EXISTENCE RESULTS FOR LINEAR EVOLUTION EQUATIONS
OF PARABOLIC TYPE

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Abstract. We study a stochastic parabolic evolution equation of the form
\[ dX + AXdt = F(t)dt + G(t)dW(t) \]
in Banach spaces. Existence of mild and strict solutions and their space-time regularity are shown in both the deterministic and stochastic cases. Abstract results are applied to a nonlinear stochastic heat equation.

1. Introduction. We consider the Cauchy problem for a stochastic linear evolution equation with additive noise
\[ \begin{cases}
    dX + AXdt = F(t)dt + G(t)dW(t), & 0 < t \leq T, \\
    X(0) = \xi
\end{cases} \]
in a UMD Banach space \( E \) of type 2 with norm \( \| \cdot \| \). Here, \( W \) denotes a cylindrical Wiener process on a separable Hilbert space \( H \), defined on a filtered, complete probability space \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P})\). Operators \( G(t), 0 \leq t \leq T \), are \( \gamma \)-radonifying operators from \( H \) to \( E \), whereas \( F \) is an \( E \)-valued measurable function on \([0, T]\). Initial value \( \xi \) is an \( E \)-valued \( \mathcal{F}_0 \)-measurable random variable. And, \( A \) is a sectorial operator in \( E \), i.e. it is a densely defined, closed linear operator satisfying the condition

(A) The spectrum \( \sigma(A) \) of \( A \) is contained in an open sectorial domain \( \Sigma_\varpi \):
\[ \sigma(A) \subset \Sigma_\varpi = \{ \lambda \in \mathbb{C} : |\arg \lambda| < \varpi \} \]
with some angle \( 0 < \varpi < \frac{\pi}{2} \). The resolvent satisfies the estimate
\[ \| (\lambda - A)^{-1} \| \leq \frac{M_\varpi}{|\lambda|}, \quad \lambda \notin \Sigma_\varpi \]
with some constant \( M_\varpi > 0 \) depending only on the angle \( \varpi \).

The equation (1) has been extensively explored in different settings. The deterministic case (i.e. \( G \equiv 0 \)) has been investigated by many researchers (see, e.g., [1, 3, 7, 11, 15, 17], [19]-[22], [32]-[35]). Not only weak solutions but also strict
solutions have been studied. Three main approaches are known for the study, namely, the semigroup methods, the variational methods, and the methods of using operational equations.

For the stochastic case (i.e. $G \neq 0$), three main approaches are also known for the study of weak solutions in $L_2$ spaces. They are the semigroup methods (see, e.g., [4]-[6]), the variational methods (see, e.g., [16]), and the martingale methods ([31]). Some extensions of the first two methods in weighted Sobolev spaces or weighted Hölder spaces can be found in [9, 10, 12]. In addition, some nonautonomous versions of (1) are studied in [24, 26, 30]. Let us review some maximal regularity results on the equation.

In [6], Da Prato-Lunardi considered a stochastic convolution $W_G$ in separable Banach space $E$ of martingale type 2:

$$W_G(t) = \int_0^t S(t-s)G(t)dW(t),$$

where $S(t) = e^{-tA}$ is an analytic semigroup generated by a linear operator $A$ on $E$. They showed maximal regularity for $W_G$ as follows. If $G \in D_A(\theta,q)(0 < \theta < 1, 2 \leq q < \infty)$, then

$$W_G \in D_A(\theta + \frac{1}{2}, q).$$

Here, $D_A(\cdot, \cdot)$ stands for the real interpolation space between $D(A)$ and $E$. In detail,

$$\|x\|_{D_A(\theta,q)} = \|x\| + \int_0^1 \|t^{1-\theta}AS(t)x\|^q dt.$$

A generalization of this maximal regularity to stochastic convolutions driven by time homogeneous Poisson random measures and cylindrical infinite dimensional Wiener processes can be found in [2].

In [28], van Neerven et al. showed maximal $L_p$-regularity of the stochastic convolution $W_G$ with operator $A$ having a bounded $H_\infty$-calculus of angle less than $\frac{\pi}{2}$ in $L_q(\mathcal{O})(q \geq 2)$, where $\mathcal{O}$ is an arbitrary $\sigma$-finite measure space. It is shown that $W_G$ takes values from the fractional domain $D(A^{\frac{1}{2}})$, and

$$A^{\frac{1}{2}}W_G \in L_p(\mathbb{R}^+; L_q(\mathcal{O})) \quad (p \geq 2),$$

provided $G \in L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}, H))$. Furthermore, if $A$ is invertible, then for every $0 \leq \theta < \frac{1}{2}$,

$$W_G \in D_A(\frac{1}{2} - \frac{1}{p}, p) \cap H^{\theta,p}(\mathbb{R}^+; D(A^{\frac{1}{2}-\theta})).$$

where $D_A(\frac{1}{2} - \frac{1}{p}, p) = (L_q(\mathcal{O}), D(A))^{\frac{1}{2} - \frac{1}{p}, p}$ is a real interpolation space.

In [29], these authors continued their study in [28] by showing maximal $\gamma$-regularity for both the deterministic and stochastic convolutions. It is shown that the deterministic convolution $U$ defined by

$$U(t) = \int_0^t S(t-s)F(t)dt,$$

possesses maximal regularity

$$AU \in \gamma(\mathbb{R}^+, E),$$
provided $F \in \gamma(\mathbb{R}^+; E)$ (see Subsection 2.1 for the definition of the space of $\gamma$-radonifying operators). When $G \in L_p(\Omega; \gamma(L_2(\mathbb{R}^+; H); E))(0 < p < \infty)$, the stochastic convolution $W_G$ possesses maximal regularity

$$W_G \in L_p(\Omega; \gamma(H^{-\theta}(\mathbb{R}^+); D(A^{1/2-\theta})))$$

for $0 \leq \theta < 1/2$.

where $H^{-\theta}(\mathbb{R}^+)$ denotes the usual Bessel potential space.

In [4, 5], it is shown that when $A$ is a bounded linear operator, the equation (1) considered in Hilbert spaces possesses strict solutions (under certain conditions on coefficients and initial value).

In the present paper, we want to consider the equation in Banach spaces (in the deterministic case), and in UMD Banach spaces of type 2 (in the stochastic case), where both $F$ and $G$ have temporal and spatial regularity. In the deterministic case, our results improve those in [23], and generalize a maximal regularity theorem in [35]. In the stochastic case, we show existence and regularity of mild and strict solutions, provided $A$ is a (unbounded) sectorial operator. Let us give a brief description of these results.

Assume that

$$A^{-\alpha_1} F \in \mathcal{F}^{\beta,\sigma}((0,T]; E), \quad A^{-\alpha_2} G \in \mathcal{F}^{\beta,\sigma}((0,T]; \gamma(H; E))$$

for some $-\infty < \alpha_1, \alpha_2 < \infty, 0 < \sigma < \beta \leq 1$. (Spaces $\mathcal{F}^{\beta,\sigma}$ of weighted Hölder continuous functions are defined in Section 2.3.) In the deterministic case, we show the maximal regularity for both initial value $\xi$ and function $F$ (see Theorem 3.2).

Loosely speaking, if $\xi \in D(A^{\beta-\alpha_1})$ and $A^{-\alpha_1} F \in \mathcal{F}^{\beta,\sigma}((0,T]; E)$, then (1) has a mild solution in the spaces:

$$X \in \mathcal{C}((0,T]; D(A^{\beta-\alpha_1})) \quad \text{and} \quad A^{1-\alpha_1} X \in \mathcal{F}^{\beta,\sigma}((0,T]; E).$$

In particular, if $\alpha_1 \leq 0$, $X$ becomes a strict solution possessing the regularity:

$$A^{-\alpha_1} \frac{dX}{dt} \in \mathcal{F}^{\beta,\sigma}((0,T]; E).$$

In the stochastic case (see Corollary 1), if

$$\xi \in D(A^{\beta-\alpha_1}), \quad A^{-\alpha_1} F \in \mathcal{F}^{\beta,\sigma}((0,T]; E),$$

and

$$A^{-\alpha_2} G \in \mathcal{F}^{\beta,\sigma}((0,T]; \gamma(H; E))$$

with some $\alpha_1 \leq 0, \alpha_2 < \alpha_1 - 1/2$, then there exists a unique strict solution $X$ to (1) such that

$$X \in \mathcal{C}([0,T]; D(A^{\beta-\alpha_1})), \quad AX \in \mathcal{C}((0,T]; E) \quad \text{a.s.}$$

For the study, we use the semigroup methods. In particular, we very often use an identity:

$$\int_s^t (t-u)^{\alpha-1} (u-s)^{\beta-1} du = (t-s)^{\alpha+\beta-1} B(\beta, \alpha), \quad 0 \leq s < t < \infty,$$

where $B(\cdot, \cdot)$ is the Beta function, and $0 < \alpha, \beta < 1$ are some constants. Notice that when $\alpha + \beta = 1$, we have

$$\int_s^t (t-u)^{\alpha-1} (u-s)^{-\alpha} du = \frac{\pi}{\sin(\pi \alpha)}, \quad 0 \leq s < t < \infty.$$
This identity has been used as a key point in the so-called factorization method introduced by Da Prato et al. (see [4, 5]).

As an application, we consider a special case, namely \( \sigma_1 = -a(x)u(t,x) + b(t,x) \) and \( \sigma_2 = \sigma(t,x) \) (see (54) and (61)), of the nonlinear stochastic heat equation

\[
\frac{\partial u}{\partial t} = \Delta u + \sigma_1(t,x,u(t,x)) + \sigma_2(t,x,u(t,x)) \dot{W}(t,x).
\]

We should mention that weak solutions to (4) have been studied in [13, 14, 18, 31] and references therein. By using our abstract results, strict solutions to (54) and (61) can be obtained (see Theorems 5.1 and 5.2).

The paper is organized as follows. Section 2 is preliminary. Section 3 studies the deterministic case of (1). The stochastic case is investigated in Section 4. Finally, Section 5 gives an application to heat equations.

2. Preliminary.

2.1. UMD Banach spaces of type 2 and \( \gamma \)-radonifying operators. Let us first recall the notion of UMD Banach spaces of type 2.

**Definition 2.1.** (i) A Banach space \( E \) is called a UMD space if for some (equivalently, for all) \( 1 < p < \infty \), there is a constant \( c_p(E) \) such that for any \( L_p \)-integrable \( E \)-valued martingale difference \( \{M_n\} \) (i.e. \( \sum_{i=1}^{n} M_i \) is an \( L_p \)-integrable \( E \)-valued martingale) on a complete probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \), and any function \( \epsilon : \{1, 2, 3, \ldots \} \rightarrow \{-1, 1\} \),

\[
\mathbb{E}_{\mathbb{P}'} \| \sum_{i=1}^{n} \epsilon(i)M_i \|_p \leq c_p(E) \mathbb{E}_{\mathbb{P}'} \| \sum_{j=1}^{n} M_j \|_p,
\]

\( n = 1, 2, 3, \ldots \).

(ii) A Banach space \( E \) is said to be of type 2 if there exists \( c_2(E) > 0 \) such that for any Rademacher sequence \( \{\epsilon_i\} \) on a complete probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \) and any finite sequence \( \{x_k\}_{k=1}^{n} \) of \( E \), it holds that

\[
\mathbb{E}_{\mathbb{P}'} \| \sum_{i=1}^{n} \epsilon_i x_i \|^2 \leq c_2(E) \sum_{i=1}^{n} \|x_i\|^2.
\]

(Recall that a Rademacher sequence is a sequence of independent symmetric random variables taking values from the set \( \{1, -1\} \).)

**Remark 1.** All Hilbert spaces and \( L_p \) spaces \( (2 \leq p < \infty) \) are UMD spaces of type 2. When \( 1 < p < 2 \), \( L_p \) spaces are UMD spaces.

From now on, if not specified, \( E \) and \( H \) always denote a UMD Banach space of type 2, and a separable Hilbert space, respectively.

Let us now review the notion of \( \gamma \)-radonifying operators. For more details on the subject, see [27].

Suppose that \( \{e_n\}_{n=1}^{\infty} \) is an orthonormal basis of \( H \). And, \( \{\gamma_n\}_{n=1}^{\infty} \) is a sequence of independent standard Gaussian random variables on a probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \). A \( \gamma \)-radonifying operator from \( H \) to \( E \) is an operator, denoted by \( \phi \) for example, in \( L(H; E) \) such that the Gaussian series \( \sum_{n=1}^{\infty} \gamma_n \phi e_n \) converges in \( L_2(\Omega'; E) \).
The set of $\gamma$-radonifying operators is denoted by $\gamma(H;E)$. It is equipped with a norm, which is independent of $\{e_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$:

$$
\|\phi\|_{\gamma(H;E)} = \left[ \mathbb{E}\|\sum_{n=1}^\infty \gamma_n \phi e_n\|^2 \right]^{\frac{1}{2}}, \quad \phi \in \gamma(H;E).
$$

Then, $(\gamma(H;E), \| \cdot \|_{\gamma(H;E)})$ is complete. It is isometrical to the space of Hilbert-Schmidt operators from $H$ to $E$ if $E$ is a Hilbert space, too.

The following lemma is frequently used in Section 4.

**Lemma 2.2.** Let $\phi_1 \in L(E)$ and $\phi_2 \in \gamma(H;E)$. Then, $\phi_1 \phi_2 \in \gamma(H;E)$ and

$$
\|\phi_1 \phi_2\|_{\gamma(H;E)} \leq \|\phi_1\|_{L(E)} \|\phi_2\|_{\gamma(H;E)}.
$$

### 2.2. Stochastic integrals

This subsection introduces some properties of stochastic integrals of $\gamma(H;E)$-valued processes with respect to cylindrical Wiener process.

In the remain of this paper, when we mention a stochastic process, we mean it is defined on a filtered, complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. In addition, when a stochastic process is said to be continuous, we mean that it admits a continuous modification. Let us first recall the definition of cylindrical Wiener processes.

**Definition 2.3.** A family $W = \{W(t)\}_{t \geq 0}$ of bounded linear operators from $H$ to $L^2(\Omega)$ is called a cylindrical Wiener process on $H$ if

1. $W h = \{W(t)h\}_{t \geq 0}$ is a scalar Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ for every $h \in H$.
2. $\mathbb{E}[W(t_1)h_1 W(t_2)h_2] = \min\{t_1, t_2\} \langle h_1, h_2 \rangle_H$, $0 \leq t_1, t_2 < \infty$, $h_1, h_2 \in H$.

