PSEUDO-CONFORMAL QUATERNIONIC CR STRUCTURE ON
(4n+3)-DIMENSIONAL MANIFOLDS

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Abstract. We study a geometric structure on a (4n+3)-dimensional smooth manifold \(M\) which is an integrable, nondegenerate codimension 3-subbundle \(\mathcal{D}\) on \(M\) whose fiber supports the structure of 4n-dimensional quaternionic vector space. It is thought of as a generalization of the quaternionic CR structure. In order to study this geometric structure on \(M\), we single out an \(\mathfrak{sp}(1)\)-valued 1-form \(\omega\) locally on a neighborhood \(U\) of \(M\) such that \(\text{Null} \omega = \mathcal{D}|_U\). We shall construct the invariants on the pair \((M, \omega)\) whose vanishing implies that \(M\) is uniformized with respect to a finite dimensional flat quaternionic CR geometry. The invariants obtained on \((4n+3)\)-\(M\) have the same formula as the curvature tensor of quaternionic (indefinite) Kähler manifolds. From this viewpoint, we shall exhibit a quaternionic analogue of Chern-Moser’s CR structure.

Introduction

The Weyl curvature tensor is a conformal invariant of Riemannian manifolds and the Chern-Moser curvature tensor is a CR invariant on strictly pseudo-convex CR-manifolds. A geometric significance of the vanishing of these curvature tensors is the appearance of the finite dimensional Lie group \(G\) with homogeneous space \(X\). The geometry \((G, X)\) is known as conformally flat geometry \((\text{PO}(n+1), S^n)\), spherical CR-geometry \((\text{PU}(n+1), S^{2n+1})\) respectively. Similarly, the complete simply connected quaternionic \((n+1)\)-dimensional quaternionic hyperbolic space \(\mathbb{H}^{n+1}_\mathbb{H}\) with the group of isometries \(\text{PSp}(n+1, 1)\) has the natural compactification homeomorphic to a \((4n+4)\)-ball endowed with an extended smooth action of \(\text{PSp}(n+1, 1)\). When the boundary sphere \(S^{4n+3}\) of the ball is viewed as the real hypersurface in the quaternionic projective space \(\mathbb{H}P^{n+1}\), the elements of \(\text{PSp}(n+1, 1)\) act as quaternionic projective transformations of \(S^{4n+3}\). Since the action of \(\text{PSp}(n+1, 1)\) is transitive on \(S^{4n+3}\), we obtain a flat (spherical) quaternionic CR geometry \((\text{PSp}(n+1, 1), S^{4n+3})\). (Compare [14].) Combined with the above two geometries, this exhibits parabolic geometry on the boundary of the compactification of rank one-symmetric space of noncompact type. (See [3], [11], [21], [15].) This observation naturally leads us to the problem of the existence of a geometric structure on a \((4n+3)\)-dimensional manifold \(M\) and the geometric invariant whose vanishing implies that \(M\) is locally equivalent to the flat quaternionic CR manifold \(S^{4n+3}\). For this purpose we shall introduce a notion of pseudo-conformal quaternionic CR structure \((\mathcal{D}, \{\omega_\alpha\}_{\alpha=1,2,3})\) on a \((4n+3)\)-dimensional manifold \(M\). First of all we recall a pseudo-conformal quaternionic structure which is discussed in [3]. Contrary to the nondegenerate CR structure, the almost complex structure on \(\mathcal{D}\) is not assumed to be integrable. However, by the requirement of our structure equations, we can prove the integrability of quaternionic structure in [21].
**Theorem A.** Each almost complex structure $J_\alpha$ of the quaternionic CR structure is integrable on the codimension 1 contact subbundle $\text{Null} \omega_\alpha$ ($\alpha = 1, 2, 3$).

A $(4n+3)$-dimensional complete simply connected quaternionic pseudo-Riemannian space form $\Sigma^{3+4p,4q}_H$ of type $(3+4p,4q)$ with constant curvature 1 has been introduced in $\text{(22)}$ ($p+q = n$). In $\text{(3)}$, we show that this is a model space of nondegenerate quaternionic CR structure. There exists a canonical pseudo-Riemannian metric $g$ associated to the nondegenerate pseudo-conformal quaternionic CR structure. Then we see in $\text{(4)}$ that the integrability of three almost complex structures $\{J_\alpha\}_{\alpha=1,2,3}$ is equivalent with the condition that $(M,g)$ is a pseudo-Sasakian 3-structure. (Compare $\text{(3)}$.) Using the pseudo-Riemannian connection of the pseudo-Sasakian 3-structure, we can define a quaternionic CR curvature tensor (cf. $\text{(5)}$). Based on this curvature tensor, we shall establish a curvature tensor $T$ which is invariant under the equivalence of pseudo-conformal quaternionic CR structures in $\text{(7)}$. The $(4n+3)$-dimensional manifold $S^{3+4p,4q}$ introduced in $\text{(7)}$ is a pseudo-conformal quaternionic CR manifold with all vanishing pseudo-conformal QCR curvature tensor. The model $S^{3+4p,4q}$ contains $\Sigma^{3+4p,4q}_H$ as an open dense subset. In $\text{(1)}$ we shall describe an explicit formula of our tensor $T$ (cf. Theorem $\text{(1)}$).

**Theorem B.** Let $T = (T^i_{jk\ell})$ be the fourth-order curvature tensor on a nondegenerate pseudo-conformal QCR manifold $M$ in dimension $4n+3$ ($n \geq 0$). If $n \geq 2$, then $T = (T^i_{jk\ell}) \in \mathcal{R}_0(\text{Sp}(p,q) \cdot \text{Sp}(1))$ which has the formula:

$$T^i_{jk\ell} = R^i_{jk\ell} - \left\{ (g_{jk} \delta^i_k - g_{jk} \delta^i_j) + \left[ I_j I^i_k - I_j I^i_j - 2I^i_j I^i_k \right] + J^i_j J^i_k - J^i_j J^i_k + 2J^i_j J^i_k + K^i_j K^i_k - K^i_j K^i_k + 2K^i_j K^i_k \right\}.$$ 

When $n = 1$, $T = (W^i_{jk\ell}) \in \mathcal{R}_0(\text{SO}(4))$ which has the same formula as the Weyl conformal curvature tensor. When $n = 0$, there exists the fourth-order curvature tensor $TW$ on $M$ which has the same formula as the Weyl-Schouten tensor.

We shall prove that the vanishing of curvature tensor $T$ on a nondegenerate pseudo-conformal quaternionic CR manifold $M$ of type $(3+4p,4q)$ gives rise to a uniformization with respect to the flat (spherical) pseudo-conformal quaternionic CR geometry in $\text{(3)}$ see Theorem $\text{(1)}$. (Compare $\text{(21)}$ for uniformization in general.)

**Theorem C.**

(i) If $M$ is a $(4n+3)$-dimensional nondegenerate pseudo-conformal quaternionic CR manifold of type $(3+4p,4q)$ ($p+q = n \geq 1$) whose curvature tensor $T$ vanishes, then $M$ is uniformized over $S^{3+4p,4q}$ with respect to the group $\text{PSp}(p+1,q+1)$.

(ii) If $M$ is a 3-dimensional pseudo-conformal quaternionic CR manifold whose curvature tensor $TW$ vanishes, then $M$ is conformally flat (i.e. locally modelled on $S^3$ with respect to the group $\text{PSp}(1,1)$).

In particular, if $p = n, q = 0$, then $S^{3+4n,0} = S^{4n+3}$ is the positive-definite flat (spherical) quaternionic CR and our pseudo-conformal quaternionic CR geometry is the spherical quaternionic CR geometry ($\text{PSp}(n+1,1), S^{4n+3}$) as explained in the beginning of Introduction. By Theorem C, if $M$ is a flat (spherical) pseudo-conformal positive definite quaternionic CR manifold, then there exists a developing map $\text{dev} : \hat{M} \to S^{4n+3} = S^{4n+3,0}$ from the universal covering space $\hat{M}$. It is an immersion preserving the pseudo-conformal quaternionic CR structure such that $\text{dev}^* \omega_0 = \lambda \cdot \hat{\omega} \cdot \hat{\lambda}$ where $\hat{\omega}$ is the lift of $\omega$ to $\hat{M}$. As the global case, positive-definite pseudo-conformal quaternionic
CR manifolds contain the class of 3-Sasakian manifolds. (Refer to [5], [7], [29], [30] for Sasakian structure.) However, we emphasize that the converse is not true. We shall recall two typical classes of compact (spherical) pseudo-conformal quaternionic CR manifolds but not Sasakian 3-manifolds [14]: one is a quaternionic Heisenberg manifold \( \mathcal{M}/\Gamma \). Some finite cover of \( \mathcal{M}/\Gamma \) is a Heisenberg nilmanifold which is a principal 3-torus bundle over the flat quaternionic n-torus \( T^n_{\text{H}} \). (Compare §7.3.) Another manifold is a pseudo-Riemannian standard space form \( \Sigma^{3,4n}_n \), \( \text{H}/\Gamma \) of constant negative curvature of type \((4n, 3)\). It is a compact quotient of the homogeneous space \( \Sigma^{3,4n}_n = \text{Sp}(1, n)/\text{Sp}(n) \). Some finite cover of \( \Sigma^{3,4n}_n/\Gamma \) is a principal \( S^3 \)-bundle over the quaternionic hyperbolic space form \( \text{H}/\Gamma^* \). Those manifolds are not positive-definite compact 3-Sasakian manifolds. (Compare [14], [16] more generally.)

When a geometric structure is either contact structure or complex contact structure, it is known that the first Stiefel-Whitney class or the first Chern class is the obstruction to the existence of global 1-forms representing their structures respectively. As a concluding remark to the pseudo-conformal quaternionic structure but not necessarily pseudo-conformal quaternionic CR structure, we verify that the obstruction relates to the first Pontrjagin class \( p_1(M) \) of a \((4n + 3)\)-dimensional pseudo-conformal quaternionic manifold \( M \) \((n \geq 1)\). In §11 we prove that the following relation of the first Pontrjagin classes. (See Theorem 11.3.)

**Theorem D.** Let \((M, D)\) be a \((4n + 3)\)-dimensional pseudo-conformal quaternionic manifold. Then the first Pontrjagin classes of \( M \) and the bundle \( L = TM/D \) has the relation that \( 2p_1(M) = (n + 2)p_1(L) \). Moreover, if \( M \) is simply connected, then the following are equivalent.

1. \( 2p_1(M) = 0 \). In particular, the first rational Pontrjagin class vanishes.
2. There exists a global \( \text{Im}\mathbb{H} \)-valued 1-form \( \omega \) on \( M \) which represents a pseudo-conformal quaternionic structure \( D \). In particular, there exists a hypercomplex structure \( \{I, J, K\} \) on \( D \).

1. **Pseudo-conformal quaternionic CR structure**

When \( \mathbb{H} \) denotes the field of quaternions, the Lie algebra \( \text{sp}(1) \) of \( \text{Sp}(1) \) is identified with \( \text{Im}\mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \). Let \( M \) be a \((4n + 3)\)-dimensional smooth manifold \( M \). A \( 4n \)-dimensional orientable subbundle \( D \) equipped with a quaternionic structure \( Q \) is called a pseudo-conformal quaternionic structure on \( M \) if it satisfies that

(i) \( D \cup [D, D] = TM \).

(ii) The 3-dimensional quotient bundle \( TM/D \) at any point is isomorphic to the Lie algebra \( \text{Im}\mathbb{H} \).

(iii) There exists a \( \text{Im}\mathbb{H} \)-valued 1-form \( \omega = \omega_1 i + \omega_2 j + \omega_3 k \)
locally defined on a neighborhood of \( M \) such that

\[
\mathcal{D} = \text{Null} \omega = \bigcap_{\alpha=1}^3 \text{Null} \omega_\alpha \text{ and } d\omega_\alpha|\mathcal{D} \text{ is nondegenerate.}
\]

Here each \( \omega_\alpha \) is a real valued 1-form \((\alpha = 1, 2, 3)\).

(iv) The endomorphism \( J_\gamma = (d\omega_\beta|\mathcal{D})^{-1} \circ (d\omega_\alpha|\mathcal{D}) : \mathcal{D} \to \mathcal{D} \)
constitutes the quaternionic structure \( Q \) on \( D \):

\[
J_\gamma^2 = -1, \ J_\alpha J_\beta = J_\gamma = -J_\beta J_\alpha, \ (\gamma = 1, 2, 3) \text{ etc.}
\]
We note the following from the condition (iv).

**Lemma 1.1.** Put $\sigma_\alpha = (d\omega|D)$ on $\mathcal{D}$. There is the following equality:

$$
\sigma_1(J_1X,Y) = \sigma_2(J_2X,Y) = \sigma_3(J_3X,Y) \quad (\forall X,Y \in \mathcal{D}).
$$

Moreover, the form

$$(1.2) \quad g^D = \sigma_\alpha \circ J_\alpha$$

is a nondegenerate $Q$-invariant symmetric bilinear form on $\mathcal{D}$ \quad ($\alpha = 1, 2, 3$), i.e. $g^D(X,Y) = g^D(J_\alpha X, J_\alpha Y)$, $g^D(X,J_\alpha Y) = \sigma_\alpha(X,Y)$, etc., ($\alpha = 1, 2, 3$).

**Proof.** By (iv), it follows that

$$(1.3) \quad \sigma_\alpha(J_\alpha X,Y) = \sigma_\alpha(J_\beta J_\alpha X), Y) = \sigma_\gamma(J_\alpha X,Y)
= \sigma_\gamma(J_\alpha X,Y) = \sigma_\beta(J_\alpha X,Y).$$

Put $g^D(X,Y) = \sigma_\alpha(J_\alpha X,Y)$ for $X,Y \in \mathcal{D}$ ($\alpha = 1, 2, 3$), which is nondegenerate by (iii). As $-J_\beta = \sigma_\beta^{-1} \circ \sigma_\alpha$ by (iv), calculate that $g^D(Y,X) = -\sigma_\alpha(X,J_\alpha Y) = \sigma_\gamma(J_\beta X, J_\alpha Y) = -\sigma_\beta(Y, J_\beta X) = g^D(X,Y)$. It follows that $g^D(X,Y) = \sigma_\alpha(J_\alpha X,Y) = \sigma_\alpha(J_\alpha (J_\alpha Y), J_\alpha X) = g^D(J_\alpha Y, J_\alpha X).$ \quad $\square$

In general, there is no canonical choice of $\omega$ which annihilates $\mathcal{D}$. The fiber of the quotient bundle $TM/\mathcal{D}$ is isomorphic to $\text{Im } \mathbb{H}$ by $\omega$ on a neighborhood $U$ by (ii). The coordinate change of the fiber $\mathbb{H}$ is described as $v \rightarrow \lambda \cdot v \cdot \mu$ for some nonzero scalars $\mu, \nu \in \mathbb{H}$. If $\omega'$ is another 1-form such that $\text{Null } \omega' = \mathcal{D}$ on a neighborhood $U'$, then it follows that $\omega' = \lambda \cdot \omega \cdot \mu$ for some functions $\lambda, \mu$ locally defined on $U \cap U'$. This can be rewritten as $\omega' = u \cdot a \cdot \omega \cdot b$ where $a, b$ are functions with valued in $\text{Sp}(1)$ and $u$ is a positive function. Since $\bar{\omega} = -\omega'$, it follows that $a \cdot \omega \cdot b = \bar{b} \cdot \omega \cdot \bar{a}$, i.e. $(\bar{b}a) \cdot \omega \cdot (\bar{b}a) = \omega$. As $\omega : T(U \cap U') \rightarrow \text{Im } \mathbb{H}$ is surjective, $ba$ centralizes $\text{Im } \mathbb{H}$ so that $ba \in \mathbb{R}$. Hence, $b = \pm \bar{a}$. As we may assume that $\mathcal{D}$ is orientable, $\omega'$ is uniquely determined by

$$(1.4) \quad \omega' = u \cdot a \cdot \omega \cdot b \text{ for some functions } a \in \text{Sp}(1), u > 0 \text{ on } U \cap U'.$$

We must show that the definition of $\omega'$ does not depend on the choice of any form $\omega'$ of locally conjugate to $\omega$, i.e. it satisfies (iii), (iv). Differentiate the equation (1.1) which yields $d(\omega') = u \cdot a \cdot d\omega \cdot \bar{a}$ on $\mathcal{D}|U \cap U'$. Let $A = (a_{ij}) \in \text{SO}(3)$ be the matrix function determined by

$$(1.5) \quad \text{Ad}_a \left( \begin{array}{c} i \\ j \\ k \end{array} \right) = a \left( \begin{array}{c} i \\ j \\ k \end{array} \right) \bar{a} \quad \text{Ad}_a \left( \begin{array}{c} i \\ j \\ k \end{array} \right) = A \left( \begin{array}{c} i \\ j \\ k \end{array} \right) .$$

When $\omega$ is replaced by $\omega'$, a new quaternionic structure on $\mathcal{D}$ is obtained as

$$(1.6) \quad \left( \begin{array}{c} J_1' \\ J_2' \\ J_3' \end{array} \right) = \bar{A} \left( \begin{array}{c} J_1 \\ J_2 \\ J_3 \end{array} \right) .$$

Note from (1.3) that

$$(1.7) \quad (\omega', \omega', \omega') = (\omega_1, \omega_2, \omega_3) u \cdot A = u (\sum_{\beta=1}^{3} a_{\beta 1} \omega_\beta, \sum_{\beta=1}^{3} a_{\beta 2} \omega_\beta, \sum_{\beta=1}^{3} a_{\beta 3} \omega_\beta) .$$
Differentiate \(1.7\) and restricting to \(D\),
\[
d\omega'_{\alpha}(X,Y) = u \sum_{\beta} a_{\beta\alpha} d\omega_{\beta}(X,Y) \quad \text{(Lemma 1.4)}
\]
\[
= -u(a_{1\alpha}q^D(J_1X,Y) + a_{2\alpha}q^D(J_2X,Y) + a_{3\alpha}q^D(J_3X,Y))
\]
\[
= -ug^D((a_{1\alpha}J_1 + a_{2\alpha}J_2 + a_{3\alpha}J_3)X,Y) = -ug^D(J'_{\alpha}X,Y).
\]
It follows
\[
(1.8) \quad d\omega'_{\alpha}(J'_{\alpha}X,Y) = ug^D(X,Y) \quad (\alpha = 1, 2, 3).
\]
In particular, \(d\omega'_\alpha|D\) is nondegenerate, proving \((iii)\). Put \(\sigma'_\alpha = d\omega'_\alpha|D\). As in \((iv)\), the endomorphism is defined by the rule: \(I'_\alpha = (\sigma'_\alpha|D)^{-1} \circ (\sigma'_\alpha|D)\), i.e. \(\sigma'_\alpha(I'_\alpha X, Y) = \sigma'_\alpha(X,Y) \quad (\forall X,Y \in D)\). Then we show that \(\{I'_\alpha\}_{\alpha=1,2,3}\) coincides with \(\{J'_\alpha\}_{\alpha=1,2,3}\) on \(D\). For this, as \(\sigma'_\alpha(X,Y) = -ug^D(J'_\alpha X,Y)\) by \((1.8)\), it follows that \(\sigma'_\beta(I'_\alpha X,Y) = -ug^D(J'_\beta(I'_\alpha X),Y)\) and the above equality implies that \(J'_\beta(I'_\alpha X) = J'_\alpha X\) \((\forall X \in D)\). Hence, \(I'_\alpha = -J'_\beta J'_\alpha = J'_\gamma\). This proves \((iv)\).

The nondegenerate bilinear form \(g^D\) is locally defined on \(\mathcal{D}U\) of signature \((4p, 4q)\) with \(4p\)-times positive sign and \(4q\)-times negative sign such that \(p + q = n\). As above put \(g^D(X,Y) = d\omega'_{\alpha}(J'_{\alpha}X,Y)\) \((X,Y \in \mathcal{D})\). We have

**Corollary 1.2.** If \(\omega' = u\tilde{\alpha} \cdot \omega \cdot a\) on \(U \cap U'\), then
\[
g^D = u \cdot g^D.
\]
In particular, the signature \((p,q)\) is constant on \(U \cap U'\) \((and \ hence \ everywhere \ on \ M)\) under the change \(\omega' = u\tilde{\alpha} \cdot \omega \cdot a\).

Consider locally the structure equation:
\[
(1.9) \quad \rho_{\alpha} = d\omega_{\alpha} + 2\omega_{\beta} \wedge \omega_{\gamma}
\]
where \((\alpha, \beta, \gamma) \sim (1, 2, 3)\) up to cyclic permutation. If the skew symmetric 2-form \(\rho_{\alpha}\) satisfies that
\[
(1.10) \quad \text{Null } \rho_1 = \text{Null } \rho_2 = \text{Null } \rho_3,
\]
then the pair \((\omega, Q)\) is a local quaternionic CR structure on \(M\). See [5], [1]. If the quaternionic CR structure is globally defined on \(M\), i.e. there exists a \(\text{Im} \mathbb{H}\)-valued 1-form \(\omega\) defined entirely on \(M\), then it coincides with the pseudo-Sasakian 3-structure of \(M\), see [14].

We introduce the following definition due to the manner of Libermann [25].

**Definition 1.3.** The pair \((\mathcal{D}, Q)\) on \(M\) is said to be a pseudo-conformal quaternionic CR structure if there exists locally a 1-form \(\eta\) with \(\text{Null } \eta = \mathcal{D}\) on a neighborhood \(U\) of \(M\) such that \(\eta\) is conjugate to a quaternionic CR structure on \(U\). Namely there exists a \(\text{Im} \mathbb{H}\)-valued 1-form \(\omega\) representing the quaternionic CR structure of \(U\) for which \(\eta = \lambda \cdot \omega \cdot \tilde{\lambda}\) where \(\lambda : U \to \mathbb{H}\) is a function and \(\tilde{\lambda}\) is the conjugate of the quaternion.

2. QUATERNIONIC CR STRUCTURE

Suppose that \(\omega\) is a quaternionic CR structure on a neighborhood of \(M\). Let \(\rho_{\alpha} = d\omega_{\alpha} + 2\omega_{\beta} \wedge \omega_{\gamma}\) be as in \((1.9)\). Put \(V = \text{Null } \rho_{\alpha} \quad (\alpha = 1, 2, 3)\) \((cf. \ (1.10))\). Since \(\text{dim } \mathcal{D} = 4n\), let \(\{v_1, v_2, v_3\}\) be a basis of \(V\). Put \(\omega_1(v_j) = a_{ij}\). As \(\omega_1 \wedge \omega_2 \wedge \omega_3|V \neq 0\), the \(3 \times 3\)-matrix \((a_{ij})\) is nonsingular. Put \(b_{ij} = (a_{ij})^{-1}\) and \(\xi_j = \sum b_{jk}v_k\). Then \(\omega_\alpha(\xi_\beta) = \delta_\alpha\beta\) and locally,
\[
(2.1) \quad V = \{\xi_\alpha, \alpha = 1, 2, 3\}.
\]
Lemma 2.1. Let $\mathcal{L}$ be the Lie derivative. Then, $\mathcal{L}_{\xi_{\alpha}}(\mathcal{D}) = \mathcal{D}$ ($\alpha = 1, 2, 3$).

Proof. For $X \in \mathcal{D}$, $\omega_{\beta}(\mathcal{L}_{\xi_{\alpha}}(X)) = \omega_{\beta}(\{\xi_{\alpha}, X\})$. As

$$0 = \rho_{\beta}(\xi_{\alpha}, X) = d\omega_{\beta}(\xi_{\alpha}, X) + 2\omega_{\gamma} \wedge \omega_{\alpha}(\xi_{\alpha}, X) = \frac{1}{2}(-\omega_{\beta}(\{\xi_{\alpha}, X\})),$$

we have $\omega_{\beta}(\{\xi_{\alpha}, X\}) = 0$ for $\beta = 1, 2, 3$. Hence, $\mathcal{L}_{\xi_{\alpha}}(X) \in \mathcal{D} = \bigcap_{\beta=1}^{3} \text{Null } \omega_{\beta}$. \hfill \square

We prove also that $\mathcal{L}_{\xi} V = V$ for $\xi \in V$.

Lemma 2.2. The distribution $V$ is integrable. The vector fields $\xi_{\alpha}$ determined by (2.1) generates the Lie algebra isomorphic to $\mathfrak{so}(3)$, i.e. $[\xi_{\alpha}, \xi_{\beta}] = 2\xi_{\gamma}$. $(\alpha, \beta, \gamma) \sim (1, 2, 3)$.

