New $p$-adic hypergeometric functions concerning with syntomic regulators

M. Asakura

Abstract

We introduce new functions, which we call the $p$-adic hypergeometric functions of logarithmic type. We show the congruence relations that are similar to Dwork’s. This implies that they are convergent functions, so that the special values at $t = \alpha$ with $|\alpha|_p = 1$ are defined under a mild condition. We then show that the special values appear in the syntomic regulators for hypergeometric curves. We expect that they agree with the special values of $p$-adic $L$-functions of elliptic curves in some cases.

1 Introduction

Let $s \geq 1$ be an integer. For a $s$-tuple $\underline{a} = (a_1, \ldots, a_s) \in \mathbb{Z}_p^s$ of $p$-adic integers, let

$$F_{\underline{a}}(t) = sF_{s-1} \left( \begin{array}{c} a_1, \ldots, a_s \\ 1, \ldots, 1 \end{array} : t \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{n!} \cdots \frac{(a_s)_n}{n!} t^n$$

be the hypergeometric power series where $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ denotes the Pochhammer symbol. This is just a formal power series with $\mathbb{Z}_p$-coefficients, and one cannot define special values at $t = \alpha$ for $|\alpha| = 1$ (more strongly, it cannot be a convergent function in general, cf. Lemma 4.9 below). In his seminal paper [Dw], B. Dwork introduced the $p$-adic hypergeometric functions, which are defined as ratios of hypergeometric power series.

Let $\alpha'$ denote the Dwork prime, which is defined to be $(\alpha + l)/p$ where $l \in \{0, 1, \ldots, p - 1\}$ is the unique integer such that $\alpha + l \equiv 0 \pmod{p}$. Put $\underline{a}' = (a'_1, \ldots, a'_s)$. Then Dwork’s $p$-adic hypergeometric function is defined to be

$$\mathcal{F}_{Dw}(\underline{a}) = F_{\underline{a}}(t)/F_{\underline{a}'}(t^p).$$

This is a convergent function in the sense of Krasner. More precisely Dwork proved the congruence relations

$$\mathcal{F}_{Dw}^{<p^n}(\alpha) \equiv \frac{F_{\underline{a}}^{<p^n}(t)}{[F_{\underline{a}'}^{<p^n}(t)]^{<p^n}} \pmod{p^n\mathbb{Z}_p[[t]]}$$

where for a power series $f(t) = \sum c_n t^n$, we write $f(t)^{<m} := \sum_{n<m} c_n t^n$ the truncated polynomial (warning: $[F(t^p)]^{<p^n}$ is not the substitution of $t$ with $t^p$ in $F(t)^{<p^n}$).
In this paper, we introduce new $p$-adic hypergeometric functions, which we call the $p$-adic hypergeometric functions of logarithmic type. Let $W = W(\mathbb{F}_p)$ be the Witt ring of $\mathbb{F}_p$. Let $\sigma$ be a $p$-th Frobenius on $W[[t]]$ given by $\sigma(t) = ct^p$ with $c \in 1 + pW$. Then our new functions are defined to be power series

$$F^\sigma_{\mathfrak{a}}(t) := \frac{1}{F_{\mathfrak{a}}(t)} \left[ \psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p - p^{-1} \log(c) + \int_0^t (F_{\mathfrak{a}}(t) - F_{\mathfrak{a}}(t^p)) \frac{dt}{t} \right]$$

where $\psi_p(z)$ is the Iwasawa logarithmic function and $\psi_p(z)$ is the $p$-adic digamma function defined in §2.2 below. Notice that $F^\sigma_{\mathfrak{a}}(t)$ is also $p$-adically continuous with respect to $\mathfrak{a}$. In case $a_1 = \cdots = a_s = c = 1$, one has $F^\sigma_{\mathfrak{a}}(t) = (1 - t) \ln_1^{(p)}(t)$ the $p$-adic logarithm. In this way, we can regard $F^\sigma_{\mathfrak{a}}(t)$ as a deformation of the $p$-adic logarithm.

There are congruence relations for $F^\sigma_{\mathfrak{a}}(t)$ that are similar to Dwork’s. Let us write $F^\sigma_{\mathfrak{a}}(t) = G_{\mathfrak{a}}(t)/F_{\mathfrak{a}}(t)$. Then our congruence relations are the following

$$F^\sigma_{\mathfrak{a}}(t) \equiv G_{\mathfrak{a}}(t) \pmod{p^n W[[t]]}.$$

Thanks to this, $F^\sigma_{\mathfrak{a}}(t)$ is a convergent function, and the special value at $t = \alpha$ is defined for $|\alpha| \leq 1$ such that $F_{\mathfrak{a}}(\alpha) \pmod{p^n} \neq 0$ mod $p$ for all $n$.

Dwork showed a geometric aspect of his $p$-adic hypergeometric functions by his unit root formula. Namely, for a smooth ordinary elliptic curve $y^2 = x(1-x)(1-\alpha x)$ over $\mathbb{F}_p$, he proved that the unit root $\epsilon_p$ (i.e. the Frobenius eigenvalue such that $|\epsilon_p| = 1$) agrees with the special value of his $p$-adic hypergeometric function,

$$\epsilon_p = (-1)^{\frac{p-1}{2}} \mathcal{F}_{\frac{1}{2}, \frac{1}{2}}^{\text{Dw}}(\overset{\circ}{\alpha})$$

where $\overset{\circ}{\alpha} \in \mathbb{Z}_p^\times$ is the Teichmüller lift of $\alpha \in \mathbb{F}_p^\times$. We give a geometric aspect of our $F^\sigma_{\mathfrak{a}}(t)$, which concerns with the syntomic regulator map. Let $\alpha \in W$ satisfy that $\alpha \neq 0, 1 \text{ mod } p$. Let $X_\alpha$ be the hypergeometric curve $X_\alpha : y^N = x^A(1-x)^B(1-(1-\alpha)x)^{N-B}$, and

$$\text{reg}_{\text{syn}} : K_2(X_\alpha) \rightarrow H^2_{\text{syn}}(X_\alpha, \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(X_\alpha/K), \quad K := \text{Frac}W(\mathbb{F}_p)$$

the syntomic regulator map from Quillen’s $K_2$. Then for a certain $K_2$-symbol $\xi$, we shall show the following (see Theorem [4.12] for the notation)

$$\langle \text{reg}_{\text{syn}}(\xi|_{X_\alpha}), e_{\text{unit}}^{(\cdot)} \rangle = \frac{\zeta^1_n - \zeta^1_n}{N} \mathcal{F}_{\alpha, \text{un}}^{(\sigma_\alpha)}(\alpha) \langle \omega_n, e_{\text{unit}}^{(\cdot)} \rangle.$$

Similar results hold for other curves (see [4.6, 4.7 and 4.8]. In case $(N, A, B) = (2, 1, 1)$, the curve $X_\alpha$ is an elliptic curve. One can expect the $p$-adic counterpart of the Rogers-Zudilin type formula in view of the $p$-adic Beilinson conjecture by Perrin-Riou [P, 4.2.2] (see also [CG, Conj.2.7]). For example, we conjecture

$$(1 - p\epsilon_p^{-1}) \mathcal{F}_{\frac{1}{2}, \frac{1}{2}}^{(\sigma_\alpha)}(\alpha) \sim_{Q^n} L_p(X_\alpha, \omega^{-1}, 0).$$
if $\alpha = -1, \pm 2, \pm 4, \pm 8, \pm 16, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{16}$ where $x \sim_{Q^\times} y$ means $x = ay$ for some $a \in Q^\times$. See Conjecture 4.28 for the detail. As long as the author knows, this is the first formulation toward the $p$-adic Rogers-Zudilin formula.

This paper is organized as follows. §2 is the preliminary section on Diamond’s $p$-adic polygamma functions. More precisely we shall give a slight modification of Diamond’s polygamma (though it might be known to the experts). We give a self-contained exposition, because the author does not find a suitable reference, especially concerning with our modified functions. In §3 we introduce the $p$-adic hypergeometric functions of logarithmic type, and prove the congruence relations. In §4 we show that our new $p$-adic hypergeometric functions appear in the syntomic regulators of the hypergeometric curves. A number of conjectures on $p$-adic Rogers-Zudilin formula are provided in §4.9.

Acknowledgement. The origin of this work is the discussion with Professor Masataka Chida about the paper [B] by Brunault. We tried to understand it from the viewpoint of [A] or [AM]. We computed a number of examples with the aid of computer, and finally arrived at the definition of $F(\sigma)(a(t))$. We should say, the half of the credit belong to him.

Notation. Throughout this paper, we write by $\mu_n(K)$ the group of $n$-th roots of unity in a field $K$. We write $\mu_\infty(K) = \cup_{n \geq 1} \mu_n(K)$. If there is no fear of confusion, we drop “$K$” and simply write $\mu_n$.

2. $p$-adic polygamma functions

The complex analytic polygamma functions are the $r$-th derivative

$$\psi^{(r)}(z) := \frac{d^r}{dz^r} \left( \frac{\Gamma'(z)}{\Gamma(z)} \right), \quad r \in \mathbb{Z}_{\geq 0}. $$

In his paper [D], Jack Diamond gave a $p$-adic counterpart of the polygamma functions $\psi^{(r)}_{D,p}(z)$ which are given in the following way.

$$\psi^{(0)}_{D,p}(z) = \lim_{s \to \infty} \frac{1}{p^s} \sum_{n=0}^{p^s-1} \log(z + n), \quad (2.1)$$

$$\psi^{(r)}_{D,p}(z) = (-1)^{r+1} r! \lim_{s \to \infty} \frac{1}{p^s} \sum_{n=0}^{p^s-1} \frac{1}{(z + n)^r}, \quad r \geq 1, \quad (2.2)$$

where $\log(z)$ is the Iwasawa logarithmic function which is characterized as a continuous function on $\mathbb{C}_p^\times$ such that $\log(z_1 z_2) = \log(z_1) + \log(z_2)$, $\log(z) = 0$ if $z \in \mu_\infty$ or $z = p$ and $\log(z) = -\sum_{n=1}^{\infty} \frac{(1 - z)^n}{n}, \quad |z - 1| < 1$.

It should be noticed that the series $(2.1)$ and $(2.2)$ converge only when $z \notin \mathbb{Z}_p$, and hence $\psi^{(r)}_{D,p}(z)$ turn out to be locally analytic functions on $\mathbb{C}_p \setminus \mathbb{Z}_p$. This causes inconvenience in
our discussion. In this section we give a continuous function \( \psi_p^{(r)}(z) \) on \( \mathbb{Z}_p \) which is a slight modification of \( \psi_{D,p}(z) \). See \[2.2\] for the definition and also \[2.4\] for alternative definition in terms of \( p \)-adic measure.

2.1 \( p \)-adic polylogarithmic functions

Let \( x \) be an indeterminate. For an integer \( r \in \mathbb{Z} \), the \( r \)-th \( p \)-adic polylogarithmic function \( \ln_p^{(r)}(x) \) is defined as a formal power series

\[
\ln_p^{(r)}(x) := \sum_{k \geq 1, p \nmid k} \frac{x^k}{k^r} = \lim_{s \to \infty} \left( \frac{1}{1 - x^{sp}} \sum_{1 \leq k < p^r, p \nmid k} \frac{x^k}{k^r} \right) \in \mathbb{Z}_p[[x]]
\]

which belongs to the ring

\[
\mathbb{Z}_p \left[ x, \frac{1}{1 - x} \right] := \lim_{s \to \infty} \left( \mathbb{Z}/p^s\mathbb{Z} \left[ x, \frac{1}{1 - x} \right] \right)
\]

of convergent power series. If \( r \leq 0 \), this is a rational function, more precisely

\[
\ln_0^{(p)}(x) = \frac{1}{1 - x} - \frac{1}{1 - x^p}, \quad \ln_1^{(p)}(x) = \left( x \frac{d}{dx} \right) \ln_0^{(p)}(x).
\]

If \( r > 0 \), this is known to be an overconvergent function, more precisely it has a (unique) analytic continuation to the domain \( |x - 1| > |1 - \zeta_p| \) where \( \zeta_p \in \overline{\mathbb{Q}}_p \) is a primitive \( p \)-th root of unity (e.g. [AM 2.2]).

Let \( W(\mathbb{F}_p) \) be the Witt ring of \( \mathbb{F}_p \) and \( F \) the \( p \)-th Frobenius endomorphism. Define the \( p \)-adic logarithmic function

\[
\log_p^{(p)}(z) := \frac{1}{p} \log \left( \frac{z^p}{F(z)} \right) := -\sum_{n=1}^{\infty} \frac{p-1}{n} \left( 1 - \frac{z^p}{F(z)} \right)^n
\]

on \( W(\mathbb{F}_p)^\times \). This is different from the Iwasawa \( \log(z) \) in general, but one can show \( \log_p^{(p)}(1 - z) = -\ln_1^{(p)}(z) \) for \( z \in W(\mathbb{F}_p)^\times \) such that \( F(z) = z^p \) and \( z \not\equiv 1 \mod p \).

Proposition 2.1 (cf. \[C\] IV Prop.6.1, 6.2) Let \( r \in \mathbb{Z} \) be an integer. Then

\[
\ln_r^{(p)}(x) = x \frac{d}{dx} \ln_{r+1}^{(p)}(x), \quad (2.3)
\]

\[
\ln_r^{(p)}(x) = (-1)^{r+1} \ln_r^{(p)}(x^{-1}), \quad (2.4)
\]

\[
\sum_{\zeta \in \mu_N} \ln_r^{(p)}(\zeta x) = \frac{1}{N^{r-1}} \ln_r^{(p)}(x^N) \quad \text{(distribution formula).} \quad (2.5)
\]
Proof. (2.3) and (2.5) are immediate from the power series expansion \( \ln_p(x) = \sum_{k \geq 1, p \nmid k} x^k / k^r \).

On the other hand (2.4) follows from the fact

\[
\frac{1}{1 - x^{p^s}} \sum_{1 \leq k < p^s, p \nmid k} x^{-k} k^r = \frac{-1}{1 - x^{p^s}} \sum_{1 \leq k < p^s, p \nmid k} x^{p^s - k} k^r = (1 - x^{p^s}) \sum_{1 \leq k < p^s, p \nmid k} x^{k^r} (p^s - k)^r
\]

modulo \( p^s \mathbb{Z}[x, (1 - x)^{-1}] \). □

Lemma 2.2 Let \( m, N \geq 2 \) be integers prime to \( p \). Let \( \varepsilon \in \mu_m \setminus \{1\} \). Then for any \( n \in \{0, 1, \ldots, N - 1\} \), we have

\[
N^r \sum_{\nu N = \varepsilon} \nu^{-n} \ln_{r+1}^{(p)}(\nu) = \lim_{s \to \infty} \frac{1}{1 - \varepsilon^{p^s}} \sum_{0 \leq k < p^s, k + n/N \not\equiv 0 \mod p} \frac{\varepsilon^k}{(k + n/N)^{r+1}}.
\]

Proof. Note \( \sum_{\nu N = \varepsilon} \nu^i = N\varepsilon^{i/N} \) if \( N \mid i \) and \( = 0 \) otherwise. We have

\[
N^r \sum_{\nu N = \varepsilon} \nu^{-n} \ln_{r+1}^{(p)}(\nu x) = N^r \sum_{k \geq 1, p \nmid k} \sum_{\nu N = \varepsilon} \frac{\nu^{k-n} x^k}{k^{r+1}} = N^{r+1} \sum_{N \mid (k-n), p \nmid k} \frac{\varepsilon(k-n)/N x^k}{k^{r+1}} = \sum_{k+n/N \not\equiv 0 \mod p, k \geq 0} \frac{(\varepsilon x)^k}{(k + n/N)^{r+1}} = \frac{1}{1 - (\varepsilon x)^p^s} \sum_{0 \leq k < p^s, k+n/N \not\equiv 0 \mod p} \frac{(\varepsilon x)^k}{(k + n/N)^{r+1}}
\]

modulo \( p^s \mathbb{Z}[x, (1 - \varepsilon x^N)^{-1}, (1 - \varepsilon x)^{-1}] \). Since \( \varepsilon \neq 1 \), the evaluation at \( z = 1 \) makes sense, and then we have the desired equation. □

Lemma 2.3 Let \( r \neq 1 \) be an integer. Then

\[
L_N := \frac{N^{r-1}}{1 - N^{r-1}} \sum_{\varepsilon \in \mu_N \setminus \{1\}} \ln^{(p)}_r(\varepsilon)
\]

does not depend on an integer \( N \geq 2 \) prime to \( p \). We define \( \zeta_p(r) := L_N \). Note \( \zeta_p(r) = 0 \) if \( r \) is an even integer.

\footnote{This agrees with the special value of the \( p \)-adic zeta function \( \zeta_p(s) \) (\cite{Li}, (3))}.
Proof. Set \( S_N := \sum_{\varepsilon \in \mu N \setminus \{1\} \ln_r(p)} \). Let \( N_1, N_2 \geq 2 \) be integers prime to \( p \).

\[
S_{N_1 N_2} = \sum_{\nu \in \mu N_1 \setminus \{1\} \ln_r(p)} + \sum_{\nu \in \mu N_2 \setminus \{1\} \ln_r(p)}
\]

\[
= S_{N_1} + \sum_{\varepsilon \in \mu N_2 \setminus \{1\}} \frac{1}{N_1 - 1} \ln_r(p) \quad \text{(distribution (2.5))}
\]

Reversing \( N_1 \) and \( N_2 \), we get

\[
S_{N_1} + \frac{1}{N_1 - 1} S_{N_2} = S_{N_2} + \frac{1}{N_2 - 1} S_{N_1} \iff \frac{N_1^{r-1}}{1 - N_1^{r-1}} S_{N_1} = \frac{N_2^{r-1}}{1 - N_2^{r-1}} S_{N_2}
\]
as required. □

2.2 \( p \)-adic polygamma functions

Let \( r \in \mathbb{Z} \) be an integer. For \( z \in \mathbb{Z}_p \), define

\[
\tilde{\psi}_p^{(r)}(z) := \lim_{n \to \infty} \sum_{1 \leq k < n, p \nmid k} \frac{1}{k^{r+1}}
\]  

(2.6)

The existence of the limit follows from the fact that

\[
\sum_{1 \leq k < p^s, p \nmid k} k^m = \begin{cases} 
-1^{p-1} & p \geq 3 \text{ and } (p-1)|m \\
2^{r-1} & p = 2 \text{ and } 2|m \\
1 & p = 2 \text{ and } s = 1 \\
0 & \text{otherwise}
\end{cases}
\]  

(2.7)

modulo \( p^s \). Thus \( \tilde{\psi}_p^{(r)}(z) \) is a \( p \)-adic continuous function on \( \mathbb{Z}_p \). More precisely

\[
z \equiv z' \mod p^s \implies \tilde{\psi}_p^{(r)}(z) - \tilde{\psi}_p^{(r)}(z') \equiv \begin{cases} 
0 \mod p^s & p \geq 3 \text{ and } (p-1)|(r+1) \\
0 \mod p^s & p = 2, s \geq 2 \text{ and } 2|(r+1) \\
0 \mod p^{s-1} & \text{otherwise}
\end{cases}
\]  

(2.8)

Define the \textit{\( p \)-adic Euler constant} by

\[
\gamma_p := -\lim_{s \to \infty} \frac{1}{p^s} \sum_{0 \leq j < p^s, p \nmid j} \log(j), \quad (\log = \text{Iwasawa log}).
\]

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This is different from Diamond’s \( p \)-adic Euler constant. His constant is \( p/(p-1) \gamma_p \). [1] §7.
We define the $r$-th $p$-adic polygamma function to be

$$
\psi_p^{(r)}(z) := \begin{cases} 
-\gamma_p + \bar{\psi}_p^{(0)}(z) & r = 0 \\
-\zeta_p(r + 1) + \bar{\psi}_p^{(r)}(z) & r \neq 0
\end{cases}
$$

(2.9)

where $\zeta_p(r+1)$ is the constant defined in Lemma 2.3. If $r = 0$, we also write $\psi_p(z) = \psi_p^{(0)}(z)$ and call it the $p$-adic digamma function.

### 2.3 Formulas on $p$-adic polygamma functions

**Theorem 2.4**

1. $\bar{\psi}_p^{(r)}(0) = \bar{\psi}_p^{(r)}(1) = 0$ or equivalently $\psi_p^{(r)}(0) = \psi_p^{(r)}(1) = -\gamma_p$ or $= -\zeta_p(r + 1)$.

2. $\bar{\psi}_p^{(r)}(z) = (-1)^r \bar{\psi}_p^{(r)}(1 - z)$ or equivalently $\psi_p^{(r)}(z) = (-1)^r \psi_p^{(r)}(1 - z)$ (note $\zeta_p(r + 1) = 0$ for odd $r$).

3. $\bar{\psi}_p^{(r)}(z + 1) - \bar{\psi}_p^{(r)}(z) = \psi_p^{(r)}(z + 1) - \psi_p^{(r)}(z) = \begin{cases} 
z^{-r-1} & z \in \mathbb{Z}_p^\times \\
0 & z \in p\mathbb{Z}_p.
\end{cases}$

Compare the above with [NIST] p.144, 5.15.2, 5.15.5 and 5.15.6.

