Optimal No-Regret Learning in General Games: Bounded Regret with Unbounded Step-Sizes via Clairvoyant MWU

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Abstract

In this paper we solve the problem of no-regret learning in general games. Specifically, we provide a simple and practical algorithm that achieves constant regret with fixed step-sizes. The cumulative regret of our algorithm provably decreases linearly as the step-size increases. Our findings depart from the prevailing paradigm that vanishing step-sizes are a prerequisite for low regret as championed by all state-of-the-art methods to date [8, 29, 33, 9, 10, 1, 11].

We shift away from this paradigm by defining a novel algorithm that we call Clairvoyant Multiplicative Weights Updates (CMWU). CMWU is Multiplicative Weights Updates (MWU) equipped with a mental model (jointly shared across all agents) about the state of the system in its next period. Each agent records its mixed strategy, i.e., its belief about what it expects to play in the next period, in this shared mental model which is internally updated using MWU without any changes to the real-world behavior up until it equilibrates, thus marking its consistency with the next day’s real-world outcome. It is then and only then that agents take action in the real-world, effectively doing so with the “full knowledge” of the state of the system on the next day, i.e., they are clairvoyant. CMWU effectively acts as MWU with one day look-ahead, achieving bounded regret. At a technical level, we establish that self-consistent mental models exist for any choice of step-sizes and provide bounds on the step-size under which their uniqueness and linear-time computation are guaranteed via contraction mapping arguments. Our arguments extend well beyond normal-form games with little effort.

Figure 1: “Vanilla” MWU vs. SOTA OMWU [10] vs. CMWU in a 4-player 10-strategy game. We plot the max over agents’ cumulative regret for several common random initializations. The shaded region represents the max/min regret range across runs. For a zoom-in on the cumulative regret of CMWU for larger step-sizes \( \eta \) see Fig. (2). For the asymptotic behavior of OMWU see Appendix C.
1 Introduction

Regret is one of the most standard performance measures in the online learning and learning in games literature [8, 29, 32, 39, 23, 21]. In this paper, we solve the problem of no-regret learning in general games. In doing so, we bring to conclusion a line of work that can be traced all the way back to the seminal papers of Blackwell and Hannan [5, 20]. Arguably, however, the most exciting aspect of our result is the manner in which it is achieved. We design an algorithm that achieves constant regret in general games with fixed step-sizes. In fact, the cumulative regret of our algorithm decreases linearly as the step-size increases, i.e., the larger the step-size the better. These findings depart from the conventional wisdom that vanishing step-sizes are a prerequisite for low regret. How is this possible?

At its core, our result relies on a fundamental but underappreciated difference between online learning and online learning in games: whereas online learning relates to the study of open loop systems (i.e., algorithms), learning in games relates to the study of closed loop systems (i.e., dynamical systems). In the former, absolutely nothing can be inferred about the future states of the system which in principle may evolve arbitrarily. In the latter, the future cannot evolve arbitrarily as it is a function of its current state, making it at least in principle more predictable.

This realization has recently been explored in a class of algorithms that are known as “Optimistic” [30, 12]. This class of algorithms makes the optimistic assumption that the state of the system tomorrow will be identical to today, resulting in a slight recency bias in the learning behavior. Optimistic Multiplicative Weights Update (OMWU) (also referred to as Optimistic Hedge), for example, is a variant of the ubiquitous MWU algorithm [2] with the sole difference that the payoff contributions of the last period are taken into account twice. Such algorithms and numerous variants thereof have enjoyed wide adoption in recent years, showing strong performance gains over their non-optimistic counterparts in various settings (e.g. [13, 12, 10, 27, 1, 19, 37, 38, 3, 18, 25, 16, 36, 11, 15, 14, 28]).

In a technical tour de force, [10] recently established that with appropriately chosen decreasing step-sizes the cumulative regret diverges at a rate of $O(\log^4(T))$, which improves significantly upon the previous state-of-the-art bound of $O(T^{1/4})$ [33].

Unfortunately, Optimistic algorithms suffer from an intrinsic design flaw by fusing together predictability and exploitability. Their assumption about the state of the system in the next period is only correct if there is no adaptation/exploitation on the part of the agents. Thus, a tension is introduced between predictability (which decreases with the step-size) and exploitability (which increases with the step-size), resulting in performance losses that are hard to keep under control.

