The mass insertion approximation without squark degeneracy

Guy Raz

Particle Physics Department
Weizmann Institute of science
Rehovot 76100, Israel

We study the applicability of the mass insertion approximation (MIA) for calculations of neutral meson mixing when squark masses are not degenerate and, in particular, in models of alignment. We show that the MIA can give results that are much better than an order of magnitude estimate as long as the masses are not strongly hierarchical. We argue that, in an effective two-squark framework, $\tilde{m}_q = (\tilde{m}_1 + \tilde{m}_1)/2$ is the best choice for the MIA expansion point, rather than, for example, $\tilde{m}_q^2 = (\tilde{m}_1^2 + \tilde{m}_2^2)/2$.

I. INTRODUCTION

The mass insertion approximation (MIA) is often used to simplify expressions involving supersymmetric contributions to flavour changing neutral current processes from loop diagrams [1, 2, 3]. The simplification is achieved by the replacement of a sum over all possible internal propagators and the appropriate mixing at the vertices, with a single (small) off-diagonal mass insertion in a basis where all gauge couplings are diagonal. The resulting expressions are formulated in terms of parameters which can be estimated in various supersymmetric models.

It may seem, naively, that the smallness of the off-diagonal mass-squared matrix element would justify the approximation. The true picture, however, is that these off-diagonal elements are the product of mixing angles at the vertices and mass-squared differences between intermediate squarks. The MIA, on the other hand, is a Taylor expansion only with respect to the latter, namely, the mass-squared difference (we give an exact formulation of these statements in section II). A small off-diagonal element does not necessarily imply a small
mass difference. Instead, it may be related to small mixing angles. But then the validity of the MIA is questionable.

This is exactly the situation in the framework of quark-squark alignment (QSA) models \[4, 5\]. In this class of supersymmetric models the squark masses-squared are all of the same order of magnitude, a free parameter denoted by $\widetilde{m}^2$, and the mass squared differences between them are also of the same order of magnitude, that is: $|\widetilde{m}_i^2 - \widetilde{m}_j^2|/\widetilde{m}^2 = O(1)$. Apriori, this is a problematic situation for using the MIA. Yet, it is frequently used in the literature.\(^1\)

We therefore study the validity of the MIA in the context of QSA models. We confirm that the approximation is applicable and useful for such models. We also clarify the connection between the (unknown) details of the squark mass spectrum and the MIA parameters.

The organization of this work is as follows: We formulate the details of the MIA in section II. The analysis and our results for non-degenerate squarks masses are presented in section III.

II. FORMULATION OF THE MASS INSERTION APPROXIMATION.

Let us first formulate the details of the MIA. To illustrate it in the context of QSA, we study a specific example: The supersymmetric contribution to neutral $K$ meson mixing from gluino box diagrams with two intermediate squarks. The relevant diagrams are shown in figure [4]. We focus on the following term arising from these diagrams [4]:

$$
M_{12}^K \supset C \left( Z_{2i}^d Z_{i1}^d Z_{2j}^d Z_{j1}^d \right) J_4(\widetilde{m}_g^2, \widetilde{m}_i^2, \widetilde{m}_j^2). \tag{1}
$$

Here $C$ is a numerical factor, given in terms of the $K$ meson parameters:

$$
C \equiv \left( \frac{4\pi}{i} \right)^2 \frac{\alpha_s^2 m_K f_K^2 \hat{B}_K \eta}{2} \sim \left( \frac{4\pi}{i} \right)^2 5.4 \times 10^4 \text{ MeV}^3, \tag{2}
$$

$Z_{ij}^d$ are the quark-squark mixing angles, and $J_4$ is given by

$$
J_4(\widetilde{m}_g^2, \widetilde{m}_i^2, \widetilde{m}_j^2) \equiv \frac{11}{54} I_4(\widetilde{m}_g^2, \widetilde{m}_i^2, \widetilde{m}_j^2) + \frac{2}{27} \widetilde{m}_g^2 I_4(\widetilde{m}_g^2, \widetilde{m}_i^2, \widetilde{m}_j^2), \tag{3}
$$

