Anomalous dimensions
in $\mathcal{N}=4$ SYM theory at order $g^4$

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Abstract

We compute four-point correlation functions of scalar composite operators in the $\mathcal{N}=4$ supercurrent multiplet at order $g^4$ using the $\mathcal{N}=1$ superfield formalism. We confirm the interpretation of short-distance logarithmic behaviours in terms of anomalous dimensions of unprotected operators exchanged in the intermediate channels and we determine the two-loop contribution to the anomalous dimension of the $\mathcal{N}=4$ Konishi supermultiplet.

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1 Introduction and summary of the results

The conjectured AdS/CFT correspondence \[1, 2, 3, 4\] has renewed the interest in conformal field theories (CFT's), in particular $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) in $D=4$, and has raised the issue of finding to what extent high order perturbative computations are feasible in the weak coupling regime.

The interest is twofold. On the one hand, given the explicit analytic expressions of certain four-point amplitudes in the AdS context, one may ask whether it is possible to recognize some systematic pattern in the perturbative $\mathcal{N}=4$ field-theoretic results. On the other hand, insights gained in such finite theories may prove to be useful in theories with (partial) supersymmetry breaking.

In CFT the dependence of two- and three-point functions of scalar primary operators on the external insertion points is completely fixed by conformal symmetry. For protected operators non-renormalisation theorems hold that prevent quantum corrections to the lowest order contribution. Explicit computations have confirmed the validity of these theorems in the context of the AdS/CFT correspondence \[5, 6, 7\]. Except for extremal \[8, 9, 10\] and sub-extremal correlators \[11\], no such non-renormalisation theorems are believed to hold for higher-point functions.

A particularly interesting class of higher-point functions are the four-point functions of the lowest chiral primary operators, i.e. dimension-two gauge-invariant scalar composites in the $\mathcal{N}=4$ supercurrent multiplet. Perturbative computations involving such operators have been performed at order $g^2$ in \[12, 13\]. Motivated by the presence of string corrections to the AdS effective action \[14, 15\], instanton computations have been carried out in \[16, 17, 18, 13\]. All such computations show logarithmic behaviours at short-distances that allow an interpretation in terms of anomalous dimensions of unprotected operators in long supermultiplets exchanged in intermediate channels \[13\]. Similar analyses have been performed in the AdS context in the supergravity limit \[19\]. A detailed comparison of the two classes of results is problematic because of the different regimes in the $(g^2, N)$ parameter space that are explored by the two approaches. Nevertheless, explicit AdS computations in the scalar sector that are amenable to a perturbative analysis in the field-theoretic weak coupling regime, such as the one presented here, have been recently completed in \[20\] in the supergravity approximation with the help of the results of \[21\].

In this paper, we compute at order $g^4$ four-point correlation functions of the lowest dimension scalar composite operators in the $\mathcal{N}=4$ supercurrent multiplet using the $\mathcal{N}=1$ superfield formalism. We confirm the interpretation of the short-distance logarithmic behaviour in terms of anomalous dimensions of unprotected operators exchanged in intermediate channels and we extract the two-loop contribution to the anomalous dimension of the $\mathcal{N}=4$ Konishi supermultiplet. In order to illustrate our point we will concentrate in most of our presentation on the simplest of the six independent four-point functions of chiral primary operators \[12, 13\] (see below). Identities that have been proved to hold both in perturbation theory in \[12, 22\] and at the one-instanton level in \[13\] should be enough to determine all four-point functions from the one we consider.

A recent interesting paper by Eden, Schubert and Sokatchev has reported on similar computations at order $g^4$ from a different vantage point and confirmed the validity of the above identities \[23\]. Their approach is based on the less familiar $\mathcal{N}=2$ harmonic
superspace formalism. It is remarkable that, although we adopt a different approach, we find the same result for the four-point function on which we focus our attention, up to an overall constant that was not fixed in \[23\]. We would like to stress that, contrary to expectations, the $\mathcal{N}=1$ superfield approach we have pursued is not prohibitively complicated and it is of wider applicability. In particular we have in mind applications to some interesting finite $\mathcal{N}=1$ superconformal theories that are promising candidates for realistic theories after soft supersymmetry breaking.

The plan of the paper is as follows. In section 2, for the sake of completeness, we will write down the action, propagators and vertices in the $\mathcal{N}=1$ superfield formalism and identify the six independent four-point functions of chiral primary operators. In section 3 we draw the relevant Feynman superdiagrams and sketch the computation for the simplest possible four-point function. In section 4 we discuss the final result and interpret the dominant logarithmic behaviours at short distance in terms of the anomalous dimension of the lowest dimensional operator, $\mathcal{K}_1$, in the $\mathcal{N}=4$ Konishi supermultiplet. We confirm the one-loop results of \[13\] and extract the two-loop contribution\[4\] to the anomalous dimension of $\mathcal{K}_1$. We finally comment on possible extensions of our results to finite $\mathcal{N}=1$ supersymmetric theories that arise after soft breaking of $\mathcal{N}=4$ supersymmetry.

