Hints from the Standard Model
for Particle Masses and Mixings

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Abstract:
The standard model taken with a momentum space cut-off may be viewed as an effective low energy theory. The structure of it and its known parameters can give us hints for relations between these parameters. In the present investigation the Higgs problem will be discussed, the possible connection of the Higgs meson with the heavy top quark, and the geometric structure of the quark and lepton mass matrices.

\footnote{Invited talk given at the meeting on “What comes beyond the Standard Model?”, Bled (Slovenia), July 1998}
1 Introduction

The standard model is likely to describe the effective interaction at low energy of an underlying more fundamental theory. One may speculate that some parameters which emerge at long distances are insensitive to details of what is going on at much higher scales. For instance, their values may be given by the fix points of renormalization group equations, thus being rather independent of the starting numbers at small distances [1]. Or, these parameters could arise in a bootstrap-type scenario [2].

In this talk I do not want to discuss specific models of this type, but I will simply look at the measured parameters of the standard model and at its divergence structure, in order to find hints for possible connections between these parameters. I will concentrate on the vacuum expectation value of the Higgs field, which is defined without reference to external particles and their momenta. Thus its divergence property, depending solely on the structure of the vacuum, can be quite different from those of ordinary coupling constants, which can be renormalized using a momentum subtraction scheme.

By taking the standard model as an effective theory, one should use a momentum cut-off. The dependence of measurable quantities on the cut-off reflects the influence of new physics on the low energy domain. The minimization of this influence provides suggestions for the sought relations.

2 The vacuum expectation value of the Higgs field and the invariant Higgs potential

We write the Higgs part of the Lagrangian in the form

\[ \mathcal{L}_H = (D_\mu \Phi)\dagger D_\mu \Phi + \frac{J}{2} \Phi^\dagger \Phi - \frac{\lambda}{2} (\Phi^\dagger \Phi)^2 + \frac{1}{\sqrt{2}} j \cdot \Phi + \text{Yukawa couplings} \]

\[ \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i \varphi_2 \\ \varphi_0 + i \varphi_3 \end{pmatrix}. \quad (2.1) \]

The quantity \( J \) is taken to be \( J = J_0 + J_1 \) with \( J_0 > 0 \) describing the Higgs mass parameter responsible for spontaneous symmetry breaking. \( J_1 \) can be viewed as an outside field. It is used for generating a gauge invariant potential and will finally be set to zero. The quantity \( j \) determines the field direction of the spontaneous symmetry breaking. It could be due to a light
quark condensate and may be neglected after the occurrence of the symmetry breaking. Accordingly, the potential in the tree-graph approximation takes the form

\[ V_0 = \frac{\lambda}{8} (\sum_i \varphi^2_i)^2 - \frac{J}{4} \sum_i \varphi^2_i - \frac{1}{2} j \varphi_0 . \]  

(2.2)

For \( J > 0 \) the minimum of \( V_0 \) occurs for \( \varphi_i = \hat{\varphi}_i(J, j) \) with

\[ \hat{\varphi}_{1,2,3} = 0, \quad \lambda \hat{\varphi}_0^3 - J \hat{\varphi}_0 = j . \]  

(2.3)

We have to select the real solution of the cubic equation. In the limit \( j \to 0, \lambda \neq 0 \) one gets

\[ \hat{\varphi}_0(J) = \sqrt{\frac{J}{\lambda}}, \quad m_H^2(J) = \frac{\partial^2 V_0}{\partial \varphi_0^2} \big|_{\varphi=\hat{\varphi}} = J \]

\[ m_i^2 = \frac{\partial^2 V_0}{\partial \varphi_i^2} \big|_{\varphi=\hat{\varphi}} = 0 \quad i = 1, 2, 3 . \]  

(2.4)

By replacing as usual \( \varphi_0(x) \) by

\[ \varphi_0(x) = \hat{\varphi}_0(J, j) + H(x) \]  

(2.5)

the interaction part \( \mathcal{L}'_{int} \) of the shifted Higgs Lagrangian allows one to evaluate \( < H > \). To lowest-order the expression is

\[ < H > = i \int d^4x < 0 | T(H(0), \mathcal{L}'_{int}(x)) | 0 > . \]  

(2.6)

The result [3] obtained from (2.6) can be used to write the gauge-invariant vacuum expectation value of the square of the Higgs field \( \sigma(J) = < \sum_i \varphi_i^2 > \) for \( J > 0 \) and \( j = 0 \) in the form

\[ \sigma(J) = \frac{J}{\lambda} - 2 < H^2 > + 2 g^2 + g'^2 \frac{g^2}{4 \lambda} < Z^\mu Z^\mu > \]

\[ + 4 g^2 \frac{g^2}{4 \lambda} < W^+_\mu W^-\mu > - \frac{g_t^2}{\lambda} < \bar{t} t > / m_t . \]  

