ON AN EXAMPLE OF ASPINWALL AND MORRISON

BALÁZS SZENDRŐI

ABSTRACT. In this paper, a family of smooth multiply connected Calabi–Yau threefolds is investigated. The family presents a counterexample to global Torelli as conjectured by Aspinwall and Morrison.

INTRODUCTION

The aim of this paper is to prove

Theorem 0.1. The one-parameter family of smooth, multiply connected Calabi–Yau threefolds \( \mathcal{Y} \rightarrow B \) over the base \( B = \mathbb{P}^1 \setminus \{ 1, \xi, \ldots, \xi^4, \infty \} \), constructed by Aspinwall–Morrison in [1] (cf. Section 1), with \( \xi \) a primitive fifth root of unity, has the following properties:

- For any \( t \in B \), there exists an isomorphism
  \[ H^3(Y_t, \mathbb{Q}) \cong H^3(Y_{\xi t}, \mathbb{Q}) \]
  preserving rational polarized Hodge structures (for a stronger statement, see Theorem 2.3).
- There is a Zariski-open set \( U \subset B \) such that for \( t \in U \), \( i = 0, \ldots, 4 \), the fibres \( Y_{\xi^it} \) are pairwise non-isomorphic as algebraic varieties.

The family \( \mathcal{Y} \rightarrow B \) is a quotient of a family of quintics, manufactured in such a way that a certain symmetry of a cover \( \mathcal{Z} \rightarrow B \) of \( \mathcal{Y} \rightarrow B \) fails to descend in any obvious way to a symmetry of \( \mathcal{Y} \rightarrow B \). The existence of this symmetry on the cover implies the statement about Hodge structures (Theorem 2.3). On the other hand, an isomorphism between \( Y_t \) and \( Y_{\xi t} \) for general \( t \) would force, via a specialization argument (Theorem 4.2), the existence of an automorphism \( \sigma \) on the fibre \( Y_0 \) over 0 of a special kind. However, the automorphism group of \( Y_0 \) can be computed explicitly (Theorem 3.1), and such a \( \sigma \) does not exist. For technical reasons, the argument runs on a family of singular models \( \overline{\mathcal{Y}} \rightarrow B \) of \( \mathcal{Y} \rightarrow B \). (See Section 4.)

Theorem 1.3 establishes the fact, conjectured by Aspinwall and Morrison, that the family \( \overline{\mathcal{Y}} \rightarrow B \) provides a counterexample to global Torelli for Calabi–Yau threefolds. Previous counterexamples to Torelli were given in [13]; there I considered families of birationally equivalent Calabi–Yau threefolds. By [1, Theorem 4.12], birational equivalence implies isomorphism between (rational) Hodge structures. However, in the present case the situation should be entirely different.

Conjecture 0.2. For general \( t \in B \), the threefolds \( Y_{\xi^it} \) for \( i = 0, \ldots, 4 \) are not birationally equivalent to one another.

Date: 23 June 2002.
1991 Mathematics Subject Classification. 14J32, 14C34, 14M25.
Research partially supported by an Eastern European Research Bursary from Trinity College, Cambridge and an ORS Award from the British Government.
One obvious direct approach to this conjecture is to aim to understand the various birational models of a fixed fibre $Y_t$. Birational models of minimal threefolds can be studied via their cones of nef divisors in the Picard group; so this method requires an explicit understanding of the nef cone of $Y_t$. An étale cover $Z_t$ of $Y_t$ is a toric hypersurface. A recent conjecture [3, Conjecture 6.2.8] of Cox and Katz aimed at giving a complete understanding of the nef cone of toric Calabi–Yau hypersurfaces. However, I prove in [14] that in fact the conjecture of Cox and Katz fails for $Z_t$. At this point the computation of the nef cone of $Y_t$ seems rather hopeless. A different approach to Conjecture 0.2 is required.

To conclude the introduction, let me point out that the varieties $Y_t$ are multiply connected with fundamental group $\mathbb{Z}/5\mathbb{Z}$ (Proposition 1.4 and Proposition 1.7). This is a curious fact. The construction of Aspinwall and Morrison requires in an essential way that members of the mirror Calabi–Yau family should have nontrivial (and in fact non-cyclic) fundamental group. Computations of Gross [3, Section 3] connect torsion in the integral cohomologies of mirror Calabi–Yau threefolds, and these computations imply that the cohomology (and hence homology) of $Y_t$ should have torsion of some kind. However, the direct relationship between failure of Torelli and the fundamental group seems rather mysterious; compare also Remark 2.4.

Acknowledgments I wish to thank Pelham Wilson, Mark Gross and Peter Newstead for comments and help.

Notation and conventions All schemes and varieties are defined over $\mathbb{C}$. A Calabi–Yau threefold is a normal projective threefold $X$ with canonical Gorenstein singularities, satisfying $K_X \sim 0$ and $H^1(X, O_X) = 0$. Some statements use the language of toric geometry; my notation follows Fulton [5] and Cox–Katz [3, Chapter 3]. If $A$ is a $\mathbb{Z}$-module then $A_{\text{free}}$ denotes the torsion free part.

