Research of radon transport processes in anisotropic media

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Abstract. The study of radon migration in geological environments is relevant for the search and contouring of oil and gas deposits, the search for uranium and thorium ores, environmental mapping when choosing sites for the construction of industrial and residential structures, and for predicting events in seismic activity zones. The paper considers a mathematical model of the three-dimensional problem of radon diffusion-advection in piecewise constant layered media with inclusions, taking into account the anisotropy of the diffusion properties of subregions of the geological environment. A combined method for solving the problem is described, based on a combination of the methods of Laplace integral transformations, integral representations with the construction of the Green function of the host layered medium, and Fredholm integral equations of the second kind arising at the boundaries of local inclusions. The results of a comparison of the data of computational and natural experiments for some special cases are presented. A mathematical model of the inverse geometric problem of search for the boundary of a local inclusion is presented.

1. Introduction
The transportation of matter in diffusing layered media is the subject of research by both theorists and practitioners in various fields. One of the tasks is to study the migration of radon and its daughter decay products in the geological environment.

Mathematical modeling of the processes of radon distribution in the soil and its runoff into the surface layer of the atmosphere is associated with the solution of parabolic boundary-value problems in mathematical physics. The development of mathematical models, algorithms solution, and programs for calculating the processes of radon dissipation is an urgent task of practical importance in many scientific schools and fields.

Review of the literature on the topic of research shows a large number of works on the construction and study of mathematical models of radon transport processes. However, they are based only on one-dimensional diffusion or diffusion-convective mathematical models in homogeneous isotropic geological environments (for example, [1-9]).

2. Statement of the problem of diffusion-adveective transfer of radon and the way to solve it
Without limiting generality of reasoning, we will consider a horizontal layered model of a medium with local inclusions (figure 1).
Let a horizontal layered medium be divided by smooth parametrically defined boundaries
\[ \gamma_{i,0} = \left\{ \gamma_{i,1}(x, y) \right\} \rightarrow z_i = \text{const} \ n_p u \sqrt{x^2 + y^2} \rightarrow \infty \] 
(i = 0, N) into horizontal layers \( \Omega_{0,0}, \Omega_{1,0}, \ldots, \Omega_{N,0} \), filled with matter, the diffusion properties of which are described by symmetric diffusion tensors \( D_{i,0} = \begin{pmatrix} d_{xx}^{i,0} & d_{xy}^{i,0} & d_{xz}^{i,0} \\ d_{xy}^{i,0} & d_{yy}^{i,0} & d_{yz}^{i,0} \\ d_{xz}^{i,0} & d_{yz}^{i,0} & d_{zz}^{i,0} \end{pmatrix} \) and speeds of advection \( v_{0,0}, v_{1,0}, v_{2,0}, \ldots, v_{N,0} \) respectively. Each layer \( \Omega_{i,0} \) contains \( M_i \) of local inclusions \( \Omega_{i,j} \) (j = 1,1) with filled with matter, the physical properties of which are described by constant symmetric diffusion tensors \( D_{i,j} = \begin{pmatrix} d_{xx}^{i,j} & d_{xy}^{i,j} & d_{xz}^{i,j} \\ d_{xy}^{i,j} & d_{yy}^{i,j} & d_{yz}^{i,j} \\ d_{xz}^{i,j} & d_{yz}^{i,j} & d_{zz}^{i,j} \end{pmatrix} \) and speeds of advection \( v_{i,j} \), i = 0, N; j = 1, \( M_i \).

The mathematical model of diffusion-advecutive radon transfer in the field of research
\[ \Omega = \bigcup_{i=0}^{N} \bigcup_{j=0}^{M_i} \Omega_{i,j} \subset \mathbb{R}^3 \] 
can be represented by an initial-boundary-value problem of the form:

\[
\frac{\partial A_{i,j}(P,t)}{\partial t} = \text{div}(D_{i,j} \nabla A_{i,j}(P,t)) + v_{i,j} \frac{\partial A_{i,j}(P,t)}{\partial z} - \lambda(A_{i,j}(P,t) - A_{i,\infty}),
\]

\[ P = P(x, y, z) \in \Omega_{i,j}, i = 0, N; j = 0, M_i; \]

\[
((D_{i,0} \nabla A_{i,0}(P,\bar{t})) + v_{i,0} A_{i,0}(P,t))\big|_{\gamma_{i,0}} = ((D_{i+1,0} \nabla A_{i+1,0}(P,\bar{t})) + v_{i+1,0} A_{i+1,0}(P,t))\big|_{\gamma_{i,0}}, i = 0, N - 1;
\]

