STRING FIELD ACTIONS FROM $W_\infty^*$

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ABSTRACT

Starting from $W_\infty$ as a fundamental symmetry and using the coadjoint orbit method, we derive an action for one dimensional strings. It is shown that on the simplest nontrivial orbit this gives the single scalar collective field theory. On higher orbits one finds generalized KdV type field theories with increasing number of components. Here the tachyon is coupled to higher tensor fields.

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1. Introduction

Our recent investigations of the collective field formulation of d=1 string theory [1,2] emphasized a number of characteristic symmetry properties. These were described by a $w_\infty$–algebra of observables which was exhibited first at the classical and then at the quantum level. The theory is described by a scalar field representing the density of fermions, the chiral components of which obey decoupled equations arising from the Hamiltonian

$$\mathcal{H} = \int (\frac{\alpha^3}{3} - \frac{\alpha}{3}) dx + \int v(x)(\alpha_+ - \alpha_-) dx$$

with the Poisson structure: \( (u(1) \times u(1) \text{ current algebra}) \)

$$\{\alpha_+, \alpha_+\} = -\{\alpha_-, \alpha_-\} = \delta'(x - y)$$

$$\{\alpha_+, \alpha_-\} = 0$$

It was shown that the theory was classically Liouville–integrable for any potential \( v(x) \); moreover, due to the Poisson structure (2), a \((w_\infty)^2\) algebra structure arose, generated by

$$h^\pm_{m n} \equiv \int x^{m-1} \frac{\alpha^m_n - \alpha^{n-m}_m}{m-n} dx, \quad (m \geq n)$$

and reading

$$\{h^+_m, h^+_p\} = \left((m-1)q - (n-1)p\right) h^{n+q}_{m+p-2}$$

$$\{h^+, h^-\} = 0$$

For each chirality we therefore have a \(w_\infty\) algebra isomorphic to the Poisson bracket algebra of polynomials \(x^m p^{m-n}\).

The \((w_\infty)^2\)–algebra survives quantization. In the case when \(v(x) = x^2\), it allows the construction [2] of an exact set of discrete eigen-operators for the Hamiltonian (1). These were shown to also realize a \((w_\infty)^2\)-algebra. In short the \((w_\infty)^2\) algebra was established to be the spectrum–generating algebra [2,4] for discrete states of the d=1 string theory. A similar conclusion was reached directly from the study of vertex operators in [5,6].
It appears therefore that the $w_\infty$-algebra and its possible “quantum” deformations such as the algebra of differential operators with polynomial coefficients $\{x^n \partial_x^m\}$ (of which $w_\infty$ is the 1-contraction algebra), plays a basic symmetry role in any field theory of strings [1-10]. The collective field theory described above or equivalently the free fermion theory is just a first example of a more general structure.

We wish to present here a construction of an extended class of field theories, using the coadjoint orbit construction [11] applied to the algebra $w_\infty$, using its Lie-Baxter [15] structure. We will show that the theory corresponding to the lowest nontrivial orbit is precisely the collective tachyon field theory. This will clarify the Poisson structure (2), the origin of the Hamiltonian (1), and the associated representation of the $w_\infty$-algebra (3) which will be understood as a subalgebra of the algebra of observables on a symplectic manifold canonically generated by the coadjoint action of the algebra of fields $w_\infty$ on its dual $w_\infty^*$. The construction of more general orbits then leads to field theories with an increasing number of fields. These fields include the tachyon, which interacts with the other fields in a $w_\infty$-covariant way. In general one has candidate field theories for any number of these supplementary degrees of freedom.

2. The Coadjoint Orbit Construction

It is a well-known feature of dynamical systems having the dual $G^*$ of the Lie algebra $G$ of a group $G$ as phase space, that the canonical way of obtaining a symplectic manifold in this phase space (i.e. a bona fide phase space with non-degenerate Poisson brackets) is by constructing orbits of the coadjoint action of $G$ on $G^*$ [11]. Representing the action of $G^*$ on $G$ by the duality bracket $< >$, the coadjoint action is then given by:

$$\forall g \in G, \ g^* \in G^*; h \in G : < Ad^* h.g^*, g > = < g^*, Ad(h) \cdot g >$$

(5)

where the adjoint action is defined by the Lie bracket:

$$Ad(h) \cdot g = [g, h].$$

(6)

The integrated coadjoint action of the group $G$ on an initial point $b \in G^*$ then spans a coadjoint orbit in $G^*$.
We recall the following results:

1. On each coadjoint orbit there exists a natural symplectic 2-form, obtained by reduction of the canonical Kirillov-Poisson bracket (dual of the Lie bracket) [11].

