A note on abelian cubulated groups

Zachary Munro

McGill University, Montreal, QC, Canada

Correspondence
Zachary Munro, McGill University, 1017 Burnside Hall 805 Sherbrooke Street West, Montreal QC H2T 2T4, Canada.
Email: zachary.munro@mail.mcgill.ca

Abstract
Any abelian group acting freely on a CAT(0) cube complex is free abelian.

MSC 2020
20F65 (primary), 20K20 (secondary)

The aim of this note is to prove the following.

Main Theorem. Any abelian group acting freely on a CAT(0) cube complex is free abelian.

We consider actions by combinatorial automorphisms, and an action is free as long as no group element stabilizes a (finite-dimensional) cube. Note the converse of the theorem is obviously true, because the collection of groups acting freely on CAT(0) cube complexes is closed under direct sums. Also note that many non-free abelian groups act freely on CAT(0) polygonal complexes, for example, \( \mathbb{Q} \) acts freely on a CAT(0) 2-complex \( K \) so that \( \mathbb{Q} \setminus K \) has finite volume [1, II.7.15 Exercises].

We emphasize that the theorem relies crucially on the geometry of CAT(0) cube complexes. If an abelian group \( G \) acts freely on a CAT(0) cube complex, then all finitely generated subgroups are free abelian. This can easily be deduced independently of the theorem. However, this is not enough to conclude that \( G \) is free abelian. Indeed, a direct product \( \mathbb{Z}^\infty \) of infinite cyclic groups is not free abelian though all countable subgroups are free abelian.

In our following discussion, we will assume some familiarity with CAT(0) cube complexes. For a more detailed discussion, see [3]. \( X \) will always denote a CAT(0) cube complex, and \( d \) the combinatorial metric on the 0-cubes \( X^0 \). We allow \( X \) to be infinite-dimensional and even contain infinite-dimensional cubes, that is, infinite ascending sequences of cubes.

Definition 1. A wall \( \{ A, B \} \) of a set \( S \) is a partition of \( S \) into two nonempty subsets. An element of a wall is a halfspace. A wall \( \{ A, B \} \) of a set \( S \) separates points \( x, y \in S \) if \( x \) and \( y \) belong to different elements of the partition \( \{ A, B \} \). A wallspace \( Y = (S, \mathcal{W}) \) is a set \( S \) and a set of walls \( \mathcal{W} \) of \( S \) such that any two points of \( S \) are separated by at most a finite number of walls. Any subset \( M \subset S \) has an induced wallspace structure, where the walls of \( M \) are the restrictions of walls of \( S \). We call the...
dual CAT(0) cube complex $CY$ of a wallspace $Y$ the \textit{cubulation} of $Y$: Each 0-cube $CY^0$ corresponds to a choice of halfspaces for each wall such that (i) all pairs of chosen halfspaces intersect and (ii) there are no infinite descending chains $A_0 \supset A_1 \supset \cdots$ of chosen halfspaces. Each 1-cube $CY^1$ joins 0-cubes with a single difference in their choice of halfspaces. The construction of $CY$ is completed by adding an $n$-cube wherever the 1-skeleton of an $n$-cube exists in $CY^1$.

**Definition 2.** A \textit{median algebra} $Y = (S, \mu)$ is a set $S$ with a function $\mu: S \times S \times S \to S$ such that:

1. $\mu(x, y, y) = y$ for all $x, y \in S$.
2. $\mu(x, y, z) = \mu(z, x, y) = \mu(x, z, y)$ for all $x, y, z \in S$.
3. $\mu(\mu(x, w, y), w, z) = \mu(x, w, \mu(y, w, z))$ for all $w, x, y, z \in S$.

A median algebra is \textit{discrete} if for all $x, y \in S$, the set $I(x, y) := \{z \in S : \mu(x, y, z) = z\}$ is finite. A set $B \subset S$ is \textit{convex} if $I(x, y) \subset B$ for all $x, y \in B$. One can give any discrete median algebra $Y$ a wallspace structure by letting the walls be all $\{B, S - B\}$ with $B, S - B$ convex. The cubulation $CY$ of a median algebra is the cubulation of this wallspace. The 0-cells $CY^0$ are in bijection with the elements of $S$. A \textit{median subalgebra} is a subset $B \subset S$ closed under the $\mu$ operation.

For a CAT(0) cube complex $X$, the 0-skeleton $X^0$ is naturally a discrete median algebra, where $\mu(x, y, z)$ is the unique 0-cube lying in the intersection of all halfspaces containing at least two of $x, y, z \in X^0$. We thus think of CAT(0) cube complexes as discrete median algebras. We say that a hyperplane $H$ of $X$ \textit{intersects} a median subalgebra $Y \subset X$ if each halfspace of $H$ contains points of $Y$.

