ALGEBRAIC ELEMENTS OVER THE RING OF POWER SERIES

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Abstract

We give a necessary and sufficient condition for a type of generalized power series to be algebraic over the ring of power series with coefficients in a finite field. This result extend a classical theorem of Huang-Stefănescu.

1. Introduction

This work is concerning to the problem of describing algebraic elements over the ring of power series. It is well known that when \( k \) is an algebraically closed field of characteristic zero, the algebraic closure of the field of power series \( k((t)) \) is the so-called field of Puiseux series. That is, the field formed by the union of all the fields \( k((t^{1/d})) \) where \( d \) is a positive integer. This result is known as the Newton-Puiseux theorem (see for example [2] for a formal presentation). When \( k \) is an algebraically closed field of characteristic \( p > 0 \), Chevalley [3] noted that the polynomial

\[
Z^p - Z - t^{-1} \in k((t))[Z]
\]

has no root in the field of Puiseux series. In fact, the Abhyankar’s paper [1] shows that this polynomial can be factored as follows

\[
Z^p - Z - t^{-1} = \prod_{i=0}^{p-1} (Z - i - \sum_{j=1}^{\infty} t^{-\frac{j}{p^i}}).
\]

Using this factorization Huang [4] considered generalized power series of the form

\[
f(t) = \sum_i a_i t^{-\frac{s_i}{mp^{n_i}}},
\]

where \( m \in \mathbb{Z}_{>0} \), \( n_i \in \mathbb{Z}_{\geq 0} \) and \( s_i \in \mathbb{Z} \) and the support of \( f \) is a well-ordered set. The set of all generalized power series of this type is a field and Huang [4] proved that contains an algebraic closure of the field \( k((t)) \). If \( k \) is a perfect field of positive characteristic, the algebraic closure of \( k((t)) \) consist of the field of the so-called twist-recurrent series, and it is a result of Kedlaya [6] (see also [7]). The twist-recurrent series are generalized power series that hold two technical conditions, one over the exponents of the series and another one over the coefficients (see definition in [6]). As example of this type of series are the series that appear in the following theorem given independently by Huang [4] and Stefănescu [11] (see also [12]).

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Theorem 1.1. (Huang, Stefănescu) The series \( f(t) = \sum_{i=1}^{\infty} a_i t^{-i} \in \mathbb{F}_p((t^{Q,p})) \) is algebraic over \( \mathbb{F}_p((t)) \) if and only if the sequence \( \{a_i\} \) is eventually periodic.

Theorem 1.1 can be also deduced from the main result of [6]. S. Vaidya [13] extend the criterion of Huang-Stefănescu to a certain type of functions, in the case that \( k \) is not equal to algebraic closure of its prime field.

In this paper we present the analogue to Theorem 1.1 for generalized power series in two variables with coefficients in a finite field (Corollary 3.7).

2. Generalized power series

Let \( \Gamma \) be a totally ordered group and let \( k \) be a field. The field of Hahn series \( k((t^\Gamma)) \) is defined to be the collection of all elements of the form

\[
f = \sum_{\alpha \in \Gamma} c_\alpha t^\alpha
\]

with \( c_\alpha \in k \) such that the set of exponents of \( f \) is a well-ordered set. The sum and product are given by

\[
\sum_{\alpha \in \Gamma} c_\alpha t^\alpha + \sum_{\alpha \in \Gamma} d_\alpha t^\alpha = \sum_{\alpha \in \Gamma} (c_\alpha + d_\alpha) t^\alpha
\]

and

\[
(\sum_{\alpha' \in \Gamma} c_{\alpha'} t^{\alpha'}) (\sum_{\alpha'' \in \Gamma} d_{\alpha''} t^{\alpha''}) = \sum_{\alpha \in \Gamma} \sum_{\alpha' + \alpha'' = \alpha} c_{\alpha'} d_{\alpha''} t^\alpha.
\]

The support of a series \( f \) is the set \( \{\alpha \in \Gamma \mid c_\alpha \neq 0\} \). The field \( k((t^\Gamma)) \) is also called the field of **generalized power series** over \( k \) with support in \( \Gamma \). It is known that when \( k \) is algebraically closed and \( \Gamma \) is a divisible group, the field \( k((t^\Gamma)) \) is algebraically closed [5].

Rayner noted that we can take the set of series with support in a proper subfamily of the family of all well-ordered subsets of \( \Gamma \) and still lead a field. He called this subfamily a field-family (see definition in [8]). With the notion of field-family, a family of of algebraically closed fields of series containing the ring of power series in several variables is given in [10, Theorem 5.3].

Let us denote by \( k((t_1^{Q_1}, t_2^{Q_2})) \), the field of generalized power series with coefficients in \( k \) and support a well-ordered subset of \( \mathbb{Q} \times \mathbb{Q} \).

Lemma 2.1. Suppose that \( f_1, ..., f_l \in k((t_1^{Q_1}, t_2^{Q_2})) \) with support in \( (-1, 0] \times (-1, 0] \) and that \( f_1, ..., f_l \) are linearly dependent over \( k((t_1, t_2)) \). Then \( f_1, ..., f_l \) are also linearly dependent over \( k \).

Proof. Since \( f_1, ..., f_l \) are linearly dependent over \( k((t_1, t_2)) \), we can obtain a nonzero linear relation of the form,

\[
\varphi_1 f_1 + \cdots + \varphi_l f_l = 0,
\]

where \( \varphi_k \in k[[t_1, t_2]] \) \( \forall k = 1, ..., l \). We can write \( \varphi_k = \sum_{m,n} \varphi_k(n,m) t_1^m t_2^n \) for some \( \varphi_k(n,m) \in k \). Therefore
\[ 0 = \sum_{n,m} \left( \sum_{k=1}^l \phi_k(n,m) f_k \right) i_1^m i_2^m. \] (2.2)

Now note that the support of \( (\sum_{k=1}^l \phi_k(n,m) f_k) i_1^m i_2^m \) is contained in \((n - 1, n] \times (m - 1, m]\) and thus these supports are disjoint for different \(n\) and \(m\). This implies that the summand in \([22]\) must be zero for each \((n, m)\) and thus \(\sum_{k=1}^l \phi_k(n,m) f_k = 0\) for each \((n, m)\). Note that the \(\phi_k(n,m)\) cannot all be zero because \(\phi_1, ..., \phi_l\) would have all been zero. It follows that \(f_1, ..., f_l\) are linearly dependent over \(k\).

From now on \(\mathbb{F}\) will be a finite field.

Denote \(A_p := \{0, 1, ..., p-1\}\) and for \(c\) a nonnegative integer, let \(T_c\) be the subset given by

\[ T_c := \left\{ -\frac{b_1}{p} - \frac{b_2}{p^2} - \cdots | b_i \in A_p, \sum b_i \leq c \right\}. \]

We recall that a sequence \(a_n\) is eventually periodic if there exist \(s\) and \(m\) such that \(a_{n+s} = a_n\) for all \(n \geq m\).

From the proof of Lemma 2.6 in \([11]\), we can extract the following lemma.

