A note on computations of the D-brane superpotential

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Abstract

We develop some computational methods for the integrals over the 3-chains on the compact Calabi–Yau 3-folds that play a prominent role in the analysis of the topological B-model in the context of the open mirror symmetry. We discuss such 3-chain integrals in two approaches. In the first approach, we provide an algorithm to obtain the inhomogeneous Picard–Fuchs equations. In the second approach, we discuss the analytic continuation of the period integral to compute the 3-chain integral directly. The latter direct integration method is applicable for both on-shell and off-shell formalisms.

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1. Introduction

The progress of the research on mirror symmetry has been remarkably active in recent years. Mirror symmetry is a powerful tool to study the non-perturbative aspects of effective field theory arising from type II string compactifications, and the field theoretic derivation of mirror symmetry is discussed in [54, 55]. In the recent developments, the mirror symmetry of the open string sector has become tractable, and it is possible to compute the effective superpotential on D-brane, which wraps around the cycles in the Calabi–Yau 3-fold. The concrete study of the mirror symmetry for the open string sector was initiated in [1], and the enumerative structure predicted in [2] is confirmed in the toric Calabi–Yau case. These results are also consistent with the topological vertex computations [3].

Recently, the open mirror symmetry has been extended to certain compact Calabi–Yau manifolds. Walcher [4] predicted the disk instanton numbers for the quintic 3-fold with an involution brane as a natural extension of the closed mirror symmetry [5]. This was proven rigorously by localization calculation on the A-model side in [6]. On the other hand, the B-model analysis is developed further in [7]. In particular, for the B-model side, the computation of the 3-chain integrals, which is obtained by the reduction of the holomorphic Chern–Simons action, plays a prominent role. They are the solutions of the Picard–Fuchs equation with an inhomogeneous term resulting from the boundary contribution. The open
mirror symmetry with an involution brane for the other compact Calabi–Yau 3-folds is further studied in [8–10].

The relative period for compact Calabi–Yau is studied in [11] by replacing curves with a divisor and adding a logarithmic factor in the period. Related to this work, the study of the toric branes in the compact Calabi–Yau geometry is also developed in [12] along similar lines as [1]. There are some related works [13–20]. The toric brane is specified by the extra toric charges, and an open string modulus is introduced. The effective superpotential on the toric brane is given by the relative period integrals, which are the integrals of holomorphic 3-form over the 3-chains with the boundaries. The relative period satisfies the extended Picard–Fuchs equation [21–23] which depends on both closed and open string moduli. See also [24] for a review. Extremizing the effective superpotential with respect to the open moduli, one finds the same results as the involution brane [11, 12]. Therefore, the effective superpotential for the involution brane is called on-shell.

The aim of this paper is to study the B-model side of open mirror symmetry. First, we will discuss the computation of the inhomogeneous term [7, 20] in the Picard–Fuchs equation for the 3-chain integral associated with the holomorphic curves which is mirror to the involution brane. The inhomogeneous Picard–Fuchs equation is usually derived by performing the Griffiths–Dwork algorithm [32]. The algorithm itself is clear and straightforward, but the explicit computation needs some effort. In this paper, we propose a more efficient computational method for the Griffiths–Dwork algorithm by considering the 3-chain integral more precisely. Taking the integration by parts for the 3-chain integral successively, we find a ring structure which is generated by the boundary terms. The ring structure makes the Griffiths–Dwork algorithm more manifest, and one can obtain the inhomogeneous Picard–Fuchs equation rather efficiently.

Second, we discuss the direct integration of the 3-chain integral. In the study of the original mirror symmetry [5], the period integral is computed by the direct integration of the holomorphic 3-form over the 3-cycle. In general, these periods are obtained systematically as the solutions of the Picard–Fuchs equation [37], but the direct computation is still meaningful because of its clear geometric picture. In this paper, we consider the direct integration of the 3-chain integral via analytic continuation. The main point of our work is the direct evaluation of the 3-chain integral itself without computing the other periods which form the Gauss–Manin system for the \( \mathcal{N} = 1 \) special geometry. In the analytic continuation, we replace the formal power sum with respect to the complex structure moduli in the period integral by the residue integrals. In the evaluation of the fundamental period, the poles in the residue integrals only appear at the even-integer values, and we obtain the power sum solution of the fundamental period around the large complex structure point. For 3-chain integral, the poles at odd-integer points appear in the integrand of the residue integral. Picking up all the odd-integer points in the residue integrals, we obtain the celebrated solution for the superpotential on the involution brane. Our method is advantageous for the computations of the Calabi–Yau 3-folds with multiple moduli, because in these cases, the inhomogeneous Picard–Fuchs equation becomes too complicated.

Finally, we extend our method to the direct computation for the evaluation of the relative period integrals which appear in the so-called off-shell formalism. The Poincaré residue theorem implies that the relative period integral can also be computed by introducing the logarithmic factor which restricts the integral to the divisor locus [11]. Therefore, we are able to apply our method to the integral of the holomorphic 3-form with a logarithmic factor.  

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3 In [14], a similar computation is discussed for the relative periods.
4 The original framework of \( \mathcal{N} = 1 \) special geometry is discussed in, e.g., [22, 23, 25] and the application to the compact Calabi–Yau is discussed in [11].
The results of our computation coincide with those of [11, 12, 17], and we can check that they satisfy the extended Picard–Fuchs equation for the toric brane. We can also verify that at the critical locus of the open moduli, the relative period coincides with the on-shell results.

This paper is organized as follows. In section 2, we discuss the method to compute the inhomogeneous term of the Picard–Fuchs equation. The inhomogeneous terms are computed explicitly for some one-parameter complete intersection models. These examples are already considered in [4, 28, 10] and our computation correctly recovers their results. In section 3, we discuss the direct computation of the 3-chain integral via analytic continuation. We first consider the two basic examples, quintic hypersurface in \(\mathbb{CP}^4\) and double cubic complete intersection in \(\mathbb{CP}^5\), and find the solutions of the inhomogeneous Picard–Fuchs equations. We also consider one of the two-parameter examples considered in [10], and one of the non-Fermat-type one-parameter complete intersections which has not yet been discussed in the context of the open mirror symmetry. We check that the result of the former case coincides with [10] and the latter case also shows a consistent result in a sense that it predicts integral invariants. In section 4, we extend our analysis of the direct computation to the relative period integrals. We first discuss how the integral of the holomorphic 3-form with a logarithmic factor yields a similar form as the 3-chain integral, and then we evaluate the integrals for the above three one-parameter models. In section 5, we try to fix the normalization ambiguities resulted from analytic continuations. In section 6, we give conclusions. In the appendix, we present the details of computations in section 2.

2. Inhomogeneous Picard–Fuchs equations

2.1. An alternative technique to Griffiths’ reduction

In order to consider the chain integrals of holomorphic 3-forms and derive the Picard–Fuchs equations, there is a useful method known as the Griffiths–Dwork method. We will introduce this method here briefly by adopting the mirror quintic as an example. The mirror quintic Calabi–Yau 3-fold \(Y_5\) is defined by the degree-5 homogeneous polynomial in \(\mathbb{CP}^4/\mathbb{Z}_5^3\),

\[
W = \frac{1}{5} \left( x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \right) - \psi x_1 x_2 x_3 x_4 x_5 = 0,
\]

(2.1)

where \(x_1, x_2, \ldots, x_5\) are the homogeneous coordinates of \(\mathbb{CP}^4\) and \(\psi\) is a complex structure modulus of the hypersurface. The holomorphic 3-form \(\Omega(z)\) is given as the residue at the locus \(W = 0\) on the ambient \(\mathbb{CP}^4\) by

\[
\Omega(z) = \text{Re} s_W \tilde{\Omega}(z), \quad \tilde{\Omega}(z) = \psi \frac{\omega_0}{W},
\]

(2.2)

where \(z = (5\psi)^{-5}\). The holomorphic 4-form \(\omega_0\) is defined by

\[
\omega_0 = \sum_{i=1}^{5} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_5,
\]

(2.3)

where \(\hat{dx}_i\) implies the absence of \(dx_i\). For the small tubular neighborhood \(T_\epsilon(\Gamma)\) around \(\Gamma\), one can express the integral of the holomorphic 3-form

\[
\int_{\Gamma} \Omega(z) = \frac{1}{2\pi i} \int_{T_\epsilon(\Gamma)} \tilde{\Omega} = \frac{\psi}{2\pi i} \int_{T_\epsilon(\Gamma)} \frac{\omega_0}{W}.
\]

(2.4)

To derive the Picard–Fuchs equation, the systematic algorithm proposed by Griffiths is applied for the reduction of the pole order in \(\tilde{\Omega}\). In this paper, instead of adopting the Griffiths reduction algorithm, we will develop an alternative way to derive the inhomogeneous Picard–Fuchs equation. Our method is an extension of the procedure which is studied for the period integrals without boundaries [36, 41, 46].
We will consider the mirror quintic defined (2.1) as an example. Let us represent the defining polynomial (2.1) with the redundant coefficient parameters \(a_i\) of each monomial:

\[
W' = a_1x_1^5 + a_2x_2^5 + a_3x_3^5 + a_4x_4^5 + a_5x_5^5 + a_0x_1x_2x_3x_4x_5.
\] (2.5)

The difference between (2.1) and (2.5) can be compensated by the transformations\(^5\)

\[
x_i \to a_i^{-1/5}x_i \quad (i = 1, \ldots, 5)
\]

with \(\psi = -a_0(a_1a_2a_3a_4a_5)^{-1/5}\). (2.6)

In this parametrization (2.5), one notes an obvious differential relation,

\[
\left(\frac{\partial}{\partial a_0}\right)^5 \frac{\omega_0}{W'} = \prod_{i=1}^5 \left(\frac{\partial}{\partial a_i}\right) \frac{\omega_0}{W'}.
\] (2.7)

We rewrite this equation with (2.6), \(z = 1/\psi^5\) and \(\theta = z\partial/\partial z\):

\[
[\theta^5 - z(5\theta + 5)(5\theta + 4)(5\theta + 3)(5\theta + 2)(5\theta + 1)]\Omega(z) = 0,
\] (2.8)

and factorize the differential operator \(\theta \mathcal{L} = \theta [\theta^4 - z \prod_{i=1}^4 (\theta + i)] \mathcal{L}\). In the case when \(\Gamma\) has no boundary, the period integrals are determined by the four independent solutions of the Picard–Fuchs equation

\[
\left[\theta^4 - z \prod_{i=1}^4 (\theta + i)\right] \int_{\Gamma} \Omega(z) = 0,
\] (2.9)

because \(\mathcal{L}\Omega(z)\) is \(\theta\)-exact.

In the case when \(\Gamma\) has boundaries, i.e. 3-chain, the boundary contributions give rise to the inhomogeneous term of the differential equation\(^6\). It is not so easy to extend the above algorithm to the integral with boundaries, because the Picard–Fuchs equation is obtained by the fifth-order equation rather than the fourth-order equation. Here we discuss the direct derivation of the fourth-order differential equation for \(\Omega\). We introduce an integral

\[
\mathcal{F} = \int \frac{\omega_0}{x_1x_2x_3x_4x_5} \log W',
\] (2.10)

This integral is invariant under the transformation \(x_i \to \lambda x_i\), \(i = 1, \ldots, 5\) up to a moduli-independent term. The derivative of \(\mathcal{F}\) with respect to the moduli is well defined as an integral over \(\mathbb{C}\mathbb{P}^4\). It is easy to see

\[
\left(\frac{\partial}{\partial a_0}\right)^5 \mathcal{F} = \prod_{i=1}^5 \left(\frac{\partial}{\partial a_i}\right) \mathcal{F}.
\] (2.11)

Rewriting this equation using (2.6), \(\theta = z\partial/\partial z\), and a relation, \(\Omega = -\theta \mathcal{F}\), one can find the fourth-order differential equation

\[
[\theta^4 - z(5\theta + 4)(5\theta + 3)(5\theta + 2)(5\theta + 1)]\Omega(z) = 0.
\] (2.12)

We have seen that a rather trivial relation leads to the Picard–Fuchs equation by scaling the integration variables. However, in the case when \(\Gamma\) has boundaries, we cannot expect that the boundaries do not depend on the moduli. Then we should pick up the contributions from the boundaries when we consider the rescaling of the variables.

\(^5\) Strictly speaking, in the \(x_1 = 1\) patch, we transform as \(x_i \to (a_i/a_1)^{-1/5}x_i\), \(i = 2, 3, 4, 5\). Transformation for other patches can be obtained quite similarly.

\(^6\) The physical meaning of this integral over the chain is discussed a little in the next section.
2.2. Fourth-order differential equation with an inhomogeneous term

The boundaries of the 3-chain are specified explicitly after the coordinate transformations (2.6) [7]. So as to treat the chain integral explicitly, we have to use the expression of $\Omega$ defined by

$$\Omega = -\theta \int \frac{\omega_0}{x_1 x_2 x_3 x_4 x_5} \log W,$$

$$W = \frac{1}{5} (x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5) - \psi x_1 x_2 x_3 x_4 x_5. \tag{2.13}$$

In the following, we consider a certain scaling method, which is essentially equivalent to the above derivation of the Picard–Fuchs equation. To perform the rescaling of the coordinates $x_i$, we separate $\omega_0$ and $\Omega$ into two parts:

$$\omega_0 = \omega_0(1234) + \omega_0(5), \quad \Omega = \Omega(1234) + \Omega(5), \tag{2.14}$$

$$\omega(1234) = -x_5 \, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4, \tag{2.15}$$

$$\omega(5) = x_4 \, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 - x_3 \, dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 + x_2 \, dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 - x_1 \, dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \tag{2.16}$$

$$\Omega(1234) = \psi \int \frac{\omega(1234)}{W}, \quad \Omega(5) = \psi \int \frac{\omega(5)}{W}. \tag{2.17}$$

For $\Omega(1234)$, we rescale $x_i$ ($i = 1, 2, 3, 4$) by $\tilde{x}_i$:

$$\Omega(1234) = \psi \int \frac{\omega(1234)}{W} = -\psi \frac{\partial}{\partial \psi} \int \frac{\omega(1234)}{x_1 x_2 x_3 x_4 x_5} \log W = -\psi \frac{\partial}{\partial \psi} \int \frac{\omega(1234)}{x_1 x_2 x_3 x_4 x_5} \log W$$

$$+ \int \frac{\partial}{\partial x_5} \log W \frac{\omega(1234)}{x_1 x_2 x_3 x_4 x_5} + \int \frac{\partial}{\partial x_4} \log W \frac{\omega(1234)}{x_1 x_2 x_3 x_5}. \tag{2.18}$$

In this computation, we adopted the following integration formula to take the boundary effects into account properly:

$$\psi \frac{\partial}{\partial \psi} \int_{a}^{b} \frac{f(x)}{x} \, dx = \psi \frac{\partial}{\partial \psi} \int_{a}^{b \psi^m} \frac{f(\psi^m \tilde{x})}{\tilde{x}} \, d\tilde{x}$$

$$= \int_{a \psi^m}^{b \psi^m} \psi \frac{\partial}{\partial \psi} \frac{f(\psi^m \tilde{x})}{\tilde{x}} \, d\tilde{x} - m \int_{a}^{b} \frac{\partial}{\partial x} f(x) \, dx, \tag{2.19}$$

where $\tilde{x} := x / \psi^m$.

For $\Omega(5)$, we rescale only $x_5$ by $\tilde{x}_5 / \psi$:

$$\Omega(5) = \psi \int \frac{\omega(5)}{W} = -\psi \frac{\partial}{\partial \psi} \int \frac{\omega(5)}{x_1 x_2 x_3 x_4 x_5} \log W$$

$$+ \int \frac{\partial}{\partial x_5} \log W \, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$$

$$- \int \frac{\partial}{\partial x_5} \log W \, dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5$$

$$+ \int \frac{\partial}{\partial x_5} \log W \, dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5. \tag{2.20}$$
where we adopted (2.19). Combining these two contributions, we obtain

\[ \Omega = \Omega_{\text{even}} + \Omega_{\text{odd}} \]

\[ = \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} + 5 \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} \]

\[ + \int d \log W \wedge (d \log x_1 \wedge d \log x_2 \wedge d \log x_3 - d \log x_1 \wedge d \log x_2 \wedge d \log x_4 \]

\[ + d \log x_1 \wedge d \log x_3 \wedge d \log x_4 - d \log x_2 \wedge d \log x_3 \wedge d \log x_4). \]  

(2.21)

To derive the fourth-order differential equation with respect to \( \psi \), we consider the action of \( \theta \) on the \( \int \frac{x^5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} \) term in \( \Omega \) first:

\[ \theta \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} = \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} + \int d \left( \frac{x_5}{W} \right) \wedge (-d \log x_2 \wedge d \log x_3 \wedge d \log x_4 \]

\[ + d \log x_2 \wedge d \log x_3 \wedge d \log x_4 - d \log x_2 \wedge d \log x_3 \wedge d \log x_5 \]

\[ + d \log x_3 \wedge d \log x_4 \wedge d \log x_5). \]  

(2.22)

In the computation, we separated \( \omega_0 \) and rescaled coordinates as above. To proceed, we consider the action of \( \theta \) on \( \int \frac{x^5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} \) which appears in \( \theta^2 \Omega \):

\[ \theta \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} = \frac{2}{3} \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} \]

\[ + \int d \left( \frac{x_5}{W} \right) \wedge (-d \log x_1 \wedge d \log x_3 \wedge d \log x_4 \]

\[ + d \log x_1 \wedge d \log x_3 \wedge d \log x_4 - d \log x_1 \wedge d \log x_2 \wedge d \log x_5 \]

\[ + d \log x_2 \wedge d \log x_3 \wedge d \log x_5). \]  

(2.23)

In the same manner, we find a term in \( \theta^3 \Omega \),

\[ \theta \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} = \frac{3}{4} \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} \]

\[ + \int d \left( \frac{x_5}{W} \right) \wedge (-d \log x_1 \wedge d \log x_2 \wedge d \log x_4 \]

\[ - d \log x_1 \wedge d \log x_2 \wedge d \log x_5 + d \log x_1 \wedge d \log x_3 \wedge d \log x_5 \]

\[ - d \log x_2 \wedge d \log x_3 \wedge d \log x_5). \]  

