Dynamical Analysis of Cylindrically Symmetric Anisotropic Sources in $f(R,T)$ Gravity

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In this paper, we have analyzed the stability of cylindrically symmetric collapsing object filled with locally anisotropic fluid in $f(R,T)$ theory, where $R$ is the scalar curvature and $T$ is the trace of stress-energy tensor of matter. Modified field equations and dynamical equations are constructed in $f(R,T)$ gravity. Evolution or collapse equation is derived from dynamical equations by performing linear perturbation on them. Instability range is explored in both Newtonian and post-Newtonian regimes with the help of adiabatic index, which defines the impact of physical parameters on the instability range. Some conditions are imposed on physical quantities to secure the stability of the gravitating sources.

Keywords: Collapse; $f(R,T)$ gravity; Dynamical equations; Instability range; Adiabatic index.

I. INTRODUCTION

Astrophysics and theories regarding gravity bear two emerging issues: aftermath of gravitational collapse and explorations regarding stability of celestial bodies. Collapse of a star depends on availability of its fuel, exhaustion of all of its fuel makes the gravitational collapse indispensable because inward gravitational pull, meanwhile overpowers the outward drawn force. Undoubtedly, the size of collapsing star determines the end state of evolution, the life span of huge stars having mass equivalent to ten to twenty solar masses is not comparable to that of stars assuming sufficiently little mass. Moreover, the more massive stars are the more vulnerable to instability due to heat flux emanating because of high energy dissipation during collapsing phenomenon.

The astrophysical bodies seek attention only if they show resistance against fluctuations and remain stable. In 1964, Chandrasekhar [4] presented investigations of primary level on dynamical stability of spherical bodies. He pinpointed the instability range of a star assuming mass $'M'$ and radius $'r'$ with the help of adiabatic index $\Gamma$ comprising the inequality $\Gamma \geq \frac{2}{3} + n \frac{\bar{M}}{4}$. Adiabatic index is a tool which is defined as pressure to density ratio and used to determine the stability range of celestial objects. It imposes some conditions on physical parameters to meet stability criterion which is defined for stellar objects. Instability problems in the theory of general relativity (GR) coupled with dissipation, shear, zero expansion, radiation, isotropy and local anisotropy was addressed by Herrera and his companions in [3]-[10]. They established the fact that instability range is vulnerable to drastic changes if slight variations take place in isotropic profile and shearing effects. Sharif and Abbas [11] analyzed the collapse of charged cylindrical celestial bodies for non-adiabatic and perfect fluid. Sharif and Azam [12] presented stability analysis for cylindrically symmetric thin-shell wormholes.

Researchers contributed substantially in addressing the instability problems with the help of GR but it is no longer helpful to explain the cosmological and astrophysical phenomena in a satisfactory way in the presence of dark matter. The concept of exotic matter like dark energy and observational evidences of expanding universe has put the theoretical cosmology into crisis. Due to limitations of GR on large scales, gravitational modified theories grab the attention of astrophysicists. With the help of these theories, they put great efforts to analyze collapsing phenomenon and stability of astronomical bodies. Among different modified theories, $f(R)$ gravity admits one of the basic modifications in Einstein-Hilbert action which includes high order curvature $R$ to explain the above mentioned exotic matter. With the help of various observations like cosmic microwave background, clustering spectrum and weak lensing [13]-[16], it was concluded that the stability range is increased by the inclusion of curvature terms of higher order.

In modified theories of gravity, collapsing phenomenon has been widely studied. Collapse of self gravitating dust particles has been discussed in [17], where authors found that analysis of gravitational collapse is an important tool to constrain the modified models that present late time cosmological acceleration. Meanwhile, Gosh and Maharaj [18] established exact solutions of null dust non-static cluster of particles in $f(R)$ gravity, constrained by constant curvature describing anti de-sitter background. Some highly important prospects of celestial collapse for $f(R)$ theory

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are worked out in [19]-[22]. Sharif and Bhatti [23] discussed instability conditions for cylindrically symmetric self-gravitating objects surrounding in charged expansion-free anisotropic environment. Kausar and Noureen [24] discussed the evolution of gravitating sources in the context of $f(R)$ theory and concluded that adiabatic index has its dependence upon the electromagnetic background, mass and radius of spherically symmetric bodies.

In 2011, another modification to the theory of GR was introduced by Harko et al. in [25], which is extension of $f(R)$ theory, such theory is named as $f(R,T)$ theory. The $f(R,T)$ theory of gravity covers curvature and matter coupling and its action includes an arbitrary function of Ricci scalar $'R'$ and trace of energy momentum tensor $T$. After its introduction, this gained significant attention and authors discussed its various properties including reconstructions schemes, energy conditions, cosmological and thermodynamical implications, neutron stars, scalar perturbations, wormholes and analysis of anisotropic universe models and stability etc. in [20]. Shabani and Farhoudi [27] applied dynamical system approach to elaborate weak field limit and presented analysis of cosmological implications of $f(R,T)$ models with various cosmological parameters like Hubble parameter, equation of state parameter, snap parameter.

Recently, dynamical analysis of self gravitating sources has been discussed in $f(R,T)$ theory. Noureen and Zubair [28] discussed dynamical instability of spherically symmetric collapsing star in the presence ofien anisotropic fluid. The implications of shear free and expansion free conditions on dynamical instability are also discussed in the framework of $f(R,T)$ [29]. Motivating from the significance of non-spherical symmetries, impact of axially symmetric gravitating sources has also been explored in context of $f(R)$ and $f(R,T)$ [30]. Yousaf and Bhatti [31] identified dynamical instability conditions of a self gravitating cylindrical object in $f(R,T)$ theory of gravity.

In this paper, we have chosen the model “$f(R,T) = R + \alpha R^2 + \gamma R^n + \lambda T$” to present the dynamical analysis of cylindrically symmetric object. In order to present this analysis we employ perturbation approach on collapse equations and explore the instability range of the model under consideration with the help of adiabatic index $\Gamma$ in both Newtonian and post-Newtonian regimes. The paper has been organized as follows: The next section contains modified field equations and dynamical equations. In section III, we have provided the adopted $f(R,T)$ model and presented perturbation scheme along with corresponding collapse equation. Discussion about the instability in the form of adiabatic index in both Newtonian and post-Newtonian regimes is also presented in the same section. The last section IV concludes our main findings which is followed by Appendix A.

