Abstract

We consider complementary dynamical systems related to stationary Korteweg-de Vries hierarchy of equations. A general approach for finding elliptic solutions is given. The solutions are expressed in terms of Novikov polynomials in general quasi-periodic case. For periodic case these polynomials coincide with Hermite and Lamé polynomials. As a byproduct we derive $2 \times 2$ matrix Lax representation for Rosochatius-Wojciechowski, Rosochatius system, second flow of stationary nonlinear vector Schrödinger equations and complex Neumann systems.

1 Introduction

The problem of integrability and nonintegrability of Hamiltonian system with $g$ degrees of freedom has been a subject of considerable interest for many years. Recently remarkable progress has been achieved in connection of study of stationary soliton type equations. The method of restricted flows was introduced in [1, 2, 3] as a non-linearization of the Korteweg-de Vries (KdV) spectral problem and was generalized in [4, 5]. The coupled Neumann system, the Neumann system, the Garnier type systems, the Rosochatius-Wojciechowski, Rosochatius and the Hénon-Heiles type systems are examples of this type. In this relation important results about the algebro-geometrical interpretation of these systems are obtained in [6]. General approach to completely integrable dynamical systems of Neumann type is discussed in [7, 8, 9]. Quasi-periodic solutions and spectral interpretation using both algebro-geometrical and spectral methods are
given in \[11\]. Quasi-periodic solutions \((g = 2)\) and periodic solutions associated to Lamé and Treibich-Verdier potentials are obtained in \[16, 17\]. New method of constructing elliptic finite-gap solutions of the stationary KdV hierarchy, based on a theorem due to Picard, is proposed in \[19, 20, 21, 22, 23, 24\].

In this paper we are concerned with the following different approaches

- integrable dynamical systems related to Hill’s equation in the case of finite-gap potential \[2, 31, 8, 10, 12, 9, 11, 47\]
- method of stationary flows and restricted flows \[37, 7\]
- method of non-linearization of the KdV spectral problem \[5\]
- method of separation of variables \[57, 59, 60\]
- algebro-geometrical construction \[8, 9, 11\].

We give unified construction based on algebro-geometrical approach. New solutions in terms of Novikov and Gelfand-Dickey (GD) polynomials are given in explicit form.

The paper is organized as follows. In section 2 we construct the stationary flows associated to the KdV hierarchy strictly following \[37, 41, 42, 43\]. Some maps between completely integrable dynamical systems are presented. In section 3 we analyse the relation between restricted KdV flows and Garnier type systems following \[42\]. The map between stationary and restricted KdV flows are given in section 4 \[42\]. The list of known dynamical systems related to stationary KdV equations are presented in section 3.

In section 4 we formulate Baker-Akhiezer function approach from algebro-geometrical and spectral point of view respectively. New solutions of integrable dynamical system associated to stationary KdV equations in terms of Novikov polynomials are presented. In particular these polynomials associated to Lamé potentials coincide with Hermite polynomials and Lamé polynomials. In section 5 \((2 \times 2)\) Lax representations of Garnier, Rosochtius I and stationary second flow of vector nonlinear Schrödinger equation are given. Following \[60, 59, 61\] \(r\)-matrix approach of these systems is discussed.

2 KdV hierarchy and Gelfand-Dickey polynomials

In this section we follow the geometrical construction of \[37, 11, 12, 13\] with only little changes of signs and different spectral parameter. Let \(M\) be a bi-Hamiltonian manifold: if the associated Poisson operator \(P^\lambda := P_1 + 4\lambda P_0\) admits as a Casimir a formal Laurent series \(h(\lambda)\)

\[
h(\lambda) := \sum_{j \geq 0} h_j \lambda^{-j}
\]
then $h_0$ is a Casimir of $P_0$ and the coefficients $h_j$ ($j \geq 0$) are the Hamiltonian functions of a hierarchy of bi-Hamiltonian vector fields $X_j$:

$$X_j = P_j dh_j = P_0 dh_{j+1}, \quad (j \geq 0) \tag{2.2}$$

At any point $u \in M$, the bi-Hamiltonian flows are given by $\frac{du}{dt}_j = X_j(u)$, $t_j$ being the evolution parameter of the $j$th flow. The vector field (2.2) are Hamiltonian also with respect to the Poisson operator $P_0$. In fact the recursion relation (2.2) can be written as

$$X_j = P_0^{\lambda} dh^{(j)}(\lambda), \quad h^{(j)}(\lambda) := (\lambda^j h(\lambda))_+ \tag{2.3}$$

where the index $+$ means the projection of a Laurent series onto the purely polynomial part.

Let $M$ be the algebra of polynomials in $u, u_x, u_{xx}, \ldots$ ($u = u(x)$ is a $C^\infty$ function of $x$ and the subscript $x$ means the derivative with respect to $x$), and let $P_0$ and $P_1$ be the two Poisson operators of the KdV hierarchy (2.4):

$$P_0 := \frac{d}{dx}, \quad P_1 := \frac{d^3}{dx^3} - 4u \frac{d}{dx} - 2u_x. \tag{2.4}$$

The gradients of the Casimirs of the associated Poisson operator $P_0^{\lambda}$ can be obtained searching the functions $v(\lambda) := \sum_{j \geq 0} v_j \lambda^{-j}$ which are solutions of the following equation:

$$B^{\lambda}(v(\lambda), v(\lambda)) = a(\lambda) \tag{2.5}$$

where $a(\lambda) = \sum_{j \geq 0} a_j \lambda^{-j}$, $a_j$ are constant parameters and $B^{\lambda}$ is the bilinear function

$$B^{\lambda}(w_1, w_2) := w_1 w_2 + 4u w_1 w_2x - w_1x w_2x - 4(u - \lambda)w_1 w_2. \tag{2.6}$$

In fact $B^{\lambda}$ is related to the Poisson operator through the relation

$$\frac{d}{dx} B^{\lambda}(w_1, w_2) = w_1 P^{\lambda} w_2 + w_2 P^{\lambda} w_1. \tag{2.7}$$

Equation (2.5) can be solved developing the left-hand side as a Laurent series

$$B^{\lambda}(v(\lambda), v(\lambda)) = \sum_{k \geq -1} B_k \lambda^{-k} \tag{2.8}$$

so that, for each $a(\lambda)$, it furnishes the coefficient of the solution $v(\lambda)$ (unique up to a sign). The solution corresponding to $a(\lambda) = -\lambda$ is the so-called basis solution $\bar{v}(\lambda)$; its first coefficients are

$$v_0 = 1, \quad v_1 = 1, \quad v = 2(u_{xx} + 3u^2), \tag{2.9}$$
$$v_3 = 2(u_{xxxx} + 5u_x^2 + 10u_{xx}u + 10u^3), \tag{2.10}$$
and so on, namely the gradients of the first KdV Hamiltonians. In what follows we shall consider also the function $v(\lambda) = c(\lambda)\bar{v}(\lambda)$, which is a solution of (2.3) for
\begin{equation}
\begin{aligned}
a(\lambda) = -\lambda c^2(\lambda), \quad c(\lambda) = 1 + \sum_{j\geq 1} c_j \lambda^{-j},
\end{aligned}
\end{equation}
where the coefficients $c_j$ are free parameters. In this case the first 1-forms of the hierarchy are $v_0 = 1$, $v_1 = \bar{v}_1 + c_1$, $v_2 = \bar{v}_2 + c_1 \bar{v}_1 + c_2$, and so on.

The coefficient $B_k$ can be expressed through the GD polynomials. For each Laurent series $v(\lambda)$ let us consider the functions $B_k(\lambda) := B_{\lambda}(v(\lambda), v^{(k)}(\lambda))$, where these functions have the form
\begin{equation}
\begin{aligned}
B_k(\lambda) = \lambda^{k+1} v_0^2 + \sum_{j=1}^{k-1} \lambda^{k-j} (p_{0j} + v_0 v_{j+1}) + \sum_{j=0}^{\infty} \lambda^{-j} p_{jk}.
\end{aligned}
\end{equation}
It can be shown that
\begin{equation}
\begin{aligned}
B_{-1} = -v_0^2, \quad B_k = p_{0k} - v_0 v_{k+1}.
\end{aligned}
\end{equation}
Furthermore, if $v(\lambda)$ is a solution of (2.3), the coefficients $p_{jk}$ in (2.12) are polynomials in $u$ and its $x$-derivatives. They will be referred to as Gelfand-Dickey (GD) polynomials and the function $B_{\lambda}$ as their generating function.

The fundamental property of the GD polynomials, stemming from (2.3), (2.12) is the following relation with the gradients $v_j = d\bar{h}_j$ and the bi-Hamiltonian vector field $X_k$:
\begin{equation}
\begin{aligned}
d \frac{d}{dx} p_{jk} = v_j X_k.
\end{aligned}
\end{equation}
We report some GD polynomials to be used in what follows ($v_0 = 1$):
\begin{equation}
\begin{aligned}
p_{00} &= 4u - v_1, \\
p_{01} &= 8uv_1 - v_1^2 - v_2 + 2v_{1xx}, \\
p_{02} &= 4uv_1^2 + 8uv_2 - 2v_1v_2 - v_3 - v_1^2 + 2v_1v_{1xx} + 2v_{2xx}, \\
p_{12} &= 8uv_1v_2 - v_2^2 + 4uv_3 - v_1v_3 - v_4 + 2v_1v_2v_2 + 2v_2v_{1xx} + 2v_1v_{2xx} + v_{3xx}, \\
p_{kk} &= 2v_{kxx}v_k - v_k^2 + 4uv_k^2.
\end{aligned}
\end{equation}
The GD polynomials corresponding to the basis solution $\bar{v}(\lambda)$ are the polynomials defined in [37], proposition 12.1.12.

