The Annihilating-Ideal Graph of Commutative Rings II*

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Abstract

In this paper we continue our study of annihilating-ideal graph of commutative rings, that was introduced in Part I (see [4]). Let \( R \) be a commutative ring with \( \mathcal{A}(R) \) its set of ideals with nonzero annihilator and \( Z(R) \) its set of zero divisors. The annihilating-ideal graph of \( R \) is defined as the (undirected) graph \( \mathcal{A}\mathcal{G}(R) \) that its vertices are \( \mathcal{A}(R)^* = \mathcal{A}(R) \setminus \{0\} \) in which for every distinct vertices \( I \) and \( J \), \( I-J \) is an edge if and only if \( IJ = (0) \). First, we study the diameter of \( \mathcal{A}\mathcal{G}(R) \). A complete characterization for the possible diameter is given exclusively in terms of the ideals of \( R \) when either \( R \) is a Noetherian ring or \( Z(R) \) is not an ideal of \( R \). Next, we study coloring of annihilating-ideal graphs. Among other results, we characterize when either \( \chi(\mathcal{A}\mathcal{G}(R)) \leq 2 \) or \( R \) is reduced and \( \chi(\mathcal{A}\mathcal{G}(R)) \leq \infty \). Also it is shown that for each reduced ring \( R \), \( \chi(\mathcal{A}\mathcal{G}(R)) = \text{cl}(\mathcal{A}\mathcal{G}(R)) \). Moreover, if \( \chi(\mathcal{A}\mathcal{G}(R)) \) is finite, then \( R \) has a finite number of minimal primes, and if \( n \) is this number, then \( \chi(\mathcal{A}\mathcal{G}(R)) = \text{cl}(\mathcal{A}\mathcal{G}(R)) = n \). Finally, we show that for a Noetherian ring \( R \), \( \text{cl}(\mathcal{A}\mathcal{G}(R)) \) is finite if and only if for every ideal \( I \) of \( R \) with \( I^2 = (0) \), \( I \) has finite number of \( R \)-submodules.

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0. Introduction

The present paper is a sequel to [4] and so the notations introduced in Introduction of [4] will remain in force. Thus throughout the paper, $R$ denotes a commutative ring with identity, $Z(R)$ denotes the set of all zero divisors of $R$ and $\mathfrak{I}(R)$ denotes the set of all proper ideals of $R$. If $X$ is either an element or a subset of $R$, then the annihilator of $X$ is $Ann(X) = \{ r \in R | rX = 0 \}$. The zero divisor graph of $R$, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^* := Z(R)$ in which for every two vertices $x$ and $y$, $x \rightarrow y$ is an edge if and only if $x \neq y$ and $xy = 0$. As [4], we say that the ideal $I$ of $R$ is an annihilating-ideal if $Ann(I) \neq (0)$ (i.e., there exists a nonzero ideal $J$ of $R$ such that $IJ = (0)$). Let $\mathfrak{A}(R)$ be the set of all annihilating-ideals of $R$. Then the annihilating-ideal graph of $R$, denoted by $\mathfrak{AG}(R)$, is a undirected simple graph with the vertex set $\mathfrak{A}(R)^* := \mathfrak{A}(R) \setminus \{(0)\}$ in which every two distinct vertices $I$ and $J$ are adjacent if and only if $IJ = (0)$ (see Part I [4] for more details).

Recall that a graph $G$ is connected if there is a path between every two distinct vertices. For distinct vertices $x$ and $y$ of $G$, let $d(x, y)$ be the length of the shortest path from $x$ to $y$ and if there is no such path we define $d(x, y) = \infty$. The diameter of $G$ is $\text{diam}(G) = \sup\{d(x, y) : x$ and $y$ are distinct vertices of $G\}$. The girth of $G$, denoted by $g(G)$, is defined as the length of the shortest cycle in $G$ and $g(G) = \infty$ if $G$ contains no cycles. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. Also, if a graph $G$ contains one vertex to which all other vertices are joined and $G$ has no other edges, is called a star graph. In [2, Theorem 2.1], it is shown that for every ring $R$, $\mathfrak{AG}(R)$ is a connected graph and $\text{diam}(\mathfrak{AG}(R)) \leq 3$, and if $\mathfrak{AG}(R)$ contains a cycle, then $g(\mathfrak{AG}(R)) \leq 4$ (see [7]). In Section 1 of this paper, we study the diameter of the annihilating-ideal graphs. By using the papers [4] and [6], we determine the relationship between the diameter of $\mathfrak{AG}(R)$ and $\Gamma(R)$. In particular, a complete characterization for the possible diameter is given exclusively in terms of the ideals of $R$ when either $R$ has finitely many minimal primes or $Z(R)$ is not an ideal.

A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph $G$, denoted by $cl(G)$, is called the clique number of $G$. Let $\chi(G)$ denote the chromatic number of the graph $G$, that is, the minimal number of colors needed to color the vertices of $G$ so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$. Beck in [3] conjectured that $\chi(\Gamma(R)) = cl(\Gamma(R))$. But, D.D. Anderson and M. Naseer...
gave a counterexample to this conjecture in [1]. In fact, the counterexample is the ring $R = \mathbb{Z}_4 [X, Y, Z]/(X^2 - 2, Y^2 - 2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ - 2)$ for which $\chi(\Gamma(R)) = 5$ but $cl(\Gamma(R)) = 4$ (note that in Beck’s coloring all elements of the ring were vertices of the graph but the vertices of $\Gamma(R)$ are nonzero zero divisors of $R$). In Section 2, we look at the coloring of the annihilating-ideal graph of rings. First, we show that for the counterexample above we have $\chi(\mathcal{A}_G(R)) = cl(\mathcal{A}_G(R)) = 4$. On the other hand, however we have not found any example where $\chi(\mathcal{A}_G(R)) > cl(\mathcal{A}_G(R))$. The lack of such counterexamples together with the fact that we have been able to establish the equality $\chi(\mathcal{A}_G(R)) = cl(\mathcal{A}_G(R))$ for Anderson-Naseer’s counterexample and also for reduced rings motivates the following conjecture.

**Conjecture 0.1.** For every commutative ring $R$, $\chi(\mathcal{A}_G(R)) = cl(\mathcal{A}_G(R))$.

In Section 2, among other results, we characterize rings $R$ for which $\chi(\mathcal{A}_G(R)) \leq 2$. It is shown that for a reduced ring $R$ the following conditions are equivalent: (1) $\chi(\mathcal{A}_G(R)) < \infty$, (2) $cl(\mathcal{A}_G(R)) < \infty$, (3) $\mathcal{A}_G(R)$ does not have an infinite clique and (4) $R$ has finite number of minimal primes. Moreover, if $R$ a non-domain reduced ring, then $\chi(\mathcal{A}_G(R))$ is the number of minimal primes of $R$. Also, it is shown that for a Noetherian ring $R$, $cl(\mathcal{A}_G(R))$ is finite if and only if every ideal $I$ of $R$ with $I^2 = (0)$ has finite number of $R$-submodules. Finally we conjecture that “if $\mathcal{A}_G(R)$ does not have an infinite clique, then $\chi(\mathcal{A}_G(R))$ is finite”.

