Theory of the Maxwell Pressure Tensor and the Tension in a Water Bridge

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A water bridge refers to an experimental “flexible cable” made up of pure deionized water which can hang across two supports maintained with a sufficiently large voltage difference. The resulting electric fields within the deionized water flexible cable, maintain a tension which sustains the water against the downward force of gravity. A detailed calculation of the water bridge tension will be provided in terms of the Maxwell pressure tensor in a dielectric fluid medium. General properties of the dielectric liquid pressure tensor are discussed along with unusual features of dielectric fluid Bernoulli flows in an electric field. Analogies between dielectric fluid Bernoulli flows in strong electric fields and quantum Bernoulli flows in superfluids are explored.

PACS numbers: 92.40.Bc, 03.50.De

I. INTRODUCTION

Recent observations[1, 2, 3] have been made of water bridges stretched across supports which are maintained at large voltage differences. A water bridge is a “flexible cable” made up of pure deionized water which has an electric field $\mathbf{E}$ in virtue of an applied voltage across the supports at the ends of the fluid cable. Previously to this work, it was not fully understood what forces hold up the water bridge against the force of gravity. It will here be shown that the forces responsible for holding up the water bridge follow from the Maxwell electric field pressure tensor in dielectric polar fluids. In particular, the water bridge viewed as a flexible cable has an electric field induced tension sufficiently large so as to explain the water bridge support. In discussing the general theory of the pressure tensor for isotropic dielectric fluids, in Sec.II the thermodynamic laws applied to a fluid dielectric are described in detail. It is shown that in the presence of an electric field, there are two different thermodynamic pressures, $P$ and $\bar{P}$. These turn out to be eigenvalues of the full pressure tensor as discussed in Sec.IV. If an infinitesimal surface area $\delta A_\perp$ has a normal perpendicular to the electric field lines, then the pressure force is $P\delta A_\perp$. If an infinitesimal surface area $\delta A_\parallel$ has a normal parallel to the electric field lines, then the pressure force is $\bar{P}\delta A_\parallel$. It is shown that these results completely characterize the pressure tensor.

In Sec.IV we compute the tension in a fluid dielectric cylinder. The physics of that calculation is easily explained. Consider a simple cylinder of length $L$ and cross sectional area $A = \pi R^2$. Suppose that a spatially uniform electric field $\mathbf{E}$ exists in a direction parallel to the cylinder axis. Since the tangential component of the electric field is continuous, the electric field inside the cylinder is the same as the electric field outside the cylinder. On the other hand, the displacement field $\mathbf{D} = \varepsilon \mathbf{E}$ is discontinuous at the endpoints of the cylinder; i.e. the ends of the cylinder have a charge density of $\pm \sigma$ wherein

$$4\pi \sigma = \Delta D = (\varepsilon - 1)E = 4\pi \frac{Q}{A}. \quad (1)$$

The end of the cylinder which is at the tail of the electric field vector has charge $+Q$ while the end of the cylinder which is at the arrow of the electric field vector has charge $-Q$. The tension in the cylinder is then evidently given by $\tau = QE$ which in virtue of Eq.(1) reads

$$\frac{\tau}{A} = \left[\frac{\varepsilon - 1}{4\pi}\right] E^2. \quad (2)$$

In what follows, the stress Eq.(2) will be rigorously derived from the Maxwell pressure tensor within the cylinder. The ratio of the tension to cylinder weight, $Mg = \rho ALg$, obeys

$$\frac{\tau}{Mg} = \left[\frac{\varepsilon - 1}{4\pi \rho g L}\right] E^2, \quad (3)$$

wherein $\rho$ and $L$ represent, respectively, the mass density and length of the cylinder. Although we have employed free end boundary conditions to the cylinder, the tension $\tau$ is a local stress quantity independent of global boundary conditions. In Sec.VI we exhibit a plot of the hanging water bridge flexible cable for experimental values of the parameters in Eq.(3) and find the agreement between theory and experiment to be satisfactory.

