EXTREMIZERS FOR FOURIER RESTRICTION INEQUALITIES: CONVEX ARCS

By

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Abstract. We establish the existence of extremizers for a Fourier restriction inequality on planar convex arcs without points with collinear tangents whose curvature satisfies a natural assumption. More generally, we prove that any extremizing sequence of nonnegative functions has a subsequence which converges to an extremizer.

Contents

1 Introduction 338
2 The cap estimate 341
3 The decomposition algorithm 344
4 A geometric property of the decomposition 347
5 Upper bounds for extremizing sequences 352
6 A concentration compactness result 354
7 Exploring concentration 357
8 Comparing optimal constants 362
   8.1 Introducing local coordinates. . . . . . . . . . . . . . . . . . . 362
   8.2 The unperturbed case. . . . . . . . . . . . . . . . . . . . . . . . 364
   8.3 A variational calculation. . . . . . . . . . . . . . . . . . . . . . 365
      8.3.1 The term $\int f^6 \sigma$. . . . . . . . . . . . . . . . . . . . 366
      8.3.2 The term $\int f^6 \sigma$. . . . . . . . . . . . . . . . . . . . 370

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1 Introduction

Consider a compact arc \( \Gamma \subset \mathbb{R}^2 \) of a smooth, convex curve equipped with arc-length measure \( \sigma \). Assume the curvature \( \kappa \) of \( \Gamma \) to be positive everywhere; equivalently, assume that \( \lambda := \min_\Gamma \kappa \) is a positive real number. Let \( \ell := \sigma(\Gamma) \) and parametrize \( \Gamma \) by arclength:

\[
\gamma : [0, \ell] \rightarrow \mathbb{R}^2 \quad s \mapsto \gamma(s) = (x(s), y(s)).
\]

For \( s \in [0, \ell] \), let \( t(s) = (x'(s), y'(s)) \) be the tangent indicatrix and let \( \theta(s) \in S^1 \) measure the net rotation described by the vector \( t(s) \) as we run the curve \( \gamma \) from 0 to \( s \). In other words, if we let \( e_1 = (1, 0) \), then \( \theta \) is the unique continuous function satisfying \( t(s) = (\cos \theta(s), \sin \theta(s)) \) for every \( s \in [0, \ell] \), and such that \( \theta(0) = \arccos(e_1 \cdot t(0)) \). The function \( \theta \) is related to the curvature \( \kappa \) via

\[
\theta(s) = \int_0^s \kappa(t)dt.
\]

We further assume that the arc \( \Gamma \) has no points with collinear tangents\(^1\), i.e., points \( \gamma(s_0), \gamma(s_1) \in \Gamma \) for which \( t(s_0) = -t(s_1) \). By compactness, this implies that there exists some constant \( \delta_0 > 0 \) such that

\[
|t(s) + t(s')| \geq \delta_0, \quad \forall s, s' \in [0, \ell].
\]

Certain subsets of \( \Gamma \) are of special interest to us. A \textbf{cap} \( \mathcal{C} \subset \Gamma \) is a set of the form

\[
\mathcal{C} = \mathcal{C}(s, r) = \{ \gamma(s') \in \Gamma : |s - s'| < r \}
\]

for some \( s \in [0, \ell] \) and \( r > 0 \). We write \( |\mathcal{C}| := \sigma(\mathcal{C}) \).

The space \( L^2(\sigma) \) consists of all functions \( f : \Gamma \rightarrow \mathbb{C} \) for which the quantity

\[
\|f\|_{L^2(\sigma)}^2 := \int_{\Gamma} |f(z)|^2 d\sigma(z)
\]

\(^1\)We hope to remove this assumption in a later work.
is finite. Given \( f \in L^2(\sigma) \), the Fourier transform of the measure \( f\sigma \) is defined as

\[
\hat{f}\sigma(x, t) := \int_{\Gamma} f(z) e^{-i(x,t) \cdot z} d\sigma(z).
\]

The Tomas-Stein inequality [26], whose proof we recall in the next section, states that there exists a finite constant \( C[\Gamma] \) such that

\[
\|\hat{f}\sigma\|_{L^6(\mathbb{R}^2)} \leq C[\Gamma]\|f\|_{L^2(\sigma)}
\]

for every \( f \in L^2(\sigma) \). Henceforth, we take \( C[\Gamma] \) to be the best constant in (2), defined by

\[
C[\Gamma] := \sup_{0 \neq f \in L^2(\sigma)} \frac{\|\hat{f}\sigma\|_6}{\|f\|_{L^2(\sigma)}}.
\]

**Definition 1.** An extremizing sequence for (2) is a sequence \( \{f_n\} \) of functions in \( L^2(\sigma) \) satisfying \( \|f_n\|_{L^2(\sigma)} \leq 1 \) such that \( \|\hat{f_n}\sigma\|_6 \to C[\Gamma] \) as \( n \to \infty \). An extremizer for (2) is a nonzero function \( f \in L^2(\sigma) \) which satisfies \( \|\hat{f}\sigma\|_6 = C[\Gamma]\|f\|_{L^2(\sigma)} \).

**Definition 2.** A nonzero function \( f \in L^2(\sigma) \) is said to be a \( \delta \)-near extremizer for the inequality (2) if \( \|\hat{f}\sigma\|_6 \geq (1 - \delta)C[\Gamma]\|f\|_{L^2(\sigma)} \).

A natural question is whether extremizers exist. More generally, one can ask if extremizing sequences are precompact in \( L^2(\sigma) \). Previous work includes the study of extremizers for Strichartz/Fourier restriction inequalities in [15], [13] and [7]. Kunze [15] proved the existence of extremizers for the parabola in \( \mathbb{R}^2 \) by showing that any nonnegative extremizing sequence is precompact. Foschi [13], whose work we recall in greater detail in Section 8, showed that Gaussians are extremizers for this situation and computed the corresponding optimal constant. The best constant and extremizers for the paraboloid in \( \mathbb{R}^3 \) were also computed in [13]. The existence of extremizers for the restriction on the sphere \( S^2 \) was proved by Christ and Shao in [7] and, to the best of our knowledge, is the only result concerning existence of extremizers for the endpoint restriction problem on a compact manifold.

Other results on (non-)existence of extremizers and/or computation of sharp constants for Fourier restriction operators and Strichartz inequalities can be found in [4, 9, 11, 12, 14, 21, 22].

Here is our main result.

**Theorem 1.** Let \( \Gamma \) be a compact arc of a smooth, convex curve in the plane without points with collinear tangents, equipped with arclength measure \( \sigma \).
Assume that the curvature $\kappa$ of $\Gamma$ is a strictly positive function. If the second derivative of the curvature with respect to arclength satisfies

$$
\frac{d^2 \kappa}{ds^2}(p_0) < \frac{3}{2} \kappa(p_0)^3
$$

at every $p_0 \in \Gamma$ which is a global minimum of the curvature, then any extremizing sequence of nonnegative functions in $L^2(\sigma)$ for the inequality (2) is precompact.

Note that no generality is lost in assuming that extremizing sequences consist of nonnegative functions only. This can easily be seen by rewriting inequality (2) in the following equivalent convolution form:

$$
\| f * \sigma * f \|_{L^2(\mathbb{R}^2)} \leq \frac{C[\Gamma]^3}{2\pi} \| f \|_{L^2(\sigma)}^3.
$$

It is natural to ask about the significance of the geometric condition (3). For instance, if it is not satisfied, does this mean that extremizers fail to exist? While we are unable to provide a complete answer to this question at the moment, we analyze the situation in which condition (3) fails in a rather strong sense in the companion paper [19], and establish a complementary (negative) result along these lines.

We conclude this section by briefly outlining the structure of this paper and giving an idea of the proof of Theorem 1, which follows the lines of the proof in [7] of precompactness of extremizing sequences for the adjoint Fourier restriction operator on the sphere $S^2 \subset \mathbb{R}^3$.

In the next section, we follow the classical argument of Carleson and Sjölin [3] to prove the Tomas-Stein inequality (2). Using the analysis of a certain bilinear form from [17, 21], we establish the following refinement:

$$
\| \hat{f} \sigma \|_{L^6(\mathbb{R}^2)} \lesssim \| f \|_{L^2(\sigma)}^{1-\beta/2} \sup_{\mathcal{C} \subset \Gamma} \left( |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} d\sigma \right)^{\beta/3},
$$

where the supremum ranges over all caps $\mathcal{C} \subset \Gamma$ and $\beta > 0$ is a small universal constant.

In Section 3, we use estimate (4) to describe an iterative procedure which takes a nonnegative function $f \in L^2(\sigma)$ as input and produces a sequence of functions $\{f_n\}$ associated with disjoint caps $\{\mathcal{C}_n\} \subset \Gamma$ for which $f = \sum_n f_n$ in the $L^2$-sense. This decomposition enjoys certain geometric properties which are described in Section 4. After introducing a suitable metric on the set of all caps, we establish the fact that distant caps interact weakly. Together with the decomposition algorithm, this implies an inequality of geometric nature which is a key step toward gaining control of extremizing sequences.
In Section 5, we prove that any near extremizer satisfies appropriately scaled upper bounds with respect to some cap; and in Section 6, we use this to obtain a result of concentration compactness [16] flavor. This result basically states that a nonnegative extremizing sequence behaves in one of two possible ways (up to extraction of a subsequence and up to a small $L^2$ error): it is either uniformly integrable, or it concentrates at a point. Precompactness can be derived in the former case, since the main obstruction pointed out in [11] is easy to rule out: in fact, $L^2$ weak limits of nonnegative, uniformly integrable sequences of functions are nonzero.

The proof is therefore finished once we show that concentration cannot occur. Aiming at a contradiction, we explore some of the properties that an extremizing sequence which concentrates at a point would have to enjoy. In Section 7, we compute a certain limiting operator norm exactly and, in particular, show that an extremizing sequence which concentrates must do so at a point of minimal curvature. A second ingredient consists of comparing the constant $C[Γ]$ from inequality (2) with the optimal constant for the adjoint Fourier restriction inequality on an appropriately dilated parabola equipped with projection measure, as studied in [13]. We obtain this ingredient in Section 8, postponing some of the more technical estimates to Appendix 10. We derive the desired contradiction in Section 9, completing the proof of Theorem 1.

**Notation.** For reals $x, y ∈ \mathbb{R}$, we write $x = O(y)$ or $x ≲ y$ if there exists a real constant $C$ such that $|x| \leq C|y|$, and $x \asymp y$ if $C^{-1}|y| \leq |x| \leq C|y|$ for some finite constant $C \neq 0$. When we want to make explicit the dependence of $C$ on some parameter $α$, we write $x = O_α(y)$ or $x \lesssim_α y$. As is customary, the value of $C$ is allowed to change from line to line. For $A ⊆ \mathbb{R}^d$ and $λ ∈ \mathbb{R}$, we denote $\{λx : x ∈ A\}$, the $λ$-dilation of $A$ by $λ · A$. The Minkowski sum of $A$ with itself $\{x + x' : x ∈ A$ and $x' ∈ A\}$ is denoted by $A + A$. Sharp constants always appear in bold face. We denote the real and imaginary parts of the complex number $z ∈ \mathbb{C}$, respectively, by $\Re z$ and $\Im z$.

## 2 The cap estimate

Let $f, g ∈ L^2(σ)$. We seek to estimate the $L^3$ norm of the product

$$\tilde{fσ} \cdot \tilde{gσ}(x, t) = \int_0^\ell \int_0^\ell f(γ(s))g(γ(s'))e^{-i(x,t)-(γ(s)+γ(s'))}dsds'.$$
For that purpose, it suffices to estimate the $L^{3/2}$ norm of the convolution of the measures $f\sigma * g\sigma$, which is defined by duality as
\[ \langle f\sigma * g\sigma, \varphi \rangle = \int_0^\ell \int_0^\ell f(\gamma(s))g(\gamma(s'))\varphi(\gamma(s) + \gamma(s'))dsds' \]
for every test function $\varphi \in C_0^\infty(\Gamma, \sigma)$.

To analyze the integral (5), we make the change of variables
\[ (s, s') \mapsto (u, v) = (x(s) + x(s'), y(s) + y(s')). \]

Splitting
\[ f(\gamma(s))g(\gamma(s')) = f(\gamma(s))g(\gamma(s'))(\chi_{[s,s']} + \chi_{[s',s]}) \]
and using the triangle inequality, we lose no generality in assuming that $s > s'$ in the support of $f(\gamma(s))g(\gamma(s'))$. As a consequence, the transformation (6) is injective in the support of $f(\gamma(s))g(\gamma(s'))$. It follows that
\[ \widehat{f\sigma} \cdot \widehat{g\sigma}(x, t) = \int_{\Gamma + \Gamma} f(\gamma(s(u, v)))g(\gamma(s'(u, v)))e^{-i(x, t)\cdot(u, v)}J^{-1}du dv, \]
where $J = J(s(u, v), s'(u, v))$ is the Jacobian of the transformation (6) in the region $\{ s > s' \}$, i.e.,
\[ J(s, s') = \left| \frac{\partial(u, v)}{\partial(s, s')} \right| = |x'(s)y'(s') - x'(s)y'(s)| = |\sin(\theta(s) - \theta(s'))|. \]

Note that, for $(u, v) \in \mathbb{R}^2$,
\[ f\sigma * g\sigma(u, v) = \begin{cases} f(\gamma(s(u, v)))g(\gamma(s'(u, v)))J^{-1} & \text{if } (u, v) \in \Gamma + \Gamma, \\ 0 & \text{otherwise.} \end{cases} \]

The Hausdorff-Young inequality implies that
\[ \|\widehat{f\sigma} \cdot \widehat{g\sigma}\|_3 \leq \|f\sigma * g\sigma\|_{3/2} \]
\[ \lesssim \left( \int_0^\ell \int_0^\ell |f(\gamma(s))|^{3/2} |g(\gamma(s'))|^{3/2} |\sin(\theta(s) - \theta(s'))|^{-1/2}dsds' \right)^{2/3}. \]

Since $\Gamma$ has no points with collinear tangents (i.e., condition (1) holds),
\[ |\sin(\theta(s) - \theta(s'))| \geq \min \left\{ \frac{2}{\pi} |\theta(s) - \theta(s')|, \delta_0(1 + O(\delta_0^2)) \right\} \]
for every $s, s' \in [0, \ell]$. On the other hand, since $\lambda = \min_{\Gamma} \kappa$,
\[ |\theta(s) - \theta(s')| = \left| \int_{s'}^s \kappa(t)dt \right| \geq \lambda |s - s'|. \]
It follows that
\[
\| f \sigma * g \sigma \|_{3/2}^{3/2} \lesssim \int_0^\ell \int_0^\ell |f(\gamma(s))|^{3/2} |g(\gamma(s'))|^{3/2} |\sin(\theta(s) - \theta(s'))|^{-1/2} ds ds'
\]
\[
\lesssim_{\lambda, \delta_0} \int_0^\ell \int_0^\ell |f(\gamma(s))|^{3/2} |g(\gamma(s'))|^{3/2} |s - s'|^{-1/2} ds ds'.
\]
Note that the implicit constant blows up as \( \lambda \downarrow 0^+ \); this is why we assume that \( \Gamma \) has everywhere positive curvature.

For \( 0 < \alpha < 1 \), consider the bilinear form
\[
B_\alpha(F, G) := \int \int_{\mathbb{R}^2} F(x)G(x')|x - x'|^{-\alpha} dx dx'.
\]
The case \( \alpha = 1/2 \) is related to the preceding discussion. In fact, setting \( F := |f \circ \gamma|^{3/2} \) and \( G := |g \circ \gamma|^{3/2} \), we already know that
\[
(8) \quad \| f \sigma * g \sigma \|_{3/2}^{3/2} \lesssim B_{1/2}(F, G).
\]

For \( 0 < \alpha < 1 \) and \( p = 2/(2 - \alpha) \), the Hardy-Littlewood-Sobolev inequality implies that \( |B_\alpha(F, F)| \lesssim_p \| F \|_{L^p(\mathbb{R})} \). In particular, estimate (8) combines with the \( L^{4/3} \) bound for \( B_{1/2} \) to yield the Tomas-Stein inequality (2)
\[
\| \widehat{f \sigma} \|_6 = \| (\widehat{f \sigma})^2 \|_{3/2}^{3/2} \lesssim \| f \sigma * f \sigma \|_{3/2}^{3/2} \lesssim B_{1/2}(F, F)^{1/3} \lesssim \| F \|_{4/3}^{2/3} = \| f \|_{L^2(\sigma)}.
\]

Following previous work from [18] and [17], Quilodrán proved in [21, Proposition 4.5] that, for the same range of \( \alpha \) and value of \( p \), there exists a constant \( \beta > 0 \) such that
\[
(9) \quad |B_\alpha(F, F)| \lesssim \| F \|_{p}^{2-\beta} \sup_I (\| |I|^{-1+1/p} \int_I |F| \|^{\beta}
\]
for every \( F \in L^p(\mathbb{R}) \). Here, the supremum ranges over all compact intervals \( I \) of \( \mathbb{R} \). If instead of the \( L^{4/3} \) bound for \( B_{1/2} \) we use the more refined estimate (9), then reasoning in a similar way as before leads to the following improved estimate.

**Proposition 2.** (Cap estimate) There exist \( C < \infty \) and \( \beta > 0 \) such that for every \( f \in L^2(\sigma) \),
\[
(10) \quad \| \widehat{f \sigma} \|_{L^p(\mathbb{R})} \leq C \| f \|_{L^2(\sigma)}^{1-\beta/2} \sup_{e \in \Gamma} (|e|^{-1/4} \int_e |f|^{3/2} d\sigma)^{\beta/3}.
\]

**Proof.** As before, set \( F(s) := |f(\gamma(s))|^{3/2} \). Then (8) and (9) imply
\[
\| \widehat{f \sigma} \|_6 \lesssim \| f \sigma * f \sigma \|_{3/2}^{1/2} \lesssim B_{1/2}(F, F)^{1/3} \lesssim \| F \|_{4/3}^{2/3} \beta/3 \sup_I (|I|^{-1+1/p} \int_I |F| ds)^{\beta/3}
\]
\[
= \| f \|_{L^2(\sigma)}^{1-\beta/2} \sup_{e \in \Gamma} (|e|^{-1/4} \int_e |f|^{3/2} d\sigma)^{\beta/3},
\]
as desired. \(\)
3 The decomposition algorithm

The cap estimate (10) is the only ingredient we need to prove the analog of [7, Lemma 2.9], which establishes a weak connection between functions satisfying modest lower bounds $\|\hat{f}\|_6 \gtrsim \delta \|f\|_2$ and characteristic functions of caps.

**Lemma 3.** For each $\delta > 0$, there exist $C_\delta < \infty$ and $\eta_\delta > 0$ such that for $f \in L^2(\sigma)$ satisfying $\|\hat{f}\|_6 \geq \delta C[\Gamma]\|f\|_{L^2(\sigma)}$, there exist a decomposition $f = g + h$ and a cap $\mathcal{C} \subset \Gamma$ satisfying

\begin{align}
&0 \leq |g|, |h| \leq |f|, \\
&g, h \text{ have disjoint supports,} \\
&|g(\gamma(s))| \leq C_\delta \|f\|_{L^2(\sigma)}|\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(\gamma(s)), \text{ for all } s \in [0, \ell], \\
&\|g\|_{L^2(\sigma)} \geq \eta_\delta \|f\|_{L^2(\sigma)}.
\end{align}

**Proof.** The proof is analogous to that of [7, Lemma 2.9], but we reproduce it here for the convenience of the reader. We can, without loss of generality, normalize so that $\|f\|_{L^2(\sigma)} = 1$. By Proposition 2, there exists a cap $\mathcal{C}$ such that

$$\int_{\mathcal{C}} |f|^{3/2} d\sigma \geq \frac{1}{2} c(\delta)|\mathcal{C}|^{1/4}.$$ 

Here, $c(\delta) = c_0 \cdot \delta^{3/\beta}$ for some absolute constant $c_0 > 0$ whose exact value is not important for the analysis. Let $R \geq 1$, and let $E := \{ \gamma(s) \in \mathcal{C} : |f(\gamma(s))| \leq R \}$. Set $g = f\chi_E$ and $h = f - f\chi_E$. Then $g$ and $h$ have disjoint supports, $g + h = f$. $g$ is supported on $\mathcal{C}$, and $\|g\|_{L^1} \leq R$. Since $|h(\gamma(s))| \geq R$ for almost every $\gamma(s) \in \mathcal{C}$ for which $h(\gamma(s)) \neq 0$, we have

$$\int_{\mathcal{C}} |h|^{3/2} d\sigma \leq R^{-1/2} \int_{\mathcal{C}} |h|^2 d\sigma \leq R^{-1/2} \|f\|_2^2 = R^{-1/2}.$$ 

Define $R$ by $R^{-1/2} = c(\delta)|\mathcal{C}|^{1/4}/4$. Then

$$\int_{\mathcal{C}} |g|^{3/2} d\sigma = \int_{\mathcal{C}} |f|^{3/2} d\sigma - \int_{\mathcal{C}} |h|^{3/2} d\sigma \geq \frac{1}{4} c(\delta)|\mathcal{C}|^{1/4}.$$ 

By Hölder’s inequality, since $g$ is supported on $\mathcal{C}$,

$$\|g\|_2 \geq |\mathcal{C}|^{-1/6} \left( \int_{\mathcal{C}} |g|^{3/2} d\sigma \right)^{2/3} \geq c'(\delta) = c'(\delta)\|f\|_2 > 0. \qedhere$$

Conditions 13 and 14 easily imply the following lower bound on the $L^1$ norm of $g$. 