We introduce a radically different philosophy in the design of learning algorithms. We shift away from the prevailing paradigm by defining a novel class of algorithms that we call Clairvoyant. It is instructive to focus on the case of Clairvoyant MWU (CMWU). CMWU is MWU equipped with a mental/simulated/synthetic model (jointly shared across all agents) about the state of the system in its next period. Each agent records its mixed strategy, i.e., its belief about what it expects to play in the next period in this shared mental model, which is internally updated using MWU without any changes to the real-world behavior up until it equilibrates, thus marking its consistency with the next day’s real-world outcome. It is then and only then that agents take action in the real-world, effectively doing so with the “full knowledge” of the state of the system on the next day, i.e., they are clairvoyant. CMWU acts as MWU with one day look-ahead, achieving bounded regret. At a technical level, we establish that self-consistent mental/simulated models exist for any choice of step-sizes, and provide bounds on the step-size under which their uniqueness and linear-time computation are guaranteed via contraction mapping arguments. Our arguments extend well beyond normal form games with little effort (e.g. time-evolving games, regularized games/learning, concave utilities with smooth dependencies on opponent play, a.o.).
Our results. In the case of the natural generation of (CMWU), Clairvoyant-Follow-the-Regularized-Leader (CFTRL) we show a constant, universal bound on the regret of each agent $i$ equal to $\max_x h_i(x) - \min_x h_i(x)$ for all games, all strictly convex regularizers $h_i$ and all step-sizes $\eta_i > 0$. In the case of CMWU, we show how to compute the fixed point in linear time as long as $\eta_i < \frac{2}{(N-1)mV}$ where $N$ the number of agents, $m$ the number of strategies and $V$ the maximum absolute payoff of the game. Combining these two results, we have shown that CMWU can efficiently be computed and achieve regret $\frac{(N-1)m^2 \ln(m)V}{2}$ in all games.

A note about bounded vs. unbounded regret. In economics or game theory, amortizing over performance losses/regret over time and reporting on the time average regret is arguably well justified. Relatively quickly, the per day performance losses vanish as pennies on the dollar, an acceptable compromise when lacking alternatives. In ML, payoffs are gradients, i.e., akin to monopoly money. We do not really care about our regret/payoffs per se, but about the performance of the system on the task at hand (e.g., image generation). We use regret/payoffs only as a guide to test how accurately our training algorithm tracks the incentive path laid out by the ML architecture. If the regret diverges to infinity, this means that the training algorithm drifts uncontrollably and catastrophically in the parameter space with the error/uncertainty growing possibly at a rate $\text{Regret} \propto \text{number of parameters}$. Thus, in ML we are primarily interested in cumulative regret and not time-average regret and in fact in keeping it as close to zero as possible.

A note about fixed vs. decreasing step-sizes. In ML, we naturally wish to move quickly in the parameter space of our models, exploring the space of available solutions as efficiently as possible. Decreasing the step-size over time is akin to wasting GPU cycles that should have been allocated more expeditiously. An analogous point applies as well to game theory and economics. It is not particularly clear why a decreasing rate/step-size is a natural behavioral assumption to make. Experimental work in the behavioral economics literature (e.g., the seminal works [6, 7] and numerous follow-ups) does not seem to justify it, finding that fixed and sometimes large intensity of choice/payoff intensity (i.e., step-size) fits data well (e.g., $\eta = 2.579$, Table VI [8]). The relative ease of producing time-average regret bounds under this modelling assumption, however, seems to have largely ossified it in the online learning in games literature. Experienced-Weighted-Attraction (EWA), the canonical learning model in behavioral game theory, which performs exponential discounting over past payoffs, is (in a slightly stripped-down form) MWU with Shannon entropy as the regularizer of the daily utilities [17]. Thus, we capture Clairvoyant EWA as a special case of the regularized Clairvoyant MWU, resulting once again in constant regret with fixed step-sizes.
2 Preliminaries & Model

2.1 Normal Form Games

We begin with basic definitions from game theory. A finite normal-form game \( \Gamma \equiv \Gamma(N,S,u) \) consists of a set of players \( N = \{1, ..., N\} \) where player \( i \) may select from a finite set of actions or pure strategies \( S_i \). Each player has a payoff function \( u_i : S \equiv \prod_i S_i \rightarrow \mathbb{R} \) assigning reward \( u_i(s) \) to player \( i \). It is common to describe \( u_i \) with a payoff tensor \( A^{(i)} \) where \( u_i(s) = A^{(i)}_{s} \). Let \( m = \max_i |S_i| \) and \( V = \max_{i,s} |A^{(i)}_s| \).