2. This symplectic 2-form allows the construction of action functionals on the coadjoint orbit, exhibiting a global coadjoint-invariance plus a local, gauge-type coadjoint invariance group obtained from the $Ad^*$-invariance group of the original point $b$ in the orbit [12]. They read [13,14]:

$$S = \int d\lambda d\tau < Ad^* (g(\lambda, \tau)) \cdot b, [u_\lambda, u_\tau] > - \int_{\partial M_1} H d\tau$$  \hspace{1cm} (7)

where:

. $b$ is an arbitrarily chosen origin on the orbit $\mathcal{O}$

. $g(\lambda, \tau)$ is a 2-parameter element of $G$, spanning a 2d surface $M_1$ in the orbit $\mathcal{O}$.

. $u_\lambda, u_\tau$ are elements of $\mathcal{G}$ respectively describing the coadjoint infinitesimal action of $g^{-1}\partial_\lambda g$ and $g^{-1}\partial_\tau g$.

. $<>$ is the duality bracket, $[]$ the Lie bracket.

. $\tau$ will play the role of time for the Lagrangian (7).

. $\lambda$ is a supplementary variable, integrated over whenever possible, allowing to consider topologically non-trivial phase spaces varying from 0 to 1.

. $H$ is an arbitrary Hamiltonian function on $\mathcal{O}$.

. $\partial M_1$ is the boundary at fixed $\lambda = 1$ of the 2-dimensional surface $M_1$ in $\mathcal{O}$ spanned by $Ad^* g(\lambda, \tau) \cdot \sim Ad^* g(1, \tau) \cdot b$

3. The Lagrange equations of motion from the action $S$ in (7) are equivalent to the Hamiltonian equations of motion generated by $H$ on the boundary $\partial M_1$ [12].

The coadjoint orbit method is crucial in the Lie-Baxter construction of integrable systems, which we are now going to describe. A Lie-Baxter algebra is a Lie algebra.
endowed with two Lie algebra structures, namely:
\[
\text{ad}(x) \cdot y = [x, y] \quad \text{(usual Lie bracket)}
\]
\[
\text{ad}_R(x) \cdot y = [Rx, y] + [x, Ry] \quad (R - \text{commutator}); R \in \text{End}(\mathcal{G})
\]
(8)

In order for (b) to satisfy the Jacobi identity, \( R \) must obey the classical Yang-Baxter algebra [15]. In this case, it was shown that

1. \( Ad^* \)-invariant functions of \( \mathcal{G}^* \) are in involution under the two Poisson-Kirillov brackets generated by the two Lie algebra structures (8) [16].

2. The equation of motion on \( \mathcal{G}^* \) (or in fact on a \( Ad^*_R \)-coadjoint orbit of \( \mathcal{G}^* \)), given by a \( Ad^* \)-invariant Hamiltonian (typically a Gelfand-Dikii [17] hamiltonian \( \mathrm{Tr}(g^n)^{m/m}|_{g^* \in S^*} \)), has a Lax representation, and is Liouville–integrable due to 1).

Finally, there exists a well-known method, known as Adler-Kostant-Symes scheme, to obtain such Lie-Baxter algebras [16]. Whenever the Lie algebra \( \mathcal{G} \) can be decomposed as a vector space into a direct sum of two sub-algebras \( \mathcal{N} \) and \( \mathcal{K} \), the endomorphism \( P_N - P_K \) (projections on \( \mathcal{N}, \mathcal{K} \)) is a solution of the Yang-Baxter equation. In this case, one can compute the second coadjoint action of \( \mathcal{G}_R^* \) on \( \mathcal{G}_R^* \):

\[
\text{ad}_R^*(n_2 + k_2) \cdot (n_1^*) = \text{ad}^*(n_2) \cdot n_1^* + \text{ad}^*(k_2) \cdot k_1^*
\]
(9)

We now apply this formalism to the case when \( \mathcal{G} \) is a \( w_{\infty} \)-algebra.