\[
A_{i,0}(P,t)\big|_{\gamma_{i,0}} = A_{i+1,0}(P,t)\big|_{\gamma_{i,0}}, i = 0, N - 1;
\]

\[
((D_{i,j} \nabla A_{i,j}(P,\bar{t}))) + v_{i,j} A_{i,j}(P,t))\big|_{\gamma_{i,j}} = ((D_{i,j} \nabla A_{i,j}(P,\bar{t}))) + v_{i,j} A_{i,j}(P,t))\big|_{\gamma_{i,j}}, i = 0, N; j = 1, M_i;
\]

As \( \lambda_{\infty} A_{i,j}(P,t) = A_{i,j}(P,t) \big|_{\gamma_{i,j}}, i = 0, N; j = 1, M_i; \)

as \( z \rightarrow \infty, \lim_{z \rightarrow \infty} A_{N,0}(P,t) = 0; \)

as \( m_{\infty} A_{i,0}(P,t) = A_{i,0}(P,t), i = 0, N; A_{i,j}(P,0) = 0, i = 0, N; j = 0, M_i. \)
Here $A_{ij}(P,t)$ is the function of the volumetric activity of radon (VAR) in the soil, $A_{ij}(P,t) \in C^2_\Omega \cap C^1_\Omega \cap C^0_\Omega$; $\lambda$ – decay constant of radon; $A_{ij,\infty}$ – VAR, in radioactive equilibrium with radium ($^{226}$Ra) at a given depth in the soil of the $i$ th layer, which is equal to $A_{ij,\infty} = K_{i,em} A_{i,r0} \rho_{t,ij} (1-\eta_i)$, where $K_{i,em}$ is the radon emanation coefficient, $A_{i,r0}$ – is the specific activity of a $^{226}$Ra, $\rho_{t,ij}$ – is the density of solid particles, $\eta_i$ – is the porosity of the soil, $A_i(P,t)$ – is the normal radon field describing diffusion-advection of radon in a layered medium under the assumption of the absence of inclusions. Variable $t \geq 0$ – time.

If the area $\Omega_{0,0}$ – is a surface layer of the atmosphere, then in the problem (1) let us assume $A_{0,\infty} = 0$. When $M_0 > 0$ inclusions $\Omega_{0,1}, \ldots, \Omega_{0,M_0}$ can describe residential and industrial buildings.

We apply the combined method for solving this problem based on the methods of integral transformations, integral representations and boundary integral equations, and construct an algorithm for calculating the VAR field [10].

We represent the desired function of the VAR in the soil $A_{ij}(P,t)$ as the sum of two auxiliary functions of the normal $A_i(P,t)$ and anomalous $\tilde{A}_{ij}(P,t)$ fields:

$$A_{ij}(P,t) = A_i(P,t) + \tilde{A}_{ij}(P,t), i = 0,N, j = 0,M_j,$$

where the normal radon field is determined by the boundary value problem and describes the radon field in a horizontal layered medium without inclusions:

$$\frac{\partial A_i(P,t)}{\partial t} = div(D_{i,0} \nabla A_i(P,t)) + \nu_{i,0} \frac{\partial A_i(P,t)}{\partial z} - \lambda (A_i(P,t) - A_{i,\infty}), P \in \Omega_{i,0}, i = 0,N;$$

$$\left( D_{i,0} \nabla A_{i}(P,t), \vec{n} \right) = \left( D_{i,1+0} \nabla A_{i+1}(P,t), \vec{n} \right) + \nu_{i,1+0} A_{i+1}(P,t), i = 0,N-1;$$

$$A_i(P,t) \bigg|_{\gamma_i} = A_{i+1}(P,t) \bigg|_{\gamma_i}, i = 0,N-1;$$

$$\lim_{z \to \infty} A_N(P,t) = A_{N,\infty}, \lim_{z \to \infty} A_0(P,t) = 0;$$

$$\lim_{P \in \Omega_{i,0}, \vec{x}^2 + \vec{y}^2 \to \infty} A_{i}(P,t) = \tilde{A}_i(z,t), i = 0,N;$$

$$A_i(0,0) = 0, i = 0,N.$$

Here $\tilde{A}_i(z,t)$ – the VAR in a piecewise homogeneous horizontal layered medium with plane-parallel boundaries $\vec{z} = \vec{z}_i, i = 0,N-1$ and diffusion coefficients $\tilde{d}_i = d_{i,0}, i = 0,N$. It should be noted that in the case of a homogeneous isotropic medium with plane-parallel boundaries, the function of the normal radon field $A_i(P,t)$ coincides with the function $\tilde{A}_i(z,t)$. The method of determination $\tilde{A}_i(z,t)$ is described in [2].

