1. Introduction

These lectures deal with a particular problem in the theory of anyons—particles obeying statistics that is neither Bose–Einstein or Fermi–Dirac, but something in between. Like so many other developments in theoretical physics nowadays, the concept of anyons has so far had disappointingly few applications to observed phenomena—the fractional quantum Hall effect being the only candidate at the moment. I sincerely hope that this typical trait of postmodern theory will not apply to anyons in all future. Anyway, the subject possesses considerable elegance and is intellectually rewarding, and thus perhaps worth spending some time upon.

I will start by explaining how the possibility for exotic statistics arises, and then concentrate on the problem of how to describe a many–anyon system. By no means will the treatment cover all the attempts in this direction, nor will the list of references exhaust the vast literature on the subject. I have only quoted works I have directly consulted when preparing the lectures and these notes. I offer my apologies to those whose work is not discussed or cited.

My involvement in the problem arose from an attempt to describe comprehensively the statistical mechanics of anyons—an effort that eventually led to other things. I wish to thank my collaborators Masud Chaichian and Ricardo Felipe Gonzalez for many enlightening discussion. An excellent set of lectures given at the University of Helsinki in 1993 by Finn Ravndal gave additional inspiration.

2. The Symmetry Group Approach to the Quantum Mechanics of Identical Particles

The average student of quantum mechanics, when first faced with a treatment of identical particles, would have encountered an argument that runs somewhat like this:

Consider a system of $N$ particles described by a Hamiltonian

$$H = H(1, 2, ..., N),$$  \hspace{1cm} (2.1)
where the label $i$ denotes operators (coordinates, momenta, spins,...) relating to the $i$th particle. The statement that the particles are identical is taken to mean that any conceivable Hamiltonian (2.1) describing a system of such particles is invariant under a permutation of the operators relating to different particles: Denote by $\pi$ the permutation

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & N \\ \pi(1) & \pi(2) & \ldots & \pi(N) \end{pmatrix}$$

belonging to the group $S_N$ of permutations of $N$ objects, then the particles are identical if

$$U(\pi)H(1, 2, \ldots, N)U^{-1}(\pi) \equiv H(\pi(1), \pi(2), \ldots, \pi(N)) = H(1, 2, \ldots, N). \quad (2.2)$$

If this is the case, then according to the general principles of quantum mechanics, the eigenstates of $H$ should transform according to some representation of $S_N$:

$$U(\pi)|\psi_j\rangle = \sum_k |\psi_k\rangle D_{kj}(\pi). \quad (2.3)$$

Denoting by $|1, 2, \ldots, N\rangle$ the eigenstates of a complete set of commuting one-particle operators, on which the permutation operators act as follows:

$$U(\pi)|1, 2, \ldots, N\rangle = |\pi(1), \pi(2), \ldots, \pi(N)\rangle, \quad (2.4)$$

the wavefunctions

$$\psi(1, 2, \ldots, N) = \langle 1, 2, \ldots, N|\psi\rangle \quad (2.5)$$

transform in the following way:

$$\langle 1, 2, \ldots, N|U(\pi)|\psi_j\rangle = \psi_j(\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(N)) = \sum_k \psi_k(1, 2, \ldots, N)D_{kj}(\pi),$$

or, by a relabelling of the arguments of the wave functions:

$$\psi_j(1, 2, \ldots, N) = \sum_k \psi_k(\pi(1), \pi(2), \ldots, \pi(N))D_{kj}(\pi). \quad (2.6)$$

A textbook in group theory would tell us that there are exactly two one-dimensional representations of any $S_N$, $N \geq 2$:

- The trivial representation $D(\pi) = 1$. Particles transforming according to this representation are called bosons and their wavefunctions are completely symmetric.

- The alternating representation $D(\pi) = (-1)^{|\pi|}$, where $|\pi|$ denotes the number of exchanges needed to build the permutation (although not unique, this number is always either even or odd for a given $\pi$). Particles transforming in this way are called fermions, and their wavefunctions are completely antisymmetric.

Irreducible representations of higher dimensions do occur for $N \geq 3$, and the term parastatistics has been introduced to describe this situation. Parastatistics will not be treated further in this course, be it either because it does not seem to occur in
nature or because general theorems say that parastatistics can always be replaced by hidden ("colour") degrees of freedom.

The alternative between Bose or Fermi statistics, which the above argument led us to, can be expressed in a simple way, which, however, is an extremely powerful tool for computations. I am referring to the formalism of second quantization. Let \( \psi^\dagger(x) \) be the operator creating a particle at \( x \), and \( \psi(x) \) the operator annihilating a particle at \( x \) (for simplicity we assume that the particle number is conserved as e.g. in nonrelativistic many-body theory), and \( |0\rangle \) the vacuum (no particle) state. Then the particle statistics is all contained in the algebraic relations:

\[
[\psi(x), \psi(x')]_\mp = [\psi^\dagger(x), \psi^\dagger(x')]_\mp = 0, \tag{2.7}
\]

\[
[\psi(x), \psi^\dagger(x')]_\mp = \delta(x - x'), \tag{2.8}
\]

\[
\psi(x)|0\rangle = 0, \tag{2.9}
\]

where the - sign applies to bosons, the + sign to fermions. The state with particles localized at \( x_1, ..., x_N \) is represented by the state vector

\[
|x_1, ..., x_N\rangle = \psi^\dagger(x_1) \cdots \psi^\dagger(x_N)|0\rangle \tag{2.10}
\]

and the relations (2.7) automatically ensure the correct symmetry properties.