1. The diameter of an annihilating-ideal graph

By Anderson and Livingston [2, Theorem 2.3], for every ring $R$, the zero divisor graph $\Gamma(R)$ is a connected graph and $diam(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 4$ (see [7]). Moreover, Lucas in [6] characterized the diameter of $\Gamma(R)$ in terms of the ideals of $R$. As we have seen in [4, Theorem 2.1], for a ring $R$, $\mathcal{A}_G(R)$ is also a connected graph with $diam(\mathcal{A}_G(R)) \leq 3$. In general, Lucas’s results are not true for annihilating-ideal graphs, but the following proposition more or less summarizes the overall situation for the relationship between the diameter of $\mathcal{A}_G(R)$ and $\Gamma(R)$.

**Proposition 1.1.** Let $R$ be a ring.

(a) If $diam(\Gamma(R)) = 0$, then $diam(\mathcal{A}_G(R)) = 0$.
(b) If $diam(\Gamma(R)) = 1$, then $diam(\mathcal{A}_G(R)) = 0$ or 1.
(c) If $diam(\Gamma(R)) = 2$, then $diam(\mathcal{A}_G(R)) = 1, 2$ or 3.
(d) If \(\text{diam}(\Gamma(R)) = 3\), then \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 3\).
(e) If \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 0\), then \(\text{diam}(\Gamma(R)) = 0\) or 1.
(f) If \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 1\), then \(\text{diam}(\Gamma(R)) = 1\) or 2.
(g) If \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 2\), then \(\text{diam}(\Gamma(R)) = 2\).
(h) If \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 3\), then \(\text{diam}(\Gamma(R)) = 2\) or 3.

**Proof.** (a). Let \(\text{diam}(\Gamma(R)) = 0\) i.e., \(\Gamma(R)\) has one vertex. Thus by [6, Theorem 2.6.], \(R\) is isomorphic to either \(\mathbb{Z}_4\) or \(\mathbb{Z}_2[y]/(y^2)\). Clearly, in any case \(\mathcal{A}\mathcal{G}(R)\) has also one vertex and so \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 0\).

(b). Clearly, if \(\text{diam}(\Gamma(R)) = 1\) then \(\Gamma(R)\) is a complete graph with more than one vertex. Thus by [6, Theorem 2.6], either (i) \(R\) is reduced and isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2\), or (ii) \(R\) is non-reduced, \(Z(R)^2 = (0)\) and \(R\) is not isomorphic to either \(\mathbb{Z}_2\) or \(\mathbb{Z}_2[y]/(y^2)\). If \(R \cong \mathbb{Z}_2 \times \mathbb{Z}_2\), then \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 1\). In the later case, \(Z(R)^2 = (0)\) implies that \(\mathcal{A}\mathcal{G}(R)\) is also a complete graph (see [4, Theorem 2.7]. Now, if \(\mathcal{A}\mathcal{G}(R)\) has one vertex (as \(\mathbb{Z}_p\) where \(p\) is an odd prime number), then \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 0\), otherwise, \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 1\).

(c). By [6, Theorem 2.6], \(\text{diam}(\Gamma(R)) = 2\) if and only if either \(R\) is reduced with exactly two minimal primes and at least three nonzero zero divisors, or \(Z(R)\) is an ideal whose square is not \(0\) and each pair of distinct zero divisors has a nonzero annihilator. If \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 0\), then \(\mathcal{A}\mathcal{G}(R)\) has only one vertex, say \(I\), and \(I^2 = (0)\). It follows that \(Z(R) = I\) and so by [2, Theorem 2.8], \(\Gamma(R)\) is a complete graph. Thus \(\text{diam}(\Gamma(R)) = 0\) or 1, contradicting with our hypothesis. Thus \(\text{diam}(\mathcal{A}\mathcal{G}(R)) > 0\), i.e., \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 1, 2\) or 3 (since by [4, Theorem 2.1], \(\text{diam}(\mathcal{A}\mathcal{G}(R)) \leq 3\)). We claim that all these three cases may happen. For see this, let \(R_2 = F_1 \times F_2\) where \(F_1\) and \(F_2\) are fields and \(F_1 \not\cong \mathbb{Z}_2\) and \(R_1 = \mathbb{Z} \times \mathbb{Z}_2\). Then \(\text{diam}(\Gamma(R_1)) = \text{diam}(\Gamma(R_2)) = 2\) but \(\text{diam}(\mathcal{A}\mathcal{G}(R_1)) = 1\) and \(\text{diam}(\mathcal{A}\mathcal{G}(R_2)) = 2\). For an example of a ring \(R\) for which \(\text{diam}(\Gamma(R)) = 2\) but \(\text{diam}(\mathcal{A}\mathcal{G}(R_2)) = 3\) see the Example 1.7 of this paper.

(d). Let \(\text{diam}(\Gamma(R)) = 3\). Then there exist \(a, b \in Z(R)\) such that \(d(a, b) = 3\). Clearly, \(Ra\) and \(Rb\) are vertices of \(\mathcal{A}\mathcal{G}(R)\) and \(RaRb \neq (0)\). We claim that \(Ra \neq Rb\), for if not, then \(Ra = Rb\) implies that \(a\) and \(b\) have the same annihilator i.e., \(d(a, b) = 2\), a contradiction. Now by [4, Theorem 2.1], \(d(Ra, Rb) \leq 3\) in \(\mathcal{A}\mathcal{G}(R)\). If \(d(Ra, Rb) \neq 3\), then \(d(Ra, Rb) = 2\), i.e., there exists \(0 \neq c \in R\) such that \(RcRa = RcRb = (0)\). It follows \(d(a, b) = 2\) in \(\Gamma(R)\), a contradiction. Thus \(d(Ra, Rb) = 3\) and so \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 3\).

(e). Clearly, \(\text{diam}(\mathcal{A}\mathcal{G}(R)) = 0\) if and only if the ring \(R\) has only one nonzero proper ideal, if and only if \(Z(R)\) is the only nonzero proper ideal of \(R\). In this case \(Z(R)^2 = (0)\) and so \(\Gamma(R)\) is a complete graph. If \(\Gamma(R)\) has one vertex, then by [6, Theorem 2.6 (1)], \(R\) is isomorphic to either \(\mathbb{Z}_4\) or \(\mathbb{Z}_2[y]/(y^2)\).
and so $\text{diam}(\Gamma(R)) = 0$, otherwise, when $\Gamma(R)$ has more than one vertices (as $R = \mathbb{Z}_{p^2}$ where $p$ is an odd prime number), then $\text{diam}(\Gamma(R)) = 1$.

