A BOUNDEDNESS CRITERION FOR SINGULAR INTEGRAL OPERATORS OF CONVOLUTION TYPE ON THE FOCK SPACE

GUANGFU CAO, JI LI, MINXING SHEN, BRETT D. WICK AND LIXIN YAN

Abstract. We show that for an entire function \( \varphi \) belonging to the Fock space \( \mathcal{F}^2(\mathbb{C}^n) \) on the complex Euclidean space \( \mathbb{C}^n \), the integral operator

\[
S_{\varphi}F(z) = \int_{\mathbb{C}^n} F(w) e^{z \cdot \bar{w}} \varphi(z - \bar{w}) \, d\lambda(w), \quad z \in \mathbb{C}^n,
\]

is bounded on \( \mathcal{F}^2(\mathbb{C}^n) \) if and only if there exists a function \( m \in L^\infty(\mathbb{R}^n) \) such that

\[
\varphi(z) = \int_{\mathbb{R}^n} m(x) e^{-2(z - ix)^2} \, dx, \quad z \in \mathbb{C}^n.
\]

Here \( d\lambda(w) = \pi^{-n} e^{-|w|^2} \, dw \) is the Gaussian measure on \( \mathbb{C}^n \). With this characterization we are able to obtain some fundamental results of the operator \( S_{\varphi} \), including the normality, the \( C^* \) algebraic properties, the spectrum and its compactness. Moreover, we obtain the reducing subspaces of \( S_{\varphi} \).

In particular, in the case \( n = 1 \), we give a complete solution to an open problem proposed by K. Zhu for the Fock space \( \mathcal{F}^2(\mathbb{C}) \) on the complex plane \( \mathbb{C} \) (Integr. Equ. Oper. Theory 81 (2015), 451–454).

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1. Introduction

The Fock space $F^2(\mathbb{C}^n)$ consists of all entire functions $F$ on the complex Euclidean space $\mathbb{C}^n$ such that

$$\|F\|_{F^2(\mathbb{C}^n)} = \left( \int_{\mathbb{C}^n} |F(z)|^2 d\lambda(z) \right)^{\frac{1}{2}} < \infty,$$

where

$$d\lambda(z) = \pi^{-n} e^{-\|z\|^2} dz$$

is the Gaussian measure on $\mathbb{C}^n$. The Fock space $F^2(\mathbb{C}^n)$ is a Hilbert space, whose inner product is inherited from $L^2(\mathbb{C}^n, d\lambda)$. This space is a convenient setting for many problems in functional analysis, mathematical physics, and engineering. We refer to [2, 3, 5, 15, 17, 30, 31] for an introduction to the theory of Fock spaces and some connections with other areas of mathematics and engineering.

For $\varphi \in F^2(\mathbb{C}^n)$, consider the integral operator

$$S_{\varphi} F(z) = \int_{\mathbb{C}^n} F(w)e^{z \cdot \bar{w}} \varphi(z - \bar{w}) d\lambda(w). \tag{1.1}$$

In 2015, K. Zhu proposed the following problem for the Fock space $F^2(\mathbb{C})$ on the complex plane $\mathbb{C}$ (see [31]): Characterize those functions $\varphi \in F^2(\mathbb{C})$ such that the integral operator $S_{\varphi}$ in (1.1) is bounded on $F^2(\mathbb{C})$.

Two natural conjectures arise from Zhu’s question and are related to the “reproducing kernel thesis”, which roughly says that the behavior of $S_{\varphi}$ is determined by its action on the normalized reproducing kernels $k_z$ of the Fock space. Two possible versions of this reproducing kernel thesis one might hope to be true are:

$$\sup_{z \in \mathbb{C}} \left| \langle S_{\varphi} k_z, k_z \rangle_{F^2(\mathbb{C})} \right| = \sup_{z \in \mathbb{C}} |\varphi(z - \bar{z})| < \infty.$$

This strategy is a common, and successful, one to try when working on operator theoretic questions in complex analysis, see [1, 4, 7, 20, 22, 25, 29]. While natural, this is unfortunately not true since it is possible to provide a counterexample (provided in Remark 3.5 below) to the reproducing kernel thesis in this context, meaning that the exact answer to Zhu’s question is more subtle. In this article, we obtain a complete solution to this open problem using harmonic analysis methods and are further able to resolve the question for the Fock space in all dimensions.

In [31], via an example, Zhu suggests that there should be some connection between resolving his question and harmonic analysis since he demonstrates that the Hilbert transform is unitarily equivalent to $S_{\varphi}$ for a special choice of $\varphi$. From this one example we were lead to guess that Fourier multiplier operators, which are in correspondence with bounded functions, should in fact provide the answer to Zhu’s question. Indeed, we have the following result on the Fock space $F^2(\mathbb{C}^n)$.
Theorem 1.1. The integral operator $S_\varphi$ in (1.1) is bounded on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if there exists an $m \in L^\infty(\mathbb{R}^n)$ such that

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x)e^{-2(x - i\frac{z}{2})^2} dx, \quad z \in \mathbb{C}^n. \quad (1.2)$$

Moreover, we have that

$$\|S_\varphi\|_{\mathcal{F}^2(\mathbb{C}^n) \to \mathcal{F}^2(\mathbb{C}^n)} = \|m\|_{L^\infty(\mathbb{R}^n)}.$$

The idea of the proof is to utilize the Bargmann transform to reformulate the question as one about a certain operator on $L^2(\mathbb{R}^n)$ that is translation invariant. Then for the operator we have in this context, it will fall into a category of operators well-studied in the harmonic analysis literature, the Fourier multiplier operators, to which we apply the Bargmann transform again and provide the answer to Zhu’s question.

With the characterization in Theorem 1.1 we are able to obtain some fundamental operator theory results about $S_\varphi$. In particular, we are able to determine the normality of $S_\varphi$, the spectrum of an individual $S_\varphi$ and the reducing subspaces of $S_\varphi$. A particular corollary of our work is:

Theorem 1.2. Suppose $\varphi \in \mathcal{F}^2(\mathbb{C}^n)$ such that $S_\varphi$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then $S_\varphi^* = S_\varphi$, where $\varphi$ is as in (1.2) and

$$\tilde{\varphi}(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x)e^{-2(x - i\frac{z}{2})^2} dx.$$

Furthermore, $S_\varphi$ is normal.

In the last decades, Toeplitz operators, Hankel operators and composition operators on several analytic function spaces (Hardy spaces, Bergman spaces, Dirichlet spaces and Fock spaces) have been widely studied. For example, one may consult the references [5, 6, 12, 23]. It is well-known that these operators are never normal if their symbols are analytic. For example, if $\varphi$ is a bounded analytic function on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$, or unit ball $\mathbb{B}_n$ in the complex space $\mathbb{C}^n$, then $T_\varphi$, the Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ or $H^2(\mathbb{B}_n)$, is normal if and only if $\varphi$ is a constant. However, $S_\varphi$ is always normal although $\varphi$ is analytic, this is a surprising phenomenon. For the other operator theory results that are immediate corollaries of Theorem 1.1 and Theorem 1.2 we refer to Section 5.

We provide two remarks regarding our main results Theorem 1.1 and 1.2, on the extension to the Fock space $\mathcal{F}_2(\mathbb{C}^n)$ and on the boundedness on the Fock space $\mathcal{F}_p(\mathbb{C}^n)$ for $p \in [1, \infty)$, respectively.

Remark 1.3. There are natural extensions of the results in Theorem 1.1 and Theorem 1.2 to the Fock space $\mathcal{F}_2(\mathbb{C}^n)$, where

$$\|F\|_{\mathcal{F}_2(\mathbb{C}^n)} = \left(\int_{\mathbb{C}^n} |F(z)|^2 d\lambda_\alpha(z)\right)^{1/2} < \infty,$$

and

$$d\lambda_\alpha(z) = \pi^{-n} e^{-\alpha|z|^2} d^nz$$

with $\alpha > 0$. We don’t precisely formulate these results since the modifications necessary to do so are standard.
Remark 1.4. It is natural to ask whether the characterization of $S_\psi$ as in Theorem 1.1 can imply boundedness of $S_\psi$ on the Fock space $\mathcal{F}^p(\mathbb{C}^n)$ for $p \in [1, \infty)$, where $\mathcal{F}^p(\mathbb{C}^n)$ consists of all entire functions $F$ on the complex Euclidean space $\mathbb{C}^n$ such that

$$
\|F\|_{\mathcal{F}^p(\mathbb{C}^n)} = \left( \int_{\mathbb{C}^n} |F(z)|^p d\lambda(z) \right)^{\frac{1}{p}} < \infty.
$$

However, this is not true for $p \in [1, 2)$. We will provide a counterexample in Section 3. The reader should not confuse the definition we use here with the other definition in the literature of those entire functions such that $f(z)e^{-\frac{|z|^2}{2}}$ belong to $L^p(\mathbb{C}^n)$. See for example [9]. We point out that the operator $S_\psi$ may not be well-defined in the other Fock space for $2 < p < \infty$. Explanations will be provided in Section 3.

The outline of the remainder of the paper is as follows. In Section 2 we collect the basic definitions and concepts that we will need to prove the main result. In Section 3 we give the proof of the main result and in Section 4 we show how the main result can recover the known examples in the literature and can further recover some canonical Calderón–Zygmund operators. In Section 5 we study operator theoretic properties of the operator $S_\psi$, including the normality, $C^*$ algebraic properties, the compactness, the spectrum and the reducing subspaces. In the final section we provide some concluding remarks.

2. Preliminaries

We now set the notation and some common concepts to be used throughout the course of the paper. $\mathbb{R}^n$ denotes the real Euclidean space and $\mathbb{C}^n$ denotes the complex Euclidean space. To simplify the dot product notation, we will denote by simple juxtaposition: $xy = x \cdot y = \sum_{j=1}^{n} x_j y_j$. In particular, this implies that $x^2 = x \cdot x = \sum_{j=1}^{n} x_j^2$. The Hermitian inner product in $\mathbb{C}^n$ will be denoted by $\bar{z}w$ when $z, w \in \mathbb{C}^n$; this then gives $|z|^2 = \bar{z}z = \sum_{j=1}^{n} |z_j|^2$. The standard norm on the Lebesgue space $L^2(\mathbb{R}^n)$ will be denoted by $\|f\|_2 = \|f\|_{L^2(\mathbb{R}^n, dx)}$. And, as introduced earlier, the Fock space on $\mathbb{C}^n$ will be denoted by $\mathcal{F}^2(\mathbb{C}^n)$ with the norm:

$$
\|f\|_{\mathcal{F}^2(\mathbb{C}^n)} = \left( \int_{\mathbb{C}^n} |F(z)|^2 d\lambda(z) \right)^{1/2}
$$

where $d\lambda(z) = \pi^{-n} e^{-|z|^2} d\lambda.$

A fundamental tool in our analysis is the Fourier transform of a function $f$, i.e.

$$
\mathcal{F} f(x) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{2\pi i x \cdot y}{2}} f(y) dy, \quad x \in \mathbb{R}^n.
$$

The inverse of the Fourier transform $\mathcal{F}$ will be denoted by $\mathcal{F}^{-1}$, i.e., $\mathcal{F}^{-1} \mathcal{F} = \mathcal{F}^{-1} = \mathbb{I}$, the identity operator on $L^2(\mathbb{R}^n)$.

