Global existence of weak solutions to dissipative transport equations with nonlocal velocity

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Abstract
We consider 1D dissipative transport equations with nonlocal velocity field:
\[ \theta_t + uu_x + \delta u_x \theta + \Lambda^\gamma \theta = 0, \quad u = \mathcal{N}(\theta), \]
where \( \mathcal{N} \) is a nonlocal operator given by a Fourier multiplier. We especially consider two types of nonlocal operators:
1. \( \mathcal{N} = \mathcal{H} \), the Hilbert transform,
2. \( \mathcal{N} = (1 - \partial_x)^{-\alpha} \).

In this paper, we show several global existence of weak solutions depending on the range of \( \gamma, \delta \) and \( \alpha \). When \( 0 < \gamma < 1 \), we take initial data having finite energy, while we take initial data in weighted function spaces (in the real variables or in the Fourier variables), which have infinite energy, when \( \gamma \in (0,2) \).

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1. Introduction

In this paper, we consider transport equations with nonlocal velocity. Here, the non-locality means that the velocity field is defined through a nonlocal operator that is represented in terms of a Fourier multiplier. For example, in the two dimensional Euler equation in vorticity form,

\[ \omega_t + u \cdot \nabla \omega = 0, \]

the velocity is recovered from the vorticity \( \omega \) through

\[ u = \nabla^\perp (-\Delta)^{-\frac{1}{2}} \omega \quad \text{or equivalently} \quad \hat{u}(\xi) = \frac{i\xi^\perp}{|\xi|^2} \hat{\omega}(\xi). \]

Other nonlocal and quadratically nonlinear equations appear in many applications. Prototypical examples are the surface quasi-geostrophic equation, the incompressible porous medium equation, Stokes equations, and magneto-geostrophic equation in multi-dimensions. For more details on nonlocal operators in these equations, see [1].

We here study 1D models of physically important equations. The 1D reduction idea was initiated by Constantin–Lax–Majda [8]: they proposed the following 1D model

\[ \theta_t = \theta \mathcal{H} \theta \]

for the 3D Euler equation in the vorticity form and proved that \( \mathcal{H} \theta \) blows up in finite time under certain conditions. Motivated by this work, other similar models were proposed and analyzed in the literature [1–6, 12, 13, 16, 19, 20, 23]. In this paper, we consider the following 1D equation:

\[ \theta_t + u \theta_x + \delta \theta \Lambda^\gamma \theta + \nu \Lambda \theta = 0, \quad u = \mathcal{N}(\theta). \tag{1.1} \]

Depending on a nonlocal operator \( \mathcal{N} \), (1.1) has structural similarity of several important fluid equations as described below. The goal of this paper is to show the existence of weak solutions with rough initial data. To this end, we will choose functionals carefully to extract more information from the structure of the nonlinearity to construct weak solutions.

1.1. The case \( \mathcal{N} = \mathcal{H} \)

We first take the case \( \mathcal{N} = \mathcal{H} \), the Hilbert transform. Then, (1.1) becomes

\[ \theta_t + (\mathcal{H} \theta) \theta_x + \delta \theta \Lambda^\gamma \theta + \nu \Lambda \theta = 0, \tag{1.2} \]

where the range of \( \gamma \) and \( \delta \) will be specified below. We note that (1.2) is considered as an 1D model of the dissipative surface quasi-geostrophic equation. The surface quasi-geostrophic equation describes the dynamics of the mixture of cold and hot air and the fronts between them in 2D [10, 26]. The equation is of the form

\[ \theta_t + u \cdot \nabla \theta + \nu \Lambda^\gamma \theta = 0, \quad u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), \tag{1.3} \]

where the scalar function \( \theta \) is the potential temperature and \( \mathcal{R}_j \) is the Riesz transform

\[ \mathcal{R}_j f(x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{(x_j - y_j)f(y)}{|x - y|^3} \, dy, \quad j = 1, 2. \]

As Constantin–Lax–Majda did for the Euler equation, the equation (1.2) is derived by replacing the Riesz transforms with the Hilbert transform. The case \( \delta = 0 \) and \( \delta = 1 \) correspond to (1.3) in non-divergence and divergence form, respectively. We take a parameter \( \delta \in [0, 1] \) to cover more general nonlinear terms in (1.2). We note that there are several singularity formation
results when \( \nu = 0 \): \( 0 < \delta < \frac{1}{\nu} \) and \( \delta = 1 \) [23], \( 0 < \delta \leq 1 \) [6] and \( \delta = 0 \) [12, 19, 28]. By contrast, we look for weak solutions of (1.2) globally in time (see e.g. [14]). From now on, we set \( \nu = 1 \) for notational simplicity.

We now consider (1.2). In the direction of seeking a weak solution, we assume that \( \theta_0 \) satisfies the conditions

\[
\theta_0(x) > 0, \quad \theta_0 \in L^1 \cap H^\frac{1}{2}.
\]

Since (1.2) satisfies the minimum principle (see section 2) when \( \delta \geq 0 \), \( \theta(t,x) \geq 0 \) for all time. This sign condition combined with the structure of the nonlinearity enables us to use the following function space

\[
A_T = L^\infty \left( 0, T; L^p \cap H^\frac{1}{2} \right) \cap L^2 \left( 0, T; H^{\frac{2+\gamma}{2}} \right) \quad \text{for all } p \in (1, \infty).
\]

We note that we add the sign condition of \( \theta_0 \) for the better understanding of the structure of the nonlinearity in the sense that the sign condition and the dissipative term (of any order) deplete the nonlinear effect. If we remove the sign condition of \( \theta_0 \), we would have to control higher other norms, but then we will obtain strong solutions locally in time which are not the notion of solutions we are going to obtain in this paper. Note that the positivity condition is something that is also assumed in other works (see e.g. [12, 14, 16]).

**Definition 1.1.** We say \( \theta \) is a weak solution of (1.2) if \( \theta \in A_T \) and (1.2) holds in the following sense: for any test function \( \psi \in C^\infty_c ([0, T) \times \mathbb{R}) \),

\[
\int_0^T \int_\mathbb{R} \left[ \theta \psi_t + (\mathcal{H} \theta) \theta \psi_x + (1 - \delta) \Lambda \theta \theta \psi - \theta \Lambda^\gamma \psi \right] \, dx \, dt = \int_\mathbb{R} \theta_0(x) \psi(0,x) \, dx
\]

holds for any \( 0 < T < \infty \).

**Theorem 1.1.** Let \( \gamma \in (0, 1) \) and \( \delta \geq \frac{1}{\nu} \). Then, for any \( \theta_0 \) satisfying (1.4), there exists a weak solution of (1.2) in \( A_T \) for all \( T > 0 \). Moreover, a weak solution is unique when \( \gamma = 1 \).

For \( \gamma \in (0, 2) \), we consider infinite energy solutions of (1.2). More precisely, we take a family of weights \( w_\beta = \left( 1 + |x|^2 \right)^{-\frac{\beta}{2}}, 0 < \beta < \gamma \) and we shall prove various existence theorems. For the critical case, we take initial data in the following weighted Sobolev space

\[
\theta_0 \in H^\frac{1}{4} (w_\beta \, dx) \cap L^\infty.
\]

These weighted spaces are defined in section 2. We note that \( \theta_0 \) can decay (slowly) at infinity. For example, as long as \( \beta - 2\eta > 1 \), \( |\theta_0(x)| \approx |x|^{-\eta}, \eta > 0 \) is allowed to stay in \( L^2 (w_\beta \, dx) \)

\[
\int_{|x| \geq 1} \frac{|x|^{2\eta}}{\left( 1 + |x|^2 \right)^{\frac{\beta}{2}}} \, dx < \infty.
\]

But, we can still use the energy method to obtain a weak solution of (1.2). Let

\[
B_T = L^\infty \left( 0, T; H^\frac{1}{4} (w_\beta \, dx) \right) \cap L^2 \left( 0, T; H^1 (w_\beta \, dx) \right).
\]
Theorem 1.2. Assume $\gamma = 1$, then, for any $\theta_0$ satisfying (1.5) with $\|\theta_0\|_{L^\infty}$ being sufficiently small, there exists a unique weak solution of (1.2) in $\mathcal{B}_T$ for all $T > 0$.

Remark 1. It is worth mentioning that this theorem is also true in the unweighted setting, this is an important point because one would need to use the existence of solutions in the unweighted setting to prove theorem 1.2. Indeed, one has nice this is an important point because one would need to use the existence of solutions in the unweighted setting to prove theorem 1.2. Indeed, one has nice this is an important point because one would need to use the existence of solutions in the unweighted setting to prove theorem 1.2. Indeed, one has nice this is an important point because one would need to use the existence of solutions in the unweighted setting to prove theorem 1.2. Indeed, one has nice this is an important point because one would need to use the existence of solutions in the unweighted setting to prove theorem 1.2.

For the construction using compactness we refer to [18].

We also define $A^\alpha = \left\{ f \in L^1_{\text{loc}} : \|f\|_{A^\alpha} = \int_\mathbb{R} (1 + |\xi|^{\alpha})|\hat{f}(\xi)|d\xi < \infty \right\}$.

We also define

$\mathcal{W}_T = L^\infty \left( 0, T; W^{1, \infty} \right) \cap W^{1, \infty} \left( 0, T; L^\infty \right) \cap L^1 \left( 0, T; W^{2, \infty} \right)$.
Theorem 1.5. Let $\gamma = 1$ and $\delta \in \mathbb{R}$. Then, for any $\theta_0 \in A^1$ with
\[
\|\theta_0\|_{A^1} < \frac{\sqrt{\pi}}{\sqrt{2}(1 + |\delta|)},
\]
there exists a unique weak solution of (1.2) verifying the following inequality for all $T > 0$
\[
\theta \in W_T, \quad \sup_{t \in [0,T]} \|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}(1 + |\delta|)\|\theta_0\|_{A^1}}{\sqrt{\pi}}\right) \int_0^T \|\theta(t)\|_{A^1} dt \leq \|\theta_0\|_{A^1}.
\]

We note that $\theta_0 \in A^1$ can have infinite energy. For example, we take $\tilde{\theta}_0(\xi) = \frac{1}{\sqrt{\pi} |\xi|} e^{-|\xi|^2}$ for $\xi \neq 0$. Then, $\theta_0 \in A^1$ but $\theta_0 \notin L^2$.

We observe that the proof of theorem 1.5 is due to the perfect balance (in the critical case $\gamma = 1$) between the derivatives in the nonlinearity and the diffusive linear operator. This is due to the fact that, in Wiener spaces, the parabolic gain of regularity for $\Lambda$ is $L^1 t A^1_x$ (i.e. a full derivative in the Wiener space). This is in contrast with the case of $L^2$, where the parabolic gain of regularity for $\Lambda$ is $L^2 t H^{1/2}$ (i.e. just half derivative in $L^2$). In particular, the proof is based in an inequality of the type
\[
\frac{d}{dt} E(t) + E(t) D(t) \leq -D(t),
\]
where $E$ and $D$ are the appropriate energy and dissipation. From such an inequality the decay of the energy for small enough initial energy can be easily obtained. In the case where $\gamma > 1$, that balance is broken and the previous inequality has to be replaced by
\[
\frac{d}{dt} E(t) + E^{2-\alpha}(t) D^\alpha(t) \leq -D(t),
\]
with $0 < \alpha < 1$. In our setting, the lack of Poincaré inequality that relates $E$ and $D$, makes this new inequality less suited for our approach.

1.2. The case $\mathcal{N} = (1 - \partial_{xx})^{-\alpha}$ and $\delta = 0$

In this case, (1.1) is changed to the equation
\[
\theta_t + u\theta_x + \delta \theta A \theta + \Lambda^\gamma \theta = 0, \quad u = (1 - \partial_{xx})^{-\alpha} \theta.
\]
This equation is closely related to a generalized Proudman–Johnson equation [25, 27, 31]:
\[
f_{xx} + ff_{xx} + \delta f f_{xx} = \nu f_{xxxx},
\]
which is derived from the 2D incompressible Navier–Stokes equations via the separation of space variables when $\delta = 1$. By taking $w = f_{xx}$,
\[
w_t + fw_x + \delta f w = \nu w_{xx}, \quad f = (\partial_{xx})^{-1} w.
\]
The inviscid case with $\delta = 2$ is equivalent to the Hunter–Saxton equation arising in the study of nematic liquid crystals [15]. The equation (1.7) is also considered as a model equation of the Lagrangian averaged Navier–Stokes equations [21] which are given by
\[
\partial_t (1 - \sigma^2 \Delta) u + u \cdot \nabla (1 - \sigma^2 \Delta) u + (\nabla u)^T \cdot (1 - \sigma^2 \Delta) u = -\nabla p + \nu \Delta (1 - \sigma^2 \Delta) u, \quad \nabla \cdot u = 0.
\]
But, we here consider (1.7) with $\delta = 0$ to see how the regularizing effect in $u$ overcomes difficulties from the quadratic term $u\theta_x$. Along this direction, we look closely to see if there is a solution when $\alpha$ and $\gamma$ meet certain conditions.

We first deal with (1.7) with initial data in $L^2 \cap L^\infty$. Let

$$C_T = L^\infty (0, T; L^p) \cap L^2 (0, T; H^{\gamma})$$

for all $p \in [2, \infty]$.

**Definition 1.2.** We say $\theta$ is a weak solution of (1.7) if $\theta \in C_T$ and (1.7) holds in the following sense: for any test function $\psi \in C^\infty_c ([0, T) \times \mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} \left[ \theta \psi_t + u_x \theta \psi + u \theta_x \psi - \theta A^{\gamma} \psi \right] \, dx \, dt = \int_{\mathbb{R}} \theta_0(x) \psi(0, x) \, dx$$

holds for any $0 < T < \infty$.

**Theorem 1.6.** Let $\gamma \in (0, 2)$ and $\alpha = 1/4 - \gamma/4$. Then, for any $\theta_0 \in L^2 \cap L^\infty$, there exists a weak solution of (1.7) in $C_T$ for all $T > 0$. Moreover, such a weak solution is unique if $\gamma > 1$ and $\alpha > 1/4$.

We note that $\theta_0 \in L^2 \cap L^\infty$ is enough to construct a weak solution in theorem 1.6, but we need to strengthen $\gamma$ and $\alpha$ to get the uniqueness of weak solutions.

