Sums of weighted averages of gcd-sum functions II

Isao Kiuchi and Sumaia Saad Eddin

Abstract

In this paper, we establish the following two identities involving the Gamma function and Bernoulli polynomials, namely

\[ \sum_{k \leq x} \frac{1}{k^s} \left( \sum_{j=1}^{k^s} \log \Gamma \left( \frac{j}{k^s} \right) \sum_{d | k} f \mu(d) \right) \quad \text{and} \quad \sum_{k \leq x} \frac{1}{k^s} \sum_{j=0}^{k^s-1} B_m \sum_{d | k, d^s | j} f \mu(d) \]

with any fixed integer \( s > 1 \) and any arithmetical function \( f \). We give asymptotic formulas for them with various multiplicative functions \( f \). We also consider several formulas of Dirichlet series associated with the above identities. This paper is a continuation of an earlier work of the authors.

1 Introduction

Throughout the paper we use the following notations: Let \( \mathbb{N} \) be the set of positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( f \) and \( g \) be two arithmetical functions. The Dirichlet convolution of \( f \) and \( g \) is defined by \( f \ast g(n) = \sum_{d|n} f(d)g(n/d) \) for \( n \in \mathbb{N} \). The functions \( \mu, \phi \) and \( \psi \), as usual, denote the Möbius function, the Euler totient function and the Dedekind function. We define arithmetic functions \( 1 \) and \( \text{id} \) by \( 1(n) = 1 \) and \( \text{id}(n) = n \) for all \( n \). We recall that Bernoulli polynomials are defined by the generating function

\[ \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \]

for \( |t| < 2\pi \). For \( x = 0 \), the numbers \( B_n = B_n(0) \) are always called Bernoulli numbers. Other notations will be given in the next section.

For \( j, k \in \mathbb{N} \), let \( \gcd(j, k) \) denote their greatest common divisor. For a fixed integer \( s \in \mathbb{N} \), let \( (a, b)_s \) denote the greatest common \( s \)-power divisor of \( a \) and \( b \). If \( a = j \) and \( b = k^s \), then it is

\[ (j, k^s)_s = \max \{ d^s : d^s | j, d | k \}. \]

Mathematics Subject Classification 2010: 11A25, 11N37, 11Y60.

Keywords: gcd-sum functions; Gamma function; Bernoulli polynomials.
Therefore, \((j, k^1)_1 = \gcd(j, k)\). The Ramanujan sum \(c_k\) is an arithmetic function which is defined as
\[
c_k(j) = \sum_{\substack{m=1 \\ \gcd(m, k) = 1}}^k \exp \left(2\pi i m j/k \right) = \sum_{d | \gcd(j, k)} d \mu \left(\frac{k}{d} \right),
\]
where \(i = \sqrt{-1}\). This function was first introduced by Ramanujan in 1918. In more recent years, various generalizations of the Ramanujan sum have been constructed. One of the most known generalizations of \(c_k\) is according to Cohen \([3],[4],[5]\), who defined the arithmetic function \(c_k^{(s)}\) for \(k, j \in \mathbb{N}\) and \(s \in \mathbb{N}\) by
\[
c_k^{(s)}(j) = \sum_{\substack{m=1 \\ (j, k^s)_s = 1}}^k \exp \left(2\pi i m j/k^s \right).
\]
The author proved that
\[
c_k^{(s)}(j) := \sum_{d | (j, k^s)_s} d^s \mu \left(\frac{k}{d} \right).
\]
In 1952, Anderson and Apostol \([1]\) introduced more general arithmetic function \(s_k\) defined by the identity
\[
s_k(j) = \sum_{d | \gcd(k, j)} f(d) g \left(\frac{k}{d} \right),
\]
with any arithmetical functions \(f\) and \(g\). In case that \(f = \text{id}\) and \(g = \mu\), the formula (1) becomes the Ramanujan sum \(c_k\). In the same paper, the authors gave several analytic and algebraic properties of \(s_k\). This latter is generalized to
\[
s_k^{(s)}(j) := \sum_{d | (j, k^s)_s} f(d) g \left(\frac{k}{d} \right).
\]
Again, when \(f = \text{id}\) and \(g = \mu\), the above formula gives the Cohen-Ramanujan sum \(c_k^{(s)}(j)\). There has been a good amount of work to study both functions \(s_k\) and \(s_k^{(s)}\). Recently, the first author derived identities for the partial sum of weighted averages of \(s_k(j)\) and \(s_k^{(s)}(j)\) with weights being logarithms, Gamma function \(\Gamma\), Bernoulli polynomials and others in \([7]\) and \([8]\). For any real number \(x > 1\) and any fixed integers \(r, s, m \geq 1\), he proved that:
\[
\sum_{k \leq x} \frac{1}{k^{s(r+1)}} \sum_{j=1}^{k^s} j^r s_k^{(s)}(j) = \frac{1}{2} \sum_{d \ell \leq x} f(d) g(\ell) + \frac{1}{r+1} \sum_{d \ell \leq x} \frac{f(d)}{d^s} g(\ell) \\
+ \frac{1}{r+1} \sum_{m=1}^{\lfloor r/2 \rfloor} \left( r+1 \right) B_{2m} \sum_{d \ell \leq x} \frac{f(d)}{d^s} g(\ell) \ell^{2ms},
\]
\[ \sum_{k \leq x} \frac{1}{k^s} \sum_{j=1}^{k^s} s_k^{(s)}(j) \log \left( \frac{j}{k^s} \right) = \log \sqrt{2\pi} \sum_{d | k} \frac{f(d)}{d^{s}} g(\ell) \]
\[ - \log \sqrt{2\pi} \sum_{d | k} \frac{f(d)}{d^{s}} g(\ell) \ell^s - \frac{s}{2} \sum_{d | k} \frac{f(d)}{d^{s}} g(\ell) \log \ell, \quad (4) \]
and
\[ \sum_{k \leq x} \frac{1}{k^s} \sum_{j=0}^{k^s-1} B_m \left( \frac{j}{k^s} \right) s_k^{(s)}(j) = B_m \sum_{d | k} \frac{f(d)}{d^{s}} g(\ell) \ell^s, \quad (5) \]

For any real or complex number \( a \), arithmetic functions \( \phi_a \) and \( \psi_a \) denote the Jordan totient function and the generalized Dedekind function defined by \( \text{id}_a \ast \mu \) and \( \text{id}_a \ast |\mu| \), respectively where \( \text{id}_a(n) = n^a \) for \( n \in \mathbb{N}_0 \). The von Mangoldt function is denoted by \( \Lambda \).

