A GEOMETRIC INTERPRETATION OF RANICKI DUALITY

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To the memory of Andrew Ranicki

Abstract. Consider a commutative ring \( R \) and a simplicial map, \( X \to K \), of finite simplicial complexes. The simplicial cochain complex of \( X \) with \( R \) coefficients, \( \Delta^*X \), then has the structure of an \( (R, K) \) chain complex, in the sense of Ranicki [15]. Therefore it has a Ranicki-dual \( (R, K) \) chain complex, \( T\Delta^*X \). This (contravariant) duality functor \( T \): \( \mathcal{B}_R \mathcal{K} \to \mathcal{B}_R \mathcal{K} \) was defined algebraically on the category of \( (R, K) \) chain complexes and \( (R, K) \) chain maps.

Our main theorem, 7.1, provides a natural \( (R, K) \) chain isomorphism:
\[ T\Delta^*X \cong C(X_K) \]
where \( C(X_K) \) is the cellular chain complex of a (nonsimplicial) subdivision, \( X_K \), of the complex \( X \). The \( (R, K) \) structure on \( C(X_K) \) arises geometrically.

1. Introduction; Description of Results

This article is an addition to a theory of blocked surgery, pioneered by Ranicki and augmented by others in [15], [16], [1], [7], [8], [4], [5]. It is still in a developing state. It gives a new geometric interpretation of Ranicki’s notion of the dual, \( T\Delta^*X \) when \( C \) itself arises geometrically – in particular when \( C = \Delta^*X \), if \((X, \pi)\) is a \( K \)-space (defined in 3.2). This result, Theorem 7.1, is the main theorem. It says, roughly, that \( T\Delta^*X = C(X_K) \) for the CW-complex \( X_K \).

It is also our aim to give a transparent definition of this duality functor \( T \), a clear treatment of Ranicki’s natural transformation \( e : T^2 \to \text{id} \). and a simple proof that \( e_C : T^2C \to C \) is an \( (R, K) \) chain equivalence for all \( C \).

Our larger goal is to facilitate applications of Ranicki’s theory to geometric questions such as the topological rigidity of non-positively curved groups as in [8], [9], [4], [5] when those groups have elements of finite order.

The typical input of Ranicki’s theory is a “\( K \)-blocked normal map”. By this we mean a degree-one normal map of pl-manifolds, \( (f, b) : M^n \to X^n \), (as in [2] or [19]) plus a “control map”, \( \pi : X \to K \), where \( K \) is a finite simplicial complex. One seeks to understand the obstruction to obtaining a normal cobordism of \( (f, b) \) to a “\( K \)-blocked homotopy equivalence”, \( N^n \to X \) (a map which is a homotopy equivalence over each “block” in \( K \)).

In the classical case \((K = \text{point}; \ [2], \ [18], [19])\) one has the “surgery obstruction” \( \sigma(f, b) \in L_n(\mathbb{Z}\pi_1(X)) \) to such a normal cobordism. This functor \( L_n(\cdot) \), was generalized in [15] to yield obstruction groups \( L_n(A) \) for any “category-with-chain-involution” \((A, *, \epsilon)\). Here \( A \) is an additive category, \( \mathcal{B}A \to \mathcal{B}A \) is a contravariant functor satisfying certain conditions, on the category \( \mathcal{B}A \), of finite chain complexes in \( A \), and \( \epsilon : (*)^2 \to \text{id} \), is an equivalence in the homotopy category of \( \mathcal{B}A \).

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Ranicki, in [15], starts with a finite complex $K$ and a category with chain involution, $\mathcal{A} = (A, *, e)$ as above. He then constructs the additive category $A_K$ of $K$-blocked objects from $A$, and $K$-blocked $A$-maps. From $*$, and $e$, he defines the Ranicki Duality Functor $T : \mathcal{B}(A_K) \rightarrow \mathcal{B}(A_K)$, and the natural transformation $e : T^2 \rightarrow id_{\mathcal{B}(A_K)}$. This construction allows one to define the surgery obstruction groups, $L_n(A_K)$ where $A_K = (A_K, T, e)$.

This applies directly to a $K$-blocked normal map, $M^n \xrightarrow{(f, k)} X^n \xrightarrow{\sigma} K$. Here $A = A(R)$, the category of finitely generated free modules over a fixed commutative ring $R$. We write $AR_K$ for $(AR)K$ and $BR_K$ for $B(AR_K)$. Its objects are $(R, K)$-chain complexes. So the simplicial cochain complexes of $X$ and $M$ denoted $\Delta^*X$ and $\Delta^*M$, and the simplicial chain complexes, $\Delta X'$ and $\Delta M'$, are $(R, K)$-chain complexes. (See Section 3).

However, Ranicki’s original definition of $BR_K \xrightarrow{T} BR_K$ is rather scattered and nonconceptual. Indeed his assertion in [15] of the crucial theorem that $(AR_K, T, e)$ is a category with chain involution was only proved completely in 2018 (by Macko and Adams-Florou, [1]).

This paper interprets Ranicki’s notions geometrically. Section 2 fixes chain-complex conventions. Section 3 reviews Ranicki’s concepts concerning $(R, K)$ complexes while attempting to simplify notation. In Section 4 we introduce the $(R, K)$ chain complex $C \otimes_K D$, defined if $D$ is an $(R, K)$ complex and $C$ is an $(R, K^{op})$ complex. This complex $C \otimes_K D$ is a certain quotient of $C \otimes_R D$.

Our definition (see 5.1) of the Ranicki dual $TC$, of an $(R, K)$ complex $C$, is:

$$TC = C^* \otimes_K \Delta^*K.$$  

In Section 6 we show, using work of M. Cohen [3], that each $K$-space $(X, \pi)$ defines a certain regular CW-complex $X_K$, whose cellular chain complex has a natural $(R, K)$ structure. Therefore from each $K$-space $(X, \pi)$ we obtain three $(R, K)$ chain complexes:

1. $\Delta^*X$, the simplicial cochain complex of $X$ (Definition 3.2).
2. $C(X_K)$, the cellular chain complex of the CW complex $X_K$ (Section 7).
3. $\Delta X'$, the simplicial chain complex of $X'$, the barycentric subdivision of $X$ (Definition 6.2).

This paper shows that these three are closely related by $T$. Our main result, Theorem 7.1, exhibits an isomorphism of $(R, K)$ chain complexes:

$$\Phi_X : T\Delta^*X \cong C(X_K)$$

When $X$ is a $pl$-manifold, Poincare duality then becomes an $n$-cycle in the $(R, K^{op})$ complex, $Hom_{(R, K)(TC(X_K), C(X_K))}$.

We then use the work of McCrory, [12], to prove there are $(R, K)$ chain homotopy equivalences: $T\Delta X' \cong \Delta^*X$; $C(X_K) \cong \Delta X'$.

This regular CW complex $X_K$ is a subdivision of $X$. The derived complex $X'$, is a simplicial subdivision of $X_K$. In fact, for each simplex $T$ of $X$ and each face $\sigma$ of $\pi(T) \in K$, there is a single cell $T_\sigma$ of $X_K$. This cell $T_\sigma$ turns out to be simply $(\pi \mid T)^{-1}D(\sigma, \pi(T))$, where $D(\sigma, \pi(T))$ is the dual cell of $\sigma$ in $\pi(T)$.

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2. Chain Complex Conventions

Throughout this paper, $R$ denotes a fixed commutative ring; $AR$ is the additive category of finitely generated free $R$ modules.