3. Coadjoint Orbits in \( w_{\infty}^* \)- Algebra

The \( w_{\infty} \)-algebra \( \mathcal{W} \) is generated by the elements \( w^m_n \) with the commutator:

\[
[w^m_n, w^q_p] = ((m - 1)q - (p - 1)n)w^{n+q\, m+p-2}_m
\]
(10)

To any element of \( \mathcal{W} \) we associate the 2-variable function:

\[
h \in \mathcal{W} = \sum h^m_n w^m_n \rightarrow h(x, y) = \sum h^m_n x^{n+m-1} p^{m-1}
\]
(11)

The commutator of two elements \( h_1, h_2 \) of \( \mathcal{W} \) is represented under the correspondance (11) by the Poisson bracket of \( h_1(x, y) \) with \( h_2(x, y) \), generated by the one–dimensional symplectic structure \( \{x, y\} = 1 \).
The dual algebra $W^*$ is generated by dual elements $w_m^{*n}$; the functional representation reads:

$$\bar{h} \in W^* = \sum \bar{h}_m^n w_m^{*n} \rightarrow \bar{h}(x,y) = \sum \bar{h}_m^n x^{-n-m} p^{-m}$$

(12)

so that the duality bracket $<\bar{h}|h>$ becomes the contour integral

$$\oint dx \oint dy \quad \bar{h}(x,y)h(x,y).$$

(13)

from where the coadjoint action immediately becomes, in the functional representation

$$Ad^*(h(x,y)) \cdot \bar{k}(x,y) = \{\bar{h},k\}$$

(14)

This allows to identify $Ad$ and $Ad^*$ under the identification $w_m^{*n} = w_{-m-s}^{n}$. Now we have an obvious Adler-Kostant-Symes decomposition of $W$ as:

$$W = W^+ \bigoplus W^-$$

$$W^+ = \{w_m^n, m \geq 1\}$$

$$W^- = \{w_m^n, m \leq 0\}$$

(15)

The Lie-Baxter algebra structure reads:

$$[w_1^+ + w_1^-, w_2^+ + w_2^-] = [w_1^+, w_1^+] + [w_1^-, w_2^-]$$

(16)

The identification of $W$ and $W^*$ becomes an identification between $(W^+)^*$ and $W^-$, and $(W^-)^*$ and $W^+$. The coadjoint action then reads:

$$Ad_R^*(w_1^+ + w_1^-) \cdot (w^+ + w^-) = [w_1^+, w^-]_- + [w_1^-, w^+]_+$$

(17)

We shall now restrict ourselves to coadjoint orbits in $W^+ \simeq (W^-)^*$. The coadjoint action (14) reduces to the projected (standard) coadjoint action of $W^- \sim [W^-, W^+]_+$. Let us first describe in detail the simplest non-trivial example of a coadjoint orbit. It turns out to correspond to the string collective field theory [1,2]. Consider the subset of $(W^-)^*$ parametrized by the functions:

$$h(x,y) = y^2 + u(x)$$

(18)

$u(x)$ being any function of $x$, formally $u = \sum_{n \in \mathbb{Z}} u^{(n)} x^n$. 
We now apply the construction scheme described in Section 2:

(a) The set \( h(x, y) \) parametrizes a coadjoint orbit of \( W^+ \). Indeed, the coadjoint action of \( W^- \) under (14) reads:

\[
Ad^* (W^-) \cdot h = \{ y^2 + u(x), y^{-1}v_1(x) + y^{-2}v_2(x) + \cdots \}_+ = -2 \frac{\partial v_1}{\partial x}
\]  

We see that \( u(x) \) transforms into \( u(x) - 2v_{1,x} \), hence any function \( u(x) \) can be reached from a given \( u_0(x) \) provided that \( \int dx \ u_0(x) = \int u(x) \). We recover also the argument that any integral of the form \( \int dx \frac{\partial v}{\partial x} \equiv 0 \) for any meromorphic function \( v(x) \); this argument was used [1,2] in our derivation of the \( w_\infty \) - algebra.

(b) The representatives of \( g^{-1}\partial_\tau g \) and \( g^{-1}\partial_\lambda g \in W^- \), defined in (7), can be explicitly evaluated. Indeed, the coadjoint action simply translates into a “deformation” of \( u(x) \) as \( u(x, \lambda, \tau) \). In particular it follows from (19) that

\[
\frac{\partial u}{\partial \tau, \lambda} = -2 \frac{\partial v^\tau_1}{\partial x}
\]  

where \( g^{-1}\partial_{\tau,\lambda} g \sim y^{-1}v_{1}^{(\tau,\lambda)}(x) + \cdots \) Hence, inside (7):

\[
u_\tau \equiv \frac{1}{2}y^{-1}\partial_{x}^{-1}\frac{\partial u}{\partial \tau} + \cdots
\]

\[
u_\lambda \equiv \frac{1}{2}y^{-1}\partial_{x}^{-1}\frac{\partial u}{\partial \lambda} + \cdots
\]  

