THE SCALED RELATIVE GRAPH OF A LINEAR OPERATOR

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Abstract. The scaled relative graph (SRG) of an operator is a subset of the complex plane. It captures several salient features of an operator, such as contractiveness, and can be used to reveal the geometric nature of many of the inequality based arguments used in the convergence analyses of fixed point iterations. In this paper we show that the SRG of a linear operator can be determined from the numerical range of a closely related linear operator. Furthermore we demonstrate that the SRG of a linear operator has a range of spectral and convexity properties, and satisfies an analogue of Hildebrant’s theorem.

1. Introduction

The scaled relative graph (SRG) was introduced by Ryu, Hannah and Yin in [11] as a geometric tool for the modular analysis of operators. The SRG of an operator is a subset of the complex plane that captures a number of important features of the operator, such as whether or not it is contractive. The SRGs of simpler operators can be combined in an intuitive graphical manner to bound the SRGs of the operators resulting from their algebraic composition. These rules for combining operators can be used as geometric analogues of the inequalities typically used in, for example, the convergence proofs of fixed point iterations. This has been used to give a unified geometric treatment of the convergence rates of a wide range of algorithms, including gradient descent, Douglas-Rachford splitting and the method of alternating projections.

The promise of the SRG extends far beyond the analysis of algorithms from convex optimization. As already noted in [2], the modular fashion in which the SRG can be manipulated makes it an ideal candidate for dynamical system analysis, and the authors additionally give preliminary results connecting the SRG to classical tools from control theory. In order to unlock this potential, a better understanding of how to determine the SRG of an operator is required. For example, even the question of how to determine the SRG when the operator is a square matrix with real entries has only been fully resolved in the case that the matrix is normal, or of dimension 2 [7].

Our primary motivation is to better understand the geometry of the SRG, building our intuition from the finite dimensional linear case, where the operators in
question are matrices. However, from a theoretical perspective, the results from
the matrix case can be pushed through to the case of linear operators on Hilbert
spaces with little to no changes. Since such operators are relevant in a wide range
of applications, particularly in the study of differential equations, this is the setting
we will consider. Our main result is to show that the SRG of a linear operator can
be determined from the numerical range of a closely related linear operator. This
allows much of the machinery that has been developed to understand the numerical
range to be applied in the SRG setting. We use this to show that the SRG, like
the numerical range, has a range of convexity and spectral properties, and satsifies
an analogue of Hildebrandt’s theorem [5]. Despite these similarities, the convexity
properties of the SRG are rooted in hyperbolic geometry, and its spectral prop-
erties capture information about the approximate point spectrum rather than the
spectrum.

Section 2 introduces the relevant concepts from the theory of linear operators and
hyperbolic geometry, and also reviews the definition of the SRG and known results
on the SRG of a matrix. In section 3 we relate the SRG to the numerical range.
Section 3.1 establishes the connections in the case of complex Hilbert spaces. In this
subsection we also characterise the spectral and convexity properties of the SRG,
derive the analogue of Hildebrant’s theorem, and show how to plot the boundary of
the SRG of an operator defined either by a matrix or a linear differential equation.
Finally in section 3.2 we show how to determine the SRG of a linear operator on a
real Hilbert space using the results from section 3.1.

2. Notation and preliminaries

2.1. Basic notation. Throughout F will denote either the real field, R, or the
complex field, C. When speaking geometrically we will also refer to C as the
complex plane. The complex conjugate of \( z \in C \) will be denoted by \( \bar{z} \). A set \( s \)
is said to be convex if \( ts_1 + (1-t)s_2 \in s \) for all \( 0 \leq t \leq 1 \) and \( s_1, s_2 \in s \), and
the closure of \( s \) is denoted by \( \text{cl} s \). Furthermore, the convex hull \( \text{co}(s) \) is defined
to be the smallest convex set containing \( s \), and the boundary of a set \( s \subseteq C \) will
be denoted by \( \partial s = (\text{cl} s \cap \{C \setminus s\}) \). We will overload notation as appropriate to
apply to sets, for example \( \{z_1, z_2\} \) will denote the set \( \{z_1, z_2\} \), and more generally
\( h(s) = \{h(z) : z \in s\} \).

2.2. Operators on Hilbert spaces. \( \mathcal{H} \) denotes a Hilbert space over the field
F, equipped with an inner product \( \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to F \) which defines a norm
\( \|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle} \). \( T : \mathcal{H} \to \mathcal{H} \) will be called a linear operator if it is linear, and
\( \sup \{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\} < \infty \). The identity operator will be denoted by \( I \)
(\( Ix = x \) for all \( x \in \mathcal{H} \)).

To illustrate our results we will primarily consider the cases that

(1) \( \mathcal{H} \) is \( \mathbb{R}^n \), equipped with the inner product \( \langle y, x \rangle = x^T y \);
(2) \( \mathcal{H} \) is \( \mathbb{C}^n \), equipped with the inner product \( \langle y, x \rangle = x^T y \);

in which case the linear operators correspond to the square matrices with entries
in \( \mathbb{R} \) and \( \mathbb{C} \) respectively. We will also consider the Hilbert space of complex valued
Lebesgue square integrable functions \( L^2(\mathbb{R}) \), with inner product

\[ \langle y, x \rangle = \int_{\mathbb{R}} x(t)^* y(t) \, dt, \quad x, y \in L^2(\mathbb{R}), \]
and the Hilbert space of complex valued square summable sequences $\ell^2(\mathbb{N})$, with inner product
\[
\langle y, x \rangle = \sum_{j \in \mathbb{N}} x_j y_j, \quad x, y \in \ell^2(\mathbb{N}) .
\]

We define the graph of a linear operator $T$ as
\[
\text{gra} T = \{ (x, Tx) : x \in \mathcal{H} \},
\]
and denote the adjoint of $T$ as $T^*$ ($\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{H}$). $T$ is said to be invertible if there is a linear operator $S$ such that $TS = ST = I$, and we denote this inverse as $T^{-1}$. The spectrum of $T$ defined to be the subset of the complex plane
\[
\sigma(T) = \{ z : z \in \mathbb{F}, (T - zI) \text{ is not invertible} \} .
\]

We additionally say that $\lambda \in \sigma(T)$ is in the approximate point spectrum ($\lambda \in \sigma_{\text{ap}}(T)$) if there exist a sequence of unit vectors such that $\lim_{n \to \infty} \| (T - \lambda I) x_n \| = 0$. In the matrix case $\sigma(T) = \sigma_{\text{ap}}(T)$, and $\lambda$ is an eigenvalue of $T$ if and only if $\lambda \in \sigma(T)$.

