Two photon conditional phase gate based on Rydberg slow light polaritons

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Abstract
We analyze the fidelity of a deterministic quantum phase gate (QPG) for two photons counterpropagating as polaritons through a cloud of Rydberg atoms under the condition of electromagnetically induced transparency (EIT). We provide analytical results for the phase shift of the quantum gate, and provide an estimation for all processes leading to a reduction to the gate fidelity. Especially, the influence of losses from the intermediate level, dispersion of the photon wave packet, scattering into additional polariton channels, finite lifetime of the Rydberg state, as well as effects of transverse size of the wave packets are accounted for. We show that the flatness of the effective interaction, caused by the blockade phenomena, suppresses the corrections due to the finite transversal size. This is a strength of Rydberg-EIT setup compared to other approaches. Finally, we provide the experimental requirements for the realization of a high fidelity QPG using Rydberg polaritons.

Keywords: Rydbergs, Rydberg polaritons, quantum gate, quantum optics, amo

(Some figures may appear in colour only in the online journal)

Introduction
Photons interact extremely weakly with each other, propagate with the speed of light and provide a high bandwidth. These three features make photons an excellent carrier of information. However, for applications in quantum information processing [1], interactions on the level of single quanta are necessary. Such interactions can be achieved by coupling photons to matter [2–6]. Especially, Rydberg-EIT (rEIT) has emerged a promising approach [7–13] visible in great experimental [4, 14–28] and theoretical [29–55] progress. A photonic quantum phase gate (QPG) using Rydberg-EIT in a counter-propagating setup was first discussed by Friedler et al [56] and an extended description was shown by Gorshkov et al [30]. However, a study including all effects that decrease the fidelity of the phase gate, as well as, a discussion of a specific microscopic setup is still missing. Here, we attempt to fill this gap by analyzing within a microscopic description the different source for a reduction in gate fidelity in a realistic setup.

Recently, a few alternative approaches for a QPG have emerged [57, 58], which were motivated (at least partially) by the belief that the link between propagation and interaction [59, 60] precludes high-fidelity gates whenever a cross-phase modulation (XPM) on a single photon level is used. However, for the Rydberg-EIT setup the interaction between Rydberg polaritons has a finite range and Rydberg polaritons acquire an effective mass in addition to the linear slow light velocity, and therefore the proposed no-go theorems [59–61] do not apply [55]. The effective interaction between Rydberg polaritons has an important feature, namely that the interaction potential is flat [40] for distances smaller than the interaction range—the so-called blockade radius, which is on the order of 10 μm. This feature could in principle enable a phase gate in a copropagating configuration, which in general is characterized by longer interaction times and therefore could enable greater phase shifts. However, the condition of a homogeneous phase shift requires compression of photons to the size of the blockade radius. This violates the conditions for neglecting the mass term which depicts the mass.

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Passing each other, the photons pick-up the interaction-induced phase shift only if both have a specific polarization. This phase shift can be on the order of $\pi$ because of the strong interactions between Rydberg atoms. We will discuss the microscopic setup in detail at the end of the manuscript, see figure 4. We define the fidelity $F$ and phase shift $\phi$ of the gate using the overlap between the two-photon wave-functions with $(\mathcal{E}\mathcal{E})$ and without $(\mathcal{E}\mathcal{E}^{V=0})$ the interaction $V$ between the polaritons:

$$\sqrt{F} e^{i\phi} = \langle \mathcal{E}\mathcal{E}^{V=0} | \mathcal{E}\mathcal{E} \rangle.$$

Our definition of the fidelity includes all detrimental effects and is more restrictive than commonly used [36, 66] conditional-fidelity in which two output states are normalized (for details see appendix)

$$\sqrt{F_{\text{cond}}} e^{i\phi} = \langle \mathcal{E}\mathcal{E}^{V=0} | \mathcal{E}\mathcal{E}^{V=0} \rangle / \langle \mathcal{E}\mathcal{E}^{V=0} | \mathcal{E}\mathcal{E}^{V=0} \rangle.$$ (2)

We will identify the leading contributions to the fidelity by different physical phenomena

$$F = (\beta_\gamma \beta_{\text{sc}} \beta_{\text{wp}} \beta_{\text{at}} \beta_{\text{Ry}} \beta_{\text{at}})^2.$$ (3)

Here,

- $\beta_\gamma$ accounts for the spontaneous emission from the intermediate $p$-level caused by the interaction,
- $\beta_{\text{sc}}$ accounts for scattering into additional polariton channels,
- $\beta_{\text{wp}}$ includes the impact of the finite length of the wave-packets on (a) the losses from $p$-state during propagation and (b) the inhomogeneous interaction-induced phase-shift
- $\beta_{\text{at}}$ describes the distortion of the phase shift due to spatially varying atomic density,
- $\beta_{\text{Ry}}$ denotes losses due to the finite lifetime of the Rydberg state, whereas
- $\beta_{\text{at}}$ depicts finite beam waist effects leading to scattering of photons to other transversal photonic modes.