(21)

(c) The action \( S \) (7) therefore becomes:

\[
S = \int dx \ d\lambda \ dt \ \frac{1}{4} \left( \partial_x^{-1} u_\tau u_\lambda - \partial_x^{-1} u_\lambda u_\tau \right) - H(u)
\]

Introducing the 1+1 dimensional field \( u(x, \tau) \equiv u(x, \lambda = 1, \tau) \) we have after integration

\[
S = -\frac{1}{4} \int dx \ dt \ u \partial_x^{-1} \frac{du}{dt} - H(u).
\]  

(22)

From (22), the canonical Poisson structure of any field theory on this orbit is
easily read:

\[ \Pi_u \equiv \frac{\partial L}{\partial \dot{u}} = \frac{1}{4} \partial_x^{-1} u \Rightarrow \{u(x), u(y)\} = 4\delta'(x - y) \] (23)

We have recovered the Poisson structure (2) of the collective field theory. Now it follows from the Lie-Baxter algebra construction that there exists a set of commuting Hamiltonians on the \( Ad_R^* \)-invariant orbits, i.e. the \( Ad^* \)-invariant functions. Such functions are easily obtained, once one introduces the \( Ad^* \)-invariant trace:

\[ \text{Tr}(w \in W) = \int dx dy w(x, y) \] (24)

Its crucial \( Ad^* \)-invariance properly follows from:

\[ \text{Tr}([w_1, w_2]) = \int dx dy \{w_1, w_2\} = 0 \text{ by part integration.} \]

It can therefore be used to define \( Ad^* \)-invariant functions of \( W^+ \), namely the equivalent of Gelfand-Dikii hamiltonians [17]. \( \text{Tr} L^{p/n} \) for \( L \sim y^n + \cdots \) in \( W^\infty \). In particular, on the orbit in \( W^+ \sim y^2 + u(x) \), the \( Ad^* \)-invariant functions reduce to:

\[ I_k = \text{Tr} \left( [y^2 + u(x)]^{2k+1/2} \right) \sim \int u^{k+1}(x) dx \] (25)

In this way we recover naturally the hierarchy of commuting Hamiltonians for the zero-potential case.

The first nontrivial Hamiltonian with \( k = 2 \) gives the cubic interaction and the Lagrangian

\[ \mathcal{L} = \frac{1}{4} (u_+ \partial_x^{-1} \dot{u}_+ - u_- \partial_x^{-1} \dot{u}_-) - \frac{1}{12\sqrt{2}} (u_+^3 - u_-^3) \] (26)

where we have added an identical \( w_\infty \)-coadjoint action for the other chirality component. Defining \( u^\pm = \sqrt{2}(\Pi \pm \phi) \) one ends up with the cubic collective field theory.

\[ \mathcal{L}_{\text{coll}} = \Pi \dot{\phi} - \left( \frac{1}{2} \Pi \dot{x} \phi \Pi_x x + \frac{1}{6} \phi^3 \right) \] (27)

Let us now comment on the Poisson structure (23). It is easy to check the consistency of the procedure by comparing this structure with the restriction to the orbit of
the Kirillov-Poisson structure (for $W$ or $W_R$, since there is no difference when one considers solely the subalgebra $W^+$). The $KP$ bracket on $W_*$ reads [3],

$$\{h^*_m, h^*_p\} = ((p-1)n - (m-1)q) h^{*n+q}_{m+p-2} \quad (28)$$

Reduced to the orbit $y^2 + u(x)$, which corresponds to the constraint $\{h^2_2 = 1, h^p_m = 0 \forall m \neq 0, 2\}$, it gives:

$$\{h^n_0, h^m_0\} = (m-n) h^{m+n}_{-2} = (m-n) \delta_{m+n,2} \quad (29)$$

d thereby identifying the canonically conjugate variables on the orbit as $\{h^n_0, h^2_0-n\}$ for $n > 1$}, plus the invariant $h^n_0 \sim \int u \, dx$. The coordinates on the orbit $h^n_0$ are the coefficients of the formal expansion of $u$ as $u = \sum h^n_0 x^{-n}$, or $h^n_0 = \oint x^{n-1} u(x) \, dx$. This is a very simple realization of the Darboux theorem for this symplectic manifold. From (29) and using:

$$\sum_{n \geq 0}^{n \leq 0} n x^{-n-1} y^{n-1} = \Delta(x, y) = \delta'(x - y) \quad \text{(under $\oint$ integration)} \quad (30)$$

the Poisson structure (23) follows.