(2.7)

Here \( g, g' \) are the gauge couplings for the vector bosons \( W \) and \( Z \), and \( g_t \) denotes the Yukawa coupling for the top quark. Fermions of lower mass are neglected, but could easily be added. The “vacuum leaks” \( < H^2 >, < Z^\mu Z^\mu > , \ldots \) could be finite in the full theory with a correspondingly modified
vacuum structure. By taking for the effective theory a particle momentum cut-off chosen to be universal for all propagators, Eq. (2.7) leads to

\[
\sigma(J) = \frac{J}{\lambda} + \frac{\Lambda^2}{8\pi^2} \left[ 3 + \frac{3g^2 + g'^2}{4\lambda} + \frac{6g^2}{4\lambda} - \frac{12g^2}{2\lambda} \right] 
\]

\[
+ \frac{J}{8\pi^2} \left( 1 + 3 \left( \frac{g^2 + g'^2}{4\lambda} \right)^2 + 6 \left( \frac{g^2}{4\lambda} \right)^2 \right) \ln \frac{\Lambda^2}{J/\lambda} 
\]

\[
- \frac{J}{8\pi^2} \left( \ln \lambda + 3 \left( \frac{g^2 + g'^2}{4\lambda} \right)^2 \ln \frac{g^2 + g'^2}{4} \right) 
\]

\[
+ 6 \left( \frac{g^2}{4\lambda} \right)^2 \ln \frac{g^2}{4} - 12 \left( \frac{g^2}{2\lambda} \right)^2 \ln \frac{g^2}{2} \right) .
\]

(2.8)

The term \( \frac{\Lambda^2}{8\pi^2} \) which represents the free field part of \( <\Sigma_i\phi_i^2> \) is written separately. It also appears in the case \( J < 0 \) where one has \( \hat{\phi}_0(J) = 0 \) and no spontaneous symmetry breaking. For \( J < 0 \) one finds to one loop order

\[
\sigma(J) = \frac{\Lambda^2}{8\pi^2} + \frac{J}{8\pi^2} \ln \frac{2\Lambda^2}{-J} .
\]

(2.9)

Thus, in our approximation, \( \sigma(J) \) is discontinuous at \( J = 0 \), indicating a first-order phase transition with strength proportional to \( \Lambda^2 \):

\[
\Delta\sigma = -\frac{\Lambda^2}{8\pi^2} \left[ 3 + \frac{3g^2 + g'^2}{4\lambda} + \frac{6g^2}{4\lambda} - \frac{12g^2}{2\lambda} \right] 
\]

(2.10)

If we would keep \( j \neq 0 \), the jump of \( \sigma(J) \) would be replaced by a rapid, but now continuous, change in the region \(-j^{2/3} \leq J \leq j^{2/3}\). The right-hand side of (2.10) with its quadratic divergence also shows up in the conventional (non-gauge invariant) Higgs potential \( V(<\varphi>) \). Its appearance constitutes an essential part of the hierarchy problem and necessitates large fine-tuned subtractions depending in a specific way on the standard model parameters. This is quite unnatural and is known as the hierarchy problem. Quadratic divergencies do not occur in supersymmetric models where the super partners of each particle provide for a cancellation of such divergencies. In spite

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\[2\text{At } J = 0 \text{ the one-loop approximation for } \sigma(J) \text{ is insufficient. However, (2.10) is expected to hold for the change of } \sigma \text{ within a larger region around } J = 0.\]
of this elegant solution, attempts have been made to obtain a cancellation of divergencies by special choices of the standard model parameters which could make the introduction of superpartners at the weak scale unnecessary \[4, 3\]. It is speculated that the particle couplings arrange themselves such as to stabilize the particle masses. Expressed in terms of masses, the relation for which (2.10) vanishes is

\[
S_{\Lambda^2} = (3m_H^2 + 3m_Z^2 + 6m_W^2 - 12m_t^2)/<\varphi_0>^2 = 0.
\]  
(2.11)

It is known as the Veltman condition \[4\]. A sofar unsolved problem is the extension of this relation which includes higher loop effects. To my knowledge no gauge invariant scheme is known which allows one to isolate the quadratic divergencies with the help of a single cut-off parameter (except lattice calculations).

