The operator $\sqrt{-1}\hat{c}(V)(d + \delta)$ and the Kastler-Kalau-Walze type theorems

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Abstract
In this paper, we obtain two Lichnerowicz type formulas for the operators $\sqrt{-1}\hat{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\hat{c}(V)$. And we give the proof of Kastler-Kalau-Walze type theorems for the operators $\sqrt{-1}\hat{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\hat{c}(V)$ on 3,4-dimensional oriented compact manifolds with (resp.without) boundary.

Keywords: Lichnerowicz type formulas; the operators $\sqrt{-1}\hat{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\hat{c}(V)$; Kastler-Kalau-Walze type theorems.

1. Introduction

Until now, many geometers have studied noncommutative residues. In [5, 16], authors found noncommutative residues are of great importance to the study of noncommutative geometry. In [2], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes showed us that the noncommutative residue on a compact manifold $M$ coincided with the Dixmier’s trace on pseudodifferential operators of order $-\dim M$ in [3]. And Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein-Hilbert action. Kastler [7] gave a brute-force proof of this theorem. Kalau and Walze proved this theorem in the normal coordinates system simultaneously in [6]. Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator $\text{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of $D^2$ in [1].

On the other hand, Wang generalized the Connes’ results to the case of manifolds with boundary in [11, 12], and proved the Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [13]. In [13, 14], Wang computed $\text{Wres}[\pi^+D^{-1} \circ \pi^+D^{-1}]$ and $\text{Wres}[\pi^+D^{-2} \circ \pi^+D^{-2}]$, where the two operators are symmetric, in these cases the boundary term vanished. But for $\text{Wres}[\pi^+D^{-1} \circ \pi^+D^{-3}]$, Wang got a nonvanishing boundary term [10], and give a theoretical explanation for gravitational action on boundary. In others words, Wang provides a kind of method to study the Kastler-Kalau-Walze type theorem for manifolds with boundary. In [8], Lópe and his collaborators introduced an elliptic differential operator which is called the Novikov operator. In [13], Wei and Wang proved Kastler-Kalau-Walze type theorem for modified Novikov operators on compact manifolds. In [13], the leading symbol of the Dirac operator $D$ is $\sqrt{-1}\hat{c}(\xi)$. To get the leading symbol of the operator which is not $\sqrt{-1}\hat{c}(\xi)$, we consider the operator $\sqrt{-1}\hat{c}(V)(d + \delta)$ in this paper, which is motivated by the sub-signature operator in [13].

In this paper, we obtain two Lichnerowicz type formulas for the operators $\sqrt{-1}\hat{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\hat{c}(V)$, and prove the following main theorems.

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Theorem 1.1. Let $M$ be a 4-dimensional oriented compact manifold with boundary $\partial M$ and the metric $g^{TM}$ as in Section 3, the operators $D_V = \sqrt{-1} \tilde{c}(V)(d + \delta)$ and $D_V^* = -\sqrt{-1}(d + \delta)\tilde{c}(V)$ be on $\bar{M}$ ($\bar{M}$ is a collar neighborhood of $M$), then

$$
\text{Wres}[\pi^+ D_V^{-1} \circ \pi^+(D_V^*)^{-1}]
= 32\pi^2 \int_M \left( -\frac{4}{3}K - 8 \sum_{q=1}^{4} |\nabla^L_{\tilde{c}}(V)|^2 \right) d\text{Vol}_M + \int_{\partial M} \left( \frac{-3ih'(0)}{2} - \frac{27\pi^2h'(0)}{8} - \frac{\pi^2}{4} \right) \pi\Omega_3 d\text{Vol}_M,
$$

where $K$ is the scalar curvature, $h$ is defined by (3.1), and $\Omega_3$ is the canonical volume of $S^3$.

Theorem 1.2. Let $M$ be a 4-dimensional oriented compact manifold with boundary $\partial M$ and the metric $g^{TM}$ as in Section 3, the operator $D_V = \sqrt{-1} \tilde{c}(V)(d + \delta)$ be on $\bar{M}$ ($\bar{M}$ is a collar neighborhood of $M$), then

$$
\text{Wres}[\pi^+ D_V^{-1} \circ \pi^+(D_V^*)^{-1}]
= 32\pi^2 \int_M \left( -\frac{4}{3}K - 8 \sum_{q=1}^{4} |\nabla^L_{\tilde{c}}(V)|^2 \right) d\text{Vol}_M + \int_{\partial M} \left( \frac{-3ih'(0)}{2} - \frac{27\pi^2h'(0)}{8} - \frac{\pi^2}{4} \right) \pi\Omega_3 d\text{Vol}_M,
$$

where $K$ is the scalar curvature, $h$ is defined by (3.1), and $\Omega_3$ is the canonical volume of $S^3$.

We note that two operators in Theorem 1.2 are symmetric, but we still get the non-vanishing boundary term.

Theorem 1.3. Let $M$ be a 3-dimensional oriented compact manifold with boundary $\partial M$ and the metric $g^{TM}$ as in Section 3, the operator $D_V = \sqrt{-1} \tilde{c}(V)(d + \delta)$ be on $\bar{M}$ ($\bar{M}$ is a collar neighborhood of $M$), then

$$
\text{Wres}[(\pi^+ D_V)^{-1}] = 4i\pi^2\text{vol}_{\partial M},
$$

where $\text{vol}_{\partial M}$ denotes the canonical volume form of $\partial M$.

The paper is organized in the following way. In Section 2 by using the definition of the operators $\sqrt{-1} \tilde{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\tilde{c}(V)$, we compute the Lichnerowicz formulas for the operators $\sqrt{-1} \tilde{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\tilde{c}(V)$. In Section 3 we prove the Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary for the operators $\sqrt{-1} \tilde{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\tilde{c}(V)$. In Section 4, we compute $\text{Wres}[(\pi^+ (D_V)^{-1})^2]$ for 3-dimensional oriented manifolds with boundary.

2. The operator $\sqrt{-1} \tilde{c}(V)(d + \delta)$ and its Lichnerowicz formula

Firstly we introduce some notations about the operators $\sqrt{-1} \tilde{c}(V)(d + \delta)$ and $-\sqrt{-1}(d + \delta)\tilde{c}(V)$. Let $M$ be an $n$-dimensional ($n \geq 3$) oriented compact Riemannian manifold with a Riemannian metric $g^{TM}$.

Let $\nabla^L$ be the Levi-Civita connection about $g^{TM}$. In the fixed orthonormal frame $\{e_1, \cdots, e_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$
\nabla^L(e_1, \cdots, e_n) = (e_1, \cdots, e_n)(\omega_{s,t}).
$$

Let $\epsilon(e^*_j), \iota(e^*_j)$ be the exterior and interior multiplications respectively, where $e^*_j = g^{TM}(e_j, \cdot)$. And $\hat{c}(e_j)$ be the Clifford action, write

$$
\hat{c}(e_j) = \epsilon(e^*_j) + \iota(e^*_j); \ c(e_j) = \epsilon(e^*_j) - \iota(e^*_j),
$$

where $e^*_j$ is the Hodge dual of $e_j$.
which satisfies
\[ \partial(e_i)\partial(e_j) + \partial(e_j)\partial(e_i) = 2g^{TM}(e_i, e_j); \]
\[ c(e_i)c(e_j) + c(e_j)c(e_i) = -2g^{TM}(e_i, e_j); \]
\[ c(e_i)\partial(e_j) + \partial(e_j)c(e_i) = 0. \]  
(2.3)

By [17], we have
\[ D = d + \delta = \sum_{i=1}^{n} c(e_i) \left[ e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)[\partial(e_s)\partial(e_t) - c(e_s)c(e_t)] \right]. \]  
(2.4)

Let \( e_1, e_2, \ldots, e_n \) be the orthonormal basis of \( TM \), the operators \( D_V \) and \( D_V^\ast \) acting on \( \wedge^s T^*M \otimes \mathbb{C} \) are defined by
\[ D_V = \sqrt{-1}c(V)(d + \delta) \]
\[ = \sqrt{-1}c(V) \sum_{i=1}^{n} c(e_i) \left[ e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)[\partial(e_s)\partial(e_t) - c(e_s)c(e_t)] \right] \]  
(2.5)

\[ D_V^\ast = -\sqrt{-1}(d + \delta)c(V) \]
\[ = -\sqrt{-1} \sum_{i=1}^{n} c(e_i) \left[ e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)[\partial(e_s)\partial(e_t) - c(e_s)c(e_t)] \right] \]  
(2.6)

where \( V \) is a vector field, and \( |V| = 1 \).

Next, we get the following theorem about Lichnerowicz formulas,

**Theorem 2.1.** The following equalities hold:
\[ D_V^\ast D_V = -\left[ g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i} \partial_j}) \right] - \frac{1}{8} \sum_{ijkl} R_{ijkl} \partial(e_i)\partial(e_j)c(e_k)c(e_l) + \frac{1}{4} K; \]
\[ D_V^2 = -\left[ g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i} \partial_j}) \right] - \frac{1}{8} \sum_{ijkl} R_{ijkl} \partial(e_i)\partial(e_j)c(e_k)c(e_l) + \frac{1}{4} K \]
\[ + \frac{1}{4} \sum_{i=1}^{n} c(V) \sum_{q=1}^{n} \partial(\nabla_{e_q} V)(c(e_q)c(e_i)) \right)^2 \]  
(2.7)

where \( K \) is the scalar curvature.

**Proof.** Let \( M \) be a smooth compact oriented Riemannian \( n \)-dimensional manifolds without boundary and \( N \) be a vector bundle on \( M \). If \( P \) is a differential operator of Laplace type, then it has locally the form
\[ P = -(g^{ij}\partial_i \partial_j + A^i \partial_i + B), \]  
(2.8)

where \( \partial_i \) is a natural local frame on \( TM \) and \((g^{ij})_{1 \leq i,j \leq n} \) is the inverse matrix associated to the metric matrix \((g_{ij})_{1 \leq i,j \leq n} \) on \( M \), and \( A^i \) and \( B \) are smooth sections of \( \text{End}(N) \) on \( M \) (endomorphism). If a Laplace type
operator \( P \) satisfies (2.8), then there is a unique connection \( \nabla \) on \( N \) and a unique endomorphism \( E \) such that
\[
P = \left[ g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i} \partial_j}) + E \right],
\]
where \( \nabla^L \) is the Levi-Civita connection on \( M \). Moreover (with local frames of \( T^*M \) and \( N \)), \( \nabla_{\partial_i} = \partial_i + \omega_i \) and \( E \) are related to \( g^{ij} \), \( A^i \) and \( B \) through
\[
\omega_i = \frac{1}{2} g_{ij} (A^i + g^{kl} \Gamma^j_{ki} \text{id}),
\]
\[
E = B - g^{ij} (\partial_i \omega_j + \omega_i \omega_j - \omega_k \Gamma^k_{ij}),
\]
where \( \Gamma^j_{ki} \) is the Christoffel coefficient of \( \nabla^L \).