\section*{2.1 The method of stationary flows}

The method of stationary flows was developed in order to reduce the flows of the KdV hierarchy onto the set $M_g$ of fixed points of the $g$th flow $X_g$ of the hierarchy
\begin{equation}
\begin{aligned}
M_g := \{ u|X_g(u, u_x, \ldots, u^{(2g+1)}) = 0 \}.
\end{aligned}
\end{equation}
As $M_g$ is odd-dimensional it can not be a symplectic manifold; nevertheless we will show that it is a bi-Hamiltonian manifold; it will be referred to as extended phase space. Moreover, $M_g$ is naturally foliated, on account of (2.2) and (2.4), by a one-parameter family of $2g$-dimensional submanifolds $S_g$ given by

$$S_g := \{ u|v_{g+1}(u, u_x, \ldots, u^{(2g)} = c \} \tag{2.17}$$

($c$ being a constant parameter), which are invariant manifolds with respect to each vector field of the KdV hierarchy, due to the invariance of the functions $v_k$. So $M_g$ can be parametrized naturally by $v_1, \ldots, v_{g+1}$ and by their $x$-derivatives $v_1 x, \ldots, v_{2g} x$. We shall use these coordinates in what follows.

From the computational point of view, one proceeds as follows.

(i) Due to (2.3) and (2.5), the manifold $M_g$ is defined by the solutions $u$ of the equation

$$B^\lambda(v(\lambda), v^{(g)}(\lambda)) = \lambda^g a(\lambda) \tag{2.18}$$

where $v(\lambda) = \sum_{j=0}^{g} v_j \lambda^{-j}$, $a(\lambda) = \sum_{j=-1}^{2g} a_j \lambda^{-j}$. In particular, if $a(\lambda) = -\lambda c^2(\lambda)$, $M_g$ is given by

$$M_n = \left\{ u|\bar{X}_n + \sum_{j=1}^{n} c_j \bar{X}_{n-j} = 0 \right\} \tag{2.19}$$

i.e. by the solutions of the Lax-Novikov equations. Taking into account (2.12) and choosing $a_{-1} = -1$, by equation (2.18) the coefficients of $\lambda^{g+1}$ we get $v_0^2 = 1$; from now on we put $v_0 = 1$. Moreover, equating the coefficients of the other powers of $\lambda$ we get the following system:

$$p_{0k} - v_{k+1} = a_k, \quad (k = 0, \ldots, n - 1) \quad p_{jn} = a_{n+j}, \quad (j = 0, \ldots, n). \tag{2.20}$$

(ii) In order to obtain the first Poisson tensor $P_0$, we eliminate $u = v_1/2 + a_0/4$ from (2.20) using the first equation ($k = 0$) and we extract the system of $n$ second-order ODEs in the $v_j$ ($j = 1, \ldots, n$):

$$p_{0k} - v_{k+1} = a_k, \quad (k = 1, \ldots, g - 1) \quad p_{0g} = a_g \tag{2.21}$$

which will be referred as $P_0$-system. The remaining equations (2.20) will furnish a set of $g$ independent integrals of motion. In order to obtain a second Poisson structure, we consider the following system: ($P_1$-system)

$$p_{0k} - v_{k+1} = a_k, \quad (k = 1, \ldots, g - 1) \quad p_{g0} = a_g \tag{2.22}$$

with $u$ as above.

(iii) The $P_0$-system (2.21) and the $P_1$-system (2.22) can be written as canonical Hamiltonian systems

$$r_{kk} = \frac{\partial H_{(0)}^{(0)}}{\partial s_k}, \quad s_{kx} = -\frac{\partial H_{(0)}^{(0)}}{\partial r_k}. \tag{2.23}$$
have $n$ integrals of motion given by

$$K_j = -\frac{1}{8} p_{jg}, (j = 1, \ldots, g),$$

$$H_j = -\frac{1}{8} p_{jg} x = a_{g+j}, (j = 0, \ldots, g - 1).$$

Moreover, the map $\Phi : M_g \to M_g$ in the extended phase space generates a second Poisson structure.

### 3 Integrable dynamical systems related to hierarchy of stationary KdV equations

In the recent years, remarkable progress has been achieved in the description of those quasi(periodic) potentials which belong to a given spectrum. Many integrable systems of differential equations are shown to be closely connected with Hill’s equation in the case of a finite gap potential. The coupled Neumann system, The Neumann system, and the Rosochatius systems are examples of this type.

In this paragraph we are concerned with the following completely integrable systems.

The Garnier system

$$\xi_{ixx} = \left(2 \sum_{j=1}^{g} \xi_j \eta_j + \tilde{a}_i\right) \xi_i, \quad \eta_{ixx} = \left(2 \sum_{j=1}^{g} \xi_j \eta_j + \tilde{a}_i\right) \eta_i.$$  \hspace{1cm} (3.1)

The $g$-dimensional anisotropic harmonic oscillator in radial quartic potential, is obtained when $\xi_i = \eta_i, i = 1, \ldots, g$

$$\xi_{ixx} = \left(2 \sum_{j=1}^{g} \xi_j^2 + \tilde{a}_i\right) \xi_i.$$  \hspace{1cm} (3.2)

Another interesting integrable system was proposed recently \[48\], we call it the Rosochatius-Wojciechowski system. In our context, this system is obtained by the Deift elimination procedure. Let

$$\xi_i = \psi_i \exp(\theta_i), \eta_i = \psi_i \exp(-\theta_i), \sqrt{f_i} = \psi_i^2 \theta_i,$$

then equations (3.1) transform to the Rosochatius-Wojciechowski system

$$\psi_{ixx} = \left(2 \sum_{j=1}^{g} \psi_j^2\right) \psi_i + \tilde{a}_i \psi_i - f_i / \psi_i^3, i = 1, \ldots, g.$$  \hspace{1cm} (3.4)
and the Hamiltonian is given by

\[
H = \sum_{j=1}^{g} \chi_j^2 - \left( \sum_{k=1}^{g} \psi_k^2 \right)^2 - \sum_{j=1}^{n} \hat{a}_j \psi_j^2 - \sum_{j=1}^{g} \hat{f}_j \psi_j^2,
\]

(3.5)

where \( \chi_j = \psi_{jx} \) are canonical momenta.

(ii) The coupled Neumann system

\[
\tilde{\xi}_{ixx} + 2 \sum_{j=0}^{g} b_j \tilde{\xi}_j \tilde{\eta}_j + \tilde{\xi}_{jx} \tilde{\eta}_{jx} = b_i \tilde{\xi}_i, \]

(3.6)

\[
\tilde{\eta}_{ixx} + 2 \sum_{j=0}^{g} b_j \tilde{\xi}_j \tilde{\eta}_j + \tilde{\xi}_{jx} \tilde{\eta}_{jx} = b_i \tilde{\eta}_i.
\]

(3.7)

with constraint \( \sum_{i=0}^{g} \tilde{\xi}_i \tilde{\eta}_i = 1 \), where \( b_0 < b_1 < \ldots < b_g \) are fixed real numbers. The Neumann system is obtained when \( \xi_i = \eta_i \)

\[
\tilde{\xi}_{ixx} + 2 \sum_{j=0}^{g} b_j \tilde{\xi}_j \tilde{\xi}_j^2 + \tilde{\xi}_{jx} \tilde{\xi}_{jx} = b_i \tilde{\xi}_i.
\]

(3.8)

This system describes the motion of uncoupled harmonic oscillators \( \tilde{\xi}_{ixx} = b_i \tilde{\xi}_i \), constrained by the force \( \sum_{i=0}^{g} (b_i \tilde{\xi}_i^2 + \tilde{\xi}_{ix}^2) \) to move on the unit sphere \( \sum_{i=0}^{g} \tilde{\xi}_i^2 = 1 \).

Let

\[
\tilde{\xi}_i = \tilde{\psi}_i \exp(\tilde{\theta}_i), \quad \tilde{\eta}_i = \tilde{\psi}_i \exp(-\tilde{\theta}_i), \quad \sqrt{\tilde{f}_i} = \tilde{\psi}_i^2 \tilde{\theta}_{ix},
\]

(3.9)

then by Deift procedure (3.9), equations (3.7) transform to Rosochatius system

\[
\tilde{r}_{ixx} = - \left( \sum_{j=0}^{g} b_j \tilde{r}_j^2 + \tilde{r}_{jx}^2 - \frac{\tilde{f}_j}{\tilde{r}_j} \right) \tilde{r}_i - \frac{\tilde{f}_j}{\tilde{r}_j} + b_i \tilde{r}_i.
\]

(3.10)

where \( \sum_{i=0}^{g} \tilde{r}_i^2 = 1 \).

4 Baker-Akhiezer function

We review in this section some basic facts about Baker–Akhiezer function which will be used in the sequel.

Let \( K \) be the hyperelliptic Riemann surface \( \mu^2 = \prod_{i=0}^{2g} (\lambda - \lambda_i) = R(\lambda) \). The points of \( K \) are pairs \( P = (\lambda, R) \) and \( \lambda(P) \) is the value of the natural projection \( P \to \lambda(P) \) of \( K \) to the complex projective line \( \mathbb{C}P^1 \).

For given nonspecial divisor \( D \), there is an unique Baker-Akhiezer (BA) function \( \Psi(t, \lambda) \), such that

(i) the divisor of the poles of \( \Psi \) is \( D \),
(ii) $\Psi$ is meromorphic on $K \backslash \infty$

(iii) when $P \to \infty$

\[
\Psi(x, P) \exp(-kx) = 1 + \sum_{s=1}^{\infty} m_s(x) k^{-s}, \quad (4.1)
\]

is holomorphic and $k = \sqrt{\lambda(P)}$ is a local parameter near $P = \infty$.

