The quantum Witten-Kontsevich series and one-part double Hurwitz numbers

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Abstract

We study the quantum Witten-Kontsevich series introduced by Buryak, Dubrovin, Guéré and Rossi in [BDGR19] as the logarithm of a quantum tau function for the quantum KdV hierarchy. This series depends on a genus parameter $\epsilon$ and a quantum parameter $\hbar$. When $\hbar = 0$, this series restricts to the Witten-Kontsevich generating series for intersection numbers of psi classes on moduli spaces of stable curves.

We establish a link between the $\epsilon = 0$ part of the quantum Witten-Kontsevich series and one-part double Hurwitz numbers. These numbers count the number of non-equivalent holomorphic maps from a Riemann surface of genus $g$ to $\mathbb{P}^1$ with a complete ramification over 0, a prescribed ramification profile over $\infty$ and a given number of simple ramifications elsewhere. Goulden, Jackson and Vakil proved in [GJV05] that these numbers have the property to be polynomial in the orders of ramification over $\infty$. We prove that the coefficients of these polynomials are the coefficients of the quantum Witten-Kontsevich series.

We also present some partial results about the full quantum Witten-Kontsevich power series.

Contents

1 Introduction 2
1.1 Overview ........................................... 2
1.2 The quantum Witten-Kontsevich series ...................... 5
1.3 One-part double Hurwitz numbers .......................... 12
1.4 Statement of the results ................................ 13
1.5 Plan of the paper ..................................... 17
1.6 Acknowledgments ..................................... 17

2 The string equation 17
2.1 On the substitution $u_i = \delta_{i,1}$ ......................... 17
2.2 A proof of the string equation .......................... 18

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The discovery of deep connections between enumerative geometry and integrable systems was initiated by the Witten conjecture [Wit90] proved by Kontsevich [Kon92]. The Witten conjecture states that a generating series of intersection numbers of the so-called ψ-classes on the moduli spaces of curves is the logarithm of a tau function of the Korteweg–de Vries (KdV) hierarchy. This result also revealed the important role played by tau functions at the interplay of these two fields. Many examples of such connections formulated in terms of tau function followed, see [Wit93, FSZ10, Oko00, OP06] for various examples. The statement has always the same formulation; the logarithm of a tau function of some integrable hierarchy is a generating series of numbers with a geometrical content. This paper is the first step in the extension of these results to quantum integrable hierarchies.

In a more systematic approach, Dubrovin and Zhang [DZ01] constructed integrable hierarchies associated to semisimple cohomological field theories (CohFT). In their construction, the potential of the CohFT is the logarithm of a tau function of the corresponding Dubrovin-Zhang (DZ) hierarchy. For example, when the CohFT is trivial, the DZ hierarchy is KdV and its potential is the Witten-Kontsevich series.

In 2014, Buryak [Bur15] constructed another type of integrable hierarchies associated to CohFTs called the double ramification (DR) hierarchies. It was conjectured [Bur15, BGR19] that the DR hierarchies can be obtained from the DZ hierarchies by a change of coordinates. A stronger version of the DR/DZ conjecture states that a tau function of the DR hierarchy is given by the potential of the CohFT up to a term related to this change of variable.

Remarkably, the DR hierarchies were quantized by Buryak and Rossi in [BR16]. Moreover, Buryak, Dubrovin, Guéré and Rossi introduced their quantum tau functions in [BDGR19]. In the classical and
quantum settings, the tau functions of the DR hierarchies were constructed indirectly using the tau symmetric hamiltonian structure of the DR hierarchies [BDGR18, BDGR19]. In the classical setting, the DR/DZ conjecture gives an interpretation of a DR tau function as the potential of the corresponding CohFT. However, in the quantum setting, nothing was known about the tau functions (except that we recover the classical DR tau function in the classical limit $\hbar = 0$), in particular an interpretation of their coefficients was missing.

In this paper, we study the first example of a quantum tau function by investigating the logarithm of a quantum tau function associated to the trivial CohFT, we denote this series by $F(q)$. In this case, the quantum DR hierarchy corresponds to a quantization of KdV and the classical limit of the logarithm of the tau function, $F(q)|_{\hbar=0}$, is given by the Witten-Kontsevich series. The series $F(q)$ is then called the quantum Witten-Kontsevich series.

### Table 1: First terms of the quantum Witten-Kontsevich.

| $\epsilon^0$ | $\hbar^0$ | $\hbar^1$ | $\hbar^2$ |
|--------------|------------|------------|------------|
| $t^3/6 + t^3t_0/6 + t^3t_2/24 + \cdots$ | $t^4_{24} + t^4t_0_{24} + t^4t_2_{24} + \cdots$ | $\frac{1}{1920} t_6 + \frac{1}{1920} t_0 t_7 + \cdots$ | $\frac{1}{480} t_4 + \frac{1}{480} t_0 t_5 + \cdots$ |
| $t^2_{24} + t^2t_0_{24} + t^2t_2_{24} + \cdots$ | $\frac{1}{120} t_5 + \frac{1}{120} t_0 t_6 + \frac{1}{240} t_1 t_5 + \cdots$ | $\frac{1}{384} t_3 + \frac{1}{384} t_0 t_4 + \frac{1}{192} t_1 t_3 + \cdots$ | $\frac{7}{5760} t_2 + \frac{7}{5760} t_0 t_3 + \frac{7}{1920} t_1 t_2 + \cdots$ |
| $t^1_{24} + t^1t_0_{24} + t^1t_2_{48} + \cdots$ | $\frac{1}{5760} t_3 + \frac{1}{5760} t_0 t_4 + \frac{1}{288} t_1 t_3 + \cdots$ | $\frac{1}{2880} t_2 + \frac{1}{2880} t_0 t_3 + \frac{1}{1920} t_1 t_2 + \cdots$ | $\frac{7}{5760} t_2 + \frac{7}{5760} t_0 t_3 + \frac{7}{1920} t_1 t_2 + \cdots$ |
| $t^0_{52} + t^0t_0_{52} + t^0t_1_{1152} + \cdots$ | $\frac{1}{1152} t_4 + \frac{1}{1152} t_0 t_5 + \frac{1}{384} t_1 t_4 + \cdots$ | $\frac{29}{5760} t_2 t_3 + \cdots$ | $\frac{1}{1920} t_1 + \frac{1}{1920} t_0 t_2 + \frac{1}{2880} t_4 + \cdots$ |

In Table 1, we give the first terms of $F(q)$. This series depends on two parameters $\epsilon$ and $\hbar$. The
coefficients of $\epsilon^2 (-i\hbar)^k$ appear in line $l$ and column $k$. The left column $k = 0$ contains the coefficients of $F^{(q)}|_{\epsilon = 0}$ and one can recognize the coefficients of the Witten-Kontsevich series. In this column, the box of line $l$ corresponds to the genus $l$ intersection numbers of $\psi$-classes on the moduli space of curves. Starting from the second column, the boxes are divided by dashed lines into levels. This subdivision corresponds to some vanishings of the coefficients of $F^{(q)}$ and will be explained in Section 1.4.1. The reader familiar with intersection numbers on the moduli space of curves will recognize typical genus $g$ intersection numbers in the boxes of the diagonal $l + k = g$ of the array.

The main result of the paper is a combinatorial interpretation of the coefficients of $F^{(q)}|_{\epsilon = 0}$, corresponding to the top line $l = 0$ of the table, in terms of Hurwitz numbers. More precisely, consider the one-part double Hurwitz numbers studied in [GJV05] and subject to a conjectural ELSV formula. These numbers count the number of non-equivalent holomorphic maps from a genus $g$ Riemann surface to the sphere with a complete ramification over 0, a prescribed ramification profile over $\infty$ and a fixed number of simple ramification elsewhere. Goulden, Jackson and Vakil showed that these numbers depends polynomially in the orders of ramification over $\infty$. The coefficients of these polynomials are the coefficient $F^{(q)}|_{\epsilon = 0}$.

This relation was completely unexpected since the coefficients of $F^{(q)}|_{\epsilon = 0}$ are built from a combination of a certain type of intersection numbers on moduli spaces of curves and their relation with Hurwitz theory is non trivial. Various results relating Hurwitz theory and intersection theory on the moduli space are known see [ELSV01, OP06], however this one appears in the quantum context and is completely new.

An interpretation of the rest of $F^{(q)}$ is still missing. However, we point out a conjecture which may interest the reader. Let $g, l, n \geq 0$ such that $g \geq l$ and let $(d_1, \ldots, d_n)$ be a list of nonnegative integers such that $\sum_{i=1}^{n} d_i = 2g - 3 + n - l$. The coefficient of $\epsilon^2 \hbar^g - l$ in $\frac{\partial^n F^{(q)}}{\partial t_{d_1} \cdots \partial t_{d_n}}|_{t_* = 0}$ is the Hodge integral

$$\int_{\overline{M}_{g,n}} \lambda_1 \lambda_2 \cdots \lambda_n d_1 \cdots d_n,$$

where we denoted by $\overline{M}_{g,n}$ the Deligne-Mumford compactification of the moduli space of curves and $\lambda_j$ is the $j$-th Chern class of the Hodge bundle, complete definitions will be given below. The form of this expression suggests a hidden localization formula.

We also prove a deformed version of the string equation

$$\frac{\partial}{\partial t_0} F^{(q)} = \sum_{i \geq 0} t_{i+1} \frac{\partial}{\partial t_i} F^{(q)} + \frac{t_0^2}{2} - \frac{i \hbar}{24},$$

and conjecture a dilaton equation

$$\frac{\partial}{\partial t_1} F^{(q)} = \sum_{i \geq 0} t_i \frac{\partial}{\partial t_i} F^{(q)} + \epsilon \frac{\partial}{\partial \epsilon} F^{(q)} + 2\hbar \frac{\partial}{\partial \hbar} F^{(q)} - 2F^{(q)} + \frac{\epsilon^2}{24}.$$

Generally, when the coefficients of the power series have a simple geometrical interpretation, the string and dilaton equations follow from elementary geometrical properties of the $\psi$-classes. However, the definition of the quantum Witten-Kontsevich series $F^{(q)}$ is non geometric in nature and quite elaborate. This makes the proof of the string relation much more complicated. And we have not been able to prove the dilaton
relation so far. We can nevertheless remark that the dilaton equation is proved for \( F^{(q)}|_{\epsilon=0} \) as a consequence of our main theorem since it is proved in [GJV05] for the coefficients of one-part double Hurwitz numbers.

This work suggests that quantum tau functions introduced by Buryak, Dubrovin, Guéré and Rossi [BDGR19] have an interesting interpretation in terms of enumerative geometry. The interpretation of the rest of \( F^{(q)} \) and other quantum tau functions will be the object of future research. Deformed version Virasoro constraints for \( F^{(q)} \) will also be the subject of further studies.

### 1.2 The quantum Witten-Kontsevich series

We introduce a quantum extension of the Witten-Kontsevich power series. We use the quantum deformation of the Korteweg–de Vries (KdV) hierarchy constructed by Buryak and Rossi [BR16]. Based on the construction of this quantum integrable hierarchy, the quantum Witten-Kontsevich series was introduced in [BDGR19].

#### 1.2.1 The classical Witten-Kontsevich series

Let \( \overline{M}_{g,n} \) be the moduli space of stable curves of genus \( g \) with \( n \) marked points. Let \( \pi : \overline{C}_{g,n} \to \overline{M}_{g,n} \) be the universal curve. Denote by \( \omega_{rel} \) the relative cotangent line bundle over \( \overline{C}_{g,n} \) and let \( \psi_i = c_1 (\sigma_i^* (\omega_{rel})) \), where \( \sigma_i : \overline{M}_{g,n} \to \overline{C}_{g,n} \) is the \( i \)-th section of the universal curve. We also define the classes \( \lambda_i = c_i (\pi_* \omega_{rel}) \) which will appear in Section 1.2.6. The Witten-Kontsevich series is

\[
F (\epsilon, t_0, t_1, \ldots) = \sum_{g,n \geq 0} \frac{\epsilon^{2g}}{2g-2+n} \sum_{d_1, \ldots, d_n \geq 0} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle g \ t_{d_1} \cdots t_{d_n},
\]

where \( \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \). Note that specifying the genus in the notation \( \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \) is redundant since this number is non-zero only if \( \sum d_i = 3g - 3 + n \). We use this notation in view of its quantum generalization.

An alternative definition of \( F \) is given by the famous Witten-Kontsevich theorem [Wit90, Kon92] : \( F \) is the logarithm of the tau function of the KdV hierarchy associated to the solution \( u(x, t_0, t_1, \ldots) \) with the initial condition \( u(x, 0, 0, \ldots) = x \). This particular solution of the KdV hierarchy is called the string solution. The definition of the quantum Witten-Kontsevich series is a generalization of this point of view.

#### 1.2.2 A formal Poisson structure of the KdV hierarchy

We define an algebra of power series and a Poisson structure on it, we use them to describe each equation of the KdV hierarchy as a Hamiltonian equation. To motivate these definitions, we introduce the infinite dimensional space of periodic functions \( P := \{ u : S^1 \to \mathbb{C} \} \). Suppose these periodic functions have a Fourier transform \( u(x) = \sum_{a \in \mathbb{Z}} p_a e^{iax} \), this gives a system of coordinates \( \{ p_a, a \in \mathbb{Z} \} \) on \( P \). We define an algebra of power series in the indeterminates \( p_a, a \in \mathbb{Z} \), and interpret it as the algebra of functions on \( P \).

**Definition 1.1.** Let \( \mathcal{F}(P) \) be the algebra \( \mathbb{C}[p_{>0}] [p_{\leq 0}, \epsilon] \), where the indeterminates \( p_{>0} \) (resp. \( p_{\leq 0} \)) stands for \( p_a \) with \( a \in \mathbb{Z}_{>0} \) (resp. \( a \in \mathbb{Z}_{\leq 0} \)).
Definition 1.2. The Poisson structure on $\mathcal{F}(P)$ is given by
\[
\{p_a, p_b\} = i\alpha \delta_{a+b,0},
\]
and we extend it to $\mathcal{F}(P)$ by the Leibniz rule.

The Hamiltonians of the KdV hierarchy are elements of $\mathcal{F}(P)$, they will be introduced in Section 1.2.6 as the classical limit $\hbar = 0$ of their quantum counterparts. We denote by $\mathcal{H}_d$ with $d \geq -1$ these classical Hamiltonians. The $d$-th equation of the KdV hierarchy is then given by
\[
\frac{\partial u}{\partial t_d} = \{u, \mathcal{H}_d\}.
\]
The whole system of equations forms the KdV hierarchy.

Example 1.3. From the Hamiltonian $\mathcal{H}_1 = \frac{1}{3} \sum_{a+b+c=0} p_a p_b p_c + \frac{\epsilon^2}{24} \sum_{a \in \mathbb{Z}} (ia)^2 p_{ap-a}$, we recover the KdV equation which is the first equation of the KdV hierarchy
\[
\frac{\partial u}{\partial t_1} = \sum_{a \in \mathbb{Z}} \{p_a, \mathcal{H}_1\} e^{iax} = u \partial_x u + \frac{\epsilon^2}{12} \partial_x^3 u.
\]

In the next sections, we construct the quantum KdV hierarchy. It is a quantum deformation of the KdV hierarchy. In Section 1.2.3, we deform the product of $\mathcal{F}(P)$ in the direction given by the Poisson bracket to get a noncommutative star product. The space of functions endowed with this star product will be denoted by $\mathcal{F}_\hbar(P)$. Then, in Section 1.2.5 we introduce the so-called double ramification cycle in $\overline{M}_{g,n}$ and use intersection numbers with this cycle to introduce the quantum Hamiltonians of the quantum KdV hierarchy. These quantum Hamiltonians are elements of $\mathcal{F}_\hbar(P)$. They commute with respect to the star product as mentioned in Section 1.2.7. Hence, they form a quantum integrable hierarchy. We finally present the quantum KdV equations in Section 1.2.8.

Once the quantum KdV equations introduced, we are able to define the quantum Witten-Kontsevich series, this is done in Section 1.2.9.

1.2.3 The star product

In the quantization deformation setting, we enlarge the space of functions to $\mathcal{F}(P)[[\hbar]]$ and endow it with a new product, the star product.

Definition 1.4. Let $W$ be the free algebra generated by the $p_a$ modulo the commutations relations $[p_a, p_b] = i\alpha \delta_{a+b,0}$. The normal ordering of $f \in \mathcal{F}(P)[[\hbar]]$ is the element of $W[[\epsilon, \hbar]]$ obtained by first sorting each monomial of $f$ with the $p_{<0}$ on the left and then replacing the product of the $p_a$ by the non-commutative product of $W[[\epsilon, \hbar]]$. We denote by $:\ f$ the normal ordering of $f$.

Definition 1.5. Let $f, g \in \mathcal{F}(P)[[\hbar]]$. The star product $f \star g$ is an element of $\mathcal{F}(P)[[\hbar]]$ defined in the following way. Organize the product $:\ f :: g$ : in $W[[\epsilon, \hbar]]$ with the $p_{<0}$ on the left using the commutation relation $[p_a, p_b] = i\alpha \delta_{a+b,0}$ of $W[[\epsilon, \hbar]]$. This process is well defined according to the polynomiality in the $p_{>0}$ of $f$. This organization of $:\ f :: g$ : is the normal ordering of a unique element of $\mathcal{F}(P)[[\hbar]]$. This element is the star product $f \star g$.

We denote by $\mathcal{F}_\hbar(P)$ the deformed algebra obtained by endowing $\mathcal{F}(P)[[\hbar]]$ with the star product.
Remark 1.6. Let \( f, g \in \mathcal{F}^h (P) \). The star product is a quantum deformation of the usual product on \( \mathcal{F} (P) \) in the direction given by the Poisson bracket, that is
\[
 f \star g = fg + O (h)
\]
and
\[
 [f, g] = h \{ f, g \} + O (h^2).
\]
In particular, when we substitute \( h = 0 \) in \( \mathcal{F}^h (P) \) we obtain \( \mathcal{F} (P) \).

Certain special elements of \( \mathcal{F}^h (P) \) will be of particular interest to us. We define them now.

1.2.4 Differential polynomials

We give two equivalent definitions of differential polynomials and explain how to identify them. Differential polynomials appear in the construction of the quantum Witten-Kontsevich series, in particular this identification will be necessary.

Notation 1.7. From now on, we denote by \( u_i \) the \( i \)-th derivative of the function \( u : S^1 \to \mathbb{C} \), i.e. \( u_s = \partial_s^u u \) with \( s \geq 0 \).

Definition 1.8. A differential polynomial is an element of \( \mathcal{A} := \mathbb{C} [u_0, u_1, \ldots] [[\epsilon, h]] \).

Definition 1.9. Let \( d \) be a positive integer. Let \( (\phi_0, \ldots, \phi_d) \) be a list where \( \phi_k (a_1, \ldots, a_k) \in \mathbb{C} [a_1, \ldots, a_k] [[\epsilon, h]] \) is a symmetric polynomial in its \( k \) indeterminates \( a_1, \ldots, a_k \) for \( 0 \leq k \leq d \). The formal Fourier series associated to \( (\phi_0, \ldots, \phi_d) \) is
\[
 \phi (x) = \sum_{A \in \mathbb{Z}} \left( \sum_{k=0}^{d} \sum_{a_1, \ldots, a_k \in \mathbb{Z}} \phi_k (a_1, \ldots, a_k) p_{a_1} \cdots p_{a_k} \right) e^{ix.A} \in \mathcal{F}^h (P) [[e^{-ix}, e^{ix}]].
\]

The set of formal Fourier series associated to any \( d \in \mathbb{N} \) and any \( (\phi_0, \ldots, \phi_d) \) is an algebra that we denote by \( \tilde{\mathcal{A}} \).

Lemma 1.10. The algebras \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) are isomorphic.

Indeed, by substituting the formal Fourier series \( u_s (x) = \sum_{a \in \mathbb{Z}} (ia)^s p_a e^{iax} \), with \( s \in \mathbb{N} \), of \( u \) and its derivative in a differential polynomial, we obtain an element of \( \tilde{\mathcal{A}} \). By this application, the differential monomial \( u_{s_1} \cdots u_{s_n} \) yields the formal Fourier series associated to \( \phi_n (a_1, \ldots, a_n) = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)}^{s_1} \cdots a_{\sigma(n)}^{s_n} \) and \( \phi_i = 0 \) if \( i \neq n \).

The elements of \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) will be called differential polynomials. However, we will keep the notations \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) in order to indicate our point of view.

Remark 1.11. The Fourier mode of frequency \( A \in \mathbb{Z} \) in a differential polynomial is an element of \( \mathcal{F}^h (P) \). In this text, we will only use the elements of \( \mathcal{F}^h (P) \) obtained in this way.
Definition 1.12. The derivative $\partial_x$ of a differential polynomial $\phi$ is the differential polynomial obtained by multiplying the $A$-th mode of $\phi$ by $A$.

The integration along $S^1$ of a differential polynomial $\phi$ is the 0-th mode of $\phi$. We denote by $\int_{S^1} \phi(x)\,dx = \overline{\phi}$ this integral.

The primitive of a differential polynomial $\phi$ is a differential polynomial $\psi$ such that $\partial_x \psi = \phi$.

Proposition 1.13. Let $\phi$ and $\psi$ be two differential polynomials. The commutator $[\phi, \psi]$ is a differential polynomial.

A proof can be found in [BR16] where the authors give an expression for this commutator in terms of $u$ and its derivative. It is clear from their expression that it is a differential polynomial.

1.2.5 The double ramification cycle

Fix a list of $n$ integers $A = (a_1, \ldots, a_n)$ which add up to zero. To this list we associate a space of rubber stable maps to $\mathbb{P}^1$ relative to $0$ and $\infty$ in the following way.

Let $n_+$ be the number of positive $a_i$’s in $A$. Let $\mu = (\mu_1, \ldots, \mu_{n_+})$ be the partition made from these positive $a_i$’s. Similarly, let $n_-$ be the number of negative $a_i$’s, after changing their signs we make the partition $\nu = (\nu_1, \ldots, \nu_{n_-})$. Note that $\mu$ and $\nu$ are two partitions of $\frac{1}{2} \sum_{i=1}^{n} |a_i|$. Let $n_0$ be the number of vanishing $a_i$’s. We denote by

$$\overline{\mathcal{M}}_g(a_1, \ldots, a_n) := \overline{\mathcal{M}}_{g,n_0}(\mathbb{P}^1, \mu, \nu)$$

the moduli space of rubber stable maps to $\mathbb{P}^1$ relative to $0$ and $\infty$ with profile given by $\mu$ and $\nu$, where the genus $g$ source curve has $n_0$ additional marked points with image in $\mathbb{P}^1\setminus\{0, \infty\}$. Rubber means that two relative stable maps are identified is they differ by a $\mathbb{C}^*$ action in the target $\mathbb{P}^1$ (see e.g. [FP05] where these maps are defined as unparametrized relative stable maps). This space is endowed with the map $st : \overline{\mathcal{M}}_g(a_1, \ldots, a_n) \rightarrow \overline{\mathcal{M}}_{g,n}$ that forgets everything except the source curve and stabilizes it. Moreover, $\overline{\mathcal{M}}_g(a_1, \ldots, a_n)$ has a virtual fundamental class of virtual dimension $2g - 3 + n$.

Definition 1.14. The double ramification cycle $\text{DR}_g(a_1, \ldots, a_n)$ is defined by

$$\text{DR}_g(a_1, \ldots, a_n) := st_* \left( [\overline{\mathcal{M}}_{g,n}(a_1, \ldots, a_n)]^{\text{virt}} \right) \in H_{2(2g-3+n)}(\overline{\mathcal{M}}_{g,n}).$$

It is conjectured that the double ramification cycle is a polynomial in the $a_i$’s and a proof was announced by Pixton and Zagier. However, we will only need that the intersection number of the double ramification cycle with a tautological class (see e.g. [FP15] for a definition) is a polynomial in the $a_i$’s.

Proposition 1.15 ([BR16], Proposition B.1). Let $\alpha$ be a tautological class in $H^*(\overline{\mathcal{M}}_{g,n})$. There exists a polynomial $P_{g,n}(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$\int_{\text{DR}_g(a_1, \ldots, a_n)} \alpha = P_{g,n}(a_1, \ldots, a_n)$$

for all $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i = 0$. Moreover $P_{g,n}$ is even and of degree $2g$. 

8
1.2.6 Quantum Hamiltonian densities and quantum Hamiltonians

Definition 1.16. Fix $d \geq -1$. The quantum Hamiltonians density $H_d$ of the quantum KdV hierarchy is

$$H_d (x) = \sum_{g \geq 0, m \geq 0} \frac{(i \hbar)^g}{m!} \sum_{a_1, \ldots, a_m \in \mathbb{Z}} \left( \int_{\text{DR}_g (0, a_1, \ldots, a_m, - \sum a_i)} \psi_{d+1}^d \Lambda \left( \frac{-\epsilon^2}{i \hbar} \right) p_{a_1} \cdots p_{a_m} e^{ix \sum a_i} \right) \in \tilde{A},$$

where $\Lambda \left( \frac{-\epsilon^2}{m} \right) := 1 + \left( \frac{-\epsilon^2}{m} \right) \lambda_1 + \cdots + \left( \frac{-\epsilon^2}{m} \right)^g \lambda_g$.

The Hamiltonian density $H_d (x)$ is an element of $\tilde{A}$. Indeed, the degree of the class $\text{DR}_g (0, a_1, \ldots, a_m, - \sum a_i)$ is $2 (2g - 1 + n)$, hence the summation over $g$ and $n$ is finite. Moreover the $\psi$- and $\lambda$- classes are tautologicals [FP15], then Proposition 1.15 implies that the integral is a polynomial in the $a_i$’s.

