Hamiltonicity of 2-block intersection graphs of TS($v, \lambda$): $v \equiv 0$ or 4 (mod 12)

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Abstract
A TS($v, \lambda$) is a pair ($V, B$) where $V$ contains $v$ points and $B$ contains 3-element subsets of $V$ so that each pair in $V$ appears in exactly $\lambda$ blocks. A 2-block intersection graph (2-BIG) of a TS($v, \lambda$) is a graph where each vertex is represented by a block from the TS($v, \lambda$) and each pair of blocks $B_i, B_j \in B$ are joined by an edge if $|B_i \cap B_j| = 2$. Using constructions for TS($v, \lambda$) given by Schreiber [10], we show that there exists a TS($v, \lambda$) for $v \equiv 0$ or 4 (mod 12) whose 2-BIG is Hamiltonian.

1 Introduction
There are many links between design theory and graph theory, and often times considering a problem from one field in the context of the other aids in solving the problem. In this paper, we will focus on constructing a particular type of graph directly from a design and investigating the Hamiltonicity of such a graph.

A pairwise balanced design is a set of points with a collection of blocks such that each pair of points lie in a unique block. If we require each block to have size 3 but relax the balancing condition, we have a triple system. A triple system (TS($v, \lambda$)) is a pair ($V, B$) where $V$ is a set of $v$ points and $B$ is a set of 3-element subsets of $V$ (called blocks) such that each pair of points in $V$ appears in exactly $\lambda$ blocks. A TS($v, 1$) is called a Steiner triple system. A TS($v, \lambda$) is called simple if there are no repeated blocks in $B$. Dehon [3] showed that there exists a TS($v, \lambda$) if and only if $\lambda \leq v - 2$, $\lambda(v - 1) \equiv 0$ (mod 6) and $\lambda(v - 1) \equiv 0$ (mod 2). We say that $v, \lambda$ are admissible if there exists a TS($v, \lambda$).

A block intersection graph (BIG) of a TS($v, \lambda$) ($V, B$) is a graph where each block in $B$ represents a vertex and two vertices in the BIG are adjacent if the corresponding blocks share at least one point. This can be extended by varying the adjacency rules as follows. A $k$-block intersection graph ($k$-BIG) of a TS($v, \lambda$) ($V, B$) has the same vertex set as a BIG, but two vertices in the $k$-BIG are adjacent if they share exactly $k$ vertices in common.

Our investigation focuses on 2-BIGs of TS($v, \lambda$)s. In particular, we are interested in studying which 2-BIGs are Hamiltonian. Dewar [4] showed that there exists a TS($v, 2$) whose 2-BIG is Hamiltonian if $v \equiv 3$ or 7 (mod 12) and $v \geq 7$, or $v \equiv 1$ or 4 (mod 12) and $v \neq 0$ (mod 5). This result was extended by Erzurumluoğlu and Pike in [6] where the complete spectrum was given for the existence of TS($v, 2$)s whose 2-BIGs are Hamiltonian.
It is important to realize that there can be many distinct TS\((v, \lambda)\) with the same parameters. Thus it is possible for there to exist a TS\((v, \lambda)\) that is Hamiltonian (as shown in this paper) and for a TS\((v, \lambda)\) to be non-Hamiltonian (as shown in [3]). On the other hand, Horák [7] showed that the 2-BIG of all TS\((v, 1)\) are Hamiltonian. Later, Alspach et al. [1] showed that under certain conditions all pairwise balanced designs with the same parameters have a Hamiltonian BIG. In [8], it was shown that all TS\((v, \lambda)s\) have Hamiltonian 1-BIGs for arbitrary index \(\lambda\). Despite the fact that the 1-BIG of a TS\((v, \lambda)\) and the 2-BIG of the same TS\((v, \lambda)\) are subgraphs of the BIG of the same TS\((v, \lambda)\), not all TS\((v, \lambda)s\) have Hamiltonian 2-BIGs. In [3], but all a finite number of elements of the spectrum were determined for which there exists a TS\((v, 2)\) whose 2-BIG is connected but has no Hamilton path (and therefore no Hamilton cycle). Mahmoodian made the observation that the 2-BIG of the unique TS\((6, 2)\) is the Petersen graph [9]. Additionally, Colbourn and Johnstone [2] showed that there is a TS\((19, 2)\) whose 2-BIGs is connected but is not Hamiltonian. To date, this is the extent of what is known on the Hamiltonicity of the 2-BIGs of TS\((v, \lambda)s\). We aim to show that when \(v \equiv 0\) or \(4\) (mod 12) there exists a TS\((v, \lambda)\) whose 2-BIG is Hamiltonian for any admissible \(v, \lambda\).

Recall that a TS\((v, \lambda)\) exists if and only if \(\lambda \leq v - 2, \lambda v(v - 1) \equiv 0\) (mod 6) and \(\lambda(v - 1) \equiv 0\) (mod 2). Thus when \(v\) is even, \(\lambda\) must be even. Suppose \(n \equiv 1, 5\) (mod 6) and \(v = 2n + 2\). In [10], Schreiber gives a construction (see Section 2) for a set of \(\binom{n}{3}\) triples (blocks) on \(v\) points that can be partitioned into \(n\) sets, each forming a TS\((v, 2)\). Then the union of any \(t\) of these sets is a simple TS\((v, 2t)\) for \(1 \leq t \leq n\). In this paper, we show that the 2-BIG of any TS\((v, \lambda)\) formed by taking \(t = \frac{n}{2}\) sets from the Schreiber construction is Hamiltonian. Thus the main result is as follows.

**Theorem 1.** If \(v \equiv 0\) or \(4\) (mod 12), then there exists a TS\((v, \lambda)\) whose 2-BIG contains a Hamilton cycle for all admissible \(v, \lambda\).

# The Construction

Let \(n \equiv 1\) or \(5\) (mod 6) and \(v = 2n + 2\) for the remainder of the paper. The following construction is a restatement of what was given by Schreiber in [10] for forming \(v/3\) triples on \(v\) points that can be partitioned into \(n\) sets, each forming a TS\((v, 2)\).

**Construction 1.** [10] Let \(g \in \mathbb{Z}_n = \{0, 1, \ldots, n - 1\}\), and let \(\tau\) denote the automorphism of \(\mathbb{Z}_n\) such that 
\[
\tau(h) = -2h \quad \text{for each } h \in \mathbb{Z}_n.
\]
Then \(\tau(0) = 0\), and since \(\mathbb{Z}_n\) can contain no element of order 3 (as \(\gcd(n, 3) = 1\)), \(\tau(h) \neq h\) for \(h \neq 0\). Thus \(\tau\) permutes the elements of \(\mathbb{Z}_n^* (\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\})\) into cycles. Call the ordered pair 
\[
(h, \tau(h)), \quad g \in \mathbb{Z}_n^*,
\]
an arc and \((-h, \tau(-h))\) the opposite arc in a directed graph on the vertex set \(\mathbb{Z}_n\). Thus there are \(\frac{n(n - 1)}{2}\) pairs of opposite arcs for each \(h\). Let \(D_g^*\) be the directed graph with vertex set \(\mathbb{Z}_n\) and formed by the set of arcs in \(\{(h + g, \tau(h) + g), (-h + g, \tau(-h) + g) : h \in \mathbb{Z}_n\}\). We form the set of blocks \(S_g\) for each \(g \in \mathbb{Z}_n\) in the following manner:

**Step 0.** For each pair of opposite arcs (on the same cycle or not) arbitrarily color one red, one blue.

**Step 1a.** Take each triple of different elements \(\{(a+g, 0), (b+g, 0), (c+g, 0)\}\), where \(a, b, c \in \mathbb{Z}_n\), if \(a+b+c \equiv 3g\) (mod \(n\)).

**Step 1b.** For each triple formed in Step 1a, add 7 more triples by independently adding \((0, 1)\) to \((a, 0), (b, 0), (c, 0)\) (recall that each point in the block is from \(\mathbb{Z}_n \times \mathbb{Z}_2 \cup \{\infty_0, \infty_1\}\)).

**Step 1c.** For each arc \((a, b)\) (that is, \(b = -2a\)), we add the two triples \(\{(a, 0), (b, 0), (a, 1)\}\) and \(\{(a, 0), (b, 1), (a, 1)\}\).

**Step 2a.** If \((a, b)\) is a red arc, add the 4 triples \(\{\infty_0, (a, 0), (b, 0)\}\), \(\{\infty_0, (a, 1), (b, 1)\}\), \(\{\infty_1, (a, 0), (b, 1)\}\), and \(\{\infty_1, (a, 1), (b, 0)\}\).

**Step 2b.** If \((a, b)\) is a blue arc, add the 4 triples \(\{\infty_1, (a, 0), (b, 0)\}\), \(\{\infty_1, (a, 1), (b, 1)\}\), \(\{\infty_0, (a, 0), (b, 1)\}\), and \(\{\infty_0, (a, 1), (b, 0)\}\).
Step 3. Add the 4 triples \(\{\infty_0, \infty_1, (g, 0)\}, \{\infty_0, \infty_1, (g, 1)\}, \{\infty_0, (g, 0), (g, 1)\}\) and \(\{\infty_1, (g, 0), (g, 1)\}\).

