Holography-based quantum projector in a state space of linear photon momentum

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Keywords: projective measurements, quantum information processing, coupled wave theory, volume holography, mutually unbiased bases, quantum mechanics

Abstract

A transmission volume hologram is evaluated as a quantum projector operating on linear momentum states of individual photons in a four-dimensional state space. A state space is defined by four momentum vectors that are either efficiently diffracted or transmitted by the hologram. The influence of the volume hologram on the complex amplitudes of the basis states is modeled using coupled-wave theory. Measurement probabilities obtained with a second hologram introduced as a state-space analyzer are compared to measurement probabilities associated with a projection operator in quantum mechanics. A compact sequential hologram configuration for projecting superposition states onto all of the basis states is demonstrated for individual photons prepared in all 20 states of the five mutually unbiased bases of the four-dimensional state space. The complex amplitudes associated with superpositions of the basis states are imparted to individual photons via computer-generated holography and a liquid-crystal spatial light modulator. The coupled-wave theory analysis indicates that relative phase relationships in transmitted superposition states can be preserved to within 1/28th of an optical cycle and measured detection probabilities compare favorably to those associated with a quantum projector in a basis of discrete orthonormal states. Small departures of the measured statistics from theoretical expectations are quantified and attributed to imperfections in the hologram and experimental setup.

1. Introduction

Central to quantum physics are the principles of superposition and indeterminacy associated with projective measurements in a basis of discrete orthonormal states that define a state space [1]. The two-dimensional state space of photon spin, understood in classical optics as polarization, is broadly applied to quantum computation and quantum information where projective measurements play an important role [2, 3]. The transverse momentum of the photon, understood in classical optics as transverse modulation of amplitude and phase, has also been considered as a mechanism for encoding photons in higher-dimensional state spaces for applications in the quantum information sciences [4–6].

The recent interest in transverse momentum has largely centered on photon orbital angular momentum and considerable attention has been given to the task of de-multiplexing, or sorting, photons according to their angular momentum while minimizing cross talk between adjacent orthogonal field modes [7–11]. De-multiplexing is a projection operation that occurs when the incident photon states are matched to the states that form the measurement basis. Quantum projection operations with matched bases result in deterministic measurements. In the more general case where projections are performed on superpositions of the measurement basis states, probabilistic outcomes result from the random nature of wavefunction collapse.

Linear transverse momentum (LTM) is potentially useful to applications in quantum information processing. With the two degrees of freedom associated with wavefront tilt, the number of LTM states that can couple efficiently between two apertures increases quadratically with increasing aperture sizes [12]. Also, LTM
states naturally decompose into a spectrum of plane waves in a Fourier transform plane of an optical system and, in principle, a simple lens and photon-counting detector can yield measurement probabilities associated with the angular power spectrum [13]. Diffraction precludes the de-multiplexing of adjacent LTM field modes but, non-adjacent modes naturally and effectively de-multiplex at the focus of an optical system and in the far field of propagation, albeit with a fixed and typically small angular separation.

In analogy to polarization, where one can project an arbitrary polarization state onto two orthogonal states in any one of three mutually unbiased bases (MUBs), one can define MUBs in higher dimensional state spaces for projecting transverse momentum states. MUBs of transverse momentum consist of superpositions of transverse momentum states with specific relative phases. While a simple lens/detector combination could perform projective measurements in a plane-wave basis, it cannot discern the relative phases within the superposition states in order to perform these measurements in all of the MUBs of a LTM state space. Previously, we presented preliminary results from a demonstration that suggests a transmission volume hologram can yield diffraction efficiencies corresponding to projective measurement outcomes in any of the MUBs of a LTM state space [14]. Volume holograms possess a number of characteristics that may be of utility to applications in quantum information processing. These include a response to both the amplitude and phase of an incident optical field, high optical efficiency, and high angular separation in a collimated beam space. Volume holograms also offer a technology path for compact robust components with design controls that may enhance compatibility with photonic integrated circuits. It is therefore of interest to understand quantitatively how volume holograms may be optimized and implemented as quantum projectors in higher dimensional state spaces of LTM.

