Connection formulas for the $\lambda$ generalized Ising correlation functions

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Abstract
We derive and prove the connection formulas for the $\lambda$ generalized diagonal Ising model correlation functions.

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1. Introduction

The $\lambda$ generalized Ising model correlations may be defined for $T < T_c$ by the expression [1]

$$C^-(N,t;\lambda) = (1 - k^2)^{1/4} \exp \sum_{n=1}^{\infty} \lambda^{2n} F_N^{(2n)}$$

$$F_N^{(2n)} = \frac{(-1)^{n+1}}{n(2\pi)^{2n}} \oint \prod_{j=1}^{2n} \frac{d z_j \bar{z}_j}{1 - \bar{z}_j z_j} \prod_{j=1}^{n} Q(z_{2j-1}) Q(z_{2j-1}^{-1}) P(z_j) P(z_j^{-1})$$

where the contours of integration are $|z_j| = 1 - \epsilon$, $z_{2n+1} \equiv z_1$ and

$$P(z) = 1/Q(z) = (1 - k z)^{1/2}.$$ 

The parameter $k$ satisfies $0 \leq k \leq 1$ and we will use

$$t = k^2.$$ 

When $\lambda = 1$ the expression (1) reduces to the diagonal correlation function of the Ising model given by the $N \times N$ Toeplitz determinant [3]

$$C(N,t;1) = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}$$
with
\[ a_n = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\omega \theta} \left[ \frac{1 - ke^{-i\theta}}{1 - ke^{i\theta}} \right]^{1/2} \] (6)
and
\[ k = (\sinh 2E_v/kT \sinh 2E_h/kT)^{-1}. \]

The expression (1) and (2) is obtained in [1] for \( \lambda = 1 \) by extending to all orders the procedure used by Wu [4] to compute the leading order expansion of (5) for \( k \) fixed and \( N \to \infty \). It should be noted that an equivalent form of (1) and (2) was given in the 1976 paper of [2]. The integrals of the two expressions are presumably seen to be equal by adding appropriate total derivatives to the integrands but this has never been explicitly demonstrated.

The generalized correlation (1) for \( 0 < t < 1 \) was shown in [5] by extensive computer computations to satisfy the sigma form of the Painlevé VI equation first derived by Miwa and Jimbo [6] for the diagonal Ising correlation
\[ (t(t-1)h''')^2 + 4h'(t-1)h' - h - 1/4)(h' - h) = N^2 ((t-1)h' - h)^2 \] (7)
where
\[ h(t) = t(t-1) \frac{d}{dt} \ln C^-(1-t)^{1/4} \] with \( t = (\sinh 2E_v/kT \sinh 2E_h/kT)^{-2}. \) (8)
The validity of (7) for the generalized correlation (1) with \( \lambda \neq 1 \) has been studied analytically in [7] and related work is in [8] but a mathematically rigorous proof is so far lacking. We will here follow [5] and use (7) for (1) for all \( \lambda \).

It is readily seen from the definition (1) that these generalized correlations are analytic at \( t = 0 \) and that for \( t \to 0 \)
\[ C^-(N,t;\lambda) = (1-t)^{1/4} \{ 1 + \lambda^2 (1/2)_N (3/2)_N N^{N+1} (1 + O(t)) \} \] (9)
where by noting that
\[ t(t-1) \frac{d}{dt} \ln(1-t)^{1/4} - \frac{t}{4} = 0 \] (10)
we have for \( t \to 0 \) the one parameter boundary condition for (7)
\[ h(t) \to -\lambda^2 N^{N-1} (1/2)_N (3/2)_N \frac{2N}{4N!(N+1)!} \] (11)
The question of interest is to determine the behaviour of this one parameter family at \( t = 1 \). It is readily seen by direct substitution into (7) that as \( t \to 1 \) there is a two parameter solution
\[ h(t) = -\frac{1 - \sigma^2}{4} + \frac{1}{8} (1 - \sigma^2)(1-t) \]
\[ + \frac{s}{16\sigma} (1 + \sigma)(2N + \sigma)(1-t)^{1-\sigma} \]
\[ - \frac{s}{16\sigma} (1 - \sigma)(2N - \sigma)(1-t)^{1+\sigma} \] (12)
and thus to order \( (1-t)^{1+\sigma} \)
\[ C^-(N,t;\sigma,\hat{s}) = K(N;\sigma)(1-t)^{\sigma/4} \left[ 1 - \frac{1 - \sigma^2}{8} (1-t) \right] \]
\[ + \frac{\hat{s}(N;\sigma)}{16\sigma} (2N + \sigma)(1-t)^{1-\sigma} \]
\[ - \frac{\hat{s}(N;\sigma)^{-1}}{16\sigma} (2N - \sigma)(1-t)^{1+\sigma} + O((1-t)^{2-\sigma}) \]  \hfill (13)

where \( \sigma(\lambda) \) and \( \hat{s}(N,\sigma) \) are two integration constants for the second order equation (7) and \( K(N,\sigma) \) is a normalizing constant which can be determined from the original definition (1).

