Kantowski-Sachs bulk viscous string cosmological model in $f(R)$ theory of gravity

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Abstract. The present work deals with a spatially homogeneous and anisotropic Kantowski-Sachs space-time filled with bulk viscous fluid, containing one-dimensional cosmic strings. To obtain a deterministic solution of the field equations, we used the assumption of shear scalar is proportional to expansion scalar of the model. We have investigated geometric and kinematic properties of our bulk viscous string model and discussed the role of bulk viscosity in the evolution of Kantowski-Sachs universe within the framework of $f(R)$ modified theory gravity. A detailed physical discussion of the dynamical parameters is presented through a graphical representation.

1. Introduction
Recent observational data on the cosmic expansion history (Perlmutter et al. [1]; Tegmark et al. [2]; Riess et al. [3]) has lead to the discovery of accelerated expansion of the universe. It is believed that the reason for this is an exotic type of unknown force with huge negative pressure dubbed as dark energy (DE). However, the nature and behavior of DE is still a mystery. There are two major approaches to address this problem of cosmic acceleration either by introducing a dark energy component in the Universe and study its dynamics (Caldwell [4]; Sahni and Starobinsky [5]; Padmanabhan [6]; Sharif and Zubair [7]; Santhi et al. [8, 9]) or by interpreting it as a failure of general relativity and consider modifying Einstein’s theory of gravitation termed as “modified gravity approach”. Among the various modifications of Einstein’s theory, $f(R,T)$ theory of gravity (Harko et al. [10]) and $f(R)$ theory of gravity which provides a natural unification of early-time inflation and late-time acceleration (Capozziello and Francaviglia [11]; Nojiri and Odintsov [12]) are attracting more and more attention.

String cosmological models are important during structure formation in the early stages of evolution of the universe. Spontaneous symmetry breaking through the phase transition in the early universe result in a random network of stable line like topological defects familiar as cosmic strings. It is generally known that massive strings serve as starting point for the large structures like cluster of galaxies in the universe. So these string models have gained considerable attention of research workers. Several authors in the literature have investigated important aspects of string cosmological models in various theories of gravitation ([13]-[18]). Recently, Sahoo et al. [19] discussed Bianchi type string cosmological models in $f(R,T)$ gravity. Aditya and Reddy [20] have studied Bianchi type-I string cosmological models in $f(R)$ theory.
of gravity.

During the neutrinos coupling in the early stages of universe, the matter behaved as a viscous fluid. The coefficient of viscosity decreases as the universe expands. The effect of viscosity on the evolution of the universe and the strong dissipation, due to the neutrino viscosity, may considerably reduce the anisotropy of the black body radiation have been discussed by Misner [21, 22]. The effective total negative pressure which leads to a repulsive gravity in bulk viscosity overcomes the attractive gravity of the matter and gives an impulsion for rapid expansion of the universe. Rao et al. [23, 24] studied FRW bulk viscous cosmological model in some scalar-tensor theories of gravitation. Santhi et al. [25, 26] have discussed some Bianchi type bulk viscous string models in \( f(R) \) theory of gravity.

It is found that some large-angle anomalies (Eriksen et al. [27]) appear in cosmic microwave background radiations which violate the statistical isotropy of the universe. The universe may have achieved a slight anisotropic geometry in cosmological models regardless of the inflation. Recently, many authors have studied various cosmological models with anisotropic background within the framework of modified theories of gravitation ([28]-[33]).

Motivated by the above investigations, in this paper, we consider Kantowski-Sachs space-time in presence of bulk viscosity with one dimensional cosmic strings within the framework of \( f(R) \) theory of gravitation. The plan of the work as follows: Sect. 2 is devoted to the derivation of field equations and solutions of field equations. Sect. 3 contains a detailed physical discussion of the model. Summary and conclusions are presented in the last section.

2. Field equations and the model

Here we consider the Kantowski-Sachs space-time in the form

\[
\text{ds}^2 = dt^2 - X^2 dr^2 - Y^2 (d\psi^2 + \sin^2 \psi d\varphi^2),
\]

where \( A \) and \( B \) are functions of cosmic time \( t \) only. Kantowski-Sachs class of metrics represent anisotropic and homogeneous but expanding (or contracting) cosmologies. They also provide models where the effects of anisotropies can be estimated and compared with the FRW class of cosmologies (Thorne, [34]).

