ON NON-ISOTOPIC SPANNING SURFACES FOR A CLASS OF ARBORESCENT KNOTS

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Abstract. We use the methods of Hedden, Juhasz, and Sarkar to exhibit a set of arborescent knots that bound large numbers of non-isotopic minimal genus spanning surfaces. In particular, we describe a sequence of prime knots $K_n$ which will bound at least $2^{2n-1}$ non-isotopic minimal spanning surfaces of genus $n$.

1. Equivalence of surfaces

In [4], M. Hedden, A. Juhasz, and S. Sarkar show that the knot 8_9, with some orientation, is the oriented boundary of two distinct spanning surfaces, where we take two compact, connected, oriented surfaces $\Sigma_1, \Sigma_2 \subset S^3$ to be equivalent if one is ambiently isotopic to the other. By taking the $n$-fold connected sum of 8_9 with itself, they obtain a sequence of knots $K_n, n \in \mathbb{N}$ which bound at least $n$ distinct spanning surfaces. The increasing number of surfaces arises, however, from the independent choices possible for each summand.

In this paper, we will extend their technique of combining sutured Floer homology ([7]) with the Seifert form to improve this result to prime knots. Namely,

Theorem 1. For each $n \in \mathbb{N}$, there exists an oriented, prime knot $K$ in $S^3$ which bounds at least $2^{2n-1}$ oriented minimal genus spanning surfaces, each with genus $n$.

Our examples will be arborescent knots which are also alternating. By a theorem of Menasco, a reduced, alternating, prime knot diagram represents a prime knot. $K$ being arborescent allows us to easily guarantee these conditions. The reader can find an example of the prime knots we construct in Figure 3. That knot bounds at least 8 surfaces of genus 2.

Throughout this paper we will assume that:

1. $\Sigma$ is an oriented, compact, connected surface embedded in $S^3$, with a single boundary knot.
2. $N(\Sigma)$ is a tubular neighborhood of $\Sigma$ equipped with a product structure by an orientation preserving diffeomorphism $\Sigma \times [-1, 1] \hookrightarrow N(\Sigma) \subset S^3$
3. $\Sigma^{\pm}$ is the image of $\Sigma \times \{\pm 1\}$ under the inclusion of $N(\Sigma)$. Furthermore, if $a \subset \Sigma$ is a simple closed curve, then $a^{\pm}$ is the image of $a \times \{\pm 1\}$ in $\Sigma^{\pm}$. We will call these the positive/negative push-off(s) of $a$. 

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(4) $S^3_\Sigma = S^3 \setminus \text{Int} N(\Sigma)$

We will use the following notions for the equivalence of oriented surfaces:

**Definition 2.** A smooth map $f : (Y_1, \Sigma_1) \rightarrow (Y_2, \Sigma_2)$ is an orientation preserving diffeomorphism of pairs if $f : Y_1 \rightarrow Y_2$ is an orientation preserving diffeomorphism which induces an orientation preserving diffeomorphism from $\Sigma_1 \subset Y_1$ to $\Sigma_2 \subset Y_2$.

**Definition 3.** Two compact, connected, oriented, surfaces, $\Sigma_1$ and $\Sigma_2$, embedded in $S^3$ are equivalent if there is an orientation preserving isotopy $\Phi_t$ of $S^3$ with 1) $\Phi_0 = \text{Id}$ and 2) $\Phi_1$ restricting to a diffeomorphism from $\Sigma_1$ to $\Sigma_2$.

After this paper first appeared, the author learned that one can find examples among 2-bridge knots where there are $2^{2k-1}$ “inequivalent” incompressible Seifert surfaces for a knot of genus $k$. This follows from the work of Hatcher and Thurston in [3]. However, their notion of equivalence is different from that in this paper: in [3] two spanning surfaces are equivalent if they are isotopic in the complement of the knot. Similarly, Jessica Banks notes that M. Sakuma classified minimal genus Seifert surfaces for special arborescent links, which are very similar to the examples in this paper, again using isotopy in the complement of the link to provide the notion of equivalence. Using this classification she describes examples of arborescent knots bounding $2^{2k-1}$ surfaces of minimal genus $k$ which are different up to isotopy in the complement of the knot, [1]. In addition, this approach to classifying spanning surfaces use entirely different techniques that we will use in this paper.

However, the notion of equivalence in definition Section 3 is stronger than that in [3], [11], and [5]. Instead of requiring that the boundary of the surface $\Sigma$ lies on a fixed link $L$ – in particular during an isotopy – we instead use allow the boundary and the surface to be isotoped. Our notion of equivalence corresponds, therefore, more closely to that required for the isotopy classification of surfaces in $S^3$.

As an illustration of the difference, J. R. Eisner provided examples of composite knots with infinitely many spanning surfaces, up to isotopies preserving the knot, and these examples give rise to Kakimizu complexes corresponding to the complex structure on $\mathbb{R}$ with $\mathbb{Z}$ as vertices, see section 3 in [5]. However, the different surfaces come from spinning one summand around the arc in the other that is used for the connect sum. This spinning can be undone if we allow the knot to move, so all these surfaces are equivalent under our definition.

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2. Matched trees

2.1. Definitions.

Let $T$ be a finite, connected tree with vertices $V$ and edges $E$. In addition, suppose $T$ possesses this additional structure:

1. $T$ is bipartite: $V$ is the disjoint union of $B$ and $W$ and any edge $e \in E$ has one vertex in $B$ and the other in $W$.
2. $T$ has a matching: there is a bijection $m : B \to W$ such that for each $b \in B$, $b$ and $m(b)$ are the endpoints of an edge in $T$. (We will think of $m$ also as the subset of edges $\{b, m(b)\}$ in $E$)
3. $(T, m)$ is directed according to the convention that 1) $e \in m$ is oriented from its endpoint in $B$ to that in $W$, while 2) $e \not\in m$ will be directed oppositely, from $W$ to $B$.

