Fluid phonons and inflaton quanta at the protoinflationary transition

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Abstract
Quantum and thermal fluctuations of an irrotational fluid are studied across the transition regime connecting a protoinflationary phase of decelerated expansion to an accelerated epoch driven by a single inflaton field. The protoinflationary inhomogeneities are suppressed when the transition to the slow roll phase occurs sharply over space-like hypersurfaces of constant energy density. If the transition is delayed, the interaction of the quasi-normal modes related, asymptotically, to fluid phonons and inflaton quanta leads to an enhancement of curvature perturbations. It is shown that the dynamics of the fluctuations across the protoinflationary boundaries is determined by the monotonicity properties of the pump fields controlling the energy transfer between the background geometry and the quasi-normal modes of the fluctuations. After corroborating the analytical arguments with explicit numerical examples, general lessons are drawn on the classification of the protoinflationary transition.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In the conventional lore, the large-scale temperature and polarization anisotropies of the Cosmic Microwave Background are caused by curvature inhomogeneities with typical wavelengths exceeding the Hubble radius at the time of matter radiation equality [1, 2]. A nearly flat spectrum of Gaussian fluctuations of the spatial curvature naturally arises from the quantum inhomogeneities of a single inflaton field evolving during a quasi-de Sitter stage of expansion. Although the simplest scenario is consistent with the observational signatures, different sets of initial conditions have been explored through the years.

Initial states different from the vacuum can modify the temperature and polarization anisotropies at large scales. This general approach has been scrutinized along various
perspectives (see, e.g., [3–11]). Temperature-dependent phase transitions [3, 4] lead to an initial thermal state for the metric perturbations [5–7, 9, 10]. If the initial state is not thermal (but it is not the vacuum either), curvature phonons can be similarly produced via stimulated emission. Second-order correlation effects of the scalar and tensor fluctuations of the geometry can be used to explore the statistical properties of the initial quantum state [8] by applying the tenets of Hanbury–Brown–Twiss interferometry [12] which is employed, in quantum optics, to infer the bunching properties of visible light.

The modifications of the initial state are subjected to a number of constraints all originating, directly or indirectly, from the comparison between the energetic content of the initial fluctuations and the energy density of the background geometry. The criterion for the avoidance of severe backreaction effects is not unique. Single-field quasi-de Sitter inflationary models with general initial states of primordial quantum fluctuations have been examined in [9, 11] with the aim of deriving constraints from the study of higher-order correlation functions and from the renormalizability of the energy–momentum tensor of the fluctuations. It is equally plausible to demand that the energy–momentum pseudo-tensor of the scalar and tensor fluctuations does not exceed the energy density and pressure of the background geometry, as argued in [13].

In this paper a complementary and novel approach to the problem of the initial conditions of cosmological perturbations is pursued. The ever expanding inflationary backgrounds are geodesically incomplete and inflation cannot be eternal in the past. Thus it is legitimate to suppose the existence of a protoinflationary phase where the dynamics of the background was not yet accelerated. As the terminology indicates, the purpose here is not to test the universality of inflation given a set of arbitrary and widely different choices of the preinflationary dynamics. While under certain conditions inflation can be dynamically realized, it is not eternal in the past either. The modest purpose here is not to challenge inflation but to analyze the initial conditions of large-scale inhomogeneities in an improved dynamical framework. Just to avoid potential misunderstandings, it should be clear that the transition from a deceleration to acceleration has nothing to do with the so-called bouncing behavior where the background passes from contraction to expansion or vice versa (see, e.g., [14] and references therein). In the present framework the universe will always be expanding even during the protoinflationary phase.

The approach suggested here is pragmatic and the attention is focused on single-field inflationary models leading to a nearly flat spectrum of curvature inhomogeneities [15]. During the protoinflationary phase of decelerated dynamics the energy–momentum tensor is dominated by a single perfect fluid. The analysis can be generalized to include various inflaton fields and protoinflationary fluids but this will not be the primary goal of this investigation. Unlike the standard scenario, during the protoinflationary phase the seeds of curvature inhomogeneities are fluid phonons, i.e. the quantum excitations of an irrotational and relativistic fluid discussed by Lukash [16] (see also [17, 18]) right after one of the first formulations of inflationary dynamics [19]. The whole irrotational system can be reduced to a single (decoupled) normal mode which is promoted to field operator in case the initial fluctuations are required to minimize the quantum Hamiltonian of the phonons [16]. The canonical normal mode identified in [16] is invariant under infinitesimal coordinate transformations as required in the context of the Bardeen formalism [20] (see also [17]). The subsequent analyses of [21] and [22] follow the same logic of [16] but in the case of scalar field matter; the normal modes identified in [16, 21, 22] coincide with the (rescaled) curvature perturbations on comoving orthogonal hypersurfaces [23, 24] (see also the beginning of section 3).

The fluid phonons can be treated quantum mechanically but the initial state does not need to be the vacuum: if the protoinflationary phase is dominated by radiation, the fluid
phonons are more likely to follow a Bose–Einstein distribution as it happens in inflationary models based on temperature-dependent phase transitions [4–6]. When the protoinflationary inhomogeneities are suppressed across the boundary, the initial normalization of curvature perturbations is set, most likely, by the quantum mechanical fluctuations generated during the inflationary phase and, depending on the duration of inflation, by the stimulated emission from the protoinflationary relics. The conditions for the suppression or for the enhancement of protoinflationary curvature perturbations are related to the evolution of the pump fields controlling the transfer between the energy density of the background and the quasi-normal modes of the system.

This paper is organized as follows. After introducing the governing equations, in section 2, the quasi-normal mode of the system are derived in section 3. The fate of the large-scale curvature perturbations across the protoinflationary transition is investigated in section 4 where the normalization of the fluctuations during the protoinflationary phase is also discussed. In section 5 the nature of the transition is clarified in terms of the monotonicity properties of the pump fields accounting for the contribution of the inflaton and of the protoinflationary fluid to the total curvature perturbations. Explicit analytical and numerical examples are used to draw some general lessons on the dynamical features of the protoinflationary transition. Section 6 contains some concluding remarks. The derivation of the coupled evolution equations of the quasi-normal modes related, asymptotically, to fluid phonons and inflaton quanta is reported in the appendix.

2. Governing equations

2.1. General consideration

The minimal set of assumptions characterizing the framework of the present investigation stipulates that the four-dimensional geometry is determined by the Einstein equations, supplemented by the conservation equations accounting for the dynamics of the inflaton and of the protoinflationary sources:

\[ R^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta R = 8\pi G \left[ T^\alpha_\beta (\phi) + T^\alpha_\beta (\rho_{pr} , p_{pr}) \right], \quad (2.1) \]

\[ g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + \frac{\partial V}{\partial \phi} = 0, \quad (2.2) \]

\[ \nabla^\alpha T^\alpha_\beta = 0, \quad g^{\alpha\beta} u_\alpha u_\beta = 1, \quad (2.3) \]

where \( T^\alpha_\beta (\phi) \) and \( T^\alpha_\beta (\rho_{pr} , p_{pr}) \) are, respectively, the energy–momentum tensors of the inflaton field \( \phi \) and of the protoinflationary fluid:

\[ T^\alpha_\beta (\phi) = \partial_\alpha \phi \partial_\beta \phi - \left[ \frac{1}{2} g^{\alpha\beta} \partial_\gamma \phi \partial_\gamma \phi - V(\phi) \right] \delta^\alpha_\beta, \quad (2.4) \]

\[ T^\alpha_\beta (\rho_{pr} , p_{pr}) = (p_{pr} + \rho_{pr}) u_\alpha u_\beta - p_{pr} \delta^\alpha_\beta. \quad (2.5) \]

The subscripts in the energy density and pressure remind of the protoinflationary origin of the fluid variables. In a conformally flat background metric of the type \( g_{\alpha\beta} = a^2(\tau) \eta_{\alpha\beta} \) (where \( a(\tau) \) is the scale factor in conformal time and \( \eta_{\alpha\beta} \) is the Minkowski metric), equations (2.1)–(2.3) lead to a set of four equations

\[ \mathcal{H}^2 = \frac{8\pi G}{3} \left[ \frac{\dot{a}^2 \rho_{pr} + \dot{\phi}^2}{2} + V a^2 \right], \quad (2.6) \]

1 Greek indices run from 0 to 3; the signature of the metric \( g_{\alpha\beta} \) is mostly minus and \( \nabla_\alpha \) denote the covariant derivative with respect to \( g_{\alpha\beta} \).
\[ H^2 - H' = 4\pi G[a^2(\rho_{\text{pe}} + p_{\text{pe}}) + \psi'^2], \]  
\[ \psi'' + 2H\psi' + \frac{\partial V}{\partial \psi}a^2 = 0, \]  
\[ \rho_{\text{pe}}' + 3H(\rho_{\text{pe}} + p_{\text{pe}}) = 0, \]

which are not all independent and whose specific form is dictated by the fluid content of the primordial plasma. In equations (2.6)–(2.9) the prime denotes a derivation with respect to the conformal time coordinate \( \tau \); furthermore \( H = (\ln a)' \). The connection between \( H \) and the Hubble parameter is \( H = H/a \). The effective energy and pressure densities of \( \psi \) are given by

\[ \rho_{\psi} = \frac{\psi^2}{2a^2} + V(\psi), \quad p_{\psi} = \frac{\psi^2}{2a^2} - V(\psi). \]

Note that equation (2.8) is equivalent to

\[ \rho_{\psi}' + 3H(\rho_{\psi} + p_{\psi}) = 0, \quad c_{\psi}^2 \equiv \frac{\rho_{\psi}'}{\rho_{\psi}} = 1 + \frac{2a^2}{3H\psi'} \left( \frac{\partial V}{\partial \psi} \right) \frac{H}{\psi'}. \]

### 2.2. Uniform curvature gauge

The most general scalar fluctuation of the four-dimensional metric is parametrized by four different functions whose number can be eventually reduced by specifying (either completely or partially) the coordinate system:

\[ \delta_{i,j}g_{00} = 2a^2\phi, \quad \delta_{i,j}g_{ij} = 2a^2(\psi \delta_{ij} - \partial_i \partial_j E), \quad \delta_{i,j}g_{0j} = -a^2 \partial_i B, \]

where \( \delta_i \) denotes the scalar mode of the corresponding tensor component; the full metric (i.e. background plus inhomogeneities) is given, in these notations, by \( g_{\alpha\beta}(\vec{x}, \tau) = \tilde{g}_{\alpha\beta}(\tau) + \delta_{i,j}g_{ij}(\vec{x}, \tau) \) where, as already mentioned prior to equations (2.6)–(2.9) \( \tilde{g}_{\alpha\beta}(\tau) = a^2(\tau) \eta_{\alpha\beta} \).

