Game representations of classes of piecewise definable functions

Luca Motto Ros

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Vienna, Austria

Key words Borel function, Baire function, projective function, measurable function, game, determinacy, Wadge hierarchy

MSC (2000) 03E15, 03E60

We present a general way of defining various reduction games on \( \omega \) which "represent" corresponding topologically defined classes of functions. In particular, we will show how to construct games for piecewise defined functions, for functions which are pointwise limit of certain sequences of functions and for \( \Gamma \)-measurable functions. These games turn out to be useful as a combinatorial tool for the study of general reducibilities for subsets of the Baire space [10].

1 Introduction

The first reduction games which have appeared in the literature are perhaps the Lipschitz game \( G_L \) and the Wadge game \( G_W \) (both were defined by Wadge in his Ph.D. Thesis, see [17]). They are a special kind of infinite two-player zero sum games on \( \omega \) with perfect information, and are designed in such a way that if player II has a legal strategy \( \tau \) in \( G_L \) (resp. \( G_W \)) then from \( \tau \) it can be recovered in a canonical and fixed way a function \( f_\tau \) from the Baire space \( \omega^\omega \) into itself which is Lipschitz with constant 1 (resp. continuous). Conversely, given a Lipschitz with constant 1 (resp. continuous) function \( f : \omega^\omega \to \omega^\omega \) one can construct a legal strategy \( \tau \) for player II in the corresponding game such that \( f = f_\tau \). These games were introduced to study the relations (also called reducibilities) \( \leq_L \) and \( \leq_W \), where for every \( A, B \subseteq \omega^\omega \)

\[ A \leq_L B \Leftrightarrow A = f^{-1}(B) \text{ for some Lipschitz with constant 1} \]

(resp. continuous) function \( f \).

The link between these preorders and the corresponding games is the following: given \( A, B \subseteq \omega^\omega \), a payoff set for \( G_L \) (resp. \( G_W \)) is canonically constructed (see Section 3) in such a way that player II has a winning strategy in \( G_L(A, B) \) (resp. \( G_W(A, B) \)) if and only if \( A \leq_L B \) (resp. \( A \leq_W B \)). Assuming the Axiom of Determinacy AD, or even just the determinacy of the corresponding games \( G_L \) and \( G_W \), Wadge proved that both \( \leq_L \) and \( \leq_W \) induce well-behaved stratifications of the subsets of \( \omega^\omega \) which have turned out to be very useful in various parts of Set Theory (see e.g. [3, 10]).

Some years later, Van Wesep defined, building on work of Wadge, another reduction game, the backtrack game \( G_{bt} \), but at that time it was not clear which should be the “topological” class of functions \( \mathcal{F} \) corresponding to legal strategies for player II in \( G_{bt} \). It was Andretta who solved this problem in [2], by showing that such \( \mathcal{F} \) is exactly the collection of those \( f : \omega^\omega \to \omega^\omega \) for which there is a partition \( \langle P_n \mid n \in \omega \rangle \) of \( \omega^\omega \) into closed sets such that \( f \upharpoonright P_n \) is continuous for every \( n \in \omega \) (see [2, Theorem 21]), which in turn coincide with the collection of the \( \Delta^0_2 \)-functions by a theorem of Jayne and Rogers (see e.g. [12, Theorem 1.1] for a proof of this last result). Another reduction game, namely the eraser game \( G_E \), was defined (essentially) by Duparc in such a way that \( f : \omega^\omega \to \omega^\omega \) is a Baire class 1 function if and only if there is some legal strategy \( \tau \) for II in \( G_E \) such that \( f = f_\tau \). Finally, some work related to this topic was developed in [6] (although in this case there are no reduction games directly involved).
Having all these useful reduction games, it is quite natural to ask if one could also define reduction games for other “natural” collections of functions (this question was explicitly posed by Andretta in his [3]: “Is there a Wadge-style game for higher levels of reducibility, like $\Delta^0_3$ and such?”). More precisely: say that a set of functions $F$ is playable if there is some reduction game $G_*$ such that for every $f: \omega^\omega \to \omega^\omega$, $f \in F$ if and only if there is a legal strategy $\tau$ for $\text{II}$ in $G_*$ for which $f = f_\tau$ (this notion will be completely formalized in Section 3). Clearly not every set of functions is playable: for example, the collection of all functions from $\omega$ into itself is not playable, as a simple cardinality argument shows (the strategies for any game on $\omega$ are always at most $2^{2\omega}$). Nevertheless we can ask the following:

**Question 1.** Which (topologically defined) classes of functions are playable?

The motivation for this problem mainly relies on the fact that the presence of a reduction game provides combinatorial tools for the study of the reducibility induced by the corresponding set of functions — see e.g. [17, 2].

The first partial answer to this general problem was given by Semmes in [14] and in his Ph.D. thesis [13]: there he proposed a game (called tree game) which corresponds to the Borel functions, and some other games (the multitape game $G_M$, the multitape eraser game $G_{ME}$, and the game $G_{1,3}(f)$) which correspond, respectively, to the functions strictly continuous on a $\Pi^0_2$-partition, to the functions which are of Baire class 1 on a $\Pi^0_2$-partition, and to the Baire class 2 functions — see next section for the definitions of these classes of functions.

In this paper we somewhat extend these results providing a positive answer to Question 1 for a wide class of subsets of the Borel functions, and for $\Gamma$-measurable functions (where $\Gamma$ is any boldface pointclass closed under countable unions and finite intersections): therefore the paper is in some respect unusual for a research publication in mathematics, as it mainly consists of definitions and of proofs that these definitions are correct. Nevertheless, in the last section we will also provide some applications of these games which motivate our interest in this subject.

The material of this paper (except for Section 5) mainly comes from Sections 2.2. and 4.8 of the author’s Ph.D. thesis [11] or is obtained via minor variations of the constructions contained therein, but for the reader’s convenience (and to avoid confusions) in the present paper we have adapted most of the terminology and notation used in [11] to the one already used in [14], with the following exception: because of the applications of reduction games to reducibilities for sets of reals given in Section 6, in this paper the payoff set of a reduction game will be defined starting from two sets of reals (see Section 3), whereas in [14] it was defined starting from a (partial) function from the reals into the reals (nevertheless, it is quite easy to see how to modify one kind of presentation into the other).

The constructions we are going to present rely on a very general way of defining games for sets of functions which are piecewise defined, for sets of functions which are (pointwise) limits of certain sequences of functions, and for $\Gamma$-measurable functions: most of the proofs involve some sort of operation for games which allows to transform a sequence of already known reduction games (representing some classes of functions) into a new reduction game which represent the larger class of those functions piecewise in the old classes on a definable partition, or the class of the pointwise limits of the old functions.

The paper is reasonably self-contained and is organized as follows: in Section 2 we will fix some notation, while in Section 3 we will give a precise definition of what should be meant by reduction game and playable set of functions, and give some basic examples (both old and new). In Sections 4 and 5 we will prove our main results, showing how to construct new reduction games (building on other known games): this will give a “uniform” solution to our problem for almost all sets of functions involved in (generalizations of) Wadge’s theory, for $\Gamma$-measurable functions, and for some other related sets of functions. In Section 6 we will give some examples of how to apply the techniques arising from these games to the study of various reducibilities, and we will prove some relationships between the corresponding determinacy axioms. Finally, in Section 7 we will give the optimal condition under which constructions like those presented in Section 4 can be carried out: even if this technical improvement allows to deal with a strictly larger class of sets of functions, for the sake of simplicity we have postponed
it to the last section because it complicates very much the presentation without adding relevant ideas for the construction of the new reduction games.

2 Preliminaries and notation

In most of the applications involving games, one usually assumes AD (or some other axiom of this kind) and then uses the combinatorics arising from the winning strategies in the games under consideration to prove the desired results. However, AD (and, in general, all known determinacy principles which are not restricted to the context of a small definable pointclass, like the pointclass of Borel sets) contradicts the full axiom of choice AC, and therefore in presenting new games and their applications one has to be careful and just use choice principles which do not contradict AD. In this paper, we will always work in $ZF + AC_\omega(\mathbb{R})$ except for Section 6, in which we will sometimes need the Axiom of Dependent Choice (over the reals) $DC(\mathbb{R})$ and the axiom BP, that is the statement “every set of reals has the Baire property”.

Our notation is quite standard and we refer the reader to the monograph [5] for all the undefined symbols and notions. Given two sets $A$ and $B$, we will denote by $B^A$ the collection of all functions from $B$ to $A$. Thus we will denote by $\omega A$ the set of all $\omega$-sequences of elements of $A$, while the collection of the finite sequences of elements of $A$ will be denoted by $<\omega A$ (we will refer to the length of a finite sequence $s$ with the symbol $lh(s)$). As usual in Descriptive Set Theory, the elements of the Baire space $\omega \omega$ will be called reals. If $n, k \in \omega$ we will write $n^{(k)}$ for the sequence $(n, \ldots, n)$ and $\bar{n}$ for the $\omega$-sequence $(n, n, n, \ldots)$. For simplicity of notation, we will also put $\Sigma^0_{<\xi} = \bigcup_{\mu<\xi} \Sigma^0_\mu$, $\Pi^0_{<\xi} = \bigcup_{\mu<\xi} \Pi^0_\mu$ and $\Delta^0_{<\xi} = \bigcup_{\mu<\xi} \Delta^0_\mu$. Let $(\cdot, \cdot): \omega \times \omega \to \omega$ be the bijection $(n, m) = 2^n(2m + 1) - 1$. Then we can define the homeomorphism (where $\omega A$ is endowed with the product topology of the discrete topology on $A$)

$$\bigotimes: \omega A \to \omega A: (x_n \mid n \in \omega) \to \bigotimes_n x_n = (x_n(m) \mid (n, m) \in \omega),$$

and, conversely, the “projections” $\pi_n: \omega A \to \omega A$ defined by $\pi_n(x) = (x(\langle n, m \rangle) \mid m \in \omega)$: clearly, every $\pi_n$ is surjective, continuous and open.

Unless otherwise specified, in what follows all functions should be intended as partial functions, i.e. defined just on some $X \subseteq \omega \omega$ (endowed with the relative topology inherited from $\omega \omega$), and not necessarily on the whole space $\omega \omega$. We denote by $\text{lip}(2^\mathcal{K})$ the collection of the Lipschitz functions with constant less or equal than $2^\mathcal{K}$, and put $\text{lip} = \bigcup_{k \in \mathcal{K}} \text{lip}(2^k)$. Since they played a special role as reducibilities, we will denote by $L$ the set $\text{lip}(1)$, and by $W$ the set of all continuous functions. Moreover, the collection of the Baire class $\xi$ functions (equivalently, $\Sigma^0_{\xi+1}$-measurable functions — see the next paragraph for the definition) will be denoted by $\mathcal{B}_\xi$. Finally, given any nonzero countable ordinal $\xi$, we will denote by $\mathcal{D}_\xi$ the collection of all $\Delta^0_\xi$-functions, i.e. of those $f: X \to \omega \omega$ such that $f^{-1}(D) \in \Delta^0_\xi(X)$ for every $D \in \Delta^0_\xi$.

