SWITCHING OPERATIONS FOR HADAMARD MATRICES

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Abstract. We define several operations that switch substructures of Hadamard matrices thereby producing new, generally inequivalent, Hadamard matrices. These operations have application to the enumeration and classification of Hadamard matrices. To illustrate their power, we use them to greatly improve the lower bounds on the number of equivalence classes of Hadamard matrices in orders 32 and 36 to 3,578,006 and 18,292,717.

1. Introduction

Two matrices, $A$ and $B$, with entries in the set $\{-1, 1\}$ are Hadamard equivalent if $B$ can be obtained from $A$ by some sequence of

- row negations,
- column negations,
- row permutations, and
- column permutations.

Hadamard equivalence is so named because of its connection with Hadamard matrices, defined as square matrices with elements equal to $\pm 1$ whose rows are mutually orthogonal. The listed moves all preserve the property of being a Hadamard matrix.

In this paper, we describe some additional moves, called switching operations, that preserve the property of being a Hadamard matrix. These operations, when applied over and over again to a seed matrix, generally produce many inequivalent Hadamard matrices.

Furthermore, adjoining the new operations to the list above gives new notions of equivalence. These weaker notions of equivalence may be useful in the classification of Hadamard matrices since they partition the set of Hadamard matrices into a much smaller number of equivalence classes than does Hadamard equivalence, but at the same time provide an effective method for enumerating the elements of these newly defined equivalence classes.

Extensive calculation indicates that the number of Hadamard equivalence classes that can be constructed using the new operations is enormous. This is a big step forward since, although complete enumerations up to order 28 suggest that the number of equivalence classes grows rapidly in higher order, up till now there has been no general method for producing the vast numbers of equivalence classes that we expect to exist. The many

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known Hadamard matrix construction techniques typically apply only in scattered orders, or tend to produce Hadamard matrices with special features such as large automorphism groups, large Hadamard submatrices, or self-duality.

The most prolific method for constructing Hadamard matrices has been to use two Hadamard matrices of size \( n \), \( A \) and \( B \), to build Hadamard matrices of size \( 2n \)

\[
H = \begin{bmatrix} A & PB \\ A & -PB \end{bmatrix} \quad \text{and} \quad \tilde{H} = \begin{bmatrix} A & A \\ BP & -BP \end{bmatrix}
\]

(1.1)

where \( P \) is any permutation matrix. Both \( A \) and \( B \) can be taken from any equivalence class. In order 32, Lin, Wallis, and Lie \[25\] produced at least 66099 inequivalent matrices from the five equivalence classes in order 16. Since the resulting matrices contain Hadamard submatrices of order 16, however, they cannot be considered generic. In contrast, the new operations produce at least 3.57 million equivalence classes, most of which do not contain Hadamard submatrices of order 16.

Lam, Lam, and Tonchev have exercised great ingenuity in deriving lower bounds on the number of Hadamard matrices of size \( 2n \) of the form (1.1), and have produced spectacularly large bounds in orders 40 and higher \[22, 23\]. If the lessons learned from order 32 are any guide, the true numbers of Hadamard equivalence classes in these orders are far greater still.

Our results are even more striking in orders congruent to 4 (mod 8) since the construction (1.1) does not apply. The previously known equivalence classes in order 36 numbered in the hundreds. By the new methods, at least 18.29 million classes can be produced.

The seed matrices used to obtain all these new equivalence classes were derived from the Hadamard matrix literature up to 2005. After this work was substantially complete, Bouyukliev, Fack, and Winne announced the classifications of 2-(31, 15, 7) and 2-(35, 17, 8) designs with automorphisms of odd prime order. From these designs, they found tens of thousands of new Hadamard equivalence classes in orders 32 and 36 \[1, 5\]. Most of these matrices have not yet been analyzed by our method. Compared with the analysis of the dozens of previously known Hadamard equivalence classes in order 32 (excluding the matrices from construction (1.1)), and the hundreds of previously known \( H \)-classes in order 36, analyzing these new matrices is a major undertaking, and will require considerable optimization of our methods. Therefore, with one important exception, we have not used the matrices of Bouyukliev, Fack, and Winne in our enumeration, although we make a few remarks on our preliminary analysis in the next paragraph. The exception is a matrix of order 36 in Smith class 16 (defined in Section 4.3), no previous example of which appears to have been known. This was used as a seed matrix to produce a new family containing at least five million Hadamard equivalence classes.

In orders 4, 8, 12, 16, 20, 24, 28 the numbers of Hadamard equivalence classes are known to be 1, 1, 1, 5, 3, 60, 487 \[11, 12, 13, 18, 19\]. We define a weaker notion of equivalence, which we call \( Q \)-equivalence, by adjoining the new operations to the operations that define Hadamard equivalence. The numbers of \( Q \)-equivalence classes are 1, 1, 1, 1, 1, 2, 2. In order 32, we find that the 3.57 million known Hadamard equivalence classes
are grouped into 11 Q-equivalence classes, and that in order 36, the 18.29 million known equivalence classes are grouped into 21 Q-equivalence classes. As mentioned above, these numbers do not include Hadamard equivalence classes or Q-equivalence classes derived from the recently discovered matrices of Bouyukliev, Fack, and Winne. An analysis of their matrices should provide a good test of the ideas of this paper regarding using Q-equivalence in classifying Hadamard matrices. Preliminary analysis of a sampling of the new matrices does not turn up any new large Q-classes, but does indicate the presence of a large number of new small Q-classes (perhaps in the hundreds or more). We intend to make a complete enumeration of these, and a full analysis of all the new matrices. The results will be presented in a follow-up to the present paper.

2. Overview of switching

Suppose that an $n \times n$ Hadamard matrix can be put in the form

$$
\begin{bmatrix}
1 & \cdots & 1 & - \cdots & - & \cdots & - & - & - & 1 & \cdots & 1 \\
1 & \cdots & 1 & - \cdots & - & 1 & \cdots & 1 & - & \cdots & - \\
1 & \cdots & 1 & 1 & \cdots & 1 & - & \cdots & - & - & \cdots & - \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
a_5 & b_5 & c_5 & d_5 \\
\vdots & \vdots & \vdots & \vdots \\
a_n & b_n & c_n & d_n
\end{bmatrix}
$$

(2.1)

where $a_i, b_i, c_i,$ and $d_i$ are $\{-1, 1\}$-vectors of length $n/4$. The columns of the matrix have been grouped into four sets of $n/4$ columns each. A new, generally inequivalent, Hadamard matrix can be obtained by negating the $4 \times \frac{n}{4}$ block of 1s in the upper left corner (shown in boldface). We call this operation switching a closed quadruple.

Suppose instead that we can put the matrix in the form of Figure 1 where the $A_{ij}$ are square matrices of size $(n - 4)/4$. A new, often inequivalent, matrix can be obtained by negating the all 1 block of size $4 \times \frac{n-4}{4}$ contained in the first four rows, and the all 1 block of size $\frac{n-4}{4} \times 4$ contained in the first four columns (both shown in boldface). We call this operation switching a Hall set.

Justification for these claims and further elaboration are given in the subsequent sections.

3. Closed quadruples and Hall sets

3.1. 3-normalization. Let $H$ be a Hadamard matrix of size $n$. Denote its rows by $h_i$ and its elements by $h_{ij}$. Define the Hadamard product of two vectors to be

$$(a_1, \ldots, a_n) \circ (b_1, \ldots, b_n) := (a_1b_1, \ldots, a_nb_n).$$

Let $j_k$ be the all 1 vector of length $k$. 
Definition. A Hadamard matrix of size $n$ is $3$-normalized on rows $(i,j,k)$ if, in every column $\ell$, the set $\{h_{i\ell}, h_{j\ell}, h_{k\ell}\}$ contains an even number of $-1$s, or equivalently if $h_i \circ h_j \circ h_k = j_n$.

