On the Incompatibility of Connectivity and Local Pooling in Random Graphs

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Abstract—For a wireless communications network, Local Pooling (LoP) is a desirable property due to its sufficiency for the optimality of low-complexity greedy scheduling techniques. However, LoP in network graphs with a primary interference model requires an edge sparsity that may be prohibitive to other desirable properties in wireless networks, such as connectivity. In this paper, we investigate the impact of the edge density on both LoP and the size of the largest component under the primary interference model, as the number of nodes in the network grows large. For both Erdős-Rényi (ER) and random geometric (RG) graphs, we employ threshold functions to establish critical values for either the edge probability or communication radius necessary for these properties to hold. These thresholds demonstrate that LoP and connectivity (or even the presence of a giant component) cannot both hold asymptotically as the network grows in size for a large class of edge probability or communication radius functions. We then use simulation to explore this problem in the regime of small network sizes, which suggests the probability that an ER or RG graph satisfies LoP and contains a giant component decays quickly with the size of the network.

Index Terms—local pooling; greedy maximal scheduling; primary interference; random graphs; connectivity; giant component.

I. INTRODUCTION

The stability region of a queueing network is often defined as the set of exogenous traffic arrival rates for which a stabilizing scheduling policy exists. A scheduling policy is optimal if it stabilizes the network for the entire stability region. In [2], Tassiulas and Ephremides proved the optimality of the Maximum Weight Scheduling (MWS) policy, which prioritizes backlogged queues in the network. However, for arbitrary communication networks and interference models, employing MWS tends to incur large computation and/or communication costs. Under the assumption of graph-based networks with primary interference, the MWS policy simplifies to that of the Maximum Weighted Matching (MWM) problem, for which there are polynomial-time algorithms.

Greedy and heuristic scheduling can help reduce these operating costs further, usually at the expense of optimality. The relative performance of these policies is often defined by the fraction $\gamma$ of the stability region they attain. For example, Sarkar and Kar [3] provide a $O(\Delta \log \Delta \log n)$-time (where $\Delta$ is the max degree of the network) scheduling policy that attains at least $2/3$ of the stability region for tree graphs under primary interference. Lin and Shroff [4] prove that a maximal scheduling policy on arbitrary graphs can do no worse than $1/2$ of the stability region under primary interference. Maximal matching policies can be implemented to run in $O(\log^2 n)$-time [5]. Lin and Rasool [6] propose a constant, $O(1)$-time algorithm that asymptotically achieves at least $1/3$ of the stability region under primary interference. This naturally leads to the question of whether or not greedy scheduling techniques may in fact be optimal ($\gamma = 1$).

A. Related Work

Sufficient conditions for the optimality of Greedy Maximal Scheduling (GMS) employed on a network graph $G(V,E)$ were produced by Dimakis and Walrand [7] and called Local Pooling (LoP). The GMS algorithm (called Longest Queue First (LQF) in [7]) consists of an iterated selection of links in order of decreasing queue lengths, subject to pair-wise link interference constraints. Computing whether or not an arbitrary graph $G$ satisfies LoP consists of solving an exponential number of linear programs (LPs), one for each subset of links in $G$. Trees are an example of one class of graphs proved to satisfy LoP. However, while LoP is sufficient, a full characterization of the set of graphs for which GMS is optimal is unknown.

The work by Birand et al. [8] produced a simpler characterization of all LoP-satisfying graphs under the primary interference model using forbidden subgraphs on the graph topology. Even more remarkably, they provide an $O(n)$-time algorithm for computing whether or not a graph $G$ satisfies LoP. Concerning general interference models, the class of co-strongly perfect interference graphs are shown to satisfy LoP conditions. The definition of co-strongly perfect graphs is equated with the LoP conditions of Dimakis and Walrand [7]. Additionally, both Joo et al. and Zussman et al. [9], [10] prove that GMS is optimal on tree graphs for $k$-hop interference models.

For graphs that do not satisfy local pooling, Joo et al. [9], [11] provide a generalization of LoP, called $\sigma$-LoP. The LoP factor of a graph, $\sigma$, is essentially formulated from the original LPs of Dimakis and Walrand [7]. It is shown that GMS is stable for $\gamma = \sigma$ of the network’s stability region. The LoP factor is a single-parameter scaling of the network stability-region. Li et al. [12] generalize LoP further to that of $\Sigma$-LoP, which includes a per-link LoP factor $\sigma_i$ that scales each dimension of $\Lambda$ independently and recovers a superset of the provable GMS stability region under the single parameter LoP factor.
As mentioned previously, checking for LoP conditions can be computationally prohibitive, particularly under arbitrary interference models. Therefore, algorithms to easily estimate or bound \( \sigma \) and \( \sigma_k \) are of interest and immediate use in studying GMS stability. Joo et al. [9] provide a lower bound on \( \sigma \) by the inverse of the largest interference degree of a nested sequence of increasing subsets of links in \( G \), and provide an algorithm for computing the bound. Li et al. [12] refine this algorithm to provide individual per-link bounds on \( \sigma_k \). Under the primary interference model, Joo et al. [11] show that \( \Delta/(2\Delta - 1) \) is a lower bound for \( \sigma \). Both Birand et al. and Li et al. [8], [12] note that a lower bound for \( \sigma \) can be computed using the ratio of the minimum to maximum cardinality maximal schedules.

Joo et al. [9] define the worst-case LoP over a class of graphs, and in particular find bounds on the worst-case \( \sigma \) for geometric-unit-disk graphs with a \( k \)-distance interference model. Birand et al. [8] relate particular topologies that admit arbitrarily low \( \sigma \), and provide upper and lower bounds on \( \sigma \) for several classes of interference graphs, other than primary interference. The body of work by Brzezinski et al. [10], [13], [14] brings some attention to multi-hop (routing) definitions and necessary conditions for LoP under primary interference, and giant components with that of numerical results for finite network sizes. We establish a regular threshold function and the graph properties of interest. We establish useful and easier to analyze separate sufficient conditions for LoP under primary interference model. Joo et al. [11] also treat the case of multi-hop traffic and LoP conditions.

### II. Model & Definitions

Let \( G_n \) be the set of all \( 2^{(\binom{n}{2})} \) simple graphs on \( n \) nodes. A common variant of an Erdős-Rényi (ER) graph is constructed from \( n \) nodes where undirected edges between pairs of nodes are added using \( i.i.d. \) Bernoulli trials with edge probability \( p \in [0,1] \). For each choice of \( p \), let \( G_{n,p} \) denote the finite probability space formed over \( G_n \).

We will also consider a common variant of a random geometric (RG) graph, in which \( n \) node positions are modeled by a Binomial Point Process (BPP) within a unit square \([−1/2,1/2]^2 \subset \mathbb{R}^2\). Undirected edges between pairs of nodes are added if the Euclidean distance between the two nodes is less than a given, fixed distance \( r \in [0,\infty) \). For each choice of \( r \), let \( G_{n,r} \) denote the finite probability space formed over \( G_n \). Note that the particular RG model we have chosen is equivalent to a Poisson Point Process (PPP) conditioned on having \( n \) nodes within the unit square, producing an ‘equivalent’ intensity \( \lambda = n \).

Interference in a graph \( G_n \in G_n \) is captured as a pairwise function between its edges. Specifically, we adopt the primary (one-hop) interference model, under which adjacent edges (sharing a common node) interfere with one another. Under this assumption, we can employ the forbidden subgraph characterization of LoP conditions found in [8].

Let \( P \) refer to both i) a specific property or condition of a graph \( G_n \), as well as ii) the subset of graphs of \( G_n \) for which the property holds, as described by Def. [1].

**Definition 1** (Graph Property [15]). A graph property \( P \) is a subset of \( G_n \) that is closed under isomorphism \( (\sim_{iso}) \): i.e., \( G \in P \), \( H \in G_n \), \( G \sim_{iso} H \Rightarrow H \in P \).

**Definition 2** (Monotone Graph Property [16]). Graph property \( P \) is monotone increasing if \( G \in P \), \( H \supset G \Rightarrow H \in P \). Correspondingly, graph property \( P \) is monotone decreasing if \( G \in P \), \( H \subset G \Rightarrow H \in P \).
Let $\mathbb{P}\{G_{n,p} \in \mathcal{P}\}$ denote the probability that a random graph $G_{n,p}$ generated according to $G_{n,p}$ satisfies graph property $\mathcal{P}$. For a monotone (increasing or decreasing) graph property, increasing the edge probability $p \in [0, 1]$ will cause a corresponding transition of $\mathbb{P}\{G_{n,p} \in \mathcal{P}\}$ between 0 and 1. Similarly, $\mathbb{P}\{G_{n,p} \in \mathcal{P}\}$ (analogously defined using $G_{n,r}$ and $G_{n,r}$) for a monotone graph property will also experience a transition as the edge distance $r \in [0, \infty)$ increases. In this case, it is of interest to study the behavior of the limiting probability $\lim_{n \to \infty} \mathbb{P}\{G_{n,p(n)} \in \mathcal{P}\}$ and $\lim_{n \to \infty} \mathbb{P}\{G_{n,r(n)} \in \mathcal{P}\}$ in response to the choice of $p(n)$ and $r(n)$, respectively. (We note that it is more convenient to express the thresholds of RG graphs on $\mathbb{R}^2$ in the form of the square of the edge distance $r(n)^2$ as opposed to $r(n)$.) In the rest of this paper, we will use $\mathbb{P}\{\cdot\}$ as a short form for $\mathbb{P}\{G_{n,p(n)} \in \mathcal{P}\}$ or $\mathbb{P}\{G_{n,r(n)} \in \mathcal{P}\}$ and use a general edge function $e(n)$ as a stand in for either $p(n)$ or $r(n)^2$. A threshold function $e^*(n)$ for graph property $\mathcal{P}$, when it exists, helps determine the limiting behavior of $\mathbb{P}\{\cdot\}$ for choices of edge function $e(n)$ relative to the threshold function $e^*(n)$. Further, like [17], we will use the phrase ‘$\mathcal{P}$ holds asymptotically almost surely (a.a.s.)’ to mean $\lim_{n \to \infty} \mathbb{P}\{\cdot\} = 1$ and the phrase ‘$\mathcal{P}$ holds asymptotically almost never (a.a.n.)’ to mean $\lim_{n \to \infty} \mathbb{P}\{\cdot\} = 0$. We will denote asymptotic equivalence of two functions as $f(n) \sim g(n)$, that is $\lim_{n \to \infty} f(n)/g(n) = 1$. Finally, let $\Phi(x)$ be the c.d.f. of a standard normal r.v., and let $n^k = n!/(n-k)!$ denote the falling factorial.