For infinitesimal coordinate shifts \( \tau \to \bar{\tau} = \tau + \epsilon_0 \) and \( x' \to \bar{x}' = x' + \partial' \epsilon \) the functions \( \phi(\vec{x}, \tau), B(\vec{x}, \tau), \psi(\vec{x}, \tau) \) and \( E(\vec{x}, \tau) \) introduced in equation (2.12) transform as

\[ \phi \to \bar{\phi} = \phi - H\epsilon_0 - \epsilon', \quad \psi \to \bar{\psi} = \psi + H\epsilon_0, \]
\[ B \to \bar{B} = B + \epsilon_0 - \epsilon', \quad E \to \bar{E} = E - \epsilon. \]

In the uniform curvature gauge two out of the four functions of equation (2.12) are set to zero \([25]\):

\[ E = 0, \quad \psi = 0. \]

Starting from a gauge where \( E \) and \( \psi \) do not vanish, the perturbed line element can always be brought in the form (2.15) by demanding \( \bar{E} = 0 \) and \( \bar{\psi} = 0 \) in equations (2.13) and (2.14). More specifically, if \( E \neq 0 \) and \( \psi \neq 0 \), the uniform curvature gauge condition can be recovered by fixing the gauge parameters as \( \epsilon = E \) and \( \epsilon_0 = -\psi/H \). This choice guarantees that, in the transformed coordinate system, \( \bar{\psi} = \bar{E} = 0 \).

The gauge condition of equation (2.15) implies that the fluctuations of the spatial curvature vanish but, in this case, the perturbed metric also contains off-diagonal elements. With the condition (2.15) the gauge freedom is totally fixed without the need of further conditions: because of this property the functions \( \phi(\vec{x}, \tau) \) and \( B(\vec{x}, \tau) \) bear an extremely simple relation to one of the conventional sets of gauge-invariant variables, as it will be shown later in this section. Finally, the off-diagonal coordinate system will prove very useful in section 3 and in the appendix for the analysis of the coupled system of quasi-normal modes. In the gauge
(2.15) the inhomogeneities of the energy–momentum tensors $T^\beta_\mu(\psi)$ and $T^\beta_\alpha(\rho_{pr}, p_{pr})$ are, respectively,

$$
\delta s \ T^0_0 = \frac{1}{a^2} \left( -\phi \psi'^2 + \frac{\partial V}{\partial \psi} \chi ' + \chi ' \psi' \right), \quad \delta s \ T^0_i = -\frac{1}{a^2} \phi \psi \partial^i \chi - \frac{\psi'^2}{a^2} \partial^i B, \quad (2.16)
$$

$$
\delta s \ T^i_0 = \frac{1}{a^2} \left( \phi \psi'^2 + \frac{\partial V}{\partial \psi} \chi ' - \chi ' \psi' \right) \delta^i, \quad (2.17)
$$

$$
\delta s \ T^0_0 = \frac{2}{a^2} [-H \nabla^2 B - 3H^2 \phi].
$$

The combination of equations (2.16)–(2.18) with equations (2.19)–(2.21) implies that the (00) and (0i) components of the perturbed Einstein equations with mixed indices become

$$
(H^2 - \delta_{\mu} \ n \delta_{\nu} B) = -4\pi G a^2 (\delta \rho_{pr} + \delta \rho_{\psi}).
$$

(2.22)

$$
(H^2 - \delta_{\mu} \ n \delta_{\nu} B - 4\pi G a^2 \frac{1}{a^2} [\phi \nabla^2 \chi + \psi' \nabla^2 B]),
$$

(2.23)

where $\theta_{pr}(\chi, \tau) = \partial_i v^i$. The variables $\delta \rho_{\psi}$ and $\delta p_{\psi}$ correspond to the fluctuations of the energy density and of the pressure of the inflaton field:

$$
\delta \rho_{\psi} = \frac{1}{a^2} \left( -\phi \psi'^2 + \chi ' + \frac{\partial V}{\partial \psi} \chi \right),
$$

(2.24)

$$
\delta p_{\psi} = \frac{1}{a^2} \left( -\phi \psi'^2 - \chi ' - \frac{\partial V}{\partial \psi} \chi \right).
$$

(2.25)

To avoid lengthy notations we wrote $\delta \rho_{\psi}$ (instead of $\delta s \rho_{\psi}$), $\delta \rho_{pr}$ (instead of $\delta s \rho_{pr}$) and similarly for the corresponding pressures; this notation is fully justified and unambiguous once the scalar nature of the fluctuations has been established, as specified by the general formulae written above. Bearing in mind this specification, the (ii) component of the perturbed Einstein equations reads

$$
[\left( -\delta_{\mu} \ n \delta_{\nu} B - \frac{1}{a^2} \nabla^2 (\phi + B' + 2\nabla B) \right) \delta^i + \frac{1}{a^2} \partial_i \partial^i [\phi + B' + 2\nabla B] ] = 4\pi G a^2 \left( -\delta_{\mu} \ n \delta_{\nu} \Pi^i + \delta \rho_{pr} \delta^i + \delta p_{pr} \delta^i \right). \quad (2.26)
$$

The separation of the traceless part from the trace in equation (2.26) leads to two independent relations:

$$
(\delta_{\mu} \ n \delta_{\nu} B) \phi + H \phi' + \frac{1}{a^2} \nabla^2 (\phi + B' + 2\nabla B) = 4\pi G a^2 \left( \delta \rho_{pr} + \delta \rho_{\psi} \right), \quad (2.27)
$$

$$
\partial_i \partial^i [\phi + B' + 2\nabla B] - \frac{1}{a^2} \nabla^2 (\phi + B' + 2\nabla B) \delta^i = 8\pi G a^2 \Pi^i. \quad (2.28)
$$
If the anisotropic stresses $\Pi^i_j$ is neglected, equation (2.26) is the equivalent to the following pair of conditions:
\[
(H^2 + 2\mathcal{H}^2)\phi + H\phi' = 4\pi G\alpha^2(\delta p_x + \delta \rho), \tag{2.29}
\]
\[
\phi + B' + 2\mathcal{H}B = 0. \tag{2.30}
\]
The evolution equation for the perturbed inflaton is:
\[
\chi'' + 2\mathcal{H}\chi' - \nabla^2 \chi + \frac{\partial V}{\partial \psi} a^2 + 2\phi \frac{\partial V}{\partial \psi} a^2 - \psi' \phi' - \psi^2 B = 0. \tag{2.31}
\]
Finally, by perturbing the covariant conservation equation of the fluid energy–momentum tensor the evolution equation for the density fluctuation is
\[
\delta \rho + (p + \rho) \theta + 3H(\delta p + \delta \rho) = 0, \tag{2.32}
\]
while the equation for the three-divergence of the velocity becomes
\[
(\theta + \nabla^2)B' + \left[\frac{\mathcal{L} + H(p + \rho)}{p + \rho}\right] (\theta + \nabla^2)B + \nabla^2 \delta p + \nabla^2 \phi = 0. \tag{2.33}
\]
Equations (2.31) and (2.33) represent the starting point for the derivation of the quasi-normal modes of the system.

### 2.3. Gauge-invariant observables

The coordinate system defined by equation (2.15) completely fixes the gauge freedom without the need of further subsidiary conditions. The absence of spurious gauge modes is then guaranteed as it happens for other choices of coordinates removing completely the gauge freedom such as the conformally Newtonian gauge. As a consequence, the perturbation variables defined in the gauge (2.15) bear a simple relation to the various gauge-invariant observables. The specific connection between the degrees of freedom defined in the uniform curvature gauge and other common gauge-invariant combinations will now be outlined.

The curvature perturbation on comoving orthogonal hypersurfaces (conventionally denoted by $\mathcal{R}$) and the Bardeen potential (conventionally denoted by $\Psi$) coincide, up to the result of equation (2.34)
\[
\Psi = -\mathcal{H}B, \quad \mathcal{R} = -\frac{H^2}{H^2 - \mathcal{H}^2}\phi. \tag{2.34}
\]

The result of equation (2.34) can be easily derived from the explicit gauge transformation relating the uniform curvature hypersurfaces with the comoving orthogonal hypersurfaces. Conversely, from the customary expression of $\mathcal{R}$ in terms of the gauge-invariant Bardeen potentials $\Phi$ and $\Psi$ the result of equation (2.34) can be cross-checked. The curvature perturbations on comoving orthogonal hypersurfaces are given by
\[
\mathcal{R} = -\Psi - \frac{\mathcal{H}(\mathcal{H}\Phi + \Psi')}{\mathcal{H}^2 - \mathcal{H}^2}. \tag{2.35}
\]
But in terms of the variables introduced in equation (2.12) the expression of $\Phi$ and $\Psi$ is:
\[
\Phi = \phi + (B - E')' + \mathcal{H}(B - E'), \quad \Psi = \psi - \mathcal{H}(B - E'). \tag{2.36}
\]
According to equation (2.36), in the gauge (2.15) $\Phi = \phi + B' + \mathcal{H}B$ and $\Psi = -\mathcal{H}B$. By inserting into equation (2.35) the expressions for $\Phi$ and $\Psi$ written in the gauge (2.15) the results of equation (2.34) are immediately recovered. The total density contrast on uniform curvature hypersurfaces can be expressed as
\[
\zeta = -\frac{\delta \rho}{\rho}, \quad \delta \rho = (\delta \rho_x + \delta \rho_y), \quad \rho_t = \rho + \rho_{pr}. \tag{2.37}
\]
From equation (2.22) recalling equations (2.34) and (2.37) we can also obtain the relation between $\zeta$, $\Psi$ and $\Psi_1$:

$$\zeta = \Psi + \frac{\nabla^2 \Psi}{12\pi G a^2 (\rho_1 + \rho_1)}.$$ \hspace{1cm} (2.38)

It is relevant to remind that $\zeta$ and $\Psi$ are often used interchangeably. This is justified provided the wavelengths under considerations are sufficiently larger than the Hubble radius at the corresponding time. Otherwise the two variables $\zeta$ and $\Psi$ are physically different.