3-normalization is a normalization of the columns. A 3-normalized matrix remains 3-normalized if any two of the rows $i$, $j$, $k$ or of any single row other than $i$, $j$, $k$ is negated. Note that 3-normalization was introduced in [31]. The definition given here is slightly weaker in that it makes no stipulation that the row sums be positive, and does not impose any particular ordering on the columns. In the next paragraph we restate some needed results from [31].

The field structure $(C_1, C_2, C_3, C_4)$ of a 3-normalized Hadamard matrix of size $n$ is the partition of the set of columns $c$ into four classes, $C_i$, accordingly as $(h_{jc}, h_{kc}, h_{tc}) = (1,1,1), (-1,-1,1), (-1,1,-1)$, or $(1,-1,-1)$. The four classes are called fields and are all of length $n/4$. In a row $r \notin \{j,k,\ell\}$ the sum of the elements in a field is the same for each of the four fields in the row. This follows from orthogonality of row $r$ with rows $j$, $k$, $\ell$. Since the sum of the entries in a field is even if $n/4$ is even, and odd if $n/4$ is odd, the row sum of row $r \notin \{j,k,\ell\}$ must be congruent to $n$ (mod 8).

A quadruple of rows, $(i,j,k,\ell)$ of a Hadamard matrix $H$ of size $n$ is said to be of type $r$, $0 \leq r \leq n/8$, if exactly $4r$ of the entries in $h_i \circ h_j \circ h_k \circ h_\ell$ equal $-1$ or exactly $4r$ entries equal $+1$. This notion was introduced by Kimura [19].

\begin{figure}
\begin{align*}
\begin{bmatrix}
1 & - & - & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & - & \cdots & - \\
- & 1 & - & - & 1 & \cdots & 1 & - & \cdots & - & \cdots & - & \cdots & - & - & \cdots & - \\
- & - & 1 & - & 1 & \cdots & 1 & - & \cdots & - & 1 & \cdots & 1 & 1 & \cdots & 1 \\
- & - & - & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & - & \cdots & - & 1 & \cdots & 1 \\
1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & A_{11} & A_{12} & A_{13} & A_{14} \\
1 & 1 & 1 & 1 & - & 1 & 1 & - \\
\vdots & \vdots & \vdots & A_{21} & A_{22} & A_{23} & A_{24} \\
1 & 1 & 1 & - \\
\vdots & \vdots & \vdots & A_{31} & A_{32} & A_{33} & A_{34} \\
- & 1 & - & 1 \\
1 & 1 & - & - \\
\vdots & \vdots & \vdots & A_{41} & A_{42} & A_{43} & A_{44} \\
1 & 1 & - & - \\
\end{bmatrix}
\end{align*}
\end{figure}
**Definition.** A quadruple of rows, \((i, j, k, \ell)\) of a Hadamard matrix \(H\) of size \(n\), is closed if \(h_i \circ h_j \circ h_k \circ h_\ell = \pm j_n\).

A closed quadruple is a quadruple of type 0. Thus if \(H\) is 3-normalized on three rows of a closed quadruple, then the fourth will consist entirely of 1s or entirely of -1s. The field structure is independent of which three rows of the closed quadruple are chosen.

Quadruples of type 1 will also play an important role in what follows. They were used extensively by Hall in the classification of Hadamard matrices of order 20 [12] and by Kimura in the classification for order 28 [20, 19]. If \(H\) is 3-normalized on three rows of a type-1 quadruple, then the fourth row will contain one odd-sign entry in each of the fields induced by the 3-normalization. Kimura and Ohmori referred to such quadruples as Hall sets [21].

**Proposition 3.1.** If a Hadamard matrix of size \(n\) has a closed quadruple, then \(n = 4\) or \(n \equiv 0 \pmod{8}\).

**Proof.** Let \((i, j, k, \ell)\) be the closed quadruple. 3-normalize the matrix on rows \(i, j, k\) so that \(h_\ell = \pm j_n\). Orthogonality implies that all rows except for \(h_\ell\) have row sum 0. All row sums of rows other than \(i, j, k\) must be congruent to \(n\) \(\pmod{8}\). If \(n > 4\) this can only happen when \(n \equiv 0 \pmod{8}\). \(\square\)

### 3.2. Obtaining new Hadamard matrices by switching closed quadruples.

**Definition.** Let \(H\) be a Hadamard matrix of size \(n\) which has a closed quadruple, \(Q\). Let \((C_1, C_2, C_3, C_4)\) be the partition of columns induced by 3-normalization on \(Q\). Switching the closed quadruple \(Q\) means negating all the elements \(h_{rc}\), where \(r \in Q\) and \(c \in C_i\) for some \(i \in \{1, 2, 3, 4\}\).

**Proposition 3.2.** The matrix produced by switching a closed quadruple \(Q\) in a Hadamard matrix \(H\) is a Hadamard matrix.

**Proof.** Any matrix containing a closed quadruple is Hadamard equivalent to one of the form (2.1). It is evident that switching preserves orthogonality of the columns in that matrix. Since column orthogonality is preserved under the operations needed to put \(H\) in the form (2.1), the conclusion holds generally. \(\square\)

It appears that when \(n > 8\), switching always produces a Hadamard matrix that is inequivalent to the original Hadamard matrix.

Note that the equivalence class of the Hadamard matrix produced by switching \(Q\) is independent of which of the four fields \(C_i\) we choose to negate. To see this, note that negating the closed quadruple elements in \(C_2\) is equivalent to first negating the closed quadruple elements in \(C_1\), then negating all four rows of the closed quadruple, and finally performing a certain permutation of the rows of \(Q\). The same holds for \(C_3\) and \(C_4\).
3.3. More general row switching operations. It was observed by Denniston [7] in connection with symmetric (25, 9, 3) designs that, starting from a design, a new inequivalent design can be obtained by switching a substructure known as an oval. Denniston’s switching operation can be thought of as an operation that permutes certain elements of the incidence matrix of the design. In fact, if zeroes are replaced with $-1$s in the incidence matrix of a (25, 9, 3) design, ovals satisfy our definition of a closed quadruple, and our switching operation is equivalent to Denniston’s.

We can formulate more general switching operations acting on more general structures, which we will refer to generically as “designs.” Consider a set, $\mathcal{M}$, of matrices of a fixed size which represent the designs in question. If $R \in \mathcal{M}$, we suppose that the elements of $R$ are taken from some set $S$, that the rows satisfy some set of properties $P$, and furthermore, that $R$ satisfies $R^T R = M$ where $M$ is some fixed matrix. For example, if $\mathcal{M}$ represents (25, 9, 3) designs, then $S = \{0, 1\}$, the set $P$ contains the property that every row of $R \in \mathcal{M}$ has exactly nine $1$s, and $M = 6I + 3J$ where $J$ is the $25 \times 25$ all $1$ matrix. Many other types of matrices and designs, including certain D-optimal designs can also be defined within this framework.

Let $R \in \mathcal{M}$ and partition the incidence matrix into two submatrices, $A$ and $X$, 

$$R = \begin{bmatrix} A \\ X \end{bmatrix}.$$ 

Now suppose that $B$ is a matrix of the same dimensions as $A$, with elements taken from the same set $S$, satisfying the same properties $P$, and that $B^T B = A^T A$. Then the matrix obtained from $R$ by replacing $A$ with $B$ is also a matrix of the original type.