**Proof.** (1) and (3) are immediate from definition on noting (2.7). We show (2). Since $\mathbb{Z}_{>0}$ is a dense subset in $\mathbb{Z}_p$, it is enough to show in case $z = n > 0$ an integer. Let $s > 0$ be arbitrary such that $p^s > n$. Then

$$
\begin{align*}
\bar{\psi}_p^{(r)}(n) &\equiv \sum_{1 \leq k < n, p \nmid k} \frac{1}{kr+1} \equiv (-1)^{r+1} \sum_{-n < k \leq -1, p \nmid k} \frac{1}{kr+1} \equiv (-1)^{r+1} \sum_{p^s-n+1 < k \leq p^s, p \nmid k} \frac{1}{kr+1} \\
&\equiv (-1)^{r+1} \sum_{0 \leq k < p^s, p \nmid k} \frac{1}{kr+1} - (-1)^{r+1} \sum_{0 \leq k < p^s-n+1, p \nmid k} \frac{1}{kr+1} \\
&\equiv (-1)^r \sum_{0 \leq k < p^s-n+1, p \nmid k} \frac{1}{kr+1} \\
&\equiv (-1)^r \bar{\psi}_p^{(r)}(1 - n)
\end{align*}
$$

modulo $p^s$ or $p^{s-1}$. Since $s$ is an arbitrary large integer, this means $\bar{\psi}_p^{(r)}(n) = (-1)^r \bar{\psi}_p^{(r)}(1 - n)$ as required. 

**Theorem 2.5** Let $0 \leq n < N$ be integers and suppose $p \nmid N$. Then

$$
\bar{\psi}_p^{(r)} \left( \frac{n}{N} \right) = N^r \sum_{\varepsilon \in \mathbb{Z}_N \setminus \{1\}} (1 - \varepsilon^{-n}) \ln_{r+1}^{(p)}(\varepsilon).
$$

(2.10)
For example
\[ \psi_p^{(r)} \left( \frac{1}{2} \right) = -\zeta_p(r + 1) + 2^{r+1} \ln_{r+1}(p)(-1) = (1 - 2^{r+1})\zeta_p(r + 1). \]

Compare this with [NIST] p.144, 5.15.3.

**Proof.** We may assume \( n > 0 \). Let \( s > 0 \) be an integer such that \( p^s \equiv 1 \mod N \). Write \( p^s - 1 = lN \).

\[
S := \sum_{\varepsilon \in \mu \setminus \{1\}} \left( 1 - \varepsilon^{-n} \right) \ln_{r+1}(\varepsilon) \equiv \sum_{1 \leq k < p^s, \, p \nmid k} \left( \frac{1 - \varepsilon^{-n}}{1 - \varepsilon^k} \right) \varepsilon^k \equiv \sum_{1 \leq k < p^s, \, p \nmid k} \left( \frac{\varepsilon^k + \cdots + \varepsilon^{k+N-n-1}}{k^{r+1}} \right)
\]

modulo \( p^s \). Note \( \sum_{\varepsilon \in \mu \setminus \{1\}} \varepsilon^i = N - 1 \) if \( N \mid i \) and \( = -1 \) otherwise. By (2.7), we have

\[
S \equiv \sum_k \frac{N}{k^{r+1}} \mod p^{s-1}
\]

where \( k \) runs over the integers such that \( 0 \leq k < p^s, \, p \nmid k \) and there is an integer \( 0 \leq i < N - n \) such that \( k + i \equiv 0 \mod N \). Hence

\[
N^r S \equiv \sum_{k \equiv 0 \mod N} \frac{1}{(k/N)^{r+1}} + \sum_{k \equiv -1 \mod N} \frac{1}{(j - 1/N)^{r+1}} + \cdots + \sum_{k \equiv -N+1 \mod N} \frac{1}{(j - (N - n - 1)/N)^{r+1}}
\]

\[
\equiv \sum_{1 \leq j \leq l} \frac{1}{j^{r+1}} + \sum_{1 \leq j \leq l} \frac{1}{(j + l)^{r+1}} + \cdots + \sum_{1 \leq j \leq l} \frac{1}{(j + l(N - n - 1))^{r+1}}
\]

\[
= \sum_{1 \leq j \leq l} \frac{1}{j^{r+1}} = \sum_{j \equiv 0 \mod p} \frac{1}{j^{r+1}}.
\]

Since \( l(N - n) + 1 \equiv n/N \mod p^s \), the last summation is equivalent to \( \tilde{\psi}^{(r)}(n/N) \) modulo \( p^{s-1} \) by definition. \( \square \)

**Remark 2.6** The complex analytic analogy of Theorem 2.5 is the following. Let \( \ln_r(z) = \)
\[ \ln_r^n(z) = \sum_{n=1}^{\infty} z^n / n^r \] be the analytic polylog. Then

\[ \begin{align*}
N^r \sum_{k=1}^{N-1} (1 - e^{-2\pi i kn/N}) \ln_{r+1}(e^{2\pi i k/N} - e^{2\pi i (m-n)/N}) &= \sum_{n=1}^{\infty} \sum_{k=1}^{N-1} \frac{N^r}{m^r+1} (e^{2\pi i km/N} - e^{2\pi i (m-n)/N}) \\
&= \sum_{k=1}^{\infty} \frac{N^{r+1}}{(kN)^{r+1}} - \frac{N^{r+1}}{(kN - N + n)^{r+1}} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{r+1}} - \frac{1}{(k - 1 + n/N)^{r+1}}.
\end{align*} \]

If \( r = 0 \), then this is equal to \( \psi(z) - \psi(1) \) ([NIST] p.139, 5.7.6). If \( r \geq 1 \), then this is equal to \( \zeta(r+1) + (-1)^r / r! \psi^{(r)}(n/N) \) ([NIST] p.144, 5.15.1).

**Theorem 2.7** Let \( m \geq 1 \) be an positive integer prime to \( p \).

1. Let \( \psi_p(z) = \psi^{(0)}_p(z) \) be the \( p \)-adic digamma function. Then

\[ \psi_p(mz) - \log^{(p)}(m) = \frac{1}{m} \sum_{i=0}^{m-1} \psi_p(z + \frac{i}{m}). \]

2. If \( r \neq 0 \), we have

\[ \psi_p^{(r)}(mz) = \frac{1}{m^{r+1}} \sum_{i=0}^{m-1} \psi_p^{(r)}(z + \frac{i}{m}). \]

Compare the above with [NIST] p.144, 5.15.7.

**Proof.** By Lemma 2.3, the assertions are equivalent to

\[ \frac{1}{m^{r+1}} \sum_{i=0}^{m-1} \ln_{r+1}^{(p)}(z + \frac{i}{m}) = \sum_{\varepsilon \in \mu_N \setminus \{1\}} \ln_{r+1}^{(p)}(\varepsilon) \quad (2.11) \]

for all \( r \in \mathbb{Z} \). Since \( \mathbb{Z}_p(p) \cap [0, 1) \) is a dense subset in \( \mathbb{Z}_p \), it is enough to show the above in case \( z = n/N \) with \( 0 \leq n < N, p \not| N \). By Theorem 2.5

\[ \frac{1}{m^{r+1}} \sum_{i=0}^{m-1} \ln_{r+1}^{(p)}(z + \frac{i}{m}) = \frac{1}{m^{r+1}} \sum_{i=0}^{m-1} \ln_{r+1}^{(p)}(nm + iN/mN) \]

\[ = \frac{N^r}{m} \sum_{i=0}^{m-1} \sum_{\nu \in \mu_N \setminus \{1\}} (1 - \nu^{-nm-iN}) \ln_{r+1}^{(p)}(\nu). \]

The last summation is divided into the following 2-terms

\[ \sum_{i=0}^{m-1} \sum_{\nu \in \mu_N \setminus \{1\}} (1 - \nu^{-nm}) \ln_{r+1}^{(p)}(\nu) = m \sum_{\nu \in \mu_N \setminus \{1\}} (1 - \nu^{-nm}) \ln_{r+1}^{(p)}(\nu), \]

9
\[
\sum_{i=0}^{m-1} \sum_{\varepsilon \in \mu_m \setminus \{1\}} \sum_{\nu^N = \varepsilon} (1 - \nu^{-nm} \varepsilon^{-i}) \ln_{r+1}^{(p)}(\nu) = m \sum_{\varepsilon \in \mu_m \setminus \{1\}} \sum_{\nu^N = \varepsilon} \ln_{r+1}^{(p)}(\nu) = \frac{m}{N^r} \sum_{\varepsilon \in \mu_m \setminus \{1\}} \ln_{r+1}^{(p)}(\varepsilon)
\]

where the last equality follows from the distribution formula (2.5). Since the former is equal to \(\tilde{\psi}_{p}^{(r)}(nm/N)\) by Theorem 2.5, the equality (2.11) follows. \(\square\)

### 2.4 \(p\)-adic measure

For a function \(g : \mathbb{Z}_p \to \mathbb{C}_p\), the Volkenborn integral is defined by

\[
\int_{\mathbb{Z}_p} g(t) dt = \lim_{s \to \infty} \frac{1}{p^s} \sum_{0 \leq j < p^s} g(j).
\]

**Theorem 2.8** Let \(\log : \mathbb{C}_p^\times \to \mathbb{C}_p\) be the Iwasawa logarithmic function. Let

\[
1_{\mathbb{Z}_p^\times}(z) := \begin{cases} 1 & z \in \mathbb{Z}_p^\times \\ 0 & z \in p\mathbb{Z}_p \end{cases}
\]

be the characteristic function. Then

\[
\psi_{p}(z) = \int_{\mathbb{Z}_p} \log(z + t) 1_{\mathbb{Z}_p^\times}(z + t) dt.
\]

**Proof.** Let \(Q(z) := \int_{\mathbb{Z}_p} 1_{\mathbb{Z}_p^\times}(z + t) \log(z + t) dt\). Then

\[
Q(z + 1) - Q(z) \equiv \begin{cases} p^{-s}(\log(z) - \log(z + p^s)) & z \in \mathbb{Z}_p^\times \\ 0 & z \in p\mathbb{Z}_p \end{cases} \mod p^s.
\]

For \(z \in \mathbb{Z}_p^\times\), since

\[
p^{-s}(\log(z) - \log(z + p^s)) = -p^{-s} \log(1 + z^{-1}p^s) \equiv z^{-1}
\]

modulo \(p^{s-1}\) (or \(p^s\) in case \(p \geq 3\)), it follows from Theorem 2.4 (3) that \(Q(z)\) differs from \(\psi_{p}(z)\) by a constant. Since

\[
Q(0) \equiv \frac{1}{p^s} \sum_{0 \leq j < p^s, p \nmid j} \log(j) \equiv -\gamma_p,
\]

we obtain \(Q(z) = \psi_{p}(z)\). \(\square\)
Theorem 2.9 If \( r \neq 0 \), then
\[
\psi_p^{(r)}(z) = -\frac{1}{r} \int_{\mathbb{Z}_p} (z + t)^{-r} 1_{\mathbb{Z}_p^\times}(z + t) dt
\]
where \( 1_{\mathbb{Z}_p^\times}(z) \) denotes the characteristic function as in Theorem 2.8.

Proof. Let \( Q(z) \) be the right hand side. Then
\[
Q(z) \equiv -\frac{1}{rp^s} \sum_{0 \leq k < p^s, p \not| k} \frac{1}{(z + k)^r} \mod p^s.
\]
If \( z \in \mathbb{Z}_p^\times \), then
\[
Q(z + 1) - Q(z) \equiv -\frac{1}{r} \left( \frac{1}{(z + p^s)^r} - \frac{1}{z^r} \right) \equiv z^{-1-r} \mod p^s,
\]
and if \( z \in p\mathbb{Z}_p \), then \( Q(z + 1) \equiv Q(z) \). This shows that \( Q(z) - \psi_p^{(r)}(z) \) is a constant by Theorem 2.4 (3). Let \( S_a(x) \) be the unique polynomial such that \( S_a(n) = \sum_{k=1}^n k^a \) for any \( n \). As is well-known (e.g. [NIST, 24.4.7]),
\[
S_a(x) = \frac{1}{a+1} \sum_{j=1}^{a+1} (-1)^{a+1-j} \binom{a+1}{j} B_{a+1-j} x^j, \quad a \in \mathbb{Z}_{\geq 0}
\]
where \( B_j \) denotes the \( j \)-th Bernoulli number (\( B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, \ldots \)). Then
\[
\frac{1}{p^s} \sum_{0 \leq k < p^s, p \not| k} \frac{1}{k^r} \equiv \frac{1}{p^s} \sum_{0 \leq k < p^s, p \not| k} k^{p^s-1(p-1)-r} = S_{p^s-1(p-1)-r}(p^s) - p^{p^s-1(p-1)-r} S_{p^s-1(p-1)-r}(p^{s-1}) \\
\equiv (-1)^r B_{p^s-1(p-1)-r} = B_{p^s-1(p-1)-r}
\]
where the last equality follows from \( B_{2k+1} = 0 \). We thus have
\[
Q(0) \equiv -\frac{B_{p^s-1(p-1)-r}}{r} \mod p^s,
\]
and hence
\[
Q(0) = -\lim_{s \to \infty} \frac{B_{p^s-1(p-1)-r}}{r} = -\zeta_p(r + 1) = \psi_p^{(r)}(0)
\]
as required. \(\square\)
3 \textit{p}-adic hypergeometric functions of logarithmic type

For an integer \( n \geq 0 \), we denote by \( (a)_n \) the Pochhammer symbol,
\[
(a)_0 := 1, \quad (a)_n := a(a + 1) \cdots (a + n - 1), \quad n \geq 1.
\]
For \( a \in \mathbb{Z}_p \), we denote by \( a' := (a + l)/p \) the Dwork prime where \( l \in \{0, 1, \ldots, p - 1\} \) is the unique integer such that \( a + l \equiv 0 \mod p \). We denote the \( i \)-th Dwork prime by \( a^{(i)} \) which is defined to be \( (a^{(i-1)})' \) with \( a^{(0)} = a \).

3.1 Definition

Let \( a_i, b_j \in \mathbb{Q}_p \) with \( b_j \not\in \mathbb{Z}_{\leq 0} \). Let
\[
sF_{s-1} \left( \frac{a_1, \ldots, a_s}{b_1, \ldots, b_{s-1}} : t \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_s)_n}{(b_1)_n \cdots (b_{s-1})_n} n^n t^n.
\]
be the hypergeometric power series with \( \mathbb{Q}_p \)-coefficients. In what follows we only consider the cases \( a_i \in \mathbb{Z}_p \) and \( b_j = 1 \), and then the above has \( \mathbb{Z}_p \)-coefficients.

Definition 3.1 (\( p \)-adic hypergeometric functions of logarithmic type) Let \( s \geq 1 \) be a positive integer. Let \( \underline{a} = (a_1, \ldots, a_s) \in \mathbb{Z}_p^s \) and \( \underline{a}' = (a_1', \ldots, a_s') \) where \( a_i' \) denotes the Dwork prime. Put
\[
F_{\underline{a}}(t) := sF_{s-1} \left( \frac{a_1, \ldots, a_s}{1, \ldots, 1} : t \right), \quad F_{\underline{a}'}(t) := sF_{s-1} \left( \frac{a_1', \ldots, a_s'}{1, \ldots, 1} : t \right).
\]
Let \( W = W(\mathbb{F}_p) \) denote the Witt ring of \( \mathbb{F}_p \). Let \( \sigma : W[[t]] \to W[[t]] \) be the \( p \)-th Frobenius endomorphism given by \( \sigma(t) = ct^p \) with \( c \in 1 + pW \), compatible with the Frobenius on \( W \). Then we define a power series
\[
\mathcal{F}_{\underline{a}}(\sigma)(t) := \frac{1}{F_{\underline{a}}(t)} \left[ \psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p - p^{-1} \log(c) + \int_0^t (F_{\underline{a}}(t) - F_{\underline{a}'}(t^\sigma)) \frac{dt}{t} \right]
\]
where \( \psi_p(z) \) is the \( p \)-adic digamma function defined in \( \S2.2 \) and \( \log(z) \) is the Iwasawa logarithmic function. We call this the \( p \)-adic hypergeometric functions of logarithmic type.

We first note that \( \mathcal{F}_{\underline{a}}(\sigma)(t) \) is a power series with \( W \)-coefficients. Indeed letting \( \mathcal{F}_{\underline{a}}(\sigma)(t) = G_{\underline{a}}(t)/F_{\underline{a}}(t) \) and \( G_{\underline{a}}(t) = \sum B_i t^i \), it is enough to see that \( B_i \in W \) for all \( i \). Let \( F_{\underline{a}}(t) = \sum A_i t^i \) and \( F_{\underline{a}'}(t) = \sum A_i' t^i \). If \( p \nmid i \), then \( B_i = A_i/i \) is obviously a \( p \)-adic integer. For \( i = mp^k \) with \( k \geq 1 \) and \( p \nmid m \), one has
\[
B_i = B_{mp^k} = A_{mp^k} - c_{mp^k-1} A_{mp^k-1}^{(1)}.
\]
Since \( c_{mp^k-1} \equiv 1 \mod p^k \), it is enough to see \( A_{mp^k} \equiv A_{mp^k-1}^{(1)} \mod p^k \). However this follows from [Dw] p.36, Cor. 1.
3.2 Congruence relations

For a power series \( f(t) = \sum_{n=0}^{\infty} A_n t^n \), we denote \( f(t)_{<m} := \sum_{n<m} A_n t^n \) the truncated polynomial.

**Theorem 3.2** Suppose that \( a_i \notin \mathbb{Z}_{\leq 0} \) for all \( i \). Let us write \( F_{\sigma}^{(a)}(t) = G_{\sigma}^{(a)}(t)/F_{\sigma}^{(a-1)}(t) \). If \( p \equiv 1 + 2W \), then for all \( n \geq 1 \)

\[
F_{\sigma}^{(a)}(t) \equiv G_{\sigma}^{(a)}(t)/F_{\sigma}^{(a-1)}(t) \mod p^n W[[t]]. \tag{3.1}
\]

If \( p = 2 \) and \( c \equiv 1 + 2W \) (not necessarily \( c \equiv 1 + 4W \)), then the above holds modulo \( p^{n-1} \).

**Corollary 3.3** Suppose that there exists an integer \( r \geq 0 \) such that \( a_i^{(r+1)} = a_i \) for all \( i \) where \((-)^{(r)}\) denotes the \( r \)-th Dwork prime. Then

\[
F_{\sigma}^{(a)}(t) \in W(t, F_{\sigma}^{(a-1)}(t)) := \lim_{n \to \infty} (W/p^n[t, F_{\sigma}^{(a-1)}(t)])
\]

is a convergent function. For \( \alpha \in W \) such that \( F_{\sigma}^{(a-1)}(\alpha) \neq 0 \mod p \) for all \( i \), the special value of \( F_{\sigma}^{(a)}(t) \) at \( t = \alpha \) is defined, and it is explicitly given by

\[
F_{\sigma}^{(a)}(\alpha) = \lim_{n \to \infty} G_{\sigma}^{(a)}(\alpha)/F_{\sigma}^{(a-1)}(\alpha).
\]

3.3 Proof of Congruence relations : Reduction to the case \( c = 1 \)

Throughout the sections 3.3, 3.4 and 3.5, we use the following notation. Fix \( s \geq 1 \) and \( a = (a_1, \ldots, a_s) \) with \( a_i \notin \mathbb{Z}_{\leq 0} \). Let \( \sigma(t) = cl^p \) be the Frobenius. Put

\[
F_{\sigma}^{(a)}(t) := \sum_{n=0}^{\infty} A_n^{(a)} t^n, \quad A_n^{(a)} := \frac{(a_1^{(a)})_n \cdots (a_s^{(a)})_n}{n!}
\]

where \( a_k^{(a)} \) denotes the \( i \)-th Dwork prime. Letting \( F_{\sigma}^{(a)}(t) = G_{\sigma}^{(a)}(t)/F_{\sigma}^{(a-1)}(t) \), we put

\[
G_{\sigma}^{(a)}(t) = \sum_{n=0}^{\infty} B_n t^n
\]

or explicitly

\[
B_0 = \psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p, \tag{3.3}
\]

\[
B_n = \frac{A_n}{n}, \quad (p \nmid n), \quad B_{mp^k} = \frac{A_{mp^k} - c_{mp^k-1} A_{mp^k-1}}{mp^k}, \quad (m, k \geq 1). \tag{3.4}
\]

**Lemma 3.4** The proof of Theorem 3.2 is reduced to the case \( \sigma(t) = l^p \) (i.e. \( c = 1 \)).
Proof. Write $f(t)_{\geq m} := f(t) - f(t)_{< m}$. Put $n^* := n$ if $c \in 1 + 2pW$ and $n^* := n - 1$ if $p = 2$ and $c \notin 1 + 4W$. Theorem 3.2 is equivalent to saying

$$F_c(t)G_c(t)_{\geq n^*} \equiv F_c(t)_{\geq n^*}G_c(t) \mod p^nW[[t]],$$

namely

$$\sum_{i+j=m} A_{i+p^n} B_j - A_{j+p^n} B_i \equiv 0 \mod p^{n^*}$$

for all $m \geq 0$. Suppose that this is true when $c = 1$, namely

$$\sum_{i+j=m} A_{i+p^n} B_j^o - A_{j+p^n} B_i^o \equiv 0 \mod p^{n^*} \quad (3.5)$$

where $B_j^o$ are the coefficients $(3.3)$ or $(3.4)$ when $c = 1$. We denote by $B_i$ the coefficients for an arbitrary $c \in 1 + pW$. We then want to show

$$\sum_{i+j=m} A_{i+p^n} (B_j^o - B_j) - A_{j+p^n} (B_i^o - B_i) \equiv 0 \mod p^{n^*}. \quad (3.6)$$