Let \( g^{ij} = g(dx_i, dx_j) \), \( \xi = \sum_j \xi_j dx_j \) and \( \nabla^L_{\partial_i} \partial_j = \sum_k \Gamma^L_{ij} \partial_k \), we denote that
\[
\sigma_i = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)c(e_s)c(e_t); \quad a_i = \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i)\tilde{c}(e_s)\tilde{c}(e_t);
\]
\[
\xi^j = g^{ij} \xi_i; \quad \Gamma^k = g^{ij} \Gamma^k_{ij}; \quad \sigma^j = g^{ij} \sigma_i; \quad a^j = g^{ij} a_i.
\]

Then the operators \( D_V \) and \( D_V^* \) can be written as
\[
D_V = \sqrt{-1} \hat{c}(V) \sum_{i=1}^n c(e_i)[e_i + a_i + \sigma_i];
\]
\[
D_V^* = -\sqrt{-1} \sum_{i=1}^n c(e_i)[e_i + a_i + \sigma_i]\hat{c}(V).
\]
By (1) and (17), we have
\[
(d + \delta)^2 = -\Delta_0 - \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i)\tilde{c}(e_j)c(e_k)c(e_l) + \frac{1}{4} K;
\]
\[
-\Delta_0 = \Delta = -g^{ij}(\nabla^L_i \nabla^L_j - \Gamma^k_{ij} \nabla^L_k).
\]
By (2.8) and (2.9), we have
\[
D_V^* D_V = (d + \delta)^2
= -\sum_{ij} g^{ij} \left[ \partial_i \partial_j + 2\sigma_i \partial_j + 2a_i \partial_j - \Gamma^k_{ij} \partial_k + (\partial_i \sigma_j) + (\partial_j a_i) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma^k_{ij} \sigma_k \right]
- \Gamma^k_{ij} a_k
- \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i)\tilde{c}(e_j)c(e_k)c(e_l) + \frac{1}{4} K.
\]

Similarly, by \( d + \delta = \sum_{q=1}^n c(e_q) \nabla^L_{e_q} - T^*M \), we have
\[
D_V^2 = (d + \delta)^2 + \sum_{q=1}^n \hat{c}(V)\tilde{c}(\nabla^L_{e_q} V)c(e_q)(d + \delta)
= -\sum_{ij} g^{ij} \left[ \partial_i \partial_j + 2\sigma_i \partial_j + 2a_i \partial_j - \Gamma^k_{ij} \partial_k + (\partial_i \sigma_j) + (\partial_j a_i) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma^k_{ij} \sigma_k \right]
- \Gamma^k_{ij} a_k
+ \sum_{ij} g^{ij} \left[ \hat{c}(V) \sum_{q=1}^n \tilde{c}(\nabla^L_{e_q} V)c(e_q)\tilde{c}(\partial_i)\partial_j + \tilde{c}(V) \sum_{q=1}^n \hat{c}(\nabla^L_{e_q} V)c(e_q)c(\partial_i)\sigma_i + \tilde{c}(V) \sum_{q=1}^n \hat{c}(\nabla^L_{e_q} V)c(e_q) \right].
\]
\[
c(d_i a_i) - \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k)c(e_l) + \frac{1}{4} K.
\]

(2.18)

By (2.12)-(2.18), we have

\[
(\omega_i)_{D_V^*D_V} = \sigma_i + a_i,
\]

(2.19)

\[
E_{D_V^*D_V} = \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k)c(e_l) - \frac{1}{4} K.
\]

(2.20)

Since \(E\) is globally defined on \(M\), taking normal coordinates at \(x_0\), we have \(\sigma_i(x_0) = 0, a_i(x_0) = 0, \partial j[c(\partial_j)](x_0) = 0, \Gamma^k(x_0) = \delta^k_i, \) then

\[
E_{D_V^*D_V}(x_0) = \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k)c(e_l) - \frac{1}{4} K.
\]

(2.21)

Similarly, we have

\[
E_{D_V^2}(x_0) = \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k)c(e_l) - \frac{1}{4} K - \frac{1}{4} \sum_{i=1}^{n} \hat{c}(V) \sum_{q=1}^{n} \hat{c}(\nabla_{e_q} V) c(e_q) c(e_i)]^2
+ \frac{1}{2} \sum_{j=1}^{n} [\nabla_{e_j}^* T^* M (\hat{c}(V) \sum_{q=1}^{n} \hat{c}(\nabla_{e_q} V) c(e_q)) c(e_j)],
\]

(2.22)

then by (2.34), we get Theorem 2.1.

Similarly, we have

\[
E_{D_V^2}(x_0) = \frac{1}{8} \sum_{ijkl} R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k)c(e_l) - \frac{1}{4} K - \frac{1}{4} \sum_{i=1}^{n} \hat{c}(V) \sum_{q=1}^{n} \hat{c}(\nabla_{e_q} V) c(e_q) c(e_i)]^2
+ \frac{1}{2} \sum_{j=1}^{n} [\nabla_{e_j}^* T^* M (\hat{c}(V) \sum_{q=1}^{n} \hat{c}(\nabla_{e_q} V) c(e_q)) c(e_j)],
\]

(2.22)

From [1], we konw that the noncommutative residue of a generalized laplacian \(\Delta\) is expressed as

\[
(n - 2)\Phi_2(\Delta) = (4\pi)^{-\frac{n}{2}} \Gamma(\frac{n}{2}) \text{Wres}(\Delta - E_{D_V^*D_V}),
\]

(2.23)

where \(\Phi_2(\Delta)\) denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of \(\Delta\). Now let \(\Delta = D_V^*D_V\), since \(D_V^*D_V\) is a generalized laplacian, we can suppose \(D_V^*D_V = \Delta - E_{D_V^*D_V}\), then we have

\[
\text{Wres}(D_V^*D_V)^{\frac{n-2}{2}} = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}(\frac{1}{6} K + E_{D_V^*D_V}) d\text{Vol}_M,
\]

(2.24)

\[
\text{Wres}(D_V^2)^{\frac{n-2}{2}} = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}(\frac{1}{6} K + E_{D_V^2}) d\text{Vol}_M,
\]

(2.25)

where Wres denotes the noncommutative residue.

Next, we need to compute \(\text{tr}(E_{D_V^*D_V})\) and \(\text{tr}(E_{D_V^2})\). Obviously, we have

(1)

\[
\text{tr}\left(-\frac{1}{4} K\right) = -\frac{1}{4} \text{tr}[\text{id}].
\]

5
\begin{align}
(2) & \sum_{ijkl} \text{tr}[R_{ijkl} \hat{c}(e_i) \hat{c}(e_j) c(e_k) c(e_l)] = 0. \\
(3) & \hat{c}(V) \hat{c}(\nabla^L_{eq} V) = -\hat{c}(\nabla^L_{eq} V) \hat{c}(V), \quad \hat{c}(V) c(e_q) = -c(e_q) \hat{c}(V), \\
& \hat{c}(e_q) \hat{c}(\nabla^L_{eq} V) + \hat{c}(\nabla^L_{eq} V) \hat{c}(e_q) = 2g^{TM}(\nabla^L_{eq} V, e_q), \quad (\hat{c}(V))^2 = |V|^2 = 1, \\
& \hat{c}(\nabla^L_{eq} V) \hat{c}(\nabla^L_{em} V) + \hat{c}(\nabla^L_{em} V) \hat{c}(\nabla^L_{eq} V) = 2g^{TM}(\nabla^L_{eq} V, \nabla^L_{em} V),
\end{align}
we also get
\begin{align}
\text{tr} \sum_{i=1}^{n} \hat{c}(V) \sum_{q=1}^{n} \hat{c}(\nabla^L_{eq} V) c(e_q) c(e_i)]^2 = & \text{tr} \sum_{i=1}^{n} \hat{c}(V) \sum_{q=1}^{n} \hat{c}(\nabla^L_{eq} V) c(e_q) c(e_i) \hat{c}(V) \sum_{m=1}^{n} \hat{c}(\nabla^L_{em} V) c(e_m) c(e_i)] \\
= & -\text{tr} \sum_{i=1}^{n} \sum_{q=1}^{n} \hat{c}(\nabla^L_{eq} V) c(e_q) c(e_i) \sum_{m=1}^{n} \hat{c}(\nabla^L_{em} V) c(e_m) c(e_i)] \\
= & -\text{tr} \sum_{i=1}^{n} \sum_{q=1}^{n} c(e_q) c(e_i) \hat{c}(\nabla^L_{eq} V) \sum_{m=1}^{n} \hat{c}(\nabla^L_{em} V) c(e_m) c(e_i)] \\
= & -2 \sum_{i,q,m=1}^{n} g^{TM}(\nabla^L_{eq} V, \nabla^L_{em} V) \text{tr}[c(e_q) c(e_i) c(e_m) c(e_i)] \\
& + \text{tr} \sum_{i=1}^{n} \sum_{q=1}^{n} c(e_q) c(e_i) \sum_{m=1}^{n} \hat{c}(\nabla^L_{em} V) \hat{c}(\nabla^L_{eq} V) c(e_m) c(e_i)],
\end{align}
then we have
\begin{align}
\text{tr} \sum_{i=1}^{n} \sum_{q=1}^{n} \hat{c}(\nabla^L_{eq} V) c(e_q) c(e_i) \sum_{m=1}^{n} \hat{c}(\nabla^L_{em} V) c(e_m) c(e_i)] = & \sum_{i,q,m=1}^{n} g^{TM}(\nabla^L_{eq} V, \nabla^L_{em} V) \text{tr}[c(e_q) c(e_i) c(e_m) c(e_i)],
\end{align}
and by $c(e_i) c(e_m) + c(e_m) c(e_i) = -2g^{TM}(e_m, e_i)$, we have
\begin{align}
\text{tr} \sum_{i=1}^{n} \sum_{q=1}^{n} \hat{c}(\nabla^L_{eq} V) c(e_q) c(e_i) \sum_{m=1}^{n} \hat{c}(\nabla^L_{em} V) c(e_m) c(e_i)] \\
= & -2 \sum_{i,q,m=1}^{n} g^{TM}(\nabla^L_{eq} V, \nabla^L_{em} V) \delta_{m,i} \text{tr}[c(e_q) c(e_i)] + n \sum_{q,m=1}^{n} g^{TM}(\nabla^L_{eq} V, \nabla^L_{em} V) \text{tr}[c(e_q) c(e_m)] \\
= & -(n-2) \sum_{q=1}^{n} |\nabla^L_{eq} V|^2 \text{tr}[\text{id}],
\end{align}
By \( \nabla_c \alpha \beta = (\nabla_c \alpha \beta) + \alpha(\nabla_c \beta) \), we have

\[
\text{tr} \sum_{j=1}^{n} [\hat{\alpha}(V) \sum_{q=1}^{n} \hat{\alpha}(\nabla_{e_q} V) c(e_q) c(e_j)]^2 = (n - 2) \sum_{q=1}^{n} |\nabla_{e_q} V|^2 \text{tr}[\text{id}]. \tag{2.32}
\]

(4) By \( \nabla_c^{\alpha \beta} (\alpha \beta) = (\nabla_c^{\alpha \beta} \alpha \beta) + \alpha(\nabla_c^{\alpha \beta} \beta) \), we have

\[
\text{tr} \sum_{j=1}^{n} [\nabla_c^{\alpha \beta} (\hat{\alpha}(V) \sum_{q=1}^{n} \hat{\alpha}(\nabla_{e_q} V) c(e_q) c(e_j))]
\]

\[
= \text{tr} \sum_{j=1}^{n} [\nabla_c^{\alpha \beta} (\hat{\alpha}(V)) \sum_{q=1}^{n} \hat{\alpha}(\nabla_{e_q} V) c(e_q) c(e_j)] + \text{tr} \sum_{j=1}^{n} [\hat{\alpha}(V) \sum_{q=1}^{n} \nabla_c^{\alpha \beta} (\hat{\alpha}(\nabla_{e_q} V)) c(e_q) c(e_j)]
\]

\[
+ \text{tr} \sum_{j=1}^{n} [\hat{\alpha}(V) \sum_{q=1}^{n} \hat{\alpha}(\nabla_{e_q} V) \nabla_c^{\alpha \beta} (c(e_q)) c(e_j)]. \tag{2.33}
\]

By

\[
\nabla_c^{\alpha \beta} (\hat{\alpha}(V)) = \hat{\alpha}(\nabla_{e_q} V), \quad \hat{\alpha}(V) \hat{\alpha}(\nabla_{e_q} V) = \hat{\alpha}(V) \hat{\alpha}(\nabla_{e_q} V) + \hat{\alpha}(\nabla_{e_q} V) \hat{\alpha}(V) = 2g^{TM}(V, \nabla_{e_q} V)
\]

\[
\nabla_c^{\alpha \beta} (\nabla_{e_q} V) = \nabla_{e_q} (\nabla_{e_q} V), \quad g^{TM}(\nabla_{e_q} V, V) + g^{TM}(\nabla_{e_q} V, \nabla_{e_q} V) = e_q(g^{TM}(\nabla_{e_q} V, V)) = 0,
\]

\[
\hat{\alpha}(V) \nabla_{e_q} V + \nabla_{e_q} V \hat{\alpha}(V) = 0, \quad \nabla_c^{\alpha \beta} (c(e_q)) = c(\nabla_{e_q} e_q)
\]

we get

(4-a)

\[
\text{tr} \sum_{j=1}^{n} [\nabla_c^{\alpha \beta} (\hat{\alpha}(V)) \sum_{q=1}^{n} \hat{\alpha}(\nabla_{e_q} V) c(e_q) c(e_j)]
\]

\[
= \text{tr} \sum_{j=1}^{n} [\hat{\alpha}(\nabla_{e_q} V) \sum_{q=1}^{n} \hat{\alpha}(\nabla_{e_q} V) c(e_q) c(e_j)]
\]

\[
= 2 \sum_{j,q=1}^{n} g^{TM}(\nabla_{e_q} V, \nabla_{e_q} V) \text{tr}[c(e_q) c(e_j)] - \sum_{j,q=1}^{n} \text{tr}[\hat{\alpha}(\nabla_{e_q} V) \nabla_{e_q} V c(e_q) c(e_j)]. \tag{2.35}
\]