Suppose for example, that $R$ is an $n \times n$ Hadamard matrix with an $m \times n$ submatrix $A$ whose columns are all identical to columns of a particular $m \times m$ Hadamard matrix $H_m$ or to negations of such columns. Let $H'_m$ be another $m \times m$ Hadamard matrix. Denote column $j$ of $H_m$ by $v_j$ and column $j$ of $H'_m$ by $v'_j$. Define $[a, b]$ to be the set of integers $i$ satisfying $a \leq i \leq b$. For $j = 1, \ldots, n$, let column $j$ of $A$ be $\sigma(j)v_{a(j)}$ where $\sigma : [1, n] \to \{-1, 1\}$ and $a : [1, n] \to [1, m]$. Let $B$ be the matrix with the same dimensions as $A$ whose columns are $\sigma(j)v'_{a(j)}$ for $j = 1, \ldots, n$. Then $B$ will satisfy $B^T B = A^T A$, and so we may use it to obtain a new Hadamard matrix of order $n$.

Note that if $m = 1$ and we let $H_1 = [1]$ and $H'_1 = [-1]$ then the above operation amounts to negation of a row. Likewise, if $m = 2$ and $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ while $H'_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ then the above operation amounts to swapping two rows.

Switching closed quadruples is an instance of the $m = 4$ case. Let $A$ be a $4 \times n$ matrix whose columns, or their negations, are columns of $H_4$, a $4 \times 4$ Hadamard matrix. Orthogonality of the rows of $A$ implies that $A$ is a closed quadruple. Now negating one
column of $H_4$ and using the resulting matrix to construct $B$, has the effect of switching the closed quadruple formed by the rows of $A$. Thus, in some sense switching closed quadruples is a natural extension of the operations of row negation and row permutation.

3.4. **Closed quadruples and Hadamard submatrices.** There is an additional sense in which switching closed quadruples is a natural extension of the operation of row permutation. Consider the matrix $H$ of size $2n$ defined in equation (1.1). One may negate or permute the columns of $A$ or $B$ without changing the equivalence class of $H$. One may also negate a row of $PB$ (or of $A$) without changing the equivalence class of $H$. The reason is that negating row $j$ of $PB$ amounts to swapping rows $j$ and $n+j$ of $H$.

On the other hand, changing the permutation $P$, for example by performing the additional row swap $(i, j)$, usually does change the equivalence class of $H$. The additional swap will affect four rows of $H$, namely $i$, $j$, $i+n$, $j+n$. These four rows form a closed quadruple. One of the four fields of this quadruple is the set of columns of $H$ in which rows $i$ and $j$ of $PB$ differ. We make the switch that negates the entries in rows $i$, $j$, $i+n$, and $j+n$ that lie within this field. The result is identical to the result of swapping rows $i$ and $j$ of $PB$. Therefore, in this context, switching a closed quadruple amounts to swapping a pair of rows in one of the two matrices from which $H$ was constructed.

3.5. **Properties of Hall sets.** Hall sets play the role for matrices of order $n \equiv 4 \pmod{8}$ that closed quadruples play for matrices of order $n \equiv 0 \pmod{8}$.

Hall sets can be found both in Hadamard matrices of order $n \equiv 0 \pmod{8}$ and in those of order $n \equiv 4 \pmod{8}$. Four columns are singled out in the definition of a Hall set, namely the columns whose sign in the Hadamard product differs from the sign of all the other columns. When $n \equiv 4 \pmod{8}$ these form a Hall set in the columns, as shown by Kimura and Ohmori [21]. For convenience of the reader, we reprove this here. We include the corresponding result for $n \equiv 0 \pmod{8}$ for good measure. Define the **Hall columns** to be the four distinguished columns. There is one Hall column in each field.

**Proposition 3.3.** Let $H$ be a Hadamard matrix of order $n$. If $n \equiv 0 \pmod{8}$ then the Hall columns form a closed quadruple. If $n \equiv 4 \pmod{8}$ then the Hall columns form a Hall set.

**Proof.** We assume without loss of generality that $H$ is 3-normalized on three rows of the Hall set. Consider a row not contained in the Hall set. Let $x_i$ denote the element of that row in the Hall column of field $i$. Let $a_i$ denote the sum of the remaining elements of field $i$. Then orthogonality with the Hall set rows implies

\[
x_1 + a_1 = x_2 + a_2 = x_3 + a_3 = x_4 + a_4
\]

\[
a_1 + a_2 + a_3 + a_4 = x_1 + x_2 + x_3 + x_4,
\]

which implies that the row sum, which must be congruent to $n \pmod{8}$, equals $2(x_1 + x_2 + x_3 + x_4)$. Hence the product $x_1 x_2 x_3 x_4$ is positive for $n \equiv 0 \pmod{8}$ and negative for $n \equiv 4 \pmod{8}$. In each row of the Hall set, the product of the four elements in Hall columns is always positive, so the result follows. \qed
Remark. When \( n \equiv 0 \pmod{8} \) the existence of a Hall set implies the existence of a closed quadruple in the columns, but the converse is not true. The existence of a closed quadruple does not imply the existence of a corresponding Hall set. \(^1\)

Henceforth we will consider the \( n \equiv 4 \pmod{8} \) case, and when we speak of a Hall set, we will mean both the four rows of the set and the four corresponding Hall columns.

By permuting the Hall rows and columns to the top- and leftmost positions and normalizing appropriately we obtain the form

\[
H = \begin{bmatrix}
H_4 & F_1 & F_2 & F_3 & F_4 \\
1 & A_{11} & A_{12} & A_{13} & A_{14} \\
G_1 & A_{21} & A_{22} & A_{23} & A_{24} \\
G_2 & A_{31} & A_{32} & A_{33} & A_{34} \\
G_3 & A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
\]  

(3.3)

where

\[
H_4 = \begin{bmatrix}
1 & - & - & - \\
- & 1 & - & - \\
- & - & 1 & - \\
- & - & - & 1
\end{bmatrix}, \quad F_1 = \begin{bmatrix}
1 & \ldots & 1 \\
1 & \ldots & 1 \\
1 & \ldots & 1 \\
1 & \ldots & 1
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
1 & \ldots & 1 \\
- & \ldots & - \\
- & \ldots & - \\
1 & \ldots & 1
\end{bmatrix}, \quad F_3 = \begin{bmatrix}
1 & \ldots & 1 \\
- & \ldots & - \\
1 & \ldots & 1 \\
- & \ldots & -
\end{bmatrix}, \quad F_4 = \begin{bmatrix}
- & \ldots & - \\
- & \ldots & - \\
1 & \ldots & 1 \\
1 & \ldots & 1
\end{bmatrix}, \quad (3.4)
\]

\( G_1 = F_1^T, G_j = -F_j^T \) for \( j \in \{2, 3, 4\} \), and \( A_{ij} \) are submatrices whose row and column sums equal 2 when \( i = j \) and 0 when \( i \neq j \).

Definition. By switching a Hall set in the matrix \( H \) defined in eqn. (3.3) we mean the operation of replacing \( F_i \) by its negation and \( G_i \) by its negation for one of the choices \( i = 1, 2, 3, 4 \).