For any additive category $A$ we will write $\mathcal{B}A$ for the additive category of finite chain complexes, $C = \{C_q, \partial_q\}_{q \in \mathbb{Z}}$ and chain maps $f = \{f_q : C_q \to D_q\}_{q \in \mathbb{Z}}$ from $A$. (Finite means: $C_q = 0$ for all but finitely many $q$). We abbreviate $\mathcal{B}(AR)$ to $\mathcal{B}R$.

As usual two chain maps $f, g : C \to D$ are chain homotopic if there is a sequence of $A$ maps, $h = \{h_q : C_q \to D_{q+1}\}$, for which $d_{q+1}h_q + h_{q-1}d_q^C = g_q - f_q \ \forall q$.

We regard $A$ as the full subcategory of $\mathcal{B}A$ consisting of chain complexes concentrated in degree zero.

Let $C, D \in \text{Ob}(\mathcal{B}R)$. The complexes $C \otimes R D$, and $\text{Hom}_R(C, D)$ in $\text{Ob}(\mathcal{B}R)$, are:

$$(C \otimes R D)_q = \sum_{r \in \mathbb{Z}} C_r \otimes R D_{q-r}; \quad \text{Hom}_R(C, D)_q = \sum_{r \in \mathbb{Z}} \text{Hom}_R(C_r, D_{q+r})$$

and:

$$d^{C \otimes R D}(x \otimes y) = d^C x \otimes y + (-1)^{|x|} x \otimes d^D y; \quad d^{\text{Hom}_R} = d^D \circ \phi - (-1)^{|\phi|} \phi \circ d^C$$

The evaluation map, $\text{eval}_{C,D} : \text{Hom}_R(C, D) \otimes_R C \to D$ is the R-chain map:

$$\text{eval}_{C,D}(f \otimes x) = f(x).$$

Write $\text{ev}_C : C^* \otimes C \to R$ for $\text{eval}_{C,R}$.

The contravariant functor $\mathcal{B}R \xrightarrow{\text{ev}} \mathcal{B}R$ is : $C^* = \text{Hom}(C, R); \quad f^* = \text{Hom}(f, 1_R)$.

Therefore we have:

$$(C^*)_q = \text{Hom}_R(C_q, R); \quad d^C_{q-1} = (-1)^{q+1}(d^C_{q+1})^* : (C^*)_{q-1} \to (C^*)_q.$$

The functor $*$ comes with a natural equivalence, $\varepsilon : (\ast)^2 \to 1_{\mathcal{B}R}$. Specifically, the chain isomorphism $\varepsilon_C : C^{**} \to C$ is characterized by the identity:

$$a(\varepsilon_C(a)) = (-1)^q a(a) \quad \forall a \in (C^{**})_q, \quad a \in (C^*)_q.$$

3. Basic Definitions For $(R, K)$ Chain Complexes

**Definition 3.1.** Let $K$ be a finite poset with partial order $\leq$. For example, $K$ can be a finite simplicial complex where $\sigma \leq \tau$ means $\sigma$ is a face of $\tau$. $K^{op}$ denotes the same set with the opposite partial order.

1. An $(R, K)$ module is an ordered pair $M = (M(K), \{M(\sigma)\}_{\sigma \in K})$ such that:
   a. $M(K)$ and each $M(\sigma)$ are $R$-modules in $\text{Ob}(AR)$;
   b. $M(K) = \bigoplus_{\sigma \in K} M(\sigma)$.

   More generally, for any $S \subset K$ we write: $M(S) = \bigoplus_{\sigma \in S} M(\sigma)$.

2. An $(R, K)$ map $M \xrightarrow{f} N$ of $(R, K)$ modules is a map $M(K) \xrightarrow{f(\tau, \sigma)} N(K)$ of $R$ modules, whose components, $f(\tau, \sigma) : M(\sigma) \to N(\tau)$, satisfy:

   $$f(\tau, \sigma) = 0 \text{ unless } \tau \geq \sigma.$$

3. The additive category of $(R, K)$ maps and modules is written $AR_K$.

   We abbreviate the category of chain complexes, $\mathcal{B}(AR_K)$, to $\mathcal{B}R_K$.

4. An object $C = \{C_q, \partial_q\}_{q \in \mathbb{Z}}$ of $\mathcal{B}R_K$ is an $(R, K)$ chain complex. We then write $C(K)$ for $\{C_q(K), \partial_q\}_{q \in \mathbb{Z}}$, an $R$-chain complex in $\text{ob}(\mathcal{B}R)$.

   Note: $C \in \text{ob}(\mathcal{B}R_K)$ is specified by specifying the $R$ complex $C(K)$ and the required collection $\{C_q(\sigma)\}_{\sigma \in K, q \in \mathbb{Z}}$ of $R$ submodules.

5. Let $C, D \in \text{ob}(\mathcal{B}R_K)$. $\text{Hom}_{AR_K}(C, D)$ is the $(R, K^{op})$ complex such that:
(a) \( \text{Hom}_{(R,K)}(C,D)(K) \) is the subcomplex of \( \text{Hom}_R(C(K), D(K)) \) given by those \( f = \{ f_q : C_q \to D_q + |f| \}_{q \in \mathbb{Z}} \) for which each \( f_q \) is an \((R,K)\) map.
(b) \( \text{Hom}_{(R,K)}(C,D)_p(\sigma) \) is the set of \( f \in \text{Hom}_{(R,K)}(C,D)(K)_p \) satisfying:
\[
f_q|_{|\epsilon_q(\tau)|} = 0 \quad \text{if} \quad \tau \neq \sigma, \forall q.
\]

(6) We say a sequence of chain maps \( 0 \to C' \xrightarrow{\partial'} C \xrightarrow{\partial} C'' \to 0 \) in \( \mathcal{B}_R \) is exact if for each \( \sigma \neq \tau, i(\sigma, \tau) = 0, j(\sigma, \tau) = 0 \), and, for all \( q \), the corresponding sequence, \( 0 \to C'_q(\sigma) \to C_q(\sigma) \to C''_q(\sigma) \to 0 \). is an exact sequence in \( AR \).
We then say \( i \) is an \((R,K)\) monomorphism and \( j \) is an \((R,K)\) epimorphism.

(7) Note that \( * \) specifies a contravariant functor, \( \mathcal{B}_R \xrightarrow{\ast} \mathcal{B}_{R^{op}} \), provided that we define \((C^\ast)_q(\sigma)\) as \((C_{-q}(\sigma))^\ast\) and \( d^C \ast \) as \( d^C(K)^\ast \) for \( C \in \text{ob}(\mathcal{B}_R) \).

\( \mathcal{B}_R \xrightarrow{\ast} \mathcal{B}_{R^{op}} \) preserves exactness and homotopy. The transformation \( \varepsilon_C : C^\ast \to C \) of Section 2 is an \((R,K)\) isomorphism, for all \( C \in \text{ob}(\mathcal{B}_R) \).

(8) We say \( S \subset K \) is full if, whenever \( \rho, \tau \in K \), then \( \{ \sigma \mid \rho \leq \sigma \leq \tau \} \subset S \).

Let \( C \) be an \((R,K)\) complex. Let \( S \) be full. Define \( \partial_C^q(S) : C_q(S) \to C_{q-1}(S) \) by:
\[
\partial_C^q(S)_x = \sum_{\tau \in S} \sigma^C(\tau, \sigma)x \quad \text{for all} \quad \tau, \sigma \in S \quad \text{and all} \quad x \in C(\sigma).
\]
Then \( C(S) := \{ C_q(S), \partial_C^q(S) \}_{q \in \mathbb{Z}} \) is an \( R \)-chain complex.

