The energy-momentum tensor (EMT) distribution in a closed system is an important measure of the underlying dynamics, and according to Einstein’s general relativity, is responsible for the space-time geometry nearby. For a stable nuclear system, such as the proton, deeply-virtual Compton scattering currently studied at JLab 12 GeV facility and future Electron Ion Collider. We define the leading momentum-current multipoles, tensor monopole \(\tau\) (\(T_0\)) and scalar quadrupole \(\sigma^2\) (\(S_2\)) moments, relating to the so-called \(D\)-term in the literature. We calculate the momentum current distribution in hydrogen atom and its monopole moment in the basic unit of \(\tau_0 = \hbar^2/4M\), showing that the sign of \(D\)-term has little to do with mechanical stability. The momentum current distribution also strongly modifies the static gravitational field inside hadrons.

In this section, we consider multipole expansion of the momentum currents in hadrons, with three series \(S^{(J)}\), \(\tilde{T}^{(J)}\), and \(T^{(J)}\), in connection with the gravitational fields generated nearby. The momentum currents are related to their energy-momentum form factors, which in principle can be probed through processes like deeply-virtual Compton scattering currently studied at JLab 12 GeV facility and future Electron Ion Collider. We define the leading momentum-current multipoles, tensor monopole \(\tau\) (\(T_0\)) and scalar quadrupole \(\sigma^2\) (\(S_2\)) moments, relating to the so-called \(D\)-term in the literature. We study multipole expansion of the momentum currents in hadrons, with three series corresponding to three degrees of freedom of the conserved current. The multipole expansion for gravitational systems focus on generation of gravitational waves. Our interest is in static systems which only involves time-independent moments. We study the moments of \(T^{(J)}\) in the second subsection, which have three independent series corresponding to three degrees of freedom in the momentum currents. Even though most of the results in this section are not new, they help us to understand the physical significance of the EMT distributions in a quantum mechanical bound state systems such as hadrons or atoms.

We study multipole expansion of the static EMT distribution \(T^{\mu\nu}(\vec{r})\) in a finite system, particularly the momentum current, \(T^{(J)}(\vec{r})\). Multipole expansion for electromagnetic systems is well known [25] and we repeat it in the first subsection for the energy current or momentum density \(T^{0\mu}\), which defines two moment series corresponding to two degrees of freedom of the conserved current. The multipole expansion for gravitational systems has also been worked out in the literature to considerable details [15–16]. However, most of the studies in gravitational systems focus on generation of gravitational waves. Our interest is in static systems which only involves time-independent moments. We study the moments of \(T^{(J)}\) in the second subsection, which have three independent series corresponding to three degrees of freedom in the momentum currents. Even though most of the results in this section are not new, they help us to understand the physical significance of the EMT distributions in a quantum mechanical bound state systems such as hadrons or atoms.

We study of the physics of the EMT moments in the context of the linearized Einstein’s equation in the weak...
gravitational limit, in which the symmetric metric tensor $g^{\mu\nu}$ can be approximated by the flat space metric $\eta^{\mu\nu} = (1, -1, -1, -1)$ (this choice is opposite to the standard convention in the gravitation literature) plus a small perturbation $h^{\mu\nu}$,

$$
g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{2}$$

The rank-2 tensor $h^{\mu\nu}$ has 10 independent components. Using the coordinates re-parameterization invariance $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu c_\nu + \partial_\nu c_\mu - \eta_{\mu\nu} \partial^\rho c_\rho$ where $c_\mu$ is an arbitrary vector field, only 6 components are independent. It is common to define the trace-reversed metric perturbation

$$
h_{\mu\nu} = h^{\mu\nu} - \frac{\eta^{\mu\nu}}{2} h_\rho^\rho, \tag{3}$$

and a convenient gauge choice is then the harmonic or Lorenz gauge defined by four conditions [16]

$$
\partial_\mu h^{\mu\nu} = 0, \tag{4}$$

which are manifestly consistent with the conservation of the EMT.

An important reason to introduce the harmonic gauge is that the trace-reversed metric satisfies the linearized Einstein’s equation

$$
\Box h^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu}. \tag{5}
$$

where $\Box = \partial^\mu \partial_\mu$, $G$ is Newton’s constant, $c$ the speed of light, and $T^{\mu\nu}$ the EMT of matter fields. Since Eq. (5) is just the standard wave equation, it can be solved as [16]

$$
h^{\mu\nu}(t, \vec{r}) = \frac{4G}{c^4} \int d^3\vec{r}' \frac{1}{|\vec{r}' - \vec{r}|} T^{\mu\nu} \left(t - \frac{|\vec{r}' - \vec{r}|}{c}, \vec{r}'\right). \tag{6}$$

For a time-dependent source, the above will lead to the generation of gravitational waves ($\sim 1/r$) in regions far away from the source. Among the six independent physical components of the metric, only two correspond to gravitational waves. In the harmonic gauge, they are defined by the transverse-traceless condition [16]. For example, if the wave vector is along the $z$ direction, then the two independent components are $h_{xy} = h_{yx}$ and $h_{xx} = -h_{yy}$.

We will not consider the moment series for the energy/mass density $T^{00}$, which defines

$$
M_{i_1...i_l} = \int d^3\vec{r} T^{00}(\vec{r}), \tag{7}
$$

where (...) are symmetric and trace-free (STF) parts of the tensor. The generation of gravity by mass multipoles are well-known [13][16].

### A. Multipole Expansion for Energy Current

Consider the conserved energy current or momentum density $T^{0\mu}$, in a static system. Here for simplicity, we use notation similar to electromagnetism, with $j^{\mu}$ standing for $T^{0\mu}$. The static conservation law becomes,

$$
\partial_\mu j^\mu = 0, \tag{8}
$$

where $\vec{j}(\vec{r})$ is a static current distribution. Given the vector current, the vector field $\hat{A}$ (standing for $h^{\mu\nu} c^4 / (4G)$), that satisfies the Laplace equation

$$
\nabla^2 \hat{A}(\vec{r}) = -4\pi \vec{j}(\vec{r}), \tag{9}
$$

can be solved as

$$
A_i(\vec{r}) = \int \frac{d^3\vec{r}}{|\vec{r}' - \vec{r}|} \frac{1}{r} j_{i,i_1...i_l} \partial_{i_1}...\partial_{i_l} \frac{1}{r} . \tag{10}
$$

At large distance $r \gg r'$, $A^i$ follows the following multipole expansion

$$
A_i(\vec{r}) = \sum_{l=0}^\infty \sum_{i_1...i_l} (-1)^l \frac{1}{l!} j_{i,i_1...i_l} \partial_{i_1}...\partial_{i_l} \frac{1}{r} , \tag{11}
$$

where the moments of the vector current

$$
j_{i,i_1...i_l} = \int d^3\vec{r} r_{i_1...i_l} j_i(\vec{r}). \tag{12}
$$

and the symbol (...) again will be used to denotes symmetric and traceless part between tensor indices $i_1$ to $i_l$. Clearly, the subtraction of trace removes all the moments weighted with $r^2$, $r^4$, etc, which do not yield any new tensor structure, nor contribute to the vector field at large distances.