The four possible negations of the definition produce equivalent matrices. The proof of this is similar to the proof of the analogous property of closed quadruples given in the discussion following Proposition (3.2). Switching is well defined even when the Hall rows and columns do not appear in positions 1–4 or when the normalization is different from the one in (3.3). We need only apply a signed permutation to put the matrix into the form (3.3), switch as in the definition, and then apply the inverse signed permutation.

\(^1\)Sylvester matrices exhibit this in extreme form. One can show by induction on \( k \) that the Sylvester matrix of size \( 2^k \) has \( \frac{1}{4} \binom{2^k}{k} \) closed quadruples, and that any of its other row quadruples is of type \( 2^{k-3} \) (which means its Hadamard product has as many entries +1 as −1). Therefore, if \( k \geq 4 \), the Sylvester matrix has many closed quadruples, but no Hall sets. Since Sylvester matrices are self dual, the same is true of column quadruples.
Proposition 3.4. The matrix produced by switching a Hall set in a Hadamard matrix is a Hadamard matrix.

Proof. We will assume the form (3.3) since the conclusion is unaffected by the permutations and negations needed to convert the matrix to that form. When $i \neq j$, the rows of $F_j$ are orthogonal to the rows of $A_{ij}$ as the latter have row sum 0. Therefore, negating $F_j$ does not alter the orthogonality of rows 1–4 of $H$ with the rows of $H$ contained in the block $[G_i A_{i1} A_{i2} A_{i3} A_{i4}]$. Row $k$ ($k = 1, 2, 3, 4$) of $F_j$ has inner product $\pm 2$ with any of the rows of $A_{jj}$ while row $k$ of $H_4$ has inner product $\mp 2$ with any of the rows of $G_j$. Negating both $F_j$ and $G_j$ produces sign changes in these inner products that produce opposite contributions to any of the inner products of rows 1–4 of $H$ with the rows of $H$ contained in the block $[G_j A_{j1} A_{j2} A_{j3} A_{j4}]$. □

Examples are known where switching a Hall set in a Hadamard matrix $H$ produces a matrix equivalent to $H$. In general, however, one obtains an inequivalent matrix.

4. Invariants

First we prove that the number of closed quadruples in a Hadamard matrix of size $16k + 8$ is invariant under switching closed quadruples. Second, we show that the binary, doubly even, self-dual code associated to the transpose of a Hadamard matrix of size $16k + 8$ is unchanged by switching a closed quadruple of that matrix. Finally we show that the integer equivalence class is preserved under switching of Hall sets.

4.1. A closed quadruple switching invariant for $n \equiv 8 \pmod{16}$. We will need to understand the ways that closed row quadruples may overlap within a Hadamard matrix.

Proposition 4.1. Suppose $(i, j, k, \ell)$ and $(i', j', k', \ell')$ are distinct closed quadruples with nonempty intersection. Then the number of rows common to the two quadruples is 2 if $n \equiv 8 \pmod{16}$ and 1 or 2 if $n \equiv 0 \pmod{16}$.

Proof. The number of common rows cannot be 3 since the fourth row of a closed quadruple is determined, up to sign, by the other three, and the two quadruples are assumed distinct. Therefore the number of common rows must be either 1 or 2.

We will show that if the number of common rows is 1, then $n \equiv 0 \pmod{16}$.

Assume the number of common rows to be 1 and let $n = 8r$. Take the two quadruples to be $(1, 2, 3, 4)$ and $(1, 5, 6, 7)$, and 3-normalize the matrix on rows 2, 3, 4. Normalize row 1 to have positive entries. By suitable column permutations, the structure of the first five rows can be brought to the form:

1. $1_r \ 1_r \ 1_r \ 1_r \ 1_r \ 1_r \ 1_r \ 1_r$
2. $1_r \ 1_r \ -1_r \ -1_r \ -1_r \ -1_r \ 1_r \ 1_r$
3. $1_r \ 1_r \ -1_r \ -1_r \ 1_r \ 1_r \ -1_r \ -1_r$
4. $1_r \ 1_r \ 1_r \ 1_r \ -1_r \ -1_r \ -1_r \ -1_r$
5. $1_r \ -1_r \ 1_r \ -1_r \ 1_r \ -1_r \ 1_r \ -1_r$
The form of row 5 is a consequence of the fact that the sum of elements in each of the four fields must be zero. Since \((1, 5, 6, 7)\) is closed, the Hadamard product of rows 6 and 7 equals either row 5 or its negation. By normalizing row 7 appropriately we may assume the former. Consider the two subfields that compose the first field in the above structure. They will be further subdivided as

- 5. \(1_a \quad 1_{r-a} \quad -1_b \quad -1_{r-b} \quad \ldots\)
- 6. \(1_a \quad -1_{r-a} \quad 1_b \quad -1_{r-b} \quad \ldots\)
- 7. \(1_a \quad -1_{r-a} \quad -1_b \quad 1_{r-b} \quad \ldots\)

The subfields composing the remaining three fields will be subdivided similarly. Because there are \(r\) 1s per field in rows 6 and 7, just as in row 5, we have the constraints \(a + b = r\) and \(a + (r - b) = r\). Therefore \(a = b = r - a = r - b = r/2\) and hence \(r\) is even. Consequently \(n \equiv 0 \pmod{16}\). □

Note that all of the degrees of overlap between closed quadruples allowed by the Proposition occur in practice.

**Proposition 4.2.** Let \(n \equiv 8 \pmod{16}\). Let \(H\) be a Hadamard matrix of size \(n\) which has a closed row quadruple \(Q\). Switching \(Q\) does not change the number of closed row quadruples in \(H\).

**Proof.** In the matrix obtained from \(H\) by switching \(Q\), the rows of \(Q\) still form a closed quadruple. Also, any quadruple, whether closed or not, that doesn’t involve any rows of \(Q\) is unaffected by switching. The only way the number of closed quadruples could change is if a closed quadruple were created or destroyed by switching \(Q\). Such a closed quadruple would have to overlap \(Q\) (either before or after switching) and would therefore share exactly two of \(Q\)'s rows. However, the Hadamard product of any pair of rows in \(Q\) is not altered by negation of any of the fields of \(Q\). Hence the Hadamard product of the four rows of a putative overlapping quadruple would be unchanged by such a negation. Therefore, any closed quadruple overlapping \(Q\) in two rows remains closed after switching \(Q\). Likewise, any quadruple overlapping \(Q\) in two rows which is not closed initially, will not be closed after switching \(Q\). □

It is worth pointing out that switching a closed *column* quadruple does change the number of closed row quadruples in general. Furthermore, switching closed row quadruples generally does change the number of closed row quadruples when \(n \equiv 0 \pmod{16}\). For example, when \(n = 16\), the five equivalence classes of Hadamard matrices have 140, 76, 44, 28, and 28 closed row quadruples. Each of these five classes can be obtained starting from any of the others and performing a series of switches of closed row quadruples.

**4.2. Invariant codes.** Codes can be associated with Hadamard matrices, and are useful in their classification. For our purposes, codes can be thought of as collections of vectors over some finite field. The vectors in a code are called *codewords*, and the *weight* of a codeword is the number of its entries that are non-zero. The *support* of a codeword is the set of positions in which it has a non-zero entry.
One way to associate a linear code with a Hadamard matrix of size $n$ is to normalize the columns of the matrix so that all entries in the first row equal 1, then to change all $-1$ entries to 0, and finally to take the linear span of the rows of the resulting matrix over some finite field $\mathbb{F}_p$ where $p$ is a prime. One could equally well normalize on a row other than the first and the resulting code would be the same. The dimension of such a linear code is its dimension as a subspace of $\mathbb{F}_p^n$. If $C$ is such a linear code, then its dual code, $C^\perp$ is the subspace of $\mathbb{F}_p^n$ consisting of all vectors orthogonal to all codewords in $C$. Basic linear algebra implies that the dimensions of a code and its dual satisfy $\dim(C) + \dim(C^\perp) = n$. If $C \subseteq C^\perp$ then $C$ is said to be self-orthogonal. If $C = C^\perp$ then $C$ is said to be self-dual.

