Orthogonal Polynomial Projection Error Measured in Sobolev Norms in the Unit Disk

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Abstract We study approximation properties of weighted $L^2$-orthogonal projectors onto the space of polynomials of degree less than or equal to $N$ on the unit disk where the weight is of the generalized Gegenbauer form $x \mapsto (1-|x|^2)^{\alpha}$. The approximation properties are measured in Sobolev-type norms involving canonical weak derivatives, all measured in the same weighted $L^2$ norm. Our basic tool consists in the analysis of orthogonal expansions with respect to Zernike polynomials. The sharpness of the main result is proved in some cases. A number of auxiliary results of independent interest are obtained including some properties of the uniformly and nonuniformly weighted Sobolev spaces involved, connection coefficients between Zernike polynomials, an inverse inequality, and relations between the Fourier–Zernike expansions of a function and its derivatives.

Keywords Zernike polynomials · Connection coefficients · Orthogonal projection · Weighted Sobolev space

Mathematics Subject Classification 41A25 · 41A10 · 42C10 · 46E35

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1 Introduction

The main purpose of this work is proving the analogue on the unit disk of a well-known fact in the case of the interval; namely, in its simplest manifestation, the orthogonal projector \( \text{Proj}_N \) mapping \( L^2((-1, 1)) \) onto the space of polynomials of degree less than or equal to \( N \), equivalently defined as the operator returning the truncation at degree \( N \) of the Fourier–Legendre series of its argument, obeys

\[
(\forall u \in H^l((-1, 1))) \quad \| u - \text{Proj}_N(u) \|_{H^1((-1, 1))} \leq C N^{3/2 - l} \| u \|_{H^l((-1, 1))}, \tag{1.1}
\]

where \( C > 0 \) depends only on \( l \) and \( H^1((-1, 1)) \) denote standard Sobolev spaces (this was first proved in [10]; see [9, Chapter 5] for detailed proofs of (1.1), its analogues for the Chebyshev weight and the periodic unweighted case; see [18] for its analogue for general Gegenbauer weights on the unit interval). Our main result (Theorem 3.11) is

\[
(\forall u \in H^l_w(B^2)) \quad \| u - \text{Proj}_N(u) \|_{H^r_w(B^2)} \leq C N^{-1/2 + 2r - l} \| u \|_{H^r_w(B^2)}, \tag{1.2}
\]

where \( B^2 \) is the unit disk, \( \text{Proj}_N \) is the \( L^2_w(B^2) \)-orthogonal projector onto the space of bivariate polynomials of total degree less than or equal to \( N \), and \( C > 0 \) depends only on the integers \( 1 \leq r \leq l \) and the weight \( w \), which in turn is of the generalized Gegenbauer form \( x \mapsto (1 - |x|^2)^\alpha \), \( \alpha > -1 \). The crucial role Fourier–Legendre expansions play in the cited proofs of (1.1) will be taken up here by Fourier–Zernike expansions; in particular, \( \text{Proj}_N \) in (1.2) can be expressed as the truncation at total degree \( N \) of the Fourier–Zernike series of its argument.

The main result, besides being important on its own, has applications in the analysis of polynomial interpolation operators (this is the motivation behind (1.1) and its analogues in [10] and [9, Chapter 5]) and, because of the relative simplicity of orthogonal expansion truncation operators, has been exploited in the one-dimensional case by the present author to give partial characterizations of approximability spaces involved in the analysis of nonlinear iterative methods for the numerical solution of high-dimensional PDE [14, Chapter 4].

We expect some of the auxiliary results to be useful in the design and the analysis of spectral methods on the unit disk (cf. the survey [5]; see also [32]).

We emphasize that neither (1.1) nor (1.2) are best or quasi-best approximation results; in particular, in each case the restriction of the weighted \( L^2 \)-orthogonal projector \( \text{Proj}_N \) does not result in the \( H^1(-1, 1) \)- or the \( H^r_w(B^2) \)-orthogonal projector, respectively. Such weighted Sobolev best approximation results can be found in [9, Chapter 5] and [18] in the one-dimensional case and in [25, § 4] for balls (constant weight). See also [12, § 5] for related results set in a different kind of Sobolev-type space.
1.1 Structure of this Work

In the rest of this introductory section, we briefly provide pointers to relevant literature on the Zernike families of orthogonal polynomials (Sect. 1.2), introduce some basic notation (Sect. 1.3) and some results concerning the Jacobi family of univariate polynomials we will use later (Sect. 1.4). In Sect. 2, we present those auxiliary results that do not depend on the two-dimensional character of our main problem. In the main Sect. 3, we introduce the exact normalization and indexing scheme of Zernike polynomials we will adopt—i.e., that of [34], obtain connection coefficients between Zernike polynomials of different parameters sometimes involving their derivatives and, as a consequence, relations between the expansion coefficients of a function and that of its derivatives (Sect. 3.1). Then, we prove our main result and extend it by complex interpolation (Sect. 3.2) and later prove where we can and otherwise conjecture the sharpness of our main result (Sect. 3.3).

1.2 Zernike Polynomials

The families of Zernike or disk polynomials ([4,21,22], [13, Chapter 2], [34]) are pairwise $L^2_w$-orthogonal in the unit disk, with $w$ of the form $x \mapsto (1 - |x|^2)\alpha$, and play there the role the Gegenbauer or symmetric Jacobi families of polynomials play in the unit interval. Sometimes (but not in this work) the words “complex” or “generalized” are prepended if otherwise the names Zernike/disk polynomials are deemed to correspond exclusively to the real-valued or $\alpha = 0$ cases. These families of polynomials have been used as basis functions for the approximation of functions and the numerical solution of partial differential equations (see the references in [5, §4] to which we would add [26]). Just like their one dimensional counterparts, the Zernike polynomials are subject to a wealth of useful and sometimes quite elegant identities (cf. [34] mainly; see also [2,17,19,20,29,33]). As it is bound to happen with multivariate orthogonal polynomials, the Zernike polynomials are not the only possible family of orthogonal polynomials with respect to the abovementioned weights; cf. [13, § 2.3].

1.3 Notation

We denote by $\mathbb{N}$ the set of strictly positive integers $\{1, 2, \ldots\}$ and let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We denote by $\Pi^d$ the space of complex polynomials in $d$ variables and by $\Pi^d_n$ the subspace of $\Pi^d$ consisting of polynomials of degree at most $n$. Let $B^d := \{x \in \mathbb{R}^d \mid |x| < 1\}$, i.e., the unit ball of $\mathbb{R}^d$.

We will denote the Lebesgue $d$-dimensional measure of subsets $\Omega \subset \mathbb{R}^d$ simply by $|\Omega|$ and integrals of functions $f : \Omega \to \mathbb{C}$ with respect to this measure simply by $\int_{\Omega} f$ or $\int_{\Omega} f(x) \, dx$.

Given an open subset $\Omega$ of a Euclidean space $\mathbb{R}^d$, a measurable and almost-everywhere nonnegative and finite weight function $w : \Omega \to \mathbb{R}$, and $m \in \mathbb{N}_0$, let
\[ L_w^2(\Omega) := \left\{ u : \Omega \to \mathbb{C} \text{ Lebesgue measurable} \mid \| u \|_{L_w^2(\Omega)} := \left( \int_{\Omega} |u|^2 w \right)^{1/2} < \infty \right\}, \]
\[
H_m^w(\Omega) := \left\{ u \in L_w^2(\Omega) \mid \| u \|_{H_m^w(\Omega)} := \left( \sum_{k=0}^m \| \partial_\alpha u \|_{L_w^2(\Omega)}^2 \right)^{1/2} < \infty \right\}, \quad (1.3a)
\]
where in turn the seminorms \(| \cdot |_{H_k^w(\Omega)}\) are defined by
\[
| u |_{H_k^w(\Omega)} := \left( \sum_{|\alpha|=k} \| \partial_\alpha u \|_{L_w^2(\Omega)}^2 \right)^{1/2}.
\] (1.3b)

The \( L_w^2(\Omega) \) are Hilbert spaces and under the additional condition \( w^{-1} \in L_{\text{loc}}^1(\Omega) \) so are the \( H_m^w(\Omega) \) (cf. [24]). All the weight functions used in this work satisfy these conditions.

Given \( a \in \mathbb{C} \) and \( n \in \mathbb{N}_0 \), the Pochhammer symbol \((a)_n\) is defined as \( \prod_{k=0}^{n-1} (a+k) \). Due to the empty product convention, \((a)_0 = 1\) for any \( a \in \mathbb{C} \). Also, \((a)_{m+n} = (a)_m (a+m)_n\). If \( a \neq -\mathbb{N}_0 \), \((a)_n = \Gamma(a+n)/\Gamma(a)\), where \( \Gamma \) is the gamma function (cf. [3, § 1]), which in turn is finite and nonzero on \( \mathbb{C} \setminus (-\mathbb{N}_0) \) and obeys \( \Gamma(z+1) = z \Gamma(z) \) with \( z \) over the same set. Besides these properties, we will also use the asymptotic formula (cf. [27, § 4.5])
\[
\left( \forall (a, b) \in \mathbb{C} \times \mathbb{C} \right) \quad \frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \quad \text{as } \Re(z) \to +\infty; \quad (1.4)
\]
i.e., the limit of the ratio of both sides is 1.

We denote the forward difference operator with respect to some index \( j \) by \( \Delta_j \); that is, \( \Delta_j(f_j) = f_{j+1} - f_j \). Lastly, we will denote generic positive constants by \( C \) with or without sub- and superscripts, tildes, hats, etc., and they may vary from line to line and even from expression to expression.

### 1.4 Jacobi Polynomials

Let \( \alpha, \beta > -1 \), and let \( \chi^{(\alpha, \beta)} : (-1, 1) \to \mathbb{R} \) be the function defined by \( \chi^{(\alpha, \beta)}(t) = (1-t)\alpha (1+t)^\beta \). The Jacobi polynomial of parameter \((\alpha, \beta)\) and degree \( n \), denoted by \( P_n^{(\alpha, \beta)} \), is defined as the member of said degree of the orthogonalization of the sequence of monomials \( (x \mapsto x^n)_{n \in \mathbb{N}_0} \) with respect to the \( L_w^2(\chi^{(\alpha, \beta)}(-1, 1)) \) inner product together with the normalization condition \( P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n} \) (cf. [30, § 4.1]). \( P_n^{(\alpha, \beta)} \) is also a polynomial with respect to \( \alpha \) and \( \beta \) [30, Paragraph 4.22.1].

In [3, Theorem 7.1.3] we find the connection coefficients which allow for expressing \( P_n^{(\gamma, \beta)} \) in terms of the \( P_k^{(\alpha, \beta)} \), \( k \in \{0, \ldots, n\} \); namely,
We note that on account of the continuity of the Jacobi polynomials with respect to their parameters, this relation is still valid if $\alpha + \beta = -1$ if the above coefficients are replaced by their corresponding limits.

2 Polynomial Eigenfunctions on the Euclidean Unit Ball

Let $d \in \mathbb{N}$, and let $\rho : B^d \to \mathbb{R}$ be the function defined by

$$\forall x \in B^d \quad \rho(x) := 1 - |x|^2.$$ We will often make silent use of the fact that for all $x \in B^d$, $\text{dist}(x, \partial B^d) \leq \rho(x) \leq 2\text{dist}(x, \partial B^d)$, for many results in the cited literature are stated in terms of spaces weighted with powers of the distance-to-the-boundary function.

