Generalizations of normal ordering and applications to quantization in classical backgrounds

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Abstract
A nonlocal method of extracting the positive (or the negative) frequency part of a field, based on knowledge of a 2-point function, leads to certain natural generalizations of the normal ordering of quantum fields in classical gravitational and electromagnetic backgrounds and illuminates the origin of the recently discovered nonlocalities related to a local description of particles. A local description of particle creation by gravitational backgrounds is given, with emphasis on the case of black-hole evaporation. The formalism reveals a previously hidden relation between various definitions of the particle current and those of the energy-momentum tensor. The implications to particle creation by classical backgrounds, as well as to the relation between vacuum energy, dark matter, and cosmological constant, are discussed.

KEY WORDS: Current of particle density ; classical background ; particle creation

1 Introduction

The distinction between the positive frequency solutions and the negative frequency solutions of field equations plays a fundamental role in quantum field theory. In particular, their distinction is closely related to the distinction between annihilation and creation operators, which determine the
representation of the field algebra, the particle content related to quantum fields, and the normal ordering related to the renormalization of various operators bilinear in fundamental fields. However, the distinction between the positive and the negative frequencies has no invariant meaning in general relativity. In gauge theories, like electrodynamics, it also depends on the choice of gauge. This is closely related to the particle creation theoretically predicted to occur in classical gravitational [1, 2, 3] and electromagnetic [4, 5] backgrounds. This is also closely related to the noncovariance (with respect to general coordinate transformations) of the concept of particles [1, 6, 7]. The noncovariance is closely related to the fact that annihilation and creation operators are not local objects. In particular, the horizon plays a fundamental role in the black-hole evaporation [3] and the Unruh effect [7], which raised serious doubts on the correctness of the formalism that describes these effects [8, 9, 10, 11].

Recently, progress in establishing the covariance of the concept of particles has been achieved by constructing an operator that represents the local current of particle density [12, 13]. This local current is covariant with respect to general coordinate transformations and invariant with respect to gauge transformations. Nevertheless, this local current possesses a nonlocal property related to the fact that the determination of the current at a point \( x \) requires knowledge of the field on a whole Cauchy surface at which \( x \) lies. Besides, this current depends on the choice of a 2-point function. As shown in [12, 13], there exist a choice that is consistent with the usual results of particle creation described by a Bogoliubov transformation [1, 2, 3, 4, 5].

In this paper we show that the nonlocality appearing in the calculation of the particle current is related to a nonlocal procedure of extracting the positive frequency part \( \phi^+(x) \) and the negative frequency part \( \phi^-(x) \) from the field \( \phi(x) \). It appears that the particle current can be written in a completely local form as an operator bilinear in the fields \( \phi^+(x) \) and \( \phi^-(x) \). By introducing certain generalizations of normal ordering, the current can be written even as an (suitably ordered) operator bilinear in the field \( \phi(x) \). Different orderings of the particle current correspond to different orderings of other well-known local quantities, such as the energy-momentum tensor \( T_{\mu\nu}(x) \). In particular, an ordering that retains the infinite vacuum energy leads also to an infinite number of particles in the vacuum. This suggests that it might not be meaningless to talk about the system in which the vacuum is at rest and, consequently, that this vacuum energy might not contribute to the cosmological constant.

It also appears that the definitions of \( \phi^+ \) and \( \phi^- \) depend on the choice of the 2-point function. Therefore, the normal ordering of \( T_{\mu\nu} \) also depends on this choice. If the 2-point function is chosen such that the particle creation occurs, then \( T_{\mu\nu} \) is not conserved. This suggests that one should choose
the 2-point function such that the particle creation does not occur.

In Sec. 2, a method of extracting \( \phi^+ \) and \( \phi^- \) from a hermitian field \( \phi \) is presented and used to define the normal ordering and some generalizations of it. The method is based on a particular choice of 2-point functions \( W^+ \) and \( W^- \). This is applied in Sec. 3 to write the particle current in a very elegant and purely local form. A generalization to other choices of \( W^\pm \) is studied in Sec. 4 where the particle creation by a gravitational background is described in a local way, with emphasis on the description of particle creation by black holes. The formalism is generalized to complex scalar and spinor fields (interacting with a classical gravitational or electromagnetic background) in Sec. 5. The relation between the particle current and the corresponding energy-momentum tensor is discussed in Sec. 6 where the implications to the particle creation and to the relation between vacuum energy, dark matter, and cosmological constant, are discussed. The conclusions are drawn in Sec. 7.

