C-SYMPLECTIC POSET STRUCTURE ON A SIMPLY CONNECTED SPACE

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ABSTRACT. For a field $k$ of characteristic zero, we introduce a cohomologically symplectic poset structure $P_k(X)$ on a simply connected space $X$ from the viewpoint of $k$-homotopy theory. It is given by the poset of inclusions of subgroups preserving c-symplectic structures in the group $E(X)_k$ of $k$-homotopy classes of $k$-homotopy self-equivalences of $X$, which is defined by the $k$-Sullivan model of $X$. We observe that the height of the Hasse diagram of $P_k(X)$ added by 1, denoted by c-s-depth$_k(X)$, is finite and often depends on the field $k$. In this paper, we will give some examples of $P_k(X)$.

1. Introduction

Compact Kähler manifolds are well-known to be symplectic manifolds and they are also formal spaces from the viewpoint of minimal models [4]. However, symplectic manifolds are not necessarily formal spaces (e.g. [7]). Dennis Sullivan posed the following problem [29] (cf. [30, Problem 4], [9, Question 4.99]): Are there algebraic conditions on the minimal model of a compact manifold, implying the existence of a symplectic structure on it? There are interesting approaches to answer this question for certain cases (e.g. see [19], [20], [30]). In this paper we discuss simply connected cohomologically symplectic structures, instead of the usual symplectic structures. A Poincaré duality space (cf. [9, Definition 3.1]) $Y$ is said to have a cohomologically symplectic (abbr., c-symplectic) structure if there is a rational cohomology class $\omega \in H^2(Y; \mathbb{Q})$ such that $\omega^n$ is a top class for $Y$ (cf. [9, Definition 4.87]). Since $H^2(Y; \mathbb{Z}) \cong [Y, K(\mathbb{Z}, 2)]$ and $H^2(Y; \mathbb{Q}) \cong H^2(Y; \mathbb{Z}) \otimes \mathbb{Q}$, we can consider the following situation:

(1) an integral cohomology class $\mu \in H^2(Y; \mathbb{Z})$ corresponds to the homotopy class of a map $\mu : Y \to K(\mathbb{Z}, 2)$ and
(2) $\mu \otimes 1_\mathbb{Q}$ in $H^2(Y; \mathbb{Z}) \otimes \mathbb{Q}$ is a c-symplectic class on $Y$.

Furthermore it follows from Serre’s result that in the homotopy category any map is “the same” as a fibration $p : Y' \to K(\mathbb{Z}, 2)$. Thus from the beginning we can assume that $Y$ is the total space of a fibration to $K(\mathbb{Z}, 2)$. Let $X$ be the fibre of the fibration $\mu : Y \to K(\mathbb{Z}, 2)$. Then we rationalize this fibration and study it by minimal models.

In [27], J. Sato and the second named author treated the above c-symplectic analogue of Sullivan’s question, considering the fiber space $X$ of the fibration $\mu$. They asked the following question under the additional condition that the fiber space $X$ is simply connected: What conditions on $X$ induce a c-symplectic structure

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of the total space \( Y_\mu \) of the fibration
\[
\mu : X \rightarrow Y_\mu \xrightarrow{\mu} K(\mathbb{Z}, 2). \tag{*}
\]

They call such a space \( X \) pre-c-symplectic (i.e., pre-cohomologically symplectic). For example, they gave necessary and sufficient conditions for that the product \( S^{k_1} \times S^{k_2} \times \cdots \times S^{k_n} \) of odd-dimensional spheres is pre-c-symplectic. In this paper, we furthermore consider the following

**Problem 1.1.** For a simply connected space \( X \), how can we classify c-symplectic spaces \( Y_\mu \) in (\(*\))?

When \( S^1 \) acts on \( X \), we have a Borel fibration (\(*\)) with \( K(\mathbb{Z}, 2) = BS^1 \). In our case, it is always a Borel fibration of a free \( S^1 \)-action on a finite \( CW \)-complex \( X' \) in the rational homotopy type of \( X \) \[14\] Proposition 4.2. In \[20\], V. Puppe gives a classification of \( S^1 \)-fibrations on a certain space \( X \) having fixed points via the algebraic deformation theory of M. Gerstenhaber over an algebraically closed field \( k \). This classification gives a Hasse diagram (e.g. see \[20\] page 350)). However, in the case of almost free \( S^1 \)-actions, which have empty fixed points, we cannot give such a classification using the algebraic deformation theory of Gerstenhaber. Thus we cannot get such a Hasse diagram. In this paper, we can get a classification of the rational homotopy types of products of odd-dimensional spheres. In our case, it is always a Borel fibration of a free \( S^1 \)-action on a finite \( CW \)-complex \( X' \) in the rational homotopy type of \( X \) \[14\] Proposition 4.2. In \[20\], V. Puppe gives a classification of \( S^1 \)-fibrations on a certain space \( X \) having fixed points via the algebraic deformation theory of M. Gerstenhaber over an algebraically closed field \( k \). This classification gives a Hasse diagram (e.g. see \[20\] page 350)). However, in the case of almost free \( S^1 \)-actions, which have empty fixed points, we cannot give such a classification using the algebraic deformation theory of Gerstenhaber. Thus we cannot get such a Hasse diagram. In this paper, we can get a classification of the total spaces \( Y_\mu \) by using the groups of \( k \)-homotopy classes of \( k \)-homotopy self-equivalences, by which we can get a poset denoted as \( \mathcal{P}_k(X) \) via the inclusions of the subgroups over a field of characteristic zero \( k \). See \[31\] for some examples of such a poset of free \( S^1 \)-actions on some rational types with \( X \) when \( k = \mathbb{Q} \). Refer to \[6\] for the other general classification given by a \( k \)-deformation of Sullivan model.

In \[2\] we recall \( k \)--Sullivan models according to \[3\]. In \[3\] we define the “c-symplectic poset structure” \( \mathcal{P}_k(X) \) for \( X \) over \( k \) and a \( c \)-symplectic depth of \( X \) over \( k \), denoted by \( c \)-depth\(_k(X) \) (which is defined by adding 1 to the height or the length of the Hasse diagram of the poset \( \mathcal{P}_k(X) \)). We give some results in \[3\] and their proofs are given in \[4\]. In \[5\] we treat \( \mathcal{P}_k(X) \) mainly in the case that spaces \( X \) have the rational homotopy types of products of odd-spheres.

## 2. \( k \)-Sullivan Models

Let \( X_\mathbb{Q} \) and \( f_\mathbb{Q} \) be the rationalizations \[18\] of a simply connected space \( X \) with finite rational cohomology and a map \( f \), respectively. Let \( M(X) = (\Lambda V, d) \) be the Sullivan model of \( X \). It is a freely generated \( \mathbb{Q} \)-commutative differential graded algebra (\( \mathbb{Q} \)-DGA) with a \( \mathbb{Q} \)-graded vector space \( V = \bigoplus_{i \geq 2} V^i \) where \( \dim V^i < \infty \) and a decomposable differential; i.e., \( d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1} \) and \( d \circ d = 0 \). Here \( \Lambda^+ V \) is the ideal of \( \Lambda V \) generated by elements of positive degree. Denote the degree of a homogeneous element \( x \) of a graded algebra by \( |x| \). Then \( xy = (-1)^{|x||y|}yx \) and \( d(xy) = d(x)y + (-1)^{|x|}x d(y) \). Note that \( M(X) \) determines the rational homotopy type of \( X \). In particular, \( H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}) \) and \( V^i \cong \text{Hom}(\pi_i(X), \mathbb{Q}) \). Refer to \[7\] for details.

Let \( k \) be a field of characteristic zero (then \( \mathbb{Q} \subset k \)). The \( k \)-minimal model of a space \( X \) is \( M(X_k) := (\Lambda V_k, d_k) \) where \( \Lambda V_k := \Lambda V \otimes k \) and \( d_k \) is the \( k \)-extension of \( d \). We say that \( X \) and \( Y \) have the same \( k \)-homotopy type (denote \( X_k \cong Y_k \) when there is a \( k \)-DGA isomorphism \((\Lambda V_k, d_k) \cong (\Lambda U_k, d_k') \) for \( M(Y) = (\Lambda U, d') \). Refer to \[3\] Definition 1]. Notice that, for \( k_1 \supset k_2 \), \((\Lambda V_k, d_k) \cong (\Lambda U_k, d_k') \) if
(AV_k, d_k) \cong (AU_k, d'_k) \text{ ([3], [23]).}

The k-model of the above fibration (**) is given as the k-relative model

\((k[t], 0) \rightarrow (k[t] \otimes AV_k, D_k) \xrightarrow{\mu} (AV_k, d_k) = M(X_k)\) (**) where \(|t| = 2\) and \((k[t], 0)\) is the k-model of \(K(\mathbb{Z}, 2)\).