**Lemma 2.2.** Consider a sequence \(\{a_n\} \subset \mathbb{F}\). Suppose that there is \(d\), and \(c_0, ..., c_d\) not all zero, such that \(c_0 a_n + c_1 a_{n+1} + \cdots + c_d a_{n+d} = 0, \forall n \geq k_0\) for some \(k_0 \in \mathbb{Z}_{\geq 0}\). Then \(\{a_n\}\) becomes eventually periodic.

**Proof.** We may suppose that \(c_d \neq 0\). Dividing by the constant \(c_d\) we get

\[ a_{n+d} = -c_d^{-1}(c_0 a_n + c_1 a_{n+1} + \cdots + c_{d-1} a_{n+d-1}) \]

Thus for every \(n \in \mathbb{Z}_{\geq 0}\), we have that \(a_{n+d}\) is completely determined by the \(d\)--tuple \((a_n, a_{n+1}, ..., a_{n+d-1})\). Since \(a_n \in \mathbb{F}\) \(\forall n\) and \(\mathbb{F}\) is finite, we get that the set

\[ \{(a_n, a_{n+1}, ..., a_{n+d-1}) | n \in \mathbb{Z}_{\geq 0}\} \]

is finite. Therefore there are \(r, t \in \mathbb{Z}_{\geq 0}\), \(r \neq t\) such that \((a_r, a_{r+1}, ..., a_{r+d-1}) = (a_t, a_{t+1}, ..., a_{t+d-1})\). This implies that \(a_{r+d} = a_{t+d}\) and then \(a_{r+d} = a_{t+d}\). But this implies that \((a_{r+1}, ..., a_{r+d}) = (a_{t+1}, ..., a_{t+d})\) and then \(a_{r+d+1} = a_{t+d+1}\). In general it follows that \(a_{r+d+k} = a_{t+d+k}\) for every \(k \in \mathbb{Z}_{\geq 0}\). We may suppose that \(r < t\) and let \(s := t - r\). Then \(a_{n+s} = a_{n+t-r} = a_{d+t+n-r-d}\). Therefore if \(n \geq r + d\), we get that \(a_{d+t+n-r-d} = a_{d+t+n-r-d} = a_n\). That is, \(a_{n+s} = a_n\) for every \(n \geq m := \max(r + d, k_0)\).

**Lemma 2.3.** Let \(f = \sum_{i,j} f_{(i,j)} i_1^m i_2^m \in \mathbb{F}(\langle i_1^Q, i_2^Q \rangle)\) be a series with support in \(T_c \times T_c\). Suppose that there exist positive integers \(M, N, R\) such that there are \(d_0, ..., d_{RN-1} \in \mathbb{F}\) not all zero such that every sequence \(\{a_n\}_{n=0}^\infty\) of the form,

\[ a_n = f\left( -\frac{\nu_1}{p_1^{1+n}}, ..., -\frac{\nu_k}{p_k^{1+n}} \right) \]

satisfies

\[ d_0 a_n^2 + d_1 a_{n+1}^2 + \cdots + d_{RN-1} a_{n+RN-1}^2 = 0 \] (2.3)
for all $n \geq M$. If $v$ and $p^{RN-1+M}v$ are in $T_c \times T_c$, then
\[ d_0f^p_{p^{RN-1}v} + d_1f^p_{p^{RN-2}v} + \cdots + d_{RN-1}f^p_{v} = 0. \]

**Proof.** Let say that $v := (-\frac{v_1}{p^1} - \frac{v_2}{p^2} - \cdots - \frac{v_n}{p^n}, -\frac{w_1}{p^1} - \frac{w_2}{p^2} - \cdots - \frac{w_k}{p^k})$, therefore
\[ p^{RN-1+M}v = ((-\frac{v_1}{p^1} - \frac{v_2}{p^2} - \cdots - \frac{v_n}{p^n}, -\frac{w_1}{p^1} - \frac{w_2}{p^2} - \cdots - \frac{w_k}{p^k})) \]
Consider the sequence
\[ a_n = f(-\frac{v_1}{p^1} \cdots - \frac{v_n}{p^n}, -\frac{w_1}{p^1} \cdots - \frac{w_k}{p^k}). \]
We get that
\[ d_0f^p_{p^{RN-1}v} + d_1f^p_{p^{RN-2}v} + \cdots + d_{RN-1}f^p_{v} = d_0a^p_M + d_1a^p_{M+1} + \cdots + d_{RN-1}a^p_{M+RN-1} = 0, \]
by hypothesis.

\[ \square \]

3. Algebraic series

**Lemma 3.1.** Let \( -\frac{v_1}{p^1} - \cdots - \frac{v_n}{p^n} \) be an element of $T_c$. Let
\[ f = \sum_i f(i, -\frac{v_1}{p^1} \cdots - \frac{v_n}{p^n}) t_1^{i_1} t_2^{i_2} \]
be a series with support in $T_c \times T_c$. Then $f$ is algebraic over $\mathbb{F}((t_1, t_2))$, if and only if every sequence of the form $a_n = f(-\frac{v_1}{p^1} \cdots - \frac{v_n}{p^n}, -\frac{w_1}{p^1} \cdots - \frac{w_k}{p^k})$ is eventually periodic.

**Proof.** Suppose that every sequence of the form
\[ a_n = f(-\frac{v_1}{p^1} \cdots - \frac{v_n}{p^n}, -\frac{w_1}{p^1} \cdots - \frac{w_k}{p^k}) \]
is eventually periodic. Is enough to see that $f' := \sum f(i, -\frac{v_1}{p^1} \cdots - \frac{w_k}{p^k}) t_1^l$ is algebraic over $\mathbb{F}((t_1, t_2))$. Note that $f' \in \mathbb{F}((t_1^{\frac{1}{p^n}}))$, so by \[ 6 \] Theorem 15 $f'$ is algebraic over $\mathbb{F}((t_1))$ and then over $\mathbb{F}((t_1, t_2))$.
Now suppose that $f$ is algebraic over $\mathbb{F}((t_1, t_2))$. Then $f'$ is algebraic over $\mathbb{F}((t_1, t_2))$. So we can write,
\[ \varphi_0 f' + \varphi_1 f'^p + \cdots + \varphi_l f'^p_l = 0, \]
for some $l \in \mathbb{Z}_{\geq 0}$ and for some $\varphi_k \in \mathbb{F}[[t_1, t_2]], \forall k = 0, \cdots, l$. Denote
\[ m := \min\{ \alpha_2 | \exists \alpha_1 \text{ with } (\alpha_1, \alpha_2) \in \text{ supp}(\varphi_k) \text{ for some } k \in \{0, \cdots, l\} \}. \]
Multiply by $t_2^{-m}$ both sides of (3.1), we get
\[ \varphi_0 f' + \varphi_1 f'^p + \cdots + \varphi_l f'^p_l = 0, \]
(3.2)
where some of the $\varphi'_k$ have some terms just depending on $t_1$, that is, terms with support of the form $\{(\alpha_1,0)\}$. For $k = 0, \ldots, l$, we can write

$$\varphi'_k = \varphi_k^{(1)} + \varphi_k^{(2)}$$

where $\varphi_k^{(1)}$ just contains the terms of $\varphi_k'$ with support of the form $\{(\alpha_1,0)\}$ and $\varphi_k^{(2)}$ contains the remaining terms. By (3.2), we can write,

$$\varphi'_0 f' + \cdots + \varphi'_1 f^p l = - \varphi_0^{(2)} f' - \cdots - \varphi_1^{(2)} f^p l$$