Based on this construction, for each \(g \in \mathbb{Z}_n\), \(S_g\) is a set of triples that forms a TS\((v, 2)\) and \(S_{g_1} \cap S_{g_2} = \emptyset\) for each \(g_1, g_2 \in \mathbb{Z}_n\). Thus, \(\bigcup_{i=1}^{\lambda/2} S_i\) is a set of triples that forms a TS\((v, \lambda)\). Furthermore, the union of any \(t\) of these \(S_g\)'s is a simple TS\((v, 2t)\) for \(1 \leq t \leq n\).

3 Hamilton Path in Steps 1a and 1b

To show that the 2-BIG of a TS\((v, \lambda)\) formed by Construction 1 is Hamiltonian, we will simply build vertex disjoint paths that together span the vertex set and whose end vertices are adjacent. Then we will concatenate the paths to form a Hamilton cycle. We begin in this section by finding a Hamilton path through the portion of the 2-BIG that is formed from the blocks of the TS\((v, \lambda)\) that are given in Steps 1a and 1b in Construction 1.

The Cartesian product \(G \square H\) of graphs \(G\) and \(H\) is a graph such that the vertex set of \(G \square H\) is the Cartesian product \(V(G) \times V(H)\); and any two vertices \((u, u')\) and \((v, v')\) are adjacent in \(G \square H\) if and only if either \(u = v\) and \(u'\) is adjacent with \(v'\) in \(H\) or \(u' = v'\) and \(u\) is adjacent with \(v\) in \(G\). We refer to the 3-cube illustrated in each of the graphs in Figure 1 as \(Q_3\). The Cartesian product of an edge and the 3-cube is illustrated in Figure 2. The following results are easily verified by examination of Figures 1 and 2.

**Lemma 2.** For any vertex \(v_1 \in V(Q_3)\), there are four different vertices at distance 1 or 3 from \(v_1\) that we could choose as \(v_2\) such that there is a Hamilton path through \(Q_3\) with endpoints \(v_1\) and \(v_2\).

**Lemma 3.** Let \(M = K_2 \square Q_3\). There exists a Hamilton path on \(M\) with endpoints \(v_1\) and \(v_2\) where \(v_1, v_2 \in V(M), v_1 = (k_1, q_1),\) and \(v_2 = (k_2, q_2), q_1 \neq q_2\).

Figure 1: Four Hamilton path from \(v_1\) to \(v_2\)

Figure 2: A Hamilton path from \(v_1\) to \(v_2\)

We call a path with \(k\) edges a \(k\)-path. We denote a \(k\)-path with edges \(\{a_0a_1, a_1a_2, \ldots, a_{n-1}a_n\}\) as \([a_0, a_1, \ldots, a_n]\).

**Lemma 4.** Let \(P\) be a 2-path and let \(M = P \square Q_3\). There exists a Hamilton path on \(M\) with endpoints \(v_1\) and \(v_2\), where \(v_1, v_2 \in V(M), v_1 = (k_1, q_1),\) \(v_2 = (k_1, q_2),\) and \(k_1\) is an endpoint in \(P\).
Proof. Let $P = [k_1, k_2, k_3]$ and $V(Q_3) = \{q_1, \ldots, q_8\}$. Label the vertices of $M$ as in Figure 3. Then

$$[(k_1, q_1), (k_2, q_1), (k_2, q_2), (k_2, q_3), (k_2, q_4), (k_3, q_3), (k_3, q_2), (k_3, q_1), (k_3, q_5), (k_3, q_6), (k_3, q_7), (k_3, q_8), (k_2, q_5), (k_2, q_7), (k_2, q_6), (k_1, q_5), (k_1, q_6), (k_1, q_7), (k_1, q_8), (k_1, q_4), (k_1, q_3), (k_1, q_2)]$$

is a Hamilton path in $M$ through the 24 vertices represented by $M$ (see Figure 3). Additionally, this path begins and ends in the same 3-cube, thus proving our result.

We denote a walk on the edges $\{x_1 x_2, x_2 x_3, x_3 y_1, y_1 y_2, y_2 y_1, y_1 x_3, x_3 x_4, x_4 x_5\}$ as $(x_1, x_2, x_3, y_1, y_2, y_1, x_3, x_4, x_5)$. Let $W$ be a walk on a graph $G$. We say that $W$ is a Hamilton walk in $G$ if the following conditions are satisfied:

- $W$ has one or more subwalks; each subwalk is formed by traversing a 2-path in both directions.
- If we remove all such subwalks from $W$ along with the interior vertices from each corresponding subwalk, we are left with a Hamilton path on the remaining vertices.

Notice that by the second condition, $W$ is a spanning subgraph of $G$. For example, let $W = (x_1, x_2, x_3, y_1, y_2, y_1, x_3, x_4, x_5)$ be a spanning walk in a graph $G$. The subwalk $(x_3, y_1, y_2, y_1, x_3)$ is formed by traversing the 2-path $[x_3, y_1, y_2]$ in both directions. Thus if we remove the vertices $y_1$ and $y_2$, we are left with the Hamilton path $(x_1, x_2, x_3, x_4, x_5)$ on the vertices $\{x_1, x_2, x_3, x_4, x_5\}$.

If $P_1$ and $P_2$ are two paths where an endpoint, $x$, of $P_1$ is adjacent to an endpoint, $y$, of $P_2$ then the concatenation of the two paths, $P_1 \circ P_2$, is $P_1 \cup \{xy\} \cup P_2$.

Lemma 5. Let $W$ be a Hamilton walk in a graph with vertex set $Z_s$ and endpoints 0 and $s - 1$, and let $M = W \boxtimes Q_3$. Then $M$ contains a Hamilton path with endpoints $v_1$ and $v_s$ where $v_1, v_s \in V(M)$, $v_1 = (0, q_1)$, and $v_s = (s - 1, q_s)$ for some $q_1, q_s \in V(Q_3)$.

Proof. Lemma 3 shows that there exists a Hamilton path on any subgraph of $M$ induced by the Cartesian product of two adjacent vertices of $W$ and $Q_3$, and Lemma 4 shows there exists a Hamilton path on any subgraph of $M$ induced by a subwalk $W$ formed from traversing a 2-path in both directions with endpoints in the same 3-cube. It remains to be shown that these paths can be joined together in $M$. Write $W = w_1 \circ w_2 \circ \ldots \circ w_x$ for some $x \in Z$ where for each $i \in \{1, \ldots, x\}$, $w_i$ is either a 1-path or a 2-path. The subgraph of $M$ induced by the Cartesian product of $w_i$ and $Q_3$ contains a Hamilton path $P_i$. One endpoint of $P_i$ is $v_1$. Then for $i = 2, 3, \ldots, x$, simply choose one endpoint of $P_i$ to be the vertex that is adjacent to an endpoint of $P_{i-1}$. \qed
Consider how the triples (blocks) are described in Construction 1. Because the second coordinate of each point in the triples described in Step 1a is 0, we will let \( (\alpha, \beta, \gamma)_g \) denote the triple \( \{(a+g, 0), (b+g, 0), (c+g, 0)\} \) where \( \alpha = a + g, \beta = b + g, \gamma = c + g \) for some \( g \in \mathbb{Z}_n \). Let \( g_1, g_2 \in \mathbb{Z}_n \), and let \( G_1 \) be the subgraph of the 2-BIG of the TS\((v, 4)\) that is obtained by Step 1a of Construction 1 using \( S_{g_1} \) and \( S_{g_2} \). Then \( G_1 \) can be drawn in a honeycomb configuration (See Figures 4 and 5). Thus we will refer to this subgraph as the honeycomb graph. We provide this honeycomb graph when \( g_1 = 0 \) and \( g_2 = 1 \) for \( n = 11 \) and \( n = 13 \) in Figures 4 and 5 respectively.

**Lemma 6.** Let \( v = 2n+2, n \equiv 1 \text{ or } 5 \pmod{6} \), and \( g_1, g_2 \in \mathbb{Z}_n \) such that \( g_1 < g_2 \) and \( (g_2-g_1) \neq 0 \pmod{n} \). Let \( G \) be the subgraph of the 2-BIG of the TS\((v, 4)\) formed from \( S_{g_1} \) and \( S_{g_2} \) from Steps 1a and 1b of Construction 1. Then \( G \) contains a Hamilton path with endpoints \( Z_1 = \{(-2g_1 + 3g_2, x_1), (g_1, y_1), (4g_1 - 3g_2, z_1)\} \) and \( Z_2 = \{((3g_1 - 2g_2, x_2), (g_2, y_2), (-3g_1 + 4g_2, z_2)\} \) where \( x_1, x_2, y_1, y_2, z_1, z_2 \in \{0, 1\} \). Furthermore, if the path starts at \( Z_2 \), the first seven vertices in the Hamilton path after \( Z_2 \) are all of the form \( \{(3g_1 - 2g_2, x_2'), (g_2, y_2'), (-3g_1 + 4g_2, z_2')\} \) where \( x_2', y_2', z_2' \in \{0, 1\} \).

