Fusion rules for Quantum Transfer Matrices as a Dynamical System on Grassmann Manifolds

O. Lipan \* P. Wiegmann \dagger A. Zabrodin \‡

November 1, 2018

Abstract

We show that the set of transfer matrices of an arbitrary fusion type for an integrable quantum model obey these bilinear functional relations, which are identified with an integrable dynamical system on a Grassmann manifold (higher Hirota equation). The bilinear relations were previously known for a particular class of transfer matrices corresponding to rectangular Young diagrams. We extend this result for general Young diagrams. A general solution of the bilinear equations is presented.

\* James Franck Institute of the University of Chicago, 5640 S. Ellis Avenue, Chicago, IL 60637, USA
\dagger James Franck Institute and and Enrico Fermi Institute of the University of Chicago, 5640 S. Ellis Avenue, Chicago, IL 60637, USA and Landau Institute for Theoretical Physics
\‡ Joint Institute of Chemical Physics, Kosygina str. 4, 117334, Moscow, Russia and ITEP, 117259, Moscow, Russia
1. Introduction

One of the key objects in the theory of quantum integrable systems is the family of transfer matrices ($T$-matrices). They are operators acting in the Hilbert space of the quantum model and represent a commutative set of integrals of motion.

A $T$-matrix for a quantum model on a one-dimensional lattice is constructed out of a Lax operator $L_i,Y(u)$, where $i$ is a lattice site and $u$ is a spectral parameter. The Lax operator is characterized by a Young diagram $Y$ and acts in the tensor product $V_i \otimes V_Y$ of the Hilbert space on the site $i$ and the space of the representation $Y$ (the auxiliary space). The $T$-matrix is then a trace of the product of Lax operators along the chain, taken over the auxiliary space:

$$T^{(Y)}(u) = \text{tr}_Y L_N,Y(u - v_N) \ldots L_2,Y(u - v_2)L_1,Y(u - v_1).$$  \hspace{1cm} (1)

It follows from the Yang-Baxter equation that these operators commute for different Young diagrams and spectral parameters: $[T^{(Y_1)}(u_1), T^{(Y_2)}(u_2)] = 0$.

The $T$-matrices are linearly independent but obey a set of functional relations called fusion rules. The fusion procedure allows one to construct Lax operators and $T$-matrices for higher Young diagrams out of a set of Lax operators for lower ones and, ultimately, out of the one-box diagram. (For the basic material on the quantum inverse scattering method and the fusion procedure we refer the reader to the papers \cite{1,2}, respectively.)

The fusion rules are especially simple and are closed for rectangular Young diagrams \cite{3,4}. Let $T_{a,s}(u)$ be the transfer matrix for a rectangular diagram with height $a$ and length $s$. Then the fusion rule may be written in bilinear form

$$T_{a,s}(u + 1)T_{a,s}(u - 1) - T_{a,s+1}(u)T_{a,s-1}(u) = T_{a+1,s}(u)T_{a-1,s}(u).$$  \hspace{1cm} (2)

This relation has been identified \cite{5} with Hirota’s celebrated bilinear difference equation \cite{6} for the $\tau$-function for the hierarchy of classical difference integrable equations. Since $T$-matrices commute at different $u$, $a$, $s$, the same relation holds for their eigenvalues, so one can treat $T$ in eq. (2) as number-valued functions. These relations can be treated as an integrable classical equation and it proves to be useful to obtain a complete solution of the quantum problem \cite{5}.

The bilinear relation (2) reveals an intimate connection between the fusion rules and the geometry of Grassmann manifolds. The point is that the Hirota equation can be viewed as a particular case of the general Plücker relations for coordinates on a Grassmannian manifold \cite{7}.

In this Letter we extend this result and show that the fusion rules for general Young diagrams (not necessarily rectangular) have this bilinear form and are equivalent to the general Plücker relations. These are equivalent to higher Hirota equations – the hierarchy of discrete integrable equations. We restrict ourselves to the series $A_{k-1}$.