A pointclass $\mathcal{G} \subseteq \mathcal{P}(\omega \omega)$ is said to be boldface if it is closed under continuous preimages, and is called $\Sigma$-pointclass if it is boldface and closed under countable unions and finite intersections. A set $S \subseteq \omega \omega \times \omega \omega$ is said to be universal for $\mathcal{G}$ if $\mathcal{G}^{\omega \omega} = \{ S_x \mid x \in \omega \omega \}$, where $S_x = \{ y \in \omega \omega \mid (x, y) \in S \}$. A function $f: X \to \omega \omega$ is $\mathcal{G}$-measurable if $f^{-1}(U) \in \mathcal{G}(X)$ for every open set $U$. The collection of such functions is denoted by $\mathcal{F}_\mathcal{G}$. Note that if $\mathcal{F}$ is a $\Sigma$-pointclass then $f \in \mathcal{F}_\mathcal{G}$ if and only if $f^{-1}(U_n) \in \Delta f(X) = \mathcal{G}(X) \cap \Gamma(X)$ for any clopen subbasis $\{ U_n \mid n \in \omega \}$ of the topology of $\omega \omega$. A $\Gamma$-partition of $X \subseteq \omega \omega$ is a family $\langle D_n \mid n \in \omega \rangle$ of pairwise disjoint sets of $\Gamma(X)$ such that $X = \bigcup_{n \in \omega} D_n$. If $\mathcal{G} = \Sigma^0_\xi$ then every $\Gamma$-partition of $X$ is automatically a $\Delta^0_\xi$-partition. Given a sequence $\mathcal{F} = \mathcal{F}_0, \mathcal{F}_1, \ldots$ of sets of functions and a $\Sigma$-pointclass $\mathcal{G}$, we will denote by $\mathcal{D}\mathcal{F}$ the collection of those $f: X \to \omega \omega$ for which there is a $\Gamma$-partition (equivalently, a $\Delta \Gamma$-partition) $\langle D_n \mid n \in \omega \rangle$ of $X$ and a sequence $f_0, f_1, \ldots$ of functions with domain $X$ such that $f_n \in \mathcal{F}_n$ and $f \restriction D_n = f_n \restriction D_n$ for every $n \in \omega$. If $\mathcal{G} = \Sigma^0_\xi$ for some $\xi$, we will simply write $\mathcal{D}\mathcal{F}$ instead of $\mathcal{D}\Sigma^0_\xi$ (in this case we can replace

---

1 When $\mathcal{F}_n = \mathcal{F}$ for every $n$ we will systematically use the symbol $\mathcal{F}$ instead of $\mathcal{F}_n$ in all the notation.
“\(\Delta^0_\zeta\)-partition” with “\(\Pi^0_{<\zeta}\)-partition” in the definition above). In particular, a function in \(D^W_\zeta\) will be said \textit{continuous on a \(\Delta^0_\zeta\)-partition} (equiv. on a \(\Pi^0_{<\zeta}\)-partition). The following minor variation of the previous definition in general gives a different set of functions — see [10, Remark 6.2]: \(\tilde{D}^F_\Gamma\) (resp. \(\tilde{D}^F_\zeta\)) denotes the collection of those \(f : X \to \omega\) for which there is a \(\Delta_\Gamma\)-partition (resp. a \(\Delta^0_\zeta\)-partition or, equivalently, a \(\Pi^0_{<\zeta}\)-partition) \(\langle D_n \mid n \in \omega \rangle\) of \(X\) such that \(f \upharpoonright D_n \in \mathcal{F}_n\) for every \(n \in \omega\). A function in \(\tilde{D}^W_\zeta\) will be said \textit{strictly continuous on a \(\Delta^0_\zeta\)-partition} (equiv. on a \(\Pi^0_{<\zeta}\)-partition). Both the previous definitions are useful: for instance \(D^W_\zeta\) has been used in [10, Section 6] as a natural example of Borel-amenable reducibility, while \(\tilde{D}^W_\zeta\) has been used in [13, Section 5.2] to find a generalization for the level 3 of the theorem of Jayne and Rogers mentioned in the introduction, i.e. to show that \(D_3 = \tilde{D}^W_\zeta\).

Finally, given \(\tilde{F}\) as above, we will denote by \(\lim \tilde{F}\) the collection of those \(f : X \to \omega\) for which there is a sequence of functions \(f_0, f_1, \ldots\) with domain \(X\) such that \(f_n \in \mathcal{F}_n\) and \(f\) is the \textit{pointwise} limit of the sequence \(\langle f_n \mid n \in \omega \rangle\).

In Section 6, we will deal with various reducibilities for sets of reals and with the corresponding hierarchies of complexity of \(\mathcal{P}(\omega)\). Therefore for all the terminology and the results about these concepts we refer the reader to [10] — in fact we suggest to keep a copy of that paper while reading that section in order to compare the various results with the combinatorial arguments proposed here. The unique modification is that here we will sometimes consider reductions from some \(X \subseteq \omega\) to \(\omega\): given such an \(X\), a set of functions \(\mathcal{F}\) with domain \(X\), and sets \(A, B \subseteq \omega\), we say that \(A\) is \(\mathcal{F}\)-reducible to \(B\) (in symbols \(A \leq_T \mathcal{F} B\)) just in case there is some \(f \in \mathcal{F}\) such that for every \(x \in X\)

\[
x \in A \iff f(x) \in B,
\]

that is \(A \cap X = f^{-1}(B)\). Finally, given a set \(\mathcal{F}\) of \textit{totally defined} functions, recall that the \textit{Semi-Linear Ordering Principle} for \(\mathcal{F}\) (denoted by \(\text{SLO}\mathcal{F}\)) is the statement

\[
\forall A, B \subseteq \omega (A \leq_T \mathcal{F} B \lor \neg B \leq_T \mathcal{F} A),
\]

where \(\neg B\) denotes \(\omega \setminus B\).

### 3 Reduction games

As recalled in the introduction, the first examples of reduction games are \(G_L\), \(G_W\), \(G_{br}\) and \(G_E\) (defined in [17, pp. 72 and 64], [15, p. 86], and [4, p. 69], respectively), which represent, respectively, the classes of functions \(L, \mathcal{W}, D_2\) and \(B_1\) (see e.g. [17, Theorem B8] and [14, Theorems 3.1, 4.1, and 5.1]). Here we just give a brief and informal description of the rules of these games. In \(G_L\), both \(I\) and \(II\) have to play a natural number at each of their turns. The game \(G_W\) is a variation of \(G_L\) in which \(II\) has the further option of “passing” (i.e. skipping her move at some turn), but with the condition that at the end of the run she has played infinitely many natural numbers, i.e. she has enumerated a real. The backtrack game \(G_{br}\) is a further variation of \(G_W\) in which \(II\) can still pass (with the same condition above), but even backtrack; i.e. she can delete all her previous moves at once and start to play natural numbers (or pass) anew, with the restriction that in each run she can use this option only finitely many times (this guarantees that at the end of the run she has indeed played some real). Finally, the eraser game \(G_E\) can be described in the following way: \(I\) must play a natural number on each of his turn, while \(II\) can either play a natural number or erase the last natural number which appears on her board, but to guarantee that at the end of each run \(II\) has indeed played a real, we require that for each \(x \in X\) and each \(n \in \omega\) there must be some \(m\) such that for every \(k \geq m\) we have \(lh(t_k) \geq n\), where \(t_k\) is the sequence of natural numbers that \(II\) has played (after possible erasings) when \(I\) has enumerated \(x \upharpoonright k\). In other words, \(II\) has to enumerate a real \(y \in \omega\) and she has the option of changing the \(n\)-th digit of \(y\) at any time, but for each \(n\) she can take this option only finitely many times.

Many other games (both old and new) can be obtained by modifying one of these games, for example:

- a simple variation of \(G_W\) leads to the game \(G_{k,\text{Lip}}\), in which \(II\) can still pass but \textit{at most} \(k\) times in a run (this is also equivalent to requiring that \(II\) pass for the first \(k\) turns and then plays the rest
of the game without passing any more): as we will see in Proposition 3.6, the strategies for $\Pi$ in $G^{X}_{k,\text{Lip}}$ induce exactly the functions in Lip$(2^{k})$;

- a variation of $G_{\text{br}}$ leads instead to the game $G_{\text{Lip-br}}$ in which $\Pi$ can still backtrack finitely many times but is no more allowed to pass: legal strategies for $\Pi$ in $G_{\text{Lip-br}}$ induce exactly the functions in $D^{\text{Lip}}_{2}$ (which is a proper subset of $D_{2}$ — see [10, pp. 45-46]);

- another variation of $G_{W}$ leads to the multitape game $G_{W}$ defined by Semmes in [14, p. 202], which can be described as follows: $I$ has to play his numbers on a single row, while $\Pi$ has to play her numbers on a table containing $\omega$-many rows. At each turn, $I$ plays a natural number on his unique row, while $\Pi$ has first to choose one of her rows and then either pass or play a new natural number on it, with the condition that in each run of the game she has to play an infinite amount of numbers on exactly one of her rows: by Andretta-Semmes’ [14, Theorem 6.1], legal strategies for $\Pi$ in this game induce the functions in $D_{2}$;

- the game $G_{E}$ can be “iterated” by using $n$ eraser operators ranked with a priority in order to obtain games $G_{B,n}$ whose legal strategies for $\Pi$ induce exactly the Baire class $n$ functions.

Various other reduction games, like the multitape eraser game $G_{ME}$ defined in [14, Section 7] which represents $D_{2}^{\text{S1}}$, can be obtained in a similar way. However, each of these games seems to be strictly related to the particular presentation of the game itself, and therefore it seems difficult to guess which should be the definition of a game representing a more complex class of functions, like e.g. $D_{\omega+\omega}^{\text{W5}}$, $D_{\omega^{2}+9}^{\text{B2}}$, or $F_{\Sigma_{1}}$. In order to have a uniform approach to the problem of representing classes of functions by means of games, it is useful to first abstractly define the notion of reduction game (see Subsection 3.1), to isolate some basic examples of such games (like the already defined $G_{W}$, or games representing class of the form $F_{T}$ — see Subsection 3.2 and Section 5), and then find some operations which correspond to the analogous topological operations used in the definition of the new classes of functions, like the operation of taking pointwise limits, or of giving piecewise definitions on a definable partition of the space (this is done in Sections 4 and 7).

### 3.1 Reduction games and playable sets of functions

A reduction game is a tuple $G_{s} = (X, M_{s}, R_{s}, \iota_{s})$ (where $*$ is some symbol which identify the game) such that $X \subseteq \omega$, $M_{s}$ is a countable set disjoint from $\omega$ (called set of moves), $R_{s} \subseteq \omega^{*} \times \omega^{*}(\omega \cup M_{s})$ (called set of rules) and $\iota_{s}$ is a function from $R_{s}$ into $\omega$ (called interpretation function).

The rules of the game are as follows: $I$ plays elements of $\omega$, while $\Pi$ plays elements of $\omega \cup M_{s}$, so that after $\omega$-many turns $I$ will have produced a real $x \in \omega$ while $\Pi$ will have produced $y \in \omega(\omega \cup M_{s})$ (y is called complete play of $\Pi$). Then we give the following two conditions:

1. if $x \notin X$ then $I$ loses;
2. if $x \in X$ but $(x, y) \notin R_{s}$ then $\Pi$ loses.

If $\sigma$ is a strategy for $I$ and $y$ is the complete play of $\Pi$ in a run of $G_{s}$, then $\sigma * y$ denotes the real enumerated by $I$ while following $\sigma$ against $y$. Similarly, if $\tau$ is a strategy for $I$ and $x \in \omega$, we denote by $x * \tau$ the complete play produced by $\Pi$ in the run of $G_{s}$ in which $I$ enumerates $x$ and $\Pi$ plays according to $\tau$ ($\sigma * t$ and $s * \tau$ are defined in a similar way for every $t \in \omega(\omega \cup M_{s})$ and $0 \neq s \in <\omega$). A strategy $\sigma$ for $I$ is said to be legal if $\sigma * y \in X$ for every $y \in \omega(\omega \cup M_{s})$, while a strategy $\tau$ for $\Pi$ is said to be legal if $(x, x * \tau) \in R_{s}$ for every $x \in X$. The collection of legal strategies for $\Pi$ in $G_{s}$ will be denoted by $\text{LS}_{s}$.

Given $A, B \subseteq \omega$, $G_{s}(A, B)$ is defined by the following winning condition: If neither (1) or (2) have occurred, then $\Pi$ wins if and only if $x \in A \iff \iota_{s}(x, y) \in B$ (in this case $A, B$ are called payoff sets of $G_{s}(A, B)$ and $\iota_{s}(x, y)$ is called play or output real of $\Pi$). A strategy (for either $I$ or $\Pi$) is said to be winning in the game $G_{s}(A, B)$ if it is legal and always guarantees the victory of the corresponding player, whatever his or her opponent plays.
Notice that every $\tau \in \text{LS}_\omega$ canonically induces the unique function

$$f_\tau : X \rightarrow \omega^\omega : x \mapsto \iota_\omega(x, x \cdot \tau),$$

(in this case we will say that $\tau$ represents $f$), whereas for some function $f : X \rightarrow \omega^\omega$ there can be distinct $\tau, \tau' \in \text{LS}_\omega$ such that $f = f_\tau = f_{\tau'}$. We will put $F_* = \{ f : X \rightarrow \omega^\omega : f = f_\tau \text{ for some } \tau \in \text{LS}_\omega \}$. With this notation, $\text{II}$ has a winning strategy in the game $G_*(A, B)$ if and only if $A$ is $F_* \text{- reducible}$ to $B$: this is why the games described above are called reduction games.