3. Quasi-normal modes of the system

The system of section 2 describing the evolution across the protoinflationary boundary has two asymptotic limits corresponding to the situation where one of the two components is either absent or dynamically negligible. If the protoinflationary fluid and the inflaton are simultaneously present the evolution is characterized by a pair of (interacting) quasi-normal modes which are the generalization of the normal modes obtainable in the case of a single component. After swiftly summarizing what happens in the two asymptotic limits, the derivation of the quasi-normal modes of the whole system will be presented. The interested reader may also consult the appendix where some of the technical results involved in the derivation are collected.

In the limit $\phi'(\tau) \to 0$ and $\chi(\vec{x}, \tau) \to 0$ the fluctuations of the inflaton energy density and of the inflaton pressure are both vanishing, i.e. $\delta \rho_{\phi} = \delta p_{\phi} = 0$. Since $\delta p_{pr} = c_s^2 \delta \rho_{pr}$, equation (2.22) can be multiplied by $c_s^2$ and summed to equation (2.29). After simple algebra the following result will be obtained:

$$H \phi'' + \left[ H^2 \left( 1 + 3c_s^2 \right) + 2H' \right] \phi = -Hc_s^2 \nabla^2 B.$$ \hspace{1cm} (3.1)

Introducing the variables $\alpha$ and $\gamma$ defined as:

$$\gamma = \frac{3}{2} \left( 1 + \frac{\rho_{pr}}{\rho_{pr}} \right), \quad \alpha^2 = \frac{\gamma}{4\pi G c_s^2}.$$ \hspace{1cm} (3.2)

Equation (3.1) becomes

$$\left( \frac{\phi}{\gamma} \right)' = -\frac{\nabla^2 B}{4\pi G \alpha^2 \gamma}.$$ \hspace{1cm} (3.3)

Differentiating both sides of equation (3.3) with respect to $\tau$, two kinds of terms (proportional to $\nabla^2 B$ and to $\nabla^2 B'$) will arise; using then equation (2.29) to eliminate the terms proportional to $\nabla^2 B'$ and equation (3.3) to eliminate the terms containing $\nabla^2 B$, a decoupled equation for $\phi$ is readily obtained. Recalling the simple relation between $\phi$ and $R$ mentioned in (2.34) the resulting equation becomes

$$R_{pr}'' + 2 \frac{\nabla}{\nabla} R_{pr}' - c_s^2 \nabla^2 R_{pr} = 0,$$ \hspace{1cm} (3.4)

where $R_{pr}$ is the curvature perturbation on comoving orthogonal hypersurfaces and

$$R_{pr} = -\frac{\phi}{\gamma} = -\frac{q_e}{\zeta_{pr}}, \quad \zeta_{pr} = \alpha a \frac{a^2 \sqrt{\rho_{pr} + \rho_{pr}}}{H c_s}.$$ \hspace{1cm} (3.5)

2 The background-dependent functions $\alpha^2$ and $\gamma$ mentioned in equations (3.1)–(3.2) illustrate, for convenience, the notations [16, 18]. In the rest of the paper it will be more practical to adopt a slightly different set of variables which are introduced in equations (3.4) and (3.5).
The function $q_v$ introduced in equation (3.5) is actually the normal mode of the system obeying
\[
q_v'' - c_s^2 \nabla^2 q_v - \frac{\gamma}{z_{\phi}} q_v = 0.
\] (3.6)

By using the momentum constraint in the purely hydrodynamical case (i.e. $\phi'(\tau) \to 0$ and $\chi(\vec{x}, \tau) \to 0$ in equation (2.23)) the following chain of equalities holds:
\[
\nabla^2 R_{\phi} = -\nabla^2 \left( \frac{\phi}{\gamma} \right) = \frac{4\pi G \delta^2 (p_{\phi} + \rho_{\phi})}{\gamma H} (\theta_{\phi} + \nabla^2 B),
\] (3.7)

which also implies, always neglecting the inflaton, that $\nabla^2 R_{\phi} = \mathcal{H}(\theta_{\phi} + \nabla^2 B)$. The same steps leading to equations (3.4) and (3.5) can be applied to the case of scalar field matter in the absence of protoinflationary component (i.e. $\rho_{\phi} = P_{\phi} = 0$ and $\delta \rho_{\phi} = \delta P_{\phi} = \theta_{\phi} = 0$). The evolution equation of the curvature perturbation will be given, in this case, by:
\[
R_v'' + 2 \frac{\chi'}{z_{\phi}} R_v' - \nabla^2 R_v = 0, \quad \frac{z_{\phi}}{z_{\omega}} = \frac{\theta_{\phi}}{H}.
\] (3.8)

The separate use of the momentum constraint in the scalar field case (i.e. $\theta_{\phi} = 0$ in equation (2.23)) leads to the analog of equation (3.7):
\[
R_v'' - \frac{4\pi G \chi'}{H} \chi = - \frac{q_x}{z_{\phi}},
\] (3.9)

where $q_x = a \chi \equiv -z_\phi R_v$ obeys, from equation (3.8), the following equation:
\[
q_x'' - \nabla^2 q_x = \frac{z_{\phi}'}{z_{\omega}} q_x = 0.
\] (3.10)

which is the analog of equation (3.6) holding in the absence of inflaton contribution. When dealing with the process of parametric amplification, the variables $z_{\phi}(t)$ and $z_{\omega}(t)$ are dubbed pump fields since they control the rate of energy transfer from the background to the fluctuations. This terminology is often used in quantum optics [12] (see also [8]) and shall also be employed in the forthcoming considerations.

The results obtained in equations (3.4)–(3.7) assume the absence of the inflaton field. Conversely equations (3.8) and (3.9) assume the absence of the protoinflationary fluid. If the contributions of the fluid and of the scalar field are simultaneously taken into account, the evolution equations of the resulting system can be reduced to a pair of coupled equations whose solution gives directly the curvature fluctuations on comoving orthogonal hypersurfaces. Indeed, by keeping both contributions, equation (2.23) implies
\[
\phi = \frac{4\pi G \chi'}{H} \chi + \frac{4\pi G \delta^2 (p_{\phi} + \rho_{\phi})}{\mathcal{H} u},
\] (3.11)

where the notation $(\theta_{\phi} + \nabla^2 B) = -\nabla^2 u$ has been introduced. To simplify the discussion, we shall assume that the protoinflationary fluid is characterized by a constant barotropic index $\omega$ so that $c_s = \sqrt{\omega}$. The derivation of the coupled system of the quasi-normal modes is reported in the appendix. The final equations obeyed by $q_v$ and $q_x$ are
\[
q_v'' - c_s^2 \nabla^2 q_v + A \omega (\tau) q_v + B \omega (\tau) q_x + C \omega (\tau) q_x' = 0,
\] (3.12)
\[
q_x'' - \nabla^2 q_x + \overline{A} \omega (\tau) q_x + \overline{B} \omega (\tau) q_v + \overline{C} \omega (\tau) q_v' = 0.
\] (3.13)

The coefficients $A \omega (\tau)$, $B \omega (\tau)$ and $C \omega (\tau)$ depend on the conformal time coordinate and are given by the following expressions:

3 In equations (3.14)–(3.16) and (3.17)–(3.19) natural Planckian units $M_p = 1$ are used. The same units are also used in the second part of the appendix where the explicit derivation of equations (3.12) and (3.13) is reported.
The coefficients \( \rho \) of constant energy density when the slow-roll dynamics starts off. This choice for the matching space-time the energy–momentum tensor experiences a finite discontinuity on the hypersurface irrotational fluid dominates the total energy density and pressure. In a slightly inhomogeneous 4. Across the protoinflationary transition it is immediate to show from equations (3 21) that \( \mathcal{R} \rightarrow \mathcal{R}_\psi \) when \( \psi \rightarrow 0 \). In the same way when \( \psi \rightarrow 0 \) equations (3 9) and (3 21) imply that \( \mathcal{R} \rightarrow \mathcal{R}_\psi \). The quasi-normal modes \( q_v \) and \( q_x \) describe, asymptotically, the excitations corresponding to fluid phonons and inflaton quanta.

4. Across the protoinflationary transition

The crudest model for the transition stipulates that prior to the onset of inflation the perfect and irrotational fluid dominates the total energy density and pressure. In a slightly inhomogeneous space-time the energy–momentum tensor experiences a finite discontinuity on the hypersurface of constant energy density when the slow-roll dynamics starts off. This choice for the matching
The sudden approximation is to assume that $\zeta_\tau(t) \to 0$ during the protoinflationary phase. Conversely during the slow-roll epoch $\zeta_{\tau\mathcal{E}}(t) \to 0$. The sudden approximation (together with the required continuity of the extrinsic curvature) implies the suppression of the density contrast and of the metric fluctuation across the protoinflationary boundary. The potential limitations of the sudden approximation are scrutinized in section 5 where the protoinflationary dynamics is described in terms of a class of exact solutions of the background equations which will be presented later.

