Subleading-$N_c$ corrections in non-linear small-$x$ evolution

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We explore the subleading-$N_c$ corrections to the large-$N_c$ Balitsky–Kovchegov (BK) evolution equation by comparing its solution to that of the all-$N_c$ Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) equation. In earlier simulations it was observed that the difference between the solutions of JIMWLK and BK is unusually small for a quark dipole scattering amplitude, of the order of $0.1\%$, which is two orders of magnitude smaller than the naively expected $1/N_c^2 \approx 11\%$. In this paper we argue that this smallness is not accidental. We provide analytical arguments showing that saturation effects and correlator coincidence limits fixed by group theory constraints conspire with the particular structure of the dipole kernel to suppress subleading-$N_c$ corrections reducing the difference between the solutions of JIMWLK and BK to $0.1\%$. We solve the JIMWLK equation with improved numerical accuracy and verify that the remaining $1/N_c$ corrections, while small, still manage to slow down the rapidity-dependence of JIMWLK evolution compared to that of BK. We demonstrate that a truncation of JIMWLK evolution in the form of a minimal Gaussian generalization of the BK equation captures some of the remaining $1/N_c$ contributions leading to an even better agreement with JIMWLK evolution. As the $1/N_c$ corrections to BK include multi-reggeon exchanges one may conclude that the net effect of multi-reggeon exchanges on the dipole amplitude is rather small.

1 Introduction

Little is known about the features of small-$x$ evolution in the Color Glass Condensate (CGC) picture [1–28] beyond the Balitsky–Kovchegov (BK) truncation [21–25] of the Balitsky hierarchy of evolution equations [23–25]. Besides the theoretical work deriving the Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) equations that summarize the Balitsky hierarchies in a compact form, only a single numerical study of generic properties of the full evolution equations is available, carried out by Rummukainen and Weigert [29]. All other studies employ some additional approximation, typically in form of the BK truncation or even more schematic approximation. The BK truncation, as the Mueller dipole model [3–5] it is based on, explicitly neglects $1/N_c$ corrections to the full $\ln(1/x)$ evolution of QCD observables at high energy. Nevertheless, both JIMWLK evolution and its BK truncation correctly reproduce the $N_c$-dependence of the linear Balitsky–Fadin–Kuraev–Lipatov (BFKL) [30, 31] evolution equation in their respective low density limits. This implies that in the linear, low density (BFKL) domain subleading $1/N_c$ corrections are manifestly absent from JIMWLK evolution. The influence of $1/N_c$ corrections on the non-linear part of the full, untruncated evolution equations is much harder to estimate.
The only study of the full leading-ln(1/x) JIMWLK equation available [29] has established, albeit only summarily, that the 1/Nc corrections appear to be much smaller than the 1/Nc² naively expected for the gluon-dominated evolution. Instead of expected 1/Nc² ≈ 10% corrections, the JIMWLK solution for the scattering amplitude of a quark dipole on a target nucleus found in [29] differs from the solution of the BK equation for the same quantity by only 0.1%. This has established the BK equation as a reasonable tool to predict the energy dependence of CGC cross sections, at least after running coupling and some DGLAP corrections are included [32–35]. However, the question remains whether the unexpectedly small difference found in [29] is accidental, being perhaps due to either some intrinsic properties of the calculated dipole amplitude or to some features of the numerical setup used in [29]. In this paper we argue that the smallness of the 1/Nc corrections found in [29] is not accidental. In fact it is imposed by an interplay of group theoretical properties and saturation effects of the CGC. As a result non-linear small-x evolution turns out to be an example of a system in which the 1/Nc corrections are much smaller than naively expected.

We should emphasize that our discussion remains strictly within the context of JIMWLK evolution and within that only explores the contextual neighborhood of the BK truncation. The JIMWLK evolution equation is valid for scattering on a large target that provides a strong gluon field, e.g. for a nucleus with a large atomic number A. It does not include contributions of diagrams which are not enhanced by the strong target field or, for a large nucleus, are subleading in powers of A. This excludes, right from the start any discussion of pomeron loop contributions, as they are not included in the JIMWLK framework. Indeed for small targets with a weaker gluon field like a proton, which has A = 1, pomeron loops are not parametrically suppressed anymore. While pomeron loop induced fluctuations have also recently been identified in [36] as a source of possible large factorization violations for such small targets with some parametric uncertainty, [37] had found that running coupling corrections tend to strongly numerically suppress such fluctuations, so that we feel that our exclusion of pomeron loops from the analysis should not lead to a very severe restriction for the applicability of our results.

In addition to the large gluon field in the target required for JIMWLK evolution, the BK evolution equation induces a correlator factorization assumption that is valid only in the large-Nc limit. [See Eq. (6) below.] Hence the BK factorization (6) has two types of corrections: those suppressed by the powers of A and those suppressed by powers of Nc. In this paper we are interested in the second kind of corrections only, in 1/Nc corrections, which are resummed to all orders in the JIMWLK equation but are excluded in the BK equation.

Even within the purview of JIMWLK evolution we restrict ourselves to a subset of phenomena: We only discuss how 1/Nc suppressed contributions affect dipole evolution. Quantities that have no good approximation in terms of (multi-) dipole projectiles scattering on dense targets are beyond the scope of our discussion. An example not addressed here would be pA scattering at high energies: any realistic description of a proton projectile lies far outside the standard dipole large Nc approximation, despite the fact that JIMWLK evolution does cover this example faithfully. Obviously, in situations like this, where even a leading order large Nc dipole description is unavailable any discussion of the size of 1/Nc-corrections is moot.

One should also note that JIMWLK and BK equations were both first derived at leading order αs ln(1/x), but have a whole tower of αs-suppressed corrections, of which only the next to leading order (NLO) terms are partially available. Running coupling corrections [32–34] have been calcu-

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1 Kuokkanen, Rummukainen, Weigert, in preparation.
lated and partial results for the remaining contributions (new physics channels) are available [34,38]. These corrections will change quantitative features to some extent, but should not completely distort the qualitative structures found at leading order. Our discussion and simulations will therefore focus on the leading order situation and only comment on NLO corrections where possible.

In Sect. 2 we prepare the ground for our arguments, reminding the reader about the differences between the BK equation and the JIMWLK equation for a $q\bar{q}$ dipole scattering amplitude. The removal of subleading $1/N_c$ corrections in the BK equation is operationally achieved by factorizing the expectation value of the product of a pair of dipole operators into a product of their expectation values. For this reason we refer to the BK equation as a factorized truncation of the JIMWLK equation. The difference of the unfactorized and the factorized expectation values measures the size of the factorization violations. The factorization violation $\Delta$ is defined in Sect. 2 [see (8)], where we also present its main features. At one loop accuracy, a vanishing $\Delta$ would imply a complete decoupling of all $1/N_c$ corrections from dipole correlators and is thus the crucial quantity to explore.\(^2\)

In Sect. 3 we will clarify the reason for the smallness of the factorization violations observed in [29], for “typical” factorization violations $\Delta$. A more in depth discussion of this issue than that offered in [29] must first note that the correlator $\Delta$ measuring the factorization violation itself has in fact contributions that do reach all the way up to their natural size of $1/N_c^2$ in certain regions of configuration space. (Note that $\Delta$ depends on three transverse coordinates: the positions of the original quark and anti-quark, and of the emitted gluon. Varying those coordinates gives different values of $\Delta$.) However, as observed in [29], the typical contributions to $\Delta$ in the majority of configuration space are in fact tiny compared to $1/N_c^2$. In Sect. 3 we will systematically map out configuration space to identify all regions with factorization violations. We will argue on general grounds that the factorization violation $\Delta$ is indeed much smaller than $1/N_c^2$ in the majority of its configuration space, in agreement with the result of numerical simulations presented in [29]. We will also demonstrate analytically that the evolution kernel wipes out all contributions from the only region where the factorization violation $\Delta$ is of the naively expected order $1/N_c^2$. We thus will complete the proof of the statement that $1/N_c$ corrections to BK evolution, which are consistently included into JIMWLK evolution, are indeed much smaller than $1/N_c^2$. This constitutes our first main result.

The basis of our mapping out of configuration space in a systematic way is the insight that the origin of the factorization violations is to be found in a set of group theoretical identities that apply to coincidence points of (s-channel) $n$-point functions involved in the Balitsky hierarchies (i.e., the limits in which any pair of the transverse coordinates overlaps). The identities are shown in (12). They are, by construction, respected in JIMWLK evolution, but automatically broken at the $1/N_c^2$ level by the correlator factorization assumption underlying the BK truncation.

In Sect. 4 we note that it is possible to extend the BK equations in a minimal manner that reinstates these group theoretical constraints for all eikonal correlators in high energy scattering. The inspiration comes from calculating all involved Wilson line correlators in the quasi-classical approximation known as the McLerran-Venugopalan (MV) model [6–8]. One can sum up all Glauber-Mueller (GM) multiple rescatterings [39] to calculate various 2- and 3-point functions (see e.g. [12, 40–43]). Using the resulting correlation functions one can construct the factorization violation $\Delta$ and study

\(^2\)At NLO, running coupling corrections primarily modify the evolution kernel and thus mainly modify how strongly a non-vanishing $\Delta$ affects evolution speed [see also Sect. 5]. Other NLO corrections generically introduce new $1/N_c$ suppressed contributions but are accompanied by an additional power of $\alpha_s$.\(^3\)
its properties. This allows us to revisit our earlier general observation on the structure of $\Delta$ in configuration space and amend it with explicit expressions for the correlators, albeit within a model. However, as was noted in [43] and as we will explain in Sect. 4.1, one can also insert the 2- and 3-point correlators obtained in the GM/MV approximation into the JIMWLK evolution equation for the 2-point correlator (the lowest order equation in the corresponding Balitsky hierarchy). One can then suggest treating the resulting equation as an evolution equation in its own right [43], though no parametric justification/proof of this statement exists. This equation (see (22) below) is thus only a guess for the evolution equation beyond the leading-$N_c$ BK equation. The result will be referred to as a Gaussian truncation (GT) of the Balitsky hierarchies or equivalently the JIMWLK equation. This Gaussian truncation had been introduced originally in [43] as an “exponential parametrization” for $q\bar{q}$ dipoles and a certain set of other correlators, with an evolution equation derived explicitly for the $q\bar{q}$ dipole operator. On this level it was also explicitly used in [27] to unify a diversity of “McLerran-Venugopalan models.”

The relationship of the Gaussian truncation to the BK equation turns out to be unexpectedly subtle: On the one hand it extends the BK truncation in the sense that it includes a set of subleading $1/N_c$ corrections, those “minimally” required to reinstate the coincidence limits violated in the BK factorization. Consistently, the Gaussian truncation reduces to BK in the large $N_c$ limit. On the other hand, Eq. (22), the evolution equation in the Gaussian truncation turns out to be equivalent to the BK evolution equation with respect to dynamical content. The only changes occur in the way this content is mapped onto the expressions for correlators.

In Sect. 4.2 we compare the factorization violation given by GT and by JIMWLK, and find them similar qualitatively, but still quite different quantitatively. Since we view GT as a truncation of JIMWLK evolution and hence the Balitsky hierarchies, we will also clarify where GT breaks consistency with JIMWLK: GT remains only an approximation to the full JIMWLK evolution.

Our analytical arguments are complemented in Sect. 5 by a new numerical study of the JIMWLK evolution equation that goes beyond that of [29] with simulations on much larger lattices in the transverse space, extending the $48^2 - 512^2$-range covered earlier with simulations on $512^2 - 4096^2$ lattices. We emphasize that the simulations presented here are done for fixed coupling only: at present the numerical simulation of the exact JIMWLK kernel with the running coupling corrections found in [32-34] would render the numerical cost prohibitive. To efficiently include them would require us to find an alternative representation that allows a factorized form of the JIMWLK Hamiltonian akin to that used at leading order. This remains beyond the scope of this paper. Nevertheless, the additional numerical effort allows us to reduce extrapolation errors considerably (they arise mostly from the infinite volume limit as it turns out) and establish reliably that JIMWLK evolution is in fact slightly slower than factorized BK evolution: Subleading $1/N_c$ corrections indeed slow down evolution, just as was observed earlier with running coupling corrections. Evolution speed turns out to be particularly sensitive to factorization violations, which is in keeping with the integral expressions of Eq. (31) below. At one loop order, we observe numerically a 3-5% slowdown induced by factorization violations where our simulations approach the scaling region. We argue that running coupling corrections should suppress the UV part of phase space leading to a strong reduction of this difference of evolution speeds between JIMWLK and BK. We cannot estimate the influence of other NLO corrections which may well have their own offsetting effects, but they should not completely distort the leading order picture. We conclude that, while the net effect remains

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3This is different in content as well as in spirit from the Gaussian approximation discussed in [44].
small, $1/N_c$ corrections pull our predictions towards evolution speeds compatible with experiment, not in the opposite direction. This qualitative slowdown effect is our second main conclusion.

In Sect. 6 we concentrate on the physical origin of $1/N_c$ corrections to the BK evolution equation. As noted above, the BK truncation reproduces the $N_c$-dependence of the linear BFKL evolution equation, corresponding to a two-reggeon state in the $t$-channel. However, among other $1/N_c$ corrections, the BK truncation neglects contributions of multiple reggeon exchanges [45–48]. Some of those omitted higher reggeon exchanges, like the odderon contribution corresponding to a $C$-odd three-reggeon exchange [49–53], have been included in the BK-truncated CGC formalism by a minimal modification of the truncated evolution equations [54–56]. Higher-order reggeon exchanges usually require substantial modification of Mueller’s dipole model to a more generic $s$-channel picture as one generically is required to include $1/N_c$ suppressed multipole correlators on top of simple dipoles: see [57] for an analogue of the Bartels–Jareelszewicz–Kwieciński–Praszalowicz (BJKP) evolution equation [45–48] in the $s$-channel formalism. Generically not much has been done to identify the contributions of higher $n$-reggeon exchange contributions [58] to nonlinear JIMWLK evolution in any systematic way.

Nevertheless, the fact that the odderon [54–56] and 4-reggeon [57] exchanges are included in the $s$-channel evolution picture allows us to conjecture that all multi-reggeon exchanges are included in the JIMWLK evolution equation. The JIMWLK equation also probably includes some multi-reggeon vertices containing more legs on the target side of the evolution than on the projectile side. If this conjecture is true, one concludes that the difference between the dipole amplitude given by BK and by JIMWLK is at least partially due to an aggregate of multiple-reggeon effects. The smallness of this difference then, in turn, would indicate the smallness of multiple-reggeon exchange effects.