A nice and important fact is that the Bose–Fermi–alternative looks the same in any representation: Indeed, in Eqs. (2.4) and (2.5) we did not specify which representation we were using. In the language of second quantization, we see this as follows:

Let \( \{u_n(x)\} \) be a complete set of orthonormal functions (eigenfunctions of a one–particle operator):

\[
\int dx u_n^*(x)u_m(x) = \delta_{nm},
\]

\[
\sum_n u_n(x)u_m^*(x') = \delta(x - x').
\]

Expanding \( \psi(x), \psi^\dagger(x) \):

\[
\psi(x) = \sum_n a_n u_n(x),
\]

\[
\psi^\dagger(x) = \sum_n a_n^\dagger u_n^*(x),
\]

one easily derives the algebra of the operators \( a_n, a_n^\dagger \), which is formally identical to (2.7)–(2.8):

\[
[a_n, a_m]_\mp = [a_n^\dagger, a_m^\dagger]_\mp = 0,
\]

\[
[a_n, a_m^\dagger]_\mp = \delta_{nm}. \tag{2.11}
\]

The fact that the second quantization is equally simple in any representation is of crucial importance e.g. in the relativistic case, where (asymptotic) states of sharp momenta make sense, but states of sharp localization do not.
Can more exotic possibilities for the statistics of identical particles be envisaged? The answer, gleaned at in partial results for 1+1-dimensional field theories, was definitively shown to be yes by Leinaas and Myrheim in 1977, provided the dimension of space is 1 or 2. Subsequently, the name anyons (in Spanish: cualquierones, according to Eduardo Fradkin) was given to particles obeying such exotic statistics. It is evident that all parts of the previous reasoning cannot apply to anyons (since our argument uniquely led to bosons or fermions), but it is of interest to investigate how much of it can be saved, not least because of the computational ease of the second quantized formalism.

As an example, consider generalizing the relations (2.7) to

\[ \psi(x)\psi(x') = \alpha \psi(x')\psi(x). \]

(2.12)

Exchanging \( x \) and \( x' \) we see that consistency requires \( \alpha = \alpha^{-1} \), i.e. \( \alpha = \pm 1 \). Thus we are back at the Bose-Fermi alternative. Hence \( \alpha \) cannot be a constant, rather should we take

\[ \psi(x)\psi(x') = \alpha(x, x')\psi(x')\psi(x), \]

(2.13)

with \( \alpha(x, x') = \alpha^{-1}(x', x) \). But the form (2.13) is now peculiar to \( x \)-space; in \( p \)-space, e.g., the relation will look completely different!

3. How Come Anyons?

In order to clearly see what new features are implied in the reasoning leading to anyon statistics, I will here present a strict, orthodox party line. Like all party lines, it should be constantly challenged, and ways to overthrow the orthodoxy should be sought. In this way new discoveries can be made, and what survives of the orthodoxy will stand on more secure ground. So here we go:

By identical particles we shall mean the following: Firstly, the configuration space for \( N \) identical particles in \( D \)-dimensional Euclidean space is not \( (\mathbb{R}^D)^N \), instead \( (x_1 \ldots x_N) \) should be identified with \( (x_{\pi(1)} \ldots x_{\pi(N)}) \) for any \( \pi \in S_N \). The space obtained after such an identification, denoted by \( (\mathbb{R}^D)^N/S_N \), has the awkward property of possessing potentially singular points. The candidates for such points are the fixed points of the action of \( S_N \) on \( (\mathbb{R}^D)^N \), i.e. the diagonal \( \Delta \equiv \{ (x_1 \ldots x_N) \in (\mathbb{R}^D)^N | x_i = x_j \text{ for at least one pair} \} \). So, to stay clear of trouble we should remove the diagonal (we could imagine that there is a hard core interaction between the particles keeping them apart). The first statement of our dogma is thus that the configuration space of \( N \) identical particles in \( D \)-dimensional space is

\[ M_D^N = \frac{(\mathbb{R}^D)^N - \Delta}{S_N}. \]

(3.1)

Secondly, we will allow as observables only symmetric operators. This means e.g. that we are not allowed to consider one-particle operators such as \( H_0(i) = p_i^2/2m \), although in the symmetry group way of looking at things a quantity like \( \langle H_0(1) \rangle \)
makes sense (it is perfectly calculable), by symmetry, of course, it equals \( \langle H_0(2) \rangle = \cdots = \langle H_0(N) \rangle \).

How can we then introduce the concept of particle statistics, since by adopting the above dogma we have banished all talk about "interchanging particles" and the like? The key point is that the configuration manifold (3.1) is (for \( D \geq 2 \)) not simply connected: There are closed loops in \( M^D_N \) that cannot be continuously shrunk to a point. A simple example makes this clear. \( M^2_2 \) can be constructed as follows: The coordinates \( x_1, x_2 \) of \( \mathbb{R}^2 \times \mathbb{R}^2 \) are replaced by the center of mass coordinate \( X = \frac{1}{2}(x_1 + x_2) \) and the relative coordinate \( x = x_1 - x_2 \). Removing the diagonal \( x_1 = x_2 \) means leaving out the origin of the \( x \)-plane. "Modding" by \( S_2 \) means identifying \( x \) and \(-x\). This we can do by restricting us to the left half \( x \)-plane \( x_1 \geq 0 \). On the \( x^2 \)-axis we still have to modify the points \((0,x^2)\) and \((0,-x^2)\), i.e. glue the negative \( x^2 \)-axis to the positive \( x^2 \)-axis. The resulting construction is the surface of a cone with the tip (\( x = 0 \)) excluded:

\[
M^2_2 = \mathbb{R}^2 \times \{ \text{cone without the tip} \}.
\]

Evidently, any closed loop on the mantle of the cone encircling the tip cannot be shrunk to a point; thus \( M^2_2 \) is multiply connected.