In this paper, we consider the influence of a simple plane-wave-interference volume hologram on individual photons prepared in LTM states and their superpositions and compare the influence of the hologram to the characteristics of an idealized quantum projector acting within a basis of discrete orthonormal states. Since diffraction precludes the discrimination of adjacent orthogonal field modes, the approach is based on the angular selectivity of the hologram rather than the orthogonality of modes. A state space is defined by four photon momentum vectors and their superpositions. Three of the basis states are efficiently transmitted by the hologram and one is efficiently diffracted by the hologram. The complex amplitudes of the transmitted and diffracted components are modeled using coupled-wave theory. The results indicate the relative phases of the transmitted states are preserved to within 1/28th of an optical cycle. The 10° diffraction angle introduced by the hologram is sufficiently large that the transmitted and diffracted components are easily isolated for subsequent operations. Cross talk in the transmitted and diffracted wavefunctions resulting from an individual photon prepared in a superposition state is quantified via statistical measurements using a second hologram as a state-space analyzer. Measured detection probabilities compare favorably to those associated with a projection operator in quantum mechanics. A compact sequential hologram arrangement for projecting superposition states onto the four basis states is demonstrated with individual photons prepared in all 20 states of the five MUBs of the four-dimensional state space.

Photo-thermo-refractive (PTR) glass plane-wave-interference holograms are chosen for empirical measurements due to their high optical quality, high optical efficiency, stability, and well-defined characteristics that follow the predictions of coupled-wave theory [9, 15, 16]. The spatially varying amplitude and phase associated with the LTM states and their superpositions are imparted to individual photons via computer-generated holography (CGH) and a liquid-crystal (LC) spatial light modulator (SLM) [17]. Small departures in measured statistics from those associated with an ideal projector are quantified and attributed to imperfections in the hologram and experimental setup.

2. Characteristics of an idealized projection operator in a discrete orthonormal basis

This section describes the characteristics of an idealized projection operator in an orthonormal basis. Consider a state space defined by four mutually orthonormal basis vectors, \( |1\rangle, |2\rangle, |3\rangle, |4\rangle \) and superpositions of these vectors. Owing to orthonormality, \( \langle 3 | 4 \rangle = \delta_{ij} \) with \( i \) and \( j \) representing integer values 1 through 4. Consider the projector \( P_{22} \) onto \( |2\rangle \) described by the outer product, \( P_{22} = |2\rangle \langle 2| \) and consider a superposition state \( |a\rangle = |1\rangle + |2\rangle + |3\rangle + |4\rangle \) with normalization \( \langle a | a \rangle = 1 \) and complex amplitudes \( a, b, c, \) and \( d \).

Figure 1 illustrates conceptually an optical component performing the function of the projector \( P_{22} \) on \( |a\rangle \). The projector removes the component of the state vector associated with state \( |2\rangle \) and directs it along a different path. If the complex amplitudes of the remaining states are preserved in transmission, then the transmission path can be described by the operator \( 1 - |2\rangle \langle 2| \). The probability of detecting the photon in each path is given by the squared modulus of the transmitted and projected wavefunctions as \( 1 - \langle 2 | 2 \rangle = |a|^2 + |b|^2 + |c|^2 + |d|^2 \) and \( \langle 2 | 2 \rangle = |b|^2 \), respectively. The detection probabilities given by the squared moduli are insensitive to both absolute and relative phases. The absolute phase associated with an isolated state is usually inconsequential in the absence of an
absolute reference. The relative phases in a superposition however are important to defining a given superposition state.

In practice, measured probabilities after a single projecting element cannot discern cross contamination from state-independent optical efficiencies. Cross contamination can be evaluated with a second projector acting as a state-space analyzer in each path. Owing to orthogonality of the basis states, projecting twice in succession onto a given state is theoretically equivalent to projecting a single time. Figure 2 illustrates an idealized projector followed by an idealized projector/analyzer in each path. Applying an idealized projector/analyzer to the transmitted state preserves this state in transmission and, since state $|2\rangle$ is absent, the projected output is the vacuum state $|\rangle$. In the projected output path, the analyzer projects state $|2\rangle$ once again and, since the input consists exclusively of state $|2\rangle$, the transmitted state is the vacuum state $|\rangle$. Contamination in either output of the first projector leads to deviations in measurement statistics at the outputs of the secondary projectors.