(9) In particular, a simplicial complex \( K \) is a poset. As usual \( \Delta_*(K;R) = \{ \Delta_q(K;R), \partial_q \}_{q \in \mathbb{Z}} \) denotes the simplicial chain complex of \( K \). One can choose a basis, \( bK \) for \( \Delta_*(K;R) \) consisting of one oriented \( q \)-simplex, \( \sigma = (v_0, \ldots, v_q) \in \Delta_q(K;R) \) for each \( q \)-simplex with vertices \( v_0, \ldots, v_q \) of \( K \).
Recall: \( (v_0, \ldots, v_q) = \text{sgn}(\pi)(v_{\pi(0)}, \ldots, v_{\pi(q)}) \) for each \( \pi \in S_{q+1} \). The oriented \( q \)-simplex \( \sigma \in \Delta_q(K;R) \) defines a dual cochain \( \sigma^* \in \Delta^*(K;R)_{-q} \) such that \( \sigma^*(\tau) = 0 \) for all \( \tau \neq \pm \sigma \), and \( \sigma^*(\sigma) = 1 \).

One then defines \( \sigma^{**} \in \Delta_q(K;R)^{**} \) by: \( \varepsilon(\sigma^{**}) = \sigma \).
Here \( \Delta^*(K;R) \) is the simplicial cochain complex of \( K \).

Each simplex \( \sigma \in K \) defines subcomplexes, \( \overline{\sigma} \) and \( \partial \sigma \), and a subset \( st(\sigma) \):
\[
\overline{\sigma} = \{ \tau \in K \mid \tau \leq \sigma \}; \quad \partial \sigma = \{ \tau \in K \mid \tau < \sigma \}; \quad st(\sigma) = \{ \tau \in K \mid \tau \geq \sigma \}
\]

(10) The incidence number \( [\tau, \sigma] \in \{ 1, -1, 0 \} \) is defined for any oriented simplices \( \sigma, \tau \) of \( K \). It satisfies: \( \partial_q(\sigma) = \sum_{\tau \in bK} [\sigma, \tau] \tau \) for any basis, \( bK \) of oriented simplices of \( K \). \( [\sigma, \tau] \neq 0 \) iff \( \tau \) is a codimension-one face of \( \sigma \).

**Definition 3.2.** \((K\text{-spaces}, \Delta^* X \text{ and } \Delta X)\)

Let \( K \) be a finite simplicial complex. A \( K \)-space is a pair \((X, \pi)\) where \( X \) is a finite simplicial complex and \( |X| \xrightarrow{\pi} |K| \) is a simplicial map, \( X \to K \). A map of \( K \)-spaces, \((X, \pi_X) \to (Y, \pi_Y)\) is a simplicial map \( f : |X| \to |Y| \) satisfying: \( \pi_Y f = \pi_X \).

Let \((X, \pi)\) be a \( K \)-space.

\( \Delta X \) denotes the \((R,K^{op})\) complex for which \( \Delta X(K) = \Delta_*(X;R) \). For each \( \sigma \in K \), \( \Delta X_p(\sigma) \) is the submodule generated by oriented \( p \)-simplices in \( \Delta_p(X;R) \) whose underlying \( p \)-simplex, \( S \in X \), satisfies \( \sigma = \pi(S) \in K \).

By definition, \( \Delta^* X = \ast(\Delta X) \). Therefore \( \Delta^* X(K) = \text{Hom}_R(\Delta_*(X;R); R) = \Delta^*(X;R) \), the simplicial cochain complex of \( X \). For each \( \sigma \in K \), \( \Delta^*(X)_p(\sigma) \) is therefore the submodule spanned by all \( S^* \) for which \( S \in \Delta_p(X;R) \) is an oriented simplex and \( \sigma = \pi(S) \in K \).

\[
\]
A map \( f : X \to Y \) of \( K \)-spaces induces an \((R, K)\) chain map \( f^* : \Delta^* Y \to \Delta^* X \) and an \((R, K^{op})\) chain map \( f_* : \Delta Y \to \Delta X \).

The next lemma will be used in section 5.

**Lemma 3.3.** Suppose \( S \in K \) and there is no \( \tau \in K \) for which \( S < \tau \). The \( K \)-space \((\mathcal{S}, \text{inclusion})\) specifies the \((R, K)\) complex \( \Delta^* \mathcal{S} \). Then \( \Delta^* \mathcal{S} (st(\sigma)) \) is a contractible \( R \)-complex for all \( \sigma \in K \) such that \( \sigma \neq S \). Also \( \Delta^* \mathcal{S} (st(S)) = RS^* \).

**Proof.** It is obvious that \( \Delta^* \mathcal{S} (st(S)) = RS^* \) (after orienting \( S \)) and that \( \Delta^* \mathcal{S} (st(\sigma)) = 0 \) if \( \sigma \) is not a face of \( S \). So we assume \( \sigma < S \). Let \( \tau \) be the complementary face of \( \sigma \) in \( S \). Then the joins, \( \mathcal{S} = \sigma * \tau \) and \( \partial \sigma * \tau \) are contractible simplicial complexes. Note \( st(\sigma) = \mathcal{S} - \partial \sigma * \tau \). Consequently, \( \Delta^* \mathcal{S} (st(\sigma)) = \Delta^* (\sigma * \tau, \partial \sigma * \tau; R) \) is a contractible chain complex. \( \square \)

4. \( C \otimes_K D \) and the Isomorphism \( \text{Hom}_{(R,K)}(D, C^*) \cong (C \otimes_K D)^* \)

For the rest of this paper, \( K \) denotes a finite simplicial complex. Moreover, throughout this section, \( C \) denotes an \((R, K^{op})\) complex and \( D \) denotes an \((R, K)\) complex.

In \( K \), the star of any simplex, \( st(\sigma) \), as well as \( K - st(\sigma) \) are full in \( K \). Moreover the chain complex \( C(K - st(\sigma)) \) is a subcomplex of \( C(K) \) and the resulting inclusion fits into a short exact sequence of chain maps in \( \mathcal{BR} \):

\[
0 \to C(K - st(\sigma)) \xrightarrow{i_{st(\sigma)}} C(K) \xrightarrow{p_{st(\sigma)}} C(st(\sigma)) \to 0
\]

Here \( C(K) \xrightarrow{p_{st(\sigma)}} C(st(\sigma)) \) is defined by: \( p_{st(\sigma)}|_{C_q(st(\sigma))} = 1C_q(st(\sigma)) \); and \( p_{st(\sigma)}|_{C(K - st(\sigma))} = 0 \).

**Definition 4.1.** \((C \otimes_K D, C \otimes_R D, \text{ and } C \otimes_R D \xrightarrow{\pi_{C,D}} C \otimes_K D)\).

Let \( C \) be an \((R, K^{op})\) complex and \( D \) be an \((R, K)\) complex.

