ON THE PERFECTION OF SCHEMES

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Abstract. We present a detailed and elementary construction of the inverse perfection of a scheme and discuss some of its main properties. We also establish a number of auxiliary results (for example, on inverse limits of schemes) which do not seem to appear in the literature.

1. Introduction

Let \( p \) be a prime number and let \( \mathbb{F}_p \) denote the field with \( p \) elements. In the course of our review of the construction of the perfect Greenberg functor in [BGA], we were hampered by the lack of an adequate reference work on the subject of (inverse) perfections of \( \mathbb{F}_p \)-schemes. Although the classical reference [Gre] presents in some detail the construction of the inverse perfection \( Y_{pf} \) of an \( \mathbb{F}_p \)-scheme \( Y \), it does not discuss its main properties. On the other hand, the relatively recent preprint [BS] briefly discusses some of the main properties of the indicated construction (see [BS, Lemmas 3.4 and 3.8], parts of which overlap with some of the results presented here), but it does not address the perfection of \( \mathbb{F}_p \)-group schemes. Our aim in this paper is to present a detailed and elementary construction of the inverse perfection of an \( \mathbb{F}_p \)-scheme and discuss some of its properties. The (inverse) perfection functor has played, and continues to play, a significant role in algebraic geometry (see, for example, [Ser1, Ser2, BD, BW, Pep, KL]). We believe that our presentation will be useful to all students and researchers that at some point in their studies will need to consider the (inverse) perfection of an \( \mathbb{F}_p \)-scheme.

We briefly indicate the contents of the individual Sections.

Section 2 presents some basic results on the fpqc and fppf topologies. These statements may be well-known to some readers but, to our knowledge, they do not appear in the literature. Section 3 discusses certain basic properties of projective limits of schemes that, to our surprise, we could not find in the standard literature on the subject. In particular, Proposition 3.8 shows that, if \( k \) is any field, then the inverse limit functor is exact on certain types of “Mittag-Leffler” short exact sequences of projective systems in the category of commutative \( k \)-group schemes. Section 4 is a detailed discussion of the construction of the perfect closure (or direct perfection) of an \( \mathbb{F}_p \)-algebra. The developments of Section 4 are then extended to

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the category of $\mathbb{F}_p$-schemes in Section 5. The final Section 6 discusses exactness properties of the inverse perfection functor on the category of group schemes over a perfect field of positive characteristic $p$.

We will write $|X|$ for the underlying topological space of a scheme $X$. Further, if $S$ is a scheme, we will write $(\text{Sch}/S)$ for the category of $S$-schemes.

2. The flat topologies

Let $S$ be a scheme and let $C$ be a full subcategory of $(\text{Sch}/S)$ which is stable under (fiber) products and contains the identity morphism of $S$. Recall from [Vis, §2.3.2, pp. 27-28] that an fpqc morphism is a faithfully flat morphism of schemes $f: X \to Y$ with the following property: if $x$ is a point of $X$, then there exists an open neighborhood $U$ of $x$ such that the image $f(U)$ is open in $Y$ and the induced morphism $U \to f(U)$ is quasi-compact. See [Vis, Proposition 2.33, p. 27] for a list of equivalent properties. Clearly, a faithfully flat and quasi-compact morphism of schemes is an fpqc morphism. Further, the class of fpqc morphisms is stable under base change [Vis, Proposition 2.35(v), p. 28]. An fppf morphism is a faithfully flat morphism locally of finite presentation. By [Vis, Proposition 2.35(iv), p. 28], every fppf morphism is an fpqc morphism. The fpqc (respectively, fppf) topology on $C$ is the topology where the coverings are collections of flat morphisms $\{X_\alpha \to X\}$ in $C$ such that the induced morphism $\coprod X_\alpha \to X$ is an fpqc (respectively, fppf) morphism. Clearly, the fpqc topology is finer than the fppf topology. If $\tau = \text{fpqc}$ or $\text{fppf}$, we will write $C^\tau$ for the corresponding site and $C^\tau_{\text{sheaf}}$ for the category of sheaves of sets on $C^\tau$. For either of the sites mentioned above, every representable presheaf is a sheaf [Vis, Theorem 2.55, p. 34] and the induced functor

$$h_S: C \to C^\tau_{\text{sheaf}}, Y \mapsto \text{Hom}_S(-, Y),$$

is fully faithful. A sequence $1 \to F \to G \to H \to 1$ of groups in $C$ will be called exact for the $\tau$ topology on $C$ if the corresponding sequence of sheaves of groups, namely $1 \to h_S(F) \to h_S(G) \to h_S(H) \to 1$, is exact, i.e., the first nontrivial map identifies $h_S(F)$ with the kernel of $h_S(G) \to h_S(H)$ and the latter map is locally surjective, i.e., for every $U \in C$ and $s \in h_S(H)(U)$, $s$ can be locally lifted to $h_S(G)(U)$ in the $\tau$ topology [Mil, p. 122, lines 22-24].

Remarks 2.1. Let $q: G \to H$ be a morphism of groups in $C$.

(a) It is not difficult to check that, if $h_S(q): h_S(G) \to h_S(H)$ is locally surjective for the $\tau$ topology, then $h_S(q)$ is an epimorphism of sheaves for the indicated topology [SGA3, IV, Definition 1.1], i.e., the induced map of sets $\text{Hom}(h_S(H), \mathcal{F}) \to \text{Hom}(h_S(G), \mathcal{F})$ is injective for every $\mathcal{F} \in C^\tau_{\text{sheaf}}$. For the converse, see [SGA3, IV, Remark 4.4.8].

(b) We will show below that, if $h_S(q)$ is locally surjective for the $\tau$ topology, i.e., an epimorphism of sheaves in the $\tau$ topology (see (a)), then $q$ is a surjective morphism of schemes. However, the converse may fail to hold. For example,
Let \( p \) be a prime number, set \( k = \mathbb{F}_p \) and let \( q: 0 \to \alpha_p \) be the canonical morphism, where \( \alpha_p = \text{Spec}(k[x]/(x^p)) \) is the kernel of the Frobenius endomorphism \( \mathbb{G}_{a,k} \to \mathbb{G}_{a,k} \). Since \( \alpha_p \) is a one-point scheme, \( q \) is clearly a surjective morphism of schemes. However, it is easy to check that \( h_S(q) \), where \( S = \text{Spec} k \), is not locally surjective.

**Lemma 2.2.** Let \( 1 \to F \to G \to H \to 1 \) be a sequence of groups in \( \mathcal{C} \) which is exact for the \( \tau \) topology on \( \mathcal{C} \), where \( \tau = \text{fppf} \) or \( \text{fpqc} \). Then \( i \) identifies \( F \) with the scheme-theoretic kernel of \( q \), i.e., \( F = \text{Ker} q = G \times_H S \), and \( q \) is a surjective morphism of schemes.

**Proof.** Since \( \text{Ker} q \) represents the functor \( \text{Ker}(h_S(q): h_S(G) \to h_S(H)) = h_S(F) \), i.e., \( h_S(\text{Ker} q) = h_S(F) \), the first assertion is clear. To show that \( q: G \to H \) is a surjective morphism of schemes, it suffices to check that, for every field \( K \) and every morphism \( h: \text{Spec} K \to H \), there exists an extension \( K' \) of \( K \) and a morphism \( g: \text{Spec} K' \to G \) such that \( q \circ g = h \circ j \), where \( j: \text{Spec} K' \to \text{Spec} K \) is the canonical morphism (see [EGA I, Proposition 3.6.2, p. 244]). Since \( h_S(q) \) is locally surjective, there exist a \( \tau \) covering \( \{ u_\alpha: Y_\alpha \to \text{Spec} K \} \) in \( \mathcal{C} \) and \( S \)-morphisms \( g_\alpha: Y_\alpha \to G \) such that \( q \circ g_\alpha = h \circ u_\alpha \) for every \( \alpha \). Now, since \( \coprod Y_\alpha \to \text{Spec} K \) is surjective, there exists an index \( \alpha_0 \) such that \( Y_{\alpha_0} \neq \emptyset \). Choose \( y \in Y_{\alpha_0} \) and let \( i: \text{Spec} \kappa(y) \to Y_{\alpha_0} \) be the canonical morphism. Then \( K' = \kappa(y) \) and \( g = g_{\alpha_0} \circ i: \text{Spec} K' \to G \) are the required extension and morphism, respectively. \( \square \)

**Lemma 2.3.** Let \( q: G \to H \) be a morphism of groups in \( \mathcal{C} \). If \( q \) is an fppf (respectively, fpqc) morphism, then the sequence

\[
1 \to \text{Ker} q \to G \xrightarrow{q} H \to 1
\]

is exact for the fppf (respectively, fpqc) topology on \( \mathcal{C} \).

**Proof.** It suffices to check that \( h_S(q) \) is locally surjective in the fppf (respectively, fpqc) topology. Let \( u: Z \to H \) be a morphism in \( \mathcal{C} \). If \( q \) is an fppf (respectively, fpqc) morphism, then \( q \times_H Z: G \times_H Z \to Z \) is an fppf (respectively, fpqc) morphism. Thus \( q \times_H Z \) is an fppf (respectively, fpqc) covering and \( q \circ \text{pr}_1 = u \circ (q \times_H Z) \), where \( \text{pr}_1: G \times_H Z \to G \) is the first projection. The lemma is now clear. \( \square \)

In the next two statements, \( \overline{k} \) is a fixed algebraic closure of a field \( k \).

**Proposition 2.4.** Let \( k \) be a field and let \( q: G \to H \) be a flat morphism of \( k \)-group schemes locally of finite type.

(i) If \( q \) is surjective, then \( q \) is an fppf morphism.

(ii) If \( q(\overline{k}): G(\overline{k}) \to H(\overline{k}) \) is surjective, then \( q \) is surjective.

**Proof.** Since \( G \) is locally of finite type, \( q \) is locally of finite presentation by [EGA I, 6.2.1.2 and Proposition 6.2.3(v), p. 298] and assertion (i) is clear. If \( G \) and \( H \) are of finite type, assertion (ii) is well-known [DG, I, §3, Corollary 6.10, p. 96]. To prove
(ii) when $G$ and $H$ are only locally of finite type, we may assume that $k = \overline{k}$ by Proposition 3.6.4, p. 245]. Let $G^0$ denote the identity component of $G$. Since $q$ is an open morphism, $q(G^0)$ is open in $H$. On the other hand, since both $G^0$ and $H^0$ are of finite type by [SGA3 new, VI, Proposition 2.5(ii)], the induced morphism $q^0: G^0 \to H^0$ is quasi-compact. Consequently, $q(G^0) = q^0(G^0)$ is closed in $H$ by [SGA3 new, VI, Proposition 2.5.2(a)]. We conclude that $q(G^0)$ is both open and closed in $H$. Now, if $C$ is a connected component of $G$, then $C = xG^0$ for some $x \in G(k)$ [SGA3 new, VI, proof of Corollary 2.4.1], whence $q(C)$ is open and closed in $H$. Now let $y \in H$ and let $C'$ be the connected component of $H$ which contains $y$. Since $H(k)$ is very dense in $H$ [EGA IV 3, Corollary 10.4.8], there exists a point $z$ in $C' \cap H(k) = C' \cap q(G(k))$. Let $x \in G(k)$ be such that $z = q(x)$. If $C'$ is the connected component of $G$ which contains $x$, then $z \in C' \cap q(C')$, whence $C' \cap q(C) \neq \emptyset$. Thus $C' \subseteq q(C)$ since $q(C)$ is open and closed in $H$. Consequently, $y \in q(C) \subseteq q(G)$, which completes the proof.