(2.24)

and a term in \( \theta^4 \Omega \),

\[ \theta \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} = 4 \int \frac{x_5}{W} \frac{\omega_0}{x_1x_2x_3x_4x_5} \]

\[ + \int d \left( \frac{x_5}{W} \right) \wedge (-d \log x_1 \wedge d \log x_2 \wedge d \log x_3 \]

\[ - d \log x_1 \wedge d \log x_2 \wedge d \log x_5 + d \log x_1 \wedge d \log x_3 \wedge d \log x_5 \]

\[ - d \log x_2 \wedge d \log x_3 \wedge d \log x_5). \]  

(2.25)

The first term on the left-hand side of (2.25) can be simply given by

\[ \int \frac{\omega_0}{W^3} (x_1x_2x_3x_4x_5)^5 = \frac{1}{24} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \frac{1}{\psi} \Omega. \]  

(2.26)
From these equations, one finds the fourth-order differential equation for $\Omega$:

$$\theta^4 \Omega = \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \Omega + 6 \int d \left( \frac{x_1^2 x_2^2 x_3}{W^4} \right) \wedge \frac{\omega_4}{x_1 x_2 x_3 x_5}$$

$$+ 2 \theta \int d \left( \frac{x_1^2 x_2^2 x_3}{W^3} \right) \wedge \frac{\omega_3}{x_1 x_2 x_4 x_5} + \theta^2 \int d \left( \frac{x_1^2 x_2^2}{W^2} \right) \wedge \frac{\omega_2}{x_1 x_3 x_4 x_5}$$

$$+ \theta^3 \int d \frac{x_1^2}{W} \wedge \frac{\omega_1}{x_2 x_3 x_4 x_5} - \theta^3 \int d \log W \wedge \frac{\omega_5}{x_1 x_2 x_3 x_4}. \quad (2.27)$$

This equation can be rewritten as follows:

$$\theta^4 \Omega - \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \Omega = \int d\tilde{\beta}, \quad (2.28)$$

$$\tilde{\beta} = 6 \frac{x_1^2 x_2^2 x_3}{W^4} \omega_4 + 2 \frac{x_1^2 x_2^2 x_3}{W^3} \frac{\omega_3}{x_1 x_2 x_3 x_5} + \theta^2 \frac{x_1^2 x_2^2}{W^2} \frac{\omega_2}{x_1 x_3 x_4 x_5}$$

$$+ \theta^3 \left( \frac{x_1^2}{W} \right) \frac{\omega_1}{x_2 x_3 x_4 x_5} - \theta^4 \log W \frac{\omega_5}{x_1 x_2 x_3 x_4}$$

$$= 6 \frac{x_1^2 x_2^2 x_3}{W^4} \omega_4 + 6 \frac{x_1^2 x_2^2 x_3}{W^3} \omega_3 + 6 \frac{x_1^2 x_2^2}{W^2} \omega_2$$

$$+ 6 \frac{x_1^2}{W} \omega_1 + 6 \frac{\omega_5}{x_1 x_2 x_3 x_4 x_5}$$

$$+ 2 \frac{x_1^2 x_2 x_3}{W^3} \omega_2 + 6 \frac{x_1^2 x_2 x_3}{W^3} \omega_1 + 12 \frac{x_1^2 x_2^2}{W^2} \omega_5$$

$$+ \psi \frac{x_1^2}{W} \omega_1 + 7 \psi \frac{x_1 x_2 x_3}{W^2} \omega_5 + \psi \frac{x_1^2}{W} \omega_5. \quad (2.29)$$

This equation is nothing but the Picard–Fuchs equation with the inhomogeneous term$^7$ that was obtained in [7].

In this way, we have described an effective approach to obtain the inhomogeneous term in the Picard–Fuchs equation for the 3-chain integral. The 3-chain on mirror quintic is specified by the matrix factorization [30]. The details of the computations for the 3-chain integral are obtained in the celebrated paper [7]. In the following section, we shall propose a method for evaluating the 3-chain integral directly via analytic continuation. Formally, the 3-chain integral from the direct computation satisfies the inhomogeneous Picard–Fuchs equation obtained by this rescaling algorithm.

### 2.3. Inhomogeneous Picard–Fuchs equation for the double cubic

It is straightforward to extend our algorithm to the complete intersection models. We now discuss the mirror $Y_{3,3}$ of the Calabi–Yau complete intersection $X_{3,3}[1^4]$. The ordinal Picard–Fuchs operator and the period of this model are discussed in [39]. $Y_{3,3}$ is defined by two homogeneous polynomials of degree 3 in $\mathbb{CP}^3/((\mathbb{Z}_3)^3 \times \mathbb{Z}_9)$:

$$W_1 = \frac{1}{3} (x_1^3 + x_2^3 + x_3^3) - \psi x_4 x_5 x_6 = 0,$$

$$W_2 = \frac{1}{3} (x_1^3 + x_2^3 + x_3^3) - \psi x_1 x_2 x_3 = 0. \quad (2.30)$$

The holomorphic 3-form is given by

$$\Omega(z) = \text{Re} \left. s_{W_1=0} \text{Re} \left. s_{W_2=0} \omega_0 \right|_{W_1 W_2} \right. \quad (2.31)$$

$^7$ Precisely speaking, we find $\tilde{\beta}$ in [7] by exchanging $x_1$ and $x_4$.  

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References:

1. Fuji et al. (2011) J. Phys. A: Math. Theor. 44 (2011) 465401.

2. [Other relevant references should be cited here for further reading.]
where we have introduced \( \omega_0 \) as the 5-form on the ambient space \( \mathbb{C}P^5 \):

\[
\omega_0 = \sum_{i=1}^{6} (-1)^i x_i \, dx_1 \wedge \cdots \wedge \, dx_i \wedge \cdots \wedge dx_6. \tag{2.32}
\]

Using a small tube \( T_\epsilon (\Gamma) \) around \( \Gamma \) of size \( \epsilon \),

\[
\int_\Gamma \Omega = \int_{T_\epsilon (\Gamma)} \omega_0 \, W_1 W_2. \tag{2.33}
\]

We apply our rescaling algorithm to this model. Representing the defining equations (2.30) as

\[
W'_1 = a_1 x_1^3 + a_2 x_2^3 + a_3 x_3^3 - a_7 x_4 x_5 x_6,
\]

\[
W'_2 = a_4 x_4^3 + a_5 x_5^3 + a_6 x_6^3 - a_8 x_1 x_2 x_3, \tag{2.34}
\]

we find the obvious differential relation

\[
\prod_{i=1}^{6} \left( \frac{\partial}{\partial a_i} \right) \Omega = \left( \frac{\partial}{\partial a_7} \right)^3 \left( \frac{\partial}{\partial a_8} \right)^3 \Omega. \tag{2.35}
\]

Therefore, in this case, we have to factorize two \( \theta \psi \)s to obtain the fourth-order Picard–Fuchs equation. Applying our method to reproduce (2.35) with \( \psi \) derivatives and performing the integration of the exact terms, we find that the inhomogeneous term of the Picard–Fuchs equation (for the detail of the evaluation, see the appendix) is

\[
\int_{T_\epsilon (\Gamma)} \, \beta = \int_{T_\epsilon (C_+ C_-)} \, \beta = \frac{4\pi^2}{3\psi^5}. \tag{2.36}
\]

From this and \( z := 1/(3\psi)^6 \), we can write down the (normalized) inhomogeneous Picard–Fuchs equation as

\[
L_{PF} T_B(z) = \frac{3^2}{4\pi^2} z^{1/2}. \tag{2.37}
\]

The normalization of the domainwall tension is given by

\[
T_B(z) = \frac{|GP|}{(2\pi i)^3 \psi^2} \int_\Gamma \Omega(z), \tag{2.38}
\]

where \( |GP| \) is the order of the Greene–Plesser orbifold group (so in this case \( |GP| = 3^4 \)). The details of the computations for (the mirror of) \( X_{3,3} \) are described in the appendix.

### 3. Direct integration via analytic continuation

In this section, we discuss the solution of the inhomogeneous Picard–Fuchs equation by a direct computation. The approach of the direct computation is studied in [14], but we rather discuss the direct computation of the superpotential (or the tension of the BPS domainwall) itself by performing the analytic continuation of the period integral. The advantage of our method is that it reproduces the whole expression of the superpotential difference, whereas the Picard–Fuchs equation only determines it up to the periods.

#### 3.1. Mirror of the quintic \( X_5 \) [15]

The defining equation of the mirror quintic Calabi–Yau 3-fold \( Y_5 \) is given in (2.1). Let \( C_\pm \) be

\[
C_\pm = \{ x_1 + x_2 = 0, \, x_3 + x_4 = 0, \, x_5^2 \pm \sqrt{5\psi} x_1 x_3 = 0 \}; \tag{3.1}
\]
then they are B-brane mirror to the real Lagrangian submanifold in A-side, which is defined by the fixed locus of the antiholomorphic involution. This B-brane can be obtained by the use of the matrix factorization method and the grade restriction rule [7, 30]. Moreover, $C_\pm$ and $C_\mp$ are homologous to each other and there exists a 3-chain $\Gamma$ which interpolates between the curves $C_\pm$. Physically, $C_\pm$ correspond to two supersymmetric vacua of an $\mathcal{N} = 1$ supersymmetric theory on the D5-brane worldvolume\(^8\) and we can find the BPS domainwall which wraps $\Gamma$, with boundaries on $C_\pm$. The tension of a BPS domainwall between the two vacua is equal to the difference of superpotentials of $C_\pm$, and given by the integral of holomorphic 3-form over 3-chain $\Gamma$ [31]. In mathematical terminology, it determines a Griffiths’ normal function of the variation of the mixed Hodge structure (see e.g. [32–34] and the references therein) and the superpotentials have information about the obstruction of the curves $C_\pm$ [35].

We shall perform the integration of the holomorphic 3-form in the patch of $x_1 = 1$. The computation of another patch $x_3 = 1$ gives the same result under the exchange of the coordinates $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$. On the $x_1 = 1$ patch, the period integral yields

$$
\Pi = \frac{S^3 \psi}{(2\pi i)^4} \int_{E(\Gamma)} dx_2 dx_3 dx_4 dx_5 W |_{x_1=1}. 
$$

(3.2)

The fundamental period $\Pi_0$ is obtained by integrating over the tubular domain of the fundamental cycle, $T(E(\Gamma)) = y_2 \times y_3 \times y_4$, where $y_i$ encircle the complex coordinates $x_i$. Because of the difficulty in finding 3-cycles whose periods include doubly logarithmic terms $(\log z)^2$, one usually calculates the periods by using the Picard–Fuchs differential equations which govern several periods [36]. On the other hand, the domainwall tension is given by the integration over the tubular domain of the 3-chain $\Gamma$ whose boundary is $\partial \Gamma = C_+-C_-$. It will be desirable to compute the 3-chain integral directly, because it is expected from the A-model that the superpotential difference contains at most a single logarithm [4],

Taking into account the resolution of the singularities [7], we introduce the good coordinates

$$
T = x_1^{-1}x_2, \quad X = x_1x_3^{-1}x_4^{-1}, \quad Y = x_1^{-5}x_5, \quad Z = x_1x_3^{-1}x_4^{-1}x_5^{-1}. 
$$

(3.3)

In these local coordinates, the defining equation in the $x_1 = 1$ patch becomes

$$
W = \frac{1}{8}[1 + T^5 + (X^2Z^3 + X^3Z^2)Y^2 + Y(1 - 5\psi TXTZ)],
$$

(3.4)

and the brane loci $C_\pm$ correspond to $T = -1, X = -Z = \pm \frac{1}{\sqrt{2}}$. For the evaluation of the 3-chain integrals, we further introduce the ‘polar coordinates’ for $X$ and $Z$ as

$$
X = -\frac{\zeta w}{(5\psi T)^{1/2}}, \quad Z = -\frac{\zeta^{-1} w}{(5\psi T)^{1/2}},
$$

(3.5)

where the coordinate $w$ covers the whole complex plane, whereas the coordinate $\zeta$ covers half of the plane\(^9\). In these coordinates, the defining equation can be rewritten as

$$
W = \frac{1}{8}[1 - T^5 + (5\psi)^{-5/2}(\zeta - \zeta^{-1})u^{-5/2}T^{-5/2}Y^2 + Y(1 - u^2)],
$$

(3.6)

and $C_\pm$ correspond to

$$
\zeta = 1, \quad w = \pm 1, \quad T = -1.
$$

(3.7)

---

\(^8\) This D5-brane locates entire non-compact $\mathbb{R}^{1,3}$ and wraps the curves $C_\pm$ in the Calabi–Yau 3-fold.

\(^9\) Basically, the good local coordinates are found heuristically. In [61], a rather simple prescription to find the good local coordinates is discussed in detail and applied to the cases of the quite complicated Pfaffian Calabi–Yau varieties. Later, in subsection 3.4, we will use this prescription effectively to find the local coordinates of the non-Fermat-type model.
The period integral (3.2) can be expressed in terms of these local coordinates as
\[ \Pi = \frac{10}{(2\pi i)^4} \int \frac{d\zeta}{\zeta} \frac{dT}{T} w \frac{dw}{1 - T^3 + (5\psi)^{-5/2}(\zeta - \zeta^{-1})w^{5}T^{-5/2}Y^{2} + Y(1 - w^{2})}. \]

Integration over \( Y \) can be easily performed (we pick up one of two simple poles) and we find
\[ \Pi = \frac{10}{(2\pi i)^3} \int \frac{d\zeta}{\zeta} \frac{dT}{T} w \frac{dw}{\sqrt{(1 - w^{2})^{2} - 4w^{5}(\zeta - \zeta^{-1})(T^{5/2} - T^{-5/2})^{2}}}. \]

where we have defined \( z = 1/(5\psi)^{5} \).

In the large modulus limit, the 3-chain integrals can be expanded as
\[ \Pi = 10 \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{dw}{2\pi i} \frac{w^{5n+1}}{(1 - w^{2})^{2n+1}} \times \int \frac{d\zeta}{(2\pi i)\zeta} (\zeta - \zeta^{-1})^{n} \int \frac{dT}{(2\pi i)T} (T^{5/2} - T^{-5/2})^{n}z^{n/2}. \]

This expression is very useful, because all the integrals are separated. This separation enables us to determine paths of the integration independently. This formula is common for cycle/chain integrals and we choose the appropriate contours of each variable according to what we want to calculate. Basically, the difference between the fundamental period and the chain integral appears as a choice of paths of the \( w \)-integration.

There are six singular points in the \( w \)-integration. In the small \( z \) limit (i.e. the large modulus limit in the A-model), four of them are located near \( w = \pm 1 \) and one of them is located near infinity and the last one is at infinity. The fundamental cycle encircles around the infinity in the large modulus limit. Therefore, we identify the contour for fundamental cycle \( \Gamma_{0} \) which encircles around \( w = -1 \) and \( w = 1 \) drawn in figure 1.
As a contour for the \( \zeta \)-coordinate, we choose \( \zeta = e^{i\theta}, -\pi/2 < \theta < \pi/2 \). For \( T \), we choose \( T = e^{i\phi}, 0 < \phi < 2\pi/5 \), because of the Greene–Plessor orbifold group action. Non-trivial \( \mathbb{Z}_5 \) actions on the \( T, X, Y, Z \) coordinates are expressed as the following charges:

\[
g_1 = (3, 1, 0, 1), \quad g_2 = (4, 3, 0, 3),
\]

and the \( Y \) coordinate is a singlet. To check the validity of the above contour, we evaluate the fundamental period. The \( \zeta \)-integral vanishes unless \( n = 2m (m = 0, 1, \ldots) \), and we find

\[
\int \frac{d\zeta}{2\pi i\zeta} (\zeta - \zeta^{-1})^{2m} = (-1)^m \frac{1}{2} \frac{(2m)!}{(m!)^2}.
\]

(3.12)

The \( T \)-integral can be performed and we find

\[
\int \frac{dT}{2\pi i T} (T^{5/2} - T^{-5/2})^{2m} = (-1)^m \frac{1}{5} \frac{(2m)!}{(m!)^2}.
\]

(3.13)

We can also perform the \( w \)-integral by changing \( w = 1/x \) and evaluating the pole at \( x = 0 \):

\[
\int_{\Gamma_0} \frac{dw}{2\pi i (1 - w^2)^{m+1}} = \frac{(5m)!}{(4m)!m!}.
\]

(3.14)

The contour \( \Gamma_0 \) is shown in figure 1. Collecting these results, we obtain the following well-known result [5]:

\[
\omega_0 = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} e^{m}.
\]

(3.15)

This fact supports the validity of our contour for each variables.

We now evaluate the 3-chain integral with boundaries \( \partial \Gamma = C_+ - C_- \). Because the brane is located at (3.7), we must be careful when considering the \( w \)-contour. Before resolving the Hirzebruch–Jung singularity of \( \{ x_1 + x_2 = 0 = x_3 + x_4 \}/\mathbb{Z}_5 \), one finds two intersection points of \( C_+ \cap C_- \). One of them is located in the \( x_1 = 1 \) patch; we denote it by \( p \). After the resolution of the Hirzebruch–Jung singularity [7], each singular point is replaced by the rational curves and we denote the intersections of \( C_\pm \) with such curves by \( p_\pm \). In the \( w \)-plane, they correspond to \( w = \pm 1 \). We can easily expect that the contour, which corresponds to the chain connecting \( p_+ \) and \( p_- \), can be written as in figure 2. The contour is bounded by the two boundaries because it cannot be removed from these singular points. Since this contour integral encircles cut, it can be represented as twice the line integral connecting two boundaries.
Now we must be careful about the covering of the coordinates. The patch $x_1 = 1$ cannot cover the entire boundary. Therefore, we should add another contribution to the boundary which can be obtained in the patch $x_3 = 1$. It is enough to cover the entire boundary by these two patches. However, the form of the other local coordinates is identical to the one used here. Eventually, we claim that the line integral representing the 3-chain integral connecting two boundaries is four times (i.e. two patches and two line integrals) the line integral connecting $w = -1$ and $w = 1$ in the large modulus expansion.