Given two median algebras $(T, \mu_1)$, $(F, \mu_2)$, the product $(T \times F, \mu_1 \times \mu_2)$ defines a median algebra. We will suppress the function $\mu$ in our discussion below with the understanding that products of median algebras come equipped with the product median function. We say that a median subalgebra $Y \subset S$ is a \textit{product} of median subalgebras $T, F \subset S$ if for some (hence any) choice of basepoint $(t_1, f_1) \in T \times F$, there exists an isomorphism $T \times F \to Y \subset S$ identically mapping $(t_1, F)$ to $F$ and $(T, f_1)$ to $T$.

For a CAT(0) cube complex $X$, the cubulations of $X$ as a wallspace and a discrete median algebra coincide. Genevois proved that this also holds for median subalgebras of $X$ [3, Lemma 2.10]:

**Lemma 3.** Let $X$ be a CAT(0) cube complex and $Y \subset X$ a median subalgebra. The induced wallspace structure on $Y$ coincides with its wallspace structure as a discrete median algebra.

Note that the cubulation of a submedian algebra $Y \subset X$ does not have to be a subcomplex of $X$. For example, if $X = \mathbb{Z}^2$ and $Y = \{(x, y) : x = y\}$, then $CY = \mathbb{Z}$, but there is no embedding $CY \to X$ which is the identity on points of $CY^0$.

Hyperplanes of CAT(0) cube complexes respect the product structure of median subalgebras. More precisely, we have the following.

**Lemma 4.** Let $X$ be a CAT(0) cube complex. Suppose that a median subalgebra $Y \subset X$ is the product $T \times F$ of median subalgebras $T, F \subset X$. Then any hyperplane of $X$ which intersects $F$ does not intersect $T$.

**Proof.** Let $(t_1, f_1) \in Y$ be the basepoint of the product. Suppose that a hyperplane $H$ intersects both $T$ and $F$. Then there must exist $(t_1, f_2)$ and $(t_2, f_1)$ separated from $(t_1, f_1)$ by $H$. 

Since $M$ is a product of median algebras, $\mu((t_1, f_1), (t_1, f_2), (t_2, f_1)) = (t_1, f_1)$. On the other hand, $\mu((t_1, f_1), (t_1, f_2), (t_2, f_1))$ should lie in the halfspace of $H$ containing $(t_1, f_2), (t_2, f_1)$, a contradiction. □

For an isometry $g$ of a CAT(0) cube complex, let $\|g\| = \min_{x \in X_0} d(x, gx)$ and $\text{Min}(g) = \{x \in X_0 : d(x, gx) = \|g\|\}$. Haglund proved the following classification of isometries of a CAT(0) cube complex $X$ [4].

**Theorem 5.** For an automorphism $g$ of a CAT(0) cube complex $X$, precisely one of the following holds.

1. $g$ is elliptic: $g$ stabilizes a finite cube of $X$. Equivalently, orbits of $g$ are bounded.
2. $g$ is inverting: Orbits of $g$ are unbounded and $g$ swaps the halfspaces of some hyperplane.
3. $g$ is loxodromic: $g$ acts as a nontrivial translation on some bi-infinite geodesic. Such a geodesic is an axis of $g$.

Furthermore, when $g$ is loxodromic, every axis gets translated by $\|g\| > 0$ and every element of $\text{Min}(g)$ lies on an axis. This implies $\|g^n\| = n\|g\|$.

Suppose a group $G$ acts on a CAT(0) cube complex $X$. Replace $X$ by its first cubical subdivision. Then all elements $g \in G$ act by either elliptic or loxodromic isometries, and these two cases can be characterized by $\|g\| = 0$ and $\|g\| > 0$, respectively. If the action of $G$ on $X$ is free, then all $g \in G$ act by loxodromic isometries.

In studying isometries of cube complexes from a median viewpoint, Genevoix essentially proved the following theorem [3, Proposition 4.9].

**Theorem 6.** Suppose that $G$ acts on a CAT(0) cube complex $X$, and a central element $g \in G$ acts loxodromically. Then $\text{Min}(g)$ is a $G$-invariant median subalgebra of $X$ which decomposes as a product $T \times F$ of median subalgebras $F, T \subset X$ such that:

1. The action of $G$ on $\text{Min}(g) = T \times F$ respects the product structure.
2. $F \subset X$ is an isometrically embedded CAT(0) cube complex that $g$ translates by $\|g\|$.
3. $g$ acts trivially on $T$.

Note that $T$ can fail to be isometrically embedded. See Example 8.

As a result of Theorem 6, $\text{Min}(g) \subset X$ is a median subalgebra and it thus makes sense to take its cubulation $\text{CMin}(g)$. This cubulation of $\text{Min}(g)$ agrees with its cubulation as an induced wallspace by Lemma 3.