Quantum mechanics on multiply connected spaces shows interesting new features not present in the familiar case when the configuration space is topologically trivial, and the possibility of exotic statistics lies hidden in these new traits. I will present the argument using Feynman’s path integral formulation of quantum mechanics. If you prefer the Schrödinger wave function formulation, I recommend the book by Morandi 5.

In Feynman’s formulation, the propagator for a configuration \( a \in M^D_N \) at time \( t_a \) to develop into a configuration \( b \) at \( t_b \) is given by the path integral

\[
K(b,t_b;a,t_a) = \int_{q(t_a)=a}^{q(t_b)=b} Dq e^{iS}.
\]

(3.2)

Here \( S \) is the action, and the integral runs over all paths \( q(t) \) in \( M^D_N \) connecting \( a \) to \( b \). As \( M^D_N \) is multiply connected, there are paths which cannot be continuously deformed into each other. All paths which can be deformed into each other we group into the same homotopy class. The set of all paths from \( a \) to \( b \) is thus divided into homotopy classes, and we denote the set of all homotopy classes by \( \pi(M^D_N,a,b) \).

As was first pointed out by Schulman 6, in this case we can a priori weight the contributions from different classes differently:

\[
K = \sum_{a \in \pi} \chi(\alpha) K^\alpha,
\]

(3.3)

where the sum runs over all classes, \( K^\alpha \) denotes the integral over all paths in the class \( \alpha \).

What consistency conditions do the weights \( \chi(\alpha) \) have to satisfy? The answer to this question was given by Laidlaw and Morette–De Witt 7. The argument is quite
subtle, and I shall only summarize the results here, referring the reader to the original paper for details.

Firstly, the weights $\chi(\alpha)$ have to be pure phases: $|\chi(\alpha)| = 1$. The reason for this is basically that when $t_b - t_a \to 0$, only one class contributes, and this term has to reproduce $K$ up to a phase.

Secondly, we make the following observation. Let us choose a fixed point $q_0 \in M^D_N$, and fixed paths $C_a, C_b$ joining $a$ and $b$ to $q_0$, respectively. Then in each class $\alpha$ there are representatives consisting of:
- the path $C_a$
- a closed loop in $M^D_N$ starting and ending at $q_0$
- the path $C_b$.

Now the set of all closed loops at $q_0$ falls into classes for which we can introduce a "multiplication": The product of two loops $\gamma, \gamma'$ is the loop formed by first going around $\gamma$ and then around $\gamma'$. With this multiplication the set of classes of closed loops at $q_0$ forms a group (the unit element being the class to which the constant loop staying at $q_0$ belongs, and the inverse of the class of the loop $\gamma$ being the class to which traversing $\gamma$ in the opposite direction belongs), the fundamental or first homotopy group $\pi_1(M^D_N)$ (we can drop the reference to $q_0$, since all the groups at different $q_0$ are isomorphic).

In this way we establish a one-to-one correspondence between $\pi(M^D_N, a, b)$ and $\pi_1(M^D_N)$ (which by the way is not unique, since it depends on the choice of $C_a$ and $C_b$), and $\alpha$ can be taken as labelling $\pi_1(M^D_N)$.

If $t_c$ is a time intermediate between $t_a$ and $t_b$, the propagator has to obey

$$K(b, t_b; a, t_a) = \int_{M^D_N} dc K(b, t_b; c, t_c) K(c, t_c; a, t_a). \quad (3.4)$$

For the classes this means

$$K^\gamma(b, t_b; a, t_a) = \sum_{\alpha, \beta: \alpha \beta = \gamma} \int dc K^\beta(b, t_b; c, t_c) K^\alpha(c, t_c; a, t_a). \quad (3.5)$$

Together, (3.4) and (3.5) imply that

$$\chi(\alpha)\chi(\beta) = \chi(\gamma = \alpha \beta), \quad (3.6)$$

i.e. the weights form a unitary, one-dimensional representation of $\pi_1(M^D_N)$.

These groups are known:

$$\pi_1(M^2_N) = B_N, \quad (3.7)$$

the $N$-string braid group, whereas

$$\pi_1(M^D_N) = S_N \quad (3.8)$$

for $D \geq 3$. Both $B_N$ and $S_N$ are generated by $N - 1$ generators $\sigma_1 \ldots \sigma_{N-1}$, obeying the constraints

$$\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2, \quad (3.9)$$
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \]  
\hspace{1cm} (3.10)

The difference between \( B_N \) and \( S_N \) arises from the fact that for \( S_N \) we require in addition to (3.9) and (3.10)
\[ \sigma_i^2 = e, \]  
\hspace{1cm} (3.11)

where \( e \) is the unit element.

The connection to particle statistics comes through recognizing that the class of closed loops \( \sigma_i \) corresponds to an interchange of particles \( i \) and \( i + 1 \). In the plane \( (D = 2) \) this can be done in two homotopically inequivalent ways, which can be represented by the loop where these two particles move on the circumference of a circle passing through their original locations either counterclockwise (corresponding to \( \sigma_i \)) or clockwise (corresponding to \( \sigma_i^{-1} \)) interchanging their places, whereas all other particles stay put. In three or more dimensions these loops can be deformed into each other e.g. by rotating the circle around a diameter, i.e. \( \sigma_i = \sigma_i^{-1} \) in accordance with (3.11), but not in two dimensions.