Now, if we denote sums on the left-hand side of (3), (4) and of (5) by \( M_r^{(s)}(x; f, g) \), \( A^{(s)}(x; f, g) \) and \( H_m^{(s)}(x; f, g) \) respectively, take \( f \ast \mu \) in place of \( f \) and \( g = 1 \) and use the fact that \( f \ast \mu \ast 1 = f \), \( f \ast \mu \ast \text{id}_s = f \ast \phi_s \) and \( f \ast \mu \ast \log = f \ast \Lambda \), we get

\[ M_r^{(s)}(x; f \ast \mu, 1) := \sum_{k \leq x} \frac{1}{k^{s(r+1)}} \sum_{j=1}^{k^s} j^r \sum_{d | k} \frac{f \ast \mu(d)}{d^{s}} \]
\[ = \frac{1}{2} \sum_{n \leq x} \frac{f(n)}{n^s} + \frac{1}{r+1} \sum_{d \leq x} \frac{f \ast \mu(d)}{d^{s}} + \frac{1}{r+1} \sum_{m=1}^{[r/2]} \left( \frac{r+1}{2m} \right) B_{2m} \sum_{d \leq x} \frac{f \ast \mu(d)}{d^{s(2m+1)}} \], \quad (6)

\[ A^{(s)}(x; f \ast \mu, 1) := \sum_{k \leq x} \frac{1}{k^s} \sum_{j=1}^{k^s} \log \left( \frac{j}{k^s} \right) \sum_{d | k} \frac{f \ast \mu(d)}{d^{s}} \]
\[ = \log \sqrt{2\pi} \sum_{n \leq x} \frac{f \ast \phi_s(n)}{n^s} - \log \sqrt{2\pi} \sum_{n \leq x} \frac{f(n)}{n^s} - \frac{s}{2} \sum_{n \leq x} \frac{f \ast \Lambda(n)}{n^s}, \quad (7) \]
and
\[ H_m^{(s)}(x; f \ast \mu, 1) := \sum_{k \leq x} \frac{1}{k^s} \sum_{j=0}^{k^s-1} B_m \sum_{d | k} \frac{f \ast \mu(d)}{d^{s}} \frac{1}{\ell^{ms}} \]

We will simply write \( M_r^{(s)}(x; f) \), \( A^{(s)}(x; f) \) and \( H_m^{(s)}(x; f) \) instead of writing \( M_r^{(s)}(x; f \ast \mu, 1) \), \( A^{(s)}(x; f \ast \mu, 1) \) and \( H_m^{(s)}(x; f \ast \mu, 1) \) respectively. When \( m = 1 \) and \( 2m \), we deduce
that

\[ H_1^{(s)}(x; f) = -\frac{1}{2} \sum_{n \leq x} \frac{f(n)}{n^s}, \]  

\[ H_2^{(s)}(x; f) = B_{2m} \sum_{d \leq x} f \ast \mu(d) \frac{1}{d^{2ms}}, \]  

respectively. In \([8]\) and \([9]\), the authors studied the special case of \( M_r^{(s)}(x; f) \), \( A^{(s)}(x; f) \) and \( H_m^{(s)}(x; f) \) when \( s = 1 \). They gave several interesting asymptotic formulas for weighted averages of gcd-sum function \( f(\gcd(k, j)) \) with \( f \) belonging to various multiplicative functions. We recall that the gcd-sum function, which is also known as Pillai’s arithmetic function, is essentially defined by

\[ P(n) = \sum_{k=1}^{n} \gcd(k, n). \]

Properties and various generalizations of \( P(n) \) have been widely studied by many authors. For a nice survey on this function, see \([11]\). One of many generalizations of \( P(n) \) is given by the identity

\[ P_f(n) = \sum_{k=1}^{n} f(\gcd(k, n)) \]

for an arbitrary arithmetical function \( f \).

For the general case when \( s > 1 \), the authors investigated, in \([9]\), the function \( M_r^{(s)}(x; f) \) and gave asymptotic formulas of it with \( f = \text{id}_{s+a}, \phi_s, \psi_s, \phi_{s+a}, \psi_{s+a}, \tau \ast \text{id}_s \), for any fixed real number \( a \) such that \(-1 < a < 0\). The aim of this paper is to give asymptotic formulas for \( A^{(s)}(x; f) \) and \( H_1^{(s)}(x; f) \) and \( H_2^{(s)}(x; f) \) defined by identities \([7]\), \([8]\) and \([9]\) with various multiplicative functions \( f \). Furthermore, we also consider several Dirichlet series associated with those functions. This work is a continuation of \([9]\).

\section{Main results}

Before proceeding further we need to fix additional notations. Let \( \tau \) and \( \sigma \) be the divisor function and the sum of divisors function (sigma function) defined by \( 1 \ast 1 \) and \( \text{id} \ast 1 \), respectively. Let \( \theta \) be the number appearing in the Dirichlet divisor problem

\[ \sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x), \]  

with \( \Delta(x) = O\left(x^{\theta+\varepsilon}\right) \) for \( \varepsilon > 0 \). The best estimate up to date for \( \Delta(x) \) due to Huxley \([6]\) is

\[ O\left(x^{131/416}(\log x)^{26947/8320}\right). \]
The constant term \( \gamma \) is the Euler constant.
More general, for any real or complex number \( a \), the function \( \sigma_a \) denotes a generalized divisor function where \( \sigma_a = id_a * 1 \). We recall that

\[
\sum_{n \leq x} \sigma_a(n) = \zeta(1-a)x + \frac{\zeta(1+a)}{1+a}x^{1+a} - \frac{\zeta(-a)}{2} + \Delta_a(x)
\]

(11)

for \(-1 < a < 0\). Here \( \zeta \) denotes the Riemann zeta-function. The problem of improving \( \Delta_a(x) \) is known as the generalized Dirichlet divisor problem. The classical estimate for \( \Delta_a(x) \) is

\[
O_a \left( \frac{x^{1+a}}{1+a} \right),
\]

(12)

for any small number \( \varepsilon > 0 \), see [10].