The field equations of \( f(R) \) gravity are obtained from the action

\[
S = \int \sqrt{-g} \left( \frac{1}{16\pi G} f(R) + L_m \right) d^4 x,
\]

where \( f(R) \) is a general function of the Ricci scalar and \( L_m \) is the matter Lagrangian. Variation of action (2) with respect to metric gives the following field equations

\[
F(R)R_{ij} - \frac{1}{2} f(R) g_{ij} - \nabla_i \nabla_j F(R) + g_{ij} \Box F(R) = \kappa T_{ij}
\]

where \( F(R) = \frac{df}{dR} \) and \( \Box = \nabla^i \nabla_i \), \( \nabla_i \) is the covariant derivative. Contracting the field equations (3), we get

\[
F(R)R - 2f(R) + 3\Box F(R) = \kappa T.
\]

Using above equation in equation (3), the field equations take the form

\[
F(R)R_{ij} - \nabla_i \nabla_j F(R) - \kappa T_{ij} = g_{ij} \left( \frac{F(R)R - \Box F(R) - \kappa T}{4} \right).
\]

Equation (4) is an important relationship between \( f(R) \) and \( F(R) \) which will be used to simplify the field equations and to evaluate \( f(R) \).
The energy momentum tensor for a bulk viscous fluid containing one-dimensional cosmic strings is given by

\[ T_{ij} = (\rho + p)u_i u_j - \bar{p}g_{ij} - \lambda x_i x_j \]  

(6)

\[ \bar{p} = p - 3\xi H = \omega \rho \]  

(7)

Here \( \bar{p} \) is the total pressure which includes the isotropic pressure \( p \), \( \lambda \) is the string tension density, \( \rho \) is the rest energy density of the system, \( \xi(t) \) is the coefficient of bulk viscosity, \( 3\xi H \) is generally known as bulk viscous pressure, \( H \) is the Hubble parameter of the model.

Equation of state (EoS) provides a relation between isotropic pressure and energy density and is given by

\[ p = \gamma \rho \]  

(8)

where \( \gamma \) is the EoS parameter and \( \omega = \gamma - \zeta \) (where \( \zeta \) is a constant). The values of \( \gamma = -1, 0, 1/3, 1 \) represent vacuum dominated, matter dominated, radiation dominated and stiff fluid era respectively.

Also, \( u_i \) is the four velocity vector, \( x_i \) is a space-like vector which represents the anisotropic directions of the string and they satisfies

\[ g^{ij}u_i u_j = -x^i x_j = 1, \quad u^i x_i = 0 \]  

(9)

We assume the string to be lying along the \( x \)-axis. The one dimensional strings are assumed to be loaded with particle and energy density is \( \rho_0 = \rho - \lambda \).

By adopting co-moving coordinates, the field equations (5) for the metric (1) yield the following equations:

\[
\left( \frac{\dddot{X}}{X} + 2\frac{\dddot{Y}}{XY} \right) F - \frac{f(R)}{2} + \dddot{F} + 2\frac{\dddot{Y}Y}{Y} = -\kappa(\bar{p} - \lambda) \\
\left( \frac{\dddot{Y}}{Y} + \frac{\dddot{X}}{X} \frac{\dddot{Y}^2}{Y^2} + \frac{1}{Y^2} \right) F - \frac{f(R)}{2} + \dddot{F} \left( \frac{\dddot{X}}{X} + \frac{\dddot{Y}}{Y} \right) = -\frac{\kappa \bar{p}}{F} \\
\left( \frac{\dddot{X}}{X} + 2\frac{\dddot{Y}}{Y} \right) F - \frac{f(R)}{2} + \dddot{F} \left( \frac{\dddot{X}}{X} + 2\frac{\dddot{Y}}{Y} \right) = \kappa \rho
\]

(10) (11) (12)

where overhead dot stands for ordinary differentiation with respect to cosmic time \( t \).

We define the following parameters for the Kantowski-Sachs model:

Hubble’s parameter of the model

\[ H = \frac{\dot{a}}{a} \]  

(13)

where

\[ a(t) = (XY^2)^{1/3} \]  

(14)

is the average scale factor. Anisotropic parameter \( A_h \) is given by

\[ A_h = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2, \]  

(15)

where \( H_1 = \dot{X}/X, H_2 = H_3 = \dot{Y}/Y \) are directional Hubble’s parameters, which express the expansion rates of the universe in the directions of \( x, y \) and \( z \) respectively.

Expansion scalar and shear scalar are defined as

\[ \theta = u^i_{;i} = \frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y} \]  

(16)
\[ \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma_{ij} = \frac{1}{3} \left( \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} \right)^2, \]  

(17)

where \( \sigma_{ij} \) is shear tensor, \( A_h \) is the deviation from isotropic expansion and the universe expands isotropically if \( A_h = 0 \). Deceleration parameter is given by

\[ q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1. \]  

(18)

The universe exhibits accelerating volumetric expansion if \(-1 \leq q < 0\), decelerating volumetric expansion if \( q > 0 \), and exhibits constant-rate volumetric expansion if \( q = 0 \).