A. Threshold Functions

First, we restate threshold function definitions in [18] for a graph property $\mathcal{P}$ using edge function $e(n)$, threshold function $e^*(n)$, and the asymptotic notation of [19].

**Definition 3 (Threshold Function).** $e^*(n)$ is a threshold function for monotonically increasing graph property $\mathcal{P}$ if:

$$\lim_{n \to \infty} \mathbb{P}\{\mathcal{P}\} = \begin{cases} 0 & \text{if } e(n) \in o(e^*(n)) \\ 1 & \text{if } e(n) \in \omega(e^*(n)) \end{cases}. \quad (1)$$

**Definition 4 (Regular Threshold Function).** $e^*(n)$ can be called a regular threshold function if there exists a distribution function $F(x)$ for $0 < x < \infty$ such that at any of $F$’s points of continuity, $x$:

$$e(n) \sim xe^*(n) \Rightarrow \lim_{n \to \infty} \mathbb{P}\{\mathcal{P}\} = F(x). \quad (2)$$

$F(x)$ is known as the threshold distribution function for graph property $\mathcal{P}$. When satisfied, Def. 3 covers the limiting behavior of $\mathbb{P}\{\mathcal{P}\}$ for all $e(n)$ that lie an order of magnitude away from threshold $e^*(n)$. Conversely, any function $e(n) \in \Theta(e^*(n))$ is also a threshold function of graph property $\mathcal{P}$. This has since been called the weak, or coarse, threshold function definition [15], [17], [20]. When Def. 4 applies, we can control the limiting value of $\mathbb{P}\{\mathcal{P}\}$ to the extent that $F(x)$ allows. This can be accomplished by choosing $e(n)$ to be a multiplicative factor $x$ of $e^*(n)$.

The two ‘statements’ of a threshold function:

$$e(n) \in o(e^*(n)) \Rightarrow \lim_{n \to \infty} \mathbb{P}\{\mathcal{P}\} = 0$$

$$e(n) \in \omega(e^*(n)) \Rightarrow \lim_{n \to \infty} \mathbb{P}\{\mathcal{P}\} = 1,$$

are commonly referred to as the 0-statements and the 1-statement, as they dictate when $\mathcal{P}$ holds with limiting probability 0 or 1. Note that for a monotone decreasing property, the 0- and 1-statements are appropriately reversed.

B. Sharp Threshold Functions

Stronger variations of the weak threshold have been defined and used, called either sharp, strong, or very strong threshold functions [17], [20], [21]. We restate sharp threshold function definitions in [18] using $e(n)$, $e^*(n)$, and the asymptotic notation of [19].

**Definition 5 (Sharp Threshold Function).** A $(e^*(n), \alpha(n))$ pair is a sharp threshold function for monotonically increasing graph property $\mathcal{P}$ if $\alpha(n) \in o(e^*(n))$, $\alpha(n)$ is asymptotically positive, and:

$$\lim_{n \to \infty} \mathbb{P}\{\mathcal{P}\} = \begin{cases} 0 & \text{if } e(n) \in e^*(n) - \omega(\alpha(n)) \\ 1 & \text{if } e(n) \in e^*(n) + \omega(\alpha(n)) \end{cases}. \quad (3)$$

When satisfied, Def. 5 covers the limiting behavior of $\mathbb{P}\{\mathcal{P}\}$ for all $e(n)$ that lie an additive factor (greater or smaller than order $\alpha(n)$) away from $e^*(n)$. Conversely, any function $e(n) \in e^*(n) + O(\alpha(n))$ is also a sharp threshold function of graph property $\mathcal{P}$. Also note: by itself, $e^*(n)$ is a regular threshold function, that is, $e^*(n)$ satisfies Def. 3 with ‘degenerate’ distribution function $F(x) = 1 \{x > 1\}$ [18]. When presented alone (without $\alpha(n)$), $e^*(n)$ is still referred to as a sharp/strong threshold function [17], perhaps prompting [21] to propose the term ‘very strong’ to denote a $(e^*(n), \alpha(n))$ pair.

**Definition 6 (Regular Sharp Threshold Function).** A sharp threshold function $(e^*(n), \alpha(n))$ is a regular sharp threshold function if there exists a distribution function $F(x)$ for $-\infty < x < \infty$ such that for any of $F$’s points of continuity, $x$:

$$e(n) \sim e^*(n) + x\alpha(n) \Rightarrow \lim_{n \to \infty} \mathbb{P}\{\mathcal{P}\} = F(x). \quad (4)$$

$F(x)$ is known as the sharp-threshold distribution function for graph property $\mathcal{P}$.

When Def. 6 applies, we again control the limiting value of $\mathbb{P}\{\mathcal{P}\}$ to the extent that $F(x)$ allows. This can be accomplished by choosing $e(n)$ to be $e^*(n)$ plus a term asymptotically equivalent to $x\alpha(n)$.

C. Graph Properties

We are interested in several graph properties, listed in Table I. First, Thm. 3.1 from [8] established a set of forbidden subgraphs that characterizes Local Pooling $\mathcal{P}_{\text{lop}}$ under primary interference constraints. Next, Lem. 3.6 from [8] proved a simple upper bound on the number of edges permitting Local Pooling, $\mathcal{P}_{\text{edge}}$. We conclude by establishing some useful properties and bounds of $\mathcal{P}_{\text{lop}}$, namely separate sufficient and
necessary properties for Local Pooling, $P_{\text{lop}}^\text{L}$ and $P_{\text{lop}}^\text{U}$. In later sections, thresholds for these three properties $P_{\text{edge}}$, $P_{\text{lop}}^\text{L}$, and $P_{\text{lop}}^\text{U}$ will be compared with thresholds for the connectivity property $P_{\text{conn}}$ and a looser notion of connectivity $P_{\text{giant}}$.

| Symbol | Property |
|--------|----------|
| $P_{\text{lop}}$ | satisfies LoP (Thm. 1) |
| $P_{\text{edge}}$ | contains no more than 2n edges |
| $P_{\text{lop}}^\text{L}$ | contains no cycles |
| $P_{\text{lop}}^\text{U}$ | contains no cycles of lengths $\{k \geq 6, k \neq 7\}$ |
| $P_{\text{conn}}$ | is connected |
| $P_{\text{giant}}(\beta)$ | largest component has normalized size $\geq \beta, \beta \in (0, 1)$ |
| $P_{\text{giant}}$ | $\exists \beta > 0$: largest component has normalized size $\geq \beta$ |

### Theorem 1 (Local Pooling $P_{\text{lop}}$)
A graph $G_n \in P_{\text{lop}}$ if and only if it contains no subgraphs within the set $F = \{C_k | k \geq 6, k \neq 7\} \cup \{D_k^{s,t} | k \geq 0; s, t \in \{5, 7\}\}$, where $C_k$ is a cycle of length $k \geq 3$ and $D_k^{s,t}$ is a union of cycles of lengths $s$ and $t$ joined by a $k$-edge path (a ‘dumbbell’).

### Lemma 1 ($P_{\text{edge}}$ Necessary for $P_{\text{lop}}$)
A graph property $P_{\text{edge}}$ is necessary for graph property $P_{\text{lop}}$.

### Lemma 2 ($P_{\text{lop}}$ Monotonicity)
$P_{\text{lop}}$ is a monotone decreasing property.

### Lemma 3 (Separate Sufficient and Necessary Conditions for $P_{\text{lop}}$)
$P_{\text{lop}}^\text{L}$ and $P_{\text{lop}}^\text{U}$ are sufficient and necessary properties for $P_{\text{lop}}$, respectively, producing nested subsets:

$$P_{\text{lop}}^\text{L} \subseteq P_{\text{lop}} \subseteq P_{\text{lop}}^\text{U}. \quad (5)$$

### Lemma 4 (Probability Bounds for $P_{\text{lop}}$)
Under any choice of $p(n)$ (or $r(n)$) used to generate ER (or RG) graphs on $n$ nodes:

$$\mathbb{P}\{P_{\text{lop}}^\text{L}\} \leq \mathbb{P}\{P_{\text{lop}}\} \leq \mathbb{P}\{P_{\text{lop}}^\text{U}\}, \forall n \in \mathbb{Z}^+. \quad (6)$$

### III. ER Graphs
In this section, we examine several properties of interest for ER graphs. We first provide a regular sharp threshold function for $P_{\text{edge}}$, a necessary property for $P_{\text{lop}}$. We also find that a regular threshold and distribution function can be directly established for property $P_{\text{lop}}$ by considering the presence of forbidden subgraphs in $F$. Known threshold functions for connectivity and giant components are re-stated for comparison with that of $P_{\text{lop}}$. We show that the threshold function for $P_{\text{lop}}$ is incompatible with the known regular threshold function for $P_{\text{giant}}(\beta)$ — that is, choosing $p(n)$ so that $P_{\text{giant}}(\beta)$ holds a.a.s. implies that $P_{\text{lop}}$ holds a.a.n. It then follows that the stricter notion of connectivity is also incompatible with $P_{\text{lop}}$.

#### A. Local Pooling
If we want to keep the expected number of edges $E(n)$ in $G_{n,p(n)}$ to be exactly $2n$, we should set $p(n) = 4/(n - 1)$. This naturally suggests a threshold function of $p^*(n) = 1/n$. This is indeed a threshold function for $P_{\text{edge}}$ (as are $p^*(n) = 4/(n - 1)$ and $p^*(n) = 4/n$). While not particularly novel, we include Prop. 1 and note that we have not come across a citation for the result.

#### Proposition 1 (Regular Sharp Threshold for $P_{\text{edge}}$ in $G_{n,p(n)}$)
The pair $(p^*(n) = 4/n, \alpha(n) = 2\sqrt{2n/n^2})$ is a regular sharp threshold function for graph property $P_{\text{edge}}$ with a sharp-threshold distribution function $F(x) = \Phi(-x)$ (flipped Normal).