2.1 The function \( A^{(s)}(x; f) \)

Among many possible applications of \( A^{(s)}(x; f) \), we study in particular the following four cases when \( f = \phi_s, \phi_{s+a}, \psi_s, \) and \( \psi_{s+a} \) with \( s \) being any fixed integer \( s \geq 2 \) and \(-1 < a < 0\). Define the functions \( D_s(x) \) and \( \widetilde{D}_s(x) \) by

\[
D_s(x) = -\sum_{d \leq x} \frac{\mu(d)}{d^s} \vartheta \left( \frac{x}{d} \right) - \frac{1}{2\zeta(s)} \]

(13)

and

\[
\widetilde{D}_s(x) = -\sum_{d \leq x} \frac{|\mu(d)|}{d^s} \vartheta \left( \frac{x}{d} \right) - \frac{\zeta(s)}{2\zeta(2s)},
\]

(14)

where \( \vartheta(x) = x - [x] - \frac{1}{2} \). From [11], we have the following main results.

**Theorem 2.1.** For any real number \( x > 1 \), fixed number \( a \) such that \(-1 < a < 0\), and fixed integer \( s \geq 2 \), we have

\[
A^{(s)}(x; \phi_s) = \frac{\log \sqrt{2\pi}}{\zeta(s+1)}x \log x + \frac{s\zeta'(s+1)}{2\zeta(s+1)^2}x
\]

\[
+ \frac{\log \sqrt{2\pi}}{\zeta(s+1)^2} \left( 2\gamma - 1 - 2\frac{\zeta'(s+1)}{\zeta(s + 1)} - \zeta(s + 1) \right) x + Y_1^{(s)}(x),
\]

(15)

\[
A^{(s)}(x; \phi_{s+a}) = \frac{\sqrt{2\pi}}{\zeta(s+1)}x + \frac{s\zeta'(s + a + 1)}{2(1 + a)\zeta(s + a + 1)^2}x^{1+a}
\]

\[
+ \frac{\log \sqrt{2\pi}}{1 + a} \left( \frac{\zeta(1 + a)}{\zeta(s + a + 1)^2} - \frac{1}{\zeta(s + a + 1)} \right) x^{1+a} + Y_2^{(s)}(x),
\]

(16)
where

\[ Y_1^{(s)}(x) = \log \sqrt{2\pi} \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \Delta \left( \frac{x}{d} \right) + \frac{s}{2} \sum_{d \leq x} \frac{\mu * \Lambda(d)}{d^s} \vartheta \left( \frac{x}{d} \right) \]

\[ - \frac{s \zeta'(s)}{4 \zeta(s)^2} - \log \sqrt{2\pi} D_\ast(x) + O_s \left( x^{1-s}(\log x)^2 \right), \]  

and

\[ Y_2^{(s)}(x) = \log \sqrt{2\pi} \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \Delta_a \left( \frac{x}{d} \right) \]

\[ - \frac{\zeta(-a)}{2\zeta(s)^2} \log \sqrt{2\pi} \frac{\zeta(-a)}{\zeta(s)} \left( \log \sqrt{2\pi} - \frac{s \zeta'(s)}{2\zeta(s)} \right) + O_{s,a} \left( x^a \right). \]

**Theorem 2.2.** Under the hypothesis of Theorem 2.1, we have

\[ A^{(s)}(x; \psi_s) = \frac{\log \sqrt{2\pi}}{\zeta(2s+2)} x \log x + \frac{s \zeta'(s+1)}{2\zeta(2s+2)} x \]

\[ + \frac{\log \sqrt{2\pi}}{\zeta(2s+2)} \left( 2\gamma - 2 \frac{\zeta'(2s+2)}{\zeta(2s+2)} - \zeta(s+1) \right) x + Y_3^{(s)}(x), \]

and

\[ A^{(s)}(x; \psi_{s+a}) = \frac{\log \sqrt{2\pi} \zeta(1-a)}{\zeta(2s+2)} x + \frac{s \zeta'(s+a+1)}{2(1+a)\zeta(2s+2a+2)} x^{1+a} \]

\[ + \frac{\log \sqrt{2\pi}}{1+a} \left( \frac{\zeta(1+a)}{\zeta(2s+2a+2)} - \frac{\zeta(s+a+1)}{\zeta(2s+2a+2)} \right) x^{1+a} + Y_4^{(s)}(x), \]

where

\[ Y_3^{(s)}(x) = \log \sqrt{2\pi} \sum_{d \leq x} \frac{\mu * |\mu|d}{d^s} \Delta \left( \frac{x}{d} \right) - \frac{s \zeta'(s)}{4 \zeta(s)^2} \]

\[ + \frac{s}{2} \sum_{d \leq x} \frac{\mu * |\mu|d}{d^s} \vartheta \left( \frac{x}{d} \right) - \log \sqrt{2\pi} \bar{D}_\ast(x) + O_s \left( x^{1-s}(\log x)^2 \right), \]

and

\[ Y_4^{(s)}(x) = \log \sqrt{2\pi} \sum_{d \leq x} \frac{\mu * |\mu|d}{d^s} \Delta_a \left( \frac{x}{d} \right) - \frac{\zeta(-a)}{2 \zeta(2s)} \log \sqrt{2\pi} \]

\[ + \frac{\zeta(-a)}{\zeta(s)} \left( \log \sqrt{2\pi} - \frac{s \zeta'(s)}{2\zeta(s)} \right) + O_{s,a} \left( x^a \right). \]
Remark 2.3. Note that, by using elementary results of $\Delta$ and $\Delta_a$, it is easy to check that our functions $Y_1(x)$ and $Y_3(x)$ are estimated by $O_s \left( x^{\frac{1}{2} + \varepsilon} \right)$, and $Y_2(x)$ and $Y_4(x)$ are estimated by $O_{s,a} \left( x^{\frac{1}{2} + \alpha + \varepsilon} \right)$. Hence we get
\[
\lim_{x \to \infty} \frac{A^{(s)}(x; \phi_s)}{x \log x} = \log 2 \pi \zeta(s+1)^2, \quad \lim_{x \to \infty} \frac{A^{(s)}(x; \psi_s)}{x \log x} = \log 2 \pi \zeta(s+2),
\]
and
\[
\lim_{x \to \infty} \frac{A^{(s)}(x; \phi_{s+a})}{x} = \frac{\log \sqrt{2\pi} \zeta(1-a)}{(s+1)^2}, \quad \lim_{x \to \infty} \frac{A^{(s)}(x; \psi_{s+a})}{x} = \frac{\log \sqrt{2\pi} \zeta(1-a)}{(2s+2)}.
\]
From the above, we conclude that it is difficult to improve the $O$-terms in our theorems, since they basically depend on estimations of $\Delta$ and $\Delta_a$.