We will call this constellation of objects a matched tree $T$. The set of matched trees will be denoted $\mathcal{T}_m$.

The simplest example of a matched tree is the tree with two vertices partitioned as $B = \{b\}$ and $W = \{w\}$, equipped with the matching $m(b) = w$, and oriented accordingly. This particular tree will be denoted $T_1$.

We will depict matched trees by drawing those edges in $m$ as double lines and edges in $E \setminus m$ as single lines. Furthermore, we identify those $v \in B$ by filled dots, while those $v \in W$ are identified by empty circles. For another example, drawn with this convention, see Figure 1.
We can drop reference to the matching from \((T, m)\) since according to the following lemma there is a unique one, if any exist at all.

**Lemma 4.** Any tree \(T\) admits at most one matching.

**Proof:** If \(m\) and \(m'\) are distinct matchings on a bipartite graph \(\Gamma\), then the symmetric difference \(m \Delta m' \subset E\) consists of edges which form cycles in \(\Gamma\). Consequently, when \(\Gamma\) is a tree, there can be at most one matching. \(\diamondsuit\)

Matched trees can also be described as those trees with the following properties

**Lemma 5.** Let \(T_m\) is the smallest set \(T\) of trees such that

1. \(T_1 \in T\), and
2. \(T \in T\) if and only if there is a subtree \(T' \in T\) such that \(V(T) = V(T') \cup \{l, v\}\), and \(E(T) = E(T') \cup \{e_1, e_2\}\) with \(e_1\) joining \(v\) to a vertex in \(T'\) and \(e_2\) joining \(l\) to \(v\).

We will see in the proof that the matched, bipartite structure on \(T_1\) extends to all trees in this set.

**Proof:** First, we show that \(T \subset T_m\). Since \(T_1\) is in both, we suppose that any tree tree in \(T\) with \(2n - 2\) vertices is also in \(T_m\). We will then show that any tree in \(T\) with \(2n\) vertices is also in \(T_m\), and draw the desired conclusion by induction. Suppose \(T\) has \(2n\) vertices, and that \(T' \subset T\) is a subtree guaranteed by the second condition. Then \(e_1\) joins \(v\) to a vertex \(v' \in T'\). First we extend the bipartite structure: if \(v' \in B\) assign \(v\) to \(W\) and \(l\) to \(B\), otherwise assign \(v\) to \(B\) and \(l\) to \(W\). The matching on \(T'\) uniquely extends by assigning \(m(l) = v\) when \(v' \in B\) (hence \(l \in B\)) and \(m(v) = l\) when \(v' \in W\). Finally, there is exists unique way to direct \(T\) according to the orientation convention. Hence \(T \in T_m\) as well.

Now need we show that \(T_m \subset T\). We again prove this by induction on the number of vertices in \(T\). For 2 vertices, the only connected, bipartite, matched tree is the starting tree \(T_1\). Suppose \(T\) has \(2n\) vertices \((n \geq 2)\) and that any matched tree in \(T_m\) with fewer than \(2n\) vertices is in \(T\). Let \(m\) be the matching for \(T\). If we contract all the edges in \(m\), we obtain a new tree \(\Gamma\) with \(\geq 2\) vertices. \(\Gamma\) has a leaf vertex \(g\) which corresponds in \(T\) to an edge \(e_2 \in m\) joining a leaf vertex \(l\) to a bivalent vertex \(v\). Let \(e_2\) be the other edge at \(v\). If we delete \(e_1, e_2, v, l\) we obtain a tree \(T'\) in \(T_m\) with \(2n - 2\) vertices. Thus \(T' \in T\), so \(T \in T\) as well. \(\diamondsuit\)

As a byproduct, this proof provides a way to iteratively construct all matched trees with a given number of vertices.

### 2.2. Partial Ordering.
Definition 6. Let \( v, v' \) be vertices of a directed tree \( T \). We will write \( v \leq v' \) if there is a (possibly empty) directed path from \( v' \) to \( v \) in \( T \). We will write \( v <_1 v' \) if the directed path from \( v' \) to \( v \) contains exactly one edge.

Lemma 7. For any directed tree, \( \leq \) is a partial order on the vertices of \( T \).

The orientation conventions for matched trees allow us to be more specific:

Lemma 8. When \( T \) is a matched tree the terminal vertices for \((T, \leq)\) are all in \( W \), while the initial vertices are all in \( B \).

For two vertices \( v, v' \) in a directed tree the set \( \{ u \in V \mid v \leq u \leq v' \} \) is either empty (if \( v \not\leq v' \)) or consists of the vertices of the linearly ordered path from \( v' \) to \( v \). Furthermore, if \( b \in B \) and \( w \in W \) in a matched tree \((T, m)\), with \( b \geq w \), then this path has a matching coming from the restriction of \( m \).

The bipartite structure on a matched tree allows us to refine some of these considerations:

Definition 9. For \( w \in W \), let \( B_w = \{ b \in B \mid w \leq b \} \) be the vertices in \( B \) that are above \( w \). For \( b \in B \), let \( W_b = \{ w \in W \mid b \geq w \} \) be the vertices in \( W \) that are below \( b \).

For a matched tree, the partial order on \( B \) and \( W \) is reflected in these sets alone.

Lemma 10. Let \( T \) be a bipartite tree oriented by \( m \). For \( b, b' \in B \), \( b \leq b' \) if and only if \( W_b \subset W_{b'} \). Likewise, for \( w, w' \in W \), \( w \leq w' \) if and only if \( B_w \supset B_{w'} \).

