Integrable Hamiltonian equations of fifth order with the Hamiltonian operator $D_x$

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ABSTRACT. All non-equivalent integrable evolution equations of the fifth order of the form $u_t = D_x \frac{\delta H}{\delta u}$ are found.

1. Introduction.

Consider integrable Hamiltonian evolution equations of the form

$$u_t = D_x \left( \frac{\delta H}{\delta u} \right). \quad (1.1)$$

Here $H(x, u, u_x, ...)$ is the Hamiltonian and $D_x$ is the total $x$-derivative. By integrability we mean the existence of infinite hierarchy of commuting flows (or the same higher symmetries) for (1.1). The celebrated KdV equation with $H = -\frac{1}{2} u_x^2 + u^3$ provides the simplest example of such an equation.

In [1] integrable equations (1.1) of the third order have been investigated. Such equations have the form

$$u_t = D_x \left( \frac{\partial H}{\partial u} - D_x \frac{\partial H}{\partial u_x} \right), \quad (1.2)$$

where $H = H(x, u, u_x)$. It follows from the simplest necessary integrability condition [2, 3] that there exist two different kinds of integrable Hamiltonians:

$$H = a_1 u_x^2 + a_2 u_x + a_3 \quad \text{and} \quad H = k_1 u_x + k_2 + \sqrt{k_3 u_x^2 + k_4 u_x} + k_5,$$

where $a_i$ and $k_i$ are some functions of $u$ and $x$. Since the function $H$ is defined up to the equivalence $H \to H + D_x f(x, u) + \lambda u$, we may put without loss of generality $a_2 = k_1 = 0$.

The following statement lists all integrable Hamiltonians up to canonical transformations (see Section 2).

**Theorem 1.** Suppose that non-linear equation (1.2) has infinite hierarchy of higher symmetries

$$u_{\tau_k} = F_k(x, u, u_x, ...), \quad k = 1, 2, \ldots.$$
Then the Hamiltonian $H$ is canonically equivalent to one of the following functions:

$$H = -\frac{u_x^2}{2Q(u)^3} + \frac{P(u)}{Q(u)}, \quad (1.3)$$

$$H = -\frac{u_x^2}{2u^3} + \frac{1}{3}P(x)u^3, \quad (1.4)$$

$$H = \sqrt{u_x + P(u)}. \quad (1.5)$$

Here $P$ and $Q$ are arbitrary polynomials of degrees not greater than 4, and 2, correspondingly.

In this paper we consider the fifth order integrable Hamiltonian evolution equations of the form

$$u_t = D_x \left( \frac{\delta H(x, u, u_x, u_{xx})}{\delta u} \right) = D_x \left( \frac{\partial H}{\partial u} - D_x \frac{\partial H}{\partial u_x} + D^2_x \frac{\partial H}{\partial u_{xx}} \right). \quad (1.6)$$

It turns out that there exist two types of integrable Hamiltonians:

$$H = a_1u_{xx}^2 + a_2u_{xx} + a_3 \quad \text{and} \quad H = k_1u_{xx} + k_2 + \sqrt[3]{u_{xx} + k_3},$$

where $a_i$ and $k_i$ are some functions of $u, u_x$ and $x$.

Using the symmetry approach to integrability [2,3], we obtain a complete list of canonical forms for integrable Hamiltonians $H$. Our proof of the classification statement contains an algorithm which allows to bring any integrable Hamiltonian to one of the canonical forms by canonical transformations.

The list contains Hamiltonians for all fifth order commuting flows of equations from Theorem 1. For example, the Hamiltonian

$$H_1 = \frac{1}{2}u_{xx}^2 - 5uu_x^2 + \frac{5}{2}u^4$$

generates the fifth order equation from the KdV hierarchy. Besides these Hamiltonians, we found the following integrable Hamiltonians of second order:

$$H = \sqrt[3]{u_{xx} + (5u^2 + 9k)u_x + 2u(u^2 + k)(u^2 + 9k)}, \quad (1.7)$$

$$H = \sqrt[3]{u_{xx} + \frac{1}{3}c_0u^3 + \frac{1}{2}(c_1x + c_2)u^2 + (c_3x^2 + c_4x)u}, \quad (1.8)$$

$$H = \frac{1}{2}u_{xx}^2 - \frac{5}{6}u_x^3 + \frac{5}{2}u^2u_x + \frac{1}{6}u^6, \quad (1.9)$$

$$H = \frac{1}{2}u_{xx}^2 - \frac{20}{3}u_x^4 + \frac{5}{3}u^6 + \frac{20}{3}u^2(ku + 2)^2 + \frac{20}{3}k^3 + 10\frac{k^2}{u} + 8\frac{k}{u^3} + 8\frac{1}{3u^4}. \quad (1.10)$$
The constant $k$ can be reduced to $k = 0$ or to $k = 1$ by a scaling. The constants $c_i$ are also not essential (see Remark 4). Notice that the equation
\[ u_\tau = D_x \frac{\delta}{\delta u} \left( \frac{1}{2} (c_1 x + c_2) u^2 + (c_3 x^2 + c_4 x) u \right) \]
is the classical symmetry for the Hamiltonian equation with $H$ given by (1.8).

It follows from [4, 5] that if the right hand side of integrable Hamiltonian equation
\[ u_t = u_n + F(x, u, u_x, u_{xx}, \ldots, u_{n-1}), \quad u_i = \frac{\partial^i u}{\partial x^i}. \tag{1.11} \]
is polynomial and homogeneous, then the hierarchy of this equation contains an equation of third or fifth order. This statement looks very credible also without any additional restrictions for the right hand side of the equation. The proof in the general case is absent and this statement has a status of the conjecture well-known for experts. No counterexamples to this conjecture are known.

Up to this conjecture, in this paper taken together with [1] we have described simplest equations for all hierarchies of the integrable Hamiltonian equations of the form (1.1). In other words, any integrable equation (1.1) is equivalent to a generalized symmetry of one of the equations presented in our papers.

2. Preliminaries.

2.1. Canonical transformations. It is easy to verify that the equation (1.6) is stable under transformations $H \to H + D_x f(x, u, u_x) + \lambda u$, where the function $f$ and the constant $\lambda$ are arbitrary.

Some transformations preserve the Hamiltonian form (1.6) changing the right hand side of equation. We call such transformations canonical.