Lemma 4. Let $g \in L^2(\sigma)$ satisfy $|g(x)| \leq a|\mathcal{C}|^{-1/2}\chi_{\mathcal{C}}(x)$ and $\|g\|_2 \geq b$ for some $a, b > 0$ and $\mathcal{C} \subset \Gamma$. Then there exists a constant $C = C(a, b) > 0$ such that

$$\|g\|_{L^1(\sigma)} \geq C|\mathcal{C}|^{1/2}.$$ 

Proof.

$$\|g\|_{L^1(\sigma)} = \int_{\mathcal{C}} |g| d\sigma \geq a^{-1}|\mathcal{C}|^{1/2}\|g\|_{L^2(\sigma)}^2 \geq a^{-1}b^2|\mathcal{C}|^{1/2}. \quad \square$$

In what follows, we restrict our attention to nonnegative functions.\(^2\) As was already mentioned, inequality (2) is equivalent, via Plancherel’s theorem, to

$$\|f* f* f\|_{L^2(\mathbb{R}^2)} \leq \frac{C[\Gamma]^3}{2\pi}\|f\|_{L^2(\sigma)}^3.$$ 

The pointwise inequality $|f* f* f| \leq |f|\sigma * |f|\sigma * |f|\sigma$ then implies that if $f$ is an extremizer for inequality (2), so is $|f|$; similarly, if $\{f_n\}$ is an extremizing sequence, so is $\{|f_n|\}$.

A decomposition algorithm analogous to that from [7, Step 6A] may be applied to any given nonnegative $f \in L^2(\sigma)$. We describe it precisely.

Decomposition algorithm. Initialize by setting $G_0 = f$ and $\epsilon_0 = 1/2$.

Step $n$: The inputs for step $n$ are a nonnegative function $G_n \in L^2(\sigma)$ and a positive number $\epsilon_n$. Its outputs are functions $f_n, G_{n+1}$ and nonnegative numbers $\epsilon^*_n, \epsilon_{n+1}$.

If $\|G_n\sigma * G_n\sigma * G_n\sigma\|_2 = 0$, then $G_n = 0$ almost everywhere. The algorithm then terminates, and we define $\epsilon^*_n = 0, f_n = 0$, and $G_m = f_m = 0, \epsilon_m = 0$ for all $m > n$.

If $0 < \|G_n\sigma * G_n\sigma * G_n\sigma\|_2 < \epsilon^*_n(2\pi)^{-1}C[\Gamma]^3\|f\|_2^3$, then replace $\epsilon_n$ by $\epsilon_n/2$; repeat until the first time that $\|G_n\sigma * G_n\sigma * G_n\sigma\|_2 \geq \epsilon^*_n(2\pi)^{-1}C[\Gamma]^3\|f\|_2^3$. Define $\epsilon^*_n$ to be this value of $\epsilon_n$. Then

$$\|G_n\sigma * G_n\sigma * G_n\sigma\|_2 \leq 8(\epsilon^*_n)^3\frac{C[\Gamma]^3}{2\pi}\|f\|_2^3.$$ 

Apply Lemma 3 to obtain a cap $\mathcal{C}_n$ and a decomposition $G_n = f_n + G_{n+1}$ with disjointly supported nonnegative summands satisfying $f_n \leq C_n\|f\|_2|\mathcal{C}_n|^{-1/2}\chi_{\mathcal{C}_n}$.

\(^2\)For much of the analysis, this makes no difference, but nonnegativity plays a crucial role in Section 6 when we establish precompactness of uniformly integrable extremizing sequences; see the proof of Lemma 18.
and \( \|f_n\|_2 \geq \eta_n \|f\|_2 \). Here, \( C_n, \eta_n \) are bounded above and below, respectively, by quantities which depend only on \( \|G_n \sigma \ast G_n \sigma \ast G_n \sigma\|_2^2 / \|G_n\|_2 \gtrsim \epsilon_n^* \). Define \( \epsilon_{n+1} = \epsilon_n^* \), and move on to step \( n + 1 \).

The following lemma contains exact analogs of [7, Lemmas 8.1, 8.3, 8.4].

**Lemma 5.** Let \( f \in L^2(\sigma) \) be a nonnegative function with positive norm. If the decomposition algorithm never terminates for \( f \), then \( \epsilon_n^* \to 0 \) as \( n \to \infty \), and \( \sum_{n=0}^{N} f_n \to f \) in \( L^2(\sigma) \) as \( N \to \infty \).

The proof of Lemma 5 is identical to the corresponding one in [7] and therefore is omitted.

This decomposition is in general very inefficient. However, if \( f \) nearly extremizes inequality (2), then more useful properties hold. Before we turn into these, let us recall a useful fact about near extremizers which already appeared in [20, Lemma 9.2].

**Lemma 6.** Let \( f = g + h \in L^2(\sigma) \). Suppose that \( g \perp h, g \neq 0 \), and that \( f \) is a \( \delta \)-near extremizer for some \( \delta \in (0, \frac{1}{4}] \). Then

\[
\|h\|_2^2 \|f\|_2^2 \leq C \max\left( \frac{\|\hat{h} \|_6}{\|h\|_2}, \delta^{1/2} \right).
\]

Here, \( C < \infty \) is a constant independent of \( g \) and \( h \).

The proof is almost identical to that of [7, Lemma 7.1] and, as such, is omitted.

As the next lemma shows, regardless of whether the decomposition algorithm terminates for \( f \), the norms of \( f_n, G_n \) enjoy upper bounds independent of \( f \) for all but very large \( n \).

**Lemma 7.** There exist a sequence of positive constants \( \gamma_n \to 0 \) and a function \( N : (0, \frac{1}{2}] \to \mathbb{N} \) satisfying \( N(\delta) \to \infty \) as \( \delta \to 0 \) such that for every nonnegative \( \delta \)-near extremizer \( f \in L^2(\sigma) \), the quantities \( \epsilon_n^* \) and the functions \( f_n, G_n \) obtained when the decomposition algorithm is applied to \( f \) satisfy

\[
\begin{align*}
\epsilon_n^* &\leq \gamma_n \text{ for all } n \leq N(\delta), \\
\|G_n\|_2 &\leq \gamma_n \|f\|_2 \text{ for all } n \leq N(\delta), \text{ and} \\
\|f_n\|_2 &\leq \gamma_n \|f\|_2 \text{ for all } n \leq N(\delta).
\end{align*}
\]

**Proof.** By (15) and (16),

\[
\frac{C[\Gamma]^3}{2\pi} \|G_n\|_2^3 \geq \|G_n \sigma \ast G_n \sigma \ast G_n \sigma\|_2 \geq \left( \epsilon_n^* \right)^3 \frac{C[\Gamma]^3}{2\pi} \|f\|_2^3
\]

\[
= \left( \frac{\epsilon_n^* \|f\|_2^3}{\|G_n\|_2^3} \right) \cdot \frac{C[\Gamma]^3}{2\pi} \|G_n\|_2^3,
\]

346  
DIogo Oliveira E Silva
so $\epsilon_n^* \leq \|G_n\|_2/\|f\|_2$. Thus the second conclusion implies the first. Since $\|f_n\|_2 \leq \|G_n\|_2$, it also implies the third.

We recall two facts. First, Lemma 6 applied to $h = G_n$ and $g = f_0 + \ldots + f_{n-1}$ asserts that there are constants $c_0, C_1 \in \mathbb{R}^+$ such that whenever $f \in L^2$ is a $\delta$-near extremizer, either $\|G_n\sigma\|_6 \geq c_0\|G_n\|_2^2\|f\|_2^{-1}$, or $\|G_n\|_2 \leq C_1\delta^{1/2}\|f\|_2$. Second, according to Lemma 3, there exists a nondecreasing function $\rho : (0, \infty) \to (0, \infty)$ satisfying $\rho(t) \to 0$ as $t \to 0$ such that for every nonzero $f \in L^2$ and any $n$, if $\|G_n\sigma\|_6 \geq t\|G_n\|_2$, then $\|f_n\|_2 \geq \rho(t)\|G_n\|_2$.

Choose a sequence $\{\gamma_n\}$ of positive numbers which tends monotonically to 0, but does so sufficiently slowly to satisfy $(n + 1)\gamma_n\rho(c_0\gamma_n) > 1$ for all $n$. Define $N(\delta)$ to be the largest integer satisfying $\gamma_N(\delta) \geq C_1\delta^{1/2}$. Note that $N(\delta) \to \infty$ as $\delta \to 0$ because $\gamma_n > 0$ for all $n$.

Let $f, \delta$ be given. Suppose that $n \leq N(\delta)$. Aiming at a contradiction, suppose that $\|G_n\|_2 > \gamma_n\|f\|_2$. Then, by definition of $N(\delta)$, $\|G_n\|_2 \leq C_1\delta^{1/2}\|f\|_2$. By the above dichotomy, $\|G_n\sigma\|_6 \geq c_0\|G_n\|_2^2\|f\|_2^{-1} \geq c_0\gamma_n\|G_n\|_2$.

By the second fact reviewed above, $\|f_n\|_2 \geq \rho(c_0\gamma_n)\|G_n\|_2 \geq \gamma_n\rho(c_0\gamma_n)\|f\|_2$. Since $\|G_n\|_2 \geq \|G_n\|_2$ for every $m \leq n$, the same lower bound follows for $\|f_m\|_2$ for every $m \leq n$. Since the functions $f_m$ are pairwise orthogonal, $\sum_{m \leq n} \|f_m\|_2^2 \leq \|f\|_2^2$, and consequently $(n + 1)\gamma_n\rho(c_0\gamma_n) \leq 1$, a contradiction.

The following lemma is a direct consequence of the decomposition algorithm coupled with Lemma 3.

**Lemma 8.** For every $\epsilon > 0$, there exist $\delta_\epsilon > 0$ and $C_\epsilon < \infty$ such that if $f \in L^2(\sigma)$ is a nonnegative $\delta_\epsilon$-near extremizer, then the functions $f_n, G_n$ associated to $f$ by the decomposition algorithm satisfy, for every $n \in \mathbb{N}$,

(i) if $\|G_n\|_2 \geq \epsilon\|f\|_2$, then there exists a cap $\mathcal{C}_n \subset \Gamma$ such that

$$f_n \leq C_\epsilon\|f\|_2|\mathcal{C}_n|^{-1/2}\mathcal{X}_{\mathcal{C}_n};$$

(ii) if $\|G_n\|_2 \geq \epsilon\|f\|_2$, then $\|f_n\|_2 \geq \delta_\epsilon\|f\|_2$.

### 4 A geometric property of the decomposition

Consider two caps $\mathcal{C}, \mathcal{C}' \subset \Gamma$, and assume without loss of generality that $|\mathcal{C}'| \leq |\mathcal{C}|$. Let $f, g \in L^2(\sigma)$ be such that $\text{supp}(f) \subset \mathcal{C}$ and $\text{supp}(g) \subset \mathcal{C}'$. The following estimate is a direct consequence of (7) and Hölder’s inequality:

$$\|f\sigma \ast g\sigma\|_{3/2} \lesssim \left(\inf_{s,s'} |\sin(\theta(s) - \theta(s'))|\right)^{-1/3} \|f\|_{L^2(\sigma)}\|g\|_{L^2(\sigma)}. \quad (21)$$
Letting $\ell(\mathcal{C}, \mathcal{C}') := \inf_{s, s' \in \mathcal{C}} |s - s'|$ and using (8) and Hölder’s inequality, we can rewrite this estimate as

$$
\|f\sigma \ast g\sigma\|_{L^3(\mathbb{R}^2)} \lesssim \left( \frac{\ell(\mathcal{C}, \mathcal{C}')}{|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}} \right)^{-1/3} \|f\|_{L^2(\mathcal{C})} \|g\|_{L^2(\mathcal{C})}.
$$

(22)

For characteristic functions of caps we have the following additional estimate.

**Lemma 9.** Let $\mathcal{C}, \mathcal{C}' \subset \Gamma$ be caps. Then

$$
\|\chi_{\mathcal{C}} \sigma \ast \chi_{\mathcal{C}'} \sigma\|_{L^3(\mathbb{R}^2)} \lesssim \left( \frac{|\mathcal{C}'|}{|\mathcal{C}|} \right)^{1/12} |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2}.
$$

(23)

**Proof.** Without loss of generality, we may assume that $10|\mathcal{C}'| \leq |\mathcal{C}|$; otherwise, estimate (23) is just a consequence of the fundamental inequality

$$
\|f\sigma \ast g\sigma\|_{L^3(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathcal{C})} \|g\|_{L^2(\mathcal{C})}.
$$

(24)

Let $\mathcal{C}^*$ be a cap neighborhood of $\mathcal{C}'$ with the same center and of size $|\mathcal{C}^*| = |\mathcal{C}|^{1/4}|\mathcal{C}'|^{1/4}$. Split $\chi_{\mathcal{C}}$ as $\chi_{\mathcal{C}} = \chi_{\mathcal{C} \cap \mathcal{C}^*} + \chi_{\mathcal{C} \setminus \mathcal{C}^*}$. Then

$$
\|\chi_{\mathcal{C}} \sigma \ast \chi_{\mathcal{C}'} \sigma\|_{L^3(\mathbb{R}^2)} \leq \|\chi_{\mathcal{C} \cap \mathcal{C}^*} \sigma \ast \chi_{\mathcal{C}'} \sigma\|_{L^3(\mathbb{R}^2)} + \|\chi_{\mathcal{C} \setminus \mathcal{C}^*} \sigma \ast \chi_{\mathcal{C}'} \sigma\|_{L^3(\mathbb{R}^2)}.
$$

The first summand can be easily estimated using (24). While $\mathcal{C} \setminus \mathcal{C}^*$ is not necessarily a cap, it is the union of at most two caps, say, $\mathcal{C}_1$ and $\mathcal{C}_2$. We can use estimate (22) to control the contribution of each of these caps. Noting that

$$
\min\{ \ell(\mathcal{C}_1, \mathcal{C}'), \ell(\mathcal{C}_2, \mathcal{C}') \} \geq \ell(\mathcal{C} \setminus \mathcal{C}^*, \mathcal{C}') \gtrsim |\mathcal{C}^*|,
$$

we have

$$
\|\chi_{\mathcal{C}} \sigma \ast \chi_{\mathcal{C}'} \sigma\|_{L^3(\mathbb{R}^2)} \lesssim \left( \frac{|\mathcal{C} \cap \mathcal{C}^*|^{1/2}}{|\mathcal{C}^*|^{1/2}} + \left( \frac{\ell(\mathcal{C}_1, \mathcal{C}')}{|\mathcal{C}_1|^{1/2}|\mathcal{C}'|^{1/2}} \right)^{-1/3} |\mathcal{C}_1|^{1/2}

\quad + \left( \frac{\ell(\mathcal{C}_2, \mathcal{C}')}{|\mathcal{C}_2|^{1/2}|\mathcal{C}'|^{1/2}} \right)^{-1/3} |\mathcal{C}_2|^{1/2} \right) |\mathcal{C}'|^{1/2}

\lesssim \left( \frac{|\mathcal{C}^*|^{1/2} + \left( \frac{|\mathcal{C}^*|}{|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}} \right)^{-1/3} |\mathcal{C}|^{1/2} \right) |\mathcal{C}'|^{1/2}

\lesssim \left( \frac{|\mathcal{C}|}{|\mathcal{C}|} \right)^{1/12} |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2},
$$

as desired.

The set of all caps can be made into a metric space. Define the distance $d$ from $\mathcal{C} = \mathcal{C}(s, r)$ to $\mathcal{C}' = \mathcal{C}(s', r')$ to be the hyperbolic distance from $(s, r)$ to $(s', r')$ in the upper half plane model. More explicitly,

$$
d(\mathcal{C}, \mathcal{C}') := \ar 
\cos h \left( 1 + \frac{(s - s')^2 + (r - r')^2}{2rr'} \right).
$$

(25)
If \( s = s' \), the distance depends only on the ratio of the two radii. When \( r = r' \), the
distance is \( \propto r^{-1}|s - s'| \), and so this distance has the natural scaling.

In our next result, we use estimates (22) and (23) to prove that the quantity
\[ \| \chi e^{s} \chi e^{s'} \|_{L^{3/2}([0,1])} \]is much smaller than the trivial bound \( |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2} \) unless \( \mathcal{C}, \mathcal{C}' \)
have comparable radii and nearby centers.

**Lemma 10.** For any \( \epsilon > 0 \) there exists \( \rho < \infty \) such that
\[ \| \chi e^{s} \chi e^{s'} \|_{L^{3/2}([0,1])} < \epsilon |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2} \]
whenever \( d(\mathcal{C}, \mathcal{C}') > \rho \).

**Proof.** Let \( \mathcal{C} = \mathcal{C}(s, r) \) and \( \mathcal{C}' = \mathcal{C}'(s', r') \). As before, assume \( r' \leq r \). We
consider three cases.

**Case 1:** Start by assuming that \( \mathcal{C} \) and \( \mathcal{C}' \) have comparable radii, say, \( \frac{1}{10} r \leq r' \leq r \). Then \( \mathcal{C} \) and \( \mathcal{C}' \) are not far apart unless the corresponding centers are far apart. We
may therefore assume that \( |s - s'| \geq 10r \), which in turn implies \( \ell(\mathcal{C}, \mathcal{C}') \gtrsim |s - s'| \).

Using estimate (22), we conclude that
\[ \| \chi e^{s} \chi e^{s'} \|_{L^{3/2}([0,1])} \lesssim \left( \frac{|s - s'|}{r} \right)^{-1/3} |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2} \lesssim (\cosh \rho)^{-1/6} |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2}, \]
provided \( d(\mathcal{C}, \mathcal{C}') > \rho \).

**Case 2:** Assume now that \( 10r' < r \) and \( |s - s'| < 10r \). We can use Lemma 9
to conclude
\[ \| \chi e^{s} \chi e^{s'} \|_{L^{3/2}([0,1])} \lesssim \left( \frac{r'}{r} \right)^{1/12} |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2}. \]
This quantity is \( \lesssim (\cosh \rho)^{-1/12} |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2} \), provided \( d(\mathcal{C}, \mathcal{C}') > \rho \). This concludes the analysis in this case.

**Case 3:** Suppose \( 10r' < r \) and \( |s - s'| \geq 10r \). For this case, \( d(\mathcal{C}, \mathcal{C}') > \rho \) implies
\[ \frac{(s - s')^2}{rr'} \gtrsim \cosh \rho \quad \text{or} \quad \frac{r}{r'} \gtrsim \cosh \rho. \]

Using, as before, estimate (22) in the former case and Lemma 9 in the latter case,
we arrive at the desired conclusion. \( \square \)

For later applications, we need a trilinear version of this lemma which follows
immediately from the previous result.

