A SEMIORTHOGONAL DECOMPOSITION FOR BRAUER SEVERI SCHEMES

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Abstract. A semiorthogonal decomposition for the bounded derived category (the category of perfect complexes in a non smooth case) of coherent sheaves on a Brauer Severi scheme is given. It relies on bounded derived categories (categories of perfect complexes in a non smooth case) of suitably twisted coherent sheaves on the base.

1. Introduction

In this paper we give a semiorthogonal decomposition of the bounded derived category (the category of perfect complexes in a non smooth case) of coherent sheaves on a Brauer Severi scheme $f : X \to S$. Very roughly, Brauer Severi schemes could be seen as a kind of twisted projective bundles. This leads us to generalize the semiorthogonal decomposition given in [O] for projective bundles, by considering twisted sheaves on the base $S$ instead of untwisted ones.

Let us recall what happens in the case of a projective bundle. Let $S$ be a smooth projective variety, $E$ a vector bundle of rank $r+1$ over $S$. We consider its projectivization $p : X = \mathbb{P}(E) \to S$. We then have the following semiorthogonal decomposition for the bounded derived category $\mathcal{D}(X)$ of coherent sheaves on $X$.

Theorem 1.1. (Orlov) Let $\mathcal{D}(S)_k$ be the full and faithful subcategory of $\mathcal{D}(X)$ whose objects are all objects of the form $p^* A \otimes \mathcal{O}_X(k)$ for an object $A$ of $\mathcal{D}(S)$. Then the set of admissible subcategories

$$(\mathcal{D}(S)_0, \ldots, \mathcal{D}(S)_r)$$

is a semiorthogonal decomposition for the bounded derived category $\mathcal{D}(X)$ of coherent sheaves on $X$.

Proof. This is [O], Theorem 2.6. \qed

The aim of the paper is to give the following generalization. Let $f : X \to S$ be a Brauer Severi scheme of relative dimension $r$ over a locally notherian scheme $S$. Let $\alpha$ be the corresponding class in $H^2(S, \mathbb{G}_m)$. Let us denote by $\mathcal{D}(X)$ the category of perfect complexes of coherent sheaves on $X$ and by $\mathcal{D}(S, \alpha)$ the category of perfect complexes of $\alpha$-twisted coherent sheaves on $S$. Notice that in the smooth case they actually correspond to the bounded derived categories.
Theorem 5.1. There exist admissible full subcategories $D(S,X)_k$ of $D(X)$, such that $D(S,X)_k$ is equivalent to the category $D(S,\alpha^{-k})$ for all $k$ in $\mathbb{Z}$.

The set of admissible subcategories

$$(D(S,X)_0, \ldots, D(S,X)_r)$$

is a semiorthogonal decomposition for the category $D(X)$ of perfect complexes of coherent sheaves on $X$.

It will be clear in the proof of the theorem that the construction of the full admissible subcategories $D(S,X)_k$ is strictly related to the definition of the full admissible subcategories $D(S)_k$ in Orlov’s theorem.

The paper is organized as follows: in section 2 we give the definition of twisted sheaves and we state the basic facts about their connection with Brauer Severi schemes. In section 3 we recall basic facts about derived categories, categories of perfect complexes and derived functors in twisted case, following [C]. In section 4 we recall the definition of admissible subcategories and semiorthogonal decomposition in a triangulated category and we state some basic results about it. The main Theorem and its proof are given in section 5, together with a simple example.

Notations.
All schemes considered are locally noetherian.

$S_{et}$ denotes the étale site of a scheme $S$. For the definition of a (étale) site, see [SGA4 2] or [M].

Given $U \to S$ in $Cov(S_{et})$, we denote $U''$ the fibered product $U \times_S U$ and $U'''$ the fibered product $U \times_S U \times_S U$. We call $p_1$ and $p_2$ the projections from $U''$ to $U$ and $q_{i,j}$ the projections from $U'''$ to $U''$.

$f : X \to S$ is a Brauer Severi scheme of relative dimension $r$, that means that $f$ is smooth and each fiber of $f$ is isomorphic to $\mathbb{P}^r$.

$X_U$ denotes $f^{-1}(U)$ for $U$ in $Cov(S_{et})$. We use notations $X_U'', X_U'''$, $p_{i,X}$ and $q_{i,j,X}$ in the natural way. Notice that $X_U'' = X_{U'''}$.

Given a scheme $S$ we denote $D(S)$ (respectively $D(S,\alpha)$) the triangulated category of perfect complexes of ($\alpha$-twisted) sheaves on $S$, where $\alpha$ is an element of the Brauer group $\text{Br}(S)$.