When $\gamma = 1$, we consider weights $w_\beta(x) = (1 + |x|^2)^{-\beta}$ with $0 < \beta < 1$, and take initial data in $H^1(w_\beta \, dx) \cap L^\infty$. Let $\alpha = 1/4$ and

$$D_T = L^\infty (0, T; H^1(w_\beta \, dx)) \cap L^2 (0, T; H^{\gamma}(w_\beta \, dx))$$. 

**Theorem 1.7.** Let $\gamma = 1$ and $\alpha = 1/4$. Then, for any $\theta_0 \in H^1(w_\beta \, dx) \cap L^\infty$, there exists a unique global weak solution of (1.7) in $D_T$ for all $T > 0$.

Compared to theorem 1.2, we do not assume that $\|\theta_0\|_{L^\infty}$ is small to prove theorem 1.7.

In theorems 1.6 and 1.7, we have restrictions on the range of $\alpha$. Again, we can remove these conditions by looking for a solution of (1.7) in function spaces defined by the Fourier variables.

**Theorem 1.8.** Let $\gamma = 1$ and $\alpha \geq 0$. Then, for any $\theta_0 \in A^1$ satisfying

$$\|\theta_0\|_{A^1} < \frac{\sqrt{\pi}}{\sqrt{2}}, \tag{1.8}$$

there exists a unique weak solution of (1.7) verifying the following inequality for all $T > 0$

$$\theta \in W_T, \quad \sup_{t \in [0, T]} \|\theta(t)\|_{A^1} + \left( 1 - \frac{\sqrt{2} \|\theta_0\|_{A^1}}{\sqrt{\pi}} \right) \int_0^T \|\theta_x(t)\|_{A^1} \, dt \leq \|\theta_0\|_{A^1}.$$

**Remark 2.** We note that theorem 1.8 remains valid with straightforward changes in the spirit of theorem 1.5 when $\delta \neq 0$. 
2. Preliminaries

All constants will be denoted by $C$ that is a generic constant. In a series of inequalities, the value of $C$ can vary with each inequality. For $s \in \mathbb{R}$, $H^s$ is a Hilbert space with

$$
\|f\|_{2^s}^2 = \int_\mathbb{R} \left(1 + |\xi|^2\right)^s |\hat{f}(\xi)|^2 d\xi.
$$

2.1. Hilbert transform and fractional Laplacian

The Hilbert transform is defined as

$$
\mathcal{H}f(x) = \text{p.v.} \int_\mathbb{R} \frac{f(y)}{x-y} dy.
$$

The differential operator $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$ is defined by the action of the following kernels [11]:

$$
\Lambda^\gamma f(x) = c_\gamma \text{p.v.} \int_\mathbb{R} \frac{f(x) - f(y)}{|x-y|^{1+\gamma}} dy,
$$

(2.1)

where $c_\gamma > 0$ is a normalized constant. When $\gamma = 1$,

$$
\Lambda f(x) = \mathcal{H}f(x).
$$

Moreover, we have the following identity:

$$
\mathcal{H} \left( \frac{\partial_t}{\partial t} \theta(x) \right) = \frac{1}{2} \left[ (\Lambda \theta)^2 - (\theta_x)^2 \right].
$$

(2.2)

We also recall the following pointwise property of $\Lambda^\alpha$.

**Lemma 2.1.** [11] Let $0 \leq \alpha \leq 2$ and $f \in S$. Then,

$$
f(x) \Lambda^\alpha f(x) \geq \frac{1}{2} \Lambda^\alpha \left( f^2(x) \right),
$$

$$
f^2(x) \Lambda f(x) \geq \frac{1}{3} \Lambda \left( f^3(x) \right) \text{ when } f \geq 0
$$

2.2. Minimum and maximum principles

In theorem 1.1, we assume $\theta_0 > 0$. To obtain global-in-time solutions, we need $\theta(t,x) \geq 0$ for all time. We first assume that $\theta(t,x) \in C^1([0,T] \times \mathbb{R})$ and $x_i$ be a point such that $m(i) = \theta(t,x_i)$. If $m(i) > 0$ for all time, nothing is left to prove. So, we check a point $(t,x_i)$ where $m(i) = 0$. Since $m(i)$ is a continuous Lipschitz function, it is differentiable at almost every $t$ by Rademacher’s theorem. From the definition of $\Lambda^\gamma$,

$$
\frac{d}{dt} m(t) = -\delta \partial_t \theta(t,x_i) \text{p.v.} \int_\mathbb{R} \frac{\theta(t,x_i) - \theta(t,y)}{|x_i - y|^{1+\gamma}} dy - \text{p.v.} \int_\mathbb{R} \frac{\theta(t,x_i) - \theta(t,y)}{|x_i - y|^{1+\gamma}} dy
$$

$$
\geq \left[ -\delta \text{p.v.} \int_\mathbb{R} \frac{\theta(t,x_i) - \theta(t,y)}{|x_i - y|^{1+\gamma}} dy \right] m(t).
$$

Since the quantity in the bracket is nonnegative when $\delta \geq 0$, we have that $m(t)$ is non-decreasing in time if $\theta_0 > 0$ and thus $\theta(t,x) \geq 0$ for all time. Similarly, maximum values of $\theta(t,x)$ are...
non-increasing in time when \( \theta_0 > 0 \) with \( \theta_0 \in L^\infty \). For general initial data satisfying (1.4) and (1.5), we can use regularization method. For such a regularized problem with smooth solution \( \theta^\epsilon \), the same argument works. Then, we construct \( \theta \) as the limit of \( \theta^\epsilon \). As \( \theta \) will be also the pointwise limit of \( \theta^\epsilon \) almost everywhere, we conclude that \( \theta(t,x) \geq 0 \).

Since (1.7) is purely a dissipative transport equation, we immediately have

\[
\| \theta(t) \|_{L^\infty} \leq \| \theta_0 \|_{L^\infty}.
\]

2.3. The Wiener spaces \( A^\alpha \)

The Wiener space is defined as

\[
A^0 = \left\{ f \in L^1_{\text{loc}} : \hat{f}(\xi) \in L^1 \right\},
\]
where \( \hat{f} \) denotes the Fourier transform of \( f \)

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix \cdot \xi} dx.
\]

\( A^0 \) is a Banach space endowed with the norm

\[
\| f \|_{A^0} = \| \hat{f} \|_{L^1}.
\]
Furthermore, using Fubini’s theorem, \( A^0 \) is a Banach algebra, i.e.

\[
\| fg \|_{A^0} \leq \| f \|_{A^0} \| g \|_{A^0}.
\]

Once we have defined \( A^0 \), we can define the full scale of homogeneous, \( \dot{A}^\alpha \), and inhomogeneous, \( A^\alpha \), Wiener spaces as

\[
\dot{A}^\alpha = \left\{ f \in L^1_{\text{loc}} : \| f \|_{\dot{A}^\alpha} = \int_{\mathbb{R}} |\xi|^\alpha |\hat{f}(\xi)| d\xi < \infty \right\},
\]

\[
A^\alpha = \left\{ f \in L^1_{\text{loc}} : \| f \|_{A^\alpha} = \int_{\mathbb{R}} (1 + |\xi|^\alpha) |\hat{f}(\xi)| d\xi \right\}.
\]

(2.3)

For these spaces, the following inequalities hold

\[
\| f \|_{C(\mathbb{R})} \leq \| f \|_{\dot{A}^\alpha(\mathbb{R}^d)} \quad \forall f \in A^0(\mathbb{R}^d).
\]

(2.4)

\[
\| f \|_{\dot{A}^\alpha(\mathbb{R})} \leq \| f \|_{\dot{A}^\alpha(\mathbb{R}^d)}^{1-\theta} \| f \|_{\dot{A}^\alpha(\mathbb{R}^d)}^\theta \quad \forall 0 < \theta < 1, \alpha \geq 0, f \in A^0(\mathbb{R}) \cap \dot{A}^\alpha(\mathbb{R}).
\]

(2.5)

As a consequence of (2.4), we obtain that if \( u \in A^0 \) has infinite energy then

\[\limsup_{|x| \to \infty} |u(x)| + \liminf_{|x| \to \infty} |u(x)| < \infty.\]

2.4. Commutator estimate

In the proof of theorem 1.1, we need to estimate a commutator term involving \( \Lambda^i_1 \). To do this, we first recall Hardy–Littlewood–Sobolev inequality in 1D. Let \( K_\alpha(x) = \frac{1}{|x|^\alpha} \) and \( T_\lambda f = K_\lambda * f \). Then,
Lemma 2.2. For \( f \in L^2 \), \( g \in L^2 \) and \( \psi \in W^{1,\infty} \),
\[
\| \left[ \Lambda^{1/2}, \psi \right] f - \left[ \Lambda^{1/2}, \psi \right] g \|_{L^2} \leq C \| \psi \|_{W^{1,\infty}} \| f - g \|_{L^2}.
\]

Proof. By the definition of \( \Lambda^{1/2} \), we have
\[
\left( \left[ \Lambda^{1/2}, \psi \right] f - \left[ \Lambda^{1/2}, \psi \right] g \right) (x) = c_1 \int \frac{(\psi(y) - \psi(x))(f(y) - g(y))}{|x - y|^\frac{3}{2}} dy
\]
and thus
\[
\left\| \left[ \Lambda^{1/2}, \psi \right] f - \left[ \Lambda^{1/2}, \psi \right] g \right\|_{L^2} \leq C \| \psi \|_{W^{1,\infty}} \int \frac{|f(y) - g(y)|}{|x - y|^\frac{3}{2}} dy.
\]
(2.6)

Using Hardy–Littlewood–Sobolev inequality, we obtain that
\[
\left\| \left[ \Lambda^{1/2}, \psi \right] f - \left[ \Lambda^{1/2}, \psi \right] g \right\|_{L^2} \leq C \| \psi \|_{W^{1,\infty}} \| f - g \|_{L^2}
\]
which completes the proof. \( \square \)

2.5. Muckenhoupt weights

We briefly introduce weighted spaces. A weight \( w \) is a positive and locally integrable function. A measurable function \( \theta \) on \( \mathbb{R} \) belongs to the weighted Lebesgue spaces \( L^p(wdx) \) with \( 1 \leq p < \infty \) if and only if
\[
\| \theta \|_{L^p(wdx)}^p = \int_{\mathbb{R}} |\theta(x)|^p w(x) dx < \infty.
\]

An important class of weights is the Muckenhoupt class \( A_p \) for \( 1 < p < \infty \) [7, 24]. Let \( 1 < p < \infty \), we say that \( w \in A_p \) if and only if there exists a constant \( C_p,w > 0 \) such that
\[
\sup_{r > 0, x_0 \in \mathbb{R}} \left( \frac{1}{2r} \int_{[x_0 - r, x_0 + r]} w dx \right)^{\frac{1}{p}} \left( \frac{1}{2r} \int_{[x_0 - r, x_0 + r]} w^{\frac{1}{p'}} dx \right)^{1 - p} \leq C_{p,w}.
\]

This class satisfies the following properties.

1. Calderón–Zygmund type operators are bound on \( L^p(wdx) \) when \( w \in A_p \) and \( 1 < p < \infty \) [29].
2. Let \( w \in A_p \). We define weighted Sobolev spaces as follows
\[
\begin{align*}
&f \in H^1(wdx) \iff f \in L^2(wdx) \text{ and } f_\alpha \in L^2(wdx), \\
&f \in H^1(wdx) \iff (1 - \partial_\alpha)^{1/2} f \in L^2(wdx) \iff f \in L^2(wdx) \text{ and } \Lambda f \in L^2(wdx), \\
&f \in H^2(wdx) \iff (1 - \partial_\alpha)^2 f \in L^2(wdx) \iff f \in L^2(wdx) \text{ and } \Lambda^2 f \in L^2(wdx).
\end{align*}
\]
(2.8)
(3) Gagliardo–Nirenberg type inequalities (see e.g. [22])

$$\|A^βf\|_{L^p(wdx)} \leq C \|f\|_{L^p(wdx)} \|A^βf\|_{L^p(wdx)},$$

$$\|\theta\|_{L^p(wdx)} \leq C \|\|\|_{L^p(wdx)} \|A^β\|_{L^p(wdx)}.$$  \hspace{1cm} (2.9)

This latter inequality can be proved for instance by using the weighted Sobolev embedding $H^1(wdx) \hookrightarrow L^4(wdx)$, and then by weighted interpolation one recover the second inequality in (2.9).

In this paper, we take weights $w_β = (1 + |x|^2)^{-\frac{β}{2}}$, $0 < β < 1$, which belongs to the $A_p$ class of Muckenhoupt for all $1 < p < ∞$. These weights also satisfy the following properties. For the proofs, see [18] (for the first two points), and [17] (for the last two points).

**Lemma 2.3.** Let $w_β(x) = (1 + |x|^2)^{-\frac{β}{2}}$.

- Let $p \geq 2$ be such that $\frac{3}{2} - β(1 - \frac{1}{p}) > 1$, then the commutator $\frac{1}{w_β}[A^{1/2}, w_β]$ is bounded from $L^p(wdx)$ to $L^p(wdx)$.
- Let $2 \leq p < ∞$, then the commutator $\frac{1}{\sqrt{w}}[A, \sqrt{w}]$ is bounded from $L^p(wdx)$ to $L^p(wdx)$.
- Let $s \in (0,1)$, then for all $β \in (0,s)$, the commutator $\frac{1}{w}[A^{1/2}, w]$ is bounded from $L^2(wdx)$ to $L^2(wdx)$.
- Let $s \in (1,2)$, then for all $β \in (0,1)$, the commutator $\frac{1}{w}[A^{1/2}, w]$ is bounded from $L^2(wdx)$ to $L^2(wdx)$.
- The commutator $\frac{1}{w}[A^{1/2}, w]$ is bounded from $L^p(wdx)$ to $L^p(wdx)$.

