THE NUMERICAL RADIUS HAAGERUP NORM AND HILBERT SPACE SQUARE FACTORIZATIONS

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Abstract. We study a factorization of bounded linear maps from an operator space $A$ to its dual space $A^*$. It is shown that $T : A \rightarrow A^*$ factors through a pair of a column Hilbert spaces $H_c$ and its dual space if and only if $T$ is a bounded linear form on $A \otimes A$ by the canonical identification equipped with a numerical radius type Haagerup norm. As a consequence, we characterize a bounded linear map from a Banach space to its dual space, which factors through a pair of Hilbert spaces.

1. Introduction

The factorization through a Hilbert space of a linear map plays one of the central roles in the Banach space theory (c.f. [17]). Also in the $C^*$-algebra and the operator space theory, many important factorization theorems have been proved related to the Grothendieck type inequality in several situations [8], [5], [18], [21].

Let $\alpha$ be a bounded linear map from $\ell^1$ to $\ell^\infty$, $\{e_i\}_{i=1}^\infty$ the canonical basis of $\ell^1$, and $B(\ell^2)$ the bounded operators on $\ell^2$. We regard $\alpha$ as the infinite dimensional matrix $[\alpha_{ij}]$ where $\alpha_{ij} = \langle e_i, \alpha(e_j) \rangle$. The Schur multiplier $S_\alpha$ on $B(\ell^2)$ is defined by $S_\alpha(x) = \alpha \circ x$ for $x = [x_{ij}] \in B(\ell^2)$ where $\alpha \circ x$ is the Schur product $[\alpha_{ij}x_{ij}]$. Let $w(\cdot)$ be the numerical radius norm on $B(\ell^2)$. In [12], it was shown that

$$\| S_\alpha \|_w = \sup_{x \neq 0} \frac{w(\alpha \circ x)}{w(x)} \leq 1$$

if and only if $\alpha$ has the following factorization with $\|a\|\|b\| \leq 1$:

$$\begin{array}{ccc}
\ell^1 & \overset{\alpha}{\longrightarrow} & \ell^\infty \\
\downarrow & & \uparrow a^t \\
\ell^2 & \underset{b}{\longrightarrow} & \ell^2^* \\
\end{array}$$

where $a^t$ is the transposed map of $a$.

Motivated by the above result, we will show a square factorization theorem of a bounded linear map through a pair of column Hilbert spaces $H_c$ between an operator space and its dual space. More precisely, let us suppose that $A$ is an operator space in $B(H)$ and $A \otimes A$ is the algebraic tensor product. We
define the numerical radius Haagerup norm of an element \( u \in A \otimes A \) by
\[
\| u \|_{wh} = \inf \left\{ \frac{1}{2} \| [x_1, \ldots, x_n, y_1^*, \ldots, y_n^*] \|^2 \mid u = \sum_{i=1}^n x_i \otimes y_i \right\}.
\]

Let \( T : A \to A^* \) be a bounded linear map. We show that \( T : A \to A^* \) has an extension \( T' \) which factors through a pair of column Hilbert spaces \( \mathcal{H}_c \) so that
\[
C^*(A) \xrightarrow{T'} C^*(A)^*
\]
\[
a \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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2. Factorization on operator spaces

Let $\mathcal{B}(\mathcal{H})$ be the space of all bounded operators on a Hilbert space $\mathcal{H}$. Throughout this paper, let us suppose that $A$ and $B$ are operator spaces in $\mathcal{B}(\mathcal{H})$. We denote by $C^*(A)$ the $C^*$-algebra in $\mathcal{B}(\mathcal{H})$ generated by the operator space $A$. We define the numerical radius Haagerup norm of an element $u \in A \otimes B$ by

$$
\|u\|_{wh} = \inf \left\{ \frac{1}{2} \| [x_1, \ldots, x_n, y_1^*, \ldots, y_n^*] \|_2^2 \mid u = \sum_{i=1}^n x_i \otimes y_i \right\},
$$

where $[x_1, \ldots, x_n, y_1^*, \ldots, y_n^*] \in M_{2n}(C^*(A + B))$, and denote by $A \otimes_{wh} B$ the completion of $A \otimes B$ with the norm $\| \cdot \|_{wh}$.

Recall that the Haagerup norm on $A \otimes B$ is

$$
\|u\|_h = \inf \{ \| [x_1, \ldots, x_n] \| \| [y_1, \ldots, y_n]^t \| \mid u = \sum_{i=1}^n x_i \otimes y_i \},
$$

where $[x_1, \ldots, x_n] \in M_{1,n}(A)$ and $[y_1, \ldots, y_n]^t \in M_{n,1}(B)$.

By the identity

$$
\inf_{\lambda > 0} \frac{\lambda \alpha + \lambda^{-1} \beta}{2} = \sqrt{\alpha \beta}
$$

for positive real numbers $\alpha, \beta \geq 0$, the Haagerup norm can be rewritten as

$$
\|u\|_h = \inf \left\{ \frac{1}{2} \| [x_1, \ldots, x_n] \|^2 + \|[y_1^*, \ldots, y_n^*] \|^2 \mid u = \sum_{i=1}^n x_i \otimes y_i \right\}.
$$

Then it is easy to check that

$$
\frac{1}{2} \|u\|_h \leq \|u\|_{wh} \leq \|u\|_h
$$

and $\|u\|_{wh}$ is a norm. We use the notation $x \alpha \odot y^t$ for $\sum_{i=1}^n \sum_{j=1}^m x_i \alpha_{ij} \otimes y_j$, where $x = [x_1, \ldots, x_n] \in M_{1,n}(A)$, $\alpha = [\alpha_{ij}] \in M_{n,m}(\mathbb{C})$ and $y^t = [y_1^*, \ldots, y_n^*] \in M_{m,1}(B)$. We note the identity $x \alpha \odot y^t = x \odot \alpha y^t$.