\(^1\) In fact, since QSA models allow estimates of $|\widetilde{m}_i^2 - \widetilde{m}_j^2|/\widetilde{m}^2$ but not of the individual masses, the MIA provides the best way to derive meaningful results.
with $\tilde{m}_g$ the gluino mass, $\tilde{m}_i$, $\tilde{m}_j$ the down squark masses and

$$I_4(\tilde{m}_g^2, \tilde{m}_i^2, \tilde{m}_j^2) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - \tilde{m}_g^2)(p^2 - \tilde{m}_i^2)(p^2 - \tilde{m}_j^2)}$$

$$= \frac{i}{(4\pi)^2} \left[ \frac{1}{(\tilde{m}_i^2 - \tilde{m}_g^2) (\tilde{m}_j^2 - \tilde{m}_g^2)} + \frac{\tilde{m}_i^2}{(\tilde{m}_i^2 - \tilde{m}_j^2) (\tilde{m}_i^2 - \tilde{m}_g^2)^2} \ln \left( \frac{\tilde{m}_i^2}{\tilde{m}_g^2} \right) \right] \left( \frac{\tilde{m}_j^4}{(\tilde{m}_j^2 - \tilde{m}_g^2) (\tilde{m}_j^2 - \tilde{m}_i^2)^2} \ln \left( \frac{\tilde{m}_j^2}{\tilde{m}_i^2} \right) \right),$$

$$\tilde{I}_4(\tilde{m}_g^2, \tilde{m}_i^2, \tilde{m}_j^2) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{(p^2 - \tilde{m}_g^2)(p^2 - \tilde{m}_i^2)(p^2 - \tilde{m}_j^2)}$$

$$= \frac{i}{(4\pi)^2} \left[ \frac{\tilde{m}_g^2}{(\tilde{m}_i^2 - \tilde{m}_g^2) (\tilde{m}_j^2 - \tilde{m}_g^2)} + \frac{\tilde{m}_i^4}{(\tilde{m}_i^2 - \tilde{m}_j^2) (\tilde{m}_i^2 - \tilde{m}_g^2)^2} \ln \left( \frac{\tilde{m}_i^2}{\tilde{m}_g^2} \right) \right] \left( \frac{\tilde{m}_j^4}{(\tilde{m}_j^2 - \tilde{m}_g^2) (\tilde{m}_j^2 - \tilde{m}_i^2)^2} \ln \left( \frac{\tilde{m}_j^2}{\tilde{m}_i^2} \right) \right).$$

The MIA is nothing more than a Taylor expansion of $J_4$. We choose to expand the squark masses around some point $\tilde{m}_q$. Owing to the specific form of $J_4$ we find

$$J_4(\tilde{m}_g^2, \tilde{m}_i^2, \tilde{m}_j^2) = \sum_{m,n=0}^{\infty} \frac{C_{m+n}(x)}{\tilde{m}_q^{2(1+m+n)}} (\Delta \tilde{m}_i^2)^m (\Delta \tilde{m}_j^2)^n.$$

FIG. 1: Gluino box diagrams contributing to $K^0 - \bar{K}^0$ mixing.
where \( x \equiv \tilde{m}_g^2/\tilde{m}_q^2 \), and \( \Delta \tilde{m}_i^2 \equiv \tilde{m}_i^2 - \tilde{m}_q^2 \). The coefficient \( C_{m+n}(x)/\tilde{m}_q^{2(1+m+n)} \) is the \((m+n)\)th derivative of \( J_4 \) evaluated at \( \tilde{m}_q \), times the symmetry factor \( 1/(m!n!) \). (The exact form of the coefficient is not important for our purpose.)

Substituting (6) in (1) we get

\[
C \sum_{m,n=0}^{\infty} \left\{ \frac{C_{m+n}(x)}{\tilde{m}_q^{2(1+m+n)}} \left[ Z_{2i}^d (\Delta \tilde{m}_i^2)^m Z_{i1}^{d\dagger} \right] \left[ Z_{2j}^d (\Delta \tilde{m}_j^2)^n Z_{j1}^{d\dagger} \right] \right\} .
\]  

(7)

Note that a sum over \( i \) and \( j \) is implied. If the MIA is valid, this expansion converges fast and we can keep only the lowest order terms. However, due to the unitarity of \( Z^d \), the terms with either \( m = 0 \) or \( n = 0 \) vanish. The first non-vanishing contribution, therefore, will be from the term with \( m = n = 1 \). This term, however, is special since we can write (again, a sum over \( j \) is implied):

\[
Z_{2j}^d (\Delta \tilde{m}_j^2) Z_{j1}^{d\dagger} = Z_{2j}^d (\tilde{m}_q^2 + \Delta \tilde{m}_j^2) Z_{j1}^{d\dagger} = (\tilde{M}^2_d)_{21} ,
\]  