At this point it is worth mentioning the family of phase-gate schemes in which at least one photon [27, 57, 58, 67–69] is stored as a collective atomic excitation. This type of scheme is superior compared to counter-propagating scheme when it comes to the $\beta_{\text{sc}}$ and $\beta_{\text{wp}}$ in equation equation (3). However, such quantum gates suffer from an additional factor decreasing the fidelity, which accounts for the finite storage and retrieval probability of the photons; for an efficient storage and retrieval [70–72], such schemes can be preferable in some circumstances.

**Propagation inside the medium**

We consider photons characterized by a single transverse mode propagating through an atomic ensemble [73], where the atomic ground state is coupled to a Rydberg $s$-state via an intermediate short-lived $p$-state, see figure 1(a). We introduce the electric field operators $\hat{E}_+^z(z)$ and $\hat{E}_-^z(z)$ creating at position $z$ photons propagating to the right and left, respectively. For
the atomic density much higher than the photonic density, the excitations of atoms generated by right- and left-moving photons into s-level and p-level are well-described by the bosonic field operators $\hat{S}_x(\epsilon)$ and $\hat{p}_x(\epsilon)$, respectively [30].

Then, we obtain the non-interacting part of the microscopic Hamiltonian [30, 40], i.e. $H_s + H_c$, within the rotating frame and under the rotating-wave approximation

$$H_{\text{sf}} = \hbar \int \text{d}z \begin{pmatrix} \hat{S}_s(z) \\ \hat{p}_s(z) \end{pmatrix} \begin{pmatrix} \mp i\hbar \partial_z & g & 0 \\ g & \Delta & \Omega \\ 0 & \Omega & 0 \end{pmatrix} \begin{pmatrix} \hat{S}_s(z) \\ \hat{p}_s(z) \end{pmatrix}. \quad (4)$$

Here, $g$ denotes the collective coupling of the photons to the matter via the excitation of ground state atoms into the p-level, while $2\Omega$ denotes the Rabi frequency of the control field between the p-level and the Rydberg state, figure 1. Note that the kinetic energy of the photons $\mp i\hbar \partial_z$, accounts for the deviation from the EIT condition. We introduced the complex detuning $\Delta = \delta - i\gamma$, which accounts for the detuning $\delta$ of the control field and the decay rate $2\gamma$ from the p-level. Since we are interested in two counter-propagating polaritons, the interaction between the Rydberg levels is described by

$$H_{\text{it}} = \int \text{d}z \int \text{d}z' \tilde{V}(\epsilon - \epsilon') \tilde{S}_{s}(z) \tilde{S}_{s}(z') \tilde{S}_{s}(z) \tilde{S}_{s}(z).$$

The full scattering of the two photons inside the media is well accounted for by the $T$-matrix. As the interaction acts only between Rydberg states, it is sufficient to study the $T$-matrix for the Rydberg states alone, which is denoted as $T_{kk'}(K, \omega)$. Here, $\hat{K} = \hbar (q_+ + q_)$ denotes the center of mass momentum and $\hbar\omega$ the total energy, while $\hat{K}' (\hat{K})$ is the incoming (outgoing) relative momentum. Note, that the total energy $\hbar\omega$ as well as the center of mass momentum $\hat{K}$ are conserved in our system. The resummation of all ladder diagrams [40] leads to the integral equation

$$T_{kk'}(K, \omega) = V_{K-K'} + \int \frac{\text{d}q}{2\pi} V_{K-q} \chi_q(K, \omega) T_{kk'}(K, \omega). \quad (5)$$

The full pair propagator of two counter-propagating polaritons and its overlap with the Rydberg state takes the form

$$\chi_q = \chi + \frac{\alpha_D}{\hbar k_{0}(K, \omega) - \hbar q + i\Omega^+} + \sum_{j=1}^{4} \frac{\alpha_j}{\hbar k_{j}(K, \omega) - \hbar q + i\Omega^+} \quad (6)$$

and is shown in figure 2(b). Here, $\chi$ accounts for the saturation of the pair propagation at large relative momenta $\hbar q \to \pm\infty$, and takes in the relevant regime $\Omega \ll |\Delta|$ the form

$$\chi(\omega) = \frac{\Delta}{2\hbar \Omega^2} \frac{1}{1 + \frac{\omega}{2\Omega^2}}. \quad (7)$$

The second term in equation (6) is the pole structure for the propagation of the two incoming dark-state polaritons. The general expression of $\alpha_D$ and $k_0$ are complicated. Therefore,

$$\text{Figure 2. (a) Dispersion relations for the right- and left-propagating fields are shown using the property that } e^{-i\epsilon_p(p)}(p) = e^{i\epsilon_p(p)}, \text{ see equation (4), and therefore the curves overlap. For each direction of propagation, three noninteracting polariton branches exist. Two dark state polaritons (denoted by stars) can scatter into four channels, each represented by the different color of arrows. Solid lines correspond to right-propagating and dashed lines to left-propagating polaritons. Note that the total momentum $\hbar K$ and energy $\hbar\omega$ are conserved during the scattering process. Strongly suppressed losses into two bright polaritons are described by light and dark brown arrows. Crucial losses into bright-dark polariton pair are depicted by red and purple arrows. (b) Full pair propagator $\chi$ in the function of relative momentum $\hbar q$. Vertical lines depict relative resonant momenta with colors corresponding to colors of arrows and stars from figure (a). Note that both light and dark brown resonances are so narrow that cannot be resolved in the plot, and that resonance for pair of dark polaritons (orange) is much wider than purple and red resonances. All results are presented for $g = 3 \delta, \delta = 3 \Omega, \gamma = 0, \omega = 0.1 \times 2\Omega^2/\delta, K = -0.4 \times g^2/\hbar c$.}