**Proof.** Let \( G_1 \) be defined as above. Let \( G_2 \) be isomorphic to any subgraph of \( G \) that is obtained by taking exactly one triple from Step 1a and its corresponding 7 triples in Step 1b of Construction 1. Thus \( G_2 \) is isomorphic to the 3-cube, \( Q_3 \) as shown in Figure 6. Then \( G \) is formed by taking the Cartesian product of the honeycomb graph \( G_1 \) and the 3-cube \( G_2 \), that is, \( G = G_1 \square G_2 \). Figure 6 shows \( G \) when \( n = 11 \). We will show the desired Hamilton path can be constructed by using mathematical induction.

![Figure 6: Correspondence between a single vertex in Step 1a and 8 vertices in Step 1b](image)

When \( n = 11 \), the corresponding honeycomb graph \( G_1 \) is given in Figure 8. Label the vertices in this honeycomb graph as in Figure 8. Each vertex in \( G_1 \) represents a total of eight vertices in \( G \). Referring to
Figure 7: Graph formed by Steps 1a and 1b in Construction 1 when $n = 11$

Figure 8 defines a walk on the honeycomb graph by $W = (a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots, a_{15}, b_{15})$. Then if we removed the portion of subwalk $(a_2, b_2, a_3, b_2, a_2)$ and the interior vertices $b_2$ and $a_3$, what remains would be a Hamilton path on the remaining vertices of $G_1$. Since $W$ is a Hamilton walk in $G_1$, Lemma 5 shows that we can find a Hamilton path through $G$. So we conclude that we can find a Hamilton path that starts at the vertex $Z_1$ (one of the eight vertices that $a_1$ represents in $G$) and ends at $Z_2$ (one of the eight vertices that $b_{15}$ represents in $G$). The Hamilton path for $n = 11$ is illustrated in Figure 10. Figure 9 and Lemma 5 show that this lemma is also true when $n = 13$.

Assume this lemma holds for all $k \equiv 1$ or $5 \pmod{6}$ such that $v = 2k+2$ and $k < n$. Suppose that $v = 2n+2$. Let $H_1$ be the graph formed from Step 1a of Construction 1 and let $H = H_1 \Box G_2$ be the subgraph of the 2-BIG of the TS($v, 4$) formed from $S_{g_1}$ and $S_{g_2}$ from Steps 1a and 1b of the construction. Then by the induction hypothesis, $H$ contains the desired Hamilton path. Let $G = G_1 \Box G_2$ be the subgraph of the 2-BIG associated with the TS($v, 4$) obtained by Construction 1. First, we will show that $H < G$. Then we will show that $G_1$ has a Hamilton walk from $\{-2g_1 + 3g_2, g_1, 4g_1 - 3g_2\}_{g_1}$ to $\{3g_1 - 2g_2, g_2, 4g_2 - 3g_1\}_{g_2}$, and thus by Lemma 5 we can find a Hamilton path through $G$ with endpoints $Z_1$ and $Z_2$.

We begin by describing the vertices of $G_1$ and then proceed to describe the edges of $G_1$. Define the vertices of $G_1$, $U_{i,j}$, as triples from Step 1a as follows. For each $i \in \{0, 1, \ldots, (n-7)/3\}$ and each $j \in \{1, 2, \ldots, 3i + 4\}$, when $n \equiv 1 \pmod{6}$ let

$$U_{i,j} = \begin{cases} 
\{g_1(-3i - 5) + g_2(3i + 6), \\
\frac{1}{2}(g_1(-6i + 3j - 7) + g_2(6i - 3j + 9)), \quad \text{if } j \equiv 1 \pmod{2}, \\
\frac{1}{2}(g_1(12i - 3j + 23) + g_2(-12i + 3j - 21))\}_{g_1} \\
\{g_1(-3i - 5) + g_2(3i + 6), \\
\frac{1}{2}(g_1(-6i + 3j - 10) + g_2(6i - 3j + 12)), \quad \text{if } j \equiv 0 \pmod{2}, \\
\frac{1}{2}(g_1(12i - 3j + 20) + g_2(-12i + 3j - 18))\}_{g_2}
\end{cases}$$
and each of the points in Figure 9: Walk used in 2-BIG formed from blocks in Step 1a in Construction \( \text{[Construction]} \) when \( n = 11 \)

For each \( i \in \{0, 1, \ldots, (n-8)/3 \} \) and each \( j \in \{1, 2, \ldots, 3i + 5\} \), when \( n \equiv 5 \pmod{6} \), let

\[
U_{i,j} = \begin{cases} 
\frac{1}{2}(g_{1}(-12i + 3j - 25) + g_{2}(12i - 3j + 27)), \\
\frac{1}{2}(g_{1}(6i - 3j + 11) + g_{2}(-6i + 3j - 9)), & \text{if } j \equiv 1 \pmod{2}, \\
g_{1}(3i + 7) + g_{2}(-3i - 6), & \text{if } j \equiv 0 \pmod{2}.
\end{cases}
\]

Let \( U_{0,5} = \{-2g_{1} + 3g_{2}, g_{1}, 4g_{1} - 3g_{2}\}_{g_{1}} \) for \( n \equiv 1 \pmod{6} \). Let \( U_{0,6} = \{-2g_{1} + 3g_{2} + 3(g_{2} - g_{1}), g_{1}, 4g_{1} - 3g_{2}\}_{g_{2}} \) and \( U_{0,7} = \{-2g_{1} + 3g_{2}, g_{1}, 4g_{1} - 3g_{2}\}_{g_{1}} \) for \( n \equiv 5 \pmod{6} \). For each \( i, j \), let \( \overline{U}_{i,j} = \{a', b', c'\} \) be a vertex where \( U_{i,j} = \{a, b, c\} \) and \( a' = g_{1} + g_{2} - a, b' = g_{1} + g_{2} - b, c' = g_{1} + g_{2} - c \). Two vertices \( U_{i,j} \) and \( U_{i,j} \) are adjacent if \( |U_{i,j} \cap U_{i,j}| = 2 \). It is clear that each element in the triples \( U_{i,j} \) and \( \overline{U}_{i,j} \) is an integer because of \( j \)'s parity. Because the sum of the elements in each triple is either \( 3g_{1} \) or \( 3g_{2} \) modulo \( n \), \( U_{i,j} \) and \( \overline{U}_{i,j} \) are indeed triples from Step 1a.

Now we will show that each \( U_{i,j} \) is unique. Elements in \( U_{i,j} \), sum to \( 3g_{1} \) or \( 3g_{2} \) when \( j_{1} \equiv 0 \pmod{2} \) and \( n \equiv 1 \) or \( 5 \pmod{6} \) respectively, while \( U_{i,j} \) sums to \( 3g_{2} \) or \( 3g_{1} \) when \( j_{2} \equiv 1 \pmod{2} \) and \( n \equiv 1 \) or \( 5 \pmod{6} \). Therefore, \( U_{i,j} \neq U_{i,j} \) when \( j_{1} \neq j_{2} \pmod{2} \). It remains to show \( U_{i,j} \neq U_{i,j} \) when \( i_{1} \neq i_{2} \). In order to show this, we assume that they are equal. The only way this can happen is if there is equivalence modulo \( n \) between each of the points in \( U_{i,j} \) and \( U_{i,j} \). We will show that in each case, a contradiction arises. Let \( j \equiv 1 \pmod{2} \) and \( n \equiv 1 \pmod{6} \) so that

\[
U_{i_{1},j} = \{g_{1}(-3i_{1} - 5) + g_{2}(3i_{1} + 6), \frac{1}{2}(g_{1}(-6i_{1} + 3j - 7) + g_{2}(6i_{2} - 3j + 9))\}, \\
U_{i_{2},j} = \{g_{1}(-3i_{2} - 5) + g_{2}(3i_{2} + 6), \frac{1}{2}(g_{1}(-6i_{2} + 3j - 7) + g_{2}(6i_{2} - 3j + 9))\}.
\]