There is a unique function $u(x)$ such that

\[
\Psi_{xx} - u(x) \Psi = \lambda(P) \Psi, \quad (4.2)
\]

where $\Psi$ is a BA function. Inserting expansion (4.1) into (4.2), we obtain

\[
\Psi_{xx} - 2m_{1x}(x) \Psi - \lambda(P) \Psi = \exp(kx) O(k^{-1}), \quad (4.3)
\]

and due to the uniqueness of $\Psi$, we prove (4.2), with $u(x) = 2m_{1x}(x)$.

By the Riemann-Roch theorem, there exists a unique differential $\tilde{\Omega}$ and a nonspecial divisor $D^r$ of degree $g$ such that the zeros of $\tilde{\Omega}$ are $D + D^r$ and the expansion at $P = \infty$, $\tilde{\Omega}(P) = (1 + O(k^{-2})) dk$.

For given nonspecial divisor $D^r$, there exists a unique dual Baker-Akhiezer (BA) function such that

(i) the divisor of the poles of $\Psi$ is $D^r$,

(ii) $\Psi$ is meromorphic on $K \backslash \infty$

(iii) when $P \to \infty$

\[
\Psi^\tau(x, P) \exp(-kx) = 1 + \sum_{s=1}^{\infty} \tilde{m}_s(x) k^{-s}, \quad (4.4)
\]

Fix $\tau$ to be the hyperelliptic involution $P = (\lambda, R) \to (\lambda, -R)$, then we have $D^r = \tau D$, $\Psi^\tau(x, P) = \Psi(x, \tau P)$. Let $\sum_{i=1}^{g} \mu_i(0)$ be the $\lambda$-projection of $D$, and $\sum_{i=1}^{g} \mu_i(x)$ be the $\lambda$-projection of the zero divisor of $\Psi(x, P)$. The function $\Psi(t, P)\Psi^\tau(t, P)$ is meromorphic on $\mathbb{CP}^1$ and the following identity takes place

\[
\Psi(x, P)\Psi^\tau(x, P) = \frac{F(x, \lambda)}{F(0, \lambda)}, \quad (4.5)
\]

where $F(x, \lambda) = \prod_{i=1}^{g} (\lambda - \mu_i(x))$. Introduce the Wronskian

\[
\{\Psi(x, P), \Psi^\tau(x, P)\} = \frac{\Psi_x(x, P)\Psi^\tau(x, P) - \Psi(x, P)\Psi^\tau_x(x, P)}{2\sqrt{R(\lambda)}} = (4.6)
\]

\[
\prod_{i=1}^{g} (\lambda - \mu_i(0))^{-1},
\]

and the differential $\tilde{\Omega}$ is given explicitly by

\[
\tilde{\Omega}(P) = \frac{1}{2} \prod_{i=1}^{g} (\lambda - \mu_i(0))/ \sqrt{R(\lambda)} d\lambda. \quad (4.7)
\]
We assume that $E(P)$ is a meromorphic function on $K$ with $g + 1$ simple poles $\infty, p_1, \ldots, p_g$ and at $P \to \infty$, $E(P) = k + \ldots$, and $\tilde{E}(P)$ is meromorphic function with $g + 1$ simple poles $q_0, q_1, \ldots, q_g$ and at $P \to \infty$, $\tilde{E}(P) = k^{-1} + \ldots$. We also suppose that the divisors of poles of $E(P)$ and $\tilde{E}(P)$ are different from $D, D^\tau$.

Let

$$\tilde{\xi}_i = \tilde{\xi}_i^0 \Psi(x, q_i), \quad \tilde{\eta}_i = \tilde{\eta}_i^0 \Psi^\tau(x, q_i),$$

$$\tilde{\xi}_i^0 \eta_i^0 = \text{Res}_{p=q_i} \tilde{\Omega}, \quad b_i = \lambda(q_i), \quad i = 0, \ldots, g \quad (4.8)$$

$$\tilde{\xi}_i^0 \eta_i^0 = \text{Res}_{p=q_i} \tilde{\Omega}, \quad a_i = \lambda(p_i), \quad i = 1, \ldots, g \quad (4.9)$$

then

$$u(x) = -\left( \sum_{i=0}^g b_i \tilde{\xi}_i \tilde{\eta}_i + \tilde{\xi}_{ix} \tilde{\eta}_{ix} \right), \quad \sum_{i=0}^g \tilde{\xi}_i \tilde{\eta}_i = 1$$

$$u(x) = 2 \sum_{i=1}^g \xi_i \eta_i + \text{const.} \quad (4.10)$$

Let us construct the meromorphic differential $\tilde{E} \Psi \Psi^\tau \tilde{\Omega}$. By direct computations we have

$$\sum_{i=0}^g \text{Res}_{p=q_i} \tilde{E} \Psi \Psi^\tau \tilde{\Omega} + \text{Res}_{p=\infty} \tilde{E} \Psi \Psi^\tau \tilde{\Omega} = \sum_{i=0}^g \tilde{\xi}_i \tilde{\eta}_i - 1 = 0, \quad (4.11)$$

where $\tilde{\xi}_i^0 \eta_i^0 = \text{Res}_{p=q_i} \tilde{\Omega}$. Differentiating $\sum_{i=0}^g \tilde{\xi}_i \tilde{\eta}_i = 1$ twice and using Eq. (4.10), we obtain first expression in (4.11). The eigenvalue equations

$$\tilde{\xi}_{i xx} = (\lambda(q_i) + u(x)) \tilde{\xi}_i, \quad (4.12)$$

$$\tilde{\eta}_{i xx} = (\lambda(q_i) + u(x)) \tilde{\eta}_i, \quad (4.13)$$

by replacing $u(x)$ from (4.10) are the coupled Neumann system (3.7). By computations of the same kind, we have

$$\sum_{i=1}^g \text{Res}_{p=p_i} \tilde{E} \Psi \Psi^\tau \tilde{\Omega} + \text{Res}_{p=\infty} \tilde{E} \Psi \Psi^\tau \tilde{\Omega} = \sum_{i=1}^g \xi_i \eta_i - u(x) + \frac{1}{2} \text{const.} = 0,$$

where $\xi_i^0 \eta_i^0 = \text{Res}_{p=p_i} \tilde{\Omega}$. The corresponding eigenvalue equations are the Garnier system (3.1).

### 4.1 Spectral interpretation

Let $p$ be a positive real divisor of degree $g$ on a real hyperelliptic curve

$$\mu^2 = R(\lambda) = \prod_{i=0}^{2g}(\lambda - \lambda_i), \quad \lambda_0 < \lambda_1 < \ldots, < \lambda_{2g}. \quad (4.15)$$
The projection $\lambda(p_i)$ lie in the closed lacunae $[\lambda_{2i-1}, \lambda_{2i}]$. The following considerations, due to Jacobi, can be used to construct such a divisor.

Each divisor $p$ determines and is determined by a system of polynomials

$$\tilde{A}(\lambda) = \prod_{i=1}^{g}(\lambda - \lambda(p_i)), \quad \tilde{C}(\lambda) = \tilde{A}(\lambda) \sum_{i=1}^{g} \frac{\sqrt{R(p_i)}}{A'(p_i)(\lambda - \lambda(p_i))},$$

$$\tilde{B}(\lambda) = \lambda^{g+1} + \ldots,$$

of degrees $g$, $g - 1$, $g + 1$, respectively, with $R = \tilde{C}^2 - \tilde{A}\tilde{B}$. The complementary divisor $q$ is also determined by this construction. This is the content of the following step.

For a given spectral data

$$\lambda_0 = 0 < \lambda_1 < \ldots < \lambda_{2g}, \quad \lambda(p_i) \in [\lambda_{2i-1}, \lambda_{2i}], \quad i = 1, \ldots, g$$

there exists

$$\lambda(q_i), \quad i = 0, \ldots, g \quad \lambda(q_0) \in (-\infty, \lambda_0], \quad \lambda(q_i) \in [\lambda_{2i-1}, \lambda_{2i}],$$

such that $R = \tilde{C}^2 - \tilde{A}\tilde{B}$, and the projections $\lambda(q_i)$ are the roots of $\tilde{B}$.

Note that the functions $E(P)$, $\tilde{E}(P)$ are meromorphic on $K$ and the following formulas are immediate

$$E(P) = \left(\sqrt{R(\lambda)} + \tilde{C}(\lambda)/\tilde{A}(\lambda), \quad \tilde{C}(p_i) = \sqrt{R(p_i)}$$

$$\tilde{E}(P) = \left(\sqrt{R(\lambda)} + \tilde{C}(\lambda)/\tilde{B}(\lambda), \quad \tilde{C}(q_i) = -\sqrt{R(q_i)}$$

Now we recall some facts from the periodic theory of Hill’s equation. We suppose that $u(x)$ is a real finite-gap potential, i.e. the operator $L$ has only $2g + 1$ simple eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_{2g}$ and the rest of the spectrum consists of double eigenvalues. The periodic spectra of $L$ is determined by the combined eigenvalues of the periodic

$$L f_{2i} = \lambda_{2i} f_{2i}, \quad f(x + 1) = f(x), \quad i = 0, \ldots, g$$

and the antiperiodic

$$L f_{2i-1} = \lambda_{2i-1} f_{2i-1}, \quad f(x + 1) = -f(x), \quad i = 1, \ldots, g$$

eigenvalue equations. The intervals $(-\infty, \lambda_0], [\lambda_{2i-1}, \lambda_{2i}]$ are termed lacunae. The Floquet solutions(periodic BA function) and the corresponding Floquet multipliers, are given by

$$\Psi(x, \lambda) = \left[F(x, \lambda)/F(0, \lambda)\right]^{1/2} \exp\left(\int_0^x \sqrt{R(\lambda)}/F(x', \lambda)dx'\right)$$

$$\Psi(x + 1, \lambda) = \rho_+(\lambda)\Psi(x, \lambda)$$

$$\Psi^\tau(x, \lambda) = \left[F(x, \lambda)/F(0, \lambda)\right]^{1/2} \exp\left(-\int_0^x \sqrt{R(\lambda)}/F(x', \lambda)dx'\right)$$

$$\Psi^\tau(x + 1, \lambda) = \rho_-(\lambda)\Psi^\tau(x, \lambda), \quad \rho_\pm = \exp(\pm\tilde{p}(\lambda))$$
where
\[
\tilde{p}(\lambda) = \int_0^1 \sqrt{R(\lambda)/F(x, \lambda)} \, dx \tag{4.27}
\]

Note that if \( \lambda \) is in the periodic spectrum, \( \Psi(x, \lambda_{2i}) = f_{2i}, i = 0, \ldots, g \) is a periodic eigenfunction, and \( \Psi(x, \lambda_{2i-1}) = f_{2i-1}, i = 1, \ldots, g \) is an antiperiodic eigenfunction. It is well known that the projections of the zeros of the Floquet solution define the auxiliary spectrum of \( L \).