2.1. The Fock Space. We start by recalling some basic facts about the Fock space. Throughout the paper, we denote the scalar product on $\mathcal{F}^2(\mathbb{C}^n)$ by $\langle \cdot, \cdot \rangle_{\mathcal{F}^2(\mathbb{C}^n)}$. It is well-known (see for example,
[15, Theorem 1.63]) that the collection of monomials of the form
\[
e_\alpha(z) = \left(\frac{1}{\alpha!}\right)^{\frac{1}{2}} z^\alpha = \prod_{j=1}^{n} \left(\frac{1}{\alpha_j!}\right)^{\frac{1}{2}} z_j^{\alpha_j}
\]
for all \(\alpha = (\alpha_1, \ldots, \alpha_n)\) with \(\alpha_j \geq 0\), forms an orthonormal basis for \(\mathcal{F}^2(\mathbb{C}^n)\). This space \(\mathcal{F}^2(\mathbb{C}^n)\) is a reproducing kernel Hilbert space, that is \(C\) is an entire function on \(\mathbb{C}^n\).

Proposition 2.1 (\cite{15}). \(T\) is an important consequence of the existence of a reproducing kernel is that every bounded operator \(T\) on \(\mathcal{F}^2(\mathbb{C}^n)\) can be written as an integral operator. More precisely we have

The reproducing kernel of \(\mathcal{F}^2(\mathbb{C}^n)\) is
\[
K(z, \bar{w}) = \sum_{\alpha} e_\alpha(z)\bar{e}_\alpha(w) = \sum_{\alpha} \frac{z^{\alpha} \cdot \bar{w}^\alpha}{\alpha!} = e^{z \cdot \bar{w}},
\]
so that \(\|K(z, \cdot)\|_{\mathcal{F}^2(\mathbb{C}^n)}^2 = e^{\|z\|^2}\) and
\[
F(z) = \int_{\mathbb{C}^n} F(w)e^{z \cdot \bar{w}}d\lambda(w), \quad z \in \mathbb{C}^n
\]
when \(F \in \mathcal{F}^2(\mathbb{C}^n)\).

An important consequence of the existence of a reproducing kernel is that every bounded operator \(T\) on \(\mathcal{F}^2(\mathbb{C}^n)\) can be written as an integral operator. More precisely we have

**Proposition 2.1 ([15]).** If \(T\) is a bounded operator on \(\mathcal{F}^2(\mathbb{C}^n)\), let \(K_T(z, \bar{w}) = TK(\cdot, \bar{w})(z)\). Then \(K_T\) is an entire function on \(\mathbb{C}^{2n}\) that satisfies
(a) \(K_T(\cdot, w) \in \mathcal{F}^2(\mathbb{C}^n)\) for all \(w\) and \(K_T(z, \cdot) \in \mathcal{F}^2(\mathbb{C}^n)\) for all \(z\);
(b) \(|K_T(z, \bar{w})| \leq e^{\|z\|^2 + \|w\|^2}\|T\|\);
(c) \(TF(z) = \int_{\mathbb{C}^n} K_T(z, \bar{w})F(w)d\lambda(w)\) for all \(F \in \mathcal{F}^2(\mathbb{C}^n)\) and \(z \in \mathbb{C}^n\).

As we can see from this proposition, the form of the kernel in (1.1) is
\[
K_T(z, \bar{w}) = e^{z \cdot \bar{w}}\varphi(z - \bar{w}).
\]

**2.2. The Bargmann Transform.** The Bargmann transform is an old tool in mathematics analysis and mathematical physics (see \([2, 3, 15, 17, 24, 31, 30, 32]\) and references therein). Consider \(f \in L^2(\mathbb{R}^n)\), and define
\[
Bf(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{2z \cdot x - z^2}dx
\]
\[
= \left(\frac{2}{\pi}\right)^{\frac{n}{2}} e^{z^2} \int_{\mathbb{R}^n} f(x)e^{-(x-z)^2}dx, \quad z \in \mathbb{C}^n.
\]

Since the function \(e^{2z \cdot x - z^2 - (z^2/2)}\) is in \(L^1(\mathbb{R}^n)\), the integral is absolutely convergent in \(L^2(\mathbb{R}^n)\). Using Morera’s theorem one may verify that \(Bf\) is an entire holomorphic function on \(\mathbb{C}^n\). From (2.4) one sees that the Bargmann transform is very closely related to the Fourier transform or the Fourier-Wiener transform (see \([15, 17]\)).

The following result is well-known (see for example, \([17]\)).
Lemma 2.2. The Bargmann transform is a unitary operator from $L^2(\mathbb{R}^n)$ onto $\mathcal{F}^2(\mathbb{C}^n)$: it is one-to-one, onto, and isometric in the sense that

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{C}^n} |Bf(z)|^2 d\lambda(z).$$

Proof. For the proof, we refer to [17, Proposition 3.4.3].

Let us now compute the inverse Bargmann transform. Since $B$ is unitary, for $F \in \mathcal{F}^2(\mathbb{C}^n)$ and $g \in L^2(\mathbb{R}^n)$, by (2.4) we have

$$\langle B^{-1}F, g \rangle_{L^2(\mathbb{R}^n)} = \langle F,Bg \rangle_{\mathcal{F}^2(\mathbb{C}^n)} = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{C}^n} F(z) \int_{\mathbb{R}^n} g(x)e^{2\pi^2 x^2 - \pi \xi^2} dx d\lambda(z),$$

and hence

$$B^{-1}F(x) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{C}^n} F(z)e^{2\pi^2 x^2 - \pi \xi^2} d\lambda(z), \quad x \in \mathbb{R}^n. \tag{2.5}$$

To prove our main result Theorem 1.1, we need to study the Bargmann transform of the Fourier transform (a bounded operator on $L^2(\mathbb{R}^n)$) and inverse Fourier transform (also a bounded operator on $L^2(\mathbb{R}^n)$).

Lemma 2.3. For every $F \in \mathcal{F}^2(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, we have

$$B\mathcal{F}B^{-1}F(z) = F(-iz), \quad \text{and} \quad B^{-1}B^{-1}F(z) = F(iz).$$

Proof. This lemma was proved in [14, Theorem 3] for the case $n = 1$. See also [32, Theorem 4]. We give a brief proof of this lemma in the higher dimensional case for completeness and the convenience of the reader.

By taking the Fourier transform, we have

$$\mathcal{F}B^{-1}F(\xi) = \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-2i\xi \cdot t} B^{-1}F(t) dt = 2^n \pi^{-\frac{1}{4}} \int_{\mathbb{C}^n} F(w) e^{-\frac{\pi}{4} \xi^2} e^{\langle \xi, \omega \rangle^2} \int_{\mathbb{R}^n} e^{-\langle t - (\xi, \omega) \rangle^2} dt d\lambda(w).$$

Recall that by a change of variables and standard calculus computations,

$$\int_{\mathbb{R}^n} e^{-\langle t - (\xi, \omega) \rangle^2} dt = \pi^{\frac{n}{2}}.$$

We then have

$$\mathcal{F}B^{-1}F(\xi) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{C}^n} F(w) e^{-\frac{\pi}{4} \xi^2} e^{\langle \xi, \omega \rangle^2} d\lambda(w)$$

$$= \left(\frac{2}{\pi}\right)^{\frac{n}{4}} e^{-\xi^2} \int_{\mathbb{C}^n} F(w) e^{\frac{\pi}{4} \omega^2} e^{-2\langle \xi, \omega \rangle} d\lambda(w). \tag{2.6}$$

Then, by taking the Bargmann transform of $\mathcal{F}B^{-1}F$ we get

$$B\mathcal{F}B^{-1}F = \left(\frac{2}{\pi}\right)^{n/2} e^{-\frac{\pi}{4} \xi^2} \int_{\mathbb{C}^n} F(w) e^{\frac{\pi}{4} \omega^2} e^{-\frac{\pi}{4} \omega^2} \int_{\mathbb{R}^n} e^{-2\langle \xi, \omega \rangle^2} d\lambda(w)$$

$$= e^{-\frac{\pi}{4} \xi^2} \int_{\mathbb{C}^n} F(w) e^{\frac{\pi}{4} \omega^2} e^{-\frac{\pi}{4} \omega^2} d\lambda(w).$$
\[
= \int_{\mathbb{C}^n} F(w) e^{(-iz)\cdot w} \, d\lambda(w) \\
= F(-iz),
\]
where the last equality follows from the reproducing formula.

By repeating the above proof, we also have
\[
B F^{-1} B^{-1} F(z) = F(iz).
\]
The proof of Lemma 2.3 is complete. \(\Box\)

3. Proof of Theorem 1.1

In this section we provide the proof of our main result Theorem 1.1. To begin with, we need the following auxiliary result.

Lemma 3.1. For any \( m \in L^\infty(\mathbb{R}^n) \), the entire function
\[
\varphi(z) = \int_{\mathbb{R}^n} m(x) e^{-2(x-z^2)^2} \, dx,
\]
\( z \in \mathbb{C}^n \)

belongs to \( \mathcal{F}^2(\mathbb{C}^n) \).

Proof. For every \( z \in \mathbb{C}^n \), we write \( z = u + iv \). Then we have
\[
\varphi(z) = \int_{\mathbb{R}^n} m(x - \frac{1}{2}v) e^{-2x^2 + 2ixu + \frac{1}{2}v^2} \, dx = \pi^{\frac{n}{2}} F^{-1} \left[ m(x - \frac{1}{2}v) e^{-2x^2} \right] (u) e^{\frac{i}{2}u^2}.
\]
By Plancherel’s theorem,
\[
\|\varphi\|^2_{\mathcal{F}^2(\mathbb{C}^n)} = \pi^{-n} \int_{\mathbb{C}^n} |\varphi(z)|^2 e^{-4|z|^2} \, dz \\
= \int_{\mathbb{R}^n} e^{-v^2} \, dv \int_{\mathbb{R}^n} |F^{-1} \left[ m(x - \frac{1}{2}v) e^{-2x^2} \right] (u)|^2 \, du \\
= \int_{\mathbb{R}^n} e^{-v^2} \, dv \int_{\mathbb{R}^n} |m(x - \frac{1}{2}v) e^{-2x^2}|^2 \, dx \\
\leq \|m\|^2_{L^\infty} \int_{\mathbb{R}^n} e^{-v^2} \, dv \int_{\mathbb{R}^n} e^{-4x^2} \, dx < \infty,
\]
and so \( \varphi \in \mathcal{F}^2(\mathbb{C}^n) \). This finishes the proof of Lemma 3.1. \(\Box\)

The proof of Theorem 1.1 relies on the following elementary fact taken from harmonic analysis characterising the translation invariant operators that are bounded on \( L^2(\mathbb{R}^n) \).