The structure of the paper is as follows. In Sec. 2, we outline the higher Hirota equations. In Sec. 3, we present the fusion relations in the bilinear form for an arbitrary Young diagram and identify them with the higher Hirota equation (6). In Sec. 4 the general solution to eq. (12) is given for a relevant class of boundary conditions. Finally, Sec. 6 contains the proofs based on the Plücker relations, reviewed in Sec. 5.
2. Higher Hirota equations

The difference soliton equations form hierarchies in complete analogy with differential soliton equations. Higher members of the hierarchy (higher Hirota difference equations) are bilinear relations for a function \( \tau(x_1, x_2, \ldots, x_r) \) of \( r \) variables. They have the form

\[
\begin{vmatrix}
1 & z_1 & z_1^2 & \cdots & z_1^{r-2} & \tau_1 \hat{\tau}_1 \\
1 & z_2 & z_2^2 & \cdots & z_2^{r-2} & \tau_2 \hat{\tau}_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & z_r & z_r^2 & \cdots & z_r^{r-2} & \tau_r \hat{\tau}_r \\
\end{vmatrix} = 0, \tag{3}
\]

where the \( z_i \) are arbitrary constants and

\[
\begin{align*}
\tau_i & \equiv \tau(x_1, x_2, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_r), \\
\hat{\tau}_i & \equiv \tau(x_1 + 1, x_2 + 1, \ldots, x_{i-1} + 1, x_i, x_{i+1} + 1, \ldots, x_r + 1). \tag{4}
\end{align*}
\]

In a more compact form they read

\[
\sum_{l=1}^{r} \alpha_l \tau_l \hat{\tau}_l = 0, \tag{5}
\]

where the \( \alpha_l \) are minors of the matrix (3). They may be treated as independent constants subject to the condition \( \sum_{l=1}^{r} \alpha_l = 0 \). For \( r = 3 \) one gets the three-term Hirota equation (3).

The transformation

\[
\tau(x_1, \ldots, x_r) \to \exp \left[ \frac{1}{2r-4} \sum_{l=1}^{r} \log \alpha_l \left( \sum_{j=1, j \neq l}^{r} x_j \right) \right] \tau(x_1, \ldots, x_r) \tag{6}
\]

sends eq. (3) to the canonical form,

\[
\sum_{l=1}^{r} \tau_l \hat{\tau}_l = 0, \tag{7}
\]

with no extra parameters.

3. Bilinear Fusion Rules

Let us define the Young diagram by the coordinates of its corners (see the Figure):

\[
\begin{align*}
Q_i &= (\lambda_i, \mu_{i-1}), & i = 1, \ldots, n + 1, \\
P_i &= (\lambda_i, \mu_i), & i = 1, \ldots, n. \tag{8}
\end{align*}
\]
We imply that $\lambda_{n+1} = \mu_0 = 0$. The coordinates $\lambda_i$ and $\mu_i$ are ordered in strictly decreasing and strictly increasing order respectively: $\lambda_i > \lambda_{i+1}$, $\mu_i < \mu_{i+1}$. The set of ladder coordinates is also useful:

\[
s_i = \lambda_i - \lambda_{i+1}, \quad i = 1, \ldots, n, \\
a_i = \mu_i - \mu_{i-1}, \quad i = 1, \ldots, n.
\]

(9)

The Young diagram $Y = Y(\lambda, \mu)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ has $a_1$ rows of length $\lambda_1$, $a_2$ rows of length $\lambda_2$ and so on. Hereafter we denote $T(Y)(u) = T^\mu_\lambda(u)$.

In refs. 10, 11 (see also 12) the transfer matrix for a general Young diagram has been expressed through transfer matrices for Young diagrams which consist of either a single row or a single column. It is given by the determinants:

\[
T^\mu_\lambda(u) = \det_{1 \leq i,j \leq \mu_n} \left( T_{y_i-i+j}(u - \mu_n + \lambda_1 - y_i + i + j - 1) \right). 
\]

(10)

\[
T^\mu_\lambda(u) = \det_{1 \leq i,j \leq \lambda_1} \left( T_{y'-i+j}(u - \mu_n + \lambda_1 + y'_i - i - j + 1) \right). 
\]

(11)

Here $y_j$ is the length of the $j$-th row, $j = 1, \ldots, \mu_n$. Similarly, $y'_k$ are lengths of rows of the transposed diagram $Y'$ i.e. the diagram $Y$ reflected with respect to its main diagonal. The entries $T_m(u)$ ($T^m(u)$) of the determinants are $T$-matrices corresponding to the one-row (one column) Young diagram of length $m$.