In the context of problem (3), the anomalous radon field satisfies the following boundary-value problem:

$$\frac{\partial \tilde{A}_{ij}(P,t)}{\partial t} = div(D_{i,j} \nabla \tilde{A}_{ij}(P,t)) + \nu_{i,j} \frac{\partial \tilde{A}_{ij}(P,t)}{\partial z} - \lambda \tilde{A}_{ij}(P,t), P \in \Omega_{ij}, i = 0,N, j = 0,M_j;$$

$$\left( D_{i,0} \nabla \tilde{A}_{i+0}(P,t), \vec{n} \right) + \nu_{i,1+0} \tilde{A}_{i+0}(P,t) = \left( D_{i,1+0} \nabla \tilde{A}_{i+1}(P,t), \vec{n} \right) + \nu_{i,1+0} \tilde{A}_{i+1}(P,t), i = 0,N-1;$$

$$\tilde{A}_{i+0}(P,t) \bigg|_{\gamma_i} = \tilde{A}_{i+1}(P,t) \bigg|_{\gamma_i}, i = 0,N-1;$$

$$\tilde{A}_i(0,0) = 0, i = 0,N-1;$$

$$\tilde{A}_i(0,0) = 0, i = 0,N.$$
\[
\left( D_{i,j} \nabla \overline{A}_{i,j}(P,t) \right) + v_{i,j} \overline{A}_{i,j}(P,t) \right) \bigg|_{\gamma_{i,j}} = \left( (D_{i,0} \nabla \overline{A}_{i,0}(P,t) + v_{i,0} \overline{A}_{i,0}(P,t) + \psi_{i,0}(P,t) ) \bigg|_{\gamma_{i,j}} ,
\]

\[
i = 0, N, j = 1, M_j, \quad \psi_{i,0}(P,t) = ((D_{i,0} - D_{i,j}) \nabla A_i(P,t) + (v_{i,0} - v_{i,j}) A_i(P,t) \quad (*)
\]

\[
\overline{A}_{i,j}(P,t) \bigg|_{\gamma_{i,j}} = \overline{A}_{i,0}(P,t) \bigg|_{\gamma_{i,j}}, \quad i = 0, N, j = 1, M_j;
\]

\[
\lim_{P \to \infty} \overline{A}_{i,0}(P,t) = 0, i = 0, N;
\]

\[
\overline{A}_{i,j}(P,0) = 0, i = 0, N, j = 0, M_j.
\]

Function \( \psi_{i,0}(P,t) \) of the form (*) is deduced by applying equality (2) to the boundary conditions of the original problem (1).

Let’s make a substitution in the problem (4):

\[
\overline{A}_{i,j}(P,t) = e^{-\gamma t} u_{i,j}(P',t),
\]

where \( P' = (x, y, z'), z' = z + v_{i,j}t \).

It gives the following problem:

\[
\frac{\partial u_{i,j}(P',t)}{\partial t} = \text{div}(D_{i,j} \nabla u_{i,j}(P',t)), \quad P' \in \Omega_{i,j}, i = 0, N, j = 0, M_j;
\]

\[
\left( (D_{i,0} \nabla u_{i,0}(P',t) + v_{i,0} u_{i,0}(P',t) \right) \bigg|_{\gamma_{i,0}} = \left( (D_{i,0} \nabla u_{i,0}(P',t) + v_{i,1,0} u_{i,0}(P',t) \bigg|_{\gamma_{i,n}}, i = 0, N - 1;
\]

\[
\left( (D_{i,j} \nabla u_{i,j}(P',t) + v_{i,j} u_{i,j}(P',t) \bigg|_{\gamma_{i,j}} = \left( (D_{i,0} \nabla u_{i,0}(P',t) + v_{i,1,0} u_{i,0}(P',t) + \psi_{i,0}(P',t) \bigg|_{\gamma_{i,j}}, \quad i = 0, N, j = 1, M_j;
\]

\[
\left. u_{i,j}(P',t) \right|_{\gamma_{i,j}} = u_{i,0}(P',t), \quad i = 0, N, j = 1, M_j;
\]

\[
\lim_{P' \to \infty} u_{i,0}(P',t) = 0, i = 0, N;
\]

\[
u_{i,j}(P',0) = 0, i = 0, N, j = 0, M_j.
\]