From group-theoretic point of view, the moments $j_{i,i_1...i_l}$ forms a tensor product of spin-1 and spin-$l$ irreducible representations of the three-dimensional rotation group and can be decomposed into a direct sum of spin-$(l-1)$, $l$ and $l+1$ representations

$$
[1] \oplus [l] = [l-1] \oplus [l] \oplus [l+1]. \tag{13}
$$

In terms of tensor notation, the above decomposition can be written as [15][16]

$$
\hat{j}_{i,i_1...i_l} = U^{(l+1)}_{i,i_1...i_l} + \tilde{V}^{(l)}_{i_1i_2...i_l} + \delta_{i,i_1} V^{(l-1)}_{i_2...i_l}, \tag{14}
$$

The spin-$l+1$, $l$ and $l-1$ parts $U^{(l+1)}$, $\tilde{V}^{(l)}$, $V^{(l-1)}$ reads explicitly

$$
U^{(l+1)}_{i,i_1...i_l} = j_{i,i_1...i_l}, \tag{15}
$$

$$
\tilde{V}^{(l)}_{i_1...i_l} = \frac{l}{l+1} j_{i,i_1...i_l}, \tag{16}
$$

$$
V^{(l-1)}_{i_2...i_l} = \frac{2l-1}{2l+1} j_{i,i_2...i_l}. \tag{17}
$$
where indices between $[...]$ are anti-symmetrized. The above decomposition applies for a generic vector current $j$ not necessarily conserved. 

For a conserved current, it is easy to show that the totally symmetric $(l + 1)$-multipole always vanishes

$$U^{(l+1)}_{ii_1...i_l} = 0 ,$$  

or

$$\int d^3 \vec{r} \ r_{i_1...i_l} j_{i_l}(\vec{r}) = 0$$  

which holds with and without trace subtraction. It generates a large number of identities among the moments after some contractions of indices, for example,

$$2 \int d^3 \vec{r} \vec{r} \cdot \vec{j} r_{i_1...i_{l-1}} = -(l - 1) \int d^3 \vec{r} \ r^2 j_{i_1...i_{l-1}}.$$  

For $l = 1$, it simply reduces to

$$\int d^3 \vec{r} \vec{r} \cdot \vec{J} = 0 ,$$  

and for $l = 2$, the identity reads

$$2 \int d^3 \vec{r} \vec{r} \cdot \vec{J} r_i = - \int d^3 \vec{x} \vec{x}^2 j_i .$$  

These identities will be useful later.

The contribution of the $V^{(l-1)}_{i_2...i_l}$ multipoles to the vector potential is

$$A^V_{l} = \frac{(-1)^l}{l!} V^{(l-1)}_{2...l} \partial_{2...l} \partial_{i_l} \frac{1}{r} ,$$  

and can be gauged away by the gauge transformation

$$A_i \rightarrow A_i - \partial_i \left( \frac{(-1)^l}{l!} V^{(l-1)}_{2...l} \partial_{2...l} \partial_{i_l} \frac{1}{r} \right) .$$  

Therefore they do not produce a physical effect in static gauge theories. However, the time-varying V-multipoles are important for time-dependent effects such as radiation, and they form also a useful series for describing the current distribution. The first such a moment is

$$V^{(0)} = \int d^3 \vec{r} \vec{r} \cdot \vec{J}(\vec{r}) = 0 ,$$  

as a consequence of the identity Eq. (21). Thus the first non-vanishing moment appears at

$$V^{(1)} = \frac{3}{5} \int d^3 \vec{r} \left( \vec{r} \vec{r} \cdot \vec{J}(\vec{r}) - \frac{1}{3} r^2 j_i \right) = - \frac{1}{2} \int d^3 \vec{r} \vec{r}^2 j_i(\vec{r}) .$$  

In the second equality we have used the identity Eq. (22). Clearly, this is not independent and relates to the current radius. The independent V-moment series starts from $V^{(2)}$, and they can all be related to moments of $\vec{r} \cdot \vec{J}$.

The physically-interesting moment series in the static case are

$$\tilde{V}^{(l)}_{ii_1...i_l} \sim \int d^3 \vec{r} \ m_l(\vec{r}) r_{i_1...i_{l-1}} ,$$  

where the $m_l$ is the well-known “magnetization density” [25] in case of the electric current or “angular-momentum density” in case of the energy current $T^{ij} = T^{0j}$,

$$\tilde{m}(\vec{r}) = \vec{r} \times \vec{J}(\vec{r}) ,$$  

$$\tilde{J}(\vec{r}) = \vec{r} \times \tilde{T}(\vec{r}) .$$  

$V^{(1)}$ is just the magnetic moment in electromagnetic and total angular momentum vector $S$ for the energy current.

B. Multipole Expansion for Momentum Currents

The momentum current $T^{ij}$ can be decomposed into the 3D trace and traceless parts, calling it tensor and scalar parts, respectively. The scalar the multipole expansion defines

$$S^{(l)}_{ii_1...i_l} = \int d^3 \vec{r} \ r_{i_1...i_{l-1}} T_{kk}(\vec{r}) ,$$  

which is similar to the multipoles of the energy/mass density.

For the tensor part, we can make the following multipole decomposition [12, 13, 16, 20],

$$[2] \otimes [l] = [l - 2] \oplus [l - 1] \oplus [l] \oplus [l + 1] \oplus [l + 2] .$$  

In terms of tensor notation, we first define the moments $T_{ij,i_1...i_l}$ similar to Eq. (12)

$$T_{ij,i_1...i_l} = \int d^3 \vec{r} \ T_{ij}(\vec{r}) r_{i_1...i_l} ,$$  

where $(i_1...i_l)$ again denotes the traceless and symmetric part. The tensor decomposition then reads [15, 16, 25]

$$T_{ij,i_1...i_l} = U^{(l+2)}_{ij,i_1...i_l} + \tilde{U}^{(l+1)}_{ij,i_1...i_l} + \tilde{S}^{(l)}_{ij,i_1...i_l} + \tilde{T}^{(l)}_{ij,i_1...i_l} + \tilde{T}^{(l-2)}_{ij,i_1...i_l} ,$$

where traceless and symmetric subtractions in $ij$ and in $i_1...i_l$ are always assumed. The multipole series denote [15, 16, 25]

$$U^{(l+2)}_{ij,i_1...i_l} = T_{ij,i_1...i_l} ,$$

$$\tilde{U}^{(l+1)}_{ij,i_1...i_l} = \frac{2l}{l + 2} T_{ij,i_1...i_l} ,$$

$$\tilde{S}^{(l)}_{ij,i_1...i_l} = \frac{6l(l - 1)}{(l + 1)(2l + 3)} T_{i_1,i_2...i_l} ,$$

$$\tilde{T}^{(l-2)}_{ij,i_1...i_l} = \frac{2l - 3}{2l + 1} T_{ij,i_1...i_l} ,$$

$$T^{(l-2)}_{ij,i_1...i_l} = \frac{2l - 3}{2l + 1} T_{ij,i_1...i_l} .$$
where we have included certain coefficients in the definition.