(f). Clearly, $\text{diam}(\mathbb{A}(R)) = 1$ if and only if $\mathbb{A}(R)$ is a complete graph with more than one vertex, i.e., either $R = F_1 \oplus F_2$, where $F_1$, $F_2$ are fields, or $Z(R)$ is an ideal of $R$ that is not minimal, $Z(R)^3 = (0)$ and for each ideal $I \subseteq Z(R)$, $IZ(R) = (0)$ (see [4, Theorem 2.7]). If $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $Z(R)^2 = (0)$, then $\Gamma(R)$ is a complete graph with more than one vertex i.e., $\text{diam}(\Gamma(R)) = 1$. On the other hand, if $R \not= \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $Z(R)^2 \not= (0)$, then $\Gamma(R)$ is not a complete graph and so $\text{diam}(\Gamma(R)) \geq 2$. If $\text{diam}(\Gamma(R)) = 3$, then by (d) above $\text{diam}(\mathbb{A}(R)) = 3$, a contradiction. Thus we have $\text{diam}(\Gamma(R)) = 2$.

(g). If $\text{diam}(\mathbb{A}(R)) = 2$, then by (a), (b) and (d) above, $\text{diam}(\Gamma(R)) \not= 0, 1$ and 3. Thus $\text{diam}(\Gamma(R)) = 2$.

(h). If $\text{diam}(\mathbb{A}(R)) = 3$, then by (a) and (b) above, $\text{diam}(\Gamma(R)) \not= 0$ and 1. Thus $\text{diam}(\Gamma(R)) = 2$ or 3. By (d) above if $\text{diam}(\Gamma(R)) = 3$, then $\text{diam}(\mathbb{A}(R))$ is also 3. For the case $\text{diam}(\mathbb{A}(R)) = 3$ but $\text{diam}(\Gamma(R)) = 2$, see Example 1.7. □

In next two theorems, we characterize when $Z(R)$ is not an ideal of $R$ and either $\text{diam}(\mathbb{A}(R)) = 2$ or $\text{diam}(\mathbb{A}(R)) = 3$, respectively.

**Theorem 1.2.** Let $R$ be a ring such that $Z(R)$ is not an ideal of $R$. Then $\text{diam}(\mathbb{A}(R)) = 2$ if and only if $R$ is reduced with exactly two minimal primes and at least three nonzero annihilating ideals.

**Proof.** ($\Rightarrow$). Let $\text{diam}(\mathbb{A}(R)) = 2$. Then $R$ has at least three nonzero annihilating ideals and by Proposition 1.1 (g), $\text{diam}(\Gamma(R)) = 2$. Now by [6, Theorem 2.6, (3)], $R$ is a reduced ring with exactly two minimal primes.

($\Leftarrow$). Suppose $R$ is a reduced ring with exactly two minimal primes $P_1$ and $P_2$. Let $D_1 := R/P_1$, $D_2 := R/P_2$. Since $R$ is reduced, $P_1 \cap P_2 = (0)$. Thus the neutral map $\varphi : R \rightarrow D_1 \times D_2$ is a monomorphism ring homomorphism. It is easy to check that if $I$ and $J$ are vertices in $\mathbb{A}(R)$ such that $IJ = (0)$, then $\bar{I} := \varphi(I)$ and $\bar{J} := \varphi(J)$ are also vertices of $\mathbb{A}(D_1 \times D_2)$ and $\bar{I} \bar{J} = (0)$. Since $D_1$ and $D_2$ are domains, each vertex of $\mathbb{A}(D_1 \times D_2)$ is the form $(A, (0))$ or $(0, B)$ for some nonzero ideals $A$ of $D_1$ or $B$ of $D_2$.

First, we show that $\text{diam}(\mathbb{A}(R)) \leq 2$. For see this, suppose $I$ and $J$ are two distinct vertices in $\mathbb{A}(R)$. We need only consider 2 cases:

Case 1: $\varphi(I) = (I, (0))$ and $\varphi(J) = (J, (0))$. It follows that $I, J \subseteq P_2$. Since $P_1P_2 = (0)$, $P_1I = P_1J = (0)$ and so $d(I, J) \leq 2$.

Case 2: $\varphi(I) = (I, (0))$ and $\varphi(J) = ((0), J)$. It follows that $I \subseteq P_2$ and
$J \subseteq P_1$. Since $P_1 P_2 = (0)$, $IJ = (0)$ and so $d(I, J) = 1$.
Thus $diam(\mathbb{M}(R)) \leq 2$. Since $P_1$ and $P_2$ are two distinct vertices of $\mathbb{M}(R)$, $diam(\mathbb{M}(R)) \geq 1$. If $diam(\mathbb{M}(R)) = 1$, then for each nonzero ideal $I$ of $R$ such that $P_1 \neq I \neq P_2$, we must have $IP_1 = IP_2 = (0)$. It follows that $I \subseteq P_1 \cap P_2 = (0)$, a contradiction. Thus $diam(\mathbb{M}(R)) = 2$. □

**Theorem 1.3.** Let $R$ be a ring such that $Z(R)$ is not an ideal. Then the following statements are equivalents.

1. $diam(\mathbb{M}(R)) = 3$.
2. $diam(\Gamma(R)) = 3$.
3. Either $R$ is non-reduced or $R$ is a reduced ring with more than two minimal primes.

**Proof.** (1) $\Rightarrow$ (2). Let $diam(\mathbb{M}(R)) = 3$. If $diam(\Gamma(R)) \neq 3$, then by Proposition 1.1 (h), $diam(\Gamma(R)) = 2$. Thus by [6, Theorem 2.6, (3)], $R$ is reduced with exactly two minimal primes. Since $diam(\mathbb{M}(R)) = 3$, there exist at least three nonzero annihilating ideals in $R$. Thus by Theorem 1.2, $diam(\mathbb{M}(R)) = 2$, a contradiction. Therefore, $diam(\Gamma(R)) = 3$.

(2) $\Rightarrow$ (1) is by Proposition 1.1, (d).

(2) $\Leftrightarrow$ (3) is by [6, Theorem 2.6]. □

Now, we are in position to give a complete characterization for the possible diameter of $\mathbb{M}(R)$ when $Z(R)$ is not an ideal of $R$.

**Theorem 1.4.** Let $R$ be a ring such that $Z(R)$ is not an ideal of $R$. Then $1 \leq diam(\mathbb{M}(R)) \leq 3$ and

1. $diam(\mathbb{M}(R)) = 1$ if and only if $R \cong F_1 \times F_2$ where $F_1$ and $F_2$ are fields;
2. $diam(\mathbb{M}(R)) = 2$ if and only if $R$ is reduced with exactly two minimal primes and at least three nonzero annihilating ideals;
3. $diam(\mathbb{M}(R)) = 3$ if and only if either $R$ is a non-reduced ring or $R$ is a reduced ring with more than two minimal primes.

