STABILITY OF HYPERSURFACES WITH CONSTANT $r$-TH ANISOTROPIC MEAN CURVATURE

YIJUN HE AND HAIZHONG LI

Abstract. Given a positive function $F$ on $S^n$ which satisfies a convexity condition, we define the $r$-th anisotropic mean curvature function $H^F_r$ for hypersurfaces in $\mathbb{R}^{n+1}$ which is a generalization of the usual $r$-th mean curvature function. Let $X: M \to \mathbb{R}^{n+1}$ be an $n$-dimensional closed hypersurface with $H^F_r = \text{constant}$, for some $r$ with $0 \leq r \leq n-1$, which is a critical point for a variational problem. We show that $X(M)$ is stable if and only if $X(M)$ is the Wulff shape.

§1. Introduction

Let $F: S^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$ (D^2 F + F1)_x > 0, \quad \forall x \in S^n, $$

where $S^n$ denotes the standard unit sphere in $\mathbb{R}^{n+1}$, $D^2 F$ denotes the intrinsic Hessian of $F$ on $S^n$ and 1 denotes the identity on $T_xS^n$, $> 0$ means that the matrix is positive definite. We consider the map

$$ \phi: S^n \to \mathbb{R}^{n+1}, $$

$$ x \mapsto F(x)x + (\text{grad}_{S^n} F)_x, $$

its image $W_F = \phi(S^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of $F$ (see [3], [7], [8], [10], [13], [17], [18]). We note when $F \equiv 1$, $W_F$ is just the sphere $S^n$.

Now let $X: M \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed, orientable hypersurface. Let $\nu: M \to S^n$ denotes its Gauss map, that is, $\nu$ is the unit inner normal vector of $M$.

Let $A_F = D^2 F + F1$, $S_F = -d(\phi \circ \nu) = -A_F \circ d\nu$. $S_F$ is called the $F$-Weingarten operator, and the eigenvalues of $S_F$ are called anisotropic principal curvatures. Let $\sigma_r$ be

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the elementary symmetric functions of the anisotropic principal curvatures \( \lambda_1, \lambda_2, \ldots, \lambda_n \):

\[
\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \leq r \leq n).
\]

We set \( \sigma_0 = 1 \). The \( r \)-th anisotropic mean curvature \( H^F_r \) is defined by \( H^F_r = \sigma_r / C^r_n \), also see Reilly \[15\].

For each \( r, 0 \leq r \leq n-1 \), we set

\[
\mathcal{A}_{r,F} = \int_M F(\nu)\sigma_r dA_X.
\]

The algebraic \((n+1)\)-volume enclosed by \( M \) is given by

\[
V = \frac{1}{n+1} \int_M \langle X, \nu \rangle dA_X.
\]

We consider those hypersurfaces which are critical points of \( \mathcal{A}_{r,F} \) restricted to those hypersurfaces enclosing a fixed volume \( V \). By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional

\[
\mathcal{F}_{r,F;\Lambda} = \mathcal{A}_{r,F} + \Lambda V(X),
\]

where \( \Lambda \) is a constant. We will show the Euler-Lagrange equation of \( \mathcal{F}_{r,F;\Lambda} \) is:

\[
(r + 1) \sigma_{r+1} - \Lambda = 0.
\]

So the critical points are just hypersurfaces with \( H^F_{r+1} = \text{const.} \)

If \( F \equiv 1 \), then the function \( \mathcal{A}_{r,F} \) is just the functional \( \mathcal{A}_r = \int_M S_r dA_X \) which was studied by Alencar, do Carmo and Rosenberg in \[1\], where \( H_r = S_r / C^r_n \) is the usual \( r \)-th mean curvature. For such a variational problem, they call a critical immersion \( X \) of the functional \( \mathcal{A}_r \) (that is, a hypersurface with \( H_{r+1} = \text{constant} \) stable if and only if the second variation of \( \mathcal{A}_r \) is non-negative for all variations of \( X \) preserving the enclosed \((n+1)\)-volume \( V \). They proved:

**Theorem 1.1.** (\[1\]) Suppose \( 0 \leq r \leq n-1 \). Let \( X: M \to \mathbb{R}^{n+1} \) be a closed hypersurface with \( H_{r+1} = \text{constant} \). Then \( X \) is stable if and only if \( X(M) \) is a round sphere.

