Thermodynamics of a finite system of classical particles with short and long range interactions and nuclear fragmentation

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Abstract

We describe a finite inhomogeneous three dimensional system of classical particles which interact through short and (or) long range interactions by means of a simple analytic spin model. The thermodynamic properties of the system are worked out in the framework of the grand canonical ensemble. It is shown that the system experiences a phase transition at fixed average density in the thermodynamic limit. The phase diagram and the caloric curve are constructed and compared with numerical simulations. The implications of our results concerning the caloric curve are discussed in connection with the interpretation of corresponding experimental data.

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1 Introduction

Up to now, our knowledge about the properties of highly excited nuclei generated by means of energetic particle or heavy ion collisions is still rather limited. In practice, the only well-established information concerns universal scaling properties \(^1\) of fragment size multiplicities which are observed in the final stage of the reaction. These observables can be reproduced by means of simple, classical microscopic models. They can be interpreted as a sign for the existence of a phase transition of percolation type.

One major handicap towards further understanding of excited nuclear matter is concerned with the distribution of the available energy among the nucleons in the excited system. At present it is not possible to say whether the highly excited expanding system can be considered to be in some thermal equilibrium or not. We know from specific experimental results that the nucleus is able to pick up and distribute huge amounts of energy, more or less equally, over internal degrees of freedom, at least in specific experimental circumstances. Theoretical interpretations in terms of so called dynamic models \(^2\) show that this is however not necessarily fully established.

If one nevertheless assumes that equilibrium is reached, the central point of interest concerns the thermodynamic properties of finite and infinite nuclear matter up to very high energies and its connection with the universality properties of fragment distributions. During the last years several interesting attempts have been performed in this direction on the theoretical side. They rest on generic models, the Ising \(^3\), the lattice gas and related models \([4-7]\) and mean field approaches \([8]\). They constitute interesting attempts towards a unified description of the global thermodynamics and fragment generation in nuclear systems.

The work which we develop here takes place in the same framework. In practice we work out a classical, fully analytic realistic 3-dimensional model in the framework of the grand canonical ensemble. The analytic treatment allows for a clear insight into the correlations between the microscopic structure and the global thermodynamic properties of finite systems and their thermodynamic limit. It is able to incorporate both short and (or) long range interactions. Hence it can be used to take the Coulomb interaction into account and to investigate the thermodynamic properties of other classical microscopic systems. Finally many-body correlation functions which allow to define fragment sizes can be worked out analytically.

In the sequel we shall essentially develop the expression of the partition function and its properties, and work out the relations between energy, temperature and density of the system. Indications about the determination of correlation functions will be given at the end of the letter and developed in the future.

2 The model: grand canonical partition function for a system with short range interactions

We consider a 3-dimensional system of classical particles confined in a volume \(V = (Nd)^3\) where \(N^3\) is the number of cubic cells of linear dimension \(d\) (the extension to a parallelepipedic volume \(V = N_1N_2N_3d^3\) is straightforward). Each cell \(k\) is occupied by a particle \((s_k = 1)\) or empty \((s_k = 0)\). Nearest neighbour particles interact by means of an attractive...
2-body potential of strength \(-V_0\) \((V_0 > 0)\). The Hamiltonian of the system reads

\[
H_0 = \sum_{k=1}^{A} \frac{p_k^2}{2m} - V_0 \sum_{\langle n \cdot n \rangle_1} s_i s_j \tag{1}
\]

where \(A\) is the total number of particles and \(\langle n \cdot n \rangle_1\) stands for “nearest-neighbour sites \((i, j)\)” in each space direction.