Then, in order to evaluate the $w$-integration, we consider the analytic continuation of the above period integral. We replace the discrete sum with respect to $n$ by the contour integral with respect to $s$. This is achieved by the Barnes integral formula, so the 3-chain integral becomes

$$\Pi = 40 \oint ds \frac{\pi \cos(\pi s)}{2\pi i} \Gamma(2s + 1) \int_{-1}^{1} dw \frac{w^{5s+1}}{2\pi i (1 - w^2)^{2s+1}} \times \int \frac{d\zeta}{2\pi i} (\zeta - \zeta^{-1})^s \int \frac{dT}{2\pi i T} (T^{5/2} - T^{-5/2})^{sz},$$

where the contour with respect to $s$ encircles non-negative integers. Next we assume that we can separate $\Gamma$ into two parts $\Gamma_+$ and $\Gamma_-$, which are intuitively considered as the contributions of $C_+$ and $C_-$, respectively (shown in figure 3). We denote the $\Gamma_+$ contribution by $\Pi_+$ and $\Gamma_-$ contribution by $\Pi_-$. Since the boundary $C_+$ corresponds to $T = 1$, $\zeta = 1$ and $w = +1$, we will identify the chain integral $\Pi_+$ as twice the line integral starting from $w = 0$ to $w = 1$. Similarly, the integral $\Pi_-$ can be written as the line integral from $-1$ to $0$ and we can obtain this $\Pi_-(z)$ by the relation $\Pi_-(z^{1/2}) = \Pi_+(z^{-1/2})$. Namely, we claim

$$\Pi = \Pi_- - \Pi_+ = \int_{\Gamma_-} - \int_{\Gamma_+} = 2 \int_{-1}^{0} dw (\cdot \cdot \cdot) - 2 \int_{0}^{1} dw (\cdot \cdot \cdot).$$

This is the assumption we should make.

Later we will adopt a similar prescription for the complete intersection Calabi–Yau case (double cubic model), in which case the situation is a little bit different. Nevertheless, we will see that this prescription reproduces the known results for the chain integral.

The integral we should evaluate is

$$\Pi_+ = 40 \oint d\zeta \frac{\pi \cos(\pi s)}{2\pi i} \Gamma(2s + 1) \int_{0}^{1} dw \frac{w^{5s+1}}{2\pi i (1 - w^2)^{2s+1}} \times \int \frac{d\zeta}{2\pi i} (\zeta - \zeta^{-1})^s \int \frac{dT}{2\pi i T} (T^{5/2} - T^{-5/2})^{sz}. $$

\hspace{10cm} (3.18)
We now evaluate each integral contained in the above formula. The integral over $w$ can be evaluated as
\[
\int_0^1 dw \frac{w^{5s+1}}{(1-w^2)^{2s+1}} = \frac{\pi}{2} \sin(2\pi s) \frac{\Gamma(s+1)}{\Gamma(s+1+\frac{5}{2})}.
\]
(3.19)
\[
\zeta \text{ has a parametrization of the half circle } \zeta = e^{i\theta}, -\pi/2 < \theta < \pi/2 \text{ and the } \zeta \text{ integral can be evaluated as half the value of the integral of the contour. The result is}
\]
\[
\frac{1}{2\pi i} \int \frac{dc}{\zeta} (\zeta - \zeta^{-1})^s = \frac{1}{2} \cos \left( \frac{\pi s}{2} \right) \frac{\Gamma(s+1)}{\Gamma(s+1+\frac{5}{2})}.
\]
(3.20)
\[
\text{As noted before, because of the orbifold structure of the variable } T, \text{ the integration region of } T \text{ is } T = e^{i\phi}, 0 < \phi < 2\pi/5. \text{ Therefore, the integral of } T \text{ leads to}
\]
\[
\frac{1}{2\pi i} \int dT (T^{5/2} - T^{-5/2})^s = \frac{\pi}{5} \frac{\Gamma(s+1)}{\Gamma(s+1+\frac{5}{2})}.
\]
(3.21)
\[
\text{It is now easy to perform the integrations to find}
\]
\[
\Pi_+ = \frac{1}{2\pi i} \int ds \frac{\pi \cos(\pi s)}{2\pi i \sin(\pi s)} \left( \frac{2\pi \cos(\frac{\pi s}{2})}{\sin(2\pi s)} \right)^\frac{\zeta(s+1)}{\zeta(s+1+\frac{5}{2})} z^s.
\]
(3.22)
\[
\text{The integral of } s \text{ has double poles for } s = 2n \\text{ and single poles for } s = 2n+1 (n = 0, 1, 2, \ldots). \\text{By evaluating these poles, we finally have}
\]
\[
\Pi_+ = \frac{1}{4\pi i} \sigma_1 + \frac{1}{4} \sigma_0 + \frac{\infty}{4} \sum_{n=0} \frac{\Gamma(n+\frac{7}{2})}{\Gamma(n+\frac{5}{2})} e^{\frac{1}{2} s+1/2}.
\]
(3.23)
\[
\text{Here } \sigma_0 \text{ is the fundamental period and } \sigma_1 \text{ is the logarithmic period given by}
\]
\[
\sigma_1(z) = \sigma_0(z) \log z + \sum_{n=1} \frac{\Gamma(5n+1)}{\Gamma(n+1)^3} (5\Psi(5n+1) - 5\Psi(n+1)) z^n,
\]
(3.24)
\[
\text{where } \Psi \text{ denotes the digamma function. By the relation } \Pi_-(z^{1/2}) = \Pi_+(z^{1/2}) \text{, we obtain}
\]
\[
\Pi_- = \frac{1}{4\pi i} \sigma_1 - \frac{1}{4} \sigma_0 - \frac{\infty}{4} \sum_{n=0} \frac{\Gamma(n+\frac{7}{2})}{\Gamma(n+\frac{5}{2})} e^{s+1/2}.
\]
(3.25)
\[
\text{This result agrees with the form given in [7]. This agreement can be used as the justification of the choice of the line integral we have made for the } w \text{-integral.}
\]
\[
\text{By the results of [4, 6, 7], under the mirror map}
\]
\[
\log q(z) = 2\pi i u(z) = \frac{\sigma_1(z)}{\sigma_0(z)},
\]
(3.26)
\[
\Pi_{\pm}(z) \text{ are identical to } \text{‘the A-model normal function’ [7, section 3.2] of the real quintic after suitable normalization of the holomorphic 3-form:}
\]
\[
\frac{\Pi_{\pm}(z(q))}{\sigma_0(z(q))} = \frac{t}{2} \pm \frac{1}{4} + \frac{1}{2\pi^2} \sum_{k,d \text{ odd}} \frac{\eta_d^{(0,\text{real})}}{k^2} q^{k/2}.
\]
(3.27)
\[
\text{where } \eta_d^{(0,\text{real})} \text{ denote real BPS numbers [2, 29] (which are half of the disk instanton numbers). This number should be an integer by the enumerative interpretation and we can exactly confirm this property [4, 6, 7].}
3.2. Mirror of the double cubic \( X_{3,3} [1^6] \)

Now we compute the period for the mirror geometry of the double cubic \( X_{3,3} [1^6] \). As is the case with mirror quintic, the computation of the fundamental period works well, so we will concentrate on the chain integral only. The mirror Calabi–Yau \( Y_{3,3} \) is the complete intersection in \( \mathbb{CP}^5 / (\mathbb{Z}_3)^2 \times \mathbb{Z}_9 \) and the defining equations are given by

\[
W_1 = \frac{1}{2} (x_1^3 + x_2^3 + x_3^3) - \psi x_4 x_5 x_6, \\
W_2 = \frac{1}{2} (x_1^3 + x_2^3 + x_3^3) - \psi x_1 x_2 x_3.
\]  

(3.28)

The D5-brane is wrapping around the curves \( C_\pm \) which are given by the intersection with hyperplanes \( x_1 + x_2 = 0 \) and \( x_3 + x_4 = 0 \),

\[
C_\pm = \{ x_1 + x_2 = 0, x_4 + x_5 = 0, x_3^3 - 3 \psi x_4 x_5 x_6 = 0, x_6^3 - 3 \psi x_1 x_2 x_3 = 0 \}.
\]  

(3.29)

Since the orbifold group is \( (\mathbb{Z}_3)^2 \times \mathbb{Z}_9 \), the 3-chain integral in the \( x_5 = 1 \) patch is

\[
\Pi = \frac{3^4}{(2\pi i)^3} \int \frac{dx_1 dx_2 dx_3 dx_4 dx_6}{W_1 W_2}. 
\]  

(3.30)

To describe the curves, we should introduce the resolved coordinates to which the Greene–Presser orbifold group acts nicely. In the patch \( x_5 = 1 \), the coordinates are

\[
X = XZ^2 Y^4, \quad X = X^2 Z Y^2, \quad x_3 = Y, \quad x_4 = T, \quad x_5 = 1, \quad x_6 = U Y.
\]  

(3.31)

The coordinate for another patch \( x_5 = 1 \) is also found by exchanging \( (x_1, x_2, x_3, x_4, x_5, x_6) \leftrightarrow (x_1, x_2, x_3) \), and the curves \( C_\pm \) are completely covered by these two patches. In these local coordinates, the location of the brane \( C_\pm \) is specified by \( T = -1, U = -\frac{1}{3\psi} \), \( X = -Z = \pm \frac{1}{(3\psi)^{1/3}} \).

Now as in the case of quintic, we consider \( \Pi_+ \), the contribution from \( C_+ \). The 3-chain integral \( \Pi_+ \) can be expressed as

\[
\Pi_+ = \frac{3^3}{(2\pi i)^3} \int \frac{dX dY dZ dT dU}{[1 - 3\psi TU + (XZ^2 + X^2 Z)Y]((U^3 - 3\psi XZ)Y + 1 + T^3)}.
\]  

(3.32)

One can perform the integration with respect to the variable \( Y \) in the 3-chain integral by picking up the rest:

\[
\Pi_+ = \frac{3^3}{(2\pi i)^3} \int \frac{dX dZ dT dU}{(U^3 - 3\psi XZ)(1 - 3\psi TU) - (XZ^2 + X^2 Z)(1 + T^3)}.
\]  

(3.33)

To perform the contour integrations for the remaining variables, we introduce the polar coordinates

\[
X = \frac{w \xi v^{3/2}}{(3\psi)^{1/2}}, \quad Z = -\frac{w \xi^{-1} v^{3/2}}{(3\psi)^{1/2}}, \quad U = -\frac{v}{3\psi t}, \quad T = -t.
\]  

(3.34)

The curve \( C_\pm \) is now

\[
t = 1, \quad v = 1, \quad \xi = 1, \quad w = \pm 1.
\]  

(3.35)

Then the 3-chain integral can be expressed as

\[
\Pi_+ = \frac{6}{(2\pi i)^3} \int \frac{w dw \, dv \, dt \, d\xi}{\xi \tau \tau^2} \frac{1}{(1 - w^2)(v - 1) - w^3 v^{3/2}(\xi - \xi^{-1})(r^{3/2} - t^{-3/2})(3\psi)^{-3}}.
\]  

(3.36)

Near the large complex structure limit point, this expression can be expanded with respect to \( \tau = 1/(3\psi)^{1/3} \), and rewritten as the Barnes integral form

\[
\Pi_+ = \frac{6}{2\pi i} \int \frac{ds}{2\pi i} \frac{d \pi \cos(\pi s)}{\sin(\pi s)} \frac{\tau^{s/2}}{2 \pi i} \int \frac{dw}{2 \pi i} \frac{w^{3s-1}}{(1 - w^2)^{s+1}} \int \frac{dv}{2 \pi i} \frac{v^{3s/2}}{(v - 1)^{s+1}} \times \int \frac{dt}{2 \pi i \xi} (\xi - \xi^{-1})^s \int \frac{d\tau}{2 \pi i} \left( r^{3s/2} - t^{-3s/2} \right)^s.
\]  

(3.37)
Contrary to the quintic case where the cut structure is present for \( w \), there is no cut structure for the integral here. However, we apply the same prescription as the quintic case and identify the integral over \( w \) as a line integral. Thus, we can find the contribution from \( \Pi_+ \) in the \( x_3 = 1 \) patch as a line integral from \( w = 0 \) to \( w = 1 \). Since we also have the contribution from the patch \( x_3 = 1 \) which can be converted to the same form as \((3.37)\), we identify \( \Pi_+ \) as twice the line integral from \( w = 0 \) to \( w = 1 \).

For the several integrations, we shall frequently use the following formula:

\[
\int_0^1 \frac{dx}{2\pi i} \frac{x^{\mu/2}}{(1-x)^{N+s+1}} = \frac{1 - e^{2N\pi i s}}{2\pi i} \int_0^1 \frac{dx}{(1-x)^{N+s+1}} = \frac{\pi}{2\sin(\pi s)} \frac{\Gamma(1/2 + s)}{\Gamma(Ns + 1)}.
\]  

where the contour is chosen as in figure 4.

Taking into account for the Greene–Presser group action \( G \simeq (\mathbb{Z}_3)^2 \times \mathbb{Z}_9 \), we specify the integral contours for \( (\zeta, t) \). Each variable is parametrized as \( (\zeta, t) = (e^{i\phi}, e^{i\phi}) \) where \(-\pi/2 \leq \theta \leq \pi/2\) and \( 0 \leq \phi \leq 2\pi/3 \). Therefore, the values of the integrals \( \zeta \) and \( t \) are just half and one-third of the result of the contour in figure 4, respectively. Then each integral in the 3-chain integral is evaluated as follows:

\[
\int_0^1 \frac{du}{2\pi i} \frac{u^{3s}}{(1-u^{3s})^{s+1}} = \frac{\pi}{2\sin(\pi s)} \frac{\Gamma(1/3 + s)}{\Gamma(Ns + 1)} \Gamma(1/2 + s),
\]

\[
\int_0^1 \frac{d\nu}{2\pi i} \frac{\nu^{3s/2}}{(1-\nu^{3s/2})^{s+1/2}} = \frac{\pi}{2\sin(\pi s)} \frac{\Gamma(1/3 + s)}{\Gamma(Ns + 1)} \Gamma(1/2 + s),
\]

\[
\int \frac{d\xi}{2\pi i} \xi^{-1/2}(1-\xi^{-1}) = \frac{\cos(\pi/3)}{2} \frac{\Gamma(1/3 + s)}{\Gamma(1/2 + s)},
\]

\[
\int \frac{d\tau}{2\pi i} e^{-1/2}(1-\tau^{-1}) = \frac{\pi}{3} \frac{\Gamma(1/3 + s)}{\Gamma(1/2 + s)}. \tag{3.39}
\]

Collecting all contributions, one finds

\[
\Pi_+ = \frac{1}{2\pi i} \int \frac{ds}{2\pi i} \frac{\pi \cos(\pi s)}{\sin(\pi s)} e^{is/2} \pi \cos(\pi s) \frac{\Gamma(1/3 + s)}{\sin(\pi s)} \frac{\Gamma(1/2 + s)}{\Gamma(1/2 + s)} \frac{\Gamma(1/3 + s)}{\Gamma(1/3 + s)} \frac{\Gamma(1/2 + s)}{\Gamma(1/2 + s)} e^{is/2}. \tag{3.40}
\]
There are simple poles at \( s = 2n + 1 \) and double poles at \( s = 2n \). After the residual integrals on \( s \), we obtain

\[
\Pi_+ = \frac{1}{4\pi i} \sigma_1 + \frac{1}{4} \sigma_0 + \frac{1}{4} \tau, \tag{3.41}
\]

where \( \sigma_0 \) is the fundamental period, \( \sigma_1 \) is the logarithmic period and the remaining term \( \tau \) is given as follows:

\[
\tau = \sum_{n=0}^{\infty} \frac{\Gamma(3n + \frac{1}{2})^2}{\Gamma(n + \frac{3}{2})^6} \omega^{n+1/2}. \tag{3.42}
\]

\( \Pi_- \) can also be obtained in the same way as the quintic case.

### 3.3. Mirror of \( X_{12}[1^2, 2^2, 6] \)

In this subsection, we discuss one of the two-parameter Calabi–Yau hypersurfaces, \( X_{12}[1^2, 2^2, 6] \).\(^{10} \) This model is expected to have the \( \mathbb{Z}_3 \)-structure of the open string vacuum, and instanton numbers are calculated in [10] by using open mirror symmetry\(^{11} \). The solution of the inhomogeneous Picard–Fuchs equation is complicated, but we will see that it can be obtained rather simply by our method. We also note that off-shell situations of this model are discussed in [14] by another technique, the simple direct integration based on the classic paper [5].

The defining polynomial of the mirror \( Y_{12} \) is given by

\[
W = \frac{1}{12} x_1^{12} + \frac{1}{12} x_2^{12} - \frac{1}{6} x_1^6 - \frac{1}{6} x_4^6 + \frac{1}{2} x_2^2 - \psi x_1 x_2 x_3 x_4 x_5 - \frac{\phi}{6} (x_1 x_2)^6, \tag{3.43}
\]

and the Greene–Plessor orbifold group is \( (\mathbb{Z}_6)^2 \times \mathbb{Z}_2 \). Note that the signs for the third and fourth monomials are changed by phase transformations for convenience. Following [10], we choose the boundary as the intersection with hyperplanes such as

\[
x_1^2 = 2^{1/6} x_3, \quad x_2^2 = 2^{1/6} x_4. \tag{3.44}
\]

Then the boundaries can be specified by

\[
x_3 = \alpha_{\pm} (x_1 x_2)^3, \tag{3.45}
\]

where \( \alpha_{\pm} \) are the two solutions of

\[
\frac{1}{2} \alpha^2 - 2^{-1/3} \psi \alpha - \frac{\phi}{6} = 0. \tag{3.46}
\]

We choose the local coordinates as

\[
x_1 = 1, \quad x_3 = 2^{-1/6} T, \quad x_2^2 = Y, \quad x_4^6 = Y X^4, \quad x_4 = \frac{1}{2} X^2 Z^6; \tag{3.47}
\]

then the defining polynomial can be written as follows:

\[
2W = \frac{1}{6} (1 - T^6) + \frac{1}{6} Y^2 (X^8 - X^2 Z^6) + Y (1 - 2^{1/3} \psi X Z T - \frac{1}{2} X^4). \tag{3.48}
\]

By introducing the following polar coordinates:

\[
X = \frac{w^{1/3} - 1}{(2^{1/3} \psi T)^{1/3}}, \quad Z = \frac{w^{1/3} - 1}{(2^{1/3} \psi T)^{1/3}}, \quad T^2 = t, \tag{3.49}
\]

the defining equation can be rewritten as

\[
2W = \frac{1}{6} (1 - t^3) + \frac{1}{6} Y^2 \left( \frac{1}{(2^{1/3} \psi)^4 t^{2/3} (1 - \zeta)^3} + Y \left( 1 - \frac{\phi}{3 (2^{1/3} \psi)^2 t^{1/3} \zeta^{1/3}} \right) \right). \tag{3.50}
\]

\(^{10} \) Several analyses of (closed) mirror symmetry for 2-modulus Calabi–Yau are in [40–45].