Definition 1.17. The quantum Hamiltonians of the quantum KdV hierarchy are obtained by the $x$-integration of the Hamiltonian densities:

$$\overline{H}_d = \int_{S^1} H_d (x) \, dx \in \mathcal{F}^h (P),$$

for $d \geq -1$.

Remark 1.18. When we substitute $\hbar = 0$, we obtain the classical Hamiltonians densities $h_d (x) := H_d (x) \big|_{\hbar = 0}$ and the classical Hamiltonians $\overline{h}_d := \overline{H}_d \big|_{\hbar = 0}$ of the KdV hierarchy, see [Bur15] for a proof.

1.2.7 Integrability and tau symmetry

Two properties are needed for the construction of the quantum Witten-Kontsevich series: the commutativity of the Hamiltonians and the tau-symmetry. These two properties are proved in [BDGR19], however the authors defined the quantum Hamiltonian densities and quantum Hamiltonians in a slightly different way from ours. We explain the equivalence between these definitions in Appendix A.

Proposition 1.19 (Quantum integrability). We have

$$[\overline{H}_{d_1}, \overline{H}_{d_2}] = 0, \text{ for } d_1, d_2 \geq -1.$$ 

We say that the quantum hierarchy is integrable.

Remark 1.20. The substitution $\hbar = 0$ in $\frac{1}{\hbar} [\overline{H}_{d_1}, \overline{H}_{d_2}] = 0$ gives the integrability condition of the classical KdV hierarchy.

Proposition 1.21 (Tau symmetry). We have

$$[H_{d_1-1} (x), \overline{H}_{d_2}] = [H_{d_2-1} (x), \overline{H}_{d_1}], \text{ for } d_1, d_2 \geq -1.$$ 

Remark 1.22. The substitution $\hbar = 0$ in $\frac{1}{\hbar} [H_{d_1-1} (x), \overline{H}_{d_2}] = \frac{1}{\hbar} [H_{d_2-1} (x), \overline{H}_{d_1}]$ gives the tau symmetry of the KdV hierarchy.
1.2.8 The quantum KdV equations

**Definition 1.23.** The time-dependent function \( f^t \in F^h (P) \) is a solution of the quantum KdV hierarchy with initial condition \( f \in F^h (P) \) if
\[
\frac{df^t}{dt_d} = \frac{1}{\hbar} [f^t, \Pi_d] \quad \text{for } d \geq 0.
\]

**Proposition 1.24.** A solution of the quantum KdV hierarchy with initial condition \( f \in F^h (P) \) is given by
\[
f^t = \exp \left( \sum_{k \geq 0} \frac{t_k}{\hbar} [\cdot, \Pi_k] \right) f.
\]

**Remark 1.25.** A classical Hamiltonian flow can be viewed in two different ways: (i) a flow \( t \to u^t \) interpreted as a flow on the phase space \( P \); (ii) a flow \( t \to f^t \) on the space \( F^h (P) \) of functions on \( P \), satisfying \( f^t (u) = f (u^t) \). The standard way of writing the classical KdV hierarchy uses the first point of view, while the classical limit \( (h = 0) \) of the quantum KdV hierarchy uses the second. Because of this, the substitution \( h = 0 \) in the quantum KdV equations does not directly yield the standard presentation of the KdV equations. To obtain the classical KdV equations, we substitute \( h = 0 \) in the quantum KdV equations to get
\[
\frac{\partial u}{\partial t} = \{u, \hbar \Pi_d\}.
\]
A solution of the KdV then has the form \( u (x, t) = \sum_{a \in \mathbb{Z}} p^a e^{iax} \).

We emphasize that in the quantum setting we have \( f^t (u) \neq f (u^t) \), the two points of view (i) and (ii) are no more equivalent, a solution is a trajectory on \( F^h (P) \).

Given a differential polynomial \( \phi \in \tilde{A} \), we denote by \( \phi^t := \exp \left( \sum_{k \geq 0} \frac{t_k}{\hbar} [\cdot, \Pi_k] \right) \phi \). In this notation, the operator \( \exp \left( \sum_{k \geq 0} \frac{t_k}{\hbar} [\cdot, \Pi_k] \right) \) acts on each mode of \( \phi \) so that each mode of \( \phi^t \) is a solution of the quantum KdV hierarchy.

**Proposition 1.26.** The time-dependent differential polynomial \( \phi^t \) is an element of \( \tilde{A} [[t_0, t_1, \ldots]] \).

This follows from Proposition 1.13.

1.2.9 The quantum Witten-Kontsevich series

**Overview.** In the classical settings, Dubrovin and Zhang [DZ01, BDGR18] constructed tau functions from tau symmetric Hamiltonian hierarchies. In this construction, one associates a tau function to any solution of the hierarchy. The KdV hierarchy is a tau symmetric Hamiltonian hierarchy (see Remark 1.22). Thus, one way to build the Witten-Kontsevich series is to construct the tau functions of KdV following Dubrovin and Zhang and then taking the logarithm of the tau function associated to the string solution, that is the solution starting at \( u (x) = x \).

The construction of Dubrovin and Zhang was generalized in [BDGR19]. In this work, the authors defined quantum tau functions from tau symmetric quantum Hamiltonian hierarchies. In this context, a quantum tau function is associated to a point of the phase space \( P \). Of course, the restriction \( h = 0 \) of a quantum tau function is a classical tau function of the related classical hierarchy. The point of
Construction of quantum tau functions. We now give the construction of quantum tau functions of the quantum KdV hierarchy and their logarithms. We follow the construction of [BDGR19]. This construction is based on two properties of the quantum KdV hierarchy: its integrability and the tau symmetry.

Let $d_1, d_2$ be two positive integers. Start from the differential polynomial $[H_{d_1}, \Pi_{d_2}]$. Due to the commutativity of the Hamiltonians $\int [H_{d_1}(x), \Pi_{d_2}] \, dx = 0$, there exists primitives. Define the two-point function $\Omega_{d_1, d_2}^\hbar \in \tilde{A}$ by

$$\partial_x \Omega_{d_1, d_2}^\hbar := \frac{1}{\hbar} \left[ H_{d_1-1}, \Pi_{d_2} \right],$$

where we fix the constant using the recursive formula $\frac{\partial \Omega_{d_1, d_2}^\hbar}{\partial p_0}\bigg|_{p_*=0} = \Omega_{d_1-1, d_2}^\hbar\bigg|_{p_*=0} + \Omega_{d_1, d_2-1}^\hbar\bigg|_{p_*=0}$ with the initial conditions $\Omega_{0, d_2}^\hbar\bigg|_{p_*=0} = \Omega_{d_2}^\hbar\bigg|_{p_*=0} = H_{d-1}\big|_{p_*=0}$, where $d \geq 0$. This convention differs from the one used in [BDGR19]. We made this choice so that the quantum Witten-Kontsevich series satisfies the string equation.

The time-dependent differential polynomial $\Omega_{d_1, d_2}^{\hbar, t}$ with initial condition $\Omega_{d_1, d_2}^{\hbar, 0}$ is an element of $\tilde{A}[\{t_0, t_1, \ldots\}]$ according to Proposition 1.26. By the commutativity of the Hamiltonians and the tau-symmetry, $\Omega_{d_1, d_2}^{\hbar, t}$ and $\frac{\partial}{\partial t_{d_3}} \Omega_{d_1, d_2}^{\hbar, t}$ are invariants under the permutations of their indices $d_1, d_2, d_3$. Thus, using twice the Poincaré lemma, we conclude that there exists a power series $\mathcal{F} \in \tilde{A}[\{t_0, t_1, \ldots\}]$ such that

$$\frac{\partial^2 \mathcal{F}}{\partial t_{d_1} \partial t_{d_2}} = \Omega_{d_1, d_2}^{\hbar, t}.$$

Definition 1.27. The logarithm of a quantum tau function of the quantum KdV hierarchy is obtained by first evaluating $\mathcal{F} \in \tilde{A}[\{t_0, t_1, \ldots\}]$ at a point $u \in \mathbb{C}[\{x, \epsilon, \hbar\}]$ (interpreted as a point of $P$) and then at $x = 0$. It is an element of $\mathbb{C}[\{\epsilon, \hbar, t_0, t_1, \ldots\}]$. It is uniquely defined up to constant and linear terms.

Remark 1.28. We can use the same construction with $\hbar = 0$ in order to define the logarithm of the classical tau functions of the KdV hierarchy (up to constant and linear terms in $t_*$). We then associate a classical tau function to any point $u \in \mathbb{C}[\{x, \epsilon\}]$. However, the evaluation $\Omega_{d_1, d_2}^{\hbar=0, t}(u)$ of the time dependant differential polynomial $\Omega_{d_1, d_2}^{\hbar=0, t}$ at $u \in \mathbb{C}[\{x, \epsilon\}]$ is equal to $\Omega_{d_1, d_2}^{\hbar=0, t=0}(u^t)$, where $t \rightarrow u^t$ is the solution of the KdV hierarchy starting at $u \in \mathbb{C}[\{x, \epsilon\}]$. This equality is due to the equivalence of the points of views (i) and (ii) of Remark 1.25. Thanks to this change of perspective, the classical limit of this construction associates a classical tau function to any solution of the hierarchy; we recover the definition of classical tau functions of Dubrovin and Zhang (see [DZ01, BDGR18]).

Remark 1.29. One may wonder why we forget about the $x$ dependancy. Let $\phi^t$ be the time-dependent differential polynomial with initial condition the differential polynomial $\phi$. One can verify using Lemma 2.3
that the 0-th quantum KdV equation is
\[ \frac{\partial \phi^t}{\partial t_0} = \frac{1}{\hbar} [\phi^t, \mathcal{H}_0] = \partial_x \phi^t, \]
that is the evolutions along \( x \) and \( t_0 \) are the same. Hence we recover the \( x \) dependancy in the quantum tau functions by setting \( t_0 := t_0 + x \).

The quantum Witten-Kontsevich series.

**Definition 1.30.** The quantum Witten-Kontsevich series \( F(q) \) is obtained by first evaluating \( F \) at the point \( u(x) = x \) of \( P \) and then \( x = 0 \). Moreover we impose that the coefficient of \( e^{2\hbar \theta - l} t_0 t_{d+1} \) is the coefficient of \( e^{2\hbar \theta - l} t_0 t_{d+1} \) for any \( 0 \leq l \leq g \) and \( d \geq 0 \). We also impose that the constant coefficient of \( e^{2\hbar \theta - l} \) is given by \( \frac{1}{2g-2} \) times the coefficient of \( e^{2\hbar \theta - l} t_1 \) when \( g \geq 2 \) and \( 0 \) otherwise.

Let \( k, g \) be two non-negative integers, and a list \((d_1, \ldots, d_n)\) of non-negative integers. The quantum correlators \( \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{l,g-l} \) is the coefficient of the quantum Witten-Kontsevich series written as
\[ F(q) = \sum \frac{\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{l,g-l}}{n!} \epsilon^{2l} (-\hbar \epsilon)^{g-l} t_{d_1} \cdots t_{d_n} \in \mathbb{C}[[\epsilon, \hbar, t_0, t_1, \ldots]]. \]

**Remark 1.31.** We made this particular choice of constant and linear terms so that \( F(q) \) satisfies the string and dilaton equations, see Section 1.4.2.

**Remark 1.32.** The power series \( F \) is an element of \( \mathcal{A}[[t_0, t_1, \ldots]] \). In order to define the quantum Witten-Kontsevich series, we have to evaluate \( F \) at the point \( u(x) = x \) and then substitute \( x = 0 \). This is equivalent to evaluate \( F \) at \( u_0 = 0 \), \( u_1 = 1 \) and \( u_i = 0 \), when \( i \geq 2 \).

**Proposition 1.33.** We have
\[ F(q) \bigg|_{\hbar = 0} = F. \]

Indeed, as mentioned in Remark 1.28, \( F(q) \big|_{\hbar = 0} \) is the logarithm of the tau function of the KdV hierarchy associated to the string solution (the solution starting at \( u(x) = x \)) with our particular choice of constant and linear terms in \( t_\ast \). Thus, \( F(q) \big|_{\hbar = 0} \) is the Witten-Kontsevich series \( F \) up to constant and linear terms. Moreover, \( F \) satisfies the string and dilaton equations, hence its constant and linear terms in \( t_\ast \) are the same as those of \( F(q) \big|_{\hbar = 0} \).

### 1.3 One-part double Hurwitz numbers

Fix three nonnegative integers \( g, n, d \) and a partition \( \mu = (\mu_1, \ldots, \mu_n) \) of \( d \). Consider the degree \( d \) branched covers of the Riemann sphere \( f : C \to \mathbb{P}^1 \) by a Riemann surface \( C \) of genus \( g \) such that
- \( f \) is completely ramified over 0,
- the ramification profile of \( f \) over \( \infty \) is given by \( \mu \),
all the ramifications of \( f \) over \( \mathbb{P}^1 \setminus \{0, \infty\} \) are simple.

By the Riemann-Hurwitz formula, we count \( r = 2g - 1 + n \) simple ramifications over \( \mathbb{P}^1 \setminus \{0, \infty\} \). There is a finite number of isomorphism classes of such covers.

**Definition 1.34.** The one-part double Hurwitz number is the weighted sum

\[
H_{(d),\mu}^g = \sum_{[f]} \frac{1}{|\text{Aut} f|}
\]

where the summation is over isomorphism classes of such covers.

**Proposition 1.35** (Goulden-Jackson-Vakil [GJV05]). For a fixed genus \( g \), the Hurwitz number \( H_{(d),\mu}^g \) depends polynomially on \( \mu_1, \ldots, \mu_n \); this polynomial is divisible by \( d = \mu_1 + \cdots + \mu_n \).

**Definition 1.36.** Let \((d_1, \ldots, d_n)\) be a list of non-negative integers. A Hurwitz correlator is the number

\[
\langle \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \rangle_g = ( -1 )^{4g-3+n-\sum d_i} \left[ \mu_1^{d_1} \cdots \mu_n^{d_n} \right] \left( \frac{H_{(d),\mu}^g}{r!d} \right),
\]

where \( \left[ \mu_1^{d_1} \cdots \mu_n^{d_n} \right] \) denotes the coefficient of \( \mu_1^{d_1} \cdots \mu_n^{d_n} \).

**Proposition 1.37** (Goulden-Jackson-Vakil [GJV05]). The Hurwitz correlators \( \langle \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \rangle_g \) vanishes if \( \sum d_i \) is outside the interval

\[ [2g - 3 + n, 4g - 3 + n] \]

or if \( \sum d_i \) has the parity of \( n \).

In [GJV05], Goulden, Jackson and Vakil prove various properties of the Hurwitz correlators: they satisfy the string and dilaton equations. The polynomial \( H_{(d),\mu}^g \) is actually divisible by \( (\sum_i \mu_i)^{r-1} \). They also conjecture an ELSV type formula for the Hurwitz correlators.

### 1.4 Statement of the results

#### 1.4.1 Correlators of the Witten-Kontsevich series

The quantum Witten-Kontsevich series \( F^{(q)} \) is an element of \( \mathbb{C}[[\epsilon, h, t_0, t_1, \ldots]] \). Its classical part, obtained by plugging \( h = 0 \), is the Witten-Kontsevich series \( F \) of Section 1.2.1.

The following theorem concerns the restriction \( \epsilon = 0 \) of the quantum Witten-Kontsevich series. The coefficients of \( F^{(q)} \big|_{\epsilon=0} \) are expressed in terms of one-part double Hurwitz numbers.

**Theorem 1.** Fix two nonnegative integers \( g, n \) and a list of nonnegative integers \((d_1, \ldots, d_n)\). We have

\[
\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g} = \langle \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \rangle_g.
\]

Thus we have a geometric interpretation for the coefficients of \( \epsilon^0 h^g \) and \( \epsilon^{2g} h^0 \) of the quantum Witten-Kontsevich series. So far there is no such interpretation for the other coefficients, but we have a conjecture for some of them. Let us first explain some vanishing properties of the correlators.
**Level structure.** The correlators satisfy some vanishing properties similar to the Hurwitz correlators (see Proposition 1.37).

**Proposition 1.38.** Fix three nonnegative integers \( g, n, l \) such that \( l \leq g \). The correlator \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{l, g-l} \) vanishes if
\[
\sum_{i=1}^{n} d_i > 4g - 3 + n - l \quad \text{or if} \quad \sum_{i=1}^{n} d_i \equiv n - l \pmod{2},
\]
where \((d_1, \ldots, d_n)\) is a list of nonnegative integers.

**Conjecture 1.** Fix three nonnegative integers \( g, n, l \) such that \( l \leq g \). The correlator \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{l, g-l} \) vanishes if
\[
\sum_{i=1}^{n} d_i < 2g - 3 + n - l,
\]
where \((d_1, \ldots, d_n)\) is a list of nonnegative integers.

Hence, the correlators are possibly nonzero only when \( \sum d_i \) takes the \( g + 1 \) values of the interval \([2g - 3 + n - l, 4g - 3 + n - l]\) with the parity of its maximum (or minimum). We say that the correlators \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{l, g-l} \) are structured in \( g + 1 \) levels.

**Minimal level.** We have the following geometrical interpretation for the correlators in the minimal levels.

**Conjecture 2.** Fix three nonnegative integers \( g, n, l \) such that \( l \leq g \). When \( \sum d_i = 2g - 3 + n - l \), the correlators are given by
\[
\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{l, g-l} = \int_{\overline{M}_{g, n}} \lambda_g \lambda_0 \psi_1^{d_1} \ldots \psi_n^{d_n}.
\]

**Remark 1.39.** Let us check the level structure and the minimal level property when \( \epsilon = 0 \). In this case, the correlators \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0, g} \), with \( g \geq 0 \) are equal to \( \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_{g} \) according to the main theorem. The level structure of the correlators follows from the similar level structure described in Proposition 1.37. The minimal level property in this case reads
\[
\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0, g} = \int_{\overline{M}_{g, n}} \lambda_g \lambda_0 \psi_1^{d_1} \ldots \psi_n^{d_n} = \int_{\overline{M}_{g, n}} \lambda_g \psi_1^{d_1} \ldots \psi_n^{d_n},
\]
which follows from the analogous equality for Hurwitz correlators [GJV05, Proposition 3.12].

**Remark 1.40.** When \( l = g \), the correlators \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g, 0} \) are the coefficients of the classical Witten-Kontsevich series. They are given by
\[
\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g, 0} = \int_{\overline{M}_{g, n}} \psi_1^{d_1} \ldots \psi_n^{d_n}.
\]

They correspond to the top level of the level structure: \( \sum d_i = 3g - 3 + n \). All the other levels vanish. In particular, the bottom level is given by
\[
\int_{\overline{M}_{g, n}} \lambda_g^2 \psi_1^{d_1} \ldots \psi_n^{d_n} = 0.
\]
Recap table. The following table presents the structure of the correlators of the quantum Witten-Kontsevich series coming from Theorem 1, Proposition 1.38, Conjecture 1 and Conjecture 2. Fix three nonnegative integers \( l, k \) and \( n \). In the box corresponding to the \( l \)-th line and the \( k \)-th column we store the correlators

\[
\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{l,k}
\]

where \( (d_1, \ldots, d_n) \) is a list of nonnegative integers, coming from the coefficients of \( \epsilon^{2l} h^k \) of \( F(q) \). We set \( g := l + k \) in the table.

The first row and column in the table present proved facts, while the boxes \( k, l \geq 1 \) present conjectures.
\[
\begin{array}{c|c|c|c}
\hline
\varepsilon^0 & h^0 & h^1 & \ldots & h^{k} \\
\hline
\int_{\mathcal{M}_{0,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} & \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_1 \quad \text{s.t.} \quad \sum d_i = 4 - 3 + n & \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g \quad \text{s.t.} \quad \sum d_i = 4g - 3 + n & \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g \quad \text{s.t.} \quad \sum d_i = 4g - 5 + n & \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g \quad \text{s.t.} \quad \sum d_i = 4g - 1 + n \\
\hline
\int_{\mathcal{M}_{1,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} & \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_1 = \int_{\mathcal{M}_{1,n}} \lambda_0 \lambda_1 \psi_1^{d_1} \cdots \psi_n^{d_n} & \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g = \int_{\mathcal{M}_{g,n}} \lambda_g \psi_1^{d_1} \cdots \psi_n^{d_n} & \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g = \int_{\mathcal{M}_{g,n}} \lambda_g \psi_1^{d_1} \cdots \psi_n^{d_n} \\
\hline
\varepsilon^2 & \int_{\mathcal{M}_{1,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} & \int_{\mathcal{M}_{2,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} & \int_{\mathcal{M}_{g,n}} \lambda_1 \lambda_2 \psi_1^{d_1} \cdots \psi_n^{d_n} \\
\hline
\varepsilon^{2l} & \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} & \int_{\mathcal{M}_{g,n}} \lambda_{g-1} \lambda_g \psi_1^{d_1} \cdots \psi_n^{d_n} & \int_{\mathcal{M}_{g,n}} \lambda_l \lambda_g \psi_1^{d_1} \cdots \psi_n^{d_n} \\
\hline
\end{array}
\]

\[k + 1\]

\[k + l + 1\]
1.4.2 String and dilaton for the quantum Witten-Kontsevich series

In [GJV05], the authors prove the string and dilaton equations for the Hurwitz correlators. Hence Theorem 1 implies that the \( \epsilon = 0 \) restriction of the quantum Witten-Kontsevich series satisfies the string and dilaton equations. These equations are actually satisfied by the full quantum Witten-Kontsevich series.

**Theorem 2.** The quantum Witten-Kontsevich series satisfies the string equation

\[
\frac{\partial}{\partial t_0} F^{(q)} = \sum_{i \geq 0} t_{i+1} \frac{\partial}{\partial t_i} F^{(q)} + \frac{t_0^2}{2} - \frac{i \hbar}{24}.
\]

**Conjecture 3.** The quantum Witten-Kontsevich series satisfies the dilaton equation

\[
\frac{\partial}{\partial t_1} F^{(q)} = \sum_{i \geq 0} t_i \frac{\partial}{\partial t_i} F^{(q)} + \epsilon \frac{\partial}{\partial \epsilon} F^{(q)} + 2 \hbar \frac{\partial}{\partial \hbar} F^{(q)} - 2 F^{(q)} + \frac{\epsilon^2}{24}.
\]

1.5 Plan of the paper

In Section 2, we prove the string equation that is Theorem 2.

In Section 3, we prove some properties of Eulerian numbers. These properties bring a key simplification in the proof of Theorem 1.

In Section 4, we prove Theorem 1.

In Section 5, we prove a combinatorial identity that is used in the last step of the proof Theorem 1.

In Section 6, we prove the vanishing properties of the correlators stated in Proposition 1.38.

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2 The string equation

The string equation for the quantum Witten-Kontsevich series does not directly come from a comparison between the \( \psi \)-classes and their pull-backs as it usually does. Indeed, the quantum Witten-Kontsevich series is defined by various commutations of Hamiltonians, that are themselves defined by integration of \( \psi \)-classes over the double ramification cycle, and then the substitution \( u_i = \delta_{i,1} \). We have to follow this definition to prove the string equation and any other property of the quantum Witten-Kontsevich series.

2.1 On the substitution \( u_i = \delta_{i,1} \)

The definition of the quantum Witten-Kontsevich series uses differential polynomials as elements of \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \). We need to explain how to substitute \( u_i = \delta_{i,1} \) in an element of \( \tilde{\mathcal{A}} \). This is the purpose of the following lemma.
Lemma 2.1. Let \( \phi \) be a differential polynomial. Write it as a formal Fourier series, that is

\[
\phi(x) = \sum_{k=0}^{d} \sum_{a_1, \ldots, a_k \in \mathbb{Z}} \phi_k(a_1, \ldots, a_k) p_{a_1} \cdots p_{a_k} e^{ix(a_1 + \cdots + a_k)}
\]

where \( \phi_k(a_1, \ldots, a_k) \in \mathbb{C}[a_1, \ldots, a_k][[\hbar, \Omega]] \) is a symmetric polynomial in its \( k \) indeterminates \( a_1, \ldots, a_k \) for \( 0 \leq k \leq d \). The substitution \( u_i = \delta_{i,1} \) in \( \phi \) is given by

\[
\phi \bigg|_{u_i = \delta_{i,1}} = \sum_{k \geq 0} (-i)^k [a_1 \cdots a_k] \phi_k(a_1, \ldots, a_k),
\]

where \([a_1 \cdots a_k]\) denotes the coefficient of \( a_1 \cdots a_k \).

Proof. The Fourier series of \( u \) and its \( s \)-th derivative is given by \( u_s(x) = \sum_{a \in \mathbb{Z}} (ia)^s p_a e^{iax} \). Hence we get

\[
\phi(x) = \sum_{k \geq 0} \sum_{a_1, \ldots, a_k \in \mathbb{Z}} \left( \sum_{s_1, \ldots, s_k \geq 0} a_1^{s_1} \cdots a_k^{s_k} [a_1^{s_1} \cdots a_k^{s_k}] \phi_k(a_1, \ldots, a_k) \right) p_{a_1} \cdots p_{a_k} e^{ix(a_1 + \cdots + a_k)}
\]

\[
= \sum_{k \geq 0} \sum_{s_1, \ldots, s_k \geq 0} (-i)^{s_1 + \cdots + s_k} u_{s_1} \cdots u_{s_k} [a_1^{s_1} \cdots a_k^{s_k}] \phi_k(a_1, \ldots, a_k).
\]

Hence \( \phi \bigg|_{u_i = \delta_{i,1}} = \sum_{k \geq 0} (-i)^k [a_1 \cdots a_k] \phi_k(a_1, \ldots, a_k). \)

\[ \square \]

2.2 A proof of the string equation

We prove in this section that the quantum Witten-Kontsevich series satisfies the string equation

\[
\frac{\partial}{\partial t_0} {F(q)} = \sum_{i \geq 0} t_{i+1} \frac{\partial}{\partial t_i} {F(q)} + \frac{t_0^2}{2} - \frac{i\hbar}{24}.
\]

Plan of the proof. The string equation is an equality of power series. In order to prove this equation, we verify that the constant and linear terms of the power series on the LHS and RHS correspond, then we show that the second derivative of this equation is true. From the definition of the quantum Witten-Kontsevich series, we find that the derivative with respect to \( t_{d_1} \) and \( t_{d_2} \) of the string equation yields

\[
\left. \frac{\partial}{\partial t_0} \Omega_{d_1,d_2}^{h,t} \right|_{u_i = \delta_{i,1}} = \sum_{k \geq 0} t_{k+1} \left. \frac{\partial}{\partial t_k} \Omega_{d_1,d_2}^{h,t} \right|_{u_i = \delta_{i,1}} + \Omega_{d_1-1,d_2}^{h,t} \bigg|_{u_i = \delta_{i,1}} + \Omega_{d_1,d_2-1}^{h,t} \bigg|_{u_i = \delta_{i,1}} + \delta_0_{d_1} \delta_0_{d_2}, \quad (3)
\]

where \( d_1 \) and \( d_2 \) are two nonnegative integers.