3. The surfaces

Let \( T \) be a matched tree as in the previous section. Equip \( T \) with

(1) a framing map \( f : V \to \mathbb{Z} \), and
(2) a plumbing map \( \epsilon : E \to \{-1, +1\} \)

Given \((T, f, \epsilon)\), we can define an oriented surface in \( S^3 \). To do this recall

Definition 11. An oriented surface \( \Sigma \subset S^3 \) is the Murasugi sum of the oriented surfaces \( \Sigma_1 \) and a surface \( \Sigma_2 \) if there is a decomposition \( S^3 = B_1 \cup \Sigma B_2 \) where \( B_1 \) and \( B_2 \) are closed balls, such that

(1) For \( i = 1, 2 \), \( B_i \cap \Sigma \subset S^3 \) is isotopic to \( \Sigma_i \), while preserving the orientations on the surfaces,
(2) \( S \cap \Sigma \) is a disc \( D \) such that \( \partial D \cap \partial \Sigma \) consists of \( 2n > 2 \) points on is boundary.

Definition 12. The Murasugi sum is positive for \( \Sigma_i \) if the orientation for \( S \) inherited from \( S \cap \Sigma \) agrees with its orientation as the boundary of \( B_i \). It is negative, if it disagrees.
We provide two local models for the $n = 2$ case of the Murasugi sum. Two diagrams illustrate these models in Figure 2. Suppose that $\Sigma \subset S^3$ is a compact, oriented surface with boundary. We assume we are given an oriented circle $h$ in $\Sigma$, and an oriented arc $\gamma$, properly embedded in $\Sigma$, with $h \cap \gamma = +1$, algebraically and geometrically.

Let $N$ be a neighborhood of $\gamma$ in $S^3$ equipped with coordinates such that

1. $N$ is homeomorphic to $[-1, 1]^3$,
2. $\gamma$ is $\{0\} \times [-1/2, 1/2] \times \{0\}$,
3. $h \cap N$ is $[-1, 1] \times \{(0,0)\}$
4. $\Sigma \cap N$ is $[-1, 1] \times [-1/2, 1/2] \times \{0\}$ with $\partial \Sigma \cap N = ([1, 1] \times \{-1/2, 0\}) \cup -([1, 1] \times \{(1, 0)\})$.

and all identifications preserve the relevant orientations.

Let $\Sigma' \subset S^3$ be another surface equipped with a closed curve $C$ and an $\delta$ from the boundary to the boundary, so that $C \cap \delta = +1$, algebraically and geometrically. Let $\epsilon \in \{1, -1\}$. Suppose $M \subset S^3$ is homeomorphic to $[-1, 1]^3$ such that

1. $\delta$ equals $-([1/2, 1/2] \times \{(0,0)\})$.
2. $C \cap [-1, 1]^2 \times \{0\} = \{0\} \times [-3/4, 3/4] \times \{0\}$. 

Figure 2. The top digram is a positive plumbing of the annulus, using the notation below. The bottom is a negative plumbing. Note that the annuli have the same number of full twists in each. The boundaries of the two surfaces on the right will represent the same link class.
Then \( \Sigma \neq \Sigma' \) is the union of \( \Sigma \) with the image of \( \Sigma' \) under the coordinate identification of \( N \) and \( M \). This surface is a Murasugi sum of \( \Sigma \) and \( \Sigma' \) with decomposing sphere equal to \( S \) for any \( \Sigma \). For a surface \( \Sigma \) containing \( M \), we will shorten to \( T, f, \epsilon \) and \( \Sigma' \). When \( \epsilon = +1 \) then \( \Sigma' \subset [-1,1]^2 \times [0,1] \), whereas if \( \epsilon = -1 \) then \( \Sigma' \subset [-1,1]^2 \times [-1,0] \).

Now let \((T, f, \epsilon)\) be a matched tree equipped with vertex and edge labellings (which we will shorten to \( T \) from here on). For each vertex \( v \) of \( T \) let \( A_v \) be an annulus in \( S^3 \), with \( \text{unknotted} \), oriented, core circle \( h_v \), making \( f(v) \) right handed twists around \( h_v \) relative to the framing provided by a disk \( h_v \) bounds. If \( v \) is the endpoint for the edges \( e_1, \ldots, e_n \), pick simple arcs \( \gamma_{e_1}, \ldots, \gamma_{e_n} \) running between the two boundary components, with \( h_v \cap \gamma_v = +1 \). As the constructions will not depend on these arcs, we will suppress them from the notation.

To construct \( \Sigma_{(T,f,\epsilon)} \) we Murasugi sum all the annuli \( A_v \) by using the arcs \( \gamma_e \subset A_v \) and \( \gamma'_{e'} \subset A_{v'} \) corresponding to edge \( e \) between \( v \) and \( v' \). To fully specify \( \Sigma_T \) we need to know whether to use the positive or negative sum for each edge. For edge \( e \) from \( v \) to \( v' \), with \( v < v' \), we use the local model from \( \epsilon(e) \) with \( \Sigma \) being the surface containing \( A_v \) and \( \Sigma' \) being the surface containing \( A_{v'} \) for the larger vertex \( v' \).

**Lemma 13.** For a surface \( \Sigma_T \) as constructed above, the following are true

1. \( S^3_{\Sigma_T} \) is homeomorphic to a handlebody.
2. \( \partial \Sigma_T = K_T \) is a knot
3. The knot type of \( K_T \) does not depend on the map \( \epsilon \), when \( T, f, \) and the arcs \( \gamma_e \) are all fixed.

