Scattering problems for the one-dimensional nonlinear Dirac equation with power nonlinearity

Hironobu Sasaki
Department of Mathematics and Informatics, Chiba University, 263–8522, Japan
E-mail: sasaki@math.s.chiba-u.ac.jp

Abstract. We study scattering problems for the one-dimensional nonlinear Dirac equation
\( (\partial_t + \alpha \partial_x + i\beta) \Phi(t, x) = F(\Phi(t, x)), \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \) (NLD)

Here, \( \Phi = \Phi(t, x) = (\phi_1(t, x), \phi_2(t, x)) \) is a \( \mathbb{C}^2 \)-valued unknown function defined on \( \mathbb{R} \times \mathbb{R} \),
\( \partial_t = (\partial/\partial t)I \), \( I \) is the \( 2 \times 2 \) identity matrix,
\( \partial_x = (\partial/\partial x)I \), \( i = \sqrt{-1} \), \( \alpha \) and \( \beta \) are \( 2 \times 2 \) Hermitian matrices satisfying that
\( \alpha^2 = \beta^2 = I \) and \( \alpha \beta + \beta \alpha = 0 \), \( F \) fulfills the following condition for some \( p > 1 \):
\( F = F(z_1, z_2) \) is a function from \( \mathbb{C}^2 \) into itself. Under the identification \( \mathbb{C}^2 \ni (z_1, z_2) \simeq (x_1, \ldots, x_4) =: x \in \mathbb{R}^4 \), \( F \) is of class \( C^2 \) and the following inequalities hold for any multi-index \( \gamma = (\gamma_1, \ldots, \gamma_4) \) with \( \gamma_1 + \cdots + \gamma_4 \leq 2 \):
\[ |\partial^\gamma_{\mathfrak{d}} F(x)| \leq C |x|^{p-|\gamma|}, \]
\[ |\partial^\gamma_{\mathfrak{d}} F(x) - \partial^\gamma_{\mathfrak{d}} F(y)| \leq C |x - y|(|x| + |y|)^{p-|\gamma|-1}. \]

Since the term \( F(z) = \lambda |z|^{p-1}z \) (\( |z| := |z_1| + |z_2| \), \( z = (z_1, z_2) \in \mathbb{C}^2 \)) satisfies (N), the nonlinear Dirac equation
\( (\partial_t + \alpha \partial_x + i\beta) \Phi(t, x) = \lambda |\Phi(t, x)|^{p-1} \Phi(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R} \)
is an example of (NLD).

Nonlinear Dirac equations (NLD) have been used to model many physical systems (see, e.g., Pelinovsky [6] and references therein). For instance, the massive Thirring model is expressed by (NLD) with
\[ F(\Phi) = \lambda \begin{pmatrix} |\phi_2|^2 \phi_1 \\ |\phi_1|^2 \phi_2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \]
and (NLD) with
\[
F(\Phi) = \lambda|\phi_1|^2|\phi_2|^2 \Phi + \lambda|\Phi|^2 \left( \frac{|\phi_2|^2}{|\phi_1|^2} \phi_1 - \frac{|\phi_1|^2}{|\phi_2|^2} \phi_2 \right), \quad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\]
occurs in the context of the Feshbach resonance for Bose–Einstein condensates.

By using decay estimates for solutions to the free Dirac equation, we can expect that \( p = 3 \) is the critical exponent for the long time behavior of solutions to (NLD). In particular, if \( p > 3 \), then we can also expect that the wave operator \( W_- \) for (NLD) is well-defined on some \( 0 \)-neighborhood of a Hilbert space. Our aim of this paper is to prove the existence of \( W_- \) for (NLD) with \( p > 3 \).

As for the long-time behavior of solutions to the one-dimensional nonlinear Schrödinger equation
\[
iu_t + u_{xx} = \lambda|u|^{p-1}u,
\]
the critical exponent is \( p = 3 \) (see, e.g., [1, 3]). Hayashi–Naumkin [5] proved the existence of \( W_- \) for (1.1) with \( p > 3 \) by using inequalities associated with \( J_S := U_S(t)xU_S(-t) \). Here, \( \{U_S(t)\}_{t \in \mathbb{R}} \) is the free Schrödinger evolution group. Since the identity
\[
J_S = \exp \left( \frac{ix^2}{4t} \right) \left( 2it \frac{\partial}{\partial x} \right) \exp \left( -\frac{ix^2}{4t} \right)
\]
holds, the operator \( J_S \) is applicable to estimates of the nonlinearity. In the case of the one-dimensional nonlinear Klein–Gordon equation
\[
u_{tt} - \nu_{xx} + \nu = \lambda|\nu|^{p-1}\nu,
\]
the critical exponent is also \( p = 3 \) (see, e.g., [4, 5]). Hayashi–Naumkin [5] proved the existence of \( W_- \) for (1.2) with \( p > 3 \) by using inequalities associated with \( J_{KG}^\pm := U_{KG}^\pm(t)xU_{KG}^\pm(-t) \), where \( U_{KG}^\pm(t) = \exp(\mp it\omega) \) and \( \omega = \sqrt{1 - (\partial^2/\partial x^2)} \). Since the free Klein–Gordon equation can be expressed by \((\partial/\partial t) - i\omega)((\partial/\partial t) + i\omega)\nu = 0\), we can call \( \{U_{KG}^\pm(t) \oplus U_{KG}^-(-t)\}_{t \in \mathbb{R}} \) the free Klein–Gordon evolution group on the Hilbert space \( H^s(\mathbb{R}^n) \oplus H^s(\mathbb{R}^n) \) \((s \in \mathbb{R})\). In order to estimate the nonlinearity, the following identity was used:
\[
J_{KG}^\pm = \pm i\omega^{-1} \left( P_{KG} - \left( \frac{\partial}{\partial t} \pm i\omega \right) x \right),
\]
where we define \( P_{KG} = t(\partial/\partial x) + x(\partial/\partial t) \), which is directly applicable to estimates of the nonlinearity.