2 $\mathcal{N}=4$ SYM in $\mathcal{N}=1$ superspace

The field content of $\mathcal{N}=4$ SYM \[24\] comprises a vector, $A_\mu$, four Weyl spinors, $\psi^A$ ($A=1,2,3,4$), and six real scalars, $\varphi^i$ ($i=1,2,\ldots,6$), all in the adjoint representation of the gauge group $\mathcal{G}$. Since no off-shell formulation is available that manifestly preserves $\mathcal{N}=4$ supersymmetry, in order to compute quantum corrections one has to resort either to the $\mathcal{N}=2$ harmonic superspace approach pursued in \[12, 23\] or to the more familiar $\mathcal{N}=1$ formalism pursued in \[13\].

In the $\mathcal{N}=1$ formalism the fundamental fields can be arranged into a vector superfield, $V$, and three chiral superfields, $\Phi^I$ ($I=1,2,3$). The six real scalars, $\varphi^i$, are combined into three complex fields, namely

$$\phi^I = \frac{1}{\sqrt{2}} (\varphi^I + i\varphi^{I+3}) \quad \phi^+_I = \frac{1}{\sqrt{2}} (\varphi^I - i\varphi^{I+3}) \quad (1)$$

that are the lowest components of the superfields $\Phi^I$ and $\Phi^+_I$, respectively. Three of the Weyl fermions, $\psi^I$, are the spinors of the chiral multiplets. The fourth spinor, $\lambda = \psi^4$, together with the vector, $A_\mu$, form the vector multiplet. In this formulation only a $SU(3) \times U(1)$ subgroup of the original $SU(4)$ R-symmetry group is manifest. $\Phi^I$ and $\Phi^+_I$ transform in the representations $3$ and $\bar{3}$ of the global $SU(3)$ “flavour” Q-symmetry, while $V$ is a singlet. Under the axial $U(1)$ R-symmetry the vector $A_\mu$ is neutral, the gaugino $\lambda$ has charge $+3/2$, the spinors of the chiral multiplets $\psi^I$ have charge $-1/2$ and the three complex scalars $\phi^I$ have charge $+1$.

\[1\] Notice that we call “$\ell$-loops” calculations that are of order $g^{2\ell}$. The authors of refs. \[12\] and \[23\] dub diagrams of this order as $\ell+1$-loop calculations.
The (Euclidean) action in the $\mathcal{N}=1$ superfield formulation reads

$$
S = -2 \text{tr} \left\{ \int d^4x \left[ \left( \int d^2\theta \frac{1}{16} W^a W_a + \text{h.c.} \right) + \left( \int d^2\theta d^2\bar{\theta} e^{-gV} \Phi^I_1 e^{gV} \Phi^I \right) \right. \\
- \frac{g}{3!\sqrt{2}} \left( \int d^2\theta \varepsilon_{IJK} \Phi^I [\Phi^J, \Phi^K] - \int d^2\bar{\theta} \varepsilon^{IJK} \Phi^I_1 [\Phi^I_1, \Phi^K_1] \right) \right\},
$$

where $W_a$ is the chiral superfield-strength of $V$

$$
W_a = -\frac{1}{4g} \bar{D}^2 \left( e^{-2gV} D_a e^{2gV} \right).
$$

The trace over the colour indices is defined by

$$
\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \quad a, b = 1, \ldots, \dim(\mathcal{G}),
$$

where $T^a$ are the generators in the fundamental representation of the gauge group.

Notice that a gauge fixing term, $S_{gf}$, has to be added to the classical action (2). As usual we decide to take

$$
S_{gf} = \frac{1}{16\alpha} \text{tr} \int d^4x \int d^2\theta d^2\bar{\theta} \left[ (D^2 V) \left( \bar{D}^2 V \right) \right],
$$

where $\alpha$ is a gauge parameter. We will not display here ghost terms since they do not contribute to the Green functions that we will consider at the order we work. The choice $\alpha = 1$ greatly simplifies all the computations, as it makes all corrections to the propagators of the fundamental superfields vanishing at order $g^2$ [25, 26].

Expanding the exponentials $e^{\pm gV}$ in (2) gives

$$
S = \int d^4x \int d^2\theta d^2\bar{\theta} \left\{ V^a \Box V_a - \Phi^I_i \Phi^I_a - ig f_{abc} \Phi^I_i V^b \Phi^I_c + \frac{g^2}{2} f_{abc} f_{ecd} \Phi^I_i V^b V^c \Phi^I_d \\
- \frac{ig}{3!\sqrt{2}} f_{abc} \left[ \varepsilon_{IJK} \Phi^I_i \Phi^J_j \Phi^K_k \delta(\bar{\theta}) - \varepsilon^{IJK} \Phi^I_1 \Phi^J_1 \Phi^K_1 \delta(\bar{\theta}) \right] + \ldots \right\},
$$

where $f_{abc}$ are the structure constants of the gauge group and we have displayed only the terms that are relevant for our subsequent computations.