Let $c = 1 + pe$ with $e \neq 0$ (if $e = 0$, there is nothing to prove). Then

$$\sum_{i+j=m} A_{i+p^n} (B_j^o - B_j) = A_{m+p^n} p^{-1} \log(c) + \sum_{1 \leq j \leq m} p^{-1} \binom{c/j/p - 1}{j/p} A_{m+p^n-j} A_{j/p}^{(1)}$$

$$= A_{m+p^n} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} p^{-1} e^i + \sum_{1 \leq j \leq m} \binom{j/p}{i} p^{-1} e^i A_{m+p^n-j} A_{j/p}^{(1)}$$

$$= \sum_{i=1}^{\infty} \left( A_{m+p^n} \frac{(-1)^{i+1}}{i} + \sum_{1 \leq j \leq m} \binom{j/p}{i} A_{m+p^n-j} A_{j/p}^{(1)} \right) p^{-i} e^i$$

$$= \sum_{i=1}^{\infty} \left( \sum_{0 \leq j \leq m} \binom{j/p - 1}{i - 1} A_{m+p^n-j} A_{j/p}^{(1)} \right) p^{-i} e^i$$

where we mean $A_{j/p}^{(k)} = 0$ for $p \nmid j$. Similarly

$$\sum_{i+j=m} A_{j+p^n} (B_i^o - B_i) = \sum_{i=1}^{\infty} \left( \sum_{0 \leq j \leq m} \binom{m+p^n-j}{i-1} A_j A_{(m+p^n-j)/p}^{(1)} \right) p^{-1} e^i$$

Therefore it is enough to show that

$$\frac{p^{-i} e^i}{i} \sum_{0 \leq j \leq m} \binom{j/p - 1}{i - 1} A_{m+p^n-j} A_{j/p}^{(1)} \equiv \frac{p^{-i} e^i}{i} \sum_{0 \leq j \leq m} \binom{m+p^n-j}{i-1} A_j A_{(m+p^n-j)/p}^{(1)} \mod p^{n^*}.$$
equivalently
\[
\sum_{0 \leq j \leq m} (1 - j/p)_{i-1} A_{m+p^n - j}^{(1)} \equiv \sum_{0 \leq j \leq m} (1 - (m + p^n - j)/p)_{i-1} A_j A_{(m+p^n - j)/p} \mod p^{n^*-i+1} le^{-i}
\]
for all \( i \geq 1 \) and \( m \geq 0 \). Recall the Dwork congruence
\[
\frac{F(t)}{F(t)} \equiv \frac{[F(t)]_{<p^n}}{F(t)_{<p^n}} \mod p^l \mathbb{Z}_p[[t]], \quad m \geq l
\]
from [Dw, p.37, Thm. 2, p.45]. This immediately implies \((3.7)\) in case \( i = 1 \). Suppose \( i \geq 2 \). To show \((3.7)\), it is enough to show
\[
\sum_{0 \leq j \leq m} (j/p)^k A_{m+p^n - j}^{(1)} A_{j/p} \equiv \sum_{0 \leq j \leq m} ((m + p^n - j)/p)^k A_j A_{(m+p^n - j)/p} \mod p^{n^*-i+1} le^{-i}
\]
for each \( k \geq 0 \). We write \( A_j^* := j^k A_j^{(1)} \), and put \( F^*(t) := \sum_{j=0}^\infty A_j^* t^j \). Then \((3.8)\) is equivalent to saying
\[
F(t)_{<p^n} F^*(t^p) \equiv F(t) [F^*(t^p)]_{<p^n} \mod p^{n^*-i+1} le^{-i} \mathbb{Z}_p[[t]].
\]
We show \((3.9)\), which finishes the proof of Lemma 3.4. It follows from [Dw, p.45, Lem. 3.4] that we have
\[
\frac{F^*(t)}{F(t)} \equiv \frac{F^*(t)_{<p^n}}{F(t)_{<p^n}} \mod p^l \mathbb{Z}_p[[t]], \quad m \geq l.
\]
This implies
\[
\frac{F^*(t^p)}{F(t^p)} \equiv \frac{F^*(t^p)_{<p^n}}{F(t^p)_{<p^n}} \mod p^{n^*-1} \mathbb{Z}_p[[t]].
\]
Therefore we have
\[
\frac{F^*(t^p)}{F(t^p)} \equiv \frac{[F(t^p)]_{<p^n} [F^*(t^p)]_{<p^n}}{F(t^p)_{<p^n}} \equiv \frac{[F(t^p)]_{<p^n}}{F(t^p)_{<p^n}} \mod p^{n^*-1} \mathbb{Z}_p[[t]].
\]
If \( p \geq 3 \), then \( \text{ord}_p(p^{n^*-i+1} le^{-i}) = \text{ord}_p(p^{n^*-i+1} le^{-i}) \leq n - 1 \) for any \( i \geq 2 \), and hence \((3.9)\) follows. If \( p = 2 \), then \( \text{ord}_p(p^{n^*-i+1} le^{-i}) \leq n \) but not necessarily \( \text{ord}_p(p^{n^*-i+1} le^{-i}) \leq n - 1 \). If \( e \in 2W \setminus \{0\} \), then \( \text{ord}_p(p^{n^*-i+1} le^{-i}) = \text{ord}_p(p^{n^*-i+1} le^{-i}) \leq n - i < n - 1 \), and hence \((3.9)\) follows. If \( e \) is a unit, then \( \text{ord}_p(p^{n^*-i+1} le^{-i}) = \text{ord}_p(p^{n^*-i} le^{-i}) \leq n - 1 \) for any \( i \geq 2 \), as required again. This completes the proof.

3.4 Proof of Congruence relations : Preliminary lemmas

Until the end of §3.5 let \( \sigma \) be the Frobenius given by \( \sigma(t) = t^p \) (i.e. \( c = 1 \)). Therefore
\[
B_0 = \psi_p(a_1) + \cdots + \psi_p(a_s) + s \gamma_p, \quad B_i = \frac{A_i - A_{1/i}^{(1)}}{i}, \quad i \in \mathbb{Z}_{\geq 1}
\]
where \( A_i^{(k)} \) are as in \((3.2)\), and we mean \( A_{i/p}^{(k)} = 0 \) unless \( i/p \in \mathbb{Z}_{\geq 0} \).
Lemma 3.5 For an $p$-adic integer $\alpha \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 1}$, we define
\[
\{\alpha\}_n := \prod_{1 \leq i \leq n, p \not| (n+i-1)} (\alpha + i - 1),
\]
and $\{\alpha\}_0 := 1$. Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$, and $l \in \{0, 1, \ldots, p-1\}$ the integer such that $a \equiv -l \mod p$. Then for any $m \in \mathbb{Z}_{\geq 0}$, we have
\[
m \equiv 0, 1, \ldots, l \mod p \implies \frac{(a)_m}{m!} \left( \frac{(a')_{[m/p]}}{[m/p]!} \right)^{-1} = \frac{\{a\}_m}{\{1\}_m},
\]
\[
m \equiv l + 1, \ldots, p-1 \mod p \implies \frac{(a)_m}{m!} \left( \frac{(a')_{[m/p]}}{[m/p]!} \right)^{-1} = \left( l + \frac{m}{p} \right) \frac{\{a\}_m}{\{1\}_m}
\]
where $a' = a^{(1)}$ is the Dwork prime.

Proof. Straightforward. $\square$

Lemma 3.6 (Dwork) For any $m \in \mathbb{Z}_{\geq 0}$, $A_m/A_m^{(1)}$ are $p$-adic integers, and
\[
m \equiv m' \mod p^n \implies \frac{A_m}{A_m^{(1)}} \equiv \frac{A_{m'}}{A_{m'}^{(1)}} \mod p^n.
\]

Proof. This is [Dw] p.36, Cor. 1, or one can easily show this by using Lemma 3.5 on noticing the fact that $\{\alpha\}_{p^n} \equiv -1 \mod p^n$ for any $\alpha \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 1}$. $\square$

Lemma 3.7 Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$ and $m, n \in \mathbb{Z}_{\geq 1}$. Then
\[
1 - \frac{(a')_{mp^n-1}}{(mp^n-1)!} \left( \frac{(a)_{mp^n}}{(mp^n)!} \right)^{-1} \equiv mp^n(\psi_p(a) + \gamma_p) \mod p^{2n}. \tag{3.11}
\]
Moreover $A_{mp^n-1}^{(1)}/A_{mp^n}$ and $B_k/A_k$ are $p$-adic integers for all $k, m \geq 0, n \geq 1$, and
\[
\frac{A_{mp^n-1}^{(1)}}{A_{mp^n}} \equiv 1 - mp^n(\psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p) \mod p^{2n}, \tag{3.12}
\]
\[
p \not| m \implies \frac{B_{mp^n}}{A_{mp^n}} = \frac{1 - A_{mp^n-1}/A_{mp^n}}{mp^n} \equiv B_0 \mod p^n. \tag{3.13}
\]

Proof. We already see that $A_{mp^n-1}/A_{mp^n} \in \mathbb{Z}_p$ in Lemma 3.5. It is enough to show (3.11). Indeed (3.12) is immediate from (3.11), and (3.13) is immediate from (3.12). Moreover (3.12) also implies that $B_k/A_k \in \mathbb{Z}_p$ for any $k \in \mathbb{Z}_{\geq 0}$. 16
Let us show (3.11). Let \( a = -l + p^n b \) with \( l \in \{0, 1, \ldots, p^n - 1\} \). Then

\[
\frac{(a')_{mp^n-1}}{(mp^n-1)!} \left( \frac{(a')_{mp^n}}{(mp^n)!} \right)^{-1} = \frac{1}{a_{mp^n}} = \prod_{l < k < mp^n, k \equiv l \pmod{p}} \frac{k - l}{k - l + p^n b} \times \prod_{0 < k < l, k \equiv l \pmod{p}} \frac{k - l + mp^n}{k - l + p^n b}
\]

by Lemma 3.5. If \((p, n) \neq (2, 1)\), we have

\[
\frac{1}{a_{mp^n}} = \prod_{l < k < mp^n, 1 \leq k < l} \left( 1 - p^n \frac{b}{k - l} \right) \prod_{0 < k < l, 1 \leq k < l} \left( 1 - \frac{p^n(b - m)}{k - l} \right)
\]

\[
\equiv 1 - p^n \left( \sum_{l < k < mp^n, 1 \leq k < l} \frac{b}{k - l} + \sum_{0 < k < l, 1 \leq k < l} \frac{b - m}{k - l} \right)
\]

\[
\equiv 1 - mp^n \sum_{l < k < mp^n, 1 \leq k < l} \frac{1}{k - l}
\]

\[
= 1 - mp^n \sum_{1 \leq k < mp^n - 1, 2 \nmid k} \frac{1}{k}
\]

\[\equiv 1 - mp^n (\psi_p(a) + \gamma_p) \quad \text{(by (2.8))}\]

modulo \( p^{2n} \), which completes the proof of (3.11). In case \((p, n) = (2, 1)\), we need another observation (since the 3rd equivalence does not hold in general). In this case we have

\[
\frac{1}{a_{2m}} \equiv 1 - 2 \left( \sum_{l < k < 2m, 2 \nmid k - l} \frac{b}{k - l} + \sum_{0 < k < l, 2 \nmid k - l} \frac{b - m}{k - l} \right) \mod 4
\]

\[
= 1 - 2 \left( \sum_{l < k < 2m, 2 \nmid k - l} \frac{m}{k - l} + \sum_{0 < k < l, 2 \nmid k - l} \frac{b - m}{k - l} \right)
\]

\[
\equiv 1 - 2m \left( \sum_{0 < k < 2m - 1, 2 \nmid k} \frac{1}{k} + b - m \right) \mod 4
\]

and

\[
\psi_2(a) + \gamma_2 \equiv \sum_{0 < k < L, 2 \nmid k} \frac{1}{k} \mod 2
\]

where \( L \in \{0, 1, 2, 3\} \) such that \( a = -l + 2b \equiv L \mod 4 \). Therefore (3.11) is equivalent to

\[
m \left( \sum_{0 < k < 2m - 1, 2 \nmid k} \frac{1}{k} - \sum_{0 < k < L, 2 \nmid k} \frac{1}{k} + b - m \right) \equiv 0 \mod 2.
\]
We may assume that \( m > 0 \) is odd and \( b = 0, 1 \) (hence \( a = 0, \pm 1, 2 \)). Then one can check this on a case-by-case analysis.

**Lemma 3.8** For any \( m, m' \in \mathbb{Z}_{\geq 0} \) and \( n \in \mathbb{Z}_{\geq 1} \), we have
\[
m \equiv m' \mod p^n \implies \frac{B_m}{A_m} \equiv \frac{B_{m'}}{A_{m'}} \mod p^n.
\]

**Proof.** If \( p \nmid m \), then \( \frac{B_m}{A_m} = \frac{1}{m} \) and hence the assertion is obvious. Let \( m = kp^i \) with \( i \geq 1 \) and \( p \nmid k \). It is enough to show the assertion in case \( m' = m + p^n \). If \( n \leq i \), then
\[
\frac{B_m}{A_m} \equiv \frac{B_{m'}}{A_{m'}} \equiv B_0 \mod p^n
\]
by (3.13). Suppose \( n \geq i \). Notice that
\[
1 - \frac{B_m}{A_m} = \frac{A_{m/p}^{(1)}}{A_m} = \prod_{r=1}^{s} \left\{ \frac{1}{a_r} \right\} \frac{m}{m_p}
\]
by (3.10) and Lemma 3.5. We have
\[
1 - m' \frac{B_{m'}}{A_{m'}} = \prod_r \left\{ \frac{1}{a_r} \right\} \frac{m_{p^i+p^n}}{m_{p^i}}
= \prod_r \left\{ \frac{1}{a_r} \right\} \frac{1 + kp^i}{m_{p^i}} \prod_r \frac{1 + kp^i_{p^n}}{m_{p^i}}
= \left( 1 - \frac{B_m}{A_m} \right) \prod_r \left\{ \frac{1}{a_r} \right\} \frac{1 + kp^i_{p^n}}{m_{p^i}}
\]
by (3.11) and Lemma 3.5. The equivalence (**) follows from (2.8) in case \( (p, i) \neq (2, 1) \), and in case \( (p, i) = (2, 1) \), it does from the fact that
\[
\psi_2(z + 2) - \psi_2(z) \equiv 1 \mod 2, \quad z \in \mathbb{Z}_2.
\]
Therefore we have
\[
kp^i \left( \frac{B_{m'}}{A_{m'}} - \frac{B_m}{A_m} \right) \equiv -p^n \frac{B_{m'}}{A_{m'}} + p^n B_0 \mod p^{i+n}.
\]
By (3.13), the right hand side vanishes. This is the desired assertion. \( \square \)
Lemma 3.9  \( S_m := \sum_{i+j=m} A_{i+p^n} B_j - A_i A_{j+p^n} \) for \( m \in \mathbb{Z}_{\geq 0} \). Then

\[
S_m \equiv \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j} \mod p^n.
\]

Proof.

\[
S_m = \sum_{i+j=m} A_{i+p^n} B_j - A_i A_{j+p^n} \frac{B_i+p^n}{A_j+p^n}
\]

\[
\equiv \sum_{i+j=m} A_{i+p^n} B_j - A_i A_{j+p^n} \frac{B_j}{A_j} \mod p^n \quad \text{(Lemma 3.8)}
\]

\[
= \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j}
\]

as required. \( \square \)

Lemma 3.10

\[
S_m \equiv \sum_{i+j=m} (A^{(1)}_{i/p} A^{(1)}_{j/p} + p^n - A^{(1)}_{i/p} A^{(1)}_{j/p} + p^n) \frac{A_i}{A^{(1)}_{i/p}} \frac{A_j}{A^{(1)}_{j/p}} B_j \mod p^n.
\]

Proof. This follows from Lemma 3.9 and Lemma 3.6. \( \square \)

Lemma 3.11  For all \( m, k, s \in \mathbb{Z}_{\geq 0} \) and \( 0 \leq l \leq n \), we have

\[
\sum_{i+j=m \atop i \equiv k \mod p^n} A_i A_{j+p^n-1} - A_j A_{i+p^n-1} \equiv 0 \mod p^l.
\] (3.14)

Proof. There is nothing to prove in case \( l = 0 \). If \( l = n \), then (3.14) is obvious as

\[
\text{LHS} = \sum_{i+j=m} A_i A_{j+p^n-1} - A_j A_{i+p^n-1} = 0.
\]

Suppose that \( 1 \leq l \leq n - 1 \). Let \( A_{i}^{(r)} \) be as in (3.2). For \( r, k \in \mathbb{Z}_{\geq 0} \) we put

\[
F^{(r)}(t) := \sum_{i=0}^{\infty} A_{i}^{(r)} t^i,
\]

\[
F_k^{(r)}(t) := \sum_{i=k \mod p^n-1} A_{i}^{(r)} t^i = p^{-n+l} \sum_{s=0}^{p^n-1} \zeta^{-sk} F(\zeta^s t)
\] (3.15)
where $\zeta$ is a primitive $p^{n-t}$-th root of unity. Then (3.14) is equivalent to

$$F_k(t)F_{m-k}(t) \equiv F_k(t)F_{m-k}(t) \mod p^n$$

where $F_k(t) = F_k^{(0)}(t)$. It follows from the Dwork congruence [Dw] p.37, Thm. 2 that one has

$$\frac{F^{(i)}(t)}{F^{(i+1)}(tp^2)} \equiv \frac{F^{(i)}(tp^2)}{[F^{(i+1)}(tp^3)]_{<p^n}} \mod p^n$$

for any $m \geq n \geq 1$. This implies

$$\frac{F^{(i)}(tp^2)}{F^{(i+1)}(tp^3)} \equiv \frac{F^{(i)}(tp^3)}{[F^{(i+1)}(tp^4)]_{<p^n+1}} \mod p^n, \quad \frac{F^{(i)}(tp^2)}{F^{(i+1)}(tp^3)} \equiv \frac{F^{(i)}(tp^2)}{[F^{(i+1)}(tp^4)]_{<p^n+1}} \mod p^n, \ldots$$

Hence we have

$$\frac{F(t)}{F^{(n-1)}(tp^n)} = \frac{F(t)}{F^{(1)}(tp^2)} \cdot \frac{F^{(1)}(tp^2)}{F^{(2)}(tp^3)} \cdot \cdots \cdot \frac{F^{(n-1)}(tp^n-1)}{F^{(n-1)}(tp^{n-1})} \equiv \left[\frac{F(t)}{[F^{(1)}(tp^2)]_{<p^n}}\right] \cdot \left[\frac{F^{(1)}(tp^2)}{[F^{(2)}(tp^3)]_{<p^n}}\right] \cdot \cdots \cdot \left[\frac{F^{(n-1)}(tp^n-1)}{[F^{(n-1)}(tp^{n-1})]_{<p^n}}\right] \mod p^{d-n+1} \mathbb{Z}_p[[t]] $$

namely there are $a_i \in \mathbb{Z}_p$ such that

$$\frac{F(t)}{F^{(n-1)}(tp^n)} = \frac{F(t)}{[F^{(n-1)}(tp^n)]_{<p^n}} + p^{d-n+1} \sum_i a_i t^i.$$

Substitute $t$ for $\zeta^s t$ in the above and multiply it by

$$\left(\frac{F(t)}{F^{(n-1)}(tp^n)}\right)^{-1} = \left(\frac{F(t)}{[F^{(n-1)}(tp^n)]_{<p^n}} + p^{d-n+1} \sum_i a_i t^i\right)^{-1}.$$

Then we have

$$F(\zeta^s t)F(t) - F(\zeta^s t)F(t) = p^{d-n+1} \sum_{i=0}^{\infty} b_i(\zeta^s) t^i$$

where $b_i(x) \in \mathbb{Z}_p[x]$ are polynomials which do not depend on $s$. Applying $\sum_{s=0}^{\infty} \zeta^{-sk}(-)$ on both side, one has

$$p^{n-1}F_k(t)F(t) - p^{n-1}F_k(t)F(t) = p^{d-n+1} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \zeta^{-sk} b_i(\zeta^s) t^i$$

20
by (3.15). Since \( \sum_{s=0}^{p^n-1} c^s j = 0 \) or \( p^{n-1} \), the right hand side is zero modulo \( p^{d+1} \). Therefore

\[
\frac{F_k(t)}{F(t)} \equiv \frac{F_k(t)_{<p^d}}{F(t)_{<p^d}} \mod p^{d-n+1} \mathbb{Z}_p[[t]].
\]

This implies

\[
\frac{F_k(t)F_j(t)_{<p^d} - F_k(t)_{<p^d}F_j(t)}{F(t)} \equiv \frac{F_k(t)_{<p^d}F_j(t)_{<p^d} - F_k(t)_{<p^d}F_j(t)_{<p^d}}{F(t)_{<p^d}} = 0 \mod p^{d-n+1}.
\]

Now (3.16) is the case \((d,j) = (n-1,s-k)\).