Consider point transformations of the form
\[ x = \varphi(y, v), \quad u = \psi(y, v). \tag{2.1} \]
The invertibility of the transformation is equivalent to the inequality $\Delta = \varphi_y \psi_v - \varphi_v \psi_y \neq 0$. Transformation (2.1) is called canonical if $\Delta = \varphi_y \psi_v - \varphi_v \psi_y = 1$. It is easy to verify that canonical transformations preserve the form of equation (1.6). The Hamiltonian of the resulting equation is given by
\[ \tilde{H}(y, v, v_y) = H \left( \varphi(y, v), \psi(y, v), \frac{D_v(\psi)}{D_y(\varphi)} \right) D_y(\varphi). \tag{2.2} \]

Example. Linear transformations of the form
\[ x = f(y), \quad u = \frac{v}{f'} + g(y), \quad \tilde{H} = H f'. \]
are canonical for arbitrary functions $f$ and $g$.

**Remark 1.** If we consider only Hamiltonians that do not depend on $x$ explicitly, we still have non-trivial canonical transformations

$$x = f(v) + y, \quad u = v, \quad \tilde{H} = (f'v_y + 1)H.$$  

$$x = v, \quad u = f(v) + y, \quad \tilde{H} = Hv_y, \quad \square$$

Besides (2.1) we use the following canonical transformations of a more general form:

1. Dilatations of the form

$$t = \alpha\tilde{t}, \quad x = \beta y, \quad u = \gamma v, \quad \tilde{H} = \frac{\alpha}{\beta\gamma^2}H(\beta y, \gamma v)$$

are admissible for any $H$.

2. If $H$ does not depend on $x$, then the Galilean transformation

$$y = x + ct, \quad v = u, \quad \tilde{H} = H - \frac{1}{2}cv^2;$$

is admissible.

3. If $H = h(u_x) + cxu$, where $c$ is a constant, then the following transformation

$$u \to u + ct, \quad H \to H - cxu$$

is admissible.

4. If $H = h(u_{xx}) + (c_1x^2 + c_2x)u$, where $c_1$ and $c_2$ are some constants, then the following transformation

$$u \to u + 2c_1xt + c_2t, \quad H \to H - (c_1x^2 + c_2x)u$$

is admissible.

**Remark 2.** Sometimes we use the point transformations (2.1) without the condition $\Delta = 1$. Using the properties of the Euler operators that are presented in [6] one can find the form of the transformed equation (1.6):

$$v_t = f D_y \left( f \frac{\delta\tilde{H}}{\delta v} \right), \quad (2.3)$$

where $f = \Delta^{-1} = f(y, v)$, $\tilde{H} = H(D_y\varphi)$. One can easily verify the operator $f \circ D_y \circ f$ is the Hamiltonian one for any $f$.

2.2. **Canonical densities.** The necessary integrability conditions for equations (1.11) have the form of an infinite chain of the canonical conservation laws

$$\frac{d}{dt}(\rho_n) = \frac{d}{dx}(\theta_n), \quad n = -1, 0, 1, \ldots,$$  

(2.4)
where the functions $\rho_i$ are defined by a recurrent formula. Following [7], we find that in the case of fifth order equations

$$u_t = F(x, u, u_x, u_2, \ldots, u_5)$$  \hspace{1cm} (2.5)$$

the recurrence can be written as follows

$$\rho_{n+4} = \frac{1}{5} \rho_{n+1} \theta_n - \frac{1}{5} \rho_n \left( F_0 \delta_{n,0} + F_1 \rho_n + F_2 D_x(\rho_n) + F_2 \sum_{i=1}^{n} \rho_i \rho_j + F_3 D_x^2(\rho_n) \right)$$

$$- \frac{1}{5} \rho_{n+1} F_3 \left( \frac{3}{5} D_x \sum_{i=1}^{n} \rho_i \rho_j \right) - \frac{1}{5} \rho_n F_4 \left( D_x^3(\rho_n) + 2 D_x \sum_{i=1}^{n} \rho_i \rho_j \right)$$

$$- \frac{1}{5} \rho_{n+1} F_4 \left( - \sum_{i=1}^{n} D_x(\rho_i) D_x(\rho_j) + 2 D_x \sum_{i=1}^{n} \rho_i \rho_j \rho_k \right)$$

$$- (\rho_{n+1})^{-4} \left( \frac{1}{5} D_x^4(\rho_n) + \frac{1}{2} D_x^3 \sum_{i=1}^{n} \rho_i \rho_j - \frac{1}{2} D_x \sum_{i=1}^{n} D_x(\rho_i) D_x(\rho_j) + \frac{2}{3} D_x^2 \sum_{i=1}^{n} \rho_i \rho_j \rho_k \right.$$  \hspace{1cm} (2.6)$$

$$\left. - \sum_{i=1}^{n} \rho_i D_x(\rho_j) D_x(\rho_k) + \frac{1}{2} D_x \sum_{i=1}^{n} \rho_i \rho_j \rho_k \rho_l + \frac{1}{5} \sum_{i=1}^{n} \rho_i \rho_j \rho_k \rho_l \rho_m - \rho_{n+4} \right) ,$$

where $n = -4, -3, -2, \ldots$, $F_n = \partial F / \partial u_n$ and

$$\rho_{n+1} = F_5^{-1/5} .$$

By definition,

$$\sum_{a}^{b} \rho_{l_1} \cdots \rho_{l_k} = \sum_{l_1 + \cdots + l_k = b} \rho_{l_1} \cdots \rho_{l_k} .$$

In particular, we have

$$\sum_{i=1}^{n} \rho_i \rho_j = \rho_{n+1}^2, \quad \sum_{i=1}^{n} \rho_i \rho_j = 2 \rho_{n+1} \rho_0, \quad \sum_{i=1}^{n} \rho_i \rho_j \rho_k = 0 ,$$

$$\sum_{i=1}^{n} \rho_i \rho_j \rho_k = 3 \rho_{n+2}^2 + 6 \rho_{n+1} \rho_{n+1} + \ldots, \quad \sum_{i=1}^{n} \rho_i \rho_j \rho_k \rho_l \rho_m = 5 \rho_{n+4}^4 + \ldots$$

The first three canonical densities are given by

$$\rho_{n+1} = F_5^{-1/5}, \quad \rho_0 = -\frac{1}{5} F_4 \rho_{n+1}^5 - 2 D_x \ln \rho_{n+1} ,$$

$$\rho_1 = \frac{1}{2} F_4 D_x(\rho_{n+1}^4) + \frac{2}{5} \rho_{n+1}^4 D_x F_4 + \frac{2}{25} \rho_{n+1}^9 F_4 - \frac{1}{5} \rho_{n+1}^4 F_3 - 3 (\rho_{n+1})^{-3} (D_x \rho_{n+1})^2 + 2 (\rho_{n+1})^{-2} D_x^2(\rho_{n+1}) .$$

For equations (1.6) we have

$$\rho_{n+1} = \left( \frac{\partial^2 H}{\partial u_{xx}^2} \right)^{-1/5} .$$  \hspace{1cm} (2.7)$$

Let us denote $\rho_{n+1} = a(x, u, u_x, u_{xx})$. Then

$$\frac{\partial^2 H}{\partial u_{xx}^2} = a^{-5} .$$  \hspace{1cm} (2.8)$$
We use the following notations: \( f \sim g \iff f - g \in \text{Im} D_x \). In particular, \( f \sim 0 \iff f \in \text{Im} D_x \). We write here \( D_x \) instead of \( \frac{d}{dx} \) and \( D_t \) instead of \( \frac{d}{dt} \) for brevity. Thus, all integrability conditions may be written in the following form \( D_t \rho_n \sim 0, n = -1, 0, \ldots \).