Due to momentum current conservation $\partial_l T^{ij} = 0$ in a static system, not all the multipoles above are non-vanishing. One can show that the following general identities are true

$$\frac{1}{k!} \sum_P \int d^3 \vec{r} T_{ij}(1) r_{iP(1)} \ldots r_{iP(k)} = 0 ,$$  \hspace{1cm} (39)

where $P$ runs over all permutations $P(1), \ldots, P(k)$ of 1, \ldots, $k$. From the above, one can show that the $l + 2$ and $l + 1$ moments all vanish

$$U^{l+2}_{ij\ldots} \equiv 0 , \quad U^{l+1}_{ij\ldots} \equiv 0 .$$  \hspace{1cm} (40)

One can form more identities from Eq. (39) by performing contractions or symmetrization/antisymmetrizations. For example, by contracting $i$ with one of the other indices under permutation one has

$$\int d^3 \vec{r} T_{ij} r_{i1} \ldots r_{ik} = - k \int d^3 \vec{r} T_{ij} r_{i1} r_{i2} \ldots r_{ik} .$$  \hspace{1cm} (41)

By contracting two of the indices under permutation, one has

$$2 \int d^3 \vec{r} T_{ij} r_{i1} r_{i2} \ldots r_{ik-2} = -(k - 2) \int d^3 \vec{r} T_{ij} r_{i1} r_{i2} \ldots r_{ik-2} ,$$  \hspace{1cm} (42)

and so on. For $k = 2$ and $k = 3$, the above reduces to

$$\int d^3 \vec{r} T_{ij} r_{i1} r_{i2} = - \frac{1}{2} \int d^3 \vec{r} T_{ij} r_{i1} r_{i2} ,$$  \hspace{1cm} (43)

and

$$\int d^3 \vec{r} r^2 T_{ij} r_{i1} r_{i2} = - 2 \int d^3 \vec{r} T_{ij} r_{i1} r_{i2} = \int d^3 \vec{r} T_{ij} r_{i1} r_{i2} .$$  \hspace{1cm} (44)

Notice that above holds without trace subtraction as well.

Moreover, by performing anti-symmetrization of $x$ with one of the indices under permutation in Eq. (39) one obtain

$$\int d^3 \vec{r} T_{ij} r_{i1} r_{j1} = 0 ,$$  \hspace{1cm} (45)

and

$$\int d^3 \vec{r} (r_{ij} T_{ij} r_{i1} r_{i2} + r_{ij} T_{ij} r_{i1} r_{i2}) = 0 .$$  \hspace{1cm} (46)

and so on.

Given these relations, we can re-express the tensor momentum current multipole $S_{ij\ldots}^{(l)}$ in terms of that of scalar momentum current multipoles. First let’s consider $l = 2$, in this case the above reduces to

$$\tilde{S}_{i1i2}^{(2)} = \frac{12}{7} \int d^3 \vec{r}$$

$$\times \left(r_{i1} r_{i2} - \frac{1}{3} r^2 T_{ij} r_{i1} r_{i2}\right)$$  \hspace{1cm} (47)

which by using Eq. (43) and Eq. (44) reduces to

$$\tilde{S}_{i1i2}^{(2)} = - 2 \int d^3 \vec{r} T_{ij} r_{i1} r_{i2} T_{ij} (\vec{r}) \equiv - 2 \sigma_{i1i2} ,$$  \hspace{1cm} (48)

where the quadrupole of the scalar momentum current $\sigma_{i1i2}$ or scalar quadrupole $S2$ is defined as

$$\sigma_{ij} \equiv S_{ij}^{(2)} = \int d^3 \vec{r} T_{kk} (\vec{r}) r_{i1} r_{j2} .$$  \hspace{1cm} (49)

In fact, for general $l$ one can show that the above relation remains valid \[15\]

$$\tilde{S}_{i1i2\ldots}^{(l)} = - 2 S_{i1i2\ldots}^{(l)} ,$$  \hspace{1cm} (50)

therefore, at given order $l$, there is only one series of linear independent spin-$l$ multipoles $S_{i1i2\ldots}^{(l)}$.

Given the moments, one can study their contribution to $\tilde{h}^{ij}$. By standard methods, the contribution of $\tilde{S}$ and $\tilde{S}$ reads \[15\] \[16\]

$$\tilde{h}_{ij}^{SI} = \frac{4G(-1)^l}{l!}$$

$$\times \left(\frac{\delta_{ij}}{3} (\tilde{S}_{i1i2}^{(l)}) - \tilde{S}_{i1i2}^{(l)} \partial_{i1} \ldots \partial_{i1} \frac{1}{r} + \tilde{S}_{i1i2i1}^{(l)} \partial_{j1} \partial_{i1} \ldots \partial_{i1} \frac{1}{r} \right) ,$$  \hspace{1cm} (51)

where symmetrization between $i$ and $j$ is assumed. Using the relation Eq. (50), the above can be written in the form

$$\tilde{h}_{ij}^{SI} = \delta_{ij} \partial_{k} \zeta_k - \partial_i \zeta_j - \partial_j \zeta_i$$  \hspace{1cm} (52)

where

$$\zeta_i^{SI} = \frac{4G(-1)^l}{l!} S_{kki1i2i3i4} \partial_{i1} \ldots \partial_{i1} \frac{1}{r} ,$$  \hspace{1cm} (53)

therefore, after a gauge transformation

$$\tilde{h}_{ij}^{SI} \rightarrow \tilde{h}_{ij}^{SI} + \partial_i \zeta_j + \partial_j \zeta_i - \eta_{ij} \partial^a \zeta_a ,$$  \hspace{1cm} (54)

we are left only with the contribution in $\tilde{h}_{00}$ \[15\] \[16\]

$$\tilde{h}_{00}^{SI} = \frac{4G(-1)^l}{l!} \tilde{S}_{i1i2i3i4} \partial_{i1} \ldots \partial_{i1} \frac{1}{r} ,$$  \hspace{1cm} (55)

which is just the standard scalar multipole expansion with $T_{kk} (\vec{r})$ as the scalar density, similar to the energy density multipoles. The leading contribution comes from the scalar quadrupole $S2$ moment.

Finally we come to the other two $\tilde{T}^{(l-1)}$ and $T^{(l-2)}$ multipole series. Their contribution to the trace-reversed metric perturbation $\tilde{h}^{ij}$ reads \[15\] \[16\]

$$\tilde{h}_{ij}^{(l-1)} = \frac{2G(-1)^l}{l!} \tilde{T}^{(l-1)}_{i1i2i3i4} \partial_{j1} \partial_{i1} \ldots \partial_{i1} \frac{1}{r} + (i \rightarrow j) ,$$  \hspace{1cm} (56)
and
\[ \tilde{h}_{ij}^{(l-2)} = \frac{4G(-1)^l}{l!} \bar{T}_{ij}^{(l-2)} \partial_i \partial_j \partial_k \ldots \partial_l \frac{1}{r^{l+1}}. \] (57)

Similar to the case of the vector current, they can all be gauged away through gauge transformations
\[ \zeta_i^{(l-1)} = \frac{2G(-1)^l}{l!} \bar{T}^{(l-1)} \partial_i \partial_j \partial_k \ldots \partial_l \frac{1}{r^{l+1}}, \] (58)
and
\[ \zeta_i^{(l-2)} = \frac{2G(-1)^l}{l!} \bar{T}^{(l-2)} \partial_i \partial_j \partial_k \ldots \partial_l \frac{1}{r^{l+1}}. \] (59)

Since \( \partial^l \zeta_i^{(l-1)} = \partial^l \zeta_i^{(l-2)} = 0 \) due to antisymmetrization in \( T^{(l-2)} \) and \( \bar{T}^{(l-2)} = 0 \) at large distance, the gauge transformation will not produce new terms in \( \tilde{h}^{(0)} \). Therefore, both series \( \bar{T}^{(l-1)} \) and \( T^{(l-2)} \) have no physical effect at large distance in static case.