**Lemma 7.** Suppose two loxodromic isometries $g, h$ of $X$ commute. Then $\text{Min}(g) \cap \text{Min}(h^m)$ is nonempty for some $m > 0$.

**Proof.** By Theorem 6, we have a decomposition $\text{Min}(g) = T \times F$, and there is induced action of $h$ on $\text{Min}(g)$, which respects the product structure. We consider the action of $h$ on $T$.

If the action of $h$ on $T$ is elliptic, then there is some power $h^m$ that fixes a vertex of $T$. Thus, the action of $h^m$ on $X$ stabilizes a translate of $F$ in $T \times F$. Replace $F$ with this translate. Since $F$ is an isometrically embedded CAT(0) cube complex and $h$ is loxodromic, there exists an axis of $h^m$ contained in $F$. 
Suppose the action of \( h \) on \( T \) has unbounded orbits. By Theorem 5, after replacing \( h \) by a power \( h^m \), we can assume the following: \( h \) acts by a loxodromic isometry on \( T \), and \( h \) either acts by a loxodromic isometry on \( F \) or fixes a 0-cube of \( F \). Let \( x = (t_1, f_1) \) be an element of \( \text{Min}(h) \) for the action of \( h \) on \( \text{CMin}(g) \). Since \( \text{Min}(g) \) is a product, the points \( t_1 \in T \) and \( f_1 \in F \) are elements of \( \text{Min}(h) \) for the action of \( h \) on \( T \) and \( F \), respectively. Thus, we can find a geodesic segment \((t_1,\ldots,t_n)\) in the cubulation \( \bar{T} \) so that \( \sigma_T = \bigcup_{k \in \mathbb{Z}} h^k \cdot (t_1,\ldots,t_n) \) is an axis of \( h \) and \((t_1,\ldots,t_n)\) is a fundamental domain of \( \sigma_T \). If \( h \) acts by a loxodromic isometry on \( F \), then there is a geodesic segment \([x, hx] = (t_1, f_1), \ldots, (t_1, f_r), (t_2, f_r), \ldots, (t_n, f_r)\), where \((f_1,\ldots,f_r) \subset F \) is a fundamental domain of an axis \( \sigma_F \) of \( h \) in \( F \). If instead \( h \) fixes a vertex of \( F \), then we can take \( r = 1 \) and \( f_1 \) a vertex fixed by \( h \). In either case, the geodesic \( \sigma = \bigcup_{k \in \mathbb{Z}} h^k \cdot [x, hx] \) is an axis of \( h \) for its action on \( \text{CMin}(g) \).

By Theorem 6, consecutive vertices \( f_i, f_{i+1} \in F \) are distance one in \( X \), but consecutive vertices \( t_i, t_{i+1} \in T \) can be distance greater than one in \( X \). For each \( i = 1,\ldots,n-1 \), let \( \gamma_i \subset X \) be a geodesic joining \( t_i \) to \( t_{i+1} \). Setting \( \gamma = \bigcup_{k \in \mathbb{Z}} h^k \cdot ((t_1, f_1),\ldots,(t_1, f_r), (\gamma_1, f_r), \ldots, (\gamma_n, f_r)) \), we claim that \( \gamma \) is an axis of \( h \) in \( X \). Indeed, suppose that a hyperplane \( H \) intersects \( \gamma \) in two distinct edges. The two edges cannot be of the form \((h^k \cdot (t_1, f_i), h^k \cdot (t_1, f_{i+1}))\) and \((h^k' \cdot (t_1, f_j), h^k' \cdot (t_1, f_{j+1}))\); otherwise, \( H \) would intersect \( \sigma_F \) twice, contradicting Lemma 3. Finally, \( H \) cannot intersect an edge of the form \((h^k \cdot (t_1, f_i), h^k \cdot (t_1, f_{i+1}))\) and an edge in some \( h^{k'} \cdot (\gamma_j, f_r) \) by Lemma 4. Thus, no hyperplane \( H \) intersects \( \gamma \) twice, and \( \gamma \) is a geodesic. Since \( \gamma \) is stabilized by \( h \), it is an axis of \( h \) intersecting \( F \subset X \).

Note that in a finite-dimensional cube complex \( X \), the CAT(0) minsets \( \text{Min}(g) \) and \( \text{Min}(h) \) intersect by the Flat Torus Theorem [1]. However, when \( X \) is infinite-dimensional, the CAT(0) minsets can be empty. Though the combinatorial minsets \( \text{Min}(g) \) and \( \text{Min}(h) \) are never empty, their intersection can be empty. Taking a power is required.