3. A linear momentum projector based on angular selectivity in a transmission volume hologram

The preceding section describes the theoretical characteristics of an idealized projection operator in a discrete orthonormal state space. This section discusses the orthonormal eigenfunctions of the LTM operators in the context of diffraction and position–momentum uncertainty. Since these phenomena preclude adjacent mode discrimination, orthogonality is not considered to be a necessary condition for discriminating LTM states. Instead, the angular selectivity of a hologram is chosen as the criterion for defining the state space and coupled-wave theory provides a theoretical description of the complex amplitudes transmitted and diffracted by the hologram. In order to minimize the range of angles necessary to define a state space, diffraction minima and maxima are considered in close proximity to the Bragg angle.

3.1. Eigenfunctions of the LTM operators and their angular spectra

Optical field modes within an aperture can be defined in terms of traveling-wave solutions to the wave equation as is done in treatments of heterodyne detection [18]. The complex amplitudes describing the transverse spatial characteristics of these modes are of the form

$E_m(x) = \text{rect}\left(\frac{x}{L_x}\right) \frac{1}{\sqrt{L_x}} e^{i m\pi x}$

and

$E_n(y) = \text{rect}\left(\frac{y}{L_y}\right) \frac{1}{\sqrt{L_y}} e^{i n\pi y}$,

where $x$ and $y$ are the transverse coordinates, $L_x$ and $L_y$ are the lengths of the aperture along $x$ and $y$, and $m$ and $n$ can assume any integer value. Considering the $y$-component $E_n(y)$, the rect function defines an aperture along the $y$ axis with $\text{rect}(y/L_y) = 1$ for $|y| < L_y/2$ and $0$ otherwise and the term $1/\sqrt{L_y}$ is a normalizing coefficient. The
nominal angle of propagation for the n\text{th} mode is n\lambda/L_x radians but, owing to diffraction, the optical power is distributed over range of propagation angles \alpha_i along y. The angular power spectrum resulting from diffraction by the aperture along the y axis is well-known and for the n\text{th} mode is given by the squared modulus of the Fourier transform of the field within the aperture to be \left|\int \left(\hat{\epsilon}(\alpha_i-n\lambda/L_x)\right)e^{i\alpha_i}d\alpha_i\right|^2 in normalized form. The peak of the power spectra occurs where \alpha_i = n\lambda/L_x and the full angular width of the central lobe is 2\lambda/L_x, corresponding to twice the mode spacing. Hence, the central lobes of adjacent modes overlap by half their width while those of alternate modes do not overlap. Similarly for \mathcal{E}_m(x) along the x axis.

These orthonormal field modes are also eigenfunctions of the transverse momentum operators \hat{p}_x = -i\hbar \frac{\partial}{\partial x} and \hat{p}_y = -i\hbar \frac{\partial}{\partial y} with eigenvalues of transverse momentum \hbar \frac{m_2}{L_x} and \hbar \frac{n_2}{L_x}, respectively. Along the y axis, these functions describe photons with quantized LTM given by \hbar \frac{m_2}{L_x} and root-mean-square (rms) uncertainties in position and momentum, \sigma_{\nu} and \sigma_{\nu}\hat{p}_x, respectively, constrained by the product inequality\sigma_{\nu}\sigma_{\nu}\hat{p}_x \geq \frac{\hbar}{2}. The rms uncertainty in position is \sigma_{\nu} = \frac{L_x}{\sqrt{2}} and the rms uncertainty in transverse momentum is \sigma_{\nu}\hat{p}_x \geq \frac{\sqrt{2}}{L_x} \hbar corresponding to a minimum rms uncertainty in mode number of about 28%. Similarly for the eigenvalues along x.

In both the classical and quantum description, adjacent mode discrimination is limited by cross talk that is on the order of the mode spacing. Orthogonality is not necessary for mode discrimination. Consequently, the angular selectivity of a volume hologram is considered independent of orthogonality both as a mechanism for projection operations and as a criterion for defining a LTM state space.

3.2. Coupled-wave theory description of a transmission volume hologram

Transmission volume holograms are well known for the high angular selectivity that may be achieved with thick media. For monochromatic light near the Bragg angle, coupled-wave theory provides a method for treating in transmission volume holograms are well known for the high angular selectivity that may be achieved with thick media. For monochromatic light near the Bragg angle, coupled-wave theory provides a method for treating in projection operations and as a criterion for de-
field $\xi$ varies nearly linearly with angle, as illustrated by the dashed line, and is equal to zero at the Bragg angle. The transmission and diffraction efficiencies are plotted in figure 3(b) as solid and dashed lines, respectively, along the scale to the left. The first and second transmission maxima are calculated to occur at $\Theta_B \pm 0.18^\circ$ and $\Theta_B \pm 0.39^\circ$, respectively. At these angles, the phase retardances of the transmitted fields are calculated to be 0.42 and 0.20 radians, respectively. The corresponding relative phase delay introduced within a superposition of these states would be only 0.22 radians or less than $1/28$th of an optical cycle. These characteristics create an opportunity to define a discrete state space where states with momentum vectors aligned with the Bragg angle are diffracted with near-unity efficiency while states with momentum vectors aligned with the transmission maxima are transmitted with near-unity efficiency. Furthermore, the relative phases of the transmitted states will be preserved to within a small fraction of an optical cycle.