Usually, the topological definition of a certain class of functions is virtually independent from the particular domain of the functions under consideration (apart from its topology, of course), meaning that the definition of such class uses $X$ just as a sort of parameter. For example, a function is said to be continuous if the preimage of an open set is still open, and this definition does not involve any other information on the domain of the function except for its topology. Therefore, to have a decent notion of representation of a (topologically defined) class of functions $F_*$ (with domain arbitrary subsets of $\omega^\omega$) by means of a certain set of reduction games $G_* = \{ G_* = (X, M_*, R_*, \iota_*) : X \subseteq \omega^\omega \}$, it seems natural to stipulate by convention that all the games in $G_*$ share the same set of moves, set of rules and interpretation function, or at least$^2$ that all sets of rules (and consequently all the interpretation functions) of the games in $G_*$ are defined by a single formula which uses $X$ as a parameter. Such classes of games are said to be parametrized. With a little abuse of notation, if $G_*$ is a parametrized class of games we will denote by $F_*$ again the collection of all functions induced by the legal strategies in one of the games of $G_*$. 

**Definition 1.** Let $F_*$ be any set of functions from subsets of $\omega^\omega$ into $\omega^\omega$. We say that $F_*$ is **playable** if there is a parametrized class $G_*$ of reduction games such that $F_* = G_*$. In this case, we also say that the class $G_*$ represents $F_*$. 

Finally, for $G_* = (\omega^\omega, M_*, R_*, \iota_*)$ we will denote by $\text{AD}(G_*)$ (or simply $\text{AD}^*$) the statement: “for all $A, B \subseteq \omega^\omega$, the game $G_*(A, B)$ is determined (i.e. either $\text{II}$ or $\text{I}$ has a winning strategy)”. $\text{AD}^*$ is obviously a consequence of $\text{AD}$ (to see this it is enough to “code” in the natural way the reduction game $G_*$ into a classical game on $\omega$). Moreover, if $R_*$ and $\iota_*$ are not too complicated, the above implications have also “local versions”. For example, if $R_*$ is a Borel subset of $\omega^\omega \times (\omega \cup M_*)$ (endowed with the product topology) and $\iota_*$ is Borel, then Borel determinacy is sufficient to have that for every Borel $A, B \subseteq \omega^\omega$ the game $G_*(A, B)$ is determined (and, more generally, local versions of $\text{AD}$ imply local versions of $\text{AD}^*$). As we will see, if $F_*$ is a playable subset of the Borel functions it is in practice always the case that the parametrized class $G_*$ of reduction games which represents $F_*$ has a Borel set of rules and a Borel interpretation function (defined independently from $X$).

### 3.2 Some examples

Now we want to give some examples on how to formalize the games presented at the beginning of this section into reduction games. As the names suggest, the set $M_*$ will be used to code the alternative moves (like “pass”, “backtrack”, “erase”, and so on) of $\text{II}$, $R_*$ will be used to code the rules of the game (that is the rules that $\text{II}$ must respect in order to have a chance of victory), and $\iota_*$ will be used to recover from the (play of $\text{I}$ and the) complete play of $\text{II}$ the real that must be used in checking the winning condition. We will present just three cases, namely continuous functions, Baire class 1 functions, and Lipschitz functions with constant $2^k$: however, we will prove that the corresponding reduction game really represents the desired class of functions just for the last case, as the other two well-known proofs can be obtained using classical arguments (see e.g. [14, Theorems 3.1 and 5.1]).

**Example 3.1.** Define the reduction game $G_W = (X, M_W, R_W, \iota_W)$ by:

- $M_W = \{ p \}$ (the symbol $p$ will be interpreted as “pass”);
- $R_W = \{ (x, y) \in \omega^\omega \times (\omega \cup \{ p \}) : \forall n \exists m \geq n(y(m)) \neq p \}$;
- $\iota_W : R_W \rightarrow \omega^\omega : (x, y) \mapsto (y(n)) | y(n) \neq p, n \in \omega$.

$^2$ We will take this second option just in Section 5, when considering games related to boldface pointclasses $\Gamma$ larger than the collection of Borel sets.
Proposition 3.2. A function \( f: X \rightarrow \omega \) is continuous if and only if \( f = f_{\tau} \) for some \( \tau \in \text{LS}_W \).

Example 3.3. Define the reduction game \( G_E = (X, M_E, R_E, \iota_E) \) by:
- \( M_E = \{ E \} \) (the symbol \( E \) will be interpreted as “erase”, and will correspond to the backspace key of a usual computer keyboard);
- for \( s \in \langle \omega \cup \{ E \} \rangle \), inductively define \( \iota(s) \in \langle \omega \rangle \) by letting \( \iota(\emptyset) = \emptyset, \iota(s \cdot n) = \iota(s) \cdot n \) (for \( n \in \omega \)), and \( \iota(s \cdot E) = \iota(s) \cup \{ \text{lh}(\iota(s)) - 1 \} \);
- \( R_E = \{ (x, y) \in \langle \omega \rangle \times \langle \omega \cup \{ E \} \rangle \mid \forall n \exists m \forall k \geq m (\text{lh}(\iota(y \cdot k)) \geq n) \} \);
- \( \iota_W: R_W \rightarrow \langle \omega \rangle: (x, y) \mapsto \bigcup_n \iota(y \cdot k_n) \), where \( k_n \) is the smallest \( m \) such that \( \forall k \geq m (\text{lh}(\iota(y \cdot k)) \geq n) \).

Proposition 3.4. A function \( f: X \rightarrow \omega \) is of Baire class 1 if and only if \( f = f_{\tau} \) for some \( \tau \in \text{LS}_E \).

Example 3.5. Define the reduction game \( G_{k,\text{Lip}} = (X, M_{k,\text{Lip}}, R_{k,\text{Lip}}, \iota_{k,\text{Lip}}) \) by:
- \( M_{k,\text{Lip}} = \{ p \} \) (the symbol \( p \) will be interpreted as “pass” again);
- \( R_{k,\text{Lip}} = \{ (x, y) \in \langle \omega \cup \{ p \} \rangle \mid \forall n (y(n) = p \iff n < k) \} \);
- \( \iota_{k,\text{Lip}}: R_{k,\text{Lip}} \rightarrow \langle \omega \rangle: (x, y) \mapsto \langle y(n + k) \mid n \in \omega \rangle \).

Proposition 3.6. For every \( k \in \omega \), a function \( f: X \rightarrow \omega \) is in Lip(\( 2^k \)) if and only if there is some \( \tau \in \text{LS}_{k,\text{Lip}} \) such that \( f = f_{\tau} \).

Proof. Let \( \tau \in \text{LS}_{k,\text{Lip}} \). Then \( \langle (s \cdot \tau)(n + k) \mid n + k < \text{lh}(s) \rangle \) has length \( \max\{0, \text{lh}(s) - k\} \), which implies that \( d(f_s(x), f_{s}(x')) \leq 2^k \) for every \( x, x' \in X \). For the other direction, let \( f: X \rightarrow \omega \) be in Lip(\( 2^k \)). Given \( s \in n + k \omega \), let \( t_s \in n \omega \) be the unique sequence such that \( N_s \cap X \subseteq f^{-1}(N_{t_s}) \) (such a sequence must exist since \( f \in \text{Lip}(2^k) \)), and define the strategy \( \tau \) for \( \text{II} \) in \( G_{k,\text{Lip}} \) by \( \tau(s) = p \) if \( \text{lh}(s) < k \) and \( \tau(s) = t_s(\text{lh}(t_s) - 1) \) otherwise. It is clear that \( \tau \) is legal and such that \( f = f_{\tau} \).

4 Constructing new reduction games

Let us start with a technical definition.

Definition 2. Let \( G_s = (X, M_s, R_s, \iota_s) \) be a reduction game and \( \mathcal{F}_s \) be the set of functions induced by the strategies in \( \text{LS}_s \). We say that \( G_s \) (or \( \mathcal{F}_s \)) is \( p \)-closed if \( \mathcal{F}_s \) coincides with the set of functions induced by the legal strategies for \( \text{II} \) in the new reduction game \( G_p \) defined by:
- \( M_p = M_s \cup \{ p \} \), where \( p \) is a new symbol not in \( \omega \cup M_s \);
- \( R_p = \{ (x, y) \in \langle \omega \rangle \times \langle \omega \cup M_p \rangle \mid \forall n \exists m \geq n (x(m) = p) \land (x, \langle y(n) \mid n \in \omega \rangle) \in R_s \} \);
- \( \iota_p: R_p \rightarrow \langle \omega \rangle: (x, y) \mapsto \iota_s(x, (y(n) \mid n \in \omega)) \).

A set of functions \( \mathcal{F} \) is adequate if it contains the identity function and there is a parametrized class \( G_s \) of \( p \)-closed reduction games which represents \( \mathcal{F} \).

Roughly speaking, the condition for a set \( \mathcal{F}_s \) of being \( p \)-closed is the natural counterpart in terms of strategies of the property of being closed under right-composition with continuous functions from \( X \) into itself, while a set of functions is adequate if it is not too small. The previous definition could seem a little bit obscure, but it covers almost all the important cases and is designed in such a way that the arguments presented in the next subsections can be carried out in a very general way. For instance, it is obvious that \( G_W \) is \( p \)-closed, but let us check as a nontrivial example that \( G_E \) (and hence \( B_1 \)) is \( p \)-closed as well. Consider the game \( G_{E}^p \): clearly \( \text{LS}_E \subseteq \text{LS}_E^p \) and, conversely, every \( \tau \in \text{LS}_E^p \) can be converted in a legal strategy for \( \text{II} \) in \( G_E \) by substituting every use of the symbol \( p \) with the pair of moves “play 0 and then play \( E \)”. However, one has also to notice that not all the sets of functions considered in this paper are \( p \)-closed — for a counterexample just take Lip(\( 2^k \)).

---

3 Notice that here we are using the (equivalent) definition of \( G_{k,\text{Lip}} \) as the game in which \( \text{II} \) pass exactly for the first \( k \) turns — see page 5.

4 Notice that, as shown in Section 7, adequateness is not the optimal condition for our purpose: nevertheless, it allows to give an easier presentation on the subsequent constructions avoiding some technical and notational complications, and thus seems to be a good compromise between generality of the arguments and clearness of exposition.
In the next subsections we will show how to construct games for the classes of functions $D^F_\xi$, $\tilde{D}^F_\xi$ and $\lim \tilde{F}$, provided that $\xi$ is some fixed countable nonzero ordinal and $\tilde{F}$ is a countable sequence of not too small (i.e. adequate) playable sets of functions (albeit for simplicity of presentation in Subsections 4.1 and 4.2 we will just deal with the simpler case in which $\tilde{F}$ is constantly equal to some fixed adequate $F$: this covers all the most important cases that one encounters in practice, and the general case can easily be recovered from these particular examples). Taking $F = W$ in Theorems 4.1 and 4.3 we get, in particular, a generalization of the games $G_W$ and $G_M$ for (simultaneously) all higher level (the existence of such games was still an open problem). Moreover, since it is easy to see that the constructions below always produce reduction games which are $p$-closed, by iterating those constructions one can get a wide class of reduction games, namely games representing $D^B_\xi$ and $\tilde{D}^B_\xi$ for every $\xi, \mu < \omega_1$.

4.1 Games for $D^F_\xi$

Fix any increasing sequence of ordinals $\langle \mu_n \mid n \in \omega \rangle$ cofinal in $\xi$ and, for each $n \in \omega$, a set $P_n$ which is $\Pi^0_{\mu_n}$-complete (we will see in Claim 4.0.1 that the choice of the $\mu_n$’s and of the $P_n$’s is not essential). Let $F$ be an adequate set of functions, and let $G_* = (X, M_*, R_*, i_*)$ be a $p$-closed reduction game representing the subset of $F$ consisting of those function which have domain $X$. Let $G_W = (\langle \omega, M_W, R_W, i_W \rangle)$ be the game defined in Example 3.1 representing the set of (totally defined) continuous functions $W$. Now define $G^F_\xi = (X, M^F_\xi, R^F_\xi, i^F_\xi)$ as follows:

- $M^F_\xi = M_* \cup M_W$;
- $R^F_\xi = \{(x, y) \in \omega \times (\omega \cup M^F_\xi) \mid \forall i((x, \pi_{2i}(y)) \in R_W \land (x, \pi_{2i+1}(y)) \in R_*) \land \exists i(i_W(x, \pi_{2i}(y)) \in P_i)\}$;
- $i^F_\xi : R^F_\xi \to \omega : (x, y) \to i_*(x, \pi_{2i+1}(y))$, where $i$ is smallest such that $i_W(x, \pi_{2i}(y)) \in P_i$.