2. Twisted sheaves

In this section, we give the definition of twisted sheaves and we state the relationship between them and Brauer Severi schemes. We are working in étale topology, but all can be defined and stated in analytic topology as well (see [C], I, 1).

Definition 2.1. Let $S$ be a scheme with étale topology, $U \to S$ in $Cov(S_{et}), \alpha \in \Gamma(U''', \mathbb{G}_m)$ a 2-cocycle.
An $\alpha$-twisted sheaf on $S$ is given by a sheaf $E$ over $U$ and an isomorphism $\phi : p_1^*E \cong p_2^*E$, such that
\[
(q_{2,3}^*\phi) \circ (q_{1,2}^*\phi) = \alpha(q_{1,3}^*\phi)
\]
We say that such a sheaf is coherent if $E$ is a coherent sheaf on $U$, and we denote $\text{Mod}(S, \alpha)$ the category of $\alpha$-twisted sheaves on $S$, $\text{Coh}(S, \alpha)$ the category of coherent $\alpha$-twisted sheaves on $S$ and $\mathbf{D}(S, \alpha)$ the category of perfect complexes of such sheaves.

The category $\text{Mod}(S, \alpha)$ does not change neither by refining the open cover $U \rightarrow X$, nor by changing $\alpha$ by a cochain.

**Lemma 2.2.** If $\alpha$ and $\alpha'$ represent the same element of $H^2(S, \mathbb{G}_m)$, the categories $\text{Mod}(S, \alpha)$ and $\text{Mod}(S, \alpha')$ are equivalent.

**Proof.** This is [C], Lemma 1.2.8. Indeed if $\alpha$ and $\alpha'$ are in the same cohomology class they differ by a 1-cochain: $\alpha = \alpha' + \delta \gamma$. But then sending any $\alpha'$-twisted sheaf $(E, \phi)$ to the $\alpha$-twisted sheaf $(E, \gamma \phi)$ gives the required equivalence. $\square$

**Remark 2.3.** Notice that in general the choice of the cochain $\gamma$ matters: different choices give different equivalences. Since we are just interested in the existence of such equivalences and not in a special one, in what follows this choice will not matter.

Now we can see how twisted sheaves arise naturally when we consider Brauer Severi schemes. Let $f : X \rightarrow S$ be a smooth morphism between schemes such that each fibre is isomorphic to $\mathbb{P}^r$. Then we call $X$ a Brauer Severi scheme of relative dimension $r$ over $S$.

We can find a covering $U \rightarrow S$ in $\text{Cov}(S_{et})$, such that $X_U = f^{-1}(U)$ is a projective bundle over $U$ and $X_U \rightarrow X$ is a covering in $\text{Cov}(X_{et})$. Then we have a local picture $\mathbb{P}(E_U) \rightarrow U$, where $E_U$ is a locally free sheaf of rank $r + 1$ on $U$ and we have an isomorphism $\rho : \mathbb{P}(E_U) \rightarrow X_U$. This fact is a classical application of descent theory ([C], I, 8).

Consider the cartesian diagram
\[
X''_U \xrightarrow{\psi} X'_U \xrightarrow{\phi} X_U
\]
and call the projections $p_{i,X}$ and $q_{i,j,X}$. We have an isomorphism
\[
\psi := p_{1,X}^* \rho^{-1} \circ p_{2,X}^* \rho : \mathbb{P}(p_1^*E_U) \rightarrow \mathbb{P}(p_2^*E_U).
\]
We would like to lift it to an isomorphism $\phi : p_1^*E_U \rightarrow p_2^*E_U$.

Consider $U$ such that $p_1^*E_U$ and $p_2^*E_U$ can be trivialized. This implies that $\psi$ is an automorphism of $U'' \times \mathbb{P}^r$ and then it gives a section of $\text{PGL}(r+1, U'')$. We can again refine $U$ in order to obtain from it a section of $\text{GL}(r+1, U'')$, which will give us the required isomorphism $\phi : p_1^*E_U \rightarrow p_2^*E_U$. Notice that this is not canonical since it can be done up to a choice of an element of $\Gamma(U'', \mathbb{G}_m)$. 

For this reason, we have \((q^*_1,2\phi) \circ (q^*_2,3\phi) = \alpha_U(q^*_1,3\phi)\), where \(\alpha_U \in \Gamma(U''', \mathbb{G}_m)\). We can see that \(\alpha_U\) gives a cocycle and then \((E_U, \phi)\) is an \(\alpha\)-twisted sheaf.