Proposition 3.2. Let \( T \) be a bounded linear transformation mapping \( L^2(\mathbb{R}^n) \) into itself. Then a necessary and sufficient condition that \( T \) commutes with translation is that there exists a bounded measurable function \( m(y) \) (a “multiplier”) so that \( \mathcal{F}(Tf)(y) = m(y) \mathcal{F}f(y) \) for all \( f \in L^2(\mathbb{R}^n) \). In this case the norm of \( T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) is equal to \( \|m\|_{L^\infty} \).

Proof. For the proof of this proposition see [26, Proposition 2, Chapter 2]. \(\Box\)
For more information on the translation invariant operators, we refer to [19] and [28, Chapter 1]. In the following we denote by $\mathcal{M}^{2,2}(\mathbb{R}^n)$ the set of all bounded linear operators on $L^2(\mathbb{R}^n)$ that commute with translations.

Recall that the operators $B$ and $B^{-1}$ are the Bargmann transform in (2.4) and the inverse Bargmann transform in (2.5), respectively. For every bounded operator $S_\varphi$ in (1.1) on the space $\mathcal{F}^2(\mathbb{C}^n)$, consider the operator

\begin{equation}
T = B^{-1}S_\varphi B.
\end{equation}

A crucial observation is that the above operator $T$ commutes with translation so that we can apply Proposition 3.2 in the proof of Theorem 1.1. To be precise, we first have the following result.

**Lemma 3.3.** If the integral operator $S_\varphi$ in (1.1) is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then there exists an operator $T \in \mathcal{M}^{2,2}(\mathbb{R}^n)$ such that

\begin{equation}
S_\varphi F(z) = BTB^{-1}F(z),
\end{equation}

for $F \in \mathcal{F}^2(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. Moreover, there exists a bounded measurable function $m(y)$ so that $\mathcal{F}(T f)(y) = m(y)\mathcal{F} f(y)$ for all $f \in L^2(\mathbb{R}^n)$.

**Proof.** Let $T$ be the operator given in (3.1). Then the operator $T$ is bounded on $L^2(\mathbb{R}^n)$ since the Bargmann transform $B$ is unitary operator from $L^2(\mathbb{R}^n)$ to $\mathcal{F}^2(\mathbb{C}^n)$ and $S_\varphi$ in (1.1) is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.

Let us show that $T$ commutes with translation. To do so, let $\tau_a$ denote the translation operator by $a \in \mathbb{R}^n$ that acts on a function $f$ as

$$(\tau_a f)(x) = f(x - a).$$

By the definition of the operators $B$ and $B^{-1}$,

$$B\tau_a B^{-1}(F)(z) = F(z - a)e^{z-a \cdot \overline{x}} =: W_a F(z).$$

Then we have

\begin{equation}
\tau_a T = B^{-1}(B\tau_a B^{-1})S_\varphi B = B^{-1}W_a S_\varphi B
\end{equation}

and

\begin{equation}
T \tau_a = B^{-1}S_\varphi (B\tau_a B^{-1})B = B^{-1}S_\varphi W_a B.
\end{equation}

A straightforward calculation shows that

$$W_a S_\varphi F(z) = \int_{\mathbb{C}^n} F(w) e^{z-a \cdot \overline{w}} \varphi((z - a) - \overline{w})e^{z-a \cdot \overline{w}} d\lambda(w)$$

\begin{align*}
&= \pi^{-n} e^{-\frac{|z|^2}{2}} \int_{\mathbb{C}^n} F(w) \varphi((z - a) - \overline{w})e^{z-a \cdot \overline{w}} + z \cdot a - |w|^2 d\lambda(w) \\
&= \pi^{-n} e^{-\frac{|z|^2}{2}} \int_{\mathbb{C}^n} F(u - a) \varphi(z - \overline{u})e^{(z-a) \cdot (\overline{u} - a)} + z \cdot a - (u-a) \cdot (\overline{u} - a) d\lambda(u) \\
&= e^{-\frac{|a|^2}{2}} \int_{\mathbb{C}^n} F(u - a) \varphi(z - \overline{u})e^{u \cdot a + z \cdot \overline{a}} d\lambda(u),
\end{align*}

where $\lambda$ is the Lebesgue measure on $\mathbb{C}^n$. Thus, we have the desired result.
and

\[ S_\varphi W_a F(z) = \int_{\mathbb{C}^n} F(w - a)e^{wz - \frac{z^2}{2}} e^{z\bar{w}} \varphi(z - \bar{w}) \, d\lambda(w) \]

\[ = e^{-\frac{z^2}{2}} \int_{\mathbb{C}} F(w - a)\varphi(z - \bar{w})e^{wz - \frac{z^2}{2}} d\lambda(w), \]

and so \( W_\varphi S = S_\varphi W_a \). This, in combination with (3.3) and (3.4), shows that \( T \) commutes with translation, and so \( T \in \mathcal{M}^{2,2} (\mathbb{R}^n) \). By Proposition 3.2, there exists a bounded measurable function \( m(y) \) so that \( F(Tf)(y) = m(y)Ff(y) \) for all \( f \in L^2(\mathbb{R}^n) \). The proof of Lemma 3.3 is complete. \( \square \)

Further, we have the following result.

**Lemma 3.4.** If \( T \in \mathcal{M}^{2,2} (\mathbb{R}^n) \) is given by convolution such that \( F(Tf)(y) = m(y)Ff(y) \) with an \( L^\infty (\mathbb{R}^n) \) function \( m \) and for all \( f \in L^2(\mathbb{R}^n) \), then for every \( F \in \mathscr{S}^2 (\mathbb{C}^n) \),

\[
BTB^{-1}F(z) = \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{C}^n} F(w)e^{z\bar{w}} \left( \int_{\mathbb{R}^n} m(x)e^{-2(x - \frac{1}{4}(z - \bar{w}))^2} dx \right) d\lambda(w), \quad z \in \mathbb{C}^n. 
\]

**Proof.** By Lemma 2.3,

\[
(B^*BT^{-1}F)(z) = F(-iz).
\]

This gives

\[
(F^{-1}B)(x) = B^{-1}(B^*BT^{-1}F)(x) = \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{C}^n} F(w)e^{-2(x - \frac{1}{4}z)^2} \, d\lambda(w),
\]

and so

\[
B(mF^{-1}B^{-1}F)(z) = \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x)(F^{-1}B^{-1}F)(x)e^{2xz - x^2 - \frac{z^2}{4}} dx 
\]

\[
= \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{C}^n} F(w)e^{-iz\bar{w}} \int_{\mathbb{R}^n} m(x)e^{A(x,z,w)} dxd\lambda(w),
\]

where

\[
A(x,z,w) = -2x^2 - \frac{z^2}{2} + 2x \cdot z + \frac{z\bar{w}}{2} - 2ix \cdot \bar{w} + iz \cdot \bar{w}
\]

\[
= -2x^2 + 2x \cdot (z - i\bar{w}) - \frac{(z - i\bar{w})^2}{2}
\]

\[
= -2 \left( x - \frac{z - i\bar{w}}{2} \right)^2.
\]

By Lemma 2.3 again,

\[
(B^*F^{-1}B^{-1}F)(z) = F(iz).
\]

Therefore,

\[
BTB^{-1}F(z) = (B^*F^{-1}(mF^{-1}B^{-1}F))(z) = (B^*F^{-1}B^{-1})B(mF^{-1}B^{-1}F)(z) = B(mF^{-1}B^{-1}F)(iz)
\]

\[
= \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{C}^n} F(w)e^{z\bar{w}} \left( \int_{\mathbb{R}^n} m(x)e^{-2(x - \frac{1}{4}(z - \bar{w}))^2} dx \right) d\lambda(w).
\]

The proof of Lemma 3.4 is complete. \( \square \)
Now we are ready to prove our main result, Theorem 1.1.

**Proof of Theorem 1.1.** Assume that the operator $S_\varphi$ in (1.1) is bounded on $\mathcal{F}^2(\mathbb{C}^n)$. Let us show that there exists an $m \in L^\infty(\mathbb{R}^n)$ such that (1.2) holds. Indeed, it follows from Lemma 3.3 and Lemma 3.4 that there exists an $L^\infty(\mathbb{R}^n)$ function $m$ such that for every $z \in \mathbb{C}^n$,

$$S_\varphi(F)(z) = B T B^{-1}(F)(z)$$

(3.6)

$$= \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{C}^n} F(w) e^{-z \cdot \overline{w}} \left(\int_{\mathbb{R}^n} m(x) e^{-\frac{1}{2} (x - \frac{i}{2} z)^2} dx\right) d\lambda(w).$$

Define

(3.7)

$$\varphi_0(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x) e^{-\frac{1}{2} (x - \frac{i}{2} z)^2} dx.$$

By Lemma 3.1, we have that $\varphi_0 \in \mathcal{F}^2(\mathbb{C}^n)$.

Let $\varphi$ be an entire function in (1.1). We now show that $\varphi = \varphi_0$. Indeed, we take $z = 0$ in (1.1) and (3.6) to see that for all $F \in \mathcal{F}^2(\mathbb{C}^n)$

(3.8)

$$\int_{\mathbb{C}^n} F(w) (\varphi(-\overline{w}) - \varphi_0(-\overline{w})) d\lambda(w) = 0.$$

Notice that $\psi(w) = \varphi(-w) - \varphi_0(-w) \in \mathcal{F}^2(\mathbb{C}^n)$. From the standard orthonormal basis $\{e_\alpha(z)\}_\alpha$ for $\mathcal{F}^2(\mathbb{C}^n)$, we decompose $\psi$ into the series

$$\psi(w) = \sum_\alpha c_\alpha e_\alpha(w) = \sum_\alpha c_\alpha \left(\frac{1}{\alpha!}\right)^{\frac{1}{2}} w^\alpha,$$

with $\sum_\alpha |c_\alpha|^2 = ||\psi||^2_{\mathcal{F}^2(\mathbb{C}^n)}$. Define

$$\Psi(w) = \sum_\alpha \overline{c_\alpha} \left(\frac{1}{\alpha!}\right)^{\frac{1}{2}} w^\alpha,$$

where $\overline{c_\alpha}$ is the complex conjugate of $c_\alpha$, so that $\psi(\overline{w}) = \overline{\Psi(w)}$ and $||\psi||^2_{\mathcal{F}^2(\mathbb{C}^n)} = ||\Psi||^2_{\mathcal{F}^2(\mathbb{C}^n)}$. By (3.8),

(3.9)

$$0 = \int_{\mathbb{C}^n} F(w) \overline{\psi(w)} d\lambda(w) = \int_{\mathbb{C}^n} F(w) \overline{\Psi(w)} d\lambda(w).$$

Letting $F = \Psi$ in (3.9), we see that $\Psi(w) = 0$ for all $w \in \mathbb{C}^n$, and so $\psi(w) = 0$. Hence,

$$\varphi(z) = \varphi_0(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x) e^{-\frac{1}{2} (x - \frac{i}{2} z)^2} dx$$

as desired.

