Cosmological Term and Fundamental Physics*

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A nonvanishing cosmological term in Einstein’s equations implies a nonvanishing spacetime curvature even in absence of any kind of matter. It would, in consequence, affect many of the underlying kinematic tenets of physical theory. The usual commutative spacetime translations of the Poincaré group would be replaced by the mixed conformal translations of the de Sitter group, leading to obvious alterations in elementary concepts such as time, energy and momentum. Although negligible at small scales, such modifications may come to have important consequences both in the large and for the inflationary picture of the early Universe. A qualitative discussion is presented which suggests deep changes in Hamiltonian, Quantum and Statistical Mechanics. In the primeval universe as described by the standard cosmological model, in particular, the equations of state of the matter sources could be quite different from those usually introduced.

1. INTRODUCTION

Recent observational data reveal the presence of a nonvanishing positive cosmological constant $\Lambda$. Although the astrophysical and cosmological consequences of this fact have already been extensively studied, an analysis of the consequences for fundamental Physics is still lacking. The main reason for such absence is, probably, that the local effects of a small cosmological term would be negligible and hardly detectable experimentally. From the conceptual point of view, however, such analysis is not only justified but highly desirable. In addition, it could find some applications in the cosmology of the early universe: inflation requires a very high value for $\Lambda$ and the physical laws applicable in the primeval universe could eventually be different from those of the ordinary Physics as we know it. With these motivations in mind, the basic purpose of this essay will be to make a qualitative discussion on how the presence of a cosmological term could eventually produce changes in ordinary fundamental Physics.

The crucial consequence of a cosmological term in the sourceless Einstein’s equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 0$$

(1)

is that Minkowski spacetime, for which $R_{\mu\nu} = 0$, is no more a solution: contracting with $g^{\mu\nu}$ leads clearly to the scalar curvature $R = -4 \Lambda$ and that, inserted back in the equation, to the Ricci tensor $R_{\mu\nu} = -\Lambda g_{\mu\nu}$. The solution is a uniformly curved de Sitter spacetime $dS^4$. For a positive $\Lambda$ this four-dimensional space can be seen as a hypersurface in the five-dimensional pseudo-Euclidean space $E^{4,1}$ with metric $\eta_{AB} = (+1, -1, -1, -1, -1)$, an inclusion whose points in Cartesian coordinates $(\xi^A) = (\xi^0, \xi^1, \xi^2, \xi^3, \xi^4)$ satisfy

$$\eta_{AB} \xi^A \xi^B = -L^2.$$

(2)

The “pseudo-radius” $L$ is a length parameter related to $\Lambda$ by

$$\Lambda = \frac{3}{L^2}$$

(3)

and, consequently, to the curvature. That hypersurface has the de Sitter group $SO(4,1)$ as group of motions and will be accordingly denoted $dS(4,1)$. Using $\eta_{ab} (a,b = 0,1, 2, 3)$ for the Lorentz metric $\eta = \text{diag} (1, -1, -1, -1)$, Eq. (2) can be put in the form

$$\eta_{ab} \xi^a \xi^b - (\xi^4)^2 = -L^2.$$

(4)

This is the equation defining the de Sitter hypersurface in terms of the five-dimensional ambient space coordinates $\xi^A$. It is possible to describe the same hypersurface in terms of four-dimensional stereographic coordinates.

2. STEREOGRAPHIC COORDINATES

Such coordinates $x^a$ are obtained through a stereographic projection from the de Sitter hypersurface into a Minkowski spacetime. When the Minkowski plane is placed at $\xi^4 = 0$, the projection is given by

$$\xi^a = h^a_\mu x^\mu \equiv \Omega(x) \delta^a_\mu x^\mu = \Omega(x) x^a$$

(5)

and

$$\xi^4 = -L \Omega(x) \left(1 + \frac{\sigma^2}{4L^2}\right),$$

(6)

where

$$\Omega(x) = \frac{1}{1 - \sigma^2 / 4L^2},$$

(7)

with $\sigma^2 = \eta_{ab} x^a x^b = \eta_{\mu\nu} x^\mu x^\nu$ the Lorentz invariant interval. In these expressions we have used the relations $x^a = \delta^a_\mu x^\mu$ and $\eta_{\mu\nu} = \delta^a_\mu \delta^b_\nu \eta_{ab}$, which means that $\delta^a_\mu$ is a kind of trivial tetrad. The $h^a_\mu$ introduced in