From now on, given a Brauer Severi scheme \(f : X \to S\), we will consider the \(\alpha\)-twisted sheaf \((E_U, \phi)\) described above and the category \(\mathbf{D}(S, \alpha)\). Notice that the choice of \(\alpha_U\) could be modified by a 1-cochain, but, by Lemma 2.2, this would give an equivalent category. In fact everything depends just on the cohomology class of \(\alpha\).

The class \(\alpha\) represents the obstruction to \(f : X \to S\) to be a projective bundle. To express this via cohomology, recall the exact sequence of sheaves over \(S\):

\[
1 \longrightarrow \mathbb{G}_m \longrightarrow GL(r+1) \longrightarrow PGL(r+1) \longrightarrow 1.
\]

It gives a long cohomology sequence:

\[
\cdots \longrightarrow H^1(S, GL(r+1)) \longrightarrow H^1(S, PGL(r+1)) \xrightarrow{\delta} H^2(S, \mathbb{G}_m) \longrightarrow \cdots
\]

and especially a connecting homomorphism \(\delta\).

Let \([X]\) be the cohomology class of \(X\) in \(H^1(S, PGL(r+1))\) and \(\alpha' := \delta([X])\) in \(H^2(S, \mathbb{G}_m)\). If \(\alpha' = 0\), the class \([X]\) would lift to an element of \(H^1(S, GL(r+1))\), that is a rank \(r+1\) vector bundle on \(S\). Since \(X\) is not a projective bundle, \(\alpha'\) is a nonzero element of the cohomological Brauer group \(\text{Br}'(S) := H^2(S, \mathbb{G}_m)\) and it is exactly the cohomology class \(\alpha\) of the \(\alpha_U\) described above.

As a projective bundle \(\mathbb{P}(E_U)\) over \(U\), on \(X_U\) there exists a tautological line bundle \(\mathcal{O}_{X_U}(1)\). We will also write \(\mathcal{O}_{X_U}(k)\) for \(k \in \mathbb{Z}\), clearly meaning \(\mathcal{O}_{X_U}(-1) = \mathcal{O}_{X_U}(1)^\vee\) and so on.

Notice that the choice of the bundle \(\mathcal{O}_{X_U}(1)\) over \(X_U\) depends on the choice of \(E_U\), moreover \(\mathcal{O}_{X_U}(1)\) does not glue as a global untwisted sheaf \(\mathcal{O}_X(1)\) on \(X\). However, the existence of a section for the morphism \(f\) ensures the existence of a global \(\mathcal{O}_X(1)\).

**Lemma 2.4.** Let \(f : X \to S\) be a Brauer Severi scheme. If \(s : S \to X\) is a section of \(f\), then there exists a vector bundle \(G\) on \(S\) such that \(\mathbb{P}(G) \cong X \to S\).

**Proof.** The result is known, but since it is hard to find a reference, we give a proof.

Consider the diagram

\[
\begin{array}{ccccccccc}
X''''_U & \xrightarrow{q_{i,j,X}} & X''_U & \xrightarrow{p_{1,X}} & X_U & \xrightarrow{f} & X \\
\downarrow{s} & & \downarrow{s} & & \downarrow{s} & & \downarrow{s} & & \downarrow{s} \\
U''' & \xrightarrow{q_{i,j}} & U'' & \xrightarrow{p_{2,X}} & U & \xrightarrow{f} & S.
\end{array}
\]

Here \(s\) and \(f\) are improperly used to mean their pull-backs to \(U\), \(U''\) and \(U'''\) in order to keep a clearer notation.
We can choose \( \mathcal{O}_{X_U}(1) \) such that \( s^*\mathcal{O}_{X_U}(1) = \mathcal{O}_U \).

Consider now \( p_{1,X}^*\mathcal{O}_{X_U}(1) \) and \( p_{2,X}^*\mathcal{O}_{X_U}(1) \), the two pull-backs of \( \mathcal{O}_{X_U}(1) \) to \( X_U'' \). There exists an invertible sheaf \( L \) on \( S \) such that \( p_{1,X}^*\mathcal{O}_{X_U}(1) \cong p_{2,X}^*\mathcal{O}_{X_U}(1) \otimes f^*L \). Since \( s^*p_{2,X}^*\mathcal{O}_{X_U}(1) = \mathcal{O}_{U''} \), we have \( L \) trivial. We choose an isomorphism
\[
\phi : p_{1,X}^*\mathcal{O}_{X_U}(1) \to p_{2,X}^*\mathcal{O}_{X_U}(1)
\]
such that \( s^*\phi = \text{Id}_{\mathcal{O}_{U''}} \).