The computation of \( \sigma(\lambda), \hat{s}(N,\sigma), K(N,\sigma) \) for (1) is the purpose of this note.

The results are as follows

\[ \sigma = \left( \frac{2}{\pi} \right) \arccos \lambda \]  \hfill (14)

\[ \hat{s}(N,\sigma) = 16\sigma \prod_{n=1}^{N} \frac{1 - \sigma/2n}{1 + \sigma/2n} \]  \hfill (15)

\[ K(N;\sigma) = 2^{-\sigma^2} \left( \frac{\sigma}{\sin \pi \sigma/2} \right) \prod_{m=1}^{N-1} \left( 1 - \frac{1}{4m^2} \right) \prod_{m=1}^{N-1} \left( 1 - \frac{\sigma^2}{4m^2} \right)^{N-m} \]  \hfill (16)

In section 2 we briefly discuss the history of this connection problem and in section 3 we present special cases which confirm the results (14)–(16). A proof of (15) and (16) is given in section 4 by use of the Toda-like relation of Mangazeev and Guttmann [10] and a brief discussion of open questions is in section 5.

2. History

In the scaling limit

\[ t \to 1, \quad N \to \infty, \quad \text{with} \quad N(1-t) = r \quad \text{fixed} \]  \hfill (17)

the scaling function

\[ G_-(r;\lambda) = \lim_{\text{scaling}} (1-t)^{-1/4} C^-(N,t;\lambda) \]  \hfill (18)

was shown in 1976 in [2] for \( \lambda = 1 \) to be expressed in terms of a Painlevé III function and the connection formulas for \( \sigma(\lambda) \) and \( \hat{s}(N,\sigma) \) for the \( \lambda \) generalized scaling function were computed the next year in [9] by a direct expansion of the integrals in the scaled version of (1) and (2) where it was also proven that the lambda extension of the scaled correlation satisfies the same PIII equation as does the Ising case of \( \lambda = 1 \). The normalization constant for the scaling limit was computed in 1991 by Tracy in [11] again starting from the explicit series expansion.

In 1982 Jimbo [12] computed the connection formulas for \( \sigma \) and \( \hat{s} \) for the solutions of the general Painlevé VI equation (1.3) of [12] in terms of monodromy data by means of deformation theory under certain generic restrictions. It is to be noted that the Ising model does not satisfy the generic constraints of [12] but, nevertheless, it is explained in [15] that the results of [12] can be justified for the Ising case `after some tedious analysis going into the depths of Jimbo’s proof’. Nongeneric cases have also been studied by Guzzetti [16]) but the case relevant for the generalized Ising correlations seems not to have been investigated.
In the generic case an expression for the normalizing constant $K$ for the tau function of the solutions of the PVI equation in terms of infinite products of Barnes $G$ functions was first conjectured in 2013 by Iorgov, Lisovyy and Tykhyy [13] and proven in 2016 by Its, Lisovyy and Prokhorov [14].

The results of [12] and of [13, 14] parametrize the behaviour at $t = 0$ and $t = 1$ in terms of the parameters in the monodromy matrices. This gives an implicit connection between the parameters. However the parametric parameters have never been eliminated to give a direct connection between the parameters at $t = 0$ and $t = 1$. On the other hand the results of connection for PIII of [9] and [11] are a direct and not a parametric connection of the parameters $\lambda$ at $t = 0$ with the parameters $\sigma, \hat{s}, K$ at $t = 1$.

3. Special cases

We here present several special cases of computations which confirm the results (14)–(16).

