Recursion Relations for Tree-level Amplitudes in the $SU(N)$ Non-linear Sigma Model

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It is well-known that the standard BCFW construction cannot be used for on-shell amplitudes in effective field theories due to bad behavior for large shifts. We show how to solve this problem in the case of the $SU(N)$ non-linear sigma model, i.e. non-renormalizable model with infinite number of interaction vertices, using scaling properties of the semi-on-shell currents, and we present new on-shell recursion relations for all on-shell tree-level amplitudes in this theory.

INTRODUCTION

Scattering amplitudes are physical observables that describe scattering processes of elementary particles. The standard perturbative expansion is based on the method of Feynman diagrams. In last two decades there has been a huge progress on alternative approaches, driven by the idea that the amplitude should be fully determined by the on-shell data with no need to access the off-shell physics. This effort has lead to amazing discoveries that have uncover many surprising properties and dualities of amplitudes in gauge theories and gravity. One of the most important breakthroughs in this field was the discovery of the BCFW recursion relations $[1,2]$ that allow us to reconstruct the on-shell amplitudes recursively from most primitive amplitudes. They are applicable in many field theories, however, in some cases like effective field theories they can not be used.

Effective field theories play important role in theoretical physics. One particularly important example is the $SU(N)$ non-linear sigma model which describes the low-energy dynamics of the massless Goldstone bosons corresponding to the chiral symmetry breaking $SU(N) \times SU(N) \to SU(N)$. In the low energy QCD they are associated with the octet of pseudoscalar mesons and the model provides leading order predictions of interactions of pions and kaons that dominate hadronic world at lowest energies. It is also a starting point for many extensions or alternatives of electroweak standard model.

In this short note we find the recursion relations for all tree-level amplitudes of Goldstone bosons in the $SU(N)$ non-linear sigma model. The importance of this result is two-fold: (i) It shows that the BCFW-like recursion relations can be applicable to much larger class of theories than expected before. This might also help to understand better properties of the theory invisible otherwise. It also tells us that the $SU(N)$ non-linear sigma model despite being an effective (and therefore non-renormalizable) field theory behaves in some cases similar to renormalizable theories. (ii) It provides an effective tool for leading order (tree-level) calculations of amplitudes with many external pions which might be important for low energy particle phenomenology. More detailed description together with other results will be presented in $[3]$.

BCFW RECURSION RELATIONS

Let us consider an $n$-pt on-shell scattering amplitude of massless particles, and denote $t^a$ the generators of the Lie algebra of corresponding global symmetry group $G$. If at tree-level each Feynman diagram carries a single trace $Tr(t^{a_1}t^{a_2}...t^{a_n})$, we can decompose the full amplitude $A_n$ into sectors with the same group factor,

$$A_{n}^{tree} = \sum_{\sigma/\mathbb{Z}_n} A_n(p_{\sigma(1)},...p_{\sigma(n)}) Tr(t^{\sigma(1)}...t^{\sigma(n)}) \quad (\text{1})$$

where the sum is over all non-cyclic permutations. For each stripped amplitude $A_n$ we have a natural ordering of momenta $p_{\sigma(1)}...p_{\sigma(n)}$ and a single term $A_n(p_1,p_2,...p_n)$ generates all the other by trivial relabeling. At the loop level we can define analogous object in the planar limit but in the general case this simple decomposition is not possible due to terms with multiple traces.

In 2004 Britto, Cachazo, Feng and Witten (BCFW) $[1,2]$ found a recursive construction of tree-level on-shell amplitudes. The stripped amplitude $A_n = A_n(p_1,...,p_n)$ is a gauge invariant object and one can try to fully reconstruct it from its poles. Because of the ordering the only poles that can appear are of the form $P_{ab}^2 = 0$ where $P_{ab} = \sum_{k=a}^{b} p_k$ for some $a,b$. On the pole the amplitude factorizes into two pieces,

$$A_{L}(p_a,...,p_b,-P_{ab}) \frac{1}{P_{ab}^2} A_{R}(P_{ab},p_{b+1},...,p_{n-1}) \quad (\text{2})$$

