COTANGENT BUNDLES FOR “MATRIX ALGEBRAS CONVERGE TO THE SPHERE”

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Dedicated to the memory of Richard V. Kadison

Abstract. In the high-energy quantum-physics literature one finds statements such as “matrix algebras converge to the sphere”. Earlier I provided a general setting for understanding such statements, in which the matrix algebras are viewed as compact quantum metric spaces, and convergence is with respect to a quantum Gromov-Hausdorff-type distance. More recently I have dealt with corresponding statements in the literature about vector bundles on spheres and matrix algebras. But physicists want, even more, to treat structures on spheres (and other spaces) such as Dirac operators, Yang-Mills functionals, etc., and they want to approximate these by corresponding structures on matrix algebras. In preparation for understanding what the Dirac operators should be, we determine here what the corresponding “cotangent bundles” should be for the matrix algebras, since it is on them that a “Riemannian metric” must be defined, which is then the information needed to determine a Dirac operator. (In the physics literature there are at least 3 inequivalent suggestions for the Dirac operators.)

Introduction

In the literature of theoretical high-energy physics one finds statements along the lines of “matrix algebras converge to the sphere” and “here are the Dirac operators on the matrix algebras that correspond to the Dirac operator on the sphere”. But one also finds that at least three inequivalent types of Dirac operator are being proposed in this context. See, for example, [2], [3], [4], [7], [11], [13], [26], [27] and the references they contain, as well as [17] which contains some useful comparisons. In [18], [19], [22], [23] I provided definitions and theorems that give a precise meaning to the convergence of matrix algebras to spheres. These results were developed in the general context of coadjoint orbits of compact Lie groups, which is the appropriate context for this topic, as is clear from the physics literature. I seek to give eventually a precise meaning to the statements about Dirac operators.

In ordinary differential geometry, Dirac operators are built from Riemannian metrics, which give a smooth assignment of an inner product to the tangent vector space at each point of the manifold. But in the non-commutative

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setting suitable “tangent bundles” are scarce, while “cotangent bundles” are relatively common. They are often called “first order differential calculi” [10]. In ordinary differential geometry it is well-known that a Riemannian metric can equivalently be specified by giving a smooth assignment of an inner product to the cotangent vector space (the dual of the tangent vector space) at each point of the manifold. The main result of this paper is to indicate what the “cotangent bundles” are for the matrix algebras that converge to the sphere and to other spaces. The appropriate context is that of connected compact semisimple Lie groups, and that is the context in which we work in this paper. The statement and proof require the detailed theory of roots and weights for semisimple Lie groups and their representations, and we prefer to state our main result (Theorem 4.1) after we have established our notation and conventions for this detailed theory. The particular case in which $G = SU(n)$ with its defining representation of $G$ on $\mathbb{C}^n$ was treated earlier in [9, 8, 16].

In the non-commutative context the “cotangent bundles” are actually bimodules, which in the commutative context are the bimodules of smooth cross-sections for the ordinary cotangent bundles. In the non-commutative context we will continue to refer to these bimodules as “cotangent bundles”.

After passing my qualifying exam I went to talk with Dick Kadison about possible research directions. He suggested that I think about the relations between groups and operator algebras. This paper is one bit of the evidence that I have been following his suggestion ever since, with great pleasure.

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1. Preliminaries on compact Lie groups and their representations