We used the following algorithm of checking whether a given function \( S(x,u,u_1,\ldots,u_n) \) belongs to \( \text{Im} D_x \). At first, \( S \) has to be linear in the highest derivative \( u_n \). If it holds true, then we can subtract from \( S \) a function of the form \( D_x Q(x,u,u_1,\ldots,u_{n-1}) \) such that the difference does nor depend on \( u_n \). Repeating this order lowering procedure, we either arrive at the situation when the function is nonlinear in its highest derivative, or we get zero.

### 3. Classification

**Theorem 2.** If non-linear equation of the form (1.6) has infinitely many generalized symmetries

\[
 u_{\tau_k} = F_k(x,u,u_x,\ldots), \quad k = 1, 2, \ldots
\]

and has no symmetries of order \( p \), where \( 1 < p < 5 \), then its Hamiltonian can be reduced to one of the canonical forms (1.7)–(1.10).

**Remark 3.** If we have \( k \neq 0 \) in (1.7), then the scaling \( u \rightarrow u k^{1/2}, \; x \rightarrow x k^{-1}, \; t \rightarrow t k^{-4/3} \) brings \( k \) to 1. If \( k \neq 0 \) in (1.10) then we can reduce \( k \) to 1 by the scaling \( u \rightarrow u k^{-1}, \; x \rightarrow x k, \; t \rightarrow t k^{-4} \). Therefore we may assume that \( k = 0 \) or \( k = 1 \) for both these Hamiltonians.

**Remark 4.** All constants in Hamiltonian (1.8) can be normalized by canonical transformations to 1 or 0. If \( c_0 \neq 0 \) then using the dilatation \( u \rightarrow \alpha u, \; t \rightarrow \alpha^{-1} t, \; \alpha = c_0^{-3/8} \), we obtain \( c_0 = 1 \). After that we can apply the transformation \( u \rightarrow u - \frac{1}{2}(c_1 x + c_2) \) to obtain the Hamiltonian

\[
 H = \sqrt[3]{u_{xx}} + \frac{1}{3} u^3 + (k_1 x^2 + k_2 x) u,
\]

where \( k_1 \) and \( k_2 \) are some constants. If \( k_1 \neq 0 \) the scaling \( u \rightarrow u \mu, \; x \rightarrow x \mu^{-4}, \; t \rightarrow t \mu^{-5} \) with \( \mu = k_1^{1/10} \) leads to \( k_1 = 1 \). Now we reduce \( k_2 \) to zero by the translation \( x \rightarrow x - k_2/2 \). If \( k_1 = 0 \) and \( k_2 \neq 0 \), we normalize \( k_2 \) by 1 with the help of a scaling. Thus, if \( c_0 \neq 0 \) we have the following three non-equivalent Hamiltonians:

\[
 H_1 = \sqrt[3]{u_{xx}} + \frac{1}{3} u^3 + x^2 u, \quad (3.1) \\
 H_2 = \sqrt[3]{u_{xx}} + \frac{1}{3} u^3 + x u, \quad (3.2) \\
 H_3 = \sqrt[3]{u_{xx}} + \frac{1}{3} u^3. \quad (3.3)
\]

If \( c_0 = 0 \) and \( c_1 \neq 0 \) then (1.8) is equivalent to

\[
 H_4 = \sqrt[3]{u_{xx}} + \frac{1}{2} x u^2. \quad (3.4)
\]
In the case \( c_0 = c_1 = 0 \) the Hamiltonian is equivalent to
\[
H_5 = \sqrt{u_{xx}}. \quad \square \tag{3.5}
\]

**Proof of Theorem 2.** Consider the first integrability condition \( D_t(a) \sim 0 \). It can be verified that
\[
D_t a \sim -\frac{5}{2} u_4^2 a^{-10} D_x \left( a^5 \frac{\partial^2 a}{\partial u_2^2} \right) + O(3), \tag{3.6}
\]
where the symbol \( O(n) \) denotes terms of differential order not greater than \( n \). Since the right hand side of (3.6) has to belong to \( \text{Im} D_x \) it should be linear in the highest derivative. Therefore we get
\[
\frac{d}{dx} \left( a^5 \frac{\partial^2 a}{\partial u_2^2} \right) = 0.
\]
Integrating this equation we obtain that
\[
\frac{\partial^2 a}{\partial u_2^2} = -2 c_0^2 a^{-5}, \tag{3.7}
\]
where \( c_0 \) is a constant.

Now we consider the third integrability condition and find that
\[
D_t \rho_1 \sim -10 u_5^2 a^{-13} \frac{\partial a}{\partial u_2} \left[ 2 u_3 \left( a^2 \frac{\partial a}{\partial u_2} - c_0 \right) \left( a^2 \frac{\partial a}{\partial u_2} + c_0 \right) + a^3 D_x \left( a^2 \frac{\partial a}{\partial u_2} \right) \right] \tag{3.8}
\]
\[
- \frac{5}{3} u_4^3 a^{-18} \left[ 2 u_3 \left( a^2 \frac{\partial a}{\partial u_2} - c_0 \right) \left( a^2 \frac{\partial a}{\partial u_2} + c_0 \right) \left( 99 a^4 \left( \frac{\partial a}{\partial u_2} \right)^2 + 20 c_0^2 \right) + O(2) \right] + \ldots
\]
Equating to zero the coefficients at \( u_5^2 u_3 \) and \( u_4^3 u_3 \), we obtain
\[
\frac{\partial a}{\partial u_2} = z a^{-2},
\]
where \( z = \pm c_0 \). If \( z \neq 0 \) we reduce \( z \) to 1 by the dilatation \( u \to \frac{u}{3z} \) and arrive at the case
\[
\text{A} : \quad a = (u_{xx} + q(x, u, u_x))^{1/3}.
\]
If \( z = 0 \), then we get
\[
\text{B} : \quad a = a(x, u, u_x).
\]