The game $G^F_\xi$ can be visualized as a two-player game in which I has to fill in a table with a single row, while II has to fill in a table with $\omega$-many rows.

| I | A | x₀ | x₁ | x₂ | \ldots | xₖ | xₖ₊₁ | \ldots | → x |
|---|---|---|---|---|---|---|---|---|---|
| P₀ | | c₀ | c₁ | c₂ | \ldots | cₖ | \ldots | → c₀ |
| B | | y₀ | y₁ | y₂ | \ldots | yₖ | \ldots | → y₀ |
| P₁ | | c₁ | c₁ | c₁ | \ldots | cₖ | \ldots | → c₁ |
| II | B | | y₁ | y₁ | y₁ | \ldots | yₖ | \ldots | → y₁ |
| \vdots | | | | | | | | |
| Pₙ | | cₙ | cₙ | cₙ | \ldots | cₙ | \ldots | → cₙ |
| B | | yₙ | yₙ | yₙ | \ldots | yₙ | \ldots | → yₙ |
| \vdots | | | | | | | | |

At the $k$-th turn, I plays a natural number on his (unique) row, while player II has two options (in what follows $n$ is the unique natural number such that $(n, m) = k$ for some/any $m \in \omega$): pass or play a natural number on the $n$-th row if $n = 2i$ is even (but at the end of the run she must have played infinitely many natural numbers on such row, i.e. she must have produced a real $c'$ on it), or else play an element of $\omega \cup M_*$ on her $n$-th rows if $n$ is odd. The even rows are control rows which can activate the rows immediately below them (the odd ones), and this happens exactly when the real $c'$ played on the $2i$-th row belongs to the control set $P_i$. In every run of the game, II has to activate at least one of the odd rows, and she has to make sure that the sequence $y'$ she has played on the $2i + 1$-st row belongs
to the set of rules $R_*$ for every $i \in \omega$ (i.e. she must “respect the rules” of $G_*$ on the odd rows). Finally, the output real played by $\Pi$ is exactly $\iota_*(x, y')$, where $i$ is least such that the $2i + 1$-st row is activated.

Every strategy $\tau$ for $\Pi$ in $G_\xi^F$ can be seen as a sequence $\langle \tau_n \mid n \in \omega \rangle$ of legal strategies for $\Pi$ in the game $G_W = \langle \omega, M_W, R_W, \omega \rangle$ or in $G_* = \langle X, M_*, R_*, \iota_* \rangle$ (depending on whether $n$ is even or odd), each of which is used on the corresponding row. In fact, if $\langle \tau_n \mid n \in \omega \rangle$ is such a sequence we can define the strategy $\tau = \bigotimes_n \tau_n$ for $\Pi$ in $G_\xi^F$ as follows: for each $\emptyset \neq s \in \omega^\omega$, let $m, n \in \omega$ be such that $lh(s) = \langle n, m \rangle + 1$, and define $\tau(s) = \tau_n(s \mid (m + 1))$. It is not hard to check that $\tau = \bigotimes_n \tau_n$ is a (non necessarily legal) strategy for $\Pi$ in $G_\xi^F$ such that $\pi_n(x * \tau) = x * \tau_n$ for every $x \in X$. Thus to define a legal strategy for $\Pi$ in $G_\xi^F$ it is enough to give a sequence $\langle \tau_n \mid n \in \omega \rangle$ of legal strategies for $\Pi$ in $G_W$ (resp. $G_*$) if $n$ is even (resp. odd), and check that for every $x \in X$ there is some $i \in \omega$ such that $\iota_W(x, x * \tau_i) \in P_i$.

Conversely, given a legal strategy $\tau$ for $\Pi$ in $G_\xi^F$ and a natural number $n \in \omega$, we can define the legal strategy $\pi_n(\tau)$ for $\Pi$ in, respectively, $G_W$ if $n$ is even or in $G_\xi^F$ if $n$ is odd (where $G_W^\emptyset$ and $G_\xi^\emptyset$ are defined as in Definition 2) as follows: for each $\emptyset \neq s \in \omega^\omega$, define $\pi_n(\tau)(s) = \emptyset$ if there is no $m \in \omega$ such that $lh(s) = \langle n, m \rangle + 1$, and $\pi_n(\tau)(s) = \tau(s)$ otherwise. Since both $G_W$ and $G_\xi^F$ are $\emptyset$-closed, with a little abuse of notation we will confuse each $\pi_n(\tau)$ (respectively, $\pi_{2n+1}(\tau)$) with any legal strategy in $G_W$ (resp. $G_\xi^F$) which induces the same function $f_{\pi_{2n+1}(\tau)}$ (resp. $f_{\pi_{2n+1}(\tau)}$) on $X$. It is not hard to check that the operations $\pi_n$ on strategies “commute” with $\bigotimes_n$: given a strategy $\tau$ for $\Pi$ in $G_\xi^F$, for every $x \in X$ and $i \in \omega$ we have that $\iota_W(x, \pi_{2n}(x * \tau)) = \iota_W(x, \pi_{2n}(x * \bigotimes_n \pi_n(\tau)))$ and $\iota_\tau(x, \pi_{2n+1}(x * \tau)) = \iota_\tau(x, \pi_{2n+1}(x * \bigotimes_n \pi_n(\tau)))$. In particular, $\tau$ is a legal strategy for $\Pi$ in $G_\xi^F$ if and only if $\bigotimes_n \pi_n(\tau)$ is, and in the positive case the two strategies induce the same function on $X$ (i.e. $f_\tau = f_{\bigotimes_n \pi_n(\tau)}$).

We will now prove that, as already announced, the choice of the ordinals $\mu_n$’s and of the sets $P_n$’s is not essential. Let $\langle \mu_n \mid n \in \omega \rangle$ be a (non necessarily increasing) sequence of ordinals cofinal in $\xi$, and for every $n \in \omega$ let $P_n$ be $\Pi_\mu_n$-complete. Let $\hat{G}_\xi^F$ be the game defined as at the beginning of this section but using the $\hat{P}_n$’s instead of the $P_n$’s.

**Claim 4.0.1.** For every legal strategy $\hat{\tau}$ for $\hat{\Pi}$ in $\hat{G}_\xi^F$ there is a legal strategy $\tau$ for $\Pi$ in $G_\xi^F$ such that $f_\tau = f_\hat{\tau}$, and, conversely, for every legal strategy $\rho$ for $\Pi$ in $G_\xi^F$ there is a legal strategy $\hat{\rho}$ for $\hat{\Pi}$ in $\hat{G}_\xi^F$ such that $f_\rho = f_{\hat{\rho}}$. Therefore, $G_\xi^F$ and $\hat{G}_\xi^F$ represent the same set of functions.

**Proof.** Since the $\mu_n$’s are cofinal in $\xi$ and the $P_n$’s are $\Pi_\mu_n$-complete, for every $k \in \omega$ there is some $n_k$ such that $P_k \subseteq P_{n_k}$, thus there is a winning strategy $\sigma_k$ for $\Pi$ in $G_W(P_k, P_{n_k})$. Moreover, we can fix some $y_n \notin P_n$ for every $n$, and for every $y \in \omega^\omega$ let $\rho_y \in LS_W$ and $\rho_{id} \in LS_\xi$ be such that $f_{\rho_y}$ is constantly equal to $y$ and $f_{\rho_{id}} = id$ is the identity function. Finally, given $\tau_0, \tau_1 \in LS_W$ let $\tau_1 * \tau_0 \in LS_W$ be any strategy such that $f_{\tau_1 * \tau_0} = f_{\tau_1} * f_{\tau_0}$. Now define $\tau_2n = \sigma_k * \pi_{2k}(\hat{\tau})$ and $\tau_{2n+1} = \pi_{2k+1}(\hat{\tau})$ if $n = n_k$ for some $k \in \omega$, and $\tau_{2n} = \rho_{y_n}$ and $\tau_{2n+1} = \rho_{id}$ otherwise. Finally put $\tau = \bigotimes_n \tau_n$.

Given $x \in \omega^\omega$, it is not hard to check that $\omega_W(x, \pi_{2k}(x * \hat{\tau})) \in \hat{P}_k \iff \omega_W(x, \pi_{2k+1}(x * \hat{\tau})) \in P_{n_k}$ and $\iota_\tau(x, \pi_{2k+1}(x * \hat{\tau})) = \iota_\tau(x, \pi_{2k+1}(x * \hat{\tau}))$, while $\omega_W(x, \pi_{2k}(x * \hat{\tau})) = y_n \notin P_n$ for every $n$ which is not of the form $n_k$ for some $k \in \omega$. Hence $\tau$ is a legal strategy for $\Pi$ in $G_\xi^F$ if and only if $\hat{\tau}$ were a legal strategy for $\hat{\Pi}$ in $\hat{G}_\xi^F$, and moreover for every $x \in \omega^\omega$ we have that $\iota_\tau(x, \pi_{2k+1}(x * \hat{\tau})) = \iota_\tau(x, \pi_{2k+1}(x * \hat{\tau}))$, where $k$ is least such that $\omega_W(x, \pi_{2k}(x * \hat{\tau})) \in \hat{P}_k$ (which implies that $n_k$ is the least $m \in \omega$ such that $\omega_W(x, \pi_{2k}(x * \hat{\tau})) \in P_m$).

Using the same argument, one can convert any $\rho \in LS_\xi$ into a $\hat{\rho} \in LS_\xi^F$ such that $f_{\hat{\rho}} = f_\rho$, hence we are done. □

Thus from this point onward we can take the option of changing the sets $P_n$’s in the definition of $G_\xi^F$ at our pleasure, provided that the new ones are $\Pi_\mu_n$-complete for some (non necessarily increasing) sequence of ordinals $\langle \mu_n \mid n \in \omega \rangle$ cofinal in $\xi$. Copyright line will be provided by the publisher.
Theorem 4.1. For every $X, A, B \subseteq \omega \omega$ and every $f : X \to \omega \omega$ we have that:

i) $f \in D^F_\xi$ if and only if there is some $\tau \in LS^F_\xi$ such that $f = f_\tau$;

ii) if $I$ has a winning strategy in $G^F(A, B)$, then $I$ has also a winning strategy in $G_L(A, B)$. In particular, in this case, $B \subseteq_c \neg A$, i.e. there is a contraction (that is a Lipschitz function with constant $< 1$) $g$ such that $x \in B \iff g(x) \notin A$ for every $x \in \omega \omega$, and range$(g) \subseteq X$.

**Proof.** First suppose that $f : X \to \omega \omega$ is in $D^F_\xi$: let $\langle D_k | k \in \omega \rangle$ be a sequence of $\Pi^0_1$-sets such that (the strategy for $I$ in the run of a second auxiliary game defined in this way is clearly legal since $\sigma \neq \top$)

\[
F_0 = \{x \in X | \mu(x, \pi_0(x * \tau)) \in P_0\} \\
F_n+1 = \{x \in X | \mu(x, \pi_{2n}(x * \tau)) \in P_{n+1} \land \forall \mu \in \mu(x, \pi_{2n+1}(x * \tau)) \notin P_n\}.
\]

Clearly the $F_n$’s form a $\Delta^0_1$-partition of $X$ and $\pi_{2n+1}(\tau) \in LS_\nu$ for every $n$. Thus each $\pi_{2n+1}(\tau)$ induces a function $f_\tau = f_{\pi_{2n+1}(\tau)} : X \to \omega \omega$ in $F$, and it is easy to check that $f_\tau = f_\tau \cap F_n$ for every $n \in \omega$, that is $f_\tau \in D^F_\xi$.

Finally, let $\rho$ be a winning strategy for $I$ in $G^F(A, B)$. We define a strategy $\sigma$ for $I$ in $G_L(A, B)$ in the following way:

Let $y \in \omega \omega$ be the real enumerated by $\Pi$ in $G_L(A, B)$. Consider the run of the auxiliary game $G_0 = G_y(\omega \omega, \omega \omega)$ in which $I$ enumerates $y$ and $\Pi$ follows $\rho_{id}$. Now fix $z \in P_0$, and consider the run of a second auxiliary game $G_1 = G^F(A, B)$ in which $I$ follows $\rho$ and $\Pi$ uses $\rho_2$ on the even rows, and “copy” the moves of $\Pi$ in the previously described run of $G_0$ on the odd ones (the strategy for $\Pi$ defined in this way is clearly legal since $z \in P_0$). Then at each turn “copy” the corresponding move made by $I$ in the run of $G_1$ described above.