Analogously, we call a critical immersion \( X \) of the functional \( \mathcal{A}_{r,F} \) stable if and only if the second variation of \( \mathcal{A}_{r,F} \) (or equivalently of \( \mathcal{F}_{r,F;\Lambda} \)) is non-negative for all variations of \( X \) preserving the enclosed \((n+1)\)-volume \( V \).

In \[13\], Palmer proved the following theorem (also see Winklmann \[18\]):

**Theorem 1.2.** (\[13\]) Let \( X: M \to \mathbb{R}^{n+1} \) be a closed hypersurface with \( H^F_1 = \text{constant} \). Then \( X \) is stable if and only if, up to translations and homotheties, \( X(M) \) is the Wulff shape.

In this paper, we prove the following theorem:
Theorem 1.3. Suppose $0 \leq r \leq n - 1$. Let $X: M \to \mathbb{R}^{n+1}$ be a closed hypersurface with $H^{F}_{r+1} = \text{constant}$. Then, $X$ is stable if and only if, up to translations and homotheties, $X(M)$ is the Wulff shape.

Remark 1.4. In the case $F \equiv 1$, Theorem 1.3 becomes Theorem 1.1. Theorem 1.3 gives an affirmative answer to the problem proposed in [8].

§2. Preliminaries

Let $X: M \to \mathbb{R}^{n+1}$ be a smooth closed, oriented hypersurface with Gauss map $\nu: M \to S^n$, that is, $\nu$ is the unit inner normal vector field. Let $X_t$ be a variation of $X$, and $\nu_t: M \to S^n$ be the Gauss map of $X_t$. We define

$$\psi = \langle \frac{dX_t}{dt}, \nu_t \rangle, \quad \xi = (\frac{dX_t}{dt})^\top,$$

where $\top$ represents the tangent component and $\psi, \xi$ are dependent of $t$. The corresponding first variation of the unit normal vector is given by (see [10], [13], [18])

$$\nu'_t = -\text{grad} \psi + d\nu_t(\xi),$$

the first variation of the volume element is (see [2], [3] or [9])

$$\partial_t dA_{X_t} = (\text{div} \xi - nH\psi)dA_{X_t},$$

and the first variation of the volume $V$ is

$$V'(t) = \int_M \psi dA_{X_t},$$

where grad, div, $H$ represents the gradients, the divergence, the mean curvature with respect to $X_t$ respectively.

Let $\{E_1, \cdots, E_n\}$ be a local orthogonal frame on $S^n$, let $e_i = e_i(t) = E_i \circ \nu_t$, where $i = 1, \cdots, n$ and $\nu_t$ is the Gauss map of $X_t$, then $\{e_1, \cdots, e_n\}$ is a local orthogonal frame of $X_t: M \to \mathbb{R}^{n+1}$.

The structure equation of $x: S^n \to \mathbb{R}^{n+1}$ is:

$$\left\{ \begin{array}{l}
\frac{dx}{dt} = \sum_i \theta_i E_i \\
\frac{dE_i}{dt} = \sum_j \theta_{ij} E_j - \theta_i x \\
\frac{d\theta_i}{dt} = \sum_j \theta_{ij} \wedge \theta_j \\
\frac{d\theta_{ij}}{dt} - \sum_k \theta_{ik} \wedge \theta_{kj} = -\frac{1}{2} \sum_{kl} \tilde{R}_{ijkl} \theta_k \wedge \theta_l = -\theta_i \wedge \theta_j
\end{array} \right.$$

where $\theta_{ij} + \theta_{ji} = 0$ and $\tilde{R}_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. 