4.1. The continuity of the extrinsic curvature

If the stress tensor undergoes a finite discontinuity on a space-like hypersurface the inhomogeneities are matched by requiring the continuity of the induced three metric and of the extrinsic curvature on that hypersurface. The extrinsic curvature is defined as

$$K_{ij} = \frac{1}{2N} [\nabla_i N_j + \nabla_j N_i - \gamma_{ij}].$$

(4.1)

Recalling equation (2.12), in a generic coordinate system the lapse function, the shift vectors and three-metric $\gamma_{ij}$ are, respectively,

$$N_i = a^2 \delta_\tau \vec{B}, \quad N^2 = (1 + 2\overline{\phi})a^2, \quad \gamma_{ij} = a^2 (1 - 2\overline{\phi}) \delta_{ij} + 2a^2 \delta_{\tau} \delta_i \vec{E}.$$  

(4.2)

Using equation (4.2) into equation (4.1) the covariant and mixed components of the extrinsic curvature read, to first order in the scalar metric perturbations,

$$K_{ij}(\vec{x}, \tau) = -a^2 \delta \theta_{ij} + a [\delta_{\tau} \delta_{ij} (\vec{B} - \vec{E}^\prime - 2\tau \vec{E}) + (\vec{\omega} + 2\tau \vec{\omega} + \vec{\omega}^\prime) \delta_{ij}],$$

$$K^i_j(\vec{x}, \tau) = -\frac{1}{a} [\delta^i_j (\vec{\omega} + \vec{\omega}^\prime) + \delta_{\tau} \delta_i (\vec{B} - \vec{E}^\prime)].$$

(4.3)

The continuity of the background extrinsic curvature implies that across the transition the scale factor and the Hubble rate must be continuous. In cosmic time, a continuous form of the scale factor can be written when, for instance, the inflationary phase is characterized by a set of constant slow-roll parameters (see, e.g., Equation (4.37) of section 4 for a general definition of the slow roll parameters). In this case we shall have that

$$a_{\text{pr}}(t) = a_* \left( \frac{t - t_*}{t_*} \right)^\alpha, \quad t \leq t_*,$$

$$a_{\text{inf}}(t) = a_* \left[ \frac{\alpha}{\beta} \left( \frac{t - t_*}{t_*} \right) + \beta - \alpha \right]^\beta, \quad t > t_*.$$  

(4.4)

In equation (4.4) the inflationary evolution is realized for $\beta \gg 1$. For some applications it is useful to recall the conformal time parametrization where the continuity of the scale factors across the protoinflationary boundary can be expressed as

$$a_{\text{pr}}(\tau) = a_* \left[ \left( \frac{\overline{\alpha}}{\overline{\beta}} + 1 \right) \frac{\tau_*}{\tau_1} + \frac{\tau}{\tau_1} \right]^\overline{\sigma}, \quad -\left( \frac{\overline{\alpha}}{\overline{\beta}} + 1 \right) \tau_* < \tau \leq -\tau_*,$$

$$a_{\text{inf}}(\tau) = \left( \frac{\overline{\alpha}}{\overline{\beta}} \frac{\tau_*}{\tau_1} \right)^\overline{\sigma} \left( \frac{-\tau_*}{\tau} \right)^\overline{\beta}, \quad -\tau_* \leq \tau \leq -\tau_1,$$

(4.5)

where $\overline{\alpha} = \alpha/(1 - \alpha)$ and $\overline{\beta} = \beta/\beta - 1$. From equation (4.4) the scale factor and its first derivative are continuous in $t_*$. By going in the conformal parametrization $a(\tau) \, d\tau = \alpha \, dt$ the time coordinate $\tau$ becomes negative and therefore the conformal time scale factor and its first
derivative with respect to $\tau$ are continuous in $-\tau_\ast$. In the parametrization of equation (4.5) the inflationary regime occurs for $\beta \gg 1$ and $\bar{B} \rightarrow 1$.

From equation (4.3) the continuity of the inhomogeneous part of $\bar{K}^i_j$ and $\bar{K}^{ij}$ implies the separate continuity of the combinations

$$[\bar{\psi}]_\pm = 0, \quad [\bar{E}]_\pm = 0, \quad [\bar{H}\phi + \bar{\psi}]_\pm = 0, \quad [\bar{B} - \bar{E}]_\pm = 0,$$

(4.6)

where, following the general treatment of sudden transitions [26], the subscript $\pm$ denotes the jump of the corresponding quantity across the transition (i.e. $[f]_\pm = f_+ - f_-$). In the coordinate system where $\tau$ is constant, the equation for the hypersurface of constant energy density becomes

$$\delta \rho_t = 0.$$

But since

$$\delta \rho_t = \delta \rho_{\rho_t} = \delta \rho_\rho + \delta \rho_\rho,$$

(4.7)

the condition $\delta \rho_\rho = 0$ implies $e_0 = \delta \rho_\rho / \rho_\rho$. Recalling now the expressions for $\bar{\psi}, \bar{\psi}, \bar{E}$ and $\bar{B}$ stemming from equations (2.13) and (2.14), the conditions of equation (4.6) become

$$\begin{align*}
\left[ \psi + \frac{\bar{H}}{\rho_\rho} \frac{\delta \rho_\rho}{\rho_\rho} \right]_\pm &= 0, \\
\left[ B - \bar{E}' + \frac{\delta \rho_\rho}{\rho_\rho} \right]_\pm &= 0, \\
\left[ \bar{H} \phi + \psi' + \left( \bar{H}' - \bar{H}^2 \right) \frac{\delta \rho_\rho}{\rho_\rho} \right]_\pm &= 0.
\end{align*}$$

(4.8)

Equations (4.8) are general and can be studied in any gauge. According to equation (4.8), the continuity of the extrinsic curvature in the gauge (2.15) implies that the following combinations must separately be continuous:

$$\begin{align*}
\left[ B + \frac{\delta \rho_\rho}{\rho_\rho} \right]_\pm &= 0, \\
\left[ \frac{\delta \rho_\rho}{\rho_\rho} \right]_\pm &= 0,
\end{align*}$$

(4.9)

$$\left[ \bar{H} \phi + \bar{H} \frac{\delta \rho_\rho}{\rho_\rho} \right]_\pm = 0.$$

(4.10)

Since $\bar{H}$ is continuous, equation (4.10) reduces to

$$\left[ \bar{H} \phi + \bar{H} \frac{\delta \rho_\rho}{\rho_\rho} \right]_\pm = 0.$$

(4.11)

The Hamiltonian constraint of equation (2.22) can be written in the form

$$\delta \rho_t = -\frac{\bar{H}}{4\pi G\alpha^2} [V^2 B + 3\bar{H}\phi].$$

(4.12)

Equations (4.9) and (4.11) are then equivalent to the two conditions

$$[B]_\pm = 0, \quad \left[ -\frac{\bar{H}^2}{\bar{H}^2 - \bar{H}'^2} \phi \right]_\pm = 0.$$

(4.13)

But according to equation (2.34) we have that $\Psi = -\bar{H}B$ and $\mathcal{R} = -\bar{H}^2 / \left( \bar{H}^2 - \bar{H}'^2 \right)$; thus the continuity of the scale factor and of $\bar{H}$ implies the continuity of $\mathcal{R}$ and $\Psi$ across the protoinflationary transition. The evolution will now be separately solved during the protoinflationary phase and during the inflationary phase. The matching conditions expressed by equation (4.13) agree with former treatments [26] but in the coordinate system defined by equation (2.15).
4.2. Protoinflationary evolution

During the protoinflationary phase and for \( z_\nu \to 0 \) the system reduces to the triplet of equations

\[
\dot{\mathcal{B}} + 2H \mathcal{B} + \left( \frac{H}{H^2} \right) \frac{\mathcal{R}}{a} = 0, \quad (4.14)
\]

\[
\mathcal{R} = -\left( \frac{H^2}{H} \right) \frac{\mathcal{R}}{a} + 2 \frac{\nabla^2 \mathcal{B}}{a}, \quad (4.15)
\]

\[
\delta_t = - \frac{2}{3} \frac{\mathcal{R}}{H} - 2 \frac{H}{H^2} \mathcal{R}, \quad \delta_t = \frac{\delta \rho_{\text{pr}}}{\rho_{\text{pr}}}, \quad (4.16)
\]

where the overdot denotes a derivation with respect to the cosmic time coordinate and \( \delta_t \) is the total density contrast which is dominated, in this case, by the protoinflationary fluid. Equation (4.14) comes from equation (2.30); equation (4.15) is equation (3.1) but written in the cosmic time coordinate; equation (4.16) derives from equation (2.22).

To enforce a correct normalization on the solutions we proceed as follows. After promoting the normal mode \( q_v (\vec{x}, \tau) \) and its conjugate momentum to the status of field operators obeying canonical commutation relations at equal times, the mode expansion for \( q_v (\vec{x}, \tau) \) becomes

\[
\hat{q}_v (\vec{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[ f_k (\tau) \hat{a}^+_k e^{-i \vec{k} \cdot \vec{x}} + f_k (\tau) \hat{a}_k e^{i \vec{k} \cdot \vec{x}} \right], \quad (4.17)
\]

where \( [\hat{a}_k, \hat{a}^+_l] = \delta^{(3)} (\vec{k} - \vec{l}) \) and the mode function \( f_k (\tau) \) obeys

\[
f_k'' + \left[ k^2 c_s^2 - \frac{z_{pr}'}{z_{pr}} \right] f_k = 0, \quad z_{pr} = \frac{a^2 \sqrt{\rho_{\text{pr}} + \rho_{\text{pr}}}}{H c_s}, \quad (4.18)
\]

\[
\frac{z_{pr}'}{z_{pr}} = \frac{\nu^2 - 1/4}{y^2}, \quad y = \left( \frac{\nu}{\beta} + 1 \right) \tau_s + \tau, \quad \nu = \frac{3(1 - w)}{2(3w + 1)}. \quad (4.19)
\]