Let $T$ be a torus group, that is, a commutative connected compact Lie group, isomorphic to a finite product of circle groups. We will denote its Lie algebra by the traditional $\mathfrak{h}$. For any finite-dimensional unitary representation $(\mathcal{H}, \pi)$ of $T$ we let $\pi$ also denote the corresponding representation of $\mathfrak{h}$. For each $H \in \mathfrak{h}$ the operator $\pi_H$ is skew-adjoint, and so its eigenvalues are purely imaginary. Since the $\pi_H$’s all commute with each other, they are
simultaneously diagonalizable. Because we need to keep track of the struc-
ture over \( \mathbb{R} \), we will use a convention for the weights of a representation
that is slightly different from the usual convention. If \( \xi \in \mathcal{H} \) is a common
eigenvector for the \( \pi_H \)'s, there will be a linear functional \( \alpha \) on \( \mathfrak{h} \) (with values
in \( \mathbb{R} \)) such that
\[
\pi_H(\xi) = i\alpha(H)\xi
\]
for all \( H \in \mathfrak{h} \). For each \( \alpha \in \mathfrak{h}' \) (where \( \mathfrak{h}' \) denotes the dual vector space to \( \mathfrak{h} \)) we set
\[
\mathcal{H}_\alpha = \{ \xi \in \mathcal{H} : \pi_H(\xi) = i\alpha(H)\xi \text{ for all } H \in \mathfrak{h} \}.
\]
If there are non-zero vectors in \( \mathcal{H}_\alpha \) then we say that \( \alpha \) is a
weight of the representation \((\mathcal{H}, \pi)\). We denote the set of all weights for this representation
by \( \Delta_{\pi} \). Then
\[
\mathcal{H} = \bigoplus \{ \mathcal{H}_\alpha : \alpha \in \Delta_{\pi} \}.
\]
Suppose, instead, that \( \mathcal{H} \) is a Hilbert space over \( \mathbb{R} \) and that \( \pi \) is a repre-
sentation of \( T \) by orthogonal transformations. The corresponding repre-
sentation of \( \mathfrak{h} \) is by skew-symmetric operators, which may have no eigenvectors.
Let \( \mathcal{H}^C \) denote the complexification of \( \mathcal{H} \), and let \( \theta \) denote the corresponding
complex conjugation operator on \( \mathcal{H}^C \), so that \( \theta \) is a conjugate linear isome-
try such that \( \theta^2 = I_{\mathcal{H}^C} \). Let \( \pi \) also denote the extension of \( \pi \) to \( \mathcal{H}^C \). Notice
that \( \theta \) commutes with each \( \pi_H \). Let \( \alpha \) be a weight of \( \pi \), and let \( \xi \in \mathcal{H}^C_\alpha \).
Then for any \( H \in \mathfrak{h} \)
\[
\pi_H(\theta \xi) = \theta(\pi_H \xi) = \theta(i\alpha(H)\xi) = -i\alpha(H)\theta(\xi).
\]
Thus \( \theta \) carries \( \mathcal{H}^C_\alpha \) into, in fact onto, \( \mathcal{H}^C_{-\alpha} \). Thus when a unitary representa-
tion is the complexification of an orthogonal representation, if \( \alpha \) is a weight
of the representation then so is \( -\alpha \). Let \( v = \xi + \theta(\xi) \) and \( w = i(\xi - \theta(\xi)) \),
so that \( v, w \in \mathcal{H} \). Then \( \pi_H(v) = i\alpha(H)w \) and \( \pi_H(w) = -\alpha(H)v \).
Now let \( G \) be a compact connected semisimple Lie group. For discussion
and proofs of the results we state below see [6, 14, 24, 25] . We make a
choice of a maximal torus, \( T \), in \( G \). Let \( \mathfrak{g} \) denote the Lie algebra of \( G \), and
let \( \mathfrak{h} \) be its subalgebra for \( T \). As in [21], we let \( \text{Kil} \) denote the negative of
the Killing form on \( \mathfrak{g} \), so that it is a (positive) inner product on \( \mathfrak{g} \). Then
the adjoint representation, \( \text{Ad} \), of \( G \) on \( \mathfrak{g} \) is by orthogonal operators for \( \text{Kil} \).
Thus the corresponding adjoint representation, \( \text{ad} \), of \( \mathfrak{g} \), which is just the
left regular representation of \( \mathfrak{g} \) on itself, is by skew-symmetric operators for \( \text{Kil} \). We let \( \mathfrak{g}^C \) denote the complexification of \( \mathfrak{g} \). The non-zero weights for
\( \text{Ad} \) or \( \text{ad} \) are called the “roots” of \( G \). We denote the set of roots simply
by \( \Delta \). By the comments made above, if \( \alpha \in \Delta \) then \( -\alpha \in \Delta \). In the
standard way [14, 24, 25] we make a choice, \( \Delta^+ \), of positive roots, and we
let \( S \) denote the corresponding set of simple roots in \( \Delta^+ \). For each root
\( \alpha \) we let \( \mathfrak{g}^C_\alpha \) denote the corresponding root space. We extend \( \text{Kil} \) to \( \mathfrak{g}^C \) by
\( \mathbb{C} \)-bilinearity (not sesquilinearity). It is a standard fact that this extended
\( \text{Kil} \) is non-degenerate, and that the root spaces \( \mathfrak{g}^C_\alpha \) and \( \mathfrak{g}^C_\beta \) are orthogonal to
each other for \( \text{Kil} \) exactly if \( \alpha - \beta \neq 0 \), while all root spaces are orthogonal
to $\mathfrak{h}^C$. It is also a standard fact that these root spaces are all of dimension 1, and that $[\mathfrak{g}_C^+, \mathfrak{g}_C^-]$ is not of dimension 0 (so is of dimension 1). We want to choose usual elements $H_\alpha$, $E_\alpha$ and $F_\alpha$ in these spaces, but we need to choose them in a careful way so that they mesh well with representations.

Let $(\mathcal{H}, \pi)$ be a finite-dimensional unitary representation of $G$. We extend the corresponding representation of $\mathfrak{g}$ to a representation (still denoted by $\pi$) of $\mathfrak{g}^C$. Let $W \in \mathfrak{g}^C$ with $W = X + iY$ for $X, Y \in \mathfrak{g}$. Then

$$(\pi W)^* = (\pi X)^* + (i\pi Y)^* = \pi(-X + iY).$$

Thus it is appropriate to define an involution on $\mathfrak{g}^C$ by $(X + iY)^* = -X + iY$, so that $(\pi W)^* = \pi W^*$ for all $W \in \mathfrak{g}^C$ (as in [12, 25]). Notice that for all $W, Z \in \mathfrak{g}^C$ we have $[W, Z]^* = [Z^*, W^*]$. The following result is certainly well-known, but I have not seen in the literature a derivation of it quite like the one below, though there are similarities with results in [12, 25].

**Proposition 1.1.** With notation as above, for each $\alpha \in \Delta^+$ we can choose $H_\alpha \in i\mathfrak{h}$ and $E_\alpha \in \mathfrak{g}_C^+$ such that $[E_\alpha, E_\alpha^*] = H_\alpha$ and $[H_\alpha, E_\alpha] = 2E_\alpha$. Setting $F_\alpha = E_\alpha^*$, we then obtain $[H_\alpha, F_\alpha] = -2F_\alpha$.

