Some Baer Invariants of Free Nilpotent Groups

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Abstract

We present an explicit structure for the Baer invariant of a free n-th nilpotent group (the n-th nilpotent product of infinite cyclic groups, \( \mathbb{Z}^n \star \mathbb{Z}^n \star \cdots \star \mathbb{Z} \)) with respect to the variety \( \mathcal{V} \) with the set of words \( \mathcal{V} = \{ [\gamma_{c_1+1}, \gamma_{c_2+1}] \} \), for all \( c_1 \geq c_2 \) and \( 2c_2 - c_1 > 2n - 2 \). Also, an explicit formula for the polynilpotent multiplier of a free n-th nilpotent group is given for any class row \( (c_1, c_2, \ldots, c_t) \), where \( c_1 \geq n \).

Key Words: Baer invariant; Free nilpotent group; Nilpotent product; Polynilpotent variety.

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1. Introduction and Preliminaries

One of the interesting problems, related to the well-known notion of the Schur multiplier and its varietal generalization, the Baer invariant, is to obtain some structures for the Baer invariant of a given group $G$ and specially of some famous products of groups, such as direct products, semidirect products, free products, nilpotent products and regular products. Determining these Baer invariants of a given group is known to be very useful for classification of groups into isologism classes. Also, structures of Baer invariants are very essential for studying generalized capability and covering groups. Some people have succeeded to find the structures as follows.

I. Schur [18] in 1907 and J. Wiegold [20] in 1971 obtained the structure of the Schur multiplier of the direct product of two finite groups as follows:

$$M(A \times B) \cong M(A) \oplus M(B) \oplus \frac{|A,B|}{[A,B,A,B]_1},$$

where $\frac{|A,B|}{[A,B,A,B]_1} \cong A_{ab} \otimes B_{ab}$.

In 1979, M. R. R. Moghaddam [15] and in 1998, G. Ellis [2], succeeded to extend the above result to obtain the structure of the $c$-nilpotent multiplier of the direct product of two groups, $N_c M(A \times B)$. Also in 1997 the first author in a joint paper [10] presented an explicit formula for the $c$-nilpotent multiplier of a finite abelian group.

K. I. Tahara [19] in 1972 and W. Haebich [6] in 1977 found some structures for the Schur multiplier of the semidirect product of two groups. Also, the first author [9,13] extended some of the above results to the variety of nilpotent groups.

In 1972 W. Haebich [5] presented a formula for the Schur multiplier of a regular product of a family of groups. It is known that the regular product is a generalization of the nilpotent product and the last one is a generalization of the direct product, so Haebich’s result is an interesting generalization of the Schur’s result. Also, M. R. R. Moghaddam [16], in 1979 gave a formula
similar to Haebich’s formula for the Schur multiplier of a nilpotent product. Moreover, in 1992, N. D. Gupta and M. R. R. Moghaddam [4] presented an explicit formula for the $c$-nilpotent multiplier of the $n$th nilpotent product $\mathbb{Z}_2^* \otimes \mathbb{Z}_2$. G. Ellis [3] remarked that there is a slip in the statement of the result in case $c \leq n - 1$ and gave the correct structure.

In 2001, the first author [11] found a structure similar to Haebich’s type for the $c$-nilpotent multiplier of a nilpotent product of a family of cyclic groups. The $c$-nilpotent multiplier of a free product of some cyclic groups was studied by the first author [12] in 2002.

Recently, the authors [14,17] concentrated on the Baer invariant with respect to the variety of polynilpotent groups, for the first time. We presented an explicit structure for some polynilpotent multipliers of the $n$th nilpotent product of some infinite cyclic groups [17] and also found explicit structures for all polynilpotent multipliers of finitely generated abelian groups [14].