Proof. By (2.1), note that

$$(2.2) \quad V = \{\xi \in TM \mid \rho_{\beta}(\xi, v) = \rho_{\beta}(\xi, v) = \rho_{\beta}(\xi, v) = 0, \forall v \in TM\} = \{\xi_{\alpha} : \alpha = 1, 2, 3\}.$$

Since $0 = \rho_{\alpha}(\xi_{\beta}, \xi_{\gamma}) = \frac{1}{2}(-\omega_{\beta}(\{\xi_{\beta}, \xi_{\gamma}\}) + 2)$, it follows that $[\xi_{\beta}, \xi_{\gamma}] - 2\xi_{\alpha} \in \text{Null } \omega_{\alpha}$. Applying

$$\rho_{\beta}(\xi_{\beta}, \xi_{\gamma}) = \frac{1}{2}(-\omega_{\beta}(\{\xi_{\beta}, \xi_{\gamma}\}) + 0) = 0,$$

it yields also that $[\xi_{\beta}, \xi_{\gamma}] - 2\xi_{\alpha} \in \text{Null } \omega_{\beta}$. Similarly as $\rho_{\gamma}(\xi_{\beta}, \xi_{\gamma}) = 0$, we obtain $[\xi_{\beta}, \xi_{\gamma}] - 2\xi_{\alpha} \in \bigcap_{\beta=1}^{3} \text{Null } \omega_{\beta} = \mathcal{D}$ for $\alpha = 1, 2, 3$. As $\rho_{\alpha}([\xi_{\beta}, \xi_{\gamma}] - 2\xi_{\alpha}, v) = \rho_{\alpha}([\xi_{\beta}, \xi_{\gamma}], v)$ for arbitrary $v \in \mathcal{D}$, By the definition of $\rho_{\alpha}$, calculate

$$\rho_{\alpha}([\xi_{\beta}, \xi_{\gamma}], v) = \frac{1}{2}\omega_{\alpha}(\{\xi_{\beta}, \xi_{\gamma}\}, [v]).$$

By Jacobi identity

$$= \frac{1}{2}(\omega_{\beta}(\{\xi_{\gamma}, [v], \xi_{\beta}\}) + \omega_{\beta}(\{\xi_{\beta}, [v], \xi_{\gamma}\})) = 0 \quad \text{(by Lemma 2.1).}$$

Since $\rho_{\alpha}$ is nondegenerate on $\mathcal{D}$ by (iii), $\{\xi_{\beta}, \xi_{\gamma}\} = 2\xi_{\alpha} \quad (\alpha = 1, 2, 3)$. Hence, such a Lie algebra $V$ is locally isomorphic to the Lie algebra of $\text{SO}(3)$. \hfill \square

We collect the properties of $\omega_{\alpha}, \rho_{\alpha}, J_{\alpha}, g^{\mathcal{D}}$. (Compare [3].)

Lemma 2.3. Up to cyclic permutation of $(\alpha, \beta, \gamma) \sim (1, 2, 3)$, the following properties hold.

1. $\mathcal{L}_{\xi_{\alpha}} \omega_{\alpha} = 0$, $\mathcal{L}_{\xi_{\alpha}} \omega_{\beta} = \omega_{\gamma} = -\mathcal{L}_{\xi_{\beta}} \omega_{\alpha}$.
2. $\mathcal{L}_{\xi_{\alpha}} \rho_{\beta} = 0$, $\mathcal{L}_{\xi_{\alpha}} \rho_{\beta} = \rho_{\gamma} = -\mathcal{L}_{\xi_{\beta}} \rho_{\alpha}$.
3. $\mathcal{L}_{\xi_{\alpha}} J_{\beta} = 0$, $\mathcal{L}_{\xi_{\alpha}} J_{\beta} = J_{\gamma} = -\mathcal{L}_{\xi_{\beta}} J_{\alpha}$.
4. $\mathcal{L}_{\xi_{\alpha}} g^{\mathcal{D}} = 0$.

Proof. (1). First note that $\epsilon_{\theta}(\omega_{\alpha}(X)) = \omega_{\alpha}(\{\xi, X\}) = 1 \quad (x \in M)$, $\epsilon_{\theta}(\omega_{\beta} \wedge \omega_{\gamma})(X) = \omega_{\beta} \wedge \omega_{\gamma}(\{\xi, X\}) = 0 \quad (\alpha \neq \beta, \gamma)$, and $\epsilon_{\theta} \rho_{\alpha}(X) = \rho_{\alpha}(\{\xi, X\})$, $\epsilon_{\theta} \rho_{\alpha}(X) = \rho_{\alpha}(\{\xi, X\})$ by (3.3).

$$\mathcal{L}_{\xi_{\alpha}} \omega_{\alpha} = (d\epsilon_{\xi_{\alpha}} + \epsilon_{\xi_{\alpha}} d)\omega_{\alpha} = \epsilon_{\xi_{\alpha}} d\omega_{\alpha} = \epsilon_{\xi_{\alpha}} (-2\omega_{\beta} \wedge \omega_{\gamma} + \rho_{\alpha}) \quad \text{by (1.9)}$$

$$(2.3) \quad = -2\epsilon_{\xi_{\alpha}} (\omega_{\beta} \wedge \omega_{\gamma}) + \epsilon_{\xi_{\alpha}} \rho_{\alpha} = 0,$$

Next,

$$\mathcal{L}_{\xi_{\alpha}} \omega_{\beta} = \epsilon_{\xi_{\alpha}} d\omega_{\beta} = \epsilon_{\xi_{\alpha}} (-2\omega_{\gamma} \wedge \omega_{\alpha} + \rho_{\beta}) = -2\epsilon_{\xi_{\alpha}} (\omega_{\gamma} \wedge \omega_{\alpha}),$$

while $-2\epsilon_{\xi_{\alpha}} (\omega_{\gamma} \wedge \omega_{\alpha})(v) = 0$ for $v \in \text{Null } \omega_{\gamma}$ and $-2\epsilon_{\xi_{\alpha}} (\omega_{\gamma} \wedge \omega_{\alpha})(\xi_{\gamma}) = 1$. Hence $\mathcal{L}_{\xi_{\alpha}} \omega_{\beta} = \omega_{\gamma}.$
Noting that \( \xi_\alpha \) is nondegenerate on \( \xi_\alpha \), recall from Lemma 1.1 that
\[
\sigma_\gamma = -\sigma_\gamma
\]
(2.4)
Similarly,
\[
\mathcal{L}_{\xi_\alpha} \rho_\beta = \mathcal{L}_{\xi_\alpha} (d\omega_\beta + 2\omega_\gamma \wedge \omega_\alpha)
\]
(2.4)
\[
= (d\xi_\alpha + \iota_{\xi_\alpha} d)\omega_\beta + 2\mathcal{L}_{\xi_\alpha} (\omega_\gamma \wedge \omega_\alpha)
\]
\[
= d\xi_\alpha \omega_\beta + 2\mathcal{L}_{\xi_\alpha} \omega_\gamma \wedge \omega_\alpha + 2\omega_\gamma \wedge \mathcal{L}_{\xi_\alpha} \omega_\alpha
\]
(2.4)
\[
= d(L_{\xi_\alpha} - d\xi_\alpha)\omega_\beta + 2L_{\xi_\alpha} \omega_\gamma \wedge \omega_\alpha \quad \text{(by (1))}
\]
(2.4)
\[
= d(L_{\xi_\alpha} \omega_\beta) - 2L_{\xi_\alpha} \omega_\alpha \wedge \omega_\alpha = d\omega_\gamma - 2\omega_\beta \wedge \omega_\alpha
\]
(2.4)
\[
= d\omega_\gamma + 2\omega_\alpha \wedge \omega_\beta = \rho_\gamma.
\]
(2.4)
Similarly,
\[
\mathcal{L}_{\xi_\alpha} \rho_\alpha = \mathcal{L}_{\xi_\alpha} (d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma)
\]
(2.5)
\[
= d\xi_\alpha \omega_\alpha + 2\mathcal{L}_{\xi_\alpha} \omega_\beta \wedge \omega_\gamma + 2\omega_\beta \wedge \mathcal{L}_{\xi_\alpha} \omega_\gamma
\]
(2.5)
\[
= d(L_{\xi_\alpha} - d\xi_\alpha)\omega_\alpha + 2\omega_\gamma \wedge \omega_\alpha + 2\omega_\beta \wedge (-\omega_\beta)
\]
(2.5)
\[
= dL_{\xi_\alpha} \omega_\alpha = 0 \quad \text{(by (1))}
\]
(2.5)
(3). As \( \mathcal{L}_{\xi_\alpha} \rho_\alpha = 0 \) by property (2),
\[
0 = (\mathcal{L}_{\xi_\alpha} \rho_\alpha)(J_\beta X, Y)
\]
(2.5)
\[
= \mathcal{L}_{\xi_\alpha} (\sigma_\alpha(J_\beta X, Y)) - \sigma_\alpha(\mathcal{L}_{\xi_\alpha} (J_\beta X), Y) - \sigma_\alpha(J_\beta X, \mathcal{L}_{\xi_\alpha} Y).
\]
(2.5)
Noting that \( J_\beta = \sigma^{-1}_\alpha \circ \sigma_\gamma \) by Lemma 1.1, we have
\[
\sigma_\alpha((\mathcal{L}_{\xi_\alpha} J_\beta) X, Y) = \sigma_\alpha(\mathcal{L}_{\xi_\alpha} (J_\beta X), Y) - \sigma_\alpha(J_\beta \mathcal{L}_{\xi_\alpha} (X), Y)
\]
(2.6)
\[
= \mathcal{L}_{\xi_\alpha} (\sigma_\alpha(J_\beta X, Y)) - \sigma_\alpha(J_\beta X, \mathcal{L}_{\xi_\alpha} Y) - \sigma_\alpha(J_\beta \mathcal{L}_{\xi_\alpha} X, Y)
\]
(2.6)
\[
= (\mathcal{L}_{\xi_\alpha} \sigma_\gamma)(X, Y) = -\sigma_\beta(X, Y) \quad \text{(by property (2))}
\]
(2.6)
\[
= \sigma_\alpha(J_\gamma X, Y)
\]
(2.6)
As \( \sigma_\alpha \) is nondegenerate on \( D \), \( \mathcal{L}_{\xi_\alpha} J_\beta = J_\gamma \). Similarly,
\[
\sigma_\gamma((\mathcal{L}_{\xi_\alpha} J_\alpha) X, Y) = \sigma_\gamma(\mathcal{L}_{\xi_\alpha} (J_\alpha X), Y) - \sigma_\gamma(J_\alpha \mathcal{L}_{\xi_\alpha} (X), Y)
\]
(2.7)
\[
= -\mathcal{L}_{\xi_\alpha} (\sigma_\gamma(J_\alpha X, Y)) + \mathcal{L}_{\xi_\alpha} (\sigma_\gamma(J_\alpha X, Y))
\]
(2.7)
\[
- \sigma_\gamma(J_\alpha X, \mathcal{L}_{\xi_\alpha} Y) - \sigma_\gamma(J_\alpha \mathcal{L}_{\xi_\alpha} X, Y)
\]
(2.7)
\[
= \sigma_\beta(J_\alpha X, Y) + \mathcal{L}_{\xi_\alpha} (\sigma_\beta(X, Y)) - \sigma_\beta(X, \mathcal{L}_{\xi_\alpha} Y) - \sigma_\beta(\mathcal{L}_{\xi_\alpha} X, Y)
\]
(2.7)
\[
= \sigma_\beta(J_\alpha X, Y) + (\mathcal{L}_{\xi_\alpha} \sigma_\beta)(X, Y)
\]
(2.7)
\[
= -\sigma_\gamma(X, Y) + \sigma_\gamma(X, Y) = 0,
\]
(2.7)
it follows that \( \mathcal{L}_{\xi_\alpha} J_\alpha = 0 \).
(4). Recall from Lemma 1.1 that \( g^D(X, Y) = \sigma_\alpha(J_\alpha X, Y) = \rho_\alpha(J_\alpha X, Y) \) \( X, Y \in D \) for each \( \alpha \).
Then
\[
(L_{\xi_\alpha} g^D)(X, Y) = \xi_\alpha(g^D(X, Y)) - g^D(\mathcal{L}_{\xi_\alpha} X, Y) - g^D(X, \mathcal{L}_{\xi_\alpha} Y)
\]
(2.8)
\[
= \xi_\alpha(\rho_\beta(J_\beta X, Y)) - \rho_\beta(J_\beta \mathcal{L}_{\xi_\alpha} X, Y) - \rho_\beta(J_\beta X, \mathcal{L}_{\xi_\alpha} Y).
\]
(2.8)
On the other hand, \( \mathcal{L}_{\xi_\alpha} \rho_\beta = \rho_\gamma \) by property (2) and so
\[
\xi_\alpha(\rho_\beta(J_\beta X, Y)) = \rho_\beta(L_{\xi_\alpha} J_\beta X, Y) + \rho_\beta(J_\beta X, L_{\xi_\alpha} Y) + \rho_\gamma(J_\beta X, Y).
\]
Substitute this into the equation \(2.3\).

\[
(L_{\xi_\alpha} g^D)(X, Y) = \rho_\beta(L_{\xi_\alpha} J_\beta X, Y) + \rho_\beta(J_\beta X, L_{\xi_\alpha} Y) \\
+ \rho_\gamma(J_\beta X, Y) - \rho_\beta(J_\beta L_{\xi_\alpha} X, Y) - \rho_\beta(J_\beta X, L_{\xi_\alpha} Y) \\
= \rho_\beta((L_{\xi_\alpha} J_\beta) X, Y) + \rho_\gamma(J_\beta X, Y) \quad \text{(by property (3))} \\
= \rho_\beta(J_\gamma X, Y) + \rho_\gamma(J_\beta X, Y) = 0,
\]

hence, \(L_{\xi_\alpha} g^D = 0\). \(\square\)

2.1. **Three CR structures.** Let \(\{\omega_\alpha\}, \{J_\alpha\}, \{\xi_\alpha\}; \quad \alpha = 1, 2, 3\) be a nondegenerate quaternionic CR structure on \(U \subset M\) such that \(D|U = \bigcap_{\alpha=1}^3 \text{Null } \omega_\alpha\). We can extend the almost complex structure \(J_\alpha\) to an almost complex structure \(\overline{J}_\alpha\) on \(\text{Null } \omega_\alpha = D \oplus \{\xi_\beta, \xi_\gamma\}\) by setting:

\[
\begin{align*}
\overline{J}_\alpha|D &= J_\alpha, \\
\overline{J}_\alpha \xi_\beta &= \xi_\gamma, \overline{J}_\alpha \xi_\gamma &= -\xi_\beta.
\end{align*}
\]

\((\alpha, \beta, \gamma)\) is a cyclic permutation of \((1, 2, 3)\). First of all, note the following formula (cf. [19]):

\[
L_X(i_\gamma d\omega_\alpha) = i_{[X,Y]} d\omega_\alpha + i_Y L_X d\omega_\alpha = i_{[X,Y]} d\omega_\alpha + i_Y L_X d\omega_\alpha \quad (\forall X, Y \in TU).
\]

Secondly, we remark the following.

**Lemma 2.4.** For \(X \in D\),

\[
i_X d\omega_a = i_{J_\alpha X} d\omega_b \quad (a, b, c) \sim (1, 2, 3).
\]

**Proof.** Let \(TU = D \oplus V\) where \(V = \{\xi_1, \xi_2, \xi_3\}\). If \(X \in D\), then \(d\omega_a(X, \xi) = 0\) for \(\forall \xi \in V\). As \(d\omega_b(J_\alpha X, \xi) = 0\) similarly, it follows that \(i_X d\omega_a = i_{J_\alpha X} d\omega_b = 0\) on \(V\). If \(Y \in D\), calculate

\[
d\omega_a(X, Y) = -d\omega_a(J_\alpha(J_\alpha X), Y) = -d\omega_b(J_\beta(J_\beta X), Y) \quad \text{(from Lemma 1.1)}
\]

\[
= d\omega_b(J_\beta X, Y), \quad \text{hence } i_X d\omega_a = i_{J_\alpha X} d\omega_b \text{ on } U.
\]

\(\square\)

In particular, we have

\[
i_X d\omega_2 = i_{J_1 X} d\omega_3 \quad \text{for } \forall X \in D.
\]

There is the decomposition with respect to the almost complex structure \(J_1\):

\[
\text{Null } \omega_1 \otimes \mathbb{C} = D \otimes \mathbb{C} \oplus \{\xi_2, \xi_3\} \otimes \mathbb{C}
\]

\[
= T^{1,0} \oplus T^{0,1}
\]

where \(T^{1,0} = D^{1,0} \oplus \{\xi_2 - i\xi_3\}\). We shall observe that the same formula as in Lemma 6.8 of Hitchin [12] can be also obtained for \(D\). (We found Lemma 6.8 when we saw a key lemma to the Kashiwada’s theorem [17].)

**Lemma 2.5.** If \(X, Y \in D^{1,0}\), then

\[
i_{[X,Y]} d\omega_2 = i_{[X,Y]} d\omega_3.
\]
Proof. Let $X \in D^{1,0}$ so that $J_1 X = i X$, then
\begin{equation}
\mathcal{L}_X d\omega_2 = (d_X + i_X d) d\omega_2 = d(i_X d\omega_2) = d(i_{J_1 X} d\omega_3) \tag{2.13}
\end{equation}

Applying $Y \in D^{1,0}$ to the equation (2.11) and using the formula (2.10) (extended to a $\mathbb{C}$-valued one),
\begin{align*}
\mathcal{L}_X (i_Y d\omega_2) &= \mathcal{L}_X (i_{J_1 Y} d\omega_3) = i\mathcal{L}_X (i_Y d\omega_3) \tag{from 2.11} \\
&= i\iota_{[X,Y]} d\omega_3 + i_Y i\mathcal{L}_X d\omega_3 \\
&= i\iota_{[X,Y]} d\omega_3 + i_Y \mathcal{L}_X d\omega_2 \tag{by 2.14}.
\end{align*}

Compared this with the equality (2.10) to $\omega_n = \omega_2$, we obtain $i\iota_{[X,Y]} d\omega_3 = i\iota_{[X,Y]} d\omega_2$.

\hfill $\square$

We prove the following equation (which is used to show the existence of a complex contact structure on the quotient of the quaternionic $CR$ manifold by $S^1$).

**Proposition 2.6.** For any $X, Y \in D^{1,0}$, there exist $a \in \mathbb{R}$ and $u \in D^{1,0}$ such that
\[ [X, Y] = a(\xi_2 - i\xi_3) + u. \]
Conversely, given an arbitrary $a \in \mathbb{R}$, we can choose such $X, Y \in D^{1,0}$ and some $u \in D^{1,0}$.

Proof. As $g(J_{\alpha^*}, J_{\alpha^*}) = g(\cdot, \cdot)$ (cf. Lemma 2.3), we note that $d\omega_1|D^{1,0}$, $d\omega_2|D^{1,0}$, $d\omega_3|D^{1,0}$ are nondegenerate. Given $X, Y \in D^{1,0}$, put $d\omega_2(X, Y) = g(X, J_2 Y) = \frac{-1}{2} a$ for some $a \in \mathbb{R}$. (Note that conversely for any $a \in \mathbb{R}$, we can choose $X, Y \in D^{1,0}$ such that $d\omega_2(X, Y) = g(X, J_2 Y) = \frac{-1}{2} a$.) Then $\omega_2([X, Y]) = a$ so that there is an element $v \in \text{Null} \omega_2 \otimes \mathbb{C}$ such that
\[ [X, Y] - a \cdot \xi_2 = v. \]
As $d\omega_2(X, Y) = g(X, J_1 J_2 Y) = -g(X, J_2 (J_1 Y)) = -ig(X, J_2 Y) = -i^2 a$, it follows that $\omega_3([X, Y]) = -i a$. Since $\omega_3(v) = \omega_3([X, Y] - \xi_2) = \omega_3([X, Y]), v = -i a \cdot \xi_3 + u$ for some $u \in \text{Null} \omega_3 \otimes \mathbb{C}$. Then we have that $[X, Y] = a(\xi_2 - i\xi_3) + u$. Obviously, $\omega_2(u) = 0$. As $X, Y \in D^{1,0}$, $\omega_1(u) = \omega_1([X, Y]) = -2d\omega_1(X, Y) = 0$ for which $u \in D \otimes \mathbb{C}$. We now prove that $u \in D^{1,0}$. First we note that
\[ \iota_{[X,Y]} d\omega_2 = a\iota_{(\xi_2 - i\xi_3)} d\omega_2 + i u d\omega_2. \tag{2.15} \]
As $\xi_2$ (respectively $\xi_3$) is characteristic for $\omega_2$ (respectively $\omega_3$) from Lemma 2.3, $\iota_{\xi_2} d\omega_2 = 0$ (respectively $\iota_{\xi_3} d\omega_3 = 0$). Using (2.3), the function satisfies $d\xi_2 \omega_2 = 0$ (respectively $d\xi_3 \omega_3 = 0$). It follows that $\iota_{\xi_2} d\omega_2 = (\mathcal{L}_{\xi_2} - d\xi_2) \omega_2 = \mathcal{L}_{\xi_2} \omega_2 = -\omega_1$. Then $\iota_{(\xi_2 - i\xi_3)} d\omega_2 = (\iota_{\xi_2} d\omega_2 - i\iota_{\xi_3} d\omega_2) = i\omega_1$ so (2.15) becomes
\[ \iota_{[X,Y]} d\omega_2 = a i \omega_1 + i u d\omega_2. \tag{2.16} \]
As $\mathcal{L}_{\xi_2} \omega_3 = \omega_1$, it follows $\iota_{\xi_2} d\omega_3 = \omega_1$. Similarly
\[ \iota_{[X,Y]} d\omega_3 = a\iota_{(\xi_2 - i\xi_3)} d\omega_3 + i u d\omega_3 = a \omega_1 + i u d\omega_3. \tag{2.17} \]
Substitute (2.16), (2.17) into the equality $\iota_{[X,Y]} d\omega_2 = i\iota_{[X,Y]} d\omega_3$ of Lemma 2.5 which concludes that
\[ i u d\omega_2 = i u d\omega_3. \tag{2.18} \]
Since $d\omega_2(u, X) = d\omega_3(J_1 u, X)$ for any $X \in D \otimes \mathbb{C}$, (2.18) implies that
\[ d\omega_3(J_1 u, X) = i u d\omega_2(X) = d\omega_3(i u, X). \tag{2.19} \]
As \( d\omega_3 \) is nondegenerate on \( \mathcal{D} \) (and so is on \( \mathcal{D} \otimes \mathbb{C} \)), we obtain that \( J_1 u = i u \). Hence, \( u \in \mathcal{D}^{1,0} \).

By definition a CR structure on an odd dimensional manifold consists of the pair \((\text{Null} \omega, J)\) where \( \omega \) is a contact structure and \( J \) is a complex structure on the contact subbundle \( \text{Null} \omega \) (i.e. \( J \) is integrable). In addition, the characteristic vector field \( \xi \) for \( \omega \) is said to be a characteristic CR-vector field if \( \mathcal{L}_\xi J = 0 \). Consider \((\text{Null} \omega_a, J_a)\) on \( U \) \((a = 1, 2, 3)\). By Lemma \( \ref{lemma:structure} \), each \( \xi_a \) is a characteristic vector field for \( \omega_a \) on \( U \). From \( \text{(3)} \) of Lemma \( \ref{lemma:structure} \), \( \mathcal{L}_{\xi_a} J_a = 0 \). It is easy to check that \( \mathcal{L}_{\xi_a} J_a = 0 \).

**Theorem 2.7.** Each \( \bar{J}_a \) is integrable on \( \text{Null} \omega_a \). As a consequence, a nondegenerate quaternionic CR structure \( \{\omega_a, J_a\}_{a=1,2,3} \) on a neighborhood \( U \) of \( M^{4n+3} \) induces three nondegenerate CR structures \( \{\text{Null} \omega_a, \bar{J}_a\} \) equipped with characteristic CR-vector field \( \xi_a \) for each \( \omega_a \) \((a = 1, 2, 3)\). In fact, \( \omega_a(\xi_a) = 1 \) and \( d\omega_a(\xi_a, X) = 0 \) \((\forall X \in TM)\) \((a = 1, 2, 3)\).