Let $(S, \Sigma)$ be a measurable space. A function $\varphi : S \to E$ is said to be strongly measurable if it is the pointwise limit of a sequence of simple functions. A function $\phi : S \to L(H;E)$ is said to be $H$-strongly measurable if $\phi(\cdot)h : S \to E$ is strongly measurable for all $h \in H$.

Denote by $\mathcal{N}^2([0,T])$ the class of all $H$-strongly measurable processes $\phi : [0,T] \times \Omega \to \gamma(H;E)$ such that

1. $\phi$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.
2. $\phi \in L^2(0,T) \times \Omega; \gamma(H;E))$.

For each $\phi \in \mathcal{N}^2([0,T])$, the stochastic integral $\int_0^T \phi(t)dW(t)$ is defined as a limit of integrals of adapted step processes. By a localization argument stochastic integrals can be extended to the class $\mathcal{N}([0,T])$ of all $H$-strongly measurable and adapted processes $\phi : [0,T] \times \Omega \to \gamma(H;E)$ which are in $L^2([0,T]; \gamma(H;E))$ a.s. (see [27]).

**Theorem 2.4.** There exists $c(E) > 0$ depending only on $E$ such that

$$
\mathbb{E}\left|\int_0^T \phi(t)dW(t)\right|^2 \leq c(E)\|\phi\|_{L^2([0,T] \times \Omega; \gamma(H;E))}^2, \quad \phi \in \mathcal{N}^2([0,T]),
$$

here $\|\phi\|_{L^2([0,T] \times \Omega; \gamma(H;E))} = \int_0^T \mathbb{E}[\phi(s)^2]_{\gamma(H;E)} ds$. In addition, for any $\phi$ in $\mathcal{N}^2([0,T])$ (resp. $\mathcal{N}([0,T])$), $\{\int_0^t \phi(s)dW(s), 0 \leq t \leq T\}$ is a $E$-valued continuous martingale (resp. local martingale) and a Gaussian process.

For the proof, see, e.g., [27].
**Proposition 1.** Let $B$ be a closed linear operator on $E$ and $\phi : [0, T] \times \Omega \rightarrow \gamma(H; E)$. If both $\phi$ and $B\phi$ belong to $\mathcal{N}^2([0, T])$, then

$$B \int_0^T \phi(t)dW(t) = \int_0^T B\phi(t)dW(t) \quad a.s.$$ 

The proof for Proposition 1 is very similar to one in [5]. So, we omit it.

Let us finally restate the Kolmogorov continuity theorem. This theorem gives a sufficient condition for a stochastic process to be Hölder continuous.

**Theorem 2.5.** Let $\zeta$ be an $E$-valued stochastic process on $[a, b]$, $0 \leq a < b < \infty$. Assume that for some $c > 0$, $\epsilon_1 > 1$ and $\epsilon_2 > 0$,

$$E\|\zeta(t) - \zeta(s)\|^c \leq c(t - s)^{1 + \epsilon_2}, \quad a \leq s < t \leq b. \quad (5)$$

Then, $\zeta$ has a version whose $P$-almost all trajectories are Hölder continuous functions with an arbitrarily smaller exponent than $\frac{\epsilon_2}{\epsilon_1}$.

When $\zeta$ is a Gaussian process, the condition (5) can be weakened.

**Theorem 2.6.** Let $\zeta$ be an $E$-valued Gaussian process on $[a, b]$, $0 \leq a < b < \infty$, such that $E\zeta(t) = 0$ for $a \leq t \leq b$. Assume that for some $c > 0$ and $0 < \epsilon \leq 1$,

$$E\|\zeta(t) - \zeta(s)\|^2 \leq c(t - s)^\epsilon, \quad a \leq s < t \leq b.$$ 

Then, there exists a modification of $\zeta$ whose $P$-almost all trajectories are Hölder continuous functions with an arbitrarily smaller exponent than $\frac{\epsilon}{2}$.

For the proofs of Theorems 2.5 and 2.6, see, e.g., [5].

2.3. **Spaces of weighted Hölder continuous functions.** For $0 < \sigma < \beta \leq 1$, denote by $\mathcal{F}^{\beta, \sigma}((0, T]; E)$ the space of all $E$-valued continuous functions $f$ on $(0, T]$ (resp. $[0, T]$) when $0 < \beta < 1$ (resp. $\beta = 1$) with the properties:

(i) When $\beta < 1$,

$$t^{1 - \beta}f(t) \text{ has a limit as } t \rightarrow 0. \quad (6)$$

(ii)

$$\sup_{0 \leq s < t \leq T} \frac{s^{1 - \beta + \sigma}\|f(t) - f(s)\|}{{(t - s)}^{\sigma}} = \sup_{0 \leq t \leq 1} \sup_{0 \leq s < t} \frac{s^{1 - \beta + \sigma}\|f(t) - f(s)\|}{{(t - s)}^{\sigma}} < \infty. \quad (7)$$

(iii)

$$\lim_{t \rightarrow 0} w_f(t) = 0, \quad (8)$$

where $w_f(t) = \sup_{0 \leq s < t} \frac{s^{1 - \beta + \sigma}\|f(t) - f(s)\|}{{(t - s)}^{\sigma}}$.

It is clear that $\mathcal{F}^{\beta, \sigma}((0, T]; E)$ is a Banach space with norm

$$\|f\|_{\mathcal{F}^{\beta, \sigma}(E)} = \sup_{0 \leq t \leq T} t^{1 - \beta}\|f(t)\| + \sup_{0 \leq s < t \leq T} \frac{s^{1 - \beta + \sigma}\|f(t) - f(s)\|}{{(t - s)}^{\sigma}}.$$ 

In addition, for $f \in \mathcal{F}^{\beta, \sigma}((0, T]; E)$,

$$\|f(t)\| \leq \|f\|_{\mathcal{F}^{\beta, \sigma}(E)} t^{\beta - 1}, \quad 0 < t \leq T,$$

$$|f(t) - f(s)| \leq w_f(t)(t - s)^{\sigma} s^{\beta - \sigma - 1} \leq \|f\|_{\mathcal{F}^{\beta, \sigma}(E)} (t - s)^{\sigma} s^{\beta - \sigma - 1}, \quad 0 < s \leq t \leq T. \quad (9)$$

For more details on weighted Hölder continuous function spaces $\mathcal{F}^{\beta, \sigma}$, see [35].
2.4. **Strict and mild solutions.** Let us restate the problem (1). Throughout this paper, we consider (1) in a UMD Banach space $E$ of type 2, where

(i) $A$ is a sectorial operator on $E$.

(ii) $W$ is a cylindrical Wiener process on a separable Hilbert space $H$, defined on a filtered, complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

(iii) $F$ is a measurable nonrandom function from $[0, T]$ to $(E, \mathcal{B}(E))$.

(iv) $G$ is $\gamma(H; E)$-valued nonrandom process in $\mathbb{N}^2([0, T])$ (see Subsections 2.1 and 2.2).

(vi) $\xi$ is an $E$-valued $\mathcal{F}_0$-measurable random variable.

Since $A$ is sectorial, we have the following result.

**Lemma 2.7.** Let $A$ be a sectorial operator in $E$. Then, $(-A)$ generates an analytical semigroup $S(t) = e^{-tA}$. Furthermore,

(i) For $\theta \geq 0$, there exists $\iota_\theta > 0$ such that

$$\|A^\theta S(t)\| \leq \iota_\theta t^{-\theta}, \quad 0 < t < \infty,$$

(10) and

$$\|A^{-\theta}\| \leq \iota_\theta.$$ (11)

In particular,

$$\|S(t)\| \leq \iota_0, \quad 0 \leq t < \infty.$$ (12)

(ii) For $0 < \theta \leq 1$,

$$t^\theta A^\theta S(t)$$ converges to 0 strongly on $E$ as $t \to 0$. (13)

For the proof, see, e.g., [35].

If the function $F$ is Bochner integrable on $[0, T]$, one can define strict solutions to (1).

**Definition 2.8.** An $E$-valued process $X$ on $[0, T]$ is called a strict solution of (1) if for $0 < t \leq T$, the following holds:

(i) $X$ is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$.

(ii) $X(t) \in \mathcal{D}(A)$ and $\left\| \int_0^t AX(s)ds \right\| < \infty$ a.s.

(iii) $X(t) = \xi + \int_0^t [F(s) - AX(s)]ds + \int_0^t G(s)dW(s)$ a.s.

In the meantime, if $F$ satisfies a condition that the deterministic convolution $\int_0^t S(t-s)F(s)ds$ is well defined, one can define mild solutions to (1).

**Definition 2.9.** An $E$-valued process $X$ on $[0, T]$ is called a mild solution of (1) if $X$ is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, and for $0 < t \leq T$,

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s)ds + \int_0^t S(t-s)G(s)dW(s)$$ a.s.

A strict (mild) solution $X$ on $[0, T]$ is said to be unique if any other strict (mild) solution $\bar{X}$ on $[0, T]$ is indistinguishable from it, i.e.

$$\mathbb{P}\{X(t) = \bar{X}(t) \text{ for every } 0 \leq t \leq T\} = 1.$$

**Remark 2.** (i) The integrals in (iii) of Definition 2.8 are well defined due to the integrability of $F$ and the assumption that $G \in \mathcal{N}^2([0, T])$. Similarly, existence of the deterministic convolution together with the continuity of the semigroup $S(t)$ also implies that the integrals in Definition 2.9 are well defined.

(ii) A strict solution is a mild solution. The inverse is, however, not true in general (see [5]).
3. **The deterministic case.** In this section, we consider the deterministic case of (1), i.e. the equation

\[
\begin{aligned}
    dX + AXdt &= F(t)dt, \quad 0 < t \leq T, \\
    X(0) &= \xi
\end{aligned}
\]

in a Banach space \( E \). (The UMD and type 2 properties are unnecessary.)

Suppose that

\( (F1) \quad A^{-\alpha_1} F \in \mathcal{F}^{\beta, \sigma}((0, T]; E) \) for some \( 0 < \sigma < \beta \leq 1 \) and \(-\infty < \alpha_1 < 1\).

Let us first consider the case where initial value \( \xi \) is arbitrary in \( E \).

**Theorem 3.1.** Let (A) and (F1) be satisfied. Then, there exists a unique mild solution \( X \) to (14) in the function space

\[ X \in C((0, T]; D(A^{1-\alpha_1})) \]

with the estimate

\[
\|X(t)\| + t^{1-\alpha_1}\|A^{1-\alpha_1}X(t)\| \\
\leq C[\|\xi\| + \|A^{-\alpha_1} F\|_{\mathcal{F}^{\beta, \sigma}(E)} \max\{t^{\beta-\alpha_1}, t^\beta\}], \quad 0 < t \leq T.
\]

Furthermore, if \( \alpha_1 \leq 0 \), then \( X \) becomes a strict solution of (14) possessing the regularity

\[ X \in C([0, T]; E) \cap C^1((0, T]; E) \]

with the estimate

\[
\|\frac{dX}{dt}\| \leq C[\|\xi\| + \|A^{-\alpha_1} F\|_{\mathcal{F}^{\beta, \sigma}(E)} \max\{t^{\beta-\alpha_1}, t^\beta\}], \quad 0 < t \leq T.
\]

Here, the constant \( C \) depends only on the exponents.

**Proof.** The proof is divided into four steps.

**Step 1.** Let us show that (14) possesses a unique mild solution in the space \( C((0, T]; D(A^{1-\alpha_1})) \).

We have

\[
\int_0^t A^{1-\alpha_1} S(t-s)F(s)ds \\
= \int_0^t AS(t-s)A^{-\alpha_1} F(s)ds \\
= \int_0^t AS(t-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)]ds + \int_0^t AS(t-s)dsA^{-\alpha_1} F(t) \\
= \int_0^t AS(t-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)]ds + [I - S(t)]A^{-\alpha_1} F(t).
\]

The integral in the right-hand side of the latter equality is well defined and continuous on \((0, T]\) because

\[
\int_0^t \|AS(t-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)]\|ds \\
\leq \int_0^t \|AS(t-s)||A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)||ds \\
\leq \iota_1\|A^{-\alpha_1} F\|_{\mathcal{F}^{\beta, \sigma}(E)} \int_0^t (t-s)^{\sigma-1}s^{\beta-\sigma-1}ds
\]
\[= t_1\|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)}B(\beta - \sigma, \sigma)t^{\beta - 1} < \infty, \quad 0 < t \leq T,\]  

where \(B(\cdot, \cdot)\) is the Beta function. Here we used (9), (10), and (F1). Hence, the integral \(\int_0^t A^{1-\alpha_1}S(t-s)F(s)ds\) is continuous on \((0, T]\).

Since \(A^{1-\alpha_1}\) is closed, we observe that

\[A^{1-\alpha_1} \int_0^t S(t-s)F(s)ds = \int_0^t A^{1-\alpha_1}S(t-s)F(s)ds.\]

Thus, \(A^{1-\alpha_1} \int_0^t S(t-s)F(s)ds\) is continuous on \((0, T]\).

On the other hand, it is clear that \(A^{1-\alpha_1}S(\cdot)\xi\) is also continuous on \((0, T]\). Thus, the function \(X\) defined by

\[X(t) = A^{\alpha_1-1}\left[A^{1-\alpha_1}S(t)\xi + A^{1-\alpha_1} \int_0^t S(t-s)F(s)ds\right]\]

is a unique mild solution of (14) in \(C([0, T]; \mathcal{D}(A^{1-\alpha_1}))\).