For very thick holograms with high angular selectivity, it can also be important to consider the angular spectra of the incident light. For the circular aperture used in the experiments, the normalized power spectrum associated with a wavevector at angle $\alpha_0$ along $y$ is given by

$$\frac{4 J_1[(\alpha_0 - \alpha_0)/\lambda]}{aD(\alpha_0 - \alpha_0)/\lambda}$$

where $D$ is the diameter of the aperture and $J_1$ is a Bessel function of the first kind, order 1. In this case, the full angular width of the central lobe is $2.44 \lambda/D$. The modes previously defined within a square aperture are not orthogonal over the circular aperture. While it is possible to define orthogonal modes within the circular aperture, this is not considered necessary for defining a state space based on angular selectivity. The dotted lines in figures 3(a) and (b) illustrate the angular power spectra in the $y$–$z$ plane calculated for the $D = 3.4$ mm diameter circular pupil at the hologram. The normalized optical power is plotted according to the scale to the right and is shown for three nominal angles of incidence corresponding to the Bragg angle and two of the transmission maxima. For the conditions of the experiment, the angular spectra are much narrower than the diffraction minima and maxima. Considering only the light that is efficiently transmitted or diffracted, the phases $\xi$ and $\xi - \delta$ within the angular spectra are equal to zero to within a small fraction of $2\pi$. In principle, the angular selectivity of the hologram could be further narrowed by increasing the thickness of the hologram. However, the diffraction efficiency will decline as the angular selectivity function becomes more narrow than the angular spectrum of the incident light [12].

### 3.3. A state space based on angular selectivity

Figure 4 illustrates one possible choice of four photon momentum vectors that will maximize the diffraction efficiency for one state by the hologram while maximizing the transmission efficiency for the remaining states and simultaneously keep the total range of angles small. In this particular case, out-of-plane angles have been introduced in order to optimize projections onto all of the basis states with the multiple hologram orientations introduced later in section 4.2. The $x'$–$y'$ plane is included as a viewing aid. The components of angles in the $y$–$z$ plane are indicated in figure 3 by the corresponding kets. The momentum vector for state $|2\rangle$ is oriented for
maximum diffraction efficiency at the Bragg angle. States $|3\rangle$ and $|4\rangle$ lie in the $x$–$z$ plane which is aligned for maximum transmission at the first maximum below the Bragg angle. State $|1\rangle$ is oriented for maximum transmission efficiency at the second transmission maximum.

The grating vector of the hologram lies in the $y$–$z$ plane and the interaction of the wavevector with the hologram occurs in this plane. The momentum of states $|1\rangle$ and $|2\rangle$ lie within this plane but, that of states $|3\rangle$ and $|4\rangle$ include a very small component in the orthogonal $x$–$z$ plane. The 0.18° out-of-plane propagation of states $|3\rangle$ and $|4\rangle$ increases the optical path through the hologram by approximately 4 nm resulting in an additional phase delay of less than 1/100th of an optical cycle due to the bulk index of the material. Within the limits of coupled-wave theory, the relative phasing of the transmitted states is very nearly preserved.

4. Projective quantum measurements with PTR glass holograms

The transmission volume holograms are 13.5 × 13.5 mm aperture by 1.27 mm thick permanent holograms fabricated by OptiGrate by interfering two planar wavefronts in PTR glass [15]. At 532 nm optical wavelength, an incident plane wave experiences peak diffraction at the Bragg angle near normal incidence and results in reconstruction of the reference plane wave at 10° in air relative to the surface normal. The holograms are optically efficient with a peak internal diffraction efficiency of about 96%, a scattering loss of less than 5%, and coated-surface reflection losses of less than 0.25% per surface. The peak external diffraction efficiency is measured to be 91%. The Bragg angle in air is approximately $\Theta_B = 0.21°$. The first diffraction minima are located at approximately ±0.19° angular separation from $\Theta_B$ and yield a measured internal diffraction efficiency of about 1.5%. This angular separation corresponds about 21 times $\lambda/D$ for the $D = 3.4$ mm diameter pupil over which the holograms are illuminated. The secondary diffraction minima result in a measured internal diffraction efficiency of about 0.5%.

