Quantum repeaters and computation by a single module

Koji Azuma, Hitoshi Takeda, Masato Koashi, and Nobuyuki Imoto
Department of Materials Engineering Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan

(Dated: February 28, 2010)

We present a protocol of remote nondestructive parity measurement (RNPM) on a pair of quantum memories. The protocol works as a single module for key operations such as entanglement generation, Bell measurement, parity check measurement, and an elementary gate for extending one-dimensional cluster states. The RNPM protocol is achieved by a simple combination of devices such as lasers, optical fibers, beam splitters, and photon detectors. Despite its simplicity, a quantum repeater composed of RNPM protocols is shown to have a communication time that scales sub-exponentially with the channel length, and it can be further equipped with entanglement distillation. With a reduction in the internal losses, the RNPM protocol can also be used for generating cluster states toward measurement-based quantum communication.

PACS numbers: 03.67.Hk, 03.67.Lx

In quantum mechanics, measuring a property of a system inevitably causes disturbance on its state. Hence an ideal measurement would be the one that leaves the measured system with only as much disturbance as is necessary. A simple nontrivial example of such a measurement is the nondestructive parity (NP) measurement on two qubits $AB$, which is the projection measurement to the subspace with even parity spanned by $\{|00\rangle_{AB}, |11\rangle_{AB}\}$ and to the odd one spanned by $\{|01\rangle_{AB}, |10\rangle_{AB}\}$. When the qubits are in state $|\psi\rangle_{AB}$ initially, the unnormalized post-measurement state is ideally either $P_e\psi_{AB}$ or $P_o\psi_{AB}$, where $P_e$ is the projection onto the even (odd) subspace. This measurement provides a powerful tool when the two qubits are quantum memories located far apart. For example, if we prepare each qubit in state $|\phi\rangle_{AB}$ and to the odd one spanned by $\{|01\rangle_{AB}, |10\rangle_{AB}\}$. When the qubits are in state $|\phi\rangle_{AB}$, the NP measurement leaves the pair in maximally entangled state (Bell state) $\sqrt{2}P_e|\phi\rangle_{AB}$ or $\sqrt{2}P_o|\phi\rangle_{AB}$, where $\sqrt{2}P_e|\phi\rangle_{AB}$ is the projection onto the even (odd) subspace. Various other nontrivial operations are also derived from the NP measurement.

In this paper, we provide a simple protocol to implement the NP measurement, which we call remote nondestructive parity measurement (RNPM) protocol. The protocol is based on an off-resonant coupling of light pulses with the quantum memories, and it works even if the quantum memories are distant. The deviation of the RNPM protocol from the ideal NP measurement mainly comes from the loss in the optical channel, whose transmission depends on its length $L$ as $\eta_L := e^{-L/L_{att}}$ with an attenuation length $L_{att}$. This makes the RNPM protocol probabilistic and noisy, but these imperfections behave in a controlled way, even with the use of threshold detectors that cannot distinguish one from two or more photons. As a result, the RNPM protocol constitutes a viable module which can be singly used to build a quantum repeater, in contrast to the other known repeater protocols [11,12]. Moreover, the local use of highly efficient RNPM protocols will also allow us to generate cluster states.

The requirement on the memory qubit for the RNPM protocol is as follows. The qubit is assumed to allow us to apply phase flip $\hat{Z} := |0\rangle\langle 0| - |1\rangle\langle 1|$. Hadamard gate $H := |+\rangle\langle +| + |\rangle\langle -|$ with $|\rangle := |+\rangle Z$, and $Z$-basis measurement. The qubit is also assumed to interact with an off-resonant laser pulse $a$ in a coherent state $|\alpha\rangle_{a} := e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} (\alpha^n/\sqrt{n!}) |n\rangle_{a}$ according to a unitary operation $U_{\theta,j} |\alpha\rangle_{a} = e^{-i(\theta/2\alpha^2/|j\rangle}|\alpha e^{i(\theta/2)}_{\alpha}\rangle_{a}$ ($j = 0,1$), where $\{|n\rangle_{a}\}$ are the number states of the mode $a$, $\phi_{a} = \alpha^2 \sin \theta$, and $\theta$ is a fixed parameter for the strength of the interaction. Since this interaction is an off-resonant coupling based on a basic Hamiltonian – Jaynes-Cummings Hamiltonian, it will be feasible with various qubits such as an individual $\Delta$-type atom, a trapped ion, a single electron trapped in quantum dots, a nitrogen-vacancy (NV) center in a diamond with a nuclear spin degree of freedom, and a neutral donor impurity in semiconductors [3,9].

