Let $J$ and $K$ be convex sets in $\mathbb{R}^n$ whose affine spans intersect at a single rational point in $J \cap K$, and let $J \oplus K = \text{conv}(J \cup K)$. We give formulas for the generating function $\sigma_{\text{cone}(J \oplus K)}(z_1, \ldots, z_n, z_{n+1}) = \sum_{(m_1, \ldots, m_n) \in \ell(J \oplus K) \cap \mathbb{Z}^n} z_1^{m_1} \cdots z_n^{m_n} z_{n+1}$ of lattice points in all integer dilates of $J \oplus K$ in terms of $\sigma_{\text{cone} J}$ and $\sigma_{\text{cone} K}$, under various conditions on $J$ and $K$. This work is motivated by (and recovers) a product formula of B. Braun for the Ehrhart series of $P \oplus Q$ in the case where $P$ and $Q$ are lattice polytopes containing the origin, one of which is reflexive. In particular, we find necessary and sufficient conditions for Braun’s formula and its multivariate analogue.

1. Introduction

Given arbitrary convex subsets $J, K \subseteq \mathbb{R}^n$, we denote the convex hull of their union by $J \oplus K := \text{conv}(J \cup K)$. We call $J \oplus K$ a free sum of $J$ and $K$ when $J$ and $K$ each contain the origin and their respective linear spans are orthogonal coordinate subspaces (i.e., subspaces spanned by subsets of the standard basis vectors $e_1, \ldots, e_n$). More generally, we will write “$J \oplus K$ is a free sum” when $J \oplus K$ is a free sum of $J$ and $K$ up to the action of $\text{SL}_n(\mathbb{Z})$ on $\mathbb{R}^n$. A familiar example is the octahedron $\text{conv}\{\pm e_1, \pm e_2, \pm e_3\}$ in $\mathbb{R}^3$, which is the free sum of the “diamond” $\text{conv}\{\pm e_1, \pm e_2\}$ and the line segment $\text{conv}\{\pm e_3\}$. Free sums arise naturally in toric geometry because the free-sum operation is dual to the Cartesian product operation under polar duality: $(P \times Q)^\vee = P^\vee \oplus Q^\vee$. For example, the free-sum decomposition above of the octahedron corresponds to the decomposition of the toric variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as the product of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$.

Our goal is to understand the integer lattice points in a free sum and its integer dilates in terms of the corresponding data for its summands. Of particular interest is the case of a free sum $P \oplus Q$ in which $P$ and $Q$ are rational polytopes. A rational

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1The free sum is sometimes called the direct sum. Diverse conditions on the summands appear in the literature. Some authors require that the origin, or at least a unique point of intersection, be in the interior of each summand. Others require no intersection, insisting only that the linear spans of the summands be orthogonal coordinate subspaces. We require each summand to contain the origin, but we allow the origin to be on the boundary.
(respectively, lattice polytope in $\mathbb{R}^n$ is a polytope all of whose vertices are in $\mathbb{Q}^n$ (respectively, the integer lattice $\mathbb{Z}^n$). Given a rational polytope $P \subseteq \mathbb{R}^n$, its Ehrhart series

$$Ehr_P(t) := 1 + \sum_{k \in \mathbb{Z}_{\geq 1}} |kP \cap \mathbb{Z}^n| t^k$$

is the generating function of the Ehrhart quasi-polynomial of $P$, which counts the integer lattice points in $kP$ as a function of an integer dilation parameter $k$. Let $\text{den} \ P$ denote the denominator of $P$, the smallest positive integer such that the corresponding dilate of $P$ is a lattice polytope. A famous theorem of Ehrhart [8] says that

$$Ehr_P(t) = \frac{\delta_P(t)}{(1 - t^{\text{den} \ P})^{\dim P + 1}}$$

for some polynomial $\delta_P$, the $\delta$-polynomial of $P$. (Common alternative names for the $\delta$-polynomial include $h^*$-polynomial and Ehrhart $h$-vector.) See, e.g., [3, 11, 16] for this and many more facts about Ehrhart series.

Our work is motivated by the following result of B. Braun, which expresses the $\delta$-polynomial of $P \oplus Q$ in terms of the $\delta$-polynomials of $P$ and $Q$ when $P$ is a reflexive polytope (defined in Section 3 below).

**Theorem 1.1** ([4]). Suppose that $P, Q \subseteq \mathbb{R}^n$ are lattice polytopes such that $P$ is reflexive, $Q$ contains the origin in its relative interior, and $P \oplus Q$ is a free sum. Then

$$\delta_{P \oplus Q} = \delta_P \delta_Q.$$  

That is, in terms of Ehrhart series,

$$Ehr_{P \oplus Q}(t) = (1 - t) Ehr_P(t) Ehr_Q(t).$$  

Our first main result, Theorem 1.2 below, gives a multivariate generalization of Theorem 1.1 for arbitrary compact convex sets. Our second main result, Theorem 1.3 below, characterizes the free sums of rational polytopes that satisfy our multivariate generalization of equation (2). A characterization of the free sums satisfying equation (2) itself is a consequence. Before stating our results, we first need to define some notation.

The Ehrhart series is a specialization of a multivariate Laurent series defined as follows. Let $\alpha : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the affine embedding $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, 1)$. Given a convex set $K \subseteq \mathbb{R}^n$, let cone $K \subseteq \mathbb{R}^{n+1}$ be the set of all nonnegative scalar multiples of elements of $\alpha(K)$. Equivalently, cone $K$ is the intersection of all linear cones containing $\alpha(K)$. Write $S_\mathbb{Z}$ for the set of integer lattice points in a set $S$. The lattice-point generating function $\sigma_S(z)$ of $S \subseteq \mathbb{R}^{n+1}$ is the formal multivariate Laurent series

$$\sigma_S(z) := \sum_{m \in S_\mathbb{Z}} z^m.$$  

(Here we follow the convention of writing $\sigma_S(z)$ for $\sigma_S(z_1, \ldots, z_{n+1})$ and $z^m$ for $z_1^{m_1} \cdots z_{n+1}^{m_{n+1}}$, where $m = (m_1, \ldots, m_{n+1})$.) The Ehrhart series $Ehr_P(t)$ then arises as a specialization of $\sigma_{\text{cone } P(z)}$:

$$Ehr_P(t) = \sigma_{\text{cone } P}(1, \ldots, 1, t).$$  

Let $e_1, \ldots, e_n, e_{n+1}$ denote the standard basis vectors in $\mathbb{R}^{n+1}$. Given a closed linear cone $C \subseteq \mathbb{R}^{n+1}$ not containing $-e_{n+1}$, define the projection $\varepsilon_C : C \to \partial C$...
where \( \partial \) denotes relative boundary) by letting
\[
\varepsilon_C(x) := x - \max \{ \lambda \in \mathbb{R} : x - \lambda e_{n+1} \in C \} e_{n+1}.
\]
Given a compact convex set \( J \subseteq \mathbb{R}^n \), we write \( \varepsilon_J \) as an abbreviation for \( \varepsilon_{\text{cone} J} \). (We require \( J \) to be compact so that \( \text{cone} J \) is closed.) The lower envelope of \( C \) is
\[
\partial C := \varepsilon_C(C).
\]
Thus, the lower envelope of \( C \) is the set of points that are “vertically minimal” within \( C \). The lower lattice envelope of \( C \) is
\[
\partial Z_C := \varepsilon_C(C_Z).
\]
Thus, the lower lattice envelope is the vertical projection of the lattice points in \( C \) onto the lower envelope of \( C \). Observe that the lower lattice envelope is not necessarily the set \( (\partial C)_Z \) of lattice points in the lower envelope of \( C \). In general, some elements of \( \partial Z_C \) may not be lattice points.

**Theorem 1.2** (proved on p. 7). Suppose that \( J, K \subseteq \mathbb{R}^n \) are convex sets such that \( J \) is compact and \( J \oplus K \) is a free sum. Suppose moreover that \( \partial Z_{\text{cone} J} = (\partial \text{cone} J)_Z \). Then
\[
\sigma_{\text{cone}(J \oplus K)}(z) = (1 - z_{n+1}) \sigma_{\text{cone} J}(z) \sigma_{\text{cone} K}(z).
\]
We call equation (3) the multivariate Braun equation. Our second main result states that, when \( J \) and \( K \) are rational polytopes, the converse of Theorem 1.2 also holds. (Whether the converse holds for free sums \( J \oplus K \) of arbitrary convex sets is still an open question.) Given a rational polytope \( P \) containing the origin, we observe in Proposition 3.2 below that \( \partial Z_{\text{cone} P} = (\partial \text{cone} P)_Z \) if and only if the polar dual \( P^\vee \) of \( P \) (relative to its linear span) is a lattice polyhedron. We show that, if a free sum of rational polytopes satisfies the multivariate Braun equation, then the dual of one of those polytopes is a lattice polyhedron.