3.5 Proof of Congruence relations: End of proof

We finish the proof of Theorem 3.2. Let \( S_m \) be as in Lemma 3.9. The goal is to show

\[
S_m \equiv 0 \mod p^n, \quad \forall m \geq 0.
\]

Let us put

\[
q_i := \frac{A_i}{A_{\lfloor i/p \rfloor}^{(1)}}, \quad A(i, j) := A_i^{(1)} A_j^{(1)}, \quad A^*(i, j) := A(j, i + p^{n-1}) - A(i, j + p^{n-1})
\]

\[
B(i, j) := A^*(\lfloor i/p \rfloor, \lfloor j/p \rfloor).
\]

Then

\[
S_m \equiv \sum_{i+j=m} B(i, j)q_i q_j \frac{B_i}{A_j} \mod p^n
\]

by Lemma 3.10. It follows from Lemma 3.8 and Lemma 3.6 that we have

\[
k \equiv k' \mod p^i \implies \frac{B_k}{A_k} = \frac{B_{k'}}{A_{k'}}, \quad q_k \equiv q_{k'} \mod p^{i+1}.
\]

(3.17)

By Lemma 3.11 we have

\[
\sum_{i+j=s \atop i \equiv k \mod p^{n-1}} A^*(i, j) \equiv 0 \mod p^l, \quad 0 \leq l \leq n
\]

(3.18)

for all \( s \geq 0 \). Let \( m = l + sp \) with \( l \in \{0, 1, \ldots, p-1\} \). Note

\[
B(i, m-i) = \begin{cases} 
A^*(k, s-k) & kp \leq i \leq kp+l \\
A^*(k, s-k-1) & kp+l < i \leq (k+1)p-1. 
\end{cases}
\]
Therefore

\[ S_m \equiv \sum_{i+j=m} B(i, j) q_i q_j B_j A_j \mod p^n \]

\[ = \sum_{i=0}^{p-1} \sum_{k=0}^{[(m-i)/p]} B(i + kp, m - (i + kp)) q_i q_j m^{(i+kp)} A_{m-(i+k)} \]

\[ = \sum_{k=0}^{s} B(i + kp, m - (i + kp)) \sum_{i=0}^{t} q_i q_j m^{(i+kp)} A_{m-(i+k)} \]

\[ + \sum_{k=0}^{s-1} B(i + kp, m - (i + kp)) \sum_{i=t+1}^{p-1} q_i q_j m^{(i+kp)} A_{m-(i+k)} \]

\[ = \sum_{k=0}^{s} A^*(k, s-k) \left( \sum_{i=0}^{t} q_i q_j m^{(i+kp)} A_{m-(i+k)} \right) \]

\[ + \sum_{k=0}^{s-1} A^*(k, s-k-1) \left( \sum_{i=t+1}^{p-1} q_i q_j m^{(i+kp)} A_{m-(i+k)} \right) \]

We show that the first term vanishes modulo \( p^n \). It follows from (3.17) that we have

\[ k \equiv k' \mod p^i \implies P_k \equiv P_{k'} \mod p^{i+1}. \] (3.19)

Therefore one can write

\[ \sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^n-1} P_i \left( \sum_{k \equiv i \mod p^n} A^*(k, s-k) \right) \mod p^n. \]

It follows from (3.18) that \((*)\) is zero modulo \( p \). Therefore, again by (3.19), one can rewrite

\[ \sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^n-2} P_i \left( \sum_{k \equiv i \mod p^{n-2}} A^*(k, s-k) \right) \mod p^n. \]

It follows from (3.18) that \((**)\) is zero modulo \( p^2 \), so that one has

\[ \sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^n-3} P_i \left( \sum_{k \equiv i \mod p^{n-3}} A^*(k, s-k) \right) \mod p^n. \]
by (3.19). Continuing the same discussion, one finally obtains
\[
\sum_{k=0}^{s} A^*(k, s - k)P_k \equiv \sum_{k=0}^{s} A^*(k, s - k) = 0 \mod p^n
\]
the vanishing of the first term. In the same way one can show the vanishing of the second term,
\[
\sum_{k=0}^{s} A^*(k, s - 1 - k)Q_k \equiv 0 \mod p^n.
\]
We thus have \( S_m \equiv 0 \mod p^n \). This completes the proof of Theorem 3.2.

4 Geometric aspect of \( p \)-adic hypergeometric functions of logarithmic type

We mean by a fibration over a ring \( R \) a projective flat morphism of quasi-projective smooth \( R \)-schemes.

4.1 Hypergeometric curves

Let \( N \geq 2 \) be an integer and \( p \) a prime number (we shall soon assume \( p > N \)). Let \( A, B \) be integers such that \( 0 < A, B < N \) and \( \gcd(N, A) = \gcd(N, B) = 1 \). Let \( f : Y \to \mathbb{P}^1 \) be a fibration over \( \mathbb{Q}_p \) whose general fiber \( X_\lambda = f^{-1}(\lambda) \) is the projective nonsingular model of the affine curve
\[
y^N = x^A(1 - x)^B(1 - \lambda x)^{N-B}.
\]
We call \( f \) a hypergeometric curve (or a hypergeometric fibration of Gauss type according to the notion of [AO2, 3.2]). This is a fibration of curves of genus \( N - 1 \), smooth outside \( \lambda = 0, 1, \infty \) and it has a totally degenerate semistable reduction at \( \lambda = 1 \) ([AO2, Prop. 3.1, Rem. 3.2]). Put \( S := \text{Spec} \mathbb{Q}_p[\lambda, (\lambda - \lambda^2)^{-1}] \subset \mathbb{P}^1 \) and \( X := f^{-1}(S) \). We assume that the divisor \( D := Y \setminus X \) is a NCD. Let \( \overline{Y} = X \times \overline{\mathbb{Q}_p} \) and \( \bar{f} : \overline{Y} \to \mathbb{P}^1_{\overline{\mathbb{Q}_p}} \) be the base change. Let \([\zeta] : \overline{Y} \to \overline{Y}\) denote the automorphism given by
\[
[\zeta](x, y, \lambda) = (x, \zeta^{-1}y, \lambda)
\]
for a \( N \)-th root \( \zeta \in \mu_N = \mu_N(\overline{\mathbb{Q}_p}) \). For a \( \mathbb{Q}[\mu_N] \)-module \( V \), we denote by \( V(n) \) the subspace on which \([\zeta]\) acts by multiplication by \( \zeta^n \) for all \( \zeta \in \mu_N : \)
\[
V(n) := \{ x \in V \mid [\zeta]x = \zeta^n x, \forall \zeta \in \mu_N \}.
\]
Then one has the eigen decomposition
\[
H^1_{dR}(\overline{X}/\overline{S}) = \bigoplus_{n=1}^{N-1} H^1_{dR}(\overline{X}/\overline{S})(n)
\]
of \(\mathcal{O}(\mathcal{S})\)-module and each eigen space is free of rank 2. A basis of \(H^1_{\text{dR}}(\mathcal{X}/\mathcal{S})(n)\) is given by
\[
\omega_n := x^{A_n} (1 - x)^{B_n} (1 - \lambda x)^{n-1} - B_n \frac{dx}{y^n}, \quad \eta_n := \frac{x}{1 - \lambda x} \omega_n
\]
where we put
\[
A_n := \left\lceil \frac{nA}{N} \right\rceil, \quad B_n := \left\lceil \frac{nB}{N} \right\rceil.
\]
One easily sees that \(\omega_n\) is the first kind (i.e. a holomorphic 1-form on \(X_\lambda\)), \(\eta_n\) the second kind.

### 4.2 Gauss-Manin connection

Let \(1 \leq n \leq N - 1\) be an integer. Put
\[
a_n := \left\lfloor \frac{-nB}{N} \right\rfloor, \quad b_n := \left\lfloor \frac{-nA}{N} \right\rfloor
\]
where \(\{x\} := x - \lfloor x \rfloor\) denotes the fractional part. In what follows, we also use another coordinate \(t = 1 - \lambda\). Let
\[
F_n(t) := 2F_1 \left( a_n, b_n ; 1 ; t \right) = \sum_{i=0}^{\infty} \frac{(a_n)_i (b_n)_i}{i!} t^i \in \mathbb{Z}_p[[t]]
\]
be the hypergeometric power series. Put
\[
\tilde{\omega}_n := \frac{1}{F_n(t)} \omega_n, \quad \tilde{\eta}_n := -t(1 - t)^{a_n + b_n} (F'_n(t) \omega_n + a_n F_n(t) \eta_n)
\]
which form a \(\mathbb{Q}_p((t))\)-basis of \(\mathbb{Q}_p((t)) \otimes H^1_{\text{dR}}(X/S)\).

**Proposition 4.1** Let \(\nabla : H^1_{\text{dR}}(X/S) \to \Omega^1_S \otimes H^1_{\text{dR}}(X/S)\) be the Gauss-Manin connection. Then
\[
(\nabla(\tilde{\omega}_n), \nabla(\tilde{\eta}_n)) = dt \otimes \left( \tilde{\omega}_n, \tilde{\eta}_n \right) \left( \begin{array}{cc} 0 & 0 \\ t^{-1} (1 - t)^{-a_n - b_n} F_n(t)^{-2} & 0 \\ \end{array} \right),
\]
\[
(\nabla(\omega_n), \nabla(\eta_n)) = dt \otimes \left( \omega_n, \eta_n \right) \left( \begin{array}{cc} 0 & -b_n (t - t^2)^{-1} \\ -a_n & ((a_n + b_n + 1) t - 1) (t - t^2)^{-1} \\ \end{array} \right).
\]

**Proof.** We may replace the base field \(\mathbb{Q}_p\) with \(\mathbb{C}\). Let \(\zeta \in \mathbb{C}^\times\) be a primitive \(N\)-th root of unity. Since \(\nabla\) commutes with the automorphism \([\zeta]\), the connection preserves the eigen components \(H^1_{\text{dR}}(X/S)(n)\),
\[
\nabla(H^1_{\text{dR}}(X/S)(n)) \subset \Omega^1_S \otimes H^1_{\text{dR}}(X/S)(n).
\]
We only show \((4.5)\) since \((4.4)\) can be derived from it. Let \(X_t = f^{-1}(t)\) denote the fiber over a complex point \(t\) of \(S\). We denote by \(X_t^{\text{Rm}} = X_t(\mathbb{C})\) the associated Riemann surface. Let \(P_0\)
(resp. \(P_1\)) be the point \((x, y) = (0, 0)\) (resp. \((x, y) = (1, 0)\)) of \(X_t^{an}\). Let \(e\) be a path in \(X_t^{an}\) from \(P_0\) to \(P_1\) such that \(x \in [0, 1]\) (real interval) and \(y = x^{A/N}(1 - x)^{B/N}(1 - (1 - t)x)^{1 - B/N}\) takes the principal values. The key formula is

\[
\int_e \omega_n = \int_0^1 \omega_n = B(a_n, b_n)_{2}F_{1}\left(\frac{a_n, b_n}{a_n + b_n}; 1 - t\right), \tag{4.6}
\]

\[
\int_e \eta_n = B(a_n, b_n + 1)_{2}F_{1}\left(\frac{a_n + 1, b_n + 1}{a_n + b_n + 1}; 1 - t\right) = -a_n^{-1} \frac{d}{dt} \left(\int_e \omega_n\right) \tag{4.7}
\]

where \(B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)\) is the beta function. The path \(e\) is not a closed path but a homology cycle in \(H_1(X_t^{an}, \{P_0, P_1\}; \mathbb{Z})\). For \(\zeta \in \mu_N\), the cycle \(\gamma(\zeta) := (1 - [\zeta])\) defines a homology cycle in \(H_1(X_t^{an}, \mathbb{Z})\) as \([\zeta]P_0 = P_0\) and \([\zeta]P_1 = P_1\). Obviously

\[
\int_{\gamma(\zeta)} \omega_n = \int_e (1 - [\zeta]) \omega_n = (1 - \zeta^n) \int_e \omega_n, \quad \int_{\gamma(\zeta)} \eta_n = (1 - \zeta^n) \int_e \eta_n. \tag{4.8}
\]

Letting \(T\) be the local monodromy at \(t = 0\), put \(\delta(\zeta) := (T - 1)\gamma(\zeta)\). Recall a formula (\([\text{NIST}\ 15.8.10])

\[
B(a_n, b_n)_{2}F_{1}\left(\frac{a_n, b_n}{a_n + b_n}; 1 - t\right) = \sum_{i=0}^{\infty} \frac{(a_n)_i (b_n)_i}{i!^2} (C_i - \log t)t^n \tag{4.9}
\]

\[
C_i := 2 \psi(1) - \psi(a_n) - \psi(b_n) + \sum_{k=1}^{i} \frac{2}{k} - \frac{1}{k + a_n - 1} - \frac{1}{k + b_n - 1}.
\]

Therefore we have

\[
\int_{\delta(\zeta)} \omega_n = 2\pi i (1 - \zeta^n)_{2}F_{1}\left(\frac{a_n, b_n}{1}; t\right), \quad \int_{\delta(\zeta)} \eta_n = -a_n^{-1} \frac{d}{dt} \left(\int_{\delta(\zeta)} \omega_n\right). \tag{4.10}
\]

Now we show \((4.5)\). Let \(\nabla_{\omega} \omega_n = f_n(t)\omega_n + g_n(t)\eta_n\). Applying \(\int_{\gamma(\zeta)}\) and \(\int_{\delta(\zeta)}\) on it, one has

\[
\int_{\gamma(\zeta)} \nabla_{\omega} \omega_n = \frac{d}{dt} \int_{\gamma(\zeta)} \omega_n = f_n(t) \int_{\gamma(\zeta)} \omega_n + g_n(t) \int_{\gamma(\zeta)} \eta_n,
\]

\[
\int_{\delta(\zeta)} \omega_n = f_n(t) \int_{\delta(\zeta)} \omega_n + g_n(t) \int_{\delta(\zeta)} \eta_n.
\]

Each of them characterizes \(f_n\) and \(g_n\), and then one can show \((4.5)\) by a direct calculus. This completes the proof. \(\Box\)

For the later use, we sum up the result on the homology cycles \(\gamma(\zeta), \delta(\zeta)\).

**Lemma 4.2** Let \(\gamma(\zeta), \delta(\zeta) \in H_1(X_t^{an}, \mathbb{Z})\) be as in the proof of Proposition \((4.4)\). Then \(\{\gamma(\zeta), \delta(\zeta) \mid \zeta \in \mu_N \setminus \{1\}\}\) forms a basis of \(H_1(X_t^{an}, \mathbb{Q})\). Furthermore the invariant part of \(H_1(X_t^{an})\) under the local monodromy \(T\) at \(t = 0\) is spanned by \(\delta(\zeta)\)'s \((N - 1\text{-dimensional})\).
Proof. Since \( \dim_Q H_1(X^m_t, \mathbb{Q}) = 2N - 2 \), it is enough to prove that \( \gamma(\zeta), \delta(\zeta) \) are linearly independent. To do this, let

\[
A_n(\zeta) := \left( \int_{\delta(\zeta)} \omega_n \int_{\delta(\zeta)} \eta_n \right) = (1 - \zeta^n) \left( P_n - a_{1-n} Q_n \right) \tag{4.11}
\]

where we put \( P_n := B(a_n, b_n) F_1(\frac{a_n, b_n}{a_n + b_n}, 1 - t) \) and \( Q_n := 2\pi i F_1(\frac{a_n, b_n}{1}, t) \). Then it is enough to show that the \((2N - 2) \times (2N - 2)\)-period matrix \( (A_n(\zeta))_{1 \leq n \leq 2N - 2, 1 \leq \zeta \leq \mu_N \setminus \{1\}} \) is invertible. This is reduced to show \( \det A_n(\zeta) \neq 0 \) for each \( n \) and \( \zeta \). However this follows from a formula

\[
P_n \frac{dQ_n}{dt} - Q_n \frac{dP_n}{dt} = 2\pi i (t^{-a_n - b_n} - 1 - t)^{-1}.
\]

Let \( V \) be the invariant part \( H_1(X^m_t, \mathbb{Q}) \) under \( T \) (i.e. \( V = \ker(T - 1|H_1(X^m_t)) \)). Then, \((4.10)\) implies that \( \delta(\zeta) \in V \). On the other hand, since \( X_t \) has a totally degenerate semistable reduction at \( t = 0 \) (\( \iff \lambda = 1 \)), one has

\[
\dim_Q V = \frac{1}{2} \dim_Q H_1(X^m_t) = N - 1.
\]

Hence the latter statement follows. \( \Box \)

4.3 de Rham symplectic basis

Let \( J(\overline{X}/S) \) be the jacobian scheme for \( \overline{X}/S \). This is a \((N - 1)\)-dimensional abelian scheme over \( S \) endowed with the principal polarization, and it has a totally degenerate semistable reduction at \( t = 1 \). Namely letting \( \Delta := \text{Spec} \overline{\mathbb{Q}}_p[[t]] \hookrightarrow S \), there is a semistable model \( J_\Delta \to \Delta \) such that the central fiber is an algebraic torus \( T \). Put \( \Delta^* := \text{Spec} \overline{\mathbb{Q}}_p((t)) \) and \( J_{\Delta^*} := J_\Delta \times_\Delta \Delta^* \). We fix coordinate functions \( u_i \) such that \( T \cong \prod \text{Spec} \overline{\mathbb{Q}}_p[u_i, u_i^{-1}] \). Using the uniformization \( \rho : \mathbb{G}_m^{N-1} \to J_\Delta \) in the rigid analytic sense, one has a surjective map

\[
\tau : H^1_{\text{dr}}(J_{\Delta^*}/\Delta^*) \longrightarrow \overline{\mathbb{Q}}_p((t))^{N-1} \tag{4.12}
\]

which is given by \( \tau(\omega) = (\text{Res}_{u_i=0}(\rho^*\omega))_{1 \leq i \leq N-1} \) (see [AM] 4.1 for more detail). We say that \( \{\tilde{\omega}_i, \tilde{\eta}_i\}_{1 \leq i \leq N-1} \) forms a de Rham symplectic basis of \( H^1_{\text{dr}}(J_{\Delta^*}/\Delta^*) \) if

\begin{enumerate}
\item[(DS1)] \( \tilde{\omega}_i \in \Gamma(J_{\Delta^*}, \Omega^1_{J_{\Delta^*}/\Delta^*}) \) and \( \{\tau\tilde{\omega}_i\} \) span the \( \mathbb{Q} \)-lattice \( \mathbb{Q}^{N-1} \subset \overline{\mathbb{Q}}_p((t))^{N-1} \). In other words, the \( \mathbb{Q} \)-linear span of \( \{\rho^*\tilde{\omega}_i\}_{1 \leq i \leq N-1} \) coincides with the \( \mathbb{Q} \)-linear span of \( \{du_j/u_j\}_{1 \leq i \leq N-1} \).
\item[(DS2)] \( \tilde{\eta}_i \in \ker(\tau) \) and they satisfy \( \langle \tilde{\omega}_i, \tilde{\eta}_j \rangle = \delta_{ij} \) where \( \delta_{ij} \) denotes the Kronecker delta, and \( \langle x, y \rangle \) denotes the cup-product pairing with respect to the principal polarization.
\end{enumerate}

Notice that \( \{\tilde{\eta}_i\}_{1 \leq i \leq N-1} \) is automatically determined by \( \{\tilde{\omega}_i\}_{1 \leq i \leq N-1} \) by (DS2).
Proposition 4.3 Put
\[ \omega(\nu) := \sum_{n=1}^{N-1} \nu^n \tilde{\omega}_n, \quad \eta(\nu) := \sum_{n=1}^{N-1} \nu^{-n} \tilde{\eta}_n \]
for \( \nu \in \mu_N \setminus \{1\} \). Then \( \tilde{\omega}_i \) are \( \mathbb{Q} \)-linear combinations of \( \omega(\nu) \)'s, and \( \tilde{\eta}_i \) are \( \mathbb{Q} \)-linear combinations of \( \eta(\nu) \)'s.

Proof. By the conditions (DS1) and (DS2) we may replace the base field with \( \mathbb{C} \). Recall from Lemma 4.2 that the homology group \( H_1(X^a_n, \mathbb{Q}) \) is spanned by \( \gamma(\zeta) \) and \( \delta(\zeta) \)'s. Moreover the invariant part of \( H_1(X^a_n) \) under the local monodromy at \( t = 0 \) is spanned by \( \delta(\zeta) \)'s. By (4.10) one has
\[ \int_{\delta(\zeta)} \tilde{\omega}_n = \text{constant}, \quad \int_{\delta(\zeta)} \tilde{\eta}_n = 0. \]
This shows that the de Rham symplectic basis is given by certain \( \mathbb{C} \)-linear combinations of \( \tilde{\omega}_n, \tilde{\eta}_n \) (\( 1 \leq n \leq N - 1 \)). The rest is to check
\[ \frac{1}{2\pi i} \int_{\delta(\zeta)} \omega(\nu) \in \mathbb{Q}, \quad \int_{\gamma(\zeta)} \eta(\nu) \in \mathbb{Q}. \]
However this is immediate from (4.8) and (4.10) (cf. the proof of [AM, Prop.4.4]).

4.4 Rigid cohomology and an exact category Fil\text{-}\text{F}\text{-}\text{MIC}(S)

Lemma 4.4 Suppose that \( p > N \). Then there is an integral regular model
\[ f_{\mathbb{Z}_p} : Y_{\mathbb{Z}_p} \rightarrow \mathbb{P}^1_{\mathbb{Z}_p} \]
once Zp such that \( Y_{\mathbb{Z}_p} \) is smooth over \( \mathbb{Z}_p \). Moreover let \( S_{\mathbb{Z}_p} := \text{Spec} \mathbb{Z}_p[\lambda, (\lambda - \lambda^2)^{-1}] \) and \( X_{\mathbb{Z}_p} := f_{\mathbb{Z}_p}^{-1}(S_{\mathbb{Z}_p}) \). Then, \( X_{\mathbb{Z}_p} \) is smooth over \( S_{\mathbb{Z}_p} \) and the reduced part of \( D_{\mathbb{Z}_p} := Y_{\mathbb{Z}_p} \setminus X_{\mathbb{Z}_p} \) is a relative NCD over \( \mathbb{Z}_p \).