We shall show that this is the inhomogeneous counterpart of the homogeneous analytic model which we developed and discussed in ref. [20]. The model \((1)\) can be brought into a form which looks more closely like the Ising model by introducing the spin variable

\[
\sigma_k = 2s_k - 1 \tag{2}
\]

Up to the kinetic term, \(H_0\) then becomes an Ising Hamiltonian in a magnetic field

\[
H_0 = \sum_{k=1}^{A} \frac{p_k^2}{2m} - \left(\frac{V_0}{4}\right) \left[ \sum_{\langle n \cdot n \rangle_1} \sigma_k \sigma_\ell + 6 \sum_{k=1}^{N^3} \sigma_k + 3N^3 \right] \tag{3}
\]

The conservation of the number of particles \(A\) is expressed through the constraint

\[
\frac{1}{2} \sum_{k=1}^{N^3} (1 + \sigma_k) = A \tag{4}
\]

In order to get an exactly solvable model we replace the Ising-like model \((3)\) by the spherical model [9]. It has been known since a long time that the spherical model is exactly solvable in any number of dimensions, in the presence of symmetry-breaking fields, for very general (including long-range) interactions and even in the presence of competing interactions or disorder [9-17].

Yet it does display, in more than two and less than four dimensions, a second order phase transition with non-trivial (i.e. non mean field) critical exponents. The spherical model represents therefore an ideal test bench on which complicated concepts can be checked analytically.

To introduce it, we replace the Ising spins \(\sigma_k = \pm 1\) in \((3)\) by continuous real variables \(\sigma_k \in [-\infty, +\infty]\) and constrain these variables through the relation

\[
\sum_{k=1}^{N^3} \sigma_k^2 = N^3 \tag{5}
\]

Practically it is easier to implement the constraints \((4), (5)\) only in the mean through Lagrange multipliers \(\lambda\) and \(\mu\). In the thermodynamic limit \(N \to \infty\) it is known that this leads to the same result [18]. The mean spherical Hamiltonian can then be written as

\[
H = H_0 + \lambda \sum_{k=1}^{N^3} \sigma_k^2 + \left(\frac{\mu}{2}\right) \left( \sum_{k=1}^{N^3} \sigma_k + N^3 \right) \tag{6}
\]

and the partition function becomes

\[
Z = \int_{-\infty}^{+\infty} \prod_{k,i} dp_k^i \int_{-\infty}^{+\infty} \prod_k d\sigma_k \exp(-\beta H). \tag{7}
\]
We remark that going over from the Ising model to the spherical model implies the replacement
\[
\prod_{k=1}^{N^3} \sum_{\{\sigma_k=\pm 1\}} \rightarrow \prod_{k=1}^{N^3} \int_{-\infty}^{+\infty} d\sigma_k.
\]
The quantity \( p_i^k \) is the momentum of particle \( k \) in the space direction \( i \) \((i = 1, \ldots, n)\). In the sequel we fix \( n = 3 \) and assume periodic boundary conditions in the 3 space directions.

The constraints
\[
-\beta^{-1} \partial \ln Z / \partial \lambda = N^3 \\
-\beta^{-1} \partial \ln Z / \partial \mu = A
\] (8)
fix the parameters \( \lambda \) and \( \mu \). This completes the definition of the model.

At first sight the spherical constraint may appear bizarre, introducing an effective long-range interaction between the spins \( \sigma_k \). But this is only apparent and can be interpreted as follows. Consider an \( m \)-component spin variable \( S_a, a = 1, \ldots, m \) of unit length \( \sum_{a=1}^{m} S_a S_a = 1 \). The \( O(m) \) vector model with Hamiltonian \( H_{O(m)} = -J \sum_{<n,n>_1} \sum_{a=1}^{m} S_i^a S_j^a \) can be shown to converge to the spherical model in the limit \( m \rightarrow \infty \) [10] (for \( m = 1, 2, 3 \) one recovers the familiar Ising, XY and Heisenberg models respectively).