In this paper, we modify \( J_{KG}^\pm \) and \( P_{KG} \) to study Dirac equations and we show that \( W_- \) for (NLD) is well-defined on some \( 0 \)-neighborhood of a Hilbert space if \( p > 3 \). Furthermore, we prove the existence of the scattering operator \( S \) defined on some \( 0 \)-neighborhood of a Hilbert space if \( p > 3 + 1/6 \).

In order to state our main results, we list some notation. For \( s, w \geq 0 \), we define the weighted Sobolev space \( H^{s,w} \) by
\[
H^{s,w} = \left\{ \Psi \in L^2(\mathbb{R}, \mathbb{C}^2); \ x^m \partial_x^k \Psi \in L^2(\mathbb{R}, \mathbb{C}^2), \ m = 0, 1, \cdots, w, \ k = 0, 1, \cdots, s \right\}.
\]
The \( H^{s,w} \)-norm is given by
\[
\|\Psi; H^{s,w}\| = \sum_{m=0}^w \sum_{k=0}^s \left\| x^m \partial_x^k \Psi; L^2(\mathbb{R}, \mathbb{C}^2) \right\|.
\]
Moreover, we denote $H^{s,0}$ by $H^s$, which is equal to the usual Sobolev space $H^s(\mathbb{R}, \mathbb{C}^2)$. For a positive number $\delta$ and for a Hilbert space $(X, \|\cdot\|)$, we denote the closed ball of radius $\delta$ centered at the origin in $X$ by $B(\delta, X)$. That is, $B(\delta, X) = \{x \in X; \|x\| \leq \delta\}$. We define the linear operator $D(t)$ for $t \in \mathbb{R}$ by

$$D(t) = \cos(t\omega)I - \omega^{-1}\sin(t\omega)(\alpha \partial_x + i\beta).$$

For any $s \geq 0$, we find that $D(t)\Phi_0 (\Phi_0 \in H^s)$ is a time-global solution to the free Dirac equation and that $\{D(t)\}_{t \in \mathbb{R}}$ is a strongly continuous one parameter unitary group on the Hilbert space $H^s$. Therefore we call $\{D(t)\}_{t \in \mathbb{R}}$ the free Dirac evolution group. For $s, w \geq 0$ and for $w_1 \in [0, w]$, we define the wave operator $W_- : B(\delta; H^{s,w}_1) \ni \Phi_0 \mapsto \Phi(0) \in H^{s,w}_1$ if the following property holds for some $\delta > 0$ and for some Banach space $Y \subset C(\mathbb{R}; H^s)$:

For any $\Phi_0 \in B(\delta; H^{s,w}_1)$, there uniquely exists a time-global solution $\Phi \in Y$ to (NLD) such that $D(-t)\Phi(t) \in C(\mathbb{R}; H^{s,w}_1)$ and

$$\lim_{t \to -\infty} \|D(-t)\Phi(t) - \Phi_0; H^{s,w}_1\| = 0.$$

Moreover, we define the inverse wave operator $V_+ : B(\delta; H^{s,w}_1) \ni \Phi_0 \mapsto \Phi_+ \in H^{s,w}_1$ if the following property holds:

For any $\Phi_0 \in B(\delta; H^{s,w}_1)$, there uniquely exist a time-global solution $\Phi \in Y$ to (NLD) and a data $\Phi_+ \in H^{s,w}_1$ such that $\Phi(0) = \Phi_0$, $D(-t)\Phi(t) \in C(\mathbb{R}; H^{s,w}_1)$ and

$$\lim_{t \to -\infty} \|D(-t)\Phi(t) - \Phi_+; H^{s,w}_1\| = 0.$$

If we see that $W_- : B(\delta; H^{s,w}_1) \to H^{s,w}_1$ and $V_+ : B(C_0\delta; H^{s,w}_1) \to H^{s,w}_1$ are well-defined and that $W_- (B(\delta; H^{s,w}_1)) \subset B(C_0\delta; H^{s,w}_1)$, then we define the scattering operator $S : B(\delta; H^{s,w}_1) \to H^{s,w}_1$ by $S = V_+ \circ W_-.$

We are ready to state our main results:

**Theorem 1.1** Let $F$ satisfy (N) with $p \geq 5$. Then there exists some positive number $\delta_0$ such that the scattering operator $S : B(\delta_0; H^1) \to H^1$ is well-defined.