Before proving Eq. (3), let us focus on the constant and linear terms of the string equation. To do so, we need the following lemma to compute the two-point function with a zero insertion.
Lemma 2.2. Fix a nonnegative integer $d$. We have
\[ \Omega^b_{0,d} = H_{d-1}. \]

Proof. This is an equality between elements of $\tilde{A}$, that is between two lists of polynomials. The equality between the first polynomial of each list follows from $\Omega^b_{0,d} \bigg|_{p^* = 0} = H_{d-1} \bigg|_{p^* = 0}$. To obtain the equality of the rest of the polynomials, it is enough to check that $\partial_x \Omega^b_{0,d} = \partial_x H_{d-1}$.

Using the definition of the two-point function and the tau symmetry, we find
\[ \partial_x \Omega^b_{0,d} = \frac{1}{\hbar} \left[ H_{d-1}, \Pi_0 \right] = \frac{1}{\hbar} \left[ H_{d-1}, \Pi_0 \right]. \]

We noticed in Remark 1.29 that the commutator of any differential polynomial with $\Pi_0$ corresponds to the derivative with respect to $x$. Hence we get the equality
\[ \partial_x \Omega^b_{0,d} = \partial_x H_{d-1}. \]

Constant term of the string equation. The constant term in the RHS of the string equation is given by $-\frac{i}{24}$. We then have to show that $-\frac{i}{24}$ is also the coefficient of $t_0$ in $F(q)$. By construction, the coefficient of $t_0$ is the coefficient of $t_0 t_1$. The coefficient of $t_0 t_1$ in $F(q)$ is given by $\Omega^b_{0,1} \bigg|_{u_i = \delta_{i,1}}$ Lemma 2.2 $H_0 \bigg|_{u_i = \delta_{i,1}}$.

We use the expression of $H_0$ given by the following lemma to deduce that this coefficient is $-\frac{i}{24}$.

Lemma 2.3. We have
\[ H_0 = \frac{u_0^2}{2} + \epsilon^2 \frac{u_2}{12} - \frac{i \hbar}{24}. \]

Proof. We have to compute the polynomials
\[ P_{m,g,k}(a_1, \ldots, a_m) = \int_{DR_g(0, a_1, \ldots, a_m, -\sum a_i)} \psi_1 \lambda_k, \]
for any $m, g, k \geq 0$. The double ramification cycle $\text{DR}_g(0, a_1, \ldots, a_m, -\sum a_i)$ is a polynomial in the parts of the ramifications profiles with coefficients in $H^{2g} \left( \mathcal{M}_{g,n+2} \right)$, hence these polynomials vanishes for dimensional reasons if $(m, g, k) \neq (2, 0, 0), (1, 1, 1), (0, 1, 0)$ or $(0, 2, 2)$. We have $P_{2,0,0} = \int_{\mathcal{M}_{0,4}} \psi_1 = 1$ and $P_{1,1,1} = \frac{a^2}{12}$ (see [Bur15, Section 4.3.1]). If $(m, g, k) = (0, 1, 0)$ and $(0, 2, 2)$ we use that $\text{DR}_g(0, 0) = (-1)^g \lambda_g$ (see [JPPZ17]) to get
\[ P_{0,1,0} = \int_{\text{DR}_1(0,0)} \psi_1 = -\int_{\mathcal{M}_{1,2}} \psi_1 \lambda_1 = -\frac{1}{24}, \quad \text{and} \quad P_{0,2,2} = \int_{\text{DR}_2(0,0)} \psi_1 \lambda_2 = 0. \]

Using the expression of the Fourier series of $u$ and its derivative, we obtain our expression of $H_0$. \qed
Linear terms of the string equation. We now focus on the linear terms. The coefficient of $t_0$ does not appear on the RHS of the string equation. Let us show that it vanishes in the LHS. The coefficient of $t_0$ in the LHS of string equation is given by the coefficient of $t_0t_0$ in $F^{(q)}$. This coefficient is

$$\Omega_{0,0}\bigg|_{u_1=\delta_{i,1}} = H_{-1}\bigg|_{u_1=\delta_{i,1}} = 0.$$  

We used $H_{-1} = u_0$ as one can verify from the definition.

Fix an integer $d \geq 1$. The coefficient of $t_d$ on the LHS of the string equation is given by the coefficient of $t_d t_d$ in $F^{(q)}$. The coefficient of $t_d$ on the RHS of the string equation is given by the coefficient of $t_d t_d$ in $F^{(q)}$, which is equal to the coefficient of $t_d t_d$ in $F^{(q)}$ by definition of the linear terms of $F^{(q)}$.

Some necessary lemmas. We prove three lemmas used for the proof of Equation (3). The first one contains the geometric origin of the string equation.

Lemma 2.4. Fix a nonnegative integer $d$. The Hamiltonian density $H_d$ satisfy the string equation

$$\frac{\partial}{\partial p_0}H_d = H_{d-1}.$$  

The proof is analogous to the one of Lemma 2.7 in [BR16].

Proof. We have

$$\frac{\partial}{\partial p_0}H_d = \sum_{g \geq 0, m \geq 0 \atop 2g + m + 1 > 0} \frac{(ih)^g}{m!} \sum_{a_1, \ldots, a_m \in \mathbb{Z}} \left( \int_{\mathcal{D} \mathcal{R}_g(0, a_1, \ldots, a_m, 0, -\sum a_i)} \psi^d \Lambda \left( -\frac{\epsilon^2}{ih} \right) \right) p_{a_1} \cdots p_{a_m} e^{ix \sum a_i}.$$  

Let $\pi : \mathcal{M}_{g,m+3} \to \mathcal{M}_{g,m+2}$ be the map defined when $(g, m) \neq (0, 0)$ that forgets the $(m + 2)$-th marked point. We use that $\pi^* \mathcal{D} \mathcal{R}_g (0, a_1, \ldots, a_m, -\sum a_i) = \mathcal{D} \mathcal{R}_g (0, a_1, \ldots, a_m, 0, -\sum a_i)$, $\pi^* \Lambda \left( -\frac{\epsilon^2}{ih} \right) = \Lambda \left( -\frac{\epsilon^2}{ih} \right)$ and $\psi^{d+1} = \pi^* \left( \psi^d \right) + D \pi^* \left( \psi^d \right)$, where $D$ is the divisor with a bubble containing the marked points 1 and $(m + 2)$, to obtain

$$\int_{\mathcal{D} \mathcal{R}_g(0, a_1, \ldots, a_m, 0, -\sum a_i)} \psi^{d+1} \Lambda \left( -\frac{\epsilon^2}{ih} \right) = \begin{cases} \int_{\mathcal{D} \mathcal{R}_g(0, a_1, \ldots, a_m, 0, -\sum a_i)} \psi^d \Lambda \left( -\frac{\epsilon^2}{ih} \right) & \text{if } 2g + m > 0 \text{ and } d > 0 \\ 0 & \text{if } 2g + m > 0 \text{ and } d = 0 \\ \delta_{d,0} & \text{if } g = 0, m = 0 \text{ and } d = 0. \end{cases}$$  

This proves the lemma.  

Lemma 2.5. Fix $d_1, d_2$ two positive integers, we have

$$\frac{\partial \Omega_h^{d_1,d_2}}{\partial p_0} = \Omega_h^{d_1-1,d_2} + \Omega_h^{d_1,d_2-1} + \delta_0 \delta_0 \cdot \delta_{0,d_1} \delta_{0,d_2},$$  

where we use the convention that $\Omega_h$ vanishes if at least one of its indices is negative.
Proof. If \((d_1, d_2) = (0, 0)\), we obtain \(\Omega_{0,0} = H_{-1} = u_0\) from Lemma 2.2. Hence the equation is satisfied.

Otherwise, we want to prove an equality of elements of \(\mathcal{A}\), that is an equality of two lists of symmetric polynomials. The equality of the first polynomial of each list follows from the choice of the constant \(\Omega_{d_1,d_2}^h|_{p_u=0}\) in the definition of \(\Omega_{d_1,d_2}^h\). To obtain the equality for the rest of the polynomials, it is enough to prove that the \(x\)-derivative of the equation is verified. From the definition of \(\Omega_{d_1,d_2}^h\), we have

\[
\partial_x \frac{\partial}{\partial p_0} \Omega_{d_1,d_2}^h = \frac{\partial}{\partial p_0} \partial_x \Omega_{d_1,d_2}^h = \frac{\partial}{\partial p_0} \frac{1}{h} \left[ H_{d_1-1}, \Omega_{d_2}^h \right] \\
= \frac{1}{h} \left[ \partial H_{d_1-1}, \Omega_{d_2}^h \right] + \frac{1}{h} \left[ H_{d_1-1}, \frac{\partial \Omega_{d_2}^h}{\partial p_0} \right] \\
= \partial_x \Omega_{d_1-1,d_2}^h + \partial_x \Omega_{d_1,d_2-1}^h.
\]

We used Lemma 2.4 to obtain the last equality.

\(\square\)

**Lemma 2.6.** Let \(\phi\) be a differential polynomial, we have

\[
\partial_x \phi \bigg|_{u_i=\delta_{i,1}} = \frac{\partial \phi}{\partial p_0} \bigg|_{u_i=\delta_{i,1}}.
\]

Proof. Write \(\phi\) as an element of \(\mathcal{A}\), that is

\[
\phi(x) = \sum_{k \geq 0} \sum_{a_1,\ldots,a_k \in \mathbb{Z}} \phi_k(a_1,\ldots,a_k) p_{a_1} \cdots p_{a_k} e^{ix(a_1+\cdots+a_k)},
\]

where \(\phi_k(a_1,\ldots,a_k) \in \mathbb{C}[a_1,\ldots,a_k][[\epsilon, h]]\) is a symmetric polynomial in its \(k\) indeterminates \(a_1,\ldots,a_k\) for \(0 \leq k \leq d\). Then, thanks to Lemma 2.1, we have

\[
\partial_x \phi \bigg|_{u_i=\delta_{i,1}} = \sum_{k \geq 0} (-1)^k t^{k+1} \left[ a_1 \cdots a_k, \phi_k(a_1,\ldots,a_k) (a_1+\cdots+a_k) \right] \\
= \sum_{k \geq 0} (-1)^k t^{k+1} \sum_{j=1}^{k} \left[ a_1 \cdots a_j \cdots a_k, \phi_k(a_1,\ldots,a_{j-1},0,a_{j+1},\ldots,a_k) \right] \\
= \frac{\partial \phi}{\partial p_0} \bigg|_{u_i=\delta_{i,1}}.
\]

\(\square\)

**Proof of Equation (3).** We first recall this equation:

\[
\frac{\partial}{\partial t} \Omega_{d_1,d_2}^{h,t} \bigg|_{u_i=\delta_{i,1}} = \sum_{k \geq 0} t_{k+1} \frac{\partial}{\partial t_k} \Omega_{d_1,d_2}^{h,t} \bigg|_{u_i=\delta_{i,1}} + \Omega_{d_1-1,d_2}^{h,t} \bigg|_{u_i=\delta_{i,1}} + \Omega_{d_1,d_2-1}^{h,t} \bigg|_{u_i=\delta_{i,1}} + \delta_{0,d_1} \delta_{0,d_2}.
\]

21
Proof. We prove this equality at every degree in the indeterminates \((t_0, t_1, \ldots)\). Recall that \(\Omega^{h,t}_{d_1,d_2} = \exp \left( \sum_{k \geq 0} \frac{t^k}{k} \left[ \cdot, \partial^k \right] \right) \Omega^h_{d_3,d_2} \). Let \(n \geq 0\) and \((d_3, \ldots, d_n)\) be a list of nonnegative integers. Then the coefficient of \(t_{d_3} \cdots t_{d_n}\) of the LHS is given by

\[
\frac{1}{h^{n-1}} \left[ \cdots \left[ \frac{\Omega^h_{d_1,d_2}}{E} \cdot \Pi_{d_3}, \Pi_{d_4}, \ldots, \Pi_{d_n} \right] \right]_{u_i = \delta_{i,1}} = \frac{1}{h^{n-2}} \left[ \cdots \left[ \frac{\Omega^h_{d_1,d_2}}{E} \cdot \Pi_{d_3}, \Pi_{d_4}, \ldots, \Pi_{d_n} \right] \right]_{u_i = \delta_{i,1}}.
\]

We act with \(\frac{\partial}{\partial p_0}\) on every elements of the commutators. Then we use Lemmas 2.4 and 2.5 to find

\[
\frac{1}{h^{n-2}} \left[ \cdots \left[ \frac{\Omega^h_{d_1,d_2}}{E} \cdot \Pi_{d_3}, \Pi_{d_4}, \ldots, \Pi_{d_n} \right] \right]_{u_i = \delta_{i,1}} = \frac{1}{h^{n-2}} \left[ \cdots \left[ \Omega^h_{d_1,d_2} \cdot \Pi_{d_3}, \Pi_{d_4}, \ldots, \Pi_{d_n} \right] \right]_{u_i = \delta_{i,1}} + \cdots + \frac{1}{h^{n-2}} \left[ \cdots \left[ \Omega^h_{d_1,d_2} \cdot \Pi_{d_3}-1, \Pi_{d_4}, \ldots, \Pi_{d_n} \right] \right]_{u_i = \delta_{i,1}}.
\]

We recognize the coefficient of \(t_{d_3} \cdots t_{d_n}\) of the RHS Equation (3).

3 Eulerian numbers

Eulerian numbers naturally appear in the proof of the main theorem. We will need some of their properties in this proof.

Definition 3.1. Fix two nonnegative integers \(k, n\) and a permutation \(\sigma \in S_n\). A descent of the permutation \(\sigma\) is an integer \(i \in \{1, \ldots, n\}\) such that \(\sigma(i) > \sigma(i + 1)\). The Eulerian number \(\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle\) is the number of permutation of \(S_n\) with \(k\) descents.

The Eulerian polynomial \(E_n(t)\) is the generating polynomial of Eulerian numbers

\[
E_n(t) := \sum_{k=0}^{n-1} \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle t^k.
\]

The following two propositions are basic properties of Eulerian numbers. They are proved in the first chapter of [Pet15].

22
Proposition 3.2 (Carlitz identity). Let \( d \) be a nonnegative integer. Let \( t \) be a formal variable. We have

\[
\sum_{k \geq 1} k^d t^k = \frac{tE_d(t)}{(1-t)^{d+1}}.
\]

Remark 3.3. In the proof of the main theorem, we need to compute the numbers

\[
\sum_{k_1+\cdots+k_q = N} k_1^{d_1} \cdots k_q^{d_q} = [t^N] \prod_{i=1}^q \left( \sum_{k \geq 1} k_i^{d_i} t^k \right),
\]

where \( d_1, \ldots, d_q \) and \( N \) are nonnegative integers numbers. According to Carlitz identity, this can be done using Eulerian number. We will use this fact to simplify our computations.

Proposition 3.4. Let \( t \) and \( z \) be two formal variables. We have

\[
\sum_{n \geq 0} E_n(t) \frac{z^n}{n!} = \frac{t - 1}{t - e^z(t-1)}.
\]

Corollary 3.5. Let \( t \) and \( z \) be two formal variables. We have

\[
\sum_{n \geq 0} \frac{tE_n(t) z^{n+1}}{(n+1)! (1-t)^{n+1}} = -z - \ln (t - e^{-z}) + \ln (t-1).
\]

Proof. Start from the formula of From Proposition 3.4, that is

\[
\sum_{n \geq 0} E_n(t) \frac{z^n}{n!} = \frac{t - 1}{t - e^z(t-1)}.
\]

First multiply both sides of this equality by \( \frac{t}{1-z} \). Then substitute in both sides \( z := \frac{t}{1-t} \). Finally choose in both sides the primitive with respect to \( z \) which vanishes when \( z = 0 \). On the LHS, we obtain \( \sum_{n \geq 0} \frac{tE_n(t) z^{n+1}}{(n+1)! (1-t)^{n+1}} \). On the RHS we obtain \(-z - \ln (t - e^{-z}) + \ln (t-1)\). \( \square \)

Notation 3.6. In the following, we use the notation

\[
S(z) = \frac{\text{sh}(z/2)}{z/2} = \sum_{l \geq 0} \frac{z^{2l}}{2^{2l} (2l+1)!}.
\]

Lemma 3.7. Let \( A, B, t \) and \( z \) be some formal variables. We have

\[
\sum_{k > 0} AB z^2 k S(kA z) S(kB z) t^k = \ln \left( \frac{1 - te^{-\frac{A-B}{2}z}}{1 - te^{-\frac{A-B}{2}z}} \right) \frac{1 - te^{-\frac{A-B}{2}z}}{1 - te^{-\frac{A-B}{2}z}},
\]

23
Proof. We start from the LHS. Use the developed expression of $S$ (see Notation 3.6) to obtain
\[
\sum_{k>0} AB z^2 k S(kA z) S(kB z) t^k = \sum_{k>0, l_1, l_2 \geq 0} \frac{A^{2l_1+1} B^{2l_2+1} z^{2(l_1+l_2)+2}}{2^{2(l_1+l_2)} (2l_1 + 1)! (2l_2 + 1)!} k^{2(l_1+l_2)+1} t^k.
\]
Then, we use the Carlitz identity (Proposition 3.2) to compute the sum running over $k$. We obtain
\[
\sum_{l_1, l_2 \geq 0} \frac{A^{2l_1+1} B^{2l_2+1} z^{2(l_1+l_2)+2}}{2^{2(l_1+l_2)} (2l_1 + 1)! (2l_2 + 1)!} \frac{t E_{2(l_1+l_2)+1}(t)}{(1-t)^{2(l_1+l_2)+2}}.
\]
Simplifying this expression using
\[
\sum_{l_1+l_2 = l} \frac{A^{2l+1} B^{2l+1}}{(2l_1 + 1)! (2l_2 + 1)!} = \frac{1}{2} \frac{(A + B)^{2l+2} - (A - B)^{2l+2}}{(2l + 2)!},
\]
we obtain
\[
\sum_{l \geq 0} \frac{(A + B)^{2l+2} - (A - B)^{2l+2}}{2^{2l+1} (2l + 2)!} z^{2l+2} \frac{t E_{2l+1}(t)}{(1-t)^{2l+2}}.
\]
Denote by $F(z,t) := -z - \ln (t - e^{-z}) + \ln (t - 1)$. According to Corollary 3.5, we have
\[
\sum_{l \geq 0} \frac{(A + B)^{2l+2}}{2^{2l+1} (2l + 2)!} z^{2l+2} \frac{t E_{2l+1}(t)}{(1-t)^{2l+2}} = F\left(\frac{A + B}{2}, z, t\right) + \left(-\frac{A + B}{2}, z, t\right)
\]
and
\[
\sum_{l \geq 0} \frac{(A - B)^{2l+2}}{2^{2l+1} (2l + 2)!} z^{2l+2} \frac{t E_{2l+1}(t)}{(1-t)^{2l+2}} = F\left(\frac{A - B}{2}, z, t\right) + \left(-\frac{A - B}{2}, z, t\right).
\]
Finally, we remark that
\[
F\left(\frac{A + B}{2}, z, t\right) + \left(-\frac{A + B}{2}, z, t\right) - \left(F\left(\frac{A - B}{2}, z, t\right) + \left(-\frac{A - B}{2}, z, t\right)\right)
\]
\[
= \ln\left(\frac{1 - te^{A-B}z}{1 - te^{A+B}z}\right) \frac{(1 - te^{A-B}z)(1 - te^{A+B}z)}{(1 - te^{A+B}z)(1 - te^{-A-B}z)}
\]
to obtain the RHS of the Lemma. \hfill \Box

Lemma 3.8. Let $A, B$ and $t$ be some formal variables. We have
\[
\frac{(1 - te^{A-B})}{(1 - te^{A+B})} = 1 + 4 \sum_{k>0} \frac{\text{sh}(A) \text{sh}(B)}{\text{sh}(A + B)} \text{sh}(k(A+B)) t^k.
\]
Proof. Start from the LHS of the equality. Develop the two geometric series of the denominators, we obtain
\[
\frac{1}{(1 - te^A + B)(1 - te^{-A - B})} = \sum_{n,m \geq 0} e^{(n-m)(A+B)t^{n+m}} = \sum_{k \geq 0} \frac{\text{sh}((k+1)(A+B))}{\text{sh}(A+B)} t^k.
\]
Express the numerator as \((1 - te^A - B)(1 - te^{-A - B}) = t^2 - 2t \text{ch}(A - B) + 1\). We then obtain the following expression for \((1 - te^A - B)(1 - te^{-A - B})\):
\[
\frac{1}{\text{sh}(A + B)} \sum_{k \geq 0} (\text{sh}((k - 1)(A + B)) - 2t \text{ch}(A - B) \text{sh}(k(A + B)) + \text{sh}((k + 1)(A + B))) t^k.
\]
We use first the hyperbolic identity \(\text{sh}((k - 1)(A + B)) + \text{sh}((k + 1)(A + B)) = 2t \text{ch}(A + B) \text{sh}(k(A + B))\) and then \(\text{ch}(A + B) \text{ch}(A - B) = 2 \text{sh}(A) \text{sh}(B)\) to obtain the result.

The following property is a key ingredient of the proof of the main theorem.

Corollary 3.9. Let \(A, B, t\) and \(z\) be some formal variables. We have
\[
\exp \left( \sum_{k \geq 0} ABz^2 kS(kAz)S(kBz)t^k \right) = \frac{(1 - te^{\frac{A+B}{2}z})(1 - te^{-\frac{A+B}{2}z})}{(1 - te^{\frac{A-B}{2}z})(1 - te^{-\frac{A+B}{2}z})} = 1 + 4 \sum_{k \geq 0} \frac{\text{sh}\left(\frac{A}{2}z\right) \text{sh}\left(\frac{B}{2}z\right)}{\text{sh}\left(\frac{A+B}{2}z\right)} \text{sh}\left(\frac{k(A+B)}{2}z\right) t^k.
\]

Proof. The first equality is obtained from Lemma 3.7. The second is given by the formula of Lemma 3.8 with \(A := \frac{A}{2}z\) and \(B := \frac{B}{2}z\).

4 Proof of the main theorem

We give in this section the proof of Theorem 1, that is we prove the equality
\[
\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_{g},
\]
where \(g, n, d_1, \ldots, d_n\) are some nonnegative integers. First, we explain the strategy of the proof.

4.1 Computing \(\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g}\) and \(\langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_{g}\)

In the next section, we show that the string equation allows one to express \(\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g}\) and \(\langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_{g}\) from the quantum and Hurwitz correlators with a \(\tau_0\) insertion. We deduce that it is enough to prove the equality
\[
\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \langle \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_{g}
\]
in order to prove Theorem 1. Then, we explain how to obtain an explicit expression for the RHS, and how to compute the LHS. The two expressions are completely different, but can be used to prove the equality.
4.1.1 String Equation

Fix a nonnegative integer \( g \). The correlators of the quantum Witten-Kontsevich and the Hurwitz correlators satisfy the string equation

\[
\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \sum_{i=1}^{n} \langle \tau_{d_1} \ldots \tau_{d_i-1} \ldots \tau_{d_n} \rangle_{0,g},
\]

\[
\langle \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g = \sum_{i=1}^{n} \langle \langle \tau_{d_1} \ldots \tau_{d_i-1} \ldots \tau_{d_n} \rangle \rangle_g.
\]

The first equation is the statement of Theorem 2 proved in Section 2. The second equation is Proposition 3.10 in [GJV05]. Let us define the following generating series

\[
\hat{G}_g^{(q)}(s_1,\ldots,s_n) := \sum_{d_1,\ldots,d_n \geq 0} \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} s_1^{d_1} \cdots s_n^{d_n}
\]

and

\[
G_g^{(q)}(s_1,\ldots,s_n) := \sum_{d_1,\ldots,d_n \geq 0} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} s_1^{d_1} \cdots s_n^{d_n}.
\]

We also define \( \hat{G}_g^H \) and \( G_g^H \) by replacing the quantum correlators by the Hurwitz correlators. According to the string equation, we have

\[
\hat{G}_g^{(q)} = (s_1 + \cdots + s_n) G_g^{(q)} \quad \text{and} \quad \hat{G}_g^H = (s_1 + \cdots + s_n) G_g^H.
\]

We can inverse these two equations in the same way, we then obtain \( G_g^{(q)} \) in terms of \( \hat{G}_g^{(q)} \) and \( G_g^H \) in terms of \( \hat{G}_g^H \). Hence, proving \( \hat{G}_g^{(q)} = \hat{G}_g^H \) is equivalent to prove \( G_g^{(q)} = G_g^H \).