Similar to (2.31), we have

\[
\text{tr} \sum_{j=1}^{n} [\hat{\alpha}(V) \sum_{q=1}^{n} \hat{\alpha}(\nabla_{e_q} V) c(e_q) c(e_j)] = - \sum_{q=1}^{n} |\nabla_{e_q} V|^2 \text{tr}[\text{id}]. \tag{2.36}
\]

(4-b)

\[
\text{tr} \sum_{j=1}^{n} [\hat{\alpha}(V) \sum_{q=1}^{n} \nabla_c^{\alpha \beta} (\hat{\alpha}(\nabla_{e_q} V)) c(e_q) c(e_j)]
\]

\[
= \sum_{j,q=1}^{n} \text{tr}[\hat{\alpha}(V) \nabla_{e_q} V c(e_q) c(e_j)]
\]
\[
= 2 \sum_{j,q=1}^{n} g^{TM}(\nabla_{e_j} \nabla_{e_q} V, V) \text{tr}[c(e_q)c(e_j)] - \text{tr} \sum_{j=1}^{n} \sum_{q=1}^{n} \tilde{c}(\nabla_{e_j} \nabla_{e_q} V) \tilde{c}(V)c(e_q)c(e_j),
\]
(2.37)

then,
\[
\sum_{j,q=1}^{n} \text{tr}[\tilde{c}(V)\tilde{c}(\nabla_{e_j} \nabla_{e_q} V)c(e_q)c(e_j)] = \sum_{j,q=1}^{n} g^{TM}(\nabla_{e_j} \nabla_{e_q} V, V) \text{tr}[c(e_q)c(e_j)]
\]
\[
= - \sum_{q=1}^{n} g^{TM}(\nabla_{e_q} \nabla_{e_q} V, V) \text{tr}[\text{id}]
\]
\[
= \sum_{q=1}^{n} |\nabla_{e_q} V|^2 \text{tr}[\text{id}] .
\]
(2.38)

Then, by (2.36), (2.38) and (2.39), we have
\[
\text{tr} \sum_{j=1}^{n} [\tilde{c}(V) \sum_{q=1}^{n} \tilde{c}(\nabla_{e_q} L) \nabla^{*}_{e_j} T^{*} M (c(e_q))c(e_j)] = \text{tr} \sum_{j=1}^{n} [\tilde{c}(V) \sum_{q=1}^{n} \tilde{c}(\nabla_{e_q} L)c(\nabla_{e_q} e_j)c(e_j)] = 0.
\]
(2.39)

Therefore, we get
\[
\text{tr} \sum_{j=1}^{n} [\nabla^{*}_{e_j} T^{*} M (\tilde{c}(V) \sum_{q=1}^{n} \tilde{c}(\nabla_{e_q} L)V)c(e_j)] = 0.
\]
(2.40)

Then by (2.24) and (2.25), we have the following theorem,

**Theorem 2.2.** If \( M \) is a \( n \)-dimensional compact oriented manifold without boundary, and \( n \) is even, then we get the following equalities :

\[
\text{Wres}(D_{\nu}^{*}D_{\nu}) - \frac{n-2}{4} K = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_\mathcal{M} 2^n \left( -\frac{1}{12} K \right) d\text{Vol}_\mathcal{M},
\]
\[
\text{Wres}(D_{\nu}^{\nu}) - \frac{n-2}{4} = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_\mathcal{M} 2^n \left( -\frac{1}{12} K - \frac{n-2}{4} \sum_{q=1}^{n} |\nabla_{e_q} V|^2 \right) d\text{Vol}_\mathcal{M},
\]
(2.43)

where \( K \) is the scalar curvature.
3. A Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary

In this section, we prove the Kastler-Kalau-Walze type theorem for 4-dimensional oriented compact manifolds with boundary. We firstly recall some basic facts and formulas about Boutet de Monvel’s calculus and the definition of the noncommutative residue for manifolds with boundary which will be used in the following. For more details, see in Section 2 in [13].

Let $M$ be a 4-dimensional compact oriented manifold with boundary $\partial M$. We assume that the metric $g^{TM}$ on $M$ has the following form near the boundary,

$$
g^{TM} = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2,
$$

(3.1)

where $g^{\partial M}$ is the metric on $\partial M$ and $h(x_n) \in C^\infty([0,1)) := \{ h|_{[0,1]} \mid \widehat{h} \in C^\infty((-\varepsilon,1)) \}$ for some $\varepsilon > 0$ and $h(x_n)$ satisfies $h(x_n) > 0$, $h(0) = 1$ where $x_n$ denotes the normal directional coordinate. Let $U \subset M$ be a collar neighborhood of $\partial M$ which is diffeomorphic with $\partial M \times [0,1)$. By the definition of $h(x_n) \in C^\infty([0,1))$ and $(\partial M \times [-\varepsilon,0])$ such that $\widehat{h}|_{[0,1]} = \hat{h}$ and $\hat{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $g'$ on $\tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon,0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon,0]$

$$
g' = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2,
$$

(3.2)

such that $g'|_M = g$. We fix a metric $g'$ on the $\tilde{M}$ such that $g'|_M = g$.

Let the Fourier transformation $\mathcal{F}'$ be

$$
\mathcal{F}' : L^2(\mathbb{R}_t) \rightarrow L^2(\mathbb{R}_e); \quad \mathcal{F}'(u)(\nu) = \int_{\mathbb{R}} e^{-i\nu t} u(t) dt
$$

and let

$$
r^+ : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}_e^+); \quad f \mapsto f|_{\mathbb{R}_e^+}; \quad \mathbb{R}_e^+ = \{ x \geq 0; x \in \mathbb{R} \}.
$$

We define $H^+ = \mathcal{F}'(\Phi(\mathbb{R}_e^+)); \quad H^- = \mathcal{F}'(\Phi(\mathbb{R}_e^-))$ which satisfies $H^+ \perp H^-$, where $\Phi(\mathbb{R}_e^+) = r^+ \Phi(\mathbb{R})$, $\Phi(\mathbb{R}_e^-) = r^- \Phi(\mathbb{R})$ and $\Phi(\mathbb{R})$ denotes the Schwartz space. We have the following property: $h \in H^+$ (resp. $H^-$) if and only if $h \in C^\infty(\mathbb{R})$ which has an analytic extension to the lower (resp. upper) complex half-plane $\{ \text{Im}\xi < 0 \}$ (resp. $\{ \text{Im}\xi > 0 \}$) such that for all nonnegative integer $l$,

$$
\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l c_k}{d\xi^l}(\xi),
$$

as $|\xi| \rightarrow +\infty$, $\text{Im}\xi \leq 0$ (resp. $\text{Im}\xi \geq 0$) and where $c_k \in \mathbb{C}$ are some constants.

Let $H'$ be the space of all polynomials and $H^+ = H^- \bigoplus H^+; \quad H = H^+ \bigoplus H^-$. Denote by $\pi^+$ (resp. $\pi^-$) respectively the projection on $H^+$ (resp. $H^-$). Let $\tilde{H} = \{ \text{rational functions having no poles on the real axis} \}$ such that $\tilde{H}$ is a dense subset of $H$. Then on $\tilde{H}$,

$$
\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{\nu \rightarrow 0^+} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + i\nu - \xi} d\xi,
$$

(3.3)

where $\Gamma^+$ is a Jordan close curve included $\text{Im}(\xi) > 0$ surrounding all the singularities of $h$ in the upper half-plane and $\xi_0 \in \mathbb{R}$. In our computations, we only compute $\pi^+ h$ for $h$ in $\tilde{H}$. Similarly, define $\pi^-$ on $\tilde{H}$,

$$
\pi^- h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi)d\xi.
$$

(3.4)

So $\pi^-(H^-) = 0$. For $h \in H \bigcap L^1(\mathbb{R})$, $\pi^+ h = \frac{1}{2\pi} \int_{\mathbb{R}} h(\nu) d\nu$ and for $h \in H^+ \bigcap L^1(\mathbb{R})$, $\pi^- h = 0$.

An operator of order $m \in \mathbb{Z}$ and type $d$ is a matrix

$$
\tilde{A} = \begin{pmatrix}
\pi^+ P + G & K \\
T & S
\end{pmatrix} : C^\infty(M, E_1) \bigoplus C^\infty(\partial M, F_1) \rightarrow C^\infty(M, E_2) \bigoplus C^\infty(\partial M, F_2),
$$
where $M$ is a manifold with boundary $\partial M$ and $E_1, E_2$ (resp. $F_1, F_2$) are vector bundles over $M$ (resp. $\partial M$). Here, $P : C^\infty_0(\Omega, \mathbb{F}_1) \to C^\infty(\Omega, \mathbb{E}_2)$ is a classical pseudodifferential operator of order $m$ on $\Omega$, where $\Omega$ is a collar neighborhood of $M$ and $\mathbb{E}_i|_M = E_i$ $(i = 1, 2)$. $P$ has an extension: $E'(\Omega, \mathbb{E}_1) \to \mathcal{D}'(\Omega, \mathbb{E}_2)$, where $E'(\Omega, \mathbb{E}_1)$ is the dual space of $C^\infty_0(\Omega, \mathbb{E}_1)$ $(C^\infty_c(\Omega, \mathbb{E}_2))$. Let $e^+: C^\infty(\Omega, E_1) \to E'_1(\Omega, \mathbb{E}_2)$ denote extension by zero from $M$ to $\Omega$ and $r^+ : \mathcal{D}'(\Omega, \mathbb{E}_2) \to \mathcal{D}'(\Omega, E_2)$ denote the restriction from $\Omega$ to $X$, then define

$$\pi^+ P = r^+ P e^+ : C^\infty(M, E_1) \to \mathcal{D}'(\Omega, E_2).$$

In addition, $P$ is supposed to have the transmission property; this means that, for all $j, k, \alpha$, the homogeneous component $p_j$ of order $j$ in the asymptotic expansion of the symbol $p$ of $P$ in local coordinates near the boundary satisfies:

$$\partial^k_{x^j} \partial^\alpha_{\xi} p_j(x', 0, 0, +1) = (-1)^{d-\lfloor|\alpha|\rfloor} \partial^k_{x^j} \partial^\alpha_{\xi} p_j(x', 0, 0, -1),$$

then $\pi^+ P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ by [12]. Let $G, T$ be respectively the singular Green operator and the trace operator of order $m$ and type $d$. Let $K$ be a potential operator and $S$ be a classical pseudodifferential operator of order $m$ along the boundary (For detailed definition, see [11]). Denote by $B^m,d$ the collection of all operators of order $m$ and type $d$, and $B$ is the union over all $m$ and $d$.

Recall that $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map: $B^m,d \times B^{m',d'} \to B^{m+m',\max(m',d')}$. Write

$$\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \tilde{S} \end{pmatrix} \in B^{m,d}, \quad \tilde{A}' = \begin{pmatrix} \pi^+ P' + G' & \tilde{K}' \\ T' & \tilde{S}' \end{pmatrix} \in B^{m',d'}.$$

The composition $\tilde{A} \tilde{A}'$ is obtained by multiplication of the matrices (For more details see [12]). For example $\pi^+ P \circ G'$ and $G \circ G'$ are singular Green operators of type $d'$ and

$$\pi^+ P \circ \pi^+ P' = \pi^+ (PP') + L(P, P').$$

Here $PP'$ is the usual composition of pseudodifferential operators and $L(P, P')$ called leftover term is a singular Green operator of type $m' + d$. For our case, $P, P'$ are classical pseudo differential operators, in other words $\pi^+ P \in B^\infty$ and $\pi^+ P \in B^\infty$.