In Sec.VII the hydrostatics of the dielectric fluid in an electric field is explored. The crucial quantity of interest is the mean isothermal polarization per molecule which
obey
\[ p = \alpha_T E \Rightarrow 4\pi \alpha_T = m \left( \frac{\partial \varepsilon}{\partial \rho} \right)_{T,E}, \] (4)

wherein \( m \) is the mass of a single molecule and \( \alpha_T \) is the isothermal polarizability. For the water liquid and vapor phases, the molecular polarizabilities, respectively, obey \( \alpha_{T,\text{gas}} \ll \alpha_{T,\text{liquid}} \). Fluid dielectric films in strong electric fields adsorbed on insulating walls tend to swell to a large thickness. In Sec. VII the theory of dielectric polar fluid Bernoulli flows in strong electric fields is discussed. Together with the effect of film thickening, it turns out that the film is capable of crawling up an insulating container wall against the force of gravity and flow as in a siphon over the top of the wall and down the other side. In the concluding Sec. VIII the role of Bernoulli flows in forming water bridges will be discussed. Analogies between dielectric polar liquid Bernoulli flows in strong electric fields and quantum Bernoulli flows in superfluids will be explored. These analogies include the ability of films to climb against gravitational forces and even to pass over the tops of these walls over to the other side.

II. THERMODYNAMIC ARGUMENTS

Let \( \tilde{f}(\rho,T,D) \) represent the Helmholtz free energy per unit volume for a dielectric fluid of mass density \( \rho \), temperature \( T \) and Maxwell displacement field \( D \):
\[ df = -sdT + \zeta d\rho + \frac{1}{4\pi} E \cdot dD, \] (5)

wherein \( s \) is the entropy per unit volume, \( \zeta \) is the chemical potential per unit mass and \( E \) is the electric field. The thermodynamic pressure which follows from Eq. (4),
\[ \tilde{P} = \zeta \rho - \tilde{f}, \]
\[ d\tilde{P} = sdT + \rho d\zeta - \frac{1}{4\pi} E \cdot dD. \] (6)

On the other hand, one may employ the free energy
\[ f = \tilde{f} - \frac{1}{4\pi} E \cdot D, \]
\[ df = -sdT + \zeta d\rho - \frac{1}{4\pi} D \cdot dE, \] (7)
yielding the pressure
\[ P = \zeta \rho - f, \]
\[ dP = sdT + \rho d\zeta + \frac{1}{4\pi} D \cdot dE. \] (8)
The two different pressures obey
\[ P = \tilde{P} + \frac{1}{4\pi} E \cdot D. \] (9)

For a dielectric fluid in an electric field, isotropy dictates that \( D \) be parallel to \( E \) even if the detailed equations of state are non-linear, i.e. isotropy yields a free energy of the form
\[ f(\rho,T,E) \equiv f(\rho,T,E^2) \] (10)
so that Eqs. (7) and (10) imply
\[ D = \varepsilon E, \] (11)

wherein \( \varepsilon(\rho,T,E^2) = -8\pi f(\rho,T,E^2)/\partial(E^2) \). Thus, the two possible fluid pressures in Eq. (10) obey
\[ \tilde{P} = P - \frac{\varepsilon}{4\pi} E^2. \] (12)

It may at first glance appear strange that there are two physically different thermodynamic pressures in a dielectric fluid subject to an electric field. However, the situation may be clarified when it is realized that due to the electric field, the pressure is in reality a tensor.

III. PRESSURE TENSOR

In order to compute the Maxwell pressure tensor in a fluid dielectric, imagine that the fluid undergoes a strain wherein a fluid particle at point \( r \) is sent to the new point \( r' \). Such a transformation induces a strained length scale \( ds^2 = dr' \cdot dr' \) described by a metric tensor
\[ ds^2 = g_{ij}(r)dr^i dr^j. \] (13)
The pressure tensor \( P_{ij} \) is then described in terms of the free energy change due to a metric strain
\[ \delta F = \frac{1}{2} \int P_{ij}\delta g^{ij}dV. \] (14)
The free energy variation is described by
\[ \delta F = \delta \int fdV = \int \delta fdV + \int \delta fdV. \] (15)

In detail, the volume element of the strained fluid is determined by \( g(r) = \det[g_{ij}(r)] \) via
\[ dV = \sqrt{g(r)}d^3r \Rightarrow \delta dV = -\frac{1}{2} g_{ij}\delta g^{ij}dV. \] (16)
The volume variational Eq. (10) together with mass conservation in turn implies a mass density variation
\[ \delta \rho = \frac{1}{2} g_{ij}\delta g^{ij}\rho. \] (17)

Furthermore, the change in the magnitude of the electric field is
\[ \delta E^2 = \delta g^{ij}E_i E_j. \] (18)