This equality implies that $\varphi'_0 f' + \varphi'_1 f^p + \cdots + \varphi'_1 f^p l = 0$, because the terms in the left side of (3.3) just depend on $t_1$. This means that $f'$ is algebraic over $\mathbb{F}[t_1]$ then again by [6, Theorem 15], we get that every sequence of the form $a_n$ is eventually periodic.

\[ \square \]

**Lemma 3.2.** Let $-\frac{\alpha_1}{p^1} - \cdots - \frac{\alpha_e}{p^e}$ be an element of $T_c$. Let

$$f = \sum_j f(-\frac{\alpha_1}{p^1}, \ldots, -\frac{\alpha_e}{p^e}, j) t_1^{p^1} \cdots t_2^{p^e}$$

be a series with support in $T_c \times T_c$. Then $f$ is algebraic over $\mathbb{F}((t_1, t_2))$, if and only if every sequence of the form $a_n = f(-\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, \ldots, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e})$ is eventually periodic.

**Proof.** Apply a similar argument as in the proof of Lemma 3.1 \[ \square \]

**Theorem 3.3.** Let $f = \sum_{i,j} f_{i,j} t_1^{i} t_2^{j} \in \mathbb{F}((t_1^0, t_2^0))$ be a series with support in $T_c \times T_c$. Suppose that there exist positive integers $M$ and $N$ such that every sequence $\{a_n\}_{n=0}^{\infty}$ of the form,

$$a_n = f(-\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, \ldots, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e})$$

has period $N$ after $M$ terms and the sequences of the form

$$b_n = f(-\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, \ldots, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e})$$

$$c_n = f(-\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e}, \ldots, -\frac{\alpha_1}{p^1} \cdots - \frac{\alpha_e}{p^e})$$

are eventually periodic. Then $f$ is algebraic over $\mathbb{F}((t_1, t_2))$.

**Proof.** Let $p^r$ be the cardinality of $\mathbb{F}$, where $r \in \mathbb{Z}_{\geq 0}$. There exist $d_0, \ldots, d_{2rN-1} \in \mathbb{F}$ not all zero such that every sequence $a_n$ satisfies

$$d_0 a_n^p + d_1 a_{n+1}^{p^2} + \cdots + d_{2rN-1} a_{n+2rN-1}^{p^{2rN}} = 0$$

for all $n \geq M$.

Indeed, note that
\[
a_n^p + a_{n+1}^p + \cdots + a_{n+rN-1}^p - a_{n+rN+1}^p - \cdots - a_{n+2rN-1}^p = 0. \text{ Thus } d_s = 1 \text{ for } s = 0, \ldots, rN - 1 \text{ and } d_s = -1 \text{ for } s = rN, rN + 1, \ldots, 2rN - 1, \text{ is a solution.}
\]

Consider one of these solutions \((d_0, \ldots, d_{2rN-1})\) and consider the series
\[
g := d_0 f^{1/p_{2rN-1}} + d_1 f^{1/p_{2rN-2}} + \cdots + d_{2rN-1} f.
\]

If \(g = 0, f\) is algebraic. So we can assume that \(g \neq 0\). We are going to show that \(g\) is a finite sum of algebraic series, from which it follows that \(g^{p_{2rN}}\) is algebraic and thus \(f\).

Note that the coefficient \(g_{j'}\) of \(g\) is
\[
g_{j'} = d_0 f^{1/p_{2rN-1}j'} + d_1 f^{1/p_{2rN-2}j'} + \cdots + d_{2rN-1} f_{j'},
\]
where \(j' \in \text{supp}(f)\).

Take
\[
j' = (-\frac{v_1}{p^i_1} - \frac{v_2}{p^i_2} - \cdots - \frac{v_e}{p^i_e}, -\frac{\omega_1}{p^j_1} - \frac{\omega_2}{p^j_2} - \cdots - \frac{\omega_k}{p^j_k}) \in \text{supp}(f).
\]

With out loss of generality suppose that \(e \leq k\). Note that there is \(m_0 \in \mathbb{Z}_{>0}\) such that
\[
\frac{b}{p^{e-2rN+1-M}} < \frac{1}{k} \forall n \geq m_0 \text{ and for any } b \in \{0, 1, 2, \ldots, p - 1\}.
\]

If \(j'\) holds that \(i_1, \ldots, i_e \geq m_0\) and \(j_1, j_2, \ldots, j_k \geq m_0\) then \(p^{e-2rN+1-M}j' \in T_e \times T_e\) because
\[
\frac{v_1}{p^{i_1-2rN+1-M}} + \cdots + \frac{v_e}{p^{i_e-2rN+1-M}} < e \frac{1}{k} \leq 1 \text{ and }
\]
\[
\frac{\omega_1}{p^{j_1-2rN+1-M}} + \cdots + \frac{\omega_k}{p^{j_k-2rN+1-M}} < 1.
\]

Thus by Lemma 2.3 we can conclude that \(g_{j'}^{p_{2rN}} = 0\) and then \(g_{j'} = 0\). We can sort out the remaining terms of \(g\) whose supports use \(\{v_1, ..., v_e\}\) in the first coordinate and \(\{\omega_1, ..., \omega_k\}\) in the second coordinate, in series with support of the following forms

i) Series with support of the form \((-\frac{v_1}{p^i_1} - \cdots - \frac{v_e}{p^i_e}, -\frac{\omega_1}{p^j_1} - \cdots - \frac{\omega_k}{p^j_k})\), where some of the indices \(i_s\) for \(s \in \{1, ..., e\}\) are variable indices (these variable indices are in \(\mathbb{Z}_{>0}\)) and all the remaining indices are constant and these constant indices satisfy that belongs to the set \(\{1, ..., m_0 - 1\}\).

ii) Series with support of the form \((-\frac{v_1}{p^i_1} - \cdots - \frac{v_e}{p^i_e}, -\frac{\omega_1}{p^j_1} - \cdots - \frac{\omega_k}{p^j_k})\), where some of the \(j_t\) for \(t \in \{1, ..., k\}\) are variable indices and all the remaining indices are constant and they satisfy that belongs to the set \(\{1, ..., m_0 - 1\}\).

iii) Series with support of the form \((-\frac{v_1}{p^i_1} - \cdots - \frac{v_e}{p^i_e}, -\frac{\omega_1}{p^j_1} - \cdots - \frac{\omega_k}{p^j_k})\) where some of the indices \(i_s\) for \(s \in \{1, ..., e\}\) are constant and they satisfy that \(i_s \in \{1, ..., m_0 - 1\}\) and where some of the \(j_t\) for \(t \in \{1, ..., k\}\) are constant and they satisfy that \(j_t \in \{1, ..., m_0 - 1\}\) and the remaining indices are variable indices.
Note that there are many finitely series with support as \( i, ii \) and \( iii \). We are going to show that these series are algebraic.