It is not hard to check that since $\rho$ is winning then $\sigma * y \in X$ and $\sigma * y \in A \iff y \notin B$: thus $\sigma$ is a winning strategy for $I$ in $G_L(A, B)$. The rest of part ii) follows by standard arguments. □

**Remark 4.2.** Although the game $G^F_\xi$ and the multitape game $G^W_\xi$ defined in [14] were developed independently, one should notice that the easiest direction of the proof of Theorem 4.1 (the one which goes from strategies to functions) presents some affinity (at least in spirit) with the corresponding direction of the proof of [14, Theorem 6.1]. However, the definition of $G_M$ (and, consequently, the whole Theorem 6.1 of [14]) does not seem to admit a simple and straightforward generalization for higher levels: this should be contrasted with the definition of the games $G^W_\xi$ (and of the games $G^W_\xi$ defined in the next subsection), which simultaneously gives a sort of “uniform” definition for all levels $\xi$ of games representing $D^W_\xi$ and $D^W_\xi$ — in fact the games $G^W_\xi$ and $G^W_\xi$ can be seen as a direct “translation” of the definitions of $D^W_\xi$ and $D^W_\xi$ into the game-theoretic formalism.

Copyright line will be provided by the publisher
4.2 Games for $\tilde{D}_F^\xi$

We now want to prove that also the collection $\tilde{D}_F^\xi$ is playable by showing how to modify the game $G_{\xi}^F$ to obtain the game $\tilde{G}_{\xi}^F$, which will represent this new set of functions. The idea is to allow II to not follow the rules on some of her rows. Here is the formal definition of $\tilde{G}_{\xi}^F = (X, \tilde{M}_F^\xi, \tilde{R}_F^\xi, \tilde{r}_F^\xi)$:

- $\tilde{M}_F^\xi = M_x \cup M_W$;
- $\tilde{R}_F^\xi = \{ (x, y) \in \omega \times \omega | \forall i((x, \pi_{2i}(y)) \in R_W) \land \exists i (\nu_W(x, \pi_{2i}(y)) \in P_i) \land \forall i (\nu_W(x, \pi_{2i}(y)) \in P_i \Rightarrow (x, \pi_{2i+1}(y)) \in R_x) \}$;
- $\tilde{r}_F^\xi: \tilde{R}_F^\xi \rightarrow \omega: (x, y) \mapsto \iota(x, \pi_{2i+1}(y))$, where $i$ is smallest such that $\iota_W(x, \pi_{2i}(y)) \in P_i$.

Thus the game $\tilde{G}_{\xi}^F$ can be visualized as the variant of the game $G_{\xi}^F$ in which II must “respect the rules” just on all the activated rows (rather than on all her odd rows).

Every strategy $\tau$ for II in $G_{\xi}^F$ can again be seen as a product $\otimes_n \tau_n$ of strategies $\tau_n$ for II in $G_W = (\omega, M_W, R_W, \nu_W)$ and $G_x = (X, M_x, R_x, \iota_x)$ (where now $X_n$ is a subset of $X$ depending on the index $n$). One can also define the projections $\tau_n$ on strategies as in the previous subsection, and check that they “commute” with the operation $\otimes_n$. Note that given a sequence of strategies $\tau_n$ as above, $\otimes_n \tau_n$ is a legal strategy for II in $G_{\xi}^F$ if and only if $X_{2n+1} \supseteq f_{\tau_{2n}}^{-1}(P_n)$ and $\{ f_{\tau_{2n}}^{-1}(P_n) | n \in \omega \}$ cover $X$.

Using this fact one can prove the following theorem in a similar way to Theorem 4.1.

Theorem 4.3. For every $X, A, B \subseteq \omega$ and every $f: X \rightarrow \omega$ we have that:

i) $f \in \tilde{D}_F^\xi$ if and only if there is some $\tau \in LS_F^\xi$ such that $f = f_{\tau}$;

ii) if I has a winning strategy in $\tilde{G}_{\xi}^F(A, B)$, then I has also a winning strategy in $G_{\xi}(A, B)$.

4.3 Games for lim $\tilde{F}$

The idea to require II to fill a table with $\omega$-many rows allows us also to define a (quite trivial) game for lim $\tilde{F}$ (hence, in particular, for all $B_\xi$’s). Since in this case considering an arbitrary sequence $\tilde{F} = \langle F_n | n \in \omega \rangle$ of adequate playable sets of functions does not significantly increase the complexity of the presentation, we will not restrict ourselves to a constant $\tilde{F}$. Suppose that the functions in $F_n$ have all domain $X$, and let $G_n = (X, M_n, R_n, \iota_n)$ be a sequence of p-closed reduction games, each representing the corresponding $F_n$. The reduction game $G_{\lim \tilde{F}} = (X, M_{\lim \tilde{F}}, R_{\lim \tilde{F}}, \iota_{\lim \tilde{F}})$ is defined as follows:

- $M_{\lim \tilde{F}} = \bigcup_n M_n$;
- $R_{\lim \tilde{F}} = \{ (x, y) \in \omega \times \omega | \forall n (x, \pi_n(y)) \in R_n \land \lim_{n} \iota_n(x, \pi_n(y)) \}$;
- $\iota_{\lim \tilde{F}}: R_{\lim \tilde{F}} \rightarrow \omega: (x, y) \mapsto \lim_{n} \iota_n(x, \pi_n(y))$.

The game $G_{\lim \tilde{F}}$ can be visualized as the game in which at each turn I must play a natural number on his (unique) row, while II has to play either a natural number or a symbol from $M_n$ on the $n$-th row of her table with $\omega$-many rows, with the condition that she must “respect the rules” of the corresponding game on each of these rows and that $\lim_{n} x_n$ must exists, where $x_n$ is the value of $\iota_n$ on (i.e. the “interpretation” of) what II has played on the $n$-th row: in this case, the output real of II is exactly $\lim_{n} x_n$.

As for the games $G_{\xi}^F$, it is easy to check that every strategy $\tau$ for II in $G_{\lim \tilde{F}}$ can be decomposed into $\omega$-many strategies $\pi_{n}(\tau)$ for II in $G_n$ (one for each row), and conversely $\omega$-many strategies $\tau_n$ for II in $G_n$ can be coded up into a unique strategy $\otimes_n \tau_n$ for II in $G_{\lim \tilde{F}}$. Moreover it is easy to check that $f: X \rightarrow \omega$ is in $\lim \tilde{F}$ if and only if there is some $\tau \in LS_{\lim \tilde{F}}$ such that $f = f_{\tau}$ (this is because the use of the table with $\omega$-many rows allows to directly code the definition of “being limit of a sequence of functions” into a single game).

5 Games for $\Gamma$-measurable functions

Let $\Gamma$ be any $\Sigma$-pointclass. The main goal of this section is to construct games representing the collection $J_{\Gamma}$ of $\Gamma$-measurable functions $f: X \rightarrow \omega$. When $\Gamma = \Sigma_{\xi+1}^0$ (for some $\xi < \omega_1$) this just give
an alternative way of defining games for Baire class $\xi$ functions (see Section 4.3), but note that since $e.g. \Sigma^i_1$ is trivially a $\Sigma^i$-pointclass (for every $n \in \omega$), the main result of this section gives also a new way (albeit less informative than the construction given in [13]) of defining a game for the class of all Borel functions (taking $\Gamma = \Sigma^i_1$), and simultaneously solves the problem of finding games for the projective functions posed by Semmes in his Ph.D. thesis [13].

Fix a universal set $S \subseteq \omega^{2} \times \omega$ for $\Gamma$ and let $G_W = (\omega, M_W, R_W, \iota_W)$ be the Wedge game representing (totally defined) continuous functions. Here is the definition of the game $G_T = (X, M_T, R_T, \iota_T)$:

- $M_T = M_W = \{p\}$;
- $R_T = \{(x, y) \in \omega^{2} \times (\omega \cup M_T) \mid \forall n((x, \pi_n(y)) \in R_W) \wedge n, m (D_{y, n, m} \cap X \in \Delta_T(X)) \wedge \forall n \exists m (x \in D_{y, n, m})\}$, where $D_{y, n, m} = \{x \in \omega \mid (\iota_W(x, \pi_{(n, m)}(y)), x) \in S\} = \{x \in \omega \mid x \in S_{iw(x, \pi_{(n, m)}(y))}\}$;
- $\iota_T: R_T \to \omega^{2}: (x, y) \mapsto \iota_T(x, y)$, where $\iota_T(x, y)(n) = m \iff m$ is smallest such that $x \in D_{y, n, m}$.

The game $G_T$ can be visualized as follows: player $I$ must fill, as usual, a single row by playing a natural number at each of his turn (thus he produces a real $x \in X$). Player $II$ is in charge of filling again a table with $\omega$-many rows: she can pass, but at the end of the round she must have enumerated a real $y_n$ on her $n$-th row (for each $n \in \omega$). The rules for $II$ are that each $y_{(n, m)}$ must code a set $D_{y, n, m} \in \Gamma$ whose intersection with $X$ is in $\Delta_T(X)$ (i.e. such that there is $P_{n, m} \in \bar{\Gamma}$ for which $P_{n, m} \cap X = D_{y, n, m} \cap X$), and for every $n$ there must be an $m$ such that $x \in D_{y, n, m}$. The output real $z$ is then defined by $z(n) = m$ if and only if $m$ is the smallest $k$ such that $x \in D_{y, n, k}$. As usual, any strategy $\tau$ for $\Gamma$ can be seen as a product $\bigotimes_{n} \tau_n$ of legal strategies for $\Gamma$ in $G_W$, and one can define the projections $\pi_n$ of strategies in $LS_T$ in such a way that they "commute" with the operation $\bigotimes_{n}$.

**Theorem 5.1.** For every $X, A, B \subseteq \omega$ and every $f: X \to \omega$ we have that:

i) $f \in \mathcal{F}_{T}$ if and only if there is some $\tau \in \mathcal{L}_{S_{T}}$ such that $f = f_\tau$;

ii) if $I$ has a winning strategy in $G_{T}(A, B)$, then $I$ has also a winning strategy in $G_{L}(A, B)$.

**Proof.** First assume that $f \in \mathcal{F}_{T}$. Since the sets $B_{n, m} = \{z \in \omega \mid z(n) = m\}$ form a clopen subbasis for the usual topology of $\omega$, we have that $f^{-1}(B_{n, m}) \in \Delta_{T}(X)$. Let $S_{n, m} \in \Gamma$ be such that $S_{n, m} \cap X = f^{-1}(B_{n, m})$, $y_{n, m}$ be a code for $S_{n, m}$, and $x \in X$ be the real enumerated by $I$: if we put $\tau = \bigotimes_{n} \tau_{n}$, where $\tau_{(n, m)} \in \mathcal{L}_{SW}$ is any strategy representing the constant function with value $y_{n, m}$, then $\tau$ is clearly a legal strategy for $II$ in $G_{T}$ such that $f = f_\tau$.

Assume now $\tau \in \mathcal{L}_{S_{T}}$. Then

$$f_\tau^{-1}(B_{n, m}) = \{x \in X \mid x \in D_{x + \tau_{n, m}} \wedge \forall k \in \mathbb{N} (x \notin D_{x + \tau_{n, k}})\} \in \Gamma(X)$$

because $x \notin D_{x + \tau_{n, k}} \iff x \notin P_{n, k}$, where $P_{n, k} \in \bar{\Gamma}$ is such that $P_{n, k} \cap X = D_{x + \tau_{n, k}} \cap X$. Hence $f_\tau \in \mathcal{F}_{T}$.

Finally, let $\rho$ be a winning strategy for $I$ in $G_{T}(A, B)$, and $c_{\omega}, c_{\emptyset}$ be codes for, respectively, $\omega$ and $\emptyset$ (as elements of $\Gamma$). Then the strategy $\sigma$ for $I$ in $G_{L}(A, B)$ defined in the following way is clearly winning (the proof being the same as in Theorem 4.1):

Let $y \in \omega$ be the real enumerated by $II$ in $G_{L}(A, B)$, and consider the run of the auxiliary game $G_{T}(A, B)$ in which $I$ plays according to $\rho$ and $II$ enumerates $c_{\omega}$ or $c_{\emptyset}$ on her $(n, m)$-th row depending on whether $y(n) = m$ or $y(n) \neq m$ (since $n \leq (n, m)$ for any $m$, this strategy for $II$ is clearly legal). Then copy at each turn the corresponding move made by $I$ in the run of $G_{T}(A, B)$ described above.