Following [13, eq. (3.1.2)], for every $\alpha > -1$, we define the space of polynomials orthogonal with respect to $L^2_{\rho^\alpha}(B^d)$ of degree exactly $N$ as

$$V_N(L^2_{\rho^\alpha}(B^d)) = \{ P \in \Pi^d_N : \forall Q \in \Pi^d_{N-1}, \langle P, Q \rangle_{L^2_{\rho^\alpha}(B^d)} = 0 \}.$$ It transpires from the theory exposed in Section 3.2 of [13] (using the fact that $\alpha > -1$) that $\bigcup_{N=0}^\infty V_N(L^2_{\rho^\alpha}(B^d))$ spans $\Pi^d$ and, consequently, $\bigcup_{N=0}^\infty V_N(L^2_{\rho^\alpha}(B^d))$ spans $\Pi^d$.

Also, in [13, Theorem 8.1.3] it was proved that members of $V_N(L^2_{\rho^\alpha}(B^d))$ satisfy a differential equation which can be readily rearranged to

$$L^{(\alpha)}(u) := -\frac{1}{\rho^\alpha}\text{div}\left(\rho^{\alpha+1}\nabla u\right) - \sum_{1 \leq i < j \leq d} D_{i,j}^2 u = \lambda^{(\alpha)}_{d,N} u,$$ where

$$\lambda^{(\alpha)}_{d,N} = N(N + d + 2\alpha),$$ and where $D_{i,j}$ denotes the differential operator (an angular derivative; cf. [11, § 2.1]) defined by

$$(D_{i,j} u)(x) := x_j \partial_i u(x) - x_i \partial_j u(x).$$ Given $N \in \mathbb{N}_0$, let $r_{d,N} := \binom{N+d-1}{N}$, and let $p^{(\alpha)}_{d,N,j} \in \{1, \ldots, r_{d,N}\}$, be the members of any $L^2_{\rho^\alpha}(B^d)$-orthonormal basis of $V_N(L^2_{\rho^\alpha}(B^d))$. It follows that the collection of all $p^{(\alpha)}_{d,N,j}$ (with fixed $d$ and $\alpha$) is $L^2_{\rho^\alpha}(B^d)$-orthonormal and spans $\Pi^d$. Some explicit such bases appear in [13, § 5.2], from where it can also be inferred that the dimension of $V_N(L^2_{\rho^\alpha}(B^d))$ is indeed $r_{d,N}$. 
We introduce now the ad hoc function space

$$HZ\alpha(B^d) := C^\infty(B^d),$$ (2.3a)

where the closure is taken with respect to the norm

$$\|v\|_{HZ\alpha(B^d)} := \left( \|v\|_{L^2(B^d)}^2 + \|\nabla v\|_{L^2(\rho^{\alpha+1}(B^d))}^2 + \sum_{1 \leq i < j \leq d} \|D_{i,j}v\|_{L^2(B^d)}^2 \right)^{1/2}.$$ (2.3b)

**Remark 2.1** The arguments put forth in [24] are readily adapted to the presence of the angular differential operators $D_{i,j}$ in order to guarantee that the space defined by the norm (2.3b) is a Hilbert space and hence so is $HZ\alpha(B^d)$.

The space $HZ\alpha(B^d)$ is the natural setting for the weak form (not explicitly needed here) of the eigenvalue Eq. (2.1). Here, we will use it to assist in connecting the membership of functions in the weighted Sobolev spaces $H^k_{\rho^{\alpha}}(B^d)$ of our interest with certain weighted square summability of their expansion coefficients with respect to the above introduced bases of orthogonal polynomials.

**Lemma 2.2** Let $d \in \mathbb{N}$ and $\alpha > -1$. For all $u \in L^2_{\rho^{\alpha}}(B^d)$,

$$u = \sum_{N=0}^{\infty} \sum_{j=1}^{r_d,N} \hat{u}_{d,N,j}^{(\alpha)} p_{d,N,j}^{(\alpha)}$$

where $\hat{u}_{d,N,j}^{(\alpha)} := \left(u, p_{d,N,j}^{(\alpha)}\right)_{L^2_{\rho^{\alpha}}(B^d)}$, the series converging in $L^2_{\rho^{\alpha}}(B^d)$. If, further, $u \in HZ\alpha(B^d)$, the above series also converges in $HZ\alpha(B^d)$. There also hold the Parseval identities

$$\left(\forall u \in L^2_{\rho^{\alpha}}(B^d)\right) \|u\|_{L^2_{\rho^{\alpha}}(B^d)}^2 = \sum_{N=0}^{\infty} \sum_{j=1}^{r_d,N} \left|\hat{u}_{d,N,j}^{(\alpha)}\right|^2,$$ (2.5)

$$\left(\forall u \in HZ\alpha(B^d)\right) \|u\|_{HZ\alpha(B^d)}^2 = \sum_{N=0}^{\infty} \sum_{j=1}^{r_d,N} (1 + \lambda_{d,N,j}^{(\alpha)}) \left|\hat{u}_{d,N,j}^{(\alpha)}\right|^2.$$ (2.6)

**Proof** Directly from the above definitions, the collection $P = \{p_{d,N,j}^{(\alpha)} \mid N \in \mathbb{N}_0, j \in \{1, \ldots, r_d,N\}\}$ is $L^2_{\rho^{\alpha}}(B^d)$-orthonormal, and its span is $\Pi^d$. As $\int_{B^d} \exp(|y|)\rho(y)^{\alpha} \, dy < \infty$, by [13, Theorem 3.2.18], $\Pi^d$ is dense in $L^2_{\rho^{\alpha}}(B^d)$. Thus, $P$ is a Hilbert basis of $L^2_{\rho^{\alpha}}(B^d)$.

Let $\hat{v}$ be the unit outward normal vector defined on $\partial B^d$. Given any $N \in \mathbb{N}_0$, $j \in \{1, \ldots, r_d,N\}$ and $t \in C^\infty(B^d)$, by integration by parts, using that $\rho^{\alpha+1}$ vanishes on $\partial B^d$, that for $1 \leq i < k \leq d$, $x_i \hat{v}_i - x_k \hat{v}_k$ vanishes on $\partial B^d$ and $D_{i,k}\rho^{\alpha}$ vanishes in $B^d$ and (2.1), we find that
\[
\left\{ t, p_{d,N,j}^{(\alpha)} \right\} \mid \mathbb{H}_{\alpha}(B^d) = \left( 1 + \lambda_{d,N}^{(\alpha)} \right) \left\{ t, p_{d,N,j}^{(\alpha)} \right\} L^2_{1\rho\alpha}(B^d).
\]

The above equality extends to \( t \in \mathbb{H}_{\alpha} \) by density (cf. (2.3)). It follows that the renormalized collection \( \tilde{P} = \left\{ \left( 1 + \lambda_{d,N}^{(\alpha)} \right)^{-1/2} p_{d,N,j}^{(\alpha)} : N \in \mathbb{N}_0 \land j \in \{1, \ldots, r_d,N \} \right\} \), its span still being \( \mathcal{P}^d \), is \( \mathbb{H}_{\alpha}(B^d) \)-orthonormal. Further, if \( s \in \mathbb{H}_{\alpha}(B^d) \) is \( \mathbb{H}_{\alpha}(B^d) \)-orthogonal to \( \mathcal{P}^d \), by the above equality and the fact that the eigenvalues in (2.2) are nonnegative, \( s \) is \( L^2_{1\rho\alpha}(B^d) \)-orthogonal to \( \mathcal{P}^d \) as well; i.e., \( s = 0 \). Thus, \( \tilde{P} \) is a Hilbert basis of \( \mathbb{H}_{\alpha}(B^d) \).

The desired result then stems from the basic properties of Hilbert bases; see, e.g., [6, Corollary 5.10]. \( \square \)

**Lemma 2.3** Let \( d \in \mathbb{N} \), \( \alpha > -1 \) and \( k \in \mathbb{N}_0 \). Then, there exists a positive constant \( C = C(d, \alpha, k) \) such that

\[
\left( \forall u \in H_{1\rho\alpha}^k(B^d) \right) \sum_{N=0}^{\infty} \sum_{j=1}^{r_d,N} \left( \lambda_{d,N}^{(\alpha)} \right)^k \left| \hat{u}_{d,N,j}^{(\alpha)} \right|^2 \leq C \left\| u \right\|^2_{H_{1\rho\alpha}^k(B^d)}.
\]

**Proof** Let us first note that from [23, Remark 11.12.(iii)], every \( u \in H_{1\rho\alpha}^1(B^d) \) is the limit in the same norm of a sequence of \( C^\infty(B^d) \) functions. The norm of \( \mathbb{H}_{\alpha}(B^d) \) being weaker than the \( H_{1\rho\alpha}^1(B^d) \) norm, the same sequence converges to \( u \) in \( \mathbb{H}_{\alpha}(B^d) \), so by definition (2.3), \( u \in \mathbb{H}_{\alpha}(B^d) \). Combining this fact with the second Parseval identity (2.6),

\[
\left\| u \right\|^2_{H_{1\rho\alpha}^k(B^d)} \geq C_1 \left[ \left\| u \right\|^2_{\mathbb{H}_{\alpha}(B^d)} - \left\| u \right\|^2_{L^2_{1\rho\alpha}(B^d)} \right] = C_1 \sum_{N=0}^{\infty} \sum_{j=1}^{r_d,N} \lambda_{d,N}^{(\alpha)} \left| \hat{u}_{d,N,j}^{(\alpha)} \right|^2 . \tag{2.7}
\]

Now, expanding the terms in (2.1), we find that

\[
L^{(\alpha)}(u) = -\rho \Delta u + (2\alpha + 1 + d)x \cdot \nabla u - \sum_{1 \leq i < j \leq d} \left( \chi_i^2 \partial_j^2 u + \chi_j^2 \partial_i^2 u - 2\chi_i \chi_j \partial_i \partial_j u \right).
\]

As the coefficients \( \rho, x, \chi_j^2 \), etc. above have \( L^\infty(B^d) \) derivatives of all orders, we infer that for all \( m \in \mathbb{N}_0 \), \( L^{(\alpha)}(H_{1\rho\alpha}^m(B^d), H_{1\rho\alpha}^m(B^d)) \). By integration by parts and (2.1), for every \( N \in \mathbb{N}_0 \), \( j \in \{1, \ldots, r_d,N \} \) and \( u \in C^\infty(B^d) \),

\[
\left\langle L^{(\alpha)}(u), p_{d,N,j}^{(\alpha)} \right\rangle_{L^2_{1\rho\alpha}(B^d)} = \left\langle u, L^{(\alpha)}(p_{d,N,j}^{(\alpha)}) \right\rangle_{L^2_{1\rho\alpha}(B^d)} = \lambda_{d,N,j}^{(\alpha)} \left\langle u, p_{d,N,j}^{(\alpha)} \right\rangle_{L^2_{1\rho\alpha}(B^d)} .
\]

Because of the density of \( C^\infty(B^d) \) in \( H_{1\rho\alpha}^2(B^d) \) [23, Remark 11.12.(iii)], the above equality extends to \( u \) in the latter space. Thus, if \( k \) is even, we obtain the desired result via the first Parseval identity (2.5) by applying the \( k/2 \)-th power of \( L^{(\alpha)} \) to \( u \). If \( k \) is odd, we similarly apply the \((k - 1)/2\)-th power of \( L^{(\alpha)} \) to \( u \) and then use (2.7). \( \square \)
Given $N \in \mathbb{N}_0$, let $\text{Proj}_{N}^{(\alpha)} : L_{\rho^\alpha}^2(B^d) \to \Pi_N^d$ be the orthogonal projection from $L_{\rho^\alpha}^2(B^d)$ onto $\Pi_N^d$. On account of (2.4) in Lemma 2.2, we can express it as a truncation operator:

$$\forall u \in L_{\rho^\alpha}^2(B^d)) \quad \text{Proj}_{N}^{(\alpha)}(u) = \sum_{n=0}^{N-1} \sum_{j=1}^{r_{d,n}} \hat{u}_{d,n,j}^{(\alpha)} P_{d,n,j}^{(\alpha)}.$$  \hfill (2.8)

**Corollary 2.4** If $d \in \mathbb{N}$, $\alpha > -1$, and $k \in \mathbb{N}_0$, there exists a positive constant $C = C(\alpha, d, k)$ such that

$$\left( \forall u \in H_{\rho^\alpha}^k(B^d) \right) \quad \left( \forall N \in \mathbb{N}_0 \right) \quad \|u - \text{Proj}_{N}^{(\alpha)}(u)\|_{L_{\rho^\alpha}^2(B^d)} \leq C(N+1)^{-k} \|u\|_{H_{\rho^\alpha}^k(B^d)}.$$  

**Proof** This is a direct consequence of the Parseval identity (2.5) in Lemma 2.2, Lemma 2.3, and the fact that the $\lambda_{d,N}^{(\alpha)}$ depend quadratically on $N$. \hfill $\square$

**Remark 2.5** The result of Corollary 2.4 is related to a particular case of [35, Corollary 4.4]. However, in that work $\alpha \geq -1/2$, and the result is stated in terms of the domains of powers of the $L^{(\alpha)}$ operator (seen as unbounded operators in $L_{\rho^\alpha}^2(B^d)$) instead of weighted Sobolev spaces such as the $H_{\rho^\alpha}^k(B^d)$ (cf. (1.3)).

