Effects of Rotation and Magnetic Field on the Revival of a Stalled Shock in Supernova Explosions

Kotaro Fujisawa1,4, Hirotada Okawa1,2,4, Yu Yamamoto1, and Shoichi Yamada1,3

1 Advanced Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
2 Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
3 Science & Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

Abstract

We investigate axisymmetric steady solutions of (magneto)hydrodynamics equations that approximately describe accretion flows through a standing shock wave onto a protoneutron star and discuss the effects of rotation and magnetic field on the revival of the stalled shock wave in supernova explosions. We develop a new powerful numerical method to calculate the two-dimensional steady accretion flows self-consistently. We first confirm the results of preceding papers that there is a critical luminosity of irradiating neutrinos, above which there exists no steady solution in spherical models. If a collapsing star is rotating and/or has a magnetic field, the accretion flows are no longer spherical owing to the centrifugal force and/or Lorentz force, and the critical luminosity is modified. In fact, we find that the critical luminosity is reduced by about 50%–70% for very rapid rotations; the rotation frequencies are 0.2–0.45 s⁻¹ at the radius of r = 1000 km (equivalent to spin periods ~0.5–0.22 ms at r = 10 km) and about 20%–50% for strong toroidal magnetic fields (the strengths of which are 1.0 × 10¹²–3.0 × 10¹⁵ G at r = 1000 km), depending on the mass accretion rate. These results may also be interpreted as the existence of a specific angular momentum or critical magnetic field, above which there exists no steady solution and the standing shock wave will be revived for a given combination of mass accretion rate and neutrino luminosity.

Key words: magnetohydrodynamics (MHD) – methods: numerical – shock waves – stars: rotation – stars: magnetic field – supernovae: general

1. Introduction

Core-collapse supernovae (CCSNe) play important roles in various fields of astrophysics such as star formation, galactic evolution, and the acceleration of cosmic-rays through nucleosynthesis, energetic shock waves, and high luminosities. The large gravitational energy liberated in the collapse makes CCSNe promising sites for the emissions of neutrinos and gravitational waves as well as heavy elements. The understanding of the physical processes in and the mechanism of CCSNe is important for the development of multimessenger astronomy.

CCSNe commence with the gravitational collapse of the core of massive stars at the ends of their lives. When the central density reaches the nuclear saturation density, core bounce occurs and a shock wave is generated. As it propagates outward, the shock loses energy via photodissociations of nuclei and neutrino cooling and eventually stagnates inside the core due to the ram pressure of accreting matter in addition to these energy losses. The stalled shock wave should be revived somehow to produce a successful explosion. It is widely expected that it will be re-energized by the irradiation of matter by neutrinos diffusing out of a protoneutron star (PNS; Wilson 1985). This neutrino-heating scenario is currently the most favored mechanism of CCSNe (Janka et al. 2012; Kotake et al. 2012; Burrows 2013; Foglizzo et al. 2015; Müller 2016; Burrows et al. 2018 for recent reviews).

Intensive and extensive efforts in numerical simulations have revealed so far that no successful explosion is obtained in spherical symmetry (e.g., Liebendörfer et al. 2001, 2005; Rampp & Janka 2002; Sumiyoshi et al. 2005), and that multidimensional effects are crucially important (Burrows et al. 2006; Bruenn et al. 2009; Marek & Janka 2009; Suwa et al. 2010; Müller et al. 2012; Takiwaki et al. 2012; Couch 2013; Couch & Ott 2013; Hanke et al. 2013; Murphy et al. 2013; Lenz et al. 2015; Melson et al. 2015; Nakamura et al. 2015; Bruenn et al. 2016; Roberts et al. 2016; O’Connor & Couch 2018). Among them are rotation (Fryer & Heger 2000; Kotake et al. 2003; Thompson et al. 2005; Marek & Janka 2009; Iwakami et al. 2014a; Nakamura et al. 2014; Takiwaki et al. 2016; Summa et al. 2018), a magnetic field (Akiyama et al. 2003; Kotake et al. 2004; Yamada & Sawai 2004; Sawai et al. 2005; Obergaulinger et al. 2006, 2014, 2018; Burrows et al. 2007; Takiwaki et al. 2009; Sawai & Yamada 2014, 2016; Mösta et al. 2015), non-spherical structures of the progenitor (Couch & Ott 2013; Takahashi & Yamada 2014; Couch et al. 2015; Takahashi et al. 2016), turbulence (Murphy & Burrows 2008; Murphy & Meakin 2011; Murphy et al. 2013; Couch & Ott 2015; Mabanta & Murphy 2018), (magneto)hydrodynamical instabilities (Blondin et al. 2003; Scheick et al. 2006; Blondin & Mezzacappa 2007; Iwakami et al. 2008; Guilet et al. 2010; Wongwathanarat et al. 2010; Fernández et al. 2014; Takiwaki et al. 2014; Fernández 2015), general relativistic gravity (Dimmelmeier et al. 2002; Shibata & Sekiguchi 2004, 2005; Kuroda et al. 2012, 2016; Müller et al. 2012; Ott et al. 2012), and neutrino transport (Nagakura et al. 2014, 2017, 2018; Dolence et al. 2015; Pan et al. 2016). It is true that large-scale dynamical simulations have played a crucial role in recent progresses in our understanding of these ingredients, but we believe that a more phenomenological approach that employs toy models still plays an indispensable and complementary role to understand each effect more deeply.
Burrows & Goshy (1993) took such an approach and introduced the concept of critical neutrino luminosity. They approximated the accretion flows after shock stagnation with spherical steady solutions of the hydrodynamics (HD) equations that incorporate neutrino heating of matter; they found then that for a given mass accretion rate, there is a critical neutrino luminosity, above which no steady solution exists; they surmised that the revival of the stalled shock wave should occur there. The existence of such a critical neutrino luminosity was confirmed both in similar phenomenological methods (Yamasaki & Yamada 2005, 2007; Keshet & Balberg 2012; Murphy & Dolence 2017) and in numerical simulations (Janka & Mueller 1996; Ohnishi et al. 2006; Iwakami et al. 2008, 2014a, 2014b; Murphy & Burrows 2008; Nordhaus et al. 2010; Hanke et al. 2013; Suwa et al. 2016). The reason for this is probably the difficulty in numerically obtaining non-spherical steady solutions of the HD equations. Even in Yamasaki & Yamada (2005), the number of solutions produced was not large. Not to mention, there has been no attempt to incorporate magnetic field in such approaches.

In this paper, we develop a new scheme to numerically obtain axisymmetric steady solutions of HD and magnetohydrodynamics (MHD) equations that describe self-consistently non-spherical post-shock accretion flows in the presence of rotation and/or magnetic field. Based on these solutions, we study the effects of rotation and magnetic field on the critical neutrino luminosity systematically. This paper is organized as follows. In Section 2, we formulate the problem and give the basic equations to be solved. We then explain our new numerical method. In Section 3, we present numerical solutions and the critical luminosities obtained with them. Finally, we give some discussions and summarize this paper in Section 4.

2. Formulations

2.1. Assumptions and Basic Equations

After the shock stalls, the mass accretion rate and neutrino luminosity change slowly, and the post-shock accretion flows may be approximated as steady states with constant mass accretion rates and neutrino luminosities (Burrows & Goshy 1993; Yamasaki & Yamada 2005). We make the following assumptions to construct idealized models in this paper.

1. Equatorial symmetry is assumed in addition to stationarity ($\partial / \partial t = 0$) and axisymmetry ($\partial / \partial \phi = 0$).
2. We ignore both viscosity and magnetic resistivity and consider only ideal HD or MHD equations.
3. We consider the accretion flows only in the domain between the stalled shock wave and the neutrinosphere, and impose the outer and inner boundary conditions there.
4. We assume that the neutrinosphere is spherical and the neutrino flux is isotropic for simplicity.
5. The PNS is treated as a point mass with $1.3 M_\odot$. Only its Newtonian gravitational forces are considered, and the self-gravity of accreting matter is ignored for simplicity.
6. We ignore the neutrino transfer again for simplicity, and the luminosity and energy spectrum of neutrinos are assumed to be constant in radius (see Equations (4) and (5)).
7. A simplified equation of state (EOS) that ignores the photodissociations of nuclei is adopted, and convection is not taken into account.

Under these assumptions, the basic equations are given as

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \rho u_\theta) = 0,$$

(1a)
where $\rho$, $u_r$, $B_r$, $p$, $\varepsilon$, and $\dot{q}$ denote, respectively, the density, velocity, magnetic field, pressure, specific internal energy, and neutrino-heating rate; $G$ and $M$ are the gravitational constant and the mass of PNS. In the absence of the magnetic field, $B$ is simply set to zero. In its presence, on the other hand, we also solve the Maxwell equations under the conditions of stationarity and ideal MHD (i.e., $\nabla \cdot B = 0$, $\nabla \times E = 0$, $E = -u \times B$) as follows:

$$u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{\partial u_\theta}{\partial r} - \frac{u_\theta^2 + u_r^2}{r} = -\frac{1}{\rho} \frac{\partial \rho}{\partial r} - \frac{GM}{r^2},$$

$$- \frac{1}{4\pi\rho} \left[ B_r \frac{\partial B_\theta}{\partial r} + B_\theta \frac{\partial B_r}{\partial r} + \frac{B_r^2 + B_\theta^2}{r} \right],$$

$$u_r \frac{\partial u_\theta}{\partial r} + u_\theta \frac{\partial u_r}{\partial r} + u_r \frac{\partial u_\theta}{\partial r} - \frac{u_\theta^2 \cot \theta}{r} = -\frac{1}{\rho \rho} \frac{\partial \rho}{\partial \theta}$$

$$+ \frac{1}{4\pi\rho} \left[ B_r \frac{\partial B_\theta}{\partial \theta} - B_\theta \frac{\partial B_r}{\partial \theta} \right]$$

$$\frac{\partial \varepsilon}{\partial r} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial r} + \frac{u_r \left( \frac{\partial \varepsilon}{\partial \theta} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial \theta} \right)}{r} = \dot{q},$$