Players are also allowed to use mixed strategies \( x_i = (x_{is_i})_{s_i \in S_i} \in \Delta(S_i) \equiv \mathcal{X}_i \). The set of mixed strategy profiles is \( \mathcal{X} = \prod_i \mathcal{X}_i \). A strategy is fully mixed if \( x_{is_i} > 0 \) for all \( s_i \in S_i \) and \( i \in N \). Individuals compute the payoff of a mixed strategy linearly using expectation. Formally,

\[
  u_i(x) = \sum_{s \in S} u_i(s) \prod_{i \in N} x_{is_i}.
\]

We also introduce additional notation to express player payouts for brevity in our analysis later. Let \( v_{is_i}(x) = u_i(s_i; x_{-i}) \) denote the reward \( i \) receives if \( i \) opts to play pure strategy \( s_i \) when everyone else commits to their strategies described by \( x \). This results in \( u_i(x) = (v_i(x), x_i) \).

2.2 Online Learning Algorithms in Games

We study games from a learning perspective where agents iteratively update their mixed strategies over time based on the performance of pure strategies in prior iterations via an online adaptive algorithm. We will start by describing one of the most classical online learning algorithms, Multiplicative Weights Update (MWU) and its variant, Optimistic Multiplicative Weights Update (OMWU). The update rule for (MWU) can be written as

\[
  x_{is_i}^{t+1} = \frac{x_{is_i}^t \exp (\eta_i \cdot u_{is_i}(x^t))}{\sum_{\bar{s}_i \in S_i} x_{is_i}^t \exp (\eta_i \cdot u_{i\bar{s}_i}(x^t))}
\]

(MWU)

It is also well known that MWU can be written as a special case of Follow-the-Regularized-Leader (FTRL) algorithms with entropic regularizer. Specifically, given a strictly convex regularizer \( h_i : \mathcal{X}_i \rightarrow \mathbb{R} \), an agent updates their strategies via

\[
  y_i^t = y_i^{t-1} + v_i(x^t);
  x_i^{t+1} = \arg \max_{x_i \in \mathcal{X}_i} \left\{ \langle y_i^t, x_i \rangle - \frac{h_i(x_i)}{\eta_i} \right\}.
\]

(FTRL)

The parameter \( \eta_i > 0 \) is typically called the learning rate or step-size. The payoff vector \( y_i^t \) represents the cumulative payout for all pure strategies since the beginning of the time. Formally, \( y_{is_i}^t - y_{is_i}^0 \) denotes the cumulative payout agent \( i \) would have received had she played pure strategy \( s_i \) from iteration 1 to iteration \( t \), whereas \( y_{is_i}^0 \) allows to initialize the dynamic \( (x_i^t) \) at a place other than the global minimizer of \( h_i \). In other words, \( y_i^t = y_i^0 + \sum_{\tau=1}^{t} v_i(x^\tau) \). Thus, agent \( i \) selects the strategy \( x_i^{t+1} \) that maximizes the difference between an expected cumulative payout term and a strictly convex regularization term. The negative entropy regularizer \( h_i(x_i) = \sum_{s_i \in S_i} x_{is_i} \log x_{is_i} \) gives rise to MWU. Finally, let \( \eta = \max_{i \in N} \eta_i \).

\(^1(s_i; x_{-i}) \) denotes the strategy \( x \) after replacing \( x_i \) with \( s_i \).
The update rule for (OMWU), also referred to as Optimistic Hedge, can be written as
\[
x_{t+1}^i = \frac{x_t^i \exp (\eta_i \cdot \{2u_{is_i}(x^t) - u_{is_i}(x^{t-1})\})}{\sum_{s_i \in S_i} x_t^{is_i} \exp (\eta_i \cdot \{2u_{is_i}(x^t) - u_{is_i}(x^{t-1})\})}
\] (OMWU)

The update of (OMWU) modifies the MWU update by replacing the utility vector \(v_i(x^t)\) with an optimistic guess of the following iteration’s payoff vector, \(v_i'(x^t) = v_i(x^t) + (v_i(x^t) - v_i(x^{t-1}))\). Another way of interpreting (OMWU) is that it allows each agent to project in their mind all of their opponents upcoming behavior in the next time-step \(t + 1\), and all agents essentially use their last observed at time \(t\) as a predictor for their next behavior at time \(t + 1\). This follows from the intuition that if the opponent in the game is using a stable/regularized algorithm, then their behavior between the two iterations will not change much. (OMWU) makes the optimistic but ultimately incorrect assumption that all opponents’ behavior will not change at all between periods \(t\) and \(t + 1\).