First we show that the numerical radius Haagerup norm has the injectivity.

**Proposition 2.1.** Let $A_1 \subset A_2$ and $B_1 \subset B_2$ be operator spaces in $\mathcal{B}(\mathcal{H})$. Then the canonical inclusion $\Phi$ of $A_1 \otimes_{wh} B_1$ into $A_2 \otimes_{wh} B_2$ is isometric.

**Proof.** The inequality $\|\Phi(u)\|_{wh} \leq \|u\|_{wh}$ is trivial. To get the converse inequality, let $u = \sum_{i=1}^n x_i \otimes y_i \in A_1 \otimes B_1$. We may assume that $\{y_1, \ldots, y_k\} \subset B_2$ is linearly independent and there exists an $n \times k$ matrix of scalars $L \in M_{nk}(\mathbb{C})$ such that $[y_1, \ldots, y_n]^t = L[y_1, \ldots, y_k]^t$. We put $z^t = [y_1, \ldots, y_k]^t$.

Then we have

$$
u = x \odot y^t = x \odot Lz^t = xL(L^*L)^{-1/2} \odot (L^*L)^{1/2} z^t.$$
and
\[ \|x L(L^* L)^{-1/2}, ((L^* L)^{1/2} z^t)^*\| \leq \|x, (y^t)^*\|. \]

So we can get a representation \( u = [x'_1, \ldots, x'_k] \odot [y'_1, \ldots, y'_k]^t \) with
\[ \|[[x'_1, \ldots, x'_k, y'_1, \ldots, y'_k]^t] \| \leq \|[[x, (y^t)]^t] \|
\]
and \( \{y'_1, \ldots, y'_k\} \) is linearly independent. This implies that \( x'_1, \ldots, x'_k \in A_1 \).

Applying the same argument for \( x'_1, \ldots, x'_k \) instead of \( \{y_1, \ldots, y_n\} \), we can get a representation \( u = [x''_1, \ldots, x''_t] \odot [y''_1, \ldots, y''_t]^t \) with
\[ \|[[x''_1, \ldots, x''_t, y''_1, \ldots, y''_t]^t] \| \leq \|[[x, (y^t)]^t] \|
\]
and \( x''_i \in A_1 \) and \( y''_i \in B_1 \). It follows that \( \|\Phi(u)\|_{wh} \geq \|u\|_{wh} \).

We also define a norm of an element \( u \in C^*(A) \otimes C^*(A) \) by
\[ \|u\|_{wh} = \inf \{ \|[[x_1, \ldots, x_n]^t]\|^2 w(\alpha) \mid u = \sum x_i^* \alpha_{ij} \otimes x_j \}, \]
where \( w(\alpha) \) is the numerical radius norm of \( \alpha = [\alpha_{ij}] \) in \( M_n(\mathbb{C}) \).

\( A \otimes_{wh} A \) is defined as the closure of \( A \otimes A \) in \( C^*(A) \otimes_{wh} C^*(A) \).

**Theorem 2.2.** Let \( A \) be an operator space in \( \mathbb{B}(\mathcal{H}) \). Then \( A \otimes_{wh} A = A \otimes_{wh} A \).

**Proof.** By Proposition 2.1 and the definition of \( A \otimes_{wh} A \), it is sufficient to show that \( C^*(A) \otimes_{wh} C^*(A) = C^*(A) \otimes_{wh} C^*(A) \).

Given \( u = \sum_{i=1}^n x_i \otimes y_i \in C^*(A) \otimes C^*(A) \), we have
\[ u = [x_1, \ldots, x_n, y_1^*, \ldots, y_n^*]\left[ \begin{array}{cc} 0_n & 1_n \\ 0_n & 0_n \end{array} \right] \odot [x_1^*, \ldots, x_n^*, y_1, \ldots, y_n]^t. \]

Since \( w(\left[ \begin{array}{cc} 0_n & 1_n \\ 0_n & 0_n \end{array} \right]) = \frac{1}{2} \), then \( \|u\|_{wh} \geq \|u\|_{wh} \).

To establish the reverse inequality, suppose that \( u = \sum_{i,j=1}^n x_i^* \alpha_{ij} \otimes x_j \in C^*(A) \otimes C^*(A) \) with \( w(\alpha) = 1 \) and \( \|[[x_1, \ldots, x_n]^t]\|^2 = 1 \). It is enough to see that there exist \( c_i, d_i \in C^*(A) (i = 1, \ldots, m) \) such that \( u = \sum_{i=1}^m c_i \otimes d_i \) with \( \|[[c_1, \ldots, c_m, d_1^*, \ldots, d_m^*]\|^2 \leq 2 \). By the assumption \( w(\alpha) = 1 \) and Ando’s Theorem [1], we can find a self-adjoint matrix \( \beta \in M_n(\mathbb{C}) \) for which
\[ P = \begin{bmatrix} 1 + \beta & \alpha \\ \alpha^* & 1 - \beta \end{bmatrix} \] is positive definite in \( M_{2n}(\mathbb{C}) \).
Set \([c_1, \ldots, c_{2n}] = [x_1^*, \ldots, x_n^*, 0, \ldots, 0]P^{1/2}\) and \([d_1, \ldots, d_{2n}]^t = P^{1/2}[0, \ldots, 0, x_1, \ldots, x_n]^t\). We note that \(u = [c_1, \ldots, c_{2n}] \odot [d_1, \ldots, d_{2n}]^t\). Then we have
\[
\|[(c_1, \ldots, c_{2n}, d_1^t, \ldots, d_{2n}^t)]^2
= \|x_1^*, \ldots, x_n^*, 0, \ldots, 0\|P[x_1, \ldots, x_n, 0, \ldots, 0]^t
+ [0, \ldots, 0, x_1^*, \ldots, x_n^*]P[0, \ldots, 0, x_1, \ldots, x_n]^t\|
= \|[x_1^*, \ldots, x_n^*](1 + \beta)[x_1, \ldots, x_n]^t
+ [x_1^*, \ldots, x_n^*(1 - \beta)[x_1, \ldots, x_n]^t\|
= 2\|[x_1, \ldots, x_n]^t\|^2 = 2.
\]