2.3. The scaled relative graph. We define the SRG of a linear operator $T$ to be the subset of the complex plane
\[
\text{SRG}(T) = \left\{ \frac{\| y \|}{\| x \|} \exp \left( \pm i \arccos \left( \frac{\text{Re} \left( \langle y, x \rangle \right)}{\| y \| \| x \|} \right) \right) : x \in \mathcal{H}, y = Tx, \| x \| = 1 \right\} .
\]

For linear operators this definition coincides with the more general definition of the SRG from [11]. It follows from the definition of the SRG that $T$ is contractive if and only if $\text{SRG}(T)$ is contained in the closed unit disk.

The SRG captures some of the geometric features of the input-output pairs of the operator. Recall that the angle $\theta$ between $x \in \mathcal{H}$ and $y \in \mathcal{H}$ is typically defined through
\[
\cos \theta = \frac{\text{Re} \left( \langle y, x \rangle \right)}{\| y \| \| x \|} .
\]

The SRG is then the union of the ‘polar representations’ of the input-output pairs $(x, y) \in \text{gra} T$, in which the magnitude is given by the ratio between the norms of the output and input, and the argument the angle between the input and output.

2.4. The Beltrami-Klein mapping. The Beltrami-Klein mapping is a tool from two dimensional hyperbolic geometry. Its importance in the context of the SRG was first recognised in [7], where it was used in the construction of the SRG for normal matrices with real entries. We will now introduce the relevant concepts and review these results. The Beltrami-Klein mapping maps the complex plane into the closed unit disk through
\[
f(z) = \frac{(z - i) (\overline{z} - i)}{1 + \overline{z} z} .
\]

This mapping sends generalised circles centred on the real axis onto chords of the unit circle, as illustrated in Figure 1. We will also need to apply this function to linear operators, in which case it will be understood that
\[
f(T) = (I + T^*T)^{-\frac{1}{2}} (T^* - iI) (T - iI) (I + T^*T)^{-\frac{1}{2}} .
\]
Figure 1. Illustration of the Beltrami-Klein mapping. The generalised circles centred on the real axis in (a) are mapped by \( z' = f(z) \) to the chords of the unit circle in (b). The chords are sent back to their corresponding generalised circles by \( z = g(z') \).

Note that \( f(z) \) is not bijective, since \( f(z) = f(\bar{z}) \). However, for any \( z \) for which \( \text{Im}(z) \geq 0 \),

\[
    z = \frac{\text{Im}(f(z)) + i \sqrt{1 - |f(z)|^2}}{\text{Re}(f(z)) - 1}.
\]

This relation motivates the definition of

\[
    g(z) = \begin{cases} 
        \frac{\text{Im}(z) \pm i \sqrt{1 - |z|^2}}{\text{Re}(z) - 1}, & \text{if } \text{Im}(z) \geq 0 \\
        \frac{\text{Im}(z) \pm i \sqrt{1 - |z|^2}}{\text{Re}(z) + 1}, & \text{if } \text{Im}(z) < 0
    \end{cases}
\]
Figure 2. The hyperbolic straight line between two points \(z_1, z_2\) consists of two circular arcs under the Poincaré half-plane model.

This map sends each point in the closed unit disk back to the corresponding complex conjugate pair \((g(f(z)) = \{z, \overline{z}\})\). Since \(\text{SRG}(T) = \overline{\text{SRG}(T)}\), this establishes that

\[
\text{SRG}(T) = g(f(\text{SRG}(T))) = g(f(\sigma(T))).
\]

As we will see in the next section, \(f(\text{SRG}(T))\) is in many ways simpler to understand than \(\text{SRG}(T)\). Equation (2.2) then shows that we can always convert a result on \(f(\text{SRG}(T))\) back to a result on \(\text{SRG}(T)\) using \(g(\cdot)\).

This pattern of obtaining a simplified analysis of \(f(\text{SRG}(T))\) can also be seen in the main result of [7]. Introducing the notation \(\text{co}_{\text{Be-Kl}}(\cdot) = g(\text{co}(f(\cdot)))\), there it was shown that if \(T\) is a matrix with real entries (acting on a real Hilbert space) and \(T = T^*\), then \(f(\text{SRG}(T)) = \text{co}(f(\sigma(T)))\), or equivalently

\[
\text{SRG}(T) = \text{co}_{\text{Be-Kl}}(\sigma(T)).
\]

Furthermore the above remains true for \(TT^* = T^*T\) with the understanding that \(\sigma(T)\) denotes the spectrum of \(T\) when viewed on the corresponding complex Hilbert space (i.e. look at all the eigenvalues of \(T\) in \(\mathbb{C}\), not just those in the underlying field \(\mathbb{R}\) of the operator).