2 Extraction of \( \phi^\pm \) from \( \phi \) and generalizations of normal ordering

A scalar hermitian field \( \phi(x) \) in a curved background satisfies the equation of motion

\[
(\nabla^\mu \partial_\mu + m^2 + \xi R)\phi = 0.
\]  

(1)

We choose a particular complete orthonormal set of solutions \( \{f_k(x)\} \) obeying the relations

\[
\begin{align*}
(f_k, f_{k'}) &= -(f_k^*, f_{k'}^*) = \delta_{kk'}, \\
(f_k^*, f_{k'}^*) &= (f_k, f_{k'}^*) = 0,
\end{align*}
\]

(2)

where the scalar product is defined as

\[
(\phi_1, \phi_2) = i \int_\Sigma d\Sigma^\mu \phi_1^{\ast \mu} \partial_\mu \phi_2.
\]  

(3)

The field \( \phi \) can be expanded as

\[
\phi(x) = \phi^+(x) + \phi^-(x),
\]

(4)

where

\[
\begin{align*}
\phi^+(x) &= \sum_k a_k f_k(x), \\
\phi^-(x) &= \sum_k a_k^\dagger f_k^*(x).
\end{align*}
\]

(5)
Introducing the 2-point functions

\[ W^+(x, x') = \sum_k f_k(x)f_k^*(x'), \]
\[ W^-(x, x') = \sum_k f_k^*(x)f_k(x'), \tag{6} \]

we find

\[ \phi^+(x) = i \int_{\Sigma} d\Sigma' W^+(x, x') \hat{\partial}_{\nu} \phi(x'), \]
\[ \phi^-(x) = -i \int_{\Sigma} d\Sigma' W^-(x, x') \hat{\partial}_{\nu} \phi(x'), \tag{7} \]

which is the curved-spacetime generalization of the standard result for flat spacetime [14]. We see that the extraction of \( \phi^+ (x) \) and \( \phi^- (x) \) from \( \phi(x) \) is a nonlocal procedure. Note that the integrals in (6) do not depend on the choice of the timelike Cauchy hypersurface \( \Sigma \) because \( W^\pm (x, x') \) satisfy the equation of motion (1) with respect to \( x' \), just as \( \phi(x') \) does. However, these integrals depend on the choice of \( W^\pm (x, x') \), i.e., on the choice of the set \( \{ f_k(x) \} \).

Having defined \( \phi^+ \) and \( \phi^- \), one can define the normal ordering in the usual way as the ordering that puts \( \phi^- \) on the left and \( \phi^+ \) on the right. Explicitly,

\[ :\phi^+ \phi^-: = \phi^- \phi^+, \tag{8} \]

while the normal ordering of the combinations \( \phi^- \phi^+ \), \( \phi^+ \phi^+ \), and \( \phi^- \phi^- \) leaves these combinations unchanged. We generalize (8) by introducing 4 different orderings \( N_{(\pm)} \) and \( A_{(\pm)} \) defined by the relations analogous to (8):

\[ N_{(+)} \phi^+ \phi^- = \phi^- \phi^+, \quad N_{(-)} \phi^+ \phi^- = -\phi^- \phi^+, \]
\[ A_{(+)} \phi^- \phi^+ = \phi^+ \phi^- , \quad A_{(-)} \phi^- \phi^+ = -\phi^+ \phi^- . \tag{9} \]

The normal ordering \( N_{(+)} \) is identical to the normal ordering in (8). The normal ordering \( N_{(-)} \) appears naturally in quantum field theory of fermion fields, but, as we shall see, it is also useful for boson fields. The antinormal orderings \( A_{(\pm)} \) are useful because one can introduce the symmetric orderings \( S_{(\pm)} \) defined by

\[ S_{(+)} = \frac{1}{2} [N_{(+)} + A_{(+)}], \]
\[ S_{(-)} = \frac{1}{2} [N_{(-)} + A_{(-)}]. \tag{10} \]

When \( S_{(+)} \) acts on a bilinear combination of fields, then it acts as the “default” ordering, i.e., \( S_{(+)} \phi \phi = \phi \phi \). The usefulness of the \( S_{(-)} \) ordering will become clear later.
3 Particle current

The particle current for scalar hermitian fields can be written as \[\text{(11)}\]

\[j_{\mu}(x) = \int_{\Sigma} d\Sigma' \frac{1}{2}(W^+(x,x') \partial_\mu \partial'_\nu \phi(x) \phi(x') + W^-(x,x') \partial_\mu \partial'_\nu \phi(x') \phi(x)).\]

Using \(\text{(7)}\), we see that it can be written in a purely local form as

\[j_{\mu}(x) = \frac{i}{2} [\phi(x) \partial_\mu \phi^+(x) + \phi^-(x) \partial_\mu \phi(x)].\]