Regardless of whether \( n \equiv 1 \) or \( 5 \pmod{6} \) or \( j \equiv 0 \) or \( 1 \pmod{2} \), if \( \alpha_{1} \equiv \alpha_{2} \pmod{n} \), \( \beta_{1} \equiv \beta_{2} \pmod{n} \), or \( \gamma_{1} \equiv \gamma_{2} \pmod{n} \), then \( (g_{1} - g_{2})(i_{1} - i_{2}) \equiv 0 \pmod{n} \). But this implies \( (g_{1} - g_{2}) \equiv 0 \pmod{n} \) or \( (i_{1} - i_{2}) \equiv 0 \pmod{n} \) and we assumed both of these cannot occur. So we may assume that \( \alpha_{1}, \beta_{1}, \gamma_{1} \) is not equivalent to \( \alpha_{2}, \beta_{2}, \gamma_{2} \) modulo \( n \) respectively when \( n \equiv 1 \) or \( 5 \pmod{6} \) or \( j \equiv 0 \) or \( 1 \pmod{2} \). Suppose that \( \alpha_{1} \equiv \beta_{2} \pmod{n} \), \( \beta_{1} \equiv \gamma_{2} \pmod{n} \), and \( \gamma_{1} \equiv \alpha_{2} \pmod{n} \). Then we have \(-3(g_{1} - g_{2})(i_{1} - i_{2} + \frac{4}{2}) \equiv 0 \pmod{n} \).
-3(g_1 - g_2)(i_1 + 2i_2 - j + 5) \equiv 0 \pmod{n}, and 3(g_1 - g_2)(2i_1 + i_2 - \frac{i_1 - i_2}{2}) \equiv 0 \pmod{n}. The first and third equivalence relations tell us \(i_1 \equiv -2 \pmod{n}, which tell us \(i_1 \geq n - 2\), but we have defined \(i_1 \leq \frac{n-2}{2}\,\), so we have a contradiction. The only other alternative is if \(a_1 \equiv \gamma_2 \pmod{n}, \beta_1 \equiv a_2 \pmod{n}, and \zeta_1 \equiv y_2 \pmod{n},\) and similar argument shows a contradiction as well. Now suppose \(n \equiv 1 \pmod{6}\) and \(j \equiv 0 \pmod{2}\). Then if we again assume \(a_1 \equiv \beta_2 \pmod{n}, \beta_1 \equiv \gamma_2 \pmod{n}, and \gamma_1 \equiv a_2 \pmod{n},\) we get \(-\frac{3}{2}(g_1 - g_2)(2i_1 - 2i_2 + j) \equiv 0 \pmod{n}, -3(g_1 - g_2)(5 + i_1 + 2i_2 - j) \equiv 0 \pmod{n}, and \frac{2}{3}(g_1 - g_2)(10 + 4i_1 + 2i_2 - j) \equiv 0 \pmod{n}.) The first and third equivalences tell us that \(3i_1 \equiv -5 \pmod{n},\) but then \(i_1 \geq \frac{n-5}{3}\,\) and we defined \(i_1 \leq \frac{n-8}{3}\,\), a contradiction. If \(n \equiv 1 \pmod{6}\) and \(j \equiv 1 \pmod{2}\), then using the same strategy as above, we get that \(-3(g_1 - g_2)(6 + 2i_1 + i_2 - j) \equiv 0 \pmod{n}, \frac{2}{3}(g_1 - g_2)(-1 + 2i_1 - 2i_2 - j) \equiv 0 \pmod{n}, and \frac{2}{3}(g_1 - g_2)(13 + 2i_1 + 4i_2 - j) \equiv 0 \pmod{n}.) The last two equivalences tell us that \(3i_2 \equiv -7 \pmod{n},\) but we assumed that \(i_2 \leq \frac{n-8}{3}\,\), a contradiction. If \(n \equiv 5 \pmod{6}\) and \(j \equiv 0 \pmod{2}\), then using the same strategy as above, we get that \(-3(g_1 - g_2)(6 + 2i_1 + i_2 - j) \equiv 0 \pmod{n}, \frac{2}{3}(g_1 - g_2)(-1 + 2i_1 - 2i_2 - j) \equiv 0 \pmod{n}, and \frac{2}{3}(g_1 - g_2)(12 + 2i_1 + 4i_2 - j) \equiv 0 \pmod{n}.) The last two equivalences tell us that \(i_2 \equiv -2\,\), but this is impossible since \(i_2 \leq \frac{n-8}{3}\,\), a contradiction. In all cases, if \(a_1 \equiv \gamma_2 \pmod{n}, \beta_1 \equiv a_2 \pmod{n}, and \gamma_1 \equiv \beta_2 \pmod{n}, a\) similar contradiction arises. Thus \(U_{i1,j} \neq U_{i2,j} \). The same methods can be used to show that \(U_{i1,j} \neq \overline{U}_{i2,j}\,\) and \(\overline{U}_{i1,j} \neq \overline{U}_{i2,j}\,\).

We will next show that in G the following holds:

- \(U_{i,j} \sim U_{i,j+1}\),
- \(U_{i,j} \sim U_{i+1,j+3}\) if \(j\) is odd,
- \(\overline{U}_{i,j} \sim \overline{U}_{i,j+1}\),
- \(\overline{U}_{i,j} \sim \overline{U}_{i+1,j+3}\) if \(j\) is odd, and
- either \(U_{(n-\gamma)/3,j} \sim \overline{U}_{(n-\gamma)/3,j}\) if \(j \equiv 1 \pmod{2}\) and \(n \equiv 1 \pmod{6}\) or \(U_{(n-\gamma)/3,j} \sim \overline{U}_{(n-\gamma)/3,j}\) if \(j \equiv 1 \pmod{2}\) and \(n \equiv 5 \pmod{6}\).

Since the parity of \(j\) is different between \(U_{i,j}\) and \(U_{i,j+1}\), it is an easy calculation to verify that \(U_{i,j}\) and \(U_{i,j+1}\) are adjacent when \(n \equiv 1\) or \(5 \pmod{6}\) by examining the definition of \(U_{i,j}\). This is true for both \(n \equiv 1 \pmod{6}\) and \(n \equiv 5 \pmod{6}\).
Suppose that $j \equiv 1 \pmod{2}$ and $n \equiv 1 \pmod{6}$. Then

$$U_{i+1,j+3} = \{g_1(-3(i+1) - 5) + g_2(3(i+1) + 6),$$

$$\frac{1}{2}(g_1(-6(i+1) + 3(j + 3) - 10) + g_2(6(i+1) - 3(j + 3) + 12)),$$

$$\frac{1}{2}(g_1(12(i+1) - 3(j + 3) + 20) + g_2(-12(i+1) + 3(j + 3) - 18))\}$$

$$= \{g_1(-3i - 8) + g_2(3i + 9),$$

$$\frac{1}{2}(g_1(-6i + 3j - 7) + g_2(6i - 3j + 9)),$$

$$\frac{1}{2}(g_1(12i - 3j + 23) + g_2(-12i + 3j - 21))\}.$$

Thus $U_{i,j} \sim U_{i+1,j+3}$. The same calculation will show that $U_{i,j} \sim U_{i+1,j+3}$ when $j \equiv 0 \pmod{2}$ or when $n \equiv 5 \pmod{6}$ for any $j$. Since $U_{i,j} = \{a', b', c'\}$ is defined by $U_{i,j} = \{a, b, c\}$ where $a' = g_1 + g_2 - a$, $b' = g_1 + g_2 - b$, and $c' = g_1 + g_2 - c$, it is clear that $\overline{U}_{i,j} \sim \overline{U}_{i,j+1}$ and $\overline{U}_{i,j} \sim \overline{U}_{i+1,j+3}$. It remains to show that when $j$ is odd either $U_{(n-7)/3,j} \sim \overline{U}_{(n-7)/3,j}$ or $U_{(n-8)/3,j} \sim \overline{U}_{(n-8)/3,j}$ if $n \equiv 1 \pmod{6}$ or if $n \equiv 5 \pmod{6}$ respectively. Suppose that $n \equiv 1 \pmod{6}$ and $j \equiv 1 \pmod{2}$. Then

$$U_{(n-7)/3,j} = \{g_1(-n+2) + g_2(n-1),$$

$$\frac{1}{2}(g_1(3j-2n+7) + g_2(-3j+2n-5)),$$

$$\frac{1}{2}(g_1(-3j+4n-5) + g_2(3j-4n+7))\}$$

and

$$\overline{U}_{(n-7)/3,j} = \{g_1(n-1) + g_2(-n+2),$$

$$\frac{1}{2}(g_1(-3j+2n-5) + g_2(3j-2n+7))$$

$$\frac{1}{2}(g_1(3j-4n+7) + g_2(-3j+4n-5))\}.$$

Since the points in the triples are calculated modulo $n$, it is clear that $U_{(n-7)/3,j} \sim \overline{U}_{(n-7)/3,j}$. The argument will be similar for $n \equiv 5 \pmod{6}$, thus it is omitted.

Based on the adjacencies given above, it is clear that $H_1 < G_1$ for any $n$. This is illustrated in Figure 11 which gives $G_1$ when $n = 19$ and $n = 17$ with $H_1$ when $n = 13$ and $n = 11$ respectively. Then, because $G = G_1 \square G_2$ and $H = H_1 \square G_2$, it follows that $H < G$. Let $R$ be the Hamilton path in $H$ on $n - 6$ vertices from the induction hypothesis. Let $R_1$ be the corresponding Hamilton walk on $H_1$.