Remark 1. Given a set \(s \subseteq \mathbb{C}\), \(\text{co}_{\text{Be-Kl}}(s)\) behaves like the convex hull, however with the notion of a straight line taken from hyperbolic geometry under the Poincaré half-plane model. First recall that \(\text{co}(s)\) is equal to the set of all points that lie on a straight line between \(z_1, z_2 \in s\). In the Poincaré half-plane model (adapting things slightly for our needs), the straight line between two points \(z_1, z_2\) consists of the two arc segments of the generalised circle centred on the real axis that passes through the points \(\{z_1, z_2, \overline{z_1}, \overline{z_2}\}\) that:

1. connect a pair of points in \(\{z_1, z_2, \overline{z_1}, \overline{z_2}\}\);
2. do not intersect the real axis.

This is illustrated in Figure 2. The set \(\text{co}_{\text{Be-Kl}}(s)\) is then the set of all points that lie on a hyperbolic straight line under the Poincaré half-plane model between \(z_1, z_2 \in s\).
2.5. The numerical range. The numerical range is a classical object in the study of linear operators on complex Hilbert spaces. For a linear operator it is defined to be the subset of the complex plane

\[ W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \} \]

A nice introduction to the numerical range can be found in [6, 12]. The following facts about the numerical range of a linear operator are standard:

i) \( W(T) = \text{co}(W(T)) \) (\( W(T) \) is convex);

ii) if \( TT^* = T^*T \), then \( \text{cl} \ W(T) = \text{co}(\sigma(T)) \) (if \( T \) is normal, the closure of \( W(T) \) equals the convex hull of the spectrum of \( T \));

iii) \( \text{cl} \ W(T) \supseteq \sigma(T) \) (the closure of \( W(T) \) contains the spectrum of \( T \)).

More generally, the similarity invariance of the spectrum implies that the convex hull of the spectrum is also contained in \( \text{cl} \ W(STS^{-1}) \), and hence in the intersection of the sets \( \text{cl} \ W(STS^{-1}) \) for all choices of \( S \). An elegant result of Hildebrant [5] shows that this containment is tight, in the sense that

iv) \( \text{co}(\sigma(T)) = \bigcap \{ \text{cl} \ W(STS^{-1}) : S, S^{-1} \text{ are linear operators} \} \).

3. Results

3.1. Connection to the numerical range. In this subsection we connect the SRG to the numerical range. The following theorem shows that the SRG of a linear operator \( T \) on a complex Hilbert space can be obtained from the numerical range of \( f(T) \). Furthermore SRG \( (T) \) is endowed with convexity and spectral properties along the lines of i)–iv) from section 2.5, with two main differences.

(1) The notion of convexity is taken with respect to the Poincaré half-plane model, as explained in Remark 1 (i.e. replace \( \text{co}(\cdot) \) with \( \text{co}_{\text{Be-Kl}}(\cdot) \)).

(2) The spectral properties pertain to the approximate point spectrum instead of the spectrum (i.e. replace \( \sigma(T) \) with \( \sigma_{ap}(T) \)).

This gives the SRG a similar geometrical flavour to the numerical range, albeit with respect to a different geometry. The fact that the SRG lifts out features of the approximate point spectrum rather than the spectrum is curious, but of no consequence if \( T \) is finite dimensional or normal, since in these cases \( \sigma_{ap}(T) = \sigma(T) \). In general, as demonstrated by the set of equivalences in the theorem statement, analogues of iii)–iv) also hold for the spectrum if and only if \( \text{cl} \ SRG(T) \supseteq \text{cl} \ SRG(T^*) \).

**Theorem 1.** Given a linear operator \( T \) on a complex Hilbert space,

\[
\text{SRG}(T) = g(W(f(T))).
\]

In addition:

i) \( \text{SRG}(T) = \text{co}_{\text{Be-Kl}}(\text{SRG}(T)) \);

ii) if \( TT^* = T^*T \), then \( \text{cl} \ \text{SRG}(T) = \text{co}_{\text{Be-Kl}}(\sigma_{ap}(T)) \);

iii) \( \text{cl} \ \text{SRG}(T) \supseteq \sigma_{ap}(T) \);

iv) \( \bigcap \{ \text{cl} \ \text{SRG}(STS^{-1}) : S, S^{-1} \text{ are linear operators} \} = \text{co}_{\text{Be-Kl}}(\sigma_{ap}(T)) \).

Furthermore the following are equivalent:

v) \( \text{cl} \ \text{SRG}(T) \supseteq \sigma(T) \);

vi) \( \bigcap \{ \text{cl} \ \text{SRG}(STS^{-1}) : S, S^{-1} \text{ are linear operators} \} = \text{co}_{\text{Be-Kl}}(\sigma(T)) \);

vii) \( \text{cl} \ \text{SRG}(T) \supseteq \text{cl} \ \text{SRG}(T^*) \);

viii) \( \sigma_{ap}(T) \supseteq \sigma(T) \cap \mathbb{R} \).
The proof of this result is given at the end of the subsection after a series of examples.

Example 1 (The SRG in the matrix case). In this example we will illustrate Theorem 1 when the operator $T$ is a matrix with entries in $\mathbb{C}$, and draw some additional conclusions that apply in this case.

1. The SRG is a compact set ($\text{cl} \, \text{SRG} \, (T) = \text{SRG} \, (T)$). This follows directly from the compactness of the numerical range in the finite dimensional case.
2. The boundary of $\text{SRG} \, (T)$ is easily computed. This is because $f \, (T)$ can be computed using standard algorithms, and inner and outer approximations of the boundary of the numerical range can be computed to arbitrary precision by solving a sequence of eigenvalue problems [6]. This is illustrated in Figure 3(1) and Figure 4(1).
3. The SRG of $T$ is equal to the SRG of its adjoint. To see this, note that the approximate point spectra of $T$ and $T^*$ are equal to their spectra ($\sigma_{ap} \, (T) = \sigma \, (T)$ and $\sigma_{ap} \, (T^*) = \sigma \, (T^*)$). Therefore statement $\psi$ in Theorem 1 is true for both $T$ and $T^*$, implying that $\text{SRG} \, (T) = \text{SRG} \, (T^*)$.
4. $\text{SRG} \, (STS^{-1})$ can be made arbitrarily close to $\text{co}_{\text{Be-Kl}} \, (\sigma \, (T))$ using a single similarity transform. To see this, note that the Jordan decomposition of $T$ ensures that there exists an invertible matrix $Q$ such that

