Partition Functions on the Euclidean Plane with Compact Boundaries in Conformal and Non-Conformal Theories

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Abstract

In this letter we calculate the exact partition function for free bosons on the plane with lacunae. First the partition function for a plane with two spherical holes is calculated by matching exactly for the infinite set of Wilson coefficients in an effective world line theory and then performing the ensuing Gaussian integration. The partition is then re-calculated using conformal field theory techniques, and the equality of the two results is made manifest. It is then demonstrated that there is an exact correspondence between the Wilson coefficients (susceptibilities) in the effective field theory and the weights of the individual excitations of the closed string coherent state on the boundary. We calculate the partition function for the case of three holes where CFT techniques necessitate a closed form for the map from the corresponding closed string pants diagrams. Finally, it is shown that the Wilson coefficients for the case of quartic and higher order kernels, where standard CFT techniques are no longer applicable, can also be completely determined. These techniques can also be applied to the case of non-trivial central charges.
I. INTRODUCTION

The partition functions for conformal field theories on surfaces with boundaries are of interest in diverse areas of physics. In soft condensed matter physics it is of interest to calculate partition functions on the Euclidean plane with compact boundaries for the purpose of understanding fluctuation induced forces \(^2\). In string theories, amplitudes correspond to such partition functions and one must sum over all moduli and Euler characteristics. However, to calculate a particular gauge slice is not always simple. Powerful methodologies in boundary conformal field theory (BCFT) \(^1\) have been utilized to determine the spectrum of primary operators in Gaussian as well as non-Gaussian theories. These methods utilize the state operator correspondence to implement the boundary conditions via the insertion of boundary operators.

In this paper we will take a different approach based on world-line effective field theories (EFT) \(^3\) as applied to statistical systems \(^4\,\,\,5\). In particular, we will be interested in the case of partition functions on the Euclidean plane with compact boundaries. The EFT approach is similar to BCFT in that the effects of the boundaries are accounted for via operator insertions. However, we will make no use of the state operators correspondence, and as such, the technique will not be limited to conformal field theories.

We will begin by considering the free bosonic action

\[
S^B_0 = \frac{1}{2} \int d^2 r \partial \phi \partial \phi ,
\]

on a plane with holes which satisfy Neumann boundary conditions. This problem is identical to finding the partition function for a film embedded with rigid disks which are allowed to bob but not tilt \(^1\), where the notion of tilting is given meaning by a particular choice of gauge. In particular the scalar field partition function will correspond to the Monge gauge choice \(^6\) where the height of the field \(\phi\) is measured with respect to a Euclidean flat base plane. In the film model there are non-linear corrections which are there as a consequence of reparameterization invariance, but it can be shown that these corrections are suppressed \(^4\) by powers of \(k/\beta\) where \(k\) is the modulus characterizing the surface energy. The fluctuation induced forces in these systems are relevant to soft condensed matter problems \(^2\).

\(^1\) This is not true for all observables but will hold for the partition source-free partition function.
In [4] it was shown that it is possible to utilize the world-line effective field theory developed for studying black hole inspirals [3] to study the Casimir forces on two dimensional surfaces (the dimensionality is actually irrelevant). The idea is to work in the point particle approximation, and then account for the finite size (read boundary conditions) by adding an infinite tower of higher dimensional operators to the action. For defects with only linear responses to external perturbations these terms are all quadratic in the field, and their dimensionality is fixed by the number of derivatives. Power counting shows that non-linear interactions on the world-line generate corrections which are suppressed at high temperatures. In the limit where the size of the objects ($R$) is small compared to the distance between them $r$, truncating the infinite sum induces errors of order $(R/r)^\tau$, where $\tau$ is the number of derivatives in the first term dropped in the series. The coefficients of these operators (Wilson coefficients) correspond to generalized polarizabilities and are fixed by performing a matching calculation between the full theory and the effective theory. The full set of Wilson coefficients can be calculated by first matching the response of the individual holes to multipole fields in the full and effective theory. We will then show that these matching coefficients can equally well be extracted by determining the boundary operators in the Hamiltonian version of the theory. Comparing the two procedures illuminates a map between closed string excitations and effective polarizabilities. The model is then solved by summing the series for the higher dimensional operators into a non-local operator which can be used to write down a closed form expression for the partition function.