In our case, the procedure of replacing the model (1) by the spherical model (6-8) leads to an extension which describes the fact that the particles do not lie at constant distance from each other, since their centres in cells of volume \( d^3 \) can be found at random inside each occupied cell. This is in fact the essence of an earlier modelisation of the system [19, 20]. Allowing \( s_k \) (resp. \( \sigma_k \)) to vary continuously can be interpreted in practice as the description of a system for which \( \sigma_k = \pm 1 \) and an interaction \( V_{<n,n>_1} \) which takes values on the interval \([0, +\infty]\). But events with large value of \( V_{<n,n>_1} \) are strongly suppressed because the largest part of their contributions are generated by \( \sigma_k \)'s lying in a strip of values, due to the quadratic form of \( H \) in (7) and the constraint (5) on the volume. Our interpretation does not take care of the fact that the positions of a set of interacting particles are correlated and hence, so are the two-body strengths \( V_{<n,n>_1} \). However, we expect that this is not a major problem because of the fact that our calculations concern the partition function \( Z \) which delivers average quantities and hence these correlations should not show a strong effect.

In order to diagonalise the free energy in the exponent of \( Z \) we Fourier-transform \( \{\sigma_k\} \) to \( \{\tau_j\} \). In each space direction
\[
\tau_\ell = N^{-1/2} \sum_{k=1}^{N} e^{-i\ell k} \sigma_k.
\]
Then
\[
\sum_k \sigma_k \sigma_{k+1} = \sum_\ell e^{-i\ell} \tau_\ell \tau_{-\ell} \\
\sum_k \sigma_k^2 = \sum_\ell \tau_\ell \tau_{-\ell}
\] (9)
The quadratic form in \( \{\tau_\ell \} \) can be explicitly integrated and

\[
Z(\beta, A, N^3) \propto (2\pi m)^{3A/2} \cdot \beta^{-(3A+N^3)/2} \cdot \exp\left(-\beta \mu N^3/2 + 3\beta V_0 N^3/4\right) \\
\cdot \exp \left\{ \beta [(\mu - 3V_0)/2] N^3 / [4(\lambda - 3V_6/2)] \right\} \\
\cdot \prod_{\{i_1, i_2, i_3\}} \left[ \lambda - V_0(\cos \xi_{i_1} + \cos \xi_{i_2} + \cos \xi_{i_3})/2 \right]^{-1/2}
\]

(10)

with \( \xi_{i_k} = 2\pi i_k/N \) \((i_k = 0, 1, \ldots, N - 1)\) and the product runs over all possible values of \((i_1, i_2, i_3)\). The first term on the r.h.s. of (10) corresponds to the kinetic energy contribution.

We define

\[
M = -[1 + 2(\beta N^3)^{-1} \partial \ln Z/\partial \mu] = -(\mu - 3V_0)/[4(\lambda - 3V_6/2)]
\]

which is constant and related to the density through

\[
M = 2\rho/\rho_0 - 1
\]

where \( \rho_0 \) is the normal density of nuclear matter. Then equations (8) and (10) lead to

\[
\sum_{i=0}^{N^3-1} [\lambda(\beta) - V_0 \alpha_i/2]^{-1} = 2N^3 \beta(1 - M^2)
\]

with \( \alpha_i \equiv \alpha_{i_1 i_2 i_3} = \sum_{k=1}^{3} \cos \xi_{i_k} \)

and \( \mu(\beta) = 4[3V_0/2 - \lambda(\beta)]M + 3V_0 \).

(11)

Equation (11) is a kind of dispersion relation with \( N^3 \) poles. We look for the solution which corresponds to \( \lambda > V_0 \alpha_{\text{max}}/2 \) which is positive (\( \alpha_{\text{max}} = 3 \)) and guarantees that the integrals in the expression of \( Z \) make sense.

From (11) one can immediately see that in the thermodynamic limit \( (N \to \infty) \)

\( \lambda(\beta) \to V_0 \alpha_{\text{max}}/2 \).

This shows a singularity in the total free energy, even though the free energy per particle remains finite.

For \( M = 0 \) \((\rho/\rho_0 = 1/2)\), this transition is continuous and occurs at the critical temperature \([13]\)

\[
\beta_c = 2W_3(0)/V_0
\]

where \( W_3(0) = (1/2) \int_0^\infty e^{-3x} I_0^3(x)dx = 0.2527 \)

\( I_0(x) \) being a modified Bessel function \([21]\).