1. Let \( C \otimes_R D \) be the \((R, K)\) complex for which:

\[
(C \otimes_R D)(K) = C(K) \otimes_R D(K); \quad (C \otimes_R D)(\rho) = (C(K) \otimes_R D(\rho))_q \quad \forall \rho \in K, q \in \mathbb{Z}
\]

2. Let \( C \otimes_K D \) be the \((R, K)\) complex for which:

\[
(C \otimes_K D)(K) = C(K) \otimes_K D(K); \quad (C \otimes_K D)(\rho) = \sum_{\rho \in K} (C(st(\rho)) \otimes_R D(\rho))_q \quad \forall q \in \mathbb{Z}
\]

\[
(C \otimes_K D)(\rho) = C(st(\rho)) \otimes_R D(\rho) \quad \forall \rho \in K
\]

3. The map \( C \otimes_R D \xrightarrow{\pi_{C,D}} C \otimes_K D \) is an \((R, K)\) chain epimorphism, if we define \( \pi_{C,D}(\sigma, \rho) = 0 \) for \( \sigma \neq \rho \) and

\[
\pi_{C,D}(\rho, \rho) = p_{st(\rho)} \otimes_R 1_{D(\rho)} : C(K) \otimes_R D(\rho) \to C(st(\rho)) \otimes_R D(\rho).
\]

Explicitly, for any \( \rho \leq \tau \) and \( x \otimes_R y \in C_{r}(\tau) \otimes_R D_{q-r}(\rho) \subset (C \otimes_K D)_q(\rho) \), we have

\[
d^{C \otimes_K D}(x \otimes y) = \sum_{\{\sigma|\rho \leq \sigma \leq \tau\}} d^C(\sigma, \tau)x \otimes y + (-1)^r x \otimes d^D(\sigma, \rho)y.
\]

We now show that \((C \otimes_K D)^*\) is a convenient expression for \( \text{Hom}_{(R,K)}(D, C^*) \):
Lemma 4.2. There is a natural isomorphism $\Psi$ of functors, denoted,

\[ \Psi_{C,D} : \text{Hom}_{(R,K)}(D,C^*) \cong (C \otimes_K D)^* \]

for any $(C, D)$ in $\text{Ob}(\mathcal{B}R_K \times \mathcal{B}R_K)$.

Proof. Suppose $f$ is in $\text{Hom}_{(R,K)}(D,C^*)(\sigma)$ for some $\sigma \in K$ and $q \in \mathbb{Z}$. Define an $R$-map, $\Psi(f) : C(st(\sigma)) \otimes D(\sigma)_{-q} \to R$, by the formula:

\[ \Psi(f)(x \otimes y) = (-1)^{|x||y|}f(y)(x) \quad \text{for } x \otimes y \in (C \otimes_K D)_{-q}(\sigma). \]

The same formula yields 0, if $x \otimes y$ is in $(C \otimes_K D)(\tau)_{-q}$ for $\tau \neq \sigma$. One easily sees that this rule (i.e. $f \mapsto \Psi(f)$ gives an isomorphism,

\[ \Psi_{C,D} : \text{Hom}_{(R,K)}(D,C^*) \xrightarrow{\cong} (C \otimes_K D)^* \]

of $(R,K)$ complexes for all such $C, D$. Naturality is straightforward. $\square$

5. Ranicki Duality and the $(R,K)$ Chain Equivalence $e : T^2 \to 1_{\mathcal{B}R_K}$

Definition 5.1. Ranicki Duality is the contravariant functor $\mathcal{B}R_K \xrightarrow{T} \mathcal{B}R_K$

defined for a chain complex $C \in \text{Ob}(\mathcal{B}R_K)$ and an $(R,K)$ chain map, $f : C \to D$ by:

\[ TC = C^* \otimes_K \Delta^* K \quad Tf = f^* \otimes_K 1_{\Delta^* K} \]

$\Delta^* K$ comes from the $K$-space, $(K,1_K)$. After examining [15], p. 75 and p.26, lines -6 to -4 one can see that this is in agreement with the definition indicated there, up to isomorphism and differences in sign conventions. In particular compare our formula for $d^{C \otimes_K D}$ with that on p.26, line -5 of [15].

Corollary 5.2. $T$ is an exact homotopy functor.

Proof. : By Lemma 4.2, $TC = C^* \otimes_K \Delta^* K$ is isomorphic to $\text{Hom}_{(R,K)}(\Delta^* K,C)^*$ (since $\varepsilon_C : C^{**} \cong C$ for all $C$). But $C \mapsto C^*$ and $C \mapsto \text{Hom}(\Delta^* K,C)$ are both exact homotopy functors. The result follows. $\square$

We now want to show that $T^2 C$ and $C$ are $(R,K)$-chain equivalent. See 5.5.

Definition 5.3. (of $E_C : \text{Hom}_{(R,K)}(\Delta^* K,C) \otimes_K \Delta^* K \to C$).

Let $C$ be an $(R,K)$ complex.

Consider the evaluation chain map, $\text{Eval}_{A,B} : \text{Hom}_R(A,B) \otimes_R A \to B$, when $A = \Delta^* K(K)$ and $B = C(K)$. Its restriction to $(\text{Hom}_{(R,K)}(\Delta^* K,C) \otimes_R \Delta^* K)(K)$, denoted $E'_C$, is an $(R,K)$ chain map,

\[ E'_C : \text{Hom}_{(R,K)}(\Delta^* K,C) \otimes_R \Delta^* K \to C \]

(by definition of an $(R,K)$ map). Moreover, for each $\sigma \in K$, $E'_C$ annihilates $\text{Hom}_{(R,K)}(\Delta^* K,C)(K - st(\sigma)) \otimes_R \Delta^* K(\sigma)$. Therefore $E'_C$ descends uniquely to an $(R,K)$ chain map,

\[ E_C : \text{Hom}_{(R,K)}(\Delta^* K,C) \otimes_K \Delta^* K \to C, \quad E_C(f \otimes \sigma^*) = f(\sigma^*). \]

satisfying: $E'_C = E_C \circ \pi_{H,\Delta^* K}$. Here $H = \text{Hom}_{(R,K)}(\Delta^* K,C)$ (see 4.1).

$E$ is obviously natural in $C$.

For each $(R,K)$ complex $C$, set $\Psi_C = \Psi_{C,\Delta^* K}$

In view of Lemma 4.2, we have an $(R,K)$ chain isomorphism:

\[ \Psi_C \otimes 1_{\Delta^* K} : \text{Hom}_{(R,K)}(\Delta^* K,C) \otimes_K \Delta^* K \xrightarrow{\cong} (C^* \otimes_K \Delta^* K)^* \otimes_K \Delta^* K = T^2 C. \]
Definition 5.4. For each \((R, K)\) complex \(C\) define \(e_C : T^2C \to C\) as the unique map for which \(E_C = e_C \circ (Ψ_C \otimes 1_{Δ^*K})\).

Note \(e_C\) is an \((R, K)\) chain epimorphism and \(e\) is a natural transformation.

Theorem 5.5. \(e_C : T^2C \overset{\cong}{\to} C\) is an \((R, K)\) chain equivalence, for each \((R, K)\) complex \(C\).

Proof. By [15] (Proposition 4.7), we need only prove that \(e_C(σ, σ) : T^2C(σ) \to C(σ)\) is an \(R\)-chain equivalence, for all \(σ \in K\).

Case I: Assume there is a simplex \(S ∈ K\) for which: \(C(σ) = 0\quad ∀σ \neq S\).

We need only show \(e_C(S, S)\) is a chain isomorphism, and \(T^2C(σ)\) is contractible for \(σ \neq S\). We compute, for all \(σ \in K\), in view of the restriction on \(C\):

\[
TC(st(σ)) = (C^* \otimes_K Δ^*K)(st(σ)) = (C^* \otimes_R Δ^*S)(st(σ))
\]

So: \(T^2C(σ) \cong C^*(S) \otimes_R Δ^{**}S(st(σ)) \otimes_R Rσ^*

So for \(σ \neq S\). \(T^2C(σ)\) is contractible because \(Δ^{**}S(st(σ))\) is contractible by 3.3.