In the effective theory the action for for $n$-bodies may be written as

\[ S = S_0 + \frac{1}{2} \sum_i C_i (\partial_{a_1} \ldots \partial_{a_n} \phi)(\partial_{a_1} \ldots \partial_{a_n} \phi), \]

where the second term is localized at the origin. Operators involving traces over indices are redundant in this theory as they can be traded for other operators via a canonical transformation. For non-spherically symmetric objects the polarizibilities will transform non-trivially under the two dimensional rotation group, but are singlets for the spherical case of interest here. These coefficients are fixed by a simple matching procedure whereby we place the objects in a background field and calculate its response. For a given polarizability of a given order we may extract the corresponding $C_i$ by placing the object in a multipole field of the same order.

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To utilize holomorphy we match in the complex plane a general n-pole background field taking the form

$$\phi_{bg}(z, \bar{z}) = \frac{\lambda}{R^n}(z^n + \bar{z}^n).$$

(3)

We will impose Neumann boundary conditions which preserve the conformal symmetry.

$$\partial_z \phi(z) |_{R} = 0$$

(4)

This choice means that the first non-zero $C_i$ will be $C_2$, a dipole polarizability. In the full theory we solve the rudimentary boundary value problem. The induced field is given by

$$\delta \phi = \lambda R^n \left( \frac{1}{z^n} + \frac{1}{\bar{z}^n} \right).$$

(5)

Expanding around the background field (3) utilizing the Greens function in complex coordinates

$$G(z, \bar{z}) = \frac{1}{4\pi} \left( \log(z) + \log(\bar{z}) \right)$$

(6)

and calculating the one point function leads to the EFT result

$$\delta \phi = -\frac{C_n \lambda}{4\pi R^n} 2^{2n-1} n! (n-1)! \left( \frac{1}{z^n} + \frac{1}{\bar{z}^n} \right).$$

(7)

We may then extract the matching coefficient to be

$$C_n = -\frac{\pi}{n!(n-1)!} 2^{n-2} R^{2n}. $$

(8)

This form of matching coefficient can also be obtained from perspective of closed string theory. If we conformally map an isolated hole into an infinite cylinder, and consider the direction along the length as being time-like, then the partition function corresponds to a closed string propagator emanating from a boundary state [7]. The boundary states are annihilated by the constraint

$$\partial_1 \phi | \psi \rangle = 0$$

(9)

where the world-sheet is parameterized by $(\sigma_1, \sigma_2)$ and $\sigma_1$ is the time like direction.

Our gauge choice (Monge) corresponds to choosing mode oscillators in the string expand-
sion to be along one space like direction in the embedding space. In terms of the mode operators the constraint reads

\[(\alpha_n + \tilde{\alpha}_{-n}) \mid \psi \rangle = 0 \quad (10)\]

where the (standard) normalization is fixed via

\[\partial \phi(z) = -i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m}{z^{m+1}}, \quad (11)\]

\[[\alpha_m, \alpha_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m \delta_{m,-n}, \quad (12)\]

and

\[S = \int \frac{1}{4\pi \alpha'} (\partial \phi \bar{\partial} \phi) d^2 \sigma. \quad (13)\]

The solution to this constraint is

\[\mid \psi \rangle \propto e^{-\sum_{p=1}^{\infty} \left(\frac{2}{p} \alpha_p \tilde{\alpha}_{-p}\right) \mid 0 \rangle}. \quad (14)\]