However, moment series \( \bar{T}^{(l-1)} \) and \( T^{(l-2)} \) still provide useful characterization of the momentum current distribution (and do have physical effects in time-varying systems). At \( l = 1 \), the \( \bar{T}^{(0)} \) vanishes. At \( l = 2 \), non-vanishing moment is related to the tensor-momentum-current monopole or tensor monopole \( T \) for short:
\[ T^{(0)} = \frac{1}{5} \int d^3 r \bar{T}_{ii}(\vec{r}) \left( r_i r_j - \frac{\delta_{ij}}{3} r^2 \right) \]
\[ = \frac{2}{15} \int d^3 \vec{r} \frac{1}{r^2} s(r), \] (60)

where the second line serves as a definition for \( s(r) \) (“shear pressure”). Using Eq. (43), it is related to the scalar momentum-current radius,
\[ T^{(0)} = -\frac{1}{6} \int d^3 \vec{r} \bar{T}_{ii}(\vec{r}) \]
\[ = -\frac{1}{2} \int d^3 \vec{r} \frac{p(\vec{r})}{r}, \] (61)

where \( T_{ii} \) is proportional to the so-called pressure \( p(r) \) in the other literature [18][20][22]. We choose to define the tensor-MC monopole moment of a system as
\[ \tau = -\frac{T^{(0)}}{2}, \] (62)

which relates to the “\( D \)-term” \( D(0) \) [14] as \( \tau = \frac{D(0)}{4M} \).

Next we come to the \( \bar{T} \_{} \_{} T_{ij} \) dipole \( \bar{T}_i{}^{(1)} \). It is anti-symmetric in \( ij \) can be written as
\[ \bar{T}_i{}^{(1)} = \frac{2}{5} \int d^3 \vec{r} \bar{T}_{k[i}(\vec{r}) r_{j)} \] (63)

If one define the dilatation current at \( t = 0 \),
\[ j_D^i = r_k T_{k[i}, \] (64)
then \( \bar{T}_i{}^{(1)} \) can be conveniently expressed as the “magnetic moment” of the dilatation current. Due to Eq. (46), \( \bar{T}_i{}^{(1)} = 0 \) identically.

To summarize, there are three series of multipoles for momentum current, \( S^{(l)}, T^{(l-1)}, \) and \( T^{(l-2)} \). The leading-order moments are tensor monopole \( T0 \) \( \tau = -\frac{T^{(0)}}{2} \), and scalar quadrupole \( S2 \) \( \sigma_{ij} = S_{ij}^{(2)} \), with vanishing tensor dipole \( \bar{T}1 \). To the next order, one has scalar octupole \( S3 \), tensor quadrupoles \( T2 \) and \( T2 \), and tensor dipole \( T1 \), and so on.

II. EMT FORM FACTORS OF HADRONS AND GRAVITATIONAL MULTIPOLES

In this section, we consider examples of the gravitational multipoles in hadrons of different spin. Not all hadrons are capable of generating all types of gravitational multipoles. For spin-0 particle such as the pion or \( ^4 \)He nucleus, only two multipoles can be generated, one corresponds to the total mass \( M \) (mass monopole \( M0 \)), and the other to the momentum-current tensor monopole \( \tau \) (\( T0 \)). For a spin-1/2 hadron such as the proton and neutron, one can generate in addition the angular-momentum dipole \( (\bar{V} \_{} 1) \). For a spin-1 resonance, such as \( \rho \) meson, one can generate the mass quadrupole \( M2 \), scalar quadrupole \( S2 \), and tensor quadrupole \( T2 \). In the following we will discuss each of them in turn.

We work in the limit that the hadron masses are large so that their Compton wavelength is negligible [27]. This is true in the large \( N_c \) limit for baryons (certainly not true for a pion). In this case, one can directly Fourier-transform the form factors to the position space to obtain the space density distributions. For particles whose Compton wavelength is not small compared to its size, an option is to go to the infinite momentum frame [28][30] where one has to be content with a 2D interpretation. In practice, we adopt the standard Breit frame approach as the definition of a spatial density. When studying the gravitational perturbation at the distance \( r \) much larger than the Compton wave length, the formula in terms of the form factors are accurate, independent of the density interpretation.

A. Spin-0 case

Let’s first consider a scalar system. The EMT matrix element between the plane wave states \( |P^\mu \rangle \) and \( |P'^\nu \rangle \) defines the gravitational form factors, \( A \) and \( C \) [14][15]
\[ \langle P'|T^\mu\nu|P \rangle \]
\[ = 2P'^\mu P^\nu A(q^2) + 2(q^\mu q^\nu - g^\mu\nu q^2)C(q^2) \] (65)
where \( q^\mu = P'^\mu - P^\mu \) is the momentum transfer. The momentum conservation requires \( A(0) = 1 \). In the Breit frame where \( \vec{P} + \vec{P}' = 0 \), the Fourier transformation of \( MA(q) \) corresponds to a part of the mass density \( \rho_m(\vec{r}) \).

The form factor \( C(q^2) \) is related to the tensor-MC monopole distribution in the system, besides contribu-
ing to the mass density $\rho_m(\vec{r})$. In fact, if we Fourier-transform $T^{ij}$ to the coordinate space, it has the form

$$T^{ij}(\vec{r}) = (\nabla^2 \delta^i_j - \nabla^i \nabla^j) \frac{C(r)}{M^2}.$$  \hfill (66)

where $C(r)$ is the Fourier version of $C(q^2)$ and we have divided a factor $2M$ from the relativist normalization ($\langle P\mid P' \rangle = (2\pi)^3 2P^0 \delta^3(\vec{P} - \vec{P}')$). A simple calculation shows that the MC monopole moment $\tau$ is just,

$$\tau = \frac{C(q = 0)}{M},$$  \hfill (67)

which relates to the $D$-term $\tau$ as $\tau = D^{(0)} \frac{\alpha}{4\pi M}$. For a free spin-$0$ boson $\bar{\tau}$,

$$\tau_{\text{boson}} = -\frac{\hbar^2}{4M}$$  \hfill (68)

when proper SI unit $\text{kg} \cdot \text{m}^4 \cdot \text{s}^{-2}$ is restored. We define a fundamental unit $\tau_0 = \frac{\hbar^2}{4\pi M}$, and write $\tau_{\text{boson}} = g_b \tau_0$, then $g_b = -1$. For a minimally-coupled interacting theory, the monopole moment remains the same $\frac{\alpha}{4\pi M}$. For more complicated examples including non-minimal coupling, QCD pseudo-Goldstone bosons, as well as large nuclei, see Refs. [13, 24, 31] for extensive discussions. Monopole density distribution is related to $s(r)$ defined in Eq. (60).

For a system with long-range force, such as a charged particle, it can be shown that $C(q \to 0)$ is infrared divergent $\frac{\alpha \pi}{16|q|^3} + \frac{\alpha}{6\pi M} \ln \frac{\hbar^2}{M^2}$. \hfill (69)

The first term is due to the large $r$ asymptotic decay of Coulomb potential and is classical in nature, while the second term is quantum in nature. It can be shown [11] that a divergent monopole moment $\tau_{\text{eff}}(\tau_{\text{eff}} = +\infty)$ will generate $1/r^2$ correction to the space-time metric

$$h^{ij}(\vec{r}) = \frac{G \alpha \hat{r}_i \hat{r}_j}{r^4} + \frac{4G\alpha}{3\pi Mr^3} (\hat{r}_i \hat{r}_j - \delta^{ij}),$$  \hfill (70)

where the first term comes form the linear divergent part $\frac{\alpha \pi}{16|q|^3}$ of $C(q)$, while the second term is due to the logarithmic divergent term $\frac{\alpha}{6\pi M} \ln \frac{\hbar^2}{M^2}$.

### B. $C(q^2)$ Contribution to Gravitational Potential

According to Sec. I, it appears that the tensor monopole $\tau$ does not contribute to the long-distance properties of the gravity, other than it produces a pure gauge contribution. However, form factor $C(q^2)$ does generate a short-distance static gravitational potential $\bar{h}^{00}$ through its contribution to the energy density.