**Proof.** Consider the case for \((\text{Null} \omega_1, \bar{J}_1)\). Let \( \text{Null} \omega_1 \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} \) where \( T^{1,0} = \mathcal{D}^{1,0} \oplus \{\xi_2 - i \xi_3\} \). By Proposition \( \ref{proposition:real} \) if \( X, Y \in \mathcal{D}^{1,0} \), there exist elements \( a \in \mathbb{R} \) and \( u \in \mathcal{D}^{1,0} \) such that \( [X, Y] = a(\xi_2 - i \xi_3) + u \). By definition,

\[
\bar{J}_1[X, Y] = a \bar{J}_1(\xi_2 - i \xi_3) + J_1 u = ai(\xi_2 - i \xi_3) + iu = i[X, Y],
\]

it follows that \([X, Y] \in T^{1,0}\). It suffices to show that the element \([\xi_2 - i \xi_3, v] \in T^{1,0}\) for \( v \in \mathcal{D}^{1,0} \). As \( \mathcal{L}_{\xi_2} J_1 = -J_3 \) and \(-J_3 v = (\mathcal{L}_{\xi_3} J_1)v = \mathcal{L}_{\xi_2} (J_1 v) = J_1(\mathcal{L}_{\xi_2} v)\),

\[
J_1(\mathcal{L}_{\xi_2} v) = J_3 v + i \mathcal{L}_{\xi_3} v.
\]

It follows that \( J_1(\mathcal{L}_{\xi_2} v) = -iJ_3 v + i \mathcal{L}_{\xi_3} v \). Using this equality and \( \ref{equation:structure} \), it follows that

\[
\bar{J}_1[\xi_2 - i \xi_3, v] = J_1(\mathcal{L}_{\xi_2} v) - i J_1(\mathcal{L}_{\xi_3} v) = i \mathcal{L}_{\xi_3} v + L_{\xi_3} v \]

Therefore, \([T^{1,0}, T^{1,0}] \subset T^{1,0}\) so that \( \bar{J}_1 \) is a complex structure on \( \text{Null} \omega_1 \), i.e., \((\text{Null} \omega_1, \bar{J}_1)\) is a CR structure on \( U \). The same holds for \((\text{Null} \omega_b, \bar{J}_b)\) \((b = 2, 3)\).

\( \square \)

3. Model of QCR Space Forms with Type \((4p + 3, 4q)\)

Suppose that \( p + q = n \). Let \( \mathbb{H}^{n+1} \) be the quaternionic number space in quaternionic dimension \( n + 1 \) with nondegenerate quaternionic Hermitian form

\[
\langle x, y \rangle = \overline{x_1 y_1} + \cdots + \overline{x_{p+1} y_{p+1}} - \overline{x_{p+2} y_{p+2}} - \cdots - \overline{x_{n+1} y_{n+1}}.
\]

If we denote \( \text{Re}(x, y) \) the real part of \( \langle x, y \rangle \), then it is noted that \( \text{Re}(\cdot, \cdot) \) is a nondegenerate symmetric bilinear form on \( \mathbb{H}^{n+1} \). In the quaternion case, the group of all invertible matrices \( \text{GL}(n+1, \mathbb{H}) \) is acting from the left and \( \mathbb{H}^* = \text{GL}(1, \mathbb{H}) \) acting as the scalar multiplications from the right on \( \mathbb{H}^{n+1} \), which forms the group \( \text{GL}(n+1, \mathbb{H}) \cdot \text{GL}(1, \mathbb{H}) = \text{GL}(n+1, \mathbb{H}) \times \mathbb{R}^* \).

Let \( \text{Sp}(p+1, q) \cdot \text{Sp}(1) \) be the subgroup of \( \text{GL}(n+1, \mathbb{H}) \cdot \text{GL}(1, \mathbb{H}) \) whose elements preserve the nondegenerate bilinear form \( \text{Re}(\cdot, \cdot) \). Denote by \( \Sigma_{\mathbb{H}}^{3p+4q} \) the \((4n+3)\)-dimensional quadric space:

\[
\left\{ (z_1, \ldots, z_{p+1}, w_1, \ldots, w_q) \in \mathbb{H}^{n+1} \right\}
\]

\[
|| (z, w) ||^2 = |z_1|^2 + \cdots + |z_{p+1}|^2 - |w_1|^2 - \cdots - |w_q|^2 = 1.
\]
In particular, the group $\Sp(p+1, q) \cdot \Sp(1)$ leaves $\Sigma_{3+4}^{3+4p,4q}$ invariant. Let $\langle \cdot, \cdot \rangle_x$ be the nondegenerate quaternionic inner product on the tangent space $T_x \mathbb{H}^{n+1}$ obtained from the parallel translation of $\langle \cdot, \cdot \rangle$ to the point $x \in \mathbb{H}^{n+1}$. Recall that $\{I, J, K\}$ is the standard quaternionic structure on $\mathbb{H}^{n+1}$ which operates as $Iz = zI$, $Jz = zJ$, or $Kz = zK$. As usual, $\{I_x, J_x, K_x\}$ acts on $T_x \mathbb{H}^{n+1}$ at each point $x$. Then it is easy to see that $g^{3+4}_X(X, Y)_x = \Re(X, Y)_x$ ($\forall X, Y \in T_x \mathbb{H}^{n+1}$) is the standard pseudo-euclidean metric of type $(p + 1, q)$ on $\mathbb{H}^{n+1}$ which is invariant under $\{I, J, K\}$. Restricted $g^{3+4}_H$ to the quadric $\Sigma^{3+4p,4q}_3$ in $\mathbb{H}^{n+1}$, we obtain a nondegenerate pseudo-Riemannian metric $g$ of type $(3 + 4p, 4q)$ where $p + q = n$.

**Definition 3.1.** ([22]) The quadric $\Sigma^{3+4p,4q}_3$ is referred to the quaternionic pseudo-Riemannian space form of type $(3 + 4p, 4q)$ with constant curvature 1 endowed with a transitive group of isometries $\Sp(p + 1, q) \cdot \Sp(1)$ for which $\Sigma^{3+4p,4q}_3 = \Sp(p + 1, q) \cdot \Sp(1)/\Sp(p, q) \cdot \Sp(1)$ where $\Sp(p, q) \cdot \Sp(1)$ is the stabilizer at $(0, 0, \cdots, 0)$.

Compare [32], [22]. When $(\Sigma^{3+4p,4q}_3, g^{3+4})$ is viewed as a real pseudo-Riemannian space form, the full group of isometries is $O(4p + 4q)$. It is noted that the intersection of $O(4p + 4q)$ with $GL(n + 1, \mathbb{H}) \cdot GL(1, \mathbb{H})$ is $Sp(p + 1, q) \cdot Sp(1)$. When $\Sigma^{3+4p,4q}_3$ is the normal vector at $x \in \Sigma^{3+4p,4q}_3$, $T_x \Sigma^{3+4p,4q}_3 = N_x^{1}$ with respect to $g^{3+4}_H$. If $N$ is a normal vector field on $\Sigma^{3+4p,4q}_3$, then $IN, JN, KN \in T\Sigma^{3+4p,4q}_3$ such that there is the decomposition $T\Sigma^{3+4p,4q}_3 = \{IN, JN, KN\} \perp \{IN, JN, KN\}^{\perp}$. Let $D = \{IN, JN, KN\}^{\perp}$ which is the 4$n$-dimensional subbundle. As $\Sigma^{3+4}_3$ is a $(I, J, K)$-invariant metric, $(D, g|D)$ is also invariant under $\{I, J, K\}$. Now, $\Sp(1)$ acts freely on $\Sigma^{3+4p,4q}_3$ as (right translations):

$$
(\lambda, (z_1, \cdots, z_{p+1}, w_1, \cdots, w_q)) = (\bar{z}_1 \bar{\lambda}, \cdots, \bar{z}_{p+1} \bar{\lambda}, w_1 \cdot \bar{\lambda}, \cdots, w_q \cdot \bar{\lambda}).
$$

**Definition 3.2.** Let $\mathbb{H}^{p,q}$ be the orbit space $\Sigma^{3+4p,4q}_3/\Sp(1)$ which is said to be the quaternionic pseudo-Kähler projective space of type $(4p, 4q)$.

See Definition 4.5 for the definition of quaternionic pseudo-Kähler manifold in general. $\mathbb{H}^{p,q}$ is shown to be a quaternionic pseudo-Kähler manifold in Theorem 4.6 provided that $4n \geq 8$. When $p = n, q = 0$, $\mathbb{H}^{n,0}$ is the standard quaternionic projective space $\mathbb{H}^{n}$. When $p = 0, q = n$, $\mathbb{H}^{0,n}$ is the quaternionic hyperbolic space $\mathbb{H}^{n}_{3}$, and $\mathbb{H}^{p,n}$ is the canonical quaternionic line bundle over the quaternionic Kähler projective space $\mathbb{H}^{p}$. There is the equivariant principal bundle:

$$(3.3) \quad \Sp(1) \rightarrow \Sp(p + 1, q) \cdot \Sp(1), \Sigma^{3+4p,4q}_3 \xrightarrow{\pi} (\Sp(p + 1, q), \mathbb{H}^{p,q})$$

On the other hand, let

$$(3.4) \quad \omega_0 = - (z_1 dz_1 + \cdots + z_{p+1} dz_{p+1} - w_1 dw_1 - \cdots - w_q dw_q).$$

Then it is easy to check that $\omega_0$ is an $\mathfrak{sp}(1)$-valued 1-form on $\Sigma^{3+4p,4q}_3$. Let $\xi_1, \xi_2, \xi_3$ be the vector fields on $\Sigma^{3+4p,4q}_3$ induced by the one-parameter subgroups $\{e^{\theta i}\}_\theta \in \mathbb{R}, \{e^{\theta j}\}_\theta \in \mathbb{R}, \{e^{\theta k}\}_\theta \in \mathbb{R}$ respectively, which is equivalent to that $\xi_1 = IN, \xi_2 = JN, \xi_3 = KN$. A calculation shows that

$$(3.5) \quad \omega_0(\xi_1) = i, \quad \omega_0(\xi_2) = j, \quad \omega_0(\xi_3) = k.$$ 

By the formula of $\omega_0$, if $a \in \Sp(1)$, then the right translation $R_a$ on $\Sigma^{3+4p,4q}_3$ satisfies that

$$(3.6) \quad R^*_a \omega_0 = a \cdot \omega_0 \cdot \bar{a}.$$ 

Therefore, $\omega_0$ is a connection form of the above bundle. Note that $\Sp(p + 1, q)$ leaves $\omega_0$ invariant. We shall check the conditions (i), (ii), (iii), (iv) and [14,10] so that $(\Sigma^{3+4p,4q}_3, \{I, J, K\}, g, \omega_0)$
will be a quaternionic CR manifold. First of all, it follows that
\[
\omega_0 \wedge \omega_0 \wedge \omega_0 \wedge (d\omega_0 \wedge d\omega_0) \wedge \cdots \wedge (d\omega_0 \wedge d\omega_0)^n \neq 0 \quad \text{at any point of } \Sigma_{\mathbb{H}}^{3+4p,4q}.
\]
(Compare [14], [28] for example). In fact, letting \( \omega_0 = \omega_1 i + \omega_2 j + \omega_3 k \) as before,
\[
\omega_0^3 \wedge d\omega_0^{2n} = 6\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge (d\omega_1^2 + d\omega_2^2 + d\omega_3^2)^n.
\]
This calculation shows (iii). In particular, each \( \omega_0 \) is a nondegenerate contact form on \( \Sigma_{\mathbb{H}}^{3+4p,4q} \).

Using \( \Sigma_{\mathbb{H}} \) and as \( \xi_1 \) generates \( \{ e^{i\theta} \}_{\theta \in \mathbb{R}} \subset \text{Sp}(1) \), \( \mathcal{L}_{\xi_1} \omega_1 = 0 \). (Similarly we have \( \mathcal{L}_{\xi_2} \omega_2 = \mathcal{L}_{\xi_3} \omega_3 = 0 \).) Noting that \( \omega_0(\xi_a) = 1 \) and \( 0 = \mathcal{L}_{\xi_a} \omega_0 = i \xi_a \circ d\omega_0 \) from (3.5), each \( \xi_a \) is the characteristic vector field for \( \omega_0 \). Moreover, note that \( \{ \xi_1, \xi_2, \xi_3 \} \) generates the fields of Lie algebra of \( \text{Sp}(1) \). It follows that \( \mathcal{D} = \bigoplus_{a=1}^3 \text{Null} \omega_a \) for which there is the decomposition \( T\Sigma_{\mathbb{H}}^{3+4p,4q} = \{ \xi_1, \xi_2, \xi_3 \} \oplus \mathcal{D} \). If \( \{ e_i \}_{i=1, \ldots, 4n} \) is the orthonormal basis of \( \mathcal{D} \), then the dual frame \( \theta^i \) is obtained as \( \theta^i(e_j) = \delta_j^i \) and \( \theta^i(\xi_1) = \theta^i(\xi_2) = \theta^i(\xi_3) = 0 \). In order to prove that the distribution uniquely determined by \( \omega_0 \) are \( \{ \xi_1, \xi_2, \xi_3 \} \) (cf. [14] also), we need the following lemma.

**Lemma 3.3.**
\[
d\omega_1(X,Y) = g(X, IY), \quad d\omega_2(X,Y) = g(X, JY), \quad d\omega_3(X,Y) = g(X, KY)
\]
where \( X, Y \in \mathcal{D} \).

**Proof.** Given \( X, Y \in \mathcal{D} \), let \( u, v \) be the vectors at the origin by parallel translation of \( X, Y \) at \( x \in \Sigma_{\mathbb{H}}^{3+4p,4q} \) respectively. Then by definition, \( g(X,Y) = \text{Re}\langle u, v \rangle \). Furthermore,
\[
g(X, IY) = \text{Re}\langle u, v \cdot i \rangle = \text{Re}\langle u, v \cdot i \rangle.
\]
From (3.5), if \( X, Y \in \mathcal{D} \), then
\[
d\omega_0(X,Y) = -(d\bar{z}_1 \wedge dz_1 + \cdots + d\bar{z}_p+1 \wedge dz_{p+1} - d\bar{w}_1 \wedge dw_1 - \cdots - d\bar{w}_q \wedge dw_q)(u,v).
\]
Then a calculation shows that \( d\omega_0(X,Y) = -\frac{1}{2}(\langle u, v \rangle - \langle u, v \rangle) \). It is easy to check that the \( i \)-part of
\[
-\frac{1}{2}(\langle u, v \rangle - \langle u, v \rangle) = \text{Re}\langle u, v \cdot i \rangle.
\]
Since \( d\omega_1(X,Y) \) is the \( i \)-part of \( d\omega(X,Y) \) and by (3.6), we obtain the equality \( g(X, IY) = d\omega_1(X,Y) \). Similarly, we have that \( g(X, JY) = d\omega_2(X,Y) \), \( g(X, KY) = d\omega_3(X,Y) \).

From this lemma, \( d\omega_0(e_i, e_j) = g(e_i, J_a e_j) = -J_a \theta^1 \).

Since \( \{ \xi_1, \xi_2, \xi_3 \} \) generates \( \text{Sp}(1) \) of the bundle \( \Sigma_{\mathbb{H}} \), we obtain \( d\omega_0 + 2\omega_0 \wedge \omega_0 = -J_a \theta^1 \wedge \theta^1 \).

Applying to \( J, K \) similarly, we obtain the following structure equation of the bundle \( \Sigma_{\mathbb{H}} \):
\[
d\omega_0 + \omega_0 \wedge \omega_0 = -(L_j i + J_{ij} j + K_{ij} k) \theta^1 \wedge \theta^1.
\]
From this equation, the condition (14.10) is easily checked so that \( \text{Null} \omega_a = \{ \xi_1, \xi_2, \xi_3 \} \). We summarize that

**Theorem 3.4.** \( \Sigma_{\mathbb{H}}^{3+4p,4q}, \{ \omega_a \}_{a=1,2,3}, \{ I, J, K, \} \) is a \((4n + 3)\)-dimensional homogeneous quaternionic CR manifold of type \((3 + 4p, 4q)\) where \( p + q = n \geq 0 \). Moreover, there exists the equivariant principal bundle \( \Sigma_{\mathbb{H}} \) of the pseudo-Riemannian submersion over the homogeneous quaternionic pseudo-Kähler projective space \( \mathbb{H}P^{p,q} \) of type \((4p, 4q)\): \( \text{Sp}(1) \rightarrow (\text{Sp}(p+1,q) \cdot \text{Sp}(1), \Sigma_{\mathbb{H}}^{3+4p,4q}, \pi) \rightarrow (\text{PSp}(p+1,q), \mathbb{H}P^{p,q}, \tilde{g}) \).
We shall prove more generally in Theorem 4.6 that \((PSp(p+1,q), \mathbb{H}^{4p,4q})\) supports an invariant quaternionic pseudo-Kähler metric \(\tilde{g}\) of type \((4p,4q)\).

**Remark 3.5.** (a) In [2], it is shown that \((\Sigma^{3+4p-4q}_\mathbb{H}, \{I,J,K\}, g)\) is a pseudo-Sasakian space form of constant positive curvature with type \((4n,4n)\).

(b) When \(q = 0\) or \(p = 0\), we can find discrete cocompact subgroups from \(Sp(n+1)\cdot Sp(4)\) of positive scalar curvature. Thus, we obtain compact nondegenerate quaternionic CR pseudo-Riemannian standard space form \(V^{4n+3}_{1}/\Gamma\) of type \((4n,3)\) with constant sectional curvature \(-1\) which is an \(Sp(1)\)-bundle over the quaternionic Kähler hyperbolic orbifold \(\mathbb{H}^{n}/\Gamma^*\) of negative scalar curvature. As we know, there exists no compact pseudo-Sasakian manifold (or quaternionic CR manifold) whose pseudo-Kähler orbifold is of zero Ricci curvature. However in our case an indefinite Heisenberg nilmanifold is a compact pseudo-conformal quaternionic CR manifold whose pseudo-Kähler orbifold is of zero Ricci curvature, see [17].

4. Local Principal bundle

Let \(\{e_i\}_{i=1,\ldots,4n}\) be the basis of \(\mathcal{D}|U\) such that \(g^\mathcal{D}(e_i, e_j) = g_{ij}\). We choose a local coframe \(\theta^i\) for which

\[(4.1) \quad \theta^i|V = 0 \quad \text{and} \quad \theta^i(e_j) = \delta_{ij}.
\]

As usual the quaternionic structure \(\{J_\alpha\}_{\alpha=1,2,3}\) can be represented locally by the matrix \(J^{\alpha j}_i\) such as \(J_\alpha e_i = J^{\alpha j}_i e_j\). Using (1.2), note that \(\rho_\alpha(e_j, e_i) = J^{\alpha k}_j g_{jk} = J^{\alpha i}_j\). Here the matrix \((g_{ij})\) lowers and raises the indices. Then we can write the structure equation (1.9) by using \(\theta^i\):

\[(4.2) \quad d\omega_\alpha + 2\omega_\beta \wedge \omega_\gamma = -J^{\alpha j}_i \theta^i \wedge \theta^j \quad (\alpha = 1,2,3).
\]

Using \(\omega\) of (iii), we have the following formula corresponding to (1.2):

\[(4.3) \quad d\omega + \omega \wedge \omega = -(J^{1 i}_j \hat{i} + J^{2 i}_j \hat{j} + J^{3 i}_j \hat{k}) \theta^j \wedge \theta^i.
\]

Denote by \(\mathcal{E}\) the local transformation groups generated by \(V\) acting on a small neighborhood \(U'\) of \(U\). As \(\mathcal{E}\) is locally isomorphic to the compact Lie group \(SO(3)\) by Lemma 2.2, it acts properly on \(U'\). (See for example [27].) If we note that each \(\xi_\alpha\) is a nonzero vector field everywhere on \(U\), then the stabilizer of \(\xi_\alpha\) is finite at every point. By the slice theorem of compact Lie groups [3], choosing a sufficiently small neighborhood \(U'\) of the identity from \(\mathcal{E}\), \(\mathcal{E}'\) acts properly and freely on \(U'\). We choose such \(U'\) (respectively \(\mathcal{E}'\)) from the beginning and replace it by \(U\) (respectively \(\mathcal{E}\)). Then there is a principal local fibration:

\[(4.4) \quad \mathcal{E} \twoheadrightarrow U \xrightarrow{\pi} U/\mathcal{E}.
\]

If we note that \(V \oplus \mathcal{D} = TM|U\), \(\pi\) maps \(\mathcal{D}\) isomorphically onto \(T(U/\mathcal{E})\) at each point of \(U\). So \(\{\pi_* e_i \mid i = 1,\ldots,4n\}\) is a basis of \(T(U/\mathcal{E})\) at each point of \(U/\mathcal{E}\). Let \(\hat{\theta}^i\) be the dual frame on \(U/\mathcal{E}\) such that

\[(4.5) \quad \hat{\theta}^i(\pi_* e_j) = \delta_{ij} \quad \text{on} \quad U/\mathcal{E}.
\]

Since \(\theta^i\) is the coframe of \(\{e_i\}\) and \(\pi^* \hat{\theta}^i|V = \theta^i|V = 0\), it follows that

\[(4.6) \quad \pi^* \hat{\theta}^i = \theta^i \quad \text{on} \quad U \quad (i = 1,\ldots,4n).
\]
Lemma 4.1. Put $J_1 = I$, $J_2 = J$, $J_3 = K$ respectively. Let $\{\phi_\theta\}_{-\varepsilon < \theta < \varepsilon}$ be a local one-parameter subgroup of the local group $E$. Then there exists an element $G_\theta \in SO(3)$ satisfying the following:

\[
(1) \quad (\phi_\theta)_* \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = G_\theta \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.
\]

\[
(4.7)
\]

\[
(2) \quad \begin{pmatrix} I_{\phi_{\theta y}} \\ J_{\phi_{\theta y}} \\ K_{\phi_{\theta y}} \end{pmatrix} \circ \phi_{\theta *} = \phi_{\theta *} \circ {}^t G(\theta) \begin{pmatrix} I_y \\ J_y \\ K_y \end{pmatrix}.
\]

Proof. Since every leaf of $V$ is locally isomorphic to $SO(3)$, $\xi_a$ is viewed as the fundamental vector field to the principal fibration $\pi : U \to U/E$. Thus we may assume that $\xi_1, \xi_2, \xi_3$ correspond to $i, j, k$ respectively so that $\phi^1_\theta = e^{i\theta}$, $\phi^2_\theta = e^{j\theta}$, $\phi^3_\theta = e^{k\theta}$ up to conjugacy by an element of $SO(3)$. A calculation shows that $(\phi^1_\theta)_* ((\xi_2)_x) = \cos 2\theta \cdot (\xi_2)_x + \sin 2\theta \cdot (\xi_3)_x$. Similarly, $(\phi^1_\theta)_* ((\xi_3)_x) = -\sin 2\theta \cdot (\xi_2)_x + \cos 2\theta \cdot (\xi_3)_x$. This holds similarly for $\phi^1_\theta, \phi^2_\theta, \phi^3_\theta$. It turns out that if $\phi_\theta \in E$, then there exists an element $G_\theta \in SO(3)$ which shows the above formula (1).

Since $\phi_t$ preserves $D (-\varepsilon < t < \varepsilon)$, using (1) we see that

\[
(4.8) \quad \phi^*_t (\omega_1, \omega_2, \omega_3) = (\omega_1, \omega_2, \omega_3) G_t.
\]

Since there exists an element $g_t \in Sp(1)$ such that $g_t \begin{pmatrix} i \\ j \\ k \end{pmatrix} g_t = G_t \begin{pmatrix} i \\ j \\ k \end{pmatrix} (\tilde{g}_t$ is the quaternion conjugate of $g_t$), (4.8) is equivalent with

\[
(4.9) \quad \phi^*_t \omega = g_t \cdot \omega \cdot \tilde{g}_t.
\]

Differentiate this equation which yields that

\[
(4.10) \quad \phi^*_t (d\omega + \omega \wedge) \equiv g_t (d\omega + \omega \wedge) \tilde{g}_t \mod \omega.
\]

Using the equation (12), it follows that

\[
\phi^*_t ((I_{ij}, J_{ij}, K_{ij}) \begin{pmatrix} i \\ j \\ k \end{pmatrix} \theta^i \wedge \theta^j) \equiv (I_{ij}, J_{ij}, K_{ij}) g_t \begin{pmatrix} i \\ j \\ k \end{pmatrix} \tilde{g}_t \theta^i \wedge \theta^j
\]

\[
= (I_{ij}, J_{ij}, K_{ij}) G_t \begin{pmatrix} i \\ j \\ k \end{pmatrix} \theta^i \wedge \theta^j.
\]

Noting that $\phi^*_t \theta^i = \phi^*_t (\pi^* \theta^i) = \theta^i$, the above equation implies that

\[
(4.11) \quad (I_{ij}(\phi_t(x)), J_{ij}(\phi_t(x)), K_{ij}(\phi_t(x))) \equiv (I_{ij}(x), J_{ij}(x), K_{ij}(x)) G_t(x) \mod \omega.
\]
Since \( \pi_* \varphi_{t_*}((e_i)_x) = \pi_*((e_i)_{\varphi_{t_*}x})\) \((x \in U)\), it follows \(\varphi_{t_*}((e_i)_x) = (e_i)_{\varphi_{t_*}x}\). Letting \(G_t = (s_{ij}) \in \text{SO}(3)\) and using \ref{4.11},
\[
I_{\varphi_{t_*}x}(\varphi_{t_*}((e_i)_x)) = I_{\varphi_{t_*}x}(\varphi_{t_*}((e_i)_{\varphi_{t_*}x})) = (P_{ij}^t(x) \cdot s_{11} + J_{ij}^t(x) \cdot s_{21} + K_{ij}^t(x) \cdot s_{31})(\varphi_{t_*}((e_j)_x))
\]
\[
= (\varphi_{t_*})(s_{11} \cdot I_{\varphi_{t_*}x}(e_i)_x + s_{21} \cdot J_{\varphi_{t_*}x}(e_i)_x + s_{31} \cdot K_{\varphi_{t_*}x}(e_i)_x))
\]
\[
= (\varphi_{t_*})(s_{11}, s_{21}, s_{31}) \begin{pmatrix} I_x \\ J_x \\ K_x \end{pmatrix} (e_i)_x.
\]

The same argument applies to \(J_{\varphi_{t_*}x}, K_{\varphi_{t_*}x}\) to conclude that \(\begin{pmatrix} I_{\varphi_{t_*}x} \\ J_{\varphi_{t_*}x} \\ K_{\varphi_{t_*}x} \end{pmatrix} \circ \varphi_{t_*} = \varphi_{t_*} \circ t G_t \begin{pmatrix} I_x \\ J_x \\ K_x \end{pmatrix}\).