The link of multi-reggeon exchanges with subleading $1/N_c$ corrections gives a natural explanation for the slowdown of JIMWLK evolution compared to BK observed in Sect. 5. Generically one would argue that nonlinear effects will work to temper any influence of multi-reggeon contributions, which would complement the power suppression of $1/N_c$ contributions via the kernel observed earlier in our line of argument. If true, this is testable numerically, but it is not a priori clear how to test this. Identifying the Gaussian truncation with iterated two-reggeon exchange gives a handle on this as well: we may filter out the multi-reggeon exchanges by comparing the Gaussian truncation with full JIMWLK evolution. It turns out that the Gaussian truncation has a distinctive feature that is naturally violated by multi-reggeon exchanges: the Gaussian truncation would predict strict Casimir scaling of dipole correlators in different representations. (Casimir scaling is defined in (23).) In Sect. 6 we illustrate this statement by extending GT to include the simplest multi-reggeon contribution in the form of an odderon exchange: we then show that it indeed violates the Casimir scaling. Therefore we argue that the size of Casimir scaling violations can quantify the net contribution of all multi-reggeon exchanges. We thus can numerically explore the effect of multi-reggeon exchanges by measuring the violations of Casimir scaling of the dipole correlators. Casimir scaling violation in the numerical solution of JIMWLK that we performed is studied in Sect. 6. It turns out that the Casimir scaling violations (which summarize the collective effect of all multi-reggeon exchanges included in JIMWLK evolution) are generically small and do not grow with energy (see e.g. Fig. 10). This is our third main result.

We review our results and methods in Sect. 7.
2 Dipole evolution in JIMWLK and BK frameworks

JIMWLK evolution is equivalent to sets of coupled infinite hierarchies of evolution equations, the simplest of which is based on the equation for the $q\bar{q}$-dipole correlators $\langle \hat{S}^{q\bar{q}}_{x\bar{y}} \rangle(\hat{Y})$ for the scattering on a target at high energies in which the scattering of the $q$ and $\bar{q}$ is expressed via light-like Wilson lines in the fundamental representation $U_x$ or $U_y$ respectively (at fixed transverse positions $x, y$),

$$ \hat{S}^{q\bar{q}}_{x\bar{y}} := \frac{\text{tr}(U_x U_y^\dagger)}{N_c}. $$

(1)

This operator is gauge invariant in the sense that the contributions that close the trace at $x^+ = \pm\infty$ are unity to leading order in $\ln(1/x)$.

Averaging the operator in Eq. (1) over all states in the target wave function yields the $Y$-dependent S-matrix for the scattering of a dipole on that specific target. The evolution equation for this average, $\langle \hat{S}_{x\bar{y}} \rangle(Y)$, involves a gluon Wilson line operator $\hat{U}$ in the adjoint representation on its right-hand side. At fixed coupling can be written either as [18, 23, 24]

$$ \frac{d}{dY} \langle \text{tr}(U_x U_y^\dagger) \rangle(Y) = \frac{\alpha_s}{\pi} \int d^2 z \ k_{x\bar{y}} \left( \langle \tilde{U}_z \rangle \text{tr}(t^a U_x t^b U_y^\dagger) \rangle(Y) - C_f \langle \text{tr}(U_x U_y^\dagger) \rangle(Y) \right) $$

(2)

or, using (1) and the Fierz identity

$$ \langle \tilde{U}_z \rangle \text{tr}(t^a U_x t^b U_y^\dagger) = \text{tr}(U_x U_y^\dagger) \text{tr}(U_x U_y) - \frac{1}{N_c} \text{tr}(U_x U_y) $$

(3)

as

$$ \frac{d}{dY} \langle \hat{S}_{x\bar{y}} \rangle(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \ k_{x\bar{y}} \langle \hat{S}_{xz} \hat{S}_{z\bar{y}} - \hat{S}_{x\bar{y}} \rangle(Y). $$

(4)

The integral kernel in both (2) and (4) is given by [3, 21]

$$ k_{x\bar{y}} := \frac{(x - y)^2}{(x - z)^2 (z - y)^2}. $$

(5)

Eqs. (2) and (4) are completely equivalent versions of the first equation in the Balitsky hierarchy of the quark dipole operator (1). Eqs. (2) and (4) obviously do not represent closed equations since the evolution of $\langle \text{tr}(U_x U_y^\dagger) \rangle(Y)$ depends on an operator with an additional gluon operator $\hat{U}$ insertion.

The evolution equation of that new operator, $\langle \langle \tilde{U}_z \rangle \text{tr}(t^a U_x t^b U_y^\dagger) \rangle(Y)$, in turn will involve yet one more insertion of a gluon operator $\hat{U}$, iteratively creating an infinite coupled hierarchy of evolution equations, the Balitsky hierarchy of the quark dipole operator (1) [23, 24]. JIMWLK evolution summarizes the totality of all such hierarchies, based on any (gauge invariant) combination of multipole operators but can only be solved numerically [29] at considerable numerical cost. The situation can be simplified for the price of introducing an additional approximation that truncates the hierarchy. The most widely used truncation is known as the BK approximation. It assumes the factorization

$$ \langle \hat{S}_{xz} \hat{S}_{z\bar{y}} \rangle(Y) \rightarrow \langle \hat{S}_{xz} \rangle(Y) \langle \hat{S}_{z\bar{y}} \rangle(Y). $$

(6)
which turns Eq. (11) into a closed equation in terms of $\langle \hat{S}_{xy} \rangle(Y)$ only and thus decouples the rest of the Balitsky hierarchy. The BK truncation is valid and is parametrically justified in the large-$N_c$ limit for scattering on a large dilute nuclear target. Using (6) in (4) we obtain the BK evolution equation

$$\frac{d}{dY} \langle \hat{S}_{xy} \rangle(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2z \ K_{xzy} \left[ \langle \hat{S}_{xz} \rangle(Y) \langle \hat{S}_{zy} \rangle(Y) - \langle \hat{S}_{zy} \rangle(Y) \right].$$

(7)

Provided that the dipole correlator shapes $\langle \hat{S}_{xy} \rangle(Y)$ are not too different (this notion with be refined in Sec. 5 in JIMWLK (without factorization (6)) and BK (with factorization, as shown in (7)) the factorization violation that creates the difference between the two

$$\Delta_{xzy}(Y) := \langle \hat{S}_{xz} \hat{S}_{zy} \rangle(Y) - \langle \hat{S}_{xz} \rangle(Y) \langle \hat{S}_{zy} \rangle(Y)$$

(8a)

can be simply interpreted as the difference of the correlators on the right-hand side of Eq. (2) and its BK counterpart (7), i.e.

$$\Delta_{xzy}(Y) = \left[ \langle \hat{S}_{xz} \hat{S}_{zy} \rangle(Y) \right] - \left[ \langle \hat{S}_{xz} \rangle(Y) \langle \hat{S}_{zy} \rangle(Y) - \langle \hat{S}_{zy} \rangle(Y) \right]$$

(8b)

(the $\langle \hat{S}_{xy} \rangle(Y)$ term is the same under these conditions and cancels trivially) or as a fluctuation away from a mean field value

$$\Delta_{xzy}(Y) = \langle \left( \hat{S}_{xz} - \langle \hat{S}_{xz} \rangle(Y) \right) \left( \hat{S}_{zy} - \langle \hat{S}_{zy} \rangle(Y) \right) \rangle(Y).$$

(8c)

The first interpretation directly leads us to consider factorization violations as a source for a difference in evolution speed in JIMWLK and BK, the second interpretation will lead us to the question of what kind of degrees of freedom (which are absent in BK but included in JIMWLK) would be associated with these fluctuations. We meet the latter question repeatedly in all remaining sections, here we will first look at the individual terms in Eq. (8b) to get a generic idea of the structure of configuration space and how it affects evolution and then give a first glimpse at how JIMWLK evolution via (4) might differ from BK evolution (7).

In both cases in an otherwise translationally invariant system with a given parent dipole the integrands (correlators and kernel separately – that is why we will leave the latter aside) have a twofold mirror symmetry in the $z$-plane: one with respect to the $(x - y)$-axis, the other with respect to an axis perpendicular to $(x - y)$, through the midpoint $(x + y)/2$. The latter only holds if $\langle \text{tr}(U_x U_y^\dagger) \rangle(Y) = \langle \text{tr}(U_y U_x^\dagger) \rangle(Y)$, i.e., it is real (and thus symmetric in $x \leftrightarrow y$) as is the case if we study its contribution to the total DIS cross section at high energy [21, 23]. In this context it is useful to introduce a $z$ coordinate with respect to $(x + y)/2$ as the origin

$$z' := z - (x + y)/2.$$  

(9)

There are strong zeroes in the correlators on the right-hand side of the evolution equations as well as in $\Delta$ when $z \to x$ or $y$. They are needed to cancel the kernel singularities at these points and have their origin in real virtual cancellations. Generically, these zeroes are not isolated but lie on lines that separate the positive from the negative contributions to the integrand of the evolution equations. Picking out the positive sign regions in the integrand, the mirror symmetries allow two situations: one in which there are two separate such regions adjacent to the $q$ and $\bar{q}$ respectively,
and another where the regions are joined together [generically in situations with dipole correlators not too dissimilar from the Golec-Biernat–Wüsthoff (GB-W) case [59] (also known as Glauber-Mueller multiple rescatterings) which serves as our initial conditions; the initial conditions are in fact radially symmetric]. The generic patterns are shown in Fig. 1, which presents contour plots of the right-hand side of the BK equation (7) (divided by \( \langle \hat{S}_{zy} \rangle(Y) \)) obtained by performing a numerical solution of that equation. The horizontal and vertical axis on each panel show \( z'_1 \) and \( z'_2 \), which are the two components of the two-dimensional vector \( \vec{z}' \). The coordinates are plotted in the units of the initial correlation length \( R_s(Y_0) \) of the system (formally defined as the distance at which the dipole correlator falls to 1/2, i.e. via \( \langle \hat{S}_{|r|=R_s(Y)} \rangle(Y) = 1/2 \)). \( R_s(Y) \) can be thought of as the inverse of the saturation scale \( Q_s(Y) \): \( R_s(Y) = 1/Q_s(Y) \). \( Y_0 \) is the rapidity of the initial conditions for (7). The dots in Fig. 1 denote the positions \( x \) and \( y \) of the quark and the anti-quark, with \( |x - y| \) taken in Fig. 1 to be equal to \( R_s(Y_0) \). They lie on the contour lines that separate positive from negative regions. The left panel of Fig. 1 corresponds to the initial conditions for the BK evolution \( (R_s(Y)/R_s(Y_0)) = 1 \), while the right panel corresponds to a higher rapidity \( Y > Y_0 \) where \( R_s(Y)/R_s(Y_0) = 0.34 \), i.e., after running the evolution for some time.

For fixed parent dipole size \( |x - y| \) the most extreme situations arise when we vary \( z' \) along the \( x - y \) axis and the axis perpendicular to it, all other directions in the \( z' \)-plane interpolate smoothly. The two axes, along with one intermediate 45° axis, are also shown in both panels of Fig. 1. Evolution speed, in either JIMWLK or BK, is then a consequence of a numerically delicate balance of the negative and positive regions of the quantity plotted in Fig. 1. Since we are talking of evolution for \( S \) in which generically \( S \) is driven to smaller values at fixed dipole sizes as rapidity \( Y \) increases,

\[
\langle \hat{S}_{xz}/\langle \hat{S}_{zy} \rangle \rangle - 1 = \frac{R_s(Y)/R_s(Y_0)}{|x - y| = R_s(Y_0)}
\]

\[
\langle \hat{S}_{zz}/\langle \hat{S}_{zy} \rangle \rangle - 1 = \frac{R_s(Y)/R_s(Y_0)}{|x - y| = R_s(Y_0)}
\]

**Fig. 1**: Contour plot of the correlators on the right-hand side of the BK equation (7) (divided by \( \langle \hat{S}_{xy} \rangle \)) to normalize the large \( z \) asymptotics to \( -1 \) in the \( z' \)-plane at different stages in the evolution, for fixed \( |x - y| = R_s(Y_0) \). Here \( z' = (z'_1, z'_2) \). The left panel shows the initial conditions for the evolution, while the right panel displays the evolved distribution (see text). The dots mark \( x \) and \( y \), the locations of the parent \( q \) and \( \bar{q} \). These points always fall on the boundaries between positive and negative contributions to the right-hand side of (7) (marked by contour lines going through the dots). The half-rays denote the angles at which the factorization violations will be plotted in Fig. 2.
the negative regions in Fig. 1 push evolution forward (these contributions are generically those at large $|z'|$), while the positive regions in Fig. 1 (generically near $x$ and $y$) slow it down. At fixed coupling, any change of evolution speed can be mapped onto a change of relative weight of these two contributions. Starting from a non-scaling initial condition like the GB-W model, evolution typically speeds up until scaling is reached and evolution speed is maximal. (Scaling here is defined as the situation in which all rapidity dependence is carried by the saturation scale so that observables like $(\hat{S}_r)(Y)$ become functions of scaling ratios like $r/R_s(Y)$ only [60].) This is mirrored perfectly in a shrinking of the positive regions from the radially symmetric situation of the GB-W initial condition (Fig. 1 left) to a situation in which there are two separate positive regions near the $q$ and $\bar{q}$ positions (Fig. 1 right).