In the case \( w = 1/3 \) (corresponding to a radiation fluid) \( \nu = 1/2 \); it is interesting to remark that, in the case \( w = 0 \) (but with \( c_s \neq 0 \), \( \nu = 3/2 \) allowing for a flat spectrum of phonons, as discussed in [16]). Equation (4.18) can be solved exactly in terms of Hankel functions and the solution is

\[
f_k (\tau) = \frac{N_v}{\sqrt{2k c_s}} \sqrt{k c_s y H_k^{(2)} (k c_s y)}, \quad N_v = \sqrt{\frac{2}{\pi}} e^{-i \pi (2\nu + 1)/4}, \quad (4.20)
\]

where, as usual, we shall focus on the case \( c_s = \sqrt{w} \). If the initial conditions are not quantum mechanical but rather thermal, then the initial state will contain thermal phonons:

\[
\langle \hat{a}^+_k \hat{a}_k \rangle = \pi K = \frac{1}{e^{\epsilon c_s T} - 1}. \quad (4.21)
\]

where \( T = aT \) denotes the comoving temperature while \( T \) is the physical temperature. The power spectrum of the fluid phonons can be easily determined from the two-point function evaluated at equal times:

\[
\langle \hat{q}_v (\vec{x}, \tau) \hat{q}_v (\vec{y}, \tau) \rangle = \int \frac{dk}{2\pi k} P_{q_v} (k, \tau) \frac{\sin k r}{k r}, \quad r = |\vec{x} - \vec{y}|, \quad (4.22)
\]

\[
P_{q_v} (k, \tau) = \frac{k^3}{2\pi^2} |f_k (\tau)|^2 (2\pi K + 1).
\]
If \( k_{c_\nu} \gg \sqrt{g} \) we have that \( n_k \to 0 \) and the quantum mechanical initial conditions dominate; conversely if \( k_{c_\nu} \ll \sqrt{g} \) the thermal initial conditions dominate against the quantum mechanical ones. Once the phonon spectrum is known the spectrum of curvature perturbations is

\[
P_R(k, \tau) = \frac{k^3}{2\pi^2} \left| f_k(\tau) \right|^2 (2n_k + 1).
\] (4.23)

From the spectrum of curvature phonons it is elementary to derive the spectrum of the metric fluctuations and of the Bardeen potential. In fact, from equation (4.14) the expression for \( B(\vec{x}, \tau) \) becomes

\[
B(\vec{x}, \tau) = -\frac{\mathcal{C}_B(w)}{M_P a^2(\tau)} \int^{\tau} a(\tau') q_v(\vec{x}, \tau') d\tau', \quad C_B(w) = \frac{\sqrt{3w(w+1)}}{2}.
\] (4.24)

Using equation (4.24) and recalling that \( \Psi(\vec{x}, \tau) = -\mathcal{H} B(\vec{x}, \tau) \) the mode expansion for \( \Psi(\vec{x}, \tau) \) is:

\[
\Psi(\vec{x}, \tau) = -\frac{\mathcal{H} C_B(w)}{M_P a^2(\tau)} \int d^3k \left\{ \hat{\alpha}_k g_k(\tau) e^{-i\vec{k} \cdot \vec{x}} + \hat{\alpha}_k^c g_k^c(\tau) e^{i\vec{k} \cdot \vec{x}} \right\}.
\] (4.25)

Finally, using equation (4.28) the power spectrum of the density contrast turns out to be

\[
P_\delta(k, \tau) = \frac{k^3}{2\pi^2 M_P} C_\delta^2(w) |h_k(\tau)|^2 (2n_k + 1), \quad C_\delta(w) = \sqrt{\frac{3(w+1)}{w}}.
\] (4.29)

Let us now suppose that the initial fluid phase is dominated by thermal phonons. In this rather realistic situation \( w = 1/3 \) and \( c_s^2 = \sqrt{w} \). From equations (4.22) and (4.23) the spectrum of curvature perturbations can be recast in the following form:

\[
P_R(k, k_{H'}, T) = \frac{N_{\text{eff}}}{1440 \sqrt{3}} \left( \frac{k}{k_{H'}} \right)^2 \left( \frac{T}{M_P} \right)^4 \coth \sqrt{\frac{\pi^2 N_{\text{eff}}}{360} \left( \frac{k}{k_{H'}} \right) \left( \frac{T}{M_P} \right)}.
\] (4.30)

where \( k_{H'} = a H = \mathcal{H} \); in equation (4.30) the physical temperature \( T \) is related to the Hubble rate as \( \mathcal{H}^2 M_P^2 = N_{\text{eff}} \pi^2 T^4 / 30 \), where \( N_{\text{eff}} \) denotes the effective number of relativistic degrees of freedom. In the limit \( (k/k_{H'}) (T/M_P) < 1 \) equation (4.30) becomes

\[
P_R(k, k_{H'}, T) = \frac{\sqrt{N_{\text{eff}}}}{24 \sqrt{30} \pi} \left( \frac{k}{k_{H'}} \right) \left( \frac{T}{M_P} \right)^3.
\] (4.31)
When the wavenumber is of the order of the particle horizon during the protoinflationary phase the amplitude of the curvature phonons is solely controlled by the temperature which must not exceed the Planck temperature. When $k \ll k_H$ the power spectrum is further suppressed. Since the transition to the fully developed inflationary phase occurs for $\tau \simeq \tau^* \sim 1/(a_* H_*)$ the spectrum computed from equations (4.30) and (4.31) stops being valid for $k \sim k_* = 1/\tau_*$. This means that $(k/k_*) < 1$ implying $k/k_H < |\tau/\tau^*|$ which simply tells that the initial conditions during the protoinflationary phase must be set not too early or, equivalently, not too close to the Planck curvature scale.

Similar considerations apply in the discussion the spectrum of the metric perturbations and of the density contrast. From equations (4.24)–(4.27) we arrive at the following explicit expression:

$$P_{\delta}(k, \tau) = \frac{3}{2\pi} \left( \frac{H_*}{M_P} \right)^2 \left( \frac{k^2 + 12 k H_*}{k_*^2} \right) \coth \left[ \sqrt{\frac{\pi^2 N_{\text{eff}} 360}{k_* H_*}} \right] \left( \frac{k}{k_*} \right) ,$$  \hspace{1cm} (4.33)

where $k_* = a_* H_*$ and, as before, $k_H = a H$. From equation (4.29) it can be argued that as long as $k < k_*$ and $H_* < M_P$ the modes inside the Hubble radius (i.e. $k \gg k_H$) do not jeopardize the validity of the perturbative expansion. During the protoinflationary phase the Hubble rate sharply increases towards the singularity and, in this situation, it can happen that large fluctuations arise for typical scales larger than the Hubble radius. This effect is caused by the nearness of the singularity and he lower limit in the time coordinate should be fixed by enforcing the validity of the perturbative expansion.

4.3. Suppression of density contrast and metric fluctuations

During the slow-roll phase and and in the limit $z_{pr} \to 0$ the analog of equations (4.14)–(4.16) can be written as

$$\dot{\delta}_1 = -2 \frac{\nabla^2 B}{3 H a} + 2\epsilon \delta \cdot \nabla \delta, \hspace{1cm} \delta_1 = \frac{\delta \rho_{\delta}}{\rho_{\delta}},$$  \hspace{1cm} (4.36)

where $\delta_1$ is now dominated by the fluctuations of the inflaton. We shall assume, for instance, the validity of the solution (4.4) with $\beta \gg 1$. This requirement is even too restrictive since the results discussed hereunder are valid in the slow-roll approximation, i.e. when both slow-roll parameters

$$\epsilon = \frac{H}{H^2} = \frac{M_P^2}{2} \left( \frac{V_{\phi}}{V} \right)^2, \hspace{1cm} \eta = \frac{\ddot{\phi}}{H \dot{\phi}} = \epsilon - \eta, \hspace{1cm} \eta = M_P^2 \frac{V_{\phi\phi}}{V}$$  \hspace{1cm} (4.37)
are much smaller than 1 but not necessarily constant. If $\mathcal{R}$ is continuous across the transition equation (4.34) implies that the expression for $B(\vec{x}, t)$ becomes

$$B(\vec{x}, t) = \frac{\mathcal{R}(\vec{x}, t)}{H a} - \frac{1}{a^2(t)} \int_{t'}^{t} \frac{\mathcal{R}(\vec{x}, t') a(t')}{H(t')} \, dt' - \frac{1}{a^2(t)} \int_{t'}^{t} \frac{\dot{\mathcal{R}}(\vec{x}, t') a(t')}{H(t')} \, dt'. \quad (4.38)$$

Conversely, the continuity of $B(\vec{x}, t)$ in equation (4.35) implies $\dot{\mathcal{R}} \simeq 0$ for typical wavelengths larger than the Hubble radius. Therefore equations (4.16), (4.36) and (4.38) imply

$$B(\vec{x}, t_{pr}) = \frac{\mathcal{R}_{*}(\vec{x})}{a_{pr} H_{pr}} \left[ 1 - \frac{H}{a} \int_{t_{pr}}^{t_{ex}} a(t') \, dt' \right] \simeq \frac{\mathcal{R}_{*}(\vec{x})}{a_{pr} H_{pr}} \left( \frac{5 + 3w}{3w + 3} \right), \quad (4.39)$$

$$B(\vec{x}, t_{inf}) = \frac{\mathcal{R}_{*}(\vec{x})}{a_{inf} H_{inf}} \left[ 1 - \frac{H}{a} \int_{t_{pr}}^{t_{inf}} a(t') \, dt' \right] \simeq \frac{\mathcal{R}_{*}(\vec{x})}{a_{inf} H_{inf}} \left( \frac{\epsilon}{\epsilon + 1} \right), \quad (4.40)$$

where $t_{pr}$ and $t_{inf}$ are, respectively, the values of the cosmic time coordinate during the protoinflationary phase and during the slow-roll phase when $\epsilon \ll 1$ and $\eta \ll 1$. Recalling that $\Psi(\vec{x}, t) = -aH B(\vec{x}, t)$, equations (4.39) and (4.40) show that

$$\Psi(\vec{x}, t_{inf}) = \frac{3(w + 1)}{5 + 3w} \left( \frac{\epsilon}{\epsilon + 1} \right) \Psi(\vec{x}, t_{pr}). \quad (4.41)$$

Since during the protoinflationary phase the evolution is by definition decelerated, $0 \leq w \leq 1$. Conversely in the inflationary regime $\epsilon \ll 1$ which demonstrates the suppression of the metric fluctuation. The same logic leads to following relation valid in the case of the density contrasts:

$$\delta(\vec{x}, t_{inf}) = \frac{3 \epsilon (w + 1)}{2} \delta(\vec{x}, t_{pr}). \quad (4.42)$$

Once more, since $3(w + 1)/2 \simeq O(1)$ and $\epsilon \ll 1$ we also have that $\delta(\vec{x}, t_{inf}) \ll \delta(\vec{x}, t_{pr})$ demonstrating the suppression of the density contrast across the protoinflationary boundary. Note that the results reported here hold for a slow-roll parameter $\epsilon$ which is not necessarily constant.