We will only consider binary codes ($p = 2$) in this paper, but it should be noted that codes over $\mathbb{F}_p$, $p$ an odd prime, are closely connected with integer equivalence, which is discussed in the next section. The 2-rank of a Hadamard matrix is the same as the dimension of its associated binary code. Two binary codes are isomorphic if one can be converted to the other by a permutation of coordinate positions. The following result is proved (in greater generality) in many places. (For example, see [21], Section 2.3.)

**Theorem 4.3.** Let $H$ be a Hadamard matrix of size $n$. Let $C$ be a binary code associated to $H$ as described above. Then,

1. If $n \equiv 4 \pmod{8}$ then $C = \{1_n\}^\perp$ which implies $\dim(C) = n - 1$;
2. If $n \equiv 0 \pmod{8}$ then $C$ is self-orthogonal which implies $\dim(C) \leq n/2$;
3. If $n \equiv 8 \pmod{16}$ then $C$ is self-dual which implies $\dim(C) = n/2$.

Since all Hadamard matrices of a size congruent to 4 (mod 8) have the same binary code, the binary code does not help with classification (although codes over other fields may). For the present, we focus on binary codes associated with matrices of size $n \equiv 0 \pmod{8}$. It is not hard to show that such codes are doubly-even, that is, all of their code words have weight divisible by 4. For an illustration that various 2-ranks allowed by Theorem 4.3 do occur in practice, we consider some results discussed by Assmus and Key in [3, 2]. They note that the five non-equivalent Hadamard matrices of size 16 have binary codes of dimensions 5, 6, 7, 8, and 8. Only the last two are self-dual, and they turn out not to be isomorphic. Contrast this with size 24 where the 60 non-equivalent Hadamard matrices must all have self-dual codes of dimension 12. Assmus and Key proved that these 60 classes of matrices are associated with six different doubly-even, self-dual, binary codes.

Jennifer Key pointed out [17] that when $H$ is a Hadamard matrix of size 24, the number of closed quadruples coincides with the number of code words of weight 4 in the binary code associated with the columns of $H$. (We might also call this the code associated with $H^T$.) We elaborate a bit on her observation, which reflects a general phenomenon for matrices of size $n \equiv 8 \pmod{16}$.

**Proposition 4.4.** Let $H$ be a Hadamard matrix of size $n \equiv 0 \pmod{8}$. Let $C$ be the linear binary code constructed from the columns of $H$. That is, $C$ is the linear span over $\mathbb{F}_2$ of the columns of a matrix $A$ formed by normalizing $H$ so that one of its columns
consists entirely of 1s and then changing $-1$ to 0s. Let \( \{i, j, k, \ell\} \) be the support of a weight 4 codeword in \( C \). Then rows \( i, j, k, \) and \( \ell \) of \( H \) form a closed quadruple.

**Proof.** Since \( C \) is self-orthogonal, the weight 4 codeword with support \( \{i, j, k, \ell\} \) is orthogonal to every column of \( A \). This means that every column of \( A \) has an even number of 1s in positions \( i, j, k, \) and \( \ell \), which implies the result. \( \square \)

This result has a partial converse with self-duality of \( C \) being the needed additional assumption.

**Proposition 4.5.** Let \( H, A, \) and \( C \) be defined as in Proposition 4.4 and suppose in addition that \( C \) is self-dual. Let \( i, j, k, \) and \( \ell \) label the rows of a closed quadruple in \( H \). Then \( \{i, j, k, \ell\} \) is the support of a weight 4 codeword in \( C \).

**Proof.** Since one column of \( A \) consists entirely of 1s, and since \( i, j, k, \) and \( \ell \) label a closed quadruple, every column of \( A \) has an even number of 1s among the positions \( i, j, k, \) and \( \ell \). Let \( c \) be the vector in \( \mathbb{F}_2^n \) with support \( \{i, j, k, \ell\} \). Then \( c \) is orthogonal to every column of \( A \) and therefore to every codeword in \( C \). Hence \( c \in C^\perp \). Since \( C \) is self-dual, we also have \( c \in C \). \( \square \)

We have established a one-to-one correspondence between the closed quadruples of a Hadamard matrix \( H \) of size \( n \equiv 0 \) (mod 8) and weight 4 code words in the binary code associated to the columns of \( H \), provided that that code is self-dual. This correspondence therefore holds for all Hadamard matrices of size congruent to 8 (mod 16).

We finally investigate the effect of switching a closed quadruple of \( H \) on the binary code associated to the columns of \( H \). In this connection, we note that the closed quadruple switching operation was defined and used in the coding theory context by Phelps, Rifà, and Villanueva [32]. They were concerned with Hadamard matrices of size \( 2^t \) whose codes can range in dimension from \( t + 1 \) to \( 2^{t-1} \). Starting with a code of minimal dimension, corresponding to the Sylvester matrix, they produced codes, and the corresponding matrices, of the next two higher dimensions by switching. For further details, see Lemmas 4.2 and 4.3 of [32]. Note that closed quadruples correspond to subcodes of dimension three.

Our focus in this paper will be on codes at the opposite end of the range of possible dimensions, that is, on the self-dual codes. We have the following inclusion of codes:

**Proposition 4.6.** Let \( H, A, \) and \( C \) be defined as in Proposition 4.4 and suppose in addition that \( C \) is self-dual. Let \( H' \) be a Hadamard matrix obtained from \( H \) by switching a closed quadruple, and let \( C' \) be the code associated to the columns of \( H' \). Then \( C' \subseteq C \). Furthermore, if \( C' \neq C \), then \( C \) is spanned by \( C' \) and a particular weight 4 vector.

**Proof.** We may assume that \( H \) has been normalized so that all entries in its first column equal 1. The matrix \( A \) is then obtained from \( H \) simply by replacing \(-1\)s with 0s. The code \( C \) is the span of the columns of \( A \).

Let \( \{i, j, k, \ell\} \) be a closed quadruple of \( H \). Proposition 4.5 asserts that \( C \) contains a codeword \( c \) with support \( \{i, j, k, \ell\} \). Let \( \{C_1, C_2, C_3, C_4\} \) be the partition into fields of the
set of columns of $H$ induced by the closed quadruple $(i, j, k, \ell)$. Switching means negating all matrix elements in rows $i, j, k, \ell$ and in the columns of one of the $C_m$, $m = 1, 2, 3, 4$. Should column 1 be one of the affected columns, the normalization of the resulting matrix, $H'$, will no longer be such that column 1 contains 1s only. To restore the normalization, we simply negate rows $i, j, k, \ell$. The net result will be that all columns but those of $C_m$ are affected by the switching.

At any rate, the matrix $A'$, obtained from $H'$ by changing −1s to 0s, will differ from $A$ only in that the elements in rows $i, j, k, \ell$ and in a certain subset of the columns will have been changed to their complements ($0 \rightarrow 1, 1 \rightarrow 0$). This change can be effected by adding the vector $c$ to the appropriate columns of $A$. Therefore, $C'$, which is the span of the columns of $A'$, is spanned by a set of linear combinations of codewords in $C$. Hence $C' \subseteq C$. Finally, $C$ is clearly the span of $C' \cup \{c\}$. □

We note that the code $C'$ obtained in the above proof depends on which of the four fields, $C_m$, was used in the switching. Nevertheless, the isomorphism class of the code will be independent of this choice.