The orthogonal projection operators defined in (2.8) allow for defining an equivalent norm for the $H_{\rho^\alpha}^k(B^d)$ spaces.

**Proposition 2.6** Let $d \in \mathbb{N}$, $\alpha > -1$, and $k \in \mathbb{N}$. Then, the functional $u \mapsto \|u\|_{H_{\rho^\alpha}^k(B^d)} + \|\text{Proj}_{k-1}^{(\alpha)}(u)\|_{L_{\rho^\alpha}^2(B^d)}$ is an equivalent norm for $H_{\rho^\alpha}^k(B^d)$.

**Proof** Setting $\Omega = B^d, \kappa = 1, p = q = 2, \beta = \alpha$, and $\alpha = \alpha$ in Theorem 8.8 of [28], we have that $H_{\rho^\alpha}^1(B^d)$ is compactly embedded in $L_{\rho^\alpha}^2(B^d)$. By standard arguments [1, Remark 6.4.4], this implies that $H_{\rho^\alpha}^k(B^d)$ is compactly embedded in $H_{\rho^\alpha}^{k-1}(B^d)$. Then, the desired result follows from the Peetre–Tartar lemma; in the formulation of [31, Lemma 11.1], it comes from setting $E_1 = H_{\rho^\alpha}^k(B^d), E_2 = [L_{\rho^\alpha}^2(B^d)]^{(k+\beta-1)}, E_3 = H_{\rho^\alpha}^{k-1}(B^d), A = \nabla^k (k\text{-fold gradient}), B$ the injection from $H_{\rho^\alpha}^k(B^d)$ onto $H_{\rho^\alpha}^{k-1}(B^d), G = L_{\rho^\alpha}^2(B^d), \text{ and } M = \text{Proj}_{k-1}^{(\alpha)}$ and noting that $\nabla^k u \equiv 0$ implies $u \in \Pi_{k-1}$, which in turn is a consequence of [31, Lemma 6.4]. \hfill $\square$

3 Truncation Projection in the Two-Dimensional Case

3.1 Zernike Polynomials and Fourier–Zernike Series

Let $\theta : B^2 \to \mathbb{R}$ and $r : B^2 \to \mathbb{R}$ be the usual components of the Cartesian-to-polar change of coordinates. Given $\alpha > -1$ and $m, n \in \mathbb{N}_0$, we adopt the following definition of the Zernike polynomial $P_{m,n}^{(\alpha)} [34, eq. (2.1)]:$

$$P_{m,n}^{(\alpha)} = \frac{\Gamma(\min(m, n) + 1)\Gamma(\alpha + 1)}{\Gamma(\min(m, n) + \alpha + 1)} r^{m-n} e^{i(m-n)\theta} P_{\min(m,n)}^{(\alpha, |m-n|)} (2r^2 - 1). \hfill (3.1)$$
In order to simplify some expressions below, we adopt the convention
\[ P_{m,n}^{(\alpha)} \equiv 0 \quad \text{if } m < 0 \text{ or } n < 0. \] (3.2)

It can be inferred from [34, § 2.1] that each \( P_{m,n}^{(\alpha)} \) defined in (3.1) is indeed a bivariate algebraic polynomial of degree \( m + n \) and that \( \{ P_{m,n}^{(\alpha)} \mid m + n = N \} \) is an orthogonal basis of \( \mathcal{V}_N(L^2_{\rho^\alpha}(B^2)) \). Thus, each \( P_{m,n}^{(\alpha)} \) is an eigenfunction of (2.1) in the \( d = 2 \) and \( N = m + n \) case with associated eigenvalue (cf. (2.2))
\[ \lambda_{m,n}^{(\alpha)} = (m + n)(m + n + 2 + 2\alpha). \] (3.3)

The attractiveness of the precise form for the basis functions given in (3.1) is apparent in light of the simplicity of the relations (cf. equations (3.4) and (5.3) of [34])
\[ h_{m,n}^{(\alpha)} := \| P_{m,n}^{(\alpha)} \|_{L^2_{\rho^\alpha}(B^2)}^2 = \frac{\pi \Gamma(\alpha + 1)^2}{m + n + \alpha + 1} \frac{\Gamma(m + 1)}{\Gamma(m + \alpha + 1)} \frac{\Gamma(n + 1)}{\Gamma(n + \alpha + 1)}, \] (3.4)
\[ \partial_z^* P_{m,n}^{(\alpha)} = \frac{(m + \alpha + 1)n}{\alpha + 1} P_{m,n+1}^{(\alpha+1)} \quad \text{and} \quad \partial_z P_{m,n}^{(\alpha)} = \frac{m(n + \alpha + 1)}{\alpha + 1} P_{m,n-1}^{(\alpha+1)}, \] (3.5)

which are valid for all \((m, n) \in \mathbb{N}_0 \times \mathbb{N}_0\); here, \( \partial_z^* = \frac{1}{2} (\partial_1 + i \partial_2) = \frac{\rho^\alpha}{2} (\partial_r + \frac{\theta}{r} \partial_\theta) \) and \( \partial_z = \frac{1}{2} (\partial_1 - i \partial_2) = \frac{\rho^\alpha}{2} (\partial_r - \frac{\theta}{r} \partial_\theta) \). For analytical purposes, these differential operators can be used in lieu of the canonical ones because for all weakly differentiable \( u \) and \( l \in \mathbb{N} \), the relation
\[ |\nabla^l u|^2 = \sum_{l_1+l_2=l} |\partial_{z_1}^{l_1} \partial_{z_2}^{l_2} u|^2 \cong_l \sum_{l_1+l_2=l} |\partial_{\varphi}^{l_1} \partial_{\varphi}^{l_2} u|^2 \] (3.6)

holds almost everywhere, with \( \cong_l \) meaning that each side is bounded by the other times some positive constant depending on \( l \) only. When \( l = 1 \), the left-hand side is exactly twice the right-hand side almost everywhere.

We now translate some of the results of Sect. 2 to the differently indexed and nonnormalized basis (3.1). Suppose that \( \alpha > -1 \). From Lemma 2.2,
\[ \left( \forall u \in L^2_{\rho^\alpha}(B^2) \right) \quad u = \sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} \hat{u}^{(\alpha)}_{m,n} P_{m,n}^{(\alpha)} \] (3.7)

in the \( L^2_{\rho^\alpha}(B^2) \) sense in general and in the \( HZ_\alpha(B^2) \) sense if, in addition, \( u \in HZ_\alpha(B^2) \); here, for all \( u \in L^2_{\rho^\alpha}(B^2) \) and \((m,n) \in \mathbb{N}_0 \times \mathbb{N}_0\),
\[ \hat{u}^{(\alpha)}_{m,n} := \left[ u, P_{m,n}^{(\alpha)} \right]_{L^2_{\rho^\alpha}(B^2)} / h_{m,n}^{(\alpha)}. \]
Further, Parseval’s identity manifests itself as
\[
\begin{cases}
(\forall u \in L^2_{\rho^\alpha}(B^2)) \quad \|u\|^2_{L^2_{\rho^\alpha}(B^2)} = \sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} \left[ \frac{1}{1 + \lambda_{m,n}} \left| \hat{u}_{m,n}^{(\alpha)} \right|^2 \right] h_{m,n}^{(\alpha)}. \\
(\forall u \in H^{k}_{\rho^\alpha}(B^2)) \quad \|u\|^2_{H^{k}_{\rho^\alpha}(B^2)} = \sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} \left( \lambda_{m,n}^{(\alpha)} \right)^k \left| \hat{u}_{m,n}^{(\alpha)} \right|^2 h_{m,n}^{(\alpha)} \leq C \|u\|^2_{H^{k}_{\rho^\alpha}(B^2)}. 
\end{cases}
\]
(3.8)

From Lemma 2.3, we know that there exists a positive constant \( C = C(\alpha, k) \) such that
\[
(\forall u \in H^{k}_{\rho^\alpha}(B^2)) \sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} \left( \lambda_{m,n}^{(\alpha)} \right)^k \left| \hat{u}_{m,n}^{(\alpha)} \right|^2 h_{m,n}^{(\alpha)} \leq C \|u\|^2_{H^{k}_{\rho^\alpha}(B^2)}. 
\]
(3.9)

The projection (truncation) operator \( \Proj^{(\alpha)}_N : L^2_{\rho^\alpha}(B^2) \rightarrow \Pi^2_N \) of (2.8) here takes the form
\[
(\forall u \in L^2_{\rho^\alpha}(B^2)) \quad \Proj^{(\alpha)}_N (u) = \sum_{m+n \leq N} \hat{u}_{m,n}^{(\alpha)} P_{m,n}^{(\alpha)}.
\]
(3.10)

**Proposition 3.1** (Connection coefficients between Zernike polynomials) If \( \alpha, \gamma > -1 \) and \((m, n) \in \mathbb{N}_0 \times \mathbb{N}_0,\)
\[
P_{m,n}^{(\alpha)} = \frac{\Gamma(m+1) \Gamma(n+1) \Gamma(\alpha+1)}{\Gamma(\alpha+m+1) \Gamma(\alpha+n+1) \gamma + 1} \times \sum_{k=0}^{\min(m,n)} \left( \frac{(\alpha - \gamma)_{k}}{\Gamma(k+1)} \right) \times \frac{\Gamma(\gamma + m - k + 1) \Gamma(\gamma + n - k + 1)(\gamma + m + n - 2k + 1)}{\Gamma(m-k+1) \Gamma(n-k+1) \gamma + m + n - 2k + 1)} 
\]
\[
\quad \times \left[ P_{m-k,n-k}^{(\gamma)} \right].
\]

**Proof** From the definition (3.1) of \( P_{m,n}^{(\alpha)} \), using (1.5) to expand \( P_{m,n}^{(\alpha,|m-n|)} \) in terms of \( P_{j}^{(\gamma,|m-n|)}, \ j \in \{0, \ldots, \min(m,n)\} \), expanding Pochhammer symbols with suitable arguments into ratios of gamma functions, using the basic property \( x \Gamma(x) = \Gamma(x+1) \), the fact that \( \min(m,n) + |m-n| = \max(m,n) \), the fact that for any commutative function \( B : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{R} \) it holds that \( B(\min(m,n), \max(m,n)) = B(m,n) \) and some cancellations, we find that
\[
P_{m,n}^{(\alpha)} = \frac{\Gamma(m+1) \Gamma(n+1) \Gamma(\alpha+1)}{\Gamma(\alpha+m+1) \Gamma(\alpha+n+1)} \times \sum_{j=0}^{\min(m,n)} \left( \frac{(\alpha - \gamma)_{\min(m,n)-j}}{\Gamma(\min(m,n) - j + 1)} \right) \Gamma(\max(m,n) + 1 + j) \times \frac{\Gamma(\gamma + |m-n| + 1 + j)(\gamma + |m-n| + 2j + 1)}{\Gamma(|m-n| + 1 + j) \Gamma(\gamma + \max(m,n) + 2 + j)} \times r^{|m-n|} \epsilon^{(\gamma,|m-n|)} P_{j}^{(\gamma,|m-n|)}(2^j - 1).
\]
On defining $m_j := j + \max(m-n, 0)$ and $n_j := j + \max(n-m, 0)$ and noting that

$$m_j \geq 0, \quad n_j \geq 0, \quad m-n = m_j - n_j \quad \text{and} \quad j = \min(m_j, n_j),$$

we find that dividing and multiplying each term of the above sum by $\frac{\Gamma(j+1)\Gamma(j+1)}{\Gamma(j+y+1)}$, we can make $P_{m_j, n_j}^{(y)}$ appear. Substituting the summation variable for $k = \min(m, n) - j$ (wherein $m_j$ and $n_j$ turn into $m-k$ and $n-k$, respectively) and using $|m-n| + 2 \min(m, n) = m+n$ plus some of the previously used identities, we obtain the desired result after a number of elementary cancellations.