II. DYNAMICAL EQUATIONS IN $f(R,T)$

The $f(R,T)$ modification of Einstein-Hilbert action is given by

$$\int dx^4\sqrt{-g} \frac{f(R,T)}{16\pi G} + L(m).$$

Here, action due to matter is described by $L(m)$, whose different choices can be taken into account, each of which leads to a particular form of fluid.

The variation of above modified action with respect to metric $g_{\alpha\beta}$ leads to the following set of field equations

$$R_{\alpha\beta} f_R(R,T) - \frac{1}{2} g_{\alpha\beta} f(R,T) + (g_{\alpha\beta} \Box - \nabla_\alpha \nabla_\beta) f_R(R,T) = 8\pi G T_{\alpha\beta}^{(m)} + f_T(R,T) T_{\alpha\beta}^{(m)} - f_T(R,T) \Theta_{\alpha\beta},$$

(2)

where $f_R(R,T) = \frac{\partial f(R,T)}{\partial R}$, $f_T(R,T) = \frac{\partial f(R,T)}{\partial T}$, while $\nabla_\beta$ and $\Box$ are derivative operators and represent covariant derivative and four-dimensional Levi-Civita covariant derivative respectively. The term $\Theta_{\alpha\beta}$ is defined as

$$\Theta_{\alpha\beta} = g^{\mu\nu} \delta T_{\alpha\mu\beta} = -2 T_{\alpha\beta} + g_{\alpha\beta} L^{(m)} - 2 g^{\mu\nu} \frac{\partial L^{(m)}}{\partial g_{\mu\nu}}.$$

(3)

Here, we have chosen $L^{(m)} = \mu, 8\pi G = 1$, then the expression $\Theta_{\alpha\beta}$ becomes

$$\Theta_{\alpha\beta} = -2 T_{\alpha\beta} + \mu g_{\alpha\beta}.$$

(4)

For aforementioned choice of matter Lagrangian and Eq. (3), the modified field equations (2) will be as follows:

$$G_{\alpha\beta} = T_{\alpha\beta}^{\epsilon ff},$$

(5)

where

$$T_{\alpha\beta}^{\epsilon ff} = \frac{1}{f_R} \left[ (f_T + 1) T_{\alpha\beta}^{(m)} - \mu g_{\alpha\beta} f_T + \frac{f_R - f_T}{2 g_{\alpha\beta} (\nabla_\alpha \nabla_\beta - g_{\alpha\beta} \Box) f_R} \right].$$

(6)
where $T_{\alpha\beta}^{(m)}$ represents the energy momentum tensor for ordinary matter.

The system that we have chosen for analysis is cylindrically symmetric object which consists on a timelike three dimensional boundary surface $\Sigma$. Under consideration boundary surface $\Sigma$ constitutes two regions termed as interior and exterior region. Interior region inside the boundary $[32]$ is

$$ds^2_{\Sigma} = A^2(t, r)dt^2 - B^2(t, r)dr^2 - C^2(t, r)d\phi^2 - dz^2,$$

(7)

whereas the line element for exterior region $[33]$ can be defined with the help of following diagonal form:

$$ds^2_{\Sigma} = \left(\frac{-2M}{r}\right)du^2 + 2drd\nu - r^2(d\phi^2 + \zeta^2dz^2),$$

(8)

where $'\nu'$ is the retarded time, $'M'$ is total gravitating mass and $\zeta$ represents an arbitrary constant. The fluid can be indicated through the following configurations of the mathematical form $[34]$

$$T_{\alpha\beta}^{(m)} = (\mu + p_r)V_\alpha V_\beta - p_r g_{\alpha\beta} + (p_\phi - p_r)S_\alpha S_\beta + (p_\phi - p_r)K_\alpha K_\beta,$$

(9)

where $\mu$ represent energy density, $p_r$, $p_\phi$ and $p_\phi$ are principal stresses, while $V_\beta$, $S_\beta$ and $K_\beta$ denote four-velocity and four-vectors respectively. Under co-moving relative motion, these four-vectors and four-velocity are defined as

$$V_\beta = A\delta^\beta_\beta, \quad S_\beta = S\delta^\beta_\beta, \quad K_\beta = C\delta^\beta_\beta,$$

(10)

and satisfy the following relations:

$$V^\beta V_\beta = -1, \quad K^\beta K_\beta = S^\beta S_\beta = 1, \quad V^\beta K_\beta = S^\beta K_\beta = V^\beta S_\beta = 0.$$  

The expansion scalar $\Theta$ defines the rate of change of matter distribution and is given by the following mathematical formula

$$\Theta = V^\alpha_{;\alpha} = \frac{1}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right),$$

(11)

where 'dot' indicates partial derivatives w.r.t time coordinate.

The Ricci invariant corresponding to the interior region given in [7] is

$$R(r, t) = \frac{2}{B^2} \left[\frac{A''}{A} + \frac{C''}{C} + \frac{A'}{A} \left(\frac{C'}{C} - \frac{B'}{B}\right) - \frac{B' C'}{B C}\right] - \frac{2}{B^2} \left[\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A}}{A} \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\ddot{B}\dot{C}}{B C}\right],$$

(12)

where 'prime' represents partial derivatives w.r.t radial coordinate.

