Anomalous Magnetothermal Resistance of High-T$_c$ Superconductors: Anomalous Cyclotron Orbits at a Dirac Point

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**Abstract**

In trying to explain the anomalously sharp dependence of thermal conductivity on magnetic field in the cuprate high $t_c$ superconductors, it was found that previous discussions of quasiparticle motion near the gap nodes were in error because of failure to take into account total current conservation. We present corrected equations of motion for quasiparticle motion and discuss their relationship to the observations.
In thermal conductivity of High $T_c$ cuprates, at all temperatures below $T_c$, there is observed a remarkably steep dependence on applied $c$-axis magnetic fields even as low as .1T. Fig (1) shows observations in both $(La - Sr)_2CuO_4$ and YBCO [1]. From thermal Hall (Righi-LeDuc) effect, it was concluded that the enhancement of thermal conductivity below $T_c$ is electronic, and that large fields seem to block this conductivity. But one of the primary mysteries has been the very sensitive dependence on magnetic fields in which any Zeeman or cyclotron frequency corresponds to an energy $< 1^\circ$K, much lower than thermal energy $\sim 30-60^\circ$K in either case.

It has been clear that the carriers responsible for electronic thermal conductivity at lower temperatures must be those near the nodes of the “$d$–symmetry” energy gap at $\pm k_F$, $\pm k_F$. Although their density is relatively small, the very rapid removal below $T_c$, as the gap opens up, of the electronic scattering mechanism responsible for the normal state “$T$-linear” resistivity could allow peaking of quasiparticle conductivities. [2] The ARPES data on electron Green’s functions near the gap nodes, however, do not support this interpretation, and it seems more plausible that the extra thermal conductivity is caused by the anomalous properties of carriers near the nodes—specifically, that they acquire velocity parallel to the Fermi surface as well as the conventional Fermi velocity. This problem will be the subject of later publications. Our first hypothesis for the $H$-field sensitivity was that spin-charge separation was somehow involved in this large conductivity enhancement and that $\kappa$ was being destroyed by the effects of Zeeman splitting, but this mechanism seems unable energetically to account for the sharp $B$-dependence. The mechanism is described below and involves quasiparticles which are almost conventional.

This problem has been discussed by Lee and Simon [3], who derived scaling laws for thermal conductivity which should be valid once the carriers are confined to the nodes. This should be true for $T < T_c/2$, at the very least. These scaling laws are

$$K_{xx} = \text{const} \times T \ F \left( \frac{\sqrt{B}}{T} \right) \quad (1a)$$
\[ K_{xy} = \text{const} \times \frac{T^2}{E_F} F' \left( \frac{\sqrt{B}}{T} \right). \]  

(1b)

These scaling laws are quite successful for \( T < 50^\circ \) for optimally doped YBCO, and clearly the arguments of Ref (3) (which depend on the result that energy levels in the nodes scale as \( \sqrt{B} \), and hence depend sensitively on magnetic field) contain the essence of the phenomena. The purpose of this paper is to support their general scaling argument with a specific picture of the nature of the energy levels at the nodes in a magnetic field; and to predict that broadened Landau-like levels will be observed, contrary to previous discussions of these energy levels which ignore the subtle properties of gap nodes. [5]

One can discuss the physics semiclassically either as a Tomasch interference effect involving Andreev scattering or as a form of “Landau level” formation for excitations around the “Dirac point” nodes of the \( d \)-wave gap. The former point of view is simpler. First we consider the effect of the uniform magnetic field and will later discuss the effect of vortices. The \( B \) field (along the \( c \)-axis, perpendicular to the plane) causes a Larmor precession of all the electrons around the Fermi surface according to

\[ \hbar \dot{k} = \frac{e}{c} v_F \times B \quad \text{(electrons and holes)} \]  

(2)