**Remark 3.** Actually, the last point of lemma 2.3 is not proved in [18] or [17]. However, a slight modification of the proof given in [18] of the second point of lemma 2.3 gives the last point of lemma 2.3. Indeed, the idea of the proof of all of those commutator estimates is to split the domain of integration into 3 regions ($Δ_1$, $Δ_2$, and $Δ_3$ according to the notation in [18]). Since we are dealing with a quite singular kernel, we need to do a sort of second Taylor expansion in the region where the kernel is very singular (the region $Δ_1$), the same argument works writing $w$ in stead of $γ$. As well, in the region $Δ_2$ one can follow [18]. In the region $Δ_3$, the estimates of the kernel change a little bit, indeed, we have the following estimate

$$|K(x, y)| \leq C \frac{w(x)^{\frac{1}{2}} + w(y)^{\frac{1}{2}}}{|x - y|^{2β}}.$$  \hspace{1cm}  \hspace{1cm} (2.5)

where we used that, on $Δ_1$ we have $1 \leq w(x)^{-1} \leq C|x - y|^β$ and $1 \leq w(y)^{-1} \leq C|x - y|^β$. To conclude, it suffices to observe that, since $0 < β < 1$ and $\max(1 - \frac{1}{p}, \frac{1}{p}) \in (0,1)$ for all $p \geq 2$, we have $K \in L^1$.

We shall also need to estimate $|A^β|_w$ for $s \in (0,2)$, the estimate of such a term will be done using the following lemma (see [18] and [17] for the proof).

**Lemma 2.4.** Let $w_β(x) = (1 + |x|^2)^{-\frac{β}{2}}$, with $β \in (0,1)$.

- $|\partial_α w_β(x)| \leq C(β)w_β(x)$
- For all $s \in (0,2)$, there exists a constant $C = C(s, β) > 0$ such that the following estimate holds
2.6. Littlewood–Paley decomposition

Consider a fixed function \( \phi_0 \in \mathcal{D}(\mathbb{R}) \) that is non-negative and radial and is such that \( \phi_0(\xi) = 1 \) if \( |\xi| \leq 1/2 \) and \( \phi_0(\xi) = 0 \) if \( |\xi| \geq 1 \). Then, we define a new function \( \psi_0 \): \( \psi_0(\xi) = \phi_0(\xi/2) - \phi_0(\xi) \) (which is supported in a corona). Then, for \( j \in \mathbb{Z} \), we define the two distributions \( S_j f = \mathcal{F}^{-1}(\phi_0(2^{-j} \xi) \hat{f}(\xi)) \) and \( \Delta_j f = \mathcal{F}^{-1}(\psi_0(2^{-j} \xi) \hat{f}(\xi)) \) and we get the so-called inhomogeneous Littlewood–Paley decomposition of \( f \in \mathcal{S}'(\mathbb{R}) \) that is for all \( K \in \mathbb{Z} \) the following inequality holds in \( \mathcal{S}'(\mathbb{R}) \)

\[
|\Lambda^j \omega_j(x)| \leq C \omega_j(x).
\]

\[
2.6. \text{Littlewood–Paley decomposition}
\]

Passing to the limit in equality (2.10) as \( K \to -\infty \) in the \( \mathcal{S}'(\mathbb{R}) \) topology one obtains

\[
f = \sum_{j \in \mathbb{Z}} \Delta_j f.
\]

The equality (2.11) is called homogeneous decomposition of \( f \) and is defined modulo polynomials. We are now ready to define the homogeneous weighted Sobolev spaces \( \dot{H}^s_w \) for \( |s| < 1/2 \), they are defined as follows

\[
f \in \dot{H}^s_w \iff f = \sum_{j \in \mathbb{Z}} \Delta_j f, \quad \text{in} \, \mathcal{S}'(\mathbb{R}) \quad \text{and} \quad \sum_{j \in \mathbb{Z}} 2^{2js} \| \Delta_j f \|^2_{L^2_w} < \infty.
\]

We shall use the Bernstein’s inequality, that is for all \( f \in \mathcal{S}'(\mathbb{R}) \) and \( (j, s) \in \mathbb{Z} \times \mathbb{R} \), and for all \( 1 \leq p \leq q \leq \infty \) and all weights \( w \in A_{\infty} \), we have

\[
\| \Lambda^s \Delta_j f \|_{L^p_w} \lesssim 2^{j/s} \| \Delta_j f \|_{L^p_w} \quad \text{also} \quad \| \Delta_j f \|_{L^p_w} \lesssim 2^{j(\frac{1}{p} - \frac{1}{q})} \| \Delta_j f \|_{L^q_w} \quad \text{and} \quad \| \Lambda^s \Delta_j f \|_{L^p_w} \lesssim 2^{j/s} \| \Delta_j f \|_{L^p_w}.
\]

Finally, we recall that for any distributions \( f \) and \( g \) that are in \( \mathcal{S}'(\mathbb{R}) \), one has the following paraproduct formula

\[
f g = \sum_{q \in \mathbb{Z}} S_{q+1} f \Delta_q g + \sum_{j \in \mathbb{Z}} \Delta_j f S_j g.
\]

2.7. Compactness

Since we look for weak solutions, we use compactness arguments when we pass to the limit in weak formulations.

**Lemma 2.5.** [30] Let \( X_0, X, X_1 \) be reflexive Banach spaces such that

\[
X_0 \subset \subset X \subset X_1,
\]

where \( X_0 \) is compactly embedded in \( X \). Let \( T > 0 \) be a finite number and let \( \alpha_0 \) and \( \alpha_1 \) be two finite numbers such that \( \alpha_i > 1 \). Then, \( Y = \{ u \in L^{\alpha_i} (0, T; X_0), \quad \partial_t u \in L^{\alpha_i} (0, T; X_1) \} \) is compactly embedded in \( L^{\alpha_0} (0, T; X) \).
Lemma 2.6 ([9]). Consider a sequence \((\theta^\epsilon) \in C([0,T] \times B_R(0))\) that is uniformly bounded in \(L^\infty([0,T], W^{1,\infty}(B_R(0)))\). Assume further that the weak derivative \(\frac{\partial \theta^\epsilon}{\partial x}\) is in \(L^\infty([0,T], L^\infty(B_R(0)))\) (not necessarily uniform) and is uniformly bounded in \(L^\infty([0,T], W^{-2,\infty}(B_R(0)))\). Finally suppose that \(\theta^\epsilon \in C([0,T] \times B_R(0))\). Then there exists a subsequence of \((\theta^\epsilon)\) that converges strongly in \(L^\infty([0,T] \times B_R(0))\).

3. Proof of theorem 1.1

3.1. A priori estimates

We first obtain a priori bounds of the equation
\[
\theta_t + (\mathcal{H}\theta) \theta_x + \delta \theta \Lambda \theta + \Lambda^2 \theta = 0. \tag{3.1}
\]
We note that by the minimum principle applied to (3.1), we have \(\theta(t,x) \geq 0\) for all \(t \geq 0\).

To obtain \(H^\frac{1}{2}\) bound of \(\theta\), we begin with the \(L^2\) bound. We multiply (3.1) by \(\theta\) and integrate over \(\mathbb{R}\). Then,
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2} + \|\Lambda^2 \theta\|^2_{L^2} = -\int [(\mathcal{H}\theta) \theta_x] \, dx - \delta \int [\theta^2 \Lambda \theta] \, dx = \left(\frac{1}{2} - \delta\right) \int [\theta^2 \Lambda \theta] \, dx.
\]
Since \(\theta \geq 0\), we have
\[
\int [\theta^2 \Lambda \theta] \, dx = \int \int \frac{(\theta(x) - \theta(y))^2}{|x-y|^2} \cdot \frac{\theta(x) + \theta(y)}{2} \, dx \, dy \geq 0
\]
and thus
\[
\|\theta(t)\|^2_{L^2} + 2 \int_0^t \|\Lambda^2 \theta(s)\|^2_{L^2} \, ds \leq \|\theta_0\|^2_{L^2}. \tag{3.2}
\]

We next estimate \(\theta\) in \(H^\frac{1}{2}\). We multiply (3.1) by \(\Lambda \theta\) and integrate over \(\mathbb{R}\):
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda \theta\|^2_{L^2} + \|\Lambda^{1+\frac{1}{2}} \theta\|^2_{L^2} = -\int [(\mathcal{H}\theta) \theta_x \Lambda \theta] \, dx - \delta \int [\theta \Lambda^2 \theta] \, dx. 
\]
By (2.2), we have
\[
-\int [(\mathcal{H}\theta) \theta_x \Lambda \theta] \, dx = \int [\theta \mathcal{H} (\theta_x (\mathcal{H}\theta_x))] \, dx = \frac{1}{2} \int \left[\theta (\Lambda^2 \theta - (\theta_x)^2)\right] \, dx,
\]
and hence
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^\frac{1}{2} \theta\|^2_{L^2} + \|\Lambda^{1+\frac{1}{2}} \theta\|^2_{L^2} = \left(\frac{1}{2} - \delta\right) \int [\theta (\Lambda^2 \theta)^2] \, dx - \frac{1}{2} \int \left[\theta (\theta_x)^2\right] \, dx \leq 0,
\]
where we use the sign conditions \(\theta \geq 0\) and \(\delta \geq \frac{1}{2}\). This leads to the inequality
\[
\|\Lambda^\frac{1}{2} \theta(t)\|^2_{L^2} + 2 \int_0^t \|\Lambda^{1+\frac{1}{2}} \theta(s)\|^2_{L^2} \, ds \leq \|\Lambda^\frac{1}{2} \theta_0\|^2_{L^2}. \tag{3.3}
\]

By (3.2) and (3.3), we obtain that
\[
\|\theta(t)\|^2_{H^\frac{1}{2}} + 2 \int_0^t \|\Lambda^\frac{1}{2} \theta(s)\|^2_{H^\frac{1}{2}} \, ds \leq \|\theta_0\|^2_{H^\frac{1}{2}}. \tag{3.4}
\]
We finally estimate $\theta$ in $L^1$. Since $\theta \geq 0$,

$$\frac{d}{dt} \|\theta\|_{L^1} = \frac{d}{dt} \int |\theta| dx = (1 - \delta) \int \theta \Delta \theta dx \leq C \|\nabla^2 \theta\|^2_{L^2},$$

and thus we conclude that

$$\|\theta(t)\|_{L^1} \leq \|\theta_0\|_{L^1} + C \int_0^t \|\theta(s)\|^2_{L^2} ds \leq \|\theta_0\|_{L^1} + Ct \|\theta_0\|^2_{H^{\frac{1}{2}}}. \quad (3.5)$$

### 3.2. Approximation and passing to limit

We first regularize initial data as $\theta_0^\epsilon = \rho \ast \theta_0$ where $\rho$ is a standard mollifier. We then regularize the equation by putting the Laplacian with the coefficient $\epsilon$:

$$\theta_t^\epsilon + (\mathcal{H} \theta^\epsilon) \theta^\epsilon + \delta \theta^\epsilon \Lambda \theta^\epsilon + \Lambda^\gamma \theta^\epsilon = e \theta_{\text{ext}}^\epsilon. \quad (3.6)$$

For the proof of the existence of a global-in-time smooth solution, see [18] (section 6). Moreover, $(\theta^\epsilon)$ satisfies that

$$\|\theta^\epsilon(t)\|_{L^1} + \|\theta^\epsilon(t)\|^2_{H^{\frac{1}{2}}} + 2 \int_0^t \|\nabla^2 \theta^\epsilon(s)\|^2_{H^{\frac{1}{2}}} ds + \epsilon \|\nabla \theta^\epsilon\|^2_{H^{\frac{1}{2}}} \leq \|\theta_0\|_{L^1} + C(1 + t) \|\theta_0\|^2_{H^{\frac{1}{2}}}.$$

Therefore, $(\theta_\epsilon)$ is bounded in $A_T$ uniformly in $\epsilon > 0$. From this, we have uniform bounds

$$\mathcal{H} \theta^\epsilon \in L^4(0, T; L^4), \quad \theta^\epsilon \in L^2(0, T; L^4)$$

and hence

$$(\mathcal{H} \theta^\epsilon)^\epsilon \in L^\frac{1}{2}(0, T; H^{-1}).$$

Moreover,

$$\Lambda^\gamma \theta^\epsilon + e \theta_{\text{ext}}^\epsilon \in L^2(0, T; H^{-1}).$$

To estimate $\theta^\epsilon \Lambda^\gamma$, we use the duality argument. For any $\chi \in L^2(0, T; H^2)$,

$$\|\langle \theta^\epsilon \Lambda^\gamma, \chi \rangle\| \leq \int |\hat{\theta}^\epsilon \Lambda^\gamma(\xi) \hat{\chi}(\xi)| d\xi \leq \int |\hat{\theta}^\epsilon(\xi - \eta)| |\eta| |\hat{\chi}(\eta)| |\hat{\gamma}(\xi)| d\eta d\xi$$

$$\leq \int |\hat{\theta}^\epsilon(\xi - \eta)| \left( |\eta|^\frac{1}{2} \left( |\xi - \eta|^\frac{1}{2} + |\xi|^\frac{1}{2} \right) \right) |\hat{\chi}(\eta)| |\hat{\gamma}(\xi)| d\eta d\xi$$

$$= \int |\xi - \eta|^\frac{1}{2} |\hat{\theta}^\epsilon(\xi - \eta)| |\eta|^\frac{1}{2} |\hat{\chi}(\eta)| |\hat{\gamma}(\xi)| d\eta d\xi$$

$$+ \int |\xi|^\frac{1}{2} |\hat{\theta}^\epsilon(\xi - \eta)| |\eta|^\frac{1}{2} |\hat{\chi}(\eta)| |\hat{\gamma}(\xi)| d\eta d\xi$$

$$\leq \|\Lambda^\frac{1}{2} \theta^\epsilon\|^2_{L^2} \|\nabla\|_{L^2} + \|\Lambda^\frac{1}{2} \theta^\epsilon\|_{L^2} \|\nabla\|_{L^2} \|\Lambda^\frac{1}{2} \chi\|_{L^2} \leq C \|(1 + \Lambda)^{\frac{3}{2} + \frac{3}{2}} \theta^\epsilon\|^2_{L^2} \|\chi\|_{H^2}.$$

Since $(1 + \Lambda)^{\frac{3}{2} + \frac{3}{2}} \theta^\epsilon \in L^4(0, T; L^4)$ uniformly in $\epsilon > 0$, we have

$$\int \|\langle \theta^\epsilon \Lambda^\gamma, \chi \rangle\| d\tau \leq C \|(1 + \Lambda)^{\frac{3}{2} + \frac{3}{2}} \theta^\epsilon\|^2_{L^2} \|\chi\|_{L^2}.$$
This implies that $\theta^\epsilon \Delta \theta^\epsilon \in L^2 (0,T; H^{-2})$. So, we conclude that from the equation of $\theta^\epsilon$

$$\theta^\epsilon \in L^2 (0,T; H^{-2})$$.