**Theorem 1.3** (proved on p. 12). Let \( P, Q \subseteq \mathbb{R}^n \) be rational polytopes such that \( P \oplus Q \) is a free sum. Then
\[
\sigma_{\text{cone}(P \oplus Q)}(z) = (1 - z_{n+1}) \sigma_{\text{cone} P}(z) \sigma_{\text{cone} Q}(z)
\]
if and only if either \( P^\vee \) or \( Q^\vee \) is a lattice polyhedron.

The univariate analogue of Theorem 1.3 is a consequence:

**Theorem 1.4** (proved on p. 13). Let \( P, Q \subseteq \mathbb{R}^n \) be rational polytopes such that \( P \oplus Q \) is a free sum. If either \( P^\vee \) or \( Q^\vee \) is a lattice polyhedron, then
\[
\text{Ehr}_{P \oplus Q}(t) = (1 - t) \text{Ehr}_P(t) \text{Ehr}_Q(t)
\]
and hence
\[
\delta_{P \oplus Q}(t) = \frac{(1 - t)(1 - \frac{1}{\text{lcm}(\text{den} P, \text{den} Q)})^{\text{dim} P + \text{dim} Q + 1}}{(1 - \frac{1}{\text{den} P})^{\text{dim} P + 1}(1 - \frac{1}{\text{den} Q})^{\text{dim} Q + 1}} \delta_P(t) \delta_Q(t).
\]
Conversely, if either equation (5) or equation (6) holds, then either \( P^\vee \) or \( Q^\vee \) is a lattice polyhedron. In particular, if \( P, Q \subseteq \mathbb{R}^n \) are lattice polytopes such that \( P \oplus Q \) is a free sum, then
\[
\delta_{P \oplus Q} = \delta_P \delta_Q
\]
if and only if either \( P^\vee \) or \( Q^\vee \) is a lattice polyhedron.
After laying the groundwork for our approach to free sums in Section 2, we prove Theorem 1.2 and various corollaries, including Theorem 1.1, in Section 3. In Section 4, we give an expression for $\sigma_{\text{cone}(P \oplus K)}$ when $P \oplus K$ is an arbitrary free sum in which $P$ is a rational polytope (Theorem 4.2). We then use this expression to prove Theorems 1.3 and 1.4.

Although Sections 3 and 4 address only the case where $J \oplus K$ is a free sum, our approach is not confined to this situation. Section 5 introduces the notion of affine free sums $J \oplus K$, where $J$ and $K$ may intersect at an arbitrary rational point. We derive formulas for the lattice-point generating functions of cones over affine free sums $J \oplus K$ under certain conditions on $J$ and $K$. One case of interest that satisfies these conditions is an affine free sum $P \oplus K$ where $P$ is a Gorenstein polytope of index $k$ intersecting an orthogonal convex set $K$ at the unique point $p \in P$ such that $kp$ is a lattice point in the relative interior of $kP$ (Corollary 5.9).

2. Decompositions of cones over free sums

We begin our study of the generating function $\sigma_{\text{cone}(J \oplus K)}$ from the vantage point of the following easy identity: Given any convex sets $J, K \subseteq \mathbb{R}^n$, the convex hull $J \oplus K$ of their union satisfies

$$\text{cone}(J \oplus K) = \text{cone } J + \text{cone } K,$$

where the sum on the right is the Minkowski sum $S + T := \{s + t : s \in S, t \in T\}$. The goal of this section is to provide two refinements to equation (8), first by making the equation “disjoint”, and then by restricting the equation to lattice points. As it stands, equation (8) “double counts” elements of $\text{cone}(J \oplus K)$, in the sense that there are many ways to express an element of the left-hand side as a sum from the right-hand side. Proposition 2.1 below gives a non-double-counting version of equation (8) under certain conditions on $J$ and $K$. Proposition 2.2 below provides a similar expression for the integer lattice points in $\text{cone}(J \oplus K)$ when $J \oplus K$ is a free sum.

First we make a few additional notational remarks: We write $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ for the orthogonal projection

$$\pi : (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n).$$

Given a subset $S$ of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$, let $\text{lin } S$ be the linear span of $S$. We say that two sublattices $\mathcal{L}, \mathcal{M} \subseteq \mathbb{Z}^n$ are complementary sublattices of $\mathbb{Z}^n$ if each element of $(\text{lin}(\mathcal{L} \cup \mathcal{M}))_\mathbb{Z}$ is the sum of a unique element of $\mathcal{L}$ and a unique element of $\mathcal{M}$. Hence, when $J \oplus K$ is a free sum, $(\text{lin } J)_\mathbb{Z}$ and $(\text{lin } K)_\mathbb{Z}$ are complementary sublattices of $\mathbb{Z}^n$.

Equation (8) says that

$$\text{cone}(J \oplus K) = \bigcup_{x \in \text{cone } J} (x + \text{cone } K).$$

Using the concept of the lower envelope (defined in Section 1), we can replace the union above by a disjoint union. This yields the desired “disjoint” version of equation (8). We use $\bigcup$ to denote disjoint union.

**Proposition 2.1.** Suppose that $J, K \subseteq \mathbb{R}^n$ are convex sets with $J$ compact and $0 \in K$. Suppose in addition that the linear spans of $J$ and $K$ intersect trivially.
Then
\[ \text{cone}(\mathcal{J} \oplus \mathcal{K}) = \bigcup_{x \in \partial \text{cone } \mathcal{J}} (x + \text{cone } \mathcal{K}). \]

Proof. We first show that the union on the right-hand side is a disjoint union. Suppose that
\[ x_1 + y_1 = x_2 + y_2 \]
for some \( x_1, x_2 \in \partial \text{cone } \mathcal{J} \) and \( y_1, y_2 \in \text{cone } \mathcal{K} \). Then we have \( \pi(x_1) + \pi(y_1) = \pi(x_2) + \pi(y_2) \). Hence
\[ \pi(x_1) - \pi(x_2) = \pi(y_2) - \pi(y_1) \in \text{lin } \mathcal{J} \cap \text{lin } \mathcal{K} \]
because the left-hand side of the equality is in \( \text{lin } \mathcal{J} \) while the right-hand side is in \( \text{lin } \mathcal{K} \). Since \( \text{lin } \mathcal{J} \cap \text{lin } \mathcal{K} = \{0\} \), it follows that \( \pi(x_1) = \pi(x_2) \). Now, the preimage \( \pi^{-1}(\pi(x_1)) \) contains exactly one point in \( \partial \text{cone } \mathcal{J} \), so \( x_1 = x_2 \), proving disjointness.

It remains only to show that
\[ \bigcup_{x \in \partial \text{cone } \mathcal{J}} (x + \text{cone } \mathcal{K}) = \bigcup_{x \in \text{cone } \mathcal{J}} (x + \text{cone } \mathcal{K}). \]
The left-hand side is contained in the right-hand side because \( \mathcal{J} \) is compact, so \( \partial \text{cone } \mathcal{J} \subseteq \text{cone } \mathcal{J} \). Conversely, if \( w \in x + \text{cone } \mathcal{K} \) for some \( x \in \text{cone } \mathcal{J} \), then
\[ w - (x - \varepsilon_{\mathcal{J}}(x)) \in \varepsilon_{\mathcal{J}}(x) + \text{cone } \mathcal{K}. \]
Now, \( x - \varepsilon_{\mathcal{J}}(x) \) is a nonnegative multiple of \( e_{n+1} \), which is in \( \text{cone } \mathcal{K} \) because \( 0 \in \mathcal{K} \). Thus, adding \( x - \varepsilon_{\mathcal{J}}(x) \) to both sides of (9) yields \( w \in \varepsilon_{\mathcal{J}}(x) + \text{cone } \mathcal{K} \). Since \( \varepsilon_{\mathcal{J}}(x) \in \partial \text{cone } \mathcal{J} \), this proves the claim. \( \square \)

Our ultimate goal is to understand the generating function \( \sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})} \), so we need a version of the disjoint union in Proposition 2.1 that is restricted to the lattice points in \( \text{cone}(\mathcal{J} \oplus \mathcal{K}) \). This is provided by the following proposition. See also Figure 4.