We now describe our RNPM protocol in detail. Suppose that the qubits $A$ and $B$ are respectively held by Alice and Bob, who are distance $L_0$ apart [see Fig. 1(a)]. Claire is located in between, connected to Alice and Bob with optical channels $a \rightarrow c_1$ and $b \rightarrow c_2$ with lengths $L_A \leq L_0$ and $L_B := L_0 - L_A$, respectively. Let $T_A := \tau_{l_{1}}$ and $T_B := \tau_{l_{2}}$ be the overall transmittance of the channels, where $\tau$ stands for the local loss. The RNPM protocol proceeds as follows: (i) Alice (Bob) prepares pulse $a$ (pulse $b$) in a coherent state $|\alpha/\sqrt{T_A}|_{a}$ ($|\alpha/\sqrt{T_B}|_{b}$) with $\alpha \geq 0$, and let it interact with qubit $A$ (qubit $B$) by $U_0$; (ii) Alice (Bob) sends Claire the pulse $a$ (the pulse $b$) through the optical channel $a \rightarrow c_1$ ($b \rightarrow c_2$); (iii) On receiving the pulses $c_1c_2$, Claire makes

*Electronic address: [azuma@qi.mp.es.osaka-u.ac.jp](mailto:azuma@qi.mp.es.osaka-u.ac.jp)
the pulses interfere by a half beam splitter; (iv) On the mode receiving the constructive interference, Claire applies displacement operation \( \hat{D}(\sqrt{2}a\cos(\theta/2)) \) by using a local oscillator (LO); (v) Claire counts photons of the output modes \( d_1d_2 \) by two photon detectors, and she announces the outcome \((m,n)\); (vi) If \( m+n \) is odd, Bob applies phase flip \( \hat{Z} \) to qubit \( B \). Events with \( m > 0 \) and \( n = 0 \) (\( m = 0 \) and \( n > 0 \)) indicates outcome ‘odd’ (‘even’), which are regarded as the success events of this protocol.

To see the back actions in the success events, we use the fact that the RNPM protocol works equivalently if we omit step (iv) and replace step (i) with the following: (i') After making pulse \( d \) (pulse \( b \) in a coherent state \( |\alpha/\sqrt{T_A}\rangle \) \( |\alpha/\sqrt{T_B}\rangle \) interact with qubit \( A \) (qubit \( B \), Alice (Bob) applies displacement operation \( \hat{D}(-\alpha/\sqrt{T_A}\cos(\theta/2)) \) \( \hat{D}(-\alpha/\sqrt{T_B}\cos(\theta/2)) \) on the pulse. In this protocol, through steps (i')-(iii), qubits \( AB \) are transformed as

\[
\begin{align*}
\ket{00}_{AB} &\rightarrow \ket{00}_{AB} |\beta_{A}\rangle_{\alpha} |\beta_{B}\rangle_{B} \rightarrow \ket{00}_{AB} |d_1\rangle |\sqrt{2}d_2\rangle, \\
\ket{01}_{AB} &\rightarrow \ket{01}_{AB} |\beta_{A}\rangle_{\alpha} |\beta_{B}\rangle_{B} \rightarrow \ket{01}_{AB} |d_1\rangle |\sqrt{2}d_2\rangle, \\
\ket{10}_{AB} &\rightarrow \ket{10}_{AB} |\beta_{B}\rangle_{\alpha} |\beta_{B}\rangle_{B} \rightarrow \ket{10}_{AB} |d_1\rangle |\sqrt{2}d_2\rangle, \\
\ket{11}_{AB} &\rightarrow \ket{11}_{AB} |\beta_{B}\rangle_{\alpha} |\beta_{B}\rangle_{B} \rightarrow \ket{11}_{AB} |d_1\rangle |\sqrt{2}d_2\rangle,
\end{align*}
\]

