Discrete Local Induction Equation

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Abstract
The local induction equation, or the binormal flow on space curves is a well-known model of deformation of space curves as it describes the dynamics of vortex filaments, and the complex curvature is governed by the nonlinear Schrödinger equation. In this paper, we present its discrete analogue, namely, a model of deformation of discrete space curves by the discrete nonlinear Schrödinger equation. We also present explicit formulas for both smooth and discrete curves in terms of τ functions of the two-component KP hierarchy.

1 Introduction
The local induction equation (LIE)
\[
\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2},
\]
(1.1)
is one of the most important models of deformation of space curves, where \(\gamma(x, t) \in \mathbb{R}^3\) is a smooth space curve parametrized by the arc-length \(x\) and \(t\) is a deformation parameter. In a physical setting, it describes the dynamics of vortex filaments driven by the self-induction in the inviscid fluid under the local induction approximation [8]. It is well-known that if \(\gamma\) obeys LIE, then the curvature and the torsion, or equivalently, the complex curvature of \(\gamma\) solves the nonlinear Schrödinger equation (NLS)
\[
\sqrt{-1} \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} |u|^2 u = 0.
\]
(1.2)
Here, $u = u(x, t) \in \mathbb{C}$ is the complex curvature defined by the Hasimoto transformation

$$u = \kappa e^{\sqrt{-1} \lambda}, \quad \Lambda = \int_x^x \lambda \, dx,$$

(1.3)

where $\kappa$ and $\lambda$ are the curvature and the torsion of $\gamma$, respectively.

In this paper, we consider the integrable discretization of LIE in both space and time; we construct the discrete deformation of discrete space curves which is described by the discrete nonlinear Schrödinger equation (dNLS)[1, 2, 29, 30, 31]

$$
\left( \sqrt{1 - \frac{\epsilon^2}{4}} - 1 \right) u_{n+1}^m - \left( \sqrt{1 - \frac{\epsilon^2}{4}} + 1 \right) u_n^m + \left( u_{n+1}^m + u_{n-1}^m \right) \left( 1 + \frac{\epsilon^2}{4} |u_n^m|^2 \right) \Gamma_n^m = 0,
$$

(1.4)

where $u_n^m \in \mathbb{C}$, $\Gamma_n^m \in \mathbb{R}$, $n, m$ are discrete independent variables corresponding to space and time, respectively, and $\epsilon, \delta \in \mathbb{R}$ are constants corresponding to the lattice intervals of space and time, respectively. dNLS (1.4) reduces to NLS (1.2) by taking the simple continuous limit $x = n \epsilon$, $t = m \delta$, $\epsilon, \delta \to 0$.

Let us explain the relationship between LIE (1.1) and NLS more concretely. We use the Frenet frame $\Phi = \Phi(x, t) = [T(x, t), N(x, t), B(x, t)] \in \text{SO}(3)$, where $T, N, B$ are the tangent, the normal, and the binormal vectors defined by

$$
T = \gamma', \quad N = \frac{\gamma''}{|\gamma''|}, \quad B = T \times N, \quad ' = \frac{\partial}{\partial x},
$$

(1.5)

respectively. Note that it follows that $|T| = |\gamma'| = 1$ since $x$ is the arc-length. Then we have the Frenet-Serret formula

$$
\frac{\partial \Phi}{\partial x} = \Phi L, \quad L = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix},
$$

(1.6)

where $\kappa = |\gamma''|$ and $\lambda = -\langle B', N \rangle$ are the curvature and the torsion, respectively. In this setting, LIE (1.1) is expressed as the deformation by the binormal flow

$$
\frac{\partial \gamma}{\partial t} = \kappa B,
$$

(1.7)

and the corresponding deformation equation of the Frenet frame is given by

$$
\frac{\partial \Phi}{\partial t} = \Phi M, \quad M = \begin{bmatrix} 0 & \kappa \lambda & 0 \\ -\kappa \lambda & 0 & -\kappa' \\ \kappa' & \frac{\kappa'}{\kappa} - \lambda^2 & 0 \end{bmatrix}.
$$

(1.8)

The compatibility condition of the system of linear partial differential equations (1.6) and (1.8)

$$
\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} = LM - ML,
$$

(1.9)
yields
\[
\frac{\partial \kappa}{\partial t} = -2 \frac{\partial \kappa}{\partial x} \lambda - \kappa \frac{\partial \lambda}{\partial x}, \quad \frac{\partial \lambda}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\kappa''}{\kappa} + \frac{\kappa^2}{2} - \lambda^2 \right). \tag{1.10}
\]

Discretization of curves and their deformations preserving underlying integrable structures is an important problem in the discrete differential geometry. For example, continuous deformations of the discrete plane and space curves have been considered in [5, 9, 10, 11, 12, 16, 22, 25] where the deformations described by the differential-difference analogue of the mKdV and the NLS equations are formulated. In particular, Doliwa and Santini formulated the binormal deformation of the discrete space curves [5] where the discrete analogue of complex curvature is governed by the semi-discrete NLS equation (sdNLS) or the Ablowitz-Ladik equation [1, 2]
\[
\sqrt{-1} \frac{du_n}{dt} + \frac{u_{n+1} - 2u_n + u_{n-1}}{\epsilon^2} + \frac{1}{4} (u_{n+1} + u_{n-1}) |u_n|^2 = 0. \tag{1.11}
\]

Nakayama formulated this deformation of discrete space curves as the dynamics of discrete vortex filaments driven by the self-induction caused by the Biot-Savart law with the discrete analogue of the local induction approximation [22].

As to the discrete deformations of discrete curves, the isoperimetric deformation of discrete plane curves described by the discrete mKdV equation (dmKdV) has been studied in [15, 18, 21]. For discrete space curves, the deformations by the discrete sine-Gordon equation (dsG) and dmKdV has been studied in [6, 17, 18], and the deformation by dNLS is formulated in [14, 27].

The dsG and dmKdV describe torsion-preserving isoperimetric and equidistant deformation of the space discrete curves with constant torsion. However, formulation of discrete deformation of space discrete curves with varying torsion is a difficult problem. The only example so far is presented by Hoffmann [13, 14], where it has been claimed that composition of certain two isoperimetric equidistant deformations, which is called the doubly discrete isotropic Heisenberg magnet (ddIHM), can be regarded as a discrete analogue of LIE. This formulation uses quarternions and its geometric meaning is clear, and it was used for numerical simulation of fluid flow [27, 34]. However, in this formulation, the role of complex curvature is not clear. In fact, the deformation is expressed as a coupled system of tangent vectors and displacement vectors of the discrete curves, but the complex curvature does not appear. Moreover, although it is shown in [13] that it is gauge equivalent to the well-known discrete analogue of NLS (1.2) proposed by Ablowitz-Ladik [1, 2]
\[
(1 + c_0) u_n^m - (1 + c_0^*) u_{n+1}^{m+1} + \left(-c_{-2} u_{n-1}^{m+1} + c_{-2} u_{n+1}^{m} \right) \left(1 + \frac{\epsilon^2}{4} |u_n^m|^2 \right) \Gamma_n^m \\
+ c_2 \left(u_{n-1}^m + \epsilon^2 u_{n-1}^m \Delta_n^m \right) - c_2^* \left(u_{n+1}^{m+1} + \epsilon^2 u_{n+1}^{m+1} \Delta_n^{m+1} \right) = 0,
\]
\[
\frac{\Gamma_{n+1}^m}{\Gamma_n^m} = 1 + \frac{\epsilon^2}{4} |u_n^m|^2, \\
\Delta_n^m - \Delta_{n+1}^m = \frac{1}{4} (u_{n+1}^m u_{n+1}^m - u_{n-1}^m u_{n-1}^m),
\]

it is not clear that \(u_n^m\) actually plays a role of complex curvature of the corresponding discrete space curves. In (1.12), * means complex conjugate, \(c_0, c_2, c_{-2}\) are complex constants, \(\epsilon\) is a real constant which plays a role of lattice interval of space, \(u_n^m, \Delta_n^m \in \mathbb{C}, \Gamma_n^m \in \mathbb{R}\), and \(n, m\) are discrete variables corresponding to space and time, respectively.
On the other hand, from the viewpoint of discretization of LIE (1.1), we may consider dNLS (1.4) as a simpler discrete analogue of NLS, which is obtained from (1.12) by putting the parameters as

\[ c_0^* = c_{-2} = \sqrt{\frac{\delta}{\epsilon^2}}, \quad c_2 = 0. \]  

(1.13)

The purpose of this paper is to formulate the discrete model of LIE where the complex curvature of the discrete curves is governed by dNLS (1.4). In our formulation, the deformation of discrete curves is expressed in terms of the discrete Frenet frame with the coefficients given by the curvature and torsion of the curves explicitly. dNLS arises as the governing equation of the complex curvature, which is the same as the case of smooth curves. Based on this formulation, we present explicit formulas for the deformations of both smooth and discrete curves in terms of \( \tau \) functions of the two-component KP hierarchy. It is well-known that the matrix of the Frenet-Serret formula gives the Lax matrix of AKNS type by SO(3)–SU(2) correspondence [20] and that the AKNS hierarchy arises as a reduction of the two-component KP hierarchy [19, 33]. These formulas are consistent with this fact. For completeness, we also discuss the case of continuous deformation of discrete curves described by sdNLS (1.11).

2 Discrete models of LIE

2.1 Discrete space curves

Let \( \gamma_n \in \mathbb{R}^3 \) be a discrete space curve with

\[ |\gamma_{n+1} - \gamma_n| = \epsilon, \]  

(2.1)

where \( \epsilon \) is a constant. We introduce the \textit{discrete Frenet frame} \( \Phi_n = [T_n, N_n, B_n] \in \text{SO}(3) \) by

\[ T_n = \frac{\gamma_{n+1} - \gamma_n}{\epsilon}, \quad B_n = \frac{T_{n-1} \times T_n}{|T_{n-1} \times T_n|}, \quad N_n = B_n \times T_n. \]  

(2.2)

Then it follows that the discrete Frenet frame satisfies the \textit{discrete Frenet-Serret formula}

\[ \Phi_{n+1} = \Phi_n L_n, \quad L_n = R_1(-\nu_{n+1})R_3(\kappa_{n+1}), \]  

(2.3)

where

\[ R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, \quad R_3(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]  

(2.4)

and \( \nu_n, \kappa_n \) are defined by

\[ \langle T_{n-1}, T_n \rangle = \cos \kappa_n, \quad \langle B_n, B_{n-1} \rangle = \cos \nu_n, \quad \langle B_n, N_{n-1} \rangle = \sin \nu_n, \]  

\[ -\pi \leq \nu_n < \pi, \quad 0 < \kappa_n < \pi. \]  

(2.5)

We introduce the \textit{discrete complex curvature} \( u_n \) by

\[ u_n = \frac{2}{\epsilon} \tan \frac{\kappa_n}{2} e^{\sqrt{-1} \Lambda_n}, \quad \Lambda_n - \Lambda_{n-1} = -\nu_n. \]  

(2.6)
2.2 Continuous binormal flow

The binormal flow on the discrete space curves can be introduced by \([5, 10]\)

\[
\frac{dy_n}{dt} = \frac{2}{\epsilon} \tan \frac{\kappa_n}{2} B_n.
\] (2.7)

**Theorem 2.1.** If \(\gamma_n\) is deformed according to (2.7), the discrete complex curvature \(u_n\) satisfies

\[
\frac{1}{\sqrt{-1}} \frac{du_n}{dt} = \frac{1}{\epsilon^2} (1 + \frac{\epsilon^2}{4} |u_n|^2) (u_{n+1} + u_{n-1}) + \frac{c(t)}{\epsilon^2} u_n,
\] (2.8)

where \(c(t)\) is an arbitrary function in \(t\) depending on the boundary condition.

**Proof.** From (2.2) and (2.7), the Frenet frame \(\Phi_n\) satisfies

\[
\frac{d\Phi_n}{dt} = \Phi_n M_n,
\] (2.9)

where

\[
M_{21} = \frac{2}{\epsilon^2} \tan \frac{\kappa_{n+1}}{2} \sin \nu_{n+1},
\]

\[
M_{31} = \frac{2}{\epsilon^2} \left( \tan \frac{\kappa_{n+1}}{2} \cos \nu_{n+1} - \tan \frac{\kappa_n}{2} \right),
\] (2.10)

\[
M_{32} = \frac{2}{\epsilon^2} \sin \kappa_n \left( \cos \kappa_n \cos \nu_{n+1} \tan \frac{\kappa_{n+1}}{2} - (1 + \cos \kappa_n) \tan \frac{\kappa_n}{2} \cos \nu_n \tan \frac{\kappa_{n-1}}{2} \right).
\]

It follows that \(\kappa_n\) and \(\nu_n\) should satisfy

\[
\frac{d\kappa_n}{dt} = \frac{2}{\epsilon^2} \left( \sin \nu_{n+1} \tan \frac{\kappa_{n+1}}{2} - \sin \nu_n \tan \frac{\kappa_{n-1}}{2} \right),
\] (2.11)

so that the diagonal elements in \(M_n\) are zero and that \(M_n \in \mathfrak{so}(3)\). Then the compatibility condition of the Frenet-Serret formula (2.3) and the deformation equation (2.9)

\[
\frac{dL_n}{dt} = L_n M_{n+1} - M_n L_n
\] (2.12)

yields

\[
\frac{dv_n}{dt} = -(\chi_n - \chi_{n-1}),
\]

\[
\chi_n = \frac{1}{\epsilon^2} \left( \cot \frac{\kappa_n}{2} + \tan \frac{\kappa_n}{2} \right) \left( \cos \nu_{n+1} \tan \frac{\kappa_{n+1}}{2} + \cos \nu_n \tan \frac{\kappa_{n-1}}{2} \right),
\] (2.13)

which is summed up as

\[
\frac{d\Lambda_n}{dt} = \frac{1}{\epsilon^2} \left( \cot \frac{\kappa_n}{2} + \tan \frac{\kappa_n}{2} \right) \left( \cos \nu_{n+1} \tan \frac{\kappa_{n+1}}{2} + \cos \nu_n \tan \frac{\kappa_{n-1}}{2} \right) + c(t).
\] (2.14)

Here, \(c(t)\) is an arbitrary function in \(t\) depending on the boundary condition. Then (2.11) and (2.14) can be rewritten as a single equation in terms of \(u_n\), which is nothing but (2.8). This completes the proof. \(\Box\)
2.3 Discrete LIE

We now formulate the discrete deformation of discrete space curve as a discrete model of LIE (1.1). We start from dNLS (1.4), and identify \( u_n \) as the complex curvature of discrete space curve \( \gamma_n \in \mathbb{R}^3 \) where \( m \) is the number of iterations of discrete deformations. We impose the boundary condition as

\[
u_n^m \rightarrow 0 \ (n \rightarrow \pm \infty), \quad \Gamma_n^m \rightarrow \Gamma_{\pm \infty}^m \ (n \rightarrow \pm \infty) \tag{2.15}\]

Then one of the main statements of this paper is given as follows:

**Theorem 2.2** (discrete LIE). For a fixed \( m \), let \( \gamma_m^m \in \mathbb{R}^3 \) be a discrete space curve satisfying

\[
|\gamma_{n+1}^m - \gamma_n^m| = \epsilon, \tag{2.16}
\]

and \( \Phi_n^m = [T_n^m, N_n^m, B_n^m] \in SO(3) \) be the discrete Frenet frame defined in (2.2) satisfying the discrete Frenet-Serret formula

\[
\Phi_n^m = \Phi_n^m L_n^m, \quad L_n^m = R_1(-\gamma_n^m) R_2(\kappa_{n+1}^m). \tag{2.17}
\]

Let \( \rho_n^m \) be a complex discrete curvature of \( \gamma_n^m \). We compute \( \rho_n^{m+1} \) by dNLS (1.4) under the boundary condition (2.15) and determine \( \kappa_{n+1}^m \) and \( \Lambda_{n+1}^m \) by \( \rho_n^{m+1} = \frac{2}{\epsilon} \tan \frac{\kappa_{n+1}^m}{2} e^{\sqrt{2} \Lambda_{n+1}^m} \). We define a new curve \( \gamma_{n+1}^m \in \mathbb{R}^3 \) by

\[
\frac{\gamma_{n+1}^m - \gamma_n^m}{\delta} = \frac{2}{\epsilon^3} \left( P_n^m T_n^m + Q_n^m N_n^m + R_n^m B_n^m \right), \tag{2.18}
\]

\[
P_n^m = \delta \left( -1 + \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}} \right),
\]

\[
Q_n^m = \delta \left[ \tan \frac{\kappa_n^m}{2} - \tan \frac{\kappa_{n-1}^m}{2} \cos(\Lambda_{n-1}^m - \Lambda_n^m) \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}} \right], \tag{2.19}
\]

\[
R_n^m = \epsilon^2 \tan \frac{\kappa_n^m}{2} - \delta \tan \frac{\kappa_{n-1}^m}{2} \sin(\Lambda_{n-1}^m - \Lambda_n^m) \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}}.
\]

Suppose that \( \Gamma_{\infty}^m \) and \( \Gamma_{-\infty}^m \) are either 1 or \( 1 + \frac{\epsilon}{\delta^2} \). Then, it follows that

1. \( |\gamma_{n+1}^m - \gamma_n^{m+1}| = \epsilon \). Namely, \( \gamma_n^{m+1} \) is an isoperimetric deformation of \( \gamma_n^m \).
2. \( \rho_n^{m+1} \) gives the complex discrete curvature of \( \gamma_{n+1}^m \).

Equations (2.18) and (2.19) can be regarded as a discrete analogue of LIE (1.1), which will be referred to as the discrete LIE (dLIE). We note that dLIE is an implicit scheme in a sense that \( \gamma_n^{m+1} \) is determined by using \( \kappa_{n+1}^m \) and \( \Lambda_{n+1}^m \) which incorporate the information of \( \gamma_n^m \). This is resolved by using dNLS (1.4) to compute \( \kappa_{n+1}^m \) and \( \Lambda_{n+1}^m \), as described in Theorem 2.2. We will explain the details of how to compute numerically the deformation of curves given by dLIE in Section 2.5. We also remark that the deformation given by dLIE is not an equidistant deformation in contrast with the deformation described by dmKdV [17]. In fact, one can show the following proposition:

**Proposition 2.3.** Let \( \gamma_n^m \in \mathbb{R}^3 \) be the family of discrete space curves given in Theorem 2.2. Then it follows that

\[
\left| \frac{\gamma_{n+1}^m - \gamma_n^m}{\delta} \right|^2 = \frac{4}{\epsilon^2} \left( -1 + \frac{\Gamma_n^m}{\cos^2 \frac{\kappa_n^m}{2}} \right). \tag{2.20}
\]
Equation (2.20) also implies that the solution of dNLS (1.4) should satisfy the condition \( \Gamma_n^m \geq \cos^2 \frac{\kappa_n}{2} \) in order to be consistent with the curve deformation.

Continuous limit with respect to time can be simply taken as \( t = m \delta \) and \( \delta \to 0 \). Then dNLS (1.4) and corresponding deformation equation (2.18) and (2.19) yields the sdNLS (2.8) with \( c(t) = -2 \) and the binormal flow (2.7).

The dLIE (2.18) and (2.19) implies the following deformation of discrete Frenet frame:

**Proposition 2.4.** Let \( \gamma_n^m \in \mathbb{R}^3 \) be the family of discrete space curves given in Theorem 2.2, and \( \Phi_n^m \) be its discrete Frenet frame. Then \( \Phi_n^m \) satisfies

\[
\begin{align*}
\Phi_{n+1}^m &= \Phi_n^m M_n^m, \\
M_n^m &= \frac{1}{\Gamma_{n+1}^m} \begin{bmatrix}
|\alpha_n^m|^2 - |\beta_n^m|^2 & 2\Re(\alpha_n^m \beta_n^{m*}) & -2\Im(\alpha_n^m \beta_n^{m*}) \\
-2\Re(\alpha_n^m \beta_n^{m*}) & \Re((\alpha_n^m)^2 - (\beta_n^m)^2) & -\Im((\alpha_n^m)^2 + (\beta_n^m)^2) \\
-2\Im(\alpha_n^m \beta_n^{m*}) & -\Im((\alpha_n^m)^2 + (\beta_n^m)^2) & \Re((\alpha_n^m)^2 - (\beta_n^m)^2)
\end{bmatrix} \in \text{SO}(3),
\end{align*}
\]  

(2.21)

where \( \alpha_n^m, \beta_n^m \in \mathbb{C} \) are given by

\[
\begin{align*}
\alpha_n^m &= \sqrt{-1} \frac{\delta}{\epsilon^2} \left[ \left( 1 - \sqrt{-1} \frac{\epsilon^2}{\delta} \right) - \left( 1 + \frac{\epsilon^2}{4} u_{n+1}^m u_{n+1}^{m*} \Gamma_{n+1}^m \right) \right] e^{\frac{\sqrt{\epsilon}}{2}(\Lambda_{n+1}^m - \Lambda_n^m)}, \\
\beta_n^m &= \sqrt{-1} \frac{\delta}{2\epsilon} (u_n^{m+1} - u_{n+1}^m) \Gamma_{n+1}^m e^{-\frac{\sqrt{\epsilon}}{2}(\Lambda_{n+1}^m + \Lambda_{n+1}^m)},
\end{align*}
\]  

(2.22)

respectively. The compatibility condition with the discrete Frenet formula (2.17) \( L_n^m M_{n+1}^m = M_n^m L_{n+1}^m \) yields dNLS (1.4).

Note that the Frenet-Serret formula (2.17) and the deformation equation (2.21) can be transformed to the SU(2) version by the standard correspondence of SO(3) and SU(2) (see, for example, [17, 28]) as

\[
\begin{align*}
\phi_{n+1}^m &= \phi_n^m L_n^m, \\
L_n^m &= \begin{bmatrix}
\cos \frac{\kappa_n}{2} e^{-\frac{\sqrt{\epsilon}}{2}\gamma_n^m} & -\sin \frac{\kappa_n}{2} e^{-\frac{\sqrt{\epsilon}}{2}\gamma_n^m} \\
\sin \frac{\kappa_n}{2} e^{\frac{\sqrt{\epsilon}}{2}\gamma_n^m} & \cos \frac{\kappa_n}{2} e^{\frac{\sqrt{epsilon}}{2}\gamma_n^m}
\end{bmatrix} \in \text{SU}(2),
\end{align*}
\]  

(2.23)

\[
\begin{align*}
\phi_{n+1}^m &= \phi_n^m M_n^m, \\
M_n^m &= \frac{1}{\sqrt{\Gamma_{n+1}^m}} \begin{bmatrix}
\frac{\alpha_n^m}{\beta_n^{m*}} \\
-\frac{\beta_n^m}{\alpha_n^{m*}}
\end{bmatrix} \in \text{SU}(2),
\end{align*}
\]

which is known as the Lax pair of dNLS [1, 2].

### 2.4 Proof

In this section we give the proof of the results in Section 2.3. The first statement of Theorem 2.2 may be verified directly in principle, by computing \( \gamma_{n+1}^m - \gamma_n^m \) and its length from (2.18), (2.19) and the discrete Frenet-Serret formula (2.17) under the assumption that \( u_n^{m+1} \) is determined by dNLS (1.4). In fact, \( \gamma_{n+1}^m - \gamma_n^m \) can be expressed from (2.2), (2.17) and (2.18) as

\[
\begin{align*}
\gamma_{n+1}^m - \gamma_n^m &= (\gamma_{n+1}^m - \gamma_n^m) - (\gamma_{n+1}^m - \gamma_n^m) + (\gamma_n^m - \gamma_n^m) = \Phi_n^m \begin{bmatrix}
\epsilon \\
0
\end{bmatrix} + \frac{2\delta}{\epsilon^3} L_n^m \begin{bmatrix}
P_n^{m+1} \\
Q_n^{m+1}
\end{bmatrix} - \frac{2\delta}{\epsilon^3} \begin{bmatrix}
P_n^m \\
R_n^m
\end{bmatrix}.
\end{align*}
\]

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However, each entry has a complicated expression in terms of $\kappa_n^m$, $\gamma_n^m$ and $\Lambda_n^m$ to carry out further calculations. To make it feasible, we change the Frenet frame to an alternate frame used in \([8, 20]\), which we call the complex parallel frame in this paper. Let $F_n^m = [T_n^m, U_n^m, \Gamma_n^m] \in U(3)$ be the complex parallel frame defined by

$$U_n^m = e^{\frac{i}{\sqrt{2}} \Lambda_n^m} \left( \Lambda_n^m + \sqrt{1} \Gamma_n^m \right).$$  \hspace{1cm} (2.24)$$

The complex parallel frame $F_n^m$ is related to the discrete Frenet frame $\Phi_n^m$ as

$$F_n^m = \Phi_n^m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \sqrt{1} & -\sqrt{1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{i}{\sqrt{2}} \Lambda_n^m} & 0 \\ 0 & 0 & e^{-\frac{i}{\sqrt{2}} \Lambda_n^m} \end{bmatrix}.$$  \hspace{1cm} (2.25)$$

Note that for two real vectors $v, w \in \mathbb{R}^3$ expressed in terms of $F_n^m$ as

$$v = F_n^m \begin{bmatrix} p \\ q \\ q^* \end{bmatrix}, \quad w = F_n^m \begin{bmatrix} x \\ y \\ y^* \end{bmatrix},$$

their scalar product and vector product are given by

$$\langle v, w \rangle = px + qy^* + q^*y, \quad v \times w = F_n^m \sqrt{1} \begin{bmatrix} q^*y - qy^* \\ qx - py \\ -q^*x + py^* \end{bmatrix},$$  \hspace{1cm} (2.27)$$

respectively. In particular, we have

$$|v|^2 = p^2 + 2|q|^2.$$  \hspace{1cm} (2.28)$$

The advantage of complex parallel frame is that the complex curvature $u_n^m$ naturally arises in this framework; the discrete Frenet-Serret formula (2.17) and the deformation of the discrete curve are rewritten in terms of $u_n^m$ as

$$F_{n+1}^m = F_n^m X_n^m, \quad X_n^m = \frac{1}{1 + \frac{\epsilon^2}{4} |u_n^m|^2} \begin{bmatrix} 1 - \frac{\epsilon^2}{4} |u_{n+1}^m|^2 & -\frac{\epsilon}{\sqrt{2}} u_{n+1}^m & -\frac{\epsilon^2}{4} u_{n+1}^m \\ \frac{\epsilon}{\sqrt{2}} u_n^m & 1 & -\frac{\epsilon}{\sqrt{2}} u_n^m \\ \frac{\epsilon^2}{4} u_n^m & -\frac{\epsilon^2}{4} (u_{n+1}^m)^2 & 1 \end{bmatrix},$$  \hspace{1cm} (2.29)$$

and

$$\gamma_{n+1}^m = \gamma_n^m + \frac{2\delta^2}{\epsilon^3} F_n^m \left[ -1 + \left( 1 + \frac{\epsilon^2}{4} |u_n^m|^2 \right) \Gamma_n^m \\
\frac{\epsilon}{2\sqrt{2}} \left( -\sqrt{1} \frac{\epsilon^2}{\delta} + 1 \right) u_{n+1}^m - u_{n+1}^m + \left( 1 + \frac{\epsilon^2}{4} |u_n^m|^2 \right) \Gamma_n^m \\
\frac{\epsilon}{2\sqrt{2}} \left( \frac{\epsilon^2}{\delta} - 1 \right) u_n^m - u_n^m + \left( 1 + \frac{\epsilon^2}{4} |u_n^m|^2 \right) \Gamma_n^m \right],$$  \hspace{1cm} (2.30)$$

\text{Page 8}
respectively. We have

$$
\gamma_{n+1}^{m+1} - \gamma_{n+1}^{m} = \gamma_{n+1}^{m} - \gamma_{n}^{m} - \frac{\sqrt{2} \delta^2}{\epsilon^2} F_n^m \left[ \frac{\sqrt{\frac{\epsilon}{4}}}{\epsilon^2} \left( 1 + \sqrt{1 - \frac{\epsilon^2}{\delta}} \right) \right] \left[ \frac{1}{1 + \sqrt{1 - \frac{\epsilon^2}{\delta}}} \right] F_n^m \left[ \left( \frac{1}{1 + \sqrt{1 - \frac{\epsilon^2}{\delta}}} \right) \right] \left[ \left( \frac{1}{1 + \sqrt{1 - \frac{\epsilon^2}{\delta}}} \right) \right] \left[ \left( \frac{1}{1 + \sqrt{1 - \frac{\epsilon^2}{\delta}}} \right) \right] \\
= \epsilon F_n^m \left[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] - \frac{\sqrt{2}}{\Gamma_{n+1}^m} \left[ \begin{bmatrix} \sqrt{\frac{\epsilon}{4}} \beta_n^m \\ \alpha_n^m \beta_n^{m+1} \\ \alpha_n^m \beta_n^m \end{bmatrix} \right],
$$

(2.31)

where $\alpha_n^m$ and $\beta_n^m$ are given by

$$
\alpha_n^m = \alpha_n^m e^{-\frac{\sqrt{\epsilon}}{2} (\Lambda_n^{m+1} - \Lambda_n^{m})} = \sqrt{1 - \frac{\epsilon^2}{\delta^2}} \left( \left( 1 + \sqrt{1 - \frac{\epsilon^2}{\delta}} \right) \right) \left( \left( 1 + \sqrt{1 - \frac{\epsilon^2}{\delta}} \right) \right) \left( \left( 1 + \sqrt{1 - \frac{\epsilon^2}{\delta}} \right) \right) \left( \left( 1 + \sqrt{1 - \frac{\epsilon^2}{\delta}} \right) \right) \left( \left( 1 + \sqrt{1 - \frac{\epsilon^2}{\delta}} \right) \right),
$$

(2.32)

respectively. Then we obtain

$$
|\gamma_{n+1}^{m+1} - \gamma_{n+1}^{m}|^2 = \epsilon^2 \left( 1 + \frac{4|\beta_n^m|^2}{\Gamma_{n+1}^m} \right) \left( |\alpha_n^m|^2 + |\beta_n^m|^2 - \Gamma_{n+1}^m \right).
$$