The isomorphism \( \phi \) satisfies an untwisted cocycle condition. Indeed,
\[
s^*((q_{1,2,X}^*\phi) \circ (q_{2,3,X}^*\phi) \circ (q_{1,3,X}^*\phi)^{-1}) = \text{Id}_{\mathcal{O}_{U''}}.
\]
This shows that \( \mathcal{O}_{X_U}(1) \) gives a global untwisted sheaf \( \mathcal{O}_X(1) \) and that means \( X \) is a projective bundle over \( S \).

\[\square\]

3. Derived categories and functors in twisted case

In this section we show what happens to most common derived functors when we consider the category of perfect complexes of twisted sheaves on a scheme. We will state theorems we need for the rest of the paper. Proofs and a more satisfying description can be found in [HRD]. It is in fact an adaptation to twisted case of the results of [HRD].

**Remark 3.1.** In a non smooth case, the following theorems can not be stated if we work in the bounded derived category of coherent sheaves. In order to extend the ideas to a more general context, we will deal with categories of perfect complexes of \((\alpha\text{-twisted})\) sheaves. In the smooth case, they turn out to be the same as bounded derived categories of \((\alpha\text{-twisted})\) coherent sheaves, but keep in mind that in the non smooth case what we call here \( \mathsf{D}(S) \) (resp. \( \mathsf{D}(S;\alpha) \)) is not the bounded derived category of \((\alpha\text{-twisted})\) coherent sheaves on \( S \) but just a full triangulated subcategory.

A complete treatement of perfect complexes on a site is given in [SGA6]. Everything is defined in the very general context of fibered categories, hence all definitions fit for twisted sheaves.

**Theorem 3.2.** Let \( f : X \to S \) be a morphism between schemes, let \( \alpha, \alpha' \) be in \( H^2(S,\mathbb{G}_m) \), and \( \mathsf{AB} \) be the category of abelian groups. Then the following derived functors are defined:

\[
\begin{align*}
\mathsf{R}\mathsf{Hom} &: \mathsf{D}(S,\alpha)^\circ \times \mathsf{D}(S,\alpha') \to \mathsf{D}(S,\alpha^{-1}\alpha') \\
\mathsf{R}\mathsf{Hom} &: \mathsf{D}(S,\alpha)^\circ \times \mathsf{D}(S,\alpha) \to \mathsf{D}^b(\mathsf{AB}) \\
\otimes_S &: \mathsf{D}(S,\alpha) \times \mathsf{D}(S,\alpha') \to \mathsf{D}(S,\alpha\alpha') \\
Lf^* &: \mathsf{D}(S,\alpha) \to \mathsf{D}(X, f^*\alpha)
\end{align*}
\]

If \( f : X \to S \) is a projective lci (locally complete intersection) morphism, then we can define:

\[
Rf_* : \mathsf{D}(X, f^*\alpha) \to \mathsf{D}(S,\alpha).
\]
Proof. [C], Theorem 2.2.6. Remark that asking $f$ to be lci projective is too restrictive, but sufficient in our case. See [SGA6], III, 4 for details.

Theorem 3.3. (Projection Formula). Let $f : X \to S$ be a projective lci morphism between schemes, $\alpha, \alpha' \in H^2(S, \mathbb{G}_m)$. Then there is a natural functorial isomorphism
\[ Rf_*(F) \otimes_S G \cong Rf_*(F \otimes_X Lf^*G) \]
for $F \in D(X, f^*\alpha)$ and $G \in D(S, \alpha')$.

Proof. [C], Theorem 2.3.5.

Theorem 3.4. (Adjoint property of $Rf_*$ and $Lf^*$). Let $f : X \to S$ be a projective lci morphism between schemes, $\alpha \in H^2(S, \mathbb{G}_m)$. Then we have
\[ R\text{Hom}(Lf^*F, G) \cong R\text{Hom}(F, Rf_*G) \]
for $F \in D(S, \alpha)$ and $G \in D(X, f^*\alpha)$.

Proof. [C], Theorem 2.3.9.

Theorem 3.5. (Flat Base Change). Let $f : X \to S$ be a projective lci morphism between schemes, $\alpha \in H^2(S, \mathbb{G}_m)$. Let $u : S' \to S$ be a flat morphism, let $X' = X \times_S S'$ and $v, g$ projections in the cartesian square:

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
g \downarrow & & \downarrow f \\
S' & \xrightarrow{u} & S.
\end{array}
\]

Then there is a natural functorial isomorphism:
\[ u^*Rf_*F \cong Rg_*v^*F \]
for $F \in D(X, f^*\alpha)$.