Now trying to extend the above results to the vast variety of outer commutators we concentrate, as a first step, on a variety $\mathcal{V}$ with a set of words $\{[\gamma_{c_1+1}, \gamma_{c_2+1}]\}$. In this paper we intend to extend the last above results in two directions. First, to obtain an explicit formula for some Baer invariants of the $n$th nilpotent product of some infinite cyclic groups,

$$\mathcal{V}M(\mathbb{Z}^n \otimes \mathbb{Z}^n \otimes \ldots \otimes \mathbb{Z})$$

in which $\mathcal{V}$ is the above variety for all $c_1 \geq c_2$ and $2c_2 - c_1 > 2n - 2$. Note that the first restriction on the parameters, $c_1 \geq c_2$, is very natural and the second one help us to calculate the Baer invariant. Second, to present an explicit formula for the polynilpotent multiplier of the $n$th nilpotent product of some infinite cyclic groups

$$\mathcal{N}_{c_1, \ldots, c_t}M(\mathbb{Z}^n \otimes \mathbb{Z}^n \otimes \ldots \otimes \mathbb{Z})$$

for any class row $(c_1, c_2, \ldots, c_t)$, where $n \leq c_1$. 

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In the following you can find some preliminaries which are used in our method.

**Definition 1.1.** Let $G$ be any group with a free presentation $G \cong F/R$, where $F$ is a free group. Then, after R. Baer [1], the *Baer invariant* of $G$ with respect to a variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^{*}F]} ,$$

where $V$ is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of $F$ with respect to $\mathcal{V}$ and

$$[RV^{*}F] = \langle v(f_1, \ldots, f_{i-1}, f_{i}r, f_{i+1}, \ldots, f_{n})v(f_1, \ldots, f_{i}, \ldots, f_{n})^{-1} | r \in R, 1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbb{N} \rangle.$$

In special case of the variety $\mathcal{A}$ of abelian groups, the Baer invariant of $G$ will be the well-known notion the *Schur multiplier*

$$\frac{R \cap F''}{[R, F]}.$$

If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer invariant of $G$ with respect to $\mathcal{N}_c$ which is called the *c-nilpotent multiplier* of $G$, will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]} ,$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of $F$ and $[R, 1F] = [R, F], [R, cF] = [[R, c-1F], F]$, inductively.

**Lemma 1.2** (J. A. Hulse and J. C. Lennox 1976). If $u$ and $w$ are any two words and $v = [u, w]$ and $K$ is a normal subgroup of a group $G$, then

$$[Kv^{*}G] = [[Ku^{*}G], w(G)]u(G), [Kw^{*}G]].$$
Proof. See [8, Lemma 2.9].

Now, using the above lemma, let $\mathcal{V}$ be the outer commutator variety of groups defined by the set of words $\{[\gamma_{c_1+1}, \gamma_{c_2+1}]\}$, then the Baer invariant of a group $G$ with respect to this variety, is as follows:

$$\mathcal{V}M(G) \cong \frac{R \cap [\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[R, _{c_1}F, \gamma_{c_2+1}(F)]} \cdot [R, _{c_2}F, \gamma_{c_1+1}(F)]^\star.$$  (\star)

**Definition 1.3.** Basic commutators are defined in the usual way. If $X$ is a fully ordered independent subset of a free group, the basic commutators on $X$ are defined inductively over their weight as follows:

(i) the members of $X$ are basic commutators of weight one on $X$;

(ii) assuming that $n > 1$ and that the basic commutators of weight less than $n$ on $X$ have been defined and ordered;

(iii) a commutator $[b, a]$ is a basic commutator of weight $n$ on $X$ if $wt(a) + wt(b) = n$, $a < b$, and if $b = [b_1, b_2]$, then $b_2 \leq a$. The ordering of basic commutators is then extended to include those of weight $n$ in any way such that those of weight less than $n$ precede those of weight $n$. The natural way to define the order on basic commutators of the same weight is lexicographically, $[b_1, a_1] < [b_2, a_2]$ if $b_1 < b_2$ or if $b_1 = b_2$ and $a_1 < a_2$.

The next two theorems are vital in our investigation.

**Theorem 1.4 (P. Hall [7]).** Let $F = \langle x_1, x_2, \ldots, x_d \rangle$ be a free group, then

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)}, \quad 1 \leq i \leq n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, \ldots, x_d\}$.