2.2 Dirichlet series associated with $A^{(s)}(x; f)$.

Let $F(w)$ and $G(w)$ be two functions represented by Dirichlet series as follows
\[
F(w) = \sum_{k=1}^{\infty} \frac{f(k)}{kw}, \quad \text{Re}(w) > \sigma_1,
\]
\[
G(w) = \sum_{k=1}^{\infty} \frac{g(k)}{kw}, \quad \text{Re}(w) > \sigma_2,
\]
which are absolutely convergent in the half-plane $\text{Re}(w) > \sigma_1$ and $\text{Re}(w) > \sigma_2$, respectively. Then Dirichlet series of the first derivative $G'(w)$ of $G(w)$ is given by
\[
G'(w) = -\sum_{k=1}^{\infty} \frac{g(k)}{kw} \log k, \quad \text{Re}(w) > \sigma_2.
\]

For $s, k \in \mathbb{N}_0$, define the weighted average $\kappa^{(s)}(k; f, g)$ by
\[
\kappa^{(s)}(k; f, g) = \frac{1}{k^s} \sum_{j=1}^{k^s} s_j^{(s)}(j) \log \Gamma \left( \frac{j}{k^s} \right)
\]
In [8], the first author showed that
\[
\kappa^{(s)}(k; f, g) = \log \sqrt{2\pi} \frac{f}{\text{id}_s} * g(k) - \log \sqrt{2\pi} \frac{f * g(k)}{k^s} - \frac{s}{2k^s} (f * g \log)(k).
\]
(23)

Let $K^{(s)}(w; f, g)$ be a function represented by Dirichlet series as follows:
\[
K^{(s)}(w; f, g) := \sum_{k=1}^{\infty} \frac{\kappa^{(s)}(k; f, g)}{kw},
\]
where this summation is absolutely convergent for \( \text{Re}(w) > \alpha \). We use a property of Dirichlet series to obtain

\[
K^{(s)}(w; f, g) = \log \sqrt{2\pi} \sum_{k=1}^{\infty} \sum_{\ell n = k}^{\infty} \frac{f(\ell) g(n)}{\ell^s} \frac{1}{n^w}
\]

\[
- \log \sqrt{2\pi} \sum_{k=1}^{\infty} \sum_{\ell n = k}^{\infty} f(\ell) g(n) \frac{1}{\ell^w n^{w+s}} - \frac{s}{2} \sum_{k=1}^{\infty} \sum_{\ell n = k}^{\infty} f(\ell) g(n) \log n \frac{1}{\ell^w n^{w+s}}
\]

\[
= \log \sqrt{2\pi} \sum_{\ell = 1}^{\infty} \frac{f(\ell)}{\ell^{w+s}} \sum_{n=1}^{\infty} \frac{g(n)}{n^w}
\]

\[
- \log \sqrt{2\pi} \sum_{\ell = 1}^{\infty} \frac{f(\ell)}{\ell^{w+s}} \sum_{n=1}^{\infty} \frac{g(n) \log n}{n^{w+s}}.
\]

This leads to the following result

\[
K^{(s)}(w; f, g) = \log \sqrt{2\pi} F(w + s) (G(w) - G(w + s)) + \frac{s}{2} F(w + s) G'(w + s).
\]

Taking \( f \ast \mu \) in place of \( f \) and \( g = 1 \), the above identity becomes

\[
K^{(s)}(w; f \ast \mu, 1) = \log \sqrt{2\pi} F(w + s) \left( \frac{\zeta(w)}{\zeta(w + s)} - 1 \right) + \frac{s}{2} F(w + s) \frac{\zeta'(w + s)}{\zeta(w + s)}.
\]

Again, we write \( K^{(s)}(w; f) \) instead of writing \( K^{(s)}(w; f \ast \mu, 1) \). For \( f = \phi_s, \psi_s, \phi_{s+a} \) and \( \psi_{s+a} \) successively, we deduce the following identities:

**Corollary 2.4.** Under the above notations, we have

\[
K^{(s)}(w; \phi_s \ast \mu) = \log \sqrt{2\pi} \frac{\zeta(w)}{\zeta(w + s)} \left( \frac{\zeta(w)}{\zeta(w + s)} - 1 \right) + \frac{s}{2} \frac{\zeta(w) \zeta'(w + s)}{\zeta(w + s)^2},
\]

\[
K^{(s)}(w; \psi_s \ast \mu) = \log \sqrt{2\pi} \zeta(w) \frac{\zeta(w + s)}{\zeta(2w + 2s)} \left( \frac{\zeta(w)}{\zeta(w + s)} - 1 \right) + \frac{s}{2} \frac{\zeta(w) \zeta'(w + s)}{\zeta(2w + 2s)},
\]

\[
K^{(s)}(w; \phi_{s+a} \ast \mu) = \log \sqrt{2\pi} \frac{\zeta(w - a)}{\zeta(w + s)} \left( \frac{\zeta(w)}{\zeta(w + s)} - 1 \right) + \frac{s}{2} \frac{\zeta(w - a) \zeta'(w + s)}{\zeta(w + s)^2},
\]

\[
K^{(s)}(w; \psi_{s+a} \ast \mu) = \log \sqrt{2\pi} \frac{\zeta(w - a) \zeta(w + s)}{\zeta(2w + 2s)} \left( \frac{\zeta(w)}{\zeta(w + s)} - 1 \right) + \frac{s}{2} \frac{\zeta(w - a) \zeta'(w + s)}{\zeta(2w + 2s)}.
\]

### 2.3 The function \( H^{(s)}_m(x; f) \)

Now, we consider the partial sums of the weighted average of \( s_k^{(s)}(j) \) involving Bernoulli polynomials. We establish asymptotic formulas of \( \text{(8)} \) and \( \text{(9)} \) for \( f = \phi_s, \phi_{s+a}, \psi_s, \psi_{s+a} \), with \( -1 < a < 0 \). Our results are precisely the following:
Theorem 2.5. For any real number $x > 1$, fixed integers $s \geq 2$, $m \geq 1$, we have
\begin{align*}
H_1^{(s)}(x; \phi_s) &= -\frac{1}{2\zeta(s+1)} x - \frac{D_s(x)}{2} + O_s \left( x^{1-s} \right), \\
H_1^{(s)}(x; \psi_s) &= -\frac{\zeta(s+1)}{2\zeta(2s+2)} x - \frac{\widetilde{D}_s(x)}{2} + O_s \left( x^{1-s} \right),
\end{align*}
where $D_s(x)$ and $\widetilde{D}_s(x)$ are given by (13) and (14). Moreover, we have
\begin{align*}
H_2^{(s)}(x; \phi_s) &= B_{2m} \frac{\zeta(2ms+1)}{\zeta(s+1)^2} x - B_{2m} \frac{\zeta(2ms)}{2\zeta(s)^2} + O_{m,s} \left( x^{1-s} \log x \right), \\
H_2^{(s)}(x; \psi_s) &= B_{2m} \frac{\zeta(2ms+1)}{2\zeta(2s+2)} x - B_{2m} \frac{\zeta(2ms)}{2\zeta(2s)} + O_{m,s} \left( x^{1-s} \log x \right).
\end{align*}