$\square$

**Corollary 2.5.** Let $k$ be a field and let $q: G \to H$ be a flat morphism of $k$-group schemes locally of finite type. Assume that $q(\overline{k}): G(\overline{k}) \to H(\overline{k})$ is surjective. Then the sequence of $k$-group schemes

$$1 \to \text{Ker} q \to G \xrightarrow{q} H \to 1$$

is exact for both the fppf and fpqc topologies on $(\text{Sch}/k)$.

**Proof.** This is immediate from the proposition and Lemma 2.3 $\square$

### 3. Projective limits of schemes

To our knowledge, the results presented in this Section do not appear in the standard literature on projective limits of schemes (e.g., in [EGA IV 3, §8]). These results will be used in Section 5 to extend to the category of schemes the ring-theoretic constructions of Section 4.

In this Section, $k$ denotes an arbitrary field.

If $S$ is a scheme, $\Lambda$ is a directed set and $(X_\lambda, u_{\lambda, \mu}; \lambda, \mu \in \Lambda)$ is a projective system of $S$-schemes with affine transition morphisms, then $X = \lim X_\lambda$ exists in the category of $S$-schemes. Further, for every $S$-scheme $Z$, there exists a canonical bijection

$$\text{Hom}_S(Z, \lim X_\lambda) = \lim \text{Hom}_S(Z, X_\lambda).$$

See [EGA IV 3, Proposition 8.2.3 and Lemma 8.2.4].

**Proposition 3.2.** Let $(X_n)$ and $(Y_n)$ be projective systems of $k$-schemes with index set $\mathbb{N}$ and affine transition morphisms and let $X = \lim X_n$ and $Y = \lim Y_n$ be the corresponding limits in the category of $k$-schemes. Further, let $(f_n): (X_n) \to (Y_n)$ be a morphism of projective systems and let $f = \lim f_n: X \to Y$ be its limit. Consider, for a morphism of $k$-schemes, the property of being:
(i) quasi-compact;
(ii) quasi-separated;
(iii) separated;
(iv) affine;
(v) a closed immersion;
(vi) flat.

If \( P \) denotes one of above properties and there exists an \( n_0 \in \mathbb{N} \) such that the \( k \)-morphisms \( f_n : X_n \to Y_n \) have property \( P \) for all \( n \geq n_0 \), then \( f : X \to Y \) has property \( P \) as well.

**Proof.** For every \( n \in \mathbb{N} \), there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X_n & \xrightarrow{f_n} & Y_n,
\end{array}
\]

where the vertical morphisms are affine \([EGA, IV_3 (8.2.2)]\) and therefore quasi-compact and separated. Consequently, if a single morphism \( f_n \) has one of the properties (i)-(iv), then \( f \) has the same property by \([EGA, II, Proposition 1.6.2]\) and \([EGA I_{new}, Propositions 5.3.1(v), p. 279, 6.1.5(v), p. 291, and 6.1.9(v), p. 294]\).

For (v), we may localize \([EGA I_{new}, Corollary 4.2.4(ii), p. 262]\) to reduce to the case \( (f_n) = (\text{Spec} \phi_n) \), where \( (\phi_n) : (B_n) \to (A_n) \) is a morphism of direct systems of \( k \)-algebras and the maps \( \phi_n : B_n \to A_n \) are surjective homomorphisms of \( k \)-algebras for every \( n \geq n_0 \) \([EGA I_{new}, Remark 4.2.1.1, p. 260]\). Since the direct limit functor is exact on the category of \( k \)-algebras, the homomorphism of \( k \)-algebras \( \varinjlim \phi_n : \varinjlim B_n \to \varinjlim A_n \) is surjective. This completes the proof for property (v).

In the case of property (vi), the proposition follows (after localizing, as above) from \([Bou, I, §2, no. 7, Proposition 9, p. 20]\). □

**Remarks 3.3.**

(a) In the setting of the proposition, if \((X_n)\) and \((Y_n)\) are projective systems of \( k \)-group schemes (in particular, their transition morphisms are morphisms of \( k \)-group schemes, i.e., homomorphisms), then \( f : X \to Y \) is a morphism of \( k \)-group schemes. This follows from \((3.1)\).

(b) Clearly, properties of a morphism whose definition includes the condition of being (locally) of finite type (e.g., proper, smooth, quasi-projective, etc.) are not preserved in the limit.

In general, open immersions are not preserved in the limit, as Example 3.5 below shows. However, open and arbitrary immersions are preserved in the limit in a particular case that will be relevant in the next section.
Proposition 3.4. In the setting of Proposition 3.2 assume in addition that the transition maps of the systems \((X_n)\) and \((Y_n)\) are homeomorphisms. If, for some \(n_0 \in \mathbb{N}\), \(f_n: X_n \to Y_n\) is an immersion (respectively, open immersion) for every \(n \geq n_0\), then the limit morphism \(f: X \to Y\) is an immersion (respectively, open immersion).

Proof. By [EGA IV], Proposition 8.2.9, we may identify \(|X|\) and \(|X_n|\) for every \(n \in \mathbb{N}\), and similarly for \(Y\). Under these identifications, \(|f|: |X| \to |Y|\) is identified with \(|f_n|: |X_n| \to |Y_n|\) for every \(n \in \mathbb{N}\). Thus, by [EGA I, Proposition 4.2.2, p. 260], it suffices to check that \(f\) induces a surjection (respectively, bijection) \(\mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X, x}\) for every \(x \in X\). By [EGA IV, Corollary 8.2.12] and the identifications made above, there exists a canonical isomorphism

\[\mathcal{O}_X = \lim_{n \geq n_0} \psi_n^*(\mathcal{O}_{X_n}),\]

where \(\psi_n: X \to X_n\) is the canonical projection, and similarly for \(Y\). Thus, since \(\mathcal{O}_{Y_n, f(x)} \to \mathcal{O}_{X_n, x}\) is a surjection (respectively, bijection) for every \(n \geq n_0\), the direct limit morphism \(\mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X, x}\) has the same properties. \(\square\)

Example 3.5. In general, as noted above, open immersions are not preserved in the limit, as the following example shows (compare with [EGA IV, comment preceding (8.3.9)]). Let \(A\) be an integral domain which is not a field and let \(K\) be its field of fractions. Note that \(K\) is the inductive limit of the localizations \(A_f\) as \(f\) ranges over the set \(A \setminus \{0\}\) ordered by divisibility. Set \(X_f = \text{Spec } A_f\) and \(Y_f = \text{Spec } A\) for every \(f \in A \setminus \{0\}\) and consider the morphism of projective systems (with evident transition morphisms) \((X_f) \to (Y_f)\), where each map \(X_f \to Y_f\) is the canonical open immersion \(\text{Spec } A_f \to \text{Spec } A\). Then the limit morphism is the canonical morphism \(\text{Spec } K \to \text{Spec } A\), which is not, in general, an open immersion. To obtain an example with index set \(\mathbb{N}\), choose \(A = \mathbb{Q}[t_m; m \in \mathbb{N}]\) and, for every \(n \in \mathbb{N}\), set \(f_n = \prod_{m=1}^n t_m\). Then, choosing \(X_{f_n} = \text{Spec } A_{f_n}\) and \(Y_{f_n} = \text{Spec } A\) as above, we obtain the desired example.

Proposition 3.6. Let \((X_n)\) be a projective system of finite \(k\)-schemes with index set \(\mathbb{N}\). Then \(X = \varprojlim X_n\) is an affine \(k\)-scheme of dimension zero and every residue field of \(X\) is an algebraic extension of \(k\).

Proof. For every \(n \in \mathbb{N}\), set \(X_n = \text{Spec } A_n\) and let \(A = \varprojlim A_n\), so that \(X = \text{Spec } A\). Since each \(k\)-algebra \(A_n\) is integral over \(k\), \(A\) is integral over \(k\) as well. Thus, by [Liu] Proposition 5.10(a), p. 69, \(\dim X = \dim A = 0\). Finally, if \(m\) is a maximal ideal of \(A\), then \(A/m\) is integral and therefore algebraic over \(k\). \(\square\)

The next statement concerning abelian groups is well-known. We sketch its proof in order to adapt it to the setting of commutative \(k\)-group schemes in Proposition 3.8 below.
Lemma 3.7. Let

\[ 0 \rightarrow (F_n) \rightarrow (G_n) \rightarrow (H_n) \rightarrow 0 \]

be an exact sequence of projective systems of abelian groups with index set \( \mathbb{N} \). If the transition morphisms of the system \((F_n)\) are surjective, then the sequence of abelian groups

\[ 0 \rightarrow \varprojlim F_n \rightarrow \varprojlim G_n \rightarrow \varprojlim H_n \rightarrow 0 \]

is exact.

Proof. (Sketch. For more details, see [EGA I new, Lemma 7.2.8, p. 175]). For every \( n \in \mathbb{N} \), let \( f_n : G_n \rightarrow H_n \) be the given homomorphism. It is well-known that the nontrivial part of the present proof is to establish the surjectivity of \( \varphi = \varprojlim \varphi_n \in H \), where \((\varphi_n) \in (H_n)\) is a coherent sequence, i.e., \( w_n(\varphi_n) = \varphi_{n-1} \). Since \( f_1 \) is surjective, there exists a lifting \( \psi_1 \in G_1 \) of \( \varphi_1 \), i.e., \( f_1(\psi_1) = \varphi_1 \). Now, since \( f_2 \) is surjective, there exists a lifting \( \psi_2' \in G_2 \) of \( \varphi_2 \), i.e., \( f_2(\psi_2') = \varphi_2 \), but \( \psi_2' \) may not be coherent with \( \psi_1 \), i.e., \( v_2(\psi_2') \neq \psi_1 \). To obtain a lifting \( \psi_2 \) of \( \varphi_2 \) which is, in fact, coherent with \( \psi_1 \), we proceed as follows. Since \( f_1(\psi_1) = \varphi_1 = w_2(\varphi_2) = v_2(f_2(\psi_2')) = f_1(v_2(\psi_2')) \), the difference \( \vartheta = \psi_1 - v_2(\psi_2') \in \text{Ker} f_1 = F_1 \). Now, since \( u_2 : F_2 \rightarrow F_1 \) is surjective by hypothesis, there exists \( \psi_2'' \in F_2 \) such that \( u_2(\psi_2'') = \vartheta \). Now set \( \psi_2 = \psi_2' + \psi_2'' \in G_2 \). Then \( \psi_2 \) is a lifting of \( \varphi_2 \) which is coherent with \( \psi_1 \), i.e., \( f_2(\psi_2) = \varphi_2 \) and \( v_2(\psi_2) = \psi_1 \).