This proves (2).

**Lemma 4.2.** The quaternionic structure \(\{I, J, K\}\) on \(D|U\) induces a family of quaternionic structures \(\{I_t, J_t, K_t\}_{t \in \Lambda}\) on \(U/E\).

**Proof.** Choose a small neighborhood \(V_i \subset U/E\) and a section \(s_i : V_i \rightarrow U\) for the principal bundle \(\pi : U \rightarrow U/E\). Let \(\tilde{x} \in V_i\) and a vector \(\tilde{X}_\tilde{x} \in TV_i\). Choose a vector \(X_{s_i(\tilde{x})} \in D_{s_i(\tilde{x})}\) such that \(\pi_*(X_{s_i(\tilde{x})}) = \tilde{X}_\tilde{x}\). Define endomorphisms \(I_t, J_t, K_t\) on \(V_i\) to be
\[
(\tilde{I}_t)_{\tilde{x}}(\tilde{X}_\tilde{x}) = \pi_* I_{s_t(\tilde{x})} X_{s_t(\tilde{x})},
\]
\[
(\tilde{J}_t)_{\tilde{x}}(\tilde{X}_\tilde{x}) = \pi_* J_{s_t(\tilde{x})} X_{s_t(\tilde{x})},
\]
\[
(\tilde{K}_t)_{\tilde{x}}(\tilde{X}_\tilde{x}) = \pi_* K_{s_t(\tilde{x})} X_{s_t(\tilde{x})}.
\]

Since \(\pi_* : D_{s_t(\tilde{x})} \rightarrow T_{\tilde{x}}(U/E)\) is an isomorphism, \(\tilde{I}_t, \tilde{J}_t, \tilde{K}_t\) are well-defined almost complex structures on \(V_i\). So we have a family \(\{\tilde{I}_t, \tilde{J}_t, \tilde{K}_t\}_{t \in \Lambda}\) of almost complex structures associated to an open cover \(\{V_i\}_{i \in \Lambda}\) of \(U/E\). Suppose that \(V_i \cap V_j \neq \emptyset\). If \(\tilde{x} \in V_i \cap V_j\), then there is an element \(\varphi_\theta \in E\) such that \(s_j(\tilde{x}) = \varphi_\theta \cdot s_i(\tilde{x})\). As \(\varphi_\theta\) preserves \(D\), \(\varphi_\theta X_{s_i(\tilde{x})} \in D_{s_j(\tilde{x})}\) and \(\pi_*(\varphi_\theta X_{s_i(\tilde{x})}) = \tilde{X}_\tilde{x}\). Then
\[
X_{s_j(\tilde{x})} = \varphi_\theta X_{s_i(\tilde{x})}.
\]

Let \(\{\tilde{I}_j, \tilde{J}_j, \tilde{K}_j\}\) be almost complex structures on \(V_j\) obtained from \ref{4.12}. Using Lemma \ref{4.11} and \ref{4.13}, calculate at \(s_j(\tilde{x}) \in V_i \cap V_j\),
\[
\begin{pmatrix} (\tilde{I}_j)_{\tilde{x}} \\ (\tilde{J}_j)_{\tilde{x}} \\ (\tilde{K}_j)_{\tilde{x}} \end{pmatrix} \tilde{X}_\tilde{x} = \pi_* \begin{pmatrix} I_{s_j(\tilde{x})} \\ J_{s_j(\tilde{x})} \\ K_{s_j(\tilde{x})} \end{pmatrix} X_{s_j(\tilde{x})} = \pi_* \begin{pmatrix} I_{\varphi_\theta s_i(\tilde{x})} \\ J_{\varphi_\theta s_i(\tilde{x})} \\ K_{\varphi_\theta s_i(\tilde{x})} \end{pmatrix} \varphi_\theta X_{s_i(\tilde{x})}
\]
\[
= \pi_* \varphi_\theta \circ t G_\theta \begin{pmatrix} I_{s_i(\tilde{x})} \\ J_{s_i(\tilde{x})} \\ K_{s_i(\tilde{x})} \end{pmatrix} X_{s_i(\tilde{x})}
\]
\[
= t G_\theta (\varphi_\theta) \pi_* \begin{pmatrix} I_{s_i(\tilde{x})} \\ J_{s_i(\tilde{x})} \\ K_{s_i(\tilde{x})} \end{pmatrix} X_{s_i(\tilde{x})} = t G_\theta \begin{pmatrix} (\tilde{I}_i)_{\tilde{x}} \\ (\tilde{J}_i)_{\tilde{x}} \\ (\tilde{K}_i)_{\tilde{x}} \end{pmatrix} \tilde{X}_\tilde{x},
\]

hence \(\begin{pmatrix} (\tilde{I}_j)_{\tilde{x}} \\ (\tilde{J}_j)_{\tilde{x}} \\ (\tilde{K}_j)_{\tilde{x}} \end{pmatrix} = t G_\theta \begin{pmatrix} (\tilde{I}_i)_{\tilde{x}} \\ (\tilde{J}_i)_{\tilde{x}} \\ (\tilde{K}_i)_{\tilde{x}} \end{pmatrix}\) on \(\tilde{x} \in V_i \cap V_j\). Thus, \(\{\tilde{I}_t, \tilde{J}_t, \tilde{K}_t\}_{t \in \Lambda}\) defines a quaternionic structure on \(U/E\). \(\square\)
4.1. Pseudo-Sasakian 3-structure and Pseudo-Kähler structure. We now take \( \{e_i\}_{i=1,\ldots,4n} \) of \( D|U \) as the orthonormal basis, i.e. \( g_{ij} = \delta_{ij} \). Then the bilinear form \( g^D = \sum_{i=1}^{4p} \theta^i \cdot \theta^i - \sum_{i=p+1}^{4n} \theta^i \cdot \theta^i \)
defined on \( D \) induces a pseudo-Riemannian metric on \( U/E \):

\[
\hat{g} = \sum_{i=1}^{4p} \hat{\theta}^i \cdot \hat{\theta}^i - \sum_{i=p+1}^{4n} \hat{\theta}^i \cdot \hat{\theta}^i
\]
such that \( g^D = \pi^* \hat{g} \). Let \( \hat{\nabla} \) be the covariant derivative on \( U/E \). If \( \hat{\omega}_j^i \) is the Levi-Civita connection with respect to \( \hat{g} \), then it follows that

\[
\hat{\nabla} e_i = \hat{\omega}_i^j e_j.
\]

Then \( \hat{\omega}_j^i \) satisfies that

\[
d\hat{\theta}^i = \hat{\theta}^j \wedge \hat{\omega}_j^i, \quad \hat{\omega}_{ij} + \hat{\omega}_{ji} = 0.
\]

Put

\[
\hat{\Omega}_j^i = d\hat{\omega}_j^i - \hat{\omega}_j^\sigma \wedge \hat{\omega}_\sigma^j = \frac{1}{2} \hat{R}^i_{jkl} \hat{\theta}^k \wedge \hat{\theta}^l.
\]

Consider the following pseudo-Riemannian metric on \( U \):

\[
\hat{g}_\alpha(X, Y) = \sum_{a=1}^{3} \omega_a(X) \cdot \omega_a(Y) + \hat{g}_{\pi(x)}(\pi_*X, \pi_*Y) \quad (X, Y \in T_x U).
\]

(4.18)

(4.19)

(Equivalently \( \hat{g} = \sum_{a=1}^{3} \omega_a \cdot \omega_a + \sum_{i=1}^{4p} \theta^i \cdot \theta^i - \sum_{i=p+1}^{4n} \theta^i \cdot \theta^i \).)

Then we have shown in (4.19) that the local principal fibration \( E \to (U, \hat{g}) \to (U/E, \hat{g}) \) is a pseudo-Sasakian 3-structure. In fact the following equation is equivalent with the normality condition of the pseudo-Sasakian 3-structure. (Compare (29), (5).)

**Proposition 4.3.** Let \( \{\omega_\alpha\}, \{J_\alpha\}, \{\xi_\alpha\}_{\alpha=1,2,3} \) be a nondegenerate quaternionic CR structure on \( U \) of a \((4n+3)\)-manifold \( M \). If \( \nabla \) is the Levi-Civita connection on \((U, \hat{g})\), then

\[
(\nabla_X \tilde{J}_\alpha)Y = \hat{g}(X, Y)\xi_\alpha - \omega_\alpha(Y)X \quad (\alpha = 1, 2, 3).
\]

**Proof.** For \( X, Y \in TU \), consider the following tensor

\[
N^{\omega_\alpha}(X, Y) = N(X, Y) + (X \omega_\alpha(Y) - Y \omega_\alpha(X))\xi_\alpha
\]

where \( N(X, Y) = [\tilde{J}_\alpha X, \tilde{J}_\alpha Y] - [X, Y] - \tilde{J}_\alpha[\tilde{J}_\alpha X, Y] - \tilde{J}_\alpha[X, \tilde{J}_\alpha Y] \) is the Nijenhuis torsion of \( \tilde{J}_\alpha \) \((\alpha = 1, 2, 3)\). A direct calculation for a contact metric structure \( \hat{g} \) (cf. (3)) shows that

\[
2\hat{g}((\nabla_X \tilde{J}_\alpha)Y, Z) = \hat{g}(N^{\omega_\alpha}(Y, Z), \tilde{J}_\alpha X) + \langle \tilde{L}_{\tilde{J}_\alpha X} \omega_\alpha \rangle(Y)
- (\tilde{L}_{\tilde{J}_\alpha Y} \omega_\alpha)(X) + 2\hat{g}(X, Y)\omega_\alpha(Z) - 2\hat{g}(X, Z)\omega_\alpha(Y).
\]

We have shown in Theorem (2.7) that each \( \tilde{J}_\alpha \) is integrable on \( \text{Null } \omega_\alpha \). It follows from the famous theorem that the Nijenhuis torsion of \( \tilde{J}_\alpha \), \( N(X, Y) = 0 \) \((\forall X, Y \in \text{Null } \omega_\alpha)\). By the formula \((4.20)\), \( N^{\omega_\alpha}(X, Y) = 0 \) for \( \forall X, Y \in \text{Null } \omega_\alpha \). To obtain \((4.19)\), it has to show that \( N^{\omega_\alpha}(X, Y) \) vanishes for all \( X, Y \in TU \). Noting the decomposition \( TU = \{\xi_1\} \oplus \text{Null } \omega_1 \), it suffices to show that \( N^{\omega_1}(\xi_1, X) = 0 \) (similarly for \( \alpha = 2, 3 \)). Since \( \xi_\alpha \) is a characteristic CR-vector field for \((\omega_\alpha, \tilde{J}_\alpha)\) \((\alpha = 1, 2, 3)\), i.e. \( \mathcal{L}_{\xi_\alpha} \tilde{J}_\alpha = 0 \), it follows that \( \tilde{J}_1[\xi_1, Y] = [\xi_1, \tilde{J}_1 Y] \) \((\forall Y \in \text{Null } \omega_1)\). In particular, \( \tilde{J}_1\xi_1 = [\xi_1, \tilde{J}_1 X] = -[\xi_1, X] \). Hence, \( N^{\omega_1}(\xi_1, X) = 0 \). As a consequence, we see that \( N^{\omega_3}(X, Y) = 0 \)
(∀ X, Y ∈ TU). On the other hand, if \( N^\omega(X, Y) = 0 \) (∀ X, Y ∈ TU), then it is easy to see that 
(\( \mathcal{L}_{J_a} \omega_a \))(Y) = (\( \mathcal{L}_{J_a} Y \omega_a \))(X) = 0. (See [5].) From (4.18), note that \( \omega_\alpha(X) = \tilde{g}(\xi_\alpha, X) \). The above equation (4.18) follows.

As \( \{\omega_\alpha, \theta^b\}_{\alpha=1,2,3; i=1...4n} \) are orthonormal coframes for the pseudo-Sasakian metric \( \tilde{g} \) (cf. (4.18)), the structure equation says that there exist unique 1-forms \( \varphi^i_j \), \( \tau^i_{\alpha} \) \( i, j = 1, \ldots, 4n; \alpha = 1, 2, 3 \) satisfying:

\[
d\theta^i = \theta^i \wedge \varphi^i_j + \sum_{\alpha=1}^{3} \omega_\alpha \wedge \tau^i_{\alpha} \quad (\varphi^i_j + \varphi^i_j = 0).
\]

Then the normality condition for the pseudo-Sasakian 3-structure is reinterpreted as the following structure equation.

**Theorem 4.4.** There exists a connection form \( \{\omega^i_j\} \) such that

\[
d\tilde{\mathcal{J}}^a_{ij} - \tilde{\omega}^a_\sigma \tilde{\mathcal{J}}^a_{ij} - \tilde{J}^a_{i_0} \omega^a_j = 2\tilde{J}^b_{ij} \cdot \omega_c - 2\tilde{J}^b_{ij} \cdot \omega_b \quad ((a, b, c) \sim (1, 2, 3)).
\]

**Proof.** It follows from Proposition 4.3 that (\( \nabla X \tilde{J}_a \))\( e_i = \tilde{g}(X, e_i) \xi_\alpha \) for the orthonormal basis \( e_i \in B \). From the structure equation (4.21), let \( \nabla X e_i = \varphi^i_j(X)e_j + \sum_{b=1}^{3} (\tau_b)_i \xi_b \), which is substituted into the equation (\( \nabla X \tilde{J}_a \))\( e_i = \nabla X (\tilde{J}_a e_i) - \tilde{J}_a (\nabla X e_i) \):

\[
(\nabla X \tilde{J}_a) e_i = (d(\tilde{J}^a_{i_j}) - \varphi^i_j(X)(\tilde{J}^a_{i_j}) + (\tilde{J}^a_{i}) \varphi^j(X)) e_\ell \\
+ \sum_{b=1}^{3} (\tilde{J}^a_{i}) (\tau_b)_i(X) \xi_b - \sum_{b \neq a}^{3} (\tau_b)_i(X) \xi_c.
\]

As \( \tilde{g}(X, e_i) = \tilde{g}_{\alpha} \theta^k(X) \) (cf. (4.18)), this implies that \( d(\tilde{J}^a_{i}) - \varphi^i_j(X)(\tilde{J}^a_{i}) + (\tilde{J}^a_{i}) \varphi^j(X) = 0 \) and \( (\tilde{J}^a_{i}) (\tau_b)_i(X) \xi_a = \tilde{g}_{\alpha} \theta^k(X) \xi_a \). It follows \( -(\tau_b)_i = (\tilde{J}^a_{i}) \theta^a \). Then \( (\tau_a)_i \tilde{g}^{jk} = -(\tilde{J}^a_{i}) \tilde{g}^{jk} \theta^j = (\tilde{J}^a_{i}) \tilde{g}^{jk} \theta^j \), so that \( (\tau_a)_i = (\tilde{J}^a_{i}) \theta^j \). As \( \tilde{g}^{ij} = \pm \delta^{ij} \), use \( \tilde{g}^{ij} \) to lower the above equations:

\[
d(\tilde{J}^a_{i}) - \varphi^i_j(X)(\tilde{J}^a_{i}) + (\tilde{J}^a_{i}) \varphi^j(X) = 0, \\
(\tau_a)_i = (\tilde{J}^a_{i}) \theta^j.
\]

Putting

\[
\omega^i_j = \varphi^i_j - \sum_{a=1}^{3} (\tilde{J}^a_{i}) \omega_a,
\]
the equation (4.21) reduces to

\[
d\theta^i = \theta^i \wedge \omega^i_j \quad (\omega_{ij} + \omega_{ji} = 0).
\]

Differentiate our equation (4.22) \( d\omega_a + 2\omega_b \wedge \omega_c = -\tilde{J}^a_{i_0} \theta^i \wedge \theta^j \) \((a, b, c) \sim (1, 2, 3)) \) and substitute (4.20). Then it becomes (after alternation):

\[
(d\tilde{J}^a_{ij} - \tilde{\omega}^a_\sigma \tilde{J}^a_{ij} - \tilde{J}^a_{i_0} \omega^a_j + \omega_b \cdot 2\tilde{J}^c_{ij} - \omega_c \cdot 2\tilde{J}^b_{ij}) \wedge \theta^i \wedge \theta^j = 0.
\]

If we note (4.21) and (4.26), \( d\tilde{J}^a_{ij} - \tilde{\omega}^a_\sigma \tilde{J}^a_{ij} - \tilde{J}^a_{i_0} \omega^a_j = 0 \) mod \( \omega_1, \omega_2, \omega_3 \). As the forms \( \omega_\alpha \wedge \theta^i \wedge \theta^j \) are linearly independent, the result follows. \( \square \)
Definition 4.5. Let $\hat{\nabla}$ be the Levi-Civita connection on an almost quaternionic pseudo-Riemannian manifold $(X, \hat{g})$ of type $(4p, 4q)$ $(p + q = n)$. Then $X$ is said to be a quaternionic pseudo-Kähler manifold if for each quaternionic structure $\{\hat{J}_a; a = 1, 2, 3\}$ defined locally on a neighborhood of $X$, there exists a smooth local function $A \in \mathfrak{so}(3)$ such that

$$\nabla \left( \begin{array}{c} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{array} \right) = A \cdot \left( \begin{array}{c} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{array} \right)$$

provided that $\dim X = 4n \geq 8$. Equivalently if $\hat{\Omega}$ is the fundamental 4-form globally defined on $X$, then $\hat{\nabla} \hat{\Omega} = 0$.

We have shown the following result in [2] when $\dim U/\mathcal{E} = 4n \geq 12$ by Swann’s method.

Theorem 4.6. The set $(U/\mathcal{E}, \hat{g}, \{\hat{I}_i, \hat{J}_a, \hat{K}_i\}_{i \in \Lambda})$ is a quaternionic pseudo-Kähler manifold of type $(4p, 4q)$ provided that $\dim U/\mathcal{E} = 4n \geq 8$. Moreover, $(U/\mathcal{E}, \hat{g})$ is an Einstein manifold of positive scalar curvature $(4n \geq 4)$ such that

$$\hat{R}_{ij} = 4(n+2)\hat{g}_{ij}.$$  

Proof. As we put $\theta^i = \pi^s \hat{\theta}^i$, the equation (4.16) implies that $d\theta^i = \theta^j \wedge \pi^s \hat{\omega}^j_i$, $\pi^s \hat{\omega}^j_i + \pi^s \hat{\omega}^i_j = 0$.

Compared this with the equation (4.20) and by skew-symmetry, it is easy to check that

$$\pi^s \hat{\omega}^j_i = \hat{\omega}^j_i.$$  

Put $\hat{V} = V_i$ and $\hat{J}_1 = \hat{I}_i$, $\hat{J}_2 = \hat{J}_i$, $\hat{J}_3 = \hat{K}_i$ on $\hat{V}$. Let $s = s_i : \hat{V} \rightarrow U$ be the section as before. Since $\pi^s s_i ((\hat{e}_j)_x) = (\hat{e}_j)_x = \pi^s ((e_j)_s(x)), \pi^s ((\hat{e}_j)_x) - (e_j)_s(x) \in V = \{\xi_1, \xi_2, \xi_3\}$. Then $\theta^i s_i ((\hat{e}_j)_x) = \theta^i ((e_j)_s(x))$ from (4.11). A calculation shows that $(\hat{J}_a)_\hat{x} \hat{e}_i = \pi^s (J_a)_s(\hat{x}) e_i = \pi^s ((\hat{J}^a)_j (s(\hat{x})) e_j) = (\hat{J}^a)_j (s(\hat{x})) \hat{e}_j$ (cf. (4.11)). As we put $\hat{J}^a_{ij} \hat{e}_i = (\hat{J}^a)_j (s(\hat{x})) \hat{e}_j$, note that

$$\hat{J}^a_{ij} (s(\hat{x})) = \hat{J}^a_{ij} (\hat{x}) ~ (a = 1, 2, 3).$$

In particular,

$$d(\hat{J}^a)_{ij} \circ s_i (\hat{X}_\hat{x}) = d(\hat{J}^a)_{ij} (\hat{X}_\hat{x}) ~ (\forall \hat{X}_\hat{x} \in T_{\hat{x}}(\hat{V})) ~ (a = 1, 2, 3).$$

Since $\pi^s s_i (\hat{X}_\hat{x}) = \hat{X}_\hat{x} (\hat{X}_\hat{x} \in T_{\hat{x}}(\hat{V}))$,

$$\omega^a_{ij} (\hat{X}_\hat{x}) = \omega^a_{ij} (s_i (\hat{X}_\hat{x})).$$

Plug these equations (4.11), (4.20) and (4.30) into (4.22):

$$d(\hat{J}^a)_{ij} (s \hat{X}) - \omega^a_{ij} (s \hat{X}) \cdot (\hat{J}^a)_{s \sigma} (s (\hat{x})) - (\hat{J}^a)_{\sigma} (s (\hat{x})) \cdot \omega^a_{ij} (s \hat{X})$$

$$= d(\hat{J}^a)_{ij} (\hat{x}) - \omega^a_{ij} (\hat{x}) \cdot (\hat{J}^a)_{s \sigma} (\hat{x}) - (\hat{J}^a)_{\sigma} (\hat{x}) \cdot \omega^a_{ij} (\hat{x})$$

$$= 2(\hat{J}^a)_{ij} (s (\hat{x})) \cdot \omega_c (s \hat{X}) - 2(\hat{J}^a)_{ij} (\hat{x}) \cdot \omega_c (s \hat{X})$$

$$= 2(\hat{J}^a)_{ij} (\hat{x}) \cdot \omega_c (s \hat{X}) - 2(\hat{J}^a)_{ij} (\hat{x}) \cdot \omega_c (s \hat{X}).$$

Using these,

$$\hat{\nabla}_{\hat{x}} (\hat{J}_a) ((\hat{e}_i)_\hat{x}) = \hat{\nabla}_{\hat{x}} (\hat{J}_a) \hat{e}_i - (\hat{J}_a) (\hat{\nabla}_{\hat{x}} \hat{e}_i)$$

$$= (d(\hat{J}^a)_{ij} (X)) - (\hat{J}^a)_{s \sigma} (\hat{x}) \cdot \omega^a_{ij} (\hat{x}) - (\hat{J}^a)_{\sigma} (\hat{x}) \cdot \omega^a_{ij} (\hat{x})$$

$$= 2(\hat{J}^a)_{ij} (\hat{x}) (\hat{e}_j) \cdot s^a \omega_c (\hat{x}) - 2(\hat{J}^a)_{ij} (\hat{x}) (\hat{e}_j) \cdot s^a \omega_c (\hat{x})$$

$$= (2(\hat{J}_a)_{ij} \cdot s^a \omega_c (\hat{x}) - 2(\hat{J}_a)_{ij} \cdot s^a \omega_c (\hat{x}) \cdot (\hat{e}_i)_\hat{x}).$$
Therefore, \( \hat{\nabla}_X (\hat{J}_a) = 2(\hat{J}_b) \hat{\omega}_c (\hat{X}) - 2(\hat{J}_c) \hat{\omega}_b (\hat{X}) \). This concludes that

\[
(4.34) \quad \hat{\nabla} \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix} = 2 \begin{pmatrix} 0 & s^* \omega_3 & -s^* \omega_2 \\ -s^* \omega_3 & 0 & s^* \omega_1 \\ s^* \omega_2 & -s^* \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \end{pmatrix}.
\]

As we put \( \hat{J}_1 = \hat{I}_1 \), \( \hat{J}_2 = \hat{I}_2 \), \( \hat{J}_3 = \hat{K}_i \) on \( \mathbb{V} \), \( (U/E, \hat{g}, \{\hat{I}_i, \hat{J}_i, \hat{K}_i\} \in \Lambda) \) is a quaternionic pseudo-Kähler manifold for \( \text{dim } U/E \leq 8 \). Using the Ricci identity (cf. \( (2.11), (2.12) \) of \([13],[30]\)) , a calculation shows that

\[
(5.2) \quad \mathcal{R}_{jl} = -4(n + 2) \left( s^* (d\omega_1 + 2\omega_2 \wedge \omega_3) \right) (\hat{e}_j, \hat{e}_k) \hat{K}^k_l (\hat{x}).
\]

\[
(5.4) \quad \mathcal{R}_{jl} = -4(n + 2) \left( s^* (d\omega_1 + 2\omega_2 \wedge \omega_3) \right) (\hat{e}_j, \hat{e}_k) \hat{K}^k_l (\hat{x}).
\]

Using \( d\omega_a + 2\omega_b \wedge \omega_c = -J^a_{ij} \theta^j \wedge \theta^i \) and \([13],[29]\), it follows that \( s^* (d\omega_a + 2\omega_b \wedge \omega_c) (\hat{e}_j, \hat{e}_k) = -J^a_{jk} (s(\hat{x})) = -J^a_{jk} (\hat{x}) \). Since \( (\hat{J}^a)^k_j, (\hat{J}^a)^k_J = -\delta^k_j \), \( \hat{R}_{jl} = +4(n + 2)(\hat{J}^a)^k_j (\hat{x}) \cdot (\hat{J}^a)^k_J (\hat{x}) = 4(n + 2)g_{jf} \) when \( n > 1 \) and \( \hat{R}_{jl} = +4(\hat{I}_{jk} (\hat{x}) \cdot \hat{I}^k_j (\hat{x}) + \hat{J}_{jk} (\hat{x}) \cdot \hat{J}^k_j (\hat{x}) + \hat{K}_{jk} (\hat{x}) \cdot \hat{K}^k_j (\hat{x})) = 4.3g_{jf} \) when \( n = 1 \). 