The scalar curvature for the metric (1) is given by

\[ R = 2 \left( \frac{\dddot{X}}{X} + 2 \frac{\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + 2 \frac{\dot{X} \dot{Y}}{XY} + \frac{1}{Y^2} \right). \]  

(19)

Equations (10)-(12) are a set of three independent equations with six unknown variables \( X, Y, F(R), \rho, \sigma \) and \( \lambda \). Therefore, we need some additional constraints to solve the above system of equations. We use the following physically viable assumptions:

(i) We consider that shear scalar \( (\sigma) \) is proportional to the expansion scalar \( (\theta) \). This leads to a relation between the metric potentials (Collins et al. [35]), i.e.,

\[ X = Y^k \]  

(20)

where \( k > 1 \) is a constant and takes care of anisotropy of the space-time (we have taken the integration constant as a unity). The physical reason for this assumption is warranted from the observations of the velocity redshift relation for extragalactic sources which suggest that the Hubble expansion of the universe may achieve isotropy when \( \frac{\sigma}{\theta} \) is constant (Kantowski and Sachs [36]). Collins [35] have studied the physical significance of this condition for a perfect fluid.

(ii) Chiba et al. [37] have shown that \( f(R) \) theory of gravity equivalent to scalar-tensor theory of gravity. The power-law relation between scalar field and average scale factor has already been used by Johri and Sudharsan [38] in the context of FRW Brans-Dicke models with bulk viscosity. Uddin et al. [39] have established a result in the context of \( f(R) \) theory of gravity which shows that

\[ F(R) \propto a^m \]  

(21)

where \( m \) is an arbitrary constant. From equation (21), we have

\[ F(R) = F_0 a^m \]  

(22)

where \( F_0 \) is proportionality constant.

(iii) The varying deceleration parameter proposed by Mishra et al. [40] is given by

\[ q = -\frac{\ddot{a}}{a \dot{a}} = b(t) \]  

(23)

where \( a(t) \) is the average scale factor of the universe and \( b(t) \) is an arbitrary function of time. By solving the above equation (23) using some suitable assumptions, the average scale factor obtained as (Mishra et al. [40])

\[ a(t) = [\sinh(\alpha t)]^{\frac{1}{2}} \]  

(24)
where $\alpha$ is an arbitrary constant and $\beta$ is a positive constant. This average scale factor yields a deceleration parameter ($q$) varying from early decelerated phase to current accelerated phase. Mishra et al. [41], Santhi et al. [42] and Rao and Prasanthi [43] have investigated some cosmological models using this average scale factor. So, in this paper we consider average scale factor given in equation (24).

From Eqs. (14), (20), (22) and (24) we get the metric potentials as

$$X = [\sinh(\alpha t)]^{\frac{3}{2(1+\beta)}}; \quad Y = [\sinh(\alpha t)]^{\frac{3}{2(1+\beta)}}$$

and the function $F(R)$ as

$$F = F_0[\sinh(\alpha t)]^{\frac{3}{2(1+\beta)}}.$$  \hspace{1cm} (26)

Now the metric (1) can be written as

$$ds^2 = dt^2 - [\sinh(\alpha t)]^{\frac{6k}{2(1+\beta)}} dt^2 - [\sinh(\alpha t)]^{\frac{6k}{2(1+\beta)}} (d\psi^2 + \sin^2 \psi d\varphi^2).$$  \hspace{1cm} (27)

3. Cosmological parameters and Discussion

Equation (27) along with Eq. (26) represents Kantowski-Sachs universe with bulk viscous fluid with one dimensional cosmic string in $f(R)$ modified theory of gravity along with the following physical and geometrical parameters which are very important in the discussion of cosmology.

From equations (10)-(12), we obtain the string density $\lambda$ as

$$\lambda = \frac{F_0(k-1)(\sinh(\alpha t))^{\frac{3}{2(1+\beta)}}}{\kappa(1+\omega)} \left\{ \frac{9\alpha^2 F_0}{(k+2)^2 \beta^2} + \frac{3\alpha^2(3(k+1) + m(k+2))}{(k+2)^2 \beta^2} (\coth(\alpha t))^2 + \frac{3\alpha^2(3 - \beta(k+2))}{(k+2)^2 \beta^2} (\text{cosech}(\alpha t))^2 - (\sinh(\alpha t))^{\frac{6k}{2(1+\beta)}} \right\}. $$  \hspace{1cm} (28)