The structure equation of $X_t$ is (see [11], [12]):

\begin{equation}
\begin{aligned}
\text{d}X_t &= \sum_i \omega_i e_i \\
\text{d}\nu_t &= -\sum_i h_{ij}\omega_j \\
\text{d}e_i &= \sum_j \omega_{ij}e_j + \sum_j h_{ij}\omega_j\nu_t \\
\text{d}\omega_i &= \sum_j \omega_{ij} \wedge \omega_j \\
\text{d}\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} &= -\frac{1}{2} \sum_{kl} R_{ijkl} \theta_k \wedge \theta_l \\
\end{aligned}
\end{equation}

where $\omega_{ij} + \omega_{ji} = 0$, $R_{ijkl} + R_{ijlk} = 0$, and $R_{ijkl}$ are the components of the Riemannian curvature tensor of $X_t(M)$ with respect to the induced metric $dX_t \cdot dX_t$. Here we have omitted the variable $t$ for some geometric quantities.

From $d\nu_t = d(E_i \circ \nu_t) = \nu_t^*dE_i = \sum j \nu_t^*\theta_{ij}e_j - \nu_t^*\theta_i\nu_t$, we get

\begin{equation}
\begin{cases}
\omega_{ij} = \nu_t^*\theta_{ij} \\
\nu_t^*\theta_i = -\sum_j h_{ij}\omega_j,
\end{cases}
\end{equation}

where $\omega_{ij} + \omega_{ji} = 0$, $h_{ij} = h_{ji}$.

Let $F: S^n \to \mathbb{R}^+$ be a smooth function, we denote the coefficients of covariant differential of $F$, $\text{grad}_{S^n} F$ with respect to $\{E_i\}_{i=1,\ldots,n}$ by $F_i, F_{ij}$ respectively.

From (2.7), $d(F(\nu_t)) = \nu_t^*dF = \nu_t^*\left(\sum_i F_i\theta_i\right) = -\sum_{ij}(F_i \circ \nu_t)h_{ij}\omega_j$, thus

\begin{equation}
\text{grad}(F(\nu_t)) = -\sum_{ij}(F_i \circ \nu_t)h_{ij}e_j = d\nu_t(\text{grad}_{S^n} F).
\end{equation}

Through a direct calculation, we easily get

\begin{equation}
d\phi = (D^2 F + F1) \circ dx = \sum_{ij} A_{ij} \theta_i E_j,
\end{equation}

where $A_{ij}$ is the coefficient of $A_F$, that is, $A_{ij} = F_{ij} + F\delta_{ij}$.

Taking exterior differential of (2.9) and using (2.5) we get

\begin{equation}
A_{ijk} = A_{jik} = A_{ikj},
\end{equation}

where $A_{ijk}$ denotes coefficient of the covariant differential of $A_F$ on $S^n$.

We define $(A_{ij} \circ \nu_t)_k$ by

\begin{equation}
d(A_{ij} \circ \nu_t) + \sum (A_{kj} \circ \nu_t)\omega_{ki} + \sum_k (A_{ik} \circ \nu_t)\omega_{kj} = \sum_k (A_{ij} \circ \nu_t)_k \omega_k.
\end{equation}

By a direct calculation using (2.7) and (2.11), we have

\begin{equation}
(A_{ij} \circ \nu_t)_k = -\sum_l h_{kl}A_{ijl} \circ \nu_t.
\end{equation}

We define $L_{ij}$ by

\begin{equation}
\left(\frac{d e_i}{d t}\right)^\top = -\sum_j L_{ij} e_j,
\end{equation}
where \( \top \) denote the tangent component, then \( L_{ij} = -L_{ji} \) and we have (see [2], [4] or [9])

\[
h'_{ij} = \psi_{ij} + \sum_k \{h_{ijk}\xi_k + \psi h_{ik} h_{jk} + h_{ik} L_{kj} + h_{jk} L_{ki}\}.
\]