The $z'$ plane symmetries of the dipole evolution equations translate directly into the factorization violations $\Delta_{xzy}(Y)$ from Eqs. (8), and also the zeroes at the $q$ and $\bar{q}$ positions carry over. In [29] two of us (Rummukainen and Weigert) had observed numerically that all factorization violations tested were positive (i.e., qualitatively acted to slow down evolution compared to BK), and unexpectedly small, at least in the regions that contribute to evolution: instead of $\Delta \sim 1/N_c^2 \sim 10\%$ at $N_c = 3$ one found contributions roughly another magnitude smaller. Fig. 2 re-illustrates the observation

$$|z'| = 0.4 \cdot R_s(Y) \quad \angle = 0^\circ \quad |z'| = 0.4 \cdot R_s(Y) \quad \angle = 45^\circ \quad |z'| = 0.4 \cdot R_s(Y) \quad \angle = 90^\circ$$

![Fig. 2: Factorization violations from JIMWLK evolution (scaled up by $N_c^2$) plotted against varying parent dipole size at fixed $|z'| = 0.4 \cdot R_s(Y)$. The angles $0^\circ$ (left), $45^\circ$ (middle), and $90^\circ$ (right) are the angles between $z'$ and $r$ and refer to rays in the $z'$-plane as indicated in Fig. 1 for one fixed $|r| = |x - y|$. Shown are three different rapidities each. One obtains a reduction by a factor of 10 compared to the natural size $\Delta^J \sim 1/N_c^2$ (or $\Delta^J N_c^2 \sim 1$). This was observed earlier in [29], with any differences being due to the slightly different correlator geometries chosen here for ease of comparison with the discussion below. Shown are “typical” regions that contribute to evolution, see Sect. 3 for details. Note also that only the $0^\circ$ ray shows special structure since it contains strict coincidence limits (i.e. the limits where $x$, $y$ or $z$ overlap), all other angles are qualitatively well represented by the $90^\circ$ case.

of [29], namely that the factorization violation $\Delta$ is about an order of magnitude smaller than the naively expected $10\%$ in regions relevant for evolution. Fig. 2 plots $\Delta_{xzy}(Y)$ (henceforth referred to as simply $\Delta$ without the arguments) from the numerical solution of JIMWLK evolution equation which we will describe below. $\Delta_{xzy}(Y)$ is plotted in Fig. 2 as a function of the parent dipole size

$$r = x - y$$

(10)
for fixed $|z'|$ with the angle between $z'$ and $r$ being $0^\circ$ (left panel), $45^\circ$ (middle panel) and $90^\circ$ (right panel). These directions were also shown in Fig. 1. In each panel of Fig. 2 the factorization violation $\Delta$ is plotted for three different rapidities: $Y_1$, $Y_1 + 3$, and $Y_1 + 5$, with the exact numerical value of $Y_1$ being irrelevant here (along with the value of the fixed coupling constant $\alpha_s$), as our goal in this Section is only to demonstrate the size of the typical factorization violations.

While the correlator geometries in Fig. 2 differ slightly from those shown in [29], the magnitudes are comparable. We see again that instead of naively expected $\Delta N_c^2 \sim 1$ one gets $\Delta N_c^2 \sim 0.1$, which is an order of magnitude smaller. As rapidity increases beyond the values shown in Fig. 2 the factorization violation $\Delta$ does not grow significantly beyond the values achieved in the figure. In [29] no attempt was made to clarify in which regions of configuration space the factorization violation $\Delta$ is small, and no generic discussion of relative importance of configuration space regions was given. To fully understand the statement of the smallness of corrections one must expand on the discussion given there and first gain a better understanding of where to expect sizable contributions, since mapping out all of the configuration space in $z'$, $x − y$ and $Y$ is otherwise not feasible. This will also provide the underlying reason for the observed smallness.

3 Origin and smallness of the factorization violation: an interplay of saturation and coincidence limits

Smallness of the specific factorization violations (8) are only one facet of a more generic question: what kind of deviations from full JIMWLK evolution are caused by the factorization assumption (6) with its associated truncation of the Balitsky hierarchy of the quark dipole operator?

Full JIMWLK evolution does not only couple in a full hierarchy of evolution equations for the quark dipole operator, it has an even wider scope: It consistently incorporates hierarchies based on any $n$-point correlator. Examples for such distinct hierarchies are obtained by considering the infinite set of dipole correlators labeled by all finite dimensional unitary representations $\mathcal{R}$. Each of them has its own distinct evolution equation, that can be summarily written as

$$
\frac{d}{dY} \langle \bar{R} tr(\bar{R} U_{x, y}) \rangle(Y) = \frac{\alpha_s}{\pi^2} \int d^2 z \mathcal{K}_{xzy} \left( \langle [\tilde{U}_z]^{ab} t^c t^a t^b \rangle \langle \bar{R} tr(\bar{R} U_{x, y}) \rangle(Y) \right) - C_{\mathcal{R}} \langle \bar{R} tr(\bar{R} U_{x, y}) \rangle(Y).
$$

Here $\tilde{U}_z$ refers to the group element in the representation $\mathcal{R}$, with analogous notations for the trace, generators and conjugate representation. $C_{\mathcal{R}}$ denotes the second Casimir of the representation of the dipole, i.e., for the $q\bar{q}$ correlator of the BK case it equals $C_f = \frac{N_c^2 - 1}{2N_c}$ or for a $gg$ dipole it would be $C_A = N_c$. The gluon produced in the evolution step is denoted by $\tilde{U}_z$ and is of course always in the adjoint representation.

The hierarchies based on the dipole equations (11) are by no means all independent (group constraints and coincidence limits may reveal that the $Y$ dependence of the same multi-$\tilde{U}$ correlator does appear in several hierarchies), nor do they exhaust all the information contained in JIMWLK evolution (for instance operators with non-vanishing triality are absent from the family of dipole hierarchies). What is important here is that JIMWLK evolution treats this multitude of correlator equations consistently – as long as no truncation assumptions are made.
The BK approximation greatly simplifies this intricately interlinked set of hierarchies and may not capture all of its features in the process: since there is no simple generalization of the Fierz identity \(^3\) for \(\langle \tilde{U}_{ab} \rangle^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t} (Y)\) in an arbitrary representation \(\mathcal{R}\), it may become impossible to consistently generalize the BK approximation to even this class of equations, despite the fact that one can write expressions for the BK (large \(N_c\)) limit of generic dipole operators. (For the cases of \(\mathcal{R}\) being fundamental or adjoint representations the BK approximation is indeed possible and is done routinely.) To consistently include all equations \(^{11}\) is to go at least one step beyond the BK approximation and below, in Sect. \(^4\) we shall see that this can indeed be achieved quite elegantly.

The key feature satisfied by JIMWLK evolution that is violated by BK factorization beyond the leading-\(N_c\) limit is a set of group identities for the three point correlators \(\langle \tilde{U}_{ab} \rangle^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t} (Y)\) on the right-hand side of the evolution equations \(^{11}\). In what follows we will generically use the term coincidence limits to refer to the limits were any pair of points \((x, y, z)\) or all three of them \((x, y, z)\) coincide with each other. At the coincidence limits the correlator \(\langle \tilde{U}_{ab} \rangle^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t} (Y)\) should inherit relationships that JIMWLK evolution respects on the operator level (see Appendix \(\mathcal{A}\) for their derivation):

\[
\begin{align*}
\lim_{y \to x} \tilde{U}_{z}^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t} = C_{\mathcal{R}} \frac{d_{\mathcal{R}}}{d_{A}} \tilde{t} \tilde{U}_{z}^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t},
\end{align*}
\]

\[\quad (12a)\]

\[
\begin{align*}
\lim_{z \to y \text{ or } x} \tilde{U}_{z}^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t} = C_{\mathcal{R}} \frac{d_{\mathcal{R}}}{d_{A}} \tilde{t} \tilde{U}_{z}^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t},
\end{align*}
\]

\[\quad (12b)\]

\[
\begin{align*}
\lim_{z \to y, y \to x} \tilde{U}_{z}^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t} = C_{\mathcal{R}} d_{\mathcal{R}},
\end{align*}
\]

\[\quad (12c)\]

where \(d_{\mathcal{R}}\) stands for the dimension of the representation \((d_f = N_c \text{ for the fundamental representation, } d_{A} = N_c^2 - 1 \text{ for adjoint, etc.})\) and \(\tilde{t}\) denotes the trace in the adjoint representation. While the third statement is merely a normalization statement, the first two are remarkable: we read off that in the limit of small parent dipole the three point operator on the left-hand side reduces to a gluon dipole, no matter what representation \(\mathcal{R}\) refers to, while in the limit \(z \to x\) or \(y\) it reduces to an \(\mathcal{R}\mathcal{R}\)-dipole. The latter, \((12b)\), is crucial to ensure the real virtual cancellations in \((11)\).

For correlators, the implications of \((12)\) go far beyond the isolated points featuring in the limits shown. Since the correlation (saturation) length \(R_s = 1/Q_s\) is the only dimensionful parameter, the only scale in the problem, \((12)\) determines the generic behavior of \(\langle \tilde{U}_{ab} \rangle^{ab}_{\frac{\alpha c}{\alpha a} \frac{\beta \alpha}{\beta b} \frac{r}{r} t} (Y)\) in all of configuration space.

Configuration space is first divided into two classes in which \(r\) is either smaller or larger than \(R_s\). For each of these classes one has to distinguish two cases according to whether the distance between the gluon and the nearest quark is larger or smaller than \(R_s\). The configurations are shown in Fig.\(^3\) and exhaust all physically different situations (labels “a” through “d” in the figure are in correspondence to the equation labels in \((13)\) below). One infers from \((12)\) that in regions “a” and
It also vanishes trivially in region “c”, where \( \langle \bar{q} q \rangle \) the falloff is dipole-like 

\[
|\mathbf{r}| \ll R_s \ll |\mathbf{z}'| \quad \text{region “a”}
\]

\[
|\mathbf{r}| \cong 2 |\mathbf{z}'| \gg R_s : \quad \langle \tilde{U}_z \rangle \approx C R \frac{d_R}{d_A} \langle \tilde{U}_z U_x^\dagger \rangle(Y), \quad (13a)
\]

\[
|\mathbf{r}| \cong 2 |\mathbf{z}'| \gg R_s : \quad \langle \tilde{U}_z \rangle \approx C R \langle \tilde{U}_z U_x^\dagger \rangle(Y), \quad (13b)
\]

\[
|\mathbf{r}| \gg R_s, |\mathbf{z}'| \neq |\mathbf{r}|/2 : \quad \langle \tilde{U}_z \rangle \approx C R \langle \tilde{U}_z U_x^\dagger \rangle(Y) \to 0. \quad (13c)
\]

This leaves only one region, labeled “d” in Fig. 3 in which the contributions are not suppressed. In this remaining region, all scales are small simultaneously, as one would naively expect in a system with a finite correlation length \( R_s \). In region “d” we have

\[
|\mathbf{z}'|, |\mathbf{r}| \ll R_s : \quad \langle \tilde{U}_z \rangle \approx C R \langle \tilde{U}_z U_x^\dagger \rangle(Y) \lesssim C R. \quad (13d)
\]

Fig. 4 illustrates this theoretical discussion with contour plots of the three point function as obtained from actual JIMWLK simulations. The plots show dependence on \( q\bar{q} \) separation \(|\mathbf{r}|\) and the distance of the gluon location with respect to the \( q\bar{q} \) midpoint \(|\mathbf{z}'|\), with \( \mathbf{z}' \) perpendicular to and parallel to \( \mathbf{r} \), i.e., along two of the lines indicated in Fig. 4. One may notice that the contributions on the
axes, $|\mathbf{r}| = 0$ and $|z'| = 0$ have no angular dependence: the first corresponds to zero size parent dipoles in which case $\mathbf{r}$ does not single out any direction to refer to, the second keeps $\mathbf{z}$ firmly in the middle of the $q\bar{q}$ pair while varying its size so that again the angle does not play a role.

Fig. 4: Behavior of three point correlator as discussed in Eq. (13) taken from our numerical solution of the JIMWLK equation at some intermediate $Y$. $z'$ is varied along rays of fixed angle with respect to $\mathbf{r}$ ($0^\circ$ and $90^\circ$), c.f. Fig. 1. The regions are labeled in correspondence to Fig. 3, the correlators display the generic behavior anticipated in Eq. (13). Note that region “b” is only present near $0^\circ$ and completely disappears for $90^\circ$.

Eqs. (12) and (13) represent but one example of a much larger set of group constraints for more complicated correlators that are all inherently true in the full JIMWLK setting, but broken by the BK factorization. The BK truncation is geared towards quark dipoles (where $\mathcal{R}$ is the fundamental representation), where it approximates the Fierz identity (3) by dropping the $1/N_c$ term

$$\langle [\tilde{U}_z]^{ab} \text{tr}(t^a U_x t^b U_y^\dagger) \rangle \approx \text{tr}(U_x U_z^\dagger) \text{tr}(U_z U_y^\dagger) + \mathcal{O}(1/N_c)$$

This distorts the coincidence limit of the quark dipole version of Eqs. (12) into their large $N_c$ approximations

$$\lim_{y \to x} \text{tr}(U_x U_y^\dagger) = |\text{tr}(U_x U_z^\dagger)|^2$$

$$\lim_{z \to y \text{ or } x} \text{tr}(U_x U_y^\dagger) = \text{tr}(U_x U_y^\dagger) N_c$$

$$\lim_{z \to -x, y \to -x} \text{tr}(U_x U_y^\dagger) = N_c^2,$$

i.e., it approximates the gluon dipole operator on the right-hand side of Eq. (12a) by the square of the quark dipole operator, and replaces the constants in the remaining equations by their large $N_c$ counterparts.

For correlators, the implications of (15) mirror the conclusions drawn in Eq. (13) up to corrections of order $1/N_c^2$, hence one naively expects the factorization violations to be of precisely that order,
unless there is a stronger cancellation at work in the coefficient of that $1/N_c^2$ term. Eq. (13) contains all that is needed to assess this issue if one uses (3) to recast $\Delta$ in terms of two and three point correlators only (the first two terms represent the unfactored correlator):

$$\Delta_{xyz}(Y) = \frac{1}{N_c^2} \left[ \langle [\hat{U}_z]^{ab} 2 \text{tr}(t^a U_x t^b U_y^\dagger) \rangle + \frac{\langle \text{tr}(U_x U_z^\dagger) \rangle}{N_c} - \langle \text{tr}(U_x U_z) \rangle \langle \text{tr}(U_x U_y^\dagger) \rangle \right](Y). \quad (16)$$

The four distinct regions of Fig. 3 and Eq. (13) can then be addressed in turn (all individual correlators are real and positive):

- **Region “a”, $|z'| \gg R_s \gg |r|$**: Both the first and the last term inside the brackets of (16) are exponentially small, but the second term approaches 1. In the extreme case $|z'| \to \infty, |r| \to 0$ one finds $\Delta_{xyz}(Y) \to 1/N_c^2$. Contrary to $\langle [\hat{U}_z]^{ab} 2 \text{tr}(t^a U_x t^b U_y^\dagger) \rangle$, $\langle \text{tr}(U_x U_z^\dagger) \rangle \langle \text{tr}(U_x U_y^\dagger) \rangle$ contains a $z$-independent additive term that survives this limit. If region “a” were to contribute to evolution at all, this would destroy infrared safety of JIMWLK evolution (see below).

- **Region “b”, $|r| \approx 2 |z'| \gg R_s$ (gluon near $q$ or $\bar{q}$)**: Since there are always two large distances involved, all three of the terms in (16) are exponentially suppressed and the contribution is naturally much smaller than $1/N_c^2$.

- **Region “c”, $|r| \gg R_s, |z'| \approx |r|/2$**: With all three inter-particle distances large, all terms are exponentially suppressed individually, rendering their sum much smaller than $1/N_c^2$.