Let $M$ be a $n$-dimensional compact oriented manifold with boundary $\partial M$. Denote by $B$ the Boutet de Monvel’s algebra. We recall that the main theorem in [1, 13].

**Theorem 3.1.** ([Fedosov-Golse-Leichtnam-Schrohe]) Let $M$ and $\partial M$ be connected, $\dim M = n \geq 3$, and let $\tilde{S}$ (resp. $\tilde{S}'$) be the unit sphere about $\xi$ (resp. $\xi'$) and $\sigma(\xi)$ (resp. $\sigma(\xi')$) be the corresponding canonical $n-1$ (resp. $(n-2)$) volume form. Set $\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \tilde{S} \end{pmatrix} \in B$, and denote by $p$, $b$ and $s$ the local symbols of $P, G$ and $S$ respectively. Define:

$$\overline{\text{Wres}}(\tilde{A}) = \int_X \int_{\tilde{S}} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi)dx + 2\pi \int_{\partial X} \int_{\tilde{S}'} \left[ \text{tr}_E [s_{-n}(x', \xi')] + \text{tr}_F [s_{-n}(x', \xi')]ight] \sigma(\xi')dx',$$

where $\overline{\text{Wres}}$ denotes the noncommutative residue of an operator in the Boutet de Monvel’s algebra. Then a) $\text{Wres}(\tilde{A}, B) = 0$, for any $\tilde{A}, B \in B$; b) It is the unique continuous trace on $B/B^\infty$.

**Definition 3.2.** ([13]) Lower dimensional volumes of spin manifolds with boundary are defined by

$$\text{Vol}_n^{(p_1, p_2)} M := \overline{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}],$$

(3.6)
Lemma 3.5. \[\text{We get} \]
\[
\overline{\text{Wres}}[\pi^+D^{-p_1} \circ \pi^+D^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{\nabla^*} \bigotimes \mathbb{C} [\pi_n(D^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \tag{3.7}
\]
and
\[
\Phi = \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \left(-\frac{\alpha}{\sigma(j+k+1)}\right) \times \text{trace}_{\nabla^*} \bigotimes \mathbb{C} [\partial_{x_n} \partial_{\xi} \partial_{x_n} \partial_{\xi} (D^{-p_2})(x', 0, \xi', \xi_n)]
\]
\[
\times \partial_{x_n} \partial_{\xi} \partial_{x_n} \partial_{\xi} (D^{-p_2})(x', 0, \xi', \xi_n) dx' \tag{3.8}
\]
where the sum is taken over \(a + l - k - |a| - j - 1 = -n, r \leq -p_1, l \leq -p_2\).

Since \([\pi_n(D^{-p_1-p_2})]|_{\partial M}\) has the same expression as \(\pi_n(D^{-p_1-p_2})\) in the case of manifolds without boundary, so locally we can compute the first term by \([1, 7, 9, 13]\).

For any fixed point \(x_0 \in \partial M\), we choose the normal coordinates \(U\) of \(x_0\) in \(\partial M\) (not in \(M\)) and compute \(\Phi(x_0)\) in the coordinates \(\bar{U} = U \times [0, 1) \subset M\) and the metric \(\frac{1}{h(x_0)} g_{\partial M} + dx_n^2\). The dual metric of \(g_{\partial M}\) on \(\bar{U}\) is \(h(x_0) g_{\partial M} + dx_n^2\). Write \(g^i_{ij} = \frac{h(x_0)}{\partial x_i \partial x_j}; g^{ij}_{\partial M} = \frac{h(x_0)}{\partial x_i \partial x_j}\), then
\[
[g^{ij}_{\partial M}] = \begin{bmatrix} \frac{h(x_0)}{\partial x_i \partial x_j} & 0 \\ 0 & 1 \end{bmatrix}; \quad [g^{ij}_{\partial M}] = \begin{bmatrix} \frac{h(x_0)}{\partial x_i \partial x_j} & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.9}
\]
and
\[
\partial_x g^{ij}_{\partial M}(x_0) = 0, i, j \leq n - 1; \quad g^{ij}_{\partial M}(x_0) = \delta_{ij}. \tag{3.10}
\]

From \([13]\), we can get the following three lemmas,

**Lemma 3.3.** \([13]\) With the metric \(g_{\partial M}\) on \(M\) near the boundary,
\[
\partial_{x_j} (\xi^j_{g_{\partial M}})(x_0) = \begin{cases} 0, & \text{if } j < n, \\ h'(0) \xi^j_{g_{\partial M}}, & \text{if } j = n, \end{cases} \tag{3.11}
\]
\[
\partial_{x_j} (c(\xi))(x_0) = \begin{cases} 0, & \text{if } j < n, \\ \partial_{x_n} (c(\xi))(x_0), & \text{if } j = n, \end{cases} \tag{3.12}
\]
where \(\xi = \xi' + \xi_n dx_n\).

**Lemma 3.4.** \([13]\) With the metric \(g_{\partial M}\) on \(M\) near the boundary,
\[
\omega_{s,t}(e_i)(x_0) = \begin{cases} \omega_{s,n}(e_i)(x_0) = \frac{1}{2} h'(0), & \text{if } s = n, t = i, i < n, \\ \omega_{i,n}(e_i)(x_0) = -\frac{1}{2} h'(0), & \text{if } s = i, t = n, i < n, \\ \omega_{s,i}(e_i)(x_0) = 0, & \text{other cases}, \end{cases} \tag{3.13}
\]
where \((\omega_{s,t})\) denotes the connection matrix of Levi-Civita connection \(\nabla^L\).

**Lemma 3.5.** \([13]\) When \(i < n\), then
\[
\Gamma^k_{ij}(x_0) = \begin{cases} \Gamma^k_{ij}(x_0) = \frac{1}{2} h'(0), & \text{if } s = t = i, k = n, \\ \Gamma^k_{ni}(x_0) = -\frac{1}{2} h'(0), & \text{if } s = n, t = i, k = i, \\ \Gamma^k_{in}(x_0) = -\frac{1}{2} h'(0), & \text{if } s = i, t = n, k = i, \end{cases} \tag{3.14}
\]
in other cases, \(\Gamma^k_{ij}(x_0) = 0\).

By \([37]\) and \([38]\), we firstly compute
\[
\overline{\text{Wres}}[\pi^+D^{-1} \circ \pi^+D^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\nabla^*} \bigotimes \mathbb{C} [\pi_n((D^{-1}D^{-1})^{-1})] \sigma(\xi) dx + \int_{\partial M} \Phi, \tag{3.15}
\]
Lemma 3.7. The following identities hold:

\[
\Phi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} \frac{(-i)^{\alpha+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\Lambda \cdot T \cdot M} \bigotimes c[D_{\xi_n}^{\alpha} \partial_{\xi_n}^k \tilde{e} \cdot (D_V^{-1})(x', 0, \xi', \xi_n)] \\
\times \partial_{\xi_n}^k \tilde{e} \cdot (D_V^{-1})(x', 0, \xi', \xi_n)]d\xi_n d\xi(\xi') dx',
\]

(3.16)

and the sum is taken over \( r + l - k - j - |\alpha| = -3, \ r \leq -1, l \leq -1. \)

By Theorem 2.2, we can compute the interior of \( \tilde{\text{Wres}}[\pi^+ D_V^{-1} \circ \pi^+ (D_V^*)^{-1}] \), so

\[
\int_M \int_{|\xi'|=1} \text{trace}_{\Lambda \cdot T \cdot M} [\sigma_{-4}((D_V^*)^{-1})] \sigma(\xi) d\xi = 32\pi^2 \int_M \left( -\frac{4}{3} K \right) d\text{Vol}_M.
\]

(3.17)

Now we need to compute \( \int_{\partial M} \Phi \). Since, some operators have the following symbols.

Lemma 3.6. The following identities hold:

\[
\sigma_1(D_V) = \sigma_1(D_V^*) = -\tilde{c}(V)c(\xi);
\]

\[
\sigma_0(D_V) = \frac{\tilde{c}(V)}{4} \left( \sum_{i, s, t} \omega_{i, s, t}(e_i)c(e_i)\tilde{c}(e_s)\tilde{c}(e_t) - \sum_{i, s, t} \omega_{i, s, t}(e_i)c(e_i)c(e_s)c(e_t) \right);
\]

\[
\sigma_0(D_V^*) = \frac{\tilde{c}(V)}{4} \left( \sum_{i, s, t} \omega_{i, s, t}(e_i)c(e_i)\tilde{c}(e_s)\tilde{c}(e_t) - \sum_{i, s, t} \omega_{i, s, t}(e_i)c(e_i)c(e_s)c(e_t) \right) - \frac{1}{q} \sum_{q=1}^{\infty} c(\xi)\tilde{c}(\nabla_{\xi_n}L V).
\]

(3.18)

Write

\[
D_{\xi_n}^0 = (-i)^{\alpha} \partial_{\xi_n}^\alpha; \ \sigma(D_V) = p_1 + p_0; \ (\sigma(D_V)^{-1}) = \sum_{j=1}^{\infty} q_j.
\]

(3.19)

By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma(D_V \circ D_V^{-1}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi_n}^\alpha [\sigma(D_V)] D_{\xi_n}^\alpha [\sigma(D_V^{-1})]
\]

\[
= (p_1 + p_0)(q_1 + q_2 + q_3 + \cdots) + \sum_j (\partial_{\xi_n} p_1 + \partial_{\xi_n} p_0)(D_{x_j} q_1 + D_{x_j} q_2 + D_{x_j} q_3 + \cdots)
\]

\[
= p_1 q_1 + (p_1 q_2 + p_0 q_1 + \sum_j \partial_{\xi_n} p_1 D_{x_j} q_1) + \cdots,
\]

(3.20)

so

\[
q_1 = p_1; \ q_2 = -p_1[p_0 p_1 + \sum_j \partial_{\xi_n} p_1 D_{x_j} (p_1^{-1})].
\]

(3.21)

Lemma 3.7. The following identities hold:

\[
\sigma_{-1}(D_V^{-1}) = \sigma_{-1}((D_V^*)^{-1}) = -\frac{\tilde{c}(V)c(\xi)}{|\xi|^2};
\]

\[
\sigma_{-2}(D_V^{-1}) = \frac{c(\xi)\sigma_0(D_V) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right];
\]

\[
\sigma_{-2}((D_V^*)^{-1}) = \frac{c(\xi)\sigma_0(D_V^*) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right].
\]

(3.22)
Theorem 3.8. Let $M$ be a 4-dimensional oriented compact manifold with boundary $\partial M$ and the metric $g^{TM}$ as in Section 3, the operators $D_V = \sqrt{-\text{Tr}(V)}(d + \delta)$ and $D_V^* = -\sqrt{-1}(d + \delta)\bar{c}(V)$ be on $M$ ($\overline{M}$ is a collar neighborhood of $M$), then

$$
\overline{\text{Wres}}[\pi^+ D_V^{-1} \circ \pi^+(D_V^*)^{-1}]
= 32\pi^2 \int_M \left( -\frac{4}{3} K \right) d\text{Vol}_M + \int_{\partial M} \left( -\frac{3i\eta'(0)}{2} - \frac{27\pi^2 \epsilon'(0)}{8} - \frac{\pi^2}{4} \right) \pi \Omega d\text{Vol}_M,
$$

(3.23)

where $K$ is the scalar curvature.