The following expressions hold
\[
\xi_i \eta_i = \prod_{j=1}^{g} (\lambda(p_i) - \mu_j(x))/\tilde{A}'(p_i), \tag{4.28}
\]
\[
\xi_i^0 \eta_i^0 = \prod_{j=1}^{g} (\lambda(p_i) - \mu_j(0))/\tilde{A}'(p_i),
\]
\[
\tilde{\xi}_i \tilde{\eta}_i = \prod_{j=1}^{g} (\lambda(q_i) - \mu_j(x))/\tilde{B}'(q_i), \tag{4.29}
\]
\[
\tilde{\xi}_i^0 \tilde{\eta}_i^0 = \prod_{j=1}^{g} (\lambda(q_i) - \mu_j(0))/\tilde{B}'(q_i).
\]

Using (4.19), (4.20) we obtain
\[
\text{Res}_{P=p_i} E(P)\Psi \Psi^* \tilde{\Omega} = x_i^0 y_i^0 \Psi(x, p_i) \Psi^*(x, p_i)
\]
\[
= \prod_{j=1}^{g} (\lambda(p_i) - \mu_j(x))/\tilde{A}'(p_i),
\]
where \( \xi_i^0 \eta_i^0 \) is given by (4.28).

Let
\[
e_{2i}^2 = \frac{\prod_{j \neq i}^{g} (\lambda_{2i} - \lambda_2)}{\prod_{j \neq 0}^{g} (\lambda_{2i} - \mu_j(0))}, \quad f_{2i}^2 = \frac{\prod_{j \neq i}^{g} (\lambda_{2i} - \mu_j(t))}{\prod_{j \neq 0}^{g} (\lambda_{2i} - \mu_j(0))} = \Psi^2(\lambda_{2i}), \tag{4.30}
\]
i = 0, \ldots, g, then the expressions (4.10), (4.29) are the famous McKean-Moerbeke expansion of the potential \( u(x) \) in terms of squares of the eigenfunctions
\[
u(x) = - \left( 2 \sum_{i=0}^{g} \lambda_{2i} f_{2i}^2/e_{2i} + \sum_{i=1}^{g} \lambda_{2i-1} - \sum_{i=1}^{g} \lambda_{2i} + \lambda_0 \right), \tag{4.31}
\]
where the following identity among the squares of eigenfunctions hold on
\[
\sum_{i=0}^{g} e_{2i}^{-2} f_{2i}^2 = 1. \tag{4.32}
\]

The results of this section, may be summarized by following:
Let $u(x)$ be a real nonsingular finite-gap potential. There exists $g$ eigenfunctions $\Psi(p_1), \ldots, \Psi(p_g)$ and $g + 1$ eigenfunctions $\Psi(q_0), \ldots, \Psi(q_g)$ of Hill's equation, corresponding to the eigenvalues $\lambda(p_1), \ldots, \lambda(p_g)$ and $\lambda(q_0), \ldots, \lambda(q_g)$, respectively, such that

(i) 

$$u(x) = 2 \sum_{i=1}^{g} \Psi(p_i) \Psi^\tau(p_i) e_i^{-2} + 2 \sum_{i=1}^{g} \lambda(p_i) - 2 \sum_{i=0}^{2g} \lambda_i,$$  

$$e_i^{-2} = \prod_{j=1}^{g} (\lambda(p_i) - \mu_j(0))/\tilde{A}' \quad \Psi(p_i) \equiv \Psi(x, \lambda)|_{\lambda=p_i}, \ i = 1, \ldots, g.$$  

(ii) 

$$u(x) = 2 \sum_{i=0}^{g} \lambda(q_i) \Psi(q_i) \Psi^\tau(q_i) e_i^{-2} - 2 \sum_{i=0}^{g} \lambda(q_i) + \sum_{i=0}^{2g} \lambda_i,$$  

$$e_i^{-2} = \prod_{j=1}^{g} (\lambda(q_i) - \mu_j(0))/\tilde{B}' \quad \Psi(q_i) \equiv \Psi(x, \lambda)|_{\lambda=q_i}, \ i = 0, \ldots, g,$$  

$$\sum_{i=0}^{g} e_i^{-2} \Psi(q_i) \Psi^\tau(q_i) = 1.$$  

The corresponding eigenvalue equations are the Garnier and coupled Neumann system.

Let 

$$e_{2i-1}^{2} = \prod_{i \neq j}^{g} (\lambda_{2i-1} - \lambda_{2j-1})/ \prod_{j=1}^{g} (\lambda_{2i-1} - \mu_j(0)),$$  

$$f_{2i-1}^{2} = \prod_{j=1}^{g} (\lambda_{2i-1} - \mu_j(x))/ \prod_{j=1}^{g} (\lambda_{2i-1} - \mu_j(0)), \ i = 1, \ldots, g$$  

then we have the following expansion of the potential $u(x)$ in terms of squares of antiperiodic eigenfunctions

$$u(x) = 2 \sum_{i=1}^{g} f_{2i-1}^{2} e_{2i-1}^{2} + 2 \sum_{i=1}^{g} \lambda_{2i-1} - \sum_{i=0}^{2g} \lambda_i.$$  

We call the dynamical systems such in (i), (ii), complementary dynamical systems.
4.2 Solutions in terms of auxiliary spectrum of Hill’s equation

The solutions of the Garnier system in terms of auxiliary spectrum $\mu_j(x)$, $j = 1, \ldots, g$ are

$$\xi_i = \xi_i^0[F(x, a_i)/F(0, a_i)]^{1/2} \exp \left( \int_0^x \sqrt{R(a_i)/F(x', a_i)}dx' \right), \quad (4.37)$$

$$\eta_i = \eta_i^0[F(x, a_i)/F(0, a_i)]^{1/2} \exp \left( -\int_0^x \sqrt{R(a_i)/F(x', a_i)}dx' \right), \quad (4.38)$$

where $\mu_j(x)$ satisfies the following system of differential equations

$$\frac{d}{dx} \mu_j(x) = 2 \sqrt{R(\mu_j)/g \prod_{j \neq k} (\mu_j(x) - \mu_k(x))}, \quad (4.39)$$

with initial conditions

$$\mu_j(0) \in [\lambda_{2i-1}, \lambda_{2i}], \quad \xi_i^0 \eta_i^0 = F(0, a_i)/\prod_{i \neq j} (a_i - a_j). \quad (4.40)$$

Differentiating expressions

$$\xi_i \eta_i = \xi_i^0 \eta_i^0 \Psi(x, p_i) \Psi^\tau(x, p_i) = \prod_{j=1}^g (\lambda(p_i) - \mu_j(x)) / \tilde{A}'(p_i), \quad (4.41)$$

and

$$\xi_i x \eta_i - \xi_i \eta_i x = \{ \Psi(x, p_i), \Psi^\tau(x, p_i) \} \xi_i^0 \eta_i^0 , \quad (4.42)$$

we have

$$\Upsilon(x, P)|_{\lambda=\lambda(p_i)} = \frac{d}{dx} \log \xi_i(t)$$

$$= \left[ \frac{1}{2} \frac{d}{dx} \prod_{j=1}^g (\lambda - \mu_j(x)) + \sqrt{R(\lambda)} / \prod_{j=1}^g (\lambda - \mu_j(x)) \right] |_{\lambda=\lambda(p_i)} \quad (4.43)$$

$$\Upsilon^\tau(x, P)|_{\lambda=\lambda(p_i)} = \frac{d}{dx} \log \eta_i(t)$$

$$= \left[ \frac{1}{2} \frac{d}{dx} \prod_{j=1}^g (\lambda - \mu_j(x)) - \sqrt{R(\lambda)} / \prod_{j=1}^g (\lambda - \mu_j(x)) \right] |_{\lambda=\lambda(p_i)} \quad (4.44)$$

direct integration of (4.43), (4.44) gives the solutions (4.37), (4.38). The function $\Upsilon(x, P)$ has $g$ poles at $\mu_j(x)$, then the numerator of (4.43) is zero when $\lambda = \mu_j(x)$ and the following system takes place

$$\frac{d}{dx} \left( \prod_{j=1}^g (\lambda - \mu_j(x)) \right) |_{\lambda=\mu_j(x)} = 2 \sqrt{R(\mu_j(x))}. \quad (4.45)$$
This is another form of the system (4.39). In the same way, we can obtain the solutions of the coupled Neumann system by replacing \( \lambda(p_i), \ i = 1, \ldots, g \) with \( \lambda(q_i), \ i = 0, \ldots, g \) in (4.37), (4.38).