**Proof:** All three are proved by induction. First, they are true for triples \((T_1, f_1, \epsilon_1)\) for any \( f_1 \) and \( \epsilon_1 \) can be verified by directly checking. Now suppose that for any \( T' \in T_m \) with fewer than \( 2n \) vertices, all three properties are true. Let \( T \in T_m \) have \( 2n \) vertices. Then there are two edges \( e_1 \) and \( e_2 \) with \( e_2 \in m \), joining \( l \) to \( v \). And \( e_1 \) joining \( v \) to \( v' \), such that \( T \) minus \( e_1, e_2, v, l \) is in \( T_m \). By construction \( e_1 \) corresponds to a sphere \( S \subset S^3 \) which effects the sum of \( \Sigma_{T'} \) to the surface specified by \( v, l \), and \( e_2 \), which independently form a copy of \( T_1 \) in one of the balls \( B \) with boundary \( S \). The complement of this surface in \( B \) consists of two one handles attached to a thickening of \( S \), and thus when glued to the complement of \( \Sigma_{T'} \) in the other ball, will still be a handlebody (since \( B \) can be thought of as a portion of the neighborhood of \( \Sigma_{T'} \). Since \( \partial \Sigma_{T_1} \) and \( \partial \Sigma_{T'} \) are both knots, the sum will also be a knot. Finally, the link that is the boundary of the sum of the annulus for \( v \) to the surface \( \Sigma_{T'} \) will not
depend on the choice of $\epsilon$ from the local models above, and likewise for the sum of $l$ to $\Sigma_{T_0 U(v)}$. The proposition is thus proved. ♦

Note that $K_T$ is an arborescent knot, by construction. There is an example of a tree $T$ and the corresponding knot $K_T$ in Figure 3.

4. Seifert Forms

For each $\Sigma_{T,\epsilon}$ there is a Seifert form $\theta_{T,\epsilon} : H_1(\Sigma; \mathbb{Z}) \otimes H_1(\Sigma; \mathbb{Z}) \to \mathbb{Z}$ defined by

$$\theta_{T,\epsilon}([a],[b]) = \text{lk}(a,b^+)$$

This form is invariant under orientation preserving diffeomorphisms of pairs $f : (S^3, \Sigma_1) \to (S^3, \Sigma_2)$.

Indeed, such a diffeomorphism of pairs maps the product structure of $N(\Sigma_1)$ to one for $N(\Sigma_2)$. Consequently, $f(a^+) = (f(a))^+$ for any cycle $a \subset \Sigma_1$. As it will likewise map surfaces with boundary $a$ or $a^+$ we see that

$$\theta_{\Sigma_2}(f_*(\alpha_1), f_*(\alpha_2)) = \theta_{\Sigma_1}(\alpha_1, \alpha_2)$$

where $f_*$ is the map induced on the homology of $\Sigma_1$.

To find the Seifert form for $\Sigma_{T,\epsilon}$ we will use the local models for positive/negative Murasugi sum. Note that the construction of $\Sigma_{T,\epsilon}$ also provides a basis $\{h_v | v \in V\}$ for $H_1(\Sigma_{T,\epsilon})$ with the property that

$$h_v \cap h_{v'} = \begin{cases} +1 & v <_1 v' \\ 0 & \text{otherwise} \end{cases}$$
Lemma 14. The Seifert form for $\Sigma_{T, \epsilon}$ is the form $\theta_{T, \epsilon}$ where

1. for each $v \in V$ let $\theta_T(h_v, h_v') = f(v)$,
2. for $v, v' \in V$ and $\{v, v'\} \notin E$ let $\theta_T(h_v, h_{v'}) = \theta_T(h_{v'}, h_v) = 0$
3. for $v, v' \in V$ with $v < v'$, if
   a. $\epsilon(\{v, v'\}) = +1$, then $\theta_T(h_v, h_{v'}) = 0$ while $\theta_T(h_{v'}, h_v) = 1$
   b. $\epsilon(\{v, v'\}) = -1$, then $\theta_T(h_v, h_{v'}) = -1$ while $\theta_T(h_{v'}, h_v) = 0$

Proof: If $v$ and $v'$ are not joined by an edge in $T$ then $lk(h_v, h_v') = lk(h_{v'}, h_v') = 0$, since the annuli $A_v$ and $A_{v'}$ are located in disjoint balls. Furthermore, for $lk(h_v, h_v')$ we need only consider the situation for the annulus $A_v$, where it is clear that we recover the framing $f(v)$. For $v < v'$ we will use the local models. If $\epsilon = +1$ then the surface for $v'$ lies in a ball on the positive side of the image of $A_v$. Consequently, $h_{v'}$ lies entirely within this ball, and bounds a disc there. So $lk(h_v, h_v') = 0$. When $\epsilon = -1$, the surface lies on the negative side. Thus $h_{v'}$ loops around $h_v$ in the same manner as $(\cos(\theta), 0, \sin(\theta))$ loops around the $y$-axis in $\mathbb{R}^3$, namely in a left-handed manner. Thus, when $\epsilon = -1$ we have $lk(h_v, h_v') = -1$. By the properties of the Seifert form $\theta_{\Sigma}(h_v, h_{v'}) - \theta(h_{v'}, h_v) = -h_v \cap h_{v'} = +1$. Thus when $\epsilon = +1$, $\theta(h_{v'}, h_v) = 1$, but when $\epsilon = -1$, $\theta(h_{v'}, h_v) = -1 + 1 = 0$. This is exactly the form $\theta_T$ defined previously. $\Diamond$