Let us perform the following shift on the external data:

$$p_i(z) = p_i + zq, \quad p_j(z) = p_j - zq \quad (\text{3})$$

where $i$ and $j$ are two randomly chosen indices, $z$ is a complex parameter and $q$ is a fixed null vector which is also orthogonal to $p_i$ and $p_j$, $q^2 = (q \cdot p_i) = (q \cdot p_j) = 0$. Note that the shifted momenta remain on-shell and still satisfy momentum conservation. The original amplitude
\( A_n \) becomes a meromorphic function \( A_n(z) \) with only simple poles and if it vanishes for \( z \to \infty \) we can use Cauchy theorem to reconstruct it,
\[
A_n(z) = \sum_i \frac{\text{Res}(A_n, z_i)}{z - z_i},
\]
where \( z_i \) are poles of \( A_n(z) \).

\[
(P_{ab})^2 = (p_a + \cdots + p_n + \cdots p_b)^2 = 0,
\]
located in \( z_{ab} = -P_{ab}^2/2(q \cdot P_{ab}) \). Note that \( A_n(z) \) has a pole only if \( i \in (a, \ldots, b) \) or \( j \in (a, \ldots, b) \) (not both or none). There exists a convenient choice \( j = i+1 \) which minimizes a number of terms in \([4]\). According to \([2]\) \( \text{Res}(A_n, z_i) \) is a product of two lower point amplitudes with shifted momenta and the Cauchy theorem \([4]\) can be rewritten as
\[
A_n(z) = \sum_{a,b} A_L(z) \frac{1}{P_{ab}} A_R(z),
\]
where the sum is over all poles \( P_{ab}(z)^2 = 0 \) and
\[
A_L(z) = A_L(p_a, \ldots, p_i(z), \ldots, p_b),
A_R(z) = A_R(-p_{ab}, p_{b+1}, \ldots, p_j(z), \ldots, p_{a-1}).
\]

In the physical case we set \( z = 0 \). \( A_L \) and \( A_R \) in \([6]\) are lower point amplitudes, \( n_R, n_L < n \) and therefore we can reconstruct \( A_n(z) \) recursively from simple on-shell amplitudes not using the off-shell physics at any step. BCFW recursion relations were originally found for Yang-Mills theory \([11,12]\), and proven to work in gravity \([3,6]\). There are many works showing validity in other theories (e.g. for coupling to matter see \([7]\)).

If the amplitude \( A_n(z) \) is constant or grows for large \( z \), the prescription \([4]\) cannot be used directly. The constant behavior was studied e.g. in \([9]\) on the cases of \( \lambda \phi^4 \) and Yukawa theory. In the generic situation of a power behavior \( A_n(k) \approx z^k \) for \( z \to \infty \) we can use the following formula \([9]\)
\[
A_n(z) = \sum_{i=1}^n \frac{\text{Res}(A_n, z_i)}{z - z_i} \prod_{j=1}^{k+1} \frac{z - a_j}{z_i - a_j} + \sum_{j=1}^{k+1} A_n(a_j) \prod_{l=1, l \neq j}^{k+1} \frac{z - a_j}{a_j - a_l},
\]
which reconstructs the amplitude in terms of its residues and its values at additional points \( a_j \) different from \( z_i \).

This is a generalization of formula first written in this context in \([10]\) and further discussed in \([11,12]\) where \( a_i \) are chosen to be roots of \( A_n(z) \).

The other option is to use the all-line shift, i.e. deforming all external momenta. This was inspired by the work by Risager \([12]\) and recently used for studying the on-shell constructibility of generic renormalizable theories in \([8]\). This approach will be useful for our purpose.

**SEMI-ON-SHELL AMPLITUDES**

The Lagrangian of the \( SU(N) \) non-linear sigma model can be written as
\[
\mathcal{L} = F^2 4 \text{Tr}(\partial_a U \partial^a U^\dagger),
\]
where \( F \) is a constant and \( U \in SU(N) \). In the most common exponential parametrization \( U = \exp(i \phi/F) \) where \( \phi = \sqrt{2} \phi^a t^a \). The \( t^a \)'s are generators of \( SU(N) \) Lie algebra normalized according to \( \text{Tr}(t^a t^b) = \delta^{ab} \). Note that for \( N = 2 \), \([10]\) is a leading \( O(p^2) \) term in the Lagrangian for the Chiral Perturbation Theory \([13]\), which provides a systematic effective field theory description for low energy QCD with two massless quarks. In this case \( \phi^a \) represent the pion triplet.