We now extract a subsequence of $(\theta^\epsilon)$, using the same index $\epsilon$ for simplicity, and a function $\theta \in A_T$ such that

$$\theta^\epsilon \to \theta \quad \text{in} \quad L^\infty \left(0,T; L^p \cap H^{1} \right) \quad \text{for all} \ p \in (1, \infty),$$

$$\theta^\epsilon \to \theta \quad \text{in} \quad L^2 \left(0,T; H^{2} \right),$$

$$\theta^\epsilon \to \theta \quad \text{in} \quad L^2 \left(0,T; H^{\frac{1}{2}} \right),$$

$$\theta^\epsilon \to \theta \quad \text{in} \quad L^2 (0,T; L^p_{\text{loc}}) \quad \text{for all} \ p \in (1, \infty)$$

(3.7)

where we use lemma 2.5 for the strong convergence.

We now multiply (3.6) by a test function $\psi \in C_c^\infty ([0,T) \times \mathbb{R})$ and integrate over $\mathbb{R}$. Then,

$$\int_0^T \int \left[ \theta^\epsilon \psi_t + (\mathcal{H}\theta^\epsilon) \theta^\epsilon \psi_x + (1 - \delta)\Lambda \theta^\epsilon \theta^\epsilon \psi - \theta^\epsilon \Lambda \gamma \psi + e\theta^\epsilon \psi_{xx} \right] dx dt = \int \theta^\epsilon_0 (x) \psi (0,x) dx,$$

which can be rewritten as

$$\int_0^T \int \left[ \theta^\epsilon \psi_t + (\mathcal{H}\theta^\epsilon) \theta^\epsilon \psi_x - \theta^\epsilon \Lambda \gamma \psi + e\theta^\epsilon \psi_{xx} \right] dx dt = \int \theta^\epsilon_0 (x) \psi (0,x) dx$$

$$= - (1 - \delta) \int_0^T \int \Lambda \frac{\theta^\epsilon}{t} \theta^{\frac{1}{2}} \Lambda \frac{\psi}{x} \theta^\epsilon \psi_{xx} dx dt - (1 - \delta) \int_0^T \int \Lambda \frac{\theta^\epsilon}{t} \theta^\epsilon \psi_{xx}^2 dx dt.$$

(3.8)

By lemma 2.5 with

$$X_0 = L^2 \left(0,T; H^{\frac{1}{2}} \right), \quad X = L^2 \left(0,T; L^2_{\text{loc}} \right), \quad X_1 = L^2 \left(0,T; H^{-2} \right),$$

we can pass to the limit to I. Moreover, since

$$\left[ \Lambda \frac{\theta^\epsilon}{t} \theta^\epsilon \to \left[ \Lambda \frac{\theta}{t} \theta \right] \right.$$

strongly in $L^2 (0,T; L^6)$ by lemma 2.2 and $\Lambda \frac{\theta^\epsilon}{t} \theta^\epsilon$ converges weakly in $L^2 \left(0,T; L^2 \right)$ by (3.7), we can pass to the limit to II. Lastly, lemma 2.5 with

$$X_0 = L^2 \left(0,T; H^{\frac{1}{2}} \right), \quad X = L^2 \left(0,T; H^{\frac{1}{2}}_{\text{loc}} \right), \quad X_1 = L^2 \left(0,T; H^{-2} \right),$$

allows to pass to the limit to III. Combining all the limits together, we obtain that

$$\int_0^T \int \left[ \theta \psi_t + (\mathcal{H}\theta) \theta \psi_x + (1 - \delta)\Lambda \theta \theta \psi_{xx} \right] dx dt = \int \theta_0 (x) \psi (0,x) dx.$$

(3.9)

3.3. Uniqueness when $\gamma = 1$

To show the uniqueness of a weak solution, let $\theta = \theta_1 - \theta_2$. Then, $\theta$ satisfies the following equation:
\[ \theta_t + \Lambda \theta = -(\mathcal{H} \theta) \theta_x - (\mathcal{H} \theta_x) \theta_x - \delta \theta \Lambda \theta_t - \delta \theta_x \Lambda \theta_x - \delta \theta \Lambda \theta_x - \delta \theta_x \Lambda \theta_x, \quad \theta(0, x) = 0. \]  

We multiply \( \theta \) to (3.10) and integrate over \( \mathbb{R} \). Then,

\[ \frac{1}{2} \frac{d}{dt} \| \theta \|_{L^2}^2 + \left\| \Lambda^{1/2} \theta \right\|_{L^2}^2 = \int \left[ -(\mathcal{H} \theta) \theta_t - (\mathcal{H} \theta_x) \theta_x - \delta \theta \Lambda \theta_t - \delta \theta_x \Lambda \theta_x - \delta \theta \Lambda \theta_x - \delta \theta_x \Lambda \theta_x \right] \theta \, dx. \]

The first three terms in the right-hand side are easily bounded by

\[ C \| \theta_x \|_{L^2} \| \theta \|_{L^2} + C \| \theta_{2x} \|_{L^2} \| \theta \|_{L^2}^2. \]

Moreover, the last term is bounded by using lemma 2.1

\[ -\delta \int \theta_t \theta \Lambda \theta \, dx \leq -\frac{\delta}{2} \int \theta_x \Lambda \theta_x \, dx = \frac{\delta}{2} \int \theta_x \Lambda \theta_x \, dx \leq C \| \theta_x \|_{L^2} \| \theta \|_{L^2}^2. \]

Hence we derive that

\[ \frac{d}{dt} \| \theta \|_{L^2}^2 + \left\| \Lambda^{1/2} \theta \right\|_{L^2}^2 \leq C \| \theta_x \|_{L^2} \| \theta \|_{L^2} + C \| \theta_{2x} \|_{L^2} \| \theta \|_{L^2}^2 \]

\[ \leq C \left( \| \theta_x \|_{L^2}^2 + \| \theta_{2x} \|_{L^2}^2 \right) \| \theta \|_{L^2}^2 + \frac{1}{2} \left\| \Lambda^{1/2} \theta \right\|_{L^2}^2. \]

Since

\[ \theta_x \in L^2 \left( 0, T : L^2 \right), \quad \theta_{2x} \in L^2 \left( 0, T : L^2 \right) \]

when \( \gamma = 1 \), we conclude that \( \theta = 0 \) in \( L^2 \) and thus a weak solution is unique. This completes the proof of theorem 1.1.

4. Proof of theorem 1.2

4.1. A priori estimate

We consider the equation

\[ \theta_t + (\mathcal{H} \theta) \theta_x + \delta \theta \Lambda \theta + \Lambda \theta = 0. \]  

(4.1)

Since (4.1) satisfies the minimum and maximum principles, we have

\[ \theta(t, x) \geq 0, \quad \| \theta(t) \|_{L^\infty} \leq \| \theta_0 \|_{L^\infty}. \]

We begin with the \( L^2 \) bound. For notational simplicity, we suppress the dependence of \( \beta \). We multiply (4.1) by \( \beta \theta \) and integrate in \( x \). Then,

\[ \frac{1}{2} \frac{d}{dt} \int \left( \left\| \theta \right\|_{L^2(\eta, \infty)}^2 + \left\| \Lambda^{1/2} \theta \right\|_{L^2(\eta, \infty)}^2 \right) \, dx = -\int (\mathcal{H} \theta) \theta \theta \theta \, dx - \int \theta (\Lambda \theta) \theta \theta \, dx - \int \Lambda^{1/2} \theta \left( \Lambda^{1/2} \theta \right) \theta \, dx \]

\[ = \frac{1}{2} \int (\mathcal{H} \theta) (\theta^2) \, dx - \frac{1}{2} \int \theta (\Lambda \theta) \theta \, dx \]

\[ - \int \Lambda^{1/2} \theta \left( \Lambda^{1/2} \theta \right) \theta \, dx \]

\[ \leq C \left( \| \theta_0 \|_{L^\infty, \delta} \right) \left( \| \sqrt{\theta} \|_{L^2} \| \theta \|_{L^2} + \| \sqrt{\theta} \|_{L^2}^2 \right) + \int \frac{C \left( \| \theta_0 \|_{L^\infty, \delta} \right)}{\sqrt{\theta}} \left( \| \Lambda^{1/2} \theta \|_{L^2} \right) \, dx. \]

Then, using lemma 2.3 one obtains, for \( C \left( \| \theta_0 \|_{L^\infty, \delta} \right) = C_1 = (1 + \delta) \| \theta_0 \|_{L^\infty} \) for all \( \eta > 0 \) and
\[
\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{L^2(\text{vdx})} + \| A^{1/2} \theta \|^2_{L^2(\text{vdx})} \leq \left( \| \theta \|_{L^2(\text{vdx})} \| \Lambda \theta \|_{L^2} + \| \theta \|^2_{L^2(\text{vdx})} \right) C_1 + \| \theta \|^2_{L^2(\text{vdx})} \leq \left( \frac{C_1}{2\eta_1} + C_1 + 1 \right) \| \theta \|^2_{L^2(\text{vdx})} + \frac{C_1 \eta_1}{2} \| \Lambda \theta \|^2_{L^2}.
\]

In particular,
\[
\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{L^2(\text{vdx})} \leq \left( \frac{C_1}{2\eta_1} + C_1 + 1 \right) \| \theta \|^2_{L^2(\text{vdx})} + \frac{C_1 \eta_1}{2} \| \Lambda \theta \|^2_{L^2}.
\]  

(4.2)

We next multiply (4.1) by \( A^{1/2} (w A^{1/2} \theta) \) and integrate in \( x \). We first focus one the term \( \theta \Lambda \theta \), the resulting term in the computation of the evolution of the \( \dot{H}^{1/2}(\text{vdx}) \) norm is

\[
T_\delta \theta = \delta \int A^{1/2} (w A^{1/2} \theta) \Lambda \theta \, dx
= \delta \int \theta \sqrt{w} A \theta \left( \frac{1}{\sqrt{w}} \left[ A^{1/2}, w \right] A^{1/2} \theta \right) \, dx + \int \theta w |\Lambda \theta|^2 \, dx
\leq \delta \| \theta_0 \|_{L^\infty} \left( \| \theta \|_{\dot{H}^{1/2}(\text{vdx})} \| \theta \|_{H^{1/2}(\text{vdx})} + \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})} \right)
\]

hence
\[
| T_\delta \theta | \leq \left( \frac{\eta_2}{2} + 1 \right) C_2 \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})} + \frac{2C_2}{\eta_2} \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})}.
\]  

(4.3)

where \( C_2 = \delta \| \theta_0 \|_{L^\infty} \).

For the transport part (which is the case \( \delta = 0 \)), we only need to use inequality 4.4 of [18], indeed, it is shown that, for some constants \( C_3 > 0, C_4 > 0 \) one has
\[
\frac{1}{2} \frac{d}{dt} \int \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})} \, dx \leq (C_3 \| \theta_0 \|_{L^\infty} - 1) \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})}
+ C_4 \left( \int \theta^2 \, dx + \int |A^{1/2} \theta|^2 \, w \, dx \right).
\]  

(4.4)

Hence, we get the following control for the full equation (4.1) by collecting the estimates (4.2)–(4.5)
\[
\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})} \leq \left( C_3 \| \theta_0 \|_{L^\infty} + \left( \frac{\eta_2}{2} + 1 \right) C_1 + \frac{C_1 \eta_1}{2} - 1 \right) \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})}
+ (C_4 + \frac{2C_2}{\eta_2}) \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})}.
\]  

(4.5)

Hence, choosing \( \eta_1, \eta_2 \) and \( \| \theta_0 \|_{L^\infty} \) small enough (for instance less than \( \frac{1}{1 - \frac{1}{100}} \) so that \( C_1 \) would be also small), one gets
\[
\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})} \, dx + C_5 \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})} \leq C_7 \| \theta \|^2_{\dot{H}^{1/2}(\text{vdx})}.
\]  

(4.6)

Hence, integrating in time \( s \in [0, T] \) and using Gronwall’s inequality one obtains that for all finite \( T > 0 \)
\[
\| \theta(T) \|^2_{\dot{H}^{1/2}(\text{vdx})} + \| \theta \|^2_{L^2([0,T],\dot{H}^{1/2}(\text{vdx}))} \leq \| \theta_0 \|^2_{\dot{H}^{1/2}(\text{vdx})} \exp \left( C_2 \| \theta_0 \|_{L^\infty} T \right).
\]  

(4.7)
4.2. Approximation and passing to limit

To show the existence of a weak solution in $\mathcal{D}_T$, we first approximate the initial data $\theta_0$. Let $\chi$ be a smooth positive function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Let $\chi_R(x) = \chi(x/N), N \in \mathbb{N}$, and consider truncated initial data $\theta^N_0(x) = \theta_0(x)\chi_N(x)$. Then, a direct computation shows that

$$
\lim_{N \to \infty} \|\theta^N_0 - \theta_0\|_{H^1(dx)} = 0.
$$

Moreover, this truncation does not alter the non-negativity and does not increase the $L^\infty$ norm. So, if $\|\theta_0\|_{L^\infty}$ is sufficiently small, there is a global-in-time solution of

$$
\partial_t \theta^N + \mathcal{H} \partial_x \theta^N + \delta \partial_x \Lambda \theta^N + \Lambda \theta^N = 0, \quad \theta^N(0,x) = \theta^N_0(x). \quad (4.8)
$$

From the a priori estimates, the sequence $(\theta^N)$ is bounded in $L^\infty([0, T], H^1(dx)) \cap L^2([0, T], H^1(dx))$ uniformly with respect to $N$. We now take a test function $\psi \in C^\infty_c([0, T] \times \mathbb{R})$. Then, $\psi \theta^N$ is bounded in $L^2([0, T], H^1)$. Moreover, since $\theta^N \in L^\infty([0, T] \times \mathbb{R})$

$$
\psi (\mathcal{H} \theta^N \partial_x \theta^N + \delta \partial_x \Lambda \theta^N + \Lambda \theta^N) = (\psi \mathcal{H} \theta^N \partial_x \theta^N + (\delta - 1) \psi \theta^N \Lambda \theta^N + \psi \Lambda \theta^N) \in L^2([0, T], H^{-1}).
$$

By lemma 2.5, we can pass to the limit in the weak formulation,

$$
\int_0^T \int \left[ \theta^N \psi_x + (\mathcal{H} \theta^N) \theta^N \psi_x + (1 - \delta) \Lambda \theta^N \theta^N \psi - \theta^N \Lambda \psi \right] \, dx \, dt = \int_0^T \psi(0,x) \, dx,
$$

to obtain a weak solution $\theta$ which is also in $L^\infty([0, T], H^1(dx)) \cap L^2([0, T], H^1(dx))$.