Proof. This is done by constructing the integral model explicitly. Since it is a long and tedious argument, I just sketch it.

The integral model over a neighborhood of \( \lambda = 1 \) can be obtained in the same way as the proof of [A, Thm.4.1] (indeed the desingularization there works over \( \mathbb{Z}_p \) as \( p > N \)). Let us construct the integral model over a neighborhood of \( \lambda = 0 \). We begin with a scheme \( U = U_0 \cup U_1 \) where
\[ U_0 = \text{Spec} \mathbb{Z}_p[[\lambda]][x, y] / (y^N - x^A(1 - x)^B(1 - \lambda x)^{N-B}), \]
\[ U_1 = \text{Spec} \mathbb{Z}_p[[\lambda]][u, v] / (v^N - u^{N-A}(u - 1)^B(u - \lambda)^{N-B}) \]
given by \( u = x^{-1} \) and \( v = yx^{-2} \). Then \( U \to \text{Spec} \mathbb{Z}_p[[\lambda]] \) is projective. Both of \( U_i \) are not normal. One easily sees that the normalization of \( U_0 \) is smooth over \( \mathbb{Z}_p \) while the normalization of \( U_1 \) has a singular locus over \( u = 0 \). Consider a neighborhood

\[
\hat{U}_1 := \text{Spec} \mathbb{Z}_p[[\lambda, u, v]]/(v^N - u^{N-A}(u-1)^B(u-\lambda)^{N-B}) \hookrightarrow U_1.
\]

Since \( p > N \), the power series expansion of \((1-u)\hat{\lambda}\) belongs to \( \mathbb{Z}_p[[u]] \). Therefore we may replace the variable \( v \) with \( v(1-u)^{B/N} \), and hence we have

\[
\hat{U}_1 \cong \text{Spec} \mathbb{Z}_p[[\lambda, u, v]]/(v^N - (-1)^Bu^{N-A}(u-\lambda)^{N-B})
= \text{Spec} \mathbb{Z}_p[[w, u, v]]/(v^N - (-1)^Bu^{N-A}w^{N-B})
\]

with \( w = u - \lambda \). It is a simple exercise to resolve the singular point of \( x^a \pm y^b z^c = 0 \) where \( 0 < a, b, c < p \) integers. This completes the construction of the integral model over \( \lambda = 0 \).

To construct the integral model over a neighborhood of \( \lambda = \infty \), let \( s = \lambda^{-1} \). We begin with a scheme \( U = U_0 \cup U_1 \) where

\[
U_0 = \text{Spec} \mathbb{Z}_p[[s]][x, y]/(s^{N-B}y^N - x^A(1-x)^B(s-x)^{N-B})
\]

\[
U_1 = \text{Spec} \mathbb{Z}_p[[\lambda]][u, v]/(s^{N-B}v^N - u^{N-A}(u-1)^B(su-1)^{N-B})
\]

glued by \( u = x^{-1} \) and \( v = yx^{-2} \). Then \( U \to \text{Spec} \mathbb{Z}_p[[s]] \) is projective. We resolve the singularities of \( U_0 \) (we omit it for \( U_1 \) as it is similar). The singular locus is \( \{x = s = 0\} \) and \( \{x - 1 = s = 0\} \). In a neighborhood of the locus \( \{x = s = 0\} \), there is an embedding

\[
V_0 = \text{Spec} \mathbb{Z}_p[[s, x]][u]/(s^{N-B}u^N - x^A(s-x)^{N-B}) \hookrightarrow U_0
\]

given by \( u = y(1-x)^{-B} \), and in a neighborhood of the locus \( \{x - 1 = s = 0\} \), there is an embedding

\[
V_1 = \text{Spec} \mathbb{Z}_p[[s, v]][u]/(s^{N-B}u^N - v^B) \hookrightarrow U_0
\]

given by \( v = 1 - x \) and \( u = y(x^A(s-x)^{N-B})^{-1} \). Then it is not hard to resolve the singularities of \( V_0 \) and \( V_1 \) if we note that all exponents of the monomials are less than \( p \). This completes the proof.

Let \( \sigma \) be a \( p \)-th Frobenius on \( \mathbb{Z}_p[t, (t - t^2)^{-1}] \), the ring of overconvergent power series, which naturally extends on \( \mathbb{Q}_p[t, (t - t^2)^{-1}] := \mathbb{Q}_p \otimes \mathbb{Z}_p[t, (t - t^2)^{-1}] \). Write \( X_{F_p} := X_{\mathbb{Z}_p \times \mathbb{Z}_p} \mathbb{F}_p \) and \( S_{F_p} := S_{\mathbb{Z}_p \times \mathbb{Z}_p} \mathbb{F}_p \). Then the rigid cohomology groups

\[
H^\bullet_{\text{rig}}(X_{F_p}/S_{F_p})
\]

are defined. We refer the book [LS] for the general theory of rigid cohomology. The required properties in below is the following.

- \( H^\bullet_{\text{rig}}(X_{F_p}/S_{F_p}) \) is a finitely generated \( \mathcal{O}(S)^1 \sim \mathbb{Q}_p[t, (t - t^2)^{-1}]^1 \)-module.
• (Frobenius) The p-th Frobenius \( \Phi \) on \( H^*_{\rig}(X_{\mathbb{F}_p}/S_{\mathbb{F}_p}) \) (depending on \( \sigma \)) is defined in a natural way. This is a \( \sigma \)-linear endomorphism:

\[
\Phi(f(t)x) = \sigma(f(t)) \Phi(x), \quad \text{for } x \in H^*_{\rig}(X_{\mathbb{F}_p}/S_{\mathbb{F}_p}), \ f(t) \in \mathcal{O}(S)\, .
\]

• (Comparison) There is the comparison isomorphism with the algebraic de Rham cohomology,

\[
c : H^*_{\rig}(X_{\mathbb{F}_p}/S_{\mathbb{F}_p}) \cong H^*_{\dR}(X/S) \otimes \mathcal{O}(S)\, .
\]

In [AM, 2,1] we introduce a category \( \text{Fil}^*\text{-MIC}(S) = \text{Fil}^*\text{-MIC}(S, \sigma) \). It consists of collections of datum \((H_{\dR}, H_{\rig}, c, \Phi, \nabla, \text{Fil}^*)\) such that

- \( H_{\dR} \) is a finitely generated \( \mathcal{O}(S) \)-module,
- \( H_{\rig} \) is a finitely generated \( \mathcal{O}(S)\, \cdot \)-module,
- \( c : H_{\rig} \cong H_{\dR} \otimes \mathcal{O}(S)\, \cdot \), the comparison
- \( \Phi : \sigma^* H_{\rig} \xrightarrow{\cong} H_{\rig} \) is an isomorphism of \( \mathcal{O}(S)\, \cdot \)-module,
- \( \nabla : H_{\dR} \to \Omega^1_{S/\mathbb{Q}_p} \otimes H_{\dR} \) is an integrable connection that satisfies \( \Phi \nabla = \nabla \Phi \).
- \( \text{Fil}^* \) is a finite descending filtration on \( H_{\dR} \) of locally free \( \mathcal{O}(S) \)-module (i.e. each graded piece is locally free), that satisfies \( \nabla(\text{Fil}^i) \subset \Omega^1 \otimes \text{Fil}^{i-1} \).

Let \( \text{Fil}^* \) denote the Hodge filtration on the de Rham cohomology, and \( \nabla \) the Gauss-Manin connection. Write

\[
H^i(X/S) := (H^*_{\dR}(X/S), H^*_{\rig}(X_{\mathbb{F}_p}/S_{\mathbb{F}_p}), c, \Phi, \nabla, \text{Fil}^*)
\]
an object of \( \text{Fil}^*\text{-MIC}(S) \).

For an integer \( r \), the Tate object \( \mathcal{O}_S(r) \in \text{Fil}^*\text{-MIC}(S) \) is defined in a customary way (loc.cit.). We simply write

\[
M(r) = M \otimes \mathcal{O}_S(r)
\]

for an object \( M \in \text{Fil}^*\text{-MIC}(S) \).

Let \( W = W(\mathbb{F}_p) \) be the Witt ring, and \( K = \text{Frac}W \) the fractional field. Write \( Y_W := Y_{\mathbb{Z}_p} \times \mathbb{Z}_p W \) etc. Let \( J(X_W/S_W) \to S_W \) be the jacobian fibration. Let \( \Delta_{W}^* := \text{Spec}W[[t]][t^{-1}] \to S_W \) and \( J_{W}^* := J(X_W/S_W) \times_{S_W} \Delta_{W}^* \). Let \( \{ \widehat{w}_i, \widehat{n}_i \} \) be the de Rham symplectic basis in \( \Delta_{W}^* \). Then one can see (from the proof of Lemma 4.4) that \( J(X_W/S_W) \to S_W \) has a split multiplicative reduction. Moreover it is not hard to see that \( \{ \widehat{w}_i, \widehat{n}_i \} \) forms a free basis of \( H^1_{\dR}(J_{W}^*/\Delta_{W}^*) \).

Let \( \sigma \) be the Frobenius on \( W[[t]] \) compatible with the Frobenius on \( W \), such that \( \sigma(t) = c t^p \) with \( c \in 1 + pW \). Then the Frobenius \( \Phi_{X/S} \) on \( H^1_{\dR}(X/S) \otimes \mathcal{O}(S)\, \cdot \) naturally extends on \( H^1_{\dR}(X/S) \otimes K((t)) = H^1_{\dR}(J_{W}^*/\Delta_{W}^*) \otimes K((t)) \). We shall later use the following lemma.

29
Lemma 4.5 Let $\tilde{\omega}_n, \tilde{\eta}_n$ be as in (4.3). Let $m \in \{1, 2, \ldots, N - 1\}$ be the unique integer such that $pm \equiv n \bmod N$. Then

$$\Phi_{X/S}(\tilde{\eta}_m) \in K \tilde{\eta}_n, \quad \Phi_{X/S}(\tilde{\omega}_m) \equiv p \tilde{\omega}_n \pmod{K((t)) \tilde{\eta}_n}.$$

**Proof.** Let $\nabla : H^1_{dR}(X/K((t))) \rightarrow \Omega^1_{K((t))/K} \otimes H^1_{dR}(X/K((t)))$ be the Gauss-Manin connection. Since $\Phi_{X/S} \nabla = \nabla \Phi_{X/S}$, we have $\Phi_{X/S} \text{Ker}(\nabla) \subset \text{Ker}(\nabla)$. Since $\{\tilde{\eta}_n\}_n$ forms a $K$-basis of $\text{Ker}(\nabla)$ by Proposition 4.1, we have

$$\Phi_{X/S}(\tilde{\eta}_m) \in \bigoplus_{n=1}^{N-1} K \tilde{\eta}_n.$$

Since $\Phi_{X/S} \omega = [\zeta^p] \Phi_{X/S}$, we further have $\Phi_{X/S}(\tilde{\eta}_m) \in K \tilde{\eta}_n$. Put

$$M := H^1_{dR}(X/K((t)))/(\tilde{\eta}_n)_{1 \leq n \leq N-1} \cong \bigoplus_{n=1}^{N-1} K((t)) \tilde{\omega}_n$$

on which the Frobenius $\Phi_{X/S}$ acts. Since $\Phi_{X/S} \omega = [\zeta^p] \Phi_{X/S}$, we have $\Phi_{X/S}(\tilde{\omega}_m) = h(t) \tilde{\omega}_n$ for some $h(t) \in K((t))$. Moreover since $\nabla$ induces the connection $\nabla$ on $M$, and it satisfies $\nabla(\tilde{\omega}_n) = 0$ for all $n$ (Proposition 4.1), we have $\nabla(\Phi_{X/S} \tilde{\omega}_m) = \Phi_{X/S} \nabla(\tilde{\omega}_n) = 0$. Therefore, we have

$$\Phi_{X/S}(\tilde{\omega}_m) \equiv \alpha \tilde{\omega}_n \pmod{K((t)) \tilde{\eta}_n}$$

(4.13)

with some $\alpha \in K$.

We show $\alpha = p$ in (4.13). Let $\tilde{f} : \mathbb{P}^1_{\mathbb{Z}_p} \rightarrow \mathbb{P}^1$ be the integral model in Lemma 4.4. Let $\Delta_W := \text{Spec} W[[t]] \hookrightarrow \mathbb{P}^1_W$ and put $\mathcal{W} := \tilde{f}^{-1}(\Delta_W)$. Let $D_W \subset \mathcal{W}$ be the fiber over $t = 0$, and $D_{W,i}$ the irreducible components. Since $\tilde{f}$ has a totally degenerate semistable reduction at $t = 0$, $D_W$ is reduced and each $D_{W,i}$ is isomorphic to $\mathbb{P}^1_W$. Let $Z_W$ be the intersection locus of $D_W$. This is a disjoint union of $(N - 1)$-copies of $\text{Spec} W$. More precisely the components $\{P_\nu\}$ of $Z_W$ are indexed by $\nu \in \mu_N \setminus \{1\}$, and each $P_\nu$ corresponds to the point $u = \nu$ where $u$ is the parameter such that $u^\lambda = y/(1-x)|_{D_W}$. We consider the log-crystalline cohomology groups

$$H^\bullet_{\log-cryst}((\mathcal{T}_p, D_\mathcal{T}_p)/(\Delta_W, 0)) \cong H^\bullet(\mathcal{W}, \Omega^{\bullet + 1}_{\mathcal{W}/W[[t]]}(\log D_W)).$$

The composition of morphisms

$$\Omega^{\bullet + 1}_{\mathcal{W}/W[[t]]}(\log D_W) \xrightarrow{\partial_W} \bigoplus_{\nu \in \mu_N \setminus \{1\}} \mathcal{O}_W[-1] \cdot P_\nu$$

of complexes gives rise to the natural map

$$R : H^1(\mathcal{W}, \Omega^{\bullet + 1}_{\mathcal{W}/W[[t]]}(\log D_W)) \rightarrow \bigoplus_{\nu \in \mu_N \setminus \{1\}} W(-1) \cdot P_\nu$$

(4.14)
which turns out to be the quotient map by the monodromy weight filtration on the log-
crystalline cohomology. The map (4.14) is compatible with respect to the Frobenius $\Phi_Y$ on
the left and the Frobenius $\Phi_Z$ on the right. Notice that $\Phi_Z$ is given by $\Phi_Z(\alpha P_\nu) = pF(\alpha)P_\nu$
where $F$ is the Frobenius on $W$.

We turn to the proof of $\alpha = p$ in (4.13). There are the natural maps

\[
H^\bullet_{\log\text{-}\text{crys}}(\mathcal{O}_\mathcal{F}, D_{\mathcal{F}}) / (\Delta_{\mathcal{F}}, \{0\}) \otimes \mathbb{Q} \rightarrow H^\bullet_{\text{rig}}(\mathcal{O}_\mathcal{F} / S_{\mathcal{F}}) \otimes \mathcal{O}(S) K((t))
\]

compatible with the Frobenius actions. Notice that the elements $\{\widetilde{\omega}_n\}$ lie in the left top term.
By a direct computation, one has $R(\widetilde{\omega}_i) = \sum_{\nu} \nu^i P_\nu$. We then have

\[
R(\Phi_Y(\widetilde{\omega}_m)) = \Phi_Z(R(\widetilde{\omega}_m)) = \Phi_Z \left( \sum_{\nu \in \mu_N \setminus \{1\}} \nu^m P_\nu \right) = \sum_{\nu \in \mu_N \setminus \{1\}} p\nu^m P_\nu = pR(\widetilde{\omega}_n).
\]

Since $\Phi_Z$ and $\Phi_{X/S}$ are compatible, this implies

\[
R(\alpha \omega_n) = pR(\widetilde{\omega}_n)
\]

by (4.13), and hence $\alpha = p$ as required.

4.5 Syntomic Regulators of hypergeometric curves

Lemma 4.6 Let $\zeta_i \in \mu_N(K)$ be $N$-th roots of unity such that $\zeta_1 \neq \zeta_2$ (possibly $\zeta_i = 1$). Then there exists a $K_2$-symbol

\[
\xi \in K_2(X_{\mathbb{Z}_p})
\]

such that

\[
d\log(\xi) = \sum_{n=1}^{N-1} \frac{\zeta_1^n - \zeta_2^n}{N} \frac{d\lambda}{1 - \lambda} \omega_n = -\sum_{n=1}^{N-1} \frac{\zeta_1^n - \zeta_2^n}{N} dt \omega_n
\]

(4.15)

where $t = 1 - \lambda$.

Proof. We can construct $\xi$ in the same way as the proof of [A Theorem 4.1], if we replace the Deligne-Beilinson cohomology in loc.cit. with the syntomic cohomology, and if we note that the desingularization there also works over $\mathbb{Z}_p$.

Remark 4.7 In [AM] we only consider the case $(A, B) = (1, N - 1)$. In this case there is an explicit description of $\xi$,

\[
\xi = \left\{ \frac{y - \zeta_1(1 - x)}{y - \zeta_2(1 - x)} \frac{(1 - \lambda) x^2}{(1 - x)^2} \right\} \in K_2(X).
\]
Let \( \xi \in K_2(X_{Z_p}) \) be the element as in Lemma 4.6. According to [AM, §2], one can associate a 1-extension
\[
0 \rightarrow H^1(X/S)(2) \rightarrow M_\xi(X/S) \rightarrow \mathcal{O}_S \rightarrow 0
\]
in the exact category Fil-F-MIC(S) (loc.cit. Prop.2.1). Let \( e_\xi \in \text{Fil}^0 M_\xi(X/S)_{\text{dR}} \) be the unique lifting of \( 1 \in \mathcal{O}_S(S) \). Define \( \varepsilon_i^{(n)}(t) \) and \( E_i^{(n)}(t) \) by
\[
\varepsilon_i^{(n)}(t) = E_i^{(n)}(t) F_n(t)^{-1} - t(1 - t)^{a_n + b_n} F_i'(t) E_i^{(n)}(t) - a_n t(1 - t)^{a_n + b_n} F_n(t) E_i^{(n)}(t).
\]
\[
\varepsilon_2^{(n)}(t) = -a_n t(1 - t)^{a_n + b_n} F_n(t) E_2^{(n)}(t).
\]
By the definition \( \varepsilon_i^{(n)}(t) \) are automatically overconvergent functions:
\[
\varepsilon_i^{(n)}(t) \in K[t, (t - t^2)^{-1}]^+.\]
Moreover since \( F_i'(t)/F_n(t) \) is an overconvergent function by [Dw, p.45, Lem. 3.4] we have
\[
\frac{E_i^{(n)}(t)}{F_n(t)} \in K[t, (t - t^2)^{-1}, h(t)^{-1}]^+, \quad h(t) := \prod_m F_m(t) < p
\]
where \( m \) runs over all integers in \( \{1, \ldots, N - 1\} \) such that for some \( i \in \mathbb{Z}_{\geq 0} \), \( a_n^{(i)} = \{-mB/N\} \) and \( b_n^{(i)} = \{-mA/N\} \), or equivalently \( mp^i \equiv n \mod N \).

**Theorem 4.8** Assume that \( \sigma \) is given by \( \sigma(t) = ct^p \) with \( c \in 1 + pW \). Then
\[
\frac{E_1^{(n)}(t)}{F_n(t)} = \mathcal{F}^{(\sigma)}_{a_n, b_n}(t)
\]
where the right hand side is the \( p \)-adic hypergeometric function of logarithmic type defined in [3.1].

**Proof.** The Frobenius \( \sigma \) extends on \( K((t)) \), and \( \Phi \) also extends on \( K((t)) \otimes H^1_{\text{dR}}(X/S) \) in the natural way. Apply the Gauss-Manin connection \( \nabla \) on (4.18). Since \( \nabla \Phi = \Phi \nabla \) and \( \nabla(e_\xi) = \text{dlog}\xi \), we have
\[
- (1 - \Phi) \left( F_n(t)\frac{dt}{t} \wedge \tilde{\omega}_n \right) = \nabla(E_1^{(n)}(t)\tilde{\omega}_n + E_2^{(n)}(t)\tilde{\eta}_n).
\]
Let $\Phi_{X/S}$ denote the $p$-th Frobenius on $H_{\text{rig}}^1(X_0/S_0)$. Then the $\Phi$ on $H_{\text{rig}}^1(X/S)(2)$ agrees with $p^{-2}\Phi_{X/S}$ by definition of Tate twists. It follows from Lemma 4.5 that we have

$$\Phi_{X/S}(\tilde{\omega}_m) \equiv p\tilde{\omega}_n \mod K((t))\tilde{\eta}_n.$$ 

Therefore

$$\text{LHS of (4.23)} \equiv -(F_n(t) - F_n(t^\sigma))\frac{dt}{t} \wedge \tilde{\omega}_n \mod K((t))\tilde{\eta}_n.$$ 

On the other hand, it follows from Proposition 4.1 that we have

$$\text{RHS of (4.23)} \equiv \left(E_{(n)}(1)(t)\right)' dt \wedge \tilde{\omega}_n \mod K((t))\tilde{\eta}_n.$$ 

We thus have

$$\frac{d}{dt} E_{(n)}(1)(t) = F_n(t) - F_n(t^\sigma)$$

namely

$$E_{(n)}(1)(t) = C + \int_0^t F_n(t) - F_n(t^\sigma)\frac{dt}{t}$$

for some constant $C \in K$. We determine the constant $C$ in the following way. Firstly $E_{(n)}(1)(t)/F_n(t)$ is an overconvergent function by (4.21). If $C = \psi_p(a_n) + \psi_p(b_n) + 2\gamma_p$, then $E_{(n)}(1)(t)/F_n(t) = \mathcal{F}_{a_n,b_n}(t)$ is a convergent function by Corollary 3.3. If there is another $C'$ such that $E_{(n)}(1)(t)/F_n(t)$ is a convergent function, then it follows

$$\frac{C - C'}{F_n(t)} \in K\langle t, (t - t^2)^{-1}, h(t)^{-1} \rangle.$$ 

This is impossible by Lemma 4.9 below. This means that there is no possibility other than $C = \psi_p(a_n) + \psi_p(b_n) + 2\gamma_p$. This completes the proof. \qed

In the above proof, we use the following lemma.