Theorem 2.6. Under the hypothesis of Theorem 2.5. For any fixed real number $a$ such that $-1 < a < 0$, we have
\begin{align*}
H_1^{(s)}(x; \phi_{s+a}) &= -\frac{1}{2(1+a)\zeta(s+a+1)} x^{1+a} - \frac{\zeta(-a)}{2\zeta(s)} + O_{s,a} \left( x^a \right), \\
H_1^{(s)}(x; \psi_{s+a}) &= -\frac{\zeta(s+a+1)}{2(1+a)\zeta(2s+2a+2)} x^{1+a} - \frac{\zeta(-a)\zeta(s)}{2\zeta(2s)} + O_{s,a} \left( x^a \right), \\
H_2^{(s)}(x; \phi_{s+a}) &= \frac{B_{2m}\zeta(2ms+a+1)}{(1+a)\zeta(s+a+1)^2} x^{1+a} + \frac{B_{2m}\zeta(-a)\zeta(2ms)}{\zeta(s)^2} + O_{a,m,s} \left( x^a \right),
\end{align*}
and
\begin{align*}
H_2^{(s)}(x; \psi_{s+a}) &= \frac{B_{2m}\zeta(2ms+a+1)}{(1+a)\zeta(2s+2a+2)} x^{1+a} + \frac{B_{2m}\zeta(-a)\zeta(2ms)}{\zeta(2s)} + O_{a,m,s} \left( x^a \right).
\end{align*}

2.4 Dirichlet series associated with $H_m^{(s)}(x; f)$

From \cite{K}, we recall that
\begin{equation}
\frac{1}{k^s} \sum_{j=0}^{k^s-1} B_m \left( \frac{j}{k^s} \right) \nu^{(s)}(j) = B_m \sum_{d \ell = k} \frac{f(d)}{d^s} g(\ell d) \overline{\nu^{(s)}}.
\end{equation}

We introduce the notation $\nu_m^{(s)}(k; f, g)$ on the left-hand side of the above formula for convenience. Then, Dirichlet series associated with $\nu_m^{(s)}(k)$ is given by
\begin{equation}
L^{(s)}(w; f, g) := \sum_{k=1}^{\infty} \frac{\nu_m^{(s)}(k; f, g)}{k^w},
\end{equation}

9
which is absolutely convergent for Re$(w) > \beta$. We use identity (32) to proceed:

$$L_m^{(s)}(w; f, g) = B_m \sum_{k=1}^{\infty} \left( \sum_{\ell=n}^{\infty} \frac{f(\ell) g(n)}{n^{m s}} \right) \frac{1}{k^w} = B_m \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{f(\ell) g(n)}{n^{w+m s}}.$$  

This leads at once to

$$L_m^{(s)}(w; f, g) = B_m F(w + s) G(w + m s). \tag{34}$$

Now, we replace $f \ast \mu$ by $f$ and $g = 1$ into (34) to obtain

$$L_m^{(s)}(w; f, \mu, 1) := L_m^{(s)}(w; f) = B_m F(w + s) \frac{\zeta(w + m s)}{\zeta(w + s)}.$$ 

Then, we deduce the following results:

**Corollary 2.7.** Under the above notations, we have

$$L_m^{(s)}(w; \phi_s \ast \mu) = B_m \frac{\zeta(w) \zeta(w + m s)}{\zeta(w + s)^2}, \quad L_m^{(s)}(w; \phi_{s+a} \ast \mu) = B_m \frac{\zeta(w - a) \zeta(w + m s)}{\zeta(w + s)^2},$$

$$L_m^{(s)}(w; \psi_s \ast \mu) = B_m \frac{\zeta(w) \zeta(w + m s)}{\zeta(2w + 2s)}, \quad L_m^{(s)}(w; \psi_{s+a} \ast \mu) = B_m \frac{\zeta(w - a) \zeta(w + m s)}{\zeta(2w + 2s)}.$$

### 3 Auxiliary results

Before going into the proof of main results, we need to give some auxiliary lemmas.

**Lemma 3.1.** For any sufficiently large number $x > 1$ and fixed integer $s \geq 2$, we have

$$\sum_{n \leq x} \phi_s(n) = \frac{1}{\zeta(s + 1)} x + D_s(x) + O_s \left( x^{1-s} \right), \tag{35}$$

$$\sum_{n \leq x} \psi_s(n) = \frac{\zeta(s + 1)}{\zeta(2s + 2)} x + \tilde{D}_s(x) + O_s \left( x^{1-s} \right), \tag{36}$$

where $D_s(x)$ and $\tilde{D}_s(x)$ are defined by (13) and (14).

**Proof.** The formula (35) follows from (2.9) in [7]. On the other hand, we have

$$\sum_{n \leq x} \psi_s(n) = \sum_{d \leq x} \frac{|\mu(d)|}{d^s} \sum_{\ell \leq x/d} 1$$

$$= x \sum_{d \leq x} \frac{|\mu(d)|}{d^{s+1}} - \sum_{d \leq x} \frac{|\mu(d)|}{d^s} \vartheta \left( \frac{x}{d} \right) - \frac{1}{2} \sum_{\ell \leq x} \frac{|\mu(\ell)|}{\ell^s}$$

$$= \frac{\zeta(s + 1)}{\zeta(2s + 2)} x - \sum_{d \leq x} \frac{|\mu(d)|}{d^s} \vartheta \left( \frac{x}{d} \right) - \frac{\zeta(s)}{2 \zeta(2s)} + O_s \left( x^{1-s} \right),$$

which completes the proof of (36). \qed
Lemma 3.2. For any sufficiently large number $x > 1$, fixed integer $s \geq 2$ and fixed number $a$ such that $-1 < a < 0$, we have

$$\sum_{n \leq x} \frac{\phi_{s+a}(n)}{n^s} = \frac{1}{(1+a)\zeta(s+a+1)} x^{1+a} \frac{\zeta(-a)}{\zeta(s)} + O_{a,s}(x^a), \quad (37)$$

$$\sum_{n \leq x} \frac{\psi_{s+a}(n)}{n^s} = \frac{\zeta(s+a+1)}{(1+a)\zeta(2s+2a+2)} x^{1+a} \frac{\zeta(-a)\zeta(s)}{\zeta(2s)} + O_{a,s}(x^a). \quad (38)$$