where $\rho$, $u_r$, $B_r$, $p$, $\varepsilon$, and $\dot{q}$ denote, respectively, the density, velocity, magnetic field, pressure, specific internal energy, and neutrino-heating rate; $G$ and $M$ are the gravitational constant and the mass of PNS. In the absence of the magnetic field, $B$ is simply set to zero. In its presence, on the other hand, we also solve the Maxwell equations under the conditions of stationarity and ideal MHD (i.e., $\nabla \cdot B = 0$, $\nabla \times E = 0$, $E = -u \times B$) as follows:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 B_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta B_\theta \right) = 0,$$

$$- u_\theta \frac{\partial B_r}{\partial \theta} + u_r \frac{\partial B_\theta}{\partial \theta} - B_r \frac{\partial u_\theta}{\partial \theta}$$

$$+ B_\theta \frac{\partial u_r}{\partial \theta} + u_r B_\theta \cot \theta = 0,$$

$$ru_\theta \frac{\partial B_r}{\partial r} - ru_r \frac{\partial B_\theta}{\partial r} - r B_r \frac{\partial u_r}{\partial r}$$

$$+ r B_\theta \frac{\partial u_\theta}{\partial r} + B_r u_\theta - B_\theta u_r = 0,$$

$$ru_r \frac{\partial u_\theta}{\partial r} - ru_\theta \frac{\partial u_r}{\partial r} + ru_\theta \frac{\partial B_r}{\partial r} - ru_r \frac{\partial B_\theta}{\partial r}$$

$$+ B_r \frac{\partial u_\theta}{\partial \theta} - B_\theta \frac{\partial u_r}{\partial \theta} + u_r \frac{\partial \varepsilon}{\partial \theta}$$

$$- u_\theta \frac{\partial \varepsilon}{\partial \theta} + B_r u_\theta - B_\theta u_r = 0,$$

where $E_r$ is the electric field. Following Yamasaki & Yamada (2005), we employ a simplified EOS given as follows:

$$p = \frac{11\pi^2}{180} \frac{k^4}{c^3 h^3} T^4 + \frac{\rho k T}{m_N},$$

$$\varepsilon = \frac{11\pi^2}{60} \frac{k^4}{c^3 h^3} T^4 + \frac{3 \rho T}{2 m_N},$$

where $k$, $c$, $h$, $m_N$, and $T$ are the Boltzmann constant, speed of light, reduced Planck constant, nucleon mass, and matter temperature, respectively. Note that this EOS ignores photodissociations of nuclei and, as a consequence, the critical luminosity tends to be overestimated. Since we are interested in the qualitative effects of rotation and/or magnetic field on the critical luminosity in this paper, the simplification is justified. The net neutrino-heating rate is also simplified as in Herant et al. (1992),

$$\dot{q} = 4.8 \times 10^{32} \left[ 1 - \sqrt{1 - \frac{\rho^2}{\rho^2}} \right] \frac{L_\nu}{2\pi r_\nu^2} T_\nu^2$$

$$- 2.0 \times 10^{48} T^6 \text{ (erg s}^{-1} \text{ g}^{-1}),$$

where the neutrino luminosity $L_\nu$, and temperature $T_\nu$, are model parameters, which are constant both in time and space, and common to $\nu_e$ and $\nu_\mu$; we assume further that they are related to the radius of the neutrinosphere $r_\nu$, which is coincident with the inner boundary in our models, as

$$L_\nu = \frac{7}{4} \pi r_\nu^2 \sigma_s T_\nu^4,$$

where $\sigma_s$ denotes the Stefan–Boltzmann constant. The neutrino temperature, which characterizes the neutrino energy spectrum, is set to $T_\nu = 4.5$ MeV, following Yamasaki & Yamada (2005).

2.2. Boundary Conditions

The region of our interest is the one between the neutrinosphere and the stalled shock wave, and the inner and outer boundary conditions are imposed there, respectively. Following Yamasaki & Yamada (2005), we impose $\rho = 10^{14}$ g cm$^{-3}$ on the inner boundary, which approximately corresponds to the condition that the optical depth to the neutrinosphere from infinity is equal to 2/3. The latter condition was adopted by Burrows & Goshy (1993) and Murphy & Burrows (2008; see Keshet & Balberg 2012 for a comparison of these two conditions). The radius of the neutrinosphere is obtained from the neutrino luminosity and temperature in Equation (5).

The main difficulty in obtaining steady accretion flows through the stalled shock wave in multiple dimensions stems from the simple fact that the shock surface is non-spherical. In this paper, we assume that the shape of the shock surface expressed as $r = r_s(\theta)$ in polar coordinates is given with the Legendre function $P_2$ as

$$r_s(\theta) = r_{sph} \{ 1 - \epsilon P_2(\cos \theta) \},$$

$$P_2(\cos \theta) = \frac{1}{2} (3 \cos^2 \theta - 1),$$

where $r_{sph}$ and $\epsilon$ are the equatorial radius and deformation amplitude of the shock surface, respectively. Instead of attempting to obtain the shock surface directly, we employ these two degrees of freedom alone, which turn out to be sufficient (see Section 2.4). Once the shock shape is determined, the post-shock quantities are obtained from the pre-shock quantities via the Rankine–Hugoniot relations, which are given in Appendix A. We assume that the pre-shock flows are spherically symmetric free falls from infinity with different mass accretion rates. We do not attempt in this paper to elaborate on the outer flow so that rotation and/or magnetic field would be taken into account consistently.
Other boundary conditions are also administered there at \( \theta = 0 \) and \( \theta = \pi/2 \). As in Yamasaki & Yamada (2005), we impose Neumann conditions for \( \rho, \ u_r, \ T, \ B_r \) at \( \theta = 0, \pi/2 \), and for \( u_\varphi, \ B_\varphi \) at \( \theta = \pi/2 \) while we set Dirichlet conditions for \( u_\theta \), \( B_\theta \) at \( \theta = 0 \) and for \( u_\theta \) and \( B_\theta \) also at \( \theta = \pi/2 \) as follows:

\[
\begin{align*}
\frac{\partial \rho}{\partial \theta} &= \frac{\partial u_r}{\partial \theta} = \frac{\partial T}{\partial \theta} = \frac{\partial B_r}{\partial \theta} = 0, \\
\frac{\partial u_\theta}{\partial \theta} &= B_\theta = u_\varphi = B_\varphi = 0, \quad (7a) \\
\frac{\partial \rho}{\partial \theta} &= \frac{\partial u_r}{\partial \theta} = \frac{\partial T}{\partial \theta} = \frac{\partial B_r}{\partial \theta} = 0, \quad (7b)
\end{align*}
\]

at \( \theta = 0 \) and

\[
\begin{align*}
\frac{\partial \rho}{\partial \theta} &= \frac{\partial u_r}{\partial \theta} = \frac{\partial T}{\partial \theta} = \frac{\partial B_r}{\partial \theta} = \frac{\partial u_\varphi}{\partial \theta} = \frac{\partial B_\varphi}{\partial \theta} = 0, \quad (8a) \\
\frac{\partial u_\theta}{\partial \theta} &= B_\theta = 0, \quad (8b)
\end{align*}
\]

at \( \theta = \pi/2 \).

2.3. Rotation Law and Magnetic Field Profiles

In the presence of a magnetic field, the rotation law and the magnetic field profile should be given consistently with each other, since there are some conserved quantities along the streamline that constrain the variations of the rotation and magnetic field for axisymmetric and steady ideal MHD flows (see Lovelace et al. 1986; Fujisawa et al. 2013). The details of their derivations are described in Appendix B.

If a magnetic field is purely toroidal \( (B_r = B_\theta = 0) \), the specific angular momentum \( \ell \) and a quantity related to the magnetic torque \( \sigma \) are conserved along the streamline:

\[
\begin{align*}
\ell(\psi) &= r \sin \theta \dot{u}_r, \\
\sigma(\psi) &= \frac{B_\varphi}{r \sin \theta \rho}, \quad (9a) \\
\end{align*}
\]

where \( \psi \) is the stream function defined so that the following relations should be

\[
\begin{align*}
u_r &= \frac{1}{4 \pi \rho r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = \frac{1}{4 \pi \rho r \sin \theta} \frac{\partial \psi}{\partial r}. \quad (10)
\end{align*}
\]

Instead of fixing the functional forms of \( \ell \) and \( \sigma \) explicitly, we give the rotational velocity and magnetic field strength at the outer boundary to satisfy Equations (7) and (8). Following Yamasaki & Yamada (2005), we assume that a spherical shell is rotating rigidly at \( r = 1000 \text{ km} \) as

\[
\dot{u}_r(r_{1000 \text{ km}}, \theta) = 2 \pi r_{1000 \text{ km}} \sin \theta \theta f_{1000 \text{ km}}. \quad (11)
\]

Since the specific angular momentum \( \ell \) is conserved along an individual streamline, the rotational velocity just above the shock surface is given as

\[
\dot{u}_r(r, \theta) = 2 \pi r^2 r_{1000 \text{ km}} \sin \theta \theta f_{1000 \text{ km}}. \quad (12)
\]

if the streamlines are assumed to be radial. Yamasaki & Yamada (2005) obtained steady solutions only for \( f_{1000 \text{ km}} = 0.03 \text{ s}^{-1} \) (slow rotation) and \( f_{1000 \text{ km}} = 0.1 \text{ s}^{-1} \) (fast rotation). We consider a larger variety in this paper.

As for the toroidal magnetic field, we assume the following profile at the same radius of 1000 km as

\[
B_\varphi(r_{1000 \text{ km}}, \theta) = B_{1000 \text{ km}} \sin \theta, \quad (13)
\]

where \( B_{1000 \text{ km}} \) gives the strength of the toroidal magnetic field there. Since \( \sigma \) in Equation (9) is a conserved quantity, the toroidal magnetic field just outside the shock surface is determined as

\[
B_\varphi(r, \theta) = B_{1000 \text{ km}} \sqrt{r_{1000 \text{ km}} \sin \theta}, \quad (14)
\]

if we again assume a radial infall upstream of the shock wave (see Appendix B). Note that these functional forms in Equations (11) and (13) satisfy the boundary conditions given in Equations (7) and (8).