Photons are prepared in individual linear momentum states and their superposition with a phase modulating SLM. The complex amplitudes of the basis states are described by simple linear phase functions along $x$ and $y$ but, superpositions of these states include transverse modulation in both the amplitude and phase. Both the individual basis states and their superpositions are prepared using the CGH–SLM technique described previously [17]. The complex field calculated for a given state is pre-conditioned to compensate for artifacts of phase modulation holography and aberrations intrinsic to the SLM. The SLM used in this work is a Boulder Nonlinear Systems 512 × 512 element LC SLM that operates in reflection and introduces phase retardations of up to $\pi$ radians at the 532 nm wavelength.

4.1. Detection probabilities with a projector and a state-space analyzer

Figure 5 shows the experimental setup configured to measure transmission and diffraction probabilities associated with a single hologram. Spatially coherent light from a Coherent 10 mW continuous-wave frequency-doubled Nd: YAG laser passes through an attenuator system (AS) to reduce the photon rate to approximately 1 MHz. The attenuated light is then spatially filtered, collimated and propagated to the SLM. The SLM is illuminated at an angle of 7.5° relative to the surface normal. The CGH functions associated with the basis states and their superposition are calculated and displayed on the SLM over a 7.68 mm diameter circular aperture. A circular pupil function was chosen to be consistent with the Zernike polynomial description of the optical path function introduced to compensate aberrations in the SLM. An iris, placed in close proximity to the SLM, prevents light from illuminating pixels beyond the pupil region. While the angular resolution of the transmission volume hologram is limited by the thickness of the hologram, the range of angles that may be achieved with the CGH–SLM technique is limited by the resolution of the SLM. In order to achieve the range of
angles required to accommodate the angular selectivity of the PTR glass hologram, the SLM is used in conjunction with a nominally 2:1 afocal pupil relay (APR) that serves to magnify the wavefront tilts by a factor of two. The −1 diffracted order containing the desired modulation characteristics is selectively passed by a spatial filter (not shown) in the 2:1 APR and propagated to a pupil that is co-located with the volume hologram. The transmission and diffraction probabilities are measured with an avalanche photo diode (APD) operated in Geiger-mode (GM) for photon counting. In order to accurately measure probabilities associated with a wavefunction that potentially includes any of the four basis states, the pupil at the hologram is imaged onto the 175 micron diameter sensor of the APD with a near-diffraction-limited 25:1 APR relay as shown. The GM–APD detection system is operated free running with a maximum detection rate of 30 million counts per second. This is sufficiently larger than the detected photon rates to ensure the data are dominated by individual photon events. Detector counts are acquired over a period of one second. For consistency in optical efficiency and detector quantum efficiency, a single 25:1 APR and GM–APD combination is re-positioned for each of the measurements shown.

For this measurement realization, the hologram function is calculated for an equally weighted superposition state |ψ⟩ = a|1⟩ + b|2⟩ + c|3⟩ + d|4⟩ with real-valued coefficients a = b = c = d = 1/2. Momentum states |1⟩, |2⟩, |3⟩, and |4⟩ are defined by symmetric wavevector angles of ±0.188° along x and y at the 3.4 mm reduced pupil at the hologram. This angle was chosen to match the state space momentum vectors to the peak and first minimum of the diffraction efficiency function. Symmetric wavefront tilts were chosen to optimize the results presented later in section 4.3. This choice of angles resulted in a small misalignment of momentum state |1⟩ to the secondary minimum. With the hologram aligned for maximum diffraction efficiency for state |2⟩, the measured internal diffraction efficiencies for states |1⟩, |2⟩, |3⟩ and |4⟩ are 3.4%, 96.0% 1.5% and 1.5%, respectively.