Let \( s_{ij} = \sum_k A_{ik} h_{kj} \), then from (2.7) and (2.9), we have

\[
d(\phi \circ \nu_t) = \nu_t^* d\phi = -\sum_{ij} s_{ij} \omega_j e_i.
\]

We define \( S_F \) by

\[
S_F = -d(\phi \circ \nu) = -A_F \circ d\nu,
\]

then we have \( S_F(e_j) = \sum_i s_{ij} e_i \). We call \( S_F \) to be F-Weingarten operator. From the positive definite of \( (A_{ij}) \) and the symmetry of \( (h_{ij}) \), we know the eigenvalues of \( (s_{ij}) \) are all real. We call them anisotropic principal curvatures, and denote them by \( \lambda_1, \ldots, \lambda_n \).

Taking exterior differential of (2.15) and using (2.6) we get

\[
s_{ijk} = s_{ikj},
\]

where \( s_{ijk} \) denotes coefficient of the covariant differential of \( S_F \).

We have \( n \) invariants, the elementary symmetric function \( \sigma_r \) of the anisotropic principal curvatures:

\[
\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \leq r \leq n).
\]

For convenience, we set \( \sigma_0 = 1 \) and \( \sigma_{n+1} = 0 \). The \( r \)-th anisotropic mean curvature \( H^F_r \) is defined by

\[
H^F_r = \sigma_r / C^r_n, \quad C^r_n = \frac{n!}{r!(n-r)!}.
\]

We have by use of (2.22) and (2.6)

\[
\sum_{ij} \frac{d((A_{ij} E_i \otimes E_j) \circ \nu_t)}{dt} = \sum_{ij} \langle D(A_{ij} E_i \otimes E_j)\nu_t, \nu'_t \rangle
\]

\[
= -\sum_{ijk} A_{ijk}(\psi_k + \sum_l h_{kl} \xi_l) e_i \otimes e_j,
\]

where \( D \) is the Levi-Civita connection on \( S^n \).

On the other hand, we have

\[
\sum_{ij} \frac{d((A_{ij} E_i \otimes E_j) \circ \nu_t)}{dt} = \sum_{ij} \{A'_{ij} e_i \otimes e_j + A_{ij} \left(\frac{de_i}{dt}\right)^\top \otimes e_j + A_{ij} e_i \otimes \left(\frac{de_j}{dt}\right)^\top\}.
\]

By use of (2.13), we get from (2.19) and (2.20)

\[
\frac{d(A_{ij} \circ \nu_t)}{dt} = A'_{ij}(t) = \sum_k \{-A_{ijk} \psi_k - \sum_l A_{ijk} h_{kl} \xi_l + A_{ik} L_{kj} + A_{jk} L_{ki}\}.
\]

By (2.12), (2.14), (2.21) and the fact \( L_{ij} = -L_{ji} \), through a direct calculation, we get the following lemma:
Lemma 2.1.  \( \frac{ds_{ij}}{dt} = s'_{ij}(t) = \sum_k \{(A_{ik} \psi_k)_j + s_{ijk} \xi_k + \psi s_{ik} h_{kj} + s_{kj} L_{ki} + s_{ik} L_{kj}\}. \)

As \( M \) is a closed oriented hypersurface, one can find a point where all the principal curvatures with respect to \( \nu \) are positive. By the positiveness of \( A_F \), all the anisotropic principal curvatures are positive at this point. Using the results of Gårding (5), we have the following lemma:

**Lemma 2.2.** Let \( X: M \to \mathbb{R}^{n+1} \) be a closed, oriented hypersurface. Assume \( H^{F}_{r+1} > 0 \) holds on every point of \( M \), then \( H^{F}_k > 0 \) holds on every point of \( M \) for every \( k = 1, \ldots, r \).