Proof. From see [2], Theorem 3.2 (b), for $-1 < a < 0$, we have

$$\sum_{n \leq x} n^a = \frac{x^{1+a}}{1+a} + \zeta(-a) + O_a(x^a), \quad (39)$$

We use the above formula to get

$$\sum_{n \leq x} \frac{\phi_{s+a}(n)}{n^s} = \sum_{d \leq x} \frac{\mu(d)}{d^s} \sum_{\ell \leq x/d} \ell^a$$

$$= \frac{x^{1+a}}{1+a} \sum_{d \leq x} \frac{\mu(d)}{d^{s+a+1}} + \zeta(-a) \sum_{d \leq x} \frac{\mu(d)}{d^s} + O_a \left( x^a \sum_{\ell \leq x} \frac{1}{\ell^{s+a}} \right)$$

$$= \frac{1}{(1+a)\zeta(s+a+1)} x^{1+a} \frac{\zeta(-a)\zeta(s)}{\zeta(2s)} + O_{a,s}(x^a),$$

which completes the proof of (37). Similarly we get (38). \(\square\)

4 Proofs

4.1 Proof of Theorem 2.1

For $f = \phi_s$, we use identity (7) to write

$$A(s)(x; \phi_s) = \log \sqrt{2\pi} \sum_{n \leq x} \frac{\phi_s * \phi_s(n)}{n^s} - \log \sqrt{2\pi} \sum_{n \leq x} \frac{\phi_s(n)}{n^s} - \frac{s}{2} \sum_{n \leq x} \frac{\phi_s * \Lambda(n)}{n^s}.$$

To produce an asymptotic formula of $A(s)(x; \phi_s)$, we need to estimate the first and third sums above. Using the identities

$$\frac{\phi_s * \phi_s}{\text{id}_s} = \left( \frac{\mu}{\text{id}_s} * \frac{\phi_s}{\text{id}_s} \right) * 1 = \frac{\mu * \mu}{\text{id}_s} * \tau,$$

and (10), and formulas

$$\sum_{d \leq x} \frac{\mu * \mu(d)}{d^{s+1}} = \frac{1}{\zeta(s+1)^2} + O_s \left( x^{-s} \log x \right), \quad (40)$$

$$\sum_{d \leq x} \frac{\mu * \mu(d)}{d^{s+1}} \log d = 2 \frac{\zeta(s+1)}{\zeta(s+1)^3} + O_s \left( x^{-s}(\log x)^2 \right), \quad (41)$$

11
we get
\[ \sum_{n \leq x} \frac{\phi_s \ast \phi_s(n)}{n^s} = \sum_{d \leq x} \frac{\mu \ast \mu(d)}{d^s} \sum_{\ell \leq x/d} \tau(\ell) \]
\[ = x \log x \sum_{d \leq x} \frac{\mu \ast \mu(d)}{d^s+1} - x \sum_{d \leq x} \frac{\mu \ast \mu(d)}{d^s+1} \log d \]
\[ + (2\gamma - 1)x \sum_{d \leq x} \frac{\mu \ast \mu(d)}{d^s+1} + \sum_{d \leq x} \frac{\mu \ast \mu(d)}{d^s} \Delta \left( \frac{x}{d} \right) \]
\[ = \frac{1}{\zeta(s+1)^2} x \log x + \frac{1}{\zeta(s+1)^2} \left( 2\gamma - 1 - 2\frac{\zeta'(s+1)}{\zeta(s+1)} \right) x \]
\[ + \sum_{d \leq x} \frac{\mu \ast \mu(d)}{d^s} \Delta \left( \frac{x}{d} \right) + O_s \left( x^{1-s} (\log x)^2 \right). \] (42)

For the third sum of \( A^{(s)}(x; \phi_s) \), we use the fact that
\[ \phi_s \ast \Lambda = \mu \ast \Lambda \ast 1, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mu \ast \Lambda(n)}{n^{s+1}} = \frac{-\zeta'(s+1)}{\zeta(s+1)^2}, \]
to write
\[ \sum_{n \leq x} \frac{\phi_s \ast \Lambda(n)}{n^s} = \sum_{d \leq x} \frac{\mu \ast \Lambda(d)}{d^s} \sum_{\ell \leq x/d} 1 \]
\[ = x \sum_{d \leq x} \frac{\mu \ast \Lambda(d)}{d^s+1} - \sum_{d \leq x} \frac{\mu \ast \Lambda(d)}{d^s} \vartheta \left( \frac{x}{d} \right) - \frac{1}{2} \sum_{d \leq x} \frac{\mu \ast \Lambda(d)}{d^s} \]
\[ = -\frac{\zeta'(s+1)}{\zeta(s+1)^2} x - \sum_{d \leq x} \frac{\mu \ast \Lambda(d)}{d^s} \vartheta \left( \frac{x}{d} \right) + \frac{\zeta'(s)}{2\zeta(s)^2} + O_s \left( \frac{(\log x)^2}{x^{s-1}} \right). \]

From the latter formula, identity (35), and (43), we get the identity (15).

Now, we recall that
\[ A^{(s)}(x; \phi_{s+a}) = \log \sqrt{2\pi} \sum_{n \leq x} \frac{\phi_s \ast \phi_{s+a}(n)}{n^s} - \log \sqrt{2\pi} \sum_{n \leq x} \frac{\phi_{s+a}(n)}{n^s} - \frac{s}{2} \sum_{n \leq x} \frac{\phi_{s+a} \ast \Lambda(n)}{n^s}. \]