In the induction step, we will concatenate collections of 3-paths. If $n \equiv 1 \pmod{6}$, for $k \in \{1, \ldots, \frac{2n-17}{3}\}$, we define the $k^{th}$ 3-path to be

$$P_k = \begin{cases} [U_{k,3k-1}, U_{k,3k}, U_{k,3k+1}, U_{k,3k+2}] & \text{if $k$ is odd,} \\ [U_{k,3k+2}, U_{k,3k+1}, U_{k,3k}, U_{k,3k-1}] & \text{if $k$ is even.} \end{cases}$$

If $n \equiv 5 \pmod{6}$, for $k \in \{1, \ldots, \frac{2n-25}{3}\}$ we define the $k^{th}$ 3-path to be

$$P_k = \begin{cases} [U_{k,3k+3}, U_{k,3k+2}, U_{k,3k+1}, U_{k,3k}] & \text{if $k$ is odd,} \\ [U_{k,3k+3}, U_{k,3k+1}, U_{k,3k+2}, U_{k,3k+3}] & \text{if $k$ is even.} \end{cases}$$

The paths $P_k$ are highlighted in Figure 12. In a similar manner, we have the following 3-paths for $k' \in \{1, \ldots, \frac{n-7}{3}\}$ when $n \equiv 1 \pmod{6}$ and for $k \in \{1, \ldots, \frac{n-11}{3}\}$ when $n \equiv 5 \pmod{6}$:

$$Q_{k'} = \begin{cases} [U_{2k'-2,6k'-3}, U_{2k'-1,6k'-4}, U_{2k'-1,6k'-5}, U_{2k',6k'-4}] & \text{if $n \equiv 1 \pmod{6}$,} \\ [U_{2k'-1,6k'-4}, U_{2k',6k'-5}, U_{2k'+1,6k'+8}] & \text{if $n \equiv 5 \pmod{6}$.} \end{cases}$$
These paths are highlighted in Figure 13. Then we can define $W$ to be the following Hamilton walk, which covers all of the vertices in $G_1$ of the induction step. (The parts of this walk not including $R_1$ are highlighted in Figure 14.)

$$W = \begin{cases} 
[U_{0,5}, U_{0,4}, U_{0,3}, U_{0,2}, U_{0,1}, U_{0,2}] \circ Q_1 \circ Q_2 \circ \cdots \circ Q_{(n-7)/3} \circ [U_{1,6}, U_{1,5}] \circ P_{(2n-17)/3} \circ \cdots \circ P_2 \circ P_1 \circ R_1 \circ [U_{1,1}, U_{1,3}, U_{1,4}, U_{0,1}, U_{0,3}, U_{0,4}, U_{0,5}, U_{0,5}, U_{0,5}] & \text{if } n \equiv 1 \pmod{6}, \\
[U_{0,7}, U_{0,6}, U_{0,4}, U_{0,5}, U_{1,8}, U_{0,5}, U_{0,4}, U_{0,3}, U_{0,2}, U_{0,1}, U_{1,4}, U_{1,5}, U_{1,6}] \circ Q_1 \circ Q_2 \circ \cdots \circ Q_{(n-1)/3} \circ [U_{2,10}, U_{2,9}] \circ P_{(2n-25)/3} \circ \cdots \circ P_2 \circ P_1 \circ [U_{1,3}] \circ R_1 \circ [U_{1,3}, U_{2,6}, U_{2,7}, U_{2,8}, U_{1,5}, U_{1,4}, U_{0,1}, U_{0,2}, U_{0,3}, U_{1,6}, U_{1,7}, U_{1,8}, U_{0,5}, U_{0,4}, U_{0,6}, U_{0,7}] & \text{if } n \equiv 5 \pmod{6}.
\end{cases}$$

If the endpoints of a path are identified as $a$ and $b$, then we say that it is an $(a, b)$-path. Since $\hat{\gamma}$ can be odd, we may not be able to pair together each $S_g$ with $S_{g-1}$ in Lemma 6. To address this issue, the next lemma shows three consecutive $S_g$'s can be used to build a Hamilton path.

**Lemma 7.** Let $v = 2n + 2$, $n \equiv 1$ or 5 (mod 6), and $g \in \mathbb{Z}_n$. Let $\hat{G}$ be the subgraph of the 2-BIG of the TS($v, \lambda$) formed by the blocks of $S_{g-1} \cup S_g \cup S_{g+1}$, given in Steps 1a and 1b of Construction 1. There is a Hamilton path through $\hat{G}$ with endpoints $Z_1 = \{(g + 2, x_1), (g - 1, y_1), (g - 4, z_1)\}$ and $Z_2 = \{(g - 3, x_2), (g, y_2), (g + 3, z_2)\}$ where $x_1, x_2, y_1, y_2, z_1, z_2 \in \{0, 1\}$. Furthermore, if the path starts at $Z_2$, the first seven vertices in the Hamilton path after $Z_2$ are all of the form \{(3g_1 - 2g_2, x_2'), (g_2, y_2'), (-3g_1 + 4g_2, z_2')\} where $x_2', y_2', z_2' \in \{0, 1\}$.

**Proof.** By Lemma 6 there is a Hamiltonian path through the 2-BIG, $G$, of $S_{g-1} \cup S_g$. Suppose $\{\alpha, \beta, \gamma\}_{g+1} \in S_{g+1}$ is a triple obtained by Step 1a. Then $\alpha + \beta + \gamma = 3(g + 1)$. But then $\alpha + \beta + (\gamma - 3) = 3g$ and so $\{\alpha, \beta, \gamma - 3\}_g \in S_g$. Furthermore, these two triples correspond to adjacent vertices in $G$. Thus any vertex $v_i \in V(\hat{G})$ corresponding to a triple from $S_g$ is adjacent to some vertex $v'_i \in V(\hat{G})$ corresponding to a triple from $S_{g+1}$.

Recall that any single vertex from Step 1a yields a total of eight vertices in Steps 1a and 1b (see Figure 6). Since $n$ is odd, $(g + 1) - (g - 1) = 2$, $g - (g - 1) = 1$, and $(g + 1) - g = 1$, it follows from Lemma 6 that there
Figure 12: Paths $P_i$ are highlighted in $n \equiv 1 \pmod{6}$ case (n = 19 on left) and in $n \equiv 5 \pmod{6}$ case (n = 17 on right)

is a Hamilton $(Z_1, Z_2)$-path through $G$. Furthermore, this path contains a Hamiltonian subpath on these eight vertices. Suppose the Hamilton path through $G$ is as follows:

$$P = (Z_1, \ldots, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, \ldots, Z_2)$$

where $v_1$ is the vertex corresponding to $\{\alpha, \beta, \gamma - 3\}$ from Step 1a and $v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are the other vertices from Step 1b. Then we form a Hamilton path on $V(G) \cup \{v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8'\}$ as follows:

$$P' = (Z_1, \ldots, v_1, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8, \ldots, Z_2).$$

We may follow the same process for each of the vertices in $V(\hat{G})$ that correspond to triples from $S_{g+1}$, thus creating a Hamilton $(Z_1, Z_2)$-path through $\hat{G}$.

4 Hamilton Path in Steps 1c, 2a, 2b, and 3

In this section we show that there exists a Hamilton path through the portion of the 2-BIG that is formed from the blocks of the TS($v, \lambda$) that are given in Steps 1c, 2a, 2b, and 3 of Construction 1.

It is well known that any integer can be represented using negative bases. Negative bases were first studied by Vittorio Grünwald in 1885. For the purposes of our construction, we only care about one particular negative base.

**Theorem 8.** Let $s$ be any integer. Then $s$ can be written as

$$s = \sum_{k=1}^{\infty} (-2)^{k-1} y_k$$

where $y_k \in \{0, 1\}$ for each $k \geq 1$.

The underlying graph $H$ of a directed graph $H'$ is a graph that has the same vertex set and edge set as $H'$, but all of the edges in $H$ have no direction. A directed path in $H'$ is a path with the property that for each pair of consecutive arcs $e_1$ and $e_2$, the head of $e_1$ is adjacent to the tail of $e_2$. We define $G \cup H$ as the union of $G$ and $H$, not necessarily vertex disjoint. We say that a directed graph is strongly connected if there exists a directed path from any vertex to any other vertex.
Lemma 9. Let \( g \in \mathbb{Z}_n \) for \( n \equiv 1 \text{ or } 5 \mod 6 \). Let \( D'_g \) be a directed graph with vertex set \( \{0, \ldots, n-1\} \) such that there is a directed arc from \( i \) to \( i + g \) for each \( i \in \mathbb{Z}_n \) where calculations are done modulo \( n \). Then \( D'_g \cup D'_{g+1} \) is strongly connected.

Proof. Let \( x \) be a vertex. Since all calculations will be identical for \( D'_g \) as they are for \( D'_0 \), without loss of generality, let \( g = 0 \). We define a mapping \( \alpha : \mathbb{Z}_n \to \mathbb{Z}_n \) by \( \alpha(x) = -2x \). Notice that this mapping represents the arcs in \( D'_0 \). Then in \( D'_0 \), there is a directed edge from \( x \) to \( -2x \) for each \( x \in \mathbb{Z}_n \). Therefore, in \( D'_1 \), we can define the mapping \( \beta : \mathbb{Z}_n \to \mathbb{Z}_n \) to represent the arcs in \( D'_1 \) by \( \beta(x) = -2(x-1) + 1 = -2x + 3 \). Thus we must show that for any pair of vertices \( \{x, y\} \), there is a directed path from \( x \) to \( y \). Equivalently, we must show there is a composition of some number of mappings \( \alpha \) and \( \beta \) that will send \( x \) to \( y \).