Take one of this series, let say \( f^{(1)} \). Suppose that the series \( f^{(1)} \) has some of the indices \( i_s \) as variable indices and some of the \( j_t \) also as variable indices. Without loss of generality suppose that \( i'_1, \ldots, i'_l \) and \( j'_1, \ldots, j'_m \) are the constant indices. Note that it is enough to show that the following series is algebraic,

\[
\sum g(-\frac{u_1}{p^l}, \ldots, -\frac{u_l}{p^l}, -\frac{u_{l+1}}{p^{l+1}}, \ldots, \frac{u_{l+m}}{p^{l+m+1}}, \ldots, \frac{u_k}{p^k}),
\]

where the sum is running over all \( i_{l+1}, \ldots, i_e, j_{m+1}, \ldots, j_k \in \mathbb{Z}_{>0} \) such that the 2-tuple

\[
(-\frac{u_1}{p^l}, \ldots, -\frac{u_l}{p^l}, \frac{1}{p^l}(-\frac{u_{l+1}}{p^{l+1}}, \ldots, -\frac{u_k}{p^k}), -\frac{1}{p^l}, \ldots, -\frac{1}{p^l}(-\frac{u_{l+m+1}}{p^{l+m+1}}, \ldots, -\frac{u_k}{p^k})) \in \text{supp}(g).
\]

We denote this series again by \( f^{(1)} \). Note that sequences of the form

\[ e_n = g(-\frac{v_1}{p^l}, \ldots, -\frac{v_l}{p^l}, -\frac{v_{l+1}}{p^{l+1}}, \ldots, -\frac{v_k}{p^k}), -\frac{1}{p^l}, \ldots, -\frac{1}{p^l}(-\frac{v_{l+m+1}}{p^{l+m+1}}, \ldots, -\frac{v_k}{p^k}) \]

are eventually periodic because \( e_n \) is a sum of sequences of period \( N \) after \( M \) terms, explicitly we have

\[ e_n = d_{2rN-1}f^{(1)}(-\frac{v_1}{p^l}, \ldots, -\frac{v_l}{p^l}, -\frac{v_{l+1}}{p^{l+1}}, \ldots, -\frac{v_k}{p^k}), -\frac{1}{p^l}, \ldots, -\frac{1}{p^l}(-\frac{v_{l+m+1}}{p^{l+m+1}}, \ldots, -\frac{v_k}{p^k})), \]

\[ d_{2rN-2}f^{(1)}(-\frac{v_1}{p^l}, \ldots, -\frac{v_l}{p^l}, -\frac{1}{p^l}(-\frac{v_{l+1}}{p^{l+1}}, \ldots, -\frac{v_k}{p^k})), -\frac{1}{p^l}, \ldots, -\frac{1}{p^l}(-\frac{v_{l+m+1}}{p^{l+m+1}}, \ldots, -\frac{v_k}{p^k})), \]

\[ \cdots + d_0f^{(1)}(-\frac{v_1}{p^l}, \ldots, -\frac{v_l}{p^l}, -\frac{1}{p^l}(-\frac{v_{l+1}}{p^{l+1}}, \ldots, -\frac{v_k}{p^k})), \]

where \( * \) is \( -\frac{u_1}{p^l}(-\frac{1}{p^l}) - \cdots - \frac{u_m}{p^m}(-\frac{1}{p^l}) - \frac{1}{p^l}(-\frac{1}{p^l}(-\frac{u_{l+m+1}}{p^{l+m+1}} - \cdots - \frac{u_k}{p^k})) \)

At this point, we can repeat all the above argument for \( f^{(1)} \) instead of \( f \) and \( e_n \) instead of \( a_n \). That is, we can find \( d_0^{(1)}, \ldots, d_{2rN-1}^{(1)} \in \mathbb{F} \) not all zero such that

\[ d_0^{(1)}e_n^0 + d_1^{(1)}e_n^{p^2N} + \cdots + d_{2rN-1}^{(1)}e_n^{p^{2rN}N} = 0 \quad (3.5) \]

for all \( n \geq M \). Consider the series

\[ g^{(1)} := d_0^{(1)}f^{(1)}1/p^{2rN-1} + d_1^{(1)}f^{(1)}1/p^{2rN-2} + \cdots + d_{2rN-1}^{(1)}f^{(1)} \]

If \( g^{(1)} = 0 \) then \( f^{(1)} \) is algebraic. So we can assume that \( g^{(1)} \neq 0 \).

We are going to show that \( g^{(1)} \) is algebraic, from this it will follows that \( f^{(1)} \) is algebraic. Reasoning analogously as before we can see now that \( g^{(1)} \) is a finite sum of series such that the support of these series hold that at least one of the following is true: the number of variable indices \( i_s \) is less than \( e - l \) or the number of variable indices \( j_t \) is less than \( k - m \). Take one of this series, let say \( f^{(2)} \).
\[ f^{(2)} = h(t_1, t_2) \sum g_{\left( \frac{v_1}{p_1} \ldots \frac{v_i}{p_i} \right) \frac{v_{i+1}}{p_{i+1}} \ldots \frac{v_k}{p_k}} \frac{t_1^{v_1}}{p_1^{v_1} t_2^{v_{i+1}} p_{i+1}^{v_{i+1}}} \ldots \frac{t_1^{v_k}}{p_k^{v_k}}, \]

where \( a > 1 \) or \( b > 1 \), and the sum is running over all \( i_1, \ldots, i_e, j_{m+b}, \ldots, j_k \in \mathbb{Z}_{>0} \) and

\[ h(t_1, t_2) = t_1^{v_1} \ldots t_2^{v_k} \frac{t_1^{v_{i+1}}}{p_{i+1}^{v_{i+1}}} \ldots \frac{t_1^{v_k}}{p_k^{v_k}}, \]

Reasoning as before we have that sequences of the form

\[ g^\nu = g_{\left( \frac{v_1}{p_1} \ldots \frac{v_i}{p_i} \right) \frac{v_{i+1}}{p_{i+1}} \ldots \frac{v_k}{p_k}} \frac{t_1^{v_1}}{p_1^{v_1} t_2^{v_{i+1}} p_{i+1}^{v_{i+1}}} \ldots \frac{t_1^{v_k}}{p_k^{v_k}}, \]

is algebraic. In general, we get

\[ g^{(n)} := d_0^{(n)} f^{(n)1/p^{2rN-1}} + d_1^{(n)} f^{(n)1/p^{2rN-2}} + \ldots + d_{2rN-1}^{(n)} f^{(n)}, \]

where \( g^{(n)} \) is a finite sum of series and the support of these series holds that at least one of the following is true: the number of variable indices \( i_s \) is less than the number of variable indices of the exponents of \( f^{(n)} \) in the first coordinate or the number of variable indices \( j_t \) is less than the number of variable indices of the exponents of \( f^{(n)} \) in the second coordinate. So eventually, in many finite steps we will get sequences of the following forms