We show that this function satisfies the bilinear equation:

\[
T^\mu_\lambda(u-1)T^\mu_\lambda(u+1) - T^\mu_{\lambda+1}(u)T^\mu_{\lambda-1}(u) = \sum_{i=1}^{n} T^\mu_{\lambda+\theta^i}(u)T^{\mu-\theta^i}_{\lambda+\theta^i}(u),
\]

(12)

*We thank A. Kuniba for communication on this point.
where
\[ \lambda + 1 \equiv (\lambda_1 + 1, \ldots, \lambda_n + 1) \]
and the step function \( \theta^i = (\theta^i_1, \ldots, \theta^i_n) \) is defined by \( \theta^i_j = 0 \) if \( j < i \) and \( \theta^i_j = 1 \) if \( j \geq i \), so
\[
\mu + \theta^i \equiv (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1} + 1, \ldots, \mu_n + 1),
\]
\[
\lambda + \theta^{i+1} \equiv (\lambda_1, \ldots, \lambda_i, \lambda_{i+1} + 1, \lambda_{i+2} + 1, \ldots, \lambda_n + 1).
\]

In terms of the ladder coordinates \( T_{\lambda_1, \ldots, \lambda_n}^a(u) \equiv T^a_\lambda(u) \) the equation acquires the form
\[
T_{\lambda_1, \ldots, \lambda_n}^a(u - 1)T_{\lambda_1, \ldots, \lambda_n}^a(u + 1) - T_{\lambda_1, \ldots, \lambda_n+1}^a(u)T_{\lambda_1, \ldots, \lambda_n-1}^a(u) = \sum_{i=1}^{n-1} T_{\lambda_1, \ldots, \lambda_i+1, \ldots, \lambda_n}^a(u)T_{\lambda_1, \ldots, \lambda_i-1, \ldots, \lambda_n}^a(u).
\]

In these relations the following boundary conditions are imposed:
\[ T_{\lambda_1, \ldots, \lambda_n}^a(u) = 0 \quad \text{if at least one of } a_i = -1. \] (14)

Under this condition eq. (13) consists of diagrams with the number of corners \( \leq n \). As soon as \( a_i = 0 \), the term \( T_{\lambda_1, \ldots, \lambda_i-1, \ldots, \lambda_n}^a(u) \) is zero and drops out from the sum. All the remaining terms have \( a_i = 0 \) and correspond to diagrams with \( n - 1 \) corners.

Eq. (13) can be transformed to a higher Hirota form (5) by a linear change of variables. First of all, let us notice that the equation actually depends on \( n + 2 \) variables rather than \( 2n + 1 \) variables \( a_i, s_i \). Indeed, the combinations
\[
q_i = \frac{1}{2}(a_i + s_i), \quad i = 1, \ldots, n - 1,
\]
are the same in all \( T \)-functions involved in eq. (13) and, therefore, can be considered as parameters.

It is convenient to change the variables in two steps. First, introduce the variables \( p_j, j = 0, 1, \ldots, n + 1 \):
\[
p_0 = u,
\]
\[
p_i = \frac{1}{2}(a_i - s_i), \quad 1 \leq i \leq n - 1,
\]
\[
p_n = a_n,
\]
\[
p_{n+1} = s_n - \frac{1}{2} \sum_{i=1}^{n-1} (a_i - s_i).
\]

Note that \( \sum_{i=0}^{n+1} p_i = u + a_n + s_n \). In the new variables \( (p_0, p_1, \ldots, p_{n+1}) \equiv T_{\lambda_1, \ldots, \lambda_n}^a(u) \) the equation acquires the form
\[
T(p_0 + 1)T(p_0 - 1) = \sum_{i=1}^{n+1} T(p_i + 1)T(p_i - 1),
\]
(16)
where we have dropped the variables that do not undergo shifts.

Next introduce the variables:

\[ x_i = \frac{1}{2} \left( p_i - \sum_{j=0,\neq i}^{n+1} p_j \right), \quad i = 0, 1, \ldots, n + 1. \tag{17} \]

In terms of the initial variables they are:

\[ x_0 = \frac{1}{2} (u - a_n - s_n), \]
\[ x_i = \frac{1}{2} (-u - a_n - s_n + a_i - s_i), \quad 1 \leq i \leq n - 1, \]
\[ x_n = \frac{1}{2} (-u + a_n - s_n), \]
\[ x_{n+1} = \frac{1}{2} (-u + \lambda_1 - \mu_n). \tag{18} \]

Finally, passing to the function \( \tau(x_0, \ldots, x_{n+1}) = T(p_0, \ldots, p_{n+1}) \), we arrive at eq. (3) with \( r = n + 2, \alpha_1 = -1, \alpha_i = 1, i = 2, 3, \ldots, n + 2. \)

This proves that the bilinear fusion rules are equivalent to the higher Hirota equations.