We recall the column (resp. row) Hilbert space \(\mathcal{H}_c\) (resp. \(\mathcal{H}_r\)) for a Hilbert space \(\mathcal{H}\). If \(\xi = [\xi_{ij}] \in M_n(\mathcal{H})\), then we define a map \(C_n(\xi)\) by
\[
C_n(\xi) : \mathbb{C}^n \ni [\lambda_1, \ldots, \lambda_n] \longmapsto \sum_{j=1}^n \lambda_j \xi_{ij}] \in \mathcal{H}^n
\]
and denote the column matrix norm by \(\|\xi\|_c = \|C_n(\xi)\|\). This operator space structure on \(\mathcal{H}\) is called the column Hilbert space and denoted by \(\mathcal{H}_c\).

To consider the row Hilbert space, let \(\overline{\mathcal{H}}\) be the conjugate Hilbert space for \(\mathcal{H}\). We define a map \(R_n(\xi)\) by
\[
R_n(\xi) : \overline{\mathcal{H}}^n \ni [\eta_1, \ldots, \eta_n] \longmapsto \sum_{j=1}^n (\xi_{ij}|\eta_j)] \in \mathbb{C}^n
\]
and the row matrix norm by \(\|\xi\|_r = \|R_n(\xi)\|\). This operator space structure on \(\mathcal{H}\) is called the row Hilbert space and denoted by \(\mathcal{H}_r\).

Let \(a : C^*(A) \rightarrow \mathcal{H}_c\) be a completely bounded map. We define a map \(d : C^*(A) \rightarrow \overline{\mathcal{H}}\) by \(d(x) = a(x^*)\). It is not hard to check that \(d : C^*(A) \rightarrow \overline{\mathcal{H}}_r\) is completely bounded and \(\|a\|_{cb} = \|d\|_{cb}\) when we introduce the row Hilbert space structure to \(\overline{\mathcal{H}}\). In this paper, we define the adjoint map \(a^*\) of \(a\) by the transposed map of \(d\), that is, \(d^t : (\overline{\mathcal{H}})_r^* = ((\mathcal{H}_r)^*)_r^* = (\mathcal{H}_c)^* = \mathcal{H}_c \rightarrow C^*(A)^*\) (c.f. \([5]\)). More precisely, we define
\[
\langle a^*(\eta), x \rangle = \langle \eta, d(x) \rangle = \langle \eta|a(x^*) \rangle \quad \text{for} \quad \eta \in \mathcal{H}, x \in C^*(A).
\]

A linear map \(T : A \rightarrow A^*\) can be identified with the bilinear form \(A \times A \ni (x, y) \longmapsto \langle x, T(y) \rangle \in \mathbb{C}\) and also the linear form \(A \otimes A \rightarrow \mathbb{C}\). We use \(T\) also to denote both of the bilinear form and the linear form, and \(\|T\|_{\beta*}\) to denote the norm when \(A \otimes A\) is equipped with a norm \(\|\|_{\beta}^\|\|\).

We are going to prove the main theorem. The proof will be given by the similar way as in the case of the original Haagerup norm in \([4]\).
Theorem 2.3. Suppose that $A$ is an operator space in $\mathcal{B}(\mathcal{H})$, and that $T : A \times A \to \mathbb{C}$ is bilinear. Then the following are equivalent:

1. $\|T\|_{wh^*} \leq 1$.
2. There exists a state $p_0$ on $C^*(A)$ such that
   $$|T(x, y)| \leq p_0(xx^*)^{\frac{1}{2}}p_0(yyy^*)^{\frac{1}{2}}$$
   for $x, y \in A$.
3. There exists a $\ast$-representation $\pi : C^*(A) \to \mathcal{B}(\mathcal{K})$, a unit vector $\xi \in \mathcal{K}$ and a contraction $b \in \mathcal{B}(\mathcal{K})$ such that
   $$T(x, y) = (\pi(x)b\pi(y)\xi \mid \xi)$$
   for $x, y \in A$.
4. There exist an extension $T' : C^*(A) \to C^*(A)^\ast$ of $T$ and completely bounded maps $a : C^*(A) \to \mathcal{K}_c$, $b : \mathcal{K}_c \to \mathcal{K}_c$ such that
   $$\xymatrix{\mathcal{K}_c \ar[r]^b & \mathcal{K}_c \\
C^*(A) \ar[u]^{a^*} \ar[r]^{T'} & C^*(A)^\ast \ar[u]_a \ar[r] & C^*(A)^\ast}
$$
   i.e., $T' = a^*b$ with $\|a\|^2_{cb}\|b\|_{cb} \leq 1$.