**Proof.** Suppose that $Z(R)$ is not an ideal of $R$. Then by [4, Theorem 2.1], $diam(\mathbb{M}(R)) \leq 3$. If $diam(\mathbb{M}(R)) = 0$, then $\mathbb{M}(R)$ has only one vertex, i.e., $Z(R)$ is the only nonzero proper ideal of $R$, a contradiction. Thus we conclude that $diam(\mathbb{M}(R)) = 1$, 2 or 3.

For (1), we note that $\mathbb{M}(R)$ is a complete graph (i.e., $diam(\mathbb{M}(R)) = 0$ or 1) if and only if either $R = F_1 \oplus F_2$, where $F_1$, $F_2$ are fields, or $Z(R)$ is an
ideal of \( R \), \( Z(R)^3 = (0) \) and for each ideal \( I \subsetneq Z(R) \), \( IZ(R) = (0) \) (see [4, Theorem 2.7]). Thus \( diam(\mathbb{G}(R)) = 1 \) if and only if \( R \cong F_1 \times F_2 \) where \( F_1 \) and \( F_2 \) are fields.

Finally statements (2) and (3) are from Theorems 1.2 and 1.3 above. □

In the next theorem, we provide a sufficient condition for \( \mathbb{G}(R) \) to have diameter 3 when \( R \) is a non-reduced ring. First we need the following lemma (see also [6, Lemma 2.3]).

**Lemma 1.5.** Let \( R \) be a ring and let \( I \) be an annihilating-ideal of \( R \). If \( N \) is a nilpotent ideal of \( R \), then \( N + I \) is an annihilating-ideal of \( R \).

**Proof.** Let \( N \) be a nonzero nilpotent ideal and assume \( cI = (0) \) where \( c \neq 0 \). Since \( N \) is nilpotent, there is a positive integer \( m \) such that \( cN^m = 0 \) with \( cN^{m-1} \neq 0 \). Clearly, \( cN^{m-1} \subseteq Ann(N + I) \). □

**Theorem 1.6.** Let \( R \) be a non-reduced ring. If there is a pair of annihilating-ideals \( I, J \) of \( R \) such that \( I + J \) is not an annihilating-ideal, then \( \text{diam}(\mathbb{G}(R)) = 3 \).

**Proof.** Let \( I, J \in \mathbb{A}^*(R) \) be such that \( Ann(I + J) = (0) \). Then \( d(I, J) \neq 2 \). By Lemma 1.5, neither \( I \) nor \( J \) can be nilpotent. If \( IJ \neq (0) \), then \( d(I, J) = 3 \). Thus we may assume \( IJ = (0) \). Since \( IJ = (0) \), \( (I + J)^2 = I^2 + J^2 \) has no nonzero annihilator. Since \( R \) is not reduced, there exists a nonzero nilpotent \( q \in R \). Thus without loss of generality we may assume that \( qJ^2 \neq (0) \). Since \( I \) is an annihilating-ideal and \( qJ \) is nilpotent, \( I + qJ \) is an annihilating-ideal by Lemma 1.5. On the other hand \( I + qJ \neq J \), for if not, then \( I \subseteq J \) and so \( Ann(I + J) = Ann(J) \neq (0) \), a contradiction. Now consider the pair \( I + qJ \) and \( J \). Since \( I + J = I + qJ + J \) and \( Ann(I + J) = (0) \), \( d(I + qJ, J) \neq 2 \). But \( (I + qJ)J = qJ^2 \neq (0) \). Thus \( d(I + qJ, J) = 3 \) and \( diam(\mathbb{G}(R)) = 3 \). □

**Example 1.7.** Let \( R \) be the ring in [6, Example 5.5]. Then \( R \) is a non-reduced ring and there is a pair of annihilating-ideals \( I, J \) of \( R \) such that \( I + J \) is not an annihilating-ideal (see [6, Example 5.5, (6)]). Thus by Theorem 1.6, \( diam(\mathbb{G}(R)) = 3 \), but since \( R \) is a McCoy ring (see [6, Example 5.5, (4)] \( diam(\Gamma(R)) = 2 \). (Note: a ring \( R \) is said to be a McCoy ring if each finitely generated ideal contained in \( Z(R) \) has a nonzero annihilator).

Let \( R \) be a ring with finitely many minimal primes \( P_1, P_2, \ldots, P_n \). Then it is easy to see that an ideal \( I \) of \( R \) is an annihilating-ideal if and only if \( I \) is contained in at least one minimal prime.
Lemma 1.8. Let $R$ be a ring with finitely many minimal primes. If $R$ has more than two minimal primes and there are nonzero annihilating-ideals $I$, $J$ such that $I + J$ is not an annihilating-ideal, then $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 3$.

Proof. By Theorem 1.6, we can assume that $R$ is a reduced ring. If $IJ \neq (0)$, then $d(I, J) = 3$ and therefore $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 3$. Thus we may assume $R$ has more than two minimal primes and that there are zero annihilating-ideals $I$ and $J$, such that $IJ$ is not an annihilating-ideal. Each minimal prime of $R$ contains at least one of $I$ and $J$, but with $\text{Ann}(I + J) = (0)$, and not both of them. Thus without loss of generality we may assume there are minimal primes $P_1$, $P_2$, $P_3$, $I \subseteq P_1 \cap P_2$, $P_3$, $J \subseteq P_3$, $J \not\subseteq P_1$ and $J \not\subseteq P_2$. Let $q \in P_1 \cap P_3 \setminus P_2$ and consider the pair $I + qJ$ and $J$. Since $R$ is reduced, $IJ = (0)$ and neither $I$ nor $q$ is contained in $P_2$. $(0) \neq qJ^2 = J(I + qJ)$. Since $I + qJ \subseteq P_1$, that $I + qJ$ is a nonzero annihilating-ideal of $R$. On the other hand, $I + J = I + qJ + J$ is an ideal with no nonzero annihilator. Thus $d(I + qJ, J) = 3$ and $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 3$. $\square$

Now, we are in position to give a complete characterization for the possible diameter of $\mathbb{A}\mathbb{G}(R)$ when $R$ has finitely many minimal primes (as Noetherian rings).