Let \( \lambda(p_i) \) be the antiperiodic eigenvalues \( \lambda_{2i-1}, \ i = 1, \ldots, g \). Then the exponential function in (4.37), (4.38) cancel, \( \xi^0_i = \eta^0_i \) and the solutions of the \( g \)-dimensional oscillator are

\[
\xi^2_i = \prod_{j=1}^g (\lambda_{2i-1} - \mu_j(x))/\prod_{i \neq j} (\lambda_{2i-1} - \lambda_{2j-1}). \tag{4.46}
\]

Let \( \lambda(p_i) = a_i \) be in a general position, i.e. \( \lambda(p_i) \in [\lambda_{2i-1}, \lambda_{2i}] \) and by Deift elimination procedure we may identify \( \xi_i \) with \( \xi^0_i[F(x, a_i)/F(0, a_i)]^{1/2} \) and

\[
\theta_i = \int_0^x \sqrt{R(a_i)/\prod_{i=1}^g (a_i - \mu_j(x'))} dx', \quad \xi^0_i = \eta^0_i, \tag{4.47}
\]

and, hence, the solutions of the Rosochatius-Wojciechowski system are

\[
\xi^2_i = \prod_{j=1}^g (a_i - \mu_j(x))/\prod_{i \neq j} (a_i - a_j). \tag{4.48}
\]

Inserting explicit expression of BA-function given by (4.23) in Hill’s equation we have

\[
\frac{1}{2} F_{xx}(x, \lambda)F(x, \lambda) - \frac{1}{4} F^2_x(x, \lambda) - (u(x) + \lambda)F^2(x, \lambda) = -R(\lambda). \tag{4.49}
\]

The polynomial solution of (4.49) below we will call Novikov polynomial \([25]\). Assuming that Novikov polynomial dependts on time \( t \), the zero curvature representation for KdV hierarchy of equations have the following form

\[
M_t(\lambda') - L_x(\lambda') + [M(\lambda'), L(\lambda')], \tag{4.50}
\]

where matrices \( L \) and \( M \) are given by

\[
L(\lambda') = \begin{pmatrix}
-F_x(x, t, \lambda')/2 & F(x, t, \lambda') \\
-F_{xx}(x, t, \lambda')/2 + Q(x, t, \lambda')F(x, t, \lambda') & F_x(x, t, \lambda')/2
\end{pmatrix},
\]

\[
M'(\lambda') = \begin{pmatrix}
0 & 1 \\
Q(x, t, \lambda') & 0
\end{pmatrix}.
\]

The equation (4.50) is equivalent to

\[
\frac{\partial Q}{\partial t} = -2 \left[ \frac{1}{4} \partial_x^3 - Q(x, t, \lambda') \partial_x - \frac{1}{2} Q_x(x, t, \lambda') \right] \cdot F(x, \lambda'). \tag{4.51}
\]

where \( Q(x, t, \lambda') = u(x) + \lambda' \) in the case of KdV hierarchy. Equation (4.51) is called the generating equation. For a different choices of the form of \( F(x, t, \lambda') \)
and \( Q(x, t, \lambda') \), this procedure leads to different hierarchies of integrable equations, as an example to the KdV, nonlinear Schrödinger and sine-Gordon hierarchies or to the Dym hierarchy. The Lax representation \( L_x = [M, L] \) yields the hyperelliptic curve \( K = (\mu', \lambda') \)

\[
\text{Det}(L(\lambda') - \mu'I) = 0,
\]

\[
\mu^2 = -\frac{1}{2} F F_{xx} + \frac{1}{4} F_t^2 + (\lambda' + u) F = R(\lambda'),
\]

generating the integrals of motion for stationary KdV hierarchy.

Using the equation (4.49) and the following expansion of potential \( u(x) \) in terms of squares of eigenvalue functions

\[
\xi_k^2(x) = \beta_k(x)
\]

have the form

\[
\frac{1}{2} \beta_{kxx} \beta_k - \frac{1}{4} \beta_k^2 - (u(x) + a_k) \beta_k = \frac{R(a_k)}{a_{kj}} = -f_k
\]

where we use the solutions \( F(x, a_k)/a_{kj} \) of Rosochatius-Wojciechowski system and \( a_{kj} \equiv \prod_{k \neq j} (a_k - a_j) \). Denoting \( f_k = R(a_k)/a_{kj} \) and \( d = \sum_{i=1}^g a_i - \frac{1}{2} \sum_{k=0}^{2g} \lambda_k \) for the original variable \( \beta_k = \xi_k^2 \) we have the following equation

\[
\xi_{kxx} = 2 \left( \sum_{i=1}^{g} \xi_i^2 + d \right) \xi_k - \frac{f_k}{\xi_k^3}.
\]

To understand the role of GD polynomials and of their generating function in the construction of a map between stationary and restricted flows of KdV equation and exact solution of completely integrable systems related to Hill’s equation let us consider the following system:

\[
p_{00} - v_1 = a_0, \quad P_0 \left( v_1 - \sum_{j=1}^{n} \beta_j \right),
\]

\[
P^{\lambda_k} \beta_k = 0, \quad (k = 1, \ldots, n)
\]

where \( \lambda_1, \ldots, \lambda_n \) are fixed parameters, \( P^{\lambda_k} := P_1 + 4 \lambda_k P_0 \) (\( P_0 \) and \( P_1 \) being the two KdV Poisson operators). This is a system of \( (g + 2) \) equations in \( u, v_1, \beta_1, \ldots, \beta_g \). The second equation will be referred to as the \( P_0 \)-restriction of the first KdV flow \( X_0 = P_0 v_1 = v_1x \), and the last \( n \) equations define the kernel of \( g \) Poisson operators extracted from the Poisson operator. On account of (2.16), (2.4) and (2.6) this system is equivalent to the following one:

\[
u = \frac{v_1 + a_0}{4}, \quad v_1 = \sum_{j=1}^{n} \beta_j + c, \quad B^{\lambda_k}(\beta_k, \beta_k) = f_k,
\]
where $c$ and $f_k$ are free parameters and $B^\lambda$ is just the generating function of the GD polynomials.

Using the first two equations to eliminate $u$ and $v_1$ from the last $g$ equations, one gets a system of $n$ second-order ODEs for $\beta_1, \ldots, \beta_g$:

$$2\beta_{kxx}\beta_k - \beta_{kxx}^2 + 2\beta_k^2 \left( \sum_{j=1}^{n} \beta_j + d \right) - \lambda_k \beta_k^2 = f_k, \quad k = 1, \ldots, g \quad (4.59)$$

where $d := c + a_0/2$. Introducing the so-called eigenfunction variables $\psi_j^2 = \beta_j$ and the momenta $\chi_j = \psi_{jx}$, equations $(4.59)$ can be written in canonical Hamiltonian form

$$\psi_{jx} = \frac{\partial K_G}{\partial \chi_j}, \quad \chi_{jx} = -\frac{\partial K_G}{\partial \psi_j}, \quad j = 1, \ldots, n \quad (4.60)$$

with Hamiltonian

$$K_G = \sum_{j=1}^{n} \chi_j^2 - \left( \sum_{k=1}^{n} \psi_j^2 \right)^2 - \sum_{j=1}^{n} a_j \psi_j^2 - \sum_{j=1}^{n} \frac{f_j}{\psi_j^2}. \quad (4.61)$$

A set of integrals of motion is

$$I_j = \chi_j^2 - \psi_j^2 \left( a_j + \sum_{k=1}^{g} \psi_k^2 \right) - \frac{f_j}{\psi_j^2} + \sum_{k \neq j}^{n} \frac{1}{a_{jk}} \left( - \frac{f_j \psi_k^2}{\psi_j^2} - \frac{f_k \psi_j^2}{\psi_k^2} + (\psi_j \chi_k - \psi_k \chi_j)^2 \right). \quad (4.62)$$

where we denote $a_{jk} = a_j - a_k$.

Now we shall construct a map between the $g$-th stationary flow and the previous restricted flow of the KdV hierarchy. To this end we extend the corresponding phase spaces, regarding some free parameters in the Hamiltonian functions as additional dynamical variables. As for the $P_1$-formulation of the stationary flow we extend its phase space to a $(3g+1)$-dimensional space, $\tilde{M}_n$, with coordinates $(q_k, p_k; a_0, \ldots, a_{g-1}, a_{2g})$; analogously we consider the $P_0$-formulation of the first restricted flow in the extended space $\tilde{M}_g$ with coordinates $(\psi_k, \chi_k; f_1, \ldots, f_k, d)$.

Let us consider the solutions $q_k$ of the dynamical equations $(2.24)$; then $v^{(g)}(\lambda)$ given by

$$v^{(g)}(\lambda) = \lambda (q^2(\lambda))^{(g-1)} - q^g, \quad (4.63)$$

with $q(\lambda) = 1 + \sum_{j=1}^{g} q_j \lambda^{-j}$, satisfies $(2.17)$, and consequently satisfies the following equation:

$$B^\lambda(v^{(g)}, v^{(n)}) = \lambda^{2g} d(\lambda), \quad (4.64)$$

where, as above, we put $u = v_1/2 + a_0/4$. So, for each $g$-tuple of distinct complex parameters $a_j$, any solution $v^{(g)}(\lambda)$ fulfills the system

$$B^{a_k}(v^{(g)}(a_k), v^{(g)}(a_k)) = a_k^{2g} d(a_k), \quad (k = 1, \ldots, g) \quad (4.65)$$
where \( v^{(g)}(a_k) := v^{(λ)}|_{λ=a_k} \). In order to have a solution also satisfying the second equation \( v_1 = \sum_{j=1}^{g} β_j + c \), the Lagrange interpolation formula can be used. It allows us to represent the polynomial \( v^n(λ) \) by

\[
v(n)(λ) = a(λ) \left( 1 + \sum_{j=1}^{g} \frac{β_j}{λ - a_j} \right), \tag{4.66}
\]

where \( a(λ) = \prod_{j=1}^{g} (λ - a_j) \), and

\[
β_j = \frac{v^{(g)}(a_k)}{a'(a_k)}(k = 1, \ldots, g). \tag{4.67}
\]

\((a'(λ)) means the derivative of \( a(λ) \) with respect to \( λ \).