The $f(R, T)$ field equations for the interior of cylindrically symmetric system are

$$G_{00} = \frac{A^2}{f_R} \left[\mu + \frac{f - Rf}{2} + \eta_{00}\right], \quad G_{01} = \frac{1}{f_R} \left[\dot{f}_R - \frac{f'}{A} \dot{r} - \frac{\dot{B}}{B} f_R\right],$$

(13)

$$G_{11} = \frac{B^2}{f_R} \left[\mu f_T - \frac{f - Rf}{2} + \eta_{11}\right], \quad G_{22} = \frac{C^2}{f_R} \left[\mu f_T - \frac{f - Rf}{2} + \eta_{22}\right],$$

(14)

$$G_{33} = \frac{1}{f_R} \left[\mu f_T - \frac{f - Rf}{2} + \eta_{33}\right],$$

(15)

where

$$\eta_{00} = \frac{\dot{f}_R}{A^2} + \frac{\dot{f}_R}{A^2} \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{\ddot{f}_R}{B^2} \left(\frac{B'}{B} + \frac{C'}{C}\right), \quad \eta_{11} = \frac{\dot{f}_R}{A^2} + \frac{\dot{f}_R}{A^2} \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\right) - \frac{\ddot{f}_R}{B^2} \left(\frac{A'}{A} + \frac{C'}{C}\right),$$

(16)

$$\eta_{22} = \frac{\dot{f}_R}{A^2} - \frac{\dot{f}_R}{A^2} \left(\frac{A}{B} - \frac{\dot{B}}{B}\right) - \frac{\ddot{f}_R}{B^2} \left(\frac{A'}{A} - \frac{B'}{B}\right),$$

(17)

$$\eta_{33} = \frac{\dot{f}_R}{A^2} - \frac{\dot{f}_R}{A^2} \left(\frac{A}{B} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) - \frac{\ddot{f}_R}{B^2} \left(\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C}\right).$$

(18)
The dynamical equations are significant for the establishment of the stability range of relativistic bodies, so we are interested in their construction. In \( f(R, T) \) framework, it is observed that the divergence of energy-momentum tensor is non-vanishing and is found to be

\[
\nabla^\alpha T_{\alpha\beta} = \frac{f_T}{1-f_T} \left[ (\Theta_{\alpha\beta} - T_{\alpha\beta}) \nabla^\alpha \ln f_T - \frac{1}{2} g_{\alpha\beta} \nabla^\alpha + \nabla^\alpha \Theta_{\alpha\beta} \right].
\]

(19)

The divergence of “effective energy-momentum” tensor which is mentioned in above expression yields the following two equations:

\[
\dot{\mu} \left( \frac{1-f_T + f_R f_T}{f_R(1-f_T)} \right) - \frac{f_R}{f_R} + \frac{\dot{B}}{B} \frac{1}{f_R} (1 + f_T) (\mu + P_r) + \frac{\dot{C}}{C} \frac{1}{f_R} (1 + f_T) (\mu + P_\phi) + \frac{T}{2} \frac{f_T}{1-f_T} + H_0(r, t) = 0,
\]

(20)

\[
P_r' \left( \frac{1-f_T^2 + 2f_R f_T}{f_R(1-f_T)} \right) + \frac{P_r}{f_R} \left( f_T' - f_R^2 (1 + f_T) \right) + \mu' \frac{f_T}{f_R} + \frac{\mu}{f_R} \left( f_T' - f_R f_R' \right) + \frac{A'}{A} \frac{1}{f_R} (1 + f_T) (\mu + P_r)
\]

\[
+ \frac{C'}{C} \frac{1}{f_R} (1 + f_T) (P_r - P_\phi) + \frac{f_T' - f_T f_R'}{1-f_T} (\mu + P_r) + \frac{f_T}{1-f_T} \frac{T'}{2} + \frac{f_T}{1-f_T} \mu' + H_1(r, t) = 0.
\]

(21)

These are the required dynamical equations that will lead to collapse equation. Here, \( H_0(r, t) \) and \( H_1(r, t) \) represent extra curvature dark source terms emerging from the \( f(R, T) \) gravitational field and these quantities are given in Appendix. The variation of physical parameters of gravitating system with the passage of time can be observed with the help of perturbation scheme as given in the below section.

### III. \( f(R, T) \) MODEL AND PERTURBATION SCHEME

Perturbation scheme is a mathematical tool that is used to find approximate solution of a differential equation. After applying perturbation scheme, the corresponding equation can be broken into parts i.e., static and perturbed parts. We employ this theory to analyze the effects of \( f(R, T) \) model on the evolution of celestial body under consideration. We have applied perturbation theory in such a way that initially all the quantities are in static equilibrium but with the passage of time perturbed quantities have both radial and time dependence. The selection of model is very important for analysis. The \( f(R, T) \) model we have considered for evolution analysis is combination of extended Starobinsky model \[35\] and linear term of trace \( T \), written mathematically as

\[
f(R, T) = R + \alpha R^2 + \gamma R^n + \lambda T,
\]

(22)

where \( n \geq 3, \alpha \) and \( \gamma \) corresponds to the positive real values, while \( \lambda \) is coupling parameter and \( \lambda T \) represent modification to \( f(R) \) gravity. Assuming \( 0 < \varepsilon \ll 1 \), the functions may be written in the following form:

\[
A(t, r) = A_0(r) + \varepsilon D(t)a(r),
\]

(23)

\[
B(t, r) = B_0(r) + \varepsilon D(t)b(r),
\]

(24)

\[
C(t, r) = C_0(r) + \varepsilon D(t)c(r),
\]

(25)

\[
\mu(t, r) = \mu_0(r) + \varepsilon \mu_0(t, r),
\]

(26)

\[
p_\rho(t, r) = p_\rho_0(r) + \varepsilon p_\rho_0(t, r),
\]

(27)

\[
p_\phi(t, r) = p_\phi_0(r) + \varepsilon p_\phi_0(t, r),
\]

(28)

\[
p_z(t, r) = p_z_0(r) + \varepsilon p_z_0(t, r),
\]

(29)

\[
m(t, r) = m_0(r) + \varepsilon m_0(t, r),
\]

(30)

\[
R(t, r) = R_0(r) + \varepsilon D_1(t)e_1(r),
\]

(31)

\[
T(t, r) = T_0(r) + \varepsilon D_2(t)e_2(r),
\]

(32)

\[
f(R, T) = [R_0(r) + \alpha R_0^2(r) + \gamma R_0^n(r) + \lambda T_0] + \varepsilon D_1(t)e_1(r) \times [1 + 2\alpha R_0(r) + \gamma n R_0^{n-1}] + \varepsilon \lambda D_2(t)e_2(r),
\]

(33)

\[
f_R = (1 + 2\alpha R_0(r) + \gamma n R_0^{n-1}) + \varepsilon D_1(t)e_1(r)
\]

\[
\times (2\alpha + \gamma n(n-1) R_0^{n-2}),
\]