**Proposition 2.2.** Suppose that \( \mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n \) are convex sets such that \( \mathcal{J} \) is compact and \( \mathcal{J} \oplus \mathcal{K} \) is a free sum. Then
\[ \text{cone}(\mathcal{J} \oplus \mathcal{K})_\mathbb{Z} = \bigcup_{x \in \partial \text{cone } \mathcal{J}} (x + \text{cone } \mathcal{K})_\mathbb{Z}. \]

Proof. The elements of the right-hand side are lattice points that are contained in \( \text{cone}(\mathcal{J} \oplus \mathcal{K}) \) by the previous proposition. Hence, such elements are in the left-hand side.

To prove the converse containment, let \( w \in \text{cone}(\mathcal{J} \oplus \mathcal{K})_\mathbb{Z} \) be given. By Proposition 2.1, there exist \( x \in \partial \text{cone } \mathcal{J} \) and \( y \in \text{cone } \mathcal{K} \) such that \( w = x + y \). Thus, \( \pi(w) = \pi(x) + \pi(y) \). Now, \( \pi(w) \) is an integer lattice point in \( \text{lin } (\mathcal{J} \cup \mathcal{K}) \), while \( \pi(x) \in \text{lin } \mathcal{J} \) and \( \pi(y) \in \text{lin } \mathcal{K} \). Since \( \text{lin } (\mathcal{J})_\mathbb{Z} \) and \( \text{lin } (\mathcal{K})_\mathbb{Z} \) are complementary sub-lattices of \( \mathbb{Z}^n \), it follows that \( \pi(x) \in \mathbb{Z}^n \). Furthermore, \( x \in \text{cone } \mathcal{J} \) and \( 0 \in \mathcal{J} \), so there exists an integer \( \lambda \) such that \( \pi(x) + \lambda e_{n+1} \in (\text{cone } \mathcal{J})_\mathbb{Z} \). Therefore, \( x = \varepsilon_{\mathcal{J}}(\pi(x) + \lambda e_{n+1}) \in \partial \text{cone } \mathcal{J} \). \( \square \)
Figure 1. A depiction of cone($\mathcal{J} \oplus \mathcal{K}$). The dots indicate elements of $\partial \mathcal{Z}$ cone $\mathcal{J}$. The shaded regions represent translations of cone $\mathcal{K}$ by elements of $\partial \mathcal{Z}$ cone $\mathcal{J}$. The import of Proposition 2.2 is that all lattice points in cone($\mathcal{J} \oplus \mathcal{K}$) are within these shaded regions.

Remark 2.3. If $\mathcal{J} \oplus \mathcal{K}$ is not a free sum, then Proposition 2.2 does not hold. For example, let $\mathcal{J} \subseteq \mathbb{R}^2$ be the segment $[(-1, 0), (1, 0)]$, and let $\mathcal{K} \subseteq \mathbb{R}^2$ be the segment $[(-1, -2), (1, 2)]$. Note that $(\text{lin} \mathcal{J})_\mathbb{Z}$ and $(\text{lin} \mathcal{K})_\mathbb{Z}$ are not complementary sublattices in $\mathbb{Z}^2$, so $\mathcal{J} \oplus \mathcal{K}$ is not a free sum. The equation in Proposition 2.2 fails to hold in this case because, for example, the lattice point $(1, 1, 1)$ appears in cone($\mathcal{J} \oplus \mathcal{K}$)$_\mathbb{Z}$ but not in $\bigcup_{x \in \partial \mathcal{Z}} \text{cone} \mathcal{J} (x + \text{cone} \mathcal{K})_\mathbb{Z}$.

3. Sufficient conditions for the multivariate Braun equation

The multivariate Braun equation (3) does not hold for all free sums $\mathcal{J} \oplus \mathcal{K}$ of convex sets. In this section, we give conditions on $\mathcal{J}$ and $\mathcal{K}$ that suffice to imply equation (3). The conditions we give generalize those originally given by Braun in [4]. In the next section, we will show that, conversely, our conditions are necessary in the case where $\mathcal{J}$ and $\mathcal{K}$ are rational polytopes.

To apply Proposition 2.2, we need to get our hands on the set $\partial \mathcal{Z}$ cone $\mathcal{J}$. The next proposition considers the case where all the elements of this set are integer lattice points.

Proposition 3.1. Let $\mathcal{J} \subseteq \mathbb{R}^n$ be a compact convex set containing the origin. Then the following conditions are equivalent:

(a) $\partial \mathcal{Z}$ cone $\mathcal{J} = (\partial \text{cone} \mathcal{J})_\mathbb{Z}$,
(b) $(\partial \text{cone} \mathcal{J})_\mathbb{Z} = (\text{cone} \mathcal{J})_\mathbb{Z} \setminus (\text{cone} \mathcal{J} + e_{n+1})_\mathbb{Z}$,
(c) $\sigma_{\partial \text{cone} \mathcal{J}} (z) = (1 - z_{n+1}) \sigma_{\text{cone} \mathcal{J}} (z)$.

Proof. We start by proving that (a) and (b) are equivalent. First, note that the set containments

$$\partial \mathcal{Z} \text{ cone } \mathcal{J} \supseteq (\partial \text{ cone } \mathcal{J})_\mathbb{Z}$$

and

$$(\partial \text{ cone } \mathcal{J})_\mathbb{Z} \subseteq (\text{ cone } \mathcal{J})_\mathbb{Z} \setminus (\text{ cone } \mathcal{J} + e_{n+1})_\mathbb{Z}$$

always hold. To see that the respective converse containments are equivalent, observe that $x \mapsto \mathcal{E} \mathcal{J} (x)$ is a bijection between non-lower-envelope points in $(\text{ cone } \mathcal{J})_\mathbb{Z} \setminus (\text{ cone } \mathcal{J} + e_{n+1})_\mathbb{Z}$ and non-lattice points in $\partial \mathcal{Z} \text{ cone } \mathcal{J}$, with inverse bijection $(a, \lambda) \mapsto (a, [\lambda])$. Thus, if either containment above is an equality, then so too is the other.
Finally, the left- (resp. right-) hand side of (c) lists the points of the left- (resp. right-) hand side of (b) in generating-function form, so (b) and (c) are equivalent.

Theorem 1.2 is now an easy corollary of the previous proposition.

Proof of Theorem 1.2 (stated on p. [3]). Since \( \partial_z \text{cone} \mathcal{J} = (\partial \text{cone} \mathcal{J})_Z \), the set-theoretic equation in Proposition 2.2 can be restated in terms of generating functions as follows:

\[
\sigma_{\text{cone}(J \oplus K)}(z) = \sigma_{\text{cone} J}(z) \sigma_{\text{cone} K}(z).
\]

The theorem now follows from the equivalence of (a) and (c) in Proposition 3.1.

The conditions in Proposition 3.1 take on an especially nice form when the convex set \( \mathcal{J} \) is a rational polytope. We now show that, in this case, these conditions are equivalent to the condition that the polar dual of \( \mathcal{J} \) is a lattice polyhedron. We recall the relevant definitions.

The (polar) dual of a polytope \( \mathcal{P} \subseteq \mathbb{R}^n \) containing the origin is defined to be the polyhedron

\[
\mathcal{P}^\vee := \{ \varphi \in (\text{lin} \mathcal{P})^* : \varphi(a) \leq 1 \text{ for all } a \in \mathcal{P} \},
\]

where \( V^* \) denotes the set of all real-valued linear functionals on a vector space \( V \).

Let \( \varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_\ell \in (\text{lin} \mathcal{P})^* \) be linear functionals such that

\[
\mathcal{P} = \{ a \in \text{lin} \mathcal{P} : \varphi_1(a), \ldots, \varphi_k(a) \leq 1 \text{ and } \psi_1(a), \ldots, \psi_\ell(a) \leq 0 \}.
\]

Then \( \mathcal{P}^\vee \) can be expressed as the Minkowski sum of a polytope and a polyhedral cone in the dual space \( (\text{lin} \mathcal{P})^* \) as follows:

\[
\mathcal{P}^\vee = \text{conv} \{ \varphi_1, \ldots, \varphi_k \} + \text{pos} \{ \psi_1, \ldots, \psi_\ell \},
\]

where \( \text{pos} S \) denotes the positive hull \( \{ \lambda a : a \in \text{conv} S \text{ and } \lambda \geq 0 \} \) of a set \( S \). We call \( \mathcal{P}^\vee \) a lattice polyhedron if its vertices are in the dual integer lattice defined by

\[
(\text{lin} \mathcal{P})_Z^* := \{ \varphi \in (\text{lin} \mathcal{P})^* : \varphi(a) \in \mathbb{Z} \text{ for all } a \in (\text{lin} \mathcal{P})_Z \}.
\]

A polytope \( \mathcal{P} \) is reflexive if both \( \mathcal{P} \) and \( \mathcal{P}^\vee \) are lattice polytopes. Reflexive polytopes were introduced by Victor Batyrev to study mirror symmetry in string theory [1].