**Lemma 4.9** Let $s \geq 1$ be an integer, and let $a = (a_1, \ldots, a_s) \in \mathbb{Z}_{p^s}$. Suppose that there are infinitely many $k \in \mathbb{Z}_{\geq 0}$ such that $a_1^{(k)} \cdots a_s^{(k)} \not\equiv 0 \mod p$ where $(-)^{(k)}$ denotes the $k$-th Dwork prime. Then, for all $i \in \mathbb{Z}_{\geq 0}$ the hypergeometric power series

$$F^{(i)}(t) = F_{a^{(i)}}(t) = \sum_{n=0}^{\infty} \frac{(a_1^{(i)})_n}{n!} \cdots \frac{(a_s^{(i)})_n}{n!} t^n$$

cannot be a convergent function.

**Proof.** Thanks to the Dwork congruence, one has

$$\frac{F^{(i)}(t)_{<p^{s+1}}}{(F^{(i+1)}(t)_{<p^s})^p} \equiv F^{(i)}(t)_{<p} \mod p\mathbb{Z}_{p}[[t]]$$

33
for any $i, n \in \mathbb{Z}_{\geq 0}$. This implies

$$F^{(i)}(t)_{<p^n} \equiv F^{(i)}(t)_{<p}(F^{(i+1)}(t)_{<p})^p \cdots (F^{(i+n-1)}(t)_{<p})^{p^{n-1}} \mod p\mathbb{Z}_p[t]. \quad (4.25)$$

By the assumption, there are infinitely many $k \in \mathbb{Z}_{\geq 0}$ such that $F^{(k)}(t)_{<p} \in \mathbb{F}_p[t]$ is not a constant. Therefore, the degree of $F^{(i)}(t)_{<p^n} \in \mathbb{F}_p[t]$ goes to infinity as $n \to \infty$.

Now we show that $F^{(i)}(t)$ cannot be a convergent function. If it were, then there is a nonzero polynomial $g(t) \in \mathbb{F}_p[t]$ such that $g(t)F^{(i)}(t) \in \mathbb{F}_p[t]$ turns out to be a polynomial. Hence

$$g(t)F^{(i)}(t) \in \mathbb{F}_p[t]$$

for all sufficiently large $n$, and the degree of the right hand side does not depend on $n$. This is obviously impossible as $\deg F^{(i)}(t)_{<p^n} \to \infty$.

**Remark 4.10** In case $N | (p - 1)$, the main theorem of [AM] gives the complete description of the syntomic regulator. More precisely, let $\lambda = 1 - t$ and let $\sigma_\lambda: W[[\lambda]] \to W[[\lambda]]$ be the $p$-th Frobenius given by $\sigma_\lambda(\lambda) = c\lambda^p$. Let $E^{(n)}_{i, AM}(\lambda)$ be defined in the same way as (4.18) but we take $\sigma_\lambda$ as the Frobenius. Then

$$\frac{d}{d\lambda} E^{(n)}_{1, AM}(\lambda) = \frac{F_n(\lambda)}{1 - \lambda} - (-1)^{\frac{p-1}{2}n} p^{-1} F_n(\lambda^\sigma) \frac{d\lambda^\sigma}{d\lambda}$$

$$\frac{d}{d\lambda} E^{(n)}_{2, AM}(\lambda) = \frac{E^{(n)}_{1, AM}(\lambda) F_n(\lambda)^2}{\lambda - \lambda^2} + (-1)^{\frac{p-1}{2}n} p^{-1} \tau_n(\sigma)(\lambda) \frac{F_n(\lambda^\sigma)}{1 - \lambda^\sigma} \frac{d\lambda^\sigma}{d\lambda}$$

where $\tau_n(\sigma)(\lambda)$ is the log of the period (see [AM] (3.10)), and

$$E^{(n)}_{1, AM}(0) = 0, \quad E^{(n)}_{2, AM}(0) = 2N \sum_{\nu | N} \nu^{-n} \text{ln}_2(p)(\nu).$$

Notice that one can rewrite $E^{(n)}_{2, AM}(0) = 2\psi^{(1)}_p(\frac{1}{t^2}) - \psi^{(1)}_p(\frac{1}{2^2})$ by Theorem 2.5.

Let us compare the proof of Theorem 4.8 with the proof in [AM]. The discussion to obtain (4.24) is the same. Moreover, if $N | (p - 1)$, then one can also obtain

$$\frac{d}{dt} E^{(n)}_2(t) = - \frac{E^{(n)}_1(t)}{t(1-t)(a_n + b_n) F_n(t)^2} + t^{-1} \tau_n(\sigma)(t) F_n(t^\sigma)$$

in the same way as [AM]. On the other hand, the discussion to obtain $E^{(n)}_1(0)$ is completely different (the reader finds that here is much simpler). It seems difficult to determine $E^{(n)}_2(0)$. Indeed the author expects

$$E^{(n)}_2(0) = \frac{1}{2} \left[ -2\gamma_p - \psi_p(a_n) - \psi_p(b_n) + p^{-1} \log c \right]^2 + \frac{1}{2} (\psi^{(1)}_p(a_n) + \psi^{(1)}_p(b_n))$$

with the aid of computer, though he has not succeeded to prove it.
Theorem 4.11 Let \( \alpha \in W \) such that \( \alpha \not\equiv 0, 1 \mod p \). Let \( \sigma_\alpha \) be the Frobenius given by \( t^\sigma = F(\alpha)\alpha^{-p} \) where \( F \) is the Frobenius on \( W \). Let \( f_{\mathbb{Z}_p} : Y_{\mathbb{Z}_p} \to \mathbb{F}_{\mathbb{Z}_p} \) be the integral model in Lemma 2.4. Let \( X_\alpha \) be the fiber at \( t = \alpha \) (\( \Leftrightarrow \lambda = 1 - \alpha \)), which is a smooth projective variety over \( W \). Let

\[
\text{reg}_{\text{syn}} : K_2(X_\alpha) \longrightarrow H^2_{\text{syn}}(X_\alpha, \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(X_\alpha/\mathbb{K}), \quad \mathbb{K} := \text{Frac}(\mathbb{F}_p)
\]

be the syntomic regulator map. Then

\[
\text{reg}_{\text{syn}}(\xi|_{X_\alpha}) = \sum_{n=1}^{N-1} \frac{\zeta_1^n - \zeta_2^n}{N} (\varepsilon_1^{(n)}(\alpha)\omega_n + \varepsilon_2^{(n)}(\alpha)\eta_n).
\]

Proof. This is a direct consequence of the compatibility of 1-extensions in \( \text{Fil}_{\text{F-MIC}}(\mathbb{S}) \) and the rigid syntomic regulator map (see [AM] §6) (especially Prop. 6.4) for the detail).

Theorem 4.12 Let the notation and assumption be as in Theorem 4.11. Suppose further that \( X_\alpha \) has an ordinary reduction. Let \( \langle - , - \rangle : H^1_{\text{dR}}(X_\alpha/\mathbb{K}) \otimes H^1_{\text{dR}}(X_\alpha/\mathbb{K}) \to H^2_{\text{dR}}(X_\alpha/\mathbb{K}) \cong \mathbb{K} \) denote the cup-product pairing. Then we have

\[
\langle \text{reg}_{\text{syn}}(\xi|_{X_\alpha}), \varepsilon_{\text{unit}}^{(-n)} \rangle = \frac{\zeta_1^n - \zeta_2^n}{N} \mathcal{F}_{a_n,b_n}(\alpha)(\omega_n, \varepsilon_{\text{unit}}^{(-n)})
\]

for a unit root vector \( \varepsilon_{\text{unit}}^{(-n)} \in H^1_{\text{dR}}(X_\alpha/\mathbb{K})(-n) \) (i.e there is some \( \varepsilon_\alpha \in W^\times \) such that \( \Phi(\varepsilon_{\text{unit}}^{(-n)}) = \varepsilon_\alpha \varepsilon_{\text{unit}}^{(-n)} \).

Proof. Notice that \( \varepsilon_{\text{unit}}^{(n)} \) agrees with \( \tilde{\eta}_n \) up to constant. Then the desired assertion is immediate from Theorems 4.8 and 4.11.

4.6 Hypergeometric fibrations of Fermat type

Let \( N, M \geq 2 \) be integers. Let \( f : Y \to \mathbb{P}^1 \) be the fibration over \( \mathbb{Q}_p \) whose general fiber \( X_t = f^{-1}(t) \) is the nonsingular projective model of an affine equation

\[
(x^N - 1)(y^M - 1) = t.
\]

We call this a hypergeometric fibration of Fermat type according to [AO2, 3.3]. This is a fibration of curves of genus \( (N-1)(M-1) \), smooth outside \( t = 0, 1, \infty \) and it has a totally degenerate semistable reduction at \( t = 0 \). Put \( S := \text{Spec}(\mathbb{Q}_p[\lambda, (\lambda - \lambda^2)^{-1}]) \subset \mathbb{P}^1 \) and \( X := f^{-1}(S) \). We assume that the divisor \( D := Y \setminus X \) is a NCD. Let \( \overline{Y} = X \times \mathbb{Q}_p \) and \( \overline{f} : \overline{Y} \to \mathbb{P}^1_{\mathbb{Q}_p} \) be the base change. The group \( \mu_N \times \mu_M = \mu_N(\mathbb{Q}_p) \times \mu_M(\mathbb{Q}_p) \) acts on \( \overline{Y} \) in the following way

\[
[\zeta, \nu] \cdot (x, y) = (\zeta x, \nu y), \quad (\zeta, \nu) \in \mu_N \times \mu_M.
\]
We denote by $V(i, j)$ the subspace on which $(\zeta, \nu)$ acts by multiplication by $\zeta^i \nu^j$ for all $(\zeta, \nu)$. Then one has the eigen decomposition

$$H^1_{dr}(\mathcal{X}/\mathcal{S}) = \bigoplus_{i=1}^{N-1} \bigoplus_{j=1}^{M-1} H^1_{dr}(\mathcal{X}/\mathcal{S})(i, j),$$

and each eigenspace $H^1_{dr}(\mathcal{X}/\mathcal{S})(i, j)$ is free of rank 2 over $\mathcal{O}(\mathcal{S})$. Let

$$\omega_{i, j} := x^{i-1}y^{j-1} \frac{M^{-1} dx}{y^{M-1}(x^N - 1)} = -x^{i-1}y^{j-1} \frac{N^{-1} dy}{x^{N-1}(y^M - 1)},$$

(4.26)

$$\eta_{i, j} := \frac{1}{y^M - 1 + t\omega_{i, j}}$$

(4.27)

for integers $i, j$ such that $1 \leq i \leq N - 1$, $1 \leq j \leq M - 1$. Then $\omega_{i, j}$ is the 1st kind, and $\eta_{i, j}$ is the 2nd kind. They form a $\mathcal{O}(\mathcal{S})$-free basis of $H^1_{dr}(\mathcal{X}/\mathcal{S})(i, j)$. Put

$$a_i := 1 - \frac{i}{N}, \quad b_j := 1 - \frac{j}{M}.$$  

(4.28)

and

$$\tilde{\omega}_{i, j} := \frac{1}{F_{a_i, b_j}(t)} \omega_{i, j}, \quad \tilde{\eta}_{i, j} := -t(1-t)^{a_i+b_j}(F_{a_i, b_j}(t)\omega_{i, j} + a_i F_{a_i, b_j}(t)\eta_{i, j})$$

(4.29)

where $F_{a_i, b_j}(t) := {}_2F_1\left(\begin{matrix} a_i, b_j \\ \end{matrix} \right; t)$ is the hypergeometric power series.

**Proposition 4.13**

$$\left( \nabla(\omega_{i, j}) \quad \nabla(\eta_{i, j}) \right) = dt \otimes \left( \omega_{i, j} \quad \eta_{i, j} \right) \left( \begin{array}{ccc} 0 & -b_j(t - t^2)^{-1} \\ -a_i & -a_i + b_j(t - t^2)^{-1} \end{array} \right),$$

$$\left( \nabla(\tilde{\omega}_{i, j}) \quad \nabla(\tilde{\eta}_{i, j}) \right) = dt \otimes \left( \tilde{\omega}_{i, j} \quad \tilde{\eta}_{i, j} \right) \left( \begin{array}{ccc} 0 & -a_i - b_j(t - t^2)^{-2} \\ t^{-1}(1-t)^{-a_i-b_j}F_{a_i, b_j}(t) & 0 \end{array} \right).$$

**Proof.** We may replace the base field with $\mathbb{C}$. For $(\varepsilon_1, \varepsilon_2) \in \mu_N \times \mu_M$, let $\delta(\varepsilon_1, \varepsilon_2)$ be the homology cycles defined in [A] (2.2)). Then it follows from [A] Lem. 2.3 that we have

$$\frac{1}{2\pi \sqrt{-1}} \int_{\delta(\varepsilon_1, \varepsilon_2)} \omega_{i, j} = -\frac{\varepsilon_1^i \varepsilon_2^j}{NM} F_{a_i, b_j}(t), \quad \frac{1}{2\pi \sqrt{-1}} \int_{\delta(\varepsilon_1, \varepsilon_2)} \eta_{i, j} = a_i^{-1} \frac{\varepsilon_1^i \varepsilon_2^j}{NM} F_{a_i, b_j}(t).$$

(4.30)

Thus the proof goes in the same as that of Proposition [4.11].

**Lemma 4.14** Suppose $p > \max(N, M)$. Let $W = W(\mathbb{F}_p)$ be the Witt ring and $K = \text{Frac}(W)$ the fractional field. Then there exists a regular model $f_W : Y_W \to \mathbb{P}^1_W$ over $W$ such that the reduced part of $D_W := Y_W \setminus X_W$ is a relative NCD over $W$, where we put $S_W := \text{Spec}W[t, (t - t^2)^{-1}]$ and $X_W := f_W^{-1}(S_W)$.  

36
\textbf{Proof.} The affine equation

\begin{equation}
y^M = 1 + \frac{t}{x^N - 1}
\end{equation}

defines a regular scheme in $\text{Spec}W[x, y, t, (1 - x^N)^{-1}]$. Letting $z = x^{-1}$, the equation

\begin{equation}
y^M = 1 + \frac{t z^N}{1 - z^N}
\end{equation}

also defines a regular scheme in $\text{Spec}W[z, y, t, (1 - z^N)^{-1}]$. Let $\zeta \in \mu_N$ and $y = w^{-1}$. Then the equation is

\begin{equation}
x - \zeta = \left( x - \zeta + \frac{t}{u(x)} \right) w^M, \quad u(x) := \frac{x^N - 1}{x - \zeta} \in W[[x - \zeta]]^\times
\end{equation}

and this defines a regular scheme in $\text{Spec}W[[x - \zeta]][w, t, t^{-1}]$. We thus have a projective flat morphism $f_W^t : Y_W^t \rightarrow \text{Spec}W[t, t^{-1}]$ with $Y_W^t$ regular. As is easily seen, $f_W^t$ is smooth over $\text{Spec}W[t, (t - t^2)^{-1}]$. The fiber $D_{W^t}^t = (f_W^t)^{-1}(1)$ is not a NCD. More precisely, at the point $(x, y, t) = (0, 0, 1)$ in $\text{Spec}W[x, y, t, (1 - x^N)^{-1}]$, the embedding $D_{W^t}^t \hookrightarrow Y_W^t$ is locally isomorphic to $\{ y^M = x^N \} \hookrightarrow \text{Spec}W[x, y]$. Take the embedded resolution such that the reduced part of the inverse image of $\{ y^M = x^N \}$ is a NCD. We thus have a projective flat morphism $f_W^t : Y_W^t \rightarrow \text{Spec}W[t, t^{-1}]$ with $Y_W^t$ regular, such that it is smooth over $\text{Spec}W[t, (t - t^2)^{-1}]$ and the reduced part of the divisor $(f_W^t)^{-1}(1)$ is a NCD.

Next, we construct a model at $t = 0$. The affine equations (4.31) and (4.32) define the regular scheme around $t = 0$. The equation (4.33) can be written

\begin{equation}
(y^M - 1)(x - \zeta) = \frac{t}{u(x)}
\end{equation}

and this defines a regular scheme in $\text{Spec}W[[x - \zeta, t]][y]$. We thus have a projective flat model $Y_W^0 \rightarrow \text{Spec}W[[t]]$ and one can easily see that the central fiber is already a reduced and normal crossing.

Finally we construct a model at $t = \infty$. Let $s = t^{-1}$ and $z = x^{-1}, y = w^{-1}$. Then

\begin{equation}
(x^N - 1)(y^M - 1) = t \quad \iff \quad w^M = s(x^N - 1)(1 - w^M)
\end{equation}

defines a scheme in $\text{Spec}W[[s]][x, w]$ with singular locus $\{ x^N - 1 = w = s = 0 \}$ which is isomorphic to the $A_M$-singularity $x_1 x_2 = x_3^3$. One can resolve the singularities such that the reduced part of the central fiber at $s = 0$ is a NCD. Moreover

\begin{equation}
(x^N - 1)(y^M - 1) = t \quad \iff \quad z^N w^M = s(1 - z^N)(1 - w^M)
\end{equation}

defines a scheme in $\text{Spec}W[[s, z]][w]$ with singular locus $\{ z = w^N - 1 = s = 0 \}$ which is isomorphic to the $A_N$-singularity $x_1 x_2 = x_3^N$. Hence one can resolve the singularities. Patching the above schemes, we have a projective flat model $f_W^\infty : Y_W^\infty \rightarrow \text{Spec}W[[s]]$.

The desired scheme $Y_W \rightarrow \mathbb{P}_W^1$ is obtained by patching $Y_W^t, Y_W^0$ and $Y_W^\infty$. This completes the proof. □

37
Lemma 4.15 Let \( J(X_W/S_W) \to S_W \) be the jacobian fibration. Let \( \Delta^*_W := \text{Spec} W[[t]][t^{-1}] \to S_W \) and \( J\Delta^*_W := J(X_W/S_W) \times_{S_W} \Delta^*_W \). Let \( \{ \tilde{\omega}_k, \tilde{\eta}_k \}_k \) be a free basis of \( H^1_{\text{dr}}(J\Delta^*_W/\Delta^*_W) \) such that it forms a de Rham symplectic basis of \( K((t)) \otimes H^1_{\text{dr}}(J\Delta^*_W/\Delta^*_W) \) in the sense of \((4.35)\). Then \( \tilde{\omega}_k \) are \( \mathbb{Q} \)-linear combinations of

\[
\tilde{\omega}(\varepsilon_1, \varepsilon_2) = \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} \varepsilon_1^{-i} \varepsilon_2^{-j} \tilde{\omega}_{i,j}, \quad (\varepsilon_1, \varepsilon_2) \in \mu_N \times \mu_M,
\]

and \( \tilde{\eta}_k \) are \( \mathbb{Q} \)-linear combinations of

\[
\tilde{\eta}(\varepsilon_1, \varepsilon_2) = \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} \varepsilon_1^{-i} \varepsilon_2^{-j} \tilde{\eta}_{i,j}, \quad (\varepsilon_1, \varepsilon_2) \in \mu_N \times \mu_M.
\]

Proof. Thanks to Proposition \([4.13]\) together with \([4.30]\), the proof goes in the same way as that of Proposition \([4.3]\) (detail is left to the reader).

We keep the assumption \( p > \max(N, M) \). For \( (\nu_1, \nu_2) \in \mu_N(K) \times \mu_M(K) \), we consider a \( K_2 \)-symbol

\[
\xi = \xi(\nu_1, \nu_2) = \left\{ \frac{x-1}{x-\nu_1}, \frac{y-1}{y-\nu_2} \right\} \in K_2(X_W).
\]

(4.34)

One immediately has

\[
d\log(\xi) = -\sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (1-\nu_1^{-i})(1-\nu_2^{1-j}) \frac{dt}{t} \omega_{i,j}.
\]

(4.35)

Let \( \sigma \) be a \( p \)-th Frobenius on \( W[[t]] \) given by \( \sigma(t) = ct^p \) with \( c \in 1 + pW \). The symbol \( \xi \) defines the 1-extension

\[
0 \to H^2(X/S)(2) \to M_\xi(X/S) \to \mathcal{O}_S \to 0
\]

in the category of Fil-F-MIC(\( S \)). Let \( e_\xi \in \text{Fil}^0 M_\xi(X/S)_{\text{dr}} \) be the unique lifting of \( 1 \in \mathcal{O}_S(S) \). Let \( e^{(i,j)}_k(t) \) and \( E^{(i,j)}_k(t) \) be defined by

\[
e_\xi - \Phi(e_\xi) = \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (1-\nu_1^{1-i})(1-\nu_2^{1-j}) [e^{(i,j)}_1(t) \omega_{i,j} + e^{(i,j)}_2(t) \eta_{i,j}]
\]

(4.36)

\[
e_\xi - \Phi(e_\xi) = \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (1-\nu_1^{1-i})(1-\nu_2^{1-j}) [E^{(i,j)}_1(t) \tilde{\omega}_{i,j} + E^{(i,j)}_2(t) \tilde{\eta}_{i,j}]
\]

(4.37)

where \( \{ \tilde{\omega}_k, \tilde{\eta}_k \} \) is the de Rham symplectic basis as in Lemma \([4.15]\).