Theorem 1.9. Let $R$ be a ring with finitely many minimal primes. Then

1. $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 0$ if and only if $Z(R)$ is the only nonzero proper ideal of $R$.
2. $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 1$ if and only if $Z(R)$ is an ideal of $R$, $Z(R)^3 = (0)$ and for each ideal $I \not\subseteq Z(R)$, $IZ(R) = (0)$.
3. $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 2$ if and only if either (i) $R$ is reduced with exactly two minimal primes and at least three nonzero annihilating-ideals, or (ii) $Z(R)$ is an ideal whose square is not $(0)$ and for each pair of annihilating-ideals $I$ and $J$, $I + J$ is an annihilating-ideal.
4. $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 3$ if and only if there are annihilating-ideals $I \neq J$, such that $I + J$ is not an annihilating-ideal and either (i) $R$ is reduced with more than two minimal primes, or (ii) $R$ is non-reduced.

Proof. (1) is evident.
The statement (2) is from [4, Theorem 2.7].
(3) $\Rightarrow$. Suppose that $\text{diam}(\mathbb{A}\mathbb{G}(R)) = 2$. Then by Proposition 1.1 $\text{diam}(\Gamma(R)) = \ldots$
2. Thus by [6, Theorem 2.6 (3)], either (i) $R$ is reduced with exactly two minimal primes and at least three nonzero zero divisors, or (ii) $Z(R)$ is an ideal whose square is not $(0)$ and each pair of distinct zero divisors has a nonzero annihilator. The Case (i) implies that $Z(R)$ is not an ideal of $R$ and so by Theorem 1.2, $R$ is a reduced ring with exactly two minimal primes and at least three nonzero annihilating-ideals. Assume that the Case (ii) holds. If $R$ is not reduced, then by Theorem 1.6, for each pair of annihilating-ideals $I$ and $J$, $I + J$ is an annihilating-ideal. Thus we can assume that $R$ is reduced and $R$ has more than two minimal primes. Hence by Lemma 1.8, for each pair of annihilating-ideals $I$ and $J$, $I + J$ is an annihilating-ideal. (3) ($\Leftarrow$) is evident.

Finally statement (4) is from Theorems 1.3, 1.6, Lemma 1.8 and statement (3) above. □

2. Coloring of the annihilating ideal graphs

The goal of this section is to study of coloring of the annihilating ideal graphs of rings, in particular the interplay between $\chi(\mathcal{AG}(R))$ and $cl(\mathcal{AG}(R))$.

Beck [3] showed that if $R$ is a reduced ring or a principal ideal ring, then $\chi(\Gamma(R)) = cl(\Gamma(R))$. He also showed that for $n = 1, 2$ or $3$, $\chi(\Gamma(R)) = n$ if and only if $cl(\Gamma(R)) = n$. Based on these positive results, Beck conjectured that $\chi(\Gamma(R)) = cl(\Gamma(R))$ for each ring $R$ when $\chi(\Gamma(R)) < \infty$. In [1], D.D. Anderson and M. Naseer gave a counterexample to this conjecture. We start this section with the following interesting result about Anderson-Naseer’s counterexample.

**Proposition 2.1.** Let $$R = \mathbb{Z}_4[X, Y, Z]/(X^2 - 2, Y^2 - 2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ - 2).$$
Then $\chi(\mathcal{AG}(R)) = cl(\mathcal{AG}(R)) = 4$.

**Proof.** Clearly, $R$ is a finite local ring with 32 elements and $J(R) = Z(R) = \{0, 2, x + 2, y + 2, x + y, x + y + 2, z, z + 2, x + z, x + z + 2, y + z, y + z + 2, x + y + z, x + y + z + 2\}$. One can easily see that the following 15 ideals are the only nonzero proper ideals of $R$.

$$\{(2), (x), (y), (z), (x + y), (x + z), (y + z), (x + y + z), (x,y), (x, z), (y, z), (x,y + z), (y, x + z), (z, x + y), (x, y, z)\}.$$ 

Thus the graph $\mathcal{AG}(R)$ has 15 vertices and so is the graph in Figure 1 below.
Now by using Figure 1, it easy to see that \{(2), (x), (y), (y + z)\} is a maximal clique and each other clique of \(\Lambda G(R)\) has at most 3 elements. Thus \(cl(\Lambda G(R)) = 4\). Also, the Figure 1, shows that the minimal number of colors needed to color \(\Lambda G(R)\) is 4 colors, which we label as 1, 2, 3 and 4. Thus \(\chi(\Lambda G(R)) = cl(\Lambda G(R)) = 4\). □

Next, we characterize rings \(R\) for which \(\chi(R) = 1\) or 2.

**Proposition 2.2.** Let \(R\) be a ring. Then \(\chi(\Lambda G(R)) = 1\) if and only if \(R\) has only one nonzero proper ideal.

**Proof.** Since \(\Lambda G(R)\) is a connected graph and \(\chi(\Lambda G(R)) = 1\), it can not have more than one vertex, and so by [4, Theorem 1.4], \(R\) has only one nonzero proper ideal. The converse is clear. □

**Theorem 2.3.** For a ring \(R\) the following statements are equivalent:
(1) \( \chi(\mathbb{A}_G(R)) = 2 \).
(2) \( \mathbb{A}_G(R) \) is a bipartite graph.
(3) \( \mathbb{A}_G(R) \) is a complete bipartite graph.
(4) Either \( R \) is a reduced ring with exactly two minimal primes or \( \mathbb{A}_G(R) \) is a star graph.

**Proof.** (1) \( \Leftrightarrow \) (2) and (3) \( \Rightarrow \) (2) are clear.
(2) \( \Rightarrow \) (4). Suppose that \( \mathbb{A}_G(R) \) is bipartite with two parts \( V_1 \) and \( V_2 \).