In addition, after orienting \(S\) we conclude

\[
T^2C(S) = C^{**}(S) \otimes_R Rσ^{**} \otimes_R Rσ^*,
\]

\[
e_C(S, S)(c \otimes S^{**} \otimes S^*) = ±ε_{C}(c) \quad ∀c \in C^{**}(S).
\]

So \(e_C(σ, σ)\) is a chain isomorphism for \(σ = S\) and a chain equivalence for \(σ \neq S\). This completes the proof in Case I.

Case II (the general case): For any \(C \neq 0\) in \(BR_K\) one can choose some \(S ∈ K\) for which \(C(S) \neq 0\), and an exact sequence \(0 \to C' \to C \to C'' \to 0\) for which \(i(S, S) : C'(S) \to C(S)\) is an isomorphism, and \(C''(σ) = 0\) for \(σ \neq S\). For example, choose \(S\) to be of maximum dimension among \(\{σ \in K \mid C(σ) \neq 0\}\).

The argument is by induction on the number \(n\), of \(σ \in K\), for which \(C(σ) \neq 0\).

If \(n = 1\), Case I applies. If \(n > 1\), by induction, \(e_{C''}(σ, σ)\) and \(e_{C'}(σ, σ)\) are \(R\)-chain equivalences. Also the commuting diagram below has exact rows.

\[
\begin{array}{cccccc}
0 & \to & C'(σ) & \to & C(σ) & \to & C''(σ) & \to & 0 \\
\downarrow e_{C''}(σ, σ) & & \downarrow e_C(σ, σ) & & \downarrow e_{C'}(σ, σ) & \\
0 & \to & T^2C'(σ) & \to & T^2C(σ) & \to & T^2C''(σ) & \to & 0 \\
\end{array}
\]

Therefore \(e_C(σ, σ)\) is an \(R\)-chain equivalence for all \(σ\). This completes the proof. □

Note: The first proof of the above theorem appeared in [1].

6. Construction of the Ball Complex \(X_K\)

The purpose of this section is to construct the complex \(X_K\) advertised in the introduction and establish its properties.

Notation 6.1. Let \(X\) be a finite simplicial complex in a euclidean space, with vertex set \(V_X\). Its underlying polyhedron is: \(|X| = ∪\{σ \mid σ ∈ X\}\). For each \(p ≥ 0\), \(X_p\) denotes the set of \(p\)-simplices of \(X\).

If \(|X|\) is pl-homeomorphic to \(I^n\) we say \(|X|\) is a pl \(n\)-ball and write \(∂X\) for the subcomplex for which \(|∂X| = ∂|X|\).
Each \( p \)-simplex \( \sigma \in X \) is the convex hull, \([v_0, v_1, \ldots, v_p]\), of its vertices in \( V_X \). Its barycenter is \( \hat{\sigma} := \frac{1}{p+1} \sum_{i=0}^{p} v_i \in \sigma^o \).

Choose \( b \sigma \in \sigma^o \) for each \( \sigma \in X \). The derived complex \( X' \) is defined as the unique simplicial subdivision of \( X \) for which \( V_X' = \{ b \sigma \mid \sigma \in X \} \). \( X' \) has one \( p \)-simplex, \([b \sigma_0, b \sigma_1 \ldots, b \sigma_p]\), for each decreasing sequence \( \sigma_0 > \cdots > \sigma_p \) of simplices of \( X \). The ordered \( p+1 \) tuple \( (b \sigma_0, b \sigma_1, \ldots, b \sigma_p) \) specifies an oriented \( p \)-simplex denoted \( [b \sigma_0, b \sigma_1 \ldots, b \sigma_p] \in \Delta_p(X'; R) \) which we abbreviate to \( \langle \sigma_0; \sigma_1, \ldots, \sigma_p \rangle \). These form a canonical basis for \( \Delta_p(X'; R) \) (in contrast to \( \Delta_p(X; R) \)).

Because we want to use the McCrory cap product, we follow the orderings of [12] regarding simplices of \( X' \).

**Definition 6.2.** Let \((X, \pi)\) be a \( K \)-space. The derived complexes of \((X, \pi)\) are the simplicial subdivisions \( X' \) of \( X \), and \( K' \) of \( K \) whose vertex sets \( \{ b \sigma \mid \sigma \in K \} \) and \( \{ b S \mid S \in X \} \) are chosen as follows:

If \( \sigma \in K \), \( b \sigma := \hat{\sigma} \in \sigma^o \);

If \( S \in X \) and \( \sigma = \pi(S) \), \( b S := \text{centroid of } (S \cap \pi^{-1}(\hat{\sigma})) \in S^o \).

By construction, \( \pi(V_{X'}) \subset V_{X'} \). So \( \pi \) is also a simplicial map from \( X' \) to \( K' \), because \( \pi \) is linear on each simplex of \( X' \).

\( X' \) provides a second geometric example, \( \Delta X' \), of an \((R, K)\) complex:

We define \( \Delta X' \) by,

(1) \( \Delta X'(K) = \Delta_s(X'; R) \).

(2) For each \( \sigma \in K, p \in \mathbb{Z}, (\Delta X')_p(\sigma) \) is the submodule of \( \Delta_p(X'; R) \) spanned by all \( \langle Q^0, \ldots, Q^p \rangle \) in \( X' \) for which \( \sigma = \pi(Q^p) \).

It is straightforward to see that \( \Delta X' \) is an \((R, K)\) complex.

The dual cone of a simplex \( \sigma \in K \), denoted \( D(\sigma, K) \), was first defined in [14], Section 7. It is a subcomplex of \( K' \) (and a pl ball if \( K \) is a pl-manifold). It gives rise to several "dual" subcomplexes in \( K' \) and \( X' \) which we define now.

**Definition 6.3.** Let \((X, \pi)\) be a \( K \)-space. Suppose \( \sigma, \tau \in K, T \in X \).

(1) \( D(\sigma, K) := \{ (\sigma_0, \sigma_1, \ldots, \sigma_p) \in K' \mid \sigma \leq \sigma_p \} \)

(2) \( D(\sigma, \tau) := \{ (\sigma_0, \sigma_1, \ldots, \sigma_p) \in K' \mid \sigma \leq \sigma_p, \sigma_0 \leq \tau \} \)

(3) \( D_\sigma T := \{ (S_0, S_1, \ldots, S_p) \in X' \mid \sigma \leq \pi(S_p), S_0 \leq T \} \)

(4) \( T_\sigma := |D_\sigma T| \).

Of course, \( D(\sigma, \tau) = \emptyset \) unless \( \sigma \leq \tau \), and \( D_\sigma T = \emptyset \) unless \( \sigma \leq \pi(T) \).

\( D_\sigma T \) is a subcomplex of \( X' \). \( D(\sigma, K) \) and \( D(\sigma, \tau) \) are subcomplexes of \( K' \).

**Lemma 6.4.** Let \((X, \pi)\) be a \( K \)-space. Suppose \( \sigma \in K, T \in X, \) and \( \sigma \leq \pi(T) \).

(1) \( T_\sigma = |D_\sigma T| \) is a pl ball. \( \dim(T_\sigma) = \dim(T) - \dim(\sigma) \).