As an illustration of the use of Proposition 4.6 consider the codes associated with the five equivalence classes of $16 \times 16$ Hadamard matrices. Matrices in either of the two classes associated with self-dual codes have 28 closed quadruples. Switching any of these 28 quadruples produces a matrix in the class corresponding to the code of dimension 7. We therefore conclude that the code of dimension 7 is a subspace of both of the codes of dimension 8, and that each of the latter can be obtained by augmenting the code of dimension 7 with a weight 4 vector whose support corresponds to a suitable closed quadruple.

A corollary of Proposition 4.6 is immediate.

**Corollary 4.7.** Let $H'$ be obtained from a Hadamard matrix $H$ by switching a closed quadruple. If the linear binary codes associated to the columns of $H$ and $H'$ are both self-dual, then they are equal. In particular, if $H$ is of size $n \equiv 8 \pmod{16}$, then the linear binary codes associated to the columns of $H$ and $H'$ are equal.

Corollary 4.7 will be important when we discuss classification of Hadamard matrices of size 24. Since in the setting of Corollary 4.7 there is a one-to-one correspondence between closed quadruples and weight 4 code words, the corollary also provides an alternative proof of Proposition 4.2.

### 4.3. A Hall set switching invariant

An important notion used in the classification of Hadamard matrices is that of integer equivalence.

**Definition.** Two integer matrices $A$ and $B$ are integer equivalent if $A$ can be converted to $B$ by some sequence of the following row and column operations:

- permutation of rows (columns)
- negation of rows (columns)
- addition of an integer multiple of a row (column) to another row (column).
Associated to the integer equivalence class of a matrix $A$ of size $n$ is a set of integers $s_1, \ldots, s_n$ called invariant factors satisfying:

1. The matrix $\text{diag}(s_1, \ldots, s_n)$ is integer equivalent to $A$.
2. There exists $r$ such that $1 \leq r \leq n$ and $s_i | s_{i+1}$ for $1 \leq i \leq r-1$ and $s_{r+1} = \ldots = s_n = 0$.
3. The product $s_1 s_2 \ldots s_r$ equals the GCD of the $i \times i$ minors of $A$.

The matrix $\text{diag}(s_1, \ldots, s_n)$ is called the Smith normal form of $A$. Two integer equivalent matrices have the same Smith normal form.

A number of properties of the Smith normal form of a Hadamard matrix have been proved [38, 30]:

1. $s_1 = 1$; $s_2 = \ldots = s_{\alpha+1} = 2$, for some $\alpha \geq \lfloor \log_2 n \rfloor + 1$;
2. $s_i s_{n+1-i} = n$.

In order 36, for example, we have [6]

- $s_1 = 1$
- $s_i = 2$ for the next $\alpha$ values of $i$. ($2 \leq i \leq \alpha + 1$)
- $s_i = 6$ for the next $34 - 2\alpha$ values of $i$
- $s_i = 18$ for the next $\alpha$ values of $i$
- $s_{36} = 36$

where $6 \leq \alpha \leq 17$. The single parameter $\alpha$ determines the integer equivalence class of a Hadamard matrix $H$ in order 36, and we say that $H$ is in Smith class $\alpha$.

That the Smith class is invariant under switching Hall sets is implied by the following:

**Proposition 4.8.** If $B$ is obtained from $A$ by switching a Hall set, then $B$ is integer equivalent to $A$.

**Proof.** Switching a Hall set can be achieved by a sequence of integer row and column operations. Let the order of the matrix in (3.3) be $4k + 4$. Adding each of rows 1 through 4 to each of the $k$ rows 5 through $k + 4$, and then adding each of columns 1 through 4 to each of columns 5 through $k + 4$ has the effect of negating $F_1$ and $G_1$. \qed

5. **Equivalence relations**

Hadamard equivalence, usually simply called “equivalence,” was defined in the introduction. We will call Hadamard equivalence classes $H$-classes. By adjoining additional operations to the list of operations given there, we can define new equivalence relations. We already did this in the previous section when we defined integer equivalence, whose equivalence classes are the Smith classes. The considerations of the previous section also allow us to define equivalence with respect to the associated binary linear code, or code equivalence for short: Two Hadamard matrices are code equivalent if their binary linear codes, as defined in the statement of Proposition 4.4 are isomorphic. In this section, we define further notions of equivalence.
Definition. If \( n \equiv 0 \pmod{8} \) then two Hadamard matrices \( A \) and \( B \) of size \( n \) are \( Q \)-equivalent if \( B \) can be obtained from \( A \) by some sequence of the operations

- row or column negation
- row or column permutation
- switching a closed quadruple of rows
- switching a closed quadruple of columns.

If the last operation is disallowed, then \( A \) and \( B \) are said to be \( QR \)-equivalent; if the third operation is disallowed then \( A \) and \( B \) are said to be \( QC \)-equivalent. When \( n \equiv 4 \pmod{8} \), \( Q \)-equivalence is defined by replacing the last two operations with

- switching a Hall set.

Associated with these equivalence relations are equivalence classes, called \( Q \)-classes, \( QR \)-classes, and \( QC \)-classes.

Hadamard equivalence is stronger than \( Q \)-equivalence and therefore has a more refined equivalence class structure. In other words, there are at least as many \( H \)-classes as there are \( Q \)-classes, and each \( H \)-class is contained entirely within a particular \( Q \)-class. \( QR \)-equivalence (or \( QC \)-equivalence) is intermediate in strength between \( H \)-equivalence and \( Q \)-equivalence, and will therefore have an intermediate number of equivalence classes. When \( n \equiv 8 \pmod{16} \), Corollary 4.7 implies that \( QR \)-equivalence is a refinement of code equivalence: two Hadamard matrices in the same \( QR \) class have isomorphic codes; the converse does not necessarily hold as will be seen in the case \( n = 24 \), which is discussed in the next section.

By Proposition 4.8, \( Q \)-equivalence is stronger than integer equivalence when \( n \equiv 4 \pmod{8} \) which implies that there are at least as many \( Q \)-classes as there are Smith classes in those orders.

An equivalence class, of any type, may or may not be self-dual. The dual of a set of matrices is the set containing their transposes. A set that equals its own dual is self-dual. Many but not all \( Q \)-classes turn out to be self-dual. In other words, many matrices are \( Q \)-equivalent to their transposes. From the row-column symmetry in the definition of \( Q \)-equivalence it follows that if a \( Q \)-class contains at least one self-dual matrix, then that \( Q \)-class is self-dual. We will see examples of these phenomena in the next section.

6. Application to the enumeration of inequivalent Hadamard matrices

We remind the reader that Hadamard matrices have been completely classified up to order 28. There are five \( H \)-classes in order 16 [11], three in order 20 [12], 60 in order 24 [13, 18], and 487 in order 28 [19]. Using the available lists of \( H \)-classes, which can be obtained from a number of sources [33, 34, 36], we will be able to determine the structure of the \( Q \)-classes in these orders. The classification of \( H \)-classes in orders 32 and higher appears to be very difficult. We will content ourselves with identifying the \( Q \)-classes of all Hadamard matrices in orders 32 and 36 that were known before the recent work.
of Bouyukliev, Fack, and Winne (see Introduction), and completely enumerating those Q-classes that are small enough for this to be feasible.