We apply to problem (6) the solution method described in [11] using the Laplace integral transformation

\[
F(P',s) = \int_0^\infty u(P',t)e^{-\sigma t} dt
\]

with the Riemann-Mellin inversion formula

\[
u(P',t) = \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} F(P',s)e^{\sigma t} ds.
\]
\[ F_{i,0}(P',s) \big|_{\gamma_i} = F_{i+1,0}(P',s) \big|_{\gamma_i}, \quad i = 0, N-1; \]
\[ ((D_{i,j} \nabla F_{i,j}(P',s), \vec{n}) + v_{i,j} F_{i,j}(P',s)) \big|_{\gamma_{i,j}} = ((D_{i,j} \nabla F_{i,j}(P',s), \vec{n}) + v_{i,j} F_{i,j}(P',s) + F_{\psi,0}(P',s)) \big|_{\gamma_{i,j}}, \]
\[ i = 0, N, j = 1, M_i, \quad F_{\psi,0}(P',s) = ((D_{i,j} - D_{i,j}) \nabla F_{i,j}(P',s), \vec{n}) + (v_{i,j} - v_{i,j}) F_{i,j}(P',s); \]
\[ F_{i,j}(P',s) \big|_{\gamma_{i,j}} = F_{i,0}(P',s) \big|_{\gamma_i}, \quad i = 0, N, j = 1, M_i; \]
\[ \lim_{P' \to \infty} F_{i,j}(P',s) = 0, \quad i = 0, N, \]

where the functions \( F_{\psi,0}(P',s) \) and \( F_{i,j}(P',s) \) are the images of the functions \( \psi_{i,0}(P',t) \) and \( A_i(P',t) \) in the transformation (7) respectively.

The problem (9) – is a boundary value problem for differential equations in partial derivatives of elliptic type. We solve it by the method of integral representations and integral equations. To do this, we analyze the auxiliary problem for the Green's function \( G(P,Q) \) – the function of a point source located at an arbitrary point \( Q(x_q, y_q, z_q) \) and generating a diffusion field of ordinary force in the surrounding space (in a layered medium without inclusions):

\[ \text{div}(D_{i,j} \nabla G_{i,j}(P',Q)) - sG_{i,j}(P',Q) = -\delta(P',Q), \quad P' \in \Omega_{i,j}, \quad i = 0, N; \]
\[ ((D_{i,j} \nabla G_{i,j}(P',Q), \vec{n}) + v_{i,j} G_{i,j}(P',Q)) \big|_{\gamma_{i,j}} = ((D_{i,j} \nabla G_{i,j}(P',Q), \vec{n}) + v_{i,j} G_{i,j}(P',Q)) \big|_{\gamma_{i,j}}, \]
\[ i = 0, N - 1; \]
\[ G_{i,0}(P',Q) \big|_{\gamma_i} = G_{i+1,0}(P',Q) \big|_{\gamma_{i+1}}, \quad i = 0, N - 1; \]
\[ \lim_{P' \to \infty} G_{i,j}(P',Q) = 0, \quad i = 0, N. \]

Then the integral representation of the solution of problem (9) in the area \( \Omega \) will have the form:

\[ F(P',s) = \sum_{i=0}^{N} \sum_{j=1}^{M_i} \int_{\gamma_{i,j}} F_{i,j}(Q,s) (v_{i,j} - v_{i,j}) G_{i,0}(P',Q) + ((D_{i,j} - D_{i,j}) \nabla G_{i,j}(P',Q), \vec{n}_0)) d\gamma'_{i,j} + \]
\[ + \sum_{i=0}^{N} \sum_{j=1}^{M_i} \int_{\gamma_{i,j}} F_{\psi,0}(Q,s) G_{i,0}(P',Q) d\gamma'_{i,j} \]

Here \( \vec{n}_0 \) – is the vector of the external normal to the inclusion boundary at the point \( Q \).

It follows from (11), that the solution of the problem (9) can be obtained at any point \( P' \) in a piecewise constant anisotropic medium \( \Omega_{i,j} (i = 0, N, j = 1, M_i) \), if the solution of the problem (10) – the Green's function \( G_{i,j}(P',Q) \), \( i = 0, N \), and the boundary values of the function \( F_{i,j}(Q,s) \) are known at the inner boundaries of the subdomains not included to the problem for the Green's function..