Hibi [12] showed that a lattice polytope \( \mathcal{P} \) containing the origin in its interior is reflexive if and only if \( (k \mathcal{P} \setminus (k - 1)\mathcal{P})_Z = (\partial(k \mathcal{P}))_Z \) for all integers \( k \geq 2 \). This latter condition, in turn, is equivalent to \( \partial_z \text{cone} \mathcal{P} = (\partial \text{cone} \mathcal{P})_Z \). Hibi’s proofs carry over with virtually no change if we merely assume that \( \mathcal{P} \) is rational and contains the origin (not necessarily in its interior). Hibi’s arguments then show that \( \mathcal{P}^\vee \) is a lattice polyhedron if and only if \( \partial_z \text{cone} \mathcal{P} = (\partial \text{cone} \mathcal{P})_Z \). We include a proof of this equivalence for completeness (Proposition 3.2 below). Non-lattice rational polytopes with lattice duals have appeared, e.g., in [9], which gives a rational analogue of a theorem of Hibi on the Ehrhart series of reflexive polytopes [12].

**Proposition 3.2.** Let \( \mathcal{P} \) be a rational polytope with \( 0 \in \mathcal{P} \). Then \( \mathcal{P}^\vee \) is a lattice polyhedron if and only if \( \partial_z \text{cone} \mathcal{P} = (\partial \text{cone} \mathcal{P})_Z \).
**Proof.** Suppose that \( P^\vee \) is a lattice polyhedron. It is clear that \( \partial_Z \text{cone} P \supseteq (\partial_Z \text{cone} P)_Z \). To prove the converse containment, let \( \mathbf{x} \in \partial_Z \text{cone} P \) be given. By definition of the lower lattice envelope, we have that \( \pi(x) \in \mathbb{Z}^n \). Let \( \varphi_1, \varphi_2, \ldots, \varphi_k \) be the vertices of \( P^\vee \), and let \( \lambda := \max \{ \varphi_1(\pi(x)), \ldots, \varphi_k(\pi(x)) \} \). Then \( \pi(x) \in \lambda P \) while \( \pi(x) \notin (\lambda - \varepsilon)P \) for \( 0 < \varepsilon < \lambda \). Thus, \( \mathbf{x} = (\pi(x), \lambda) \in \partial_Z \text{cone} P \). Furthermore, since each \( \varphi_i \) is a dual integer lattice point, we have that \( \lambda \in \mathbb{Z} \), which implies that \( \mathbf{x} \in (\partial_Z \text{cone} P)_Z \), proving the desired containment.

Conversely, suppose that \( P^\vee \) has a vertex \( \varphi_j \notin (\text{lin} P)_Z^* \). Let \( \Lambda \subseteq (\text{lin} P)_Z \) be the sublattice of \( (\text{lin} P)_Z \) on which \( \varphi_j \) evaluates as an integer. Thus, \( \Lambda \) is a full-rank proper sublattice of \( (\text{lin} P)_Z \). Let \( F \) be the facet of \( P \) supported by the hyperplane \( \varphi_j = 1 \). Then there exists a lattice point \( \mathbf{a} \in (\text{pos} F)_Z \setminus \Lambda \). (This may be seen by observing that \( \text{pos} F \) contains \( \Lambda \)-translate of a fundamental domain of \( \Lambda \), which in turn contains elements of \( \mathbb{Z}^n \setminus \Lambda \).) We then have that \( \varphi_j(\mathbf{a}) \notin \mathbb{Z} \) but \( (\mathbf{a}, \varphi_j(\mathbf{a})) \in \partial_Z \text{cone} P \), so that \( \partial_Z \text{cone} P \not\subseteq (\partial_Z \text{cone} P)_Z \). \( \square \)

As a corollary of Propositions 3.1 and 3.2, we find that the multivariate Braun equation \([3]\) holds when one of the summands is a rational polytope whose polar dual is a lattice polyhedron.

**Corollary 3.3.** Let \( P \subseteq \mathbb{R}^n \) be a rational polytope such that \( P^\vee \) is a lattice polyhedron, and let \( K \subseteq \mathbb{R}^n \) be a convex set such that \( P \oplus K \) is a free sum. Then

\[
\sigma_{\text{cone}(P \oplus K)}(z) = (1 - z_{n+1}) \sigma_{\text{cone} P}(z) \sigma_{\text{cone} K}(z). \]

By applying the specialization \( \text{Ehr}_P(t) = \sigma_{\text{cone} P}(1, \ldots, 1, t) \), we arrive at the following generalization of Braun’s Theorem 1.1.

**Corollary 3.4.** If \( P, Q \subseteq \mathbb{R}^n \) are rational polytopes such that \( P^\vee \) is a lattice polyhedron and \( P \oplus Q \) is a free sum, then

\[
\text{Ehr}_{P \oplus Q}(t) = (1 - t) \text{Ehr}_P(t) \text{Ehr}_Q(t)
\]

and hence

\[
\delta_{P \oplus Q}(t) = \frac{(1 - t)(1 - \frac{\text{lcm}(\text{den } P, \text{den } Q)}{\text{dim } P + \text{dim } Q + 1})}{(1 - \frac{\text{den } P}{\text{dim } P + 1})(1 - \frac{\text{den } Q}{\text{dim } Q + 1})} \delta_P(t) \delta_Q(t).
\]

In particular, if \( P, Q \subseteq \mathbb{R}^n \) are lattice polytopes such that \( P^\vee \) is a lattice polyhedron and \( P \oplus Q \) is a free sum, then

\[
\delta_{P \oplus Q} = \delta_P \delta_Q.
\]

**Remark 3.5.** Corollary 3.4 recovers the following generalization of Theorem 1.1 due to Braun [4, Corollary 1]: Let \( P \) and \( Q \) be as in Theorem 1.1 and let \( \Lambda' \) (respectively, \( \Lambda'' \)) be a lattice polytope equal to the intersection of \( P \) (resp., \( Q \)) with a finite collection of half-spaces in \( \text{lin } P \) (resp., \( \text{lin } Q \)) bounded by hyperplanes passing through the origin. Then \( \delta_{P' \oplus Q'} = \delta_{P'} \delta_{Q'} \).

There are lattice polytopes covered by our Corollary 3.4 that do not satisfy the conditions of Braun’s [4, Corollary 1]. For example, let \( P \subseteq \mathbb{R}^2 \) be the polygon \( \text{conv } \{(-1,0), (1,0), (3,1), (-3,1)\} \). Then \( P \) is not contained in any reflexive polytope, but the dual of \( P \) is a lattice polyhedron. (The polygon \( P \) is a 2-dimensional analogue of a so-called **top** polytope. Top polytopes, like reflexive polytopes, originally arose in string theory [7].)
4. Necessary conditions for the multivariate Braun equation

In this section, we prove Theorem 1.3, the converse of Theorem 1.2, in the case where the summands are rational polytopes. That is, we show that, if \( P \) and \( Q \) are rational polytopes containing the origin such that

\[
\sigma_{\text{cone}(P \oplus Q)}(z) = (1 - z_{n+1}) \sigma_{\text{cone} P}(z) \sigma_{\text{cone} Q}(z),
\]

then either \( P^\vee \) or \( Q^\vee \) is a lattice polyhedron. We also prove Theorem 1.4, the univariate version of Theorem 1.3.

Fix a rational polytope \( P \subseteq \mathbb{R}^n \) such that \( 0 \in P \), and let \( K \subseteq \mathbb{R}^n \) be a convex set such that \( P \oplus K \) is a free sum. As in the previous section, we approach the generating function \( \sigma_{\text{cone}(P \oplus K)} \) via the decomposition of cone(\( P \oplus K \))\(_{\mathbb{Z}} \) given by Proposition 2.2. The first step, therefore, is to find a useful description of the lower lattice envelope \( \partial_z \text{cone} P \) in the case where we do not necessarily have \( \partial_z \text{cone} P = (\partial \text{cone} P)_{\mathbb{Z}} \).