**Case A.** Integrating equation (2.8) we find that the Hamiltonian is equivalent to
\[
H = f(x, u, u_x) - \frac{9}{2} a.
\]
The first integrability condition yields \( \frac{\partial^2 f}{\partial u_2^2} = 0 \). Since the Hamiltonian is defined up to functions from \( \text{Im} D_x \) we set without loss of generality
\[
H = h(x, u) - \frac{9}{2} a, \quad a = \sqrt[3]{u_{xx} + q(x, u, u_x)}.
\]
It follows from the third integrability condition that
\[ q = q_0 + q_1 u_x + q_2 u_x^2 + q_3 u_x^3, \quad q_i = q_i(x, u). \]

Applying (non-canonical) transformation \( y = \varphi(x, u), \ v = \psi(x, u) \), we reduce to zero \( q_2 \) and \( q_3 \). To do this we can take for \( \varphi \) and \( \psi \) any functionally independent solutions of the system of PDEs
\[
\begin{align*}
\varphi_{xx} &= 2 \frac{\varphi_{xx} \varphi_x}{\varphi_u} - \frac{\varphi_x^2 \varphi_{uu}}{\varphi_u^2} + q_0 \varphi_u - q_1 \varphi_x + q_2 \frac{\varphi_x^2}{\varphi_u} - q_3 \frac{\varphi_x^3}{\varphi_u^2}, \\
\psi_{xx} &= 2 \frac{\psi_{xx} \varphi_x}{\varphi_u} - \frac{\varphi_x^2 \psi_{uu}}{\varphi_u^2} + 2 \frac{h}{\varphi_u^3} (\varphi_{uu} \varphi_x - \varphi_{ux} \varphi_u) + q_0 \psi_u - q_1 \psi_x \\
&\quad + q_2 \frac{\varphi_x^2}{\varphi_u^2} (\varphi_u \psi_x - \Delta) + q_3 \frac{\varphi_x^2}{\varphi_u^3} (2\Delta - \varphi_u \psi_x),
\end{align*}
\]
where \( \Delta = \varphi_x \psi_u - \varphi_u \psi_x \).

The resulting Hamiltonian has the following structure:
\[
\hat{H} = h(x, u) - \frac{9}{2} 3^{\frac{3}{2}} f(x, u) u_{xx}^3 + \tilde{q}_0(x, u) + \tilde{q}_1(x, u) u_x,
\]
where \( f = \Delta^{-1} \). The Hamiltonian form of the corresponding equation (2.5) is given by (2.3).

The first canonical density for this equation has the form
\[
\rho_{-1} = 3^{\frac{3}{2}} f u_{xx}^3 + \tilde{q}_0 + \tilde{q}_1 u_x f^{-1/5}.
\]

The integrability condition \( D_1(\rho_{-1}) \sim 0 \) leads to \( f = c \), where \( c \) is a non-zero constant. We reduce \( c \) to 1 by the transformation \( t \to t \cdot c^{-7/3} \). This means that for integrable equations the above transformation \( y = \varphi, v = \psi \) is a canonical one. Thus we have shown that the Hamiltonian can be reduced to
\[
H = h(x, u) - \frac{9}{2} a, \quad a = 3^{\frac{3}{2}} u_{xx}^3 + q_0 + q_1 u_x,
\]
where the Hamiltonian form of the corresponding equation (2.5) is given by (1.6).

It follows from the third integrability condition that
\[
q_1 = s_2(x) u^2 + s_1(x) u + s_0(x).
\]

Under canonical transformations of the form \( y = f(x), v = u/f' + g(x) \) this function changes as follows: \( q_1 \to \tilde{q}_1 = \tilde{s}_2 u^2 + \tilde{s}_1 u + \tilde{s}_0 \), where
\[
\tilde{s}_2 = s_2 f', \quad \tilde{s}_1 = s_1 - 2 g s_2 f', \quad \tilde{s}_0 = (f')^{-2} (s_0 f'' + f'' - g s_1 f' + g^2 s_2 f'^3).
\]

If \( s_2 \neq 0 \), we take \( f = 5 \int s_2^{-1} dx \) and \( g = s_1/10 \) to get \( \tilde{s}_2 = 5, \tilde{s}_1 = 0, \tilde{s}_0 = s_1 \). If \( s_2 = 0 \), then \( \tilde{s}_1 = s_1 \).

Choosing \( g = 0 \) and taking for \( f \) any non-constant solution of the equation \( f'' + s_0 f' = 0 \),
we arrive at $s_0 = 0$. So we are to consider the cases

\[ \text{A.1. } q_1 = 5u^2 + s(x), \quad \text{A.2. } q_1 = s(x)u. \]

**Case A.1.** It follows from the third integrability condition that $s(x)$ is a constant. We denote it by $9k$. Moreover we find that

$$q_0 = 2u(u^2 + k)(u^2 + 9k).$$

For such $q_1$ and $q_0$ the third integrability condition turns out to be equivalent to $h = c_2u^2 + c_1u$ where $c_1$ and $c_2$ are some constants. As it was mentioned in Section 2.1, $c_1$ is a trivial constant and the term $c_1u^2$ can be removed by the Galilean transformation. So, we obtain the integrable Hamiltonian (1.7).

**Case A.2.** From the first and third integrability conditions we find that

$$H = \frac{1}{3}p_0(x)u^3 + \frac{1}{2}p_1(x)u^2 + p_2(x)u - \frac{9}{2}\sqrt{u_{xx} + r_1(x)u + r_0(x)},$$

To simplify this Hamiltonian we use linear (non-canonical) transformations of the form $y = \varphi(x)$, $v = u g(x) + \psi(x)$. Taking a solution of the PDE system

$$\varphi' = g^2, \quad g'' = 2g^{-1}g'^2 + r_1g, \quad \psi'' = 2g^{-1}g'\psi' + r_0g$$

for $g, \varphi$ and $\psi$, we arrive at the transformed Hamiltonian

$$H = F(x, u) - \frac{9}{2}\left(u_{xx}f(x)\right)^{1/3},$$

where $f = g^{-3}$ and $F = h/g^2$ is the third degree polynomial with respect to $u$.

Since the transformation was not canonical, the Hamiltonian equation now takes the form (2.3). The first canonical density now reads as $\rho_{-1} = u_{xx}^{1/3}f^{-7/15}$. The first integrability condition immediately leads to the fact that $f$ is a constant. By a dilatation of the form $t \rightarrow \lambda t$ we reduce $f$ to 1. This means that for any integrable Hamiltonian we can reduce the coefficients $r_1, r_0$ to zero by a canonical transformation. Now we find that the polynomial $F$ has the form

$$F = \frac{1}{3}c_0u^3 + \frac{1}{2}(c_1x + c_2)u^2 + (c_3x^2 + c_4x)u,$$

where $c_i$ are some constants. Thus, we obtain Hamiltonian (1.8). All constants $c_i$ are inessential (see Remark 4).