We now repeat the above argument starting with \( \varphi_2 \) in place of \( \varphi_1 \) to obtain a lifting \( \psi_3 \) of \( \varphi_3 \) which is coherent with \( \psi_2 \), and so forth. In this way we obtain a coherent sequence \((\psi_n) \in (G_n)\) which is a lifting of \((\varphi_n) \in (H_n)\). Then \( \psi = \varprojlim \psi_n \in G \) is a lifting of \( \varphi \in H \), i.e., \( f(\psi) = \varphi \), which completes the proof. \( \square \)

The following statement extends the previous lemma to the category of commutative group schemes over a field \( k \).

Proposition 3.8. Let

\[ 0 \rightarrow (F_n) \rightarrow (G_n) \rightarrow (H_n) \rightarrow 0 \]

be a sequence of projective systems of commutative \( k \)-group schemes indexed by \( \mathbb{N} \) with affine transition morphisms. Assume that the following conditions hold.

(i) For every \( n \in \mathbb{N} \), the sequence of commutative \( k \)-group schemes

\[ 0 \rightarrow F_n \xrightarrow{i_n} G_n \xrightarrow{f_n} H_n \rightarrow 0 \]

is exact for the fpqc topology on \((\text{Sch}/k)\).

(ii) For every \( n \in \mathbb{N} \), \( f_n : G_n \rightarrow H_n \) is flat and quasi-compact.

(iii) The transition morphisms of the system \((F_n)\) are surjective.

Then the sequence of commutative \( k \)-group schemes

\[ 0 \rightarrow \varprojlim F_n \rightarrow \varprojlim G_n \rightarrow \varprojlim H_n \rightarrow 0 \]
is exact for the fpqc topology on \((\text{Sch}/k)\).

**Proof.** By (i) and Lemma 2.2, \(f_n\) is surjective for every \(n \in \mathbb{N}\). Now, for each integer \(n \geq 2\), let \(u_n : F_n \to F_{n-1}\), \(v_n : G_n \to G_{n-1}\) and \(w_n : H_n \to H_{n-1}\) be the given transition morphisms. Then \(v_n \circ u_n = i_{n-1} \circ u_n\) for every \(n \geq 2\). Now set \(F = \lim \leftarrow F_n\), \(G = \lim \leftarrow G_n\), \(H = \lim \leftarrow H_n\) and \(f = \lim \leftarrow f_n : G \to H\). By (i), \((\ref{3.1})\) and the left-exactness of the inverse limit functor on the category of abelian groups, the sequence

\[
0 \to F \to G \xrightarrow{f} H
\]

is exact for the fpqc topology on \((\text{Sch}/k)\). Further, by (ii) and Proposition 3.2, \(f\) is flat and quasi-compact. Thus, by Lemma 2.3, we are reduced to checking that \(f\) is a surjective morphism of schemes. Recall that, by [EGA I new, Proposition 3.6.2, p. 244], \(f\) is surjective if, and only if, for every field \(K\) and every morphism \(\varphi : \text{Spec} \, K \to H\), there exist an extension \(K'\) of \(K\) and a morphism \(\psi : \text{Spec} \, K' \to G\) such that the following diagram commutes:

(3.9)

\[
\begin{array}{ccc}
\text{Spec} \, K' & \rightarrow & \text{Spec} \, K \\
\psi \downarrow & & \varphi \\
G & \xrightarrow{f} & H.
\end{array}
\]

We will construct \(\psi\) and \(K'\) by adapting to the present context the proof of Lemma 3.7.

For every \(n \in \mathbb{N}\), let \(\varphi_n : \text{Spec} \, K \to H_n\) be the composition of \(\varphi : \text{Spec} \, K \to H\) and the canonical morphism \(H \to H_n\). Then \((\varphi_n)\) is a coherent sequence, i.e., \(w_n \circ \varphi_n = \varphi_{n-1}\) for every \(n \geq 2\), and \(\varphi = \lim \varphi_n\) by \((\ref{3.1})\). Now, since \(f_1 : G_1 \to H_1\) is surjective, there exist an extension \(K_1\) of \(K\) and a morphism \(\chi_1 : \text{Spec} \, K_1 \to G_1\) such that the following diagram commutes:

(3.10)

\[
\begin{array}{ccc}
\text{Spec} \, K_1 & \rightarrow & \text{Spec} \, K \\
\chi_1 \downarrow & & \varphi_1 \\
G_1 & \xrightarrow{f_1} & H_1.
\end{array}
\]

Next, since \(f_2\) is surjective, there exist an extension \(K'_2\) of \(K_1\) and a morphism \(\chi'_2 : \text{Spec} \, K'_2 \to G_2\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Spec} \, K'_2 & \rightarrow & \text{Spec} \, K_1 \\
\chi'_2 \downarrow & & \varphi_2 \circ g_1 \\
G_2 & \xrightarrow{f_2} & H_2.
\end{array}
\]

We have

\[
f_1 \circ \chi_1 \circ g'_2 = \varphi_1 \circ g_1 \circ g'_2 = w_2 \circ \varphi_2 \circ g_1 \circ g'_2 = w_2 \circ f_2 \circ \chi'_2 = f_1 \circ v_2 \circ \chi'_2.
\]
Thus $\chi_1 \circ g'_2 - v_2 \circ \chi'_2$: Spec $K_2' \to G_1$ factors through a morphism $\vartheta_2$: Spec $K_2' \to F_1$, i.e.,

$$\chi_1 \circ g'_2 = v_2 \circ \chi'_2 + i_1 \circ \vartheta_2.$$  

(3.11)

Now, since $u_2: F_2 \to F_1$ is surjective by (iii), there exist an extension $K_2$ of $K_2'$ and a morphism $\chi''_2$: Spec $K_2 \to F_2$ such that the following diagram commutes

$$\begin{array}{ccc}
\text{Spec } K_2 & \xrightarrow{g''_2} & \text{Spec } K_2' \\
\chi''_2 & \downarrow & \vartheta_2 \\
F_2 & \xrightarrow{u_2} & F_1.
\end{array}$$

Set $\chi_2 = \chi'_2 \circ g'' + i_2 \circ \chi''_2$: Spec $K_2 \to G_2$ and $g_2 = g_1 \circ g'_2 \circ g''_2$: Spec $K_2 \to \text{Spec } K$. Since $f_2 \circ i_2: F_2 \to H_2$ is the zero morphism (i.e., $f_2 \circ i_2$ factors through the zero section Spec $k \to H_2$), we have $f_2 \circ \chi_2 = f_2 \circ \chi'_2 \circ g'' = \varphi_2 \circ g_1 \circ g'_2 \circ g''_2 = \varphi_2 \circ g_2$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
\text{Spec } K_2 & \xrightarrow{g_2} & \text{Spec } K \\
\chi_2 & \downarrow & \varphi_2 \\
G_2 & \xrightarrow{f_2} & H_2.
\end{array}$$

(3.12)

Further, by (3.11), we have

$$\chi_1 \circ g'_2 \circ g''_2 = v_2 \circ \chi'_2 \circ g''_2 + i_1 \circ \vartheta_2 \circ g''_2 = v_2 \circ \chi'_2 \circ g''_2 + i_1 \circ u_2 \circ \chi''_2$$

$$= v_2 \circ \chi'_2 \circ g'' + v_2 \circ i_2 \circ \chi''_2 = v_2 \circ \chi_2,$$

i.e., the following diagram commutes

$$\begin{array}{ccc}
\text{Spec } K_2 & \xrightarrow{g_2 \circ g''_2} & \text{Spec } K_1 \\
\chi_2 & \downarrow & \chi_1 \\
G_2 & \xrightarrow{u_2} & G_1.
\end{array}$$

(3.13)

We now repeat the above argument starting with diagram (3.12) in place of diagram (3.10) and obtain diagrams similar to (3.12) and (3.13) associated to $\varphi_3$, and so forth. In this way we obtain field extensions $K \subseteq K_1 \subseteq K_2 \subseteq \ldots$ and morphisms $\chi_n$: Spec $K_n \to G_n$ for $n = 1, 2, \ldots$, such that, for every $n \in \mathbb{N}$, the following diagrams commute:

$$\begin{array}{ccc}
\text{Spec } K_n & \xrightarrow{f_n} & \text{Spec } K \\
\chi_n & \downarrow & \varphi_n \\
G_n & \xrightarrow{f_n} & H_n
\end{array}$$
Let $K' = \bigcup_{n \geq 1} K_n$ and, for every $n \in \mathbb{N}$, let $\psi_n : \text{Spec } K' \to G_n$ be the composition of $\chi_n : \text{Spec } K_n \to G_n$ and the canonical morphism $\text{Spec } K' \to \text{Spec } K_n$. Then the sequence $(\psi_n)$ is a coherent lifting of $(\varphi_n)$, i.e., for every $n \in \mathbb{N}$ we have $u_{n+1} \circ \psi_{n+1} = \psi_n$ and the following diagram commutes:

\[
\begin{array}{c}
\text{Spec } K' \\
\downarrow \psi_n \\
G_n \\
\downarrow \varphi_n \\
\text{Spec } K_p \\
\end{array}
\quad
\begin{array}{c}
H_n \\
\end{array}
\]

The field $K'$ and the morphism $\psi = \lim \psi_n : \text{Spec } K' \to G$ are the ones required to make (3.9) commute. This completes the proof. $\square$

Remarks 3.14.

(a) As is well-known, Lemma 3.7 holds, more generally, if $(F_n)$ satisfies the Mittag-Leffler condition [Hart, Proposition 9.1(b), p. 192] (recall that the Mittag-Leffler condition holds if the transition maps of the system $(F_n)$ are surjective). Thus Proposition 3.8 may be regarded as a partial extension of [Hart, Proposition 9.1(b), p. 192] to the category of commutative $k$-group schemes. Note also that, by Remark 3.3(b), the fppf topology cannot be used in place of the fpqc topology in Proposition 3.8. For the same reason, we cannot prove Proposition 3.8 by working only with $\mathbb{F}_p$-rational points since it seems unlikely that Proposition 2.4(ii) remains valid when the morphism $q$ in that statement is quasi-compact rather than locally of finite type.

(b) The statements 3.2–3.4 are valid in the more general setting of [EGA, IV.3, §8.2], i.e., when $	ext{Spec } k$ is replaced by an arbitrary scheme and $\mathbb{N}$ is replaced by an arbitrary directed set.