where \( \beta := \alpha \sin(\theta/2) \) and \( \beta_X := \beta/\sqrt{T_A} (X = A, B) \). Since this protocol does not use LO after (i'), we are allowed to assume that the total number \( k \) of photons in modes \( ab \) was measured after step (i'), without affecting the protocol at all.

We start with the ideal case where \( T_A = T_B = 1 \) and the detectors at modes \( d_1d_2 \) are the ideal photon-number-resolving detectors. Then, the \( k \) photons in modes \( ab \) are preserved throughout steps (ii) and (iii), which leads to \( m + n = k \). Combined with Eq. 1, this suggests that all the \( k \) photons are captured by one of the detectors. Hence, if photon detector \( d_1 \) (\( d_2 \)) announces the arrival of \( k(>0) \) photons, from \( \langle k|0\rangle = 0 \) and \( \langle k|\sqrt{2}\beta\rangle = (-1)^k/k!\sqrt{2}\beta \), we see that the back action of the RNPM protocol is \( \tilde{P}_{odd} \) \( (P_{even}) \) after Bob’s phase flip at step (vi).

We can easily describe the back actions of the RNPM protocol with practical channels and detectors, as long as the dark counting are negligible, namely, \( |0\rangle_{d_1} \) always produces \( m = 0 \). This guarantees that the success outcome still gives the correct parity, but \( l := m + n \) is no longer equal to \( k \). Since the back action depends only on \((-1)^k\), we see the following. If \( l \equiv k \) (mod 2), the final state is the same as the ideal case. Otherwise, the final state suffers from a phase flip error \( \hat{Z} \). This observation means that the success probability \( p \) and the phase error probability \( \epsilon \) (conditioned on the success) are solely determined from the joint probability \( Q(k,l) \) as follows,

\[
p = \sum_{l \geq 1} \chi^+_l, \quad \epsilon = \frac{1}{2p} \sum_{l \geq 1} (\chi^+_l - \chi^-_l),
\]

with \( \chi^+_l := \sum_k (\pm 1)^k Q(k,l) \).

Let us derive the explicit forms of \((p, \epsilon)\) with various types of detectors with quantum efficiency \( \eta \). Here we consider the case \( T_A = T_B(= T) \) for simplicity, and the general cases are treated in Appendix A. Since \( k \) is the total number of photons in two coherent states with amplitudes \( i\beta/\sqrt{T} \), it follows the Poissonian distribution \( P_\lambda(k) \) \( := (e^{-\lambda}\lambda^k/k! \) with \( \lambda = 2\beta^2/T \). When photon-number resolving detectors are used, \( l = m + n \) is the number of photons that has passed through a channel with transmittance \( \eta T \). Hence we have \( Q(k,l) = Q_\infty(k,l) \) \( := B_{2\beta^2/T}(l)P_{2\beta^2/T} \) with a binomial distribution \( B_p(l) := \left[p^l(1-p)^{k-1-l}\right]/[l!(l-k)!] \). Using Eq. 2, we have \( p(\beta) = 1 - e^{-2\beta^2\eta} \) and \( \epsilon(\beta, T) = (1 - e^{-2\beta^2\eta}[2(\eta T)^{-1} - 1])/2 \). When we use single photon detectors, we are informed of detection of exactly one photon. Hence we have \( Q(k,1) = Q_\infty(k,1) \) and \( Q(k,0) = Q_\infty(k,0) + \sum_{l \geq 2} Q_\infty(k,l), \) leading to \( p(\beta) = P_{2\beta^2/T}(1) \) and \( \epsilon(\beta, T) = (1 - e^{-2\beta^2\eta}[2(\eta T)^{-1} - 1])/2 \). When threshold detectors are used, from \( Q(k,1) = \sum_{l \geq 2} Q_\infty(k,l) \), we obtain \( p(\beta) = 1 - e^{-2\beta^2\eta} \) and \( \epsilon(\beta, T) = (1 - e^{-2\beta^2\eta}[2(\eta T)^{-1} - 1])/2 \).