**Case B.** Integrating equation (2.8), we find that without loss of generality the Hamiltonian can be written in the form

$$H = h(x, u, u_x) + \frac{u_{xx}^2}{2a^3} \quad (3.9)$$
It follows from the first integrability condition that
\[ D_x \left( a^3 \frac{\partial^2 a}{\partial u_x^2} \right) = 0. \]

The solution of this equation is given by
\[ a = \sqrt{a_0 + a_1 u_x + a_2 u_x^2}, \quad a_1^2 - 4 a_2 a_0 = c^2, \]
where \( a_i = a_i(x, u) \), and \( c \) is a constant. It can be verified (see [1]) that \( a_2 \) can be reduced to zero by a canonical transformation \( y = \varphi(x, u), \ v = \psi(x, u) \). Now the condition \( a_1^2 - 4 a_2 a_0 = c^2 \) implies that \( a_1 \) is a constant. Using a scaling of the form \( u \rightarrow \lambda u \), we arrive at the following two subcases:

**B.1.** \( a = a(x, u) \), \quad **B.2.** \( a = \sqrt{u_x + q(x, u)} \).

**Case B.1.** It follows from the first integrability condition that
\[ \frac{\partial^3 a}{\partial u^3} = 0, \quad \frac{\partial}{\partial x} \left( \frac{\partial^2 a}{\partial u^2} - 3 \left( \frac{\partial a}{\partial u} \right)^2 \right) = 0, \]
which means that
\[ a = s_0 + s_1 u + s_2 u^2, \quad \delta = s_1^2 - 4 s_2 s_0 = \text{constant}, \]
where \( s_i = s_i(x) \). Using linear canonical transformations
\[ y = \varphi(x), \quad v = u(\varphi')^{-1} + f(x) \]
and scalings, we can simplify the function \( a \). As a result we arrive at the subcases

**B.1.1.** \( a = 1 \), \quad **B.1.2.** \( a = u \), \quad **B.1.3.** \( a = u^2 + z \),
where \( z \) is a constant.

**Case B.1.1.** It follows from the third integrability condition that
\[ H = \frac{1}{2} u_{xx}^2 + \frac{1}{3} f_1 u_x^3 + \frac{1}{2} f_2 u_x^2 + f_3, \quad f_i = f_i(x, u). \]

The fifth integrability condition implies \( f_1 = c_0, \ f_2 = c_1 u^2 + a_2 u + a_3 \), where \( a_i = a_i(x) \) and \( c_i \) are constants such that \( c_0(4 c_0^2 - 5 c_1) = 0 \).

**Subcase B.1.1.a.** If \( c_0 \neq 0 \), then \( c_1 = \frac{4}{5} c_0^2 \neq 0 \). Using a canonical transformation of the form \( u \rightarrow u - f(x) \), we reduce \( a_2 \) to zero. It follows from the fifth integrability condition that
\[ H = \frac{1}{2} u_{xx}^2 + \frac{1}{3} u_x^3 c_0 + \frac{2}{5} u_x^2 c_0^2 u^2 + \frac{8}{1875} u^6 c_0^4. \]

Using a scaling of the form \( u \rightarrow \lambda u \), we put \( c_0 = -5/2 \) getting Hamiltonian (1.9).
**Subcase B.1.1.b.** If \( c_0 = 0 \) and \( c_1 \neq 0 \), we reduce the function \( a_2 \) to zero by the canonical transformation \( u \to u - a_2/(2c_1) \). Assuming \( a_2 = 0 \), we find from the fifth integrability condition that

\[
H = \frac{1}{2} u_{xx}^2 + \frac{1}{2} (c_1 u^2 + c_2) u_x^2 + \frac{c_1^2}{100} u^6 + \frac{c_1 c_2}{20} u^4 + \frac{1}{2} c_3 u^2 + c_4 u.
\]

The constant \( c_4 \) is trivial (see Section 2.1) and \( c_3 \) can be annihilated by the Galilean transformation. By a transformation of the form \( u \to \lambda u \) we reduce \( c_1 \) to 10 and get

\[
H = \frac{1}{2} u_{xx}^2 + 5 u_x^2 u_x^2 + u^6 + \frac{1}{2} c(u_x^2 + u^4).
\]

The corresponding evolution equation

\[
u_t = D_x (u_4 - 10 u(u_2 + u_x^2) + 6 u^5 + c(2 u^3 - u_{xx}))
\]

is a higher symmetry of the mKdV equation \( u_t = D_x (u_{xx} - 2u^3) \). This equation has Hamiltonian (1.3) with \( Q = 1 \) and \( P = -\frac{1}{2} u^4 \).

**Subcase B.1.1.c.** If \( c_0 = c_1 = 0 \), integrability conditions imply \( a_2 = k = constant \) and

\[
H = \frac{1}{2} u_{xx}^2 + \frac{1}{2} (k u + a_3) u_x^2 + \frac{40 c_3 + 3 k^2}{40} u^4 + \frac{1}{3} a_4 u^3 + \frac{1}{2} a_5 u^2 + a_6 u, \quad a_t = a_i(x),
\]

where \( k(k^2 + 20 c_3) = 0 \).

**I.** If \( k \neq 0 \), then \( c_3 = -k^2/20 \). Using the shift \( u \to u - a_3/k \), we reduce \( a_3 \) to zero. After that integrability conditions lead to the fact that \( a_5 \) and \( a_6 \) are constants and that \( a_4 = 0 \). The constant \( a_6 \) is trivial and \( a_5 \) can be reduced to 0 by the Galilean transformation. Using the transformation \( u \to -(10/k) u \), we obtain

\[
H = \frac{1}{2} u_x^2 - 5 u_x^2 + \frac{5}{2} u^4, \quad u_t = D_x (u_4 + 10 u(u_2 + u_x^2) + 5 u_x^2).
\]

The latter equation belongs to the hierarchy of the KdV equation \( u_t = D_x (u_{xx} + 3 u^3) \). This equation has Hamiltonian (1.3) with \( Q = 1 \) and \( P = u^3 \).