Write \( d(P) \) for the denominator \( \text{den}(P^\vee) \) of \( P^\vee \). For each nonnegative integer \( i \), let

\[
\text{cone}^i P := \text{cone} P + \frac{i}{d(P)} e_{n+1},
\]

\[
\text{cone}^i K := \text{cone} K - \frac{i}{d(P)} e_{n+1}.
\]

(Observe that the definition of \( \text{cone}^i K \) depends upon the choice of \( P \), although this is not reflected in the notation.) We similarly define the shifted lower envelopes \( \partial \text{cone}^i P := \partial \text{cone} P + \frac{i}{d(P)} e_{n+1} \) and \( \partial \text{cone}^i K := \partial \text{cone} K - \frac{i}{d(P)} e_{n+1} \) of these shifted cones. The rind of \( \text{cone} P \) is \( (\text{cone} P) \setminus (\text{cone} P + e_{n+1}) \).

Proposition 4.1 below is a generalization of Proposition 3.1 as applied to any rational polytope containing the origin. Before giving the formal statement of Proposition 4.1, we give an informal summary. See also Figure 2.

- Each point in the lower lattice envelope is the result of taking a unique lattice point on some shifted lower envelope contained in the rind of \( \text{cone} P \) and projecting that lattice point down to the lower envelope.
- No lattice point lies between consecutive shifted lower envelopes.
- Hence, every lattice point in the rind lies on exactly one of the shifted lower envelopes.

**Proposition 4.1.** Suppose that \( P \subseteq \mathbb{R}^n \) is a rational polytope with \( 0 \in P \), and let \( d(P) := \text{den}(P^\vee) \). Define the shifted cones \( \text{cone}^i P \) for \( 0 \leq i \leq d(P) \) as above. Then we have the following:

1. \( \partial_z \text{cone} P = \bigcup_{i=0}^{d(P)-1} \left( \partial \text{cone}^i P \right)_{\mathbb{Z}} - \frac{i}{d(P)} e_{n+1} \),
2. \( \left( \partial \text{cone}^i P \right)_{\mathbb{Z}} = \left( \text{cone}^i P \right)_{\mathbb{Z}} \setminus \left( \text{cone}^{i+1} P \right)_{\mathbb{Z}} \) for \( 0 \leq i \leq d(P) - 1 \),
3. \( \sigma_{\partial \text{cone}^i P} = \sigma_{\text{cone}^i P} - \sigma_{\text{cone}^{i+1} P} \) for \( 0 \leq i \leq d(P) - 1 \),
4. \( (1 - z_{n+1}) \sigma_{\text{cone} P}(z) = \sum_{i=0}^{d(P)-1} \sigma_{\partial \text{cone}^i P}(z) \).

**Proof.** The right-hand side of part (a) is contained in the left-hand side because elements of the right-hand side are points in \( \partial_z \text{cone} P \) that are directly beneath lattice points. To see that the left-hand side of part (a) is contained in the right-hand side, let \( x \in \partial_z \text{cone} P \) be given. It suffices to show that \( x + \frac{i}{d(P)} e_{n+1} \in \mathbb{Z}^{n+1} \) for some \( i \in \{0, \ldots, d(P) - 1\} \). Let

\[
\lambda := \max \{ \varphi(\pi(x)) : \varphi \text{ is a vertex of } P^\vee \},
\]
and let $k := \lfloor \lambda \rfloor$. Thus, $\pi(x) \in \lambda P$, but $\pi(x) \notin (\lambda - \varepsilon)P$ for all $0 < \varepsilon < \lambda$. Hence, $(\pi(x), \lambda) \in \partial \text{cone} \ P$, so $x = (\pi(x), \lambda)$. Now, every vertex $\varphi$ of $P$ satisfies $\varphi(a) \in \mathbb{Z}^n$ for all $a \in \mathbb{Z}^n$. Since $\pi(x) \in \mathbb{Z}^n$, we thus have that $k = \lambda + i d(P)$ for some $i \in \{0, \ldots, d(P) - 1\}$. Therefore, $x + \frac{i}{d(P)} e_{n+1} = (\pi(x), k) \in \mathbb{Z}^{n+1}$, as required.

To see that the union in part (a) is disjoint, suppose that $x_1 - i d(P) e_{n+1} = x_2 - j d(P) e_{n+1}$ for some $x_1 \in (\partial \text{cone}^i P)_\mathbb{Z}$ and $x_2 \in (\partial \text{cone}^j P)_\mathbb{Z}$, where, without loss of generality, $0 \leq i < j < d(P)$. Then $x_2 - x_1 = \left(\frac{j}{d(P)} - \frac{i}{d(P)}\right) e_{n+1}$ is a lattice point and $0 \leq \frac{j}{d(P)} - \frac{i}{d(P)} < 1$. This implies that $i = j$, showing disjointness and proving part (a).

To prove part (b), suppose that there is an element $x$ on the right-hand side that is not on the left-hand side. Then, for some integer $i$ such that $0 \leq i < d(P) - 1$ and some $\lambda$ such that $\frac{i}{d(P)} < \lambda < \frac{i+1}{d(P)}$, we have $x - \lambda e_{n+1} \in \partial \text{cone} \ P$. Hence, by part (a), there exist $y \in \mathbb{Z}^{n+1}$ and $j \in \{0, \ldots, d(P) - 1\}$ such that $y - \frac{j}{d(P)} e_{n+1} = x - \lambda e_{n+1}$. This implies that $\lambda - \frac{j}{d(P)}$ is an integer, which is a contradiction. This proves part (b).

Part (c) follows immediately, since it is a restatement of part (b) in terms of generating functions. Part (d) results from summing both sides of part (c) over all integers $i$ such that $0 \leq i < d(P) - 1$.

Using the previous proposition, we can write down a version of Proposition 2.2 in which the sets in the disjoint union are indexed by lattice points. This allows us to translate the resulting set equality directly into an equality of generating functions.

**Theorem 4.2.** Suppose that $P \subseteq \mathbb{R}^n$ is a rational polytope and $K \subseteq \mathbb{R}^n$ is a convex set such that $P \oplus K$ is a free sum. Then

$$\text{cone}(P \oplus K)_{\mathbb{Z}} = \bigcup_{i=0}^{d(P)-1} \bigcup_{x \in (\partial \text{cone}^i P)_{\mathbb{Z}}} (x + \text{cone}_i K)_{\mathbb{Z}}.$$
Therefore,

\[
\sigma_{\text{cone}}(P \oplus K) = \sum_{i=0}^{d(P)-1} \sigma_{\partial \text{cone}}^i P \sigma_{\text{cone}} K
\]

(10)

\[
= \sum_{i=0}^{d(P)-1} \left( \sigma_{\text{cone}}^i P - \sigma_{\text{cone}}^{i+1} P \right) \sigma_{\text{cone}} K.
\]

(11)

Proof. By Proposition 2.2,

\[
\text{cone}(P \oplus K) = \bigsqcup_{x \in \partial \text{cone} P} (x + \text{cone} K).
\]

By Proposition 4.1(a), this becomes

\[
\text{cone}(P \oplus K) = \sum_{i=0}^{d(P)-1} \bigsqcup_{x \in (\partial \text{cone})^i P} (x - \frac{i}{d(P)} e_{n+1} + \text{cone} K).
\]

Equation (10) is the restatement of this equality in terms of generating functions, and equation (11) follows from Proposition 4.1(c).

Remark 4.3. Some of the terms in equation (10) may be zero. For example, if \( P \) is the interval \([-2, 3]\), then \( \sigma_{\partial \text{cone}}^0 P = \sigma_{\partial \text{cone}}^5 P = 0 \). Nonetheless, if \( d(P) > 1 \), then \( \sigma_{\partial \text{cone}}^i P \neq 0 \) for some \( i \in \{1, \ldots, d(P) - 1\} \) by Proposition 3.2.

Before proving Theorem 1.3, we need two lemmas constraining when lattice points can appear in the shifted lower envelopes of cones over compact convex sets.

Lemma 4.4. Let \( J \subseteq \mathbb{R}^n \) be a compact convex set, and let \( \rho \) be a rational number. Then \( (\partial \text{cone} J + \rho e_{n+1})_Z \neq \emptyset \) if and only if \( (\partial \text{cone} J - \rho e_{n+1})_Z \neq \emptyset \).

Proof. Since \( \rho \in \mathbb{Q} \), a ray \( R \) originating at \( \rho e_{n+1} \) contains a lattice point if and only if the inversion of \( R \) through \( \rho e_{n+1} \) also contains a lattice point. Hence, the set \( \partial \text{cone} J + \rho e_{n+1} \), which is a union of rays originating at \( \rho e_{n+1} \), contains a lattice point if and only if its inversion through \( \rho e_{n+1} \) contains a lattice point. But \( \partial \text{cone} J - \rho e_{n+1} \) is just the inversion of this latter set through the origin. That is,

\[
\partial \text{cone} J - \rho e_{n+1} = -((\partial \text{cone} J + \rho e_{n+1}) - \rho e_{n+1}) + \rho e_{n+1}.
\]

Since inversion through the origin is a lattice-preserving operation, the claim follows.