Proof. When $n = 4$, then $\text{tr}_{\wedge^* T M}[\text{id}] = \text{dim} (\wedge^*(\mathbb{R}^4)) = 16$, the sum is taken over $r + l - k - j - |\alpha| = -3$, $r \leq -1$, $l \leq -1$, then we have the following five cases:

**Case a)** $I)$ $r = -1$, $l = -1$, $k = j = 0$, $|\alpha| = 1$.

By (3.16), we get

$$
\Phi_1 = -\int_{|\xi| = 1} \int_{-\infty}^{+\infty} \sum_{|\alpha| = 1} \text{tr}[\partial^\alpha \pi^+_\xi \sigma_{-1}(D_V^{-1}) \times \partial^\alpha \xi_i \xi_{-1}((D_V^*)^{-1})] (x_0) d\xi_n \sigma(\xi') d\xi'.
$$

(3.24)

By Lemma 3.3 for $i < n$, we have

$$
\partial_{x_i} \left( -\frac{\bar{c}(V) c(\xi)}{|\xi|^2} \right) (x_0) = -\frac{\partial_{x_i} (\bar{c}(V) c(\xi)(x_0))}{|\xi|^2} - \bar{c}(V) \partial_{x_i} \left[ \frac{c(\xi)}{|\xi|^2} \right] (x_0)
$$

$$
= -\sum_{i=1}^{n} \partial_{x_i} (V_i) \bar{c}(\xi_i) \frac{c(\xi)}{|\xi|^2} (x_0),
$$

(3.25)

where $\bar{c}(V) = \sum_{i=1}^{n} V_i \bar{c}(\xi_i), V_i = g^{TM}(V, \xi_i)$.

Then

$$
\partial^\alpha \xi_i \sigma_{-1}((D_V^*)^{-1})
= -\sum_{i=1}^{n} \partial_{x_i} (V_i) \bar{c}(\xi_i)
\frac{\xi_n^2 \sum_{i=1}^{n} \partial_{x_i} (V_i) \bar{c}(\xi_i)}{(1 + \xi_n^2)^2}
\frac{c(dx_n)}{(1 + \xi_n^2)^2} + \frac{2 \xi_n \sum_{i=1}^{n} \partial_{x_i} (V_i) \bar{c}(\xi_i)}{(1 + \xi_n^2)^2} c(\xi').
$$

(3.26)

By $c(\xi) = \sum_{i=1}^{n} \xi_i c(dx_j), |\xi|^2 = \sum_{ij} g^{ij} \xi_i \xi_j$, for $i < n$, we have

$$
\partial_{x_i} \pi^+_\xi \left( -\frac{\bar{c}(V) c(\xi)}{|\xi|^2} \right) (x_0)
$$

$$
= \pi^+_\xi \partial_{x_i} \left( -\frac{\bar{c}(V) \sum_{j=1}^{n} \xi_j c(dx_j)}{|\xi|^2} \right) (x_0)
$$

$$
= \pi^+_\xi \left( \frac{-\bar{c}(V) c(dx_i) + 2 \sum_{j=1}^{n} \xi_j \bar{c}(V) c(dx_j)}{|\xi|^2} \right) (x_0)
$$

$$
= \frac{i}{2(\xi_n - i)} \left[ \bar{c}(V) c(dx_i) - \sum_{j=1}^{n-1} \xi_j \bar{c}(V) c(dx_j) \right] - \frac{1}{2(\xi_n - i)^2} \sum_{j=1}^{n-1} \xi_j \bar{c}(V) c(dx_j) - \frac{i}{2(\xi_n - i)^2} \xi_i \bar{c}(V) c(dx_n).
$$

(3.27)

Then

$$
\sum_{|\alpha| = 1} \text{tr}[\partial^\alpha \pi^+_\xi \sigma_{-1}(D_V^{-1}) \times \partial^\alpha \xi_i \sigma_{-1}((D_V^*)^{-1})](x_0)
$$

13
\[
\begin{align*}
&= -\frac{i(1 - \xi^2)}{2(\xi_n - i)^2(\xi_n + i)^2} \sum_{i=1}^{n-1} \sum_{l=1}^{n-1} \text{tr}[\partial_{x_i} (V) \tilde{c}(V) c(dx_l) \tilde{c}(e_l) c(dx_n)] \\
&+ \frac{i\xi_n}{(\xi_n - i)^2(\xi_n + i)^2} \sum_{i=1}^{n-1} \sum_{ij=1}^{n-1} \text{tr}[\partial_{x_i} (V) \tilde{c}(V) c(dx_j) \tilde{c}(e_i) c(\xi')] \\
&+ \frac{i(1 - \xi^2)}{2(\xi_n - i)^4(\xi_n + i)^2} \sum_{i=1}^{n-1} \sum_{ij=1}^{n-1} \text{tr}[\xi_j \partial_{x_i} (V) \tilde{c}(V) c(dx_j) \tilde{c}(e_i) c(dx_n)] \\
&- \frac{\xi_n}{(\xi_n - i)^2(\xi_n + i)^2} \sum_{i=1}^{n-1} \sum_{ij=1}^{n-1} \text{tr}[\xi_j \partial_{x_i} (V) \tilde{c}(V) c(dx_j) \tilde{c}(e_i) c(\xi')] \\
&+ \frac{i(1 - \xi^2)}{2(\xi_n - i)^4(\xi_n + i)^2} \sum_{i=1}^{n-1} \sum_{ij=1}^{n-1} \text{tr}[\xi_i \partial_{x_i} (V) \tilde{c}(V) c(dx_n) \tilde{c}(e_i) c(\xi')].
\end{align*}
\]

By \(\tilde{c}(V) \tilde{c}(e_l) \tilde{c}(V) = 2g^{TM}_{V,e_l} = 2V_l\) and \(\text{tr}ab = \text{tr}ba\),

\[
\sum_{i=1}^{n-1} \sum_{l=1}^{n-1} \text{tr}[\partial_{x_i} (V) \tilde{c}(V) c(dx_l) \tilde{c}(e_l) c(dx_n)] \\
= \sum_{i=1}^{n-1} \sum_{l=1}^{n-1} \partial_{x_i} (V_i) \text{tr}[^{\tilde{c}}(V) c(dx_l) \tilde{c}(e_l) c(dx_n)] \\
= -\sum_{i=1}^{n-1} \sum_{l=1}^{n-1} \partial_{x_i} (V_i) \text{tr}[c(dx_l) \tilde{c}(V) \tilde{c}(e_l) c(dx_n)] \\
= -\sum_{i=1}^{n-1} \sum_{l=1}^{n-1} 2V_i \partial_{x_i} (V_l) \text{tr}[c(dx_l) c(dx_n)] + \sum_{i=1}^{n-1} \sum_{l=1}^{n-1} \partial_{x_i} (V_l) \text{tr}[c(dx_l) \tilde{c}(e_i) \tilde{c}(V) c(dx_n)] \\
= -\sum_{i=1}^{n-1} \sum_{l=1}^{n-1} 2V_i \partial_{x_i} (V_l) \text{tr}[c(dx_l) c(dx_n)] - \sum_{i=1}^{n-1} \sum_{l=1}^{n-1} \partial_{x_i} (V_l) \text{tr}[\tilde{c}(V) c(dx_l) \tilde{c}(e_l) c(dx_n)],
\]

then,

\[
\sum_{i=1}^{n-1} \sum_{l=1}^{n-1} \partial_{x_i} (V_l) \text{tr}[^{\tilde{c}}(V) c(dx_l) \tilde{c}(e_l) c(dx_n)] = -\sum_{i=1}^{n-1} \sum_{l=1}^{n-1} V_i \partial_{x_i} (V_l) \text{tr}[c(dx_l) c(dx_n)] = 0.
\]

(3.30)
By (3.16), we get

\[
\sum_{l=1}^{n} \sum_{ij=1}^{n-1} \text{tr}[\xi_j \xi_i \partial_{x_l}(V_l) \widehat{c}(V) c(dx_j) \widehat{c}(e_l)c(\xi')]
\]

\[
= \sum_{l=1}^{n} \sum_{ij=1}^{n-1} \xi_j \xi_i \partial_{x_l}(V_l) \text{tr}[\widehat{c}(V) c(dx_j) \widehat{c}(e_l)c(\xi')]
\]

\[
= - \sum_{l=1}^{n} \sum_{ij=1}^{n-1} \xi_j \xi_i \xi_k \partial_{x_l}(V_l) \text{tr}[c(dx_j) \widehat{c}(V)\widehat{c}(e_l)c(dx_k)]
\]

\[
= - \sum_{l=1}^{n} \sum_{ij=1}^{n-1} \xi_j \xi_i \xi_k 2V_l \partial_{x_l}(V_l) \text{tr}[c(dx_j) c(dx_k)] 
+ \sum_{l=1}^{n} \sum_{ij=1}^{n-1} \xi_j \xi_i \xi_k \partial_{x_l}(V_l) \text{tr}[c(dx_j) \widehat{c}(e_l)\widehat{c}(V)c(dx_k)],
\]

(3.31)

then, by \( \partial_{x_l}(V_l) \frac{V_l}{2} = \frac{1}{2} \partial_{x_l}((V_l)^2) = 0 \),

\[
\sum_{l=1}^{n} \sum_{ij=1}^{n-1} \xi_j \xi_i \partial_{x_l}(V_l) \text{tr}[\widehat{c}(V) c(dx_j) \widehat{c}(e_l)c(\xi')]
= - \sum_{l=1}^{n} \sum_{ij=1}^{n-1} \xi_j \xi_i \xi_k \partial_{x_l}(V_l) \text{tr}[c(dx_j) c(dx_k)] = 0.
\]

(3.32)

Similarly, we have the following equalities:

\[
\sum_{l=1}^{n} \sum_{ij=1}^{n-1} \text{tr}[\partial_{x_l}(V_l) \widehat{c}(V) c(dx_j) \widehat{c}(e_l)c(\xi')] = 0; \quad \sum_{l=1}^{n} \sum_{ij=1}^{n-1} \text{tr}[\xi_j \partial_{x_l}(V_l) \widehat{c}(V) c(dx_n) \widehat{c}(e_l)c(\xi')] = 0;
\]

\[
\sum_{l=1}^{n} \sum_{ij=1}^{n-1} \text{tr}[\xi_j \xi_i \partial_{x_l}(V_l) \widehat{c}(V) c(dx_j) \widehat{c}(e_l)c(dx_n)] = 0; \quad \sum_{l=1}^{n} \sum_{ij=1}^{n-1} \text{tr}[\xi_j \xi_i \partial_{x_l}(V_l) \widehat{c}(V) c(dx_j) \widehat{c}(e_l)c(dx_n)] = 0.
\]

(3.33)

Therefore,

\[
\Phi_1 = - \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{x_l}^2 \pi_{\xi_l}^+ \sigma_{-1}(D_{V^{-1}}) \times \partial_{x_n}^2 \sigma_{-1}((D_{V^*})^{-1})] (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= 0.
\]

(3.34)

case a) II) \( r = -1, \ l = -1, \ k = |\alpha| = 0, \ j = 1. \)

By (3.16), we get

\[
\Phi_2 = - \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_l}^+ \sigma_{-1}(D_{V^{-1}}) \times \partial_{x_n}^2 \sigma_{-1}((D_{V^*})^{-1})] (x_0) d\xi_n \sigma(\xi') dx'.
\]

(3.35)

By Lemma 3.7, we have

\[
\partial_{x_n}^2 \sigma_{-1}(D_{V^{-1}})(x_0) = \partial_{x_n} \sigma_{-1}(D_{V^{-1}})(x_0) = \frac{\partial_{x_n}(\widehat{c}(V)) c(\xi) c(x_0)}{|\xi|^2} - \frac{\widehat{c}(V) \partial_{x_n} c(\xi')(x_0)}{|\xi|^2} + \frac{\widehat{c}(V) c(\xi')}{|\xi|^2} h'(0)(x_0).
\]

(3.36)

(3.37)
By (3.3), (3.4) and the Cauchy integral formula we have
\[
\pi_{\xi_n} \left[ \frac{\partial_{x_n} (\hat{c}(\xi)) c(\xi)}{|\xi|^2} \right] (x_0)|_{|\xi'|=1} = -\partial_{x_n} (\hat{c}(\xi)) \pi_{\xi_n}^+ \left[ \frac{(i\xi_n + 2)c(\xi') + ic(dx_n)}{1 + \xi_n^2} \right] \\
= -\partial_{x_n} (\hat{c}(\xi)) \lim_{n \to 0^-} \int_{\Gamma^+} \frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n - i)} \, d\eta_n \\
= i\partial_{x_n} (\hat{c}(\xi)) \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}.
\] (3.38)