we provide its analytical form only in the experimentally relevant regimes. The last sum accounts for the resonant scattering of the two incoming dark polaritons into four outgoing channels containing at least one bright polariton, see figure 2. Note that in contrast to the copropagating equation, where the massive-like behavior can be neglected only in certain regimes [55], here, the kinetic part is always linear in the relative momentum. Thus, the large phase shift is possible without a drop of the fidelity caused by the distortion of the wave-packet shape.

In the following, we derive the scattering phase shift during the collision of two dark polaritons. The first important step is to notice the expression for the scattering wave function $\psi_{kk'}^{\text{sc}}(r)$

$$\psi_{kk'}^{\text{sc}}(r) = \frac{1}{V(r)} \int \frac{\text{d}k}{2\pi} e^{ikr} T_{kk'}(r) \equiv T_{kk'}(r). \quad (8)$$

It describes the amplitude to find two Rydberg excitations for the incoming relative momentum $k'$, and follows from the close analogy of equation (5) to the Lippmann–Schwinger equation in the scattering theory [40]. In the next step, we absorb the term $\chi$ by introducing the effective interaction
potential [40]

\[ V_{\text{eff}} (r) = \frac{V(r)}{1 - \chi (\omega) V(r)}. \]  

(9)

The effective interaction potential saturates at \(-1/\chi \) for distances shorter than the blockade radius \( \xi \); the latter reduces to \( \xi = (|C_a|/\hbar \Omega^2)^{1/6} \) for van der Waals interaction. Then, equation (5) for the \( T \)-matrix simplifies to

\[ T_{ik} = V_{\text{eff}} k \gamma + \int \frac{d\eta}{2\pi} V_{\text{eff}} k^{\gamma} [\xi_{k} - \chi] T_{qk} \]  

(10)

and includes the contribution from all different poles in the two-particle propagator.

Finally, we focus on the experimentally relevant regime described by slow light polaritons \( g \gg \Omega \) with large single photon detuning \( |\Delta| \gg \Omega \). Then, the last term in equation (6) describing scattering into a bright polariton is strongly suppressed for small energies \( |\omega| < 2\Omega^2/|\Delta| \); the influence of these additional processes on the gate fidelity can be discussed within perturbation theory, see below. In the leading order, we can therefore neglect these terms and study the \( T \)-matrix with the dominant pole for the propagation of the dark polaritons alone. Within this regime, we find

\[ \alpha_D (K, \omega) = \frac{g^2}{\Omega^2 c} \left( \frac{1}{2} \left[ \frac{(\Delta K)^2}{\hbar^2} \left( \frac{\Delta \omega}{\Delta K^2} + 1 \right)^2 + 1 \right] + \frac{1}{2} \right), \]

\[ k_D (K, \omega) = \frac{g^2}{2\Delta c} \left( \frac{1 + \Delta \omega}{1 + \Delta \omega/(2\Omega^2)} \right). \]

Furthermore, we are interested in scattering processes, where the incoming momentum is on-shell and describes two dark polaritons, thus, the incoming momentum \( \hbar k \) reduces to \( \hbar K \) with conserved energy \( \hbar \omega \) and center of mass momentum \( \hbar K \).

Then, the equation for the \( T \)-matrix can be analytically solved and takes the form

\[ T_{ik} (r) = V_{\text{eff}} (r) e^{ikr} \exp \left[ -\frac{i\alpha_D}{\hbar} \int^r_\infty dz \; V_{\text{eff}} (z) \right]. \]  

(11)

With the relation (8) between the scattering wave function and the \( T \)-matrix, the final result for the phase shift due to the collision of the two dark polaritons takes the form

\[ \begin{align*}
\psi_{k0}^<(r) &= \frac{1}{1 - \chi V} e^{ikr} \exp \left[ -\frac{i\alpha_D}{\hbar} \int^r_\infty dz V_{\text{eff}} (z) \right], \\
&= \begin{cases} 
\text{e}^{ikr} & r \to -\infty, \\
\text{e}^{-ikr} & r \to +\infty,
\end{cases}
\end{align*} \]  

(12)

with the phase shift

\[ \varphi = -\frac{i\alpha_D}{\hbar} \int_{-\infty}^{\infty} dz \; V_{\text{eff}} (z). \]  

(13)