1. The construction

Following [3], define maps $g_i : \mathbb{P}^4 \to \mathbb{P}^1$ by

$$g_1 : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_0 : z_1 : z_2 : z_3 : z_4]$$
$$g_2 : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_0 : z_1 : z_2 : z_3 : z_4]$$
$$g_3 : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_1 : z_2 : z_3 : z_4 : z_0]$$

where $\xi$ is a fixed primitive fifth root of unity. Let

$$G = \langle g_1, g_2, g_3 \rangle, \quad H = \langle g_1, g_2 \rangle$$

be subgroups of $\text{PGL}(5, \mathbb{C})$ generated by the transformations $g_i$. As abstract groups $H \cong (\mathbb{Z}/5\mathbb{Z})^2, G \cong \mathbb{Z}/5\mathbb{Z} \ltimes (\mathbb{Z}/5\mathbb{Z})^2$.

I will be interested in hypersurfaces in the varieties $\mathbb{P}^4/G$ and $\mathbb{P}^4/H$; the latter is a toric variety and its toric description will be useful in the sequel.

Proposition 1.1. In the contravariant description, $\mathbb{P}^4/H \cong \mathbb{P}_{M, \Delta}$, where $M \cong \mathbb{Z}^4$ and $\Delta \subset M_\mathbb{R}$ is the polyhedron

$$\Delta = \text{span}\{ (1, 0, 0, 0), (-3, 5, -4, -2), (0, 0, 1, 0), (0, 0, 0, 1), (2, -5, 3, 1) \}.$$ 

With $N = \text{Hom}(M, \mathbb{Z})$, the dual polyhedron $\Delta^* \subset N_\mathbb{R}$ of $\Delta$ is

$$\Delta^* = \text{span}\{ (-1, -2, -1, -1), (4, 1, -1, -1), (-1, -1, -1, -1), (-1, 2, 4, -1), (-1, 0, -1, 4) \}.$$ 

The polyhedron $\Delta^*$ has no interior lattice points apart from the origin, has no lattice points in the interiors of its three- or one-dimensional faces, and has precisely
two lattice points $P_{2i-1}, P_{2i}, i = 1, \ldots, 10$ in the interiors of each of its ten two-dimensional faces.

Proof. This is a standard toric calculation; for details see [14 Proposition 1.1].

Let $\Sigma$ be the fan consisting of cones over faces of $\Delta^*$ in $N_\mathbb{R}$. This fan defines the toric variety $\mathbb{X}_{N, \Sigma} \cong \mathbb{P}_{M, \Delta}$.

**Proposition 1.2.** $\mathbb{P}_{M, \Delta}$ is a $\mathbb{Q}$-factorial Gorenstein variety, with ten curves of canonical singularities. Every permutation $\eta$ of the lattice points $\{P_i\}$ gives rise to a partial resolution $\mathbb{X}_{\Sigma_\eta} \to \mathbb{P}_{M, \Delta}$. The varieties $\mathbb{X}_{\Sigma_\eta}$ have isolated singularities only.

Proof. This is basic toric geometry. The curves of singularities correspond to the ten two-dimensional faces of $\Delta^*$. The singularities can be partially resolved by subdividing the fan $\Sigma$ using the lattice points $\{P_i\}$ in any order. Any permutation $\eta$ of these points gives a fan $\Sigma_\eta$ in the space $N_\mathbb{R}$ and a corresponding toric partial resolution $\mathbb{X}_{\Sigma_\eta}$ with isolated singularities.

The family of hypersurfaces of interest in this paper is constructed from

$$Q = \left\{ \sum_{i=0}^{4} z_i^5 - 5t \prod_{i=0}^{4} z_i = 0 \right\} \subset \mathbb{P}^4 \times B,$$

where $B = \mathbb{C} \setminus \{1, \xi, \ldots, \xi^4\}$. Second projection gives a smooth family $p : Q \to B$ of Calabi–Yau quintics $Q_t$. The groups $G$ and $H$ act on $\mathbb{P}^4 \times B$ by acting trivially on $B$, and hence on $Q$; these actions preserve holomorphic three-forms in the fibres. Let

$$\bar{Z} = Q / H,$$

$$\bar{Y} = Q / G = \bar{Z} / K.$$

Here $K \cong \mathbb{Z}/5\mathbb{Z}$ is the group generated by the image of $g_3$ in $\text{Aut}(\bar{Z})$. Both $\bar{Z}$ and $\bar{Y}$ are naturally families over $B$ with fibres $\bar{Z}_t$ and $\bar{Y}_t$ respectively.

**Proposition 1.3.** For $t \in B$, $\bar{Z}_t$ is a canonical Calabi–Yau threefold with ten isolated $\frac{1}{5}(1, 1, 3)$ quotient singularities. The group $K$ acts freely on $\bar{Z}_t$. The variety $\bar{Y}_t$ is a canonical Calabi–Yau threefold with two isolated $\frac{1}{5}(1, 1, 3)$ quotient singularities.

Proof. Easy explicit check.

The family $\bar{Z} \to B$ is a family of non-degenerate anti-canonical hypersurfaces in the toric variety $\mathbb{P}_{\Delta}$. The partial resolutions $\mathbb{X}_{\Sigma_\eta} \to \mathbb{P}_{M, \Delta}$ give rise to morphisms $\mathbb{Z}_\eta \to \bar{Z}$ over $B$, with $\mathbb{Z}_\eta \to B$ a family of nonsingular threefolds as $\mathbb{X}_{\Sigma_\eta}$ is nonsingular in codimension three.