For calculations of on-shell scattering amplitudes within this model we use stripped amplitudes \( A_n(p_1, \ldots p_n) \). The Lagrangian \([10]\) contains only terms with the even number of \( \phi \), therefore \( A_{2n+1} = 0 \) and only \( A_{2n} \) are non-vanishing. It is easy to show that it makes no difference whether we use \( SU(N) \) or \( U(N) \) symmetry group because the \( U(1) \) piece decouples \([8]\). For our purpose it is convenient to use Cayley parametrization of \( U(N) \) non-linear sigma model,
\[
U = 1 + i \frac{\phi}{F} 1 - \frac{i}{2F} \phi = 1 + 2 \sum_{n=1}^\infty \left( i \frac{F}{2F} \phi \right)^n.
\]
Plugging for \( U \) into \([10]\) we get an infinite tower of terms with two derivatives and an arbitrary number of \( \phi \). This is common for any parametrization, however, in this parametrization, the stripped Feynman rule for the interaction vertex is particularly simple,
\[
V_{2n+1} = 0, \quad V_{2n+2} = \left( \frac{-1}{2F^2} \right)^n \left( \sum_{i=0}^n p_{2i+1} \right)^2.
\]
It is easy to see that the shifted amplitudes \( A_n(z) \approx z \) for \( z \to \infty \). Without additional information on the values at two points \( a_i \) the relation \([9]\) cannot be used. Therefore, we will follow different strategy to determine \( A_n(z) \) recursively.

Let us define a semi-on-shell current
\[
J^{a_1, a_2, \ldots, a_n}_{\sigma} (p_1, \ldots p_n) = \langle 0 | \phi^{a_1}(0) | \pi^{a_1}(p_1) \ldots \pi^{a_n}(p_n) \rangle
\]
as a matrix element of the field \( \phi^{a_1}(0) \) between vacuum and the \( n \)-particle state \( |\pi^{a_1}(p_1) \ldots \pi^{a_n}(p_n)\rangle \). The momentum \( p_{n+1} \) attached to off-shell satisfying \( p_{n+1} = - \sum_{j=1}^n p_j = -P_{n+1} \). At the tree-level the current can be written as a sum of stripped currents \( J_n(p_{\sigma(1)} \ldots p_{\sigma(n)}) \) as
\[
J^{a_1, a_2, \ldots, a_n}_{\sigma} (p_1, \ldots p_n) = \sum_{\sigma \in S_n} \text{Tr}(t^{a_1} t^{a_2} \ldots t^{a_n}) J_n(p_{\sigma(1)} \ldots p_{\sigma(n)}).
\]
The on-shell amplitude \( A_{n+1}(p_1, \ldots, p_{n+1}) \) can be extracted from \( J_n(p_1, \ldots, p_n) \) by means of the LSZ formulas

\[
A_{n+1}(p_1, \ldots, p_{n+1}) = -\lim_{p_{n+1}^2 \to 0} p_{n+1}^2 J_n(p_1, \ldots, p_n). \tag{15}
\]

The one particle states are normalized according to \( J_1(p) = 1 \). Note that \( J_{2n} = 0 \) in agreement with \( A(p_1, \ldots, p_{2n+1}) = 0 \) via \( \sum \). For currents \( J(1, \ldots, n) = J_n(p_1, \ldots, p_n) \) we can write generalized Berends-Giele recursion relations \( \sum \) (n.b. \( P_{ab} = \sum_{k=0}^{b} P_k \)),

\[
J(1, \ldots, n) = \frac{i}{p_{n+1}^2} \sum_{m=3}^{n} \sum_{j_0<j_1<\cdots<j_m} iV_{m+1}(P_{j_0j_1}, \ldots, -P_{n})
\times \prod_{k=0}^{m-1} J(j_k+1, \ldots, j_{k+1}), \tag{16}
\]

where \( j_0 = 0 \) and \( j_m = n \). This equation can be equivalently graphically represented as