Note, that the wave function is strongly suppressed within the blockade region \( |r| < \xi \) as the excitation of two Rydberg atoms is quenched at short distances. In the presented leading order, the scattering into additional channels is suppressed and therefore we obtain only a phase shift. The phase shift contains an imaginary part accounting for the losses during the scattering process by spontaneous emission from the intermediate \( p \)-level. Performing the remaining integration for a microscopic van der Waals interaction, the phase reduces to

\[ \varphi (K, \omega) = \alpha_{p0} (K, \omega) \frac{2\pi g^2}{3|\Delta|} \left( \frac{\Delta \omega}{\Delta K^2} + \frac{1}{2} \right)^{3/2}. \]  

(14)

Note, that the phase still depends on the center of mass momentum and the total energy, and therefore, we expect a weakly inhomogeneous phase shift for a wave packet. The leading contribution is determined on the two-photon resonance, i.e. \( \omega = 0 \) and \( K = 0 \), where the solution agrees with [30]. For \( \gamma \ll |\delta| \) the exponent \( \phi \) simplifies to

\[ \phi + i|\eta| = \frac{2\pi g^2}{3|\Delta|} \left( 1 + \frac{5}{6} \frac{\gamma}{|\delta|} \right). \]  

(15)

The \( \beta \), is the main contribution to the reduction of the gate fidelity, and it requires the far detuned regime \( |\delta| \gg \gamma \) in combination with large optical depth per blockade radius.

### Scattering into additional open channels

Next, we continue with a detailed analysis of all additional contributions leading to a reduction of the gate fidelity. We start with the study of the influence of the additional poles in the two polariton propagator \( \chi \), see figure 2. These poles describe additional open channels characterized by the relative momenta \( k_f \). Therefore, the interaction between the polaritons leads to scattering of the incoming dark polaritons into these open channels. Consequently, the outgoing wave function takes the form

\[ \psi_{k0}^>(r) = \beta_{\infty} e^{ikr} + \sum_{j=1}^{4} e^{ik_j r}, \]  

(17)

with \( |\beta_{\infty}|^2 \) a reduced probability to remain in the dark polariton state, and \( |e_j|^2 \) a probability to be in the other channel. The analysis can be performed straightforwardly in lowest order perturbation theory in \( \alpha_j/\alpha_D \). Then, the correction \( \delta T \) to the \( T \)-matrix is determined by the equation

\[ \delta T_{ik} (r) = \frac{i\alpha_D}{\hbar} V_{\text{eff}} (r) e^{ikr} \int^r_\infty dz e^{-ikz} \delta T_{ik} (z) \]

\[ -\sum_{j=1}^{4} \frac{i\alpha_j}{\hbar} V_{\text{eff}} (r) \int^r_\infty dz e^{ik_j (r - z)} \delta T_{ik} (z). \]  

(18)
The general solution takes the form
\[
\delta T_{\text{sc}}(r) = \sum_{j=1}^{4} \frac{\alpha_j}{\alpha_D} \left\{ e^{i\frac{\theta(v)}{\hbar}} \int_{-\infty}^{\infty} d\tau e^{i\theta(k_0-\kappa)\tau} \partial_{\tau} e^{i\theta(\tau)} \right\} d\tau - T_{\text{sc}}(r) \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} \psi_{\text{in}}(\tau) \partial_{\tau} \psi_{\text{out}}(\tau),
\]
where \( \theta(\tau) = -\alpha_D \int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\tau} d\tau '' \psi_{\text{in}}(\tau) \partial_{\tau} \psi_{\text{out}}(\tau) \). The first term describes the weight in the additional open channels, i.e.
\[
e_j = \frac{\alpha_j}{\alpha_D} \int_{-\infty}^{\infty} d\tau e^{i\theta(k_0-\kappa)\tau} \partial_{\tau} e^{i\theta(\tau)},
\]
whereas the second term accounts for the reduction of the outgoing dark polariton state, and can be written in leading order \( \alpha_1/\alpha_D \) as
\[
1 - |\beta_{\text{sc}}|^2 = -2 \sum_{j=1}^{4} \frac{\alpha_j}{\alpha_D} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \psi_{\text{in}}(\tau) \partial_{\tau} \psi_{\text{out}}(\tau).
\]
In order to estimate the value of the suppression, it is sufficient to focus on the resonant regime with \( \omega = 0 \) and \( K = 0 \). Then, the values of momenta \( k_1 \) and the weight of the residues \( \alpha_j \) take a simple analytical form. Specifically, in the experimentally relevant regime with \( \Omega \ll g \), we find that the poles group into pairs of equal weight
\[
k_1 = k_2 = \frac{g^2}{\Delta_c}, \alpha_1 = \alpha_2 = \frac{g^2 + \Delta^2}{4 \Delta c},
k_3 = k_4 = -\Delta/c, \alpha_3 = \alpha_4 = \frac{g^2 \Omega^2}{4(g^2 + \Delta^2) c}.
\]
The first two poles describe excitations at high relative momentum, whereas the second pair of poles at low relative momenta which is always strongly suppressed for slow light polaritons, see figure 2. Therefore, the reduction due to the scattering into the additional open channels can be estimated to satisfy the inequality
\[
|\beta_{\text{sc}}|^2 \geq 1 - \phi^2 |2\alpha_1/\alpha_D| = 1 - \phi^2 \frac{\Omega^2}{\Delta^2} \left| \frac{\Delta^2 + g^2}{2g^2} \right|^2.
\]
This term is suppressed for \( \Omega \ll |\Delta| \), which provides an additional constraint on the experimental parameters.