**Theorem 1.5 (Witt Formula [7]).** The number of basic commutators of weight $n$ on $d$ generators is given by the following formula:

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m)d^{m/n},$$
where \( \mu(m) \) is the Möbius function.

**Definition 1.6.** Let \( V \) be a variety of groups defined by a set of laws \( V \). Then the **verbal product** of a family of groups \( \{G_i\}_{i \in I} \) associated with the variety \( V \) is defined to be

\[
V \prod_{i \in I} G_i = \frac{\prod^* G_i}{V(G) \cap [G_i]^*},
\]

where \( G = \prod_{i \in I} G_i \) is the free product of the family \( \{G_i\}_{i \in I} \) and \( [G_i]^* = < [G_i, G_j] | i, j \in I, i \neq j \rangle^G \) is the cartesian subgroup of the free product \( G \).

The verbal product is also known as **varietal product** or simply **\( V \)-product**. If \( V \) is the variety of all groups, then the corresponding verbal product is the free product; if \( V = A \) is the variety of all abelian groups, then the verbal product is the direct product. Also, if \( V = N_c \) is the variety of nilpotent groups of class at most \( c \geq 1 \), then the verbal product is called the **\( c \)th nilpotent product** of the \( G_i \)'s.

## 2. The Main Results

Let \( G \cong \mathbb{Z} \ast \mathbb{Z} \ast \ldots \ast \mathbb{Z} \) be the \( n \)th nilpotent product of \( m \) copies of the infinite cyclic group \( \mathbb{Z} \). Using Definition 1.6, it is easy to see that \( G \) is the free \( n \)th nilpotent group of rank \( m \), and so has the following free presentation

\[
1 \rightarrow \gamma_{n+1}(F) \rightarrow F \rightarrow G \rightarrow 1,
\]

where \( F \) is the free group on a set \( X = \{x_1, x_2, \ldots, x_m\} \).

Now, we try to obtain the structure of some outer commutator multipliers of \( G \) of the form

\[
\mathcal{V}M(G),
\]

where \( \mathcal{V} \) is defined by the set of words \( \{[\gamma_{c_1+1}, \gamma_{c_2+1}] | c_1 \geq c_2, \text{ and } 2c_2 - c_1 > 2n - 2 \} \) (Note that \( \gamma_c = [x_1, \ldots, x_c] \)).
Using (⋆) we have
\[\mathcal{V}M(G) \cong \frac{\gamma_{n+1}(F) \cap [\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)]}[\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)].\]

Now if we have \(c_1 + c_2 + 1 \geq n\), then
\[\mathcal{V}M(G) \cong \frac{[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)]}[\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)].\]

In order to find the structure of \(\mathcal{V}M(G)\), we need the following notation and lemmas. Using Definition and Notation 1.2, we defined the following sets when \(c_1 \geq c_2\).

- **A** = \{\([\beta, \alpha]\) \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, c_1 + 1 \leq \text{wt}(\beta) \leq c_1 + n, c_2 + 1 \leq \text{wt}(\alpha) \leq c_2 + n\};

- **B** = \{\([\beta, \alpha]\) \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, c_1 + n + 1 \leq \text{wt}(\beta), c_2 + 1 \leq \text{wt}(\alpha), \text{wt}(\beta) + \text{wt}(\alpha) \leq 2n + c_1 + c_2 + 1\};

- **C** = \{\([\beta, \alpha]\) \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, c_2 + n + 1 \leq \text{wt}(\beta), c_1 + 1 \leq \text{wt}(\alpha), \text{wt}(\beta) + \text{wt}(\alpha) \leq 2n + c_1 + c_2 + 1\}.