If the magnetic field has poloidal and toroidal components, both of them should be fixed at the outer boundary. If we again assume a radial free fall outside the shock, the poloidal magnetic field lines should be also radial. Then the continuity equation of the magnetic field gives the poloidal magnetic field just outside the shock surface as

\[
B_r(r, \theta) = \frac{r_{1000 \text{ km}}}{r} B_\theta, \quad B_\theta(r, \theta) = 0, \quad (15)
\]

where \( B_\theta \) is a constant giving the strength of the poloidal magnetic field at \( r = 1000 \text{ km} \). When the poloidal magnetic field is non-zero, the specific angular momentum and the toroidal magnetic field are no longer independent of each other (Fujisawa et al. 2013), and Equations (12) and (14) are not valid. The values of \( u_\varphi(r_s, \theta) \) and \( B_\varphi(r_s, \theta) \) just ahead of the shock surface are then obtained by solving Equations 37(a) and 39(a) in Appendix B numerically.

2.4. Numerical Scheme and Parameter Settings

We briefly summarize our new numerical scheme to obtain steady solutions here. More details are given in Appendix C. Each steady solution is characterized by five parameters, \( M, L_\nu, f_{1000 \text{ km}}, B_{1000 \text{ km}}, \) and \( B_\theta \). We discretize Equations (1) and (2) and obtain nonlinear systems of equations. Setting the above five parameters and assuming the values of \( r_s \) and \( \epsilon \) in Equation (6), we solve these equations numerically from the shock surface down to the neutrinosphere. If the density \( \rho \) at the neutrinosphere so obtained does not satisfy the requirement \( \rho = 10^{13} \text{ g cm}^{-3} \), we modify the values of \( r_s \) and \( \epsilon \) and repeat the procedure. \( r_s \) and \( \epsilon \) are hence regarded as the eigenvalues in the boundary-value problem for this system. In order to obtain the value of the critical neutrino luminosity, we calculate a sequence of solutions for fixed values of four parameters \( (M, f_{1000 \text{ km}}, B_{1000 \text{ km}}, \) and \( B_\theta) \), changing the value of \( L_\nu \) until a steady-state solution is no longer obtained. We develop a new numerical method dubbed the W4 method (Okawa et al. 2018), which is meant to solve the nonlinear systems of equations. The details of the W4 method are described in Appendix D (see also our recent paper Okawa et al. 2018). The accuracy and convergence of numerical results are checked in Appendix E.

Information on the rotation and magnetic field deep inside the progenitor is scarce. In general, massive stars, such as O- and B-type stars, tend to rotate rapidly (e.g., Hunter et al. 2008). Approximately 10% of them have surface rotational velocities larger than 300 km s\(^{-1}\) (Ramírez-Aguadero et al. 2013). On the other hand, recent surveys for massive stars indicate that some Galactic O- and B-type stars have magnetic fields of 100–1000 G at their surfaces (Wade & MiMeS Collaboration 2015). Their total magnetic fluxes roughly
coincide with those of magnetars whose magnetic field is about $10^{14}$–$10^{15}$ G on their surfaces. According to the stellar evolution models, the toroidal magnetic field is likely to be larger than the poloidal one by orders of magnitude, due to the differential winding inside massive stars (Heger et al. 2005). Some progenitors may hence have a rapidly rotating and/or strongly magnetized core.

On the other hand, stellar evolution models also indicate that the transport of angular momentum during the quasi-static phase of the progenitor reduces the angular momentum of the core, particularly if the magnetic torque is taken into account (e.g., Heger et al. 2005; Potter et al. 2012). In fact, the specific angular momentum contained in between the enclosed masses of 1.3 $M_\odot$ and 2 $M_\odot$, where we calculate the steady solution in this paper, is estimated to be $j \sim 10^{16}$–$10^{17}$ cm$^2$ s$^{-1}$ if the magnetic field is not taken into account and $j \sim 10^{14}$–$10^{15}$ cm$^2$ s$^{-1}$ if it is (Heger et al. 2005). These specific angular momenta roughly correspond to $f_{1000_{\text{km}}} \sim 0.1$–1 s$^{-1}$ and $\sim 0.001$–0.01 s$^{-1}$ for the non-magnetized and magnetized cases, respectively. From observations, some young pulsars are estimated to have been born with $P \sim 20$ ms (e.g., Marshall et al. 1998), the spin period that corresponds to $f_{1000_{\text{km}}} \sim 0.005$ s$^{-1}$ if angular momentum is conserved during the collapse. Note that the rotation of the completely newly born pulsar may have been modestly spun down, however, by the dynamical motion of the accreting matter (Kazeroni et al. 2017) and other mechanisms such as wind and magnetic braking, although they may not be efficient for rapidly rotating progenitors (Ott et al. 2006). The rotation might not be so rapid and does not play an important role in the dynamics of core collapse after all. One should bear in mind, however, that almost all stellar and PNS evolution calculations are based on spherically averaged one-dimensional models and have uncertainties in the mechanism and formulation of angular momentum transport and magnetic field.

The aim of this paper is to systematically study the effects of rapid rotation and/or a strong magnetic field, if any exists, on the critical neutrino luminosity in CCSNe using the new numerical scheme. We set the model parameters as $f_{1000_{\text{km}}} \sim 0$–0.45 s$^{-1}$, $B_{1000_{\text{km}}} \sim 0$–$3.0 \times 10^{12}$ G, and $B_{B} \sim 0$–$3 \times 10^{14}$ G. They are almost the same as those in the non-magnetized stellar evolution models and are much faster than those in the magnetized counterparts. In fact, they roughly correspond to the rotation frequency of $f \sim 0$–5 m s$^{-1}$ and the magnetic field strength of $B \sim 0$–$5 \times 10^{14}$ G on the PNS surface. If the angular momentum were conserved completely during the collapse and a rigidly rotating neutron star (NS) with a radius of 10 km were born as a result, the spin period of the NS would be $P_{\text{NS}} \gtrsim 0.2$ ms. Note that the spin period of NS $\sim 0.2$ ms almost corresponds to the breakup, at which the rotational energy becomes comparable to the gravitational energy and the rotational equilibrium ceases to exist. The magnetic field strength of $\gtrsim 10^{14}$ G is canonical on the surface of magnetars. The neutrino temperature and the PNS mass are fixed to $T_{\nu} = 4.5$ MeV and $M = 1.3 M_\odot$, respectively, in this paper.

3. Numerical Results

3.1. Streamlines of Steady Accretion Flow

Figure 1 displays streamlines in a meridian plane for a model either with rotation (left panel) or with a toroidal magnetic field (right panel). In these models, the neutrino luminosity and accretion rate are set as $L_{\nu} = 26 \times 10^{51}$ erg s$^{-1}$ and $M = 0.5 M_\odot$ s$^{-1}$. The innermost black dotted curve indicates the neutrinosphere, whereas the outermost black dashed curve indicates the shock surface in each panel. In the left panel, the dashed-dotted lines denote the profile of the rotational velocity $u_{\text{rot}}$, the rotational frequency is $f_{1000_{\text{km}}} = 0.2$ s$^{-1}$, and the value of $\epsilon$ for shock deformation is $\epsilon \sim 2.0 \times 10^{-3}$. In the right panel, the dashed-dotted lines show the profile of the toroidal magnetic field $B_{B}$; the strength of the toroidal magnetic field at the outer boundary is $B_{1000_{\text{km}}} = 2 \times 10^{12}$ G and $\epsilon \sim 1.8 \times 10^{-3}$.
the right panel, on the other hand, the flow is bent toward the symmetry axis. This is due to the Lorentz force exerted by the toroidal magnetic field, which indeed behaves as a negative centrifugal force (e.g., Fujisawa 2015). Note that the flow patterns of both models are similar near the rotation axis because of the boundary condition for $u_0$ on the axis (Equation (7)).

The left panel in Figure 2 shows the result for a model with both rotation and a toroidal magnetic field. The mass accretion rate, neutrino luminosity, and strength of the toroidal magnetic field are set to the same values as those in Figure 1, but the rotation frequency is somewhat smaller. We find $\epsilon = 2.3 \times 10^{-3}$, slightly larger than in the previous cases in Figure 1. The centrifugal force and the Lorentz force almost cancel each other in this solution. The streamlines are nearly radial in this model except for the inner region where the Lorentz force is dominant over the centrifugal force, and the streamlines look similar to those for the purely toroidal magnetic field. This is understood from the conserved quantities $\ell$ and $\sigma$ along the streamline. As a matter of fact, $\ell$ does not depend on the density while $\sigma$ does, as is clear in Equation (9). Since the density is low near the shock surface $\rho \sim 10^9 \text{ g cm}^{-3}$ and gets higher toward the neutrinosphere, where $\rho \sim 10^{11} \text{ g cm}^{-3}$, the Lorentz force tends to be dominant in the inner region and vice versa.

The right panel in Figure 2 shows streamlines for a model that incorporates poloidal magnetic fields in addition to toroidal ones. The neutrino luminosity and accretion rate are set to $L_\nu = 70 \times 10^{51} \text{ erg s}^{-1}$ and $\dot{M} = 2.0 \times 10^3 M_\odot \text{ s}^{-1}$. The rotational frequency and the strength of magnetic fields are $f_{1000 \text{ km}} = 0.2 \text{ s}^{-1}$, $B_{1000 \text{ km}} = 10^{12} \text{ G}$, and $B_0 = 10^{11} \text{ G}$. Since the value of $B_0$ is smaller than $B_{1000 \text{ km}}$, the poloidal magnetic field in this model is much weaker than the toroidal magnetic field, the situation observed in realistic simulations. We find that the value of $\epsilon$ is $\epsilon = 2.3 \times 10^{-3}$ again. Note that the poloidal magnetic field lines are aligned with the streamlines in ideal MHD. They are slightly curved near the neutrinosphere similarly to the left panel in Figure 1 because of the Lorentz force mainly exerted by the toroidal magnetic field.