Next, assume that (1.2) holds for some $m \in L^\infty(\mathbb{R}^n)$. Then Lemma 3.1 shows that the function $\varphi$ as in (1.2) is an entire function in $\mathcal{F}^2(\mathbb{C}^n)$. For the operator $S_\varphi$ in (1.1), we apply Lemma 3.4 to obtain

$$S_\varphi = BTB^{-1},$$

where $T \in \mathcal{M}^{2,2}(\mathbb{R}^n)$ is given by convolution such that $(FTf)(y) = m(y)Ff(y)$ for an $L^\infty(\mathbb{R}^n)$ function $m$ and for all $f \in L^2(\mathbb{R}^n)$. From the properties of the operators $B$ and $B^{-1}$, the operator $S_\varphi$ is bounded on the space $\mathcal{F}^2(\mathbb{C}^n)$.
To conclude, we point out that by using $S \psi = BTB^{-1}$, one obtains
\[ \|S \psi\|_{\mathcal{F}^2(C^n) \to \mathcal{F}^2(C^n)} = \|BTB^{-1}\|_{\mathcal{F}^2(C^n) \to \mathcal{F}^2(C^n)} = \|T\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = \|m\|_{L^\infty(\mathbb{R}^n)}. \]
The proof of Theorem 1.1 is complete. \[\square\]

**Remark 3.5.** From [31, Proposition 2], we know that when $n = 1$, a necessary condition for $S \psi$ to be bounded on $\mathcal{F}^2(\mathbb{C})$ is that $\psi(z - \bar{z})$ is bounded. In other words, the boundedness of $S \psi$ implies that the function $\psi$ is bounded on the imaginary axis. However, this is not a sufficient condition, showing that the reproducing kernel thesis fails for this problem. Indeed, we consider
\[ \varphi(z) = \int_{\mathbb{R}} \psi(x)e^{-2(x - \frac{i}{2}z)^2} dx, \]
where $\psi(x)$ belongs to $L^4(\mathbb{R}) \setminus L^\infty(\mathbb{R})$. Following the proof of Lemma 3.1, we can see $\varphi \in \mathcal{F}^2(\mathbb{C})$, hence $\varphi$ can gives rise to the operator $S \varphi$. Hölder’s inequality shows that $\varphi(z - \bar{z})$ is bounded on the imaginary axis. But it can not be given by
\[ \varphi(z) = \int_{\mathbb{R}} m(x)e^{-2(x - \frac{i}{2}z)^2} dx \]
for any bounded function $m$. If this were possible, then there would exist a bounded function $m$ such that $\varphi$ has the above representation. Then for all $z$,
\[ \int_{\mathbb{R}} (\psi(x) - m(x))e^{-2(x - \frac{i}{2}z)^2} dx = 0. \]
Set $z = u$ to be an arbitrary real number, then it becomes
\[ \int_{\mathbb{R}} (\psi(x) - m(x))e^{-2x^2 + 2xu} dx = 0, \]
which means $\mathcal{F}^{-1}[(\psi(x) - m(x))e^{-2x^2}](u) = 0$. Since $(\psi(x) - m(x))e^{-2x^2}$ is an $L^2$ function, then we have $\psi(x) = m(x)$, which is a contradiction. Therefore, by the theorem $S \varphi$ is not bounded on $\mathcal{F}^2(\mathbb{C})$, although $\varphi$ is bounded on the imaginary axis.

From Theorem 1.1, we see that from the multiplier function $m$ we obtain the analytic function $\varphi$. We now show how $\varphi$ gives rise to $m$.

**Proposition 3.6.** Suppose $\varphi \in \mathcal{F}^2(\mathbb{C}^n)$ such that $S \varphi$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$. Then for $T f := B^{-1}S \varphi Bf$, $f \in L^2(\mathbb{R}^n)$, we have $\mathcal{F}(T f)(x) = m(x)\mathcal{F} f(x)$ with
\[ m(x) = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \varphi(z - \bar{w})e^{x \cdot w - 2ix \cdot \bar{w} + \frac{w^2}{2}} dwdz. \]

**Proof.** Consider $T f := B^{-1}S \varphi Bf$. Taking the Fourier transform it becomes $m\mathcal{F} f = \mathcal{F} B^{-1}S \varphi Bf$. Let
\[ f_0(x) = e^{-x^2}, \]
then we have $Bf_0(z) = (2/\pi)^{-n/4}$. It follows that
\[ S \varphi Bf_0(z) = \left(\frac{2}{\pi}\right)^{-4} \int_{\mathbb{C}^n} e^{z \cdot w}\varphi(z - \bar{w})d\lambda(w). \]
By Lemma 2.3, we have

\[(B^* B^{-1})S_\varphi B f_0(z) = S_\varphi B f_0(-iz) = \left(\frac{2\lambda}{\pi}\right)^{-\frac{1}{2}} \int_{C^\nu} e^{-iz\bar{w}} \varphi(-iz - \bar{w}) d\lambda(w).\]

Now we get

\[
\mathcal{F} B^{-1} S_\varphi B f_0(x) = B^{-1}(B^* B^{-1}) S_\varphi B f_0(x) = \int_{C^\nu} \int_{C^\nu} e^{-iz\bar{w}} \varphi(-iz - \bar{w}) e^{-x^2 + 2x\bar{y} - \frac{\bar{y}^2}{4}} d\lambda(w)d\lambda(z)
\]

\[
= \int_{C^\nu} \int_{C^\nu} e^{\bar{w}z} \varphi(z - \bar{w}) e^{-2ix\bar{y} + \frac{\bar{y}^2}{2}} d\lambda(w)d\lambda(z) \cdot f_0(x).
\]

Since \(\mathcal{F} f_0(x) = f_0(x)\), we get the relation

\[
m(x) = \int_{C^\nu} \int_{C^\nu} \varphi(z - \bar{w}) e^{-2ix\bar{y} + \frac{\bar{y}^2}{2}} d\lambda(w)d\lambda(z).
\]

The proof of Proposition 3.6 is complete. \(\square\)

As from the first comment in Remark 1.4, it is natural to ask whether the characterization of \(S_\varphi\) as in Theorem 1.1 can imply some boundedness on the Fock space \(\mathcal{F}^p(C^n)\) for \(p \in [1, \infty)\). For \(p > 2\), for \(S_\varphi\) defined in (1.1) with \(\varphi\) as in (1.2), by using Hölder’s inequality one can verify that \(S_\varphi\) is bounded from \(\mathcal{F}^p(C^n)\) to \(\mathcal{F}^p(C^n)\). We omit the details here. However, this is not true in general when \(p \in [1, 2)\). We now provide a counterexample in dimension \(n = 1\) with \(S_\varphi = BHB^{-1}\), where \(H\) is the Hilbert transform on \(\mathbb{R}\) (we refer to Example 2 in Section 4 for details, see also [32, Section 8]).

**Proposition 3.7.** Let \(S_\varphi = BHB^{-1}\), where \(H\) is the Hilbert transform on \(\mathbb{R}\). Suppose \(1 \leq p < 2\). Then \(S_\varphi\) is not well-defined on \(\mathcal{F}^p(\mathbb{C})\).

**Proof.** For \(S_\varphi = BHB^{-1}\), we see that from Example 2 in Section 4, the function \(\varphi\) is as in (1.2) with \(m(x) := -i\text{sgn}(x)\). Consider \(F(w) := e^{\frac{w^2}{2}}\). Note that this function \(F\) is in \(\mathcal{F}^p(\mathbb{C})\) for all \(1 \leq p < 2\) but is not in \(\mathcal{F}^p(\mathbb{C})\) for any \(p \geq 2\). Then

\[
S_\varphi F(z) = -i \int_0^\infty \int_{C} F(w) e^{\frac{w^2}{2}} e^{-2(x - \frac{1}{2}(z - \bar{w}))^2} d\lambda(w) dx
\]

\[
+ i \int_{-\infty}^0 \int_{C} F(w) e^{\frac{w^2}{2}} e^{-2(x - \frac{1}{2}(z - \bar{w}))^2} d\lambda(w) dx
\]

\[
= -i \int_0^\infty \int_{C} e^{\frac{w^2}{2}} e^{-2(x - \frac{1}{2}(z - \bar{w}))^2} d\lambda(w) dx
\]

\[
- i e^{\frac{2}{x^2 + \frac{1}{2}z^2}} e^{2x\bar{y}(z - \bar{w})} d\lambda(w) e^{-2x^2} dx.
\]

Now we see that by writing \(w = a + ib\),

\[
\int_{C} e^{\frac{w^2}{2} + \frac{\bar{y}^2}{4}} (e^{2x\bar{y}(z - \bar{w})} - e^{-2x\bar{y}(z - \bar{w})}) d\lambda(w)
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{2xiz - 2xb} e^{-2xia} - e^{-2xia} e^{2xia}) da e^{-2b^2} db.
\]
But it is obvious that for each \( z \in \mathbb{C}, x \in (0, \infty), \) and \( b \in \mathbb{R} \), the integral
\[
\int_{\mathbb{R}} (e^{2ixz-2xb} e^{-2xia} - e^{-2ixz+2xb} e^{2xia}) \, da
\]
is not convergent. Thus, we see that \( S_{\varphi} \) is not well-defined on \( \mathcal{F}^p(\mathbb{C}) \). \( \square \)

As from the second comment in Remark 1.4, there is another possible definition the Fock space \( F^p \) for \( 1 < p < \infty \), which consists of all entire functions \( f \) on \( \mathbb{C} \) such that the function \( f(z)e^{-\frac{|z|^2}{p}} \) belongs to \( L^p(\mathbb{C}) \) with the norm
\[
\|f\|_{F^p} := \left( \frac{p}{2\pi} \int_{\mathbb{C}} \left| f(z) e^{-\frac{|z|^2}{p}} \right|^p \, dz \right)^{\frac{1}{p}}.
\]
It is clear that \( F^2 \) is the same as \( \mathcal{F}^2 \). See for example [9]. The main result in [9] states that when \( 2 < p < \infty \), the Bargmann transform maps \( L^p(\mathbb{R}) \) boundedly into \( F^p \), but NOT onto. Hence, if we consider the operator \( S_{\varphi} = BHB^{-1} \) as in Proposition 3.7, which is well defined on \( F^2 \), then \( S_{\varphi} \) is not well-defined on \( F^p \) for \( 2 < p < \infty \).

In the theory of singular integrals in harmonic analysis, it is well-known (see [10, 27]) that the famous “\( T(1) \)” theorem of David and Journé gives necessary and sufficient conditions for generalized Calderon-Zygmund operators to be bounded on \( L^2(\mathbb{R}^n) \). We propose the following open problem on the Fock space \( \mathcal{F}^2(\mathbb{C}^n) \) (see also Proposition 2.1).

**Open problem:** Characterize those entire functions \( K_T(z, w) \) on \( \mathbb{C}^{2n} \) such that the integral operator
\[
TF(z) = \int_{\mathbb{C}^n} K_T(z, w) F(w) \, d\lambda(w), \quad z \in \mathbb{C}^n
\]
is bounded on \( \mathcal{F}^2(\mathbb{C}^n) \).