(2) \( \partial D_\sigma T = \partial^0 D_\sigma T \cup \partial^\sigma D_\sigma T \), (the inner and outer boundaries) where:

\( \partial^\sigma D_\sigma T = \cup\{D_{\sigma T} \mid \sigma < \rho\}; \quad \partial^\sigma D_\sigma T = \{D_{\sigma S} \mid S < T\} \)

(3) Suppose \( \sigma < \pi(T) \). Then \( |\partial^\sigma D_\sigma T| \) and \( |\partial^0 D_\sigma T| \) are pl balls of dimension \( \dim(D_\sigma T) - 1 \), and

\( \partial(\partial^\sigma D_\sigma T) = \partial(\partial^\sigma D_\sigma T) = \partial^\sigma D_\sigma T \cap \partial^\sigma D_\sigma T \).
The induced map $f$ of first derived complexes is then a map of ball complexes, $\mathcal{X}$. Theorem 6.7.

Proof. If for each ball $bT$ of $(3)$: The equation in $(3)$, and the fact that $bT$ is a pl manifold, are proved in [3] (Proposition 5.6 (2), applied to $\pi_T : T \to \tau$ is simplicial. So $T$ is a compact convex polyhedron and therefore a pl ball. Since $|D(\sigma, \tau)| \cap \tau = \emptyset$, this operator $(\pi_T)^{-1}$ preserves codimension:

$$\dim(\tau) - \dim(D(\sigma, \tau)) = \dim(T) - \dim(D(\sigma, T)).$$

Since $\dim(D(\sigma, \tau)) = \dim(\tau) - \dim(\sigma)$, we get: $\dim(D(\sigma, T) = \dim(T) - \dim(\sigma)$. □

Proof. of (2): See [3], Proposition 5.6(2), applied to $\pi_T : T \to \pi(T)$. □

Proof. of (3): The equation in (3), and the fact that $|\partial D_\sigma T|$ and $|\partial^o D_\sigma T|$ are both pl manifolds, are proved in [3] (Proposition 5.6 (3),(4)). To show $|\partial D_\sigma T|$ is a pl ball, it suffices to note that it collapses to the vertex $bT$, and so $|\partial D_\sigma T|$ is a regular neighborhood of $bT$ in $|\partial D_\sigma T|$ (by 3.30 of [13]). Then by 3.13 of[13], $\partial D_\sigma T$ is also a pl ball. □

Definition 6.5. ([13] p.27) A ball complex is a finite collection $Z = \{ B_i \}_{i \in I}$ of pl balls in a euclidean space, such that each point of $|Z| := \bigcup \{ B \mid B \in Z \}$ lies in the interior of precisely one ball of $Z$, and the boundary of each $B \in Z$ is a union of balls of lesser dimension of $Z$. Therefore $\langle |Z|, Z \rangle$ is a regular CW-complex.

Let $Z$ and $Y$ be ball complexes $A$ be a map of ball complexes if for each ball $B$ of $Z$, $f(B)$ is a ball of $Y$.

Definition 6.6. Let $(X, \pi)$ be a $K$-space. We define

$$X_K = \{ \tau \mid \sigma \in K, T \in X, \sigma \leq \pi(T) \}$$

Theorem 6.7. Let $(X, \pi)$ be a $K$-space. Then $X_K$ is a ball complex. Moreover $X'$ is a simplicial subdivision of $X_K$. Also, $X_K$ is a subdivision of $X$.

Let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map of $K$-spaces. The induced map $f' : X' \to Y'$ of first derived complexes is then a map of ball complexes, $f_K : X_K \to Y_K$.

Proof. (The induced map $f'$ means the simplicial map $f' : X' \to Y'$ for which $f'(bS) = b(f(S))$ for each $S \in X$.) By Lemma 6.4 the boundary of each $T_\sigma$ is a union of balls of $X_K$ with smaller dimension and

$$T_\sigma = \bigcap \{ A^o \mid A = \langle S_0, \ldots, S_p \rangle \in D_\sigma T, A \notin \partial D_\sigma T, A \notin \partial^o D_\sigma T \}. $$

This can be rewritten as:

$$T_\sigma = \bigcap \{ A^o \mid A = \langle S_0, \ldots, S_p \rangle \in X', \sigma = \pi(S_p), T = S_0 \}. $$

By equation (6.1), for each $A \in X'$ there is a unique $T_\sigma \in X_K$ for which $A^o \subset T_\sigma$. Therefore:

$$|X'| = \bigcap \{ T_\sigma \mid T_\sigma \in X_K \} = |X_K|.$$  

This proves that $X_K$ is a ball complex and that $X'$ is a subdivision of $X_K$. Because $T_\sigma \subset T$, we see $X_K$ is a subdivision of $X$.

Now let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map of $K$-spaces. For each simplex $S \in X$ we see $f(S) \in Y$ because $f$ is simplicial. For each face $\sigma$ of $\pi_X(S)$ in $K$, we see
from the definitions that $f'(D_\sigma S) = D_\sigma f(S)$. So $f'$ is a map of ball complexes, $f_K : X_K \to Y_K$.

\[\square\]

7. **The Isomorphism $\Phi_X : T\Delta^* X \cong C(X_K)$**

Our main theorem is:

**Theorem 7.1.** For each $K$-space $(X, \pi)$ the cellular chain complex of $X_K$ with $R$ coefficients, denoted $C(X_K)$, comes with a natural $(R, K)$ complex structure. There is defined (below) an isomorphism of $(R, K)$ chain complexes:

$$\Phi_X : T\Delta^* X \cong C(X_K).$$

For each map $f : (X, \pi_X) \to (Y, \pi_Y)$ of $K$-spaces, the square below commutes.

\[
\begin{array}{ccc}
T(\Delta^* X) & \xrightarrow{T(f')} & T(\Delta^* Y) \\
\Phi_X & & \Phi_Y \\
C(X_K) & \xrightarrow{f_K} & C(Y_K)
\end{array}
\]

**Proof:** Choose a basis $bK$ of oriented cells for $\Delta_*(K; R)$. Choose next, a basis $bX$ of oriented cells for $\Delta_*(X; R)$. But choose the orientations in $bX$ so that if $T \in bX$ and $\sigma \in bK$ are both $q$-cells, and if $\pi_*(T) = \pm \sigma \in \Delta_q(K; R)$, then:

$$\pi_*(T) = (-1)^{\dim \sigma} \sigma \in \Delta_q(K; R).$$

We call such a pair, $(bK, bX)$ an orientation for $(X, \pi)$.

Our first task is to construct the cellular chain complex $C_*(X_K; R)$ as the underlying $R$-complex of an $(R, K)$ complex $C(X_K)$ Define

$$C(X_K) = \Delta X \otimes_K \Delta^* K; \quad C_*(X_K; R) = (\Delta_*(X; R) \otimes_K \Delta^*(K; R))(K)$$

For each oriented simplex $\rho$ of $K$ and oriented simplex $T$ of $X$, define

$$[T_\rho] = T \otimes_K \rho^* \in C_{|T| - |\sigma|}(X_K; R)$$

Define $bX_K = \{[T_\rho] \mid T \in bX, \rho \in bK, T_\rho \in X_K\}$. Then $bX_K$ is an $R$-basis for $C_*(X_K; R)$ in bicomplex correspondence with the cells of $X_K$. Write $\partial_q$ for the boundary map in $C_*(X_K; R)$, namely $(d_{\Delta X \otimes_K \Delta^* K})_q$.

But to justify these definitions, we must check that $C_*(X_K; R)$ does compute the cellular homology of $X_K$. It suffices to check, for any $[T_\rho] \in bX_K$, that $\partial_q([T_\rho])$ is a sum with $\pm 1$ coefficients of those $[S_\sigma] \in bX_K$ which are $(q - 1)$-faces of $T_\rho$. (See [6], for example.)