The right hand side is a sum of products of lower point currents with Feynman vertices \( \sum \). The current \( J_n \) is obviously a homogeneous function of momenta of degree 0. It is not cyclic because there is a special off-shell momentum \( p_{n+1} \). Note, however, \( J_n \) is unphysical object and can be different in different parametrizations. From now on we will use only Cayley parametrization where it has interesting properties under the re-scaling of all even or all odd on-shell momenta. Namely for \( t \to 0 \):

\[
J_{2n+1}(tp_1, tp_2, tp_3, \ldots, tp_{2n}, tp_{2n+1}) = O(t^2), \tag{17}
\]

\[
J_{2n+1}(p_1, tp_2, tp_3, \ldots, tp_{2n}, tp_{2n+1}) \to \frac{1}{(2F^2)^n}. \tag{18}
\]

We postpone the detailed discussion to \( \sum \). The proof is by induction using Berends-Giele recursion relations \( \sum \) which are more suitable for this purpose than the analysis of Feynman diagrams used to show scaling properties of Yang-Mills theory and gravity in \( \sum \).

### NEW RECURRENCE RELATIONS

The scaling properties \( \sum \) and \( \sum \) are our guide for finding recursion relations for \( J_{2n+1} \). Let us define the complex deformation of the current \( J_{2n+1}(z) \):

\[
J_{2n+1}(z) = J_{2n+1}(p_1,zp_2,\ldots,zp_{2n},p_{2n+1}), \tag{19}
\]

i.e. the momenta are shifted according to

\[
p_{2k}(z) = zp_k, \quad p_{2k+1}(z) = p_{2k+1}. \tag{20}
\]

Note that the momentum conservation is hold because the off-shell momentum \( p_{2n+2} = -\sum_{k=1}^{n} p_k \) becomes also shifted. In the limit \( z \to 0 \) using \( \sum \) we get

\[
\lim_{z \to 0} J_{2n+1}(z) = \frac{1}{(2F^2)^n}. \tag{21}
\]

On the other hand for \( z \to \infty \) we get as a consequence of homogeneity and \( \sum \) the current \( J_{2n+1}(z) \) vanishes like

\[
J_{2n+1}(z) = O\left(\frac{1}{z^2}\right), \tag{22}
\]

and we can use the standard BCFW recursion relations to reconstruct it from its poles. The singularities of the physical current \( J_{2n+1}(1) \) are determined by condition \( P_{ij}^2 = 0 \) which implies the following condition for the poles of \( J_{2n+1}(z) \)

\[
P_{ij}^2(z) = (zp_{ij} + q_{ij})^2 = 0, \tag{23}
\]

where \( j-i \) is even and we have decomposed \( P_{ij} = p_{ij} + q_{ij} \) where \( p_{ij} \) and \( q_{ij} \) is the sum of even and odd momenta respectively between \( i \) and \( j \),

\[
p_{ij} = \sum_{i \leq k \leq j} p_k, \quad q_{ij} = \sum_{i \leq k+1 \leq j} p_{k+2}. \tag{24}
\]

For \( j-i > 2 \) we find two solutions of \( \sum \), namely

\[
z_{ij}^+ = \frac{-\lambda (p_{ij} \cdot q_{ij} \pm \sqrt{(p_{ij} \cdot q_{ij})^2 - P_{ij}^2 q_{ij}^2})}{P_{ij}}. \tag{25}
\]

For the special case of three-particle pole, \( j-i = 2 \), either \( q_{ij} = 0 \) or \( P_{ij} = 0 \). For the first case \( z_{ij}^+ = 0 \) and the corresponding residue does vanish, \( \text{Res}(J_{2n+1, z_{ij}^+}) = 0 \), while \( z_{ij}^- = -2(p_{ij} \cdot q_{ij})/P_{ij}^2 \). In the second case there is only one solution of \( \sum \),

\[
z_{ij} = -a_{ij}^2/2(p_{ij} \cdot q_{ij}). \tag{26}
\]