### Wave packet propagation

The next step is to study the influence of realistic wave packets onto the gate fidelity. There are two different contributions: first one is a distortion of the wave packet due to the non-linear corrections to the dispersion relation of the dark polariton, which is also present for noninteracting polaritons. Second contribution comes from the dependence of phase factor \( \varphi \) on \( K \) and \( \omega \) leading to the longitudinally inhomogeneous phase shift for wave packets. In the following, both phenomena will be discussed in detail.

The right and left moving photonic wave packets inside the homogeneous atomic media are described by the slowly varying envelope for the electric field
\[
E_{\pm}(\tau, t) = \int \frac{d\nu}{2\pi} E_{\pm}(\nu)e^{i\nu(\tau - L_{\pm}) - i\nu t}.
\]
Here, \( E_{\pm}(L_{\pm}, t) \) describes the electric field of the incoming photonic wave packet at the boundary of the atomic media with \( L = L_+ - L_- \) the length of the media, see figure 1. In addition, the momentum \( p_{\nu}(\nu) \) of a polariton is related to frequency \( \nu \) by the dispersion relation for dark polaritons. The momentum exhibits the general form
\[
p(\nu) = \nu \frac{g^2 + \Omega^2 + \Delta^2 - \nu^2}{\nu^2 - \nu^2/c^2},
\]
where \( c = c/\sqrt{\Delta} \) and the effective mass \( m = \hbar g^2/(2c^2\Delta\Omega^2) \). The mass leads to a wave packet dispersion as well as losses from the intermediate p-level due to the imaginary part of the mass. Note that even though it looks as if the mass term contributes to the dynamics of the pair of polaritons, the physics is more subtle. Namely, in equation (6) we showed that two-photon-dynamics for conserved \( K, \omega \) has a linear dependence on the relative momentum (which allows simple analytical solution, equation (11)). However, a conserved relative momentum \( K_0 \) is a function of \( K, \omega \) and \( m \) which corresponds to equation (24) which we analyze here.

The probability for the polariton to pass through the media is given by
\[
P_{\pm} = \int \frac{d\nu}{2\pi} |E_{\pm}(\nu)|^2 e^{-2p_{\nu}(\nu)L},
\]
where \( p_{\nu}(\nu) \) denotes the imaginary part of \( p(\nu) \). Combining this contribution with the interaction effects between the polaritons, we observe two effects: first, a weak shift for the scattering phase \( \Delta \phi \) due to the averaging over different energy and momentum states, secondly, a reduction in the gate fidelity \( \beta_{\text{wp}} \). Both quantities are determined by the equation
\[
\beta_{\text{wp}} e^{i\Delta \phi} = \int \frac{d\nu d\nu'}{(2\pi)^2} |E_{\pm}(\nu) E_{\mp}(\nu')|^2 e^{-2L[p_{\nu}(\nu)+p_{\nu'}(\nu')]}
\times \exp[i\varphi(K, \omega) - i\Delta \phi + \eta].
\]
Note that \( \varphi(K, \omega) \) depends on the total energy \( \hbar \omega = \hbar \nu_1 + \hbar \nu_2 \) and the center of mass momentum \( \hbar K = \hbar \nu_1 - \hbar \nu_2 \).

In order to illustrate the phenomena of the wave packet propagation, we focus on a Gaussian incoming wave packet
\[
E_{\pm}(\nu) = \exp[-\nu^2/2\sigma^2] / (\pi \sigma^2)^{1/4}.
\]
with a small bandwidth \( \sigma \ll \Delta \Omega^2/|\Delta| \). Then, it is sufficient to keep the leading quadratic corrections in \( \omega \) and \( K \). First, the impact on the fidelity simplifies to
\[
\beta_{\text{wp}} = P.P. \exp \left[ - \frac{4}{\sigma^2} \phi^2 \frac{|\Delta|}{12\Delta^2} \right]
\]
with
\[ P_{\pm} = \exp \left[ -2i\frac{Lg}{\delta c} \left( \frac{\delta\sigma}{\Omega^2} \right)^2 \gamma \right]. \]  
(29)

Second, the shift in the coherent phase takes the form
\[ \Delta\phi = \frac{19}{2} \phi \left( \frac{\sigma\Delta}{12\Omega^2} \right)^2. \]  
(30)

**Impact of atomic distribution**

Additional contribution to the fidelity comes from the inhomogeneity of the atomic distribution. To illustrate this effect, we consider Gaussian atomic distribution \( n_a(z) = n_0 \exp[-z^2/L^2] \), where \( L \) characterizes the cloud’s length. In the relevant regime of \( \xi \ll L \), the scattering phase shift depends locally on the atomic density at which two polaritons interact with each other. Inside the medium photons are compressed to the size \( \sigma_p = cL^2/g^2\sigma \), where \( g \) in this paragraph denotes the value of collective coupling at the center of the cloud. For short photons \( \sigma_c \ll L \), the corrections to the fidelity and phase shift take the form (for details see appendix)
\[ \beta_{\text{at}}e^{i\Delta\phi_{\text{at}}} = 1 - i\phi \left( \frac{\sigma_c}{2L} \right)^2 - \frac{3}{2} \phi^2 \left( \frac{\sigma_c}{2L} \right)^4. \]  
(31)

This puts additional constraint on the length of the medium: \( L \gg \sigma_c \).