3. C-SYMPLECTIC POSET STRUCTURE

Let \(E(X_\mathbb{Q})\) be the group of homotopy classes of (unpointed) homotopy self-equivalences of a rationalized space \(X_\mathbb{Q}\) (e.g. [11, 22, 21, 22, 25]). Let \(autM\) be the group of k-DGA-autmorphisms of a k-DGA \(M\). Denote \(f \sim_k g\) when two maps \(f\) and \(g\) of \(autM\) are k-DGA-homotopic; i.e., there is a k-DGA-map \(H : M \rightarrow M \otimes \Lambda(s, ds)_k\) such that \(|s| = 0\) and \(|ds| = 1\) with \(H |_{s=0} = f\) and \(H |_{s=1} = g\). The group of k-DGA-homotopy classes of k-DGA-autmorphisms of \(M(X_k) = (AV_k, d_k)\)

\[Aut(AV_k, d_k) = aut(AV_k, d_k) / \sim_k\]

is denoted by \(E(X_k)\) and it is equal to \(aut(AV_k, d_k) / I\) for the subgroup \(I\) of inner automorphisms of \((AV_k, d_k)\) [29 Proposition 6.5].

For a fibration \(\mu\) (see the above (**) ) where \(Y = Y_\mu\) is c-symplectic, let \(E(p_\mathbb{Q})\) denote the group of (unpointed) fibrewise self-equivalences \(f_\mathbb{Q} : Y_\mathbb{Q} \rightarrow Y_\mathbb{Q}\) of the rationalized fibration \(p_\mathbb{Q} : Y_\mathbb{Q} \rightarrow K(\mathbb{Z}, 2)_\mathbb{Q} = K(\mathbb{Q}, 2)\) of (**). Thus \(p_\mathbb{Q} \circ f_\mathbb{Q} = p_\mathbb{Q}\). Furthermore, let \(E(p_k)\) denote the group of fibrewise self-equivalences \(f_k\) of \(p_k : Y_k \rightarrow K(\mathbb{Z}, 2)_k =: K(k, 2)\). Thus \(p_k \circ f_k = p_k\) in

\[\xymatrix{ X_k \ar[r]^{\mu} & Y_k \ar[r]^{p_k} & K(k, 2) \ar@{=}[d] \ar@{=}[d] \\
X_k \ar[r]^{\mu} & Y_k \ar[r]^{p_k} & K(k, 2), \ar@{=}[d] \ar@{=}[d]}
\]

where \(\bar{f}_k\) is the induced map of \(f_k\). Let \(E_\mu(X_k)\) denote the image of the natural homomorphism induced by fibre restrictions

\[F_\mu : E(p_k) \rightarrow E(X_k)\]

with \(F_\mu(f_k) = \bar{f}_k\). We let \(Aut_t(k[t] \otimes AV_k, D_k)\) denote the groups of k-DGA-homotopy classes of k-DGA-autmorphisms \(f\) of \((k[t] \otimes AV_k, D_k)\) in the above (**) with \(|f| = t\). Then

\[E(p_k) = Aut_t(k[t] \otimes AV_k, D_k)\]

and \(F_\mu : E(p_k) \rightarrow E(X_k)\) is equivalent to the restriction map

\[F'_\mu : Aut_t(k[t] \otimes AV_k, D_k) \rightarrow Aut(AV_k, d_k)\]

with \(F'_\mu(f) = p_t(f(v))\) for \(v \in V\) and \(p_t\) in (**). Then

\[E_\mu(X_k) = \text{Im} F'_\mu.\]

Refer to [28 Proposition 2.2] (also see [10, 8]).

Remark 3.1. Recall that, for a fibration \(X \rightarrow E \rightarrow B\), D. Gottlieb posed the problem [12, 55]: Which homotopy equivalences of \(X\) into itself can be extended to fibre homotopy equivalences of \(E\) into itself? In our case, we can propose a k-realization problem on c-symplectic spaces: For which subgroup \(G\) of \(E(X_k)\) does or does not there exist a fibration \(\mu\) such that \(E_\mu(X_k) = G\)?
Lemma 3.2. If $\mu \cong_k \mu'$ (over $K(k,2)$), then $E_\mu(X_k)$ and $E_{\mu'}(X_k)$ are naturally identified in $E(X_k)$.

We are interested in the set $S := \{E_\mu(X_k)\}_{\mu \in \mathcal{I}}$ of subgroups of $E(X_k)$ for the set $I$ of isomorphism classes of $Y_k$ in ($*$) where $Y_k$ are c-symplectic.

Definition 3.3. We define an equivalence relation $\sim$ for $\mu$ and $\tau$ of $I$ if $E_\mu(X_k) = E_\tau(X_k)$ in $E(X_k)$. For equivalence classes $[\mu]$ and $[\tau]$, put $[\mu] \leq [\tau]$ when there is an inclusion $i : E_\mu(X_k) \rightarrow E_\tau(X_k)$ in $E(X_k)$. We define

$$P_k(X) := (S/\sim, \leq) = ([\mu]), \leq),$$

which is called the c-symplectic poset structure on $X$ over $k$.

Definition 3.4. For a field $k$ of characteristic zero, we define the c-symplectic depth (abbr., c-s-depth) of a simply connected space $X$ over $k$ by

$$c\text{-}s\text{-depth}_k(X) = \max \{n \mid [\mu_i] > [\mu_{i+1}] > \cdots \}$$

for $[\mu_i] \in P_k(X)$.

Here we note that

$$c\text{-}s\text{-depth}_k(X) = \text{the height of } P_k(X) + 1.$$

Suppose that $\mu_1 \cong_k \mu_1'$ and $\mu_2 \cong_k \mu_2'$. If $E_{\mu_1}(X_k) \subset E_{\mu_2}(X_k)$, then we can regard as $E_{\mu_1'}(X_k) \subset E_{\mu_2'}(X_k)$ from Lemma 3.4. Thus we have

Lemma 3.5. The non-negative integer $c\text{-}s\text{-depth}_k(X)$ is a $k$-homotopy invariant.

Proposition 3.6. When $\dim H^*(X; \mathbb{Q}) < \infty$, $c\text{-}s\text{-depth}_k(X) < \infty$ for all $k$.

If $X$ is not pre-c-symplectic, then $P_k(X) = \emptyset$, in which case we set $c\text{-}s\text{-depth}_k(X) := 0$. If $X$ is pre-c-symplectic, then $P_k(X) \neq \emptyset$, whence clearly we have that $c\text{-}s\text{-depth}_k(X) \geq 1$. Thus it is obvious that $X$ is pre-c-symplectic if and only if $c\text{-}s\text{-depth}_k(X) > 0$ for any $k$.

Remark 3.7. It is expected that $c\text{-}s\text{-depth}_k(X)$ measures the abundance of c-symplectic structures associated to a simply connected space $X$.

Proposition 3.8. If $X$ is pre-c-symplectic, then the formal dimension $fd(X)$ of $X$ is odd, where $fd(X) := \max \{n \mid H^n(X; \mathbb{Q}) \neq 0\}$.

Proof. Since $X$ is pre-c-symplectic, we have a fibration $\mu : X \rightarrow Y_\mu \rightarrow K(\mathbb{Z},2)$. Then we have a fibration one step back in the Barratt-Puppe sequence, which is an $\Omega K(\mathbb{Z},2) = S^1$-fibration over $Y$ with the total space $X$, $S^1 \rightarrow X \rightarrow Y$. Let $fd(Y) := m$. Then it follows from the Gysin sequence for cohomology that $H^{m+1}(X; \mathbb{Q}) \cong H^m(Y; \mathbb{Q}) \neq 0$ and $H^{k+1}(X; \mathbb{Q}) \cong H^k(Y; \mathbb{Q}) = 0$ for $k > m$, thus we get $fd(X) = m+1 = fd(Y)+1$. Since $fd(Y) = m$ is even, the formal dimension $fd(X)$ of $X$ is odd.

Note that $fd(X) = \max \{n \mid H^n(X; k) \neq 0\}$ for any $k$ and that if $X$ is a compact manifold, then the formal dimension is the same as the (real) dimension of the manifold.

Corollary 3.9. If the formal dimension of $X$ is even, then $c\text{-}s\text{-depth}_k(X) = 0$.

Corollary 3.10. (1) For any $n > 0$, $c\text{-}s\text{-depth}_k(S^{2n}) = 0$.
(2) If $X$ is a c-symplectic space, then $c\text{-}s\text{-depth}_k(X) = 0$.
(3) If $c\text{-}s\text{-depth}_k(G/H) > 0$ for a homogeneous space $G/H$, $\text{rank} G > \text{rank} H$.
(4) If $c\text{-}s\text{-depth}_k(X) > 0$ and $c\text{-}s\text{-depth}_k(Y) > 0$, then $c\text{-}s\text{-depth}_k(X \times Y) = 0$. 