Using \(\text{(11)}\) and the identities \(\phi^+ \partial_\mu \phi^+ = \phi^- \partial_\mu \phi^- = 0\), this can be written in a very elegant form as

\[j_{\mu} = i\phi^- \partial_\mu \phi^+ + \partial_\mu \phi.\]

Similarly, using \(\text{(13)}\), this can be written in another elegant form without an explicit use of \(\phi^+\) and \(\phi^-\), as

\[j_{\mu} = N_{(-)} \frac{i}{2} \phi \partial_\mu \phi.\]

Note that the expression on the right-hand side of \(\text{(14)}\) without the ordering \(N_{(-)}\) vanishes identically. Nevertheless, the ordering \(N_{(-)}\) makes this expression nonvanishing. This peculiar feature is probably the reason that the particle current has not been discovered earlier.

The normal ordering \(N_{(-)}\) provides that \(j_{\mu}|0\rangle = 0\). This is related to the fact that the total number of particles is

\[N = \int_{\Sigma} d\Sigma' j_{\mu} = \sum_k a_k^\dagger a_k.\]

Alternatively, one can choose the symmetric ordering \(S_{(-)}\) defined in \(\text{(10)}\), i.e., one can define the particle current as

\[j_{\mu} = S_{(-)} \frac{i}{2} \phi \partial_\mu \phi.\]

This leads to the total number of particles

\[N = \sum_k \frac{1}{2}(a_k^\dagger a_k + a_k a_k^\dagger) = \sum_k \left(a_k^\dagger a_k + \frac{1}{2}\right).\]

We see that this ordering generates the vacuum particle number equal to \(\sum_k 1/2\), in complete analogy with the vacuum energy which, in Minkowski spacetime, can be written as \(\sum_k \omega_k/2\). We discuss the physical implications of this in Sec. 6.
4 Other choices of $W^\pm$ and particle creation

When the gravitational background is time dependent, one can introduce a new set of solutions $u_l(x)$ for each time $t$, such that $u_l(x)$ are positive-frequency modes at that time. This leads to functions with an extra time dependence $u_l(x; t)$ that do not satisfy (1) [12, 13]. Here $t$ is the time coordinate of the spacetime point $x = (t, x)$. We define $\phi^+$ and $\phi^-$ as in (7), but with the 2-point functions

\begin{align}
W^+(x, x') &= \sum_l u_l(x; t)u_l^*(x'; t'), \\
W^-(x, x') &= \sum_l u_l^*(x; t)u_l(x'; t'),
\end{align}

(18)

used instead of (6). As shown in [12, 13], such a choice of the 2-point functions leads to a local description of particle creation consistent with the conventional global description based on the Bogoliubov transformation. Putting

\begin{align}
\phi(x) &= \sum_k a_k f_k(x) + a_k^\dagger f_k^*(x),
\end{align}

(19)

in (7) with (18), we find

\begin{align}
\phi^+(x) &= \sum_l A_l(t)u_l(x; t), \\
\phi^-(x) &= \sum_l A_l^\dagger(t)u_l^*(x; t),
\end{align}

(20)

where

\begin{align}
A_l(t) &= \sum_k \alpha_{lk}(t)a_k - \beta_{lk}(t)a_k^\dagger, \\
\alpha_{lk}(t) &= (f_k, u_l), \\
\beta_{lk}(t) &= -(f_k^*, u_l).
\end{align}

(21)

By putting (20) in (18), we find

\begin{align}
\bar{j}_\mu(x) &= i\sum_{l,l'} A_l^\dagger(t)u_l^*(x; t)\partial_\mu A_{l'}(t)u_{l'}(x; t).
\end{align}

(23)

Note that, owing to the extra time dependence, the fields $\phi^+$ and $\phi^-$ in (20) do not satisfy the equation of motion (1). Consequently, the current (23) is not conserved, i.e., the quantity $\nabla^\mu j_\mu$ is a nonvanishing local scalar function describing the creation of particles in a local and invariant way, similarly as in [12, 13].