Notice that, contrarily to the games defined in all the previous sections, it is no more true that e.g. if $A, B \subseteq \Delta^1_1$ then Borel determinacy implies that $G_{T}(A, B)$, where $G_{T} = (\omega, M_{T}, R_{T}, \iota_{T})$, is determined. This is because of the use of codes for sets in $\Delta_{T}$, which generally makes the set of rules more complicated than $\Gamma$ itself: in fact, in most cases, to say that "$x$ codes a $\Delta_{T}$-set" require roughly speaking at least one real quantifier over a predicate of the same complexity as $\Delta_{T}$ (it is well-known e.g. that the set of codes for the Borel sets forms a $\Pi^1_1$-complete set).
Nevertheless, if $\Gamma \subseteq \Delta^1_1$ (that is if $\Gamma = \Sigma^0_2$ for some countable $\xi$, being $\Gamma$ a $\Sigma$-pointclass) one can redefine the games $G^\Gamma_{\Sigma^0_2}$ in such a way that the new sets of rules and the interpretation functions remain Borel (so Borel determinacy will imply that these new games are determined whenever $A, B \subseteq \Delta^1_1$). This can be obtained by fixing in advance a sequence of $\Pi^0_{\mu_n}$-complete sets $P_n$ (where $\langle \mu_n \mid n \in \omega \rangle$ is an increasing sequence of countable ordinals cofinal in $\xi$) as in the definition of the games $G^\xi_{\Sigma^0_2}$, and then using the fact that for every $\Delta^1_\xi(X)$ set $D \subseteq X$ (hence also for each $f^{-1}(\Delta_{B,m})$, where $f \in F^\xi_{\Sigma^0_2}$ and the $B_{n,m}$’s are defined as above) there is a $\Pi^0_{\xi}(X)$-partition $\langle \xi_n \mid n \in \omega \rangle$ of $X$ such that $D = \bigcup_{i \in \xi} C_i$ for some $I \subseteq \omega$: roughly speaking, in the new games player $I$ will have again to completely fill a board with $\omega$-many rows, but the function $f$ will be determined by checking which of the reals that appears on the rows of $\Pi$’s table (instead of the real $x$ enumerated by $I$) belongs to the corresponding set $P_n$.

We leave to the reader the exact definition of these games, as well as the proof that they represent $\mathcal{F}_{\Sigma^0_2}$.

The same kind of construction introduced for the games $G^\Gamma$ allows also to define games for $D^\Gamma_F$ or $\overline{D}^\Gamma_F$ (where $\overline{F}$ is a sequence of adequate playable sets of functions) for an arbitrary $\Sigma$-pointclass $\Gamma$. For simplicity of presentation, we will deal again only with the case $D^\Gamma_F$. The idea is simply to take the game $G^\Gamma_F$ and, instead of fixing in advance the control sets $P_n$, require $I$ to produce on each control row the code for some control set in $\Delta^\Gamma_F$: a (non control) row will be activated just in case the real $x \in X$ enumerated by $I$ belongs to the set $D \in \Delta^\Gamma_F$ coded on the corresponding control row.

More precisely, given a $\pi$-closed reduction game $G_\pi = (X, \mathcal{M}_\pi, \mathcal{R}_\pi, \mathcal{I}_\pi)$ representing the functions of $\mathcal{F}$ with domain $X$, define $G^\Gamma_F = (X, M^\Gamma_F, R^\Gamma_F, I^\Gamma_F)$ as follows (where $G_\mathcal{W} = (\omega, \mathcal{M}_\mathcal{W}, \mathcal{R}_\mathcal{W}, \mathcal{I}_\mathcal{W})$ is the Wadge game representing continuous functions):

- $M^\Gamma_F = \mathcal{M}_\pi \cup \mathcal{M}_\mathcal{W};$
- $R^\Gamma_F = \{ (x, y) \in \omega \times \omega \mid \forall n \in \omega ([x, \pi_{2n}(y)]) \in \mathcal{R}_\mathcal{W} \land D_{y,n} \cap X \in \Delta^\Gamma_F \land (x, \pi_{2n+1}(y)) \in \mathcal{R}_\pi \}$
- $I^\Gamma_F : X \rightarrow \omega : (x, y) \mapsto \mathcal{I}_\pi(x, \pi_{2i}(y)),$ where $i$ is smallest such that $x \in D_{y,i}$.

Combining the proofs of Theorem 4.1 and Theorem 5.1 it is not hard to check that:

**Theorem 5.2.** For every $X, A, B \subseteq \omega$ and every $f : X \rightarrow \omega$ we have that:

i) $f \in D^\Gamma_F$ if and only if there is some $\tau \in LS^\Gamma_F$ such that $f = f_\tau$;

ii) if $I$ has a winning strategy in $G^\Gamma_F(A, B)$, then $I$ has a winning strategy in $G_\mathcal{W}(A, B)$ as well.

## 6 Determinacy and applications to reducibilities

In this section we will analyze the relationships among some determinacy axioms, and show how to apply the techniques arising from reduction games to the study of the reducibilities between sets of reals induced by the corresponding sets of functions. Here we will just present two cases, namely the cases corresponding to Lip and $D^\xi_W$ (for any fixed $\xi$). Notice that all reduction games used in this section are always intended to be of the form $G_\pi = (X, \mathcal{M}_\pi, \mathcal{R}_\pi, \mathcal{I}_\pi)$ with $X = \omega$.

### 6.1 Lip-reducibilities

We first consider the following axioms which are related to the games $G_{k,\text{Lip}} (k \in \omega)$: this will lead in Theorem 6.2 to a slight extension of the results concerning the equivalence of some determinacy axioms obtained by Andretta in his [1] and [2], although we must note that the most difficult implication involved in such extension was already proved in those papers.

$\text{AD}(G_{k,\text{Lip}})$: For every $A, B \subseteq \omega$ the game $G_{k,\text{Lip}}(A, B)$ is determined.

$\text{AD}^{\text{Lip}}$: For every $A, B \subseteq \omega$ there is some $k \in \omega$ such that $G_{k,\text{Lip}}(A, B)$ is determined.

$\text{AD}^{\text{LP}}$: For every $A, B \subseteq \omega$ and for every $k \in \omega$ the game $G_{k,\text{Lip}}(A, B)$ is determined.
Lemma 6.1.  i) $\mathsf{AD} \Rightarrow \mathsf{AD}^{Lip}$;  
ii) $\mathsf{AD}^{Lip} \iff \forall k \in \omega(\mathsf{AD}(G_{k,Lip})) \Rightarrow \mathsf{AD}(G_{k,Lip}) \Rightarrow \mathsf{AD}^{Lip} \Rightarrow \mathsf{SLO}^{Lip} \Rightarrow \mathsf{SLO}^{W}$ for every $k \in \omega$;  
iii) $\mathsf{AD}^{L} \Rightarrow \mathsf{AD}^{Lip}$.  

\textit{Proof.} The first part is obvious since $\mathsf{AD}$ easily implies that every reduction game is determined, and the equivalence and the first two implications of part ii) are obvious as well. The third implication of part ii) can be proved using the trivial observation that winning strategies for $I$ in any of the games $G_{k,Lip}$ induce contractions, while the last implication follows from $\mathsf{Lip} \subseteq W$. It remains only to prove part iii). Fix $A,B \subseteq \omega$ and $k \in \omega$. If $B = \omega$ then $G_{k,Lip}(A,B)$ is trivially determined ($II$ has a winning strategy if $A = \omega$, and $I$ has a winning strategy if $A \neq \omega$); hence we can assume $B \neq \omega$ and fix some $y \notin B$. Consider the auxiliary game $G = G_{L}(A,0^{(k)} \upharpoonright B)$: if $I$ has a winning strategy $\sigma$ for $G$, $I$ can also win $G_{k,Lip}(A,B)$ simply playing $\sigma(0^{(i)})$ for the first $k$ turns (i.e. for $i \leq k$), and then playing $\sigma(0^{(k)} \upharpoonright s)$ if $II$ has enumerated a sequence of the form $p^{(k)} \upharpoonright s$ (for some $s \in \omega$) in the game $G_{k,Lip}(A,B)$, and 0 otherwise. Conversely, if $II$ has a winning strategy $\tau$ in the game $G$, then she can also win $G_{k,Lip}(A,B)$ by playing $p$ for the first $k$ rounds, and then playing $\tau(s)$, where $s$ is the sequence enumerated by $I$ in $G_{k,Lip}(A,B)$, if $(s \upharpoonright k)* = 0^{(k)}$, or enumerating $y$ otherwise. \hfill $\Box$  

Theorem 6.2 (BP+DC($\mathbb{R}$)). Let $A_L$ be one of the axioms $\mathsf{AD}(G_{k,Lip})$, $\mathsf{AD}^{Lip}$ and $\mathsf{AD}^{Lip}$. Then $A_L \iff \mathsf{SLO}^{W}$.  

\textit{Proof.} By Lemma 6.1 we have $\mathsf{AD}^{L} \Rightarrow A_L \Rightarrow \mathsf{SLO}^{W}$, and since under $\mathsf{BP} + \mathsf{DC}(\mathbb{R})$ we have from [2, Proposition 15 and Theorem 18] that $\mathsf{SLO}^{W} \Rightarrow \mathsf{AD}^{L}$, we get the desired equivalence. \hfill $\Box$  

In particular, this theorem implies that all the results about the $\mathsf{Lip}$-hierarchy obtained in [8] (such as the fact that the structure induced by $\leq_{Lip}$ can be completely determined and is a well-founded semi-linear order whose antichains have size at most two, or the relationship between this hierarchy and the ones induced by $\leq_{L}$ and $\leq_{W}$) hold under any of the axioms listed above (together with $\mathsf{BP} + \mathsf{DC}(\mathbb{R})$).

6.2 $\mathcal{D}_{\xi}^{W}$-reducibilities  

We now turn our attention to the $\mathcal{D}_{\xi}^{W}$-hierarchies (for some fixed nonzero $\xi < \omega_1$). Recall from [10, pp. 47-48] that there are two operations $\Sigma^{\xi}$ and $\Pi^{\xi}$ such that $\{\Sigma^{\xi}(A), \Pi^{\xi}(A)\}$ are the successors of $A$ in the $\mathcal{D}_{\xi}^{W}$-hierarchy whenever $A \leq_{\mathcal{D}_{\xi}^{W}} A$. Here are the definitions:  

$$\Sigma^{\xi}(A) = \{x \in \omega \mid \exists n(\pi_{2n}(x) \in P_n \wedge \forall i < n(\pi_{2i}(x) \notin P_i) \wedge \pi_{2n+1}(x) \in A)\}$$

and  

$$\Pi^{\xi}(A) = \Sigma^{\xi}(A) \cup R_{\xi},$$

where the $P_n$’s are $\Pi^{\mu_n}_{\omega}$-complete for an increasing sequence of ordinals $\langle \mu_n \mid n \in \omega \rangle$ cofinal in $\xi$ and $R_{\xi} = \{x \in \omega \mid \forall n(\pi_{2n}(x) \notin P_n)\}$. There is a strict relationship between the games $G^{\xi}_{\xi}$ (in particular when $F = W$) and these successor operations — in fact our definition of $G^{\xi}_{\xi}$ was originally motivated by the definitions of $\Sigma^{\xi}$ and $\Pi^{\xi}$.

Proposition 6.3. For every $A,B \subseteq \omega$, the following are equivalent:  

i) $A \leq_{\mathcal{D}_{\xi}^{W}} B$;  
ii) $A \leq_{W} \Sigma^{\xi}(B)$ and $A \leq_{W} \Pi^{\xi}(B)$;  
iii) $A \leq_{W} \Sigma^{\xi}(B)$ via some function $f$ such that $\text{range}(f) \cap R_{\xi} = \emptyset$;  
iv) $A \leq_{W} \Pi^{\xi}(B)$ via some function $f$ such that $\text{range}(f) \cap R_{\xi} = \emptyset$.  