**Proposition 1.4.** The families $\mathbb{Z}_\eta$ are all canonically isomorphic to a unique toric resolution $\mathbb{Z} \to \bar{Z}$ over $B$. For $t \in B$, the fibre $\mathbb{Z}_t$ is a smooth Calabi–Yau threefold with Hodge numbers $h^{1,1}(\mathbb{Z}_t) = 21$, $h^{2,1}(\mathbb{Z}_t) = 1$. In the resolution $\mathbb{Z}_t \to \bar{Z}_t$ there are two exceptional divisors over every singular point $S_t$, a Hirzebruch surface $E_t \cong \mathbb{F}_3$ and a projective plane $F_t \cong \mathbb{P}^2$ intersecting in a $\mathbb{P}^1$ which is the negative section in the Hirzebruch surface and a line in $\mathbb{P}^2$.

Proof. Let $\eta_1, \eta_2$ be two permutations of the interior lattice points. There is a corresponding birational map $\mathbb{X}_{\Sigma_{\eta_1}} \dasharrow \mathbb{X}_{\Sigma_{\eta_2}}$ whose exceptional sets are disjoint from the families $\mathbb{Z}_\eta$. This implies the first part. The other statements follow from easy toric calculations.
Proposition 1.5. The action of the group $K \cong \mathbb{Z}/5\mathbb{Z}$ on $\tilde{Z}$ extends to a free action on the resolution $\tilde{Z}$ over $B$. Thus there is an etale cover $\tilde{Z} \to \tilde{Y} = \mathbb{Z}/K$ over $B$. The fibre $Y_t$ for $t \in B$ is a Calabi–Yau resolution of $\tilde{Y}$ with Hodge numbers $h^{1,1}(Y_t) = 5$, $h^{2,1}(Y_t) = 1$.

Proof. The action of $K$ is generated by the symmetry $g_3$ of $\mathbb{P}^4$. This symmetry descends to the toric variety $\mathbb{P}_\Delta$ as a toric symmetry induced by a lattice isomorphism $\alpha_3 : M \to M$ fixing the polyhedron $\Delta$ and permuting the lattice points $\{P_i\}$. Composition with the permutation induced by $\alpha_3$ gives a correspondence $\eta \to \eta'$ between permutations of the set $\{P_i\}$ and $\alpha_3$ gives rise to an isomorphism $\tilde{g}_3 : X_{\chi,\eta} \to X_{\chi,\eta'}$. This isomorphism restricts to anti-canonical families as an isomorphism $\mathbb{Z}_\eta \to \mathbb{Z}_{\eta'}$, or, by Proposition 1.4, as an automorphism $\mathbb{Z} \to \mathbb{Z}$. By construction, this automorphism is the required extension of $g_3$ and it clearly generates a free group action on $\tilde{Z}$ over $B$. \qed

I conclude this section by proving two auxiliary statements.

Proposition 1.6. The family $\tilde{Y} \to B$ restricted to a neighbourhood of $0 \in B$ is the universal deformation space of its central fibre $\tilde{Y}_0$ in the analytic category.

Proof. By general theory, the projective variety $\tilde{Y}_0$ has a versal deformation space $\mathcal{X} \to S$ in the analytic category. Thus $H^0(\tilde{Y}_0, T_{\tilde{Y}_0}) = 0$ and this implies that $\mathcal{X} \to S$ is in fact universal. By Ran’s extension [2] of the Bogomolov–Tian–Todorov theorem, unobstructedness holds for $\tilde{Y}_0$. Thus $S$ is smooth. Further, the codimension of the singularities of $\tilde{Y}_0$ is three. By the argument of [2, A.4.2], it follows that the first-order tangent space of $S$ at the base point is isomorphic to $H^1(\tilde{Y}_0, T_{\tilde{Y}_0})$, a one-dimensional complex vector space.

In order to prove that $\tilde{Y} \to B$ is the universal deformation space, all I need to show is that its Kodaira–Spencer map is injective. Recall the family $Q \to B$, a deformation of the Fermat quintic $Q_0$ over $B$. Choosing a $(G$-invariant) three-form on $Q_0$ gives rise to a commutative diagram

$$
\begin{array}{ccc}
T_0(B) & \xrightarrow{k} & H^1(Q_0, T_{Q_0}) \\
\parallel & & \xrightarrow{\sim} \uparrow \downarrow j \\
T_0(B) & \xrightarrow{l} & H^1(\tilde{Y}_0, T_{\tilde{Y}_0}) \xrightarrow{\sim} H^1(\tilde{Y}_0, \tilde{\Omega}^2_{\tilde{Y}_0}).
\end{array}
$$

Here $k$ and $l$ are the Kodaira–Spencer maps, whereas the map $j$ is given by pullback of (orbifold) two-forms (the sheaf of orbifold two-forms $\tilde{\Omega}^2_{\tilde{Y}_0}$ is defined carefully in [2, A.3]). The map $k$ is injective, as $Q$ is a nontrivial first-order deformation of $Q_0$. By commutativity, $l$ is also injective. This proves the Proposition. \qed

Proposition 1.7. For $t \in B$, the Calabi–Yau manifold $Z_t$ is simply connected.

Proof. The variety $Z_t$ is a resolution of the threefold $\tilde{Z}_t = Q_t/H$. Let $Q^0_t$ be the open set of $Q_t$ on which the action of $H$ is free; it is the complement of a finite set of points and hence is simply connected. Let $Z^0_t = Q^0_t/H$; $\pi_1(Z^0_t) \cong H$.