Let us now consider the questions where and when the particles are created. (Note that the space localization of the particle creation process cannot be directly considered in the conventional global approach based
on the Bogoliubov transformation, simply because the local density of particles is not defined in this approach.) It is clear that $\nabla^\mu j_\mu(x) = 0$ at the spacetime points $x$ at which the modes $u_\ell$ do not have the extra time dependence. Therefore, in general, the particles are created at the points at which the modes $u_\ell$ have this extra time dependence. One could choose the modes $u_\ell$ as highly nonlocal modes, such as the plane wave modes in Minkowski spacetime are. However, a question such as “Where a particle with a definite momentum and a completely undetermined position is created?” does not make sense. Therefore, we assume that $u_\ell$ are some localized wave packets that, at a given instant of time, are negligible everywhere except in a small space volume $\mathcal{L}$. Assume that $u_\ell(x)$ is a linear combination of modes that are all positive frequency modes at some instant of time. If the metric does not depend on time, then, during the time evolution, these modes remain positive frequency modes. If the metric depends on time, then, during the time evolution, $u_\ell(x)$ ceases to be a linear combination of positive frequency modes. During an infinitesimal change of time, the modes $u_\ell(x)$ suffer an extra infinitesimal change related to the choice of new modes that are positive frequency modes at the new time. These infinitesimally modified new modes are also negligible everywhere except in the small (infinitesimally translated due to a finite group velocity of the packet) space volume. Therefore, the modes $u_\ell(x; t)$ have a nonnegligible extra time dependence only inside this small volume and only when the metric is time dependent. This implies that, in general, the particles are created at the spacetime points at which the metric is time dependent.

As a particular example, let us discuss the particle creation caused by a spherically symmetric gravitational collapse. Assume that all collapsing matter is contained in a ball with a radius $R(t)$. From the Birkhoff theorem it follows that the metric is time independent outside the ball, so all particles are created inside the ball. A certain amount of particles is created before the matter approaches a state in which all matter is trapped by an apparent horizon. These particles have not a thermal distribution. For the distribution to be approximately thermal, it is essential that the waves suffer an approximately exponential red shift, which occurs when the waves propagate close to the horizon. Therefore, since the particle production is a local process, the Hawking thermal radiation results from particles that are created near the horizon.

The space components of the particle current $j_\mu$ determine the direction of the particle motion. Let us use them to confirm that the Hawking radiation is outgoing, as is usually argued by less direct arguments. Asymptotically, i.e., at late times and large distances from the horizon, we can approximate the modes $u_\ell$ with the usual plane wave modes. Therefore, in
the asymptotic region we can make the replacement

\[ u_l(x; t) \rightarrow u_q(x) = \frac{e^{-iqx}}{\sqrt{V 2\omega_q}}, \quad (24) \]

where \( \omega_q = (m^2 + q^2)^{1/2} \). For convenience, \( u_q(x) \) are normalized in a finite volume \( V \). We integrate the current over the whole space. Since the integral is dominated by the contributions from the large distances, we use (23) and (24) to obtain

\[ J_\mu \equiv \int d^3x j_\mu \simeq \sum_q \frac{q_\mu}{\omega_q} A_q^\dagger A_q. \quad (25) \]

In particular, in the vacuum \( |0\rangle \) defined by \( a_k|0\rangle = 0 \), (24) and (21) give

\[ \langle 0|J_\mu|0\rangle = \sum_q \frac{q_\mu}{\omega_q} n_q, \quad (26) \]

where

\[ n_q = \sum_k |\beta_q^k|^2. \quad (27) \]

The \( \beta \)-coefficients in (21) vanish for asymptotically ingoing modes because such modes have not experienced the black-hole gravitational field. Therefore, (27) has a form

\[ n_q = \theta(q^r) n(q^r), \quad (28) \]

where \( q^r \) is the radial component of the 3-momentum \( q \) and \( \theta \) is the step function. For massless fields, \( n(q^r) \) is the thermal distribution equal to \( \frac{1}{\exp(8\pi M q^r) - 1} \), where \( M \) is the mass of the black hole. We see that \( \langle 0|J_0|0\rangle = \sum_q n_q \) represents the total number of produced particles. On the other hand, the Cartesian components \( \langle 0|J_i|0\rangle \) vanish due to the cancellation of contributions from the opposite \( q_i \)'s. However, from (28) we see that the radial component

\[ \langle 0|J^r|0\rangle = \sum_q \frac{q^r}{\omega_q} n_q \quad (29) \]

is positive, which confirms that the flux of created particles is outgoing.

## 5 Generalization to complex fields

A complex scalar field \( \phi(x) \) and its hermitian conjugate field \( \phi^\dagger(x) \) in an arbitrary gravitational background can be expanded as

\[ \phi = \phi^{(P)} + \phi^{(A)}^-, \quad \phi^\dagger = \phi^{(P)}^- + \phi^{(A)}^+, \quad (30) \]
\[
\phi^{(P)+}(x) = \sum_k a_k f_k(x), \quad \phi^{(P)-}(x) = \sum_k a_k^\dagger f_k^\dagger(x), \\
\phi^{(A)+}(x) = \sum_k b_k f_k(x), \quad \phi^{(A)-}(x) = \sum_k b_k^\dagger f_k^\dagger(x).
\]

(31)