**Lemma 2.1.** If \(n - 1 \leq 2c_2 - c_1\), then every element of **A** is a basic commutator on \(X\).

**Proof.** Every element of **A** has the form \([\beta, \alpha]\), where \(\beta \text{ and } \alpha \text{ are basic commutators on } X, \beta > \alpha \text{ and } c_1 + 1 \leq \text{wt}(\beta) \leq c_1 + n, c_2 + 1 \leq \text{wt}(\alpha) \leq c_2 + n\). Now, let \(\beta = [\beta_1, \beta_2]\), then in order to show that \([\beta, \alpha]\) is a basic commutator on \(X\), it is enough to show that \(\beta_2 \leq \alpha\). Since \(\beta = [\beta_1, \beta_2]\) is a basic commutator on \(X\), so \(\beta_1 > \beta_2\) and hence \(\text{wt}(\beta_2) \leq \frac{1}{2}\text{wt}(\beta)\). Now, if \(n - 1 \leq 2c_2 - c_1\), then \(\frac{1}{2}(c_1 + n) < c_2 + 1\). Thus, since \(\text{wt}(\beta) \leq c_1 + n\), we have
\[\text{wt}(\beta_2) \leq \frac{1}{2}\text{wt}(\beta) \leq \frac{1}{2}(c_1 + n) < c_2 + 1 \leq \text{wt}(\alpha)\].
Therefore \( \beta_2 < \alpha \) and hence the result holds. \( \square \)

**Lemma 2.2.** With the above notation,

(i) if \( 2n - 2 < 2c_2 - c_1 \), then every element of \( B \) is a basic commutator on \( X \);

(ii) if \( 2n - 2 < 2c_1 - c_2 \), then every element of \( C \) is a basic commutator on \( X \).

**Proof.** (i) Let \([\beta, \alpha]\) be an element of \( B \). By definition of \( B \), \( \beta \) and \( \alpha \) are basic commutators on \( X \) and \( \beta > \alpha \). Now, let \( \beta = [\beta_1, \beta_2] \), where \( \beta_1 \) and \( \beta_2 \) are basic commutators on \( X \). It is enough to show that \( \beta_2 \leq \alpha \). Since \( \beta = [\beta_1, \beta_2] \) is a basic commutator, so \( \beta_1 > \beta_2 \) and hence \( wt(\beta_2) \leq \frac{1}{2} wt(\beta) \).

By definition of \( B \), we have \( wt(\beta_2) \leq \frac{1}{2} wt(\beta) \leq \frac{1}{2} (2n + c_1 + c_2 + 1 - c_2 - 1) = \frac{1}{2} (2n + c_1) < c_2 + 1 \leq wt(\alpha) \).

(ii) A similar argument can be applied for elements of \( C \). \( \square \)

**Lemma 2.3.** With the above notation, the following statements hold.

(i) If \( c_2 + n < c_1 + 1 \), then

\[
|A| = \left( \sum_{i=c_1+1}^{c_1+n} \chi_i(m) \right) \left( \sum_{c_2+1}^{c_2+n} \chi_i(m) \right).
\]

(ii) If \( c_2 + n \geq c_1 + 1 \), then

\[
|A| = \left( \sum_{i=c_1+1}^{c_1+n} \chi_i(m) \right) \left( \sum_{c_2+1}^{c_1} \chi_i(m) \right) + \\
( \sum_{i=c_2+n+1}^{c_1+n} \chi_i(m) ) ( \sum_{c_1+1}^{c_2+n} \chi_i(m) ) + \chi_2( \sum_{i=c_1+1}^{c_2+n} \chi_i(m) ).
\]

**Proof.** (i) We have \( c_2 + 1 \leq wt(\alpha) \leq c_2 + n < c_1 + 1 \leq wt(\beta) \leq c_1 + n \), so \( \beta > \alpha \) and hence all possible of \( \beta \) and \( \alpha \) is accepted. Clearly the number of \( \beta \)'s is \( (\sum_{i=c_1+1}^{c_1+n} \chi_i(m)) \) and the number of \( \alpha \)'s is \( (\sum_{c_2+1}^{c_2+n} \chi_i(m)) \). Thus the
result holds.