Here \( e \) is the electron charge, negative in this case, \( v_F \) the Fermi velocity at the nodal point, and \( \hbar k \) the electron momentum. Since we speak of an electron, \( |k| \) may be assumed to be somewhat (\( \delta k \)) greater than \( k_F \), giving an excess kinetic energy \( \delta E = v_F \delta k \). Holes obey the same equation with the signs of both \( e \) and \( v_F \) changed, hence with the same direction of precession. The \( k \)-vector of an electron near a node is rotated away from the node, and therefore eventually encounters a gap equal to its excess kinetic energy. We assume that in the superconducting state there are point nodes in the gap, which in this region varies as

\[ \Delta = \frac{d\Delta}{dk} (k_\perp - k_{\text{node}}) = \hbar v_\Delta (k_\perp - k_{\text{node}}) = \hbar v_\Delta \Delta k. \]  

(3)

where \( k_\perp \) is the momentum perpendicular to \( v_F = \frac{\partial E}{\partial k} \) and hence parallel to the Fermi surface, and \( v_\Delta \) is defined by (3). At this point the electron must be Andreev reflected as
a hole in the state of opposite momentum, supplying \((-2e)\) of charge to the condensate. (Essentially, Fermion number as well as charge must be conserved.) This hole Larmor precesses in the same direction in \(k\)-space.

This at first appears to scotch the possibility of a periodic process. However, it must be realized that the Andreev reflection takes place not on a zero-momentum condensate but on a condensate which macroscopically obeys London’s equation

\[ p = \frac{eA}{c} \]

so that the momentum carried off by the pair is \(\frac{2eA}{c}\). This means that the hole reappears not at the same momentum but at a momentum shifted back by

\[ \delta p = -2h \Delta k \]

(it is easily verified that the change in \(\frac{2eA}{c} = 2h\Delta k\).) Now the hole Larmor precesses from one barrier to the other, is again Andreev reflected as an electron, and the process begins again.

The period of this process defines a frequency \(f\) which will be the excess energy level separation for electrons of that energy. It is easy to estimate. For electrons of energy \(\hbar\omega\), the separation of the two barriers is

\[ \Delta k = 2\frac{dk}{d\Delta} \cdot \hbar \omega \quad (4) \]

and the velocity in \(k\)-space gives

\[ \frac{dk_\perp}{dt} = f \cdot \frac{\Delta k}{2} = \frac{1}{2} \frac{ev_F}{\hbar c} \cdot B \]

or,

\[ f = \frac{ev_F}{4\omega} \left| \frac{d\Delta}{dk} \right| B \]

To find the lowest energy state permitted by the uncertainty principle, we set \(f = \frac{\pi}{2\omega}\) and obtain
\[ h^2 \omega_0^2 = \frac{\pi}{2} \frac{\hbar e}{c} v_F B \nu = \frac{\pi}{2} \hbar^2 \left( \frac{v_F \Delta}{\nu} \right) \]

\[ = \frac{\pi}{2} (\hbar \omega_c) E_F \cdot \frac{\nu}{v_F} \]

(5)

We have defined a velocity \( v_\Delta = \frac{d\Delta}{d\nu} \) to represent the steepness of the energy gap at the nodes. This has been estimated by Lee [4] to be \( \sim \frac{1}{8} v_F \). Using that figure, and \( E_F \sim 1 \text{eV} \), at 1T \( h \omega_0 \sim 4.5 \text{meV} \), which is of the right order of magnitude to explain the data.

A very physical way to think of this resonance is as a Landau level in the Dirac points at each of the four nodes of the energy gap, at which the quasiparticle excitations act like massless Fermions of energy

\[ E_k^2 = \hbar^2 v_F^2 (k_\parallel - k_F)^2 + \left( \frac{d\Delta}{d\nu} \right)^2 k_\perp^2 \]

(6)

These excitations change character as they precess around the elliptical energy contour, from electrons through neutral electron-hole mixtures to holes, but by the same argument given above the quasiparticle precesses everywhere as though it had charge \( e \) and a velocity given by

\[ v_{Qp} = \frac{1}{\hbar} \nabla_k E_k \]