The energy density $\rho$ is

$$\rho = \frac{F_0(\sinh(\alpha t))^{\frac{3}{2(1+\beta)}}}{\kappa(1+\omega)} \left\{ \frac{9\alpha^2 k + 1}{(k+2)^2 \beta^2} - \frac{m^2}{\beta} + \frac{3\alpha^2(3(k+1) + m(k+2))}{(k+2)^2 \beta^2} - \frac{m\alpha(m - \alpha\beta)}{\beta^2} \right\} (\coth(\alpha t))^2 + \frac{3m^2}{(k+2)\beta} (\coth(\alpha t))^2 - (\sinh(\alpha t))^{\frac{6k}{2(1+\beta)}} \right\}. $$  \hspace{1cm} (29)

Proper pressure $p$ is

$$p = \frac{F_0(\sinh(\alpha t))^{\frac{3}{2(1+\beta)}}}{\kappa(1+\omega)} \left\{ \frac{9\alpha^2(k+1)}{(k+2)^2 \beta^2} - \frac{m^2}{\beta} + \frac{3\alpha^2(k+1)(3 - \beta(k+2))}{(k+2)^2 \beta^2} - \frac{m\alpha(m - \alpha\beta)}{\beta^2} \right\} (\coth(\alpha t))^2 + \frac{3m^2}{(k+2)\beta} (\coth(\alpha t))^2 - (\sinh(\alpha t))^{\frac{6k}{2(1+\beta)}} \right\}. $$  \hspace{1cm} (30)

Particle energy density is

$$\rho_p = \frac{F_0(\sinh(\alpha t))^{\frac{3}{2(1+\beta)}}}{\kappa(1+\omega)} \left\{ \frac{9\alpha^2((k+1) - (k-1)(1+\omega))}{(k+2)^2 \beta^2} - \frac{m^2}{\beta} + \frac{\alpha(\coth(\alpha t))^2}{\beta^2} \left\{ \frac{3\alpha(3 - \beta(k+2))}{(k+2)^2} \right\} (\coth(\alpha t))^2 - (\sinh(\alpha t))^{\frac{6k}{2(1+\beta)}} \right\}. $$  \hspace{1cm} (31)
Coefficient of bulk viscosity $\xi$ is

$$\xi = \frac{F_0(\gamma - \omega)(\sinh(\alpha t))^{\frac{2\beta}{3\alpha\omega(1 + \omega)}}}{\coth(\alpha t)} \left\{ \frac{9a^2(k+1)}{(k+2)^2\beta^2} - \frac{m^2}{\beta} + \frac{3ma^2}{(k+2)\beta} (\coth(\alpha t))^2 - (\sinh(\alpha t))^{\frac{3\alpha}{\beta}} \right\} \coth(\alpha t)^2 \right\}

Spatial volume of the model

$$V = (\sinh(\alpha t))^{3/\beta}. \hspace{1cm} (33)$$

Hubble’s parameter

$$H = \frac{\alpha}{\beta} coth(\alpha t). \hspace{1cm} (34)$$

Expansion scalar is

$$\theta = \frac{3\alpha}{\beta} coth(\alpha t). \hspace{1cm} (35)$$

Shear scalar is

$$\sigma^2 = \frac{3\alpha^2(k-1)^2}{\beta^2(2k+1)^2} \coth^2(\alpha t). \hspace{1cm} (36)$$

Anisotropic parameter is

$$A_h = \frac{2(k-1)^2}{(2k+1)^2}. \hspace{1cm} (37)$$

where $H_1$, $H_2$ and $H_3$ are the directional Hubble parameters. The deceleration parameter is

$$q = -1 + \beta(sech^2(\alpha t)). \hspace{1cm} (38)$$

Jerk parameter is given by

$$j = 1 + \beta(2\beta - 3)sech^2(\alpha t). \hspace{1cm} (39)$$

It is believed that in cosmology the cosmic jerk parameter can explain the transition of the universe from the decelerating to accelerating phase. This transition of the universe occurs for different models with positive value of the jerk parameter and the negative value of the deceleration parameter (Chiba and Nakamura [44]; Visser [45]). For example, $\Lambda CDM$ models have a constant jerk parameter and equal to unity.

Nojiri and Odintsov [46] have obtained the solution of the field equations for FRW model using reconstruction technique. Also, they have studied different cosmological epochs, matter dominated phase, transition phase and accelerated phase by assuming the average scale factor proportional to exponential function. In this work, we have considered Kantowski-Sachs bulk viscous string cosmological model in $f(R)$ gravity. We have plotted the behavior of energy density at different epochs like matter dominated, radiation and stiff fluid and discussed their physical significance. This is quite similar to the discussion presented by Nojiri and Odintsov [46].