In a similar way as in Sec. 3, we find
\[
\phi^{(P)+}(x) = i \int \Sigma d\Sigma W^+(x,x') \partial_{\nu} \phi(x'), \\
\phi^{(A)+}(x) = i \int \Sigma d\Sigma W^+(x,x') \partial_{\nu}^\dagger \phi^\dagger(x'), \\
\phi^{(P)-}(x) = -i \int \Sigma d\Sigma W^-(x,x') \partial_{\nu} \phi(x'), \\
\phi^{(A)-}(x) = -i \int \Sigma d\Sigma W^-(x,x') \partial_{\nu}^\dagger \phi^\dagger(x').
\]

(32)

The particle current \(j^{(P)}_{\mu}\) and the antiparticle current \(j^{(A)}_{\mu}\) are
\[
j^{(P)}_{\mu}(x) = \int \Sigma d\Sigma' \frac{1}{2} \left( W^+(x,x') \partial_{\nu} \phi(x') \phi(x) + W^-(x,x') \partial_{\nu}^\dagger \phi^\dagger(x') \phi(x) \right), \\
j^{(A)}_{\mu}(x) = \int \Sigma d\Sigma' \frac{1}{2} \left( W^+(x,x') \partial_{\nu}^\dagger \phi^\dagger(x') \phi(x) \phi(x') + W^-(x,x') \partial_{\nu} \phi(x') \phi(x) \phi(x) \right).
\]

(33)

Therefore, they can be written in a purely local form similar to (13) as
\[
j^{(P)}_{\mu} = i \phi^{(P)-} \partial_{\mu} \phi^{(P)+} + j^{\text{mix}}_{\mu}, \\
j^{(A)}_{\mu} = i \phi^{(A)-} \partial_{\mu} \phi^{(A)+} - j^{\text{mix}}_{\mu},
\]

(34)

where
\[
j^{\text{mix}}_{\mu} = \frac{i}{2} \left[ \phi^{(P)-} \partial_{\mu} \phi^{(P)+} - \phi^{(P)-} \partial_{\mu} \phi^{(A)+} \right].
\]

(35)

The current of charge \(j^{(-)}_{\mu}\), defined as
\[
j^{(-)}_{\mu} = j^{(P)}_{\mu} - j^{(A)}_{\mu},
\]

(36)

can be written in more familiar forms as
\[
j^{(-)}_{\mu} = :i \phi^\dagger \partial_{\mu} \phi: \\
= \frac{i}{2} \left[ \phi^\dagger \partial_{\mu} \phi - \phi \partial_{\mu} \phi^\dagger \right].
\]

(37)
Using (9), we see that this can also be written as
\[
\begin{align*}
    j_{\mu}^{(-)} &= N_{(+)}i\phi^{+}\partial_{\mu}\phi \\
    &= N_{(+)}\frac{i}{2}[(\phi^{+})^{\dagger}\partial_{\mu}\phi - \phi^{+}\partial_{\mu}(\phi^{+})^{\dagger}] .
\end{align*}
\] (38)

The current of total number of particles \( j_{\mu}^{(+)} \) is defined as
\[
    j_{\mu}^{(+)} = j_{\mu}^{(P)} + j_{\mu}^{(A)} .
\] (39)

It is shown in [13] that \( j_{\mu}^{(+)} \) can be written as a sum of two particle currents attributed to the hermitian fields \( \phi_{1} \) and \( \phi_{2} \) defined by
\[
    \phi = \frac{\phi_{1} + i\phi_{2}}{\sqrt{2}} ,
\] (40)
as
\[
    j_{\mu}^{(+)} = j_{\mu}^{(1)} + j_{\mu}^{(2)} ,
\] (41)
where \( j_{\mu}^{(1)} \) and \( j_{\mu}^{(2)} \) are two currents of the form (11). Therefore, using (14), we can write (11) as
\[
    j_{\mu}^{(+)} = N_{(-)}\frac{i}{2}[(\phi^{+})^{\dagger}\partial_{\mu}\phi_{1} + \phi_{2}\partial_{\mu}\phi_{2}] .
\] (42)

Using (40), it is straightforward to show that (12) can be written in a form analogous to (38) as
\[
    j_{\mu}^{(+)} = N_{(-)}\frac{i}{2}[(\phi^{+})^{\dagger}\partial_{\mu}\phi_{1} + \phi_{2}\partial_{\mu}\phi_{2}] .
\] (43)

The results above can be summarized by defining the currents
\[
    q_{\mu}^{(\pm)} = \frac{i}{2}[(\phi^{+})^{\dagger}\partial_{\mu}\phi \pm \phi^{+}\partial_{\mu}(\phi^{+})^{\dagger}] ,
\] (44)
which leads to
\[
    j_{\mu}^{(\pm)} = N_{(\mp)}q_{\mu}^{(\pm)} .
\] (45)

The current \( q_{\mu}^{(+)} \) vanishes, but the current \( N_{(-)}q_{\mu}^{(+)} \) does not vanish.