- **Region “d”, $|z'|, |r| \lesssim R_s$**: The terms inside the brackets are order $N_c^2 - 1, 1, \text{and } N_c^2$ respectively. Moreover, in the strict coincidence limit $x = y = z$, they cancel exactly! This guarantees a very strong (albeit not exponential) reduction of the coefficient in front of $1/N_c^2$. The cancellation is slightly less pronounced farther from exact coincidence, for scales $|z'|, |r|$ of order $R_s$, before large distance damping at the boundary to the previous regions sets in.

One concludes that $\Delta$ is strictly bounded from above by $1/N_c^2$, but there is only one region left in which this bound is actually reached – region “a”. In all other regions strong cancellations reduce the contributions to values significantly below this bound.

So far we have used general arguments based on coincidence limits (12) and on the effects of saturation on the dipole scattering amplitude (two-Wilson line correlators) to argue that $\Delta$ is in fact much smaller than $1/N_c^2$ for much of its phase space. To understand the impact of $\Delta$ on the evolution let us rewrite (4) using (5a):

$$\frac{d}{dY} \langle \hat{S}_{xy}(Y) \rangle = \frac{\alpha_s N_c}{2\pi^2} \int d^2z \ K_{zzz} \left[ \langle \hat{S}_{xz}(Y) \rangle \langle \hat{S}_{zy}(Y) \rangle - \langle \hat{S}_{zy}(Y) \rangle \langle \hat{S}_{xy}(Y) \rangle + \Delta_{xyz}(Y) \right]. \quad (17)$$

Eq. (17) shows how $\Delta$ enters the full, untruncated evolution equations.

As we saw above, somewhat surprisingly, region “a” where the maximal possible factorization violation occurs is characterized by small parent dipole size $|r|Q_s \to 0$ but with the gluon produced far away, with $|z'|Q_s \gg 1$. This region, however, has no impact on evolution at all: it is completely power-suppressed by the evolution kernel (5) in (17) which goes to zero as the sixth power of distances involved:

$$\frac{(x - y)^2}{(x - z)^2(z - y)^2} \approx \frac{r^2}{(z')^4} \to 0. \quad (18)$$
Were it not for this kernel suppression, JIMWLK evolution (represented via the $q\bar{q}$ Balitsky hierarchy) would receive large distance contributions from this region – infrared safety would be lost. Notably, region “a” is the only large distance region that requires suppression from the kernel. The remaining large distance regions (“b” and “c”) are exponentially suppressed on the correlator level and automatically decouple from evolution. That leaves the last region (region “d”) with its strong cancellation of contributions as dictated by the properties of the coincidence limits: it remains as the sole channel through which the non-factorized contributions affect the energy dependence of the $q\bar{q}$-dipole. It is this region which was quoted in [29] to contribute the “typical” factorization violations without connecting this to the coincidence limits (12).

While the argument given in this section does not give a parametric estimate for the size of the factorization violations, it does explain why the contributions are naturally much smaller than $1/N_c^2$. We see that factorization violation $\Delta_{xyz}$ is bounded by $1/N_c^2$ from above. However this $1/N_c^2$ value is reached only in a small subset of $(x, y, z)$ configuration space (in region “a”), which is suppressed by the evolution kernel. The relative suppression of the integral of the factorization violation over all z’s in (17) compared to the first term on the right-hand side of the equation is therefore much stronger than the $1/N_c^2$ one would naively expect. Note that the generic arguments given here also do not allow to determine the sign of the contribution and thus do not allow to infer if one should expect JIMWLK evolution to be slower or faster than the factorized BK truncation.

We might now just push ahead and map out configuration space of the JIMWLK 3-point correlators using numerical results from our simulations, to systematically supplement the numerical results of [29] and Fig. 2 with contributions from the regions not shown there (numerical results will be shown in Figs. 6 and 7). Let us instead first give a simple generalization of BK factorization that treats the set of equations (11) consistently and respects the coincidence limits (12). This generalization will restore at least part of the true factorization violation and respect the configuration space pattern deduced from (13) and (16). This will likely improve agreement with JIMWLK evolution and give some insight into the question in which direction evolution speed is changed by the factorization violations.

4 Gaussian truncation of JIMWLK

4.1 A step beyond BK

Our argument for the suppression of $1/N_c^2$ corrections in the previous section were based on saturation effects and coincidence limits. We observed that the BK equation, while incorporating the saturation effects, violates the coincidence limits at the subleading $N_c$ level.

Approaches that both incorporate saturation physics and respect the coincidence limits of the general argument given in Sec. 3 are well established in the literature. They take the form of variants of the McLerran-Venugopalan model [7–9] and the closely related Glauber-Mueller approximation to high energy scattering [2–4], which can be rigorously established by summing QCD diagrams without taking into account small $x$ evolution. All these descriptions fall into a class of approximations of the JIMWLK average $\langle \ldots \rangle(Y)$ over Wilson lines $U_x$, that is characterized by a longitudinally local
Gaussian averaging procedure that can be cast as

\[
\langle \ldots \rangle(Y) = \exp\left\{ -\frac{1}{2} \int dY' d^2x d^2y \ G_{Y',xy} \ \frac{\delta}{\delta A_{x,Y'}^\alpha} \ \frac{\delta}{\delta A_{y,Y'}^\beta} \right\} \ldots . \tag{19}
\]

We refer the reader to [27] for a discussion of how various well known models can be recovered from the generic form shown in Eq. (19) by choosing specific expressions for \( G_{Y',xy} \). In the quasi-classical limit \( G \) encodes a two gluon exchange with the target in the \( t \)-channel. That this same generic approach automatically satisfies the coincidence limits has also been demonstrated in [27] for the case of \( \mathcal{R} \) being the fundamental representation.

Stepping beyond the quasi-classical limit in [43], Kovner and Wiedemann have suggested an all \( N_c \) evolution equation that merges BK principles with the Gaussian treatment of correlators incorporated in Eq. (19) by what amounts to applying the averaging prescription to quark and gluon dipoles.

In fact, Eq. (19) allows us to extend the treatment of [43] beyond the specific case of quark (fundamental) and gluon dipole evolution. This results in a self-consistent treatment of the evolution of all generic dipole operators in which one replaces the \( q \) and \( \bar{q} \) by colored objects in arbitrary representations \( \mathcal{R} \) and \( \bar{\mathcal{R}} \). Doing so, one finds completely generic expressions for the previously discussed correlators (see Appendix B for calculational details and also [12, 40–43] for similar calculations):

\[
\langle \text{tr}(U_x \bar{U}_y) \rangle(Y) = \mathcal{R} \ e^{-C_R \mathcal{G}_{Y,xy}} \tag{20a}
\]

\[
\langle \text{tr}(t^a U_x t^b \bar{U}_y) \rangle(Y) = C_R \mathcal{R} \ e^{-\frac{N_c}{2} (G_{Y,xx} + G_{Y,yy}) - C_R \mathcal{G}_{Y,xy}} \tag{20b}
\]

For convenience we have introduced

\[
\mathcal{G}_{Y,xy} := \int dY' \left( G_{Y',xy} - \frac{1}{2} (G_{Y',xx} + G_{Y',yy}) \right) \tag{21}
\]

to denote the combination in which the t-channel gluons enter these expressions. Note that \( \mathcal{G}_{Y,xx} \equiv 0 \) as required by consistency in (20a). Quick inspection reveals that (20b) indeed complies with (12) as advertised. In fact, this property is not specific to this particular set of correlators. Any correlator calculated using (19) (or any generalization thereof) will automatically satisfy all necessary group constraints by construction.

This procedure then is a candidate to generalize the BK factorization in which one simply trades an evolution equation for \( \langle \hat{S}_{xy} \rangle(Y) \) for an evolution equation for \( \mathcal{G}_{Y,xy} \). The procedure at least qualitatively repairs the flaw that is the source of factorization violation in the BK equation. We will find below that it provides quite good qualitative insights on factorization violation but is not sufficient to obtain quantitatively correct results.

The equation for \( \mathcal{G} \) has already been derived in [27] (and in a somewhat different form earlier in [43]5), starting from the \( q\bar{q} \)-dipole evolution equation (2). We reiterate that this average treats

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5 To connect with the form given in [43], Eq. (5.3), one should reconstruct the evolution equation for the \( q\bar{q} \)-dipole operator by multiplying (22) with \( \exp(-C_f \mathcal{G}_{Y,xy}) \) and note that our \( \mathcal{G}_{Y,xy} \) corresponds to \( e(x, y) \) in [43].
all dipole equations consistently: Inserting (20) into the generic dipole evolution equation (11) yields one and the same equation for \( G \),

\[
\frac{d}{dY} G_{Y,xy} = \frac{\alpha_s}{\pi^2} \int d^2 z \ K_{xxz} \left( 1 - e^{-\frac{N_c}{2} \left( \hat{G}_{Y,xz} + \hat{G}_{Y,yz} - \hat{G}_{Y,xy} \right)} \right),
\]

(22)

irrespective of the representation \( R \). Note that (22) is similar to (though not exactly the same as) the Ayala-Gay Ducati-Levin (AGL) evolution equation [61–63].

Contrary to BK evolution which systematically discards all \( 1/N_c^2 \) suppressed terms contained in JIMWLK, the Gaussian truncation has no expansion parameter justifying the approximation. Nevertheless we expect it to lead to a good approximation of JIMWLK evolution since

• the equation incorporates a subset of these \( 1/N_c^2 \) corrections that is sufficient to restore the coincidence limits;

• the low density limit of Eq. (22) (viewed as its small \( G \) limit) reduces to the BFKL equation;

• it has a large \( N_c \) limit that is compatible with the BK equation as will be seen below.

Despite (22) being surprisingly more generic than the BK equation, in the sense that the procedure addresses arbitrary dipoles irrespective of representation, one remains with an approximation to the true JIMWLK evolution, and does not obtain an exact solution of the JIMWLK equation: the evolution equation for \( \langle \left[ [U_z^{-1}]^{ab \mathcal{R}} \left( \rho_a \mathcal{R} \rho_b \mathcal{R} \right)^\dagger \right] \rangle(Y) \) resulting from JIMWLK would impose additional conflicting conditions on \( \mathcal{G} \), and can only be satisfied by introducing degrees of freedom beyond \( \mathcal{G} \). The Gaussian truncation still deviates from JIMWLK evolution at the level of evolution equations for three point functions.

It is worth noting two particular features that the average (19) entails. First, Eq. (20a) implies Casimir scaling for dipole correlators:

\[
\frac{1}{d_{\mathcal{R}_1}} \langle \left( \rho_1 \mathcal{R}_1 \rho_1^\dagger \right) \rangle(Y) = \left( \frac{1}{d_{\mathcal{R}_2}} \langle \left( \rho_2 \mathcal{R}_2 \rho_2^\dagger \right) \rangle(Y) \right)^{C_{\mathcal{R}_2}/C_{\mathcal{R}_1}}.
\]

(23)

Second, somewhat surprisingly, Eq. (22) can be mapped back onto the BK equation. This implies that the dynamical content of the Gaussian truncation is the same as that of the BK equation. The main improvement is how this information is mapped onto the correlators. As we shall see, this leads to a slightly better approximation of JIMWLK results. On the practical side, this turns into a time saver: one can recycle the numerical tools written to solve the BK equation, provided one relates correlators and initial conditions accordingly.

One way to see that the dynamical content is the same is based on a simple re-parametrization of the BK \( S \)-matrix in as close an analogy to (20a) as possible. We write

\[
1 - N_{Y,xy}^{BK} = s_{Y,xy}^{BK} = \langle \left( \rho_1 \mathcal{R}_1 \rho_1^\dagger \right) \rangle(Y)/N_c = e^{-\frac{N_c}{2} \hat{G}_{Y,xy}}.
\]

(24)

\footnote{Approximate Casimir scaling has been observed for Wilson line correlators in the context of heavy quark potentials in [64,65].}
\( \tilde{G}_{Y,xy} \) should be thought of as simply being defined by the solutions of the BK equation through (24). The \( N_c \) dependent constant is a convention chosen in keeping with the \( N_c \) lore of the Mueller dipole model and the BK equation. Next one inserts this into the BK equation for \( S \),

\[
\frac{d}{dY} S_{Y,xy}^{\text{BK}} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z \ K_{xzy} \left( S_{Y,xz}^{\text{BK}} S_{Y,zy}^{\text{BK}} - S_{Y,xy}^{\text{BK}} \right),
\]

and obtains

\[
\frac{d}{dY} \tilde{G}_{Y,xy} = \frac{\alpha_s}{\pi^2} \int d^2 z \ K_{xzy} \left( 1 - e^{-N_c \left( \tilde{G}_{Y,xz} + \tilde{G}_{Y,yz} - \tilde{G}_{Y,xy} \right)} \right)
\]

which is identical to (22), thus establishing our claim of identical dynamical content, despite the different treatment of correlators.

The procedure to obtain solutions for the Gaussian truncation that would serve to determine, say, the \( q\bar{q} \)-proton cross section for DIS at HERA would then be to choose an initial condition for \( S \), read off \( G \) via (20a), insert it in place of \( \tilde{G} \) in (24) to determine the initial condition on \( S_{BK} \) to be used in the BK equation (25). After solving Eq. (25) to obtain \( S_{BK} \) at all rapidities one reverses the procedure to recover \( G \) via (24) and (20a) at each rapidity \( Y \). This \( G \) then determines all correlators of the Gaussian truncation through (19) and the special case shown in Eq. (20).

As an immediate consequence we conclude that the asymptotic scaling shape for the dipole correlators in the Gaussian truncation can be obtained from those of the BK equation using a simple power law relationship (23). As an immediate corollary also evolution speeds of the Gaussian truncation and BK evolution must coincide in that region. This link does not extend to the pre-asymptotic regime and we will see that GT tends to be slower that BK in that range in Sec. 5.

Two further points are worth noting: (i) one may recover BK factorization (wherever it can be meaningfully applied) as the leading \( N_c \) contribution of the new procedure. (This can be verified for the contributions entering the BK equation by taking the large \( N_c \) contributions in the exponents in (20).) (ii) In the small density limit, i.e., the limit of weak target fields where \( G \) is small its evolution equation, Eq. (22), consistently reduces to the BFKL equation for \( G \), in keeping with the underlying interpretation.

In summary, one might think of the Gaussian truncation as a minimal extension of the BK factorization to consistently incorporate group constraints with an associated set of “minimal” subleading \( 1/N_c \) corrections without changing the dynamical content of the associated evolution equation. Below we will frequently abbreviate Gaussian truncation as GT.