5. Enhanced protoinflationary inhomogeneities

5.1. Monotonicity properties

The dynamics of the protoinflationary transition depends on the behavior of the pump fields $z_{pr}(t)$ and $z_{p}(t)$. In the sudden treatment of the transition discussed in section 4 the pump fields $z_{pr}(t)$ and $z_{p}(t)$ evolve monotonically in cosmic time and the corresponding rates are positive definite (i.e. $\dot{z}_{pr}/z_{pr} > 0$ and $\dot{z}_{p}/z_{p} > 0$). Furthermore the evolution of $z_{pr}(t)$ is decelerated (i.e. $\ddot{z}_{pr}(t) < 0$). These requirements are satisfied in section 4 where $z_{pr}(t) \propto a(t)$, the evolution is decelerated as long as the perfect barotropic fluid dominates (i.e. $\dot{z}_{pr}(t) > 0$ and $\ddot{z}_{pr}(t) < 0$); furthermore, recalling equation (4.37), during the slow-roll phase

$$\frac{\dot{z}_{p}}{z_{p}} = H \left[ 1 + \frac{\dot{\Phi}}{H \dot{\Phi}} - \frac{H}{H^{2}} \right] = H(1 + \eta + \epsilon) > 0, \quad (5.1)$$

since, by definition of slow-roll, $\epsilon \ll 1$ and $\eta \ll 1$ with $\epsilon > 0$. In the particular case of exponential potentials the slow-roll parameters are constant. The monotonicity requirements define the conventional protoinflationary dynamics and they seem to be naively sufficient to implement a successful transition. Are they at all necessary? Is it possible to implement the transition in a different way without relying on the monotonicity of the pump fields? If this is the case, which are the consequences for the evolution of curvature perturbations?

If the transition to the slow-roll phase is not sudden the behavior of $z_{pr}(t)$ and $z_{p}(t)$ is not necessarily monotonic. Whenever $z_{pr}(t)$ and $z_{p}(t)$ do not evolve monotonically they can
develop either a global maximum or a global minimum. There can be even more cumbersome evolutions where a finite number of maxima and minima arise. These situations can be discussed after having addressed the basic case where either \(\dot{z}_{\text{pe}}(t)\) or \(\dot{z}_{\text{p}}(t)\) (or both) vanish for a finite value of the cosmic time coordinate \(t\). The absence of monotonic behavior is dynamically realized in different ways by appropriately modifying the relative weight of the inflationary and fluid components in the total energy–momentum tensor of the system. A class of analytic solutions exhibiting non-monotonic behavior for the pump fields can be obtained by solving equations (2.6)–(2.9) whose explicit form, in the cosmic time coordinate, is:

\[
3H^2 M_p^2 = \frac{\dot{\phi}^2}{2} + V(\phi) + \rho_{\text{pr}},
\]

\[
2H M_p^2 = -\dot{\phi}^2 - (p_{\text{pr}} + \rho_{\text{pr}}),
\]

\[
\dot{\phi} + 3H\dot{\phi} + \frac{3V}{\dot{\phi}} = 0,
\]

\[
\dot{\rho}_{\text{pr}} + 3H(p_{\text{pr}} + \rho_{\text{pr}}) = 0.
\]

We shall be interested in solutions where \(z_{\text{pe}}(t)\) and \(z_{\text{p}}(t)\) are not monotonic but satisfy the correct boundary conditions typical of a protoinflationary dynamics. Equation (5.3) can be written in terms of \(z_{\text{pe}}(t)\) and \(z_{\text{p}}(t)\) and using the definition of \(\epsilon(t)\) given in equation (4.37)

\[
a^2 M_p^2 \epsilon(t) = z_{\text{pe}}^2(t) + \epsilon^2 z_{\text{p}}^2(t), \quad \epsilon(t) = -\frac{H}{H^2}.
\]

5.2. Exact solutions

By requiring that \(z_{\text{pe}}(t)\) and \(z_{\text{p}}(t)\) are proportional it is possible to obtain a suitable ansatz for the solution of the whole system subjected to the requirement that the evolution of \(z_{\text{pe}}(t)\) and \(z_{\text{p}}(t)\) is not monotonic. In the case of perfect barotropic fluid with constant sound speed the full solution of equations (2.6)–(2.7) and (2.8)–(2.9) can be expressed in terms of the scale factor and of the inflaton field:

\[
a(t) = a_0 [\sinh (\beta H_0 t)]^{1/\beta}, \quad \beta = \frac{3(w + 1)}{2},
\]

\[
\varphi(t) = \varphi_0 \pm \sqrt{\frac{2}{\beta} M_p \sqrt{\Omega_\ast}} \ln \left[ \tanh \left( \frac{\beta H_0 t}{2} \right) \right],
\]

where the parameter \(\Omega_\ast\) and the protoinflationary energy density are defined, respectively, as

\[
\Omega_\ast = \frac{\rho_\ast}{3H^2 M_p^2}, \quad \rho_\ast = \rho_0 \left( \frac{a_\ast}{a} \right)^{3(w+1)}.
\]

Finally the inflaton potential is

\[
V(\varphi) = 3H^2 M_p^2 + \frac{3}{2} (1 - w)H^2 M_p^2 (1 - \Omega_\ast) \sinh^2 \left[ \sqrt{\frac{\beta}{2} \left( \frac{\varphi - \varphi_0}{\Omega_\ast} \right)} \right].
\]

Equations (5.7)–(5.10) solve equations (5.2)–(5.5) in the case of a constant barotropic index. Furthermore, as anticipated, the solution satisfies the boundary conditions characterizing the protoinflationary transition. In particular for \(\beta H_0 t < 1\) the solution is decelerated and from equation (5.7) we have

\[
a(t) \simeq a_\ast (\beta H_0 t)^{1/\beta}, \quad H_i = \frac{1}{\beta t_i} = \frac{2}{3(w + 1)t_i}.
\]
In the opposite limit (i.e. $\beta H_\ast t \gg 1$) the solution is accelerated with $H(t) \simeq H_\ast$:

$$H(t) = \frac{H_\ast}{\tanh (\beta H_\ast t)}, \quad \dot{H} = -\frac{\beta H_\ast^2}{\sinh^2 (\beta H_\ast t)}. \quad (5.12)$$

The parameter $\Omega_\ast = \rho_\ast / (3H_\ast^2M_p^2) < 1$ measures the amount of radiation at the onset of inflation. From equation $(5.11)$, $H_\ast$ is not the initial curvature scale but rather the curvature scale at the onset of the inflationary evolution. The conditions $\dot{a} > 0$ and $\ddot{a} > 0$ imply, from equation $(5.7)$,

$$\dot{a} = \frac{a H_\ast^2}{2} \left[ \sinh (H_\ast t/\beta) \right]^{1/\beta-2} \left[ 1 - 2\beta + \cosh \left[ 2H_\ast \beta t \right] \right] > 0. \quad (5.13)$$

From equation $(5.13)$ $\dot{a} > 0$ iff $\cosh^2 (\beta H_\ast t) > (\beta - 1)$. The beginning of the inflationary phase $t_\ast$ is then given by

$$H_\ast t_\ast = \frac{1}{\beta} \ln \left[ \sqrt{\beta} + \sqrt{\beta - 1} \right]. \quad (5.14)$$

This requirement shows that $t_\ast$ ranges between $0.54 H_\ast^{-1}$ in the case $w = 1/3$ and and $0.49 H_\ast^{-1}$ in the case $w = 1$. For numerical purposes related to the evolution of the inhomogeneities (see the discussion hereunder) it is practical to use $H_\ast t$ as new time coordinate. In fact $H_\ast t$ approximately coincides with the natural logarithm of the total number of inflationary e-folds. Hence $H_\ast t_{\text{max}} \approx \mathcal{O}(\ln N_{\text{tot}})$ where we can take, for illustrative purposes $N_{\text{tot}}$ between 70 and 100. Equations $(4.37)$ and $(5.7)$–$(5.8)$ imply that $\epsilon(t)$ gets progressively much smaller than 1 for $\beta H_\ast t \gg 1$ while $\eta(t)$ and $\bar{\eta}(t)$ are nearly constant in the same limit:

$$\epsilon(t) = \frac{\beta}{\cosh^2 (\beta H_\ast t)}, \quad \eta(t) = -\beta, \quad \bar{\eta}(t) = \beta \left[ 1 + \cosh^2 (\beta H_\ast t) \right] / \cosh^2 (\beta H_\ast t). \quad (5.15)$$

In conclusion the solution of equations $(5.7)$–$(5.10)$ satisfies the physical properties of a protoinflationary regime and can be used to describe the transition to from a decelerated epoch to an accelerated phase where the slow roll condition is verified for $\epsilon$ but not for $\eta$. Notice, finally, that when $\Omega_\ast \ll 1$ the potential $V(\varphi)$ of equation $(5.10)$ can always be expanded in powers of $\Omega_\ast$ as

$$V(\varphi) = V_0(\varphi) + V_1(\varphi)\Omega_\ast + V_2(\varphi)\Omega_\ast^2 + \cdots \quad (5.16)$$

where the functions $V_0(\varphi), V_1(\varphi), V_2(\varphi)$ are uniquely fixed from the exact form of the potential.