Theorem 4.16 Suppose \( p > \max(N, M) \). We have

\[
\frac{E^{(i,j)}_1(t)}{F_{a_1,b_j}(t)} = \mathcal{F}^{(\sigma)}_{a_1,b_j}(t).
\]

38
Hence
\[ \langle \text{reg}_{\text{syn}}(\xi), e_{\text{unit}}^{(-i,-j)} \rangle = (1 - \nu_1^{-i})(1 - \nu_2^{-j}) \prod_{(\alpha, \beta)} \langle \omega_{i,j}, e_{\text{unit}}^{(-i,-j)} \rangle \]
for \( \alpha \in W \) such that \( \alpha \not\equiv 0 \mod p \) where \( \sigma_\alpha(t) = F(\alpha)\alpha^{-ptp} \).

Proof. In the same way as Lemma 4.5, one can show
\[ \Phi(\tilde{\omega}_{i',j'}) \equiv \tilde{\omega}_{i,j} \mod \sum K((t)) \eta_{i,j} \]
where \((i', j')\) are the pair of integers such that \(1 \leq i' \leq N - 1, 1 \leq j' \leq M - 1\) and \(pi' \equiv i \mod N, pj' \equiv j \mod M\). The rest is the same proof as that of Theorems 4.8 and 4.12. \( \square \)

4.7 Syntomic regulator of Fermat curves

We apply Theorem 4.16 to the study of the syntomic regulator of the Fermat curve
\[ F : z^N + w^M = 1, \quad p \nmid NM. \]
The group \( \mu_N \times \mu_M \) acts on \( F \) by \((\varepsilon_1, \varepsilon_2) \cdot (z, w) = (\varepsilon_1 z, \varepsilon_2 w)\). Let \( H^1_{\text{dR}}(F/K)(i, j) \) denote the subspace on which \((\varepsilon_1, \varepsilon_2)\) acts by multiplication by \( \varepsilon_1^i \varepsilon_2^j \). Let
\[ I = \left\{ (i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq N - 1, 1 \leq j \leq M - 1, \frac{i}{N} + \frac{j}{M} \neq 1 \right\}, \]
then
\[ H^1_{\text{dR}}(F/K) = \bigoplus_{(i,j) \in I} H^1_{\text{dR}}(F/K)(i, j) \quad (4.38) \]
and each eigen space \( H^1_{\text{dR}}(F/K)(i, j) \) is one-dimensional with basis \( z^{i-1}w^{j-M}dz \) (e.g. [G] §2). Moreover
\[ H^1_{\text{dR}}(F/K)(i, j) \subset \Gamma(F, \Omega^1_{F/K}) \iff \frac{i}{N} + \frac{j}{M} < 1. \]
In particular, the genus of \( F \) is \( 1 + \frac{1}{2}(NM - N - M - \gcd(N, M)) \).

Theorem 4.17 Suppose that \( p > \max(N, M) \). Let \( F : z^N + w^M = 1 \) be the Fermat curve over \( W \). Let \( \{1 - z, 1 - w\} \in K_2(F) \otimes \mathbb{Q} \) be Ross’ element. Let
\[ \text{reg}_{\text{syn}} : K_2(F) \otimes \mathbb{Q} \longrightarrow H^2_{\text{syn}}(F, \mathbb{Q}_p(2)) \cong H^1_{\text{dR}}(F/K) \]
be the syntomic regulator map and let \( A^{(i,j)} \in K \) be defined by
\[ \text{reg}_{\text{syn}}(\{1 - z, 1 - w\}) = \sum_{(i,j) \in I} A^{(i,j)} M^{-1} z^{i-1}w^{j-M}dz. \]
Suppose that \((i, j) \in I\) satisfies the following (see also Lemma 4.18 below)

\[(i) \ \frac{i}{N} + \frac{j}{M} < 1; \quad (ii) \ F_{\frac{i}{N}, \frac{j}{M}}(1)_{<p^n} \neq 0 \pmod{p}, \forall n \geq 1. \quad (4.39)\]

Then we have

\[A^{(i,j)} = F^{(\sigma)}_{\frac{i}{N}, \frac{j}{M}}(1) \quad (4.40)\]

where \(\sigma = \sigma_1\) (i.e. \(\sigma(t) = t^p\)).

Notice that the special value \(F^{(\sigma)}_{\frac{i}{N}, \frac{j}{M}}(1)\) is defined under the condition \((4.39)\) (ii).

**Lemma 4.18**

1. Let \(a, b \in \mathbb{Z}_p\). Then \(F_{a, b}(1)_{<p^n} \neq 0 \pmod{p}\) for all \(n \geq 1\) if and only if \(F_{a(k), b(k)}(1)_{<p^n} \neq 0 \pmod{p}\) for all \(k \geq 0\) where \(a(k)\) denotes the Dwork k-th prime.

2. Let \(a_0, b_0 \in \{0, 1, \ldots, p-1\}\) satisfy \(a \equiv -a_0\) and \(b \equiv -b_0 \pmod{p}\). Then

\[F_{a, b}(1)_p \equiv \frac{\Gamma(1 + a_0 + b_0)}{\Gamma(1 + a_0)\Gamma(1 + b_0)} = \frac{(a_0 + b_0)!}{a_0!b_0!} \pmod{p}.\]

In particular

\[F_{a, b}(1)_p \neq 0 \iff a_0 + b_0 \leq p - 1.\]

3. Suppose that \(N \mid (p - 1)\) and \(M \mid (p - 1)\). Then for any \((i, j)\) such that \(0 < i < N\) and \(0 < j < M\) and \(i/N + j/M < 1\), the conditions \((4.39)\) hold.

**Proof.** (1) is a consequence of the Dwork congruence (see also \((4.25)\)). We show (2). Obviously \(F_{a, b}(t)_{<p} \equiv F_{-a_0, -b_0}(t)_{<p} \pmod{p}\), and \(F_{-a_0, -b_0}(t)_{<p} = F_{a_0, b_0}(t)\) as \(a_0\) and \(b_0\) are non-positive integers greater than \(-p\). Then apply Gauss’ formula (e.g. \([\text{NIST}]\) 15.4.20)

\[2F_1\left( \begin{array}{c} a, b \\ c \\ \end{array} ; 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re}(c-a-b) > 0.\]

To see (3), letting \(a = i/N\) and \(b = j/M\), we note that \(a(k) = a, b(k) = b\) and \(a_0 = i(p-1)/N, b_0 = j(p-1)/M\). Then the condition \((4.39)\) (ii) follows by (1) and (2).

Recall from \((4.34)\) the \(K_2\)-symbol \(\xi = \xi(\nu_1, \nu_2)\). Let \(\sigma\) be the \(p\)-th Frobenius on \(K[t, (t - t^2)^{-1}]\) given by \(\sigma(t) = t^p\), and let

\[0 \rightarrow H^2(X/S)(2) \rightarrow M_\xi(X/S) \rightarrow \mathcal{O}_S \rightarrow 0 \quad (4.41)\]

be the 1-extension in Fil-F-MIC(S, \(\sigma\)) associated to \(\xi\). Let \(e_\xi \in \text{Fil}^0M_\xi(X/S)_{\text{dR}}\) be the unique lifting of \(1 \in \mathcal{O}(S)\). Let

\[e_\xi - \Phi_\sigma(e_\xi) = \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (1 - \nu_1^{-i})(1 - \nu_2^{-j})[\varepsilon_{1, \sigma}^{(i,j)}(t)\omega_{i,j} + \varepsilon_{2, \sigma}^{(i,j)}(t)\eta_{i,j}]\]

\[= \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} (1 - \nu_1^{-i})(1 - \nu_2^{-j})[E_{1, \sigma}^{(i,j)}(t)\tilde{\omega}_{i,j} + E_{2, \sigma}^{(i,j)}(t)\tilde{\eta}_{i,j}].\]

40
be as in (4.36) and (4.37) where we write “σ” to emphasize that they depend on σ. Take another Frobenius τ on $K[t, (t - t^2)]$ given by $τ(t) = 1 - (1 - t)^p$. In other words, letting $λ := 1 - t$ be another parameter, τ is the Frobenius on $K[λ, (λ - λ^2)^{-1}]$ given by $τ(λ) = λ^p$.

Define $ε_{k,τ}(λ)$ by

$$e_ξ - Φ_τ(e_ξ) = \sum_{i=1}^{M} \sum_{j=1}^{N-1} (1 - ν_1^{-i})(1 - ν_2^{-j})[ε_{i,τ}(λ)ω_{i,j} + ε_{i,j}(λ)η_{i,j}].$$

arising from the 1-extension in Fil-F-MIC($S, τ$). The relation to $ε^{(i,j)}(t)$ is the following (e.g. [EK, 6.1], [Ke, 17.3.1])

$$Φ_τ(e_ξ) - Φ_σ(e_ξ) = \sum_{n=1}^{∞} \frac{(t - t^σ)^n}{n!} Φ_σ δ^t_τ e_ξ$$

where $δ^t_i = \nabla_{\frac{d}{dt}}$ is the differential operator on $M_ξ(X/S)_{dR}$.

**Lemma 4.19** Let $1 \leq i \leq N - 1$ and $1 \leq j \leq M - 1$ be integers, and put $a_i := 1 - i/N$ and $b_j := 1 - j/M$. Put

$$f_n(t) = f_{n,i,j}(t) := \frac{1}{F_{a_i,b_j}(t)} \left( \frac{d^{n-1}}{dt^{n-1}} \frac{F_{a_i,b_j}(t)}{t} \right)^{τ}. $$

for $n \in \mathbb{Z}_{≥1}$. Then

$$ε^{(i,j)}_{1,τ}(λ) - F_{a_i,b_j}(t)^{τ} = \sum_{n=1}^{∞} \frac{(t - t^σ)^n}{n!} F_{a_i,b_j}(t)^{τ} f_n(t) + a_i^{-1} \frac{F_{a_i,b_j}(t)^{τ}}{F_{a_i,b_j}(t)} ε_{2,τ}^{(i,j)}(λ).$$

Notice that $f_n(t)$ is a convergent function by [Dw] p.37, Thm. 2, p.45 Lem. 3.4]

**Proof.** By (4.35),

$$δ_τ(e_ξ) = \sum_{i=1}^{M} \sum_{j=1}^{N-1} (1 - ν_1^{-i})(1 - ν_2^{-j}) \frac{1}{t} ω_{i,j} = \sum_{i=1}^{M} \sum_{j=1}^{N-1} (1 - ν_1^{-i})(1 - ν_2^{-j}) \frac{F_{a_i,b_j}(t)}{t} \tilde{ω}_{i,j}. $$

By Proposition 4.13

$$δ^t_τ(e_ξ) = \sum_{i=1}^{M} \sum_{j=1}^{N-1} (1 - ν_1^{-i})(1 - ν_2^{-j}) \frac{d^{n-1}}{dt^{n-1}} \left( \frac{F_{a_i,b_j}(t)}{t} \right) \tilde{ω}_{i,j} + G_{n,i,j} \tilde{η}_{i,j}$$

with some $G_{n,i,j}$. Apply this to (4.45). Then we have

$$ε^{(i,j)}_{1,τ}(λ) - ε^{(i,j)}_{1,σ}(t) = \sum_{n=1}^{∞} \frac{(t - t^σ)^n}{n!} F_{a_i,b_j}(t)^{τ} f_n(t) + a_i^{-1} \frac{F_{a_i,b_j}(t)^{τ}}{F_{a_i,b_j}(t)} ε_{2,τ}^{(i,j)}(λ) - ε_{2,σ}^{(i,j)}(t)$$

and hence

$$ε^{(i,j)}_{1,τ}(λ) - Φ_{a_i,b_j}(t)^{τ} = \sum_{n=1}^{∞} \frac{(t - t^σ)^n}{n!} F_{a_i,b_j}(t)^{τ} f_n(t) + a_i^{-1} \frac{F_{a_i,b_j}(t)^{τ}}{F_{a_i,b_j}(t)} ε_{2,τ}^{(i,j)}(λ)$$

by (4.29) and (4.43). Since $F_{a_i,b_j}(t) = E_{1,σ}^{(i,j)}(t)/F_{a_i,b_j}(t)$ by Theorem 4.16 the assertion follows
Lemma 4.20 Let $L$ be the least common multiple of $N, M$. Let $\lambda = 1 - t$. Let $\pi : \text{Spec} W((\lambda_0)) \to S_W$ be given by $\lambda_0^L = \lambda$ and put $\hat{X}_W := X_W \times_{S_w} \text{Spec} W((\lambda_0))$. Then there is a Cartesian diagram

$$\begin{array}{ccc}
\hat{X}_W & \longrightarrow & \hat{Y}_W \\
\downarrow & & \downarrow \omega_w \\
\text{Spec} W((\lambda_0)) & \longrightarrow & \text{Spec} W[[\lambda_0]]
\end{array}$$

such that $\hat{Y}_W$ is regular and the central fiber $Z$ over $\lambda_0 = 0$ is a relative NCD over $W$ containing two Fermat curves $z^N + w^M = 1$. Moreover let $J := J(\hat{X}_W/W((\lambda_0))) \to \text{Spec} W((\lambda_0))$ be the jacobian fibration, and $J(\alpha) \subset J$ the component associated to the eigen space $\sum_{i/N + j/M \neq 1} H^1_{\text{dR}}(J/W((\lambda_0)))(i, j)$. Then $J(\alpha)$ has a good reduction at $\lambda_0 = 0$.

Proof. We begin with the scheme

- $U_1 = \text{Spec} W[[\lambda_0]][x, y, (x^N - 1)^{-1}]/(y^M - (x^N - \lambda_0^L)(x^N - 1)^{-1})$
- $U_2 = \text{Spec} W[[\lambda_0]][x, w]/(w^M - (x^N - \lambda_0^L)(x^N - 1)^{M-1})$
- $U_3 = \text{Spec} W[[\lambda_0]][z, y, (z^N - 1)^{-1}]/(y^M - (1 - \lambda_0^L z^N)(1 - z)^{-1})$

glued by $w = y(x^N - 1)$ and $x = z^{-1}$. Then $U = U_1 \cup U_2 \cup U_3 \to \text{Spec} W[[\lambda_0]]$ is projective. $U_3$ is regular. $U_2$ has a singular locus $\{x^N - 1 = w = 0\}$, which one can resolve by normalization. Let $U' \to U$ be the normalization. Then $U'$ has an isolated singularity $(x, y, \lambda_0 = 0)$ in $U_1$, which is locally isomorphic to $\{y^M = x^N - \lambda_0^L \}$ in $\text{Spec} W[[x, y, \lambda_0]]$. Thus one can resolve the singularity $\hat{Y}_W \to U'$ in a standard way, and there is one Fermat curve given by

$$E : (y/\lambda_0^m)^M = (x/\lambda_0^n)^N - 1, \quad mM = nN = L$$

in the exceptional divisor. On the other hand, the proper transform of the curve

$$F : U_1 \cap \{\lambda_0 = 0\} = \{y^M = x^N(x^N - 1)^{-1}(\Leftrightarrow x^{-N} + y^{-M} = 1)\}$$

is also the Fermat curve, so that there are two Fermat curves in the fiber at $\lambda_0 = 0$.

The jacobian fibration $J$ has a semistable reduction. Let $J(\alpha)_0 \to \text{Spec} W$ be the semistable scheme at $\lambda_0 = 0$. Then it follows from (4.38) that one sees that the natural homomorphism

$$J(E) \times J(F) \longrightarrow J(\alpha)_0$$

is surjective so that there is no torus part of $J(\alpha)_0$. This means that $J(\alpha)$ has a good reduction at $\lambda_0 = 0$. \qed

Lemma 4.21 Let $\hat{Y}_W$ be as in Lemma 4.20. Put $\hat{Y}_K := \hat{Y}_W \times_{W[[\lambda_0]]} K[[\lambda_0]]$ where $K = \text{Frac} W$ is the fractional field. Let $H^1_{\text{dR}}(\hat{Y}_K/K[[\lambda]])(i, j)$ be the eigen component. If $a_i + b_j < 1$, then it has a basis $\omega_{i,j}$ and $\lambda^k \eta_{i,j}$ where $a_i := 1 - i/N$, $b_j := 1 - j/M$ and $k = L(a_i + b_j)$.
**Proof.** Notice that if \( a_i + b_j \neq 1 \), then \( H^1_{\text{dR}}(\hat{Y}_K/K[[\lambda_0]])(i,j) \) is a free \( K[[\lambda_0]] \)-module of rank 2 by Lemma 4.20. The basis is obtained by Deligne’s canonical extension, namely it is enough to check that the residue

\[
\text{Res}(\nabla) : H/\lambda_0 H \to H/\lambda_0 H, \quad H := K[[\lambda_0]]\omega_{i,j} + K[[\lambda_0]]\eta_{i,j}
\]

of the Gauss-Manin connection is zero when \( a_i + b_j < 1 \). However this is immediate from Proposition 4.13. \( \square \)

**Lemma 4.22** If \( a_i + b_j < 1 \), then \( \text{ord}_{\lambda=0}(\varepsilon_{1,\tau}^{(i,j)}(\lambda)) \geq 0 \) and \( \text{ord}_{\lambda=0}(\varepsilon_{2,\tau}^{(i,j)}(\lambda)) \geq 1 \).

**Proof.** Since the \( K_2 \)-symbol \( \xi \) has no boundary at \( \lambda = 0 \), the right hand side of (4.44) belongs to \( H^1_{\text{dR}}(\hat{Y}_K/K[[\lambda_0]])(i,j) \). By Lemma 4.21 this implies

\[
\varepsilon_{1,\tau}^{(i,j)}(\lambda), \lambda_0^{-k}\varepsilon_{2,\tau}^{(i,j)}(\lambda) \in K[[\lambda_0]],
\]

so that the assertion follows. \( \square \)

**Lemma 4.23** If \( a_i + b_j < 1 \) and \( F_{a_i,b_j}(1) \equiv 0 \mod p \) for all \( n \geq 1 \), then

\[
\varepsilon_{1,\tau}^{(i,j)}(0) = \varepsilon^{(\sigma)}_{a_i,b_j}(1)
\]

where the left hand side denotes the evaluation at \( \lambda = 0 \) (\( \Leftrightarrow t = 1 \)) and the right hand side denotes the evaluation at \( t = 1 \). Note that the left value is defined by Lemma 4.22.

**Proof.** This is straightforward from Lemma 4.19 on noticing that \( F'_{a_i,b_j}(t)/F_{a_i,b_j}(t) \) and \( f_n(t) \) are convergent at \( t = 1 \) by [Dw] p.45, Lem. 3.4 ] under the condition that \( F_{a_i,b_j}(1) \equiv 0 \mod p \) for all \( n \geq 1 \). \( \square \)

**Proof of Theorem 4.17** Let \( F \subset \hat{Y}_W \) be the Fermat curve in the central fiber given in (4.46). Put \( z = x^{-1} \) and \( w = y^{-1} \). Then

\[
\omega_{i,j}\mid F = -M^{-1}z^{N-i-1}w^{-j}dz
\]

(4.47)

where \((-)\mid F\) denotes the pull-back

\[
H^1_{\text{dR}}(\hat{Y}_K/K[[\lambda_0]]) \otimes K[[\lambda_0]]/(\lambda_0) \to H^1_{\text{dR}}(F/K).
\]

The symbol \( \xi \) in (4.34) can be regarded as an element of \( K_2(\hat{Y}_W) \). Let \( 0 < i < N, 0 < j < M \) and let

\[
\text{reg}_{\text{syn}}(\xi\mid F)(i,j) \in H^1_{\text{dR}}(F/K)(i,j) = K\omega_{i,j}\mid F
\]

denote the eigen component of the syntomic regulator. Then, by Lemma 4.21

\[
\text{reg}_{\text{syn}}(\xi\mid F)(i,j) = -(1 - \nu_1^{-i})(1 - \nu_2^{-j})[\varepsilon_{1,\tau}^{(i,j)}(\lambda_0^L)\omega_{i,j} + \lambda_0^{-k}\varepsilon_{2,\tau}^{(i,j)}(\lambda_0) \cdot (\lambda_0^k\eta_{i,j})]_F
\]

43
for \((i, j)\) such that \(a_i + b_j < 1\). By Lemma 4.22, the second term vanishes. Hence by Lemma 4.23, we have

\[
\text{reg}_{\text{syn}}(\xi | F)(i, j) = \text{reg}_{\text{syn}}(\xi(\nu_1, \nu_2) | F)(i, j) = -(1 - \nu_1^{-i})(1 - \nu_2^{-j})\mathcal{F}^{(\sigma)}_{a_i, b_j}(1)\omega_{i,j}|_F
\]

for \((i, j)\) such that \(a_i + b_j < 1\) and \(F_{a_i, b_j}(1) < p^n \not\equiv 0 \mod p\) for all \(n \geq 1\). Taking the summation over \((\nu_1, \nu_2) \in \mu_N \times \mu_M\), we have

\[
\text{reg}_{\text{syn}}(\Xi | F)(i, j) = -N M \mathcal{F}^{(\sigma)}_{a_i, b_j}(1)\omega_{i,j}|_F
\]  

(4.48)

where we put

\[
\Xi := \sum_{(\nu_1, \nu_2) \in \mu_N \times \mu_M} \xi(\nu_1, \nu_2) = \left\{ \frac{(x - 1)^N}{x^N - 1}, \frac{(y - 1)^M}{y^M - 1} \right\} = \left\{ \frac{(1 - z)^N}{1 - z^N}, \frac{(1 - w)^M}{1 - w^M} \right\}.
\]

The symbol \(\Xi | F \in K_2(F)\) is

\[
\{ (1 - z)^N, (1 - w)^M \} - \{ (1 - z)^N, 1 - w^M \} - \{ 1 - z^N, (1 - w)^M \} + \{ 1 - z^N, 1 - w^M \} = \{ (1 - z)^N, (1 - w)^M \} - \{ (1 - z)^N, z^N \} - \{ w^M, (1 - w)^M \} + \{ w^M, 1 - w^M \} = N M \{ 1 - z, 1 - w \}.
\]

This is Ross’ element. Hence (4.47) and (4.48) gives (4.40). This completes the proof of Theorem 4.17.