Notice that if we can show that any \( x \) can be mapped to \( x+1 \) by a sequence of compositions of \( \alpha \) and \( \beta \), we can map \( x \) to any \( z \) by mapping \( x \) to \( x+1 \), mapping \( x+1 \) to \( x+2 \), mapping \( x+2 \) to \( x+3 \), and so on. In this manner, we will form a directed walk. But since we could form a directed path from \( x \) to \( z \) from this directed walk, it is enough to show that there is a directed walk from \( x \) to \( x+1 \) for any \( x \). Without loss of generality, we may assume \( x = 1 \). Thus we aim to show that we can map \( 1 \) to \( 2 \). Represent the mapping from \( x \) to \( z \) as a composition of \( t \) mappings, where each mapping is either \( \alpha \) or \( \beta \). So we define \( z \) as a function of \( x \):

\[
z(x) = -2(-2(\cdots(-2(-2x+3y_{t-1})+3y_{t-2})+3y_{t-3})\cdots)+3y_1+3y_0
\]

\[
= (-2)^t + 3(-2)^{t-1}y_{t-1} + 3(-2)^{t-2}y_{t-2} + \cdots + 3(-2)y_1 + 3y_0
\]

\[
= (-2)^t x - 6 \sum_{k=1}^{t-1} (-2)^{k-1} y_k + 3y_0
\]

where \( y_0, y_1, \ldots, y_{t-1} \) are either 0 or 1, and

\[
y_i = \begin{cases} 0 & \text{if } \alpha \text{ is applied, or} \\ 1 & \text{if } \beta \text{ is applied.} \end{cases}
\]

Because we assume \( x = 1 \), we may choose \( y_0 = 0 \) and thus we map \( 1 \) to \( z \) by

\[
z(x) \equiv (-2)^t + (-6) \sum_{k=1}^{t-1} (-2)^{k-1} y_k \pmod{n}
\]

where \( y_1, y_2, \ldots, y_{t-1} \in \{0, 1\}, y_0 = 0 \). Let \( t \) be large enough to be a multiple of order of \( -2 \) in \( \mathbb{Z}_n \); denote this order as \( a \). We can say this without loss of generality since writing a base-\( t \) number as 100 is the same as
construction shows that if two adjacent arcs \((a, b)\) to one of the graphs in Figure 15, depending on the color of the edge applied to \(D\) Construction 1, each edge in a pair of consecutive directed edges in our directed trails of the form of the color of the two arcs chosen in Step 0. Since we deal exclusively with directed trails, we will never have which correspond to these arcs are connected and form the subgraph in Figure 16 up to isomorphism, regardless

\[
U \equiv 0 \pmod{6}, \quad U \equiv 2 \pmod{6}, \quad U \equiv 0 \pmod{6}, \quad U \equiv 1 \pmod{6}, \quad U \equiv 0 \pmod{6}, \quad U \equiv 1 \pmod{6}
\]

Figure 14: Walk taken during induction step for \(n \equiv 1 \pmod{6}\) \((n = 19\) on left) and \(n \equiv 5 \pmod{6}\) \((n = 17\) on right)

writing this base-\(\ell\) number as 000100. Since \(t\) is a multiple of \(a\), we have:

\[
z(x) \equiv 1 + (-6) \left( \sum_{k=1}^{t-1} (-2)^{k-1} y_k \right) \pmod{n}.
\]

Since \(n \equiv 1 \pmod{6}\), we can write \(n = 6s + 1\) for some \(s\), so \(s = \frac{n - 1}{6}\). By Lemma 8, choose \(y_1, y_2, \ldots, y_{t-1}\) so that

\[
s = \sum_{k=1}^{t-1} (-2)^{k-1} y_{k-1}.
\]

Then we have

\[
z(x) \equiv 1 + (-6)s \equiv 1 + (-6) \left( \frac{n - 1}{6} \right) \equiv 2 \pmod{n}.
\]

Since we only used mappings \(\alpha\) and \(\beta\), and not their inverses, the walk described using the mappings \(\alpha\) and \(\beta\) is directed and so the directed graph \(D'_y \cup D'_{y+1}\) is strongly connected. A similar argument can be made when \(n \equiv 5 \pmod{6}\). \(\square\)

Let \(g \in \mathbb{Z}_n\) for \(n \equiv 1\) or \(5 \pmod{6}\). Let \(D'_y\) be the directed graph given in Construction 1 and \(G_y\) be the subgraph of the 2-BIG of the TS\((v, 2)\) formed by the blocks of \(S_y\) given in Steps 1c, 2a, 2b, and 3. By Construction 1, each edge in \(D'_y \cup D'_{y+1}\) corresponds to a subgraph of \(G_y \cup G_{y+1}\) on 6 vertices that is isomorphic to one of the graphs in Figure 15 depending on the color of the edge applied to \(D'_y \cup D'_{y+1}\) in Step 0. Note that the vertices of degree 1 in either graph in Figure 15 are from either Step 2a or Step 2b. Careful inspection of the construction shows that if two adjacent arcs \((a, b)\) and \((b, c)\) are in \(D'_y \cup D'_{y+1}\), then the subgraphs of \(G_y \cup G_{y+1}\) which correspond to these arcs are connected and form the subgraph in Figure 15 up to isomorphism, regardless of the color of the two arcs chosen in Step 0. Since we deal exclusively with directed trails, we will never have a pair of consecutive directed edges in our directed trails of the form \(\{(a, b), (c, b)\}\) or \(\{(a, b), (a, c)\}\). Because there is a directed arc from \(i + g\) to \(-2i + g\) for each \(i \in \mathbb{Z}_n\), we get that \(D'_y \cup D'_{y+1}\) is strongly connected by Lemma 8. Because \(D'_y \cup D'_{y+1}\) is connected, it follows that \(G_y \cup G_{y+1}\) is connected as well. Thus we have the following result.

**Lemma 10.** Let \(g \in \mathbb{Z}_n\) for \(n \equiv 1\) or \(5 \pmod{6}\). Let \(G_y \cup G_{y+1}\) be the subgraph of the 2-BIG of the TS\((v, 4)\) formed by the blocks of \(S_y\) and \(S_{y+1}\) given in Steps 1c, 2a, 2b, and 3 of Construction 1. Then \(G_y \cup G_{y+1}\) is connected.
Hamilton path of the prescribed type is

Because the two edges

Proof.

Let

versus

Lemma 11. Let \( v = 2n + 2 \) and \( n \equiv 1 \) or 5 (mod 6). Let \( D' \) be the directed graph given in Construction 11 and let \( G \) be the subgraph of the 2-BIG of \( TS(v, \lambda) \) formed by the blocks given in Steps 1c, 2a, and 2b. Let \( H < G \) be the graph formed by selecting two edges from \( D' \) that form a directed path \([e_1, e_2]\) in \( D' \). There is a Hamilton path in \( H \) that begins at a vertex in \( H \) corresponding to \( e_1 \) in Step 2a or 2b and ends at a vertex in \( H \) corresponding to \( e_2 \) in Step 2a or 2b.

Proof. Because the two edges \( e_1, e_2 \) form a directed trail in \( D' \), \( H \) is isomorphic to the graph in Figure 16. A Hamilton path of the prescribed type is

\[
\{a_{1,1}, b_{1,1}, a_{2,1}, a_{3,1}, b_{2,2}, a_{4,2}, a_{4,1}, b_{2,1}, a_{3,1}, a_{2,2}, b_{1,2}, a_{1,2}\}.
\]

Let \( D' \) be the directed graph formed from Step 0 of Construction 11 and let \( H \) be the subgraph of the 2-BIG of \( TS(v, \lambda) \) formed by the blocks given in Steps 1c, 2a, and 2b. The following lemma shows that we can find a Hamilton path in the subgraph of \( H \) induced by a directed trail in \( D' \) with both endpoints corresponding to the same edge in the trail.

Lemma 12. Let \( v = 2n + 2 \) and \( n \equiv 1 \) or 5 (mod 6). Let \( D' \) be the directed graph given by Step 0 in Construction 11 and let \( H \) be the subgraph of the 2-BIG of \( TS(v, \lambda) \) formed by the blocks given in Steps 1c, 2a, and 2b. Let \( H < H \) be formed by selecting a directed trail of length \( k \geq 3 \), \( T = [e_0, e_1, \ldots, e_{k-1}] \), from \( D' \). Then
there is a Hamilton path \( P \) in \( \tilde{H} \) that begins at a vertex in \( \tilde{H} \) corresponding to \( e_0 \) (or \( e_{k-1} \)) in Step 2a or Step 2b and ends at a vertex in \( \tilde{H} \) corresponding to \( e_0 \) (or \( e_{k-1} \)) in Step 2a or Step 2b.