**Proof.** Let $\alpha \in \Delta$ be given, and choose a non-zero $E \in \mathfrak{g}_C^+$. Then $E^* \in \mathfrak{g}_C^-$ and $E^* \neq 0$. Then $[E, E^*] \neq 0$ since $[\mathfrak{g}_C^+, \mathfrak{g}_C^-]$ is not of dimension 0. Furthermore, $[E, E^*]$ is self-adjoint for $^*$, and so is in $i\mathfrak{h}$. We must relate all this to Kil. For any $H \in \mathfrak{h}$ we have

$$\text{Kil}(H, [E, E^*]) = \text{Kil}([-E, H], E^*) = \alpha(H) \text{Kil}(E, E^*).$$

It is easily calculated that Kil$(Z, Z^*)$ is strictly negative for any non-zero $Z \in \mathfrak{g}^C$. Rescale $E$ so that Kil$(E, E^*) = -1$. Then for all $H \in \mathfrak{h}$

$$\alpha(H) = \text{Kil}(H, i[E, E^*]).$$

Set $\tilde{H}_\alpha = [E, E^*]$. Then

$$[\tilde{H}_\alpha, E] = \alpha(\tilde{H}_\alpha)E = -\text{Kil}(\tilde{H}_\alpha, \tilde{H}_\alpha)E.$$

Notice that the coefficient of $E$ on the right side is positive. This equation says that

$$[[E, E^*], E] = -\text{Kil}([E, E^*], [E, E^*])E.$$

It is then clear that we can rescale $E$ so that the coefficient of $E$ on the right side has value 2. Denote the resulting $E$ by $E_\alpha$ and set $H_\alpha = [E_\alpha, E_\alpha^*]$. We see that $[H_\alpha, E_\alpha] = 2E_\alpha$ as desired.

2. **Highest weight vectors**

Let $(\mathcal{H}, \pi)$ be an irreducible unitary representation of $G$. By the standard theory [14, 24, 25], for our choice of $\Delta^+$ made in the previous section there is a highest weight vector, $\xi_\alpha \in \mathcal{H}$, for $\pi$, with $\|\xi_\alpha\| = 1$. It is unique up to phase. As a weight vector it is an eigenvector for all the $\pi_H$ for $H \in \mathfrak{h}$. The
fact that it is a highest weight vector means exactly that \( \pi_{E_{\alpha}}\xi_0 = 0 \) for all \( \alpha \in \Delta^+ \). Define \( \lambda \) on \( \mathfrak{g} \) by

\[
\lambda(X) = -i\langle \xi_0, \pi_X\xi_0 \rangle.
\]

(We take the inner product on \( \mathcal{H} \) to be linear in the second variable, as done in [21, 12, 10].) Up to sign \( \lambda \) is exactly the “equivariant momentum map” of equation 23 of [15] evaluated on the highest weight vector. Because \( \pi_X \) is skew-symmetric for all \( X \in \mathfrak{g} \), we see that \( \lambda \) is \( \mathbb{R} \)-valued on \( \mathfrak{g} \). Extend \( \lambda \) to \( \mathfrak{g}^C \) in the usual way. Notice that for any \( \alpha \in \Delta^+ \) we have

\[
i\lambda(H_\alpha) = \langle \xi_0, [\pi_{E_{\alpha}}^{\ast}, \pi_{E_{\alpha}}] \xi_0 \rangle = \langle \xi_0, \pi_{E_{\alpha}}\pi_{E_{\alpha}}^{\ast}\xi_0 \rangle \geq 0,
\]

so that \( \lambda \) is "dominant". Note that \( \lambda \) does not depend on the phase of \( \xi_0 \).

From now on we will denote \( \xi_0 \) by \( \xi_\lambda \).

Because \( \xi_\lambda \) is a highest weight vector, we clearly have \( \lambda(E_\alpha) = 0 \) for all \( \alpha \in \Delta^+ \), and \( \lambda(F_\alpha) = 0 \) for all \( \alpha \in \Delta^+ \) because \( F_\alpha = E_\alpha^{\ast} \). Furthermore, because \( [E_\alpha, E_\alpha^{\ast}] = H_\alpha \) and \( [H_\alpha, E_\alpha] = 2E_\alpha \) and \( [H_\alpha, F_\alpha] = -2F_\alpha \), the triplet \( (H_\alpha, E_\alpha, F_\alpha) \) generates via \( \pi \) a representation of \( \mathfrak{sl}(2, \mathbb{C}) \), for which the spectrum of \( \pi_{H_\alpha} \) must consist of integers. In particular, \( i\lambda(H_\alpha) \) is an integer, necessarily non-negative, in fact equal to \( \|F_\alpha\xi_\lambda\|^2 \). We see in this way that \( \lambda \) is a quite special element of \( \mathfrak{g}' \).

Let \( \mu \) denote the weight of \( \xi_\lambda \), so that \( \pi_H(\xi_\lambda) = i\mu(H)\xi_\lambda \) for all \( H \in \mathfrak{h} \). Comparison with the definition of \( \lambda \) shows that \( \mu \) is simply the restriction of \( \lambda \) to \( \mathfrak{h} \). It is clear that \( \lambda \) is determined by \( \mu \) in the sense that \( \lambda \) has value 0 on the Kil-orthogonal complement of \( \mathfrak{h}^C \). Thus from now on we will let \( \lambda \) also denote the weight of \( \xi_\lambda \). (Thus the special properties of \( \lambda \) mean that, as a weight, \( \lambda \) is a “dominant integral weight”.)