The results above can be easily generalized to the case in which the field interacts with a background electromagnetic field, in a way similar to that in [13]. The equations are essentially the same, but the derivatives \( \partial_{\mu} \) are replaced by the corresponding gauge-covariant derivatives and the particle 2-point functions \( W^{(P)\pm} \) are not equal to the antiparticle 2-point functions \( W^{(A)\pm} \).
Similarly to the gravitational case, in the case of interaction with an electromagnetic background, different choices for the 2-point functions exist. One is a generalization of (6) based on a particular choice of a complete orthonormal set of solutions to the equations of motion. The other is a generalization of (13) and leads to a local description of the particle-antiparticle pair creation consistent with the conventional global description based on the Bogoliubov transformation. The third choice is based on the Schwinger-DeWitt Green function and leads to the conservation of the particle currents in classical electromagnetic backgrounds.

The results of this section can also be generalized to anticommuting fermion fields (see also [16]). As the analysis is very similar to the case of complex scalar fields, we simply note the final results. The particle and antiparticle currents can be written in a form similar to (33) [13]. In particular, a similar integration over $x'$ occurs, which is related to the extraction of $\psi^{(P)+}$, $\psi^{(P)-}$, $\psi^{(A)+}$, and $\psi^{(A)-}$ from the fermion fields $\psi$ and $\bar{\psi}$. Introducing the currents

$$ q^{(\pm)}_\mu = \frac{1}{2} [\bar{\psi} \gamma^\mu \psi \pm \psi^T \gamma^\mu \bar{\psi}^T], $$

(46)

the currents

$$ j^{(\pm)}_\mu = j^{(P)}_\mu \pm j^{(A)}_\mu $$

(47)

can be written as

$$ j^{(\pm)}_\mu = N^{(\pm)} q^{(\pm)}_\mu. $$

(48)

The current $q^{(+)}_\mu$ vanishes (due to the anticommutation relations among the fermion fields), but the current $N^{(+)} q^{(+)}_\mu$ does not vanish. The current $j^{(-)}_\mu$ can also be written in more familiar forms as

$$ j^{(-)}_\mu = :\bar{\psi} \gamma^\mu \psi: $$

$$ = \frac{1}{2} [\bar{\psi} \gamma^\mu \psi - \psi^T \gamma^\mu \bar{\psi}^T]. $$

(49)

## 6 Relation between the particle current and the energy-momentum tensor

In classical field theory, the energy-momentum tensor of a real scalar field is

$$ T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - g_{\mu\nu} \frac{1}{2} [g^\alpha^\beta (\partial_\alpha \phi)(\partial_\beta \phi) - m^2 \phi^2]. $$

(50)

Contrary to the conventional concept of particles in quantum field theory, the energy-momentum is a local quantity. Therefore, the relation between the definition of particles and that of the energy-momentum is not clear in
the conventional approach to quantum field theory in curved spacetime [1]. In this section, we exploit our local and covariant description of particles to find a clearer relation between particles and their energy-momentum.

In quantum field theory, one has to choose some ordering of the operators in (50), just as a choice of ordering is needed in order to define the particle current. Although it is not obvious how to choose these orderings, it seems natural that the choice of ordering for one quantity determines the ordering of the other one. For example, if the quantum energy-momentum tensor is defined as $T_{\mu\nu} := N_i T_{\mu\nu}$, then the particle current should be defined as $N_i = N_{(+)} T_{\mu\nu}$. The nonlocalities related to the extraction of $\phi^+$ and $\phi^-$ from $\phi$, needed for the definition of the normal orderings $N_{(+)}$ and $N_{(-)}$, appear both in the energy-momentum and in the particle current. Similarly, if $W^\pm$ is chosen as in (6) for one quantity, then it should be chosen in the same way for the other one. The choices as above lead to a consistent picture in which both the energy and the number of particles vanish in the vacuum $|0\rangle$ defined by $a_k |0\rangle = 0$.