4. The perfect closure of a ring

Let $p$ be a prime number and let $\mathbb{F}_p$ be the field with $p$ elements. If $A$ is a ring of characteristic $p$, i.e., an $\mathbb{F}_p$-algebra, let $F_A$ denote the Frobenius endomorphism of $A$, i.e., $F_A(a) = a^p$ for every $a \in A$. The $\mathbb{F}_p$-algebra $A$ is said to be perfect if $F_A$ is an isomorphism. For every $\mathbb{F}_p$-algebra $A$, set $F_A^n = F_A \circ F_A \circ \cdots \circ F_A$ ($n$ times) if $n > 0$ and let $F_A^n$ be the identity morphism of $A$ if $n = 0$. If $A$ is perfect and $n < 0$, set $F_A^n = (F_A^{-1})^{-n}$. For every $n$ such that $F_A^n$ is defined and every $x \in A$, we will write $x^{p^n} = F_A^n(x)$. 

\[
\begin{array}{c}
\text{Spec } K_{n+1} \\
\downarrow \chi_{n+1} \\
G_{n+1} \\
\end{array}
\quad
\begin{array}{c}
\text{Spec } K_n \\
\downarrow \chi_n \\
G_n \\
\end{array}
\]

Let $K' = \bigcup_{n \geq 1} K_n$ and, for every $n \in \mathbb{N}$, let $\psi_n : \text{Spec } K' \to G_n$ be the composition of $\chi_n : \text{Spec } K_n \to G_n$ and the canonical morphism $\text{Spec } K' \to \text{Spec } K_n$. Then the sequence $(\psi_n)$ is a coherent lifting of $(\varphi_n)$, i.e., for every $n \in \mathbb{N}$ we have $u_{n+1} \circ \psi_{n+1} = \psi_n$ and the following diagram commutes:

\[
\begin{array}{c}
\text{Spec } K' \\
\downarrow \psi_n \\
G_n \\
\downarrow \varphi_n \\
\text{Spec } K \\
\end{array}
\quad
\begin{array}{c}
H_n \\
\end{array}
\]
The perfect closure of the $\mathbb{F}_p$-algebra $A$ is a pair $(A^{\text{pf}}, \phi_A)$ consisting of a perfect $\mathbb{F}_p$-algebra $A^{\text{pf}}$ and a homomorphism of $\mathbb{F}_p$-algebras $\phi_A: A \to A^{\text{pf}}$ which has the following universal property: for every perfect $\mathbb{F}_p$-algebra $B$ and every homomorphism of $\mathbb{F}_p$-algebras $\psi: A \to B$, there exists a unique homomorphism of $\mathbb{F}_p$-algebras $\psi^{\text{pf}}: A^{\text{pf}} \to B$ such that $\psi^{\text{pf}} \circ \phi_A = \psi$, i.e., the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow{\phi_A} & \searrow{\psi^{\text{pf}}} \\
A^{\text{pf}} & \downarrow{\phi_A^{\text{pf}}} & \\
\end{array}
\]

In other words, the assignment $A \mapsto A^{\text{pf}}$ defines a covariant functor from the category of $\mathbb{F}_p$-algebras to the category of perfect $\mathbb{F}_p$-algebras which is left adjoint to the inclusion functor, i.e., for every perfect $\mathbb{F}_p$-algebra $B$, there exists a canonical bijection

\[
\text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B) \cong \text{Hom}_{\text{Perf-}\mathbb{F}_p\text{-alg}}(A^{\text{pf}}, B), \psi \mapsto \psi^{\text{pf}},
\]

whose inverse is given by

\[
\text{Hom}_{\text{Perf-}\mathbb{F}_p\text{-alg}}(A^{\text{pf}}, B) \cong \text{Hom}_{\mathbb{F}_p\text{-alg}}(A, B), \omega \mapsto \omega \circ \phi_A.
\]

We will show below that $(A^{\text{pf}}, \phi_A)$ exists. The above universal property will then show that $(A^{\text{pf}}, \phi_A)$ is unique up to a unique isomorphism.

Let $C$ be any $\mathbb{F}_p$-algebra (e.g., $C = \mathbb{F}_p$) and let $A$ be a $C$-algebra with structural morphism $\alpha: C \to A$. Set $\alpha^{\text{pf}} = (\phi_A \circ \alpha)^{\text{pf}}$. Thus the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & A \\
\downarrow{\phi_C} & \searrow{\phi_A} \\
C^{\text{pf}} & \xrightarrow{\alpha^{\text{pf}}} & A^{\text{pf}}. \\
\end{array}
\]

Then $A^{\text{pf}}$ is a $C$-algebra via the composite homomorphism $\alpha^{\text{pf}} \circ \phi_C = \phi_A \circ \alpha$. Further, $\phi_A: A \to A^{\text{pf}}$ is a homomorphism of $C$-algebras such that, if $\psi: A \to B$ is a homomorphism of $C$-algebras, where $B$ is perfect, then $\psi^{\text{pf}}: A^{\text{pf}} \to B$ is the unique homomorphism of $C^{\text{pf}}$-algebras such that $\psi^{\text{pf}} \circ \phi_A = \psi$. Thus there exists a canonical bijection

\[
\text{Hom}_{C\text{-alg}}(A, B) \cong \text{Hom}_{\text{Perf-}C^{\text{pf}}\text{-alg}}(A^{\text{pf}}, B), \psi \mapsto \psi^{\text{pf}},
\]

(which generalizes (4.2)) whose inverse is given by

\[
\text{Hom}_{\text{Perf-}C^{\text{pf}}\text{-alg}}(A^{\text{pf}}, B) \cong \text{Hom}_{C\text{-alg}}(A, B), \omega \mapsto \omega \circ \phi_A.
\]

(which generalizes (4.3)). In particular, if $C$ is perfect, then there exists a canonical bijection

\[
\text{Hom}_{\text{Perf-}C\text{-alg}}(A^{\text{pf}}, B) \cong \text{Hom}_{C\text{-alg}}(A, B), \omega \mapsto \omega \circ \phi_A.
\]
which agrees with (4.3) when $C = \mathbb{F}_p$. The pair $(A^{\text{pf}}, \phi_A)$ can be constructed as follows (see [Gre, p. 314] or [Bou2, V, §1, no. 4, pp. A.V.5-6]): set

$$A^{\text{pf}} = \lim_{n \geq 0} A_n,$$

where $A_n = A$ for every $n \geq 0$ and each transition map $A_n \to A_{n+1}$ is the Frobenius endomorphism $F_A$, and let $\phi_A$ be the canonical homomorphism $A = A_0 \to A^{\text{pf}}$. Then (4.7) is a perfect $C^{\text{pf}}$-algebra with the required universal property (4.1). Further, the Frobenius automorphism $F_A^{\text{pf}}: A^{\text{pf}} \to A^{\text{pf}}$ can be described as follows: if $\alpha \in A^{\text{pf}} = \lim_{n \geq 0} A_n$ is represented by $a \in A_n$, where $n \geq 1$, then $F_A^{\text{pf}}(\alpha)$ is represented by $a$ regarded as an element of $A_{n-1}$. On the other hand, $F_A^{-1}(\alpha)$ is represented by $a$ regarded as an element of $A_{n+1}$. See [Gre, p. 314, last paragraph].

**Remark 4.8.** The perfect closure $A^{\text{pf}}$ of an $\mathbb{F}_p$-algebra $A$ should not be confused with the perfection $A^{\text{perf}}$ of $A$, which is defined as the projective (rather than inductive) limit $\lim \leftarrow A$, where the transition maps are all equal to the Frobenius endomorphism $F_A$. The latter object is relevant, for example, in the constructions of Fontaine’s rings of periods [Fon, 1.2.2] and of the tilting functor in the theory of perfectoid algebras [Sch, Theorem 5.17]. The assignment $A \mapsto A^{\text{perf}}$ defines a covariant functor from the category of $\mathbb{F}_p$-algebras to the category of perfect $\mathbb{F}_p$-algebras which is right (rather than left) adjoint to the inclusion functor. To distinguish $A^{\text{pf}}$ from $A^{\text{perf}}$, some authors (e.g., Kedlaya and Liu [KL, 3.1.2 and 3.4.1]) call $A^{\text{pf}}$ (respectively, $A^{\text{perf}}$) the direct (respectively, inverse) perfection of $A$.

Now let $k$ be a perfect field of characteristic $p$, so that $k$ is naturally an $\mathbb{F}_p$-algebra, and let $A$ be a $k$-algebra. Then $A^{\text{pf}} = \lim_{n \geq 0} A_n$ is an $A$-algebra via $\phi_A$ and the induced $k$-algebra structure on $A^{\text{pf}}$ can be described as follows: if $\lambda \in k$ and $\alpha \in A^{\text{pf}}$ is represented by $a \in A_n$, then $\lambda \alpha$ is represented by $\lambda^p a \in A_n$. By (4.6), the map

$$\text{Hom}_{k,\text{alg}}(A^{\text{pf}}, B) \sim \text{Hom}_{k,\text{alg}}(A, B), \omega \mapsto \omega \circ \phi_A,$$

is bijective.

The following lemma shows that the operations of perfect closure and tensor product are compatible up to a canonical isomorphism.

**Lemma 4.10.** Let $A$ be a $k$-algebra and let $B$ and $C$ be $A$-algebras. Then the canonical map

$$(\phi_B \otimes_A \phi_C)^{\text{pf}} : (B \otimes_A C)^{\text{pf}} \to B^{\text{pf}} \otimes_{A^{\text{pf}}} C^{\text{pf}}$$

is an isomorphism of perfect $k$-algebras.

**Proof.** Let $D$ be an arbitrary perfect $k$-algebra. By the universal property of the tensor product [Liu, Proposition 1.14, p. 5], there exists a canonical bijection between the set of $k$-algebra homomorphisms $B \otimes_A C \to D$ and the set of pairs of
$k$-algebra homomorphisms $B \to D$ and $C \to D$ which induce the same $k$-algebra homomorphism $A \to D$. Thus, using (4.9), we obtain canonical bijections

$$
\text{Hom}_{k\text{-alg}}((B \otimes_A C)^{\text{pf}}, D) \cong \text{Hom}_{k\text{-alg}}(B \otimes_A C, D) \\
\cong \text{Hom}_{k\text{-alg}}(B, D) \times \text{Hom}_{k\text{-alg}}(A, D) \text{Hom}_{k\text{-alg}}(C, D) \\
\cong \text{Hom}_{k\text{-alg}}(B^{\text{pf}}, D) \times \text{Hom}_{k\text{-alg}}(A^{\text{pf}}, D) \text{Hom}_{k\text{-alg}}(C^{\text{pf}}, D) \\
\cong \text{Hom}_{k\text{-alg}}(B^{\text{pf}} \otimes A^{\text{pf}} C^{\text{pf}}, D),
$$

whence the lemma follows. \qed

Note that, if $A$ is a $k$-algebra, the Frobenius endomorphism of $A$ is a homomorphism of $\mathbb{F}_p$-algebras which is not, in general, a homomorphism of $k$-algebras. Consequently, (4.7) only defines $A^{\text{pf}}$ as an inductive limit in the category of $\mathbb{F}_p$-algebras. In order to extend the operation of perfect closure to the category of $k$-schemes in the next Section, we need to represent $A^{\text{pf}}$ as an inductive limit in the category of $k$-algebras. To this end, we introduce the following notions.