**Step 2.** Let us verify the estimate (15).

If \(\alpha_1 < 0\), (9), (11), and (12) give

\[\int_0^t \|S(t-s)F(s)\|ds \leq \int_0^t \|A^{\alpha_1}S(t-s)\|\|A^{-\alpha_1}F(s)\|ds\]

\[\leq \int_0^t \|t^{-\alpha_1}t_0\|\|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)}s^{\beta - 1}ds\]

\[= \frac{t^{-\alpha_1}t_0\|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)}t^{\beta}}{\beta} < \infty, \quad 0 \leq t \leq T.\]

If \(\alpha_1 \geq 0\), (10) and (11) give

\[\int_0^t \|S(t-s)F(s)\|ds \leq \int_0^t \|A^{\alpha_1}S(t-s)\|\|A^{-\alpha_1}F(s)\|ds\]

\[\leq \int_0^t \|t_{\alpha_1}\|\|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)}(t-s)^{-\alpha_1}s^{\beta - 1}ds\]

\[= t_{\alpha_1}\|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)}B(\beta, 1 - \alpha_1)t^{\beta - \alpha_1} < \infty, \quad 0 < t \leq T.\]

Hence, in any case of \(\alpha_1\), we have

\[\int_0^t \|S(t-s)F(s)\|ds \leq C\|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)}\max\{t^{\beta-\alpha_1}, t^\beta\}, \quad 0 < t \leq T,\]  

where \(C\) is some positive constant depending only on the exponents.

We now have

\[\|X(t)\| + t^{1-\alpha_1}\|A^{1-\alpha_1}X(t)\|\]

\[= \|S(t)\xi + \int_0^t S(t-s)F(s)ds\|\]

\[+ t^{1-\alpha_1}\|A^{1-\alpha_1}S(t)\xi + \int_0^t A^{1-\alpha_1}S(t-s)F(s)ds\|\]

\[\leq \|S(t)\xi\| + t^{1-\alpha_1}\|A^{1-\alpha_1}S(t)\xi\| + \int_0^t \|S(t-s)F(s)\|ds\]

\[+ t^{1-\alpha_1}\int_0^t \|AS(t-s)[A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t)]\|ds\]

\[+ t^{1-\alpha_1}\|[I - S(t)]A^{-\alpha_1}F(t)\|.\]
Therefore, (17) and (18) give
\[ |X(t)| + t^{1-\alpha_1}||A^{1-\alpha_1}X(t)|| \]
\[ \leq ||S(t)|| + t^{1-\alpha_1}||A^{1-\alpha_1}S(t)|| + C||A^{-\alpha_1}F||_{F_{\beta, \sigma(E)}} \max\{\beta, \alpha_1, t\} \]
\[ + t^{1-\alpha_1}||I - S(t)||A^{-\alpha_1}F(t)|| + t_1||A^{-\alpha_1}F||_{F_{\beta, \sigma(E)}} B(\beta - \sigma, \sigma) t^{\beta - \alpha_1}, \]
\[ 0 < t \leq T. \]

It is then seen that
\[ \|X(t)\| + t^{1-\alpha_1}||A^{1-\alpha_1}X(t)|| \leq t_0\|\xi\| + t_1\|\xi\| + C||A^{-\alpha_1}F||_{F_{\beta, \sigma(E)}} \max\{\beta, \alpha_1, t\} \]
\[ + (1 + t_0)||A^{-\alpha_1}F||_{F_{\beta, \sigma(E)}} \alpha \]
\[ + t_1||A^{-\alpha_1}F||_{F_{\beta, \sigma(E)}} B(\beta - \sigma, \sigma) t^{\beta - \alpha_1}, \]
\[ 0 < t \leq T \]
due to (9), (10), and (12). Thus, (15) has been verified.

**Step 3.** Let us show that if \( \alpha_1 \leq 0 \), then
- \( X \in C([0, T]; E) \cap C^1((0, T]; E) \).
- \( X \) is a strict solution of (14).

In view of (18), \( \int_0^t S(t - s)F(s)ds \) is continuous on \([0, T]\). Since
\[ X(t) = S(t)\xi + \int_0^t S(t - s)F(s)ds, \]
we obtain that
\[ X \in C([0, T]; E). \]

Let \( A_n = A(1 + \frac{A}{n})^{-1} \) (\( n = 1, 2, 3 \ldots \)) be the Yosida approximation of \( A \). Then, \( A_n \) satisfies (A) uniformly and generates an analytic semigroup \( S_n(t) \) (see, e.g., [35]). Furthermore, for any \( 0 \leq \nu < \infty \) and \( 0 < t \leq T \),
\[ \lim_{n \to \infty} A_n^\nu S_n(t) = A^\nu S(t) \quad \text{in } L(E), \]
\[ \lim_{n \to \infty} A_n^{-\nu} = A^{-\nu} \quad \text{in } L(E), \]
and
\[ ||A_n^\nu S_n(t)|| \leq \varsigma_\nu t^{-\nu} \quad \text{if } \nu > 0, 0 < t \leq T, \]
\[ ||A_n^\nu S_n(t)|| \leq \varsigma_\nu e^{-\varsigma_\nu t} \quad \text{if } \nu = 0, 0 \leq t \leq T, \]
\[ ||A_n^{-\nu}|| \leq \varsigma_\nu, \]
where \( \varsigma_\nu > 0 \) is some constant independent of \( n \).

Consider a function \( X_n \) defined by
\[ X_n(t) = S_n(t)\xi + \int_0^t A_n^{\alpha_1} S_n(t - s)A^{-\alpha_1}F(s)ds, \quad 0 \leq t \leq T. \]

We have
\[ A_nX_n(t) = A_n S_n(t)\xi + \int_0^t A_n^{1+\alpha_1} S_n(t - s)[A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t)]ds \]
\[ + \int_0^t A_n^{1+\alpha_1} S_n(t - s)dsA^{-\alpha_1}F(t) \]
\[ = A_n S_n(t)\xi + \int_0^t A_n^{1+\alpha_1} S_n(t - s)[A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t)]ds \]
\[ + A_n^{\alpha_1}[I - S_n(t)]A^{-\alpha_1}F(t). \]
The norm of $A_nX_n(t)$ is then estimated as follows. When $-1 \leq \alpha_1 \leq 0$, (9), (21), and (22) give

$$
\|A_nX_n(t)\| \leq \xi_1 t^{-1} \|\xi\| + \xi_1 + \alpha_1 \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} \int_0^t (t-s)^{\sigma-\alpha_1-1} s^{\beta-\sigma-1} ds
$$

$$
+ \xi_1 (1 + \varsigma_0 e^{-\omega_0 t}) \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} t^{\beta-1}
$$

$$
= \xi_1 t^{-1} \|\xi\| + \xi_1 + \alpha_1 \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} B(\beta - \sigma, \sigma - \alpha_1) t^{\beta-\alpha_1-1}
$$

$$
+ \xi_1 (1 + \varsigma_0 e^{-\omega_0 t}) \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} t^{\beta-1}, \quad 0 < t \leq T.
$$

On the other hand, when $\alpha_1 < -1$,

$$
\|A_nX_n(t)\| \leq \xi_1 t^{-1} \|\xi\| + \|A_{n1}^1\| \int_0^t \|A_nS_n(t-s)\| A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t) ds
$$

$$
+ \xi_1 (1 + \varsigma_0 e^{-\omega_0 t}) \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} t^{\beta-1}
$$

$$
\leq \xi_1 t^{-1} \|\xi\| + \xi_1 \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} \int_0^t (t-s)\|s^{\sigma-\alpha_1-1} s^{\beta-\sigma-1} ds
$$

$$
+ \xi_1 (1 + \varsigma_0 e^{-\omega_0 t}) \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} t^{\beta-1}
$$

$$
= \xi_1 t^{-1} \|\xi\| + \xi_1 \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} B(\beta - \sigma, \sigma) t^{\beta-1}
$$

$$
+ \xi_1 (1 + \varsigma_0 e^{-\omega_0 t}) \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)} t^{\beta-1}, \quad 0 < t \leq T.
$$

Therefore, there always exists $C_1 > 0$ independent of $n$ such that

$$
\|A_nX_n(t)\| \leq C_1 \|\xi\| t^{-1} + C_1 \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta, \sigma}(E)}
$$

$$
\times \max\{t^{\beta-\alpha_1-1}, t^{\beta-1}\}, \quad 0 < t \leq T.
$$

Using (20), (22), and (23), we have

$$
\lim_{n \to \infty} A_nX_n(t) = Y(t),
$$

where

$$
Y(t) = AS(t)\xi + \int_0^t A^{1+\alpha_1} S(t-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)] ds
$$

$$
+ A^{\alpha_1}[I - S(t)]A^{-\alpha_1} F(t).
$$

Let us verify that $Y$ is continuous on $(0, T)$. Take $0 < t_0 \leq T$. We have for every $t \geq t_0$,

$$
\|Y(t) - Y(t_0)\| = \|\{AS(t)\xi - AS(t_0)\xi\|
$$

$$
+ \{A^{\alpha_1}[I - S(t)]A^{-\alpha_1} F(t) - A^{\alpha_1}[I - S(t_0)]A^{-\alpha_1} F(t_0)\}
$$

$$
+ \int_{t_0}^t A^{1+\alpha_1} S(t-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)] ds
$$

$$
+ \int_0^{t_0} A^{1+\alpha_1} S(t-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)] ds
$$

$$
- \int_0^{t_0} A^{1+\alpha_1} S(t_0-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t_0)] ds
$$

$$
\leq \|AS(t_0)\|S(t - t_0) - I\|\xi\|
$$

$$
+ \|A^{\alpha_1}[I - S(t)]A^{-\alpha_1} F(t) - A^{\alpha_1}[I - S(t_0)]A^{-\alpha_1} F(t_0)\|.$$
Hence, (9) and (10) give

\[
\|Y(t) - Y(t_0)\|
\]

\[
\leq \tau_1 t_0^{-1} \|S(t - t_0)\xi - \xi\|
\]

\[
+ \|A^{\alpha_1}[I - S(t)]A^{-\alpha_1}F(t) - A^{\alpha_1}[I - S(t_0)]A^{-\alpha_1}F(t_0)\|
\]

\[
+ \int_{t_0}^{t} \|A^{1+\alpha_1}S(t - s)\|A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t)\| ds
\]

\[
+ \|A^{\alpha_1}[S(t - t_0) - S(t)]A^{-\alpha_1}F(t_0) - A^{-\alpha_1}F(t)\|
\]

\[
+ \int_{t_0}^{t} \|S(t - t_0) - I\|A^{-\epsilon}A^{1+\alpha_1+\epsilon}S(t - s)\|A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t_0)\| ds
\]

\[
\leq \tau_1 t_0^{-1} \|S(t - t_0)\xi - \xi\|
\]

\[
+ \|A^{\alpha_1}[I - S(t)]A^{-\alpha_1}F(t) - A^{\alpha_1}[I - S(t_0)]A^{-\alpha_1}F(t_0)\|
\]

\[
+ \|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)} \int_{t_0}^{t} \|A^{1+\alpha_1}S(t - s)\|(t - s)^{\sigma}S^{\beta - \sigma - 1} ds
\]

\[
+ \|A^{\alpha_1}[S(t - t_0) - S(t)]A^{-\alpha_1}F(t_0) - A^{-\alpha_1}F(t)\|
\]

\[
+ \|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)} \int_{t_0}^{t} A^{1+\alpha_1+\epsilon}S(u)du
\]

\[
\times \int_{t_0}^{t} \|A^{1+\alpha_1+\epsilon}S(t - s)\|(t_0 - s)^{\sigma}S^{\beta - \sigma - 1} ds,
\]

where 0 < \epsilon < 1 is some small constant. Therefore,

\[
\lim_{t \searrow t_0} \|Y(t) - Y(t_0)\|
\]

\[
\leq \lim_{t \searrow t_0} \left[ \|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)} \int_{t_0}^{t} \|A^{1+\alpha_1}S(t - s)\|(t - s)^{\sigma}S^{\beta - \sigma - 1} ds
\]

\[
+ \|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)} \|S(t - t_0) - I\| \int_{t_0}^{t_0} \|A^{1+\alpha_1}S(t_0 - s)\|(t_0 - s)^{\sigma}S^{\beta - \sigma - 1} ds \right].
\]

If -1 \leq \alpha_1 \leq 0, then this inequality together with (10) implies that

\[
\lim_{t \searrow t_0} \|Y(t) - Y(t_0)\|
\]

\[
\leq \|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta,\sigma}(E)} \lim_{t \searrow t_0} \left[ \int_{t_0}^{t} (t - s)^{\sigma - \alpha_1 - 1}S^{\beta - \sigma - 1} ds
\]

\[
+ \int_{t_0}^{t_0} u^{\epsilon - 1} du \int_{t_0}^{t_0} (t_0 - s)^{\sigma - \alpha_1 - \epsilon - 1}S^{\beta - \sigma - 1} ds \right].
\]
\[ \leq \|A^{-\alpha_1}F\|_{\mathcal{F}_{\beta,\sigma}(E)} \limsup_{t \to t_0} \left[ \frac{t_{1+\alpha_1}t_0^{\beta-\sigma-1}(t-t_0)^{\sigma-\alpha_1}}{\sigma-\alpha_1} \right. \\
+ \left. \frac{t_1-t_0}{\epsilon} \int_0^t (t-s)^{\sigma-\alpha_1-\epsilon-1}(s-t_0)^{\beta-\sigma-1}ds \right] \]

\[ = \|A^{-\alpha_1}F\|_{\mathcal{F}_{\beta,\sigma}(E)} \limsup_{t \to t_0} \left[ \frac{t_{1+\alpha_1}t_0^{\beta-\sigma-1}(t-t_0)^{\sigma-\alpha_1}}{\sigma-\alpha_1} \right. \\
+ \left. \frac{t_1-t_0}{\epsilon} B(\beta-\sigma, \sigma-\alpha_1-\epsilon) t_0^{\beta-\alpha_1-\epsilon-1}(t-t_0)^{\epsilon} \right] = 0. \]

In the meantime, if \( \alpha_1 < -1 \), then by taking \( 0 < \epsilon < -(1 + \alpha_1) \) and using (10), (11), and (12), we have

\[ \limsup_{t \to t_0} \|Y(t) - Y(t_0)\| \]

\[ \leq \|A^{-\alpha_1}F\|_{\mathcal{F}_{\beta,\sigma}(E)} \limsup_{t \to t_0} \left[ \frac{t_{1+\alpha_1}t_0^{\beta-\sigma-1}(t-t_0)^{\sigma-\alpha_1}}{\sigma-\alpha_1} \right. \\
+ \left. \frac{t_1-t_0}{\epsilon} \int_0^t u^{\sigma-1}du \int_0^t (t-s)^{\sigma-\alpha_1-\epsilon}(t_0-s)^{\beta-\sigma-1}ds \right] \]

\[ \leq \|A^{-\alpha_1}F\|_{\mathcal{F}_{\beta,\sigma}(E)} \limsup_{t \to t_0} \left[ \frac{t_{1+\alpha_1}t_0^{\beta-\sigma-1}(t-t_0)^{\sigma+1}}{\sigma+1} \right. \\
+ \left. \frac{t_1-t_0}{\epsilon} B(\beta-\sigma, \sigma+1) (t-t_0)^{\beta+1} \right] = 0. \]

In this way, we observe that

\[ \lim_{t \to t_0} Y(t) = Y(t_0). \]

Similarly, we obtain that

\[ \lim_{t \to t_0} Y(t) = Y(t_0). \]

Thus, the function \( Y \) is continuous at \( t = t_0 \) and then at every point in \((0, T]\).