As seen in the above examples, the success probability \( p \) and the phase error probability \( \epsilon \) of the RNPM protocol are under a trade-off relation, which is control-
In stabilizing the relative phase between pulses and in the Bell measurement, we have a contribution of the phase errors in the two initial pairs $L - \rho L$ the other hand, the choice of $L_0$ is nearly zero and the local loss $\tau$ determines the trade-off relation. We describe various applications of the RNPM protocol below.

Long-distance quantum communication over lossy channels: The goal here is to share an entangled pair of qubits between two end stations separated by distance $L$. With direct transmission of single photons, the communication time would increase exponentially with distance $L$ according to $e^{L/L_{\text{att}}}$. Disposition of relaying stations with quantum memories helps to avoid the exponential increase by using a quantum repeater protocol [1]. Let us see how a repeater protocol is built up from the RNPM protocol. Suppose that the stations are placed at $l_0 := L/2^n$ intervals (see Fig. 1 (c)). Each station has at least two qubits.

The first step is entanglement generation between neighboring stations separated by $l_0$. The RNPM protocol is applied to the two qubits in state $|+\rangle |+\rangle$, and is repeated until it is successful. Assuming the time $l_0/c$ for each trial, it takes time $(l_0/c)p(\beta_g)^{-1}$ on average, and the Bell state is produced with phase error probability $\epsilon_0 := \epsilon(\beta_g, \tau l_0/2)$. Here we consider the case with $L_A = L_B = l_0/2$ for simplicity of the notations. (The cases with $L_A = l_0$ are found in Appendix B)

Next, the repeater protocol proceeds to entanglement connection [F]. Suppose that two stations separated by $2^j l_0$ ($j = 0, 1, \ldots, n - 1$) can share a qubit pair in the Bell state with phase error probability $\epsilon_j$ and with average time $t_j$. After creating two such pairs connecting three stations, the middle one executes the Bell measurement by locally applying the RNPM protocol as in Fig. 1 (d), which succeeds with probability $p(\beta_s)$ and produces entangled qubits $2^{j+1}l_0$ apart. Adding up the contribution of the phase errors in the two initial pairs and in the Bell measurement, we have that $1 - 2\epsilon_j = (1 - 2\epsilon_j)^2(1 - 2\epsilon(\beta_s, \tau))$. Since it approximately takes time $(3/2)t_j$ per trial [2], we have $t_{j+1} \sim (3/2)t_j p(\beta_s)^{-1}$ for the average time for success. Solving these recursive relations, we see that the average total time $T = t_n$ is approximately written as

$$T \sim \frac{l_0}{c} \left(\frac{3}{2}\right)^{j} p(\beta_g)^{-1} p(\beta_s)^{-\log_2(L/l_0)},$$

and the final state is $\hat{\rho}^{AB} = \hat{F}[\Phi^+] |\Phi^+\rangle_{AB} + (1 - \hat{F})[\Phi^-] |\Phi^-\rangle_{AB}$ with

$$2\hat{F} - 1 = (1 - 2\epsilon(\beta_g, \tau l_0/2))^{L/l_0} (1 - 2\epsilon(\beta_s, \tau))^{L/l_0} - 1. \quad (4)$$

For large $L$, it should be chosen as $\beta_g^2 \sim \beta_s^2 \sim O(l_0/L)$. Then, noticing that $p(\beta) \sim O(\beta^2)$ and $\epsilon(\beta, T) \sim O(\beta^2)$ hold regardless of the types of the photon detectors, we have $\hat{F} \sim O(1)$ and $T \sim O((3/2)^{\log_2(L/l_0)} (L/l_0)^{\log_2(L/l_0)} + 1)$. Hence, $T$ increases only sub-exponentially with $L$. We also numerically optimized $T$ for fixed values of final fidelity $F$ and the distance $L$, which are shown in Fig. 2.