Since $K_T = \partial \Sigma_T$ is a knot the groups $H_1(\Sigma_T; \mathbb{Z})$ and $H_1(S^3_{\Sigma_T}; \mathbb{Z})$ are isomorphic (as in [4] or [5]). In fact, there is an explicit isomorphism provided in [6]: $\Phi_{\Sigma}: H_1(\Sigma; \mathbb{Z}) \to H_1(S^3_{\Sigma}; \mathbb{Z})$ where

$$\Phi_{\Sigma}([a]) = [a^+] - [a^-]$$

This isomorphism has the following naturality: if $f: (S^3, \Sigma_1) \to (S^3, \Sigma_2)$ is an orientation preserving diffeomorphism of pairs, where the surfaces have a single boundary component, then the following commutes

$$H_1(\Sigma_1; \mathbb{Z}) \xrightarrow{\Phi_{\Sigma_1}} H_1(Y_1, \Sigma_1; \mathbb{Z})$$

$$\downarrow (f|_{\Sigma_1})_* \downarrow (f|_{Y_1, \Sigma_1})_*$$

$$H_1(\Sigma_2; \mathbb{Z}) \xrightarrow{\Phi_{\Sigma_2}} H_1(Y_2, \Sigma_2; \mathbb{Z})$$

since $f$ preserves the positive and negative push-offs. Consequently,

$$\overline{\theta}_{\Sigma}([\alpha], [\beta]) = \theta_{\Sigma}(\Phi_{\Sigma}^{-1}([\alpha]), \Phi_{\Sigma}^{-1}([\beta]))$$

is a pairing on $H_1(S^3_{\Sigma}; \mathbb{Z})$ which will be preserved by the restriction to $S^3_{\Sigma_1}$ of an orientation preserving diffeomorphism of pairs, in the same manner as $\theta_{\Sigma}$.

We now compute $\overline{\theta}_{T, \epsilon}$. First we describe the map $\Phi$. 
Lemma 15. Given a matched tree $T$ with framing $f$ and plumbing $\epsilon$, let $\{c_v|v \in V\}$ be the basis for $H_1(S^3_{\Sigma_T,\epsilon})$ given by meridians to the annuli $A_v$, and oriented so that $\text{lk}(c_v, h_v) = +1$. Using these bases, the isomorphism $\Phi_{\Sigma_T,\epsilon}$ is defined by

$$h_v \mapsto \sum_{v < v'} c_{v'} - \sum_{v' < v} c_{v'}$$

Proof: Let $h_v$ be a basis element for $H_1(\Sigma_T)$. We saw above that $h_v \cap h_{v'} = +1$ if $v <_1 v'$ and $h_v \cap h_{v''} = -1$ if $v'' <_1 v$. Then $h_v^+ - h_v^-$ bounds an annulus $C$ which intersects each $h_{v'}$ and $h_{v''}$. This annulus has orientation given by a positive normal to $A_v$, followed by a tangent to $h_v$. Consequently, for $h_v \cap h_{v'} = +1$, the triple $N, h_v, h_{v'}$ is a positively oriented basis. Therefore, $h_{v'}$ intersects $C$ positively. Similarly, for $v'' <_1 v$ the intersection is negative. Thus $C$ minus open neighborhoods of the points of intersection provides a homology relation showing that $\Phi_{\Sigma_T}(\{h_v\})$ is homologous to the sum of the meridians around $h_{v'}$ for $v < v'$ and minus the meridians around $h_{v''}$ for $v'' < v$. These meridians are precisely $c_v$, so

$$\Phi_{\Sigma_T}(h_v) = \sum_{v < v'} c_{v'} - \sum_{v' < v} c_{v'}$$

as required. ◇

To compute $\bar{\theta}_{T,\epsilon}$, however, we need $\Phi^{-1}$.

Lemma 16. In the bases described above, $\Phi^{-1}$ is generated by

$$c_b \mapsto \sum_{w \in W_b} h_w \quad \text{for } b \in B$$

$$c_w \mapsto -\sum_{b \in B_w} h_b \quad \text{for } w \in W$$

Proof: Let $b \in B$ and $w = m(b)$. Then $h_{m(b)} \rightarrow c_b - \sum_{v < m(b)} c_v$. Since $T$ is bipartite, each $v$ with $v <_1 m(b)$ will also be in $B$. In particular, $v < b$ so $W_v \subset W_b$. If we rewrite this as $c_b \rightarrow h_{m(b)} + \sum_{v < m(b)} c_v$ and repeat with each of the $v$’s we will obtain $c_b \rightarrow \sum_{w \in W_b} h_w$. For $w \in W$ we know that $w = m(b)$ for some $b$ and that $w$ is the only vertex satisfying $b >_1 v$. Consequently, $h_b \rightarrow \sum_{b <_1 v} c_v - c_w$. Rearranging we obtain $c_w \rightarrow \sum_{b <_1 v} c_v - h_b$. Each $v$ with $b <_1 v$ is in $W$, so we can repeat with each term in the first sum to obtain $c_w \rightarrow -\sum_{b \in B_w} h_b$. ◇

From this description we can calculate $\theta_T(\Phi^{-1}(c_w), \Phi^{-1}(c_{w'}))$: for $b, b' \in B$

$$\bar{\theta}_T(c_b, c_{b'}) = \sum_{w \in W_b \cap W_{b'}} f(w)$$
while for \( w, w' \in W \)
\[
\overline{\theta}_T(c_w, c_{w'}) = \sum_{b \in B_w \cap B_{w'}} f(b)
\]
Furthermore, for \( v, v' \) let \( \gamma \) be the directed path from \( v' \) to \( v \) (or be empty if there isn’t one). Let \( N_{\pm,m}(\gamma) \) be the number of \( \epsilon = \pm 1 \) edges on \( \gamma \) which are also in \( m \), and \( N_{\pm,\sim,m} \) be the number that are not in \( m \). Then for \( b \in B \) and \( w \in W \), let \( \gamma \) be the directed path from \( b \) to \( w \), or else empty if no such path exists. We have
\[
\overline{\theta}_T(c_w, c_b) = N_{-1,\sim,m}(\gamma) - N_{+1,m}(\gamma)
\]
\[
\overline{\theta}_T(c_b, c_w) = N_{-1,m}(\gamma) - N_{+1,\sim,m}(\gamma)
\]
where it is to be understood that if \( w > b \) then both of these are 0.