4.3. Uniqueness

To show the uniqueness of a weak solution, we consider the equation of $\theta = \theta_1 - \theta_2$ given by

$$
\partial_t \theta + \Lambda \theta = - (\mathcal{H} \theta) \theta_1 - (\mathcal{H} \theta_2) \theta_2 - \delta \partial \Lambda \theta_1 - \delta \partial \Lambda \theta_2, \quad \theta(0,x) = 0. \quad (4.9)
$$

We multiply $\theta$ by (4.9) and integrate over $\mathbb{R}$. Then,

$$
\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2(dx)} + \|\Lambda \theta\|^2_{L^2(dx)} = \int \left[ - (\mathcal{H} \theta) \theta_1 - (\mathcal{H} \theta_2) \theta_2 - \delta \partial \Lambda \theta_1 - \delta \partial \Lambda \theta_2 \right] \theta \, dx
$$

$$
- \int \Lambda \theta \left[ \Lambda \theta, w \right] \, dx.
$$

As before, the last term is bounded by

$$
\int \Lambda \theta \left[ \Lambda \theta, w \right] \, dx \leq \frac{1}{2} \|\Lambda \theta\|^2_{L^2(dx)} + C \|\theta\|^2_{L^2(dx)}.
$$

The first three terms in the right-hand side are easily bounded by

$$
C \left( \|\theta_1\|^2_{L^1(dx)} + \|\theta_2\|^2_{L^1(dx)} + \|\theta_2\|^2_{L^1(dx)} \right) \|\theta\|^2_{L^1(dx)}.
$$
Moreover, since $\delta > 0$, $\theta_2 \geq 0$ and $w \geq 0$, the fourth term is bounded by using lemma 2.1

$$- \delta \int \theta_2 \Lambda \theta w \, dx \leq - \frac{\delta}{2} \int \theta_2 w \Lambda \theta^2 \, dx = - \frac{\delta}{2} \int \theta^2 \Lambda (\theta_2 w) \, dx = \frac{\delta}{2} \int \mathcal{H}(\theta^2) (\theta_2 w) \, dx$$

$$\leq C \left( \|\theta_2 w\|_{L^2(w \, dx)} + \|\theta_2\|_{L^2(w \, dx)} \right) \|\theta\|^2_{L^2(w \, dx)}.$$ 

Hence we obtain that

$$\frac{d}{dt} \|\theta\|^2_{L^2(w \, dx)} + \|\Lambda^\frac{1}{2} \theta\|^2_{L^2(w \, dx)}$$

$$\leq C \left( \|\theta_{1x}\|_{L^2(w \, dx)} + \|\theta_{2x}\|_{L^2(w \, dx)} + \|\theta_2\|_{L^2(w \, dx)} \right) \|\theta\|^2_{L^2(w \, dx)} + C \|\theta\|^2_{L^2(w \, dx)}$$

$$\leq C \left( 1 + \|\theta_{1x}\|^2_{L^2(w \, dx)} + \|\theta_{2x}\|^2_{L^2(w \, dx)} + \|\theta_2\|^2_{L^2(w \, dx)} \right) \|\theta\|^2_{L^2(w \, dx)} + \frac{1}{2} \|\Lambda^\frac{1}{2} \theta\|^2_{L^2(w \, dx)},$$

where we use (2.9) to obtain the last inequality. Since

$$\theta_{1x} \in L^2(0, T ; L^2(w \, dx)), \quad \theta_{2x} \in L^2(0, T ; H^1(w \, dx)),$$

we conclude that $\theta = 0$ in $L^2(w \, dx)$ and thus a weak solution is unique. This completes the proof of theorem 1.2.

In the two next subsections we shall prove a global existence theorem and a local existence theorem, for respectively the subcritical case and the supercritical case), we shall just focus on the \textit{a priori} estimates since the construction by compactness is classical (see [17]).

5. Proof of theorem 1.3

5.1. A priori estimate

Since the case $\delta = 0$ has been already done in [17], one observes that it suffices to estimate the $L^2(w \, dx)$ part

$$T_1 = \delta \int w_{\beta,k} \theta^2 \Lambda \theta \, dx,$$

and the weighted homogeneous Sobolev part, that is

$$T_2 = \delta \int w_{\beta,k} \Lambda^\frac{1}{2} \theta \Lambda^\frac{1}{2} (\Lambda \theta) \, dx.$$ 

We shall control $T_1$ and $T_2$, for the sake of readability, we shall just write $w$ in stead of $w_{\beta,k}$. We first control the $L^2$ part

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2(w \, dx)} + \|\Lambda^\frac{1}{2} \theta\|^2_{L^2(w \, dx)}$$

$$= - \frac{1}{2} \int (\mathcal{H}(\theta)) (\theta^2) \, w \, dx - \delta \int \theta^2 \Lambda \theta \, w \, dx - \int \Lambda^\frac{1}{2} \theta \Lambda^\frac{1}{2} [\Lambda^\frac{1}{2}, w] \theta \, dx$$

$$= \left( \frac{1}{2} - \delta \right) \int \theta^2 \Lambda \theta \, w \, dx + \frac{1}{2} \int (\mathcal{H}(\theta)) \theta^2 w \, dx - \int \Lambda^\frac{1}{2} \theta \Lambda^\frac{1}{2} [\Lambda^\frac{1}{2}, w] \theta \, dx$$

$$\leq C(\|\theta\|_{L^\infty}, \delta) \left( \|\sqrt{w} \theta\|_{L^2} \|\sqrt{w} \Lambda \theta\|_{L^2} + \|\sqrt{w} \theta\|^2_{L^2} \right) + \int \sqrt{w} \Lambda^\frac{1}{2} \theta \frac{1}{\sqrt{w}} \left( \|\Lambda^\frac{1}{2}, w\| \theta \right) \, dx.$$
Then, using lemma 2.3 one obtains, for $C(\|\theta_0\|_{L^\infty}, \delta) = C_1 = (1 + \delta)\|\theta_0\|_{L^\infty}$ for all $\eta_1 > 0$ and

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2(\eta_1 dt)} + \|\Lambda^2 \theta\|^2_{L^2(\eta_1 dt)} \leq \left( \|\theta\|_{L^2(\eta_1 dt)} \|\Lambda \theta\|_{L^2(\eta_1 dt)} + \|\theta\|^2_{L^2(\eta_1 dt)} \right) C_1 + \|\theta\|_{L^2(\eta_1 dt)} \|\theta\|_{\dot{H}^{s/2}(\eta_1 dt)}$$

$$\leq C_2 \|\theta\|^2_{\dot{H}^s(\eta_1 dt)} + \frac{1}{100} \|\theta\|^2_{\dot{H}^{s/2}(\eta_1 dt)}.$$ 

Now, we study the contribution coming from the homogeneous Sobolev part $\dot{H}^1(w)$. The transport part (corresponding to $\delta = 0$) has been treated in [17], indeed, it is shown that

$$\frac{1}{2} \partial_t \|\theta\|^2_{\dot{H}^1(\eta_1 dt)} \leq C \|\theta\|^2_{\dot{H}^1(\eta_1 dt)}.$$ 

Therefore, it just remains to estimate

$$I_2 \theta = -\delta \int w \Lambda \theta (\theta \Lambda \theta) \, dx.$$ 

In order to take advantage of the dissipation, we start by rewriting $I_2$ as follows,

$$I_2 \theta = -\delta \int \Lambda^s (w \Lambda \theta) \Lambda^{1-s} (\theta \Lambda \theta) \, dx$$

$$= - \int \sqrt{w} \Lambda^{1-s} (\theta \Lambda \theta) \frac{1}{\sqrt{w}} [\Lambda^s, w] \Lambda \theta \, dx - \int \sqrt{w} \Lambda^{1-s} (\theta \Lambda \theta) \, dx = I_1 + I_2.$$

In order to estimate these two terms, we shall use the weighted Littlewood–Paley decomposition. We start with $I_1$, we have using the commutator lemma 2.3 that

$$|I_1| \leq \|\theta \Lambda \theta\|_{\dot{H}^{s-\nu}(\eta_1 dt)} \|\theta\|_{\dot{H}^1(\eta_1 dt)},$$

and then we estimate the right hand side by using the paraproduct formula (19) and we get

$$\|\theta \Lambda \theta\|_{\dot{H}^{s-\nu}(\eta_1 dt)} \leq \sum_{q \in \mathbb{Z}} 2^{q(1-\mu)} \|S_{q+1} \theta \Delta_q \mathcal{H} \theta_q\|_{L^2(\eta_1 dt)} + \sum_{j \in \mathbb{Z}} 2^{j(1-\mu)} \|\Delta_j S_q \mathcal{H} \theta_q\|_{L^2(\eta_1 dt)}.$$ 

The first sum is controlled as follows

$$\sum_{q \in \mathbb{Z}} 2^{q(1-\mu)} \|S_{q+1} \theta \Delta_q \mathcal{H} \theta_q\|_{L^2(\eta_1 dt)} \leq \sum_{q \in \mathbb{Z}} 2^{(1-\mu)q} \|S_{q+1} \theta\|_{L^\infty} \|\Delta_q \mathcal{H} \theta_q\|_{L^2(\eta_1 dt)}$$

$$\leq \|\theta\|_{L^\infty} \sum_{q \in \mathbb{Z}} 2^{(1-\mu)q} \|\Delta_q \theta_q\|_{L^2(\eta_1 dt)}$$

$$\leq \|\theta\|_{L^\infty} \|\theta_0\|_{\dot{H}^{s-\nu}(\eta_1 dt)}$$

$$\leq \|\theta\|_{L^\infty} \|\theta_0\|_{\dot{H}^{s-\nu}(\eta_1 dt)}$$

where we used Berstein’s inequality and the continuity of the Hilbert transform on $L^2(\eta_1 dt)$.

The other sum has to be treated in a different manner since we do not want to put $L^\infty$ in the term $S_j \mathcal{H} \theta_q$. For instance, we can use Hölder’s inequality and then Bernstein’s inequality to recover some sufficiently nice Sobolev norm. We obtain,
\[
\sum_{j \in \mathbb{Z}} 2^{(1-\mu)} \| \Delta \theta S_j \mathcal{H} \theta_s \|_{L^2(w)} \leq \sum_{j \in \mathbb{Z}} 2^{(1-\mu)} \| \Delta \theta \|_{L^p(w)} \| S_j \mathcal{H} \theta_s \|_{L^q(w)} \\
\leq C \sum_{j \in \mathbb{Z}} 2^{(1-\mu)} 2^{j(\frac{1}{2} - \frac{1}{p})} \| \Delta \theta \|_{L^2(w)} \| \mathcal{H} \theta_s \|_{L^2(w)} \\
\leq C \sum_{j \in \mathbb{Z}} 2^{(1-\mu)} 2^{j(\frac{1}{2} - \frac{1}{p})} \| \Delta \theta \|_{L^2(w)} 2^j \| \mathcal{H} \theta_s \|_{L^2(w)} \\
\leq C \| \theta \|_{L^p(w)} \sum_{j \in \mathbb{Z}} 2^{j(2-\mu + \frac{k}{2})} \| \Delta \theta \|_{L^2(w)} \\
\leq C \| \theta \|_{L^p(w)} \| \theta \|_{H^{2-\mu + \frac{k}{2}}(w)} \| \theta \|_{H^{2-\mu + \frac{k}{2}}(w)}.
\]

Hence,
\[
|I_1| \leq \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2-\mu}(w)} \\
\leq \| \theta_0 \|_{L^\infty} \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2-\mu}(w)} + C \| \theta \|_{H^{2-\mu}(w)} \| \theta_0 \|_{L^\infty} \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2-\mu}(w)}.
\]

For \(I_2\), following \(I_1\), one gets
\[
|I_2| \leq \| \theta_0 \|_{L^\infty} \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2-\mu}(w)} + \| \theta_0 \|_{L^\infty} \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2-\mu}(w)}.
\]

The idea is then to choose \(q\) very small. Indeed, the associated Lebesgue space \(L^q\) would be then a good substitute for \(L^\infty\). It is worth recalling that this latter space is not well suited to estimate \(\mathcal{H} \theta\) since it does not map \(L^\infty\) to \(L^\infty\) but \(L^\infty\) to \(BMO\).