We stress that the generated state $\hat{\rho}^{AB}$ includes only one-type of error, which is a good property for quantum communication. For example, for the state $\hat{\rho}^{AB}$, the formula of secure key rate of the entanglement-based protocol [16, 17] is proportional to $1 - h(F)$ with the binary entropy function $h(x) := -x \log_2 x - (1 - x) \log_2(1 - x)$, which implies that the secret key is distillable for any $F > 1/2$.

Entanglement distillation: While the optical losses considered above are the dominant obstacle in long-distance communication, other types of small noises will be also present. Entanglement distillation not only helps to counter such general errors, but also reduces the scaling of the communication time to be polynomial in distance $L$ [1]. In a simple method of distillation called the recurrence method [18], Alice and Bob first transform each pair of qubits locally into the so-called Werner state while keeping the fidelity $F$ to a Bell state. Suppose that they have two such pairs $A_1B_1$ and $A_2B_2$ with $F > 1/2$. Alice applies a C-NOT gate on her qubit $A_1$ as the control and on $A_2$ as the target, and measures $A_2$ on Z-basis (the whole process is called parity check measurement). Bob also applies the same measurement on his qubits. Their outcomes will agree with a probability $P_{\text{rec}}(\hat{F})$, and then the remaining pair $A_1B_1$ will have an improved fidelity.

Since the outcome of each party is the parity of the two qubits, it can also be obtained via the RNPM protocol. In addition, if the RNPM protocol succeeds, by subsequently measuring $A_2$ on X basis to produce outcome $x$ and then by applying $Z^x$ on $A_1$, the post-measurement state of $A_1$ is also simulated except the phase error $\epsilon(\beta, \tau)$ [see Fig. 1 (e)]. The overall success probability $P_{\text{rec}} := P_{\text{rec}}(\hat{F}) p^2(\beta)$, which is in a trade-off relation with the fidelity $F'$ of the final state and is controllable through $\beta$.
single photon detectors.

Generation of cluster states: One promising way for implementing quantum computing is the so-called measurement-based quantum computation, where computation proceeds with sequential one-qubit measurements on a system in a highly entangled state – the cluster state \( [19, 20] \). The addressing of individual qubits is easier when they are located not so close to each other. Such a sparse configuration also helps to reduce correlated errors from the environment. In this case, the RNPM protocol works as an entangler for qubits that are not in close proximity. In fact, the gate shown in Fig. 1 (f) can be used for extending one-dimensional cluster states, and the parity check measurement in Fig. 1 (e) can be used for fusing two cluster states \( [21, 22] \). The combination of these two types of gates enables us to build up a large cluster state. Hence, with future development of good detectors and reduction of internal losses, the RNPM protocol will also work as a tool for implementing quantum computing.

We have proposed a versatile protocol, called the RNPM protocol, for measuring the parity of two separated qubits in a non-destructive way. The performance of the RNPM protocol is simply related to the optical loss and the characteristics of photon detectors. We have shown that, even with threshold detectors, the protocol can be used as a module to build up quantum repeaters for long-distance quantum communication. Efficient single photon detectors will allow us to equip the repeaters with entanglement distillation, a countermeasure against arbitrary types of noises. With further improvement of the performance, more general quantum computation will be brought within the scope through the generation of cluster states via the RNPM protocol. We believe that the existence of such a versatile protocol puts a renewed interest in developing efficient photon detectors and quantum memories off-resonantly coupled to light.

