A definable nonstandard model of the reals

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Abstract

We prove, in ZFC, the existence of a definable, countably saturated elementary extension of the reals.

Introduction

It seems that it has been taken for granted that there is no distinguished, definable nonstandard model of the reals. (This means a countably saturated elementary extension of the reals.) Of course if V = L then there is such an extension (just take the first one in the sense of the canonical well-ordering of L), but we mean the existence provably in ZFC. There were good reasons for this: without Choice we cannot prove the existence of any elementary extension of the reals containing an infinitely large integer.1 2 Still there is one.

Theorem 1 (ZFC). There exists a definable, countably saturated extension *R of the reals R, elementary in the sense of the language containing a symbol for every finitary relation on R.

The problem of the existence of a definable proper elementary extension of R was communicated to one of the authors (Kanovei) by V. A. Uspensky.

A somewhat different, but related problem of unique existence of a nonstandard real line *R has been widely discussed by specialists in nonstandard analysis.3 Keisler notes in [3, §11] that, for any cardinal κ, either inaccessible or satisfying 2κ = κ+, there exists unique, up to isomorphism, κ-saturated nonstandard real line *R of cardinality κ, which means that a reasonable level of uniqueness modulo isomorphism can be

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1In fact, from any nonstandard integer we can define a non-principal ultrafilter on N, even a Lebesgue non-measurable set of reals [4], yet it is consistent with ZF (even plus Dependent Choices) that there are no such ultrafilters as well as non-measurable subsets of R [5].
2It is worth to be mentioned that definable nonstandard elementary extensions of N do exist in ZF. For instance, such a model can be obtained in the form of the ultrapower F/U, where F is the set of all arithmetically definable functions f : N → N while U is a non-principal ultrafilter in the algebra A of all arithmetically definable sets X ⊆ N.
3“What is needed is an underlying set theory which proves the unique existence of the hyperreal number system [...]” (Keisler [3] p. 229).
achieved, say, under GCH. Theorem [1] provides a countably saturated nonstandard real line $^*\mathbb{R}$, unique in absolute sense by virtue of a concrete definable construction in ZFC. A certain modification of this example also admits a reasonable model-theoretic characterization up to isomorphism (see Section [4]).

The proof of Theorem [1] is a combination of several known arguments. First of all (and this is the key idea), arrange all non-principal ultrafilters over $\mathbb{N}$ in a linear order $A$, where each ultrafilter appears repetitiously as $D_a$, $a \in A$. Although $A$ is not a well-ordering, we can apply the iterated ultrapower construction in the sense of [1, 6.5] (which is “a finite support iteration” in the forcing nomenclature), to obtain an ultrafilter $D$ in the algebra of all sets $X \subseteq \mathbb{N}^A$ concentrated on a finite number of axes $\mathbb{N}$. To define a $D$-ultrapower of $\mathbb{R}$, the set $F$ of all functions $f : \mathbb{N}^A \to \mathbb{R}$, also concentrated on a finite number of axes $\mathbb{N}$, is considered. The ultrapower $F/D$ is OD, that is, ordinal-definable, actually, definable by an explicit construction in ZFC, hence, we obtain an OD proper elementary extension of $\mathbb{R}$. Iterating the $D$-ultrapower construction $\omega_1$ times in a more ordinary manner, i.e., with direct limits at limit steps, we obtain a definable countably saturated extension.

To make the exposition self-contained and available for a reader with only fragmentary knowledge of ultrapowers, we reproduce several well-known arguments instead of giving references to manuals.

1 The ultrafilter

As usual, $\mathfrak{c}$ is the cardinality of the continuum.