3.1. $C^-(N, t; \lambda)$ for $N = 0, 1$

In [17] a prescription is given to express $C^-(N, t; \lambda)$ in terms of the theta functions

$$\theta_2(u; q) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos([2n + 1]u)$$

(19)

$$\theta_3(u; q) = 1 + 2 \sum_{n=1}^{\infty} q^n \cos 2nu$$

(20)

$$\theta_4(u; q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^n \cos 2nu$$

(21)

where $q$ is related to the variable $t$ by

$$q = e^{-\pi K'(k)/K(k)}$$

(22)

with $k^2 = t$ and $K(k)$ and $K'(k)$ are the complete elliptic integrals of the first kind. In [5, 10] and [18] this prescription was used to obtain explicit expressions for $N = 0, 1$

$$C^-(0, t; \lambda) = \frac{\theta_3(u; q)}{\theta_3(0, q)}$$

(23)

$$C^-(1, t; \lambda) = \frac{-\theta'_3(u; q)}{\sin(u)\theta_2(0, q)\theta_3(0; q)}$$

(24)

where prime indicates the derivative with respect to $u$ and

$$\lambda = \cos u.$$  

(25)

The expressions (23) and (24) are expanded at $t = 0$ by the direct use of (19) and (20) to obtain the form (9). The expansion at $t = 1$ is obtained from (23) and (24) by use of the identities (on page 370 of [19] with $v \to u/\pi$)

$$\theta_3(uK(k); e^{-\pi K'(k)/K(k)})e^{-u^2 K(k)/\pi K'(k)} = \left(\frac{K'(k)}{K(k)}\right)^{1/2} \theta_3(u; e^{-\pi K'(k)/K(k)})$$

(26)
\[ \theta_4 \left( \frac{uK(k)}{iK'(k)}; e^{-\pi K(k)/K'(k)} \right) = \left( \frac{K'(k)}{K(k)} \right)^{1/2} \theta_2 (u; e^{-\pi K'(k)/K(k)}). \]

The results (14)–(16) are obtained by comparing these explicit expansions at \( t = 1 \) with the general form (13).

### 3.2. Algebraic cases for \( \lambda = \cos(m\pi/n) \) where \( \sigma = 2m/n \)

When

\[ \lambda = \cos(m\pi/n) \]

the function \( C^-(N, t; \lambda) \) is an algebraic function and in [5] the explicit results are given for \( N = 0, 1, 2 \) that for \( \lambda = \cos(\pi/4) \) with \( \sigma = 1/2 \)

\[ C^-(0, t; \cos(\pi/4)) = 2^{-1/4}(1 - t)^{1/16}[1 + (1 - t)^{1/2}]^{1/4} \]  
\[ C^-(1, t; \cos(\pi/4)) = 2^{-3/4}(1 - t)^{1/16}[1 + (1 - t)^{1/2}]^{3/4} \]  
\[ C^-(2, t; \cos(\pi/4)) = 2^{-5/4}(1 - t)^{1/16}[1 + (1 - t)^{1/2}]^{5/4}[5 - (1 - t)^{1/2}]^2 \]

When \( t = 0 \) these expressions are expanded as

\[ C^-(0, t; \cos(\pi/4)) = (1 - t)^{1/4}(1 + \frac{1}{8}t + \frac{5}{64}t^2 + \frac{15}{256}t^3 + O(t^4)) \]  
\[ C^-(1, t; \cos(\pi/4)) = (1 - t)^{1/4}(1 + \frac{3}{128}t^2 + \frac{3}{128}t^3 + O(t^4)) \]  
\[ C^-(2, t; \cos(\pi/4)) = (1 - t)^{1/4}(1 + \frac{5}{512}t^3 + O(t^4)) \]

which agree with the expansion at \( t = 0 \) of (9) with \( \lambda = \cos(\pi/4) = 2^{-1/2} \).

When \( t \to 1 \) these expressions are expanded as

\[ C^-(0, t; \cos(\pi/4)) = 2^{-1/4}(1 - t)^{1/16}[1 + \frac{1}{4}(1 - t)^{1/2} \]  
\[ - \frac{3}{32}(1 - t) + \frac{7}{128}(1 - t)^{3/2} + O(t^2)] \]  
\[ C^-(1, t; \cos(\pi/4)) = 2^{-3/4}(1 - t)^{1/16}[1 + \frac{3}{4}(1 - t)^{1/2} \]  
\[ - \frac{3}{32}(1 - t) + \frac{5}{128}(1 - t)^{3/2} + O(t^2)] \]  
\[ C^-(2, t; \cos(\pi/4)) = 2^{-5/4}\frac{5}{4}(1 - t)^{1/16}[1 + \frac{21}{20}(1 - t)^{1/2} \]  
\[ - \frac{3}{32}(1 - t) - \frac{9}{128}(1 - t)^{3/2} + O(t^2)]. \]
The only other case where an explicit result is given in [5] is for
\[
\lambda = \cos(\pi/3) = \frac{1}{2} \quad \text{with } \sigma = \frac{2}{3}
\]
where for \( N = 0 \) the function \( C^- (0, t; \cos(\pi/3)) \) satisfies
\[
16C^{12} - 16C^9 + 8t(1-t)C^3 + t(1-t) = 0.
\]
(38)