Figure 6 shows portions of the experimental setup configured to measure transmission and diffraction probabilities after a second hologram is added as a state-space analyzer. In figure 6(a), the second hologram is placed in the path of the wavefunction transmitted by the first hologram and in figure 6(b), the second hologram is placed in the path of the wavefunction diffracted by the first hologram. In both cases, the second hologram is aligned to maximize diffraction of momentum state |2⟩. Detection probabilities associated with individual photons transmitted and diffracted by the second hologram are measured with the APR GM–APD arrangement described above. In order to account for all of the transmitted and diffracted components, detection probabilities are also measured in the alternate output path of the first hologram. The 100 mm depth of focus at the pupil is sufficiently large that both holograms remain within the reduced pupil of the system.
Table 1 compares the measured probabilities to those of a theoretical projector for each of the optical paths shown in figures 5, 6(a) and (b). For each configuration, the measured probabilities are internal diffraction probabilities calculated by dividing the measured detection rate at each detector location by the sum of the detection rates at all detector locations after subtracting background count rates. The fourth column shows the measured detection probabilities for photons transmitted and diffracted by the first hologram to be 0.74 and 0.26, respectively compared to the theoretical values of $0.75$ and $0.25$, respectively. The final column shows the theoretical and measured detection probabilities after the secondary hologram projectors. The measured detection probabilities of 0.01 and 0.02 associated with the theoretical vacuum states suggest there is only small contamination of the states projected by the first hologram. The measured detection probabilities for the twice transmitted and twice diffracted photons are 0.72 and 0.22 respectively and are in approximate agreement with the theoretical probabilities of 0.75 and 0.25, respectively.

The small discrepancies between the measured and theoretical probabilities can be largely accounted for by considering the measured transmission and diffraction probabilities associated with each of the four momentum states at each hologram. If $t_1, t_2, t_3, t_4$ and $d_1, d_2, d_3, d_4$ represent the individual photon transmission and diffraction probabilities at the first hologram for states $|1\rangle, |2\rangle, |3\rangle, |4\rangle$, respectively, then the photon detection probabilities may be estimated as follows. The internal diffraction probabilities for the first hologram are measured to be $d_1 = 0.034, d_2 = 0.960, d_3 = 0.015, d_4 = 0.015$. The corresponding internal transmission probabilities are $t_1 = 0.967, t_2 = 0.041, t_3 = 0.985, t_4 = 0.985$. From these values, the individual photon detection probabilities after the first hologram are calculated to be $\sum_{\text{t}} t_i = 0.74$ in transmission and $\sum_{\text{d}} d_i = 0.26$ in diffraction in agreement with the measured values of 0.74 and 0.26, respectively. Similarly, if $t_1', t_2', t_3', t_4'$ and $d_1', d_2', d_3', d_4'$ represent the individual transmission and diffraction probabilities at the second hologram, then the detection probabilities after the secondary hologram can also be estimated. The internal transmission probabilities for the second hologram are measured to be $d_1' = 0.042, d_2' = 0.942, d_3' = 0.026, d_4' = 0.025$. The corresponding internal transmission probabilities are, $t_1' = 0.958, t_2' = 0.058, t_3' = 0.975, t_4' = 0.975$. From these values, the individual photon detection probabilities after the second hologram are calculated to be $\sum_{\text{t}} t_i' = 0.71$ versus 0.72 measured for transmission by two consecutive holograms, $\sum_{\text{d}} d_i' = 0.03$ versus 0.01 measured for transmission followed by diffraction, $\sum_{\text{d}} d_i' = 0.03$ versus 0.02 measured for diffraction followed by transmission, and $\sum_{\text{d}} d_i' = 0.23$ versus 0.22 measured for diffraction by two consecutive holograms. Discrepancies between the calculated and measured probabilities are generally small and are attributed to experimental errors including drifts in the laser power and background illumination during the measurements. The fractional deviation from the calculated values is most significant for the vacuum states where the error can be comparable to the expected value.

### 4.2. Projections onto the basis states with cascaded holograms

When a photon-counting detector is placed in the diffracted output path of a single hologram that is aligned to one of the four LTM basis states, the pair can serve as a state detector signaling the arrival of an individual photon in that particular state. Photons in the remaining three states are efficiently transmitted by the hologram and do not result in a detection event. Holograms can then be placed in sequence to construct a state detector for all of the four basis states. Owing to the random nature of wavefunction collapse, photons in superposition states will yield random outcomes.
Figure 7 illustrates a compact geometry for projecting states of the state space onto the four basis states. The linear momentum vectors of the four basis states $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ are designated by angles $\Theta_1$, $\Theta_2$, $\Theta_3$, $\Theta_4$, respectively. These angles are symmetric about a common axis as illustrated in figure 4. Four PTR glass holograms, fabricated to the same specifications described in sections 2 and 3, are placed in sequence and rotated in the transverse plane at 90° angles relative to one another with the hologram grating vectors $\mathbf{K}$ oriented as shown. Each hologram is tilted for maximum diffraction efficiency for one of the basis states and minimum diffraction efficiency at the angle defined by the axis of symmetry. For each hologram, one of the four basis states experiences high diffraction efficiency and the remaining three experience high transmission efficiency.