4.1.2 An explicit expression for \( \langle \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g \)

In [GJV05], Theorem 3.1 gives the following explicit expression for the one-part double Hurwitz numbers

\[
H_{(d),\mu}^g = r!d^{r-1} \left[ z^{2g} \right] \prod_{i=1}^{n} S \left( \mu_i z \right) \frac{S(0)}{S(z)},
\]

where \( d = \mu_1 + \cdots + \mu_n \) is the degree of the ramified cover, \( r = 2g - 1 + n \) is the number of simple ramifications and \( \left[ z^{2g} \right] \) denotes the coefficient of \( z^{2g} \). We also used Notation 3.6, that is \( S(z) = \frac{\sh(z/2)}{z/2} \).

Note that the polynomiality of the one-part double Hurwitz numbers \( H_{(d),\mu}^g \) in their ramifications \( \mu_1,\ldots,\mu_n \) is clear from this expression. From the definition of the Hurwitz correlators given in Eq. (2), we find

\[
\langle \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g = (-1)^{-2+n-\sum_{i=1}^{n} d_i} \left( \mu_1^{d_1} \cdots \mu_n^{d_n} \right) \left[ z^{2g} \right] \left( \mu_1 + \cdots + \mu_n \right)^{2g-2+n} \frac{S(\mu_1 z) \cdots S(\mu_n z)}{S(z)},
\]

where we used \( S(0) = 1 \).
4.1.3 Computing $\langle \tau_0 \tau_1 \ldots \tau_n \rangle_{0,g}$

From the construction of the quantum Witten-Kontsevich series (see Section 1.2.9), we get the following expression of its correlators

$$
\langle \tau_0 \tau_1 \ldots \tau_n \rangle_{0,g} = i^g \left[ \epsilon^0 \hbar^g \left( \frac{\partial^{n-1} \Omega_0^{d_1}}{\partial t_d \ldots \partial t_{d_n}} \right) \right]_{t_* = 0, u_* = \delta_{1,i}} = i^g \left[ \epsilon^0 \hbar^g \left( \frac{1}{\hbar^{n-1}} \sum \left[ \Omega_0 \tau_1, \overline{H}_2 \ldots, H_{d_n} \right] \right) \right]_{u_* = \delta_{1,i}}.
$$

Lemma 2.2 gives $\Omega_{0,d} = H_{d-1}$. Thus we have to study

$$
\langle \tau_0 \tau_1 \ldots \tau_n \rangle_{0,g} = i^g \left[ \hbar^{g+n-1} \sum \left[ H_{d_{1-1}}, \overline{H}_2 \ldots, \overline{H}_{d_n} \right] \right]_{u_* = \delta_{1,i}, \epsilon = 0}. \quad (6)
$$

We will compute this expression in the proof. To do so, we need a computable expression of $H_d$ and a computable expression of the commutator. We give these expressions in the next two paragraphs. The computation will be carried out in $\bar{A}$. We will then perform the substitution $u_i = \delta_{1,i}$ using Lemma 2.1.

**A computable expression of $H_p$.** In [BSSZ15], Theorem 1 gives an explicit expression for the intersection number of a DR-cycle with the maximal power of a $\psi$-class. From this theorem we get

$$
\int_{DR_g(0, a_1, \ldots, a_m, -\sum a_i)} \psi_{p+1}^{1} = \delta_{p+1, 2g-1+m} \left[ z^{2g} \frac{S(a_1 z) \ldots S(a_m z) S(\sum_{i=1}^{m} a_i z)}{S(z)} \right].
$$

We then obtain from the definition of $H_p$ given by Eq. (1),

$$
H_p(x) \bigg|_{\epsilon = 0} = \sum_{g \geq 0, m \geq 0} \frac{(i \hbar)^g}{m!} \sum_{a_1, \ldots, a_m \in \mathbb{Z}} \left( \int_{DR_g(0, a_1, \ldots, a_m, -\sum a_i)} \psi_1^{d+1} A \left( \frac{-\epsilon^2}{i \hbar} \right) \right) p_{a_1} \cdots p_{a_m} e^{ix \sum_{i=1}^{m} a_i} = \sum_{g \geq 0} \frac{(i \hbar)^g}{m!} \sum_{a_1, \ldots, a_m \in \mathbb{Z}} \left[ z^{2g} \frac{S(a_1 z) \ldots S(a_m z) S(\sum_{i=1}^{m} a_i z)}{S(z)} \right] p_{a_1} \cdots p_{a_m} e^{ix \sum_{i=1}^{m} a_i}, \quad (7)
$$

with $m = p + 2 - 2g$.

**Explicit expression of the star product.** We will use the following expression of the star product.

**Proposition 4.1.** Let $f, g \in \mathcal{F}^h(P)$, we have

$$
f \star g = f \exp \left( \sum_{k>0} \frac{i \hbar k}{\partial p_k \partial p_{-k}} \right) g. \quad (8)
$$

The notations $\frac{\partial}{\partial p_k}$ and $\frac{\partial}{\partial p_{-k}}$ mean that the derivative acts on the left or on the right, that is $f \star g = fg + \sum_{k>0} i \hbar k \frac{\partial^2 f}{\partial p_k \partial p_{-k}} + \sum_{k_1, k_2 > 0} (i \hbar)^2 k_1 k_2 \frac{\partial^2 f}{\partial p_{k_1} \partial p_{k_2}} \frac{\partial^2 g}{\partial p_{-k_1} \partial p_{-k_2}} + \cdots$.

This property is a consequence of Wick’s theorem. To be exhaustive, we give a short proof.

27
Proof. Recall from Section 1.2.3 that the star product is defined by \( f \ast g := f \cdot g : W[[\epsilon, \hbar]] \). In order to obtain the explicit expression of \( f \ast g \) in Eq. (8), we must write \( f : g : \) as a sum of normally ordered terms. To do so, we commute the \( p \)’s of \( g : \) to the left of \( f : \) using the commutation relation \([p_a, p_b] = i\hbar a \delta_{a+b,0}\).

First, if all the \( p \)’s of \( g : \) commute with the \( p \)’s of \( f : \), we obtain the term \( f g : \). Then, if one contraction between a \( p < 0 \) of \( g : \) with a \( p > 0 \) of \( f : \) occurs, we obtain the term \( \sum_{k>0} i\hbar \frac{\partial f}{\partial p_k} \frac{\delta_{p-k}}{\partial p} \cdot \). Similarly, if \( q \geq 1 \) contractions occur, we obtain the term \( \sum_{k_1,\ldots,k_q>0} (i\hbar)^q q! k_1 \cdots k_q \frac{\partial^q f}{\partial p_{k_1} \cdots \partial p_{k_q}} \cdot \).

4.2 Proof of the equality \( \langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g} = \langle \langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle \rangle_g \)

In Section 4.2.1, we prove the equality of Eq. (4) for \( n = 1 \).

In Section 4.2.2, we prove the equality of Eq. (4) for \( n = 2 \). This particular case is included as an example to illustrate the proof of the general case.

In Section 4.2.3, we prove the equality of Eq. (4) for \( n \geq 2 \).

Convention. In the rest of the proof, we focus on the restriction \( \epsilon = 0 \). Hence, we forget the formal variable \( \epsilon \) and always suppose it to be zero.

4.2.1 Proof for \( n = 1 \)

Fix two nonnegative integers \( d \) and \( g \). We prove in this section that

\( \langle \tau_0 \tau_d \rangle_{0,g} = \langle \langle \tau_0 \tau_d \rangle \rangle_g \).

We start from the LHS, \( \langle \tau_0 \tau_d \rangle_{0,g} \). Recall that Eq. (6) gives \( \langle \tau_0 \tau_d \rangle_g = i^g \hbar^g S(az) \delta_{i,1} \) \((a_i = \delta_{i,1})\). Then we use the expression of \( H_p \) in Eq. (7) and perform the evaluation \( u = \delta_{i,1} \) with Lemma 2.1. We find

\[
\langle \tau_0 \tau_d \rangle_{0,g} = i^g \frac{i^g}{m!} (-i)^m [a_1 \cdots a_m] [z^{2g}] S(az) \delta_{i,1} \sum_{i=1}^m a_iz \frac{S(az)}{S(z)}
\]

with \( m = d + 1 - 2g \). Note that there is no \( a \)-linear term in \( S(az) \) because \( S \) is an even function, hence the expression of \( \langle \tau_0 \tau_d \rangle_{0,g} \) simplifies to

\[
\langle \tau_0 \tau_d \rangle_{0,g} = \frac{(-i)^{d+1}}{m!} [a_1 \cdots a_m] [z^{2g}] S(\sum_{i=1}^m a_iz) \frac{S(az)}{S(z)}.
\]

Let \( A = \sum_{i=1}^m a_i \). It is easy to check \([a_1 \cdots a_m] S(\sum_{i=1}^m a_iz) = m! [A^m] S(az) \). Hence we get

\[
\langle \tau_0 \tau_d \rangle_{0,g} = (-i)^{d+1} [z^{2g} A^m] S(az) \frac{S(az)}{S(z)} = (-i)^{d+1} [z^{2g} A^{d+1-2g}] S(az) \frac{S(az)}{S(z)},
\]

and we rewrite it as

\[
\langle \tau_0 \tau_d \rangle_{0,g} = i^{-1-d} [A^d] [z^{2g}] A^{2g-1} S(az) \frac{S(az)}{S(z)}.
\]

We recognize the expression of \( \langle \langle \tau_0 \tau_d \rangle \rangle_g \) given by Eq. (5).
4.2.2 Proof for $n = 2$

Fix three nonnegative integers $d_1, d_2$ and $g$. We prove in this section that

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = \langle \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle \rangle_g.$$ 

We start from the LHS. Recall that Eq. (6) gives

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = [h^g] \left( \frac{i}{\hbar} \left[ H_{d_1}, \mathbf{P}_{d_2} \right] \right) \big|_{u_i = \delta_{i,1}}.$$ 

In Step 1, we obtain an expression of $\frac{i}{\hbar} \left[ H_{d_1}, \mathbf{P}_{d_2} \right]$ using the formulas of $H_{d_1-1}$, $\mathbf{P}_{d_2}$ and of the star product given in Section 4.1.3.

In Step 2, we first extract the coefficient of $h^g$ in $\frac{i}{\hbar} \left[ H_{d_1-1}, \mathbf{P}_{d_2} \right]$. Then we perform the substitution $u_i = \delta_{i,1}$ in $[h^g] \frac{i}{\hbar} \left[ H_{d_1-1}, \mathbf{P}_{d_2} \right]$ and get a first expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$. This expression will be totally different from the expression of $\langle \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle \rangle_g$ given by Eq. (5).

In Steps 3,4 and 5, we will transform this last expression of $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$ into the expression of $\langle \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle \rangle_g$ given by Eq. (5).

**Step 1.** We compute $\frac{i}{\hbar} \left[ H_{d_1-1}, \mathbf{P}_{d_2} \right] = \frac{i}{\hbar} \left( H_{d_1-1} * \mathbf{P}_{d_2} - \mathbf{P}_{d_2} * H_{d_1-1} \right)$. Recall that

$$H_{d_1-1}(x) = \sum_{g_1 \geq 0} \sum_{a_1, \ldots, a_{m_1} \in \mathbb{Z}} \frac{(ih)^{g_1}}{m_1!} \left( \sum_{z^g_1} S(a_1 z) \cdots S(a_{m_1} z) S(\sum_{i=1}^{m_1} a_i z) \right) p_{a_1} \cdots p_{a_{m_1}} e^{ix \sum_{i=1}^{m_1} a_i}$$

and

$$\mathbf{P}_{d_2} = \sum_{g_2 \geq 0} \sum_{b_1, \ldots, b_{m_2} \in \mathbb{Z}} \frac{(ih)^{g_2}}{m_2!} \left( \sum_{w^{g_2}} S(b_1 w) \cdots S(b_{m_2} w) \right) p_{b_1} \cdots p_{b_{m_2}}.$$ 

The expression of $\mathbf{P}_{d_2}$ is obtained by a formal $x$-integration along $S^1$ of $H_{d_2}$. Hence this imposes the condition $\sum_{i=1}^{m_2} b_i = 0$ and then $S(\sum_{i=1}^{m_2} b_i w) = 1$.

From the expression of the star product (8), we get

$$H_{d_1-1} * \mathbf{P}_{d_2} = H_{d_1-1} \exp \left( \sum_{k>0} \frac{i\hbar}{\partial p_k} \frac{\partial}{\partial p_k} \right) \mathbf{P}_{d_2}$$

$$= H_{d_1-1} \sum_{q \geq 0} \frac{(ih)^q}{q!} \left( \sum_{k_1, \ldots, k_q > 0} \frac{\partial}{\partial p_{k_1}} \cdots \frac{\partial}{\partial p_{k_q}} \right) \mathbf{P}_{d_2}.$$ 

We first describe the action of the left-derivatives of the star product on $H_{d_1-1}$. Fix $g_1$ in $H_{d_1-1}$. The product of $q$ left-derivatives $\frac{\partial}{\partial p_{k_1}} \cdots \frac{\partial}{\partial p_{k_q}}$ acts on the formal power series

$$\sum_{a_1, \ldots, a_{m_1} \in \mathbb{Z}} S(a_1 z) \cdots S(a_{m_1} z) S(\sum_{i=1}^{m_1} a_i z) p_{a_1} \cdots p_{a_{m_1}}$$

yielding

$$m_1 \cdot (\tilde{m}_1 + 1) \sum_{a_1, \ldots, a_{\tilde{m}_1} \in \mathbb{Z}} S(a_1 z) \cdots S(a_{\tilde{m}_1} z) S(k_1 z) \cdots S(k_q z) S\left( (\tilde{A} + K) z \right) p_{a_1} \cdots p_{a_{\tilde{m}_1}}$$

yielding
where $\tilde{m}_1 = m_1 - q$, $\tilde{A} = \sum_{i=1}^{\tilde{m}_1} a_i$ and $K = \sum_{i=1}^{q} k_i$. Indeed, each derivative $\frac{\partial}{\partial p_{a_j}}$ may act on each factor $p_{a_i}$. Without loss of generality we can assume that $i = \tilde{m}_1 + j$, multiplying the result by $m_1 \cdots (\tilde{m}_1 + 1)$ to account for the number of equivalent choices. The derivative yields a nonvanishing result if and only if $a_i = k_j$.

Similarly we describe the action of the right-derivatives of the star product on $H_{d_2}$. Fix $g_2$ in $H_{d_2}$. The product of $q$ right-derivatives $\frac{\partial}{\partial p_{-k_1}} \cdots \frac{\partial}{\partial p_{-k_q}}$ acts on the formal series $\sum_{b_1,\ldots,b_{m_2} \in \mathbb{Z}} S(b_1 w) \cdots S(b_{m_2} w) p_{b_1} \cdots p_{b_{m_2}}$ yielding

$$m_2 \cdots (\tilde{m}_2 + 1) \sum_{b_1,\ldots,b_{\tilde{m}_2} \in \mathbb{Z}} S(b_1 w) \cdots S(b_{\tilde{m}_2} w) S(-k_1 w) \cdots S(-k_q w) p_{b_1} \cdots p_{b_{\tilde{m}_2}}$$

where $\tilde{m}_2 = m_2 - q$ and $\tilde{B} = \sum_{i=1}^{\tilde{m}_2} b_i$.

Recall that $S$ is an even function, hence $S(-k_i w) = S(k_i w)$. Note that the condition $\sum_{i=1}^{m} b_i = 0$ becomes $K = \tilde{B}$.

Finally, the expression of $H_{d_1 - 1} \star \overline{H}_{d_2}$ becomes

$$\sum_{g \geq 0} \sum_{g_1 + g_2 + q = g} \frac{(ih)^g}{\tilde{m}_1! \tilde{m}_2! q!} \left[ z^{2g_1} w^{2g_2} \right] \times \sum_{a_1,\ldots,a_{\tilde{m}_1} \in \mathbb{Z}} \sum_{b_1,\ldots,b_{\tilde{m}_2} \in \mathbb{Z}} \sum_{k_1,\ldots,k_q > 0} k_1 \cdots k_q$$

$$\times S(a_1 z) \cdots S(a_{\tilde{m}_1} z) \times S(k_1 z) \cdots S(k_q z) \times S\left(\left(\tilde{A} + \tilde{B}\right) z\right) \
\times S(b_1 w) \cdots S(b_{\tilde{m}_2} w) S(-k_1 w) \cdots S(-k_q w)$$

$$\times \frac{S(z)}{S(w)} \times p_{a_1} \cdots p_{a_{\tilde{m}_1}} p_{b_1} \cdots p_{b_{\tilde{m}_2}} e^{i\alpha(A+B)} ,$$

where $\tilde{m}_1 = d_1 + 1 - 2g_1 - q$ and $\tilde{m}_2 = d_2 + 2 - 2g_2 - q$.

We can re-do this exercise to compute $\overline{H}_{d_2} \star H_{d_1 - 1}$. The main difference is the condition $K = \tilde{B}$ which
becomes $K = -\tilde{B}$. Thus, the expression of \( \frac{i^g}{\hbar} \left[ H_{d_1-1}, \overline{H}_{d_2} \right] \) is

\[
\sum_{g \geq 0} \sum_{g_1 + g_2 + q - 1 = g} \frac{i^{2g+1} \hbar^g}{m_1! m_2! q!} \left[ z^{2g_1} w^{2g_2} \right] \sum_{a_1, \ldots, a_{\tilde{m}_1} \in \mathbb{Z}} \sum_{b_1, \ldots, b_{\tilde{m}_2} \in \mathbb{Z}} \frac{S(a_1 z) \cdots S(a_{\tilde{m}_1} z) S(\left( \tilde{A} + \tilde{B} \right) z) S(b_1 w) \cdots S(b_{\tilde{m}_2} w)}{S(z) S(w)} \times \left( \sum_{k_1 + \cdots + k_q = \tilde{B}} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right) \times p_{a_1} \cdots p_{a_{\tilde{m}_1}} p_{b_1} \cdots p_{b_{\tilde{m}_2}} e^{i x (\tilde{A} + \tilde{B})},
\]

where $\tilde{m}_1 = d_1 + 1 - 2g_1 - q$ and $\tilde{m}_2 = d_2 + 2 - 2g_2 - q$.

**Remark 4.2.** The term $q = 0$ of the expression of $H_{d_1-1} \star \overline{H}_{d_2}$ in Eq. (9) corresponds to the commutative part of the star product. This term disappears in the commutator $\frac{1}{\hbar} \left[ H_{d_1-1}, \overline{H}_{d_2} \right]$. We can then suppose that $q \geq 1$ in Eq. (10).

**Change of notation.** For convenience, we change the notation by removing the tildes, i.e. we set $m_1 := \tilde{m}_1$, $m_2 := \tilde{m}_2$, $A := \tilde{A}$ and $B := \tilde{B}$.

**Step 2.** We first extract the coefficient of $\hbar^g$ from $\frac{i^g}{\hbar} \left[ H_{d_1-1}, \overline{H}_{d_2} \right]$. Then we evaluate this coefficient, which is a differential polynomial, at $u_i = \delta_{i,1}$. We will then get an expression for $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = [h^g] \frac{i^g}{\hbar} \left[ H_{d_1}, \overline{H}_{d_2} \right]|_{u_i = \delta_{i,1}}$.

We extract the coefficient of $\hbar^g$ in $\frac{i^g}{\hbar} \left[ H_{d_1-1}, \overline{H}_{d_2} \right]$ from its expression obtained in Eq. (10). This only
removes the summation over $g$. With our new notations, this coefficient is

$$
[h^g] \frac{i^g}{\hbar} \left[ H_{d_{1-1}}, H_{d_2} \right] = \sum_{g_1+g_2+q-1=g} \sum_{g_1, g_2 \geq 0, q \geq 1} \frac{i^{2g+1}}{m_1!m_2!q!} [z^{2g_1} w^{2g_2}]
$$

$$
\times \sum_{a_1, \ldots, a_{m_1} \in \mathbb{Z}} \sum_{b_1, \ldots, b_{m_2} \in \mathbb{Z}} S(a_1 z) \cdots S(a_{m_1} z) S((A + B) z) S(b_1 w) \cdots S(b_{m_2} w)
$$

$$
\times \left( \sum_{k_1 + \cdots + k_q = B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right)
$$

$$
\times \left( \sum_{k_1 + \cdots + k_q = -B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right)
\times p_{a_1} \cdots p_{a_{m_1}} p_{b_1} \cdots p_{b_{m_2}} e^{ix(A+B)},
$$

where $m_1 = d_1 + 1 - 2g_1 - q$ and $m_2 = d_2 + 2 - 2g_2 - q$.

This last expression is a differential polynomial thanks to Proposition 1.13. In order to substitute $u_i = \delta_{i,1}$, we use Lemma 2.1. We get

$$
\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{a,g} = \sum_{g_1+g_2+q-1=g} \frac{i^{d_1-d_2}}{m_1!m_2!q!} [z^{2g_1} w^{2g_2}] [a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}]
$$

$$
\times \sum_{a_1, \ldots, a_{m_1} \in \mathbb{Z}} \sum_{b_1, \ldots, b_{m_2} \in \mathbb{Z}} S(a_1 z) \cdots S(a_{m_1} z) S((A + B) z) S(b_1 w) \cdots S(b_{m_2} w)
$$

$$
\times \left( \sum_{k_1 + \cdots + k_q = B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right)
$$

$$
\times \left( \sum_{k_1 + \cdots + k_q = -B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right),
$$

where $m_1 = d_1 + 1 - 2g_1 - q$ and $m_2 = d_2 + 2 - 2g_2 - q$.

Remark 4.3. It can look confusing that in this expression, $b_i$ stands for a formal variable and an integer when we write $k_1 + \cdots + k_q = B = \sum_{i=1}^{m_2} b_i$. This is due to the the presence of Ehrhart polynomials. Indeed, the coefficient of any power of $z$ and $w$ in

$$
\sum_{k_1 + \cdots + k_q = B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w)
$$

is an Ehrhart polynomial in the indeterminate $B = \sum_{i=1}^{m_2} b_i$, see [BR16, Lemma A.1] for a proof. Hence, when we write $B$ as an integer and use this lemma to justify that $B$ can also be used as a formal variable. The same phenomenon applies to $\sum_{k_1 + \cdots + k_q = -B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w)$ when $B < 0$. 

32
Plan of Steps 3, 4 and 5. This expression of \( \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} \) is completely different from the one of \( \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_g \) given by Eq. (5). Moreover, it is difficult to compute the number \( \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} \) from this expression. Let us point out the difficulties. The three last lines of Eq. (11) form a series depending on the parameters \( d_1, d_2, g_1, g_2, q \) in the indeterminates \( a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w \). We denote this series by \( F_{d_1, d_2, g_1, g_2, q} (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w) \) so that Eq. (11) becomes

\[
\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = \sum_{g_1 + g_2 + q - 1 = g, g_1, g_2 \geq 0, q \geq 1} [z^{2g_1} w^{2g_2}] [a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}] F_{d_1, d_2, g_1, g_2, q} (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w).
\]

Hence, for each choice of the parameters \( d_1, d_2, g_1, g_2, q \), we have to extract the coefficient of \( z^{2g_1} w^{2g_2} a_1 \ldots a_{m_1} b_1 \ldots b_{m_2} \) in \( F_{d_1, d_2, g_1, g_2, q} \). Then we sum these coefficients over the parameters \( g_1, g_2, q \). The main difficulty is to extract the coefficient of \( b_i \) from the expression appearing in parenthesis in \( F_{d_1, d_2, g_1, g_2, q} \), that is from

\[
\sum_{k_1 + \ldots + k_q = B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) - \sum_{k_1 + \ldots + k_q = -B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w).
\]

As we explained in Remark 4.3, the coefficient of any power of \( z \) and \( w \) in each of the two sums is an Ehrhart polynomial in the indeterminate \( B = \sum_{i=1}^{m_1} b_i \). However we do not have an explicit expression for the coefficients of these Ehrhart polynomials. Luckily, Eulerian numbers appear in the computation of these coefficients, see Remark 3.3. The plan is to modify our expression of \( \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} \) in order to use known properties of Eulerian numbers.

In Step 3, we will modify our expression of \( \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} \) using simplifications arising from extracting the coefficient of \( a_1 \ldots a_{m_1} b_1 \ldots b_{m_2} \) in \( F_{d_1, d_2, g_1, g_2, q} (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}, z, w) \). These simplifications mainly come from the fact that \( z \to S(z) \) is even.

In Step 4, we use some changes of variables in order to get \( \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} \) as the coefficient of an exponential generating series. This is the exponential of an expression that can be computed using Eulerian numbers.

In Step 5, we finally we use a property of Eulerian number in order to get a simplified expression of \( \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} \). We then recover from it the expression of \( \langle \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle \rangle_g \) given by Eq. (5).

**Step 3.** The evaluation \( u_i = \delta_{i,1} \) brings many simplifications that we explain now.

- First recall that \( S \) is an even power series so that the coefficient of \( \alpha \) in \( S(\alpha z) \times F(\alpha) \) where \( F \) is a
formal power series in $\alpha$ is the coefficient of $\alpha$ in $F(\alpha)$. Hence, Expression (11) simplifies as

$$\langle \tau_0 \tau_d \rangle_{0,g} = \sum_{g_1 + g_2 + q - 1 = g} \frac{i^{d_1 - d_2}}{m_1! m_2! q!} \left[ z^{2g_1} w^{2g_2} \right]$$

$$\times [a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}] \frac{S((A + B) z)}{S(z) S(w)}$$

$$\times \left( \sum_{k_1 + \ldots + k_q = B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right) - \left( \sum_{k_1 + \ldots + k_q = -B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right).$$

• Fix $g_1, g_2$ and $q$ in Expression (12) so that $m_1$ and $m_2$ are fixed. We extract the coefficient of $a_1 \ldots a_{m_1} b_1 \ldots b_{m_2}$ from a power series that only depends of the sums $A = a_1 + \ldots + a_{m_1}$ and $B = b_1 + \ldots + b_{m_2}$. This is equivalent to extracting the coefficient of $\frac{A^{m_1} B^{m_2}}{m_1! m_2!}$ from the same power series.