3. COADJOINT ORBITS

Let \( \mu \in \mathfrak{g}' \) with \( \mu \neq 0 \). The coadjoint orbit of \( \mu \) is \( O_\mu = \{ \text{Ad}_{x}^{\ast}(\mu) : x \in G \} \). Then \( G \) acts transitively on \( O_\mu \). Let \( K = \{ x \in G : \text{Ad}_{x}^{\ast}(\mu) = \mu \} \), the stability subgroup of \( \mu \). Then \( O_\mu \) can be naturally identified with the homogeneous space \( G/K \). As in [21] we will usually work with \( G/K \) rather than directly with \( O_\mu \). Let \( \mathfrak{k} \) be the Lie algebra of \( K \). Then it is evident that \( \mathfrak{k} = \{ Y \in \mathfrak{g} : \mu([Y, X]) = 0 \text{ for all } X \in \mathfrak{g} \} \).

Since Kil is definite on \( \mathfrak{g} \), there is a (unique) element in \( \mathfrak{g} \), denoted by \( Z_0 \) in [21], such that

\[
\lambda(X) = \text{Kil}(X, Z_0)
\]

for all \( X \in \mathfrak{g} \). It is easily seen that the Ad-stability subgroup of \( Z_0 \) is again \( K \). Let \( T_0 \) be the closure in \( G \) of the one-parameter group \( r \mapsto \exp(r Z_0) \), so that \( T_0 \) is a torus subgroup of \( G \). Then it is easily seen that \( K \) consists exactly of all the elements of \( G \) that commute with all the elements of \( T_0 \). Note that \( T_0 \) is contained in the center of \( K \) (but need not coincide with the center). Since each element of \( K \) will lie in a torus subgroup of \( G \) that contains \( T_0 \), it follows that \( K \) is the union of the tori that it contains, and so \( K \) is connected (corollary 4.22 of [14]). Thus for most purposes we can
just work with the Lie algebra, \( \mathfrak{k} \), of \( K \) when convenient. In particular, 
\[ \mathfrak{k} = \{ X \in \mathfrak{g} : [X, Z_\alpha] = 0 \} \]
and \( \mathfrak{k} \) contains the Lie algebra, \( \mathfrak{t}_\alpha \), of \( T_\alpha \).

Let us apply the above considerations to the \( \lambda \) of the previous section. We view \( \lambda \) as extended to \( \mathfrak{g}^C \). We saw that for all \( \alpha \in \Delta^+ \) we have \( \lambda(E_\alpha) = 0 = \lambda(F_\alpha) \). It follows that \( Z_\alpha \) is \( K \)-orthogonal to all the root spaces of \( \mathfrak{g}^C \), and so is in \( \mathfrak{h}^C \). But also \( Z_\alpha \in \mathfrak{g} \), and so \( Z_\alpha \in \mathfrak{h} \). It follows that \( T_\alpha \) is contained in the maximal torus \( T \) that we had chosen in the previous section. But \( K \) is the centralizer of \( T_\alpha \), and so \( K \) contains \( T \). Consequently \( \mathfrak{h} \subseteq \mathfrak{t} \) and \( \mathfrak{h}^C \subseteq \mathfrak{t}^C \).

As in [21] let \( \mathfrak{m} = \mathfrak{t}^\perp \) (for Kil). As seen there (and in many other places), \( \mathfrak{m} \) is naturally identified with the tangent space at the coset \( mK \) of \( \mathbb{G}/K \), and we will use this later. We have further that \( \mathfrak{m}^C = \mathfrak{t}^C \perp \mathfrak{m} \). We now make more precise for our special situation some results in section 3 of [5].

**Proposition 3.1.** With notation as above, \( \mathfrak{t}^C \) is the direct sum of \( \mathfrak{h}^C \) with the span of \( \{ E_\alpha, F_\alpha : \lambda(H_\alpha) = 0 \} \), while \( \mathfrak{m}^C \) is the span of \( \{ E_\alpha, F_\alpha : \lambda(H_\alpha) \neq 0 \} \).

**Proof.** We saw above that \( \mathfrak{h}^C \subseteq \mathfrak{t}^C \). Since \( T \) is clearly a maximal torus in \( K \) it follows that \( \mathfrak{t}^C \) is the direct sum of \( \mathfrak{h}^C \) and the weight spaces that it contains. The proof of the first statement is then completed by:

**Lemma 3.2.** Let \( \alpha \in \Delta^+ \). If \( \lambda(H_\alpha) = 0 \) then \( E_\alpha, F_\alpha \in \mathfrak{t}^C \). Conversely, if either \( E_\alpha \in \mathfrak{t}^C \) or \( F_\alpha \in \mathfrak{t}^C \) then \( \lambda(H_\alpha) = 0 \).