### Proof
Note, the condition $P_{\text{edge}}$ is not sufficient for $P_{\text{lop}}$ and only provides an upper bound on a threshold function for $P_{\text{lop}}$. We improve upon this by considering established thresholds for the presence of individual forbidden subgraphs (such as cycles and dumbbells) in $G_{n,p(n)}$. Note, the threshold for the existence of edge-induced subgraphs in ER graphs is related to the maximum density of edges to vertices of the subgraph $G$. Cycles of a given length, being less ‘dense’, will tend to occur at a lower threshold $p(n) \sim 1/n$ than dumbbells. By focusing on just the set of forbidden cycles, we find that these individual thresholds combine to form a ‘semi-sharp’ regular threshold function for $P_{\text{lop}}$, similar in form to the threshold for all cycles (Thm. 5b [18]). This is formalized by Thm. 2.

#### Theorem 2 (Regular Threshold for $P_{\text{lop}}$ in $G_{n,p(n)}$)
$p^*(n) = 1/n$ is a regular threshold function for graph property $P_{\text{lop}}$, with distribution function:

$$F(x) = \begin{cases} \sqrt{1 - x} \exp \left( \sum_{k \in K} \frac{x^k}{k!} \right), & x < 1 \\ 0, & x \geq 1 \end{cases} \quad (7)$$

where $K = \{1, 2, 3, 4, 5, 7\}$.

### Proof
See App. F

Thm. 2 provides the limiting behavior of $\mathbb{P}\{P_{\text{lop}}^\text{L}\}$ when $p(n)$ is chosen relative to $1/n$. In the case that $p(n)$ is asymptotically larger than $1/n$, we have that $P_{\text{lop}}$ is satisfied a.a.n.. However, in order to guarantee that $P_{\text{lop}}$ is satisfied a.a.s., $p(n)$ must be chosen $o(1/n)$. Thus, we have established how to choose $p(n)$ in order to asymptotically satisfy $P_{\text{lop}}$ with probability between 0 and 1. Correspondingly, Thm. 2 can be weakened to provide a threshold function for property $P_{\text{lop}}$.

#### Corollary 1 (Threshold Function for $P_{\text{lop}}$ in $G_{n,p(n)}$)
$p^*(n) = 1/n$ is a threshold function for $P_{\text{lop}}$.

### Proof
See App. G
B. Connectivity and Giant Components

Previously established results provide a sharp threshold function for connectivity in ER graphs:

Lemma 5 (Regular Sharp Threshold for \( P_{\text{conn}} \) in \( G_{n,p(n)} \) [16, 24]). The pair \( (p^{*}(n) = \log(n)/n, \alpha(n) = 1/n) \) is a regular sharp threshold function for graph property \( P_{\text{conn}} \) with sharp-threshold distribution function \( F(x) = \exp(-\exp(-x)) \) (Gumbel).

We can also loosen our restriction that \( G \) be connected and look at threshold functions for the formation of giant components in random graphs. A giant component exists if the largest connected components contains a positive fraction of the vertices of \( G \) as \( n \to \infty \). Thm. 5.4 of [17] provides a relevant threshold function \( p^{*}(n) = c(\beta)/n \) for the existence of a giant component with normalized size \( \beta \in (0, 1) \). We find that the same threshold function easily applies to the existence of a giant component of size greater than or equal to \( \beta \).

Corollary 2 (Regular Threshold for \( P_{\text{giant}}(\beta) \) in \( G_{n,p(n)} \) [17]). Let \( \beta^{*} \in (0, 1) \), then \( p^{*}(n) = c(\beta^{*})/n \) is a regular threshold function for graph property \( P_{\text{giant}}(\beta) \), with distribution function \( F(x) = \mathbb{1}\{x > 1\} \), where:

\[
    c(\beta) = \frac{1}{\beta} \ln \left( \frac{1}{1 - \beta} \right).
\]

Proof: See App. I.

Given the facts that i) \( P_{\text{top}} \) and \( P_{\text{giant}} \) are monotone decreasing and increasing properties respectively, and ii) their respective threshold functions do not ‘overlap’ (recall that \( c(\beta) > 1 \)), we present a statement of mutual exclusion between the two properties:

Theorem 3 (Mutual Exclusion of \( P_{\text{top}} \) and \( P_{\text{giant}}(\beta) \) in \( G_{n,p(n)} \)). In ER graphs with edge probability function \( p(n) \) and desired giant component size \( \beta \in (0, 1) \):

\[
    \lim_{n \to \infty} \frac{p(n)}{1/n} \geq 0 \Rightarrow \lim_{n \to \infty} \mathbb{P}(P_{\text{top}} \cap P_{\text{giant}}(\beta)) = 0.
\]

Proof: See App. J.

Note that the threshold for connectivity has a higher order than that of giant components (\( \log(n)/n \) vs. \( c(\beta)/n \)), thus, we expect (and find) that properties \( P_{\text{top}} \) and \( P_{\text{conn}} \) exhibit an identical mutual exclusion:

Corollary 3 (Mutual Exclusion of \( P_{\text{top}} \) and \( P_{\text{conn}} \) in \( G_{n,p(n)} \)). In ER graphs with edge probability function \( p(n) \):

\[
    \lim_{n \to \infty} \frac{p(n)}{1/n} \geq 0 \Rightarrow \lim_{n \to \infty} \mathbb{P}(P_{\text{top}} \cap P_{\text{conn}}) = 0.
\]

Proof: See App. J.

We note that the set of \( p(n) \) covered by Thm. 3 and Cor. 3 is a rather large class, covering all functions that can be placed into an asymptotic relationship with \( 1/n \). This includes \( o(1/n) \) and \( \omega(1/n) \), but leaves out certain functions that contain periodic components (e.g., \( \sin(n) + 1/n \)). We note that these ‘sinusoidal’ functions may oscillate across the threshold \( 1/n \) for certain graph properties of interest and do not make sense to employ when attempting to satisfy monotone properties in ER graphs. We also note that a more elegant, larger characterization of the set of \( p(n) \) that satisfy this mutual exclusion may exist (particularly for \( P_{\text{conn}} \) whose threshold lies at a higher order than that of \( P_{\text{top}} \)). Refer to Fig. 1 for a visual comparison of the limiting behavior of the properties in Table I in ER graphs.

IV. RG Graphs

In this section, we examine several properties of interest for RG graphs. We first provide a regular sharp threshold function for \( P_{\text{top}} \), a necessary property and threshold upper bound for \( P_{\text{top}} \). We obtain a tighter threshold upper bound for \( P_{\text{top}} \) by considering the presence of forbidden subgraphs in \( F \). This upper bound is sufficient to prove the threshold for \( P_{\text{top}} \) is incompatible with known regular threshold function \( r^{*}(n^{2}) = \log(n)/(\pi n) \) for \( P_{\text{conn}} \) — that is, choosing \( r(n^{2}) \) so that \( P_{\text{top}} \) holds a.a.s. implies that \( P_{\text{conn}} \) holds a.a.s. Further, relaxing our desire for connectivity from \( P_{\text{conn}} \) to \( P_{\text{giant}} \) lowers the regular threshold function from \( \log(n)/(\pi n) \) to \( \lambda_{c} n/n \) with \( \lambda_{c} \in (0, \infty) \). However, we find that this is insufficient to prevent the incompatibility of \( P_{\text{top}} \) with \( P_{\text{giant}} \).

A. Local Pooling

Proposition 2 (Regular Sharp Threshold for \( P_{\text{top}} \) in \( G_{n,r(n)} \)). The pair \( (r^{*}(n^{2}) = 4/(\pi n), \alpha(n) = 2\sqrt{2n}/(\pi n^{2}) \) is a regular sharp threshold function for graph property \( P_{\text{top}} \) with sharp-threshold distribution function \( F(x) = \Phi(-x) \) (flipped Normal).

Proof: See App. K.

Remark 1. The leading term of the threshold in Prop. 2 was motivated by solving an expression for the expected number of edges in \( G_{n,r(n)} \) for \( r(n)^{2} \). The second term of the threshold is specifically chosen such that all multiplicative factors other than \(-x\) cancel out from scaling (a) and subsequent standardization (d) in the proof.

Proposition 3 (Upper Bound for \( P_{\text{top}} \) in \( G_{n,r(n)} \)). When \( r^{*}(n^{2}) \sim c/n^{6/5} \), the upper bound for LoP can be expressed as:

\[
    \limsup_{n \to \infty} \mathbb{P}(P_{\text{top}}) \leq \exp \left( \frac{-(\pi c/4)^{5}}{6!} \right)
\]

Proof: See App. L.

Remark 2. Unlike the case of ER graphs where cycles of all orders began appearing at the same threshold \( p(n) \sim 1/n \), the RG thresholds of forbidden subgraphs in \( F \) are more...
spread out (order \(k\) vertex-induced subgraphs yielding an order \(k\) edge-induced forbidden subgraph begin to appear at \(r(n)^2 \sim n^{-k/(k-1)}\)). For Prop. 3, we wished to find the tightest upper bound for \(\mathcal{P}_{\text{lop}}\) that was amenable to asymptotic analysis. Thus, we first restricted our attention to the lowest order vertex-induced subgraphs (subgraphs of order 6). Second, we noted that evaluating \(\mu_6\) for all feasible, order 6 graphs that contain the forbidden edge-induced \(K_6\) appears to be neither analytically tractable nor computationally viable, so we apply a second upper bound by focusing on a specific vertex-induced subgraph, the complete graph \(K_6\), and derive an easy upper bound for \(\mu_6\).

The upper bound in Prop. 3 yields the following 0-statement:

**Corollary 4** (0-statement for \(\mathcal{P}_{\text{lop}}\) in \(G_{n,r(n)}\)). When \(r(n)^2 \in \omega(1/n^{6/5})\), \(\lim_{n \to \infty} \mathbb{P}\{\mathcal{P}_{\text{lop}}\} = 0\).

**Proof:** This follows immediately from Prop. 3. \(\square\)

Due to fact that all forbidden subgraphs contain at least 6 or more vertices, it does not seem likely that a corresponding 1-statement would hold at a lower threshold than \(r^*(n)^2 \sim 1/n^{6/5}\). Thus, we are led to make the following conjecture:

**Conjecture 1** (Threshold for \(\mathcal{P}_{\text{lop}}\) in \(G_{n,r(n)}\)). \(r^*(n)^2 = 1/n^{6/5}\) is a threshold function for graph property \(\mathcal{P}_{\text{lop}}\).