It can shown by solving the linearized Einstein equation that

$$h_C^{00}(\vec{r}) = -\frac{8\pi G}{c^4 M} C(r),$$  \hfill (71)

$$h_C^{ij}(\vec{r}) = \frac{8\pi G}{c^4 M} C(r) \delta^{ij},$$  \hfill (72)

thus $C(r)$ contributes a part of the gravitational potential at short distance! The total static potential will be added by the form factor $A(q^2)$ contribution

$$h_A^{00}(\vec{r}) = \frac{2}{c^4} V(r),$$  \hfill (73)

$$h_A^{ij}(\vec{r}) = \frac{2}{c^4} V(r) \delta^{ij},$$  \hfill (74)

where

$$V(r) = \int d^3 \vec{r}' \frac{GM}{|\vec{r} - \vec{r}'|} A(\vec{r}').$$  \hfill (75)

At large $r$, $h_A^{00}$ decays exponentially whereas $h_A^{ij}$ reduces to a point-mass Newton’s potential. For a non-relativistic probe, only $h_C^{00}$ matters.

### C. Spin-$\frac{1}{2}$ case

For a spin-$1/2$ system, the matrix element of the EMT is $[37, 39]$

$$\langle P' \mid T^\mu_{\nu} \mid P \rangle = \bar{u} (P' S') \left[ A(q^2) \gamma^\mu \tilde{P}^\nu + B(q^2) \frac{\delta^{(\mu \nu)}}{2M} + C(q^2) \frac{g^\mu q^\nu - g^\nu q^\mu}{2M} \right] u(PS),$$  \hfill (76)

where $\tilde{P}^\mu = (P + P')^\mu / 2$. There are now three dimensionless form factors $A(q^2)$, $B(q^2)$ and $C(q^2)$. The physics of $A(q^2)$ is the same as that for the spin-$0$ case.

The form factor $B(q^2)$ is related to the angular momentum distribution in the system. Indeed, the momentum density $T^{0i}$ is

$$\tilde{T}^i(\vec{r}) = -\frac{1}{2} \vec{S} \times \nabla (A(r) + B(r)).$$  \hfill (77)

where $S^i$ is the spin moment of the momentum density $\vec{S}(\vec{r}) = \vec{r} \times \vec{T}(\vec{r})$. The $V$ moment of the momentum density yields total angular momentum $\vec{S}$. Angular momentum conservation requires $B(q^2 = 0) = 0$.

The MC monopole moment is zero for a free fermion $\textbf{10}$. In general, a spin-$1/2$ system has

$$\tau = \frac{C(0)}{M}.$$  \hfill (78)
density matrix

to simplify the expression, we also use symmetric polarization

\( \epsilon \)

where

\[ \epsilon = \epsilon' \]

Following form

quadrupole generates \( 1 \)

independent gravitational form factors [12, 43, 44]

where \( \hat{\epsilon} \)

different from other conventions. An example of

\( \rho \)

definition of the dimensionless form factors is slightly dif-

For the convenience of calculating multipoles, the above

model calculation.

states have small momenta, and \( \epsilon \cdot P = \epsilon' \cdot P' = 0 \). To simplify the expression, we also use symmetric polarization density matrix

\[ E^{\mu\nu} = \frac{1}{2} \left( \epsilon^{\mu} \epsilon'^{\nu} + \epsilon^{\nu} \epsilon'^{\mu} \right) \]

and the anti-symmetric polarization tensor

\[ S^{\mu\nu} = i \left( \epsilon^{\mu} \epsilon'^{\nu} - \epsilon^{\nu} \epsilon'^{\mu} \right) \]

For the convenience of calculating multipoles, the above

definition of the dimensionless form factors is slightly dif-

EMT form factors can be found in [45] using chiral perturbation theory and in [46] from Nambu-Jona-Lasinio model calculation.

To relate the above form factors to gravitational multi-

poles, we consider the static limit where initial and final states have small momenta, and \( \epsilon^0 = 0 \) and \( \epsilon' = \epsilon' = \epsilon' \) is arbitrary. Looking at the \( T^{00} \), \( A(q^2 = 0) = 1 \) gives rise to the mass-monopole contribution, while \( \hat{A}(q^2 = 0) \) contributes to the mass quadrupole \( M2 \),

\[ M_{ij} = \int d^3r \left( \delta r^i r^j - \frac{1}{3} \delta_{ij} r^2 \right) \rho_m(\vec{r}) = \frac{2 \hat{A}(0)}{M} \bar{E}_{ij} \]

where \( \bar{E}_{ij} \) is the traceless part of \( E^{ij} \). The mass

quadrupole generates \( 1/r^3 \) perturbation in \( \bar{h}^{00} \) in the following form

\[ \bar{h}^{00} = \frac{2G M_{ij}}{r^3} \left( 3 \hat{\rho}^i \hat{\rho}^j - \delta^{ij} \right) \]

The momentum density \( T^{0i} \) in Fourier space is,

\[ T^{0i} = \frac{i}{2} S^{ij} q_j J(q^2) = \frac{i}{2} \left( \vec{S} \times \vec{q} \right)^i J(q^2) \]

where the axial vector \( \vec{S} = \text{Re}(i\vec{e} \times \vec{e}') \), from which one identify \( J(\vec{r}) \) as the angular momentum dipole density. Angular momentum conservation constraints \( J(q^2 = 0) = 1 \).

From the expression for \( T^{ij} \), one reads off the tensor-

MC monopole \( T00 \) moment

\[ \tau = \frac{C(0)}{M} , \]

which is zero for a free photon. The monopole moment of the \( \rho \) meson appears close to that of the pion. For other spin-1 systems including deuteron, see Ref. [18].

There is also a new tensor-MC quadrupole \( T22 \) moment

\[ T_{ij}^{(2)} = - \frac{\hat{C}(0)}{48M^2} \bar{E}_{ij} \]

where the multiple series \( T_{ij}^{(2)} \) is defined in Eq. [88]. The contribution \( \hat{C}(0) \) to the scalar momentum current is new for spin-1 systems.

Finally, the scalar-MC quadrupole moment \( S2 \) can be calculated as,

\[ \sigma_{ij} = \frac{D(q^2 = 0)}{M} \bar{E}_{ij} \]

Thus the tensor quadrupole is proportional to \( D(0) \) form factor defined above. After gauge transformation, it will generate contribution to \( \bar{h}^{00} \) as

\[ \bar{h}^{r,00} = \frac{2G \sigma_{ij}}{r^3} \left( 3 \hat{\rho}^i \hat{\rho}^j - \delta^{ij} \right) \]

which can be combined with the one from mass-

quadrupole into the form

\[ \bar{h}^{00} = \frac{2G (M_{ij} + \sigma_{ij})}{r^3} \left( 3 \hat{\rho}^i \hat{\rho}^j - \delta^{ij} \right) \]

in consistent with the general results in Refs. [15] [16].

III. SCALAR MOMENTUM-CURRENT DISTRIBUTION AND T0 MOMENT IN HYDROGEN ATOM

In this section we study the EMT of the hydrogen-like atom. Contrary to the single charged electron, hydrogen-like atoms are charge neutral and are expected to have finite scalar-MC monopole moment \( \tau \). We first show that in the quantum mechanics it is possible to construct a conserved EMT using quantum-mechanical wave functions. However, it still has long-range Coulomb tail due to the interaction between the electron and the proton. We then show that the above conserved EMT can be
justified in the field theoretic framework and identified as the leading-order electron kinetic contribution plus the leading order Coulomb photon exchange contribution. By adding the single electron and single proton contributions, the Coulomb tail gets removed and the resulting monopole moment \( \tau = \hbar^2/4M \) and positive. We argue that the result is accurate to leading order in \( \alpha \). This example shows that the sign of the EMT has little to do with the “mechanical stability”.