Proof. We will first give an intuitive explanation of the proof and then give the explicit construction. Though each pair of consecutive directed edges in \( T \) forms a graph isomorphic to the graph in Figure 16 there are many ways that three directed edges can form a subgraph in \( H \). Label the vertices in \( T \) formed by \( e_0 \) and \( e_1 \) as in Figure 16 and suppose without loss of generality that the Hamilton path \( P \) we wish to construct begins at \( a_1 \). Then the first six vertices of \( P \) will be \( a_{1,1}, b_{1,1}, a_{2,1}, a_{3,2}, b_{2,2}, a_{4,2} \). Depending on the colors of \( e_0, e_1, \) and \( e_2 \), the subgraph in \( H \) formed by the 18 vertices represented by \( P \) is isomorphic to the subgraph formed by the 18 vertices represented by \( e_1, e_2, e_3 \). So we may assume that any set of 18 vertices formed from \( e_i, e_{i+1}, e_{i+2} \) is isomorphic to the graph on the left in Figure 17. We can form a path, \( P_1 \), through half of the vertices represented by \( e_0, e_1, \ldots, e_{k-3} \) as shown by the solid black path in Figure 18. Depending on the value of \( k \) (mod 4), there are four possibilities for a path, \( P_3 \), through the 12 vertices represented by \( e_{k-2} \) and \( e_{k-1} \) as shown in Figure 19. Whichever path is chosen for \( P_3 \), join \( P_3 \) to the dotted path \( P_2 \) through \( e_{k-3}, e_{k-4}, \ldots, e_0 \) shown in Figure 18. The dashed edges represent edges that are not used in any path. Thus there is a Hamilton path \( P_1 \circ P_3 \circ P_2 \) in \( \tilde{H} \) using all edges in the directed trail \( T \) that begins at vertex \( a_1 \) and ends at vertex \( a_2 \) in \( \tilde{H} \) (see Figure 18).

Now we give the explicit construction of this path. For \( i = 0, 1, \ldots, k-1 \), label the vertices in \( \tilde{H} \) that correspond to \( e_i \) with the elements in the set \( \{a_{1,i}, a_{2,i}, a_{3,i}, a_{4,i}, b_{1,i}, b_{2,i}\} \) as in Figure 16. The Hamilton path through \( H \) will consist of three subpaths \( P_1, P_2, \) and \( P_3 \), which we now describe. Define the 3-paths \( p_i \) and \( \overline{p}_i \) for any \( i \) depending on the value of \( i \) modulo 4 as follows:

\[
p_i = \begin{cases} 
[a_{1,i}, b_{1,i}, a_{2,i}] & \text{if } i \equiv 0 \pmod{4} \\
[a_{3,i}, b_{2,i}, a_{4,i}] & \text{if } i \equiv 1 \pmod{4} \\
[a_{4,i}, b_{2,i}, a_{3,i}] & \text{if } i \equiv 2 \pmod{4} \\
[a_{2,i}, b_{1,i}, a_{1,i}] & \text{if } i \equiv 3 \pmod{4}
\end{cases}
\]

and

\[
\overline{p}_i = \begin{cases} 
p_{i-1} & \text{if } i \equiv 1, 3 \pmod{4} \\
p_{i+1} & \text{if } i \equiv 0, 2 \pmod{4}
\end{cases}
\]

Then \( P_1 = p_0 \circ p_1 \circ \ldots \circ p_{k-3} \) and \( P_2 = \overline{p}_{k-3} \circ \overline{p}_{k-4} \circ \ldots \circ \overline{p}_0 \); these are illustrated in Figure 18 when \( k = 5 \). We define \( P_3 \) based on one of the endpoints of \( P_1 \). This is illustrated in Figure 19 and is defined as follows.

\[
P_3 = \begin{cases} 
[a_{4,k-3}, a_{4,k-2}, a_{4,k-1}, b_{2,k-1}, a_{3,k-1}, a_{2,k-2}, b_{1,k-2}] & \text{if } k \equiv 0 \pmod{4}, \\
[a_{3,k-3}, a_{2,k-2}, a_{3,k-1}, b_{2,k-1}, a_{4,k-1}, a_{4,k-2}, b_{2,k-2}] & \text{if } k \equiv 1 \pmod{4}, \\
[a_{1,k-3}, a_{1,k-2}, a_{2,k-1}, a_{1,k-1}, a_{2,k-2}, a_{3,k-2}, a_{2,k-3}] & \text{if } k \equiv 2 \pmod{4}, \text{ and} \\
[a_{2,k-3}, a_{3,k-2}, a_{2,k-1}, a_{1,k-1}, a_{1,k-2}, b_{1,k-2}, b_{2,k-2}, a_{4,k-2}, a_{4,k-1}, a_{4,k-2}, a_{4,k-3}] & \text{if } k \equiv 3 \pmod{4},
\end{cases}
\]

Then \( P = P_1 \circ P_3 \circ P_2 \) is the desired Hamilton path. \( \square \)
Recall that because \( v \) is even, \( \lambda \) is also even. The next lemma puts all of the pieces in this section together to show that there is a Hamilton path with specific endpoints in the subgraph of the 2-BIG of \( TS(v, \lambda) \) formed by the blocks from Steps 1c, 2a, 2b, and 3.

**Lemma 13.** Let \( v \equiv 0 \) or \( 4 \pmod{12} \) and \( 4 \leq \lambda \leq v - 2 \). There exists a Hamilton path in the subgraph of the 2-BIG of \( TS(v, \lambda) \) formed from the blocks in Steps 1c, 2a, 2b, and 3 of Construction \( \spadesuit \) with endpoints \( A \) and \( B \) where \( B = \{ \infty, x, (4, y_1), (n - 2, z_1) \} \) for some \( x, y, z_1 \in \{0, 1\} \) and \( A \) satisfies the following conditions.

- If \( \lambda/2 \) is even then \( A = \{ \infty, x_2, (\lambda + 1, y_2), (\lambda - 5, z_2) \} \) for any \( x_2, y_2, z_2 \in \{0, 1\} \).
- If \( \lambda/2 \) is odd then \( A = \{ \infty, x_2, (\lambda, y_2), (\lambda - 6, z_2) \} \) for any \( x_2, y_2, z_2 \in \{0, 1\} \).

**Proof.** Let \( G \) be the subgraph of the 2-BIG of \( TS(v, \lambda) \) formed from the blocks in Steps 1c, 2a, and 2b. Let \( D'_g \) be a directed graph formed from Step 0 in Construction \( \spadesuit \) with vertex set \( \mathbb{Z}_n \) such that there is a directed arc from \( i + g \) to \( -2i + g \) for each \( i \in \mathbb{Z}_n \) where indices are calculated modulo \( n \). Let \( D' = \bigcup_{g \in \mathbb{Z}_{\lambda/2}} D'_g \). Notice that the entire structure of the subgraph of the 2-BIG of \( TS(v, \lambda) \) for Steps 1c, 2a, and 2b in Construction \( \spadesuit \) is dictated by the choice of which edges in \( D' \) are red or blue (see Step 0 in Construction \( \spadesuit \)). We will form \( G \) by using the graph \( D' \), and we will handle the vertices in Step 3 of the Construction \( \spadesuit \) by modifying the path we form in \( G \).

By Lemma \( \spadesuit \) the directed graph \( D' \) is strongly connected, i.e. there is a directed walk from any vertex in \( D' \) to any other vertex in \( D' \). For each vertex \( x \notin \{g, g + 1\} \) in \( D'_g \cup D'_{g+1} \), there are two arcs from \( x \) to two other distinct vertices based on the mapping described in Lemma \( \spadesuit \) and similarly, there are two arcs that end at \( x \). We denote these mappings as \( \alpha \) and \( \beta \) in Lemma \( \spadesuit \). If \( x \in \{g, g + 1\} \), then \( x \) has exactly one in-degree and one out-degree. Thus, since the in-degree of each vertex is the same as its out-degree and \( D' \) is strongly connected, there is a directed Euler tour \( E \) on the edges in \( D' \). The graph \( G \) formed from \( D' \) is connected by Lemma \( \spadesuit \) so we will use \( E \) to find a Hamilton path. Each edge of \( E \) corresponds to a directed colored edge (red or blue).
and this directed colored edge corresponds to 6 vertices in $G$ depending on Steps 1c, 2a, or 2b. Without loss of generality, let $E$ end on an edge $e_k \in \{(4, n - 2), (n - 2, 4)\} \subseteq E(D')$ that represents 6 vertices in $G$; one of these 6 vertices in $G$ is vertex $B$. We know that we can have $e_k$ be the last edge in $E$ since we can think of the $E$ as the union of edge-disjoint directed closed trails all containing the vertex $n - 2$ or 4, and we can just choose one edge of a closed trail to be the end of the Euler tour. This does mean that we forfeit some ability to choose on which edge our Euler tour begins. Fortunately, we will not simply follow the directed Euler when creating a Hamilton path in $G$. Notice that vertex $A$ is from either Step 2a or 2b, and is in $S(\lambda - 4)/2$ if $\lambda/2$ is even or $S(\lambda - 8)/2$ if $\lambda/2$ is odd, so one of $(\frac{\lambda}{2} + 1, \frac{\lambda}{2} - 5), (\frac{\lambda}{2} - 5, \frac{\lambda}{2} + 1)$ is an arc in $D'$ or one of $(\frac{\lambda}{2}, \frac{\lambda}{2} - 6), (\frac{\lambda}{2} - 6, \frac{\lambda}{2})$ is an arc in $D'$ respectively. Let $e_k$ be the arc in $E$ that represents 6 vertices, one of which is vertex $A$.