$$QTQ^{-1} = D + N,$$

where $D$ is a diagonal matrix consisting of the eigenvalues of $T$, and $N$ is a strictly upper triangular matrix ($N_{jk} = 0$ if $j \leq k$). Hence if $S_\gamma = \text{diag} \, (\gamma, \gamma^2, \ldots) \, Q$, where $\text{diag} \, (\gamma, \gamma^2, \ldots)$ denotes the diagonal matrix with entries $\gamma, \gamma^2, \ldots$, then

$$S_\gamma TS_\gamma^{-1} = D + \text{diag} \, (\gamma, \gamma^2, \ldots) \, N \text{diag} \, (\gamma, \gamma^2, \ldots)^{-1}.$$

Since

$$\lim_{\gamma \to \infty} \left\| \text{diag} \, (\gamma, \gamma^2, \ldots) \, N \text{diag} \, (\gamma, \gamma^2, \ldots)^{-1} \right\| = 0$$

and $\text{SRG} \, (D) = \text{co}_{\text{Be-Kl}} \, (\sigma \, (T))$, it follows that by making $\gamma$ sufficiently large the difference between $\text{SRG} \, (S_\gamma TS_\gamma^{-1})$ and $\text{co}_{\text{Be-Kl}} \, (\sigma \, (T))$ can be made arbitrarily small.

Example 2 (The SRG in the differential equation case). In this example we study the SRG of an operator defined by a differential equation. This example can be viewed as a generalisation of [2, Theorem 1]. In the following we will consider

$$\frac{d^p}{dt^p} y + \ldots + \alpha_{p-1} \frac{d}{dt} y + \alpha_p y = \beta_0 \frac{d^q}{dt^q} x + \ldots + \beta_{q-1} \frac{d}{dt} x + \beta_q x,$$

where $\alpha_j, \beta_k \in \mathbb{C}$ and $x, y \in L^2 (\mathbb{R})$, though the approach we describe works just as well when these coefficients are square matrices and $x, y$ are vectors of functions.
\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix} \mathbf{x}
\]

\[
y =
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix} \mathbf{x}
\]

(a) \( T : \mathbb{C}^4 \to \mathbb{C}^4, \sigma (T) = \{1, 1 + \sqrt{2}, 1 - \sqrt{2} \pm \sqrt{27}/16\} \).

(b) \( T : \mathcal{L}^2 (\mathbb{R}) \to \mathcal{L}^2 (\mathbb{R}), \sigma (T) = \left\{2 / (\omega + 1)^2 : \omega \in \mathbb{R} \cup \{\infty\}\right\} \).

**Figure 3.** Illustration of \text{cl SRG}(T) (the orange region), \text{co}_{\text{Be-Kl}}(\sigma(T)) (the grey region), and \(\sigma(T)\) (the black dots or thick black line) for two different operators.

in \(\mathcal{L}^2 (\mathbb{R})\). Note that in applications it might seem more natural to work on a real Hilbert space, where \(\alpha_j, \beta_k \in \mathbb{R}\), and \(\mathbf{x}, \mathbf{y}\) are real valued functions. In the next subsection it will be shown that from the perspective of the SRG this distinction is unimportant, and we may as well consider the case of complex Hilbert spaces.
Figure 4. The Beltrami-Klein mapping of the regions from Figure 3. In both cases \( f(\text{cl SRG}(T)) \) is convex, as guaranteed by Theorem 1i).

It is possible to associate a range of different operators \( x \mapsto y \) with (3.2) depending on the time interval or the boundary conditions that are being studied. A perspective that has been particularly profitable both in theory and in practice has been to associate (3.2) with a linear operator \( T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) defined through a multiplication operator \( T_h : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) in the frequency domain. In this setting, denoting the Fourier transform as \( F : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \), we have

\[
T_h \hat{x}(\omega) = h(\omega) \hat{x}(\omega),
\]

and

\[
h(\omega) = \frac{\beta_0 (i\omega)^q + \ldots \beta_{q-1}i\omega + \beta_q}{(i\omega)^p + \ldots \alpha_{p-1}i\omega + \alpha_p}.
\]

The function \( h(\omega) \) is often referred to as a multiplier or transfer function. We will now show how to determine \( \text{cl SRG}(T) \). The first thing to note is that both the SRG and the numerical range are unitarily invariant. That is given any linear operator \( U \) such that \( UU^* = U^*U = I \), \( \text{SRG}(U^*TU) = \text{SRG}(T) \) and \( W(U^*TU) = W(T) \). Therefore

\[
(3.3) \quad \text{SRG}(T) = \text{SRG}(T_h) = g(W(S^* (T_h^* - iI)(T_h - iI)S)),
\]

where \( S \) is any invertible linear operator such that \( SS^* = (I + T_h^*T_h)^{-1} \). The first equality follows from the properties of the Fourier transform, and to see the second, observe that

\[
S^{-1} (I + T_h^*T_h)^{-\frac{1}{2}}
\]