Ultrafilters on $\mathbb{N}$ hardly admit any definable linear ordering, but maps $a : \mathfrak{c} \to \mathcal{P}(\mathbb{N})$, whose ranges are ultrafilters, readily do. Let $A$ consist of all maps $a : \mathfrak{c} \to \mathcal{P}(\mathbb{N})$ such that the set $D_a = \text{ran} a = \{a(\xi) : \xi < \mathfrak{c}\}$ is an ultrafilter on $\mathbb{N}$. The set $A$ is ordered lexicographically: $a <_{\text{lex}} b$ means that there exists $\xi < \mathfrak{c}$ such that $a(\xi) = b(\xi)$ and $a(\xi) < b(\xi)$ in the sense of the lexicographical order $<$ on $\mathcal{P}(\mathbb{N})$ (in the sense of the identification of any $u \subseteq \mathbb{N}$ with its characteristic function).

For any set $u$, $\mathbb{N}^u$ denotes the set of all maps $f : u \to \mathbb{N}$.

Suppose that $u \subseteq v \subseteq A$.

If $X \subseteq \mathbb{N}^v$ then put $X \downarrow u = \{x \upharpoonright u : x \in X\}$.

If $Y \subseteq \mathbb{N}^u$ then put $Y \uparrow v = \{x \in \mathbb{N}^v : x \upharpoonright u \in Y\}$.

We say that a set $X \subseteq \mathbb{N}^A$ is concentrated on $u \subseteq A$, if $X = (X \downarrow u) \uparrow A$; in other words, this means the following:

$$\forall x, y \in \mathbb{N}^A \left( x \upharpoonright u = y \upharpoonright u \implies (x \in X \iff y \in X) \right). \quad (*)$$

We say that $X$ is a set of finite support, if it is concentrated on a finite set $u \subseteq A$. The collection $\mathcal{X}$ of all sets $X \subseteq \mathbb{N}^A$ of finite support is closed under unions, intersections, complements, and differences, i.e., it is an algebra of subsets of $\mathbb{N}^A$. Note that if $(*)$ holds for finite sets $u, v \subseteq A$ then it also holds for $u \cap v$. (If $x \upharpoonright (u \cap v) = y \upharpoonright (u \cap v)$ then consider $z \in \mathbb{N}^A$ such that $z \upharpoonright u = x \upharpoonright u$ and $z \upharpoonright v = y \upharpoonright v$.) It follows that for any $X \in \mathcal{X}$ there is a least finite $u = |X| \subseteq A$ satisfying $(*)$.

In the remainder, if $U$ is any subset of $\mathcal{P}(I)$, where $I$ is a given set, then $Ui \Phi(i)$ (generalized quantifier) means that the set $\{i \in I : \Phi(i)\}$ belongs to $U$.
The following definition realizes the idea of a finite iteration of ultrafilters. Suppose that $u = a_1 < \cdots < a_n \subseteq A$ is a finite set. We put
\[
D_u = \{X \subseteq \mathbb{N}^u : D_{a_n} k_n \cdots D_{a_2} k_2 D_{a_1} k_1 ((k_1, k_2, \ldots, k_n) \in X)\};
\]
\[
D = \{X \in \mathcal{X} : X \downarrow \|X\| \in D_{\|X\|}\}.
\]
The following is quite clear.

**Proposition 2.**

(i) $D_u$ is an ultrafilter on $\mathbb{N}^u$;

(ii) if $u \subseteq v \subseteq A, v$ finite, $X \subseteq \mathbb{N}^u$, then $X \in D_u$ iff $X \uparrow v \in D_v$;

(iii) $D \subseteq \mathcal{X}$ is an ultrafilter in the algebra $\mathcal{X}$;

(iv) if $X \in \mathcal{X}, u \subseteq A$ finite, and $\|X\| \subseteq u$, then $X \in D \iff X \downarrow u \in D_u$.

\[\square\]

**2 The ultrapower**

To match the nature of the algebra $\mathcal{X}$ of sets $X \subseteq \mathbb{N}^A$ of finite support, we consider the family $F$ of all $f : \mathbb{N}^A \to \mathbb{R}$, concentrated on some finite set $u \subseteq A$, in the sense that
\[
\forall x, y \in \mathbb{N}^A \ (x \uparrow u = y \uparrow u \implies f(x) = f(y)). \tag{\dagger}
\]
As above, for any $f \in F$ there exists a least finite $u = \|f\| \subseteq A$ satisfying (\dagger).