We then employ
\[ \delta f = \left( \frac{\partial f}{\partial \rho} \right)_{T,E^2} \delta \rho + \left( \frac{\partial f}{\partial E^2} \right)_{T,\rho} \delta E^2, \]
\[ \delta f = \zeta \delta \rho - \frac{\varepsilon}{8\pi} \delta E^2. \] (19)
In virtue of Eqs. (15)–(19),
\[
\delta F = \frac{1}{2} \int \delta g_{ij} \left[ g_{ij} (\zeta \rho - f) - \frac{\varepsilon}{4\pi} E_i E_j \right] dV.
\]  (20)

From Eqs. (8), (14) and (20) we have the final form for the pressure tensor
\[
P_{ij} = P g_{ij} - \frac{\varepsilon}{4\pi} E_i E_j.
\]  (21)

In dyadic notation, the pressure tensor is given by
\[
P = P1 - \frac{\varepsilon}{4\pi} \varepsilon E \cdot E,
\]
\[
P = P1 - \frac{\varepsilon E^2}{4\pi} n n \quad \text{with} \quad n = \frac{E}{E'},
\]
\[
P = P1 + (\hat{P} - P) n n,
\]  (22)

wherein Eq. (12) has been invoked.

It is now clear as to why there are two thermodynamic pressures, \( P \) and \( \hat{P} \), in Sec. IV. For an infinitesimal surface area \( \delta A \) whose normal is perpendicular to the electric field lines, the pressure force is \( \hat{P} \delta A \). For an infinitesimal surface area \( \delta A \) whose normal is parallel to the electric field lines, the pressure force is \( P \delta A \). These results describe completely when to use the pressure \( P \) and when to use the pressure \( \hat{P} \).

IV. TENSION IN A FLUID CYLINDER

For a fluid dielectric cylinder of length \( L \) and cross sectional area \( A \) in a uniform electric field parallel to the axis, let us consider the work done in changing the volume \( V = L A \)
\[
dV = AdL + LdA.
\]  (23)

Employing the pressure tensor Eq. (22), one finds that both pressures, \( \hat{P} \) and \( P \) are required
\[
dW = -\hat{P} AdL - P L dA.
\]  (24)

When a fluid is stretched at constant volume \( dV = AdL + LdA \) \( = 0 \)
\[- LdA = AdL \quad \Rightarrow \quad dW = (P - \hat{P}) AdL = \hat{\tau} dL.
\]  (25)

The effective tension is thereby
\[
\hat{\tau} = (P - \hat{P}) A = \frac{\varepsilon E^2 A}{4\pi},
\]  (26)

wherein Eq. (12) has been invoked. Subtracting the tension that would be present for an electric field in the vacuum, \( \tau = \hat{\tau} - \tau_{vac} \), yields our final result for the tension,
\[
\tau = \frac{(\varepsilon - 1) E^2 A}{4\pi}
\]  (27)
in agreement with Eq. (2) of Sec. I.

V. HANGING FLEXIBLE CABLE

The water bridge consists of a flexible fluid cable which can be suspended by its endpoints. There is a slight sag in the cable as befits the equilibrium of the total gravitational force \( M g \) downward and the total Maxwell tension force \( 2 \tau \sin \theta \) upward wherein \( \theta \) is the angle between the cable tangent at the support and the horizontal:
\[
M g = 2 \tau \sin \theta \quad \Rightarrow \quad \sin \theta = \frac{2 \pi \rho g L}{(\varepsilon - 1) E^2}
\]  (28)

wherein Eq. (3) has been invoked. Eq. (28) allows for the theoretical computation of \( \theta \) in terms of experimental dielectric constants, mass densities, water bridge lengths and electric fields. With regard to the electric field, one notes the exact conversion between the Gaussian units here employed and the usual engineering units,
\[
1 \text{ Gaussian} \equiv 299.792458 \text{ volt/cm}.
\]  (29)

The agreement between theory and experiment in predicting the slight hang of the water bridge as a flexible cable, as in the above Fig. 1, is satisfactory.

VI. HYDROSTATICS IN AN ELECTRIC FIELD

The force density on the fluid as described by the pressure tensor Eq. (22) is given by
\[
f = -\text{div} P = -\text{grad} P + \text{div} \left( \frac{\varepsilon E E}{4\pi} \right).
\]  (30)

Within the bulk liquid \( \text{div} \mathbf{D} = \text{div}(\varepsilon \mathbf{E}) = 0 \) so that Eq. (30) thereby reads
\[
f = -\text{grad} P + \frac{\varepsilon}{4\pi} \mathbf{E} \cdot \text{grad} \mathbf{E}.
\]  (31)
On the other hand, from the Gibbs-Duhem Eq. (8) under equilibrium isothermal conditions $dT = 0$,

$$\text{grad}P = \rho \text{grad}\zeta + \frac{\varepsilon}{4\pi}(E \cdot \text{grad})E \tag{32}$$

so that the force per unit volume can be computed from the chemical potential per unit mass; i.e.