(ii) Let $c_1 + 1 \leq c_2 + n$. Then $c_2 + 1 \leq c_1 + 1 \leq c_2 + n \leq c_1 + n$. Put

$$A_1 = \{[\beta, \alpha] \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, \quad c_1 + 1 \leq wt(\beta) \leq c_1 + n, \quad c_2 + 1 \leq wt(\alpha) < c_1 + 1\};$$

$$A_2 = \{[\beta, \alpha] \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, \quad c_1 + 1 \leq wt(\beta) \leq c_2 + n, \quad c_1 + 1 \leq wt(\alpha) \leq c_2 + n\};$$

$$A_3 = \{[\beta, \alpha] \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, \quad c_2 + n < wt(\beta) \leq c_1 + n, \quad c_1 + 1 \leq wt(\alpha) \leq c_2 + n\}.$$

Clearly $A_1, A_2, A_3$ are mutually disjoint and $A = A_1 \cup A_2 \cup A_3$. Therefore $|A| = |A_1| + |A_2| + |A_3|$. Also, it is easy to see that

$$|A_1| = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m))(\sum_{i=c_2+1}^{c_1} \chi_i(m));$$

$$|A_2| = \chi_2(\sum_{i=c_1+1}^{c_2+n} \chi_i(m));$$

$$|A_3| = (\sum_{i=c_2+n+1}^{c_1+n} \chi_i(m))(\sum_{i=c_1+1}^{c_2+n} \chi_i(m)).$$

Hence the result holds. \(\square\)

**Lemma 2.4.** With the above notation and assumption,

(i) if $c_2 + n < c_1 + 1$, then $A \cap C = \emptyset$;

(ii) if $c_2 + n \geq c_1 + 1$, then $|A \cap C| = (\sum_{i=c_2+n+1}^{c_1+n} \chi_i(m))(\sum_{i=c_1+1}^{c_2+n} \chi_i(m)).$

**Proof.** By definition of $A$ and $C$ we have

$$A \cap C = \{[\beta, \alpha] \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha, \quad \max\{c_1+1, c_2+n+1\} \leq wt(\beta) \leq c_1+n, \quad \max\{c_1+1, c_2+1\} \leq wt(\alpha) \leq c_2+n\}.$$

(i) Since $c_2 \leq c_1$, $c_1 + 1 \leq wt(\alpha) \leq c_2 + n$ which is a contradiction to the assumption $c_2 + n < c_1 + 1$. Hence in this case we have $A \cap C = \emptyset$. 

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If \( c_2 + n \geq c_1 + 1 \), then \( c_1 + 1 \leq \text{wt}(\alpha) < c_2 + n + 1 \leq \text{wt}(\beta) \leq c_1 + n \). Thus we have always \( \beta > \alpha \) and hence the result holds. □

The following corollary is an immediate consequence of the last two lemmas.

**Corollary 2.5.** With the above notation and assumption,

1. if \( c_2 + n < c_1 + 1 \), then
   \[
   |A - C| = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m)) (\sum_{c_2+1}^{c_1} \chi_i(m));
   \]
2. if \( c_2 + n \geq c_1 + 1 \), then
   \[
   |A - C| = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m)) (\sum_{c_2+1}^{c_1} \chi_i(m)) + \chi_2 (\sum_{i=c_1+1}^{c_2+n} \chi_i(m)).
   \]

**Lemma 2.6.** With the above notation and assumption, if \( n - 1 \leq c_2 \leq c_1 \), then we have

\[
[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)] \equiv A - C \mod H,
\]

where \( H = [\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)] \).

**Proof.** Let \( [\alpha, \beta] \) be a generator of \([\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]\), so \( \alpha \in \gamma_{c_1+1}(F) \) and \( \beta \in \gamma_{c_2+1}(F) \). By P. Hall’s Theorem (1.3) we can put \( \alpha = \alpha_1 \alpha_2 \ldots \alpha_t \eta \) and \( \beta = \beta_1 \beta_2 \ldots \beta_s \mu \), where \( \alpha_1, \ldots, \alpha_t \) are basic commutators on \( X \) of weights \( c_1 + 1, \ldots, c_1 + n \), and \( \beta_1, \ldots, \beta_s \) are basic commutators of weights \( c_2 + 1, \ldots, c_2 + n \) on \( X \) and \( \eta \in \gamma_{c_1+n+1}(F) \) and \( \mu \in \gamma_{c_2+n+1}(F) \).