• Consider the expression in parenthesis in Eq. (12). This is a power series in the indeterminate $B$. Moreover, when $B \geq 0$ this power series becomes

$$\sum_{k_1 + \ldots + k_q = B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w)$$

and when $B < 0$ it becomes

$$- \sum_{k_1 + \ldots + k_q = -B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w).$$

We are interested in the coefficients of this power series, so for simplicity we can suppose $B > 0$. Expression (12) becomes

$$\langle \tau_0 \tau_d \rangle_{0,g} = \sum_{g_1 + g_2 + q - 1 = g} \frac{i^{d_1 - d_2}}{m_1! m_2! q!} \left[ z^{2g_1} w^{2g_2} \right]$$

$$\times [A^{m_1} B^{m_2}] \frac{S((A + B) z)}{S(z) S(w)}$$

$$\times \left( \sum_{k_1 + \ldots + k_q = B} k_1 \cdots k_q S(k_1 z) \cdots S(k_q z) S(k_1 w) \cdots S(k_q w) \right).$$

34
Step 4. We perform the changes of variables $z := Az$ and $w := Bw$ in Expression (13). Recall that $m_1 = d_1 + 1 - 2g_1 - q$ and $m_2 = d_2 + 2 - 2g_2 - q$, these changes of variables yield

$$
\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = \sum_{g_1 + g_2 + q - 1 = g}^{g+1} \frac{i^{-d_1 - d_2}}{q!} \left[ z^{2g_1} w^{2g_2} A^{d_1 + 1 - q} B^{d_2 + 2 - q} \right] \sum_{k_1 + \cdots + k_q = B} k_1 \cdots k_q S(k_1 Az) \cdots S(k_q Az) S(k_1 Bw) \cdots S(k_q Bw).
$$

We re-write this as

$$
\sum_{q=1}^{g+1} \frac{i^{-d_1 - d_2}}{q!} \left[ A^{d_1 + 1 - q} B^{d_2 + 2 - q} \right] \sum_{g_1 + g_2 = g - q + 1} z^{2g_1} w^{2g_2} \frac{S((A + B) Az)}{S(Az) S(Bw)} \sum_{k_1 + \cdots + k_q = B} k_1 \cdots k_q S(k_1 Az) \cdots S(k_q Az) S(k_1 Bw) \cdots S(k_q Bz).
$$

Note that for any formal power series $G(z, w) = \sum_{i,j \geq 0} G_{i,j} z^i w^j$, we have $\sum_{g_1 + g_2 = h} [z^{g_1} w^{g_2}] G(z, w) = G_{g_1, g_2} = [z^h] G(z, z)$. Using this remark in Expression (14) with $h = g + 1 - q$ and using that $G(z, w)$ is even in $z$ and $w$, we get the following expression for $\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g}$,

$$
\sum_{q=1}^{g+1} \frac{i^{-d_1 - d_2}}{q!} \left[ A^{d_1 + 1 - q} B^{d_2 + 2 - q} z^{2g_1} w^{2g_2} \right] \sum_{k_1 + \cdots + k_q = B} k_1 \cdots k_q S(k_1 Az) \cdots S(k_q Az) S(k_1 Bz) \cdots S(k_q Bz).
$$

Re-write this expression as

$$
\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = i^{-d_1 - d_2} A^{d_1} B^{d_2} z^{2g} \frac{S((A + B) Az)}{S(Az) S(Bz)} \sum_{q=1}^{g+1} \frac{1}{q!} A^{q-1} B^{q-2} z^{2q-2} \sum_{k_1 + \cdots + k_q = B} \prod_{i=1}^{q} k_i S(k_i Az) S(k_i Bz).
$$

We can extend the range of summation to $q$ running from 1 to $\infty$. Indeed, it is clear from the expression that the terms with $q > g + 1$ vanishes, since we extract the coefficient of $z^{2g}$ from a power series with a
factor $z^{2q-2}$. Hence, the second line of Expression (15) can be re-written as the coefficient of an exponential series. Expression (15) becomes

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = i^{-d_1-d_2} \left[ A^{d_1} B^{d_2} z^{2g} \right] \frac{S((A+B)Az)}{S(Az)S(Bz)} \times \frac{1}{AB^2 z^2} \left[ t^B \right] \left( \exp \left( \sum_{k>0} AB z^2 k S(kAz) S(kBz) t^k \right) - 1 \right)$$

Step 5. We use properties of Eulerian numbers in order to simplify the expression of the exponential power series in Expression (16). Using first Proposition 3.9 and then the definition $S(z) = \frac{\text{sh}(z/2)}{z/2}$, we get

$$\exp \left( \sum_{k>0} AB z^2 k S(kAz) S(kBz) t^k \right) = 1 + 4 \sum_{k>0} \frac{\text{sh}(\frac{A}{2}z) \text{sh}(\frac{B}{2}z)}{\text{sh}(\frac{A+B}{2}z)} \text{sh} \left( \frac{kA+B}{2} z \right) t^k$$

Thus, Expression (16) becomes after extracting the coefficient of $t^B$

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = i^{-d_1-d_2} \left[ A^{d_1} B^{d_2} z^{2g} \right] \frac{S((A+B)Az)}{S(Az)S(Bz)} \times 4 \frac{\text{sh}(\frac{A}{2}z) \text{sh}(\frac{B}{2}z)}{\text{sh}(\frac{A+B}{2}z)} \text{sh} \left( \frac{B (A+B)}{2} z \right).$$

We re-write the second line as $\frac{S(Az)S(Bz)}{S((A+B)z)} S(B (A+B) z)$ so that

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = i^{-d_1-d_2} \left[ A^{d_1} B^{d_2} z^{2g} \right] \frac{S((A+B)Az) S(B(A+B)z)}{S((A+B)z)}.$$

Finally, the change of variable $z := \frac{x}{A+B}$ in this last expression gives $\langle \langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle \rangle_g$ as expressed in Eq. (5), that is

$$\langle \tau_0 \tau_{d_1} \tau_{d_2} \rangle_{0,g} = (-1)^{\frac{d_1-d_2}{2}} \left[ A^{d_1} B^{d_2} z^{2g} \right] (A+B)^{2g} \frac{S(Az)S(Bz)}{S(z)}.$$

4.2.3 Proof for $n \geq 2$

Convention. Because of the multiple use of the index $i$, we choose to denote the imaginary unit $i$ as $\sqrt{-1}$ in this section.

Fix $n+1$ nonnegative integers $d_1, \ldots, d_n$ and $g$, we prove in this section that

$$\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g} = \langle \langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle \rangle_g.$$

36
for \( n \geq 2 \). We start from the LHS. As explained in the strategy of the proof, we have

\[
\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = [h^g] \frac{\sqrt{-1} h^n}{h^{n-1}} \left[ \ldots \left[ H_{d_{1-1}}, \overline{H}_{d_2} \right], \ldots, \overline{H}_{d_n} \right] \bigg|_{u_i = \delta_{1,i}} .
\]

We follow the steps of the previous section in this more general setting. The main difference occurs in Step 5, where we need a combinatorial lemma which was obvious when \( n = 2 \). Let us recall these steps.

In Step 1, we obtain an expression of \( \left[ \ldots \left[ H_{d_{1-1}}, \overline{P}_{d_2} \right], \ldots, \overline{P}_{d_n} \right] \) using the formulas of \( H_{d_{1-1}}, \overline{P}_{d_2}, \ldots, \overline{P}_{d_n} \) and of the developed expression of the star product.

In Step 2, we first extract the coefficient of \( h^g \) in \( \frac{\sqrt{-1} h^n}{h^{n-1}} \left[ \ldots \left[ H_{d_{1-1}}, \overline{P}_{d_2} \right], \ldots, \overline{P}_{d_n} \right] \). Then we perform the substitution \( u_i = \delta_{1,i} \) in it and get a first expression of \( \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} \). However this expression will be totally different from the one of \( \langle \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g \) given by Eq. (5).

In Steps 3,4 and 5, we will transform this last expression of \( \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} \) into the expression of \( \langle \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g \) given by Eq. (5).

**Step 1.** We compute in the first step \( \left[ \ldots \left[ H_{d_{1-1}}, \overline{P}_{d_2} \right], \ldots, \overline{P}_{d_n} \right] \). From the expression of the Hamiltonian density Eq. (7), we set

\[
H_{d_{1-1}}(x) = \sum_{g_1,m_1 \geq 0} \frac{(\sqrt{-1} h)^{g_1}}{m_1!} \sum_{a_1, \ldots, a_{m_1} \in \mathbb{Z}} \left( \left[ \sum_{z_i} \frac{S(a_1 \cdots a_{m_1} z)^S(A_1 z)}{S(z)} \right] \right) p_{a_1} \cdots p_{a_{m_1}} e^{\sqrt{-1} x A_1},
\]

and

\[
\overline{P}_{d_i} = \sum_{g_i,m_i \geq 0} \frac{(\sqrt{-1} h)^{g_i}}{m_i!} \sum_{a_i, \ldots, a_{m_i} \in \mathbb{Z}} \left( \left[ \sum_{z_i} \frac{S(a_1 \cdots a_{m_i} z)^S(A_i z)}{S(z)} \right] \right) p_{a_1} \cdots p_{a_{m_i}}, \text{ with } 2 \leq i \leq n,
\]

where \( A_i := \sum_{j=1}^{m_i} a_j^i \) and \( 1 \leq i \leq n \). In these notations, we use the variables of summations \( a_j^i \) in \( H_{d_{1-1}} \) and the variables of summations \( a_j^i \) in \( \overline{P}_{d_i} \), with \( 2 \leq i \leq n \). Note that in the notation \( a_j^i \), \( i \) is just an upper index. The expression of \( \overline{P}_{d_i} \) is obtained by a formal \( x \)-integration along \( S^1 \) of \( H_{d_i} \). Hence, the sum over \( a_1^i, \ldots, a_{m_i}^i \) has the constraint \( A_i = 0 \).

In Step 1.1, we give an expression for \( H_{d_{1-1}} \star \overline{P}_{d_2} \star \cdots \star \overline{P}_{d_n} \). In Step 1.2 we explain why this is the only term of the commutator \( \left[ \ldots \left[ H_{d_{1-1}}, \overline{P}_{d_2} \right], \ldots, \overline{P}_{d_n} \right] \) needed to compute \( \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} \).

**Step 1.1.**

37
Proposition 4.4. We have

\[ H_{d_1-1} \star \prod_{d_2} \star \cdots \star \prod_{d_n} = \sum_{q_1 \geq 0, I \in C} \sum_{g_1, \ldots, g_n \geq 0} \sum_{\tilde{m}_1, \ldots, \tilde{m}_n \geq 0} \text{with conditions } \alpha \]

\[ \prod_{i=1}^{n} \left( \frac{\sqrt{-1})^{q_i}}{m_i!} \right)^2 \sum_{a_1^i, \ldots, a_{m_i}^i \in \mathbb{Z}} W_i (a_1^i, \ldots, a_{m_i}^i, z_i) p_{a_1} \cdots p_{a_{m_i}} e^{\sqrt{-1} x A_i} \]

\[ \times \prod_{l \in C} \left( \frac{\sqrt{-1})^{q_l}}{q_l!} \sum_{k_j^l, \ldots, k_{i_l}^l > 0} k_j^l \cdots k_{i_l}^l W^l (k_j^l, \ldots, k_{i_l}^l, z_l) \right), \]

where

- \( C \) is the set of pairs (2-element subsets) of \( \{1, \ldots, n\} \); we also denote by \( C_i \subset C \) the subset of pairs that contain \( i \),
- the conditions \( \alpha \) on the summations running over \( g_1, \ldots, g_n, \tilde{m}_1, \ldots, \tilde{m}_n \) and \( q_l, I \in C \) are
  \[ 2g_1 + \tilde{m}_1 + \sum_{l \in C_1} q_l = d_1 + 1 \]
  and
  \[ 2g_j + \tilde{m}_j + \sum_{l \in C_j} q_l = d_j + 2, \text{ with } 2 \leq j \leq n, \]
- the weight \( W_i \) with \( 1 \leq i \leq n \) is defined by
  \[ W_i (a_1^i, \ldots, a_{m_i}^i, z_i) := \prod_{j=1}^{m_i} S \left( a_j^i z_i \right) S \left( \tilde{A}_i z_i + \cdots + \tilde{A}_n z_i \right) \]
  and
  \[ W_i (a_1^i, \ldots, a_{m_i}^i, z_i) := \prod_{j=1}^{m_i} S \left( a_j^i z_i \right) S \left( z_i \right), \text{ for } 2 \leq i \leq n, \]
- \( \tilde{A}_j = \sum_{i=1}^{\tilde{m}_j} a_i \),
- the conditions \( \beta \) are the following \( (n-1) \) constraints over the summations:
  \[ \tilde{A}_i - \sum_{j=1}^{i-1} K^{(j,i)} + \sum_{j=i+1}^{n} K^{(i,j)} = 0, \text{ for } 2 \leq i \leq n \]
  where \( K^l = k_1^l + \cdots + k_{q_l}^l, \text{ for } I \in C, \)
• the weight \( W^I \) for \( I = \{i, j\} \in \mathcal{C} \) is defined by

\[
W^I (k_1^I, \ldots, k_q^I, z_I) := S (k_1^I z_i) \cdots S (k_q^I z_i) \times S (k_1^I z_j) \cdots S (k_q^I z_j),
\]

note that the notation \( z_I \) means \( z_i, z_j \).

**Proof.** We use the expression of the star product given by

\[
f \ast g = f \sum_{q \geq 0} \frac{(-\hbar)^q}{q!} \left( \sum_{k_{1, \ldots, q} > 0} k_1 \cdots k_q \frac{\partial}{\partial p_{k_1}} \cdots \frac{\partial}{\partial p_{k_q}} \cdots \frac{\partial}{\partial p_{k_{-1}}} \right) g.
\]  

(19)

The star product is associative as one can check from Eq. (19). We use this associativity in the following way

\[
H_{d_1-1} \ast \Pi_{d_2} \ast \cdots \ast \Pi_{d_n} = (\cdots (H_{d_1-1} \ast \Pi_{d_2}) \ast \cdots \ast \Pi_{d_n}).
\]

Each of the \( n-1 \) star products has couples of derivatives acting on the left and on the right with opposite indices. Let \( 2 \leq i \leq n \). The \((i-1)\)th star product acts on the left on \( H_{d_1-1}, \Pi_{d_2}, \ldots, \Pi_{d_{i-1}} \) and on the right only on \( \Pi_{d_i} \). Fix a nonnegative integer \( q \) and \( q \) positive integers \( k_1, \ldots, k_q \). Consider the term

\[
\left( \frac{-\hbar}{q!} \right)^q k_1 \cdots k_q \frac{\partial}{\partial p_{k_1}} \cdots \frac{\partial}{\partial p_{k_q}} \cdots \frac{\partial}{\partial p_{k_{-1}}}
\]

in the development in \( \hbar \) of the \((i-1)\)th star product. Among these \( q \) left derivatives, we denote by \( q_{i,j} \), with \( j < i \), the number of derivatives acting on \( \Pi_{d_j} \) (or \( H_{d_1-1} \) if \( j = 1 \)). We furthermore add an upper index \( \{i, j\} \) on the corresponding \( k \) variables so that the \( q_{i,j} \) left derivatives coming from the \((i-1)\)th star product and acting on the \( j \)th Hamiltonian are denoted by

\[
\frac{\partial}{\partial p_{k_{i,j}}} \cdots \frac{\partial}{\partial p_{k_{i,j}}}
\]

The associate right derivatives \( \frac{\partial}{\partial p_{k_{-1,j}}} \cdots \frac{\partial}{\partial p_{k_{-1,j}}} \) act on \( \Pi_{d_i} \) with opposite indices. With this notation, we obtain

\[
H_{d_1-1} \ast \Pi_{d_2} \ast \cdots \ast \Pi_{d_n}
\]

\[
= \sum_{q_i \geq 0, i \in \mathcal{C}} \sum_{k_1^I, \ldots, k_q^I \geq 0, i \in \mathcal{C}} \Pi_{i \in \mathcal{C}} \left( \frac{-\hbar}{q!} \right)^q k_1^I \cdots k_q^I
\]

\[
\times \prod_{j \in \mathcal{C}_1} \frac{\partial^{q_{i,j}}}{\partial p_{k_{i,j}}^1 \cdots \partial p_{k_{i,j}}^q} H_{d_1-1}
\]

\[
\times \prod_{i=2}^n \prod_{j=1}^{i-1} \left( \frac{\partial^{q_{i,j}}}{\partial p_{k_{i,j}}^1 \cdots \partial p_{k_{i,j}}^q} \right) \prod_{i=1}^n \left( \frac{\partial^{q_{i,i}}}{\partial p_{k_{i,i}}^1 \cdots \partial p_{k_{i,i}}^q} \right) \Pi_{d_i}.
\]

(20)
Let us explain this formula. The derivatives acting on $\mathbf{H}_{d_i}$ have two different origins; the derivatives coming from the $(i-1)$th star product, these are the derivatives with negative indices in the product running over the variable $j$, and the derivatives coming from the $i$th to the $(n-1)$th star product, these are the derivatives with positive indices in the product running over the variable $l$. Similarly, the derivatives acting on $H_{d_{i-1}}$ come from all the star products and have positive indices. Moreover, when we develop the star products, we have to choose which derivative acts on which Hamiltonian so that multinomial coefficients appear and simplify the factorials.

We now describe the action of the $\sum_{j=1,j\neq i}^{n} q_{(i,j)}$ derivatives of the last line of Eq. (20) on $\mathbf{H}_{d_i}$. We find

$$
\prod_{j=1}^{i-1} \left( \frac{\partial q_{(i,j)}}{\partial p_{-k_{(i,j)}}} \right) \prod_{l=i+1}^{n} \left( \frac{\partial q_{(i,l)}}{\partial p_{k_{(i,l)}}} \right) \mathbf{H}_{d_i} = \sum_{g_{i,m_i} \geq 0, 2g_i + m_i + \sum_{l \in \mathcal{C}_1} q_l = d_i + 2} \frac{(\sqrt{-1} \hbar)^{g_i}}{m_i!} \left[ z^{2g_i} \right] \times \sum_{a_1^i, \ldots, a_{m_i}^i \in \mathbb{Z}} W_i \left( a_1^i, \ldots, a_{m_i}^i, z_i \right) p_{a_1^i} \cdots p_{a_{m_i}^i} \times \prod_{j=1}^{i-1} S \left( k_1^{(i,j)} z_i \right) \cdots S \left( k_{q_{(i,j)}}^{(i,j)} z_i \right) \prod_{l=i+1}^{n} S \left( k_1^{(i,l)} z_i \right) \cdots S \left( k_{q_{(i,l)}}^{(i,l)} z_i \right). \tag{21}
$$

Indeed, when the $\sum_{j=1,j\neq i}^{n} q_{(i,j)}$ derivatives act on $S(a_1^i z_i) \cdots S(a_{m_i}^i z_i) p_{a_1^i} \cdots p_{a_{m_i}^i}$ in $\mathbf{H}_{d_i}$, it remains $m_i = m_i - \sum_{j=1,j\neq i}^{n} q_{(i,j)}$ variables $p$. The part of this expression which is not reached by the derivatives is contained in the third line while the part reached by the derivatives with negative and positive indices is the content of the last line. Finally, the condition $A_i = 0$ in $\mathbf{H}_{d_i}$ becomes $\hat{A}_i + \sum_{j=i+1}^{n} K^{(i,j)} - \sum_{j=1}^{n-1} K^{(j,i)} = 0$, this is the $i$th equation of conditions $\beta$.

Similarly, there are only derivatives with positive indices acting on $H_{d_{i-1}}$. We find

$$
\prod_{j \in \mathcal{C}_1} \left( \frac{\partial q_j}{\partial p_{k_j}} \right) H_{d_{i-1}} = \sum_{g_{1,m_1} \geq 0, 2g_1 + m_1 + \sum_{l \in \mathcal{C}_1} q_l = d_{i-1} + 1} \frac{(\sqrt{-1} \hbar)^{g_1}}{m_1!} \left[ z^{2g_1} \right] \times \sum_{a_1^1, \ldots, a_{m_1}^1 \in \mathbb{Z}} W_1 \left( a_1^1, \ldots, a_{m_1}^1, z_1 \right) p_{a_1^1} \cdots p_{a_{m_1}^1} e^{\sqrt{-1} \alpha} \sum_{i=1}^{n} \hat{A}_i \times \prod_{j \in \mathcal{C}_1} S \left( k_1^j z_1 \right) \cdots S \left( k_{q_j}^j z_1 \right). \tag{22}
$$

Note that $A_1$ becomes after the action of the derivatives

$$
\hat{A}_1 - \sum_{j=2}^{n} K^{(1,j)} = \sum_{i=1}^{n} \hat{A}_i.
$$

40
We obtained this equality by summing the \((n - 1)\) equations of conditions \(\beta\) (see Eq. (18)).

By combining Eq. (20), (21) and (22) we obtain Eq. (17). This proves the proposition.

**Step 1.2.** Similarly to the expression of \(H_{d_1 - 1} \star \overline{1} \cdots \overline{n} \) obtained in Step 1.1, we can get the expressions of the \(2^{n-1} - 1\) others terms appearing in \([\ldots [H_{d_1 - 1}, \overline{1}_d], \ldots, \overline{n}_d]\). To each term we associate a permutation \(\sigma \in S_n\) such that \(\sigma(i)\) is the index of the Hamiltonian appearing at the \(i\)th position (the index of \(H_{d_1 - 1}\) is 1 and the index of \(\overline{1}_d\) is \(i\)). Then, the expression of the term in \([\ldots [H_{d_1 - 1}, \overline{1}_d], \ldots, \overline{n}_d]\) corresponding to the permutation \(\sigma\) is given by Expression (17) with one modifications: conditions \(\beta\) become

\[
\tilde{A}_i - \sum_{j=1}^{\sigma^{-1}(i)-1} K^{(i,\sigma(j))} + \sum_{j=\sigma^{-1}(i)+1}^{n} K^{(i,\sigma(j))}, \quad 2 \leq i \leq n.
\]

However, it is not necessary to compute these \(2^{n-1} - 1\) other terms in order to obtain \(\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{h,g}\). Indeed, \([\ldots [H_{d_1 - 1}, \overline{1}_d], \ldots, \overline{n}_d]\) is a power series in the indeterminates \(a_1^1, \ldots, a_{m_1}^1, \ldots, a_1^n, \ldots, a_{m_n}^n\). In Step 2 we extract one coefficient of this power series. Then, we can restrict to compute the terms in \([\ldots [H_{d_1 - 1}, \overline{1}_d], \ldots, \overline{n}_d]\) such that

\[
\tilde{A}_i > 0, \quad \text{with } i \geq 2.
\]

The only term in \([\ldots [H_{d_1 - 1}, \overline{1}_d], \ldots, \overline{n}_d]\) satisfying these inequalities is \(H_{d_1 - 1} \star \overline{1}_d \cdots \overline{n}_d\). Indeed, in the other terms, the condition coming from the Hamiltonian on the leftmost in the star product, say \(\overline{1}_d\), is

\[
\tilde{A}_i + \sum_{1 < j} K^{(1, j)} = 0,
\]

that is \(\tilde{A}_j < 0\).