Copyright line will be provided by the publisher
Proof. Obviously, iii) $\iff$ iv) since $\Sigma^\xi(B) \setminus R_\xi = \Pi^\xi(B) \setminus R_\xi$ for every set $B \subseteq \omega_\omega$. Moreover, iii) and iv) together trivially imply ii), and i) implies iii) and iv) since, by definition of $G^W_\xi$, every winning strategy for $\II$ in $G^W_\xi(A, B)$ can be obviously converted into a winning strategy for $\II$ in both $G_W(A, \Sigma^\xi(B))$ and $G_W(A, \Pi^\xi(B))$. To see that ii) implies i), let $\sigma^0$ and $\sigma^1$ be, respectively, winning strategies for $\II$ in $G_W(A, \Sigma^\xi(B))$ and $G_W(A, \Pi^\xi(B))$. As already observed in Claim 4.0.1, we can change the sets $P_n$ in the definition of $G^W_\xi$ with some suitable $\hat{P}_n$’s, and it will suffice to show that $\II$ has a winning strategy in $G^W_\xi(A, B)$, where $G^W_\xi$ is the game defined using the $\hat{P}_n$’s instead of the $P_n$’s. Choose for every $n \in \omega$ and $i = 0, 1$ a strategy $\sigma^i_n \in L^S_W$ representing $\pi_n \circ f_{\sigma^i}$. Then put $\hat{P}_2n = \hat{P}_2n+1 = P_n$ for every $n \in \omega$ and set $\tau = \bigotimes_n \tau_n$, where $\tau_{1k} = \sigma^0_{2k}$, $\tau_{4k+1} = \sigma^0_{2k+1}$, $\tau_{4k+2} = \sigma^0_{2k}$ and $\tau_{4k+3} = \sigma^0_{2k+1}$ ($k \in \omega$). Notice that each $\hat{P}_n$ is $\Pi^\xi_n$-complete, where $\mu_k$ for $k \in \omega$, $i = 0, 1$ (so that $\langle \mu_n \mid n \in \omega \rangle$ is a sequence of ordinals cofinal in $\xi$). We claim that $\tau \in L^W_{\xi}$. Let $x \in \omega_\omega$. Since the $\sigma^i$’s are winning strategies in the corresponding games, we have that for every real $x$

$$f_{\sigma^0}(x) \in R_\xi \Rightarrow x \notin A \Rightarrow f_{\sigma^1}(x) \notin R_\xi,$$

thus there must be some $n$ such that either $\pi_{2n}(f_{\sigma^0}(x)) \in P_n$ or else $\pi_{2n}(f_{\sigma^1}(x)) \in P_n$. But this implies that either $\iota_W(x, \pi_{4n}(x + \tau)) \in \hat{P}_{2n}$ or $\iota_W(x, \pi_{4n+2}(x + \tau)) \in \hat{P}_{2n+1}$, hence $\tau$ is legal. To finish the proof, let $n$ be the smallest natural number such that $\iota_W(x, \pi_{2n}(x + \tau)) \in \hat{P}_n$: then if $n = 2k + i$ ($k \in \omega$, $i = 0, 1$) we clearly have $\iota_W(x, \pi_{2n+1}(x + \tau)) = \pi_{2k+1}(f_{\sigma^i}(x))$ and

$$\pi_{2k+1}(f_{\sigma^i}(x)) \in B \iff f_{\sigma^i}(x) \in B_i \iff x \in A,$$

where $B_0 = \Sigma^\xi(B)$ and $B_1 = \Pi^\xi(B)$. Therefore $\tau$ is a winning strategy for $\II$ in $G^W_\xi(A, B)$.

As for the Lipschitz games $G_{k\text{-Lip}}$, the games $G^W_\xi$ allow us to introduce new determinacy axioms (one for each $\xi$):

$AD^W_\xi$: For every $A, B \subseteq \omega_\omega$ the game $G^W_\xi(A, B)$ is determined.

Clearly $\AD \Rightarrow AD^W_\xi \Rightarrow SLO^D_\xi$, but Proposition 6.3 allows us to prove the following stronger corollary.

Corollary 6.4. $AD^W_\xi$ implies $AD^W_\xi$.

Proof. Let $A$ and $B$ be two subsets of $\omega_\omega$. By $AD^W$, the games $G_W(A, \Sigma^\xi(B))$ and $G_W(A, \Pi^\xi(B))$ are determined. If $I$ has a winning strategy in one of these two games, then he can convert this strategy into a winning strategy for $I$ in $G^W_\xi(A, B)$ in the obvious way, hence we can assume that $II$ wins both the games. But in this case $II$ has a winning strategy in $G^W_\xi(A, B)$ by Proposition 6.3, hence we are done.

There is a natural question arising from the previous corollary, namely:

Question 2. Assume $BP + DC(\mathbb{R})$. Given a countable ordinal $\xi > 1$, does the converse to Corollary 6.4 hold?

This question was answered positively for $\xi = 2$ by Andretta in his [2], where it is shown that in fact $SLO^D_\xi$ (which is a direct consequence of $AD^W_\xi$) implies $AD^W$ if we assume $BP + DC(\mathbb{R})$. The proof is carried out with an induction on the $D^2$-hierarchy of degrees (which can be determined under $SLO^D_\xi + BP + DC(\mathbb{R})$), but even if we will show that for any $\xi$ the axioms $AD^W_\xi + BP + DC(\mathbb{R})$ are indeed strong enough to determine the $D^\xi$-hierarchy of degrees as well (see Theorem 6.7 below), it seems that the argument used by Andretta does not generalize in a straightforward way to higher levels. Therefore Question 2 is still completely open for $\xi \geq 3$.  

Copyright line will be provided by the publisher
We will now prove that, as announced in the previous paragraph, the axiom \( \text{AD}_\xi^W \) is strong enough\(^5\) to determine (together with \( \text{BP} \) and \( \text{DC}(\mathbb{R}) \)) the degree-structure induced by \( \text{D}_\xi^W \), or even by any Borel-amenable set of reductions \( \mathcal{F} \supseteq \text{D}_\xi^W \) (see [10] for a general introduction to such degree-structures). This shows that to study a Borel-amenable reducibility \( \mathcal{F} \) we just need to assume an axiom which is “of the same level” of \( \mathcal{F} \), rather than the seemingly stronger \( \text{SLO}^W \).

Toward our goal, we will simply modify the arguments presented in [10] whenever an axiom stronger than \( \text{AD}_\xi^W \) was required. We start by proving a lemma (analogous to [10, Lemma 2.1]) under the new axiomatization.

**Lemma 6.5.** Assume \( \text{AD}_\xi^W \). For every set of reductions \( \mathcal{F} \supseteq \text{D}_\xi^W \) and every \( A, B \subseteq \omega^\omega \) we have \( A <_\mathcal{F} B \Rightarrow A \leq_\mathcal{F} B \).

**Proof.** Since \( \text{AD}_\xi^W \Rightarrow \text{SLO}^\omega \Rightarrow \text{SLO}^\mathcal{F} \), from \( A <_\mathcal{F} B \) we have \( A <_\mathcal{F} \neg B \). But then \( \Pi \) cannot win \( G_\xi^\mathcal{F}(\neg B, A) \) (if this would happen, then \( \neg B \leq^\mathcal{F} A \) and hence also \( B \leq^\mathcal{F} A \)). Therefore by \( \text{AD}_\xi^W \), we have that \( \Pi \) has a winning strategy in the same game, and hence \( A \leq_\mathcal{F} B \) by Theorem 4.1. Moreover \( B \not\leq_\mathcal{F} A \) since otherwise \( B \leq^\mathcal{F} A \) (here we use the fact that \( \mathcal{L} \subseteq \text{D}_\xi^W \subseteq \mathcal{F} \)), and thus \( A \leq_\mathcal{F} B \).

**Lemma 6.6.** Assume \( \text{AD}_\xi^W + \text{BP} + \text{DC}(\mathbb{R}) \). For every set of reductions \( \mathcal{F} \supseteq \text{D}_\xi^W \) the relation \( \leq^\mathcal{F} \) is well-founded.

**Proof.** It clearly suffices to prove that there is no \( \leq^\mathcal{F} \) descending chain — the equivalence between this statement and well-foundedness can be obtained in the usual way using the existence of a surjection \( j : \omega^\omega \to \mathcal{F} \) (see [3, Corollary 2.2]). So assume towards a contradiction that \( A_0 >^\mathcal{F} A_1 >^\mathcal{F} \ldots \) is such a chain. We claim that \( A_{n+1} \leq_\mathcal{F} A_n \) and \( A_{n+1} \leq_\mathcal{F} \neg A_n \) for every \( n \in \omega \), i.e. that \( \Pi \) wins both \( G_\mathcal{L}(A_n, A_{n+1}) \) and \( G_\mathcal{L}(\neg A_n, A_{n+1}) \): applying then the classic Martin-Monk argument to these winning strategies, we can construct the flip-set which contradicts \( \Pi \), finishing our proof. First note that for every \( n \in \omega \), we have \( A_{n+1} <^\mathcal{F} \neg A_n \) by \( \text{SLO}^\mathcal{F} \) (which follows from \( \text{AD}_\xi^W \)). From this fact one can conclude, arguing as in Lemma 6.5, that \( \Pi \) cannot win either \( G_\xi^\mathcal{F}(A_n, A_{n+1}) \) nor \( G_\xi^\mathcal{F}(\neg A_n, A_{n+1}) \): but then \( \Pi \) wins both games by \( \text{AD}_\xi^W \), and hence \( A_{n+1} \leq_\mathcal{F} A_n \), \( \neg A_n \) by Theorem 4.1.

**Theorem 6.7.** Assume \( \text{AD}_\xi^W + \text{BP} + \text{DC}(\mathbb{R}) \), and let \( \mathcal{F} \supseteq \text{D}_\xi^W \) be any Borel-amenable set of reductions. Then the degree-structure induced by \( \leq^\mathcal{F} \) is completely determined and looks like the Wadge one.

**Proof.** Since \( \text{AD}_\xi^W \Rightarrow \text{SLO}^\mathcal{F} \) and \( \leq^\mathcal{F} \) is well-founded by Lemma 6.6, we have that Theorem 3.1 and Theorem 4.6 of [10] are provable under our new axiomatization (for part \( \text{viii} \) of Theorem 3.1, assume \( \mathcal{B} \equiv^\mathcal{F} A \), \( \mathcal{B} \not\leq^\mathcal{F} \neg A \)) since \( \Pi \) cannot win either \( G_\xi^\mathcal{F}(B, \neg A) \) nor \( G_\xi^\mathcal{F}(\neg A, B) \), we have \( \mathcal{A} \leq \mathcal{B} \) and \( \mathcal{B} \leq \mathcal{A} \) by Theorem 4.1). Moreover, [10, Theorem 5.3] follows from \( \Pi \) alone, hence we are done.

Finally, note that we can also reprove Theorem 4.7 of [10] in this new context using Lemma 6.5 instead of [10, Lemma 2.1]: therefore under \( \text{AD}_\xi^W + \text{BP} + \text{DC}(\mathbb{R}) \) we have that for every pair \( \mathcal{F}, \mathcal{G} \supseteq \text{D}_\xi^W \) of Borel-amenable sets of reductions, \( \mathcal{F} \) is equivalent to (i.e. induces the same hierarchy of degrees as) \( \mathcal{G} \) just in case they have the same characteristic set, that is just in case

\[
\Delta_\mathcal{F} = \{ D \subseteq \omega^\omega \mid D \leq^\mathcal{F} \mathcal{N}_{\{0\}} \} = \{ D \subseteq \omega^\omega \mid D \leq^\mathcal{G} \mathcal{N}_{\{0\}} \} = \Delta_\mathcal{G}.
\]

### 7 Non adequate playable set of functions

This final section is devoted to a technical refinement of the notion of being adequate for a certain set of functions in relationship to the possibility of representing such set by means of reduction games (using the ideas coming from Sections 4 and 5).

---

\(^5\) All the following results can also be proved assuming that for every \( A, B \subseteq \omega^\omega \) either \( A \leq_\mathcal{F} B \) or \( \neg B \leq_\mathcal{F} A \), which by Theorem 4.1 is a (seemingly weaker) consequence of \( \text{AD}_\xi^W \).
We start with a significative example. As observed after Definition 2, the class of all Lipschitz functions Lip is not adequate (being not p-closed), but still it is possible (and useful) to consider e.g. classes of the form $D_{Lip}^\xi$, which are proper subsets of $D^{\Omega}$ (see [10, pp. 45-46]). Roughly speaking, in order to define the game $G_{Lip}^\xi$ (whose legal strategies for II will induce the functions in $D_{Lip}^\xi$), it is enough to modify the algorithm\(^6\) that II must follow to fill in the $\omega$-many rows of her table in the game $G_{Lip}^\xi$: in fact, in the game $G_{Lip}^\xi$, II will have to simultaneously play a new natural number on a certain finite set of rows at each of her turns. More precisely: for $0 \neq k \in \omega$ let $\langle s_n^k \mid n \in \omega \rangle$ be an enumeration without repetitions of $k$, and let $\langle \mu_n \mid n \in \omega \rangle$ and $\langle P_n \mid n \in \omega \rangle$ be chosen as in Subsection 4.1. Then define $G_{Lip}^\xi = (X, M^\xi_{Lip}, R^\xi_{Lip}, Lip^\xi)$ by:

- $M^\xi_{Lip} = \emptyset$;
- for $y \in \omega$, $k \in \omega$ and $i = 0, 1$ define $y_{2k+1} = \langle s^{2n+2}_y(k+i) \mid n \geq k \rangle$;
- $R^\xi_{Lip} = \{(x, y) \in \omega \times \omega \mid 3n(y_{2n} \in P_n)\}$;
- $\xi_{Lip} : R^\xi_{Lip} \to \omega$, $\langle x, y \rangle \mapsto y_{2n+1}$ where $n$ is smallest such that $y_{2n} \in P_n$.