\[ f^{(no)} = h(t_1, t_2) \sum g_{\left( \frac{v_1}{p_1} \ldots \frac{v_i}{p_i} \right) \frac{v_{i+1}}{p_{i+1}} \ldots \frac{v_k}{p_k}} \frac{t_1^{v_1}}{p_1^{v_1} t_2^{v_{i+1}} p_{i+1}^{v_{i+1}}} \ldots \frac{t_1^{v_k}}{p_k^{v_k}}, \]

where the sum is running over all \( i_e, j_k \in \mathbb{Z}_{>0} \), and

\[ h(t_1, t_2) = t_1^{v_1} \ldots t_2^{v_k} \frac{t_1^{v_{i+1}}}{p_{i+1}^{v_{i+1}}} \ldots \frac{t_1^{v_k}}{p_k^{v_k}}, \]

because on each step the number of variable indices is strictly decreasing. So as before we can get a series

\[ g^{(no)} := d_0^{(no)} f^{(no)1/p^{2rN-1}} + d_1^{(no)} f^{(no)1/p^{2rN-2}} + \ldots + d_{2rN-1}^{(no)} f^{(no)}, \]

The series \( g^{(no)} \) is a sum of a finite number of series of the following forms

\[ \sum g_{\left( \frac{v_1}{p_1} \ldots \frac{v_i}{p_i} \right) \frac{v_{i+1}}{p_{i+1}} \ldots \frac{v_k}{p_k}} \frac{t_1^{v_1}}{p_1^{v_1} t_2^{v_{i+1}} p_{i+1}^{v_{i+1}}} \ldots \frac{t_1^{v_k}}{p_k^{v_k}}, \]

where \( i \) is a constant, or

\[ \sum g_{\left( \frac{v_1}{p_1} \ldots \frac{v_i}{p_i} \right) \frac{v_{i+1}}{p_{i+1}} \ldots \frac{v_k}{p_k}} \frac{t_1^{v_1}}{p_1^{v_1} t_2^{v_{i+1}} p_{i+1}^{v_{i+1}}} \ldots \frac{t_1^{v_k}}{p_k^{v_k}}, \]
where \( j \) is a constant. We can apply the hypothesis that sequences as \((b_n)\) and \((c_n)\) are eventually periodic and Lemma 3.1 or Lemma 3.2 to conclude that these series are algebraic over \( F((t_1, t_2)) \). It follows that series with support as i), ii) and iii) are algebraic over \( F((t_1, t_2)) \). Since there are many finitely elections of the \( v_i \)'s and \( \omega_j \)'s such that \( \sum v_i \leq c \) and \( \sum \omega_j \leq c \), it follows that \( g \) is a finite sum of algebraic series. Thus \( f \) is algebraic over \( F((t_1, t_2)) \).

Let \( A \) be a ring. Let \( \Gamma \) be an abelian totally ordered group and let \( \infty \) be an element such that \( x < \infty \) for every \( x \) in \( \Gamma \). Extend the law on \( \Gamma \cup \{\infty\} \) by \( \infty + x = \infty + \infty = \infty \). A map

\[
\nu: A \rightarrow \Gamma \cup \{\infty\}
\]

is said to be a valuation of \( A \) if satisfies the following properties for all \( x, y \in A \):

i) \( \nu(xy) = \nu(x) + \nu(y) \),

ii) \( \nu(x+y) \geq \min(\nu(x), \nu(y)) \).

iii) \( \nu(x) = \infty \) if and only if \( x = 0 \).

Remark 3.4. [14, Remark 1.3] If we have a mapping \( \nu: A \rightarrow \Gamma \cup \{\infty\} \) with conditions i), ii) and with \( \nu(0) = \infty \), but if we don’t assume that \( \nu \) takes the value \( \infty \) only for 0, the set \( \mathcal{P} = \nu^{-1}(\infty) \) is an ideal prime of \( A \) and \( \nu \) induces a valuation on the integral domain \( A/\mathcal{P} \).

**Proposition 3.5.** The second property of a valuation \( \nu \) can be generalized for any set \( \{x_1, \ldots, x_n\} \) in \( A \) by \( \nu(\sum_{i=1}^n x_i) \geq \min(\nu(x_1), \ldots, \nu(x_n)) \). If the minimum is reached by only one of the \( \nu(x_i) \) we get the equality: \( \nu(\sum_{i=1}^n x_i) = \min(\nu(x_1), \ldots, \nu(x_n)) \).

**Proof.** See for example [14, Proposition 1.3] \( \square \)

**Theorem 3.6.** Suppose that \( v_1, \ldots, v_n \), and \( \omega_1, \ldots, \omega_b \) belong to \( \sum_p \). Let

\[
f = \sum f(\frac{v_1}{p_t}, \ldots, \frac{v_n}{p_t}, \frac{\omega_1}{p_t}, \ldots, \frac{\omega_b}{p_t})_1 \frac{w_1}{p_t} \cdots \frac{w_a}{p_t} t_1^{a} \frac{w_i}{p_t} \cdots \frac{w_j}{p_t} t_2^j
\]

be a series with support in \( \sum_{i=1}^n v_i \times \sum_{j=1}^b \omega_j \) and suppose that \( f \) is algebraic over \( F((t_1, t_2)) \). Then sequences of the form,

\[
a_n = f(\frac{v_1}{p_t} \cdots \frac{v_n}{p_t}, \frac{\omega_1}{p_t} \cdots \frac{\omega_b}{p_t}), \quad b_n = f(\frac{v_1}{p_t} \cdots \frac{v_n}{p_t}, \frac{\omega_1}{p_t} \cdots \frac{\omega_b}{p_t}), \quad c_n = f(\frac{v_1}{p_t} \cdots \frac{v_n}{p_t}, \frac{\omega_1}{p_t} \cdots \frac{\omega_b}{p_t})
\]

are eventually periodic.
Proof. Consider a sequence

\[ a_n = f\left(\frac{u_1}{p^1}, \ldots, \frac{u_{r-1}}{p^{r-1}}, \frac{h - \omega_1}{p^t}, \ldots, \frac{h - \omega_r}{p^t}\right) \cdot \frac{u_1}{p^1} - \frac{h - \omega_1}{p^t} \cdot \frac{u_2}{p^2} + \ldots + \frac{h - \omega_r}{p^t} \cdot \frac{u_a}{p^a} \]

Since \( f \) is algebraic over \( \mathbb{F}((t_1, t_2)) \), so is

\[ g := t_1^{\frac{u_1}{p^1} - 1} t_2^{\frac{u_2}{p^2} - 1} \ldots f. \]

Since \( g \) is algebraic over \( \mathbb{F}((t_1, t_2)) \), the extension \( \mathbb{F}((t_1, t_2))(g)/\mathbb{F}((t_1, t_2)) \) is finite. So we can choose any linear dependence among \( g, g^p, g^{p^2}, \ldots, g^{p^n}, \ldots \). Thus there are \( d \in \mathbb{Z}_{>0} \) and \( \varphi_0, \varphi_1, \ldots, \varphi_d \in \mathbb{F}((t_1, t_2)) \) not all zero such that

\[ \varphi_0 g + \varphi_1 g^p + \cdots + \varphi_d g^{p^d} = 0. \]