5. QUATERNIONIC CR CURVATURE TENSOR

Recall from \([4.20]\) that \( d\theta^j = \theta^j \wedge \omega_j, \omega_j \wedge \omega_j = 0 \) where \( \pi^* \omega_j = \omega_j, \pi^* \hat{\theta}^j = \theta^j \) from \([4.21]\), \([4.30]\) respectively \((i, j = 1, \cdots, 4n)\). Define the fourth-order tensor \( \hat{R}^i_{jk} \) on \( U \) by putting

\[
(5.1) \quad d\omega_j^j - \omega_j^j \wedge \omega_a^i \equiv \frac{1}{2} \hat{R}^i_{jk} \theta^k \wedge \theta^l \text{ mod } \omega_1, \omega_2, \omega_3.
\]

By \([4.24]\), it follows that

\[
(5.2) \quad \hat{R}^i_{jk} = \pi^* \hat{R}^i_{jk}.
\]

The equality \([4.27]\) implies that

\[
(5.3) \quad R_{jk} = 4(n + 2)g_{jk}.
\]

Differentiate the structure equation \([4.24]\).

\[
(5.4) \quad 0 = d\theta^j \wedge \varphi_j^j - \theta^j \wedge d\varphi_j^j + \sum_a d\omega_a \wedge \tau_a^i - \sum_a \omega_a \wedge d\tau_a^i.
\]

Substitute \([4.24]\) and \([4.21]\) into \([5.4]\):

\[
\theta^j \wedge (d\varphi_j^j - \varphi_j^k \wedge \varphi_j^k - \sum_a J^a_{kj} \theta^k \wedge \tau_a^j) + \sum_a \omega_a \wedge (d\tau_a^j - \tau_a^k \wedge \varphi_j^k) + 2\omega_2 \wedge \omega_3 \wedge \tau_1^j + 2\omega_3 \wedge \omega_1 \wedge \tau_2^j + 2\omega_1 \wedge \omega_2 \wedge \tau_3^j = 0.
\]

This implies that

\[
(5.5) \quad \theta^j \wedge (d\varphi_j^j - \varphi_j^k \wedge \varphi_j^k - \sum_a J^a_{kj} \theta^k \wedge \tau_a^j) \equiv 0 \text{ mod } \omega_1, \omega_2, \omega_3.
\]
We use (5.5) to define the curvature form:

\begin{equation}
\Phi_j^i = d\varphi_j^i - \varphi_j^k \wedge \varphi_k^i + \sum_{a=1}^{3} \theta^k \wedge J_{jk}^a \tau_a^i - \theta^i \wedge \theta_j.
\end{equation}

Set

\begin{align*}
1\Phi^i &= d\tau_1^i - \tau_1^k \wedge \varphi_k^i + \omega_2 \wedge \tau_3^i - \omega_3 \wedge \tau_2^i, \\
2\Phi^i &= d\tau_2^i - \tau_2^k \wedge \varphi_k^i + \omega_3 \wedge \tau_1^i - \omega_1 \wedge \tau_3^i, \\
3\Phi^i &= d\tau_3^i - \tau_3^k \wedge \varphi_k^i + \omega_1 \wedge \tau_2^i - \omega_2 \wedge \tau_1^i,
\end{align*}

which satisfy the following relation.

\begin{equation}
\theta^j \wedge \Phi_j^i + \omega_1 \wedge 1\Phi^i + \omega_2 \wedge 2\Phi^i + \omega_3 \wedge 3\Phi^i = 0.
\end{equation}

We may define the fourth-order curvature tensor $T_{ijkl}^j$ from $\Phi^i_j$:

\begin{equation}
\Phi^i_j = \frac{1}{2} T_{ijkl}^j \theta^k \wedge \theta^l \mod \omega_1, \omega_2, \omega_3.
\end{equation}

**Remark 5.1.** In view of (5.9), there exist the fourth-order curvature tensors $W_{jka}^i (a = 1, 2, 3)$ and $V_{jbc}^i (1 \leq b < c \leq 3)$ for which we can describe:

\begin{equation}
\Phi^i_j = \frac{1}{2} T_{ijkl}^j \theta^k \wedge \theta^l + \frac{1}{2} \sum_a W_{jka}^i \theta^k \wedge \omega_a + \frac{1}{2} \sum_{b < c} V_{jbc}^i \omega_b \wedge \omega_c.
\end{equation}

6. Transformation of pseudo-conformal $QCR$ structure

6.1. $G$-structure. When $\{\theta^i\}_{i=1, \ldots, 4n}$ are the 1-forms locally defined on a neighborhood $U$ of $M$, we form the $\mathbb{H}$-valued 1-form $\{\omega^i\}_{i=1, \ldots, n}$ such as

\begin{equation}
\omega^i = \theta^i + \theta^{2n+i} + \theta^{3n+i}.
\end{equation}

We shall consider the transformations $f: U \to U$ of the following form:

\begin{align*}
\omega^i &= \lambda \cdot \omega \cdot \lambda (= u^2 \cdot \omega \cdot \bar{a}), \\
f^*(\omega^j) &= U^j_\ell \omega^\ell \cdot \bar{\lambda} + \lambda \bar{\ell} \cdot \omega \bar{\lambda}
\end{align*}

such that $\lambda = u \cdot a$ for some smooth functions $u > 0$, $a \in \text{Sp}(1)$ and $U' \in \text{Sp}(p, q)$ with $p + q = n$. Let $G$ be the subgroup of $\text{GL}(n+1, \mathbb{H}) \cdot \mathbb{H}^*$ consisting of matrices

\begin{equation}
\begin{pmatrix}
\lambda & 0 \\
\lambda \cdot \bar{\ell} & U' \end{pmatrix} \cdot \lambda.
\end{equation}

Recall that $\text{Sim}(\mathbb{H}^n) = \mathbb{H}^n \times (\text{Sp}(p, q) \cdot \mathbb{H}^*)$ is the quaternionic affine similarity group where $\mathbb{H}^* = \text{Sp}(1) \times \mathbb{R}^+$. Then note that $G$ is anti-isomorphic to $\text{Sim}(\mathbb{H}^n)$ given by the map

\begin{equation}
\begin{pmatrix}
\lambda x^j \\
0 \end{pmatrix} \cdot \lambda \rightarrow (X x^j, X \cdot \lambda) \in \mathbb{H}^n \times (\text{Sp}(p, q) \cdot \mathbb{H}^*).
\end{equation}
(Here \(x^* = \bar{x}\).) We represent \(G\) as the real matrices. Let \(\tilde{v}\) be a vector of the quaternionic vector space \(\mathbb{H}^n\). The group \(\text{Sp}(p, q) \cdot \mathbb{H}^*\) is the subgroup of \(\text{GL}(4n, \mathbb{R})\) acting on \(\mathbb{H}^n\) by

\[
(U' \cdot \lambda)\tilde{v} = U'\tilde{v} \cdot \bar{\lambda}
\]

where \(U' \in \text{Sp}(p, q)\), \(\lambda \in \mathbb{H}^*\). Write \(\lambda = u \cdot a \in \mathbb{R}^+ \times \text{Sp}(1)\) so that \(\text{Sp}(p, q) \cdot \mathbb{H}^*\) is embedded into \(\mathbb{R}^+ \times \text{SO}(4p, 4q)\) in the following manner:

\[
U' \cdot \lambda(\tilde{v}) = uU'\tilde{a} = uU'\tilde{a} \circ (a\bar{a}) = u(U'\tilde{a}) \circ \text{Ad}_a(\tilde{v}) = u \cdot U\tilde{v} \quad (\tilde{v} \in \mathbb{H}^n = \mathbb{R}^{4n})
\]

in which

\[
U = U'\tilde{a} \circ \text{Ad}_a \in \text{SO}(4p, 4q),
\]

\[
\text{Ad}_a \begin{pmatrix} i \\ j \\ k \end{pmatrix} = a \begin{pmatrix} i \\ j \\ k \end{pmatrix} \tilde{a} = A \begin{pmatrix} i \\ j \\ k \end{pmatrix} \text{ for some } A \in \text{SO}(3).
\]

We put the vector \(\tilde{v}^j \in \mathbb{H}^n\) in such a way that \(\tilde{v}^j = v^j + v^{n+j}i + v^{2n+j}j + v^{3n+j}k\) \((j = 1, \ldots, n)\). Form the real \((4 \times 3)\)-matrix

\[
\begin{pmatrix} -v^j+n & -v^j+2n & -v^j+3n \\ v^j & -v^j+3n & v^j+2n \\ v^j+3n & v^j & -v^j+n \\ -v^j+2n & v^j & v^j+n \end{pmatrix}
\]

It is easy to check that

\[
\lambda\tilde{v}^j \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \bar{\lambda} = \lambda((1 \ i \ j \ k)V^j \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix})u^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_A & 0 \end{pmatrix} V^j \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.
\]

Then \(G\) is isomorphic to the subgroup of \(\text{GL}(4n + 3, \mathbb{R})\) consisting of matrices

\[
\begin{pmatrix} u^2 \cdot t_A & 0 \\ u^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_A & 0 \end{pmatrix} V^1 & u \cdot U \\ \vdots \\ u^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_A & 0 \end{pmatrix} V^n \end{pmatrix}
\]
Here $A \in \text{SO}(3)$, $U = (U^i_j) \in \text{SO}(4p, 4q)$.

Using the coframe field $\{\omega_1, \omega_2, \omega_3, \theta^1, \cdots, \theta^{4n}\}$, $f$ is represented by

$$f^*(\omega_1, \omega_2, \omega_3) = u^2(\omega_1, \omega_2, \omega_3)A,$$

$$f^*\theta^i = u^k U^i_k + \sum_{\alpha=1}^3 \omega_\alpha v^i_\alpha,$$

where

$$ \begin{pmatrix} v^{i_j}_{j-3} & v^{i_j}_{j-2} & v^{i_j}_{j-1} & v^{i_j}_{j} \\ v^{i_{j-2}}_{j-2} & v^{i_{j-2}}_{j-1} & v^{i_{j-2}}_{j} & v^{i_{j-2}}_{j+1} \\ v^{i_{j-1}}_{j-2} & v^{i_{j-1}}_{j-1} & v^{i_{j-1}}_{j} & v^{i_{j-1}}_{j+1} \\ v^{i_j}_{j} & v^{i_j}_{j+1} & v^{i_j}_{j+2} & v^{i_j}_{j+3} \end{pmatrix} = u^2 \begin{pmatrix} 1 & 0 & tA \\ 0 & tA \end{pmatrix} V^j (j = 1, \cdots, n).$$

Let $\mathcal{F}(M)$ be the principal coframe bundle over $M$. A subbundle $P$ of $\mathcal{F}(M)$ is said to be a bundle of the nondegenerate integrable $G$-structure if $P$ is the total space of the principal bundle $G \to P \to M$ whose points consist of such coframe fields $\{\omega_1, \omega_2, \omega_3, \theta^1, \cdots, \theta^{4n}\}$ satisfying the condition (1.1), (1.9), (1.10). A diffeomorphism $f : M \to M$ is a $G$-automorphism if the derivative $f^* : \mathcal{F}(M) \to \mathcal{F}(M)$ induces a bundle map $f^* : P \to P$ in which $f^*$ has the form locally as in (6.2) (equivalently (6.12)).

**Definition 6.1.** Let $\text{Aut}_{\text{QC}}(M)$ be the group of all $G$-automorphisms of $M$.

**6.2. Automorphism group $\text{Aut}(M)$.** Let $W$ be the $(n + 2)$-dimensional arithmetic vector space $\mathbb{H}^{p+1,q+1}$ over $\mathbb{H}$ equipped with the standard Hermitian metric $\mathcal{B}$ of signature $(p + 1, q + 1)$ where $p + q = n$. Then note that the isometry group $\mathcal{G} = \text{Aut}(W, \mathcal{B}) = \text{Sp}(p + 1, q + 1)$ and $W$ has the gradation $W = W^{-1} + W^0 + W^1$, where $W^{\pm 1}$ are dual 1-dimensional isotropic subspaces and $W^0$ is (h-non-degenerate) orthogonal complement to $W^{-1} + W^1$. The gradation $W$ induces the gradation of the Lie algebra $\mathfrak{g}$ of depth two, i.e.

$$\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2.$$

Here $\mathfrak{g}^0 = \mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(n)$.

In $\mathbb{K}$ we introduced a notion of pseudo-conformal quaternionic structure. This geometry is defined by a codimension three distribution $\mathcal{H}$ on a $(4n + 3)$-dimensional manifold $M$, which satisfies the only one condition that the associated graded tangent space $\text{gr} T_x M = T_x M/\mathcal{H}_x + \mathcal{H}_x$ at any point is isomorphic to the quaternionic Heisenberg Lie algebra $\mathfrak{M}(p, q) \cong \mathfrak{g}^+ = \mathfrak{g}^{-2} + \mathfrak{g}^{-1}$, i.e. the Iwasawa subalgebra of $\text{Sp}(p+1, q+1)$. Put $\text{Sp}(W) = \text{Sp}(p+1, q+1)$. We proved that such a geometry is a parabolic geometry so that it admits a canonical Cartan connection and its automorphism group $\text{Aut}(M)$ is a Lie group. More precisely, if $P^+(\mathbb{H})$ is the parabolic connected subgroup of the symplectic group $\text{Sp}(W)$ corresponding to the dual parabolic subalgebra $\mathfrak{p}^+(\mathbb{H}) = \mathfrak{g}^+ + \mathfrak{g}^0$ of $\mathfrak{sp}(W)$, then there is a $P^+(\mathbb{H})$-principal bundle $\pi : B \to M$ with a normal Cartan connection $\kappa : TB \to \mathfrak{sp}(W)$ of type $\text{Sp}(W)/P^+$. There exists the canonical pseudo-conformal quaternionic structure $\mathcal{H}^\text{can}$ on $\text{Sp}(p+1, q+1)/P^+(\mathbb{H})$ with all vanishing curvature tensors (cf. 6.2). A pseudo-conformal quaternionic manifold $(M, \mathcal{H})$ is locally isomorphic to a $(\text{Sp}(p+1, q+1)/P^+(\mathbb{H}), \mathcal{H}^\text{can})$ if and only if the associated Cartan connection $\kappa$ is flat (i.e. has zero curvature). Put $S^{4p+3,4q} = \text{Sp}(p+1, q+1)/P^+(\mathbb{H})$. Then $S^{4p+3,4q}$ is the flat homogeneous model diffeomorphic to $S^{4p+3} \times S^{4q+3}/\text{Sp}(1)$ where the product of spheres $S^{4p+3} \times S^{4q+3} = \{(z^+, z^-) \in \mathbb{H}^{p+1,q+1} | B(z^+, z^-) = 1, B(z^-, z^-) = -1\}$ is the subspace of $W = \mathbb{H}^{p+1,q+1}$ and the action of $\text{Sp}(1)$ is induced by the diagonal right action on $W$. The group of all automorphisms $\text{Aut}(S^{4p+3,4q})$ preserving this flat structure is $\text{PSp}(p+1, q+1)$. 


Suppose that \( M \) is a pseudo-conformal quaternionic CR manifold. By definition, \( T_xM \cong T_xM/D_x \cong D_x = \text{Im}H + H^n \cong \mathcal{M}(p,q) \) at \( \forall x \in M \). Then each \( G \)-automorphism of \( \text{Aut}_{\text{QCR}}(M) \) preserves \( \mathcal{M}(p,q) \) by the above formula (6.12). Since a pseudo-conformal quaternionic CR structure is a pseudo-conformal quaternionic structure by Definition 1.3, note that \( \text{Aut}_{\text{QCR}}(M) \) is a (closed) subgroup of \( \text{Aut}(M) \) which is a Lie group as above.

**Corollary 6.2.** The group \( \text{Aut}_{\text{QCR}}(M) \) is a finite dimensional Lie group for a pseudo-conformal quaternionic CR manifold \( M \).

### 7. Pseudo-conformal QCR structure on \( S^{3+4p,4q} \)

We shall prove that the pseudo-conformal CR homogeneous model \( \Sigma_{\mathbb{H}}^{3+4p,4q} \) induces a pseudo-conformal quaternionic CR structure on \( S^{3+4p,4q} \) which coincides with the flat pseudo-conformal quaternionic structure.

#### 7.1. Quaternionic pseudo-hyperbolic geometry

Let

\[
B(z, w) = \bar{z}_1w_1 + \bar{z}_2w_2 + \cdots + \bar{z}_{p+1}w_{p+1} - \bar{z}_{p+2}w_{p+2} - \cdots - \bar{z}_{n+2}w_{n+2}
\]

be the above Hermitian form on \( \mathbb{H}^{n+2} = \mathbb{H}^{p+1,q+1} \) \((p+q = n)\). We consider the following subspaces in \( \mathbb{H}^{n+2} - \{0\} \):

\[
V_0^{4n+7} = \{ z \in \mathbb{H}^{n+2} | B(z, z) = 0 \},
\]

\[
V_-^{4n+8} = \{ z \in \mathbb{H}^{n+2} | B(z, z) < 0 \}.
\]

Let \( \mathbb{H}^* \rightarrow ((\text{Sp}(p+1,q+1) \cdot \mathbb{H}^*, \mathbb{H}^{n+2} - \{0\}) \xrightarrow{P} (\text{PSp}(p+1,q+1), \mathbb{H}P^{n+1}) \) be the equivariant projection. The quaternionic pseudo-hyperbolic space \( \mathbb{H}^{p+1,q}_{\mathbb{H}} \) is defined to be \( P(V_-^{4n+8}) \) (cf. [10]). Let \( \text{GL}(n+2, \mathbb{H}) \) be the group of all invertible \((n+2) \times (n+2)\)-matrices with quaternion entries. Denote by \( \text{Sp}(p+1,q+1) \) the subgroup consisting of

\[
\{ A \in \text{GL}(n+2, \mathbb{H}) | B(Az, Aw) = B(z, w), z, w \in \mathbb{H}^{n+2} \}.
\]

The action \( \text{Sp}(p+1,q+1) \) on \( V_-^{4n+8} \) induces an action on \( \mathbb{H}^{{p+1,q}_\mathbb{H}} \). The kernel of this action is the center \( \mathbb{Z}/2 = \{ \pm 1 \} \) whose quotient is the pseudo-quaternionic hyperbolic group \( \text{PSp}(p+1,q+1) \).

It is known that \( \mathbb{H}^{{p+1,q}_\mathbb{H}} \) is a complete simply connected pseudo-Riemannian manifold of negative sectional curvature from \(-1\) to \(-1/4\), and with the group of isometries \( \text{PSp}(p+1,q+1) \) (cf. [13]). Remark that when \( q = 0, p = n \), \( P(V_-^{4n+8}) = \mathbb{H}^{{p+1,q}_\mathbb{H}} \) is the quaternionic Kähler hyperbolic space with the group of isometries \( \text{PSp}(n+1,1) \). The projective compactification of \( \mathbb{H}^{p+1,q}_{\mathbb{H}} \) is obtained by taking the closure \( \overline{\mathbb{H}^{p+1,q}_{\mathbb{H}}} \) in \( \mathbb{H}P^{n+1} \). Then it is easy to check that \( \overline{\mathbb{H}^{p+1,q}_{\mathbb{H}}} = \mathbb{H}^{p+1,q}_{\mathbb{H}} \cup P(V_-^{4n+7}) \).

The boundary \( P(V_-^{4n+7}) \) of \( \mathbb{H}^{p+1,q}_{\mathbb{H}} \) is identified with the quadric \( S^{3+4p,4q} \) by the correspondence:

\[
[z_+, z_-] \mapsto \begin{pmatrix} z_+ \\ z_- \\ \overline{z_+} \\ \overline{z_-} \end{pmatrix}.
\]

Since the pseudo-hyperbolic action of \( \text{PSp}(p+1,q+1) \) on \( \mathbb{H}^{p+1,q}_{\mathbb{H}} \) extends to a smooth action on \( S^{3+4p,4q} = P(V_-^{4n+7}) \) as projective transformations because the projective compactification \( \overline{\mathbb{H}^{p+1,q}_{\mathbb{H}}} \) is an invariant domain of \( \mathbb{H}P^{n+1} \).
7.2. Existence of pseudo-conformal QCR structure on $S^{3+4p,4q}$. Recall that $\Sigma_{\mathbb{H}}^{3+4p,4q} = \{(z_1, \ldots, z_{p+1}, w_1, \ldots, w_q) \in \mathbb{H}^{n+1} \mid |z_1|^2 + \cdots + |z_{p+1}|^2 - |w_1|^2 - \cdots - |w_q|^2 = 1\}$ equipped with the quaternionic CR structure $\omega_0$ (cf. 5.3). The embedding $\iota$ of $\Sigma_{\mathbb{H}}^{3+4p,4q}$ into $S^{4p+3,4q}$ is defined by $(z_1, \ldots, z_{p+1}, w_1, \ldots, w_q) \mapsto ([z_1, \ldots, z_{p+1}, w_1, \ldots, w_q, 1])$. Then $\iota(\Sigma_{\mathbb{H}}^{3+4p,4q})$ is an open dense submanifold of $S^{4p+3,4q}$ since it misses the subspace $S^{4p+3,4q-1} = S^{4p+3} \times S^{4q-1}/\text{Sp}(1)$ in $S^{4p+3,4q}$. We know that $\Sigma_{\mathbb{H}}^{3+4p,4q}$ has the transitive isometry group $\text{Sp}(p+1, q) \cdot \text{Sp}(1)$ (cf. Definition 3.1). Then this embedding implies that $\text{Sp}(p+1, q) \cdot \text{Sp}(1)$ is identified with the subgroup $P(\text{Sp}(p+1, q) \times \text{Sp}(1))$ of $\text{PSp}(p+1, q+1)$ leaving the last component $z_{n+2}$ invariant in $V_0^{4n+7} \subset \mathbb{H}^{n+2}$.

By pullback, each element $h$ of $\text{PSp}(p+1, q+1)$ gives a quaternionic CR structure $h^{-1} \omega_0$ on the open subset $h(\Sigma_{\mathbb{H}}^{3+4p,4q})$ of $S^{3+4p,4q}$. Noting that $h^{-1} \mathcal{H}^{\text{can}} = \mathcal{H}^{\text{can}}_h$ and Definition 3.1, we shall prove that $(S^{3+4p,4q}, \mathcal{H}^{\text{can}}_h)$ admits a pseudo-conformal quaternionic CR structure by showing that $\text{Null} h^{-1} \omega_0$ coincides with the restriction of $\mathcal{H}^{\text{can}}_h | h(\Sigma_{\mathbb{H}}^{3+4p,4q})$.