Obviously the \( g \) functions \( β_k \) are solutions of the following system

\[
2β_{kxx}β_k - β_{kx}^2 + 2β_k^2 \left( \sum_{j=1}^{g} β_j - λ \right) - λ_k β_k^2 = \frac{λ_k^2 \hat{d}(a_k)}{(a'(a_k))^2}, \quad k = 1, \ldots, g \tag{4.68}
\]

Furthermore, \( β_k \) satisfy the so-called Bargmann constraint

\[
\sum_{j=1}^{g} (β_j - a_j) = v_1, \tag{4.69}
\]

as one can verify by means of (4.66).

The function \( B^λ \) is also a generating function of integrals of motion for Garnier system. Indeed evaluating the function \( B^λ \) by means of (4.66) and eliminating the first \( x \)-derivatives of \( χ_k \) by means of Hamilton equations \((2.24)\), one gets

\[
4 \sum_{j=1}^{g} \frac{I_j}{λ - λ_j} + \sum_{j=1}^{g} \frac{f_j}{(λ - λ_j)^2} + 2d - λ = \frac{λ^2 \hat{d}(λ)}{(a(λ))^2}, \tag{4.70}
\]

where \( I_j \) are the functions. Taking in this equation the residues at \( λ = a_j \) it follows that the functions \( I_j \) are integrals of motion along the flow \((2.24)\).

Let \( λ(q_i) = b_i \) be in a general position, i.e. \( λ(q_i) = b_i \in (-∞, λ_0], [λ_{2i-1}, λ_{2i}] \) and by Deift elimination procedure we may identify \( ψ_i \) with \( \xi^0[F(t, b_i)/F(0, b_i)]^{1/2} \) and

\[
\hat{θ}_i = \int_{0}^{x} \sqrt{R(b_i)} \prod_{i=1}^{g} (b_i - μ_j(x')) dx', \tag{4.71}
\]

and, hence, the solutions of the Rosochatius system are

\[
\hat{ψ}_i^2 = \prod_{j=0}^{g} (b_i - μ_j(x))/ \prod_{i ≠ j}^g (b_i - b_j). \tag{4.72}
\]
where $i, j = 0, \ldots, g$. Now we illustrate the general approach with some simple examples.

**Example 1** Let $u(x) = 6\wp(x + \omega')$ be the two-gap Lamé potential with simple periodic spectrum (see for example [16])

$$
\lambda_0 = -\sqrt{3g_2}, \quad \lambda_1 = -3e_0, \quad \lambda_2 = -3e_1, \quad \lambda_3 = -3e_2, \quad \lambda_4 = \sqrt{3g_2},
$$

and the corresponding Hermite polynomial have the form

$$
F(\wp(x + \omega'), \lambda) = \lambda^2 - 3\wp(x + \omega')\lambda + 9\wp^2(x + \omega') - \frac{9}{4}g_2.
$$

Consider the following genus 2 nonlinear anisotropic oscillator with Hamiltonian

$$
H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}(q_1^2 + q_2^2)^2 - \frac{1}{2}(a_1q_1^2 + a_2q_2^2),
$$

where $(q_i, p_i), i = 1, 2$ are canonical variables with $p_i = q_{ix}$ and $a_1, a_2$ are arbitrary constants. The simple solutions of this system are given in terms of Hermite polynomial

$$
q_1^2 = 2F(x, \lambda_1), \quad q_2^2 = 2F(x, \lambda_2),
$$

Let us list the corresponding solutions

- Periodic solutions expressed in terms of single Jacobian elliptic functions

The nonlinear anisotropic oscillator admit the following solutions:

$$
q_1 = C_1 \text{sn}(\alpha x, k), \quad q_2 = C_2 \text{cn}(\alpha x, k),
$$

where amplitudes $C_1, C_2$ and temporal pulsewidth $1/\alpha$ of are defined by parameters $a_1$ and $a_2$ as

$$
\alpha^2 k^2 = a_2 - a_1, \quad C_1^2 = a_2 + \alpha^2 - \alpha^2 k^2, \quad C_2^2 = a_1 + \alpha^2,
$$

where $0 < k < 1$.

Following our spectral method it is clear, that the solutions [4.78] are associated with eigenvalues $\lambda_2 = -e_2$ and $\lambda_3 = -e_3$ of one-gap Lamé potential.

- Periodic solutions expressed in terms of products of Jacobian elliptic functions
\[ q_1 = C \text{dn}(\alpha x, k) \text{sn}(\alpha x, k), \quad (4.79) \]
\[ q_2 = C \text{dn}(\alpha x, k) \text{cn}(\alpha x, k), \quad (4.80) \]

where \( \text{sn}, \text{cn}, \text{dn} \) are the standard Jacobian elliptic functions \( [54] \), \( k \) is the modulus of the elliptic functions \( 0 < k < 1 \), \( a \) the wave characteristic parameters: amplitude \( C \), temporal pulsewidth \( 1/\alpha \) and \( k \) are related to the physical parameters and, \( k \) through the following dispersion relations

\[
C^2 = \frac{2(4a_2 - a_1)}{5},
\]
\[
k^2 = \frac{(4a_2 - a_1)}{15},
\]
\[
\alpha^2 = \frac{5(a_2 - a_1)}{4a_2 - a_1}.
\]

We have found the following solutions of the nonlinear oscillator

\[
q_1 = C \alpha^2 k^2 \text{cn}(\alpha x, k) \text{sn}(\alpha x, k) \quad (4.81)
\]
\[
q_2 = C \alpha^2 \text{dn}^2(\alpha x, k) + C_1 \quad (4.82)
\]

where \( C, C_1, \alpha \) and \( k \) are expressed through parameters \( a_1 \) and \( a_2 \) by the following relations

\[
C^2 = \frac{18}{a_2 - a_1},
\]
\[
C_1 = \frac{C(4a_1 - a_2)}{5},
\]
\[
k^2 = \frac{2\sqrt{\frac{2}{3}(a_2^2 - a_1^2)}}{2\sqrt{\frac{2}{3}(a_2^2 - a_1^2) + aa_2 - 3a_1}},
\]
\[
\alpha^2 = \frac{1}{10}(2a_2 - 3a_1 + \sqrt{\frac{5}{3}(a_2^2 - a_1^2)}). \quad (4.83)
\]

- Periodic solutions associated with the two-gap Treibich-Verdier potentials

Below we construct the two periodic solutions associated with the Treibich-Verdier potential. Let us consider the potential

\[
u(x) = 6\psi(x + \omega') + 2\frac{(e_1 - e_2)(e_1 - e_3)}{\psi(x + \omega') - e_1} \quad (4.84)\]

and construct the solution in terms of Lamé polynomials associated with the eigenvalues \( \lambda_1, \lambda_2, \lambda_1 > \lambda_2 \)

\[
\lambda_1 = e_2 + 2e_1 + 2\sqrt{(e_1 - e_2)(7e_1 + 2e_2)},
\]
\[
\lambda_2 = e_3 + 2e_1 + 2\sqrt{(e_1 - e_3)(7e_1 + 2e_3)}. \quad (4.85)\]
The finite and real solutions \( q_1, q_2 \) have the form
\[
q_1 = C_1 \text{sn}(z, k) \text{dn}(z, k) + C_2 \text{sd}(z, k),
\]
\[
q_2 = C_3 \text{cn}(z, k) \text{dn}(z, k) + C_4 \text{cd}(z, k),
\]
(4.86)
where \( C_i, i = 1, \ldots, 4 \) are constants and have important geometrical interpretation \( [16] \). The concrete expressions in terms of \( k, \tilde{\lambda}_1, \tilde{\lambda}_2 \) are given in \([17]\).

Analogously we can find the elliptic solution associated with the eigenvalues
\[
\tilde{\lambda}_1 = e_2 + 2e_1 + 2\sqrt{(e_1 - e_2)(7e_1 + 2e_2)}, \quad \tilde{\lambda}_2 = -6e_1,
\]
(4.87)

We have
\[
q_1 = \tilde{C}_1 \text{dn}^2(z, k)
\]
(4.88)
\[
q_2 = C_1 \text{sn}(z, k) \text{dn}(z, k) + C_2 \text{sd}(z, k),
\]
(4.89)
where \( C \) is given in \([17]\).

The general formula for elliptic solutions of genus 2 nonlinear anisotropic oscillator is given in \([17]\)
\[
q_1 = 1 \tilde{\lambda}_2 - \tilde{\lambda}_1 \sum_{i=1}^{N} \phi(x - x_i)
\]
\[
+ 6 \sum_{1 \leq i < j \leq N} \phi(x - x_i)\phi(x - x_j) - \frac{Ng_2}{4} + \sum_{1 \leq i < j \leq 5} \lambda_i \lambda_j
\]
(4.90)
\[
q_2 = \frac{1}{\tilde{\lambda}_1 - \lambda_2} \left( 2\tilde{\lambda}_2^2 + 2\tilde{\lambda}_1 \sum_{i=1}^{N} \phi(x - x_i)
\right.
\]
\[
+ 6 \sum_{1 \leq i < j \leq N} \phi(x - x_i)\phi(x - x_j) - \frac{Ng_2}{4} + \sum_{1 \leq i < j \leq 5} \lambda_i \lambda_j
\]
(4.91)
where \( x_i \) are solutions of equations \( \sum_{i \neq j} \phi'(x_i - x_j) = 0, j = 1, \ldots, N \) and \( N \) is positive integer.