(2.33)

In order to evaluate $|\alpha_n^m|^2 + |\beta_n^m|^2$, we introduce $\psi_n^m$ by

$$
\psi_n^m = \phi_n^m R_1(-\Lambda_n^m), \quad R_1(\theta) = \begin{bmatrix} e^{\frac{\sqrt{\epsilon}}{2} \theta} & 0 \\ 0 & e^{-\frac{\sqrt{\epsilon}}{2} \theta} \end{bmatrix},
$$

(2.34)

and consider a Lax pair of dNLS

$$
\psi_{n+1}^{m+1} = \psi_n^m \tilde{L}_n^m, \quad \tilde{L}_n^m = \frac{1}{\sqrt{1 + \frac{\epsilon^2}{\delta^2} u_{n+1}^{m+1}^2}} \begin{bmatrix} 1 & -\frac{\epsilon}{\delta} u_{n+1}^{m+1} \\ \frac{\epsilon}{\delta} u_{n+1}^{m+1} & 1 \end{bmatrix},
$$

$$
\psi_{n+1}^m = \psi_n^m \tilde{M}_n^m, \quad \tilde{M}_n^m = \frac{1}{\sqrt{\Gamma_{n+1}^m}} \begin{bmatrix} \alpha_n^m & \beta_n^m \\ -\beta_n^{m+1} & \alpha_n^m \end{bmatrix}.
$$

(2.35)

Note that the gauge transformation (2.34) preserves the compatibility conditions of associated linear problems; $\tilde{L}_n^m \tilde{M}_n^m = \tilde{M}_n^m \tilde{L}_n^{m+1}$ is equivalent to $L_n^m M_n^{m+1} = M_n^m L_n^{m+1}$. Indeed, one can verify directly that dNLS (1.4) is obtained from those compatibility conditions. Taking the determinant of both sides of $L_n^m M_n^{m+1} = M_n^m L_n^{m+1}$, we see from $\det \tilde{L}_n^m = 1$ that $\det \tilde{M}_n^{m+1} = \det \tilde{M}_n^m$, which implies

$$
\frac{1}{\Gamma_{n+1}^m} \left( |\alpha_n^m|^2 + |\beta_n^m|^2 \right) = C_m,
$$

(2.36)

for some $C_m$ depending only on $m$. Substituting (2.32) and taking the limit $n \to \pm \infty$, we see from the boundary condition (2.15) that $C_m = 1$, namely

$$
|\alpha_n^m|^2 + |\beta_n^m|^2 = \Gamma_{n+1}^m,
$$

(2.37)
and thus (2.33) gives $|\gamma_n^{m+1} - \gamma_n^{m+1}| = \epsilon$. This completes the proof of the first statement of Theorem 2.2. Consequently (2.31) yields

$$T_n^{m+1} = \frac{\gamma_n^{m+1} - \gamma_n^{m+1}}{\epsilon} = F_n^m \frac{1}{\Gamma_n^{m+1}} \begin{bmatrix} |\hat{\alpha}_n^m|^2 - |\hat{\beta}_n^m|^2 \end{bmatrix} - \sqrt{2} \hat{\alpha}_n^m \hat{\beta}_n^m. \quad (2.38)$$

We next proceed to the proof of the second statement. Our goal is to show

$$\langle T_n^{m+1}, T_n^{m+1} \rangle = \cos \kappa_n^{m+1}, \quad \langle B_n^{m+1}, B_n^{m+1} \rangle = \cos \nu_n^{m+1}, \quad \langle N_n^{m+1}, B_n^{m+1} \rangle = \sin \nu_n^{m+1}, \quad (2.39)$$

where $\kappa_n^{m+1}$ and $\nu_n^{m+1}$ are determined from dNLS (1.4), $T_n^{m+1}$, $B_n^{m+1}$ and $B_n^{m+1}$ are constructed from the Frenet frame $\Phi_n^m$ by using the deformation equation (2.18) and (2.19). Note that from the compatibility condition $\tilde{T}_n \tilde{M}_n^{m+1} = \tilde{M}_n \tilde{T}_n^{m+1}$ of (2.35) we have the following equality:

$$\begin{bmatrix} \hat{\alpha}_n^{m+1} \\ \hat{\beta}_n^{m+1} \end{bmatrix} = \frac{1}{1 + \frac{\epsilon^2}{4} |u_n^m|^2} \begin{bmatrix} 1 & \frac{\epsilon}{2} u_n^{m+1} \\ -\frac{\epsilon}{2} u_n^{m+1} & 1 \end{bmatrix} \frac{1}{\Gamma_n^{m+1}} \begin{bmatrix} |\hat{\alpha}_n^{m+1}|^2 - |\hat{\beta}_n^{m+1}|^2 \\ -\sqrt{2} \hat{\alpha}_n^{m+1} \hat{\beta}_n^{m+1} \end{bmatrix}, \quad (2.40)$$

which will be frequently used in the calculations below. Shifting $n \to n - 1$ in (2.38), the discrete Frenet-Serret formula (2.29) and (2.40) yields

$$T_n^{m+1} = F_n^m \frac{1}{1 + \frac{\epsilon^2}{4} |u_n^m|^2} \begin{bmatrix} 1 - \frac{\epsilon^2}{4} |u_n^m|^2 & \frac{\epsilon}{\sqrt{2}} u_n^m \\ -\frac{\epsilon}{\sqrt{2}} u_n^m & 1 \end{bmatrix} \frac{1}{\Gamma_n^{m+1}} \begin{bmatrix} |\hat{\alpha}_n^{m+1}|^2 - |\hat{\beta}_n^{m+1}|^2 \\ -\sqrt{2} \hat{\alpha}_n^{m+1} \hat{\beta}_n^{m+1} \end{bmatrix}, \quad (2.41)$$

where

$$x_n^m = \epsilon \left( u_n^{m+1} \hat{\alpha}_n^m \beta_n^m + u_n^{m+1} \hat{\beta}_n^m \beta_n^m \right), \quad y_n^m = \frac{\epsilon}{2} \left( u_n^{m+1} \hat{\alpha}_n^m \beta_n^m - u_n^{m+1} \hat{\beta}_n^m \beta_n^m \right). \quad (2.42)$$

Then we compute $\langle T_n^{m+1}, T_n^{m+1} \rangle$ by using (2.27), (2.38) and (2.41) as

$$\langle T_n^{m+1}, T_n^{m+1} \rangle = \frac{1 - \frac{\epsilon^2}{4} |u_n^m|^2}{1 + \frac{\epsilon^2}{4} |u_n^m|^2} \frac{1}{\Gamma_n^{m+1}} \begin{bmatrix} x_n^m \\ \sqrt{2} y_n^m \end{bmatrix} = \cos \kappa_n^{m+1},$$
which is the first equation of (2.39). In order to show the second equation of (2.39), first we need to compute $B_{n+1}^{m+1}$. Noticing (2.27) and (2.37), we have from (2.38), (2.41)

$$
T_{m+1}^{n} - T_{m+1} = \frac{1}{1 + \frac{e^2}{4} |u_{n+1}^m|^2} \Gamma_n^{m+1} \left[ -2y_n m \overline{\alpha}_n m \overline{\beta}_n + 2y_n m \overline{\alpha}_n m \overline{\beta}_n \right]
$$

We have

$$
\epsilon \sqrt{2} \left( u_{n+1} m \alpha_n m \overline{\beta}_n - u_{n+1} m \alpha_n m \overline{\beta}_n \right) + \epsilon \sqrt{2} \left( u_{n+1} m \alpha_n m \overline{\beta}_n \right) + \epsilon \sqrt{2} \left( u_{n+1} m \alpha_n m \overline{\beta}_n \right) + \epsilon \sqrt{2} \left( u_{n+1} m \alpha_n m \overline{\beta}_n \right)
$$

Then we obtain by using (1.4)

$$
|T_{m+1}^{n} - T_{m+1}| = \frac{\epsilon |u_{m+1}^n|}{1 + \frac{e^2}{4} |u_{m+1}^n|^2},
$$

which gives

$$
B_{n+1}^{m+1} = \frac{F_{n}^{m} \sqrt{-1}}{\sqrt{2} T_{m+1}^{n}} \left[ \sqrt{2} \left( e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n \right) \right] - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n
$$

Computing $B_{n+1}^{m+1}$ by using (2.29) and (2.40), we have after long but straightforward calculations

$$
B_{n+1}^{m+1} = \frac{F_{n}^{m} \sqrt{-1}}{\sqrt{2} T_{m+1}^{n}} \left[ \sqrt{2} \left( e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n \right) \right] - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n
$$

where

$$
C_{n} = \overline{\alpha}_n m - \frac{e}{2} u_{n+1} m \overline{\beta}_n, \quad D_{n} = \overline{\beta}_n + \frac{e}{2} u_{n+1} m \overline{\alpha}_n m.
$$

Then we obtain $\langle B_{n+1}^{m+1}, B_{n+1}^{m+1} \rangle$ by using (2.27) and (2.37) as

$$
\langle B_{n+1}^{m+1}, B_{n+1}^{m+1} \rangle = \frac{e^{\sqrt{-1} \Lambda m+1 n - \Lambda m+1 n}}{2} = \frac{e^{\sqrt{-1} \Lambda m+1 n} + e^{\sqrt{-1} \Lambda m+1 n}}{2} = e^{\sqrt{-1} \Lambda m+1 n} = \cos \nu_{m+1},
$$

which proves the second equation of (2.39). The third equation can be proved in a similar manner starting from

$$
\Lambda_{m+1} = B_{n+1}^{m+1} \times T_{n+1} = \frac{F_{n}^{m} \sqrt{-1}}{\sqrt{2} T_{m+1}^{n+1}} \left[ \sqrt{2} \left( e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n + e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n \right) \right] - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n - e^{\sqrt{-1} \Lambda m+1 n} \overline{\alpha}_n m \overline{\beta}_n
$$

(2.47)
This proves the second statement of Theorem 2.2.

Now let us next prove Proposition 2.4. From \( T_n^{m+1} \) (2.38), \( B_n^{m+1} \) (2.43), \( N_n^{m+1} \) (2.47), one can write \( F_n^{m+1} = [T_n^{m+1}, U_n^{m+1}, U_n^{m+1+}] \) as

\[
F_n^{m+1} = F_n^{m, y_n^m}, \quad = \frac{1}{\Gamma_n^{m}} \begin{bmatrix}
|\alpha_n|^2 - |\beta_n|^2 & \alpha_n^{m+} \beta_n^{m} - \alpha_n^{m+} \beta_n^{m} & \sqrt{-1} \left(\alpha_n^{m+} \beta_n^{m} - \alpha_n^{m+} \beta_n^{m}\right) \\
-\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m} & \left(\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m}\right) & \sqrt{-1} \left(\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m}\right) \\
-\sqrt{-1} \left(\alpha_n^{m+} \beta_n^{m} - \alpha_n^{m+} \beta_n^{m}\right) & -\sqrt{-1} \left(\alpha_n^{m+} \beta_n^{m} - \alpha_n^{m+} \beta_n^{m}\right) & \left(\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m}\right)
\end{bmatrix},
\]

(2.48)

Noticing from (2.32) that

\[
\alpha_n^{m} \beta_n^{m} = -\sqrt{-1} \Lambda_n^{m+1}, \quad (\alpha_n^{m})^2 = (\alpha_n^{m+})^2 e^{-\sqrt{-1}(\Lambda_n^{m+1} - \Lambda_n^{m})}, \quad (\beta_n^{m})^2 = (\beta_n^{m+})^2 e^{-\sqrt{-1}(\Lambda_n^{m+1} + \Lambda_n^{m})},
\]

(2.49)

and using (2.25), we obtain the deformation of the Frenet frame as

\[
\Phi_n^{m+1} = \Phi_n^{m} M_n^{m},
\]

(2.50)

\[
M_n^{m} = \frac{1}{\Gamma_n^{m+1}} \begin{bmatrix}
|\alpha_n|^2 - |\beta_n|^2 & 2 \Re(\alpha_n^{m+} \beta_n^{m}) & -2 \Im(\alpha_n^{m+} \beta_n^{m}) \\
-\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m} & \left(\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m}\right) & \left(\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m}\right) \\
-\Im(\alpha_n^{m+} \beta_n^{m}) & -\Im(\alpha_n^{m+} \beta_n^{m}) & \left(\alpha_n^{m+} \beta_n^{m} + \alpha_n^{m+} \beta_n^{m}\right)
\end{bmatrix},
\]

(2.51)

which proves Proposition 2.4.

Finally, Proposition 2.3 can be verified by direct computation by using (2.28), (2.30) and (2.37). Therefore we have proved all the statements in Section 2.3.

### 2.5 Numerical computations

Let us describe how to compute numerically the deformation of curves given by dLIE from a given initial curve. We first consider the case of sufficiently long curve with \( n_0 \leq n \leq n_1 \) for some fixed \( n_0, n_1 \) under the vanishing boundary condition, where the calculation is carried out from small \( n \) to large \( n \). Then, for a given \( m \), an algorithm to compute the deformation \( \gamma_n^{m+1} \) from a given initial curve \( \gamma_n^m \) is described as follows:

1. Give an initial curve \( \gamma_n^m \) for \( n_0 \leq n \leq n_1 \) for some fixed \( n_0 \) and \( n_1 \). At boundaries, give \( B_n^m, u_n^{m+1}, \Gamma_n^m \) and \( T_n^m, \Gamma_n^m \) may be chosen close to either 1 or 1 + \( \epsilon^2/\delta^2 \).

At the left edge \( n = n_0 \):

2. Compute \( \Phi_n^{m} = [T_n^m, N_n^m, B_n^m] \) from \( \gamma_n^m, \gamma_n^{m+1} \) and \( B_n^m \) by using (2.2).

3. Compute \( \gamma_n^{m+1} \) from \( u_n^m, u_n^{m+1}, \Gamma_n^m \) and \( \Phi_n^m \) by using (2.18) and (2.19).
Repeat from \( n = n_0 + 1 \) to \( n_1 - 1 \):

4. Compute \( \Phi^m_n = [T^m_n, N^m_n, B^m_n] \) from \( \gamma^m_{n+1}, \gamma^m_{n}, \Phi^m_{n-1} \) by using (2.2).

5. Compute \( u^m_n \) from \( \Phi^m_n \) and \( \Phi^m_{n-1} \).
   More precisely, compute \( \kappa^m_n \) and \( \nu^m_n \) by using (2.5). Then compute \( u^m_n \) by (2.6).

6. Compute \( u^m_{n-1} \) and \( \Gamma^m_n \) from \( u^m_{n-1}, u^m_{n-2}, u^m_{n-1} \) and \( \Gamma^m_{n-1} \) by using dNLS (1.4).

7. Compute \( \gamma^m_{n+1} \) from \( u^m_n, u^m_{n-1}, \Gamma^m_n \) and \( \Phi^m_n \) by using (2.18) and (2.19).

At the right edge \((n = n_1)\):

8. Compute \( \Phi^m_{n_1} = [T^m_{n_1}, N^m_{n_1}, B^m_{n_1}] \) from \( T^m_{n_1} \) and \( \Phi^m_{n_1-1} \) by using (2.2).