We will need the following auxiliary result, which is of independent interest (see Proposition 4.26 and Theorem 4.29 of [14] for its one-dimensional analogue and one application, respectively).

**Proposition 3.2** For all $\alpha > -1$ and $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$,

$$\left| P_{m,n}^{(\alpha)} \right|_{H^{1/2}_{\rho_\alpha}(B^2)}^2 = \frac{2 \pi \Gamma(\alpha + 1)^2 \Gamma(m + 1) \Gamma(n + 1)}{(\alpha + 1)\Gamma(m + \alpha + 1)\Gamma(n + \alpha + 1)}(2m n + (m + n)(\alpha + 1)).$$

**Proof** We first observe that for all $\alpha > -1$ and $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$,

$$\left\| P_{m,n}^{(\alpha+1)} \right\|_{L^2_{\rho_\alpha}(B^2)}^2 = \frac{\pi \Gamma(m + 1) \Gamma(n + 1) \Gamma(\alpha + 1) \Gamma(\alpha + 2)}{\Gamma(m + \alpha + 2) \Gamma(n + \alpha + 2)}.$$ (3.11)

Indeed, using Proposition 3.1 to expand $F_{m,n}^{(\alpha+1)}$ in terms of the $P_{m-k,n-k}^{(\alpha)}$ with $k \in \{0, \ldots, \min(m, n)\}$, noting that the latter are $L^2_{\rho_\alpha}(B^2)$-orthogonal, and using (3.4), we obtain

$$\left\| P_{m,n}^{(\alpha+1)} \right\|_{L^2_{\rho_\alpha}(B^2)}^2 = \frac{\pi \Gamma(m + 1)^2 \Gamma(n + 1)^2 \Gamma(\alpha + 2)^2}{\Gamma(\alpha + m + 2)^2 \Gamma(\alpha + n + 2)^2} \sum_{k=0}^{\min(m,n)} \theta_{m,n,k}^{(\alpha)},$$

where $\theta_{m,n,k}^{(\alpha)} = (\alpha + m + n - 2k + 1)\frac{\Gamma(\alpha+m-k+1)\Gamma(\alpha+n-k+1)}{\Gamma(m-k+1)\Gamma(n-k+1)}$. Using that $\theta_{m,n,k}^{(\alpha)} = \Delta_k(\zeta_{m,n,k}^{(\alpha)})$, where $\zeta_{m,n,k}^{(\alpha)} = \frac{\Gamma(m-k+\alpha+2)\Gamma(n-k+\alpha+2)}{(\alpha+1)\Gamma(m-k+1)\Gamma(n-k+1)}$, the above sum telescopes and (3.11) follows. Then the desired result is a direct consequence of the relations in (3.5) and (3.11).

The following result is an inverse or Markov-type inequality.

**Proposition 3.3** Let $\alpha > -1$. Then there exists $C = C(\alpha) > 0$ such that for all $N \in \mathbb{N}_0$ and $p \in \Pi^2_N$,

$$\|\nabla p\|_{L^2_{\rho_\alpha}(B^2)}^2 \leq C N^2 \|p\|_{L^2_{\rho_\alpha}(B^2)}.$$
Proposition 3.4  If with adequate shifts of the relations in (3.5) yields (3.13) and (3.14).

By dividing Proposition 3.2 by (3.4), for all \(m, n \in \mathbb{N}_0\),

\[
|P_{m,n}^{(\alpha)}|^2_{H_{\rho,\alpha}^1(B^2)} = \frac{2(2m + (m + n)(\alpha + 1))(m + n + \alpha + 1)}{\alpha + 1} \left\| P_{m,n}^{(\alpha)} \right\|_{L^2_{\rho,\alpha}(B^2)}^2.
\]

From the polar separated form of \(P_{m,n}^{(\alpha)}\) in (3.1), it is clear that its derivative with respect to \(r\) or \(\theta\) results in a function that is the multiplication of a function depending on \(r\) and the angular mode \(e^{i(m-n)\theta}\). Then, for every degree \(k \in \mathbb{N}_0\), the collection \(\{\nabla P_{m,n}^{(\alpha)} \mid m + n = k\}\) is pairwise \([L^2_{\rho,\alpha}(B^2)]^2\)-orthogonal. Therefore, expanding \(p \in \mathcal{H}_{\rho,\alpha}^1\) as in (3.7), using standard inequalities and the fact that \(4mn \leq (m+n)^2\),

\[
|p|^2_{H_{\rho,\alpha}^1(B^2)} = \left| \sum_{k=0}^{N} \sum_{m+n=k} \hat{p}_{m,n}^{(\alpha)} P_{m,n}^{(\alpha)} \right|_{H_{\rho,\alpha}^1(B^2)}^2 \leq (N+1) \sum_{k=0}^{N} \sum_{m+n=k} \hat{p}_{m,n}^{(\alpha)} P_{m,n}^{(\alpha)} \right|_{H_{\rho,\alpha}^1(B^2)}^2 \leq (N+1) \sum_{k=0}^{N} \sum_{m+n=k} \frac{k(k+\alpha+1)(k+2\alpha+2)}{\alpha + 1} \left\| \hat{p}_{m,n}^{(\alpha)} \right\|_{L^2_{\rho,\alpha}(B^2)}^2 \left\| P_{m,n}^{(\alpha)} \right\|_{L^2_{\rho,\alpha}(B^2)}^2.
\]

Bounding \(k\) by \(N\) and using Parseval’s identity (3.8), we obtain the desired result. \(\square\)

We will now deduce some simple relations between Zernike polynomials that will be useful later to express the expansion coefficients of the derivatives of a function in terms of the expansion coefficients of the function itself. Related identities including three-term recurrences appear in [34, §5]; (3.13) and (3.14) appear in [20] in the case \(\alpha = 0\).

Proposition 3.4  If \(\alpha > -1\), then for \((m, n) \in \mathbb{N}_0 \times \mathbb{N}_0\), we have the parameter-raising expansion

\[
(m + n + \alpha + 1) P_{m,n}^{(\alpha)} = \frac{(m + \alpha + 1)(n + \alpha + 1)}{\alpha + 1} P_{m,n}^{(\alpha+1)} - \frac{m n}{\alpha + 1} P_{m-1,n-1}^{(\alpha+1)},
\]

and the same-parameter expansions with respect to first order derivatives

\[
(m + n + \alpha + 1) P_{m,n}^{(\alpha)} = \frac{n + \alpha + 1}{n + 1} \partial_z P_{m,n+1}^{(\alpha)} - \frac{m}{m + \alpha} \partial_z P_{m-1,n}^{(\alpha)},
\]

and

\[
(m + n + \alpha + 1) P_{m,n}^{(\alpha)} = \frac{m + \alpha + 1}{m + 1} \partial_{z^*} P_{m+1,n}^{(\alpha)} - \frac{n}{n + \alpha} \partial_{z} P_{m,n-1}^{(\alpha)}.
\]

Proof  We obtain (3.12) from Proposition 3.1 by setting \(\gamma = \alpha + 1\). Combining (3.12) with adequate shifts of the relations in (3.5) yields (3.13) and (3.14). \(\square\)
Proposition 3.5 Let $\alpha > -1$, and let $u \in H^k_{P_\alpha}(B^2)$. Then,

\[(\forall (m, n) \in \mathbb{N}_0 \times \mathbb{N}_0) \lim_{L \to \infty} L^{k-\alpha-1/2} \hat{u}_{m+L, n+L}^{(\alpha)} = 0.\]

Proof Using the forms of $\lambda_{m+L, n+L}^{(\alpha)}$ and $h_{m+L, n+L}^{(\alpha)}$ that stem from (3.3) and (3.4), respectively, and applying the asymptotic formula (1.4) on the ratio of gamma functions therein, we obtain

\[
\lambda_{m+L, n+L}^{(\alpha)} \sim 4L^2 \quad \text{and} \quad h_{m+L, n+L}^{(\alpha)} \sim \pi \Gamma(\alpha + 1)^2 2^{-1} L^{-1-2\alpha} \quad \text{as } L \to \infty.
\]

Combining this with the fact (coming from (3.9)) that

\[
\lim_{L \to \infty} \left( \lambda_{m+L, n+L}^{(\alpha)} \right)^k \left| u_{m+L, n+L}^{(\alpha)} \right|^2 h_{m+L, n+L}^{(\alpha)} = 0,
\]

we obtain the desired result. \[\square\]

Lemma 3.6 Let $\alpha > -1$ and

\[u \in H^k_{P_\alpha}(B^2) \quad \text{with} \quad k = \begin{cases} 1 & \text{if } \alpha \in [-1/2, \infty), \\ 2 & \text{if } \alpha \in (-1, -1/2). \end{cases} \tag{3.15}\]

Then, the coefficients of the Fourier–Zernike series (3.7) of $\partial_{z^*} u$ and $\partial_z u$ can be expressed in terms of the coefficients of the corresponding series of $u$ according to

\[
\widehat{(\partial_{z^*} u)}_{m, n}^{(\alpha)} = (m + n + \alpha + 1) \sum_{l=0}^{\infty} \frac{(m + 1)_l}{(m + \alpha + 1)_l} \frac{(n + 1)_{l+1}}{(n + \alpha + 1)_{l+1}} \hat{u}_{m+l, n+l+1}^{(\alpha)},
\]

and

\[
\widehat{(\partial_{z} u)}_{m, n}^{(\alpha)} = (m + n + \alpha + 1) \sum_{l=0}^{\infty} \frac{(m + 1)_{l+1}}{(m + \alpha + 1)_{l+1}} \frac{(n + 1)_l}{(n + \alpha + 1)_l} \hat{u}_{m+l+1, n+l}^{(\alpha)}.
\]

for $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Proof Let us abbreviate $v_{m, n} = \widehat{(\partial_{z^*} u)}_{m, n}^{(\alpha)}$. Then, using (3.13), the fact that $P_{-1, n}^{(\alpha)} \equiv 0$ and $\partial_{z^*} P_{m, 0}^{(\alpha)} \equiv 0$ (cf. (3.2) and (3.5)), and careful index tracking, we have that given $M, N \in \mathbb{N}$,
\[
\sum_{m=0}^{M} \sum_{n=0}^{N} v_{m,n} P^{(\alpha)}_{m,n} = \sum_{m=0}^{M-1} \sum_{n=0}^{N} \left[ \frac{v_{m,n-1}}{m + n + \alpha} \frac{n + \alpha}{n} - \frac{v_{m+1,n}}{m + n + \alpha + 2} \frac{m + 1}{m + \alpha + 1} \right] \partial z u^{(\alpha)}_{m,n} \\
+ \sum_{m=0}^{M} \frac{v_{m,N}}{m + N + \alpha + 1} \frac{N + \alpha + 1}{N + 1} \partial z u^{(\alpha)}_{m,n+1}
\]