Proof. (1)$\Rightarrow$(2) By Proposition 2.1, we can extend $T$ on $C^*(A) \otimes_{wh} C^*(A)$ and also denote it by $T$. We may assume $\|T\|_{wh^*} \leq 1$. By the identity $(\ast)$, it is sufficient to show the existence of a state $p_0 \in S(C^*(A))$ such that
   $$|T(x, y)| \leq \frac{1}{2}p_0(xx^* + y^*y)$$
   for $x, y \in C^*(A)$.

Moreover it is enough to find $p_0 \in S(C^*(A))$ such that
   $$\text{Re}T(x, y) \leq \frac{1}{2}p_0(xx^* + y^*y)$$
   for $x, y \in C^*(A)$.

Define a real valued function $T_{\{x_1, \ldots, x_n, y_1, \ldots, y_n\}}(\cdot)$ on $S(C^*(A))$ by
   $$T_{\{x_1, \ldots, x_n, y_1, \ldots, y_n\}}(p) = \sum_{i=1}^n \frac{1}{2}p(x_ix_i^* + y_i^*y_i) - \text{Re}T(x_i, y_i),$$
   for $x_i, y_i \in C^*(A)$. Set
   $$\Delta = \{T_{\{x_1, \ldots, x_n, y_1, \ldots, y_n\}} \mid x_i, y_i \in C^*(A), n \in \mathbb{N}\}.$$  

It is easy to see that $\Delta$ is a cone in the set of all real functions on $S(C^*(A))$. Let $\nabla$ be the open cone of all strictly negative functions on $S(C^*(A))$. For any $x_1, \ldots, x_n, y_1, \ldots, y_n \in C^*(A)$, there exists $p_1 \in S(C^*(A))$ such that
\[ p_1(\sum x_i^* x_i + y_i^* y_i) = \| \sum x_i^* x_i + y_i^* y_i \|. \]  

Since
\[
T_{\{x_1, \ldots, x_n, y_1, \ldots, y_n\}}(p_1) = \frac{1}{2} p_1(\sum x_i^* x_i + y_i^* y_i) - \text{Re} \sum T(x_i, y_i)
\]
\[
= \frac{1}{2} \| \sum x_i x_i^* + y_i y_i^* \| - \text{Re} \sum T(x_i, y_i)
\]
\[
\geq \frac{1}{2} \| \sum x_i x_i^* + y_i y_i^* \| - | \sum T(x_i, y_i) |
\]
\[
\geq 0,
\]

then \( \Delta \cap \nabla = \phi \).

By the Hahn-Banach Theorem, there exists a measure \( \mu \) on \( S(C^*(A)) \) such that \( \mu(\Delta) \geq 0 \) and \( \mu(\nabla) < 0 \). So we may assume that \( \mu \) is a probability measure. Now put \( p_0 = \int p d\mu(p) \). Since \( T_{\{x, y\}} \in \Delta \), then
\[
\frac{1}{2} p_0(xx^* + y^* y) - \text{Re} T(x, y) = \int T_{\{x, y\}}(p) d\mu(p) \geq 0.
\]

(2) \( \Rightarrow \) (1) Since
\[
| \sum T(x_i, y_i) | \leq \sum p_0(x_i x_i^*)^{\frac{1}{2}} p_0(y_i y_i^*)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{2} \sum p_0(x_i x_i^* + y_i y_i^*)
\]
\[
\leq \frac{1}{2} \| [x_1, \ldots, x_n, y_1^*, \ldots, y_n^*] \|^2
\]
for \( x, y \in A \), then we have that \( T \in (A \otimes \text{wh} A)^* \) with \( \| T \|_{\text{wh}^*} \leq 1 \).

(1) \( \Rightarrow \) (3) As in the proof of the implication (1) \( \Rightarrow \) (2), we can find a state \( p \in S(C^*(A)) \) such that \( |T(x, y)| \leq p(xx^*)^{\frac{1}{2}} p(y^* y)^{\frac{1}{2}} \) for \( x, y \in C^*(A) \). By the GNS construction, we let \( \pi : C^*(A) \rightarrow \mathbb{B}(K) \) is the cyclic representation with the cyclic vector \( \xi \) and \( p(x) = \langle \pi(x) \xi \mid \xi \rangle \) for \( x \in C^*(A) \). Define a sesquilinear form on \( K \times K \) by \( \langle \pi(y) \xi, \pi(x) \xi \rangle = T(x^*, y) \). This is well-defined and bounded since
\[
| \langle \pi(y) \xi, \pi(x) \xi \rangle | \leq p(x^* x)^{\frac{1}{2}} p(y^* y)^{\frac{1}{2}} = \| \pi(x) \xi \| \| \pi(y) \xi \|.
\]

Thus there exists a contraction \( b \in \mathbb{B}(K) \) such that \( T(x^*, y) = (b\pi(y) \xi | \pi(x) \xi) \).