### 4.2 Factorization violations in the Gaussian truncation

Now that, with JIMWLK, BK and GT, we have accumulated three different ways to simulate small \( x \) evolution for all of which the averaging procedure \( \langle \ldots \rangle (Y) \) leads to different results we need to refine our notations to avoid confusion by distinguishing \( \langle \ldots \rangle_J(Y) \), \( \langle \ldots \rangle_B(Y) \) and \( \langle \ldots \rangle_G(Y) \) respectively. We also are faced with different factorization violations and define

\[
\Delta^J_{xyz}(Y) := \langle (\hat{S}_{xz} - \langle \hat{S}_{xz} \rangle_J(Y))(\hat{S}_{zy} - \langle \hat{S}_{zy} \rangle_J(Y)) \rangle_J(Y)
\]

(27)

to distinguish it from the analogous (non-vanishing) quantity taken in the Gaussian truncation which we will denote \( \Delta^G_{xyz}(Y) \). Only in BK this is set to zero by fiat: \( \Delta^B_{xyz}(Y) \equiv 0 \).
\( \Delta^{J}_{xzy}(Y) \) is the only channel through which higher order correlations of JIMWLK feed into the dipole equation of the Balitsky hierarchy as can be made explicit by splitting \( \langle \hat{S}_{xy} \rangle \) into two parts as

\[
\frac{d}{dY} \langle \hat{S}_{xy} \rangle_J(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z K_{xzy} \left( \langle \hat{S}_{xz} \rangle_J(Y) \langle \hat{S}_{zy} \rangle_J(Y) - \langle \hat{S}_{xy} \rangle_J(Y) \right) + \frac{\alpha_s N_c}{2\pi^2} \int d^2 z K_{xzy} \Delta^J_{xzy}(Y) .
\]  

(28)

Without the \( \Delta^J \) term on the right-hand side this would reduce to the BK equation and decouple from the rest of its Balitsky hierarchy. As such it is also the only source of possible differences in evolution speed and correlator shape between JIMWLK, BK and the Gaussian approximation. The evolution equation of the latter can also be rendered in the form (28) with all \( J \) replaced by \( G \).

The Gaussian truncation provides explicit expressions for its associated factorization violation \( \Delta^G_{xzy}(Y) \) which manifestly respects the regional patterns outlined in Sec. [3] on general grounds. Using (20) and the Fierz identity (3) one finds

\[
\Delta^G_{xzy}(Y) = \frac{N_c^2}{N_c} \left[ 1 - e^{-Cf B_{Y,xzy}} \right] - \left( N_c^2 - 1 \right) \left[ 1 - e^{-\frac{N_c}{2} B_{Y,xzy}} \right] e^{-Cf G_{Y,xy}} ,
\]  

(29)

where \( B_{Y,xzy} \) is a shorthand notation for the correlator combination already known from the evolution equation (22), the expression for the three point function (20b) or the right-hand side of the BFKL equation

\[
B_{Y,xzy} := G_{Y,xz} + G_{Y,zy} - G_{Y,xy} .
\]  

(30)

\( \Delta^G \) is positive semidefinite and strictly vanishes only where \( B \) vanishes. \( B = 0 \) is the hyper-surface on which the integrand of (22) changes sign, in analogy to what was shown for the BK equation in Fig. [1]. At large positive \( B \), the fractional expression in (29) approaches \( 1/N_c^2 \), and it grows exponentially at negative \( B \). The region of large, positive \( B \) corresponds to \( |z'| \gg |r| \) (region “a” in the notation of Sect. [3]), the region power suppressed by the evolution kernel. On the other hand, the region of maximally negative \( B \) corresponds to large parent dipoles \( |r| \gg R_s \) (regions “b” and “c” in Fig. [3]) and is is strongly suppressed by the overall \( e^{-Cf G_{Y,xy}} \). This leaves the region near \( B = 0 \) to contribute. This region coincides with the region “d” singled out in Sect. [3] to yield the main contributions. Hence \( \Delta^G \) in (29) illustrates the main feature of the factorization violations derived in Sect. [3].

The factorization violation in the Gaussian truncation Eq. (29) also explicitly vanishes in the low density (BFKL) limit: Contributions to Eq. (29) in fact start off at order \( B^2 \), i.e., are manifestly a nonlinear effect.

Since the right-hand side of the BK equation for \( \langle \hat{S} \rangle \) is negative, one immediately concludes that the positive contribution of the factorization violations included in the Gaussian truncation slow down evolution for any parent dipole size \( r \). One would expect this to carry over to JIMWLK evolution, i.e. one would expect JIMWLK evolution to be slower than BK.

In the following we will show plots that map out factorization violations and their influence on evolution. Since we want to show both the dependence on parent dipole size \( |x-y| \) and the position of the produced gluon \( z \) we can not simultaneously plot \( \Delta \) against all \( z \) degrees of freedom. We
rely on the $z'$-plane symmetries shown in Fig. 1 and restrict ourselves to half rays at $0^\circ$ and $90^\circ$ as done earlier and remind the reader that other half rays will smoothly interpolate these extreme cases.

Contour plots of the factorization violations in the Gaussian truncation and the full JIMWLK simulation in the continuum limit (see Sect. 5 below) at some fixed rapidity are shown in Fig. 5. They confirm that the regional pattern deduced in Sect. 3 is indeed seen in both JIMWLK evolution and the Gaussian truncation. Despite this qualitative agreement, it becomes evident that the Gaussian truncation underestimates the magnitude of the contributions significantly. Fig. 6 shows the corresponding contributions to the integrand of the evolution equation (28), i.e. after the kernel has been multiplied in. The kernel both power suppresses the large $|z'|$ region and enhances the contributions of small $|z'|$ and, as already anticipated, the results become quite sensitive to the short range behavior as $B$ goes to zero in the lower left hand corners of the plots.

5 Quantitative consequences: slowdown of evolution

The notion that the presence (or absence) of a factorization violation term in Eq. (28) would affect evolution speed can be made more precise. By extension of an argument in [66] we define evolution speed\footnote{Evolution speed as defined in [66] refers to the scaling regime with a uniquely defined $Q_s(Y)$ and is defined as $\lambda(Y) := \frac{d}{dY} \ln Q_s^2(Y)$. Outside the scaling region the starting point $Q_s(Y)$ is no longer uniquely defined and one may take this as one of many possible definitions for a well behaved measure of evolution speed.} as a $\int d^2r/r^2$-integral of the right-hand side of the evolution equation (4)

$$\lambda(Y) := -\frac{\alpha_s N_c}{2\pi^3} \int d^2r/r^2 \int d^2z \ K_{xyz} \langle \hat{S}_{xz} \hat{S}_{zy} - \hat{S}_{xy} \rangle(Y)$$

(31)

and then proceed to split the contributions according to (28). (Again $r = x - y$.)

To obtain a quantitative comparison of JIMWLK and BK equations, one should take note that due to non-vanishing $\Delta^J$\footnote{As will be the case in all the numerical comparisons shown.}

$$\langle \hat{S}_{xy} \rangle_J(Y) \neq \langle \hat{S}_{xy} \rangle_B(Y)$$

(32)

even if one chooses them to be equal at the initial rapidity $Y_0$:

$$\langle \hat{S}_{xy} \rangle_J(Y_0) = \langle \hat{S}_{xy} \rangle_B(Y_0)$$

(33)

as will be the case in all the numerical comparisons shown.

To calculate the difference of evolution speeds at some finite $Y - Y_0$ one has to compare Eq. (28) with the right-hand side of

$$\frac{d}{dY} \langle \hat{S}_{xy} \rangle_B(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2z K_{xyz} \left( \langle \hat{S}_{xz} \rangle_B \langle \hat{S}_{zy} \rangle_B - \langle \hat{S}_{xy} \rangle_B \right)(Y).$$

(34)

The difference between the JIMWLK and BK evolution speeds from Eqs. (28) and (34) is due to the difference in the JIMWLK and BK averagings of the $S$-matrices and to the presence of the
Fig. 5: Contour plots of the factorization violation $\Delta$ scaled up by $N_c^2$ at a fixed rapidity in JIMWLK (left) and in the Gaussian truncation (right). The plots scan parent dipole size $|x - y|$ and distance $|z'|$ from the midpoint $(x + y)/2$ at 90° (top row) and 0° (bottom row) with respect to $x - y$. The 0° case is special since it contains contributions where $\Delta$ strictly vanishes due to the coincidence limit $z = x$ or $y$ which appears here as a line with $|z'| = |x - y|/2$. The bulk of the contributions is similar to the 90° case. As was the case for the three-point-functions of Fig. 4, the contributions along the axes $|r| = 0$ and $|z'| = 0$ are identical for all angles. Fig. 4 shows cuts along horizontal lines near the bottom of the two JIMWLK plots.
Fig. 6: Factorization violations shown in Fig. 5 multiplied by the BK kernel (i.e. the integrand of the $\Delta^J$ term in the evolution equation) that determines how much they contribute to evolution, by $Q_s^2 N_c^2$, and by a scaling factor with varies from panel to panel. The notation and geometry are the same as explained in the caption of Fig. 5. In both cases the region where $\Delta$ reaches $1/N_c^2$ is completely suppressed by the kernel. Note the marked difference in size of the rescaling factor: the contribution of factorization violations in JIMWLK (left, with $10^4$) is an order of magnitude larger than in the Gaussian truncation (right, with $10^5$).
Indeed these two causes of difference are interconnected. Introducing the shape dependent correlator difference

$$\Delta^J(Y) := \left( \langle \hat{S}_{xz} \rangle_J \langle \hat{S}_{zy} \rangle_J - \langle \hat{S}_{xz} \rangle_B \langle \hat{S}_{zy} \rangle_B - \left( \langle \hat{S}_{zy} \rangle_J - \langle \hat{S}_{zy} \rangle_B \right) \right) (Y) \quad (35)$$

the difference in evolution speed arises from two non-vanishing contributions according to

$$\Delta \lambda^J(Y) := \lambda_J(Y) - \lambda_B(Y) \quad (36)$$

$$= -\frac{\alpha_s N_c}{2\pi^3} \int \frac{d^2 r}{r^2} \int d^2 z K_{xzy} \Delta^J_{xzy}(Y) - \frac{\alpha_s N_c}{2\pi^3} \int \frac{d^2 r}{r^2} \int d^2 z K_{xzy} \Delta^J_{xzy}(Y).$$

Given identical initial conditions, the $\Delta^J$-term is the sole reason for a non-vanishing $\Delta \lambda^J(Y)$ to be generated at all, but the second term may in some cases be quantitatively not less important once that has happened.

It is worth noting that in the definition of $\lambda$, Eq. (31), the contributions at small $r^2$ are enhanced by the conformal measure $d^2 r / r^2$. This carries over to the differences of evolution speed between factorized and unfactorized evolution, Eq. (36). We have already shown in Fig. 6 that this is where the main contributions to the product of kernel $K$ times $\Delta$ lie, so that one might expect that $\Delta \lambda$ receives an enhanced contribution from any non-vanishing $\Delta$ observed in our simulations.

Since it is the product of kernel and $\Delta$ that enters, one would expect that $\lambda$ should also be sensitive to any modifications to the kernel as we step beyond leading order by, for example, inclusion of running coupling effects. We would expect running coupling effects to generically suppress contributions at small $|r|$ and thus reduce some of the speed difference visible at leading order. At the same time one should be aware that contributions beyond running coupling would also affect evolution speed and the way $1/N_c$ corrections couple into dipole evolution. Those might be of the same size as the running coupling contributions, but it would be very peculiar if they were to affect the general pattern of the leading order behavior observed in the simulations shown below.

We have performed a numerical solution of the JIMWLK evolution equation at one loop accuracy with fixed coupling along the lines of and using the techniques developed in [29]. The method is based on the fact that the JIMWLK equation takes the form of a functional Fokker-Planck equation which has an equivalent Langevin formulation [18,67] in which the averages $\langle \ldots \rangle_J(Y)$ are expressed as ensemble averages of fields $U_{Y,x}$, with the $Y$-dependence implemented by a Langevin equation for the ensemble members. To implement such a description numerically one is forced to discretize transverse space to represent the ensemble member fields $U_{Y,x}$ in terms of a finite number of degrees of freedom. This automatically introduces a UV regulator in the guise of the lattice spacing $a$ and a IR regulator, the lattice size $L$. The reader interested in further details of the numerical solution and in the implementation of the numerical procedure is referred to [29]. In order to be able to directly compare our JIMWLK simulations with BK and GT results at fixed $a$ and $L$, we have chosen to run our BK and GT simulation on the same type of regular lattice with identical $a$ and $L$ values although that is decidedly not the most efficient way to perform standalone BK or GT simulations.

Fig. 7 shows numerical results for evolution speeds $\lambda(Y)$ as a function of $R_s(Y) = 1/Q_s(Y)$, for lattices sizes varying from $256^2$ to $4096^2$. The JIMWLK results of the first two plots are compiled from the simulations presented in [29], this were the largest lattice sizes taken into account then. All the others are based on new simulations, with measurements taken at smaller $Y$ intervals (which
Fig. 7: Comparing evolution speed in JIMWLK, BK and GT on different size lattices, starting from initial conditions with identical dipole correlators $\langle \hat{S}_{xy} \rangle_J(Y_0) = \langle \hat{S}_{xy} \rangle_B(Y_0) = \langle \hat{S}_{xy} \rangle_G(Y_0)$. The vertical line in the first four plots marks $L/R_s(Y) = 20$. The bands show statistical (Jackknife) errors.

explains the more fine-grained raggedness of the JIMWLK results – adding additional intermediate steps would further enhance the phenomenon).

For the new runs, initial conditions were chosen to carefully explore the convergence to both the
continuum limit and the infinite volume limit, i.e., to scan for UV- and IR-cutoff artifacts. IR phase space available in the simulation at the initial condition at $Y_0$ is varied by increasing lattice size $L$ compared to initial correlation length $R_s(Y_0)$. UV phase space is varied by increasing lattice size at (approximately) fixed $L/R_s(Y_0)$. During evolution active phase space, which is centered around $1/R_s(Y)$, moves towards the UV, so that available IR phase space grows with $L/R_s(Y)$ while the UV shrinks with $R_s(Y)/a$.

The most striking feature of the JIMWLK simulations are the large fluctuations of evolution speed on all but the largest lattices. The fluctuations turn out to be IR dominated, they average out as we increase the number of points in the IR. UV cutoff effects manifest themselves only for the longest runs, as a relatively sharp turn downwards as one follows the curves from right to left as $R_s$ shrinks while $Y$ grows. This downturn (where present) indicates that the simulation is running out of UV phase space with $R_s(Y)/a \lesssim 10$. While this behavior is not physical, it affects all our simulations in the same way and one does not prevent us from comparing the behavior of the simulations with each other.