(34)

\[
f_T = \lambda,
\]

(35)

\[
\Theta(t, r) = \varepsilon \Theta.
\]

(36)
In above equations, $R_0$ represents the static part of Ricci scalar whose value is given below

$$R_0(r) = \frac{2}{B_0^2} \left[ \frac{A_0''}{A_0} + \frac{B_0'}{B_0} \left( \frac{1}{r} - \frac{B_0'}{B_0} \right) \right],$$

while the value of perturbed part of Ricci scalar is given as

$$-De = \frac{2 \dot{B}}{B_0} \left( \frac{\bar{b}'}{B_0} + \bar{r} \right) + D \left[ \frac{4 b}{B_0^2} R_0 - \frac{2}{B_0^2} \left\{ \frac{a''}{A_0} - \frac{a A_0''}{A_0^2} + \frac{c''}{r} - \frac{A_0'}{A_0} \left( \frac{1}{r} - \frac{\bar{r}}{B_0} \right) \right\} \right] + \frac{B_0'}{B_0} \left( \frac{\bar{b}'}{r} \right) - \frac{1}{r} \left( \frac{b}{B_0^2} \right).

(38)$$

The static configuration of $f(R, T)$ field equations \[15\] with assumption $C_0(r) = r$ is as:

$$\frac{1}{B_0^2} \frac{A_0'}{A_0} - \frac{B_0'}{B_0} \left( \frac{1}{r} + \frac{A_0'}{A_0} \right) = \frac{1}{Z} \left( P_{z0} + \lambda (P_{z0} + \mu_0) - \frac{\lambda T_0 - \alpha R_0^2 - \gamma (n - 1) R_0^2}{2} \right),$$

$$+ \frac{1}{Z} \left( P_{z0} + \lambda (P_{z0} + \mu_0) - \frac{\lambda T_0 - \alpha R_0^2 - \gamma (n - 1) R_0^2}{2} \right) + \eta^{(s)}_{11},

(42)

where

$$\eta^{(s)}_{11} = \frac{Z R_0}{B_0^2} \left[ R_0' - R_0 \left( \frac{B_0'}{B_0} - \frac{1}{r} \right) \right], \quad \eta^{(s)}_{11} = - \frac{Z R_0}{B_0^2} \left( \frac{A_0'}{A_0} + \frac{1}{r} \right),

(43)

After applying the static configuration, the first dynamical equation \[20\] is identically satisfied, while the 2nd equation \[21\] has the following form:

$$\frac{1}{Z} \left[ \frac{1 - \lambda^2 + 2 \lambda \delta}{1 - \lambda} \frac{P_{z0}'}{Z} - \frac{(1 + \lambda) Z'}{Z} P_{z0} + \lambda \left( \frac{1 + \lambda}{Z} \mu_0 - \frac{Z'}{Z^2} \mu_0 \right) + (1 + \lambda) \frac{A_0'}{A_0} (\mu_0 + P_{z0}) + \frac{1 + \lambda}{r} (P_{z0} - P_{z0}) \right]

+ \frac{1}{1 - \lambda} \left( \frac{T_0'}{2} + \mu_0' \right) + H_1^{(s)} = 0,

(45)

where $H_1^{(s)}$ represents the static part of $H_1$ and is given below:

$$H_1^{(s)} = \frac{1}{Z} \left( \frac{\lambda T_0 + \alpha R_0^2 + \gamma (n - 1) R_0^2}{2} + \eta^{(s)}_{11} \right) + \frac{1}{Z} \left( \frac{A_0'}{A_0} (\eta^{(s)}_{11} + \eta^{(s)}_{11}) + \frac{1}{r} (\eta^{(s)}_{11} - \eta^{(s)}_{11}) \right).

(46)$$
1 \frac{\hat{\mu}(1 - \lambda + \lambda Z)}{1 - \lambda} + \left\{ \mu_0 e Z R_0 + \frac{b}{B_0} (1 + \lambda) (\mu_0 + P_{\phi 0}) + \frac{\bar{\phi}}{r} (1 + \lambda) (\mu_0 + P_{\phi 0}) + \frac{\lambda e Z}{2(1 - \lambda)} + Z H_0^{(p)} \right\} \hat{D} = 0,

\begin{aligned}
\frac{\hat{D}}{A_0^2} &\left[ \frac{\hat{\phi}}{Z} \right] + \left\{ \frac{\hat{\phi}'}{Z} \right\} &= 0,

\frac{\hat{D}}{A_0^2} &\left[ \frac{\hat{\phi}}{Z} \right] + \left\{ \frac{\hat{\phi}'}{Z} \right\} &= 0,

\frac{\hat{D}}{A_0^2} &\left[ \frac{\hat{\phi}}{Z} \right] + \left\{ \frac{\hat{\phi}'}{Z} \right\} &= 0,

\frac{\hat{D}}{A_0^2} &\left[ \frac{\hat{\phi}}{Z} \right] + \left\{ \frac{\hat{\phi}'}{Z} \right\} &= 0.
\end{aligned}

where $Z = 1 + 2\alpha R_0 + \gamma n R_0^{-1}$, $Z_{R_0} = \hat{Z}_{R_0} R_0$, $Z_{R_0 R_0} = \hat{Z}_{R_0} R_0^2$, $H_0^{(p)}$, $H_1^{(p)}$ represent perturbed parts of $H_0$ and $H_1$ respectively which are addressed in Appendix. It is also assumed that $D_1 = D_2 = D$ and $c_1 = c_2 = e$.