By using commutator calculus, it is easy to see that

\[
[\alpha, \beta] = \Pi_{i,j} [\alpha_i, \beta_j] f_{ij} [\alpha_i, \mu] g_i [\eta, \beta_j] h_j,
\]

where \( f_{ij}, g_i, h_j \in \gamma_n(F) \) (Note that, \( n - 1 \leq c_2 \leq c_1 \)). Now, it is easy to see that
\[
[\beta_i, \eta] \in [\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)] \\
[\mu, \alpha_i] \in [\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)] \\
[\beta_i, \alpha_j, f_{ij}] \in [\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]
\]

(Note that the last holds by the three subgroups lemma).

Therefore we have

\[
[\alpha, \beta] \equiv \Pi_{i,j}[\alpha_i, \beta_j] \pmod{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}.
\]

Note that if \(\alpha_i < \beta_j\), then we have \(c_1 + 1 \leq wt(\alpha_i) \leq wt(\beta_j) \leq c_2 + n\) and so we can consider \([\alpha_i, \beta_j] = [\beta_j, \alpha_i]^{-1}\), where \(c_1 + 1 \leq wt(\beta_j) \leq c_1 + n\) and \(c_2 + 1 \leq wt(\alpha_i) \leq c_2 + n\). Thus we have \([\beta_j, \alpha_i] \in A\) and so it is easy to see that \(\Pi_{i,j}[\alpha_i, \beta_j] \in < A >\). Since \(C \subseteq [\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)] \subseteq H\), hence the result holds. \(\square\)

**Lemma 2.7.** With the above notation and assumption, if \(n - 1 \leq c_2 \leq c_1\), then

(i) \([\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)] \equiv < B > \mod{\gamma_{c_1+c_2+2n+2}(F)}\);

(ii) \([\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)] \equiv < C > \mod{\gamma_{c_1+c_2+2n+2}(F)}\).

**Proof.** (i) Let \([\beta, \alpha]\) be a generator of \([\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)]\), so, \(\beta \in \gamma_{c_1+n+1}(F)\) and \(\alpha \in \gamma_{c_2+1}(F)\). By P. Hall’s Theorem (1.3) we can write

\(\alpha = \alpha_1\alpha_2 \ldots \alpha_n\eta\) and \(\beta = \beta_1\beta_2 \ldots \beta_s\mu\), where \(\alpha_1, \ldots, \alpha_i\) are basic commutators of weights \(c_2 + 1, \ldots, c_2 + n\) on \(X\) and \(\eta \in \gamma_{c_2+n+1}(F)\), and \(\beta_1, \ldots, \beta_s\) are basic commutators of weights \(c_1+n+1, \ldots, c_1+2n\) on \(X\) and \(\mu \in \gamma_{c_1+2n+1}(F)\).

By using commutator calculus, it is easy to see that

\[
[\beta, \alpha] = \Pi_{i,j}[\beta_i, \alpha_j][f_{ij}][\mu, \alpha_j][g_i][\beta_i, \eta][h_j],
\]

where \(f_{ij}, g_i, h_j \in \gamma_{c_2+1}(F)\). Now we have

\[
wt(\alpha_j) + wt(\mu) \geq (c_2 + 1) + (c_1 + 2n + 1) \geq c_1 + c_2 + 2n + 2
\]

\[
wt(\eta) + wt(\beta_j) \geq (c_2 + n + 1) + (c_1 + n + 1) \geq c_1 + c_2 + 2n + 2
\]
\(\text{wt}(\alpha_j) + \text{wt}(\beta_i) + \text{wt}(f_{ij}) \geq (c_2 + 1) + (c_1 + n + 1) + (c_2 + 1) \geq c_1 + c_2 + 2n + 2\)

for all \(1 \neq f_{ij} \in \gamma_{c_2+1}(F)\) since \(c_2 \geq n - 1\).