This can be read off from Fig. 7 by tracing the following systematic features: With the saturation scale safely more than an order of magnitude smaller than the inverse lattice spacing, only varying IR phase space affects $\lambda$. As $L/R_s(0)$ increases from 8.35 to 99.46 JIMWLK evolution becomes less and less affected by IR fluctuations. Evolution speeds (as compared to BK and GT, which both are not affected by fluctuations) slow down until they settle at their infinite volume limit at around $L/R_s(0) \approx 50$.

Note that in the first four panels (with smallest $L/R_s(0)$) JIMWLK is initially faster than both BK and the Gaussian truncation, before this becomes less and less pronounced as $L/R_s(Y)$ grows with evolution. This is a direct consequence of the impact of fluctuations becoming less pronounced as shrinking $R_s(Y)$ cuts off contributions from the infrared. It turns out that for very small $L/R_s(Y)$ where IR fluctuations contribute most, the $\Delta J_B$ contribution completely overwhelms the $\Delta J$ contribution which in all cases gives a contribution that slows evolution down. The relative size of this contributions shrinks strongly when $L/R_s(Y) > 25$ and is no longer able to overwhelm $\Delta J$ in the runs with $L/R_s(Y_0) > 25$.

Similarly, for all other runs, where $L/R_s(Y) > 25$ from the outset, we observe (on average) a clear hierarchy of evolution speeds with BK the fastest, the Gaussian truncation in the middle and JIMWLK the slowest. In this range this is simply a reflection of the relative size of factorization violations: the larger these are in the regions enhanced by the kernel, the slower the evolution becomes. This hierarchy is already visible in the rescaling factors of Figs. 5 and 6, scaling violations in GT are consistently an order of magnitude smaller than in JIMWLK across the $Y$ range explored. This becomes evident again, if we contrast the JIMWLK results of Fig. 2 with their counterpart in the Gaussian truncation, Fig. 8. Were we to extend our comparison of BK and GT evolutions into the asymptotic range, however, evolution speeds would necessarily become identical as discussed in Sec. 4.1. We can use this to assess how closely the simulations shown in Fig. 7 approach the asymptotic scaling region. With the regular grid necessary to compare with JIMWLK this is not practical, but a simulation that only compares BK and GT can make be implemented more efficiently and in fact reach the asymptotic limit. Using this freedom we find that this occurs just beyond the region where we loose UV accuracy in the longest JIMWLK simulations such as that in the bottom middle plot of Fig. 7. The result of this comparison is shown in Fig. 9. Comparing the ratios of evolution speeds in Fig. 9 we conclude that in the asymptotic scaling region (at fixed coupling!) one should expect a factorization violation induced 3-5% slowdown in evolution speed.
Fig. 8: Factorization violations in the Gaussian truncation (scaled up by $N_s^2$) against varying parent dipole size at fixed $|z'| = 0.4 \cdot R_s(Y)$ depicted here for comparison with JIMWLK results shown in Fig. 4. The larger factorization violations in JIMWLK lead to slower evolution. The strong change from $Y_1$ to higher rapidities is mirrored by a convergence of evolution speed between GT and BK with evolution towards asymptotic regime shown in Fig. 9.

in JIMWLK evolution compared to BK evolution at one loop accuracy. There is no sign from the simulation and no theoretical reason to argue that factorization violations in JIMWLK should disappear in the asymptotic region. As already noted we would expect this relative slowdown effect to become less pronounced at NLO.

6 JIMWLK beyond the Gaussian truncation: higher order correlators

The simulation results shown in the previous section show a persistently small but measurable improvement of the Gaussian truncation over the BK approximation. Still, JIMWLK evolution is much more general than either truncation. Both approximations restrict the information retained in evolution to two point functions that, in the low density limit matches up with double reggeon exchange as incorporated in BFKL evolution. JIMWLK evolution, on the other hand, allows for multi–reggeon exchange in evolution and even the limited set of correlators discussed in the above is sensitive to their contributions. An example of this is the simplistic form in which dipole correlators of higher representations are mapped back onto the quark dipole correlator. This is a direct consequence of the fact that the Gaussian truncation only iterates two reggeon exchange into Glauber exponents.

One might attempt to include multi–reggeon exchanges by generalizing (19) to include higher order
Fig. 9: Ratios of evolution speed with finite lattice spacing and in the continuum limit. The plots correspond one to one to the bottom row in Fig. 7. [The sharp up- or downturn of the curves with finite a towards the left indicate complete breakdown of the simulations in the UV.] Shown are in all plots, from bottom to top (excepting the dash-dotted lines): $\lambda_f(Y)/\lambda_B(Y)$ and $\lambda_G(Y)/\lambda_B(Y)$ (both at finite $a$) and $\lambda_G(Y)/\lambda_B(Y)$ (in the continuum limit). A comparison of the two upper curves indicates the size of the UV cutoff effects which remain fairly small compared to the error on the JIMWLK results. Evolution in JIMWLK is slower than in both BK and GT in all plots. The left- and rightmost plots serve to assess lattice artifacts in JIMWLK evolution. Compared to the middle plot where the JIMWLK-with the middle plot within errors. (To facilitate this comparison the solid line from the left panel to factorization violations. This is expected to be strongly reduced by running coupling effects.

terms; naive inclusion of three–reggeon terms would modify (37) to

$$
\langle \ldots \rangle(Y) = \exp \left\{ -\frac{1}{2} \int dY' \int d^2x \int d^2y \ G_{Y',xy} \ \frac{\delta}{\delta \lambda_{B_y,Y'}} \frac{\delta}{\delta \lambda_{B_{x,Y'}}} \right. \\
\left. -\frac{1}{3!} \int dY' \int d^2x \int d^2y \int d^2z \ \left( G_{Y',xy} f_{abc} + G_{Y',xy} d_{abc} \right) \frac{\delta}{\delta \lambda_{B_{x,Y'}}} \frac{\delta}{\delta \lambda_{B_{y,Y'}}} \frac{\delta}{\delta \lambda_{B_{z,Y'}}} \cdots \right\} 
$$

and include odderon contributions [49]. However, starting with three reggeon terms, locality in $Y$ is an assumption that might prove to be too restrictive and will generally not lead to a consistent treatment of the Balitsky hierarchies. Moreover, one has to be careful in simply exponentiating the 3-reggeon terms, as is done in (37). Here the best way to keep the calculations under parametric control is to employ the power counting developed for the classical gluon fields in [10, 11]. In the quasi-classical limit the leading term in the exponent of (37) corresponds to a two-gluon exchange with a nucleon in the nucleus, such that $G_{Y,xy} \sim \alpha_s^2 A^{1/3}$ with $A$ the atomic number of the nucleus. For $\alpha_s \ll 1$ and $A \gg 1$ there exists a regime where $\alpha_s^2 A^{1/3} \sim 1$ and the GT approximation of Sect. 4 resums all powers of $\alpha_s^2 A^{1/3}$. From the standpoint of this quasi-classical power counting
the second term in the exponent of (37) corresponds to a 3-gluon \( t \)-channel exchange with a single nucleon, such that \( G^I_{Y'xyz} \sim G^d_{Y'xyz} \sim \alpha_s^3 A^{1/3} \), i.e., it is suppressed by one power of the coupling \( \alpha_s \) compared to the leading term. Iteration of such term more than once would be beyond the precision of the approximation: two 3-gluon exchanges are of the same order in \( \alpha_s \) and \( A \) as a 2-gluon exchange combined with a 4-gluon exchange. From this perspective the contributions of Eq. 37 are only under parametric control up to linear order in \( G^I \) and \( G^d \):

\[
\langle \ldots \rangle (Y) = \exp \left\{ -\frac{1}{2} \int dY' \int d^2 x \int d^2 y \ G_{Y',xyz} \frac{\delta}{\delta A_{y',y}^a \delta A_{y',y}^b} \right\} \times \left[ 1 - \frac{1}{3!} \int dY'' \int d^2 x \int d^2 y \int d^2 z \ (G^I_{Y'xyz} f_{abc} + G^d_{Y'xyz} d_{abc}) \right]
\]

A discussion of truncations that systematically include multi–reggeon contributions goes beyond the scope of this paper, but it is not hard to play with (37) as an ansatz to explore the consequences of the inclusion of an odderon term in this manner (see Appendix C): it becomes quite manifest that such multi–reggeon contributions naturally break Casimir scaling. The reason for this is that in general, higher representation contributions in the \( t \)-channel start to pick up on the more complicated decomposition of a general \( s \)-channel \( R \bar{R} \)-dipole into irreducible representations. (Higher representations were also considered in [68].) For the odderon contribution they account for the fact that the \( q \bar{q} \) dipole acquires an imaginary part (with a specific \( N_c \) dependence) while a \( gg \)-dipole remains real since the adjoint representation is real by definition.

With the numerical results from JIMWLK at hand it is straightforward look for the actual presence of Casimir scaling-violating effects in JIMWLK evolution. This is explicitly shown in Fig. 10. We note in particular that the violations of Casimir scaling do not grow with energy: they seem to qualitatively scale with \( Q_s \) and might, if anything even be slowly erased, but any firm conclusion to that effect is beyond the present numerical accuracy, in particular because of the short lever arm available before the simulation starts to “fall through the lattice” (i.e. runs out of UV phase space). Note that this qualitative \( Q_s \)-scaling behavior occurs in a region far outside the \( Q_s \)-scaling region of the dipole correlator itself. Presently we have no systematic explanation for this observation. (The \( Q_s \)-scaling region of the dipole correlator is not reachable in the current simulations of JIMWLK equation due to the limited UV phase space on the lattices used.)

To illustrate that the violations of Casimir scaling are driven by nontrivial coordinate dependence we show in Fig. 11 that no power law relationship modeled on Eq. 23 can provide a good explanation for the differences observed in Fig. 10. By itself, this does not exclude a more intricate functional relationship, but that in turn would require its own explanation. Our earlier arguments would lead us to believe that it is much more likely and natural that new degrees of freedom (starting with four point contributions to the average 37) are needed to explain this difference.

7 Conclusions

We have explored the size and nature of \( 1/N_c \) corrections in the JIMWLK equation and have found that quite natural cancellations lead to the much stronger suppression than the naively expected
Fig. 10: Violation of Casimir scaling for fundamental and adjoint two point correlators for three different rapidities plotted as a function of dipole size $r$ in units of correlation length $R_s(Y)$. Left: adjoint correlators compared to the Casimir-rescaled fundamental correlators. Error bands indicating numerical uncertainty are too narrow to be clearly visible. Right: differences of the two. (These are more stable than ratios which tend to become uncontrollable at large distances.) The bands indicate the size of the errors. The violation follows correlation length $R_s(Y)$, indicated by vertical lines. It would appear to scale with $Q_s$ well within errors. Note that this is in a regime where the dipole cross section has not yet reached its scaling regime and decidedly does not scale yet.

$1/N_c^2 \approx 10\%$ (for $N_c = 3$) observed in [29]. The argument is based first on both group theoretical coincidence limits for singlet Wilson line correlators and scaling with the saturation scale $Q_s(Y)$ to establish suppression of contributions in most of configuration space. All remaining contributions are then shown to be then decoupled from evolution after suppression by the BK kernel has been taken into account.

We have shown that $1/N_c$ corrections enter through the factorization violation $\Delta$ and have explored its properties. It is bounded by $1/N_c^2$ from above, but is much smaller than $1/N_c^2$ for most of its phase space, due to saturation effects controlled by coincidence limits leading to extra suppression. The argument is generic and can be easily applied beyond the leading $\ln(1/x)$ approximation used here. Thus saturation effects provide an extra suppression of $1/N_c^2$ corrections to the BK evolution that reduces the difference between the JIMWLK and BK results for the dipole scattering amplitude considerably at any accuracy. While NLO corrections will have some quantitative impact, we do not expect them to grossly change the LO result for $\Delta$ or correlator differences (when compared at the same $R_s(Y)$ or $Q_s(Y)$). At one loop we find typical contributions to $\Delta$ of the order of $10^{-3}$ or $0.1\%$ of the individual correlator values for correlator differences.

To complement the qualitative discussion for correlators in JIMWLK evolution not only numerically but also analytically, we have made a step beyond the leading $1/N_c$ BK truncation by exploring an
Fig. 11: The violation of Casimir scaling in JIMWLK evolution for two lattice sizes with the same infrared phase space as indicated by the $L/R_s(Y_0)$ value. The plots explore the correlator difference $S_{gy} - (S_{gq}^q)^b$ as a function of dipole size $r$ for values of $b = 2$ (the BK result) up to the GT value of $N_c/C_f = 9/4$ at $N_c = 3$. Intermediate values of $b$ fall into the shaded areas. The result for “a best fit” is indicated as a dashed (red) line. Clearly a simple modification of the power alone is not sufficient to remove the mismatch. To remed y the situation one would either need a more complex functional relationship between dipole correlators or, more likely, new degrees of freedom to alter the $(x - y)^2$ dependence in at least one of the correlators shown.

alternative truncation of JIMWLK evolution in the spirit of a Glauber-Mueller iterated two-reggeon exchange truncation that we dubbed the Gaussian truncation. This truncation includes a minimal set of subleading $1/N_c$ corrections necessary to restore the coincidence limits for correlators that are at the core of our cancellation argument for subleading contributions. Correspondingly, it includes a minimal set of factorization violations. They turn out to have the right qualitative structure but are numerically noticeably smaller than the factorization violation of full JIMWLK evolution. The Gaussian truncation allows access to a larger subset of the Balitsky hierarchies than BK truncation by treating evolution equations for dipole operators in arbitrary representations consistently. As a result the Gaussian truncation proves to be a somewhat better approximation to full JIMWLK evolution. Accordingly, one of its main consequences, Casimir scaling between dipole operators of different representations, turns out not to be strongly violated in full JIMWLK.

We have firmly established that factorization violations slow down evolution compared to the BK truncation, both in the minimalist from introduced in the Gaussian truncation and in full JIMWLK. One of main differences between the two is that factorization violation in JIMWLK persist during evolution on a level comparable with the factorization violation present in the GB-W–like initial condition used, while they become notably smaller in the Gaussian truncation.

Evolution speed is somewhat more sensitive to $1/N_c$ corrections than the factorization violations in the correlators due to an enhancement of contributions from small parent “dipoles.” At one loop
accuracy this leads to a relative slowdown of JIMWLK evolution that approaches 3-5% near the scaling region. Running coupling corrections are known to reduce evolution speed by suppressing the relative importance of small parent dipoles. We have argued that the same mechanism is likely to also reduce the relative slowdown, i.e. the impact of $1/N_c$ corrections on evolution speed. The subleading-$N_c$ terms present in JIMWLK but absent in BK can likely be attributed to multi–reggeon $t$-channel exchanges and multi–reggeon splitting vertices. The smallness of the difference between the dipole scattering amplitude obtained from JIMWLK and BK appears to indicate that multi–reggeon effects are not important for this observable. Further investigation is needed to clarify if this is indeed the case.