Our procedure requires that we maintain a database of inequivalent matrices. As new matrices are generated, they are put in a canonical form and compared with known matrices to prevent duplication in the database. To put the matrices in canonical form, we followed the suggestion of Brendan McKay [26], converting $n \times n$ matrices to graphs on $4n$ vertices and then using the graph isomorphism program nauty that he developed [27].

The canonical form of the graph computed by nauty was then converted back into a matrix. As suggested in the nauty User’s Guide [28], we used the vertex invariant cellquads at level 2, which improves the efficiency in processing this type of graph.

To generate lists of inequivalent Hadamard matrices of order $n$ we carried out the following procedure, which requires a seed Hadamard matrix of order $n$ as input:

(1) Initialize hadList to null list.
(2) Compute canonical form of seed matrix using nauty. Append it to hadList.
(3) Compute canonical form of transpose of seed matrix. If it differs from canonical form of seed matrix, append it to hadList.
(4) Initialize ctr to 1.
(5) Let $H$ be matrix number ctr on hadList. If $n \equiv 4 \pmod{8}$ and this matrix is in the H-class of the transpose of the previous one, skip to Step 7.
(6) For each closed row quadruple ($n \equiv 0 \pmod{8}$) or Hall set ($n \equiv 4 \pmod{8}$) in $H$,
   (a) Switch the quadruple (Hall set) and compute the canonical form of the resulting matrix to obtain $H'$.
   (b) If $H'$ differs from all matrices on hadList, append it to hadList. Then if the canonical form of the transpose of $H'$ differs from $H'$, append it to hadList as well.
(7) Increment ctr. If hadList is not exhausted, return to Step 5.

Note that this procedure generates the Q-class of the seed matrix unless the Q-class happens to be non-self-dual, in which case it generates the union of the Q-class and its dual. This is due to the use of the transposition operation in Step 6(b). Non-self-dual Q-classes always turn out to be small, and when the situation arises, we partition the union into two Q-classes by hand. (We could use column quadruple switching in the $n \equiv 0 \pmod{8}$ case and dispense with transposition in both cases, thereby avoiding this issue, but we found it convenient to use transposition to keep track of duality.) We can also modify the procedure by simply eliminating the transposition step, in which case the procedure generates the QR-class of the seed matrix in the $n \equiv 0 \pmod{8}$ case.

Here are the results on the Q-classes and QR-classes for orders 16 and 24:

- $n = 16$: The five H-classes are all Q-equivalent. More strikingly, they are all QR-equivalent.
- $n = 24$: Of the 60 H-classes, 59 are Q-equivalent. The H-class missing from the main Q-class is that of the Paley matrix which has no closed quadruples and is self-dual. It forms a Q-class all by itself.
As stated in Section 4.2, Assmus and Key classified the 60 H-classes according to the doubly-even binary codes associated to the columns of the matrices. (See Table 1 in [3] or Table 7.1 in [2], but beware that 42\(^D\), listed with the code D, should be listed with the code C, and that 32\(^C\) in line 3 of the table should be changed to 32\(^D\).) We use QR-equivalence to refine this classification.

Assmus and Key found that six codes, labeled A, C, D, E, F, and G, occur. They are distinguished by the number of code words of weight 4 and therefore, according to Propositions 4.4 and 4.5, by the number of closed row quadruples in the associated matrices. We now see Corollary 4.7, on the invariance of codes under switching, in action. For example, the matrices associated with the code D all have 12 closed row quadruples. Switching any of these quadruples produces another matrix with code D. Depending on which of these matrices one starts with, switching row quadruples produces a QR-class of size 5 or of size 10. These two QR-classes together account for all 15 H-classes associated with the code D.

Results for all the codes appear in Table 1.

| code | # weight-4 code words | size of code class | sizes of QR-classes |
|------|------------------------|--------------------|---------------------|
| A    | 30                     | 8                  | 8                   |
| C    | 18                     | 17                 | 17                  |
| D    | 12                     | 15                 | 5, 10               |
| E    | 66                     | 8                  | 8                   |
| F    | 6                      | 10                 | 5, 5                |
| G    | 0                      | 2                  | 1, 1                |

Table 1. The 6 codes associated with the 60 Hadamard matrices of order 24.

Note that the matrices associated with the [24, 12] extended Golay code G do not contain closed row quadruples. One class of such matrices must be that of the Paley Hadamard matrix as we have already stated that it has no closed quadruples. There is a second class of matrices with no closed row quadruples. The matrices in this class, however, do each have 66 closed column quadruples, since their duals turn out to be in the class of the code E.

The results on Q-classes in orders 20 and 28 are:

- \(n = 20\): The 3 H-classes are Q-equivalent.
- \(n = 28\): Of the 487 H-classes, 486 of them (the ones containing Hall sets [20]) are Q-equivalent. The Paley matrix (generated from quadratic residues in GF(3\(^3\))) contains no Hall set and therefore its H-class forms a Q-class by itself.

Before presenting our results in orders 32 and 36, we ask what might the results so far lead us to expect in higher order? It is striking that except for a small number of exceptions (the H-classes of the Paley matrices in orders 24 and 28), all Hadamard matrices of given order are Q-equivalent. Could this be a general phenomenon?
In order 36, a difficulty arises. By Proposition 4.8 the Smith class is invariant under the defining operations of Q-equivalence. We will see that at least six different Smith classes occur, and so there must be at least six Q-classes, each possibly containing many H-classes. The reason the multiplicity of Smith classes was not an issue in order 28 is that \(7 = 28/4\) is an odd square free number. By a result in [39] this implies that all Hadamard matrices in order 28 lie in a single Smith class. From the foregoing discussion, the best we can hope for for general \(n \equiv 4 \pmod{8}\) is that within each Smith class there will be a single dominant Q-class, and that the total number of Q-classes will still be small.

The results we have obtained so far appear to support the idea of a single dominant Q-class in order 32, and of a single dominant Q-class within each Smith class in order 36. The total number of Q-classes also appears to be very small relative to the number of H-classes. We found only a few tens of Q-classes in our analysis of the matrices known prior to the work of Bouyukliev, Fack, and Winne, but a preliminary analysis of their matrices suggests that the number will rise into the hundreds, if not higher. Our method was to collect as many Hadamard matrices as possible from the literature or using known construction techniques, and then to apply our algorithm to each of these matrices in order to obtain its Q-class. In fortunate cases our program terminated in a reasonable time, giving us a complete enumeration of the elements of the Q-class of the given seed matrix. In less fortunate cases—and if our speculations are correct, this is expected to be the usual situation—the Q-class was too big to enumerate completely. Instead, we compared partially constructed Q-classes with each other, and looked for overlaps. By so doing, we managed to identify unambiguously the Q-class of every Hadamard matrix in orders 32 and 36 known to us prior to the work of Bouyukliev, Fack, and Winne, to enumerate the smaller of these Q-classes, and to obtain lower bounds on the sizes of the larger Q-classes.

6.1. Order 32.

**Proposition 6.1.** All Hadamard matrices of either of the forms
\[
H = \begin{bmatrix} A & B \\ A & -B \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} A & A \\ B & -B \end{bmatrix},
\]
where \(A\) and \(B\) are any Hadamard matrices of order 16, are Q-equivalent.

*Proof.* From the discussion in Section 3.4 it follows that, from the matrix \(\begin{bmatrix} A & B \\ A & -B \end{bmatrix}\), with \(A\) and \(B\) fixed, we may obtain any matrix of the form \(\begin{bmatrix} A & PB \\ A & -PB \end{bmatrix}\), by switching closed row quadruples.

