Topological Winding and Unwinding in Metastable Bose-Einstein Condensates

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Topological winding and unwinding in a quasi-one-dimensional metastable Bose-Einstein condensate are shown to be manipulated by changing the strength of interaction or the frequency of rotation. Exact diagonalization analysis reveals that quasidegenerate states emerge spontaneously near the transition point, allowing a smooth crossover between topologically distinct states. On a mean-field level, the transition is accompanied by formation of grey solitons, or density notches, which serve as an experimental signature of this phenomenon.

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Metastability of a physical system leads to a rich variety of quantum phases and transport properties that are not present in the ground state phase. An illustrative example is superflow and phase slip in a narrow superconducting channel [1]. Other examples include Feshbach molecules formed in high rotational states [2] and metastable quantum phases in higher Bloch bands [3]. Recent experimental advances in cold atoms/molecules have made it possible to realize excited, metastable states which persist for a long time. These states provide an excellent medium in which to investigate fundamental aspects of condensed matter systems such as topological excitations and superfluidity [4, 5, 6, 7].

It is widely believed that the angular momentum per particle in a weakly repulsive one-dimensional (1D) superfluid ring system [8, 9] is quantized at $T = 0$ and that there are discontinuous jumps between states having different values of the phase winding number. In this Letter, we point out that this applies only to the ground state; continuous transitions do in fact occur between metastable states of repulsive condensates. The underlying physics behind this phenomenon is the emergence of a dark or grey soliton train [10] which bifurcates from the plane-wave solution and carries a fraction of the quantized value of the angular momentum.

Starting with mean-field theory for scalar bosons subject to rotation, we proceed through progressively deeper levels of insight into the quantum many-body nature of this problem, making a link between semiclassical and quantum solitons in metastable states. We find that the phase slip, which allows a smooth crossover between topologically distinct states, is caused by a quantum soliton. The latter consists of a linear superposition of the rotationally-invariant many-body eigenstates of the Hamiltonian [8]. In both Bogoliubov theory and quantum many-body theory the broken-symmetry soliton state is shown to be stable against perturbation.

This phenomenon can be realized by hot atoms confined in fast-rotating circular waveguides or toroidal traps [11]. First, to obtain a metastable uniform condensate one quickly stops the rotation and then lowers the temperature. Second, one adiabatically changes the angular frequency of the trap in the presence of a small arbitrary perturbation in the trapping potential. This causes atoms to adiabatically take the higher-energy path of a metastable soliton state, as we will show. Third, one stops the adiabatic change in the frequency at the correct point to arrive at a different winding number. All of these processes can occur continuously.

We consider a system of $N$ bosonic atoms in a quasi-1D torus with radius $R$, under an external rotating drive with angular frequency $2\Omega$. The length, angular momentum, and energy are measured in units of $R$, $\hbar$, and $\hbar^2/(2mR^2)$, respectively. The Hamiltonian is given by the Lieb-Liniger Hamiltonian in a rotating frame of reference [11, 12],

$$
\hat{H} = \int_0^{2\pi} d\theta \hat{\psi}^\dagger (-i\partial_\theta - \Omega)^2 \hat{\psi} + g_{1D}\hat{\psi}^2/2, \tag{1}
$$

where $g_{1D}$ characterizes the strength of the s-wave interatomic collisions in 1D [13] rescaled by $\hbar^2/(2mR)$, $\theta$ is the azimuthal angle, and the bosonic field operator satisfies periodic boundary conditions: $\hat{\psi}(\theta) = \hat{\psi}(\theta + 2\pi)$. Since the Hamiltonian is periodic with respect to $\Omega$, the properties of the system are periodic in $\Omega$ with period $1[14]$, in direct analogy to the reduced Brillouin zone in a Bloch band [11, 12]. Without loss of generality we will henceforth restrict ourselves to $\Omega \in [0, 1)$. The Hamiltonian is integrable via the boson-fermion mapping in the Tonks-Girardeau (TG) limit $g_{1D} \gg N$ [15], and via the Bethe ansatz in the weak-interaction limit $g_{1D}N \lesssim O(1)$ as well as intermediate-interaction regimes.