Let us notice that the bilinear relations for the \( T \)-matrices (12) are not unique. There exist other bilinear relations of a slightly different structure.

4. General Solution to the Higher Hirota Equation (3.5)

The higher Hirota equation with particular boundary conditions (14) is solved explicitly along the lines of ref. [7].

For the series \( A_{k-1} \), the number of rows in Young diagrams does not exceed \( k \). The boundary conditions (14) in this case are:

\[ T^\mu_1(u) = 0 \quad \text{as} \quad \mu_n > k \quad \text{or} \quad \mu_n < 0. \tag{19} \]

In what follows, it is convenient to set \( \mu_{n+1} = k \).

The general solution depends on \((n+1)k\) arbitrary functions \( h_{l}\)(\(u\)), \( l = 1, \ldots, n+1, j = 1, \ldots, k \). Let us form the \( k \times k \) matrix:

\[ H^{(\mu,\lambda)}_{i,j} = h_{l(i)}(u - \lambda_1 + \mu_n + 2\lambda_{l(i)} - 2i + 2), \tag{20} \]

where the function \( l(i) \) is determined by the condition

\[ \mu_{l(i)-1} + 1 \leq i \leq \mu_{l(i)} + 1, \]

and \( i = 1, 2, \ldots, k \). This matrix has a horizontal strip structure: the \( l \)-th strip consists of \( \mu_l - \mu_{l-1} \) rows.

Then the general solution has the form:

\[ T^\mu_1(u) = \det_{1 \leq i,j \leq k} H^{(\mu,\lambda)}_{i,j}. \tag{21} \]
This form of the solution is convenient for deriving the Bethe Ansatz equations for a general $A_{k-1}$ quantum integrable model. The derivation is along the lines of ref. Other determinant representations also exist.

5. Grassmannians and Plücker Relations

The proof of the bilinear relations for the $T$-matrices is based on the Plücker identities. They appear as relations between coordinates on Grassmann manifolds (see 14, 15, 16). The connection between Hirota’s bilinear equations and the geometry of Grassmann manifolds is well known 7, 17, 18. The Grassmann manifolds related to general solutions of a soliton equation are infinite dimensional. Remarkably, the solutions of interest, specified by the boundary conditions (19), correspond to finite dimensional Grassmannians. This allows one to write down a general solution in terms of determinants. Numerous determinant formulas like (10), (11), (21) may be obtained in this way.

The Grassmannian $G_{M+1}^{N+1}$ is a collection of all $(M + 1)$-dimensional linear subspaces of the complex $(N + 1)$-dimensional vector space $\mathbb{C}^{N+1}$. In particular, $G_{1}^{N+1}$ is the complex projective space $\mathbb{P}^{N}$. Let $X \in G_{N+1}^{M+1}$ be such a $(M + 1)$-dimensional subspace spanned by vectors $x^{(j)} = \sum_{i=0}^{N} x_{i}^{(j)} e_{i}, \ j = 1, \ldots, M + 1$, where the $e_{i}$ are the basis vectors of $\mathbb{C}^{N+1}$. The collection of their coordinates form a rectangular $(N + 1) \times (M + 1)$-matrix $x_{i}^{(j)}$. Let us consider its $(M + 1) \times (M + 1)$ minors:

$$\det_{pq}(x_{i}^{(q)}) \equiv (i_0, i_1, \ldots, i_M), \quad p, q = 0, 1, \ldots, M,$$

obtained by choosing $M + 1$ lines $i_0, i_1, \ldots, i_M$. These $C_{N+1}^{M+1}$ minors are called the Plücker coordinates of $X$. They are defined up to a common scalar factor and provide the Plücker embedding of the Grassmannian $G_{N+1}^{M+1}$ into the projective space $\mathbb{P}^{d}$, where $d = C_{N+1}^{M+1} - 1$ ($C_{N+1}^{M+1}$ is the bimomial coefficient).