Similarly, we have,
\[
\pi_{\xi_n}^+ \left[ \frac{\hat{c}(\xi)c(\xi')|\xi'|^2 h'(0)}{|\xi|^4} \right] (x_0)|_{|\xi'|=1} = \frac{\hat{c}(\xi)\partial_{x_n}(c(\xi'))(x_0)}{2(\xi_n - i)}; \tag{3.39}
\]
\[
\pi_{\xi_n}^+ \left[ \frac{\hat{c}(V)c(\xi)c(\xi')|\xi|^2|\xi'|^2 h'(0)}{|\xi|^4} \right] (x_0)|_{|\xi'|=1} = -\frac{i\partial_{x_n}(\hat{c}(\xi))c(\xi') + ic(dx_n)}{4(\xi_n - i)^2}.
\] (3.40)

By (3.27), then
\[
\pi_{\xi_n}^+ \partial_{x_n}(\sigma_{-1}(D_{V^{-1}}))|_{|\xi'|=1} = \frac{i\partial_{x_n}(\hat{c}(\xi))c(\xi') + ic(dx_n)}{2(\xi_n - i)} - i\frac{\partial_{x_n}(\hat{c}(\xi))c(\xi')}{4(\xi_n - i)^2}.
\] (3.41)

Similar to (3.29) - (3.32), we have the following equalities:
\[
\begin{align*}
\text{tr}[\partial_{x_n}(\hat{c}(\xi))c(\xi')\hat{c}(\xi)c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n}(\hat{c}(\xi))c(\xi')\hat{c}(\xi)c(\xi') ] = 0; \\
\text{tr}[\partial_{x_n}(\hat{c}(\xi))c(dx_n)\hat{c}(\xi)c(dx_n)] &= 0; \quad \text{tr}[\partial_{x_n}(\hat{c}(\xi))c(dx_n)\hat{c}(\xi)c(\xi')] = 0; \\
\text{tr}[\hat{c}(\xi)\partial_{x_n}(c(\xi'))\hat{c}(\xi)c(\xi')] &= 8h'(0) \sum_{k=1}^{n-1} \xi_k^2; \quad \text{tr}[\hat{c}(\xi)c(\xi')\hat{c}(\xi)c(\xi)] = 16 \sum_{k=1}^{n-1} \xi_k^2; \\
\text{tr}[\hat{c}(\xi)c(dx_n)\hat{c}(\xi)c(\xi') ] &= 0; \quad \text{tr}[\hat{c}(\xi)c(dx_n)\hat{c}(\xi)c(dx_n)] = 0.
\end{align*}
\] (3.42)

By (3.36), (3.31) and (3.32), we have
\[
\text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_{V}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((D_{V}^{-1}))]|(x_0) = \\
\frac{(24\xi_n^2 + 8\xi_n^2 - 8 - 16i)h'(0)}{(\xi_n - i)^2(\xi_n + i)^2} + \frac{(64\xi_n^2 i - 32\xi_n^2)h'(0)}{(\xi_n - i)^3(\xi_n + i)^3} \sum_{k=1}^{n-1} \xi_k^2 + \frac{(24\xi_n^2 - 8\xi_n^2)h'(0)}{(\xi_n - i)^3(\xi_n + i)^3} \sum_{k=1}^{n-1} \xi_k^2 \, dx_n\sigma(\xi')dx'.
\] (3.43)

Considering \(\int_{\mathbb{R}} \xi \xi_i = \frac{1}{2} \delta^{ij}\) see (2), then
\[
\Phi_2 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}((D_{V}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((D_{V}^{-1}))]|(x_0)d\xi_n\sigma(\xi')dx' = \\
-\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left[ \frac{(24\xi_n^2 + 8\xi_n^2 - 8 - 16i)h'(0)}{(\xi_n - i)^2(\xi_n + i)^2} + \frac{(64\xi_n^2 i - 32\xi_n^2)h'(0)}{(\xi_n - i)^3(\xi_n + i)^3} \right] \sum_{k=1}^{n-1} \xi_k^2 \, dx_n\sigma(\xi')dx'.
\]
By (3.16), we get

\[- \frac{1}{2} \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} (2\xi_n^2 - 8\xi_n^3)\xi_n^2 d\xi_n \sigma(\xi') dx' = -4h'(0)\Omega_{3} \frac{2\pi i}{2} (2\xi_n^2 + \xi_n - 1 - 2i)(\xi_n + i)^3 |\xi_n| = dx' - 16h'(0) \frac{\pi^2}{2} \frac{2\pi i}{4!} (2\xi_n^2 - \xi_n^3)(\xi_n + i)^3 |\xi_n|= dx'\]

\[- 4h'(0)\Omega_{3} \frac{2\pi i}{2} (3\xi_n^2 + \xi_n - 1 - 2i)(\xi_n + i)^3 |\xi_n|= dx'\]

\[- 4h'(0)\Omega_{3} \frac{2\pi i}{2} (3\xi_n^2 - \xi_n^3)(\xi_n + i)^3 |\xi_n|= dx'\]

\[= \left( \frac{(1 - 6i)h'(0)}{4} + \frac{(1 - 6i)\pi^2 h'(0)}{8} \right) \pi \Omega_{3} dx',\]

(3.44)

where \(\Omega_{3}\) is the canonical volume of \(S^3\).

**case a) III** \(r = -1, \ l = -1, \ j = |\alpha| = 0, \ k = 1.\)

By (3.16), we get

\[\Phi_{3} = - \frac{1}{2} \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^{+} \sigma_{-1}(D\nu^{-1})] (3.45)\]

By Lemma 3.7, we have

\[\partial_{\xi_n} \partial_{\xi_n} \sigma_{-1}(D\nu^{-1}) (3.46)\]

By (3.47), (3.48) and (3.49), we have

\[\text{trace}[\partial_{\xi_n} \pi_{\xi_n}^{+} \sigma_{-1}(D\nu^{-1})] (3.48)\]

Then,

\[\Phi_{3} = - \frac{1}{2} \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^{+} \sigma_{-1}(D\nu^{-1})] (3.49)\]

\[= - \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} 2h'(0) (3\xi_n^2 - 4\xi_n^3 + \xi_n^4)(\xi_n + i)^3 \sum_{k=1}^{n-1} \xi_k^2 d\xi_n \sigma(\xi') dx' - \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{4h'(0)}{1} (\xi_n^3 + i)^3 \sum_{k=1}^{n-1} \xi_k^2 d\xi_n \sigma(\xi') dx'\]

\[= -2h'(0)\Omega_{3} \frac{2\pi i}{2} (3\xi_n^2 - 4\xi_n^3 + \xi_n^4)(\xi_n + i)^3 |\xi_n|= dx' - 4h'(0)\Omega_{3} \frac{2\pi i}{4!} \left( \frac{1}{(\xi_n + i)^2} \right)^{(3)} |\xi_n|= dx'\]

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By Lemma 3.7 we have
\[ \frac{c(\xi)\sigma_0(D\varphi)(x_0)\xi(\xi)}{\xi^2} + \frac{c(\xi)}{\xi^0} (dx_n)[\partial_{x_n}(c(\xi'))(x_0)]\xi^2 - c(\xi'h'(0)]\xi_{\partial t}^2, \]
(3.51)
where
\[ \sigma_0(D\varphi)(x_0) = \frac{i\hat{c}(V)}{4} \sum_{s,t,i} \omega_{s,t}(e_i(x_0)c(e_i)e_n) - \frac{i\hat{c}(V)}{4} \sum_{s,t,i} \omega_{s,t}(e_i(x_0)c(e_i)c(e_i)). \]
(3.52)
We denote
\[ A_0^1(x_0) = \frac{i\hat{c}(V)}{4} \sum_{s,t,i} \omega_{s,t}(e_i(x_0)c(e_i)e_n) = i\hat{c}(V)a_0^1(x_0); \]
\[ A_0^2(x_0) = \frac{-i\hat{c}(V)}{4} \sum_{s,t,i} \omega_{s,t}(e_i(x_0)c(e_i)c(e_i)) = i\hat{c}(V)a_0^2(x_0), \]
(3.53)
where \( a_0^1 = c_0c(dx_n) \) and \( c_0 = -\frac{1}{4}\pi h'(0). \)
Then
\[ \pi_{\xi_n}^+ \sigma_{-2}(D\varphi^{-1})(x_0))|_{\xi| = 1} = \pi_{\xi_n}^+ \left[ \frac{c(\xi)A_0^1(x_0)\xi(\xi)}{(1 + \xi_n^2)} \right. \]
\[ \left. + \pi_{\xi_n}^+ \left[ \frac{\xi_n c(dx_n)A_0^1(x_0)\xi(\xi)}{(1 + \xi_n^2)} \right] \right. \]
\[ \left. - h'(0) \frac{c(\xi)c(dx_n)\xi(\xi)}{(1 + \xi_n^2)} \right]. \]
(3.54)
By computations, we have
\[ \pi_{\xi_n}^+ \left[ \frac{c(\xi)A_0^1(x_0)\xi(\xi)}{(1 + \xi_n^2)} \right] \]
\[ = \pi_{\xi_n}^+ \left[ \frac{c(\xi')A_0^2(x_0)\xi(\xi')}{(1 + \xi_n^2)} \right] + \pi_{\xi_n}^+ \left[ \frac{\xi_n c(dx_n)A_0^1(x_0)\xi(\xi)}{(1 + \xi_n^2)} \right] \]
\[ + \pi_{\xi_n}^+ \left[ \frac{\xi_n c(dx_n)A_0^2(x_0)\xi(\xi')}{(1 + \xi_n^2)} \right] + \pi_{\xi_n}^+ \left[ \frac{\xi_n c(dx_n)A_0^1(x_0)\xi(\xi)}{(1 + \xi_n^2)} \right] \]
\[ = - \frac{c(\xi')A_0^2(x_0)\xi(\xi')(2 + i\xi_n)}{4(\xi_n - i)^2} + \frac{ic(\xi')A_0^2(x_0)c(dx_n)}{4(\xi_n - i)^2} \]
\[ + \frac{ic(dx_n)A_0^1(x_0)c(\xi')}{4(\xi_n - i)^2} + \frac{i\xi_n c(dx_n)A_0^1(x_0)c(dx_n)}{4(\xi_n - i)^2}. \]
(3.55)
Since
\[ c(dx_n)a_0^1(x_0) = \frac{-1}{4}h'(0) \sum_{i=1}^{n-1} c(e_i)e_i c(e_i)e_n, \]
(3.56)
then by the relation of the Clifford action and $trab = trba$, we have the following equalities:

$$tr[c(e_i)c(e_i)c(e_n)c(e_{n})] = 0 \quad (i < n); \quad tr[a_n^0c(dx_n)] = 0; \quad tr[\tilde{c}(\xi')c(dx_n)] = 0. \tag{3.57}$$

Since

$$\partial_{\xi_n}\sigma^{-1}(DV^*)^{-1} = -\tilde{c}(V)\left[\frac{c(dx_n)}{1 + \xi_n} - \frac{2\xi_n c(\xi') + 2\xi_n^2 c(dx_n)}{(1 + \xi_n)^2}\right]. \tag{3.58}$$

By (3.57) and (3.62), we have

$$tr[\pi^+_{\xi_n}\left[\frac{c(\xi)c(V)0_1(x_n)0_1(c(\xi))}{(1 + \xi_n)^2}\right] \times \partial_{\xi_n}\sigma^{-1}(DV^*)^{-1}(x_0)\right]|_{\xi' = 1} = -\frac{1}{2(1 + \xi_n^2)}tr[c(\xi')a_0^0(x_0)] - \frac{i}{2(1 + \xi_n^2)}tr[c(dx_n)a_0^0(x_0)]$$

$$= -\frac{1}{2(1 + \xi_n^2)}tr[c(\xi')a_0^0(x_0)]. \tag{3.59}$$

We note that $i < n$, $\int_{\xi' = 1} [\xi_1, \xi_2, \ldots, \xi_{i+1}] \sigma(\xi') = 0$, so $tr[c(\xi')a_0^0(x_0)]$ has no contribution for computing case b).