Therefore, by a similar argument at the end of the proof of the previous lemma, we have

\[
[\beta, \alpha] = \Pi_{i,j} [\beta_i, \alpha_j] \mod \gamma_{c_1+c_2+2n+2}(F)
= \Pi_{\text{wt}(\alpha_j)+\text{wt}(\beta_i) \leq c_1+c_2+2n+1} [\beta_i, \alpha_j] \in B \mod \gamma_{c_1+c_2+2n+2}(F).
\]

(ii) Similar to part (i). \(\Box\)

Now, we are in a position to state and proof the first main result of the paper.

**Theorem 2.8.** With the above notation and assumption, if \(2c_2 - c_1 > 2n - 2\) and \(c_1 \geq c_2\), then \(\mathcal{V}M( Z^n \ast Z^n \ast \ldots \ast Z)\) is a free abelian group with the following basis:

\[D = \{ a[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)] \mid a \in A - C \} \].

**Proof.** Note that if \(c_1 \geq c_2\) and \(2c_2 - c_1 > 2n - 2\), then it is easy to see that the following inequalities hold:

\[c_1 + c_2 + 1 \geq n, \ c_1 \geq 2c_2 \geq n - 1, \ 2c_2 - c_1 \geq n - 1, \ and \ 2c_1 - c_2 > 2n - 2.\]

Therefore all of the previous lemmas hold. Clearly

\[\mathcal{V}M( Z^n \ast Z^n \ast \ldots \ast Z) = \frac{[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}\]

is an abelian group which is generated by \(D\), using Lemma 2.6. So it is enough to show that \(D\) is linearly independent. Consider the free abelian
group \( \gamma_{c_1+c_2+2}(F)/\gamma_{c_1+c_2+2n+2}(F) \), with the basis of all basic commutators of weights \( c_1 + c_2 + 2, \ldots, c_1 + c_2 + 2n + 1 \) and the following fact
\[
\frac{H\gamma_{c_1+c_2+2n+2}(F)}{\gamma_{c_1+c_2+2n+2}(F)} = \langle \hat{w} \mid w \in B \cup C \rangle \leq \frac{\gamma_{c_1+c_2+2}(F)}{\gamma_{c_1+c_2+2n+2}(F)}.
\]

Now, suppose
\[
\sum_{i=1}^{\ell} k_i \hat{a}_i = 0 \quad \text{in} \quad \frac{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]},
\]
where \( a_1, \ldots, a_\ell \in A - C \), \( k_i \in \mathbb{Z} \). So we have
\[
\sum_{i=1}^{\ell} k_i a_i \in [\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)].
\]

By Lemma 2.7 the group \( [\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)] \) is generated by \( B \cup C \) modulo \( \gamma_{c_1+c_2+2n+2}(F) \). Thus considering the free abelian group \( \gamma_{c_1+c_2+2}(F)/\gamma_{c_1+c_2+2n+2}(F) \) we have \( \sum_{i=1}^{\ell} k_i \hat{a}_i = \sum_{j=1}^{l} d_j \hat{w}_j \) for some \( d_j \in \mathbb{Z} \) and some \( w_j \in B \cup C \). This implies that \( \sum k_i \hat{a}_i - \sum d_j \hat{w}_j = 0 \), where \( a_i \)'s and \( w_j \)'s are basic commutators of weights \( c_1 + c_2 + 2, \ldots, c_1 + c_2 + 2n + 1 \).

By the form of elements of \( A, B \) and \( C \), we have \( (B \cup C) \cap (A - C) = \emptyset \) so \( k_i = 0 \) and \( d_j = 0 \), for all \( i, j \). Hence the result holds. \( \square \)

**Corollary 2.9.** With the above notation and assumption, if \( c_1 \geq c_2 \) and \( 2c_2 - c_1 > 2n - 2 \), then the following hold.

(i) If \( c_2 + n < c_1 + 1 \), then \( \mathcal{V}M(\mathbb{Z}^n \ast \mathbb{Z}^n \ast \ldots \ast \mathbb{Z}^{m-copies}) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} (k \text{-copies}) \),

where \( k = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m))(\sum_{i=c_2+1}^{c_2+n} \chi_i(m)) \).