**II.** If \( k = 0 \), then the Hamiltonian

\[
H = \frac{1}{2} u_{xx}^2 + \frac{1}{2} a_3 u_x^2 + c_3 u^4 + \frac{1}{3} a_4 u^3 + \frac{1}{2} a_5 u^2 + a_6 u, \quad a_t = a_i(x)
\]

can be simplified by a canonical transformation of the form \( u \to u - f(x) \). The coefficients of the Hamiltonian are transformed as follows:

\[
\tilde{a}_3 = a_3, \quad \tilde{c}_3 = c_3, \quad \tilde{a}_4 = a_4 - 12 c_3 f, \quad \tilde{a}_5 = a_5 + 12 c_3 f^2 - 2 a_4 f, \quad \tilde{a}_6 = a_6 - f^{(4)} - 4 c_3 f^3 + a_3 f^2 - a_5 f + (f a_6)'.
\]

Consider the following alternatives:

a) If \( c_3 \neq 0 \), then we take \( f = a_4/(12 c_3) \) to get \( \tilde{a}_4 = 0 \).

b) If \( c_3 = 0, a_4 \neq 0 \), then choosing \( f = a_5/(2 a_4) \) we get \( \tilde{a}_5 = 0 \).

c) If \( c_3 = 0, a_4 = 0 \), then we consider the normalization \( \tilde{a}_6 = 0 \).
In the cases a) and b) we obtain non-integrable equations $u_t = D_x(u_4 + 4c_3u^3 + cu_2)$ and $u_t = D_x(u_4 + c_4u^2 + cu_2)$, $c_4 \neq 0$, correspondingly. For the first of them the ninth integrability condition is not fulfilled and for the second the condition number eleven is broken.

In the case c) a linear equation appears with the Hamiltonian $H = \frac{1}{2}(u^2_{xx} + a(x)u^2_x + b(x)u^2)$.

**Case B.1.2.** Here we consider the Hamiltonians of the form

$$H = h(x,u,u_x) + \frac{u^2_{xx}}{2u^3}.$$  

It follows from the third integrability condition that

$$D_x \left( u^7 \frac{\partial^4 h}{\partial u^4_x} \right) = 0 \quad \text{or} \quad h = c_1 \frac{u^4}{u^7} + q_1 u^3 + q_2 u^2 + q_3 u_x + q_4, \quad q_i = q_i(x,u).$$

The dependence of the functions $q_i$ on $u$ can be specified by the third and fifth integrability conditions:

$$H = \frac{u^2_{xx}}{2u^3} - \frac{15}{8} u^4 + \frac{1}{2} u_x \left( \frac{s_1}{u^5} + \frac{s_2}{u} \right) + \frac{s_2^2}{50} u^5 \frac{1}{2} s_3 u^2 - \frac{u^{-3}}{150} (10 s''_1 - 3 s''_1) - \frac{u}{15} (5 s''_2 + s_1 s_2),$$

where $s_i = s_i(x)$. The functions $s_1$, $s_2$ and $s_3$ are changed under the canonical transformation $y = \varphi(x)$, $v = u/\varphi'$ as follows:

$$\tilde{s}_1 = \frac{1}{2} \left( \varphi' \right)^{-4} (10 \varphi''' \varphi' - 15 \varphi''^2 + 2 s_1 \varphi''), \quad \tilde{s}_2 = s_2 \varphi'' , \quad \tilde{s}_3 = s_3 \varphi'.$$

Choosing $\varphi$ as a nonzero solution of the equation $10 \varphi''' \varphi' - 15 \varphi''^2 + 2 s_1 \varphi'' = 0$, we obtain $\tilde{s}_1 = 0$. In this case the integrability conditions 1-7 are equivalent to equations

$s_2^{(5)} = 0, \quad s_3^{(5)} = 0, \quad s_3 s'_2 = 2s_2 s'_3.$

Two solutions: (i) $s_3 \neq 0$, $s_3 = c_0 + c_1 x + c_2 x^2$, $s_2 = k s_3^2$, and (ii) $s_3 = 0$ and $s_2(x)$ is a polynomial of degree not greater than 4, lead to

$$H_1 = \frac{u^2_{xx}}{2u^3} - \frac{15}{8} u^4 + \frac{s_2^2}{2u} + \frac{k^2}{50} s_3 u^5 - \frac{1}{2} s_3 u^2 - \frac{2}{3} k u (s_3 s'' + s''_3),$$

and

$$H_2 = \frac{u^2_{xx}}{2u^3} - \frac{15}{8} u^4 + \frac{s_2^2}{2u} + \frac{s_2^2}{50} u^5 - \frac{u}{3} s''.$$  

Equations (3.6) corresponding to both $H_1$ and $H_2$ have third order symmetries of the form

$$u_\tau = D_x \left( \frac{u_{xx}}{u^3} - \frac{3}{2} u_x + u^2 P(x) \right),$$

where $P = -\frac{3}{5} k s_3^2$ and $P = -\frac{3}{5} s_2$ correspondingly. These symmetries are generated by Hamiltonians of the form (1.4).

**Case B.1.3.** For Hamiltonians of the form

$$H = h(x,u,u_x) + \frac{u^2_{xx}}{2 \mu^3}, \quad \mu = u^2 + z \quad (3.11)$$
the first integrability condition leads to
\[ \frac{\partial h}{\partial x} = D_x f(x, u) \]
for some function \( f \). Therefore \( h \) has the form \( h = D_x g + h_0(u, u_x) \) and without loss of generality we assume that \( h = h(u, u_x) \).

It follows from the third integrability condition that
\[ D_x \left( \mu^7 \frac{\partial^4 h}{\partial u_x^4} + 160 \mu \right) = 0, \]
and therefore the Hamiltonian is equivalent to
\[ H = \frac{u_x^2}{2 \mu^5} - \frac{20}{3} u_x^4 (c_1 + \mu) + \frac{1}{3} q_1 u_x^3 + \frac{1}{2} q_2 u_x^2 + q_3, \tag{3.12} \]
where \( q_i = q_i(u) \) and \( c_1 \) is a constant. From the third and fifth integrability conditions we derive two relations
\[ (8 c_1 + 9 z)(4 c_1 + 5 z) = 0 \]
and\( (8 c_1 + 9 z)(6 c_1 + 7 z) = 0 \), which implies \( c_1 = -9 z/8 \). Taking this into account, we find that the third integrability condition is equivalent to the following relations:
\[ \mu^3 q_3''' + 12 \mu(14 \mu - 11 z) q_2' + 24 u(14 \mu - 5 z) q_2 = 0, \tag{3.13} \]
\[ z(5 q_3''' - 2 \mu^5 q_3 q_2' - 18 u \mu^4 q_2^2) = 0, \quad z q_1 = 0. \tag{3.14} \]
From the fifth integrability condition we additionally find
\[ u q_1' = -10 q_1, \quad q_1(5 u^2 q_2'' + 70 u q_2' + 210 q_2 - 8 u^{12} q_1^2) = 0. \tag{3.15} \]
Consider the following two subcases corresponding to \( z = 0 \) and \( z \neq 0 \) in (3.12)–(3.14).