On the other hand, due to (20) and (21),

\[ \lim_{n \to \infty} X_n(t) = X(t). \]

Thus, we arrive at

\[ X(t) = \lim_{n \to \infty} X_n(t) = \lim_{n \to \infty} A^{-1}A_nX_n(t) = A^{-1}Y(t). \]

As a consequence,

\[ X(t) \in \mathcal{D}(A), \quad 0 < t \leq T, \]

and

\[ AX = Y \in \mathcal{C}((0, T]; E). \]

Furthermore, since \( A_n^{\alpha_1} \) is bounded, by some direct calculations,

\[ \frac{dX_n}{dt} = -A_nX_n + A_n^{\alpha_1}A^{-\alpha_1}F(t), \quad 0 < t \leq T. \]

From this equation, for any \( 0 < \epsilon \leq T \),

\[ X_n(t) = X_n(\epsilon) + \int_\epsilon^t [A_n^{\alpha_1}A^{-\alpha_1}F(s) - A_nX_n(s)]ds, \quad \epsilon \leq t \leq T. \quad (24) \]
Using (23), the Lebesgue dominate convergence theorem applied to (24) provides that
\[ X(t) = X(\epsilon) + \int_{\epsilon}^{t} [F(s) - AX(s)] ds, \quad \epsilon \leq t \leq T. \]  
(25)

This shows that \( X \) is differentiable on \([\epsilon, T]\). Since \( \epsilon \) is arbitrary in \((0, T]\), we conclude that
\[ X \in C^1((0, T]; E). \]  
(26)

Combining (19) and (26), the first statement of this step has been verified, i.e.
\[ X \in C([0, T]; E) \cap C^1((0, T]; E). \]

On the other hand, taking \( \epsilon \to 0 \) in (25), we have
\[ X(t) = \xi + \int_{0}^{t} [F(s) - AX(s)] ds, \quad 0 < t \leq T. \]

Since
\[ \int_{0}^{t} \|F(s)\| ds \leq \int_{0}^{t} \|A^{\alpha_1}\| \|A^{-\alpha_1} F(s)\| ds \leq \|A^{\alpha_1}\| \|A^{-\alpha_1} F\|_{\mathcal{F}_{E}} \int_{0}^{t} s^{\beta-1} ds < \infty, \]
the integral \( \int_{0}^{t} F(s) ds \) is well defined. The latter equality then shows that \( \int_{0}^{t} AX(s) ds \) is well defined, and that
\[ X(t) = \xi - \int_{0}^{t} AX(s) ds + \int_{0}^{t} F(s) ds, \quad 0 < t \leq T. \]

Thus, \( X \) is a strict solution of (14). The second statement of this step has been also proved.

**Step 4.** Let us prove that the strict solution \( X \) satisfies the estimate (16) (with \( \alpha_1 \leq 0 \)).

Thanks to (23),
\[
\|AX(t)\| = \|Y(t)\| = \lim_{n \to \infty} \|A_nX_n(t)\|
\leq C_1 \|\xi\| t^{\beta-1} + C_1 \|A^{-\alpha_1}F\|_{\mathcal{F}_{E}} \max\{t^{\beta-\alpha_1-1}, t^{\beta-1}\}, \quad 0 < t \leq T.
\]
Therefore,
\[
t\|AX(t)\| \leq C_1 \|\xi\| + \|A^{-\alpha_1}F\|_{\mathcal{F}_{E}} \max\{t^{\beta-\alpha_1}, t^\beta\}, \quad 0 < t \leq T.
\]

This together with (25) gives
\[
\left\| \frac{dX}{dt} \right\| = \|F(t) - AX(t)\|
\leq \|A^{\alpha_1}\| \|A^{-\alpha_1} F(t)\| + \|AX(t)\|
\leq t^{-\alpha_1} \|A^{-\alpha_1} F\|_{\mathcal{F}_{E}} t^{\beta-1} + C_1 \|\xi\| t^{-1}
+ C_1 \|A^{-\alpha_1} F\|_{\mathcal{F}_{E}} \max\{t^{\beta-\alpha_1-1}, t^{\beta-1}\}, \quad 0 < t \leq T.
\]

Hence, there exists \( C > 0 \) depending only on the exponents and the constants in (10), (11) and (12) such that
\[
t\left\| \frac{dX}{dt} \right\| \leq C \|\xi\| + C \|A^{-\alpha_1} F\|_{\mathcal{F}_{E}} \max\{t^{\beta-\alpha_1}, t^\beta\}, \quad 0 < t \leq T.
\]

Thus, (16) has been verified.

The proof of the theorem is complete thanks to Steps 1-4. \( \Box \)
Let us now consider the case where initial value $\xi$ belongs to a subspace of $E$, namely $D(A^{\beta-\alpha_1})$. The below theorem shows maximal regularity for both initial value $\xi$ and function $F$. It generalizes a maximal regularity theorem of Yagi ([35, Theorem 3.5]) in which he considered the case $\alpha_1 = 0$. Furthermore, it improves our previous result by eliminating the condition $\frac{1+\sigma}{2} < \alpha_1 \leq \frac{\sigma}{2}$ in [23, Theorem 1].

**Theorem 3.2.** Let (A) and (F1) be satisfied. Let $\xi \in D(A^{\beta-\alpha_1})$. Then, there exists a unique mild solution of (14) possessing the regularity

$$X \in C([0,T]; D(A^{1-\alpha_1})) \cap C([0,T]; D(A^{\beta-\alpha_1})),$$

and

$$A^{1-\alpha_1}X \in F^{\beta,\sigma}((0,T]; E).$$

In addition, $X$ satisfies the estimate

$$\|A^{\beta-\alpha_1}X\|_C + \|A^{1-\alpha_1}X\|_{F^{\beta,\sigma}(E)} \leq C[\|A^{\beta-\alpha_1}\xi\| + \|A^{-\alpha_1}F\|_{F^{\beta,\sigma}(E)}],$$

(27)

where $\| \cdot \|_C$ is the supremum norm. Furthermore, when $\alpha_1 \leq 0$, $X$ becomes a strict solution of (14) possessing the regularity

$$X \in C^1([0,T]; E)$$

and

$$A^{-\alpha_1}\frac{dX}{dt} \in F^{\beta,\sigma}((0,T]; E)$$

with the estimate

$$\left\| A^{-\alpha_1}\frac{dX}{dt} \right\|_{F^{\beta,\sigma}(E)} \leq C[\|A^{\beta-\alpha_1}\xi\| + \|A^{-\alpha_1}F\|_{F^{\beta,\sigma}(E)}].$$

(28)

Here, $C$ is some positive constant depending only on the exponents.

**Proof.** The proof is divided into three steps.

**Step 1.** Let us verify that

$$X \in C([0,T]; D(A^{1-\alpha_1})) \cap C([0,T]; D(A^{\beta-\alpha_1})).$$

In Theorem 3.1, we have already shown that

$$X \in C([0,T]; D(A^{1-\alpha_1})).$$

We now have

$$A^{\beta-\alpha_1} \int_0^t S(t-s)F(s)ds = A^\beta \int_0^t S(t-s)[A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t)]ds$$

$$+ A^{\beta-1} \int_0^t AS(t-s)A^{-\alpha_1}F(t)ds$$

$$= \int_0^t A^\beta S(t-s)[A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t)]ds$$

$$+ A^{\beta-1}[I - S(t)]A^{-\alpha_1}F(t).$$

Hence, (9), (10), and (F1) give

$$\|A^{\beta-\alpha_1} \int_0^t S(t-s)F(s)ds\|$$

$$\leq \int_0^t \|A^\beta S(t-s)||A^{-\alpha_1}F(s) - A^{-\alpha_1}F(t)||ds + \|A^{\beta-1}[I - S(t)]A^{-\alpha_1}F(t)\|$$
In view of (8) and (13), it follows that

\[ \text{we observe that} \]

\[ \text{Let us show that} \]

\[ \text{The condition (6) is hence fulfilled.} \]

We use a decomposition:

\[ A \]

\[ \text{In addition, (10) and (11) give} \]

\[ \text{Let us now prove that} \]

\[ \text{we observe that} \]

\[ \text{Step 2. Let us now prove that} \]

\[ A^{1-\alpha} X \in F^{\beta,\sigma}((0,T]; E), \]

and that (27) holds true.

We use a decomposition:

\[ A^{1-\alpha} X(t) = A^{1-\alpha} S(t) \xi + \int_0^t A S(t-s)[A^{1-\alpha} F(s) - A^{1-\alpha} F(t)] ds \]

\[ + \int_0^t A S(t-s) ds A^{1-\alpha} F(t) \]

\[ = A^{1-\alpha} S(t) \xi + \int_0^t A S(t-s)[A^{1-\alpha} F(s) - A^{1-\alpha} F(t)] ds \]

\[ + [I - S(t)] A^{1-\alpha} F(t) \]

\[ = J_1(t) + J_2(t) + J_3(t). \]

Let us show that \( J_1, J_2, \) and \( J_3 \) belong to \( F^{\beta,\sigma}((0,T]; E). \)

Proof for \( J_1. \) Using (13) and the expression

\[ t^{1-\beta} A^{1-\alpha} S(t) \xi = t^{1-\beta} A^{1-\beta} S(t) A^{\beta-\alpha} \xi, \]

it is easily seen that

\[ \lim_{t \to 0} t^{1-\beta} J_1(t) = 0. \]

The condition (6) is hence fulfilled.

In addition, (10) and (11) give

\[ \sup_{0 \leq t \leq T} t^{1-\beta} ||J_1(t)|| \leq \sup_{t \in [0,T]} t^{1-\beta} ||A^{1-\beta} S(t)|| ||A^{\beta-\alpha} \xi|| \]

\[ \leq t^{1-\beta} ||A^{\beta-\alpha} \xi||. \]

On the other hand, for \( 0 < s < t \leq T, \)

\[ \frac{s^{1-\beta+\sigma}}{(t-s)^\sigma} ||J_1(t) - J_1(s)|| = \frac{s^{1-\beta+\sigma} ||A^{1-\alpha} [S(t) - S(s)] \xi||}{(t-s)^\sigma} \]

\[ \leq \frac{||A^{-\sigma} [S(t-s) - I]||}{(t-s)^\sigma} s^{1-\beta+\sigma} ||A^{1-\beta+\sigma} S(s) A^{\beta-\alpha} \xi|| \]
Therefore, (10) gives
\[ f(s) = s^{1-\beta+\sigma} \| A^{1-\beta+\sigma} S(s) A^{\beta-\alpha_1} \xi \|. \]

Therefore, (10) gives
\[
\frac{s^{1-\beta+\sigma} \| J_1(t) - J_1(s) \|}{(t-s)^\sigma} \leq \frac{\int_0^s (t_1 - s) u^{\sigma-1} du}{(t-s)^\sigma} f(s)
\]
\[
= \frac{t_1 - \sigma}{\sigma} f(s)
\]
\[
\leq \frac{t_1 - \sigma}{\sigma} t_1^{1-\beta+\sigma} \| A^{\beta-\alpha_1} \xi \|, \quad 0 \leq s < t \leq T. \tag{30}
\]

Note that \( f(\cdot) \) is continuous on \([0, T]\) and
\[
\lim_{t \to 0} \sup_{0 \leq s \leq t} f(s) = 0 \quad \text{(see (13)).}
\]

Thus,
\[
\sup_{0 \leq s < t \leq T} \frac{s^{1-\beta+\sigma} \| J_1(t) - J_1(s) \|}{(t-s)^\sigma} < \infty,
\]
and
\[
\lim_{t \to 0} \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma} \| J_1(t) - J_1(s) \|}{(t-s)^\sigma} = 0.
\]

The conditions (7) and (8) are then satisfied.