The difficulty in proving this conjecture lies in establishing a sufficient condition whose probability lower bounds \(\mathbb{P}\{\mathcal{P}_{\text{lop}}\}\) while maintaining enough tractability to take its limit as \(n \to \infty\). We note that none of the results presented in this paper depends on this conjecture.

**B. Connectivity and Giant Components**

Previously established results provide a regular sharp threshold function for connectivity and a regular threshold for giant components in RG graphs:

**Lemma 6** (Regular Sharp Threshold for \(\mathcal{P}_{\text{conn}}\) in \(G_{n,r(n)}\) [25]). The pair \((r^*(n)^2 = \log(n)/(\pi n), \alpha(n) = 1/(\pi n))\) is a regular sharp threshold function for graph property \(\mathcal{P}_{\text{conn}}\) with sharp-threshold distribution function \(F(x) = e^{-e^{-x}}\) (Gumbel).

**Lemma 7** (Regular Threshold for \(\mathcal{P}_{\text{giant}}\) in \(G_{n,r(n)}\) [25]). \(r^*(n)^2 = \lambda_c/n\) is a regular threshold function for graph property \(\mathcal{P}_{\text{giant}}\) with threshold distribution function \(F(x) = 1\{x > 1\}\), where \(\lambda_c \in (0, \infty)\) is the critical percolation threshold.

Given the facts that that i) \(\mathcal{P}_{\text{lop}}\) and \(\mathcal{P}_{\text{giant}}\) are monotone decreasing and increasing properties respectively, and ii) their respective 0-statements ‘overlap’, we present a statement of mutual exclusion between the two properties:

**Theorem 4** (Mutual Exclusion of \(\mathcal{P}_{\text{lop}}\) and \(\mathcal{P}_{\text{giant}}\) in \(G_{n,r(n)}\)). In RG graphs with edge radius function \(r(n)^2\):

\[
\lim_{n \to \infty} \frac{r^*(n)^2}{1/n} \geq 0 \implies \lim_{n \to \infty} \mathbb{P}\{\mathcal{P}_{\text{lop}} \cap \mathcal{P}_{\text{giant}}\} = 0.
\]

**Proof:** See App. \(\Box\)

Again, in the case of RG graphs, the threshold for connectivity has a higher order than that of giant components \((\log(n)/(\pi n)\) vs. \(\lambda_c/n)\), thus, we expect (and find) that properties \(\mathcal{P}_{\text{lop}}\) and \(\mathcal{P}_{\text{conn}}\) exhibit an identical mutual exclusion:

**Corollary 5** (Mutual Exclusion of \(\mathcal{P}_{\text{lop}}\) and \(\mathcal{P}_{\text{conn}}\) in \(G_{n,r(n)}\)). In RG graphs with edge radius function \(r(n)^2\):

\[
\lim_{n \to \infty} \frac{r^*(n)^2}{1/n} \geq 0 \implies \lim_{n \to \infty} \mathbb{P}\{\mathcal{P}_{\text{lop}} \cap \mathcal{P}_{\text{conn}}\} = 0. \quad (13)
\]

**Proof:** See App. \(\Box\)

In the case of RG graphs, we note that the threshold for \(\mathcal{P}_{\text{lop}}\) must lie at a lower order than both that of \(\mathcal{P}_{\text{giant}}\) and \(\mathcal{P}_{\text{conn}}\), whereas in ER graphs, \(\mathcal{P}_{\text{lop}}\) and \(\mathcal{P}_{\text{giant}}\) were both located at \(1/n\). Refer to Fig. 2 for a visual comparison of the limiting behavior of the properties in Table I in RG graphs.

**V. ALGORITHMS FOR BOUNDING \(\mathcal{P}_{\text{lop}}\)**

Birand \textit{et al.} [9] outline an \(O(n)\)-time exact algorithm checking whether or not a graph with \(n\) vertices satisfies \(\mathcal{P}_{\text{lop}}\) under primary interference constraints. At a high-level, the algorithm involves decomposition of the graph into bi-connected components and checking each component for certain characteristics; among these is a test for ‘long’ cycles (in order to exclude forbidden cycle lengths). Our analytical results suggest that the formation of cycles are the major factor prohibiting LoP in ER and RG random graphs, so we have implemented simpler algorithms to check for necessary and sufficient conditions for LoP in random graphs (\(\mathcal{P}_{\text{lop}}\) and \(\mathcal{P}_{\text{lop}}^U\)). Our simulations are performed in Matlab, where we make use of MatlabBGL [26] for graph decomposition into connected components and depth-first-search. These following algorithms and their supporting functions are listed in Listing I and are centered around the detection of long cycles.

\textsc{HasCycleEq} accepts an input graph \(G\), a cycle-length \(k\), and a maximum number of iterations \(I\) and reports whether or not a cycle of length \(k\) exists within \(G\). \textsc{HasCycleEq} relies directly upon a randomized algorithm, denoted AYK, proposed by Alon \textit{et al.} (Thm. 2.2) [27], which iteratively generates random, acyclic, directed subgraphs of \(G\) and tests for cycles via the subgraph’s adjacency matrix. If no cycles of length \(k\) are found after the \(I\)th iteration, we have \textsc{HasCycleEq} report that no length \(k\) cycles exist in \(G\), which may be a false negative. As a result, \textsc{HasCycleEq} is suitable for use in upper-bounding the probability of the non-existence of forbidden cycles, namely in \(\mathcal{P}_{\text{lop}}\).

\textsc{HasCycleEq} accepts an input graph \(G\), a minimum cycle-length \(K\), and a maximum number of iterations \(I\) and...
reports whether or not a cycle of length $K$ or greater exists within $G$. In general, the decision problem formulation (also known as the long-cycle problem) is NP-hard, but polynomial for fixed-parameter $k$. We make use of a result by Gabow and Nie (Thm. 4.1) [28]: for $K > 3$, depth-first-search DFS may be used to detect the existence of cycles of length longer than $2K - 4$ by examining the back-edges discovered by DFS. Note, a DFS back-edge of length $K - 1$ implies the existence of a length $K$ cycle. Thus, if a ‘long’ back-edge is found by DFS, we may report that such a cycle exists (line 7). In the event that DFS fails to detect long back-edges, Thm. 4.1 of [28] says a long simple cycle, if it exists, will have length between $K$ and $2K - 4$. For each length $k$ within this range, we call the randomized algorithm in HASCYCLEEQ, thus HASCYCLEEQ may also report false negatives. Alternately, when $K = 3$, HASCYCLEEQ is an exact algorithm (lines 8-10 involving HASCYCLEEQ are short-circuited) that checks for the existence of any cycle. This is accomplished by running DFS and examining the resulting tree for back-edges of length 2 or longer. In the event no such back-edges are found, we may conclude that graph $G$ is cycle-free.

Finally, we discuss PLOPL and PLOPU. PLOPL checks for the existence of any cycles and calls HASCYCLEEQ directly. For the reasons discussed above, PLOPL is an exact (not randomized) algorithm and suitable for lower bounding the probability of satisfying LoP conditions. PLOPU checks for the existence of forbidden cycles. For forbidden cycles of length 6, we call HASCYCLEEQ, while for forbidden cycles of length 8 or longer, we call HASCYCLEEQ. For this reason, the curves displayed for $P_{lop}^U$ in later figures are an upper bound for $P_{lop}^U$ (which can be improved by increasing the number of allowed iterations, $I$), but nevertheless yield valid upper bounds for $P_{lop}$ and additionally demonstrate the mutual exclusivity between $P_{lop}$ and $P_{giant}$ in ER and RG graphs.

**Remark 3.** One could obtain a tighter sufficient condition $P_{lop}^{\text{L}}$ by restricting cycles of length $k \geq 5$ instead of all cycles. We have not done so for the following reasons: i) the use of HASCYCLEEQ with $K = 5$ will not produce an exact answer (but instead an upper bound on $P_{lop}^{\text{L}}$), and ii) we are more concerned and satisfied with characterizing an upper bound for $P_{lop}$ and its interaction with connectivity requirements.

**VI. Numerical Results**

The analytical results presented thus far are asymptotic ($n \to \infty$). In this section, we compare the analytical mutual exclusion of LoP and giant components with that of numerical results for finite network sizes and find that convergence to this exclusion between properties is rather quick as the network grows in size.

**A. ER Graphs**

In Fig. 3, we see that the numerical (simulation) results generally match their analytical limits at $n = 10^4$. In particular, as $n \to \infty$, the numerical curves associated with $P_{\text{giant}}(c)$ become increasingly sigmoidal about $c \approx 1.15$ when $c$ is set to a rather conservative value of 0.25. Also note that the effect of increasing the minimum required giant component size $\beta$ serves to shift the associated curves in Fig. 3 to the right, further negating any chance of both satisfying local pooling and having a giant component. Regarding the properties that bound $P_{lop}$, we see that there is good agreement between analytical and numerical results for $c < 1$, but for the $c > 1$ regime, there are noticeable ‘tails’ on the numerical curves. This is largely, if not fully, attributable to the fact that the numerical curves can only check for cycles of lengths up to $n$ while the analytical curves require not having cycles for arbitrarily large lengths in the limit.

In Fig. 4 we focus on edge probability functions $p(n) = c/n$ with parameter $1 \leq c \leq 1.15$, which falls between the asymptotic thresholds for $P_{lop}$ and $P_{\text{giant}}(0.25)$ (see Fig. 3). For each edge probability function within this regime, we plot the probability that an ER graph satisfies both $P_{lop}^U$ and $P_{\text{giant}}(0.25)$ as a function of the network size, $n$. We observe that the

### Listing 1 Pseudo-code checking for $P_{lop}^U$ and $P_{lop}^L$

```plaintext
function HASCYCLEEQ(G,k,I)
    return AYK(G,k,I)

function HASCYCLEEQ(G,K,I)
    T_DFS ← DFS(G)
    if LONGESTBACKEDGE(T_DFS) ≥ (K - 1) then
        return TRUE
    else
        for k = K to 2K - 4 do
            if HASCYCLEEQ(G,k,I) then
                return TRUE
        return FALSE

function PLOPU(G,I)
    return HASCYCLEEQ(G,3,I)

function PLOPU(G,I)
    if HASCYCLEEQ(G,8,I) then
        return TRUE
    else
        return HASCYCLEEQ(G,6,I)
```

![Fig. 3. Probabilities of graph properties occurring in ER graphs are plotted as a function of $c$ where the edge probability is chosen according to $p(n) = c/n$. Asymptotic (as $n \to \infty$), analytical (A) probabilities are plotted in dashed lines. Numerical (N) probabilities at a specific finite $n = 10^4$ are plotted in solid lines with 95% confidence intervals generated from $S = 10^3$ i.i.d. graphs of size $n = 10^4$. PLOPU was configured to use a maximum of $I = 10^5$ iterations.](image-url)
exclusion between $P_{\text{lop}}$ and $P_{\text{giant}}$ develops rather rapidly.