(7)

i.e.,

\[ \hbar \dot{k} = \frac{ev_{Qp}}{c} \times B \]

(8)

As with the Andreev reflection picture, (8) seems a necessary consequence of basic conservation laws. So long as the Larmor frequency remains small compared with the maximum gap \( \Delta_0 \sim 50 \text{meV} \) tunneling through the gap maxima will play a minor role. (8) leads to the same result, (5), for the cyclotron orbit frequency except for constants of order 1. Although in momentum-space the orbit is slightly complicated by the momentum shifts due to Andreev reflection on a moving condensate, these shifts occur only parallel to the Fermi surface, and as far as real space is concerned it is a conventional elliptical orbit.

In order to go beyond the semiclassical picture, it is necessary to solve the actual Bogoliubov-de Gennes equations for the quasiparticle wave function near one of the \( k \)-space
nodes of the gap. These are

$$\mathcal{H} \tilde{\psi} = \hbar \omega \tilde{\psi}$$

(9)

where $\tilde{\psi}$ is a two-component vector and $\mathcal{H}$ a two-by-two tensor

$$\mathcal{H} = \begin{pmatrix} v_F \cdot (p - \frac{2e}{c} A - mv_s) - \mu & \Delta^* \\ \Delta & ( - v_F \cdot (p + \frac{2e}{c} A + mv_s) + \mu) \end{pmatrix}$$

(10)

We will specialize to a single node of the gap and rotate this node to $k_x = k_F$, $k_y = 0$.

$$\Delta_0(k_x, k_y) = \hbar k_y \cdot v_\Delta .$$

We have included the superfluid velocity in (10), thus reintroducing the vortices. The superfluid velocity is equivalent to a displacement of the entire Fermi surface (even near the nodes) by $mv_s = p_s$ in momentum space, which leads to a “doppler” shift in the energy relative to the chemical potential. Its sign agrees with that of $A$.

But it is also essential to include the position dependence of $\Delta$ in the presence of a magnetic field, which is not irrelevant because the off-diagonal, anomalous terms of (10) represent processes in which a particle Andreev scatters into a hole, with the charge of the missing pair going into the condensate. Such a process must also be momentum-conserving, so that if the condensate has been accelerated by the magnetic field the self-energy scattering must reflect that fact, and the pair created in this process must carry the momentum of condensate pairs.

What is the space-dependence of $\Delta$? We can get a reasonably good characterization in the “London” regime $H_{c1} << B << H_{c2}$ where the vortices are dense enough to overlap strongly so that the field is fairly uniform, while the vortex cores still comprise a very small relative volume $\propto \frac{B}{H_{c2}}$. Thus almost everywhere the field and current satisfy London’s equation as modified by the presence of a vortex lattice. $\Delta$ is constant in magnitude, while its phase $\varphi$ satisfies

$$j_s = n_s e v_s = \frac{n_s e}{m} (\hbar \nabla \varphi - \frac{2e}{c} A) .$$

(11)

$j_s$ is zero averaged over the interior of the sample, where the field is nearly uniform. But near a vortex $j_s$ behaves like $\frac{\vec{\varphi}}{r}$ and $\varphi = \theta$, the actual angle measured around the vortex.
core. Note that \( \Delta \) is single-valued everywhere except at the vortex cores, though \( \varphi \) of course is not.

Now let us insert \( \varphi \) into the Hamiltonian (10). There are two ways to transform away the spatial dependence of \( \Delta \): to transform unitarily by

\[
    U = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & 1 \end{pmatrix} \quad \mathcal{H}' = U^{-1} \mathcal{H} U \tag{12a}
\]

or by

\[
    U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \tag{12b}
\]