The orbit is now explicitly described as a direct sum of $\mathbb{N}$-labeled phase spaces, therefore the algebra of functions on the orbit or, loosely speaking, the algebra of observables in the corresponding quantum theory, is a “product” of $w_\infty$-algebras, precisely,

$$\mathcal{W} = \bigoplus_{n=1}^{\infty} \bigotimes_{i=1}^{n} w_{(i)} \quad (31)$$

where

$$w_{(i)} = \{x_{(i)}^m, p_{(i)}^n \mid m, n \in \mathbb{Z}\} \quad (32)$$

denotes the $w_\infty$ algebra associated to the $i^{th}$ phase space. We recognize here the structure of the matrix model where the eigenvalues $\lambda_i$ and their conjugates $p_i$ generate a product of $w_\infty$ algebras. Our field-theoretical $(w_\infty)^2$-algebra $\int x^m \alpha^m_{n}$ is now
understood as being a subalgebra of this large $W$-algebra. $W$ would in fact be described by all generators of the form: \( \int x^m \alpha^{m_1} \alpha^{m_2} \cdots \alpha^{(p)m_p}, p = 1 \cdots \infty \). The “fundamental” algebra or algebra of fields is in fact the original $w_\infty$ algebra. Note that this construction indicates that the \( \int dx \) symbol in the construction of the algebra \( (\int x^m \alpha^{n-m}) \) must be understood as $\oint$, and it was therefore allowed to suppress all integrated terms. There is one exception, however, when one computes

\[
\{ \int x^n \alpha, \int x^{-n} \alpha \} = \int x^n y^{-n} \delta' dxdy = \int x^{-1} dx \equiv 1
\]

(and not 0, but \( x^{-1} = \partial^{-1}_x \ln x \) and \( \ln x \) is not a meromorphic function)

Any other commutator, however, was evaluated in \([1,2]\) using formal partial integrations, whenever the final order in $\alpha$ was non–zero (which is always consistent with the definition of $\int dx$ as $\oint$, except in the above case, when one considers generators with a positive power of $\alpha$), and is therefore correctly given by the standard (4)-formula.

We have now exhibited the structure of the collective string theory in $d = 1$ as a coadjoint construction on the algebra $w_\infty$. We have found that it is given by the lowest orbit, defined as $y^2 + u(x)$. It is clearly interesting to study the more general orbits, which lead to increasing number of fields and the extension of the simple tachyon collective field theory.

4. High-Dimensional Orbits and Extended Lagrangians

The higher–dimensional orbits follow from the same type of construction which we have extensively described in the lowest–dimensional case. We shall now describe the main characteristic features and give some typical examples for Lagrangian field theory associated with higher $w_\infty$ hierarchy.

First of all, a set in $W^*_-$ represented by functions of the form $F_n(y, z) = y^n + u^{(n-2)} y^{n-2} + \cdots + u^{(0)}$ is invariant under $ad^*(W_-)$. One ignores the possible term $u^{(n-1)} y^{n-1}$ since it is left invariant by the coadjoint action and is therefore irrelevant in the description of a coadjoint orbit. The above manifold, with arbitrary functions $u^{(n-2)} \cdots u^{(0)}$ (i.e. arbitrary coordinates \( \{ h_{-n}^-, \cdots, h_0^- \} \) on the basis $w_{-n}^*$ of $W_{-n}^*$), is a coadjoint orbit (up to a finite number of degrees of freedom corresponding to
normalization constants $\sim \int u^{(p)}$. Indeed, the coadjoint action reads:

$$ad^*W_\cdot F_n = \{y^n + u^{(n-2)} y^{n-2} + \cdots + u^{(0)}, y^{-(n+1)} b^{(1)} + \cdots y^{-n+1} b^{(n-1)} + \cdots\}$$

$$= n y^{n-2} \frac{\partial b^{(1)}}{\partial x} + n y^{n-3} \frac{\partial b^{(2)}}{\partial x} + \cdots \left( n \frac{\partial b^{(n-1)}}{\partial x} + \cdots 2 \frac{\partial u^{(2)}}{\partial x} b^{(1)} \right)$$

(34)

hence the form of the function $F_n$ is $ad^*$-invariant, and all functions $u^{(n)}$ can be reached by $ad^*$-action from arbitrary initial functions $u^{(n)}_0$, provided that normalization invariances such as $\int u^{(n-2)}_0 = \int u^{(n-2)} \cdots$ be respected. The Poisson bracket structure on this orbit (which is a symplectic structure up to a finite number of central degrees of freedom) is readily obtained from the reduction of the Kirillov-Poisson structure of $W^*_+$, given in (28), to the orbit $\{h_{m-p} = 0 \text{ for } p \neq 0 \cdots -n-2\}$.