(ii) If \( c_2 + n \geq c_1 + 1 \), then \( \mathcal{V}M(\mathbb{Z}^n \ast \mathbb{Z}^n \ast \ldots \ast \mathbb{Z}^{m-copies}) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}(k' \text{-copies}) \),

where \( k' = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m))(\sum_{i=c_2+1}^{c_1+n} \chi_i(m)) + \chi_2(\sum_{i=c_1+1}^{c_2+n} \chi_i(m)) \).

Note that the above result is a generalization of the main result of [17], since if \( c > 2n - 2 \), then similar formula can be obtained for \( \mathcal{N}_{c,1}M(\mathbb{Z}^n \ast \mathbb{Z}^n \ast \ldots \ast \mathbb{Z}) \).
In the rest, we try to generalize the main result of [17] in another direction. Let \( G = Z^n \ast Z^n \ast \ldots \ast Z^n \) be the free \( n \)th nilpotent group of rank \( m \) and \( N_{c_1, \ldots, c_t} \) be the variety of polynilpotent groups of class \( (c_1, \ldots, c_t) \). Consider the free presentation \( F/\gamma_{n+1}(F) \) for \( G \), where \( F \) is the free group of rank \( m \). Using Lemma 1.2 we have

\[
N_{c_1, \ldots, c_t}(G) \cong \frac{\gamma_{n+1}(F) \cap \gamma_{c_1+1}(\ldots (\gamma_{c_1+1}(F)) \ldots)}{[\gamma_{n+1}(F), c_1 F, c_2 \gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_1+1}(\ldots (\gamma_{c_1+1}(F)) \ldots)]}.
\]

If \( c_1 \geq n \), then we can consider \( \gamma_{c_1+1}(F)/[\gamma_{n+1}(F), c_1 F] \) as a free presentation for \( N_{c_1}(G) \) thus we have

\[
N_{c_t}(G) \cong \frac{\gamma_{c_1+1}(\ldots (\gamma_{c_1+1}(F)) \ldots)}{[\gamma_{n+1}(F), c_1 F, c_2 \gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_1+1}(\ldots (\gamma_{c_1+1}(F)) \ldots)]}.
\]

Hence the following useful isomorphism holds

\[
N_{c_1, \ldots, c_t}(G) \cong N_{c_1}(G) \cdots (N_{c_t}(G)) \ldots.
\]

Clearly \( N_{c_1}(G) \cong \gamma_{c_1+1}(F)/\gamma_{n+c_1+1}(F) \) if the free abelian group of rank \( \sum_{i=c_1+1}^{n} \chi_i(m) \). We need the following theorem which can be easily proved similar to the proof of the main result in a joint paper of the first author [10].

**Theorem 2.10.** Let \( G = Z^{(m)} \oplus Z_{n_1} \oplus Z_{n_2} \oplus \ldots \oplus Z_{n_k} \) be a finitely generated abelian group, where \( n_{i+1} | n_i \) for all \( 1 \leq i \leq k - 1 \). Then, for all \( c \geq 1 \)

\[
N_c(G) \cong Z^{(b_m)} \oplus Z_{n_1}^{(b_{m+1}-b_m)} \oplus Z_{n_3}^{(b_3-b_2)} \oplus \ldots \oplus Z_{n_k}^{(b_{m+k}-b_{m+k-1})},
\]

where \( b_i = \chi_{c+1}(i) \) and \( X^{(m)} \) denotes the direct sum of \( m \)-copies of \( X \).

Now, using induction and the above facts we can present an explicit formula for the polynilpotent multiplier of free nilpotent groups as follows which extends somehow the results of [10,14,15,16,17,18,20].
Theorem 2.11. Let $G = \mathbb{Z}^n \ast \mathbb{Z}^n \ast \ldots \ast \mathbb{Z}$ be the free nth nilpotent group of rank $m$ and $\mathcal{N}_{c_1, \ldots, c_t}$ be the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$. If $c_1 \geq n$, then $\mathcal{N}_{c_1, \ldots, c_t} M(G)$ is the free abelian group of rank $\chi_{c_t+1}(\ldots (\chi_{c_2+1}(\sum_{i=c_1+1}^{c_1+n} \chi_i(m))) \ldots )$.

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