Eliminating $\hat{\mu}$ from perturbed equation (47) and integrating this with respect to “t”, we get

$$\hat{\mu} = \frac{\lambda - 1}{1 - \lambda + \lambda Z} \left\{ \mu_0 e Z R_0 + \frac{b}{B_0} (1 + \lambda) (\mu_0 + P_{r 0}) + \frac{\bar{\phi}}{r} (1 + \lambda) (\mu_0 + P_{\phi 0}) + \frac{\lambda e Z}{2(1 - \lambda)} + Z H_0^{(p)} \right\} D,$$

The 2nd law of thermodynamics relates $\hat{\mu}$ and $\hat{\mu}_r$ as ratio of specific heat with assumption of Harrison-Wheeler type equation of state expressed in the following expression (36)

$$\hat{\mu}_r = \frac{P_{r 0}}{\mu_0 + P_{r 0}} \hat{\mu}.$$ (50)

The adiabatic index is a measure to recognize pressure variation with changing density. Substituting the value of $\hat{\mu}$ given in Eq. (47) in above equation, we obtain

$$\hat{P}_r = \left[ \frac{\lambda^2 - 1}{1 - \lambda + \lambda Z} \Gamma \left\{ \frac{b}{B_0} P_{r 0} + \frac{\bar{\phi}}{r} P_{r 0} \mu_0 + P_{\phi 0} \right\} + \Gamma \frac{P_{r 0}}{\mu_0 + P_{r 0}} \left( \frac{\lambda - 1}{1 - \lambda + \lambda Z} \right) \left\{ \mu_0 e Z R_0 + \frac{\lambda e Z}{2(1 - \lambda)} + Z H_0^{(p)} \right\} \right] D,$$ (51)

The expressions $\hat{P}_\phi$ and $\hat{P}_z$ can be obtained from the perturbed forms of last two field equations. Applying perturbation on field equations and eliminating $\hat{P}_\phi$ and $\hat{P}_z$ provides

$$\hat{P}_\phi = -\frac{\hat{D}}{(1 + \lambda) A_0^2} \left( e + \frac{b}{r^2 B_0} \right) - \frac{\lambda}{1 + \lambda} \hat{\mu} + DH_2,$$ (52)

$$\hat{P}_z = -\frac{\hat{D}}{(1 + \lambda) A_0^2} \left( e + \frac{b}{B_0} + \frac{\bar{\phi}}{r} \right) - \frac{\lambda}{1 + \lambda} \hat{\mu} + DH_3,$$ (53)

where $H_2$ and $H_3$ are provided in Appendix.
After substitution of values of $\bar{\mu}, \bar{P}_r, \bar{P}_\phi$, 2nd dynamical equation takes the form

$$
\frac{D}{A_0^2}[e' + c\left(\frac{1}{r} - \frac{A_0'}{A_0}\right) - \frac{1}{Z} Z_{R_0} \left(e' - \frac{A_0'}{A_0} e - \frac{b}{B_0} R_{00}\right) + Z_{R_0 R_0} e R_{00}' + \frac{\lambda e Z_{R_0}}{(1 - \lambda) Z} + Z H_0^{(p)} f] + \frac{\lambda}{Z} \left(\frac{1 - \lambda}{\lambda - 1 + \lambda Z}\right) \{\mu_0 e Z_{R_0}
+ \frac{b}{B_0} (1 + \lambda)(\mu_0 + P_{r0}) + \bar{c} \frac{1}{r} (1 + \lambda)(\mu_0 + P_{\phi 0}) + \frac{\lambda e Z_{R_0}}{2(1 - \lambda)} + Z H_0^{(p)} fr\} + \left(\frac{1 - \lambda}{1 - \lambda + \lambda Z}\right) \{\mu_0 e Z_{R_0}
+ \frac{b}{B_0} (1 + \lambda)(\mu_0 + P_{r0}) + \bar{c} \frac{1}{r} (1 + \lambda)(\mu_0 + P_{\phi 0}) + \frac{\lambda e Z_{R_0}}{2(1 - \lambda)} + Z H_0^{(p)} fr\} - \bar{c} \frac{1}{r} (1 + \lambda)(\mu_0 + P_{\phi 0}) + \frac{\lambda e Z_{R_0}}{2(1 - \lambda)} + Z H_0^{(p)} fr\}
\times \left(\frac{1 - \lambda^2 + 2\lambda Z}{Z} \right) \{\mu_0 e Z_{R_0} + \frac{b}{B_0} (1 + \lambda)(\mu_0 + P_{r0}) + \bar{c} \frac{1}{r} (1 + \lambda)(\mu_0 + P_{\phi 0}) + \frac{\lambda e Z_{R_0}}{2(1 - \lambda)} + Z H_0^{(p)} fr\} \right) + \Gamma \left(1 - 2\lambda^2 + 2\lambda Z\right) \{\lambda - \lambda^2 + 2\lambda Z\} \{\mu_0 e Z_{R_0}
+ \frac{b}{B_0} (1 + \lambda)(\mu_0 + P_{r0}) + \bar{c} \frac{1}{r} (1 + \lambda)(\mu_0 + P_{\phi 0}) + \frac{\lambda e Z_{R_0}}{2(1 - \lambda)} + Z H_0^{(p)} fr\} \right) - \Gamma \left(1 - 2\lambda^2 + 2\lambda Z\right) \{\lambda - \lambda^2 + 2\lambda Z\} \{\mu_0 e Z_{R_0}
+ \frac{b}{B_0} (1 + \lambda)(\mu_0 + P_{r0}) + \bar{c} \frac{1}{r} (1 + \lambda)(\mu_0 + P_{\phi 0}) + \frac{\lambda e Z_{R_0}}{2(1 - \lambda)} + Z H_0^{(p)} fr\} \right) - \frac{1 + \lambda}{Z} H_2 + H_1^{(p)} D = 0.
$$

(54)

A 2nd order differential equation is obtained after some manipulation in perturbed part of Ricci scalar, which takes the form as:

$$
D(t) - H_4(r) D(t) = 0,
$$

(55)

where $H_4$ is addressed in Appendix. Here, it is presumed that all the terms in $H_4$ are positive. The solution of above differential equation is of the form:

$$
D(t) = -e^{\sqrt{\Gamma} t}.
$$

(56)

To estimate the instability range in Newtonian and post-Newtonian regimes, Eq.\,(56) can be used in Eq.\,(55). The subsections followed by this section provides the dynamical analysis in both regimes.