Proof. (1) $fd(S^{2n})$ is even.  
(2) The formal dimension of a c-symplectic space is even (cf. [7] page 218, Theorem 32.6(i)).  
(3) Suppose that $\text{rank } G = \text{rank } H$. Let $T$ be a maximal torus for both $G$ and $H$. Since root theory tells us that $\dim G - \dim T$ and $\dim H - \dim T$ are both even, then $\dim G/H = \dim G - \dim H$ is also even. Hence, $c$-s-depth$_k(G/H) = 0$.  
(4) $c$-s-depth$_k(X) > 0$ and $c$-s-depth$_k(Y) > 0$ imply that their formal dimensions $fd(X)$ and $fd(Y)$ are both odd. By the Künneth theorem we have that $H^i(X \times Y, \mathbb{Q}) = \sum_{i+j=n} H^i(X, \mathbb{Q}) \otimes H^j(Y, \mathbb{Q})$. Hence $fd(X \times Y) = fd(X) + fd(Y)$, thus $fd(X \times Y)$ is even. Therefore $c$-s-depth$_k(X \times Y) = 0$. \hfill $\Box$

**Theorem 3.11.** For any $n > 0$, $c$-s-depth$_k(S^{2n+1}) = 1$.

Proof. The result $c$-s-depth$_k(S^{2n+1}) = 1$ follows from the fact that the $k$-Sullivan minimal model of $Y$ in $(*)$ is uniquely determined as $(k[t] \otimes \mathcal{A}(v)_k, D_k)$ with $D_k t = 0$ and $D_k v = t^{n+1}$ for $M(S^{2n+1}) = (\mathcal{A}(v), 0)$.

One might expect or think that $fd(X)$ is even if and only if $c$-s-depth$_k(X) = 0$, or equivalently that $fd(X)$ is odd if and only if $c$-s-depth$_k(X) > 0$. However this turns out not to be the case, due to the following result:

**Theorem 3.12.** ([27] Theorem 1.2) When $H^*(X; \mathbb{Q}) \cong \Lambda(v_1, v_2, \ldots, v_n)$ with all $|v_i| \geq 1$, then $X$ is pre-c-symplectic if and only if $n$ is odd and $|v_1| + |v_{n-1}| < |v_n|$, $|v_2| + |v_{n-2}| < |v_n|$, $|v_3| + |v_{n-3}| < |v_n|$, $\cdots$, $|v_{n-1}| + |v_{n+1}| < |v_n|$.

**Example 3.13.** Let $n$ be an odd integer $\geq 5$ and consider the following space:

$$X := S^{11} \times S^{15} \times \cdots \times S^{4k-1} \times \cdots \times S^{4n-1} \quad (k > 3).$$

Since $n$ is odd, the formal dimension $fd(X)$ is odd, but it follows from Theorem 3.12 that $c$-s-depth$_k(X) = 0$.

Theorem 5.13 at the end of §5 says for the symplectic group $Sp(n)$ whose dimension $\dim Sp(n)$ is $n(2n + 1)$ that $c$-s-depth$_k(\mathcal{A}(v)) \geq \frac{n+1}{2}$ for any odd integer $n > 3$. Thus we have the following

**Proposition 3.14.** A c-symplectic depth can be arbitrarily large.

The above Corollary 3.10(4) suggests that c-symplectic depth would not behave well with respect to taking the product of spaces. In fact, using Theorem 5.13 and Example 3.13 we can make the following

**Proposition 3.15.** (1) Even if $c$-s-depth$_k(X) = 0$ and $c$-s-depth$_k(Y) = 1$, $c$-s-depth$_k(X \times Y)$ can be arbitrarily large.

(2) Even if $c$-s-depth$_k(X) = c$-s-depth$_k(Y) = 0$, $c$-s-depth$_k(X \times Y)$ can be arbitrarily large.

Proof. (1) Indeed, let $n$ be an odd integer $\geq 5$ and let us consider the rational homotopy decomposition

$$Sp(n) \simeq Q S^3 \times S^7 \times \cdots \times S^{4(n-1)-1} \times S^{4n-1} \simeq Q Sp(n-1) \times S^{4n-1}.$$ 

We now let $X := Sp(n-1)$ and $Y := S^{4n-1}$. Then $c$-s-depth$_k(Y) = 1$. When $n$ is odd, $\dim Sp(n) = n(2n + 1)$ is odd and $\dim Sp(n-1) = fd(Sp(n-1)) = fd(X)$
is even, thus $c\text{-s-depth}_k(X) = 0$. However it follows from Theorem 5.13 that $c\text{-s-depth}_k(X \times Y) \geq \frac{n+1}{2}$.

(2) Now consider the following:

$$Sp(n) \simeq \left(S^3 \times S^7\right) \times \left(S^{11} \times \cdots \times S^{4k-1} \times \cdots \times S^{4n-1}\right) \quad (k > 3)$$

and set $X := S^3 \times S^7$ and $Y := S^{11} \times \cdots \times S^{4n-1}$. Then we have that $c\text{-s-depth}_k(X) = c\text{-s-depth}_k(Y) = 0$, but we have $c\text{-s-depth}_k(X \times Y) \geq \frac{n+1}{2}$. \hfill \qed

The following theorem indicates an example that a $c\text{-symplectic depth}$ strongly depends on a field $k$.

**Theorem 3.16.** When $X = \mathbb{C}P^n \times S^{2n+3}$ with $n$ even, $c\text{-s-depth}_0(X) = 1$ but $c\text{-s-depth}_k(X) = c(n+1)$ where $c(n+1) := n_1 + n_2 + \cdots + n_m + 1$ for the prime decomposition $n + 1 = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$. Here $\overline{\mathbb{Q}}$ means the algebraic closure of $\mathbb{Q}$. Moreover, there is a sequence of field extensions

$$\mathbb{Q} \subset k_1 \subset k_2 \subset \cdots \subset k_{c(n+1)} \subset \overline{\mathbb{Q}}$$

such that

$$c\text{-s-depth}_0(X) < c\text{-s-depth}_{k_1}(X) < c\text{-s-depth}_{k_2}(X) < \cdots < c\text{-s-depth}_{k_{c(n+1)}}(X).$$

We see the following result from the proof of Theorem 3.16.

**Corollary 3.17.** When $X = \mathbb{C}P^n \times S^{2n+3}$ with $n$ even, the poset $\mathcal{P}_k(X)$ contains the subgroup poset of the group $\mathbb{Z}/(n+1)\mathbb{Z}$.

We remark that $\text{cat}_0(\mathbb{C}P^n \times S^{2n+3}) = \text{cat}(\mathbb{C}P^n \times S^{2n+3}) = n + 1$. Here $\text{cat}$ and $\text{cat}_0$ are respectively the Lusternik-Schnirelmann (LS) category and the rational LS category $\text{cat}_0(X) := \text{cat}(X_\mathbb{Q})$ of a space $X$ (e.g., see [7]). Notice that $\text{cat}_0(X) = \text{cat}(X_\mathbb{Q})$ for all field $k$. On the other hand, for a cyclotomic field $k$, the proof of Theorem 3.16 indicates that $\mathcal{P}_k(X)$ often presents more informations on a classification of $\{Y_\mu\}_\mu$ in (§1) than $\mathcal{P}_0(X)$. In connection with LS categories, we would like to pose the following

**Question 3.18.** For any field $k$, is $c\text{-s-depth}_k(X) \leq \text{cat}_0(X)$?

**Remark 3.19.** Let $Y$ be a simply connected $c\text{-symplectic}$ space. We define

$$l_X(Y)_k := \begin{cases} \max \{ n \mid |\mu_1 > |\mu_2 > \cdots > |\mu_{n-1}| > |\mu_n| = |Y| \} \text{ in } \mathcal{P}_k(X), \\ 0, \text{ if there exists no fibration } X \to Y \to K(\mathbb{Z}, 2). \end{cases}$$

If $\dim \pi_2(Y) \otimes \mathbb{Q} = 1$, such a space $X$ uniquely exists. We speculate that $l_X(Y)_k$ must reflect a certain complexity of $c\text{-symplectic}$ structure of $Y$. This definition is something like the co-height of a prime ideal (e.g., [22]) by adding 1. Here the co-height of a prime ideal $p$ in a ring $R$ is defined as the largest $n$ for which there exists a chain of different prime ideals $p \subset p_1 \subset \cdots \subset p_n \neq R$.

4. Proofs

**Proof of Lemma**. Let $\mu : X \to Y_{\mu} \to K(\mathbb{Z}, 2)$ be a fibration with $\mu \equiv_k \mu'$. Suppose that

$$\psi : M(Y_{\mu k}) = (k[t] \otimes \Lambda V, D) \to (k[t] \otimes \Lambda V, D') = M(Y_{\mu' k})$$

Proof of Lemma
is an isomorphism over \((k[t], 0)\). Then the restriction map \(\overline{\psi} : (\Lambda V_k, D_k) \to (\Lambda V_k, D'_k)\) is an isomorphism. We define

\[
ad_{\psi} : Aut_t(k[t] \otimes \Lambda V_k, D_k) \to Aut_t(k[t] \otimes \Lambda V_k, D'_k)
\]

by \(ad_{\psi}(f) = \psi \circ f \circ \psi^{-1}\), which is well-defined. Then we get the following commutative diagram of groups:

\[
\begin{array}{ccc}
\mathcal{E}(X_k) & \xrightarrow{ad_{\psi}} & \mathcal{E}(X_k) \\
\text{res.} & & \text{res.} \\
\mathcal{E}(p_k) & \xrightarrow{ad_{\psi}} & \mathcal{E}(p'_k),
\end{array}
\]

where the vertical maps are the restriction maps. Thus \(E_{\mu}(X_k)\) is identified with \(E_{\mu'}(X_k)\) by the isomorphism \(ad_{\psi}\) given by \(ad_{\psi}(f) = \psi \circ f \circ \psi^{-1}\). \(\square\)