Using the fact that

\[\alpha_{-m} = \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(m-1)!} \partial^m \phi(0) \quad (15)\]

the state is written as

\[\mid \psi \rangle \propto e^{-\sum_{p=1}^{\infty} \frac{2}{\alpha_p' (p-1)!} (\partial^p \phi \bar{\partial}^p \phi) \mid 0 \rangle}. \quad (16)\]

rescaling the fields so that they are normalized according to (1) leads to

\[\mid \psi \rangle \propto \text{Exp} \left[ -\sum_{p=1}^{\infty} \frac{4\pi R^{2p}}{p! (p-1)!} (\partial^p \phi \bar{\partial}^p \phi) \right] \mid 0 \rangle. \quad (17)\]

Taking this as an operator insertion in the path integral reproduces the Wilson coefficients $C_n$. It is interesting to note that there is a correspondence between the string excitation and the generalized susceptibilities. For instance the dipole polarizability $(\partial \phi)^2$ on the world line maps to the dilaton excitation of the string. Higher order multipoles map to massive string excitations.
Given the action the two body problem can be solved. We utilize the holomorphic nature of the fields to re-write the action as

\[
S_{\text{int}} = \sum_n C_n (2^n (\partial_z^2 \phi)(\partial_{\bar{z}}^2 \phi))
\]  

(18)

Then using the result we have

\[
S = \int d^2z \left[ \frac{1}{2} \partial_z \phi \partial_{\bar{z}} \phi + \sum_a (4\pi) R_A \sqrt{\partial_z \partial_{\bar{z}}} I_1(2R_A \sqrt{\partial_z \partial_{\bar{z}}}) \phi(\bar{z}) \delta^2(z - z_a, \bar{z} - \bar{z}_a) \right].
\]  

(19)

The exact result for the partition function follows from treating these operators as mass insertions and recognizing that the contractions between two identical world lines lead to trivial renormalizations that are pure counter-terms. Thus the only relevant vacuum diagrams are those in which no two mass insertions are adjacent.

Let us consider the case of two particles. To calculate the free energy we draw all possible vacuum diagrams which have no disconnected sub-components. We must always add two insertions at a time since for an odd number of insertions there would have to be a self-energy graph inserted which we renormalize to zero. Diagrams which are related by a reflection symmetry are not distinct. At second order in the interaction the diagram reduces down to

![Diagram](image)

FIG. 1. All connected vacuum diagrams can be written in term of this topology. The different colored dots represent insertions at each of the two particles respectively.

\[
-\beta F = \sum_{n=1}^{\infty} \prod_{a=1,3,5,..}^{2n-1} (R_A R_B)^n \frac{1}{n} \sqrt{\partial_{z_a} \partial_{\bar{z}_a}} \sqrt{\partial_{z_{a+1}} \partial_{\bar{z}_{a+1}}} I_1(2R_A \sqrt{\partial_{z_a} \partial_{\bar{z}_a}}) I_1(2R_B \sqrt{\partial_{z_{a+1}} \partial_{\bar{z}_{a+1}}}) \times \log(z_a - z_{a+1}) \log(\bar{z}_a - \bar{z}_{a+1})
\]  

(20)

Each \(n\) represents two insertions, one of each type, and \(z_{2n+1} = z_1\). Upon expanding this
expression and differentiating one sets $z_a = z_1$ and $z_{a+1} = z_2$. Let us now expand this result in $R/r$ to compare with the literature on the lower order contributions. The leading order contribution is the dipole-dipole interaction. Expanding out \( (20) \) gives

\[
- \beta F_{d-d}^{(2)} = R_A^2 R_B^2 \partial_{z_1} \partial_{\bar{z}_1} \partial_{z_2} \partial_{\bar{z}_2} \log((z_1 - z_2) \log(\bar{z}_1 - \bar{z}_2))
\]

\[
= \frac{1}{r^4} R_A^2 R_B^2
\]

(21)

We also generate non-linear terms such as the case with two dipoles of each type in which case $i = 2$.