Eq. (2.8) can also be obtained in a more general context. By defining the “free energy” \( W(J,j) \) as the logarithm of the partition function with the action given by (2.1), one can obtain \( \sigma \) from

\[
\sigma = -4 \frac{\partial W(J,j)}{\partial J}.
\]  
(2.12)

The Legendre transformation of \( W(J,j) \) with respect to \( J \) defines the \( \sigma \)-dependent effective potential \[8\]

\[
V(\sigma,j,J_0) = W(J(\sigma,j),j) + \frac{\sigma}{4}(J(\sigma,j) - J_0).
\]  
(2.13)

It has an extremum at \( \bar{\sigma} \), for which \( J(\bar{\sigma},j) = J_0 \) where \( J_0 \) is the mass parameter in the Higgs potential.

Calculating \( W(J,j) \) by the saddle point method up to one-loop order, one gets

\[
W(J,j) = \frac{\lambda}{8} \phi_0^4(J,j) - \frac{J}{4} \phi_0^2(J,j) - \frac{j}{2} \phi_0(J,j)
\]

\[
+ \frac{1}{(4\pi)^2} \sum_p r_p \int_0^{\Lambda^2} dK^2 K^2 \ln \left(1 + \frac{m_p^2(J,j)}{K^2}\right).
\]  
(2.14)

The sum is over the particles of the standard model with their \( J \)- and \( j \)-dependent masses. \( r_p \) is a statistical factor (3 for the \( Z \), 6 for the \( W \), -12 for the top, 1 for the Higgs). For \( j = 0 \) the result is gauge-invariant. To consider the dependence of \( W(J,j = 0) \) on \( J \) rather than on \( j \) has the additional advantage that for \( J > 0 \) the Goldstone particles remain massless.
and thus do not contribute. Furthermore, to one-loop order, the potential $V(\sigma, J_0)$ remains real for all values of $\sigma$. The derivative of $W(J)$ with respect to $J$ according to (2.12) reproduces eq. (2.8). As long as $\Delta \sigma$ is not very small and $\Lambda$ is of order TeV or larger, $V(\sigma, J_0)$ calculated from (2.12–2.14), changes its shape significantly with a small change of $J_0$ for $J_0$ near zero (and small $j$).

For the purpose of renormalization we can add to $W(J, j)$ a polynomial in $J$ up to second order. The linear piece could be used to cancel the quadratic divergence in (2.8). However, it would then reappear in (2.9). Instead, I will normalize $\sigma$ such that it is zero in the limit of all particle masses going to zero, starting from $J < 0$: $\sigma(J \to 0_-, j \to 0) = 0$. To achieve this, we have to subtract the $2\Lambda^2/8\pi^2$ part in (2.8) and (2.9) by replacing $W(J, j)$ in (2.14) by $W(J, j) + \frac{\Lambda^2}{32\pi^2} J$. A further change of $W(J, j)$ using a subtraction term proportional to $J^2$ can remove the logarithmic divergence in (2.8) and (2.13). This subtraction can be interpreted as a renormalization of the Higgs coupling constant $\lambda$. Again, I do not perform such a complete subtraction since the corresponding term would then appear in (2.9) where it has no physical basis. But we can remove the logarithmic divergence for regions of $J$ where the gauge bosons and the fermions remain massless. Since I am not completely certain about the necessity of this subtraction I will consider two cases: i) no subtraction proportional to $J^2$ and – the more appealing one – ii) a subtraction such that, besides the quadratic divergence, also the logarithmic divergence is removed in (2.9). Accordingly, the factor which governs the logarithmic divergence of $\sigma$ and $V(\sigma, J_0)$ in the region of spontaneous symmetry breaking is (to one loop order, and expressed in terms of masses)

$$S_{Log \Lambda} = (\zeta m_H^4 + 3m_Z^4 + 6m_W^4 - 12m_t^4)/<\varphi_0>^4.$$ \number{2.15}

$\zeta = 1$ corresponds to no subtraction, while $\zeta = 0$ is valid when the subtraction is performed according to ii) . (I do not consider here the conventional non gauge invariant potential $V(<\varphi_0>, J_0)$. It would lead to $\zeta = 3/2$).

As a speculation I will now assume a minimum influence of new physics on the standard model. The idea is that the coupling constants of the standard model may take preferred values which stabilize the vacuum expectation value of the Higgs field and therefore also the particle masses. In particular, $\Delta \sigma$ of eq. (2.10) should be independent of $\Lambda^2$, i.e. the square bracket in (2.10) should be proportional to $1/\Lambda^2$, or vanish. If this is the case, the particle couplings are not independent of each other, but satisfy – at least
approximately – the Veltman condition. Here one encounters the problem of the scale ($\mu$) at which the particle couplings should be taken. In particular, the Yukawa coupling of the top quark is sensitive to it. The vacuum expectation value of the unrenormalized Higgs field is scale-invariant. But to take advantage of this fact, higher order calculations, and a knowledge of the scale dependence of $\Lambda$ ($\Lambda$ may be related to high mass states) would be needed. However, it is clear that the natural scale for the couplings occurring in the loop integrals is $\mu \approx \Lambda$.

