Power-Law distributions and Fisher’s information measure

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Abstract

We show that thermodynamic uncertainties (TU) preserve their form in passing from Boltzmann-Gibbs’ statistics to Tsallis’ one provided that we express these TU in terms of the appropriate variable conjugate to the temperature in a nonextensive context.

KEYWORDS: Fisher information, escort probabilities, uncertainty relations.
INTRODUCTION

Thermodynamics’ “uncertainty” (TU) relations have been the subject of much interesting work over the years (see, for instance, [1, 2, 3, 4]). An excellent, recent review is that of Uffink & van Lith [5]. A possible starting point for the derivation of TU is statistical inference [6]. The pioneer work of Mandelbrot is in the sense an obligatory reference [2]. The interest in TU derives from the fact that for two pillars of 20-th century science, 1) Heisenberg’s uncertainty relations and 2) Bohr’s complementarity principle, the existence of a classical analogue has been suggested. Since such a proposal comes from none other than Heisenberg and Bohr, the matter has been subjected to careful scrutiny, specially in the case of a putative complementarity between temperature and energy [7]. Although these ideas have not received general acceptance, several renowned authors have defended them. We can cite, among others, Refs. [1, 2, 3], whose claims remain still somewhat controversial (see [5, 8, 9]).

The purpose of the present effort is to make a (hopefully useful) contribution to the ongoing discussion by concentrating attention on particular aspects of Mandelbrot’s thermal uncertainty derivation [2], that includes, as an essential ingredient, the information measure introduced by Fisher in the twenties [6, 10]. Mandelbrot [2] was one of the first authors that linked statistical physics with the theory of statistical inference, adopting the viewpoint that one can work in statistical mechanics directly with probability distributions over macroscopic variables, the phase space microscopic substructure being largely superfluous.

Let $U$ stand for the internal energy. Mandelbrot [2] established the form of the probability distribution $p_\gamma(U)$ that allows for an appropriate description of the energy fluctuations of a system in contact with a heat bath at the temperature $T = 1/\gamma$. The ensuing distribution turns out to be the celebrated Gibbs’ canonical one [11], namely, an exponential probability density $\exp[-\gamma U]$. A quite interesting uncertainty relation between mean energy and inverse temperature can then be obtained, as detailed below. The leading role in Mandelbrot’s treatment is played by Fisher’s information measure [6, 10, 12, 13]

$$ I = \int d\mathbf{x} p_\theta(\mathbf{x}) \left( \frac{\partial p_\theta(\mathbf{x})}{\partial \theta} \right)^2 \left( \frac{1}{p_\theta} \frac{\partial p_\theta}{\partial \theta} \right)^2. $$

In order to get some insight into the significance of this measure consider a system that is specified by a physical parameter $\theta$. Let $\mathbf{x}$ be a stochastic variable and $p_\theta(\mathbf{x})$ the proba-
probability density for this variable, which depends on the parameter \( \theta \). An observer makes a measurement of \( x \) and has to best infer \( \theta \) from this measurement, calling the resulting estimate \( \tilde{\theta} = \tilde{\theta}(x) \). One wonders how well \( \theta \) can be determined. Estimation theory asserts that the best possible estimator \( \tilde{\theta}(x) \), after a very large number of \( x \)-samples is examined, suffers a mean-square error \( e^2 \) from \( \theta \) that obeys a relationship involving Fisher’s \( I \), namely, \( I e^2 = 1 \). This “best” estimator is called the efficient estimator. Any other estimator must have a larger mean-square error. The only proviso to the above result is that all estimators be unbiased, i.e., satisfy \( \langle \tilde{\theta}(x) \rangle = \theta \). Thus, Fisher’s information measure has a lower bound, in the sense that, no matter what parameter of the system we choose to measure, \( I \) has to be larger or equal than the inverse of the mean-square error associated with the concomitant experiment. This result, i.e.,

\[
I e^2 \geq 1,
\]

is referred to as the Cramer-Rao bound, and constitutes a very powerful statistical result.

The central point of the present considerations is that, in addition to the exponential distributions considered in [14], we often encounter power-law distributions (PLD) as well. PLD are certainly ubiquitous in physics (critical phenomena are just a conspicuous example [15]). It is well known that in a statistical mechanics’ context power-law distributions arise quite naturally if the information measure one maximizes (subject to appropriate constraints) in order to arrive at the equilibrium distribution is not Shannon’s one but a generalized one. Much effort in this respect has lately been reported. People employ in this type of extremizing processes Tsallis’ information measure as the quantity of interest (see [16, 17] and references therein).

Taking into account the importance of the concomitant results [16], it is almost obligatory to revisit the Fisher-Mandelbrot link by examining non-exponential distributions of the power-law kind. A first step in this direction was taken in [4], where it was shown that the above mentioned “exponential” Fisher-Mandelbrot link cannot straightforwardly be generalized to power-law distributions so as to immediately yield thermal uncertainty relations. Here we wish to explore into some more depth the issues investigated in [4] by introducing the concept of “effective” energy into the pertinent discussion.