\(^{11} \) A similar structure (\( Z_3 \)-vacua \( k \neq 2 \)) is also observed in \( X_{4,4}[1^4, 2^4] \) and \( X_{6,6}[1^2, 2^2, 3^2] \). Although the A-model picture has so far been missing, there are the enumerative predictions for real BPS numbers [10].
In these variables, the locations of the boundaries are specified by

\[ t = \eta (\eta^3 = 1), \quad \zeta = 1, \quad w = 1. \]  

(3.51)

The period integral becomes

\[ \Pi = \frac{6^2 \times 2}{(2\pi i)^4} \int \frac{dx_1 \, dx_3 \, dx_4 \, dx_5}{W} = \frac{1}{(2\pi i)^4} \int \frac{dt}{t} \int \frac{d\zeta}{\zeta} \int dw \int dY \frac{1}{2W}. \]  

(3.52)

By making integral over \( Y \), we have

\[ \Pi = \frac{1}{(2\pi i)^4} \int \frac{dt}{t} \int \frac{d\zeta}{\zeta} \int dw \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)}{\Gamma(n+1)^2} \left( \frac{1 - t^3 y + 2n (1 - \zeta)^n \zeta^{-2n/3} w^{4n}}{(1 - w - \frac{\phi}{3^2 \cdot 2^{4/3} \psi})^{2n+1}} \right)^n \cdot \left( \frac{1}{12^2 2^{8/3} \psi^2} \right)^n. \]  

(3.53)

Expanding with respect to the variable \( \phi \) and considering the analytic continuation of the summation with respect to \( k + 2n \) to the integration with respect to \( s \), we have

\[ \Pi = \frac{1}{(2\pi i)^4} \int \frac{ds}{s} \frac{\pi \cos(\pi s)}{2\pi i \sin(\pi s)} \sum_{n=0}^{\infty} \frac{\Gamma(s+1)}{\Gamma(s-2n+1)\Gamma(n+1)^2} \left( \frac{1 - r^3 y + 2n (1 - \zeta)^n \zeta^{-2n/3} w^{4n}}{(1 - w - \frac{3\phi}{3^2 \cdot 2^{4/3} \psi})^{2n+1}} \right)^n \left( \frac{3^2}{12^2 \psi^2} \right)^n. \]  

(3.54)

We perform the \( t \)-integral as a line integral from 0 to \( \eta^{-1} (\eta^3 = 1) \). For \( \zeta \), we choose the integral from 0 to 1. The \( w \)-integral is the contour integral around \( w = 1 \). In this way, we obtain

\[ \Pi = 3 \sum_{n=0}^{\infty} \int \frac{ds}{s} \frac{\pi \cos(\pi s)}{2\pi i \sin(\pi s)} \left( \frac{\eta y^{1/3} z_2^2}{\sin(\frac{\pi}{2\psi})} \right)^2 \frac{\Gamma(2s+1)}{\Gamma(\frac{s}{2} + 1) \Gamma(n - \frac{s}{2} + 1) \Gamma(s - 2n + 1) \Gamma(s+1)} \eta^s y^{s/3} z_2^2. \]  

(3.55)

where \( y = \frac{\phi}{3^2 \cdot 2^{4/3} \psi} \) and \( z_2 = \frac{3^2}{12^2 \psi^2} \).

By evaluating poles of the \( s \)-integration, we obtain a rather complicated result obtained in [10]. It is interesting that we find a rather simple expression by using an integral representation of Barnes’ type. We expect that we can also obtain a similar expression for \( X_6[1^2, 2^1] \).\(^{12}\)

### 3.4. Mirror of \( X_{3,4}[1^5, 2^1] \): non-Fermat-type example

In this subsection, we discuss the quite non-trivial example to which we apply our direct integration method effectively. The example we treat here is called the non-Fermat type, which has a very interesting property in the sense that the B-model mirror Calabi–Yau cannot be obtained as the orbifolding quotient of the A-model Calabi–Yau. The formula of the period integrals for this type Calabi–Yau has a non-trivial numerator as shown in (3.59). These kinds of models have not yet been discussed in the context of open mirror symmetry.

Concretely, we consider the mirror geometry of \( X_{3,4}[1^5, 2^1] \), which is a one-parameter compete intersection Calabi–Yau 3-fold. It is known that using the toric method, we find the

\(^{12}\) \( X_{12}[1^2, 2^1, 6] \) and \( X_6[1^2, 2^1] \) are both 2-modulus models and have the K3-fibration structure [48, 49].
following defining equations on the B-model side:

\[ W_1 = \psi - (X_1 + X_2 + X_3), \]
\[ W_2 = \psi - \left( X_4 + X_5 + \frac{1}{X_1^2 X_2 X_3 X_4 X_5} \right), \]  
(3.56)

and by using the homogeneous coordinates, this is given by

\[ W_1 = x_1^2 + x_2^2 + x_3^2 - \psi x_1^2 x_2 x_3 x_4 x_5, \]
\[ W_2 = x_1^2 + x_2^2 + x_5^2 - \psi x_1^2 x_2 x_3 x_4 x_5, \]  
(3.57)

We take B-branes \( C_{\pm} \) as the following curve:

\[ C_{\pm} = \{ x_2 + x_3 = 0, x_5 + x_6 = 0, x_1^2 - \psi x_1^2 x_2 x_3 x_4 x_5 = 0, x_4^2 - \psi x_1^2 x_2 x_3 x_4 x_5 = 0 \}. \]  
(3.58)

The period integral on the patch \( \{ x_6 = 1 \} \) is given by

\[ \Pi|_{x_6=1} = \frac{\psi^2}{(2\pi i)^3} \int \frac{\prod dX_1 \, dX_2 \, dX_3 \, dX_4 \, dX_5}{W_1 W_2} \left( x_1^2 x_2 x_3 x_4 x_5 \right) \]  
(3.59)

Now, we introduce the following new local coordinates\(^{13}\):

\[ x_1 = \psi^{1/4} T^{-1/4} w^{3/7} S^{1/28}, \quad x_2 = T^{2/7} \zeta, \quad x_3 = T^{2/7} \zeta^{-1}, \]
\[ x_4 = \psi^{1/4} T^{-1/4} w^{1/17} S^{5/28}, \quad x_5 = T, \quad x_6 = 1. \]  
(3.60)

In these new coordinates, the defining equations and the period integral are expressed by

\[ W_1 = T^{2/7} (\zeta^7 + \zeta^{-7}) - \psi^{7/4} T^{7/4} w S^{1/4} (1 - w^2), \]
\[ W_2 = 1 + T^7 - \psi^{7/4} T^{7/4} w S^{1/4} (1 - S), \]  
(3.61)

\[ \Pi = 2 \cdot 7^2 \frac{\psi^2}{(2\pi i)^3} \int \frac{dT \, d\zeta \, dS \, dw}{T \zeta S^{1/2}} \left[ \frac{\psi^{7/4} T^{7/4} w}{[Y(\zeta^7 + \zeta^{-7}) - \psi^{7/4} T^{7/4} w S^{1/4} (1 - w^2)]} \right] \]  
\[ \times \left[ 1 + T^7 - \psi^{7/4} T^{7/4} w S^{1/4} (1 - S) \right]. \]  
(3.62)

The pre-factor 2 means a contribution from the other patch \( \{ x_6 = 1 \} \).

The following computations are done straightforwardly in a similar way to other examples. First, we perform \( Y \)-integration by picking up a pole at \( Y = \psi^{1/17} T^{4/17} S^{1/17} (1 - S)^{1/17} \). Then, we expand this around the large complex structure limit \( \zeta = \frac{1}{T} = 0 \). By performing an analytic continuation to the Barnes forms, we obtain the factorized formula

\[ \Pi = 2 \cdot 7^2 \int \frac{ds}{2\pi i} \frac{\pi \cos(\pi s)}{\sin(\pi s)} z^{2/7} \int \frac{dT}{2\pi i} T^{-7s/2 - 1} (1 + T^7)^t \int \frac{d\zeta}{2\pi i} (\zeta^7 + \zeta^{-7})^t \]  
\[ \times \int \frac{dw}{2\pi i} w^{-2r - 1} (1 - w^2)^{-r - 1} \int \frac{dS}{2\pi i} S^{-s/2 - 1} (1 - S)^{-s/2 - 1}. \]  
(3.63)

Considerations similar to the other cases lead us to the following integral paths: for \( T \), \( T = e^{i\theta} \) (\( 0 \leq \theta \leq 2\pi/7 \)); for \( \zeta, \xi = e^{i\phi} (-\pi/2 \leq \phi \leq -\pi/2 + \pi/7) \); and for \( S \), we take the contour encircling 0 and 1 one time. For \( w \), which is important since the boundary of chain is expressed by \( w = \pm 1 \), we treat them in a similar way to the double cubic case. Then we obtain

\[ \int \frac{dT}{2\pi i} T^{-7s/2 - 1} (1 + T^7)^t = \frac{(-1)^{t/2} \cos(7\pi s/2)}{7} \frac{\Gamma(s + 1)}{\Gamma(s/2 + 1)^2}. \]  
(3.64)

\(^{13}\) Here we adopt the rather systematic prescription for obtaining suitable local coordinates, which is discussed in [61].
\[
\int \frac{d\zeta}{2\pi i}(\xi^7 + \xi^{-7})^t = \frac{(-1)^7}{14} \frac{\Gamma(s + 1)}{\Gamma(s/2 + 1)^2}.
\] (3.65)

\[
\oint \frac{dS}{2\pi i} S^{-s/2-1}(1 - S)^{-s-1} = e^{-\pi i s} \frac{\sin(3\pi s/2)}{\sin(\pi s/2)} \frac{\Gamma(3s/2 + 1)}{\Gamma(s + 1)\Gamma(s/2 + 1)}.
\] (3.66)

\[
\oint \frac{dw}{2\pi i} w^{-2s-1}(1 - w^2)^{-s-1} = e^{-2\pi i s} \frac{\cos(\pi s)}{\Gamma(s + 1)^2}.
\] (3.67)

\[
\int_0^1 \frac{dw}{2\pi i} w^{-2s-1}(1 - w^2)^{-s-1} = -\frac{1}{4i} \frac{1}{\sin(\pi s)} \frac{\Gamma(2s + 1)}{\Gamma(s + 1)^2}.
\] (3.68)

Altogether, we obtain the formulas of the fundamental period and the domainwall tension. Similar to the other examples, the fundamental period has simple poles at \(s = 2n\), and the domainwall tension has simple poles at \(s = 2n + 1\) and double poles at \(s = 2n\). By picking up such poles, we obtain the fundamental period\(^{14}\)

\[
\sigma_0 = \oint \frac{ds}{2\pi i} \frac{\pi \cos^2(\pi s) \cos(7\pi s/2) \sin(3\pi s/2)}{\sin(\pi s/2)} \frac{z^{s/2}(-1)^{i/2}}{\Gamma(2s + 1)\Gamma(3s/2 + 1)}
\]

\[
= \sum_{n=0}^{\infty} \frac{\Gamma(3n + 1)\Gamma(4n + 1)}{\Gamma(n + 1)^5\Gamma(2n + 1)} z^n,
\] (3.69)

and the domainwall tension

\[
\Pi_+ = -\frac{1}{4i} \oint \frac{ds}{2\pi i} \frac{\pi \cos(\pi s)}{\sin(\pi s/2)} \frac{z^{s/2}(-1)^{i/2} \cos(7\pi s/2) \sin(3\pi s/2)}{\sin(\pi s/2)} \frac{\Gamma(2s + 1)\Gamma(3s/2 + 1)}{\Gamma(s + 1)\Gamma(s/2 + 1)^5}
\]

\[
= \frac{\sigma_1}{4\pi i} + \frac{\sigma_0}{4} + \frac{1}{4} \tau,
\] (3.70)

where \(\sigma_0\) is the fundamental period, \(\sigma_1\) is the logarithmic period and

\[
\tau = \sum_{n=0}^{\infty} \frac{\Gamma(3n + 5/2)\Gamma(4n + 3)}{\Gamma(n + 3/2)^5\Gamma(2n + 2)} z^{n+1/2}.
\] (3.71)

\(\Pi_-\) can be evaluated similarly.

We can verify that this function predicts the integral real BPS invariants in the A-model side, after transforming by the mirror map and re-summing by the Ooguri–Vafa formula. The first few numbers (with the normalization in \([4]\)) are shown as

\[
n_1^{0,\text{real}} = 12, \quad n_3^{0,\text{real}} = 348, \quad n_5^{0,\text{real}} = 133,008, \quad n_7^{0,\text{real}} = 66,515,628, \ldots.
\] (3.72)

Before concluding this section, we note some comments on this direct integration method. We have obtained the domainwall tensions for several other models by the same method. For other models, it sometimes happens that the \(Y\)-coordinate dependence in the chain integration is not quadratic but cubic or a polynomial of higher degrees. Such complications seem to be present in the situation where the weights of the ambient projective space are not all the same. At least for the models we have checked, after performing the \(Y\)-integral and expanding around the large complex structure limit, the period integral becomes factorized with respect to the remaining variables. In the computation, it is not clear whether we have the correct normalization for generic cases since we use the analytic continuation. However, the above examples \(X_5[15], X_3[16]\) and \(X_{12}[1, 2, 2, 6]\) show that we can certainly reproduce

\(^{14}\) We neglect an overall factor which is common for \(\sigma_0\) and \(\Pi_+\).
the known results. Moreover, we can obtain the prediction for the integral invariants of the new example $X_{3,4}$ [15, 2]. We can also find that this method is successfully applied to the quite nontrivial Pfaffian cases in [61]. It is true that our approach has heuristic aspects in choosing local coordinates and in taking contours of the integrals; the above facts show that the direct integration via analytic continuation is a powerful method to obtain the correct form of the solutions of the inhomogeneous Picard–Fuchs equations.

4. Application to the off-shell effective superpotential

4.1. Relative period integral

Next we extend our analysis to the off-shell formalism, typically computations of superpotentials for toric branes [11, 14, 12, 15, 13, 16, 17]. The research of the mirror symmetry of the toric brane in the non-compact Calabi–Yau 3-fold is initiated in [1, 52]. The toric brane is specified by the extended toric charge vectors $\ell^{(a)}$. One can construct the mirror pair of A- and B-branes from the extended set of the toric vectors systemically.

In [11], it is proposed that the relative period of $H^3(X_3, S)$ for the holomorphic curve $S$ is assumed to be equivalent to that of $H^3(X_3, V)$ where $(S \subset V) \subset X_3$ is a divisor, i.e. a complex codimension 1 variety in the Calabi–Yau 3-fold $X_3$. This aspect has recently been studied further in relation to the non-compact Calabi–Yau 4-fold [17] and F-theory on it [15, 16].

The original open–closed duality of this kind is discussed in [25]. The relation to the heterotic theory by the further duality chain is discussed in [26, 27].

The divisor $V$ is given by the single defining equation $Q(x_i, \phi) = 0$ which is defined by one of the extended toric charge vectors. For example, in the case of the quintic 3-fold, the defining equation for the divisor is given by

$$Q(x_i, \phi) = x_5^5 - \phi x_1 x_2 x_3 x_4 x_5,$$

(4.1)

corresponding to the toric charge vector $\ell^{(2)} = (−1, 0, 0, 0, 1)$ [1]. Here we introduce the parameter $\phi$ and call it an open string modulus.

The chain integral $\int_{\Gamma} \Omega$ is regarded as the relative period integral. The holomorphic 3-form $\Omega$ is extended to the relative 3-form $\Xi_3$ in the relative cohomology class $H^3(X_3, V)$. The relative 3-form $\Xi_3$ is a pair of the closed 3-form $\Xi$ on $X_3$ and closed 2-form $\xi$ on $V$, namely $\Xi_3 = (\Xi, \xi)$. Let $\iota$ be an embedding map $\iota : V \hookrightarrow X_3$. The relative 3-form $\Xi_3 \in H^3(X_3, V)$ satisfies the equivalence relation

$$\Xi_3 \sim \Xi + (d\alpha, \iota^* \alpha - d\beta),$$

(4.2)

where $\alpha$ is a 2-form on $X_3$ and $\beta$ is a 1-form on $V$.

The relative 3-form $\Xi_3$ is integrated over the relative 3-cycle $\Gamma \in H_3(X_3, V)$. The relative 3-cycle is denoted as $\Gamma = (\iota^* \Gamma, \partial \Gamma)$. The relative 3-forms and 3-cycles are paired by the integration,

$$\int_{\Gamma} \Xi_3 := \int_{\Gamma} \Xi - \int_{\partial \Gamma} \xi.$$  

(4.3)

This definition is consistent with the above equivalence relations.

The relative period is also evaluated by the Griffiths residue integral formula [32]. Now we assumed that the original Calabi–Yau 3-fold $X_3$ is a complete intersection in a weighted projective space $\mathbb{P}^P_n$. $X_3$ is specified by $(n - 3)$ defining equations $W_a$ ($a = 1, \ldots, n - 3$).