Lemma 4.5. Let \( Q \) be a rational polytope, and let \( \rho \) be a real number. If

\[
(\partial \text{cone} Q + \rho e_{n+1})_Z \neq \emptyset,
\]

then \( \rho \) is a rational number.

Proof. Let \( x \in (\partial \text{cone} Q + \rho e_{n+1})_Z \), and let \( F \) be a facet of \( \text{cone} Q \) containing \( x - \rho e_{n+1} \). Then the supporting hyperplane \( H \) of \( \text{cone} Q \) at \( F \) is a rational hyperplane containing \( \rho e_{n+1} - x \). Therefore, the translation \( H + x \) by an integer lattice point must meet the \( e_{n+1} \)-axis at a rational point.

We are now ready to prove Theorem 1.3.
Proof of Theorem 1.3 (stated on p. 3). The “if” direction follows immediately from Corollary 3.3. To prove the converse, suppose that equation (4) holds but that \( P \) is not a lattice polyhedron. Then, by Proposition 3.2, \( \partial Z \text{cone} P \neq \partial (\text{cone} P) z \).

Hence, by Proposition 4.1(a), there exists a maximum integer \( j \) with \( 1 \leq j \leq d(P) - 1 \) such that \( (\partial \text{cone}^j P) Z \neq \emptyset \).

We claim that the nonemptiness of \( (\partial \text{cone}^j P) Z \), in combination with equation (4), implies that
\[
(12) \quad (\text{cone}^j Q) Z \setminus (\text{cone} Q) Z = \emptyset.
\]

To see this, apply Proposition 4.1(d) to rewrite equation (4) as follows:
\[
\sigma \text{cone}(P \oplus Q) = d(P) - 1 \sum_{i=0}^{d(P) - 1} \sigma \partial \text{cone}^i P \sigma \text{cone} Q.
\]

Together with Theorem 4.2, this yields
\[
\sum_{i=0}^{d(P) - 1} \sigma \partial \text{cone}^i P \sigma \text{cone} Q = \sum_{i=0}^{d(P) - 1} \sigma \partial \text{cone}^i P \sigma \text{cone} Q,
\]
or, equivalently,
\[
d(P) - 1 \sum_{i=0}^{d(P) - 1} \sigma \partial \text{cone}^i P (\sigma \text{cone} Q - \sigma \text{cone} Q) = 0.
\]

Since \( Q \subseteq \text{cone} Q \), the monomials on the left-hand side all have nonnegative coefficients. In particular, since \( (\partial \text{cone}^j P) Z \neq \emptyset \), we must have that \( \sigma \text{cone} Q - \sigma \text{cone} Q = 0 \), proving equation (12).

We now show that \( j/d(P) \geq \frac{1}{2} \). The maximality of \( j \) implies that
\[
\bigcup_{i=j+1}^{d(P) - 1} (\partial \text{cone}^i P) Z = \emptyset,
\]
which, by Lemma 4.4, becomes
\[
\bigcup_{i=j+1}^{d(P) - 1} (\partial \text{cone}^i P) Z = \emptyset.
\]

Translating by \( e_{n+1} \) and then reversing the order of the disjoint union yields
\[
\bigcup_{i=j+1}^{d(P) - 1} (\partial \text{cone}^{d(P) - i} P) Z = \bigcup_{i=1}^{d(P) - j - 1} (\partial \text{cone}^i P) Z = \emptyset.
\]

Since \( (\partial \text{cone}^j P) Z \neq \emptyset \) and \( j \geq 1 \), we must have \( j > d(P) - j - 1 \), or \( j/d(P) \geq \frac{1}{2} \), as claimed.

We now apply similar reasoning to \( Q \). Equation (12) implies that
\[
(13) \quad \bigcup_{0 < \rho \leq j/d(P)} (\partial Q - \rho e_{n+1}) Z = \emptyset.
\]

Once again applying Lemma 4.4, we get
\[
(14) \quad \bigcup_{0 < \rho \leq j/d(P)} (\partial Q + \rho e_{n+1}) Z = \emptyset,
\]
while, translating the sets in equation (13) by $e_{n+1}$ and then reversing the order of the disjoint union, we have

\[(\partial \text{cone } Q + (1-\rho) e_{n+1})_Z = \bigcup_{\rho \in \mathbb{Q} : 0 < \rho \leq j/d(P)} (\partial \text{cone } Q + \rho e_{n+1})_Z = \emptyset.\]

Since $j/d(P) \geq 1/2$, we can combine equalities (14) and (15) to conclude that

\[\bigcup_{\rho \in \mathbb{Q} : 0 < \rho < 1}(\partial \text{cone } Q + \rho e_{n+1})_Z = \emptyset.\]

Hence, by Lemma 4.5, \((\text{cone } Q)_Z \setminus (\text{cone } Q + e_{n+1})_Z = (\partial \text{cone } Q)_Z\).

Thus, by Proposition 3.1, we have that \(\partial \text{cone } Q = (\partial \text{cone } Q)_Z\). Therefore, by Proposition 3.2, \(Q^\vee\) is a lattice polyhedron.

It is now straightforward to prove Theorem 1.4, the univariate analogue of Theorem 1.3.

**Proof of Theorem 1.4 (stated on p. 3)***

Corollary 3.4 already established that, if either \(P^\vee\) or \(Q^\vee\) is a lattice polyhedron, then equations (5) and (6) hold. To prove the converse, suppose that neither \(P^\vee\) nor \(Q^\vee\) is a lattice polyhedron. Then, by Theorem 1.3,

\[\sigma_{\text{cone}(p \oplus Q)}(z) \neq (1 - z_{n+1}) \sigma_{\text{cone } p}(z) \sigma_{\text{cone } Q}(z).\]

By Theorem 4.2 and Proposition 4.1(d), this becomes

\[\sum_{i=0}^{d(P)-1} \sigma_{\text{cone } p^i} \sigma_{\text{cone } Q} \neq \sum_{i=0}^{d(P)-1} \sigma_{\text{cone } p^i} \sigma_{\text{cone } Q}.\]

Now, since \(\text{cone } Q \subseteq \text{cone } Q_i\) for all \(i\), every monomial on the right-hand side appears on the left-hand side. Thus,

\[\sigma_{\text{cone}(p \oplus Q)}(z) = (1 - z_{n+1}) \sigma_{\text{cone } p}(z) \sigma_{\text{cone } Q}(z) + \tau(z)\]

for some nonzero Laurent series \(\tau(z)\) with nonnegative coefficients. Hence, specializing equation (16) at \(z = (1, \ldots, 1, t)\) yields

\[\text{Ehr}_{p \oplus Q}(t) = (1 - t) \text{Ehr}_{p}(t) \text{Ehr}_{Q}(t) + F(t),\]

where \(F(t)\) is a nonzero power series. In particular, equation (5) does not hold. Multiplying through by the denominator of the rational function \(\text{Ehr}_{p \oplus Q}(t)\) shows that equation (6) also does not hold. (Equation (7) is just the case of equation (6) in which \(\text{den}(P) = \text{den}(Q) = 1\).)

5. Sums of polytopes intersecting at rational points

In previous sections, we considered the generating function \(\sigma_{\text{cone}(J \oplus K)}\) where \(J \oplus K\) was a free sum. In particular, \(J\) and \(K\) intersected only at the origin. Matters are essentially the same if \(J\) and \(K\) intersect at an arbitrary lattice point \(p\) in \(\mathbb{Z}^n\), since we can reduce the computation of \(\sigma_{\text{cone}(J \oplus K)}\) to the previous case via the equation

\[\sigma_{\text{cone}(J \oplus K)}(z) = \sigma_{\text{cone}((J - p) \oplus (K - p))}(z_1, \ldots, z_n, z^{\alpha(p)}).\]
(Here, in accordance with the convention mentioned in Section 1, \(z^\alpha(p)\) denotes the monomial \(z_1^{p_1} \cdots z_n^{p_n} z_{n+1}^{p_{n+1}}\), where \(p = (p_1, \ldots, p_n)\).)

We now turn to the case where \(J\) and \(K\) intersect in an arbitrary rational point in \(\mathbb{Q}^n\). Our results in this section generalize the propositions in Section 2 and some of the results in Section 3. We begin by extending our earlier definitions of lower (lattice) envelopes to accommodate projections that are not in the vertical direction.