Proof. [C], Theorem 2.3.10.

4. Semiorthogonal decompositions

Let $k$ be a field and $D$ a $k$-linear triangulated category.

Definition 4.1. A full triangulated subcategory $D' \subset D$ is admissible if the inclusion functor $i : D' \to D$ admits a right adjoint.

Definition 4.2. The orthogonal complement $D'^\perp$ of $D'$ in $D$ is the full subcategory of all objects $A \in D$ such that $\text{Hom}(B, A) = 0$ for all $B \in D'$.
We remark firstly that the orthogonal complement of an admissible subcategory is a triangulated subcategory.

It can be shown that a full triangulated subcategory $D' \subset D$ is admissible if and only if for all object $A$ of $D$, there exists a distinguished triangle $B \to A \to C$ where $B \in D'$ and $C \in D'^\perp$, see [B]. We also have the following Theorem.

**Theorem 4.3.** Let $D'$ be a full triangulated subcategory of a triangulated category $D$. Then $D'$ is admissible if and only if $D$ is generated by $D'$ and $D'^\perp$.

*Proof.* [BK], Proposition 1.5, or [B], Lemma 3.1. \hfill \Box

Admissible subcategories occur when we have a fully faithful exact functor $F : D' \to D$ that admits a right adjoint. To be precise, this functor defines an equivalence between $D'$ and an admissible subcategory of $D$.

**Definition 4.4.** An ordered sequence of admissible triangulated subcategories $\sigma = (D_1, \ldots, D_n)$ is semiorthogonal if, for all $i > j$, one has $D_j \subset D_i^\perp$. If $\sigma$ generates the category $D$, we call it a semiorthogonal decomposition of $D$.

**Lemma 4.5.** Let $\sigma = (D_1, \ldots, D_n)$ be a set of ordered full subcategories of $D$ such that $D_j \subset D_i^\perp$ for all $i > j$ and $\sigma$ generates $D$. Then $D_i$ is admissible for $i = 1, \ldots, n$, and then $\sigma$ is a semiorthogonal decomposition of $D$.

*Proof.* Consider $D_n$ and $D_n^\perp$: they generate the category $D$ and then they are admissible. In general, consider $D_i$ and $D_i^\perp$ for $1 \leq i < n$: they generate the category $D_{i+1}^\perp$ and then they are admissible. \hfill \Box

For further information about admissible subcategories and semiorthogonal decomposition, see [B] [BK] [BO].

5. The main Theorem

Let now $f : X \to S$ be a Brauer Severi scheme of relative dimension $r$ and $\alpha$ in $\text{Br}(S)$ the element associated to it as explained in section 2. This section is dedicated to the proof of the following Theorem.

**Theorem 5.1.** There exist admissible full subcategories $D(S, X)_k$ of $D(X)$, such that $D(S, X)_k$ is equivalent to the category $D(S, \alpha^{-k})$ for all $k$ in $\mathbb{Z}$.

The set of admissible subcategories

$$\sigma = (D(S, X)_0, \ldots, D(S, X)_r)$$

is a semiorthogonal decomposition for the category $D(X)$ of perfect complexes of coherent sheaves on $X$. 
Recall that there exist a rank $r + 1$ locally free sheaf $E_U$ on $U$, such that $X_U = \mathbb{P}(E_U)$ and that $E_U$ gives an $\alpha$-twisted sheaf on $S$. Moreover on $X_U$ we have a tautological line bundle $\mathcal{O}_{X_U}(1)$.

We split this section in three parts: in the first one we define the full subcategories $\mathbf{D}(S, X)_k$ of $\mathbf{D}(X)$ and we show the equivalence between $\mathbf{D}(S, X)_k$ and $\mathbf{D}(S, \alpha^{-k})$; all this is inspired by a construction by Yoshioka [Y]. In the second one we show that the sequence $\sigma$ is indeed a semiorthogonal decomposition. In the third one we give a simple example.

5.1. Construction of $\mathbf{D}(S, X)_k$.

**Definition 5.2.** We define $\mathbf{D}(S, X)_k$, for $k \in \mathbb{Z}$, to be the full subcategory of $\mathbf{D}(X)$ generated by objects $A$ such that

$$A|_{X_U} \simeq_{q,iso} f^* A_U \otimes \mathcal{O}_{X_U}(k)$$

where $A_U$ is an object in $\mathbf{D}(U)$.