We illustrate the latter calculation. Suppose \( w < b \). Starting from \( w \) we will enumerate the vertices up the path until we arrive at \( b \). Thus we get \( w = v_1, \ldots, v_2g = b \) for some \( g \). Since \( \theta_T \) is zero on basis elements, unless they correspond to consecutive vertices, we can see that only the edges in this path contribute to \( \overline{\theta}_T(c_w, c_b) \), thus
\[
\overline{\theta}_T(c_w, c_b) = -(\theta_T(h_2, h_1) + \theta_T(h_2, h_3) + \theta_T(h_4, h_3) + \theta_T(h_4, h_5) + \ldots
\]
\[
+ \theta_T(h_{2g-2}, h_{2g-1}) + \theta_T(h_{2g}, h_{2g-1}))
\]
The edge from \( v_{2i} \) to \( v_{2i-1} \) is in \( m \), while that from \( v_{2i+1} \) to \( v_{2i} \) is not. Consequently, those edge in \( m \) contribute when they evaluate to +1 under \( \epsilon \) since the even number comes first. In this case, the value of \( \theta_T \) will be +1 for this edge, so with the minus sign in front the edges in \( m \) contribute \(-N_{+1,m}(\gamma) \). The edges not in \( m \) contribute a -1 when \( \epsilon = -1 \), so altogether they contribute \(-(-N_{-1,\sim,m}(\gamma)) = N_{-1,\sim,m}(\gamma) \).

5. Sutured manifolds and Floer homology

\( S^3_{\Sigma} = S^3 \setminus \text{Int} N(\Sigma) \) is a sutured manifold in the sense of Gabai (see [4]). Indeed from the product structure on \( N(\Sigma) \) we have \( \partial S^3_{\Sigma} = -\Sigma^+ \cup \Sigma^- \cup \partial \Sigma \times [-1, 1] \). Thus \((S^3_{\Sigma}, \Sigma^+, \Sigma^-)\) is a balanced sutured manifold (4) with suture equal to the annulus \( \partial \Sigma \times [-1, 1] \) and \( R(\gamma) = \Sigma^+ \cup \Sigma^- \).\footnote{However, in the notation of [4] \( R_+(\gamma) = \Sigma^- \) and \( R_-(\gamma) = \Sigma^+ \) since the oriented normal to \( \Sigma^+ \) points into \( S^3_{\Sigma} \)}

An orientation preserving diffeomorphism \( f : M_1 \to M_2 \) between sutured manifolds \((M_1, \gamma_1)\) and \((M_2, \gamma_2)\) is called an identification of sutured manifolds if it restricts to \( R(\gamma_1) = (\partial M_1) \setminus \gamma_1 \) in a manner preserving the orientation assigned to each component by the sutures. Applied to our setting, \( f : S^3_{\Sigma_1} \to S^3_{\Sigma_2} \) must restrict to each of \( \Sigma_1^\pm \) as an orientation preserving diffeomorphism with \( \Sigma_2^\pm \). Consequently, orientation
preserving diffeomorphisms of pairs \((S^3, \Sigma)\) induce identifications of the corresponding sutured manifolds \((S^3_\Sigma, \Sigma^\pm)\).

A. Juhasz, [7], associates a \(\mathbb{Z}/2\mathbb{Z}\) vector space \(\text{SFH}(M, \gamma; s)\) to any balanced sutured manifold \((M, \gamma)\) and a choice of (relative) Spin\(^c\) structure \(s \in \text{Spin}^c(M, \gamma)\). We will not review the definition of sutured Floer homology here. Instead we list several properties of this invariant.

First, recall that there is an action of \(H_1(M; \mathbb{Z})\) on \(\text{Spin}^c(M, \gamma)\) such that for any \(s_1, s_2 \in \text{Spin}^c(M, \gamma)\) there exists a unique \(h \in H_1(M; \mathbb{Z})\) with \(h \cdot s_1 = s_2\). We will denote this by \(h = s_2 - s_1\). Then

1. \(\text{SFH}(M, \gamma; s) \not\approx \{0\}\) for finitely many \(s \in \text{Spin}(M, \gamma)\).
2. Any diffeomorphism \(f : (M_1, \gamma_1) \to (M_2, \gamma_2)\) of sutured manifolds induces a bijection \(F^{\text{Sp}} : \text{Spin}^c(M_1, \gamma_1) \to \text{Spin}^c(M_2, \gamma_2)\) such that \(F^{\text{Sp}}(\alpha \cdot s) = f_*(\alpha) \cdot F^{\text{Sp}}(s)\), where \(\alpha \in H_1(M_1; \mathbb{Z})\),
3. For a diffeomorphism \(f\) as in item [2] and any \(s \in \text{Spin}^c(M_1, \gamma_1)\) there is an isomorphism \(\text{SFH}(M_1, \gamma_1; s) \xrightarrow{fs} \text{SFH}(M_2, \gamma_2; F^{\text{Sp}}(s))\)

We note that \(\text{SFH}(M, \gamma; s)\) also comes with an invariant relative grading, but we will not need this in the sequel.

We will now describe \(\text{SFH}(S^3_{\Sigma_T})\) for a surface from a labeled, matched tree. Since these surfaces arise from repeated Murasugi sums we will use proposition 8.6 of [8], which shows that

\[
\text{SFH}(S^3_{\Sigma_T}) \approx \text{SFH}(S^3_{\Sigma_1}) \otimes \text{SFH}(S^3_{\Sigma_2})
\]

when \(\Sigma\) is the Murasugi sum of \(\Sigma_1\) and \(\Sigma_2\).