For every $\mathbb{F}_p$-algebra $C$ and every integer $n$ such that $F^n_C$ is defined, we will write $(C, F^n_C)$ for the ring $C$ regarded as a $C$-algebra via $F^n_C$. Note that, if $M$ is a $C$-module, then the abelian group $M \otimes_C (C, F^n_C)$ can be endowed with two distinct $C$-module structures, namely the standard $C$-module structure defined by $c(m \otimes d) = (cm) \otimes d = m \otimes c^n d$, and the non-standard $C$-module structure defined by $c(m \otimes d) = m \otimes cd$, where $m \in M$ and $c, d \in C$. If $M$ is a $C$-algebra, then the preceding structures are, in fact, $C$-algebra structures. We will write $M^{(F^n_C)}$ for the abelian group $M \otimes_C (C, F^n_C)$ endowed with its non-standard $C$-module structure. Thus

$$M^{(F^n_C)} = M \otimes_C (C, F^n_C) \quad \text{(as abelian groups)}$$

and the $C$-module structure on $M^{(F^n_C)}$ is given by $c(m \otimes d) = m \otimes cd \in M^{(F^n_C)}$ for every $c \in C$ and $m \otimes d \in M^{(F^n_C)}$ (where $m \in M$ and $d \in C$). For every pair of integers $m, n$ such that $F^n_C$ and $F^m_C$ are defined, there exists a canonical isomorphism of $C$-modules

$$
(M^{(F^n_C)})^{(F^m_C)} \cong M^{(F^{n+m}_C)},
$$

which is defined on generators by $(m \otimes c) \otimes d \mapsto m \otimes c^{p^m} d$, where $m \in M$ and $c, d \in C$ (its inverse is defined on generators by $m \otimes c \mapsto (m \otimes 1) \otimes c$). If $n = 0$, then (4.11) is the identity map on $M^{(F^n_C)}$.

Now let $n$ be an integer such that $F^n_C$ is defined and let $F^n_C M$ denote the abelian group $M$ equipped with the new $C$-module structure defined by $c \cdot m = c^{p^n} m$, where $c \in C$ and $m \in M$. For every pair of $C$-modules $M, N$, there exist canonical bijections

$$
\text{Hom}_{C\text{-mod}}(N^{(F^n_C)}, M) \cong \text{Hom}_{C\text{-mod}}(N, F^n_C M), \quad f \mapsto f \circ \lambda_{N,n},
$$

where $\lambda_{N,n} : N \to N^{(F^n_C)}$ maps $x$ to $x \otimes 1$. Note that, if $M$ is a $C$-algebra, then both $M^{(F^n_C)}$ and $F^n_C M$ are $C$-algebras.
If \( C \) is perfect, then \( F^n_C \) is defined for every \( n \in \mathbb{Z} \) and the canonical map
\[
F^n_C M \rightarrow M^{(p^n)}, \ m \mapsto m \otimes 1,
\]
is an isomorphism of \( C \)-modules with inverse given by \( m \otimes c \mapsto c^p m, \) where \( m \in M \) and \( c \in C. \) In this case (4.12) and (4.13) induce a bijection
\[
\text{Hom}_{C\text{-mod}}(N^{(p^n)}, M) \cong \text{Hom}_{C\text{-mod}}(N, M^{(p^n)}).
\]

Now, if \( M \) is a \( k \)-module and \( n \in \mathbb{Z} \) is arbitrary, the \( k \)-modules \( M^{(p^n)} \) and \( F^n_k M \) will be denoted by \( M^{(p^n)} \) and \( F^n k M, \) respectively. If \( C \) is a \( k \)-algebra, \( M \) is a \( C \)-module and \( n \) is an integer such that \( F^n C \) is defined, then the \( C \)-module \( F^n C M \) endowed with the \( k \)-module structure induced by the map \( k \rightarrow C \) is canonically isomorphic to the \( k \)-module \( p^n M. \) Henceforth these \( k \)-modules will be identified. Further, we will write \( p^n M \) for \( F^n C M. \) No confusion should result since it will always be clear from the context whether \( p^n M \) is being regarded as a \( C \)-module or as a \( k \)-module.

**Caveat 4.14.** If \( C \) is a \( k \)-algebra and \( F^n_C \) is defined, then the above identification of \( k \)-modules \( F^n M = p^n M \) does not extend to \( M^{(p^n)} \) and \( M^{(p^n)} \), as we now explain. Clearly the \( C \)-module \( M^{(p^n)} \) can be regarded as a \( k \)-module via the structural map \( \iota: k \rightarrow C, \) which induces a homomorphism of abelian groups \( \iota': M^{(p^n)} \rightarrow M^{(p^n)}, \ m \otimes a \mapsto m \otimes \iota(a). \) However, \( \iota' \) is not in general an isomorphism of \( k \)-modules since it may fail to be bijective. Indeed, since \( k \) is perfect, every element of \( M^{(p^n)} = M \otimes_k (k, F^n_k) \) can be written in the form \( m \otimes 1. \) This is also the case for \( M^{(p^n)} \) if \( C \) is perfect, whence \( \iota' \) is bijective in this case. However, if \( C \) is not perfect, then \( \iota' \) may fail to be surjective, e.g., for \( M = C. \) Indeed, if \( M = C \) and \( n = 1, \) then \( \iota': C \otimes_k (k, F_k) \rightarrow C \otimes_C (C, F_C) = (C, F_C) \) maps \( c \otimes 1 \) to \( c p. \)

Now let \( A \) be a \( k \)-algebra. By (4.11) and (4.13) (respectively), for every pair of integers \( m, n \) there exist canonical isomorphisms of \( k \)-algebras
\[
(A^{(p^n)})^{(p^m)} \cong A^{(p^{n+m})}
\]
and
\[
J^{(m)}_A: p^m A \cong A^{(p^{-m})}, \ a \mapsto a \otimes 1.
\]
Now, for every integer \( n, \) there exists a canonical homomorphism of \( k \)-algebras
\[
A^{(p^{n+1})} \rightarrow A^{(p^n)}
\]
which is defined on generators by \( a \otimes x \mapsto a p \otimes x. \) The composition of the map (4.15) for \( m = 1 \) and (4.17) is a homomorphism of \( k \)-algebras
\[
F_{A^{(p^n)/k}}: (A^{(p^n)})^{(p)} \rightarrow A^{(p^n)}
\]
which is called the relative Frobenius homomorphism of \( A^{(p^n)} \) over \( k. \) We have
\[
F_{A^{(p^n)/k}}((a \otimes y) \otimes x) = x(a \otimes y)p = a p \otimes xy^p \text{ for every } a \in A \text{ and } x, y \in k.
\]
Now, for every $n \in \mathbb{Z}$, there exists a canonical isomorphism of $\mathbb{F}_p$-algebras

\begin{equation}
(4.19) \quad \iota_{A,n} : A^{(p^n)} \xrightarrow{\sim} (A^{(p^n)})^{(p)}
\end{equation}

defined on generators by $\iota_n(a \otimes y) = (a \otimes y) \otimes 1$. Its inverse is defined on generators by $(a \otimes y) \otimes x \mapsto x^{p^{-1}}(a \otimes y) = a \otimes x^{p^{-1}}y$ ($a \in A, x, y \in k$). We have

\begin{equation}
(4.20) \quad F_{A^{(p^n)}}/k \circ \iota_{A,n} = F_{A^{(p^n)}}.
\end{equation}

In particular, $A$ is perfect (i.e., $F_A$ is an isomorphism) if, and only if, $F_{A/k}$ is an isomorphism.

Note that the composition

\begin{equation}
p^{-1}A \xrightarrow{\zeta_n} A^{(p)} \xrightarrow{\iota^{-1}_{A,0}} A,
\end{equation}

where $\zeta_n$ is the isomorphism of $k$-algebras (4.16) and $\iota^{-1}_{A,0}$ is the inverse of the isomorphism of $\mathbb{F}_p$-algebras (4.19) when $n = 0$, is the identity map on the underlying abelian groups. On the other hand, the composition

\begin{equation}
p^{-1}A \xrightarrow{\zeta_n} A^{(p)} \xrightarrow{F_{A/k}} A,
\end{equation}

where $F_{A/k}$ is the map (4.18) for $n = 0$, is the homomorphism of $k$-algebras $p^{-1}A \to A, a \mapsto a^p$. Similar results hold for $A^{(p^n)}$ in place of $A$.

We can now represent $A^{\text{pf}}$ as an inductive limit in the category of $k$-algebras:

**Lemma 4.21.** Let $A$ be a $k$-algebra. Then there exists a canonical isomorphism of $k$-algebras

\[ A^{\text{pf}} = \lim_{\longrightarrow} A^{(p^{-n})}, \]

where the transition maps are the maps (4.17).

**Proof.** For every $n \geq 0$, the map $\zeta_n : A \to A^{(p^{-n})}, a \mapsto a \otimes 1$, is an isomorphism of $\mathbb{F}_p$-algebras whose inverse is defined on generators by $a \otimes x \mapsto x^{p^{n}}a$ ($a \in A, x \in k$). Now, if $\alpha \in A^{\text{pf}} = \lim_{\longrightarrow} A_n$ is represented by $a_n \in A_n$, let $\varphi(\alpha) \in \lim_{\longrightarrow} A^{(p^{-n})}$ be represented by $\zeta_n(a_n) \in A^{(p^{-n})}$. A straightforward verification shows that the map $\varphi : \lim_{\longrightarrow} A_n \to \lim_{\longrightarrow} A^{(p^{-n})}$ just defined is an isomorphism of $k$-algebras.

The next lemma shows that the perfect closure of a $k$-algebra coincides with the perfect closure of its largest reduced quotient.

**Lemma 4.22.** Let $A$ be a $k$-algebra. Then the canonical projection $A \to A_{\text{red}}$ induces an isomorphism of $k$-algebras $A^{\text{pf}} = (A_{\text{red}})^{\text{pf}}$. 

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**ON THE PERFECTION OF SCHEMES**
Proof. The $p$-radical $A_{p^n}$ of $A$ considered in [Gre, p. 314] coincides with the nil-radical $\text{Nil}(A)$ of $A$. Now the arguments in [Gre, 4, p. 315] show that the map $A_{pf} \to (A_{red})_{pf}$ induced by the canonical projection $A \to A_{red} = A/\text{Nil}(A)$ is surjective with kernel contained in $\text{Nil}(A_{pf})$. But $\text{Nil}(A_{pf}) = 0$ since a perfect ring is reduced [Bon2, V, §1, no. 4, p. A.V.5].

Remark 4.23. The preceding constructions remain valid if $k$ is replaced by any perfect $\mathbb{F}_p$-algebra $C$. In particular, for every $C$-algebra $A$, there exists a canonical isomorphism of $C$-algebras

$$A_{pf} = \lim_{n \to 0} A^{(F_C^{-n})},$$

where the transition maps are given by $a \otimes c \mapsto a^p \otimes c$, where $a \in A$ and $c \in C$. When $C = \mathbb{F}_p$, the preceding isomorphism agrees with the case $C = \mathbb{F}_p$ of (4.7) (indeed, $F_C$ is the identity morphism of $C$, $A^{(F_C^{-n})} = A$ for every $n$ and the corresponding transition morphisms are all equal to $F_A$).