The class of solutions reported in equations $(5.7)$–$(5.10)$ can be used to determine the explicit forms of $z_\varphi(t)$ and of $z_\psi(t)$:

$$z_\psi(t) = \sqrt{\frac{3(1 + w)}{w}} \frac{a(t)}{\cosh (\beta H_\ast t)} \sqrt{\Omega_\ast} - \frac{1}{\sqrt{w}} \frac{\sqrt{\Omega_\ast}}{1 - \Omega_\ast}, \quad z_\varphi(t) = -\frac{1}{\sqrt{w}} \frac{\sqrt{\Omega_\ast}}{1 - \Omega_\ast}. \quad (5.17)$$

Equations $(5.17)$ are illustrated in figure 1 where the non-monotonic behavior of the pump fields is evident. In figure 1, the scale is linear on both axes and the three curves correspond to different choices of the barotropic indices. Since the plots of figure 1 are presented in terms of $H_\ast t$, they hold for any value of $H_\ast$. In spite of this the value of $H_\ast$ must be assigned in the numerical integration of the evolution of the inhomogeneities (see the discussion reported hereunder) since $t$ controls, ultimately, the absolute normalization of the power spectra. As equation $(5.17)$ shows, the value of $\Omega_\ast$ determines the relative magnitude of $z_\varphi$ and $z_\psi$. The difference in the overall sign of $z_\varphi(t)$ and $z_\psi(t)$ comes from the sign of $\dot{\psi}$ which has been taken to be negative in equation $(5.8)$. A flip in the sign of $\dot{\psi}$ entails a flip in the sign of $z_\varphi(t)$.\n
5.3. Numerical integrations

The system of equations (3.12) and (3.13) will now be integrated in Fourier space and in the cosmic time parametrization which is preferable since the class of exact solutions reported in equations (5.7)–(5.10) has a simpler expression in cosmic time. Equations (3.12) and (3.13) can be written in the form of a plane autonomous system:

\[ \dot{q}_v = p_v, \]  
\[ \dot{q}_\chi = p_\chi, \]  
\[ \dot{p}_v = -H p_v + A_{v,v}(k,t)q_v + B_{v,\chi}(t)q_\chi + C_{v,\chi}(t)p_\chi, \]  
\[ \dot{p}_\chi = -H p_\chi + A_{\chi,\chi}(k,t)q_\chi + B_{\chi,v}(t)q_v + C_{\chi,v}(t)p_v. \]  

Equations (5.18)–(5.21) hold in Fourier space, i.e. \( q_v \equiv q_v(k,t), \) \( q_\chi \equiv q_\chi(k,t) \) and similarly for \( p_v \) and \( p_\chi. \) For notational convenience and to avoid the proliferation of subscripts the reference to the modulus of the wavenumber \( k \) has been omitted unless strictly necessary. The coefficients \( A_{v,v}(k,t), B_{v,\chi}(t) \) and \( C_{v,\chi}(t) \) are given, respectively, by

\[ A_{v,v}(k,t) = \frac{3w - 1}{2}H + \frac{(3w - 1)(3w + 1)}{4}H^2 - \frac{c_s^2 k^2}{a^2} + \frac{\rho + \rho_i}{4H^2}[(w - 1)\psi^2 - 2(3(w + 1)H^2 + 2H)], \]  
\[ B_{v,\chi}(t) = \frac{\sqrt{\rho + \rho_i}}{4H \sqrt{w}} \left[ \frac{\dot{\psi}}{H}[(w - 1)\psi^2 - 2(3(w + 1)H^2 + 2H)] - 2(w + 1)\frac{\partial V}{\partial \psi} + 2(1 - w)H\dot{\psi} \right], \]  
\[ C_{v,\chi}(t) = -\frac{\sqrt{\rho + \rho_i}}{2H \sqrt{w}}(1 - w)\dot{\psi}. \]
The coefficients $\overline{A}_{v}(k, t)$, $\overline{B}_{v}(t)$ and $\overline{C}_{v}(t)$ are instead given by
\begin{align}
\overline{A}_{v}(k, t) &= -\frac{\partial^2 V}{\partial \varphi^2} + (\dot{H} + 2H^2) - k^2/a^2 \\
&\quad - \frac{\dot{\varphi}}{4H} \left[ \frac{\partial V}{\partial \varphi} + \left( \frac{\dot{\varphi}}{H} \right) \left( 4(\dot{H} + 3H^2) + (p_{\text{pr}} + \rho_{\text{pr}}) \left( \frac{w - 1}{w} \right) \right) \right] \quad (5.25) \\
\overline{B}_{v}(t) &= -\frac{\sqrt{p_{\text{pr}} + \rho_{\text{pr}}}}{4H} \left[ \frac{(w - 1)(3w - 1)}{w} \varphi H + 4 \frac{\partial V}{\partial \varphi} \right. \\
&\quad + \left. \left( \frac{\dot{\varphi}}{H} \right) \left[ 4(\dot{H} + 3H^2) + (p_{\text{pr}} + \rho_{\text{pr}}) \left( \frac{w - 1}{w} \right) \right] \right] \sqrt{w} \quad (5.26) \\
\overline{C}_{v}(t) &= \frac{\sqrt{p_{\text{pr}} + \rho_{\text{pr}}}}{2H} \left( \frac{w - 1}{\sqrt{w}} \right) \dot{\varphi}. \quad (5.27)
\end{align}

Equations (5.22)–(5.24) and equations (5.25)–(5.27) are related to equations (3.14)–(3.16) and (3.17)–(3.19) modulo an overall sign difference and the obvious algebraic changes related to the differences between the conformal and the cosmic time parametrization. Notice, furthermore, that the coefficients $A_{v}(k, t)$ and $\overline{A}_{v}(k, t)$ depend both on the cosmic time coordinate and on the wavenumber which comes from the three-dimensional Laplacian of $\varphi$ and $q_{v}$.

The initial conditions for the numerical integration of equations (5.18)–(5.21) are fixed by requiring that, at the initial integration time, $q_{v} = p_{v} = 0$ while $q_{v}$ and $p_{v}$ are determined from the thermal phonons $\overline{\eta}_{v}$ during a radiation-dominated regime (i.e. $w = 1/3$). The considerations of section 3 and the exact solutions presented in this section allow for different values of the barotropic indices and for a wider set of initial conditions. In spite of this it seems less confusing, for the purposes of the presentation, not to indulge in an excessive multiplication of examples which could cast shadow over the main motivation of the present analysis.

Since the initial conditions are given during the protoinflationary phase (i.e. $H_{v}t \ll 1$), nearly all the relevant modes are larger than the particle horizon since $k/a(t_{i}) \ll H(t_{i})$. This aspect can be understood by looking at the sharp increase of $H(t)$ in the limit $t \to 0$ where the Hubble rate dominates over $1/a(t)$ in spite of the possible largeness of the wavenumber. The full, dashed and dot–dashed curves appearing in the plots of figures 2 and 3 correspond to different values of the wavenumbers $k$ which are expressed in units of $H_{v}$ in terms of the rescaled wavenumber $\kappa = k/H_{v}$ (see also the legends of each plot). The values of $H_{v}$ used in each numerical integration are reported, in natural Planckian units $\overline{M}_{P} = 1$, in the title of each plot illustrated in figures 2 and 3.

As explained in sections 3 and 4, once the spectra of $q_{v}$ and $q_{v}$, are known, all the other quantities can be computed such as the spectra of $B$ and $\varphi$ or the spectra of $\Psi$ and $\mathcal{R}$. On both axes of figures 2 and 3 the common logarithm of the corresponding quantity is reported. Instead of illustrating the power spectrum it is more practical to compute the square root of it, as indicated on the vertical axes of figures 2 and 3. During the protoinflationary phase (i.e. roughly $H_{v}t < 1$) the spectrum goes as $\kappa^{1/2}$ as it can be deduced from the different curves holding for decreasing values of $\kappa$. Recall that in figures 2 and 3 the common logarithm of the corresponding quantity is reported on both axes. The slope $\kappa^{1/2}$ arises since, initially, $k/(2T_{a}) \ll 1$ and therefore $\mathcal{P}^{1/2}_{q_{v}} \propto k\sqrt{T_{a}/k}$. As $H_{v}t \gg 1$ the spectrum tends to change even

\[5\] Note that in equations (3.12) and (3.13) the coefficients all appear at the left-hand side in the corresponding equations. In equations (5.20) and (5.21) the coefficients appear at the right-hand side.
if this aspect is more visible by looking directly at the spectrum of $\mathcal{R}$ reported in figure 3. From figure 3 the spectrum of curvature perturbations exhibits a sudden growth which can even be understood on a qualitative basis. Recall, in this respect, the general expression of $\mathcal{R}(t)$ in terms of $q_\chi$ and $q_\nu$ (see equation (3.21)). When the spectrum of $q_\chi$ and $q_\nu$ is bounded (see figure 2) $P_R$ increases when either $z_{\text{pe}}(t)$ or $z_{\text{p}}(t)$ decrease. The slope of the spectrum goes as $\kappa^{1/2}$ for $H_* t < 1$ while it is quasi-flat for $H_* t > 1$. In the left plot of figure 3 the parameters $\Omega_\star$ and $H_*$ coincide with those adopted in figure 2; in the right plot of figure 3 the values of the fiducial parameters are instead $\Omega_\star = 10^{-4}$ and $H_* = 10^{-8}$. The examples discussed here are just meant to show that the lack of monotonicity of $z_{\text{pe}}(t)$ and $z_{\text{p}}(t)$ always entails a potential decrease of the pump fields and, hence, a potential increase of the total curvature perturbations. It is interesting to point out that the possibility of anomalous growth of curvature perturbations in single-field inflationary models has been discussed some time
ago but within a different perspective [27] and not in connection with the protoinflationary dynamics.

The analysis of this section has been conducted in the case of a single protoinflationary fluid and a single inflaton; there seems to be however no obstruction for the possible generalization of this approach to a finite number of perfect fluids simultaneously present together with a finite number of inflaton fields. Finally, the considerations of this paper just assumed the validity of the perturbative expansion prior to the onset of the inflationary evolution. The overall consistency of the approach, as already stressed in section 4, implies that valid conclusions can only be drawn in the regions of the parameter space where the perturbative expansion is valid. At the same time there are techniques, like the gradient expansion, which can be applied in situations where the conventional perturbative expansion fails [28–30]. This analysis is anyway beyond the scopes of this paper.

6. Concluding remarks

The effects of a dynamical phase preceding inflation are customarily parametrized in terms of an initial number of inflaton quanta modifying the standard vacuum initial conditions for the evolution of the large-scale inhomogeneities. In this paper the implications of a protoinflationary phase of decelerated expansion have been examined with the purpose of relaxing the standard lore. The present approach stipulates that, initially, the energy–momentum tensor is dominated by an irrotational fluid. After a transient regime (which may be either sharp or delayed) the inflaton potential dominates the total energy–momentum tensor and the slow-roll dynamics starts off. The initial conditions for the large-scale inhomogeneities during the protoinflationary phase can be either quantum mechanical or, more realistically, thermal if the protoinflationary phase is dominated by radiation.