In addition, when \(\Sigma\) is the Murasugi sum of \(\Sigma_1\) and \(\Sigma_2\), \(\text{Spin}^c(S^3_{\Sigma_T}) = \text{Spin}^c(S^3_{\Sigma_1}) \times \text{Spin}^c(S^3_{\Sigma_2})\), and this is a torsor of \(H_1(S^3_{\Sigma_1}; \mathbb{Z}) \cong H_1(S^3_{\Sigma_1}; \mathbb{Z}) \oplus H_1(S^3_{\Sigma_2}; \mathbb{Z})\), with \(H_1(S^3_{\Sigma_1}; \mathbb{Z})\) acting solely on the \(\text{Spin}^c(S^3_{\Sigma_2})\) factor, and \(H_1(S^3_{\Sigma_2}; \mathbb{Z})\) acting solely on the \(\text{Spin}^c(S^3_{\Sigma_1})\) factor.

To apply this to the sums defining \(\Sigma_T\) we also need the sutured Floer homology for the complement of an unknotted, twisted annulus in \(S^3\). Fortunately, A. Juhasz computed this in [9].

Let \(A_p \subset S^3\) be the unknotted band with \(p \in \mathbb{Z}\) right handed full twists. Then \(N(A_p) \cong S^1 \times D^2\) and \(S^3 \setminus \text{Int}N(A_p) \cong D^2 \times S^1\). The two components of \(\partial A_p\) give two curves on \(\partial N(A_p)\) which go around the longitude once, and \(p\) times around the meridian, of the unknotted circle forming the core of \(A_p\). Thinking of the the boundary torus as the boundary of \(S^3 \setminus \text{Int}N(A_p)\) switches the meridian and longitude. Thus, in \(S^3 \setminus \text{Int}N(A_p) \cong D^2 \times S^1\), the two curves provide sutures which wind
around the boundary of $D^2$ once and $p$ times around the $S^1$ factor.

Thus the sutured manifold $S^3 \setminus \text{Int} N(A_p)$ will be diffeomorphic to $T(p, 1; 2)$, where, following A. Juhasz, $[9]$, $T(p, q; n)$ is the sutured manifold $(S^1 \times D^2, \gamma)$ with suture $\gamma$ consisting of $n = 2k + 2$ parallel disjoint curves in the homology class $(p, q) \in H_1(S^1 \times S^1; \mathbb{Z})$. Proposition 9.1 of $[9]$ computes the sutured Floer homology of $T(p, q; n)$ in general. For $T(p, 1; 2)$ this proposition states that there is an identification $\text{Spin}^c(T(p, 1; 2)) \leftrightarrow \mathbb{Z}$ such that

$$\text{SFH}(T(p, 1; 2), i) \cong \begin{cases} 
\mathbb{Z} & 0 \leq i < p \\
0 & \text{otherwise}
\end{cases}$$

Furthermore, the difference of two structures in $\text{Spin}^c(T(p, 1; 2))$ is a multiple of the meridian $c$ of the twisted band.

Repeatedly applying these propositions to the Murasugi sums defining $\Sigma_T$, we see that

$$Q_T = \{ s \in \text{Spin}^c(S^3_{\Sigma_B}) | \text{SFH}(S^3_{\Sigma_B}) \neq 0 \} \cong \prod_{v \in V} [0, |f(v)| - 1]$$

The product structure reflects the action of the meridional basis $\{c_v | v \in V\}$. Namely, there is a $\text{Spin}^c$ structure $s$ so that all the other structures with non-zero sutured Floer homology can be found by taking $s + \sum a_v c_v$ with $a_v$ in the interval for $[0, |f(v)| - 1]$. In particular, the lengths of the intervals come from the identifications in the calculation of $\text{SFH}(T(p, 1; 2))$.

6. Symmetries

Finally we prove

**Theorem 17.** Let $T$ be a matched tree with $2n$ vertices, equipped with an framing map $f : V \to \mathbb{Z} \setminus \{0, \pm 1\}$ such that $v \to |f(v)|$ is injective. Suppose further that gluing arcs are fixed for each annulus $A_v$, sufficient to implement the construction of $\Sigma_{T, \epsilon}$ for any edge labeling $\epsilon : E \to \{\pm 1\}$. Then, using the same choice of arcs, $\Sigma_{T, \epsilon}$ is not equivalent to $\Sigma_{T, \epsilon'}$ unless $\epsilon = \epsilon'$.

**Note:** We disallow $f(v)$ to have absolute value 1 so that $Q_{\Sigma_T}$ will have dimension $2n$.

**Corollary 18.** For $T, f$ as above, the knot $K_T$ is the boundary of $2^{2n-1}$ distinct embedded surfaces of genus $n$.

**Proof:** We show that there is no orientation preserving diffeomorphism of pairs $f : (S^3, \Sigma_{T, \epsilon}) \to (S^3, \Sigma_{T, \epsilon'})$. To do this, we start by describing the possible maps $f_* : H_1(S^3_{\Sigma_{T, \epsilon}}; \mathbb{Z}) \to H_1(S^3_{\Sigma_{T, \epsilon'}}; \mathbb{Z})$ in terms of the meridional bases $\{c_v\}$ and $\{c'_v\}$. These maps are severely constrained by the properties of sutured Floer homology
and the injectivity of $f$.