9. Compute \( u^m_{n_1} \) from \( \Phi^m_{n_1} \) and \( \Phi^m_{n_1-1} \).

10. Compute \( u^m_{n_1-1} \) and \( \Gamma^m_{n_1} \) from \( u^m_{n_1-1}, u^m_{n_1-2}, u^m_{n_1} \) and \( \Gamma^m_{n_1-1} \) by using dNLS (1.4).

11. Compute \( \gamma^m_{n_1} \) from \( u^m_{n_1}, u^m_{n_1-1}, \Gamma^m_{n_1} \) and \( \Phi^m_{n_1} \) by using (2.18) and (2.19).

Figure 1 illustrates a result of numerical computation according to this algorithm.

![Figure 1: Numerical simulation of dLIE.](image)

We next describe how to compute the deformation of a closed curve \( \gamma^m_n \) of period \( l \) in \( n \), namely \( \gamma^m_{n+l} = \gamma^m_n \). We note that it is possible to simulate such a periodic case by using dLIE (2.18)–(2.19), although we derived it under the vanishing boundary condition (2.15). Since the computation requires a solution to dNLS which has the same period as \( \gamma^m_n \), we fix a positive number \( c \) that plays a role of tolerance in constructing \( l \)-periodic numerical solution \((u^{m+1}_n, \Gamma^{m+1}_n)\) to dNLS. An algorithm to compute \( \gamma^{m+1}_n \) is described as follows:
(1) Give an initial closed curve $\gamma^m_n$ and positive numbers $\delta, c$.

(2) Compute $\Phi^m_n = [T^m_n, N^m_n, B^m_n]$ from $\gamma^m_{n+1}, \gamma^m_n$ and $\gamma^m_{n-1}$ by using (2.2).

(3) Compute $u^m_n$ from $\Phi^m_n$ and $\Phi^m_{n-1}$ by using (2.6) with $\Lambda^0_n = 0$.

(4) Give a pair of numbers $(u^{m,+1}_1, \Gamma^m_2) \in \mathbb{C} \times \mathbb{R}_{>0}$ which satisfies (2.37) with $n = 1$, that is,

\[
\begin{align*}
\frac{\delta^2}{\epsilon^4} & \left[ \left( 1 - \sqrt{-1} \frac{\epsilon^2}{\delta} \right) - \left( 1 + \frac{\epsilon^2}{4} u^m_{n} u^{m,+1}_{n} \right) \Gamma^m_n \right] \left[ \left( 1 + \sqrt{-1} \frac{\epsilon^2}{\delta} \right) - \left( 1 + \frac{\epsilon^2}{4} u^m_{n} u^{m,+1}_{n} \right) \Gamma^m_n \right] \\
& + \frac{\delta^2}{4 \epsilon^2} (u^{m,+1}_1 - u^m_2) (u^{m,+1}_1 - u^m_2) (\Gamma^m_2)^2 = \Gamma^m_2.
\end{align*}
\]

This procedure incorporates the isoperimetricity of the deformation.

(5) Compute an $l$-periodic solution $(u^{m,+1}_n, \Gamma^m_n)$ to dNLS (1.4) as follows: For $2 \leq n \leq l + 1$, compute $(u^{m,+1}_n, \Gamma^m_n)$ by

\[
\begin{align*}
u^{m,+1}_n &= \frac{1}{\sqrt{-1} \frac{\epsilon^2}{\delta} - 1} \left[ \left( \sqrt{-1} \frac{\epsilon^2}{\delta} + 1 \right) u^m_n - \left( u^{m,+1}_n + u^{m,+1}_{n-1} \right) \left( 1 + \frac{\epsilon^2}{4} |u^m_n|^2 \right) \Gamma^m_n \right], \\
\Gamma^m_{n+1} &= \frac{1 + \frac{\epsilon^2}{4} |u^m_{n+1}|^2}{1 + \frac{\epsilon^2}{4} |u^m_{n+1}|^2} \Gamma^m_n
\end{align*}
\]

with the initial value $(u^{m,+1}_1, \Gamma^m_2)$. If $|u^{m,+1}_1 - u^{m,+1}_1| > c$, we recompute $(u^{m,+1}_n, \Gamma^m_n)$ for $2 \leq n \leq l + 1$ with the revised initial value $(u^{m,+1}_1, \Gamma^m_2) = (u^{m,+1}_{l+1}, \Gamma^m_{l+2})$.

(6) Compute $\gamma^{m+1}_n$ from $u^m_n, u^{m,+1}_n, \Gamma^m_n$ and $\Phi^m_n$ by using (2.18) and (2.19).

Figure 2 and Figure 3 illustrate the result of numerical computations according to this algorithm.

Figure 2: Numerical simulation of dLIE for a closed curve of period $l = 4$. 

3 Explicit formulas

It is well-known that the \( N \)-soliton solution of NLS (1.2) and sdNLS (1.11) can be expressed in terms of double Wronskians or double Casorati determinants which are the (reduced) \( \tau \) functions of the two-component KP (or Toda) hierarchy [3, 26, 32]. However, dNLS (1.4) is not studied well compared to NLS (1.2) and sdNLS (1.11). Hirota and Ohta presented (1.4) in [29] and constructed soliton solutions by the perturbational technique. In [31], solutions in terms of double Casorati determinants have been given without complex structure. In this section, we first establish a parametrization of both smooth and discrete space curves in terms of \( \tau \) functions of two-component KP hierarchy with suitable reduction. Introducing deformation parameters appropriately, we construct explicit formulas of smooth/discrete space curves deformed by NLS, sdNLS and dNLS, together with regular soliton type solutions to those equations.

3.1 Explicit formulas for space curves

The following Proposition can be verified by direct calculation:

**Proposition 3.1.** For arbitrary parameters \( f, g, v \in \mathbb{C} \) satisfying \( ff^* + gg^* \neq 0 \) and \( |v| = \frac{1}{2} \), we define \( T, N, B \in \mathbb{R}^3 \) as

\[
T = \frac{1}{ff^* + gg^*} \begin{bmatrix}
  f^*g + fg^* \\
  (f^*g - fg^*)/\sqrt{-1} \\
  ff^* - gg^*
\end{bmatrix},
\]

\[
N = \frac{1}{ff^* + gg^*} \begin{bmatrix}
  \{(f^*)^2 - (g^*)^2\}v + \{(f)^2 - (g)^2\}v^* \\
  \left[\{(f^*)^2 + (g^*)^2\}v - \{(f)^2 + (g)^2\}v^*/\sqrt{-1}\right] \\
  -2(f^*g^*v + fgv^*)
\end{bmatrix},
\]

\[
B = \frac{1}{ff^* + gg^*} \begin{bmatrix}
  \{-\{(f^*)^2 - (g^*)^2\}v + \{(f)^2 - (g)^2\}v^*\}/\sqrt{-1} \\
  \{(f^*)^2 + (g^*)^2\}v + \{(f)^2 + (g)^2\}v^* \\
  2(f^*g^*v - fgv^*)/\sqrt{-1}
\end{bmatrix}.
\]
Then $\Phi = [T, N, B] \in \text{SO}(3)$.

Then it is possible to regard $\Phi$ as the Frenet frame of space curves by introducing appropriate dependence on arc-length parameter of the curves.

**Proposition 3.2.** Let $f, g, G \in \mathbb{C}$, $F \in \mathbb{R}_{>0}$ be the $\tau$ functions satisfying the following bilinear equations:

\begin{align*}
F^2 &= ff^* + gg^*, \\
D_x f \cdot F &= -g^* G, \\
D_x g \cdot F &= f^* G.
\end{align*}

Under the identification

\[ v = \frac{1}{2} \frac{G}{|G|}, \]

in equations (3.1)–(3.3), it follows that

\[ N = \frac{T'}{|T'|}, \quad B = T \times N, \]

so that $\Phi$ satisfies the Frenet-Serret formula (1.6).

**Remark 3.3.** Due to (3.4)–(3.7), the binormal vector $B$ admits an alternative expression as

\[ B = \frac{1}{2|G|F} \begin{bmatrix} (D_x (g^* \cdot f - g \cdot f^*))/\sqrt{-1} \\ D_x (g^* \cdot f + g \cdot f^*) \\ (D_x (f^* \cdot f - g^* \cdot g))/\sqrt{-1} \end{bmatrix}, \]

which will be used later.

Then we have an explicit formula in terms of the $\tau$ functions for the position vector of the curve $\gamma$ as follows:

**Proposition 3.4.** Let $H \in \mathbb{C}$ be the $\tau$ function satisfying the following bilinear equation

\[ D_x H \cdot F = f^* g. \]

We also introduce an auxiliary independent variable $z$ by

\[ \frac{1}{2} D_z D_x F \cdot F = gg^*. \]

Defining $\gamma$ as

\[ \gamma = \begin{bmatrix} H + H^* \\ \frac{1}{\sqrt{-1}} F \\ x - 2 \frac{\partial \log F}{\partial z} \end{bmatrix}, \]

it follows that $\gamma' = T$. 
The curvature $\kappa$, the torsion $\lambda$, and the complex curvature $u = \kappa e^{\sqrt{\gamma^2 T f} dx}$ also admit the explicit formulas in terms of the $\tau$ functions.

**Proposition 3.5.** It follows that
\[
\kappa = 2 \frac{|G|}{F}, \quad \lambda = \frac{1}{2 \sqrt{-1}} \frac{\partial}{\partial x} \left( \log \frac{G}{G^*} \right), \quad u = 2 \frac{G}{F}.
\] (3.13)

It is possible to construct the similar formulas for discrete space curves as follows.

**Proposition 3.6.** Let $f_n, g_n, G_n \in \mathbb{C}, \ F_n \in \mathbb{R}_{>0}$ be the $\tau$ functions satisfying the following bilinear equations:
\[
F_{n+1} F_n = f_n f_n^* + g_n g_n^*, \quad (3.14)
\]
\[
f_n F_n - f_{n-1} F_{n+1} = -g_n^* G_n, \quad (3.15)
\]
\[
g_n F_n - g_{n-1} F_{n+1} = f_n^* G_n. \quad (3.16)
\]

Under the identification
\[
f = f_n, \quad g = g_n, \quad v = \frac{1}{2} \frac{G_n}{|G_n|}, \quad T = T_n, \quad N = N_n, \quad B = B_n,
\] (3.17)
in equations (3.1)–(3.3), it follows that
\[
B_n = \frac{T_{n-1} \times T_n}{|T_{n-1} \times T_n|}, \quad N_n = B_n \times T_n,
\] (3.18)
so that $\Phi_n$ satisfies the discrete Frenet-Serret formula (2.3)–(2.5).

**Remark 3.7.** By using (3.15) and (3.16) and their complex conjugate, the binormal vector $B_n$ admits an alternative expression as
\[
B_n = \frac{1}{2 |G_n| F_n} \left[ \frac{(g_n^* f_{n-1} - f_n g_{n-1} + f_n^* g_{n-1})/\sqrt{-1}}{g_n^* f_{n-1} - f_n g_{n-1} + g_n f_n - f_n^* g_{n-1}} \right], \quad \tau_{n+1} \text{ is an arbitrary constant in } n,
\] (3.19)

Then we have an explicit formula in terms of the $\tau$ functions for the position vector of the discrete curve $\gamma_n$ as follows.

**Proposition 3.8.** Let $H_n \in \mathbb{C}$ be the $\tau$ function satisfying the following bilinear equation
\[
H_{n+1} F_n - H_n F_{n+1} = f_n^* G_n. \quad (3.20)
\]

We also introduce an auxiliary independent variable $z \in \mathbb{R}$ by
\[
D_z F_{n+1} \cdot F_n = g_n g_n^*.
\] (3.21)

Defining $\gamma_n$ as
\[
\gamma_n = \frac{1}{\sqrt{-1}} \frac{H_n - H_n^*}{F_n} + \frac{H_n + H_n^*}{F_n} + \frac{1}{\sqrt{-1}} \frac{H_n - H_n^*}{F_n} + \frac{H_n + H_n^*}{F_n} + n + \rho - 2 \frac{\partial \log F_n}{\partial z}.
\] (3.22)

where $\rho$ is an arbitrary constant in $n$, it follows that $(\gamma_{n+1} - \gamma_n)/\epsilon = T_n$. 

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\textbf{Proposition 3.9.} It follows that
\[ \kappa_n = 2 \arctan \frac{G_n}{F_n}, \quad \nu_n = \frac{1}{2 \sqrt{-1}} \log \frac{G_n^* G_{n-1}}{G_n G_{n-1}^*}, \quad u_n = \frac{2 G_n}{\epsilon F_n}. \] (3.23)

3.2 Proof of explicit formulas

\textit{Proof of Proposition 3.2}

We first replace the denominators of (3.1)–(3.3) by $F^2$ by using (3.4) and compute $T'$ by using the bilinear equations (3.5), (3.6) and their complex conjugate. For instance, the first entry is computed as:
\[
\left( \frac{f^* g + g^* f}{F^2} \right)' = \left( \frac{D_x f^* \cdot F}{F^2} \right) \frac{g}{F} + \frac{f}{F} \left( \frac{D_x g \cdot F}{F^2} \right) + \frac{f}{F} \left( \frac{D_x g^* \cdot F}{F^2} \right).
\]

Similarly, the second and third entries can be computed and we obtain $|T'|$ as
\[ |T'| = 2 \frac{|G|}{F}, \] (3.24)

which proves the first equation of (3.8). The second equation of (3.8) can be justified by the fact that $\Phi \in SO(3)$. \hfill \square

\textit{Proof of Proposition 3.4}

The derivative of the first and second entries of $\gamma$ can be computed by using (3.10) and its complex conjugate. The third entry is computed by using (3.11). Equation (3.4) is used to rewrite the denominators. \hfill \square

\textit{Proof of Proposition 3.5}

The first equation in (3.13) follows from (3.24) since $\kappa = |T'|$. Adding (3.5) multiplied by $f^*$, (3.6) multiplied by $g^*$ and (3.4) differentiated by $x$ and multiplied by $F/2$, we get
\[ D_x (f^* \cdot f + g^* \cdot g) = 0. \] (3.25)

Denoting $B = \{B_1, B_2, B_3\}$ and $N = \{N_1, N_2, N_3\}$ we have
\[
\lambda = -\langle B', N \rangle = -\frac{1}{2} \sum_{i=1}^{3} D_x B_i \cdot N_i
\]
\[
= \frac{1}{\sqrt{-1} F^4} D_x \left( (f^*)^2 - (g^*)^2 \right) v \cdot (f)^2 - (g)^2 \right) v^* + (f^*)^2 + (g^*)^2 \right) v^* + 4f^* g^* v \cdot f g v^*
\]
\[
= \frac{vv^*}{\sqrt{-1} F^4} D_x \left( (f^*)^2 - (g^*)^2 \right) \cdot (f)^2 - (g)^2 \right) + (f^*)^2 + (g^*)^2 \right) \cdot (f)^2 + (g)^2 \right) + 4f^* g^* \cdot f g
\]
\[
+ \frac{1}{\sqrt{-1} F^4} \left( (f^*)^2 - (g^*)^2 \right) \left( (f)^2 - (g)^2 \right) + (f^*)^2 + (g^*)^2 \left( (f)^2 + (g)^2 \right) + 4f^* g^* f g \right) (D_x v \cdot v^*),
\]
where the first term of the most right hand side vanishes due to (3.25). Thus we have

\[
\lambda = \frac{2(\mathbf{f}^\ast \mathbf{f}^\ast + \mathbf{g}^\ast \mathbf{g}^\ast)^2}{\sqrt{-1}F^4} \mathbf{D}_x (\mathbf{v} \cdot \mathbf{v}^\ast) = \frac{2\mathbf{v}\mathbf{v}^\ast}{\sqrt{-1}} \left( \frac{\log \mathbf{v}}{\mathbf{v}^\ast} \right)',
\]

where we used (3.4). By using (3.7) we obtain the second equation of (3.13)

\[
\lambda = \frac{1}{2\sqrt{-1}} \left( \frac{\log \mathbf{G}}{\mathbf{G}^\ast} \right)'.
\] (3.26)

The third equation follows automatically from the first and second equations.

\[\square\]

**Proof of Proposition 3.6**

Adding (3.15) multiplied by \(f_n^\ast\) and (3.16) multiplied by \(g_n^\ast\), we get

\[
F_n f_n = f_n^\ast f_{n-1} + g_n^\ast g_{n-1},
\] (3.27)

Similarly, subtracting (3.15) multiplied by \(g_{n-1}\) from (3.16) multiplied by \(f_{n-1}\), we get

\[
g_n f_{n-1} - f_n g_{n-1} = G_n F_n,
\] (3.28)

where we used (3.27).