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on the tetrad field, actually of the 1-form basis members
$h^a = h^a_\mu dx^\mu = \Omega(x) \delta^a_\mu dx^\mu = \Omega(x) dx^a$. In terms of the stereographic coordinates, the de Sitter line element $ds^2 = \eta_{AB} d\xi^A d\xi^B$ is found to be $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, with

$$g_{\mu\nu} = h^a_\mu h^\nu_a \eta_{ab} \equiv \Omega^2(x) \eta_{\mu\nu}. \quad (8)$$

The de Sitter space is, therefore, conformally flat, with the conformal factor $\Omega^2(x)$. Notice that we are carefully using the Latin alphabet for the algebra (non-holonomous) indices, and the Greek alphabet for the (holonomous) homogeneous space fields and cofields. Tangent-space and spacetime indices are, as usual, interchanged with the help of the tetrad field.

3. KINEMATIC GROUP

A. Generators

The kinematic group of any spacetime will always have a subgroup accounting for both the isotropy of space (rotation group) and the equivalence of inertial frames (boosts). The remaining transformations, generically called translations, can be either commutative or not, and are responsible for homogeneity. The best known relativistic example is the Poincaré group $\mathcal{P}$, naturally associated with the Minkowski spacetime $M$ as its group of motions. It contains, in the form of a semi-direct product, the Lorentz group $\mathcal{L} = SO(3,1)$ and the translation group $\mathcal{T}$. The latter acts transitively on $M$ and its manifold can be identified with $M$. Minkowski spacetime is a homogeneous space under $\mathcal{P}$, actually the quotient $M \equiv \mathcal{T} = \mathcal{P}/\mathcal{L}$. The invariance of $M$ under the transformations of $\mathcal{P}$ reflects its uniformity. The Lorentz subgroup provides for isotropy around a given point of $M$, and translation invariance enforces that isotropy around any other point. This is the usual meaning of “uniformity”, in which $\mathcal{T}$ is responsible for the equivalence of all points.

Now, $dS(4,1)$ is also a homogeneous spacetime:

$$dS(4,1) = SO(4,1)/SO(3,1).$$

The de Sitter group is consequently a bundle with $dS(4,1)$ as base space and the Lorentz group as fiber $\mathcal{L}$. In the Cartesian coordinates $\xi^A$, the generators of the infinitesimal de Sitter transformations are

$$J_{AB} = \eta_{AC} \xi^C \frac{\partial}{\partial \xi^B} - \eta_{BC} \xi^C \frac{\partial}{\partial \xi^A}. \quad (9)$$

In terms of the stereographic coordinates $x^a$, these generators are written as

$$J_{ab} = \eta_{ac} x^c P_b - \eta_{bc} x^c P_a \quad (10)$$

and

$$J_{4a} = L P_a - (4L)^{-1} K_a. \quad (11)$$

where

$$P_a = \frac{\partial}{\partial x^a} \quad \text{and} \quad K_a = (2\eta_{ab} x^b x^c - \sigma^2 \delta_a^c) P_c \quad (12)$$

are, respectively, the generators of translations and proper conformal transformations. The $J_{ab}$’s generate the Lorentz subgroup, whereas the $J_{4a}$’s establish the transitivity of the homogeneous de Sitter space.

B. Transitivity

The crucial point in the considerations above is the interplay of distinct notions of space-distance and time-interval: those usual, related to translations, and new ones, defined by conformal transformations. In fact, as can be seen from the definition of $J_{4a}$, the transitivity of the de Sitter spacetime is defined by a mixture of usual translations and proper conformal transformations. Since $L = (3/\Lambda)^{1/2}$, the relative importance of each one of these transformations is ultimately determined by the value of the cosmological constant.