15 Mathematical justification is argued in [20].

16 The relation of the relative period and the period for the non-compact 4-fold is discussed in the appendix.

17 We choose the toric vector for the quintic 3-fold as $\ell^{(1)} = (−5, 1, 1, 1, 1)$. 

20
For the embedding map $\iota : V \hookrightarrow X_3$, the pull back of a form $\alpha$ on $X_3$ is computed by inserting the Poincaré residue map $[53]$:

$$\iota^* \alpha = \frac{1}{(2\pi i)^2} \int_{T(V)} \frac{dQ}{Q} \wedge \alpha. \quad (4.4)$$

Then one finds that all the relative period integrals arise from a relative period $\Pi_r$ with a log $Q$ factor.

$$\Pi_r = \int \log Q(x, \phi) \prod_a W_a(x, \psi) \Delta, \quad (4.5)$$

and $\Delta$ is an $n$-form on $\mathbb{P}^n$ given by

$$\Delta = \sum_{i=1}^{n+1} (-1)^{i+1} u_i \, dx^1 \wedge \cdots \wedge \hat{dx}_i \cdots \wedge dx^{n+1}, \quad (4.6)$$

where $u_i$ are weights of $\mathbb{P}^n$. This period satisfies the extended Picard–Fuchs equation. One of the solutions which depend on the open string modulus $\phi$ is the effective superpotential $W_{\text{eff}}(\psi, \phi)$. Extremizing this effective superpotential, one can fix the open string moduli as a function of the closed string moduli $\psi$. In fact, for the case of the quintic, the effective superpotential is extremized at $\phi = 5\psi$. This logarithmic factor prescription is proposed first in [11]. Next let us discuss the relation of the superpotentials in more detail.

### 4.2. Relative period and 4-fold

In the recent works [15–17], it is discussed that the type IIB theory with D5-brane on the Calabi–Yau 3-fold $X_3$ is related to F-theory on the non-compact Calabi–Yau 4-fold $X_4$. The corresponding non-compact Calabi–Yau 4-fold $X_4$ is given by the complete intersection of $W_a(x, \psi) = 0$ and the defining equation $Q_4(\phi) = x_{n+2}x_{n+3} + Q(x, \phi) = 0$, (4.7)

in the ambient space $\mathbb{P}^{n+2}$ whose weights of coordinates $x_{n+2}$ and $x_{n+3}$ are determined by the defining equations$^{18}$. We also eliminate the locus $(0 : \cdots : 0 : x_{n+2} : x_{n+3})$ in $\mathbb{P}^{n+2}$. The period of the holomorphic 4-form on this geometry is

$$\Pi_4 = \int_{T_4(\Gamma_4)} \Delta_{n+2} \prod_a W_a(x, \psi) Q_4(\phi), \quad (4.8)$$

where $T_4(\Gamma_4)$ is the tubular neighborhood of the 4-cycle $\Gamma_4$ [17] and $\Delta_{n+2}$ is the appropriate $(n + 2)$-form on ambient space $\mathbb{P}^{n+2}$.

Therefore, we have three different-looking formulas for D5-brane superpotentials:

1. the chain integral of 3-fold,
2. the relative period of 3-fold with the logarithmic factor,
3. the period of 4-fold.

Various analyses and discussions show that these three formulas are essentially the same things. Now we try to confirm this equivalence formally.

We first discuss the connection between 4-fold periods and logarithmic periods. In order to evaluate the 4-fold period integral without suffering from the divergence, we consider its derivative with respect to the open string modulus $\phi$. Here we change the variables $(x_{n+2}, x_{n+3})$ to the polar coordinates such that

$$x_{n+2} = r \, e^{i\theta}, \quad x_{n+3} = r \, e^{-i\theta}. \quad (4.9)$$

$^{18}$ If the weight of $x_{n+2}$ or $x_{n+3}$ is zero, the resulting space becomes $\mathbb{P}^{n+1} \times \mathbb{C}$. 


The derivative of the period $\Pi_4$ is

$$\partial_\phi \Pi_4 = \int_{T_4(\Gamma_4)} \partial_\phi \frac{\partial \Phi_Q(x_i, \phi) \Delta_n \wedge d(\gamma^2) \wedge d\theta}{\prod_a W_a(x_i, \psi)}.$$  \hspace{1cm} (4.10)

We define $n$-form $\Delta_n$ by the relation $\Delta_{n+2} = \Delta_n \wedge d(\gamma^2) \wedge d\theta$. The integral over $\gamma$ and $\theta$ can be performed by taking them as the coordinates of the whole two-dimensional plane. As a result, we find

$$\partial_\phi \Pi_4 = -\frac{1}{\pi} \int_{T_4(\Gamma_4)} \frac{\partial_\phi \Phi_Q(x_i, \phi) \Delta_n}{\prod_a W_a(x_i, \psi)} = -\frac{1}{\pi} \int_{T_4(\Gamma_4)} \log \Phi_Q(x_i, \phi) \Delta_n \wedge \partial_\phi \Pi_4. \hspace{1cm} (4.11)$$

Although this derivation is formal, we find that the 4-fold period integral $\Pi_4$ is equivalent to the relative period $\Pi_r$ up to the terms which do not depend on the open string moduli.

Now we discuss the relationship between the 4-fold period integrals and the 3-chain integrals on compact 3-fold. We start with the formula in the first line of (4.11). Let $s$ be a local coordinate normal to the locus $Q(x_i, \phi) = 0$ and $\Delta_r = ds \wedge \Delta'$. One finds

$$\partial_\phi \Pi_4 = -\frac{1}{\pi} \int_{T_4(\Gamma_4)} \frac{\partial_\phi \Phi_Q(x_i, \phi) ds \wedge \Delta'}{\prod_a W_a(x_i, \psi)} Q(x_i, \phi).$$

$$= -2i \int_{T_4(\Gamma_4)} \frac{\partial_\phi \Phi_Q(x_i, \phi) \Delta'}{\prod_a W_a(x_i, \psi)} \partial_\phi \Phi_Q(x_i, \phi) \frac{ds}{ds}.$$  \hspace{1cm} (4.12)

The second equality follows by performing the residue integral with respect to $s$. Here we define the 2-cycle $C = \Gamma_3 \cap \{Q(x_i, \phi) = 0\}$ and we assume that this can be identified with the boundary of 3-chain in 3-fold.

For the complete intersection Calabi–Yau 3-fold, the three chain integral $\Pi_3$ takes the form

$$\Pi_3 = \int_{T_3(\Gamma_3)} \frac{\Delta_n}{\prod_a W_a(x_i, \psi)},$$  \hspace{1cm} (4.13)

where $T_3(\Gamma_3)$ is the tubular domain around the 3-chain $\Gamma_3$ whose boundary $\partial T_3(\Gamma_3) = C_\phi$ is specified by $Q(x_i, \phi) = 0$. Since the boundary deforms as

$$\partial_x Q \frac{dx}{ds} ds + \partial_\phi Q \phi \frac{d\phi}{ds} = 0,$$  \hspace{1cm} (4.14)

we have

$$\partial_\phi \Pi_3 = \partial_\phi \int_{T_3(\Gamma_3)} \frac{ds \wedge \Delta'}{\prod_a W_a(x_i, \psi)} = \int_{T_3(\Gamma_3)} \frac{\Delta'}{\prod_a W_a(x_i, \psi)} \frac{ds}{d\phi} \sim \partial_\phi \Pi_4$$  \hspace{1cm} (4.15)

with identification $C = C_\phi$. This tells us that the chain integrals are directly related to the 4-fold period integrals.

Thus, we formally realized the equivalence of the relative integrals with logarithmic differential for the compact 3-fold, the non-compact 4-fold period and the chain integral$^{19}$. In the following, we will evaluate (4.5) directly via analytic continuation.

$^{19}$ The direct residue integrals of $x_{n+2}$ and $x_{n+3}$ are easily performed, because the locus $(0 : \cdots : 0 : x_{n+2} : x_{n+3})$ are removed. The residue integral of $\oint d_{n+3}/Q_4$ gives rise to a simple integral for the $x_{n+2}$ coordinate: $\oint d_{n+2}/Q_{n+2}$. This residue integral picks up a point $x_{n+2} = 0$; then we obtain the restricted 3-fold period integral

$$\Pi_4 \sim \int_{T_3(\Gamma_3)} \frac{\Delta}{\prod_a W_a(x_i, \psi)} \bigg|_{Q(x_i, \phi)=0}.$$  \hspace{1cm} (4.16)

Such a restriction can be rewritten as the insertion of the logarithmic form by the Poincaré residue map. Therefore, we can check the validity of the above discussion.
4.3. Direct computation of relative period for mirror quintic

Now we compute the relative period integral for the mirror quintic 3-fold $Y_5$ via analytic continuation. The relative 3-form is integrated over the tubular domain of the complete intersection of

$$W(x_i, \psi) = \sum_{i=1}^{5} x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0$$

(4.17)

and the divisor$^{20}$

$$Q(x_i, \phi) = x_5^5 - \phi x_1 x_2 x_3 x_4 x_5 = 0.$$  

(4.18)

The complete intersection is also covered completely by two patches $x_1 = 1$ and $x_3 = 1$. Since the contributions to the residue integrals from these two patches are the same, we multiply them with the factor 2 for the computation in the $x_1 = 1$ patch.

In the $x_1 = 1$ patch, we can use the parametrization $(X, Y, Z, T)$ in the previous section (3.3). In terms of this parametrization, the relative period $\Pi_r$ is given by

$$\Pi_r = 2 \cdot \int \frac{\omega_0 \log[Y^{4/5}(1 - \phi T X Z)]}{W}.$$  

(4.19)

Here we are only interested in the $\phi$-dependent term, so we neglect the term $\log Y^{4/5}$ in the numerator. Changing the parameters to polar ones, $\zeta$ and $w$, we can rewrite this integral as

$$\Pi_r = 2 \cdot \frac{10}{(2\pi i)^2} \int \frac{d\zeta}{\zeta} \frac{dT}{T} \frac{dW}{w} \frac{dY}{Y} \log \left(1 - \frac{\phi}{\pi w^2}\right)$$

$$\times \frac{1}{1 - T^5 + (5\psi)^{-5/2}(\zeta - \zeta^{-1})w^5 T^{-5/2}Y^2 + Y(1 - w^2)}.$$  

(4.20)

After the residue integral of $Y$ and the analytic continuation to the Barnes form, we obtain

$$\Pi_r = 2 \cdot 40 \int \frac{dx}{2\pi i} \frac{\cos(\pi s)}{\sin(\pi s)} \frac{\Gamma(2s + 1)}{\Gamma(s + 1)^2} \frac{2^{s/2}}{\sqrt{\pi}} \int \frac{dw}{2\pi i} \frac{w^{5s+1} \log \left(1 - \frac{\phi}{\pi w^2}\right)}{(1 - w^2)^{2s+1}}$$

$$\times \int \frac{d\zeta}{2\pi i \zeta} (\zeta - \zeta^{-1})^s \int \frac{dT}{2\pi i T} (T^{5/2} - T^{-5/2})^s.$$  

(4.21)

The integrals of $\zeta$ and $T$ can be performed as before, so let us concentrate on the integration of $w$.

To discuss the case of $|\frac{\phi}{\pi w}| < 1$, we change the integration variable to $y = 1/w$:

$$\frac{1}{2\pi i} \int_{C_w} dw \frac{w^{5s+1} \log \left(1 - \frac{\phi}{\pi w^2}\right)}{(1 - w^2)^{2s+1}} = \frac{1}{2\pi i} \int_{C_y} dy \frac{\log \left(1 - \frac{5\phi}{\pi y^2}\right)}{(y^2 - 1)^{2s+1} y^{s+1}}.$$  

(4.22)

In the integrand, there exist logarithmic branch cuts which arise from the points $y = \pm \sqrt{\frac{\phi}{\pi}}$.

We choose the contour $C_y$ surrounding $y = \pm \sqrt{\frac{\phi}{\pi}}$ as described in figure 5.

Taking into account the logarithmic branch, we can rewrite the integral $y$ as an integral over $[-\sqrt{\frac{\phi}{\pi}}, \sqrt{\frac{\phi}{\pi}}]$.

$^{20}$ Of course other choices of divisor are possible. We choose one of the defining equations of the curve and change one of the coefficients into the open moduli. In [17], the authors show that in their method all results coincide with each other at the critical locus of open moduli (on-shell), for the quintic case.
Changing the integration variable to 
where Re
The integrals with respect to 
are on-shell chain integral. As a result, the relative period becomes
From this expression of the incomplete beta function, the
Here we adopt a formula for the incomplete beta function to evaluate the above integral:

\[ B_z (p, q) = \int_0^1 t^{p-1} (1 - t)^{q-1} dt \]

where Re \( z < 1 \) and the hypergeometric function is expanded as

\[ F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{\ell=0}^{\infty} \frac{\Gamma(\alpha + \ell)\Gamma(\beta + \ell) z^\ell}{(p + \ell)!} z^\ell \]

From this expression of the incomplete beta function, the \( w \)-integration can be expressed as

\[ B_{\phi/5\psi} (p, q) \] with \( p = -s/2 \) and \( q = -2s \). Finally we obtain

\[ \frac{1}{2\pi i} \int_{C_w} \frac{w^{s+1} \log \left(1 - \frac{\phi}{5\psi} w^2\right)}{(1 - w^2)^{2s+1}} \phi_{5\psi} \frac{(\pi s)\sin(\pi s)}{\sin(\pi s)} \frac{1}{(\frac{s}{2} + 1)^{\frac{s}{2} + 1}} e^{\pi s/2} \cos \left(\frac{\pi s}{2}\right) \]

The integrals with respect to \( \xi \) and \( T \) are given by (3.20) and (3.21) as was the case of the on-shell chain integral. As a result, the relative period becomes

\[ \Pi_r = 2 \cdot \frac{1}{2\pi i} \int_{C_{\xi}} (5\psi)^{-s/2} \pi \cos(\pi s) \frac{1}{(\frac{s}{2} + 1)^{\frac{s}{2} + 1}} e^{\pi s/2} \cos \left(\frac{\pi s}{2}\right) \times (-1)^{2s+1} \sum_{\ell=0}^{\infty} \frac{\Gamma(\frac{s}{2} + 1 + \ell)}{(\ell - \frac{s}{2})!} \left(\frac{\phi}{5\psi}\right)^{-s/2 - \ell} \]
Our normalization of the holomorphic 3-form

\[ J \in \text{J. Phys. A: Math. Theor.} \]

Comparing this to the defining equation of the curve in an on-shell situation:

This result coincides with the relative period which is obtained via the extended Picard–Fuchs equation [11].

Moreover, at the critical locus of the open moduli, \( \phi \to 5\psi \), we recover an on-shell situation:

\[ \Pi_r \bigg|_{(\phi) = \pm \sqrt{5\psi}} = 2 \pi^2 \sum_{k=0}^{5k+1} \frac{(-1)^{n-k} \Gamma(k + \frac{1}{2})^4}{(2n + 1)(n - k)!(5k - n + 1)!} \phi^{n+1/2}(5\psi)^{5k-n+2}. \quad (4.29) \]

The second equality follows by the relation

\[ \sum_{n=0}^{5k+1} \frac{(-1)^{n-k}}{(2n + 1)(n - k)!(5k - n + 1)!} = \frac{1}{2} \sum_{l=0}^{4k+1} \frac{(-1)^{l}}{(k + l + \frac{1}{2})!(4k - l + 1)!} \]

and this can be obtained by the formula

\[ \frac{\Gamma(x)}{\Gamma(x + a)} = \sum_{l=0}^{\infty} \frac{(-1)^{l}}{(x + l)\Gamma(l + 1)\Gamma(a - l)} \]

with \( x = k + 1/2 \) and \( a = 4k + 2 \).

Here in (4.30), the expression \( (\phi)^{1/2} = \pm \sqrt{5\psi} \) means that \( (\phi)^{1/2} = +\sqrt{5\psi} \) can be interpreted as \( \Pi_+ \) (the superpotential of \( C_+ \)) and \( (\phi)^{1/2} = -\sqrt{5\psi} \) as \( \Pi_- \) (that of \( C_- \)) [11].

Recall that the divisor equation \( Q \) is given in (4.1) and with \( x_1 + x_2 = x_3 + x_4 = 0 \),

\[ Q = x^5 - \phi x_1 x_2 x_3 x_4 x_5 x_1 x_2 x_3 x_1 x_3 = x_5 \left( x^2 + (\phi)^{1/2} x_1 x_3 \right) \left( x^2 - (\phi)^{1/2} x_1 x_3 \right). \quad (4.33) \]

Comparing this to the defining equation of the curve in \( C_\pm (3.1) \) leads to the above statement.

Here we consider that the locus \( \{ x_5 = 0 \} \) is irrelevant. The domainwall tension can be expressed as \( \Pi_r \big|_{(\phi)^{1/2} = \pm \sqrt{5\psi}} = \Pi_r \big|_{(\phi)^{1/2} = -\sqrt{5\psi}} \) and it is nothing but (4.30).

\[ 25 \]

\[ 21 \] Our normalization of the holomorphic 3-form \( \Omega \) differs from that of [11] by a factor \( 5\psi \).
4.4. Mirror of the double cubic $X_{3,3}[1^4]$

We further apply our analytic continuation method for the mirror of the double cubic Calabi–Yau complete intersection $X_{3,3}[1^4]$. The defining equations for the mirror Calabi–Yau 3-fold $Y_{3,3}$ are given by $W_1$ and $W_2$ in (3.28). Now we consider the B-brane which is again given by the intersection with the hyperplanes $x_1 + x_2 = 0$, $x_4 + x_5 = 0$. Then, to introduce the open string modulus $\phi$, we will consider a divisor

$$Q(\phi) := x_3^2 - \phi x_4 x_5 x_6 = 0.$$  \hfill (4.34)

Now we compute the relative period $\Pi_r(\psi, \phi)$,

$$\Pi_r(\psi, \phi) = \frac{3^4 \psi^2}{(2\pi i)^6} \int w \frac{dw}{\ell} \frac{d\zeta}{v} \log \left(1 - \frac{\phi}{3\psi} w^2\right)$$

$$= \frac{2 \cdot 6}{(2\pi i)^6} \sum_{n=0}^\infty (1 - w^2)^n (v - 1) - w^3 v^{3/2} (\zeta - \zeta^{-1})(t^{3/2} - t^{-3/2})(3\psi)^{-3}$$

$$\times \int \frac{d\zeta}{\ell} \log \left(1 - \frac{\phi}{3\psi} w^2\right),$$  \hfill (4.36)

where we denote $z := (3\psi)^{-6}$.