We then conclude that
\[ J_1 \in \mathcal{F}^{\beta,\sigma}((0, T]; E). \]

Furthermore, thanks to (29) and (30),
\[
\| J_1 \|_{\mathcal{F}^{\beta,\sigma}(E)} \leq (t_1 - \beta + \frac{t_1 - \sigma}{\sigma} t_1^{1-\beta+\sigma}) \| A^{\beta-\alpha_1} \xi \|.
\]

**Proof for \( J_2 \).** The norm of \( J_2 \) is evaluated by using (9) and (F1):
\[
\| J_2(t) \| \leq \int_0^t \| AS(t - s) \| \| A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t) \| ds
\]
\[
\leq t_1 w A^{-\alpha_1} F(t) \int_0^t (t-s)^{\sigma-1}s^{\beta-\sigma-1} ds
\]
\[
= t_1 B(\beta-\sigma, \sigma)t^{\beta-1} w A^{-\alpha_1} F(t), \quad 0 < t \leq T.
\]

Therefore,
\[
t^{1-\beta} \| J_2(t) \| \leq t_1 B(\beta-\sigma, \sigma)w A^{-\alpha_1} F(t)
\]
\[
\leq t_1 B(\beta-\sigma, \sigma)\| A^{-\alpha_1} F \|_{\mathcal{F}^{\beta,\sigma}(E)}, \quad 0 < t \leq T, \tag{31}
\]

and
\[
\lim_{t \to 0} t^{1-\beta} J_2(t) = 0. \tag{32}
\]

We now observe that for \( 0 < s < t \leq T \),
\[
J_2(t) - J_2(s) = \int_s^t AS(t-u)[A^{-\alpha_1} F(u) - A^{-\alpha_1} F(t)] du
\]
\[
+ [S(t-s) - I] \int_0^s AS(s-u)[A^{-\alpha_1} F(u) - A^{-\alpha_1} F(s)] du
\]
\[\begin{align*}
&+ \int_0^s AS(t-u)[A^{-\alpha_1}F(s)-A^{-\alpha_1}F(t)]du \\
&= J_{21}(t, s) + J_{22}(t, s) + J_{23}(t, s).
\end{align*}\]

The norm of \(J_{21}(t, s)\) is estimated by using (9) and (10):
\[
\|J_{21}(t, s)\| \leq \int_s^t \|AS(t-u)\|\|A^{-\alpha_1}F(u) - A^{-\alpha_1}F(t)\|du
\]
\[
\leq \int_s^t \iota_1 w_{A^{-\alpha_1}F}(t)(t-u)^{\sigma-1}u^{\beta-\sigma-1}du
\]
\[
\leq \iota_1 w_{A^{-\alpha_1}F}(t)s^{\beta-\sigma-1}\int_s^t (t-u)^{\sigma-1}du
\]
\[
= \frac{\iota_1 w_{A^{-\alpha_1}F}(t)s^{\beta-\sigma-1}}{\sigma} (t-s)^{\sigma}.
\] (33)

The norm of \(J_{22}(t, s)\) is evaluated as follows:
\[
\|J_{22}(t, s)\| = \left\| \int_0^{t-s} AS(r)dr \int_0^s AS(s-u)[A^{-\alpha_1}F(u) - A^{-\alpha_1}F(s)]du \right\|
\]
\[
= \left\| \int_0^{t-s} \int_0^s A^2S(r+s-u)[A^{-\alpha_1}F(u) - A^{-\alpha_1}F(s)]du dr \right\|
\]
\[
\leq \iota_2 w_{A^{-\alpha_1}F}(s) \int_0^{t-s} \int_0^s (r+s-u)^{-2}(s-u)^{\sigma}u^{\beta-\sigma-1}du dr
\]
\[
= \iota_2 w_{A^{-\alpha_1}F}(s) \int_0^{s} [(s-u)^{-1} - (t-u)^{-1}](s-u)^{\sigma}u^{\beta-\sigma-1}du
\]
\[
= \iota_2 w_{A^{-\alpha_1}F}(s)(t-s) \int_0^{s} (t-u)^{-1}(s-u)^{\sigma-1}u^{\beta-\sigma-1}du.
\]

By changing variable in the latter integral \((u \to s-u)\), we have
\[
\|J_{22}(t, s)\| \leq \iota_2 w_{A^{-\alpha_1}F}(s)(t-s) \int_0^{s} (t-s+u)^{-1}u^{\sigma-1}(s-u)^{\beta-\sigma-1}du
\]
\[
= \iota_2 w_{A^{-\alpha_1}F}(s)(t-s) \int_{\frac{s}{2}}^{s} (t-s+u)^{-1}u^{\sigma-1}(s-u)^{\beta-\sigma-1}du
\] (34)
\[
+ \iota_2 w_{A^{-\alpha_1}F}(s)(t-s) \int_0^{\frac{s}{2}} (t-s+u)^{-1}u^{\sigma-1}(s-u)^{\beta-\sigma-1}du.
\]

For the first term in the right-hand side of the latter equality, we have
\[
(t-s) \int_{\frac{s}{2}}^{s} (t-s+u)^{-1}u^{\sigma-1}(s-u)^{\beta-\sigma-1}du
\]
\[
= (t-s)^{\sigma} \int_{\frac{s}{2}}^{s} (t-s+u)^{1-\sigma}(t-s+u)^{-1}u^{\sigma-1}(s-u)^{\beta-\sigma-1}du
\]
\[
\leq 2(t-s)^{\sigma}s^{-1} \int_{\frac{s}{2}}^{s} [(t-s)^{1-\sigma}(t-s+u)^{-1}u^{\sigma}](s-u)^{\beta-\sigma-1}du.
\]

Note that for every \(\frac{s}{2} \leq u \leq s\),
\[
(t-s)^{1-\sigma}(t-s+u)^{-1}u^{\sigma} = \left(\frac{t-s}{t-s+u} \right)^{1-\sigma} \left(\frac{u}{t-s+u} \right)^{\sigma} \leq 1.
\]
Hence,
\[
(t - s) \int_s^x (t - s + u)^{-1} u^{\sigma - 1} (s - u)^{\beta - \sigma - 1} du \leq 2(t - s)^\sigma s^{-1} \int_0^x (s - u)^{\beta - \sigma - 1} du
\]
\[
= \frac{2(t - s)^\sigma s^{\beta - \sigma}}{\beta - \sigma}.
\]
(Notice that \( \int_0^\infty (1 + r)^{-1} r^{\sigma - 1} dr < \infty \).)

In the meantime, for the second term, we have
\[
(t - s) \int_0^x (t - s + u)^{-1} u^{\sigma - 1} du
\]
\[
\leq 2^{1 - \beta + \sigma} s^{\beta - \sigma - 1}(t - s) \int_0^x (t - s + u)^{-1} u^{\sigma - 1} du
\]
\[
= 2^{1 - \beta + \sigma} \int_0^s (1 + r)^{-1} r^{\sigma - 1} dr s^{\beta - \sigma - 1}(t - s)^\sigma
\]
\[
\leq 2^{1 - \beta + \sigma} \int_0^\infty (1 + r)^{-1} r^{\sigma - 1} dr s^{\beta - \sigma - 1}(t - s)^\sigma.
\]
(36)

Thanks to (31), (32), (33), (37), and (38), we conclude that
\[
\| J_{22}(t, s) \| \leq C_2 w_{A - \alpha F}(s)s^{\beta - \sigma - 1}(t - s)^\alpha.
\]
(37)

The norm of the last term, \( J_{23}(t, s) \), is evaluated by using (9) and (12):
\[
\| J_{23}(t, s) \| = \|[S(t - s) - S(t)][A^{-\alpha} F(s) - A^{-\alpha} F(t)]\|
\]
\[
\leq \|[S(t - s) - S(t)][w_{A - \alpha F}(t)s^{\beta - \sigma - 1}(t - s)^\sigma
\]
\[
\leq 2u_0 w_{A - \alpha F}(t)s^{\beta - \sigma - 1}(t - s)^\sigma.
\]
(38)

Proof for \( J_3 \). Since \( t^{1 - \beta} A^{-\alpha} F(t) \) has a limit as \( t \to 0 \),
\[
\lim_{t \to 0} t^{1 - \beta} J_3(t) = \lim_{t \to 0} [I - S(t)]t^{1 - \beta} A^{-\alpha} F(t) = 0.
\]
Furthermore, (9), (11), and (12) give
\[
t^{1 - \beta} \| J_3(t) \| \leq \| I - S(t) \| t^{1 - \beta} \| A^{-\alpha} F(t) \|
\]
\[
\leq (1 + u_0) \| A^{-\alpha} F \|_{\mathfrak{F}^{\beta, \sigma}(E)}, \quad 0 \leq t \leq T.
\]

We now write
\[
J_3(t) - J_3(s) = [I - S(t)][A^{-\alpha} F(t) - A^{-\alpha} F(s)]
\]
\[
+ [I - S(t - s)]S(s)A^{-\alpha} F(s).
\]
(39)

The norm of the first term in the right-hand side of the equality is estimated by using (9), (11), and (12):
\[
\|[I - S(t)][A^{-\alpha} F(t) - A^{-\alpha} F(s)]\| \leq \|[I - S(t)]\| w_{A - \alpha F}(s)s^{\beta - \sigma - 1}(t - s)^\sigma
\]
Therefore, there exists $C > 0$ such that
\begin{align*}
\|A^{\bar{\beta}-1}[S(t-s) - I]S(s)A^{-\alpha_1}F(s)\| & \leq C_4(t-s)^\sigma s^{\bar{\beta}-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha_1}F(s)\| \\
& \leq C_4(t-s)^\sigma s^{\bar{\beta}-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha_1}F(s)\| \\
& \leq C_4(t-s)^\sigma s^{\bar{\beta}-\sigma-1}\|s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha_1}F(s)\|.
\end{align*}
In addition, since $s^{1-\beta}A^{-\alpha_1}F(s)$ has a limit as $s \to 0$, (13) gives
\begin{equation}
\lim_{s \to 0} s^{\sigma}A^{\sigma}S(s)s^{1-\beta}A^{-\alpha_1}F(s) = 0.\tag{42}
\end{equation}
According to (39), (40), (41), and (42), it is seen that
\[J_3 \in \mathcal{F}^{\sigma,\sigma}((0,T]; E),\]
and
\[\|J_3\|_{\mathcal{F}^{\sigma,\sigma}(E)} \leq C_5\|A^{-\alpha_1}F\|_{\mathcal{F}^{\sigma,\sigma}(E)}\]
with some $C_5 \geq 0$.
We have thus proved that
\[A^{1-\alpha_1}X \in \mathcal{F}^{\sigma,\sigma}((0,T]; E),\]
and that there exists $C > 0$ depending only on the exponents such that
\begin{equation}
\|A^{1-\alpha_1}X\|_{\mathcal{F}^{\sigma,\sigma}(E)} \leq \sum_{i=1}^{3} \|J_i\|_{\mathcal{F}^{\sigma,\sigma}(E)} \leq C[\|A^{\bar{\beta}-\alpha_1}\xi\| + \|A^{-\alpha_1}F\|_{\mathcal{F}^{\sigma,\sigma}(E)}]. \tag{43}
\end{equation}
On the other hand, thanks to (9), (10), and (12), we have
\begin{align*}
\|A^{\bar{\beta}-\alpha_1}X(t)\| & = \|S(t)A^{\bar{\beta}-\alpha_1}\xi + \int_{0}^{t} A^{\bar{\beta}-\alpha_1}S(t-s)F(s)ds\| \\
& \leq t_0\|A^{\bar{\beta}-\alpha_1}\xi\| + \int_{0}^{t} \|A^{\bar{\beta}}S(t-s)\|\|A^{-\alpha_1}F(s)\|ds \\
& \leq t_0\|A^{\bar{\beta}-\alpha_1}\xi\| + t_\beta\|A^{-\alpha_1}F\|_{\mathcal{F}^{\sigma,\sigma}(E)} \int_{0}^{t} (t-s)^{\sigma-\bar{\beta}}s^{\beta-\sigma-1}ds \\
& = t_0\|A^{\bar{\beta}-\alpha_1}\xi\| + t_\beta B(\beta - \sigma, 1 + \sigma - \bar{\beta})\|A^{-\alpha_1}F\|_{\mathcal{F}^{\sigma,\sigma}(E)}, \quad 0 \leq t \leq T. \tag{44}
\end{align*}
The estimate (27) then follows from (43) and (44).
Step 3. Let us show the remain of the theorem.
Assume that $\alpha_1 \leq 0$. Thanks to Theorem 3.1, $X$ is a strict solution of (14) in $C^1((0,T]; E)$. On account of (25), it is seen that
\[
A^{-\alpha_1} \frac{dX}{dt} = A^{-\alpha_1} F(t) - A^{1-\alpha_1} X(t), \quad 0 < t \leq T.
\]
Since both $A^{-\alpha_1} F$ and $A^{1-\alpha_1} X$ belong to $\mathcal{F}^{\beta,\sigma}((0,T]; E)$, we have
\[
A^{-\alpha_1} \frac{dX}{dt} \in \mathcal{F}^{\beta,\sigma}((0,T]; E).
\]
In addition, (28) follows from (27) and the estimate:
\[
\|A^{-\alpha_1} \frac{dX}{dt}\|_{\mathcal{F}^{\beta,\sigma}(E)} \leq \|A^{-\alpha_1} F\|_{\mathcal{F}^{\beta,\sigma}(E)} + \|A^{-\alpha_1} X\|_{\mathcal{F}^{\beta,\sigma}(E)}.
\]
Thus, the remain of the theorem has been proved.

4. The stochastic case. In this section, we consider the stochastic evolution equation (1), where $F$ and $G$ satisfy the following conditions:
(F2) For some $0 < \sigma < \beta - \frac{1}{2} \leq \frac{1}{2}$ and $-\infty < \alpha_1 < 1$,
\[
A^{-\alpha_1} F \in \mathcal{F}^{\beta,\sigma}((0,T]; E).
\]
(G) With the $\sigma$ and $\beta$ as above and some $-\infty < \alpha_2 < \frac{1}{2} - \sigma$,
\[
A^{-\alpha_2} G \in \mathcal{F}^{\beta,\sigma}((0,T]; \gamma(H; E)).
\]
Throughout this section, the notation $C$ stands for a universal constant which is determined in each occurrence by the exponents.
Let us recall the stochastic convolution $W_G$ defined in (2):
\[
W_G(t) = \int_0^t S(t-s)G(s)dW(s), \quad 0 \leq t \leq T.
\]
The next two theorems show the regularity of $W_G$.

Theorem 4.1. Let (A) and (G) be satisfied. Let $-\infty < \kappa_1 < \frac{1}{2} - \alpha_2$ and $-\infty < \kappa_2 < \min\{\frac{1}{2} - \sigma - \alpha_2, 1\}$. Then,
\[
W_G \in C((0,T]; D(A^{\kappa_1})) \quad \text{a.s.},
\]
\[
A^{\kappa_2} W_G \in C([\epsilon,T]; E) \quad \text{a.s.},
\]
and
\[
\mathbb{E}\|A^{\kappa_2} W_G\|_{\mathcal{F}^{\beta,\sigma}((0,T]; \mathbb{R})}
\]
for any $0 < \gamma < \sigma$ and $0 < \epsilon \leq T$. In addition,
\[
\|A^{\kappa_1} W_G(t)\| \leq C \|A^{-\alpha_2} G\|_{\mathcal{F}^{\beta,\sigma}(\gamma(H; E))} \times \max\{t^{\beta-\alpha_2-\kappa_1-\frac{1}{2}}, t^{\beta-\frac{1}{2}}\}, \quad 0 < t \leq T,
\]
where $C$ is some constant depending only on the exponents. Furthermore, if $\kappa_1 \leq \beta - \alpha_2 - \frac{1}{2}$, then
\[
W_G \in C((0,T]; D(A^{\kappa_1})) \quad \text{a.s.}
\]

Proof. We divide the proof into three steps.