Let us first assume that the cut off $\Lambda$ is very low and the cancellation of the coupling terms in (2.10) occurs already at the weak scale of $\approx 250 \text{ GeV}$, where the top mass is still big. Then, using (2.11), and for the mass of the top $m_t(m_Z) = 173 \text{ GeV}$, the Higgs mass is predicted to be $m_H(m_Z) \approx 280 \text{ GeV}$. Now we can take this value of the Higgs mass to look at the logarithmic divergence and calculate $S_{\text{Log} \Lambda}$. Using $\zeta = 1$ and again the scale $\mu \approx 250 \text{ GeV}$, we find that $S_{\text{Log} \Lambda} = -0.2$. We thus have the surprising result that for a Higgs mass of about 300 GeV the factors responsible for the quadratic and the logarithmic divergence of the Higgs potential $V(\sigma, J_0)$ are both small. Let us now consider the extreme case of strictly vanishing factors in front of $\Lambda^2$ and $\log \Lambda$ for the one-loop Higgs potential $V(\sigma, J_0)$. This provides us with two equations which allow a calculation of $m_t$ and $m_H$ in terms of the gauge couplings for $W$ and $Z$. The result is (for $\mu = 250 \text{ GeV}$ and a correspondingly very low cut off value)

$$m_t(m_Z) = 198 \text{ GeV}, \quad m_H(m_Z) = 320 \text{ GeV} \quad (2.16)$$

The fact that the value of $m_t$ obtained this way is not far away from the experimental result is presumably a fortuitous coincidence, since for the low cut-off considered a suppression of the logarithmic divergence does not appear plausible. But if not, it would indicate a very close connection of Higgs and top with a Higgs mass not much different from $2m_t \approx 350 \text{ GeV}$! On the other hand, the preferred value $\zeta = 0$ gives here no admissible solution.

Owing to the quadratic form of the two equations which simultaneously suppress quadratic and logarithmic divergences, another type of solution exists with smaller values for the Higgs mass. For this solution the relevant scale for the particles running in the loops must be extremely high (and hence the value of the cut off $\Lambda$) in order to get a large enough value for the top mass at the weak scale. We therefore take $\mu$ equal to the Planck mass, determine the Higgs and top couplings at this scale, and apply the two loop renormalization group equations to predict their values at the weak scale.
This implies, of course, that physics beyond the standard model, which could influence the standard model couplings, can occur only near and above the Planck scale. The calculation, using $\alpha_s(m_Z) = 0.12$ and $\zeta = 1$, gives

$$m_t(m_Z) = 169 \text{ GeV} \ , \ m_H(m_Z) = 140 \text{ GeV} \quad (2.17)$$

while for $\zeta = 0$,

$$m_t(m_Z) = 168 \text{ GeV} \ , \ m_H(m_Z) = 137 \text{ GeV} \quad (2.18)$$

Both solutions do not differ much since at the Planck scale $\lambda$ is found to be small (but not zero). I prefer the solution with $\zeta = 0$. Firstly, it requires a very high scale which can be identified with a cut-off near the Planck mass. This cut-off is large enough that even the logarithm of it can provide for an order of magnitude suppression. Secondly, the divergencies are eliminated in the unbroken phase as well and thirdly, $S_{\log \Lambda}$ can be expressed in terms of the $\beta$-function of $-\frac{\beta}{8\pi^2}$, i.e. the $\beta$-function of the zero order expression of $W(J_0)$:

$$4\pi^2 S_{\log \Lambda} = -\frac{\lambda^2(\mu)}{J_0^2(\mu)} \frac{\beta(J_0^2(\mu))}{\lambda(\mu)} \quad (2.19)$$

In other words, the condition for the vanishing of the logarithmic divergence of $\tilde{\sigma}$ can formally be viewed as a requirement for a fixed point for $W(J_0)$. The numbers obtained in (2.17) and (2.18) differ little from the result obtained by Bennett, Nielsen and Froggatt in the framework of their anti grand unification model. This model requires $\lambda(m_{\text{Planck}}) = 0$ (not far away from $\lambda(m_{\text{Planck}}) = 0.04$ obtained here), and the vanishing of $S_{\log \Lambda}$.