The elements of the group \( B_N \) (\( S_N \)) are formed by taking all possible products of all possible powers (positive and negative) of the generators \( \sigma_i \), taking into account the constraints (3.9), (3.10) (\( (3.9), (3.10), (3.11) \)). \( B_N \) is a group of infinitely many elements, but the inclusion of the powerful constraints (3.11) reduces the number of elements of \( S_N \) to \( N! \).

Our general result, Eq. (3.6), instructs us to look for unitary, one-dimensional representations of \( B_N \) or \( S_N \). Posing
\[ \chi(\sigma_i) = e^{i\phi_i}, \]
we see that Eq. (3.10) requires \( \phi_i = \phi_{i+1} \), i.e. all phases are equal. It is customary to write
\[ \chi(\sigma_i) = e^{-i\nu \pi} \]
\[ \chi(\sigma_i^{-1}) = e^{i\nu \pi}, \]  
\hspace{1cm} (3.12)

where \( \nu \in [0, 2) \) is the statistical parameter.

In three or more dimensions, Eq. (3.11) requires \( \chi(\sigma_i)^2 = 1 \), i.e. \( \nu = 0, 1 \) are the only possibilities (we have in fact derived the result on the one-dimensional representations of \( S_N \) mentioned in section 2!). But for \( D = 2 \) there is no restriction on \( \nu \), and anyon statistics is possible.

This, then, is one version of the accepted orthodoxy on unorthodox statistics. Parts of it can be challenged. For instance, one might ask what happens if we do not remove the diagonal but work with \( (\mathbb{R}^D)^N/S_N \) as the configuration space (which then is no longer a manifold, but rather an ”orbifold” ). As is evident from our example \( M^2_2 \), this will change the fundamental group of the configuration space, and our previous argument breaks down. In simple cases, at least, it seems that Hamiltonians on \( M^2_N \) can be extended to self-adjoint operators on \( \mathbb{R}^{2N}/S_N \) (”colliding anyons”) in many ways 9. What this means is still uncertain.
Although we in the sequel will be exclusively concerned with exotic statistics in two-dimensional space, let us briefly stop to consider what happens in one dimension. In this case the configuration space is simply connected:

$$\pi_1(M_1^N) = 0,$$

and our previous argument seems to imply that no statistics is possible. In a certain sense this is true: To exchange two particles on a line, they have to be moved past each other, and what then happens depends on any contact interaction between the particles. In other words, there is a possibility of introducing statistics through the boundary conditions to be imposed at the edges of $M_1^N$. The following example, taken from Leinaas and Myrheim, illustrates this point:

Take two particles on a line, with coordinates $x_1$ and $x_2$. $M_2^1$ is e.g. the region to the right of the diagonal $x_1 = x_2$ of the $(x_1, x_2)$-plane, and we have to decide what boundary conditions to impose on the diagonal. The normal derivative of the wave function on the boundary is the partial derivative with respect to the relative coordinate $x = x_1 - x_2$, and a natural condition on the wave function would be that the probability current vanishes at the boundary (no probability "flows out of" $M_2^1$):

$$j_n \propto i(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x})|_{x=0} = 0.$$

The solution to this condition is that

$$\frac{\partial \psi}{\partial x}|_{x=0} = \eta \psi(x = 0)$$

for any real $\eta$. Choosing $\eta = 0$ allows $\psi$ to be extended to an even function of $x$ in the whole plane, i.e. Bose statistics; if $\eta = \pm \infty, \psi(x = 0) = 0$, and $\psi$ can be extended into an antisymmetric function, i.e. we get Fermi statistics. For any other value of $\eta$ we get statistics intermediate between Bose and Fermi; i.e. what corresponds to anyons in $D = 1$.

4. The Transmutation of Statistics into a Topological Interaction

A direct attack on the anyon problem using the boundary conditions on the wave function of $N$ anyons implied by the propagator (3.3) with weights (3.12) is easy for $N = 2$, but already for $N = 3$ it becomes a very difficult problem, and significant progress in solving the three-anyon problem was achieved only very recently. Another approach, whereby the exotic statistics is transformed into a peculiar interaction between ordinary bosons or fermions, has become more popular. In this section, we shall derive this statistical interaction, following Wu, and in the next section we shall show that this same statistical interaction can be generated by introducing a gauge field with a very special kinetic term, the famous Chern–Simons term.

Define

$$z_{ij} = (x_j^1 - x_i^1) + i(x_j^2 - x_i^2) = |z_{ij}|e^{i\phi_{ij}}.$$  (4.1)
As a loop representing the class $\sigma_i$ we can take the loop in $M^2_N$ where particles $i$ and $i+1$ exchange their places by rotating counterclockwise through $\pi$ around the center point of the line joining their original positions, whereas all other particles stay where they are. Their relative polar angle changes by $\pi$:

$$\Delta \phi_{i,i+1} = \int_{t_a}^{t_b} dt \frac{d}{dt} \phi_{i,i+1}(t) = \pi,$$

(4.2)
i.e. we can write

$$\chi(\sigma_i) = e^{-i\nu \pi} = \exp(-i\nu \int_{t_a}^{t_b} dt \frac{d}{dt} \phi_{i,i+1}).$$