Taking the square of the \( L^2_{\rho^2} (B^2) \) norm of \( S_{M,N} \), using the \( L^2_{\rho^2+1} (B^2) \)-orthogonality of the terms that comprise it (which comes from (3.5)), substituting the resulting \( h_{m,N}^{(\alpha)} \) with the products \( h_{m,N}^{(\alpha)} / h_{m,N}^{(\alpha)} \), and simplifying the gamma functions appearing in the second factors (cf. (3.4)), we obtain

\[
\| S_{M,N} \|_{L^2_{\rho^2+1} (B^2)}^2 = \sum_{m=0}^{M} \left| v_{m,N} \right|^2 \frac{(N + \alpha + 1)(m + \alpha + 1)}{(m + N + \alpha + 1)(m + N + \alpha + 2)} h_{m,N}^{(\alpha)} \\
\leq \sum_{m=0}^{M} \left| v_{m,N} \right|^2 h_{m,N}^{(\alpha)}.
\]

As the \( v_{m,N} \) are the coefficients of the expansion of the \( L^2_{\rho^2} (B^2) \) function \( \partial z u \), it follows from the above inequality and Parseval’s identity (3.8) that \( S_{M,N} \xrightarrow{M,N \to \infty} 0 \) in \( L^2_{\rho^2+1} (B^2) \). The same argument leads to \( T_{M,N} \xrightarrow{M,N \to \infty} 0 \) in the same space. As the left-hand side of (3.18) tends to \( \partial z u \) as \( M, N \to \infty \) in \( L^2_{\rho^2+1} (B^2) \) (because it does so in the stronger \( L^2_{\rho^2} (B^2) \) norm), we conclude that

\[
\partial z u = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[ \frac{v_{m,n-1}}{m + n + \alpha} \frac{n + \alpha}{n} - \frac{v_{m+1,n}}{m + n + \alpha + 2} \frac{m + 1}{m + \alpha + 1} \right] \partial z u^{(\alpha)}_{m,n},
\]

(3.19)

the series converging in the \( L^2_{\rho^2+1} (B^2) \) sense.

On the other hand, per the case \( k = 1 \) of Lemma 2.3, \( u \) itself is a member of \( H_{\alpha} (B^2) \), and thus, per Lemma 2.2, its Fourier–Zernike series as defined in (3.7) converges to \( u \) in \( H_{\alpha} (B^2) \), and because of the structure of that norm (cf. (2.3)), we have, again using the fact that \( \partial z u^{(\alpha)}_{m,0} \equiv 0 \),

\[
\partial z u = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \hat{u}_{m,n}^{(\alpha)} \partial z u^{(\alpha)}_{m,n},
\]

(3.20)
the series also converging in the \( L^2_{\rho^{\alpha+1}}(B^2) \) sense.

Since in the index range involved the \( \partial_{x^*} P_{m,n}^{(\alpha)} \) are nonzero and pairwise \( L^2_{\rho^{\alpha+1}}(B^2) \)-orthogonal, we can compare the coefficients of the series (3.19) and (3.20) so as to obtain for \( m \in \mathbb{N}_0 \) and \( n \in \mathbb{N}_0 \),

\[
\frac{1}{m + n + \alpha + 1} v_{m,n} = \frac{n + 1}{n + \alpha + 1} u_{m,n+1}^{(\alpha)} + \frac{(n+1)(m+1)}{(m+n+\alpha+3)(n+\alpha+1)(m+\alpha+1)} v_{m+1,n+1}.
\]

(3.21)

An induction argument based on (3.21) can then justify that, for all \((m, n) \in \mathbb{N}_0 \times \mathbb{N}_0 \) and \( L \in \mathbb{N}_0 \),

\[
\frac{1}{m + n + \alpha + 1} v_{m,n} = \sum_{l=0}^{L} \frac{(n+1)_l}{(n+\alpha+1)_l} \frac{(m+1)_l}{(m+\alpha+1)_l} u_{m+l,n+1+l}^{(\alpha)} + R_{m,n,\alpha,L+1},
\]

where

\[
R_{m,n,\alpha,L} := \frac{(n+1)_L}{(n+\alpha+1)_L} \frac{(m+1)_L}{(m+\alpha+1)_L} \frac{1}{m+n+2L+\alpha+1} v_{m+L,n+L}.
\]

(3.22)

Now, expressing the Pochhammer symbols above as ratios of gamma functions and using the asymptotic relation (1.4), we find

\[
R_{m,n,\alpha,L} \sim \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \frac{\Gamma(m+\alpha+1)}{\Gamma(m+1)} 2^{-1-2\alpha} v_{m+L,n+L} \text{ as } L \to \infty.
\]

So far, we have only used that \( u \in H^1_{\rho^{\alpha}}(B^2) \), which is weaker than the hypothesis (3.15) when \( \alpha \in (-1, -1/2) \).

As the \( v_{m,n} \) are the Fourier–Zernike coefficients of the expansion of \( \partial_{x^*} u \) and the latter belongs to \( H^{k-1}_{\rho^{\alpha}}(B^2) \), we infer from Proposition 3.5 that \( \lim_{L \to \infty} L^{k-\alpha-3/2} v_{m+L,n+L} = 0 \), which, together with the fact that \( k \geq 1/2 - \alpha \) (only now we are making use of the full hypothesis (3.15)), implies that \( \lim_{L \to \infty} R_{m,n,\alpha,L} = 0 \). Thus, (3.16) is obtained from (3.22).

Let the reflection \( A : B^2 \to B^2 \) be defined by \( A(x) = (x_1, -x_2) \) for all \( x \in B^2 \). Then, \( u \circ A \in H^k_{\rho^{\alpha}}(B^2) \) as well and \( \partial_{x^*} u \circ A = \partial_{x^*} (u \circ A) \). This, together with (3.16), the readily verifiable formulae \( \rho \circ A = \rho, P_{m,n}^{(\alpha)} \circ A = P_{m,n}^{(\alpha)}, \) and \( h_{m,n}^{(\alpha)} = h_{m,n}^{(\alpha)} \), and the invariance of the Lebesgue measure with respect to reflections give (3.17). \( \Box \)

The hypothesis \( u \in H^2_{\rho^{\alpha}}(B^2) \) adopted in Lemma 3.6 when \( \alpha \in (-1, -1/2) \) can be relaxed to \( u \) belonging to certain interpolation spaces between \( H^1_{\rho^{\alpha}}(B^2) \) and \( H^2_{\rho^{\alpha}}(B^2) \) as long as the residual \( R_{m,n,\alpha,L+1} \) of (3.22) can be shown to tend to 0 as \( L \to \infty \). However, the example below makes it clear that we cannot relax the hypothesis all the way to the hypothesis \( u \in H^1_{\rho^{\alpha}}(B^2) \) befitting the case in which \( \alpha \in [-1/2, \infty) \).
Proposition 3.7 Let $\alpha \in (-1, -1/2)$.

1. For all $(m_0, n_0) \in \mathbb{N}_0 \times \mathbb{N}_0$, there exists $u \in H_{\rho^{\alpha}}^1(B^2)$ such that (3.16) fails for $(m, n) = (m_0 + 1, n_0)$.

2. For all $(m_0, n_0) \in \mathbb{N}_0 \times \mathbb{N}_0$, there exists $u \in H_{\rho^{\alpha}}^1(B^2)$ such that (3.17) fails for $(m, n) = (m_0, n_0 + 1)$.

Proof For all $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$, let

$$v_{m,n} := \begin{cases} (2^j)^{2\alpha - 1}(m_0 + 2^j + \alpha + 1)(n_0 + 2^j + 1) & \text{if there exists } j \in \mathbb{N}_0 \text{ such that } (m, n) = (m_0 + 2^j + 1, n_0 + 2^j), \\ 0 & \text{otherwise.} \end{cases}$$

Then, on account of (3.4) and the asymptotic formula (1.4), the sum

$$\sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} \left| v_{m,n} \right|^2 \| P_{m,n}^{(\alpha)} \|_{L_{\rho^{\alpha}}^2(B^2)}^2 = \sum_{j=0}^{\infty} (2^j)^{4\alpha-2}(m_0 + 2^j + \alpha + 1)^2(n_0 + 2^j + 1)^2 h_{m_0+2j+1,n_0+2j}$$

is finite or infinite together with $\sum_{j=0}^{\infty} (2^j)^{(4\alpha-2)+2+2-1-\alpha-\alpha} = \sum_{j=0}^{\infty} (2^j)^{2\alpha+1}$; this last expression being, indeed, finite (as $2\alpha + 1 < 0$), it transpires that

$$v := \sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} v_{m,n} P_{m,n}^{(\alpha)} \in L_{\rho^{\alpha}}^2(B^2).$$

The same argument goes on to show that, on defining for all $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$,

$$w_{m,n} := \begin{cases} (2^j)^{2\alpha - 1}(m_0 + 2^j + 1)(n_0 + 2^j + \alpha + 1) & \text{if there exists } j \in \mathbb{N}_0 \text{ such that } (m, n) = (m_0 + 2^j, n_0 + 2^j + 1), \\ 0 & \text{otherwise,} \end{cases}$$

one then has

$$w := \sum_{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0} w_{m,n} P_{m,n}^{(\alpha)} \in L_{\rho^{\alpha}}^2(B^2).$$

Using the differentiation identities in (3.5), it can be checked that the choice of the coefficients $v_{m,n}$ and $w_{m,n}$ yields $\partial_z v - \partial_z w = 0$ in the sense of distributions. From this and the definition (3.5) of the differential operators involved, it follows that the curl of $(v + w, i(v - w)) \in [L_{\rho^{\alpha}}^2(B^2)]^2 \subseteq [L^2(B^2)]^2$ is null in the sense of distributions. Therefore there exists $u \in H^1(B^2)$, unique up to an additive constant, such that $\nabla u = (v + w, i(v - w))$ (cf. [16, Theorem 2.9]). Now, $H^1(B^2) \subseteq H_{\rho^{\alpha+2}}^1(B^2) \subseteq L_{\rho^{\alpha}}^2(B^2)$ (see [23, Theorem 8.2] for the latter inclusion, which holds with continuous
embedding). In this way, we have constructed \( u \in H^1_{\rho \alpha}(B^2) \) such that \( \partial_z u = v \) and \( \partial_z u = w \).

On the other hand, from (3.22) in the proof of Lemma 3.6, we know that (3.16) holds in the case \((m, n) = (m_0 + 1, n_0)\) if and only if

\[
R_{m_0+1, n_0, \alpha, L} = \frac{(n_0 + 1)}{(n_0 + \alpha + 1)} \frac{(m_0 + 2)}{(m_0 + \alpha + 2)} \frac{1}{m_0 + n_0 + 2L + \alpha + 2} \lim_{L \to \infty} v_{m_0+1+L, n_0+L} \to 0.
\]

However, restricting our attention to the subsequence of indices \( L \) of the form \( 2^j, j \in \mathbb{N}_0 \), and using the asymptotic relation (1.4),

\[
R_{m_0+1, n_0, \alpha, 2^j} \to \frac{\Gamma(n_0 + \alpha + 1)}{\Gamma(n_0 + 1)} \frac{\Gamma(m_0 + \alpha + 2)}{\Gamma(m_0 + 2)} \frac{1}{2} \neq 0,
\]

so (3.16) cannot hold in this case and part 1 of this proposition is proved.