As a last point I like to comment on the question of the possible participation of a 4th generation. If the particles of this generation obtain their masses through the coupling to the standard model Higgs particle, one cannot have $S_{\Lambda^2}$ and $S_{\log \Lambda}$ simultaneously sufficiently small or zero. The reason is that fermion masses for the $b'$ and $t'$ would enter with masses which are larger or roughly equal to the mass of the top. One obtains imaginary solutions of the equations or, for $\zeta = 0$, too small values for the fermion masses.

We have seen that the possibility exists that the particles of the standard model have arranged their couplings such that the influence from physics
beyond the standard model is suppressed, stabilizing thereby the particle masses. This is not a trivial statement. Furthermore, it can only occur for the known three generations\(^3\). When we minimize simultaneously the quadratic and the logarithmic dependence of the vacuum expectation value of the Higgs field we find interesting relations between the Higgs and the top mass. For a very low, and perhaps too low, value of the cut-off scale we found \( m_H \approx 2m_t \). A much more interesting solution is the one with a very high value of the cut-off. In this case a properly subtracted form of \( W(J) \) is taken such that there is no divergence in the parameter region of no spontaneous symmetry breaking. Then the required absence of quadratic and logarithmic divergences in the physical region, and a cut-off value of the order of the Planck scale, leads to the correct value of the mass of the top and to a Higgs mass \( m_H \approx 140 \text{ GeV} \).

### 3 Masses and Mixings of Quarks and Leptons

The possible intimate relation between Higgs and top discussed in the previous paragraph also suggests a dominant role of the top for the structure of the quark and lepton mass matrices. The masses of the lighter particles can then be expected to be related to the top mass by powers of a small constant \([7, 8]\). Let us look at the quark and charged lepton masses at the common scale \( m_Z \) in the \( \overline{\text{MS}} \) scheme (in GeV) \([9]\):

\[
\begin{align*}
    m_t &= 173 \pm 6 \\
    m_b &= 2.84 \pm 0.10 \\
    m_{\tau} &= 1.78 \\
    m_c &= 0.58 \pm 0.06 \\
    m_s &= (70 \pm 14) \times 10^{-3} \\
    m_{\mu} &= 106 \times 10^{-3} \\
    m_u &= (2.0 \pm 0.5) \times 10^{-3} \\
    m_d &= (3.6 \pm 0.8) \times 10^{-3} \\
    m_e &= 0.51 \times 10^{-3}
\end{align*}
\]

We take as the small parameter \( \epsilon \approx \sqrt{m_c/m_t} = 0.058 \pm 0.004 \). Then, the rule \( m_t : m_c : m_u = 1 : \epsilon^2 : \epsilon^4 \) may be supposed to hold up to \( O(\epsilon) \) corrections. Similarly, the ratio of down-quark masses and the ratio of charged lepton masses are taken to be simple rational numbers times integer powers of \( \epsilon \). It is then plausible to use a corresponding geometric structure also for the off-diagonal elements of the mass matrices. Here, I will not go into details, since the paper containing these suggestions is published \([8]\). I will only quote the results: The up-quark matrix can be taken real and symmetric. For the down-quark and the charged lepton matrices the simplest possible textures are used for which the 3rd generation decouples from the first and

\(^3\)Of course, nothing can be said about particles with a different origin of their masses.
second. The first and second generation can mix by a complex entry. Taking
this mixing coefficient of the light generations purely imaginary, one obtains
– for fixed $\epsilon$ – a maximal CP-violation. It manifests itself by making
the unitarity triangle to be a right-angle one: $\gamma \simeq 90^\circ$. Besides the top mass
and $\epsilon$ there are only two additional parameters: the beauty to top, and the
$\tau$ to beauty mass ratios. With these parameters one gets the following, so
far quite successful, numbers (in GeV):

\[
\begin{align*}
m_t &= 174 & m_b &= 2.8 & m_\tau &= 1.77 \\
m_c &= 0.58 & m_s &= 84 \times 10^{-3} & m_\mu &= 103 \times 10^{-3} \\
m_u &= 1.95 \times 10^{-3} & m_d &= 4.2 \times 10^{-3} & m_\epsilon &= 0.52 \times 10^{-3}
\end{align*}
\]

(3.2)

\[
|V_{us}| = 0.216, \quad |V_{cb}| = 0.041, \quad |V_{ub}| = 0.0034, \quad |V_{td}| = 0.0094,
\] (3.3)

and $\alpha = 70^\circ$, $\beta = 20^\circ$, $\gamma = 90^\circ$.

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