By clearing denominators, we may suppose that \( \varphi_0, \varphi_1, \ldots, \varphi_d \in \mathbb{F}[[t_1, t_2]]. \)

Now consider the following subseries of \( g \):

\[ \bar{g} := \sum_{n=1}^{\infty} a_n t_1^{\frac{u_1}{p^1} + n} \ldots t_2^{\frac{u_2}{p^2} + n} \ldots . \]

Let \( h := g - \bar{g}. \) We have,

\[ 0 = \varphi_0 g + \varphi_1 g^p + \cdots + \varphi_d g^{p^d} = \varphi_0 \bar{g} + \varphi_1 \bar{g}^p + \cdots + \varphi_d \bar{g}^{p^d} + \varphi_0 h + \varphi_1 h^p + \cdots + \varphi_d h^{p^d}. \]

Let \( \psi := \varphi_0 \bar{g} + \varphi_1 \bar{g}^p + \cdots + \varphi_d \bar{g}^{p^d} \) and \( \psi' := \varphi_0 h + \varphi_1 h^p + \cdots + \varphi_d h^{p^d}. \)

The exponents of the terms from \( \varphi_i \bar{g}^{p^t} \) for \( t \in \{0, 1, \ldots, d\} \) are of the form

\[ (m_1 - \frac{u_1}{p^{l_1+n+t}} - \cdots - \frac{u_a}{p^{l_a+n+t}}), m_2 - \frac{\omega_r}{p^{l_1+n+t}} - \cdots - \frac{\omega_b}{p^{l_b+n+t}}) \] (3.7)

Fix \( t \in \{0, 1, \ldots, d\} \). Note that there is \( k_t \) such that

\[ 0 < \frac{u_1}{p^{l_1+n+t}} + \cdots + \frac{u_a}{p^{l_a+n+t}} < 1 \quad \text{and} \quad 0 < \frac{\omega_r}{p^{l_1+n+t}} + \cdots + \frac{\omega_b}{p^{l_b+n+t}} < 1, \quad \text{for all} \quad n \geq k_t. \]

We are going to show that the terms from \( \varphi_i \bar{g}^{p^t} \) whose exponents satisfy their expressions (3.7) that \( n \geq k_t \), cannot cancel with the terms from \( \varphi_i h^{p^t} \).

Consider a point in the support of \( \varphi_i h^{p^t} \). The first coordinate of this point is of the form

\[ l_1 + \frac{u_1}{p^{l_1-t}} + \cdots + \frac{u_{l_1-1}}{p^{l_1-t}} - \frac{u_1}{p^{l_1-t}} - \cdots - \frac{u_a}{p^{l_a-t}} \]

and the second coordinate is of the form

\[ l_2 + \frac{\omega_1}{p^{l_1-t}} + \cdots + \frac{\omega_{l_1-1}}{p^{l_1-t}} - \frac{\omega_1}{p^{l_1-t}} - \cdots - \frac{\omega_b}{p^{l_b-t}}. \]

We will refer to this point as the point \( A \).

Here the \( i_{1}', \ldots, i_{l_1-1}', i_1, \ldots, i_a \) and \( j_1', \ldots, j_{l_1-1}', j_1, \ldots, j_b \) that appears in (3.5) satisfy that
\[
\left( \frac{u_1}{p_1^{i_1}} + \cdots + \frac{u_{l-1}}{p_1^{i_{l-1}}} - \frac{u_1}{p_1^{i_1}} - \cdots - \frac{v_a}{p_a^{i_a}} \right) + \left( \frac{\omega_1}{p_1^{j_1}} + \cdots + \frac{\omega_{r-1}}{p_1^{j_{r-1}}} - \frac{\omega_1}{p_1^{j_1}} - \cdots - \frac{\omega_h}{p_h^{j_h}} \right) \in \text{supp}(h).
\]

Fix a point as \((3.7)\) with \(n \geq k_i\) (we will refer to this point as the point \(\mathbf{B}\)) and the point \(\mathbf{A}\). Suppose that these points are equal.

i) Suppose that (3.5) holds that \(i_1', \ldots, i_{l-1}', i_1, \ldots, i_a > t\) and \(j_1', \ldots, j_{r-1}', j_1, \ldots, j_b > t\).

- Suppose that at least one of the following is true: not any numbers \(\frac{u_1}{p_1^{i_1}} - \frac{v_1}{p_1^{i_1}} - \cdots - \frac{u_{l-1}}{p_1^{i_{l-1}}} - \frac{v_a}{p_a^{i_a}}\) at all are sumands in \(\frac{u_1}{p_1^{i_1}} + \cdots + \frac{v_a}{p_a^{i_a}}\), or not any numbers \(\frac{\omega_1}{p_1^{j_1}} - \frac{\omega_1}{p_1^{j_1}} - \cdots - \frac{\omega_{r-1}}{p_1^{j_{r-1}}} - \frac{\omega_h}{p_h^{j_h}}\) at all are sumands in \(\frac{\omega_1}{p_1^{j_1}} + \cdots + \frac{\omega_h}{p_h^{j_h}}\).

Without loss of generality suppose that we are in the first case. Note that
\[
-1 < \frac{u_1}{p_1^{i_1}} - \cdots - \frac{u_{l-1}}{p_1^{i_{l-1}}} - \cdots - \frac{v_a}{p_a^{i_a}} < 1
\]
and
\[
-1 < \frac{\omega_1}{p_1^{j_1}} - \cdots - \frac{\omega_{r-1}}{p_1^{j_{r-1}}} - \cdots - \frac{\omega_h}{p_h^{j_h}} < 1.
\]

Clearly if
\[
\frac{u_1}{p_1^{i_1}} + \cdots + \frac{u_{l-1}}{p_1^{i_{l-1}}} - \frac{v_a}{p_a^{i_a}} = 0
\]
or
\[
\frac{\omega_1}{p_1^{j_1}} + \cdots + \frac{\omega_{r-1}}{p_1^{j_{r-1}}} - \frac{\omega_h}{p_h^{j_h}} = 0,
\]
the points \(\mathbf{A}\) and \(\mathbf{B}\) cannot be equal.

If \(\frac{u_1}{p_1^{i_1}} + \cdots + \frac{u_{l-1}}{p_1^{i_{l-1}}} - \frac{v_a}{p_a^{i_a}} < 0\), the equality of the first coordinate of \(\mathbf{A}\) and \(\mathbf{B}\) implies that
\[
\frac{u_1}{p_1^{i_1}} + \cdots + \frac{v_a}{p_a^{i_a}} = \frac{u_1}{p_1^{i_1}} + \cdots + \frac{u_{l-1}}{p_1^{i_{l-1}}} - \frac{v_a}{p_a^{i_a}} = \frac{v_1}{p_1^{i_1}} + \cdots + \frac{v_a}{p_a^{i_a}}.
\]

and then
\[
\frac{v_1}{p_1^{i_1}} + \cdots + \frac{v_a}{p_a^{i_a}} = \frac{v_1}{p_1^{i_1}} + \cdots + \frac{v_{l-1}}{p_1^{i_{l-1}}} + \frac{v_1}{p_1^{i_1}} + \cdots + \frac{v_a}{p_a^{i_a}}.
\]

Thus we obtain a contradiction.