Figure 3 displays the radial and azimuthal components of velocity (left) and magnetic field (right) at $\theta = \pi/4$ and $\theta = \pi/2$. The radial components of the flow velocity $u_r$ and of the poloidal magnetic field $B_r$ are almost identical at these zenith angles. This is mainly because of our assumption that the flow and the poloidal magnetic field outside the shock surface are radial and independent of $\theta$. If we had assumed a $\theta$-dependent functional form in Equation (15), for example, we would have found $\theta$-dependent radial profiles of $B_r$. In contrast, both $u_\varphi$ and $B_\varphi$ depend on $\theta$, being larger at the equator ($\theta = \pi/2$) than at $\theta = \pi/4$. This $\theta$ dependence again ($\propto \sin \theta$) simply reflects the functional forms for $u_\varphi$ and $B_\varphi$ in Equations (11) and (13), though.

The rotational velocity and toroidal magnetic field both increase inwards from the shock surface to the neutrinosphere. It is apparent, however, that the toroidal magnetic field rises more steeply than the rotational velocity. This trend is almost independent of the functional forms for $u_\varphi$ and $B_\varphi$. In fact, it is dictated by the conservation of $\ell$ and $\sigma$, the latter of which depends on the density profile as is explicit in Equation (35) in Appendix B. One may hence roughly say that the $\theta$ dependence of the flow velocity and magnetic field is mainly determined by their dependence just ahead of the shock wave, while their radial profiles are largely constrained by the conservation laws (Equation 35), being almost independent of the outer boundary conditions.

The values of $\epsilon$ found in our models may look very small, actually much smaller than naive expectations from the rotation rates we assumed here. Note, however, that in the presence of the non-vanishing flows in the meridian section, the oblateness is not simply determined by rotation alone but is affected by the ram pressure as well. In fact, the advection in the meridian section tends to reduce the oblateness produced by the centrifugal force (Fujisawa & Eriguchi 2014). Note also that the rotational
velocities in our models are much smaller than the meridional flow velocities even when \( f_{1000 \text{ km}} \sim 0.1 \text{ s}^{-1} \) as shown in Figure 3. It is hence not surprising that the oblateness is small in our models. It should be also mentioned, however, that our assumption that the neutrinospheres are spherically symmetric should also contribute to the small oblateness in our models.

3.2. Effects of Rotation and Magnetic Field on the Critical Luminosity

Now, we look into the changes in the critical luminosity that rotation and/or the magnetic field produce. In this subsection, we ignore poloidal magnetic fields \((B_0 = 0)\) because the toroidal magnetic field is supposed to be dominant. Figure 4 displays the critical luminosities either for models with rotation alone (left panel) or for models with a toroidal magnetic field alone with no rotation (right panel), which are plotted as a function of the mass accretion rate. Note that the typical mass accretion rate is less than \( \sim 1.0 \ M_\odot \text{ s}^{-1} \) (e.g., 0.2 \( M_\odot \text{ s}^{-1} < M < 0.6 \ M_\odot \text{ s}^{-1} \) for the silicon or oxygen layer; Suwa et al. 2016; Yamamoto & Yamada 2016) at a few 100 ms after bounce when shock revival is expected. We hence set the mass accretion rate to 0.25 \( M_\odot \text{ s}^{-1} \)–1.5 \( M_\odot \text{ s}^{-1} \) in this study. As a reference, we also plot the critical luminosity of the spherically symmetric models without rotation and a toroidal magnetic field as black circles.

It is evident among other things that the critical luminosity for all these models with either rotation or a toroidal magnetic field is lower than that for the spherical models. Non-spherical forces, i.e., the centrifugal force and hoop stresses, tend to reduce the critical luminosity, although the reduction rate depends on the mass accretion rate. As a matter of fact, for a given rotation velocity or magnetic field strength at the outer boundary, the critical luminosity gets lowered more strongly as the mass accretion rate becomes smaller: the reduction rate is as high as about 50%–70% for the rapid rotation with \( f_{1000 \text{ km}} \sim 0.2 \text{ s}^{-1} \) and 20%–50% for the strong toroidal magnetic field with \( B_{1000 \text{ km}} = 10^{12} \ G \) at mass accretion rates lower than 1.0 \( M_\odot \).

Figure 5 displays the profiles of the radial velocity \( u_r \) along three radial rays with \( \theta = 0 \) (red solid line), \( \pi/4 \) (orange dotted line), and \( \pi/2 \) (blue dashed line) for representative models on the critical curve either with rotation alone (left panel) or with toroidal magnetic field alone (right panel). The three profiles are almost the same at large radii in both panels, whereas they are different from each other in the inner region. In the left panel, the green triangle and red squares correspond to rotation rates \( f_{1000 \text{ km}} = 0.1 \text{ s}^{-1} \) and \( f_{1000 \text{ km}} = 0.2 \text{ s}^{-1} \). In the right panel, the field strength of \( B_{1000 \text{ km}} = 10^{11} \ G \) and \( B_{1000 \text{ km}} = 10^{12} \ G \).
The neutrino-heating rate is normalized by the strength of the toroidal magnetic neutrino luminosities for the models with rotation alone. In the panels for pure rotation, the radial infall velocity decreases more rapidly near the neutrinosphere along the rotation axis \((\theta = 0)\) than at the equator \((\theta = \pi/2)\). This tendency was already observed in a previous work \cite{Yamasaki & Yamada 2005}. The radial profile of \(u_r\) on the equator is convex at large radii but turns concave near the neutrinosphere, whereas it remains convex along the polar axis as in the spherically symmetric case. Considering the fact that in spherical symmetry the profile of the radial velocity is still convex near the inner boundary at the critical luminosity \cite{Yamasaki & Yamada 2005}, one may say that only the polar flow reaches a critical state locally and shock revival will commence there.

Interestingly, the situation is the other way around in the right panel for a purely toroidal magnetic field. This is because such fields exert hoop stress, which behaves like a negative centrifugal force. As a result, the radial inflow velocity is smaller at the equator than at the axis near the neutrinosphere. According to the same argument, this may imply that only the equatorial flow becomes critical locally, and shock revival will be initiated on the equator. It is intriguing that the existence/non-existence of steady solutions that satisfy a certain boundary condition is itself a global issue while the critical state is realized locally as a result. Although some recent studies strongly deduce the presence of global conditions for the critical state in spherical symmetry \cite[e.g.,][]{Murphy & Dolence 2017}, one may hence claim that the critical neutrino luminosity is determined both globally and locally.

Figure 6 displays the streamlines (left panel) and the radial profiles of the net heating rates along some radial rays (right panel) in the meridian section for one of the purely rotational models at its critical luminosity. The mass accretion rate and neutrino luminosity are set to \(M = 1.0 \, M_\odot \text{ s}^{-1}\) and \(L_\nu = 52 \times 10^{51} \text{ erg s}^{-1}\), respectively. The rotational parameter is \(f = 0.2 \, \text{s}^{-1}\), and the neutrino-heating rate is normalized by \(\epsilon\). In the left panel, the thick curve indicates the boundary between the heating region and the cooling region. In the right panel, the three lines correspond to different zenith angles: \(\theta = \pi/8\) (red solid line), \(\theta = \pi/4\) (green dashed line), and \(\theta = \pi/2\) (blue dotted line).
for simplicity in this paper that the neutrinosphere is spherical (see also Equation (4)), which is certainly not true and tends to reduce the difference. We emphasize, however, that this slightly deformed cooling region results in the reduction of the critical neutrino luminosity as observed earlier.

Figure 7 shows the distribution of the plasma $\beta$ defined as

$$\beta = \frac{8\pi p}{|B| s^2},$$

in the meridian section for one of the non-rotating but purely toroidally magnetized models close to its critical luminosity. The value of $\beta$ is very high, $>10^3$, almost everywhere in the computational domain, meaning that the magnetic pressure is much smaller than the matter pressure except near the equatorial area on the neutrinosphere, where the value of $\beta$ is much smaller. This locally strong magnetic pressure (or locally low $\beta$) modifies the accretion flow there and gives rise to the critical state locally, leading to the reduction of the critical luminosity, as shown in the right panel of Figure 5.

Finally, we show in Figure 8 the critical luminosity as a function of the rotational rate $f_{1000 \text{ km}}$ (left) and the strength of the toroidal magnetic field $B_{1000 \text{ km}}$ (right). The mass accretion rate is fixed to $M = 0.5 M_\odot \text{ s}^{-1}$. On the top horizontal axis in the left panel is also given the spin period that the fluid element would have on the NS surface if the specific angular momentum were conserved.
comes too close to the neutrinosphere. These numerical results may imply the existence of a critical specific angular momentum and a critical strength of the toroidal magnetic field, above which there actually exist no steady solutions.

4. Discussion and Summary

We have numerically derived steady, non-spherical accretion flows through the standing shock wave onto the PNS in the CCSNe core and studied the effects of rotation and magnetic field on the shock revival in this paper. In order to obtain these steady solutions, we have developed the new numerical scheme to solve a system of generic nonlinear equations, which we named the W4 method. We have indeed succeeded in generating various accretion flows, with both rotation and magnetic field being incorporated self-consistently in axisymmetric 2D. It should be noted that our new method can handle both poloidal and toroidal magnetic fields simultaneously.

Our main findings are summarized as follows.

1. The shock surface and the flow pattern become non-spherical by rotation and/or a magnetic field in general. The shock surface is always oblate \((\epsilon > 0)\), whereas the streamlines are bent either toward the equatorial plane by rotation and a poloidal magnetic field or toward the symmetric axis by a toroidal magnetic field (Figures 1 and 2).

2. The toroidal magnetic field is dominant in the inner region (Figure 7) while the rotation is dominant in the outer region (Figures 2 and 3) because of the conserved quantities \(\sigma\) and \(\ell\) in Equations (9) and (35).