$$f = \rho \text{grad}\zeta. \tag{33}$$

Under a Newtonian gravitational field

$$g = -\text{grad}\Phi, \tag{34}$$

one finds the total force density equilibrium condition

$$f + \rho g = -\text{grad}(\zeta + \Phi) = 0 \tag{35}$$

yielding the uniform chemical potential condition

$$\mu = m[\zeta(\rho, T, E^2) + \Phi] = \text{const.} \tag{36}$$

To compute the electric field dependence of the chemical potential per unit mass, one may apply a Maxwell relation to the thermodynamic Eq. (7) which reads

$$\left(\frac{\partial \zeta}{\partial E}\right)_{T, \rho} = -\frac{1}{4\pi} \left(\frac{\partial D}{\partial \rho}\right)_{T, E} = -\frac{\alpha_T}{m}E, \tag{37}$$

wherein the mean molecular dipole moment $p = \alpha_T E$ defines the polarizability $\alpha_T$ as in Eq. (4). Integrating Eq. (37) completes the calculation of the chemical potential

$$\zeta(\rho, T, E) = \zeta_0(\rho, T) - \frac{\alpha_T}{2m} \int_0^{E^2} \alpha_T((\rho, T, F^2))d(F^2),$$

$$\zeta(\rho, T, E) = \zeta_0(\rho, T) - \frac{\alpha_T(\rho, T, E^2 = 0)}{2m}E^2 + \ldots \tag{38}$$

The central Eq. (38) of this section implies that the chemical potential is lowered when strong electric fields are applied.

It follows from Eq. (33) that the application of an electric field lowers the chemical potential of films of water adsorbed on insulating substrates such as glass which is often employed in physical chemistry experiments. When the chemical potential of a liquid film is lowered, the water film thickness increases; e.g. in the presence of electric fields, water in a glass beaker will have a film which appears to climb higher up the walls than would be possible in the zero electric field case. The fabrication of a water bridge begins by applying a potential difference across the water contained in two neighboring glass beakers which just touch each other. One expects and finds experimentally a thickening film layer all around the points at which the water horizontal surfaces meet the two beaker walls. The water climbs the walls of both beakers and near the touch point of the beakers splashes over the tops. When the hydrostatic calm after the splash begins, a bridge is formed at the point wherein the beakers just touch. A longer water bridge is formed after slowly separating the beakers. Let us now turn to the hydrodynamic features of the polar liquid flows in a strong electric field.

**VII. BERNOULLI FLOWS IN STRONG FIELDS**

We here employ the usual notion of a fluid derivative operator,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (v \cdot \text{grad}), \tag{39}$$

which expresses the time rate of change operator as seen by an observer moving locally with the fluid velocity $v$. For example, mass conservation reads $\dfrac{d\rho}{dt} = -\rho \text{div}v. \tag{40}$

Bernoulli flows are adiabatic, i.e. viscous entropy production is ignored. Conservation of energy then amounts to a local entropy conservation law

$$\frac{ds}{dt} = -s \text{div}v. \tag{41}$$

wherein $s$ is the entropy per unit volume. From Eqs. (39) and (40) it follows that

$$\frac{d(s)}{dt} = \frac{1}{\rho^2} \left(\frac{ds}{dt} - s \frac{d\rho}{dt}\right) = 0, \tag{42}$$

which implies a conservation law for the entropy per unit mass

$$s^* \equiv \frac{s}{\rho} \Rightarrow \frac{ds^*}{dt} = 0. \tag{43}$$

It is here that the enthalpy per unit mass $w$ makes it’s way into the adiabatic polar dielectric liquid Bernoulli flows in an electric field;

$$\zeta = w - Ts^*, \tag{44}$$

$$dw = Tds^* + \frac{1}{\rho}dP - \frac{1}{4\pi \rho}D \cdot \text{grad}E, \tag{45}$$

wherein Eq. (8) has been invoked. For an adiabatic flow with $ds^* = 0$ as in Eq. (43), Eq. (44) implies

$$\rho \text{grad}w = \text{grad}P - \frac{1}{4\pi}D \cdot \text{grad}E. \tag{45}$$