(3) \( \Rightarrow \) (4) Set \( a(x) = \pi(x) \xi \) for \( x \in C^*(A) \) and consider the column Hilbert structure for \( K \). Then it is easy to see that \( a : C^*(A) \rightarrow K_c \) is a completely contraction. Define that \( T' = a^* ba \), then it turns out \( T' \) is an extension of \( T \) and \( \| a \|^2 \| b \|_{cb} \leq 1 \).
(4)⇒(1) Since \( T'(x, y) = (ba(y)|a(x^*)) \) for \( x, y \in C^*(A) \), then we have
\[
| \sum_{i,j=1}^n T'(x_i^* \alpha_{ij}, x_j) | = | \sum_{i,j=1}^n (ba_{ij} a(x_j) | a(x_i)) |
\]
\[
= \left| \left[ \begin{array}{cc}
 b & 0 \\
 0 & b \\
 \end{array} \right] \left[ \begin{array}{c}
 \alpha_{ij} \\
 a(x_1) \\
 \vdots \\
 a(x_n) \\
 \end{array} \right] \right| \left| \left[ \begin{array}{c}
 a(x_1) \\
 \vdots \\
 a(x_n) \\
 \end{array} \right] \right| 
\]
\[
\leq w \left( \left[ \begin{array}{cc}
 b & 0 \\
 0 & b \\
 \end{array} \right] \left[ \begin{array}{c}
 \alpha_{ij} \\
 a(x_1) \\
 \vdots \\
 a(x_n) \\
 \end{array} \right] \right) \left\| \left[ \begin{array}{c}
 a(x_1) \\
 \vdots \\
 a(x_n) \\
 \end{array} \right] \right\|^2
\]
\[
\leq \|b\|_cb w(\alpha) \|a\|_cb^2 \left\| \left[ \begin{array}{c}
 x_1 \\
 \vdots \\
 x_n \\
 \end{array} \right] \right\|^2
\]
for \( \sum_{i,j=1}^n x_i^* \alpha_{ij} \otimes x_j \in C^*(A) \otimes C^*(A) \). At the last inequality, we use two facts which \( w(cd) \leq \|c\|w(d) \) for double commuting operators \( c, d \), and \( \mathbb{B}(K, K) \) is completely isometric onto \( CB(K, K) \). Hence we obtain that
\[
\|T\|_{wh^*} \leq \|T'\|_{wh^*} \leq 1. \tag*{\square}
\]

Remark 2.4. (i) If we replace the linear map \( \langle T(x), y \rangle = T(x, y) \) with \( \langle x, T(y) \rangle = T(x, y) \) in Theorem 2.3, then we have a factorization of \( T \) through a pair of the row Hilbert spaces \( \mathcal{H}_r \). More precisely, the following condition (4)' is equivalent to the conditions in Theorem 2.3.

(4)' There exist an extension \( T' : C^*(A) \to C^*(A)^* \) of \( T \) and completely bounded maps \( a : C^*(A) \to K_r, b : K_r \to K_r \) such that
\[
\begin{array}{ccc}
C^*(A) & \xrightarrow{T} & C^*(A)^* \\
\downarrow a & & \uparrow a^* \\
K_r & \xrightarrow{b} & K_r
\end{array}
\]
i.e., \( T' = a^*ba \) with \( \|a\|_{cb} \|b\|_{cb} \leq 1 \).

(ii) Let \( \ell^2_n \) be an \( n \)-dimensional Hilbert space with the canonical basis \( \{e_1, \ldots, e_n\} \). Given \( \alpha : \ell^2_n \to \ell^2_n \) with \( \alpha(e_j) = \sum_i \alpha_{ij} e_i \), we set the map \( \hat{\alpha} : \ell^2_n \to \ell^2_n^* \) by \( \hat{\alpha}(e_j) = \sum_i \alpha_{ij} \hat{e}_i \) where \( \{\hat{e}_i\} \) is the dual basis. For notational convenience, we shall also denote \( \hat{\alpha} \) by \( \alpha \). For \( \sum_{i=1}^n x_i \otimes e_i \in C^*(A) \otimes \ell^2_n \), we define a norm by \( \| \sum_{i=1}^n x_i \otimes e_i \| = \| [x_1, \ldots, x_n]^t \| \). Let \( T : C^*(A) \to C^*(A)^* \) be a bounded linear map. Consider \( T \otimes \alpha : C^*(A) \otimes \ell^2_n \to C^*(A)^* \otimes \ell^2_n^* \) with a numerical radius type norm \( w(\cdot) \) given by
\[
w(T \otimes \alpha) = \sup \{ | \sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i) | \mid \| \sum x_i \otimes e_i \| \leq 1 \}.
\]
Then we have
\[
\sup \left\{ \frac{w(T \otimes \alpha)}{w(\alpha)} : \alpha : \ell_n^2 \rightarrow \ell_n^2, \ n \in \mathbb{N} \right\} = \|T\|_{\text{wh}^*},
\]
since \(T(\sum x^*_i \alpha_{ij} \otimes x_j) = (\sum x^*_i \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i)).\)

(iii) Let \(u = \sum x_i \otimes y_i \in C^*(A) \otimes C^*(A).\) It is straightforward from Theorem 2.3 that
\[
\|u\|_{\text{wh}} = \sup \left\{ \varphi(x_i) b\varphi(y_i) : \varphi \text{ and } b \text{ are } \ast\text{-preserving completely contractions} \right\}
\]
where the supremum is taken over all \(*\)-preserving completely contractions \(\varphi\) and contractions \(b.\)

3. A variant of the numerical radius Haagerup norm

In this section, we study a factorization of \(T: A \rightarrow A^*\) through a column Hilbert space \(K_c\) and its dual operator space \(K_c^*\). Since the arguments and proofs of this section are almost the same as those given in section 2, we only indicate the places where the changes are needed.

We define a variant of the numerical radius Haagerup norm of an element \(u \in A \otimes B\) by
\[
\|u\|_{\text{wh}'} = \inf \left\{ \frac{1}{2} \| [x_1, \ldots, x_n, y_1, \ldots, y_n]^t \|^2 : u = \sum_{i=1}^n x_i \otimes y_i \right\},
\]
where \([x_1, \ldots, x_n, y_1, \ldots, y_n]^t \in M_{2n,1}(A + B),\) and denote by \(A \otimes_{\text{wh}'} B\) the completion of \(A \otimes B\) with the norm \(\|\|_{\text{wh}'}\).