To study the limit of a vanishing cosmological term ($L \to \infty$), it is convenient to define the generators

$$\Pi_a \equiv L^{-1} J_{4a} = P_a - (2L)^{-2} K_a, \quad (13)$$

from where we see that, in this limit, $\Pi_a$ reduces to ordinary translations, and the de Sitter group reduces to the Poincaré group. At the same time, the de Sitter space becomes the Minkowski spacetime $M$, which is transitive under ordinary translations. On the other hand, for studying the limit of large values of the cosmological term ($L \to 0$), it is convenient to define the generators

$$K_a \equiv 4L J_{4a} = 4L^2 P_a - K_a, \quad (14)$$

from where we see that, in this limit, $K_a$ reduces to (minus) the proper conformal generator, and the de Sitter group reduces to the conformal Poincaré group $\mathcal{P}$. In this case, the de Sitter space becomes a four-dimensional cone $N$, which is transitive under proper conformal transformations. It is impossible to move between two arbitrary points of such a spacetime by ordinary translations, but they can always be connected by some proper conformal transformation. The cone-space $N$ can be seen as singular from the point of view of ordinary spacetime translations, but it is smoothly connected through conformal transformations.

4. EXPLORING THE CHANGES: NEW FUNDAMENTAL PHYSICS?

An immediate consequence of replacing Minkowski by de Sitter as the spacetime underlying the universe in the presence of a nonvanishing $\Lambda$ concerns the concept of relativistic field. In Physics as we know it relativistic fields,
and the particles which turn up as their quanta, are classified by the representations of the Poincaré group. Each representation is fixed by the values of two Casimir invariants. As any function of two invariants is also invariant, it is possible to work with those which have a clear relationship with simple physical characteristics: mass and spin. From all the families of representations of the Poincaré group \[ E \], Nature seems to have given preference to one of the so-called discrete series, whose representations are fixed by two invariants with values

\[ C_2 = -P_a P^a = \Box = -m^2; \quad C_4 = -m^2 s(s+1). \quad (15) \]

The first, involving only translation generators, fixes the mass. It defines the 4-dimensional Laplacian operator and, in particular, the Klein-Gordon equation

\[ (\Box + m^2)\phi = 0, \quad (16) \]

which all relativistic fields satisfy. The second invariant is the square of the Pauli-Lubanski operator, used to fix the spin.

The invariant \( C_2 \) changes in the presence of a nonvanishing \( \Lambda \). The de Sitter group representations appear in analogous series, one of which tends to the Poincaré series above in the \( L \to \infty \) limit, but the corresponding values are \[ C_2 = \Box = -m^2 + L^{-2} [s(s+1) - 2], \quad (17) \]

where now \( \Box \) is the Laplace-Beltrami operator on de Sitter space. A scalar \( (s = 0) \) field would now obey

\[ \left[ \Box + m^2 \right. \left. \frac{R}{6} \right] \phi = 0. \quad (18) \]

This, by the way, could be the solution to the famous controversy on the \( R/6 \) factor: the field \( \phi \) above is not a Poincaré scalar, but a de Sitter scalar, which means to be invariant under a transformation including, in addition to Lorentz and translations, also (proper) conformal transformations. Of course, in the presence of gravitation, \( R \) will represent the total (gravitation plus background) scalar curvature. Mass, in particular, will be defined according to the field behavior under de Sitter translations, not under Poincaré translations. This change is directly related with the transitivity property of the de Sitter spacetime, whose generators (appropriate for studying the limit \( L \to \infty \)) are defined by \( \Pi_a \). It is important to notice that, not only in the case of a scalar field but for any matter field, the kinetic term of the Lagrangian will necessarily be written with the “de Sitter derivative” \( \Pi_a \).

Notice also that, for studying the limit of large values of the cosmological term \( (L \to 0) \), the corresponding field equations must be written in terms of the appropriate “de Sitter derivative” \( K_a \).