(8)

where \( \tilde{M}^2_d \) is the squark mass-squared matrix in the basis where quarks masses and gluino couplings are diagonal. Thus, if the MIA holds we can replace (6) with:

\[
C \times \frac{C_2(x)}{\tilde{m}_q^6} \left( (\tilde{M}^2_d)_{21} \right)^2 .
\]  

(9)

We stress that a small \( (\tilde{M}^2_d)_{21} \) is not enough, by itself, to justify the use of the MIA. The question of validity should be considered in the context of eq. (6). We note, however, that the quality of the MIA is not completely equivalent to the quality of the approximation that is obtained by keeping only the lowest terms in (6). The reason is that the zeroth order term \((m = n = 0)\), the first order terms \((m = 1, n = 0 \text{ and } m = 0, n = 1)\) and some of the second order terms \((m = 2, n = 0 \text{ and } m = 0, n = 2)\), while appearing in (6), do not contribute to the mixing amplitude in (7) due to the unitarity of \( Z^d \). In other words, the validity of (7) has to do with one of the second order terms in (6), rather than with all lowest order terms.

Nonetheless, it is obvious that when the expansion parameter is small, \( |\Delta \tilde{m}_i^2/\tilde{m}_q^2| \ll 1 \), the approximation is good. It is the condition in QSA models, \( |\Delta \tilde{m}_i^2/\tilde{m}_q^2| \sim 1 \) which needs a special consideration.

### III. THE CASE OF NON-DEGENERATE MASSES

In order to study quantitatively the non-degenerate case, we simplify the form of \( Z^d \) by assuming that only one mixing angle, namely \( Z^d_{12} \), is large. This is usually the case
in QSA models. Such an assumption allows us to consider only the first two generations. Expression (11) then simplifies to

$$C \times \cos^2 \theta \sin^2 \theta \times \left[ J_4(\tilde{m}_q^2, \tilde{m}_1^2, \tilde{m}_2^2) + J_4(\tilde{m}_q^2, \tilde{m}_2^2, \tilde{m}_1^2) - J_4(\tilde{m}_q^2, \tilde{m}_1^2, \tilde{m}_2^2) - J_4(\tilde{m}_q^2, \tilde{m}_2^2, \tilde{m}_1^2) \right],$$

(10)

where $\sin \theta \approx Z_{12}^d$ is the single large mixing angle. Equivalently, (8) can be written as

$$C \times \cos^2 \theta \sin^2 \theta \times \left( \frac{C_{m+n}(x)}{m_q^2(1+m+n)} \right) \times \left( (\Delta \tilde{m}_1^2)^m - (\Delta \tilde{m}_2^2)^m \right) \times \left( (\Delta \tilde{m}_1^2)^n - (\Delta \tilde{m}_2^2)^n \right).$$

(11)

The above expression manifestly demonstrates the vanishing of terms with either $m = 0$ or $n = 0$. The MIA is obtained by keeping in (11) only the term with $m = n = 1$

$$C \times \cos^2 \theta \sin^2 \theta \times \left( \frac{C_2(\tilde{m}_q^2/\tilde{m}_q^2)}{\tilde{m}_q^2} \right) \times \left( \frac{\tilde{m}_1^2 - \tilde{m}_2^2}{\tilde{m}_q^2} \right)^2.$$

(12)

We can now test the accuracy of the MIA by defining the deviation parameter (using the symmetry of $J_4$ with respect to $\tilde{m}_i^2$ and $\tilde{m}_j^2$)

$$r \equiv 1 - \frac{\left( \frac{C_2(\tilde{m}_q^2/\tilde{m}_q^2)}{\tilde{m}_q^2} \right) \left( \frac{\tilde{m}_1^2 - \tilde{m}_2^2}{\tilde{m}_q^2} \right)^2}{J_4(\tilde{m}_q^2, \tilde{m}_1^2, \tilde{m}_2^2) + J_4(\tilde{m}_q^2, \tilde{m}_2^2, \tilde{m}_1^2) - 2J_4(\tilde{m}_q^2, \tilde{m}_1^2, \tilde{m}_2^2)}.$$

(13)

Figure 2 shows $r$ as a function of $\tilde{m}_1$ and $\tilde{m}_2$ for $\tilde{m}_q = 2$ TeV, $\tilde{m}_q = 1$ TeV.