A. Hydrogen atom: quantum mechanics

The electron wave function \( \phi \) of an hydrogen atom satisfies the the Schrödinger equation

\[
\left( -\frac{1}{2m} \nabla^2 - eV_p(r) \right) \phi(\vec{r}) = E\phi(\vec{r}) .
\]

(90)

where \( e \) is the proton charge and positive)

\[
\nabla^2 V_p(r) = -e\delta^3(\vec{r}) ,
\]

(91)

is the potential of the static charge of the proton, \( V_p = e/(4\pi r) \). For convenient we chose \( m = 1 \). One further defines the static potential \( V_e \) for the electron

\[
\nabla^2 V_e(r) = e|\phi(\vec{r})|^2 ,
\]

(92)

which can be solved for the ground state as

\[
V_e(r) = \frac{ee^{-2\alpha r}(1 + \alpha r - e^{2\alpha r})}{4\pi r}.
\]

(93)

where \( \alpha = e^2/4\pi \).

By non-relativistic reduction of the Dirac equation, one can construct the following EMT \( T_{QM}^{ij} \) which consists of a kinetic term

\[
T^{ij}_K = -\frac{1}{4m} (\phi^\dagger \partial^i \partial^j \phi - \partial^i \phi^\dagger \partial^j \phi + c.c) ,
\]

(94)

plus a potential term made of interacting electric fields of the proton and electron,

\[
T^{ij}_V = \delta^{ij} \nabla V_p \cdot \nabla V_e - \partial^i V_e \partial^j V_p - \partial^i V_p \partial^j V_e .
\]

(95)

The trace of \( T_{QM}^{ij} \) is \( T^{ij}_K + T^{ij}_V \) can be calculated as

\[
T_{QM}^{ij} = |\phi|^2 (2E + eV_p) + \nabla \cdot (V_p \nabla V_e) + \frac{1}{4m} \nabla^2 |\phi|^2 .
\]

(96)

It is easy to show that \( T_{QM}^{ij} \) is conserved for the ground state, \( \partial_i T_{QM}^{ij} = 0 \).

Therefore, it can be written in a normalized state as

\[
T_{QM}^{ij}(\vec{r}) = (\delta^{ij}\nabla^2 - \nabla^i \nabla^j) \frac{C_{QM}(r)}{M} ,
\]

(97)

\[
T_{QM}^{ij}(r) = 2\nabla^2 \frac{C_{QM}(r)}{M} ,
\]

(98)

from which the \( C_{QM} \) can be calculated as

\[
\frac{C_{QM}(r)}{M} = \frac{1}{2\nabla^2} T_{QM}^{ij} = \frac{e^{-2\alpha r}(2\alpha r + 1)}{16\pi^2} - \frac{\alpha}{16\pi r^2} .
\]

(99)

Clearly, the Coulomb tail \(-\alpha/16\pi r^2 \) prevents a finite \( C(q = 0) \). Physically, the self-energies of the proton and electron will generate opposite contributions which cancel the Coulomb tail from the above expression (the EMTs of the electron and proton are separately conserved). However, this piece of physics is outside the usual non-relativistic quantum mechanics.

For the time being, we can subtract this Coulomb tail and define an effective the \( C_{eff}(r) \) in the infrared region where \( r \sim \frac{1}{\alpha} \)

\[
\frac{C_{eff}(r)}{M} = \frac{e^{-2\alpha r}(2\alpha r + 1)}{16\pi r^2} ,
\]

(100)

which is of order 1 when the momentum transfer is of order of the inverse Bohr radius. In particular, the scalar MC monopole moment \( \tau = \frac{C(0)}{M} \) for the hydrogen atom reads

\[
\tau = \frac{C_{eff}(q = 0)}{M} = \tau_0 [1 + O(\alpha \ln \alpha)] ,
\]

(101)

where \( \tau_0 = \frac{\hbar^2}{4m} \) is the basic unit defined before. Below we show that in quantum field theory, the long range Coulomb tail is indeed removed and Eq. (101) is the correct MC monopole moment.

B. Hydrogen atom: field theory

The above calculation can in fact be justified in the field theoretical framework. Let us consider the bound state in quantum electrodynamics (QED) between two types of fermions, the standard negative charged electron with mass \( m \) and positive charged “proton” with mass \( M \). At energy scale much smaller than the proton mass \( M \), the proton can be approximated by an infinitely heavy static source \( N \) represented by an auxiliary field \( N \). The Lagrangian density of the system reads

\[
\mathcal{L} = i \bar{N} \gamma^\nu \cdot DN + \mathcal{L}_{QED} ,
\]

(102)

where \( N \) represents the infinitely heavy proton moving along the \( \nu^\mu = (1, 0, 0, 0) \) direction. The Lagrangian preserves Lorentz invariance if \( \nu^\mu \) is treated also as an auxiliary field. The EMT of the above system can be shown as

\[
T^{\mu\nu} = \frac{i}{4} \bar{N} \gamma^\nu (\mu D^\nu) N + T_{QED}^{\mu\nu} .
\]

(103)

More precisely, the heavy-source only contributes to the \( T^{00} \) part of the EMT. To proceed, we fix the Coulomb
that this contribution is exactly the T
tron kinetic contribution shown in Fig. 2. One can show
ing order O
matrix element of
where to leading order in
energy
Schrodinger equation Eq. (90) in non-relativistic limit
It can be verified that it is nothing but the standard
Salpeter equation \[48\] induced by Coulomb-photon ex-
changes, see Fig. 1 for a depiction of the equation.

\[ H = \frac{1}{2} \int d^3\bar{x} \left( \bar{\psi} \partial_t \psi - \bar{\psi} \right) + \int d^3\bar{x} \bar{\psi} \left( -i \bar{\alpha} \cdot \bar{D} + m \gamma^0 \right) \psi \]
\[ + \frac{e^2}{2} \int d^3\bar{x} d^3\bar{y} \frac{\bar{\psi} \psi |\bar{N} - \bar{N}|(\bar{x}) \bar{\psi} \psi |\bar{N} - \bar{N}|(\bar{y})}{4\pi|\bar{x} - \bar{y}|} , \]
where the last term represents the Coulomb interaction.

We now consider the bound state formed by a pair of electron and the heavy-proton. The leading wave function reads
\[ |\bar{p}| = \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{\phi(\bar{k})}{2E_k} |\bar{k} e| - \bar{k} + \bar{p} \rangle_N , \]
where to leading order in \( \alpha \), \( \phi(\bar{k}) \) satisfies the Bethe-Salpeter equation \[48\] induced by Coulomb-photon exchanges, see Fig. 1 for a depiction of the equation. It can be verified that it is nothing but the standard Schrodinger equation Eq. (90) in non-relativistic limit \(|\bar{k}| \ll m\), and is characterized by two scales, the binding energy \( \alpha^2m \) and the inverse Bohr radius \( \alpha m \).

Given the above wave function, one can calculate the matrix element of \( T^{ii}(\bar{r}) \) using standard method. At leading order of \( 0(1) \) in \( \alpha \), there is only one diagram, the electron kinetic contribution shown in Fig. 2. One can show that this contribution is exactly the \( T_{ii}^{\mathrm{K}} \) in Eq. (94).