If \(\frac{u_1}{p_1^{i_1}} - \frac{u_{l-1}}{p_1^{i_{l-1}}} - \cdots - \frac{v_a}{p_a^{i_a}} > 0\), we get that
\[
\frac{v_1}{p_1^{i_1}} + \cdots + \frac{v_{l-1}}{p_1^{i_{l-1}}} - \frac{u_1}{p_1^{i_1}} - \cdots - \frac{v_a}{p_a^{i_a}} + \frac{v_1}{p_1^{i_1}} + \cdots + \frac{v_a}{p_a^{i_a}} = 1.
\]
Thus the terms from $\frac{v_1}{p^1_i^{n-t}} + \cdots + \frac{v_{l-1}}{p^{l-1}_i^{n-t}} + \frac{v_l}{p^l_i^{n-t}} + \cdots + \frac{v_{a}}{p^a_i^{n-t}} = 1 + \frac{v_1}{p^1_i^{t}} + \cdots + \frac{v_{a}}{p^a_i^{t}},$ 
a contradiction.

- Suppose that $\frac{v_1}{p^1_i^{t}} + \cdots + \frac{v_{l-1}}{p^{l-1}_i^{t}}$ appear as sumands in $\frac{v_1}{p^1_i^{n-t}} + \cdots + \frac{v_{a}}{p^a_i^{n-t}},$

and $\frac{\omega_r}{p^r_i^{n-t}} + \cdots + \frac{\omega_b}{p^b_i^{n-t}}$ appear as sumands in $\frac{\omega_r}{p^r_i^{t}} + \cdots + \frac{\omega_b}{p^b_i^{t}}$

Then

$$-\frac{v_l}{p^l_i^{n-t}} - \cdots - \frac{v_a}{p^a_i^{n-t}} = -\frac{v_l}{p^l_i^{t}} - \cdots - \frac{v_a}{p^a_i^{t}}$$

and

$$-\frac{\omega_r}{p^r_i^{n-t}} - \cdots - \frac{\omega_b}{p^b_i^{n-t}} = -\frac{\omega_r}{p^r_i^{t}} - \cdots - \frac{\omega_b}{p^b_i^{t}}.$$

Then we get $i_l = i'_l + n, \ldots, i_a = i'_a + n$ and $j_r = j'_r + n, \ldots, j_b = j'_b + n.$

That is, we get a contradiction because the term with exponent

$$(\frac{v_1}{p^1_i} + \cdots + \frac{v_{l-1}}{p^{l-1}_i} - \frac{v_l}{p^l_i} - \cdots - \frac{v_a}{p^a_i}, \frac{\omega_r}{p^r_i}, \ldots + \frac{\omega_{r-1}}{p^{r-1}_i} - \frac{\omega_r}{p^r_i} - \cdots - \frac{\omega_b}{p^b_i})$$

belongs to $h$ and not to $\overline{g}.$

It follows that the terms from $\varphi_i \overline{g}^{p^s}$ whose exponents satisfy in their expressions (3.7) that $n \geq k_t,$ cannot cancel with the terms from $\varphi_i h^{p^s}$ for which the exponents satisfy that $i'_1, \ldots, i'_{l-1}, i_1, \ldots, i_a > t$ and $j'_1, \ldots, j'_{r-1}, j_1, \ldots, j_b > t.$

ii) Now suppose that at least one of the $i'_1, \ldots, i'_{l-1}, i_1, \ldots, i_a$ is less than $t$ and $j'_1, \ldots, j'_{r-1}, j_1, \ldots, j_b > t.$

Without loss of generality suppose that $i'_1, \ldots, i'_k < t, i'_{k+1}, \ldots, i'_{l-1}, i_1, \ldots, i_a > t$ and $j'_1, \ldots, j'_{r-1}, j_1, \ldots, j_b > t.$

Now we can apply a similar argument as in the previous case replacing the set formed by $i'_1, \ldots, i'_{l-1}, i_1, \ldots, i_a$ by the set formed by $i'_{k+1}, \ldots, i'_{l-1}, i_1, \ldots, i_a$ to get a contradiction.

iii) For the case where at least one of the $j'_1, \ldots, j'_{r-1}, j_1, \ldots, j_b$ is less than $t$ and $i'_1, \ldots, i'_{l-1}, i_1, \ldots, i_a$ are greater than $t$ and the case where at least one of the $i'_1, \ldots, i'_{l-1}, i_1, \ldots, i_a$ is less than $t$ and at least one of the $j'_1, \ldots, j'_{r-1}, j_1, \ldots, j_b$ is less than $t$ we can apply a similar argument as in the previous cases to obtain a contradiction.

Thus the terms from $\varphi_i \overline{g}^{p^s}$ whose exponents satisfy in their expressions (3.7) that $n \geq k_t,$ cannot cancel with the terms from $\varphi_i h^{p^s}.$
For any \( t \in \{0, 1, \ldots, d\} \), let \( n_t := t + \max\{k_t' - t' \mid t' \in \{0, \ldots, d\}\} \). Here \( k_t' \) is defined as above and we can suppose that \( k_t' > d \). We define
\[
G_t := \sum_{n=n_t}^{\infty} a_n t_1^{\frac{v_1}{p_1}-\frac{v_t}{p_t'}+m} \cdots t_2^{\frac{v_2}{p_2'}+m} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+m}.
\]
Note that for any \( t \in \{0, 1, \ldots, d\} \), we can write
\[
\varphi_t G^p_t = \varphi_t a_1^p t_1^{\frac{v_1}{p_1'}-\frac{v_t}{p_t'}+1} \cdots t_2^{\frac{v_2}{p_2'}+1} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+1} + \cdots +
\]
\[
+ \varphi_t a_{m_t-1}^p t_1^{\frac{v_1}{p_1'}-\frac{v_{m_t-1}}{p_{m_t-1}'}+1} \cdots t_2^{\frac{v_2}{p_2'}-\frac{v_{m_t-1}}{p_{m_t-1}'}+1} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+1} + \varphi_t G^p_t.
\]
Let us write
\[
\psi'' := \varphi_0 G_0 + \varphi_1 G^p_1 + \cdots + \varphi_d G^p_d.
\]
We are going to see that the terms from \( \psi'' \) cannot cancel with the terms from \( \psi' \).