As for the games $G_{Lip}^\xi$, a strategy $\tau = \bigotimes_n \tau_n$ for II in $G_{Lip}^\xi$ can be constructed from a sequence $\langle \tau_n \mid n \in \omega \rangle$ of strategies for II in $G_{k_n,Lip}$, where $k_n$ is the unique natural number such that either $n = 2k_n$ or $n = 2k_n + 1$: in fact it is enough to define $\tau(s) = m$, where $m$ is such that $\langle \tau_n(s) \mid n < 2lh(s) \rangle = s^{2lh(s)}_m$, and it is easy to check that $\tau \in L_{k_n,Lip}^\xi$ just in case for every $x \in \omega$ there is some $n \in \omega$ such that $\tau_{k_n,Lip}(x * 2^m) \in P_n$. Conversely, given a strategy $\tau \in L_{k_n,Lip}^\xi$ one can construct the strategies $\tau^\prime_{2k+1}(\tau)(s) = p$ is $lh(s) < k$ and $\tau^\prime_{2k+1}(\tau)(s) = \langle 2lh(s), 2k+i \rangle$ otherwise.

**Theorem 7.1.** For every $X, A, B \subseteq \omega$ and every $f : X \to \omega$, we have that:

i) $f \in D_{Lip}^\xi$ if and only if there is some $\tau \in L_{k_n,Lip}^\xi$ such that $f = \tau$;

ii) if I has a winning strategy in $G_{Lip}^\xi(A, B)$, then I has also a winning strategy in $G(A, B)$.

**Proof.** Assume first that $f \in D_{Lip}^\xi$, and let $\{f_k \mid k \in \omega\} \subseteq Lip$ and $\langle D_k \mid k \in \omega \rangle$ be a sequence of $\Pi^0_2$-sets such that $\langle D_k \cap X \mid k \in \omega \rangle$ is a partition of $X$ such that $f \upharpoonright D_k = f_k \upharpoonright D_k$. By Borel determinacy\(^7\), we can find an increasing sequence $\langle n_k \mid k \in \omega \rangle$ of natural numbers such that $D_k \subseteq P_{n_k}$. Moreover, let $i_k$ be smallest such that $f_k \in Lip(2^{n_k})$, and inductively define $m_0 = \max\{n_0, i_0\}$ and $m_{k+1} = \max\{n_{k+1}, i_{k+1}, m_k + 1\}$, so that $D_k \subseteq P_{m_k}$ (hence also $D_k \subseteq Lip(2^{m_k})$) and $f_k \in Lip(2^{m_k})$ for every $k \in \omega$. Let $\sigma_k$ be a winning strategy for II in $G_{m_k,Lip}(D_k, P_{m_k})$ and $\hat{\tau}_k \in L_{m_k,Lip}$ be such that $f_k = \hat{\tau}_k$. Finally, fix $y_n \notin P_n$ and for every $y \in \omega$ let $\rho^y_k \in L_{k_n,Lip}$ be such that $\rho^y_k$ is constantly equal to $y$. Now define $\tau_{2n} = \sigma_k$ and $\tau_{2n+1} = \hat{\tau}_k$ if $n = m_k$, and $\tau_{2n} = \rho^y_k \tau_{2n+1}$ otherwise. If we construct the strategy $\tau = \bigotimes_n \tau_n$ for II in $G_{Lip}^\xi$ as explained above, it is not hard to check that $x \in D_k$ if and only if $(x * \tau)_{2m_k} \notin P_{m_k}$, and that in this case $(x * \tau)_{2m_k} \notin P_n$ for $n \neq m_k$ and $f_{\tau}(x) = (x * \tau)_{2m_k+1}$. Conversely, let $F_n$ be the set of those $x$ for which $n$ is least such that $(x * \tau)_{2n} \notin P_n$. Clearly these $F_n$’s form a $\Delta^0_\infty$-partition of $X$ and, as already observed, $\pi^\prime_{2n+1}(\tau)$ induces a function $f_n \in Lip(2^n)$. Thus $f_{\tau} = \bigcup_n f_n \upharpoonright F_n$ and we are done.

Finally, the second part of the theorem can be proved as in Theorem 4.1 (although the coding of the strategies involved is more complicated). □

---

\(^6\) Just forbidding II to pass (that is making II always play a natural number on some of her rows) does not give the desired result, because legal strategies for II in such game would induce functions uniformly continuous (rather than Lipschitz) on a definable partition and these two sets of functions are distinct by [10, pp. 45-46].

\(^7\) Using Borel determinacy, if $n_k$ is such that $D_k \in \Pi^0_{n_k}$ then $D_k \subseteq L_{P_{n_k}}$; otherwise $P_{n_k} \subseteq L \neg D_k \in \Sigma^0_{m_k}$ by SLO for Borel sets, contradicting the $\Pi^0_{m_k}$-properness of $P_{n_k}$.

Copyright line will be provided by the publisher
Theorem 7.1 clearly allows us to reprove all the results of Section 6 (except for Proposition 6.3 and its corollary), but using $G_{\xi}^{\text{Lip}}$ and $D_{\xi}^{\text{Lip}}$ instead of $G_{\xi}^{W}$ and $D_{\xi}^{W}$. The obstacle in reproving also Proposition 6.3 simply relies in the definitions of the operations $\Sigma_{\xi}$ and $\Pi_{\xi}$: nevertheless, using the "coding" introduced in the definition of $G_{\xi}^{\text{Lip}}$ it is possible to define a set $\tilde{R}_{\xi}$ and two new operations $\tilde{\Sigma}_{\xi}$ and $\tilde{\Pi}_{\xi}$ such that $\tilde{\Sigma}_{\xi}(A) \equiv W \tilde{\Sigma}_{\xi}(A)$ and $\tilde{\Pi}_{\xi}(A) \equiv W \tilde{\Pi}_{\xi}(A)$ for every $A \subseteq \omega$, and with the further property that Proposition 6.3 holds whenever we replace all occurrences of $D_{\xi}^{W}$, $\Sigma_{\xi}$, $\Pi_{\xi}$ and $R_{\xi}$ in its statement with $D_{\xi}^{\text{Lip}}$, $\tilde{\Sigma}_{\xi}$, $\tilde{\Pi}_{\xi}$ and $\tilde{R}_{\xi}$.

From the construction of $G_{\xi}^{\text{Lip}}$ we can now infer which are the minimal conditions on the $F_{n}$’s under which one can carry out the constructions above and define the games $G_{\xi}^{F}$ which represent $D_{\xi}^{F}$.

**Definition 3.** Let $G_{*} = (X, M_{*}, R_{*}, \iota_{*})$ be a reduction game and $F_{*}$ be the set of functions induced by legal strategies for $\Pi$ in $G_{*}$. We say that $G_{*}$ (or $F_{*}$) is **delayable** if for every $n \in \omega$ the set $F_{n}$ is still represented by each of the new reduction games $G_{n}^{*} = (X, M_{*}^{n}, R_{*}^{n}, \iota_{*}^{n})$ defined by:

- $M_{*}^{n} = M_{*} \cup \{p\}$, where $p$ is a new symbol not in $M_{*}$;
- $R_{*}^{n} = \{ (x, y) \in \omega \times \omega \cup M_{*}^{n} \mid \forall k (y(k) = p \iff k < n) \land (x, (y(n + k) \mid k \in \omega)) \in R_{*} \}$;
- $\iota_{*}^{n} : R_{*}^{n} \to \omega \omega : (x, y) \mapsto \iota_{*}(x, (y(n + k) \mid k \in \omega))$.

As for $p$-closure, one could note that the property of being delayable corresponds to the property of being closed under right-composition with Lipschitz functions from $X$ into itself. From this definition and from the construction above, it turns out that we can still define reduction games representing $D_{\xi}^{F}$, $\tilde{D}_{\xi}^{F}$ and $\lim \tilde{F}$ whenever each element $F_{n}$ of the sequence $\tilde{F}$ contains the identity function and is represented by a parametrized class $G_{*}$ of delayable (rather than $p$-closed) reduction games.

This technical condition is optimal if we want to define reduction games like those presented in Section 4, in which $\Pi$ has to fill in a table with $\omega$-many rows: in fact, any reduction game is by definition formalizable as a game on $\omega$, and this essentially means that in each turn $\Pi$ can make at most a finite numbers of moves on a finite number of rows of her table, condition which easily leads to our definition of delayability. However, there are still examples of natural playable sets of functions (and even of reducibilities for sets of reals, like the set $L$) which are clearly non-delayable. This leaves open the following question:

**Question 3.** Given $\xi > 1$ and a non-delayable playable set of functions $F$, are there reduction games representing the classes of functions $D_{\xi}^{F}$ and $\tilde{D}_{\xi}^{F}$? In particular, are there reduction games representing $D_{\xi}^{L}$ and $\tilde{D}_{\xi}^{L}$?

**Acknowledgements** Research partially supported by FWF (Austrian Research Fund) through Project number P 19898-N18.

**References**

[1] A. Andretta, Equivalence between Wadge and Lipschitz determinacy, Ann. Pure Appl. Logic 123, 163–192 (2003).
[2] A. Andretta, More on Wadge determinacy, Ann. Pure Appl. Logic 144, 2–32 (2006).
[3] A. Andretta, The SLO principle and the Wadge hierarchy, Foundations of the formal sciences V, Stud. Log. (Lond.) 11, 1–38 (2007).
[4] J. Duparc, Wadge hierarchy and Veblen hierarchy part I: Borel sets of finite rank, J. Symb. Logic 66, 56–86 (2001).
[5] A. S. Kechris, Classical Descriptive Set Theory, No. 156 in Graduate Text in Mathematics (Springer-Verlag, Heidelberg, New York, 1995).
[6] A. S. Kechris, Determinacy with complicated strategies, Proc. Am. Math. Soc., 94, 333–336 (1985).
[7] L. Motto Ros, A new characterization of Baire class 1 functions, Real Anal. Exchange 34, 29–48 (2008/2009).
[8] L. Motto Ros, Baire reductions and good Borel reducibilities, J. Symb. Logic 75, 323–345 (2010).
[9] L. Motto Ros, Beyond Borel amenability: scales and superamenable reducibilities, Ann. Pure Appl. Logic 161, 829–836 (2010).
[10] L. Motto Ros, Borel-amenable reducibilities for sets of reals, J. Symb. Logic 74, 27–49 (2009).
[11] L. Motto Ros, General reducibilities for sets of reals, Ph.D. Thesis, Polytechnic of Turin, Italy, 2007.
[12] L. Motto Ros and B. Semmes, A new proof of a theorem of Jayne and Rogers, Real Anal. Exchange 35, 1–9 (2010).
[13] B. Semmes, Games, trees, and Borel functions, Ph.D. Thesis, ILLC, University of Amsterdam, Holland, 2008.
[14] B. Semmes, Multitape games, in: Interactive logic, edited by J. Van Benthem, D. Gabbay and B. Löwe, No. 1 in Texts in Logic and Games, 195–207, (Amsterdam University press, 2007).
[15] R. A. Van Wesep, Subsystems of second-order arithmetic and descriptive set theory under the axiom of determinateness, Ph.D. Thesis, University of California, Berkeley, 1977.
[16] R. A. Van Wesep, Wadge degrees and descriptive set theory, in: Cabal Seminar 76-77, edited by A. S. Kechris and Y. N. Moschovakis, No. 689 in Lecture Notes in Mathematics (Springer-Verlag, 1978).
[17] W. W. Wadge, Reducibility and determinateness on the Baire space, Ph.D. Thesis, University of California, Berkeley, 1983.