2.3 Clairvoyant MWU and FTRL

In this section, we introduce a novel learning algorithm for games that we call Clairvoyant Multiplicative Weights Updates (CMWU). Critically, CMWU, unlike OMWU, forms self-confirming predictions/beliefs about what all opponents will play in the next time instance. Namely, all agents will form the same belief about what agent \(i\) will play in the next period \(t + 1\) (\(x_{t+1}^i\)). These beliefs/estimates are such that when agents simulate an extra period of play in their mind and update their current strategies using MWU, the resulting strategy for each agent \(i\) is \(x_{t+1}^i\). All agents accurately predict the behavior of all other agents tomorrow, in other words they are clairvoyant! The update rule for (CMWU) is as follows:
\[
x_{t+1}^{is_i} = \frac{x_t^{is_i} \exp (\eta_i \cdot u_{is_i}(x^{t+1}))}{\sum_{s_i \in S_i} x_t^{is_i} \exp (\eta_i \cdot u_{is_i}(x^{t+1}))}
\] (CMWU)

(CMWU) is an implicit method: the new strategy \(x^{t+1}\) appears on both sides of the equation, and thus the method needs to solve an algebraic/ fixed point equation for the unknown \(x^{t+1}\). As a result, the update rule of Clairvoyant MWU is not properly defined, in the sense that the fixed point equation of (CMWU) may admit multiple fixed-point solutions or even worse, admit no solution. In Section 2.4 we address all of these concerns. First, we prove that (CMWU) always admits a solution (Theorem 1). Then, we show that if each \(\eta_i\) is upper bounded by some game-dependent parameters, (CMWU) is a contraction map. The latter not only implies that there exists a unique fixed-point solution satisfying (CMWU) (and thus Clairvoyant MWU is well-defined), but also establishes the fact that \(x^{t+1}\) can be computed in linear-time.

Finally, we introduce Clairvoyant Follow-the-Regularized-Leader (CFTRL), the natural generalization of (CMWU) in the case of general strictly convex regularizers.
\[
y_t^i = y_{t-1}^i + v_i(x^{t+1}); \]
\[
x_{t+1}^i = \arg \max_{x_i \in X_i} \left\{ \langle y_t^i, x_i \rangle - \frac{h_i(x_i)}{\eta_i} \right\}
\] (CFTRL)

In the case of (CFTRL), we will establish constant universal upper bounds on regret for all regularizers and step-sizes. We leave the issue of their efficient implementation open for future work.
2.4 Existence of Fixed Point For Unbounded Step-Sizes

**Theorem 1.** The algebraic system of equations in (CMWU) defined by an arbitrary game \( \Gamma \), an arbitrary tuple of learning rates \( \eta_i \), and any state \( x^t \), always admits a solution.

**Proof.** We will start from the system of equations (CMWU) and denote the denominator as \( Z_i := \sum_{s_i \in S_i} x^t_{is_i} \exp(\eta_i \cdot u_{is_i}(x^{t+1})) \). By taking logarithms on both sides of the equation (CMWU) we have for all agents \( i \) and strategies \( s_i \in S_i \):

\[
\ln(x^{t+1}_{is_i}) - \ln(x^t_{is_i}) - \eta_i \cdot u_{is_i}(x^{t+1}) + \ln(Z_i) = 0 \quad \text{for all } i \in \mathcal{N}, s_i \in S_i. \tag{2}
\]

We fix for every agent an arbitrarily chosen benchmark strategy (e.g., for each agent \( i \) their first strategy, strategy 1, in their set of available strategies) and subtract that equation from all other equations of agent \( i \):

\[
\ln \left( \frac{x^{t+1}_{is_i}}{x^t_{is_i}} \right) - \ln \left( \frac{x^t_{is_i}}{x^t_{is_i}} \right) - \eta_i \cdot \left( u_{is_i}(x^{t+1}) - u_{i1}(x^{t+1}) \right) = 0 \quad \text{for all } i \in \mathcal{N}, s_i \in S_i \setminus \{1\}. \tag{3}
\]