We first show how continuous changes in the angular momentum occur in the weak-interaction regime for solutions of the Gross-Pitaevskii equation (GPE) $[-i\partial_\theta - \Omega]^2 g_{1D}N |\psi(\theta)|^2 \psi(\theta) = \mu \psi(\theta)$, where $\psi$ is the order parameter normalized to unity and $\varphi \equiv \text{Arg}(\psi)$ is its phase. The single-valuedness of the wave function requires $\varphi(\theta +
2π = ϕ(θ) + 2πJ, where J ∈ {0, ±1, ±2, ...} is the topological winding number. Stationary solutions of the GPE for g_{1D} ≥ 0 are either plane-wave states ψ(θ) = e^{iJθ}/√2π or a grey soliton train ξ whose amplitude and phase are given by |ψ(θ)| = A[1 + η dn²( jK(θ - θ₀)/π, k)]^{1/2} and ϕ(θ) = Ωθ + B Π(ξ, jK(θ - θ₀)/π, k), respectively.

Here the amplitude A = √K/[2πK(π + ηE)], the phase pre-factor B = (S/jK)√g_{sn}h_{sn}/2f_{sn}; there are j density notches in the soliton train; η = -2j²K²/g_{sn} ∈ [-1, 0] characterizes the depth of each density notch; k ∈ [0, 1] is the elliptic modulus; K(k), E(k), Π(ξ, u, k), are elliptic integrals of the first, second, and third kinds; and dn(u, k) is the Jacobi dn function. The degeneracy parameter θ₀ indicates that the soliton solutions are broken-symmetry states. We define f_{sn} = πg_{1D}N/2 - 2j²K² + 2j²KE, g_{sn} = f_{sn} + 2j²K², h_{sn} = f_{sn} + 2k²j²K², and S = 1 for 0 ≤ Ω < 0.5, S = -1 for 0.5 ≤ Ω < 1. Then ξ = -2(kjK²)/f_{sn} ≤ 0, and only when soliton solutions exist \[ 1/2 \] is k ≠ 0. In the limit η → -1, f_{sn} approaches zero, and g_{sn} and h_{sn} both approach finite positive values; consequently, the wave function approaches the Jacobi sn function, which corresponds to a dark soliton with a π-phase jump at θ₀. In the opposite limit η → 0, both the amplitude and phase approach those of the plane-wave solution with the same phase winding J. These limiting behaviours indicate that continuous change in topology of the condensate wave function is possible, as illustrated in Fig. 1(i)-(vi). Henceforth, we consider the single soliton j = 1 for simplicity, but our discussion holds for arbitrary soliton trains j > 1.

Bifurcation of the soliton train from the plane wave constitutes a second-order phase transition with respect to g_{1D} and/or Ω. Figure 2(a) shows the energy difference \[ E_j^{(sol)} - E_j^{(pw)} \] between the two solutions

\[ E_j^{(pw)} = (Ω - J)^2 + g_{1D}N/(4π), \]
\[ E_j^{(sol)} = g_{1D}N/(2π) + [3KE - K²(2 - k²)]/π² + 4K²[3E² - 2(2 - k²)KE + K²(1 - k²)]/(3π³g_{1D}N). \]

This kind of bifurcation does not occur from the ground-state energy. However, for metastable states a bifurcation can occur between the plane-wave state and the soliton state with the same winding number J. After bifurcation, the soliton energy \[ E_j^{(sol)} \] becomes larger than \[ E_j^{(pw)} \]. Furthermore, at Ω = 0.5, \[ E_j^{(sol)} \] and \[ E_j^{(pw)} \] are degenerate with a ±π-phase jump in the condensate wave function. The derivatives of the energies \[ ∂E_j^{(sol)}/∂Ω \] and \[ ∂E_j^{(pw)}/∂Ω \] have a kink at the boundary as can be verified analytically. This identifies the second-order quantum phase transition \[ \Omega_{crit} \], which occurs along a curve in the \( Ω-g_{1D} \) plane.