**B. RG Graphs**

Unlike the case of ER graphs, we note that the RG graph bounds and thresholds for $P_{\text{lop}}$ and $P_{\text{giant}}(\beta)$ (respectively) must necessarily occur at edge radius functions of different orders of $n$. For this reason, we provide two subplots in Fig. 5 that are analogous to Fig. 3 and separately consider edge radius functions $r(n)^2 = c/n^6/5$ and $r(n)^2 = c/n$. Intuitively, for edge radius function $r(n)^2 = c/n^{6/5}$, we expect to see (and also observe) two phenomena as the parameter $n$ increases: the probability of $P_{\text{lop}}$ should show convergence towards a non-zero threshold distribution function (if Conj. 4 is true) while the probability of $P_{\text{giant}}(\beta)$ should converge to zero. Similarly, for edge radius function $r(n)^2 = c/n$ we observe the opposite phenomenon: the probability of $P_{\text{giant}}(\beta)$ begins to converge to a non-zero threshold distribution function (near $c = 1.5$) when $r(n)^2 = c/n$, while the probability of $P_{\text{lop}}$ converges to zero for all $c$ at this choice of $r(n)^2$.

While we lack threshold distribution functions for both $P_{\text{lop}}$ and $P_{\text{giant}}$, we include the established upper bound for $P_{\text{lop}}$ (Prop. 3) for comparison and plot each numerical curve for increasing network sizes $n = \{10^2, 10^3, 10^4\}$. We note that the bound in Prop. 3 forbids only vertex-induced complete graphs of order 6 ($K_6$) which is looser than $P_{\text{lop}}$, which forbids edge-induced cycles of lengths $k \geq 6, k \neq 7$ from the set $F$. The combination of plots in Fig. 5 serve to demonstrate the mutual exclusion between $P_{\text{lop}}$ and $P_{\text{giant}}$ as $n \to \infty$.

In Fig. 6 we focus on both edge radius functions selected for Fig. 5 and instead parameterize by $c$ and plot versus $n$. Appropriate parameter values are chosen to explore the area in the gaps presented in Fig. 5. For each edge radius function within this regime, we plot the probability that an RG graph satisfies both $P_{\text{lop}}$ and $P_{\text{giant}}(0.25)$ as a function of the network size, $n$. We observe that the exclusion between $P_{\text{lop}}$ and $P_{\text{giant}}$ develops even more quickly than in the case of ER graphs. The increase in speed at which this exclusion develops is likely due to the separation in order between the thresholds functions that give rise to $P_{\text{lop}}$ and $P_{\text{giant}}$, which was not present in ER graphs.

**VIII. Conclusion**

In this work, we investigate the impact of edge density on the likelihood of satisfying both Local Pooling and notions of network connectivity. Threshold functions help establish critical functions for edge probability and edge radius ($p(n)$ and $r(n)$) that dictate when these desirable network properties will and will not hold as the network size increases in ER and RG graphs, respectively. We find that for both types of graphs, a large class of edge probability and edge radius functions either prohibit LoP due to the existence of forbidden cycles or prohibit giant components and full connectivity due to edge sparsity. These results are confirmed in simulations of both ER and RG graphs, and the mutual exclusion of both properties is shown to develop rather quickly as the graphs increase in size. While we find that LoP conditions are not compatible with notions of connectivity in both random graph families studied, we note that LoP has only been proven to be sufficient for the optimality of Greedy Maximal Scheduling (GMS); there may be as of yet other, unknown means of proving GMS optimality that are amenable to finding edge densities.
The expected number of copies of a $k$-length cycle, $C_k$, in $G_{n,p(n)}$ can be expressed as a product between the number of possible unlabelled cycles and the probability that each forms the desired cycle, $\mathbb{E}[G(C_k)] = n^{k-1}/(2k) * p(n)^k$. Incorporating the bounds in (15) yields:

$$\frac{nk (c-\delta)^k}{2k} \leq \mathbb{E}[G(C_k)] \leq \frac{nk (c+\delta)^k}{n^k}. \quad (16)$$

We next evaluate the following series when $c < 1$ and $\delta \in (0, 1-c)$:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n^{k} (c+\delta)^k}{2^{k}} = \sum_{k=1}^{\infty} \frac{(c+\delta)^k}{2^{k}} \leq -\log(\sqrt{1 - (c+\delta)}) \quad (17)$$

$$(b) \quad \leq -\log(\sqrt{1 - c}) + \epsilon. \quad (19)$$

where $(a)$ follows from the monotone convergence theorem, $(b)$ follows from the series converging when $c+\delta < 1$, and $(c)$ is the result of the continuity and monotonicity of $\log(\sqrt{1 - c})$ at $c$.

By a similar process on the series corresponding to the lower bound, and by controlling $\epsilon$ via decreasing choices of $\delta$, we establish:

$$\lim_{n \to \infty} \sum_{0 \leq k \leq n, k \neq 7} \mathbb{E}[G(C_k)] = -\log(\sqrt{1 - c}) - \sum_{k \in K} \frac{c^k}{2k}. \quad (20)$$

where we subtract out a finite number of terms ($K = \{1, 2, 3, 4, 5, 7\}$) that were originally included in (17) but do not correspond to forbidden cycle lengths.

**Lemma 9** (Expected Forbidden Dumbbells in $G_{n,p(n)}$). When $p(n) \sim c/n, c < 1$, the expected number of forbidden dumbbells of $F$ in $G_{n,p(n)}$ obeys:

$$\lim_{n \to \infty} \sum_{k=0}^{n} \mathbb{E}\left[G(D_k^{b,5})\right] + \mathbb{E}\left[G(D_k^{b,7})\right] + \mathbb{E}\left[G(D_k^{7,7})\right] = 0. \quad (21)$$

**Proof:** Given $p(n)$, it follows that:

$$\forall \delta > 0, \exists n_\delta > 0 : p(n) \leq \frac{c+\delta}{n}, \forall n > n_\delta. \quad (22)$$

The expected number of dumbbells, $D_k^{s,t}$ (unions of cycles of lengths $s$ and $t$ joined by a $k$-edge path), assuming $s \neq t$ and $k \geq 1$ is:

$$\mathbb{E}[G(D_k^{s,t})] = \frac{n^{k-1}/(2s)}{2^{k-1}t} s(n-s-t)^{k-1} p(n)^{s+t+k} < (c+\delta)^{s+t+k}. \quad (23)$$

where there are $n^{k}/(2s)$ unlabelled cycles $C_s$, $(n-s)^{2}/(2t)$ unlabelled cycles $C_t$ from the remaining $n-s$ vertices, and $(n-s-t)^{k-1}t$ ways of connecting $C_s$ to $C_t$ with a $k$-edge path using the remaining $n-s-t$ vertices. The probability that such a selection of vertices forms $D_k^{s,t}$ is $p(n)^{s+t+k}$.
In the event the path contains no edges, \( k = 0 \), then the cycles share a common vertex:

\[
\mathbb{E}[G(D_{0}^{s,t})] = \frac{n^2}{2s} (n-s)^{t-1} 2^{s}(n-s)^{t} \leq \frac{(c + \delta)^{s+t}}{4n} \tag{24}
\]

In this case, \( C_{t} \) is created using one vertex from \( C_{s} \) and a \((t-1)\)-edge path from the remaining \( n-s \) vertices.

Finally, if \( s = t \), then \( \mathbb{E}[G(D_{k}^{s,t})] \) contains an additional factor of 1/2 due to symmetry, but nevertheless upper bounded by the expressions in \( \tag{25} \) and \( \tag{24} \).

It remains to show that expected number of all forbidden dumbbells is zero. Let \( \hat{c} = c + \delta < 1 \) for an appropriate choice of \( \delta \in (0, 1-c) \):

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \left( \mathbb{E}[G(D_{k}^{5,5})] + \mathbb{E}[G(D_{k}^{5,7})] + \mathbb{E}[G(D_{k}^{7,7})] \right) < \lim_{n \to \infty} \frac{\hat{c}^{10} + \hat{c}^{12} + \hat{c}^{14}}{4n} \sum_{k=0}^{n} \hat{c}^{k} \equiv 0, \tag{25}
\]

where \( (a) \) follows from applying the bounds derived above and collecting common factors, while \( (b) \) follows from the convergence of the geometric series to a finite value and taking the limit \( n \to \infty \).