In the following we will carry on the calculations using $\mathcal{N}=1$ formalism in coordinate superspace, that is more suitable for the study of CFT’s. The superfield propagators in the conventions of eq. (6) are

$$
\langle \Phi^I_i(x_i, \theta_i, \bar{\theta}_i) \Phi^J_j(x_j, \theta_j, \bar{\theta}_j) \rangle = -\frac{\delta^J_I}{4\pi^2} \frac{\delta_{ab}}{e^{i\xi_{ij}} - 2\xi_{ij}} \frac{1}{x^2_{ij}},
$$

$$
\langle V_a(x_i, \theta_i, \bar{\theta}_i) V_b(x_j, \theta_j, \bar{\theta}_j) \rangle = \frac{\delta_{ab}}{8\pi^2} \frac{\delta(\theta_{ij}) \delta(\bar{\theta}_{ij})}{x^2_{ij}},
$$

where

$$
x_{ij} = x_i - x_j \quad \theta_{ij} = \theta_i - \theta_j \quad \xi^\mu_{ij} = \theta_i^\mu \sigma^\mu_{ij} \bar{\theta}_j^\mu.
$$
In the next section we will describe the calculation of the order $g^4$ perturbative correction to the four-point correlation function

\[ G^{(H)}(x_1, x_2, x_3, x_4) = \langle C^{11}(x_1)C^\dagger_{11}(x_2)C^{22}(x_3)C^\dagger_{22}(x_4) \rangle, \tag{10} \]

where the gauge-invariant composite operators

\[ C^{IJ} = \text{tr}(\phi^I\phi^J) \quad C^\dagger_{IJ} = \text{tr}(\phi^\dagger_I\phi^\dagger_J) \tag{11} \]

are the lowest components of the (anti-)chiral superfields

\[ C^{IJ} = \text{tr}(\Phi^I\Phi^J) \quad C^\dagger_{IJ} = \text{tr}(\Phi^\dagger_I\Phi^\dagger_J). \tag{12} \]

In turn $C^{IJ}$ and $C^\dagger_{IJ}$ appear in the decomposition of $Q^{ij}_{20}$, the lowest scalar components of the $\mathcal{N}=4$ current supermultiplet, under $SU(4) \to SU(3) \times U(1)$. The $Q^{ij}_{20}$ belong to the real representation $20$ of $SU(4)$ and are defined as

\[ Q^{ij}_{20} = \text{tr}(\varphi^i\varphi^j - \frac{\delta^{ij}}{6}\varphi^k\varphi^k). \tag{13} \]

The most general four-point function of the $Q^{ij}_{20}$

\[ G^{(Q)}(x_1, x_2, x_3, x_4) = \langle Q^{i_1j_1}(x_1)Q^{i_2j_2}(x_2)Q^{i_3j_3}(x_3)Q^{i_4j_4}(x_4) \rangle \tag{14} \]

can be expressed as a linear combination of $G^{(H)}$, defined in eq. (10), and the following five independent four-point functions

\[
\begin{align*}
G^{(V)}(x_1, x_2, x_3, x_4) &= \langle C^{11}(x_1)C^\dagger_{11}(x_2)C^{11}(x_3)C^\dagger_{11}(x_4) \rangle \tag{15} \\
G^{(1)}(x_1, x_2, x_3, x_4) &= \langle C^{11}(x_1)C^\dagger_{22}(x_2)C^{22}(x_3)C^\dagger_{11}(x_4) \rangle \\
G^{(2)}(x_1, x_2, x_3, x_4) &= \langle C^{11}(x_1)C^\dagger_{22}(x_2)C^{11}(x_3)C^\dagger_{22}(x_4) \rangle \\
G^{(3)}(x_1, x_2, x_3, x_4) &= \langle C^{11}(x_1)C^\dagger_{11}(x_2)C^{11}(x_3)C^\dagger_{11}(x_4) \rangle \\
G^{(4)}(x_1, x_2, x_3, x_4) &= \langle C^{11}(x_1)C^\dagger_{11}(x_2)C^{11}(x_3)C^\dagger_{11}(x_4) \rangle.
\end{align*}
\]

We stress that the above five linearly independent four-point functions have been shown to be functionally related to (10) both in perturbation theory [12, 22, 23] and non-perturbatively at the one-instanton level [13]. For instance one finds

\[ G^{(H)}(x_1, x_2, x_3, x_4) = s \ G^{(V)}(x_1, x_2, x_3, x_4) \tag{16} \]

where $s$ is one of the two independent conformally invariant cross-ratios

\[ r = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \quad s = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} \tag{17} \]

that can be constructed out of four points.
3 Superdiagrams and calculations

The four-point function $G^{(H)}(x_1, x_2, x_3, x_4)$ defined in (10) is the lowest component of the supercorrelation function

$$\Gamma^{(H)}(z_1, z_2, z_3, z_4) = \langle C_{11}^{H}(z_1) C_{11}^{\dagger}(z_2) C_{22}^{H}(z_3) C_{22}^{\dagger}(z_4) \rangle$$  \hspace{1cm} (18)

viz.