**Transversal size of the photons**

Two colliding wave-packets in the lowest Laguerre–Gauss mode are described by \( E_{\pm}(z, R_\perp) = E(z, R_\perp)u(R_\perp) \) with \( u(R) = \exp[-(\alpha^2 + \gamma^2)/w_0^2]e^{i2\pi\gamma}/w_0 \) where \( w_0 \) is the probe beam waist. The interaction potential generalized to a quasi-1D geometry depends on the relative transversal distance, \( V_{\text{eff}}(r, R) = V(r)/(1 - \gamma \omega V(r)) \), (for details see appendix). For beam waist comparable to the blockade radius, this leads to the transversely inhomogeneous phase shift [44, 56, 74]. It provides a contribution \( \beta_{Ry} \) to the gate fidelity given by
\[ \int \sum_{R_\perp} \left[ \psi(r, R) \right]^2 \left[ \psi(r, R_\perp) \right]^2 \exp[i\phi(k + R_\perp - R, \xi)] \].

Note, that we neglect transversal part of the photonic paraxial wave equation (for details see appendix). From \( \beta_{Ry} \) in the limit of \( \gamma/\delta \rightarrow 0 \) (figure 3(b)) we find that the drop of fidelity can be neglected for \( w_0/\xi \lesssim 0.35 \), which for exemplary experimental parameters, \( w_0 = 4.2 \mu m, |\Delta S| = 100S, \Delta = \gamma, \) and \( \delta = 5\gamma \), gives \( w_0/\xi = 0.17 \). Intuitively, the reason for this behavior is that polaritons interact via \( V_{\text{eff}}(r) \) which is nearly constant for the distances shorter than the blockade radius. It is an important feature of Rydberg polaritons resulting in the phase shift being nearly homogeneous also in the transverse direction. Note that this feature is no longer present in the proposals based on the pulses propagating in two parallel but spatially-separated photonic modes [36] and should be carefully taken into account.

**Rydberg lifetime**

Can be estimated using delay time \( \tau \), the polariton spends inside the medium during which it is primarily of Rydberg character, which leads to \( \beta_{Ry} = \exp[-4\gamma \tau \xi]. \)

**Experimental realization**

Two-qubit conditional QPG [2, 75, 76] is universal because combined with rotations of individual qubits enables any quantum computation. The QPG transformation reads
\[ |b, j\rangle \rightarrow \exp[i\phi \delta_b, \delta_{j, j}][|b, j\rangle], \]  
(32)

where \( |b, j\rangle \) depict basis states \( |\uparrow\rangle \) and \( |\downarrow\rangle \) of the two qubits of interest and \( \delta_b, \delta_{j, j} \) are standard Kronecker symbols. It is important to stress that QPG leaves basis states unchanged except a homogeneous phase shift \( \phi \) when both are in \( |\downarrow\rangle \) state. In order to implement the gate, it is important that both states \( |\uparrow\rangle \) and \( |\downarrow\rangle \) acquire the same time delay inside the medium, but only the state \( |\downarrow\rangle \) becomes a Rydberg polariton. In figure 4, we show the atomic scheme which satisfies this condition. As a ground state we take \( |g\rangle = |5S5/2, F = 2, m_F = 1\rangle \) from which we couple with \( \sigma_+ \) polarized photons to \( |p_+\rangle = |5P_3/2, F = 2, m_F = 2\rangle \), and with \( \sigma_- \)-polarized to \( |p_-\rangle = |5P_3/2, F = 2, m_F = 0\rangle \). We choose control field to be \( \sigma_+ \) polarized, which ensures that coupling to any Rydberg \( s \)-states from \( |p_+\rangle \) is zero because
\[ |p_+\rangle = \frac{1}{\sqrt{2}} \left[ m_j = \frac{1}{2}, m_l = \frac{1}{2} \right] - m_j = \frac{1}{2}, m_l = -\frac{1}{2} \].

Whereas from \( |p_-\rangle \) we can couple with \( \sigma_- \) control field to two
Figure 5. The $\sqrt{F}$ including all present corrections for $\phi = \pi/2$ as a function of collective coupling $g$ and length of the medium $L$, with parameters characteristic for experiments presented in [20, 21]: $L_c = 160 \text{ m}$ and $g_c/2\pi = 4.4 \text{ GHz}$. All our estimates assume that the corrections are small and a pulse is long, $\gamma g^2/|\Delta|c \gg 1$. Therefore, the estimates are not valid in the regimes where $F$ drops much below 1, therefore, we do not plot fidelity there.