**Corollary 11.** For any \( \epsilon > 0 \), there exists \( \rho < \infty \) such that
\[ \| \chi e^{s} \chi e^{s'} \chi e^{s''} \|_{L^{2}([0,1])} < \epsilon |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2} |\mathcal{C}''|^{1/2} \]
whenever \( \max\{ d(\mathcal{C}, \mathcal{C}'), d(\mathcal{C}', \mathcal{C}''), d(\mathcal{C}'', \mathcal{C}) \} > \rho \).
The inequalities of Plancherel and Cauchy-Schwarz imply that
\[
\| \chi_{C \sigma} \ast \chi_{C' \sigma} \|_2 \leq \| \chi_{C \sigma} \|^{1/2} \| \chi_{C' \sigma} \|^{1/2}.
\]
Without loss of generality, we may assume that the caps \( C \) and \( C' \) are far apart: in view of Lemma 10, we can choose \( \rho < \infty \) so that \( d(C, C') > \rho \) implies
\[
\| \chi_{C \sigma} \ast \chi_{C' \sigma} \| < \epsilon^2 |C|^{1/2} |C'|^{1/2}.
\]
Bound the first factor appearing in (26) as follows:
\[
\| \chi_{C \sigma} \ast \chi_{C' \sigma} \| \leq \| \hat{\chi}_{C \sigma} \| \| \chi_{C' \sigma} \| < \epsilon^2 |C|^{1/2} |C'|^{1/2}.
\]
The proof is then complete in view of the trivial estimate
\[
\| \chi_{C' \sigma} \ast \chi_{C'' \sigma} \| < \epsilon^2 |C'|^{1/2} |C''|.
\]

Corollary 11 allows us to establish the following additional inequality of geometric nature, which can be proved in a way similar to [20, Lemma 2.38].

**Lemma 12.** For any \( \epsilon > 0 \), there exist \( \delta > 0 \) and \( \lambda < \infty \) such that for each nonnegative \( f \in L^2(\sigma) \) which is a \( \delta \)-near extremizer, the summands \( f_n \) produced by the decomposition algorithm and the associated caps \( \mathcal{C}_n \) satisfy
\[
d(C_j, C_k) \leq \lambda \text{ whenever } \| f_j \|_2 \geq \epsilon \| f \|_2 \text{ and } \| f_k \|_2 \geq \epsilon \| f \|_2.
\]
We provide a proof which follows that of [20, Lemma 2.38] more closely.

**Proof.** Let \( f \) be a nonnegative \( L^2 \) function which satisfies \( \| f \|_2 = 1 \) and is a \( \delta \)-near extremizer for a sufficiently small \( \delta = \delta(\epsilon) \), and let \( \{ f_n, G_n \} \) be associated to \( f \) via the decomposition algorithm. Set \( F = \sum_{n=0}^N f_n \).

Suppose that \( \| f_{j_0} \|_2 \geq \epsilon \) and \( \| f_{k_0} \|_2 \geq \epsilon \). Let \( N \) be the smallest integer such that \( \| G_{N+1} \|_2 < \epsilon^3 \). Since \( \| G_n \|_2 \) is a nonincreasing function of \( n \) and \( \| f_n \|_2 \leq \| G_n \|_2 \), necessarily \( j_0, k_0 \leq N \). Moreover, by Lemma 7, there exists \( M_\epsilon < \infty \) depending only on \( \epsilon \) such that \( N \leq M_\epsilon \). By Lemma 8, if \( \delta \) is chosen to be a sufficiently small function of \( \epsilon \) then, since \( \| G_n \|_2 \geq \epsilon^3 \) for all \( n \leq N \), \( f_n \leq \theta(\epsilon)|\mathcal{C}_n|^{-1/2} \chi_{C_n} \) for all such \( n \), where \( \theta \) is a continuous, strictly positive function on \((0, 1]\).

Now let \( \lambda \) be a large real to be specified. It suffices to show that if \( \delta(\epsilon) \) is sufficiently small, the assumption \( d(\mathcal{C}_j, \mathcal{C}_k) > \lambda \) implies an upper bound for \( \lambda \) depending only on \( \epsilon \).
As proved in [7, Lemma 9.1], \( F \) can be decomposed as
\[
F = F_1 + F_2 = \sum_{n \in S_1} f_n + \sum_{n \in S_2} f_n,
\]
where \([0, N] = S_1 \cup S_2\) is a partition of \([0, N]\), \( j_0 \in S_1, k_0 \in S_2\), and \( d(C_j, C_k) \geq \lambda/2N \geq \lambda/2M_e\) for all \( j \in S_1 \) and \( k \in S_2\). Certainly, \( \|F_1\|_2 \geq \|f_{j_0}\|_2 \geq \epsilon \); similarly, \( \|F_2\|_2 \geq \epsilon \). One of the cross term satisfies
\[
\|F_1 \sigma \ast F_1 \sigma \ast F_2 \sigma\|_2 \leq \sum_{i \in S_1} \sum_{j \in S_1} \sum_{k \in S_2} \|f_i \sigma \ast f_j \sigma \ast f_k \sigma\|_2
\]
\[
\leq \theta(\epsilon)^3 \sum_{i \in S_1} \sum_{j \in S_1} \sum_{k \in S_2} |C_i|^{-1/2} |C_j|^{-1/2} |C_k|^{-1/2} \|\chi \sigma \ast \chi \sigma \ast \chi \sigma\|_2
\]
\[
\leq M_\epsilon^3 \gamma(\lambda/2M_e) \theta(\epsilon)^3
\]
where \( \gamma(\lambda) \to 0 \) as \( \lambda \to \infty \) by Corollary 11. The other cross term \( F_1 \sigma \ast F_2 \sigma \ast F_2 \sigma \) can be estimated in an identical way. It follows that
\[
\|F_2 \sigma \ast F_2 \sigma \ast F_2 \sigma\|_2 \leq \|F_1 \sigma \ast F_1 \sigma \ast F_1 \sigma\|_2 + \|F_2 \sigma \ast F_2 \sigma \ast F_2 \sigma\|_2
\]
\[
+ 3 \|F_1 \sigma \ast F_1 \sigma \ast F_2 \sigma\|_2 + 3 \|F_1 \sigma \ast F_2 \sigma \ast F_2 \sigma\|_2
\]
\[
\leq (2\pi)^{-1} C[\Gamma]^3 \|F_1\|_2^3 + 6M_\epsilon^3 \gamma(\lambda/2M_e) \theta(\epsilon)^3.
\]
Since \( F_1 \) and \( F_2 \) have disjoint supports, \( \|F_1\|_2^3 + \|F_2\|_2^3 \leq \|f\|_2^3 = 1 \), and consequently
\[
\|F_1\|_2^3 + \|F_2\|_2^3 \leq \max(\|F_1\|_2, \|F_2\|_2) \cdot (\|F_1\|_2^2 + \|F_2\|_2^2) \leq (1 - \epsilon^2)^{1/2} \cdot 1 \leq (1 - \epsilon^2)^{1/2}.
\]
Thus
\[
\|F_2 \sigma \ast F_2 \sigma \ast F_2 \sigma\|_2 \leq (2\pi)^{-1} C[\Gamma]^3 (1 - \epsilon^2)^{1/2} + 6M_\epsilon^3 \gamma(\lambda/2M_e) \theta(\epsilon)^3.
\]
Since
\[
(2\pi)^{-1} C[\Gamma]^3 (1 - \delta)^3 \leq \|f \sigma \ast f \sigma \ast f \sigma\|_2 \leq \|F \sigma \ast F \sigma \ast F \sigma\|_2 + C \|F\|_2 \|G_{N+1}\|_2
\]
\[
\leq \|F \sigma \ast F \sigma \ast F \sigma\|_2 + C \epsilon^3,
\]
we have by transitivity,
\[
(2\pi)^{-1} C[\Gamma]^3 (1 - \delta)^3 \leq C \epsilon^3 + (2\pi)^{-1} C[\Gamma]^3 (1 - \epsilon^2)^{1/2} + 6M_\epsilon^3 \gamma(\lambda/2M_e) \theta(\epsilon)^3.
\]
Since \( \gamma(t) \to 0 \) as \( t \to \infty \), this implies, for all sufficiently small \( \epsilon > 0 \), an upper bound for \( \lambda \) which depends only on \( \epsilon \), as was to be proved. \( \square \)
5 Upper bounds for extremizing sequences

The decomposition algorithm and Lemma 12 allow us to prove that every near extremizer satisfies appropriate scaled upper bounds with respect to some cap. We need the following definition.

**Definition 3.** Let \( \Theta : [1, \infty) \to (0, \infty) \) satisfy \( \Theta(R) \to 0 \) as \( R \to \infty \). A function \( f \in L^2(\sigma) \) is said to be **upper normalized** (with gauge function \( \Theta \)) with respect to the cap \( C = C(\gamma(s_0), r_0) \subset \Gamma \) of radius \( r_0 \) and center \( \gamma(s_0) \) if

\[
\|f\|_{L^2(\sigma)} \leq C < \infty, \\
\int_{\{x : |f(\gamma(s))| \geq Rr_0^{-1/2}\}} |f(\gamma(s))|^2 ds \leq \Theta(R) \text{ for all } R \geq 1, \\
\int_{\{x : |s-s_0| \geq Rr_0\}} |f(\gamma(s))|^2 ds \leq \Theta(R) \text{ for all } R \geq 1.
\]

**Proposition 13.** There exists \( \Theta : [1, \infty) \to (0, \infty) \) satisfying \( \Theta(R) \to 0 \) as \( R \to \infty \) with the following property. For every \( \epsilon > 0 \), there exist a cap \( C \subset \Gamma \) and a threshold \( \delta > 0 \) such that any nonnegative \( f \in L^2(\sigma) \) which is a \( \delta \)-near extremizer with \( \|f\|_2 = 1 \) may be decomposed as \( f = F + G \), where

\[
G, F \geq 0 \text{ have disjoint supports}, \\
\|G\|_2 < \epsilon, \\
F \text{ is upper normalized with respect to } C.
\]

Proposition 13 is actually equivalent to the following superficially weaker statement.

**Lemma 14.** There exists \( \Theta : [1, \infty) \to (0, \infty) \) satisfying \( \Theta(R) \to 0 \) as \( R \to \infty \) with the following property. For every \( \epsilon > 0 \) and \( \tilde{R} \geq 1 \), there exist a cap \( C = C(s_0, r_0) \) and a threshold \( \delta > 0 \) such that any nonnegative \( f \in L^2(\sigma) \) which is a \( \delta \)-near extremizer with \( \|f\|_2 = 1 \) may be decomposed as \( f = F + G \), where

\[
G, F \geq 0 \text{ have disjoint supports}, \\
\|G\|_2 < \epsilon, \\
\int_{\{x : F(\gamma(s)) \geq Rr_0^{-1/2}\}} F(\gamma(s))^2 ds, \int_{\{x : |s-s_0| \geq Rr_0\}} F(\gamma(s))^2 ds \leq \Theta(R)
\]

for all \( R \in [1, \tilde{R}] \).

The proof that Lemma 14 implies Proposition 13 is the exactly the same as in [7, p. 287], and so we do not include it here.
Proof of Lemma 14. Let \( \eta : [1, \infty) \to (0, \infty) \) be a function to be chosen below, independent of \( \tilde{R} \) and satisfying \( \eta(t) \to 0 \) as \( t \to \infty \). Let \( R \in [1, \tilde{R}] \), and \( \epsilon > 0 \) be given. Let \( \delta = \delta(\epsilon, \tilde{R}) > 0 \) be a small quantity also to be chosen below. Let \( 0 \leq f \in L^2(\sigma) \) be a \( \delta \)-near extremizer, with \( \|f\|_2 = 1 \). Let \( \{f_n\} \) be the sequence of functions obtained by applying the decomposition algorithm to \( f \). Choose \( \delta = \delta(\epsilon) > 0 \) sufficiently large to guarantee that \( \|G_{M+1}\|_2 < \epsilon/2 \) and that \( f_n, G_n \) satisfy all the conclusions of Lemma 8 and Lemma 7 for \( n \leq M \). Set \( F = \sum_{n=0}^{M} f_n \). Then \( \|f - F\|_2 = \|G_{M+1}\|_2 < \epsilon/2 \).

Let \( N \in \mathbb{N}_0 \) be the minimum of \( M \) and the smallest number such that \( \|f_{N+1}\|_2 < \eta \). The number \( N \) is majorized by a quantity which depends only on \( \eta \). Set \( \mathcal{F} = \mathcal{F}_N = \sum_{k=0}^{N} f_k \). It follows from part (ii) of Lemma 8 that

\[
(36) \quad \|F - \mathcal{F}\|_2 \leq \|G_{N+1}\|_2 < \zeta(\eta), \text{ where } \zeta(\eta) \to 0 \text{ as } \eta \to 0.
\]

This function \( \zeta \) is independent of \( \epsilon \) and \( \tilde{R} \).

To prove the lemma, we must produce an appropriate cap \( \mathcal{C} = \mathcal{C}(s_0, r_0) \), and establish the existence of \( \Theta \). To do the former is simple. Indeed, to \( f_0 \) is associated a cap \( \mathcal{C}_0 \) such that \( f_0 \leq C|\mathcal{C}_0|^{-1/2} \chi_{\mathcal{C}_0} \). Then \( \mathcal{C} = \mathcal{C}_0 \) is the required cap. Note that by Lemma 4, \( \|f_0\|_1 \geq c \) for some positive universal constant \( c \).

Suppose that functions \( R \mapsto \eta(R) \) and \( R \mapsto \Theta(R) \) have been chosen so that

\[
(37) \quad \eta(R) \to 0 \text{ as } R \to \infty \quad \text{ and } \quad \eta(R) \to 0 \text{ as } R \to \infty
\]

\[
(38) \quad \zeta(\eta(R))^2 \leq \Theta(R) \text{ for all } R.
\]

Then, by (36), \( F - \mathcal{F} \) already satisfies the desired inequalities in \( L^2(\sigma) \); so it suffices to show that \( \mathcal{F}(\gamma(s)) \equiv 0 \) whenever \( |s - s_0| \geq Rr_0 \) and that \( \|\mathcal{F}\|_\infty < Rr_0^{-1/2} \).

Each summand of \( \mathcal{F} \) satisfies \( f_k \leq C(\eta)|\mathcal{C}_k|^{-1/2} \chi_{\mathcal{C}_k} \), where \( C(\eta) < \infty \) depends only on \( \eta \) and, in particular, \( f_k \) is supported in \( \mathcal{C}_k \). Also, \( \|f_k\|_2 \geq \eta \) for all \( k \leq N \), by definition of \( N \). Therefore, by Lemma 12, there exists a function \( \eta \mapsto \lambda(\eta) < \infty \), such that if \( \delta \) is sufficiently small as a function of \( \eta \), \( d(\mathcal{C}_k, \mathcal{C}_0) \leq \lambda(\eta) \) for every \( k \leq N \). This is needed for \( \eta = \eta(R) \) for all \( R \) in the compact set \( [1, \tilde{R}] \), so such a \( \delta \) may be chosen as a function of \( \tilde{R} \) alone; conditions already imposed on \( \delta \) above make it a function of both \( \epsilon \) and \( \tilde{R} \).

In the region \( \{ \gamma(s) \in \Gamma : |s - s_0| \geq Rr_0 \} \), either \( f_k \equiv 0 \), or \( \mathcal{C}_k \) has radius \( \geq \frac{1}{2} Rr_0 \), or the center \( \gamma(s_k) \) of \( \mathcal{C}_k \) is such that \( |s_k - s_0| \geq \frac{1}{2} Rr_0 \). Choose a function \( R \mapsto \eta(R) \) which tends to 0 sufficiently slowly so that the latter two cases would contradict the inequality \( d(\mathcal{C}_k, \mathcal{C}_0) \leq \lambda \) and therefore cannot arise. Then \( \mathcal{F}(\gamma(s)) \equiv 0 \) when \( |s - s_0| \geq Rr_0 \).
With the function \( \eta \) having been specified, \( \Theta \) can be defined by \( \Theta(R) := \zeta(\eta(R))^2 \). Then

\[
(39) \quad \int_{\{s : |s - s_0| \geq Rr_0 \}} F(\gamma(s))^2 ds \leq \Theta(R) \text{ for all } R \in [1, \bar{R}].
\]

We claim next that \( \|F\|_{\infty} < Rr^{-1/2}_0 \) if \( R \) is sufficiently large as a function of \( \eta \). Indeed, because the summands \( f_k \) have pairwise disjoint supports, it suffices to control \( \max_{k \leq N} \|f_k\|_{\infty} \). Again by Lemma 8, \( \|f_k\|_{\infty} \leq C(\eta)|\mathcal{C}_k|^{-1/2} \).

If \( \eta(R) \) is chosen to tend to zero sufficiently slowly as \( R \to \infty \) to ensure that \( 2C(\eta(R)) \cosh \lambda(\eta(R)) < R \) for all \( k \leq N \), then

\[
\|f_k\|_{\infty} < R(2 \cosh \lambda(\eta(R)))^{-1/r} r_k^{-1/2} \leq Rr_0^{-1/2}
\]

since \( d(\mathcal{C}_k, \mathcal{C}_0) \leq \lambda(\eta(R)) \). It follows that

\[
(40) \quad \int_{\{s : F(\gamma(s)) \geq Rr^{-1/2}_0 \}} F(\gamma(s))^2 ds \leq \Theta(R) \text{ for all } R \in [1, \bar{R}],
\]

provided that \( \Theta \) is defined as above.

The final function \( \eta \) must be chosen to tend to zero slowly enough to satisfy the requirements of the proofs of both (39) and (40).

\( \square \)

6 A concentration compactness result

Let us start by making precise the previously mentioned notions of uniform integrability and concentration at a point.

**Definition 4.** Let \( (X, S, \mu) \) be a measure space, and let \( p \in [1, \infty) \). A subset \( \mathcal{U} \) of \( L^p(X) \) is called **uniformly integrable** of order \( p \) if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for every measurable subset \( A \) of \( X \) for which \( \mu(A) < \delta \),

\[
\int_A |f|^p d\mu < \epsilon \text{ for every } f \in \mathcal{U}.
\]

It is straightforward to check that if \( \mathcal{U} \) is a bounded subset of \( L^p(X) \), \( \mathcal{U} \) is uniformly integrable of order \( p \) if and only if

\[
\lim_{R \to \infty} \int_{\{|f| > R\}} |f|^p d\mu = 0
\]

uniformly with respect to \( f \in \mathcal{U} \).

It is well known that if the measure space is finite, uniform integrability coupled with a weaker form of convergence is enough to ensure strong convergence.
Proposition 15. Suppose $\mu(X) < \infty$, and let $p \in [1, \infty)$. Let $\{f_n\}$ be a sequence in $L^p(X)$ and let $f \in L^p(X)$. The sequence $\{f_n\}$ converges to $f$ in $L^p$ if and only if

(i) the sequence $\{f_n\}$ converges in measure to $f$, and
(ii) the family $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable of order $p$.

Proof. We prove only the direction that is of use to us (the if part). The assumptions, together with the fact that any family consisting of a single function is automatically uniformly integrable, make it clear that the family $\{f_n - f : n \in \mathbb{N}\}$ is also uniformly integrable of order $p$. Given $\epsilon > 0$,

$$\int_{|f - f_n| < \epsilon} |f - f_n|^p d\mu \leq \epsilon^p \mu(X).$$

On the other hand, $\mu(\{|f - f_n| \geq \epsilon\}) \to 0$ as $n \to \infty$ because of the convergence in measure, and so

$$\lim_{n \to \infty} \int_{|f - f_n| \geq \epsilon} |f - f_n|^p d\mu = 0$$

by definition of uniform integrability. The conclusion follows. \qed

Definition 5. For $p = \gamma(s_0) \in \Gamma$, we say that a sequence $\{f_n\}$ of functions in $L^2(\sigma)$ satisfying $\|f_n\|_2 \to 1$ as $n \to \infty$ concentrates at $p$ if for every $\epsilon, r > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\int_{|s - s_0| \geq r} |f_n(\gamma(s))|^2 ds < \epsilon.$$

The following concentration-compactness result is a consequence of Proposition 13.