**Proof.** If \( \lambda(H_\alpha) = 0 \) then \( \pi_{H_\alpha}\xi_\lambda = \lambda(H_\alpha)\xi_\lambda = 0 \). Also \( \pi_{E_\alpha}\xi_\lambda = 0 \) since \( \xi_\lambda \) is a highest weight vector. Since \( \{ H_\alpha, E_\alpha, F_\alpha \} \) generate a representation of \( \mathfrak{sl}(2, \mathbb{C}) \) with the usual relations, the facts about such representations (see [14, 24, 12]) imply that \( \pi_{E_\alpha}\xi_\lambda = 0 \). But then for any \( X \in \mathfrak{g}^C \) we have

\[ \lambda([E_\alpha, X]) = \langle \pi_X\xi_\lambda, \pi_{E_\alpha}\xi_\lambda \rangle = \langle \pi_X\pi_{E_\alpha}\xi_\lambda, \xi_\lambda \rangle = 0, \]

so that \( E_\alpha \in \mathfrak{t}^C \). A similar argument shows that \( F_\alpha \in \mathfrak{t}^C \). Conversely, if \( E_\alpha \in \mathfrak{t}^C \) then \( \langle [E_\alpha, X]\xi_\lambda, \xi_\lambda \rangle = 0 \) for any \( X \in \mathfrak{g}^C \). On setting \( X = F_\alpha \) we find that \( \lambda(H_\alpha) = \langle [E_\alpha, F_\alpha]\xi_\lambda, \xi_\lambda \rangle = 0 \). A similar argument applies if it is \( F_\alpha \) that is in \( \mathfrak{t}^C \).

We return to the proof of Proposition 3.1. Suppose that \( \lambda(H_\alpha) \neq 0 \). Then for every \( \beta \in \Delta \) such that \( \lambda(H_\beta) = 0 \) we have \( \alpha - \beta \neq 0 \) and so \( E_\alpha \) and \( F_\alpha \) are orthogonal to \( E_\beta \) and \( F_\beta \). Thus \( E_\alpha \) and \( F_\alpha \) are orthogonal to \( \mathfrak{t}^C \). From this the second statement follows quickly.

4. THE COTANGENT BUNDLES FOR THE MATRIX ALGEBRAS

With notation as used earlier, we let \( \mathcal{B} = \mathcal{B}(H_\lambda) \), and we let \( \alpha \) be the action of \( G \) on \( \mathcal{B} \) defined by \( \alpha_x(T) = \pi_xT\pi_x^\ast \). The corresponding representation of \( \mathfrak{g} \) is given by \( \alpha_X(T) = [\pi_X, T] \). As a first approximation to the cotangent bundle we take \( \mathcal{B} \otimes \mathfrak{g}' = \mathcal{B} \otimes (\mathfrak{g}^C)' \), viewed as a \( \mathcal{B} \)-bimodule in the evident way. For any \( T \in \mathcal{B} \) we define \( dT \) by \( (dT)(X) = \alpha_X(T) = [\pi_X, T] \). Then \( d \) is a derivation of \( \mathcal{B} \) into the bimodule \( \mathcal{B} \otimes \mathfrak{g}' \). But the definition of
the cotangent bundle (or first order calculus [10]) includes the requirement that it be generated as a bimodule by the range of \( d \). So our task is to determine for our situation what this sub-bimodule of \( \mathcal{B} \otimes \mathfrak{g}' \) is.

The representation \((\mathcal{H}_\lambda, \pi)\) need not be faithful. Its kernel at the Lie-algebra level is an ideal of \( \mathfrak{g} \). But \( \mathfrak{g} \), as a semisimple Lie algebra, is the direct sum of its minimal ideals, each of which is a simple Lie algebra (non-commutative). Denote the kernel of \( \pi \) by \( \mathfrak{g}_o \). It must be the direct sum of some of these minimal ideals. Denote the direct sum of the remaining minimal ideals by \( \mathfrak{g}_\lambda \), so that \( \mathfrak{g} = \mathfrak{g}_\lambda \oplus \mathfrak{g}_o \). Clearly \( \pi \) is faithful on \( \mathfrak{g}_\lambda \). We identify \( \mathfrak{g}_\lambda' \) with the subspace of \( \mathfrak{g}' \) consisting of linear functionals on \( \mathfrak{g} \) that take value 0 on \( \mathfrak{g}_o \).

From the definition of \( dT \) it is clear that \((dT)(X)\) is 0 for any \( X \) in \( \mathfrak{g}_o \). Consequently, the range of \( d \) is contained in the \( \mathcal{B} \)-bimodule \( \mathcal{B} \otimes \mathfrak{g}'_\lambda \). The main theorem of this section, and of this paper, is:

**Theorem 4.1.** With notation as above, the \( \mathcal{B} \)-bimodule generated by the range of \( d \) is \( \mathcal{B} \otimes \mathfrak{g}'_\lambda \). Thus \( \mathcal{B} \otimes \mathfrak{g}'_\lambda \) is the cotangent bundle for \( \mathcal{B} \) for the action \( \alpha \).

**Proof.** It is clear from the discussion above that it is sufficient to prove that if \( \pi \) is a faithful representation of \( \mathfrak{g} \) then the \( \mathcal{B} \)-bimodule generated by the range of \( d \) is \( \mathcal{B} \otimes \mathfrak{g}' \). Thus we assume that \( \pi \) is faithful for the rest of the proof.

For notational simplicity, in the rest of the proof we will use module notation for the action of \( \mathfrak{g} \) on \( \mathcal{H}_\lambda \), not mentioning \( \pi \). Thus we will write \( X\eta \) for \( \pi X(\eta) \), for example.