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Appendix A. Corrections due to inhomogeneous distribution of atoms

Analogously to [55] we can include a shape of atomic distribution in the solution of the two-body counterpropagating problem. In this section, we will use the notation from [55], i.e. $\beta(x)$ describes the amplitude of the polariton to be in a photonic state and is related to the slow light velocity $v_p = c\beta(x)^2$, while $n(x) = 1 - \beta(x)^2$ is the probability
for the polaron to be in the Rydberg state. These quantities are determined by the atomic density $n_{\text{at}}(x)$ via $\beta(x) = \Omega/\sqrt{\Omega^2 + g_0^2 n_{\text{at}}(x)}$ with $g_0$ the single atom coupling. Then, the solution to two-body problem

$$\phi(x, y, t) = e^{-i\omega t(x,y)} \phi_0(x - ct, y + ct),$$

(1.1)

where $t^\prime = t - \Delta t$ accounts for the delay of the polaritons inside the medium with $\Delta t = \int_0^t dy (1/\beta(y))^2 - 1/c$. The phase factor $\varphi$ describes the correlations built up between the photons during the propagation through the medium and takes the form

$$\varphi(z, z', t) = \frac{1}{\hbar c} \int_{z - ct}^{z'} dw \tilde{n}(w) \hat{n}(z + w - z'),$$

(2.2)

where $\tilde{n}(z) = n(\zeta(z))$, $\hat{V}(z, w) = V(\zeta(z) - \zeta(w))$ with the coordinate transformation taking the form $z = \zeta^{-1}(x) = \int_0^x dy (1/\beta(y))^2$. For large $t$ and $z$ this integral is equivalent to the integral from $-\infty$ to $\infty$. For Gaussian distribution of atoms much longer than the blockade radius, $L \gg \xi$, the phase factor $\varphi$ simplifies to

$$\varphi(z, z') = \frac{1}{\hbar c} \int dw \tilde{n}(w) \left( \frac{z + z'}{2} \right)^2 V(\zeta(z + z' - w) - \zeta(w))$$

(2.3)

and can be furthermore transformed using $u = \frac{z + z'}{2}$ and $y = \zeta(u)$

$$\varphi(z, z') = \frac{1}{\hbar c} \tilde{n}(u)^2 \int dw V(\partial_u \zeta(u) 2w),$$

(2.4)

$$\varphi(z, z') = \frac{1}{\hbar c} \tilde{n}(u)^2 \int dw \beta(y)^2 2w).$$

(2.5)

The integral can be calculated analytically, leading to

$$\varphi(z, z') \approx \frac{1}{\hbar c} n(y)^2 \frac{2\pi}{3} \frac{\xi}{2 \beta(y)^2} \frac{2\Omega^2}{\Delta}. $$

(2.6)

For $g \gg \Omega$ and compressed pulses shorter than the cloud’s length, we can set $n(y) = 1$, leading to

$$\varphi(z, z') \approx \frac{1}{\hbar c} \frac{2\pi}{3} \frac{g_0^2 n_{\text{at}}(y)}{\Omega^2} \frac{2\Omega^2}{\Delta}. $$

(2.7)

For small $y$, using that $u = \zeta^{-1}(y) = \frac{g_0^2 n_{\text{at}}(y)}{\Omega^2} L \text{Erf}(y/L) + y \approx (1 + \frac{g_0^2 n_{\text{at}}(y)}{\Omega^2}) y$, we can write

$$y = \zeta(u) \approx \left(1 + \frac{g_0^2 n_{\text{at}}(y)}{\Omega^2}\right)^{-1} u,$$

(2.8)

which gives

$$\varphi\left(u = \frac{z + z'}{2}\right) = \varphi_0 \exp\left[-\frac{\nu_0}{c L}\right]^2, $$

(2.9)

from which

$$\beta_0 e^{i\phi_0 + i\Delta \phi_0} = \lim_{t \rightarrow \infty} \langle L^2 \rangle$$

$$= \lim_{t \rightarrow \infty} \int dz \int dz' e^{i\phi_0} N^2 e^{-i(z - z' - \alpha)^2/\delta^2} e^{-i(z + z' + \alpha)^2/\delta^2} \times \exp\left[i\phi_0\left(e^{-i/\beta^2} - 1\right)\right],$$

(10.1)

where as initial conditions we have chosen two Gaussian wavepackets centered at positions $z_0$ with width $\delta$ and normalization factor $N$. This integral can be calculated analytically for $\delta/c \ll L/v_c$, leading to

$$\beta_0 e^{i\Delta \phi_0} = \left(1 + \frac{i\phi_0 v_c^2 \delta^2}{2c^2} \right)^{-1/2}.$$ (10.2)

Expanding it in small parameter $\delta/c$ we get

$$\beta_0 = 1 - \frac{3}{2} \frac{\delta^2}{2c L} \left(\frac{v_c}{c}\right)^4,$$ (11.1)

$$\Delta \phi_0 = -\phi_0 \left(\frac{v_c}{2c L}\right)^2,$$ (12.1)

which corresponds to the expressions from the main text because $s_c = \delta_c \approx v_c$.