\[
- \beta F_{d^2-d^2}^{(2)} = R_A^2 R_B^2 \frac{1}{2} \sqrt{\partial_{z_1} \partial_{\bar{z}_1} \sqrt{\partial_{z_2} \partial_{\bar{z}_2}}} I_1(2R_A \sqrt{\partial_{z_1} \partial_{\bar{z}_1}}) I_1(2R_B \sqrt{\partial_{z_2} \partial_{\bar{z}_2}}) \log(\bar{z}_1 - \bar{z}_2) \log(z_1 - z_2)
\]

\[
\times \sqrt{\partial_{z_3} \partial_{\bar{z}_3} \sqrt{\partial_{z_4} \partial_{\bar{z}_4}}} I_1(2R_A \sqrt{\partial_{z_3} \partial_{\bar{z}_3}}) I_1(2R_B \sqrt{\partial_{z_4} \partial_{\bar{z}_4}}) \log(\bar{z}_3 - \bar{z}_4) \log(z_3 - z_4)
\]

\[
= R_A^4 R_B^4 \frac{1}{2r^8}
\]

(22)

All of these perturbative results agree with those in the literature [8, 9]. New results can be read off by going to higher order.

The case of $N$ holes follows in a similar fashion. If we are interested in the $N$-body potential we have a minimum of $N$ insertions and we may add one vertex insertion at a time.

\[
- \beta F_N = \sum_{n=N}^{\infty} \sum_{\pi} \prod_{a=1}^{n} R_1^{i_1} R_2^{i_2} ... R_N^{i_N} \sqrt{\partial_{z_a} \partial_{\bar{z}_{a+1}}} I_1(2R_a \sqrt{\partial_{z_a} \partial_{\bar{z}_{a+1}}}) \times G(z_a - z_{a+1}, \bar{z}_a - \bar{z}_{a+1})
\]

(23)

The second sum is over all possible choice non-equivalent permutation of vertices. Where two permutations are in the same equivalence class if they can be rotated into each other. Defining $G(0) = 0$ implies that all choices with two adjacent identical vertices vanish. For each choice of vertices, $(i_1, i_2, i_3...)$ correspond to the number of vertices of (hole) type $(1, 2, 3...)$ in the particular term in the sum. The expression \( (23) \) is easily evaluated to arbitrary order in MATHEMATICA, though this particular form is most probably not the most efficient since it entails redundancies as compared to the two body results presented above.
II. CFT SOLUTION FOR TWO HOLES

The determinant of the Laplacian on Riemann surfaces in many instances can recovered by conformally mapping surfaces for which the partition is easily calculated to surfaces which present more of a challenge, such as the case of interest in this paper. The techniques used here are well known though it seems that the explicit result for the case of interest, two spheres in a plane, is absent in the voluminous literature on the subject of functional determinants in two dimensions.

The partition function on the plane with two holes can be calculated by a series of conformal maps starting from the cylinder. One could of course map directly from the cylinder to the plane with holes, but it is perhaps simpler to use the annulus as an intermediate step. In each step we work in a conformal gauge

\[ g_{ab} \sim e^{2\phi}\hat{g}_{ab}, \]

in which case the following quantity is an invariant \[10, 11\]

\[ I_N = \frac{\text{Det}'_N(\nabla^2)}{A} (e^{\frac{l}{2\pi}} - \frac{1}{2} \int_{\partial M} \kappa ds). \]  

(25)

Where we have chosen Neumann boundary conditions for the purpose of illustration. \( L \) is defined as

\[ L(\phi, \hat{g}) = \int_M d\hat{\mu} \hat{K}\phi + \int_{\partial M} d\hat{s}\hat{\kappa}\phi + \frac{1}{2} \int_M d\hat{\mu} \hat{g}^{ab}(\partial_a\phi \partial_b\phi), \]  

(26)

and hatted quantities are defined by the hatted metric, \( K \) is the Gaussian curvature on \( M \), \( \kappa \) is the geodesic curvature on the boundary, and \( A \) is the area of \( M \). The prime above the determinant denotes the fact that the zero mode is excluded.