All proper faces of $T_\rho$ have the form $T_\sigma$, for $\rho < \sigma$, or $S_\rho$, for $S < T$.

Suppose $[T_\rho] \in bX_K$. So $T \in bX, \rho \in bK$. Set $\tau = \pi(T) \in K$. By (4.1):

$$\partial_q[T_\rho] = d_{\Delta X \otimes_K \Delta^* K}(T \otimes_K \rho^*) = \sum_{\{\sigma | \rho \leq \sigma \leq \tau\}} \{(d_{\Delta X}(\sigma, \tau)T) \otimes \rho^* + (-1)^{|T|} T \otimes d_{\Delta^* K}(\sigma, \rho)\rho^*\}$$

$$= \sum_{S < T} [T, S][S_\rho] + (-1)^{1 + |T_\rho|} \sum_{\rho < \sigma}[\sigma, \rho][T_\sigma]$$

which is as required.
This completes the construction of the cellular chain complex of $X_K$, as an $(R, K)$ complex, $C(X_K)$.

The $(R, K)$ isomorphism, $\Phi_X : T\Delta^* X \cong C(X_K)$ is simply:

$$\Phi_X := (\varepsilon_{\Delta X} \otimes_K 1_{\Delta^* K}) : T\Delta^* X = \Delta^* X \otimes_K \Delta^* K \to \Delta X \otimes_K \Delta^* K = C(X_K).$$

Naturality of $\Phi$ is obvious from the naturality of $\varepsilon$.

\[\square\]

8. The McCrory Cap Product, $\Delta^* X$ and $\Delta X'$.

We now use the work of McCrory [12] to construct, for any $K$-space, $(X, \pi)$, an $(R, K)$ chain monomorphism serving two purposes:

$$C(X_K) \xrightarrow{c_K} \Delta X'.$$

First, it defines an $(R, K)$ chain homotopy equivalence, $T\Delta^* X \simeq \Delta X'$. Second, $C_X$ identifies $C(X_K)$ with that $(R, K)$ subcomplex of $\Delta X'$ which admits a basis consisting of one fundamental $q$-cycle, in $\Delta_q(D_\pi T, \partial D_\pi T) \subset \Delta_q(X')$, for each $q$-cell $T_\pi$ of $X_K$. (This will complete our geometric interpretation of $T$).

Let $K$ be a finite simplicial complex. McCrory (see [12], and also [10]) defines a map, $c' : \Delta_\ast(K; R) \otimes_R \Delta^* (K; R) \to \Delta_\ast(K'; R)$ which he shows is chain homotopic to the composite,

$$\Delta_\ast(K; R) \otimes_R \Delta^* (K; R) \xrightarrow{\cap} \Delta_\ast(K; R) \xrightarrow{Sd} \Delta_\ast(K'; R)$$

where $\cap$ denotes the Whitney-Cech cap product. We will write $c_K$ for $c'$. We repeat his definition here with appropriate sign changes because McCrory’s sign conventions differ slightly from ours.

For any $q$-simplex, $Q = \langle Q^0, Q^1, \ldots, Q^q \rangle$ of $K'$ in which each $Q_i$ is oriented, McCrory then defines

$$\varepsilon(Q) = [Q^0, Q^1][Q^1, Q^2] \ldots [Q^{q-1}, Q^q].$$

This is independent of the orientations on $Q_1, Q_2, \ldots, Q_{q-1}$. If $q = 0$, set $\varepsilon(Q) = 0$.

For any $n$-simplex $\tau$ and $(n-q)$-simplex $\sigma$ of $K$, each simplex $Q = \langle Q^0, Q^1, \ldots, Q^q \rangle$ of $D(\sigma, \tau)_q$ satisfies: $Q_0 = \tau; Q_q = \sigma$. Therefore, $\varepsilon(Q)$ makes sense if $\tau$ and $\sigma$ are oriented simplices chosen from some basis $bK$ of oriented simplices for $\Delta_\ast(K; R)$ (but not if $\sigma = -\tau$).

The McCrory Cap Product, $\Delta_\ast(K; R) \otimes_R \Delta^* (K; R) \xrightarrow{c_K} \Delta_\ast(K'; R)$ is the map defined by:

$$c_K(\tau \otimes \sigma^*) = \sum_{Q \in D(\sigma, \tau)_q} (-1)^{dim(\sigma)} \varepsilon(Q)Q$$

for any oriented simplices $\sigma, \tau$ in some basis $bK$. Here $q = dim(\tau) - dim(\sigma)$. Note this is zero unless $\sigma \leq \tau$. Note that $c_K$ does not change if we change the basis.

$c_K$ is a chain map. We reprove this in the Appendix, Section 9, because of the sign changes and because McCrory’s proof, [12] p.155 lines 7-8, is only a sketch.

Now suppose $(X, \pi)$ is a $K$-space.
Remark 8.2. If we choose an orientation (\(C\)) for each \(q\)-cell \(T\) of \(X\) and \(K\) and \(q = \text{dim}(T) - \text{dim}(\sigma) \neq 0:\)
\[
c_{X}(T \otimes_{R} \pi^{*}\sigma^{*}) = \sum_{Q \in (D_{\sigma}T)_{q}} (-1)^{\text{dim}(\sigma)}\varepsilon(Q)Q \in \Delta_{q}X'(\sigma)
\]
(because \(D_{\sigma}T = \cup\{D(S,T) \mid S \in X, \text{dim}(S) = \text{dim}(\sigma), \pi(S) = \sigma\}\)). This formula still makes sense and is true if \(q = 0\) and \(\pi_{*}(T) \neq -\sigma\).

In this way, \(c_{X} \circ (1 \otimes \pi^{*})\) defines an \((R,K)\) chain map,
\[
c_{X} \circ (1 \otimes \pi^{*}) : \Delta X \otimes_{R} \Delta^{*}K \to \Delta X'
\]

Proposition 8.1. There is a unique \((R,K)\) chain map
\[
C_{X} : C(X_{K}) = \Delta X \otimes_{K} \Delta^{*}K \to \Delta X'
\]
satisfying:
\[
c_{X} \circ (1 \otimes \pi^{*}) = C_{X} \circ \pi_{\Delta X, \Delta^{*}K}
\]
\(C_{X}\) is an \((R,K)\) monomorphism. For all \(q\)-cells \(T_{\sigma}\) of \(X_{K}\), with \(q \neq 0,\)
\[
C_{X}(T \otimes_{K} \sigma^{*}) = \sum_{Q \in (D_{\sigma}T)_{q}} (-1)^{\text{dim}(\sigma)}\varepsilon(Q)Q,
\]
For \(a 0\)-cell \(T_{\sigma}\) of \(X_{K}\), with \(T \in \Delta_{n}(X;R), \sigma \in \Delta_{n}(K;R)\) are oriented so that \(\pi_{*}(T) = \sigma\), then
\[
C_{X}(T \otimes_{K} \sigma^{*}) = (-1)^{\text{dim}(T)} \langle bT \rangle, \quad (bT\text{ is the barycenter of }T).
\]

Proof. Note that \(c_{X}(T \otimes_{R} \pi^{*}\sigma^{*}) = 0\) unless \(\pi(T) \geq \sigma\). Also \(c_{X}(T \otimes_{R} \pi^{*}\sigma^{*}) \in \Delta X'(\sigma)\) for all \(\sigma \in K\) and \(T \in X\) because each \(q\)-cell \(Q \in D_{\sigma}T\) is in \(\Delta_{q}X'(\sigma)\) if \(q = \text{dim}(T_{\sigma})\).