The perfect closure of some rings can be described explicitly. For example:

Example 4.24. Let $A$ be an integral domain endowed with an $\mathbb{F}_p$-algebra structure, let $L$ be the field of fractions of $A$ and let $\overline{L}$ be a fixed algebraic closure of $L$. Clearly $A$ is a subring of the perfect ring $\overline{L}$ and $a^{p^n} \in \overline{L}$ for every $a \in A$ and $n \geq 0$. Then

$$A_{pf} = \bigcup_{n \geq 0} \{a^{p^n}; a \in A \} = A[a^{p^n}; a \in A, n \in \mathbb{N}] \subseteq \overline{L}.$$

See [Bon2, V, §1, no. 4, Proposition 3, p. A.V.6]. Note that the map $\phi_A: A \to A_{pf}$ is the canonical inclusion. In particular, let $A = k[\{x_i\}]$ be the polynomial $k$-algebra on a (possibly infinite) family of independent indeterminates $\{x_i\}_{i \in I}$. Then

$$A_{pf} = k[x_i^{p^n}; i \in I, n \geq 0] = \lim_{n \to 0} k[x_i^{p^n}; i \in I]$$

inside an algebraic closure of $k(\{x_i\})$. Since every ring $k[x_i^{p^n}; i \in I]$ is a free $A$-module, we conclude that $A_{pf}$ is flat over $A$ [Bon1, §2, no. 3, Proposition 2, p. 14]. Further, since $A_{pf}$ is integral over $A$, $A_{pf}$ is, in fact, faithfully flat over $A$ [Mat2, Chapter 2, Theorem 3(2), p. 28, and Theorem 5(i), p. 33].

Remarks 4.25.

(a) In connection with the preceding Example, there exist $\mathbb{F}_p$-algebras $A$ such that $A_{pf}$ is not flat over $A$. Indeed, let $A$ be any $\mathbb{F}_p$-algebra such that $A_{red}$ is perfect and $\text{Nil}(A)^2 \neq \text{Nil}(A)$, e.g., $A = \mathbb{F}_p[x]/(x^2)$. Then $A_{pf} = (A_{red})_{pf} = A_{red}$ by Lemma [4.22], but $A_{red}$ is not flat over $A$ (otherwise $\text{Nil}(A) \otimes_A A_{red} = \text{Nil}(A)/\text{Nil}(A)^2$ would inject as a nonzero nilideal in $A_{red}$).

(b) If $A$ is any $k$-algebra, then $A_{pf}$ can be obtained from $A$ in two steps, the first of which is faithfully flat over $A$ but the second may fail to be flat over $A$. Indeed, fix a polynomial $k$-algebra $B = k[\{x_i\}]$, where $\{x_i\}_{i \in I}$ is
a (possibly infinite) family of independent indeterminates, and a surjective homomorphism of $k$-algebras $q: B \to A$. Set

$$\tilde{A} = B^{\text{pf}} \otimes_B A$$

with its natural $k$-algebra structure. Then $(\tilde{A})^{\text{pf}} = B^{\text{pf}} \otimes_{B^{\text{pf}}} A^{\text{pf}} = A^{\text{pf}}$ by Lemma 4.10. Now, since the map $B^{\text{pf}} \to \tilde{A}$ induced by $q$ is surjective, the Frobenius endomorphism $F_{\tilde{A}}: \tilde{A} \to \tilde{A}$ is surjective as well, i.e., $\tilde{A} = (\tilde{A})^p$. It follows that $(\tilde{A})_{\text{red}}$ is a perfect $k$-algebra, whence $(\tilde{A})_{\text{red}} = (\tilde{A})^{\text{pf}} = A^{\text{pf}}$. Further, since $B^{\text{pf}}$ is faithfully flat over $B$ by Example 4.24, $\tilde{A}$ is faithfully flat over $A$. Thus $A^{\text{pf}}$ can be obtained from $A$ in two steps, as claimed:

$$A \hookrightarrow \tilde{A} \twoheadrightarrow (\tilde{A})_{\text{red}} = A^{\text{pf}},$$

where the first extension $\tilde{A}/A$ is faithfully flat but the second one may not be flat, as noted in (a). Compare [Lip, Lemma 0.1, p. 18], where the algebra $\tilde{A}$ was denoted by $\bar{A}$.

(c) In general, the $k$-algebra $\tilde{A}$ considered in (b) depends on the choice of the pair $(B, q)$. For example, if $A = \mathbb{F}_p$ and we choose successively $(B, q) = (\mathbb{F}_p, 1_{\mathbb{F}_p})$, where $1_{\mathbb{F}_p}$ is the identity map of $\mathbb{F}_p$, and $(B, q) = (\mathbb{F}_p[x], \text{ev}_0)$, where $\text{ev}_0: \mathbb{F}_p[x] \to \mathbb{F}_p$ is the natural evaluation-at-zero map $f(x) \mapsto f(0)$, then we obtain correspondingly $\tilde{A} = \mathbb{F}_p$ and $\tilde{A} = \mathbb{F}_p[x^{p^{-n}}; n \in \mathbb{N}_0]/(x)$. Note that the latter algebra is non-reduced, which shows that $\tilde{A}$ can be non-reduced even if $A$ is reduced.

5. The inverse perfection of a scheme

In this Section we extend the constructions of the previous Section to the category of schemes.

Let $p$ be a prime number. For every $\mathbb{F}_p$-scheme $Y$, let $F_Y: Y \to Y$ denote the absolute Frobenius endomorphism of $Y$ [SGA3 new, VII A, §4.1]. If $A$ is an $\mathbb{F}_p$-algebra, then

(5.1) $F_{\text{Spec } A} = \text{Spec } F_A$.

The scheme $Y$ is said to be perfect if $F_Y$ is an isomorphism. Given an $\mathbb{F}_p$-scheme $Y$, there exists a pair $(Y^{\text{pf}}, \phi_Y)$ consisting of a perfect $\mathbb{F}_p$-scheme $Y^{\text{pf}}$ and a morphism of $\mathbb{F}_p$-schemes $\phi_Y: Y^{\text{pf}} \to Y$ such that, for every perfect $\mathbb{F}_p$-scheme $Z$ and every morphism of $\mathbb{F}_p$-schemes $\psi: Z \to Y$, there exists a unique morphism of $\mathbb{F}_p$-schemes $\psi^{\text{pf}}: Z \to Y^{\text{pf}}$ such that $\phi_Y \circ \psi^{\text{pf}} = \psi$. Following [KL, Theorem 8.5.5(c)], we call $Y^{\text{pf}}$ the inverse perfection of $Y$. We should note that $Y^{\text{pf}}$ is called the perfection of $Y$ in both [Mi2, p. 226] and [BW, p. 3], and the perfect closure of $Y$ in [Gre, p. 317], where it is denoted by $Y^{1/p\infty}$. In the latter reference, $Y^{\text{pf}}$ is constructed by
globalizing the functor on \(F_p\)-algebras \(A \mapsto A^{pf}\). In particular, \((\text{Spec } A)^{pf} = \text{Spec } A^{pf}\) and \(\phi_{\text{Spec } A} = \text{Spec } \phi_A\).

We will write \((\text{Perf} / F_p)\) for the category of perfect \(F_p\)-schemes. It follows from the universal property described above (or, alternatively, from (4.2)) that the \textit{inverse perfection functor}

\[(5.2) \quad (\text{Sch} / F_p) \to (\text{Perf} / F_p), \quad Y \mapsto Y^{pf},\]

is covariant and right-adjoint to the inclusion functor \((\text{Perf} / F_p) \to (\text{Sch} / F_p)\), i.e., for every perfect \(F_p\)-scheme \(Z\), there exists a canonical bijection

\[(5.3) \quad \text{Hom}_{\text{Sch} / F_p}(Z, Y) \sim \text{Hom}_{\text{Perf} / F_p}(Z, Y^{pf}), \; \psi \mapsto \psi^{pf}.\]

**Remark 5.4.** It is shown in [Gre, p. 317] that the canonical morphism \(\phi_{\text{Spec } A} = \text{Spec } \phi_A : \text{Spec } A^{pf} \to \text{Spec } A\) is a homeomorphism. Further, for every \(\alpha \in A^{pf}\) there exists an integer \(n \geq 0\) such that \(\alpha^{p^n} \in \text{Im } [\phi_A : A \to A^{pf}]\) by [Gre, p. 314]. Consequently, \(\phi_{\text{Spec } A}\) is integral and radicial [EGA I new, Proposition 3.7.1(b), p. 246]. Thus \(\phi_{\text{Spec } A}\) is, in fact, a universal homeomorphism [EGA IV, Corollary 18.12.11]. It now follows from [EGA I new, Proposition 3.8.2(v), p. 249] that \(\phi_Y : Y^{pf} \to Y\) is a universal homeomorphism for every \(F_p\)-scheme \(Y\).

More generally, if \(W\) is any perfect \(F_p\)-scheme (in lieu of \(W = \text{Spec } F_p\)) and \(Y\) is a \(W\)-scheme, then there exists a canonical bijection

\[(5.5) \quad \text{Hom}_{\text{Sch} / W}(Z, Y) \sim \text{Hom}_{\text{Perf} / W}(Z, Y^{pf}), \; \psi \mapsto \psi^{pf},\]

whose inverse is given by

\[(5.6) \quad \text{Hom}_{\text{Perf} / W}(Z, Y^{pf}) \sim \text{Hom}_{\text{Sch} / W}(Z, Y), \; \omega \mapsto \phi_Y \circ \omega.\]

The bijections (5.5) and (5.6) are induced by (4.4) and (4.6), respectively.

In the following, we focus on the case \(W = \text{Spec } k\), where \(k\) is a perfect field of characteristic \(p\).

Note that, if \(Y\) is a \(k\)-scheme, then Lemma 4.22 yields a canonical isomorphism of \(k\)-schemes

\[(5.7) \quad Y^{pf} = (Y_{\text{red}})^{pf}.\]

The following alternative construction of \(Y^{pf}\) follows from Lemma 4.21.

For every integer \(n\), let \((\text{Spec } k, \text{Spec } F_k^n)\) denote the scheme \(\text{Spec } k\) regarded as a \(k\)-scheme via \(\text{Spec } F_k^n\). If \(Y\) is a \(k\)-scheme, set

\[(5.8) \quad Y^{(p^n)} = Y \times_{\text{Spec } k} (\text{Spec } k, \text{Spec } F_k^n).\]

Then, for every \(k\)-scheme \(Z\), (4.12) induces a bijection

\[(5.9) \quad \text{Hom}_{\text{Sch} / k}(Y^{(p^{-n})}, Z) \simeq \text{Hom}_{\text{Sch} / k}(Y, Z^{(p^n)}).\]
Now, for every pair of integers \( m, n \), the isomorphism of \( k \)-algebras \( 4.15 \) induces an isomorphism of \( k \)-schemes

\[
Y^{(p^{n+m})} \sim (Y^{(p^n)})^{(p^n)}.
\]

Setting \( m = 1 \) above, we obtain an isomorphism of \( k \)-schemes

\[
Y^{(p^{n+1})} \sim (Y^{(p^n)})^{(p)}.
\]

On the other hand, by \( 4.17 \), there exists a canonical morphism of \( k \)-schemes

\[
Y^{(p^n)} \to Y^{(p^{n+1})}.
\]

The composition of \( 5.11 \) and \( 5.12 \) is the relative Frobenius morphism of \( Y^{(p^n)} \) over \( k \):

\[
F_{Y^{(p^n)}/k} : Y^{(p^n)} \to (Y^{(p^n)})^{(p)}.
\]

By \( 4.20 \), we have

\[
pr_{Y^{(p^n)}} \circ F_{Y^{(p^n)}/k} = F_{Y^{(p^n)}},
\]

where \( pr_{Y^{(p^n)}} : (Y^{(p^n)})^{(p)} \to Y^{(p^n)} \) is the first projection. It follows from \( 5.14 \) that \( F_{Y^{(p^n)}/k} \) is a universal homeomorphism (see \( SGA5 \) XV, §1, Proposition 2(a), p. 445]). Consequently, the transition morphisms \( Y^{(p^{-n})} \to Y^{(p^{-n+1})} \) of the projective system of \( k \)-schemes \( (Y^{(p^n)})_{n \geq 0} \) are universal homeomorphisms (in particular they are affine \( EGA \) II, (6.1.2)]). Now Lemma \( 4.21 \) and \( 5.7 \) show that there exist canonical isomorphisms of \( k \)-schemes

\[
Y^{pf} = \lim_{\substack{n \geq 0 \\downarrow}} Y^{(p^{-n})} = \lim_{\substack{n \geq 0 \\downarrow}} (Y_{red})^{(p^{-n})}.
\]

Remark 5.16. By Remark \( 4.23 \) the preceding considerations extend to a relative setting where \( \text{Spec } k \) is replaced by any nonempty perfect \( \mathbb{F}_p \)-scheme \( T \). Compare \( Mi2 \) pp. 226-227]. Note also that the inverse perfection \( Y^{pf} \) of a \( T \)-scheme \( Y \) depends only on its structure as a scheme over \( \mathbb{F}_p \), whereas the presentation \( 5.15 \) depends on the structure of \( Y \) as a scheme over \( T \).