This paper is organized as follows. In Section II we introduce the notion of “effective energy”, the leitmotif of the present considerations, and show that, with its help, a thermal
uncertainty relation can be derived. The meaning of this effective energy is discussed in Section III and some conclusions are drawn in Section IV.

THE EFFECTIVE ENERGY AND THERMAL UNCERTAINTY

Escort distributions [18] are a typical feature of Tsallis’ thermostatistics and of its concomitant power-law distributions [19]. It is then necessary for our present purposes to deal with the Fisher information notion as adapted to a escort probability environment, i.e., with distributions of the form $P_\theta(x) = p_\theta(x)/\int dxp_\theta(x)$. Following [4, 6, 10, 20] we cast such a measure in the fashion

$$I = \int d\mathbf{x} P_\theta(\mathbf{x})^{-1} \left[ \frac{\partial P_\theta(\mathbf{x})}{\partial \theta} \right]^2 = \left\langle \left[ \frac{1}{P_\theta} \frac{\partial P_\theta}{\partial \theta} \right]^2 \right\rangle_{esc},$$

(3)

where $p_\theta(x)$ is, again, the probability density for the stochastic variable $x \in \mathbb{R}^N$, $\theta$ a thermal parameter of the system, for example the inverse temperature $\beta$, and $q$ a real parameter that can be identified with Tsallis’ nonextensivity index [21, 22]. We speak then of “Fisher measures in a nonextensive context” [20]. The associated Cramer-Rao bound takes the form $I \Delta \theta \geq q^2$ [4, 20].

We start our considerations by writing down the probability distribution $P_\beta(x)$ that extremizes Tsallis’ information measure [21] subject to appropriate constraints posed by our a priori knowledge. In a canonical thermodynamical system, the inverse temperature $\beta$ becomes a most appropriate parameter. In the present instance the piece of information supposedly known a priori is the generalized expectation value $U_q$ of the internal energy $U(x)$

$$U_q = \langle U \rangle_{esc} = \int d\mathbf{x} P_\beta^q(\mathbf{x}) U(\mathbf{x}).$$

(4)

According to the Tsallis’ formalism, as encapsulated by its optimal Lagrange multipliers version [23] we face the following probability distribution

$$p_\beta(x) = Z_q^{-1} e_q \{-\beta[U(x) - U_q]\},$$

(5)

where $\beta$ is the variational Lagrange multiplier associated to $U_q$ and $Z_q$ is the accompanying partition function (as we sum over microstates no structure constant is needed [24])

$$Z_q = \int d\mathbf{x} e_q (-\beta (U(\mathbf{x}) - U_q)).$$

(6)
We have made use of the so-called generalized exponential [22]
\[
e_q(x) = [1 + (1 - q)x]^{1/q} \quad \text{if} \quad [1 + (1 - q)x] \geq 0
= 0 \quad \text{otherwise},
\]
(7)
a generalization of the exponential function, which is recovered when \(q \to 1\).

For the sake of an easier notation we shall try to omit hereafter, as far as possible, writing
down explicitly the variable \(x\). In order to accomplish our purposes we need to evaluate the
integrand in (3). The definition of escort distribution is now given by
\[
P_\beta = p^q_{\beta} / \int d\mathbf{x} p^q_{\beta}.
\]
Returning to our present task and taking derivatives in \(P_\beta\) we find
\[
\frac{\partial P_\beta}{\partial \beta} = q P_\beta \left\{ p^{-1}_\beta \frac{\partial p_\beta}{\partial \beta} - \left\langle p^{-1}_\beta \frac{\partial p_\beta}{\partial \beta}\right\rangle_{\text{esc}} \right\},
\]
(8)
and proceed then to take the pertinent derivatives in Eq. (5), so as to confront
\[
p^{-1}_\beta \frac{\partial p_\beta}{\partial \beta} = -\bar{Z}^{-1}_q p^{-1}_\beta (U - U_q),
\]
(9)
where we made use of the fact that \(d \ln \bar{Z}_q / d \beta = 0\), a result that one can derive by recourse
to Eqs. (5) and (6) and/or can be encountered in [23].