We first replace the denominators of (3.1)–(3.3) by \(F_n + 1 F_n\) and compute \(T_n - 1\times T_n\). For example, the first entry turns out to be

\[
\frac{1}{\sqrt{-1}F_n^2 F_{n-1}} \left[ ((g_n^\ast f_{n-1} + f_n^\ast g_{n-1})(f_n f_n^\ast + g_n g_n^\ast) - (g_n f_{n-1} + f_n g_{n-1})(f_n^\ast f_{n-1} + g_n^\ast g_{n-1})) \right]
\]

\[
= \frac{1}{\sqrt{-1}F_n^2 F_{n-1}} (g_n^\ast f_{n-1} + f_n^\ast g_{n-1} - g_n f_{n-1}^\ast - f_n g_{n-1}),
\]

where we used (3.27) and its complex conjugate. Similarly, the second and third entries are computed, and we obtain

\[
|T_n - 1\times T_n|^2 = \frac{4}{(F_n + 1 F_n)^2} (g_n f_{n-1} - f_n g_{n-1}) (g_n f_{n-1}^\ast - f_n g_{n-1}^\ast) = \frac{4G_n G_n^\ast F_n^2}{(F_n + 1 F_n)^2},
\] (3.29)

where we have used (3.28). This proves that \((T_n - 1\times T_n)/|T_n - 1\times T_n|\) coincides with the expression of \(B_n\) (3.19) in Remark 3.7, thus the first equation of (3.18) is justified. The second equation of (3.18) is just a consequence of \(\Phi_n = [T_n, N_n, B_n] \in SO(3)\).

\[\square\]

**Proof of Proposition 3.8**

The first and second entries of \(\gamma_{n+1} - \gamma_n\) can be computed by using (3.20) and its complex conjugate. The third entry is computed by using (3.21). Equation (3.14) is used to rewrite the denominators.

\[\square\]
Proof of Proposition 3.9
From (3.14), (3.27) and (3.28), we have

\[ F_{n+1} F_n^2 F_{n-1} - F_n^4 = (g_n f_n - f_n g_n) (g_n^* f_n^* - f_n^* g_n^*) = G_n G_n^* F_n^2. \]  

(3.30)

Thus we get

\[ F_{n+1} F_{n-1} - F_n F_n = G_n G_n^*. \]  

(3.31)

We prove the first equation of (3.23) by showing (2.5). We have

\[
\langle T_{n-1}, T_n \rangle = \frac{1}{F_{n+1} F_n^2 F_{n-1}} \times \{(f_n f_n^* + g_n g_n^*) (f_{n-1} f_{n-1}^* + g_{n-1} g_{n-1}^*) - 2 (g_n f_{n-1} - f_n g_{n-1}) (g_n^* f_{n-1}^* - f_n^* g_{n-1}^*) \},
\]

which is rewritten by using (3.14), (3.28) and (3.31) as

\[
\langle T_{n-1}, T_n \rangle = 1 - 2 \frac{G_n G_n^*}{F_{n+1} F_{n-1}} = 1 - \frac{G_n^* (F_n)^2}{(F_n)^2} = \cos \kappa_n,
\]

so that

\[
\frac{|G_n|^2}{(F_n)^2} = \tan^2 \frac{\kappa_n}{2},
\]

(3.32)

which is the first equation of (3.23). Similarly, we get

\[
\langle B_n, B_{n-1} \rangle = \frac{G_n^* G_{n-1} + G_{n-1}^* G_n}{2 |G_n G_{n-1}|}, \quad \langle B_n, N_{n-1} \rangle = \frac{G_n^* G_{n-1} - G_{n-1}^* G_n}{2 \sqrt{-1} |G_n G_{n-1}|},
\]

which proves the second equation of (3.23). The third equation automatically follows from (2.6).

\[ \square \]

3.3 Parametrization by \( \tau \) functions of two-component KP hierarchy

The bilinear equations (3.4) – (3.6), (3.10) and (3.11) are simultaneously solved by the \( \tau \) functions of two-component KP hierarchy. Also, equations (3.14) – (3.16), (3.20) and (3.21) are solved by those of discrete two-component KP hierarchy.

Proposition 3.10. We define a \( 2N \times 2N \) determinant \( \tau(v, r) \) by

\[
\tau(v, r) = \begin{vmatrix}
\phi_1^{(0)}(r) & \phi_1^{(1)}(r) & \cdots & \phi_1^{(N+v-1)}(r) & -1 & -(-p_1) & \cdots & -(-p_1)^{N-v-1} \\
\phi_2^{(0)}(r) & \phi_2^{(1)}(r) & \cdots & \phi_2^{(N+v-1)}(r) & -1 & -(-p_2) & \cdots & -(-p_2)^{N-v-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_N^{(0)}(r) & \phi_N^{(1)}(r) & \cdots & \phi_N^{(N+v-1)}(r) & -1 & -(-p_N) & \cdots & -(-p_N)^{N-v-1} \\
1 & -p_1^* & \cdots & -p_1^* & 1 & -(-p_1^*)^{N+v-1} & \cdots & \phi_1^{(0)}(r) & \phi_1^{(1)}(r) & \cdots & \phi_1^{(N-v-1)}(r) \\
1 & -p_2^* & \cdots & -p_2^* & 1 & -(-p_2^*)^{N+v-1} & \cdots & \phi_2^{(0)}(r) & \phi_2^{(1)}(r) & \cdots & \phi_2^{(N-v-1)}(r) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -p_N^* & \cdots & -p_N^* & 1 & -(-p_N^*)^{N+v-1} & \cdots & \phi_N^{(0)}(r) & \phi_N^{(1)}(r) & \cdots & \phi_N^{(N-v-1)}(r)
\end{vmatrix},
\]

(3.34)
Proposition 3.12. In (3.34) and (3.35), we introduce the dependence on time $t$ by
\[
\zeta_i = \frac{1}{p_i} z + p_i x + \sqrt{-1} p_i^2 t + \zeta_{i0},
\] (3.40)
We define the \( \tau \) functions as (3.36). Then
\[
u = \frac{2}{\Gamma} \frac{G}{F},
\] (3.41)
satisfy NLS (1.2). Moreover, the Frenet frame \( \Phi \) in Proposition 3.1 and \( \gamma \) in Proposition 3.4 satisfy the deformation equation (1.7).

**Proposition 3.13.** In (3.37) and (3.38), we introduce the time \( t \) dependence by
\[
\zeta_i = \frac{p_i + 1}{p_i - 1} \frac{z}{2} + \frac{\sqrt{-1}}{\epsilon^2} \left( \frac{p_i + 1}{p_i} \right) t + \zeta_0.
\] (3.42)
We define the \( \tau \) functions as
\[
G_n = A \tau_n(1, 0), \quad g_n = -A \tau_{n+1}(1, 1), \quad H_n = A \tau_{n+1}(1, 2),
\]
\[
f_n = \tau_n(0, -1), \quad F_n = \tau_n(0, 0), \quad f_n^* = \tau_{n+1}(0, 1),
\] (3.43)
\[
H_n^* = A^{-1} \tau_{n-1}(-1, -2), \quad g_n^* = A^{-1} \tau_{n}(-1, -1), \quad G_n^* = A^{-1} \tau_{n}(-1, 0),
\]
where \( A = e^{\frac{2 \sqrt{\epsilon \delta \tau}}{\epsilon^2}} \). Then
\[
u_n = \frac{2}{\epsilon} \frac{G_n}{F_n},
\] (3.44)
satisfy sdNLS (1.11). Moreover, the Frenet frame \( \Phi_n \) in Proposition 3.1 and \( \gamma_n \) in Proposition 3.8 with \( \rho = 0 \) satisfy the deformation equation (2.7).

**Proposition 3.14.** In (3.37) and (3.38), we introduce the discrete time \( m \) dependence by replacing \( \zeta_i \) with
\[
\zeta_i = \frac{p_i + 1}{p_i - 1} \frac{z}{2} + m \log \left( \frac{1 - ap_i}{1 - \frac{z}{p_i}} \right) + \zeta_0,
\] (3.45)
where
\[
a = \frac{\Gamma_\infty}{\sqrt{-1 \delta^2} + 1}.
\] (3.46)
We suppose that all the variables depend also on \( m \), and write as \( \tau_n^m(\nu, r) \). We define the \( \tau \) functions as
\[
G_n^m = A^m \tau_n^m(1, 0), \quad g_n^m = -A^m B^m \tau_{n+1}^m(1, 1), \quad H_n^m = A^m B^{2m} \tau_{n+1}^m(1, 2),
\]
\[
f_n^m = B^{-m} \tau_{n+1}^m(0, -1), \quad F_n^m = \tau_n^m(0, 0), \quad f_n^m = B^m \tau_{n+1}^m(0, 1),
\] (3.47)
\[
H_n^{m*} = A^{-m} B^{-2m} \tau_{n-1}^m(-1, -2), \quad g_n^{m*} = A^{-m} B^{-m} \tau_{n}^m(-1, -1), \quad G_n^{m*} = A^{-m} \tau_{n}^m(-1, 0),
\]
where
\[
A = \sqrt{-1 \frac{\epsilon^2}{\delta}} + 1, \quad B = \left( \frac{\sqrt{-1 \frac{\epsilon^2}{\delta}} - 1 + \Gamma_\infty}{\sqrt{-1 \frac{\epsilon^2}{\delta}} + 1 - \Gamma_\infty} \right)^{1/2},
\] (3.48)
and \( \Gamma_\infty \) is either 1 or \( 1 + \epsilon^4 / \delta^2 \). Then
\[
u_n^m = \frac{2}{\epsilon} \frac{G_n^m}{F_n^m}, \quad \Gamma_n^m = \frac{\Gamma_\infty}{F_n^m F_{n-1}^m F_{n+1}^m},
\] (3.49)
satisfy dNLS (1.4). Moreover, the Frenet frame \( \Phi_n^m \) in Proposition 3.1 and \( \gamma_n^m \) in Proposition 3.8 with \( \rho = \rho_m = \frac{2 \delta^2}{\epsilon^2} (\Gamma_\infty - 1) m \) satisfy the deformation equation (2.18) and (2.19).
It is noted that in the above Proposition, only the solutions for \( \Gamma_{\infty}^m = \Gamma_{-\infty}^m \) are given and those with the different constant boundary condition, for instance, \( \Gamma_{\infty}^m = 1 \) and \( \Gamma_{-\infty}^m = 1 + \frac{c}{\delta^2} \), are not yet known. Proofs of Proposition 3.12, 3.13 and 3.14 will be given in Appendix.

### 3.5 Regularity of solutions

The formulas given in Section 3.4 give rise to regular solutions of NLS (1.2), sdNLS (1.11) and dNLS (1.4) so that the corresponding curve deformations are also regular. In this section, we establish the regularity of solutions which is guaranteed by the strict positivity of the \( \tau \) functions \( F, F_n \) and \( F_{mn} \) in Propositions 3.12, 3.13 and 3.14, respectively.

**Lemma 3.15.** Define a \( 2N \times 2N \) matrix \( M \) as

\[
M = \begin{pmatrix}
  a_1 & p_1 a_1 & \cdots & p_1^{N-1} a_1 & -1 & -q_1 & \cdots & -q_1^{N-1} \\
  a_2 & p_2 a_2 & \cdots & p_2^{N-1} a_2 & -1 & -q_2 & \cdots & -q_2^{N-1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_N & p_N a_N & \cdots & p_N^{N-1} a_N & -1 & -q_N & \cdots & -q_N^{N-1} \\
  1 & q_1^* & \cdots & (q_1^*)^{N-1} & a_1^* & p_1^* a_1^* & \cdots & (p_1^*)^{N-1} a_1^* \\
  1 & q_2^* & \cdots & (q_2^*)^{N-1} & a_2^* & p_2^* a_2^* & \cdots & (p_2^*)^{N-1} a_2^* \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & q_N^* & \cdots & (q_N^*)^{N-1} & a_N^* & p_N^* a_N^* & \cdots & (p_N^*)^{N-1} a_N^*
\end{pmatrix},
\]

where \( p_i, q_i, a_i \) (\( i = 1, \ldots, N \)) are arbitrary complex parameters. Then we have

\[
\det M = \begin{vmatrix}
  \mu_i \\
  p_i - q_j^* \\
\end{vmatrix}
\begin{vmatrix}
  -\delta_{ij} \\
  \delta_{ij} \\
\end{vmatrix}
\prod_{1 \leq i < j \leq N} (q_i - q_j)(q_i^* - q_j^*),
\]

(3.51)

where

\[
\mu_i = a_i \prod_{k=1}^{N} \frac{(p_i - q_k^*)}{(q_i - q_k)}.
\]

(3.52)

**Proof.** We introduce \( 2N \times 2N \) matrices \( P_k \) and \( Q_k \) as

\[
P_k = \begin{bmatrix}
    J_k^* & O \\
    O & J_k
\end{bmatrix}, \quad Q_k = \begin{bmatrix}
    K_k^* & O \\
    O & K_k
\end{bmatrix},
\]

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for \(1 \leq k \leq N - 1\), where \(O\) is zero matrix of size \(N\), \(J_k\) and \(K_k\) are \(N \times N\) matrices defined by

\[
J_k = \begin{bmatrix}
1 & -q_k & & \\
-1 & 1 & -q_k & \\
& & \ddots & -q_k \\
& & & 1
\end{bmatrix}, \quad K_k = \begin{bmatrix}
q_1 - q_{k+1} & \ddots & \\
& \ddots & \ddots & \\
& & 1 & 1 \\
0 & \ddots & \ddots & 0 \\
& \vdots & \vdots & \ddots \\
0 & \ddots & \ddots & 0
\end{bmatrix}
\]

respectively. Here, \(I_k\) is unit matrix of size \(k\). Then we have

\[
MP_1P_2 \cdots P_{N-1} = \begin{bmatrix}
\prod_{k=1}^{j-1} (p_i - q_k^*) & -\prod_{k=1}^{j-1} (q_i - q_k) \\
\prod_{k=1}^{j-1} (q_i^* - q_k^*) & a_i^* \prod_{k=1}^{j-1} (p_i^* - q_k)
\end{bmatrix},
\]

and

\[
MP_1P_2 \cdots P_{N-1}Q_1Q_2 \cdots Q_{N-1} = \begin{bmatrix}
\prod_{k=1, k \neq j}^{N} (p_i - q_k) & -\delta_{ij} \prod_{k=1, k \neq j}^{N} (q_i - q_k) \\
\delta_{ij} \prod_{k=1, k \neq i}^{N} (q_i^* - q_k^*) & a_i^* \prod_{k=1, k \neq j}^{N} (p_i^* - q_k)
\end{bmatrix}
\]

Thus we get (3.51). \(\square\)

We note that \(\det M\) is nonnegative in Lemma 3.15 because \(M\) has the form of

\[
M = \begin{bmatrix}
A & -B \\
B^* & A^*
\end{bmatrix}
\]

with \(N \times N\) matrices \(A\) and \(B\). In the following Propositions we give two cases of \(\det M\) being strictly positive.
Proposition 3.16. We assume $p_i \neq p_j$ for $1 \leq i \neq j \leq N$ and $\Re p_i > 0$ for $1 \leq i \leq N$. Then $F$ in Proposition 3.12 is strictly positive.