Proposition 5.17. Consider, for a morphism of \( k \)-schemes, the property of being:

(i) separated;
(ii) quasi-compact;
(iii) quasi-separated;
(iv) a closed immersion;
(v) affine;
(vi) flat;
(vii) an open immersion;
(viii) an immersion.
If \( P \) denotes one of the above properties and \( f : X \to Y \) has property \( P \), then \( f^{pf} : X^{pf} \to Y^{pf} \) also has property \( P \).

**Proof.** Properties (i)-(vi) are stable under base change. Thus, if \( P \) denotes one of these properties and \( f \) has property \( P \), then the morphism \( X^{(p^n-\cdot)} \to Y^{(p^n-\cdot)} \) induced by \( f \) has property \( P \) as well for every integer \( n \geq 0 \). Consequently, by (5.15) and Proposition 3.2, \( f^{pf} \) has property \( P \). In the case of properties (vii) and (viii), the proposition follows from Proposition 3.3 since the transition morphisms of the systems \( (X^{(p^n-\cdot)}) \) and \( (Y^{(p^n-\cdot)}) \) are (universal) homeomorphisms. \( \square \)

**Remarks 5.18.**

(a) Let \( f : X \to Y \) be a morphism of \( k \)-schemes. Then, for every perfect \( k \)-scheme \( Z \), there exists a canonical commutative diagram of sets

\[
\begin{array}{ccc}
\text{Hom}_{Sch/k}(Z,X) & \longrightarrow & \text{Hom}_{Sch/k}(Z,Y) \\
\downarrow & & \downarrow \cong \\
\text{Hom}_{Perf/k}(Z,X^{pf}) & \longrightarrow & \text{Hom}_{Perf/k}(Z,Y^{pf})
\end{array}
\]

where the vertical maps are the bijections (5.6) for \( W = \text{Spec} \ k \) and the top (respectively, bottom) horizontal map is given by \( \psi \mapsto f \circ \psi \) (respectively, \( \varphi \mapsto f^{pf} \circ \varphi \)). It follows that, if the top horizontal map in the above diagram is a bijection for every affine perfect \( k \)-scheme \( Z \), then \( f^{pf} : X^{pf} \to Y^{pf} \) is an isomorphism of perfect \( k \)-schemes.

(b) Since the canonical morphism \( Y^{pf} \to Y \) is a universal homeomorphism by Remark 5.4, it is clear that if \( P \) is a purely topological property of a morphism of \( k \)-schemes (e.g., that of being injective, surjective, dominant, etc.) and \( f : X \to Y \) has property \( P \), then \( f^{pf} : X^{pf} \to Y^{pf} \) also has property \( P \). Further, if \( \{U_i\}_{i \in I} \) is an open covering of \( Y \), then \( \{U_i^{pf}\}_{i \in I} \) is an open covering of \( Y^{pf} \). In particular, if \( Y = \coprod_{i \in I} Y_i \), then \( Y^{pf} = \coprod_{i \in I} Y_i^{pf} \).

(c) Let \( f : X \to Y \) be an fpqc morphism of \( k \)-schemes, i.e., \( f \) is faithfully flat and every quasi-compact open subset of \( Y \) is the image of a quasi-compact open subset of \( X \) \cite[Proposition 2.33, p. 27]{Vis}. It is clear from (b) and part (vi) of the proposition that \( f^{pf} : X^{pf} \to Y^{pf} \) is an fpqc morphism as well.

(d) It follows from Lemma 4.10 that, if \( Y \) is a \( k \)-scheme and \( X \) and \( Z \) are \( Y \)\(-\)schemes, then \( (X \times_Y Z)^{pf} = X^{pf} \times_{Y^{pf}} Z^{pf} \).

(e) Let \( Y \) be a \( k \)-group scheme. It follows from (d) and (5.3) that \( Y^{pf} \) is a \( k \)-group scheme. On the other hand, for every integer \( n \), \( Y^{(p^n-\cdot)} \) is a \( k \)-group scheme and the relative Frobenius morphism \( (5.13) \) is a homomorphism \[SGA3_{new} VIII_A, 4.1.2\]. We conclude that \( (Y^{(p^n-\cdot)})_{n \geq 0} \) is a projective system in the category of \( k \)-group schemes and \( (5.15) \) is an isomorphism of \( k \)-group schemes. In particular, for every \( m \geq 0 \), the composition of (5.15) and the canonical
proof.

By \[\text{[Mac, IX, §1, no. 4, p. A.V.5]}\] that a perfect \(k\)-scheme is reduced. In particular, \(Y^{\text{pf}}\) is a reduced \(k\)-scheme for every \(k\)-scheme \(Y\).

**Proposition 5.19.** Let \(X\) be a \(k\)-scheme.

- (i) If \(X\) is étale over \(k\), then \(X\) is perfect.
- (ii) If \(X\) is finite over \(k\), then \(X^{\text{pf}}\) is finite and étale over \(k\).

**Proof.** If \(X\) is étale over \(k\), then \(X \simeq \prod_{i \in I} \text{Spec} \, k_i\) for some family \(\{k_i\}\) of finite separable field extensions of \(k\) \([\text{EGA, IV}_4, \text{Corollary 17.4.2(d')}]\). Since \(k\) is perfect, each \(k_i\) is perfect as well and assertion (i) follows from Remark 5.18(b). Now assume that \(X = \text{Spec} \, k\) is finite over \(k\). Then \(X^{\text{red}} = \text{Spec} \, B^{\text{red}}\) is finite and étale over \(k\) by \([\text{Bon2}, \text{V, §6, no. 7, Lemma 5, p. A.V.35}]\). Thus, by (i) and (5.7), \(X^{\text{pf}} = (X^{\text{red}})^{\text{pf}} = X^{\text{red}}\) is finite and étale over \(k\), as claimed. \(\square\)

**Lemma 5.20.** Let \(G\) be a \(k\)-group scheme locally of finite type such that \(G(\bar{k})\) is the trivial group. Then \(G^{\text{pf}}\) is the trivial \(k\)-group scheme.

**Proof.** By \([5.7]\), it suffices to check that \(G^{\text{red}}\) is the trivial \(k\)-group scheme. Since \(k\) is perfect, \(G^{\text{red}} \times_{\text{Spec} \, k} \text{Spec} \, \bar{k}\) is reduced whence \(G^{\text{red}} \times_{\text{Spec} \, k} \text{Spec} \, \bar{k} = (G \times_{\text{Spec} \, k} \text{Spec} \, \bar{k})^{\text{red}}\) by \([\text{EGA I}_{\text{new}}, \text{Corollary 4.5.12, p. 271}]\). Thus we may assume that \(k = \bar{k}\). By \([\text{SGA3}_{\text{new}}, \text{VI}_A, 0.2 \text{ and Lemma 0.5.2}]\), \(G^{\text{red}}\) is a reduced closed \(k\)-subgroup scheme of \(G\). Further, the hypothesis implies that \(G^{\text{red}}(k)\) is the trivial group. Now \([\text{DG}, \text{II, §5, Proposition 4.3(a), p. 245}]\) shows that \(G^{\text{red}}\) is the trivial \(k\)-group scheme. \(\square\)

**Proposition 5.21.** Let \((Y_\lambda)_{\lambda \in \Lambda}\) be a projective system of \(k\)-schemes, where \(\Lambda\) is a directed set containing an element \(\lambda_0\) such that the transition morphisms of \(k\)-schemes \(Y_\mu \to Y_\lambda\) are affine for \(\mu \geq \lambda \geq \lambda_0\). Then there exists a canonical isomorphism of perfect \(k\)-schemes

\[\left(\lim_{\lambda} Y_\lambda\right)^{\text{pf}} = \lim_{\lambda} Y_\lambda^{\text{pf}}.\]

**Proof.** By \([\text{Mac, IX, §3, dual of Theorem 1}]\), we may replace \(\Lambda\) with the cofinal subset \(\{\lambda \in \Lambda \mid \lambda \geq \lambda_0\}\) and hence assume that \(\lambda_0\) is an initial element of \(\Lambda\). For \(\lambda \geq \lambda_0\), let \(f_{\lambda, \lambda'}\) denote the transition morphisms of the system \((Y_\lambda)\). For each \(\lambda \in \Lambda\), consider the quasi-coherent \(\mathcal{O}_{Y_\lambda}\)-algebra \(\mathcal{A}_\lambda = f_{\lambda, \lambda_0}^* \mathcal{O}_{Y_\lambda}\). Then \(Y_\lambda = \text{Spec} \, \mathcal{A}_\lambda\) by \([\text{EGA I}_{\text{new}}, \text{p. 356, line -5}]\). Further, \(\mathcal{A} = \lim \mathcal{A}_\lambda\) is a quasi-coherent \(\mathcal{O}_{Y_\lambda}\)-algebra and \(Y = \lim Y_\lambda = \text{Spec} \, \mathcal{A}\) by \([\text{EGA I}_{\text{new}}, \text{IV}_3, \text{Proposition 8.2.3}]\). Now, by \([5.1]\) and the construction of \(Y_\lambda = \text{Spec} \, \mathcal{A}_\lambda\) in \([\text{EGA I}_{\text{new}}, \text{proof of Corollary 9.1.7, p. 356}]\), we have \(F_{Y_\lambda} = \text{Spec} \, F_{\mathcal{A}_\lambda}\), where \(F_{\mathcal{A}_\lambda}\) is the absolute Frobenius endomorphism of the \(\mathcal{O}_{Y_\lambda}\)-algebra \(\mathcal{A}_\lambda\). Similarly, \(F_Y = \text{Spec} \, F_{\mathcal{A}}\). Since \(F_{\mathcal{A}} = \lim F_{\mathcal{A}_\lambda}\), we conclude that \(F_Y = \lim F_{Y_\lambda}\). Consequently, if \(Y_\lambda\) is perfect for every \(\lambda\) (i.e., \(F_{Y_\lambda}\) is an isomorphism), then \(Y\) is perfect as well. Now, by Proposition \([5.17(v)]\), \((Y_\lambda^{\text{pf}})\) is a projective system.
of \( k \)-schemes with affine transition morphisms. Thus, by the preceding discussion, its limit \( \lim \lambda Y^\text{pf}_\lambda \) is a perfect \( k \)-scheme. Further, if \( Z \) is an arbitrary perfect \( k \)-scheme then, by \((5.3)\) and \((3.1)\),

\[
\text{Hom}_{\text{Perf}/k}(Z, \lim \lambda Y^\text{pf}_\lambda) = \lim \lambda \text{Hom}_{\text{Perf}/k}(Z, Y^\text{pf}_\lambda) = \lim \lambda \text{Hom}_{\text{Sch}/k}(Z, Y^\lambda) = \text{Hom}_{\text{Sch}/k}(Z, \lim Y^\lambda) .
\]

Consequently, \( (\lim \lambda Y^\lambda)^{\text{pf}} = \lim \lambda Y^\lambda \), as claimed. \( \square \)

To conclude this Section, we show below that, in an appropriate sense, the inverse perfection functor commutes with Weil restriction.