The image of $G_{N+1}^{M+1}$ in $\mathbb{P}^{d}$ is realized as an intersection of quadratic curves. This means that the coordinates $(i_0, i_1, \ldots, i_M)$ are not independent but obey the Plücker relations 14, 15:

$$(i_0, i_1, \ldots, i_M)(j_0, j_1, \ldots, j_M) = \sum_{p=0}^{M} (j_p, i_1, \ldots, i_M)(j_0, \ldots, j_{p-1}, i_0, j_{p+1}, \ldots, j_M)$$

(23)

for all $i_p, j_p, p = 0, 1, \ldots, M$.

6. The Plücker Relations and Fusion Rules

Here we outline the proof of eq. (23). It turns out that the bilinear fusion rules are realized as the Plücker relations. In order to compare them let us put $i_p = j_p$ for $p \neq 0, 1, \ldots, n$ in (23). Then all terms but the first $n + 1$ terms in the r.h.s. of (23) vanish. Then the Plücker relations read

$$[i_0, i_1, \ldots, i_n][j_0, j_1, j_2, \ldots, j_n] = \sum_{p=0}^{n} [j_p, i_1, i_2, \ldots, i_n][j_0, \ldots, j_{p-1}, i_0, j_{p+1}, \ldots, j_n],$$

(24)
where we denote
\[ i_0, \ldots, i_n \equiv (i_0, \ldots, i_{n-1}, i_n, \ldots, i_M), \]
in order to stress that all arguments with a subscript greater than \( n \) are the same in all terms of eq. (24). Only the first \( n \) arguments are shown explicitly.

Given a Young diagram \( Y(\lambda, \mu) \) and the corresponding function \( T_{\mu}^\lambda(u) \), introduce a rectangular \((\mu_n + n + 2) \times (\mu_n + 1)\)-matrix \( M_{i,j} \), \( i = -1, 0, 1, \ldots, \mu_n + n \), \( j = 1, 2, \ldots, \mu_n + 1 \). Its matrix elements are:

\[
M_{i,j} = \delta_j, \mu_n + 1, \quad M_{0,j} = \delta_j, 1 \quad (25)
\]

and

\[
M_{i,j} = T_{\lambda-i+j+l(i)-1}^\mu(u + \lambda_1 - \mu_n - \lambda_l + i + j - l) \quad (26)
\]

where \( l(i) \) is given by

\[
\mu_{l(i)-1} + l(i) \leq i \leq \mu_{l(i)} + l(i), \quad l = 1, 2, \ldots, n
\]

and \( \delta_{i,j} \) is Kronecker’s symbol. The matrix consists of \( n + 1 \) horizontal strips. Each strip except the first one consists of \( \mu_l - \mu_{l-1} \) rows. The first strip has only two rows (25).

Let us apply the Plücker relation (24) to the minors of this matrix. Choose \( i_p(j_p) \) to be the first (the last) row of the \( p \)-th strip:

\[
i_0 = -1, \quad j_0 = 0, \quad i_l = \mu_{l-1} + l, \quad j_l = \mu_l + l, \quad l = 1, \ldots, n.
\]

Now, using eq. (10), we identify the minors with the \( T \)-matrices:

\[
[i_0, i_1, \ldots, i_n] = \omega (-1)^{\mu_n} T_{\lambda}^\mu(u),
\]

\[
[j_0, j_1, \ldots, j_n] = \omega' T_{\lambda}^\mu(u + 1),
\]

(27)

where \( \omega, \omega' \) are irrelevant sign factors such that \( \omega \omega' = (-1)^{\mu_n+n} \). Computing the remaining minors, we obtain:

\[
[j_0, i_1, \ldots, i_n] = \omega T_{\lambda+1}^\mu(u + 1),
\]

\[
[i_0, j_1, \ldots, j_n] = \omega' (-1)^{\mu_n} T_{\lambda-1}^\mu(u + 1),
\]

\[
[j_p, i_1, \ldots, i_n] = \omega (-1)^{\mu_p} T_{\lambda+\theta p+1}^\mu(u + 1),
\]

\[
[j_0, \ldots, j_{p-1}, i_0, j_{p+1}, \ldots, j_n] = \omega' (-1)^{\mu_p+\mu_n} T_{\lambda-\theta p+1}^\mu(u + 1).
\]

Since the minors of the matrix (26) obey the Plücker relations (24), the \( T \)-matrices satisfy eq. (12).

The proof of eq. (27) is similar.