Proposition 16. Let $\{f_n\}$ be an extremizing sequence for (2) of nonnegative functions in $L^2(\sigma)$. Then there exists a subsequence, again denoted $\{f_n\}$, and a decomposition $f_n = F_n + G_n$, where $F_n$ and $G_n$ are nonnegative with disjoint supports, $\|G_n\|_2 \to 0$, and $\{F_n\}$ satisfies either

(41) $\{F_n : n \in \mathbb{N}\}$ is uniformly integrable of order 2, or

(42) $\{F_n\}$ concentrates at a point of $\Gamma$.

Proof. Apply Proposition 13 to each element of the sequence $\{f_n\}$ to get a decomposition $f_n = F_n + G_n$, where $F_n, G_n \geq 0$ have disjoint supports and $\|G_n\|_2 \to 0$. For each $n$, there exists a cap $\mathcal{C}_n = \mathcal{C}(\gamma(s_n), r_n)$ such that $F_n$ is upper normalized with respect to $\mathcal{C}_n$. It is important to note that Proposition 13 yields a
uniform statement in \( n \), i.e., the gauge function \( \Theta \) in conditions (28) and (29) can be chosen independently of \( n \). Let \( r^* := \limsup_{n \to \infty} r_n \).

If \( r^* > 0 \), let \( \{ r_{n_k} \} \) be a subsequence which converges to \( r^* \). Renaming, if necessary, we may assume that \( \lim_{n \to \infty} r_n = r^* \). Choosing \( N \in \mathbb{N} \) so large that \( n \geq N \) implies \( r_n \geq r^*/4 \), we have

\[
\int_{\{ F_n > R \}} F_n(\gamma(s))^2 ds \leq \int_{\{ F_n > \sqrt{r^*} r_n^{-1/2} \}} F_n(\gamma(s))^2 ds \leq \Theta \left( \frac{R \sqrt{r^*}}{2} \right),
\]

which tends to 0 as \( R \to \infty \), uniformly in \( n \). In other words, the sequence \( \{ F_n \} \) is uniformly integrable of order 2.

If \( r^* = 0 \), choose a subsequence \( \{ s_{n_k} \} \) converging to some \( s^* \in [0, \ell] \). Again renaming, if necessary, we may assume that \( \lim_{n \to \infty} s_n = s^* \). Let \( \epsilon, r > 0 \) be given. Start by choosing \( N_1 = N_1(\epsilon) \) such that \( |s_n - s^*| \leq r/2 \) if \( n > N_1 \). Then \( |s - s'| \geq r \) implies \( |s - s_n| \geq r/2 \) if \( n > N_1 \). Choose \( R = R(\epsilon) \) such that \( \Theta(R) < \epsilon \). Finally, choose \( N_2 = N_2(\epsilon, r) \) such that \( r_n \leq r/2R \) if \( n > N_2 \). Such \( N_2 \) exists since \( \lim_{n \to \infty} r_n = 0 \). If \( n > \max\{ N_1, N_2 \} \),

\[
\int_{|s - s'| \geq r} F_n(\gamma(s))^2 ds \leq \int_{|s - s_n| \geq \frac{r}{2}} F_n(\gamma(s))^2 ds \leq \int_{|s - s_n| \geq r_n} F_n(\gamma(s))^2 ds \leq \Theta(R) < \epsilon,
\]

i.e., the sequence \( \{ F_n \} \) concentrates at \( \gamma(s^*) \).

Fanelli, Vega and Visciglia [11] proved the following interesting modification of a well-known result of Brézis and Lieb [2] which does not require the a.e. pointwise convergence of the sequence of functions \( \{ h_n \} \).

**Proposition 17** ([11]). Let \( \mathcal{H} \) be a Hilbert space and \( T \) be a bounded linear operator from \( \mathcal{H} \) to \( L^p(\mathbb{R}^d) \) for some \( p \in (2, \infty) \). Let \( \{ h_n \} \in \mathcal{H} \) be such that

(i) \( \| h_n \|_{\mathcal{H}} = 1 \),

(ii) \( \lim_{n \to \infty} \| Th_n \|_{L^p(\mathbb{R}^d)} = \| T \| \),

(iii) \( h_n \to h \neq 0 \),

(iv) \( Th_n \to Th \) a.e. in \( \mathbb{R}^d \).

Then \( h_n \to h \) in \( \mathcal{H} \); in particular, \( \| h \|_{\mathcal{H}} = 1 \) and \( \| Th \|_{L^p(\mathbb{R}^d)} = \| T \| \).

We apply Proposition 17 to the adjoint Fourier restriction operator on \( \Gamma \) with \( \mathcal{H} = L^2(\sigma) \). We lose no generality in assuming that conditions (i) and (ii) are automatically satisfied by any extremizing sequence \( \{ f_n \} \). After passing to a subsequence if necessary, we may assume that \( \{ f_n \} \) converges weakly in \( L^2(\sigma) \) by Alaoglu’s theorem. If \( f_n \to f \) in \( L^2(\sigma) \), then condition (iv) follows because \( \sigma \) is compactly supported. Thus Proposition 17 states that for compactly supported measures, the only obstruction to the existence of extremizers is the possibility that every \( L^2 \) weak limit of any extremizing sequence be 0.
The advantage of working with nonnegative extremizing sequences in this context first appeared in the work of Kunze [15]. The following is the sole step in the analysis which works only for nonnegative extremizing sequences.

**Lemma 18.** Let \( \{f_n\} \) and \( \{F_n\} \) be as in Proposition 16. Suppose that \( \{F_n\} \) satisfies condition (41). Then \( \{f_n\} \) is precompact in \( L^2(\sigma) \).

**Proof.** By assumption, the sequence \( \{F_n\} \) consists of nonnegative functions and is uniformly integrable of order 2. Moreover, \( \|F_n\|_2 \to 1 \) as \( n \to \infty \).

We first show that every \( L^2 \) weak limit of \( \{F_n\} \) is nonzero. The set of \( L^2 \) weak limits of \( \{F_n\} \) is clearly nonempty. We can assume, possibly after extraction of a subsequence, that \( F_n \rightharpoonup F \) for some \( F \in L^2(\sigma) \). Suppose that \( F = 0 \) a.e. on \( \Gamma \). Then

\[
\int_\Gamma F_n d\sigma \to \int_\Gamma F d\sigma = 0 \quad \text{as} \quad n \to \infty.
\]

Since \( F_n \geq 0 \), this means that the sequence \( \{F_n\} \) converges to 0 in \( L^1(\sigma) \), and thus \( F_n \to 0 \) in measure. In view of Proposition 15, \( F_n \to 0 \) in \( L^2(\sigma) \), and so \( 1 = \|F_n\|_2 \to 0 \) as \( n \to \infty \), a contradiction. Thus \( F \neq 0 \), as was to be shown.

We use this to prove that the sequence \( \{f_n\} \) is precompact. Since \( f_n = F_n + G_n \),

\[
\|\hat{F_n}\|_6 \geq \|\hat{f_n}\|_6 - \|\hat{G_n}\|_6 \geq \|\hat{f_n}\|_6 - C[\Gamma]\|G_n\|_2.
\]

It follows that \( \|\hat{F_n}\|_6 \to C[\Gamma] \) as \( n \to \infty \). This means that \( \{F_n\} \) is itself an extremizing sequence. By the previous paragraph, we can assume, possibly after extraction of a subsequence, that \( F_n \rightharpoonup F \) for some nonzero \( F \in L^2(\sigma) \). Then an application of Proposition 17 (with \( d = 2, p = 6, H = L^2(\sigma) \) and \( T = \text{Fourier extension operator on } \Gamma \) defined by \( Tf := \hat{f}\sigma ) \) allows us to conclude that \( f_n \to F \) in \( L^2(\sigma) \) as \( n \to \infty \), and so \( \{f_n\} \) is precompact. \( \square \)

We are done with the proof of Theorem 1 once we show that condition (42) in Proposition 16 cannot be satisfied, and this is the subject of the next two sections.

### 7 Exploring concentration

We start by recalling some aspects of Foschi’s work [13]. Consider the parabola

\[
\mathbb{P} := \{ (y, z) \in \mathbb{R}^2 : z = y^2 \}
\]

equipped with projection measure\( ^3d\sigma_{\mathbb{P}} := dy \) instead of arclength measure. Then

\[
\|f_{\mathbb{P}}\|_{L^q(\mathbb{R}^2)} \leq C_F \|f\|_{L^2(\sigma_{\mathbb{P}})},
\]

\( ^3 \)See [7] and the references therein for a discussion of why this measure is natural from a geometric point of view.
where again $C_F$ denotes the optimal constant. Foschi showed that extremizers exist for the inequality (43) and computed the optimal constant

$$C_F = \left(\frac{2\pi}{12}\right)^{1/2}. \tag{44}$$

An example of one such extremizer is given by the Gaussian $G(y) := e^{-y^2}$. Other extremizers are obtained from $G$ by space-time translations, parabolic dilations, space rotations, phase shifts and Galilean transformations.

A straightforward scaling argument shows that for the dilated parabola $P_\mu := \{ (y, z) \in \mathbb{R}^2 : z = \frac{\mu y^2}{2} \}$, again equipped with projection measure $d\sigma_{P_\mu} = dy$, the optimal constant in the inequality

$$\| \widehat{f_{P_\mu}} \|_{L^6(\mathbb{R}^2)} \leq C_F[\mu] \| f \|_{L^2(\sigma_{P_\mu})} \tag{45}$$

satisfies $C_F[\mu] = C_F[1] \mu^{-1/6}$; in particular, $C_F[1] = (2\pi)^{1/2}3^{-1/12}$.

Since projection measure can be regarded as a limit of arclength measures, the analysis of extremizers for inequality (45) is of significance for our discussion. If an extremizing sequence $\{ f_n \}$ for inequality (2) concentrates at a point $\gamma(s) \in \Gamma$, then the sequence consisting of certain natural transformations of $f_n$ to functions $\tilde{f}_n$ (each $\tilde{f}_n$ being defined on the limiting parabola) is also an extremizing sequence for (45) with $\mu = \kappa(\gamma(s))$. To see why this is the case, denote by $T_{s,r}$ the restriction of the Fourier extension operator to a given cap $\mathcal{C} = \mathcal{C}(s, r) \subset \Gamma$:

$$T_{s,r}f(x, t) = \int_{\mathcal{C}} f(y)e^{-i(x,t) \cdot y}d\sigma(y) = \int_{s-r}^{s+r} f(\gamma(s))e^{-i(x,t) \cdot \gamma(s)}ds.$$

We are interested in the operator norm $\|T_{s,r}\| := \sup_{0 \neq f \in L^2(\sigma)} \|T_{s,r}f\|_6 \|f\|^{-1}_{L^2(\sigma)}$, and prove the following result.

**Proposition 19.** For every $s \in (0, \ell)$,

$$\lim_{r \to 0^+} \|T_{s,r}\| = C_F[\kappa(\gamma(s))].$$

**Proof.** Fix $s \in (0, \ell)$ and let $\kappa := \kappa(\gamma(s))$. That the left hand side is greater than or equal to the right hand side can be easily seen by taking the function $G(y) = e^{-\kappa y^2/2}$ and considering the dilated family $G_\delta(y) = \delta^{-1/2}G(\delta^{-1}y)$ for $\delta > 0$; for details, see Section 8.3 below.
So we focus on proving the reverse inequality. Let \( \sigma_{s,r} \) denote the restriction of arclength measure \( \sigma \) to the cap \( \mathcal{C}(s, r) \), and denote the triple convolution of \( \sigma_{s,r} \) with itself by \( \sigma_{s,r}^{(s)} := \sigma_{s,r} \ast \sigma_{s,r} \ast \sigma_{s,r} \). Then

\[
\|T_{s,r} f\|_6^6 = (2\pi)^2 \iint_{\mathbb{R}^2} |f\sigma_{s,r} \ast f\sigma_{s,r} \ast f\sigma_{s,r}(\xi, \tau)|^2 d\xi d\tau \\
\leq (2\pi)^2 \iint |f|^2 \sigma_{s,r} \ast |f|^2 \sigma_{s,r} \ast |f|^2 \sigma_{s,r}(\xi, \tau) \sigma_{s,r} \ast \sigma_{s,r} \ast \sigma_{s,r}(\xi, \tau) d\xi d\tau \\
\leq (2\pi)^2 \sup_{(\xi, \tau) \in \text{supp}(\sigma_{s,r}^{(s)})} \sigma_{s,r}^{(s)}(\xi, \tau) \cdot \iint |f|^2 \sigma_{s,r} \ast |f|^2 \sigma_{s,r} \ast |f|^2 \sigma_{s,r}(\xi, \tau) d\xi d\tau \\
= (2\pi)^2 \|\sigma_{s,r}^{(s)}\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^2(s,r)}^6,
\]

where we have used Hölder’s inequality twice. It is enough to show that

\[
\|\sigma_{s,r} \ast \sigma_{s,r} \ast \sigma_{s,r}\|_\infty \to \frac{C_F[\kappa]^6}{(2\pi)^2} \text{ as } r \to 0^+.
\]

After applying a rigid motion\(^4\) of \( \mathbb{R}^2 \) to a cap \( \mathcal{C} = \mathcal{C}(s, r) \), we may parametrize it by

\[
\tilde{\gamma}_{s,r} : I_r \to \mathbb{R}^2, \quad y \mapsto \left( y, g(y) = \frac{\kappa}{2}y^2 + \phi(y) \right),
\]

where \( I_r \) is an interval centered at the origin of length \( \asymp r, \kappa = g''(0)(1+g'(0)^2)^{-3/2} \) is the curvature of \( \Gamma \) at \( \gamma(s) \), and \( \phi \) is a smooth real-valued function satisfying \( \phi(y) = O(|y|^3) \) as \( |y| \to 0 \). Let \( \eta_r \in C^\infty_0(\mathbb{R}) \) be a mollified version of the characteristic function of the interval \( I_r \); specifically, fix \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta \equiv 1 \) on \([-1, 1]\) and \( \eta(\xi) = 0 \) if \( |\xi| \geq 2 \) and define \( \eta_r := \eta(2|I_r|^{-1}) \).

With these definitions, for \( |\xi| \leq |I_r|/2 \), we have

\[
\sigma_{s,r}(\xi, \tau) = G_r(\xi)\delta(\tau - g(\xi))d\xi d\tau,
\]

where \( G_r(\xi) := (1 + g'(\xi)^2)^{1/2} \eta_r(\xi) \) is a smooth function supported on the set \( \{ \xi \in \mathbb{R} : |\xi| \leq r \} \). Observe that \( G_r \) (and therefore \( \sigma_{s,r} \)) depends also on \( s \), even though this is not explicitly indicated by the notation, with uniform bounds on \( s \) and appropriately uniform dependence on \( r \) after dilations. Following [13] and [22], we compute

\[
\sigma_{s,r}^{(s)}(\xi, \tau) = \iint G_r(\omega_1)G_r(\omega_2 - \omega_1)G_r(\xi - \omega_2)\delta(\tau - g(\omega_1) - g(\omega_2 - \omega_1) - g(\xi - \omega_2))d\omega_1 d\omega_2 d\omega_3 \\
= \iint_{|\omega| \leq r} G_r(\omega_1)G_r(\omega_2)G_r(\omega_3)\delta\left( \frac{\tau - g(\omega_1) - g(\omega_2) - g(\omega_3)}{\xi - \omega_1 - \omega_2 - \omega_3} \right)d\omega_1 d\omega_2 d\omega_3.
\]

\(^4\) For a more detailed discussion of this procedure, see Section 8.1 below.
Change variables $\omega = O \cdot \zeta$, where $O \in SO(3)$ is the orthogonal matrix

$$O = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$ 

Under this transformation,

$$\langle (1, 1, 1), \omega \rangle = \langle (1, 1, 1), O \cdot \zeta \rangle = \langle O^* \cdot (1, 1, 1), \zeta \rangle = \sqrt{3} \langle (1, 0, 0), \zeta \rangle = \sqrt{3} \zeta_1.$$

The integral becomes

$$\sigma_{s,r}^{(s^3)}(\xi, \tau) = \iint_{|\zeta| \leq r} G_r \left( \frac{\xi_1}{\sqrt{3}} - \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) G_r \left( \frac{\xi_1}{\sqrt{3}} + \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) G_r \left( \frac{2\xi_3}{\sqrt{6}} \right) \delta \left( \tau - g \left( \frac{\xi_1}{\sqrt{3}} - \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) - g \left( \frac{\xi_1}{\sqrt{3}} + \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) - g \left( \frac{2\xi_3}{\sqrt{6}} \right) \right) d\xi_1 d\xi_2 d\xi_3.$$

Renaming variables and setting

$$G_r(\xi_1; \xi_2, \xi_3) = G_r \left( \frac{\xi_1}{\sqrt{3}} - \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) G_r \left( \frac{\xi_1}{\sqrt{3}} + \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) G_r \left( \frac{2\xi_3}{\sqrt{6}} \right)$$

simplifies this to

$$\sigma_{s,r}^{(s^3)}(\xi, \tau) = \frac{\chi(|\xi| \leq r)}{\sqrt{3}} \int_{|\xi_1, \xi_2, \xi_3| \leq r} G_r(\xi_1; \xi_2, \xi_3) \cdot \delta \left( \tau - \frac{\kappa}{6} \xi_2^2 - \frac{\kappa}{2} |\xi|^2 - \phi \left( \frac{\xi_1}{\sqrt{3}} - \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) - \phi \left( \frac{\xi_1}{\sqrt{3}} + \frac{\xi_2}{\sqrt{2}} - \frac{\xi_3}{\sqrt{6}} \right) - \phi \left( \frac{2\xi_3}{\sqrt{6}} \right) \right) d\xi_2 d\xi_3.$$

Introducing polar coordinates on the $(\xi_2, \xi_3)$-plane yields

$$\sigma_{s,r}^{(s^3)}(\xi, \tau) = \frac{\chi(|\xi| \leq r)}{\sqrt{3}} \int_0^{2\pi} \int_0^r \tilde{G}_r(\xi; \rho, \theta) \cdot \delta \left( \tau - \frac{\kappa}{6} \xi_2^2 - \frac{\kappa}{2} \xi_3^2 - 3\phi \left( \frac{\xi_1}{\sqrt{3}} \right) - \psi(\xi; \rho, \theta) \right) \rho d\rho d\theta,$$

where

$$\tilde{G}_r(\xi; \rho, \theta) = G_r \left( \frac{\xi_1}{\sqrt{3}} - \rho \left( \frac{\cos \theta}{\sqrt{2}} + \frac{\sin \theta}{\sqrt{6}} \right) \right) G_r \left( \frac{\xi_1}{\sqrt{3}} + \rho \left( \frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{6}} \right) \right) \cdot G_r \left( \frac{\xi_1}{\sqrt{3}} + \frac{2\rho}{\sqrt{6}} \right),$$

and

$$\psi(\xi; \rho, \theta) = \phi \left( \frac{\xi_1}{\sqrt{3}} - \rho \left( \frac{\cos \theta}{\sqrt{2}} + \frac{\sin \theta}{\sqrt{6}} \right) \right) - \phi \left( \frac{\xi_1}{\sqrt{3}} + \rho \left( \frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{6}} \right) \right).$$
and

\begin{equation}
\psi(\xi; \rho, \theta) = \frac{\kappa}{2} \rho^2 - 3\phi\left(\frac{\xi}{3}\right) + \phi\left(\frac{\xi}{3} - \rho\left(\frac{\cos \theta}{\sqrt{2}} + \frac{\sin \theta}{\sqrt{6}}\right)\right) + \\
\phi\left(\frac{\xi}{3} + \rho\left(\frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{6}}\right)\right) + \phi\left(\frac{\xi}{3} + \frac{2}{\sqrt{6}} \rho \sin \theta\right).
\end{equation}

We prepare to change variables again. Recall the assumption \(\kappa > 0\). Note that \(\psi(\xi; 0, \theta) = 0\) and \(\psi(\xi; \rho, \theta) > 0\) for sufficiently small \(\rho > 0\). Moreover, a calculation shows that the same thing happens with first derivatives: \(\partial_\rho \psi(\xi; 0, \theta) = 0\) and \(\partial_\rho \psi(\xi; \rho, \theta) > 0\) if \(\rho > 0\) is sufficiently small. We also have \(\partial^2_\rho \psi(\xi; 0, \theta) = \kappa + \phi''(\xi/3) = g''(\xi/3)\).