Casimir scaling violations present in JIMWLK evolution provide a means to assess multi–reggeon exchange contributions. We have illustrated this both with a sketch model that includes odderon contributions and a numerical comparison of $gg$ and $q\bar{q}$ dipoles. While the odderon contributions play no role for the observables considered here and are generically suppressed by evolution, the multi–reggeon contributions present here are small but contribute throughout evolution. We have numerical hints at $Q_s(Y)$-scaling behavior of these contributions that sets in much earlier than geometric scaling of dipole correlators. At present we have no systematic explanation for this observation other than “naturalness.”

Our whole discussion was carried out at the leading log level, without any NLO contributions taken into account, despite the fact that they are known to strongly influence evolution speed. This is partly due to necessity: no numerically practical way has been devised to include higher order effects, for example the running coupling corrections obtained in [32–34]. However, since the arguments given for the suppression of factorization violations are completely generic, one expects the observations made here to persist to higher orders in the perturbative expansion, even though quantitative modifications are expected to affect the precise numerical result of the cancellations observed at leading order.

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A Generic coincidence limits

To understand the coincidence limits for the operator $\tilde{U}_z^{ab} \tau^a \tau^b \tau^c \tau^d$, generic properties of the adjoint representation are useful. They immediately give rise to the identities

$$[\tilde{U}_z]^{ab} \tau^a \tau^b \tau^c \tau^d = [\tilde{U}_x \tilde{U}_z]^{ab} \tau^a \tau^b \tau^c \tau^d \quad \text{tr}(t \ U_x \ U_y \ U_z \ U_y) = [\tilde{U}_z \ U_y]^{ac} \tau^a \tau^c \tau^b \tau^d \quad \text{tr}(t \ U_x \ U_y \ U_z \ U_y)$$

(A1)
by pulling adjoint factors out of the trace.

In the limit \( x = y \) this is proportional to

\[
\frac{\alpha_R a^b}{\text{tr}(t^t)} = \alpha_R \delta^{ab},
\]

and the first task is to understand the constant. Clearly it has to be proportional to the Casimir of the representation, but normalization conventions do also play a role: With standard conventions the fundamental representation has \( \alpha_{\text{fund}} = \alpha_F = \frac{1}{2} \), while in the adjoint representation one obtains \( \alpha_{\text{adjoint}} = \alpha_A = N_c \). This can, in fact, be expressed via the Casimir values and the dimension of the representation. The usual definition of the quadratic Casimir,

\[
\frac{\text{tr}(a^a a^b \delta^{ab})}{\text{d}_A} = \frac{\text{tr}(a^a a^b \delta^{ab})}{\text{d}_A} = C_R \delta^{ac} \text{tr}(a^a a^c),
\]

implies

\[
\frac{\text{tr}(t^t)}{\text{d}_A} = C_R d_R
\]

and thus, together with the definition of \( \alpha_R \) above, \( \alpha_R d_A = C_R d_R \) or

\[
\alpha_R = C_R \frac{d_R}{d_A}
\]

with \( d_A \) the dimension of the adjoint representation. This readily leads to \( \alpha_A = C_A = N_c \) and \( \alpha_F = C_f \frac{N_c}{2N_c C_f} = \frac{1}{2} \) as obtained from direct calculation.

This allows to write the \( x = y \) limit as

\[
\lim_{y \to x} [\tilde{U}_y]^{ab} = \lim_{y \to x} [\tilde{U}_y]^{ac} \text{tr}(t^t U_x U_y t^t) = C_R \delta^{ac} \text{tr}(t^t t^t) = \text{tr}(t^t t^t).
\]

Note that this leads to a correlator in the adjoint representation, irrespective of \( R \). The only reference to \( R \) is the proportionality factor.

The limit \( z \to y \) or \( x \) is simpler:

\[
\lim_{z \to y \text{ or } x} [\tilde{U}_z]^{ab} = \lim_{z \to y \text{ or } x} [\tilde{U}_z]^{ac} \text{tr}(t^t U_x U_y t^t) = C_R \text{tr}(t^t t^t).
\]

The completely local limit can be obtained directly from \( (A6a) \) to be \( C_R d_R \) or via

\[
\lim_{z \to y \text{ or } x} [\tilde{U}_z]^{ab} = \lim_{z \to y \text{ or } x} [\tilde{U}_z]^{ac} \text{tr}(t^t U_x U_y t^t) = C_R \text{tr}(t^t t^t).
\]

This establishes Eqs. (12) of Sect. 3.

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B Gaussian target averages

While direct calculation of specific correlators using the averaging procedure \([19]\) is in many cases straightforward, the calculation can be often simplified significantly by using differential equations. This relies on the observation that

\[
\frac{d}{dY} \langle \ldots \rangle(Y) = -\frac{1}{2} \left( \int d^2 u \int d^2 v \ G_{Y,uv} \frac{\delta}{\delta A_{u,y}^a} \frac{\delta}{\delta A_{v,y}^a} \ldots \right)(Y) . \tag{B1}
\]

Applied to concrete examples this takes its simplest form if the right-hand side is directly proportional to \(\langle \ldots \rangle(Y)\) itself, but can be useful even in more general cases. We will explicitly address the examples needed in Sect. 3.

**Two point projectile \(R-\bar{R}\) correlators:** Using the notation \([21]\) and a prime to denote a \(Y\) derivative, straightforward algebra leads to

\[
\frac{d}{dY} \langle \text{tr}(U_x U_y) \rangle = -G'_{Y,xy} \langle \text{tr}(t \ t \ U_x U_y) \rangle(Y) = -C_R G'_{Y,xy} \langle \text{tr}(U_x U_y) \rangle(Y) \tag{B2}
\]

which is readily solved to obtain

\[
\langle \text{tr}(U_x U_y) \rangle(Y) = d_R e^{-C_R \varphi_{Y,xy}} . \tag{B3}
\]

The freedom in the initial condition was used to accommodate the normalization factor \(d_R\).

**Three point projectile adjoint-\(R-\bar{R}\) correlators:** These involve several distinct color structures.

\[
\frac{d}{dY} \langle \text{tr}([\bar{U}_z]^{ab} \ t \ t \ U_x U_y) \rangle(Y) = -G'_{Y,xyz} \langle \text{tr}(t \ t \ U_x U_y) \rangle(Y) \tag{B4}
\]

\[
= - \left[ \frac{N_c}{2} \ (G'_{Y,zy} + G'_{Y,zy}) + G'_{Y,xy} \left( C_R - \frac{N_c}{2} \right) \right] \langle \text{tr}(U_x U_y) \rangle(Y)
\]

where we have used

\[
\sum_{i,a,k} \ t \ \bar{t} = t \ t \ t + [t \ t \ t] = t \ C_R + i f_{iak} t \ t = t \ C_R + i f_{iak} \frac{1}{2} i f_{kij} \ t = t \ (C_R - \frac{N_c}{2}) . \tag{B5}
\]

The nontrivial point is that this holds for any representation \(R\). Integrating \([B4]\) one finds

\[
\langle \text{tr}([\bar{U}_z]^{ab} \ t \ t \ U_x U_y) \rangle(Y) = C_R d_R e^{-N_c \ \varphi_{Y,zy}} , \tag{B6}
\]

again with the free initial condition used to set the normalization properly.
C Odderon contributions lead to Casimir scaling violations

Here we explore the results of applying (37) to the calculation of dipole and 3-point correlators, as was done earlier with the Gaussian truncation.

It turns out that starting from this level of three-point $t$-channel correlators generic expressions for arbitrary representations $R$ cannot be given. This is a consequence of the arbitrarily complicated decomposition of a general $s$-channel $RR$-dipole into irreducible representations. These start to mix in nontrivial $R$-dependent ways beyond the Gaussian truncation.

We have therefore restricted ourselves to $R$ being either the fundamental or the adjoint representation. Here it turns out that $G^f$ in (37) does not contribute at all to correlators of Wilson lines, and that $G^d$, as expected, can be thought of as an odderon contribution. This will allow us to compare the results we are about to obtain to [54] by counting $G^d$ as $\mathcal{O}(\alpha_s)$ (using the quasi-classical counting) and correspondingly expanding the equations we get to the lowest order in $G^d$, as was done in (38). Our results can also be compared to [55] if we keep the $G^d$ contributions to all orders.

For the correlators in question, this contribution generates imaginary parts wherever the representation $R$ is not intrinsically real, such as the adjoint representation. For the limited set of correlators we are looking at, only the $x \leftrightarrow y$ antisymmetric combination

$$\int dY(Y) G^d_{Y,yyx} - G^d_{Y,yxx}$$

enters. For compactness, we will also absorb a constant in the shorthand to be used below. We define

$$G^O_{Y,xy} := \frac{C_d}{4} \int dY' (G^d_{Y',yyx} - G^d_{Y',yxx}) \quad \text{(C1)}$$

with

$$C_d := \frac{N_c^2 - 4}{N_c} \quad \text{(C2)}$$

characterizing the symmetric “octet” (in SU(3) parlance) in the decomposition of a $gg$-dipole into invariant multiplets (see. [69], Sec 9.12 for a systematic treatment that is much more practical than most). Note that $G^O_{Y,xx} = 0$.

Let us begin with the three point $q \bar{q}g$ function. With $R$ the fundamental representation one finds

$$\langle [\hat{U}_z]^{ab} \text{tr}(t^a U_{xa} t^b U_{yb}) \rangle(Y) = N_c C_f e^{-\left\{ \left[ \frac{N_c}{2}(G_{xx} + G_{yy}) + G_{xy}(C_f - \frac{N_c}{2}) \right] + \frac{i}{2} \left[ -\frac{G^O_{xy}}{N_c} + N_c (G^O_{xz} + G^O_{zy}) \right] \right\} (Y)}$$

$$= N_c C_f e^{-\left[ \frac{N_c}{2} (G_{xx} + G_{yy}) + G_{xy}(C_f - \frac{N_c}{2}) \right] \left\{ 1 - \frac{i}{2} \left[ -\frac{G^O_{xy}}{N_c} + N_c (G^O_{xz} + G^O_{zy}) \right] + o(G^O) \right\} (Y)} \quad \text{(C3)}$$

The expanded expression in the second line serves to recall that higher orders in this expansion in powers of $G^O$ are beyond the control of our approximation.

The coincidence limits (12), which are at the heart of factorization violations provide relationships with dipole correlators also here. Eq. (C3) in the limit $x = y$ provides the expression for the adjoint correlator

$$\langle \text{tr}(\hat{U}_z \hat{U}_z) \rangle(Y) = \langle [\hat{U}_z]^{ab} 2 \text{tr}(t^a U_{xa} t^b U_{xa}^\dagger) \rangle(Y) = 2 N_c C_f e^{-N_c G^O_{xx}} \quad \text{(C4)}$$
which turns out to be unmodified from the Gaussian truncation and remains completely independent of the odderon term \( G^O \) even without expanding in the odderon contribution. Note that this is more stringent than the group theoretical requirement that the \( gg \)-dipole has to be real, which would have allowed even powers of \( G^O \) to appear in an all orders expression in terms of \( G^O \).

The fundamental correlator, on the other hand, does acquire modifications both to real and imaginary parts. In accordance with the limits \( z = x \) and \( z = y \) of (C3) one finds

\[
\langle \text{tr}(U_x U_y) \rangle(Y) = \frac{1}{C_f} \langle ([\hat{U}_x]^{ab}) \text{tr}(t^a U_x t^b U_y) \rangle(Y) = \frac{1}{C_f} \langle ([\hat{U}_y]^{ab}) \text{tr}(t^a U_x t^b U_y) \rangle(Y)
\]

\[
= N_c e^{-C_f (\hat{g}_{Y,xy} + i \hat{g}_{Y,xy}^{O})} = N_c e^{-C_f g_{Y,xy}} [1 - i C_f G_{Y,xy}^O + o(G^{O,2})].
\]  

(C5)

Contrary to what happens in the adjoint representation, this result would emerge from our earlier expression for the \( q\bar{q} \)-dipole using the substitution

\[
\hat{g}_{Y,xy} \rightarrow g_{Y,xy} + i G_{Y,xy}^O.
\]  

(C6)

Comparing Eqs. (C5) and (C4) we can see that the odderon contribution introduces violation of the Casimir scaling of (23). We therefore can conjecture that Casimir scaling violating effects are due to multiple reggeon exchanges.

The evolution equation for the \( q\bar{q} \) dipole (Eq. 33) with \( R \) the fundamental representation, after insertion of (C5) and (C3) leads to

\[
\frac{d}{dY} (g_{Y,xy} + i G_{Y,xy}^O) = \frac{\alpha_s}{\pi^2} d^2 z \kappa_{xzy} \left( 1 - e^{-\frac{N_c}{2} (g_{xz} + i g_{xz}^O) + (g_{zy} + i g_{zy}^O)} \right).
\]

(C7)

which repeats the structure of (22), again with the simple substitution (C6).

As we have control only over the terms linear in \( G^O \) we expand (C7) and use (22) to obtain

\[
\frac{d}{dY} G_{Y,xy}^O = \frac{\alpha_s N_c}{2 \pi^2} d^2 z \kappa_{xzy} e^{-\frac{N_c}{2} (g_{xz} + g_{zy} - g_{zy}^O)} \left[ G_{Y,xy}^O + G_{Y,zy}^O - G_{Y,xz}^O \right].
\]

(C8)

Defining the real part of the \( S \)-matrix and the odderon exchange amplitude \( O \) by

\[
S_{Y,xy} = e^{-C_f g_{xy}}, \quad O_{Y,xy} = -i C_f G_{Y,xy}^O e^{-C_f g_{xy}}
\]

(C9)

and using these definitions in (C8) in the large-\( N_c \) limit (which is needed here just like it was needed to derive BK equation (25) from the GT truncation (22) in Sect. 4.1) one derives the non-linear evolution equation for the odderon found in [54] (see also [55])

\[
\frac{d}{dY} O_{Y,xy} = \frac{\alpha_s N_c}{2 \pi^2} d^2 z \kappa_{xzy} \left[ O_{Y,xy} S_{Y,xy} + S_{Y,xy} O_{Y,zy} - O_{Y,xz} \right].
\]

(C10)

In [54,55] the authors discuss how this equation maps onto the BJKP hierarchy [45–48], i.e., onto the systematic inclusion of multi–reggeon exchanges in the t-channel. Beyond the low density limit where \( G^O, G \) (and all higher \( n \)-point t-channel insertions) are small our procedure provides a
generalization consistent with JIMWLK evolution. In the linear regime our solution for $O$ stays constant with energy in agreement with [53, 54].