At this point it is helpful to perform the following change of variables \( w^{t+1}_{is_i} = \ln \left( \frac{x^{t+1}_{is_i}}{x^t_{is_i}} \right) \) for \( s_i \in S_i \setminus \{1\} \). This change of variables is a diffeomorphism\(^2\) between the interior of the simplex for each agent and \( \mathbb{R}^{|S_i| - 1} \), so there is a “1-1” mapping between solutions of systems (3) and (4). Its inverse function is \( x_i = Q(w_i) \) and its inverse as \( Q^{-1}(w) \). We will also overload notation and denote \( x = Q(w) \) for the concatenation of the maps for all agents.

\[
\rho_{is_i}(w^{t+1}) := w^{t+1}_{is_i} - w^t_{is_i} - \eta_i \cdot \left( u_{is_i}(Q(w^{t+1})) - u_{i1}(Q(w^{t+1})) \right) = 0 \quad \text{for all } i \in \mathcal{N}, s_i \in S_i \setminus \{1\}. \tag{4}
\]

Next, we will complete the proof by showing that the system of equations (4) always admits a solution. First, note that the quantity \( w^t_{is_i} - \eta_i \cdot (u_{is_i}(Q(w^{t+1})) - u_{i1}(Q(w^{t+1})) \) is bounded (both above and below). The first part is bounded because it is fixed, the second because it expresses a scaled up version of the difference between two payoffs in the game. On the other hand the quantities \( w^{t+1}_{is_i} \) are unconstrained and can take values anywhere in \( \mathbb{R} \). Hence for any, \( i \in \mathcal{N}, s_i \in S_i \setminus \{1\} \) we can find two values MIN\(_{is_i} \) and MAX\(_{is_i} \) such that when \( w^{t+1}_{is_i} \) is set to MIN\(_{is_i} \) (MAX\(_{is_i} \), respectively) \( \rho_{is_i}(w^{t+1}) < 0 \) (resp. \( > 0 \)) for all other possible choices of the remaining variables. By applying the Poincaré-Miranda theorem (see Appendix [A]) on the compact set \( x_{is_i}[\text{MIN}_{is_i}, \text{MAX}_{is_i}] \) we have that the system always admits a solution. \( \square \)

2.5 Uniqueness of Fixed Point via Map Contraction

Next, we will establish uniqueness of the fixed point for a specific range of step-sizes. The proof will be based on an application of the Banach fixed-point theorem (mapping theorem or contraction mapping theorem)\(^3\). Thus, we simultaneously provide a constructive method to compute those

\(^2\) A diffeomorphism is a differentiable function that is a bijection between two manifolds and its inverse is differentiable as well.

\(^3\) In the language of convex analysis, this corresponds to the gradient of the convex conjugate of the FTRL regularizer and its inverse respectively.
fixed points with linear convergence rate. In what follows, it will be useful to consider \((\text{MWU})\) as a map from a vector of payoffs \(v_i = (u_{i1}, \ldots, u_{i|S_i|})\) to mixed strategies parameterized by the current initial position \(x^t_i\):

\[
 f_{x^t_i}(v_i) := \left( \frac{x^t_{i1} \exp(\eta_i \cdot u_{i1})}{\sum_{s_i \in S_i} x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i})}, \ldots, \frac{x^t_{i|S_i|} \exp(\eta_i \cdot u_{i|S_i|})}{\sum_{s_i \in S_i} x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i})} \right) \quad (\text{MWU}_f)
\]

**Lemma 2.** For any choice of \(x^t_i \in \Delta(S_i)\), the \((\text{MWU}_f)\) map \(f_{x^t_i} : \mathbb{R}^{|S_i|} \rightarrow \Delta(S_i)\) is Lipschitz continuous for the Euclidean metric with Lipschitz constant \(\eta_i/2\).

**Proof.** Since \(f\) is smooth, it suffices to establish the corresponding bound for the matrix induced norm (i.e., its largest eigenvalue in absolute value) of its Jacobian. Below we drop the dependence on \(x^t_i\) to simplify notation and derive that for any \(s_i, s'_i \in S_i\) with \(s_i \neq s'_i\) we have:

\[
 Df_{s_i, s_i} = \eta_i \frac{x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i}) (\sum_{s_i \in S_i} x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i})) - (x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i}))^2}{(\sum_{s_i \in S_i} x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i}))^2} = \eta_i x^t_{i,s_i} (1 - x^t_{i,s_i})
\]

\[
 Df_{s_i, s'_i} = -\eta_i \frac{x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i}) x^t_{i,s'_i} \exp(\eta_i \cdot u_{i,s'_i}) - x^t_{i,s'_i} \exp(\eta_i \cdot u_{i,s'_i})^2}{(\sum_{s_i \in S_i} x^t_{i,s_i} \exp(\eta_i \cdot u_{i,s_i}))^2} = -\eta_i x^t_{i,s_i} x^t_{i,s'_i}
\]