It is now possible to construct the caloric curve for fixed density. The energy per particle is given by

\[
\epsilon = \frac{3}{2\beta} - \frac{V_0}{A} \left( \frac{\partial \ln Z}{\partial \theta} \right) = 3 - V_0 \left\{ \frac{3N^3}{4A} [2M(M + 1) + 1] \\
+ \frac{1}{4A} \sum_{i=0}^{N^3-1} \frac{\alpha_i}{(\lambda \beta - \theta \alpha_i/2)} \right\}
\]

(12)
with $\theta = \beta V_0$. The first term in the expression corresponds to the kinetic energy contribution.

Contrary to the model introduced in ref \[13\] one sees that the solutions $(\lambda, \mu)$ obtained through relations (11) are of the same type for any value of $M$ because here $M$ is a constant quantity. Hence one concludes that the phase transition is of the same order for any fixed density.

The energy $\epsilon$ given by (12) is a continuous function of $\beta$. The specific heat $C_V$ reads

$$C_V = -\beta^2 (d\epsilon/d\beta) = 3/2 - (V_0/4A) \left\{ \sum_{i=0}^{N^3-1} \frac{\alpha_i}{(\lambda - V_0\alpha_i/2)} + \beta (d\lambda/d\beta) \sum_{i=0}^{N^3-1} \frac{\alpha_i}{(\lambda - V_0\alpha_i/2)^2} \right\}$$

(13)

This quantity is again continuous in $\beta$. It is in fact well known \[22, 9\] that the system shows a phase transition at $\beta_c(M)$ where $C_V \sim (\beta_c - \beta)^{-\alpha}$ and $\alpha = -1$. There appears however a discontinuity in the derivative of $C_V$ with respect to $\beta$ in the thermodynamic limit. This can indeed be guessed numerically in Fig. 1 for increasing sizes of the system. We also remark that the convergence towards the $N \to \infty$ limit appears to be relatively rapid, even around the critical point. For $\beta > \beta_c(M)$ the specific heat is constant.

3 Extension of the description to a system with short and (or) long range interactions

The model can be easily extended to the case where particles interact by means of short and (or) long range two-body interactions. Hence one can for instance investigate the effects of the Coulomb interaction in the excited nuclear system.

The Hamiltonian can be written in the general form

$$H = K + V = K - [V_{01} \sum_{<n\cdot n>_1} s_i s_j + V_{02} \sum_{<n\cdot n>_2} s_i s_j + \cdots$$

$$\cdots + V_{0n} \sum_{<n\cdot n>_n} s_i s_j] + \mu \sum_i s_i + \lambda \sum_i (2s_i - 1)^2$$

(14)

where $K$ is the kinetic energy and $<n\cdot n>_k$ stands for $k^{th}$ nearest neighbour in all space directions and

$$V_{01} = V_S + V_{\ell}^{(1)}$$
$$V_{02} = V_{\ell}^{(2)}$$
$$\vdots$$
$$V_{0n} = V_{\ell}^{(n)}$$

with $V_S > 0$ and $V_{\ell}^{(i)} < 0$ $(i = 1, \ldots, n)$. $V_S$ stands for the short range potential and $V_{\ell}^{(k)}$ for the long (Coulomb) contribution. We parametrise this strength in the following way

$$V_{\ell}^{(k)} = (Z_p/A)e^2/kd$$

(15)
where $Z_p$ is the total number of protons. Here again we introduce periodic boundary conditions on the variables $\sigma_i = 2s_i - 1$.

The present procedure smears out the charge over all existing particles. Hence the interaction is not taken into account exactly, but in the average over the occupied parts of the volume, since protons are not explicitly distinguished from neutrons. This should not affect strongly the calculations since one expects that the charges are effectively distributed approximately uniformly in the ratio of $Z_p/A$ over the set of occupied cells. It should also be mentioned that, as in the case of the short range interaction, the particles interact only along the three space directions and not the directions of the diagonals. These contributions can be approximately taken into account by a consequent renormalization of the interaction strength.