Beginning with the partition function on a cylinder with length \( l \) and unit radius. A rudimentary calculation yields the result

\[ Z = \frac{1}{\sqrt{2l}} e^{\frac{l}{2\pi}} \prod_{n=1}^{\infty} \frac{1}{1 - e^{-4\pi ln}}. \]  

(27)
We then map this result to the annulus using the conformal transformation

\[ z = r_2 e^{2\pi i w} \]  

(28)

with

\[ 2\pi l = \ln(r_2/r_1). \]  

(29)

The outer (inner) radii of the annulus are \( r_2 \)(\( r_1 \)). The conformal factor is given by

\[ \phi = \frac{1}{2} \log(2\pi z \bar{z}). \]  

(30)

Given that both spaces are flat we find

\[ L_{ann} = \pi \log(r_2/r_1). \]  

(31)

The annulus partition function is then given by

\[ Z_{ann} = f(r_2/r_1^{2}) \frac{r_1/r_2}{\sqrt{2(r_2^2 - r_1^2)}}^{-1/6} \]  

(32)

\[ f(x) = \prod_{n=1}^{\infty} \frac{1}{1 - x^n} \]  

(33)

Then we can arrive at case of interest (z plane with two holes) via a conformal map from the annulus (w plane). The mapping shown in figure (2) is given by

\[ w = \frac{z - a}{az - 1} \]  

(34)
where
\[ a = 1 + x_1 x_2 + \sqrt{(x_1^2 - 1)(x_2^2 - 1)} \]
\[ R = \frac{x_1 x_2 - 1 - \sqrt{(x_1^2 - 1)(x_2^2 - 1)}}{x_1 - x_2}, \]
under the conditions \( x_2 < a < x_1 \).

The metric on the annulus is then mapped to the flat \( z \) plane
\[ d\bar{z}dz = dwd\bar{w} \frac{(a^2 - 1)^2}{(aw - 1)^2(a\bar{w} - 1)^2} \]
so that
\[ \phi = \frac{1}{2} \ln \left( \frac{(a^2 - 1)^2}{(aw - 1)^2(a\bar{w} - 1)^2} \right) \]
and \( \hat{g} = \delta_{ab} \) is the metric on the flat plane so that \( \hat{K} = 0 \). The annulus is flat \((R \sim \nabla^2 \phi)\) so \( \int_{\partial M} \kappa = 0 \) since its Euler characteristic \( \chi = 0 \). However, the integral
\[ \int_{\partial M} d\hat{s}\hat{\kappa}\phi = \int_0^{2\pi} d\theta (\phi(1, \theta) - \phi(R, \theta)) = 2\pi \log[R] \]
does not vanish since \( \phi \) is not constant along the boundary.

The bulk contribution to the anomaly must vanish since the conformal map is an element of the \( SL(2, C) \) subgroup with vanishing Schwarzian. Hence we are left the the following partition function for the plane with two holes.
\[ -\beta F = \log[f[R^2]] \]
where
\[ f[x] = \prod_{n=1}^{\infty} \frac{1}{1 - x^n} \]
and
\[ R = 1/2(r^2 - r\sqrt{r^2 - 4} - 2). \]
Expanding in the ratio of the size of the holes relative to the distance between them gives
\[ \beta F \approx \frac{1}{r^4} + \frac{4}{r^6} + \frac{31}{2r^8} + \frac{60}{r^{10}} + ... \]
Which agrees with the expansion of (20). The expression (40) certainly has many advantages over the result (20). In particular, though the expansion in (20) has a finite radius of convergence we can not expand it around to the osculating limit. Whereas the Dedekind function which results from (40), (41) has multiple representations which can be easily expanded around this limit. The result (40) may be used be compared to the so-called “proximity force approximation” [12] which has recently been systematized in terms of a derivative expansion [13].