### A. Newtonian Regime

To arrive at Newtonian-approximation, we assume $\mu_0 \gg p_{r0}, \mu_0 \gg p_{\phi 0}, \mu_0 \gg p_{r0}$, and $A_0 = 1, B_0 = 1$. Insertion of these assumptions along with the equations (55) and (56) leads to the following stability conditions

$$
\Gamma < \frac{H_4 U + V + \lambda - \frac{\ln Z}{Z(1 - \lambda)} W' + \lambda \left(\frac{1}{Z} + \frac{Z_{R_0 R_0}'}{Z^2}\right) W + \{(1 + \lambda) P_{r0} + \lambda \mu_0\} X + \mu_0 \frac{(1 + \lambda) a'}{Z} + (P_{r0} - P_{\phi 0}) Y + H_5}{\frac{(1 - \lambda^2 + 2\lambda Z)}{(1 - \lambda) Z} J' + \frac{1 + \lambda}{Z} \left(\frac{Z_{R_0 R_0}'}{Z} + \frac{1}{r}\right) J},
$$

(57)
where $H_{2N}$, $H_{0N}^{(p)}$ and $H_{1N}^{(p)}$ are the terms of $H_2$, $H_0^{(p)}$ and $H_1^{(p)}$ respectively that belong to Newtonian-approximation, and

$$U = e' + \frac{1}{r} - \frac{1}{Z} \left\{ Z R_0 (e' - b R_0) + e Z R_0 R_0' \right\} + \frac{b}{r^2},$$

$$V = \frac{e Z R_0}{Z (1 - \lambda)} \left\{ 2 \lambda - \frac{1 - \lambda^2 + 2 \lambda Z}{Z} P''_r + \left( \lambda - \frac{\lambda - \lambda^2 + \lambda Z}{Z} \right) \mu'_r \right\},$$

$$W = \frac{\mu_0 (\lambda - 1)}{1 - \lambda + \lambda Z} \left\{ e Z R_0 + (1 + \lambda) \left( b + \frac{\bar{e}}{r} \right) \right\}, \quad X = \frac{1}{Z^2} \left( e' Z R_0 + e Z R_0 R_0' - \frac{2 e Z^2 R_0'}{Z} \right),$$

$$Y = \frac{1 + \lambda}{Z} \left\{ \left( \frac{\bar{c}}{r} \right)' - \frac{e Z R_0}{Z r} \right\}, \quad J = P_0 + \frac{1 - \lambda^2}{1 - \lambda + \lambda Z} \left( b + \frac{\bar{e}}{r} \right).$$

The above condition describes the stability range of gravitating sources. All terms mentioned in the above inequality are presumed in a way that whole expression on right side of the adiabatic index $\Gamma$ remains positive. For the maintenance of the positivity, following constraints must be satisfied

$$e Z R_0 R_0' < Z R_0 (b R_0 - e'), \quad \left( \frac{\bar{c}}{r} \right)' > \frac{e Z R_0}{Z r}, \quad (58)$$

$$e' Z R_0 + e Z R_0 R_0' > \frac{2 e Z^2 R_0'}{Z}, \quad p_{r 0} > p_{\phi 0}. \quad (59)$$

### B. Post-Newtonian Regime

In this approximation, we choose

$$A_0 = 1 - \frac{m_0}{r} \quad \text{and} \quad B_0 = 1 + \frac{m_0}{r}, \quad \text{then}$$

$$\frac{A'_0}{A_0} = \frac{m_0}{r(r - m_0)}, \quad \frac{B'_0}{B_0} = - \frac{m_0}{r(r + m_0)}.$$ 

Substitution of above relations in Eq. (58) leads to the following inequality

$$\Gamma < \frac{H_4 H_8 + V + \frac{\lambda - \lambda^2 + \lambda Z}{Z (1 - \lambda)} \left( p_{\phi 0} + p_{\phi r} H_6 \right) + \frac{\lambda}{r} \left( e Z R_0 + \frac{m_0}{r(r - m_0)} \times \frac{1 + \lambda}{Z} \right) H_6}{\left( 1 - \frac{\lambda^2 + 2 \lambda Z}{Z (1 - \lambda)} \left( p_{\phi 0} + p_{\phi r} H_6 \right) + \frac{\lambda}{r} \left( e Z R_0 + \frac{m_0}{r(r - m_0)} + \frac{1}{r} \right) \left( p_{\phi 0} + p_{\phi r} H_6 \right) \right)}$$

$$+ \left( 1 + \lambda \right) P_{r 0} + \lambda \mu_0 \right) X + H_7 - \frac{1 + \lambda}{Z} H_{2N} + H_{1N}^{(p)} + H_{2N}^{(p)} + H_{1N}^{(p)} \right) \right)$$

where $H_{2pN}$, $H_{0pN}^{(p)}$ and $H_{1pN}^{(p)}$ are the terms of $H_2$, $H_0^{(p)}$ and $H_1^{(p)}$ that lie in post-Newtonian era.

$$H_6 = \left( \frac{\lambda - 1}{1 - \lambda + \lambda Z} \right) \left( \mu_0 e Z R_0 + \frac{r b}{r - m_0} (1 + \lambda) (\mu_0 + P_{r 0}) + \frac{\bar{c}}{r} (1 + \lambda) (\mu_0 + P_{\phi 0}) + \frac{\lambda e Z R_0}{2 (1 - \lambda)} + Z H_{0(pN)}^{(p)} \right),$$

$$H_7 = \left( 1 + \lambda \right) \left( \frac{\mu_0 + P_{r 0}}{Z^2} \right) \left( \frac{r a}{r - m_0} \right)' - \frac{e Z R_0}{Z^2} \times \frac{m_0}{r(r - m_0)} + (P_{r 0} - P_{\phi 0}) \left( \left( \frac{\bar{c}}{r} \right)' - \frac{e Z R_0}{Z r} \right) + \frac{\lambda e'}{2 (1 - \lambda)},$$

$$H_8 = \frac{1}{r^2} \left( \frac{r a}{r - m_0} \right)' \left[ e' + e + \frac{b}{r^2 (r + m_0)} - \left( \frac{Z R_0}{Z} \left( e' - e \frac{m_0}{r(r - m_0)} - \frac{br}{r - m_0} R_0' \right) + \frac{Z R_0 R_0' e R_0'}{Z} \right) + \frac{em_0}{r(r - m_0)} \right].$$