**Lemma 5.3.** For all $k$ in $\mathbb{Z}$, there is a functor

$$f_k^*: \mathbf{D}(S, \alpha^{-k}) \to \mathbf{D}(S, X)_k$$

given by the association

$$A_U \mapsto f^* A_U \otimes \mathcal{O}_{X_U}(k).$$

**Proof.** Firstly, $X_U$ is the projective bundle $\mathbb{P}(E_U)$ over $U$. We then have on $X_U$ the surjective morphism $f^* E_U \to \mathcal{O}_{X_U}(1)$. Given $F$ an $\alpha^{-1}$-twisted sheaf on $S$, we have the surjective morphism:

$$f^*(F_U \otimes E_U) = f^* F_U \otimes f^* E_U \to f^* F_U \otimes \mathcal{O}_{X_U}(1).$$

Since $F_U$ and $E_U$ give respectively an $\alpha^{-1}$-twisted and an $\alpha$-twisted sheaf on $S$, their tensor product $F_U \otimes E_U$ gives an untwisted sheaf on $S$. Hence we see that $f^* F_U \otimes \mathcal{O}_{X_U}(1)$ gives a quotient of an untwisted sheaf and hence an untwisted sheaf on $X$. It is now clear that given an object $A$ in $\mathbf{D}(S, \alpha^{-1})$, the object given locally by (2) belongs to $\mathbf{D}(S, X)_1$.

The proof is similar for any $k$ in $\mathbb{Z}$. \hfill \qed

**Theorem 5.4.** The functor $f_k^*$ defined in Lemma 5.3 is an equivalence between the category $\mathbf{D}(S, \alpha^{-k})$ and the category $\mathbf{D}(S, X)_k$.

**Proof.** Given $A$ in $\mathbf{D}(S, X)_1$, consider the association over $U$

$$A|_{X_U} \mapsto Rf_*(A|_{X_U} \otimes \mathcal{O}_{X_U}(-1)).$$

We show that it gives a functor $\Lambda$ from $\mathbf{D}(S, X)_1$ to $\mathbf{D}(S, \alpha^{-1})$ and that is the quasi-inverse functor of $f_k^*$.

Firstly, since $A$ is in $\mathbf{D}(S, X)_1$, on $X_U$ we have $A|_{X_U} = f^* A_U \otimes \mathcal{O}_{X_U}(1)$, with $A_U$ in $\mathbf{D}(U)$. Evaluating $\Lambda$ on $A|_{X_U}$ we get

$$Rf_*(A|_{X_U} \otimes \mathcal{O}_{X_U}(-1)) = Rf_* f^* A_U.$$
Now use projection formula:

\[ Rf_* f^* A_U = Rf_* O_X \otimes A_U. \]

We have \( R^i f_* O_X = 0 \) for \( i > 0 \) and \( f_* O_X = O_S \), and then

\[ Rf_* f^* A_U \simeq_{\text{q.iso}} A_U. \]

It follows that \( \Lambda \) associates to \( A |_{X_U} \) the object \( A_U \) in \( \mathbf{D}(U) \).

At a level of coherent sheaves, by the same reasoning used in Lemma 5.3, we have the surjective morphism

\[ f^*(F_U \otimes E_U) \rightarrow f^* F_U \otimes O_X(U)(1). \]

Since \( E_U \) is an \( \alpha \)-twisted sheaf on \( S \), we can give to \( F_U \) the structure of \( \alpha^{-1} \)-twisted sheaf over \( S \). This shows that \( \Lambda \) is actually a functor from the subcategory \( \mathbf{D}(S, X)_1 \) to the category \( \mathbf{D}(S, \alpha^{-1}) \).

It is now an evidence by \( \square \) that \( \Lambda \) and \( f^1_* \) are each other quasi-inverse.

The proof for \( k \in \mathbb{Z} \) is similar. \( \square \)

We then have constructed full subcategories \( \mathbf{D}(S, X)_k \) of \( \mathbf{D}(X) \), each one equivalent to a category of perfect complexes of suitably twisted sheaves on \( S \).

Notice that we have \( f^0_* = Lf^* = f^* \) since \( f \) is flat. In this case, everything is untwisted and we can recover a Lemma by Orlov.

**Lemma 5.5.** The functor \( f^* : \mathbf{D}(S) \rightarrow \mathbf{D}(X) \) is a full and faithful embedding.

**Proof.** \( \square \), Lemma 2.1. \( \square \)

The full subcategory of \( \mathbf{D}(X) \) which is the image of \( \mathbf{D}(S) \) under the functor \( f^* \) is in fact the category \( \mathbf{D}(S, X)_0 \) defined earlier.