Let \( \Omega_\lambda \) be the linear span of all the linear functionals from \( \mathfrak{g} \) into \( \mathcal{B} \) of the form \( X \mapsto R(dT(X))S \) for \( R, S, T \in \mathcal{B} \). Clearly from the definition, \( \Omega_\lambda \) is the cotangent bundle that we seek. Thus our task is to show that \( \Omega_\lambda = \mathcal{B} \otimes \mathfrak{g}' \). Now every operator in \( \mathcal{B} \) is the sum of rank-one operators. Thus \( \Omega_\lambda \) is the linear span of the functionals of the above form for which \( R \) and \( S \) are of rank one. For the purpose of examining these operators we use the following notation. For \( \xi, \eta \in \mathcal{H}_\lambda \) we let \( \langle \xi, \eta \rangle_o \) denote the rank-one operator defined by

\[
\langle \xi, \eta \rangle_o(\zeta) = \xi \langle \eta, \zeta \rangle
\]

for \( \zeta \in \mathcal{H}_\lambda \), where the inner product on the right side is that of \( \mathcal{H}_\lambda \) (assumed linear in its second variable). Thus for \( \xi, \eta, \zeta, \omega \in \mathcal{H}_\lambda \) and for \( T \in \mathcal{B} \) we consider linear functionals from \( \mathfrak{g} \) into \( \mathcal{B} \) of the form

\[
X \mapsto \langle \xi, \eta \rangle_o[T, X)(\zeta, \omega)_o = \langle \langle \xi, \eta \rangle_o[T, X]\zeta, \omega \rangle_o = \langle \eta, [T, X]\zeta \rangle \langle \xi, \omega \rangle_o.
\]

Fixing \( \eta, \zeta \) and \( T \) and taking linear combinations for various \( \xi, \omega \), we see that we obtain in this way all of \( \langle \eta, [T, X]\zeta \rangle \mathcal{B} \). So we see that it is sufficient
for us to consider linear combinations of linear functionals of the form

$$X \mapsto \langle \eta, [T, X] \zeta \rangle.$$ 

We denote the linear span of such functionals by $Q_\lambda$, and we see that our task is to show that $Q_\lambda = (g^C)'$. Now each of $\eta$ and $\zeta$ is a linear combination of weight vectors, and so it suffices for us to examine the case in which $\eta$ and $\zeta$ are weight vectors. Thus, if $\mu$ and $\nu$ are weights and if $\xi_\mu$ and $\xi_\nu$ are weight vectors for them, it suffices to consider functionals of the form

$$X \mapsto \langle \xi_\mu, [T, X] \xi_\nu \rangle.$$ 

Let $\alpha \in \Delta^+$ be given. Since $\pi$ is faithful and weight vectors span $H_\lambda$, there is a weight vector, $\xi_\mu$, such that $F_\alpha \xi_\mu \neq 0$. Then the representation of the $sl(2)$-subalgebra spanned by $\{H_\alpha, E_\alpha, F_\alpha\}$ generated by $\xi_\mu$ has dimension at least 2. We can change $\xi_\mu$ to be a highest weight vector for this $sl(2)$-representation. Then $i \mu(H_\alpha) > 0$, while $E_\alpha \xi_\mu = 0$ and $F_\alpha \xi_\mu \neq 0$.

Let $\phi$ be the linear functional on $g^C$ defined by

$$\phi(X) = \langle \xi_\mu, [E_\alpha, X] F_\alpha \xi_\mu \rangle.$$ 

If $H \in h^C$, then

$$\phi(H) = -i \alpha(H) \langle \xi_\mu, E_\alpha F_\alpha \xi_\mu \rangle = -i \alpha(H) \langle F_\alpha \xi_\mu, F_\alpha \xi_\mu \rangle,$$

which is a non-zero multiple of $\alpha(H)$ since $F_\alpha \xi_\mu \neq 0$. On the other hand, if $X = E_\beta$ or $X = F_\beta$ for some $\beta \in \Delta$ then $\phi(X) = 0$ because weight vectors for different weights are orthogonal. We see in this way that $Q_\lambda$ contains all linear functionals on $g^C$ that take value 0 on the $Kil$-orthogonal complement of $h^C$.

Now let us define $\phi$ instead by

$$\phi(X) = \langle \xi_\mu, [E_\alpha, X] \xi_\mu \rangle.$$ 

By considering the weights of the vectors involved, it is immediate that $\phi(H) = 0$ for all $H \in h^C$, and that $\phi(E_\beta) = 0$ for all $\beta \in \Delta^+$. Furthermore, by similar considerations, $\phi(F_\beta) = 0$ if $\beta \neq \alpha$, while $\phi(F_\alpha)$ is a non-zero multiple of $\mu(H_\alpha)$ ($\neq 0$), so that $\phi(F_\alpha) \neq 0$. So we see that $Q_\lambda$ contains a non-zero linear functional that is 0 on the $Kil$-orthogonal complement of $F_\alpha$. By replacing $F_\alpha$ by $E_\alpha$ in the formula for $\phi$, one finds in the same way that $Q_\lambda$ contains a non-zero linear functional that is 0 on $(E_\alpha)^\perp$. Putting all of this together, we see that $Q_\lambda = (g^C)'$, as desired. \[ \square \]