Although, we will use one simplification coming from the commutators. A commutator simplifies the constant term in \(h\) in the star product, that is the term coming from the commutative product. These terms correspond in the expression of \(H_{d_1 - 1} \star \overline{1}_d \cdots \overline{n}_d\) given by Eq. (17) to the terms satisfying

\[
\sum_{j=1}^{i-1} q^{(i,j)} = 0, \quad \text{for } 2 \leq i \leq n.
\]

Indeed, \(\sum_{j=1}^{i-1} q^{(i,j)}\) counts the number of left (or right) derivatives coming from the \((i - 1)\)th star product and the commutative term in the star product is the one without derivatives. We call conditions \(\gamma\) the inequalities

\[
\sum_{j=1}^{i-1} q^{(i,j)} \geq 1, \quad \text{for } 2 \leq i \leq n.
\]

**Change of notation.** We remove the tildes in our notations, i.e. we set \(m_i := \tilde{m}_i\) and \(A_i := \tilde{A}_i\) for any \(1 \leq i \leq n\).
Step 2. We first extract the coefficient of $h^g$ from $\sqrt{\frac{\tau_0}{\hbar}} \left[ \ldots [H_{d_1-1}, \mathcal{P} \mathcal{P}_{d_2}], \ldots, \mathcal{P} \mathcal{P}_{d_n} \right]$. Then we evaluate this coefficient, which is a differential polynomial, at $u_i = \delta_{i,1}$. We will then get an expression for

$$(\tau_0 \tau_{d_1} \ldots \tau_{d_n})_{0,g} = [h^g] \sqrt{\frac{\tau_0}{\hbar}} \left[ \ldots [H_{d_1-1}, \mathcal{P} \mathcal{P}_{d_2}], \ldots, \mathcal{P} \mathcal{P}_{d_n} \right]_{u_i=\delta_{i,1}}.$$

We want to extract the coefficient of $h^g$ in $\sqrt{\frac{\tau_0}{\hbar}} \left[ \ldots [H_{d_1-1}, \mathcal{P} \mathcal{P}_{d_2}], \ldots, \mathcal{P} \mathcal{P}_{d_n} \right]$. As explained in Step 1.2, we only need to study the term $H_{d_1-1} \star \mathcal{P} \mathcal{P}_{d_2} \star \ldots \star \mathcal{P} \mathcal{P}_{d_n}$ in $[\ldots [H_{d_1-1}, \mathcal{P} \mathcal{P}_{d_2}], \ldots, \mathcal{P} \mathcal{P}_{d_n}]$. The coefficient of $h^g$ in $\sqrt{\frac{\tau_0}{\hbar}} H_{d_1-1} \star \mathcal{P} \mathcal{P}_{d_2} \star \ldots \star \mathcal{P} \mathcal{P}_{d_n}$ is easily obtained from Expression (17). We get with our new notations

$$[h^g] \sqrt{\frac{\tau_0}{\hbar}} \left[ \ldots [H_{d_1-1}, \mathcal{P} \mathcal{P}_{d_2}], \ldots, \mathcal{P} \mathcal{P}_{d_n} \right] = \sum_{g_1+\ldots+g_n+\sum_{I \in \mathcal{C}} q_I = g+n-1} \sqrt{-1}^{2g+(n-1)}$$

$$\times \prod_{i=1}^{n} \left( \frac{1}{m_i!} \left[ z_i^{2q_i} \right] \sum_{a_{i_1}^1, \ldots, a_{m_i}^i \in \mathbb{Z}} W_i \left( a_{i_1}^1, \ldots, a_{m_i}^i, z_i \right) p_{a_{i_1}} \cdots p_{a_{m_i}} e^{\sqrt{-1}x A_i} \right)$$

$$\times \prod_{I \in \mathcal{C}} \left( \frac{1}{q_I!} \sum_{k_{I_1}^I, \ldots, k_{I_{q_I}}^I > 0} k_{I_1}^I \cdots k_{I_{q_I}}^I W^I \left( k_{I_1}^I, \ldots, k_{I_{q_I}}^I, z_I \right) \right)$$

$$+ 2^{n-1} - 1 \text{ other terms},$$

where we used conditions $\alpha$ to fix

$$m_1 = d_1 + 1 - 2g_1 - \sum_{I \in \mathcal{C}_1} q_I, \text{ and } m_i = d_i + 2 - 2g_i - \sum_{I \in \mathcal{C}_i} q_I, \text{ when } 2 \leq i \leq n.$$

This last expression is a differential polynomial thanks to Proposition 1.13. In order to substitute $u_i = \delta_{i,1}$, we use Lemma 2.1. We get

$$(\tau_0 \tau_{d_1} \ldots \tau_{d_n})_{0,g} = \sum_{g_1+\ldots+g_n+\sum_{I \in \mathcal{C}} q_I = g+n-1} \sqrt{-1}^{l-2-|d|}$$

$$\times \prod_{i=1}^{n} \left( \frac{1}{m_i!} \left[ a_{i_1}^1 \cdots a_{m_i}^i \right] \left[ z_i^{2q_i} \right] W_i \left( a_{i_1}^1, \ldots, a_{m_i}^i, z_i \right) \right)$$

$$\times \prod_{I \in \mathcal{C}} \left( \frac{1}{q_I!} \sum_{k_{I_1}^I, \ldots, k_{I_{q_I}}^I > 0} k_{I_1}^I \cdots k_{I_{q_I}}^I W^I \left( k_{I_1}^I, \ldots, k_{I_{q_I}}^I, z_I \right) \right)$$

$$+ 2^{n-1} - 1 \text{ other terms},$$

42
where we simplified the power of $\sqrt{-1}$ using $m_1 := d_1 + 1 - 2g_1 - \sum_{l \in \mathcal{C}} q_l$ and $m_i := d_i + 2 - 2g_i - \sum_{l \in \mathcal{C}} q_l$, when $2 \leq i \leq n$. We used the notation $|d| = \sum_{i=1}^n d_i$.

**Remark 4.5.** It can look confusing that in this expression, $a^j_i$ for $2 \leq i \leq n$ and $1 \leq j \leq m_i$ stands for a formal variable and an integer when we write the $i$th constraint

$$A_i - \sum_{l=1}^{i-1} K^{[l,i]} + \sum_{l=i+1}^n K^{[l,i]} = 0$$

of conditions $\beta$. This is due to the presence of Ehrhart polynomials. Indeed, according to [BR16, Lemma A.1], for any list of integers $A_2, \ldots, A_n$, the coefficient of any power in $z_1, \ldots, z_n$ of

$$\prod_{l \in \mathcal{C}} \left( \frac{1}{q_l} \sum_{k_1^l, \ldots, k_{q_l}^l > 0 \text{ with conditions } \beta} k_1^l \cdots k_{q_l}^l W_l^I (k_1^l, \ldots, k_{q_l}^l, z_l) \right)$$

is a polynomial in the variables $A_2, \ldots, A_n$. We will then use the $A_i$'s and the $a^j_i$'s as integers and formal variables.

**Step 3.** We use the same simplification than Step 3 in Section 4.2.2, that is we consider the simplifications coming from extracting the coefficient of $a^1_i \cdots a^m_i$ in each factor of the product over $i$.

- Recall that $S$ an even power series so that the coefficient of $\alpha$ in $S(\alpha z) \times F(\alpha)$ is the coefficient of $\alpha$ in $F$. Hence, we can replace, in our expression of $\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g}$, $W_1$ by $\frac{S(\sum_{i=1}^n A_i z_1)}{S(\sum_{i=1}^n z_1)}$ and $W_i$ by $\frac{1}{S(\sum_{i=1}^n z_1)}$, when $2 \leq i \leq n$.

- Thanks to these simplifications, we see that we extract the coefficient of $\prod_{i=1}^n a^1_i \cdots a^m_i$ from a power series which only depends in the $a^j_i$'s through their sums $A_i = \sum_{j=1}^{m_i} a^j_i$, for $1 \leq i \leq n$. This is equivalent to extracting the coefficient of $\prod_{i=1}^n \frac{A_i^{m_i}}{m_i!}$ from the same power series.

- For simplicity (see Step 1.2), we can suppose that $A_i > 0$, with $i \geq 2$.

Thanks to these three points, we get

$$\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g} = \sum_{g_1 + \cdots + g_n + \sum_{l \in \mathcal{C}} q_l = g + n - 1 \text{ with conditions } \gamma} \sqrt{-1}^{n-2-|d|}$$

$$\times \prod_{i=1}^n \left( [A_i^{m_i}] \left[ z_i^{2g_i} \right] \frac{1}{S(z_i)} \right) S(\left| A \right| z_1)$$

$$\times \prod_{l \in \mathcal{C}} \left( \frac{1}{q_l} \sum_{k_1^l, \ldots, k_{q_l}^l > 0 \text{ with conditions } \beta} k_1^l \cdots k_{q_l}^l W_l^I (k_1^l, \ldots, k_{q_l}^l, z_l) \right), \quad (23)$$
where we used the notations $|A| = \sum_{i=1}^{n} A_i$.

**Step 4.** We perform some change of variable in our expression of $\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g}$. We will then organize this expression to see $\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g}$ as the coefficient of a product of exponential power series. This will allow us to use known properties of Eulerian numbers.

We perform the change of variables $z_i := A_i z_i$, with $1 \leq i \leq n$ in Expression (23). We get

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \sum_{g_1+\ldots+g_n+\sum_{I\in\mathcal{C}} q_I = g+n-1} \sqrt{-1}^{n-2-|d|} \times \prod_{i=1}^{n} \left( \left[ A_i^{m_i+2g_i} \right] z_i^{2g_i} \right) S(A_i z_i) \times \prod_{I\in\mathcal{C}} \left( \frac{1}{q_I^{k_I}} \sum_{k_I^{1} \ldots k_I^{r_I} > 0}^{k_I^{1} \ldots k_I^{r_I}} \tilde{W}^I(k_I^{1}, \ldots, k_I^{r_I}, z_I) \right)$$

where we used the notation

$$\tilde{W}^I(k_I^{1}, \ldots, k_I^{r_I}, z_I) = W^I(k_I^{1}, \ldots, k_I^{r_I}, A_i z_i, A_j z_j) = S(k_I^{1} A_i z_i) \cdots S(k_I^{r_I} A_i z_i) S(k_I^{1} A_j z_j) \cdots S(k_I^{r_I} A_j z_j)$$

for any pair $I = \{i,j\}$ of $\mathcal{C}$.

Using that $m_1 = d_1 + 1 - 2g_1 - \sum_{I\in\mathcal{C}_1} q_I$ and $m_i = d_i + 2 - 2g_i - \sum_{I\in\mathcal{C}_i} q_I$, when $2 \leq i \leq n$, we re-write this expression as

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \sum_{\sum_{I\in\mathcal{C}} q_I \leq g+n-1} \sqrt{-1}^{n-2-|d|} \times \prod_{i=1}^{n} \left( \left[ z_i^{2g_i} \right] \frac{1}{S(A_i z_i)} \right) \prod_{I\in\mathcal{C}} \left( \frac{1}{q_I^{k_I}} \sum_{k_I^{1} \ldots k_I^{r_I} > 0}^{k_I^{1} \ldots k_I^{r_I}} \tilde{W}^I(k_I^{1}, \ldots, k_I^{r_I}, z_I) \right)$$

$$G(z_1, \ldots, z_n)$$

We use in this expression that

$$\sum_{g_1+\ldots+g_n = g+n-1-\sum_{I\in\mathcal{C}} q_I} \left[ z_i^{2g_i} \ldots z_n^{2g_n} \right] G(z_1, \ldots, z_n) = \left[ z^{2g+2n-2-\sum_{I\in\mathcal{C}} 2q_I} \right] G(z, \ldots, z)$$

44
to obtain
\[
\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g} = \sum_{\sum_{i \in \mathcal{C}} q_i \leq g + n - 1} \sqrt{-1}^{n-2-|d|} \left[ A_1^{d_1 + 1 - \sum_{i \in \mathcal{C}^1} q_i} A_2^{d_2 + 2 - \sum_{i \in \mathcal{C}^2} q_i} \cdots A_n^{d_n + 2 - \sum_{i \in \mathcal{C}^n} q_i} \right] \times \left[ z^{2g + 2n - 2 - \sum_{i \in \mathcal{C}^i} q_i} \right] \times S(|A| A_1 z) \prod_{i=1}^{n} \left( \frac{1}{S(A_i z_i)} \right) \prod_{I \subseteq \mathcal{C}} \left( \frac{1}{q_I!} \sum_{k^I_1 \cdots k^I_q > 0} k^I_1 \cdots k^I_q \tilde{W}^I (k^I_1, \ldots, k^I_q, z_I) \right).
\]

Then, we rewrite this expression as
\[
\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{n-2-|d|} \left[ A_1^{d_1} A_2^{d_2} \cdots A_n^{d_n} z^{2g} \right] S(|A| A_1 z) \prod_{i=1}^{n} \left( \frac{1}{S(A_i z_i)} \right) \frac{1}{A_1 A_2 \cdots A_n z^{2n-2}} \times \sum_{\sum_{i \in \mathcal{C}} q_i \leq g + n - 1} \prod_{I \subseteq \mathcal{C}} \left( \frac{A^{q_I} A^{q_I^I} z^{2q_I}}{q_I!} \right) \sum_{k^I_1 \cdots k^I_q > 0} k^I_1 \cdots k^I_q \tilde{W}^I (k^I_1, \ldots, k^I_q, z_I) \right).
\]

We can extend the range of summation to \( \sum_{I \subseteq \mathcal{C}} q_I \) running from 0 to \( \infty \). Indeed, it is clear from this expression that the terms with \( \sum_{I \subseteq \mathcal{C}} q_I > g + n - 1 \) vanishes, since we extract the coefficient of \( z^{2g} \) from a power series with a factor \( \frac{z^{2n-2}}{z^{2m-2}} \). Hence, we rewrite the second line of the expression in the following way
\[
\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{n-2-|d|} \left[ A_1^{d_1} A_2^{d_2} \cdots A_n^{d_n} z^{2g} \right] S(|A| A_1 z) \prod_{i=1}^{n} \left( \frac{1}{S(A_i z_i)} \right) \frac{1}{A_1 A_2 \cdots A_n z^{2n-2}} \times \prod_{i=2}^{n} \prod_{j=1}^{i-1} \left( \sum_{q(i,j) \geq 0} \frac{A_i^{q(i,j)} A_j^{q(i,j)} z^{2q(i,j)}}{q(i,j)!} \sum_{k^I_1 \cdots k^I_q > 0} k^I_1 \cdots k^I_q \tilde{W}^I (k^I_1, \ldots, k^I_q, z_i) \right) - 1 \right).
\]

Note that when \( \sum_{j=1}^{i-1} q(i,j) = 0 \), for any \( 2 \leq i \leq n \), the product running over \( j \) equals 1 so that conditions \( \gamma \) are satisfied. Then we rewrite the last line of this expression as the coefficient of an exponential series.
Using the expression of $W^{i,j}(k_1^{i,j}, \ldots, k_q^{i,j}, z_i, z_j)$ and conditions $\beta$, we get

$$
\prod_{j=2}^{n} \left( \prod_{i=1}^{j-1} \left( \sum_{q(i,j) \geq 0} A_{q(i,j)}^{j-1} A_{q(i,j)}^{j} z A_{q(i,j)}^{j} \sum_{k_{1}, \ldots, k_{q(i,j)} > 0} k_1^{i,j} \ldots \sum_{k_{1}, \ldots, k_{q(i,j)} > 0} k_q^{i,j} W^{i,j}(k_1^{i,j}, \ldots, k_q^{i,j}, z_i, z_j) \right) - 1 \right)
$$

$$
= \left[ t_2^{A_2} \ldots t_n^{A_n} \right] \prod_{i=2}^{n} \left( \prod_{j=1}^{i-1} \exp \left( A_i A_j z^2 \sum_{k > 0} k S(k A_i z) S(k A_j z) \left( t_i \frac{t_j}{t} \right)^k \right) - 1 \right) \bigg|_{t_1=1}
$$

Substituting this expression in Eq. (24), we get

$$
\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{-n - 2 - |d|} \left[ A_1^{d_1} A_2^{d_2} \ldots A_n^{d_n} z^{2g} \right] S(|A| |A| z_n) \prod_{i=1}^{n} \left( \frac{1}{S(A_i z)} \right) \frac{1}{A_1 A_2^2 \ldots A_n^{2n-2}}
$$

$$
\times \left[ t_2^{A_2} \ldots t_n^{A_n} \right] \prod_{i=2}^{n} \left( \prod_{j=1}^{i-1} \exp \left( A_i A_j z^2 \sum_{k > 0} k S(k A_i z) S(k A_j z) \left( t_i \frac{t_j}{t} \right)^k \right) - 1 \right) \bigg|_{t_1=1}.
$$

Step 5. The second line of Eq. (25) is simplified using the following property.

**Proposition 4.6 (The products of exponentials formula).** Fix $n$ positive integers $A_1, \ldots, A_n$. Fix $n$ formal variables $t_1, \ldots, t_n$; by convention, let $t_1 = 1$. We have

$$
\left[ t_2^{A_2} \ldots t_n^{A_n} \right] \prod_{i=2}^{n} \left( \prod_{j=1}^{i-1} \exp \left( A_i A_j z^2 \sum_{k > 0} k S(k A_i z) S(k A_j z) \left( t_i \frac{t_j}{t} \right)^k \right) - 1 \right)
$$

$$
= A_1 A_2^2 \ldots A_n^{2n-2} \left( \sum_{i=1}^{n} A_i \right)^{-n-2} \prod_{i=1}^{n} S(A_i z) \prod_{r=2}^{n} S(A_r (A_1 z + \ldots + A_n z)).
$$

This proposition is proved in Section 5 using properties of Eulerian numbers. According to this proposition, we get

$$
\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{-n - 2 - |d|} \left[ A_1^{d_1} A_2^{d_2} \ldots A_n^{d_n} z^{2g} \right] S(|A| |A| z_n) \prod_{i=1}^{n} \left( \frac{1}{S(A_i z)} \right) \frac{1}{A_1 A_2^2 \ldots A_n^{2n-2}}
$$

$$
\times A_1 A_2^2 \ldots A_n^{2n-2} |A|^{n-2} \prod_{i=1}^{n} S(A_i z) \prod_{r=2}^{n} S(A_r |A| z).
$$

46
that we simplify as
\[ \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{n-2|\mathfrak{d}|} \left[ A_1^{d_1} A_2^{d_2} \ldots A_n^{d_n} z^{2g} \right] \frac{|A|^{n-2}}{S(|A| z)} \prod_{r=1}^{n} S(A_r |A| z). \]

Finally, with the change of variable \( z := \frac{z}{|A|} \), we get
\[ \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} = \sqrt{-1}^{n-2|\mathfrak{d}|} \left[ A_1^{d_1} A_2^{d_2} \ldots A_n^{d_n} z^{2g} \right] |A|^{2g+n-2} \frac{1}{S(z)} \prod_{r=1}^{n} S(A_r z) \]

we recognize the expression of \( \langle \langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle_g \) given by Expression (5).

5 Proof of the products of the exponentials formula

The purpose of this section is to prove Proposition 4.6 which ends the proof of the main theorem. To do so, we first use Corollary 3.9 which follows from Eulerian numbers properties. We get
\[
\left[ t_2^{A_2} \cdots t_n^{A_n} \right] = \prod_{i=2}^{n} \left( \prod_{j=1}^{i-1} \exp \left( A_i A_j z^2 \sum_{k>0} k S(k A_i z) S(k A_j z) \left( \frac{t_i}{t_j} \right)^k \right) - 1 \right) 
\]
\[
= \left[ t_2^{A_2} \cdots t_n^{A_n} \right] \prod_{i=2}^{n} \left( \prod_{j=1}^{i-1} \left( 1 + 4 \sum_{k>0} \frac{\text{sh} \left( \frac{A_i z}{2} \right)}{\text{sh} \left( \frac{A_i + A_j}{2} \right)} \frac{\text{sh} \left( \frac{A_j z}{2} \right)}{\text{sh} \left( \frac{A_i + A_j}{2} \right)} \left( k \left( \frac{A_i + A_j}{2} \right) \left( \frac{t_i}{t_j} \right)^k \right) - 1 \right) \right),
\]
where we used the convention \( t_1 = 1 \). Recall that \( A_1, \ldots, A_n \) are positive integers and \( t_2, \ldots, t_n, z \) are formal variables.

**Proposition 5.1.** Fix \((n-1)\) positive integers \( a_2, \ldots, a_n \) and \( n \) formal variables \( \bar{A}_1, \ldots, \bar{A}_n \). Fix \((n-1)\) more formal variables \( t_2, \ldots, t_n \); by convention, let \( t_1 = 1 \). We have
\[
\left[ t_2^{a_2} \cdots t_n^{a_n} \right] \prod_{i=2}^{n} \left( \prod_{j=1}^{i-1} \left( 1 + 4 \sum_{k>0} \frac{\text{sh} \left( \bar{A}_i \right)}{\text{sh} \left( \bar{A}_i + \bar{A}_j \right)} \text{sh} \left( \bar{A}_j \right) \left( k \left( \bar{A}_i + \bar{A}_j \right) \left( \frac{t_i}{t_j} \right)^k \right) - 1 \right) \right) 
\]
\[
= 2^{2(n-1)} \frac{\prod_{i=1}^{n} \text{sh} \left( \bar{A}_i \right)}{\text{sh} \left( \bar{A}_1 + \cdots + \bar{A}_n \right)} \left( \prod_{r=2}^{n} \left( \text{sh} \left( a_r \left( \bar{A}_1 + \cdots + \bar{A}_r \right) + \bar{A}_r \left( a_{r+1} + \cdots + a_n \right) \right) \right) \right). 
\]
Before proving this proposition, let us end the proof of the products of exponentials formula. According
to this proposition by substituting $\tilde{A}_i := \frac{A_i}{2} z$ and $a_i := A_i$, we find

$$\left[ t_{a_2}^{2} \ldots t_{a_n}^{n} \right] \prod_{i=2}^{n} \prod_{j=1}^{i-1} \left( 1 + 4 \sum_{k>0} \frac{\text{sh} \left( \frac{A_i}{2} z \right) \text{sh} \left( \frac{A_j}{2} z \right)}{\text{sh} \left( A_i + A_j \right)} \text{sh} \left( k \frac{A_i + A_j}{2} z \right) \left( \frac{t_i}{t_j} \right)^k \right) - 1 \right)$$

$$= 2^{2(n-1)} \prod_{i=1}^{n} \frac{\text{sh} \left( \frac{A_i}{2} z \right)}{\text{sh} \left( \frac{A_1 + \ldots + A_n}{2} z \right)} \prod_{r=2}^{n} \frac{\text{sh} \left( A_r (A_1 + \ldots + A_n) z \right)}{\text{sh} \left( A_r z \right)}$$

$$= A_1 A_2^2 \ldots A_n^2 \frac{z^{2n-2}}{2} \left( \sum_{i=1}^{n} A_i \right) ^{n-2} \prod_{r=1}^{n} S \left( \frac{A_r (A_1 z + \ldots + A_n z)}{S (A_1 z + \ldots + A_n z)} \right),$$

where we used the notation $S \left( z \right) = \frac{\text{sh} (z/2)}{z/2}$ to obtain this equality. This proves the products of exponentials formula.

**Convention.** In the rest of this section, we make no use of the positive integers $A_1, \ldots, A_n$. However we will intensively use the formal variables $\tilde{A}_1, \ldots, \tilde{A}_n$. For convenience, we change the notation by removing the tildes on these formal variables.

**Proof of Proposition 5.1.** We prove this formula by induction over $n$. The first step $n = 2$ is obvious. Suppose this induction is proved until step $n$. We prove the $(n+1)$-th step. Start from the LHS

$$\left[ t_{a_2}^{2} \ldots t_{a_{n+1}}^{n+1} \right] \prod_{i=2}^{n+1} \prod_{j=1}^{i-1} \left( 1 + 4 \sum_{k>0} \frac{\text{sh} \left( A_i \right) \text{sh} \left( A_j \right)}{\text{sh} \left( A_i + A_j \right)} \text{sh} \left( k \left( A_i + A_j \right) \right) \left( \frac{t_i}{t_j} \right)^k \right) - 1 \right) .$$

We decompose the product in order to use the induction hypothesis; we move the terms corresponding to $i = n+1$ on a second line and get

$$\left[ t_{a_2}^{2} \ldots t_{a_{n+1}}^{n+1} \right] \times \prod_{i=2}^{n} \prod_{j=1}^{i-1} \left( 1 + 4 \sum_{k>0} \frac{\text{sh} \left( A_i \right) \text{sh} \left( A_j \right)}{\text{sh} \left( A_i + A_j \right)} \text{sh} \left( k \left( A_i + A_j \right) \right) \left( \frac{t_i}{t_j} \right)^k \right) - 1 \right) \times \left( \prod_{s=1}^{n+1} 1 + 4 \sum_{t>0} \frac{\text{sh} \left( A_s \right) \text{sh} \left( A_{n+1} \right)}{\text{sh} \left( A_s + A_{n+1} \right)} \text{sh} \left( k \left( A_s + A_{n+1} \right) \right) \left( \frac{t_{n+1}}{t_s} \right)^t \right) - 1 \right).$$

We simplify the series on the second line of the expression using the induction hypothesis. We get

$$\left[ t_{a_2}^{2} \ldots t_{a_{n+1}}^{n+1} \right] \times 2^{2(n-1)} \prod_{i=1}^{n} \frac{\text{sh} \left( A_i \right)}{\text{sh} \left( A_1 + \ldots + A_n \right)} \sum_{i_2, \ldots, i_n > 0} \prod_{r=2}^{n} \text{sh} \left( i_r (A_1 + \ldots + A_r) + A_r (i_{r+1} + \ldots + k_n) \right) t_i^r$$

$$\times \left( \prod_{s=1}^{n+1} 1 + 4 \sum_{k>0} \frac{\text{sh} \left( A_s \right) \text{sh} \left( A_{n+1} \right)}{\text{sh} \left( A_s + A_{n+1} \right)} \text{sh} \left( k \left( A_s + A_{n+1} \right) \right) \left( \frac{t_{n+1}}{t_s} \right)^k \right) - 1 \right) .$$

48
We obtain the result using the following proposition with

\[ u := t_{n+1}, \quad b := a_{n+1}, \quad B := A_{n+1}, \quad \text{and} \quad X_r = 0 \text{ for } 2 \leq r \leq n. \]

**Proposition 5.2 (The sinh formula).** Fix \( n \) positive integers \( a_2, \ldots, a_n, b \) and \( 2n \) formal variables \( A_1, \ldots, A_n, B, X_2, \ldots, X_n \). Fix \( n \) more formal variables \( t_2, \ldots, t_n \), \( u \); by convention, let \( t_1 = 1 \). Then the coefficient of \( t_2^{a_2} \cdots t_n^{a_n} u^b \) in the formal power series

\[
\sum_{i_2, \ldots, i_n > 0} \prod_{r=2}^{n} \left( \prod_{s=1}^{n} \left( 1 + 4 \sum_{j_s > 0} \frac{\text{sh}(A_s) \text{sh}(B)}{\text{sh}(A_s + B)} \text{sh}(j_s(A_s + B)) \left( \frac{u}{t_s} \right)^{j_s} \right) - 1 \right)
\]

is

\[
4 \frac{\text{sh}(A_1 + \cdots + A_n) \text{sh}(B)}{\text{sh}(A_1 + \cdots + A_n + B)} \prod_{r=2}^{n} \left( \text{sh}(a_r(A_1 + \cdots + A_r) + A_r(a_{r+1} + \cdots + a_n + b) + X_r) \right) \text{sh}(b(A_1 + \cdots + A_n + B)).
\]

The purpose of the rest of this section is to prove the sinh formula. The proof goes by induction. We prove the case \( n = 2 \) in Section 5.1. The heredity is proved in Section 5.2.

### 5.1 Proof by induction of the sinh formula: initialization

We prove in this section, the case \( n = 2 \) of the sinh formula. We begin by a series of lemmas.