Theorem 7.1. The $(4n+3)$-dimensional pseudo-conformal quaternionic manifold $(S^{4p+3,4q}, \mathcal{H}^{\text{can}})$ supports a pseudo-conformal quaternionic CR structure, i.e. there exists locally a quaternionic CR structure $\omega$ on a neighborhood $U$ such that

$$\mathcal{H}^{\text{can}} | U = \text{Null} \omega.$$  

Moreover, the automorphism group $\text{Aut}_{\text{QCR}}(S^{4p+3,4q})$ with respect to this pseudo-conformal quaternionic CR structure is $\text{PSp}(p+1, q+1)$.

Proof. First we describe the canonical pseudo-conformal quaternionic structure $\mathcal{H}^{\text{can}}$ on $S^{3+4p,4q}$ explicitly. Choose isotropic vectors $x, y \in V_0$ such that $\mathcal{B}(x, y) = 1$ and denote by $V$ the orthogonal complement to $\{x, y\}$ in $\mathbb{H}^{n+1+q+1}$. Then it follows that $T_x V_0 = \mathfrak{sp}(W)x = y \mathbb{I} \mathbb{M} + V + x \mathbb{H}$ where $T_x (x \mathbb{H}^*) = x \mathbb{H}$. Then

$$T_x S^{4k+3,4q} = P_*(T_x V_0) = (y \mathbb{I} \mathbb{M} + V + x \mathbb{H})/x \mathbb{H}.$$  

We associate to each $[x] \in S^{4k+3,4q}$ the orthogonal complement $x^\perp = V + x \mathbb{H}$. It does not depend on the choice of points from $[x]$. In fact, if $x' \in [x]$, then $x' = x \cdot \lambda$ for some $\lambda \in \mathbb{H}^*$. By the definition choosing $y'$ such that $T_{x'} V_0 = y' \mathbb{I} \mathbb{M} + V' + x' \mathbb{H}$ where the orthogonal complement $V'$ to $\{x', y'\}$ in $\mathbb{H}^{n+1+q+1}$ is uniquely determined. Let $v'$ be any vector of $V'$ which is described as $v' = y \cdot a + v + x \cdot b$ for some $a, b \in \mathbb{H}$. Then

$$0 = \mathcal{B}(x', v') = \mathcal{B}(x', y)a + \mathcal{B}(x', v) + \mathcal{B}(x', x)b$$

$$= \lambda \mathcal{B}(x, y)a + \lambda \mathcal{B}(x, v) + \lambda \mathcal{B}(x, x)b = \lambda \cdot a.$$  

Since $\lambda \neq 0$, $a = 0$ and so $v' = v + x \cdot b$. Hence $x^\perp = V + x \mathbb{H} = V + x \mathbb{H}$. Therefore the orthogonal complement $x^\perp = V + x \mathbb{H}$ in $\mathbb{H}^{n+1+q+1}$ determines a codimension three subbundle

$$\mathcal{H}^{\text{can}} = \bigcup_{[x] \in S^{4k+3,4q}} P_*(x^\perp).$$  

(7.3)

$$P_*(x^\perp) = V + x \mathbb{H}/x \mathbb{H} \subset T S^{4p+3,4q}.$$  

On the other hand, recall that if $N_p$ is the normal vector at $p \in \Sigma_{\mathbb{H}}^{3+4p,4q}$, then $(\text{Null} \omega)_p = D_p = \{IN_p, JN_p, KN_p\}$ by the definition (cf. 5.3). Since $T_p \Sigma_{\mathbb{H}}^{3+4p,4q} = N_p^\perp$ with respect to $g^\mathcal{B}$, it follows that $T_p \mathbb{H}^{n+1+3+4p,4q} = \{N_p, IN_p, JN_p, KN_p\} = D_p$. If we note that $\{N_p, IN_p, JN_p, KN_p\} = p \mathbb{H}$, then we have $D_p = p \mathbb{H}^\perp$. It is easy to see that the orthogonal complement to $p \mathbb{H}^\perp$ with respect to $g^\mathcal{B}$ coincides with the orthogonal complement to $p$ with respect to the inner product $\mathcal{B} = \langle \cdot, \cdot \rangle$. Hence, $D_p = p^\perp$. As the tangent subspace $\iota_*(D_p)$ at $\iota(p)$ in $T_{\iota(p)} V_0$ is $(D_p, 0)$ which is parallel
to $D_p$ in $T_p V_0$, it implies that $B(\iota_*(D_p), \iota(p)) = B((D_p, 0), (p, 1)) = (D_p, p) - (0, 1) = 0$. Hence $\iota_*(D_p) \subset \iota(p)^{\perp}$ (with respect to $B$). As $\iota(p)^{\perp} = V + \iota(p)\mathbb{H}$, $\iota_*(D_p) \subset V + \iota(p)\mathbb{H}$. As above $\iota_*(D_p) = (D_p, 0)$ at $\iota(p)$, but $\iota(p)\mathbb{H} = (p, 1) \cdot H$. The intersection $\iota_*(D_p) \cap \iota(p)\mathbb{H} = \{0\}$. It implies that $\iota_*(D_p) = \iota_*(D_p)/\iota(p)\mathbb{H} \subset V + \iota(p)\mathbb{H}/\iota(p)\mathbb{H}$. By $\iota_*(\text{Null } \omega_0)_p = P_\omega(\iota(p)^{\perp}) = H_{\iota(p)}$. Therefore $S^{4p+3.4q}$ admits a pseudo-conformal quaternionic $CR$ structure. Then $\text{Aut}_{QR}(S^{4p+3.4q})$ is a subgroup of $\text{Aut}(S^{4p+3.4q}) = \text{PSp}(p + 1, q + 1)$ from Proposition 7.2.

To prove $\text{Aut}_{QR}(S^{4p+3.4q}) = \text{PSp}(p + 1, q + 1)$, we recall the quaternionic Heisenberg Lie group.

### 7.3. Pseudo-conformal quaternionic Heisenberg geometry

Let $\text{PSp}(p + 1, q + 1)$ be the group of all automorphisms preserving the flat pseudo-conformal quaternionic structure of $S^{4p+3.4q} = \text{PSp}(p + 1, q + 1)/P^+(\mathbb{H})$ (cf. [13]). We consider the stabilizer of the point at infinity $\{\infty\} = \left\{1, 0, \ldots, 0, 1\right\} \in \Sigma_{\mathbb{H}}^{3+4p,4q} \subset S^{4p+3.4q}$. Recall the (indefinite) Heisenberg nilpotent Lie group $M = M(p, q)$ from [13]. It is the product $\mathbb{R}^3 \times \mathbb{H}^n$ with group law:

$$\langle a, y \rangle \cdot \langle b, z \rangle = (a + b - \text{Im}\langle y, z \rangle, y + z).$$

Here $\langle \cdot, \cdot \rangle$ is the Hermitian inner product of signature $(p, q)$ on $\mathbb{H}^n$ and $\text{Im}(\langle \cdot, \cdot \rangle)$ is the imaginary part $(p + q = n)$. It is nilpotent because the commutator subgroup $[M, M] = \mathbb{R}^3$ which is the center consisting of the form $(a, 0)$. In particular, there is the central extension:

$$\mathbb{R}^3 \to M \to \mathbb{H}^n \to 1.$$

Denote by $\text{Sim}(M)$ the semirective product $M \rtimes (\text{Sp}(p, q) \cdot \text{Sp}(1) \times \mathbb{R}^+)$ where the action $(A \cdot g, t) \in \text{Sp}(p, q) \cdot \text{Sp}(1) \times \mathbb{R}^+$ on $(a, y) \in M$ is given by:

$$(A \cdot g, t) \circ (a, y) = (t^2 \cdot gag^{-1}, t \cdot Ayg^{-1}).$$

Denote the origin by $O = \{1, 0, \ldots, 0, -1\} \in \Sigma_{\mathbb{H}}^{3+4p,4q} - \{\infty\}$. Then, the stabilizer $\text{Aut}(S^{3+4p,4q})_\infty$ is isomorphic to $\text{Sim}(M)$ (cf. [13]). The orbit $M \cdot O$ is a dense open subset of $S^{4p+3.4q}$. The embedding $\iota$ is defined by:

$$\iota(a, b, c, (z_+, z_-)) \in M \to \begin{bmatrix} \frac{|z_+|^2 - |z_-|^2}{2} - 1 + ia + jb + kc \\ \sqrt{2}z_+ \\ \sqrt{2}z_- \\ \frac{|z_+|^2 - |z_-|^2}{2} + 1 + ia + jb + kc \end{bmatrix}$$

Then the pair $(\text{Sim}(M), M)$ is said to be pseudo-conformal quaternionic Heisenberg geometry which is a subgeometry of the flat pseudo-conformal quaternionic geometry $(\text{Aut}(S^{3+4p,4q}), S^{3+4p,4q})$. We prove the rest of Theorem 7.1.

### Proposition 7.2

$\text{Aut}_{QR}(S^{4p+3.4q}) = \text{PSp}(p + 1, q + 1)$.

**Proof.** First note that $\text{PSp}(p + 1, q + 1)$ decomposes into $\text{Sim}(M) \cdot (\text{Sp}(p + 1, q) \cdot \text{Sp}(1))$. We know from [13] that each element $f = (A, a) \in \text{Sp}(p + 1, q) \cdot \text{Sp}(1)$ satisfies that $f^* \omega_0 = a \omega_0 a$, obviously $f \in \text{Aut}_{QR}(S^{4p+3.4q})$. On the other hand, it is shown that an element $h$ of $\text{Sim}(M)$ satisfy $h^* \omega_0 = \lambda \omega_0 \bar{\lambda}$ for some function $\lambda \in \mathbb{H}^*$ by using the explicit formula of $\omega_0$. (See [13].) When $h \in \text{Sim}(M)$, note that $h(\infty) = \infty$. Let $\tau : \text{PSp}(p + 1, q + 1, \infty) \to \text{Aut}(T(\infty)(S^{3+4p,4q}))$ be the tangential representation at $\infty$. Since the elements of the center $\mathbb{R}^3$ of $M$ are tangentially identity maps at $T(\infty)(S^{3+4p,4q})$, $\tau(\text{PSp}(p + 1, q + 1, \infty)) = \mathbb{R}^n \times (\text{Sp}(p, q) \cdot \text{Sp}(1) \times \mathbb{R}^+)$ which is isomorphic to the structure group $G$ (cf. [6.11]). As $\tau(h) = h_*$, $h \in \text{Aut}_{QR}(S^{3+4p,4q})$ by Definition 7.1. We have $\text{PSp}(p + 1, q + 1) \subset \text{Aut}_{QR}(S^{3+4p,4q})$. \qed
8. PSEUDO-CONFORMAL QUATERNIONIC CR INVARIANT

We shall consider the equivalence of pseudo-conformal quaternionic CR structure. Let \( d\omega + \omega \wedge \omega = -(I_{ij}i + J_{ij}j + K_{ij}k)\theta^i \wedge \theta^j \) be the equation (4.3) as before. We examine how this equation behaves under the transformation \( f \in \text{Aut}_{\text{CR}}(M) \); \( f^*\omega = \lambda \cdot \omega \cdot \dot{\lambda} \). Put \( \omega' = f^*\omega \). By (6.12),

\[
d\omega' + \omega' \wedge \omega' = f^*(d\omega + \omega \wedge \omega) = -(I_{ij}i + J_{ij}j + K_{ij}k)f^*\theta^i \wedge f^*\theta^j
\]

\[
= -(I_{ij}i + J_{ij}j + K_{ij}k)((u^2 U^i_k U^j_k \theta^k \wedge \theta^\ell + \\
\sum_a \omega_a \wedge (u^a_k U^j_k \theta^\ell - u^a_k U^j_k \theta^\ell) + \sum_{a<b} \omega_a \wedge \omega_b(u^a_k v^b_k v^i_k v^j_k))
\]

\[
= -(I_{ij}i + J_{ij}j + K_{ij}k)((u^2 U^i_k U^j_k \theta^k \wedge \theta^\ell + \sum_a \omega_a \wedge 2u^a_k U^j_k \theta^\ell + \sum_{a<b} \omega_a \wedge \omega_b(2u^a_k U^j_k w^b_k v^i_k)).
\]

Choosing \( w^a_k (a = 1, 2, 3) \) such that \( U^i_k w^k_a = v^i_a \), the above equation becomes

\[
d\omega' + \omega' \wedge \omega' = -(I_{ij}i + J_{ij}j + K_{ij}k)((u^2 U^i_k U^j_k \theta^k \wedge \theta^\ell + \\
\sum_a \omega_a \wedge 2u^a_k U^j_k U^j_k \theta^\ell + \sum_{a<b} \omega_a \wedge \omega_b(2U^i_k U^j_k w^b_k v^i_k)).
\]

Let \( U = U' \bar{a} \circ \text{Ad}_a \in \text{SO}(4p, 4q) \) be the matrix as in (6.7) so that \( Uz = U'z \bar{a} (z \in \mathbb{H}^n) \) (cf. (6.6)).

If \( \{I, J, K\} \) is the set of the standard quaternionic structure, then

\[
IU(z) = I(U'z \bar{a}) = U'z \bar{a} = U'z(a \bar{i} a) \bar{a}
\]

\[
= U'z(a_{11}i + a_{21}j + a_{31}k)\bar{a} = a_{11}U'z\bar{i}a + a_{21}U'z\bar{j}a + a_{31}U'z\bar{k}a
\]

\[
= a_{11}U(z\bar{i}) + a_{21}U(z\bar{j}) + a_{31}U(z\bar{k}) = a_{11}UJ(z) + a_{21}UI(z) + a_{31}UK(z).
\]

This follows that \( IU = a_{11}UI + a_{21}UI + a_{31}UK \). Since \( IU(e_i) = U^i_j e_i e_j \), a calculation shows that \( U^i_k U^j_k = a_{11}I^i_k I^j_k + a_{21}J^i_k J^j_k + a_{31}K^i_k K^j_k \), similarly for \( J, K \). As

\[
(8.1)
\]

\[
\begin{pmatrix}
I' \\
J' \\
K'
\end{pmatrix} = ^tA
\begin{pmatrix}
I \\
J \\
K
\end{pmatrix}
\]

is a new quaternionic structure (cf. (1.6)), it follows that

\[
I_{ij}U^i_k U^j_k = a_{11}I_{kl} + a_{21}J_{kl} + a_{31}K_{kl} = I'_{kl}.
\]

\[
(8.2)
\]

\[
J_{ij}U^i_k U^j_k = a_{12}I_{kl} + a_{22}J_{kl} + a_{32}K_{kl} = J'_{kl}.
\]

\[
K_{ij}U^i_k U^j_k = a_{13}I_{kl} + a_{23}J_{kl} + a_{33}K_{kl} = K'_{kl}.
\]

Then we obtain that

\[
d\omega' + \omega' \wedge \omega' = -(I'_{ij}i + J'_{ij}j + K'_{ij}k)((u^2 \theta^i \wedge \theta^j + \\
\sum_a \omega_a \wedge 2u^a_k \theta^j + \sum_{a<b} \omega_a \wedge \omega_b(2u^a_k w^b_k v^i_k)).
\]
We shall derive an invariant under the change $\omega' = \lambda \cdot \omega \cdot \bar{\lambda}$. Recall from (5.12) that

\begin{equation}
(\omega'_1, \omega'_2, \omega'_3) = (\omega_1, \omega_2, \omega_3)u^2 \cdot A.
\end{equation}

Let $d\theta^i = \theta^i \wedge \varphi_j^i + \sum_a \omega_a \wedge \tau_a^i$ be the structure equation (8.21). We define 1-forms $\nu_a^i$ by setting

\begin{equation}
\left( \begin{array}{c}
\nu_1^i \\
\nu_2^i \\
\nu_3^i
\end{array} \right) = u^{-2} \cdot A \left( \begin{array}{c}
\tau_1^i \\
\tau_2^i \\
\tau_3^i
\end{array} \right).
\end{equation}

Since $\tau_a^i \equiv 0 \mod \theta^k (k = 1, \ldots, 4n)$ by (4.21), note that

\begin{equation}
\nu_a^i \equiv 0 \mod \theta^k.
\end{equation}

Using (8.4) and (8.5),

\begin{equation}
\sum_a \omega_a \wedge \tau_a^i = (\omega'_1, \omega'_2, \omega'_3) \wedge \left( \begin{array}{c}
\nu_1^i \\
\nu_2^i \\
\nu_3^i
\end{array} \right) = \sum_a \omega'_a \wedge \nu_a^i,
\end{equation}

the equation (8.21) becomes

\begin{equation}
d\theta^i = \theta^i \wedge \varphi_j^i + \sum_a \omega'_a \wedge \nu_a^i.
\end{equation}

Differentiate (8.4), and then substitute (8.5), we obtain that

\begin{equation}
\theta^i \wedge (d\varphi_j^i - \varphi_j^a \wedge \varphi_a^i + u^2 J^1_{jk} \theta^k \wedge \nu_a^i + u^2 J^2_{jk} \theta^k \wedge \nu_a^i + u^2 J^3_{jk} \theta^k \wedge \nu_a^i) \equiv 0 \mod \omega_a.
\end{equation}

Taking into account (8.8) (which corresponds to (5.9)), we have the fourth-order tensor up to the terms $\omega_1, \omega_2, \omega_3$:

\begin{equation}
\frac{1}{2} T_{jk}^i \theta^k \wedge \theta^j \equiv d\varphi_j^i - \varphi_j^a \wedge \varphi_a^i + \sum_a u^2 \cdot J^a_{jk} \theta^k \wedge \nu_a^i - \theta^i \wedge \theta_j.
\end{equation}

Since $(I'_{ij}, J'_{ij}, K'_{ij}) = (I_{ij}, J_{ij}, K_{ij})A$ from (8.1) and (8.3),

\begin{equation}
\sum_a u^2 \cdot J^a_{jk} \theta^k \wedge \nu_a^i = \theta^k \wedge u^2 (I'_{jk}, J'_{jk}, K'_{jk}) \left( \begin{array}{c}
\nu_1^i \\
\nu_2^i \\
\nu_3^i
\end{array} \right) = \theta^k \wedge \sum_a J_{jk}^a \tau_a^i.
\end{equation}

The equation (8.9) can be reduced to the following:

\begin{equation}
T_{jk}^i \theta^k \wedge \theta_j^i \equiv d\varphi_j^i - \varphi_j^a \wedge \varphi_a^i + \theta^k \wedge \sum_a J_{jk}^a \tau_a^i - \theta^i \wedge \theta_j.
\end{equation}

From (8.4) and (8.5), we have shown

\textbf{Proposition 8.1.} If $\omega' = \lambda \cdot \omega \cdot \bar{\lambda}$ for which $\omega$ is a quaternionic CR structure, then the curvature tensor $T'$ satisfies that $T'_{jk}^i = T_{jk}^i$. In particular, $T = (T_{jk}^i)$ is an invariant tensor under the pseudo-conformal quaternionic CR structure.
Remark 8.2. 1. Similarly, the quaternionic structures \( \{ I', J', K' \} \) extends to almost complex structures \( \{ I', J', K' \} \) respectively.

2. Let \( f \in \text{Aut}_{QC}(M) \) be an element satisfying (6.12). Then, \( f_* e_i = uU^j_i e_j \). Using (8.2),

\[
I f_* e_i = uU^j_i I^j_k e_j = u(a_{11} I^m_i + a_{21} J^m_i + a_{31} K^m_i) U^j_m e_j
\]

\[
= f_*((a_{11} I^m_i + a_{21} J^m_i + a_{31} K^m_i)e_m)
\]

\[
= f_*((a_{11} I + a_{21} J + a_{31} K)e_i).
\]

The similar argument to \( J, K \) yields that

\[
(8.11) \quad \begin{pmatrix} f_*^{-1}f_* \\ f_*^{-1}Jf_* \\ f_*^{-1}Kf_* \end{pmatrix} = \ ^t A \begin{pmatrix} I \\ J \\ K \end{pmatrix} \quad \text{on } D.
\]

8.1. Formula of Curvature tensor. We shall find a formula of the tensor \( T \). Substitute (6.26, 1.24) into (8.10):

\[
T_{jkl}^i \theta^k \wedge \theta^l = d(\omega_j^i + \sum_a (J^a)_j \omega_a) - (\omega_k^i + \sum_a (J^a)_k \omega_a) \wedge (\omega_j^i + \sum_a (J^a)_j \omega_a)
\]

\[
+ \theta^k \wedge (I_{jk} \cdot I_{jl} \theta^l + J_{jk} \cdot J_{jl} \theta^l + K_{jk} \cdot K_{jl} \theta^l) - \theta^i \wedge \theta_j \mod \omega_a
\]

\[
= d\omega_j^i + \sum_a (J^a)_j d\omega_a - \omega_k^i \wedge \omega_j^i + \sum_a (J^a_{jk} (J^a)_l) \theta^k \wedge \theta^l - \theta^i \wedge \theta_j \mod \omega_a
\]

\[
= (d\omega_j^i - \omega_k^i \wedge \omega_j^i)
\]

\[
+ \sum_a (J^a)_j (-J^a_{kl}) \theta^k \wedge \theta^l + \sum_a (J^a_{jk} (J^a)_l) \theta^k \wedge \theta^l - \theta^i \wedge \theta_j \mod \omega_a
\]

\[
= \left( \frac{1}{2} R_{jkl}^i - \sum_a (J^a)_j J^a_{kl} + \sum_a J^a_{jk} J^a_{l} (\theta^k \wedge \theta^l) + (g_{jl} \delta^i_k - g_{jk} \delta^i_l) \right) \theta^k \wedge \theta^l \mod \omega_a.
\]

By alternation, we have

\[
(8.12) \quad T_{jkl}^i = R_{jkl}^i - \left( 2 \sum_a (J^a)_j J^a_{kl} - \sum_a J^a_{jk} (J^a)_l + \sum_a J^a_{jl} J^a_{k} + (g_{jl} \delta^i_k - g_{jk} \delta^i_l) \right).
\]

Recall the space of all curvature tensors \( R(\text{Sp}(p, q) \cdot \text{Sp}(1)) \). (See [1] for example.) It decomposes into the direct sum \( R_0(\text{Sp}(p, q) \cdot \text{Sp}(1)) \oplus R_{\text{HP}}(\text{Sp}(p, q) \cdot \text{Sp}(1)) \) \((n \geq 2)\). Here \( R_0 \) is the space of those curvatures with zero Ricci forms and \( R_{\text{HP}} \approx \mathbb{R} \) is the space of curvature tensors of the quaternionic pseudo-Kähler projective space \( \mathbb{H}^{p, q} \) (cf. Definition 8.2).

Case \( n \geq 2 \). Since we know that \( R_{jkl}^i = R_{jkl} = (4n + 8)g_{jl} \) from (6.33), the curvature tensor \( T = (T_{jkl}^i) \) satisfies the tracefree condition:

\[
T_{jkl} = (T_{jkl}^i) = (4n + 8)g_{jl} - \left( 3 \cdot 3g_{jl} + (4n - 1)g_{jl} \right) = 0.
\]

This implies that our curvature tensor \( T \) belongs to \( R_0(\text{Sp}(p, q) \cdot \text{Sp}(1)) \) when \( n \geq 2 \).
Case n = 1. When \( \dim M = 7 \), either \( p = 1, q = 0 \) or \( p = 0, q = 1 \). Choose the orthonormal basis \( \{ e_i \}_{i=1,2,3,4} \) with \( e_1 = e, e_2 = Ie, e_3 = J e, e_4 = K e \). Form another curvature tensor:

\[
R^i_{jkt} = (g_{ij} \delta^k_t - g_{jk} \delta^i_t) + \left[ I_{ij} I^k_t - I_{jk} I^i_t + 2I^i_j I^k_t \right]
+ J_{ij} I^k_t - J_{jk} I^i_t + 2J^i_j I^k_t + K_{ij} K^k_t - K_{jk} K^i_t + 2K^i_j K^k_t].
\]

(8.13)

For any two distinct \( e_i, e_j \),

\[
R^i_{jj} = \frac{4 \cdot 12}{4 \cdot 3} (g_{ij} \delta^i_k - g_{jk} \delta^i_t).
\]

(8.14)

When \( n = 1 \), we conclude that

\[
T^i_{jkt} = R^i_{jkt} - R^i_{jtk} = R^i_{jkt} - 4(g_{ij} \delta^i_k - g_{jk} \delta^i_t).
\]

(8.15)

As the curvature \( R^i_{jkt} \) satisfies the Einstein property from \( \text{Box} \), \( R^i_{jkt} = 4 \cdot 3 g_{ijt} \), the scalar curvature \( \sigma = 4 \cdot 12 \). On the other hand, the curvature tensor \( R^i_{jkt} \) has the decomposition:

\[
R^i_{jkt} = W^i_{jkt} + \frac{4 \cdot 12}{4 \cdot 3} (g_{ij} \delta^i_k - g_{jk} \delta^i_t)
\]

in the space \( \mathcal{R}(SO(4)) \) where \( SO(4) = Sp(1) \cdot Sp(1) \). Hence,

\[
T^i_{jkt} = W^i_{jkt} \in \mathcal{R}_0(SO(4))
\]

(8.16)

for which \( W^i_{jkt} \) is the Weyl curvature tensor (of \( (U/E, \bar{g}) \)).

