Crossover from isotropic to directed percolation

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Australian Government
Australian Research Council
Outline

1. Percolation

2. Directed Percolation

3. Biased Directed Percolation
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Figure 1: A typical percolation configuration, with four clusters.
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- Occupation probability $p$
- Cluster consists of nearest-neighboring occupied sites
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For infinite system ($L \to \infty$), there is a critical $p$ value ($p_c$),

- when $p \leq p_c$, no infinite cluster exists
- when $p > p_c$, an infinite cluster exists with non-zero probability.

Define the order parameter $P_\infty$, which is the probability for the existence of an infinite cluster and

$$P_\infty = \begin{cases} 
0 & \text{for } p \leq p_c \\
(p - p_c)^{\beta_P} & \text{for } p \to p_c^+.
\end{cases}$$  \hspace{1cm} (1)

$\beta_P$ is a critical exponent, and $\beta_P = 5/36$ for 2D.
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Around $p_c$, the correlation length $\xi$ diverges as

$$\xi \sim |p - p_c|^{-\nu},$$

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- $\xi$ can be intuitively viewed as the averaged cluster radius
- $\nu$ is another critical exponent and $\nu = 4/3$ for 2D

Critical exponents $\beta_p$ and $\nu$ label percolation universality class. Above two dimensions, no exact results for the thresholds and the critical exponents.
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Reformulate percolation as a stochastic process.

Use an algorithm to generate the cluster from the origin.

Let an active seed affect its nearest neighbors with probability $p$.

Distinguish different shells (time) by colors.

Figure 2: Percolation illustrated as a stochastic process.
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When growing a cluster from an active seed using breadth first scheme, the shortest-path between an occupied site and the seed is $\ell$, one has

$$\langle \ell \rangle \propto r^{d_{\text{min}}}.$$  \hfill (3)

At $p_c$, one expects:

- $P(r) \sim r^{-\beta/\nu}$.
- $P(\ell) \sim \ell^{-\beta/(\nu d_{\text{min}})}$.

Around $p_c$ ($\epsilon = p - p_c$), according to scaling theory, one has:

- $P(r, p) = r^{-\beta/\nu} f(\epsilon L^{1/\nu})$.
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Directed Percolation

Figure 3: two-dimensional directed percolation

- Rotate the square lattice
- Infection probability $p$
- Only along the time axis
- Use the same order parameter $P_\infty$
- With an exponent $\beta_{DP}$.
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Compared with Percolation,

- DP is a non-equilibrium statistical mechanics model
- have anisotropic correlation

\[ \xi_\parallel \sim |p - p_c|^{-\nu_\parallel}, \xi_\perp \sim |p - p_c|^{-\nu_\perp}. \]  \hspace{1cm} (4)

- Independent exponents \((\beta_{\text{DP}}, \nu_\parallel, \nu_\perp)\) label the DP universality class.
- DP class is a fundamental class in non-equilibrium statistical mechanics.
- No exact results even for \((1+1)\) dimensions.
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Biased Directed Percolation

Compared with DP,

- **no direction limit**
- but with anisotropic infection probabilities
  - Along time axis: \( p_{\downarrow} = pp_d \)
  - Against time axis: \( p_{\uparrow} = p(1 - p_d) \)

- When \( p_d = 1/2 \), BDP = Percolation
- When \( p_d = 0, 1 \), BDP = DP

Figure 4: Rules for BDP
Biased Directed Percolation

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Consider a new region $1/2 < p_d < 1$,

- we still use order parameter $P_\infty$, what is $\beta$ characterizing $P_\infty$?
- Correlation length

$$\xi_\parallel \sim |p - p_c|^{-\nu_\parallel}, \quad \xi_\perp \sim |p - p_c|^{-\nu_\perp}.$$  \hspace{1cm} (5)

what are the values of $\nu_\parallel$ and $\nu_\perp$?
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Biased Directed Percolation

With some fixed $1/2 < p_d < 1$ values, we sample quantities

- Percolation probability $P(t, p)$,
- $N(t)$, number of sites becoming active at time $t$,
- A revised radius of gyration $R(t) = \langle \sqrt{\sum_i r_i^2/N(t)} \rangle$.
- A ratio $Q = N(2t)/N(t)$.