We would like to thank Naoya Sota for valuable discussions. We acknowledge the support of a MEXT Grant-in-Aid for Scientific Research on Innovative Areas 21102008, a MEXT Grant-in-Aid for the Global COE Program, JSPS Grant-in-Aid for Scientific Research (C) 20540389. K.A. is supported by JSPS.
Appendix A: The performance of the RNPM protocol

Here, for arbitrary values of $T_A$ and $T_B$, we derive the performance $(p, \epsilon)$ of the RNPM protocol with various types of detectors. As shown in the main body of this paper, the performance is determined by calculating the joint probability $Q(k, l)$ with which modes $ab$ have $k$ photons in total and the arrival of $l$ photons is announced by photon detectors $d_1$ and $d_2$. Let $k_a$ and $k_b$ be the numbers of photons in modes $a$ and $b$, respectively. Since mode $a$ is in a coherent state with amplitude $i\beta/\sqrt{T_A}$, $k_a$ follows the Poissonian distribution $P_{\beta^2/T_A}(k_a)$ with $P_{\lambda}(k) := (e^{-\lambda}\lambda^k)/k!$. Similarly, $k_b$ obeys the Poissonian distribution $P_{\beta^2/T_B}(k_b)$.

Suppose that we use photon-number-resolving detectors with quantum efficiency $\eta$. Suppose that we use photon-number-resolving detectors with quantum efficiency $\eta$. Hence, the probability of detecting $l_a$ photons among $k_a$ photons in mode $a$ is given by $B_{\eta T_A}(l_a|k_a)P_{\beta^2/T_A}(k_a)$, where $B_{\eta}(l|k) := [p^l(1-p)^{k-l}]!/l!(k-l)!$ is the binomial distribution. Similarly, the probability of detecting $l_b$ photons among $k_b$ photons in mode $b$ is given by $B_{\eta T_B}(l_b|k_b)P_{\beta^2/T_B}(k_b)$. Since $Q(k, l)$ is given by the sum of all probabilities with $k = k_a + k_b$ and $l = l_a + l_b$, we have

\[
Q(k, l) := \sum_{l_a=0} l \sum_{k_a=l_a} B_{\eta T_A}(l_a|k_a)P_{\beta^2/T_A}(k_a)B_{\eta T_B}(l - l_a|k - k_a)P_{\beta^2/T_B}(k - k_a)
\]

\[
e^{-\beta^2(T_A + T_B)/2}(\eta\beta^2)\sum_{l_a=0} l \sum_{k_a=l_a} \frac{1}{l_a!(l - l_a)!} \sum_{k_a=l_a} \frac{1 - \eta T_A}{T_A} \beta^2 k_a - l_a \frac{1 - \eta T_B}{T_B} \beta^2 k - l - (k_a - l_a)! \]

\[
e^{-\beta^2(T_A + T_B)/2}(\eta\beta^2)\sum_{l_a=0} l \sum_{k_a=l_a} \frac{1}{l_a!(l - l_a)!} \sum_{k_a=l_a} \frac{1 - \eta T_A}{T_A} \beta^2 k_a - l_a \frac{1 - \eta T_B}{T_B} \beta^2 k - l - (k_a - l_a)! \]

\[
e^{-\beta^2(T_A + T_B)/2}(\eta\beta^2)\frac{1}{l!(l - l)!} \sum_{k_a=l_a} \frac{1 - \eta T_A}{T_A} \beta^2 k_a - l_a \frac{1 - \eta T_B}{T_B} \beta^2 k - l - (k_a - l_a)! \]

\[
e^{-\beta^2(T_A + T_B)/2}(\eta\beta^2)\frac{1}{l!(l - l)!} \sum_{k_a=l_a} \frac{1 - \eta T_A}{T_A} \beta^2 k_a - l_a \frac{1 - \eta T_B}{T_B} \beta^2 k - l - (k_a - l_a)! \]