Let $\mathcal{R}$ be the set of all finitary relations on $\mathbb{R}$. For any $n$-ary relation $E \in \mathcal{R}$ and any $f_1, \ldots, f_n \in F$, define
\[
E^D(f_1, \ldots, f_n) \iff D x \in \mathbb{N}^A E(f_1(x), \ldots, f_n(x)).
\]
The set $X = \{x \in \mathbb{N}^A : E(f_1(x), \ldots, f_n(x))\}$ is obviously concentrated on $u = \|f_1\| \cup \cdots \cup \|f_n\|$, hence, it belongs to $\mathcal{X}$, and $\|X\| \subseteq u = \|f_1\| \cup \cdots \cup \|f_n\|$.

In particular, $f =^D g$ means that $D x \in \mathbb{N}^A (f(x) = g(x)).$ The following is clear:

**Proposition 3.** $=^D$ is an equivalence relation on $F$, and any relation on $F$ of the form $E^D$ is $=^D$-invariant.

\[\square\]

Put $[f]_D = \{g \in F : f =^D g\}$, and $\mathcal{R} = F/D = \{[f]_D : f \in F\}$. For any $n$-ary ($n \geq 1$) relation $E \in \mathcal{R}$, let $^*E$ be the relation on $^*\mathcal{R}$ defined as follows:
\[
^*E([f_1]_D, \ldots, [f_n]_D) \iff E^D(f_1, \ldots, f_n) \iff D x \in \mathbb{N}^A E(f_1(x), \ldots, f_n(x)).
\]
The independence on the choice of representatives in the classes $[f_i]_D$ follows from Proposition 3. Put $^*\mathcal{R} = \{^*E : E \in \mathcal{R}\}$. Finally, for any $r \in \mathbb{R}$ we put $^*r = [c_r]_D$, where $c_r \in F$ satisfies $c_r(x) = r, \forall x$.

Let $\mathcal{L}$ be the first-order language containing a symbol $E$ for any relation $E \in \mathcal{R}$. Then $\langle \mathbb{R}; \mathcal{R} \rangle$ and $\langle ^*\mathcal{R}; ^*\mathcal{R} \rangle$ are $\mathcal{L}$-structures.

**Theorem 4.** The map $r \mapsto ^*r$ is an elementary embedding (in the sense of the language $\mathcal{L}$) of the structure $\langle \mathbb{R}; \mathcal{R} \rangle$ into $\langle ^*\mathcal{R}; ^*\mathcal{R} \rangle$.
Proof. This is a routine modification of the ordinary argument. By $L[F]$ we denote the extension of $L$ by functions $f \in F$ used as parameters. It does not have a direct semantics, but if $\varphi$ is a formula of $L[F]$ and $x \in \mathbb{N}^A$ then $\varphi[x]$ will denote the formula obtained by the substitution of $f(x)$ for any $f \in F$ which occurs in $\varphi$. Thus, $\varphi[x]$ is an $L$-formula with parameters in $R$.

**Lemma 5 (Łoś).** For any closed $L[F]$-formula $\varphi(f_1, ..., f_n)$ (all parameters $f \in F$ indicated), we have:

$$\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi([f_1]_D, ..., [f_n]_D) \iff D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(f_1, ..., f_n)[x]).$$

Proof. We argue by induction on the logic complexity of $\varphi$. For $\varphi$ an atomic relation $E(f_1, ..., f_n)$, the result follows by the definition of $^*E$. The only notable inductive step is $\exists$ in the direction $\leftarrow$. Suppose that $\varphi$ is $\exists y \psi(y, f_1, ..., f_n)$, and

$$D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(f_1, ..., f_n)[x]), \quad \text{that is,} \quad D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \exists y \psi(y, f_1, ..., f_n)[x]).$$