Then, one has to choose \(\mu\) close enough to \(1/2\) because we control \(H^{1+\frac{k}{2}}(w)\) with \(\alpha \in (1, 2)\). A good choice is to take for instance \(\mu = \frac{1}{2} + \frac{\epsilon}{2}\), this choice allows us to interpolate and conclude the estimates. Indeed, for all \(\delta \in (0, 1)\)
\[
\| \theta \|_{H^{2-\mu + \frac{k}{2}}(w)} \leq \| \theta \|_{H^{2+\frac{1}{2}-\mu - \delta}(w)} \| \theta \|_{H^{2+\frac{1}{2}+\mu - \delta}(w)}.
\]

Then, one observes that by choosing \(\delta\) very close to 1 (essentially for the worst term, which is the second estimate), we get
\[
\| \theta \|_{H^{2+\frac{1}{2}-\mu - \delta}(w)} = \| \theta \|_{H^{2+\frac{1}{2}-\mu - \delta}(w)} \leq \| \theta \|_{H^{2-\mu + \frac{k}{2}}(w)}
\]
and if we choose \(\delta = 1 - \epsilon\), with \(\epsilon > 0\) is chosen small enough so that \(\frac{1}{2} + \epsilon \leq \frac{\alpha}{2}\), this is possible since \(\alpha \in (1, 2)\).
\[
\| \theta \|_{H^{2+\frac{1}{2}+\mu - \delta}(w)} = \| \theta \|_{H^{2+\frac{1}{2}+\mu - \delta}(w)} \leq \| \theta \|_{H^{2+\frac{1}{2}+\mu - \delta}(w)}
\]
hence,
\[
\| \theta \|_{H^{2+\frac{1}{2}+\mu - \delta}(w)} \leq \| \theta \|_{H^{2-\mu + \frac{k}{2}}(w)} \| \theta \|_{H^{2+\frac{1}{2}+\mu - \delta}(w)}. \tag{5.1}
\]

Therefore, since \(\frac{1}{2} + \frac{1}{2} \leq \frac{\alpha}{2}\), one obtains
\[
|I_1| \leq \| \theta \|_{H^{2+\frac{k}{2}}(w)} \| \theta \|_{H^{2+\frac{k}{2}}(w)} + \| \theta \|_{L^2(w)} \| \theta \|_{H^{2-\mu}(w)} \| \theta \|_{H^{2+\frac{k}{2}}(w)}. \tag{5.2}
\]
Then, using Young’s inequality with $p_1 = \frac{2 - \nu}{2 - \nu}$ and its conjugate $p_2 = \frac{2 - \nu}{3 - \nu}$, with $0 < \nu < \delta$ where $\nu$ will be chosen later. One gets, for all $\epsilon_1 > 0$

$$|I_1| \lesssim \frac{\epsilon_1}{2} \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} + \frac{1}{2\epsilon_1} \| \theta \|^2_{H^{\frac{1}{2} - \nu}(\text{w}dx)} + \frac{1}{2\epsilon_1} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^{\frac{1}{2} - \nu}(\text{w}dx)} + \frac{\epsilon_1}{2} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)}.$$ 

And once again, we use Young’s inequality with $p = \frac{2}{2 - \nu}$ hence $q = \frac{2}{\nu}$

$$|I_1| \lesssim \frac{\epsilon_1}{2} \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} + \frac{1}{2\epsilon_1} \| \theta \|^2_{H^{\frac{1}{2} - \nu}(\text{w}dx)} + C_1(\delta, \nu) \| \theta \|^2_{L^2(\text{w}dx)} + C_2(\delta, \nu) \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} + \frac{\epsilon_1}{2} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)}.$$ 

Since $q$ is chosen large, then $\frac{\delta}{2 - \nu} 2^\frac{\nu}{2} \leq 2$, we also have that $(2 - \nu) \frac{\nu}{2} \leq 2$ (this is a consequence of $(\nu - 2)^2 \geq 0$). We can choose any value of $\nu \in (0, \delta)$ so that $\frac{\delta}{2 - \nu} (2 - \nu) \leq 2$. For instance, if $\nu = 1/2$ then we obviously have $\frac{\delta}{2 - \nu} \leq 4/3$ for all $\delta \in (0, 1)$ (importantly, there is no restriction on $\delta$, the previous estimates hold for all $\delta \in (0, 1)$, we shall use this fact to control the second term, see (5.4) below).

One finally obtains

$$|I_1| \lesssim \frac{\epsilon_1}{2} \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} + \frac{2}{\epsilon_1} \| \theta \|^2_{H^\frac{1}{2}(\text{w}dx)}.$$ 

For $I_2$, we have seen that

$$|I_2| \lesssim \| \theta_0 \|_{L^\infty} \| \theta \|_{H^{\frac{1}{2} + \nu}(\text{w}dx)} \| \theta \|_{H^{\frac{1}{2} - \nu}(\text{w}dx)} + \| \theta_0 \|^2_{L^\infty} \| \theta \|^2_{H^{\frac{1}{2} + \nu}(\text{w}dx)} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^{\frac{1}{2} - \nu}(\text{w}dx)}.$$ 

Since $\mu = \frac{1}{2} + \frac{\nu}{2}$ and $q$ is such that $q = \frac{1}{2} + \frac{\nu}{2} \leq 2$, then by using (5.1), the latter inequality becomes

$$|I_2| \lesssim \| \theta_0 \|_{L^\infty} \| \theta \|_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \| \theta \|_{H^{\frac{1}{2} - \nu}(\text{w}dx)} + \| \theta_0 \|^2_{L^\infty} \| \theta \|_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^{\frac{1}{2} - \nu}(\text{w}dx)}$$

$$\lesssim \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \| \theta \|^2_{H^\frac{1}{2}(\text{w}dx)} + \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^\frac{1}{2}(\text{w}dx)}.$$ 

where, in the last step, we used the following interpolation inequality

$$\| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \lesssim \| \theta \|_{L^\infty} \| \theta \|^2_{H^{\frac{1}{2}}(\text{w}dx)}.$$ 

Then, we use Young’s inequality with $p_3 = 2q$ and its conjugate $p_4 = 2q - 1$ in the first product

$$\| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \| \theta \|^2_{L^2(\text{w}dx)} \lesssim \epsilon_1 \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} + \frac{1}{\epsilon_1} \| \theta \|^2_{H^\frac{1}{2}(\text{w}dx)}.$$ 

it remains to estimate

$$L = \| \theta \|_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^\frac{1}{2}(\text{w}dx)}.$$ 

but $\delta = 1 - \epsilon$, so that

$$L = \| \theta \|^2_{H^{\frac{1}{2} + \frac{\nu}{2}}(\text{w}dx)} \| \theta \|^2_{L^2(\text{w}dx)} \| \theta \|^2_{H^\frac{1}{2}(\text{w}dx)}.$$ 

(5.4)
but, we have already controlled the same term with \( \delta > 0 \) in stead of \( \epsilon > 0 \) (see the second term in the right hand side of (5.2)), therefore following the same steps, one arrives at the same conclusion as in \( I_1 \) that is to say, for \( i = 1, 2 \), one has

\[
|I| \lesssim \epsilon_1 \|\theta\|_{H^{1/6}(\omega;\text{dx})}^2 + \frac{2}{\epsilon_1} \|\theta\|_{\dot{H}^{1/3}(\omega;\text{dx})}^2. \tag{5.5}
\]

Hence, to conclude it suffices to choose \( \epsilon_1 \) sufficiently small.

\[\square\]

6. Proof of theorem 1.4

The control of the \( L^2(\omega) \) norm is straightforward, indeed, using the lemma 2.3 and the weighted Sobolev embedding \( H^{1/6}(\omega) \hookrightarrow L^2(\omega) \), one obtains

\[
\partial_t \int \frac{\theta^2}{2} \, d\omega + \int |\Lambda^{\alpha/2} \theta|^2 \, d\omega = -\int \theta^2 \partial_\omega \theta \, d\omega + \int \partial_t \theta \partial_\omega \theta \, d\omega + \frac{\delta}{2} \int w \theta^2 \Lambda \theta \, d\omega + \int \Lambda^{\alpha/2} \theta \theta \, d\omega
\]

\[
\leq \|\theta\|_{H^{1/6}(\omega;\text{dx})}^2 + \|\theta\|_{H^{1/3}(\omega;\text{dx})}^2 + \|\theta\|_{H^{1/3}(\omega;\text{dx})}^2 \lesssim \|\theta\|^3_{H^1(\omega;\text{dx})}.
\]

The control of the homogeneous part \( \dot{H}^2(\omega;\text{dx}) \) is done as follows.

\[
\frac{1}{2} \partial_t \|\theta\|^2_{\dot{H}^2(\omega;\text{dx})} + \int |\Lambda^{\alpha/2} \theta|^2 \, d\omega = -\frac{1}{2} \int \theta \partial_\omega \theta \, d\omega - \frac{3}{2} \int \theta \, d\omega \Lambda \theta \, d\omega + \int w \theta \Lambda \theta \, d\omega - \delta \int \theta \Lambda \theta \, d\omega - \delta \int \theta \Lambda \theta \, d\omega - \frac{3}{2} \int \Lambda \theta \theta \, d\omega - \frac{3}{2} \int \Lambda \theta \theta \, d\omega - \frac{3}{2} \int \Lambda \theta \theta \, d\omega
\]

We already know from [17] that

\[
\sum_{j=1}^{4} I_j \leq \frac{1}{\epsilon_1} \|\theta\|^4_{\dot{H}^1(\omega;\text{dx})} + \epsilon_1 \|\theta\|_{H^1(\omega;\text{dx})}^4 + \epsilon_1 \|\theta\|_{\dot{H}^{1/3}(\omega;\text{dx})}^4
\]

where we crucially used the first point of lemma 2.3 to estimate \( I_4 \). Futhermore, one observes that \( I_3 \) and \( I_6 \) are the same as respectively \( I_2 \) and \( I_1 \), hence

\[
\sum_{j=1}^{6} I_j \leq \frac{1}{\epsilon_1} \|\theta\|^4_{\dot{H}^1(\omega;\text{dx})} + \epsilon_1 \|\theta\|_{H^1(\omega;\text{dx})}^4 + \epsilon_1 \|\theta\|_{\dot{H}^{1/3}(\omega;\text{dx})}^4. \tag{6.1}
\]

The most singular term is \( I_7 \). Since \( \delta \geq 0 \) and \( \theta \geq 0 \) we may use the pointwise inequality lemma 2.1 and then by integrating by parts we get

\[
I_7 = -\delta \int w \theta \Lambda \theta \, d\omega
\]

\[
\leq -\delta \int w \Lambda (\theta \theta) \, d\omega
\]

\[
= -\delta \int \Lambda (w \theta) (\theta \theta) \, d\omega.
\]
At this step, it is important to note that one cannot take advantage of the $L^\infty$ maximum principle, indeed, this would imply to estimate the remaning term in $L^1(w)$ and it is well-known that the Hilbert transform is not continuous on that space. The idea here is to write the term $\Lambda(w \theta)$ as a controlled commutator (via lemma 2.3) plus another term which is easy to control since the weight will be outside the differential term. More precisely, we write

$$I_7 = -\delta \int \theta_{xx}^2 \Lambda \theta \, dx - \delta \int w(\theta_{xx})^2 \Lambda \theta \, dx = I_{7,1} + I_{7,2}.$$ 

To estimate $I_{7,1}$ we use Hölder’s inequality with $\frac{2}{3} + \frac{1}{2} = \frac{5}{3}$, and the two following weighted Sobolev embeddings $H^{\alpha/2}(w \, dx) \hookrightarrow L^{\frac{2}{1+\alpha}}(w \, dx)$ and $H^{\frac{1}{3}}(w \, dx) \hookrightarrow L^{\frac{3}{5}}(w \, dx)$, and the last point of lemma 2.3, one gets

$$|I_{7,1}| \leq \|\theta_{xx}\|_{L^2(w \, dx)} \|\theta_{xx}\|_{L^{\frac{2}{1+\alpha}}(w \, dx)} \left\| \frac{1}{w} \Lambda w \theta \right\|_{L^{\frac{3}{5}}(w \, dx)} \leq \|\theta\|_{H^1(w \, dx)} \|\omega_{xx}\|_{H^{\alpha/2}(w \, dx)} \left\| \frac{1}{w} \Lambda w \theta \right\|_{L^{\frac{3}{5}}(w \, dx)}.$$

Therefore, for all $\epsilon_1 > 0$,

$$|I_{7,2}| \leq \|w\omega_{xx}\|_{L^2(w \, dx)} \|w^{\frac{1}{2}} \omega_{xx}\|_{L^{\frac{2}{1+\alpha}}(w \, dx)} \left\| \frac{1}{w} \Lambda w \theta \right\|_{L^{\frac{3}{5}}(w \, dx)} \leq \|\theta\|_{H^1(w \, dx)} \|\theta\|_{H^{\frac{1}{3}}(w \, dx)} \left\| \frac{1}{w} \Lambda w \theta \right\|_{L^{\frac{3}{5}}(w \, dx)} \leq \frac{1}{\epsilon_1} \|\theta\|_{H^1(w \, dx)}^4 + \epsilon_1 \|\theta\|_{H^{\frac{1}{3}}(w \, dx)}^2.$$

Therefore,

$$|I_7| \leq \frac{2}{\epsilon_1} \|\theta\|_{H^1(w \, dx)}^4 + 2\epsilon_1 \|\theta\|_{H^{\frac{1}{3}}(w \, dx)}^2.$$ 

Combining (6.3) and (6.2) we conclude that

$$\sum_{j=1}^{4} I_j \lesssim \frac{1}{\epsilon_1} \|\theta\|_{H^2(w \, dx)}^4 + \epsilon_1 \|\theta\|_{H^2(w \, dx)}^2 + \epsilon_1 \|\theta\|_{H^{\frac{1}{3}}(w \, dx)}^2.$$ 

hence, by choosing $\epsilon_1 > 0$ sufficiently small, one finds

$$\frac{1}{2} \partial_t \|\theta\|_{H^{\frac{1}{3}}(w \, dx)}^2 + C \int |\Lambda^{\frac{2}{3}} \theta|^2 \, w \, dx \lesssim \|\theta\|_{H^2(w \, dx)}^4 + \|\theta\|_{H^{\frac{1}{3}}(w \, dx)}^4.$$ 

By using Grönwall’s inequality, and the classical truncation and compactness arguments (see [17] for instance), one gets the theorem.
7. Proof of theorem 1.5

7.1. A priori estimates

Taking the Fourier transform of (1.2), we have that
\[
\hat{\partial_t|\hat{\theta}(\xi)|} = \frac{\hat{\theta}(\xi)\hat{\theta}(\xi) + \hat{\theta}(\xi)\hat{\theta}(\xi)}{2|\hat{\theta}(\xi)|} = \Re \left( \frac{\hat{\theta}(\xi)\hat{\theta}(\xi)}{|\hat{\theta}(\xi)|} \right) = -\Re \left[ \frac{-i\xi}{|\xi|} \hat{\theta}(\xi)\hat{\theta}(\xi) \right] = -\hat{\Re} \left[ \frac{-i\xi}{|\xi|} \hat{\theta}(\xi)\hat{\theta}(\xi) \right].
\]