5. THE COTANGENT BUNDLE FOR G

In this short section, as a prelude to discussing the cotangent bundle for coadjoint orbits, we examine the cotangent bundle for $G$. Here we only need to assume that $G$ is a connected compact Lie group, with Lie algebra $g$. In this section we will not need to take complexifications of $g$ and other vector spaces, so all vector spaces will be over $\mathbb{R}$. 
We let $A = C^\infty(G)$, and we let $\alpha$ denote the action of $G$ on $A$ by left translation. We let $\alpha$ also denote the corresponding action of $\mathfrak{g}$ on $A$. According to our consistent approach to cotangent bundles, we first consider the $A$-bimodule $A \otimes \mathfrak{g}' = C^\infty(G, \mathfrak{g}')$, and the derivation $d$ into it defined by $df(X) = \alpha_X(f)$ for $f \in A$ and $X \in \mathfrak{g}$. The cotangent bundle is then the sub-$A$-bimodule generated by the range of $d$. Since it is well-known that for the usual definition of cotangent bundles the fibers of the usual cotangent bundle of $G$ are just copies of $\mathfrak{g}'$, it is no surprise that we have:

**Theorem 5.1.** For notation as above, the cotangent bundle for $G$, i.e. for $A$, is $A \otimes \mathfrak{g}'$ itself.

**Proof.** Let $\{X_j\}_{j=1}^n$ be a basis for $\mathfrak{g}$ (so the dimension of $\mathfrak{g}$ is $n$). For any fixed $r \in \mathbb{R}, r > 0$ let $C_r$ denote the open hypercube $(-r, r)^n$. Let exp be the exponential map from $\mathfrak{g}$ into $G$, and let $\Phi : C_r \to G$ be defined by $\Phi(t_1, \ldots, t_n) = \exp(t_1X_1 + \cdots + t_nX_n)$. Choose $r$ sufficiently small that $\Phi$ is a diffeomorphism from $C_r$ onto an open neighborhood of the identity element of $G$. For each $j$ let $x_j$ denote the standard coordinate function on $C_r$. The differentials $dx_j$ form a basis for the $C^\infty_c(C_r)$-bimodule of smooth cross-sections of the usual cotangent bundle, i.e. differential forms.

Any $1$-form $\omega$ of compact support on $C_r$ can be expressed as a linear combination of the $dx_j$’s with coefficients in $C^\infty_c(C_r)$. Since $\omega$ has compact support in $C_r$, we can find a smooth function, $h$, on $C_r$ that takes value 1 on the support of $\omega$ but has compact support inside $C_r$. For each $j$ let $h_j = hx_j$. Then $\omega$ can be expressed as a linear combination of the $dh_j$’s with coefficients in $C^\infty_c(C_r)$. Since $\Phi$ is a diffeomorphism, this picture carries over to $\Phi(C_r)$, so any $1$-form on $G$ with compact support in $\Phi(C_r)$ will be a linear combination of the images of the $dh_j$’s with coefficients in $C^\infty_c(\Phi(C_r))$. Extending the images of the $h_j$’s and the coefficients to functions in $C^\infty_c(\Phi(C_r))$ that take value 0 outside $\Phi(C_r)$, we see that any $1$-form on $G$ with support in $\Phi(C_r)$ is in the bimodule generated by the range of $d$. We can cover $G$ by a finite number of translates of $\Phi(C_r)$, and then find a smooth partition of the identity, $\{p_k\}$, subordinate to this cover. Given a $1$-form $\omega$ on $G$, each of the $p_k \omega$’s will be in the $A$-bimodule generated by the range of $d$, and thus $\omega$ itself will be in that bimodule, as needed. \hfill \Box

The situation for homogenous spaces, in particular for coadjoint orbits, is more complicated.

**6. Cotangent bundles for homogeneous spaces**

In this section we treat the cotangent bundle for homogeneous spaces $G/K$ where $G$ is now any compact connected Lie group, and $K$ is any closed connected subgroup of $G$. In this paper we are primarily interested in the case in which $G$ is semisimple and $K$ is the stability subgroup for a point in a coadjoint orbit for $G$. But for just the construction of the cotangent bundle nothing special happens for that more special situation. What is special in
that situation is that then the coadjoint orbit has a Kahler structure. That is important when constructing a corresponding Dirac operator, as seen in [21], but we will not discuss that aspect in this paper.

In this section we will not need to complexify the Lie algebras, and so again all vector spaces will be over $\mathbb{R}$. As in the earlier sections, $\mathfrak{g}$ and $\mathfrak{k}$ will denote the Lie algebras of $G$ and $K$. A description of the (smooth cross-sections of the) tangent bundle was given in [20]. We will make use of that description here. As is frequently done in the present situation, we choose and fix an $Ad$-invariant inner product on $\mathfrak{g}$. (When $G$ is semisimple it can be our earlier Kil.) Much as done earlier, we set $m = k^\perp$.

In notation 4.2 of [20] the tangent bundle of $G/K$ was described as

$$T(G/K) = \{ W \in C^\infty(G, m) : W(xs) = Ad_{s^{-1}}(W(x)) \text{ for } x \in G, \ s \in K \}.$$ 

For this definition, elements of $T(G/K)$ act as derivations on $\mathcal{A} = C^\infty(G/K)$ by

$$\langle \delta_W f \rangle(x) = D_0^t(f(x \exp(tW(x)))),$$

where we write $D_0^t$ for $(d/dt)|_{t=0}$. Notice that this definition of $\delta_W$ involves right multiplication even though we have usually used left multiplication. Reasons for using right multiplication here are given in [20]. It is clear that $T(G/K)$ is a module over $\mathcal{A}$ for pointwise operations. We recognize $T(G/K)$ as just the induced bundle for the representation $Ad$ restricted to $K$ on $m$.