We remark that \(\|\|_{\text{wh}}\) and \(\|\|_{\text{wh}'}\) are not equivalent, since \(\|\|_h\) in [10] is equivalent to \(\|\|_{\text{wh}'}\) and \(\|\|_h\) and \(\|\|_{\text{wh}'}\) are not equivalent [10], [13].

**Proposition 3.1.** Let \(A_1 \subset A_2\) and \(B_1 \subset B_2\) be operator spaces in \(\mathcal{B}(\mathcal{H})\). Then the canonical inclusion \(\Phi\) of \(A_1 \otimes_{\text{wh}'} B_1\) into \(A_2 \otimes_{\text{wh}'} B_2\) is isometric.

**Proof.** The proof is almost the same as that given in Proposition 2.1. \(\square\)

In the next theorem, we use the transposed map \(a^t : (K_c)^* \rightarrow C^*(A)^*\) of \(a : C^*(A)^* \rightarrow K_c\) instead of \(a^* : K_c \rightarrow C^*(A)^*\). We note that \((K_c)^* = (\overline{K})_r\) and the relation \(a\) and \(a^t\) is given by
\[
\langle a^t(\overline{\eta}), x \rangle = \langle \overline{\eta}, a(x) \rangle = \langle \overline{\eta}, \overline{a(x)} \rangle_{\overline{K}} \text{ for } \overline{\eta} \in \overline{K}, x \in C^*(A).
\]

It seems that the fourth condition in the next theorem is simpler than the fourth one in Theorem 2.3, since we do not use \(\ast\)-structure.
Theorem 3.2. Suppose that $A$ is an operator space in $\mathcal{B}(\mathcal{H})$, and that $T : A \times A \rightarrow \mathbb{C}$ is bilinear. Then the following are equivalent:

1. $\|T\|_{\text{wh}^{\ast}} \leq 1$.
2. There exists a state $p_0$ on $C^*(A)$ such that

   $$|T(x, y)| \leq p_0(x^*x)^{\frac{1}{2}}p_0(y^*y)^{\frac{1}{2}} \quad \text{for } x, y \in A.$$

3. There exist a $*$-representation $\pi : C^*(A) \rightarrow \mathcal{B}(\mathcal{K})$, a unit vector $\xi \in \mathcal{K}$ and a contraction $b : \mathcal{K} \rightarrow \overline{\mathcal{K}}$ such that

   $$T(x, y) = (b\pi(y)\xi | \overline{\pi(x)\xi})_{\mathcal{K}} \quad \text{for } x, y \in A.$$

4. There exist a completely bounded map $a : A \rightarrow K_c$ and a bounded map $b : K_c \rightarrow (K_c)^*$ such that

   $$\begin{array}{cccc}
   A & \xrightarrow{T} & A^* \\
   a \downarrow & & \uparrow a' \\
   K_c & \xrightarrow{b} & (K_c)^*
   \end{array}
   $$

   i.e., $T = a'^*ba$ with $\|a\|^2 \|b\| \leq 1$.

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) We can prove these implications by the similar way as in the proof of Theorem 2.3.

(3)$\Rightarrow$(4) We note that we use the norm $\| \|$ for $b$ instead of the completely bounded norm $\| \|_{cb}$. 
(4)⇒(1) For $x_i, y_i \in A$, we have

$$| \sum_{i=1}^{n} T(x_i, y_i) | = | \sum_{i=1}^{n} (ba(y_i) | a(x_i))^x |$$

$$= \left| \begin{bmatrix} 0 & b \\ \vdots & \ddots & \ddots \\ 0 & \ddots & 0 & b \\ & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a(x_1) \\ \vdots \\ a(x_n) \\
(y_1) \\ \vdots \\ (y_n) \end{bmatrix} \right|$$

$$\leq w \left| \begin{bmatrix} 0 & b \\ \vdots & \ddots & \ddots \\ 0 & \ddots & 0 & b \\ & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a(x_1) \\ \vdots \\ a(x_n) \\
(y_1) \\ \vdots \\ (y_n) \end{bmatrix} \right|$$

$$= \frac{1}{2} ||b||^2 ||a||^2 ||[x_1, \ldots, x_n, y_1, \ldots, y_n]^t ||^2$$

$$\leq \frac{1}{2} ||[x_1, \ldots, x_n, y_1, \ldots, y_n]^t ||^2.$$

\[\square\]

4. Factorization on Banach spaces

Let $X$ be a Banach space. Recall that the minimal quantization $\text{Min}(X)$ of $X$. Let $\Omega_X$ be the unit ball of $X^*$, that is, $\Omega_X = \{ f \in X^* | ||f|| \leq 1 \}$. For $[x_{ij}] \in M_n(X)$, $||[x_{ij}]||_\min$ is defined by

$$||[x_{ij}]||_\min = \sup \{ ||f(x_{ij})|| | f \in \Omega_X \}.$$

Then $\text{Min}(X)$ can be regarded as a subspace in the $C^*$-algebra $C(\Omega_X)$ of all continuous functions on the compact Hausdorff space $\Omega_X$. Here we define a norm of an element $u \in X \otimes X$ by

$$||u||_{wH} = \inf \{ \sup \{ \left( \sum_{i=1}^{n} |f(x_i)|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |f(y_i)|^2 \right)^{\frac{1}{2}} \} \},$$

where the supremum is taken over all $f \in X^*$ with $||f|| \leq 1$ and the infimum is taken over all representation $u = \sum_{i=1}^{n} x_i \otimes y_i$. 