The expansion near \( t = 0 \) of the solution of (39) which does not vanish at \( t = 0 \) is
\[
C^-(0, t; \cos(\pi/3)) = (1-t)^{1/4} [1 + \frac{1}{16} t + O(t^2)]
\]
(40)

and for \( t \to 1 \)
\[
C^-(0, t; \cos(\pi/3)) = 2^{-4/9}(1-t)^{1/9} [1 + 2^{-4/3}(1-t)^{1/3} - \frac{5}{72} (1-t) + \frac{7 \cdot 2^{2/3}}{288} (1-t)^{4/3} + O(t^2)].
\]
(41)

These results for \( \lambda = \cos(\pi/4) \) and \( \cos(\pi/3) \) all agree with (14)–(16).

3.3. The Ising case \( \lambda = 1 (\sigma = 0) \)

The Ising case \( \lambda = 1 \) has been extensively studied in [17] where it is shown that as \( t \to 1 \)
\[
C^-(N, t; 1) = C(N, t = 1)[1 - \frac{N}{4} (1-t) \left( \ln(1-t) - \ln 16 + \sum_{n=1}^{N} n^{-1} \right) + \cdots]
\]
(42)

with
\[
C(N, t = 1; 1) = \left( \frac{2}{\pi} \right)^N \prod_{m=1}^{N} \left( 1 - \frac{1}{4m^2} \right)^{m-N}
\]
(43)

which agrees with (14)–(16) in the limit \( \sigma \to 0 \).

3.4. The case \( \lambda = 0 (\sigma = 1) \)

When \( \sigma = 1 \) we find from (15) and (16) that
\[
\hat{s}(N, 1) = \frac{16}{2N+1}, \quad K(N, 1) = \frac{1}{2}
\]
(44)

and thus (13) reduces to
\[
C^-(N, t; 0) = (1-t)^{1/4}
\]
(45)
as required by (1).

3.5. \( N \to \infty \) for \( K(N; \sigma) \)

To obtain the behavior of \( K(N; \sigma) \) for \( N \to \infty \) we use the identity
\[
\frac{\sin \pi \delta}{\pi \delta} = \prod_{m=1}^{\infty} \left( 1 - \frac{\delta^2}{m^2} \right)
\]
(46)
to write
\[ \frac{\sin \pi \sigma / 2}{\sigma} = \frac{\pi}{2} \prod_{m=1}^{\infty} \left( 1 - \frac{\sigma^2}{4m^2} \right) = \prod_{l=1}^{\infty} \left( 1 - \frac{1}{4l^2} \right) \prod_{m=1}^{\infty} \left( 1 - \frac{\sigma^2}{4m^2} \right) \]  
(47)
which we use in (16) to obtain
\[ K(N; \sigma) = 2^{\sigma^2} \prod_{m=1}^{N-1} \left( 1 - \frac{\sigma^2}{4m^2} \right) \prod_{m=N}^{\infty} \left( 1 - \frac{\sigma^2}{4m^2} \right) \]
\[ \times \prod_{l=1}^{N-1} \left( 1 - \frac{1}{4l^2} \right) \prod_{l=N}^{\infty} \left( 1 - \frac{1}{4l^2} \right) - N \]  
(48)
To now expand \( K(N; \sigma) \) for \( N \to \infty \) we use for the products running \( N \) to infinity
\[ \ln \prod_{m=N}^{\infty} \left( 1 - \frac{\sigma^2}{4m^2} \right) = N \sum_{m=N}^{\infty} \ln(1 - \sigma^2/4m^2) \]
\[ \sim -N\sigma^2/4 \int_{N}^{\infty} \frac{4m}{m^2} = -\sigma^2/4. \]  
(49)
For the products from 1 to \( N - 1 \) we write
\[ \prod_{m=1}^{N-1} \left( 1 - \frac{\sigma^2}{4m^2} \right)^{-m} = \prod_{m=1}^{N-1} e^{\sigma^2/4m} \prod_{m=1}^{N-1} \left( 1 - \frac{\sigma^2}{4m^2} \right) e^{-\sigma^2/4m} \]  
(50)
where for \( N \to \infty \) the second product converges and the first product is expanded using the definition of Euler's constant \( \gamma \)
\[ \lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{1}{n} - \ln N \right) = \gamma \]  
(51)
to find for \( N \to \infty \)
\[ \ln \prod_{m=1}^{N-1} e^{\sigma^2/4m} = \sigma^2 \sum_{m=1}^{N-1} \frac{1}{m} \to \frac{\sigma^2}{4} (\ln N + \gamma). \]  
(52)
Thus we have for \( N \to \infty \)
\[ K(N; \sigma) \to N^{(\sigma^2-1)/4} e^{-\sigma^2 (\sigma^2-1)(1+\gamma)/4} \]
\[ \times \prod_{m=1}^{\infty} \left( 1 - \frac{\sigma^2}{4m^2} \right)^{-m} e^{-\sigma^2/4m} \prod_{m=1}^{\infty} \left( 1 - \frac{1}{4m^2} \right) e^{1/4m}. \]  
(53)
When this is rewritten in terms of Barnes G functions and the derivative of the zeta function at \(-1\) this agrees with the result obtained by Tracy [11] for the scaling limit of \( C^-(N, t; \lambda) \)