is unitary, and compare (3.3) with the definition of \( f(\cdot) \) from section 2.4. A suitable \( S \) can then be obtained by applying factorisation techniques for rational functions. More specifically, the process of spectral factorisation can be used to find a bounded
rational function \( s : \mathbb{R} \to \mathbb{R} \) such that for all \( \omega \in \mathbb{R} \),
\[
\frac{1}{h(\omega) h(\omega) + 1} = s(\omega) s(\omega).
\]
Such a factorisation is always possible, and can be obtained directly from \( h(\omega) \) using a normalised coprime factorisation \([13]\). For example, if \( h(\omega) = 2/(i\omega + 1)^2 \) (as in Figure 3(a)), then a suitable \( s(\omega) \) is given by
\[
s(\omega) = \frac{(i\omega + 1)^2}{(i\omega)^2 + 2\sqrt{2} i\omega + \sqrt{5}}.
\]
The multiplication operator
\[
T_s \hat{\mathbf{v}}(\omega) = s(\omega) \hat{\mathbf{v}}(\omega)
\]
then satisfies \( T_s T_s^* = (I + T_s^* T_s)^{-1} \), and therefore
\[
\text{cl SRG}(T) = g\left( \left\{ \text{W} \left( s(\omega) \left( \frac{1}{h(\omega)} - i \right) (h(\omega) - i) s(\omega) \right) : \omega \in \mathbb{R} \cup \{ \infty \} \right\} \right).
\]
This is illustrated in Figure 3(a) and Figure 4(a). The above process is easily generalised to the case that \( \alpha_j, \beta_k \) are square matrices \( (h(\omega)) \) becomes a matrix of rational functions, and \( s(\omega) \) can be obtained through the process of normalised right coprime factorisation. Note that in this setting \( T \) is not guaranteed to be normal, and so unlike in the case of scalar coefficients \( \text{cl SRG}(T) \) is not necessarily equal to \( \text{co}_{\text{Be-K}}(\sigma(T)) \).

Example 3 (The SRG of the right shift operator). We have now seen two examples of operators for which the statements \( v)–viii) \) in Theorem 1 were true, and the SRG gave information on both the approximate point spectrum and the spectrum. We will now study the SRG of an operator for which this is not the case. To this end, consider the right shift operator \( T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) given by
\[
T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).
\]
The adjoint of \( T \) is the left shift operator \( (x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots) \). It is possible to compute \( \text{SRG}(T) \) and \( \text{SRG}(T^*) \) directly. The steps for \( T \) are particularly simple since \( T^* T = I \), from which it follows that
\[
f(\text{SRG}(T)) = \frac{1}{2i} \text{W}(T + T^*) = \{ z : z \in \mathbb{C}, \text{Re}(z) = 0, |z| < 1 \}.
\]
Applying the function \( g(\cdot) \) from section 2.4 then shows that
\[
\text{SRG}(T) = \{ z : z \in \mathbb{C} : |z| = 1, \text{Re}(z) \neq 0 \}\.
\]
A similar but slightly more involved calculation shows that
\[
\text{SRG}(T^*) = \{ z : z \in \mathbb{C} : |z| < 1, \text{Re}(z) \neq 0 \}.
\]
We therefore see that \( \text{cl SRG}(T^*) \supseteq \text{cl SRG}(T) \). It then follows that the statements \( v)–viii) \) in Theorem 1 are false for \( T \), but true for \( T^* \). This means for example that \( \text{cl SRG}(T) \not\supseteq \text{co}_{\text{Be-K}}(\sigma(T)) \), but \( \text{cl SRG}(T^*) \supseteq \text{co}_{\text{Be-K}}(\sigma(T)) \). This is easily confirmed directly (in fact \( \sigma_{\text{ap}}(T) \) is the unit circle and \( \sigma(T) \) is the closed unit disk, meaning that \( \text{cl SRG}(T) = \sigma_{\text{ap}}(T) \) and \( \text{cl SRG}(T^*) = \sigma(T) \)).

We now give the proof of Theorem 1.
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\[ p_1 \quad p_2 \quad p_3 \alpha \quad d_{s}(\alpha,s) \quad d_{l}(\alpha,s) \]

\[ (\lambda) \quad (\nu) \]

Figure 5. Characterisation of the points \( \gamma \in \text{co}_{\text{Be-Kl}}(s) \), for an example with \( s = \{p_1, p_2, p_3\} \).

Proof. We start by establishing (3.1). First note that considering the polar representation of a complex number \( z = r \exp(i\theta) \) shows that

\[ f(r \exp(i\theta)) = \frac{r^2 - 1 - 2ir \cos \theta}{1 + r^2}. \]

In light of our discussion from section 2.4 (c.f. (2.1)), we then see that for any \((x, y) \in \text{gra} T\),

\[
(3.4) \quad f \left( \frac{\|y\|}{\|x\|} \exp \left( \pm i \arccos \left( \frac{\text{Re} \left( \langle y, x \rangle \right)}{\|y\| \|x\|} \right) \right) \right) = \frac{\|y\|^2 - 1 - 2i \text{Re}(\langle y, x \rangle)}{1 + \|y\|^2},
\]

\[ = \frac{\|y\|^2 - \|x\|^2 - i (\langle y, x \rangle + \langle x, y \rangle)}{\|x\|^2 + \|y\|^2}, \]

where \( R(x, y) = (-i y - x, y - ix) \). Consider now the linear map

\[ Uv = \left( (I + T^*T)^{-\frac{1}{2}} v, T (I + T^*T)^{-\frac{1}{2}} v \right). \]

It is easily checked that for all \( v \in \mathcal{H} \), \( \langle Uv, Uv \rangle = \langle v, v \rangle \), \( \text{gra} T = \{ Uv : v \in \mathcal{H} \} \), and

\[ \langle RUv, Uv \rangle = \langle f(T) v, v \rangle. \]

Therefore \( f(\text{SRG}(T)) = W(f(T)) \), which shows (3.1). Point \( i) \) is then immediate from the Toeplitz-Hausdorff theorem.