So \(c_{X} \circ (1 \otimes \pi^{*}) : \Delta X \otimes_{R} \Delta^{*}K \to \Delta X'\) is an \((R,K)\) chain map annihilating each \(\Delta X(K - \text{st}(\sigma)) \otimes_{R} \Delta^{*}K(\sigma)\). Hence there is a unique \((R,K)\) chain map monomorphism, \(\Delta X \otimes_{K} \Delta^{*}K \xrightarrow{C_{X}} \Delta X'\) such that \(c_{X} \circ (1 \otimes \pi^{*}) = C_{X} \circ \pi_{\Delta X, \Delta^{*}K}\).

The calculation follows if \(q \neq 0\). If \(q = 0\), then \((\pi_{|T})^{*}\sigma^{*} = T^{*}\), so
\[
C_{X}(T \otimes_{K} \sigma^{*}) = c_{X}(T \otimes T^{*}) = (-1)^{\text{dim}T} \sum_{Q \in (D(T,T))_{0}} Q = (-1)^{\text{dim}T} \langle bT \rangle
\]
Clearly \(C_{X}\) is natural in \((X,\pi)\). \(\Box\)

Remark 8.2. If we choose an orientation \((bK, b_{*}X)\) for \(X_{K}\), then for each \(0\)-cell \(T_{\sigma} = T_{\pi(T)}\) of \(X_{K}\) with \([T_{\sigma}] \in bX_{K}\), we have \(C_{X}([T_{\sigma}]) = \langle bT \rangle\).

Corollary 8.3. For each \(q\)-cell \(T_{\sigma}\) of \(X_{K}\), \(C_{X}(T \otimes \sigma^{*})\) is a fundamental cycle, in \(\Delta_{q}(D_{\sigma}T, \partial D_{\sigma}T)\) for the \(q\)-manifold \(D_{\sigma}T\).

Proof. \(C_{X}(T \otimes \sigma^{*})\) is a fundamental cycle in \(\Delta_{q}(D_{\sigma}T, \partial D_{\sigma}T)\) since \(C_{X}\) is a chain map and since each \(Q \in (D_{\sigma}T)_{q}\) appears with coefficient \(\pm 1\) in \(C_{X}(T \otimes \sigma^{*})\). \(\Box\)

Theorem 8.4. For each \(K\)-space \((X,\pi)\), the map \(C(X_{K}) \xrightarrow{C_{X}} \Delta X'\) is an \((R,K)\) chain homotopy equivalence.
Proof. By 8.3, for all $T_\sigma$, $C_X$ restricts to a homotopy equivalence,

$$C_*(T_\sigma, \partial T_\sigma; R) \to \Delta_*(D_\sigma(T), \partial D_\sigma(T); R)$$

and it takes chains on any subcomplex of $X_K$ to chains on its subdivision. By an induction-division argument on the number of cells in the subcomplex one sees $C_X$ yields a homology equivalence and then a chain homotopy equivalence on each such subcomplex. So $C_X(\sigma, \sigma)$ is an $R$-chain equivalence for each $\sigma$. Therefore $C_X$ is an $(R, K)$ chain equivalence.

Together, 8.4 and 7.1 clearly prove:

**Corollary 8.5.** $T\Delta^*X \xrightarrow{C_X\Phi_X} \Delta X'$ is an $(R, K)$ chain homotopy equivalence. Consequently $e_{\Delta^*X} \circ T(C_X\Phi_X)$ is an explicit $(R, K)$ chain homotopy equivalence, $T\Delta X' \simeq \Delta^*X$.

9. Appendix

We must prove:

**Proposition 9.1.** $\Delta_*(K; R) \otimes_R \Delta^*(K; R) \xrightarrow{c_K} \Delta_*(K'; R)$ is a chain map. That is to say, for any oriented simplices $\sigma, \tau$ in some basis $bK$ for $\Delta K$, with $p = \dim(\tau) - \dim(\sigma)$,

$$d^K c_K(\tau \otimes \sigma^*) = c_K(d^K \tau \otimes \sigma^* + (-1)^{\dim(\tau)} \tau \otimes d^\Delta^*(K) \sigma^*)$$

where, by the definitions,

$$d^K \tau = \sum_{\rho \in bK} [\tau, \rho] \rho, \quad d^\Delta^*(K) \sigma^* = (-1)^{\dim(\sigma)+1} \sum_{\rho \in bK} [\rho, \sigma] \rho^*$$

and for any $p$-simplex $Q = \langle Q^0, Q^1, \ldots Q^p \rangle$ of $K'$,

$$d^K Q = \sum_{i=0}^{p} (-1)^i d^i(Q); \quad d^i(Q) = \langle Q^0, Q^1, \ldots, \hat{Q}^i, \ldots Q^p \rangle$$

Proof. We first prove: $d^0 c(\tau \otimes \sigma^*) = c(d^K \tau \otimes \sigma^*)$, where $c = c_K$.

$$d^0 c(\tau \otimes \sigma) = (-1)^{\dim \sigma} \sum_{Q \in D(\sigma, \tau)_p} \varepsilon(Q) \langle Q^1, \ldots Q^p \rangle = (-1)^{\dim \sigma} \sum_{\rho \in bK} [\tau, \rho] \sum_{P \in D(\sigma, \rho)} \varepsilon(P) P = c(\sum_{\rho \in bK} [\tau, \rho] \rho \otimes \sigma^*) = c(d^K \tau \otimes \sigma^*).$$

Next we show: $(-1)^p d^p c(\tau \otimes \sigma^*) = (-1)^{\dim(\tau)} c(\tau \otimes d^\Delta^*(K) \sigma^*)$:

$$(9.1) \quad (-1)^p d^p c(\tau \otimes \sigma^*) = (-1)^{p+\dim \sigma} \sum_{Q \in D(\sigma, \tau)_p} \varepsilon(Q) \langle \tau, Q^1, \ldots Q^{p-1} \rangle = (-1)^{p+1} c(\tau \otimes \sum_{\rho \in bK} [\rho, \sigma] \rho^*) = (-1)^{\dim(\tau)} c(\tau \otimes d^\Delta^*(K) \sigma^*)$$

Finally we prove $d^i c(\tau \otimes \sigma^*) = 0$ for $0 < i < p$.

For such $i$ and for $Q \in D(\sigma, \tau)$ note $d^i Q = \langle \tau, \ldots, \sigma \rangle \in D(\sigma, \tau) - \partial D(\sigma, \tau)$. So suppose $P$ is a $p - 1$ simplex of the form $d^i Q$ in the $p$ manifold $D(\sigma, \tau)$. Then
there is exactly one other \( S \in D(\sigma, \tau)_p \) having \( Q \) as a face. We can identify \( S \) by listing the vertices of \( \tau \) as \( v_0, \ldots, v_n \) so that \( Q^j = [v_j, \ldots, v_n] \) for all \( j \). Define \( S^i = [v_0, \ldots, v_{i-1}, v_i, \ldots, v_n] \) and define \( S^j = Q^j \) for \( j \neq i \). Then \( S := \langle S^0, S^1, \ldots, S^p \rangle \) in \( D(\sigma, \tau)_p \) satisfies \( d_i S = P; \, \varepsilon(S) = -\varepsilon(Q) \) so \( P \) must appear with zero coefficient in \( d^i c(\tau \otimes \sigma^*) \) for all \( p - 1 \) simplices \( P \). So \( d^i c(\tau \otimes \sigma^*) = 0. \)

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