To show that all matrices of the form \(H\) are Q-equivalent we need only to show that we can change the H-class of \(A\) or of \(B\) to any of the five classes in order 16 by switching closed quadruples. Since all Hadamard matrices of order 16 are QR- and QC-equivalent, we can achieve this by switching closed column quadruples in the \(A\) columns of \(H\) only.
or in the $B$ columns of $H$ only. (Closed column quadruples of $A$ or of $B$ extend to closed column quadruples of $H$ and switching transforms the top and bottom halves of a column the same way.)

Analogous arguments, with rows and columns interchanged, show that all matrices of the form $\tilde{H}$ are $Q$-equivalent. To show that matrices of the form $H$ and of the form $\tilde{H}$ are $Q$-equivalent to each other, simply note that both sets contain the Sylvester Hadamard matrix. □

Thus the 66099 $H$-classes identified in [25] are $Q$-equivalent. We call the $Q$-class of these matrices the Sylvester $Q$-class. We now turn to other known Hadamard matrices in order 32:

- the Paley matrix,
- 13 matrices from generalized Legendre (GL) pairs [9],
- four matrices listed in [1] and their transposes,
- the maximal excess matrix in [8],
- four matrices from Construction II in [25],
- two Williamson matrices,
- eight Goethals-Seidel matrices constructed from circulant blocks,
- 18 Goethals-Seidel matrices constructed from negacyclic blocks,
- 10 matrices constructed from two circulants,
- 17 matrices constructed from two negacyclic matrices,
- a matrix from the appendix of [23] and its transpose.

Some of these matrices were provided by Hadi Kharaghani. Discarding duplicates (which occur due to accidental equivalences) and matrices that happen to have one of the forms in Proposition 6.1, we are left with a list of 59 matrices. Of these, 49 are in the Sylvester $Q$-class. Using these matrices, and some matrices from Proposition 6.1 as seeds, we have managed to generate 3,577,996 $H$-classes in the Sylvester $Q$-class by using our program and then piecing together the results. This is certainly a gross underestimate of the actual number.

The ten exceptional matrices among the 59 all lack closed quadruples either in rows or in columns, and therefore form $Q$-classes by themselves. Of the ten exceptional matrices, six are constructed from GL pairs, and four are constructed from two negacyclic blocks. The matrices from GL pairs are listed on the web page [33] as P12–P19 (with transposes of non-self-dual matrices omitted). The exceptional GL pair matrices are P13, P15 and its transpose, P17, and P19 and its transpose. Matrix P17 is Hadamard equivalent to the Paley matrix. Of the matrices constructed from two negacyclic blocks, the exceptional ones come in two dual pairs.

The Sylvester $Q$-class and the ten singleton $Q$-classes give total of 11 known $Q$-classes in order 32, containing at least 3,578,006 Hadamard equivalence classes.
6.2. **Order 36.** As noted above, in order 36 we must consider each Smith class separately. Although Smith classes $\alpha = 6, 7, \ldots, 17$ are allowed, the only Smith classes known to be nonempty are $\alpha = 11, 12, 13, 14, 15, 16, 17$.

A complete summary of the seed matrices we compiled in order 36 follows:

- Ted Spence’s 180 matrices related to regular 2-graphs (S1–S180) \[35, 29, 36\],
- the 24 matrices of Goethals-Seidel type classified by Spence and Turyn (GS1–GS24) \[36\],
- the 11 matrices with automorphism of size 17 classified by Tonchev (T1–T11) \[37\],
- the Bush-type Hadamard matrix found by Janko (B1) \[14\],
- a regular Hadamard matrix found by Jennifer Seberry and listed on her web page (R1) \[33\], (She actually lists four, but two are duplicates, and two are of Goethals-Seidel type.)
- four Williamson Hadamard matrices (W1–W4), (There is a fifth, but it is equivalent to one of Tonchev’s.)
- the (35, 17, 8)-difference set construction (D1),
- seven matrices of the type defined by Whiteman (a Goethals-Seidel array bordered by a Hall set) (Wh1–Wh7) \[40\],
- two block negacyclic Bush-type Hadamard matrices, the first given in the paper of Janko and Kharaghani (NB1, NB2) \[15\],
- a matrix in Smith class 11, found in the course of a (fruitless) search for block circulant Bush-type matrices (O1),
- a skew Bush-type Hadamard matrix found by Leif Jørgensen and its transpose (J1, J2) \[16\],
- a matrix listed in the appendix of \[23\] (LLT1),
- the first matrix known (to us) in Smith class 16, found by Bouyukliev, Fack, and Winne \[4\] (BFW1).

Reference \[6\] was helpful in assembling the above list, but the reader should note that the 80 matrices from Steiner triple systems, which are a subset of Spence’s 180 matrices, are in Smith class 13, not 12 as stated there. We have not made a serious effort to credit the original author of every matrix on our list, as we were more concerned with compiling as complete a list as possible from readily obtainable sources. We should note, however, that many of these matrices derive from the important work of Goethals and Seidel \[10\], including the 80 matrices from Steiner triple systems mentioned above, and 11 matrices derived from Latin squares of order 6, which are also a subset of Spence’s list.

The structure we have uncovered in Smith class 13 is interesting, so we describe it in detail. Hadamard matrices in this class include 179 of Ted Spence’s 180 matrices. (His matrix 137 is in Smith class 11.) Two other matrices in Smith class 13 were previously known: the regular Hadamard matrix constructed by Seberry, and the block negacyclic Bush-type Hadamard matrix constructed by Janko and Kharaghani. Seberry’s matrix and 172 of Spence’s fall into the same Q-class which we found has size 3425. Two of Spence’s
matrices (179 and 180) and the Bush-type matrix form singleton Q-classes. They have no Hall sets. The remaining five of Spence’s matrices lie in a Q-class of size 6.

Spence’s matrix 137, which is in Smith class 11 and is one of the matrices derived from a Latin square of order 6, is intriguing. It has nine Hall sets, but switching any of these produces a matrix H-equivalent to the original.

Only two matrices on our list are in Smith class 14, B1 and LLT1. They are Q-equivalent. A major success of our program has been the complete enumeration of their Q-class, which has 954,254 elements. In each of Smith classes 15, 16, and 17 there is one known Q-class of size above five million, while all other known Q-classes are of size no greater than 5. The three large Q-classes have not yet been completely enumerated. At present, there is no evidence for more than one large Q-class in any Smith class.

The 236 matrices we compiled represent seven different Smith classes, and lie in 21 different Q-classes. Some details are given in Table 2. The union of the known Q-classes contains at least 18,292,717 Hadamard equivalence classes of order 36.

Table 2. Sizes of known Q-classes in order 36 for the 6 known Smith classes, α. A representative matrix is listed for each Q-class.

| α | Q-classes |
|---|---|
| 11 | 1 (S137), 1 (O1) |
| 12 | 1 (D1) |
| 13 | 1 (S179), 1 (S180), 1 (NB1), 6 (S172), 3425 (S1) |
| 14 | 954,254 (B1) |
| 15 | 5 (W3), 5 (W4), ≥ 5,520,880 (GS1) |
| 16 | ≥ 5,814,129 (BFW1) |
| 17 | 1 (GS11), 1 (GS12), 1 (T1), 1 (T2), 1 (T5), 1 (T6), 1 (T7), ≥ 6,000,000 (GS4) |

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