We will now show \( ii) – iv) \). First denote the shortest and longest distances from a point \( \gamma \in \mathbb{C} \) to a set \( s \subseteq \mathbb{C} \) as

\[ d_s(\gamma, s) = \inf \{|z - \gamma| : z \in s\} \text{ and } d_l(\gamma, s) = \sup \{|z - \gamma| : z \in s\} \]
respectively. We will start by showing that given any \( s \subseteq \mathbb{C} \),
\[
\text{cl} \, \text{co}_{\text{Be-Kl}} (s) = \bigcap_{\alpha \in \mathbb{R}} \{ z : z \in \mathbb{C}, d_s (\alpha, s) \leq |z - \alpha| \leq d_1 (\alpha, s) \}.
\]
To see this, observe that for any value of \( \alpha \in \mathbb{R} \), the inequalities in (3.5) characterise the points that lie outside a circle centred on \( \alpha \) with radius \( d_s (\alpha, s) \) and lie inside a circle centred on \( \alpha \) with radius \( d_1 (\alpha, s) \). This is illustrated in Figure 5(a), and the region in question corresponds to the orange annulus. Figure 5(a) shows the Beltrami-Klein mapping of these regions. Since the Beltrami-Klein mapping bijectively maps circles centred on the real axis to chords of the unit circle, and \( f (\text{co}_{\text{Be-Kl}} (s)) \) is convex, this annulus contains \( \text{cl} \, \text{co}_{\text{Be-Kl}} (s) \). Conversely every supporting hyperplane for the set \( f (\text{co}_{\text{Be-Kl}} (s)) \) corresponds to a circle centred on some value of \( \alpha \in \mathbb{R} \), and so the intersection of these regions gives \( \text{cl} \, \text{co}_{\text{Be-Kl}} (s) \).

Next note that \( \text{SRG} \left( T - \alpha I \right) = \text{SRG} \left( T \right) - \alpha \). It then follows from the definition of the SRG that
\[
\begin{align*}
\text{d}_s (\alpha, \text{SRG} (T)) &= m (T - \alpha I), \\
\text{d}_1 (\alpha, \text{SRG} (T)) &= \|T - \alpha I\|,
\end{align*}
\]
where in the first equation we have introduced the notation
\[
m (A) = \inf \{ \|Ax\| : x \in \mathcal{H}, \|x\| = 1 \}.
\]

It is then easily shown that
\[
\begin{align*}
m (T - \alpha I) &\leq \text{d}_s (\alpha, \sigma_{\text{ap}} (T)), \\
\|T - \alpha I\| &\geq \text{d}_1 (\alpha, \sigma_{\text{ap}} (T)).
\end{align*}
\]
The second of these inequalities is most usually stated in terms of the spectral radius (i.e. replace \( \sigma_{\text{ap}} (\cdot) \) with \( \sigma (\cdot) \)). However, as shown in [4, Problem 63], \( \partial \sigma (T) \subseteq \sigma_{\text{ap}} (T) \) and so this substitution incurs no loss. We therefore see that
\[
\begin{align*}
\text{d}_s (\alpha, \text{SRG} (T)) &\leq \text{d}_s (\alpha, \sigma_{\text{ap}} (T)) \quad \text{and} \quad \text{d}_1 (\alpha, \text{SRG} (T)) \geq \text{d}_1 (\alpha, \sigma_{\text{ap}} (T)).
\end{align*}
\]
When combined with (3.5) this shows that \( \text{co}_{\text{Be-Kl}} (\sigma_{\text{ap}} (T)) \subseteq \text{cl} \, \text{SRG} (T) \) (the approximate point spectrum is always a closed set), which shows iii). This claim can be strengthened to an equality whenever (3.8) and (3.9) are equalities for all \( \alpha \in \mathbb{R} \). This is the case if \( TT^* = T^*T \), which shows ii). To show iv) we are required to show that if \( \gamma \notin \text{co}_{\text{Be-Kl}} (\sigma_{\text{ap}} (T)) \), then there there exists an invertible linear operator \( S \) such that \( \gamma \notin \text{cl} \, \text{SRG} (STS^{-1}) \). In light of eqs. (3.6) to (3.9) this is equivalent to showing that given any \( \epsilon > 0 \), there exists an invertible linear operator \( S_1 \) such that
\[
\begin{align*}
m (S_1 (T - \alpha I) S_1^{-1}) &> \text{d}_s (\alpha, \sigma_{\text{ap}} (T)) - \epsilon \\
\text{and there exists an invertible linear operator} \ S_2 \quad \text{such that}
\end{align*}
\]
\[
\begin{align*}
\|S_2 (T - \alpha I) S_2^{-1}\| &< \text{d}_1 (\alpha, \sigma_{\text{ap}} (T)) + \epsilon.
\end{align*}
\]
In fact (3.11) is a well known consequence of Rota’s theorem [10], so we will only show (3.10). By [8, Theorem 1],
\[
\lim_{n \to \infty} m ((T - \alpha I)^n)^{\frac{1}{n}} = \text{d}_s (\alpha, \sigma_{\text{ap}} (T)).
\]
Therefore there exists a natural number \( n \) such that
\[
m ((T - \alpha I)^n)^{\frac{1}{n}} > \text{d}_s (\alpha, \sigma_{\text{ap}} (T)) - \epsilon.$
Now let

\[ A = \frac{1}{d_n(\alpha, \sigma_{\text{ap}}(T)) - \varepsilon}(T - \alpha I), \]

and note that \( m(A^n) > 1 \). Defining \( X = I + A^*A + \ldots + (A^n)^{-1}A^{n-1} \) we then see that for any non-zero \( x \in \mathcal{H} \),

\[ \langle (A^*X - X)x, x \rangle = \langle ((A^n)^*A^n - I)x, x \rangle \geq \left(m(A^n)^2 - 1\right)\|x\|^2 > 0. \]

Furthermore, since \( \langle Xx, x \rangle \geq \|x\|^2 \), there exists an invertible linear operator \( S_1 \) such that \( X = S_1^*S_1 \). Putting \( S_1x = y \) we now see that

\[ \frac{\langle (A^*X - X)x, x \rangle}{\langle S_1x, S_1x \rangle} = \frac{\langle S_1AS_1^{-1}y, S_1AS_1^{-1}y \rangle}{\|y\|^2} - 1 > 0. \]

Therefore \( m(S_1AS_1^{-1}) > 1 \), and so \((3.10)\) holds.