Using the characteristic polynomial of \( S_F \), \( \sigma_r \) is defined by

\[
\det(tI - S_F) = \sum_{r=0}^{n} (-1)^r \sigma_r t^{n-r}.
\]

So, we have

\[
\sigma_r = \frac{1}{r!} \sum_{i_1, \ldots, i_r, j_1, \ldots, j_r} \delta_{i_1, \ldots, i_r}^{j_1, \ldots, j_r} s_{i_1 j_1} \cdots s_{i_r j_r},
\]

where \( \delta_{i_1, \ldots, i_r}^{j_1, \ldots, j_r} \) is the usual generalized Kronecker symbol, i.e., \( \delta_{i_1, \ldots, i_r}^{j_1, \ldots, j_r} \) equals +1 (resp. -1) if \( i_1 \cdots i_r \) are distinct and \( (j_1 \cdots j_r) \) is an even (resp. odd) permutation of \( (i_1 \cdots i_r) \) and in other cases it equals zero.

We introduce two important operators \( P_r \) and \( T_r \) by

\[
P_r = \sigma_r I - \sigma_{r-1} S_F + \cdots + (-1)^r S^r_F, \quad r = 0, 1, \ldots, n,
\]

\[
T_r = P_r A_F, \quad r = 0, 1, \ldots, n - 1.
\]

Obviously \( P_n = 0 \) and we have

\[
P_r = \sigma_r I - P_{r-1} S_F = \sigma_r I + T_{r-1} d\nu, \quad r = 1, \ldots, n.
\]

From the symmetry of \( A_F \) and \( d\nu \), \( S_F A_F \) and \( d\nu \circ S_F \) are symmetric, so \( T_r = P_r A_F \) and \( d\nu \circ P_r \) are also symmetric for each \( r \).

**Lemma 2.3.** The matrix of \( P_r \) is given by:

\[
(P_r)_{ij} = \frac{1}{r!} \sum_{i_1, \ldots, i_r, j_1, \ldots, j_r} \delta_{i_1, \ldots, i_r}^{j_1, \ldots, j_r} s_{i_1 j_1} \cdots s_{i_r j_r}.
\]

**Proof.** We prove Lemma 2.3 inductively. For \( r = 0 \), it is easy to check that (2.27) is true. Assume (2.27) is true for \( r = k \), then from (2.26),

\[
(P_{k+1})_{ij} = \sigma_{k+1} \delta_{ij} - \sum_l (P_k)_{il} s_{lj} = \frac{1}{(k+1)!} \sum_l (\delta_{i_1, \ldots, i_{k+1}}^{j_1, \ldots, j_l} \delta_{j_1, \ldots, j_l}^{i_1, \ldots, i_{k+1}} - \sum_l (\delta_{i_1, \ldots, i_{k+1}}^{j_1, \ldots, j_l} \delta_{j_1, \ldots, j_l}^{i_1, \ldots, i_{k+1}}) s_{i_1 j_1} \cdots s_{i_{k+1} j_{k+1}}) = \frac{1}{(k+1)!} \sum_l \delta_{i_1, \ldots, i_{k+1}}^{j_1, \ldots, j_{k+1}} s_{i_1 j_1} \cdots s_{i_{k+1} j_{k+1}}.
\]
Lemma 3.1. For each \( r \), we have

(i) \( \sum_j (P_r)_{jj} = 0 \),

(ii) \( \text{tr}(P_r S_F) = (r + 1)\sigma_{r+1} \),

(iii) \( \text{tr}(P_r) = (n - r)\sigma_r \),

(iv) \( \text{tr}(P_r S_F^2) = \sigma_1 \sigma_{r+1} - (r + 2)\sigma_{r+2} \).

Proof. We only prove (ii), the others are easily obtained from (2.23), (2.26) and (2.27).