The grand canonical partition function can now be worked out by following the same procedure as before. Notice that the extension of the spin variables $\{\sigma_k\}$ to the continuum limit introduces, as previously in section 2, values of $|V_{0k}|$, in the interval $[0, +\infty]$. If one follows the interpretation developed there, the long range interaction in (13) can in principle take large values. However, like there, large values of $V_{0k}$ are suppressed because of the quadratic form of $H$ and the constraint (5). After some algebra one gets

$$ Z(\beta, A, N^3) \propto (2\pi m)^{3A/2} \cdot \exp(-\beta\mu N^3/2 + 3\beta W_0 N^3/4) \cdot \exp \left\{ \beta \left[ (\mu - 3W_0)/2 \right]^2 N^3/4(\lambda - 3W_0/2) \right\} \cdot \prod_{\{i_1, i_2, i_3\}} \left[ \lambda - \sum_{p=1}^n V_{0p}(\cos \xi_{i_1}^{(p)} + \cos \xi_{i_2}^{(p)} + \cos \xi_{i_3}^{(p)})/2 \right]^{-1/2} $$

with $\xi_{i_\ell}^{(p)} = \frac{2\pi N p i_\ell}{\lambda - \sum_{p=1}^n V_{0p}(\cos \xi_{i_1}^{(p)} + \cos \xi_{i_2}^{(p)} + \cos \xi_{i_3}^{(p)})/2}$

In the same way as before, $\lambda(\beta)$ and $\mu(\beta)$ are obtained through

$$ \sum_{\{i_1, i_2, i_3\}} \left[ \lambda(\beta) - \sum_{p=1}^n V_{0p}(\cos \xi_{i_1}^{(p)} + \cos \xi_{i_2}^{(p)} + \cos \xi_{i_3}^{(p)})/2 \right]^{-1} = 2N^3\beta(1 - M^2) $$

$$ M = -1 + 2A/N^3 $$

and

$$ \mu(\beta) = 4(3W_0/2 - \lambda(\beta))M + 3W_0 $$

The energy per particle $\epsilon(\beta)$ is the same as the one given by (12) with $V_0$ replaced by $W_0$,

$$ \epsilon = \frac{3}{2\beta} - \frac{1}{A} \cdot \sum_{p=1}^n V_{0p} \frac{\partial \ln Z}{\partial \theta_p} \quad \text{with} \quad \theta_p = \beta V_{0p} $$

and an expression similar to (13) for the specific heat $C_V$.

4 Results

The typical behaviour of the caloric curve for different densities is shown in Figs. 2-4 for the case of an attractive short range, an attractive long range and a combination of
an attractive short range with a repulsive long range interaction respectively. In each case, the energy increases with respect to the temperature and one does not observe the presence of a plateau, even though the description predicts a phase transition at every density. One observes simply some more or less pronounced kink in the slope of the curve which corresponds in practice to the temperature at which the transition occurs. This is observed for all types of 2-body interactions. As already mentioned in section 3, the reason for this smooth behaviour is related to the fact that in the spherical model the specific heat remains finite at the transition point even at the thermodynamic limit. Only the derivative of the specific heat shows a discontinuity at the transition points. From the knowledge of these transition points it is possible to construct the phase diagram in the \((T/T_c, \rho/\rho_0)\) plane, Fig. 5. Here \(T_c\) corresponds to the critical point at \(\rho/\rho_0 = 0.5\) where the usual spherical model undergoes a second order transition working at zero magnetic field \(h\) \([13]\). In the spherical model, the transition for \(\rho/\rho_0 \neq 0.5\) is of first order, owing to the discontinuity in the curve \(h\) vs \(M\) at \(T < T_c\). This means that when \(h = 0\), the system can pass from a magnetization \(+M\) to \(-M\). But in the present model we fix \(M\) to a certain value (constraint (8)), so flips from \(+M\) to \(-M\) are not possible here. Therefore, one expects a behaviour for every \(M\) similar to what happens at \(M = 0\). This is why we obtain the same smooth behaviour at every value of \(\rho/\rho_0\).