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Proposition 4.1. Let \( X \) be a Banach space. Then
\[
\min(X) \otimes_{wh} \min(X) = \min(X) \otimes_{wh} \min(X) = X \otimes_{wH} X.
\]
Proof. Let \( u = \sum_{i=1}^{n} x_i \otimes y_i \in \min(X) \). Then, using the identity (\( \ast \)), we have
\[
\|u\|_{wh} = \inf \left\{ \frac{1}{2} \| [x_1, \ldots, x_n, y_1, \ldots, y_n] \| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}
\]
\[
= \inf \left\{ \sup \left\{ \frac{1}{2} \| [f(x_1), \ldots, f(x_n), f(y_1), \ldots, f(y_n)] \| : f \in \Omega_X \right\} : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}
\]
\[
= \inf \left\{ \sup \left\{ \left( \sum_{i=1}^{n} |f(x_i)|^2 + |f(y_i)|^2 \right)^{\frac{1}{2}} : f \in \Omega_X \right\} : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}
\]
\[= \|u\|_{wH}.
\]
The equality \( \|u\|_{wh} = \|u\|_{wH} \) is obtained by the same way as above. \( \square \)

Let \( T : X \to X^* \) be a bounded linear map. As in Remark 2.4(ii), we consider the map \( T \otimes \alpha : X \otimes \ell^2_{\mathbb{N}} \to X^* \otimes \ell^2_{\mathbb{N}} \) and define a norm for \( \sum x_i \otimes e_i \in X \otimes \ell^2_{\mathbb{N}} \) by
\[
\| \sum x_i \otimes e_i \| = \sup \left\{ \left( \sum |f(x_i)|^2 \right)^{\frac{1}{2}} : f \in \Omega_X \right\}.
\]
We note that, given \( x \in X, x^* \) is regarded as \( \langle x^*, f \rangle = \overline{f(x)} \) for \( f \in X^* \) in the definition of \( w(T \otimes \alpha) \), that is,
\[
w(T \otimes \alpha) = \sup \left\{ \| \langle x^* \otimes e_i, T \otimes \alpha \left( \sum x_i \otimes e_i \right) \rangle \| : \| \sum x_i \otimes e_i \| \leq 1 \right\}.
\]
Let \( a : X \to Y \) be a linear map between Banach spaces. \( a \) is called a 2-summing operator if there is a constant \( C \) which satisfies the inequality
\[
(\sum \|a(x_i)\|^2)^{\frac{1}{2}} \leq C \sup \left\{ \left( \sum |f(x_i)|^2 \right)^{\frac{1}{2}} : f \in \Omega_X \right\}
\]
for any finite subset \( \{x_i\} \subset X \). \( \pi_2(a) \) is the smallest constant of \( C \), and is called the 2-summing norm of \( a \). The following might be well known.

Proposition 4.2. Let \( X \) be a Banach space. If \( a \) is a linear map from \( X \) to \( \mathcal{H} \), then the following are equivalent:

1. \( \|a : \min(X) \to \mathcal{H}_c\|_{cb} \leq 1 \).
2. \( \|a : \min(X) \to \mathcal{H}_t\|_{cb} \leq 1 \).
3. \( \pi_2(a : X \to H) \leq 1 \).
Proof. (1) $\Rightarrow$ (3) For any $x_1, \ldots, x_n \in X$, we have
\[
\sum_{i=1}^{n} \|a(x_i)\|^2 = \|[a(x_1), \ldots, a(x_n)]^t\|^2_{\text{Min}} \\
= \|a\|_{cb}^2 \|[x_1, \ldots, x_n]^t\|_{\text{Min}} \\
= \sum_{i=1}^{n} x_i^t x_i \|_{\text{Min}} \\
= \sup\{\sum_{i=1}^{n} |f(x_i)|^2 \mid f \in \Omega_X\}.
\]

(3) $\Rightarrow$ (1) For any $[x_{ij}] \in M_n(\text{Min}(X))$, we have
\[
\|[a(x_{ij})]\|^2_{M_n(H_c)} = \sup\{\sum_i \sum_j \lambda_j a(x_{ij}) \mid \sum |\lambda_j|^2 = 1\} \\
\leq \sup\{\pi_2(a)^2 \sup\{\sum_i |f(\sum_j \lambda_j x_{ij})|^2 \mid f \in \Omega_X\} \mid \sum |\lambda_j|^2 = 1\} \\
\leq \sup\{|[f(x_{ij})]|^2 \mid f \in \Omega_X\} \\
\leq \|[x_{ij}]\|^2_{M_n(\text{Min}(X))}.
\]

(2) $\Leftrightarrow$ (3) It follows from the same way as above. $\square$

Finally we can state the following result as a corollary of Theorem 2.3 and Theorem 3.2.

**Corollary 4.3.** Suppose that $X$ is a Banach space, and that $T : X \to X^*$ is a bounded linear map. Then the following are equivalent:

1. $w(T \otimes \alpha) \leq w(\alpha)$ for all $\alpha : \ell^2_n \to \ell^2_n$ and $n \in \mathbb{N}$.
2. $\|T\|_{wH^*} \leq 1$.
3. $T$ factors through a Hilbert space $\mathcal{K}$ and its dual space $\mathcal{K}^*$ by a 2-summing operator $a : X \to \mathcal{K}$ and a bounded operator $b : \mathcal{K} \to \mathcal{K}^*$ as follows:

\[
\begin{array}{ccc}
X & \overset{T}{\longrightarrow} & X^* \\
a \downarrow & & a^t \\
\mathcal{K} & \overset{b}{\longrightarrow} & \mathcal{K}^*
\end{array}
\]

i.e., $T = a^t ba$ with $\pi_2(a)^2\|b\| \leq 1$.