The fundamental group of $Z_t$ is a quotient group of $H$. Let $T_t$ be the universal cover of $Z_t$; by the generalized Riemann existence theorem, $T_t$ is an algebraic variety and it clearly has trivial canonical bundle. Let $T^0_t$ be the preimage of $Z^0_t$ under the covering map. Then $T^0_t$ has finite fundamental group; let $\tilde{T}^0_t$ be its universal cover.
\[ \tilde{T}_t^0 \text{ is an algebraic variety again. Notice however, that } Q_t^0, \tilde{T}_t^0 \text{ are both universal covers of the variety } Z_t^0, \text{ and thus by the uniqueness part of the generalized Riemann existence theorem they must be isomorphic. Thus there exists a diagram} \]
\[
\begin{array}{ccc}
Q_t & \supset & Q_t^0 \\
\downarrow & & \downarrow \\
T_t^0 & \subset & T_t \\
\downarrow & & \downarrow \\
Z_t & \supset & Z_t^0 \subset Z_t.
\end{array}
\]

The covering \( Q_t^0 \to T_t^0 \) corresponds to a group \( L \) of holomorphic automorphisms of \( Q_t^0 \). An automorphism of \( Q_t^0 \) can be thought of as a birational self-map of \( Q_t \). However, as \( Q_t \) is a minimal Calabi–Yau threefold with Picard number one, it has no birational self-maps with a nontrivial exceptional locus. So \( L \) consists of automorphisms of \( Q_t \). The fact that the map \( Q_t^0 \to T_t^0 \) factors the map \( Q_t^0 \to Z_t^0 \) implies that \( L \) must be a subgroup of \( H \).

Thus I conclude that \( T_t \) is birational to a quotient \( Q_t/L \) for a subgroup \( L \) of \( H \).

Moreover, \( \chi(Z_t) = 40 \) so \( \chi(T_t) \) equals either 40, 200 or 1000. On the other hand, for every subgroup \( L \) of \( H \), the quotient \( Q_t/L \) has a Calabi–Yau desingularization. As the Euler number is a birational invariant of smooth Calabi–Yau threefolds, the Euler number of this desingularization must be equal to that of \( T_t \). Finally, it is easy to check that \( H \) has no subgroup \( L \) such that a Calabi–Yau desingularization of \( Q_t/L \) has Euler number 200 or 1000. Thus \( L = H \) and so \( T_t = Z_t \) is its own universal cover.

2. Hodge structures

Let \( Z, Y \) denote the differentiable manifolds underlying the fibres \( Z_t, Y_t \). Let \( V_Z = H^3(Z, \mathbb{Z})_{\text{free}}, V_Y = H^3(Y, \mathbb{Z})_{\text{free}}, \) with antisymmetric pairings \( Q_Z, Q_Y \) given by cup product.

**Lemma 2.1.** Pullback by the map \( \pi : Z \to Y \) induces an injection
\[
\pi^* : V_Y \hookrightarrow V_Z
\]
with image of index at most 25. Under this embedding,
\[
Q_Z(\pi^* C_1, \pi^* C_2) = 5 Q_Y(C_1, C_2).
\]
Consequently, there is an embedding of groups
\[
\text{Aut}_Z(V_Z, Q_Z) \to \text{Aut}_Q(V_Y, Q_Y).
\]

**Proof.** The group \( K \cong \mathbb{Z}/5\mathbb{Z} \) acts without fixed points on \( Z \), so the map \( \pi \) induces a spectral sequence
\[
E_2^{p,q} = H^p(K; H^q(Z, \mathbb{Z})) \Rightarrow H^{p+q}(Y, \mathbb{Z}).
\]
The terms \( E_2^{p,q} \) for \( p > 0 \) are torsion, so \( V_Y = (E_\infty^{0,3})_{\text{free}} \). On the other hand, \( (E_\infty^{0,3})_{\text{free}} = H^0(K, H^3(Z, \mathbb{Z})_{\text{free}}) = (V_Z)^K \). There are two differentials from \( E_2^{0,3} \), both having image \( \mathbb{Z}/5\mathbb{Z} \). So there is an injection
\[
\pi^* : V_Y \hookrightarrow (V_Z)^K
\]
with image of index at most 25. This map is an isomorphism when tensored by \( \mathbb{Q} \). As both \( V_Z \) and \( V_Y \) have rank four, \( K \) must act trivially on \( V_Z \) and this proves the first part. The other two statements are immediate. \( \square \)
Let \( D_Y \) be the period domain parameterizing weight 3 polarized Hodge structures on \((V_Y, Q_Y)\). Fixing a point \( t \in B \), a marking \( H^3(Y_t, Z)_{\text{free}} \cong V_Y \) and a universal cover \( \tilde{B} \) of \( B \) leads to holomorphic period maps

\[
\begin{array}{ccc}
\tilde{B} & \xrightarrow{\tilde{\psi}} & D_Y \\
\downarrow & & \downarrow \\
B & \xrightarrow{\psi} & D_Y/\Gamma
\end{array}
\]

where \( \Gamma \) is any subgroup of \( \text{Aut}_Q(V_Y \otimes \mathbb{Q}, Q_Y) \) containing all geometric monodromies and acting properly discontinuously on \( D \). Choose

\[ \Gamma = j(\text{Aut}_Z(V_Z, Q_Z)) \subset \text{Aut}_Q(V_Y \otimes \mathbb{Q}, Q_Y) \]

under the embedding \( j \) of Lemma 2.1.