Obviously there exists a function $f \in F$, concentrated on $u = ||f_1|| \cup \cdots \cup ||f_n||$, such that, for any $x \in \mathbb{N}^A$, if there exists a real $y$ satisfying $\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(y, f_1, ..., f_n)[x]$, then $y = f(x)$ also satisfies this formula, i.e., $\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, ..., f_n)[x]$. Formally,

$$\forall x \in \mathbb{N}^A \left( \exists y \in \mathbb{R} (\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(y, f_1, ..., f_n)[x]) \implies \langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, ..., f_n)[x] \right).$$

This implies $D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, ..., f_n)[x])$. Then, by the inductive assumption, $\langle \mathbb{R}; \mathcal{R} \rangle \models \psi([f]_D, [f_1]_D, ..., [f_n]_D)$, hence $\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi([f_1]_D, ..., [f_n]_D)$, as required. \[\square\] (Lemma)

To accomplish the proof of Theorem 4 consider a closed $L$-formula $\varphi(r_1, ..., r_n)$ with parameters $r_1, ..., r_n \in \mathbb{R}$. We have to prove the equivalence

$$\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(r_1, ..., r_n) \iff \langle \mathbb{R}^*; \mathcal{R} \rangle \models \varphi(*r_1, ..., *r_n).$$

Let $f_i = c_{r_i}$, thus, $f_i \in F$ and $f_i(x) = r_i, \forall x$. Obviously $\varphi(f_1, ..., f_n)[x]$ coincides with $\varphi(r_1, ..., r_n)$ for any $x \in \mathbb{N}^A$, hence $\varphi(r_1, ..., r_n)$ is equivalent to $D x \varphi(f_1, ..., f_n)[x]$. On the other hand, by definition, $*r_i = [f_i]_D$. Now the result follows by Lemma 5. \[\square\]

3 The Iteration

Theorem 4 yields a definable proper elementary extension $\langle \mathbb{R}^*; \mathcal{R} \rangle$ of the structure $\langle \mathbb{R}; \mathcal{R} \rangle$. Yet this extension is not countably saturated due to the fact that the ultra-power $\mathbb{R}^*$ was defined with maps concentrated on finite sets $u \subseteq A$ only. To fix this problem, we iterate the extension used above $\omega_1$-many times.

Suppose that $\langle M; \mathcal{M} \rangle$ is an $L$-structure, so that $\mathcal{M}$ consists of finitary relations on a set $M$, and for any $E \in \mathcal{R}$ there is a relation $E^\mathcal{M} \in \mathcal{M}$ of the same arity, associated with $E$. Let $F_M$ be the set of all maps $f : \mathbb{N}^A \rightarrow M$ concentrated on finite sets $u \subseteq A$. The structure $F_M / D = \langle \mathcal{M}_M; \mathcal{M}_D \rangle$, defined as in Section 2 but with the modified $F$, will be called the $D$-ultrapower of $\langle M; \mathcal{M} \rangle$. Theorem 4 remains true in this general setting: the map $x \mapsto [x](x \in M)$ is an elementary embedding of $\langle M; \mathcal{M} \rangle$ into $\langle M_M; \mathcal{M}_D \rangle$.