By computations, we have

$$\pi^+_{\xi_n}\left[\frac{c(\xi)c(V)0_1(x_n)0_1(c(\xi))}{(1 + \xi_n)^2}\right] - h'(0)\pi^+_{\xi_n}\left[\frac{c(\xi)c(V)0_1(c(\xi))}{(1 + \xi_n)^3}\right] := N_1 - N_2, \tag{3.60}$$

where

$$N_1 = \frac{-1}{4(\xi_n - i)^2}[(2 + i\xi_n)c(\xi')\tilde{c}(V)a_0^0(x_0)c(\xi') - \xi_n c(dx_n)\tilde{c}(V)a_0^0(x_0)c(dx_n)$$

$$+ (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n}(c(\xi')) - c(dx_n)c(V)a_0^0(x_0)c(\xi') - c(\xi')c(V)a_0^0(x_0)c(dx_n) - i\partial_{x_n}c(\xi')] \tag{3.61}$$

and

$$N_2 = \frac{h'(0)}{2} \left[\frac{c(dx_n)}{4(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}ic(\xi') - c(dx_n)\right]. \tag{3.62}$$

then by the relation of the Clifford action and $trab = trba$, we have the following equalities:

$$tr[c(\xi')\tilde{c}(V)c(\xi')] = 0; \quad tr[c(dx_n)\tilde{c}(V)c(\xi')] = 0; \quad tr[c(\xi')\tilde{c}(V)c(dx_n)] = 0; \quad tr[c(dx_n)\tilde{c}(V)c(dx_n)] = 0. \tag{3.63}$$

By (3.58), (3.62) and (3.63), we have

$$tr[N_2 \times \partial_{\xi_n}\sigma^{-1}(DV^*)^{-1}]|_{\xi' = 1}$$

$$= \frac{i\hbar'(0)(\xi_n - 3i)}{4(\xi_n - i)^3(\xi_n + i)^2}tr[c(\xi')\tilde{c}(V)c(\xi')] + \frac{h'(0)(4\xi_n i - \xi_n^3 i - 3\xi_n^2)}{4(\xi_n - i)^3(\xi_n + i)^2}tr[c(dx_n)\tilde{c}(V)c(\xi')]$$

$$+ \frac{i\hbar'(0)(\xi_n^2 - 1)(\xi_n - 3i)}{8(\xi_n - i)^5(\xi_n + i)^2}tr[c(\xi')\tilde{c}(V)c(dx_n)] + \frac{ih'(0)(\xi_n - 3i)}{4(\xi_n - i)^5(\xi_n + i)^2}tr[c(dx_n)\tilde{c}(V)c(\xi')]$$

$$= 0.$
By (3.58), (3.61) and (3.63), we have
\[
\text{tr}[N_1 \times \partial_{\xi_n} \sigma_{-1}((D_V^*)^{-1})]|_{\xi'|=1} = \frac{3h'(0)\xi_n}{(\xi_n - i)^3(\xi_n + i)} + \frac{3h'(0)\xi_n^3 - 2i\xi_n^2 - 5\xi_n + 2i}{(\xi_n - i)^4(\xi_n + i)^2} \sum_{k=1}^{n-1} \xi_k^2.
\]  
(3.65)

By (3.64) and (3.65), we have
\[
\Phi_4 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-2}(D_V^{-1}) \times \partial_{\xi_n} \sigma_{-1}((D_V^*)^{-1})]|(\hat{x}_0) d\xi_n \sigma(\xi') dx'.
\]  
(3.66)
\[
\Phi_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-2}(D_V^{-1}) \times \partial_{\xi_n} \sigma_{-2}((D_V^*)^{-1})]|(\hat{x}_0) d\xi_n \sigma(\xi') dx'.
\]  
(3.67)

By (3.16), we get
\[
\Phi_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+ \sigma_{-1}((D_V^{-1})^{-1})]|(\hat{x}_0) d\xi_n \sigma(\xi') dx'.
\]  
(3.68)

Since
\[
\text{By (3.64) and (3.3), Lemma 3.7, we have}
\]
\[
\pi_{\xi_n}^+ \sigma_{-1}((D_V^{-1})^{-1})|_{\xi'|=1} = \frac{i\hat{c}(V)c(\xi') - \hat{c}(V)c(\hat{x}_0)}{2(\xi_n - i)}.
\]  
(3.69)

where
\[
\sigma_0(D_V^*)(x_0) = \frac{e(\xi)\sigma_0(D_V^*)(x_0)c(\xi)}{[\xi']^4} + \frac{e(\xi)}{[\xi']^0} \left[ \partial_{x_n}(c(\xi'))(x_0) \right] [\xi']^2 - c(\xi)h'(0)\|\xi\|^2_{\mathbb{H}}.
\]  
(3.70)

By (3.64) and (3.3), Lemma 3.7, we have
\[
\pi_{\xi_n}^+ \sigma_{-1}((D_V^{-1})^{-1})|_{\xi'|=1} = \frac{i\hat{c}(V)c(\xi') - \hat{c}(V)c(\hat{x}_0)}{2(\xi_n - i)}.
\]  
(3.68)

\[
\sigma_0(D_V^*)(x_0) = \frac{e(\xi)\sigma_0(D_V^*)(x_0)c(\xi)}{[\xi']^4} + \frac{e(\xi)}{[\xi']^0} \left[ \partial_{x_n}(c(\xi'))(x_0) \right] [\xi']^2 - c(\xi)h'(0)\|\xi\|^2_{\mathbb{H}}.
\]  
(3.69)

where
\[
\sigma_0(D_V^*)(x_0) = \frac{e(\xi)\sigma_0(D_V^*)(x_0)c(\xi)}{[\xi']^4} + \frac{e(\xi)}{[\xi']^0} \left[ \partial_{x_n}(c(\xi'))(x_0) \right] [\xi']^2 - c(\xi)h'(0)\|\xi\|^2_{\mathbb{H}}.
\]  
(3.70)

By (3.64), (3.61) and (3.63), we have
\[
\text{tr}[N_1 \times \partial_{\xi_n} \sigma_{-1}((D_V^*)^{-1})]|_{\xi'|=1} = \frac{3h'(0)\xi_n}{(\xi_n - i)^3(\xi_n + i)} + \frac{3h'(0)\xi_n^3 - 2i\xi_n^2 - 5\xi_n + 2i}{(\xi_n - i)^4(\xi_n + i)^2} \sum_{k=1}^{n-1} \xi_k^2.
\]  
(3.65)
By (3.68) and (3.72), we have

\[
\partial_n \left\{ \frac{c(\xi)A^1_0(x_0)}{\xi^4} + \frac{c(\xi)A^0_0(x_0)}{\xi^6} c(dx_n)[\partial_{x_n}(c(\xi'))(x_0)|\xi|^2 - c(\xi)h'(0)] \right\}
\]

\[
= \partial_n \left\{ \frac{c(\xi)A^1_0(x_0)c(\xi)}{\xi^4} + \frac{c(\xi)A^0_0(x_0)c(\xi)}{\xi^6} c(dx_n)[\partial_{x_n}(c(\xi'))(x_0)|\xi|^2 - c(\xi)h'(0)] \right\}
+ \partial_n \frac{c(\xi)A^2_0(x_0)c(\xi)}{\xi^4} + \partial_n \frac{c(\xi)[i \sum_{q=1}^n c(e_q)\hat{\partial}(\nabla_{e_q} V)](x_0)c(\xi)}{\xi^4}
\]

\[
= M_1 + M_2 - M_3,
\]

(3.71)

where

\[M_1 = \partial_n \frac{c(\xi)A^1_0(x_0)c(\xi)}{\xi^4}\]

\[M_2 = \partial_n \left\{ \frac{c(\xi)A^1_0(x_0)c(\xi)}{\xi^4} + \frac{c(\xi)A^0_0(x_0)c(\xi)}{\xi^6} c(dx_n)[\partial_{x_n}(c(\xi'))(x_0)|\xi|^2 - c(\xi)h'(0)] \right\},\]

\[M_3 = \partial_n \frac{c(\xi)[i \sum_{q=1}^n c(e_q)\hat{\partial}(\nabla_{e_q} V)](x_0)c(\xi)}{\xi^4}.
\]

By computations, we have

\[
M_1 = \partial_n \frac{c(\xi)A^1_0(x_0)c(\xi)}{\xi^4} = \frac{c(dx_n)c(V)A^0_1(x_0)c(\xi)}{\xi^4} + i \frac{c(\xi)c(V)A^0_1(x_0)c(dx_n)}{\xi^4} - i \frac{4\xi_n c(\xi)c(V)A^0_1(x_0)c(\xi)}{\xi^6};
\]

(3.72)

\[
M_2 = \frac{1}{(1 + \xi_n^2)^3} \left[ (2\xi_n - 2\xi_n^3)c(dx_n)A^0_0(c(\xi')) + (1 - 3\xi_n^2)c(dx_n)A^0_0 c(\xi')
+ (1 - 3\xi_n^2)c(\xi') A^0_0 c(dx_n) - 4\xi_n c(\xi')A^0_0 c(\xi') + (3\xi_n^2 - 1) \partial_{x_n}(c(\xi'))
- 4\xi_n c(\xi') c(dx_n) \partial_{x_n}(c(\xi')) + 2h'(0)c(\xi') + 2h'(0)\xi_n c(dx_n) \right]
+ 6\xi_n h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^4};
\]

(3.73)

\[
M_3 = \partial_n \frac{c(\xi)[i \sum_{q=1}^n c(e_q)\hat{\partial}(\nabla_{e_q} V)](x_0)c(\xi)}{\xi^4} = \frac{c(dx_n)[i \sum_{q=1}^n c(e_q)\hat{\partial}(\nabla_{e_q} V)](x_0)c(\xi)}{\xi^4} + \frac{c(\xi)[i \sum_{q=1}^n c(e_q)\hat{\partial}(\nabla_{e_q} V)](x_0)c(dx_n)}{\xi^4}
- \frac{4\xi_n c(\xi)[i \sum_{q=1}^n c(e_q)\hat{\partial}(\nabla_{e_q} V)](x_0)c(\xi)}{\xi^4}.
\]

(3.74)

By (3.68) and (3.72), we have

\[
\text{tr}[\pi^+_{\sigma_{-1}}(D_V^{-1}) \times \partial_n \frac{c(\xi)A^1_0 c(\xi)}{\xi^4}](x_0)|_{\xi^4 = 1}
\]

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Then, we get
\[\frac{1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')a_0^1(x_0)] - \frac{i}{(\xi - i)(\xi + i)^3} \text{tr}[c(dx_n)a_0^1(x_0)] \]
\[= \frac{1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')a_0^1(x_0)].\] (3.75)

We note that \(i < n, \) \(\int_{|\xi'|=1} \{\xi_1, \xi_2, \cdots, \xi_{i+1}\} \sigma(\xi') = 0,\) so \(\text{tr}[c(\xi')a_0^1(x_0)]\) has no contribution for computing case c). By (3.68) and (3.73), we have
\[\text{tr}[\pi^*_\xi \sigma_{-1}(D_{\xi}^{-1}) \times M_2]|_{|\xi'|=1} = \frac{12h'(0)(\xi_0^3 - \xi_n)}{(\xi - i)^4(\xi + i)^3} + \frac{6h'(0)(1 - 3\xi_n^2 - 4\xi_n)}{(\xi - i)^4(\xi + i)^3} \sum_{k=1}^{n-1} \xi_k^2.\] (3.76)

By (3.68) and (3.74), we have
\[\text{tr}[\pi^*_\xi \sigma_{-1}(D_{\xi}^{-1}) \times M_3]|_{|\xi'|=1} = \frac{1}{2(\xi - i)^4(\xi + i)^2} \text{tr}[\hat{c}(V)c(\xi')c(dx_n)] \sum_{q=1}^{n} c(e_q)\hat{c}(\nabla_{e_q} L V)c(\xi)]
\[- \frac{i}{2(\xi - i)^4(\xi + i)^2} \text{tr}[\hat{c}(V)c(\xi')c(dx_n)] \sum_{q=1}^{n} c(e_q)\hat{c}(\nabla_{e_q} L V)c(\xi)]
+ \frac{(1 - 4\xi_n)}{2(\xi - i)^4(\xi + i)^2} \text{tr}[\hat{c}(V)c(\xi')c(dx_n)] \sum_{q=1}^{n} c(e_q)\hat{c}(\nabla_{e_q} L V)c(\xi)]
+ \frac{(1 - 4\xi_n)i}{2(\xi - i)^4(\xi + i)^2} \text{tr}[\hat{c}(V)c(\xi')c(dx_n)] \sum_{q=1}^{n} c(e_q)\hat{c}(\nabla_{e_q} L V)c(\xi)]].\] (3.77)