We let $m'$ denote the vector-space dual of $m$, but we will also view $m'$ as the subspace of $\mathfrak{g}'$ consisting of linear functionals on $\mathfrak{g}$ that take value 0 on $\mathfrak{k}$, so that it is $\mathfrak{k}^\perp$ in the sense of duality. Note that since our inner product on $\mathfrak{g}$ is $Ad$-invariant, and $Ad$ restricted to $K$ carries $\mathfrak{k}$ into itself, $Ad$ restricted to $K$ also carries $m$ into itself. Consequently, $Ad'$ restricted to $K$ carries $m'$ into itself. Since the fibers of a cotangent bundle are just the vector-space duals of the fibers of the tangent bundle, it is appropriate for us to set:

**Notation 6.1.** We describe the cotangent bundle, $\Omega(G/K)$, of $G/K$ by:

$$\Omega(G/K) = \{ \omega \in C^\infty(G, m') : \omega(xs) = Ad'_{s^{-1}}(\omega(x)) \text{ for } x \in G, \ s \in K \}.$$ 

The pairing between $T(G/K)$ and $\Omega(G/K)$ is given by

$$\langle W, \omega \rangle_{\mathcal{A}}(x) = \langle W(x), \omega(x) \rangle$$

where the pairing on the right is that between $m$ and $m'$. It is clear that $\Omega(G/K)$ is a bimodule over $\mathcal{A}$ for “pointwise multiplication”. The differential $d$ from $\mathcal{A}$ to $\Omega(G/K)$ is of course given by $df(W) = \delta_W(f)$.

Our task is to show that, consistent with our general approach to defining cotangent bundles for actions of $G$ on C*-algebras, $\Omega(G/K)$ is generated as a bimodule by the range of the derivation. This is well-known by the usual methods of differential geometry using coordinate charts. We show here how this works in our setting.

**Theorem 6.2.** With notation as above, the sub-$\mathcal{A}$-bimodule of $\Omega(G/K)$ generated by the range of the derivation $d$ is $\Omega(G/K)$ itself.
Proof. We need to use how the smooth structure on $G/K$ relates to that of $G$. We use the “slice lemma”, lemma 11.21, of [12]. Define a function $\Phi$ from $m \times K$ to $G$ by $\Phi(Y, x) = \exp(Y)x$. The slice lemma says that there is an open neighborhood, $U$ of $0 \in m$ such that $\Phi$ restricted to $U \times K$ is a diffeomorphism onto an open neighborhood of the identity element, $e$, of $G$. In particular, each left coset of $K$ meets $\Phi(m \times \{e\}) = \exp(U)$ in at most one point. Let $C = \exp(U)$. Thus $C$ is a submanifold of $G$, and is a local cross-section for the canonical projection, $p$, of $G$ onto $G/K$. Let $O = p(C)$, so that $p \circ \exp$ restricted to $U$ is a diffeomorphism from $U$ onto $O$ by the definition of the smooth structure on $G/K$.

Let $\Omega_c(O)$ be the subspace of $\Omega(G/K)$ consisting of elements $\omega$ of compact support in $O$, that is, such that there is an open subset $O'$ which contains the support of $\omega$ and whose closure $\overline{O'}$ is contained in $O$. Here, for our notation, by $\omega$ having support in $O'$ we really mean that as a function on $G$ it has support in $p^{-1}(O')$. Notice that $\omega$ is entirely determined by its restriction to $C$ (since it takes value 0 outside of $CK$).

The pull-back, $\bar{\omega}$, of $\omega$ by $\exp$ is a smooth function from $U$ into $m'$, and is thus a differential form on $U$. Let $b_1, \ldots, b_m$ be the basis for $m'$ dual to our basis $X_1, \ldots, X_m$ for $m$. As functions on $m$ they are the coordinate functions. Then

$$\bar{\omega} = \sum \tilde{g}_j db_j$$

for certain smooth functions $\tilde{g}_j$ that are supported in $U'$, where $U'$ is the preimage of $O'$ under $p \circ \exp$. Each $db_j$ is just the constant function with value $b_j \in m'$. Since $U'$ has compact closure in $U$, we can find a smooth function, $h$, on $U$ that takes value 1 on $U'$ but has compact support inside $U$. For each $j$ set $\tilde{h}_j = h\tilde{b}_j$. Then $\tilde{g}_j d\tilde{h}_j = \tilde{g}_j db_j$ so that

$$\bar{\omega} = \sum \tilde{g}_j d\tilde{h}_j$$

while each $\tilde{h}_j$ has compact support in $U$. For each $j$ let $g_j$ and $h_j$ be the pullbacks of $\tilde{g}_j$ and $\tilde{h}_j$ to $C$ by the inverse of $\exp$. Then we have

$$\omega = \sum g_j dh_j$$

on $C$. Extending $g_j$ and $h_j$ to $CK$ and then to functions on $G$ that are in $C^\infty(G/K)$, we see that

$$\omega = \sum g_j dh_j$$

on $G/K$. Thus $\omega$ is in the bimodule generated by the range of $d$.

Since $G/K$ is compact, it can be covered by a finite number of translates of $\Omega$. By use of a partition of the identity subordinate to such a cover, it follows easily that the bimodule generated by the range of $d$ is all of $O(G/K)$.

\[\square\]

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