Proof. By taking $q_i = -p_i$ and $a_i = \phi^{(0)}_i(0)$ in Lemma 3.15, where $\phi^{(n)}_i(r)$ is given in (3.35), det $M$ coincides with $F$. Then (3.51) implies

$$
\det M = \begin{vmatrix}
p_i p_j & -\delta_{ij} \\
\mu \mu^* & \frac{1}{p_i + p_j}
\end{vmatrix}
\prod_{1 \leq i < j \leq N} (p_i - p_j)(p^*_i - p^*_j).
$$

When $\Re p_i > 0$, we can show that both of two $N \times N$ Hermite matrices

$$
\begin{bmatrix}
\mu \mu^* \\
p_i + p^*_j
\end{bmatrix}_{1 \leq i, j \leq N} \text{ and } \begin{bmatrix}
1 \\
p_i + p_j
\end{bmatrix}_{1 \leq i, j \leq N}
$$

are positive definite. Actually, for arbitrary non-zero vector $[v_1, v_2, \ldots, v_N]$ we have

$$
[v_1, v_2, \ldots, v_N] \begin{bmatrix}
\mu \mu^* \\
p_i + p^*_j
\end{bmatrix} \begin{bmatrix}
v_1^* \\
v_2^* \\
\vdots \\
v_N^*
\end{bmatrix} = \sum_{i,j=1}^{N} \frac{\mu \mu^*}{p_i + p^*_j} v_i v^*_j = \int_{-\infty}^{0} \sum_{i,j=1}^{N} \mu \mu^* v_i v^*_j e^{(p_i + p^*_j)y} \ dy
$$

$$
= \int_{-\infty}^{0} \left| \sum_{i=1}^{N} \mu v_i e^{(p_i + p^*_j) y} \right|^2 \ dy \geq 0.
$$

Therefore det $M$ is strictly positive, i.e., det $M > 0$. This completes the proof. \qed

Proposition 3.17. We assume $p_i \neq p_j$ for $1 \leq i \neq j \leq N$ and $|p_i| > 1$ for $1 \leq i \leq N$. Then $F_n$ in Proposition 3.13 and $F^m_n$ in Proposition 3.14 are strictly positive.

Proof. By taking $q_i = 1/p_i$ and $a_i = \phi^{(n)}_i(0)$ in Lemma 3.15, where $\phi^{(n)}_i(r)$ is given in (3.38) with (3.42) or (3.45), then det $M$ coincides with $F_n$ or $F^m_n$. Then we have from (3.51)

$$
\det M = \begin{vmatrix}
p_i \mu \mu^* p^*_j & -\delta_{ij} \\
p_i p^*_j - 1 & 1
\end{vmatrix}
\prod_{1 \leq i < j \leq N} \left( \frac{1}{p_i} - \frac{1}{p_j} \right) \left( \frac{1}{p^*_i} - \frac{1}{p^*_j} \right).
$$

\[25\]
When $|p_i| > 1$, we can show that both of two $N \times N$ Hermite matrices

$$
\begin{bmatrix}
  p_i \mu_i \mu_j \mu^*_j \\
  p_i p_j^* - 1
\end{bmatrix}_{1 \leq i, j \leq N},
\begin{bmatrix}
  \frac{1}{p_i^* p_j - 1}
\end{bmatrix}_{1 \leq i, j \leq N}
$$

are positive definite. Actually, for arbitrary non-zero vector $[v_1, v_2, \ldots, v_N]$ we have by noting $|p_i| > 1$

$$
[v_1, v_2, \ldots, v_N] \begin{bmatrix}
  p_i \mu_i \mu_j \mu^*_j \\
  p_i p_j^* - 1
\end{bmatrix} \begin{bmatrix}
  v_1^* \\
  v_2^* \\
  \vdots \\
  v_N^*
\end{bmatrix} = \sum_{i, j=1}^{N} \frac{\mu_i \mu^*_j}{1 - \frac{1}{p_i p_j^*}} v_i v_j^* = \sum_{i, j=1}^{N} \mu_i v_i \mu^*_j v_j^* \sum_{n=0}^{\infty} \left( \frac{1}{p_i p_j} \right)^n
$$

$$
= \sum_{n=0}^{\infty} \left( \sum_{i=1}^{N} \mu_i v_i \frac{1}{p_i^n} \right)^2 \geq 0.
$$

Therefore $\det M$ is strictly positive, i.e., $\det M > 0$. This completes the proof. □

4 Concluding Remarks

In this paper, we have constructed a discrete analogue of local induction equation describing the deformation of discrete space curves whose complex curvature is governed by the dNLS equation. We have shown some numerical simulations for both open and closed curves. We have constructed explicit formulas of space curves and related quantities in terms of the $\tau$ functions of the two-component KP hierarchy. By introducing appropriate time dependence, we have constructed regular soliton type solutions to the NLS, sdNLS and dNLS equations and explicit formulas for the deformation of space curves associated with those solutions.

The relationship to ddIHIM [13, 14] is not yet clear in this paper. Actually, it can be shown that ddIHIM is equivalent to the isotropic version of the discrete Landau-Lifschitz equation presented by Nijhoff et al [24]. Then the complex curvature of the discrete curves deformed by ddIHIM is governed by a variant of dNLS type equation which is closely related to the dNLS (1.4). Since the Landau-Lifschitz equation belongs to the BKP hierarchy [4], it is expected that we have an alternate explicit formulas of space curves in terms of the Pfaffians, namely, $\tau$ functions of the BKP hierarchy. These results will be reported in the forthcoming paper.

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A Proofs of Proposition 3.10 and 3.11

We give a proof of Proposition 3.10. Firstly we observe that \( \tau(v, r) \) defined by (3.34) and (3.35) satisfies

\[
\tau(v, r) = (-1)^r \tau(-v, -r),
\]

which verifies that \( \tau(0, 0) \) is real, \( \tau(0, -1) = \tau(0, 1) = \tau(1, -1) = \tau(1, 0) = \tau(-1, 0) \) and \( \tau(1, 2) = \tau(-1, -2) \) in (3.36). Let us define a \( 2N \times 2N \) determinant \( \sigma(v, r, s) \) by

\[
\sigma(v, r, s) = \begin{vmatrix}
\phi_1^{(0)}(r) & \phi_1^{(1)}(r) & \cdots & \phi_1^{(N+1-1)}(r) & \psi_1^{(0)}(s) & \psi_1^{(1)}(s) & \cdots & \psi_1^{(N-1)}(s) \\
\phi_2^{(0)}(r) & \phi_2^{(1)}(r) & \cdots & \phi_2^{(N+1-1)}(r) & \psi_2^{(0)}(s) & \psi_2^{(1)}(s) & \cdots & \psi_2^{(N-1)}(s) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_{2N}^{(0)}(r) & \phi_{2N}^{(1)}(r) & \cdots & \phi_{2N}^{(N+1-1)}(r) & \psi_{2N}^{(0)}(s) & \psi_{2N}^{(1)}(s) & \cdots & \psi_{2N}^{(N-1)}(s)
\end{vmatrix},
\]

(A.2)

where

\[
\phi_i^{(n)}(r) = p_i^{n-r} e^i, \quad \psi_i^{(n)}(s) = (-p_i)^{n-s} e^i, \quad \zeta_i = \frac{1}{p_i} z + p_i x + \zeta_i,
\]

and \( \eta_i \) are constants. It is easy to see that \( \sigma(v, r, s) \) satisfies

\[
\sigma(v, r - 1, s - 1) = \sigma(v, r, s) (-1)^{N-1} \prod_{i=1}^{2N} p_i.
\]

(A.4)

Now the following bilinear equations can be proved straightforwardly by the Laplace expansion technique,

\[
\sigma(v, r + 1, s + 1) \sigma(v, r, s) - \sigma(v, r + 1, s) \sigma(v, r, s + 1)
= \sigma(v + 1, r + 1, s) \sigma(v - 1, r, s + 1),
\]

(A.5)

\[
D_\tau \sigma(v, r, s + 1) \cdot \sigma(v, r, s) = -\sigma(v + 1, r, s) \sigma(v - 1, r, s + 1),
\]

(A.6)

\[
D_\tau \sigma(v + 1, r + 1, s) \cdot \sigma(v, r, s) = \sigma(v + 1, r, s) \sigma(v, r + 1, s),
\]

(A.7)

\[
D_\tau \sigma(v + 1, r + 1, s - 1) \cdot \sigma(v, r, s) = \sigma(v + 1, r, s - 1) \sigma(v, r + 1, s),
\]

(A.8)

\[
\left(\frac{1}{2} D_\tau^2 - 1\right) \sigma(v, r, s) \cdot \sigma(v, r, s) = -\sigma(v, r + 1, s) \sigma(v, r - 1, s).
\]

(A.9)

For notational simplicity, let us denote columns of determinant as

\[
\sigma(v, r, s) = | -r \quad 1 - r \quad \cdots \quad N + v - 1 - r; \quad -s \quad 1 - s \quad \cdots \quad N - v - 1 - s |
\]

(A.10)

where “\( n - r \)” in the left block means the column vector \( i(\phi_1^{(n)}(r), \phi_2^{(n)}(r), \cdots, \phi_{2N}^{(n)}(r)) \) and “\( n - s \)” in the right block means \( i(\psi_1^{(n)}(s), \psi_2^{(n)}(s), \cdots, \psi_{2N}^{(n)}(s)) \). In this notation, we have the expressions

\[
\frac{\partial}{\partial x} \sigma(v, r, s) = | -r \quad \cdots \quad N + v - 2 - r \quad N + v - r; \quad -s \quad 1 - s \quad \cdots \quad N - v - 1 - s |
\]

(A.11)

\[
\frac{\partial}{\partial z} \sigma(v, r, s) = | -1 - r \quad \cdots \quad N + v - 1 - r; \quad -s \quad 1 - s \quad \cdots \quad N - v - 1 - s |
\]

(A.12)

\[
\left(\frac{\partial^2}{\partial z \partial x} - 1\right) \sigma(v, r, s) = | -1 - r \quad \cdots \quad N + v - 2 - r \quad N + v - r; \quad -s \quad 1 - s \quad \cdots \quad N - v - 1 - s |
\]

(A.13)
For example (A.6) is proved by applying the Laplace expansion to the $4N \times 4N$ vanishing determinant in the left-hand side of the following identity,

\[
\begin{vmatrix}
-r \cdots -N+v-2-r & N+v-r & -1-s & -s \cdots -N-v-2-s \\
\phantom{0} & \mathcal{O} & \phantom{0} & \phantom{0} \\
\phantom{0} & \phantom{0} & \mathcal{O} & \phantom{0} \\
\phantom{0} & \phantom{0} & \phantom{0} & \mathcal{O}
\end{vmatrix}
= 0. \quad \text{(A.14)}
\]

In each term of the bilinear equation (A.6), two determinants $\sigma$’s have common columns

\[ -r, \cdots, N + v - 2 - r; -s, \cdots, N - v - 2 - s, \quad \text{(A.15)} \]

and distinct columns

\[ N + v - 1 - r, N + v - r; -1 - s, N - v - 1 - s \quad \text{(A.16)} \]

which are dealt to each $\sigma$. The other bilinear equations are proved in the same way with the different choice of common and distinct columns as shown below.

| Equation | common columns | distinct columns |
|----------|----------------|-----------------|
| (A.5)    | $-r, \cdots, N + v - 2 - r; -s, \cdots, N - v - 2 - s$ | $-1 - r, N + v - 1 - r; -1 - s, N - v - 1 - s$ |
| (A.7)    | $-r, \cdots, N + v - 2 - r; -s, \cdots, N - v - 2 - s$ | $-1 - r, N + v - 1 - r, N + v - r; -N - v - 1 - s$ |
| (A.8)    | $-r, \cdots, N + v - 2 - r; 1 - s, \cdots, N - v - 1 - s$ | $-1 - r, N + v - 1 - r, N + v - r; -s$ |
| (A.9)    | $1 - r, \cdots, N + v - 2 - r; -s, \cdots, N - v - 1 - s$ | $-1 - r, -r, N + v - 1 - r, N + v - r;$ |

By taking

\[ p_{N+i} = -p_i^*, \quad \zeta_{N+i, 0} = -\zeta_{i, 0}, \quad e_{n_1} = -1, \quad e_{n_{N+1}} = 1, \quad \text{(A.17)} \]

for $1 \leq i \leq N$, $\sigma$ defined by (A.2) and (A.3) reduces to $\tau$ in (3.34) up to gauge, i.e.,

\[ \tau(v, r) = \sigma(v, r, 0) (-1)^v \prod_{i=1}^{N} (-p_i^*)^\nu e_{i}^\nu. \quad \text{(A.18)} \]

Thus with the help of reduction condition (A.4), the bilinear equations (3.4)–(3.6) and (3.10) are derived from (A.5)–(A.8), respectively, and (3.11) is derived from (A.5) and (A.9). \[ \square \]