We then consider radiative corrections starting from \( 0(\alpha) \). Due to the fact that the velocities are of order \( \alpha \), only the Coulomb-photon contribution needs to be included. Furthermore, the one-loop contributions can be classified into interference and single electron/single proton diagrams, see Fig. 2 for a depiction. The interference diagram can be calculated as
\[ T^{ii}(\bar{q}) = -e^2 \int \frac{d^3k d^3k'}{(2\pi)^6} \bar{\phi}^i(k') \phi^i(k - k') \frac{\bar{k} \cdot (\bar{k} - \bar{q})}{k^2(k - \bar{q})^2} , \]
which is equivalent to
\[ T^{ii}_V = \nabla V_e \cdot \nabla V_p . \]

Therefore, we found that the leading-order electron kinetic and interference contributions in Fig. 2 will exactly lead to the conserved \( T^{ii}_{Q,M} \) in Eq. (96) with a Coulomb tail \( \alpha e^3/8m^2 \).

Clearly, in order to cancel the Coulomb tail one must add the electron and proton self-energy contributions. The electron contribution is also shown in Fig. 2 and can be calculated in the region \( q \sim 0(\alpha m) \) as
\[ C_e(q) = \frac{\alpha \pi}{16|q|} \times \frac{16\lambda^4}{(2\pi m_e + 4\alpha^2)^2} , \]
where the first factor \( \frac{\alpha \pi}{16|q|} \) is just the standard Coulomb tail, and the second factor comes from the dressing in the

FIG. 1. The Bethe-Salpeter equation for the wave function \( \phi \) denoted by the oval blob. Double line represents propagator of proton field and single line represents the electron propagator. The dashed line represents the exchange of a Coulomb photon.

FIG. 2. The order-\( 0(1) \) electron kinetic contribution (upper), the Coulomb photon interference (middle) and single electron (lower) contributions to \( T^{ii} \). Dashed lines represent Coulomb photons and crossed circles denote the operator insertions. Notice the infrared divergences for \( C(q) \) at \( q = 0 \) are cancelled between the interference and single electron and proton (not shown) contributions.

FIG. 3. Mixed contributions between the radiative (wavy line) and Coulomb photons. IR divergences are regulated by binding energy differences.
bound-state wave function. Similarly, the proton contribution is of the form in the $M \gg m$ limit

$$\frac{C_p(p)}{M} = \frac{\alpha \pi}{16|q|} \times \frac{16\alpha^4}{(\frac{q^2}{M^2} + 4\alpha^2)^2}, \quad (110)$$

and can be approximated simply by the Coulomb tail when $q \sim \alpha m$. Clearly, the above formulas can be justified only when the momentum transfer is small, and is invalid in ultraviolet region when $q$ is comparable with particle masses.

In conclusion, in the region $|q| \leq O(\alpha m)$, the $C$ form factor of the hydrogen atom reads

$$\frac{C_H(q)}{M} = \frac{1}{2m(\frac{q^2}{\alpha m^2} + 4)} - \frac{\alpha}{4|q|} \left( \frac{\pi}{2} - \text{Arctan} \frac{q}{2\alpha m} \right) + \frac{\alpha \pi}{|q|} \left( \frac{\frac{q^2}{\alpha m^2} + 4}{4} \right). \quad (111)$$

From these, the monopole moment for the hydrogen atom is

$$\tau_H = \frac{C_H(0)}{M} = \tau_0[1 + O(\alpha \ln \alpha)]. \quad (112)$$

The $\tau_H = 1$ except for a small correction of order $\alpha$, a result with opposite sign from a point-like boson.

Given the form factor $C(q^2)$, we can obtain the scalar momentum current or “pressure” [18] [19],

$$p(r) = \frac{1}{3} T^{ii} = \frac{2}{3M} \frac{1}{r^2} \frac{d}{dr} r^2 dC(r) \frac{dr}{dr}, \quad (113)$$

which can be shown to be positive for small $r$ and negative for large $r$. On the other hand, the momentum current monopole density distribution

$$\tau(r) = -\frac{2\pi}{5} r^2 \left( r^2 \tau^j - \frac{1}{3} r^2 \delta^j \right) T_{ij}(r) = -\frac{4\pi}{15} r^4 s(r). \quad (114)$$

where $s(r) = -\frac{r}{M} \frac{d}{dr} \left( \frac{1}{r^2} \frac{dC(r)}{dr} \right)$ has been called “shear pressure”. Unfortunately, there is no simple expression for it in positional space without using the Meijer $G$ function. A numerical result for $\tau(r)$ is shown in Fig. 4, which is positive definite at all $r$ and finite at $r = 0$. A plot of $C(r)$ is shown in Fig. 5.

We also show a plot of the momentum flow $T^{ij}$ in 3-space in Fig. 6. Any surface integral of the vector field yields the flux of $x$-component momentum through the surface, or force along the $x$-direction, which would produce a pressure in this direction if the momentum current gets totally absorbed. Any closed surface integral yields zero, indicating momentum conservation or net null force through any size volume.

Before ending the section, one briefly comment on the $O(\alpha)$ corrections to $C(0)$. By including radiative photons, the degree of infrared divergences is reduced due to the fact that the radiative photon couples to the 3-velocity $\vec{v}$ of the electron in the Coulomb gauge.

Therefore, the mixing diagram where Coulomb and radiative photons couple to each other can be logarithmically divergent, confirmed by the single electron calculation [11] [22] [33] [35] [36]. However, the divergences is expected to be regulated by the binding energy differences in a way similar to the famous Lamb shift, leading to finite $C(0)$ at order $O(\alpha)$. See Fig. 3 for a depiction. We will leave the calculation of order $\alpha$ corrections in a separate publication [19].
condition
\[ \int dr \, r^2 p(r) = 0, \] (115)
is trivially related to the conservation of the momentum current \[ 18, 53 \]. Using “pressure” or “force” to characterize the momentum current may generate confusion because they are not acting on any part of the system itself, but on some fictitious surfaces which would absorb the current entirely through some interactions. In our view, the most interesting way to characterize \( T^{ij} \) is by its multipole generating characteristic space-time perturbation. This is similar to use the magnetic moment etc. to describe a current distribution, which generates a particular type of magnetic field.

Further mechanical stability conditions on \( T^{ij} \) derive from comparing it the pressure and shear pressure distributions with a mechanical system. For example, it has been speculated that a negative \( D \)-term \( (C(q = 0)) \) is needed for mechanical stability \[ 51 \]. However, the sign of the scalar-MC monopole moment depends the flow pattern of the momentum currents. Reversing the direction of the momentum currents at every point in space in classical physics will reverse the \( D \)-term but should lead to another stable flow pattern. Thus the sign of \( D \)-term cannot be related to mechanical stability. In fact, in the example of hydrogen atom, the “force” from the momentum flow is directed towards the center (Using the \( C_{\text{eff}}(r) \) defined in Eq. \[ 100 \]) and the \( D \)-term is positive. On the other hand, we know perfectly well that hydrogen atom is stable due to quantum physics which has already a well-defined sense of stability. Thus using the momentum current flow to judge the stability of a quantum system appears not useful and furthermore unnecessary.

To summarize, we have revisited the gravitational fields generated by a static source by performing the multipole expansion \[ 15, 16 \] of the corresponding energy, energy current and momentum current densities. For a static and conserved EMT, there are 6 series of non-vanishing multipoles, one series for \( T^{00} \), two series for \( T^{0i} \) and three series for \( T^{ij} \). They are important characterization of spatial distribution of the corresponding energy and momentum current distributions, although at large distance only three series will contribute to the physical metric perturbation in the static case.