Figure 8 illustrates a superposition state $|\psi\rangle = a|1\rangle + b|2\rangle + c|3\rangle + d|4\rangle$, with complex coefficients $a$, $b$, $c$ and $d$, incident upon the holograms. Conceptually, wavefunction components $a|1\rangle$, $b|2\rangle$, $c|3\rangle$, and $d|4\rangle$ are diffracted at 10° angles along the $-y$, $+y$, $-x$ and $+x$ directions respectively. In the experiment, the four holograms are separated from one another by approximately 6.5 mm, well within the 100 mm depth of focus for the pupil. Components of the wavefunction diffracted by a given hologram transit the remaining holograms with high optical efficiency before propagating through the 25:1 APRs (not shown) to the GM APD detectors. Assuming 100% efficient optics and detectors, the detection probabilities for each diffracted path are $|a|^2$, $|b|^2$, $|c|^2$ and $|d|^2$.

4.3. Experimental demonstration with states of the MUBs

Projections onto the basis states are demonstrated with the experimental setup described in figure 5 and the cascaded hologram geometry described in figures 7 and 8. Individual photons are prepared in superposition states given by the relationships that define the MUBs of a four-dimensional state space [19]. Two bases of orthonormal states $\left\{ |s^m_i\rangle \right\}$ and $\left\{ |s^m_j\rangle \right\}$, where the superscript integers $m$ and $n$ denote the bases and the subscript integers $i$ and $j$ denote the states within the bases, in a $N$-dimensional complex inner-product space are said to be mutually unbiased if all inner products across their elements have the same magnitude. Namely, for all $m \neq n$, $\left| \left< s^m_i | s^n_j \right> \right| = \frac{1}{\sqrt{N}}$ holds for all $i$ and $j$. The inner products that define MUBs correspond to detection probabilities associated with projection operations. Projecting the $j$th state from the $n$th basis onto the $i$th state from the $m$th basis yields detection probabilities as follows. When the bases are matched, $m = n$ and the measurement probabilities are given by $\left| \left< s^m_i | s^m_j \right> \right|^2 = \delta_{ij}$ corresponding to deterministic outcomes. When the photon is prepared and measured unmatched bases, then $m \neq n$ and the measurement probabilities are given by
corresponding to uniformly distributed random measurement outcomes with the measurement probability given by the reciprocal of the state space dimension.

In the two dimensions of polarization, there are three MUBs; namely, rectilinear, diagonal, and circular, in which one may perform projective quantum measurements. In a four-dimensional state space, there exist five MUBs. Denoting the basis of LTM states in figure 4 as MUB, and using vector notation \((a,b,c,d)\) to describe the complex coefficients that form the superposition state \(|s^m\rangle = a|1\rangle + b|2\rangle + c|3\rangle + d|4\rangle\), the states of the four-dimensional MUBs may be written as [20],

\[
\begin{align*}
\text{MUB}_0 & = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\} \\
\text{MUB}_1 & = \{1/2(1,1,1,1), 1/2(1,1,−1,−1), 1/2(−1,−1,1,1), 1/2(1,−1,1,−1)\} \\
\text{MUB}_2 & = \{1/2(1,−1,−i,−i), 1/2(1,−1, i, i), 1/2(1, i, i, −i), 1/2(1,−i, i, i)\} \\
\text{MUB}_3 & = \{1/2(1,−i,−i,−i), 1/2(−1,1,−i,−i), 1/2(1, i, i,−i), 1/2(−1, i, i,−i)\} \\
\text{MUB}_4 & = \{1/2(1,−i,−1,−i), 1/2(−1,1,−i, i), 1/2(1, i, i,−i), 1/2(−1, i, i,−i)\},
\end{align*}
\]

where, the minus sign represents a \(\pi\) optical phase shift and complex \(i\) represents a \(\pi/2\) optical phase shift. The vectors within each basis are orthonormal \(\langle s^m_i | s^n_j \rangle = \delta_{ij}\) and across bases the vectors satisfy the inner product relationship \(\langle s^m_i | s^n_j \rangle = \delta_{ij}\).