**Proof of Proposition 3.6** Suppose that there is a sequence

\[
\mathcal{E}(X_k) \supseteq E_{\mu_1}(X_k) \supseteq E_{\mu_2}(X_k) \supseteq \cdots \supseteq E_{\mu_n}(X_k)
\]

for some \(m\). It is equivalent to a sequence

\[
Aut(\Lambda V_k, d_k) \supseteq \text{Im} F'_{\mu_1} \supseteq \text{Im} F'_{\mu_2} \supseteq \cdots \supseteq \text{Im} F'_{\mu_{m-1}}
\]

for the restriction maps \(F'_{\mu_i} : Aut_t(k[t] \otimes \Lambda V_k, D_{t,k}) \to Aut(\Lambda V_k, d_k)\) in §3. Then we have a sequence of \(k\)-algebraic groups

\[
\text{aut}(\Lambda V_k, d_k) \supseteq G_{\mu_1} \supseteq G_{\mu_2} \supseteq \cdots \supseteq G_{\mu_{m-1}}
\]

where \(G_{\mu_i}\) are the images of the restrictions sending \(t\) to zero

\[
\text{aut}_t(k[t] \otimes \Lambda V_k, D_{t,k}) \to \text{aut}(\Lambda V_k, d_k)
\]

with \(G_{\mu_i}/_k = \text{Im} F'_{\mu_i}\). Since \(\dim H^*(\Lambda V_k, d_k) = \dim H^*(X; k) < \infty\), being \(k\)-DGA-autmorphisms induces

\[
\text{aut}(\Lambda V_k, d_k) = \text{aut}(\Lambda V_k^{\leq n}, d_k|_{V_k^{\leq n}})
\]

for a sufficiently large \(n\). Here \(V_k^{\leq n}\) means the subspace \(\{v \in V_k : |v| \leq n\}\) of \(V_k\), which is finite-dimensional over \(k\). The latter is an algebraic matrix group (defined by polynomial equations in the entries) in the general \(k\)-linear group \(GL(N, k)\) for a sufficiently large \(N\) [29, page 294]. From the Noetherian property of descending chain condition, the integer \(m\) is bounded by \(N\). Thus we have \(c\)-s-depth\(_k(X) < \infty\). \(\square\)

**Proof of Theorem 3.16** Let \(M(X_k) = (\Lambda(x, y, z, k), d_k)\) with

\[
|x| = 2, \ |y| = 2n + 1, \ |z| = 2n + 3 \quad \text{and} \quad d_k(x) = d_k(z) = 0, \ d_k(y) = x^{n+1}.
\]

Then we have

\[
\mathcal{E}(X_k) = \text{Aut}(\Lambda(x, y, z, k), d_k) = \left\{ \begin{pmatrix} a & a^{n+1} \\ b \\ d \end{pmatrix} \mid a, b \in k - 0 \right\},
\]

where \(f(x) = ax, \ f(y) = a^{n+1}y\) and \(f(z) = bz\) for \(f \in \text{Aut}(\Lambda(x, y, z, k), d_k)\). Let us denote the \(k\)-relative models of \((*)\) by \((k[t] \otimes \Lambda(x, y, z, k), D_{\mu}) = M(Y_{\mu, k})\) with

\[
D_\mu(t) = D_\mu(x) = 0, \quad D_\mu(y) = x^{n+1} + x^i t^{n-i+1} + t^{n+1} \quad \text{and} \quad D_\mu(z) = xt^{n+1}
\]
where \( i \) is one of the divisors of \( n + 1 \) but not \( n + 1 \) itself or \( i = 0 \). Then \( f_d(Y_\mu) = 4n + 2 \) and \([t^{2n+1}] \neq 0 \) in \( H^*(Y_\mu; k) = k[t, x]/(x^{n+1} + x^i t^{n-i+1} + t^{n+1}, x t^{n+1}) \).

Let

\[
N_\mu := \begin{cases} i & \text{when } i \neq 0 \\ n + 1 & \text{when } i = 0. \end{cases}
\]

Then we have

\[
\mathcal{E}_\mu(X_k) = \text{Im}(F'_\mu : Aut_1(k[t] \otimes \Lambda(x, y, z)_k, D_\mu) \to Aut(\Lambda(x, y, z)_k, d_k)) = \begin{cases} \begin{pmatrix} a \\ 1 \end{pmatrix} \mid a^N = 1 & \text{for } a \in k - 0 \end{cases}
\]

\[
\cong \{ a \in k - 0 \mid a^N = 1 \}
\]
as groups. Suppose that \( k \supseteq \mathbb{Q}(e^{2 \pi i /(n+1)}) \). Then \( \mathcal{E}_\mu(X_k) \) is isomorphic to \( \mathbb{Z}_{N_\mu} = \mathbb{Z}/N_\mu \mathbb{Z} \). Notice that \( \mathcal{E}_{\mu_1}(X_k) \cap \mathcal{E}_{\mu_2}(X_k) \) if and only if \( N_j | N_i \), i.e., \( N_j \) is a divisor of \( N_i \). Therefore there is a sequence

\[
([a \in k \mid a^N = 1]) = \mathcal{E}_{\mu_1}(X_k) \supseteq \mathcal{E}_{\mu_2}(X_k) \supseteq \cdots \supseteq \mathcal{E}_{\mu_{(n+1)}}(X_k) = \{ \text{id}_{X_k} \}.
\]

Since any subgroup \( \mathcal{E}_\mu(X_k) \) of \( \mathcal{E}(X_k) \) is isomorphic to some subgroup of \( \mathbb{Z}_{n+1} \) from the degree argument of \( t, x, y \) and \( z \), it has the maximal length of inclusions of subgroups. Thus we have that

\[
\text{c-s-depth}_{k}(X) ( = \text{c-s-depth}_{\mathbb{Q}}(X) ) = c(n + 1).
\]

Moreover the sequence of length \( c(n + 1) \)

\[
\mathbb{Z}_{p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}} \supseteq \mathbb{Z}_{p_1^{n_1-1} p_2^{n_2} \cdots p_m^{n_m}} \supseteq \mathbb{Z}_{p_1^{n_1-2} p_2^{n_2} \cdots p_m^{n_m}} \supseteq \cdots \supseteq \mathbb{Z}_{p_m} \supseteq \{ 0 \}
\]

identified with the proper sequence of subgroups for the fibration \( \mu \) of the differential \( D_\mu \) with \( D_\mu(t) = D_\mu(x) = 0, D_\mu(y) = x^{n+1} + t^{n+1} \) and \( D_\mu(z) = x t^{n+1} \),

\[
\mathcal{E}_\mu(X_{\mathbb{Q}(p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m})}) \supseteq \mathcal{E}_\mu(X_{\mathbb{Q}(p_1^{n_1-1} p_2^{n_2} \cdots p_m^{n_m})}) \supseteq \cdots \supseteq \mathcal{E}_\mu(X_{\mathbb{Q}(p_m)}) \supseteq \mathcal{E}_\mu(X_{\mathbb{Q}}),
\]

where \( \mathbb{Q}(q) \) means the extension field \( \mathbb{Q}(e^{2 \pi i/q}) \) of \( \mathbb{Q} \) by adding a primitive \( q \)th root of unity. Thus there is the sequence

\[
\text{c-s-depth}_{\mathbb{Q}}(X) < \text{c-s-depth}_{k_1}(X) < \text{c-s-depth}_{k_2}(X) < \cdots < \text{c-s-depth}_{k_{(n+1)}}(X)
\]
for \( k_1 = \mathbb{Q}(p_1), k_2 = \mathbb{Q}(p_1 p_2) \) or \( \mathbb{Q}(p_1^2), \cdots, k_{(n+1)} = \mathbb{Q}(p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}) \). \( \square \)

5. Examples

In this section, for odd integers \( n_1 \leq n_2 \leq \cdots \leq n_k \), let

\[
M(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) = (\Lambda V, 0) = (\Lambda(v_1, v_2, \ldots, v_k), 0)
\]
with \(|v_i| = n_i \) for all \( i \). In the following Examples 5.1, 5.2, and 5.3(1), the poset structure of \( P_k(X) \) does not depend on \( k \).