Omitting in (11) the point \( P' \) on each of such boundaries, we get a system of Fredholm integral equations of the second kind relatively unknown boundary values \( F_{i,j}(Q,s) \) of the form:

\[ F(P',s) - \sum_{i=0}^{N} \sum_{j=1}^{M_i} \int_{\gamma_{i,j}} F_{i,j}(Q,s) (v_{i,j} - v_{i,j}) G_{i,0}(P',Q) + ((D_{i,j} - D_{i,j}) \nabla G_{i,j}(P',Q), \vec{n}_0)) d\gamma'_{i,j} = \]
\[ = \sum_{i=0}^{N} \sum_{j=1}^{M_i} \int_{\gamma_{i,j}} F_{\psi,0}(Q,s) G_{i,0}(P',Q) d\gamma'_{i,j}, \]
Thus, the algorithm for solving the original problem (1) has the form:

Step 1. We determine the normal radon field $\vec{A}_i(z,t)$ in a horizontal layered piecewise homogeneous medium with plane-parallel boundaries $z = z_i = \text{const}, i = 0, N - 1$ and diffusion coefficients $\vec{d}_i = d_{zz}^{ij}, i = 0, N$, and speeds of advection $\nu_{i,0}, i = 0, N$ according to the algorithm described in [2].

Step 2. If the boundaries of the layers $z = \gamma_{i,0}(x, y) = z_i = \text{const}$, that is, the medium has plane-parallel boundaries, then the solution for the problem (3) for the normal radon field is found: $A_i(P, t) = \vec{A}_i(z, t)$. Otherwise, the problem (3) should be solved, for example, by the method of integral equations, forming them into sections $\gamma_{i,0}(x, y) \neq z_i$.

Step 3. We calculate the functions $\psi^{i,0}(P', t)$ at the boundaries of inclusions $\gamma_{i,j}, i = 0, N, j = 1, M_i$ by the formula (*).

Step 4. For each of the parameter value $s$ of the set of quadrature nodes of the numerical inversion of the Laplace transformation (in accordance with the algorithm in [12]) by the formula (7):

Step 4.1. We find images $F_{\nu_{i0}}(P', s)$ of functions $\psi^{i,0}(P', t)$ under transformations (5) and (7).

Step 4.2. We find solution for the problem (10) with the Green’s function. It can be worked out analytically for the case of homogeneous layers with plane-parallel boundaries, for example, with the help of the Hankel integral transformation.

Step 4.3. We form the system (12) and find its solution – the boundary (at the boundaries of the inclusions) values of the function $F_{i,j}(Q, s)$.

Step 4.4. Using the formula (11), we find solution for the problem (9) – the function $F_{i,j}(P', s)$.

Step 4.5. We state the summand of the quadrature formula for the integral (8), calculating the functions $u_{i,j}(P', t)$.

Step 5. We find the anomalous field $\vec{A}_{i,j}(P, t)$ by the formula (5).

Step 6. Solution of the original problem (1) – the function $A_{i,j}(P, t)$ we work out according to the formula (2).

The proposed combined methods and algorithms are the development of the theory of solving boundary value problems for heat and mass transfer equations in piecewise constant anisotropic media and let us solve the practical problems in studying the processes of radon transport in three-dimensional piecewise constant anisotropic layered media with anisotropic inclusions.

3. The results of the natural and computational experiments

In accordance with the proposed method for solving the problem by means of the Maple computer system, programs have been developed that implement numerical algorithms for finding the functions of the normal radon field, the Green function in a piecewise homogeneous horizontal layered medium with plane-parallel boundaries, the inversion of the Laplace integral transformation, and the function of the anomalous radon field which takes into account the influence of inclusions.

The adequacy of the theoretical solutions obtained for particular cases has been tested on laboratory models, that makes it possible to take into account the initial calculated parameters [13].

Figure 2 shows the experimental device which is a cylindrical tank (diameter – 0.55 m, height – 1.30 m), filled with granite screenings (fraction 1-5 mm). The specific activity of radium was 55 Bq/kg (determined by a Gamma-1C gamma spectrometer). The emanation coefficient was taken from studies of these granite screenings when developing a standard radon sample. The diffusion coefficient was worked out during experimental studies to determine the radon flow density by the screen method [14]. The density of granite screenings was determined by the standard method.
To measure VAR horizontally, air intakes are inserted on two sides in diameter, consisting of two pieces of metal tubes, hermetically sealed at both ends. The bottom of the tank is hermetically sealed; the top is open into the atmosphere. The measurements were carried out in convective mode [15, 16]. Pore air was drawn in by a pump from one end of the tube, passed through a measuring device (Alpha Guard PQ 2000) and returned through the other end of the tube back to the container, to the sampling point.