The structure of $Q_{T,\epsilon}$ does not depend on $\epsilon$. Consequently, a map $f : (S^3, \Sigma_{T,\epsilon}) \to (S^3, \Sigma_{T,\epsilon'})$ must induce a map $F^{Sp}$ which takes the supporting parallelepiped for $S_3^{\Sigma_{T,\epsilon}}$ to that for $S_3^{\Sigma_{T,\epsilon'}}$. In fact $F^{Sp}$ must take vertices of $Q_{T,\epsilon}$ to vertices of $Q_{T,\epsilon'}$, edges to edges, faces to faces, etc. Indeed, $s' - s = h$ with $l(h) > 0$, we know that since $F^{Sp}(s') - F^{Sp}(s) = f_*(h)$, thus $F^{Sp}(s') - F^{Sp}(s) = h'$ with $(f^*)^{-1}(l)(h') > 0$. In particular, a supporting hyperplane for the set $Q_{T,\epsilon}$ must be mapped to a supporting hyperplane for $Q_{T,\epsilon'}$. Since vertices, edges and such are distinguished by the dimension of these supporting hyperplanes, we have the result we desired.

To make use of the injectivity of $|f|$, note also that if $s' - s = k[h]$ where $k \in \mathbb{Z}$ and $[h]$ is primitive, then $F^{Sp}(s') - F^{Sp}(s) = kf_*([h])$ as well. Thus, the divisibility of the difference of two Spin structures is preserved by $F^{Sp}$.

Suppose vertex $s$ for $Q_{T,\epsilon}$ is taken to vertex $u$ for $Q_{T,\epsilon'}$. Then each edge out of $s$ for $Q_{T,\epsilon}$ must be taken to an edge out of $u$. Furthermore, since the values of $|f(v)|$ are the divisibilities of the edges of the parallelepiped $Q_{T,\epsilon}$ out of $s$, when $|f|$ is injective each edge must be mapped to the unique edge emerging from $u$ with the same divisibility. Since the difference of the two vertices along this edge is a multiple of the primitive element $c_v$, we see that $c_v \to \pm c'_v$ under $f_*$ for every vertex $v$. As these span $H_1(S^3_{\Sigma_{T,\epsilon}})$, these described all the possibilities for $f_*$.

However, by comparing $\overline{\theta}_{T,\epsilon}$ and $\overline{\theta}_{T,\epsilon'}$, we can see that none of these possibilities can occur for an orientation preserving diffeomorphism of pairs $f : (S^3, \Sigma_{T,\epsilon}) \to (S^3, \Sigma_{T,\epsilon'})$.

First, suppose there is an edge $e \in m$ with $\epsilon(e) = -\epsilon'(e)$. We may assume that $\epsilon(e) = +1$. Since $e \in m$, $e$ must be an edge directed from $b \in B$ to $w = m(b) \in W$. By our calculation in section 2, $\overline{\theta}_{T,\epsilon}(c_b, c_w) = N_{-1,m}(\gamma) - N_{+1,\sim_m}(\gamma)$. In our case $\gamma = e$, so $N_{-1,m}(\gamma) = 0$ while $N_{+1,\sim_m}(\gamma) = 0$. For $\epsilon'(e) = -1$, we have $N_{-1,m}(\gamma) = 1$ while $N_{+1,\sim_m}(\gamma) = 0$. Thus $\overline{\theta}_{T,\epsilon}(c_b, c_w) = 0$ while $\overline{\theta}_{T,\epsilon'}(c'_b, c'_w) = 1$. From above the map $f$ can only map $c_b \to \pm c'_b$ and $c_w \to \pm c'_w$. As none of these choices can make the pairings equal, we see that there can be no map $f$ in this case.

Second, assume that $\epsilon(e) = \epsilon'(e)$ for any edge $e \in m$. If $\epsilon \neq \epsilon'$ there is an edge $e \not\in m$ with $\epsilon(e) = -\epsilon'(e)$. Again, we assume that $\epsilon(e) = +1$. Since the edge $e$ is not in $m$, it must be directed from a vertex $w_2 \in W$ to a vertex $b_1 \in B$. Let $b_2 = m^{-1}(w_2)$ and $w_1 = m(b_1)$. We consider the directed path from $b_2$ to $w_1$:

$$b_2 \xrightarrow{g} w_2 \xrightarrow{c} b_1 \xrightarrow{d} w_1$$
By assumption $\epsilon(g) = \epsilon'(g)$ and $\epsilon(d) = \epsilon'(d)$. Once again we compute, $\vartheta_{T,\epsilon}(c_b, c_w) = N_{-1,m}(\gamma) - N_{+1,-m}(\gamma) = N - 1$ where $N$ is the number of $-1$’s in $\epsilon(g)$, $\epsilon(d)$. On the other hand, $\vartheta_{T,\epsilon'}(c'_b, c'_w) = N'_{-1,m}(\gamma) - N'_{+1,-m}(\gamma) = N - 0 = N$. Changing the signs of $c_b$ and/or $c_w$ can only change the value from $N - 1$ to $1 - N$. Thus, $f$ can only exist if $N$ satisfies either $N - 1 = N$, which has no solutions, or $1 - N = N$. The latter has no integer solutions, so we are done.

We note that using an $f$ assigning negative numbers to $b \in B$ and positive numbers to $w \in W$ makes $K_T$ have an alternating, prime, reduced diagram. By a well known theorem of Menasco, this means $K_T$ is also prime. Consequently,

**Corollary 19.** The prime, alternating knot $K_{T_g}$ defined by the matched tree $T_g$:

$$w_1 \leftarrow b_1 \leftarrow w_2 \leftarrow \cdots \leftarrow w_{2g} \leftarrow b_{2g}$$

equipped with framing $f(w_i) = 2i + 1$ and $f(b_i) = -2i$, bounds at least $2^{2g-1}$ inequivalent oriented surfaces of genus $g$.

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