Example 5.1. (c-s-depth\(_k(X) = 1\). When \( X \) is

(a) \( S^3 \times S^3 \times S^7 \),

(b) \( S^7 \times S^9 \times S^{11} \times S^{13} \times S^{23} \),

(c) \( S^9 \times S^9 \times S^{11} \times S^{13} \times S^{15} \times S^{17} \times S^{29} \),

(d) \( S^9 \times S^{11} \times S^{13} \times S^{15} \times S^{17} \times S^{19} \times S^{31} \),
then the Hasse diagrams of $\mathcal{P}_k(X)$ are respectively one point, two points, three points and four points:

\[(a) \bullet, \quad (b) \bullet\bullet, \quad (c) \bullet\bullet\bullet, \quad (d) \bullet\bullet\bullet\bullet\]

For example, the differential $D$ in the case of $(d)$, $\{\bullet\bullet\bullet\bullet\} = \{\mu_1, \ldots, \mu_4\}$, is given by

\[
Dv_1 = \cdots = Dv_6 = 0 \quad \text{and} \\
\mu_1: Dv_7 = v_1v_6t^2 + v_2v_5t + v_3v_4t^2 + t^{16}, \\
\mu_2: Dv_7 = v_1v_6t^2 + v_2v_4t^3 + v_3v_5t + t^{16}, \\
\mu_3: Dv_7 = v_1v_5t^3 + v_2v_6t + v_3v_4t^2 + t^{16}, \\
\mu_4: Dv_7 = v_1v_4t^4 + v_2v_6t + v_3v_5t + t^{16}.
\]

Notice that $Dv_1 = \cdots = Dv_{k-1} = 0$ when $c$-s-depth$_k(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) = 1$ in general. Let

\[
\mathcal{E}(X_k) = \text{Aut}(AV_k, 0) = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \\ g \end{pmatrix} \mid a, b, c, d, e, f, g \in k - 0 \right\}
\]

where $h(v_1) = av_1$, $h(v_2) = bv_2$, $h(v_3) = cv_3$, $h(v_4) = dv_4$, $h(v_5) = ev_5$, $h(v_6) = fv_6$ and $h(v_7) = gv_7$ for $h \in \mathcal{E}(X_k)$. Then

\[
\mathcal{E}_{\mu_1}(X_k) = \{ \text{diag}(a, b, c, b^{-1}, a^{-1}, 1) \} \\
\mathcal{E}_{\mu_2}(X_k) = \{ \text{diag}(a, b, c, b^{-1}, a^{-1}, 1) \} \\
\mathcal{E}_{\mu_3}(X_k) = \{ \text{diag}(a, b, c, a^{-1}, b^{-1}, 1) \} \\
\mathcal{E}_{\mu_4}(X_k) = \{ \text{diag}(a, b, c, a^{-1}, b^{-1}, 1) \}
\]

for $a, b, c \in k$ as subgroups of $\mathcal{E}(X_k)$.

**Example 5.2.** (c-s-depth$_k(X) = 2$). (1) Let $X(n) = S^1 \times S^5 \times S^7 \times S^9 \times S^n$.

When (a) $n = 13$, (b) $n = 15$ and (c) $n = 17$, the Hasse diagrams of $\mathcal{P}_k(X(n))$ are respectively given as

\[
\begin{array}{cccccccc}
(a) & \mu_1 & (b) & \mu_1 & \mu_3 & (c) & \mu_1 & \mu_3 & \bullet \mu_5 \\
\mu_2 & \mu_2 & \mu_4 & \mu_2 & \mu_4 \\
\end{array}
\]

Here the point $\mu_1$ of $(a)$ is given by $Dv_1 = Dv_2 = Dv_3 = Dv_4 = 0$ and $Dv_5 = v_1v_4t + v_2v_3t + t^7$. On the other hand, $\mu_2$ is given by $Dv_4 = v_1v_2t$ and $Dv_i$ ($i \neq 4$) are the same as $\mu_1$.

Next, the points $\mu_k$ with $k = 1, 2, 3, 4, 5$ of $(b)$ and $(c)$ are given by the following differentials

\[
Dv_1 = Dv_2 = Dv_3 = 0 \quad \text{and}
\]
(2) Let us consider (a) \( X = S^3 \times S^5 \times S^7 \times S^{11} \times S^{15} \) and (b) \( X = S^7 \times S^9 \times S^{11} \times S^{13} \times S^{41} \). Then the Hasse diagrams of \( P_k(X) \) are respectively as follows:

\[
\begin{array}{c}
(a) & (b) \\
\begin{array}{c}
1 \\
2 & 3 \\
4
\end{array} & \\
\begin{array}{c}
1 \\
2 & 3 \\
4
\end{array}
\end{array}
\]

Here the differentials are given by

\[
Dv_1 = Dv_2 = Dv_3 = 0 \quad \text{and}
\]

\[
\begin{array}{c|ccc}
(a) & Dv_4 & Dv_5 \\
1 & 0 & v_1v_4 + v_2v_3t^4 + t^8 \\
2 & v_1t^2v_4 + v_2v_3t^4 + t^8 \\
3 & v_1v_3t + v_2v_3t^2 + t^8
\end{array}
\]

\[
\begin{array}{c|ccc}
(b) & Dv_4 & Dv_5 \\
1 & 0 & v_1v_4v_3t^4 + t^{21} \\
2 & v_1v_2t^4 + v_3v_4t^2 + t^{21} \\
3 & v_1v_3t^2 + v_2v_4t^5 + t^{21} \\
4 & v_1v_4t + v_2v_3t^2 + t^{21}
\end{array}
\]

(Note: From here on we simply denote \( k \) for \( \mu_k \) in the Hasse diagram and the left column of the table.)

In the case of (a), by degree arguments we have

\[
\mathcal{E}(X_k) = \{(\text{diag}(a,b,c,d,e),\lambda) \mid a,b,c,d,e \in k - 0, \lambda \in k\},
\]
where \( f(v_1) = av_1, f(v_2) = bv_2, f(v_3) = cv_3, f(v_4) = dv_4 \) and \( f(v_5) = ev_5 + \lambda v_1 v_2 v_3 \) for \( f \in \mathcal{E}(X_k) \). As subgroups of \( \mathcal{E}(X_k) = \{ \text{diag}(a, b, c, d, e) \} \), we have

\[
\mathcal{E}_{n_1}(X_k) = \{ (\text{diag}(a, b, b^{-1}, a^{-1}, 1), \lambda) \}
\]
\[
\mathcal{E}_{n_2}(X_k) = \{ (\text{diag}(a^{-2}, a^2, a^{-1}, 1), \lambda) \}
\]
\[
\mathcal{E}_{n_3}(X_k) = \{ (\text{diag}(a, a^2, a^{-2}, a^{-1}, 1), \lambda) \}
\]

In the case of \( b \), similarly as subgroups of \( \mathcal{E}(X_k) \), we have

\[
\mathcal{E}_{n_1}(X_k) = \{ (\text{diag}(a, b, c, d, 1) | abcd = 1) \}
\]
\[
\mathcal{E}_{n_2}(X_k) = \{ (\text{diag}(a, a^{-1}, c, c^{-1}, 1) \}
\]
\[
\mathcal{E}_{n_3}(X_k) = \{ (\text{diag}(a, b, a^{-1}, b^{-1}, 1) \}
\]
\[
\mathcal{E}_{n_4}(X_k) = \{ (\text{diag}(a, b, b^{-1}, a^{-1}, 1) \}
\]

for \( a, b, c, d \in k - 0 \).

**Example 5.3.** (c-s-depth\(_k\)(\(X\)) = 3).

(1) Let us consider (a) \( X = S^9 \times S^9 \times S^{13} \times S^{17} \) and (b) \( X = S^7 \times S^9 \times S^{11} \times S^{17} \times S^{15} \). Then we will show that the Hasse diagrams of \( P_k(X) \) are respectively as follows:

(a) \[ \begin{array}{c}
2 \quad 3 \\
\downarrow \quad \downarrow \\
5 \quad 4
\end{array} \]

(b) \[ \begin{array}{c}
2 \quad 3 \\
\downarrow \quad \downarrow \\
6 \quad 7
\end{array} \]

The differentials in (a) and (b) are given by

\[
Dv_1 = Dv_2 = 0 \ (Dv_3 = 0 \ for \ (b)) \ and \ Dv_4 = Dv_5
\]

| (a) |  |  |  |
|-----|---|---|---|
| \( Dv_3 \) | \( Dv_4 \) | \( Dv_5 \) |
| 1 0 0 | \( v_1 v_4 t + v_2 v_3 t^2 + t^3 \) |
| 2 \( v_1 v_2 t \) | \( v_1 v_4 t + v_2 v_3 t^2 + t^3 \) |
| 3 0 \( v_1 v_3 t \) | \( v_1 v_4 t + v_2 v_3 t^2 + t^3 \) |
| 4 0 \( v_1 v_2 t^2 \) | \( v_1 v_4 t + v_2 v_3 t^2 + t^3 \) |
| 5 \( v_1 v_2 t \) | \( v_1 v_4 t + v_2 v_3 t^2 + t^3 \) |

| (b) |  |  |
|-----|---|---|
| \( Dv_4 \) | \( Dv_5 \) |
| 1 0 | \( v_1 v_2 v_3 v_4 t + t^{23} \) |
| 2 0 | \( v_1 v_2 t^{13} + v_3 v_4 t^9 + t^{23} \) |
| 3 0 | \( v_1 v_3 t^{14} + v_2 v_4 t^{10} + t^{23} \) |
| 4 0 | \( v_1 v_4 t^{11} + v_2 v_3 t^{13} + t^{23} \) |
| 5 \( v_1 v_2 t \) | \( v_1 v_2 v_3 v_4 t + t^{23} \) |
| 6 \( v_1 v_2 t \) | \( v_1 v_3 t^{12} + v_2 v_4 t^{10} + t^{23} \) |
| 7 \( v_1 v_2 t \) | \( v_1 v_4 t^{11} + v_2 v_3 t^{13} + t^{23} \) |