Consequently,
\[
\frac{d}{dt} \|\theta\|_{L^\infty} \leq (1 + |\delta|) \int \|\hat{\theta}(\xi)|\xi - \zeta\| |\hat{\theta}(\xi)| d\xi \frac{1}{1 + |\delta|} - \|\theta\|_{L^2}
\]
\[
\leq (1 + |\delta|) \int \|\hat{\theta}(\xi)|\xi - \zeta\| |\hat{\theta}(\xi)| d\xi \frac{1}{1 + |\delta|} - \|\theta\|_{L^2} \leq (1 + |\delta|)\frac{1}{|\delta|^2} - 1 \|\theta\|_{L^2}.
\]

Thus, if \( \theta_0 \) satisfies the condition (1.6), we have
\[
\|\theta(t)\|_{L^\infty} + \left( 1 - \frac{(1 + |\delta|)\|\theta_0\|_{L^\infty}}{\|\theta\|_{L^2}} \right) \int_0^\infty \|\theta\|_{L^2} d\xi \leq \|\theta_0\|_{L^\infty}.
\]

Similarly,
\[
\hat{\partial_t|\hat{\theta}(\xi)|} = -\Re \left[ \int \frac{-i\xi}{|\xi|} \hat{\theta}(\xi)(i\xi - \zeta) \hat{\theta}(\xi - \zeta) d\xi \right] = -\delta \int \frac{i\xi}{|\xi|} \hat{\theta}(\xi)(i\xi - \zeta) \hat{\theta}(\xi - \zeta) d\xi \hat{\theta}(\xi - \zeta)
\]
\[
\hat{\partial_t|\hat{\theta}(\xi)|} = -\Re \left[ \int \frac{-i\xi}{|\xi|} \hat{\theta}(\xi)(i\xi - \zeta) \hat{\theta}(\xi - \zeta) d\xi \hat{\theta}(\xi - \zeta)
\]
\[
\leq (1 + |\delta|)\frac{1}{|\delta|^2} - 1 \|\theta\|_{L^2}.
\]

Thus, using (2.5), we have that
\[
\frac{d}{dt} \|\theta\|_{L^\infty} \leq (1 + |\delta|)\frac{1}{|\delta|^2} - 1 \|\theta\|_{L^2} \leq (2(1 + |\delta|)\|\theta_0\|_{L^\infty}^2 - 1) \|\theta\|_{L^2}.
\]

As a consequence, we obtain that, if \( \theta_0 \) satisfies the condition (1.6), we also have
\[
\|\theta(t)\|_{L^\infty} + \left( 1 - \frac{2(1 + |\delta|)\|\theta_0\|_{L^\infty}}{\|\theta\|_{L^2}} \right) \int_0^\infty \|\theta\|_{L^2} d\xi \leq \|\theta_0\|_{L^\infty}.
\]

By (7.1) and (7.2), we conclude that
\[
\|\theta(t)\|_{L^\infty} + \left( 1 - \frac{2(1 + |\delta|)\|\theta_0\|_{L^\infty}}{\|\theta\|_{L^2}} \right) \int_0^t \|\theta\|_{L^2} d\xi \leq \|\theta_0\|_{L^\infty}
\]

for all \( t \geq 0 \).

7.2. Approximation and passing to the limit

Define \( e^{\epsilon \hat{\partial_t}} \) the heat semigroup, i.e.
\[
e^{\epsilon \hat{\partial_t}} f(x) = e^{-\epsilon \xi^2} f(\xi),
\]
and \( g_\varepsilon(x) = e^{-\varepsilon^2 x^2} \). Note that
\[
\tilde{g}_\varepsilon(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\zeta^2}{2}}, \quad \|\tilde{g}_\varepsilon\|_{L^1} = \sqrt{2\pi}.
\]
Given \( \theta_0(x) \in A^1 \), we consider \( \theta_0^\varepsilon(x) = g_\varepsilon(x) e^{\varepsilon^2 \theta_0}(x) \). As \( \theta_0(x) \) is a bounded function, we have that \( \theta_0^\varepsilon \) is infinitely smooth and has finite total mass:
\[
\|\theta_0^\varepsilon\|_{L^1} \leq \|g_\varepsilon\|_{L^1} \left\| e^{\varepsilon^2 \theta_0} \right\|_{L^\infty} \leq \frac{\sqrt{2\pi}}{\varepsilon} \|\theta_0\|_{L^\infty}.
\]
Furthermore, using Young’s inequality and the definition of \( g_\varepsilon \),
\[
\|\theta_0^\varepsilon\|_{L^1} = \frac{1}{\sqrt{2\pi}} \left\| g_\varepsilon * (e^{-\epsilon^2 \hat{\theta}_0}) \right\|_{L^1} \leq \left\| e^{-\epsilon^2 \hat{\theta}_0} \right\|_{L^1} \leq \|\theta_0\|_{A^1}.
\]
Similarly,
\[
\|\theta_0^\varepsilon\|_{A^1} = \|\partial_\theta \theta_0^\varepsilon\|_{A^1} \leq \|\partial_\theta \theta_0\|_{A^1} + \|\partial_\delta g_\varepsilon e^{\epsilon^2 \theta_0}\|_{A^1} = \|\theta_0\|_{A^1} + c\epsilon \|\theta_0\|_{A^1}.
\]
Now we define the approximated problems
\[
\theta_0^\varepsilon + (H \theta^\varepsilon) \theta_0^\varepsilon + \delta \theta^\varepsilon \Lambda \theta^\varepsilon + \Lambda \theta^\varepsilon = \epsilon \sqrt{\delta} \theta^\varepsilon,
\] (7.4)
with finite energy approximated initial data \( \theta_0^\varepsilon \). These problems have unique smooth solutions denoted by \( \theta^\varepsilon \). Moreover, \( (\theta^\varepsilon) \) satisfies a uniform bound
\[
\|\theta^\varepsilon(t)\|_{A^1} \leq \left( 1 - \frac{2(1 + |\delta|)}{\sqrt{2\pi}} \right) \int_0^t \|\theta^\varepsilon(s)\|_{A^1} \, ds + \epsilon \int_0^t \|\theta^\varepsilon(s)\|_{A^1} \, ds \leq \|\theta_0\|_{A^1},
\]
uniformly in \( \epsilon \). Thus, the \textit{a priori} estimates lead to the following uniform-in-\( \epsilon \) bounds
\[
\sup_{\epsilon \in [0, \infty)} \left\| \theta^\varepsilon(t) \right\|_{C^2(R)} \leq \sup_{\epsilon \in [0, \infty)} \left\| \theta^\varepsilon(t) \right\|_{A^1} \leq \|\theta_0\|_{A^1} < \frac{\sqrt{2\pi}}{1 + |\delta|},
\]
\[
\sup_{\epsilon \in [0, \infty)} \left\| \theta^\varepsilon(t) \right\|_{C^2(R)} \leq \sup_{\epsilon \in [0, \infty)} \left\| \theta^\varepsilon(t) \right\|_{A^1} \leq \|\theta_0\|_{A^1} + c\epsilon \|\theta_0\|_{A^1},
\]
\[
\sup_{\epsilon \in [0, \infty)} \left\| \Lambda \theta^\varepsilon(t) \right\|_{C^2(R)} \leq \|\theta_0\|_{A^1} + c\epsilon \|\theta_0\|_{A^1}.
\] (7.5)
Moreover, from the equation (7.4) we also obtain uniform bounds
\[
\sup_{\epsilon \in [0, \infty)} \left\| \partial_\theta \theta^\varepsilon(t) \right\|_{C^2(R)} + \sup_{\epsilon \in [0, \infty)} \left\| \partial_\delta H \theta^\varepsilon(t) \right\|_{C^2(R)} \leq 2 \sup_{\epsilon \in [0, \infty)} \left\| \partial_\theta \theta^\varepsilon(t) \right\|_{A^1} \leq F_1(\|\theta_0\|_{A^1}, \delta),
\]
\[
\left\| H \theta^\varepsilon \right\|_{C^2([0, \infty) \times R)} + \left\| \theta^\varepsilon \right\|_{C^2([0, \infty) \times R)} \leq F_2(\|\theta_0\|_{A^1}, \delta)
\] (7.6)
where \( F_1 \) and \( F_2 \) only depend on the quantities in the right-hand side of (7.5). Due to Banach–Alaoglu theorem, there exists a subsequence (denoted by \( \epsilon \)) and a limit function \( \theta \in W^{1,\infty}(0, \infty) \) such that
\[
\theta^\varepsilon \rightharpoonup^* \theta \quad \text{in} \quad L^\infty(0, T; W^{1,\infty}), \quad \Lambda \theta^\varepsilon \rightharpoonup^* \Lambda \theta \quad \text{in} \quad L^\infty(0, T; L^\infty)
\]
for all \( T > 0 \). Using lemma 2.6, we have the following strong convergence
\[
\lim_{\epsilon \to 0} \|\theta^\varepsilon - \theta\|_{L^\infty(K)} + \|H \theta^\varepsilon - H \theta\|_{L^\infty(K)} = 0
\]
for $K = [0,T] \times [-R,R]$ for any $R > 0$. Since $L^\infty(K) \subset L^2(K)$, we can pass to the limit to the weak formulation
\[
\int_0^T \left[ \phi' \psi_t + (H\phi') \phi' \psi_x + (1 - \delta) \Lambda \phi' \phi' \psi - \phi' \Lambda \psi + \epsilon \phi' \psi_{xx} \right] \, dx \, dt = \int \delta_0(x) \psi(0,x) \, dx
\]
to obtain a weak solution $\theta$ satisfying
\[
\theta \in W_T, \quad \sup_{t \in [0,T]} \|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}}{\sqrt{\pi}} \|\theta_0\|_{A^0}\right) \int_0^T \|\theta_x(t)\|_{A^1} \, dt \leq \|\theta_0\|_{A^1}
\]
for all $T > 0$.

7.3. Uniqueness

To show the uniqueness of a weak solution, we consider the equation of $\theta = \theta_1 - \theta_2$ given by
\[
\theta_t + \Lambda \theta = -(H\theta) \theta_1 + (H\theta_2) \theta_x - \delta \Lambda \theta_1 - \delta \theta_2 \Lambda \theta, \quad \theta(0,x) = 0. \tag{7.7}
\]
Taking the Fourier transform of (7.7) and multiply by $\hat{\theta} |\hat{\theta}|$, we have
\[
\frac{d}{dt} \|\theta\|_{A^0} + \|\theta\|_{A^1} \leq \frac{1 + |\delta|}{\sqrt{2\pi}} \|\theta_1\|_{A^1} \|\theta\|_{A^0} + \frac{1 + |\delta|}{\sqrt{2\pi}} \|\theta_2\|_{A^0} \|\theta\|_{A^1}. \tag{7.8}
\]
Since
\[
\frac{(1 + |\delta|)}{\sqrt{2\pi}} \|\theta_2\|_{A^0} < 1, \quad \|\theta_1(t)\|_{A^1} \in L^1([0,\infty)),
\]
we have $\theta(t,x) = 0$ in $A^0$ for all time. This implies that a weak solution is unique.

8. Proof of theorem 1.6

8.1. A priori estimate

We consider the equation
\[
\theta_t + u \theta_x + \Lambda^\gamma \theta = 0, \quad u = (1 - \partial_{xx})^{-\alpha} \theta. \tag{8.1}
\]
Since (8.1) satisfies the maximum principle, we have
\[
\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.
\]
We also obtain that
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \|\Lambda \tilde{\theta}\|_{L^2}^2 = - \int u \theta \theta_t \, dx \leq \|\theta\|_{L^\infty} \|u_x\|_{L^2} \|\theta\|_{L^2}
\]
\[
\leq C \left( \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2 \right) \|\theta\|_{L^2}^2 + \frac{1}{2} \|\Lambda \tilde{\theta}\|_{L^2}^2
\]
where we use the condition $\alpha = \frac{1}{2} - \frac{\gamma}{2}$ to bound $u_x$ as
\[
\|u_x\|_{L^2} \leq C \left( \|\theta\|_{L^2} + \|\Lambda \tilde{\theta}\|_{L^2} \right).
\]
Therefore, we obtain that
\[ \|\theta(t)\|_{L^2}^2 + \int_0^t \left\| \Lambda^s \theta(s) \right\|_{L^2}^2 \, ds \leq \|\theta_0\|_{L^2}^2 e^{C(1+\|\theta_0\|_{L^2}^2)} t. \]

8.2. Approximation and passing to limit

We consider the following equation with regularized initial data:

\[ \theta'' + u' \theta' + \Lambda^s \theta = 0, \quad \theta_0 = \rho_c \ast \theta_0. \]  

Then, there exists a global-in-time smooth solution \( \theta'' \). Moreover, \( \theta'' \) satisfies that

\[ \|\theta''(t)\|_{L^\infty} + \|\theta''(t)\|_{L^2}^2 + \int_0^t \left\| \Lambda^s \theta'(s) \right\|_{L^2}^2 \, ds + \epsilon \|\nabla \theta''\|_{L^2}^2 \leq \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^2}^2 e^{C(1+\|\theta_0\|_{L^2}^2)} t. \]  

This implies the uniform bounds

\[ u'' \in L^4(0,T;L^4), \quad \theta'' \in L^4(0,T;L^4), \quad u'' \in L^2(0,T;L^2). \]

Moreover, these bounds with the equation

\[ \theta'' = -u' \theta' - \Lambda^s \theta + \epsilon \theta'' \]

we also have that

\[ \theta'' \in L^2(0,T;H^{-2}). \]

We now extract a subsequence of \( \theta'' \), using the same index \( \epsilon \) for simplicity, and a function \( \theta \in C_T \) such that

\[ \theta'' \rightharpoonup \theta \quad \text{in} \quad L^\infty(0,T;L^p) \quad \text{for all} \quad p \in (1, \infty) \]

\[ \theta'' \to \theta \quad \text{in} \quad L^2(0,T;H^2) \]

\[ \theta'' \to \theta \quad \text{in} \quad L^2(0,T;L^p_{\text{loc}}) \quad \text{for all} \quad p \in (1, \infty), \]

\[ u'' \to u, \quad \text{in} \quad L^2(0,T;L^2). \]  

where we use lemma 2.5 to obtain the strong convergence.