In particular, the \( C(q^2) \) form factor or the “\( D \)-term” is related to the tensor-MC monopole moment \( T_0 \), which has a basic unit \( \tau_0 = \hbar^2/4M \). As a concrete example, we have calculated the \( C(q^2) \) form factor for the hydrogen atom at small \( q \) region and found that the MC monopole moment is positive, which is opposite to that of a point-like boson. Moreover, we argue that notion of “mechanical stability” or “pressure” is of limited significance when applied to bound states in quantum field theories.

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[1] X.-D. Ji, Phys. Rev. D 55, 7114 (1997) arXiv:hep-ph/9609381
[2] A. Airapetian et al. (HERMES), Phys. Rev. Lett. 87, 182001 (2001) arXiv:hep-ex/0106068
[3] S. Stepanyan et al. (CLAS), Phys. Rev. Lett. 87, 182002 (2001) arXiv:hep-ex/0107043
[4] S. Chekanov et al. (ZEUS), Phys. Lett. B 573, 46 (2003) arXiv:hep-ex/0305028
[5] A. Aktas et al. (H1), Eur. Phys. J. C 44, 1 (2005) arXiv:hep-ex/0505061
[6] A. V. Radyushkin, Phys. Lett. B 385, 333 (1996) arXiv:hep-ph/9605431
[7] J. M. Zinn-Justin, Quantum Field Theory and Critical Phenomena, 2nd ed. (New York: Oxford University Press, 1989)
[8] L. Blanchet, Class. Quant. Grav. 19, 3965 (2002) arXiv:gr-qc/0309037
[9] Y. Hatta and D.-L. Yang, Phys. Rev. D 98, 074030 (2018) arXiv:1808.02163 [hep-ph]
[10] Y. Guo, X. Ji, and Y. Liu, Phys. Rev. D 103, 096010 (2021) arXiv:2103.11506 [hep-ph]
[11] J. F. Donoghue, B. R. Holstein, B. Garbrecht, and T. Konstandin, Phys. Lett. B 529, 132 (2002) Erratum: Phys. Lett. B 612, 311–312 (2005), arXiv:hep-th/0112237
[12] B. R. Holstein, Phys. Rev. D 74, 084030 (2006) arXiv:gr-qc/0607051
[13] K. S. Thorne, Rev. Mod. Phys. 52, 299 (1980)
[14] L. Blanchet, Class. Quant. Grav. 15, 1971 (1998) arXiv:gr-qc/9801101
[15] T. Damour and B. R. Iyer, Phys. Rev. D 43, 3259 (1991)
[16] M. Maggiore, Gravitational waves. Volume I: theory and experiments (2007).
[17] M. V. Polyakov and C. Weiss, Phys. Rev. D 60, 114017 (1999) arXiv:hep-ph/9903451
[18] M. V. Polyakov and P. Schweitzer, Int. J. Mod. Phys. A 33, 1830025 (2018) arXiv:1805.06506 [hep-ph]
[19] C. Lorcé, H. Moutarde, and A. P. Traivinski, Eur. Phys. J. C 79, 89 (2019) arXiv:1810.09837 [hep-ph]
[20] M. V. Polyakov, Phys. Lett. B 555, 57 (2003) arXiv:hep-ph/0210165
[21] V. D. Burkert, L. Elouadrhiri, and F. X. Girod, Nature 557, 396 (2018)
[22] P. E. Shanahan and W. Detmold, Phys. Rev. Lett. 122, 072003 (2019) arXiv:1810.07589 [nucl-th]
[23] A. Noguchi, R. Yamazaki, Y. Tabuchi, and Y. Nakamura, Nature Communications 11, 1183 (2020)
[24] C. G. Callan, Jr., S. R. Coleman, and R. Jackiw, Annals Phys. 59, 42 (1970).
[25] J. D. Jackson, Classical Electrodynamics (Wiley, 1998).
[26] L. Blanchet and T. Damour, Phil. Trans. Roy. Soc. Lond. A 320, 379 (1986)
[27] R. L. Jaffe, Phys. Rev. D 103, 016017 (2021) arXiv:2010.15887 [hep-ph]
[28] M. Burkardt, Int. J. Mod. Phys. A 18, 173 (2003) arXiv:hep-ph/0207047
[29] G. A. Miller, Phys. Rev. Lett. 99, 112001 (2007) arXiv:0705.2409 [nucl-th]
[30] A. Freese and G. A. Miller, (2021), arXiv:2108.03301 [hep-ph]
[31] J. Hudson and P. Schweitzer, Phys. Rev. D 96, 114013 (2017) arXiv:1712.05316 [hep-ph]
[32] F. A. Berends and R. Gastmans, Annals Phys. 98, 225 (1976)
[33] K. A. Milton, Phys. Rev. D 15, 538 (1977)
[34] B. Kubis and U.-G. Meissner, Nucl. Phys. A 671, 332 (2000) [Erratum: Nucl.Phys.A 692, 647–648 (2001)], arXiv:hep-ph/9908261
[35] M. Warma and P. Schweitzer, Phys. Rev. D 102, 014047 (2020) arXiv:2006.06602 [hep-ph]
[36] A. Metz, B. Pasquini, and S. Rodini, Phys. Lett. B 820, 136501 (2021) arXiv:2104.04207 [hep-ph]
[37] I. Y. Kobzarev and L. B. Okun, Zh. Eksp. Teor. Fiz. 43, 1904 (1962).
[38] H. Pagels, Phys. Rev. 144, 1250 (1966)
[39] X.-D. Ji, Phys. Rev. Lett. 78, 610 (1997) arXiv:hep-ph/9603249
[40] J. Hudson and P. Schweitzer, Phys. Rev. D 97, 056003 (2018) arXiv:1712.05317 [hep-ph]
[41] M. J. Neubelt, A. Sampino, J. Hudson, K. Tezgin, and P. Schweitzer, Phys. Rev. D 101, 034013 (2020) arXiv:1911.08906 [hep-ph]
[42] P. Hagler et al., (LHPC), Phys. Rev. D 77, 094502 (2008) arXiv:0705.4295 [hep-lat]
[43] S. K. Taneja, K. Kathuria, S. Liuti, and G. R. Goldstein, Phys. Rev. D 86, 036008 (2012) arXiv:1101.0581 [hep-ph]
[44] W. Cosyn, S. Cotogno, A. Freese, and C. Lorcé, Eur. Phys. J. C 79, 476 (2019) arXiv:1903.00408 [hep-ph]
[45] E. Epelbaum, J. Gelliga, U. G. Meißner, and M. V. Polyakov, (2021), arXiv:2109.10826 [hep-ph]
[46] A. Freese and I. C. Cloet, Physical Review C 100 (2019), 10.1103/physrevc.100.015201
[47] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations* (Cambridge University Press, 2005).
[48] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951)
[49] X. Ji and Y. Liu, in preparation.
[50] M. V. Polyakov and A. G. Shuvaev, (2002), arXiv:hep-ph/0207153
[51] I. A. Perevalova, M. V. Polyakov, and P. Schweitzer, Phys. Rev. D 94, 054024 (2016) arXiv:1607.07008 [hep-ph]
[52] C. Lorcé, Eur. Phys. J. C 78, 120 (2018) arXiv:1706.05853 [hep-ph]
[53] M. Laue, Annalen Phys. 340, 524 (1911)