In the case of (a), we have

\[ \mathcal{E}(X_k) = \{ (\text{diag}(a, b, c, d, e), \lambda) | a, b, c, d, e \in k - 0, \lambda \in k \} \]

where \( f(v_1) = av_1, f(v_2) = bv_2, f(v_3) = cv_3, f(v_4) = dv_4, f(v_5) = ev_5 + \lambda v_1 v_2 v_3 \) for \( f \in \mathcal{E}(X_k) \). In the case of (b), we have

\[ \mathcal{E}(X_k) = \{ \text{diag}(a, b, c, d, e) | a, b, c, d, e \in k - 0 \} \]

The subgroups of \( \mathcal{E}(X_k) \) for (a) and (b) are as follows:
Hence the Hasse diagram contains the following:

| (a) | $\mathcal{E}_{\mu_i}(X_K)$ | (b) | $\mathcal{E}_{\mu_i}(X_K)$ |
|-----|--------------------------|-----|--------------------------|
| 1   | $(a, b, b^{-1}, a^{-1}, 1, \lambda)$ | 1   | $(a, b, c, d, 1), \ abcd = 1$ |
| 2   | $(b^{-2}, b, b^{-1}, b^2, 1, \lambda)$ | 2   | $(a, a^{-1}, c, c^{-1}, 1)$ |
| 3   | $(a, a^2, b, b^{-1}, 1, \lambda)$ | 3   | $(a, b, a^{-1}, b^{-1}, 1)$ |
| 4   | $(a, a^2, a^2, a^{-1}, 1, \lambda)$ | 4   | $(a, b, b^{-1}, a^{-1}, 1)$ |
| 5   | $(a, a^2, a^2, a^{-1}, 1, \lambda), a^\tau = 1$ | 5   | $(a, a, a, a, 0), a^\tau b^2 c = 1$ |
| 6   | $(a, a^2, a^2, a^{-1}, 1, \lambda), \ a^\tau = 1$ | 6   | $(b^{-2}, b, b^2, b^{-1}, 1)$ |
| 7   | $(a, a^{-2}, a^2, a^{-1}, 1)$ | 7   | $(a, a^{-2}, a^2, a^{-1}, 1)$ |

for $a, b, c \in k - 0$ and $\lambda \in k$. In the case (a), $\mathcal{E}_{\mu_i}(X_K) \cong \mathbb{Z}_5 \times k$ when $k \supseteq \mathbb{Q}(\zeta_{2n})$ and it is isomorphic to $k$ when $k \nsubseteq \mathbb{Q}(\zeta_{2n})$.

(2) Let $X = \mathbb{C}P^1 \times S^3$. It is the case of $n = 14$ in Theorem 3.16. Let

$$M(X_k) = (\Lambda(x, y, z)_k, d_k) \quad \text{with} \quad |x| = 2, |y| = 29, |z| = 31, d_ky = x^{15}.$$ 

Since $n + 1 = 15(3 \cdot 5)$, assume $k \supseteq \mathbb{Q}(\zeta_{15})$. Let $(k[t] \otimes \Lambda(x, y, z)_k, D_i)$ ($i = 1, 2, 3, 4$) be the relative model with

$$D_1(x) = 0, \quad D_1(z) = x t^{15}$$

and the differential of $y$ being one of the following:

$$D_1(y) = x^{15} + t^{15},$$
$$D_2(y) = x^{15} + x^3 t^{12} + t^{15},$$
$$D_3(y) = x^{15} + x^5 t^{10} + t^{15},$$
$$D_4(y) = x^{15} + x t^{14} + t^{15}.$$ 

Then, we have the subgroups of $\mathcal{E}(X_k) = \{ \text{diag}(a, a^{15}, b) \mid a, b \in k - 0 \} \cong (k - 0)^\times$, which are the following:

$\mathcal{E}_{\mu_1}(X_K) = \{ \text{diag}(a, 1, a) \mid a^{15} = 1 \} = \{ a \in k - 0 \mid a^{15} = 1 \} \cong \mathbb{Z}/15\mathbb{Z},$

$\mathcal{E}_{\mu_2}(X_K) = \{ \text{diag}(a, 1, a) \mid a^{5} = 1 \} = \{ a \in k - 0 \mid a^{5} = 1 \} \cong \mathbb{Z}/3\mathbb{Z},$

$\mathcal{E}_{\mu_3}(X_K) = \{ \text{diag}(a, 1, a) \mid a^{5} = 1 \} = \{ a \in k - 0 \mid a^{5} = 1 \} \cong \mathbb{Z}/5\mathbb{Z},$

$\mathcal{E}_{\mu_4}(X_K) = \{ \text{diag}(1, 1, 1) \} \cong \{0\}.$

Hence the Hasse diagram contains the following:

```
\begin{tikzpicture}
  \node (z15) at (0,0) {$\mathbb{Z}_{15}$};
  \node (z3) at (-1,-1) {$\mathbb{Z}_3$};
  \node (z5) at (1,-1) {$\mathbb{Z}_5$};
  \node (z) at (0,-2) {$\{0\}$};
  \draw (z15) -- (z3);
  \draw (z15) -- (z5);
  \draw (z3) -- (z);
  \draw (z5) -- (z);
\end{tikzpicture}
```
(3) Let $X = \mathbb{C}P^8 \times S^{19}$. It is the case of $n = 8$ in Theorem 3.10. Then, when $k \supset \mathbb{Q}(e^{2\pi i/9})$, the Hasse diagram contains the following:

$$
\vcenter{\hbox{
\begin{tabular}{c|c}
$\mathbb{Z}_9$ & $\mathbb{Z}_3$ \\
\end{tabular}\
\{0\}}}
$$

**Example 5.4.** $(c\text{-}s\text{-}depth}_k(X) = 4)$. When $X = S^3 \times S^5 \times S^9 \times S^{15} \times S^{33}$ (cf. [27] Example 2.8), we have the following two cases:

1. Case of $k \supset \mathbb{Q}(e^{2\pi i/9})$. Then $\mathcal{P}_k(X) = 20$.
2. The other fields $k$. Then $\mathcal{P}_k(X) = 19$.

Indeed, the 20 (19) types of c-symplectic models $\{[\mu_1], ..., [\mu_{20}]\}$ are given as

$$M(Y_n) = (k[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5)_k, D_n)$$

with $|v_1| = 3$, $|v_2| = 5$, $|v_3| = 9$, $|v_4| = 15$, $|v_5| = 33$ and the differentials are given by $D_nv_1 = D_nv_2 = 0$ and

| $n$ | $D_nv_3$ | $D_nv_4$ | $D_nv_5$ | $\dim H^\ast(Y_n; k)$ |
|-----|-----------|-----------|-----------|----------------------|
| 1   | 0         | 0         | $v_1v_4t^{8} + v_2v_3t^{10} + t^{17}$ | 272              |
| 2   | 0         | $v_1v_2t^{4}$ | $v_1v_4t^{8} + v_2v_3t^{10} + t^{17}$ | 220              |
| 3   | 0         | $v_1v_3t^{2}$ | $v_1v_4t^{8} + v_2v_3t^{10} + t^{17}$ | 212              |
| 4   | $v_1v_2t$ | 0         | $v_1v_4t^{8} + v_2v_3t^{10} + t^{17}$ | 209              |
| 5   | $v_1v_2t$ | $v_1v_3t^{2}$ | $v_1v_4t^{8} + v_2v_3t^{10} + t^{17}$ | 149              |
| 6   | 0         | 0         | $v_1v_2t^{13} + v_3v_4t^{9} + t^{17}$ | 272              |
| 7   | 0         | $v_1v_3t^{2}$ | $v_1v_2t^{13} + v_3v_4t^{9} + t^{17}$ | 212              |
| 8   | 0         | $v_2v_3t$ | $v_1v_2t^{13} + v_3v_4t^{9} + t^{17}$ | 204              |
| 9   | 0         | 0         | $v_1v_2t^{13} + v_2v_4t^{8} + t^{17}$ | 272              |
| 10  | 0         | $v_1v_2t^{4}$ | $v_1v_2t^{13} + v_2v_4t^{8} + t^{17}$ | 220              |
| 11  | 0         | $v_2v_3t$ | $v_1v_2t^{13} + v_2v_4t^{8} + t^{17}$ | 204              |
| 12  | $v_1v_2t$ | 0         | $v_1v_2t^{13} + v_2v_4t^{8} + t^{17}$ | 209              |
| 13  | $v_1v_2t$ | $v_2v_3t$ | $v_1v_2t^{13} + v_2v_4t^{8} + t^{17}$ | 144              |
| 14  | 0         | 0         | $v_1v_2v_3v_4t + t^{17}$ | 272              |
| 15  | 0         | $v_1v_2t^{4}$ | $v_1v_2v_3v_4t + t^{17}$ | 220              |
| 16  | 0         | $v_1v_3t^{2}$ | $v_1v_2v_3v_4t + t^{17}$ | 212              |
| 17  | 0         | $v_2v_3t$ | $v_1v_2v_3v_4t + t^{17}$ | 204              |
| 18  | $v_1v_2t$ | 0         | $v_1v_2v_3v_4t + t^{17}$ | 209              |
| 19  | $v_1v_2t$ | $v_1v_3t^{2}$ | $v_1v_2v_3v_4t + t^{17}$ | 149              |
| 20  | $v_1v_2t$ | $v_2v_3t$ | $v_1v_2v_3v_4t + t^{17}$ | 144              |