Note that the terms from \( \varphi_t G^p_t \) are part of the terms of \( \varphi_t G^p_t \) whose exponents hold that \( n \geq k_t \). It follows that the terms from \( \varphi_t G^p_t \) cannot cancel with the terms from \( \varphi_t h^p_t \). Since the exponents of the terms from \( \varphi_t G^p_t \) have the same form for any \( t \in \{0, \ldots, d\} \), that is, they are of the form
\[
(m_1 - m_2 - \cdots - m_{p_2''} - \frac{\omega_1}{p_1'} - \cdots - \frac{\omega_2}{p_2'} - \cdots - \frac{\omega_{p_2''}}{p_{p_2''}'})
\]
where \( m \geq \max\{k_t' - t' \mid t' \in \{0, \ldots, d\}\} \), we obtain that the terms from \( \varphi_t G^p_t \) cannot cancel with the terms from \( \varphi_v h^p_t \) for any \( t' \in \{0, \ldots, d\} \). It follows that the terms from \( \psi'' \) cannot cancel with the terms from \( \psi' \).

We can write
\[
0 = \psi' + \psi'' + \varphi_d a_1^p t_1^{\frac{v_1}{p_1'}-\frac{v_t}{p_t'}+1} \cdots t_2^{\frac{v_2}{p_2'}+1} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+1} + \cdots + [\varphi_d a_{d-1}^p + \cdots + \varphi_2 a_1^p] t_1^{\frac{v_1}{p_1'}-\frac{v_{d-1}}{p_{d-1}'}+1} \cdots t_2^{\frac{v_2}{p_2'}-\frac{v_{d-1}}{p_{d-1}'}+1} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+1} + \cdots + [\varphi_d a_2^p + \cdots + \varphi_2 a_2^p + \varphi_1 a_1^p] t_1^{\frac{v_1}{p_1'}-\frac{v_{d-1}}{p_{d-1}'}+1} \cdots t_2^{\frac{v_2}{p_2'}-\frac{v_{d-1}}{p_{d-1}'}+1} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+1} + \cdots + [\varphi_d a_{d+1}^p + \cdots + \varphi_2 a_3^p + \varphi_1 a_2^p + \varphi_0 a_1^p] t_1^{\frac{v_1}{p_1'}+1} \cdots t_2^{\frac{v_2}{p_2'}+1} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+1} + \cdots + [\varphi_d a_{d-1}^p + \cdots + \varphi_1 a_{n_1-1}^p + \varphi_0 a_{n_0-1}] t_1^{\frac{v_1}{p_1'}+d_0-1} \cdots t_2^{\frac{v_2}{p_2'}+d_0-1} \cdots t_{p_2''}^{\frac{v_{p_2''}}{p_{p_2''}'}+d_0-1},
\]
where \( d_0 := \max\{k_t' - t' \mid t' \in \{0, \ldots, d\}\} \).
We are going to show that terms from $\psi''$ cannot cancel with terms from $-\psi' - \psi'''$:

suppose that

$$\left(m_1 - \frac{v_l}{p_l^{i_l+m}} - \cdots - \frac{v_a}{p_a^{i_a+m}}, m_2 - \frac{\omega_r}{p_r^{i_r+m}} - \cdots - \frac{\omega_b}{p_b^{i_b+m}}\right),$$

where $m \geq d_0$ is equal to

$$\left(l_1 - \frac{v_l}{p_l^{i_l+n}} - \cdots - \frac{v_a}{p_a^{i_a+n}}, l_2 - \frac{\omega_r}{p_r^{i_r+n}} - \cdots - \frac{\omega_b}{p_b^{i_b+n}}\right),$$

where $-(d - 1) \leq n \leq d_0 - 1$. If some of the $i'_l + n, \ldots, i'_a + n$ is $\leq 0$ clearly we get a contradiction. Similarly we get a contradiction, if some of the $j'_r + n, \ldots, j'_b + n$ is $\leq 0$.

In other case we get that $m = n \leq d_0 - 1$, a contradiction. It follows that $\psi'' = 0$. Since $G_0, G_1^p, \ldots, G_d^{p^d}$ have support in $(-1, 0] \times (-1, 0]$, by Lemma 2.1 there are $c_0, \ldots, c_n \in \mathbb{F}$ not all zero such that

$$c_0 G_0 + c_1 G_1^p + \cdots + c_d G_d^{p^d} = 0.$$
Consider the valuation induced by $\nu_\omega$ (Remark 3.4), we will denote it by $\nu_\omega$. Since $f$ is algebraic over the ring of power series there is a polynomial

$$P(Z) = \varphi_d Z^{p^d} + \varphi_{d-1} Z^{p^{d-1}} + \cdots + \varphi_1 Z + \varphi_0 \in F[[t_1, t_2]][Z]$$

such that $P(f) = 0$. So we can write,

$$\infty = \nu_\omega(0) = \nu_\omega(\varphi_0 + \cdots + \varphi_d f^{p^d}) \geq \min(\nu_\omega(\varphi_0), \cdots, \nu_\omega(\varphi_d f^{p^d})) \quad (3.8)$$

From this inequality it follows that the minimum happen at least twice. Then

$$\sum_{r \in \Lambda} \ln_\omega(\varphi_r)(\ln_\omega(f))^{\nu_r} = 0, \quad (3.9)$$

where $\Lambda$ is the set of indices $r$ such that $\nu_\omega(\varphi_r f^{p^r})$ is the minimum in $\ln_\omega(f)$. Note that,

$$\ln_\omega(f) = \sum_{a_2 = \nu_\omega(f)} a_1 t_1^{a_1} t_2^{a_2}$$

and $\ln_\omega(\varphi_r) = t_2^{\nu_r(\varphi_r)} A_r(t_1)$. By the equality $(3.9)$ we get that $\sum_{a_2 = \nu_\omega(f)} a_1 t_1^{a_1} t_2^{a_2}$ is algebraic over $F((t_1, t_2))$. Since $f - \ln_\omega(f)$ is algebraic over $F((t_1, t_2))$, we can repeat the above argument for $f - \ln_\omega(f)$ and so on and so on to conclude that series of the form

$$\sum_{i} f(i, -\frac{a_1}{p^i}, \cdots, -\frac{a_k}{p^k}) t_1^{p^i} t_2^{p^k}$$

are algebraic over $F((t_1, t_2))$. Now we can apply Lemma 3.1 to conclude that sequences of the form $b_n$ becomes eventually periodic. For sequences of the form $c_n$ we can use a similar argument as above but now with $\omega = (1, 0)$.

By combining Theorem 3.3 and Theorem 3.6 we get the following corollary.

**Corollary 3.7.** Let $F$ be a finite field and $s_1, s_2 \in A_p$. Consider the series

$$f = \sum_{i, j > 0} f_i \left( -\frac{a_1}{p^i}, -\frac{a_2}{p^j} \right) t_1^{p^i} t_2^{p^j} \in F((t_1^0, t_2^0)).$$

Then $f$ is algebraic over $F((t_1, t_2))$ if and only if there exist positive integers $M$ and $N$ such that every sequence of the form $a_n = f_i \left( -\frac{a_1}{p^i}, -\frac{a_2}{p^j} \right)$ where $i_0, j_0 \in \mathbb{Z}_{>0}$ has period $N$ after $M$ terms and the sequences of the form $b_n = f_i \left( -\frac{a_1}{p^i}, -\frac{a_2}{p^j} \right)$ and $c_n = f_i \left( -\frac{a_1}{p^i}, -\frac{a_2}{p^j} \right)$ are eventually periodic.

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