$$G^{(H)}(x_1, x_2, x_3, x_4) = \Gamma^{(H)}(z_1, z_2, z_3, z_4) |_{\theta_i = \bar{\theta}_i = 0} ,$$ \hspace{1cm} (19)

where $z_i = (x_i^\mu, \theta_\alpha^i, \bar{\theta}_\dot{\alpha}^i)$ as usual. The $\mathcal{N}=1$ superfield approach greatly simplifies the calculations. The choice of external flavours guarantees that there are no connected supergraphs at tree level. The potential corrections to the propagators at order $g^4$ would only contribute to disconnected supergraphs \footnote{Order $g^4$ corrections to two-point functions of operators belonging to ultra-short supermultiplets vanish \footnote{\cite{27}}.} since the choice $\alpha = 1$ in the gauge-fixing term makes all propagator corrections vanish at order $g^2$ \cite{25, 26}.

The connected superdiagrams contributing to (18) are reported in Figs. 1, 2 and 3. We have only displayed superdiagrams that do not vanish by colour contractions. There are 4 superdiagrams with only chiral lines, 21 superdiagrams with one internal vector line and 10 superdiagrams with two internal vector lines. Notice that none of them involves cubic or quartic pure vector vertices. This explains why we have not explicitly displayed the corresponding couplings in (6).

The computation of the overall weights, due to combinatorial factors and colour and flavour contractions, is greatly simplified if fake different coupling constants are introduced for each interaction term in the action. Only at the very end one imposes the relations among the couplings that guarantee $\mathcal{N}=4$ supersymmetry. As a result one finds\footnote{In what follows we will only display group theory factors relevant for the case $\mathcal{G} = SU(N)$. The generalization to an arbitrary gauge group amounts to substituting $g^2 N$ with $g^2 C_2(A)$ and $N^2 - 1$ with $\text{dim}(\mathcal{G})$, $C_2(A)$ being the quadratic Casimir of the adjoint representation.} for the superdiagrams with only chiral lines

$$A_{\text{tot}} = \frac{g^4}{8} N^2 (N^2 - 1) (2A_1 + A_2 + A_3 + A_4) ,$$ \hspace{1cm} (20)

for those with one vector line

$$B_{\text{tot}} = -\frac{g^4}{4} N^2 (N^2 - 1) (2B_1 + 2B_2 + B_3 + B_4 - B_5 - 2B_6 + 2B_7 + 2B_8 + B_9$$

$$+ B_{10} - B_{11} - 2B_{12} + B_{13} + B_{14} + B_{15} + B_{16} + B_{17} - B_{18} - B_{19} - B_{20} - B_{21})$$ \hspace{1cm} (21)

and for the superdiagrams with two vector lines

$$C_{\text{tot}} = \frac{g^4}{2} N^2 (N^2 - 1) (2C_1 + 2C_2 - 2C_3 - 2C_4 + C_5 - C_6 + 2C_7 + 2C_8 - C_9) .$$ \hspace{1cm} (22)

Note that $C_{10}$ is zero due to Grassmann integration over the $\theta$’s of the interaction vertices. In general, the integration over the $\theta$ variables of the interaction vertices is
Figure 1: Diagrams with only chiral lines.

Figure 2: Diagrams with two vector lines.
Figure 3: Diagrams with one vector line.
greatly simplified by our choice of computing the lowest component of the supercorrelation function (18). This amounts to put to zero the \( \theta \) variables of the external insertions.

The final result is a complicated combination of double, triple and quadruple integrals of convolutions of massless scalar propagators and derivatives thereof.

The general strategy for computing the resulting integrals is the following. Exploiting the conformal invariance of the full correlation function we send one of the external points (say \( x_1 \)) to infinity after multiplying the correlator by \((x_1^2)^\Delta\), where \( \Delta \) is the conformal dimension of the operator inserted at \( x_1 \). Note that this simple prescription only works for operators (like \( \mathcal{O} \)) with protected conformal dimension (\( \Delta = 2 \) in the case at hand), i.e. independent of the coupling constant \( g^2 \). For operators that acquire anomalous dimensions, one has to carefully subtract divergent terms that behave like powers of \( \log(x_1^2) \) as \( x_1 \to \infty \). In this limit the final answer will depend on the three variables \( x_{23}, x_{24} \) and \( x_{34} \).