**Lemma 10** (Expected Edges in \( G_{n,r(n)} \)). If \( r(n)^2 \in 2c/(\pi n^2) + 2\sqrt{\pi}x/(\pi n^2) \) with \( x \in \mathbb{R} \), then the mean number of edges \( M_{n,r(n)} \) in \( G_{n,r(n)} \) is:

\[
\mathbb{E}[M_{n,r(n)}] = cn + x\sqrt{c} + o(\sqrt{n}). \tag{27}
\]

**Proof:** This follows from a specialization of Proposition 3.1 in \( \tag{25} \), which provides the asymptotic mean of a subgraph count of \( G_{n,r(n)} \) when \( r(n) \in o(1) \). We will not recreate the theory here, but instead provide enough direction to allow the reader to follow along with \( \tag{25} \). The expected number of edges \( \mathbb{E}[M_{n,r(n)}] \) is given as:

\[
\mathbb{E}[M_{n,r(n)}] \sim K_{2,r^2}r(n)(d(k-1))n^k, \tag{28}
\]

where the subgraph \( K_{2} \) (the complete graph on 2 vertices, i.e., an edge) has \( k = 2 \) vertices, \( d = 2 \) is the dimension of the space in which the points of \( G_{n,r(n)} \) reside, and \( K_{2,r^2} \) is computed as follows:

\[
\mu_{K_{2,r^2}} \equiv \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x) f(y) h_{K_{2}}(\{0, x_1\}) h_{K_{2}}(\{0, x_2\}) dx_1 dy_1 \tag{29}
\]

\[
\equiv \frac{1}{2} \int_{\mathbb{R}^2} h_{K_{2}}(\{0, x_1\}) dx_1 \tag{30}
\]

\[
\equiv \frac{\pi}{2}. \tag{31}
\]

where \( (a) \) is simplified from \( \tag{25} \) for the subgraph type \( K_{2} \), \( (b) \) follows from \( f(x) \) being the uniform distribution over the unit square \([-1/2, 1/2]^2\) used to generate i.i.d. vertex positions, and \( (c) \) follows from \( h_{K_{2}}(0, x_1) \) being the indicator function on whether or not two vertices (one at the origin and the other at \( x_1 \)) with unit edge distance form \( K_{2} \). To form \( K_{2}, x_1 \) must be within the unit disk centered at the origin to connect to the vertex at the origin.

Finally, expanding \( \mu_{K_{2}}r(n)^2n^2 \) and grouping \( o(\sqrt{n}) \) terms is sufficient.

**Lemma 11** (CLT for Edges in \( G_{n,r(n)} \)). If \( r(n)^2 \in 2c/(\pi n) + 2\sqrt{\pi}x/(\pi n^2) \) with \( x \in \mathbb{R} \), then the centered and scaled number of edges \( M_{n,r(n)} \) in \( G_{n,r(n)} \) converges in distribution to that of a centered normal \( \mathcal{N} \):

\[
\frac{M_{n,r(n)} - \mathbb{E}[M_{n,r(n)}]}{n^{1/2}} \Rightarrow \mathcal{N}(0, c). \tag{32}
\]

**Proof:** This follows from a specialization of Thm. 3.13 in \( \tag{25} \), which provides a central limit theorem for collections of subgraph counts of \( G_{n,r(n)} \) when \( \lim_{n \to \infty} nr(n)^d \to \rho \in (0, \infty) \). The distribution of the centered and scaled number of edges \( (M_{n,r(n)} - \mathbb{E}[M_{n,r(n)}])/\sqrt{n} \) is an asymptotic centered normal with variance:

\[
\left( \sum_{j=1}^{k} \rho^{2k-j-1} \Phi_j(K_2, K_2) \right) - k^2 \rho^{2k-2} \mu_{K_{2}}^2, \tag{33}
\]

where the subgraph \( K_{2} \) (the complete graph on 2 vertices, i.e., an edge) has \( k = 2 \) vertices, \( d = 2 \) is the dimension of the space in which the points of \( G_{n,r(n)} \) reside, \( \mu_{K_{2},r^2} = \pi/2 \) is computed as shown in the proof of Lem. 10. \( \Phi_1(K_2, K_2) \) simplifies to:

\[
\Phi_1(K_2, K_2) = \int_{\mathbb{R}^2} h_{K_2}(0, x_2) dx_2 \int_{\mathbb{R}^2} h_{K_2}(0, x_3) dx_3 \tag{34}
\]

\[
= \pi^2, \tag{35}
\]

and \( \Phi_2(K_2, K_2) = \mu_{K_{2}} \).

Finally, note that for the given \( r(n)^2, nr(n)^2 \sim \rho = 2c/\pi \). Substituting \( \Phi_1, \Phi_2, \mu_{K_{2}} \), and \( \rho \) into \( \tag{33} \), we obtain the asymptotic variance:

\[
\rho^{2} \Phi_1(K_2, K_2) + \rho \Phi_2(K_2, K_2) - 4\rho^2 \mu_{K_2}^2 = \rho \mu_{K_{2}} = c. \tag{36}
\]

**B. Lem. 2 (\( P_{\text{Log Monotonicity}} \))**

**Proof:** Let \( G \in \mathcal{P}_{\text{Log}} \). From Thm. 1 \( G \) contains no edge-induced forbidden subgraphs from \( F \). Let \( H \subset G \) by an appropriate removal of edges. The removal of edges from \( G \) cannot possibly create edge-induced forbidden subgraphs where none existed before, therefore \( H \in \mathcal{P}_{\text{Log}} \) and \( P_{\text{Log}} \) is monotone decreasing as described by Def. 2.

**C. Lem. 3 (Separate Sufficient and Necessary Conditions for \( P_{\text{Log}} \))**

**Proof:** Since all forbidden subgraphs in \( F \) (Thm. 1) contain cycles of length 5 or larger, it immediately follows that forbidding such cycles (\( P_{\text{Log}}^L \)) is sufficient for \( P_{\text{Log}} \). Separately, forbidding any subset of subgraphs in \( F \) is a necessary condition for \( P_{\text{Log}} \), therefore forbidding cycles of lengths \( k \geq 6, k \neq 7 \) (\( P_{\text{Log}}^L \)) is necessary for \( P_{\text{Log}} \). Thus, the subsets of graphs on \( n \) vertices that satisfy \( P_{\text{Log}}^L, P_{\text{Log}}^U, P_{\text{Log}}^U \) can be nested in that order.

**D. Lem. 4 (Probability Bounds for \( P_{\text{Log}} \))**

**Proof:** For any choice of \( p(n) \) (or \( r(n) \)) with \( n \in \mathbb{Z}^+ \), \( G_{n,p(n)} \) (or \( G_{n,r(n)} \)) is a random graph generated according to a distribution on \( G_n \). The subsets \( P_{\text{Log}}^L, P_{\text{Log}}^U, P_{\text{Log}}^U \) can then be interpreted as events, and their nesting by Lem. 3 provides the desired ordering of probabilities.
E. Prop. 1 (Reg. Sharp Threshold for $P_{\text{edge}}$ in $G_{n,p(n)}$)

Proof: Let the r.v. $M_{n,p(n)}$ (shortened to $M$) be the number of edges in graph $G_{n,p(n)}$. We show for the given choice of $(p^*(n) = 2c/n, \alpha(n) = 2/\sqrt{n})$ and $F(x) = \Phi(-x)$ that:

$$p(n) \sim p^*(n) + x\alpha(n) \Rightarrow \lim_{n \to \infty} P\{M \leq cn\} = F(x), \quad (37)$$

holds for every point of continuity of $F(x), x \in \mathbb{R}$. $M_{n,p(n)}$ has a binomial $p.d.f.$ for $p(n) \sim p^*(n) + x\alpha(n), M_{n,p(n)}$ has mean and variance:

$$\mathbb{E}[M_{n,p(n)}] = \binom{n}{2} p(n) = cn + \sqrt{cn} + o(\sqrt{n}) \quad (38)$$

$$\text{Var}(M_{n,p(n)}) = \binom{n}{2} p(n)(1 - p(n)) = cn + o(n), \quad (39)$$

by using the additional facts $p(n) = 2c/n + o(1/n)$ and $(p(n))^2 = o(1/n)$.

Finally, for $p(n) \sim p^*(n) + x\alpha(n)$:

$$P\{M \leq cn\} \equiv P \left[ \frac{M - \mathbb{E}[M]}{\sqrt{\text{Var}(M)}} \leq \frac{cn - \mathbb{E}[M]}{\sqrt{\text{Var}(M)}} \right] \quad (40)$$

where $(a)$ comes from standardization of $M_{n,p(n)}$, $(b)$ follows from substitution of $(38)$ and $(39)$, $(c)$ results with the asymptotic simplification of the inequality’s r.h.s. and convergence in distribution of the standardized $M_{n,p(n)}$ to that of a standard normal via the CLT, and $(d)$ results from the continuity of the standard normal $c.d.f., \Phi(x)$.

Thus, for the specific case when $c = 2$, we conclude:

$$\lim_{n \to \infty} P\{G_{n,p(n)} \in P_{\text{edge}}\} = \lim_{n \to \infty} P\{M \leq 2n\} = \Phi(-x). \quad (44)$$

F. Thm. 2 (Reg. Threshold for $P_{\text{lop}}$ in $G_{n,p(n)}$)

Proof: Let $\Gamma$ be a connected graph. Let the r.v. $G(\Gamma)$ be the number of copies of $\Gamma$ in graph $G_{n,p(n)}$. Let $A_T = \{G(\Gamma) > 0\}$ be the event that there are one or more copies of $\Gamma$ in $G_{n,p(n)}$. We show for the given $p^*(n)$ and $F(x)$, that:

$$p(n) \sim x p^*(n) \Rightarrow \lim_{n \to \infty} P\{P_{\text{lop}}\} = F(x), \quad (45)$$

holds for every point of continuity of $F(x), x \in \mathbb{R}$.

Suppose $p(n) \sim x p^*(n)$. We first upper bound $P\{P_{\text{lop}}\}$:

$$\lim_{n \to \infty} P\{P_{\text{lop}}\} \leq \lim_{n \to \infty} P \left[ \bigcap_{\Gamma \in \mathcal{F}} \overline{A}\Gamma \right] \quad (46)$$

where $(a)$ follows by forbidding only cycles in $\mathcal{F}$ up to length $K \leq n$, and $(b)$ is a consequence of Cor. 4.9 of [16] which shows that when $p(n) \sim x/n, a$ finite-length random vector of cycle subgraph counts $\{G(C_k)\}$ converges in distribution to that of independent Poisson r.v.’s with means $\{\lambda_k = x^k/(2k)\}$.

Now, considering the upper bound, suppose $x \geq 1$. The series $\sum_{k=1}^\infty x^k/(2k)$ converges to $-\log(\sqrt{1-x})$. Thus, for arbitrarily small $\epsilon$, a sufficiently large choice for $K$ will yield:

$$- \sum_{0 \leq k \leq K} \frac{x^k}{2k} \leq -\log(\sqrt{1-x}) + \sum_{k \in K} \frac{x^k}{2k} + \epsilon \quad (49)$$

where $K = \{1, 2, 3, 4, 5, 7\}$. Substituting $(49)$ into $(47)$, we obtain the following upper bound for $\lim_{n \to \infty} P\{P_{\text{lop}}\}$:

$$\lim_{n \to \infty} P\{P_{\text{lop}}\} < \sqrt{1-x} \exp \left( \sum_{k \in K} \frac{x^k}{2k} + \epsilon \right), \quad (50)$$

where $\exp(\epsilon) \leq 1 + (e-1)\epsilon$ and the constants in front of $\epsilon$ can be rolled into $\epsilon'$ > 0.