Let \( f: Z' \rightarrow Z \) be a morphism of schemes and let \( X' \) be a \( Z' \)-scheme. We will say that the Weil restriction of \( X' \) along \( f \) exists or, more concisely, that \( \mathcal{R}_{Z'/Z}(X') \) exists, if the contravariant functor \((\text{Sch}/Z) \rightarrow (\text{Sets}), T \mapsto \text{Hom}_Z(T \times_Z Z', X')\)

is represented by a \( Z \)-scheme \( \mathcal{R}_{Z'/Z}(X') \) endowed with a morphism of \( Z' \)-schemes \( q_{X'}: \mathcal{R}_{Z'/Z}(X') \times_Z Z' \rightarrow X' \) such that the map

\[
(5.22) \quad \text{Hom}_Z(T, \mathcal{R}_{Z'/Z}(X')) \cong \text{Hom}_{Z'}(T \times_Z Z', X'), \quad g \mapsto q_{X'} \circ g_{Z'}
\]

is a bijection. This is the case if \( f \) is finite and locally free and \( X' \) is quasi-projective over \( Z' \). See [BLR, §7.6] and [CGP, Appendix A.5] for more details. Assume now that \( f \) is a morphism of perfect \( \mathbb{F}_p \)-schemes and let \( X' \) be a perfect \( Z' \)-scheme. Note that, if \( T \) is a perfect \( Z \)-scheme, then \( T \times_Z Z' \) is a perfect \( Z' \)-scheme by Remark 5.18(d). We will say that the the Weil restriction of \( X' \) along \( f \) exists in \((\text{Perf}/Z')\) or, more concisely, that \( \mathcal{R}_{Z'/Z}^{\text{pf}}(X') \) exists, if the contravariant functor

\[
(5.23) \quad (\text{Perf}/Z) \rightarrow (\text{Sets}), \quad T \mapsto \text{Hom}_{\text{Perf}/Z'}(T \times_Z Z', X'),
\]

is represented by a perfect \( Z \)-scheme \( \mathcal{R}_{Z'/Z}^{\text{pf}}(X') \).

**Lemma 5.24.** Let \( f: Z' \rightarrow Z \) be a morphism of perfect \( \mathbb{F}_p \)-schemes and let \( Y \) be a \( Z' \)-scheme. If \( \mathcal{R}_{Z'/Z}(Y) \in (\text{Sch}/Z) \) exists, then \( \mathcal{R}_{Z'/Z}^{\text{pf}}(Y^{\text{pf}}) \) exists as well and

\[
\mathcal{R}_{Z'/Z}^{\text{pf}}(Y^{\text{pf}}) = \mathcal{R}_{Z'/Z}(Y)^{\text{pf}} .
\]

**Proof.** If \( T \) is a perfect \( Z \)-scheme, then, by \((5.6)\) and \((5.22)\),

\[
\text{Hom}_{\text{Perf}/Z'}(T \times_Z Z', Y^{\text{pf}}) = \text{Hom}_{\text{Sch}/Z'}(T \times_Z Z', Y) = \text{Hom}_{\text{Sch}/Z}(T, \mathcal{R}_{Z'/Z}(Y)) = \text{Hom}_{\text{Perf}/Z}(T, \mathcal{R}_{Z'/Z}^{\text{pf}}(Y)),
\]

whence the lemma follows. \( \square \)

6. **Exactness properties**

Let \( k \) be a perfect field of positive characteristic and let \( \overline{k} \) be an algebraic closure of \( k \).
Let \((\text{Perf}/k)_{\text{fpqc}}\) be the category \((\text{Perf}/k)\) endowed with the fpqc topology, i.e., a family of morphisms \(\{Y_\alpha \to Y\}\) in \((\text{Perf}/k)\) is a covering in \((\text{Perf}/k)_{\text{fpqc}}\) if it is a covering in \((\text{Sch}/k)_{\text{fpqc}}\). The presheaf represented by a perfect and commutative \(k\)-group scheme is a sheaf on \((\text{Sch}/k)_{\text{fpqc}}\) [VI, Theorem 2.55, p. 34] and therefore also on \((\text{Perf}/k)_{\text{fpqc}}\).

**Theorem 6.1.** Let \(1 \to F \to G \to H \to 1\) be a sequence of \(k\)-group schemes which is exact for the fpqc topology on \((\text{Sch}/k)\). Then the sequence of perfect \(k\)-group schemes

\[
1 \to F^{\text{pf}} \to G^{\text{pf}} \to H^{\text{pf}} \to 1
\]

is exact for the fpqc topology on \((\text{Perf}/k)\).

**Proof.** Left-exactness follows from (5.3). Thus we are reduced to checking that \(h_S(q^{\text{pf}})\) is locally surjective in \((\text{Perf}/k)_{\text{fpqc}}\), where \(S = \text{Spec} k\) and \(q: G \to H\) is the given morphism. Let \(Z\) be a perfect \(k\)-scheme and let \(f \in \text{Hom}_{\text{Perf}/k}(Z, H^{\text{pf}})\). By (5.3), \(f = e^{\text{pf}}\) for a unique \(e \in \text{Hom}_{\text{Sch}/k}(Z, H)\). Now, since \(h_S(q^{\text{pf}})\) is locally surjective in \((\text{Sch}/k)_{\text{fpqc}}\), there exist an fpqc covering \(\{u_i: Y_i \to Z\}\) in \((\text{Sch}/k)\) and \(k\)-morphisms \(g_i: Y_i \to G\) such that \(q \circ g_i = e \circ u_i\) for every \(i\). Since \(\{u_i^{\text{pf}}: Y_i^{\text{pf}} \to Z\}\) is an fpqc covering in \((\text{Perf}/k)\) by Remark 5.18(c) and \(q^{\text{pf}} \circ g_i^{\text{pf}} = e^{\text{pf}} \circ u_i^{\text{pf}} = f \circ u_i^{\text{pf}}\) for every \(i\), the proof is complete. \(\square\)

**Corollary 6.2.** Let \(f: G \to H\) be an isogeny of \(k\)-group schemes of finite type with geometrically connected kernel. Then \(f^{\text{pf}}: G^{\text{pf}} \to H^{\text{pf}}\) is an isomorphism.

**Proof.** Let \(F = \text{Ker} f\) and write \(\bar{F}\) for the connected \(\bar{k}\)-scheme \(F \times_k \text{Spec} \bar{k}\). By the theorem and Lemma 5.20 it suffices to check that \(F(\bar{k})\) is the trivial group. Since \(\bar{F} \to \text{Spec} \bar{k}\) a finite morphism of schemes, \(|\bar{F}|\) is a finite connected topological space [EGA, II, Corollary 6.1.7], i.e., a one-point space. Since \(F(\bar{k})\) may be identified with a subset of \(|\bar{F}|\) by [EGA I new, (3.5.5), p. 243], the corollary is clear. \(\square\)

**Proposition 6.3.** Let \(0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0\) be a complex of commutative \(k\)-group schemes locally of finite type. Assume that the following conditions hold:

(i) \(f\) is quasi-compact,

(ii) \(g\) is flat,

(iii) the induced sequence of abelian groups \(0 \to F(\bar{k}) \to G(\bar{k}) \to H(\bar{k}) \to 0\) is exact.

Then the induced complex of perfect and commutative \(k\)-group schemes

\[
0 \to F^{\text{pf}} \to G^{\text{pf}} \to H^{\text{pf}} \to 0
\]

is exact for the fpqc topology on \((\text{Perf}/k)\).

**Proof.** By faithfully flat and quasi-compact descent [EGA, IV2, Proposition 2.7.1], it suffices to check the exactness of the indicated complex after extending the base
from \( k \) to \( \overline{k} \). Then, using Remark 5.18(d) and noting that conditions (i)-(iii) are stable under this base change, we may assume that \( k = \overline{k} \). Since \( g \) is flat and \( g(k) : G(k) \to H(k) \) is surjective, the sequence \( 0 \to (\text{Ker} g)^{pf} \to G^{pf} \to H^{pf} \to 0 \) is exact by Corollary 2.5 and Proposition 6.1. On the other hand, by (i) and \([\text{SGA3}_{\text{new}}], \text{VI}, \text{comments following Proposition 5.4.1}\], \( F/\text{Ker} f \) and \( \text{Im} f \) exist in the category of commutative \( k \)-group schemes and \( f \) induces an isomorphism \( F/\text{Ker} f \cong \text{Im} f \), where \( \text{Im} f \) is a closed subgroup scheme of \( \text{Ker} g \). Thus, since \( (\text{Ker} f)(k) = 0 \) by (iii), Proposition 6.1 and Lemma 5.20 show that \( f^{pf} \) induces an isomorphism of perfect and commutative \( k \)-group schemes \( F^{pf} \cong (\text{Im} f)^{pf} \). It remains to show that \( (\text{Im} f)^{pf} = (\text{Ker} g)^{pf} \). Since \( \text{Ker} g \to \text{Ker} g/\text{Im} f \) is faithfully flat and locally of finite presentation \([\text{EGA}, \text{IV}_{1}, \text{Proposition 1.3.7}]\), the sequence \( 0 \to \text{Im} f \to \text{Ker} g \to \text{Ker} g/\text{Im} f \to 0 \) is exact for the fpqc topology on \( (\text{Sch}/k) \). Now Proposition 6.1 reduces the proof to checking that \( (\text{Ker} g/\text{Im} f)^{pf} = 0 \).

Since \( (\text{Ker} g)(k) = \text{Ker} g(k) = \text{Im} f(k) \subseteq (\text{Im} f)(k) \) by (iii), the map \( (\text{Im} f)(k) \to (\text{Ker} g)(k) \) is surjective and therefore \( (\text{Ker} g/\text{Im} f)(k) = 0 \). Now Lemma 5.20 completes the proof. □

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