Given \(p \in \mathbb{Q}^n\), define \(\pi_p : \mathbb{R}^{n+1} \to \mathbb{R}^n\) via \(\pi_p(x) = \pi(x - x_{n+1}\alpha(p))\) where \(x_{n+1}\) is the last coordinate of \(x\). Thus, instead of projecting vertically down to \(\mathbb{R}^n\) (as in previous sections), \(\pi_p\) projects parallel to \(\alpha(p)\). Note that \(\pi = \pi^0\). However, in general we may not have \(\pi_p(\mathbb{Z}^{n+1}) = \mathbb{Z}^n\).

Given a closed linear cone \(C \subseteq \mathbb{R}^{n+1}\) not containing \(-\alpha(p)\), define \(\varepsilon^p_C : C \to \partial C\) via

\[
\varepsilon^p_C(x) := x - \max \{ \lambda \in \mathbb{R} : x - \lambda \alpha(p) \in C \} \alpha(p).
\]

Given a compact convex set \(J \subseteq \mathbb{R}^n\), we will write \(\varepsilon^p_J\) as an abbreviation for \(\varepsilon_{\text{cone} J}\).

We then define the \(p\)-lower envelope \(\partial^p C\) of \(C\) via

\[
\partial^p C := \varepsilon^p_C(C).
\]

Similar to the lower envelope, the \(p\)-lower envelope of \(C\) is the set of points in \(C\) that are “minimal in the direction of \(\alpha(p)\)”. Finally, we introduce the notion of \(p\)-lower lattice envelope of \(C\), defined as

\[
\partial^p_C := \varepsilon^p_C(C)\cap\mathbb{Z}^n.
\]

Thus the \(p\)-lower lattice envelope is the projection of the lattice points in \(C\) in the direction parallel to \(\alpha(p)\) onto the \(p\)-lower envelope of \(C\). The lower (lattice) envelope of previous sections reappears as the special case \(p = 0\).

We are now ready to state the generalizations of the propositions from Section 2.

**Proposition 5.1.** Suppose that \(J, K \subseteq \mathbb{R}^n\) are convex sets with \(J\) compact. Suppose in addition that the affine spans of \(J\) and \(K\) intersect in exactly one rational point \(p \in K\). Then

\[
\text{cone}(J \oplus K) = \bigcup_{x \in \partial^p \text{cone} J} (x + \text{cone} K).
\]

Once we note that, for \(x \in \text{cone} J\), \(\pi_p(x)\) is in \(\text{lin}(J - p)\), the proof of this proposition is the same as the proof of Proposition 2.1 with the appropriate replacements (such as \(p\) replaced by \(\pi_p\), and \(\varepsilon_J\) replaced by \(\varepsilon^p_J\)).

We now seek a restriction of equation (17) to lattice points that is in the spirit of Proposition 2.2. To this end, we define an analogue of the free-sum operation, which we call an affine free sum. Recall that, for \(J \oplus K\) to be a free sum, we required that \(\text{lin}(J)_Z\) and \(\text{lin}(K)_Z\) be complementary sublattices of \(\mathbb{Z}^n\). One complication of our present case is that \(\pi_p(x)\) is not necessarily a lattice point for every lattice point \(x\) in \(\text{cone} J\). Thus, we consider the refinement \(\Lambda^p := \pi_p(\mathbb{Z}^{n+1})\) of \(\mathbb{Z}^n\). There are several equivalent characterizations of this lattice:

1. \(\Lambda^p = \pi_p(\mathbb{Z}^{n+1})\).
2. \(\Lambda^p\) is the lattice in \(\mathbb{R}^n\) generated by \(\{e_1, \ldots, e_n, p\}\) under integer linear combinations.
3. \(\Lambda^p = \bigcup_{k=0}^{\lfloor \varepsilon_p \rfloor} (\mathbb{Z}^n - kp), \) where \(r := \text{den}(p)\) is the least common multiple of the denominators of the coordinates of \(p\).
We adapt our earlier notation and terminology to work with the lattice \( \Lambda^p \) as follows. For a subset \( S \) of \( \mathbb{R}^n \), let \( S_{\Lambda^p} \) denote the set of points in \( S \cap \Lambda^p \). We say that two sublattices \( \mathcal{L}, \mathcal{M} \subseteq \Lambda^p \) are complementary sublattices of \( \Lambda^p \) if each element of \( (\text{lin}(\mathcal{L} \cup \mathcal{M}))_{\Lambda^p} \) is the sum of a unique element of \( \mathcal{L} \) and a unique element of \( \mathcal{M} \).

Given convex sets \( \mathcal{J} \) and \( \mathcal{K} \) in \( \mathbb{R}^n \), we call \( \mathcal{J} \oplus \mathcal{K} \) an affine free sum if \( \mathcal{J} \) and \( \mathcal{K} \) intersect at a point \( p \in \mathbb{Q}^n \) such that \( (\text{lin}(\mathcal{J} - p))_{\Lambda^p} \) and \( (\text{lin}(\mathcal{K} - p))_{\Lambda^p} \) are complementary sublattices of \( \Lambda^p \). Equivalently, \( \mathcal{J} \oplus \mathcal{K} \) is an affine free sum if \( \mathcal{J} \) and \( \mathcal{K} \) intersect at a unique rational point and

\[
\text{lin}(\text{cone}(\mathcal{J} \oplus \mathcal{K})) = \text{lin}(\text{cone}\mathcal{J}) + \text{lin}(\text{cone}\mathcal{K}),
\]

where the sum on the right is the Minkowski sum.

**Proposition 5.2.** Suppose that \( \mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n \) are convex sets such that \( \mathcal{J} \) is compact and \( \mathcal{J} \oplus \mathcal{K} \) is an affine free sum of convex sets intersecting at \( p \in \mathbb{Q}^n \). Then

\[
\text{cone}(\mathcal{J} \oplus \mathcal{K})_\mathbb{Z} = \bigcup_{x \in \text{lin}(\text{cone}\mathcal{J})} (x + \text{cone}\mathcal{K})_\mathbb{Z}.
\]

**Proof.** Elements on the right-hand side are integer lattice points that are contained in \( \text{cone}(\mathcal{J} \oplus \mathcal{K}) \) by Proposition 5.1. Hence, such elements are in the left-hand side.

To prove the converse containment, let \( w = x + y \) where \( x \in \partial^p \text{cone}\mathcal{J} \) and \( y \in \text{cone}\mathcal{K} \). Thus, \( \pi^p(w) = \pi^p(x) + \pi^p(y) \). Now, \( \pi^p(w) \) is in \( \text{lin}(\text{cone}\mathcal{J}) - p \), while \( \pi^p(x) \in \text{lin}(\text{cone}\mathcal{J} - p) \) and \( \pi^p(y) \in \text{lin}(\text{cone}\mathcal{K} - p) \). Thus, the complementarity of \( (\text{lin}(\mathcal{J} - p))_{\Lambda^p} \) and \( (\text{lin}(\mathcal{K} - p))_{\Lambda^p} \) implies that \( \pi^p(x) \) is in \( \Lambda^p \). Hence there exists a non-negative integer \( \lambda \) such that \( (\pi^p(x), 0) + \lambda p = (\pi^p(x) + \lambda p, \lambda) \) is an integer lattice point in \( \text{cone}\mathcal{J} \). Since \( z^p_J((\pi^p(x) + \lambda p, \lambda)) = x \), we have \( x \in z^p_J((\text{cone}\mathcal{J})_\mathbb{Z}) \), and the result follows. \( \square \)

We now turn to the rational generating function \( \sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})} \) and state the generalizations of Proposition 3.1 and Theorem 1.2.

**Proposition 5.3.** Fix a compact convex set \( \mathcal{J} \subseteq \mathbb{R}^n \) containing \( p \in \mathbb{Q}^n \). Let \( r := \text{den}(p) \). Then the following are equivalent:

(a) \( \partial^p \text{cone}\mathcal{J} = (\partial^p \text{cone}\mathcal{J})_\mathbb{Z} \).
(b) \( (\partial^p \text{cone}\mathcal{J})_\mathbb{Z} = (\text{cone}\mathcal{J})_\mathbb{Z} \setminus (\text{cone}\mathcal{J} + r\alpha(p))_\mathbb{Z} \).
(c) \( \sigma_{\partial^p \text{cone}\mathcal{J}}(z) = (1 - z^{r\alpha(p)}) \sigma_{\text{cone}\mathcal{J}}(z) \).