4. The Toda-like equation
In 2010 Mangazeev and Guttmann [10] proved that \( C^-(N, t; \lambda) \) satisfies the following Toda-like equation
The verification of this identity in the limit $t \to 1$ using the expansion (13) with (14)–(16) combined with the previous results for $N = 0$, $1$ constitutes an inductive proof of the connection formulas (15) for $\hat{s}(N, \sigma)$ and (16) for $K(N; \sigma)$.

It is straightforward to see from (16) that

$$\frac{K(N + 1; \sigma)K(N - 1; \sigma)}{K(N; \sigma)^2} = \frac{N^2 - \sigma^2/4}{N^2 - 1/4}$$

and thus for $x = 1 - t \to 0$ the right hand side of (54) to order $x^{1-\sigma}$ is

$$(N^2 - 1/4)\frac{C^-(N + 1, t; \lambda)C^-(N - 1, t; \lambda)}{C^-(N, t; \lambda)^2} = \frac{N^2 - \sigma^2/4}{N^2 - 1/4} \left\{ 1 + \frac{x^{1-\sigma}}{16\sigma} [\hat{s}(N + 1; \sigma)(2N + 2 + \sigma) + \hat{s}(N - 1; \sigma)(2N - 2 + \sigma) - 2\hat{s}(N; \sigma)(2N + \sigma)] + O(x) \right\}.$$  

(56)

Using (13) we find to order $O(x^{1-\sigma})$ that

$$(1 - t)^2 \frac{d}{dt} \frac{d}{dt} \ln C^-(N, t; \lambda) + N^2 = (N^2 - \sigma^2/4) \frac{\tilde{s}(N; \sigma)(1 - \sigma)(2N + \sigma)}{16} x^{1-\sigma} + O(x)$$

(57)

and thus the leading order terms in (54) cancel because of the connection formulas (16) and from the terms of order $x^{1-\sigma}$ we obtain the recursion relation which must be satisfied by $\hat{s}(N; \sigma)$

$$- \hat{s}(N; \sigma)\sigma(1 - \sigma)(2N + \sigma) = (N^2 - \sigma^2/4) \left\{ \hat{s}(N + 1; \sigma)(2N + 2 + \sigma) + \hat{s}(N - 1; \sigma)(2N - 2 + \sigma) - 2\hat{s}(N; \sigma)(2N + \sigma) \right\}$$

(58)

which by direct substitution is easily seen to be satisfied by the expression (15) for $\hat{s}(N; \sigma)$. Thus we have proven by induction that (15) and (16) are correct.

5. Discussion

There are several open questions which need to be clarified before the results reported here can be considered as being completely proven in a mathematically rigorous manner. These questions originate in the problem that the specification of a Fredholm determinant by means of giving its kernel is not unique because of the possibility of similarity transformations and we have already noted for the expression (1) the addition of terms which do not change the traces of the powers of the kernel.

There are several different Fredholm determinants which reduce to the diagonal Ising correlation and which satisfy the equation (7). One such Fredholm determinant is the Âppell kernel (2.9) of [7] and a second is the kernel (3.2) of [7] which is obtained from the Toeplitz determinant for the diagonal Ising correlation using the identity of Borodin–Okounkov [20]. A third Fredholm determinant which satisfies the equation of Jimbo and Miwa for the Ising correlation is a special case of the hypergeometric kernel of Borodin and Deift [21] which
can also be obtained from the more general matrix kernel of [8]. It would be most satisfactory if these Fredholm determinants are all in fact equal and all have the expansion (1) but this has not been proven.

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Note added in proof. Subsequent to the submission of this paper Prof O Lisovyy has pointed out in a private communication that the result for the connection constant (16) can be seen to be a specialization of the result (1.19) in [14] which was conjectured as (4.20) of [13].

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