5 Numerical simulations in the framework of a cellular model

The present description is close to former numerical approaches (cellular model) which were introduced in order to study the thermodynamic properties and cluster content of an excited disordered system of particles \([19, 20]\). However in these simulations the system was chosen to be homogeneous \((\rho/\rho_0 = 1)\) and the numerical procedure did not imply that the system was in thermodynamic equilibrium. There was no sign for the existence of a thermodynamic phase transition.

We present here the results of numerical simulations which generalise in the framework of the canonical ensemble the former calculations. We consider a finite volume of space \(V\) in 3D made of unit cells of linear dimensions \(d = 1.8\,\text{fm} , V = N^3 d\). Each cell is either occupied or unoccupied, the total number of particles is \(A\) such that the normalized density \(\rho/\rho_0 = A/N^3\). The particles interact by means of a nuclear \([23]\) and Coulomb interaction.

For fixed density and temperature the equilibrium energy of the system is obtained by means of the following algorithm \([24]\). Assign an initial configuration with \(A\) particles in \(N^3\) cells. Select randomly with equal probability one of the two following procedures. In the first procedure one selects randomly a particle among the \(A\) occupied cells and chooses a new position of this particle inside its cell. This changes the energy of the system by an amount \(\Delta E\) which can be positive or negative. At this stage one applies the Metropolis Monte Carlo algorithm before returning to the procedure selection step. In the second procedure one selects randomly a particle \(\text{“a”}\) in one of the occupied cells and an empty cell \(\text{“b”}\) among the \(N^3 - A\) empty cells. One creates a particle at the center of the cell \(\text{“b”}\) with the same charge (proton or neutron) as the charge of the particle at \(\text{“a”}\). One destroys the particle at \(\text{“a”}\) and hence guarantees total mass and charge
conservation. This changes the energy of the system by an amount $\Delta E$ and one applies again the Metropolis Monte Carlo algorithm before going back to the procedure selection step. The average number of measurements is 4000 corresponding to ten times more Metropolis steps.

Typical results are shown in Fig. 6. The fluctuation of the results is not larger than the size of the dots. The energy per particle in the abscissa is the sum of the interaction and the kinetic energy which gives the trivial contribution $3T/2$. We checked the consistency of the procedure by means of numerical simulations of the kinetic energy of a system of free particles from a Boltzmann distribution. The fluctuations in the kinetic energy are negligible.

The behaviour of the caloric curve is very similar to the curve obtained by means of the analytic model, see Fig. 4. Indeed, the line which goes through the points corresponds to a calculation with the spherical model and $V_0 = 5.5 \text{ MeV}$ for the attractive short range interaction. It is worthwhile to notice that there is no sign for the existence of a phase transition. If however we interpret the results in the framework of the spherical model, the observed smooth behaviour would correspond indeed to a continuous transition reflected on the discontinuity in the derivative of the specific heat at the temperature predicted by the model.

6 Discussion and final remarks

The fact that the description leads to a continuous phase transition can a priori seem somewhat surprising. Indeed, other models which are related to spin systems like the lattice gas model clearly show a first order transition [4-7] which is not reproduced here. One could argue that we have ‘softened’ the strength of the transition by working in the framework of the spherical model, because it corresponds to the limit $m \to \infty$ in the $O(m)$ model. However, we remark that this is not the case. We obtain a continuous transition owing to the constraint fixing the magnetization, which prevents it from having the typical fluctuations which happen at the first order transition points in these other models. In fact, one can consider this constraint in the Ising model and see numerically that the whole transition line corresponds to a second order phase transition [25]. We have considered here the spherical model approximation in order to obtain analytical expressions for the different quantities and to incorporate short and long range interactions. The controversy between the order of the transition should work as an incentive to experimental physics to look for the best way to identify the specific properties of excited nuclei and nuclear matter and try to confirm or infirm the existence of a discontinuous phase transition [26-29].