**Case 1:** The ring \( R \) is a reduced ring. We put \( P_1 = \bigcup_{x \in V_1} I, P_2 = \bigcup_{x \in V_2} J \).
We claim that \( P_1 \) and \( P_2 \) are prime ideals of \( R \). Let \( x \in P_1 \) and \( r \in R \). Then there exists \( I \in V_1 \) such that \( Rx \subseteq I \). Since \( I \in V_1 \) and \( \mathbb{A}_G(R) \) is bipartite, there exists \( J \in V_2 \) such that \( IJ = (0) \). Thus \( RrxJ = (0) \) and so \( Rrx \in V_1 \), and hence \( rx \in P_1 \). Now let \( x, y \in P_1 \). Then there exist \( I_1, J_2 \in V_1 \) and \( J_1, J_2 \in V_2 \) such that \( x \in I_1 \), \( y \in I_2 \) and \( I_1J_1 = I_2J_2 = (0) \). Thus \( Rx_1J_1 = Ry_1J_2 = (0) \). Since \( \mathbb{A}_G(R) \) is bipartite and \( R \) is reduced, \( J_1J_2 \neq (0) \). On the other hand, \( Rx_1J_2 = Ry_1J_1 = (0) \) and so \( R(x + y)J_1J_2 = (0) \). It follows that \( R(x + y) \in V_1 \) and so \( x + y \in P_1 \). Thus \( P_1 \) is an ideal of \( R \). Similarly, \( P_2 \) is an ideal of \( R \). Let \( xy \in P_1 \) and suppose that \( x \notin P_1 \). So there exist vertices \( I \in V_1, J \in V_2 \) such that \( xy \in I \) and \( IJ = (0) \). We claim that \( yJ = (0) \). For see this let \( yJ \neq (0) \). Since \( \mathbb{A}_G(R) \) is bipartite and \( R \) is reduced, \( IyJ = (0) \) implies that \( yJ \in V_2 \) and also \( RxyJ = (0) \), implies that \( Rxy \in V_1 \). It follows that \( x \in P_1 \), a contradiction. Therefore \( yJ = (0) \) and \( Rxy \in V_1 \) i.e., \( y \in P_1 \). Thus \( P_1 \) is a prime ideal of \( R \). Similarly, \( P_2 \) is a prime ideal of \( R \). Now let \( 0 \neq t \in P_1 \cap P_2 \). Then there exist \( I \in V_1, J \in V_2 \) such that \( t \in I \cap J \). Also, there exist \( L \in V_1, K \in V_2 \) such that \( IK = (0) \) and \( JL = (0) \). It follows that \( RtK = (0) \) and \( RtL = (0) \). Since \( \mathbb{A}_G(R) \) is bipartite and \( R \) is reduced, \( Rt \in V_1 \cap V_2 = \emptyset \), a contradiction. Therefore \( P_1 \cap P_2 = (0) \).

**Case 2:** The ring \( R \) is not reduced. Assume \( x^2 = (0) \), where \( 0 \neq x \in R \). Without loss of generality we can assume that \( Rx \in V_1 \). We claim that either \( Rx \) is a minimal ideal of \( R \) or for each \( 0 \neq z \in R \) such that \( Rz \subsetneq Rx \), \( Rz \) is a minimal ideal of \( R \). For see this let \( 0 \neq Rz_1 \subsetneq Rx \), \( 0 \neq Rz_1 \subsetneq Rx,(0) \neq Rz_2 \subsetneq Rz_1 \) for some \( z_1, z_2 \in R \). Since \( (Rx)^2 = (0) \), \( Rz_1Rx = Rz_2Rx = (0) \), and so \( Rz_1, Rz_2 \in V_2 \) since \( \mathbb{A}_G(R) \) is bipartite. But \( Rz_1 \subsetneq (Rx)^2 = (0) \), a contradiction. Thus without loss of generality we can assume that \( Rx \) is a minimal ideal of \( R \). Thus \( P = \text{Ann}(x) \) is a maximal ideal of \( R \) and also \( Rx \subsetneq P \). We claim that \( V_1 = \{ Rx \} \).

**Subcase 1:** \( P = Rx \). Then \( P \) is both minimal and maximal. Let \( I \in V_1 \) such that \( I \neq Rx \). Then there exists \( L \in V_2 \) such that \( IL = (0) \). Since \( P \) is prime and \( I \not\subseteq P, L \subseteq P \) i.e., \( L = P = Rx \), a contradiction.
Subcase 2: \( P \neq Rx \). Since \( PRx = (0) \) and \( \AG(R) \) is bipartite, \( P \in V_2 \). Now let \( V_1 \neq \{Rx\} \) and \( L \in V_1 \setminus \{Rx\} \). If \( LP = (0) \), \( LRx \subseteq LP = (0) \), a contradiction. Thus \( LP \neq (0) \) and since \( \AG(R) \) is connected, there exists \( K \in V_2 \) such that \( K \neq P \) and \( LK = (0) \). It follows that \( L \cap P \neq (0) \), \( (L \cap P)Rx = (0) \). Therefore \( L \cap P \in V_2 \). But \( (L \cap P)K = (0) \), a contradiction. Thus in each case \( V_1 = \{Rx\} \) and it follows that \( \AG(R) \) is a star graph.

(4) \( \Rightarrow \) (3). Clearly if \( \AG(R) \) is a star graph, then \( \AG(R) \) is a complete bipartite graph. Thus we assume that there exist two distinct prime ideals \( P_1 \) and \( P_2 \) of \( R \) such that \( P_1 \cap P_2 = (0) \). We claim that \( Z(R) = P_1 \cup P_2 \). It is clear that \( P_1 \cup P_2 \subseteq Z(R) \). On the other hand, if \( x \in Z(R) \setminus P_1 \cup P_2 \), then there exists \( 0 \neq y \in R \) such that \( xy = 0 \), and hence \( y \in P_1 \cap P_2 = (0) \), which is a contradiction. Thus \( Z(R) = P_1 \cup P_2 \) and so by Prime Avoidance Theorem (see [8, Theorem 3.61]), for every \( I \in \mathfrak{a}^*(R) \), either \( I \in P_1 \) or \( I \in P_2 \). Also if \( I_1, I_2 \subseteq P_1, I_1I_2 \neq (0) \). Because otherwise \( I_1I_2 \subseteq P_2 \) and so \( I_1 \subseteq P_2 \) or \( I_2 \subseteq P_2 \). Similarly for each two ideal included in \( P_2 \). Now, let \( f \) be a coloring assigns to each ideal \( I \) of \( \mathfrak{a}(R) \) color \( i, i = 1, 2 \) if \( I \subseteq P_i, i = 1, 2 \). So by above argument, \( \AG(R) \) is bipartite. On the other hand, for vertices \( I, J \) such that \( I \subseteq P_1 \) and \( J \subseteq P_2 \), we have \( IJ \subseteq P_1 \cap P_2 = (0) \). Therefore, \( \AG(R) \) is a complete bipartite graph. \( \square \)

**Corollary 2.4.** Let \( R \) be an Artinian ring. Then the following statements are equivalent:

1. \( \chi(\AG(R)) = 2 \).
2. \( \AG(R) \) is a bipartite graph.
3. \( \AG(R) \) is a complete bipartite graph.
4. Either \( R \cong F_1 \times F_2 \) for some fields \( F_1 \) and \( F_2 \) or \( R \) is a local ring such that \( \AG(R) \) is a star graph.