To complete the proof we focus on the equivalence of \( v \)-\( viii \).

\( vii \) \( \Rightarrow \) \( vi \): First note that \( \sigma(T) \subseteq \sigma_{\text{ap}}(T) \cup \sigma_{\text{ap}}(T^*) \). Since by definition \( \text{SRG}(T) = \overline{\text{SRG}(T)} \), this shows that \( \sigma(T) \subseteq \text{cl \ SRG}(T) \cup \text{cl \ SRG}(T^*) \), and so by the hypothesis of \( vii \) \( \sigma(T) \subseteq \text{cl \ SRG}(T) \).

\( v \) \( \Rightarrow \) \( viii \): We proceed by contraposition. Assume that \( \alpha \in \sigma(T) \cap \mathbb{R} \) is not in \( \sigma_{\text{ap}}(T) \), and so \( m(T - \alpha I) > 0 \). From the definition of the SRG, this implies that \( 0 \notin \text{cl \ SRG}(T - \alpha I) \). Hence \( \alpha \notin \sigma(T) \), and so \( \sigma(T) \notin \text{cl \ SRG}(T) \) as required.

\( viii \) \( \Rightarrow \) \( vii \): First note that \( \|T - \alpha I\| = \|T^* - \alpha I\| \), and if \( \alpha \notin \sigma(T) \), then

\[ m(T - \alpha I) = 1/\|T - \alpha I\|^{-1} = 1/\|T^* - \alpha I\|^{-1} \].

Consider again \((3.6)\) and \((3.7)\). Observe in particular that given any \( \alpha \in \mathbb{R} \), under the hypothesis of \( viii \) \( d_n(\alpha, \sigma_{\text{SRG}}(T)) \neq 0 \) only if \( \alpha \notin \sigma(T) \). We therefore see from \((3.5)\) that \( \gamma \notin \text{cl \ SRG}(T) \) only if \( \gamma \notin \text{cl \ SRG}(T^*) \) as required.

\( viii \) \( \Rightarrow \) \( vi \): Recall that \( \partial \sigma(T) \subseteq \sigma_{\text{ap}}(T) \). Therefore under the hypothesis of \( viii \), if \( \alpha \in \mathbb{R} \), then \( d_n(\alpha, \sigma_{\text{ap}}(T)) = d_n(\alpha, \sigma(T)) \), and so \( \text{co}_{\text{Be-K}}(\sigma(T)) = \text{co}_{\text{Be-K}}(\sigma_{\text{ap}}(T)) \). \( vi \) now follows from \( iv \).

\( vi \) \( \Rightarrow \) \( vii \): Immediate.

\[ \square \]

3.2. Real Hilbert spaces. In the previous subsection we showed that for a linear operator acting on a complex Hilbert space, the concept of the SRG is closely related to the numerical range. However, largely motivated by applications from convex optimization, the SRG has primarily been studied in the context of Hilbert spaces over \( \mathbb{R} \). At first sight, it might seem like there are fundamental differences between the real and complex case. For example when viewed as an operator on a real Hilbert space with inner product \( \langle y, x \rangle = x^Ty \),

\[ (3.12) \quad f \left( \text{SRG} \left[ \frac{0}{0} \right] \right) = \left\{ z : \left| z + \frac{1+i}{2} \right| + \left| z + \frac{1-i}{2} \right| = \sqrt{2}, z \in \mathbb{C} \right\}. \]

This is not a convex set (it is the boundary of an ellipse), and therefore Theorem 1 \( i \) fails. However when we view the same matrix as an operator on \( \mathbb{C}^2 \) with inner product \( \langle y, x \rangle = \overline{x}^Ty \) we obtain

\[ f \left( \text{SRG} \left[ \frac{0}{0} \right] \right) = \left\{ z : \left| z + \frac{1+i}{2} \right| + \left| z + \frac{1-i}{2} \right| \leq \sqrt{2}, z \in \mathbb{C} \right\}. \]
Figure 6. (a) shows SRG ($T$) (the black circles) and SRG ($T_C$) (the orange region) for the matrix in (3.12). (b) shows the Beltrami-Klein mapping of these regions.

That is the SRG of the operator on the real Hilbert space is equal to the boundary of the SRG of its complexified counterpart, suggesting the two objects are in fact closely related. This is illustrated in Figure 6.

A similar behaviour is seen when studying tuples of Hermitian forms (of which the numerical range is a special case). More specifically, given two $n \times n$ symmetric matrices $A$ and $B$ with real entries, it was shown in [1] that

$$\{ x^T A x + i x^T B x : x^T x = 1, x \in \mathbb{R}^n \} = \begin{cases} \partial W(A + iB) & \text{if } n = 2; \\ W(A + iB) & \text{otherwise.} \end{cases}$$

This result relates the joint numerical range $\{ (\langle Ax, x \rangle, \langle Bx, x \rangle) : x \in \mathcal{H}, \|x\| = 1 \}$ of two operators on a finite dimensional real Hilbert space, to the numerical range of a related operator acting on a finite dimensional complex Hilbert space. Moreover, it shows that the two are different only if the Hilbert space has dimension 2, where instead the real case equals the boundary of the complex case. The main result of this subsection is an adaptation of the above that shows that the SRG behaves in an analogous manner. Before stating the result, let us first formalise the notion of complexification beyond the matrix case. The following, which can be found in [3, Chapter I], gives the suitable notion of the complexification of a Hilbert space.