And by the relation of the Clifford action and \(\text{tr}_{ab} = \text{tr}_{ba},\) we have the following equalities:
\[\text{tr}[\hat{c}(V)c(\xi')c(dx_n)] \sum_{q=1}^{n} c(e_q)\hat{c}(\nabla_{e_q} L V)c(\xi') = 0, \quad \text{tr}[\hat{c}(V)c(dx_n)] \sum_{q=1}^{n} c(e_q)\hat{c}(\nabla_{e_q} L V)c(\xi') = 0.\] (3.78)

Then, we get
\[\Phi_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi^*_\xi \sigma_{-1}(D_{\xi}^{-1}) \times \partial_{\xi_n} \sigma_{-2}((D_{\xi}^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'
= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{12h'(0)(\xi_0^3 - \xi_n)}{(\xi - i)^4(\xi + i)^3} d\xi_n dx' - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{6h'(0)(1 - 3\xi_n^2 - 4\xi_n)}{(\xi - i)^4(\xi + i)^3} \sum_{k=1}^{n-1} \xi_k^2 d\xi_n dx'
= -12h'(0)\Omega_3 \left( \frac{2\pi i}{3} \frac{\xi_0^3 - \xi_n}{(\xi_0 + i)^3} \right) |_{\xi_n=1} - 6h'(0)\Omega_3 \left( \frac{2\pi i}{3} \frac{1 - 3\xi_n^2 - 4\xi_n}{(\xi_0 + i)} \right) |_{\xi_n=1} d\xi_n dx'
= \frac{3(i - 2)^2 h'(0)}{4} \pi \Omega_3 dx'.\] (3.79)

So,
\[\Phi = \sum_{i=1}^{5} \Phi_i = \left( - \frac{3ih'(0)}{2} - \frac{27\pi^2 h'(0)}{8} - \frac{\pi^2}{4} \right) \pi \Omega_3 dx'.\] (3.80)

Then, by (3.19)-(3.17), we obtain Theorem 3.8. \(\Box\)
Next, we also prove the Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary associated to $D_V$. By (3.7) and (3.8), we will compute

$$
\text{Wres}[\pi^+ D_V^{-1} \circ \pi^+ D_V^{-1}] = \int_M \int_{|\xi'|=1} \text{trac}_{\nabla^* T M} \otimes C[\sigma_4 (D_V^{-2})] \sigma(\xi) dx + \int_{\partial M} \overline{\Phi},
$$

(3.81)

where

$$
\overline{\Phi} = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum_{\alpha} (-i)^{\alpha+j+k+1} \frac{1}{2\alpha(\alpha+1)} \times \text{trac}_{\nabla^* T M} \otimes C[\langle j, k \rangle] \sigma_r (D_V^{-1})(\xi', 0, \xi, \xi_n)
\times \partial_n \xi_n^i \partial_n \xi_n^j \sigma_r (D_V^{-1})(\xi', 0, \xi, \xi_n) d\xi_n \sigma(\xi') dx',
$$

(3.82)

and the sum is taken over $r+l-k-j-|\alpha| = -3$, $r \leq -1, l \leq -1$.

By Theorem 2.2, we compute the interior of $\text{Wres}[\pi^+ D_V^{-1} \circ \pi^+ D_V^{-1}]$, then

$$
\int_M \int_{|\xi'|=1} \text{trac}_{\nabla^* T M} \otimes C[\sigma_4 (D_V^{-2})] \sigma(\xi) dx = 32\pi^2 \int_M \left( -\frac{4}{3} K - 8 \sum_{q=1}^{4} \langle \nabla_{\xi_q}^2 V \rangle^2 \right) d\text{Vol}_M.
$$

(3.83)

**Theorem 3.9.** Let $M$ be a 4-dimensional oriented compact manifold with boundary $\partial M$ and the metric $g^{TM}$ as in Section 3, the operator $D_V = \sqrt{-1} \partial \Omega(\omega + \delta)$ be on $\overline{M}$ ($\overline{M}$ is a collar neighborhood of $M$), then

$$
\text{Wres}[\pi^+ D_V^{-1} \circ \pi^+ D_V^{-1}] = 32\pi^2 \int_M \left( -\frac{4}{3} K - 8 \sum_{q=1}^{4} \langle \nabla_{\xi_q}^2 V \rangle^2 \right) d\text{Vol}_M + \int_{\partial M} \left( -\frac{3i h'(0)}{2} - \frac{27\pi^2 h'(0)}{8} - \frac{\pi^2}{4} \right) \pi \Omega d\text{Vol}_M,
$$

(3.84)

where $K$ is the scalar curvature.

**Proof.** When $n = 4$, by Lemma 3.7, $\sigma_-(D_V^{-1}) = \sigma_-(\langle D_V \rangle^{-1})$, then we have the following five cases:

**case a I)** $r = -1$, $l = -1$, $k = j = 0$, $|\alpha| = 1$.

$$
\overline{\Phi}_1 = -\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trac}_{\nabla^* T M} \otimes C[\sigma_4 (D_V^{-2})] d\xi_n \sigma(\xi') dx' = 0.
$$

(3.85)

**case a II)** $r = -1$, $l = -1$, $k = |\alpha| = 0$, $j = 1$.

$$
\overline{\Phi}_2 = \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trac}_{\nabla^* T M} \otimes C[\sigma_4 (D_V^{-2})] d\xi_n \sigma(\xi') dx' = \left( \frac{1-6i}{4} h'(0) + \frac{1-6i}{8} \pi^2 h'(0) \right) \pi \Omega d\text{Vol}_M.
$$

(3.86)

**case a III)** $r = -1$, $l = -1$, $j = |\alpha| = 0$, $k = 1$.

$$
\overline{\Phi}_3 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trac}_{\nabla^* T M} \otimes C[\sigma_4 (D_V^{-2})] d\xi_n \sigma(\xi') dx' = 23
$$
By (3.3) and (3.4), Lemma 3.7, we have
\[
(3.87)
\]
then
\[
(3.88)
\]
case b) \( r = -2, \ l = -1, \ k = j = |\alpha| = 0. \n\]
By (3.89), we get
\[
(3.90)
\]
By (3.90), Lemma 3.7 we have
\[
(3.91)
\]
where
\[
(3.92)
\]
then
\[
(3.93)
\]
By (3.94) and (3.95), we have
\[
\Phi_5 = -i \int_{|\xi'|=1}^{+\infty} \text{trace}[\pi_{\xi'}^+ \sigma_{-1}(D_V^{-1}) \times \partial_{\xi'} \sigma_{-2}(D_V^{-1})](x_0) d\xi' dx'
\]
\[
= -i \int_{|\xi'|=1}^{+\infty} \frac{12h'((0)\xi_4 - \xi_3)}{(\xi - i)^4(\xi + i)^3} d\xi' dx' - \int_{|\xi'|=1}^{+\infty} \sum_{k=1}^{n-1} \xi_3^k d\xi' dx'
\]
By Lemma 3.6, we have

\[ \frac{2\pi i}{3!} \left[ \frac{\xi_n^3 - \xi_n}{(\xi_n + i)^3} \right]_{\xi_n = 1} = 6h'/(0) i\Omega_3 \frac{2\pi i \frac{\pi^2}{2} \left[ 1 - 3\xi_n^2 - 4\xi_n \right]_{\xi_n = 1}}{(\xi_n + i)^3} \]

\[ = \frac{3(i - 2)\pi^2 h'(0)}{4} \pi \Omega_3 dx'. \]

(3.94)

Therefore, we get

\[ \Phi = \sum_{i=1}^{5} \Phi_i = \left( -\frac{3ih'(0)}{2} - \frac{27\pi^2 h'(0)}{8} - \frac{\pi^2}{4} \right) \pi \Omega_3 dx'. \]

(3.95)

By (3.81)-(3.83), we obtain Theorem 3.9.

\[ \square \]

4. The operator \( \sqrt{-1} \mathcal{C}(V)(d + \delta) \) for 3-dimensional Spin Manifolds with Boundary

For an odd-dimensional manifolds with boundary, as in section 5.6 and 7 in [11], we have the formula

\[ \text{Res}[(\pi^+ D_{V}^{-1})^2] = \int_{\partial M} \Psi. \]

(4.1)

When \( n = 3 \), then in (3.8), \( r - k - |\alpha| + l - j - 1 = -3 \), \( r, l \leq -1 \), so we get \( r = l = -1, k = |\alpha| = j = 0 \), then

\[ \Psi = \int_{|\xi'|=1}^{+\infty} \text{trace}_{S(TM)}[\sigma_{+1}^{\perp}(D_{V}^{-1})(x', 0, \xi', \xi_n)] \times \partial_{\xi_n} \sigma_{-1}(D_{V}^{-1})(x', 0, \xi', \xi_n)] d\xi_3 \sigma(\xi') dx'. \]

(4.2)

By Lemma 3.6 we have

\[ \sigma_{+1}^{\perp}(D_{V}^{-1})|_{\xi' = 1} = -\frac{\mathcal{C}(V)[c(\xi') + ic(dx_n)]}{2i(\xi_n - i)}; \]

\[ \partial_{\xi_n} \sigma_{-1}^{\perp}(D_{V}^{-1})|_{\xi' = 1} = -\frac{\mathcal{C}(V)[c(dx_n)]}{1 + \xi_n^2} + \frac{2\xi_n \mathcal{C}(V)[c(\xi)]}{(1 + \xi_n^2)^2}. \]

(4.3)

For \( n = 3 \), we take the coordinates in Section 2. Locally \( S(TM)|_{\tilde{U}} \cong \tilde{U} \times \Lambda_{\text{even}}^{\text{even}}(2) \). Let \( \{ \tilde{f}_1, \tilde{f}_2 \} \) be an orthonormal basis of \( \Lambda_{\text{even}}^{\text{even}}(2) \) and we will compute the trace under this basis.

By \( \text{tr}[c(\xi')c(dx_3)] = 0 \); \( \text{tr}[c(dx_3)^2] = -8 \); \( \text{tr}[c(\xi')^2]|_{\xi' = 1} = -8 \), we get

\[ \text{tr}[\mathcal{C}(V)[c(\xi')\mathcal{C}(V)c(dx_3)]] = 0; \quad \text{tr}[\mathcal{C}(V)[c(dx_3)\mathcal{C}(V)c(dx_3)]] = 8; \]

\[ \text{tr}[\mathcal{C}(V)[c(\xi')\mathcal{C}(V)c(\xi)]] = 8; \quad \text{tr}[\mathcal{C}(V)[c(dx_3)\mathcal{C}(V)c(\xi)]] = 8\xi_n. \]

(4.4)

Then, by (4.3) and (4.4), we have

\[ \text{trace}_{S(TM)}[\sigma_{+1}^{\perp}(D_{V}^{-1}) \times \partial_{\xi_n} \sigma_{-1}(D_{V}^{-1})]|_{\xi' = 1} = -\frac{4}{(\xi_n + i)^2(\xi_n - i)}. \]

(4.5)

By (4.2) and (4.3) and the Cauchy integral formula, we get

\[ \Phi = 2i\pi \Omega_3 \text{vol}_{\partial M} = 4i\pi^2 \text{vol}_{\partial M}. \]

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where \( \text{vol}_{\partial M} \) denotes the canonical volume form of \( \partial M \).

Therefore, we get the following theorem

**Theorem 4.1.** Let \( M \) be a 3-dimensional oriented compact manifold with boundary \( \partial M \) and the metric \( g^M \) as in Section 3, the operator \( D_V = \sqrt{-1}c(V)(d + \delta) \) be on \( \tilde{M} \) (\( \tilde{M} \) is a collar neighborhood of \( M \)), then

\[
\text{Wres}[\pi^+ D_V^{-1}]^2 = 4i\pi^2 \text{vol}_{\partial M},
\]

(4.7)

where \( \text{vol}_{\partial M} \) denotes the canonical volume form of \( \partial M \).

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