**Proof.** By Theorem 2.3, \( (1) \Leftrightarrow (2) \Leftrightarrow (3) \).

(2) \( \Rightarrow \) (4). Assume that \( \AG(R) \) is a bipartite graph with parts \( V_1 \) and \( V_2 \). Note that since \( R \) is Artinian, \( R = R_1 \oplus \ldots \oplus R_n \) for some integer \( n \in \mathbb{N} \) and some local Artinian rings \( R_1, \ldots, R_n \). Suppose that \( n > 2 \). Since \( (R_1 \oplus 0 \oplus \ldots \oplus 0)(0 \oplus R_2 \oplus 0 \oplus \ldots \oplus 0) = (0) \), and \( \AG(R) \) is a bipartite graph, \( R_1 \oplus 0 \oplus \ldots \oplus 0 \) and \( 0 \oplus R_2 \oplus 0 \oplus \ldots \oplus 0 = (0) \) are in different parts of vertices, say in \( V_1 \) and \( V_2 \), respectively. Now \( 0 \oplus \ldots \oplus R_n \) is adjacent to both above vertices and so \( 0 \oplus \ldots \oplus R_n \in V_1 \cap V_2 = \emptyset \), a contradiction. Thus \( n \leq 2 \). Also Theorem 2.3 implies that either there exist two distinct prime ideals \( P_1 \) and
$P_2$ of $R$ such that $P_1 \cap P_2 = (0)$ or $\mathbb{A}G(R)$ is a star graph. Let $n = 2$ and suppose that there exist two distinct prime ideals $P_1$ and $P_2$ of $R$ such that $P_1 \cap P_2 = (0)$. Note that $P_1 = Q_1 \oplus Q_2$ for some ideals $Q_1$ of $R_1$ and $Q_2$ of $R_2$. But $(0 \oplus R_2)(Q_1 \oplus Q_2) = (0)$ which implies $R_2Q_2 = (0)$ and from there $Q_2 = (0)$. Now, since $(R_1 \oplus (0))(0 \oplus R_2) = (0) \subseteq P_1$, we should have $(R_1 \oplus (0)) \subseteq P_1$ and so $Q_1 = R_1$. In other words, $R_1$ is a field. Similarly, we can prove that $R_2$ is a field, too. If $n = 2$ and $\mathbb{A}G(R)$ is a star graph, say with $|V_1| = 1$. It is evident that the only vertex of $V_1$ is $R_1 \oplus (0)$. Also, since $R_2$ is an Artinian local ring, $Z(R_2) = J(R_2)$ is an ideal of $R_2$. But $(0) \oplus R_2 \in V_2$ and so $R_2 \subseteq Z(R_2)$, i.e., $R_2$ is a field. Thus anyway, if $n = 2$, $R = F_1 \oplus F_2$. Let $n = 1$. Then $Z(R) = J(R)$. If there exist two distinct prime ideals $P_1$ and $P_2$ of $R$ such that $P_1 \cap P_2 = (0)$, then $R \cong R/P_1 \oplus R/P_2$ and since $R$ is Artinian, $R/P_1$ and $R/P_2$ are fields and we are done.

$(4) \Rightarrow (2)$ is clear. □

In [4, Corollary 2.4], it is shown that for a reduced ring $R$, $\mathbb{A}G(R)$ is a star graph if and only if $R \cong F \oplus D$, where $F$ is a field and $D$ is an integral domain. It follows that $R$ has exactly two minimal primes $F \times (0)$ and $(0) \times D$. Also, for a reduced ring $R$, $\chi(\Gamma(R)) = 2$ if and only if $R$ has exactly two minimal primes (see [3, Theorem 3.8]). Thus by using these facts and Theorem 2.3, we have the following interesting result for reduced rings.

**Corollary 2.5.** Let $R$ be a reduced ring. Then the following statements are equivalent:

1. $\chi(\mathbb{A}G(R)) = 2$.
2. $\chi(\Gamma(R)) = 2$.
3. $\mathbb{A}G(R)$ is a bipartite graph.
4. $\mathbb{A}G(R)$ is a complete bipartite graph.
5. $R$ has exactly two minimal primes.

One can easily see that for every ring $R$,

$\chi(\Gamma(R)) = 2 \implies \chi(\mathbb{A}G(R)) = 2$.

But the following example shows that in general, if $R$ is not reduced,

$\chi(\mathbb{A}G(R)) = 2 \nRightarrow \chi(\Gamma(R)) = 2$.

**Example 2.6.** Let $R = \mathbb{Z}_{p^3}$ where $p$ is a positive prime number. Then it is easy to check that $\chi(\mathbb{A}G(R)) = 2$, but $\chi(\Gamma(R)) \geq p$. 

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The following proposition shows that if $\chi(\Gamma(R))$ is finite, then $\chi(\mathbb{AG}(R))$ is also finite.

**Proposition 2.7.** Let $R$ be a ring. If $\mathbb{AG}(R)$ has an infinite clique, then $\Gamma(R)$ has an infinite clique.

**Proof.** Suppose that $C$ is an infinite clique of $\mathbb{AG}(R)$. Let $\Upsilon$ be the set of all vertices $I$ of $C$ such that $I^2 = (0)$.  

Case 1: If $\bigcup \Upsilon$ is infinite, then for every $x, y \in \Upsilon$, $xy = (0)$ and so $\Gamma(R)$ has an infinite clique. 

Case 2: If $\bigcup \Upsilon$ is finite. Then there is finitely many vertices $I$ of $C$ such that $I^2 = (0)$. Thus $C \setminus \Upsilon$ is infinite. W e claim that in this case, for every $I \in C \setminus \Upsilon$, $I \not\in \bigcup \phi \in \Phi J_\phi$, where $\{J_\phi | \phi \in \Phi\}$ is an arbitrary subset of $C \setminus \Upsilon$. Let $\{I_i | i \in \mathbb{N}\}$ be a subset of distinct vertices of $C \setminus \Upsilon$. Since $I_2 \not\subset I_1$, there exists $x_2 \in I_2 \setminus I_1$ such that $x_2 \neq x_1$. Again, since $I_3 \not\subset (I_1 \cup I_2)$, there exists $x_3 \in I_3 \setminus (I_1 \cup I_2)$ such that $x_3 \neq x_1, x_2$. Continuing this way, we can get an infinite clique of $\Gamma(R)$. □

We are going to characterize reduced rings $R$ for which $\chi(\mathbb{AG}(R))$ is finite. To see that, we need some prefaces.

**Lemma 2.8.** Let $R$ be a reduced ring such that $\mathbb{AG}(R)$ does not have an infinite clique. Then $R$ is has ACC on ideals of the form $\text{Ann}(I)$ where $I$ is an ideal of $R$.

**Proof.** Let $Ann(I_1) \subset Ann(I_2) \subset Ann(I_3) \subset ...$ be a chain in $\mathbb{A}(R)$. Clearly, $I_i Ann(I_{i+1}) \neq (0)$ for each $i \geq 1$. Thus for each $i \geq 1$, there exists $x_i \in I_i$ such that $x_i Ann(I_{i+1}) \neq (0)$. Let $J_i = x_i Ann(I_{i+1}), i = 2, 3,...$. Then if $i \neq j$, $J_i \neq J_j$ because if $J_i = J_j$, then $J_i^2 = J_j^2 = 0$, contradiction. □

**Lemma 2.9.** ( [3, Lemma 3.6.]) Let $P_1 = Ann(x_1)$ and $P_2 = Ann(x_2)$ be two distinct elements of $\text{Spec}(R)$. Then we have $x_1 x_2 = 0$.