Proposition 3.11 can be proved in the same manner as above. We observe that $\tau_n(v, r)$ defined by (3.37) and (3.38) satisfies

\[ \tau_n(v, r)^* = (-1)^v \tau_{n-r}(-v, -r), \quad \text{(A.19)} \]

which verifies that $\tau_n(0, 0)$ is real, $\tau_n(0, -1)^* = \tau_{n+1}(0, 1)$, $\tau_{n+1}(1, 1)^* = -\tau_n(-1, -1)$, $\tau_n(1, 0)^* = \tau_n(-1, 0)$ and $\tau_{n+1}(1, 2)^* = \tau_{n-1}(-1, -2)$ in (3.39). We define a $2N \times 2N$ determinant $\sigma_{nm}(v, r, s)$ by

\[
\sigma_{nm}(v, r, s) = \begin{bmatrix}
\phi_1^{(n)}(r) & \phi_1^{(n+1)}(r) & \cdots & \phi_1^{(n+N-v-1)}(r) & \psi_1^{(n)}(s) & \psi_1^{(n+1)}(s) & \cdots & \psi_1^{(n+N-v-1)}(s) \\
\phi_2^{(n)}(r) & \phi_2^{(n+1)}(r) & \cdots & \phi_2^{(n+N-v-1)}(r) & \psi_2^{(n)}(s) & \psi_2^{(n+1)}(s) & \cdots & \psi_2^{(n+N-v-1)}(s) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_{2N}^{(n)}(r) & \phi_{2N}^{(n+1)}(r) & \cdots & \phi_{2N}^{(n+N-v-1)}(r) & \psi_{2N}^{(n)}(s) & \psi_{2N}^{(n+1)}(s) & \cdots & \psi_{2N}^{(n+N-v-1)}(s)
\end{bmatrix}, \quad \text{(A.20)}
\]

where

\[
\phi_i^{(n)}(r) = p_i^n (1 - p_i)^{-r} e_i^\nu, \quad \psi_i^{(n)}(s) = \left(1 - p_i\right)_{n}^\nu (1 - p_i)^{-s} e_i^\nu, \quad \zeta_i = \frac{p_i + 1}{p_i - 1} z + \zeta_{i, 0}, \quad \text{(A.21)}
\]

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and $\eta_i$ are constants. Then $\sigma_{nr'}(v, r, s)$ satisfies

$$
\sigma_{n+1,n'-1}(v, r, s) = \sigma_{nr'}(v, r, s) \prod_{i=1}^{2N} p_i, \quad \sigma_{nr'}(v, r-1, s-1) = \sigma_{nr'}(v, r, s) \prod_{i=1}^{2N} (1 - p_i).
$$

(A.22)

The following bilinear equations can be proved by the same technique as Proposition 3.10:

$$
\begin{align*}
\sigma_{n+1,n'}(v, r + 1, s) \sigma_{nr'}(v, r, s + 1) - \sigma_{n+1,n'}(v, r + 1, s + 1) \sigma_{nr'}(v, r, s) \\
= \sigma_{n,n'+1}(v + 1, r + 1, s) \sigma_{n+1,n'-1}(v - 1, r, s + 1),
\end{align*}
$$

(A.23)

$$
\begin{align*}
\sigma_{n+1,n'+1}(v, r + 1, s) \sigma_{nr'}(v, r, s) - \sigma_{n+1,n'}(v + 1, r + 1, s) \sigma_{n+1,n'-1}(v, r, s) \\
= \sigma_{n,n'+1}(v + 1, r + 1, s) \sigma_{n+1,n'}(v - 1, r, s),
\end{align*}
$$

(A.24)

$$
\begin{align*}
\sigma_{nr'}(v + 1, r + 1, s) \sigma_{n+1,n'}(v, r, s) - \sigma_{nr'}(v + 1, r + 1, s + 1) \sigma_{n+1,n'}(v, r, s) \\
= \sigma_{n,n'+1}(v + 1, r + 1, s) \sigma_{n,n'+1}(v, r + 1, s + 1),
\end{align*}
$$

(A.25)

$$
\begin{align*}
\sigma_{nr'}(v + 1, r + 1, s) \sigma_{n+1,n'}(v, r, s + 1) - \sigma_{n+1,n'}(v + 1, r + 1, s + 1) \sigma_{nr'}(v, r, s) \\
= \sigma_{n,n'+1}(v + 1, r + 1, s) \sigma_{n,n'-1}(v, r, s + 1),
\end{align*}
$$

(A.26)

$$
D_z \sigma_{n',n''}(v, r, s) \cdot \sigma_{nr'}(v, r, s) = -\sigma_{n,n'+1}(v + 1, r + 1, s) \sigma_{nr'}(v - 1, r - 1, s).
$$

(A.27)

We denote columns of determinant as

$$
\sigma_{nr'}(v, r, s) = \begin{vmatrix} n_{r,s} & n_{r+1,s} & \cdots & n_{N-1,s} & n'_{r,s} & n'_{r+1,s} & \cdots & n'_{N-1,s} \end{vmatrix},
$$

(A.28)

where $n'$ in the left block means the column vector $^{(n)}(\phi_1(r), \phi_2(r), \cdots, \phi_{2N}(r))$ and $n'$ in the right block means $^{(n)}(\psi_1(s), \psi_2(s), \cdots, \psi_{2N}(s))$. In this notation, we have the expressions

$$
\begin{align*}
\sigma_{nr'}(v, r + 1, s) &= \begin{vmatrix} n_{r,s} & n_{r+1,s} & \cdots & n_{N-2,s} & n_{r,s} & n_{r+1,s} & \cdots & n_{N-1,s} & n'_{r,s} & n'_{r+1,s} & \cdots & n'_{N-1,s} \end{vmatrix}, \\
\sigma_{nr'}(v, r, s + 1) &= \begin{vmatrix} n_{r,s} & n_{r+1,s} & \cdots & n_{N-1,s} & n_{r,s} & n_{r+1,s} & \cdots & n_{N-1,s} & n'_{r,s} & n'_{r+1,s} & \cdots & n'_{N-1,s} \end{vmatrix},
\end{align*}
$$

(A.29) (A.30) (A.31) (A.32)

$$
\sigma_{nr'}(v, r + 1, s + 1) = \begin{vmatrix} n_{r,s} & n_{r+1,s} & \cdots & n_{N-2,s} & n_{r,s} & n_{r+1,s} & \cdots & n_{N-2,s} & n'_{r,s} & n'_{r+1,s} & \cdots & n'_{N-2,s} \end{vmatrix}.
$$

(A.33)

For example (A.23) is proved by applying the Laplace expansion to the $4N \times 4N$ vanishing determinant in the left-hand side of the following identity,

$$
\begin{vmatrix}
\begin{array}{cccc}
\ast & \cdots & \ast & \ast \\
\ast & \cdots & \ast & \ast \\
\ast & \cdots & \ast & \ast \\
\ast & \cdots & \ast & \ast \\
\end{array}
\end{vmatrix}
= 0.
$$

(A.34)

The common and distinct columns of each bilinear equation are shown below.

| Equation | common columns | distinct columns |
|----------|----------------|-----------------|
| (A.23)   | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ |
| (A.24)   | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ |
| (A.25)   | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ |
| (A.26)   | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ |
| (A.27)   | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ | $n_{r,s}$, $n_{r+1,s}$, $\cdots$, $n_{N-1,s}$, $n'_{r,s}$, $n'_{r+1,s}$, $\cdots$, $n'_{N-1,s}$ |
Note that we use the following formulas for proving (A.27):

\[
\sigma_{nn'}(\nu, r, s) = \left| n_{r_{N+1}} \cdots n_{r_1} n'_{s_{N+1}} \cdots n'_{s_1} \right|, \tag{A.35}
\]

\[
\left( -\frac{\partial}{\partial \zeta} + \frac{N + \nu}{2} \right) \sigma_{nn'}(\nu, r, s) = \left| n_{r_{N+1}} \cdots n_{r_1} n'_{s_{N+1}} \cdots n'_{s_1} \right|. \tag{A.36}
\]

By taking

\[
p_{N+i} = \frac{1}{p_i^*}, \quad \zeta_{N+i,0} = -\zeta_i^*, \quad e^{i\theta} = -1, \quad e^{in+i'\theta} = 1, \tag{A.37}
\]

for \(1 \leq i \leq N\), \(\sigma_{nn'}\) defined by (A.20) and (A.21) reduces to \(\tau_n\) in (3.37) up to gauge, i.e.,

\[
\tau_n(\nu, r) = \sigma_{n0}(\nu, r, 0) \prod_{i=1}^{N} (p_i^*)^{n_i} \left( 1 - \frac{1}{p_i^*} \right)^{n_i} e^{i\zeta_i}. \tag{A.38}
\]

Thus with the help of reduction condition (A.22), the discrete bilinear equations (3.14)–(3.16), (3.20) and (3.21) are derived from (A.23)–(A.27), respectively.

\[\square\]

### B  Proofs of Proposition 3.12, 3.13 and 3.14

We first prove Proposition 3.12. Under the \(t\) dependence introduced in (3.40), it can be shown that the \(\tau\) functions satisfy the following bilinear equations:

\[
\sqrt{-1}D_t H \cdot F = D_x f^* \cdot g, \tag{B.1}
\]

\[
\frac{\sqrt{-1}}{2} D_z D_t F \cdot F = D_x f \cdot f^*, \tag{B.2}
\]

\[
\left( \sqrt{-1}D_t + D_x^2 \right) G \cdot F = 0, \tag{B.3}
\]

\[
\frac{1}{2} D_x^2 F \cdot F = GG^*. \tag{B.4}
\]

Equations (1.2) and (1.7) are verified directly from (B.1)–(B.4) by using (3.9), (3.12) and (3.13). For example, dividing (B.1) by \(\sqrt{-1}F^2\) we have

\[
\frac{\partial}{\partial t} \left( \frac{H}{F} \right) = \kappa \frac{1}{2|G|F} \frac{1}{\sqrt{-1}} D_x f^* \cdot g. \tag{B.5}
\]

Taking the real part of (B.5), we obtain

\[
\frac{\partial}{\partial t} \left( \frac{H + H^*}{F} \right) = \kappa \frac{1}{2|G|F} \frac{1}{\sqrt{-1}} D_x (g^* \cdot f - g \cdot f^*), \tag{B.6}
\]

which gives the first entry of (1.7) by using (3.9) and (3.12). The second entry of (1.7) is verified in the same way. The third entries is obtained by dividing (B.2) by \(F^2\) and using (3.25). Note that (3.25) can be derived from (3.4)–(3.6). NLS (1.2) follows from (B.3) and (B.4).

We consider the determinant \(\sigma\) given in (A.2) with \(\zeta_i\) in (3.40). We also introduce \(y\) dependence in \(\eta_i\) as

\[
\eta_i = -p_i y. \tag{B.7}
\]
Then we see that \( \sigma \) satisfies
\[
\left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \sigma(v, r, s) = \left( \sum_{i=1}^{2N} p_i \right) \sigma(v, r, s). \tag{B.8}
\]

The bilinear equations (B.1)–(B.4) are derived from the following bilinear equations for \( \sigma \) through the conditions (A.17), (A.18) and the identification given in (3.36):
\[
\sqrt{-1} D_i \sigma(v + 1, r + 1, s - 1) \cdot \sigma(v, r, s) = D_x \sigma(v, r + 1, s) \cdot \sigma(v + 1, r, s - 1), \tag{B.9}
\]
\[
\frac{\sqrt{-1}}{2} D_x D_i \sigma(v, r, s) \cdot \sigma(v, r, s) = D_x \sigma(v, r - 1, s) \cdot \sigma(v + 1, r, s), \tag{B.10}
\]
\[
\left( \sqrt{-1} D_i + D^2 \right) \sigma(v + 1, r, s) \cdot \sigma(v, r, s) = 0, \tag{B.11}
\]
\[
\frac{1}{2} D^2 \sigma(v, r, s) \cdot \sigma(v, r, s) = \sigma(v + 1, r, s) \sigma(v - 1, r, s). \tag{B.12}
\]

Here, (B.12) is equivalent to
\[
\frac{1}{2} D_x D_y \sigma(v, r, s) \cdot \sigma(v, r, s) = \sigma(v + 1, r, s) \sigma(v - 1, r, s), \tag{B.13}
\]
because of (B.8).

The bilinear equations (B.9)–(B.11) and (B.13) are obtained by the technique of Laplace expansion by choosing the common and distinct columns as follows.

| common columns | distinct columns |
|----------------|------------------|
| (B.9) \( -r, \cdots, N + v - 2 - r; 1 - s, \cdots, N - v - 1 - s \) | \( -1 - r, N + v - 1 - r; N + v + 1 - r; -s \) |
| (B.10) \( 1 - r, \cdots, N + v - 2 - r; -s, \cdots, N - v - 1 - s \) | \( -1 - r, -r, N + v - 1 - r, N + v + 1 - r; \) |
| (B.11) \( -r, \cdots, N + v - 2 - r; -s, \cdots, N - v - 2 - s \) | \( N + v - 1 - r, N + v - r, N + v + 1 - r; N - v - 1 - s \) |
| (B.13) \( -r, \cdots, N + v - 2 - r; -s, \cdots, N - v - 2 - s \) | \( N + v - 1 - r, N + v - r; N - v - 1 - s, N - v - s \) |

We remark that some of the bilinear equations are not directly obtained from the Laplace expansion of the determinant specified by the above table. For example, the Laplace expansion for (B.9) in the table gives
\[
\left( \sqrt{-1} \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \sigma(v + 1, r + 1, s - 1) \sigma(v, r, s) - \sigma(v + 1, r + 1, s - 1) \left( \sqrt{-1} \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \sigma(v, r, s) = -2 \sigma(v, r + 1, s) \frac{\partial}{\partial x} \sigma(v + 1, r, s - 1).
\]

We obtain (B.9) by adding \( x \)-derivative of (A.8) to the above equation. Similarly, (B.10) is obtained from the Laplace expansion and \( x \)-derivative of (A.9). This completes the proof of Proposition 3.12.

