -2\alpha^2 \text{tr}(b) + (2n\alpha^2 + (n - 1)\alpha - 1) \lambda_k^{-1} \right] (\nabla_k \log K)^2 \]
\[ \geq 0 \]
at each point in \( U \). (Indeed, if \( n \geq 3 \), the last inequality follows immediately from the fact that \( (n - 1)\alpha - 1 \geq 0 \). For \( n = 2 \) the last inequality follows from a straightforward calculation.)

Now, since \( \bar{p} \) is an umbilic point, we have \( Z(p) \leq nW(p) \leq nW(\bar{p}) = Z(\bar{p}) \) for each point \( p \in U \). Thus, \( Z \) attains a local maximum at \( \bar{p} \). By the strong maximum principle, \( Z(p) = Z(\bar{p}) \) for all points \( p \in U \). This implies \( W(p) = W(\bar{p}) \) for all points \( p \in U \). Thus, the set of all points where \( W \) attains its maximum is open. Consequently, \( W \) is constant. This implies that \( \Sigma \) is umbilic, hence a round sphere.

5. Proof of Theorem

Suppose we have any strictly convex solution to the flow with speed \( -K^{\alpha}\nu \), where \( \alpha \in (\frac{1}{n+2}, \infty) \). By Theorem 6.2 in [5] (see also [3]), the flow converges to a soliton after rescaling. By Theorem 4 and Theorem 6, either the limit is a round sphere, or \( \alpha = \frac{1}{n+2} \) and the limit is an ellipsoid.

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