* For example, the relative model $\mathcal{M}_r$ with $Dv_3 = Dv_4 = 0$, $Dv_5 = v_1v_2v_3v_4t + v_2v_3t^{10} + t^{17}$ has same self-equivalence as one of $i = 1$ in the above table, i.e., $\mathcal{E}_r(X_k) = \mathcal{E}_{\mu_1}(X_k)$. So $\mathcal{M}_r$ is not noted.
For degree reasons, the \( k \)-homotopy self-equivalences of \( X_k \) are given as

\[
\mathcal{E}(X_k) = \begin{cases} 
\begin{pmatrix} a & b \\ c & d \\ e \end{pmatrix} & \text{if } a, b, c, d, e \in k - 0 \\
\end{cases}
\]

\[
= \{(a, b, c, d, e) | a, b, c, d, e \in k - 0\} = (k - 0)^{\times 5}
\]

and the subgroups \( \mathcal{E}_{\mu_n}(X_k) \) for \( n = 1 \sim 20 \) are given as the following conditions on \( a, b, c, d \) with \( e = 1 \):

|   |   |   |   |   |
|---|---|---|---|---|
| 1 | \( ad = bc = 1 \) |   |   |   |
| 2 | \( a^2b = bc = 1, ab = d \) |   |   |   |
| 3 | \( a^2c = bc = 1, ac = d \) |   |   |   |
| 4 | \( ad = ab^2 = 1, ab = c \) |   |   |   |
| 5 | \( ab = c, a^2b = d, a^4b = ab^2 = 1 \) |   |   |   |
| 6 | \( ab = cd = 1 \) |   |   |   |
| 7 | \( ac = d, ab = ac^2 = 1 \) |   |   |   |
| 8 | \( bc = d, ab = bc^2 = 1 \) |   |   |   |
| 9 | \( ac = bd = 1 \) |   |   |   |
| 10 | \( ab = d, ac = ab^2 = 1 \) |   |   |   |
| 11 | \( bc = d, ac = b^2c = 1 \) |   |   |   |
| 12 | \( ab = c, a^2b = bd = 1 \) |   |   |   |
| 13 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |
| 14 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |
| 15 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |
| 16 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |
| 17 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |
| 18 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |
| 19 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |
| 20 | \( ab = c, a^2b = ab^2 = 1 \) |   |   |   |

Notice that when \( n = 5 \) and \( 13 \),

\[
\mathcal{E}_{\mu_5}(X_k) = \{(a, a^2, a^3, a^4, 1); a^5 = 1\},
\]

\[
\mathcal{E}_{\mu_{13}}(X_k) = \{(a, a^4, a^3, a^2, 1); a^5 = 1\}
\]

are both the identity \((1, 1, 1, 1, 1)\) in the case \((2)\); i.e., \([\mu_5] = [\mu_{13}] \) in \( P_k(X) \). Thus the Hasse diagram of \( P_k(X) \) is given as

![Hasse diagram](image-url)
and c-s-depth$_k(X) = 4$ in both cases. In particular, (2) is a lattice.

**Remark 5.5.** Let $F_\mu : \mathcal{E}(p_\mu) \to \mathcal{E}(X)$ be the restriction map between ordinary self-homotopy equivalence groups for the fibration $(\ast)$ in §1. From Example 5.4 (2) and the integral homotopy theory [29, §10], we see that $\text{Im}(F_\mu)$ may be at most finite even when $X$ has the rational homotopy type of the product of odd-spheres. So we would like to pose the following question.

**Question 5.6.** When is $\text{Im}(F_\mu)$ finite?

**Remark 5.7.** M.R. Hilali conjectures that $\dim \pi_\ast(Y) \otimes \mathbb{Q} \leq \dim H^\ast(Y; \mathbb{Q})$ for an elliptic simply connected space $Y$ [10, 17]. When $Y$ is c-symplectic with $\dim \pi_{\text{even}}(Y) \otimes \mathbb{Q} = 1$, it is true. Indeed, when $M(Y) = (\mathbb{Q}[t] \otimes \Lambda(v_1, \ldots, v_n), d)$ $(n > 1)$ with $|v_i|$ odd, $\dim \pi_\ast(Y) \otimes \mathbb{Q} = n + 1 < (\sum_{i=1}^n |v_i|) - 1)/2 = \max\{m \mid t^m \neq 0\} =: N$. Thus it follows from $H^\ast(Y; \mathbb{Q}) \supset \{1, t, t^2, \ldots, t^N\}$.

We first speculated that if $[\mu_i] < [\mu_j]$ in $\mathcal{P}_k(X)$, then $\dim H^\ast(Y_i; k) \leq \dim H^\ast(Y_j; k)$. But it is not true in a general field $k$. Indeed, we can see in the above Example 5.4 (2) that $[\mu_5] < [\mu_20]$ but $\dim H^\ast(Y_5; k) = 149 > 144 = \dim H^\ast(Y_{20}; k)$. Notice that the above speculation holds in the case (1). So the following question would be reasonable.

**Question 5.8.** For a sufficiently large field $k$, if $[\mu_i] < [\mu_j]$ in $\mathcal{P}_k(X)$, then is $\dim H^\ast(Y_i; k) \leq \dim H^\ast(Y_j; k)$?

**Example 5.9.** Recall Corollary 3.17. For example, the following Hasse diagrams of height 3 are contained in those of $\mathcal{P}_5(X)$ with $X = \mathbb{C}P^n \times S^{2n+3}$ of c-s-depth$_5(X) = 4$ for $n = 26$, 74 and 104, respectively:
Example 5.10. (c-s-depth$_k(X) = 5$). For the exceptional simple Lie group $E_7$, the rational type is known as (see [23])

$$(3, 11, 15, 19, 23, 27, 35)$$

namely,

$$E_7 \cong S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}.$$

Then, for $|v_1| = 3$, $|v_2| = 11$, $|v_3| = 15$, $|v_4| = 19$, $|v_5| = 23$, $|v_6| = 27$, $|v_7| = 35$,

$$\mathcal{E}(X_k) = Aut(\Lambda(v_1, v_2, v_3, v_4, v_5, v_6, v_7)_k, 0) \cong \left\{ \text{diag}(a, b, c, d, e, f, g) | a, b, c, d, e, f, g \in k - 0 \right\}$$

for degree reasons. There are the following 20-types of $k$-c-symplectic models, i.e.;

$$(k[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5, v_6, v_7)_k, D)$$

with the differentials given as

$$Dv_1 = Dv_2 = 0,$$

$$Dv_7 = v_1v_6t^3 + v_2v_5t + v_3v_4t + t^{18}$$

and

| $\mu$ | $Dv_3$ | $Dv_4$ | $Dv_5$ | $Dv_6$ | $\mathcal{E}_\mu(X_k)$ |
|------|------|------|------|------|------------------|
| 1    | 0    | 0    | 0    | 0    | $(a, b, c, c^{-1}, b^{-1}, a^{-1}, 1)$ |
| 2    | $v_1v_3t$ | 0    | 0    | 0    | $(c^{-2}, b, c, c^{-1}, b^{-1}, c^{-1}, 1)$ |
| 3    | 0    | 0    | $v_1v_2t^3$ | 0    | $(b^{-2}, b, c, c^{-1}, b^{-1}, b', 1)$ |
| 4    | 0    | 0    | 0    | $v_1v_2t'$ | $(a, a^{-2}, c, c^{-1}, a^{-1}, a^{-1}, 1)$ |
| 5    | 0    | 0    | 0    | $v_1v_3t^2$ | $(a, b, a^{-2}, a^{-2}, b^{-1}, a^{-1}, 1)$ |
| 6    | 0    | 0    | 0    | $v_1v_4t^2$ | $(a, b, a^{-2}, a^{-2}, b^{-1}, a^{-1}, 1)$ |
| 7    | 0    | 0    | 0    | $v_1v_5t^2$ | $(a, a', c, c^{-1}, a^{-2}, a^{-1}, 1)$ |
| 8    | $v_1v_3t$ | $v_1v_2t^3$ | 0    | 0    | $(b^{-2}, b, \pm b, \pm b^{-1}, b', b'^2, 1)$ |
| 9    | 0    | $v_1v_3t$ | 0    | $v_1v_2t'$ | $(c^{-2}, c^{-2}, c, c^{-1}, c^{-1}, c^{-1}, 1)$ |
| 10   | 0    | $v_1v_3t$ | 0    | $v_1v_4t^3$ | $(c^{-2}, b, c, c^{-1}, b^{-1}, c^{-1}, 1)$ : $c^2 = 1$ |
| 11   | 0    | $v_1v_3t$ | 0    | $v_1v_5t^3$ | $(c^{-2}, c^{-2}, c, c^{-1}, c^{-1}, c^{-1}, 1)$ |
| 12   | 0    | $v_1v_2t^3$ | $v_1v_3t'$ | $(b^{-2}, b, b', b^{-1}, b'^2, 1)$ |
| 13   | 0    | $v_1v_2t^3$ | $v_1v_4t'$ | $(b^{-2}, b, b', b^{-1}, b'^2, 1)$ |
| 14   | 0    | $v_1v_2t^3$ | $v_1v_5t^3$ | $(b^{-2}, b, c, c^{-1}, b^{-1}, b'^2, 1)$ : $b'^2 = 1$ |
| 15   | 0    | $v_1v_3t$ | $v_1v_2t^3$ | $v_1v_4t' | $(b^{-2}, b, \pm b, \pm b^{-1}, b'^2, 1)$ : $b'^2 = \pm 1$ |
| 16   | 0    | $v_1v_3t$ | $v_1v_2t^3$ | $v_1v_5t | $(b^{-2}, b, \pm b, \pm b^{-1}, b'^2, 1)$ : $b'^2 = \pm 1$ |
| 17   | 0    | 0    | $v_1v_3t^3$ | $v_2v_3t | $(b^{-2}, c, c, b, c^{-1}, b^{-1}, b, c, 1)$ |
| 18   | 0    | $v_1v_2t^3$ | $v_1v_3t^3$ | 0    | $(a, b, a^{-2}, b^{-1}, b'^2, a^{-1}, 1)$ |
| 19   | 0    | $v_1v_2t^3$ | 0    | $v_2v_3t | $(a, b, a^{-2}, b^{-1}, ab, a^{-1}, 1)$ |
| 20   | $v_1v_2t$ | $-v_1v_2t^3$ | 0    | $v_2v_3t | $(a, b, ab, a^{-2}, b^{-1}, a^{-1}, 1)$ |