(4.3)
The changes in all other relative polar angles add up to zero, because $\Delta \phi_{jk} = 0$, $j, k \neq i, i+1$, and $\Delta \phi_{ij} + \Delta \phi_{i+1,j} = 0, j \neq i, i+1$. For the inverse $\sigma_i^{-1}$, take a clockwise rotation $\Delta \phi_{i,i+1} = -\pi$, and

$$\chi(\sigma_i^{-1}) = e^{+i\nu \pi} = \exp(-i\nu \int_{t_a}^{t_b} dt \frac{d}{dt} \phi_{i,i+1})$$

(4.4)
again. Since any loop can be built up out of products of the generators $\sigma_i$, we see that

$$\chi(\alpha) = \exp(-i\nu \int_{t_a}^{t_b} dt \frac{d}{dt} \sum_{i<j} \phi_{ij}(t)).$$

(4.5)

Thus, supposing the dynamics of the anyons is described by a Lagrangian $L$, the propagator can be written

$$K = \sum_{\alpha} \chi(\alpha) \int_{q(t) \in \alpha} Dq e^{i \int_{t_a}^{t_b} dt L} = \sum_{\alpha} \int_{q(t) \in \alpha} Dq e^{i \int_{t_a}^{t_b} dt L_{eff}} = \int_{\text{all paths}} Dq e^{i \int_{t_a}^{t_b} dt L_{eff}},$$

(4.6)
which is the path integral of boson particles with a Lagrangian

$$L_{eff} = L - \nu \sum_{i<j} \frac{d}{dt} \phi_{ij}(t).$$

(4.7)
The last term, the statistical interaction, is a total derivative (and thus a "topological term") and will not affect the equations of motion of the system. Its only role is to provide the correct statistical weight factors in the propagator.

Let us look more closely at the case of free (nonrelativistic) anyons. Then

$$L_{eff} = \sum_{i=1}^{N} \frac{m}{2} \dot{x}_i^2 - \nu \sum_{i<j} \dot{\phi}_{ij}.$$

(4.8)
The time–derivative of the polar angle can be written

$$\dot{\phi}_{ij} = (\dot{x}_i \cdot \nabla_i + \dot{x}_j \cdot \nabla_j) \phi_{ij} = (\dot{x}_i - \dot{x}_j) \cdot \nabla_i \phi_{ij} = (\dot{x}_j - \dot{x}_i) \cdot \nabla_j \phi_{ij}.$$
Thus the canonical momentum corresponding to $x_i$ is

$$p_i = \frac{\partial L_{\text{eff}}}{\partial \dot{x}_i} = m\dot{x}_i - \nu \nabla_i \sum_{j \neq i} \phi_{ij} \equiv m\dot{x}_i + a_i, \quad (4.9)$$

where, by abusing three-dimensional notation,

$$a_i = \nu \sum_{j \neq i} \frac{e_3 \times (x_i - x_j)}{|x_i - x_j|^2}. \quad (4.10)$$

(e_3 is a unit vector perpendicular to the plane where the particles move.)

The Hamiltonian of the system is

$$H = \sum_{i=1}^{N} \dot{x}_i \cdot p_i - L_{\text{eff}} = \frac{1}{2m} \sum_{i=1}^{N} (p_i - a_i)^2, \quad (4.11)$$

which is of the same form as the minimally coupled Hamiltonian for $N$ particles moving in an abelian gauge field described by the vector potential $A(x_i) = a_i$. Indeed, $a_i$ has been given the name the statistical gauge field.

From Eq. (4.9) we see that in gauge theory language the statistical gauge field is a pure gauge, and can thus be removed by a gauge transformation:

$$\tilde{\psi} = e^{i\nu \sum_{i<j} \phi_{ij}} \psi_{\text{Bose}}(z_1, \ldots, z_N, z_1^*, \ldots, z_N^*) = \prod_{i<j} (z_{ij})^{i\nu} \Phi_{\text{Bose}}(z_1, \ldots, z_N, z_1^*, \ldots, z_N^*); \quad (4.12)$$

$$(z_k = x_k^1 + ix_k^2, z_{ij} = z_i - z_j)$$

$$\tilde{H} = e^{i\nu \sum_{i<j} \phi_{ij}} H e^{-i\nu \sum_{i<j} \phi_{ij}} = \sum_{i=1}^{N} \frac{p_i^2}{2m}. \quad (4.13)$$

The transformed wave functions are not single-valued as functions on $M_2^N$:

$$\tilde{\psi}(x_1, \ldots, x_k, \ldots, x_l, \ldots, x_N) = e^{\pm i\nu \pi} \tilde{\psi}(x_1, \ldots, x_l, \ldots, x_k, \ldots, x_N),$$

since $\phi_{kl} = \phi_{lk} \pm \pi$ (depending on which way we braid, i.e. interchange $x_k$ and $x_l$). Rather, $\tilde{\psi}$ is a proper function on the universal covering space of $M_2^N$, and interchanging particles takes us from one fundamental domain to another. Of course, the gauge transformation (4.12), (4.13) has taken us back to our starting point: Eq. (4.12) is the form of the wave function implied by the propagator (3.3).