Notice that, from (5), we immediately obtain
\[
\bar{Z}^{-1}_q p^{-1}_\beta = \{1 - (1 - q)\beta (U - U_q)\}^{-1}.
\]
(10)
At this precise stage we define the quantity \(E_{\text{eff}}\), an “effective” energy given by
\[
E_{\text{eff}} = \frac{U - U_q}{1 - (1 - q)\beta (U - U_q)},
\]
(11)
which enables one to write
\[
p^{-1}_\beta \frac{\partial p_\beta}{\partial \beta} = -E_{\text{eff}}.
\]
(12)
Replacement of the last two relations into (8) leads now to
\[
\frac{\partial P_\beta(\mathbf{x})}{\partial \beta} = q P_\beta(\mathbf{x}) (-E_{\text{eff}} + \langle E_{\text{eff}}\rangle_{\text{esc}}),
\]
(13)
and then to
\[
P^{-1}_\beta(\mathbf{x}) \left( \frac{\partial P_\beta(\mathbf{x})}{\partial \beta} \right)^2 = q^2 P_\beta(\mathbf{x}) (E_{\text{eff}} - \langle E_{\text{eff}}\rangle_{\text{esc}})^2,
\]
(14)
which, when finally replaced into (3), that is, integrating both sides of (14) over \(dx\), gives to Fisher’s information measure the appearance

\[
I = q^2 \langle (E_{\text{eff}} - \langle E_{\text{eff}} \rangle_{\text{esc}})^2 \rangle_{\text{esc}}.
\]  

(15)

A little additional algebra allows one finally to write

\[
q^{-2} I = \mu_{E_{\text{eff}}} \equiv \langle E_{\text{eff}}^2 \rangle_{\text{esc}} - \langle E_{\text{eff}} \rangle_{\text{esc}}^2,
\]

(16)

so that the Cramer-Rao bound gives

\[
\mu_{E_{\text{eff}}} \Delta \beta \geq 1,
\]

(17)

which is indeed an uncertainty relation.

MORE ON THE EFFECTIVE ENERGY

In this section we delve further into the effective energy concept. It is our aim here to show that \(E_{\text{eff}}\) is the conjugate variable to \(\beta\). For this purpose we start by reminding the reader of the result \[23\]

\[
\int dx p_{\beta}^q = \bar{Z}^{1-q},
\]

(18)

which together with the definition of escort distribution gives

\[
P_{\beta} = \bar{Z}^{q^{-1}} p_{\beta}^q
\]

(19)

Thus, it is possible to view the escort probabilities \(P_{\beta}\) in a different light by introducing Eq. (10) into above and obtaining

\[
P_{\beta} = \frac{p_{\beta}}{1 - (1 - q)\beta(U - U_q)}.
\]

(20)

We thus establish a new connection between \(P_{\beta}\) and \(p_{\beta}\) in terms of the internal energy \(U\). Further, by introducing Eq. (20) into the definition \(\int dx P_{\beta}(U - U_q) = 0\) we also find that

\[
\int dx p_{\beta} E_{\text{eff}} = 0,
\]

(21)
which tells us that \( E_{\text{eff}} \) is a proper “centered” variable. Also, by using Eq. (11) we can obtain \( U - U_q \) as function of \( E_{\text{eff}} \), namely,

\[
U - U_q = \frac{E_{\text{eff}}}{1 + (1 - q)\beta E_{\text{eff}}},
\]

so that replacing this into (5) we find

\[
p_{\beta}(x) = \frac{1}{Z_q} \frac{1}{e_q(\beta E_{\text{eff}})}.
\]

We point out now that

1. the probability (23) is to be compared to the Gibbs canonical distribution

\[
p_{\beta}(x) = \frac{1}{Z_{\text{Gibbs}}} \frac{1}{e^{\beta U}}
\]

(in the limit \( q \to 1 \) (23) tends to (24)), and

2. the \( q \)-form (Cf. Eq. (23)) of \( p_{\beta}(x) \) neatly captures the fact that the probability distribution formally depends, for fixed \( q \), only upon \( \beta \) and \( E_{\text{eff}} \).

\( \beta \) and \( E_{\text{eff}} \) are thus our two “conjugate” variables in the present nonextensive instance. It does makes sense then to apply to them the Cramer-Rao bound. Remember also that Mandelbrot never contemplated actual temperature fluctuations. He assumed throughout that \( T \) is fixed. Instead, the estimators are random quantities. Such is the meaning one is to read in a thermal uncertainty relation. And this meaning is here precisely the same, via the somewhat artificial concept of effective energy.

**CONCLUSIONS**

We have shown that a thermal uncertainty relation can be derived for power law distributions with the help of the effective energy concept \( (E_{\text{eff}}) \), that, as we have discovered here, is the conjugate variable to the inverse temperature in a nonextensive Tsallis setting.

The concept of thermal uncertainty applies to conjugate variables. These are

1. \( \beta \) and \( U \) for \( q = 1 \), and

2. \( \beta \) and \( E_{\text{eff}} \) for \( q \neq 1 \). Of course, \( E_{\text{eff}} \to U \) in the limit \( q \to 1 \).
Summing up, if one looks for the appropriate conjugate variables, the concept of thermal uncertainty continues to make sense in a nonextensive setting, contrary to what was stated in [4], where an uncertainty relation was looked for between $\beta$ and $U$, and, of course, could not be found.

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