Here we consider the analytic continuation of the above period integral. We replace the discrete sum with respect to $n$ by the contour integral with respect to $s$. The Barnes integral form of the relative period is

$$\Pi_r = \frac{2 \cdot 6}{2\pi i} \int \frac{ds}{\sin (\pi s)} \int \frac{du}{(1 - u^2)^{s+1}} \int \frac{dv}{2\pi i (v - 1)^{s+1}}$$

$$\times \int \frac{d\zeta}{(2\pi i)^2} \log \left(1 - \frac{\phi}{3\psi} w^2\right)$$

$$\times \int \frac{d\zeta}{(2\pi i)^2} \log \left(1 - \frac{\phi}{3\psi} w^2\right).$$  \hfill (4.37)

The integrals except $w$ can be performed in the same way as the on-shell formalism (3.39), and the integral for $w$ is performed as in the above quintic case. We choose the same contour as in figure 5, and the result is

$$\int \frac{du}{2\pi i} \frac{w^s \log \left(1 - \frac{\phi}{3\psi} w^2\right)}{(1 - u^2)^{s+1}} = \frac{(-1)^{s+1}}{\Gamma(s+1)} \sum_{\ell=0}^\infty \frac{\Gamma(s+1+\ell)}{\ell!} \left(\frac{\phi}{3\psi}\right)^{-\ell}.$$ \hfill (4.38)

Collecting all results and performing the residue integral of $s$ at the poles $s = -2k + 1$, the relative period in the vicinity of the orbifold point becomes

$$\Pi_r = \frac{2}{\pi^2} \sum_{k=0}^{\infty} \sum_{n=0}^{3k} \frac{(-1)^{n-k} \Gamma(k + 1/2)}{(2n+1)(n-k)! (3k-n)! \Gamma(3k+1/2)} \frac{\phi^{n+k}}{(3\psi)^{3k-n+1/2}},$$ \hfill (4.39)
where $l = n - k$.

We also check our result in the limit $\phi \to 3\psi$ (the value at the critical locus of the open modulus $\phi$). In this limit, we find

$$
\Pi_\varepsilon|_{(\phi)^{1/2}=\pm\sqrt{3}\psi} = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{1}{2})^6}{\Gamma(3k + \frac{3}{2})} (3\psi)^{6k+3}.
$$

(4.40)

In the derivation, as in the case of the quintic, we used identity (4.32) with $k = k + 1/2$ and $a = 2k + 1$.

As discussed in the quintic case, the choice $(\phi)^{1/2} = \pm\sqrt{3}\psi$ can be interpreted as $\Pi_\pm$.

The domainwall tension can be obtained as $\Pi_\varepsilon|_{(\phi)^{1/2}=\pm\sqrt{3}\psi} - \Pi_\varepsilon|_{(\phi)^{1/2}=\pm\sqrt{3}\psi}$.

4.5. Mirror of $X_{3,4}[1^5, 2^3]$: non-Fermat-type example

Now we discuss the off-shell situation of the non-Fermat complete intersections, the mirror of $X_{3,4}[1^5, 2]$. The defining equations of $X_{3,4}[1^5, 2]$ and the curve are given in (3.57) and (3.58).

Let $\phi$ be the open modulus, and we introduce the following divisor:

$$
Q = x_1^2 - \phi x_2^2 x_3 x_4 x_5.
$$

(4.41)

We consider the useful new local coordinates (3.60) again. With this setup, we evaluate the relative period by repeating the same process as performed in the other cases.

After some steps, we have the following factorized formula:

$$
\Pi_\varepsilon = 2 \cdot 7^2 \left[ \frac{\pi}{2} \int \frac{ds}{\sin(\pi s)} \frac{\cos(\pi s) \cos(7\pi s/2) \sin(3\pi s/2)}{\sin(\pi s) \sin(\pi s/2)} \right]^{1/2} \int \frac{dT}{2\pi i} T^{3s/2-1} (1 + T^2)^{1/2} \int \frac{dw}{2\pi i} \log \left( \frac{w^2 - \phi}{\psi} \right) w^{-2s-1} \left( 1 - w^2 \right)^{-s-1} \int \frac{dS}{2\pi i} S^{3s/2-1} (1 - S)^{-s-1}.
$$

(4.42)

All integrals except $w$ are evaluated in the same way as the on-shell situation. $w$-integration can be evaluated as in the cases of the quintic and the double cubic:

$$
\int \frac{dw}{2\pi i} \log \left( \frac{w^2 - \phi}{\psi} \right) w^{-2s+1} \left( 1 - w^2 \right)^{-s-1} = \frac{1}{\Gamma(1 + s)} \sum_{\ell=0}^{\infty} \frac{\Gamma(s + \ell + 1)}{\Gamma(\ell + 1)} \left( \frac{\phi}{\psi} \right)^{-\ell}.
$$

(4.43)

Thus, the relative period becomes

$$
\Pi_\varepsilon = 2 \cdot 7^2 \left[ \frac{\pi}{2} \int \frac{ds}{\sin(\pi s)} \frac{\cos(\pi s) \cos(7\pi s/2) \sin(3\pi s/2)}{\sin(\pi s) \sin(\pi s/2)} \right]^{1/2} \int \frac{dT}{2\pi i} T^{3s/2-1} (1 + T^2)^{1/2} \int \frac{dw}{2\pi i} \log \left( \frac{w^2 - \phi}{\psi} \right) w^{-2s+1} \left( 1 - w^2 \right)^{-s-1} \int \frac{dS}{2\pi i} S^{3s/2-1} (1 - S)^{-s-1}
$$

(4.44)

We evaluate the residue integral with respect to $s$ by picking up simple poles at $s = -2k - 1$ with positive integers $k$. By using identity (4.32) with $x = a = 2k + 1$, we find

$$
\Pi_\varepsilon = \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)^5}{\Gamma(3k + 3/2)} \sum_{l=0}^{2k+1} \frac{(-1)^l}{(2k + \ell + 1)\Gamma(2k - \ell + 1)\Gamma(\ell + 1)} \phi^{2k+l+1} \psi^{5k-l+5/2}
$$

$$
= \frac{1}{\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)^5}{\Gamma(3k + 3/2)\Gamma(4k + 2)} \phi^{2k+l+1} \psi^{5k-l+5/2}.
$$

(4.45)

This is the off-shell superpotential in the vicinity of the orbifold point.

Analogously to the other cases, $(\phi)^{1/2} = \pm\sqrt{3}\psi$ is regarded as $\Pi_\pm$ and the domainwall tension is $\Pi_\varepsilon|_{(\phi)^{1/2}=\pm\sqrt{3}\psi}$.
4.6. Extended Picard–Fuchs equation: consistency check

As a consistency check of the above results, we discuss the extended Picard–Fuchs equation. It is a homogeneous differential equation and its solutions are relative periods, which depend on both open and closed moduli. For the toric Calabi–Yau 3-fold, one can find the Picard–Fuchs equation from the toric data directly. Adding D-brane, we should consider the extended Picard–Fuchs system with the additional toric charge. The application of this method to the compact Calabi–Yau is discussed in [12]. See also [15, 16] for further discussions. Here, we confirm that the relative period obtained by the above arguments is a solution of the extended Picard–Fuchs equation for the quintic case. The other cases are similarly confirmed.

For a set of toric charge vectors \( \ell(a) = (\ell_a^0) \), we find the Picard–Fuchs operators \( L_a \) as follows:

\[
L_a = \prod_{k=1}^{\ell_a^0} (\theta_{a_0} - k) \prod_{\ell_{a_i} > 0}^{\ell_a^i-1} (\theta_{a_i} - k) - (-1)^{\ell_a^0} \prod_{\ell_{a_i} < 0}^{\ell_a^i-1} (\theta_{a_i} - k),
\]

where \( \ell_{a_i} \) are the coefficients of the defining equation \( W = \sum_i c_i y_i \) for the mirror Calabi–Yau 3-fold, and \( \theta_{a_0} = z_a \partial / \partial z_a \).

For the quintic Calabi–Yau 3-fold case, as previously mentioned we choose the toric vectors as \( \ell(1) = (-5, 1, 1, 1, 1, 1) \) and for divisor \( Q(x, \phi) = 0 \) expressing B-brane as \( \ell(2) = (-1, 0, 0, 0, 0, 1) \) [1]. One of the linear combinations of the toric charge vectors leads to a Picard–Fuchs operator that gives a nontrivial constraint on the relative period [12],

\[
\mathcal{L}_1' = \theta_2 \theta_1^4 + z_1 z_2 \sum_{i=1}^{5} (4\theta_1 + \theta_2 + i),
\]

where \( z_1 = (5\psi)^{-4} \phi^{-1} \) and \( z_2 = - (5\psi)^{-1} \phi \).

One can verify that the relative periods (4.29) satisfy the extended Picard–Fuchs equations

\[
\Pi_1(\psi, \phi) = 0.
\]

Thus, we can confirm that the relative periods obtained via direct integration are surely solutions for the extended Picard–Fuchs equations.

5. Normalization ambiguity

In sections 3 and 4, we have directly evaluated the D-brane superpotentials by the use of analytic continuations. This method is a powerful approach for obtaining the analytical expressions of the superpotentials for both on-shell and off-shell. The disadvantage of this approach is that we cannot determine the normalization of the superpotentials since we may have ambiguities in the analytic continuations. Therefore, in this section we try to fix the normalization by considering genus 1 amplitudes, focusing on the on-shell (i.e. involution brane) situations.

The holomorphic anomaly equation connects the tree level amplitudes with the amplitudes of the higher worldsheet topologies [56, 57]. The extension of this to the open string sector, in particular for the compact Calabi–Yau, is proposed in [28, 29]. The related works are discussed in [58, 59]. We will use the formula for one-loop amplitudes given in these works. Now we compute real BPS invariants at the Euler characteristics \(-2, 0\) (genus 0, 1) for one-parameter Calabi–Yau complete intersections in weighted projective spaces under the assumption that the formula for the Euler characteristic 0 obtained in [29] holds.
We consider $X_{d_1d_2\ldots d_k}(w_1, w_2, \ldots, w_l)$, a general complete intersection of $k$ hypersurfaces of degrees $d_1, \ldots, d_k$ in the weighted projective space $\mathbb{WP}^{l-1}(w_1, \ldots, w_l)$. Then the various ingredients are listed as follows. First, the condition of 3-fold obtained by the adjunction formula is given by $l-k=4$. The classical Yukawa coupling (triple intersection number) is

$$K_0 = \frac{\prod_{i=1}^l d_i}{\prod_{i=1}^l w_i}.$$  \hspace{2cm} (5.1)

The first and second Chern classes, $c_1, c_2$, and the Euler characteristic $\chi$ can be obtained as

$$c_1 = \sum_{i=1}^k d_i - \sum_{i=1}^l w_i = 0,$$

$$c_2 = \frac{1}{2} \frac{\prod_{i=1}^l d_i}{\prod_{i=1}^l w_i} \left( \sum_{i=1}^k d_i^2 - \sum_{i=1}^l w_i^2 \right),$$

$$\chi = \frac{1}{3} \frac{\prod_{i=1}^k d_i}{\prod_{i=1}^l w_i} \left( -\sum_{i=1}^k d_i + \sum_{i=1}^l w_i \right).$$

The discriminant of one-modulus models is

$$(\text{diss}) = \left( 1 - \frac{\prod_{i=1}^k d_i}{\prod_{i=1}^l w_i} \right).$$

The fundamental period $\varpi_0$ and logarithmic period $\varpi_1$ are

$$\varpi_0 = \sum_{n=0}^\infty \frac{\prod_{i=1}^l \Gamma(d_i n+1)}{\prod_{i=1}^l \Gamma(w_i n+1)} z^n,$$

$$\varpi_1 = \sum_{n=0}^\infty \frac{\prod_{i=1}^k \Gamma(d_i n+1)}{\prod_{i=1}^l \Gamma(w_i n+1)} \left[ \log z + \left( \sum_{i=1}^k d_i \Psi(d_i n+1) - \sum_{i=1}^l w_i \Psi(w_i n+1) \right) \right] z^n.$$  \hspace{2cm} (5.7)

where $\Psi$ denotes the digamma function.

It has turned out that the chain integral (i.e. the tension of the BPS domainwall) is the following general form (for the $\mathbb{Z}_2$ vacua case):  \hspace{2cm} (5.8)

$$\Pi(z) = \varpi_0(z) T_A(z) = \frac{c}{2\pi^2} \sum_{n=0}^\infty \prod_{i=1}^l \Gamma\left( -w_i n - \frac{d_i}{2} \right) z^{n+\frac{1}{2}}$$

where $T_A$ is the A-model domainwall tension (i.e. the disk generating function) under the transformation by the mirror map. The integer constant $c$ is the normalization factor and assumed to be given by $c = N/|S|$. Here, $N$ is the number of branes which transforms under a discrete group of the models and $S$ is the stabilizer of the curves; namely, $|S|$ counts the order of the discrete subgroup which makes the brane invariant. This constant is in some way related to the constant factor of the inhomogeneous term of the Picard–Fuchs equation in [10]. Since we do not have any A-model calculus except for $X_4(1^3)$ and $X_{3,3}(1^6)$, we do not know the

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22 All few modulus Calabi–Yau complete intersections in the weighted projective space are listed in [50]. Here we are concerned with the following specific models: $X_4(1^3)$, $X_6(1^6, 2)$, $X_6(1^4, 4)$, $X_{10}(1^5, 2, 5)$, $X_{3,3}(1^6)$, $X_{4,4}(1^6, 3^2)$, $X_{4,4}(1^7, 2)$. All these models are (closed) one-modulus models. See also [36–40, 51]. We can apply our direct integration method to these models and obtain the same enumerative numbers as [8–10] although there are subtleties we have mentioned already in the last paragraph of section 3.

23 The generalization to the $\mathbb{Z}_k$-sector ($k \neq 2$) is easy and the formula in such a case is in [10].
true normalization. In the following, we will find that most models have $c = 1$. We will also note the consistency and discrepancy with Walcher’s results for some complete intersection Calabi–Yau (CICY) models [10].

There are other constraints for the weight since they must be one-modulus models. At the level of the closed string, the periods satisfy the fourth-order homogeneous equation. Therefore, the fundamental period should have the form

$$\omega_0 = \frac{\prod_{i=1}^{k} \prod_{j=1}^{d_i} \frac{\Gamma(n+\frac{x_i}{2})}{\Gamma(\frac{x_i}{2})} \left( \frac{\prod_{i=1}^{k} d_i}{\prod_{j=1}^{d_i} w_j^{w_j}} \right)^n}{\prod_{i=1}^{d} \prod_{j=1}^{w_i} \frac{\Gamma(n+\frac{x_j}{2})}{\Gamma(\frac{x_j}{2})} \left( \frac{\prod_{i=1}^{d} d_i}{\prod_{j=1}^{d_i} w_j^{w_j}} \right)^n},$$

where $\lambda_1$ and $\lambda_2$ are determined by

$$\{\lambda_1, 1 - \lambda_1, \lambda_2, 1 - \lambda_2\} = \left\{ \frac{j}{w_i} \middle| j = 1, \ldots, w_i - 1, \ i = 1, \ldots, l \right\} - \left\{ \frac{j}{d_i} \middle| j = 1, \ldots, d_i - 1, \ i = 1, \ldots, k \right\}.$$

We can rewrite the logarithmic period in terms of $\lambda_1$ and $\lambda_2$ as follows:

$$\omega_1 = \omega_0 \log z + \tilde{\omega}_1,$$

where $\tilde{\omega}_1$ is expressed as

$$\tilde{\omega}_1 = \frac{\prod_{i=1}^{k} \prod_{j=1}^{d_i} \frac{\Gamma(n+\lambda_i) \Gamma(n+1-\lambda_i) \Gamma(n+\lambda_2) \Gamma(n+1-\lambda_2)}{\Gamma(\lambda_1) \Gamma(1-\lambda_1) \Gamma(\lambda_2) \Gamma(1-\lambda_2) (n+1)^4}}{\prod_{i=1}^{d} \prod_{j=1}^{w_i} \frac{\Gamma(n+\lambda_i) \Gamma(n+1-\lambda_i) \Gamma(n+\lambda_2) \Gamma(n+1-\lambda_2)}{\Gamma(\lambda_1) \Gamma(1-\lambda_1) \Gamma(\lambda_2) \Gamma(1-\lambda_2) (n+1)^4}} \times \left[ \sum_{i=1,2} \Psi(n+\lambda_i) - \Psi(\lambda_i) + \Psi(n+1-\lambda_i) - \Psi(1-\lambda_i) - 4(\Psi(n+1) - \Psi(1)) \right] \times \left( \frac{\prod_{i=1}^{k} \prod_{j=1}^{d_i} d_i}{\prod_{i=1}^{d} \prod_{j=1}^{w_i} w_i} \right)^n.$$

The mirror map is given by $t(z) = \frac{1}{2\pi i} \frac{\sigma_0}{\sigma_1}$ Under this map the genus 0 (disk) amplitude in the A-model, $\mathcal{T}_A(z)$, which is given by

$$\Pi(z) = \omega_0 \mathcal{T}_A(z) = \frac{c}{2\pi^2} \sum_{n=0}^{\infty} \prod_{i=1}^{k} \frac{\Gamma(-nw_i - \frac{w}{2})}{\Gamma(-nw_i - \frac{w}{2})} \frac{2n_{0,real}^{(0,real)}}{k^2} q^{\frac{ad}{2}}$$

has the enumerate structure in the A-model interpretation and we define the genus 0 real BPS invariants, $n_{0,real}^{(0,real)}$ (for $d$ odd), by

$$\mathcal{T}_A(z) = \sum_{k,d,odd} \frac{2n_{0,real}^{(0,real)}}{k^2} q^{\frac{ad}{2}},$$

where $q = e^{2\pi i z}$ [2, 4, 6]. Here $d$ is the degree and $k$ is the integer for re-summation. Note that $\tau(z)$ is a solution to the inhomogeneous Picard–Fuchs equation $L_{PF} \tau(z) = \frac{a}{2} e^{z/2}$ with

$$a = 2n_1^{(0,real)} = \left( \frac{\prod_{i=1}^{k} \frac{\Gamma(-\frac{w_i}{2}) \cdots \Gamma(-\frac{w}{2})}{2\pi^2 \Gamma(-\frac{d}{2}) \cdots \Gamma(-\frac{d}{2})} \right).$$
According to [29], the holomorphic limits $A^{\text{hol}}$ and $K^{\text{hol}}$ of the annulus amplitude $A$ and the Klein bottle amplitude $K$ are given by [29, equation (5.27)] [47]

$$
\frac{\partial}{\partial z} A^{\text{hol}} = -\frac{1}{2} (\Delta^{\text{hol}})^2 C_{zz}, \quad K^{\text{hol}} = \frac{1}{2} \log \left[ \frac{q \, dz}{z \, dq} \right].
$$