Example 2 Garnier type system. Consider the following genus 2 Garnier type system with Hamiltonian
\[
H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}(q_1^2 + q_2^2)^2 - 2\gamma_1 q_1^2 - 2\gamma_2 q_2^2 + \frac{f_1}{q_1} + \frac{f_2}{q_2},
\]
(4.92)
where \( (q_i, p_i), i = 1, 2 \) are canonical variables with \( p_i = q_{ix} \) and \( a_1, a_2 \) are arbitrary constants. The simple solutions of these system are given in terms of Hermite polynomial in the following form \([1.48]\)
\[
q_1^2 = 2F(x, a_1), \quad q_2^2 = 2F(x, a_2),
\]
(4.93)
the same settings of periodic spectra, Lamé potential and Hermite polynomi-
als (4.74) as in example 2. The main results from the general theory are the
following:

i) the parameters $f_1$ and $f_2$ are expressed in terms of algebraic curve by

\[ f_1 = \frac{R(a_1)}{a_1 - a_2}, \quad f_2 = \frac{R(a_2)}{a_2 - a_1}, \]

\[ R(\lambda) = (\lambda^2 - 3g_2)(\lambda + 3e_1)(\lambda + 3e_2)(\lambda + 3e_3). \]

ii) the parameters $a_1$ and $a_2$ must lie in one or other of the intervals

\[ [-\sqrt{3g_2}, 3e_1], [3e_2, 3e_3], [\sqrt{3g_2}, \infty) \]

(4.94)

and $\gamma_1 = 3a_1 + 2a_2$, $\gamma_2 = 2a_1 + 3a_2$.

These results are in complete agreement with solutions obtained by differ-
ent method in recent paper [58].

**Example 3** Simple solutions of the Hénon-Heiles type system.

We consider a generalized Hénon-Heiles type system with two-degres-
of freedom.

Its Hamiltonian is

\[ H_0 = \frac{1}{2}(p_1^2 + p_2^2) + q_1^3 + \frac{1}{2}q_1q_2^2 + \frac{a_0}{2q_2} + \frac{a_1}{4} \left( q_1^2 + \frac{1}{4}q_2^2 \right) - \frac{a_1}{4} q_1, \]

(4.95)

where $q_1, q_2, p_1, p_2$ are the canonical coordinates and momenta and $a_0, a_1, a_4$
are free constant parameters. This Hamiltonian encompasses the two cases
$a_0 = a_4 = 0$ and $a_0 = a_1 = 0$ introduced in [50]. Moreover $H_0$ is related with
the Hamiltonian

\[ H_H = \frac{1}{2}(\bar{p}_1^2 + \bar{p}_2^2) + \bar{q}_1^3 + \frac{1}{2}\bar{q}_1\bar{q}_2^2 + \frac{a_4}{8\bar{q}_2} + \frac{1}{2} \left( A\bar{q}_1^2 + B\bar{q}_2^2 \right), \]

(4.96)

through the map

\[ q_1 = \bar{q}_1 + \frac{A}{2} - 2B, \quad q_2 = \bar{q}_2, \quad a_0 = -2A + 12B, \quad a_1 = -A^2 + 16AB - 48B^2. \]

(4.97)

The function $H_H$ is the Hamiltonian of a classical integrable Hénon-Heiles sys-
tem with the additional term $a_4/8\bar{q}_2^2$.

The function (4.95) is the Hamiltonian of the vector field obtained reduc-
ing $X_0(u) = u_x$ to the stationary manifold $M_2$ given by the fixed points of the flow

\[ X_2 + c_1X_1 + c_2X_0 \]

\[ M_2 = u|u|^{(5)} + 10u_{xxx}u + 20u_{xx}u_x + 30u_xu^2 + c_1(u_{xxx} + 6u_xu) + c_2u_x = 0 \]

where $c_1 = -a_0/2$, $c_2 = -a_1/2 + a_0^2/4$.

It can be obtained specializing to the case $g = 2$ the Hamiltonian of the
$P_1$-system. In this case $H_2^{(1)} = H_0$ and the canonical coordinates and momenta
are, respectively, \( q_1 = v_1/2, q_2^2 = -v_2, p_1 = q_{1x}, p_2 = q_{2x} \). The integrals of motion obtained by the reduction of the GD polynomials are

\[
H_0 \equiv -\frac{1}{8}p_{01}|_x \\
H_2 \equiv -\frac{1}{8}p_{22}|_x = -\frac{a_4}{8} \\
H_1 \equiv -\frac{1}{8}p_{12}|_x =
\]

where \(-\frac{1}{8}p_{12}|_x\) is given by

\[
p_2q_1 - p_1p_2q_2 - \frac{1}{2}q_2^2q_1 - \frac{1}{8}q_4^2 + \frac{a_4q_1}{4q_2^2} - \frac{a_0}{4}q_1q_2^2 + \frac{a_1}{8}q_2^2.
\]

Next we will derive \((2 \times 2)\) matrix Lax representation for generalized Hénon-Heiles system (4.95). Using Lax representation \(L_x = [M, L]\), particular case of eq. (4.50) i.e. when there is no time dependence, we have

\[
F(x, \lambda) = \lambda^2 + \frac{1}{2}q_1\lambda - \frac{1}{16}q_2^2, \quad V = -F_x/2 = -\frac{1}{4}p_1\lambda + \frac{1}{16}q_2p_2, \\
W = -F_{xx}/2 + QF = \lambda^3 - \frac{1}{2}q_1 + \frac{1}{4}a_0)\lambda^2 + \\
\left(\frac{1}{4}q_1^2 + \frac{1}{16}q_2^2 - \frac{1}{16}a_1 + \frac{1}{8}a_0q_1\lambda + \frac{1}{16}p_2^2 + \frac{1}{64}a_4\right), \\
Q(x, \lambda) = \lambda - q_1 - \frac{1}{4}a_0.
\]

The corresponding algebraic curve have the form

\[
\mu^2 = \lambda^5 - \frac{1}{4}a_0\lambda^4 - \frac{1}{16}a_1\lambda^3 + \frac{8}{16}H_0\lambda^2 + 32H_1\lambda - \frac{1}{1024}a_4.
\]

Using explicit expression for Hermite polinomial (4.74) we obtain the following simple solutions for the system (4.95): \( q_1 = -6\psi(x + \omega'), \quad q_2^2 = -16(9\psi(x + \omega')^2 - \frac{9}{4}g_2). \)

where \(a_0 = 0, a_1 = 3.4.7g_2, A_4 = -3^4.4^4g_2g_3\).

5 2×2 Lax representation and \(r\)-matrix approach

The Lax equation for completely integrable systems discussed in the previous section

\[
L_x(\lambda) = [M(\lambda'), L(\lambda')],
\]

with matrices \(L\) and \(M\) given by

\[
L(\lambda') = \begin{pmatrix} V(x, \lambda') & U(x, \lambda') \\ W(x, \lambda') & -V(x, \lambda') \end{pmatrix} \\
M(\lambda') = \begin{pmatrix} 0 & 1 \\ Q(x, \lambda') & 0 \end{pmatrix}.
\]
is equivalent to the Garnier system, where \( U(x, \lambda'), V(x, \lambda'), W(x, \lambda'), Q(x, \lambda') \) have the form

\[
U(x, \lambda') = a(\lambda') \left( 1 - \sum_{i=1}^{g} \frac{\xi_i \eta_i}{\lambda' - a_i} \right), \quad V(x, \lambda') = -\frac{1}{2} U_x(x, \lambda') \tag{5.4}
\]

\[
W(x, \lambda') = a(\lambda') \left( \lambda' + \sum_{i=1}^{g} \xi_i \eta_i + \sum_{i=1}^{g} \frac{\xi_i \eta_i x}{\lambda' - a_i} \right), \quad Q(x, \lambda') = \lambda' + 2 \sum_{i=1}^{g} \xi_i \eta_i. \tag{5.5}
\]

Finally we point out one useful expression, which is easy to derive from Lax representation (5.1)

\[
W(x, \lambda') = U(x, \lambda') Q(x, \lambda') - \frac{1}{2} U(x, \lambda')_{xx}. \tag{5.6}
\]

The Lax representation yields the hyperelliptic curve \( K = (\mu', \lambda') \)

\[
\text{Det}(L(\lambda') - \mu'I) = 0, \tag{5.7}
\]

generating the integrals of motion \( H, F^{(i)}, i = 1, \ldots, g \). We have

\[
\mu^2 = V^2(x, \lambda') + U(x, \lambda') W(x, \lambda'), \tag{5.8}
\]

From (5.8) and explicit expressions of \( U(x, \lambda'), V(x, \lambda'), W(x, \lambda') \) we obtain

\[
\mu^2 = a(\lambda')^2 \left( \lambda' + \sum_{i=1}^{g} \frac{H_i}{\lambda' - a_i} + \frac{1}{4} \sum_{i=1}^{g} \frac{J_i^2}{(\lambda' - a_i)^2} + \frac{1}{2} \sum_{i=1}^{g} \frac{I_i}{\lambda' - a_i} \right), \tag{5.9}
\]

where

\[
I_i = \sum_{k \neq i} \frac{(\xi_k \eta_{ix} - \eta_{kix})(\eta_k \xi_{ix} - \xi_k \eta_{ix})}{a_k - a_i},
\]

\[
+ \sum_{k \neq i} \frac{(\xi_i \xi_{kx} - \xi_{kix})(\eta_k \eta_{ix} - \eta_k \eta_{ix})}{a_k - a_i},
\]

\[
H_i = \xi_{ix} \eta_i - \eta_{ix} \xi_i \left( \sum_{k=1}^{g} \xi_{k} \eta_{k} \right), \quad J_i = \xi_{ix} \eta_i - \xi_i \eta_{ix}, \tag{5.10}
\]

and \( \sum_{i=1}^{g} H_i \) is the Hamiltonian for Garnier system. Simple reduction \( \eta_i = \xi_i^* \) gives us the second flow of stationary vector nonlinear Schrödinger equation, where by * we denote complex conjugation. The complementary to the last system is complex Neumann system (see for example [63])

\[
\tilde{\xi}_{ixx} + 2 \left( \sum_{j=0}^{g} b_j |\tilde{\xi}_j|^2 + |\tilde{\xi}_{jx}|^2 \right) \tilde{\xi}_i = b_i \tilde{\xi}_i. \tag{5.11}
\]

with \( \sum_{i=1}^{g} |\xi_i|^2 = 1 \). Using the Deift elimination procedure we obtain new \( 2 \times 2 \) Lax pair for Rosochatius-Wojciechowski system. Below we list only the final results for Lax pair elements of considered in this paper dynamical systems:
• Rosochatius-Wojciechowski system

\[ U(x, \lambda') = a(\lambda') \left( 1 - \sum_{i=1}^{g} \frac{\psi_i^2}{\lambda' - a_i} \right), \quad V(x, \lambda') = -\frac{1}{2} U_x(x, \lambda') \]

\[ W(x, \lambda') = a(\lambda') \left( \lambda' + \sum_{i=1}^{g} \psi_i^2 + \sum_{i=1}^{g} \frac{1}{\lambda' - a_i} (\psi_{ix}^2 - f_i \psi_i^2) \right), \quad (5.12) \]

\[ Q(x, \lambda') = \lambda' + 2 \sum_{i=1}^{g} \psi_i^2. \]