5. The Chern–Simons Action and Anyon Statistics

The (abelian) Chern–Simons (CS) action is

$$S_{\text{CS}} = \frac{\kappa}{2} \int d^3 x \mathcal{L}_{\text{CS}} = \frac{\kappa}{2} \int d^3 x e^{\alpha \beta \gamma} A_\alpha \partial_\beta A_\gamma \quad (5.1)$$
(\epsilon_{012} = \epsilon^{012} = +1). The action (5.1) is invariant under the $U(1)$ gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$, since $\mathcal{L}_{CS}$ changes only by a total derivative. Let us couple the CS vector potential to a current $j^\mu$ describing $N$ point particles:

$$j^\mu(x) = g \sum_{n=1}^{N} v_n^\mu \delta_2(x - x_n(t)),$$

(5.2)

where the 3-velocity $v^\mu = (1, \mathbf{v})$, and $g$ is the "CS- charge". The coupling is

$$S_{int} = - \int d^3 x j^\mu(x) A_\mu(x)$$

(5.3)

($g_{\mu\nu} = \text{diag}(1, -1, -1)$). Let the particles move nonrelativistically:

$$S_{\text{matter}} = \int dt \sum_{n=1}^{N} \frac{m}{2} v_n^2.$$  

(5.4)

The total action of the model we shall study in this section is then

$$S = S_{\text{matter}} + S_{CS} + S_{int}.$$  

(5.5)

The equations of motion can be straightforwardly derived. Introducing the CS-field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

with components $E^i \equiv F_{0i}$ (CS-electric field) and $B \equiv F_{21}$ (CS-magnetic field; in 2+1 dimensions the magnetic field is a pseudoscalar), we can write the Lorentz force equations for the particles:

$$m \dot{v}_n^i = g(E^i(t, x_n) + \epsilon^{ij} v_j^i B(t, x_n))$$

(5.6)

($\epsilon^{12} = \epsilon_{12} = +1$). Varying the action with respect to $A_\mu$ we get the field equations

$$j^\mu = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho},$$

(5.7)

i.e.

$$E^i = \frac{1}{\kappa} \epsilon^{ij} j_j,$$

(5.8)

$$B = -\frac{1}{\kappa} j^0 \equiv -\frac{1}{\kappa} \rho.$$  

(5.9)

Since the CS action is of first order in the derivatives of the fields, the field equations simply express the fields as functions of the sources and allow the complete elimination of the fields from the equations of motion. Inserting Eqs (5.8) and (5.9) in (5.6) we get

$$m \dot{v}_n^i = \frac{g^2}{\kappa} \sum_{m=1}^{N} \epsilon^{ij}(v_m^j(t) - v_n^j(t)) \delta_2(x_n(t) - x_m(t)).$$  

(5.10)
We see that the Lorentz force vanishes almost everywhere, so that our system describes free particles. The Chern–Simons term is not without consequences, however. This is most clearly seen in the Hamiltonian picture. By following the standard canonical procedure we derive the Hamiltonian

$$H = \sum_{n=1}^{N} \frac{1}{2m} (p_n - gA(x_n, t))^2 + \int d^2x A_0(x)(\kappa B(x) + \rho(x)).$$

We still have the freedom to choose a suitable gauge. The clever choice is:

$$A_0 = 0, \partial_i A^i = 0.$$  \hspace{1cm} (5.12)

The latter condition allows us to solve Eq. (5.9) (which in the Hamiltonian formulation is to be imposed as a constraint, like Gauss’ law in the Hamiltonian formulation of Maxwell electrodynamics) uniquely for the CS vector potential:

$$A^i(x) = \frac{1}{2\pi\kappa} \int d^2x' \epsilon^{ij} \frac{x^j - x'^j}{|x - x'|^2} \rho(x') = \frac{g}{2\pi\kappa} \sum_{m=1}^{N} \epsilon^{ij} \frac{x^j - x'^j_m}{|x - x'|^2}. \hspace{1cm} (5.13)$$

Substituting Eq. (5.13) into Eq. (5.11), where the last term is zero in the gauge (5.12), and dropping the troublesome divergent $n = m$ terms (it has been argued that this can be justified through suitable regularization), we arrive at exactly the statistical interaction (4.11), if we identify the statistical parameter as

$$\nu = \frac{g^2}{2\pi\kappa}. \hspace{1cm} (5.14)$$

Thus, the Chern–Simons field minimally coupled to matter particles generates anyon statistics! This important observation points to a way of constructing a full-fledged field theory of anyons, a topic to which we shall now turn.

6. Nonrelativistic Chern–Simons (–Maxwell) Field Theory

Quantum field theory remains the preferred vehicle for many-body quantum theory. In the relativistic case, when particle numbers are not conserved, it is practically the only available alternative, but even in the nonrelativistic case it is computationally superior to its rivals. Thus it is natural to attempt to base a many-anyon theory on a quantum field theory of anyons.

However, we do not yet know what such a field theory should look like. In view of the results of the preceding section, according to which the minimal interaction of particles with a Chern–Simons field induces anyon statistics for the particles, it is tempting to start from a theory of a matter field in interaction with a Chern–Simons field.

Sticking to the nonrelativistic case, we might try

$$\mathcal{L}_0 = i\psi^\dagger D_0\psi + \frac{1}{2m} \psi^\dagger \mathbf{D}^2 \psi + \frac{\kappa}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma,$$  \hspace{1cm} (6.1)
where \( \psi \) is a boson field, and \( D_\mu = \partial_\mu + igA_\mu \). To the Lagrangian (6.1) we might optionally add further terms, like a Maxwell term

\[
\mathcal{L}_M = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu},
\]

(6.2)

or a contact interaction between the \( \psi \)-particles:

\[
\mathcal{L}_1 = \frac{\lambda}{2} (\psi^\dagger \psi)^2.
\]

(6.3)

The advantage in adding a Maxwell term lies in making the theory more regular in that \( A_\mu \) then becomes a physical degree of freedom (the transverse part of \( A_\mu \) describes in that case a massive ”photon” of mass \( m_\gamma = e^2 \kappa \)). The contact term (6.3) again brings in new interesting features. The classical theory based on \( \mathcal{L}_0 + \mathcal{L}_1 \) has soliton solutions \cite{13,14}. Lüscher has studied the theory \( \mathcal{L}_0 + \mathcal{L}_M \) \cite{15}. We shall here see what follows from \( \mathcal{L}_0 \) alone, mainly following \cite{12,13}. In order not to get bogged down in lots of detail, we will proceed briskly ahead with a heuristic account of the main line of argument and refer to the original papers for a more careful treatment (see also \cite{16,17}).