for $a, b, c \in k - 0$. Then, for any $k$, the Hasse diagram is
Corollary 5.11. The poset structure of $\mathcal{P}_k(E_7)$ does not depend on $k$. Moreover $c$-s-depth$_k(E_7) = 5$ for any field $k$.

Example 5.12. For the $n$-dimensional symplectic group $Sp(n)$, the rational type is given as $(3, 7, 11, \cdots, 4n - 1)$ (see [23]), namely,

$$Sp(n) \cong \mathbb{Q} S^3 \times S^7 \times \cdots \times S^{4k-1} \times \cdots \times S^{4n-1}.$$ 

It is pre-c-symplectic if $n$ is odd [27]. Let

$$M(Sp(n)_k) = (\Lambda(v_1, v_2, v_3, \cdots, v_n)_k, 0) \quad \text{with} \quad |v_i| = 4i - 1.$$ 

Then, for degree reasons [27], for the differential $D$ over a sufficiently large field $k$,

$$Dv_n = v_1v_{n-1}t + v_2v_{n-2}t + v_3v_{n-3}t + \cdots + v_{(n-1)/2}v_{(n+1)/2}t + t^{2n}$$

is uniquely defined to be $c$-symplectic. Thus by seeing $Dv_1, \cdots, Dv_{n-1}$ we have the following:

1. $c$-s-depth$_k(Sp(1)) = c$-s-depth$_k(Sp(3)) = 1$.
2. $c$-s-depth$_k(Sp(5)) = 3$ from the sequence $|\mu_1| > |\mu_2| > |\mu_3|$ for

$$M(Y_{1k}) = (k[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5)_k, D)$$

with $Dv_5 = v_1v_4t + v_2v_3t + t^5$ and

$$M(Y_{1k}) : \quad Dv_1 = Dv_2 = 0, \quad Dv_3 = 0, \quad Dv_4 = 0$$

$$M(Y_{2k}) : \quad Dv_1 = Dv_2 = 0, \quad Dv_3 = v_1v_2t, \quad Dv_4 = 0$$

$$M(Y_{3k}) : \quad Dv_1 = Dv_2 = 0, \quad Dv_3 = v_1v_2t, \quad Dv_4 = v_1v_3t.$$ 

Then we have

$$\mathcal{E}_{\mu_1}(X_k) = \{(a, b, b^{-1}a^{-1}, 1)\},$$

$$\mathcal{E}_{\mu_2}(X_k) = \{(b^{-2}, b, b^{-1}, b^2, 1)\},$$

$$\mathcal{E}_{\mu_3}(X_k) = \{(b^{-2}, b, b^{-1}, b^2, 1); \quad b^5 = 1\}$$

for $a, b \in k - 0$ and we can directly check that the sequence $\mathcal{E}_{\mu_1}(X_k) \supseteq \mathcal{E}_{\mu_2}(X_k) \supseteq \mathcal{E}_{\mu_3}(X_k)$ is maximal.

3. $c$-s-depth$_k(Sp(7)) = 4$ since $M(Y_{1k}) = (k[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5, v_6, v_7)_k, D)$ with $Dv_1 = Dv_2 = Dv_3 = 0, \quad Dv_7 = v_1v_6t + v_2v_5t + v_3v_4t + t^{14}$ and
for \((a, b, c) := \text{diag}(a, b, c, c^{-1}, b^{-1}, a^{-1}, 1)\) and we can directly check that the sequence \(E_{\mu_1}(X_k) \supseteq E_{\mu_2}(X_k) \supseteq E_{\mu_3}(X_k) \supseteq E_{\mu_4}(X_k)\) is maximal.

(4) c-s-depth\(_h\)(\(Sp(9)\)) \(\geq 5\). Indeed, \(M(Y_{k}) = (k[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)_k, D)\) with \(Dv_1 = Dv_2 = \cdots = Dv_4 = 0, Dv_5 = v_1v_8t + v_2v_7t + v_3v_6t + v_4v_5t + t^{18}\) and

\[
\begin{array}{|c|c|c|c|c|}
\hline
\mu_1 & 0 & 0 & 0 & 0 \\
\mu_2 & v_3v_1t & 0 & 0 & 0 \\
\mu_3 & v_3v_1t & v_2v_3t & 0 & 0 \\
\mu_4 & v_3v_1t & v_2v_3t & v_1v_2t^4 & 0 \\
\mu_5 & v_3v_1t & v_2v_3t & v_1v_2t^4 & v_1v_2t^{15} \\
\hline
\end{array}
\]

\(\{(a, b, c, d) \mid a, b, c, d \in k - 0\}\)

Thus we have \(E_{\mu_1}(X_k) \supseteq E_{\mu_2}(X_k) \supseteq \cdots \supseteq E_{\mu_5}(X_k)\); i.e.,

\([\mu_1] > [\mu_2] > [\mu_3] > [\mu_4] > [\mu_5]\),

which implies c-s-depth\(_h\)(\(Sp(9)\)) \(\geq 5\).

(5) c-s-depth\(_h\)(\(Sp(11)\)) \(\geq 6\). Indeed,

\(M(Y_{k}) = (k[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11})_k, D)\) with \(Dv_1 = Dv_2 = \cdots = Dv_5 = 0, Dv_{11} = v_1v_{10}t + v_2v_9t + v_3v_8t + v_4v_7t + v_5v_6t + t^{22}\) and

\[
\begin{array}{|c|c|c|c|c|}
\hline
\mu_1 & 0 & 0 & 0 & 0 \\
\mu_2 & v_5v_1t & 0 & 0 & 0 \\
\mu_3 & v_5v_1t & v_4v_1t^5 & 0 & 0 \\
\mu_4 & v_5v_1t & v_4v_1t^5 & v_3v_2t^4 & 0 \\
\mu_5 & v_5v_1t & v_4v_1t^5 & v_3v_2t^4 & v_2v_4t^2 \\
\mu_6 & v_5v_1t & v_4v_1t^5 & v_3v_2t^4 & v_2v_4t^2 & v_1v_2t^{15} \\
\hline
\end{array}
\]

\(\{(a, b, c, d, e) \mid a, b, c, d, e \in k - 0\}\)

Thus we have \(E_{\mu_1}(X_k) \supseteq E_{\mu_2}(X_k) \supseteq \cdots \supseteq E_{\mu_6}(X_k)\); i.e.,

\([\mu_1] > [\mu_2] > [\mu_3] > [\mu_4] > [\mu_5] > [\mu_6]\),

which implies c-s-depth\(_h\)(\(Sp(11)\)) \(\geq 6\).

In general, for \(Sp(n)\), by setting

\[
M(Y_{(n+1)/2}) = (\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3, \cdots, v_n)_n, D)
\]

with

\[
Dv_1 = Dv_2 = \cdots = Dv_{(n-1)/2} = 0,
Dv_i = \epsilon_i v_j v_k^{2(i-j-k)+1} \quad (\epsilon_i = 0 \text{ or } 1)
\]

\(\{(a, b, c, d, e) \mid a, b, c, d, e \in k - 0\}\)
where \( i + j = n \) for \((n+1)/2 \leq i < n\) and certain \( k \in \{ 1, 2, \cdots, (n-1)/2 \}\) with \( i \geq j + k \), we obtain the following

**Theorem 5.13.** For any odd integer \( n > 3 \),

\[
c_{s-depth}^{\mathbb{Q}}(Sp(n)) \geq \frac{n+1}{2}.
\]

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