### 4. Applications and Examples of Theorem 1.1

There are many examples to show that characterising the boundedness of \( S_{\varphi} \) is interesting and non-trivial. By choosing different functions \( \varphi \) in \( S_{\varphi} \), one can recover important operators arising in complex analysis and harmonic analysis. We now apply our main result Theorem 1.1 to a few well-known examples, such as the Riesz transform on \( \mathbb{R}^n \), the Ahlfors–Beurling operator on \( \mathbb{C} \), and a few others.

**Example 1.** If \( S_{\varphi} \) is the identity with \( \varphi(z) = 1 \), then \( \varphi \) can be written as (1.2), where \( m(x) = 1 \).

**Example 2.** Let \( S_{\varphi} = BHB^{-1} \) with \( H \) the Hilbert transform defined as
\[
H(f)(x) = \text{p.v.} \cdot \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy,
\]
where the improper integral is taken in the sense of “principle value.” Note that \( \mathcal{F}(Hf)(x) = m(x)\mathcal{F}f(x) \) with \( m(x) = -i\text{sgn}(x) \).

By Theorem 1.1, the function \( \varphi \) can be written as (1.2) with \( m(x) = -i\text{sgn}(x) \). That is,
\[
\varphi(z) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_{\mathbb{R}} -i\text{sgn}(x) e^{-2(x-\frac{1}{2}z)^2} \, dx.
\]
Note that
\[
\frac{d}{dz} \phi(z) = -i \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \left( \int_0^\infty - \int_{-\infty}^0 \right) \left( -4(x - \frac{i}{2}z) \right) \left( \frac{d}{dx} e^{-2(x + \frac{i}{4}z)^2} \right) dx
\]
\[
= - (2\pi)^{-\frac{1}{2}} e^{-2(x + \frac{i}{4}z)^2} \left[ (\infty_0 - 0_{-\infty}) \right] = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} e^{\frac{z}{2}}
\]
with \(\phi(0) = 0\). This implies
\[
\phi(z) = \frac{2}{\sqrt{\pi}} A \left( \frac{z}{\sqrt{2}} \right) \in \mathcal{F}^2(\mathbb{C}),
\]
where
\[
A(z) = \int_0^z e^{u^2} du, \quad z \in \mathbb{C},
\]
which is the antiderivative of \(e^{u^2}\) satisfying \(A(0) = 0\). See also [32, Section 8].

**Example 3.** From [31], if \(\phi(z) = e^{ax^2}\) with \(0 < a < \frac{1}{2}\), the operator \(S_\phi\) is bounded on \(\mathcal{F}^2(\mathbb{C})\). By Theorem 1.1, \(\phi\) can be written as (1.2) for some \(m \in L^\infty(\mathbb{R})\), hence
\[
\int_\mathbb{R} m(x) e^{-2(\frac{1}{4}z^2 - ax^2)} dx = \int_\mathbb{R} m(x) e^{\left\{ \frac{1}{2} i \sqrt{\frac{1}{2a}} - i \frac{1}{\sqrt{2a}} \right\}^2} e^{\frac{2}{2a} x^2} dx
\]
should be a constant. Thus we are able to choose \(m(x) = e^{-\frac{4}{\sqrt{2a}} x^2}\), which is a bounded function.

**Example 4.** Let \(\phi(z) = e^{i\theta}\). If \(\phi(z)\) has the representation (1.2), then
\[
\int_\mathbb{R} m(x) e^{-2x^2 + (2ix - \theta)z + \frac{1}{2}z^2} dx = \int_\mathbb{R} m(x) e^{2x^2 - \frac{2}{2a} x} e^{-2(\frac{1}{2}x + \frac{i}{\sqrt{a}}(0 - \theta))^2} dx
\]
should be a constant, hence
\[
\int_\mathbb{R} (m(x)e^{2x^2 - \frac{2}{2a} x} - c) e^{-2(\frac{1}{2}x + \frac{i}{\sqrt{a}}(0 - \theta))^2} dx = 0
\]
for some constant \(c\).

Thus \(m(x) = c_0 e^{-2xi\theta}\) almost everywhere, where \(c_0\) is a constant. By Theorem 1.1, \(S_\phi\) is bounded on \(\mathcal{F}^2(\mathbb{C})\) if and only if \(m\) is bounded, i.e. \(a\) is real. In fact, this is a result shown in [31] and when \(a\) is real, \(S_\phi = W_a\), which is a unitary operator defined above.

**Example 5.** Riesz transforms on \(\mathbb{R}^n\).

We now recall the Riesz transform on \(\mathbb{R}^n\): for \(f \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n, j = 1, 2, \ldots, n\), the \(j\)-th Riesz transform is defined as
\[
R_j f(x) = \lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} K_j(y) f(x - y) dy,
\]
where
\[
K_j(y) = c_n \frac{y_j}{|y|^{n+1}}, \quad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n}{2}}}.
\]

Note that for \(j = 1, 2, \ldots, n\),
\[
\mathcal{F}(R_j f)(\xi) = m_j(\xi) \mathcal{F}(f)(\xi),
\]
where
\begin{equation}
(4.1) \quad m_j(\xi) := -i \frac{\xi_j}{|\xi|}.
\end{equation}

Hence we have
\[
R_j(f)(x) = \mathcal{F}^{-1}\left( -i \frac{\xi_j}{|\xi|} \mathcal{F}(f) \right)(x).
\]

Then, by applying our main result Theorem 1.1 and Lemma 2.3, we obtain that

**Proposition 4.1.** For \( j = 1, 2, \ldots, n \), the operator \( T_j = BR_jB^{-1} : \mathcal{F}^2(\mathbb{C}^n) \to \mathcal{F}^2(\mathbb{C}^n) \) is given by
\[
T_j F(z) = \int_{\mathbb{R}^n} F(w)e^{\xi \cdot \bar{w}} \varphi_j(z - \bar{w})d\lambda(w)
\]
for all \( F \in \mathcal{F}^2(\mathbb{C}^n) \), with
\[
\varphi_j(z) := \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m_j(\xi) e^{-2i\xi \cdot \bar{w}} d\xi, \quad \text{where} \quad m_j(\xi) = -i \frac{\xi_j}{|\xi|}.
\]

From the Fourier multiplier of Riesz transform as given in (4.1), we see that \( \sum_{j=1}^n m_j^2(\xi) + 1 = 0 \), which gives a fundamental equation for Riesz transforms:
\begin{equation}
(4.2) \quad \sum_{j=1}^n R_j^2 = -Id,
\end{equation}
where \( Id \) is the identity operator on \( L^2(\mathbb{R}^n) \).

Define the operators \( S_{\varphi_j} \) by
\[
S_{\varphi_j} = BR_jB^{-1}, \quad j = 1, 2, \ldots, n.
\]

**Proposition 4.2.** The following equation holds for the operators \( \{ S_{\varphi_j} \} \)
\begin{equation}
(4.4) \quad \sum_{j=1}^n S_{\varphi_j}^2 = -Id.
\end{equation}

with \( \sum_{j=1}^n \| \varphi_j \|_{\mathcal{F}^2(\mathbb{C}^n)}^2 = 1 \), where \( Id \) is the identity operator on \( \mathcal{F}^2(\mathbb{C}^n) \).

**Proof.** Note that \( m_j \) is an odd function, so is \( \varphi_j \). Write
\[
S_{\varphi_j} F(z) = BR_jB^{-1} F(z) = \int_{\mathbb{C}^n} F(\xi)e^{\xi \cdot \bar{z}} \varphi_j(z - \bar{\xi})d\lambda(\xi).
\]

On the other hand
\[
S_{\varphi_j} F(z) = (BR_jB^{-1})(BR_jB^{-1}) F(z) = \int_{\mathbb{C}^n} F(\xi) \left( \int_{\mathbb{C}^n} \varphi_j(z - \bar{w})\varphi_j(w - \bar{\xi})e^{w \cdot \bar{z}} e^{\xi \cdot \bar{w}} d\lambda(w) \right) d\lambda(\xi).
\]

Since \( F \) is arbitrary, we get
\begin{equation}
(4.5) \quad e^{\xi \cdot \bar{z}} \varphi_j(z - \bar{\xi}) = \int_{\mathbb{C}^n} \varphi_j(z - \bar{w})\varphi_j(w - \bar{\xi}) e^{w \cdot \bar{z}} e^{\xi \cdot \bar{w}} d\lambda(w).
\end{equation}

Set \( z = \xi \) and notice \( \varphi_j(z) = \varphi_j(\bar{z}) \), then it follows that
\[
\varphi_j(z - \bar{z}) = \int_{\mathbb{C}^n} \varphi_j(z - \bar{w})\varphi_j(w - \bar{z}) e^{w \cdot \bar{z}} e^{-|z|^2} d\lambda(w)
\]
\[
= -\pi^{-n} \int_{\mathbb{C}^n} |\varphi_j(z - \bar{w})|^2 e^{w \cdot \bar{w}} e^{-|z|^2 - |w|^2} \, dw
= -\pi^{-n} \int_{\mathbb{C}^n} |\varphi_j(z - \bar{w})|^2 e^{-|z|^2} \, dw
= -\int_{\mathbb{C}^n} |\varphi_j(w + z - \bar{z})|^2 d\lambda(w).
\]

However,
\[
\sum_{j=1}^n \varphi_j(z - \bar{z}) = \sum_{j=1}^n \left(\frac{2}{\pi}\right)^\frac{n}{2} \int_{\mathbb{C}^n} m_j^2(x) e^{-2(\bar{x} - \bar{w}z)^2} \, dx = -\left(\frac{2}{\pi}\right)^\frac{n}{2} \int_{\mathbb{R}^n} e^{-2(x - \bar{w}z)^2} \, dx = -1.
\]

Then it implies
\[
\sum_{j=1}^n \int_{\mathbb{C}^n} |\varphi_j(w + it)|^2 d\lambda(w) = 1.
\]

Define translation along the imaginary axis \(\tau_{it} f(z) = f(z + it)\), where \(t\) is real. Then it says the sum
\[
\sum_{j=1}^n \|\tau_{it} \varphi_j\|_{L^2(\mathbb{C}^n)}^2 = 1
\]
under any translation along the imaginary axis. In particular, we have that \(\sum_{j=1}^n \|\varphi_j\|_{L^2(\mathbb{C}^n)}^2 = 1\).