\[
e^{-\beta^2(T_A + T_B)/2}(\eta\beta^2)\frac{1}{l!(l - l)!} \sum_{k_a=l_a} \frac{1 - \eta T_A}{T_A} \beta^2 k_a - l_a \frac{1 - \eta T_B}{T_B} \beta^2 k - l - (k_a - l_a)! \]

\[
\chi^+_l = \sum_k Q(k, l) = \sum_{k=0}^{\infty} Q(k, l) = \frac{(2\beta^2\eta)^l}{l!} e^{-2\beta^2\eta}, \quad (A3)
\]

\[
\chi^-_l = \sum_k (-1)^{k-l} Q(k, l) = \sum_{k=0}^{\infty} (-1)^{k-l} Q(k, l) = \frac{(2\beta^2\eta)^l}{l!} e^{-2\beta^2\eta}(\eta A + \eta B)^{-1}, \quad (A4)
\]

by noting $e^x = \sum_{m=0}^{\infty} x^m/m!$. Hence, the success probability $p$ and the phase error probability $\epsilon$ of the RNPM protocol with photon-number-resolving detectors are

\[
p(\beta) = \sum_{l \geq 1} \chi^+_l = \sum_{l=1}^{\infty} \chi^+_l = 1 - e^{-2\beta^2\eta}, \quad (A5)
\]

\[
\epsilon(\beta, T_A, T_B) = \frac{1}{2p} \sum_{l \geq 1} (\chi^+_l - \chi^-_l) = \frac{1}{2p} \sum_{l=1}^{\infty} (\chi^+_l - \chi^-_l) = \frac{1}{2} \left( 1 - e^{-2\beta^2\eta}(\frac{1}{T_A} + \frac{1}{T_B})^{-2} \right). \quad (A6)
\]

Note that the above expressions are reduced to the ones in the main body of the paper for $T_A = T_B (= T)$. By substituting $T_A = \tau L_A = \tau e^{-L_A/L_A}$ and $T_B = \tau L_B = \tau e^{-L_B/L_B}$ into Eqs. (A3) and (A4), one can easily confirm that, for a fixed $L_0$, the choice of $L_A = L_B = L_0/2$ gives the best performance. In other words, the RNPM protocol works best when Claire is located at the middle point between Alice and Bob.
1. Use of single photon detectors

Here we assume the use of single photon detectors with quantum efficiency $\eta$, which announce the detection of photons only when receiving exactly one photon. In this case, $Q(k, l)$ is described by

$$Q(k, 1) = Q_\infty(k, 1),$$

$$Q(k, 0) = Q_\infty(k, 0) + \sum_{l \geq 2} Q_\infty(k, l).$$

Then, $\chi^\pm_1$ are calculated to be

$$\chi^+_1 = \sum_k Q(k, 1) = \sum_{k=1}^{\infty} Q_\infty(k, 1) = 2\beta^2 \eta e^{-2\beta^2 \eta},$$

$$\chi^-_1 = \sum_k (-1)^{k-1} Q(k, 1) = \sum_{k=1}^{\infty} (-1)^{k-1} Q_\infty(k, 1) = 2\beta^2 \eta e^{-2\beta^2 \eta} \left( \frac{1}{\pi A} + \frac{1}{\pi B} \right)^{-1}$$

from the last equations in Eqs. (A3) and (A4). Hence, we conclude

$$p(\beta) = \sum_{l \geq 1} \chi^+_1 = \chi^+_1 = 2\beta^2 \eta e^{-2\beta^2 \eta},$$

$$\epsilon(\beta, T_A, T_B) = \frac{1}{2p} \sum_{l \geq 1} (\chi^+_1 - \chi^-_1) = \frac{1}{2p} (\chi^+_1 - \chi^-_1) = \frac{1}{2} \left( 1 - e^{-2\beta^2 \eta} \left( \frac{1}{\pi A} + \frac{1}{\pi B} \right)^{-1} \right).$$