• Rosochatius system

\[ U(x, \lambda') = b(\lambda') \left( \sum_{i=1}^{g} \frac{\psi_i^2}{\lambda' - b_i} \right), \quad V(x, \lambda') = -\frac{1}{2} U_x(x, \lambda') \]

\[ W(x, \lambda') = b(\lambda') \left( 1 + \sum_{i=1}^{g} \frac{1}{\lambda' - b_i} (\psi_{ix}^2 - f_i \psi_i^2) \right), \quad (5.13) \]

\[ Q(x, \lambda') = \lambda' + 2 \sum_{i=1}^{g} \psi_i^2, \]

where \( b(\lambda') = \prod_{i=0}^{g} (\lambda' - b_i) \).

• second stationary flow of vector NLSE

\[ U(x, \lambda') = a(\lambda') \left( 1 - \sum_{i=1}^{g} \frac{\lvert \xi_i \rvert^2}{\lambda' - a_i} \right), \quad V(x, \lambda') = -\frac{1}{2} U_x(x, \lambda') \]

\[ W(x, \lambda') = a(\lambda') \left( \lambda' + \sum_{i=1}^{g} \lvert \xi_i \rvert^2 + \sum_{i=1}^{g} \frac{\lvert \xi_{ix} \rvert^2}{\lambda' - a_i} \right), \quad (5.14) \]

\[ Q(x, \lambda') = \lambda' + 2 \sum_{i=1}^{g} \lvert \xi_i \rvert^2. \]

• complex Neumann system

\[ U(x, \lambda') = b(\lambda') \left( \sum_{i=1}^{g} \frac{\lvert \xi_i \rvert^2}{\lambda' - b_i} \right), \quad V(x, \lambda') = -\frac{1}{2} U_x(x, \lambda') \]

\[ W(x, \lambda') = a(\lambda') \left( 1 - \sum_{i=1}^{g} \frac{\lvert \xi_{ix} \rvert^2}{\lambda' - b_i} \right), \quad (5.15) \]

\[ Q(x, \lambda') = \lambda' + 2 \sum_{i=1}^{g} \lvert \xi_i \rvert^2. \]
Finally we want to point out that \( I \) for Rosochatius-Wojciechowski system coincide with expression given in (4.62). Another Lax equation have the following form

\[
L_x(\lambda') = [M(\lambda'), L(\lambda')] ,
\]

where matrices \( L \) and \( M \) are given by

\[
\begin{align*}
L(\lambda') &= \begin{pmatrix}
-F_x(x, \lambda')/2 & F(x, \lambda') \\
-F_{xx}(x, \lambda')/2 & F_x(x, \lambda')/2
\end{pmatrix} \\
M'(\lambda') &= \begin{pmatrix}
V(x, \lambda') & U(x, \lambda') \\
W'(x, \lambda') & -V(x, \lambda')
\end{pmatrix} \equiv \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} ,
\end{align*}
\]

where we made the following identification \( U(x, \lambda') = F(x, \lambda') \). The Poisson bracket relations for the matrix \( L(\lambda') \) \([59, 61]\) are closed into the following \( r \)-matrix algebra

\[
\{ L_1(\lambda'), L_2(\mu') \} = [r_{12}(\lambda', \mu'), L_1(\lambda')] - [r_{21}(\lambda', \mu'), L_2(\mu')] .
\]

Here the standard notations are introduced:

\[
\begin{align*}
L_1(\lambda') &= L(\lambda') \otimes I , \\
L_2(\mu') &= I \otimes L(\mu') ,
\end{align*}
\]

\[
\begin{align*}
r_{12}(\lambda, \mu) &= \frac{\Pi}{\lambda - \mu} \quad r_{21}(\lambda, \mu) = \Pi r_{12}(\mu, \lambda) \Pi ,
\end{align*}
\]

and \( \Pi \) is the permutation operator of auxiliary spaces.

The Poisson bracket relations for the Lax matrix \( L(\lambda') \) are preassigned by the initial symplectic structure. It is necessary to calculate only two brackets

\[
\{ F(x, \lambda'), F(x, \mu') \} = 0 ,
\]

and

\[
\begin{align*}
\{ V(x, \lambda'), F(x, \mu') \} &= \left\{ F(x, \lambda') \sum_{j=1}^g \frac{g_{jj} p_j(x)}{\lambda' - \mu_j(x)} \prod_{j=1}^g (\mu' - \mu_j(x)) \right\} \\
&= -F(x, \lambda') F(x, \mu') \sum_{j=1}^n \frac{g_{jj}}{(\lambda' - \mu_j(x))(\mu' - \mu_j(x))} \\
&= \frac{F(x, \lambda') F(x, \mu')}{\lambda' - \mu'} \sum_{j=1}^g \left( \frac{g_{jj}}{\lambda' - \mu_j(x)} - \frac{g_{jj}}{\mu' - \mu_j'(x)} \right) \\
&= \frac{1}{\lambda' - \mu'} [F(x, \mu') - F(x, \lambda')] ,
\end{align*}
\]
where we used a standard decomposition of rational function
\[ F(x, \lambda')^{-1} = \sum_{j=1}^{n} \frac{g_{jj}}{\lambda - \mu_j(x)}, \quad g_{jj} = \text{Res}_{\lambda = \mu_j(x)} F(x, \lambda')^{-1}(\lambda). \]
and the following definitions
\[ g_{jj} = \text{Res}_{\lambda = \mu_j(x)} F^{-1}(x, \lambda) = \frac{1}{\prod_{k \neq j} (\mu_j(x) - \mu_k(x))}, \quad (5.23) \]
\[ p_j(x) = V(x, \lambda)_{\lambda = \mu_j(x)} = \sqrt{R(\mu_j(x))}. \quad (5.24) \]

Another Poisson brackets may be directly derived from these brackets and by definition of the entries of the Lax matrix \( L(\lambda) \) via derivative of the single function \( F(x, \lambda') \)
\[ \{V(x, \lambda'), V(x, \mu')\} = 0 \]
\[ \{W(x, \lambda'), F(x, \mu')\} = \frac{d}{dx} \{V(x, \lambda'), F(x, \mu')\} = \frac{2}{\lambda' - \mu'} [V(x, \lambda') - V(x, \mu')] , \quad (5.25) \]
\[ \{W(x, \lambda'), F(x, \mu')\} = \frac{-1}{2} \frac{d^2}{dx^2} \{V(x, \lambda'), F(x, \mu')\} = \frac{1}{\lambda' - \mu'} [W(x, \lambda') - W(x, \mu')] , \]
\[ \{W(x, \lambda'), W(x, \mu')\} = \frac{-1}{2} \frac{d^3}{dx^3} \{V(x, \lambda'), F(x, \mu')\} = 0. \]

Applying the following transformation directly to the Lax representation \( L(\lambda') \) we obtain a family of the new Lax pairs \([59, 61]\)
\[ L'(\lambda') = L(\lambda') - \sigma_- \cdot [\phi(x, \lambda') F(x, \lambda')^{-1}]_N , \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad (5.26) \]
\[ M'(\lambda') = M - \sigma_- \cdot [\phi(x, \lambda') F(x, \lambda')^{-2}]_N = \begin{pmatrix} 0 & 1 \\ Q(x, \lambda') & 1 \end{pmatrix} . \]

Here \( \phi(x, \lambda') \) is a function on spectral parameter and \([z]_N \) means restriction of \( z \) onto the \( \text{ad}_{\lambda'} \)-invariant Poisson subspace of the initial \( r \)-bracket. For the rational \( r \)-matrix we can use the linear combinations of the following Taylor projections
\[ [z]_N = \left[ \sum_{k=-\infty}^{\infty} z_k \lambda^k \right]_N = \sum_{k=0}^{N} z_k \lambda^k, \quad (5.27) \]
or the Laurent projections.
New Lax matrix $L'({\lambda'})$ [58, 61] obeys the linear $r$-bracket, where constant $r_{ij}$-matrices substituted by $r'_{ij}$-matrices depending on dynamical variables.

$$r_{12}({\lambda'}, {\mu'}) \rightarrow r'_{12} = r_{12} - \left( \frac{[\phi({\lambda})F(x, {\lambda'})^{-2}, N] - [\phi({\mu})F(x, {\mu'})^{-2}, N]}{({\lambda'} - {\mu'})} \right) \cdot \sigma_{-} \otimes \sigma_{-}. \quad (5.28)$$

6 Conclusions

In this paper we have given new exact solutions of the physically significant completely integrable dynamical systems. These solutions can be interpreted as eigenfunctions of suitable differential operator. New example of complementary dynamical systems (stationary second flow of the vector nonlinear Schrödinger equation and complex Neumann system) is presented.

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