Again, to maintain the positivity of right side of the inequality (60), following conditions must be fulfilled

$$\left( \frac{ra}{r - m_0} \right)' - \frac{e Z R_0}{Z^2} \times \frac{m_0}{r(r - m_0)}, \quad \left( \frac{c}{r} \right)' > \frac{e Z R_0}{Z r},$$

$$e' + \frac{e}{r} + \frac{b}{r^2 (r + m_0)} > \frac{Z R_0}{Z} \left( e' - e \frac{m_0}{r(r - m_0)} - \frac{br}{r - m_0} R_0' \right) + \frac{Z R_0 R_0' e R_0'}{Z} - \frac{em_0}{r(r - m_0)}.$$
IV. SUMMARY

The cosmological observations from recent data-sets like cosmic microwave background, clustering spectrum, weak lensing, Planck data and supernovae type Ia unfolded that universe is expanding at an accelerated rate. Alternative gravitational theories have become a paradigm in description of gravitational interaction and its impact on expansion of universe. The alternative theories of gravity can be categorized as theories with extra gravitational fields, extra spatial dimension and higher derivatives. A large number of mechanisms have been presented to interpret the accelerated expansion of universe based on improvement of Einstein theory.

The $f(R,T)$ theory of gravity being generalization of $f(R)$ gravity representing alternative gravitation theory constituting non-minimal curvature matter coupling has gained increasing attention in recent years. Its gravitational action includes additional scalar force (trace of energy-momentum tensor) together with the function $f(R)$ of Ricci scalar, that further modifies the gravitational interaction. A scalar force is always appealing because it can reduce the time of collapse, so the addition of extra scalar term of $T$ in modified Einstein-Hilbert action provides better description of so called exotic matter.

Exploration of instability range in extended theories of gravity provide insight of gravitational interaction in current era that is expansion of universe. In this paper, we have studied the impact of $f(R,T)$ model on dynamical instability of cylindrically symmetric objects. The selection of $f(R,T)$ model for dynamical analysis is restricted to the form $f(R,T) = f(R) + \lambda T$, where $\lambda$ is an arbitrary positive constant. The $f(R,T)$ model under consideration constitutes combination of extended Starobinsky model i.e., $f(R) = R + \alpha R^2 + \gamma R^n$ for positive real values of $\alpha, \gamma$ and trace $T$ which provides a suitable replacement for dark source entities. The gravitating system chosen for analysis is assumed to be filled with anisotropic fluid in the interior.

For dynamical analysis, we have started with the construction of modified field equations within framework of $f(R,T)$ gravity for cylindrically symmetric gravitating system evolved under locally anisotropic background. Covariant divergence of effective energy-momentum tensor is taken into account to arrive at dynamical equations. The gravitating field equations describe a set of non-linear differential equations which are enough complex and their solutions are still unknown. That is why, we have chosen perturbation approach to counter this problem and considered the perturbation scheme proposed by Herrera et al.\cite{37}. The system is assumed to be static at initial stage, then gradually enters into the non-static phase depending on radial and time coordinates constituting same time dependence parameter.

In order to count with the issue of instability in a gravitating system, one may utilize numerical techniques or employ analytic approach. Highly complicated non-linear onset of modified field equations can be tackled essentially by implementing some numerical techniques and is of great importance in numerical relativity. The numerical design of Jeans analysis devised in \cite{38}-\cite{40} can be adopted for the deep insight of dynamical analysis of a particular gravitating source. However, numerical analysis may be confined to some particular model with some specific ranges of physical parameters and so turn out to be model dependent. We have chosen analytic approach to discuss the dynamics of stellar evolution for a class of models and presented general outcomes of gravitational interaction.

Linear perturbation has been applied on field equations and dynamical equations. The expressions for energy density $\mu$ and principal stresses $p_\phi$ and $p_z$ are obtained from perturbed forms of dynamical equation \cite{20} and field equations \cite{14} and \cite{15} respectively, while $p_r$ is extracted from the Harrison-Wheeler type equation. Perturbed dynamical equations together with perturbed differential equations leads to evolution equation that provides comprehensive description of celestial body of cylindrical shape for dynamical analysis. Analytic description of evolving stars can be carried through estimation of evolution equation constituting expression for adiabatic index $\Gamma$. The adiabatic index describes the stiffness in fluid distribution which is helpful in the estimation of instability eras for gravitational bodies in the presence of expansion scalar.

It is significantly important to mention here that the results of any gravitational theory must meet with the well-tested results of Newtonian (N) and post-Newtonian (pN) theories. Although, gravitational field is thought to be weak in N and pN regimes but testing the outcome in these eras is of fundamental importance. Corrections to N and pN regimes can be settled in $f(R,T)$ gravity that must be negligible or coincide with N and pN approximations. The adiabatic index $\Gamma$ requires positivity of the terms for the maintenance of stability of celestial objects in both N and pN regimes. Physical parameters involved in evolution equation are constrained to meet with the requirement of positive terms in expression for $\Gamma$ discussed in the subsections A and B of section III.

It is observed that the terms appearing in $\Gamma$ are less constrained in weak field regimes as compared to the terms that appear in case of $f(R)$ gravity, thus $f(R,T)$ gravity represents a wider class of viable models. Thus Corrections to GR can be made by assuming $\alpha \to 0$, $\gamma \to 0$, $\lambda \to 0$, while only $\lambda \to 0$ leads to the correction of $f(R)$ gravity. The local isotropy can be established by considering pressures same in $r$, $\phi$ and $z$ directions.
Appendix A

\[ H_0(r, t) = \left\{ \frac{1}{f_R} \left( \frac{f - R f_R}{2} + \eta_{00} \right) \right\}_{0} - \frac{1}{B^2} \left\{ \frac{1}{f_R} \left( f' - \frac{A'}{A} f - \frac{B'}{B} f' \right) \right\}_{1} - \frac{1}{B^2 f_R} \left( f' - \frac{A'}{A} f - \frac{B'}{B} f' \right) \]
\[ \times \left( \frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} \right) + \frac{\eta_{00}}{f_R} \left( \frac{B'}{B} + \frac{C'}{C} \right) + \frac{B \eta_{11}}{B f_R} + \frac{C \eta_{22}}{f_R} . \]