For \(u \geq 0\), set \(\rho = \rho(u) := \psi^{-1}(u)\), and compute

\begin{equation}
\sigma_{s,r}^{(s,3)}(\xi, \tau) = \frac{\chi(|s| \lesssim r)}{\sqrt{3}} \int_0^{2\pi} \int_{\rho \leq C \psi} \tilde{G}_r(\xi; \rho(u), \theta) \delta\left(\tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right) - u\right) \\
\cdot \frac{\rho(u)}{\partial_\rho \psi(\rho(u))} d\rho d\theta
\end{equation}

\begin{equation}
= \frac{\chi(|s| \lesssim r)}{\sqrt{3}} \int_0^{2\pi} \tilde{G}_r(\xi; \rho(\tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right)), \theta) \\
\cdot \frac{\rho(\tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right))}{\partial_\rho \psi(\rho(\tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right)))} d\theta.
\end{equation}

Note that for each \(\theta \in [0, 2\pi]\), the integrand in (48) is supported in the region

\[ \left\{ (\xi, \tau) \in \mathbb{R}^2 : 0 \leq \tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right) \leq C \psi(\xi; r, \theta) \right\}, \]

where the constant \(C\) is so large that the restriction \(u \leq C \psi(\xi; r, \theta)\) in the inner integral of (47) becomes redundant because of support limitations on factors present in its integrand. It is clear from expression (48) that the restriction of \(\sigma_{s,r} \ast \sigma_{s,r} \ast \sigma_{s,r}\) to its support defines a continuous function of \((\xi, \tau)\) at \((0, 0)\). Indeed, \(\tilde{G}_r\) is a smooth function of compact support in the variables \(\xi, \theta\), and \((\tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right))^{1/2}\). Additionally, as \((\xi, \tau) \to (0, 0)\),

\[ \frac{\rho(\tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right))}{\partial_\rho \psi(\rho(\tau - \frac{\kappa}{6} \xi^2 - 3 \phi\left(\frac{\xi}{3}\right)))} \to \frac{1}{\partial_\rho \psi(\rho(0))} = \frac{1}{g''(0)}. \]
If \((\xi, \tau) \in \text{supp}(\sigma_{r,s} * \sigma_{r,s} * \sigma_{r,s})\) and \(r \to 0^+\), then \((\xi, \tau) \to (0, 0)\). It follows that
\[
\lim_{r \to 0^+} \|\sigma_{r,s} * \sigma_{r,s} * \sigma_{r,s}\|_{\infty} = \frac{1}{\sqrt{3}} \int_0^{2\pi} \frac{1}{\epsilon_{\rho}^\prime(0; \rho(0), \theta)} d\theta.
\]
\[
= \frac{2\pi}{\sqrt{3}} \frac{1 + g'(0)^2}{g''(0)} = \frac{2\pi}{\sqrt{3}} \frac{1}{\kappa} = \frac{C_F[\kappa]^6}{(2\pi)^2},
\]
as desired. \(\square\)

An immediate consequence is that an extremizing sequence which concentrates must do so at a point of minimal curvature.

**Corollary 20.** Let \(\{f_n\} \subset L^2(\sigma)\) be an extremizing sequence of nonnegative functions for inequality \((2)\). Suppose that \(\{f_n\}\) concentrates at a point \(\gamma(s) \in \Gamma\). Then \(\kappa(\gamma(s)) = \lambda\).

**Corollary 21.** Let \(\{f_n\}\) and \(\{F_n\}\) be as in Proposition 16. Suppose that \(\{F_n\}\) concentrates at a point \(\gamma(s) \in \Gamma\). Then \(\kappa(\gamma(s)) = \lambda\).

### 8 Comparing optimal constants

As mentioned previously, a potential obstruction to the existence of extremizers for inequality \((2)\), and certainly to the precompactness of arbitrary nonnegative extremizing sequences, is the possibility that for an extremizing sequence satisfying \(\|f_n\|_{L^2(\sigma)} = 1, \ |f_n|^2\) could conceivably converge weakly to a Dirac mass at a point on the curve. Indeed, let \(p \in \Gamma\). The osculating parabola of \(\Gamma\) at \(p\) is \(P_{\kappa(p)}\). Foschi’s work implies that were \(C[\Gamma]\) equal to \(C_F[\kappa(p)]\), there would necessarily exist extremizing sequences of the type just described. Therefore, an essential step in our analysis is to determine under which conditions
\[
C[\Gamma] > \max_{p \in \Gamma} C_F[\kappa(p)] = C_F[\lambda].
\]
The main goal of this section is to prove the following proposition.

**Proposition 22.** Let \(\Gamma \subset \mathbb{R}^2\) be an arc satisfying the conditions of Theorem 1. Then \(C[\Gamma] > C_F[\lambda]\).

#### 8.1 Introducing local coordinates.

Let \(p_0 \in \Gamma\) be a point of minimum curvature, i.e., such that \(\kappa(p_0) = \lambda\). Assume that \(p_0\) is not an endpoint of \(\Gamma\); we postpone the discussion of the validity of this assumption to the end of this section. By translating the curve, if necessary, we may assume without loss of generality
that $p_0 = (0, 0)$. Possibly after a suitable rotation, the arc $\Gamma$ can be parametrized in a neighborhood of the origin by

$$\tilde{\gamma} : I \to \mathbb{R}^2, \ y \mapsto (y, h(y)),$$

where $h(y) = \frac{\lambda^2 y^2}{2} + ay^4 + \psi(y)$ and $\psi$ is a real-valued smooth function satisfying $\psi(y) = O(|y|^5)$ as $|y| \to 0$. The parameter $a$ is a function of the second derivative of the curvature with respect to arclength at 0; see formula (51) below. We take $I \subseteq \mathbb{R}$ to be an interval centered at the origin to be chosen later as a function of $\lambda$, $a$ and $\psi$. Finally, let $\eta_I \in C_0^\infty(\mathbb{R})$ be a mollified version of the characteristic function of $I$ such that $\eta_I \equiv 1$ on $I$ and $\eta_I \equiv 0$ outside $2 \cdot I$. As before, we accomplish this by fixing $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta \equiv 1$ on $[-1, 1]$ and $\eta(y) = 0$ if $|y| \geq 2$, and defining $\eta_I := \eta(2 |I| - 1 \cdot 2)$. These data determine a compact arc $\tilde{\gamma}(I) := \tilde{\Gamma} \subset \Gamma$ in the plane which comes equipped with arclength measure $\tilde{\sigma}$ given, for $y \in I$, by

$$d\tilde{\sigma}(y) = (1 + h'(y)^2)^{1/2} \eta_I(y) dy.$$  

We find that the first two nonzero terms in the Taylor expansion of $d\tilde{\sigma}(y)$ around $y = 0$

$$d\tilde{\sigma}(y) = \left(1 + \frac{\lambda^2}{2} y^2 + O(y^4)\right) \eta_I(y) dy$$

are independent of $a$ and $\psi$.

On the other hand, the curvature of $\tilde{\Gamma}$ at a point $\tilde{\gamma}(y)$ is given, for small $y$, by

$$\kappa(y) = \frac{h''(y)}{(1 + h'(y)^2)^{3/2}} = \lambda + (12a - \frac{3\lambda^3}{2}) y^2 + O(\lambda, a, \psi(y^4)).$$

Thus $\kappa(0) = h''(0) = \lambda > 0$. For $\kappa$ to have a minimum at $y = 0$ it is necessary that $a \geq (\lambda/2)^3$. Let $s$ denote the arclength parameter for $\tilde{\Gamma}$. By a straightforward application of the chain rule, we have

$$\frac{d^2 \kappa}{ds^2}(0) = \frac{d^2 \kappa}{dy^2}(0) = 24a - 3\lambda^3;$$

so hypothesis (3) in Theorem 1 is equivalent to $a < \frac{3}{2} \left(\frac{\lambda}{2}\right)^3$. All in all, we have

$$\left(\frac{\lambda}{2}\right)^3 \leq a < \frac{3}{2} \left(\frac{\lambda}{2}\right)^3.$$  

In Subsections 8.2 and 8.3 below, we again denote by $C[\tilde{\Gamma}] = C[\tilde{\Gamma}; \lambda, \psi, I]$ the optimal constant in the inequality

$$\| \hat{f} \tilde{\sigma} \|_{L^6(\mathbb{R}^2)} \leq C[\tilde{\Gamma}] \|f\|_{L^2(\tilde{\sigma})},$$

There is no cubic term in the expression for $h$ because, by assumption, the curvature has a minimum at $\gamma(0) = (0, 0) = p_0$. Constant and linear terms were likewise removed via the affine change of variables described above.
8.2 The unperturbed case. If \( a = \psi = 0 \), we are dealing with the unperturbed parabola \( \mathbb{P}_\lambda \). As we have already mentioned, the corresponding optimal constant satisfies \( C_F[\lambda] = C_F[1] \lambda^{-1/6} \), and examples of extremizers are given by Gaussian functions \( e^{-\rho^2/2} \) for \( \rho > 0 \).

Set \( G_0(y) := e^{-\lambda y^2/2} \), and consider functions of the form \( f = G_0 + \phi \) with \( \phi \in L^2(\sigma_{\mathbb{P}_\lambda}) \). Consider the corresponding functional

\[
\phi \mapsto C_F[\lambda]^6 \left( \int_\mathbb{R} |G_0 + \phi|^2 \, dy \right)^{3/2} - \int_\mathbb{R}^2 |G_1 + \phi|^6 \, dx \, dt \geq 0,
\]

where \( G_1(x, t) := \widetilde{G_0 \sigma_{\mathbb{P}_\lambda}}(-x, t) \) and \( \phi_1(x, t) := \widetilde{\phi \sigma_{\mathbb{P}_\lambda}}(-x, t) \). The former can be computed explicitly as

\[
G_1(x, t) = \widetilde{G_0 \sigma_{\mathbb{P}_\lambda}}(-x, t) = \int_\mathbb{R} G_0(y) e^{-itx^2} e^{ixy} \, dy
\]

\[
= \int_\mathbb{R} e^{-(1+it)x^2} e^{ixy} \, dy
\]

\[
= \left( \frac{2\pi}{\lambda} \right)^{1/2} (1+it)^{-1/2} e^{-\frac{x^2}{2it+i\lambda}}.
\]

It is easy to check that \( G_1 \in L^6(\mathbb{R}^2) \) if and only if \( p > 4 \), but we are interested in \( L^6 \) norms. Since \( \widetilde{G_0 \sigma_{\mathbb{P}_\lambda}} \) is an even function of \( x \),

\[
\|G_1\|_{L^6(\mathbb{R}^2)} = C_F[\lambda] \|G_0\|_{L^2(\mathbb{P}_\lambda)}.
\]

We follow the work of [9] and expand the functional (54) up to second order, collecting the terms that do not depend on \( \phi \) in \( I \), those that depend linearly on the real and imaginary parts of \( \phi \) in \( II \), and those that depend quadratically on the real and imaginary parts of \( \phi \) in \( III \). This yields

\[
I = C_F[\lambda]^6 \|G_0\|_6^6 - \|G_1\|_6^6,
\]

\[
II = 6C_F[\lambda]^6 \|G_0\|_2^4 \mathfrak{R} \int G_0 \phi - 6\mathfrak{R} \int |G_1|^4 \mathfrak{R} \phi_1,
\]

\[
III = 3C_F[\lambda]^6 \|G_0\|_2^2 \int |\phi|^2 + 12C_F[\lambda]^6 \|G_0\|_2^2 \left( \mathfrak{R} \int G_0 \phi \right)^2
\]

\[
- 9 \int |G_1|^4 |\phi|^2 - 6\mathfrak{R} \int |G_1|^2 G_1^2 \phi_1^2.
\]

We already know from (55) that \( I = 0 \). Since \( G_0 \) is an extremizer, \( II = 0 \) as well. Finally, note that \( III \) is, by definition, a quadratic form in \( \phi \); denote it by \( Q(\phi) \). By the symmetries of the problem (respectively, multiplication by a real number, phase shift, space translation, Galilean invariance, scaling and time translation), we have

\[
Q(G_0) = Q(iG_0) = Q(yG_0) = Q(iyG_0) = Q(y^2G_0) = Q(iy^2G_0) = 0;
\]
and it is proved in [9] that \( Q \) is positive definite in the subspace of \( L^2 \) functions which are orthogonal to the functions indicated in (59). This non-degeneracy property is not used here; rather, what is essential for our application is that \( Q(\phi) \geq 0 \) for every \( \phi \in L^2(\mathbb{R}) \). This is immediate because \( \Pi \) vanishes for every \( \phi \in L^2 \), and (54) defines a nonnegative quantity.

### 8.3 A variational calculation

In the spirit of the variational calculation in [7, Section 17], we consider the one-parameter family of trial functions given by \((G_0 + \epsilon \phi)_{0 < \epsilon \leq \epsilon_0}\) for some sufficiently small \( \epsilon_0 > 0 \) with \( G_0(y) = e^{-\lambda y^2/2} \) and \( \phi \in L^2(\mathbb{R}) \) to be chosen below such that

\[
\|\phi\|_2 = 1 \quad \text{and} \quad \int G_0 \phi = 0.
\]

For technical reasons soon to become apparent, we introduce an appropriate dilation of the cut-off \( \eta_I \) which localizes to the region \(|y| \lesssim \epsilon \log \frac{1}{\epsilon} \), and define

\[
f_\epsilon(y) := \epsilon^{-1/2}(G_0 + \epsilon \phi)(\epsilon^{-1}y)\eta_I \left( \frac{1}{\epsilon \log \frac{1}{\epsilon}} y \right).
\]

Notice that the family \((f_\epsilon)_{\epsilon > 0}\) is \( L^2 \)-normalized in the sense that

\[
\|f_\epsilon\|_{L^2(\mathcal{G})}^2 = \|G_0\|^2_2 + O(\epsilon^2) \quad \text{as} \quad \epsilon \to 0^+.
\]

Consider

\[
\Xi(\epsilon) = C_F[\lambda]^6 \|f_\epsilon\|^6_{L^2(\mathcal{G})} - \|\hat{f}_\epsilon \sigma\|^6_6.
\]

This is no longer a nonnegative expression by construction like (54). Let

\[
\varphi(u) := c_\lambda u e^{-i \lambda u^2/2},
\]

where the constant \( c_\lambda := (\pi/4\lambda^3)^{-1/4} \) is chosen to normalize \( \|\varphi\|_2 = 1 \). Then \( \varphi \) satisfies conditions (60) and \( Q(\phi) = 0 \). We claim that with this choice of \( \varphi \), for every \( \lambda > 0 \), \( a \in \mathbb{R} \) satisfying (52), and real-valued smooth \( \psi \) satisfying

\[
\psi(y) = O(|y|^5) \quad \text{as} \quad |y| \to 0,
\]

\( \Xi \) is a strictly concave function of \( \epsilon \) in a sufficiently small half-neighborhood of 0, provided the interval \( I \) is chosen sufficiently small (as a function of \( \lambda, a \) and \( \psi \)). Once we prove this, we are able to conclude that \( C_F[\lambda] < C[\hat{\Gamma}] \), and Proposition 22 follows.

Start by noting that \( \lim_{\epsilon \to 0} \Xi(\epsilon) = 0 \) and \( \Xi'(0) = 0 \). Indeed,

\[
\lim_{\epsilon \to 0} \Xi(\epsilon) = C_F[\lambda]^6 \lim_{\epsilon \to 0} \|f_\epsilon\|^6_{L^2(\mathcal{G})} - \lim_{\epsilon \to 0} \|\hat{f}_\epsilon \sigma\|^6_6 = C_F[\lambda]^6 \|G_0\|^6_2 - \|G_1\|^6_6 = 0
\]
for every \( a \in \mathbb{R} \) and \( \lambda > 0 \). On the other hand, explicit computations show that
\[
\partial \epsilon \mid_{\epsilon = 0} \| f \|_{L^2(\sigma)}^6 = 6 \| G_0 \|_2^4 \Re \int G_0 \phi
\]
and
\[
\partial \epsilon \mid_{\epsilon = 0} \| \overline{f_\epsilon \phi} \|_6^6 = 6 \Re \int |G_1|^4 \overline{G_1} \phi_1.
\]
Since \( \mathbf{II} = 0 \) and \( G_0 \perp \phi \),
\[
(65) \quad \Re \int |G_1|^4 \overline{G_1} \phi_1 = C_F [\lambda] \| G_0 \|_2^4 \Re \int G_0 \phi = 0;
\]
and it follows that \( \Xi'(0) = 0 \), as claimed.

Useful information comes from looking at second variations. Our strategy is to compute the second derivatives with respect to \( \epsilon \) (at \( \epsilon = 0 \)) of good enough approximations to the two terms appearing in the definition of \( \Xi \). We start by analyzing the more complicated one.

### 8.3.1 The term \( \| \overline{f_\epsilon \phi} \|_6 \)
To make the notation less cumbersome, we introduce the parametrization
\[
\overline{\gamma}_\epsilon : \mathbb{R} \to \mathbb{R}^2, \quad u \mapsto (u, \overline{\eta}_\epsilon(u) = \frac{\lambda u^2}{2} + a\epsilon^2 u + 2 \epsilon^2 \psi(\epsilon u)).
\]
Changing variables \( y = \epsilon u \), we have
\[
\overline{f_\epsilon \phi}(x, t) = \int \overline{f_\epsilon}(y) e^{-i(x,t) \cdot (y, \psi(y))} (1 + h'(y)^2)^{1/2} \eta_t(y) dy
\]
\[= \epsilon^{1/2} \int \epsilon^{1/2} (G_0 + \epsilon \phi)(u) e^{-i(\epsilon, \epsilon^2 \psi(\epsilon u))} (1 + h'(\epsilon u)^2)^{1/2} \eta_t \left( \frac{1}{\log \frac{1}{\epsilon}} u \right) du.
\]
Consider the approximate version of
\[
\overline{v_\epsilon}(x, t) := \epsilon^{-1/2} \overline{f_\epsilon \phi}(\epsilon^{-1} x, \epsilon^{-2} t)
\]
\[= \int (G_0 + \epsilon \phi)(u) e^{-i(\epsilon, \epsilon^2 \psi(\epsilon u))} (1 + h'(\epsilon u)^2)^{1/2} \eta_t \left( \frac{1}{\log \frac{1}{\epsilon}} u \right) du.
\]
given by a Taylor expansion of \( J_\epsilon(u) := \sqrt{1 + h'(\epsilon u)^2} \in C^\infty(I) \) and \( \overline{v_\epsilon}(x, t) \) defined by
\[
(66) \quad \overline{v_\epsilon}(x, t) := \int (G_0 + \epsilon \phi)(u) e^{-i(\epsilon, \epsilon^2 \psi(\epsilon u))} \left( 1 + \frac{\lambda^2}{2} \epsilon^2 u^2 \right) \eta_t \left( \frac{1}{\log \frac{1}{\epsilon}} u \right) du.
\]
Also, set
\[
g_\epsilon(u) := (G_0 + \epsilon \phi)(u), \quad g_\epsilon^2(u) := u^2 (G_0 + \epsilon \phi)(u), \quad \text{and} \quad d \overline{\sigma}_\epsilon(u) := \left( 1 + \frac{\lambda^2}{2} \epsilon^2 u^2 \right) du.
\]
Lemma 23. For a sufficiently small interval $I$ centered at the origin (chosen as a function of $\lambda, a, \psi$ but not $\epsilon$),

$$\|\hat{f}_\epsilon \tilde{\sigma}\|_6^6 = \|w_\epsilon\|_6^6 = \|v_\epsilon\|_6^6 + O(\epsilon^4)$$ (67)

and

$$\|f_\epsilon\|_{L^2(\tilde{\sigma})}^2 = \|g_\epsilon\|_{L^2(\tilde{\sigma}_\epsilon)}^2 + O(\epsilon^4)$$ (68)

as $\epsilon \to 0^+$.