(5.16)

Here $\Delta^{\text{hol}}$ is given by

$$
\Delta^{\text{hol}} = (\partial_z - \Gamma_z) (\partial_z + \partial_z K) \tau(z),
$$

(5.17)

where $\Gamma_z = \partial_z \log \frac{dz}{dq}$ and $\partial_z K = -\partial_z \log \sigma_0$, and the Yukawa coupling $C_{zz}$ takes the form

$$
C_{zz} = \frac{K_0}{z^{3} \cdot (\text{diss})}.
$$

(5.18)

Another form of $\Delta^{\text{hol}}$ is given in terms of $q$ as

$$
\Delta_z = \sigma_0 \left( \frac{dz}{dq} \right)^{-2} \left( \partial_z + \frac{1}{q} \right) \partial_z \sigma_T.
$$

(5.19)

Then, the genus 1 real BPS numbers $n^{(1,\text{real})}_k$ are defined by the following expansion [29, equation (5.28)]:

$$
A^{\text{hol}} + K^{\text{hol}} = \sum_{k \in \mathbb{Z}/2 \mathbb{Z}, \pm 1} \frac{2n^{(1,\text{real})}_k}{k} q^{k/2}.
$$

(5.20)

Because of the relation of the mirror map and the expression of the logarithmic periods in terms of $\lambda_1$ and $\lambda_2$, we obtain the following formula:

$$
q = \exp \left( \frac{\sigma_1}{\sigma_0} \right) = z \exp(\widetilde{\sigma}_1)
$$

$$
= z + \lambda_1 (1 - \lambda_1) \lambda_2 (1 - \lambda_2) \left[ \frac{1}{\lambda_1} + \frac{1}{1 - \lambda_1} + \frac{1}{\lambda_2} + \frac{1}{1 - \lambda_2} - 4 \right] \prod_{l=1}^{k} \frac{\prod_{i=1}^{l} d_i^{d_i}}{w_{i}^{w_i}} q^2 + \ldots.
$$

(5.21)

We can invert this as

$$
z = q - [\lambda_1 (1 - \lambda_1) + \lambda_2 (1 - \lambda_2) - 4\lambda_1 (1 - \lambda_1) \lambda_2 (1 - \lambda_2)] \prod_{l=1}^{k} \frac{\prod_{i=1}^{l} d_i^{d_i}}{w_{i}^{w_i}} q^2 + \ldots.
$$

(5.22)

Substituting the expansion into (5.20), we have

$$
K = \left[ -\frac{1}{2} (\lambda_1 (1 - \lambda_1) + \lambda_2 (1 - \lambda_2) - 4\lambda_1 (1 - \lambda_1) \lambda_2 (1 - \lambda_2)) \prod_{l=1}^{k} \frac{\prod_{i=1}^{l} d_i^{d_i}}{w_{i}^{w_i}} \right] q + \ldots.
$$

(5.23)

and we can rewrite it as

$$
K = 2^{-\gamma} \pi^{-4} \prod_{l=1}^{k} \frac{\Gamma(-\frac{w_l}{2})^{2}}{\prod_{l=1}^{k} \Gamma(-\frac{d_l}{2})^{2}} \prod_{l=1}^{k} \frac{\prod_{i=1}^{l} w_{i}^{w_i}}{\prod_{i=1}^{l} d_i^{d_i}} q + \ldots.
$$

(5.24)

On the other hand, we have

$$
A = -2^4 \pi^{-4} \prod_{l=1}^{k} \frac{\Gamma(-\frac{w_l}{2})^{2}}{\prod_{l=1}^{k} \Gamma(-\frac{d_l}{2})^{2}} \prod_{l=1}^{k} \frac{\prod_{i=1}^{l} w_{i}^{w_i}}{\prod_{i=1}^{l} d_i^{d_i}} q + \ldots.
$$

(5.25)
Therefore, we finally obtain

\[ n_2^{(\text{real})} = 2^{-8} \pi^{-4} \prod_{l=1}^{12} \Gamma \left( -\frac{c}{2} \right)^2 \left[ \prod_{l=1}^{8; \text{odd}} \Gamma \left( \frac{c}{2} \right) - \prod_{l=1}^{6; \text{odd}} \Gamma \left( \frac{c}{2} \right) \right]. \]  

(5.26)

In most cases, since we have \( c = 1 \), we can find

\[ n_2^{(\text{real})} = 0, \ 24, \ 72, \ 2^{12}, \ 0, \]  

(5.27)

for \( X_5(1^5) \), \( X_6(1^4, 2) \), \( X_8(1^4, 4) \), \( X_{10}(1^3, 2, 5) \), \( X_{13}(1^6) \), respectively. For (the \( Z_2 \)-sector of) \( X_{4,4}(1^4, 2^2) \), we have \( n_2^{(\text{real})} = 12 \) if we assume \( c = 1 \) and \( n_2^{(\text{real})} = 0 \) when \( c = 2 \). The tree level result obtained by Walcher for \( X_{4,4}(1^4, 2^2) \) corresponds to \( c = 4 \), from which we obtain negative integers for \( n_2^{(\text{real})} \). This result does not mean that \( c = 4 \) for \( X_{4,4}(1^4, 2^2) \) is not a good choice because we have holomorphic ambiguity. Moreover, we have \( n_2^{(\text{real})} = 3/2 \) for \( X_{4,4}(1^3, 2) \) if we take \( c = 1 \). Furthermore, we cannot find any good rational values for \( c \) such that we have positive integers for \( n_2^{(\text{real})} \). This result may imply that we have to add some other terms for genus 1 amplitudes by considering holomorphic ambiguities, since at present the form of holomorphic ambiguities is a kind of assumption with which we can obtain integral invariants [29].

6. Conclusions

In this paper, we discuss new methods for open mirror symmetry of the compact Calabi–Yau 3-fold. First, we have shown that the inhomogeneous Picard–Fuchs equation of the chain integral can be obtained by a rather simple algorithm. This algorithm is also effective for variousICYs. In such cases, the Griffiths–Dwork method is not so easy. Then, we have evaluated the superpotentials or domainwall tensions directly via analytic continuations. We found that our method is a powerful approach for obtaining the analytical expressions of the superpotentials for both the on-shell and off-shell formalism. We treat several models with a few moduli and reproduce the known results. In addition, our method works successfully for one of the non-Fermat complete intersections, which was not yet discussed in the context of open mirror symmetry. The disadvantage of this approach is the problem of fixing the normalization of the 3-chain integral which may result from the ambiguity of analytic continuation. So as to overcome this point, we have considered the genus 1 amplitudes, and fixed the normalization appropriately. Although our direct integration method might contain heuristic arguments, it is true that we can reproduce the known results rather easily and the computation is economical and intuitive. Furthermore, we can obtain the new results for the non-Fermat-type example and predict the real BPS invariants. We would like to stress that we can directly obtain the domainwall tension itself without treating any other relative periods. So far there has been no result of compact Calabi–Yau manifolds except for simple examples with a few (bulk) moduli. We expect that our methods work for more general classes of the Calabi–Yau 3-fold such as the Pfaffian Calabi–Yau varieties [60], which will be reported elsewhere [61].

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24 In fact, we observe that the modification of the form of holomorphic ambiguity is required for some Pfaffian Calabi–Yau cases [61].

25 While this paper was in preparation for submission, a related work appeared [19] and the analysis of open mirror symmetry on compact Calabi–Yau hypersurfaces with 2- and 3-moduli was carried out by very systematic toric and GKZ-system approach.
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Appendix. Rescaling algorithm and the inhomogeneous terms

In this appendix, we mainly discuss the rescaling algorithm for some examples.

A.1. Cubic curve

As the first example, we consider the family of the Calabi–Yau 1-fold (elliptic curve) defined as a hypersurface by the following homogeneous polynomial of degree 3 in $\mathbb{C}P^2$,

$$W = \frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - \psi x_1 x_2 x_3 = 0. \quad (A.1)$$

In this case, we define the integral of 2-form over 2-chain as

$$\int_{\Gamma} \Omega = \psi \int_{\Gamma} \frac{\omega_0}{W}, \quad (A.2)$$

where

$$\omega_0 = \sum_{i=1}^{3} (-1)^i x_i \, dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_3 = -x_3 \, dx_1 \wedge dx_2 + x_2 \, dx_1 \wedge dx_3 - x_1 \, dx_2 \wedge dx_3 \quad (A.3)$$

and $\Gamma$ has $\psi$-dependence. For simplicity, we fix here one of the homogeneous coordinates of $\mathbb{C}P^2$, $x_1$, to 1, so that $\omega_0 = -dx_2 \wedge dx_3$.

We find from (A.1) (or precisely a redundant modulus version $W'$) the obvious differential relation

$$\prod_{i} \left( \frac{\partial}{\partial a_i} \right) \frac{\omega_0}{W'} = \left( \frac{\partial}{\partial a_0} \right)^3 \frac{\omega_0}{W'}, \quad (A.4)$$

and reproduce this equation with $\psi$ derivatives in the similar way to the quintic case. To obtain the Picard–Fuchs equation which is of second order, we start from the following $\theta_\psi := \psi \, \partial_\psi$ factorized form:

$$\int \Omega = \psi \int \frac{-dx_2 \wedge dx_3}{W}$$

$$= \theta_\psi \int \log W \, (d \log x_2 \wedge d \log x_3)$$

$$= \theta_\psi \int \log W \, (d \log x_2 \wedge d \log \tilde{x}_3) \left( x_3 = \frac{\tilde{x}_3}{\psi}, W = \frac{1}{3} \left( x_1^3 + x_2^3 + \frac{\tilde{x}_3^3}{\psi^3} - 3x_1 x_2 \tilde{x}_3 \right) \right)$$

$$= \int \frac{x_3^3}{W} \left( -d \log x_2 \wedge d \log x_3 \right) + \int \frac{\partial (\log W)}{\partial x_3} \, d \log x_2 \wedge dx_3. \quad (A.5)$$

33
Here we used the following formula for the $\psi$ derivatives of the integration which has a boundary:

$$
\theta_\psi \int_a^b \frac{f(x)}{x} \, dx = \theta_\psi \int_{a/\psi^n}^{b/\psi^n} \frac{f(\psi^m x)}{x} \, d\tilde{x} = \int_{a/\psi^n}^{b/\psi^n} \theta_\psi \frac{f(\psi^m \tilde{x})}{\tilde{x}} \, d\tilde{x} - m \int_a^b \frac{d}{dx} f(x) \, dx,
$$

(A.6)

where $x = \psi^m \tilde{x}$.

It is also easy to find the differential relation using this form of $\Omega$. Next, we consider reproducing the $\partial / \partial a_i$ derivative of (A.5), using coordinate transformation,

$$
-\theta_\psi \int \frac{x_2^3}{W} \, d \log x_2 \wedge d \log x_3 = -\theta_\psi \int \frac{x_2^3}{W} \, d \log x_2 \wedge d \log x_3
$$

$$
\times \left( x_2 = \psi \tilde{x}_2, x_3 = \psi \tilde{x}_3, \tilde{W} = \frac{1}{3} \left( \frac{1}{\psi} + \tilde{x}_2^3 + \tilde{x}_3^3 - 3 \tilde{x}_2 \tilde{x}_3 \right) \right)
$$

$$
= -\int \frac{1}{W^2} x_2^3 \, d \log x_2 \wedge d \log x_3 + \int \frac{\partial}{\partial x_3} \frac{x_2^3}{W^2} \, dx_2 \wedge d \log x_3
$$

$$
+ \int \frac{\partial}{\partial x_3} \frac{x_2^3}{W} \, d \log x_2 \wedge dx_3.
$$

(A.7)

A similar procedure using the transformation $\tilde{x}_2 = x_2/\psi$ reproduces $\partial / \partial a_2$ and yields $x_2^3$,

$$
-\theta_\psi \int \frac{x_2^3}{W^2} \, d \log x_2 \wedge d \log x_3 = -2 \int \frac{x_2^3}{W^3} \, d \log x_2 \wedge d \log x_3 - \int \frac{\partial}{\partial x_2^3} \frac{x_2^3}{W^2} \, dx_2 \wedge d \log x_3.
$$

(A.8)

Therefore, we obtain

$$
\theta_\psi^2 \int \Omega = -2 \int \frac{\omega_0 (x_2, x_3)^2}{W^3} + \int d \left( \frac{x_2^3}{W^2} \right) \wedge (-d \log x_3)
$$

$$
+ \theta_\psi \int d \left( \frac{x_3^3}{W} \right) \wedge (d \log x_3 - d \log x_2) + \theta_\psi^2 \int d \log W \wedge (-d \log x_2).
$$

(A.9)

Note that from the usual derivative without coordinate transformation, we obtain

$$
\frac{1}{\psi} \theta_\psi^2 \frac{1}{\psi} \int \Omega = 2 \int \frac{\omega_0}{W^3} (x_2, x_3)^2.
$$

(A.10)

From this, we obtain the following differential equation:

$$
\frac{1}{\psi} \theta_\psi \frac{1}{\psi} \theta_\psi \frac{1}{\psi} \int \Omega + \theta_\psi^2 \int \Omega = \int d \left( \frac{x_3^3}{W} \right) \wedge (-d \log x_3)
$$

$$
+ \theta_\psi \int d \left( \frac{x_3^3}{W} \right) \wedge (d \log x_3 - d \log x_2) + \theta_\psi^2 \int d \log W \wedge (-d \log x_2).
$$

(A.11)

A.2. Double cubic

Let us recall some fundamental formulas of $X_{3,3}[16]$ as follows:

$$
W_1 = \frac{1}{3} (x_3^4 + x_2^3 + x_3^5) - \psi x_6 x_3 x_5 x_6, \quad W_2 = \frac{1}{3} (x_4^3 + x_5^3 + x_6^3) - \psi x_1 x_2 x_3.
$$

$$
\Omega = \psi^2 \int \frac{\omega_0}{W_1 W_2}, \quad \omega_0 = \sum_{i=1}^{6} (-1)^{i} x_i \, dx_1 \wedge \cdots \wedge \tilde{d}x_i \wedge \cdots \wedge dx_6.
$$

(A.12)
In the following, we mainly use the two procedures.

(I) Extract \( \theta^4 \) in the numerator by transforming as follows: transform \( x_i = \tilde{x}_j/\psi \) for the term that contains \( dx_i \) in \( \omega_0 \), and transform \( x_i = \psi \tilde{x}_j \) \( (j \neq i) \) for other terms.

(II) Extract \( x_1, x_2, x_3 \) in the numerator by transforming as follows: transform \( x_i = \psi^{1/3} \tilde{x}_i \) \((i = 1, 2, 3)\) for the term that contains all of \( dx_1, dx_2, dx_3 \) in the \( \omega_0 \), and transform \( x_i = \psi^{-1/3} \tilde{x}_i \) \((i = 4, 5, 6)\) for the term that contains all of \( dx_4, dx_5, dx_6 \) in the \( \omega_0 \).

To extract \( x_4 x_5 x_6 \), we use the same procedure by exchanging \((i = 1, 2, 3) \leftrightarrow (i = 4, 5, 6)\).

We use \( \vartheta = \psi \partial_\psi \) and the terms \( B_{1,1}, \ldots \), are contributions from boundaries and the explicit formulas are listed in the later page.