Since the Jacobian is symmetric, by the Gershgorin circle theorem all of its eigenvalues lie within \(\cup_{s_i} [Df_{s_i, s_i} - \sum_{s'_i \neq s_i} |Df_{s_i, s'_i}|, Df_{s_i, s_i} + \sum_{s'_i \neq s_i} |Df_{s_i, s'_i}|]\) \(\subseteq [0, \eta_i/2]\).

In the rest of the section we establish that once all \(\eta_i\) are selected sufficiently small then Equation \((\text{MWU}_f)\) admits a unique fixed point and is in fact a contraction map. To do so, we present a technical lemma the proof of which is deferred to Appendix B.

**Lemma 3.** Let a \(N\)-player normal form game with \(V\) being the maximum absolute payoff of the game, then for any mixed strategies \(x, x' \in \mathcal{X}\),

\[
 \sum_{i=1}^N \left\| \frac{v_i(x_{i-})}{V} - \frac{v_i(x'_{i-})}{V} \right\|_2^2 \leq (N - 1)^2 m^2 \sum_{i=1}^N \left\| x_i - x'_i \right\|_2^2
\]

**Theorem 4.** Let \((x^t_1, x^t_2, \ldots, x^t_n)\) be an arbitrary initialization of \((\text{MWU})\) dynamics and consider the map \(G : \mathcal{X} \rightarrow \mathcal{X}\) defined as follows,

\[
 G(x) := f_{x^t_1}(v(x)) = (f_{x^t_1}(v_1(x_{-1})), \ldots, f_{x^t_n}(v_n(x_{-n}))
\]

Then \(G(x)\) is Lipschitz continuous in \(\mathcal{X}\) with Lipschitz constant \(L := \frac{1}{2}\eta(N - 1)mV\), where \(\eta\) is the maximum step-size, \(N\) the number of players, \(V\) the maximum absolute value payoff of the game and \(m\) the maximum number of actions over all the agents.

**Proof.** Let \(x, x' \in \mathcal{X}\) then

\[
 \left\| G(x) - G(x') \right\|_2^2 = \sum_{i=1}^N \left\| f_{x^t_i}(v_i(x_{-i})) - f_{x'^t_i}(v_i(x'_{-i})) \right\|_2^2 \leq \frac{\eta_i^2}{4} \sum_{i=1}^N \left\| v_i(x_{-i}) - v_i(x'_{-i}) \right\|_2^2 \leq \frac{\eta^2(N - 1)^2 m^2 V^2}{4} \sum_{i=1}^N \left\| x_i - x'_i \right\|_2^2
\]

**Corollary 5.** For any \(\epsilon > 0\), the \((\text{MWU}_f)\) map with max step-size \(\eta \leq \frac{2(1-\epsilon)}{(N-1)mV}\) is a contraction and thus converges to its unique fixed point at a linear rate.
2.6 Bounded Regret Bound with Unbounded Step-Sizes

Next, we will argue that (CMWU) has bounded regret in all games for all agents. The intuition behind this proof is rather straightforward. A Follow-the-Regularized-Leader algorithm is known to be interpretable as a Follow-the-Leader algorithm with an extra (day one) payoff/cost vector capturing the cost of the strictly convex regularizer. The clairvoyance fixed-point variation results in (FTRL) becoming a “Be-The-Leader” algorithm as all agents are effectively acting with knowledge of the next day payoffs/costs and optimizing for that. “Be-The-Leader”, however, has non-positive regret and the only regret of (CMWU) or in fact, of any Clairvoyant FTRL algorithm, is due to the day one regularizer costs.

Theorem 6. (CMWU) has bounded regret

\[ \left( \leq \frac{\max_x h_i(x) - \min_x h_i(x)}{\eta_i} \right) = \frac{m \ln(m)}{\eta_i} \]

in all games for all agents and for arbitrary learning rates \( \eta_i \).