Thus, the Maxwell pressure tensor force per unit volume Eq. (21) in an adiabatic Bernoulli flow reads

$$f = -\rho \text{grad}w. \tag{46}$$

The dynamical equation of motions which accounts for momentum conservation then becomes

$$\frac{d\rho}{dt} = f + \rho g, \tag{47}$$

$$\frac{d\rho}{dt} = -\text{grad}(w + \Phi), \tag{47}$$

wherein the gravitational field Eq. (34) has been taken into account. Vorticity

$$\Omega = curlv \tag{48}$$
makes an appearance in virtue of the acceleration identities
\[
\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v},
\]
\[
\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \Omega \times \mathbf{v} - \frac{1}{2} \nabla \left( \nabla^2 \mathbf{v} \right),
\] (49)
which allow us to write Eq. (47) as
\[
\frac{\partial \mathbf{v}}{\partial t} + \Omega \times \mathbf{v} = -\nabla \left( w + \Phi + \frac{1}{2} v^2 \right). \] (50)
Employing the curl of Eq. (50) and using Eq. (48) implies the equation of motion for vorticity. It is
\[
\frac{\partial \Omega}{\partial t} + \nabla \left( \mathbf{v} \times \Omega \right) = 0,
\]
\[
\frac{d\Omega}{dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} - \Omega (\nabla \nabla \mathbf{v}). \] (51)
If at a given initial time the Bernoulli flow is irrotational, i.e. \( \Omega = 0 \), then at all later times in accordance with Eq. (51) the flow will remain irrotational. Eq. (50) then reads [1]
\[
\mathbf{v} = \nabla \varphi,
\]
\[
\frac{\partial \varphi}{\partial t} + \frac{1}{2} |\nabla \varphi|^2 + w + \Phi = 0,
\]
\[
\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( w + \Phi + \frac{1}{2} v^2 \right). \] (52)
The complete set of equations for a steady state Bernoulli flow in a strong electrostatic field \( \mathbf{E} \) and in a uniform gravitational field \( \mathbf{g} \) in the negative \( z \)-direction then follows from Eq. (52) as
\[
\text{curl} \mathbf{E} = 0,
\]
\[
\text{div} \mathbf{D} = \text{div}(\varepsilon \mathbf{E}) = 0,
\]
\[
w(s^*, P, \mathbf{E}) + \frac{1}{2} |\mathbf{v}|^2 + g z = \text{const}. \] (53)
The only difference between the normal steady state Bernoulli fluid flows in Eq. (53) and the usual case for \( \mathbf{E} = 0 \) resides in the electric field contributions to the enthalpy per unit mass \( w \). It is here useful to introduce a new thermodynamic potential per unit mass \( \varpi \) obeying
\[
w = \varpi + \frac{P}{\rho}, \] (54)
\[
d\varpi = T ds^* + \frac{P}{\rho^2} dp - \frac{1}{4 \pi \rho} \mathbf{D} \cdot \text{dE},
\]
\[
\varpi(s^*, \rho, E^2) = \varpi_0(s^*, \rho) - \frac{1}{8 \pi} \int_0^{E^2} \varepsilon(s^*, \rho, E^2) dE^2,
\]
\[
\varpi(s^*, \rho, E^2) = \varpi_0(s^*, \rho) - \frac{\varepsilon(s^*, \rho, 0)}{8 \pi} E^2 + \ldots. \] (55)
For an adiabatic \( (ds^* = 0) \) incompressible \( (dp = 0) \) steady state Bernoulli flow, Eqs. (53), (54) and (55) imply the central result of this section
\[
P + \frac{1}{2} \rho v^2 + \rho g z - \frac{\varepsilon E^2}{8 \pi} = \text{const}. \] (56)
In the case that \( \mathbf{E} = 0 \), the Bernoulli Eq. (40) indicates that water under the influence of gravity alone flows down a wall and may splash at the bottom. On the other hand, if the electric field on the bottom of the wall is negligible and the electric field on top of wall is large, then a polar dielectric fluid can crawl up a wall and may splash at the top. Such top splash processes on both of two beakers has been a precursor for building a water bridge.

VIII. CONCLUSIONS

Water subject to high electric fields can sustain structures that are more than just a bit unusual. An electric field directed parallel to the water cylinder axis can create a tension as in a stretched rubber band but with different causes. In the case of the rubber band, the tension arises from the high entropy of random knotted polymer chains. In a polar liquid the tension arises out of long ordered chains of low entropy aligned coherent dipolar domains [7, 8, 9, 10, 11]. The resulting tension in the water bridge sustains a siphon tube between two beakers without the requirement that a new external siphon tube structure of other materials be introduced.