Taking the limit \( x_1 \to \infty \) considerably simplifies the computation for two reasons. On the one hand it lowers the number of propagators to be integrated. On the other hand, before taking the limit, one can integrate by parts some of the derivatives in the chiral propagators and make them act on propagators involving the point \( x_1 \). This increases the power of \( x_1 \) in the denominator making it larger than 4 with the consequence that terms generated in this way vanish in the limit. This trick also simplifies the \( \theta \) integrals. As a result one is left with only double integrals except for the diagrams \( B_{21} \) and \( C_6 \) where the remaining integral is (naively) a triple one. A drawback of our procedure is that the limit \( x_1 \to \infty \) breaks many of the permutation symmetries of the integrand, thus diagrams that were related by rotations (or reflections) to one another must be separately computed.

An important simplification arises by observing that in the limit \( x_1 \to \infty \) certain linear combinations of diagrams contributing to (21) and (22) actually vanish. Precisely one finds \( B_1 = 0, B_9 = 0, B_{15} - B_{18} = 0, 2B_2 - B_5 - B_{20} = 0, B_3 - 2B_6 + B_{13} + B_{17} - B_{19} = 0, C_1 - C_3 = 0, C_2 + C_8 = 0, 2C_4 - C_5 + C_9 = 0 \). As already observed \( C_{10} = 0 \) due to the \( \theta \) integration even before taking the limit.

The whole result can be expressed in terms of only two functions

\[
g(i, j, k) = \int \frac{dx_5}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{20}^2} \quad (23)
\]

and

\[
f(i; j, k) = \int \frac{dx_5 dx_6}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{20}^2} \quad , \quad (24)
\]

where in the notation used in \[12\]

\[
f(i; j, k) \equiv f(i, j; i, k) \quad . \quad (25)
\]

A useful expression for the function \( g \) is \[13\]

\[
g(2, 3, 4) = \frac{x_2^2}{x_{23}^2} B \left( \frac{x_{24}^2}{x_{23}^2}, \frac{x_{24}^2}{x_{23}^2} \right), \quad (26)
\]
where \( B(r, s) \) is a box-type integral given by
\[
B(r, s) = \int d\beta_0 d\beta_1 d\beta_2 \delta(1 - \beta_0 - \beta_1 - \beta_2) \frac{1}{\beta_1 \beta_2 + r \beta_0 \beta_1 + s \beta_0 \beta_2} .
\]  
(27)
The result of the integration in (27) can be expressed as a combination of logarithms and dilogarithms as follows
\[
B(r, s) = \frac{1}{\sqrt{p}} \left\{ \ln(r) \ln(s) - \left[ \ln \left( \frac{r + s - 1 - \sqrt{p}}{2} \right) \right]^2 - 2 \text{Li}_2 \left( \frac{2}{1 + r - s + \sqrt{p}} \right) - 2 \text{Li}_2 \left( \frac{2}{1 - r + s + \sqrt{p}} \right) \right\} ,
\]  
(28)
with \( \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \) and
\[ p = 1 + r^2 + s^2 - 2r - 2s - 2rs . \]  
(29)
The explicit expression of the function \( f(i; j, k) \) has been obtained in [28]
\[
f(i; j, k) = \frac{\pi^4}{x_{ik}^2} \Phi^{(2)} \left( \frac{x_{ij}^2}{x_{jk}^2}, \frac{x_{ik}^2}{x_{jk}^2} \right) ,
\]  
(30)
where \( \Phi^{(2)}(r, s) \) involves polylogarithms of up to fourth order. For the purpose of the present investigation, however, we only need the following identities [12]
\[
\Box_j f(i; j, k) = -4\pi^2 \frac{1}{x_{ij}^2} g(i, j, k) ,
\]  
(31)
\[
\Box_i f(i; j, k) = -4\pi^2 \frac{x_{jk}^2}{x_{ij}^2 x_{ik}^2} g(i, j, k) .
\]  
(32)
The latter identity has to be used in both directions to simplify the integrals appearing in the diagrams \( A_2, B_7, B_8 \) and \( B_{11} \). The triple integrals that appear in the computation of the diagrams \( B_{21} \) and \( C_6 \) can be simplified with the help of the relation
\[
\Box_3 \int \frac{dx_5}{x_{25}^2} \left[ g(3, 4, 5) \right]^2 = -4\pi^2 \frac{x_{34}^2}{x_{24}^2} \left[ g(2, 3, 4) \right]^2 .
\]  
(33)
Eq. (33) can in turn be proved using the identity
\[
\Box_2 \left( x_{24}^2 \left[ g(2, 3, 4) \right]^2 \right) = x_{34}^2 \Box_3 \left[ g(2, 3, 4) \right]^2 ,
\]  
(34)
which can be verified by resorting to the explicit expression of \( g(2, 3, 4) \) given by eq. (26).
The total contributions of each of the three classes of diagrams of Figs. 1, 2, and 3 is
\[
A_{tot} = \frac{g^4 N^2 (N^2 - 1)}{8(4\pi^2)^8} \left\{ \frac{2}{x_{34}^2} [f(3; 2, 4) + f(2; 3, 4)] + [g(2, 3, 4)]^2 \right\} ,
\]  
(35)
\[ B_{\text{tot}} = -\frac{g^4 N^2 (N^2 - 1)}{4(4\pi^2)^8} \frac{1}{4\pi^2} \left\{ [g(2, 3, 4)]^2 (x_{24}^2 + x_{34}^2 - x_{23}^2) - 4f(3; 2, 4) - 4f(4; 2, 3) \right\}, \]  
\[ C_{\text{tot}} = \frac{g^4 N^2 (N^2 - 1)}{2(4\pi^2)^8} \frac{1}{4\pi^2} \left\{ x_{24}^2 [g(2, 3, 4)]^2 - 2f(3; 2, 4) \right\}. \]