Proof. By construction, $v_\epsilon$ is just $w_\epsilon$ with the term $J_\epsilon = (1+h'(\epsilon \cdot)^2)^{1/2}$ replaced by its Taylor approximation to order 2. For $\epsilon < 1$, set

$$G_\epsilon(u) := (1 + \frac{\lambda^2}{2} \epsilon^2 u^2)(G_0 + \epsilon \varphi)(u)$$

and

$$s_\epsilon(x, t) := -\int_\mathbb{R} G_\epsilon(u)e^{-i(x, t) \cdot \tilde{\gamma}_\epsilon(u)}(\eta\epsilon u - \eta\left(\frac{1}{\log \frac{1}{\epsilon}} u\right))du.$$ (69)

Then

$$v_\epsilon(x, t) = \hat{g}_\epsilon \sigma_{P, \epsilon}(x, t) + \frac{\lambda^2 \epsilon^2}{2} \hat{g}_\epsilon \sigma_{P, \epsilon}(x, t) + s_\epsilon(x, t),$$

where $d\sigma_{P, \epsilon} = \eta\epsilon u du$. We obtain a uniform estimate for its $L^6$ norm.

Claim 1. There exists a real constant $C$ such that $\|v_\epsilon\|_6 \leq C$ for all sufficiently small $\epsilon > 0$.

The claimed uniformity in $\epsilon$ needs to be justified: it follows from undoing the substitutions $y \mapsto \epsilon^{-1}y$ and $(x, t) \mapsto (\epsilon^{-1}x, \epsilon^{-2}t)$. Indeed,

$$\|\hat{g}_\epsilon \sigma_{P, \epsilon}\|_6 = \|\hat{f}_\epsilon \sigma P\|_6 = \|\hat{f}_\epsilon \sigma\|_6 \leq C\|G_0\|_2$$

where $d\sigma_P = \eta(y) dy$. Since $\|\epsilon^{-1/2} \hat{f}_\epsilon \sigma P(\epsilon^{-1} \cdot, \epsilon^{-2} \cdot)\|_6 = \|\hat{f}_\epsilon \sigma P\|_6$, $\|\hat{g}_\epsilon \sigma_{P, \epsilon}\|_6 = \|\epsilon^{-1/2} \hat{f}_\epsilon \sigma P(\epsilon^{-1} \cdot, \epsilon^{-2} \cdot)\|_6 = \|\hat{f}_\epsilon \sigma P\|_6 \lesssim \|\hat{f}_\epsilon\|_2 \leq C\|G_0\|_2$. 

for some $C$ independent of $\epsilon$, as claimed. Proceed similarly to get a bound $O(\epsilon^2)$
for the term involving $g_\epsilon^\flat$. Finally, define

$$g_\epsilon^\flat(u) := -G_\epsilon(u)\left(\eta_t(\epsilon u) - \eta_t\left(\frac{1}{\log \frac{1}{\epsilon}}\right)\right)$$

and notice that $s_\epsilon(x, t) = \overline{g_\epsilon^\flat \sigma P_\epsilon(x, t)}$. Estimate

$$\|s_\epsilon\|_6 \lesssim \|g_\epsilon^\flat\|_2 \lesssim \epsilon^N,$$ for all $N \in \mathbb{N},$

where the implicit constants are all independent of $\epsilon$. Thus the contribution of the third summand is likewise small, and this concludes the verification of Claim 1.

If we choose the interval $I$ (as a function of $\lambda$, $a$ and $\psi$) so small that

$$y \in I \Rightarrow |h'(y)| = |\lambda y + 4ay^3 + \psi'(y)| \leq 1,$$

the remainder

$$r_\epsilon(x, t) := w_\epsilon(x, t) - v_\epsilon(x, t)$$

will satisfy favorable bounds. Indeed, by Taylor’s theorem, we have

$$(70) \quad |r_\epsilon(x, t)| \leq C\epsilon^4 \left| \int_{\mathbb{R}} J_\epsilon'''(c_0)u^4(G_0 + \epsilon\varphi)(u)e^{-i(x,t)\cdot \tilde{\gamma}_\epsilon(u)}\eta_t\left(\frac{1}{\log \frac{1}{\epsilon}}\right)du \right|,$$

for some $c_0 \in (-\epsilon u, \epsilon u)$ and some absolute constant $C$. An argument analogous to
that used to establish Claim 1 then yields the following estimate for the remainder term.

**Claim 2.** There exists a real constant $C$ such that $\|r_\epsilon\|_6 \leq C\epsilon^4$ for all sufficiently small $\epsilon > 0$.

To finish the proof of Lemma 23, notice that $\|w_\epsilon\|_6^6 = \|v_\epsilon + r_\epsilon\|_6^6 = \|v_\epsilon\|_6^6$ plus
63 terms, all of which are $O(\epsilon^4)$ as $\epsilon \to 0^+$. This is an immediate consequence of
Hölder’s inequality together with Claims 1 and 2, for example,

$$\iint |v_\epsilon|^4 w_\epsilon r_\epsilon dxdt \leq \|v_\epsilon\|_6^5 \|r_\epsilon\|_6 \leq C\epsilon^4.$$ 

All other terms can be dealt with in a similar way, and the result follows. The
verification of (68) is easier, and we omit the details. \qed
Since we are interested only in second variations with respect to $\epsilon$ of the $L^6$ norm $\|\hat{f_\epsilon}\|_6^6$ at $\epsilon = 0$, in light of (67), it suffices to analyze $\|v_\epsilon\|_6^6$. Start by noting that

\begin{align}
(71) \quad & v_0(x, t) = G_1(x, t), \\
(72) \quad & \partial_{\epsilon} v_\epsilon|_{\epsilon=0}(x, t) = \phi_1(x, t), \\
(73) \quad & \partial_{\epsilon}^2 v_\epsilon|_{\epsilon=0}(x, t) = \lambda^2 G_2(x, t) - 2i\epsilon G_3(x, t),
\end{align}

where, as before,

$$G_1(x, t) := \hat{G_0}\sigma_{\mathcal{P}_1}(-x, t) = \left(\frac{2\pi}{\lambda}\right)^{1/2} (1 + it)^{-1/2} e^{-\frac{t^2}{2(1+it)}}$$

and

$$\phi_1(x, t) := \hat{\phi\sigma_{\mathcal{P}_1}}(-x, t) = \frac{ic}{\lambda} \left(\frac{2\pi}{\lambda}\right)^{1/2} (1 + it)^{-3/2} x e^{-\frac{\lambda x^2}{2(1+it)}}.$$ 

Additionally,

$$G_2(x, t) := [(u^2 G_0)(u)]^\wedge(-x, t) = \int_{\mathbb{R}} y^2 G_0(y)e^{-it\frac{y^2}{\lambda^2}} e^{i\lambda y} dy = 2\lambda^{-1} i \hat{\partial_t} G_1(x, t) = \left(\lambda^{-1}(1 + it)^{-1} - \lambda^{-2} x^2 (1 + it)^{-2}\right) G_1(x, t)$$

and

$$G_3(x, t) := [(u^4 G_0)(u)]^\wedge(-x, t) = \int_{\mathbb{R}} y^4 G_0(y)e^{-it\frac{y^2}{\lambda^2}} e^{i\lambda y} dy = -4\lambda^{-2} \partial_t^2 G_1(x, t) = \left(3\lambda^{-2}(1 + it)^{-2} - 6\lambda^{-3} x^2 (1 + it)^{-3} + \lambda^{-4} x^4 (1 + it)^{-4}\right) G_1(x, t).$$

**Remark 1.** The calculations to follow, which lead to formula (74) below, are largely formal and need to be justified. In particular, the fact that $\epsilon \mapsto \|v_\epsilon\|_6^6$ is twice differentiable at $\epsilon = 0$ is proved in Section 10 (Appendix 1).

As a first step in the direction of computing $\partial_{\epsilon}^2 v_\epsilon|_{\epsilon=0} \|v_\epsilon\|_6^6$, we look at

$$\partial_{\epsilon} |v_\epsilon|^6 = \partial_{\epsilon}(v_\epsilon^3 \overline{v_\epsilon^3}) = 3v_\epsilon^2(\partial_{\epsilon} v_\epsilon)\overline{v_\epsilon^3} + 3v_\epsilon^3(\partial_{\epsilon} \overline{v_\epsilon})v_\epsilon^3.$$

Recalling (71)–(72), we have $\partial_{\epsilon} v_\epsilon|_{\epsilon=0} \|v_\epsilon\|_6^6 = 6\Re(G_1^2 \overline{\phi_1 G_1^3})$; and so (65) implies

$$\partial_{\epsilon} |v_\epsilon|^6 = 6\Re \int |G_1|^4 \overline{G_1} \phi_1 dxdt = 0,$$
which we already knew. We differentiate once again and obtain
\[
\partial^2 \epsilon |v_\epsilon|^6 = 2\Re\left( 6v_\epsilon (\partial v_\epsilon)^2 |v_\epsilon|^3 + 3v_\epsilon^2 (\partial^2 v_\epsilon) |v_\epsilon|^3 + 9|v_\epsilon|^4 |\partial v_\epsilon|^2 \right),
\]
which at \( \epsilon = 0 \) equals (recall (71)–(73))
\[
\partial^2 \epsilon |_{\epsilon=0} |v_\epsilon|^6 = 2\Re\left( 6v_0 (\partial v_\epsilon) |v_\epsilon|^3 + 3v_\epsilon^2 (\partial^2 v_\epsilon) |v_\epsilon|^3 + 9|v_0|^4 |\partial v_\epsilon|^2 \right)
\]
\[
= 2\Re\left( 6G_\epsilon \varphi^2 G_1^3 + 3G_\epsilon^2 (\lambda^2 G_\epsilon - 2i\alpha G_3) G_1^3 + 9|G_1|^4 |\varphi_\epsilon|^2 \right).
\]
Thus
\[
(74) \quad \frac{1}{2} \partial^2 \epsilon |_{\epsilon=0} |v_\epsilon|^6 = 9 \int \int |G_1|^4 |\varphi_1|^2 dxdt + 6\Re \int \int |G_1|^2 \overline{G_1} \varphi_1^2 dxdt
\]
\[
+ 3\lambda^2 \Re \int \int |G_1|^4 \overline{G_1} G_2 dxdt - 6a \int \Re \left\{ it|G_1|^4 \overline{G_1} G_3 \right\} dxdt.
\]
The first two summands on the right hand side of (74) appear (with opposite signs) in the expression (58) for the quadratic form \( Q \). On the other hand, the last two summands can be explicitly evaluated, and that is our next task. The proofs of the following claims are deferred to Section 11 (Appendix 2).

Claim 3.
\[
(75) \quad 3\lambda^2 \Re \int \int_{\mathbb{R}^2} |G_1(x, t)|^4 |G_1(x, t)| G_2(x, t) dxdt = \frac{3}{2} \pi^{3/2} \lambda^{-1/2} C_F[\lambda]^6.
\]

Claim 4.
\[
(76) \quad -6a \int \int_{\mathbb{R}^2} \Re \left\{ it|G_1(x, t)|^4 \overline{G_1(x, t)} G_3(x, t) \right\} dxdt = -4a \pi^{3/2} \lambda^{-7/2} C_F[\lambda]^6.
\]

It follows from (74), (75), and (76) that
\[
(77) \quad \frac{1}{2} \partial^2 \epsilon |_{\epsilon=0} |v_\epsilon|^6 = 9 \int \int |G_1|^4 |\varphi_1|^2 dxdt + 6\Re \int \int |G_1|^2 \overline{G_1} \varphi_1^2 dxdt
\]
\[
+ \frac{3}{2} \pi^{3/2} \lambda^{-1/2} C_F[\lambda]^6 - 4a \pi^{3/2} \lambda^{-7/2} C_F[\lambda]^6.
\]

8.3.2 The term \( \|f_\epsilon\|^6_{L^2(\partial)} \). In view of (68), it is enough to compute the approximate expression \( \|g_\epsilon\|^6_{L^2(\partial)} \). Since \( G_0 \perp \varphi \),
\[
\|g_\epsilon\|^2_{L^2(\partial)} = \int \left( (G_0 + \epsilon \varphi)(u) \right)^2 |du| = \int |G_0(u)|^2 du + \left( \int |\varphi(u)|^2 du + \frac{\lambda^2}{2} \int u^2 |G_0(u)|^2 du \right) u^2
\]
\[
+ \left( \lambda^2 \Re \int u^2 G_0(u) \varphi(u) du \right) u^3 + \left( \frac{\lambda^2}{2} \int u^2 |\varphi(u)|^2 du \right) u^4.
\]
which, in turn, implies \( \partial_\epsilon \big|_{\epsilon=0} \| g_\epsilon \|_{L^2(\widetilde{a}_\epsilon)}^2 = 0 \) and

\[
\partial_\epsilon^2 \big|_{\epsilon=0} \| g_\epsilon \|_{L^2(\widetilde{a}_\epsilon)}^2 = 6 \| G_0 \|_2^4 \left( \int |\varphi(u)|^2 du + \frac{\lambda^2}{2} \int u^2 |G_0(u)|^2 du \right).
\]

One last computation shows that

\[
6 \| G_0 \|_2^4 \left( \frac{\lambda^2}{2} \int u^2 |G_0(u)|^2 du \right) = 3\lambda^2 \left( \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right)^2 \left( \int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du \right)
\]

\[
= 3\lambda^2 \frac{\pi^{1/2}}{\lambda^{23/2}} = \frac{3}{2} \pi^{3/2} \lambda^{-1/2},
\]

and so

\[
(78) \quad \frac{1}{2} \partial_\epsilon^2 \big|_{\epsilon=0} \| g_\epsilon \|_{L^2(\widetilde{a}_\epsilon)}^6 = 3 \| G_0 \|_2^4 \int |\varphi(u)|^2 du + \frac{3}{4} \pi^{3/2} \lambda^{-1/2}.
\]

**8.3.3 Putting it all together.** Using the approximations (67) and (68) given by Lemma 23, we get

\[
\Xi''(0) = \frac{1}{2} C_F[\lambda]^6 \| g_\epsilon \|_{L^2(\widetilde{a}_\epsilon)}^6 - \frac{1}{2} \partial_\epsilon^2 \big|_{\epsilon=0} \| v_\epsilon \|_{L^2(\widetilde{a}_\epsilon)}^6.
\]

The main terms in these approximations have been explicitly computed in (77) and (78). We obtain

\[
\Xi''(0) = 3 C_F[\lambda]^6 \| G_0 \|_2^4 \int |\varphi(s)|^2 ds + \frac{3}{4} C_F[\lambda]^6 \pi^{3/2} \lambda^{-1/2} - 9 \int \| G_1 \|_{L^2} \| \varphi_1 \|^2 dx dt
\]

\[
- 6\Re \int \| G_1 \|^2 \| G_1 \|^2 \varphi_1^2 dx dt - \frac{3}{2} \pi^{3/2} \lambda^{-1/2} C_F[\lambda]^6 + 4a\pi^{3/2} \lambda^{-7/2} C_F[\lambda]^6.
\]

Recalling the definition (58) of the quadratic form \( Q \) and the fact that \( G_0 \perp \varphi \), we obtain

\[
\Xi''(0) = Q(\varphi) - \frac{3}{4} \pi^{3/2} \lambda^{-1/2} C_F[\lambda]^6 + 4a\pi^{3/2} \lambda^{-7/2} C_F[\lambda]^6.
\]

It follows that

\[
\Xi(\epsilon) = \left( Q(\varphi) - \frac{3}{4} \pi^{3/2} \lambda^{-1/2} C_F[\lambda]^6 + 4a\pi^{3/2} \lambda^{-7/2} C_F[\lambda]^6 \right) \epsilon^2 + O(\epsilon^3)
\]

for sufficiently small \( \epsilon > 0 \), and so \( \Xi \) is strictly concave in a neighborhood of 0 if and only if

\[
Q(\varphi) - \frac{3}{4} \pi^{3/2} \lambda^{-1/2} C_F[\lambda]^6 + 4a\pi^{3/2} \lambda^{-7/2} C_F[\lambda]^6 < 0,
\]

i.e., if and only if

\[
a < \frac{3}{2} \left( \frac{\lambda}{2} \right)^3 - \frac{Q(\varphi)}{4\pi^{3/2} C_F[\lambda]^6} \lambda^{7/2}.
\]
The right hand side of this expression equals \( \frac{3}{2}(\lambda/2)^3 \) since \( Q(\varphi) = 0 \).

We have proved that for the choice of \( \varphi \) given by (64), for every \( \lambda > 0 \), for every \( a \in \mathbb{R} \) satisfying (52), and for every real-valued smooth \( \psi \) satisfying \( \psi(y) = O(|y|^5) \) as \( |y| \to 0 \), \( \Xi \) is a strictly concave function of \( \epsilon \), for sufficiently small \( \epsilon \). In particular, \( \Xi(\epsilon) < 0 \), which is equivalent to \( C_F[\lambda]^6 \|f_\epsilon\|_{L^2(\tilde{\sigma})}^6 < \|\hat{f}_\epsilon\tilde{\sigma}\|_6^6 \). Together with (53), this implies \( C_F[\lambda] < C[\Gamma] \). Trivially, \( C[\tilde{\Gamma}] \leq C[\Gamma] \), and this finishes the proof of Proposition 22 modulo the work deferred to the Appendices.

**Remark 2.** We have been working under the additional assumption that the curvature \( \kappa \) of \( \Gamma \) does not attain a global minimum at one of its endpoints. This represents no loss of generality. Indeed, if \( \kappa \) attained a global minimum, say at at \( p_0 = \gamma(0) \), then we could perform an identical variational calculation with functions \( f_\epsilon \) defined in a similar way to (61) but supported instead on small neighborhoods \( \tilde{\Gamma}_\epsilon \) of points \( p_\epsilon = \gamma(s_\epsilon) \in \Gamma \) with \( |s_\epsilon| \asymp \epsilon \). It is straightforward to check that the computation carries through. We omit the details.

### 9 The completion of the proof

Recall that by \( T_{s,r} \) we mean the adjoint Fourier restriction operator on a cap \( C = C(s, r) \subset \Gamma \). The following (easy) estimate is the last one we need in order to finish the argument.

**Lemma 24.** If a sequence \( \{f_n\} \subset L^2(\sigma) \) concentrates at a point \( \gamma(s) \in \Gamma \), then

\[
\lim_{n \to \infty} \|\hat{f}_n\sigma\|_6 \leq \lim_{r \to 0^+} \lim_{n \to \infty} \|T_{s,r}f_n\|_6.
\]

**Proof.** Set \( Tf := \hat{f}\sigma \), and let \( r > 0 \) be arbitrary. The Tomas-Stein inequality implies

\[
\|(T - T_{s,r})f_n\|_6 \lesssim \left( \int_{\Gamma \setminus C(s, r)} |f_n|^2 d\sigma \right)^{1/2}.
\]

It follows that

\[
\|Tf_n\|_6 \leq \|T_{s,r}f_n\|_6 + \|(T - T_{s,r})f_n\|_6 \leq \|T_{s,r}f_n\|_6 + C\left( \int_{\Gamma \setminus C(s, r)} |f_n|^2 d\sigma \right)^{1/2}.
\]

Since \( \{f_n\} \) concentrates at \( \gamma(s) \),

\[
\lim_{n \to \infty} \|Tf_n\|_6 \leq \lim_{n \to \infty} \|T_{s,r}f_n\|_6 + 0
\]

for every \( r > 0 \). The result follows. \( \square \)
We can now prove that condition (42) cannot be satisfied, i.e., the sequence \( \{ F_n \} \) given by Proposition 16 cannot concentrate. Suppose \( \{ F_n \} \) did concentrate at some point \( \gamma(s) \in \Gamma \) and assume, as before, that \( \| F_n \|_2 \to 1 \). By Corollary 21, we know that \( \kappa(\gamma(s)) = \lambda \). We may again assume that \( \gamma(s) \) is not an endpoint of \( \Gamma \).