Step 1. Let us show that
- $W_G \in C((0,T]; D(A^{\kappa_1}))$ a.s.
- $W_G$ satisfies (45).
- $W_G \in C([0,T]; D(A^{\kappa_1})$ a.s. when $\kappa_1 \leq \beta - \frac{1}{2}$. 

We have
\[ \int_0^t \| A^{\kappa_1} S(t-s) G(s) \|_{\gamma(H;E)}^2 ds \]
\[ \leq \int_0^t \| A^{\alpha_2+\kappa_1} S(t-s) \|_2^2 \| A^{-\alpha_2} G(s) \|_{\gamma(H;E)}^2 ds \]
\[ \leq \| A^{-\alpha_2} G \|_{\mathcal{F}^\beta,\sigma(\gamma(H;E))}^2 \int_0^t \| A^{\alpha_2+\kappa_1} S(t-s) \|_2^2 s^{2(\beta-1)} ds. \]
If \( \alpha_2 + \kappa_1 \geq 0 \), then
\[ \int_0^t \| A^{\kappa_1} S(t-s) G(s) \|_{\gamma(H;E)}^2 ds \]
\[ \leq t^{\frac{2}{\alpha_2+\kappa_1}} \| A^{-\alpha_2} G \|_{\mathcal{F}^\beta,\sigma(\gamma(H;E))}^2 \int_0^t (t-s)^{-2(\alpha_2+\kappa_1)} s^{2(\beta-1)} ds \]
\[ = t^{\frac{2}{\alpha_2+\kappa_1}} \| A^{-\alpha_2} G \|_{\mathcal{F}^\beta,\sigma(\gamma(H;E))}^2 B(2\beta-1,1-2\alpha_2-2\kappa_1) \]
\[ \times t^{2(\beta-\alpha_2-\kappa_1)-1} < \infty, \quad 0 < t \leq T. \]

Meanwhile, if \( \alpha_2 + \kappa_1 < 0 \), then
\[ \int_0^t \| A^{\kappa_1} S(t-s) G(s) \|_{\gamma(H;E)}^2 ds \]
\[ \leq C \| A^{-\alpha_2} G \|_{\mathcal{F}^\beta,\sigma(\gamma(H;E))}^2 \int_0^t s^{2(\beta-1)} ds \]
\[ \leq C \| A^{-\alpha_2} G \|_{\mathcal{F}^\beta,\sigma(\gamma(H;E))}^2 t^{2\beta-1} < \infty, \quad 0 \leq t \leq T. \]

Therefore, \( \int_0^t A^{\kappa_1} S(\cdot-s) G(s) dW(s) \) is well defined and continuous on \((0,T]\) (see [30, 5]). Since \( A^{\kappa_1} \) is closed, we obtain that
\[ A^{\kappa_1} W_G(t) = \int_0^t A^{\kappa_1} S(t-s) G(s) dW(s). \]
Thus, \( A^{\kappa_1} W_G \) is continuous on \((0,T]\), i.e.
\[ W_G \in C((0,T]; D(A^{\kappa_1})) \quad \text{a.s.} \]

In addition, (46) and (47) give
\[ \mathbb{E}\| A^{\kappa_1} W_G(t) \| \leq \sqrt{\mathbb{E}\left\| \int_0^t A^{\kappa_1} S(t-s) G(s) dW(s) \right\|^2} \]
\[ \leq \sqrt{c(E) \int_0^t \| A^{\kappa_1} S(t-s) G(s) \|^2 ds} \]
\[ \leq C \| A^{-\alpha_2} G \|_{\mathcal{F}^\beta,\sigma(\gamma(H;E))} \max\{t^{\beta-\alpha_2-\kappa_1-\frac{1}{2}}, t^{\beta-\frac{1}{2}}\}. \]
Hence, the estimate (45) has been proved.

Furthermore, when \( \kappa_1 \leq \beta - \alpha_2 - \frac{1}{2} \), (46) also holds true at \( t = 0 \). Thus, \( A^{\kappa_1} W_G \) is also continuous at \( t = 0 \).

**Step 2.** Let us verify that there exists an increasing function \( \eta(\cdot) \) defined on \((0,T]\) such that
\[ \lim_{t \to 0} \eta(t) = 0, \]
and
\[ \mathbb{E}\|A^{\kappa_2}W_G(t) - A^{\kappa_2}W_G(s)\|^2 \leq \eta(t)^2 s^{2(\beta - \sigma - 1)}(t - s)^{2\sigma}, \quad 0 < s \leq t \leq T. \]

From the expression
\[ A^{\kappa_2}W_G(t) = \int_0^t A^{\kappa_2}S(t - r)[G(r) - G(t)]dW(r) + \int_0^t A^{\kappa_2}S(t - r)G(t)dW(r), \]

it is seen that
\[ A^{\kappa_2}W_G(t) - A^{\kappa_2}W_G(s) \]

\[ = \int_s^t A^{\kappa_2}S(t - r)[G(r) - G(t)]dW(r) + \int_s^t A^{\kappa_2}S(t - r)[G(r) - G(t)]dW(r) \]

\[ - \int_0^s A^{\kappa_2}S(s - r)[G(r) - G(s)]dW(r) + \int_s^t A^{\kappa_2}S(t - r)G(t)dW(r) \]

\[ + \int_0^s A^{\kappa_2}S(s - r)G(t)dW(r) - \int_0^s A^{\kappa_2}S(s - r)G(s)dW(r) \]

\[ = \int_s^t A^{\kappa_2}S(t - r)[G(r) - G(t)]dW(r) \]

\[ + \int_0^s A^{\kappa_2}S(s - r)S(s - r)[G(r) - G(s) + G(s) - G(t)]dW(r) \]

\[ - \int_0^s A^{\kappa_2}S(s - r)[G(r) - G(s)]dW(r) + \int_s^t A^{\kappa_2}S(t - r)G(t)dW(r) \]

\[ + \int_0^s A^{\kappa_2}S(t - r)G(t)dW(r) + \int_0^s A^{\kappa_2}S(s - r)[G(t) - G(s) - G(t)]dW(r). \]

Hence,
\[ A^{\kappa_2}W_G(t) - A^{\kappa_2}W_G(s) \]

\[ = \int_s^t A^{\kappa_2}S(t - r)[G(r) - G(t)]dW(r) \]

\[ + \int_0^s A^{\kappa_2}S(s - r)[G(s) - G(t)]dW(r) + \int_s^t A^{\kappa_2}S(t - r)G(t)dW(r) \]

\[ + \int_0^s A^{\kappa_2}S(t - r)G(t)dW(r) \]

\[ + \int_0^s A^{\kappa_2}[S(t - r) - S(s - r)]G(t)dW(r) \]

\[ = K_1 + K_2 + K_3 + K_4 + K_5 + K_6. \]

Let us give estimates for \( \mathbb{E}\|K_i\|^2 \) \((i = 1, 2 \ldots 6)\). For \( \mathbb{E}\|K_1\|^2 \), (9) gives
\[ \mathbb{E}\|K_1\|^2 \leq c(E) \int_s^t \|A^{\kappa_2}S(t - r)[G(r) - G(t)]\|^2_{\gamma(H;E)} dr \]

\[ \leq c(E) \int_s^t \|A^{\kappa_2 + \kappa_2}S(t - r)\|^2_{\gamma(H;E)} dr \]

\[ \leq c(E) w|A^{-\alpha_2}G(t)|^2 \int_s^t \|A^{\kappa_2 + \kappa_2}S(t - r)\|^2_{\gamma(H;E)} dr, \]
where 
\[ w_{A^{-\alpha_2}G}(t) = \sup_{0 \leq s \leq t} s^{1-\beta+\sigma} \| A^{-\alpha_2}G(t) - A^{-\alpha_2}G(s) \|_{\gamma(H;E)}. \]

If \( \alpha_2 + \kappa_2 \geq 0 \), then by (10),
\[ \mathbb{E}\|K_1\|^2 \leq c(E) \sup_{t \geq 2} \| w_{A^{-\alpha_2}G}(t) \| (t-r)^2(\sigma-\alpha_2-\kappa_2) s^{2(\beta-\sigma-1)} dr \]

If \( \alpha_2 + \kappa_2 < 0 \), then by (11) and (12),
\[ \mathbb{E}\|K_1\|^2 \leq Cw_{A^{-\alpha_2}G}(t) \| \tau_{t-r}^2(\sigma-\alpha_2-\kappa_2) s^{2(\beta-\sigma-1)} (t-s)^2 \]

Hence,
\[ \mathbb{E}\|K_1\|^2 \leq Cw_{A^{-\alpha_2}G}(t) \]
Hence,
\[ E\|K_2\|^2 \leq C w_{A^{-\alpha_2}G}(s) E \left\{ s^{1-2(\alpha_2+\kappa_2)}, s^{1+2\alpha} \right\} E s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}. \]

For \( E\|K_3\|^2 \), we have
\[
E\|K_3\|^2 \leq (c(E) \int_0^s \| A^{\alpha_2+\kappa_2} S(t-r)[A^{-\alpha_2}G(s) - A^{-\alpha_2}G(t)]\|^2_{(H;E)} dr
\leq (c(E) w_{A^{-\alpha_2}G}(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma} \int_0^s \| A^{\alpha_2+\kappa_2} S(t-r)\|^2 dr.
\]

If \( \alpha_2 + \kappa_2 \geq 0 \), then
\[
E\|K_3\|^2 \leq C \int_0^s (t-r)^{-2(\alpha_2+\kappa_2)} dr w_{A^{-\alpha_2}G}(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}
\leq (t^{-2(\alpha_2+\kappa_2)} - (t-s)^{-2(\alpha_2+\kappa_2)} w_{A^{-\alpha_2}G}(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}
\leq C w_{A^{-\alpha_2}G}(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma},
\]

whereas if \( \alpha_2 + \kappa_2 < 0 \), then
\[
E\|K_3\|^2 \leq \int_0^s C dr w_{A^{-\alpha_2}G}(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}
\leq C w_{A^{-\alpha_2}G}(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.
\]

Hence,
\[
E\|K_3\|^2 \leq C w_{A^{-\alpha_2}G}(t)^2 \max\{t^{1-2(\alpha_2+\kappa_2)}, 1\} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.
\]

For \( E\|K_4\|^2 \), we observe that
\[
E\|K_4\|^2 \leq (c(E) \int_s^t \| A^{\kappa_2} S(t-r)G(t)\|^2_{(H;E)} dr
\leq (c(E) \int_s^t \| A^{\alpha_2+\kappa_2} S(t-r)\|^2 \| A^{-\alpha_2}G(t)\|^2_{(H;E)} dr
\leq (c(E) \left\| A^{-\alpha_2}G \right\|^2_{\mathcal{F}^{\beta,\sigma}(\gamma(H;E))} t^{2(\beta-1)} \int_s^t \| A^{\alpha_2+\kappa_2} S(t-r)\|^2 dr
\leq (c(E) \left\| A^{-\alpha_2}G \right\|^2_{\mathcal{F}^{\beta,\sigma}(\gamma(H;E))} t^{2\sigma} s^{2(\beta-\sigma-1)} \int_s^t \| A^{\alpha_2+\kappa_2} S(t-r)\|^2 dr,
\]
here we used the inequality
\[ t^{2(\beta-1)} = t^{2\sigma} t^{2(\beta-\sigma-1)} \leq t^{2\sigma} s^{2(\beta-\sigma-1)}. \]

The integral \( \int_s^t \| A^{\alpha_2+\kappa_2} S(t-r)\|^2 dr \) can be estimated similar to the estimate of the integral \( \int_0^s \| A^{\alpha_2+\kappa_2} S(t-r)\|^2 dr \) in the estimate for \( E\|K_3\|^2 \). Thereby,
\[
\int_s^t \| A^{\alpha_2+\kappa_2} S(t-r)\|^2 dr \leq C \max\{(t-s)^{1-2(\alpha_2+\kappa_2)}, t-s\}
= C \max\{(t-s)^{1-2(\alpha_2+\kappa_2) + \sigma}, (t-s)^{1-2\sigma}\} (t-s)^{2\sigma}
\leq C \max\{(t^{1-2(\alpha_2+\kappa_2) + \sigma}), t^{1-2\sigma}\} (t-s)^{2\sigma}.
\]