We now multiply (8.2) by a test function \( \psi \in C^\infty_c([0,T] \times \mathbb{R}) \) and integrate over \( \mathbb{R} \). Then,

\[ \int_0^T \int \left[ \theta'' \psi_t + u' \theta' \psi_x + u'' \psi + \theta'' \Lambda^s \psi + \epsilon \theta'' \psi_{xx} \right] \, dx \, dt = \int \theta_0(x) \psi(0,x) \, dx. \]

By lemma 2.5 with

\[ X_0 = L^2(0,T;H^2), \quad X = L^2(0,T;L^2_{\text{loc}}), \quad X_1 = L^2(0,T;H^{-2}) \]

and using (8.4), we can pass to the limit to obtain

\[ \int_0^T \int \left[ \theta \psi_t + u \theta \psi_x + u\psi + \theta \Lambda^s \psi \right] \, dx \, dt = \int \theta_0(x) \psi(0,x) \, dx. \]  

This completes the proof.
8.3. Uniqueness

To show the uniqueness of a weak solution, we consider the equation of \( \theta = \theta_1 - \theta_2 \) given by

\[
\partial_t + \Lambda^\gamma \theta = -u_1 \partial_x + u \partial_x, \quad \theta(0,x) = 0.
\]  

(8.6)

We multiply (8.6) by \( \theta \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{L^2} + \| \Lambda^\frac{1}{2} \theta \|^2_{L^2} = - \int u_1 \theta \theta dx + \int u \partial_x \theta dx = \frac{1}{2} \int u_1 \theta^2 dx + \int u \partial_x \theta dx
\]

\[
\leq C \| u_1 \|_{L^2} \| \theta \|_{L^\infty} + C \| u \|_{L^2} \| \theta \|_{L^2} + C \| \theta \|_{H^{\frac{1}{2}}} \| \partial_x \theta \|_{L^2} + C \| \theta \|_{H^{\frac{1}{2}}} \| \partial_x \theta \|_{L^2}
\]

\[
\leq C (\| u_1 \|_{L^2} + \| u \|_{L^2}) \| \theta \|^2_{L^2}
\]

where we use \( \gamma > 1 \) to control \( \| \theta \|_{L^\infty} \). Since \( \gamma > 1 \) and \( \alpha > \frac{1}{2} \geq \frac{1}{2} - \frac{1}{2} \), we also have

\[
\| u_1 \|_{L^2} \leq C \| \theta \|_{H^{\frac{1}{2}}}, \quad \| u \|_{L^2} \leq C \| \theta \|_{L^2}, \quad \| \theta \|_{H^{\frac{1}{2}}} \leq C \| \theta \|_{H^{\frac{1}{2}}}
\]

Hence

\[
\| \theta \|^2_{L^2} + \| \Lambda^\frac{1}{2} \theta \|^2_{L^2} \leq C \left( 1 + \| \theta \|^2_{H^{\frac{1}{2}}} + \| \theta \|^2_{H^{\frac{1}{2}}} \right) \| \theta \|^2_{L^2}
\]

Since the quantities in the parenthesis are integrable in time and \( \theta(0,x) = 0 \), we conclude that \( \theta = 0 \) in \( L^2 \) and thus a weak solution is unique. This completes the proof of theorem 1.6.

9. Proof of theorem 1.7

9.1. A priori estimate

We consider the equation

\[
\partial_t + u \partial_x + \Lambda \theta = 0, \quad u = (1 - \partial_x^2)^{-\frac{1}{2}} \theta.
\]

(9.1)

We multiply (9.1) by \( \theta \) and integrate in \( x \). Then,

\[
\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{L^2_{(\text{weak})}} + \| \Lambda^\frac{1}{2} \theta \|^2_{L^2_{(\text{weak})}} = - \int u \theta \theta dx - \int \Lambda^\frac{1}{2} \theta \left[ \Lambda^\frac{1}{2}, \theta \right] dx
\]

\[
= \frac{1}{2} \int u \theta^2 dx + \frac{1}{2} \int u \phi^2 dx
\]

As in the proof of theorem 1.2, we bound the commutator term as

\[
\int \Lambda^\frac{1}{2} \theta \left[ \Lambda^\frac{1}{2}, \theta \right] dx \leq \frac{1}{4} \| \Lambda^\frac{1}{2} \theta \|^2_{L^2_{(\text{weak})}} + C \| \theta \|^2_{L^2_{(\text{weak})}}.
\]

To estimate terms involving \( u \), we use \( \theta = (1 - \partial_x^2)^{\frac{1}{2}} u \) and (2.8) to obtain that

\[
\int u \theta^2 dx + \int u \theta^2 \theta dx \leq C \| \theta \|_{L^\infty} \left( \| u \|_{L^2_{(\text{weak})}} + \| u \|_{L^2_{(\text{weak})}} \right) \| \theta \|^2_{L^2_{(\text{weak})}}
\]

\[
\leq C \| \theta \|_{L^\infty} \left( \| \theta \|^2_{L^2_{(\text{weak})}} + \| \Lambda^\frac{1}{2} \theta \|^2_{L^2_{(\text{weak})}} \right) \| \theta \|^2_{L^2_{(\text{weak})}}
\]

\[
\leq C (\| \theta \|_{L^\infty} + \| \theta \|^2_{L^2_{(\text{weak})}}) \| \theta \|^2_{L^2_{(\text{weak})}} + \frac{1}{4} \| \Lambda^\frac{1}{2} \theta \|^2_{L^2_{(\text{weak})}}.
\]
Collecting all terms together, we obtain that
\[
\frac{d}{dt} \left\| \theta \right\|_{L^2(wdx)}^2 + \left\| \Lambda^2 \theta \right\|_{L^2(wdx)}^2 \leq C \left( \left\| \theta_0 \right\|_{L^\infty} + \left\| \theta_0 \right\|_{L^\infty} \right) \left\| \theta \right\|_{L^2(wdx)}^2
\]
and hence that
\[
\left\| \theta(t) \right\|_{L^2(wdx)}^2 + \int_0^t \left\| \Lambda^2 \theta(s) \right\|_{L^2(wdx)}^2 ds \leq \left\| \theta_0 \right\|_{L^2(wdx)}^2 \exp \left[ C \left( \left\| \theta_0 \right\|_{L^\infty} + \left\| \theta_0 \right\|_{L^\infty} \right) t \right] \cdot (9.2)
\]

We next multiply \((9.1)\) by \(-\left( \theta, w \right)\_x\), and integrate in \(x\). Then,
\[
\frac{1}{2} \frac{d}{dt} \left\| \theta_x \right\|_{L^2(wdx)}^2 + \left\| \Lambda^2 \theta \right\|_{H^1(wdx)}^2 = \int u \theta_x \left( \theta, w \right)_x dx + \int \Lambda^2 \theta \left[ \Lambda^2 \theta, w \right] \Lambda \theta dx
\]
\[
= -\frac{1}{2} \int u \lambda_\theta \left( \theta, w \right)_x dx + \frac{1}{2} \int u \lambda_\theta \left( \theta, w \right)_x^2 dx - \int \Lambda^2 \theta \left[ \Lambda^2 \theta, w \right] \theta dx + \int \Lambda \theta \theta_x dx.
\]

Following the computation in [18],
\[
\frac{d}{dt} \left\| \theta \right\|_{H^1(wdx)}^2 + \left\| \Lambda^2 \theta \right\|_{H^1(wdx)}^2 \leq C \left\| u \right\|_{H^1(wdx)} \left\| \theta \right\|_{H^1(wdx)}^2 + C \left\| \theta \right\|_{L^2(wdx)}^2 + \frac{1}{2} \left\| \Lambda^2 \theta \right\|_{H^1(wdx)}^2
\]
\[
\leq C \left\| \theta \right\|_{H^1(wdx)}^2 \left\| \theta \right\|_{L^2(wdx)}^2 + C \left\| \theta \right\|_{L^2(wdx)}^2 + \frac{1}{2} \left\| \Lambda^2 \theta \right\|_{L^2(wdx)}^2,
\]
where we use the relation in \((2.8)\) to bound \(u\) in terms of \(\theta\). Since
\[
\left\| \theta \right\|_{H^1(wdx)}^2 \left\| \theta \right\|_{L^2(wdx)} \left\| \Lambda^2 \theta \right\|_{L^2(wdx)} \leq C \left\| \theta \right\|_{H^1(wdx)} \left\| \theta \right\|_{L^2(wdx)} \left\| \Lambda^2 \theta \right\|_{L^2(wdx)}
\]
\[
\leq C \left\| \theta \right\|_{H^1(wdx)}^2 \left\| \theta \right\|_{L^2(wdx)}^2 + \frac{1}{2} \left\| \Lambda^2 \theta \right\|_{L^2(wdx)}^2
\]
by \((2.9)\), we obtain that
\[
\frac{d}{dt} \left\| \theta \right\|_{L^2(wdx)}^2 + \left\| \Lambda^2 \theta \right\|_{L^2(wdx)}^2 \leq C \left( 1 + \left\| \theta \right\|_{H^1(wdx)}^2 \right) \left\| \theta \right\|_{L^2(wdx)}^2 \cdot (9.4)
\]

Integrating in time \((9.4)\) and using \((9.2)\), we obtain that
\[
\left\| \theta \right\|_{L^2(wdx)}^2 + \int_0^t \left\| \Lambda^2 \theta(s) \right\|_{L^2(wdx)}^2 ds \leq \left\| \theta_0 \right\|_{L^2(wdx)}^2 \exp \left[ \int_0^t C \left( 1 + \left\| \theta(s) \right\|_{H^1(wdx)}^2 \right) ds \right]
\]
\[
\leq \left\| \theta_0 \right\|_{L^2(wdx)}^2 \exp \left[ Ct + \left\| \theta_0 \right\|_{L^2(wdx)}^2 \exp \left[ C \left( \left\| \theta_0 \right\|_{L^\infty} + \left\| \theta_0 \right\|_{L^\infty} \right) t \right] \right] \cdot (9.5)
\]

By \((9.2)\) and \((9.5)\), we finally obtain that
\[
\left\| \theta \right\|_{H^1(wdx)}^2 + \int_0^t \left\| \Lambda^2 \theta(s) \right\|_{H^1(wdx)}^2 ds \leq C \left\| \theta_0 \right\|_{H^1(wdx)}^2 \exp \left[ Ct + \left\| \theta_0 \right\|_{L^2(wdx)}^2 \exp \left[ C \left( \left\| \theta_0 \right\|_{L^\infty} + \left\| \theta_0 \right\|_{L^\infty} \right) t \right] \right] \cdot (9.6)
\]
9.2. Approximation and passing to limit

Since θ is more regular than a solution in theorem 1.2, we can follow the procedure in the proof of theorem 1.2.

9.3. Uniqueness

To show the uniqueness of a weak solution, we consider the equation of \( \theta = \theta_1 - \theta_2 \) given by

\[
\theta_t + \Lambda \theta = -u_1 \theta_x + u \theta_{2x}, \quad \theta(0, x) = 0.
\]

(9.7)

We multiply \( w \theta \) to (9.7) and integrate over \( \mathbb{R} \). Then,

\[
\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{L^2(\mathbb{R})} + \| \Lambda^{\frac{1}{2}} \theta \|^2_{L^2(\mathbb{R})} = \int [-u_1 \theta_x + u \theta_{2x}] w \theta \, dx \quad \left( \int \Lambda^{\frac{1}{2}} \theta \left( \Lambda^{\frac{1}{2}} w, \right) \theta \, dx \right).
\]

As before, the last term is bounded by

\[
\int \Lambda^{\frac{1}{2}} \theta \left( \Lambda^{\frac{1}{2}} w, \right) \theta \, dx \leq \frac{1}{2} \| \Lambda^{\frac{1}{2}} \theta \|^2_{L^2(\mathbb{R})} + C \| \theta \|^2_{L^2(\mathbb{R})}.
\]

The first three terms in the right-hand side are easily bounded by

\[
C \left( \| \theta_2 \|_{L^2(\mathbb{R})} + \| \theta_1 \|_{H^1(\mathbb{R})} \right) \left( \| \theta \|^2_{L^2(\mathbb{R})} + \| \theta \|^2_{H^1(\mathbb{R})} \right).
\]

By (2.9), we obtain that

\[
\frac{d}{dt} \| \theta \|^2_{L^2(\mathbb{R})} + \| \Lambda^{\frac{1}{2}} \theta \|^2_{L^2(\mathbb{R})} \leq C \left( 1 + \| \theta_2 \|^2_{L^2(\mathbb{R})} + \| \theta_1 \|^2_{H^1(\mathbb{R})} \right) \| \theta \|^2_{H^1(\mathbb{R})} + \frac{1}{2} \| \Lambda^{\frac{1}{2}} \theta \|^2_{L^2(\mathbb{R})}.
\]

Since

\[
\theta_2 \in L^2 \left( 0, T : H^1(\mathbb{R}) \right), \quad \theta_1 \in L^2 \left( 0, T : H^1(\mathbb{R}) \right),
\]

we conclude that \( \theta = 0 \) in \( L^2(\mathbb{R}) \) and thus a weak solution is unique. This completes the proof of theorem 1.7.

10. Proof of theorem 1.8

Taking the Fourier transform of (1.7), we have that

\[
\partial_t \hat{\theta} (\xi) = -\text{Re} \left[ \int \frac{1}{1 + |\xi|^2} \hat{\phi} (\xi - \zeta) \hat{\phi} (\xi - \zeta) d\zeta \right] \frac{1}{\sqrt{2\pi}} - |\xi| |\hat{\theta}|.
\]

Consequently, ignoring the factor \( \frac{1}{(1 + |\xi|^2)^{\alpha}} \), we follow the proof of theorem 1.5 with \( \delta = 0 \) and the smallness condition (1.8) to obtain that

\[
\| \theta (t) \|_{L^2} + \left( 1 - \sqrt{2} \frac{|\theta_0|_{L^2}}{\sqrt{\pi}} \right) \int_0^t \| \theta (s) \|_{L^2} \, ds \leq \| \theta_0 \|_{L^2}
\]

(10.1)
for all $t \geq 0$. We also follow the proof of theorem 1.5 to obtain a unique weak solution via the approximation procedure. This completes the proof.

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