4. $T$ has an extension $T' : C(\Omega_X) \to C(\Omega_X)^*$ which factors through a pair of Hilbert spaces $\mathcal{K}$ by a 2-summing operator $a : C(\Omega_X) \to \mathcal{K}$.
and a bounded operator \( b : K \to K \) as follows:
\[
C(\Omega_X) \xrightarrow{\mathcal{T}'} C(\Omega_X)^* \\
\xrightarrow{a} \quad \xrightarrow{a^*} \\
K \xrightarrow{b} K
\]
i.e., \( T' = a^* b a \) with \( \pi_2(a)^2 \|b\| \leq 1 \).

**Proof.** (1) \( \Rightarrow \) (2) Suppose that
\[
|\langle \sum_{i=1}^m z_i^* \otimes e_i, T \otimes \alpha(\sum_{i=1}^m z_i \otimes e_i) \rangle| \leq 1
\]
for any \( \sum_{i=1}^m z_i \otimes e_i \in X \otimes l^2_m \) with \( \|\sum_{i=1}^m z_i \otimes e_i\| \leq 1 \) and \( \alpha \in M_n(\mathbb{C}) \) with \( w(\alpha) \leq 1 \). It is easy to see that \( |\sum_{i,j=1}^m (z_i^* T(z_j)) \alpha_{ij}| \leq 1 \), equivalently \( |\sum_{i,j=1}^m (z_i^* T(z_j)) \alpha_{ij}| \leq 1 \).

Given \( \|\sum_{i=1}^n x_i \otimes y_i\|_{wH} < 1 \), we may assume that
\[
\frac{1}{2} \| [x_1, \ldots, x_n, y_1, \ldots, y_n]^t \|^2 \leq 1.
\]
Set
\[
z_i = \begin{cases} \\
\frac{1}{\sqrt{2}} x_i & i = 1, \ldots, n \\
\frac{1}{\sqrt{2}} y_{i-n} & i = n + 1, \ldots, 2n
\end{cases}
\quad \text{and } \alpha = \begin{bmatrix} 0_n & 2 \cdot 1_n \\
0_n & 0_n \end{bmatrix}.
\]
It turns out \( \|\sum_{i=1}^{2n} z_i \otimes e_i\| \leq 1 \) and \( w(\alpha) = 1 \). Then we have \( |T(\sum_{i=1}^n x_i \otimes y_i)| = |\sum_{i,j=1}^n (z_i^* T(z_j)) \alpha_{ij}| \leq 1 \). Hence \( \|T\|_{wH^*} \leq 1 \).

(2) \( \Rightarrow \) (1) Suppose that \( \|T\|_{wH^*} \leq 1 \). Then \( T \) has an extension \( T' \in (C(\Omega_X) \otimes_{wh} C(\Omega_X))^* \) with \( \|T'\|_{wh^*} \leq 1 \). Given \( \varepsilon > 0 \) and \( \alpha \in M_n(\mathbb{C}) \), there exist \( x_1, \ldots, x_n \in C(\Omega_X) \) such that \( \|\sum_{i=1}^n x_i \otimes e_i\| \leq 1 \) (equivalently \( \|x_1, \ldots, x_n\|^t \| \leq 1 \)) and \( w(T' \otimes \alpha) - \varepsilon < |\sum_{i,j=1}^n (x_i^* T'(x_j)) \alpha_{ij}| \). Hence we have
\[
w(T \otimes \alpha) \leq w(T' \otimes \alpha)
\leq |T'(\sum_{i,j=1}^n x_i^* \alpha_{ij} \otimes x_j)| + \varepsilon
\leq \| [x_1, \ldots, x_n]^t \|^2 w(\alpha) + \varepsilon
\leq w(\alpha) + \varepsilon.
\]

(2) \( \Leftrightarrow \) (3) It is straightforward from Theorem 3.2 and Proposition 4.1, 4.2.

(2) \( \Leftrightarrow \) (4) It is straightforward from Theorem 2.3 and Proposition 4.1, 4.2.

\( \square \)
Remark 4.4. Here we compare the above corollary with the classical factorization theorems through a Hilbert space. Let $X$ and $Y$ be Banach spaces. Grothendieck introduced the norm $\|\|_H$ on $X \otimes Y$ in [7] by

$$
\|u\|_H = \inf \left\{ \sup \left\{ \left( \sum_{i=1}^{n} |f(x_i)|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |g(y_i)|^2 \right)^{\frac{1}{2}} \right\} \right\}
$$

where the supremum is taken over all $f \in X^*$, $g \in Y^*$ with $\|f\|, \|g\| \leq 1$ and the infimum is taken over all representation $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. In [14], Lindenstrauss and Pelczynski characterized the factorization by using $T \otimes \alpha : X \otimes \ell^2_n \rightarrow Y \otimes \ell^2_n$ for $T : X \rightarrow Y$, however the norm on $X \otimes \ell^2$ is slightly different from the one in this paper. Their theorems with a modification are summarized for a bounded linear map $T : X \rightarrow Y^*$ as follows:

The following are equivalent:

(1) $\|T \otimes \alpha\| \leq \|\alpha\|$ for all $\alpha : \ell^2_n \rightarrow \ell^2_n$ and $n \in \mathbb{N}$.
(2) $\|T\|_{H^*} \leq 1$.
(3) $T$ factors through a Hilbert space $K$ by a 2-summing operator $a : X \rightarrow K$ and $b : K \rightarrow Y^*$ whose transposed $b^t$ is 2-summing as follows:

$$
\begin{array}{ccc}
X & \rightarrow & Y^* \\
\downarrow a & & \downarrow b \\
K & & \\
\end{array}
$$

i.e., $T = ba$ with $\pi_2(a)\pi_2(b^t) \leq 1$.

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