(These are the only two single-valued transformations) In the first case,

\[
    \mathcal{H}_{\text{holes}} = \begin{pmatrix} v_F \cdot [(p + \frac{\xi}{c} A) + mv_s] & \Delta_0 \\ \Delta_0^* & -v_F \cdot (p + \frac{\xi}{c} A + mv_s) \end{pmatrix} \tag{13a}
\]

using \( \hbar \nabla \varphi = mv_s + \frac{2e}{c} A \). This is the Hamiltonian appropriate for hole-like excitations, since when \( \Delta \) vanishes at the node, the wave-function is purely hole-like. For electrons, we use (10b), and obtain

\[
    (\mathcal{H})'_{\text{electrons}} = \begin{pmatrix} v_F \cdot (p - \frac{\xi}{c} A - mv_s) & \Delta_0 \\ \Delta_0^* & -v_F \cdot [(p - \frac{\xi}{c} A) - mv_s] \end{pmatrix} \tag{13b}
\]

If we neglect the \( (p_F \cdot v_s) \) term, we come to the conclusion that the basic form of the wave function is cyclotron orbits in momentum space around Dirac points at the gap nodes, with frequencies proportional to \( \sqrt{B} \).

The fact that these are two separate equations, one for electrons and one for holes, reflects the fact that charge is actually exactly conserved: when an electron is Andreev scattered into a hole, the charge goes into the condensate and is recovered in the inverse scattering. As Volovik has emphasized, \[5\] gap nodes are sites of chiral anomalies where supercurrent and quasiparticle current are not separately conserved.

But what of the \( p_F \cdot v_s \) terms? These are quite strong in a localized region around the vortex core. We predict that these will have only a moderate effect. The cyclotron orbits of
low order have wave-functions which occupy roughly an area of $n$ (magnetic lengths)$^2$, which is of order the size of the vortex lattice unit cell, and it will cost energy to localize them further. I suspect that the vortices act as moderately strong scattering centers, broadening the Landau levels without changing them quantitatively. (Since $v_s = 0$, these potentials will average to zero.)

Throughout this discussion, we have chosen a gauge such that $A_y = 0$, $A_x = By$. The dependence of $\Delta$ on $k_y$ is then straightforward. But if we are to assume general gauge covariance, clearly

$$\Delta(k_y) = \Delta(k_y - \frac{2eA_y}{\hbar c})$$

This in turn, implies that the $v_s$ term will also enter in the $y$ direction by symmetry. Gathering all of those various strands together, we come to the wave equation which nodal particles (electrons, for example) obey,

$$E\tilde{\psi} = \mathcal{H}\tilde{\psi}$$

$$\mathcal{H} = \begin{pmatrix}
v_F \cdot [p - \frac{e\hbar}{c} A - mv_s] & v_\Delta \cdot (p - \frac{e\hbar}{c} A - mv_s) \\
v_\Delta \cdot (p - \frac{e\hbar}{c} A - mv_s) & -v_F \cdot (p - \frac{e\hbar}{c} A - mv_s)
\end{pmatrix}$$

(14)

Thus the particles at nodal points are, unexpectedly, in all respects equivalent to Dirac fermions in a vortex lattice and a magnetic field. At this point, the equations have become essentially equivalent to those discussed by Volovik in calculating Fermionic energy levels around the gap nodes in the $He-3 \ A$ phase. Volovik incorrectly asserts that the equations for a superconductor in a magnetic field are not equivalent to his equations, for which the effective magnetic field, “$B$” = $\nabla \times \hat{\ell}$, is provided by the order parameter texture; but as we have shown above, this is not the case near a gap node and the equations are identical except for the doppler term $\vec{v} \cdot \vec{P}_s$, as is required by conservation laws.