Then, by Lemma 24 and Proposition 19,

\[
\mathcal{C}[\Gamma] = \lim_{n \to \infty} \| \hat{F}_n \sigma \|_6 \leq \lim_{r \to 0^+} \lim_{n \to \infty} \| T_{s,r} F_n \|_6 \leq \lim_{r \to 0^+} \| T_{s,r} \| = C_F[\lambda],
\]

a contradiction to Proposition 22. This concludes the proof of Theorem 1.

10 Appendix 1: \( \| v_\epsilon \|_6^6 \) is twice differentiable at \( \epsilon = 0 \)

Recall that

\[
v_\epsilon(x, t) = \int_{\mathbb{R}} (G_0 + \epsilon \varphi)(u)e^{-i(x,t) \cdot (u, \frac{\lambda^2}{2} + ac^2u^2 + \epsilon - 2\psi(u))\left(1 + \frac{\lambda^2}{2}\epsilon^2u^2\right)} \eta_I\left(\frac{1}{\log \frac{1}{\epsilon}} u\right) du,
\]

where in this formula \( 0 < \epsilon \leq \epsilon_0 \), \( \lambda = \min_{\Gamma} \kappa \), \( G_0(u) = e^{-\lambda u^2/2} \), \( \varphi(u) = c_\lambda ue^{-\lambda u^2/2} \), \( a \in [\lambda^3/8, 3\lambda^3/16] \), \( \psi \) is a real-valued smooth function satisfying \( \psi(y) = O(|y|^5) \) as \( |y| \to 0 \), and \( \eta_I \) is a mollified version of the characteristic function of the interval \( I \).

The main goal of this appendix is to prove the following result.

**Proposition 25.** The function \( \epsilon \mapsto \| v_\epsilon \|_{L_6(\mathbb{R}^2)}^6 \) is twice differentiable at \( \epsilon = 0 \), and

\[
\partial^2_{\epsilon=0} \| v_\epsilon \|_6^6 = 18 \iint |G_1|^4 |\varphi_1|^2 dxdt + 12 \Re \iint |G_1|^2 \overline{G_1} \varphi_1^2 dxdt + 3\pi^{3/2} \lambda^{-1/2} C_F[\lambda]^6 - 8a\pi^{3/2} \lambda^{-7/2} C_F[\lambda]^6.
\]

It is enough to show that

\[
\iint_{\mathbb{R}^2} \left| \partial_\epsilon v_\epsilon(x, t) \right|^6 dxdt \leq C \quad \text{and} \quad \iint_{\mathbb{R}^2} \left| \partial^2_\epsilon v_\epsilon(x, t) \right|^6 dxdt \leq C
\]

for a finite constant \( C \) (independent of \( \epsilon \)) and every \( 0 < \epsilon \leq \epsilon_0 \). The existence of \( \partial^2_{\epsilon=0} \| v_\epsilon \|_6^6 \) then follows from standard tools of analysis, and the formal computations from Subsection 8.3.1 show that its value is given by (79).

As in the proof of Proposition 23, for \( \epsilon < 1 \), set

\[
G_\epsilon(u) = \left(1 + \frac{\lambda^2}{2}\epsilon^2u^2\right)(G_0 + \epsilon \varphi)(u).
\]
We have the following expression for the oscillatory integral:

\[ v_\epsilon(x, t) = \int_{\mathbb{R}} e^{-i \phi_\epsilon(u)} G_\epsilon(u) \eta_I \left( \frac{1}{\log \frac{1}{\epsilon}} u \right) du, \]

where

\[ \phi_\epsilon(u) := \frac{x}{t} u + \frac{\lambda u^2}{2} + a \epsilon^2 u^4 + \epsilon^{-2} \psi(\epsilon u). \]  

In order to present the main estimates for \( v_\epsilon \), it is convenient to decompose the \((x, t)\)-plane in the following fashion. Let \( \eta_0, \eta \in C_0^{\infty}(\mathbb{R}) \) be even and smooth cut-off functions supported in \([-1, 1]\) and \([-2, -1/2] \cup [1/2, 2]\), respectively, with the properties that \( 0 \leq \eta_0 \leq 1, 0 \leq \eta \leq 1 \), and

\[ \eta_I \left( \frac{1}{\log \frac{1}{\epsilon}} u \right) = \eta_0(u) + \sum_{k=1}^{K(\epsilon)} \eta(2^{-k+1} u) \quad \text{for every } u \in \mathbb{R}. \]

This can be accomplished in the following fashion. Start by choosing \( K = K(\epsilon) \) satisfying \( 2^{K(\epsilon)} \lesssim \log \frac{1}{\epsilon} \). Let \( \eta_0, \eta \in C_0^{\infty}(\mathbb{R}) \) be even and smooth cut-off functions supported in \([-1, 1]\) and \([-2, -1/2] \cup [1/2, 2]\) respectively, with the properties that \( 0 \leq \eta_0 \leq 1, 0 \leq \eta \leq 1 \), and

\[ \eta_I \left( \frac{1}{\log \frac{1}{\epsilon}} u \right) = \eta_0(u) + \sum_{k=1}^{K(\epsilon)} \eta(2^{-k+1} u) \quad \text{for every } u \in \mathbb{R}. \]

We obtain, in particular, a smooth partition of unity in the interval \( \frac{1}{2} \log \frac{1}{\epsilon} \cdot I \) subordinate to the dyadic regions \( D_k := \{ u \in \mathbb{R} : 2^{k-2} \leq |u| \leq 2^k \} \). This allows us to express \( v_\epsilon \) as a sum of \( \asymp K(\epsilon) \) integrals:

\[ v_\epsilon(x, t) = v_{\epsilon, 0}(x, t) + \sum_{k=1}^{K(\epsilon)} v_{\epsilon, k}(x, t), \]

where

\[ v_{\epsilon, 0}(x, t) := \int_{\mathbb{R}} e^{-i \phi_\epsilon(u)} G_\epsilon(u) \eta_0(u) du \]

and

\[ v_{\epsilon, k}(x, t) := \int_{\mathbb{R}} e^{-i \phi_\epsilon(u)} G_\epsilon(u) \eta(2^{-k+1} u) du \]

for \( k \in \{1, 2, \ldots, K(\epsilon)\} \). Define

\[ \mathcal{C}_0 := \left\{ (x, t) \in \mathbb{R}^2 : \left| \frac{x}{t} \right| \leq \frac{1}{\lambda} \right\}; \]

\[ \mathcal{C}_k := \left\{ (x, t) \in \mathbb{R}^2 : \lambda 2^{k-2} \leq \left| \frac{x}{t} \right| \leq \lambda 2^k \right\}; \quad (k \in \{1, 2, \ldots, K(\epsilon)\}) \]

and

\[ \mathcal{C}_\infty := \left\{ (x, t) \in \mathbb{R}^2 : \left| \frac{x}{t} \right| \geq \lambda \log \frac{1}{\epsilon} \right\}. \]
This yields a decomposition of the \((x, t)\)-plane as a union of cones

\[
\mathbb{R}^2 = \mathcal{C}_0 \cup \bigcup_{k=1}^{K(\varepsilon)} \mathcal{C}_k \cup \mathcal{C}_\infty
\]

which parallels (82). For \(k \in \{2, 3, \ldots, K(\varepsilon) - 2\}\), define the “enlarged” cones

\[
\mathcal{C}_k^* := \mathcal{C}_k \cup \mathcal{C}_{k+1} \cup \mathcal{C}_{k+2}.
\]

Additionally, let

\[
\begin{align*}
\mathcal{C}_0^* &:= \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2; \\
\mathcal{C}_1^* &:= \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3; \\
\mathcal{C}_{K(\varepsilon)-1}^* &:= \mathcal{C}_{K(\varepsilon)-3} \cup \mathcal{C}_{K(\varepsilon)-2} \cup \mathcal{C}_{K(\varepsilon)-1} \cup \mathcal{C}_{K(\varepsilon)} \cup \mathcal{C}_\infty; \\
\mathcal{C}_{K(\varepsilon)}^* &:= \mathcal{C}_{K(\varepsilon)-2} \cup \mathcal{C}_{K(\varepsilon)-1} \cup \mathcal{C}_{K(\varepsilon)} \cup \mathcal{C}_\infty.
\end{align*}
\]

The estimates in the following proposition are an expression of the \textit{stationary phase principle}, which roughly states that the main contribution for an oscillatory expression like (84) comes from the information concentrated on neighborhoods of the stationary points of its phase function.

\textbf{Proposition 26.} For \(k \in \{1, \ldots, K(\varepsilon)\}\) and sufficiently small \(\varepsilon > 0\), there exist \(a_k \geq 0\) such that

\[
|v_{\varepsilon,k}(x, t)| \lesssim a_k \cdot \begin{cases} 
\langle t \rangle^{-1/2} & \text{if } (x, t) \in \mathcal{C}_k^*, \\
\langle t \rangle^{-1} & \text{if } (x, t) \in \mathcal{C}_0, \\
\langle x \rangle^{-1} & \text{if } (x, t) \in \left( \bigcup_{|j-k|\geq2} \mathcal{C}_j \right) \cup \mathcal{C}_\infty
\end{cases}
\]

and \(\sum_k 2^k a_k < \infty\). For \(k = 0\),

\[
|v_{\varepsilon,0}(x, t)| \lesssim \begin{cases} 
\langle t \rangle^{-1/2} & \text{if } (x, t) \in \mathcal{C}_0^*, \\
\langle x \rangle^{-1} & \text{otherwise.}
\end{cases}
\]

\textbf{Proof.} To make the notation less cumbersome, we limit our discussion to the case \(k \in \{3, 4, \ldots, K(\varepsilon) - 2\}\). The other cases follow similarly.

\textit{Case 1:} \((x, t) \in \mathcal{C}_k^*\). Without loss of generality, we may assume that \((x, t) \in \mathcal{C}_k^*\) is such that the phase \(\phi_\varepsilon\) has a critical point in the support of the cut-off function \(\eta_I((\log 1/\varepsilon)^{-1} \cdot )\); for otherwise, we could integrate (84) by parts and obtain a better decay in \(t\). This critical point is necessarily unique. In other words, there exists a unique \(u_0 \in 2 \log 1/\varepsilon \cdot I\) such that

\[
\frac{d\phi_\varepsilon}{du}(u_0) = \frac{x}{t} + \lambda u_0 + 4a \varepsilon^2 u_0^3 + \varepsilon^{-1} \psi'(\varepsilon u_0) = 0.
\]
In general, one cannot hope to solve this equation explicitly for $u_0$. However, since $|u_0| \lesssim \log \frac{1}{\epsilon}$,

$$u_0 = -\frac{x}{\lambda t} + O_{a, \psi}(\epsilon).$$

In particular, $|u_0| \asymp 2^k$ since $(x, t) \in \mathcal{C}_r^k$.

Translating $u \mapsto u + u_0$ and defining $\tilde{\phi}_\epsilon(u) := \phi_\epsilon(u + u_0) - \phi_\epsilon(u_0)$, we obtain

$$v_\epsilon(x, t) = e^{-it\tilde{\phi}_\epsilon(u_0)} \int_{\mathbb{R}} e^{-it\tilde{\phi}_\epsilon(u)} G_\epsilon(u + u_0) \eta I \left( \frac{1}{\log \frac{1}{\epsilon}} (u + u_0) \right) du.$$  

It suffices to find good estimates on

$$\tilde{v}_\epsilon(x, t) := e^{it\tilde{\phi}_\epsilon(u_0)} v_\epsilon(x, t) = \int_{\mathbb{R}} e^{-it\tilde{\phi}_\epsilon(u)} G_\epsilon(u + u_0) \eta I \left( \frac{1}{\log \frac{1}{\epsilon}} (u + u_0) \right) du.$$  

The new phase function $\tilde{\phi}_\epsilon$ satisfies

$$\tilde{\phi}_\epsilon(0) = 0 = \frac{d\tilde{\phi}_\epsilon}{du}(0) \quad \text{and} \quad \frac{d^2\tilde{\phi}_\epsilon}{du^2}(0) = \lambda + O(\epsilon).$$

In particular, the origin is its unique nondegenerate critical point. This property is shared by the quadratic function $v \mapsto v^2/2$. Inspired by the proof\(^6\) of the usual method of stationary phase, we change variables once again.

Recalling definition (81) and identity (86), and using Taylor’s formula, we obtain

$$\frac{v^2}{2} := \tilde{\phi}_\epsilon(u) = \phi_\epsilon(u + u_0) - \phi_\epsilon(u_0)$$

$$\quad = \frac{u^2}{2} \left( (\lambda + 12au_0^2 \epsilon^2) + 8au_0 \epsilon^2 u + 2a\epsilon^2 u^2 \right) + \epsilon^{-2} \psi(\epsilon(u + u_0))$$

$$\quad \quad - \epsilon^{-2} \psi(\epsilon u_0) - u \epsilon^{-1} \psi'(\epsilon u_0)$$

$$\quad = \frac{u^2}{2} \left( (\lambda + 12au_0^2 \epsilon^2) + 8au_0 \epsilon^2 u + 2a\epsilon^2 u^2 + \psi''(\epsilon(u_0 + \theta u)) \right)$$

for some $\theta \in (0, 1)$. Taking square roots, we have

$$v = u \left( (\lambda + 12au_0^2 \epsilon^2) + 8au_0 \epsilon^2 u + 2a\epsilon^2 u^2 + \psi''(\epsilon(u_0 + \theta u)) \right)^{1/2} =: \Phi_\epsilon(u).$$

For sufficiently small $\epsilon > 0$, $\Phi_\epsilon$ is a $C^\infty$ diffeomorphism from $2 \log \frac{1}{\epsilon} \cdot I$ onto its image, whose inverse we denote by $\Psi_\epsilon := \Phi_\epsilon^{-1}$. One can verify directly that $\frac{d\Phi_\epsilon}{du}(u) > 0$ for every $u \in 2 \log \frac{1}{\epsilon} \cdot I$. As a consequence,

$$\tilde{v}_\epsilon(x, t) = \int_{\mathbb{R}} e^{-it\frac{u^2}{2}} G_\epsilon(\Psi_\epsilon(v) + u_0) \eta I \left( \frac{1}{\log \frac{1}{\epsilon}} (\Psi_\epsilon(v) + u_0) \right) \frac{d\Psi_\epsilon}{dv}(v) dv.$$  

\(^6\)See, for instance, [23, pp. 334–337].
Rewriting (88) as
\[
\Phi_\epsilon(u) = u \left( \lambda + 2a(2u_0^2 + (2u_0 + u)^2)\epsilon^2 + \psi''(\epsilon(u_0 + \theta u)) \right)^{1/2}
\]
and recalling that \(\lambda, a > 0\) and \(\psi''(\epsilon(u_0 + \theta u)) = O(\epsilon^3)\), we see that \(\Phi_\epsilon\) is twice differentiable as a function of \(\epsilon\) for sufficiently small \(\epsilon\). The same holds for \(\frac{d\Phi_\epsilon}{du}\), and using the chain rule one can draw similar conclusions about \(\Psi_\epsilon\) and \(\frac{d\Psi_\epsilon}{dv}\).

In what follows, \(C\) is a finite nonzero constant which may change from line to line and depend on the parameters \(\lambda, a\), the function \(\psi\), and the interval \(I\) but is always be independent of \(\epsilon, x,\) and \(t\). This uniformity is crucial in our analysis.

We proceed to prove some uniform (in \(\epsilon\)) bounds for \(\frac{d\Psi_\epsilon}{dv}\) and \(\frac{d^2\Psi_\epsilon}{dv^2}\). Start by observing that for sufficiently small \(\epsilon > 0\),
\[
C^{-1} |u| \leq |\Phi_\epsilon(u)| \leq C |u| \text{ for all } u \in 2 \log \frac{1}{\epsilon} \cdot I.
\]
Since \(\Phi_\epsilon \circ \Psi_\epsilon = id\) on \(\Phi_\epsilon(2 \log \frac{1}{\epsilon} \cdot I) \subseteq C \log \frac{1}{\epsilon} \cdot I\), we have the same uniform bounds for \(\Psi_\epsilon\):
\[
(91) \quad C^{-1} |v| \leq |\Psi_\epsilon(v)| \leq C |v| \text{ for all } v \in C \log \frac{1}{\epsilon} \cdot I.
\]
We also have the uniform bounds
\[
(92) \quad C^{-1} \leq \left| \frac{d\Phi_\epsilon}{du}(u) \right| \leq C, \quad \forall u \in 2 \log \frac{1}{\epsilon} \cdot I.
\]
By the Inverse Function Theorem,
\[
(93) \quad \frac{d\Psi_\epsilon}{dv}(v) = \frac{1}{\frac{d\Phi_\epsilon}{du}(\Psi_\epsilon(v))};
\]
and so (92) implies
\[
(94) \quad C^{-1} \leq \left| \frac{d\Psi_\epsilon}{dv}(v) \right| \leq C \text{ for all } v \in C \log \frac{1}{\epsilon} \cdot I.
\]
Only the upper bound is useful to us. Similarly, one can conclude that
\[
(95) \quad \left| \frac{d^2\Psi_\epsilon}{dv^2}(v) \right| \leq C \text{ for all } v \in C \log \frac{1}{\epsilon} \cdot I.
\]
We also need to estimate the Gaussian term \(G_\epsilon\) appearing in (89) and some of its derivatives. The following claim, whose proof is straightforward and therefore omitted, provides good enough bounds.
Claim 5. The following uniform estimates hold for every sufficiently small $\epsilon > 0$, for every nonnegative integer $n$, and for every $u \in 2\log \frac{1}{\epsilon} \cdot I$:
\[
\left| \frac{d^n}{du^n} G_\epsilon(u) \right| \lesssim_n \langle u \rangle^n e^{-\frac{|u|^2}{2}}.
\]

Let us return to (89). Introducing the cut-off functions $\eta$ and $\eta_0$ as before, we can expand
\[
(96) \quad \tilde{v}_\epsilon(x, t) = \tilde{v}_{\epsilon, 0}(x, t) + \sum_{k=1}^{K(\epsilon)} \tilde{v}_{\epsilon, k}(x, t),
\]
where
\[
\tilde{v}_{\epsilon, 0}(x, t) := \int_{\mathbb{R}} e^{-it\frac{x^2}{2}} G_\epsilon(\Psi_\epsilon(v) + u_0) \frac{d\Psi_\epsilon}{dv}(v) \eta_0(\Psi_\epsilon(v) + u_0) dv
\]
and
\[
\tilde{v}_{\epsilon, k}(x, t) := \int_{\mathbb{R}} e^{-it\frac{x^2}{2}} G_\epsilon(\Psi_\epsilon(v) + u_0) \frac{d\Psi_\epsilon}{dv}(v) \eta(2^{-k+1}(\Psi_\epsilon(v) + u_0)) dv
\]
for $k \in \{1, \ldots, K(\epsilon)\}$. As before, $|\tilde{v}_{\epsilon, k}| = |v_{\epsilon, k}|$ pointwise.

Define
\[
(97) \quad b_\epsilon(v) := G_\epsilon(\Psi_\epsilon(v) + u_0) \frac{d\Psi_\epsilon}{dv}(v) \quad \text{and} \quad \eta_k(v) := \eta(2^{-k+1}(\Psi_\epsilon(v) + u_0)).
\]

Notice that $\eta_k \in C_0^\infty(\mathbb{R})$ is supported on
\[
\mathcal{E}_k := \{v \in C \log \frac{1}{\epsilon} \cdot I : 2^{-k-2} \leq |\Psi_\epsilon(v) + u_0| \leq 2^k\},
\]
and that $b_\epsilon$ is a Schwartz function on $\mathcal{E}_k$. This justifies the use of Plancherel’s Theorem, which yields
\[
\tilde{v}_{\epsilon, k}(x, t) = \int_{\mathbb{R}} e^{-it\frac{x^2}{2}} b_\epsilon(v) \eta_k(v) dv = (-2\pi i)^{1/2} t^{-1/2} \int_{\mathbb{R}} e^{it\frac{x^2}{2}} \tilde{b}_\epsilon(\eta_k(\xi)) d\xi.
\]

Set $a_k := \|\tilde{b}_\epsilon \eta_k\|_{L^1}$. We will have finished analyzing Case 1 once we verify that the sequence $\{a_k\}_k$ decays rapidly enough to force the series $\sum_k 2^k a_k$ to converge.