Is the existence of a protoinflationary phase just equivalent to a modified set of initial conditions manually imposed on the modes of the perturbations? The results of the present analysis show that the answer this question is positive insofar as the protoinflationary transition is sufficiently sharp and satisfies a number of monotonicity requirements. Across the protoinflationary boundary it is plausible to demand that the extrinsic curvature is continuous while the stress tensor undergoes a finite discontinuity on the constant energy density hypersurface. In this picture the transition is sharp and both the density contrast and the metric fluctuations are suppressed across the transition. This sudden approximation merely extends to the protoinflationary boundary the standard discussion of the various post-inflationary transitions.

The dynamics of the inhomogeneities is determined by the evolution of the two quasi-normal modes of the system. Depending on the time evolution of the pump fields controlling the contribution of the quasi-normal modes to the curvature perturbations, the protoinflationary transition is naturally classified in two broad classes. In the first category the pump fields are monotonic and the quasi-normal modes of the system are nearly not interacting. The second category contemplates the complementary situation where the transition is not monotonic and the pump fields have, at least, either a maximum or a minimum. In the latter case large-scale curvature perturbations can be enhanced in comparison with the results obtainable in the case of monotonic transitions.

The large-scale modifications of the temperature and polarization power spectra are often engineered by assuming that the onset of inflationary dynamics is only characterized by a single inflaton field possibly with non-standard initial conditions for its inhomogeneities. The exact solutions obtained in this paper, as well and the numerical treatment of the corresponding large-scale inhomogeneities, suggest that the protoinflationary dynamics is
richer than expected. In a more conservative perspective the obtained results show that the potential enhancement of large-scale curvature perturbations can be avoided by demanding that the evolution of the pump fields is strictly monotonic. It is however unclear whether or not the monotonicity of the pump fields must be considered a generic feature of the protoinflationary dynamics.

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Appendix. Explicit derivation of the quasi-normal modes

In this appendix the details of the derivations leading to equations (3.12) and (3.13) will be outlined. By taking the first time derivative of equation (2.33) the following equation arises:

\[ u'' + \mathcal{H}' (1 - 3w)u + \mathcal{H}(1 - 3w)u' = \left[ \frac{w}{w + 1} \delta' + \phi' \right], \tag{A.1} \]

where, as already mentioned in section 3, the barotropic index \( w \) has been taken to be constant so that \( c_s = \sqrt{w} \); note that \( \delta = \delta_{\rho_{\text{pr}}}/\rho_{\text{pr}} \) denotes, only in this appendix, the density contrast of the protoinflationary fluid. Now the logic is to express the right-hand side of equation (A.1) solely in terms of \( u \) and \( \chi \). The same procedure must then be applied in the case of equation (2.31).

The right-hand side of equation (A.1) is expressible in terms of \( u \), \( \chi \) and \( \chi' \). To achieve this goal, equations (2.22) and (2.32) can be summed up term by term. For a perfect barotropic fluid equation (2.32) is given by \( \delta' = -(1 + w)\theta_{\text{pr}} \); by then using the expression for \( \phi' \) obtainable from equation (2.29) it is easy to show, after some algebra, that

\[ \phi' + \frac{w}{w + 1} \delta' = \left[ w \nabla^2 u \right. + \left\{ \frac{4\pi G \rho_{\text{pr}} \phi'^2}{\mathcal{H}} - \frac{3H}{H} \right\} - \frac{\mathcal{H}' + 4\pi G \rho_{\text{pr}} \phi'^2}{\mathcal{H}} \right] \phi \]

\[ + \frac{4\pi G}{\mathcal{H}} \left[ (1 - w) \chi' \phi' - a^2 \frac{\partial V}{\partial \varphi} (1 + w) \chi \right]. \tag{A.2} \]

Equation (3.11) will then be used into equation (A.2) to eliminate \( \phi \) and, therefore, the final form of equation (A.1) is

\[ u'' + \mathcal{H}' (1 - 3w)u' - w \nabla^2 u + \left\{ (1 - 3w)\mathcal{H}' - \frac{4\pi G \rho_{\text{pr}} (p_{\text{pr}} + \rho_{\text{pr}})}{\mathcal{H}^2} \right\} u \]

\[ \times \left[ \frac{4\pi G \rho_{\text{pr}} \phi'^2}{\mathcal{H}} (w - 1) - \left( \mathcal{H}' (3w + 1) + 2H' \right) \right] \left[ \mathcal{H}^2 (3w + 1) + 2H' \right] \]

\[ - \frac{4\pi G}{\mathcal{H}} \left\{ \frac{\chi'}{H} \left[ 4\pi G \rho_{\text{pr}} \phi'^2 (w - 1) - \left( \mathcal{H}^2 (3w + 1) + 2H' \right) \right] - a^2 \frac{\partial V}{\partial \varphi} (w + 1) \right\} \chi \]

\[ + \frac{4\pi G \rho_{\text{pr}} \phi'}{\mathcal{H}} (1 - w) \chi' = 0. \tag{A.3} \]

The same strategy used to derive equation (A.3) leads to the evolution equation of the second quasinormal mode associated with equation (2.31). Here the problem is to trade the last three terms appearing at the right-hand side of equation (2.31) for appropriate combinations of \( \chi \), \( u \) and \( u' \). The relevant combination arising from the terms containing \( \phi \), \( \nabla^2 B \) and \( \phi' \) can be
The derivation of equations (A.3) and (A.4) breaks down when $c_s^2 = \sqrt{\omega} = 0$ so that the latter case must be separately treated. The starting point of the derivation will be the same except that, since $\delta \rho_{pe} = 0$, equation (2.33) is even simpler. Repeating all the steps of the derivation we obtain, after some algebra, that the evolution equation for $u$ becomes

$$u'' + \left( H - \frac{4\pi G \rho_{pe}^2}{H} \right) u' + \left[ \frac{4\pi G a^2 \rho}{H} \left( H + \frac{\mathcal{H}'}{H} \right) \right] u + \frac{4\pi G}{H} \left( \frac{\psi'}{H} \right) \left[ 2(\mathcal{H}' + 2H^2) + 4\pi Ga^2 \left( \rho_{pe} + \rho_{pe} \right) \left( \frac{w - 1}{w} \right) \right] = 0.$$  \hspace{1cm} (A.5)

Similarly, in the case $c_s^2 = \sqrt{\omega} = 0$, the evolution equation for $\chi$ becomes, after some algebra:

$$\chi'' + 2H \chi' - \nabla^2 \chi + \left\{ \frac{\partial^2 V}{\partial \phi^2} a^2 + 8\pi G \left( \frac{\psi'}{H} \right) \left[ 2 \frac{\partial V}{\partial \phi} a^2 + \left( \frac{\psi'}{H} \right) \left( 2 + \frac{\mathcal{H}'}{H^2} \right) \right] \right\} \chi + \frac{8\pi Ga^2 \rho}{H} \left[ \frac{\partial V}{\partial \phi} a^2 + \left( \frac{\psi'}{H} \right) \left( 2 + \frac{\mathcal{H}'}{H^2} \right) \right] u = 0.$$  \hspace{1cm} (A.6)

Going back to the main derivation and to equations (A.3) and (A.4), the expression for the evolution of $u$ can be modified by defining a new variable $v$ such that

$$\phi = \frac{\psi'}{2H} \chi + \frac{a\sqrt{p + \rho}_{p}}{2H} v, \quad v = a \sqrt{p_{pe} + \rho_{pe}} u.$$  \hspace{1cm} (A.7)

In equation (A.7) Planck units $\bar{M}_p^2 = 1$ have been set and, in these units, the equations for $v$ and $\chi$ become, respectively,

$$v'' + 2Hv' - w \nabla^2 v + \left\{ \frac{3(1 - w)}{2} H' + \frac{3(1 - w)}{4} H^2 \right\} v + \frac{a^2 (p_{pe} + \rho_{pe})}{4H^2} \left[ \psi'^2 (w - 1) - 2(\mathcal{H}^2 (3w + 1) + 2H') \right] v - \frac{a\sqrt{p_{pe} + \rho_{pe}}}{4H} \left[ \left( \frac{\psi'}{H} \right) \psi'^2 (w - 1) - 2(\mathcal{H}^2 (3w + 1) + 2H') \right]$$

$$- 2a^2 (w + 1) \frac{\partial V}{\partial \phi} \chi + \frac{a\sqrt{p + \rho}}{2H} \psi' (1 - w) \chi' = 0.$$  \hspace{1cm} (A.8)

and
\[
\chi'' + 2\mathcal{H}\chi' - \nabla^2\chi + \left\{ a^2 \frac{\partial^2 V}{\partial \phi^2} + \frac{\psi'}{\mathcal{H}} \left( 8a^2 \frac{\partial V}{\partial \phi} \right) \right\} \chi
+ \frac{\psi'}{\mathcal{H}} \left( 4(\mathcal{H}' + 2\mathcal{H}^2) + a^2 (\rho_{pr} + \rho_{\Psi}) \left( \frac{w - 1}{w} \right) \right) \}
+ \frac{a\sqrt{p_{pr} + \rho_{pr}}}{4\mathcal{H}} \left( \frac{w(1 - w)}{w} \right) v' - \frac{a\sqrt{p_{pr} + \rho_{pr}}}{2\mathcal{H}} \psi' \left( \frac{w - 1}{w} \right) v' = 0.
\]

From equations (A.8) and (A.9) the evolution equations for the canonical normal modes

\[ q_v = \frac{a v}{c_s}, \quad q_{\chi} = a \chi \]

leads to equations (3.12) and (3.13). The variable \( v \) is the closest analog, in the fluid case, to the inflaton fluctuation \( \chi \). It is useful to recall, as a technical remark, that the intermediate steps involving the variable \( v \) can be avoided. In the latter case \( q_v \) is directly expressible in terms of the variable \( u \) as \( q_v = z_{pr} \mathcal{H} u \), with this strategy the algebraic derivation is, though, more tedious.
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