**Lemma 2.2.** \( \Gamma \) acts properly discontinuously on \( D_Y \), so \( D_Y/\Gamma \) is an analytic space.

**Proof.** See [6, Section I.2].

After all these preparations, I can state

**Theorem 2.3.** For \( \Gamma \) chosen as above, the period map \( \psi : B \to D_Y/\Gamma \) is of degree at least five. More precisely, if \( t_1, t_2 \in B \) satisfy \( t_1^5 = t_2^5 \), then \( \psi(t_1) = \psi(t_2) \). In particular, \( Y_{t_1} \) and \( Y_{t_2} \) have isomorphic rational Hodge structure.

**Proof.** The symmetry

\[ g : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [\xi^{-1} z_0 : z_1 : z_2 : z_3 : z_4]. \]

descends to a symmetry of \( \mathbb{P}^4/H \) and maps \( \bar{Z}_t \) isomorphically to \( \bar{Z}_{\xi t} \). By an argument analogous to the proof of Proposition 1.5, this isomorphism extends to an isomorphism \( Z_t \to Z_{\xi t} \). This gives a diagram of polarized Hodge structures

\[
\begin{array}{ccc}
H^3(Y_t, Z)_{\text{free}} & \xrightarrow{\pi^*} & H^3(Z_t, Z)_{\text{free}} \\
\downarrow & \cong & \downarrow \\
H^3(Y_{\xi t}, Z)_{\text{free}} & \xrightarrow{\pi^*} & H^3(Z_{\xi t}, Z)_{\text{free}}
\end{array}
\]

Comparing this with the action of \( \Gamma \) on \( D_Y \) defined above gives the first statement. The second statement is immediate.

**Remark 2.4.** The proof of Lemma 2.1 implies that the spectral sequence

\[ E_2^{p,q} = H^p(K; H^q(Z, A)) \Rightarrow H^{p+q}(Y, A) \]

degenerates at \( E_2 \) whenever 5 is invertible in \( A \). In particular, there is an isomorphism of polarized Hodge structures

\[ H^3(Y_t, Z[1/5]) \cong H^3(Y_{\xi t}, Z[1/5]). \]

The problem is that \( \text{Aut}(V_Y \otimes \mathbb{Z}[1/5], Q_Y) \) does not act properly discontinuously on \( D_Y \), so such a statement is weaker than the one proved above. On the other hand, it seems difficult to determine the precise behavior of the spectral sequence with \( \mathbb{Z} \) coefficients, i.e. to compute the torsion in the cohomology of \( Y \).
Remark 2.5. The isomorphism of $\mathbb{Q}$-Hodge structures is due to Aspinwall and Morrison. They give a different proof coming from mirror symmetry which goes as follows. The mirror family $\mathcal{X}$ of $\mathcal{Y}$ is the quotient of a suitable family of quintic hypersurfaces by the group $\langle g_1, g_3 \rangle$. In particular, the antichiral ring of the central fibre $X_0$ of $\mathcal{X}$ with a choice of (complexified) Kähler class is isomorphic to the chiral ring of $Y_t$. On the other hand, the antichiral ring of $X_0$ can be shown to depend, via the mirror map, on $t^5$ only and not on $t$. Thus the varieties $Y_{\xi^i}$ for $i = 0, \ldots, 4$ have the same chiral ring, i.e. isomorphic rational Hodge structure.

Remark 2.6. Suppose that $Y_0$ is an $n$-fold, $G$ (a nontrivial quotient of) the fundamental group $\pi_1(Y_0)$. Then there is an étale cover $Z \to Y$; in fact there is a cover $Z_t \to Y_t$ for every deformation $Y_t$ of $Y_0$. The (primitive) cohomology $H^n_0(Z_t)$ becomes a $G$-representation, and in some cases one can recover information about $Y_t$ from the pair $(H^n_0(Z_t), \text{ action of } G)$.

A particular example of this construction is the theorem of Horikawa [8], giving a Torelli-type result for Enriques surfaces using global Torelli for K3s. However, by Proposition 1.7, the threefold $Z_t$ under investigation is simply connected. On the other hand, as the proofs above show, the Hodge structure on the middle-dimensional rational cohomology of the universal cover $Z_t$ contains no extra information, and it carries the trivial action of the fundamental group $\pi_1(Y_t)$.

3. The automorphism group of the central fibre

**Theorem 3.1.** The automorphism groups of the varieties $Y_0$, $\bar{Y}_0$ are
\[ \text{Aut}(Y_0) \cong \text{Aut}(\bar{Y}_0) \cong \langle G, g_4, g_5 \rangle / G, \]
where
\[ g_4 : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_0 : z_1 : z_2 : \xi^4 z_3 : \xi z_4], \]
\[ g_5 : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_0 : z_2 : z_4 : z_1 : z_3]. \]

In particular, every automorphism of $\bar{Y}_0$ extends to an automorphism on all (small) deformations $\bar{Y}_t$ of $\bar{Y}_0$.

**Proof.** The proof of Theorem 3.1 uses three Lemmas. The first one should certainly be well-known, but I could not find a suitable reference so I included a proof.