Moreover, we set \(\xi = 0\) in (4.5), then we get
\[
\varphi_{jj}(z) = S_{\varphi_j}(\varphi_j)(z), \quad z \in \mathbb{C}^n,
\]
hence
\[
\sum_{j=1}^n S_{\varphi_j}(\varphi_j)(z) + 1 = 0.
\]

The proof of Proposition 4.2 is complete. \(\square\)

**Example 6.** Ahlfors–Beurling operator on \(\mathbb{C}\).

The Ahlfors–Beurling operator is a very well-known Calderón–Zygmund operator on \(\mathbb{C}\), defined on \(L^p(\mathbb{C})\), \(1 < p < \infty\), as follows:
\[
\mathcal{B} \psi(z) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{C}} \frac{\psi(\xi)}{(\xi - z)^2} \, d\xi.
\]

It connects harmonic analysis and complex analysis and is of fundamental importance in several areas of mathematics including PDE and quasiconformal mappings. For example, Petermichl and Volberg [21] proved a sharp weighted estimate of \(\mathcal{B}\), which shows that any weakly quasiregular map is quasiregular. We also recall that \(\mathcal{B}\) is an isometry on \(L^2(\mathbb{C})\), and is given as a Fourier multiplier of \(\mathcal{F}(\mathcal{B}f)(\xi) = m(\xi) \mathcal{F}(f)(\xi)\), where
\[
m(\xi) = \frac{\bar{\xi}}{\xi}, \quad \xi \in \mathbb{C}.
\]

Then by applying Theorem 1.1, we have the following.
Proposition 4.3. The operator $T = B \mathcal{B} B^{-1} : \mathcal{F}^2(\mathbb{C}^2) \to \mathcal{F}^2(\mathbb{C}^2)$ is given by

$$TF(z) = \int_{\mathbb{C}^2} F(w)e^{z \bar{w}} \varphi(z - \bar{w})d\lambda(w)$$

for all $F \in \mathcal{F}^2(\mathbb{C}^2)$, with

$$\varphi(z - \bar{w}) := \frac{2}{\pi} \int_{\mathbb{R}^2} m(x) e^{-2(x - \frac{k(x_1 + x_2)}{2})^2} dx,$$

where $m(x) = \left(\frac{x_1 - ix_2}{x_1 + ix_2}\right)$, $x = (x_1, x_2) \in \mathbb{R}^2$.

Proof. For every $F \in \mathcal{F}^2(\mathbb{C}^2)$, we have

$$TF(z) = B \mathcal{B} B^{-1} F(z) = B^{-1}(\frac{\xi}{\xi}) \mathcal{F} (B^{-1} F)(z)$$

$$= B^{-1} \mathcal{F} (B^{-1} F)(z)$$

$$= B \left(\frac{\xi}{\xi}\right) \mathcal{F} (B^{-1} F)(iz),$$

where the last equality follows from Proposition 2.3.

Then from the definition of the Bargmann transform and from (2.6), we have

$$TF(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \int_{\mathbb{R}^2} \left(\frac{x_1 - ix_2}{x_1 + ix_2}\right) e^{-2x(iz)-x^2-\frac{\bar{w}^2}{2}} dx$$

$$= \frac{2}{\pi} \int_{\mathbb{R}^2} \left(\frac{x_1 - ix_2}{x_1 + ix_2}\right) e^{-2x(iz) - x^2} \int_{\mathbb{C}^2} F(w)e^{(\bar{w}^2/2)} e^{-2\bar{w}x} d\lambda(w) e^{2x(iz) - x^2 - (iz)^2/2} dx$$

$$= \frac{2}{\pi} \int_{\mathbb{R}^2} F(w)e^{z \bar{w}} \int_{\mathbb{R}^2} \left(\frac{x_1 - ix_2}{x_1 + ix_2}\right) e^{-2(x - \frac{k(x_1 + x_2)}{2})^2} dx d\lambda(w)$$

$$= \int_{\mathbb{C}^2} F(w)e^{z \bar{w}} \varphi(z - \bar{w})d\lambda(w).$$

The proof of Proposition 4.3 is complete. \qed

Parallel to the powers of Riesz transform (Proposition 4.2), we also have the following direct result of the powers of the Ahlfors–Beurling operator (see for example [13]).

Corollary 4.4. Suppose $k$ is a positive integer and $k > 1$. The operator $T^k = B \mathcal{B} B^{-1} : \mathcal{F}^2(\mathbb{C}^2) \to \mathcal{F}^2(\mathbb{C}^2)$ is given by

$$T^k F(z) = \int_{\mathbb{C}^2} F(w)e^{z \bar{w}} \varphi_k(z - \bar{w})d\lambda(w)$$

for all $F \in \mathcal{F}^2(\mathbb{C}^2)$, with

$$\varphi_k(z - \bar{w}) := \frac{2}{\pi} \int_{\mathbb{R}^2} m_k(x) e^{-2(x - \frac{k(x_1 + x_2)}{2})^2} dx,$$

where $m_k(x) = \left(\frac{x_1 - ix_2}{x_1 + ix_2}\right)^k$, $x = (x_1, x_2) \in \mathbb{R}^2$.

5. Operator Theoretic Properties of the Operator $S_{\psi}$

In this section we study operator theoretic properties of the singular integral operator $S_{\psi}$. In particular, we are able to determine the normality, $C^*$ algebraic properties, compactness, and the spectrum of the operator $S_{\psi}$. Moreover, we also obtain the reducing subspaces of $S_{\psi}$.
5.1. Normality of $S_\varphi$: Proof of Theorem 1.2.

**Proof of Theorem 1.2.** For any $f, g \in \mathcal{F}^2(C^n)$,

\[
\langle S^*_\varphi f, g \rangle_{\mathcal{F}^2(C^n)} = \langle f, S_\varphi g \rangle_{\mathcal{F}^2(C^n)} = \int_{C^n} f(z) \overline{S_\varphi g(z)} \, d\lambda(z)
\]

\[
= \int_{C^n} f(z) \int_{C^n} g(w) e^{z \overline{w}} \varphi(z - \overline{w}) \, d\lambda(w) \, d\lambda(z)
\]

\[
= \int_{C^n} f(z) \int_{C^n} \overline{g}(w) e^{z \overline{w}} \varphi(z - \overline{w}) \, d\lambda(w) \, d\lambda(z).
\]

Note that by Theorem 1.1,

\[
\varphi(z - \overline{w}) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{i}{2}(z-\overline{w}))^2} \, dx
\]

for some $L^\infty(\mathbb{R}^n)$ function $m$ such that

\[
\tilde{\varphi}(w - \overline{z}) := \overline{\varphi(z - \overline{w})} = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \overline{m(x)}e^{-2(x+\frac{i}{2}(z-\overline{w}))^2} \, dx = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{i}{2}(w-z))^2} \, dx.
\]

Thus, by Fubini’s theorem,

\[
\langle S^*_\varphi f, g \rangle_{\mathcal{F}^2(C^n)} = \int_{C^n} \int_{C^n} f(z) \overline{g}(w) e^{\overline{z}w} \overline{\varphi(z - \overline{w})} \, d\lambda(z) \, d\lambda(w)
\]

\[
= \int_{C^n} \int_{C^n} f(z) e^{wz} \tilde{\varphi}(w - \overline{z}) \, d\lambda(z) \, \overline{g}(w) \, d\lambda(w).
\]

Hence, we have

\[
S^*_\varphi f(z) = \int_{C^n} f(w) e^{z \overline{w}} \tilde{\varphi}(z - \overline{w}) \, d\lambda(w) =: S_\varphi f(z).
\]

By noting that $S_\varphi S_\psi = S_\psi S_\varphi$ for any bounded operators $S_\psi$ and $S_\varphi$, we see that $S_\varphi$ is always normal. This finishes the proof of Theorem 1.2. □

5.2. $C^*$-Algebra Generated by $S_\varphi$, Spectrum and Compactness of the Operator $S_\varphi$. As applications of Theorems 1.1 and 1.2, we can now figure out the $C^*$-algebra, the spectrum and the compactness of the operator $S_\varphi$, which were all unknown before. This in turn shows the importance of our Theorem 1.1. Here and in what follows, we denote by $M_m f = m \cdot f$ the multiplication operator $M_m$ on $L^2(\mathbb{R}^n)$ for a function $m$ in $L^\infty(\mathbb{R}^n)$.

5.2.1. $C^*$-Algebra Generated by $S_\varphi$. We first have the following result.

**Theorem 5.1.** $\mathcal{A} := \{S_\varphi : S_\varphi$ is bounded on $\mathcal{F}^2(C^n)\}$ is a commutative $C^*$-algebra.

**Proof.** By Theorem 1.1, we know that for any $\varphi \in \mathcal{F}^2(C^n)$, $S_\varphi$ is bounded if and only if there is an $m \in L^\infty(\mathbb{R}^n)$ such that (1.2) holds, and thus $S_\varphi = \mathcal{B} T \mathcal{B}^{-1}$, where $T \in \mathcal{M}^{2,2}(\mathbb{R}^n)$ with $\mathcal{F}(T f)(y) = m(y) \mathcal{F} f(y)$.

Hence, we have $S_\varphi f(z) = B \mathcal{F}^{-1} M_m \mathcal{F} B^{-1} f(z)$, where $M_m f = m \cdot f$ for $f \in L^2(\mathbb{R}^n)$. If $\varphi_1$ and $\varphi_2$ are in $\mathcal{F}^2(C^n)$ such that both $S_{\varphi_1}$ and $S_{\varphi_2}$ are bounded, then there are $m_1$ and $m_2$ in $L^\infty(\mathbb{R}^n)$ such that $S_{\varphi_1} f = B \mathcal{F}^{-1} M_{m_1} \mathcal{F} B^{-1} f$ and $S_{\varphi_2} f = B \mathcal{F}^{-1} M_{m_2} \mathcal{F} B^{-1} f$. 
Furthermore,
\[
S_{\psi_1} S_{\psi_2} f = B^T F^{-1} M_{m_1} F B^{-1} (B^T F^{-1} M_{m_2} F B^{-1} f) \\
= B^T F^{-1} M_{m_1} M_{m_2} F B^{-1} f \\
= B^T F^{-1} M_{m_1, m_2} F B^{-1} f,
\]
which shows that \( S_{\psi_1} S_{\psi_2} = S_{\psi} \), where
\[
\varphi(z) = \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m_1(x) m_2(x) e^{-2(x - \frac{z}{\sqrt{2}})^2} dx.
\]

This shows that \( \mathcal{A} \) is an algebra on \( \mathcal{F}^2(\mathbb{C}^n) \). Since \( S_{\psi}^* = S_{\bar{\psi}} \), and \( S_{\psi} S_{\psi} = S_{\psi} S_{\psi} \), for any \( S_{\psi}, S_{\bar{\psi}} \in \mathcal{A} \), we see that \( \mathcal{A} \) is a commutative \( C^* \)-algebra. In fact,
\[
\mathcal{A} \equiv L^\infty(\mathbb{R}^n)
\]
with the isomorphism map \( \mathfrak{h} : S_{\psi} \to m \) for
\[
\varphi(z) = \left( \frac{2}{\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} m(x) e^{-2(x - \frac{z}{\sqrt{2}})^2} dx.
\]
This finishes the proof of Theorem 5.1. \( \square \)

**Remark 5.2.** Note that \( L^\infty(\mathbb{R}^n) \) is a maximal commutative \( w^* \)-algebra in \( L^2(\mathbb{R}^n) \) (see for example [11, Theorem 4.58]). Moreover, from the proof of Theorem 5.1 we see that \( \mathcal{A} \equiv L^\infty(\mathbb{R}^n) \). Hence, we get that \( \mathcal{A} \) is also a maximal commutative \( w^* \)-algebra in \( \mathcal{F}^2(\mathbb{C}^n) \). Thus, for any bounded linear operator \( T \) on \( \mathcal{F}^2(\mathbb{C}^n) \), \( T \in \mathcal{A} \) if and only if \( T S_{\psi} = S_{\psi} T \) for any \( S_{\psi} \in \mathcal{A} \). It should be pointed out that \( \mathcal{A} \) has zero factors, in fact, if \( m_1, m_2 \in L^\infty(\mathbb{R}^n) \) satisfy \( |\text{supp } m_1 \cap \text{supp } m_2| = 0 \), then \( S_{\psi_1} S_{\psi_2} = 0 \), where \( \psi_1 \) and \( \psi_2 \) are defined as in (1.2) for \( m_1, m_2 \).