![Diagram](image)

**Figure 2.** The geometry of the experimental device for studying the distribution of VAR: a) excluding local inclusions; b) taking into account the local inclusion.

In accordance with the worked out mathematical model and with the geometry of this device (figure 2a), a calculation model was developed (figure 3), which consists of three layers: an air layer, a soil layer (granite screening) and a layer filled with a substance with a diffusion coefficient nonzero (it is impossible to do it equal to zero due to the mathematical model). In the last layer, the volumetric activity of radon in radioactive equilibrium with radium is zero (imitation of the impermeability of the bottom of the tank).

According to this calculation model, the following parameter values were taken:

- $n = 3$ – number of layers;
- $t = 2419200$ s (28 days) – time of test indication;
- $\lambda = 2.1 \cdot 10^{-6} \text{s}^{-1}$ – radon decay constant;
- $z_0 = 0 \text{m}, z_1 = 1.3 \text{m}$ – layer boundaries;
- $d_{1,0} = 1 \cdot 10^{-5} \text{m}^2 / \text{s}, d_{1,0} = 5 \cdot 10^{-7} \text{m}^2 / \text{s}, d_{2,0} = 1 \cdot 10^{-10} \text{m}^2 / \text{s}$ – diffusion coefficients in each layer, respectively;
- $\nu_{1,0} = \nu_{1,0} = 0 \text{m/s}$ – speeds of advection in each layer, respectively;
- $\rho_{1,0} = 2000 \text{kg/m}^3$ – density of granite screenings;
- $\eta_1 = 0.2$ – porosity of granite screenings;
$K_{1,em} = 0.15$ – coefficient of emanation of granite screenings;

$A_{1,Ra} = 55 \text{ Bq/kg}$ – specific activity of $^{226}Ra$.

The value of the VAR in radioactive equilibrium with radium, in other layers of the calculation model, $A_{1,Ra} = A_{2,Ra} = 0 \text{ Bq/m}^3$.

Figure 4 shows the results of numerical simulation (dashed line). Dots in the graph indicate the results worked out experimentally.

Figure 3. The calculation three-layer model. 

Figure 4. The graph of the VAR function.

A comparative analysis of the results at these five points without taking into account local inclusions is given in table 1, where: $z$ – depth, m; $A_{exp}$ – experimental values of the VAR, Bq/m$^3$; $A_{num}$ – VAR values found by the worked out algorithms, Bq/m$^3$; $\Delta$ – absolute error, Bq/m$^3$; $\delta$ – relative error, %.

| $z$ (m) | $A_{exp}$ (Bq/m$^3$) | $A_{num}$ (Bq/m$^3$) | $\Delta$ (Bq/m$^3$) | $\delta$ (%) |
|--------|---------------------|---------------------|---------------------|-------------|
| 0.15   | 6700                | 5186                | 1514                | 22.60       |
| 0.4    | 7600                | 8287                | 687                 | 9.04        |
| 0.65   | 9800                | 10082               | 282                 | 2.88        |
| 0.9    | 11100               | 11054               | 46                  | 0.41        |
| 1.15   | 9500                | 11465               | 1965                | 20.68       |

Thus, the results of the comparison of the data of computational and natural experiments on the study of radon transport processes in piecewise homogeneous horizontal layered media showed the adequacy and reliability of the proposed model and method for solving the problem.

The statement of the problem and the proposed algorithm for its solution allowed us to proceed to the calculation of more complex models with the inclusion of local heterogeneities and their testing on laboratory stands. The geometry of the experimental device, taking into account local inclusion, is shown in figure 2b. The inclusion is a cylindrical grid filled with quartz screenings (fraction 3-5 mm) with evenly distributed pieces of granite with uranium mineralization. The calculation model corresponds to figure 3. For this experiment, the following values of the calculated parameters were taken:

$n = 3$ – number of layers;
$t = 2419200 \text{ s (28 days)}$ – time of test indication;
$\lambda = 2.1 \cdot 10^{-6} \text{ s}^{-1}$ – radon decay constant;
$z_0 = 0 \text{ m}, z_1 = 1.3 \text{ m}$ – layer boundaries;
$d_{0,0} = 1 \cdot 10^{-5} \text{ m}^2 \text{ / s}, d_{1,0} = 1 \cdot 10^{-7} \text{ m}^2 \text{ / s}, d_{2,0} = 1 \cdot 10^{-11} \text{ m}^2 \text{ / s}$ – diffusion coefficients in each layer, respectively;
$d_{1,1} = 5 \cdot 10^{-6} \text{ m}^2 \text{ / s}$ – diffusion coefficient for cylindrical inclusion;
$v_{0,0} = v_{1,0} = v_{2,0} = 0 \text{ m / s}$ – speeds of advection in each layer, respectively;
$v_{1,1} = 0 \text{ m / s}$ – speed of advection for cylindrical inclusion;
$\rho_1 = 2600 \text{ kg / m}^3$ – density of granite screenings;
$\eta_1 = 0.1$ – porosity of granite screenings;
$K_{1,em} = 0.15$ – coefficient of emanation of granite screenings;
$A_{0,0} = 55 \text{ Bq / kg}$ – specific activity of $^{226}\text{Ra}$.

The value of the VAR, in radioactive equilibrium with radium, in other layers of the calculation model, $A_{0,0} = A_{2,0} = 0 \text{ Bq / m}^3$.

Figure 5 shows the results of numerical modeling (dashed line). Dots in the graph also indicate the results worked out experimentally.

![Figure 5. The graph of the VAR function.](image)

A comparative analysis of the results at these five points, taking into account local inclusion, is given in table 2.

| $z$ (m) | $A_{exp}$ (Bq/m$^3$) | $A_{num}$ (Bq/m$^3$) | $\Delta$ (Bq/m$^3$) | $\delta$ (%) |
|--------|----------------------|----------------------|----------------------|-------------|
| 0.15   | 8520                 | 9883                 | 1363                 | 16          |
| 0.4    | 14000                | 14893                | 893                  | 6.38        |
| 0.65   | 18500                | 18544                | 44                   | 0.24        |
| 0.9    | 22900                | 19567                | 3333                 | 14.55       |
| 1.15   | 18900                | 19241                | 341                  | 1.8         |
The results of the comparison of the data of computational and natural experiments on the study of radon transport processes in piecewise constant isotropic layered media with inclusions showed the adequacy and reliability of the proposed model and method for solving the problem.

With the help of written software package, computational experiments to study the diffusion-advection of radon in piecewise constant layered media with anisotropic inclusions were carried out. It should be noted that from a geophysical point of view, it is of particular interest to construct the surface of the diffusion field function above the inclusion, which corresponds to areal measurements.

So, for a piecewise homogeneous plane-parallel horizontal layered medium with a spherical inclusion $\Omega_{4.1}$ of radius $R = 0.5$ m centered at the point $(1.1,7)$, the graph of the surface of the desired VAR function in isolines is constructed, in the rectangle $x, y \in [-2:4]$, in a two-dimensional subspace $z_1 = 1$ m (figure 6a). The parameters values of the medium correspond to the case of the diffusion-advection model of radon transfer for a five-layer horizontal layered medium with plane-parallel boundaries [4]: $z_0 = 0$ m, $z_1 = 1$ m, $z_2 = 3$ m, $z_3 = 6$ m – boundaries; $d_{i,0} = 1 \cdot 10^5$ m$^2$/s, $d_{i,0} = d_{2,0} = d_{3,0} = d_{4,0} = 3 \cdot 10^{-6}$ m$^2$/s – diffusion coefficients in each layer, respectively; $v_{i,0} = 0$ m/s, $v_{1,0} = v_{2,0} = v_{3,0} = v_{4,0} = 4 \cdot 10^{-6}$ m/s – speeds of advection in each layer, respectively; $\rho_{1s} = \rho_{2s} = \rho_{3s} = \rho_{4s} = 2700$ kg/m$^3$ – density of solid particles of soil for each layer; $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 0.45$ – soil porosity for each layer; $K_{1,em} = K_{2,em} = K_{3,em} = K_{4,em} = 0.2$ – soil emanation coefficients for each layer; $A_{1,Ra} = 90$ Bq/kg, $A_{2,Ra} = 4$ Bq/kg, $A_{3,Ra} = 30$ Bq/kg, $A_{4,Ra} = 1000$ Bq/kg – specific activity of $^{226}\text{Ra}$ for each layer.