Figures 1(i)-(vi) illustrate a continuous change in the topology along a higher-energy, soliton path shown in Fig. 2(a) with white arrows. Following this path in Fig. 1 as Ω increases starting from (i) the plane wave with J = 1, (ii) solitons start to form past a critical point \( Ω_{crit} \), (iii) The density notch deepens for \( Ω_{crit} ≤ Ω ≤ 0.5 \). At \( Ω = 0.5 \) it forms a node, the phase of the soliton jumps by π, and the energies of the solitons with phase winding number 1 and 0 are degenerate. (iv), (v) The soliton with phase winding J = 0 deforms continuously as Ω increases. (vi) Finally, the state goes back to the plane-wave state with phase winding J = 0.

The angular momentum \[ L/N = \int dθ \psi^*(−i∂θ)ψ \] of the metastable states changes continuously along the soliton path. For the plane wave state, \[ L_j^{(pw)}/N = J \] is quantized; in contrast, for the soliton \[ L_j^{(sol)}/N = Ω + S√2f_{sn}g_{sn}h_{sn}/(g_{1D}Nπ²) \] is non-integer, as shown in Fig. 2(b). Thus a continuous change of angular momentum is possible for 1D Bose systems by taking the metastable states with energy slightly higher than that of the ground state.

We next investigate the stability of the metastable states using Bogoliubov theory \[ 20, 21 \], and identify the curve in the \( Ω-g_{1D} \) plane where the soliton solutions bifurcate from the plane-wave solutions. A stationary solution ψ of the GPE subject to a small perturbation e develops in time as \[ \tilde{ψ}(t) = e^{-i\lambda t}[\psi + \sum_n(δu_n e^{-iλ_n t} + δυ_n^* e^{iλ_n t})] \], where \( (u_n, υ_n) \) and \( λ_n \) are eigenstates and eigenvalues of the Bogoliubov-de Gennes equations (BdGE), and n denotes the index of the eigenvalues.

For the plane-wave state with phase winding J, the eigenvalues of the BdGE are obtained as \[ λ_j^{(J=1,pw)} = [n²(n² + g_{1D}N/π)]^{1/2} - 2n(Ω - J) \]. Then \[ λ_{-1}^{(J=1,pw)} \] is
ergies the total angular momentum. Figure 3(a) shows the
ature of the surface is independent of 

The index \( L \) also determined low-lying excitation energies in a finite-

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FIG. 3: (a) The lowest energy of the Hamiltonian for each angular momentum subspace for $N = 40$, $g_{\text{DD}} = 2\pi \times 7.5 \times 10^{-3}$. (b) Enlargement of (a) near the critical point. Densely packed blue curves are spectra within $L \in \{0, N\}$. Otherwise the spectra are sparse. (c) Two-body correlation function of eigenstates for each angular momentum subspace. (d) Expectation value of the angular momentum of each eigenstate, in the presence of symmetry breaking potential $V$ as a function of $\Omega$.

a noninteger value, which indeed agrees with $J_{\text{sol}}^{(1)}$. We have also calculated the one-body correlation function $g^{(1)}$ for all eigenstates in the presence of the symmetry breaking perturbation, and confirm that $g^{(1)}$ has a single density notch in the soliton regime with the depth close to the mean-field grey soliton.

In conclusion, we found a denumerably infinite set of paths to connect plane-wave states via soliton trains in a metastable system of scalar bosons on a ring. Associated with this transition, the energy of the solitons bifurcates, and a continuous change in the angular momentum becomes possible in the mean-field theory. We made a link between these mean-field results and the full quantum theory by showing that quasidegenerate energy levels are compatible with this transition, the energy of the solitons bifurcates, and confirm that $g^{(1)}$ is a noninteger value, which indeed agrees with $J_{\text{sol}}^{(1)}$. We have also calculated the one-body correlation function $g^{(1)}$ for all eigenstates in the presence of the symmetry breaking perturbation, and confirm that $g^{(1)}$ has a single density notch in the soliton regime with the depth close to the mean-field grey soliton.

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