3. The critical luminosity is lowered by rotation and/or magnetic field in general, although the degree of reduction depends on the mass accretion rate (Figure 4).

4. In the presence of rotation and/or magnetic field, the critical state is realized locally: either on the symmetry axis for rotation or on the equatorial plane for a toroidal magnetic field (Figure 5), despite the solution itself being determined globally according to the boundary conditions.

5. The gain region in the accretion flow is also deformed. The cooling region is widened near the pole by rotation (Figure 6), whereas it is affected mostly near the equator by a toroidal magnetic field (Figure 7). Although the deviation from spherical symmetry is small, it results in the substantial reduction of the critical neutrino luminosity.

6. We have demonstrated the existence of the non-vanishing critical specific angular momentum, above which no steady solution exists irrespective of the neutrino luminosity (left panel in Figure 8). Our results are roughly consistent with the results of the dynamical simulations in Iwakami et al. (2014a). We have also found that there exists a critical strength of the toroidal magnetic field, above which no steady solution exists (right panel in Figure 8).

We assumed that the neutrinosphere is spherical and the neutrino flux is isotropic because our concern in this paper is the (M)HD effect of rotation and magnetic field on the accretion flow and the critical luminosity. If the PNS rotates rapidly, the neutrinosphere will become non-spherical by centrifugal force and the neutrino flux will be anisotropic. This may have some influences on the critical luminosity, but we will defer the study of such effects to future work. Note that some of the solutions we obtained have strong rotation and/or magnetic field, and they might be unstable to various (M)HD instabilities (see, e.g., Yamasaki & Yamada 2007; Yamasaki & Foglizzo 2008; Dhang et al. 2018). As we mentioned in the introduction, we are interested in the effects of rotation and/or magnetic field on the critical luminosity, and the aim of this paper is to elucidate them qualitatively. These rather extreme cases are useful for that purpose, although they may not be realized actually.

We have also ignored other non-spherical physical processes, such as the turbulence in progenitors in this paper. Turbulence in the accretion flow expedites explosion by changing the flow property. Murphy & Burrows (2008) performed 1D and 2D dynamical simulations and suggested that the reduction of critical luminosity is caused by turbulence. Murphy & Meakin (2011) examined many turbulent models using the Reynolds decomposition and proposed a global turbulence model that reproduces the profiles and time evolutions of the simulations (see also the 2D and 3D dynamical simulations by Murphy et al. 2013). Recently, Mabanta & Murphy (2018) considered 1D spherical steady models with turbulence and contended that turbulent dissipation rather than ram-incorporated pressure from turbulence reduces the critical luminosity. Note that these works ignored the effect of magnetic fields. Masada et al. (2015), on the other hand, performed local simulations of the magneto-rotational-instability (MRI) in a 3D thin layer near the neutrinosphere with a high spatial resolution and found that the convectively stable layer around the neutrinosphere becomes fully turbulent due to the MRI. They hence concluded that magnetic field plays an important role in the generation of turbulence.

Vorticities in the accretion flow may be another important physical agent (Huete et al. 2017). In fact, Huete et al. (2018) recently studied vorticity waves in accretion flows with a perturbative approach and found that the interaction between the vorticity wave and the shock surface reduces the critical luminosity.

Another interesting multidimensional effect that has been discussed in recent years is non-spherical structures in progenitors. Violent convections in the silicon/oxygen layers affect the shock revival. Couch et al. (2015) performed 3D simulations of the last few minutes of the massive star evolution up to collapse and demonstrated that the non-spherical structures of the progenitor induced by the convection have a significant and favorable impact on explosion (see also Müller et al. 2017). As a matter of fact, Takahashi & Yamada (2014) investigated the growth of density perturbations in the accreting envelope via linear stability analysis and found that they grow indeed with the growth rates proportional to \(\ell\), the index of the Legendre functions used for the mode expansion. Takahashi et al. (2016) extended their linear stability analysis to the shock front and its downstream. They showed that the pre-shock perturbations excite instabilities in the post-shock flow efficiently and assist in the shock revival.

The state studies of these physical processes based on steady flows have been limited to 1D spherical models so far (e.g., Mabanta & Murphy 2018), since nobody has succeeded in obtaining 2D steady accretion flows. With the W4 method, we will be able to extend previous investigations to 2D self-consistently, which will be our future work.

The authors thank Dr. Kazuya Takahashi and Dr. Wakan Iwakami for helpful discussions. This work was supported by JSPS KAKENHI grant No. 16K17708, 16H03986, 17K18792. K.F. was supported by JSPS Postdoctoral Fellowship for Research Fellowship (16J10223).
Appendix A

Rankine–Hugoniot Relation on the Deformed Shock Wave

We need to impose the junction condition on the shock surface for each variable. We first define two unit vectors, \( u_\parallel \) and \( u_\perp \), each normal and tangential, respectively, to the shock surface given as \( r = r_s(\theta) \). Then, \( \chi \) is defined to be the angle between the radial unit vector \( e_r \) and \( u_\parallel \) as shown in the left panel of Figure 9. The value of \( \chi \) is obtained as follows:

\[
\begin{align}
\mathbf{u}_\parallel \cdot \mathbf{e}_r &= |\mathbf{u}_\parallel| \cos \chi, \\
\cos \chi &= \frac{d\theta}{d\theta} \left( r_s^2 + \left( \frac{d\theta}{d\theta} \right)^2 \right)^{\frac{1}{2}}.
\end{align}
\]

(17a, 17b)

It is also useful to adopt the so-called surface-fitted coordinates \((q, \theta')\) defined from the spherical coordinates \((r, \theta)\) as

\[
q \equiv \frac{r - r_v}{r_s(\theta) - r_v}, \\
\theta' = \theta.
\]

(18, 19)

This transformation maps the region between the neutrinosphere and the deformed shock surface into a domain given simply as

\[
q \in [0, 1], \quad \theta' \in [0, \pi].
\]

(20)

See the right panel of Figure 9. Note that this is the domain of our main concern in this paper. The derivatives are also transformed as follows:

\[
\begin{align}
\frac{\partial Q}{\partial r} &= \frac{1}{r_s - r_v} \frac{\partial Q}{\partial q}, \\
\frac{\partial Q}{\partial \theta} &= \frac{\partial Q}{\partial \theta'} - \frac{q}{r_s - r_v} \frac{\partial Q}{\partial q}.
\end{align}
\]

(21a, 21b)

where \( Q \) is a physical variable such as density or velocity. Henceforth, we will employ these surface-fitted coordinates alone and use the same notation \( \theta \) instead of \( \theta' \) to denote the new angle coordinate.

The MHD Rankine–Hugoniot relation (e.g., Winterhalter et al. 1984; Takahashi & Yamada 2013) is written as follows:

\[
\begin{align}
\left[ \rho \mathbf{u} \cdot \mathbf{n} \right] &= 0, \\
\begin{bmatrix} \rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u} + \left( p + \frac{B^2}{8\pi} \right) \mathbf{n} - \frac{1}{4\pi}(\mathbf{B} \cdot \mathbf{n})\mathbf{B} \end{bmatrix} &= 0,
\end{align}
\]

(22a, 22b)

where the square bracket denotes as usual the difference between the upstream and downstream values of the quantity given inside the bracket, and we use the ordinary notation \( \mathbf{n} \) for the normal vector to the shock surface. One can also express these relations as

\[
\begin{align}
\rho u_\parallel &= \hat{\rho} u_\parallel, \\
\rho u_\perp + \left( p + \frac{B^2}{8\pi} \right) \mathbf{n} - \frac{1}{4\pi} \mathbf{B} \cdot \mathbf{B} &= \hat{\rho} \hat{u}_\parallel + \hat{\rho} \hat{u}_\perp + \frac{1}{2} \hat{\rho} \hat{u}_\perp^2 + \frac{B^2}{4\pi} - \frac{1}{4\pi} \hat{\mathbf{B}} \cdot (\hat{\mathbf{B}} \cdot \hat{\mathbf{u}}),
\end{align}
\]

(23a, 23b)

where the hat \( \hat{\cdot} \) means the upstream value and the subscript \( \perp \) denotes the normal component to the shock surface. We use these relations to obtain the downstream values of physical quantities. The rotation and magnetic field upstream of the shock are given in Section 2.3 (see also Appendix B).

Since we have assumed a free-falling cold, spherical flow outside the shock surface, the radial velocity \( u_r \), density \( \rho_r \), and

\[
\begin{align}
\begin{bmatrix} \rho (u \cdot n) u + \left( p + \frac{B^2}{8\pi} \right) n - \frac{1}{4\pi} (B \cdot n) B \end{bmatrix} &= 0,
\end{align}
\]

(22b)

\[
\begin{align}
\begin{bmatrix} u \cdot n \left( \rho + \frac{1}{2} \rho u^2 + \frac{B^2}{8\pi} \right) - \frac{1}{4\pi} (B \cdot n) (B \cdot u) \end{bmatrix} &= 0,
\end{align}
\]

(22c)

\[
\begin{align}
\begin{bmatrix} |B \cdot n| = 0, \\
|n \times (u \times B)| = 0, \end{bmatrix}
\end{align}
\]

(22d, 22e)
pressure \( p_f \) just ahead of the shock surface are given as follows:

\[
    u_f(r) = -\frac{2GM}{r}, \quad (24a)
\]

\[
    \rho_f(r) = -\frac{M}{4\pi r^2 u_f} = \frac{M}{\sqrt{32\pi^2 GM}} r^{-2}, \quad (24b)
\]

\[
    p_f(r) = 0. \quad (24c)
\]

Note that the mass accretion rate is positive (\( \dot{M} > 0 \)) in this paper.