### 5.2. \( \sigma \) is a semiorthogonal decomposition.

**Lemma 5.6.** For any \( A \) in \( \mathbf{D}(S, X)_k \) and \( B \) in \( \mathbf{D}(S, X)_n \) we have \( R\text{Hom}(A, B) = 0 \) for \( r \geq k - n > 0 \).

**Proof.** We have locally \( A |_{X_U} = f^* A_U \otimes O_{X_U}(k) \) and \( B |_{X_U} = f^* B_U \otimes O_{X_U}(n) \).

We have:

\[ R\text{Hom}(A |_{X_U}, B |_{X_U}) = R\text{Hom}(f^* A_U \otimes O_{X_U}(k), f^* B_U \otimes O_{X_U}(n)) = R\text{Hom}(f^* A_U, f^* B_U \otimes O_{X_U}(n - k)). \]

We now use the adjoint property of \( f^* \) and \( Rf_* \):

\[ R\text{Hom}(f^* A_U, f^* B_U \otimes O_{X_U}(n - k)) = R\text{Hom}(A_U, Rf_*(f^* B_U \otimes O_{X_U}(n - k))). \]

Now by projection formula

\[ Rf_*(f^* B_U \otimes O_{X_U}(n - k)) = B_U \otimes Rf_*(O_{X_U}(n - k)). \]
We have $Rf_*(\mathcal{O}_X(n-k)) = 0$ for $-r \leq n-k < 0$ and hence the sheaves $R\text{Hom}(A,B)$ are zero.

Using the local to global Ext spectral sequence, we get the proof. □

We thus have an ordered set $\sigma = (\mathbf{D}(S,X)_0, \ldots, \mathbf{D}(S,X)_r)$ of orthogonal subcategories of $\mathbf{D}(X)$. Last step towards the proof of Theorem 5.1 is to show that it generates the whole category.

Consider the fiber square over $S$:

$$\begin{array}{ccc}
P := X \times_S X & \xrightarrow{p} & X \\
q & & g \\
X & \xrightarrow{f} & S.
\end{array}$$

The morphism $g$ corresponds to $f : X \to S$. We call $P$ the product $X \times_S X$.

Consider the diagonal embedding $\Delta : X \to P$. It is a section for the projection morphism $p : P \to X$. By Lemma 2.4, there exists a vector bundle $G$ on $X$ such that $P \cong \mathbb{P}(G) \to X$.

Consider now on $P$ the surjective morphism: $p^*G \twoheadrightarrow \mathcal{O}_P \to 0$. We also have the Euler short exact sequence on $P$:

$$0 \to \Omega_{P/X}(1) \to p^*G \to \mathcal{O}_P(1) \to 0.$$ Combining the exact sequence and the surjective morphism, we get a section of $\text{Hom}(\Omega_{P/X}(1), \mathcal{O}_P)$ whose zero locus is the diagonal $\Delta$ of $P$.

Remark that $\Omega_{P/X}(1) = p^*\Omega_{X/S} \otimes \mathcal{O}_P(1)$ and $\Lambda^k(p^*\Omega_{X/S} \otimes \mathcal{O}_P(1)) = p^*\Omega^k_{X/S} \otimes \mathcal{O}_P(k)$.

We get a Koszul resolution:

$$0 \to p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(r) \to \cdots \to p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(1) \to \mathcal{O}_P \to \mathcal{O}_\Delta \to 0.$$ By this complex we deduce that $\mathcal{O}_\Delta$ belongs, as an element of the category $\mathbf{D}(P)$, to the subcategory generated from

$$(4) \quad \{p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(r), \ldots, p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(1), \mathcal{O}_X \boxtimes \mathcal{O}_X\}$$

by exact triangles and shifting.

Given $A$ an element of $\mathbf{D}(X)$, we remark that $A = Rq_*(p^*A \otimes \mathcal{O}_\Delta)$. Since all involved functors (pull-back, direct image and tensor product) are exact functors, $A$ belongs to the subcategory of $\mathbf{D}(X)$ generated by

$\{Rq_*(p^*(A \otimes \Omega^r_{X/S}) \otimes \mathcal{O}_P(r)), \ldots, Rq_*(p^*(A \otimes \Omega^r_{X/S}) \otimes \mathcal{O}_P(1)), Rq_*p^*A\}$.

**Lemma 5.7.** The object $Rq_*(p^*(A \otimes \Omega^k_{X/S}) \otimes \mathcal{O}_P(k))$ in $\mathbf{D}(X)$ belongs to the subcategory $\mathbf{D}(S,X)_k$. 