7. Conclusion

We have shown that the fusion rules of quantum integrable models have the form of an integrable dynamical system on a finite dimensional Grassmann manifold. Namely, the fusion relations are identical to the hierarchy of Hirota difference equations with open boundary conditions.

This equation, supplemented by analyticity requirements \( \delta \) completely determines the spectrum of the quantum system, i.e. reveals the Bethe Ansatz equations.
In this note we just observed a connection between the fusion procedure and Grassmannian geometry. It seems to be an intimate one and is also expected for the fusion of conformal field theories. A deeper understanding of this connection would be desirable.

8. Acknowledgements

We thank H.Awata, S.Khoroshkin and A.Kuniba for discussions and J.Talstra for help. The work of O.L. was supported by the MRSEC NSF grant DMR 9400379. The work of A.Z. was supported in part by RFBR grant 95-01-01106, by ISTC grant 015 and also by NSF grant DMR-9509533. P.W. was supported by NSF grant DMR-9509533. P.W and A.Z thank the Institute for Theoretical Physics in Santa Barbara for its hospitality in April 1997 when this paper was completed.

1. L.D.Faddeev and L.A.Takhtadjan, Quantum inverse scattering method and the XYZ Heisenberg model, Uspekhi Mat. Nauk 34:5 (1979) 13-63.
2. P.P.Kulish, N.Yu.Reshetikhin and E.K.Sklyanin, Yang-Baxter equation and representation theory: I, Lett. Math. Phys. 5 (1981) 393-403; P.P.Kulish and E.K.Sklyanin, Quantum spectral transform method. Recent developments, Lecture Notes in Physics 151 61-119, Springer, 1982.
3. A.Klumper and P.Pearce, Conformal weights of RSOS lattice models and their fusion hierarchies, Physica A183 (1992) 304-350.
4. A.Kuniba, T.Nakanishi and J.Suzuki, Functional relations in solvable lattice models, I: Functional relations and representation theory, II: Applications, Int. Journ. Mod. Phys. A9 (1994) 5215-5312.
5. I.Krichever, O.Lipan, P.Wiegmann and A.Zabrodin, Quantum integrable models and discrete classical Hirota equations, preprint ESI 330 (1996), [hep-th/9604080] to appear in Commun. Math. Phys.
6. R.Hirota, Discrete analogue of a generalized Toda equation, Journ. Phys. Soc. Japan 50 (1981) 3785-3791.
7. M.Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, RIMS Kokyuroku 439 (1981) 30-46.
8. E.Date, M.Jimbo and T.Miwa, Method for generating discrete soliton equations I, II, Journ. Phys. Soc. Japan (1982) 4116-4131.
9. Y.Ohta, R.Hirota, S.Tsujimoto and T.Imai, Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy, Journ. Phys. Soc. Japan 62 (1993) 1872-1886.
10. P.P.Kulish and N.Yu.Reshetikhin, On GL3-invariant solutions of the Yang-Baxter equation and associated quantum systems, Zap. Nauchn. Sem. LOMI 120 (1982) 92-121 (in Russian), Engl. transl.: J. Soviet Math. 34 (1986) 1948-1971; N.Yu.Reshetikhin, The functional equation method in the theory of exactly soluble quantum systems, Sov. Phys. JETP 57 (1983) 691-696.
11. V.Bazhanov and N.Reshetikhin, Restricted solid on solid models connected with simply laced algebras and conformal field theory, Journ. Phys. A23 (1990) 1477-1492.
12. A.Kuniba, Y.Ohta and J.Suzuki, Quantum Jacobi-Trudi and Giambelli formulae for Uq(Br) from analytic Bethe ansatz, preprint, [hep-th/9506167].
13. A.Zabrodin, Discrete Hirota’s equation in quantum integrable models, preprint ITEP-TH-44/96 (1996), [hep-th/9610039].
14. W.V.D.Hodge and D.Pedoe, Methods of algebraic geometry, volume I, Cambridge University Press, Cambridge, 1947.
15. J.W.P. Hirschfeld and J.A. Thas, *General Galois geometries*, Claredon Press, Oxford, 1991.
16. P. Griffiths and J. Harris, *Principles of algebraic geometry*, A Wiley-Interscience Publication, John Wiley & Sons, 1978.
17. M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. RIMS, Kyoto Univ. **19** (1983) 943-1001.
18. G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Publ. IHES **61** (1985) 5-65.