### Appendix B

**The Rotation Law and Magnetic Field Profile outside the Shock Wave**

Here we explain in detail the ideas behind the outer boundary conditions for rotation and magnetic field employed in this paper. If the magnetic field is purely toroidal (\( B_r = B_\theta = 0 \)), there are two conserved quantities: specific angular momentum \( \ell \) and a quantity related to the magnetic torque \( \sigma \). The former comes from the \( \varphi \) component of the Euler equation (Equation 1 (d)), and the latter results from the \( \varphi \) component of the ideal MHD condition. They are constant along each streamline. It should be noted that the specific entropy is not conserved, since heating and cooling are taken into account in our formulation unlike previous works that assumed the barotropic (adiabatic) condition (e.g., Lovelace et al. 1986; Fujisawa et al. 2013). The conservations of \( \ell \) and \( \sigma \) are expressed as

\[
    \ell(\psi) = r \sin \theta u_\varphi, \quad (25a)
\]

\[
    \sigma(\psi) = \frac{B_\varphi}{r \sin \theta \rho}, \quad (25b)
\]

where \( \ell(\psi) \) and \( \sigma(\psi) \) are arbitrary functions of \( \psi \), the so-called stream function defined in the following relations:

\[
    u_r \equiv \frac{1}{4\pi \rho r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta \equiv -\frac{1}{4\pi \rho r^2 \sin \theta} \frac{\partial \psi}{\partial r}. \quad (26)
\]

Since we have assumed that the flow is spherical outside the shock surface, the stream function there is given explicitly as

\[
    \psi = M \cos \theta. \quad (27)
\]

We need to specify the functional forms of \( \ell \) and \( \sigma \) to fix the rotation law and the profile of the toroidal magnetic field. Following Yamasaki & Yamada (2005), we specify the angular distributions of \( u_\varphi \) and \( B_\varphi \) at \( r = 1000 \text{ km} \) in this paper. Since the stream function depends only on \( \theta \) (see Equation (27)), \( \ell \) and \( \sigma \) are also functions of \( \theta \) alone and the \( u_\varphi \) and \( B_\varphi \) at \( r = 1000 \text{ km} \) (\( \equiv r_{1000} \text{ km} \)) are expressed as follows:

\[
    u_\varphi(r_{1000} \text{ km}, \theta) = \ell(\theta) \left( \frac{r}{r_{1000} \text{ km}} \right) \sin \theta, \quad (28a)
\]

\[
    B_\varphi(r_{1000} \text{ km}, \theta) = r_{1000} \text{ km} \sin \theta \rho_{1000} \text{ km} \sigma(\theta) = \frac{M}{\sqrt{32\pi^2 GM}} \sin \theta \sigma(\theta), \quad (28b)
\]

where \( \rho_{1000} \text{ km} \) is the density at \( r_{1000} \text{ km} \). Just as in Yamasaki & Yamada (2005), we assume uniform rotation at \( r_{1000} \text{ km} \) with a rotational frequency \( \Omega_{1000} \text{ km} \). Then, \( \ell(\theta) \) is given as

\[
    \ell(\theta) = 2\pi r_{1000} \text{ km} f_{1000} \text{ km} \sin^2 \theta. \quad (29)
\]

The azimuthal component of the velocity is also obtained as follows:

\[
    u_\varphi(r, \theta) = 2\pi r_{1000} \text{ km} \sin \theta f_{1000} \text{ km}. \quad (30)
\]

As for \( \sigma \), we simply assume that it is independent of \( \theta \) at \( r_{1000} \text{ km} \):

\[
    \sigma(\theta) = -\frac{\sqrt{32\pi^2 GM}}{M} B_{1000 \text{ km}} \sqrt{r_{1000} \text{ km}}. \quad (31)
\]

Then, the profile of the toroidal magnetic field is given as follows:

\[
    B_\varphi(r, \theta) = B_{1000 \text{ km}} \left( \frac{r}{r_{1000} \text{ km}} \right) \sin \theta, \quad (32)
\]

where the \( B_{1000 \text{ km}} \) is the strength of the toroidal magnetic field at \( r_{1000} \text{ km} \). Note that the profiles of \( u_\varphi \) and \( B_\varphi \) thus obtained apparently satisfy the boundary conditions on the symmetry axis and on the equator in Equation (7).

Next, we consider the case in which the magnetic field has both poloidal and toroidal components. Then, the situation is more complicated with \( \ell \) and \( \sigma \) no longer being independent of each other (Fujisawa et al. 2013). From the continuity equation of the magnetic field (\( \nabla \cdot \mathbf{B} = 0 \)), the magnetic flux function \( \Psi \) is defined to give the poloidal magnetic field components as

\[
    B_r = \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial r}, \quad B_\theta = -\frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial \theta}. \quad (33)
\]

Since the poloidal magnetic field lines coincide with the streamlines in ideal MHD, the magnetic flux function is also constant along each streamline,

\[
    \Psi = \Psi(\psi). \quad (34)
\]

Then, \( \ell \) and \( \sigma \) are expressed in terms of the magnetic flux function, which is in turn a function of the stream function alone (see Fujisawa et al. 2013) as

\[
    \ell(\psi) = r \sin \theta u_\varphi - r \sin \theta \frac{d\Psi(\psi)}{d\psi} B_\varphi, \quad (35a)
\]

\[
    \sigma(\psi) = \frac{B_\varphi}{r \sin \theta \rho} - \frac{4\pi}{r \sin \theta} u_\varphi \frac{d\Psi(\psi)}{d\psi}. \quad (35b)
\]

Note that the specific angular momentum \( (r \sin \theta u_\varphi) \) is no longer conserved along a streamline but \( \ell \) and \( \sigma \) still remain conserved quantities, although \( \ell \) and \( \sigma \) contain terms that depend on \( \Psi \).

Just as in the previous case, in which the purely toroidal magnetic field is considered, we set the functional form of \( \Psi \) outside the shock surface from the condition that the streamlines and hence the poloidal magnetic field lines are radial as follows:

\[
    \Psi = -r_{1000} \text{ km}^2 B_0 \cos \theta, \quad (36a)
\]

\[
    B_r(r, \theta) = \frac{r^2}{r_{1000} \text{ km}} B_0, \quad (36b)
\]

\[
    \frac{d\Psi}{d\psi} = -r_{1000} \text{ km}^2 \frac{B_0}{M} = \text{constant}, \quad (36c)
\]

where \( B_0 \) is a constant corresponding to the strength of the poloidal magnetic field at \( r_{1000} \text{ km} \). It is apparent from Equation 36(b) that the radial magnetic field does not depend on \( \theta \) outside
the shock surface. Since the $\theta$ component of the magnetic field vanishes outside the shock surface, $B_\theta = 0$, the $\varphi$ component of the ideal MHD condition given in Equation (2) becomes
\[ \frac{\partial}{\partial r}(r B_\varphi u_\varphi - r B_r u_r) = 0 \] (37a)
\[ \Rightarrow (r B_\varphi u_\varphi - r B_r u_r) = C_1(\theta), \] (37b)
\[ \Rightarrow \frac{u_\varphi}{r \sin \theta} \frac{\partial}{\partial \psi} \frac{\partial \psi}{\partial \theta} - \frac{B_\varphi}{4 \pi r \rho \sin \theta} \frac{\partial \psi}{\partial \theta} = C_1(\theta). \] (37c)

In the above equations, $C_1$ is a function of $\theta$ alone and is related to $\sigma$ as follows (see Equation 35(a)):
\[ \sigma(\theta) = -4\pi \left( \frac{\partial \psi}{\partial \theta} \right)^{-1} C_1(\theta). \] (38)

The $\varphi$ component of the equations of motion (Equation 1(d)) can be recast in a similar way as
\[ u_\varphi \frac{\partial u_\varphi}{\partial r} + u_\varphi - \frac{B_\varphi}{4 \pi \rho} \left( \frac{\partial B_\varphi}{\partial r} + \frac{B_r}{r} \right) = 0 \] (39a)
\[ \Rightarrow \frac{\partial}{\partial r} (ru_\varphi) - \frac{B_\varphi}{4 \pi \rho u_r} \frac{\partial (ru_\varphi)}{\partial r} = 0 \] (39b)
\[ \Rightarrow ru_\varphi - \frac{B_\varphi}{4 \pi \rho u_r} (ru_\varphi) = C_2(\theta), \] (39c)
\[ \Rightarrow ru_\varphi - \frac{d\psi}{d\theta} B_\varphi = C_2(\theta), \] (39d)
where we use the fact that $B_\varphi/(\rho u_r)$ is constant in the current case. $C_2$ is again related to $\ell$ (see Equation (35)) as
\[ \ell(\theta) = \sin \theta C_2(\theta). \] (40)

Using $C_1$ and $C_2$ instead of $\ell$ and $\sigma$, we express $B_\varphi$ and $u_\varphi$ as
\[ B_\varphi = \left( \frac{r u_\varphi - r^3 B_r}{M} \right)^{-1} \left( C_2 - \frac{C_1}{B_r} \right), \] (41a)
\[ u_\varphi = \left( r - \frac{r^3 B_r}{M u_r} \right)^{-1} \left( C_2 - \frac{r^2 B_r}{u_r M} C_1 \right), \] (41b)

It is apparent from these expressions that there may exist a singularity on a surface, where the radial velocity becomes equal to the Alfvén velocity $u_{\varphi, A} = B_\varphi / (4\pi \rho)$. In this paper, we do not deal with this problem but simply avoid it by choosing the value of $B_0$ so that no such singular surface would be encountered.

In actual computations, we set the values of $f_{1000 \text{ km}}$ and $B_{1000 \text{ km}}$ and obtain those of $u_\varphi$ and $B_\varphi$ at $r_{1000 \text{ km}}$ as
\[ u_\varphi(r_{1000 \text{ km}}, \theta) = 2\pi r_{1000 \text{ km}} \sin \theta f_{1000 \text{ km}}, \] (42a)
\[ B_\varphi(r_{1000 \text{ km}}, \theta) = B_{1000 \text{ km}} \sin \theta, \] (42b)
and then we derive the values of $u_\varphi(r, \theta)$ and $B_\varphi(r, \theta)$ just ahead of the shock wave by solving Equations 37(a) and 39(a) numerically.