Proof. We look at it in a local situation. In this case $X_U$ is a projective bundle over $U$, and we have

$$q^* \mathcal{O}_{X_U}(k) = \mathcal{O}_{F(k)|_{X_{U'}}}.$$  

This leads us to write locally:

$$(5) \quad Rq_* (p^*(A \otimes \Omega^k_{X/S}) \otimes \mathcal{O}_{F(k)})|_{X_{U'}} = Rq_* ((p^*(A \otimes \Omega^k_{X/S}))|_{X_U} \otimes q^* \mathcal{O}_{X_U}(k)) = Rq_* (p^*(A \otimes \Omega^k_{X/S}))|_{X_U} \otimes \mathcal{O}_{X_U}(k)$$

where we used projection formula and flat base change in the last two equalities. Then we have an object locally of the form finally given in (5), and then it is an object in $D(S,S,X)$. □

We have shown that all objects $A$ in $D(X)$ belong to the subcategory generated by the orthogonal sequence $\sigma$. This implies, by Lemma 4.5, that the subcategories $D(S,S,X)_k$ are admissible and then $\sigma$ is in fact a semiorthogonal decomposition of $D(X)$. This completes the proof of Theorem 5.1.

5.3. An example. We finally treat the simplest example of a Brauer Severi scheme. Let $K$ be a field and $X$ a Brauer Severi variety over the scheme $Spec(K)$. In this case Theorem 5.1 gives a very explicit semiorthogonal decomposition of the bounded derived category $D(X)$ of coherent sheaves on $X$ in terms of central simple algebras over $K$.

The cohomological Brauer group of $Spec(K)$ is indeed the Brauer group $Br(K)$ of the field $K$. The elements of $Br(K)$ are equivalence classes of central simple algebras over $K$ and its composition law is tensor product. To each $\alpha$ in $Br(K)$ corresponds the choice of a central simple algebra over $K$.

Given the $\alpha$ corresponding to the Brauer Severi variety $X$, an $\alpha^{-1}$-twisted sheaf is then a module over a properly chosen central simple algebra $A$, and it is coherent if it is finitely generated. The category $D(Spec(K), \alpha^{-1})$ is the bounded derived category of finitely generated modules over the algebra $A$. Concerning the element $\alpha^{-k}$ in $Br(K)$, just remind that the composition law is tensor product, to see that we can choose $A^{\otimes k}$ to represent it. The construction of $D(Spec(K), \alpha^{-k})$ is then straightforward. We can state the following Corollary of the Theorem 5.1.

Corollary 5.8. Let $K$ be a field, $X$ a Brauer Severi variety over $Spec(K)$ of dimension $r$. Let $\alpha$ be the class of $X$ in $Br(K)$ and $A$ a central simple algebra over $K$ representing $\alpha^{-1}$.

The bounded derived category $D(X)$ of coherent sheaves on $X$ has a semiorthogonal decomposition $\sigma = (D(K,X)_0, \ldots, D(K,X)_r)$, where $D(K,X)_i$ is equivalent to the bounded derived category of finitely generated $A^{\otimes i}$-modules.
References

[B] A. I. Bondal, Representations of associative algebras and coherent sheaves, Math. USSR Izv. 34 (1990) No 1, 23-42.

[BK] A. I. Bondal, M. M. Kapranov, Representable functors, Serre functors and mutations, Math. USSR Izv. 35 (1990) No 3, 519-541.

[BO] A. I. Bondal, D. O. Orlov, Semiorthogonal decomposition for algebraic varieties, Math. AG/9506012.

[C] A. H. Caldararu, Derived categories of twisted sheaves on a Calabi-Yau manifold, Ph.D. Thesis (2000).

[G] A. Grothendieck, Le groupe de Brauer I, II, III, in Dix exposés sur la cohomologie des schémas, North Holland, Amsterdam (1968), 46-188.

[HRD] R. Hartshorne, Residues and Duality, Lecture notes in Math 20, Springer-Verlag (1966).

[M] J. S. Milne, Étale cohomology, Princeton Math Series, Princeton University Press (1980).

[O] D. O. Orlov, Projective bundles, monoidal transformations and derived categories of coherent sheaves, Russian Math. Izv. 41 (1993), 133-141.

[SGA4\frac{1}{2}] P. Deligne et al., Cohomologie étale (SGA 4 \frac{1}{2}), Lecture Notes in Math, 569 (1977).

[SGA6] A. Grothendieck et al., Théorie des intersections et Théorème de Riemann-Roch (SGA 6), Lecture Notes in Math, 225 (1971).

[Y] K. Yoshioka, Moduli spaces of twisted sheaves on a projective variety, Math AG/0411538 (2004).

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