Using (28) together with:

$$\sum_{n \in \mathbb{Z}} x^{-n-1} y^n \sim \delta(x-y) \quad \text{(under $\oint$)}$$

(35)

leads to the following Poisson structure:

$$\{u^{(p)}(x), u^{(q)}(y)\} = \frac{1}{2} (p-q) \delta(x-y) u^{(p+q+2)}(x) +$$

$$+ \frac{p+q+2}{2} \delta'_x(x-y) \cdot \left( u^{(p+q+2)}(x) + u^{(p+q+2)}(y) \right)$$

(36)

Although it does not seem obvious to write immediately a Lagrangian leading to such a Poisson structure, we know in principle how to do so: one has to “solve” formally (34) in order to express $b^{(r,\lambda)}$ in terms of $\partial u^{(p)}/\partial \tau$ and $\partial u^{(p)}/\partial \lambda$. Obviously, this can be achieved in a technically simple, although becoming tedious when $n$ is large, recursion scheme. Once explicitly obtained, one plugs back $u_\tau$ and $u_\lambda$ into (7), and obtains the explicit Lagrangian. We expect however that drastic simplifications occur when the fields $u^{(n)}$ are suitably redefined. Although we have no general proof, we have shown up to under $n = 5$ (4 distinct functions $a$) that (36) can be recast as:

(for instance, $n = 5$)

$$\{\tilde{u}^{(0)}, \tilde{u}^{(3)}\} = 5 \delta'(x-y) \quad \{\tilde{u}^{(1)}, \tilde{u}^{(2)}\} = 5 \delta'(x-y)$$

(37)

all other brackets vanish.
This follows from redefining in (36):

\[ \bar{u}^{(0)} = u^{(0)} - \frac{1}{5} u^{(2)} u^{(3)} \quad \bar{u}^{(1)} = u^{(1)} - \frac{3}{20} u^{(3)^2} \]  

(38)

The Lagrangian obtained from (7) then becomes

\[ \mathcal{L} = \int dx \left( \dot{u}^{(0)} \partial_x^{-1} u^{(3)} + \dot{u}^{(1)} \partial_x^{-1} u^{(2)} \right) - H \]  

(39)

If our conjecture is correct, for any \( n \) one can redefine the \( u^{(p)} \) functions in (36) so as to reduce it to

\[ \{ u^{(p)}, u^{(n-p-2)} \} \cong \delta'(x - y) \quad \forall \ p = 0 \ldots (n - 2) \]  

(40)

In particular, for \( n \) even, the field \( u^{(\frac{n-2}{2})} \) is “self-conjugate” under (40), just as the tachyon field \( \alpha \) was for \( n = 2 \). It seems natural to try an identification of this field as the tachyon field, coupled to the other fields \( u^{(p)} \) and their “momenta” \( u^{(n-p-2)} \).

The couplings are described by a specific choice of Hamiltonian. We know in principle an infinite hierarchy of commuting Hamiltonians which would lead us to integrable field theories. They read (for the \( n^{th} \) hierarchy), redefining the labeling of \( u \) as \( u^{(p)} \rightarrow u^{(n-p)} \)

\[ H^p = \int \left[ y^n + u^{(2)} y^{n-2} + \ldots + u^{(n)} \right]^{p/n} dx dy \]  

(41)

The explicit computation is a technical matter which we shall not discuss in detail. The terms that can be obtained are deduced from the following rules: (a) the \( 1/n \)-power of \( L \equiv y^n + \cdots + u^{(0)} \) has the generic form

\[ L^{1/n} = y + \sum_{q=1}^{\infty} \sum_{\{p_j\}} \frac{\prod_j u^{(p_j)}}{y^q} \]  

(42)

when for a fixed \( q \) the set \( \{ p_j \} \) is such that \( \sum p_j = q + 1 \) and \( u^{(0)} = 1, u^{(1)} = 0 \)
(b) it follows that the contributing terms to (41) of global order $y^{-1}$ are necessarily of the form $\prod_j u^{(p_j)}$ such that $\sum p_j = p + 1$ Let us now describe Hamiltonians and Lagrangians for the next orbits ($n = 3, n = 4$). The $n = 3$ orbit $y^3 + uy + v$ leads to the following Hamiltonians.