III. NON-CONFORMAL THEORIES

It is interesting to see that the effective theory methodology can still yield exact results even for non-conformal kernels. A non-conformally invariant kernel of particular interest is the bi-Laplacian which corresponds to a model of a membrane where bending dominates tension. In this case there are more operators at each order in $R$ since the equations of motion can no longer be used to eliminate operators involving a single Laplacian. In the case of the bi-Laplacian, beyond second order in derivatives we have two operators at each order in the derivative expansion. At the quadrupole level we have two possible operators. A convenient basis to choose is

$$O_a^2 = \partial_z \partial_{\bar{z}} \phi \partial_z \partial_{\bar{z}} \phi, \quad O_b^2 = \partial_z \partial_{\bar{z}} \phi \partial_{\bar{z}} \partial_z \phi.$$  \hspace{1cm} (44)

At higher orders we add one power of $\partial_z, \partial_{\bar{z}}$ to each side. So that at $n'th$ order in derivatives we have

$$O_a^{n+1} = \partial_z^{n+1} \phi \partial_{\bar{z}}^{n+1} \phi, \quad O_b^{n+1} = \partial_{\bar{z}}^{n} \partial_z \phi \partial_z^{n} \partial_{\bar{z}} \phi.$$  \hspace{1cm} (45)

In the case of the bi-Laplacian one must impose two boundary conditions to find unique solutions. Which operator has the first non-vanishing susceptibility will depend upon the particular choice. The crucial point is that the number of operators at each level in derivatives does not grow which allows again for the matching procedure to carried out to all order in the derivative expansion [19].

This fact may at first seem surprising given that the theory is not conformal. However, the theory is invariant under the $SL(2, C)$ sub-group of the Virasoro group which is revealed
once one makes a field redefinition

\[ \phi \rightarrow \left| \frac{dz}{dz'} \right| \phi'(z'). \]

(46)

In addition, higher order kernels can also be solved by the same method. However, as one goes to higher orders more (but still finite) operators appear at each order.

**IV. CONCLUSIONS**

Partition functions for on higher genus surfaces can have a rich structure. The boundaries introduce new sets of operators which make even free theories highly non-trivial. Models which are integrable on simple manifolds, such as cylinders, can be solved on more complicated surfaces through conformal mappings and the associated anomalous transformations of the stress energy in the bulk and boundary \[^2\]. However, mappings to the Euclidean plane with multiple boundaries of various shapes can become analytically intractable.

In this letter we have shown that two dimensional partition functions can also be expressed in terms of an infinite set of Wilson coefficients in a world line EFT. For Gaussian theories, such as the free bosonic one discussed here, these coefficients are exactly calculable. These coefficients which physically correspond to generalized susceptibilities, are equal to the weights of string excitations in the coherent state boundary operator in the closed string channel. Given these coefficients, the partition function follows from the vacuum energy graphs. The result is written as an infinite sum which is clearly inferior to the infinite product form which arise in the context of CFT methodology. For the case of two spherical holes we have shown the equality of the CFT and EFT results expanding to arbitrary order in the separation between the holes.

However, as the number of defects increases the only impediment to solving the theory using the world-line methodology is that the series involved grows more cumbersome. Here we only presented the sums in their most raw form, and it is surely possible to simplify them further \[^2\]. Furthermore, it is also possible to determine the Wilson coefficients for deformed holes using the fact that can take advantage of rudimentary conformal transformation techniques to calculate the infinite set of Wilson coefficients for ellipses \[^17\]. We have

\[^2\] The partition function for sections of spheres have been calculated in this fashion in \[^14\].
also shown that it is also possible to solve for the Wilson coefficients for the non-conformal bi-Laplacian kernel.

Finally it is interesting to ask how these EFT techniques can be applied to interacting theories. For any conformal theory, the conformal mapping technique is still applicable, and there is no reason that one could not find the fluctuation induced force say for an Ising model \[18\]. However, again if one is interested in more complicated geometries (such as multiple holes) the mappings will become analytically intractable. However, given that distortions of single holes to other shapes, such as ellipses, is fairly simple, the matching procedure in the effective theory technique is still relatively straightforward. Thus, given that for a simple geometry one can solve for the Wilson coefficient, at least for an integrable model, one can also determine the coefficients for more complicated shapes. The free energy for multiple holes can then be calculated via the vacuum energy diagrams as shown above, at least for theories which are integrable.

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