**Appendix C**

**Numerical Procedure**

Here we outline the procedure to solve the nonlinear differential equations that describe the steady, shocked, non-spherical accretion flows. The system of partial differential equations given in Section 2 is formally written as
\[ A(Q) \frac{\partial Q}{\partial q} + B(Q) \frac{\partial Q}{\partial \theta} + C(Q) = 0, \] (43)
where we introduce the variable vector $Q = [\rho, u_r, u_\varphi, T, B_r, B_\varphi]^T$ for brevity; $A(Q)$, $B(Q)$, and $C(Q)$ are matrix-valued nonlinear functions of $Q$. Note that the shock surface corresponds to $q = 1$ on the surface-fitted coordinates. Equations (43) are discretized at the cell center in the $q$-mesh but at the grid point in the $\theta$-mesh as follows:
\[ F_{j-1,k} = A(Q_{j-1,k}) Q_{j-1,k} - B(Q_{j-1,k}) q_{j-1} - C(Q_{j-1,k}) = 0, \] (44)
where $Q_{j-1,k}$ is defined as
\[ Q_{j-1,k} = \frac{1}{2} (Q_{j-1,k} + Q_{j,k}). \] (45)

The set of algebraic nonlinear Equations (44) is solved numerically with the W4 method described in Appendix D.

We take the following steps to obtain a solution:

1. We first fix the five model parameters $M$, $L$, $f_{1000 \text{ km}}$, $B_{1000 \text{ km}}$, and $B_0$. Then, the inner boundary $r_{c1}$ is determined from Equation (5). We give an initial guess of the shape of the shock surface $r_{j}(\theta)$, which determines the angle $\chi$ in Equation (17).

2. The density, velocity, and pressure outside the shock are given in Equation (24), and other variables such as $u_r$, $B_r$, and $B_\varphi$ are determined as explained in Appendix B.

3. Given the upstream quantities, the corresponding downstream quantities are obtained as mentioned by solving the Rankine–Hugoniot relation given in Equation (23). In particular, the radial and $\theta$ components of the velocity and magnetic field just behind the shock front are given in terms of the parallel ($u_r$, $B_r$) and perpendicular ($u_\varphi$, $B_\varphi$) components of the velocity and magnetic field as
\[ u_r = u_r \cos \chi + u_\varphi \sin \chi, \] (46a)
\[ u_\theta = u_\theta \sin \chi - u_\varphi \cos \chi, \] (46b)
\[ B_r = B_r \cos \chi + B_\varphi \sin \chi, \] (46c)
\[ B_\theta = B_\theta \sin \chi - B_\varphi \cos \chi. \] (46d)

4. We integrate the basic equations (Equations 1 and 2) inward from the outer boundary $q = 1$ to the inner boundary $q = 0$. If the density $\rho$ obtained at the inner boundary differs from the specified value $\rho = 10^{11}$ g cm$^{-3}$, we modify the shock surface, changing $r_{\text{shp}}$ and/or $\epsilon$ and repeat iteration steps 3 and 4 until the correct value of $\rho$ is obtained at the inner boundary.

We change the value of the neutrino luminosity $L_\nu$ and repeat the above procedures until we obtain a series of solutions up to the critical point.

**Appendix D**

**Details of the W4 Method**

The Newton–Raphson method is one of the simplest and most commonly used methods to solve systems of nonlinear equations numerically. It is very efficient, converging to a root
quadratically, if it really converges, which happens normally when the initial guess is sufficiently close to the root (Press et al. 1992). When the initial guess is not close to the root, however, the iteration is usually unstable and we have problems with oscillations during the iteration. A new method is then required to avoid such oscillations and the resultant non-convergence. The W4 method of our own devising is an iterative relaxation method just like the Newton–Raphson method and some of its extensions, quasi-Newton method (e.g., Broyden’s method, Broyden 1965). Unlike these Newton-type methods, the W4 method uses acceleration and damping terms in addition to the velocity term for convergence. In fact, thanks to these terms, the W4 method can avoid the non-convergent oscillation during the iteration and shows better global convergence, i.e., it obtains the solutions that the Newton-type method cannot. We explain the W4 method in detail below.

D.1. Newton–Raphson Method

In this paper, we solve a system of nonlinear equations numerically to obtain steady accretion flow through a standing shock wave onto a PNS. Such a problem can be reduced from nonlinear algebraic equations written generically as

\[ F_i(x_1, x_2, \ldots, x_N) = 0 \quad i = 1, 2, \ldots, N, \tag{47} \]

for \( N \) variables \( x_i, i = 1, 2, \ldots, N \). For notational convenience, they are also denoted by \( x \) and \( F \). In solving this type of equation, the Newton–Raphson method is probably the first choice. In the Newton–Raphson method, each function \( F_i \) is Taylor-expanded to first order at a certain \( x \) (Press et al. 1992) as

\[ F_i(x + \delta x) = F_i(x) + \sum_{j=1}^{N} \frac{\partial F_i}{\partial x_j} \delta x_j + \mathcal{O}(\delta x^2). \tag{48} \]

The Jacobian matrix is then introduced as

\[ J_{ij} = \frac{\partial F_i}{\partial x_j}. \tag{49} \]

We solve the linearized equations

\[ F_i(x + \delta x) = F_i(x) + \sum_{j=1}^{N} J_{ij} \delta x_j = 0, \tag{50} \]

for \( \delta x_j \) as

\[ \delta x_j = -\sum_{j=1}^{N} J^{-1}_{ij} F_i, \tag{51} \]

where \( J^{-1} \) is the inverse of the Jacobian matrix. The value of \( x \) is then incremented by \( \delta x \), and Equation (47) is solved for this new \( x \). The procedure is repeated until the sequence \( x^n \) determined from

\[ x^{n+1} = x^n + \delta x, \tag{52} \]

is converged. Whereas in the Newton–Raphson method the correction term \( \delta x \) is calculated with the inverse of the Jacobian matrix as in Equation (51), it is obtained in different ways (e.g., good and bad Broyden’s method; Broyden 1965) in the secant methods or other Newton-type methods. Regardless, it is important that all of these methods employ Equation (52) for a single-step evolution. We modify this in the W4 method.

D.2. W4 Method

Equation (52), which is used for the single-step evolution commonly in the Newton-type methods, may be recast into the following suggested form:

\[ \frac{x^{n+1} - x^n}{\Delta \tau} = f(x), \quad f(x) \equiv \frac{\delta x}{\Delta \tau} = - J^{-1} F. \tag{53} \]

where \( \Delta \tau \) is a fictitious time step, which is arbitrary for the moment.

Then, the resultant equation may be interpreted as an approximation to the following first-order differential equations:

\[ \frac{dx}{d\tau} = f(x). \tag{54} \]

It is probably not surprising then to consider differential equations of second-order instead of first-order given the following:

\[ \frac{d^2 x}{d\tau^2} + M_1 \frac{dx}{d\tau} + M_2 F = 0, \tag{55} \]

where \( M_1 \) and \( M_2 \) are matrices somehow related to the Jacobian matrix. Note that these equations remind us of the forced oscillation of connected springs with damping. It follows that \( M_1 \) should be positive to give damping (see Okawa et al. 2018). This analogy is important and useful in considering the behavior of solutions. Introducing “momentum” \( p \), we can decompose these second-order differential equations into two sets of first-order differential equations as follows:

\[ \frac{dx}{d\tau} = Xp, \quad \frac{dp}{d\tau} = -2p - YF, \tag{56} \]

where \( X \) and \( Y \) are matrices related to \( F \) and the Jacobian matrix. Finite-differencing these first-order differential equations, we obtain a new recurrence formula for the single-step evolution as

\[ x^{n+1} = x^n + \Delta \tau Xp^n, \quad p^{n+1} = (1 - 2 \Delta \tau)p^n - \Delta \tau YF. \tag{57} \]

These are the single-step evolution equations adopted in the W4 method. It should be apparent that this is an extension of Equation (52). It should be emphasized, however, that there is a greater degree of freedom in choosing \( X \) and \( Y \). Indeed, we have studied various choices and found that UL decomposition of the Jacobian matrix shows a nice performance in finding roots. In this decomposition \( J = UL \) and \( U \) and \( L \) are the upper and lower triangular matrices given, respectively, for the case of a \( 3 \times 3 \) matrix as follows:

\[ U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \tag{58} \]

where \( u_{ij} \) and \( \ell_{ij} \) are elements of these matrices. Setting \( X = L^{-1} \) and \( Y = U^{-1} \) as well as \( \Delta \tau = 0.5 \), we obtain in the version we refer to as the UL-W4 method (Okawa et al. 2018) the following:

\[ x^{n+1} = x^n + \frac{1}{2} L^{-1}p^n, \quad p^{n+1} = -\frac{1}{2} U^{-1}F(x^n), \tag{59} \]
where \( L_n^{-1} \) and \( U_n^{-1} \) denote \( L^{-1} \) and \( U^{-1} \) at the \( n \)th time step, respectively. Note that, Equations (59) combined,

\[
x^{n+1} = x^n - \frac{1}{4} L_n^{-1} U_n^{-1} F(x^{n-1}),
\]

(60)

differ from the single-step evolution equation in the Newton–Raphson method because \( L_n^{-1} U_n^{-1} J_n^{-1} \) and \( L_n^{-1} U_n^{-1} J_n^{-1} \) and \( J_n^{-1} \) in general. It should be also stressed that the product tends to the inverse of the Jacobian matrix \( L_n^{-1} U_n^{-1} \) as \( J_n^{-1} \) as the sequence approaches the root. These properties turn out to be crucial for both global and local convergences (Okawa et al., 2018).