\[ L^{1/3} \rightarrow \int u \, dx \quad L^{2/3} \rightarrow \int v \, dx \]
\[ L^{4/3} \rightarrow \int uv \, dx \quad L^{5/3} \rightarrow \int (9v^2 - u^3) \, dx \ldots \]  

(43)

The associated Lagrangian takes the form:

\[ \mathcal{L} = \int (\dot{u} \partial_x^{-1}v + g_1 \cdot uv + g_2(9v^2 - u^3) + \ldots) \, dx \]  

(44)

The $n = 4$ orbit $y^4 + uy^2 + vy + w$ leads to the following Hamiltonians:

\[ L^{1/4} \rightarrow \int u \, dx \quad L^{1/2} \rightarrow \int v \, dx \quad L^{3/4} \rightarrow \int (8w - u^2) \, dx \]
\[ L^{5/4} \rightarrow \int -3u^3 + 2uw + v \]  

(45)

and associated Lagrangian:

\[ \mathcal{L} = (\dot{u} \partial_x^{-1}w + \dot{w} \partial_x^{-1}u) + \dot{v} \partial_x^{-1}v \\
+ g_1(8w - u^2) + g_2(-3u^3 + 2uw + v^2) + \ldots \]  

(46)

In this way, one shall obtain integrable, $w_\infty$-invariant field theories which can be interpreted (for even $n$) as describing the coupling of the tachyon field $u^{(\frac{n}{2} - 1)}$ to other fields in a very naturally-defined way. The exact interpretation of these fields will be left for forthcoming studies; one must however underline their resemblance with the fields associated to the hierarchy of discrete states in the formalism of Klebanov and Polyakov [6]. Our sequence of fields in the $n \rightarrow \infty$ limit: $\sum_{m \geq 0} u^{(m)}(x, \tau) y^m = U(x, y, \tau)$ becomes a 2+1 field theory. However one must be careful about taking the limit $n \rightarrow \infty$. This limit would indeed correspond to taking the full $W_\infty^+$ algebra as phase space, instead of its finite $n$ orbit, and the Poisson structure is degenerate on this large phase space, preventing the construction of a Lagrangian on the previously described lines although it now reminds us of gauge–like theories. This point clearly deserves further investigations.
Finally we wish to comment on another possible extension of the previous coadjoint construction. As indicated above, $W^\infty$ is the $1$-contraction reduction of the algebra of pseudo differential operators. The above constructions have therefore the simple interpretation of long wave-length limits of the generalized KdV hierarchy obtained by application of the AKS scheme to $Ps\text{Diff}$ [18]. In fact, the phase space interpretation of $Ps\text{Diff}$ was emphasized recently by Yoneya [10]. In Wigner representation, $Ps\text{Diff}$ operators can be represented by functions of two variables $x$ and $\zeta$, of the form $F = \sum_{n \in \mathbb{Z}} u^{(n)}(x)\zeta^n$, where the coefficients $u_p^{(n)} = \oint x^{-p-1} u^{(n)}$ are the coordinates on the basis ($\sim x^n\partial_x^n$) of a given element in $Ps\text{Diff}$. The algebra is described by the product rule:

$$F_1 \cdot F_2 = \sum_{n \geq 0} \left(\frac{\partial}{\partial \zeta}\right)^n F_1 \cdot \left(\frac{\partial}{\partial x}\right)^n F_2$$

(47)

$W^\infty$ would be obtained by restricting $n$ to $1$. One then proceeds in defining the Lie–Baxter structure of $Ps\text{Diff}$ as $\{ \sum_{n \geq 0} u^{(n)}(x)\zeta^n \} \oplus \{ \sum_{n \leq 0} u^{(n)}(x)\zeta^n \}$; the duality bracket $< F_1, F_2 > = \text{Tr} F_1 \cdot F_2$ and the adjoint–invariant trace $\text{Tr} F = \oint d\zeta dx F$. Finally the Lagrangian on a given coadjoint orbit of order $n$ in $\zeta$ reads:

$$L = \oint d\zeta \oint dx \left( \zeta^n + u^{(n-2)}\zeta^{n-2} + \cdots \right) \cdot \left[ a_\tau, a_\lambda \right] d\tau d\lambda$$

(48)

when $a_{\tau,\lambda}(x,y)$ is such that:

. $a_{\tau,\lambda}(x,y)$ in ($Ps\text{ Diff}$) $= \sum_{n \geq 1} \zeta^{-n} a_{\tau,\lambda}^{(n)}$ is defined by the equation

$$\left[ \zeta^n + u^{(n-2)}\zeta^{n-2} + \cdots, a_{\tau,\lambda} \right]_+ = \frac{\partial u^{(n-2)}}{\partial \tau, \lambda} \zeta^{n-z} + \cdots$$

(49)

. $[]$ is the initial Lie bracket, $[]_+$ is the projection on $(Ps\text{Diff})_+$ corresponding to taking the second Lie-Baxter coadjoint action of $(Ps\text{Diff})_- \text{ on } Ps\text{Diff}_+$. All these features are well-known in the theory of generalized KdV hierarchies.