In order to prove the identity (16), we calculate each of three sums on the right-hand side above separately. For the first sum, we use (11), (40) and
\[ \frac{\phi_{s+a} \ast \phi_s}{\text{id}_d} = \frac{\phi_{s+a} \ast \mu}{\text{id}_d} \ast 1 = \frac{\mu \ast \mu}{\text{id}_d} \ast \sigma_a. \]
to obtain

\[ \sum_{n \leq x} \frac{\phi_{s+a} * \phi_s(n)}{n^s} = \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \sum_{\ell \leq x/d} \sigma_a(\ell) \]

\[ = \zeta(1-a) x \sum_{d \leq x} \frac{\mu * \mu(d)}{d^{s+1}} + \frac{\zeta(1+a)}{1+a} \sum_{d \leq x} \frac{\mu * \mu(d)}{d^{s+a+1}} \]

\[ - \frac{\zeta(-a)}{2} \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} + \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \Delta_a \left(\frac{x}{d}\right) = \frac{\zeta(1-a)}{\zeta(s+1)^2} x + \frac{\zeta(1+a)}{(1+a)\zeta(s+a+1)^2} x^{1+a} - \frac{\zeta(-a)}{2\zeta(s)^2} \]

\[ + \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \Delta_a \left(\frac{x}{d}\right) \]

\[ = \zeta(s+1)^2 \left( \psi_{s+a} * \phi_s \right)(x) + \zeta(s+1)(s+a+1)^2 \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \Delta_a \left(\frac{x}{d}\right) + O_{s,a} \left(x^{1-s} \log x\right). \quad (43) \]

For the third sum of \( A^{(s)}(x; \phi_{s+a}) \), we use (39) to obtain the formula

\[ \sum_{n \leq x} \frac{\phi_{s+a} * \Lambda(n)}{n^s} = \sum_{d \leq x} \frac{\mu * \Lambda(d)}{d^s} \sum_{\ell \leq x/d} \ell^a \]

\[ = - \frac{\zeta'(s+a+1)}{(1+a)\zeta(s+a+1)^2} x^{1+a} - \frac{\zeta(-a)}{\zeta(s)^2} + O_{s,a} \left(x^a\right). \]

Combining this latter, (37) and (44), we get (16).

### 4.2 Proof of Theorem 2.2

Taking \( f = \psi_s \) into (7), we have

\[ A^{(s)}(x; \psi_s) = \log \sqrt{2\pi} \sum_{n \leq x} \frac{\psi_s * \phi_s(n)}{n^s} - \log \sqrt{2\pi} \sum_{n \leq x} \frac{\psi_s(n)}{n^s} - \frac{s}{2} \sum_{n \leq x} \psi_s * \Lambda(n) n^s. \]

We notice that the first sum on the right-hand side of the above is rewritten as

\[ \frac{\psi_s * \phi_s}{\text{id}_s} = \left(\frac{\psi_s}{\text{id}_s} * \frac{\mu}{\text{id}_s}\right) * 1 = \frac{\mu * |\mu|}{\text{id}_s} * \tau. \]

Using (10) and formulas

\[ \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^{s+1}} = \frac{1}{\zeta(2s+2)} + O_s \left(x^{-s} \log x\right), \quad (44) \]

\[ \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^{s+1}} \log d = 2 \frac{\zeta'(2s+2)}{\zeta(2s+2)^2} + O_s \left(x^{-s} (\log x)^2\right), \]

Combining this latter, (37) and (44), we get (16).
we infer
\[
\sum_{n \leq x} \psi_s * \phi_s(n) \quad = \quad \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^s} \sum_{\ell \leq x/d} \tau(\ell)
\]
\[
= \quad x \log x \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^{s+1}} - x \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^{s+1}} \log d
\]
\[
+ (2\gamma - 1)x \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^{s+1}} + \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^s} \Delta \left(\frac{x}{d}\right)
\]
\[
= \quad \frac{1}{\zeta(2s + 2)} x \log x + \frac{1}{\zeta(2s + 2)} \left(2\gamma - 1 - \frac{\zeta'(2s + 2)}{\zeta(2s + 2)}\right) x
\]
\[
+ \sum_{d \leq x} \frac{\mu * |\mu|(d)}{d^s} \Delta \left(\frac{x}{d}\right) + O_s \left(x^{1-s}(\log x)^2\right). \quad (45)
\]

For the third sum of \(A^{(s)}(x; \psi_s)\), we use the fact that
\[
\frac{\psi_s * \Lambda}{\text{id}_s} = |\mu * \Lambda| \star 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|\mu * \Lambda(n)|}{n^{s+1}} = -\frac{\zeta'(s + 1)}{\zeta(2s + 2)}
\]
to obtain
\[
\sum_{n \leq x} \frac{\psi_s * \Lambda(n)}{n^s} = \sum_{d \leq x} \frac{|\mu * \Lambda(d)|}{d^s} \sum_{\ell \leq x/d} 1
\]
\[
= \quad x \sum_{d \leq x} \frac{|\mu * \Lambda(d)|}{d^{s+1}} - \sum_{d \leq x} \frac{|\mu * \Lambda(d)|}{d^s} \vartheta \left(\frac{x}{d}\right) - \frac{1}{2} \sum_{d \leq x} \frac{|\mu * \Lambda(d)|}{d^s}
\]
\[
= \quad -\frac{\zeta'(s + 1)}{\zeta(2s + 2)} x - \sum_{d \leq x} \frac{|\mu * \Lambda(d)|}{d^s} \vartheta \left(\frac{x}{d}\right) + \frac{\zeta'(s)}{2\zeta(2s)} + O_s \left(x^{1-s}(\log x)^2\right). \quad (46)
\]

From (45), (36) and (46), we get (19).

In order to prove the identity (20), we are going to estimate each term on the right-hand side of the following formula
\[
A^{(s)}(x; \psi_{s+a}) = \log \sqrt{2\pi} \sum_{n \leq x} \frac{\psi_{s+a} * \phi_s(n)}{n^s} - \log \sqrt{2\pi} \sum_{n \leq x} \frac{\psi_{s+a}(n)}{n^s} - \frac{s}{2} \sum_{n \leq x} \frac{\psi_{s+a} * \Lambda(n)}{n^s}.
\]

Since the identity
\[
\frac{\phi_s * \psi_{s+a}}{\text{id}_s} = \left(\frac{\mu}{\text{id}_s} \star \frac{\psi_{s+a}}{\text{id}_s}\right) \star 1 = \frac{\mu * |\mu|}{\text{id}_s} * \sigma_a,
\]
and the treatment of the first sum above is similar to that used in the proof of Theorem 2.1. This yields

\[
\sum_{n \leq x} \frac{\psi_{s+a} \ast \phi_s(n)}{n^s} = \frac{\zeta(1-a)}{\zeta(2s+2)} x + \frac{\zeta(1+a)}{(1+a)\zeta(2s+2a+2)} x^{1+a} - \frac{\zeta(-a)}{2\zeta(2s)} + \sum_{d \leq x} \frac{\mu \ast |\mu|(d)}{d^s} \Delta_a \left( \frac{x}{d} \right) + O_{s,a} \left( x^{1-s} \log x \right),
\]

where we used (44) instead of (40). Similarly, the third sum is

\[
\sum_{n \leq x} \frac{\psi_{s+a} \ast \Lambda(n)}{n^s} = -\frac{\zeta'(s+a+1)}{(1+a)\zeta(2s+2a+2)} x^{1+a} - \frac{\zeta(-a)\zeta'(s)}{\zeta(2s)} + O_{s,a} \left( x^{a} \right).
\]

From the above and (38), the proof is complete.