Symmetry arguments analogous to those made at the end of the proof of Lemma 3.6 show that \((x_1, x_2) \mapsto u(x_1, -x_2)\) is a function satisfying part 2. \( \square \)

**Remark 3.8** The formula analogous to (3.16) and (3.17) for symmetric Jacobi expansions, namely

\[
\hat{(u')}_{n}^{(\alpha)} = (2k + 2\alpha + 1) \sum_{n=k+1}^{\infty} \frac{(k + \alpha + 1)_{n-k}}{(2\alpha + 1)_{n-k}} \hat{u}_{n}^{(\alpha)}, \tag{3.23}
\]

where

\[
u = \sum_{n=0}^{\infty} \hat{u}_{n}^{(\alpha)} P_{n}^{(\alpha, \alpha)} \quad \text{and} \quad u' = \sum_{n=0}^{\infty} \hat{(u')}_{n}^{(\alpha)} P_{n}^{(\alpha, \alpha)},
\]

is valid for all \( u \in H^1_{\chi(\alpha, \alpha)}(-1, 1) \) if \( \alpha \geq -1/2 \) (cf. [18, eq. 2.13], where it is expressed in an equivalent way in terms of Gegenbauer polynomials). Using essentially the same arguments put forth in Lemma 3.6 and Proposition 3.7, it can be shown that if \( \alpha \in (-1, -1/2) \), then the relation (3.23) is valid under the stronger condition \( u \in H^2_{\chi(\alpha, \alpha)}(-1, 1) \) and that there are functions in \( H^1_{\chi(\alpha, \alpha)}(-1, 1) \setminus H^2_{\chi(\alpha, \alpha)}(-1, 1) \) for which the relation is false. One such example is the function defined by \( u(x) = \int_{0}^{x} v(t) \, dt \), where in turn

\[
u = \sum_{n=0}^{\infty} v_{n} P_{n}^{(\alpha, \alpha)} \quad \text{and} \quad v_{n} = \begin{cases} n^{\alpha+1} & \text{if } n \in \{2^j | j \in \mathbb{N}_0\}, \\ 0 & \text{otherwise.} \end{cases}
\]

### 3.2 Main Result

Having obtained the necessary preliminary results, we can prove our main result using roughly the same outer structure of the proof of the univariate case with \( \alpha = -1/2 \).
(Chebyshev) and $\alpha = 0$ (Legendre) on page 302 of [9]. The core of the argument lies below in Lemma 3.9 and the main result itself in Theorem 3.11. In order to express the former in a more compact form, we extend the notation $\text{Proj}_{N}^{(\alpha)}$ (cf. (3.10)) so that given any $k \in \mathbb{N}$ and $F \in [L_{\rho^{\alpha}}^{2}(B^{2})]^{k}$, $\text{Proj}_{N}^{(\alpha)}(F)$ signifies the componentwise application of $\text{Proj}_{N}^{(\alpha)}$ to $F$.

**Lemma 3.9** Let $\alpha > -1$ and $r, l \in \mathbb{N}$ with $l \geq r$. Then there exists $C = C(\alpha, l, r)$ such that for every $N \in \mathbb{N}$ and $u \in H_{\rho^{\alpha}}^{l}(B^{2})$,

$$\left\| \text{Proj}_{N}^{(\alpha)}(\nabla^{r}u) - \nabla^{r} \text{Proj}_{N}^{(\alpha)}(u) \right\|_{L_{\rho^{\alpha}}^{2}(B^{2})} \leq CN^{2r - 1/2 - l} \| u \|_{H_{\rho^{\alpha}}^{l}(B^{2})}.$$  

**Proof** Let $l \in \mathbb{N}$ and $u \in H_{\rho^{\alpha}}^{l}(B^{2})$. If we prove the existence of $C > 0$ independent of $u$ and $N$ such that

$$\left\| \text{Proj}_{N}^{(\alpha)}(\partial_{z^{*}}u) - \partial_{z^{*}} \text{Proj}_{N}^{(\alpha)}(u) \right\|_{L_{\rho^{\alpha}}^{2}(B^{2})} \leq CN^{3 - 2l} \| u \|_{H_{\rho^{\alpha}}^{l}(B^{2})}^{2}$$  

(3.24)

and the corresponding result involving the operator $\partial_{z}$, the $r = 1$ case of our desired result will follow. For the proof of (3.24), we can assume that $u$ is regular enough for the relation (3.16) between the orthogonal expansion of $u$ and that of its image under the operator $\partial_{z^{*}}$ to hold (otherwise, we can replace $u$ by the members of a sequence of $C^{\infty}(B^{2})$ functions that converges to $u$ in $H_{\rho^{\alpha}}^{l}(B^{2})$—which exists by virtue of [23, Remark 11.12.(iii)]—and once (3.24) is proved it will extend to $u$, by continuity); that is,

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{u}_{m,n}^{(\alpha)} P_{m,n}^{(\alpha)} \quad \text{and} \quad \partial_{z^{*}}u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{m,n} P_{m,n}^{(\alpha)},$$

both series converging in the $L_{\rho^{\alpha}}^{2}(B^{2})$ sense, with the $v_{m,n}$ and the $\hat{u}_{m,n}$ connected by (3.16). As $\text{Proj}_{N}^{(\alpha)}(u)$ is a polynomial, it is also regular enough to have the coefficients of its Fourier–Zernike series and the corresponding coefficients of its image under the operator $\partial_{z^{*}}$ connected by the formula (3.16). Further taking into account the fact that the expansion of $\text{Proj}_{N}^{(\alpha)}(u)$ is but a truncation of the expansion of $u$, we have

$$\text{Proj}_{N}^{(\alpha)}(u) = \sum_{m+n \leq N} \hat{u}_{m,n}^{(\alpha)} P_{m,n}^{(\alpha)} \quad \text{and} \quad \partial_{z^{*}} \text{Proj}_{N}^{(\alpha)}(u) = \sum_{m+n \leq N} v_{m,n}^{(\text{trunc})} P_{m,n}^{(\alpha)},$$

where (3.16) takes the particular form: for all $(m, n) \in \mathbb{N}_{0} \times \mathbb{N}_{0}$ with $m + n \leq N$,
\[ v_{m,n}^{(\text{trunc})} = (m + n + \alpha + 1) \sum_{l=0}^{\frac{N-m-n-1}{2}} \frac{(m + 1)_l}{(m + \alpha + 1)_l} \frac{(n + 1)_{l+1}}{(n + \alpha + 1)_{l+1}} u_{m+l,n+1+l}^{(\alpha)}. \]

In particular, \( v_{m,n}^{(\text{trunc})} = 0 \) if \( m + n = N \). Therefore, whenever \( 0 \leq m + n \leq N \) and adopting the notation \( \delta_{m,n}^{(N)} = \left\lfloor \frac{N-m-n+1}{2} \right\rfloor \),

\[
\frac{v_{m,n} - v_{m,n}^{(\text{trunc})}}{m + n + \alpha + 1} = \sum_{l=\delta_{m,n}^{(N)}}^{\infty} \frac{(m + 1)_l}{(m + \alpha + 1)_l} \frac{(n + 1)_{l+1}}{(n + \alpha + 1)_{l+1}} u_{m+l,n+1+l}^{(\alpha)}
\]

\[
= \sum_{l=0}^{\infty} \frac{(m + 1)_{l+\delta_{m,n}^{(N)}}}{(m + \alpha + 1)_{l+\delta_{m,n}^{(N)}}} \frac{(n + 1)_{l+\delta_{m,n}^{(N)}+1}}{(n + \alpha + 1)_{l+\delta_{m,n}^{(N)}+1}} u_{m+l+\delta_{m,n}^{(N)},n+1+l+\delta_{m,n}^{(N)}}^{(\alpha)}
\]

\[
= \frac{(m + 1)_{\delta_{m,n}^{(N)}}}{(m + \alpha + 1)_{\delta_{m,n}^{(N)}}} \frac{(n + 1)_{\delta_{m,n}^{(N)}+1}}{(n + \alpha + 1)_{\delta_{m,n}^{(N)}+1}} \frac{v_{m+\delta_{m,n}^{(N)},n+\delta_{m,n}^{(N)}}^{(\alpha)}}{m + n + \alpha + 2\delta_{m,n}^{(N)} + 1}, \tag{3.25}
\]

where the last equality is obtained by expanding the Pochhammer symbols of the form \((X)_{\delta_{m,n}^{(N)}+Y}\) according to the rules given in Sect. 1.3 and noting that then (3.16) can be used to make the coefficient \( v_{m+\delta_{m,n}^{(N)},n+\delta_{m,n}^{(N)}}^{(\alpha)} \) appear. Now, from (3.4),

\[
h_{m,n}^{(\alpha)} = \frac{m + n + \alpha + 2\delta_{m,n}^{(N)} + 1}{m + n + \alpha + 1} \frac{(m + 1)_{\delta_{m,n}^{(N)}}}{(m + \alpha + 1)_{\delta_{m,n}^{(N)}}} \frac{(n + 1)_{\delta_{m,n}^{(N)}+1}}{(n + \alpha + 1)_{\delta_{m,n}^{(N)}+1}} \frac{v_{m+\delta_{m,n}^{(N)},n+\delta_{m,n}^{(N)}}^{(\alpha)}}{m + \delta_{m,n}^{(N)},n+\delta_{m,n}^{(N)}}. \tag{3.26}
\]

Using (3.25) and the fact that if \( m + n = N \) then \( \delta_{m,n}^{(N)} = 0 \),

\[
\text{Proj}_{N}^{(\alpha)} (\partial_{z^+} u) - \partial_{z^+} \text{Proj}_{N}^{(\alpha)} (u) = \sum_{m+n \leq N-1} \left( v_{m,n} - v_{m,n}^{(\text{trunc})} \right) p_{m,n}^{(\alpha)}
\]

\[
+ \sum_{m+n = N} v_{m,n} p_{m,n}^{(\alpha)}
\]

\[
= \sum_{k=0}^{N} \sum_{m+n = k} (k + \alpha + 1) \frac{(m + 1)_{\delta_{m,n}^{(N)}}}{(m + \alpha + 1)_{\delta_{m,n}^{(N)}}} \frac{(n + 1)_{\delta_{m,n}^{(N)}+1}}{(n + \alpha + 1)_{\delta_{m,n}^{(N)}+1}} \frac{v_{m+\delta_{m,n}^{(N)},n+\delta_{m,n}^{(N)}}^{(\alpha)}}{k + \alpha + 2\delta_{m,n}^{(N)} + 1} p_{m,n}^{(\alpha)}. \tag{3.27}
\]

As the terms resulting in (3.27) are \( L^2_{\rho^\alpha}(B^2) \)-orthogonal to each other, taking the corresponding squared norm of both its ends, and using (3.26) results in

\[
\left\| \text{Proj}_{N}^{(\alpha)} (\partial_{z^+} u) - \partial_{z^+} \text{Proj}_{N}^{(\alpha)} (u) \right\|_{L^2_{\rho^\alpha}(B^2)}^2
\]
\[
= \sum_{k=0}^{N} \sum_{m+n=k} \frac{k + \alpha + 1}{k + \alpha + 2\delta_{m,n} + 1} \frac{(m + 1)_{\delta_{m,n}^{(N)}}}{(m + \alpha + 1)_{\delta_{m,n}^{(N)}}} \times \frac{(n + 1)_{\delta_{m,n}^{(N)}}}{(n + \alpha + 1)_{\delta_{m,n}^{(N)}}} \left| v_{m + \delta_{m,n}^{(N)}, n + \delta_{m,n}^{(N)}} \right|^2 h_{m, n + \delta_{m,n}^{(N)}, n + \delta_{m,n}^{(N)}}(\alpha).
\]