**Lemma 5.3.** Let \( \alpha, \beta \) and \( \gamma \) be some formal variables. We have

\[
\text{ch} (\alpha) \text{sh} (\beta + \gamma) - \text{ch} (\beta) \text{sh} (\alpha + \gamma) = \text{sh} (\alpha - \beta) \text{sh} (\gamma).
\]

**Proof.** One can check this formula using the basic hyperbolic identities.

**Lemma 5.4.** Let \( \mu \) and \( \nu \) be some formal variables. We have

\[
\sum_{j=0}^{b} \text{sh} (\mu j + \nu) = \frac{\text{sh} (\mu (b + 1) / 2) \text{sh} (\mu b / 2 + \nu)}{\text{sh} (\mu / 2)}
\]

\[
= \frac{\text{ch} (\mu / 2)}{\text{sh} (\mu / 2)} \text{sh} (b \mu / 2) \text{sh} (b \mu / 2 + \nu) + \text{ch} (b \mu / 2) \text{sh} (b \mu / 2 + \nu).
\]

**Proof.** Using the exponential form of the hyperbolic sine and geometric sums, we obtain the first equality. The second equality is obtained by using \( \text{sh} \left( \frac{b \mu}{2} + \frac{\nu}{2} \right) = \text{ch} \left( \frac{\mu}{2} \right) \text{sh} \left( \frac{b \mu}{2} \right) + \text{ch} \left( \frac{b \mu}{2} \right) \text{sh} \left( \frac{\nu}{2} \right) \) and simplifying.

\[ \square \]
Lemma 5.5. Fix two integers $a, b$ and four formal variables $A_1, A_2, B, X$. We have

$$\sum_{j=0}^{b} \text{sh} \left( (a+j) (A_1 + A_2) + X \right) \text{sh} \left( (b-j) (A_1 + B) \right) \text{sh} \left( (j) (A_2 + B) \right)$$

$$= \frac{\text{ch} \left( A_1 \right) \text{sh} \left( b A_1 \right) \text{sh} \left( a (A_1 + A_2) - b B + X \right)}{\text{sh} \left( A_1 \right)}$$

$$+ \frac{\text{ch} \left( A_2 \right) \text{sh} \left( b A_2 \right) \text{sh} \left( a (A_1 + A_2) + b (A_1 + A_2 + B) + X \right)}{\text{sh} \left( A_2 \right)}$$

$$+ \frac{\text{ch} \left( B \right) \text{sh} \left( b B \right) \text{sh} \left( -a (A_1 + A_2) - b A_1 - X \right)}{\text{sh} \left( B \right)}$$

$$+ \frac{\text{ch} \left( A_1 + A_2 + B \right) \text{sh} \left( b (A_1 + A_2 + B) \right) \text{sh} \left( -a (A_1 + A_2) - b A_2 - X \right)}{\text{sh} \left( A_1 + A_2 + B \right)}.$$

Proof. We start from the LHS of this formula. We first linearize the product of the three hyperbolic sines using

$$\text{sh} \left( u \right) \text{sh} \left( v \right) \text{sh} \left( w \right) = \text{sh} \left( u + v + w \right) + \text{sh} \left( u - v - w \right) + \text{sh} \left( -u + v - w \right) + \text{sh} \left( -u - v + w \right)$$  \hspace{1cm} (26)

with

$$u = (a+j) (A_1 + A_2) + X$$

$$v = (b-j) (A_1 + B)$$

$$w = j (A_2 + B).$$

Then, we use the formula of Lemma 5.4

$$\sum_{j=0}^{b} \text{sh} \left( \mu j + \nu \right) = \frac{\text{ch} \left( \mu/2 \right) \text{sh} \left( \mu b/2 + \nu \right)}{\text{sh} \left( \mu/2 \right)} \text{sh} \left( \mu b/2 + \nu \right) + \text{ch} \left( \mu/2 \right) \text{sh} \left( \mu b/2 + \nu \right)$$

to compute the finite sum of each of the four terms in the RHS of Eq. (26). We find
Finally, we prove that the sum of the second terms in the RHS of Equations (27), (28), (29), (30) vanishes. This will end the proof.

We sum the second term of the RHS of Eq. (27) and the second term of the RHS of Eq. (29). Using Lemma 5.3, that is
\[
\text{ch} (\alpha) \text{sh} (\beta + \gamma) - \text{ch} (\beta) \text{sh} (\alpha + \gamma) = \text{sh} (\alpha - \beta) \text{sh} (\gamma)
\]
with \(\alpha = bA_1\), \(\beta = -bB\) and \(\gamma = a (A_1 + A_2) + X\), we find
\[
\text{ch} (bA_1) \text{sh} (a (A_1 + A_2) - bB + X) + \text{ch} (bB) \text{sh} (-a (A_1 + A_2) - bA_1 - X) = \text{sh} (b(A_1 + B)) \text{sh} (a (A_1 + A_2) + X).
\]

We sum the second term of the RHS of Eq. (28) and the second term of the RHS of Eq. (30). Using Lemma 5.3 with \(\alpha = bA_2\), \(\beta = b (A_1 + A_2 + B)\) and \(\gamma = a (A_1 + A_2) + X\), we find
\[
\text{ch} (bA_2) \text{sh} (a (A_1 + A_2) + b (A_1 + A_2 + B) + X) + \text{ch} (b (A_1 + A_2 + B)) \text{sh} (-a (A_1 + A_2) - bA_2 - X) = -\text{sh} (b (A_1 + B)) \text{sh} (a (A_1 + A_2) + X).
\]
Hence,
\[
0 = \text{ch} (bA_1) \text{sh} (a (A_1 + A_2) - bB + X) + \text{ch} (bB) \text{sh} (-a (A_1 + A_2) - bA_1 - X) + \text{ch} (bA_2) \text{sh} (a (A_1 + A_2) + b (A_1 + A_2 + B) + X) + \text{ch} (b (A_1 + A_2 + B)) \text{sh} (-a (A_1 + A_2) - bA_2 - X).
\]
Lemma 5.6. Let $\alpha, \beta$ and $\gamma$ be some formal variables. We have
\[
\text{sh}(\alpha) \text{sh}(\beta) + \text{sh}(\gamma) \text{sh}(\alpha + \beta + \gamma) = \text{sh}(\alpha + \gamma) \text{sh}(\beta + \gamma).
\]

Proof. One can check this formula using the usual hyperbolic identities. \qed

Lemma 5.7. Let $A_1, A_2$ and $B$ be some formal variables. We have
\[
\text{ch}(A_1) \text{sh}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) + \text{sh}(A_1) \text{ch}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) + \text{sh}(A_1) \text{sh}(A_2) \text{ch}(B) \text{sh}(A_1 + A_2 + B) - \text{sh}(A_1) \text{sh}(A_2) \text{sh}(B) \text{ch}(A_1 + A_2 + B) = \text{sh}(A_1 + B) \text{sh}(A_2 + B) \text{sh}(A_1 + A_2).
\]

Proof. We start from the LHS. We use the hyperbolic identity $\text{sh}(\alpha + \beta) = \text{sh}(\alpha) \text{ch}(\beta) + \text{ch}(\alpha) \text{sh}(\beta)$. We sum the two first lines using $\alpha = A_1$ and $\beta = A_2$, we obtain
\[
\text{sh}(A_1 + A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B).
\]
We sum the two last lines using $\alpha = -B$ and $\beta = A_1 + A_2 + B$, we obtain
\[
\text{sh}(A_1) \text{sh}(A_2) \text{sh}(A_1 + A_2).
\]
Finally we sum these two terms. We factor $\text{sh}(A_1 + A_2)$ and use Lemma 5.6 to obtain the expected result. \qed

We now prove the case $n = 2$ of the sinh formula.

Proposition 5.8. Fix two integers $a_2, b$ and four formal variables $A_1, A_2, B, X$. Fix two more formal variables $t_2, u$. Then the coefficient of $t_2^{a_2} u^b$ in the formal power series
\[
\sum_{i > 0} \text{sh}(i (A_1 + A_2) + X) t_2^i \left\{ \left( 1 + 4 \sum_{j_1 > 0} \frac{\text{sh}(A_1) \text{sh}(B)}{\text{sh}(A_1 + B)} \text{sh}(j_0 (A_1 + B)) u^{j_1} \right) \left( 1 + 4 \sum_{j_2 > 0} \frac{\text{sh}(A_2) \text{sh}(B)}{\text{sh}(A_2 + B)} \text{sh}(j_1 (A_2 + B)) u^{j_2} \right) - 1 \right\}
\]
is
\[
4 \frac{\text{sh}(A_1 + A_2) \text{sh}(B)}{\text{sh}(A_1 + A_2 + B)} (\text{sh}(a_2 (A_1 + A_2) + A_2 b + X)) \text{sh}(b (A_1 + A_2 + B)).
\]
Proof. We re-write the power series of the LHS of the proposition as
\[ \Phi \{(1 + \Delta_1)(1 + \Delta_2) - 1\} \]
with
\[
\Phi = \sum_{i \geq 0} \text{sh}(i(A_1 + A_2) + X)t_2
\]
\[
\Delta_1 = 4 \sum_{j_1 \geq 0} \frac{\text{sh}(A_1)\text{sh}(B)}{\text{sh}(A_1 + B)} \text{sh}(j_0(A_1 + B))u^{j_1}
\]
\[
\Delta_2 = 4 \sum_{j_2 \geq 0} \frac{\text{sh}(A_2)\text{sh}(B)}{\text{sh}(A_2 + B)} \text{sh}(j_1(A_2 + B))\left(\frac{u}{t_2}\right)^{j_2}.
\]
We begin by expanding the expression
\[
\Phi \{(1 + \Delta_1)(1 + \Delta_2) - 1\} = \underbrace{\Delta_1 \Phi}_{\text{term (i)}} + \underbrace{\Delta_2 \Phi}_{\text{term (ii)}} + \underbrace{\Phi \Delta_1 \Delta_2}_{\text{term (iii)}}.
\]
Then we extract the coefficient of \( t_2^a u^b \):
\[
\begin{align*}
\text{in term (i)} & \quad = \frac{4}{\text{sh}(A_1 + B)} \text{sh}(b(A_1 + B)) \text{sh}(a_2(A_1 + A_2) + X), \\
\text{in term (ii)} & \quad = \frac{4}{\text{sh}(A_2 + B)} \text{sh}(b(A_2 + B)) \text{sh}((a_2 + b)(A_1 + A_2) + X), \\
\text{in term (iii)} & \quad = 16 \frac{\text{sh}(A_1)\text{sh}(B)\text{sh}(A_2)\text{sh}(B)}{\text{sh}(A_1 + B)\text{sh}(A_2 + B)} \\
& \quad \times \sum_{j=0}^{b} \text{sh}((a_2 + j)(A_1 + A_2) + X)\text{sh}((b - j)(A_1 + B))\text{sh}(j(A_2 + B)).
\end{align*}
\]
Observing that \(4\text{sh}(B)\) appears in terms (i), (ii) and (iii) but also in the result, we factor it out. For reasons that will become clear, we also factor out \(\frac{1}{\text{sh}(A_1 + B)\text{sh}(A_2 + B)\text{sh}(A_1 + A_2 + B)}\). Hence, we re-define our three terms by
\[
\begin{align*}
\text{term (i)} & \quad := \text{sh}(A_1)\text{sh}(A_2 + B)\text{sh}(A_1 + A_2 + B)\text{sh}(b(A_1 + B))\text{sh}(a_2(A_1 + A_2) + X), \\
\text{term (ii)} & \quad := \text{sh}(A_2)\text{sh}(A_1 + B)\text{sh}(A_1 + A_2 + B)\text{sh}(b(A_2 + B))\text{sh}((a_2 + b)(A_1 + A_2) + X), \\
\text{term (iii)} & \quad := 4\text{sh}(A_1)\text{sh}(B)\text{sh}(A_2)\text{sh}(A_1 + A_2 + B) \\
& \quad + \sum_{j=0}^{b} \text{sh}((a_2 + j)(A_1 + A_2) + X)\text{sh}((b - j)(A_1 + B))\text{sh}(j(A_2 + B)).
\end{align*}
\]
In Step 1, we develop terms (i) and (ii) using a basic hyperbolic identity. We also compute the sum of term (iii) using Lemma 5.5. We obtain 8 terms from terms (i), (ii) and (iii). Then in Step 2, we combine 7 of these terms using Lemma 5.6. Finally, in Step 3, we combine all the terms and we use Lemma 5.7 to obtain the result.
Step 1. Term (i) : using the hyperbolic identity \( \text{sh}(A_2 + B) = \text{sh}(A_2) \text{ch}(B) + \text{sh}(B) \text{ch}(A_2) \), we split term (i) in a sum of two terms. We get

\[
\begin{align*}
\text{sh}(A_1) & \text{sh}(A_2) \text{ch}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_1 + B)) \text{sh}(a_2(A_1 + A_2) + X) \quad (\text{ia}) \\
& + \text{sh}(A_1) \text{ch}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_1 + B)) \text{sh}(a_2(A_1 + A_2) + X). \quad (\text{ib})
\end{align*}
\]

Term (ii) : using the same hyperbolic identity for \( \text{sh}(A_1 + B) \), we split term (ii) in a sum of two terms. We get

\[
\begin{align*}
\text{sh}(A_1) & \text{sh}(A_2) \text{ch}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_2 + B)) \text{sh}((a_2 + b)(A_1 + A_2) + X) \quad (\text{iaa}) \\
& + \text{ch}(A_1) \text{sh}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_2 + B)) \text{sh}((a_2 + b)(A_1 + A_2) + X). \quad (\text{iib})
\end{align*}
\]

Term (iii) : we use Lemma 5.5 to compute the sum. This gives four terms

\[
\begin{align*}
\text{ch}(A_1) & \text{sh}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_1)) \text{sh}(a_2(A_1 + A_2) - bB + X) \quad (\text{iiiia}) \\
& + \text{sh}(A_1) \text{ch}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_2)) \text{sh}(a_2(A_1 + A_2) + b(A_1 + A_2 + B) + X) \quad (\text{iiib}) \\
& + \text{sh}(A_1) \text{sh}(A_2) \text{ch}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(B)) \text{sh}(-a_2(A_1 + A_2) - bA_1 - X) \quad (\text{iiic}) \\
& + \text{sh}(A_1) \text{sh}(A_2) \text{sh}(B) \text{ch}(A_1 + A_2 + B) \times \text{sh}(b(A_1 + A_2 + B)) \text{sh}(-a_2(A_1 + A_2) - bA_2 - X). \quad (\text{iiid})
\end{align*}
\]

Step 2. We now combine the terms (iia) until (iiic). We will do so using the formula of Lemma 5.6, that is

\[
\text{sh}(\alpha) \text{sh}(\beta) + \text{sh}(\gamma) \text{sh}(\alpha + \beta + \gamma) = \text{sh}(\alpha + \gamma) \text{sh}(\beta + \gamma).
\]

- We sum terms (iiiia) and (iib). They have the common factor \( \text{ch}(A_1) \text{sh}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \). We get

\[
\begin{align*}
\text{ch}(A_1) & \text{sh}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \\
& \times [\text{sh}(b(A_1)) \text{sh}(a_2(A_1 + A_2) - bB + X) + \text{sh}(b(A_2 + B)) \text{sh}((a_2 + b)(A_1 + A_2) + X)]
\end{align*}
\]

Then simplify the expression appearing inside the brackets using Lemma 5.6 with \( \alpha = bA_1 \), \( \beta = a_2(A_1 + A_2) - bB + X \) and \( \gamma = b(A_2 + B) \). We find

\[
\begin{align*}
\text{ch}(A_1) & \text{sh}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_1 + A_2 + B)) \text{sh}(a_2(A_1 + A_2) + bA_2 + X). \quad (31)
\end{align*}
\]

- We sum terms (iib) and (iiib) using the same computation. They have the common factor \( \text{sh}(A_1) \text{ch}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \). We get

\[
\begin{align*}
\text{sh}(A_1) & \text{ch}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \\
& \times [\text{sh}(b(A_1 + B)) \text{sh}(a_2(A_1 + A_2) + X) + \text{sh}(bA_2) \text{sh}(a_2(A_1 + A_2) + b(A_1 + A_2 + B) + X)]
\end{align*}
\]

Then simplify the expression appearing inside the brackets using Lemma 5.6 with \( \alpha = b(A_1 + B) \), \( \beta = a_2(A_1 + A_2) + X \), \( \gamma = bA_2 \). We find

\[
\begin{align*}
\text{sh}(A_1) & \text{ch}(A_2) \text{sh}(B) \text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_1 + A_2 + B)) \text{sh}(a_2(A_1 + A_2) + bA_2 + X). \quad (32)
\end{align*}
\]
• We sum the terms \((ia)\) and \((iia)\) and \((iii)\). They have the common factor \(\text{sh}(A_1)\) \(\text{sh}(A_2)\) \(\text{ch}(B)\) \(\text{sh}(A_1 + A_2 + B)\). First re-write \((ia) + (iii)\) as
\[
\text{sh}(A_1)\text{sh}(A_2)\text{ch}(B)\text{sh}(A_1 + A_2 + B) \\
\times [\text{sh}(b(A_1 + B))\text{sh}(a_2(A_1 + A_2) + X) + \text{sh}(bB)\text{sh}(-a_2(A_1 + A_2) - bA_1 - X)].
\]

We then apply Lemma 5.6 with \(\alpha = b(A_1 + B), \beta = a_2(A_1 + A_2) + X, \gamma = -bB\). We get
\[
\text{sh}(A_1)\text{sh}(A_2)\text{ch}(B)\text{sh}(A_1 + A_2 + B) \times \text{sh}(bA_1)\text{sh}(a_2(A_1 + A_2) - bB + X).
\]

Then, we add the term \((iia)\) to this expression, we get
\[
\text{sh}(A_1)\text{sh}(A_2)\text{ch}(B)\text{sh}(A_1 + A_2 + B) \\
\times [\text{sh}(bA_1)\text{sh}(a_2(A_1 + A_2) - bB + X) + \text{sh}(b(A_2 + B))\text{sh}((a_2 + b)(A_1 + A_2) + X)].
\]

Finally, we use Lemma 5.6 with \(\alpha = bA_1, \beta = a_2(A_1 + A_2) - bB + X, \gamma = b(A_2 + B)\). We find
\[
\text{sh}(A_1)\text{sh}(A_2)\text{ch}(B)\text{sh}(A_1 + A_2 + B) \times \text{sh}(b(A_1 + A_2 + B))\text{sh}(a_2(A_1 + A_2) + bA_2 + X).
\]

Step 3. We sum the three terms (31), (32) and (33) obtained in Step 2 with the remaining term of Step 1, that is term \((iiod)\). These four terms have the common factor \(\text{sh}(b(A_1 + A_2 + B))\) \(\text{sh}(a_2(A_1 + A_2) + bA_2 + X)\). We factor it. The sum of the four remaining terms is the sum of the LHS of Lemma 5.7. Using this Lemma, we get
\[
\text{sh}(A_1 + B)\text{sh}(A_2 + B)\text{sh}(A_1 + A_2) \times \text{sh}(b(A_1 + A_2 + B))\text{sh}(a_2(A_1 + A_2) + bA_2 + X).
\]

Before re-defining terms \((i), (ii)\) and \((iii)\) we factored out \(\frac{4\text{sh}(B)}{\text{sh}(A_1 + B)\text{sh}(A_2 + B)\text{sh}(A_1 + A_2 + B)}\). Multiplying this factor with the expression we just obtained, we get the result. \(\Box\)

5.2 Proof by induction of the sinh formula: heredity

Let us recall the sinh formula before proving it.

Fix \(n\) positive integers \(a_2, \ldots, a_n, b\) and \(2n\) formal variables \(A_1, \ldots, A_n, B, X_2, \ldots, X_n\). Fix \(n\) more formal variables \(t_2, \ldots, t_n, u\); by convention, let \(t_1 = 1\). The coefficient of \(t_2^{a_2} \cdots t_n^{a_n} u^b\) in the formal power series
\[
\sum_{i_2, \ldots, i_n, r > 0} \prod_{r=2}^{n} \text{sh}(i_r(A_1 + \cdots + A_r) + A_r(i_{r+1} + \cdots + i_n) + X_r) t_r^{i_r}
\times \left\{ \prod_{s=1}^{n} \left( 1 + 4 \sum_{j_s > 0} \frac{(A_s)\text{sh}(B)}{\text{sh}(A_s + B)}\text{sh}(j_s(A_s + B))\left( \frac{u}{t_s} \right)^{j_s} \right) - 1 \right\}
\]
is
\[
\frac{4\text{sh}(A_1 + \cdots + A_n)\text{sh}(B)}{\text{sh}(A_1 + \cdots + A_n + B)} \prod_{r=2}^{n} \left( \text{sh}(a_r(A_1 + \cdots + A_r) + A_r(a_{r+1} + \cdots + a_n + b) + X_r) \right) \text{sh}(b(A_1 + \cdots + A_n + B)).
\]
Proof. We prove this formula by induction over \( n \). The first step, for \( n = 2 \), is proved by Proposition 5.8 in the preceding section.

We now prove the \( n \)th step by induction. We can schematically write the formula of the LHS of the proposition as

\[
\Phi \{ \Psi (1 + \Sigma) - 1 \},
\]

with

\[
\Phi = \sum_{i_2, \ldots, i_n > 0} \prod_{r=2}^{n} \text{sh} (i_r (A_1 + \cdots + A_r) + A_r (i_{r+1} + \cdots + i_n) + X_r) t_r^{i_r},
\]

\[
\Psi = \prod_{s=1}^{n-1} \left( 1 + 4 \sum_{j_s > 0} \frac{\text{sh} (A_s) \text{sh} (B)}{\text{sh} (A_s + B)} \text{sh} (j_s (A_s + B)) \left( \frac{u}{t_s} \right)^{j_s} \right),
\]

\[
\Sigma = 4 \sum_{j_n > 0} \frac{\text{sh} (A_n) \text{sh} (B)}{\text{sh} (A_n + B)} \text{sh} (j_n (A_n + B)) \left( \frac{u}{t_n} \right)^{j_n}.
\]

We split the expression in three terms:

\[
\Psi (1 + \Sigma) - 1 = \sum_{\text{term 1}} + (\Psi - 1) + \sum (\Psi - 1).
\]

We now extract the coefficient \( t_2^{a_2} \cdots t_n^{a_n} u_b \) in the three terms coming from this development, that is from \( \Phi \Psi \), \( \Phi (\Psi - 1) \) and \( \Phi \Sigma (\Psi - 1) \). In the second and third term, we will need the \( (n - 1) \)th step of the induction to extract this coefficient. Then, we will sum these three coefficients, see this sum as the coefficient of a series and use the first step of the induction to conclude.