Summing the above three contributions yields
\[ G(x_2, x_3, x_4) \equiv \lim_{x_1 \to \infty} x_1^4 G^{(H)}(x_1, x_2, x_3, x_4) = \frac{g^4 N^2 (N^2 - 1)}{4(4\pi^2)^8 x_{34}^2} \times \]
\[ \left\{ f(2; 3, 4) + f(3; 2, 4) + f(4; 2, 3) + \frac{1}{4} (x_{24}^2 + x_{34}^2 + x_{23}^2) [g(2, 3, 4)]^2 \right\}. \]

As already explained, the actual \( x_1 \) dependence is recovered by performing in the right hand side of (38) the substitutions \( x_{23}^2 \to x_{23}^2 x_{14}^2, x_{24}^2 \to x_{24}^2 x_{13}^2 \) and \( x_{34}^2 \to x_{34}^2 x_{12}^2 \). The final result is in perfect agreement with the expression of the corresponding four-point function of hypermultiplet bilinears computed by means of \( \mathcal{N}=2 \) harmonic superspace techniques in [23].

## 4 Logarithms and anomalous dimensions

As in previous computations at order \( g^2 \) [12, 13] and at the one-instanton level [13], the function \( (38) \) shows the expected short-distance logarithmic behaviour. Indeed at short distances (\( i.e. \) in the limit in which pairs of points are taken to coincide) one finds linear and quadratic logarithms that are not incompatible with the finiteness of \( \mathcal{N}=4 \) SYM in the superconformal phase.

To illustrate this point in a simple fashion we will follow closely [13] and consider as an example the two-point function of a primary operator of scale dimension \( \Delta \)
\[ \langle O_\Delta^\dagger(x) O_\Delta(y) \rangle = \frac{A_\Delta}{(x - y)^{2\Delta}} \],
where \( A_\Delta \) is an overall normalisation constant possibly depending on the subtraction scale \( \mu \). Now suppose that \( \Delta = \Delta^{(0)} + \gamma \), \( i.e. \) the operator under consideration has an anomalous dimension. In perturbation theory \( \gamma = \gamma(g^2) \) is expected to be small and to admit an expansion in the coupling constant \( g^2 \)
\[ \gamma(g) = \gamma_1 g^2 + \gamma_2 g^4 + \ldots \].

The perturbative expansion of \( (39) \) in powers of \( g^2 \) yields
\[ \langle O_\Delta^\dagger(x) O_\Delta(y) \rangle = \frac{a_\Delta}{(x - y)^{2\Delta^{(0)}}} \times \]
\[ \left( 1 - g^2 \gamma_1 \log[\mu^2(x - y)^2] + g^4 \left\{ \frac{1}{2} \gamma_1^2 (\log[\mu^2(x - y)^2])^2 - \gamma_2 \log[\mu^2(x - y)^2] \right\} + \ldots \right). \]
where after renormalisation we have set \( A_\Delta = a_\Delta \mu^{-2\gamma} \). Thus, although the exact expression (49) is conformally invariant, at each order in \( g^2 \) eq. (41) contains powers of logarithms that seem to even violate scale invariance. Similar considerations apply to generic \( n \)-point Green functions as well.

Assuming the validity of the OPE, a four-point function can be expanded in the s-channel in the form

\[
\langle Q_A(x) Q_B(y) Q_C(z) Q_D(w) \rangle = \sum_K C_{CD}^K (z-w, \partial_w) \langle Q_A(x) Q_B(y) Q_K(w) \rangle ,
\]

where \( K \) runs over a complete set of primary operators. Descendants are implicitly taken into account by the presence of derivatives in the Wilson coefficients, \( C \)'s. An expansion like (42) is valid in the other two channels as well. To simplify formulae we assume that \( Q_A, Q_B, Q_C, Q_D \) are protected operators, i.e. they have no anomalous dimensions. In general the operators \( Q_K \) may have anomalous dimensions, \( \gamma_K \), so that \( \Delta_K = \Delta_K^{(0)} + \gamma_K \), where \( \Delta_K^{(0)} \) is the tree-level scale dimension. Similarly \( C_{IJ}^K = C_{IJ}^{(0)} + \eta_{IJ}^K \), with \( \eta_{IJ}^K \) the perturbative correction to the OPE coefficients.