Therefore,
\[
E\|K_4\|^2 \leq C \left\| A^{-\alpha_2}G \right\|^2_{\mathcal{F}^{\beta,\sigma}(\gamma(H;E))} \max\{(t^{1-2(\alpha_2+\kappa_2)}), t\} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}. \]
For $\mathbb{E}\|K_5\|^2$, it is seen that
\[
\mathbb{E}\|K_5\|^2 \leq c(E) \int_0^s \|A^{\alpha_2+\kappa_2}S(s-r)[A^{-\alpha_2}G(t) - A^{-\alpha_2}G(s)]\|^2_{\gamma(H;E)} dr
\]
\[
\leq c(E)w_{A^{-\alpha_2}G}(t)2^{2(\beta-\sigma-1)}(t-s)^{2\sigma} \int_0^s \|A^{\alpha_2+\kappa_2}S(s-r)\|^2 dr.
\]
By considering two cases: $\alpha_2 + \kappa_2 \geq 0$ and $\alpha_2 + \kappa_2 < 0$ as for $\mathbb{E}\|K_3\|^2$ and $\mathbb{E}\|K_4\|^2$, we easily arrive at
\[
\mathbb{E}\|K_5\|^2 \leq Cw_{A^{-\alpha_2}G}(t)2 \max\{s^{1-2(\alpha_2+\kappa_2)}, s\} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.
\]
Finally, for $\mathbb{E}\|K_6\|^2$, we have
\[
\mathbb{E}\|K_6\|^2 \leq c(E) \int_0^s \|A^{\alpha_2}[S(t-r) - S(s-r)]G(t)\|^2_{\gamma(H;E)} dr
\]
\[
\leq c(E) \int_0^s \|A^{\alpha_2+\kappa_2+\sigma}S(s-r)\|^2 \|S(t-s) - I\|A^{-\sigma}\|^2 \|A^{-\alpha_2}G(t)\|^2_{\gamma(H;E)} dr
\]
\[
= c(E) \int_0^s \|A^{\alpha_2+\kappa_2+\sigma}S(s-r)\|^2 dr \int_0^{t-s} A^{1-\sigma}S(\rho) d\rho
\]
\[
\times \|A^{-\alpha_2}G(t)\|^2_{\gamma(H;E)}
\]
\[
\leq c(E) \int_0^s \|A^{\alpha_2+\kappa_2+\sigma}S(s-r)\|^2 dr \left( \int_0^{t-s} \rho^{-1+\sigma} d\rho \right)^2
\]
\[
\times \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} t^{2(\beta-1)}(t-s)^{2\sigma}.
\]
If $\alpha_2 + \kappa_2 + \sigma \geq 0$, then
\[
\mathbb{E}\|K_6\|^2 \leq C \int_0^s (s-r)^{-2(\alpha_2+\kappa_2+\sigma)} dr \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} t^{2(\beta-1)}(t-s)^{2\sigma}
\]
\[
\leq C \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} s^{-2(\alpha_2+\kappa_2+\sigma)} t^{2(\beta-\sigma-1)}(t-s)^{2\sigma}
\]
\[
\leq C \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} t^{1-2(\alpha_2+\kappa_2)} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.
\]
If $\alpha_2 + \kappa_2 + \sigma < 0$, then
\[
\mathbb{E}\|K_6\|^2 \leq \int_0^s C dr \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} t^{2(\beta-1)}(t-s)^{2\sigma}
\]
\[
= Cs \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} t^{2\sigma} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}
\]
\[
\leq C \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} t^{1+2\sigma} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.
\]
Therefore,
\[
\mathbb{E}\|K_6\|^2 \leq C \|A^{-\alpha_2}G\|^2_{\mathcal{F}_{\beta,\sigma}(\gamma(H;E))} \max\{t^{1-2(\alpha_2+\kappa_2)}, t^{1+2\sigma}\} s^{2(\beta-\sigma-1)}(t-s)^{2\sigma}.
\]
In this way, we conclude that
\[
\mathbb{E}\|A^{\kappa_2}W_G(t) - A^{\kappa_2}W_G(s)\|^2 \leq 6 \sum_{i=1}^6 \mathbb{E}\|K_i\|^2
\]
\[
\leq \eta(t)^2 s^{2(\beta-\sigma-1)}(t-s)^{2\sigma},
\]
Theorem 4.2. Let

where \( \eta(\cdot) \) is some increasing function defined on \((0, T]\) such that

\[
\lim_{t \to 0^+} \eta(t) = 0.
\]

**Step 3.** Let us verify that for any \( 0 < \gamma < \sigma \) and \( 0 < \epsilon \leq T \),

\[
A^{\alpha_2}W_G \in C^\gamma([\epsilon, T]; E) \quad \text{a.s.,}
\]

and

\[
\mathbb{E}\|A^{\alpha_2}W_G\| \in \mathcal{F}^{\beta, \sigma}((0, T]; \mathbb{R}).
\]

By Theorem 2.4, \( A^{\alpha_2}W_G \) is a Gaussian process on \((0, T]\). Thanks to the estimate in Step 2, Theorem 2.6 applied to \( A^{\alpha_2}W_G \) provides that

\[
A^{\alpha_2}W_G \in C^\gamma([\epsilon, T]; E) \quad \text{a.s.}
\]

In order to prove that \( \mathbb{E}\|A^{\alpha_2}W_G\| \in \mathcal{F}^{\beta, \sigma}((0, T]; \mathbb{R}) \), we again use the estimate in Step 2. We have

\[
\mathbb{E}\|A^{\alpha_2}W_G(t) - A^{\alpha_2}W_G(s)\|^2 \leq \mathbb{E}\|A^{\alpha_2}W_G(t) - A^{\alpha_2}W_G(s)\|^2
\]

\[
\leq \eta(t)^2 \gamma^2 (t - s)^2\sigma.
\]

Then,

\[
\frac{s^{1-\beta+\sigma}}{(t-s)^\sigma} \mathbb{E}\|A^{\alpha_2}W_G(t) - A^{\alpha_2}W_G(s)\| \leq \eta(t).
\]

This implies that

\[
\sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma}}{(t-s)^\sigma} \mathbb{E}\|A^{\alpha_2}W_G(t) - A^{\alpha_2}W_G(s)\| < \infty, \tag{48}
\]

and

\[
\lim_{t \to 0} \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma}}{(t-s)^\sigma} \mathbb{E}\|A^{\alpha_2}W_G(t) - A^{\alpha_2}W_G(s)\| = 0. \tag{49}
\]

On the other hand, repeating the argument as in (46) and (47), we have

\[
\mathbb{E}\|A^{\alpha_2}W_G(t)\|^2 \leq c(E) \int_0^t \|A^{\alpha_2}S(t-s)G(s)\|^2_{\gamma(H; E)} ds
\]

\[
\leq C\|A^{-\alpha_2}G\|^2_{\mathcal{F}^{\beta, \sigma}(\gamma(H; E))} \max\{t^{2(\beta-\alpha_2-\kappa_2)-1}, t^{2\beta-1}\}.
\]

Thereby,

\[
t^{1-\beta}\mathbb{E}\|A^{\alpha_2}W_G(t)\| \leq t^{1-\beta} \sqrt{\mathbb{E}\|A^{\alpha_2}W_G(t)\|^2}
\]

\[
\leq C\|A^{-\alpha_2}G\|_{\mathcal{F}^{\beta, \sigma}(\gamma(H; E))} \max\{t^{1-\kappa_2}, t^{2}\}.
\]

Hence,

\[
\lim_{t \to 0} t^{1-\beta}\mathbb{E}\|A^{\alpha_2}W_G(t)\| = 0. \tag{50}
\]

Invoking (48), (49), and (50), we conclude that

\[
\mathbb{E}\|A^{\alpha_2}W_G\| \in \mathcal{F}^{\beta, \sigma}((0, T]; \mathbb{R}).
\]

Thanks to Steps 1 and 3, the proof of the theorem is complete. \( \square \)

**Theorem 4.2.** Let (A) and (G) be satisfied. Assume that \( \alpha_2 < \frac{1}{2} \). Then,

\[
W_G \in C((0, T]; D(A)) \quad \text{a.s.}
\]

and

\[
W_G(t) = -\int_0^t AW_G(s) ds + \int_0^t G(s)dW(s) \quad \text{a.s., } 0 < t \leq T.
\]
Proof. Theorem 4.1 for $\kappa_1=1$ provides that

$$W_G \in C((0,T]; D(A)) \quad \text{a.s.},$$

and

$$AW_G(t) = \int_0^t AS(t-s)G(s)dW(s), \quad 0 < t \leq T.$$ 

In addition, the process $\int_0^t G(s)dW(s)$ is continuous on $[0,T]$ due to Theorem 2.4.

Using the Fubini theorem, we now have

$$A \int_0^t W_G(s)ds = \int_0^t \int_0^s AS(s-u)G(u)dW(u)ds$$

$$= \int_0^t \int_u^t AS(s-u)G(u)dW(u)ds$$

$$= \int_0^t \int_0^s [G(u) - S(t-u)G(u)]dW(u)$$

$$= \int_0^t G(u)dW(u) - \int_0^t S(t-u)G(u)dW(u)$$

$$= \int_0^t G(u)dW(u) - W_G(t), \quad 0 < t \leq T.$$ 

Hence,

$$W_G(t) = -\int_0^t AW_G(s)ds + \int_0^t G(s)dW(s) \quad \text{a.s.,} \quad 0 < t \leq T.$$ 

Thus, the theorem has been proved. \[\Box\]

We are now ready to state the regularity for the equation (1).

**Theorem 4.3.** Let (A), (F2), and (G) be satisfied. Assume that $\mathbb{E}\xi < \infty$.

(i) Let $-\infty < \kappa \leq 1 - \alpha_1$ and $\kappa < \frac{1}{2} - \alpha_2$. Then, there exists a unique mild solution of (1) possessing the regularity

$$X \in C((0,T]; D(A^\kappa)) \quad \text{a.s.}$$

with the estimate

$$\mathbb{E}\|A^\kappa X(t)\| \leq \mathbb{E}\|\xi\|t^{-\kappa} + C\|A^{-\alpha_1}F\|_{F^{\beta,\sigma}(E)}t^{\beta-1} + C\|A^{-\alpha_2}G\|_{F^{\beta,\sigma}(\gamma;H;E)}$$

$$\times \max\{\nu^{\beta-\alpha_2-\kappa-\frac{1}{2}}, t^{\beta-\frac{1}{2}}\}, \quad 0 < t \leq T.$$ 

If $\alpha_1 \leq 0$ and $\alpha_2 < \frac{1}{2}$, then $X$ becomes a strict solution of (1).

(ii) Assume that $\xi \in D(A^{\beta-\alpha_1})$ a.s. Let $-\infty < \kappa \leq \min\{\beta - \alpha_1, \beta - \alpha_2 - \frac{1}{2}\}$ and $\kappa < \frac{1}{2} - \alpha_2$. Then,

$$X \in C([0,T]; D(A^\kappa)) \quad \text{a.s.}$$

with the estimate

$$\mathbb{E}\|A^\kappa X(t)\| \leq C\mathbb{E}\|A^{\beta-\alpha_1}\xi\| + \|A^{-\alpha_1}F\|_{F^{\beta,\sigma}(E)}$$

$$+ \|A^{-\alpha_2}G\|_{F^{\beta,\sigma}(\gamma;H;E)} \max\{\nu^{\beta-\alpha_2-\kappa-\frac{1}{2}}, t^{\beta-\frac{1}{2}}\}, \quad 0 < t \leq T.$$ 

Furthermore, if $\kappa < \min\{\frac{\beta}{2} - \sigma - \alpha_2, 1\}$, then for any $0 < \epsilon \leq T$ and $0 < \gamma < \sigma$,

$$A^\kappa X \in C([\epsilon,T]; E) \quad \text{a.s.}$$
and

\[
\begin{cases}
E\|A^\kappa X\| \in \mathcal{F}^{\beta,\sigma}((0,T]; \mathbb{R}), \\
E A^\kappa X, \frac{dE A^\kappa X}{dt} \in \mathcal{F}^{\beta,\sigma}((0,T]; E).
\end{cases}
\] (52)

**Proof.** Since \( G \) is adapted to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) and \( F \) is a nonrandom function, the stochastic and deterministic convolutions \( W_G \) and \( \int_0^t S(-s)F(s)ds \) are adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \). This is also true with \( S(\cdot)\xi \) because \( \xi \) is \( \mathcal{F}_0 \)-measurable random variable. Thus, Theorems 3.1 and 4.1 provide that the process \( X \) defined by

\[
X(t) = S(t)\xi + \int_0^t S(t-s)F(s)ds + W_G(t)
\]

is a unique mild solution to (1) in the space

\[
X \in C((0,T]; \mathcal{D}(A^\kappa)) \quad \text{a.s.}
\]

In addition, if \( \alpha_1 \leq 0 \) and \( \alpha_2 < \frac{1}{2} \), then \( X \) becomes a strict solution due to Theorems 3.1 and 4.2.

For Part (i), it now suffices to prove (51). Using (17) and (45), we have

\[
E\|A^\kappa X(t)\| = E\left\| A^\kappa S(t)\xi + \int_0^t A^\kappa S(t-s)F(s)ds + A^\kappa W_G(t) \right\|
\]

\[
\leq E\|A^\kappa S(t)\xi\| + \int_0^t \|A S(t-s)[A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)]\|ds
\]

\[
+ \|[I - S(t)]A^{-\alpha_1} F(t)\| + E\|A^\kappa W_G(t)\|
\]

\[
\leq \|A^\kappa S(t)\|E\|\xi\| + \int_0^t \|A S(t-s)\|\|A^{-\alpha_1} F(s) - A^{-\alpha_1} F(t)\|ds
\]

\[
+ \|[I - S(t)]A^{-\alpha_1} F(t)\| + C\|A^{-\alpha_2} G\|_{\mathcal{F}^{\beta,\sigma}(\gamma(H;E))}
\]

\[
\times \max\{t^{\beta - \alpha_2 - \kappa - \frac{1}{2}}, t^{\beta - \frac{1}{2}}\}, \quad 0 < t \leq T.
\]

Then, (9), (10), and (12) gives

\[
E\|A^\kappa X(t)\| \leq \epsilon \kappa E\|\xi\|t^{-\kappa} + [1 + \epsilon_0 + \epsilon_1 B(\beta - \sigma, \sigma)]\|A^{-\alpha_1} F\|_{\mathcal{F}^{\beta,\sigma}(E)}t^{\beta - 1}
\]

\[
+ C\|A^{-\alpha_2} G\|_{\mathcal{F}^{\beta,\sigma}(\gamma(H;E))}\max\{t^{\beta - \alpha_2 - \kappa - \frac{1}{2}}, t^{\beta - \frac{1}{2}}\}, \quad 0 < t \leq T.
\]

Thus, (51) has been verified.

It is easily seen that Part (ii) (except (52)) follows from Theorems 3.2 and 4.1 and a fact that for any \( 0 < \epsilon \leq T \) and \( 0 < \gamma < \sigma \),

\[
\mathcal{F}^{\beta,\sigma}((0,T]; E) \subset C^\gamma([\epsilon,T]; E).
\]

Let us finally prove (52). We have

\[
A^\kappa X = A^\kappa X_1 + A^\kappa W_G,
\]

where

\[
X_1 = S(t)\xi + \int_0^t S(t-s)F(s)ds.
\]

In the proofs for Theorem 3.2 (see Step 2) and Theorem 4.1 (see (50)), we already show that

\[
\lim_{t \to 0} t^{1-\beta} A^{1-\alpha_1} X_1(t) = 0,
\]

and

\[
\lim_{t \to 0} t^{1-\beta} E\|A^\kappa W_G(t)\| = 0.
\]
Since \( \kappa \leq 1 - \alpha \), we obtain that
\[
\lim_{t \to 0} t^{1-\beta} E\|A^\kappa X(t)\| = 0.
\]
This means that \( E\|A^\kappa X\| \) satisfies (6).