Alternatively, if $W^\pm$ is chosen as in (18) for the definition of particles, then it should be chosen in the same way for the definition of the energy-momentum. Owing to the extra time dependence, it is clear that both the particle current and the energy-momentum tensor are not covariantly conserved in this case:

$$\nabla_\mu j_\mu \neq 0, \quad \nabla_\mu T_{\mu\nu} \neq 0. \quad (51)$$

While the first equation in (51) is exactly what one might want to obtain, the second one represents a problem. To be more specific, assume, for simplicity, that spacetime is flat at some late time $t$. In this case, the normally ordered operator of the total number of particles at $t$ is

$$N(t) = \sum_q a_q(t) a_q(t)^\dagger, \quad (52)$$

(see (21)), while the normally ordered operator of energy is

$$H(t) = \sum_q \omega_q a_q(t) a_q(t)^\dagger. \quad (53)$$

From (52) and (53) it is clear that the produced energy exactly corresponds to the produced particles. A similar analysis can be done for the particle-antiparticle pair creation caused by a classical electromagnetic background. Since the energy should be conserved, this suggests that $W^\pm$ should not be chosen as in (18), i.e., that classical backgrounds do not cause particle creation. Of course, in a time dependent gravitational field, the energy of matter does not need to be conserved in the ordinary, noncovariant
sense: only the sum of matter and gravitational energy should be conserved. However, in the specific case above, the spacetime is flat at the late time $t$, so the gravitational energy is zero. We can choose that the metric at this late time is equal to the metric at the initial time (at which the number of particles is zero), such that the time dependence of the metric at the intermediate times is nontrivial. In such a case the contradiction between particle creation and energy conservation is obvious.

Of course, it is possible that the total energy-momentum is conserved owing to some mechanism of the back reaction that is not included in our calculation. However, just as the back reaction may prevent the creation of energy, it might also prevent the creation of particles. To support this idea, let us discuss a particular example. Consider a static electric field, the source of which is a stable charged particle. Various semiclassical calculations, based on the approximation that the electric field is static and classical, lead to pair creation. However, it is clear that pair creation is inconsistent with energy conservation. If a pair is really created, a back-reaction mechanism should provide the conservation of energy. One possibility is that the effect of the back reaction reduces to a modification of the electric field. However, the new electric field should be consistent with the Maxwell equations, so it is easy to see that it is impossible that the electric field is modified in a way consistent with energy conservation if the source of the field is not modified. On the other hand, the source cannot be modified as it is, by assumption, a stable particle. (The particle stability is a quantum property, so one cannot study it using semiclassical methods.) Therefore, we must conclude that, in this particular example, the back reaction completely prevents the pair creation. This demonstrates that a semiclassical treatment of the particle creation may lead to a completely wrong result. The formal results obtained in [12] and this paper suggest a different semiclassical approximation according to which quantum particles are never created by classical backgrounds. In this approximation, the particles are defined by using a modified Schwinger-DeWitt Green function [13] to choose the 2-point functions $W^{\pm}$. Contrary to other semiclassical approximations, such a semiclassical approximation is self-consistent in the sense that there is no particle creation that violates the energy-momentum conservation law and particles are defined in a unique way without need to choose a particular time-coordinate and a particular gauge.

Let us now choose the symmetric ordering and assume that spacetime is flat. As already discussed in Sec. 3 both the vacuum energy and the vacuum number of particles are nonvanishing in this case. Taking the Lorentz-invariant normalization of the field in an infinite volume

$$
\phi(x) = \int \frac{d^3k}{(2\pi)^32\omega(k)} [a(k)e^{-ikx} + a^\dagger(k)e^{ikx}],
$$

(54)
it is straightforward to show that the vacuum-expected value of the energy-momentum tensor is

\[ \langle 0 | S_+ T_{\mu\nu} | 0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu k_\nu}{\omega(k)}. \]  

(55)

Similarly, for the particle current we find

\[ \langle 0 | S_- i \phi \overleftrightarrow{\partial}_\mu \phi | 0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{k_\mu}{\omega(k)}. \]  

(56)

Note that the right-hand side of (56) is not only the expected value, but also the eigenvalue of \( S_- i \phi \overleftrightarrow{\partial}_\mu \phi \) in the vacuum. One can also define the energy-momentum current \( T_\mu = n^\nu T_{\mu\nu} \), where \( n^\nu \) is a unit timelike vector. We work in coordinates in which \( n^\nu = (1, 0, 0, 0) \), so in these coordinates

\[ \langle 0 | S_+ T_\mu | 0 \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} k_\mu. \]  

(57)

It is often argued that \( \langle 0 | S_+ T_{\mu\nu} | 0 \rangle \) contributes to the cosmological constant. However, the presence of a cosmological constant is equivalent to an energy-momentum tensor of the form

\[ T_{\mu\nu}^{\text{cosm}} = \lambda g_{\mu\nu}. \]  

(58)

The right-hand side of (55) does not have the form (58). (For example, \( T_{00}^{\text{cosm}} \) and \( T_{11}^{\text{cosm}} \) have the opposite sign, which is not the case for (55)). Actually, only the first term in (50) contributes to (55), while the term proportional to \( g_{\mu\nu} \) does not contribute to (55). If the term \( m^2 \phi^2 \) in (50) is replaced with a nontrivial potential \( V(\phi) \) which has a nonzero minimum at some \( \phi = \phi_{\text{vac}} \neq 0 \), then the term proportional to \( g_{\mu\nu} \) contributes to (55) even at the classical level, which is the basic idea of quintessence models. However, in the case we discuss \( \phi_{\text{vac}} = \langle 0 | \phi | 0 \rangle = 0 \), so there is no term proportional to \( g_{\mu\nu} \).