One of the main properties of the resulting Eqs. (C8) and (C10) is that $G^O \equiv 0$ (or, equivalently, $O \equiv 0$) is a stable solution of the equation, and that (as already discussed in [54, 55]) non-vanishing odderon contributions in the initial condition are erased very quickly due to nonlinear effects. This may provide a glimpse of how the Casimir scaling-violating multi–reggeon contributions may be erased by non-linear evolution even if they are present in the initial conditions.

Note that already the inclusion of $G^O$ breaks the Casimir scaling relation (23) between the dipole correlators in the fundamental and adjoint representation (see Eqs. (C5) and (C4)). It is only natural that higher t-channel $n$-point exchanges will also contribute to this breaking of Casimir scaling. The odderon contribution in our simulations, however, vanishes from the outset: the initial conditions necessary to accommodate the total cross section in DIS (as was the case for all our simulations) require $G^O \equiv 0$.

Still, this discussion does clarify the nature of what we expect to arise as one includes corrections to the Gaussian truncation of JIMWLK evolution.

References

[1] L. V. Gribov, E. M. Levin, and M. G. Ryskin, *Semihard processes in QCD*, Phys. Rept. 100 (1983) 1–150.

[2] A. H. Mueller and J.-w. Qiu, *Gluon recombination and shadowing at small values of x*, Nucl. Phys. B268 (1986) 427.

[3] A. H. Mueller, *Soft gluons in the infinite momentum wave function and the BFKL pomeron*, Nucl. Phys. B415 (1994) 373–385.

[4] A. H. Mueller and B. Patel, *Single and double BFKL pomeron exchange and a dipole picture of high-energy hard processes*, Nucl. Phys. B425 (1994) 471–488, [hep-ph/9403256].

[5] A. H. Mueller, *Unitarity and the BFKL pomeron*, Nucl. Phys. B437 (1995) 107–126, [hep-ph/9408245].

[6] L. D. McLerran and R. Venugopalan, *Gluon distribution functions for very large nuclei at small transverse momentum*, Phys. Rev. D49 (1994) 3352–3355, [hep-ph/9311205].

[7] L. D. McLerran and R. Venugopalan, *Computing quark and gluon distribution functions for very large nuclei*, Phys. Rev. D49 (1994) 2233–2241, [hep-ph/9309289].

[8] L. D. McLerran and R. Venugopalan, *Gluon distribution functions for very large nuclei at small transverse momentum*, Phys. Rev. D49 (1994) 3352–3355, [hep-ph/9311205].

[9] L. D. McLerran and R. Venugopalan, *Green’s functions in the color field of a large nucleus*, Phys. Rev. D50 (1994) 2225–2233, [hep-ph/9402335].

[10] Y. V. Kovchegov, *Non-abelian Weizsäcker-Williams field and a two- dimensional effective color charge density for a very large nucleus*, Phys. Rev. D54 (1996) 5463–5469, [hep-ph/9605448].

36
[11] Y. V. Kovchegov, Quantum structure of the non-abelian Weizsaecker-Williams field for a very large nucleus, Phys. Rev. D55 (1997) 5445–5455, hep-ph/9701229.

[12] J. Jalilian-Marian, A. Kovner, L. D. McLerran, and H. Weigert, The intrinsic glue distribution at very small x, Phys. Rev. D55 (1997) 5414–5428, hep-ph/9606337.

[13] J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, The BFKL equation from the Wilson renormalization group, Nucl. Phys. B504 (1997) 415–431, hep-ph/9701284.

[14] J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, The Wilson renormalization group for low x physics: Towards the high density regime, Phys. Rev. D59 (1999) 014014, hep-ph/9706377.

[15] J. Jalilian-Marian, A. Kovner, and H. Weigert, The Wilson renormalization group for low x physics: Gluon evolution at finite parton density, Phys. Rev. D59 (1999) 014015, hep-ph/9709432.

[16] J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, Unitarization of gluon distribution in the doubly logarithmic regime at high density, Phys. Rev. D59 (1999) 034007, hep-ph/9807462.

[17] A. Kovner, J. G. Milhano, and H. Weigert, Relating different approaches to nonlinear QCD evolution at finite gluon density, Phys. Rev. D62 (2000) 114005, hep-ph/0004014.

[18] H. Weigert, Unitarity at small Bjorken x, Nucl. Phys. A703 (2002) 823–860, hep-ph/0004044.

[19] E. Iancu, A. Leonidov, and L. D. McLerran, Nonlinear gluon evolution in the color glass condensate. I, Nucl. Phys. A692 (2001) 583–645, hep-ph/0011241.

[20] E. Ferreiro, E. Iancu, A. Leonidov, and L. McLerran, Nonlinear gluon evolution in the color glass condensate. II, Nucl. Phys. A703 (2002) 489–538, hep-ph/0109115.

[21] Y. V. Kovchegov, Small-x F^2 structure function of a nucleus including multiple pomeron exchanges, Phys. Rev. D60 (1999) 034008, hep-ph/9901281.

[22] Y. V. Kovchegov, Unitarization of the BFKL pomeron on a nucleus, Phys. Rev. D61 (2000) 074018, hep-ph/9905214.

[23] I. Balitsky, Operator expansion for high-energy scattering, Nucl. Phys. B463 (1996) 99–160, hep-ph/9509348.

[24] I. Balitsky, Operator expansion for diffractive high-energy scattering, hep-ph/9706411

[25] I. Balitsky, Factorization and high-energy effective action, Phys. Rev. D60 (1999) 014020, hep-ph/9812311.

[26] E. Iancu and R. Venugopalan, The color glass condensate and high energy scattering in QCD, hep-ph/0303204.

[27] H. Weigert, Evolution at small x_bj: The Color Glass Condensate, Prog. Part. Nucl. Phys. 55 (2005) 461–565, hep-ph/0501087.
[28] J. Jalilian-Marian and Y. V. Kovchegov, *Saturation physics and deuteron gold collisions at rhic*, *Prog. Part. Nucl. Phys.* **56** (2006) 104–231, [hep-ph/0505052].

[29] K. Rummukainen and H. Weigert, *Universal features of JIMWLK and BK evolution at small x*, *Nucl. Phys.* **A739** (2004) 183–226, [hep-ph/0309306].

[30] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *The Pomeranchuk singularity in non-Abelian gauge theories*, *Sov. Phys. JETP* **45** (1977) 199–204.

[31] Y. Y. Balitsky and L. N. Lipatov *Sov. J. Nucl. Phys.* **28** (1978) 822.

[32] E. Gardi, J. Kuokkanen, K. Rummukainen, and H. Weigert, *Running coupling and power corrections in nonlinear evolution at the high-energy limit*, *Nucl. Phys.* **A784** (2007) 282–340, [hep-ph/0609087].

[33] Y. V. Kovchegov and H. Weigert, *Triumvirate of running couplings in small-x evolution*, *Nucl. Phys.* **A784** (2007) 188–226, [hep-ph/0609090].

[34] I. Balitsky, *Quark contribution to the small-x evolution of color dipole*, [hep-ph/0609105].

[35] J. L. Albacete and Y. V. Kovchegov, *Solving high energy evolution equation including running coupling corrections*, *Phys. Rev.* **D75** (2007) 125021, arXiv:0704.0612 [hep-ph].

[36] E. Avsar and Y. Hatta, *Quantitative study of the transverse correlation of soft gluons in high energy QCD*, *JHEP* **09** (2008) 102, [0805.0710].

[37] A. Dumitru, E. Iancu, L. Portugal, G. Soyez, and D. N. Triantafyllopoulos, *Pomeron loop and running coupling effects in high energy QCD evolution*, *JHEP* **08** (2007) 062, [0706.2540].

[38] I. Balitsky and G. A. Chirilli, *Next-to-leading order evolution of color dipoles*, *Phys. Rev.* **D77** (2008) 014019, [0710.4330].

[39] A. H. Mueller, *Small x Behavior and Parton Saturation: A QCD Model*, *Nucl. Phys.* **B335** (1990) 115.

[40] B. Z. Kopeliovich, A. V. Tarasov, and A. Schafer, *Bremsstrahlung of a quark propagating through a nucleus*, *Phys. Rev.* **C59** (1999) 1609–1619, [hep-ph/9808378].

[41] B. Z. Kopeliovich, A. Schafer, and A. V. Tarasov, *Nonperturbative effects in gluon radiation and photoproduction of quark pairs*, *Phys. Rev.* **D62** (2000) 054022, [hep-ph/9908245].

[42] Y. V. Kovchegov, *Diffractive gluon production in proton nucleus collisions and in dis*, *Phys. Rev.* **D64** (2001) 114016, [hep-ph/0107256].

[43] A. Kovner and U. A. Wiedemann, *Eikonal evolution and gluon radiation*, *Phys. Rev.* **D64** (2001) 114002, [hep-ph/0106240].

[44] E. Iancu, K. Itakura, and L. McLerran, *A Gaussian effective theory for gluon saturation*, *Nucl. Phys.* **A724** (2003) 181–222, [hep-ph/0212123].

[45] J. Bartels, *High-Energy Behavior in a Nonabelian Gauge Theory. 1. Tn→m in the Leading Log Normal S Approximation*, *Nucl. Phys.* **B151** (1979) 293.
[46] J. Bartels, *High-Energy Behavior in a Nonabelian Gauge Theory. 2. First Corrections to T_n→m Beyond the Leading LNS Approximation*, Nucl. Phys. B175 (1980) 365.

[47] J. Kwiecinski and M. Praszalowicz, *Three Gluon Integral Equation and Odd c Singlet Regge Singularities in QCD*, Phys. Lett. B94 (1980) 413.

[48] T. Jaroszewicz, *Infrared Divergences and Regge Behavior in QCD*, Acta Phys. Polon. B11 (1980) 965.

[49] L. Lukaszuk and B. Nicolescu, *A Possible interpretation of p p rising total cross-sections*, Nuovo Cim. Lett. 8 (1973) 405–413.

[50] B. Nicolescu, *The Odderon at RHIC and LHC*, [0707.2923](https://arxiv.org/abs/0707.2923).

[51] R. A. Janik and J. Wosiek, *Solution of the odderon problem*, Phys. Rev. Lett. 82 (1999) 1092–1095, [hep-th/9802100](https://arxiv.org/abs/hep-th/9802100).

[52] G. P. Korchemsky, J. Kotanski, and A. N. Manashov, *Compound states of reggeized gluons in multi-colour QCD as ground states of noncompact Heisenberg magnet*, Phys. Rev. Lett. 88 (2002) 122002, [hep-ph/0111185](https://arxiv.org/abs/hep-ph/0111185).

[53] J. Bartels, L. N. Lipatov, and G. P. Vacca, *A New Odderon Solution in Perturbative QCD*, Phys. Lett. B477 (2000) 178–186, [hep-ph/9912423](https://arxiv.org/abs/hep-ph/9912423).

[54] Y. V. Kovchegov, L. Szymanowski, and S. Wallon, *Perturbative odderon in the dipole model*, Phys. Lett. B586 (2004) 267–281, [hep-ph/0309281](https://arxiv.org/abs/hep-ph/0309281).

[55] Y. Hatta, E. Iancu, K. Itakura, and L. McLerran, *Odderon in the color glass condensate*, Nucl. Phys. A760 (2005) 172–207, [hep-ph/0501171](https://arxiv.org/abs/hep-ph/0501171).

[56] A. Kovner and M. Lublinsky, *Odderon and seven Pomerons: QCD Reggeon field theory from JIMWLK evolution*, JHEP 02 (2007) 058, [hep-ph/0512316](https://arxiv.org/abs/hep-ph/0512316).

[57] Z. Chen and A. H. Mueller, *The dipole picture of high-energy scattering, the BFKL equation and many gluon compound states*, Nucl. Phys. B451 (1995) 579–604.

[58] S. E. Derkachov, G. P. Korchemsky, J. Kotanski, and A. N. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD. II: Quantization conditions and energy spectrum*, Nucl. Phys. B645 (2002) 237–297, [hep-th/0204124](https://arxiv.org/abs/hep-th/0204124).

[59] K. J. Golec-Biernat and M. Wusthoff, *Saturation effects in deep inelastic scattering at low Q^2 and its implications on diffraction*, Phys. Rev. D59 (1999) 014017, [hep-ph/9807513](https://arxiv.org/abs/hep-ph/9807513).

[60] A. M. Stasto, K. Golec-Biernat, and J. Kwiecinski, *Geometric scaling for the total \gamma^*p cross-section in the low x region*, Phys. Rev. Lett. 86 (2001) 596–599, [hep-ph/0007192](https://arxiv.org/abs/hep-ph/0007192).

[61] A. L. Ayala, M. B. Gay Ducati, and E. M. Levin, *Qcd evolution of the gluon density in a nucleus*, Nucl. Phys. B493 (1997) 305–353, [hep-ph/9604383](https://arxiv.org/abs/hep-ph/9604383).

[62] F. Ayala, A. L., M. B. Gay Ducati, and E. M. Levin, *Unitarity boundary for deep inelastic structure functions*, Phys. Lett. B388 (1996) 188–196, [hep-ph/9607210](https://arxiv.org/abs/hep-ph/9607210).
[63] A. L. Ayala Filho, M. B. Gay Ducati, and E. M. Levin, *Parton densities in a nucleon*, *Nucl. Phys.* **B511** (1998) 355–395, [hep-ph/970634].

[64] G. S. Bali, *Casimir scaling or flux counting?*, *Nucl. Phys. Proc. Suppl.* **83** (2000) 422–424, [hep-lat/9908021].

[65] G. S. Bali, *Casimir scaling of SU(3) static potentials*, *Phys. Rev.* **D62** (2000) 114503, [hep-lat/0006022].

[66] E. Iancu, K. Itakura, and L. McLerran, *Geometric scaling above the saturation scale*, *Nucl. Phys.* **A708** (2002) 327–352, [hep-ph/0203137].

[67] J.-P. Blaizot, E. Iancu, and H. Weigert, *Non linear gluon evolution in path-integral form*, *Nucl. Phys.* **A713** (2003) 441–469, [hep-ph/0206279].

[68] A. V. Popov, *Invariant color calculus*, [0805.4504].

[69] P. Cvitanović, *Group Theory, Birdtracks, Lie’s and Exceptional Groups*. Princeton University Press (2008) 285pp. [http://birdtracks.eu].