The complex optical fields associated with each of the 20 states of the five bases are created using the CGH–SLM technique and detection probabilities are measured with a single APR GM–APD combination positioned sequentially in each of the four detector positions illustrated in figure 8. The measured probabilities are summarized in figure 9 in bar graph plots for each of the 20 input photon states. Each bar graph plot shows the detection probabilities measured at each of the four detectors. The graphic on the right shows the position of the detectors in correspondence to the individual bars in each plot. The state vector labels in the right graphic indicate the positioning of each detector to a diffracted state as illustrated in figure 8. The probabilities displayed in figure 9 are the internal diffraction probabilities computed by measuring the photon count rates at the four detector locations, subtracting dark count rates, and normalizing each count rate to the sum of the four rates. The rows in the figure indicate the bases MUB, from which each photon state is derived. The columns indicate the state within each basis in correspondence with the ordering in equation (3).
In the first row of figure 9, the incident photon states are those of the MUB0 basis comprised of states efficiently diffracted by the holograms. The results demonstrate efficient state detection with low cross talk in the MUB0 basis. The probability of detecting a state matched to a corresponding hologram is near-unity and the probability of detecting any of the remaining three MUB0 states is near-zero. The average probability that the correct state is detected is 0.9988 with a standard deviation of 0.0002. The average probability that an incorrect state is detected is 0.0004 with a standard deviation of 0.0003. The remaining four rows in figure 9 illustrate the cases where the incident photon states are those associated with remaining bases MUB1, MUB2, MUB3, and MUB4. In these cases, the probabilities measured in the MUB0 basis are uniformly distributed with an average value of 0.250 with a standard deviation of only 0.008. These measured probabilities are in close agreement with the theoretical probabilities describing projective quantum measurements in MUBs. Namely, for these measurements performed in the \( m = 0 \) basis, the theoretical detection probabilities for photons in the \( n = 0 \) basis are given by \( \delta_{ij} \). For photons in the \( n \neq 0 \) bases, the theoretical probabilities are given by \( \frac{1}{4} \). Table 2 summarizes the statistics for the measurement probabilities including the minimum and maximum values.

| Matched basis | Yes | Yes | No |
|---------------|-----|-----|----|
| Matched state | Yes | No  | n/a |
| Theoretical   | 1   | 0   | 0.25 |
| Average       | 0.9988 | 0.0004 | 0.2500 |
| Stdev         | 0.0002 | 0.0003 | 0.0079 |
| Max           | 0.9992 | 0.0011 | 0.2640 |
| Min           | 0.9986 | −0.0002 | 0.2377 |

5. Discussion

The primary objective in this work is a quantitative evaluation of a hologram as a quantum projector in a state space of linear momentum. Coupled wave theory analysis of a transmission volume hologram suggests it is possible to define a discrete state space of photon linear momentum in which a hologram will perform as a projection operator as described in quantum mechanics. The hologram can efficiently diffract a given linear momentum state while efficiently transmitting the remaining states of a discrete state space and, within the limits of the theory, the relative phases of the transmitted states are preserved within a small fraction of an optical cycle. The range of angles that define the state space is minimized by exploiting diffraction extrema near the Bragg angle. In principle, smaller angular separations could be accommodated with thicker holograms however, maximizing the number of states in a system was not a priority in this work. Given the effects of diffraction with finite apertures, orthogonality is not considered to be necessary in defining the state space. Measurement probabilities obtained with a second hologram introduced as a state space analyzer compare favorably to those associated with a projection operator. A cascaded hologram geometry demonstrates measurement probabilities associated with projections onto a basis of LTM states. Based on an earlier preliminary demonstration [14], it may be possible to create transmission volume holograms to project onto other bases of LTM such as those defined by the MUBs. While the experiments presented required some complexity for the purpose of quantitative evaluation, the holograms themselves are compact, robust, and operate in the collimated space of an optical system. These features may make holography-based components of potential interest for applications in quantum information processing.

Acknowledgments

We gratefully acknowledge Leon Glebov and Vadim Smirnov of Optigrate for the design and fabrication of the hologram samples and Sami Shakir of Tau Technologies for useful theoretical discussions. This work was supported by the Air Force Office of Scientific Research.

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