It remains to provide a lower bound when $x < 1$. We start by lower bounding $P\{P_{\text{lop}}\}$:

$$P\{P_{\text{lop}}\} = P \left[ \bigcap_{\Gamma \in \mathcal{F}} \overline{A}\Gamma \right] \quad (46)$$

where $(a)$ follows from the FKG Inequality applied to the set of monotone decreasing properties $A\Gamma$ on $G_{n,p(n)}$ (Thm. 2.12 of [17]), $(b)$ is the result of applying Cor. 2.13 of [17] to each multiplicand to obtain an exponential lower bound.

First, we note that:

$$\lim_{n \to \infty} \frac{1}{1 - p(n)} = 1 \quad (54)$$

Second, by Lem. 8 and Lem. 9 (with $p(n) \sim x/n, x < 1$), the limit of the sum of the expected forbidden subgraph counts depends solely on cycles:

$$\lim_{n \to \infty} \sum_{\Gamma \in \mathcal{F}} E[G(\Gamma)] = -\log(\sqrt{1-x}) - \sum_{k \in K} \frac{x^k}{2k}, \quad (55)$$

with $K = \{1, 2, 3, 4, 5, 7\}$. Thus, by making use of $(54)$ and $(55)$ in $(53)$, the limiting probability of satisfying $P_{\text{lop}}$ is lower bounded by:

$$\lim_{n \to \infty} P\{P_{\text{lop}}\} \geq \sqrt{1-x} \exp \left( \sum_{k \in K} \frac{x^k}{2k} \right) \quad (56)$$
Finally, taking (50) and (56), we have:
\[ p(n) \sim x^p(n), x < 1 \Rightarrow \]
\[ \lim_{n \to \infty} P\{\mathcal{P}_{\text{lop}}\} = \sqrt{1-x} \exp \left( \sum_{k \in K} \frac{x^k}{2k} \right). \]
(58)

**G. Cor. 2 (Threshold Function for \( \mathcal{P}_{\text{lop}} \) in \( G_{n,p(n)} \))**

**Proof:** This follows directly from the Thm. 2 and the monotonicity of property \( \mathcal{P}_{\text{lop}} \). If \( p(n) \in \omega(p(n)) \), there exists \( x > 1 \) for which \( p(n) \) is asymptotically greater than \( x/n \). Alternatively, if \( p(n) \in o(p(n)) \), then \( p(n) \) is asymptotically less than \( x/n \) for all \( x > 0 \).

**H. Cor. 2 (Reg. Threshold for \( \mathcal{P}_{\text{giant}}(\beta) \) in \( G_{n,p(n)} \))**

**Proof:** Given \( \beta^* \in (0, 1) \), construct \( p(n) = c(\beta^*)/n \) using (8). We show that:
\[ p(n) \sim x^p(n) \Rightarrow \lim_{n \to \infty} P\{G_{n,p(n)} \in \mathcal{P}_{\text{giant}}(\beta^*)\} = F(x), \]
where \( F(x) = 1 \{x > 1\} \) for all continuity points of \( F(x) : \mathbb{R} \setminus \{1\} \).

Suppose \( p(n) \sim x^p(n) \), with \( x > 1 \). \( p(n) \) is asymptotically larger than \( p^*(n) \) and by monotonicity of (8), there exists \( \beta \in (\beta^*, 1) \) such that:
\[ \exists n_0 > 0, \forall n > n_0 : p(n) > c(\beta)/n > c(\beta^*)/n. \]

Applying part ii) of Thm. 5.4 in [17] to establish that the size of the largest component, denoted as \( L_{n,p(n)} \), converges in probability to \( \beta n \):
\[ \forall \epsilon > 0, \lim_{n \to \infty} P\left\{ \left| \frac{L_{n,p(n)}}{\beta n} - 1 \right| < \epsilon \right\} = 1. \]

By choosing \( \epsilon \) such that \( \beta^* = (1 - \epsilon)\beta \), the event \( [L_{n,p(n)}/(\beta n) - 1] < \epsilon \) is a subset of the event that \( L_{n,p(n)} \geq \beta^* n \), giving us the upper bound:
\[ P\left\{ \left| \frac{L_{n,p(n)}}{\beta n} - 1 \right| < \epsilon \right\} \leq P\left\{ \frac{L_{n,p(n)}}{n} \geq \beta^* \right\}. \]

Since \( \lim_{n \to \infty} P\{L_{n,p(n)}/(\beta n) - 1 < \epsilon\} = 1 \) and probabilities are bounded above by 1, we apply the squeeze/sandwich theorem and conclude that \( \lim_{n \to \infty} P\{L_{n,p(n)}/n > \beta^*\} = 1 \).

Alternatively, suppose \( p(n) \sim x^p(n) \), with \( x < 1 \). \( p(n) \) is asymptotically smaller than \( p^*(n) \) and by monotonicity of (8), there exists \( \beta \in (0, \beta^*) \) such that:
\[ \exists n_0 > 0, \forall n > n_0 : p(n) < c(\beta)/n < c(\beta^*)/n. \]

Again, using Thm. 5.4 in [17] to show that \( L_{n,p(n)} \) converges in probability to \( \beta n \):
\[ \forall \epsilon > 0, \lim_{n \to \infty} P\left\{ \left| \frac{L_{n,p(n)}}{\beta n} - 1 \right| < \epsilon \right\} = 1. \]

By choosing \( \epsilon \) such that \( (1 + \epsilon)\beta = \beta^* \), the event \( [L_{n,p(n)}/(\beta n) - 1] < \epsilon \) is a subset of the event that \( L_{n,p(n)} < \beta^* n \), giving us the upper bound:
\[ P\left\{ \frac{L_{n,p(n)}}{\beta n} - 1 < \epsilon \right\} \leq 1 - P\left\{ \frac{L_{n,p(n)}}{n} \geq \beta^* \right\}. \]

Since \( \lim_{n \to \infty} P\{L_{n,p(n)}/(\beta n) - 1 < \epsilon\} = 1 \) and probabilities are bounded below by 0, we apply the squeeze/sandwich theorem and conclude that \( \lim_{n \to \infty} P\{L_{n,p(n)}/n \geq \beta^*\} = 0 \).

**I. Thm. 3 (Mutual Excl. of \( \mathcal{P}_{\text{lop}} \) and \( \mathcal{P}_{\text{giant}}(\beta) \) in \( G_{n,p(n)} \))**

**Proof:** By Thm. 2, \( p(n) \sim c/n, c > 1 \) implies that \( \mathcal{P}_{\text{lop}} \) holds a.a.n. Therefore, \( p(n) \sim c/n, c \leq 1 \) is a necessary condition for \( \mathcal{P}_{\text{lop}} \) to hold a.a.s. Under this necessary condition, we see that \( p(n) \) is asymptotically less than \( c(\beta)/n \) since \( c(\beta) > 1 \) and by Cor. 2, \( \mathcal{P}_{\text{giant}}(\beta) \) holds a.a.s.

Thus, for \( p(n) \sim c/n, \forall c \leq 1 \):
\[ 0 \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}}(\beta) \cap \mathcal{P}_{\text{lop}}\} \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}}(\beta)\} = 0 \]

Alternatively, for \( p(n) \sim c/n, \forall c > 1 \):
\[ 0 \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}}(\beta) \cap \mathcal{P}_{\text{lop}}\} \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{lop}}\} = 0 \]

In both cases, we can conclude that \( \lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}}(\beta) \cap \mathcal{P}_{\text{lop}}\} = 0 \).

**J. Cor. 3 (Mutual Excl. of \( \mathcal{P}_{\text{lop}} \) and \( \mathcal{P}_{\text{conn}} \) in \( G_{n,p(n)} \))**

**Proof:** \( \mathcal{P}_{\text{giant}}(\beta) \) is a necessary condition for connectivity \( \mathcal{P}_{\text{conn}} \). Thus, \( P\{\mathcal{P}_{\text{lop}} \cap \mathcal{P}_{\text{conn}}\} \leq P\{\mathcal{P}_{\text{lop}} \cap \mathcal{P}_{\text{giant}}(\beta)\} \) for all \( \beta \in (0, 1) \). The mutual exclusion between \( \mathcal{P}_{\text{lop}} \) and \( \mathcal{P}_{\text{conn}} \) follows immediately from Thm. 3.

**K. Prop. 2 (Reg. Sharp Threshold for \( \mathcal{P}_{\text{edge}} \) in \( G_{n,r(n)} \))**

**Proof:** Let the r.v. \( M_{n,r(n)} \) (shortened to \( M \)) be the number of edges in graph \( G_{n,r(n)} \). We show that for the given choice of \( (r^* n)^2 = 2c/(\pi n) \), \( \alpha(n) = 2\sqrt{cn}/(\pi n^2) \) and \( F(x) = \Phi(-x) \) that:
\[ r(n)^2 \sim r^*(n)^2 + x\alpha(n) \Rightarrow \lim_{n \to \infty} P\{M \leq cn\} = F(x), \]
(59)
where \( F(x) = \Phi(-x) \) for all continuous points of \( F(x) : \mathbb{R} \).

For \( r(n)^2 \sim r^*(n)^2 + x\alpha(n) \):
\[ P\{M \leq cn\} \Rightarrow P\left\{ \frac{M - E[M]}{\sqrt{n}} \leq \frac{cn - E[M]}{\sqrt{n}} \right\} \]
(60)
\[ \Rightarrow P\left\{ \frac{M - E[M]}{\sqrt{n}} \leq -x\sqrt{c} + o(1) \right\} \]
(61)
\[ \Rightarrow \Phi \left( -x\sqrt{c} + o(1) \right) + o(1) \]
(62)
\[ \Rightarrow \Phi(0) + o(1) \]
(63)
\[ \Rightarrow \Phi(-x) + o(1) \]
(64)
where (a) follows from the standardization of \( M_{n,r(n)} \), (b) follows from Lem. 10 (c) follows from Lem. 11 and standardizing the argument to the c.d.f., (d) results from asymptotic simplification, and (e) follows from the continuity of the standard normal CDF \( \Phi \).