**Lemma 3.2.** Let
\[ X = \left\{ \sum_{i=0}^{n} a_i^d = 0 \right\} \subset \mathbb{P}_k^n \]
be the Fermat hypersurface. Assume that $d \geq 3$, $n \geq 2$ and that $(n, d) \neq (2, 3)$ or $(3, 4)$. Then
\[ \text{Aut}(X) \cong G_{n, d}, \]
where $G_{n, d}$ is the semi-direct product $\Sigma_{n+1} \rtimes (\mu_d)^n$ of a symmetric group and a power of the group of $d$-th roots of unity.
Proof. For $n = 2$, the result is proved in [13]. If $n \geq 3$ and $(n, d) \neq (3, 4)$, then I first claim that every automorphism comes from a projective automorphism in the given embedding. If $n \geq 4$, Lefschetz implies $\text{Pic}(X) \cong \mathbb{Z}$ and then the claim is clear. If $n = 3$ and $d \neq 4$ then the canonical class is (anti-)ample and this easily implies the claim again, see [14].

Take an element $\sigma \in \text{Aut}(X)$ represented by an invertible matrix $A = (a_{ij})$. Apply $A$ to the equation of $X$ and consider the coefficients of $x_0^{d-1}x_1$, $x_0^{d-2}x_1^2$, and $x_0^{d-2}x_1x_i$ for $i > 1$. Their vanishing shows that the set of numbers

$$\{a_{00}^{d-2}a_{01}, a_{10}^{d-2}a_{11}, \ldots, a_{n0}^{d-2}a_{n1}\}$$

solves the homogeneous system of equations given by the invertible matrix $A^T$. So all these quantities are zero. By symmetry, $a_{ij}a_{ik} = 0$ whenever $j \neq k$. Hence $A$ has at most one non-zero entry in each row. Multiplying by a suitable element in $\Sigma_{n+1}$, $A$ can be brought into diagonal form, and then all its entries are $d$-th roots of unity.

Lemma 3.3. Let $X$ be a canonical Calabi–Yau threefold with a finite number $m \geq 2$ of isolated $\frac{1}{3}(1, 1, 3)$ quotient singularities and Picard number one. Let $\pi : X \to \hat{X}$ be the Calabi–Yau resolution. Then $\text{Aut}(X) \cong \text{Aut}(\hat{X})$.

Proof. The Picard group of the resolution $X$ is

$$\text{Pic}_\mathbb{Q}(X) \cong \mathbb{Q}H \oplus \mathbb{Q}E_1 \oplus \mathbb{Q}F_1 \oplus \ldots \oplus \mathbb{Q}E_m \oplus \mathbb{Q}F_m,$$

where $H = \pi^*(\mathcal{O}_X(1))$ and $E_i$, $F_i$ are the classes of the exceptional divisors as described in Proposition 1.4. The intersection numbers are as follows:

\[
\begin{align*}
H^3 &= d > 0 & \text{the degree of } \hat{X}, \\
H \cdot E_i &= H \cdot F_i &= 0 & \text{as } H \text{ is a pullback,} \\
E_i \cdot E_j &= E_i \cdot F_j &= F_i \cdot F_j &= 0 & \text{unless } i = j, \\
E_i^3 &= (KE_i)^2 &= 8 & \text{as } E_i \cong \mathbb{P}_3, \\
F_i^3 &= (KF_i)^2 &= 9 & \text{as } F_i \cong \mathbb{P}_2, \\
E_i^2F_i &= 1, \\
F_i^2E_i &= -3.
\end{align*}
\]

Introducing the basis $H_0 = H$, $H_{2i-1} = E_i + \frac{1}{3}F_i$, $H_{2i} = F_i$ of $\text{Pic}_\mathbb{Q}(X)$, the cubic form takes the shape

$$\left(\sum_{i=0}^{2m} a_i H_i\right)^3 = d a_0^3 + 8 \frac{1}{3} \sum_{i=1}^{m} a_{2i-1}^3 + 9 \sum_{i=1}^{m} a_{2i}^3.$$

Finally, the values of the second Chern class are

$$c_2(X) \cdot E_i = -4, \quad c_2(X) \cdot F_i = -6, \quad c_2(X) \cdot H = c \geq 0,$$

where the last inequality follows from a result of Miyaoka, [11, Theorem 1.1].

Let $\sigma \in \text{Aut}(X)$ be an automorphism. It acts via pullback on $\text{Pic}_\mathbb{Q}(X)$, fixing the cubic form together with the linear form given by cup product with $c_2(X)$. I claim that the element $H_0 = H$ of $\text{Pic}_\mathbb{Q}(X)$ must be fixed under the action. To see this, note that the cubic form has been manufactured to take the shape of the Fermat cubic. Every automorphism of $\text{Pic}_\mathbb{Q}(X)$ must fix the associated (projectivized) hypersurface. The possible automorphisms are known from Lemma 3.2. Moreover, in the present case, the multiplications by roots of unity are excluded since $\sigma$ must
fix a rational vector space. The possible permutations are constrained by the fact that $c_2$ has to be fixed as well. As $c_2$ is negative on the $H_i$ for $i > 0$ and non-negative on $H = H_0$, the latter is fixed and this proves the claim.

For large and divisible $m$, the divisor class $mH$ is base-point free and as the torsion in Pic$(X)$ is finite, is the unique representative of its numerical equivalence class. As $H \in \text{Pic}_Q(X)$ is fixed by the induced action of $\sigma$, for large and divisible $m$ the space of sections of the linear system $|mH|$ is also acted on by $\sigma$. In other words, the automorphism $\sigma$ descends to the image of the associated morphism which is exactly $\bar{X}$.

For the converse, note that the quotient singularity $\frac{1}{5}(1, 1, 3)$ has a unique crepant resolution. Hence every automorphism $\bar{\sigma} \in \text{Aut}(\bar{Y})$ extends to a biregular automorphism $\sigma \in \text{Aut}(X)$ of the resolution. The Lemma follows.