Scaling behaviors

\[
P(t, \epsilon) = t^{-\delta} f_P(t^{1/\nu_{\parallel}}), \tag{6}
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\[
N(t, \epsilon) = t^{\eta} f_N(\epsilon t^{1/\nu_{\parallel}}), \tag{7}
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R(t, \epsilon) = t^{-\delta + 1/z} f_N(\epsilon t^{1/\nu_{\parallel}}), \tag{8}
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\epsilon = p - p_c.
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Scaling relation

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\delta = \beta/\nu_{\parallel}, \eta = (d\nu_{\perp} - 2\beta)/\nu_{\parallel}, z = \nu_{\parallel}/\nu_{\perp}. \tag{11}
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- A revised radius of gyration $R(t) = \langle \sqrt{\sum_i r_i^2 / N(t)} \rangle$.
- A ratio $Q = N(2t)/N(t)$.

Scaling behaviors

$$P(t, \epsilon) = t^{-\delta} f_P(t^{1/\nu_{\parallel}}),$$
$$N(t, \epsilon) = t^{\eta} f_N(\epsilon t^{1/\nu_{\parallel}}),$$
$$R(t, \epsilon) = t^{-\delta+1/z} f_N(\epsilon t^{1/\nu_{\parallel}}),$$
$$Q = 2^\eta f_Q(t^{1/\nu_{\parallel}}).$$

(6) \hspace{2cm} (7) \hspace{2cm} (8) \hspace{2cm} (9) \hspace{2cm} (10)

$\epsilon = p - p_c$.

Scaling relation

$$\delta = \beta/\nu_{\parallel}, \eta = (d\nu_{\perp} - 2\beta)/\nu_{\parallel}, z = \nu_{\parallel}/\nu_{\perp}.$$  

(11)
$N(t) \sim t^\eta$

Log-log plot $N(t)$ versus $t$. The slope corresponds to $\eta$.

Figure 5: Exponent $\eta$ with various $p_d$ values.
Biased Directed Percolation

Phase Diagram

Figure 6: Phase diagram of BDP on two (left) and three (right) dimensions.

Arrows represent the flow direction of Renormalization Group
Consider the critical region around percolation point \( (p_d = 1/2, p = 1) \).

- For \( p_d = 1/2, p \to 1 \), the percolation probability \( P(t) \) scales as

\[
P(t, \epsilon) \sim t^{-\delta} f_P(\epsilon t^{1/(\nu d_{\text{min}})}) \quad , \epsilon = p - p_c .
\]  

The exponent \( \nu = 4/3 \) for 2D.

- For \( p = 1, p_d \to 1/2 \),

\[
P(t, \epsilon_d) \sim t^{-\delta} f'_P(\epsilon_d t^{1/(\nu' d_{\text{min}})}) \quad , \epsilon_d = p_d - 1/2 .
\]  

We estimate \( 1/(\nu' d_{\text{min}}) = 0.500(5) \). With the known estimate \( d_{\text{min}} = 1.13077(2) \), we estimate \( \nu' = 1.77(1) \).
Consider the critical region around percolation point \((p_d = 1/2, p = 1)\).

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  \[
P(t, \epsilon) \sim t^{-\delta} f_P(\epsilon t^{1/(\nu d_{\text{min}})}) , \epsilon = p - p_c .
  \] (12)

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Biased Directed Percolation

Crossover exponent $\phi$ is defined as

$$(1 - p_c) \propto (p_{d,c} - 1/2)^{1/\phi}$$

(14)

- Scaling theory gives $\phi = \nu/\nu' = 0.754(6)$. 

![Graph showing the crossover from IP to DP with a slope of 0.754](image)
Biased Directed Percolation

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$$(1 - p_c) \propto (p_{d,c} - 1/2)^{1/\phi}$$

(14)

- Scaling theory gives $\phi = \nu / \nu' = 0.754(6)$. 

![Graph showing the crossover exponent for BDP, d=2 with a slope of 0.754.](image)
Crossover exponent $\phi$ is defined as

$$(1 - p_c) \propto (p_{d,c} - 1/2)^{1/\phi}$$

(14)

- Scaling theory gives $\phi = \nu / \nu' = 0.754(6)$. 

![Figure 7: Crossover exponent](image)
Biased Directed Percolation

- Use a simple BDP model to generalize Percolation and DP models
- Study the crossover effect from Percolation to DP
- Is $\nu'$ new or related to $\beta$, $\nu$, $d_{\text{min}}$?
- Is $1/((\nu'd_{\text{min}})$ exactly equal to $1/2$?
- Can $\nu'$ be derived by Stochastic Loewner Evolution (SLE), conformal field theory or Coulomb gas theory?

Zongzheng Zhou, Ji Yang, Robert M. Ziff and Youjin Deng, Phys. Rev. E 86, 021102 (2012).
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Many thanks for your attention!