We define a sequence of $L$-structures $\langle M_\alpha; \mathcal{M}_\alpha \rangle$, $\alpha \leq \omega_1$, together with a system of elementary embeddings $e_{\alpha\beta} : \langle M_\alpha; \mathcal{M}_\alpha \rangle \rightarrow \langle M_\beta; \mathcal{M}_\beta \rangle$, $\alpha < \beta \leq \omega_1$, so that
(i) \( \langle M_0; \mathcal{M}_0 \rangle = \langle \mathbb{R}; \mathcal{R} \rangle \);

(ii) \( \langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle \) is the \( D \)-ultrapower of \( \langle M_\alpha; \mathcal{M}_\alpha \rangle \), that is, \( \langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle = F_\alpha/D \), where \( F_\alpha = F_{M_\alpha} \) consists of all functions \( f: \mathbb{N}^A \to M_\alpha \) concentrated on finite sets \( u \subseteq A \). In addition, \( e_{\alpha,a+1} \) is the associated \( \ast \)-embedding \( \langle M_\alpha; \mathcal{M}_\alpha \rangle \to \langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle \), while \( e_{\gamma,a+1} = e_{\alpha,a+1} \circ e_\gamma \) for any \( \gamma < \alpha \) (in other words, \( e_{\gamma,a+1}(x) = e_{\alpha,a+1}(e_\gamma(x)) \) for all \( x \in M_\alpha \));

(iii) if \( \lambda \leq \omega_1 \) is a limit ordinal then \( \langle M_\lambda; \mathcal{M}_\lambda \rangle \) is the direct limit of the structures \( \langle M_\alpha; \mathcal{M}_\alpha \rangle, \alpha < \lambda \). This can be achieved by the following steps:

(a) \( M_\lambda \) is defined as the set of all pairs \( \langle \alpha, x \rangle \) such that \( x \in M_\alpha \) and \( x \not\in \text{ran} e_\gamma \) for all \( \gamma < \alpha \).

(b) If \( E \in \mathcal{R} \) is an \( n \)-ary relation symbol then we define an \( n \)-ary relation \( E_\lambda \) on \( M_\lambda \) as follows. Suppose that \( x_i = \langle \alpha_i, x_i \rangle \in M_\lambda \) for \( i = 1, \ldots, n \). Let \( \alpha = \sup \{ \alpha_1, \ldots, \alpha_n \} \) and \( z_i = e_{\alpha_i, \alpha}(x_i) \) for every \( i \), so that \( \alpha_i \leq \alpha < \lambda \) and \( z_i \in M_\alpha \). (Note that if \( \alpha_i = \alpha \) then \( e_{\alpha_i, \alpha} \) is the identity.) Define \( E_\lambda(x_1, \ldots, x_n) \) iff \( \langle M_\alpha; \mathcal{M}_\alpha \rangle \models E(z_1, \ldots, z_n) \).

(c) Put \( \mathcal{M}_\lambda = \{ E_\lambda: E \in \mathcal{R} \} \) then \( \langle M_\lambda; \mathcal{M}_\lambda \rangle \) is an \( \mathcal{L} \)-structure.

(d) Define an embedding \( e_{\alpha, \lambda}: M_\alpha \to M_\lambda (\alpha < \lambda) \) as follows. Consider any \( x \in M_\alpha \). If there is a least \( \gamma < \alpha \) such that there exists an element \( y \in M_\gamma \) with \( x = e_\gamma(y) \) then \( e_{\alpha, \lambda}(x) = \langle \gamma, y \rangle \). Otherwise put \( e_{\alpha, \lambda}(x) = \langle \alpha, x \rangle \).

A routine verification of the following is left to the reader.

**Proposition 6.** If \( \alpha < \beta \leq \omega_1 \) then \( e_{\alpha, \beta} \) is an elementary embedding of \( \langle M_\alpha; \mathcal{M}_\alpha \rangle \) to \( \langle M_\beta; \mathcal{M}_\beta \rangle \).

Note that the construction of the sequence of models \( \langle M_\alpha; \mathcal{M}_\alpha \rangle \) is definable, hence, so is the last member \( \langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle \) of the sequence. It remains to prove that the \( \mathcal{L} \)-structure \( \langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle \) is countably saturated.