We want to emphasize now the relation with matrix models and fermions. As before, the algebra of observables is now an infinite number of copies of operators of
the type $X^n \partial^m$:

$$X^n_i \partial^m_i \quad i = 1, 2 \cdots$$ (50)

5. The Free Fermion Formalism

It is well known [21] that classical integrable equations obtained from the coadjoint construction on the $Ps$ Diff algebra can be reformulated in terms of integral equations obeyed by free fermion correlation functions. Since this relation may help to understand the meaning of the supplementary fields $u^{(n)}$ introduced in section 4, by reformulating them in terms of free fermions (which are known to be an alternative description of matrix models [19]), we shall describe briefly its salient points:

One starts from free fermion generators $\{\psi^+_m, \psi_n, m, n \in \mathbb{Z}\}$, constructing the $Gl(\infty)$ algebra as bilinears in $\psi$:

$$g_{mn} \equiv \psi^+_m \psi_n,$$

with commutation relations $[g_{mn}, g_{pq}] = \delta_{np} g_{mq} - \delta_{mq} g_{pn}$. One introduces a hierarchy of commuting Hamiltonians $\{H^{(n)} = \sum_{n \in \mathbb{Z}} \psi^+_i \psi_{i+n}\}, n \geq 0$, and it is easy to show, by a bosonization argument, that the evolution of the free fermions is encapsulated in the following form.

$$\psi(k) = \sum_{n \in \mathbb{Z}} k^{-n} \psi_n \quad ; \quad e^{\Sigma x_n H^{(n)}}, \psi \cdot e^{-\Sigma x_n H^{(n)}} = e^{\zeta(x, k)} \psi(k)$$

$$\zeta(x, k) = \sum_{n > 0} x_n k^n$$ (51)

Indeed the bosonization formula read:

$$\alpha(k) = \psi^+(k)\psi(k) \quad ; \quad \psi(k) = e^{\partial^{-1}\alpha(k)}$$ (52)

and the Hamiltonians $H^{(n)}$ are expressed as:

$$H(x) = \sum x_n H^{(n)} = \sum x_n \alpha^{(n)} \quad ; \quad \alpha^{(n)} = \int k^{n-1} \alpha(k) dk$$ (53)

from which (51) immediately follows. The major point consists in computing particular correlation functions:

$$\tau(x \cdots)(g) = <0 | e^{-H(x)} g |0 >, \quad g \in Gl(\infty)$$

$$|0 > \text{ being the Fermi sea vacuum}$$ (54)

These correlation functions or “tau functions” obey a particular bilinear equation or
“Hirota equation” [22]:

\[
\oint e^{\zeta(x-x',k)} \tau(x_n - \frac{1}{nk^n} \cdots) \tau(x'_n + \frac{1}{nk^n} \cdots) dk = 0
\] (55)

from which the differential equations of the KP hierarchy are obtained by Taylor-expanding (55) and introducing functions \( \sim u = \frac{\partial^2}{\partial x_i^2} \ln \tau \). The KdV hierarchy is obtained in a similar way by a reduction of \( Gl(\infty) \) to the Kac-Moody algebra \( A_1^{(1)} \), or equivalently by suppressing dependence in the even variables \( x^{(2n)} \) in (55). It follows that the classical fields \( u^{(n)}(x,t) \) introduced in section 4 are interpreted as (derivatives of) correlation functions for particular operators in a free-fermion theory. Notice also that the bilinear operators:

\[
\Theta^{(n,m)} = \int dk \psi^+(k) k^n \partial_k^m \psi(k)
\] (56)

naturally realize the \( Ps \text{Diff} \) algebra, thereby providing us with an algebraic relationship between the two formalisms. In relation to these remarks, let us finally emphasize that the \( Ps \text{Diff} \) algebra, and indeed the whole KP-construction framework, was used in [10,20] to formulate effective actions for string equations, implying a possible interpretation of the higher orbit coadjoint action as an effective action for fermion correlation function.

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