Another useful choice of \( X \) and \( Y \) is derived from the LH decomposition of the Jacobian matrix, which is in turn based on QR decomposition of the transposed Jacobian matrix \( J^T \equiv QR \), where \( H \) is a Householder matrix, \( Q \) is an orthogonal matrix and \( R \) is an upper triangular matrix. The actual procedure of these decompositions is given again for a \( 3 \times 3 \) Jacobian matrix, which is expressed as

\[
J^T = A_0 = \begin{pmatrix}
a_1^{(0)} & a_2^{(0)} & a_3^{(0)} \\
a_1^{(2)} & a_2^{(2)} & a_3^{(2)} \\
a_1^{(3)} & a_2^{(3)} & a_3^{(3)}
\end{pmatrix},
\]

(61)

First, we extract the first column of this matrix as a column vector \( a_{0(0)} \equiv [a_1^{(0)} a_2^{(0)} a_3^{(0)}]^T \) and construct another vector \( b_{0(0)} \) with the norm \( |a_{0(0)}| \) as

\[
b_{0(0)} = [-\text{sign}(a_1^{(0)}), |a_{0(0)}| 0]^T,
\]

(62)

where we use the sign function. A Householder matrix is then defined as \( H_0(0) = E - 2c_0(0)c_0^T \), where \( E \) denotes the identity matrix and \( c_0(0) \) is given as

\[
c_0(0) = \frac{a_{0} - b_{0}}{|a_{0} - b_{0}|}.
\]

(63)

This matrix \( H_{0(0)} \) transforms the vector \( a_{0(0)} \) into \( b_{0(0)} \) or vice versa: \( H_{0(0)} a_{0(0)} = b_{0(0)} \) and \( H_{0(0)} b_{0(0)} = a_{0(0)} \). Multiplying this Householder matrix with \( A_0(0) \), we obtain

\[
H_{0(0)} A_0(0) = A_1 = \begin{pmatrix}
r_1 & r_2 & r_3 \\
0 & a_2^{(1)} & a_3^{(1)} \\
0 & a_3^{(1)} & a_4^{(1)}
\end{pmatrix},
\]

(64)

where \( r_{ij} \) and \( a_{ij}^{(1)} \) are the components of the matrix \( A_1 \) thus obtained, which are non-vanishing in general.

We repeat the same process on the lower right \( 2 \times 2 \) submatrix. We extract the column vector \( a_{(1)} \equiv [a_2^{(1)} a_3^{(1)}]^T \) and build a new vector \( b_{(1)} = [-\text{sign}(a_2^{(1)})|a_{(1)}| 0]^T \) and construct another Householder matrix

\[
H_{(1)} = \begin{pmatrix}
1 & 0 & 0 \\
0 & E - 2c_{(1)} c_{(1)}^T
\end{pmatrix},
\]

(65)

where \( E \) and \( c_{(1)} \equiv (a_{(1)} - b_{(1)})/|a_{(1)} - b_{(1)}| \) are now the \( 2 \times 2 \) unit vector and a vector of length 2, respectively. We multiply \( H_{(1)} \) with \( A_{(1)} \) and obtain the following:

\[
H_{(1)} A_{(1)} = A_{(2)} = H_{(1)} H_0(0) J^T = \begin{pmatrix}
r_1 & r_2 & r_3 \\
0 & r_2 & r_3 \\
0 & 0 & r_3
\end{pmatrix} = R.
\]

(66)

It is clear from these equations that the product of the two Householder matrices transforms the transposed Jacobian matrix to the upper triangular matrix \( R \). Since the Householder matrix \( H \) is orthogonal and symmetric \( (H^T = H^{-1} = H) \) by construction, the Jacobian matrix is decomposed as follows:

\[
J^T = H_{0(0)} H_{(1)} R \Rightarrow J = (H_{0(0)} H_{(1)} R)^T = L H_{(1)} H_{0(0)},
\]

(67)

where \( L \) is the transpose of \( R \) and a lower triangular matrix.

Finally, we set \( X \) and \( Y \) as \( X = H_{0(0)} \) and \( Y = H_{(1)} L^{-1} \). Then, Equations (57) become

\[
x^{n+1} = x + \frac{1}{2} H_{0(0)} p^n, \quad p^{n+1} = \frac{1}{2} H_{(1)} L^{-1} F(x).
\]

(68)

Although we explain the procedure for the \( 3 \times 3 \) Jacobian matrix, the generalization to other dimensions should be obvious: if the dimension is larger than 3, we set \( X = H_{0(0)} \) and \( Y = \cdots H_{3(3)} H_{2(2)} H_{1(1)} L^{-1} \).

We have applied the W4 method to various problems and found more often than not that it can find a root even when initial guesses are not close to the root and other Newton-type methods, such as the original Newton–Raphson method and Broyned’s method, fail to reach it. This is also the case for the problem of our interest in this paper, that is, in solving the \( M \) HD equations numerically to obtain the steady accretion flows through a stalled shock wave onto the PNS. These facts indicate that the W4 method of our own devising has a better global convergence property than those other methods. We stress that the UL and LH decompositions introduced above are just two useful possibilities and the W4 method has much greater possibilities. We refer readers to Okawa et al. (2018) for more mathematical aspects of the method and numerical tests.

### Appendix E

#### Accuracy and Convergence Test

We check here the accuracy of the numerical solutions we obtain in this paper. This may be done by inspecting how well the inner boundary condition is satisfied. Since we have imposed the condition that the inner boundary or the neutrinosphere
should be spherical, having the constant density of $10^{11}$ g cm$^{-3}$ on it, we look into the density at the inner boundary of the accretion flow obtained numerically. Figure 10 displays the relative difference between the two densities, i.e., the density derived numerically and the prescribed value of $10^{11}$ g cm$^{-3}$ for one of the solutions. The relative error is typically of the order of $10^{-4}$, implying that our iterative calculations have successfully reached the solution with a sufficient accuracy.

Next we check the convergence of the numerical solutions to the true solution by changing the number of grid points. We use the results obtained with three different numbers of grid points and check the convergence factors $Q_q$ and $Q_\theta$ (e.g., Okawa et al. 2014) defined as follows:

$$Q_q = \left| \frac{\phi_{2N_q} - \phi_{N_q}}{\phi_{N_q} - \phi_{N_q/2}} \right|,$$

$$Q_\theta = \left| \frac{\phi_{2N_\theta} - \phi_{N_\theta}}{\phi_{N_\theta} - \phi_{N_\theta/2}} \right|,$$

where $N_q$ and $N_\theta$ denote the numbers of grid points in the $q$ and $\theta$ directions, respectively, and $\phi_{N_q}$ and $\phi_{N_\theta}$ are physical quantities calculated with $N_q$ and $N_\theta$ grid points, respectively. We use the $\theta$-averaged $u_\theta$ for $\phi$:

$$\phi(q) = \frac{1}{N_\theta} \sum_{k=1}^{N_\theta} u_\theta(q, \theta_k),$$

because it is the quantity most easily influenced by the numerical errors at boundaries. Since we use the cell-centered, second-order discretization in both directions as in Equations (44) and (45), the convergence factors should be $Q \sim 4$ if it works properly. In this given convergence test, the mass accretion rate and neutrino luminosity are set to $M = 2.0 M_\odot$ s$^{-1}$ and $L_\nu = 4.5 \times 10^{52}$ erg s$^{-1}$, respectively. The rotation frequency is $f_{1000 \text{ km}} = 0.03$ s$^{-1}$, and the strengths of the poloidal and toroidal magnetic fields are given as $B_0 = 10^6$ G and $B_{1000 \text{ km}} = 10^6$ G at the outer boundary.

Figure 11 displays the convergence factor $Q_q$ (left) and $Q_\theta$ (right) as functions of $q$ for $N_q = 200$ and $N_\theta = 20$. It is obvious that our code actually has second-order convergence with respect to both $N_q$ and $N_\theta$ at these grid-point numbers. Note also that the numerical solutions are essentially unchanged already at $N_\theta \sim 20$ especially near the neutrinosphere. As for $N_q$, on the other hand, convergence is obtained at different numbers, depending on the mass accretion rate $M$ as well as on the numerical scheme employed, but the typical value is $N_q \sim 100$ for high mass accretion rates and $N_q \sim 1000$ for models with low mass accretion rates. The LH-W4 method requires smaller values of $N_q$ than the UL-W4 method. We hence set $N_\theta \sim 20$ and $N_q \sim 100$ for models with high mass accretion rates and $N_q \sim 1000$ for models with low mass accretion rates for the numerical computations in this paper.

**References**

Akiyama, S., Wheeler, J. C., Meier, D. L., & Lichtenstadt, I. 2003, *ApJ*, 584, 954

Blondin, J. M., & Mezzacappa, A. 2007, *Natur*, 445, 58

Blondin, J. M., Mezzacappa, A., & DeMarino, C. 2003, *ApJ*, 584, 971

Broyden, C. 1965, *MaCom*, 19, 577

Bruenn, S. W., Lentz, E. J., Hix, W. R., et al. 2016, *ApJ*, 818, 123

Bruenn, S. W., Mezzacappa, A., Hix, W. R., et al. 2009, *JPhCS*, 180, 012018

Burrows, A. 2013, *RvMP*, 85, 245

Burrows, A., Dessart, L., Livne, E., Ott, C. D., & Murphy, J. 2007, *ApJ*, 664, 416

Burrows, A., & Goshy, J. 1993, *ApJL*, 416, L75

Burrows, A., Livne, E., Dessart, L., Ott, C. D., & Murphy, J. 2006, *ApJ*, 640, 878

Burrows, A., Vartanian, D., Dolence, J. C., Skinner, M. A., & Radice, D. 2018, *SSRv*, 214, 33

Couch, S. M. 2013, *ApJ*, 775, 35

Couch, S. M., Chatzopoulos, E., Arnett, W. D., & Timmes, F. X. 2015, *ApJL*, 808, L21

Couch, S. M., & Ott, C. D. 2013, *ApJL*, 778, L7

Couch, S. M., & Ott, C. D. 2015, *ApJ*, 799, 5

Dhang, P., Sharma, P., & Mukhopadhyay, B. 2018, *MNRAS*, 476, 3310

Dimmelmeier, H., Font, J. A., & Müller, E. 2002, *A&A*, 388, 917

Dolence, J. C., Burrows, A., & Zhang, W. 2015, *ApJ*, 800, 10

Fernández, R. 2015, *MNRAS*, 452, 2071

Fernández, R., Müller, B., Foglizzo, T., & Janka, H.-T. 2014, *MNRAS*, 440, 2763