The numerical simulations on the inhomogeneous model have been made in the framework of the canonical ensemble, with strictly fixed volume and number of particles. The fact that the results are in qualitative agreement with those obtained analytically in the framework of the grand canonical ensemble indicates that volume and particle fluctuations do not strongly affect the physical issue. It also shows that the interpretation of the continuum limit approximation leading to the spherical model description makes sense. Notice that in the experiment, volume and exact particle number can hardly be rigorously fixed. It would nevertheless be interesting to work out the dynamics in the framework of the microcanonical ensemble where the isolated nuclear system is treated rigorously [30].
Whether this is possible has to be investigated in practice.

Finally it would be of particular interest to work out averaged $n$-body correlation functions which allows to determine the fragment size distributions. This can in principle be done by means of a straightforward extension of the partition function $Z$ to a generating function by adding terms of the type $\exp \left[ \sum_{i=1}^{N^3} (h_i s_i + g_i (1 - s_i)) \right]$ into the sum over occupation numbers. Here $h_i$ and $g_i$ are arbitrary real constants and the probability of an isolated cluster with $(i_1, \ldots, i_n)$ occupied cells can be obtained by deriving this generating function with respect to specific parameters $h_i$ and $g_i$ (which fix the surface of the cluster) and putting the parameters to zero. The determination of the cluster formation probabilities would then lead to a complete description of the fragmentation of the considered system in the $(\epsilon, \rho, \beta)$ space.

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Figure caption

Figure 1: Specific heat calculated from (13) with $\rho/\rho_0 = 0.90$, $V_0 = 8$ MeV, and different sizes of the system ($N = 10, 20, \ldots, 50$).

Figure 2: Caloric curve obtained with an attractive short range (nearest-neighbour) interaction with strength $V_0 = 8$ MeV, $A = N^3 \rho/\rho_0 = 250$.

Figure 3: Caloric curve obtained with an attractive long range interaction ($n = 20$) with strength $W_0 = 8$ MeV.

Figure 4: Caloric curve for systems which experience both an attractive short range (nearest-neighbour) with strength $V_0 = 8$ MeV and a repulsive long range interaction with strengths given by (15) ($n = 20$). Here $Z_p/A = 0.394$ and $d = 1.8$ fm, see eq.(14).

Figure 5: Phase diagram $T/T_c$ vs. $\rho/\rho_0$. The transition line is read from the behaviour of the specific heat. The point $T/T_c = 1$ corresponds to the continuous phase transition in the thermodynamic limit of the usual spherical model.

Figure 6: Caloric curve obtained by means of Metropolis Monte Carlo simulations. The points correspond to numerical results for $N^3 = 216$, $\rho/\rho_0 = 0.5$, $A = 108$ and total charge number $Z_p = 47$. The full line corresponds to the results obtained through the spherical model with $V_0 = 5.5$ MeV. The arrow labelled $T_0$ indicates the phase transition temperature. See explanations and comments in the text.
Specific heat vs. temperature

$\rho/\rho_0 = 0.90$

$N = 10, 20, ... 50$

Fig. 1
Caloric curve for attractive short range interaction

Fig. 2
Caloric curve for attractive long range interaction

\[
\rho/\rho_0 = 0.30 \quad \rho/\rho_0 = 0.50
\]

\[
\rho/\rho_0 = 0.70 \quad \rho/\rho_0 = 0.90
\]

Fig. 3
Caloric curve for attractive and repulsive interaction

\[ \frac{\rho}{\rho_0} = 0.30 \quad \frac{\rho}{\rho_0} = 0.50 \]

\[ \frac{\rho}{\rho_0} = 0.70 \quad \frac{\rho}{\rho_0} = 0.90 \]

Energy (MeV)

Temperature (MeV)

Fig. 4
Fig. 5

- Attractive short range interaction
- Attractive long range interaction
- Attr. short range + Repulsive long range

$\tau/\tau_c$ vs $\rho/\rho_0$
Fig. 6

- Simulation
- Spherical model ($V_0 = 5.5$ MeV)

Temperature (MeV) vs. Energy (MeV)

$T_0$