**Subcase B.1.3.a.** If \( z = 0 \), we find from (3.13) and (3.15) that
\[ q_1 = k_1 u^{-10}, \quad q_2 = k_2 u^{-8} + k_3 u^{-7} + k_4 u^{-6}, \]
where \( k_i \) are constants. The fifth integrability condition leads to
\[ q_3 = \frac{1}{2} k_5 u^{-2} - k_6 u^{-1} + \frac{4 k_1^2 - 15 k_2}{1500 u^4} (k_2 + 3 k_3 u), \]
where
\[ k_1(5 k_2 - 4 k_1^2) = 0, \quad (5 k_2 - 4 k_1^2)(10 k_6 + 3 k_3 k_4) = 0, \quad (5 k_2 - 4 k_1^2)(5 k_3^2 + 2 k_0 k_4 - 100 k_5) = 0. \]
Additional algebraic relations for \( k_i \) follows from the seventh integrability condition. The system of algebraic equations thus obtained has four solutions. One solution corresponds to
the integrable Hamiltonian (1.10). The remaining solutions generate the following Hamiltonians:

\[
H_1 = \frac{1}{2} \frac{u_{xx}^2}{u^{10}} - \frac{20}{3} \frac{u_x^4}{u^{12}} + \frac{u_x^2}{u^8} (c_1 u^2 + 2 c_2 u + c_3) + \frac{5}{25} \frac{c_3}{u^4} + \frac{2}{5} c_2 \frac{2 c_1 u + c_2}{u^2},
\]

\[
H_2 = \frac{1}{2} \frac{u_{xx}^2}{u^{10}} - \frac{20}{3} \frac{u_x^4}{u^{12}} + \frac{1}{2} \frac{u_x^2}{u^7} (10 c_1 + c_2 u) + \frac{5}{2} \frac{c_1^2}{u^2} + c_2 c_1,
\]

\[
H_3 = \frac{1}{2} \frac{u_{xx}^2}{u^{10}} - \frac{20}{3} \frac{u_x^4}{u^{12}} + \frac{c_1 u_x^2}{2} \frac{u^6 - c_2}{u}.
\]

The Hamiltonians \(H_1\) and \(H_2\) correspond to fifth order symmetries for equations

\[
u_t = D_x \left( \frac{u_{xx}}{u^6} - 3 \frac{u_x^2}{u^7} + \frac{1}{5} (2 c_2 u^{-2} + c_3 u^{-3}) \right)
\]

and

\[
u_t = D_x \left( \frac{u_{xx}}{u^6} - 3 \frac{u_x^2}{u^7} + \frac{c_1}{u^2} \right),
\]

correspondingly. The first of these equations has Hamiltonian (1.3) with \(Q = u^2\) and \(\forall P\). The second equation has Hamiltonian (1.3) with \(Q = u^2\) and \(P = -c_1 u\).

If \(c_2 \neq 0\), the equation corresponding to \(H_3\) does not satisfy the eleventh integrability condition. In the case \(c_2 = 0\) we get a symmetry of equation

\[
u_t = D_x \left( \frac{u_{xx}}{u^6} - 3 \frac{u_x^2}{u^7} \right).
\]

This equation has Hamiltonian (1.3) with \(Q = u^2\) and \(P = 0\). The reciprocal transformation \(dy = \rho_{-1} dx + \theta_{-1} dt, \ v(t, y) = 1/u(t, x)\) linearizes the equation. The Hamiltonian equation for \(H_3\) with \(c_2 = 0\) is also linearizable.

**Subcase B.1.3.b.** If \(z \neq 0\) in Hamiltonian (3.11), then we find from equation (3.12), (3.13) and (3.14) that

\[
H = \frac{u_{xx}^2}{2 \mu^5} - \frac{5 u_x^4}{6 \mu^7} (8 \mu - 9 z) + 5 \varphi u_x^2 + \psi,
\]

where \(\varphi = q_2/10, \ \psi = q_3, \ \mu = u^2 + z, \ \varphi = -\frac{z (2 k_1 u + k_2)}{\mu^5} + \frac{k_1 u + k_2}{\mu^4} + \frac{k_3}{\mu^3}, \ \psi = \frac{z}{2} \frac{4 k_1^2 z - 4 k_1 k_2 u - k_2^2}{\mu^3} + \frac{3 k_1 k_2 u - 4 k_1^2 z + k_2^2}{\mu^2} + \frac{5}{2} \frac{4 k_1 k_3 u + 2 k_2 k_3 + k_1^2}{\mu}.
\]

Here \(k_i\) are constants. The corresponding Hamiltonian equation is a symmetry of the following third order equation

\[
u_t = D_x \left( \frac{u_{xx}}{\mu^3} - 3 \frac{u u_x^2}{\mu^4} + \frac{k_2 u - 2 z k_1}{\mu^2} + \frac{k_1}{\mu} \right).
\]

This equation has Hamiltonian (1.3) with \(Q = \mu = u^2 + z, \ z \neq 0\) and \(\forall P\).
Case B.2. In this case \( a = \sqrt{u_x + q(x,u)} \) and the Hamiltonian has the form

\[
H = h(x,u,u_x) + \frac{1}{2} \frac{u_{xx}^2}{a^5}.
\]

The first integrability condition implies the following equation

\[
\frac{\partial}{\partial u_x} \left( a_3 \frac{\partial^2 h}{\partial u_x^2} \right) = \frac{105}{8} f_1 a^{-8} - \frac{5}{2} f_2 a^{-6} + \frac{5}{8} f_3 a^{-4},
\]

where

\[
f_1 = (q_x - q u'_u)^2, \quad f_2 = 2 q^2 q_{uu} - 4 q q_{ux} - 5 q u_x - 5 q q_u^2 + 2 q_{xx}, \quad f_3 = 8 q q_{uu} - q_u'^2 - 8 q_{ux}.
\] (3.17)

Integrating the equation for \( h \), we found that the Hamiltonian is equivalent to

\[
H = h_1(x,u) + a h_2(x,u) - \frac{1}{2} f_1 a^{-5} + \frac{1}{3} f_2 a^{-3} - \frac{5}{6} f_3 a^{-1} + \frac{1}{2} \frac{u_{xx}^2}{a^5},
\] (3.18)

where \( f_i \) are defined by (3.17), \( h_1 \) and \( h_2 \) are some functions. It follows from the first integrability condition that

\[
q^{(5)}(u) = 0, \quad h_1''(u) = 0, \quad D_x \left( h_2 - \frac{10}{3} q_{uu} \right) = 0,
\] (3.19)

\[
q_x h_{1,uu} + 2 q h_{1,uux} - q_u h_{1,ux} - h_{1,uex} = 0.
\] (3.20)

Integrating (3.19), we get

\[
q = q_1 u^4 + q_2 u^3 + q_3 u^2 + q_4 u + q_5, \quad h_1 = \frac{1}{2} s_1 u^2 + s_2 u, \quad h_2 = \frac{10}{3} (q_{uu} + c_0),
\]

where \( q_i = q_i(x) \), \( s_i = s_i(x) \) and \( c_0 \) is a constant.