**Lemma 1.** Let $\mathcal{H}$ be a real Hilbert space. Then there exists a complex Hilbert space $\mathcal{H}_C$ and a linear map $U : \mathcal{H} \rightarrow \mathcal{H}_C$ such that:

i) $\langle Ux_1, Ux_2 \rangle = \langle x_1, x_2 \rangle$ for all $x_1, x_2 \in \mathcal{H}$;

ii) for any $y \in \mathcal{H}_C$, there are unique $x_1, x_2 \in \mathcal{H}$ such that $y = Ux_1 + iUx_2$.

Given an operator $T$ on a real Hilbert space $\mathcal{H}$, we define the complexification of $T$ to be the operator $T_C$ on $\mathcal{H}_C$ which satisfies

$$T_C(Ux_1 + iUx_2) = UTx_1 + iUTx_2, \text{ for all } x_1, x_2 \in \mathcal{H}.$$
It is easy enough to check that these abstractions behave exactly as expected in the matrix case (and also in going from operators on real valued square integrable functions to \( L^2(\mathbb{R}) \)). With this definition in place we are ready to state the main result of this subsection. The following theorem shows that in all dimensions except 2 (including the infinite dimensional case), \( \text{SRG}(T) = \text{SRG}(T_C) \). Furthermore in dimension 2, \( \text{SRG}(T) \) is equal to the boundary of \( \text{SRG}(T_C) \). This means that Figures 3 and 4 also show the SRGs of the corresponding operators when viewed on a real Hilbert space, and in all cases, \( \text{SRG}(T) \) can be obtained from the numerical range of an operator on a complex Hilbert space, as described in the previous subsection.

**Theorem 2.** Let \( T \) be a linear operator on a real Hilbert space. Then

\[
\text{SRG}(T) = \begin{cases} 
\partial \text{SRG}(T_C) & \text{if } T \text{ has dimension } 2; \\
\text{SRG}(T_C) & \text{otherwise.}
\end{cases}
\]

**Proof.** Let us first slightly rework the characterisation of \( \text{SRG}(T) \) from Theorem 1 to make it suitable for operators on real Hilbert spaces. The issue is that as written, \( f(T) : \mathcal{H} \to \mathcal{H}_C \), and so we cannot define its numerical range. However the problem is only superficial, and starting from (3.4) it is easily shown that

\[
f(\text{SRG}(T)) = \{ \langle Ax, x \rangle + i \langle Bx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \},
\]

where

\[
A = (I + T^*T)^{-\frac{1}{2}}(T^*T - I) (I + T^*T)^{-\frac{1}{2}} \text{ and } B = -(I + T^*T)^{-\frac{1}{2}}(T + T^*)(I + T^*T)^{-\frac{1}{2}}.
\]

Similarly

\[
f(\text{SRG}(T_C)) = \{ \langle A_C y, y \rangle + i \langle B_C y, y \rangle : y \in \mathcal{H}_C, \|y\| = 1 \}.
\]

Direct calculation shows that for any \( y = Ux_1 + iUx_2 \in \mathcal{H}_C \),

\[
\langle A_C y, y \rangle = \langle UAx_1 + iUAx_2, Ux_1 + iUx_2 \rangle,
\]

\[
= \langle UAx_1, Ux_1 \rangle + \langle UAx_2, Ux_2 \rangle + i \langle \langle UAx_2, Ux_1 \rangle - \langle UAx_1, Ux_2 \rangle \rangle,
\]

\[
= \langle Ax_1, x_1 \rangle + \langle Ax_2, x_2 \rangle + i \langle \langle Ax_2, x_1 \rangle - \langle Ax_1, x_2 \rangle \rangle.
\]

Since \( A = A^* \) and \( \mathcal{H} \) is over \( \mathbb{R} \), \( \langle Ax_2, x_1 \rangle = \langle Ax_1, x_2 \rangle \), and so the imaginary part in the above equals zero. Using a similar argument for \( \langle B_C y, y \rangle \) therefore shows that

\[
\langle A_C y, y \rangle + i \langle B_C y, y \rangle = \langle Ax_1, x_1 \rangle + i \langle Bx_1, x_1 \rangle + \langle Ax_2, x_2 \rangle + i \langle Bx_2, x_2 \rangle
\]

\[
= p_1 \|x_1\|^2 + p_2 \|x_2\|^2,
\]

where \( p_1, p_2 \in f(\text{SRG}(T)) \). Noting that \( \|y\|^2 = \|x_1\|^2 + \|x_2\|^2 \), this implies that \( f(\text{SRG}(T)) \subseteq f(\text{SRG}(T_C)) \subseteq \text{co}(f(\text{SRG}(T))) \). By [9, Theorem 2], the joint numerical range of any two Hermitian forms on a real Hilbert space is convex unless that Hilbert space has dimension 2. Therefore \( f(\text{SRG}(T)) \) is convex unless \( T \) has dimension 2, which establishes the second case in the theorem statement. For the two dimensional case, as noted in [1], the set

\[
\{ \langle Ax, x \rangle + i \langle Bx, x \rangle : \|x\| = 1, x \in \mathcal{H} \}
\]

can only be an ellipse, circle, line or point. Since these shapes all have convex boundaries, this then implies that \( f(\text{SRG}(T)) = f(\partial \text{SRG}(T_C)) \) as required. \( \square \)
4. Conclusions

We have demonstrated that the SRG of a linear operator acting on complex Hilbert space can be determined from the numerical range of a closely related linear operator. This was used to show that Beltrami-Klein mapping of the SRG is convex, and derive an analogue of Hildebrandt’s theorem for the SRG. It was further shown how to re-purpose algorithms developed for the numerical range to plot the boundary of the SRG in the matrix and linear differential equation case. Finally these results were extended to operators on real Hilbert spaces, where it was shown that the SRG could be obtained using the results for complex Hilbert spaces through the process of complexification.

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