### 4.3 Proof of Theorems 2.5 and 2.6

By (8) with replaced \( f \) by \( \phi_s, \psi_s, \phi_{s+a}, \psi_{s+a} \) successively, and using (35), (36), (37) and (38). We get at once (24), (25), (28) and (29). To complete the proof of Theorems 2.5 and 2.6, it remains to prove the following relations

\[
\sum_{d \leq x} \frac{\phi_{s+a} \ast \mu(d)}{d^s} \frac{1}{\ell^{2ms}} = \frac{\zeta(2ms+a+1)}{(1+a)\zeta(s+a+1)^2} x^{1+a} + \frac{\zeta(-a)\zeta(2ms)}{\zeta(s)^2} + O_{s,a,m} \left( x^{a} \right),
\]

and

\[
\sum_{d \leq x} \frac{\psi_{s+a} \ast \mu(d)}{d^s} \frac{1}{\ell^{2ms}} = \frac{\zeta(2ms+a+1)}{(1+a)\zeta(s+2a+2)} x^{1+a} + \frac{\zeta(-a)\zeta(2ms)}{\zeta(2s)} + O_{s,a,m} \left( x^{a} \right).
\]

We start with (48). Using the identity

\[
\frac{\phi_s \ast \mu}{id_s} \ast \frac{1}{\ell^{2ms}} = \frac{\mu \ast \mu}{id_s} \ast \sigma_{-2ms},
\]

the formula

\[
\sum_{\ell \leq x} \sigma_{-2ms}(\ell) = \zeta(2ms+1)x - \frac{\zeta(2ms)}{2} + O_{m,s} \left( x^{1-2ms} \right)
\]
and (40), we find that

\[
\sum_{d \leq x} \frac{\phi_s * \mu(d)}{d^s} \frac{1}{\ell^{2ms}} = \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \sum_{\ell \leq x/d} \sigma_{-2ms}(\ell)
\]

\[
= \zeta(2ms + 1) x \sum_{d \leq x} \frac{\mu * \mu(d)}{d^{s+1}} - \frac{\zeta(2ms)}{2} \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s}
\]

\[
+ O \left( x^{1-2ms} \sum_{d \leq x} \frac{\tau(d)}{d^{s+1-2ms}} \right)
\]

\[
= \frac{\zeta(2ms + 1)}{\zeta(s + 1)^2} x - \frac{\zeta(2ms)}{2 \zeta(s)^2} + O_{s,m} (x^{1-s} \log x),
\]

as required. By a similar argument to the above, by using (44) instead of (40), we obtain (49). For (50), using the identity

\[
\frac{\phi_{s+a} * \mu}{id_s} * \frac{1}{id_{2ms}} = \frac{\mu * \mu}{id_s} * \frac{\sigma_{a+2ms}}{id_{2ms}},
\]

and the formula

\[
\sum_{\ell \leq x} \frac{\sigma_{a+2ms}(\ell)}{\ell^{2ms}} = \frac{\zeta(a + 2ms + 1)}{a + 1} x^{a+1} + \zeta(-a) \zeta(2ms) + O_{s,a,m} (x^a), \quad (53)
\]

we get

\[
\sum_{d \leq x} \frac{\phi_{s+a} * \mu(d)}{d^s} \frac{1}{\ell^{2ms}} = \sum_{d \leq x} \frac{\mu * \mu(d)}{d^s} \frac{\sigma_{a+2ms}(\ell)}{\ell^{2ms}}
\]

\[
= \frac{\zeta(2ms + a + 1)}{(1+a)\zeta(s + a + 1)^2} x^{1+a} + \frac{\zeta(-a) \zeta(2ms)}{\zeta(s)^2} + O_{s,a,m} (x^a).
\]

Similarly, we obtain (51). Which completes the proof of our theorems.

**Acknowledgment.** The second author is supported by the Austrian Science Fund (FWF) : Projects F5507-N26, and F5505-N26 which are parts of the special Research Program “Quasi Monte Carlo Methods : Theory and Application”. Part of this work was done while the second author was supported by the Japan Society for the Promotion of Science (JSPS). “Overseas researcher under Postdoctoral Fellowship of JSPS”.

**References**

[1] D. R. Anderson and T. M. Apostol, The evaluation of Ramanujan’s sum and generalizations, *Duke Math. J.* **20** (1952), 211–216.
[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[3] E. Cohen, An extension of Ramanujan’s sums, *Duke Math. J.* 16 (1949), 85–90.

[4] E. Cohen, An extension of Ramanujan’s sums II. Additive Properties, *Duke Math. J.* 22 (1955), 543–559.

[5] E. Cohen, An extension of Ramanujan’s sums III. Connections with totient functions, *Duke Math. J.* 23 (1956), 623–630.

[6] M. N. Huxley, Exponential sums and lattice points III, *Proc. London Math. Soc.* 87 (2003), 591–609.

[7] I. Kiuchi, On sums of averages of generalized Ramanujan sums, *Tokyo J. Math.* 40 (2017), 255–275.

[8] I. Kiuchi, Sums of averages of generalized Ramanujan sums, *J. Number Theory* 180 (2017), 310–348.

[9] I. Kiuchi and S. Saad Eddin, Sums of weighted averages of gcd-sum functions, *Int. J. Number Theory* 14 (2018), 2699-2728.

[10] Y.-F. S. Péttermann, Divisor problems and exponent pairs, *Arch. Math. (Basel)* 50 (1988), 243–250.

[11] L. Tóth, A survey of gcd-sum functions, *J. Integer Sequences* 13 (2010), Article 10.8.1.

Isao Kiuchi: Department of Mathematical Sciences, Faculty of Science, Yamaguchi University, Yoshida 1677-1, Yamaguchi 753-8512, Japan.
e-mail: kiuchi@yamaguchi-u.ac.jp

Sumaia Saad Eddin: Institute of Financial Mathematics and Applied Number Theory, Johannes Kepler University, Altenbergerstrasse 69, 4040 Linz, Austria.
e-mail: sumaia.saad_eddin@jku.at