One may be concerned that the result in [11, Theorem 4.58] is for a compact Hausdorff space \( X \) while we applied it for \( X = \mathbb{R}^n \), which is not compact. However, in this case, all we need to do is first to apply it on a large fixed ball centered at the origin with radius \( k \) in \( \mathbb{R}^n \) and then pass to \( \mathbb{R}^n \) by letting \( k \to \infty \). We omit the details here.

**Remark 5.3.** If \( m(x) \) is a real-valued function, then \( \psi = \bar{\psi} \). Thus, \( S_{\psi}^* = S_{\psi} \), that is, \( S_{\psi} \) is self-adjoint. If \( m(x) \) is the function taking purely imaginary values, then \( \bar{\psi} = -\psi \). Thus, \( S_{\psi}^* = -S_{\psi} \), that is, \( S_{\psi} \) is anti self-adjoint. For example, if \( S_{\psi} = BHB^{-1} \), then \( S_{\psi} \) is anti self-adjoint.

**5.2.2. Spectrum of the operator \( S_{\psi} \).** The computation of the spectrum of an operator \( T \) is usually a difficult problem even if \( T \) is normal (which our \( S_{\psi} \) are). But, in this particular case, using the connection with the Fourier multipliers it is possible to rather easily compute the spectrum of \( \sigma(S_{\psi}) \) in a very concise way. Perhaps the proofs of the results in this section are very difficult if one resorts to methods of analytic function theory. In general, a normal operator may have different spectrum and essential spectrum since the spectrum may contain isolated eigenvalues with finite multiplicity. However, for \( \psi \in \mathcal{F}^2(\mathbb{C}^n) \), if \( S_{\psi} \) is bounded, we can prove that the spectrums coincide. Moreover, we also study the eigenvalue of \( S_{\psi} \), as well as the approximate point spectrum.
Theorem 5.4. Suppose \( \varphi \in \mathcal{F}^2(\mathbb{C}^n) \) such that \( S_\varphi \) is bounded on \( \mathcal{B}^2(\mathbb{C}^n) \) and \( \varphi \) is defined as in (1.2) for some \( m \in L^\infty(\mathbb{R}^n) \). Then we have

1. \( \sigma(S_\varphi) = \mathcal{R}(m)(\mathbb{R}^n) \), where \( \mathcal{R}(m)(\mathbb{R}^n) \) is the essential range of \( m \);
2. \( \mu \in \mathcal{R}(m)(\mathbb{R}^n) \) is an eigenvalue of \( S_\varphi \) if and only if \( \| x : m(x) = \mu \| > 0 \);
3. \( \sigma(S_\varphi) = \sigma_a(S_\varphi) \), where \( \sigma_a(S_\varphi) \) denotes the approximate point spectrum of \( S_\varphi \);
4. \( \sigma(S_\varphi) = \sigma_e(S_\varphi) \), where \( \sigma_e(S_\varphi) \) denotes the essential spectrum of \( S_\varphi \).

Proof. We now provide the proof for these four statements.

Proof of (1): this argument is routine by the isomorphism \( b : S_\varphi \rightarrow m \).

Proof of (2): for any \( \mu \in \mathcal{R}(m)(\mathbb{R}^n) \), if \( \| x : m(x) = \mu \| > 0 \), then write \( \chi_\mu(x) = \chi_{\{x : m(x) = \mu\}}(x) \). Without loss of generality, assume \( \| x : m(x) = \mu \| < \infty \). Then \( (M_m - \mu)\chi_\mu = 0 \) and

\[
\int_{\mathbb{R}^n} \chi_\mu dx = \| x : m(x) = \mu \| > 0.
\]

This shows that \( \mu \in \sigma_p(M_m) \), further \( \mu \in \sigma(S_\varphi) \).

On the other hand, if \( \| x : m(x) = \mu \| = 0 \), we can prove that \( \mu \not\in \sigma_p(M_m) \). In fact, for any \( f \in L^2(\mathbb{R}^n) \), if \( M_m f = \mu f \), then \( f = 0 \) on \( \mathbb{R}^n \setminus \{ x : m(x) = \mu \} \). Hence, \( f = 0 \) a.e. since \( \| x : m(x) = \mu \| = 0 \). Thus \( \mu \not\in \sigma_p(M_m) \), and consequently \( \mu \not\in \sigma_p(S_\varphi) \).

Proof of (3): for any \( m \in L^\infty(\mathbb{R}^n) \), write \( M_m f = m \cdot f \), for every \( f \in L^2(\mathbb{R}^n) \).

Assume \( \mu \in \mathcal{R}(m)(\mathbb{R}^n) \), the essential range of \( m \). Then \( \| x : |m(x) - \mu| < \epsilon \| > 0 \) for any \( \epsilon > 0 \). Let \( \chi_\epsilon(x) = \chi_{\{x : |m(x) - \mu| < \epsilon\}}(x) \) be the characteristic function of \( \{ x : |m(x) - \mu| < \epsilon \} \). Choose a function \( f_\epsilon \in L^2(\mathbb{R}^n) \) such that

\[
\| \chi_\epsilon f_\epsilon \|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\chi_\epsilon f_\epsilon|^2 dx = \int_{\{ x : |m(x) - \mu| < \epsilon \}} |f_\epsilon|^2 dx = 1.
\]

We have

\[
\|(M_m - \mu)(\chi_\epsilon f_\epsilon)\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |(M_m - \mu)(\chi_\epsilon f_\epsilon)|^2 dx
\]

\[
= \int_{\{ x : |m(x) - \mu| < \epsilon \}} |(M_m - \mu)|^2 |f_\epsilon|^2 dx
\]

\[
\leq \epsilon^2 \int_{\{ x : |m(x) - \mu| < \epsilon \}} |f_\epsilon|^2 dx
\]

\[
\leq \epsilon^2.
\]

This implies that \( \mu \in \sigma_a(M_m) \), further \( \mu \in \sigma_a(S_\varphi) \).

Proof of (4): from (1) we see that \( \sigma(S_\varphi) = \mathcal{R}(m)(\mathbb{R}^n) \). Hence, without loss of generality, we now just assume that \( 0 \in \mathcal{R}(m)(\mathbb{R}^n) \). Then for any \( \epsilon > 0 \), we have

\[
\| x : |m(x)| < \epsilon \| > 0.
\]

Choose a sequence of subsets in \( \{ x : |m(x)| < \epsilon \} \) such that

\[
E_{k+1} \subset E_k \subset \{ x : |m(x)| < \epsilon \}
\]
and $|E_k| \neq 0, |E_k| \to 0$ as $k \to \infty$. Set

\[ f_k(x) = \frac{1}{\sqrt{|E_k|}} \chi_{E_k}(x), \]

where $\chi_{E_k}$ is the characteristic function of $E_k$, then

\[ \|f_k\|^2_{L^2(\mathbb{R}^n)} = \int_{E_k} \frac{1}{|E_k|} dx = 1 \]

and for any $g \in L^2(\mathbb{R}^n),

\[ \langle f_k, g \rangle_{L^2(\mathbb{R}^n)} = \frac{1}{\sqrt{|E_k|}} \langle \chi_{E_k}, g \rangle_{L^2(\mathbb{R}^n)} \leq \frac{1}{|E_k|} \|\chi_{E_k}\|_{L^2(\mathbb{R}^n)} \|\chi_{E_k} g\|_{L^2(\mathbb{R}^n)} = \|\chi_{E_k} g\|_{L^2(\mathbb{R}^n)}. \]

Note that $g \in L^2(\mathbb{R}^n)$, we have that $\|\chi_{E_k} g\| \to 0$ as $k \to \infty$. This implies that $f_k \to 0$ in $L^2(\mathbb{R}^n)$ in the weak sense.

It is not difficult to see that

\[ \|(M_m f_k)\|^2_{L^2(\mathbb{R}^n)} = \int_{E_k} |m f_k|^2 dx + \int_{\mathbb{R}^n \setminus E_k} |m f_k|^2 dx = \int_{E_k} |m f_k|^2 dx \leq \epsilon^2 \int_{E_k} |f_k|^2 dx = \epsilon^2. \]

Since $\epsilon$ is arbitrary, we see that $M_m$ is not Fredholm, that is $0 \in \sigma_{\text{e}}(M_m)$, further $0 \in \sigma_{\text{e}}(S_\varphi)$.

The proof of Theorem 5.4 is complete. \hfill \Box

5.2.3. Compactness of the Operator $S_\varphi$. Next we provide the proof of the compactness of the operator $S_\varphi$.

**Theorem 5.5.** Suppose $\varphi \in \mathcal{F}^2(\mathbb{C}^n)$ such that $S_\varphi$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$ and $\varphi$ is defined as in (1.2) for some $m \in L^\infty(\mathbb{R}^n)$. Then $S_\varphi$ is compact if and only if $\varphi = 0$.

**Proof.** We need only to prove that $S_\varphi$ can not be compact if $\varphi \neq 0$. Since $\varphi \neq 0$, we see that $m \neq 0$.

Write $E_0 = \{x : m(x) \neq 0\}$. Then $|E_0| > 0$. Thus, there is an $\epsilon_0 > 0$ such that $E_{\epsilon_0} = \{x : |m(x)| > \epsilon_0\}$ has positive measure. Without loss of generality, assume that $0 < |E_{\epsilon_0}| < \infty$. Choose a sequence of subsets in $E_{\epsilon_0}$ such that $E_{\epsilon_0} \supset E_k \supset E_{k+1}$, and $|E_k| > 0, \lim_{k \to \infty} |E_k| = 0$. Let $f_k(x)$ be defined as in (5.1). Then from the argument as in the proof of (4) of Theorem 5.4, we see that $f_k \to 0$ in $L^2(\mathbb{R}^n)$ in the weak sense.