Case n = 0. If \( \dim M = 3 \), then the above tensor is empty, so we simply set \( T = 0 \). Define the Riemannian metric on a neighborhood \( U \) of a 3-dimensional pseudo-conformal quaternionic CR manifold \( M \):

\[
g_x(X, Y) = \omega_1(X) \cdot \omega_1(Y) + \omega_2(X) \cdot \omega_2(Y) + \omega_3(X) \cdot \omega_3(Y)
\]

(\( \forall X, Y \in T_xU \)). Suppose that \( \omega' = \lambda \cdot \omega \cdot \bar{\lambda} \). Since \( (\omega_1', \omega_2', \omega_3') = u^2 \cdot (\omega_1, \omega_2, \omega_3)A \) for \( A \in SO(3) \), the metric \( g \) changes into \( g' = \omega_1' \cdot \omega_1' + \omega_2' \cdot \omega_2' + \omega_3' \cdot \omega_3' \) satisfying that

\[
g'_x(X, Y) = u^4 \cdot g_x(X, Y) \quad (\forall X, Y \in T_xU).
\]

(8.18)

Then \( g' \) is conformal to \( g \) on \( U \). Define \( TW(\omega) \) to be the Weyl-Schouten tensor \( TW(g) \) of the Riemannian metric \( g \) on \( U \). Then, it turns out that

\[
TW(\omega') = TW(\omega).
\]

(8.19)

As a consequence, \( TW(\omega) \) is an invariant tensor of \( U \) under the change \( \omega' = \lambda \cdot \omega \cdot \bar{\lambda} \).
9. Uniformization of pseudo-conformal QCR structure

If \{\omega^{(a)}, (I^{(a)}, J^{(a)}, K^{(a)}), g^{(a)}, U_a\}_{a \in A} is a pseudo-conformal quaternionic CR structure on \(M\) where \(\bigcup_{a \in A} U_a = M\), then we have the curvature tensor \(T^{(a)} = (^{(a)} T^{jk\ell})\) on each \((U_a, \omega^{(a)}) (n \geq 1)\). Similarly, \(T_{W}^{(a)} = T_{W}(\omega^{(a)})\) on \((U_a, \omega^{(a)})\) for 3-dimensional case \((n = 0)\). Then it follows from Proposition 8.1 and (8.19) that if \(\omega^{(j)} = \lambda_{a\beta} \cdot \omega^{(\alpha)} \cdot \hat{\lambda}_{\alpha\beta}\) on \(U_{\alpha} \cap U_{\beta}\), then

\[
T^{(\alpha)} = T^{(\beta)},
\]
\[
T_{W}^{(\alpha)} = T_{W}^{(\beta)}.
\]

By setting \(T|U_{\alpha} = T^{(\alpha)}\) (respectively \(T_{W}|U_{\alpha} = T_{W}^{(\alpha)}\)), the curvature \(T\) (respectively \(T_{W}\)) is globally defined on a \((4n + 3)\)-dimensional pseudo-conformal quaternionic CR manifold \(M\) \((n \geq 0)\). This concludes that

**Theorem 9.1.** Let \(M\) be a pseudo-conformal quaternionic CR manifold of dimension \(4n + 3\) \((n \geq 0)\). If \(n \geq 1\), there exists the fourth-order curvature tensor \(T = (T_{jk\ell}^{i})\) on \(M\) satisfying that:

(i) When \(n \geq 2\), \(T = (T_{jk\ell}^{i}) \in \mathcal{R}_{0}(Sp(p,q) \cdot Sp(1))\) which has the formula:

\[
T_{jk\ell}^{i} = R_{jk\ell}^{i} - \left\{ (g_{j\ell} \delta_{i}^{k} - g_{jk} \delta_{i}^{\ell}) + \left[ I_{j\ell} I_{k}^{i} - I_{jk} I_{\ell}^{i} + 2 I_{j}^{i} I_{k\ell} \right] \right. \\
+ \left. J_{j\ell} J_{k}^{i} - J_{jk} J_{\ell}^{i} + 2 J_{j}^{i} J_{k\ell} + J_{jk} K_{\ell}^{i} - K_{jk} K_{\ell}^{i} + 2 K_{j}^{i} K_{k\ell} \right\}.
\]

(ii) When \(n = 1\), \(T = (W_{jk\ell}^{i}) \in \mathcal{R}_{0}(SO(4))\) which has the same formula as the Weyl conformal curvature tensor.

If \(n = 0\), there exists the fourth-order curvature tensor \(T_{W}\) on \(M\) which has the same formula as the Weyl-Schouten curvature tensor.

We associated to a pseudo-conformal quaternionic CR structure \((\{\omega_{a}\}, \{J_{a}\}, \{\xi_{a}\})_{a=1,2,3}\) the pseudo-Sasakian metric \(g = \sum_{a=1}^{3} \omega_{a} \cdot \omega_{a} + \pi^{*} \hat{g}\) on \(U\) for which \(E \rightarrow (U, g) \overset{\pi}{\rightarrow} (U/E, \hat{g})\) is a pseudo-Riemannian submersion and the quotient \((U/E, \hat{g}, \{\hat{I}_{\alpha}, \hat{J}_{\beta}, \hat{K}_{\gamma}\}_{\alpha, \beta, \gamma})\) is a quaternionic pseudo-Kähler manifold by Theorem 1.1. Let \((g) R_{jk\ell}^{i}\) (respectively \((g) R_{jk\ell}^{i} (\text{HP})\)) denote the curvature tensor of \(g\) (respectively \(\hat{g}\)). If \(R_{\text{HP}}\) is the generator of \(\mathcal{R}_{\text{HP}}(Sp(p,q) \cdot Sp(1)) \approx \mathbb{R}\) \((n \geq 2)\), then it can be described as (cf. [1]):

\[
R_{\text{HP}} = (g_{j\ell} g_{ik} - g_{jk} g_{i\ell}) + \sum_{a=1}^{3} J_{j\ell}^{a} J_{ik}^{a} - \sum_{a=1}^{3} J_{jk}^{a} J_{i\ell}^{a} + 2 \sum_{a=1}^{3} J_{ij}^{a} J_{k\ell}^{a}
\]

where \(i,j,k,\ell\) run over \(\{1, \cdots, 4n\}\). Then the formula (12.8) of curvature tensor of \(g\) \([29]\) \((n \geq 1)\) shows the following.

**Lemma 9.2.**

\[
\pi^{*} \hat{R}_{ijk\ell} = (g) R_{ijk\ell} + \left( \sum_{a=1}^{3} J_{j\ell}^{a} J_{ik}^{a} - \sum_{a=1}^{3} J_{jk}^{a} J_{i\ell}^{a} + 2 \sum_{a=1}^{3} J_{ij}^{a} J_{k\ell}^{a} \right)
\]
\[
= (g) R_{ijk\ell} - (g_{j\ell} \delta_{ik} - g_{jk} \delta_{i\ell}) + R_{\text{HP}}.
\]

We now state the uniformization theorem.
**Theorem 9.3.** (1) Let $M$ be a $(4n+3)$-dimensional pseudo-conformal quaternionic CR manifold $(n \geq 1)$. If the curvature tensor $T$ vanishes, then $M$ is locally modelled on $S^{4p+3,4q}$ with respect to the group $\text{PSp}(p+1,q+1)$.

(2) If $M$ is a 3-dimensional pseudo-conformal quaternionic CR manifold whose curvature tensor $\text{TW}$ vanishes, then $M$ is conformally flat i.e. (locally modelled on $S^3$ with respect to the group $\text{PSp}(1,1)$).

**Proof.** Using (2) and (3), the formula of Theorem 9.1 becomes

$$T_{jkl} = \pi^* \hat{R}_{jkl} - \hat{R}_{\text{ESP}}.$$  

Compared with (3), we obtain that

$$T_{jkl} = (g) R_{jkl} - (g) \delta_{jk} \delta_{l}.$$  

The equality (5) is also true for $n = 1$. In fact, when $n = 1$, $R_{\text{ESP}} = 4(\delta_{jk} \delta_{l} - g_{jk} \delta_{l})$ (cf. (8.13), (8.14) and from (3), $(g) R_{jkl} - (g) \delta_{jk} \delta_{l} = \pi^* \hat{R}_{jkl} - \hat{R}_{\text{ESP}} = T_{jkl}$ by (3).

Suppose that $T$ (respectively $\text{TW}$) vanishes identically on $M$. First we show that $M$ is locally isomorphic to $S^{4p+3,4q}$ (respectively $M$ is locally isomorphic to $S^3$). As $T|U_\alpha = (\alpha) T_{jkl} = 0$ on $U_\alpha$, for brevity, we omit $\alpha$ so that $T = (T_{jkl})$ vanishes identically on $U$ for $n \geq 2$. As a consequence,

$$\begin{align*}
(g) R_{jkl} &= g_{jk} \delta_{l} - g_{jl} \delta_{k} \\
&\quad \text{on } \mathbb{D}[U].
\end{align*}$$

Since $(U, g)$ is a pseudo-Sasakian 3-structure with Killing fields $\{\xi_1, \xi_2, \xi_3\}$, the normality of (4.14) can be stated as $(g) R(X, \xi_\alpha)Y = g(X, Y)\xi_\alpha - g(\xi_\alpha, Y)X$ (cf. (29)). It turns out that

$$(g) R(\xi_\alpha, X, Y, Z) = g(X, Z)g(\xi_\alpha, Y) - g(X, Y)g(\xi_\alpha, Z)$$

($\forall X, Y, Z \in TU$). Then (6.6) and (7.7) imply that $(U, g)$ is the space of positive constant curvature. As $R_{jkl} = R_{\text{ESP}}$ by (3), the quotient space $(U/E, \hat{g})$ is locally isometric to the quaternionic pseudo-Kähler projective space $\mathbb{HP}^{p,q} \backslash \hat{g}_0$. (Note that if $T_{jkl} = 0$ for $n = 1$, then $\pi^* \hat{R}_{jkl} = R_{jkl} = 4(\delta_{jk} \delta_{l} - g_{jk} \delta_{l})$ from (3). When $p = 1, q = 0$, the base space $(U/E, \hat{g})$ is locally isometric to the standard sphere $S^4$ which is identified with the 1-dimensional quaternionic projective space $\mathbb{HP}^1$. If $p = 0, q = 1$, then $(U/E, \hat{g})$ is locally isometric to the quaternionic hyperbolic space $\mathbb{HP}^3 \backslash \mathbb{HP}^0$, in which we remark that the metric $\hat{g}$ is negative definite.) Hence, the bundle: $E \to (U, g) \to (U/E, \hat{g})$ is locally isometric to the Hopf bundle as the Riemannian submersion $(n \geq 1)$ (cf. Theorem 9.3):

$$\text{Sp}(1) \to (\Sigma^4_{\mathbb{HP}^{p,q}}, g_0) \to (\mathbb{HP}^{p,q}, \hat{g}_0).$$

This is obviously true for $n = 0$.

Let $\varphi : (U, g) \to (\Sigma^4_{\mathbb{HP}^{p+3,4q}}, g_0)$ be an isometric immersion preserving the above principal bundle. If $V_0 = \{\xi^0_1, \xi^0_2, \xi^0_3\}$ is the distribution of Killing vector fields which generates $\text{Sp}(1)$ of the above Hopf bundle, then we can assume that $\varphi_* \xi_\alpha = \xi^0_\alpha$ $(a = 1, 2, 3)$ (by a composite of some element of $\text{Sp}(1)$ if necessary). As $\omega_\alpha(X) = g(\xi_\alpha, X)$ ($X \in TU$) and $\omega^0_\alpha(X) = g_0(\xi_\alpha, X)$ ($X \in T\Sigma^4_{\mathbb{HP}^{p+3,4q}}$) respectively, the equality $g = \varphi^* g_0$ implies that

$$\omega_\alpha = \varphi^* \omega^0_\alpha \quad (a = 1, 2, 3), \quad \omega = \varphi^* \omega_0.$$  

If we represent $\varphi^* \theta^a = \theta^k T^a_k + \sum \omega_\alpha v^a_\alpha$ for some matrix $T^a_k$ and $v^a_\alpha \in \mathbb{R}$, then the equality $\varphi_* \xi_\alpha = \xi^0_\alpha$ shows that $v^a_\alpha = 0$ for $i = 1, \ldots, 4n$. Thus,

$$(9.8) \quad \varphi^* \theta^a = \theta^k T^a_k.$$
For each $\alpha \in \Lambda$, we have an immersion $\varphi_\alpha : U_\alpha \to \Sigma_{4p+3q}^{4p+3q}$ as above so that there is a collection of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \Lambda}$ on $M$. Put $g_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ when $U_\alpha \cap U_\beta \neq \emptyset$. It suffices to prove that $g_{\alpha\beta}$ extends uniquely to an element of $\text{PSp}(p+1,q+1) = \text{Aut}_{\text{QCR}}(S^{4p+3q})$. Suppose that

$$
(9.10) \quad \omega^{(\beta)} = \lambda \cdot \omega^{(\alpha)} \cdot \overline{\alpha} = u^2 \cdot a \cdot \omega^{(\alpha)} \cdot \overline{\alpha} \quad \text{on} \quad U_\alpha \cap U_\beta \neq \emptyset
$$

where $\lambda = u \cdot a$. The immersions $\varphi_\alpha : U_\alpha \to \Sigma_{4p+3q}^{4p+3q}$, $\varphi_\beta : U_\beta \to \Sigma_{4p+3q}^{4p+3q}$ satisfy $\omega^{(\alpha)} = \varphi_\alpha^* \omega_0$, $\omega^{(\beta)} = \varphi_\beta^* \omega_0$ as in [8N]. If we put $\mu = \lambda \circ \varphi_\alpha^{-1}$ on $\varphi_\alpha(U_\alpha \cap U_\beta)$, then the above relation shows that

$$
(9.11) \quad g_{\alpha\beta}^* \omega_0 = \mu \cdot \omega_0 \cdot 
\overline{\mu}.
$$

Using the equation that $d\omega_0^{(\alpha)}(X,Y) = g^{(\alpha)}(X,J_{\alpha}^{(\alpha)}Y)$ $(\forall X,Y \in D, a = 1, 2, 3)$ from [32], calculate that

$$
d\omega_0^{\alpha}(\varphi_\alpha \cdot J_{\alpha}^{(\alpha)}X, \varphi_\alpha \cdot Y) = d\omega_0(J_{\alpha}^{(\alpha)}X,Y) = g^{(\alpha)}(X,Y)
= g_{\alpha}(\varphi_\alpha \cdot X, \varphi_\alpha \cdot Y) = d\omega_0^{\alpha}(J_{\alpha}^{\alpha} \varphi_\alpha \cdot X, \varphi_\alpha \cdot Y).
$$

As $d\omega_0^{\alpha}$ is nondegenerate on $D$, for each $\alpha \in \Lambda$ we have

$$
(9.12) \quad \varphi_\alpha \cdot J_{\alpha}^{(\alpha)} = J_{\alpha}^{\alpha} \circ \varphi_\alpha \quad \text{on} \quad D \quad (a = 1, 2, 3).
$$

Let $\varphi_\alpha^{\beta} \theta^i = \theta_k^{\beta}(a \cdot T_k)^{i}$ for some matrix $\theta^i_k(\alpha)$ as in [99]. Then [99.12] means that $\theta^{\alpha}_k \cdot (J^a\alpha)^k = (J^\alpha)^k \cdot (\alpha)T_k^{i}$, which implies that $\theta^{\alpha}_k \in \text{GL}(n, \mathbb{H})$. As $g^{(\alpha)}(X,Y) = g_{\alpha}(\varphi_\alpha \cdot X, \varphi_\alpha \cdot Y)$ from [98], this reduces to

$$
(\alpha)T_k^i \in \text{Sp}(p,q).
$$

Let $\{\omega^{(\alpha)}, \omega^{(\beta)}\}_{i=1,\ldots,n}$, $\{\omega^{(\alpha)}, \omega^{(\beta)}\}_{i=1,\ldots,n}$ be two coframes on the intersection $U_\alpha \cap U_\beta$ where $\omega^{(\alpha)}$ is a $\text{Im}\mathbb{H}$-valued 1-form and each $\omega^{(\beta)}_i$ is a $\mathbb{H}$-valued 1-form, similarly for $\beta$. Noting [9.10], the coordinate change of the fiber $\mathbb{H}^n$ satisfies that

$$
(9.14) \quad \begin{pmatrix}
\omega^{(\beta)}_1 \\
\omega^{(\beta)}_2 \\
\vdots \\
\omega^{(\beta)}_n
\end{pmatrix} = \begin{pmatrix}
\lambda \\
\overline{\mu} \\
U'
\end{pmatrix} \begin{pmatrix}
\omega^{(\alpha)}_1 \\
\omega^{(\alpha)}_2 \\
\vdots \\
\omega^{(\alpha)}_n
\end{pmatrix} \cdot \overline{\lambda}.
$$

In order to transform them into the real forms, recall that $\text{GL}(n, \mathbb{H}) \cdot \text{GL}(1, \mathbb{H})$ is the maximal closed subgroup of $\text{GL}(4n, \mathbb{R})$ acting on $\mathbb{R}^{4n}$ preserving the standard quaternionic structure $\{I, J, K\}$. For each fiber of $D_\alpha$ (= $D_\beta$), there exists a matrix $U = (U_j^i) = U' \cdot \lambda \in \text{GL}(n, \mathbb{H}) \cdot \text{GL}(1, \mathbb{H})$ such that

$$
(9.15) \quad e_j^{(\alpha)} = \overline{U_j^i} e_i^{(\beta)}.
$$

with respect to the basis $\{e_i^{(\alpha)}\}_x \in (D_\alpha)_x$, $\{e_i^{(\beta)}\}_x \in (D_\beta)_x$. From Corollary [12]

$$
\pm u^2 \delta_{kj} = u^2 g_{\alpha}(e_k^{(\alpha)}, e_j^{(\alpha)}) = g_{\beta}(\overline{U_k^i} e_i^{(\beta)}, \overline{U_j^i} e_j^{(\beta)}) = \pm \delta_{kj} \overline{U_k^i} \overline{U_j^i},
$$

so $\{u^{-1} \overline{U_k^i}\} \in \text{Sp}(p,q) \cdot \text{Sp}(1) = \text{GL}(n, \mathbb{H}) \cdot \text{GL}(1, \mathbb{H}) \cap \text{SO}(4p, 4q)$ up to conjugacy ($n \geq 1$). Put $U = (U_k^i) = (u^{-1} \overline{U_k^i}) \in \text{Sp}(p,q) \cdot \text{Sp}(1)$, then

$$
(9.16) \quad \tilde{U} = u U = (u U_k^i) \in \text{Sp}(p,q) \cdot \text{Sp}(1) \times \mathbb{R}^+.
$$
Using coframes \( \{ \theta^i_{(\alpha)} \}, \{ \theta^i_{(\beta)} \} \) (induced from \( \{ \omega^i_{(\alpha)}, \omega^i_{(\beta)} \} \)) \( i = 1, \ldots, n \), the equation \( (9.11) \) translates into \( \theta^i_{(\beta)} = \theta^i_{(\alpha)} \bar{U}^i_k \) on \( D \). Using \( (9.14) \), it follows that

\[
\theta^i_{(\beta)} = \theta^i_{(\alpha)} \bar{U}^i_k + \sum_{a=1}^{3} \omega^i_{a(\alpha)} \cdot v_a^i \quad \text{on } U_{\alpha} \cap U_{\beta}.
\]

Here \( v_a^i \) are determined by \( \bar{v}^i \), see \( (9.12) \). Then,

\[
g_{ab \beta}^a(\theta^i) = (\varphi_{a}^{-1})^* \varphi_{b}^* (\theta^i) = (\varphi_{a}^{-1})^* (\theta^j_{(\beta)} \cdot (\beta) T^j_i)
\]

\[
= (\varphi_{a}^{-1})^* \left( (\theta^j_{(\alpha)} \bar{U}^j_k + \sum_{a=1}^{3} \omega^j_{a(\alpha)} \cdot v_a^j \cdot (\beta) T^j_i) \right)
\]

\[
= \theta^i (\sum_{a=1}^{3} \omega^j_{a(\beta)} \cdot (\beta) T^j_i) + \sum_{a=1}^{3} \omega^j_{a(\beta)} \cdot (v_a^j \cdot (\beta) T^j_i).
\]

If we put \( S = (S^j_i) = \left( (\sum_{a=1}^{3} \omega^j_{a(\beta)} \cdot (\beta) T^j_i) \right) \), then \( (9.16) \) and \( (9.17) \) imply \( S = \Sp(p, q) \cdot \Sp(1) \times \RR^+ \). Therefore, \( g_{ab \beta} \) satisfies the conditions of \( (9.12) \) from \( (9.14) \). Then the diffeomorphism \( g_{ab \beta} : \varphi_{a}(U_{\alpha} \cap U_{\beta}) \rightarrow \varphi_{b}(U_{\alpha} \cap U_{\beta}) \) is viewed locally as an element of \( \Aut_{\text{QCR}}(S^{4p+3, 4q}) = \Psp(p+1, q+1) \) because \( \Sigma_{\mathbb{H}}^{4p+3, 4q} \subset S^{4p+3, 4q} \). As \( \Psp(p+1, q+1) \) acts real analytically on \( S^{4p+3, 4q} \), \( g_{ab \beta} \) extends uniquely to an element of \( \Psp(p+1, q+1) \). Therefore, the collection of charts \( \{ U_{\alpha}, \varphi_{a} \}_{\alpha \in \Lambda} \) gives rise to a uniformization of a pseudo-conformal quaternionic \( CR \) manifold \( M \) with respect to \( (\Psp(p+1, q+1), S^{4p+3, 4q}) \).

Recall the 3-dimensional conformal geometry \( (\PO(4, 1), S^3) \) for which the orthogonal Lorentz group \( \PO(4, 1) \) is isomorphic as a Lie group to \( \Fsp(1, 1) \). Then the same is true for \( (\Fsp(1, 1), S^3) = (\PO(4, 1), S^3) \) \( n = 0 \).

\[\Box\]

**10. Quaternionic bundle**

It is known that the first Stiefel-Whitney class is the obstruction to the existence of a global 1-form of the contact structure and the first Chern class is the obstruction to the existence of a global 1-form of the complex contact structure respectively. It is natural to ask whether the first Pontrjagin class \( p_1(M) \) is the obstruction to the existence of global 1-form of the pseudo-conformal quaternionic structure (respectively pseudo-conformal quaternionic \( CR \) structure) on a \( (4n + 3) \)-manifold \( M \) \( n \geq 1 \). In order to see that, we need the elementary properties of the quaternionic bundle theory. However our structure group is \( \GL(n, \mathbb{H}) \cdot \GL(1, \mathbb{H}) \) but not \( \GL(n, \mathbb{H}) \). To our knowledge the fundamental properties of the bundle theory in this case are not proved explicitly. So we prepare the necessary facts. Let \( D \) be the \( 4n \)-dimensional bundle defined by \( D = \bigcup_{\alpha} D_{\alpha} \) where \( D_{\alpha} = D \mid U_{\alpha} = \Null \omega^{(\alpha)} \) in which there is the relation on the intersection \( U_{\alpha} \cap U_{\beta} \):

\[
\omega^{(\beta)} = \lambda \cdot \omega^{(\alpha)} \cdot \lambda = u^2 \cdot \bar{a} \omega^{(\beta)} \cdot a \quad \text{where } \lambda = u \cdot a \in \mathbb{H}^*.
\]

We have already discussed the transition functions on \( D \) in \( (9.14) \) (cf. \( (9.10) \)). In fact,

The gluing condition of the quaternionic bundle \( D \) in \( U_{\alpha} \cap U_{\beta} \) is given by

\[
\begin{pmatrix}
v_{1}^{(\alpha)} \\
v_{2}^{(\alpha)} \\
\vdots \\
v_{n}^{(\alpha)}
\end{pmatrix}
= u T
\begin{pmatrix}
v_{1}^{(\beta)} \\
v_{2}^{(\beta)} \\
\vdots \\
v_{n}^{(\beta)}
\end{pmatrix}
\cdot a,
\]

where

\[
\begin{pmatrix}
v_{1}^{(\alpha)} \\
v_{2}^{(\alpha)} \\
\vdots \\
v_{n}^{(\alpha)}
\end{pmatrix}
= u T
\begin{pmatrix}
v_{1}^{(\beta)} \\
v_{2}^{(\beta)} \\
\vdots \\
v_{n}^{(\beta)}
\end{pmatrix}
\cdot a,
\]
in which \(u(T \cdot \overline{a}) \in \text{Sp}(n) \cdot \text{Sp}(1) \times \mathbb{R}^+\).