In the absence of this term, the eigenvalues are easily obtained by squaring the Hamiltonian. Following Volovik, we write

$$\mathcal{H} = v_F \tau_x \left( \frac{\hbar}{i} \frac{\partial}{\partial x} - \frac{eA_x}{\hbar c} \right)$$

$$+ v_\Delta \tau_y \left( \frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{eA_y}{\hbar c} \right)$$
where $\vec{\tau}$ is the Nambu “isospin” matrix homologous to Pauli spin. Squaring this, we obtain

$$\mathcal{H}^2 = v_f^2 \left( \frac{\hbar}{i} \frac{\partial}{\partial x} - \frac{eA_x}{c} \right)^2$$

$$+ v_{\Delta}^2 \left( \frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{eA_y}{c} \right)^2$$

$$+ \frac{e\hbar}{c} v_f v_{\Delta} \tau \cdot B_z$$

(15)

This leads to a spectrum of eigenvalues

$$E_n^2 = \hbar^2 v_f v_{\Delta} \left( \frac{eB}{\hbar c} \right) [(2n + 1)|e| - \tau \cdot e)$$

(16)

The striking feature of this is the “zero mode” $n = 0, \tau = \frac{e}{|e|}$, which is caused, basically, by the topological effect of the vortex cores on the quasiparticles, and in fact there is just one state per vortex for holes and one for electrons. 

What do we predict, from this picture, for the thermal conductivity? The physics is that of electrons in Landau orbits whose spacing is proportional to $\sqrt{B}$, being scattered by vortex cores (also spaced by $l_B \propto \sqrt{B}$), either incoherently (at low fields, and high temperatures, where the higher orbits are relevant and the lattice is disordered) or possibly coherently, at high fields and low temperatures. In the latter case there will be gaps in the spectrum and $\kappa$ could fall to near zero: this is likely to be the cause of the plateaus observed in some samples.

The main predictions do not differ very much from those of Lee and Simon’s successful scaling laws, at least at low temperatures.

As long as we can ignore the vortex cores the same conclusion they reached, that the entire energy level structure scales as $\sqrt{B}$, seems entirely valid, so that the scaling function will be $F(\sqrt{B}/T)$. (Just by dimensional analysis: changing $B$ is equivalent to a shift in $p$ by a factor $l_B^{-1}$.) The justification given here is slightly more rigorous than that in the reference. The function which Ong and Krishana use for $K_{xx}$,

$$F = \frac{1}{1 + pB}$$
obeys the Lee-Simon scaling law over most of the range where it should, \( p \) having roughly the form \( 1/T^2 \); and as to order of magnitude, \( pB \) is of order unity where the lowest cyclotron orbit \( \omega_0 \) is of order \( T \).

At low fields and moderate temperatures, we could imagine that the classical formula

\[
\kappa \propto T \sigma \propto T \frac{1}{1 + \omega_c^2 \tau^2}
\]

would be usable. \( \omega_c^2 \) is proportional to \( B \), and it is reasonable that \( 1/\tau \) caused by scattering on the cores would be independent of \( B \) and simply proportional to the density of final states \( \propto T \) (though by no means obvious.) At high fields the states become increasingly localized and eventually only the zero mode is occupied. It is not at all clear why \( \sigma \alpha \frac{1}{\sqrt{B}} \) in this region and in fact plateaus with \( \sigma \simeq 0 \) are occasionally observed. (There are actually more low-energy states, not fewer; but they seem likely to localize better as \( B \) increases.)

The experiments show us that the vortex current scattering effect is, as expected, similar in magnitude to the cyclotron frequency itself, so that the cyclotron orbits are not sharp in energy except where the scattering is coherent.

As for \( \kappa_{xy} \), as pointed out by Lee and Simon this is smaller than \( \kappa_{xx} \) by a factor \( \sim T/E_F \) because it comes from hole-particle dissymmetry. It is significant that over a range of intermediate \( B \)'s it fits fairly well to \( \sqrt{B}/1 + pB \), (see Fig. (2),) which would follow from semiclassical arguments from the Landau orbits. But since this is in principle outside the “node” approximation, it will not be discussed in detail here.

The “spin gap” phase seems to be one in which the mode structure of the pseudogap is already developed. (This is the picture proposed by Lee and Nagaosa [4], following Baskaran and Zou, and borrowed by Fisher, Balents and Nayak [5].) Thus some of the above considerations may be relevant for critical phenomena near \( T_c \), in the spin gap phase.

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