**Theorem 1.2** Let $F$ satisfy (N) with $3 + 1/6 < p < 5$. Then there exists some positive number $\delta_0$ such that the scattering operator $S : B(\delta_0; H^{2,1}) \to H^{2,1}$ is well-defined.

**Theorem 1.3** Let $F$ satisfy (N) with $3 < p \leq 3 + 1/6$. Then there exist some positive numbers $\delta_0$ and $C_0$ satisfying the following properties:

1. The wave operator $W_- : B(\delta_0; H^{2,2}) \to H^{2,1}_1$ is well-defined. Moreover, the inverse wave operator $V_+ : B(C_0\delta_0; H^{2,1}_2) \to H^{2,1}_1$ is well-defined.

2. We have $W_- (B(\delta_0; H^{2,2}_1)) \subset B(C_0\delta_0; H^{2,2}_1)$ and $V_+ (B(C_0\delta_0; H^{2,2}_2)) \subset B(C_0^2\delta_0; H^{2,2}_1)$, and hence the composite mapping $V_+ \circ W_- : B(\delta_0; H^{2,2}_1) \to B(C_0\delta_0; H^{2,2}_1)$ is well-defined.

We give some remarks on our main results.

**Remark 1.4** Fix $p > 3$. Let $\Phi_-$ (resp. $\Phi_0$) belong to the domain of $W_-$ (resp. $V_+$). We see from the proof of Theorems 1.1–1.3 that $W_- (\Phi_-)$ and $V_+ (\Phi_0)$ are expressed by

$$W_- (\Phi_-) = \Phi_- + \int_{-\infty}^0 D(-t)F(\Phi(t))dt$$

and

$$V_+ (\Phi_0) = \Phi_0 + \int_0^\infty D(-t)F(\Phi(t))dt.$$
Here, $\Phi(t)$ is a time-global solution to (NLD). Moreover, if $p > 3 + 1/6$, then we obtain

$$S(\Phi_-) = \Phi_- + \int_{\mathbb{R}} D(-t) F(\Phi(t)) dt.$$  

**Remark 1.5** Let $3 < p \leq 3 + 1/6$. By $S'$, we denote the composite mapping $V_+ \circ W_-$ appearing in Theorem 1.3. Unfortunately, we do not see whether $S'$ becomes the scattering operator because we do not prove the existence of a time-global solution $\Phi(t)$ satisfying $\|D(-t)\Phi(t) - \Phi_\pm; H^{1,2}\| \to 0$ as $t \to \pm \infty$. However, we see that $S'$ has the same expression as the scattering operator. That is, we have

$$S'(\Phi_-) = \Phi_- + \int_{\mathbb{R}} D(-t) F(\Phi(t)) dt.$$  

**Remark 1.6** Let $p > 3$. Using the above expression of $V_+ \circ W_-$, we can obtain some results of inverse scattering problems for (NLD). For instance, for any $\Phi_- \in S(\mathbb{R}) \setminus \{0\}$, we have

$$p = \lim_{\varepsilon \to 0} \frac{\log \| (V_+ \circ W_-)(\varepsilon \Phi_-) - \varepsilon \Phi_-; L^2(\mathbb{R}; \mathbb{C}^2) \|}{\log \varepsilon}$$  

and

$$\lambda = \lim_{\varepsilon \to 0} \varepsilon^{-p} \langle (V_+ \circ W_-)(\varepsilon \Phi_-) - \varepsilon \Phi_-, \Phi_- \rangle_{L^2(\mathbb{R}; \mathbb{C}^2)} = \int_{\mathbb{R}^{1+1}} |D(t)\Phi_-(x)|^{p+1} dt(x).$$  

For the case of nonlinear Schrödinger equations, see, e.g., [2] and [8].

[1] J.E. Barab, Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation, J. Math. Phys. 25 (1984) 3270–3273.
[2] R. Carles and I. Gallagher, Analyticity of the scattering operator for semilinear dispersive equations, Comm. Math. Phys. 286 (2009) 1181–1209.
[3] T. Cazenave and F.B. Weissler, Rapidly decaying solutions of the nonlinear Schrödinger equation, Comm. Math. Phys. 147 (1992) 75–100.
[4] N. Hayashi and P.I. Naumkin, The initial value problem for the cubic nonlinear Klein-Gordon equation, Z. Angew. Math. Phys. 59 (2008) 1012–1028.
[5] N. Hayashi and P.I. Naumkin Scattering operator for nonlinear Klein-Gordon equations, Communications in Contemporary Mathematics 11 (2009) 771–781.
[6] D. Pelinovsky, Survey on global existence in the nonlinear Dirac equations in one spatial dimension, RIMS Kokyuroku Bessatsu B26 (2011) 37–49.
[7] H. Sasaki, Small data scattering for the one-dimensional nonlinear Dirac equation with power nonlinearity, preprint.
[8] W.A. Strauss, Nonlinear scattering theory, in: Scattering Theory in Math. Physics, Reidel, Dordrecht, 1974, pp. 53–78.