We want to rearrange the above sum so that those \((m', n') \in \mathbb{N}_0 \times \mathbb{N}_0\) such that \(|v_{m', n'}|^2 h_{m', n'}(\alpha)\) appears in the above sum and their accompanying coefficients become readily apparent. Let \(E_N^{(\alpha)}\) and \(O_N^{(\alpha)}\) denote the above sum restricted to the terms with \(N - k\) even and odd, respectively. In the inner sum of both resulting expressions, \(n\) can be replaced with \(k - m\) by letting \(m\) range in \(\{0, \ldots, k\}\). Applying the change of variable \((i, j) = (m + \frac{N - k}{2}, n + \frac{N - k}{2})\) in the sum defining \(E_N^{(\alpha)}\), we are left with

\[
E_N^{(\alpha)} = \sum_{i=0}^{N} \left[ \sum_{j=0}^{\min(i, N-i)} \epsilon_{N, i, j}^{(\alpha)} \right] \left| v_{i, N-i} \right|^2 h_{i, N-i}^{(\alpha)}, \tag{3.28a}
\]

where

\[
\epsilon_{N, i, j}^{(\alpha)} := \frac{N - 2j + \alpha + 1}{N + \alpha + 1} \frac{(i - j + 1)_j}{(i - j + \alpha + 1)_j} \frac{(N - i - j + 1)_j}{(N - i - j + \alpha + 1)_j}.
\]

The sum inside the square brackets in (3.28a) is invariant under the transformation \(i \mapsto N - i\). Thus, we can learn the values of all the instances of this sum by looking at the cases where \(i \leq N - i\) only. For such \(i\), it is straightforward to check that as long as \(j \in \{0, \ldots, i\}\),

\[
\epsilon_{N, i, j}^{(\alpha)} = \begin{cases} 
\Delta_j \left[ \frac{(i - j + \alpha + 1)(N - i - j + \alpha + 1)(i - j + 1)_j(N - i - j + 1)_j}{(\alpha + 1)_j(N + \alpha + 1)(i - j + \alpha + 1)_j(N - i - j + \alpha + 1)_j} \right] & \text{if } \alpha \neq 0, \\
\frac{N - 2j + \alpha + 1}{N + \alpha + 1} & \text{if } \alpha = 0.
\end{cases}
\]

Hence, the sum with respect to \(j\) telescopes if \(\alpha \neq 0\) and is well known if \(\alpha = 0\), giving (using the abovementioned invariance under the transformation \(i \mapsto N - i\) )

\[
E_N^{(\alpha)} = \sum_{i=0}^{N} \frac{(i + \alpha + 1)(N - i + \alpha + 1)}{(\alpha + 1)(N + \alpha + 1)} \left| v_{i, N-i} \right|^2 h_{i, N-i}^{(\alpha)}.
\]

Applying the change of variable \((i, j) = (m + \frac{N - k}{2}, n + \frac{N - k}{2})\) in the sum defining \(O_N^{(\alpha)}\), we obtain

\[
O_N^{(\alpha)} = \sum_{i=1}^{N} \left[ \sum_{j=1}^{\min(i, N+1-i)} \epsilon_{N, i, j}^{(\alpha)} \right] \left| v_{i, N+1-i} \right|^2 h_{i, N+1-i}^{(\alpha)}, \tag{3.29a}
\]
where

$$\theta_{N,i,j}^{(\alpha)} := \frac{N - 2j + \alpha + 2}{N + \alpha + 2} \frac{(i - j + 1)j}{(i - j + \alpha + 1)j} \frac{(N - i - j + 2)j}{(N - i - j + \alpha + 2)j}. \quad (3.29b)$$

The sum inside the square brackets in (3.29a) is invariant under the transformation $i \mapsto N + 1 - i$. Also, comparing (3.29b) with (3.29a), we find that $\theta_{N,i,j}^{(\alpha)} = \theta_{N+1,i,j}^{(\alpha)}$. Hence, we can adapt our previous argument and state

$$\theta_{N}^{(\alpha)} := \sum_{i=1}^{N} \frac{i(N+1-i)}{(\alpha+1)(N+\alpha+2)} |v_{i,N+1-i}|^2 h_{N+1-i}^{(\alpha)}. \quad (3.29a)$$

Summing the resulting expressions for $\theta_{N}^{(\alpha)}$ and $\theta_{N}^{(\beta)}$ and using the fact that $i \mapsto (i+\alpha+1)(N-i+\alpha+1)$ and $i \mapsto i(N+1-i)$, seen as functions of a real variable, attain their maxima at $N/2$ and $(N+1)/2$, respectively, we obtain

$$\| \text{Proj}_{N}^{(\alpha)} (\partial_z^* u) - \partial_z^* \text{Proj}_{N}^{(\alpha)} (u) \|_{L^2_{\rho}^{\alpha}(B^2)} \leq \frac{(N/2 + \alpha + 1)^2}{(N+\alpha+1)} \sum_{m+n=N} |v_{m,n}|^2 h_{m,n}^{(\alpha)} + \frac{((N+1)/2)^2}{(N+\alpha+2)} \sum_{m+n=N+1} |v_{m,n}|^2 h_{m,n}^{(\alpha)} \leq C_{\alpha}(N+1) \left( \| \partial_z^* u - \text{Proj}_{N-1}^{(\alpha)} (\partial_z^* u) \|_{L^2_{\rho}^{\alpha}(B^2)}^2 + \| \partial_z^* u - \text{Proj}_{N}^{(\alpha)} (\partial_z^* u) \|_{L^2_{\rho}^{\alpha}(B^2)}^2 \right) \leq C_{\alpha} C(N+1)N^{2(1-l)} \| \partial_z^* u \|_{\mathcal{H}_{\rho}^{\alpha}(B^2)}^2 + C_{\alpha} C(N+1)(N+1)^{2(1-l)} \| \partial_z^* u \|_{\mathcal{H}_{\rho}^{\alpha}(B^2)}^2,$$

where $C_{\alpha} = \sup_{N \in \mathbb{N}_0} \max\left( \frac{(N/2+\alpha+1)^2}{(N+\alpha+1)(N+\alpha+2)(N+\alpha+1)}, \frac{(N+1)/2)^2}{(N+\alpha+2)(N+\alpha+1)(N+\alpha+2)} \right) > 0$ and the last inequality comes from Corollary 2.4. Using standard inequalities, (3.24) is attained.

By using the reflection introduced at the end of the proof of Lemma 3.6, we can turn (3.24) into its analogue for the $\partial_z$ differential operator and thus conclude the proof of the $r = 1$ case of this lemma.

Starting from the inverse or Markov-type inequality in Proposition 3.3, it is readily proved by induction that there exists $C = C(\alpha,r) > 0$ such that for all $N \in \mathbb{N}_0$ and $p \in \Pi_N^2$,

$$|p|_{\mathcal{H}_{\rho}^{\alpha}(B^2)} \leq CN^{2r} \| p \|_{L^2_{\rho}^{\alpha}(B^2)}. \quad (3.30)$$

We are now in a position to obtain the general case of this lemma by induction on $r$, the initialization $r = 1$ having already been proved. Thus, let us assume that the desired result holds up to some $r \in \mathbb{N}$ and let $l \geq r + 1$ and $u \in H_{\rho}^{l}(B^2)$. Then,

$$\| \text{Proj}_{N}^{(\alpha)} (\nabla^r \partial_z^* u) - \nabla^r \partial_z^* \text{Proj}_{N}^{(\alpha)} (u) \|_{L^2_{\rho}^{\alpha}(B^2)} \leq C N^{2l} \| u \|_{L^2_{\rho}^{\alpha}(B^2)}.$$
where we have used the induction hypothesis and (3.30). Using (3.24) to bound the last term above, we obtain
\[
\|\text{Proj}_N^{(\alpha)} (\nabla^r \partial_z u) - \nabla^r \partial_z \text{Proj}_N^{(\alpha)} (u)\|_{L^2_{\rho_\alpha}(B^2)^{r+1}} \leq C N^{-1/2+2r-(l-1)} \|\partial_z u\|_{H^{l-1}_{\rho_\alpha}(B^2)} + C N^{2r} \|\text{Proj}_N^{(\alpha)} (\partial_z u) - \partial_z \text{Proj}_N^{(\alpha)} (u)\|_{L^2_{\rho_\alpha}(B^2)},
\]
Combining this with its analogue involving the \( \partial_z \) operator, we obtain the desired bound for the commutator of the projection and the \( \nabla^{r+1} \) operators.

\[\square\]

Remark 3.10 The proof of Lemma 3.9 can be significantly simplified in the \( \alpha \geq 0 \) case because then \( \mathcal{E}^{(\alpha)}_{N,i,j} \) of (3.28b) and \( \mathcal{E}^{(\alpha)}_{N,i,j} \) of (3.29b) can each be bounded by 1.

Theorem 3.11 Let \( \alpha > -1 \) and let \( r \) and \( l \) be integers with \( 1 \leq r \leq l \). Then there exists \( C = C(\alpha, l, r) > 0 \) such that for every \( N \in \mathbb{N} \) and \( u \in H^{l}_{\rho_\alpha}(B^2) \),
\[
\|u - \text{Proj}_N^{(\alpha)} (u)\|_{H^{r}_{\rho_\alpha}(B^2)} \leq C N^{-1/2+2r-l} \|u\|_{H^{l}_{\rho_\alpha}(B^2)}.
\]

Proof For every \( k \in \{1, \ldots, r\} \),
\[
\|\nabla^k (u - \text{Proj}_N^{(\alpha)} (u))\|^2_{L^2_{\rho_\alpha}(B^2)^{k+1}} \leq 2 \|\nabla^k u - \text{Proj}_N^{(\alpha)} (\nabla^k u)\|^2_{L^2_{\rho_\alpha}(B^2)^{k+1}} + 2 \|\text{Proj}_N^{(\alpha)} (\nabla^k u) - \nabla^k \text{Proj}_N^{(\alpha)} (u)\|^2_{L^2_{\rho_\alpha}(B^2)^{k+1}}.
\]

We can bound the first term using Corollary 2.4 and the second term using Lemma 3.9. As the squared \( H^{l}_{\rho_\alpha}(B^2) \) norm of \( u - \text{Proj}_N^{(\alpha)} (u) \) is the sum of the squared \( L^2_{\rho_\alpha}(B^2) \) norm of \( u - \text{Proj}_N^{(\alpha)} (u) \) (which again, can be bounded using Corollary 2.4) and the left-hand side above for \( k \in \{1, \ldots, r\} \), we obtain the desired bound upon realizing that the highest power on \( N \) that will appear is \(-1 - 2l + 4r\) and taking square roots.

\[\square\]

Given \( l \in \mathbb{N}_0 \) and \( \theta \in (0, 1) \), we use complex interpolation (see [1, Paragraph 7.51–52] for a succinct discussion that suffices for our purposes save for a strong enough statement of the reiteration theorem, which can be found in [8, Paragraph 12.3]) to define
\[
H^{l+\theta}_{\rho_\alpha}(B^2) := \left[ H^l_{\rho_\alpha}(B^2), H^{l+1}_{\rho_\alpha}(B^2) \right]_\theta.
\]
Then, by using the exact interpolation theorem, Corollary 2.4 and Theorem 3.11 are readily generalized to the above intermediate spaces:

**Corollary 3.12** Let \( \alpha > -1 \) and \( r, l \geq 0 \) with \( l \geq r \). Then there exists \( C = C(\alpha, l, r) > 0 \) such that for every \( N \in \mathbb{N} \) and \( u \in H^{l}_{\rho^m}(B^2) \),

\[
\|u - \text{Proj}_{N}^{(\alpha)}(u)\|_{H^{r}_{\rho^{m}}(B^2)} \leq CN^{e(l, r)} \|u\|_{H^{l}_{\rho^m}(B^2)},
\]

where

\[
e(l, r) := \begin{cases} 
3/2 - l & \text{if } 0 \leq r \leq 1, \\
-1/2 + 2 - r & \text{if } r \geq 1.
\end{cases}
\]

**Proof** For \( N \in \mathbb{N} \), let \( T_{N,l,r}^{(\alpha)} \) denote the operator \( I - \text{Proj}_{N}^{(\alpha)} : H^{l}_{\rho^m}(B^2) \to H^{r}_{\rho^m}(B^2) \). Let us suppose first that neither \( l \) nor \( r \) is an integer. Then, if \( \lfloor l \rfloor \geq \lfloor r \rfloor + 1 \), for \( j \in \{0, 1\} \), using the known bounds on the operator norms \( \|T_{N,l,\lfloor r \rfloor+j}^{(\alpha)}\| \) and \( \|T_{N,\lfloor l \rfloor+1,\lfloor r \rfloor+j}^{(\alpha)}\| \) and the exact interpolation theorem with interpolation parameter \( l - \lfloor l \rfloor \) results in the desired bound on \( \|T_{N,l,\lfloor r \rfloor}^{(\alpha)}\| \) and \( \|T_{N,\lfloor l \rfloor+1,\lfloor r \rfloor}^{(\alpha)}\| \); combining these estimates with the exact interpolation theorem with interpolation parameter \( r - \lfloor r \rfloor \) gives the desired bound on \( \|T_{N,l,\lfloor r \rfloor}^{(\alpha)}\| \). If \( \lfloor l \rfloor = \lfloor r \rfloor \), the bound on \( \|T_{N,l,\lfloor r \rfloor}^{(\alpha)}\| \) is obtained exactly as in the previous case but now it is combined via the exact interpolation theorem with parameter \( \theta = \frac{r-\lfloor r \rfloor}{l-\lfloor l \rfloor} \) with the desired bound on \( \|T_{N,l,l}^{(\alpha)}\| \), which, in turn, comes about by combining the known bounds on \( \|T_{N,\lfloor l \rfloor,\lfloor l \rfloor}^{(\alpha)}\| \) and \( \|T_{N,\lfloor l \rfloor+1,\lfloor l \rfloor+1}^{(\alpha)}\| \) with the exact interpolation theorem with parameter \( l - \lfloor l \rfloor \); the reiteration theorem is used to ensure that

\[
\left[ H^{\lfloor l \rfloor}_{\rho^m}(B^2), H^{\lfloor l \rfloor+1}_{\rho^m}(B^2) \right]_{\theta} \\
= \left[ H^{\lfloor l \rfloor}_{\rho^m}(B^2), H^{\lfloor l \rfloor+1}_{\rho^m}(B^2) \right]_{0} \cdot \left[ H^{\lfloor l \rfloor}_{\rho^m}(B^2), H^{\lfloor l \rfloor+1}_{\rho^m}(B^2) \right]_{l-\lfloor l \rfloor} \theta \\
= H^{\lfloor l \rfloor+1}_{\rho^m}(B^2) = H^r_{\rho^m}(B^2).
\]

The cases where either \( l \) or \( r \) is an integer are similar but simpler so we omit further details. \( \square \)

**Remark 3.13** 1. Essentially the same argument put forward in Corollary 3.12 allows for generalizing Corollary 2.4 and Theorem 3.11 using real instead of complex interpolation.

2. The proof of Corollary 3.12 works with the interpolated space defined as in (3.31) and does not depend on any further characterization of those spaces (cf. [10, Lemma 2.1], where a weighted identity of the form \( H^{\theta m} = [L^2, H^m]_\theta \), for \( (m, \theta) \in \mathbb{N} \times (0, 1) \), is tacitly used).
3.3 On the Sharpness of the Main Result

We can show the optimality with respect to the power on $N$ of Theorem 3.11 in the $r = 1$ case and that of its $r = 0$ analogue, namely Corollary 2.4, in the two-dimensional case. Note that in both cases, all the weighted Sobolev spaces involved have integer regularity parameters.

**Theorem 3.14** The powers on $N$ in the $d = 2$ case of Corollary 2.4 and the $r = 1$ case of Theorem 3.11 are sharp.

**Proof** Let $\alpha > -1$. By iterating the relations in (3.5), it is readily proved by induction that for every $m, n, l_1, l_2 \in \mathbb{N}_0$,

$$
\partial_z^{l_2} \partial_{\bar{z}}^{l_1} p_{m, n}^{(\alpha)} = \frac{(m - l_1 + 1)_{l_1} (n - l_2 + 1)_{l_2} (n + \alpha + 1)_{l_1} (m + \alpha + 1)_{l_2}}{(\alpha + 1)_{l_1 + l_2}} p_{m - l_1, n - l_2}^{(\alpha + l_1 + l_2)}.
$$

(3.32)

Given $l, j \in \mathbb{N}$ with $j \geq l$ we define the polynomial

$$
t_j^{(\alpha, l)} := \sum_{k=0}^{l} (-l)_k \Gamma(\alpha + j + l - k + 1)^2 (\alpha + 2j + 2l - 2k + 1) \frac{\Gamma(k + 1) \Gamma(j + l - k + 1)^2 (\alpha + 2j + 2l - 2k + 1)}{\Gamma(j + l + 1) \Gamma(j + 2l + 1)} p_{j+1}^{(\alpha)}.
$$

(3.33)

Let $l_1, l_2 \in \mathbb{N}_0$ be such that $l_1 + l_2 = l$. Then applying the $\partial_z^{l_2} \partial_{\bar{z}}^{l_1}$ operator to $t_j^{(\alpha, l)}$ and using (3.32) results in a linear combination of Zernike polynomials of parameter $\alpha + l$. By comparing the resulting expression term by term with the result of applying Proposition 3.1 with $(\alpha, \gamma, m, n)$ replaced by $(\alpha, \alpha + l, j + l_2, j + l_1)$ (which, because $(-l)_k = 0$ if $k \geq l + 1$, is a sum with the same number of effective terms), we find that

$$
\partial_z^{l_2} \partial_{\bar{z}}^{l_1} t_j^{(\alpha, l)} = \frac{\Gamma(\alpha + j + l_1 + 1) \Gamma(\alpha + j + l_2 + 1)}{\Gamma(j + l + 1) \Gamma(j + 2l + 1)} p_{j+1}^{(\alpha)}.
$$

whence, using (1.4), (3.4), and (3.6),

$$
\left| t_j^{(\alpha, l)} \right|^2_{H^l_{\rho^\alpha}(B^2)} \approx \sum_{q=0}^{l} \left\| \partial_z^q \partial_{\bar{z}}^{l-q} t_j^{(\alpha, l)} \right\|^2_{L^2_{\rho^\alpha}(B^2)} = \frac{\pi \Gamma(\alpha + 1)^2}{2j + l + \alpha + 1} \sum_{q=0}^{l} \Gamma(\alpha + j + l - q + 1) \Gamma(\alpha + j + q + 1) \Gamma(j + l - q + 1) \Gamma(j + q + 1).
$$

Let us define for integer $j \geq l$ the indices $N_j^{(l)}$ and the residuals $R_j^{(\alpha, l)}$ by

$$
N_j^{(l)} := 2j + 2l - 1 \quad \text{and} \quad R_j^{(\alpha, l)} := t_j^{(\alpha, l)} - \text{Proj}_{N_j^{(l)}}^{(\alpha)} (t_j^{(\alpha, l)}).
$$
As $R_{j}^{(\alpha,l)}$ is exactly the $k = 0$ term of the sum in (3.33), using (3.4),

$$
\left\| R_{j}^{(\alpha,l)} \right\|^2_{L_{\rho^2}^{2}(B^2)} = \frac{\pi \Gamma(\alpha + 1)^2}{2j + 2l + \alpha + 1} \frac{\Gamma(\alpha + j + l + 1)^2}{\Gamma(j + l + 1)^2} \frac{\Gamma(\alpha + 2j + l + 1)^2}{\Gamma(\alpha + 2j + 2l + 1)^2},
$$

and, by Proposition 3.2,

$$
\left\| R_{j}^{(\alpha,l)} \right\|^2_{H_{\rho}^{1}(B^2)} = \frac{4\pi \Gamma(\alpha + 1)^2 \Gamma(\alpha + j + l + 1)^2(\alpha + 2j + 2l + 1)^2}{(\alpha + 1)\Gamma(j + l + 1)^2(\alpha + 2j + l + 1)^2_{l+1}}(j + l)(\alpha + j + l + 1).
$$

Thus, using the asymptotic formula (1.4), for $r \in \{0, 1\}$,

$$
\left| \frac{R_{j}^{(\alpha,l)}}{H_{\rho}^{1}(B^2)} \right| \frac{t_{j}^{(\alpha,l)}}{H_{\rho}^{1}(B^2)} \sim C_{\alpha,l,r} j^{3/2r-1} \sim \tilde{C}_{\alpha,l,r} (N_{j}^{(l)})^{3/2r-l} \quad (3.34)
$$

as $j \to \infty$. By the norm equivalence of Proposition 2.6 and the fact that the Fourier–Zernike series of $t_{j}^{(\alpha,l)}$ only have nonzero terms of degrees between $2j$ and $2j + 2l$, we can replace the seminorms above by the corresponding norms. Thus, the choice $(u, N) = (t_{j}^{(\alpha,l)}, N_{j}^{(l)})$ turns the inequalities of Corollary 2.4 and that of the $r = 1$ case of Theorem 3.11 into asymptotic equalities; hence, the power on $N$ in each case is sharp. □

**Remark 3.15** We conjecture that Theorem 3.11 is optimal with respect to the power on $N$ in the $r \geq 2$ case as well and that, for every $\alpha > -1$ and $l \in \mathbb{N}$, the same sequence $(t_{j}^{(\alpha,l)}, l \geq j)$ defined in (3.33) attains the proved upper-bound rate asymptotically in the same way it does in the proof of Theorem 3.14; i.e., the right analogue of (3.34) holds. This conjecture is supported by numerical tests, details of which are provided at the end of Section 3.3 in the preprint version of this work [15].

### 3.4 Closing Remarks

We have exploited a number of old and new identities concerning Zernike polynomials and their associated orthogonal expansions in order to prove our desired bound (1.2) in Theorem 3.11, which is relevant to approximation of functions defined in the unit disk $B^2$. In doing this, we have reproduced a result already well known in the case of the unit interval $B^1 = (-1, 1)$.

The question of whether the arguments put forward in this work can be extended to the case of the $d$-dimensional unit ball $B^d$, $d \geq 3$, arises naturally. However, our attempts to answer this question were stymied by our lack of knowledge of a family of $L_{\rho^{d}}^{2}(B^d)$-orthogonal polynomials and a collection of first-order differential operators that obey identities similarly sparse to those involving the Zernike polynomials and the differential operators $\partial_{z^*}$ and $\partial_{z}$, such as (3.5), (3.16), and (3.17).
Without corresponding identities of similar simplicity, the computations involved in the differentiation-truncation commutator bound Lemma 3.9 turn out to be very complex indeed. After a preliminary computer-aided search, we suspect that corresponding simple enough identities in the $d = 3$ case in particular might very well not exist if we insist on working in the field of the complex (let alone real) numbers. We are, however, intrigued by the possibility that the quaternionic families of orthogonal polynomials on $B^3$ and differential operators described in [7] might be adapted to this end.

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