The physical properties of this inclusion are described by a symmetric diffusion tensor
\[
D_{4.1} = \begin{pmatrix}
3 \cdot 10^{-5} & 0 & 0 \\
0 & 3 \cdot 10^{-5} & 0 \\
0 & 0 & 3 \cdot 10^{-5}
\end{pmatrix}
\]
and speed of advection with a value of $v_{4.1} = 4 \cdot 10^{-5}$ m/s. In the case of an anisotropic spherical inclusion $\Omega_{4.1}$, the diffusion properties of which are described by the tensor
\[
D_{4.1} = \begin{pmatrix}
3 \cdot 10^{-5} & 10^{-5} & 0 \\
10^{-5} & 5 \cdot 10^{-5} & 0 \\
0 & 0 & 3 \cdot 10^{-5}
\end{pmatrix}
\]
the surface graph of the VAR function in the isolines has the form (figure 6b).

Figure 6. The surface graphs of the VAR function in the isolines:
a) excluding inclusion anisotropy; b) taking into account the inclusion anisotropy.
4. Statement of the inverse problem
The inverse problems are of practical interest, the theory of them, due to its theoretical and applied significance, is one of the fast developing branches of the modern theory of partial differential equations. Inverse problems for the second-order parabolic equations arise in the study of physical processes such as thermal conductivity, diffusion, propagation of electromagnetic fields in conductive media, the motion of a viscous fluid in the case when the physical characteristics of the medium are not accessible for direct measurements, but at the same time, it is possible to get additional information about the characteristics of the process itself [17].

For the discussed three-dimensional problem of radon diffusion-advection in piecewise constant layered media with inclusions (1), using additional information about the solution for each moment of time at the ground / air interface and / or at some internal points of the computational domain (in wells, mines), one can set the following types of inverse problems: identification of diffusion tensors or advection velocities (coefficient inverse problem); identification of the flow to the part of the boundary that is inaccessible to measurement (boundary inverse problem) when using additional information about the solution at the internal points of the computational domain; identification of initial conditions (evolutionary inverse problem), identification of the boundary of local inclusion (geometric inverse problem) [18].

So, the mathematical model of the inverse geometric problem of searching for the boundary \( \gamma_{i,k} \) of local inclusion \( \Omega_{i,k} \) (for example, for the problems of searching for karst caverns or the problems of delineating hydrocarbon deposits) in space \( W^2_t(H) \), is like extremals of A.N. Tikhonov’s regularized functional \( F^\alpha(\gamma_{i,k}) \) with the known, got in the surface layer of the atmosphere (on the set \( E \times [t_0, t] \), \( P \in E \subset \Omega_{0,0} \)), the experimental values \( \tilde{A}(P, t) \) of the field VAR, has the form:

\[
\gamma_{i,k} = \text{Arg min} F^\alpha(\gamma_{i,k}) = \text{Arg min} \left\{ A(P, t, \gamma_{i,k}) - \tilde{A}(P, t) \right\}^2_{L_2(E \times [t_0, t])} + \alpha \left\| \gamma_{i,k} - \gamma_{i,k}^* \right\|^2_{W^2_t(H)} \right\} \quad (13)
\]

where \( A(P, t, \gamma_{i,k}) \) – is the solution of the direct problem (1), \( \gamma_{i,k}^* \) is the apriori known description of the boundary, \( H \subset R^3 \) – is the set of changes in the parameters of the parametric description of the boundary. That is, the desired boundary is sought as a normal, relatively \( \gamma_{i,k}^* \), quasisolution of problem (1)-(13) – as an extremal of the A.N. Tikhonov’s regularizing functional (13) when solving the problem (1). Here \( F_1(\gamma_{i,k}) = \left\| A(P, t, \gamma_{i,k}) - \tilde{A}(P, t) \right\|^2_{L_2(E \times [t_0, t])} \) – is the residual functional, \( F_2(\gamma_{i,k}) = \left\| \gamma_{i,k} - \gamma_{i,k}^* \right\|^2_{W^2_t(H)} \) – is the stabilizing functional, \( \alpha \) – is the regularization parameter.

5. Conclusion
The results of a comparison of the data of computational and natural experiments showed the adequacy and reliability of the proposed model and method for solving the three-dimensional problem of radon diffusion-advection in piecewise constant layered media with inclusions. It is also shown in the work that taking anisotropy into account when modeling the processes of radon transport in geological environments leads to a significant change in the field of volumetric activity of radon and is a significant factor necessary in the description of the mathematical model of the field in real geological environments.

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