We start by estimating the $L^2$ norm of the function $b_\epsilon \eta_k$. Changing back to the original variable $u = \Psi_\epsilon(v)$ and using Hölder’s inequality together with estimate (94) and Claim 5 for $n = 0$, we get
\[
\|b_\epsilon \eta_k\|_{L^2}^2 = \int \left| G_\epsilon(\Psi_\epsilon(v) + u_0) \frac{d\Psi_\epsilon}{dv}(v) \eta(2^{-k+1}(\Psi_\epsilon(v) + u_0)) \right|^2 dv \leq \int_{|u+u_0| \leq 2^k} \left| G_\epsilon(u + u_0) \eta(2^{-k+1}(u + u_0)) \right|^2 \left| \frac{d\Psi_\epsilon}{dv}(\Phi_\epsilon(u)) \right| du \lesssim 2^k e^{-4k-1}.\]
An analogous argument, using estimate (95) instead, yields
\[ \left\| \frac{d}{db}(b_\epsilon \eta_k) \right\|_{L^2}^2 \lesssim 2^{3k} e^{-\lambda 2^{k-3}}. \]

Using Cauchy-Schwarz and Plancherel, we see that these two estimates are enough for our purposes:
\[
\begin{align*}
\hat{a}_k = \| \hat{b}_\epsilon \hat{\eta}_k \|_{L^1} & = \int_{|\xi| \leq 1} |\hat{b}_\epsilon \hat{\eta}_k(\xi)| d\xi + \int_{|\xi| \geq 1} \frac{1}{|\xi|} \left( |\xi| \| \hat{b}_\epsilon \hat{\eta}_k(\xi) \| \right) d\xi \\
& \lesssim \| \hat{b}_\epsilon \hat{\eta}_k \|_{L^2} + \left( \int_{|\xi| \geq 1} |\xi|^{-2} d\xi \right)^{1/2} \left( \int_{|\xi| \geq 1} |\xi|^2 |\hat{b}_\epsilon \hat{\eta}_k(\xi)|^2 d\xi \right)^{1/2} \\
& \lesssim \| b_\epsilon \eta_k \|_{L^2} + \left\| \frac{d}{du}(b_\epsilon \eta_k) \right\|_{L^2} \lesssim 2^{3k/2} e^{-\lambda 2^{k-3}}.
\end{align*}
\]

This completes the analysis of Case 1.

**Case 2: (x, t) ∈ C_0.** The crucial observation here is that since (x, t) ∈ C_0 and k > 2, the phase \( \phi_\epsilon \) has no critical points in the support of \( \eta(2^{-k+1} \cdot) \), i.e., the dyadic region \( D_k = \{ u \in \mathbb{R} : 2^{k-2} \leq |u| \leq 2^k \} \). Indeed, since \( |1/7| \leq \lambda \),

\[
(98) \quad \left| \frac{d\phi_\epsilon}{du}(u) \right| = \left| \frac{x}{t} + \lambda u + O(\epsilon) \right| \geq \frac{1}{2} \left( |\lambda u| - \left| \frac{x}{t} \right| \right) \geq \frac{\lambda}{2} (2^{k-2} - 1) \geq \frac{\lambda}{2}
\]

for sufficiently small \( \epsilon > 0 \).

Integrating (84) by parts, we get
\[
\begin{align*}
|v_{\epsilon,k}(x, t)| & = \left| \frac{1}{|t|} \int_{D_k} e^{-i\phi_\epsilon(u)} \frac{d}{du} \left( \frac{G_\epsilon(u) \eta(2^{-k+1} u)}{\frac{d\phi_\epsilon}{du}(u)} \right) \right| du \\
& \lesssim \left| \frac{1}{|t|} \int_{D_k} \left| \frac{d}{du}(G_\epsilon(u) \eta(2^{-k+1} u)) \right| \right| \left| \frac{d\phi_\epsilon}{du}(u) \right| du.
\end{align*}
\]

Hölder’s inequality implies
\[
|v_{\epsilon,k}(x, t)| \lesssim \frac{2^k}{|t|} \left( \left| \frac{d}{du}(G_\epsilon \eta(2^{-k+1} \cdot)) \right| \right|_{L^\infty(D_k)} + \left| \frac{G_\epsilon \eta(2^{-k+1} \cdot)}{\frac{d\phi_\epsilon}{du}} \right|_{L^\infty(D_k)},
\]

and the desired estimate\(^7\) now follows from (98), Claim 5, and the fact that \( \frac{d^2 \phi_\epsilon}{du^2} \) is uniformly bounded on \( D_k \).

**Case 3: (x, t) ∈ C_j for some j such that |k - j| > 2, or (x, t) ∈ C_\infty.** The proof is identical to that of Case 2 and is therefore omitted. \( \square \)

\(^7\)One could repeat this argument \( N \) times and obtain the bound \( |v_{\epsilon,k}(x, t)| \lesssim e^{-N} \) for \( (x, t) \in C_0 \), but this extra knowledge is of no significance to our analysis.
Let us return to the proof of Proposition 25. Observe that the estimates from Proposition 26 readily imply the following special case of the $L^2 \to L^6$ adjoint restriction inequality:

\begin{equation}
\int_{\mathbb{R}^2} |v_\epsilon(x, t)|^6 dxdt \leq C.
\end{equation}

Indeed, the expansion (83) implies

\[
\int_{\mathbb{R}^2} |v_\epsilon(x, t)|^6 dxdt = \sum_{k_1, \ldots, k_6} \int_{\mathbb{R}^2} v_{\epsilon,k_1} \tilde{v}_{\epsilon,k_2} v_{\epsilon,k_3} \tilde{v}_{\epsilon,k_4} v_{\epsilon,k_5} \tilde{v}_{\epsilon,k_6} dxdt,
\]

where, for each $j \in \{1, \ldots, 6\}$, the sum is taken over $k_j \in \{0, 1, \ldots, K(\epsilon)\}$. For fixed $(k_1, \ldots, k_6)$, the corresponding integral can be written as a sum of $K(\epsilon)$ integrals over the regions given by decomposition (85); and using the bounds given by Proposition 26 on each of these regions, one readily obtains (99). Note that

\[
\int_{\mathbb{C}_k} \langle t \rangle^{-3} dxdt \asymp 2^k,
\]

so it is crucial to know that $\sum_k 2^k b_k < \infty$.

To prove (80), it is enough to control the integrals

\[
I_0(\epsilon) := \int_{\mathbb{R}^2} |v_\epsilon(x, t)|^5 |\partial_\epsilon v_\epsilon(x, t)| dxdt;
\]

\[
I_1(\epsilon) := \int_{\mathbb{R}^2} |v_\epsilon(x, t)|^5 |\partial_\epsilon^2 v_\epsilon(x, t)| dxdt;
\]

\[
I_2(\epsilon) := \int_{\mathbb{R}^2} |v_\epsilon(x, t)|^4 |\partial_\epsilon v_\epsilon(x, t)|^2 dxdt.
\]

The reasoning just described to prove (99) can be used to establish bounds for the integrals $I_0(\epsilon)$, $I_1(\epsilon)$, and $I_2(\epsilon)$ which are uniform in $\epsilon$, provided that we have an analogue of Proposition 26 for first and second derivatives. As before, bounds for $\partial_\epsilon \tilde{v}_\epsilon$ and $\partial_\epsilon^2 \tilde{v}_\epsilon$ suffice.

Let us focus on the more involved case of second derivatives.

**Proposition 27.** For $k \in \{1, \ldots, K(\epsilon)\}$ and sufficiently small $\epsilon > 0$, there exist $b_k \geq 0$ such that

\[
|\partial_\epsilon^2 \tilde{v}_{\epsilon,k}(x, t)| \lesssim b_k \cdot \begin{cases} 
\langle t \rangle^{-1/2} & \text{if } (x, t) \in \mathbb{C}_k^*, \\
\langle t \rangle^{-1} & \text{if } (x, t) \in \mathbb{C}_0, \\
\langle x \rangle^{-1} & \text{if } (x, t) \in \left( \bigcup_{|j-k|>2} \mathbb{C}_j \right) \cup \mathbb{C}_\infty;
\end{cases}
\]

and $\sum_k 2^k b_k < \infty$. For $k = 0$,

\[
|\partial_\epsilon^2 \tilde{v}_{\epsilon,0}(x, t)| \lesssim \begin{cases} 
\langle t \rangle^{-1/2} & \text{if } (x, t) \in \mathbb{C}_0^*, \\
\langle x \rangle^{-1} & \text{otherwise}.
\end{cases}
\]
The proof follows the same steps of Proposition 26, with only one difference: we need appropriate bounds for \( \partial_\epsilon \left( \frac{d^j \Psi_\epsilon}{d \nu} \right) \) and \( \partial^2_\epsilon \left( \frac{d^j \Psi_\epsilon}{d \nu} \right) \) for \( j \in \{1, 2\} \).

Let us briefly outline how to find such bounds. Using (88), we have

\[
\Phi_\epsilon(\Psi_\epsilon(v)) = \Psi_\epsilon(v) \left( \lambda + (12au_0^3 + 8au_0\Psi_\epsilon(v) + 2a\Psi_\epsilon(v)^2)\epsilon^2 + \psi''(\epsilon(u_0 + \theta \Psi_\epsilon(v))) \right)^{1/2}.
\]

Differentiate both sides of this identity with respect to \( \epsilon \). Obviously, the left hand side vanishes. Thus the sum of all terms on the right hand side that contain a factor of the form \( \partial_\epsilon \Psi_\epsilon \) equals minus the sum of all terms that do not contain such a factor. As for the terms containing such a factor, we have the estimate

\[
|\partial_\epsilon \Psi_\epsilon(v)| \leq C \text{ for all } v \in C \log \frac{1}{\epsilon} \cdot I.
\]

It is also elementary to show that

\[
|\partial_\epsilon \left( \frac{d \Phi_\epsilon}{d u} \right)(u)| \leq C \text{ for all } u \in 2 \log \frac{1}{\epsilon} \cdot I.
\]

Differentiating both sides of (93) with respect to \( \epsilon \), we obtain

\[
\partial_\epsilon \left( \frac{d \Psi_\epsilon}{d \nu} \right)(v) = -\frac{\partial_\epsilon \left( \frac{d \Phi_\epsilon}{d u} \left( \Psi_\epsilon(v) \right) \right)}{\left( \frac{d \Phi_\epsilon}{d u} \left( \Psi_\epsilon(v) \right) \right)^2}.
\]

Using estimates (92) and (100), we similarly conclude that

\[
|\partial_\epsilon \left( \frac{d \Psi_\epsilon}{d \nu} \right)(v)| \leq C \text{ for all } v \in C \log \frac{1}{\epsilon} \cdot I.
\]

Repeating this whole procedure once again, we conclude analogously that

\[
|\partial^2_\epsilon \left( \frac{d \Psi_\epsilon}{d \nu} \right)(v)| \leq C \nu^2 \text{ for all } v \in C \log \frac{1}{\epsilon} \cdot I.
\]

The terms \( \partial_\epsilon \left( \frac{d^2 \Psi_\epsilon}{d ^2 \nu} \right) \) and \( \partial^2_\epsilon \left( \frac{d^2 \Psi_\epsilon}{d ^2 \nu} \right) \) can be dealt with similarly. Recalling what we already know from (94) and (95), we arrive at the following lemma.

**Lemma 28.** The following estimates hold for \( j \in \{1, 2\} \), for every sufficiently small \( \epsilon > 0 \) and for every \( v \in C \log \frac{1}{\epsilon} \cdot I \):

(i) \( |d^j \Psi_\epsilon(v)| \leq C \),

(ii) \( |\partial_\epsilon (d^j \Psi_\epsilon(v))| \leq C \),

(iii) \( |\partial^2_\epsilon (d^j \Psi_\epsilon(v))| \leq C \nu^2 \).

Lemma 28 can be used together with the estimates from Claim 5 to prove Proposition 27. We omit the details.
11 Appendix 2: Two explicit calculations

Recall Claim 3:

\[ 3\lambda^2 \Re \int_{\mathbb{R}^2} |G_1(x, t)|^4 \overline{G_1(x, t)} G_2(x, t) dx dt = 3 \frac{\pi^{3/2}}{2} \lambda^{-1/2} C_F [\lambda]^6. \]

**Proof.** First note that

\[ \frac{\pi}{\lambda} (1 + t^2)^{-1/2} e^{-\frac{3\pi^2}{4(1 + t^2)}}. \]

It follows that the left hand side in (75) equals

\[ 3\lambda^2 \left( \frac{2\pi}{\lambda} \right)^3 \Re \int_{\mathbb{R}^2} \left( \lambda^{-1} (1 + it)^{-1} - \lambda^{-2} x^2 (1 + it)^{-2} \right) (1 + t^2)^{-3/2} e^{-\frac{3\pi^2}{4(1 + t^2)}} dx dt. \]

Write \((1 + it)^{-1} = (1 + t^2)^{-1}(1 - it), (1 + it)^{-2} = (1 + t^2)^{-2}(1 - it)^2\) and change variables \(y = x/(1 + t^2)^{1/2}\) to compute

\[
\text{I : } \Re \int_{\mathbb{R}} (1 + t^2)^{-1/2} (1 - it) (1 + t^2)^{-3/2} e^{-\frac{3\pi^2}{4(1 + t^2)}} dx dt = \int_{\mathbb{R}} (1 + t^2)^{-5/2} e^{-\frac{3\pi^2}{4(1 + t^2)}} dx dt
\]

\[
= \left( \int_{-\infty}^{\infty} \frac{1}{(1 + t^2)^{3/2}} dt \right) \cdot \left( \int_{-\infty}^{\infty} e^{-\frac{3\pi^2}{4x^2}} dx \right) = \frac{\pi}{2} \cdot \left( \frac{\pi}{3/\lambda} \right)^{1/2} = \frac{\pi^{3/2}}{2\sqrt{3}} \lambda^{1/2}
\]

and

\[
\text{II : } \Re \int_{\mathbb{R}} x^2 (1 + t^2)^{-2} (1 - it)^2 (1 + t^2)^{-3/2} e^{-\frac{3\pi^2}{4(1 + t^2)}} dx dt
\]

\[
= \int_{\mathbb{R}} (1 + t^2)^{-7/2} (1 - t^2) x^2 e^{-\frac{3\pi^2}{4(1 + t^2)}} dx dt
\]

\[
= \left( \int_{-\infty}^{\infty} \frac{1 - t^2}{(1 + t^2)^{5/2}} dt \right) \left( \int_{-\infty}^{\infty} y^2 e^{-\frac{3\pi^2}{4y^2}} dy \right) = 0.
\]

All in all, we have

\[
3\lambda^2 \Re \int_{\mathbb{R}^2} |G_1(x, t)|^4 \overline{G_1(x, t)} G_2(x, t) dx dt = 3\lambda^2 \left( \frac{2\pi}{\lambda} \right)^3 (\lambda^{-1} \text{I} - \lambda^{-2} \text{II})
\]

\[
= 3\lambda^{-2}(2\pi)^3 \text{I} + 0
\]

\[
= \frac{3}{2} \pi^{3/2} \lambda^{-1/2} (2\pi)^3 \lambda^{-1}
\]

\[
= \frac{3}{2} \pi^{3/2} \lambda^{-1/2} C_F [\lambda]^6.
\]
Recall Claim 4:

\[
-6a \int_{\mathbb{R}^2} \Re \left\{ it |G_1(x, t)|^4 \overline{G_1(x, t)} G_3(x, t) \right\} dx dt = -4a \pi^{3/2} \lambda^{-7/2} C_F [ \lambda ].
\]

**Proof.** As with (104), we have

\[
G_1(x, t) G_3(x, t) = \left( 3 \lambda^{-2} (1 + it)^{-2} - 6 \lambda^{-3} \lambda^2 (1 + it)^{-3} + \lambda^{-4} \lambda^4 (1 + it)^{-4} \right) |G_1(x, t)|^2,
\]

and so the left hand side of (76) equals

\[
-6a \left( \frac{2 \pi}{\lambda} \right)^3 \int_{\mathbb{R}^2} \Re \left\{ it \left( 3 \lambda^{-2} (1 + it)^{-2} - 6 \lambda^{-3} \lambda^2 (1 + it)^{-3} + \lambda^{-4} \lambda^4 (1 + it)^{-4} \right) \right\} \cdot (1 + t^2)^{-3/2} e^{-\frac{\lambda y^2}{2(1+t^2)}} dx dt.
\]

Note that \( \Re \{ i (1 + it)^{-2} \} = 2t(1 + t^2)^{-2}, \Re \{ i (1 + it)^{-3} \} = (3t - t^3)(1 + t^2)^{-3} \) and \( \Re \{ i (1 + it)^{-4} \} = (4t - 4t^3)(1 + t^2)^{-4} \). Change variables \( y = x / (1 + t^2)^{1/2} \) to compute

\[
\text{I} : = \int_{\mathbb{R}^2} 2t(1 + t^2)^{-2} (1 + t^2)^{-3/2} e^{-\frac{\lambda y^2}{2(1+t^2)}} dx dt = \int_{\mathbb{R}^2} 2t(1 + t^2)^{-7/2} e^{-\frac{\lambda y^2}{2(1+t^2)}} dx dt
\]

\[
= \left( \int_{-\infty}^{\infty} \frac{2t^2}{(1 + t^2)^3} dt \right) \cdot \left( \int_{-\infty}^{\infty} e^{-\frac{\lambda y^2}{2}} dy \right) = \frac{\pi}{4} \cdot \left( \frac{\pi}{3/\lambda} \right)^{1/2} = \frac{\pi^3/2}{4\sqrt{3}} \lambda^{1/2},
\]

\[
\text{II} : = \int_{\mathbb{R}^2} x^2(3t^2 - t^4)(1 + t^2)^{-3/2} e^{-\frac{\lambda y^2}{2(1+t^2)}} dx dt
\]

\[
= \int_{\mathbb{R}^2} (3t^2 - t^4)(1 + t^2)^{-9/2} x^2 e^{-\frac{\lambda y^2}{2(1+t^2)}} dx dt
\]

\[
= \left( \int_{-\infty}^{\infty} \frac{3t^2 - t^4}{(1 + t^2)^3} dt \right) \left( \int_{-\infty}^{\infty} y^2 e^{-\frac{\lambda y^2}{2}} dy \right) = 0
\]

and

\[
\text{III} : = \int_{\mathbb{R}^2} x^4(4t^2 - 4t^4)(1 + t^2)^4(1 + t^2)^{-3/2} e^{-\frac{\lambda y^2}{2(1+t^2)}} dx dt
\]

\[
= \int_{\mathbb{R}^2} (4t^2 - 4t^4)(1 + t^2)^{-11/2} x^4 e^{-\frac{\lambda y^2}{2(1+t^2)}} dx dt
\]

\[
= \left( \int_{-\infty}^{\infty} \frac{4t^2 - 4t^4}{(1 + t^2)^3} dt \right) \left( \int_{-\infty}^{\infty} y^4 e^{-\frac{\lambda y^2}{2}} dy \right) = -\pi \cdot \left( \frac{1}{12} \left( \frac{\pi}{3/\lambda} \right)^{1/2} \right)
\]

\[
= - \frac{\pi^{3/2}}{12\sqrt{3}} \lambda^{5/2}.
\]
All in all, we have
\[
-6a \int \int \mathbb{R} \left\{ \left| it G_1(x, t) \overline{G_1(x, t)} G_3(x, t) \right| \right\} dx dt \\
= -6a \left( \frac{2\pi}{\lambda} \right)^3 (3\lambda^{-2}I - 6\lambda^{-3}II + \lambda^{-4}III) \\
= -a(18\lambda^{-2}I + 0 + 6\lambda^{-4}III) \left( \frac{2\pi}{\lambda} \right)^3 \\
= -\left( \frac{18}{4} - \frac{6}{12} \right) a\pi^{3/2} \lambda^{-7/2} (2\pi)^3 \lambda^{-1} \\
= -4a\pi^{3/2} \lambda^{-7/2} C_F [\lambda]^6,
\]
as claimed. □

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