**Term 1.** We extract the coefficient of \( t_2^{a_2} \cdots t_n^{a_n} u_b \) in \( \Phi \Sigma \). To do this, we remove the summations and substitute \( i_2 := a_2, \ldots, i_{n-1} = a_{n-1}, i_n = a_n + b, j_n = b \), we get

\[
\prod_{r=2}^{n} \text{sh} (a_r (A_1 + \cdots + A_r) + A_r (a_{r+1} + \cdots + a_n + b) + X_r)
\times 4 \frac{\text{sh} (A_n) \text{sh} (B)}{\text{sh} (A_n + B)} \text{sh} (b (A_n + B)).
\]

For reasons that will become clear later, we move the factor \( r = n \) of the product to the second line:

\[
\prod_{r=2}^{n-1} \text{sh} (a_r (A_1 + \cdots + A_r) + A_r (a_{r+1} + \cdots + a_n + b) + X_r)
\times \text{sh} ((a_n + b) (A_2 + \cdots + A_n) + X_n) \times 4 \frac{\text{sh} (A_n) \text{sh} (B)}{\text{sh} (A_n + B)} \text{sh} (b (A_n + B)).
\]
**Term 2.** We want to extract the coefficient of $t_2^{a_2} \cdots t_n^{a_n} u^b$ in $\Phi (\Psi - 1)$. First we extract the coefficient of $t_n^{a_n}$. We get

$$
\sum_{i_2, \ldots, i_{n-1} > 0} \prod_{r=2}^{n-1} \text{sh} (i_r (A_1 + \cdots + A_r) + A_r (i_{r+1} + \cdots + i_{n-1}) + A_r a_n + X_r) t_r^{i_r} \times \text{sh} (a_n (A_1 + \cdots + A_n) + X_n)
$$

and we re-arrange the product as

$$
\text{sh} (a_n (A_1 + \cdots + A_n) + X_n) \times \left[ \prod_{i_2, \ldots, i_{n-1} > 0} \prod_{r=2}^{n-1} \text{sh} (i_r (A_1 + \cdots + A_r) + A_r (i_{r+1} + \cdots + i_{n-1}) + A_r a_n + X_r) t_r^{i_r} \times \left\{ 1 + 4 \sum_{s>0} \frac{\text{sh} (A_s) \text{sh} (B)}{\text{sh} (A_s + B)} \text{sh} (k_s (A_s + B)) \left( \frac{u}{t_s} \right)^{k_s} \right\} - 1 \right].
$$

We then have to extract the coefficient of $t_2^{a_2} \cdots t_{n-1}^{a_{n-1}} u^b$ of the term in squared the bracket. Using the recursion hypothesis on this term with

$$A_1 := A_1, \ldots, A_{n-1} := A_{n-1}, B := B, X_r := X_r + A_r a_n$$

we get

$$
\text{sh} (a_n (A_1 + \cdots + A_n) + X_n) \times \left[ 4 \frac{\text{sh} (A_1 + \cdots + A_{n-1}) \text{sh} (B)}{\text{sh} (A_1 + \cdots + A_{n-1} + B)} \prod_{r=2}^{n-1} \text{sh} (a_r (A_1 + \cdots + A_r) + A_r (a_{r+1} + \cdots + a_n + b) + X_r) \text{sh} (b (A_1 + \cdots + A_{n-1} + B)) \right].
$$

Finally, we re-arrange the product as

$$
\prod_{r=2}^{n-1} \text{sh} (a_r (A_1 + \cdots + A_r) + A_r (a_{r+1} + \cdots + a_n + b) + X_r)
$$

$$
\times \left[ 4 \frac{\text{sh} (A_1 + \cdots + A_{n-1}) \text{sh} (B)}{\text{sh} (A_1 + \cdots + A_{n-1} + B)} \text{sh} (a_n (A_1 + \cdots + A_n) + X_n) \text{sh} (b (A_1 + \cdots + A_{n-1} + B)) \right].
$$

**Term 3.** We want to extract the coefficient of $t_2^{a_2} \cdots t_n^{a_n} u^b$ in $\Phi \Sigma (\Psi - 1)$. Here we start by re-arranging the product as follows :

$$
\left[ 4 \sum_{j_n>0} \frac{\text{sh} (A_n) \text{sh} (B)}{\text{sh} (A_n + B)} \text{sh} (j_n (A_n + B)) \left( \frac{u}{t_n} \right)^{j_n} \left( \sum_{i_n>0} \text{sh} (i_n (A_1 + \cdots + A_n) + X_n) t_n^{i_n} \right) \right]
$$

$$
\left[ \left( \sum_{i_2, \ldots, i_{n-1} > 0} \prod_{r=2}^{n-1} \text{sh} (i_r (A_1 + \cdots + A_r) + A_r (i_{r+1} + \cdots + i_{n-1}) + A_r i_n + X_r) t_r^{i_r} \right) \right]
$$

$$
\left\{ \prod_{s=1}^{n-1} \left( 1 + 4 \sum_{k_s>0} \frac{\text{sh} (A_s) \text{sh} (B)}{\text{sh} (A_s + B)} \text{sh} (k_s (A_s + B)) \left( \frac{u}{t_s} \right)^{k_s} \right) - 1 \right\}.
$$
Now we extract the coefficient of \( t_n^a u^b \). Note that \( t_n \) is only present in the first line of the previous expression. In \( \Sigma, 1/t_n \) appears with the same exponent as \( u \), that is \( j_n \). We then extract the coefficient of \( t_2^{a_2} \cdots t_{n-1}^{a_{n-1}} u^{b-j_n} \) from the expression in the square brackets. This is done using the recursion hypothesis with

\[
A_1 := A_1, \ldots, A_{n-1} := A_{n-1}, B := B, X_r := X_r + A_r (a_n - j_n),
\]

we get

\[
\sum_{j_n=0}^b \left( \frac{4 \text{sh} (A_n) \text{sh} (B)}{\text{sh} (A_n + B)} \text{sh} (j_n (A_n + B)) \right) \left( \sum_{i_n > 0} \text{sh} ((a_n + j_n) (A_1 + \cdots + A_n) + X_n) \right)
\]

\[
\left[ \frac{4 \text{sh} (A_1 + \cdots + A_{n-1}) \text{sh} (B)}{\text{sh} (A_1 + \cdots + A_{n-1} + B)} \right]
\]

\[
\prod_{r=2}^{n-1} \left( \text{sh} (a_r (A_1 + \cdots + A_r) + A_r (a_{r+1} + \cdots + a_n + b) + X_r) \right) \text{sh} ((b - j_n) (A_1 + \cdots + A_{n-1} + B))
\]

Again, we re-arrange the product

\[
\prod_{r=2}^{n-1} \text{sh} (a_r (A_1 + \cdots + A_r) + A_r (a_{r+1} + \cdots + a_n + b) + X_r)
\]

\[
\left( \frac{16 \text{sh} (A_1 + \cdots + A_{n-1}) \text{sh} (B) \text{sh} (A_n) \text{sh} (B)}{\text{sh} (A_1 + \cdots + A_{n-1} + B) \text{sh} (A_n + B)} \right)
\]

\[
\sum_{j_n=0}^b \text{sh} (j_n (A_n + B)) \text{sh} ((a_n + j_n) (A_1 + \cdots + A_n) + X_n) \text{sh} ((b - j_n) (A_1 + \cdots + A_{n-1} + B))
\].
We now combine terms 1, 2 and 3. We factor out \( \prod_{r=2}^{n-1} \text{sh} \left( a_r (A_1 + \cdots + A_r) + A_r (a_{r+1} + \cdots + a_n + b) + X_r \right) \) in these three terms. We present the rest as the coefficient of a formal series in \( x \) and \( y \) as follows

\[
\left( \text{sh} \left( (a_n + b) (A_2 + \cdots + A_n) + X_n \right) \times 4 \frac{\text{sh} \left( A_n \right) \text{sh} \left( B \right)}{\text{sh} \left( A_n + B \right)} \right.
\]
\[
+ \left( 4 \frac{\text{sh} \left( A_1 + \cdots + A_{n-1} \right) \text{sh} \left( B \right)}{\text{sh} \left( A_1 + \cdots + A_{n-1} + B \right)} \text{sh} \left( a_n (A_1 + \cdots + A_n) + X_n \right) \text{sh} \left( b (A_1 + \cdots + A_{n-1} + B) \right) \right)
\]
\[
+ \left( 16 \frac{\text{sh} \left( A_1 + \cdots + A_{n-1} \right) \text{sh} \left( B \right) \text{sh} \left( A_n \right) \text{sh} \left( B \right)}{\text{sh} \left( A_1 + \cdots + A_{n-1} + B \right) \text{sh} \left( A_n + B \right)} \right)
\sum_{j_n=0}^{b} \text{sh} \left( j_n (A_n + B) \right) \text{sh} \left( (a_n + j_n) (A_1 + \cdots + A_n) + X_n \right) \text{sh} \left( (b - j_n) (A_1 + \cdots + A_{n-1} + B) \right)
\]
\[
= \left[ x^a y^b \right] \sum_{i>0} \text{sh} \left( i (A_1 + \cdots + A_n) + X_n \right) x^i
\]
\[
\left\{ \begin{array}{l}
1 + 4 \sum_{j>0} \frac{\text{sh} \left( A_1 + \cdots + A_{n-1} \right) \text{sh} \left( B \right)}{\text{sh} \left( A_1 + \cdots + A_{n-1} + B \right)} \text{sh} \left( j (A_1 + \cdots + A_{n-1} + B) \right) y^j
\end{array}
\right.
\]
\[
\left( 1 + 4 \sum_{k>0} \frac{\text{sh} \left( A_n \right) \text{sh} \left( B \right)}{\text{sh} \left( A_n + B \right)} \text{sh} \left( k (A_n + B) \right) \left( \frac{y}{x} \right)^k \right)
\]
6.1 String equation

We can rewrite the string equation (Theorem 2) as the following infinite system of equations

\[(\tau_0 \tau_{d_1} \cdots \tau_{d_n})_{l,g-l} = \sum_{i=1}^{n} (\tau_{d_1} \cdots \tau_{d_{i-1}} \tau_{d_{i+1}} \cdots \tau_{d_n})_{l,g-l},\]

where \(g,l,d_1,\ldots,d_n \geq 0\) and such that a correlator vanishes if \(\tau\) has a negative index. In these equations, the quantity defined by the sum of the indices of \(\tau\) minus the number of \(\tau\) insertions does not depend on the correlator and is equal to \(\sum_{i=1}^{n} d_i - n - 1\). Moreover, the correlators of the LHS and RHS depend on the same indices \(l\) and \(g-l\). We use this system of equations to express any correlator \(\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{l,g-l}\) as a sum of correlators with a \(\tau_0\) insertion. Our two remarks are still valid: \(\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{l,g-l}\) is expressed as a sum of correlators with a \(\tau_0\) insertion such that each correlator is indexed by \(l\) and \(g-l\) and the quantity defined by the sum of the indices of \(\tau\) minus the number of \(\tau\) insertions does not depend on the correlator and is equal to \(\sum_{i=1}^{n} d_i - n - 1\) (see Section 4.1.1 to solve explicitly this system using generating series). It is then sufficient to prove that the correlators \(\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle\) vanish if \(\sum d_i > 4g - 2 + n - l\) or if \(\sum d_i\) has the parity of \(n-l+1\) in order to prove Proposition 1.38.

6.2 Properties of the Ehrhart polynomials of Buryak and Rossi

The vanishing of the correlators come from the properties the following Ehrhart polynomials.

**Lemma 6.1** ([BR16]). Fix a list of \(q\) positive integers \((r_1,\ldots,r_q)\). The function

\[C_{r_1,\ldots,r_q}(N) = \sum_{k_1+\cdots+k_q=N} k_1^{r_1} \cdots k_q^{r_q}\]

is a polynomial in \(N\) of degree \(q-1 + \sum r_i\). Moreover, this polynomial has the parity of \(q-1 + \sum r_i\).

We then deduce the following lemma.

**Lemma 6.2.** Let \(P(k_1,\ldots,k_q) \in \mathbb{C}[k_1,\ldots,k_n]\) be an even (resp. odd) polynomial. Then

\[\sum_{k_1+\cdots+k_q=N} k_1 \cdots k_q P(k_1,\ldots,k_q)\]

is an odd (resp. even) polynomial in the indeterminate \(N\) of degree \(2q-1 + \text{deg}P\).

By induction, we obtain the following lemma.

**Lemma 6.3.** Fix an integer \(n \geq 2\) and a list \(A_2,\ldots,A_n\) of nonnegative integers. Let \(C\) be the set of pairs (2-element subsets) of \(\{1,\ldots,n\}\). Fix another list of nonnegative integers \((q_I, I \in C)\). Let \(P(k_i^I, I \in C, 1 \leq i \leq q_I)\) be an even
(resp. odd) polynomial in the indeterminates $k_i^l$, where $I \in \mathcal{C}$ and $1 \leq i \leq q_I$. Then

$$
\sum_{i=1}^{n-1} \prod_{I \in \mathcal{C}_n} k_i^l \cdots k_{q_I}^l \\
\times \sum_{i=1}^{n-2} \prod_{I \in \mathcal{C}_{n-1}} k_i^l \cdots k_{q_{I}}^l \\
\times \cdots \\
\times k_{1(2)} \cdots k_{q(1,2)} \\
\times P(k_i^l, I \in \mathcal{C}, 1 \leq i \leq q_I),
$$

is a polynomial in the indeterminates $A_2, \ldots, A_n$ with the parity of $\deg P - (n - 1)$ (resp. $\deg P - n$) of degree $2\sum_{l \in \mathcal{C}} q_l - (n - 1) + \deg P$. We used the notation $K^l = \sum_{i=1}^{q_I} k_i^l$.

### 6.3 Proof of the level structure

We now prove Proposition 1.38. We first obtain an expression of the correlators $\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-l}$ in term of the polynomials of Lemma 6.3. The level structure follows from the vanishing properties of these polynomials.

#### Proposition 6.4

Fix three nonnegative integers $n, g, l$ and a list $(d_1, \ldots, d_n)$ of nonnegative integers. We have

$$
\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-l} = \sqrt{(-1)^{n-2-3l-\sum d_i}} \\
\sum_{g_1 + \cdots + g_n + \sum_{l \in \mathcal{C}} q_l = g + n - 1} \sum_{l_1 + \cdots + l_n = l} \left[ a_1^1 \cdots a_{m_1}^1 \cdots a_1^n \cdots a_{m_n}^n \right] \\
\times \prod_{I \in \mathcal{C}} \frac{1}{q_I!} \prod_{k_i^l > 0} k_i^l \\
\times \frac{1}{m_1!} P_{l_1,1,g_1, d_1} \left( a_1^1, \ldots, a_{m_1}^1, k_1^{(1,2)}, \ldots, k_{q(1,2)}^{(1,2)}, \ldots, k_1^{(1,n)}, \ldots, k_{q(1,n)}^{(1,n)}, -\sum A_i \right) \\
\times \frac{1}{m_i!} P_{l_i, g_i, l_i} \left( a_1^i, \ldots, a_{m_i}^i, -k_1^{(1,i)}, \ldots, -k_{q(1,i)}^{(1,i)}, \ldots, k_1^{(i,n)}, \ldots, k_{q(i,n)}^{(i,n)}, 0 \right),
$$

where

- $\mathcal{C}$ is the set of pairs (2-element subsets) of $\{1, \ldots, n\}$; we also denote by $\mathcal{C}_i \subset \mathcal{C}$ the subset of pairs that contain $i$,
- the conditions $\alpha$ are

$$
g_1 + (g_1 - l_1) + m_1 + \sum_{l \in \mathcal{C}_1} q_l = d_1 + 1
$$

and

$$
g_i + (g_i - l_i) + m_i + \sum_{l \in \mathcal{C}_i} q_l = d_i + 2, \text{ when } i \geq 2,
$$

61
Similarly, it remains where the first summation satisfies plug the expression of the Hamiltonian given by Eq (35) in Eq. (20). When the position 4.4: we use the associativity of the star product and its developed expression to obtain Eq. (20). We then

\[ A_i - \sum_{j=1}^{i-1} K^{(j,i)} + \sum_{j=i+1}^{n} K^{(i,j)} = 0, \quad 2 \leq i \leq n, \]

where \( K^{I} = \sum_{i=1}^{q} k_{i}^{I} \),

- the conditions \( \gamma \) are

\[ \sum_{j=1}^{n} q_{(i,j)} \geq 1, \text{ for } 2 \leq i \leq n, \]

- the polynomial \( P_{d,g,l}(x_1, \ldots, x_{m+1}) \), with \( d, g, l, m \geq 0 \), is of degree \( 2g \) and defined by

\[ P_{d,g,l}(x_1, \ldots, x_{m+1}) = \int_{\text{DR}_g(0,x_1,\ldots,x_{m+1})} \lambda_{l} \psi_{1}^{d+1}, \]

where \( \sum_{i=1}^{m+1} x_i = 0 \) and the \( \psi \)-classes sit on the marked point with weight 0 of the double ramification cycle.

**Proof.** To obtain this formula we use the definition of the correlators

\[ \langle \tau_{0} \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-l} = \left[ e^{2\theta \hat{H} - l} \right] \frac{\sqrt{-1}}{p^{n-1}} \left[ \cdots [H_{d_{1-1}}, \hat{H}_{d_2}], \ldots, \hat{H}_{d_n} \right] \bigg|_{u_i = \delta_{i1}}, \]

and proceed as in Section 4.2.2.

We first need an expression of \( H_{d_1} \star \hat{H}_{d_2} \cdots \star \hat{H}_{d_n} \). Start from the following expression of the Hamiltonians

\[ H_{d_1} = \sum_{g, m, l_i \geq 0} \frac{\epsilon^{2l_1} \left( \sqrt{-1} \hat{H} \right)^{g_1-l_1}}{m_1!} \sum_{a_1^1, \ldots, a_{m_1}^1} P_{d_1,g_1,l_1} \left( a_1^1, \ldots, a_{m_1}^1, -\sum_{j=1}^{m_1} a_j^1 \right) p_{a_1}^1 \cdots p_{a_{m_1}}^1 e^{\sqrt{-1} \sum_{j=1}^{m_1} a_j^1}, \quad (35) \]

where the first summation satisfies \( g_1 + (g_1 - l_1) + m_i = d_1 + 2 \) and \( l_i \leq g_1 \). We proceed as in the proof of Proposition 4.4: we use the associativity of the star product and its developed expression to obtain Eq. (20). We then plug the expression of the Hamiltonian given by Eq (35) in Eq. (20). When the \( \sum_{j=1}^{n} q_{i,j} \) derivatives act on \( \sum_{a_1^1 + \cdots + a_{m_1}^1} = P_{d_1,g_1,l_1} \left( a_1^1, \ldots, a_{m_1}^1, 0 \right) p_{a_1}^1 \cdots p_{a_{m_1}}^1 \) in \( \hat{H}_{d_1} \), it remains \( \tilde{m}_i = m_i - \sum_{j=1,j \neq i}^{n} q_{i,j} \) variables \( p \). Similarly, it remains \( \tilde{m}_1 := m_1 - \sum_{j=2}^{n} q_{1,j} \) variables \( p \) when the derivatives act on \( H_{d_{1-1}} \). We then obtain

\[ H_{d_1} \star \hat{H}_{d_2} \cdots \star \hat{H}_{d_n} = \sum_{g_1 \geq 0, l \in C} \sum_{g_1, \ldots, g_n \geq 0} \sum_{m_1, \ldots, m_n \geq 0} \sum_{l_i, \ldots, l_i \geq 0} \sum_{k_i^1 \geq 0, l \in C, 1 \leq i \leq q_1} \prod_{l \in C} \frac{\left( \sqrt{-1} \hat{H} \right)^{q_i}}{q_i!} k_1^l \cdots k_q^l \]

\[ \times \frac{\epsilon^{2l_1} \left( \sqrt{-1} \hat{H} \right)^{g_1-l_1}}{\tilde{m}_1!} P_{d_1-1,g_1,l_1} \left( a_1^1, \ldots, a_{\tilde{m}_1}^1, k_1^{1,2}, \ldots, k_{q_1,1}^{1,2}, \ldots, k_1^{1,n}, \ldots, k_{q_1,n}^{1,n}, -\sum_{i=1}^{n} \tilde{A}_i \right) p_{a_1}^1 \cdots p_{a_{\tilde{m}_1}}^1 e^{\sqrt{-1} \sum_{i=1}^{n} \tilde{A}_i}, \quad (36) \]

\[ \times \frac{\epsilon^{2l_i} \left( \sqrt{-1} \hat{H} \right)^{g_i-l_i}}{\tilde{m}_i!} P_{d_i,g_i,l_i} \left( a_1^i, \ldots, a_{\tilde{m}_i}^i, -k_1^{i,1}, \ldots, -k_{q_1,1}^{i,1}, \ldots, k_1^{i,n}, \ldots, k_{q_1,n}^{i,n}, 0 \right) p_{a_1}^i \cdots p_{a_{\tilde{m}_i}}^i, \]

where
\[ A_i = \sum_{j=1}^{\tilde{m}_i} a_j^i, \text{ with } 1 \leq i \leq n, \]

- the conditions \( \alpha \) are
  \[ g_1 + (g_1 - l_i) + \tilde{m}_i + \sum_{l \in \mathcal{C}_i} q_l = d_1 + 1 \]
  and
  \[ g_i + (g_i - l_i) + \tilde{m}_i + \sum_{l \in \mathcal{C}_i} q_l = d_i + 2, \text{ when } i \geq 2, \]

- the conditions \( \beta \) are
  \[ A_i - \sum_{j=1}^{i-1} K^{(j,i)} + \sum_{j=i+1}^{n} K^{(i,j)} = 0, \text{ } 2 \leq i \leq n. \]

We now modify the notations by removing the tildes, i.e. we set \( m_i := \tilde{m}_i \) and \( A_i := \tilde{A}_i \) for any \( 1 \leq i \leq n. \)

In Step 1.2 of Section 4.2.2 we explained why such an expression of \( H_{d_i} \star \mathcal{P}_{d_2} \star \cdots \star \mathcal{P}_{d_n} \) is enough to get an expression for \( \langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{l,g-l} \). We also explained that the commutators offer some simplifications given by the conditions \( \gamma \).

We now proceed as in Step 2 of Section 4.2.2: we extract the coefficient of \( e^{2l}h^{g-l} \) in the expression of \( \sqrt{\frac{-1}{\hbar}} H_{d_1} \star \mathcal{P}_{d_2} \star \cdots \star \mathcal{P}_{d_n} \) given by Eq. (36), then we use the Lemma 2.1 to substitute \( u_i = \delta_{i,1} \) in this coefficient. We get Eq. (34).

We now prove that the correlator \( \langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle_{l,g-l} \) vanishes when \( \sum d_i > 4g - 2 + n - l \) or when \( \sum d_i \) has the parity of \( n + 1 - l \). According to Section 6.1, this proves Proposition 1.38.

**Proof of Proposition 1.38.** The three last lines of Eq. (34) form a polynomial the indeterminates \( a_1^1, \ldots, a_{m_1}^1, \ldots, a_1^n, \ldots, a_{m_n}^n \) depending of the set of parameters \( S = \{ d_i, g_i, l_i, m_i \mid 1 \leq i \leq n \} \), we denote this polynomial by \( Q_S \). The parity and the degree of this polynomial is described by Lemma (6.3). Since the polynomial \( P_{d,g,l} \) is even and of degree \( 2g \), we deduce that \( Q_S \) is a polynomial of degree

\[
2 \sum_{l \in \mathcal{C}} q_l - (n - 1) + \sum_{i=1}^{n} 2q_i = 2g + n - 1
\]

and has the parity of \( n - 1 \). We used the constraint \( \sum_{l \in \mathcal{C}} q_l + \sum_{i=1}^{n} g_i = g + n - 1 \) of the first summation in formula (34) to obtain this equality.

We then extract the coefficient of \( a_1^1 \cdots a_{m_1}^1 \cdots a_1^n \cdots a_{m_n}^n \) in \( Q_S \). This coefficient vanishes if

\[
\sum_{i=1}^{n} m_i > 2g + n - 1
\]

or if \( \sum_{i=1}^{n} m_i \) has the parity of \( n \). However conditions \( \alpha \) give \( \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} d_i - 2g + l + 1 \). Hence, this coefficient vanishes if

\[
\sum_{i=1}^{n} d_i > 4g - 2 + n - l
\]

or if \( \sum d_i \) has the parity of \( n - l + 1 \). This proves the proposition. \( \square \)
A Proofs of the quantum integrability and the tau symmetry

The Hamiltonian density $H_d$ is the same than the one given in [BDGR19], although it is introduced differently. Let us verify that the two definitions lead to the same object. In particular, this will prove Propositions 1.19 and 1.21 which are proved in [BDGR19] with their definition of $H_d$.

In [BDGR19], the authors defined

$$H_d^{BDGR} := \sum_{s \geq 0} (-\partial_z)^s \frac{\partial G_{d+1}}{\partial u_s},$$

where

$$G_d := \sum_{g \geq 0, n \geq 0} \frac{(ih)^g}{n!} \sum_{a_1, \ldots, a_n \in \mathbb{Z}} \left( \int_{\text{DR}_g} (-\sum a_i a_{i+1} a_n) \psi_1^d \Lambda \left( -\frac{\epsilon^2}{4} \right) \right) p_{a_1} \cdots p_{a_n} e^{ix \sum a_i}.$$

Lemma A.1. Let $\phi \in \mathcal{A}$. We have

$$\sum_{s \geq 0} (-\partial_z)^s \frac{\partial \phi}{\partial u_s} = \sum_{b \in \mathbb{Z}} e^{-ibx} \frac{\partial \phi}{\partial p_b}. \quad (37)$$

Proof. Write $\phi$ as

$$\phi(x) = \sum_{k=0}^d \sum_{a_1, \ldots, a_k \in \mathbb{Z}} \phi_k(a_1, \ldots, a_k) p_{a_1} \cdots p_{a_k} e^{ix \sum a_i}$$

where $\phi_k(a_1, \ldots, a_k) \in \mathbb{C}[a_1, \ldots, a_k][[\epsilon, \hbar]]$ is a symmetric polynomial in its $k$ indeterminates $a_1, \ldots, a_k$ for $0 \leq k \leq d$.

We start from the RHS of Eq. (37). Using this expression of $\phi$, we get

$$\sum_{b \in \mathbb{Z}} e^{-ibx} \frac{\partial \phi}{\partial p_b} = \sum_{k \geq 1} k \left( \sum_{a_1, \ldots, a_{k-1}, b \in \mathbb{Z}} \phi_k(a_1, \ldots, a_{k-1}, b) p_{a_1} \cdots p_{a_{k-1}} e^{-ixb} \sum_{i=1}^{k-1} a_i \right).$$

We will obtain this same expression from the LHS. Note that we can rewrite $\phi$ as

$$\phi(x) = \sum_{k=0}^d \sum_{a_1, \ldots, a_k \geq 0} (-i)^k \sum_{s_1, \ldots, s_k \geq 0} u_{s_1} \cdots u_{s_k} \left( \sum_{a_1, \ldots, a_k} [a_1^{s_1} \cdots a_k^{s_k}] \phi_k(a_1, \ldots, a_k) \right).$$

Hence we find

$$\frac{\partial \phi}{\partial u_s} = \sum_{k=1}^d \sum_{j=1}^k (-i)^j [a_j^{s}] \phi_k(a_1, \ldots, a_k) p_{a_1} \cdots p_{a_{j-1}} e^{ix \sum_{i=1}^{j-1} a_i} = \sum_{k=1}^d \sum_{a_1, \ldots, a_{k-1} \in \mathbb{Z}} (-i)^k [a_k^{s}] \phi_k(a_1, \ldots, a_k) p_{a_1} \cdots p_{a_{k-1}} e^{ix \sum_{i=1}^{k-1} a_i}. $$

64
Then we get
\[
\sum_{s \geq 0} (\partial_{x})^{s} \frac{\partial \phi}{\partial u_{s}} = \sum_{d} \sum_{a_{1}, \ldots, a_{k-1} \in \mathbb{Z} \geq 0} \left( - \sum_{i=1}^{k-1} a_{i} \right)^{s} [a_{k}^{s}] \phi_{k} (a_{1}, \ldots, a_{k}) p_{a_{1}} \cdots p_{a_{k-1}} e^{ix \sum_{i=1}^{k-1} a_{i}}.
\]
Moreover we have \(\sum_{s \geq 0} \left( - \sum_{i=1}^{k-1} a_{i} \right)^{s} [a_{k}^{s}] \phi_{k} (a_{1}, \ldots, a_{k}) = \phi_{k} (a_{1}, \ldots, - \sum_{i=1}^{k-1} a_{i})\). We then obtain the expression of the RHS.

**Proposition A.2.** Fix \(d \geq 0\). We have \(H_{d} = H_{d}^{BDGR}\).

**Proof.** Using the preceding lemma and the expression of \(G_{d+1}\), we find
\[
H_{d}^{BDGR} = \sum_{b \in \mathbb{Z}} e^{-ibx} \frac{\partial G_{d+1}}{\partial p_{b}} = H_{d}.
\]

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