**Appendix B. Transversal size effects**

The propagation of photons within the paraxial approximation is described by the Hamiltonian (analogous to equation (4)) given by

$$H_\perp = t \int dz \left(\begin{array}{c} \hat{\xi}_\perp(z) \\
\hat{\xi}_\perp(z) \end{array}\right) \left(\begin{array}{c} \frac{-ic\partial_z - c}{2k_p} \nabla_\perp \frac{g_0}{\sqrt{n(z)}} \end{array}\right) \left(\begin{array}{c} \xi_\perp(z) \\
\xi_\perp(z) \end{array}\right)$$

(1.1)

$$\times \left(\begin{array}{c} \hat{\xi}_\perp(z) \\
\hat{\xi}_\perp(z) \end{array}\right) \left(\begin{array}{c} \frac{-ic\partial_z + g_0}{\sqrt{n(z)}} \end{array}\right) \hat{\xi}_\perp(z) + \hat{\xi}_\perp(z) \right) \left(\begin{array}{c} \xi_\perp(z) \\
\xi_\perp(z) \end{array}\right)$$

(1.2)

where $k_p$ is the carrier frequency of the probe photons. We are interested in an estimate of leading corrections to the fidelity, thus we neglect transversal corrections to the dispersion relation of photons. In addition, we consider atomic distribution broader than the extend of relevant photonic modes, leading to

$$H_\perp = t \int dz \left(\begin{array}{c} \hat{\xi}_\perp(z) \\
\hat{\xi}_\perp(z) \end{array}\right) \left(\begin{array}{c} \frac{-ic\partial_z - g_0}{\sqrt{n_{\text{at}}}} \end{array}\right) \left(\begin{array}{c} \xi_\perp(z) \\
\xi_\perp(z) \end{array}\right)$$

(1.3)

$$\times \left(\begin{array}{c} \hat{\xi}_\perp(z) \\
\hat{\xi}_\perp(z) \end{array}\right) \left(\begin{array}{c} \frac{-ic\partial_z + g_0}{\sqrt{n_{\text{at}}} \Delta} \end{array}\right) \hat{\xi}_\perp(z) + \hat{\xi}_\perp(z) \right) \left(\begin{array}{c} \xi_\perp(z) \\
\xi_\perp(z) \end{array}\right)$$

(1.4)
Then, the solution of interacting two-body problem can be derived analogously to equation (11) leading to

\[
\psi_{\text{int}}^n(r, \mathbf{R}_+, \mathbf{R}_-) = \frac{1}{1 - \chi V(r)} u_{00}(\mathbf{R}_+)u_{00}(\mathbf{R}_-)e^{ik_0r} \\
\times \exp \left[ -i \frac{\hbar}{\hbar} \int_{-\infty}^{r} dz V_{\text{rel}}(\mathbf{R}_+ - \mathbf{R}_-, z) \right],
\]

where \( r = \{r, \mathbf{R} = \mathbf{R}_+ - \mathbf{R}_-\} \) with \( r \) being the longitudinal component of relative distance, and \( \mathbf{R}_\pm \) are transverse components of the \( \mathbf{z}_\pm \) (\( \mathbf{z}_\pm \) corresponds to \( \mathbf{z} \) in equation (B.2) for two directions of the propagation), \( u_{00} \) is the lowest Laguerre–Gauss mode describing transverse shape of the incoming photons (in the main text \( u_{00} \) is denoted by \( u \)). This solution in the limit of \( r \to \infty \) leads to the expression for \( \beta_{tr} \) from the main text.

**Appendix C. Summary of the corrections to the fidelity**

In the table C1 we sum-up all the estimates for the corrections to the phase gate fidelity \( F \), given by equation (3). In order to generate results presented in figure 5 and figure D1, we maximize the value of \( \sqrt{F} \) for fixed \( \omega_0 \), \( \phi \), \( C_\delta \) in function of \( g, L \), by finding optimal detuning \( \delta \). Then, the value of Rabi frequency \( \Omega \) is fixed by the constraint on \( \phi \).

| \( \beta_j \) | Estimate |
|-----------------|-----------|
| \( \beta_\delta \) | \( e^{-2 \delta} \), with \( \eta = \frac{5}{6} \frac{\delta}{\gamma} \) |
| \( |\beta_{\Delta}| \) | \( 1 - \phi^2 \frac{\Delta^2}{4 \hbar^2} \) |
| \( \beta_{\text{up}} \) | \( \exp \left[ -\frac{2 \omega_0^2}{\pi} \left( \phi \frac{\Delta}{\omega_0} \right)^2 \right] \) |
| \( \beta_{\text{down}} \) | \( \exp \left[ -\frac{\omega_0^2}{2 \pi} \left( \phi \frac{\Delta}{\omega_0} \right)^2 \right] \) |
| \( \beta_u \) | \( 1 - \frac{3}{2} \phi^2 \left( \frac{\Delta}{\omega_0} \right)^2 \) |
| \( \beta_{\text{RY}} \) | \( \mathcal{F}(\omega_0/\xi) \) |
| \( \beta_{\text{RY}} \) | \( 1 - \frac{3}{2} \phi^2 \left( \frac{\Delta}{\omega_0} \right)^2 \) |

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