Thus, for the specific case when \( c = 2 \), we conclude:
\[ \lim_{n \to \infty} P\{G_{n,r(n)} \in \mathcal{P}_{\text{edge}}\} = \lim_{n \to \infty} P\{M \leq 2n\} = \Phi(-x). \]
(65)
L. Prop. 3 (Upper Bound for \( \mathcal{P}_{\text{top}} \) in \( G_{n,r(n)} \))

Proof: Let \( \Gamma_k \) be a feasible, connected, order \( k \) graph. Let \( G_e(\Gamma) \) be the edge-induced subgraph count of \( \Gamma \) on graph \( G_{n,r(n)} \). Let \( A_k = \{ G_e(\Gamma_k) \geq 1 \} \) be the event that there are one or more edge-induced copies of a \( k \)-length cycle \( (C_k) \) in \( G_{n,r(n)} \). Let \( G_v(\Gamma_k) \) be the vertex-induced subgraph count of \( \Gamma_k \) on graph \( G_{n,r(n)} \). Let \( B_{\Gamma_k} = \{ G_v(\Gamma_k) \geq 1 \} \) be the event that there are one or more vertex-induced copies of subgraph \( \Gamma_k \) in \( G_{n,r(n)} \).

A necessary condition for \( \mathcal{P}_{\text{top}} \) is that there exist no edge-induced cycles of length 6. The absence of edge-induced cycles of length 6 can be expressed as an intersection of a finite number of vertex-induced events, \( \{ B_{\Gamma_k} \} \):

\[
\mathbb{P}\{ \mathcal{P}_{\text{top}} \} \leq \mathbb{P}\{ \bigcap_{\Gamma_k \in \mathcal{R}^2} B_{\Gamma_k} \} = \prod_{\Gamma_k \in \mathcal{R}^2} \mathbb{P}(\Gamma_k \text{ feasible}) \exp(-c^5 \mu_{\Gamma_k}).
\]

By Thm. 3.5 of Penrose [25], the finite collection of vertex-induced subgraph counts \( \{ G_v(\Gamma_k) \} \) converge to independent Poisson r.v.'s. with rates \( \{ \lambda = c^5 \mu_{\Gamma_k} \} \), for our choice of \( r(n)^2 \). The null probability of the subgraph counts becomes:

\[
\lim_{n \to \infty} \mathbb{P}\left\{ \bigcap_{\Gamma_k \in \mathcal{R}^2} B_{\Gamma_k} \right\} = \prod_{\Gamma_k \in \mathcal{R}^2} \mathbb{P}(\Gamma_k \text{ feasible}) \exp(-c^5 \mu_{\Gamma_k}).
\]

where \( \mu_{\Gamma_k} \) is computed from Eq. 3.2 in [25] for each vertex-induced subgraph. We may upper bound the exponential by considering a single term in the summation where \( \Gamma_6 = K_6 \) (the complete graph on 6 vertices) has \( k = 6 \) vertices and then expressing a lower bound for \( \mu_{K_6, \mathbb{R}^2} \):

\[
\mu_{K_6, \mathbb{R}^2} = \frac{1}{6!} \int_{\mathbb{R}^2} f(x) dx \int_{\mathbb{R}^2} h_{K_6}((0, x_1, \ldots, x_5)) dx_1, \ldots, dx_5
\]

\[
\mathbb{P}\{ G_{n,r(n)} \in \mathcal{P}_{\text{conn}} \} = \lim_{n \to \infty} \mathbb{P}\{ T \leq r(n) \} = e^{-e^{-x}}.
\]

M. Lem. 6 (Reg. Sharp Threshold for \( \mathcal{P}_{\text{conn}} \) in \( G_{n,r(n)} \))

Proof: Let the r.v. \( T(G_{n,r(n)}) \) (shortened to \( T \)) be the minimum edge distance that yields a connected graph for \( G_{n,r(n)} \). Thus, the graph \( G_{n,r(n)} \) is connected iff \( T(G_{n,r(n)}) \leq r(n) \). Using a specialization of Cor. 13.21 in [25], we show that:

\[
(72)
\]

where \( F(x) = e^{-e^{-x}} \) for all continuous points of \( F(x) : \mathbb{R} \). For \( r(n)^2 \in r(n)^2 + x(\alpha(n)) + o(\alpha(n)) \):

\[
\mathbb{P}\{ T \leq r(n) \} \leq e^{-e^{-x} + o(1)}
\]

Thus, for the given choice of \( r(n)^2 \):

\[
\lim_{n \to \infty} \mathbb{P}\{ G_{n,r(n)} \in \mathcal{P}_{\text{conn}} \} = \lim_{n \to \infty} \mathbb{P}\{ T \leq r(n) \} = e^{-e^{-x}}.
\]

N. Lem. 7 (Reg. Threshold for \( \mathcal{P}_{\text{giant}} \) in \( G_{n,r(n)} \))

Proof: Given \( \lambda_c \in (0, \infty) \), construct \( r^*(n)^2 = \lambda_c/n \). We show that:

\[
(77)
\]

Suppose \( r(n)^2 \sim x r^*(n)^2 \) with \( x \geq 0 \). We have \( r(n)^2 \sim \rho/n \) with \( \rho = x\lambda_c \). Let \( h \in (0, 1/x) \) such that there exists a single, bounded population cluster at level \( h > 0 \) exactly equal to the unit square \( R_1 = [-1/2, 1/2]^2 \subset \mathbb{R}^2 \). Let \( L_1 \) be the normalized size of the largest component of \( G_{n,r(n)} \). By Thm. 11.9 of [25], we have that \( L_1 \) converges in probability to \( I(R_1; \rho) \), since complete convergence implies convergence in probability:

\[
\lim_{n \to \infty} \mathbb{P}\left\{ \frac{L_1}{n} - I(R_1; \rho) > \epsilon \right\} = 0, \forall \epsilon > 0,
\]

where \( I(R_1; \rho) = p_{\text{imp}}(x\lambda_c) \) is the percolation probability under communication radius function \( r^*(n)^2 \sim x\lambda_c/n \).

Now, suppose that \( x < 1 \). By definition, the percolation probability is zero, and:

\[
\lim_{n \to \infty} \mathbb{P}\{ |L_1/n - I(R_1; \rho)| > \epsilon \} = 0, \forall \epsilon > 0
\]

\[
\lim_{n \to \infty} \mathbb{P}\{ |L_1/n| > \epsilon \} = 0, \forall \epsilon > 0
\]

\[
\lim_{n \to \infty} \mathbb{P}\{ G_{n,r(n)} \in \mathcal{P}_{\text{giant}}(\epsilon) \} = 0, \forall \epsilon > 0
\]

\[
\lim_{n \to \infty} \mathbb{P}\{ G_{n,r(n)} \in \mathcal{P}_{\text{giant}} \} = 0.
\]
Alternately, suppose that $x > 1$. By definition, the percolation probability is positive, and:

$$1 = \lim_{n \to \infty} P\{L_1/n - I(R_1; \rho) > \epsilon\}, \forall \epsilon > 0$$  \hspace{1cm} (82)

$$\leq \lim_{n \to \infty} P\{L_1/n \geq I(R_1; \rho) - \epsilon\}, \forall \epsilon > 0$$  \hspace{1cm} (83)

$$= \lim_{n \to \infty} P\{L_1/n \geq \rho \omega(x\lambda_c) - \epsilon\}, \forall \epsilon \in (0, \rho \omega(x\lambda_c))$$  \hspace{1cm} (84)

$$= \lim_{n \to \infty} P\{G_{n,r(n)} \in \mathcal{P}_{\text{giant}}(\beta)\}, \forall \beta \in (0, \rho \omega(x\lambda_c) - \epsilon)$$  \hspace{1cm} (85)

$$= \lim_{n \to \infty} P\{G_{n,r(n)} \in \mathcal{P}_{\text{giant}}\}.$$  \hspace{1cm} (86)

By the squeeze lemma, we conclude that

$$\lim_{n \to \infty} P\{G_{n,r(n)} \in \mathcal{P}_{\text{giant}}\} = 1.$$  \hspace{1cm} ■

O. Thm. 4 (Mutual Excl. of $\mathcal{P}_{\text{top}}$ and $\mathcal{P}_{\text{giant}}$ in $G_{n,r(n)}$)

Proof: By Lem. 7, $r(n)^2 \sim c/n, c < \lambda_c$ implies that $\mathcal{P}_{\text{giant}}$ holds $a.a.n.$ Therefore, $r(n)^2 \sim c/n, c \geq \lambda_c$ is a necessary condition for $\mathcal{P}_{\text{giant}}$ to hold $a.a.s.$ Under this necessary condition, we see that $r(n)^2 \in \omega(1/n^{6/5})$ and by Cor. 4 $\mathcal{P}_{\text{top}}$ holds $a.a.n.$

Thus, for $r(n)^2 \sim c/n, \forall c \leq \lambda_c$:

$$0 \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}} \cap \mathcal{P}_{\text{top}}\} \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}}\} = 0$$

Alternately, for $r(n)^2 \sim c/n, \forall c \geq \lambda_c$:

$$0 \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}} \cap \mathcal{P}_{\text{top}}\} \leq \lim_{n \to \infty} P\{\mathcal{P}_{\text{top}}\} = 0$$

In both cases, we can conclude that

$$\lim_{n \to \infty} P\{\mathcal{P}_{\text{giant}} \cap \mathcal{P}_{\text{top}}\} = 0.$$  \hspace{1cm} ■

P. Cor. 5 (Mutual Excl. of $\mathcal{P}_{\text{top}}$ and $\mathcal{P}_{\text{conn}}$ in $G_{n,r(n)}$)

Proof: $\mathcal{P}_{\text{giant}}$ is a necessary condition for connectivity $\mathcal{P}_{\text{conn}}$; thus $P\{\mathcal{P}_{\text{top}} \cap \mathcal{P}_{\text{conn}}\} \leq P\{\mathcal{P}_{\text{top}} \cap \mathcal{P}_{\text{giant}}\}$. The mutual exclusion between $\mathcal{P}_{\text{top}}$ and $\mathcal{P}_{\text{conn}}$ follows immediately from Thm. 4.  \hspace{1cm} ■

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