**Example 8.** Consider the CAT(0) cube complex \( X = C \times \mathbb{Z} \), where \( C \) is a two-dimensional cube. Let \( g \) be an isometry of \( X \) which reflects along a diagonal of \( C \) and translates \( \mathbb{Z} \) a nonzero amount. Let \( h \) be an isometry of \( X \) which reflects across the other diagonal of \( C \) and translates \( \mathbb{Z} \) a nonzero amount. Then \( g, h \) are loxodromic and \( \text{Min}(g) \cap \text{Min}(h) \) is empty. However, \( \text{Min}(g) \cap \text{Min}(h^2) \) is nonempty.

We now turn our attention to abelian groups.

**Definition 9.** Let \( A \) be an abelian group. A **discrete norm** on \( A \) is a function \( \nu : A \to \mathbb{R} \) such that

1. There exists \( \epsilon > 0 \) such that \( \nu(g) > \epsilon \) for \( g \neq 0 \).
2. \( \nu(g h) \leq \nu(g) + \nu(h) \).
3. \( \nu(g^m) = |m| \nu(g) \) for \( m \in \mathbb{Z} \).

Steinräns proved the following characterization of free abelian groups in the category of abelian groups [6].

**Theorem 10.** An abelian group is free abelian if and only if it admits a discrete norm.
Using Stepräns's characterization, we prove the main theorem as follows.

**Proof of main theorem.** Let $A$ be an abelian group acting freely on a CAT(0) cube complex $X$. Replace $X$ with its cubical subdivision and define $\nu(g) := \|g\|$. We claim that $\nu$ is a discrete norm on $A$. Any $g \neq 0$ acts by a loxodromic isometry, so we immediately have $\nu(g) \geq 1$ and $\nu(g^m) = |m|\nu(g)$. It remains to show $\nu(gh) \leq \nu(g) + \nu(h)$ for $g, h \in A$.

By Lemma 7, there exists some $m > 0$ such that $\text{Min}(g) \cap \text{Min}(h^m)$ is nonempty. Thus, $\text{Min}(g^m) \cap \text{Min}(h^m)$ is also nonempty. Let $x \in \text{Min}(g^m) \cap \text{Min}(h^m)$. The intersection $\text{Min}(g^m) \cap \text{Min}(h^m)$ is stabilized by both $g^m$ and $h^m$. Hence, the triangle inequality applied to $x, g^m x, h^m g^m x$ yields the following:

$$|m|\nu(gh) = \nu(g^mh^m) \leq \nu(g^m) + \nu(h^m) = |m|(\nu(g) + \nu(h)).$$

Dividing by $|m|$, we retrieve our desired inequality. By Theorem 10, $A$ is free abelian. □

The main theorem naturally leads to several questions. Namely, which solvable groups act freely on CAT(0) cube complexes? A construction in [2] shows that the category of groups acting freely on CAT(0) cube complexes is closed under wreath products. Thus, there are solvable groups with arbitrarily long derived length which act freely on CAT(0) cube complexes. In contrast, Genevois proves in [3] that polycyclic groups acting freely on CAT(0) cube complexes are virtually abelian. It is thus natural to ask the following.

**Question.** Can one classify the metabelian groups acting freely on CAT(0) cube complexes? Which (infinitely generated) nilpotent groups act freely on CAT(0) cube complexes? Are they virtually free abelian?

Relevant to the above question, note that B. H. Neumann constructed uncountable class two nilpotent groups whose abelian subgroups are all countable [5].

**ACKNOWLEDGEMENTS**

Thank you Dani Wise for asking me the question which led to this note. You have been very encouraging along the way. Thank you Piotr Przytycki for reading through multiple drafts of this note and giving comments. This note is far more readable because of you. I thank the referee for many helpful suggestions and for pointing out the example of B. H. Neumann above.

**JOURNAL INFORMATION**

The Bulletin of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

**REFERENCES**

1. M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer, Berlin, 1999. MR 1744486
2. A. Genevois, *Lamplighter groups, median spaces and Hilbertian geometry*, Proc. Edinb. Math. Soc. 65 (2022), no. 2, 500–529.

3. A. Genevois, *Median sets of isometries in CAT(0) cube complexes and some applications*, Michigan Math. J. 71 (2022), no. 3, 487–532.

4. F. Haglund, *Isometries of CAT(0) cube complexes are semi-simple*, arXiv e-prints, arXiv:0705.3386, 2007.

5. B. H. Neumann, *A large nilpotent group without large abelian subgroups*, Bull. Lond. Math. Soc. 25 (1993), no. 4, 305–308. MR 1222719

6. J. Steprāns, *A characterization of free abelian groups*, Proc. Amer. Math. Soc. 93 (1985), no. 2, 347–349. MR 770551