Proof. By its construction, (CMWU) is equivalent to (MWU) adapted to the fact that on day \( \tau = 1, \ldots, t \) (MWU) receives as input the costs of the following day, \( \tau + 1 = 2, \ldots, t + 1 \). Thus, given the (FTRL) formulation of (MWU), we have that (CMWU) is equivalent to:

\[ y_i^t = y_i^{t-1} + v_i(x_i^{t+1}); \]

\[ x_i^{t+1} = \arg \max_{x_i \in X_i} \left\{ \langle y_i^t, x_i \rangle - \frac{h_i(x_i)}{\eta_i} \right\} \]

(CFTRL)

In other words, the Clairvoyant FTRL (CFTRL) solves the problem

\[ x_i^{t+1} = \arg \max_{x_i \in X_i} \left\{ \frac{-h_i(x_i)}{\eta_i} + \left\langle \sum_{\tau=2}^{t+1} v_i(x^{\tau}), x_i \right\rangle \right\} \]

By defining the concave payoff \( -\frac{h_i(x_i)}{\eta_i} \) as the day 1 payoff function of agent \( i \), we have that via a straightforward induction argument its regret is non-positive. (CFTRL), however, has to account for the fact that it does actually include the regularization payoffs in its regret calculations. Thus, we have:

\[
\text{Regret}_{\text{CFTRL}}(t+1) = \max_{x_i \in X_i} \left\{ \left\langle \sum_{\tau=2}^{t+1} v_i(x^{\tau}), x_i \right\rangle \right\} - \sum_{\tau=2}^{t+1} x_i^\tau \cdot v_i(x^{\tau}) + \\
\leq \max_{x_i \in X_i} \left\{ \left\langle \sum_{\tau=2}^{t+1} v_i(x^{\tau}), x_i \right\rangle \right\} - \left\langle \sum_{\tau=2}^{t+1} x_i^\tau \cdot v_i(x^{\tau}) \right\rangle + \\
+ \frac{\max_{x_i \in X_i} h_i(x_i) - \min_{x_i \in X_i} h_i(x_i)}{\eta_i} \leq \frac{\max_{x_i \in X_i} h_i(x_i) - \min_{x_i \in X_i} h_i(x_i)}{\eta_i}
\]

Remark 1. As in the case of the standard Follow-the-Regularized-Leader, the regret guarantees extend straightforwardly to the case where the \( u_i^t \) is a convex function of \( x_i^t \).

Remark 2. The dependency of \( u_i^t \) on \( x_{-i}^t \) needs to be \( L \)-smooth but other than that it can be arbitrary (e.g., non-convex, non-concave, etc.).
Remark 3. The dependency of the payoffs/game on time can be arbitrary as long as all agents have access to their payoff function at time $t+1$ as our regret proof needs no assumption about a connection between $v^t_i$ and $v^{t+1}_i$ and our contraction mapping proof acts on a single time instance/period at a time.

Remark 4. We can handle all strategy spaces which are equivalent up to an invertible affine map to a simplex by performing the corresponding change of variables. Specifically, we can create arbitrarily large state spaces. Furthermore, we can approximate arbitrary state spaces by adding convex regularizers (i.e., soft barriers).

Remark 5. Due to the extension of regret results to the case of convex dependence of $u_i$ on $x_i$, our results immediately extend to the case where each agent is experiencing regularized utilities (e.g., by padding their utility vector with a scaled-up/down version of the gradient of the entropy of its strategy). In accordance to prior literature, we would call this specific regularized version of CMWU, Clairvoyant Experienced-Weighted-Attraction (CEWA) (other reasonable names could be Clairvoyant Boltzmann learning, Clairvoyant (smooth) Q-learning [31, 35, 26], or merely just an instance of Clairvoyant Regularized Learning). A minor point here is that in this case the range of regularized payoffs is no longer bounded, which seems to create an issue in the proof of the contraction map, however, upon closer inspection the quantity of interest is the difference $\|v_i(x_{-i}) - v_i(x'_{-i})\|_2$ which cancels out the payoff vector related to regularization and once again it suffices to consider the maximum absolute value of the non-regularized game.

Remark 6. Regularization can also be introduced to deal with rather general games, where $u_i$ may not be a concave function of $x_i$. In fact, the Clairvoyant framework provides quite a bit of flexibility in terms of how regularization could be used in such settings. For example, regularization could be applied solely in the inner loop, the synthetic/simulated/mental model of the world, whereas when updates are implemented in the real world the agents use their non-regularized utilities. Once again, we can decouple Regularization and Optimization/Exploitation similarly to our decoupling of Predictability and Exploitation. Of course, now these models would not be perfectly Clairvoyant as there is a mismatch between the low resolution synthetic model of the world and the real world but this should still offer significant advantages. One important such advantage is that we effectively reduce the problem from Online to Offline optimization. We have full access of the function that we are trying to optimize and can thus employ any offline algorithmic technique of our choice to find a good local maximum or even approximate the global maximum.