The functions \( h_1 \) and \( q_i \) transform under canonical transformation \( y = \varphi(x), u = v/\varphi' + \psi(x) \) as follows:

\[
\tilde{h}_1 = \frac{1}{2} s_1 \varphi'^2 + (s_2 - s_1 \varphi' \psi) v
\] (3.21)

and

\[
\tilde{q}_1 = q_1 \varphi'^2, \quad \tilde{q}_2 = \varphi' (q_2 - 4 q_1 \varphi' \psi), \quad \tilde{q}_3 = q_3 - 3 q_2 \varphi' \psi + 6 q_1 (\varphi' \psi)^2,
\]

\[
\tilde{q}_4 = \varphi'^{-1} (q_4 - 2 q_3 \varphi' \psi + 3 q_2 (\varphi' \psi)^2) - 4 q_1 (\varphi' \psi)^3 - \varphi'^{-1} \varphi' 
\]

\[
\tilde{q}_5 = \varphi'^{-2} (q_5 - q_4 \varphi' \psi + q_3 (\varphi' \psi)^2 - q_2 (\varphi' \psi)^3 + q_1 (\varphi' \psi)^4 - (\varphi' \psi)'),
\] (3.22)

Let us consider the following two alternatives

**B.2.a.** \( h_1 \neq 0; \quad \textbf{B.2.b.} \quad h_1 = 0. \)

**Subcase B.2.a.** Suppose \( s_1 \neq 0; \) taking \( \varphi' = 1/s_1 \) and \( \psi = s_2 \), we obtain \( \tilde{h}_1 = \frac{1}{2} v^2 \). In this case it follows from equation (3.20) that \( q_x = 0 \). Since the Hamiltonian does not depend
on $x$ we may remove the term $\frac{1}{2}u^2$ from $\bar{H}$ by the Galilean transformation to reduced $H$ to the following form

$$
H = \frac{1}{2} u_{xx}^2 + \frac{10}{3} a (q'' + c_0) - \frac{(q_0')^2}{2 a^3} + \frac{2q'^2q'' + 5q'^2}{3 a^3} + \frac{5}{6} a^{-1} (q'^2 - 8qq''),
$$

(3.23)

where $a = \sqrt{u_x + q(u)}$ and $q^{(5)} = 0$. Equation (1.6) generated by this Hamiltonian is a fifth order symmetry for the equation (see [1])

$$
u_t = D_x \left( \frac{u_{xx}}{a^3} + q' \left( \frac{3}{a} - \frac{q}{a^3} \right) \right).
$$

This equation has Hamiltonian (1.5) with $P = q$.

**Subcase B.2.a.2.** If $s_1 = 0$ then $\bar{h}_1 = h_1 = s_2 u$. In the case $s_2' = 0$ the term $h_1 = c u$ is trivial and we have the contradiction $h_1 = 0$, hence $s_2' \neq 0$. Then equation (3.20) takes the form $q_u = -s_2''/s_2'$, therefore $q_1 = q_2 = q_3 = 0$ and $s_2'' + q_4 s_2' = 0$. It follows from formulas (3.22) that we can choose $\varphi$ and $\psi$ such that $\bar{q}_4 = \bar{q}_5 = 0$. Now we have $q = 0$ and therefore $s_2'' = 0$. The Hamiltonian takes the form

$$
H = c x u + \frac{10}{3} c_0 u_x^{1/2} + \frac{1}{2} u_{xx}^2,
$$

where $c$ is a constant. The transformation $u \to u + ct$ brings $c$ to zero and the Hamiltonian turns out to be a particular case of (3.23).

**Subcase B.2.b.** If $h_1 = 0$ then

$$
H = \frac{10}{3} (q_{uu} + c_0) a - \frac{1}{2} f_1 a^{-5} + \frac{1}{3} f_2 a^{-3} - \frac{5}{6} f_3 a^{-1} + \frac{1}{2} u_{xx}^2,
$$

(3.24)

where $f_i$ are given by (3.17). It easy to see from (3.22) that we can reduce the Hamiltonian to one of the following:

B.2.b.1. $q_1 = 1$, $q_2 = 0$; B.2.b.2. $q_1 = 0$, $q_2 = 1$, $q_3 = 0$; B.2.b.3. $q_1 = q_2 = q_4 = q_5 = 0$.

The first integrability condition results in the equation $q_x = 0$ for all of these cases. Therefore we have Hamiltonians that can be obtained from formula (3.23) as partial cases. The theorem is proved.

**Remark 5.** The equation with Hamiltonian (1.9) is well known. It is just equation (3.6) in the list from the survey [7]. The Hamiltonian equation corresponding to (1.10) can be reduced to an equation of the form

$$
u_t = u_5 + F(x, u, u_x, u_2, u_3, u_4)
$$

(3.25)

by the standard reciprocal transformation (see [3], section 1.4)

$$
d y = \rho_{-1} d x + \theta_{-1} d t, \quad v(t, y) = u(t, x),
$$

(3.26)
where $\rho_{-1}$ is the first canonical density and $\theta_{-1}$ is the correspondent flux. The resulting equation coincides with equation (3.12) from [7] up to the differential substitution $v = 2 w y^{-1/2}$. Notice that the transformation $v = (w - k/2)^{-1}$ leads to a simpler then (3.12) equation

$$w_t = D_y \left( w_4 + 10 w_y w_{yy} - 20 w_{yy} w^2 - 20 w_y^2 w + (k - 2 w)^2 (4 w^3 + 4 k w^2 + 3 k^2 w + 2 k^3) \right).$$

To prove the integrability of equations with Hamiltonians (1.7) and (1.8) one could find differential substitutions that reduce them to known equations of the form (3.25). We have verified that these two equations have local conservation laws of orders 3 and 5. Also we have found generalized symmetries of order 7 for these equations. In the both cases these symmetries have the form

$$u_\tau = D_x \frac{\delta \rho_1}{\delta u},$$

where $\rho_1$ is the canonical density for the corresponding fifth order equation. In particular, for equation with Hamiltonian (1.8) we have $\rho_1 \sim u_{xxx}^2 u_{xx}^{-7/3}$ and the seventh order symmetry is given by

$$u_\tau = D_x \frac{\delta}{\delta u} \left( \frac{u_{xxx}^2}{u_{xx}^{7/3}} \right). \quad \Box$$

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