**Proof.** We first show that (a) and (b) are equivalent. By definition of the \( p \)-lower envelope and \( p \)-lower lattice envelope, we have \( (\partial^p \text{cone}\mathcal{J})_\mathbb{Z} \subseteq (\partial^p \text{cone}\mathcal{J})_\mathbb{Z} \subseteq (\text{cone}\mathcal{J} + r\alpha(p))_\mathbb{Z} \). As in the proof of Proposition 3.1, we observe that \( x \mapsto z^p_J(x) \) is a bijection between non-lower-envelope points in \( (\text{cone}\mathcal{J})_\mathbb{Z} \setminus (\text{cone}\mathcal{J} + r\alpha(p))_\mathbb{Z} \) and non-lattice points in \( (\partial^p \text{cone}\mathcal{J})_\mathbb{Z} \). The rest of the proof is the same as the proof of Proposition 3.1. \( \square \)

**Theorem 5.4.** Suppose that \( \mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n \) are convex sets such that \( \mathcal{J} \) is compact and \( \mathcal{J} \oplus \mathcal{K} \) is an affine free sum of convex sets intersecting at \( p \in \mathbb{Q}^n \). Further suppose that \( \partial^p \text{cone}\mathcal{J} = (\partial^p \text{cone}\mathcal{J})_\mathbb{Z} \). Then

\[
(18) \quad \sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(z) = (1 - z^{r\alpha(p)}) \sigma_{\text{cone}\mathcal{J}}(z) \sigma_{\text{cone}\mathcal{K}}(z),
\]
where \( r := \text{den}(p) \).

The proof is the same as the proof of Theorem 1.2 with the appropriate replacements.

Remark 5.5. It is straightforward to adapt the arguments in Section 4 to prove a converse of Theorem 5.4 analogous to Theorem 1.3. That is, one can show that, if \( P \cup Q \) is an affine free sum of rational polytopes intersecting at \( p \in \mathbb{Q}^n \), and

\[
\sigma_{\text{cone}(P \cup Q)}(z) = (1 - z^{\text{den}(p)\alpha(p)}) \sigma_{\text{cone} P}(z) \sigma_{\text{cone} Q}(z),
\]

then \( \partial P \cap \partial Q = (\partial P \cap \partial Q)_z \). A recent preprint of W. Bruns proves this result in the general context of arbitrary affine monoids [5]. As is the case with Theorem 1.2 whether the converse of Theorem 5.4 holds for free sums \( J \cup K \) of arbitrary convex sets is still an open question.

Example 5.6. Let \( J \) be the line segment from \((0, 0)\) to \((1, 0)\) in \( \mathbb{R}^2 \) and let \( K \) be the line segment from \((\frac{1}{2}, -1)\) to \((\frac{1}{2}, 1)\) in \( \mathbb{R}^2 \). Then \( J \) and \( K \) intersect at \( p := (\frac{1}{2}, 0) \), and \( J \cup K \) is an affine free sum. The \( p \)-lower envelope of cone \( J \) is the boundary of the cone, and the set of lattice points in the boundary is precisely the set \( (\text{cone} J)_z \setminus (\text{cone} J + \alpha(p))_z \), where \( r := \text{den}(p) = 2 \). Thus, \( J \) satisfies the conditions in Proposition 5.3. Hence, Theorem 5.4 applies, yielding

\[
\sigma_{\text{cone}(J \cup K)}(z_1, z_2, z_3) = (1 - z_1 z_3^2) \sigma_{\text{cone} J}(z_1, z_2, z_3) \sigma_{\text{cone} K}(z_1, z_2, z_3) = (1 - z_1 z_3^2) \frac{1}{(1 - z_3)(1 - z_1 z_3)} \frac{1 + z_1 z_2^{-1} z_3^2 + z_1 z_2 z_3^2 + z_1 z_2^2 z_3^2}{(1 - z_2 z_3^2)(1 - z_1 z_2^2 z_3^2)}.\]

Example 5.7. Theorem 5.4 need not hold if we drop the condition \( \partial P \cap \partial Q = (\partial P \cap \partial Q)_z \). If we keep \( J \) the same set as in Example 5.6 but let \( K \) be the line segment from \((\frac{1}{2}, -1)\) to \((\frac{1}{2}, 1)\) in \( \mathbb{R}^2 \), then \( p = (\frac{1}{3}, 0) \), \( \alpha(p) = (\frac{1}{3}, 0, 1) \) and \( r = 3 \). The \( p \)-lower envelope of cone \( J \) is still the boundary of the cone, but there are now lattice points in the set \( (\text{cone} J)_z \setminus (\text{cone} J + \alpha(p))_z \) that are not in the boundary of the cone. Thus, the conditions in Proposition 5.3 are not true of \( J \), and so we would need to use generalizations of results from Section 4 to compute \( \sigma_{\text{cone}(J \cup K)}(z) \). We do not develop such generalizations here.

We now consider an important class of polytopes for which the conditions in Proposition 5.3 are true, so that Theorem 5.4 applies when one of the summands is a polytope from this class. A lattice polytope \( P \) is Gorenstein of index \( k \) if there exists a lattice point \( m \) such that \( kP - m \) is a reflexive polytope. In particular, \( m \) is the unique lattice point in \( kP \). The recent paper [14] discusses Braun’s formula in the context of Gorenstein polytopes and nef-partitions.

Proposition 5.8. Suppose that \( P \subseteq \mathbb{R}^n \) is a Gorenstein polytope of index \( k \). Let \( m \) be the unique lattice point in \( kP \), and let \( p := \frac{1}{k} m \). Then \( \partial P \cap \partial Q = (\partial P \cap \partial Q)_z \).

Proof. Since \( p \in P \), we have that \( \partial P \cap \partial Q = \partial Q \cap \partial Q \). It is well known that the Gorenstein property implies that \( (\text{cone} P)_z = (\text{cone} P + \alpha(p))_z \) (see, e.g., [6]). In particular, \( k = \text{den}(p) \). The result follows from Proposition 5.3(b).

Corollary 5.9. Suppose that \( P \subseteq \mathbb{R}^n \) is a Gorenstein polytope of index \( k \). Let \( m \) be the unique lattice point in \( kP \), and let \( p := \frac{1}{k} m \). Let \( K \subseteq \mathbb{R}^n \) be a convex set
containing $p$ such that $P \oplus K$ is an affine free sum. Then

$$
\sigma_{\text{cone}(P \oplus K)}(z) = (1 - z^{k\alpha(p)}) \sigma_{\text{cone}P}(z) \sigma_{\text{cone}K}(z).
$$

Example 5.6 is an instance of this corollary, as the line segment $J$ in that example is a Gorenstein polytope of index 2 with $m = (1, 0)$.

In Section 3, we noted that the conditions in Proposition 3.1 applied to a broader family than just the reflexive polytopes. Indeed, in that context, the integrality of the vertices of the polytope was unimportant; all that we needed was that the polar dual be a lattice polyhedron (cf. Proposition 3.2). It is natural to expect that the Gorenstein condition in Proposition 5.8 can similarly be weakened to admit non-lattice polytopes. For example, one might hope that, in Proposition 5.8, we could take $P$ to be any rational polytope such that, for some integer $k$ and some lattice point $m \in kP$, $(kP - m)^{\vee}$ is a lattice polyhedron. Unfortunately, this is not the case in general, as the following example shows.

Example 5.10. Let $P := \left[\frac{1}{4}, \frac{3}{4}\right] \subseteq \mathbb{R}^1$. Observe that $2P = \left[\frac{1}{2}, \frac{3}{2}\right]$ contains the lattice point $m := 1$ and that the polar dual of $2P - m = [-\frac{1}{2}, \frac{1}{2}]$ is the lattice polytope $[-2, 2] \subseteq (\mathbb{R}^1)^{\ast}$. Nonetheless, putting $p := \frac{1}{2}m$, the $p$-lower lattice envelope of cone $P$ contains the non-lattice point $\left(\frac{1}{2}, 2\right)$. Therefore, the conclusion of Proposition 5.8 is not true of $P$.

As mentioned in Remark 5.5, recent results by W. Bruns [5] generalize our observations to the context of general affine monoids. Nevertheless, as the example above shows, there still remains the problem of characterizing when equation (18) applies in terms of the summand polytopes, as in Proposition 3.2, rather than in terms of the cones over them, as in [5] and Remark 5.5.

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(Matthias Beck) Department of Mathematics, San Francisco State University, San Francisco, CA 94132, USA
E-mail address: mattbeck@sfsu.edu

(Pallavi Jayawant) Department of Mathematics, Bates College, Lewiston, ME 04240, USA
E-mail address: pjayawan@bates.edu

(Tyrrell B. McAllister) Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA
E-mail address: tmcallis@uwyo.edu