Partition the Euler tour $E$ into two or three directed trails: $T_1 = (e_1, \ldots, e_{\ell})$ and $T_2 = (e_{\ell+1}, \ldots, e_k)$ if $\ell \neq 1$ and $\{e_{\ell+1}, e_{\ell+2}, \ldots, e_k\}$ is even; $T_1 = (e_2, \ldots, e_\ell), T_2 = (e_{\ell+1}, e_{\ell+2}, \ldots, e_{\ell-1}),$ and $T_3 = (e_k, e_1)$ if $\ell \neq 1$ and $\{e_{\ell+1}, e_{\ell+2}, \ldots, e_k\}$ is odd; and $T_1 = (e_1, e_2, \ldots, e_{k-2})$ and $T_2 = (e_k, e_{k-1})$ if $\ell = 1$. First, suppose that $\ell \neq 1$ and $\{e_{\ell+1}, e_{\ell+2}, \ldots, e_k\}$ is even. By Lemma 12 there exists a path $P_1$ in the subgraph of $G$ represented by $T_1$ that begins at $A$ (one of the four vertices represented by $e_\ell$ in Step 2a or 2b), and ends at some other vertex $A'$ represented by $e_\ell$. By Lemma 11 and since the length of $T_2$ is even, we can pair consecutive edges in $T_2$ and find a path $P_2$ in the subgraph of $G$ that begins at a vertex adjacent to $A'$ and ends at the vertex $B$. Second, suppose that $\ell \neq 1$ and $\{e_{\ell+1}, e_{\ell+2}, \ldots, e_k\}$ is odd; then $|T_2|$ is even. By Lemma 12 there exists a path $P_1$ in the subgraph of $G$ represented by $T_1$ that begins at $A$ (one of the four vertices represented by $e_\ell$ in Step 2a or 2b), and ends at some other vertex $A'$ represented by $e_\ell$. By Lemma 11 and since the length of $T_2$ is even, we can pair consecutive edges in $T_2$ and find a path $P_2$ in the subgraph of $G$ that begins at a vertex adjacent to $A'$ and ends at a vertex $B'$ in $e_{k-1}$. Then by Lemma 12 there exists a path $P_3$ in the subgraph of $G$ represented by $T_3$ that begins at a vertex (one of the four vertices represented by $e_k$ in Step 2a or 2b) adjacent to $B'$ and ends at the vertex $B$ (one of the four vertices represented by $e_k$ in Step 2a or 2b). Finally, suppose that $\ell = 1$. By Lemma 12 there exists a path $P_1$ in $G$ that begins at $A$ (one of the four vertices represented by $e_k$ in Step 2a or 2b) and ends at another one of the four vertices in $G$, say $A'$ represented by $e_\ell$ in Step 2a or 2b. By Lemma 12 there exists a path $P_2$ in $G$ that begins at a vertex in $G$ represented by $e_k$ which is adjacent to $A'$ and ends at another one of the four vertices in $G$ represented by $e_k$, which is $B$. In each case, $P = P_1 \circ P_2$ or

Figure 19: The four possibilities for $P_3$
$P = P_1 \circ P_2 \circ P_3$ forms a Hamilton path through the subgraph of $G$ represented by Steps 1c, 2a, and 2b.

It remains to augment $P$ to form a Hamilton path that contains all of the vertices from the subgraph of the 2-BIG of $TS(v, \lambda)$ formed from the vertices in Step 3. There is an arc $e_i = (g, b)$ in $E$ (since $D'$ is strongly connected) for some $b \in V(D')$ that represents vertices in $G$ that are adjacent to the vertices in Step 3. Each of the 6 vertices represented by the arc $e_i$ is adjacent to at least two vertices in this $K_4$. Delete some edge in $P$ between any of these 6 vertices, say the edge joining $\{\infty, x_3, (g, y_5), (b, z_3)\}$ to $\{(g, x_4), (b, y_4), (g, z_4)\}$ where $x_3, x_4, y_3, y_4, z_3, z_4 \in \{0, 1\}$, and add a path through all of the vertices in this $K_4$ with endpoints at the two ends of the edge deleted from $P$. Do this for each $K_4$ formed from Step 3 to form the required Hamilton path.

\[\square\]

5 Conclusion

We are now able to prove Theorem 1.

**Theorem 1.** If $v \equiv 0$ or 4 (mod 12), then there exists a $TS(v, \lambda)$ whose 2-BIG contains a Hamilton cycle for all admissible $v, \lambda$.

**Proof.** Let $G$ be the 2-BIG of the $TS(v, \lambda)$ formed from Construction 1. Let $P_1$ be a Hamilton path created from the subgraph $G$ formed from Steps 1a and 1b of Construction 1 as follows. Partition $\mathbb{Z}_{\lambda/2}$ into $P = \{(0, 1), (2, 3), (4, 5), \ldots, (\frac{\lambda}{2} - 2, \frac{\lambda}{2} - 1)\}$ or $\{(0, 1), (2, 3), \ldots, (\frac{\lambda}{2} - 5, \frac{\lambda}{2} - 4), (\frac{\lambda}{2} - 3, \frac{\lambda}{2} - 2, \frac{\lambda}{2} - 1)\}$ if $\frac{\lambda}{2}$ is even or odd respectively. If $P$ has no 3-subsets, then the path $h_i$ between any two vertices $i, j \in \{(0, 2), 4, \ldots, \frac{\lambda}{2} - 2\}$. If $P$ has a 3-subset, then let the path $h_i$ between any two vertices $i, j \in \{(0, 2), 4, \ldots, \frac{\lambda}{2} - 2\}$ and form a final honeycomb graph as $H_{\frac{\lambda}{2} - 3, \frac{\lambda}{2} - 2, \frac{\lambda}{2} - 1}$ as described in Lemma 7. By Lemmas 2 and 3 there is a Hamilton $(Z_2, Z_1)$-path, $h_i$, in each subgraph of the 2-BIG of $TS(v, \lambda)$ represented by the honeycomb graph, $H_{i, i+1}$. Let $\ell = \frac{\lambda}{2} - 2$ if $\frac{\lambda}{2}$ is even and $\ell = \frac{\lambda}{2} - 3$ if $\frac{\lambda}{2}$ is odd. Because $Z_1$ in $H_{i, i+1}$ is adjacent to $Z_2$ in $H_{i+2, i+3}$ and $Z_1$ in $H_{\frac{\lambda}{2} - 5, \frac{\lambda}{2} - 4}$ is adjacent to $Z_2$ in $H_{\frac{\lambda}{2} - 3, \frac{\lambda}{2} - 2, \frac{\lambda}{2} - 1}$, we may define $P_1$ to be the path $P_1 = h_0 \circ h_2 \circ \ldots \circ h_{\ell}$ that begins at vertex $\alpha = \{(n - 2, x_1), (1, y_1), (4, z_1)\}$ and ends at either vertex $\{(\frac{\lambda}{2} + 1, x_2), (\frac{\lambda}{2} - 2, y_2), (\frac{\lambda}{2} - 5, z_2)\}$ or $\{(\frac{\lambda}{2}, x_2), (\frac{\lambda}{2} - 3, y_2), (\frac{\lambda}{2} - 6, z_2)\}$, if $\frac{\lambda}{2}$ is even or odd respectively, where $x_1, x_2, y_1, y_2, z_1, z_2 \in \{0, 1\}$.

By Lemma 1 there is a Hamilton path $P_2$ that begins at either $\{(\infty, x_3), (\frac{\lambda}{2} + 1, x_2), (\frac{\lambda}{2} - 5, z_2)\}$ or $\{(\infty, x_3), (\frac{\lambda}{2} - 3, x_2), (\frac{\lambda}{2} - 6, z_2)\}$ if $\frac{\lambda}{2}$ is even or odd respectively and ends at vertex $\omega = \{(\infty, x_4), (4, y_4), (n - 2, z_4)\}$ where $x_3, x_4, y_4, z_4 \in \{0, 1\}$. If $y_4 = z_1$ and $z_4 = x_1$, then $C = P_1 \circ P_2 \circ [\omega, \alpha]$ is a Hamilton cycle.

If $y_4 \neq z_1$ and/or $z_4 \neq x_1$, then we can change the first eight edges of $P_1$ without changing any of the other edges of $P_1$, and obtain a different Hamilton path $P_2$ as follows. By Lemma 5 we know what the first 7 vertices of $P_1$ are after $\alpha$. Without loss of generality, let us assume the eighth vertex of $P_1$ is $\beta = \{(n - 2, 1), (1, 0), (4, 0)\}$. Then by Lemma 2 we could choose $P_2$ to have its starting vertex be any one of the vertices in $\{(n - 2, 1), (1, 0), (4, 1)\}, \{(n - 2, 0), (1, 1), (4, 0)\}, \{(n - 2, 1), (1, 1), (4, 0)\}$ and its eighth vertex be $\beta$. Thus we may choose $\alpha \in P_2$ to be $\{(n - 2, x_1), (1, y_4), (4, z_4)\}$ where $x_1$ is determined by the choice of $y_4$ and $z_4$. Then $C = P_1 \circ P_2 \circ [\omega, \alpha]$ is a Hamilton cycle.

\[\square\]

Our future work will use Theorem 1 as a base case to show that there exists a $TS(v, \lambda)$ whose 2-BIG is Hamiltonian when $v$ is even.

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