Obviously, when $\tilde{m}_1 \approx \tilde{m}_2 \approx \tilde{m}_q$, we expect the deviation to be small. The interesting result, however, is that the deviation is small (of order 10%) even for non-degenerate $\tilde{m}_1$ and $\tilde{m}_2$, as long as $(\tilde{m}_1 + \tilde{m}_2)/2 \approx \tilde{m}_q$. This is the case, for example, when $\tilde{m}_1 = 1$ TeV, $\tilde{m}_2 = 3$ TeV and $\tilde{m}_q = 2$ TeV. In other words, the result of the exact expression (11) using $\tilde{m}_1$ and $\tilde{m}_2$, can be reproduced using the MIA of (12) with $\tilde{m}_q \approx (\tilde{m}_1 + \tilde{m}_2)/2$ and the knowledge of $\tilde{M}_{12} \sim \cos \theta (\tilde{m}_1^2 - \tilde{m}_2^2)/\tilde{m}_q^2$.

We see, therefore, that as long as the squark masses are not strongly hierarchical, the ‘right’ choice of the expansion point $\tilde{m}_q$ results with the MIA being a rather accurate approximation. This is very useful in QSA models where we have $(\tilde{m}_1^2 - \tilde{m}_2^2)/\tilde{m}_q^2 = O(1)$. The amplitudes in these models can therefore be expressed in the MIA by using $\tilde{m}_q \approx (\tilde{m}_1 + \tilde{m}_2)/2 \approx \tilde{m}$. Although this result was demonstrated here for a specific mixing contribution and using specific assumptions on the mixing angles, we found it to be quite a general result.\footnote{Moreover, the example given here directly applies to the most stringent test of QSA models, namely, the contribution to $D^0 - \overline{D^0}$ mixing.}
The result \( \tilde{m}_q \approx (\tilde{m}_1 + \tilde{m}_2)/2 \) is not trivial. Looking at (11), it seems that the best strategy is to choose \( \tilde{m}_q^2 = (\tilde{m}_1^2 + \tilde{m}_2^2)/2 \). This choice results with \( \Delta \tilde{m}_1^2 = -\Delta \tilde{m}_2^2 \), which eliminates all terms with either \( m \) or \( n \) even.\(^3\) In particular, it eliminates the next-to-leading order terms with either \( m = 1, n = 2 \) or \( m = 2, n = 1 \). Naively, one would expect faster convergence in this case and therefore obtaining a good approximation.

This naive expectation is, however, wrong. Studying the form of the coefficients \( C_{m+n} \) we find that the sign flips between even and odd \( m + n \) terms. The choice \( \tilde{m}_q^2 = (\tilde{m}_1^2 + \tilde{m}_2^2)/2 \), which eliminates all odd \( m+n \) terms, eliminates therefore all the negative sign terms in (11). Since the \( C_{m+n} \) coefficients decrease slowly, this induces a larger error and a worse approximation.

On the other hand, we find that the choice \( \tilde{m}_q \approx (\tilde{m}_1 + \tilde{m}_2)/2 \) is the most sensible one since it leads to an approximate cancellation between the next-to-leading order and the next-to-next-to-leading order terms. The extent to which such a choice is optimal, as can be seen in figure 2, is remarkable.

Although we presented here explicitly only the LL and RR contributions to the mixing

\(^3\) In other words, all terms with \( m+n \) odd are eliminated, since \( m+n \) odd implies either \( m \) even and \( n \) odd or vice versa. Some of the \( m+n \) even terms which are due to both \( m \) and \( n \) even are eliminated as well.
amplitude, the approximation also holds (over a somewhat smaller range of masses) for the LR and RL contributions.

To summarize, we confirm that the common practice of using the MIA in supersymmetric models is justified even in the case of non-degenerate squark masses, as long as the masses are not strongly hierarchical. We showed that over a wide range of masses, the best strategy is to choose the MIA expansion point $\tilde{m}_q$ to be the average of the masses involved. Using this strategy, the MIA provides a surprisingly good calculation of neutral meson mixing in models where a single mixing angle dominates, such as QSA [4, 5] and ‘effective supersymmetry’ [6, 8, 10, 11].

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