This is also a simple argument. Suppose that, for any \( k, \varphi_k(p_k, x) \) is an \( \mathcal{L} \)-formula with a single parameter \( p_k \in M_{\omega_1} \) (the case of many parameters does not essentially differ from the case of one parameter), and there exists an element \( x_k \in M_{\omega_1} \) such that \( \bigwedge_{i \leq k} \varphi_i(p_i, x_k) \) is true in \( \langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle \) — in other words, we have \( \langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle \models \varphi_i(p_i, x_k) \) whenever \( k \geq i \). Fix an ordinal \( \gamma < \omega_1 \) such that for any \( k, i \) there exist \( (\text{then obviously unique}) \) \( y_k, q_i \in M_\gamma \) with \( x_k = e_\gamma(y_k) \) and \( p_i = e_\gamma(q_i) \). Then \( \varphi_i(q_i, y_k) \) is true in \( \langle M_\gamma; \mathcal{M}_\gamma \rangle \) whenever \( k \geq i \).

Fix \( a \in A \) such that \( D_a \) is a non-principal ultrafilter, that is, all cofinite subsets of \( \mathbb{N} \) belong to \( D_a \). Consider the structure \( \langle M_{\gamma+1}; \mathcal{M}_{\gamma+1} \rangle \) as the \( D \)-ultrapower of \( \langle M_\gamma; \mathcal{M}_\gamma \rangle \).

The corresponding set \( F_\gamma \) consists of all functions \( f: \mathbb{N} \to M_\gamma \) concentrated on finite sets \( u \subseteq A \). In particular, the map \( f(x) = y_k \) whenever \( x(a) = k \) belongs to \( F_\gamma \). As any set of the form \( \{ k: k \geq i \} \) belongs to \( D_a \), we have \( D_a \cdot k \) \( \models \varphi_i(q_i, y_k) \), that is, \( D x \in \mathbb{N}^A \( \langle M_\gamma; \mathcal{M}_\gamma \rangle \models \varphi_i(q_i, f(x)) \) for \( i \in \mathbb{N} \). It follows, by Lemma 5, that \( \varphi_i(q_i, y) \) holds in \( \langle M_{\gamma+1}; \mathcal{M}_{\gamma+1} \rangle \) for any \( i \), where \( q_i = e_\gamma(q_i) \in M_{\gamma+1} \) while \( y = [f]_{D} \in M_{\gamma+1} \) is the \( D \)-equivalence class of \( f \) in \( F_\gamma \). Put \( x = e_{\gamma+1, \omega_1}(y) \); then \( \varphi_i(p_i, x) \) is true in \( \langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle \) for any \( i \) because obviously \( p_i = e_{\gamma+1, \omega_1}(q_i), \forall i \).

\[ \Box \] (Theorem 7)
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By appropriate modifications of the constructions, the following can be achieved:

1. For any given infinite cardinal $\kappa$, a $\kappa$-saturated elementary extension of $\mathbb{R}$, definable with $\kappa$ as the only parameter of definition.

2. A special elementary extension of $\mathbb{R}$, of as large cardinality as desired. For instance, take, in stage $\alpha$ of the construction considered in Section 3 ultrafilters on $\mathcal{P}_\kappa$. Then the result will be a definable special structure of cardinality $\beth_\omega$. Recall that special models of equal cardinality are isomorphic [1, Theorem 5.1.17]. Therefore, such a modification admits an explicit model-theoretical characterization up to isomorphism.

3. A class-size definable elementary extension of $\mathbb{R}$, $\kappa$-saturated for any cardinal $\kappa$.

4. A class-size definable elementary extension of the whole set universe, $\kappa$-saturated for any cardinal $\kappa$. (Note that this cannot be strengthened to $\text{Ord}$-saturation, i.e., saturation with respect to all class-size families. For instance, $\text{Ord}^+$-saturated elementary extensions of a minimal transitive model $M\models\text{ZFC}$, definable in $M$, do not exist — see [2, Theorem 2.8].)

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