First, by the use of procedure I, we consider the fourth-order derivative of \( \Omega \) with respect to \( \psi \). In the following, we apply procedure I to \( x_1, x_4, x_2, x_5 \) in turn:

\[
\begin{align*}
\theta \frac{1}{\psi} \frac{\partial \theta \varphi}{\partial \psi} & = \theta \varphi \int \frac{\omega_0}{W_1 W_2} \theta \varphi + \theta \varphi^2 \int \frac{\omega_0}{W^2} \omega_0 + B_{1,1}, \\
\theta \frac{1}{\psi} \frac{\partial \varphi}{\partial \psi} & = \varphi \int \frac{x_1}{W_1} \omega_0 + \varphi \int \frac{x_2}{W_2} \omega_0 + \theta B_{1,1} \\
& = \varphi \int \frac{x_1}{W_1} \omega_0 + \varphi \int \frac{x_2}{W_2} \omega_0 + \varphi \int \frac{x_3}{W_3} \omega_0 + \theta B_{1,1} + B_{2,1}, \\
\theta \theta \frac{1}{\psi} \frac{\partial \varphi}{\partial \psi} & = 2 \varphi \int \frac{x_1}{W_1} \omega_0 + \varphi \int \frac{x_2}{W_2} \omega_0 + \theta \varphi^2 \int \frac{x_3}{W_3} \omega_0 + \theta B_{1,1} + B_{2,1}.
\end{align*}
\]

\[
\begin{align*}
\theta \theta \theta \frac{1}{\psi} \frac{\partial \varphi}{\partial \psi} & = 4 \varphi \int \frac{x_1}{W_1} \omega_0 + \varphi \int \frac{x_2}{W_2} \omega_0 + \theta \varphi^2 \int \frac{x_3}{W_3} \omega_0 + \theta B_{1,1} + B_{2,1}.
\end{align*}
\]

(A.13)

Secondly, we try to express each term on the rhs of (A.13) as the \( \psi \)-derivative of \( \Omega \). For the first term on the rhs of (A.13), by use of procedure II, the following \( \psi \)-derivative formula can be obtained:

\[
4 \varphi \int \frac{(x_1 x_2 x_4 x_5)^3}{W_1 W_2} \omega_0 = \varphi \int \frac{x_1 x_2 x_4 x_5}{\omega_0} \omega_0 - \frac{1}{3} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} x_1 x_2 x_4 x_5 \omega_0 \omega_{(i23)}
\]

\[
+ \frac{1}{3} \sum_{i=4}^{6} \int \frac{\partial}{\partial x_i} x_1 x_2 x_4 x_5 \omega_0 \omega_{(456)},
\]

(A.14)

where \( \omega_{(i,j,k)} \) expresses the part of \( \omega_0 \) which contains only \( dx_i, \, dx_j, \) and \( dx_k \).
Next in order to extract $x^3_3$ in the numerator, we perform the $\psi$-derivation after the next transformation:

for $\omega(123)$: $x_1 = \psi^{1/3}x_1, \quad x_2 = \psi^{1/3}x_2, \quad x_3 = \psi^{-5/3}x_3$,

for $\omega(3456)$: $x_3 = \psi^{-2}x_3, \quad x_4 = \psi^{-1/3}x_4, \quad x_5 = \psi^{-1/3}x_5, \quad x_6 = \psi^{-1/3}x_6$,

for $\omega(12456)$: $x_1 = \psi^2x_1, \quad x_2 = \psi^2x_2, \quad x_4 = \psi^{5/3}x_4, \quad x_5 = \psi^{5/3}x_5, \quad x_6 = \psi^{5/3}x_6$,

$$= 4 \int \frac{x_1x_2x_3^2x_4^2}{x_0W_1^2W_2^3} \omega_0 + B_{0.1}.$$  

Then we rewrite it as the $\psi$-derivative formula by using II again:

$$= \frac{1}{\psi} \int \frac{(x_1x_2x_3x_4x_5)^2}{x_0W_1^2W_2^3} \omega_0 + \frac{1}{3\psi} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} x_0(x_1x_2x_3x_4x_5)^2}{x_0W_1^2W_2^3} \omega_{(123)}$$

$$- \frac{1}{3\psi} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_0(x_1x_2x_3x_4x_5)^2}{x_0W_1^2W_2^3} \omega_{(456)} + B_{0.1}.$$  

By the use of the transformation as follows, we perform the differentiation and extract $x^3_3$:

for $\omega(456)$: $x_4 = \psi^{1/3}x_4, \quad x_5 = \psi^{1/3}x_5, \quad x_6 = \psi^{-5/3}x_6$,

for $\omega(1236)$: $x_1 = \psi^{-1/3}x_1, \quad x_2 = \psi^{-1/3}x_2, \quad x_3 = \psi^{-1/3}x_3, \quad x_6 = \psi^{-2}x_6$,

for $\omega(12345)$: $x_1 = \psi^{5/3}x_1, \quad x_2 = \psi^{5/3}x_2, \quad x_3 = \psi^{5/3}x_3, \quad x_4 = \psi^2x_4, \quad x_5 = \psi^2x_5$,

$$= 4 \int \frac{x_1x_2x_3x_4x_5}{W_1^3W_2^3} \omega_0 + B_{0.1} + B_{0.2}.$$  

We can rewrite this as the fourth derivative of $\Omega$ with respect to $\psi$ by using transformations II:

$$= \frac{1}{16\psi^3} \left( \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \Omega - \frac{1}{16\psi^3} \left( \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} B_1 + \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} B_2 + \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} \theta \frac{1}{\psi} B_3 + B_4 \right) \right)$$

$$+ B_{0.1} + B_{0.2}.$$  

(A.15)

The fifth term can be written as the first derivative of $\Omega$ by using transformation II:

$$\psi^2 \int \frac{x_4x_5x_6}{W_1^2W_2} \omega_0 = \frac{1}{2} \theta \frac{1}{\psi} \Omega - B_{1.2}.$$  

(A.16)

The fourth term can be expressed as follows by the use of the formula which is obtained by differentiating (A.2) with respect to $\psi$ and transforming as II:

$$\psi^2 \int \frac{x_1x_2x_3x_4x_5}{W_1^3W_2^3} \omega_0 = \frac{\psi}{2} \theta \frac{1}{\psi} \Omega - \frac{\psi}{2} \theta \frac{1}{\psi} \Omega - \frac{\psi}{4} \theta \frac{1}{\psi} \Omega - \frac{\psi}{2} \theta \frac{1}{\psi} (B_{1.1} - B_{1.2}) - \frac{\psi}{2} B_{1.3}.$$  

The third term can be expressed as follows by the use of the formula which is obtained by differentiating (A.13) with respect to $\psi$ and transforming as II:

$$\psi^2 \int \frac{x_1x_2x_3x_4x_5}{W_1^3W_2^3} \omega_0 = \frac{\psi}{4} \theta \frac{1}{\psi} \Omega - \frac{\psi}{8} \theta \frac{1}{\psi} \Omega + \frac{\psi}{16} \theta \frac{1}{\psi} \Omega + \frac{\psi}{8} \theta \frac{1}{\psi} (B_{1.1} - B_{1.2})$$

$$+ \frac{\psi}{4} \theta \frac{1}{2} B_{1.3} - \frac{\psi}{4} \theta \frac{1}{2} \Omega - \frac{\psi}{4} \theta \frac{1}{2} (B_{1.1} - B_{1.2}) - \frac{\psi}{4} \theta \frac{1}{2} B_{2.1} - \frac{\psi}{4} B_{2.2}.$$  

The second term can be expressed as follows by the use of the formula which is obtained by differentiating (A.13) with respect to $\psi$ and transforming as II:
Therefore, we lead to the following deferential formula:

\[
\begin{align*}
\theta \theta \theta \frac{1}{\psi} \Omega &= \frac{1}{16} \theta \theta \theta \frac{1}{\psi^2} + \frac{1}{2} \theta \theta \theta \frac{1}{\psi^2} + \frac{1}{2} \theta \theta \theta \frac{1}{\psi^2} + B_{0,1} + B_{1,2} + \psi \theta \theta \theta \frac{1}{\psi^2} + B_{3,1} \\
&= 2 \psi \theta \theta \theta \frac{1}{\psi^2} \left[ \frac{\psi}{4} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi^2} \right] \\
&\quad + \frac{\psi}{4} \theta \theta \theta \frac{1}{\psi^2} \left[ \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{4} \theta \theta \theta \frac{1}{\psi^2} - \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi^2} \right] \\
&\quad - \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi^2} \left[ \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{4} \theta \theta \theta \frac{1}{\psi^2} - \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi^2} \right] \\
&\quad + \frac{2}{\psi^2} \left[ \frac{\psi}{4} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi^2} \right] \\
&\quad + \frac{2}{\psi^2} \left[ \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi^2} - \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi^2} \right] \\
&\quad + \frac{2}{\psi^2} \left[ \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{4} \theta \theta \theta \frac{1}{\psi^2} - \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi^2} \right] \\
&\quad + \frac{2}{\psi^2} \left[ \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{8} \theta \theta \theta \frac{1}{\psi^2} - \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi^2} \right] \\
&\quad + \theta \theta \theta \frac{1}{\psi^2} \left[ \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi} + \frac{\psi}{4} \theta \theta \theta \frac{1}{\psi^2} - \frac{\psi}{2} \theta \theta \theta \frac{1}{\psi^2} \right]
\end{align*}
\]

(A.17)
$B_{4,1} = 2\psi \int d\left( \frac{x_1^3 x_2^3 x_5}{W_1 W_2^2} \right)$

$B_{0,1} = -2 \sum_{i=1}^{2} \int \frac{\partial}{\partial x_i} \frac{x_i x_1^3 x_2^3 x_5}{x_6 W_1^2 W_2^2} \omega_{(12456)} - \frac{5}{3} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_i x_1^3 x_2^3 x_5}{x_6 W_1^2 W_2^2} \omega_{(12)}$

$\quad - \frac{1}{3} \sum_{i=1}^{2} \int \frac{\partial}{\partial x_i} \frac{x_i x_1^3 x_2^3 x_5}{x_6 W_1^2 W_2^2} \omega_{(123)} + \frac{5}{3} \int \frac{\partial}{\partial x_3} \frac{x_3 x_1^3 x_2^3 x_5}{x_6 W_1^2 W_2^2} \omega_{(123)}$

$\quad + \frac{1}{3} \sum_{i=1}^{2} \int \frac{\partial}{\partial x_i} \frac{x_i x_1^3 x_2^3 x_5}{x_6 W_1^2 W_2^2} \omega_{(3456)} + 2 \int \frac{\partial}{\partial x_3} \frac{x_3 x_1^3 x_2^3 x_5}{x_6 W_1^2 W_2^2} \omega_{(3456)}$

$B_{0,2} = \frac{1}{3\psi} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3 x_4 x_5}{x_6 W_1^2 W_2^2} \omega_{(123)} - \frac{1}{3\psi} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3 x_4 x_5}{x_6 W_1^2 W_2^2} \omega_{(12)}$

$\quad - \frac{2}{3\psi} \sum_{i=1}^{5} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3 x_4 x_5}{x_6 W_1^2 W_2^2} \omega_{(12345)} - \frac{8}{3\psi} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3 x_4 x_5}{x_6 W_1^2 W_2^2} \omega_{(12345)}$

$\quad + \frac{1}{3\psi} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3 x_4 x_5}{x_6 W_1^3 W_2^2} \omega_{(123)} + \frac{5}{3\psi} \int \frac{\partial}{\partial x_6} \frac{x_6 x_1 x_2 x_3 x_4 x_5}{x_6 W_1^3 W_2^2} \omega_{(1236)}$

$B_1 = -\frac{1}{3} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} \frac{x_i}{x_1 W_1 W_2} \omega_{(123)} + \frac{1}{3} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_i}{x_1 W_1 W_2} \omega_{(126)}$

$B_2 = -\frac{2}{3\psi} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3}{x_6 W_1^2 W_2^2} \omega_{(123)} + \frac{3}{3\psi} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3}{x_6 W_1^2 W_2^2} \omega_{(12)}$

$B_3 = \frac{8}{3\psi^2} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3}{x_1 W_1 W_2^2} \omega_{(123)} - \frac{8}{3\psi^2} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3}{x_1 W_1 W_2^2} \omega_{(12)}$

$B_4 = \frac{16}{3\psi^2} \sum_{i=1}^{3} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3}{x_1 W_1 W_2^2} \omega_{(123)} - \frac{16}{3\psi^2} \sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \frac{x_i x_1 x_2 x_3}{x_1 W_1 W_2^2} \omega_{(12)}$

So we lead to the following differential operator:

$$L = \frac{1}{16} \left( \psi^3 - \frac{1}{\psi^3} \right) \partial_\psi^4 + \frac{3}{8} \left( \psi^3 + \frac{1}{\psi^3} \right) \partial_\psi^3 + \frac{1}{16} \left( 7\psi^3 - \frac{23}{\psi^3} \right) \partial_\psi^2$$

$$+ \frac{1}{16} \left( 1 + \frac{55}{\psi^6} \right) \partial_\psi - \frac{4}{\psi^2}.$$

(A.18)

and the boundary contributions which give the inhomogeneous term of the differential equation.

Now we turn to evaluate the inhomogeneous term

$$L \int_\Gamma \omega_0 = \int_{\Omega} \omega_0 \int_{\Gamma} \frac{\omega_0}{W_1 W_2} = \int_{(C_+ - C_-)} \beta.$$  

(A.19)
We introduce the derivative \( \dot{\theta} = i \beta \), (\( z = (3\psi)^{-6} \)) to simplify equation (A.18). The standard form of the Picard–Fuchs differential operator \( L_{PF} \) is given by
\[
L_{PF} = \theta^4 - 2z(3\theta + 1)^2(3\theta + 2)^2.
\] (A.20)

\( L_{PF} \) is related to \( L \) by
\[
L_{PF} = \frac{\psi}{81} L.
\] (A.21)

We need to know the explicit form of the defining equations of the boundary curves. As noted above, the curves are defined as the intersection of hyperplanes \( P = \{ x_1 + x_2 = 0, x_4 + x_5 = 0 \} \) (3.29) and rewritten by
\[
C_\pm = \left\{ x_1 + x_2 = 0, x_4 + x_5 = 0, x_3^3 + 3\psi x_4 x_6 = 0, x_1 = \pm \frac{x_3^2}{(3\psi)^2 x_4^2} \right\}.
\] (A.22)

In addition, there are two intersection points \( C_\pm \) such as \( p_1 = \{ x_1 = x_2 = x_3 = x_6 = x_4 + x_5 = 0 \} \) and \( p_2 = \{ x_3 = x_4 = x_5 = x_6 = x_1 + x_2 = 0 \} \). Since we can apply the tubes \( T_i(C_\pm) \) into \( P \) except for the neighborhoods of \( p_1 \) and \( p_2 \), our evaluation of the inhomogeneous term of the Picard–Fuchs equation as an integration of the exact term \( d\beta \) on the tubes \( T_i(C_\pm) \) is localized around these points [7].

We choose locally resolved coordinates around the point \( p_1 \) in the \( x_3 = 1 \) patch:
\[
X = \frac{x_2^2}{x_1 x_3^4}, \quad Z = \frac{x_1^2}{x_2 x_4^4}, \quad Y = x_3, \quad T = x_4, \quad U = \frac{x_6}{x_3^4}.
\] (A.23)

In terms of these coordinates, the boundary \( C_\pm \) is parametrized as
\[
T = -1, \quad U = -\frac{1}{3\psi} \psi, \quad X = -Z = \mp \frac{1}{(3\psi)^2}, \quad Y = r e^{i\psi} (r > 0, 0 \leq \varphi < 2\pi).
\] (A.24)

The singular point \( p_1 \) is resolved and splits into two points \( p_{1,1} \). The tube around \( p_{1,1} \), \( T_i(C_{1+}; p_{1,1}) \) is generated by a vector \( v \):
\[
v = \frac{f(r)}{r} e^{i\varphi} e^{i\chi} \partial_r + \frac{\alpha}{\psi} e^{-3i\psi} e^{i\chi} \partial_x - \frac{\alpha}{\psi} e^{-3i\psi} e^{i\chi} \partial_z + e^{i\chi} e^{i\varphi} \partial_U,
\] (A.25)

\[
T = -1 + \epsilon \frac{f(r)}{r} e^{i\varphi} e^{i\chi}, \quad X = -Z = -\frac{1}{(3\psi)^2} + \epsilon \frac{\alpha}{\psi} e^{-3i\psi} e^{i\chi},
\]
\[
U = -\frac{1}{3\psi} + \epsilon e^{i\chi} e^{i\varphi}, \quad Y = r e^{i\psi},
\]
\( (0 \leq \varphi \leq 2\pi, 0 \leq \chi \leq 2\pi, 0 \leq \xi \leq 2\pi) \),
\] (A.26)

\[
dx_1 \, dx_3 \, dx_4 \, dx_6 = \frac{1}{3} Y^{5/3} \, dX \, dY \, dT \, dU = - \frac{i \alpha e^{3}}{3\psi} r^{5/3} e^{-i\psi} e^{i\chi} e^{2i\varphi} \left( f'(r) - \frac{f(r)}{r} \right) \, dr \, d\psi \, d\chi \, d\xi.
\] (A.27)

This tube satisfies the conditions that ensure to not intersect the other boundary:
\[
d_W W_{1|C_{1-}} = e^{i\chi} e^{i\psi} e^{i\varphi} \left( \frac{1}{3} f(r) + \psi \right) \neq 0 \quad (\psi > 0)
\] (A.28)
\[
d_W W_{2|C_{1-}} = e^{i\varphi} \left[ \frac{f(r)}{r} e^{i\varphi} + \frac{r^3}{(3\psi)^2} (2\alpha - e^{i\psi} e^{3i\varphi}) \right] \neq 0 \quad (\alpha > 1).
\] (A.29)
The inhomogeneous term from $T_\epsilon(C_+; p_1)$ is
\[
\int_{T_\epsilon(C_+; p_1)} \beta = (-2\psi\theta + \frac{1}{2}\psi \theta + \frac{1}{2} \psi \theta + 2\psi \theta + \frac{1}{2} \psi \theta + \psi \theta + \theta \psi + \frac{1}{2} \psi \theta) \int \frac{\partial T_\epsilon}{\partial \psi W_2} \psi_1 W_2^\alpha_1 + \left(\theta \theta \psi + \frac{1}{2} \theta \psi + \theta \psi \theta + \theta \theta \psi + \theta \theta \psi + \theta \theta \psi + \theta \theta \psi \right) \int \frac{x_1 x_2}{W_1 W_2} \psi_1 W_2^\alpha_2.
\] (A.30)

The non-zero contribution to the integral on $T_\epsilon(C_+; p_1)$ yields from
\[
\left(\frac{1}{2} \psi \theta + \theta \psi \right) \int_{T_\epsilon(C_+; p_1)} \frac{\partial T_\epsilon}{\partial \psi W_2} \frac{x_1 x_2}{W_1 W_2} d\psi d\theta d\rho_3 d\pi_4 d\pi_5 = \frac{2i\pi^3}{3\psi^3}.
\] (A.31)

In a similar way, we find that the non-zero contribution from $C_-$ is
\[
\left(\frac{1}{2} \psi \theta + \theta \psi \right) \int_{T_\epsilon(C_-; p_1)} \frac{\partial T_\epsilon}{\partial \psi W_2} \frac{x_1 x_2}{W_1 W_2} d\psi d\theta d\rho_3 d\pi_4 d\pi_5 = -\frac{2i\pi^3}{3\psi^3}.
\] (A.32)

The definition of local coordinates around $p_2$ is given by just exchanging $x_1 \leftrightarrow x_4$, $x_2 \leftrightarrow x_5$ and $x_3 \leftrightarrow x_6$ in (A.26). However, we find that there is no contribution from the integrations around $p_2$.

Note that another possibility of contribution to the inhomogeneous term, which originates from the action of the differential operator on the 3-chain, has no contribution like all other known models. We refer to [7] for details. As a result of computations, we finally find (2.36).

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