**Lemma 3.4.** Let $X$ be a smooth algebraic variety with finite fundamental group $F$. Let $Y$ be the universal cover of $X$, a smooth algebraic variety with an action of $F$ by automorphisms. Then

$$\text{Aut}(X) \cong N_{\text{Aut}(Y)}(F)/F.$$  

**Proof.** Obvious.  

To finish the proof of Theorem 3.1, let $Y_0$ be the open set of the Fermat quintic $Q_0$ on which the action of $G$ is free. Let $Y_0 = Q_0/G$. There is a sequence of maps $$\text{Aut}(Y_0) \hookrightarrow \text{Aut}(Y_0) \cong N_{\text{Aut}(Q_0)}(G)/G \cong N_{\text{Aut}(Q_0)}(G)/G.$$  

The first isomorphism follows from Lemma 3.4. The second isomorphism uses $\text{Aut}(Q_0) \cong \text{Aut}(Q_0)$; here $\text{Aut}(Q_0) \subset \text{Aut}(Q_0)$ is proved by the argument used already in Proposition 1.7 and the other direction is clear by Lemma 3.2.  

On the other hand, by Lemma 3.2 the automorphism group of $Q_0$ is the semi-direct product $G_4,5$ of the permutation and diagonal symmetries. Finding the normalizer of $G$ in $G_4,5$ is a finite search best done using a computer; a short Mathematica routine computes this normalizer to be $$N_{\text{Aut}(Q_0)}(G)/G \cong (G, g_4, g_5)/G$$  

with $g_4, g_5$ as in the statement of Theorem 3.1. So I obtain $$\text{Aut}(Y_0) \hookrightarrow \langle G, g_4, g_5 \rangle/G$$  

and it is easy to see that this is in fact an isomorphism. Finally, by Lemma 3.3, $\text{Aut}(Y_0) \cong \text{Aut}(Y_0)$. This proves the first statement. The second statement follows by inspection: every generator of the normalizer fixes $Q_0$. \hfill $\square$

4. **The proof of Theorem 0.1**

The proof is based on the following rather standard result, a version of which was used in 13 already:

**Theorem 4.1.** Let $X_i \to B$, $i = 1, 2$ be families of canonical Calabi–Yau varieties over a base scheme $B$, having simultaneous resolutions $Y_i \to X_i$ over $B$. Let $L_i$ be relatively ample relative Cartier divisors on $X_i$. Let $\text{Isom}_B(X_i, L_i)$ be the functor

$$\text{Isom}_B(X_i, L_i) : \text{Schemes} \to \text{Sets}.$$
defined by
\[ \text{Isom}_B(X_i, L_i)(S) = \{ \text{polarized } S\text{-isomorphisms} \ (X_1)_S \to (X_2)_S \} , \]
where the pullback families \((X_i)_S\) are polarized by the relatively ample line bundles \((L_i)_S\). This functor is represented by a scheme \(\text{Isom}_B(X_i, L_i)\), proper and unramified over \(B\).

Proof. By Grothendieck’s theory of the representability of Hilbert schemes and related functors, the above functor is represented by a scheme \(\text{Isom}_B(X_i, L_i)\), separated and of finite type over \(B\). The fact that the fibres have no infinitesimal automorphisms implies that \(\text{Isom}_B(X_i, L_i)\) is unramified over \(B\). Properness follows from the valuative criterion along the lines of [4, Proposition 4.4]; the existence of a simultaneous resolution is needed for this final step. □

**Theorem 4.2.** Let \(Y \to B\) be the family constructed in Section 1, \(\xi\) a primitive fifth root of unity. Then there is a Zariski dense subset \(U \subset B\), such that the fibres \(Y_t\) and \(Y_{\xi t}\) are not isomorphic as algebraic varieties for \(t \in U\).

Proof. First I work with the singular family \(\bar{Y}\); for ease of notation, let \(\bar{Y}_1 = \bar{Y}\). Fixing an ample divisor \(L\) on \(\mathbb{P}_\Delta/K\) gives by restriction a relatively ample divisor \(L\) on \(\bar{Y}_1\). Let \(L_1 = L^{\otimes 5}\).

Let \(\gamma : B \to B\) be the map of the base which is multiplication by \(\xi^{-1}\). Let \(\bar{Y}_2 \to B\) denote the pullback of \(\bar{Y}_1 \to B\) by \(\gamma\). The family \(\bar{Y}_2 \to B\) is equipped with the relatively ample line bundle \(L_2 = \gamma^*(L_1)\) and its fibre over \(t \in B\) is \(Y_{\xi t}\).

**Lemma 4.3.** Let \(t \in B\), and let \(\bar{Y}_{i,t}\) be the fibres of the two families polarized by the ample divisors \(L_{i,t}\). Then every isomorphism \(\varphi : \bar{Y}_{1,t} \to \bar{Y}_{2,t}\) satisfies \(\varphi^*(L_{2,t}) \sim L_{1,t}\).

Proof. The fibres have Picard number one, and multiplication by five annihilates every torsion element in their Picard groups. So the divisors \(L_{i,t}\) are canonical elements of the respective Picard groups. The lemma follows. □

Continuing the proof of Theorem 4.2, consider the relative isomorphism scheme
\[ \text{Isom} = \text{Isom}_B(\bar{Y}_i, L_i) \]

together with the natural map \(\text{Isom} \to B\). By Theorem 1.1, this map is proper, so its image \(V\) is a closed subvariety of the quasi-projective variety \(B\).