On the other hand, Theorem 3.2 gives
\[
A^{1-\alpha}X_1 \in F^{\beta,\sigma}((0,T]; E) \quad \text{a.s.} \quad (53)
\]
Hence, it is easily seen that
\[
E\|A^\kappa X_1\| \in F^{\beta,\sigma}((0,T]; \mathbb{R}).
\]
In addition, Theorem 4.1 provides that
\[
E\|A^\kappa W_G\| \in F^{\beta,\sigma}((0,T]; \mathbb{R}).
\]
Using the inequality:
\[
|E\|A^\kappa X(t)\| - E\|A^\kappa X(s)\|| \\
\leq |E\|A^\kappa X_1(t)\| - E\|A^\kappa X_1(s)\|| \\
+ |E\|A^\kappa W_G(t)\| - E\|A^\kappa W_G(s)\||, \quad 0 \leq t, s \leq T,
\]
it is then easily seen that \( E\|A^\kappa X\| \) satisfies (7) and (8).
In this way, we obtain that
\[
E\|A^\kappa X\| \in F^{\beta,\sigma}((0,T]; \mathbb{R}).
\]
We now have
\[
EA^\kappa X(t) = EA^\kappa X_1(t) = A^\kappa A^\alpha_1^{-1}\xi.
\]
Since \( \kappa \leq 1 - \alpha \), (53) gives
\[
EA^\kappa X \in F^{\beta,\sigma}((0,T]; E).
\]
In addition, since
\[
\frac{dE A^\kappa X}{dt} = \frac{d}{dt} [S(t)E A^\kappa \xi + \int_0^t A^\kappa S(t-s) F(s)ds] \\
= -AE A^\kappa X + A^\kappa F(t),
\]
we arrive at
\[
A^{-1} \frac{dE A^\kappa X}{dt} \in F^{\beta,\sigma}((0,T]; E).
\]
Thus, the proof is complete. \( \square \)

The following corollary is a direct consequence of Theorem 4.3.

**Corollary 1.** Let (A), (F2), and (G) be satisfied. Assume that \( \alpha_1 \leq 0, \alpha_2 < \alpha_1 - \frac{1}{2} \), and \( \xi \in \mathcal{D}(\mathcal{A}^{\beta-\alpha_1}) \) a.s. Then, \( (1) \) possesses a unique strict solution with the regularity
\[
X \in C([0,T]; \mathcal{D}(\mathcal{A}^{\beta-\alpha_1})), \quad \text{and} \quad AX \in C((0,T]; E) \quad \text{a.s.}
\]
5. An application to heat equations. In this section, we present an application to our abstract results. Consider a nonlinear heat equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - a(x)u + b(t, x) & \text{in } (0, T) \times \mathcal{O}, \\
u &= 0 & \text{on } (0, T) \times \partial \mathcal{O}, \\
u(0, x) &= u_0(x) & \text{in } \mathcal{O},
\end{align*}
\]

in a bounded domain \( \mathcal{O} \subset \mathbb{R}^d (d = 1, 2, \ldots) \) with \( C^2 \) boundary \( \partial \mathcal{O} \), where \( \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \) is the Laplace operator; \( a(\cdot), u_0(\cdot), \) and \( b(\cdot, \cdot) \) are real-valued functions in \( \mathcal{O}, \partial \mathcal{O}, \) and \([0, T] \times \mathcal{O}, \) respectively.

Consider (54) in the space \( L_p(\mathcal{O}) \) with norm \( \| \cdot \|_{L_p} \) \((2 \leq p < \infty)\), which is a UMD Banach space of type 2 (see Remark 1). Assume that \( a: \mathcal{O} \to \mathbb{R}_+ \), and satisfies

\[
\inf_{x \in \mathcal{O}} a(x) > 0.
\]

Let \( A \) be a realization of the differential operator \( -\Delta + a(x) \) in \( L_p(\mathcal{O}) \) under the Dirichlet boundary conditions on \( \partial \mathcal{O} \). Thanks to [35, Theorems 2.12 and 2.15], \( A \) is a sectorial operator of \( L_p(\mathcal{O}) \) with domain

\[
\mathcal{D}(A) = H^2_{p,D}(\mathcal{O}) = \{ u \in H^2_p(\mathcal{O}); u|_{\partial \mathcal{O}} = 0 \},
\]

here \( H^2_p(\mathcal{O}) \) stands for the space of all complex-valued functions whose partial derivatives in the distribution sense up to the second order belong to \( L_p(\mathcal{O}) \). (For the domains of its fractional powers, see [26].) As a consequence, \((-A)\) generates an analytical semigroup on \( L_p(\mathcal{O}) \).

We assume the space-time regularity of the function \( b \) and \( u_0 \) as follows:

\[
\begin{align*}
b(t, \cdot) &\in L_p(\mathcal{O}) \text{ for } 0 \leq t \leq T, \text{ and } b(\cdot, \cdot) \text{ is measurable on } [0, T] \times \mathcal{O}, \\
A^{-\alpha_1}b(t, \cdot) &\in \mathcal{F}^{\beta, \sigma}((0, T]; L_p(\mathcal{O})) \text{ for some } 0 < \sigma < \beta \leq 1, -\infty < \alpha_1 < 1, \\
u_0 &\in L_p(\mathcal{O}).
\end{align*}
\]

Putting

\[
F(t) = b(t, \cdot), \quad U(t) = u(t, \cdot) \quad 0 \leq t \leq T,
\]

and using operator \( A \), the equation (54) is formulated as a problem of the form (14) in \( L_p(\mathcal{O}) \):

\[
\begin{align*}
dU + AU dt &= F(t) dt, \quad 0 < t \leq T, \\
U(0) &= u_0.
\end{align*}
\]

The following theorem shows existence of mild and strict solutions and their regularity to the equation (54), or equivalently the equation (60).

**Theorem 5.1.** Let \( a, b \) and \( u_0 \) satisfy the regularity (55), (56), (57), and (58). Then, (60) possesses a unique mild solution in the function space:

\[
U \in C((0, T]; \mathcal{D}(A^{1-\alpha_1}))
\]

with the estimate

\[
\begin{align*}
\|U(t)\|_{L_p} + t^{-1-\alpha_1}\|A^{1-\alpha_1}U(t)\|_{L_p} \\
\leq &C\|u_0\|_{L_p} + \|A^{-\alpha_1}F\|_{\mathcal{F}^{\beta, \sigma}(L_p(\mathcal{O}))} \max\{t^{\beta-\alpha_1}, t^\beta\}, \quad 0 < t \leq T.
\end{align*}
\]

Furthermore, if \( u_0 \in \mathcal{D}(A^{\beta-\alpha_1}) \), then \( U \) enjoys stronger regularity:

\[
U \in C((0, T]; \mathcal{D}(A^{1-\alpha_1})) \cap C([0, T]; \mathcal{D}(A^{\beta-\alpha_1})),
\]
and 

\[ A^{1-\alpha_1} U \in \mathcal{F}^{\beta,\sigma}((0,T]; L_p(O)) \]

with the estimate

\[ \|A^{\beta-\alpha_1} U\|_C + \|A^{1-\alpha_1} U\|_{\mathcal{F}^{\beta,\sigma}(L_p(O))} \leq C[\|A^{\beta-\alpha_1} u_0\|_{L_p} + \|A^{-\alpha_1} F\|_{\mathcal{F}^{\beta,\sigma}(L_p(O))}]. \]

In addition, if \( \alpha_1 \leq 0 \), \( U \) becomes a strict solution of (60) possessing the regularity

\[ U \in C^1((0,T]; L_p(O)), \]

and

\[ A^{-\alpha_1} \frac{dU}{dt} \in \mathcal{F}^{\beta,\sigma}((0,T]; L_p(O)) \]

with the estimate

\[ \|A^{-\alpha_1} \frac{dU}{dt}\|_{\mathcal{F}^{\beta,\sigma}(L_p(O))} \leq C[\|A^{\beta-\alpha_1} u_0\|_{L_p} + \|A^{-\alpha_1} F\|_{\mathcal{F}^{\beta,\sigma}(L_p(O))}]. \]

Here, \( C \) is some positive constant depending only on the exponents.

Proof. As constructed above, \( A \) is a sectorial operator, i.e. it satisfies the condition (A). Thanks to the regularity (56) and (57) of the function \( b \), the function \( F \) defined by (59) satisfies the condition (F1). Hence, Theorems 3.1 and 3.2 are available for the equation (60). The conclusions of Theorem 5.1 thus follow. \( \square \)

We now want to consider the effect of space-time noise on the heat equation (54). Let us consider a stochastic heat equation:

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u - a(x)u + b(t,x) + \chi(t) \frac{\partial W}{\partial t} & \text{in } (0,T) \times O, \\
u = 0 & \text{on } (0,T) \times \partial O, \\
u(0,x) = u_0(x) & \text{in } O,
\end{cases}
\]

(61)

where everything is the same as in (54) except the white noise term \( \chi(t) \frac{\partial W}{\partial t} \). The process \( W \) is a cylindrical Wiener process on a separable Hilbert space \( H \) which is independent of the space \( L_p(O) \). And, \( \chi(t)(0 \leq t \leq T) \) are linear operators from \( H \) to \( L_p(O) \).

The equation (61) is formal. Its solution has not been defined. We now want to define solutions to (61) in the sense of Definitions 2.8 and 2.9 by formulating the equation as an abstract equation. Since (54) has been already reformed into the abstract equation (60), the equation (61) is also formulated into an equation of the form (1):

\[
\begin{cases}
dU + AU dt = F(t) dt + \chi(t) dW(t), & 0 < t \leq T, \\
U(0) = u_0
\end{cases}
\]

(62)

in \( L_p(O) \). We suppose that \( \chi \) has space-time regularity as follows:

\( \chi : [0,T] \rightarrow \gamma(H;L_p(O)) \) is \( H \)-strongly measurable and

\( \chi \in L_2((0,T); \gamma(H;L_p(O))). \)

Furthermore, with the \( \sigma \) and \( \beta \) as in (57) and some \( -\infty < \alpha_2 < \frac{1}{2} - \sigma \),

\[ A^{-\alpha_2} \chi \in \mathcal{F}^{\beta,\sigma}((0,T]; \gamma(H;L_p(O))). \]

It is now ready to state existence of mild and strict solutions and their regularity to (61) or (62).
\textbf{Theorem 5.2.} Let $a, b, u_0$ and $\chi$ satisfy the regularity (55), (56), (57), (58), and (63) with $0 < \sigma < \beta - \frac{1}{2} \leq \frac{1}{2}$.

(i) Let $\kappa \leq 1 - \alpha_1$ and $\kappa < \frac{1}{2} - \alpha_2$. Then, (62) possesses a unique mild solution in the space:
\[ U \in C([0,T]; \mathcal{D}(A^\kappa)) \quad \text{a.s.} \]
with the estimate
\begin{align*}
\mathbb{E}\|A^\alpha U(t)\|_{L_p} & \leq C\|u_0\|_{L_p} t^{-\kappa} + C\|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta,\sigma}(L_p(\mathcal{O}))} t^{\beta-1} \\
& \quad + C\|A^{-\alpha_2} \chi\|_{\mathcal{F}_{\beta,\sigma}(\gamma(H;L_p(\mathcal{O})))} \max\{t^{\beta - \alpha_2 - \kappa - \frac{1}{2}}, t^{\beta - \frac{3}{2}}\}, \quad 0 < t \leq T.
\end{align*}

If $\alpha_1 \leq 0$ and $\alpha_2 < \alpha_1 - \frac{1}{2}$, then $U$ becomes a strict solution of (61).

(ii) Assume that $u_0 \in \mathcal{D}(A^{\beta-\alpha_1})$. Let $-\infty < \kappa \leq \min\{\beta - \alpha_1, \beta - \alpha_2 - \frac{1}{2}\}$ and $\kappa < \frac{1}{2} - \alpha_2$. Then,
\[ U \in C([0,T]; \mathcal{D}(A^\kappa)) \quad \text{a.s.} \]
with the estimate
\begin{align*}
\mathbb{E}\|A^\alpha U(t)\|_{L_p} & \leq C\|A^{\beta-\alpha_1} u_0\|_{L_p} + \|A^{-\alpha_1} F\|_{\mathcal{F}_{\beta,\sigma}(L_p(\mathcal{O}))} \\
& \quad + \|A^{-\alpha_2} \chi\|_{\mathcal{F}_{\beta,\sigma}(\gamma(H;L_p(\mathcal{O})))} \max\{t^{\beta - \alpha_2 - \kappa - \frac{1}{2}}, t^{\beta - \frac{3}{2}}\}, \quad 0 < t \leq T.
\end{align*}
Furthermore, if $\kappa \leq \min\{\frac{1}{2} - \sigma - \alpha_2, 1\}$, then for any $0 < \epsilon \leq T$ and $0 < \gamma < \sigma$,
\[ A^\kappa U \in C^\gamma((\epsilon,T]; L_p(\mathcal{O})) \quad \text{a.s.}, \]
and
\[
\begin{cases}
\mathbb{E}\|A^\alpha U\| \in \mathcal{F}_{\beta,\sigma}((0,T]; \mathbb{R}), \\
\mathbb{E}A^\alpha U, \mathbb{E}A^\alpha U_t \in \mathcal{F}_{\beta,\sigma}((0,T]; L_p(\mathcal{O})).
\end{cases}
\]

\textbf{Proof.} Since $b$ and $\chi$ satisfy (56), (57), and (63) with $0 < \sigma < \beta - \frac{1}{2} \leq \frac{1}{2}$, the function $F$ defined by (59) and $\chi$ satisfy the conditions (F2) and (G). Hence, all the assumptions of Theorem 4.3 hold true with the equation (62). Theorem 5.2 thus follows from Theorem 4.3. \hfill \Box

\textbf{Remark 3.} The cylindrical Wiener process $W$ can be constructed in the following way. Let $\{e_j\}_{j=1}^\infty$ is an orthonormal and complete basis of a Hilbert space $H_0$ of real functions on $\mathbb{R}^d$. Let $\{B_j\}_{j=1}^\infty$ is a family of independent real-valued standard Wiener processes on a filtered, complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The series
\[ \sum_{j=1}^\infty e_j B_j(t) \]
then converges to a cylindrical Wiener process $\{W(t), t \in [0,T]\}$ on a separable Hilbert space $H \supseteq H_0$, where the embedding of $H_0$ into $H$ is a Hilbert-Schmidt operator (see, e.g., [5]). For the construction of the space $H$, see [8, 25]. In this way, we obtain a cylindrical Wiener process relating to the space $\mathbb{R}^d$ which we are considering (54) in.

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