Figs. 1 and 2 show the behavior of rest energy density $\rho$ and particle energy density $\rho_p$ against time $t$ for different phases of universe such as matter dominated ($\gamma = 0$), radiation dominated ($\gamma = 1/3$) and stiff fluid ($\gamma = 1$). We observed that the both $\rho$ and $\rho_p$ for three epochs are positive throughout the evolution and decrease with time. The realistic energy conditions, $\rho \geq 0$ and $\rho_p \geq 0$ are satisfied in our model. The behavior of energy tension density $\lambda$ is shown in Fig. 3 and it can be seen that $\lambda$ is negative throughout the evolution of model and increases with
Figure 1. Plot of rest energy density ($\rho$) versus $t$ for $\alpha = -0.115$, $\beta = 1.73$, $F_0 = 10$, $m = 0.9$, $k = 0.98$, $\kappa = 1$, $\gamma = 1$ and $\zeta = 0.5$.

Figure 2. Plot of particle energy density ($\rho_p$) versus $t$ for $\alpha = -0.115$, $\beta = 1.73$, $F_0 = 10$, $m = 0.9$, $k = 0.98$, $\kappa = 1$, $\gamma = 1$ and $\zeta = 0.5$.

Figure 3. Plot of string tension density versus $t$ for $\alpha = -0.115$, $\beta = 1.73$, $F_0 = 10$, $m = 0.9$, $k = 0.98$, $\kappa = 1$, $\gamma = 1$ and $\zeta = 0.5$.

Figure 4. Plot of deceleration parameter versus $t$ for $\alpha = -0.115$ and $\beta = 1.73$.

Figure 5. Plot of jerk parameter versus $t$ for $\alpha = -0.115$ and $\beta = 1.73$. 
cosmic time \( t \). The weak (\( \rho \geq \lambda \) with \( \lambda < 0 \)) and strong energy conditions (\( \rho > 0 \) with \( \lambda < 0 \)) are satisfied (Hawking and Ellis [47]).

Fig. 4 depicts the behavior of deceleration parameter (\( q \)) versus cosmic time \( t \). It can be seen that the model is evolving from early decelerating phase (\( q > 0 \)) to present accelerating phase (\( q < 0 \)). It can also be observed that the universe enters into the accelerating expansion phase at \( t \approx 6.4 \) and the present (i.e. at \( t = 13.7 \)) value of the deceleration parameter is \( q \approx -0.73 \). Both values are consistent with the observational results. Recent SNe Ia observations, explains that the present universe is accelerating and value of deceleration parameter lies within the range \(-1 \leq q < 0\) and equals to \( q \approx -0.73 \) at \( t = 13.7 \) Gyr. Fig. 5 shows the variation of jerk parameter versus cosmic time \( t \). We observe that it is positive throughout the evolution and finally equal to one. It follows that our models are consistent with the recent observations.

4. Summary and conclusions

In this paper, we have assumed the \( f(R) \) gravity field equations in the presence of bulk viscous fluid containing one dimensional cosmic strings in the background of anisotropic Kantowski-Sachs spacetime. The field equations have been solved using the variable deceleration parameter. The summary of the obtained results are follows:

Since the spatial volume of the model increases with cosmic time, the universe is expanding spatially. Rest energy density \( \rho \) and particle energy density \( \rho_p \) are positive and decreasing functions of cosmic time \( t \) and they approach to zero at late times. It can be seen that \( \lambda \) is negative throughout the evolution of universe and increases with cosmic time \( t \). Hence, we can conclude that the weak (\( \rho \geq \lambda \) with \( \lambda < 0 \)) and strong energy conditions (\( \rho > 0 \) with \( \lambda < 0 \)) are satisfied (Hawking and Ellis [47]) in our model. The Hubble’s parameter (\( H \)) and expansion scalar (\( \theta \)) become constant at late times showing a uniform expansion of the universe. There is a smooth transition from early decelerated phase to present accelerating phase (fig. 4) in the evolution of universe, which is in good agreement with recent accelerated expansion of the universe conformed by many experiments. Also, transition and the present values of deceleration parameter \( q \) are in good agreement with observational data. The jerk parameter of our model is positive and approaches to unity late times. Also, it is worthwhile to mentioning here that when \( k = 1 \) (i.e., isotropic model), we have \( \lambda = 0, A_h = 0 \) and \( \sigma^2 = 0 \), i.e., in this spacial case the strings in the universe do not survive and the model becomes isotropic and shear free.

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