**Definition 10.1.** A quaternionic \(n\)-dimensional bundle is a vector bundle over a paracompact Hausdorff space \(M\) with fiber isomorphic to the \(n\)-dimensional quaternionic vector space \(\mathbb{H}^n\). For an open cover \(\{U_\alpha\}_{\alpha \in \Lambda}\) of \(M\), if \(U_\alpha \cap U_\beta \neq \emptyset\), then there exists a transition function \(g_{\alpha \beta} : U_\alpha \cap U_\beta \to \text{GL}(n, \mathbb{H}) \cdot \text{GL}(1, \mathbb{H})\).

As a consequence, \(\mathcal{D}\) is a quaternionic \(n\)-dimensional bundle on \(M\).

As \(\text{GL}(1, \mathbb{H}) \cdot \text{GL}(1, \mathbb{H}) \approx \text{SO}(4) \times \mathbb{R}^+\), the quaternionic line bundle is isomorphic to an oriented real 4-dimensional bundle. Define the inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{H}^n\) by

\[
\langle z, w \rangle = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n.
\]

Then, \(\langle \cdot, \cdot \rangle\) satisfies that \(\langle z, w \cdot \lambda \rangle = \langle z, w \rangle \cdot \lambda\), \(\langle z \cdot \lambda, w \rangle = \bar{\lambda} \langle z, w \rangle\), \(\langle z, w \rangle = \langle w, z \rangle\) for \(\lambda \in \mathbb{H}\), and so on. By a subspace \(W\) in \(\mathbb{H}^n\) we mean a right \(\mathbb{H}\)-module. Choose \(v_0 \in \mathbb{H}^n\). Let \(V = \{v_0 \cdot \lambda \mid \lambda \in \mathbb{H}\}\) be a 1-dimensional subspace of \(\mathbb{H}^n\). Denote \(V^\perp = \{v \in \mathbb{H}^n \mid \langle v, x \rangle = 0, \forall x \in V\}\). Then it is easy to check that \(V^\perp\) is a right \(\mathbb{H}\)-module for which there is a decomposition: \(\mathbb{H}^n = V \oplus V^\perp\) as a right \(\mathbb{H}\)-module. The following is a quaternionic analogue of the splitting theorem.

**Proposition 10.2.** Given a quaternionic \(n\)-dimensional bundle \(\xi\), there exists a quaternionic line bundle \(\xi_i\) \((i = 1, \cdots, n)\) over a paracompact Hausdorff space \(N\) and a (splitting) map \(f : N \to M\) for which:

1. \(f^* \xi = \xi_1 \oplus \cdots \oplus \xi_n\).
2. \(f^* : H^*(M) \to H^*(N)\) is injective. Moreover,
3. The bundle isomorphism \(b : \xi_1 \oplus \cdots \oplus \xi_n \to \xi\) compatible with \(f\) can be chosen to preserve the inner product.

**Proof.** Let \(\mathbb{H}^n - \{0\} \to \xi_0 \xrightarrow{\pi} M\) be the subbundle of \(\xi\) consisting of nonzero sections. Noting that \(\mathbb{H}^n\) is a right \(\mathbb{H}\)-module, it induces a fiber bundle with fiber the quaternionic \(n\)-dimensional projective space \(\mathbb{HP}^{n-1}\):

\[
\mathbb{HP}^{n-1} \to Q \xrightarrow{\pi} M.
\]

Since the cohomology group \(H^*(\mathbb{HP}^{n-1}; \mathbb{Z})\) is a free abelian group, \(q^* : H^*(M) \to H^*(Q)\) is injective by the Leray-Hirsch’s theorem (cf. [20]). Put

\[
q^* \xi = \{(\ell, v) \in Q \times \xi \mid q(\ell) = \pi(v)\}.
\]

Then, \((q^* \xi, \text{pr}, Q)\) is a quaternionic bundle. Let \(\xi_1 = \{(\ell, v) \in q^* \xi \mid v \in \ell\}\) which is the 1-dimensional quaternionic subbundle of \(q^* \xi\). Choose a (right) \(\mathbb{H}\)-inner product on \(\xi\). Then it induces a (right) \(\mathbb{H}\)-inner product on \(q^* \xi\) such that the bundle projection \(\text{Pr} : q^* \xi \to \xi\) preserves the inner product obviously. Moreover, we obtain that

\[
q^* \xi = \xi_1 \oplus \xi_1^\perp.
\]

Since \(\xi_1^\perp\) is an \((n - 1)\)-dimensional quaternionic bundle over \(Q\), an induction for \(n - 1\) implies that there exist a paracompact Hausdorff space \(N\) and a splitting map \(f_1 : N \to Q\) such that \(f_1^* \xi_1^\perp = \xi_2 \oplus \cdots \oplus \xi_n\) and \(f_1^* : H^*(Q) \to H^*(N)\) is injective. If \(b_1 : \xi_2 \oplus \cdots \oplus \xi_n \to \xi_1^\perp\) is the bundle map compatible with \(f_1\), then by induction \(b_1\) preserves the inner product on the fiber between \(\xi_2 \oplus \cdots \oplus \xi_n\) and \(\xi_1^\perp\). Putting \(f = q \circ f_1 : N \to M\), we see that \(f^* : H^*(M) \to H^*(N)\) is injective and \(f^* \xi = f_1^* \xi_1 \oplus (\xi_2 \oplus \cdots \oplus \xi_n)\). Let \(\text{Pr}_1 \times b_1 : f_1^* \xi_1 \oplus (\xi_2 \oplus \cdots \oplus \xi_n) \to \xi_1 \oplus \xi_1^\perp\) be the bundle map.
Then the map $\Pr \circ (\Pr_1 \times b_1) : f_1^* \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n \to \xi$ is compatible with $f$ and preserves the inner product. This proves the induction step for $n$. \hfill \Box

Let $\xi$ be a quaternionic line bundle over $M$ with gluing condition on $U_\alpha \cap U_\beta$:

\begin{equation}
\tag{10.2}
z_\alpha = \bar{\lambda}(x)z_\beta \mu(x) = u(x) \cdot \bar{b}(x)z_\beta a(x) \quad (u > 0, a, b \in \text{Sp}(1)).
\end{equation}

Consider the tensor $\bar{\xi} \otimes \xi$ so that the gluing condition on $U_\alpha \cap U_\beta$ is given by

\[
(\bar{z}_\alpha \otimes z_\alpha) = u^2(x)\bar{a}(x)(\bar{z}_\beta b(x) \otimes \bar{b}(x))z_\beta a(x)
\]

\[
= u^2(x)\bar{a}(x)(\bar{z}_\beta \otimes z_\beta) a(x).
\]

Then $\bar{\xi} \otimes \xi$ is a quaternionic line bundle over $M$ whenever $\xi$ is a quaternionic line bundle.

**Lemma 10.3.** If $\bar{\xi} \otimes \xi$ is viewed as a 4-dimensional real vector bundle, then $p_1(\bar{\xi} \otimes \xi) = p_1(\bar{\xi}) + p_1(\xi)$. Moreover, $p_1(\bar{\xi} \otimes \xi) = p_1(\xi)$ so that $p_1(\bar{\xi} \otimes \xi) = 2p_1(\xi)$.

**Proof.** Let $\gamma$ be the canonical 4-dimensional real vector bundle over $\text{BSO}(4)$ (cf. [20]). Then, $\xi$ is determined by a classifying map $f : M \to \text{BSO}(4)$ such that $f^*\gamma = \xi$. Let $\pr_i : \text{BSO}(4) \times \text{BSO}(4) \to \text{BSO}(4)$ be the projection ($i = 1, 2$). As $\gamma$ inherits a quaternionic structure from $\xi$ through the bundle map, there is a quaternionic line bundle $\pr_1^*\bar{\gamma} \otimes \pr_2^*\gamma$ over $\text{BSO}(4) \times \text{BSO}(4)$. Now, let $h : \text{BSO}(4) \times \text{BSO}(4) \to \text{BSO}(4)$ be a classifying map of this bundle so that $h^*\gamma = \pr_1^*\bar{\gamma} \otimes \pr_2^*\gamma$.

When $\iota_1^* : \text{BSO}(4) \to \text{BSO}(4) \times \text{BSO}(4)$ is the inclusion map on each factor, $\iota_1^* \pr_2^*\gamma$ is the trivial quaternionic line bundle (we simply put $\theta^1_b$) and so $\iota_1^* h^* p_1(\gamma) = \iota_1^* p_1(\pr_1^*\bar{\gamma} \otimes \pr_2^*\gamma) = p_1(\bar{\gamma})$ and so $\iota_2^* h^* p_1(\gamma) = p_1(\gamma)$. Hence we obtain that

\[
h^* p_1(\gamma) = p_1(\bar{\gamma}) \times 1 + 1 \times p_1(\gamma).
\]

Let $f' : M \to \text{BSO}(4)$ be a classifying map for $\bar{\xi}$ such that $f'^*\gamma = \bar{\xi}$. Then the map $h(f' \times f)d$ composed of the diagonal map $d : M \to M \times M$ satisfies that

\[
(h(f' \times f)d)^*\gamma = f'^*\bar{\gamma} \otimes f^*\gamma = \bar{\xi} \otimes \xi.
\]

Therefore, $p_1(\bar{\xi} \otimes \xi) = d^*(f' \times f)^*(p_1(\bar{\gamma}) \times 1 + 1 \times p_1(\gamma)) = p_1(f'^*\bar{\gamma}) + p_1(f^*\gamma) = p_1(\bar{\xi}) + p_1(\xi).

Next, the conjugate $\bar{\xi}$ is isomorphic to $\xi$ as real 4-dimensional vector bundle without orientation. But the correspondence $(1, i, j, k) \to (1, -i, -j, -k)$ gives an isomorphism of $\xi$ onto $(-1)^3\xi$. And so, the complexification $\xi_C$ of $\xi$ (viewed as a real vector bundle) is isomorphic to $(-1)^6\xi_C = \xi_C$. By definition, $p_1(\bar{\xi}) = p_1(\xi)$.

\hfill \Box

**10.1. Relation between Pontrjagin classes.** Suppose that $\{\omega^{(\alpha)}, \{I^{(\alpha)}, J^{(\alpha)}, K^{(\alpha)}\}, g_{(\alpha)}, U_\alpha\}_{\alpha \in \Lambda}$ represents a pseudo-conformal quaternionic structure $\mathcal{D}$ on a $(4n + 3)$-dimensional manifold $M = \bigcup_{\alpha \in \Lambda} U_\alpha$. Let $L$ be the quotient bundle $TM/\mathcal{D}$. Choose the local vector fields $\{\xi^{(\alpha)}_1, \xi^{(\alpha)}_2, \xi^{(\alpha)}_3\}$ on each neighborhood $U_\alpha$ such that $\omega^{(\alpha)}_\alpha(\xi^{(\alpha)}_b) = \delta_{ab}$. Then, $L|U_\alpha$ is spanned by $\{\xi^{(\alpha)}_1\}_{i=1,2,3}$ for each
\( \alpha \in \Lambda \). Moreover, the gluing condition between \( L|U_\alpha \) and \( L|U_\beta \) is exactly given by

\[
\begin{pmatrix}
\xi_1^{(\alpha)} \\
\xi_2^{(\alpha)} \\
\xi_3^{(\alpha)}
\end{pmatrix} = u^2 A \begin{pmatrix}
\xi_1^{(\beta)} \\
\xi_2^{(\beta)} \\
\xi_3^{(\beta)}
\end{pmatrix}.
\]

(Compare Definition \( \text{[1.3]} \)) It is easy to see that \( \sum_{a=1}^{3} \omega_a^{(\alpha)} \cdot \xi_a^{(\alpha)} = \sum_{a=1}^{3} \omega_a^{(\beta)} \cdot \xi_a^{(\beta)} \). We can define a section \( \theta : TM \to L \) which is an \( L \)-valued 1-form by setting

\[
\theta|U_\alpha = \omega_1^{(\alpha)} \cdot \xi_1^{(\alpha)} + \omega_2^{(\alpha)} \cdot \xi_2^{(\alpha)} + \omega_3^{(\alpha)} \cdot \xi_3^{(\alpha)}.
\]

Then there is an exact sequence of bundles: \( 1 \to \mathcal{D} \to TM \xrightarrow{\theta} L \to 1. \)

Let \( E \) be the 1-dimensional quaternionic line bundle obtained from the union \( \bigcup_{\alpha \in \Lambda} U_\alpha \times \mathbb{H} \) by identifying

\[
(p, z_\alpha) \sim (q, z_\beta) \text{ if and only if } \begin{cases} p = q \in U_\alpha \cap U_\beta, \\ z_\alpha = \lambda \cdot z_\beta \cdot \bar{\lambda} = u^2 a \cdot z_\beta \cdot \bar{a}. \end{cases}
\]

If \( L \oplus \theta \) is the Whitney sum composed of the trivial (real) line bundle \( \theta \) on \( M \), then it is easy to see that \( L \oplus \theta \) is isomorphic to the 1-dimensional quaternionic line bundle \( E \). Then the Pontrjagin class \( p_1(E) = p_1(L \oplus \theta) \). We prove that

**Theorem 10.4.** The first Pontrjagin classes of \( M \) and the bundle \( L \) has the relation:

\[ 2p_1(M) = (n + 2)p_1(L \oplus \theta). \]

**Proof.** As \( \mathcal{D} \) is a quaternionic bundle in our sense, there is a splitting map \( f : N \to M \) such that \( f^* \mathcal{D} = \xi_1 \oplus \cdots \oplus \xi_n \) from Proposition \( \text{[10.2]} \). Let \( \Psi : \xi_1 \oplus \cdots \oplus \xi_n \to \mathcal{D} \) be a bundle map which is compatible with \( f \). Since \( \Psi \) is a right \( \mathbb{H} \)-linear map on the fiber at each point \( x \in N \), we can describe

\[
\Psi \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_x = P(x) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{f(x)}
\]

for some function \( P : N \to \text{GL}(n, \mathbb{H}) \). With respect to an appropriate inner product on \( \mathcal{D} \) and the direct inner product on \( \xi_1 \oplus \cdots \oplus \xi_n \), \( \Psi \) preserves the inner product between them by (3) of Theorem \( \text{[10.2]} \) which implies

\[
P(x) \in \text{Sp}(n).
\]

We examine the gluing condition of each \( \xi_i \) on \( f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \neq \emptyset \). For \( x \in f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \), let \( v_i^{(\alpha)} \in \xi_i|f^{-1}(U_\alpha) \). Suppose that there is an element \( v_i^{(\beta)} \in \xi_i|f^{-1}(U_\beta) \) such that \( v_i^{(\alpha)} \sim v_i^{(\beta)} \), i.e.,

\[
v_i^{(\alpha)} = \tilde{\lambda}_i v_i^{(\beta)} \mu_i \quad (\lambda_i, \mu_i \in \mathbb{H}^*; i = 1, \ldots, n).
\]
Since $\Psi(v_i^{(\alpha)}) \sim \Psi(v_i^{(\beta)})$ at $f(x)$, it follows from (10.1) that $\Psi \begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = uT \cdot \Psi \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix}$. Hence, at $f(x) \in U_a \cap U_\beta$ where $T \in \text{Sp}(n)$. As

$$P \begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = \Psi \begin{pmatrix} v_1^{(\alpha)} \\ \vdots \\ v_n^{(\alpha)} \end{pmatrix} = uT \cdot P \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix} = P(aP^{-1}TP) \begin{pmatrix} v_1^{(\beta)} \\ \vdots \\ v_n^{(\beta)} \end{pmatrix} \cdot a,$$

it follows that

$$\begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_n(x) \end{pmatrix} = u(T^{-1}TP) \begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_n(x) \end{pmatrix} \cdot a.$$
(2) \( L \) is the trivial bundle so that \( \{ \xi_\alpha \}_{\alpha=1,2,3} \) exists globally on \( M \).

(3) There exists a global \( \text{Im} \mathbb{H} \)-valued 1-form \( \omega \) on \( M \) which represents a pseudo-conformal quaternionic structure \( \mathcal{D} \). In particular, there exists a hypercomplex structure \( \{ I, J, K \} \) on \( \mathcal{D} \).

**Proof.** First note that the Whitney sum \( L \oplus \theta^1 \) is the quaternionic line bundle \( E \) with structure group lying in \( \text{SO}(3) \times \mathbb{R}^+ \subset \text{Sp}(1) \cdot \text{Sp}(1) \times \mathbb{R}^+ \). As above we have the quaternionic line bundle of \( \ell \)-times tensor \( \otimes \mathbb{H} \) with structure group \( \text{SO}(3) \times \mathbb{R}^+ \). Viewed as the 4-dimensional real vector bundle, it determines the classifying map \( g : M \to B(\text{SO}(3) \times \mathbb{R}^+) = B\text{SO}(3) \). Note that \( p : B(\text{Sp}(1) \times \mathbb{R}^+) \to B(\text{SO}(3) \times \mathbb{R}^+) \) is the two-fold covering map. As \( M \) is simply connected by the hypothesis, the map \( g \) lifts to a classifying map \( \tilde{g} : M \to B\text{Sp}(1) \) such that \( g = p \circ \tilde{g} \). Let \( \gamma \) be the 4-dimensional universal bundle over \( B\text{SO}(3) \). (Compare [21].) Then the pull back \( p^* \gamma \) is the 4-dimensional canonical bundle over \( B\text{Sp}(1) = \mathbb{HP}^\infty \) whose first Pontrjagin class \( p_1(p^* \gamma) \) generates the cohomology ring \( H^*(\mathbb{HP}^\infty; \mathbb{Z}) \). So the bundle \( \otimes \mathbb{H} \) is classified by the map \( \tilde{g} \) where

\[
[\tilde{g}] = \tilde{g}^* p_1(p^* \gamma) \in H^4(M; \mathbb{Z}),
\]
which coincides with \( p_1(\otimes \mathbb{H}) \).

1 \( \Rightarrow \) 2. If \( 2p_1(M) = 0 \), then Theorem 10.3 shows \((n + 2)p_1(L) = 0\), i.e. \( p_1((\otimes \mathbb{H}) E) = 0 \). (See Lemma 10.3.) Hence, the classifying map \( \tilde{g} : M \to B\text{Sp}(1) \) for \( \otimes \mathbb{H} \) is null homotopic so that \( \tilde{g}^* p^* \gamma = \otimes \mathbb{H} \) is trivial. There exists a family of functions \( \{ h_\alpha \} \in \text{Sp}(1) \times \mathbb{R}^+ \) such that the transition function \( g_{\alpha \beta}(x) = \delta^1 h(\alpha, \beta)(x) \) \( (x \in U_\alpha \cap U_\beta) \). As the gluing relation for \( \otimes \mathbb{H} \) is given by \( u_{\alpha \beta}^{(n+2)} \cdot a_{\alpha \beta} \cdot z \cdot a_{\alpha \beta} \), putting \( h_\alpha = a_\alpha \cdot u_\alpha \in \text{Sp}(1) \times \mathbb{R}^+ \), it follows that

\[
u_{\alpha \beta}^{(n+2)} \cdot a_{\alpha \beta} \cdot z \cdot a_{\alpha \beta} = (h_\alpha^{-1} h_\beta) z = u_\alpha^{-1} u_\beta a_\alpha a_\beta \cdot z \cdot a_\beta a_\alpha \quad (z \in \mathbb{H}).
\]

Then, \( u_{\alpha \beta}^{(n+2)} = u_\alpha^{-1} u_\beta \in \mathbb{R}^+ \) and \( a_{\alpha \beta} = \pm a_\beta a_\alpha \). As \( u_{\alpha \beta} > 0 \), \( u_{\alpha \beta} = (u_{\alpha \beta})^{\frac{1}{2(n+2)}} \cdot u_{\beta}^{\frac{1}{2(n+2)}} \). Since the gluing relation of \( E = L \oplus \theta \) is given by \( z_\alpha = u_{\alpha \beta}^2 \cdot a_{\alpha \beta} \cdot z_\beta \cdot a_{\alpha \beta} \), putting \( u_{\alpha} = (u_{\alpha})^{\frac{1}{2(n+2)}} \), a calculation shows \( z_\alpha = u_{\alpha \beta}^2 \cdot a_{\alpha \beta} \cdot z_\beta \cdot a_{\alpha \beta} = u_{\alpha}^{-1} u_{\beta}^2 \cdot a_{\beta} a_{\beta} \cdot z_\beta \cdot a_{\beta} a_\alpha \). Moreover if \( C(\alpha) \in \text{SO}(3) \) is the matrix defined by \( a_\alpha \cdot \begin{pmatrix} i & j & k \end{pmatrix} \cdot a_\alpha = C(\alpha) \begin{pmatrix} i & j & k \end{pmatrix} \) (similarly for \( C(\beta) \)), then

\[
u_{\alpha \beta}^2 \cdot A^{\alpha \beta} = u_{\alpha}^{-1} u_{\beta}^2 \cdot C(\alpha)^{-1} \circ C(\beta).
\]

Substitute this into 10.3, it follows that

\[
u' \cdot C(\alpha) \begin{pmatrix} \xi_1 \\
\xi_2 \\
\xi_3 \end{pmatrix} = u_{\alpha} \cdot C(\beta) \begin{pmatrix} \xi_1 \\
\xi_2 \\
\xi_3 \end{pmatrix}
\]
on \( U_\alpha \cap U_\beta \).

We can define the vector fields \( \{ \xi_1, \xi_2, \xi_3 \} \) on \( M \) to be

\[
u' \cdot C(\alpha) \begin{pmatrix} \xi_1 \\
\xi_2 \\
\xi_3 \end{pmatrix} = u_{\alpha} \cdot C(\alpha) \begin{pmatrix} \xi_1 \\
\xi_2 \\
\xi_3 \end{pmatrix}.
\]
Then \(\{\xi_1, \xi_2, \xi_3\}\) spans \(L\), therefore, \(L\) is trivial.

2 \(\Rightarrow\) 3. Since \((\omega_1^{(\beta)}, \omega_2^{(\beta)}, \omega_3^{(\beta)}) = (\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \omega_3^{(\alpha)})u_{\alpha}^2 \cdot A^{\alpha\beta}\), (10.8) implies that

\[
(\omega_1^{(\beta)}, \omega_2^{(\beta)}, \omega_3^{(\beta)})u_{\beta}^{-1} \cdot C^{(\beta)^{-1}} = (\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \omega_3^{(\alpha)})u_{\alpha}^{-1} \cdot C^{(\alpha)^{-1}} \text{ on } U_{\alpha} \cap U_{\beta}.
\]

Then, the \(\text{Im}\mathbb{H}\)-valued 1-form \(\omega\) on \(M\) can be defined by

\[
(10.10) \quad \omega|_{U_{\alpha}} = (\omega_1^{(\alpha)}, \omega_2^{(\alpha)}, \omega_3^{(\alpha)})u_{\alpha}^{-1} \cdot C^{(\alpha)^{-1}} \left( \begin{array}{c} i \\ j \\ k \end{array} \right).
\]

Note that \(\omega\) satisfies that \(\omega|_{U_{\alpha}} = \bar{\lambda}_{\alpha} \cdot \omega^{(\alpha)} \cdot \lambda_{\alpha}\) for some function \(\lambda_{\alpha} : U_{\alpha} \to \mathbb{H}^*\) \((\alpha \in \Lambda)\). Recall that two quaternionic structures on \(U_{\alpha} \cap U_{\beta}\) are related:

\[
A^{\alpha\beta} = C^{(\alpha)^{-1}} \circ C^{(\beta)}.
\]

As \(A^{\alpha\beta} = C^{(\alpha)^{-1}} \circ C^{(\beta)}\), it follows that

\[
(10.11) \quad C^{(\alpha)} \cdot \begin{pmatrix} I^{(\alpha)} \\ J^{(\alpha)} \\ K^{(\alpha)} \end{pmatrix} = C^{(\beta)} \cdot \begin{pmatrix} I^{(\beta)} \\ J^{(\beta)} \\ K^{(\beta)} \end{pmatrix}.
\]

Letting \(\begin{pmatrix} I \\ J \\ K \end{pmatrix}|_{U_{\alpha}} = C^{(\alpha)} \cdot \begin{pmatrix} I^{(\alpha)} \\ J^{(\alpha)} \\ K^{(\alpha)} \end{pmatrix}\), there exists a hypercomplex structure \(\{I, J, K\}\) on \(\mathcal{D}\).

3 \(\Rightarrow\) 1. If the global \(\text{Im}\mathbb{H}\)-valued 1-form \(\omega\) exists, then \(\omega\) defines a three independent vector fields isomorphic to \(L\), i.e. \(p_1(L) = 0\).

\[\Box\]

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