2. Use of threshold detectors

Here we consider the case of threshold detectors with quantum efficiency $\eta$. Since this type of detectors click only when receiving nonzero photons, we have

$$Q(k, 1) = \sum_{l \geq 1} Q_\infty(k, l),$$

$$Q(k, 0) = Q_\infty(k, 0).$$

From this, $\chi^\pm_1$ are calculated to be

$$\chi^+_1 = \sum_k Q(k, 1) = \sum_{k=1}^{\infty} Q_\infty(k, l) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} Q_\infty(k, l) = \sum_{l=1}^{\infty} \frac{(2\beta^2 \eta)^l}{l!} e^{-2\beta^2 \eta} = 1 - e^{-2\beta^2 \eta},$$

$$\chi^-_1 = \sum_k (-1)^{k-1} Q(k, 1) = \sum_{k=1}^{\infty} (-1)^{k-1} Q_\infty(k, l) = \sum_{k=1}^{\infty} (-1)^{(k-1)} \sum_{l=1}^{\infty} (-1)^{k-l} Q_\infty(k, l)$$

$$= \sum_{l=1}^{\infty} (-1)^{(l-1)} \frac{(2\beta^2 \eta)^l}{l!} e^{-\left( \frac{2-\eta T_A}{\pi A} + \frac{2-\eta T_B}{\pi B} \right) \beta^2} = (1 - e^{-2\beta^2 \eta}) e^{-2\beta^2 \eta} \left( \frac{1}{\pi A} + \frac{1}{\pi B} \right)^{-1}$$

from the last equations in Eqs. (A3) and (A4). Hence, the success probability $p$ and the phase error probability $\epsilon$ are

$$p(\beta) = \sum_{l \geq 1} \chi^+_1 = \chi^+_1 = 1 - e^{-2\beta^2 \eta},$$

$$\epsilon(\beta, T_A, T_B) = \frac{1}{2p} \sum_{l \geq 1} (\chi^+_1 - \chi^-_1) = \frac{1}{2p} (\chi^+_1 - \chi^-_1) = \frac{1}{2} \left( 1 - e^{-2\beta^2 \eta} \left( \frac{1}{\pi A} + \frac{1}{\pi B} \right)^{-1} \right).$$

Appendix B: The performance of long-distance quantum communication over lossy channels

Although the RNPM protocol has the best performance when Claire is halfway between Alice and Bob, the choice with $L_B = 0$ (Claire’s task is executed by Bob) is also worth mentioning since the stabilization of the relative phase
FIG. 4: For $L_A = l_0$ and $L_B = 0$, the minimum time $T$ needed to generate entanglement with $F = 0.9, 0.7$ over distance $L$ under the use of threshold detectors (TDs) and single photon detectors (SPDs): (a) $\tau = 0.95$ and $\eta = 0.9$; (b) $\tau = 0.98$ and $\eta = 0.95$. $c = 2 \times 10^8$ m/s, $L_{att} = 22$ km. The direct transmission time $(f \eta T_L)^{-1}$ of the photon from 10 GHz ($f = 10^{10}$) single photon source (SPS) is also described as a reference.

between pulses $c_1$ and $c_2$ is easier. Here we calculate the performance of quantum repeaters with this technical merit. More precisely, we assume the use of the RNPM protocols with $L_A = l_0 = L/2^n$ and $L_B = 0$ for the entanglement generation. In this case, the average total time $T$ and the fidelity $F$ are described by

$$T \sim \frac{l_0}{c} \left( \frac{3}{2} \right)^{\log_2(L/l_0)} p(\beta_g)^{-1} p(\beta_s)^{-\log_2(L/l_0)},$$

$$F = 1 + \frac{(1 - 2\epsilon(\beta_g, \tau, \eta l_0, \tau))^{L/l_0} (1 - 2\epsilon(\beta_s, \tau, \tau))^{L/l_0} - 1}{2}. \tag{B2}$$

By substituting Eqs. (A11) and (A12) [or Eqs. (A17) and (A18)] into these equations, we numerically optimized $T$ for fixed values of final fidelity $F$ and the distance $L$, which are shown in Fig. 4.