We are interested in a second quantized theory of anyons. Under the assumption that Eq. (6.1) does describe anyons, our strategy will be to eliminate the Chern–Simons field through an appropriate gauge transformation and to investigate the properties of the transformed field operators.

In the pure Chern–Simons case, the equations of motion again connect the field strength operators to the particle field operators:

\[
\epsilon^{\alpha\beta\gamma} F_{\beta\gamma} = \frac{2g}{\kappa} j^\alpha,
\]

(6.4)

where now

\[
j^0 \equiv \rho = \psi^\dagger \psi
\]

(6.5)

\[
j^k = \frac{i}{2m} (\psi^\dagger D^k \psi - (D^k \psi)^\dagger \psi).
\]

(6.6)

In particular, Eq. (6.4) says for \( \alpha = 0 \):

\[
B = F_{21} = -\frac{g}{\kappa} \rho.
\]

(6.7)

In a transverse gauge, \( \nabla \cdot \mathbf{A} = 0 \), this equation can be solved for the vector potential \( \mathbf{A} \):

\[
A^i(x,t) = \epsilon^{ij} \frac{\partial}{\partial x^j} \left( \frac{g}{\kappa} \int d^2 y G(x - y) \rho(y) \right).
\]

(6.8)

Here \( G(x) \) is the two-dimensional Green function

\[
\nabla^2 G(x) = \delta_2(x),
\]

13
i.e.

$$G(x) = \frac{1}{2\pi} \log(\mu|x|),$$

where $\mu$ is an arbitrary scale. Introducing again the polar angle $\phi(x)$ through

$$z(x) \equiv x^1 + i x^2 = |z| e^{i \phi(x)},$$

the Cauchy–Riemann equations for $f(z) = \log z$ read

$$\epsilon^{ij} \frac{\partial}{\partial x^j} \log |x - y| = - \frac{\partial}{\partial x^i} \phi(x - y).$$

This means that Eq. (6.8) can be written

$$A(x, t) = - \frac{g}{2\pi \kappa} \int d^2 y \nabla_x \phi(x - y) \rho(y, t).$$

(6.10)

If I am allowed to move the derivation operator outside the integral (this is not trivial: $\phi$ is not single-valued; see [3] for a discussion about the validity of this step), (6.10) is a pure gauge $A = \nabla \Lambda$, with

$$\Lambda(x, t) = - \frac{g}{2\pi \kappa} \int d^2 y \phi(x - y) \rho(y, t).$$

(6.11)

Returning to Eq. (6.4), now for $\alpha = i = 1, 2$:

$$F_{i0} = \partial_i A_0 - \partial_0 A_i = \frac{g}{\kappa} \epsilon_{ik} j^k,$$

(6.12)

we can, using the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

(6.13)

solve also for $A_0$, with the result

$$A_0(x, t) = \frac{g}{\kappa} \int d^2 y G(x - y) e^{ik} \frac{\partial j^k}{\partial y^i}.$$

(6.14)

Upon integrating by parts and using Eq. (6.13) again, we arrive at

$$A_0(x, t) = \frac{\partial}{\partial t} \frac{g}{2\pi \kappa} \int d^2 y \phi(x - y) \rho(y, t) = - \frac{\partial}{\partial t} \Lambda(x, t).$$

(6.15)

Thus we have shown that $A_\mu = - \partial_\mu \Lambda$, a pure gauge. A gauge transformation will remove it and bring (6.1) to the form

$$\mathcal{L}'_0 = i \bar{\psi}^i \partial_0 \psi_i + \frac{1}{2m} \bar{\psi}^i \nabla^2 \psi_i.$$