We next prove Proposition 3.13. First, we comment that the gauge factor \( A \) in (3.43) does not change the statement of Proposition 3.11. We consider the determinant \( \sigma_{nnr} \) given in (A.20) with \( \zeta_i \) in (3.42). Then we obtain the following bilinear equations from the Laplace expansions specified.
in the table shown below:

\[
\begin{align*}
& \left( \sqrt{-1} \epsilon^2 D_r + 2 \right) \sigma_{n,n'+1}(v + 1, r + 1, s + 1) = \sigma_{n,n'}(v, r + 1, s + 1) \sigma_{n,n'+1}(v + 1, r, s) - \sigma_{n-1,n'+1}(v + 1, r, s), \quad (B.14) \\
& \frac{\sqrt{-1}}{2} \epsilon^2 D_z D_r \sigma_{n,n'}(v, r, s) \cdot \sigma_{n,n'}(v, r, s) \\
& = \sigma_{n,n'}(v, r - 1, s) \sigma_{n,n'}(v, r + 1, s) - \sigma_{n-1,n'}(v, r - 1, s) \sigma_{n+1,n'}(v, r + 1, s), \quad (B.15) \\
& \sqrt{-1} \epsilon^2 D_r \sigma_{n,n'}(v + 1, r, s) \cdot \sigma_{n,n'}(v, r, s) \\
& + \sigma_{n+1,n'}(v + 1, r, s) \sigma_{n-1,n'}(v, r, s) + \sigma_{n-1,n'}(v + 1, r, s) \sigma_{n+1,n'}(v, r, s) = 0, \quad (B.16) \\
& \sigma_{n+1,n'}(v, r, s) \sigma_{n,n'-1}(v, r, s) - \sigma_{n+1,n'-1}(v, r, s) \sigma_{n,n'}(v, r, s) \\
& = \sigma_{n,n'}(v + 1, r, s) \sigma_{n,n'-1}(v - 1, r, s). \quad (B.17)
\end{align*}
\]

| Equation | common columns | distinct columns |
|----------|----------------|-----------------|
| (B.14)   | \[n, \ldots, n, \sigma^2 \] | \[n, \ldots, n, \sigma^2 \] |
| (B.15)   | \[n, \ldots, n, \sigma^2 \] | \[n, \ldots, n, \sigma^2 \] |
| (B.16)   | \[n, \ldots, n, \sigma^2 \] | \[n, \ldots, n, \sigma^2 \] |
| (B.17)   | \[n, \ldots, n, \sigma^2 \] | \[n, \ldots, n, \sigma^2 \] |

Note that for each of (B.14)–(B.16) we need to consider two Laplace expansions since the time evolution corresponds to superposition of a positive and a negative KP flows. By using (A.22) and (A.38) with (3.43), we obtain the following bilinear equations from (B.14)–(B.17), respectively:

\[
\begin{align*}
\sqrt{-1} \epsilon^2 D_r H_n \cdot F_n &= f_n^* g_{n-1} - f_{n-1} g_n, \quad (B.18) \\
\frac{\sqrt{-1}}{2} \epsilon^2 D_z D_r F_n \cdot F_n &= f_n^* f_{n-1}^* - f_{n-1}^* f_n, \quad (B.19) \\
\left( \sqrt{-1} \epsilon^2 D_r - 2 \right) G_n \cdot F_n + G_{n+1} F_{n-1} + G_{n-1} F_{n+1} &= 0, \quad (B.20) \\
F_{n+1} F_{n-1} - F_n^2 &= G_n^* G_n. \quad (B.21)
\end{align*}
\]

Then (B.18) and (B.19) gives (2.7), and (B.20) and (B.21) gives (1.11), respectively. This completes the proof of Proposition 3.13.

We finally prove Proposition 3.14. First it can be seen that although the factors \( A^m \) and \( B^m \) are introduced in (3.47), Proposition 3.11 is still valid for the \( \tau \) functions (3.47) with the gauge. By using (2.6), (3.1)–(3.3), (3.7) and (2.3), we have

\[
\begin{align*}
\left( \tan \frac{k_n^m}{2} \right) N_n^m - T_n^m = & -\frac{1}{2(F_n^m)^2} \left[ \left( \begin{array}{c}
\delta_n^m f_{n-1}^m + f_n^m g_{n-1}^m + g_n^m f_{n-1}^m + f_{n-1}^m g_n^m \\
- f_{n-1}^m g_{n-1}^m - f_n^m g_n^m - f_n^m g_{n-1}^m - f_{n-1}^m g_n^m
\end{array} \right) / \sqrt{-1} \right], \quad (B.22) \\
& \frac{1}{F_{n+1}^m F_n^m} \left[ \left( \frac{f_n^m}{g_n^m} \right)^2 - \left( \frac{g_n^m}{f_n^m} \right)^2 \right] \sqrt{-1} \\
e^{-1 \Lambda_n^m} (N_n^m + \sqrt{-1} B_n^m) = & \frac{1}{F_{n+1}^m F_n^m} \left[ \left( \frac{f_n^m}{g_n^m} \right)^2 - \left( \frac{g_n^m}{f_n^m} \right)^2 \right] \sqrt{-1} \\
& -2 f_n^m g_n^m.
\end{align*}
\]
and (2.18) is rewritten in terms of $\tau$ functions. The first and second entries of (2.18) are equivalent to

$$
\frac{H_{m+1}^n}{F_{m+1}^n} - \frac{H^m_n}{F^m_n} = \frac{\delta^2}{c^4} \left( \left( \sqrt{1 - \frac{\epsilon^2}{\delta^2}} - 1 \right) g_n^m f_{m+1}^n - \left( \sqrt{1 - \frac{\epsilon^2}{\delta^2}} + 1 \right) f_{m+1}^n g_n^m \right) + \Gamma_{\infty} \left( 2 f_{m+1}^m g_{m+1}^m - (f_{m+1}^m)^2 G_{m+1}^m + (g_{m+1}^m)^2 C_{m+1}^m \right),
$$

(B.24)

and its complex conjugate. From the third entry we get

$$
\frac{2\delta^2}{c^4} \left( \Gamma_{\infty} - 1 \right) - 2 \left( \log \frac{F_{m+1}^n}{F^m_n} \right) = 0.
$$

(B.25)

These equations and dNLS (1.4) are bilinearized into the following form:

$$
\sqrt{1 - \frac{\epsilon^2}{\delta^2}} H_{m+1}^n F_n^m - H^m_n F_{m+1}^n = \sqrt{\Gamma_{\infty}} \left( f_{m+1}^m g_{m+1}^m + g_{m+1}^m f_{m+1}^m \right),
$$

(B.26)

$$
\left( 1 + \sqrt{1 - \frac{\epsilon^2}{\delta^2}} \left( \Gamma_{\infty} - 1 \right) - \sqrt{1 - \frac{\epsilon^2}{\delta^2}} D \right) F_{m+1}^n \cdot F_m^m = \sqrt{\Gamma_{\infty}} \left( f_{m+1}^m f_{m+1}^m + g_{m+1}^m g_{m+1}^m \right),
$$

(B.27)

$$
\left( \sqrt{1 - \frac{\epsilon^2}{\delta^2}} - 1 \right) G_{m+1}^n F_m^m - \left( \sqrt{1 - \frac{\epsilon^2}{\delta^2}} + 1 \right) G_{m+1}^m F_{m+1}^n + \Gamma_{\infty} \left( G_{m+1}^m f_{m+1}^m + f_{m+1}^m G_{m+1}^m \right) = 0,
$$

(B.28)

$$
\sqrt{1 - \frac{\epsilon^2}{\delta^2}} \sqrt{\Gamma_{\infty}} \left( f_{m+1}^m g_{m+1}^m + g_{m+1}^m f_{m+1}^m \right) = \sqrt{\Gamma_{\infty}} \left( f_{m+1}^m f_{m+1}^m + g_{m+1}^m g_{m+1}^m \right) = 0,
$$

(B.29)

$$
\sqrt{1 - \frac{\epsilon^2}{\delta^2}} \sqrt{\Gamma_{\infty}} f_{m+1}^m F_m^m - \left( \sqrt{1 - \frac{\epsilon^2}{\delta^2}} + 1 \right) g_n^m f_{m+1}^n + \Gamma_{\infty} \left( g_n^m f_{m+1}^n - f_n^m g_m^m \right) = 0,
$$

(B.30)

$$
F_{m+1}^m f_{m+1}^m - (f_{m+1}^m)^2 = G_{m+1}^m G_m^m,
$$

(B.31)

where $\Gamma_{\infty}$ is either 1 or $1 + e^4/\delta^2$. Actually by using (B.29) and complex conjugate of (B.30), we can eliminate $g_{m+1}^m$ and $f_{m+1}^m$ from (B.26) and obtain (B.24). Similarly we take imaginary part of (B.27) and eliminate $f_{m+1}^m$ and $g_{m+1}^m$ by using (B.29) and (B.30). Then (B.25) is obtained in a straightforward way. Finally dNLS (1.4) follows from (B.28) and (B.31) through the variable transformation (3.49). Here we comment that the equations (B.26)–(B.31) seem over-determining, however they are indeed compatible.

In the following we prove that the $\tau$ functions in Proposition 3.14 actually satisfy the bilinear equations (B.26)–(B.31). We note that (B.31) is the same as (B.21) which holds irrespective of the time evolutions. Now let us define a $2N \times 2N$ determinant $\sigma_{mn'}^{mn}(v, r, s)$ by

$$
\sigma_{mn'}^{mn}(v, r, s) =
\begin{vmatrix}
\phi_1^{(n,m)}(r) & \phi_1^{(n+1,m)}(r) & \cdots & \phi_1^{(n+N-1,m)}(r) & \psi_1^{(n',m')}^{(n',m')}(s) & \psi_1^{(n',m'+1)}(s) & \cdots & \psi_1^{(n'+N-2,m')}(s) \\
\phi_2^{(n,m)}(r) & \phi_2^{(n+1,m)}(r) & \cdots & \phi_2^{(n+N-1,m)}(r) & \psi_2^{(n',m')}^{(n',m')}(s) & \psi_2^{(n',m'+1)}(s) & \cdots & \psi_2^{(n'+N-2,m')}(s) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_{2N}^{(n,m)}(r) & \phi_{2N}^{(n+1,m)}(r) & \cdots & \phi_{2N}^{(n+N-1,m)}(r) & \psi_{2N}^{(n',m')}^{(n',m')}(s) & \psi_{2N}^{(n',m'+1)}(s) & \cdots & \psi_{2N}^{(n'+N-2,m')}(s)
\end{vmatrix},
$$

(B.32)
where

$$
\phi_i^{(n,m)}(r) = p_i^n \left(1 - \frac{a^*}{p_i} \right)^m (1 - p_i)^{-r} e^{\zeta_i}, \quad \zeta_i = \frac{p_i + 1}{p_i - 1} z + \zeta_{i0},
$$

(B.33)

$$
\psi_i^{(n',m')}(s) = \left(\frac{1}{p_i} \right)^{n'} (1 - a p_i)^{-m'} (1 - p_i)^{-s} e^{\eta_i},
$$

and \( \eta_i \) are constants. Similarly to (A.28), we simply denote

$$
\sigma_{mn'}^{m'}(v, r, s) = \begin{vmatrix}
\begin{array}{cccccc}
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^* & n^* & n^* & \cdots & n^* & n^* + 1 \\
\end{array}
\end{vmatrix},
$$

(B.34)

Then we have the following difference and differential formulas:

$$
(1 - a^*) \sigma_{mn',1}^{m+1,n'+1}(v + 1, r + 1, s) = \begin{vmatrix}
\begin{array}{cccccc}
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^* & n^* & n^* & \cdots & n^* & n^* + 1 \\
\end{array}
\end{vmatrix},
$$

(B.35)

$$
(1 - a) \sigma_{mn',1}^{m+1,n'+1}(v, r + 1, s + 1) = \begin{vmatrix}
\begin{array}{cccccc}
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^* & n^* & n^* & \cdots & n^* & n^* + 1 \\
\end{array}
\end{vmatrix},
$$

(B.36)

$$
\left( (a^* - 1) \left( \frac{\partial}{\partial v} - \frac{N + v}{2} \right) - 1 \right) \sigma_{mn',1}^{m+1,n'+1}(v, r, s) = \begin{vmatrix}
\begin{array}{cccccc}
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^* & n^* & n^* & \cdots & n^* & n^* + 1 \\
\end{array}
\end{vmatrix},
$$

(B.37)

and so on. By applying the Laplace expansion to the identities,

$$
\begin{vmatrix}
\begin{array}{cccccc}
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^* & n^* & n^* & \cdots & n^* & n^* + 1 \\
\end{array}
\end{vmatrix} = 0,
$$

(B.38)

$$
\begin{vmatrix}
\begin{array}{cccccc}
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^* & n^* & n^* & \cdots & n^* & n^* + 1 \\
\end{array}
\end{vmatrix} = 0,
$$

(B.39)

$$
\begin{vmatrix}
\begin{array}{cccccc}
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\nu & \nu' & \nu'' & \cdots & n^* & n^* + 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n^* & n^* & n^* & \cdots & n^* & n^* + 1 \\
\end{array}
\end{vmatrix} = 0,
$$

(B.40)

we get the bilinear equations for \( \sigma_{mn',1}^{m+1,n'+1} \),

$$
(1 - a^*) \sigma_{mn',1}^{m+1,n'+1}(v + 1, r + 1, s) \sigma_{mn',1}^{m,n}(v, r, s + 1) - (1 - a) \sigma_{mn',1}^{m,n+1}(v + 1, r + 1, s) \sigma_{mn',1}^{m+1,n'+1}(v, r, s + 1) = a^* \sigma_{n+1,m}^{m,n+1}(v, r + 1, s + 1) \sigma_{mn',1}^{m,n+1}(v + 1, r, s) + a \sigma_{mn',1}^{m+1,n+1}(v + 1, r + 1, s) \sigma_{n+1,m+1}^{m,n+1}(v, r + 1, s) + \left( (1 - a^*)D_v \right) a^{mn',1,n'+1}(v, r, s) \cdot \sigma_{mn',1}^{m,n}(v, r, s)
$$

(B.41)

$$
\sigma_{mn',1}^{m+1,n+1}(v + 1, r, s) \sigma_{mn',1}^{m,n+1}(v, r + 1, s) - \sigma_{mn',1}^{m,n}(v + 1, r, s) \sigma_{mn',1}^{m+1,n+1}(v, r, s) = a^* \sigma_{n+1,m}^{m,n+1}(v, r, s + 1) \sigma_{mn',1}^{m,n+1}(v + 1, r, s) + a \sigma_{mn',1}^{m+1,n+1}(v + 1, r + 1, s) \sigma_{n+1,m+1}^{m,n+1}(v, r + 1, s)
$$

(B.42)
respectively. In (B.40), if we replace the column $\frac{\nu+N+1}{m}$ by $\frac{\nu+N+1}{m+1}$, we get instead of (B.43),

\[
(1-a^r)\sigma_{mn+1,m+1}^m(v+1, r+1, s) a^r_{mn}^m(v, r, s) - a^r_{mn}^m(v+1, r+1, s) a^r_{mn+1,m+1}^m(v, r, s)
+ a^r_{mn}^m(v+1, r+1, s) a^r_{mn+1,m+1}^m(v, r, s) - a^r_{mn+1,m+1}^m(v+1, r, s) a^r_{mn+1,m+1}^m(v, r+1, s) = 0.
\]

(B.44)

Similarly the replacement of column $\frac{\nu+N+1}{m}$ in (B.40) by $\frac{\nu+N}{m+1}$ gives

\[
(1-a)\sigma_{mn+1,m+1}^m(v, r, s+1) a^r_{mn}^m(v, r, s) - a^r_{mn}^m(v, r, s+1) a^r_{mn+1,m+1}^m(v, r, s)
+ a^r_{mn}^m(v, r, s+1) a^r_{mn+1,m+1}^m(v, r, s) - a^r_{mn+1,m+1}^m(v, r+1, s+1) a^r_{mn+1,m+1}^m(v+1, r, s) = 0.
\]

(B.45)

Under the specialization (A.37), $\tau_m^m(n, r)$ in Proposition 3.14 is obtained by

\[
\tau_m^m(n, r) = \sigma_{n,0}^m(v, r, 0) \prod_{i=1}^{N} (p_i^r)^\nu (1-ap_i)^m(1-a^r p_i)^m \left(1 - \frac{1}{p_i^r}\right)^{1/2} e^{i\zeta},
\]

(B.46)

and (B.26)–(B.30) are derived from (B.41)–(B.45), respectively, through the reduction condition (A.22) and the identification given in (3.47). This completes the proof of Proposition 3.14. □

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