Let us denote a generic solution of \( \sum \) by \( z_{ij} \). Then the internal momentum \( P_{ij}(z_{ij}) \) is on-shell, therefore the current \( J_{2n+1}(z_{ij}) \) factorizes into the product of lower-point semi-on-shell current \( J_{m_1} \) and the on-shell amplitude \( M_{m_2} \). Residues at the poles \( z_{ij}^+ \) are given by

\[
\text{Res}(J_{2n+1, z_{ij}^+}) = \mp [P_{ij}^2(z_{ij}^+ - z_{ij}^-)]^{-1} M_{ij}(z_{ij}^+) \times J_{2n-j+i+1}(p_{ij}, z_{ij}^+, \ldots, P_{ij}^2(z_{ij}^+), \ldots, p_{2n+1}(z_{ij}^+)) \tag{27}
\]

or graphically by
In this formula $M_{ij}(z) = P_{ij}^2(z)J_{i-1}(p_i(z), \ldots p_j(z))$. In the case of single solution $z_{ij}$ the residue is given by the similar formula where $\mp [p_{ij}(z_{ij} - z_{ij})]^{-1}$ is replaced by $[2(p_{ij} \cdot q_{ij})]^{-1}$.

Because of (22) we can write

$$J_{2n+1}(z) = \sum_{zp} \frac{\text{Res}(J_{2n+1}, zp)}{z - zp}. \quad (27)$$

The residues $\text{Res}(J_{2n+1}, zp)$ can be determined recursively from (26) as in the case of BCFW recursion relations. However, there is one difficulty. In the boundary case $i = 1, j = 2n + 1$ the equation (26) for residue $\text{Res}(J_{2n+1}, z_{1,2n+1})$ contains a current $J_{2n+1}$ on the right hand side and therefore we can not express it using lower point currents. The solution to this problem is to use two extra relations. The first is the residue theorem: because of the asymptotic behavior (22) the residue at infinity vanishes and the sum of all residues is zero,

$$\sum_{zp} \text{Res}(J_{2n+1}, zp) = 0, \quad (28)$$

while the second one is the scaling property (21) for $z \to 0$ together with (27)

$$\sum_{zp} \frac{\text{Res}(J_{2n+1}, zp)}{zp} = -\frac{1}{(2F^2)^n}. \quad (29)$$

Denoting $z_{\pm} = z_{2n+1}^\pm$ and solving for $\text{Res}(J_{2n+1}, z_{\pm})$ from (28) and (29) in terms of all other residues we can rewrite (27) in the form

$$J_{2n+1}(z) = \frac{P_{1,2n+1}^2(z)}{P_{1,2n+1}(z)^2} \frac{1}{(2F^2)^n} \left[ \sum_{zp} z_{zp} z_{zp} \frac{\text{Res}(J_{2n+1}, zp)}{zp} \quad (30) \right.$$

$$+ \frac{z_{zp} z_{zp}}{(z - z_{zp})(z - z_{zp})} \frac{\text{Res}(J_{2n+1}, zp)}{zp} \right] - z \frac{\text{Res}(J_{2n+1}, zp)}{(z - z_{zp})(z - z_{zp})} + \frac{\text{Res}(J_{2n+1}, zp)}{zp},$$

where the sum is over all solutions of (23) with the exception of $z_{\pm}$. The residues on the right-hand side depend only on lower point currents via (26). The physical case is $z = 1$ and the on-shell amplitude $A_n(p_1, \ldots p_n)$ can be obtained from $J_n(1)$ using the limit (15). Interestingly, even the fundamental 4pt case, i.e. the current $J_3$ is included in the equation (30) (here the sum is empty). Notice a very important difference between our recursion relations and the original Berends-Giele formula (10): we construct the amplitude recursively from the 4pt formula via BCFW while (10) uses critically the Lagrangian and the infinite tower of terms in the expansion of (10).

Detailed discussion of these results including the double soft-limit formula and the proof of Adler’s zeroes for stripped amplitudes $A_n$ will be discussed in [3].