(6.16)
with the transformed fields
\[
\tilde{\psi}(x, t) = e^{-ig\Lambda(x, t)}\psi(x, t)
\]
\[
\tilde{\psi}^\dagger(x, t) = \psi^\dagger(x, t)e^{ig\Lambda(x, t)}.
\] (6.17)
The gauge parameter $\Lambda$ is an operator, and this will affect the algebra of the $\tilde{\psi}$–
operators. $\psi$ and $\psi^\dagger$ were ordinary boson fields satisfying the algebra (2.7), (2.8). From Eq. (6.11) we deduce
\[
[\psi(x, t), \Lambda(y, t)] = -\frac{g}{2\pi\kappa}\phi(y - x)\psi(x, t).
\] (6.18)
with this result, it is straightforward to derive the algebra of the transformed operators (6.17). We get, using the notation (5.14), e.g.
\[
\tilde{\psi}(x, t)\tilde{\psi}(y, t) = e^{i\nu(\phi(y - x) - \phi(x - y))}\tilde{\psi}(y, t)\tilde{\psi}(x, t).
\] (6.19)
This is of the general form (2.13), but we have to ask is this really what we want?
All hinges on the argument function $\phi$ appearing in (6.19). It is a multivalued
function a priori, but can be made single–valued by introducing cuts across which $\phi$ jumps by $2\pi$. Taking the cut in the direction of the positive $x^1$–axis, so that $\phi(e_1 + \epsilon e_2) \rightarrow 0$, $\phi(e_1 - \epsilon e_2) \rightarrow 2\pi$ as $\epsilon \rightarrow 0$, we have \[13\]
\[
\phi(y - x) - \phi(x - y) = \pi\text{sgn}(x^2 - y^2), x^2 \neq y^2;
\]
\[
= \pi\text{sgn}(x^1 - y^1), x^2 = y^2.
\]
Cutting along the negative $x^1$–axis gives the opposite values for the difference of the arguments.
In either case, given the locations $x$ and $y$, the phase factor in (6.19) takes a
specific value, either $e^{i\nu\pi}$ or $e^{-i\nu\pi}$. Note that this is the phase acquired by the wave
function for two anyons in a state $|\Phi\rangle$:
\[
\Psi(x, y) = \langle 0|\tilde{\psi}(x)\tilde{\psi}(y)|\Phi\rangle
\] (6.20)
under an interchange of the arguments. But this is not enough. The phase of the wave
function should be able to change both by $+\pi\nu$ and $-\pi\nu$ in response to which way
we braid in interchanging $x$ and $y$. In other words, states built by the creation oper-
ators $\tilde{\psi}^\dagger$ acting on the vacuum, with a specific, single–valued choice of the argument
function $\phi(x)$, do not provide a representation of the braid group $B_N$.
It is evident, however, that this is the best we can achieve with local operators $\tilde{\psi}, \tilde{\psi}^\dagger$. Local information, in the meaning of initial and final positions of particles, is
simply not sufficient to code the braiding, where we have to specify also which way
around each other the particles passed in interchanging positions.
The only solution to this dilemma (and in fairness it should be added that not
all experts see this as any dilemma) within the present framework, is to give the
argument function \( \phi \) a definition that makes \( \tilde{\psi} \) a nonlocal operator. This point has been emphasized by Semenoff [12].

Let us return to the transformed field operators (6.17) with the operator-valued gauge function (6.11). We give the following definition of the argument function \( \phi \) in (6.11): To the point \( x \) we attach a curve \( C_x \) starting from some reference point \( P \) (infinity is a convenient choice) and ending at \( x \). Let \( x' \) move along \( C_x \) from \( P \) to \( x \). \( \phi_{C_x}(x, y) \) is then defined as the change in the polar angle of \( x' \), as seen from \( y \), as \( x' \) moves from \( P \) to \( x \) along \( C_x \):

\[
\phi_{C_x}(x, y) = \int_{C_x} d\mathbf{l} \cdot \nabla_{\mathbf{l}} \phi(1 - \mathbf{y}).
\]  

(6.21)

The corresponding transformed field operators now depend on the curve \( C_x \) as well:

\[
\tilde{\psi}_{[C_x]} = e^{i\nu \int d^2 y \phi_{C_x}(x, y) \rho(y)} \psi(x)
\]

\[
\tilde{\psi}^\dagger_{[C_x]} = \psi^\dagger(x) e^{-i\nu \int d^2 y \phi_{C_x}(x, y) \rho(y)}.
\]  

(6.22)

A state of two localized anyons is now given by

\[
\tilde{\psi}^\dagger_{[C_x]}\tilde{\psi}^\dagger_{[C_y]}|0\rangle = e^{-i\nu \phi_{C_x}(x, y)}|x, y\rangle,
\]  

(6.23)

where \( |x, y\rangle \) is the symmetric two-boson state

\[
|x, y\rangle = \psi^\dagger(x)\psi^\dagger(y)|0\rangle.
\]  

(6.24)

The action of an operator \( U[\sigma] \) implementing the braiding transformation \( \sigma \) is taken to mean extending the curves \( C_x, C_y \) by pieces describing the braiding (e.g. adding half a circle, clockwise or counterclockwise oriented and centered on \( y \), to \( C_x \) and then straight line pieces to both \( C_x \) and \( C_y \) so that their endpoints are interchanged). It is clear from Eq. (6.23) that this gives the two-anyon state just the correct phase \( \chi^*(\sigma) \) corresponding to the braiding \( \sigma \). In this way we have a means of representing all of \( B_N \), albeit with nonlocal states.

7. Epilogue

We have failed miserably in attaining our initial goal of finding a simple, computationally useful formulation of the many-anyon problem, like the second quantization of bosons and fermions. The solution we ended up with is complicated and not particularly workable in practical calculations. Some authors even maintain that Chern–Simons field theory has nothing to do with anyons, e.g. [19].

The rather formal manipulations we carried out in the previous section can be put on a more rigorous footing by formulating the theory on a lattice. This is especially true of the relativistic theory, which presents difficulties of its own. Again, the theory with a Maxwell term is better behaved [20][21], the pure Chern–Simons theory presents peculiar pitfalls [22]. (It is interesting to note that Fröhlich and Marchetti [23] formulate
anyon statistics for asymptotic states in p-space — probably the right thing to do in relativistic theories!

But the conclusions remain the same: Anyons are described by nonlocal operators, which are hard to deal with. If we insist on a local formulation, we have to hide the statistics in an interaction with a Chern–Simons field, and pay the price of handling the extra interaction through whatever means are available.

For lack of space, time and competence many topics of anyon physics have not been treated in these lectures. Among these are: The relation between spin and statistics, the fractional quantum Hall effect and the statistical mechanics of anyons. For these, I have to refer you to the reviews already cited \cite{bib1,bib2,bib3}, to which a few more can be added \cite{bib4}.

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