The three-point function in the right hand side of eq. (42) is determined by conformal invariance to be of the form

\[
\langle Q_A(x) Q_B(y) Q_K(z) \rangle = \frac{C_{ABK}(g^2)}{(x-y)^{\Delta_A+\Delta_B-\Delta_K}(x-z)^{\Delta_A-\Delta_B+\Delta_K}(y-z)^{\Delta_B-\Delta_A+\Delta_K}} .
\]

Eq. (43), like the two-point function (11), can be expanded in power series in \( g^2 \) giving rise to logarithmic terms. From these formulae one can extract the corrections to both the OPE coefficients and the anomalous dimensions of the operators \( Q_K \). The same procedure applies also to derivatives of the three- and four-point functions as well as to their limits for \( x_1 \to \infty \).

Let us analyze the function \( G^{(H)} \) in the limit \( x_4 \to x_3 \) that exposes the s-channel. Only operators in the singlet of the manifest \( SU(3) \times U(1) \) may be exchanged \( \mid 3 \rangle \). Barring the identity operator, that is clearly not renormalized, the leading contribution comes from the operator

\[
\mathcal{K}_1 = \frac{1}{3} : \text{tr}(\varphi^i \varphi^i) : ,
\]

that has naive dimension \( \Delta^{(0)} = 2 \) and is lowest component of the long Konishi supermultiplet.

In the case under consideration, the relative complexity of the explicit expression for \( f(i,j,k) \) makes it difficult to directly analyse the short distance behaviour of the function (58). Thus we find it more convenient to compute \( \Box_2 G(x_2, x_3, x_4) \), which can be expressed in terms of only the much simpler function \( g(i,j,k) \). Using eqs. (34), (32) and (34), one obtains

\[
\Box_2 G(x_2, x_3, x_4) = \frac{g^4 N^2 (N^2-1)}{8(4\pi^2)^8} \left[ \sum_{j=2}^4 \partial_j g(2,3,4) \cdot \partial_j g(2,3,4) - 3(4\pi^2) \left( \frac{1}{x_{23}^2 x_{34}^2} + \frac{1}{x_{24}^2 x_{34}^2} + \frac{1}{x_{23}^2 x_{24}^2} \right) g(2,3,4) \right] .
\]
The leading behaviour of (15) in the limit $x_4 \to x_3$ is given by

$$\square_2 G(x_2, x_3, x_4) \to \frac{g^4 N^2 (N^2 - 1)}{16 (4\pi^2)^6} \frac{1}{x_{23}^2 x_{34}^2} \left[ 3 \log \left( \frac{x_{34}^2}{x_{23}^2} \right) - 5 \right]. \quad (46)$$

We now compare eq. (16) with the result of the OPE analysis of the order $g^4$ contribution to the function $\square_2 G(x_2, x_3, x_4)$. The latter yields

$$\square_2 G(x_2, x_3, x_4) \to \frac{g^4 (N^2 - 1)}{(4\pi^2)^6 x_{23}^4 x_{34}^4} \left[ a_0 \gamma_1^2 \log \left( \frac{x_{34}^2}{x_{23}^2} \right) + a_0 \left( \gamma_1^2 + 2\gamma_2 + 2a_1 \right) \right], \quad (47)$$

where the coefficients $a_0$ and $a_1$ are given by

$$a_0 = \frac{1}{3} (4\pi^2)^2, \quad a_1 = -N \pi^2. \quad (48)$$

They represent the tree-level contribution of the Konishi scalar $K_1$ to the $s$-channel expansion of the function $G^{(H)}$ and its finite one-loop correction. The coefficients $a_0$ and $a_1$ are determined by the one-loop analysis performed in [13] that gives

$$G(x_2, x_3, x_4)|_{1\text{-loop}} \to \frac{g^2 (N^2 - 1)}{(4\pi^2)^6} \frac{1}{x_{23}^2 x_{34}^2} \left[ a_0 \gamma_1^2 \log \left( \frac{x_{34}^2}{x_{23}^2} \right) + a_1 \right]$$

$$= \frac{g^2 (N^2 - 1)}{(4\pi^2)^6} \frac{1}{x_{23}^2 x_{34}^2} \pi^2 N \left[ \frac{1}{2} \log \left( \frac{x_{34}^2}{x_{23}^2} \right) - 1 \right]. \quad (49)$$

We stress that the quadratic logarithmic term in $G^{(H)}$ contributes a linear logarithmic term to $\square_2 G^{(H)}$ in eq. (14).