Assume first that \(V = B\). Then \(\text{Isom}\) has a component \(I\) with a surjective unramified map onto a Zariski neighbourhood of \(0 \in B\). Now switch to the complex topology; let \(\Delta\) be a disc in \(I\) mapping isomorphically onto a neighbourhood of \(0 \in B\). Consider the pullback families \(\bar{Y}_{i,\Delta} \to \Delta\). By the definition of \(I\), these families are isomorphic under an isomorphism \(\varphi\) over \(\Delta\).

Consider the composition
\[ \bar{Y}_{1,\Delta} \xrightarrow{\varphi} \bar{Y}_{2,\Delta} \xrightarrow{\gamma^{-1}} \bar{Y}_{1,\Delta}. \]

Its restriction to the central fibre \(Y_0\) is a polarized automorphism \(\sigma\).

By Proposition 1.3, \(\bar{Y}_1 \to \Delta\) is the universal deformation space of \(Y_0\) in the analytic category. The automorphism \(\sigma\) acts on the base of the deformation space by universality. This action equals the composite of the actions of \(\varphi\) and \((\gamma^{-1})^*\)
on the base $\Delta$. However, $\varphi$ is an isomorphism over $\Delta$, so the action of $\sigma$ on $\Delta$ is multiplication by a primitive fifth root of unity, i.e. a rotation of the disc.

On the other hand, by Theorem 3.1, the action of every automorphism of $\bar{Y}_0$ on the base of the universal deformation space is trivial. Thus $\sigma$ cannot exist. So the assumption $V = B$ leads to a contradiction.

Thus $V$ is a proper closed subset of $B$. Let $U = B \setminus V$, a Zariski open subset of $B$. Over $t \in U$ the scheme $\text{Isom}$ has no points. Using Lemma 4.3, this implies that for $t \in U$ there cannot exist any isomorphism between $\bar{Y}_t$ and $\bar{Y}_{\xi_t}$.

Finally, if $Y_t \cong Y_{\xi_t}$ for some $t \in B$, then an argument analogous to the proof of Lemma 3.3 shows that the singular Calabi–Yau models $\bar{Y}_t$, $\bar{Y}_{\xi_t}$ are also isomorphic. This concludes the proof of Theorem 4.2.

Applying this theorem for $\xi^i$, $i = 1, \ldots, 4$ and taking the intersection of the resulting open sets concludes the proof of Theorem 0.1 announced in the Introduction.

Remark 4.4. Theorem 0.1 is also argued for in the paper [1]. Aspinwall and Morrison write down a power series in the coordinate $t$ of the base $B$, following [2], related to higher genus Gromov–Witten invariants of the family mirror family $\bar{X}$. This series is a function of $t$ rather than $t^5$, and this is a strong indication of the validity of Theorem 0.1. As a matter of fact, I believe that this is also an indication of the validity of Conjecture 0.2. However, a solid mathematical definition, let alone computation, of this power series has not been given to date.

References

1. P. Aspinwall and D. Morrison, Chiral rings do not suffice: $N = (2, 2)$ theories with nonzero fundamental group, Phys. Lett. B 334 (1994) 79–86.
2. M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Comm. Math. Phys. 165 (1994), 311–427.
3. D.A. Cox and S. Katz, Mirror symmetry and algebraic geometry, Mathematical Surveys and Monographs 68, American Mathematical Society, Providence, 1999.
4. B. Fantechi and R. Pardini, Automorphisms and moduli spaces of varieties with ample canonical class via deformations of abelian covers, Comm. Algebra 25 (1997), 1413–1441.
5. W. Fulton, Introduction to toric varieties, Annals of Math. Stud. 131, Princeton University Press, Princeton, 1993.
6. P.A. Griffiths, Periods of integrals on algebraic manifolds I, Amer. J. Math. 90 (1968) 568–626.
7. M. Gross, Special Lagrangian Fibrations II: Geometry. A survey of techniques in the study of special Lagrangian fibrations, in: Surveys in differential geometry: differential geometry inspired by string theory, 341–403, Surv. Differ. Geom. 5, Int. Press, Boston, MA, 1999.
8. E. Horikawa, On the periods of Enriques surfaces I–II, Math. Ann. 234 (1978), 73–88, 235 (1978), 217–246.
9. J. Kollár, Flops, Nagoya Math. J. 113 (1989), 15–36.
10. H. Matsumura and P. Monsky, On the automorphisms of hypersurfaces, J. Math. Kyoto Univ. 3 (1963/1964) 347–361.
11. Y. Miyaoka, The Chern classes and Kodaira dimension of a minimal variety, in: Algebraic geometry, Sendai 1985 (ed. T. Oda), 449–476, Adv. Stud. Pure Math. 10, North-Holland, Amsterdam–New York, 1987.
12. Z. Ran, Unobstructedness of Calabi–Yau orbifoldfolds, J. Math. Phys. 39 (1998), 625–629.
13. B. Szendrői, Calabi–Yau threefolds with a curve of singularities and counterexamples to the Torelli problem, International J. Math. 11 (2000) 449-459.
14. B. Szendrői, On a conjecture of Cox and Katz, Math. Z. 240 (2002) 233-241.
15. P. Tzermias, The group of automorphisms of the Fermat curve, J. Number Theory 53 (1995), 173–178.
Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom
E-mail address: balazs@maths.warwick.ac.uk