3 Conclusion

Clairvoyant Multiplicative Weights Updates is eerily similar to the mental model described by Jeff Hawkins in his book “On intelligence” [22]. “The brain uses vast amounts of memory to create a model of the world. Everything you know and have learned is stored in this model. The brain uses this memory-based model to make continuous predictions of future events. It is the ability to make predictions about the future that is the crux of intelligence. . . . it is the core idea in the book.” Can we think of these algorithms that we introduce here as a type of integrated proto-intelligent systems? They manage to achieve multi-goal/utility optimization with effectively no performance loss. Although maybe far fetched an idea, they definitely set overnight a new benchmark for success: bounded, if not, outright zero regret with fixed and as large as possible step-sizes. The numerous variants of our algorithms that will inevitably emerge and be tested in different settings such as extensive-form games, mean-field games, Multi-Agent Reinforcement Learning, NNs/GANs, etc. open up a totally new and exciting horizon of opportunities. Let’s see where we go from here.
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A Poincaré-Miranda Theorem

The Poincaré–Miranda theorem is a generalization of the intermediate value theorem, from a single function in a single dimension, to many functions in many dimensions. The theorem statement is as follows:

**Theorem 7.** Consider $n$ continuous functions of $n$ variables, $f_1, \ldots, f_n$. Assume that for each variable $x_i$, the function $f_i$ is constantly negative when $x_i = -1$ and constantly positive when $x_i = 1$. Then there is a point in the $n$-dimensional cube $[-1, 1]^n$ in which all functions are simultaneously equal to 0.

B Proof of Lemma 3

*Proof.* Like in Theorem 4 of [34] we show that

$$\sum_{i=1}^{N} \left\| \frac{v_i(x_{-i})}{V} - \frac{v_i(x'_{-i})}{V} \right\|_2^2 \leq (N-1)^2 m^2 \cdot \sum_{i=1}^{N} \left\| x_i - x'_i \right\|_2^2$$

Notice that

$$\left\| \frac{v_i(x)}{V} - \frac{v_i(x')}{V} \right\|_2 \leq \sqrt{n} \cdot \max_{s_i \in S_i} |v_i(s_i, x_{-i})/V - v_i(s_i, x'_{-i})/V| \leq \sqrt{m} \cdot \sum_{s_{-i}} \left| \Pi_{j \neq i} x_j, s_j - \Pi_{j \neq i} x'_j, s_j \right|$$
Since both $x, x' \in \mathcal{X}$, by known properties of total variation distance (see [24, 34]),

$$\sum_{s \neq i} |\Pi_{j \neq i} x_{j,s_i} - \Pi_{j \neq i} x'_{j,s_i}| \leq \sum_{j \neq i} \|x_j - x'_j\|_1$$

and as a result,

$$\left(\sum_{s \neq i} |\Pi_{j \neq i} x_{j,s_i} - \Pi_{j \neq i} x'_{j,s_i}|\right)^2 \leq \left(\sum_{j \neq i} \|x_j - x'_j\|_1\right)^2 \leq (N-1) \cdot \sum_{j \neq i} \|x_j - x'_j\|_1^2 \leq m(N-1) \cdot \sum_{j \neq i} \|x_j - x'_j\|_2^2$$

We overall get that

$$\sum_{i=1}^{N} \|v_i(x)/V - v_i(x')/V\|_2^2 \leq \sum_{i=1}^{N} m \cdot \left(\sum_{s \neq i} |\Pi_{j \neq i} x_{j,s_i} - \Pi_{j \neq i} x'_{j,s_i}|\right)^2 \leq m^2(N-1)^2 \cdot \sum_{i=1}^{N} \|x_i - x'_i\|_2^2$$

C Additional Figures

The theoretical polylog regret bound for general games given in [10] is plotted alongside the OMWU dynamics in Figure 3, showing that OMWU indeed achieves polylog regret in the 4-player, 10-strategy game presented in Figure 1.

Figure 3: OMWU using decreasing step-sizes as in [10] achieves polylog regret. The regret of a single orbit is shown. Its initialization was chosen uniformly at random.