The same modifications occur in the mechanics of point particles. In fact, the classical angular momentum of a particle of mass \( m \) associated with the de Sitter group is

\[ \lambda_{AB} = mc \left( \xi_A \frac{d\xi_B}{ds} - \xi_B \frac{d\xi_A}{ds} \right), \quad (19) \]

with \( ds \) the de Sitter line element. In terms of the stereographic coordinates, their components are written as

\[ \lambda_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \quad (20) \]

and

\[ \pi_\mu \equiv L^{-1} \lambda_\mu = p_\mu - (2L)^{-2} k_\mu, \quad (21) \]

where \( p_\mu = m c u_\mu \) is the ordinary momentum, and \( k_\mu = \delta_\mu^\nu k_\nu \) is the conformal momentum, with

\[ \delta_\mu^\nu = (2 \eta_{\mu\nu} x^\rho x^\lambda - \sigma^2 \delta_\mu^\lambda) \delta_\lambda^\nu \quad (22) \]

a kind of conformal Kronecker delta. The corresponding Hamiltonian is

\[ H \equiv \pi_0 = p_0 - (2L)^{-2} k_0. \quad (23) \]

We remark that \( \pi_\mu \) is the Noether conserved current under the transformations generated by \( \Pi_a \). In the limit \( L \to \infty \), the ordinary notions of momentum and energy are recovered.

Now, due to the fundamental role played by energy and momentum, we may say that these modifications will affect all branches of Physics. This is the case, for example, of Statistical Mechanics. In particular, the equations of state used in the standard cosmological model for the early Universe should be reexamined. It is not clear, for instance, whether a photon gas will retain its usual characteristics in the presence of a cosmological term. Analogous changes will also affect Quantum Mechanics. Defining the operator

\[ \hat{\pi}_\mu = -i \hbar \partial_\mu, \quad (24) \]

the modified momentum will be

\[ \hat{\pi}_\mu = -i \hbar \left[ \delta_\mu^\nu - (2L)^{-2} \delta_\mu^\nu \right] \partial_\nu. \quad (25) \]

It is then easy to verify that

\[ [\hat{\pi}_\mu, \hat{\pi}_\nu] = -i \hbar \left[ \delta_\mu^\nu - (2L)^{-2} \delta_\mu^\nu \right]. \quad (26) \]

By considering the nonrelativistic contraction limit \( c \to \infty \) \[ 10 \], which in this case yields the Newton-Hooke groups \[ 11 \], the corresponding \( \Lambda \)-modified nonrelativistic notions of momentum and energy can be obtained \[ 12 \]. It is then possible to get the modified version of the Schrödinger equation, as well as the operators commutation rules and the corresponding Heisenberg uncertainty relations. Of course, the importance of these modifications depends on the value of the cosmological term \( \Lambda \). Accordingly, although they may be negligible locally, they can eventually be important for the Physics of the very early universe. It should be mentioned that, besides the problem of adapting the usual formalisms, there are questions of principle, for example, difficulties concerning the lower bound of the quantum Hamiltonian \[ 13 \], possibly solved by localizability considerations \[ 14 \]. Anyhow, as said, Quantum Mechanics itself will be modified by the presence of \( \Lambda \), and should be reconsidered.
As we have seen, the presence of a cosmological term introduces the conformal generators in the definition of spacetime transitivity. As a consequence, the conformal transformations will naturally be incorporated in Physics, and the corresponding conformal current will appear as part of the Noether conserved current. Of course, for a small enough cosmological term, the conformal modifications become negligible and ordinary Physics remains valid. For large values of $\Lambda$, however, the conformal contributions to the physical magnitudes cannot be disregarded, and these contributions will give rise to deep conceptual changes. This could be the case, for example, of the early stages of the universe, which according to the inflationary models, is characterized by a very high value of $\Lambda$. It could also hold for large enough distances, which are also related to a remote past in the universe history. In the particular case of extremely large values of $\Lambda$, the ordinary notions of momentum and energy would become negligible, with only the corresponding conformal notions surviving. Under such extreme circumstances, a very peculiar new world emerges, whose Physics has yet to be developed from the very beginning. Of course, the whole program of rewriting so much of Physics would be a most daring enterprise. Our point here is only to call attention to the fact that facing such a program may come to be inevitable.

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