\[ H_1(r, t) = \left\{ \frac{1}{f_R} \left( \frac{R f_R - f}{2} + \eta_{11} \right) \right\}_{1} - \frac{1}{A^2} \left\{ \frac{1}{f_R} \left( f' - \frac{A'}{A} f - \frac{B'}{B} f' \right) \right\}_{0} + \frac{1}{B^2 f_R} \left( f' - \frac{A'}{A} f - \frac{B'}{B} f' \right) \]
\[ \times \left( \frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} \right) + \frac{A'}{A} \frac{1}{f_R} (\eta_{00} + \eta_{11}) + \frac{C'}{C} \frac{1}{f_R} (\eta_{11} - \eta_{22}). \]

\[ H_0^{(p)} = \frac{e}{2Z^2} \left\{ \lambda Z - Z_R (R_0 + \alpha R_0^2 + \gamma R_0^3 + \lambda T_0) \right\} - \frac{1}{B^2} \left\{ \frac{1}{Z} \left( Z_R \left( \frac{A'}{A} - \frac{A_0'}{A_0} \right) + e R_0 Z R_0 \right) \right\}_{1} + \frac{1}{Z} \frac{\eta_{00}^{(p_1)}}{Z_0} \]
\[ - \frac{1}{B^0} \left\{ \frac{1}{Z} \left( Z_R \left( \frac{A'}{A} - \frac{A_0'}{A_0} - R_0 \right) e + e R_0 Z R_0 \right) \right\}_{1} \left( \frac{A_0'}{A_0} - \frac{b}{B_0} + \frac{1}{r} \right) + \frac{1}{Z} \left( b + \frac{c}{r} - e Z R_0 \right) \frac{\eta_{00}^{(s)}}{Z_0} + \frac{b}{B_0} \]
\[ \times \frac{\eta_{11}^{(s)}}{Z} + \frac{c \eta_{22}^{(s)}}{r Z}, \]

\[ H_1^{(p)} = \left( \frac{e}{2Z^2} \right) \left\{ \lambda Z - Z_R (R_0 + \alpha R_0^2 + \gamma R_0^3 + \lambda T_0) - \frac{\eta_{00}^{(p_1)}}{Z} \right\} - \frac{1}{Z} \left\{ \left( \frac{A_0'}{A_0} - e Z R_0 \right) \frac{\eta_{00}^{(s)}}{Z} \right\}_{1} \]
\[ \times \frac{\eta_{11}^{(s)}}{Z} + \frac{\eta_{00}^{(p_1)}}{Z} + \frac{A_0'}{A_0} \left( \frac{\eta_{00}^{(p_1)}}{Z_0} + \frac{\eta_{00}^{(p_1)}}{Z_0} \right) \left\{ \left( \frac{c}{r} - \frac{e Z R_0}{Z} \right) \frac{\eta_{11}^{(s)}}{Z} + \frac{1}{r} \left( \frac{\eta_{11}^{(p_1)}}{Z_0} - \frac{\eta_{11}^{(p_1)}}{Z_0} \right) \right\}. \]

\[ H_2 = \frac{1}{1 + \lambda} \left[ \frac{Z}{A_0 B_0} + \frac{2b A'_0 B'_0}{B_0^2} + \left( \frac{A'_0}{B_0} \right)' - \frac{(a_0 b)'}{B_0^2} - \left( \frac{a}{A_0} + \frac{b}{B_0} \right) \left( \frac{A_0'}{A_0} \right)' + \frac{1}{Z} \left( e Z_R - \frac{2 c Z}{r} \right), \right] \]
\[ \times \left( \frac{A_0'}{B_0} \right)' \right\} + \frac{e}{2} \left( \lambda - R_0 Z R_0 \right) - \frac{\eta_{11}^{(p_1)}}{Z_0} \right\}, \]

\[ H_3 = \frac{1}{1 + \lambda} \left[ \frac{Z}{A_0 B_0} + \frac{2b A'_0 B'_0}{B_0^2} + \left( \frac{a_0 b}{} \right)' - \left( \frac{A_0 b'}{B_0} \right)' - \frac{1}{B_0^2} \left( \frac{a}{A_0} + \frac{b}{B_0} \right) \left( \frac{A_0'}{A_0} \right)' + \frac{e Z R_0}{Z} \right( \frac{A_0'}{B_0} \right)' \right\} - \frac{1}{r B_0^2} \]
\[ \times \left\{ e Z_R \frac{A_0 (b)'}{A_0 B_0} + \frac{1}{B_0} \left( b' - \frac{3 b (a)'}{B_0} \right) - r \left( \frac{A_0'}{A_0} - \frac{B_0'}{B_0} \right) \left( \frac{c}{r} \right)' - \left( \frac{a}{A_0} \right)' + \frac{2 b a_0'}{B_0 A_0} \right\} + e \left( \lambda - R_0 Z R_0 \right) - \frac{\eta_{11}^{(p_1)}}{Z_0} \right\}, \]

\[ H_4 = \frac{A_0 b_0}{b_0} \left[ \frac{1}{B_0^2} \left( \frac{a''}{A_0} + \frac{a_0''}{A_0} \right) + \left( \frac{c}{r} \right)' - \frac{1}{r} \left( \frac{b}{B_0} \right)' + \left( \frac{a}{A_0} \right)' + \frac{1}{r} \left( \frac{b_0}{B_0} \right)' \right] \]
\[ - \frac{2 b a_0'}{B_0 A_0} \right\} - \frac{e}{2} \right], \]

\[ H_5 = \left( \frac{\lambda - \frac{1}{1 + \lambda}}{1 - \lambda + \lambda Z} \right) \left( \frac{A_0 b_0}{A_0} \right)' \right\} \left( \frac{\lambda Z R_0}{Z} \right)' + Z H_1^{(p)} + \lambda - \lambda^2 + \lambda Z \left\{ \frac{1}{1 - \lambda + \lambda Z} \left( \frac{\lambda Z}{2(1 - \lambda)} + Z H_1^{(p)} \right) \right\} \right\}
\[ + \left. \frac{\lambda e'}{2(1 - \lambda)} + H_1^{(p)} + \frac{1 + \lambda}{Zr} H_2^{(p)} \right\} \right\} \right\}
\[ + \frac{\lambda e'}{2(1 - \lambda)} + H_1^{(p)} + \frac{1 + \lambda}{Zr} H_2^{(p)}. \]