Of course, the right-hand sides of (55)-(57) are infinite and should be regularized (and renormalized). However, there are many kinds of regularization and different kinds of regularizations are not always physically equivalent. One has to choose the regularization such that it preserves some physical property of unregularized expressions. For example, the cut-off regularization preserves the correspondence between the vacuum energy and the vacuum number of particles, but does not preserve the Lorentz invariance. On the other hand, the dimensional regularisation and the zeta-function regularization preserve the Lorentz invariance, but do not preserve the correspondence between the vacuum energy and the vacuum number of particles. Therefore, in order to obtain regularized expressions,
it seems necessary to abandon one of these two physical properties. The question is: Which one?

It is widely believed that the vacuum energy-momentum should have the form \[ (58) \] because the vacuum should be relativistically invariant. The Casimir effect [17] suggests that the vacuum energy should not be simply removed by the normal ordering. Therefore, the infinite vacuum energy-momentum is often renormalized such that it is required that the renormalized vacuum energy-momentum should have the form \[ (58) \]. However, this requirement leads to a strange and counterintuitive result that the vacuum energy \[ (2\pi)^{-3} \int d^3k \omega(k)/2 \] vanishes for massless fields and does not vanish for massive fields [18].

The discussion above suggests another possibility worthwhile to explore: Perhaps, the requirement that the vacuum should be a relativistically invariant state with an energy-momentum of the form \[ (58) \] should be abandoned. Instead, the vacuum should be viewed in a way similar to the original Dirac’s picture, in which the vacuum is filled not only with energy, but also with particles that carry this energy. The relativistic noninvariance of the vacuum reflects in flat spacetime as the existence of a preferred Lorentz frame in which the average velocity of vacuum particles is zero. Indeed, if the space components \( T_i \) of the energy-momentum vector vanish in a particular Lorentz frame but the time component \( T_0 \) does not vanish, then the space components \( T_i \) do not vanish in another Lorentz frame. This is also consistent with the explicit expression for \( T_i \) in \[ (57) \], because, if the limits of integration over space components of \( k \) are symmetric in one Lorentz frame, then they are not symmetric in another Lorentz frame, so the contributions from the opposite \( k_i \)'s cancel only in one Lorentz frame. The same is true for the particle current \[ (56) \] and for the nondiagonal components of \[ (55) \].

The interpretation above of the vacuum energy requires the existence of a preferred frame. Although this may look strange from a theoretical point of view, it is an observed fact that a preferred frame exists in the Universe. This is the frame with respect to which the expanding Universe is homogeneous and isotropic at large space scales. According to the interpretation above, the vacuum energy (not related to a nonvanishing \( \phi_{vac} \)) behaves as vacuum matter. However, this matter does not form structures (such as stars or galaxies), but is homogeneously distributed in the Universe. Such a nonclassical behavior can be understood as a consequence of the entanglement related to a very special quantum ground state \( |0\rangle \). (This is similar to various nonclassical collective effects, such as superconductivity, that appear in low-temperature solid-state physics.)

It is known that about 70% of all energy in the Universe does not form structures. It is also known that the Universe expansion accelerates, which suggests that part of the energy has the cosmological-constant form \[ (58) \].
with a negative pressure. (For a review, see, e.g., [19, 20].) Future more precise measurements of the negative pressure that causes acceleration and of the energy-density that does not form structures might show that not all energy that does not form structures can be explained by a cosmological constant, which would be an (indirect) experimental confirmation that the picture of the vacuum proposed above is qualitatively correct. To obtain a respectable quantitative picture, the renormalization is necessary. In particular, the running of the vacuum energy [21, 22] might provide useful information that can be compared with experiments.

7 Conclusion

In this paper, it is shown that the recently discovered particle currents [12, 13] can be written in purely local forms. The nonlocalities are hidden in the extraction of $\phi^+$ and $\phi^-$ from $\phi$. The formalism is applied to a local description of particle creation by gravitational backgrounds, with emphasis on the description of particle creation by black holes. The formalism also reveals a relation between particles and their energy-momentum, which suggests that it might not be consistent to use semiclassical methods for a description of particle creation. The relation between particles and their energy-momentum also suggests that the vacuum energy might contribute to dark matter that does not form structures, instead of contributing to the cosmological constant.

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