It is obvious that

\[ \|(M_m f_k)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |m \cdot f_k|^2 dx \geq \int_{E_k} |m \cdot f_k|^2 dx \geq \epsilon_0^2 \int_{E_k} |f_k|^2 dx = \epsilon_0^2 \not\to 0. \]

This shows that $M_m$ is not compact, and hence $S_\varphi$ can not be compact. \hfill \Box

5.3. Invariant subspaces of $S_\varphi$. The well-known Beurling theorem characterizes the invariant subspace lattice of the coordinate multiplier $M$, on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ (see [11, 16]). However, it is very difficult to obtain the characterization of the invariant subspace lattice of a general bounded linear operator $T$ even if $T$ is normal. One possible attempt arises from observing that the reducing subspaces of a normal operator may be determined by its spectral projections. However, in general, one does not know the explicit form of the spectral projections.
In this subsection, we characterize the reducing subspaces of $M_m$ for any $m \in L^\infty (\mathbb{R}^n)$. Moreover, based on our main result Theorem 1.1, we can further obtain the characterization of the reducing subspaces of $S_\varphi$ with $\varphi$ defined as (1.2) for some $m \in L^\infty (\mathbb{R}^n)$.

It is easy to prove that for $m \in L^\infty (\mathbb{R}^n)$, $R(M_m)$ is closed if and only if either $0 \not\in \mathcal{R}(m)$, or $0 \in \mathcal{R}(m)$, but $m$ is essentially lower bounded on $\text{supp } m$, the support of $m$. In particular, if $E \subset \mathbb{R}^n$ with $|E| > 0$, then for $\chi_E$, the characteristic function of $E$, $X_0 = \chi_E L^2 (\mathbb{R}^n)$ is an invariant subspace or zero subspace of $M_m$. Thus we have the following.

**Theorem 5.6.** Suppose $m \in L^\infty (\mathbb{R}^n)$ and $\varphi \in \mathcal{F}^2 (\mathbb{C}^n)$ is defined as in (1.2). Let $X$ be a subspace of $\mathcal{F}^2 (\mathbb{C}^n)$. Then $X$ is a reducing subspace of $S_\varphi$ if and only if there is a set $E \subset \mathcal{R}(m)$ with $|E| > 0$, such that

$$X = S_{\varphi_0} \mathcal{F}^2 (\mathbb{C}^n),$$

where $\varphi_0 = \int_{\mathbb{R}^n} \chi_E (x) e^{-2(x_{-\varphi})^2} \, dx$.

**Proof.** Let $P$ be the orthogonal projection from $\mathcal{F}^2 (\mathbb{C}^n)$ to $X$. If $X$ is a reducing subspace of $S_\varphi$, then $PS_\varphi = S_\varphi P$. Clearly, $P$ is the spectral projection of $S_\varphi$. Thus $PS_\varphi = S_\varphi P$ for any $S_\varphi \in \mathcal{A}$ since $\mathcal{A}$ is maximal commutative. We see that there is a $E \subset \mathcal{R}(m)$ with $|E| > 0$, such that $P = S_{\varphi_0}$, where $\varphi_0 = \int_{\mathbb{R}^n} \chi_E (x) e^{-2(x_{-\varphi})^2} \, dx$. Thus

$$X = P \mathcal{F}^2 (\mathbb{C}^n) = S_{\varphi_0} \mathcal{F}^2 (\mathbb{C}^n).$$

Conversely, if there is a $\varphi_0 \in \mathcal{F}^2 (\mathbb{C}^n)$ with $\chi_E, E \subset \mathcal{R}(m)$ such that $X = S_{\varphi_0} \mathcal{F}^2 (\mathbb{C}^n)$, then $X$ is a closed subspace. By noting that $S_\varphi S_{\varphi_0} = S_{\varphi_0} S_\varphi$ and $S_{\varphi_0}^2 = S_{\varphi_0}, S_{\varphi_0}^2 = S_{\varphi_0}$, we see that $S_{\varphi_0}$ is a projector which commutes with $S_\varphi$. Hence, $X = S_{\varphi_0} \mathcal{F}^2 (\mathbb{C}^n)$ is the reducing subspace of $S_\varphi$. \qed

We now recall [11, Theorem 6.9] which gives the characterization of the simple invariant subspaces of the coordinate multiplier on $L^2 (\mathbb{T})$.

**Lemma 5.7** ([11, Theorem 6.9]). If $\mu$ is a positive regular Borel measure on $\mathbb{T}$, then a non-trivial closed subspace $X$ of $L^2 (\mu)$ satisfies $M_\varphi X \subset X$ and $\cap_{n \geq 0} M_\varphi^n X = \{0\}$ if and only if there exists a Borel function $m$ on $\mathbb{T}$ such that $|m|^2 \, d\mu = d\theta / 2\pi$ and $X = m H^2 (\mathbb{T})$.

By the connection between the Hardy space $H^2 (\mathbb{T})$ and $H^2 (i\mathbb{R})$, we may characterize the simple invariant subspaces of $M_\varphi$, where $\varphi (w) = \frac{w - 1}{w + 1}$.

We say that $X$ is the simple invariant subspace of $M_\varphi$.

**Theorem 5.8.** Suppose $\varphi = \frac{w - 1}{w + 1}$ is the Riemann map from $\mathbb{C}_+$ to $\mathbb{D}$, $M_\varphi$ is the multiplier on $L^2 (i\mathbb{R})$ defined as $M_\varphi f = \varphi f$ for any $f \in L^2 (i\mathbb{R})$. Then a non-trivial closed subspace $X$ of $L^2 (i\mathbb{R})$ satisfies $M_\varphi X \subset X$ and $\cap_{n \geq 0} M_\varphi^n X = \{0\}$ if and only if there is a Borel function $m$ on $\mathbb{T}$ such that $|m| = 1 \text{ a.e.}$ and

$$X = \{(m \circ \varphi) \varphi_0 H^2 (i\mathbb{R})\} \quad \text{with} \quad \varphi_0 = \frac{1 + it}{\sqrt{1 + t^2}}.$$

Moreover, $BF^{-1} X$ is the simple invariant subspace of $S_\varphi$, where $\psi = \int_{\mathbb{R}} \varphi e^{-2(x_{-\varphi})^2} \, dx$. 


Proof. We need only to prove that $X$, a simple invariant subspace of $M_\phi$, must have the form $(m \circ \phi)\varphi_0 H^2(i\mathbb{R})$ for some $m \in L^\infty(\mathbb{T})$ with $|m| = 1$ a.e.. Write
\[ \tilde{X} = C_\phi^{-1}\left(\frac{1}{1-\phi}\right), \]
where $C_\phi^{-1}f = f \circ \phi^{-1}$. Then for any $\tilde{f} \in \tilde{X}$, there is an $f \in X$ such that
\[ \tilde{f} = C_\phi^{-1}\left(\frac{1}{1-\phi}f\right) = \frac{1}{1-z} C_\phi^{-1}f \in L^2(\mathbb{T}). \]
In fact, for any measurable function $g$ on $\mathbb{T}$, we have
\[ \int_\mathbb{T} g(e^{i\theta}) \frac{d\theta}{2\pi} = \int_\mathbb{R} g \circ \phi(it) \frac{1}{1 + t^2} \frac{dt}{\pi} \]
(see [18]), thus
\[ \|\tilde{f}\|^2_{L^2(\mathbb{T})} = \int_\mathbb{T} |\tilde{f}(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \int_\mathbb{R} |\tilde{f} \circ \phi(it)|^2 \frac{1}{1 + t^2} \frac{dt}{\pi} \]
\[ = \int_\mathbb{R} \left| C_\phi^{-1}\left(\frac{1}{1-\phi}f\right) \circ \phi \right|^2 \frac{1}{1 + t^2} \frac{dt}{\pi} \]
\[ = \int_\mathbb{R} \left| \frac{1}{1-\phi}f \right|^2 \frac{1}{1 + t^2} \frac{dt}{\pi} \]
\[ = \frac{1}{4} \int_\mathbb{R} |f(it)|^2 \frac{dt}{\pi} \]
\[ = \frac{1}{4\pi} \|f\|^2_{L^2(\mathbb{R})}, \]
that is, $\|\tilde{f}\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi\sqrt{\pi}} \|f\|^2_{L^2(\mathbb{R})}$. Hence $\tilde{X}$ is closed. For arbitrary $g \in \tilde{X}$, there is an $f \in X$ such that $g = C_\phi^{-1}\left(\frac{1}{1-\phi}f\right)$. Then
\[ M_\phi g = zC_\phi^{-1}\left(\frac{1}{1-\phi}f\right) = C_\phi^{-1}\varphi C_\phi^{-1}\left(\frac{1}{1-\phi}f\right) = C_\phi^{-1}\left[\frac{1}{1-\phi}(\varphi f)\right]. \]
It is routine to check that $M_\phi g = C_\phi^{-1}\left[\frac{1}{1-\phi}(\varphi^n f)\right]$. Since $\varphi^n f \in X$, we see that $\tilde{X}$ is a simple invariant subspace of $M_\phi$. Thus there is a $m \in L^\infty(\mathbb{T})$ with $|m| = 1$ a.e. such that
\[ \tilde{X} = mH^2(\mathbb{T}). \]
On the other hand,
\[ C_\phi \tilde{X} = C_\phi C_\phi^{-1}\left(\frac{1}{1-\phi}X\right), \]
we see that
\[ \frac{1}{1-\phi}X = (m \circ \phi)(1 + it)H^2(i\mathbb{R}), \]
since $C_\phi H^2(\mathbb{T}) = (1 + it)H^2(i\mathbb{R})$. Further,
\[ X = (m \circ \phi)(1 - \phi)(1 + it)H^2(i\mathbb{R}). \]
Note $|1 - \phi| = \frac{2}{\sqrt{1 + t^2}}$, write $\varphi_0 = \frac{1 + it}{\sqrt{1 + t^2}}$, then $|\varphi_0| = 1$, and
\[ X = (m \circ \varphi_0)\varphi_0 H^2(i\mathbb{R}). \]
By the Fourier transform and Bargmann transform, we know that $BF^{-1}X$ is a simple invariant subspace of $S_{\phi}$, completing the proof of Theorem 5.8. □

6. Concluding Remarks

This paper has studied a new class of operators on the Fock space which has totally different properties from the well-known Toeplitz operator. For example, for any $\phi \in L^2(\mathbb{C}^n)$, if the Toeplitz operator $T_{\phi}$ is bounded, then $T_{\phi}^* = T_{\phi}$; for any analytic function $\phi$, $T_{\phi}$ is subnormal, and moreover, $T_{\phi}$ is normal if and only if $\phi$ is constant. While $S_{\phi}$ has better properties as evidenced by the theorems above. Moreover, $S_{\phi}$ connects singular integrals in harmonic analysis to operators in the complex setting via the Bargmann transform. This connection enables the resolution of problems in complex analysis via techniques from harmonic analysis.

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**Guangfu Cao, Department of Mathematics, South China Agricultural University, Guangzhou, Guangdong 510640, P.R. China**

*E-mail address:* guangfucao@163.com

**Ji Li, Department of Mathematics, Macquarie University, NSW, 2109, Australia**

*E-mail address:* jili@mq.edu.au

**Minxing Shen, Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou, 510275, P.R. China**

*E-mail address:* shenmx3@163.com

**Brett D. Wick, Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130-4899 USA**

*E-mail address:* wick@math.wustl.edu

**Lixin Yan, Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou, 510275, P.R. China and Department of Mathematics, Macquarie University, NSW 2109, Australia**

*E-mail address:* mcsylx@mail.sysu.edu.cn