In [R], Ross showed the non-vanishing of the Beilinson regulator

\[
\text{reg}_B \{ 1 - z, 1 - w \} \in H^2_{\mathcal{D}}(F, \mathbb{R}(2)) \cong H^1_B(F, \mathbb{R})^{F}\sim \cong -1
\]

of his element in the Deligne-Beilinson cohomology group. We expect the non-vanishing also in the \(p\)-adic situation.

**Conjecture 4.24** Under the condition (4.39), \(\mathcal{F}^{(\sigma)}_{\frac{N}{N}, \frac{M}{M}}(1) \neq 0\).

By the congruence relation for \(\mathcal{F}^{(\sigma)}_{\frac{N}{N}, \frac{M}{M}}(i)\) (Theorem 3.2), the non-vanishing \(\mathcal{F}^{(\sigma)}_{\frac{N}{N}, \frac{M}{M}}(1) \neq 0\) is equivalent to

\[
G^{(\sigma)}_{\frac{N}{N}, \frac{M}{M}}(1) < p^n \not\equiv 0 \mod p^n
\]

for some \(n \geq 1\). A number of computations by computer indicate that this holds (possibly \(n \neq 1\)). Moreover if the Fermat curve has a quotient to an elliptic curve over \(\mathbb{Q}\), one can expect that the syntomic regulator agrees with the special value of the \(p\)-adic \(L\)-function according to the \(p\)-adic Beilinson conjecture by Perrin-Riou [P 4.2.2]. See Conjecture 4.32 below for detail.
4.8 Syntomic Regulators of elliptic curves

The method in the previous sections works not only for the hypergeometric fibrations but also for the elliptic fibrations listed in [A, §5]. We here give the results together with a sketch of the proof because the discussion is similar to the previous sections.

**Theorem 4.25** Let \( p \geq 5 \) be a prime number. Let \( f : Y \to \mathbb{P}^1 \) be the elliptic fibration defined by an affine equation \( 3y^2 = 2x^3 - 3x^2 + 1 - t \). Put \( \omega = dx/y \). Let

\[
\xi := \left\{ \frac{y - x + 1}{y + x - 1}, \frac{t}{2(x - 1)^3} \right\} \in K_2(X), \quad X := Y \setminus f^{-1}(0, 1, \infty).
\]

Let \( \alpha \in W \) satisfy that \( \alpha \not\equiv 0, 1 \mod p \) and \( X_\alpha \) has a good ordinary reduction where \( X_\alpha \) is the fiber at \( t = \alpha \). Let \( \sigma_\alpha \) denote the \( p \)-th Frobenius given by \( \sigma_\alpha(t) = F(\alpha)\alpha^{-p}t^p \). Then for a unit root \( e_{\text{unit}} \in H^1_{\text{dR}}(X_\alpha/K) \), we have

\[
\langle \text{reg}_{\text{syn}}(\xi|X_\alpha), e_{\text{unit}} \rangle = F^{(\sigma_\alpha)}(\alpha) \langle \omega, e_{\text{unit}} \rangle.
\]

**Proof.** (sketch). We first note that

\[
d\log(\xi) = \frac{dx}{y} dt/t = \omega \wedge dt/t.
\]

Let \( \mathcal{E} \) be the fiber over the formal neighborhood \( \text{Spec} \mathbb{Z}_p[[t]] \to \mathbb{P}^1_{\mathbb{Z}_p} \). Let \( \rho : \mathbb{G}_m \to \mathcal{E} \) be the uniformization, and \( u \) the uniformizer of \( \mathbb{G}_m \). Then we have

\[
\rho^*\omega = F(t) \frac{du}{u}
\]

and a formal power series \( F(t) \in \mathbb{Z}_p[[t]] \) satisfies the Picard-Fuchs equation, which is explicitly given by

\[
(t - t^2) \frac{d^2y}{dt^2} + (1 - 2t) \frac{dy}{dt} - \frac{5}{36} y = 0.
\]

Therefore \( F(t) \) coincides with the hypergeometric power series

\[
F_{\frac{1}{6}, \frac{5}{6}}(t) = 2F_1 \left( \frac{1}{6}, \frac{5}{6}; t \right)
\]

up to multiplication by a constant. Looking at the residue of \( \omega \) at the point \( (x, y, t) = (1, 0, 0) \), one finds that the constant is 1. Hence we have

\[
\rho^*\omega = F_{\frac{1}{6}, \frac{5}{6}}(t) \frac{du}{u}.
\]

Then the rest of the proof goes in the same way as Theorem 4.8. \( \square \)
Theorem 4.26 Let $f : Y \to \mathbb{P}^1$ be the elliptic fibration defined by an affine equation $y^2 = x^3 + (3x + 4t)^2$, and

$$\xi := \left\{ \frac{y - 3x - 4t + 4t}{-8t}, \frac{y + 3x + 4t}{8t} \right\}.$$  

Then, under the same notation and assumption in Theorem 4.25, we have

$$\langle \text{reg}_{\text{syn}}(\xi|_{X_\alpha}), e_{\text{unit}} \rangle = F_{\frac{3}{4}}(\sigma_\alpha) \langle \omega, e_{\text{unit}} \rangle.$$  

Proof. Let $\mathcal{E}$ be the fiber over the formal neighborhood $\text{Spec} \mathbb{Z}_p[[t]] \to \mathbb{P}^1_{\mathbb{Z}_p}$, and let $\rho : \mathbb{G}_m \to \mathcal{E}$ be the uniformization. Then one finds

$$d\log(\xi) = -3 \frac{dx \, dt}{y \, t} = -3 \omega \wedge \frac{dt}{t}$$

and

$$\rho^* \omega = \frac{1}{3} F_{\frac{3}{4}}(t) \frac{du}{u}.$$  

The rest is the same as before. \qed

Theorem 4.27 Let $f : Y \to \mathbb{P}^1$ be the elliptic fibration defined by an affine equation $y^2 = x^3 - 2x^2 + (1 - t)x$, and

$$\xi := \left\{ \frac{y - (x - 1)}{y + (x - 1)}, \frac{-tx}{(x - 1)^3} \right\}.$$  

Then, under the same notation and assumption in Theorem 4.25, we have

$$\langle \text{reg}_{\text{syn}}(\xi|_{X_\alpha}), e_{\text{unit}} \rangle = F_{\frac{3}{4}}(\sigma_\alpha) \langle \omega, e_{\text{unit}} \rangle.$$  

Proof. One finds

$$d\log(\xi) = \frac{dx \, dt}{y \, t} = \omega \wedge \frac{dt}{t}$$

and

$$\rho^* \omega = F_{\frac{3}{4}}(t) \frac{du}{u}.$$  

The rest is the same as before. \qed
4.9 Conjecture on Rogers-Zudilin type formulas

In their paper [RZ], Rogers and Zudilin give descriptions of \( L(E, 2) \) in terms of the hypergeometric functions \( \text{}_{3}F_{2} \) or \( \text{}_{4}F_{3} \). It is plausible to expect its \( p \)-adic counter part in view of the \( p \)-adic Beilinson conjecture by Perrin-Riou [P, 4.2.2], [Co, Conj.2.7]. We end this paper by formulating the \( p \)-adic Rogers-Zudilin type formulas with use of our \( p \)-adic hypergeometric functions \( F^{(p)}(\sigma) \) of logarithmic type.

Let 
\[
\begin{align*}
  f : Y & \rightarrow \mathbb{P}^{1}_{\mathbb{Q}}, \\
  X_{\lambda} &= f^{-1}(t) : y^{2} = x(1-x)(1-(1-t)x)
\end{align*}
\]
be the Legendre family of elliptic curves over \( \mathbb{Q} \) where \( t \) is the inhomogeneous coordinate of \( \mathbb{P}^{1} \). This is the hypergeometric fibration in case \((N,A,B) = (2,1,1)\). In this case one has an explicit description of the \( K_{2} \)-symbol in Lemma 4.6 (cf. [A, (4.3)], [AM, Thm. 3.1])

\[
\xi = \begin{cases} 
  y - 1 + x, \\
  y + 1 - x, \\
  \alpha x^{2}(1-x)^{2}
\end{cases} 
\in K_{2}(X_{\alpha}) \tag{4.49}
\]

In view of Theorem 4.12 together with the \( p \)-adic Beilinson conjecture by Perrin-Riou [P, 4.2.2], we expect the following.

**Conjecture 4.28** Let \( \alpha \in \mathbb{Q} \) satisfy that the symbol
\[
\xi|_{X_{\alpha}} = \begin{cases} 
  y - 1 + x, \\
  y + 1 - x, \\
  \alpha x^{2}(1-x)^{2}
\end{cases} 
\in K_{2}(X_{\alpha}) \tag{4.50}
\]
is integral in the sense of Scholl [S] where \( X_{\alpha} \) denote the fiber at \( t = \alpha \). Let \( p \) be a prime such that \( \text{ord}_{p}(\alpha) \geq 0 \) and \( X_{\alpha} \) has a good ordinary reduction at \( p \). Let \( \epsilon_{p} \in \mathbb{Z}_{p} \) denote the Frobenius eigenvalue such that \( |\epsilon_{p}| = 1 \). For a continuous character \( \chi : \mathbb{Z}_{p}^{\times} \rightarrow \mathbb{C}_{p}^{\times} \), let \( L_{p}(X_{\alpha}, \chi, s) \) denote the \( p \)-adic L-function of the elliptic curve \( X_{\alpha} \) by Mazur and Swinnerton-Dyer [MS]. Let \( \sigma_{\alpha} : \mathbb{Z}_{p}[[t]] \rightarrow \mathbb{Z}_{p}[[t]] \) be the \( p \)-th Frobenius given by \( \sigma_{\alpha}(t) = \alpha^{1-p}t^{p} \). Then there is a rational number \( C_{\alpha} \in \mathbb{Q}^{\times} \) not depending on \( p \) such that
\[
(1 - p\epsilon_{p}^{-1})F^{(p)}(\sigma_{\alpha})(\alpha) = C_{\alpha}L_{p}(X_{\alpha}, \omega^{-1}, 0)
\]
where \( \omega \) is the Teichmüller character.

Here are examples of \( \alpha \) such that the symbol (4.50) is integral (cf. [A, 5.4])
\[
\alpha = -1, \pm 2, \pm 4, \pm 8, \pm 16, \pm \frac{1}{2}, \pm \frac{1}{8}, \pm \frac{1}{4}, \pm \frac{1}{16}.
\]

From Theorems 4.25, 4.26 and 4.27 we also have the following conjectures.

**Conjecture 4.29** Let \( \alpha \in \mathbb{Q} \setminus \{0, 1\} \) and let \( X_{\alpha} \) be the elliptic curve over \( \mathbb{Q} \) defined by an affine equation \( 3y^{2} = 2x^{3} - 3x^{2} + 1 - \alpha \). Suppose that the symbol
\[
\begin{cases} 
  y - x + 1, \\
  y + x - 1, \\
  \frac{1 - \alpha}{2(x - 1)^{3}}
\end{cases} 
\in K_{2}(X_{\alpha}) \tag{4.51}
\]
Conjecture 4.30 Let \( \alpha \in \mathbb{Q} \setminus \{0, 1\} \) and let \( X_\alpha \) be the elliptic curve over \( \mathbb{Q} \) defined by an affine equation \( y^2 = x^3 + (3x + 4\alpha)^2 \). Suppose that the symbol

\[
\left\{ \frac{y - 3x - 4\alpha}{-8\alpha}, \frac{y + 3x + 4\alpha}{8\alpha} \right\} \in K_2(X_\alpha)
\]  

(4.52)
is integral in the sense of Scholl [S]. Let \( p > 3 \) be a prime such that \( \text{ord}_p(\alpha) \geq 0 \) and \( X_\alpha \) has a good ordinary reduction at \( p \). Then there is a rational number \( C_\alpha \in \mathbb{Q}^\times \) not depending on \( p \) such that

\[
(1 - p\epsilon_p^{-1}) F_{p+3}(\alpha) = C_\alpha L_p(X_\alpha, \omega^{-1}, 0).
\]

There are infinitely many \( \alpha \) such that the symbol (4.51) is integral. For example, if \( \alpha = 1/n \) with \( n \in \mathbb{Z}_{\geq 2} \) and \( n \equiv 0, 2 \mod 6 \), then the symbol (4.51) is integral (cf. [A], 5.4).

Conjecture 4.31 Let \( \alpha \in \mathbb{Q} \setminus \{0, 1\} \) and let \( X_\alpha \) be the elliptic curve over \( \mathbb{Q} \) defined by an affine equation \( y^2 = x^3 - 2x^2 + (1 - \alpha)x \). Suppose that the symbol

\[
\left\{ \frac{y - (x - 1)}{y + (x - 1)^3}, \frac{-\alpha x}{(x - 1)^3} \right\} \in K_2(X_\alpha)
\]  

(4.53)
is integral in the sense of Scholl [S]. Let \( p > 2 \) be a prime such that \( \text{ord}_p(\alpha) \geq 0 \) and \( X_\alpha \) has a good ordinary reduction at \( p \). Then there is a rational number \( C_\alpha \in \mathbb{Q}^\times \) not depending on \( p \) such that

\[
(1 - p\epsilon_p^{-1}) F_{p+3}(\alpha) = C_\alpha L_p(X_\alpha, \omega^{-1}, 0).
\]

If the denominator of \( j(X_\alpha) = 64(1 + 3\alpha)^3/(\alpha(1 - \alpha)^2) \) is prime to \( \alpha \) (e.g. \( \alpha = 1/n, n \in \mathbb{Z}_{\geq 2} \), then the symbol (4.53) is integral.

From Theorems 4.16 and 4.17 we have the following conjectures.

Conjecture 4.32 Let \( \alpha \in \mathbb{Q} \setminus \{0, 1\} \) and let \( X_\alpha \) be the elliptic curve over \( \mathbb{Q} \) defined by an affine equation \( (x^2 - 1)(y^2 - 1) = \alpha \). Suppose that the symbol

\[
\left\{ \frac{x - 1}{x + 1}, \frac{y - 1}{y + 1} \right\} \in K_2(X_\alpha)
\]  

(4.54)
is integral in the sense of Scholl [S]. Let \( p > 2 \) be a prime such that \( \text{ord}_p(\alpha) \geq 0 \) and \( X_\alpha \) has a good ordinary reduction at \( p \). Let \( \sigma = \sigma_1 \) (i.e. \( \sigma(t) = t^p \)). Then there is a rational number \( C_\alpha \in \mathbb{Q}^\times \) not depending on \( p \) such that

\[
(1 - p\epsilon_p^{-1}) F_{p+3}(\alpha) = C_\alpha L_p(X_\alpha, \omega^{-1}, 0).
\]
If the denominator of \( j(X_\alpha) = 16(\alpha^2 - 16\alpha + 16)^3/((1-\alpha)\alpha^4) \) is prime to \( \alpha \) (e.g. \( \alpha = \pm 2^n \)), then the symbol \( (4.54) \) is integral.

**Conjecture 4.33** Let \( F_{N,M} \) be the Fermat curve defined by an affine equation \( z^N + w^M = 1 \), and \( F_{2,4}^* \) the curve \( z^2 = w^4 + 1 \). Then there are rational numbers \( C_\alpha, C'_\alpha, C''_\alpha \in \mathbb{Q}^\times \) not depending on \( p \) such that

\[
(1 - p\epsilon_p^{-1}) \mathcal{F}^{(\sigma)}_{\frac{3}{5}}(1) = C_\alpha L_p(F_{3,3}, \omega^{-1}, 0),
\]

\[
(1 - p\epsilon_p^{-1}) \mathcal{F}^{(\sigma)}_{\frac{3}{5}}(1) = C'_\alpha L_p(F_{2,4}, \omega^{-1}, 0),
\]

\[
(1 - p\epsilon_p^{-1}) \mathcal{F}^{(\sigma)}_{\frac{3}{5}}(1) = C''_\alpha L_p(F_{2,4}^*, \omega^{-1}, 0).
\]

If we assume that the integral part \( K_2(E)_\mathbb{Z} \) is one-dimensional for any elliptic curve \( E \) over \( \mathbb{Q} \), some cases in the above conjectures probably follow from the main results of [BD] or [B] (the author has not checked out this). However, in the present, it seems hopeless to prove even the finite dimensionality of \( K_2(E)_\mathbb{Z} \). More direct and elementary approach would be desirable toward our conjectures.

**References**

[A] Asakura, M.: *Regulators of \( K_2 \) of Hypergeometric Fibrations*. Res. Number Theory 4 (2018), no. 2, Art. 22, 25 pp.

[AM] Asakura, M. and Miyatani, K.: *F-isocrystal and syntomic regulators via hypergeometric functions*. arXiv:1711.08854.

[AO2] Asakura, M. and Otsubo, N.: *CM periods, CM regulators and hypergeometric functions, II*, Math. Z. 289 (2018), no. 3-4, 1325–1355.

[BD] Bertolini, M. and Darmon, H.: *Kato’s Euler system and rational points on elliptic curves I: A \( p \)-adic Beilinson formula*. Israel J. Math. 199 (2014), 163–188.

[B] Brunault, F.: *Régulateurs \( p \)-adiques explicites pour le \( K_2 \) des courbes elliptiques*. Publications mathématiques de Besançon (2010), pp. 29–57.

[C] Coleman, R.: *Dilogarithms, Regulators and \( p \)-adic L-functions*. Invent. Math. 69 (1982), 171–208.

[Co] Colmez, P.: *Fonctions \( L \) \( p \)-adiques*. Séminaire Bourbaki, Vol. 1998/99, Astérisque No. 266 (2000), Exp. No. 851, 3, 21–58.

[D] Diamond, J.: *The \( p \)-adic log gamma function and \( p \)-adic Euler constants*. Trans. Amer. Math. Soc. 233 (1977), 321–337.

[Dw] Dwork, B.: *\( p \)-adic cycles*. Publ. Math. IHES, tome 37 (1969), 27–115.
[EK] Emerton, M. and Kisin, M.: An introduction to the Riemann-Hilbert correspondence for unit $F$-crystals. Geometric aspects of Dwork theory. Vol. I, II. 677–700, Walter de Gruyter, Berlin, 2004.

[G] Gross, B.: On the Periods of Abelian Integrals and a Formula of Chowla and Selberg. Invent. Math. 45 (1978), 193–211.

[Ke] Kedlaya, K.: $p$-adic differential equations. Cambridge Studies in Advanced Mathematics, 125. Cambridge University Press, Cambridge, 2010.

[MS] Mazur, B. and Swinnerton-Dyer, P.: Arithmetic of Weil curves. Invent. Math. 25 (1974) 1–61.

[P] Perrin-Riou, B.: Fonctions $L$ $p$-adiques des représentations $p$-adiques. Astérisque 229 (1995).

[RZ] M. Rogers and W. Zudilin, From $L$-series of elliptic curves to Mahler measures. Compos. Math. 148 (2012), no. 2, 385–414.

[R] Ross, R.: $K_2$ of Fermat curves and values of $L$-functions, C.R. Acad. Sci. Paris, Serie I. 312 (1991), 1–5.

[S] Scholl, A. J.: Integral elements in $K$-theory and products of modular curves. In: Gordon, B. B., Lewis, J. D., Müller-Stach, S., Saito, S., Yui, N. (eds.) The arithmetic and geometry of algebraic cycles, Banff, 1998, (NATO Sci. Ser. C Math. Phys. Sci., 548), pp. 467–489, Dordrecht, Kluwer, 2000.

[LS] Le Stum, B.: Rigid cohomology. Cambridge Tracts in Mathematics, 172. Cambridge University Press, Cambridge, 2007. xvi+319 pp.

[NIST] NIST Handbook of Mathematical Functions. Edited by Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark. Cambridge Univ. Press, 2010.

Department of Mathematics, Hokkaido University,
Sapporo 060-0810, JAPAN
E-mail: asakura@math.sci.hokudai.ac.jp