Comparison of eq. (16) with eq. (17) shows that the coefficient of the logarithmic term is related to the square of the one-loop anomalous dimension of $K_1$, and agrees with the known value $\gamma_1 = \frac{3N}{64\pi^2}$ [29]. The constant term corresponds to a combination of the one- and two-loop anomalous dimensions and the known one-loop correction to the OPE coefficient. For the $g^4$ contribution to the anomalous dimension of the Konishi supermultiplet one obtains

$$\gamma_{2\text{-loop}} = \gamma_2 g^4 = -\frac{3}{16} \frac{g^4 N^2}{(4\pi^2)^2}. \quad (50)$$

The analysis of the other two non-singlet channels ($x_{24} \to 0$ and $x_{23} \to 0$) is more subtle. The lowest dimensional operators that can be exchanged in both channels belong to the 105 and to the 84 representations of the $SU(4)$ R-symmetry. Single- and double-trace operators of dimension four in the 105 are expected to be protected [31, 26]. In order to disentangle the two $SU(4)$ representations it is necessary to consider another four-point function. We find it convenient to analyse the four-point correlator $G^{(V)}$, defined in (19). One-loop [12], one-instanton [16, 13] and two-loop [23] computations give $s G^{(V)} = G^{(H)}$. The OPE analysis of the $g^4$ contribution to $G^{(V)}$, as it is obtained from the above functional relation, confirms the non-renormalisation of the 105 operators.

The two possible operators of naive dimension four in the 84 mix at one-loop [13]. One of them, $\tilde{K}_{84}$ is a superconformal descendant of of $K_1$ and as such has the same
anomalous dimension. A careful OPE analysis, combined with the symmetry of the factor in brackets in the right hand side of eq. (38) under the exchange of $x_2$ and $x_4$, implies that this operator saturates the logarithmic behaviour in this channel. We thus conclude that the operator $\mathcal{D}_{84}$, identified in [13] as the combination orthogonal to $\mathcal{K}_{84}$, is protected also at order $g^4$. This is in agreement with the fact that it belongs to a supermultiplet satisfying a generalized shortening conditions [31, 32, 33, 34, 35, 36].

5 Comments

Let us briefly comment on the bearing of the results of this paper. First of all the very fact that it was possible to compute in closed form a four-point function of protected composite operators at order $g^4$ shows that $\mathcal{N}=4$ SYM is both non-trivial and calculable. No off-shell approach is known that preserves $\mathcal{N}=4$ supersymmetry. At the quantum level, one has to resort either to the $\mathcal{N}=1$ superspace approach pursued here or to the less familiar $\mathcal{N}=2$ harmonic superspace approach pursued in [23]. The coincidence of all known results in the two approaches gives independent support to some $\mathcal{N}=2$ harmonic superspace identities [37], based on the bonus symmetry proposed in [38], that led to drastic simplifications in the computations reported in [23].

The introduction of supersymmetric mass-terms gives rise to interesting, in some cases confining, theories that can be handled with $\mathcal{N}=1$ superfield techniques. Alternatively one might consider other finite $\mathcal{N}=1$ gauge theories some of which are conjectured to be dual to type IIB superstring on $AdS_5 \times S^5/\Gamma$, where $\Gamma$ is a discrete subgroup of an $SU(3)$ subgroup of the $SU(4)$ R-symmetry. These gauge theories govern the dynamics of the light degrees of freedom of stacks of coincident D3-branes at special orbifold singularities [39]. Other $\mathcal{N}=1$ finite theories can be obtained by perturbing $\mathcal{N}=2$ theories by mass-terms. This is the case of D3-branes at generalized conifold singularities [40], whose supergravity dual replaces $S^5$ with less trivial Einstein spaces [41]. In all such cases the $\mathcal{N}=1$ formalism pursued in this paper might allow one to compute correlation functions of protected composite operators. By OPE one would then extract anomalous dimensions and couplings of unprotected operators such as those belonging to Konishi-like long supermultiplets. These are expected to be dual to genuine string excitations and as such should decouple from the operator algebra in the supergravity limit, that is dual to the large $N$ and strong 't Hooft coupling limit. The fact that the one-loop anomalous dimension of the Konishi multiplet has been known for some time [29] to be positive was somewhat reassuring in this respect. The OPE analysis of the two-loop computations confirm the one-loop result, but at the same time yields a negative two-loop contribution to the anomalous dimension. This result certainly requires further investigation and some cross-checks [42]. Another related issue is the rôle of multi-trace operators that are expected to be dual to multiparticle states in the AdS description. We have shown and confirmed in the present paper that their mixing with single trace operators is not suppressed in the large $N$ limit at finite 't Hooft coupling. In fact protected operators belonging to supermultiplets that satisfy generalized shortening conditions [31, 32, 33, 34, 35, 36] are typically mixtures of single- and multi-trace operators. One would like to clarify their rôle in the operator algebra in relation to the “string exclusion principle” that is expected to drastically reduce the
spectrum of AdS excitations at finite “radius”, i.e. at finite $N$ [13].

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