We shall conclude with some close analogies between a polar dielectric fluid, such as water, and a quantum superfluid such as liquid \( ^4 \text{He} \). In the liquid \( ^4 \text{He} \) superfluid case, it is known that the superfluid film can act similarly to a siphon in which the film climbs over the wall and down the other side of the wall until the lowest possible gravitational energy is achieved. Similar flows can be induced in water by the application of high electric fields. In both cases, the flows are carried with little entropy production, although in the case of deionized water there is a mild Ohmic heating due to the transport of a few remaining ions.

Finally, when the vorticity \( \Omega \) becomes important [12], as in the boundary layer between the wall and the Bernoulli flow, one may expect quantum vortex lines in water with the circulation condition [13] \( \oint \mathbf{v} \cdot \text{d}r = 2 \pi h / m \). The reasoning is that in a coherent liquid flow, the velocity potential \( \varphi \) in Eq. (52) determines the phase factor \( \exp[i(m/h) \sum \varphi(r_j, t)] \) in the many body fluid wave functions. This quite general connection between the Bernoulli flow Eq. (52) and the quantum mechanical phase is not self evident [13]. For this reason we have placed the mathematical details of the proof in the attached Appendix A.
APPENDIX A: QUANTUM FLUID PHASE

The velocity potential \( \varphi(r,t) \) in the Bernoulli Eqs.\(^1\) and \(^5\),
\[
P + \rho \left[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} \left| \text{grad} \varphi \right|^2 + \Phi + \omega \right] = 0, \quad \text{(A1)}
\]
may be employed in the phase of the many body fluid quantum wave functions,
\[
U(t) = \exp \left[ \frac{im}{\hbar} \sum_j \varphi(r_j,t) \right],
\]
\[
U(t) = \exp \left[ \frac{i}{\hbar} \int \rho(r)d^3r \right], \quad \text{(A2)}
\]
wherein \( r_j \) is the position of the \( j \)th molecule and the quantum mechanical field operator for the mass density is given by\(^1\)
\[
\hat{\rho}(r) = m \sum_j \delta(r - r_j). \quad \text{(A3)}
\]
We presume a fluid microscopic Hamiltonian of the form
\[
\hat{H} = \frac{1}{2m} \sum_j |\hat{p}_j|^2 + V \quad \text{in which} \quad \hat{p}_j = -i\hbar \text{grad}_j \quad \text{(A4)}
\]
and the positions \( \{r_j\} \) all commute with \( V \). The field operator for the mass current density is thereby\(^1\)
\[
\hat{J}(r) = \frac{1}{2} \sum_j \{\hat{p}_j, \delta(r - r_j)\} \quad \text{(A5)}
\]
wherein the curly brackets indicate an anti-commutator; \( \{a,b\} \equiv ab + ba \). When viewed as a unitary transformation with the Bernoulli velocity potential \( \varphi(r,t) \) determining the phase, one notes that
\[
\hat{\rho}' = U^\dagger \hat{\rho} U = \rho, \quad \hat{J}' = U^\dagger \hat{J} U = \hat{J} + \hat{\rho} \text{grad} \varphi, \quad \text{(A6)}
\]
the Bernoulli velocity \( \mathbf{v} = \text{grad} \varphi \) contributes to the current leaving the density unchanged.

The unitary transformation on the Hamiltonian contains a time derivative term as given by
\[
\hat{H} = \hat{H} - \mathcal{E} + \int \hat{J} \cdot \text{grad} \varphi d^3r, \quad \text{(A7)}
\]
leading finally to
\[
\hat{H} = \hat{H} - \mathcal{E} + \int \hat{J} \cdot \text{grad} \varphi d^3r. \quad \text{(A8)}
\]
In the final Hamiltonian Eq.\(^A8\), the Bernoulli Eq.\(^5\) has been invoked with a total enthalpy
\[
\mathcal{E} = \int \rho(w + \Phi)d^3r. \quad \text{(A9)}
\]
The total enthalpy may be found in the microcanonical ensemble via the eigenvalue problem \( \hat{H} |n\rangle = \mathcal{E}_n |n\rangle \). Diabatic entropy producing terms producing vorticity in boundary layers from the Bernoulli irrotational flow arise from the interaction Hamiltonian
\[
\hat{H}_{\text{int}} = \int \hat{J} \cdot \text{grad} \varphi d^3r. \quad \text{(A10)}
\]

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