**Theorem 2.10.** For a reduced ring $R$, the following statements are equivalent:

1) $\chi(\mathbb{AG}(R))$ is finite.

2) $d(\mathbb{AG}(R))$ is finite.

3) $\mathbb{AG}(R)$ does not have an infinite clique.

4) $R$ has finite number of minimal prime ideals.
Proof. (1) ⇒ (2) ⇒ (3) is clear.
(4) ⇒ (1). Let \((0) = P_1 \cap P_2 \cap ... \cap P_k\), where \(P_1, ..., P_k\) are prime ideals.
Define a coloring \(f\) on \(\mathbb{A}(R)^*\) by putting \(f(J) = \min\{n \in \mathbb{N} : J \not\subseteq P_n\}\) for \(J \in \mathbb{A}(R)^*\). Then \(\chi(\mathbb{A}(R)) \leq k\).
(3) ⇒ (4). Suppose that \(R\) doesn’t have an infinite clique. So by Lemma 2.8, \(R\) has ACC on ideals of the form \(\text{Ann}(I), I \in I(R)\). Thus the set \(\{\text{Ann}(x) : 0 \neq x \in R\}\) has maximal ideals, and it is easy to see that these are prime ideals of \(R\).
Let \(\text{Ann}(x_\lambda), \lambda \in \Lambda\) be the different maximal members of the family \(\{\text{Ann}(x) : 0 \neq x \in R\}\). By Lemma 2.9, the index set \(\Lambda\) is finite. Pick \(x \in R, x \neq 0\). Then \(\text{Ann}(x) \subseteq \text{Ann}(x_\lambda)\) for some \(\lambda \in \Lambda\). Now \(x_\lambda x \neq 0\) because otherwise, \(x_\lambda^2 = 0\), which is a contradiction. Thus \(x \not\in \text{Ann}(x_\lambda)\). This means \(\bigcap_{\lambda \in \Lambda} \text{Ann}(x_\lambda) = (0)\). \(\square\)

Now we are in position to prove Conjecture 0.1 in the case of reduced rings. In fact the next corollary is an immediate conclusion of Theorem 2.10.

Corollary 2.11. Let \(R\) be a reduced ring. Then \(\chi(\mathbb{A}(R)) = \text{cl}(\mathbb{A}(R))\).
Moreover, if \(\chi(\mathbb{A}(R))\) is finite, then \(R\) has a finite number of minimal primes, and if \(n\) is this number, then
\[
\chi(\mathbb{A}(R)) = \text{cl}(\mathbb{A}(R)) = n.
\]

Adding Corollary 2.11, and [3, Theorem 3.9.] to Theorem 2.10, the next corollary is indisputable.

Corollary 2.12. Let \(R\) be a nonzero reduced ring. Then \(\chi(\mathbb{A}(R))\) is finite if and only if \(\chi(\Gamma(R))\) is finite. Furthermore, if they are finite, then
\[
\chi(\mathbb{A}(R)) = \chi(\Gamma(R)).
\]

Now, we will have a characterization of Noetherian rings \(R\) for which \(\chi(\mathbb{A}(R)) < \infty\). Before that, we need the following lemma.

Lemma 2.13. Let \(R\) be a ring. If \(\text{cl}(\mathbb{A}(R))\) is finite, then
(i) for every nonzero ideal \(I\) of \(R\) with \(I^2 = (0)\), \(I\) has finite number of \(R\)-submodules; and
(ii) for every element \(a \in \text{Nil}(R)\), \(Ra\) has finite number of \(R\)-submodules.
Proof. (i) Evident.
(ii) Suppose that $a \in \text{Nil}(R)$, $a^n = (0)$ and $n$ is the least integer with this property. If $n = 2$, (i) shows that $Ra$ has finite number of $R$-submodules. Suppose that $n > 2$. Let $k = n/2$ if $n$ is an even integer and $k = (n + 1)/2$, if $n$ is an odd one, and let $b = a^k$. Then $(Rb)^2 = (0)$. Thus by (i), $Rb$ has finite number of $R$-submodules. Assume that $Ra$ has infinite number of $R$-submodules. Define

$$\varphi : Rb \rightarrow Ra$$

$$rb \mapsto ra$$

It is easy to check that $\varphi$ is a ring epimorphism and so $Rb/\ker(\varphi) \cong Ra$. Since $Ra$ has infinite number of $R$-submodules, so $Rb/\ker(\varphi)$ and from there $Rb$ has infinite number of $R$-submodules, a contradiction. □

Theorem 2.14. Let $R$ be a Noetherian ring. Then $\text{cl}(\mathcal{AG}(R))$ is finite if and only if for every ideal $I$ of $R$ with $I^2 = (0)$, $I$ has finite number of $R$-submodules.

Proof. ($\Rightarrow$) is by Lemma 2.13 (i).

($\Leftarrow$) Suppose that for every ideal $I$ of $R$ with $I^2 = (0)$, $I$ has finite number of $R$-submodules. Let $C$ be the largest clique in $\mathcal{AG}(R)$, and let $\Upsilon$ be the set of all vertices $I$ of $C$ with $I^2 = (0)$. Then $J = \sum_{I \in \Upsilon} I$ is again a vertex of $C$ and $J^2 = (0)$. So by our hypothesis, $J$ has finite number of $R$-submodules. But if $I \in \Upsilon$, every $R$-submodule of $I$ is an $R$-submodule of $J$. Thus for every $I \in \Upsilon$, $I$ has finite number of $R$-submodules and specially $\Upsilon$ has finite element. We claim that $C \setminus \Upsilon$ has infinite member, too. Suppose that $\{I_1, I_2, \ldots\}$ is an infinite subset of $C \setminus \Upsilon$. Consider the chain $I_1 \subseteq I_1 + I_2 \subseteq I_1 + I_2 + I_3 \subseteq \ldots$. Since $R$ is Noetherian, there exists $n \in \mathbb{N}$, such that $I_1 + \ldots + I_n = I_1 + \ldots + I_n + I_{n+1}$, i.e. $I_{n+1} \subseteq I_1 + \ldots + I_n$. Since by our choice of $C$, $I_1 + \ldots + I_n \in C$, $I_{n+1}$ is adjacent to $I_1 + \ldots + I_n$ and thus $I_{n+1}^2 = (0)$, a contradiction. Thus $C$ has finite number of vertices and from there, $\text{cl}(\mathcal{AG}(R))$ is finite. □

By Theorem 2.10, one may naturally guess the following statements for the general case. We close this section with this conjecture.

Conjecture 2.15. Let $R$ be a ring. If $\mathcal{AG}(R)$ does not have an infinite clique, then $\text{cl}(\mathcal{AG}(R))$ is finite.
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