Noting \( (j, j_r) \) is symmetric in \( s_{i_1 j_1} \cdots s_{i_r j_r} \) by (2.23) and \( (j, j_r) \) is skew symmetric in \( \delta^{i_1 \cdots i_r j_r} \), we have

\[
\sum_j (P_r)_{jj} = \frac{1}{(r - 1)!} \sum_{i_1, \cdots, i_r} \delta^{i_1 \cdots i_r j_r} s_{i_1 j_1} \cdots s_{i_r j_r} = 0.
\]

\[ \square \]

Remark 2.5. When \( F = 1 \), Lemma 2.4 was a well-known result (for example, see Barbosa-Colares [2], Reilly [14], or Rosenberg [16]).

Since \( P_{r-1} S_F \) is symmetric and \( L_{ij} \) is anti-symmetric, we have

\[
(2.28) \quad \sum_{i, j, k} (P_{r-1})_{ji}(s_{kj} L_{ki} + s_{ik} L_{kj}) = 0.
\]

From (2.16), (2.26) and (i) of Lemma 2.4, we get

\[
(2.29) \quad (\sigma_r)_k = \sum_j (\sigma_r \delta_{jk})_j = \sum_j (P_r)_{jjk} + \sum_{j, l} [(P_{r-1})_{ji} s_{lk}]_j = \sum_{ij} (P_{r-1})_{ji} s_{ijk}.
\]

§3. FIRST AND SECOND VARIATION FORMULAS OF \( \mathcal{F}_{r,F,\Lambda} \)

Define the operator \( L_r : C^\infty(M) \to C^\infty(M) \) as following:

\[
(3.1) \quad L_r f = \sum_{i, j} [(T_r)_{ij} f]_i.
\]

Lemma 3.1. \( \frac{d\sigma_r}{dt} = \sigma'_r(t) = L_{r-1} \psi + \psi \langle T_{r-1} \circ d\nu, d\nu \rangle + \langle \text{grad} \sigma_r, \xi \rangle \).

Proof. Using (2.23), (2.28), (2.29), Lemma 2.1, Lemma 2.3 and (i) of Lemma 2.4, we have

\[
\sigma'_r = \frac{1}{(r - 1)!} \sum_{i_1, \cdots, i_r, j_1, \cdots, j_r} \delta^{i_1 \cdots i_r} s_{i_1 j_1} \cdots s_{i_{r-1} j_{r-1}} s'_{i_r j_r}.
\]
Lemma 3.2. For each $0 \leq r \leq n$, we have
\[(3.2) \quad \text{div}(P_r(\text{grad}_n F) \circ \nu_t) + F(\nu_t) \text{tr}(P_r \circ d\nu_t) = -(r+1)\sigma_{r+1},\]
and
\[(3.3) \quad \text{div}(P_rX^\top) + \langle X, \nu_t \rangle \text{tr}(P_r \circ d\nu_t) = (n-r)\sigma_r.\]

Proof. From (2.6), (2.15) and Lemma 2.4,

For each $0 \leq r \leq n$, we have
\[
\text{div}(P_r(\text{grad}_n F) \circ \nu_t) = \text{div}(P_r(\phi \circ \nu_t)) = \sum_{ij}((P_r)_{ji}(\phi \circ \nu_t, e_i))_j
\]
\[= - \sum_{ij}((P_r)_{ji} \delta_{ij} + (P_r)_{ji} h_{ij}) + F(\nu_t) \sum_{ij} (P_r)_{ji} h_{ij}
\]
\[= - \text{tr}(P_r S_F) - F(\nu_t) \text{tr}(P_r \circ d\nu_t)
\]
\[= -(r+1)\sigma_{r+1} - F(\nu_t) \text{tr}(P_r \circ d\nu_t),
\]

Thus, the conclusion follows.

\[\text{Theorem 3.3. (First variational formula of } \mathcal{A}_{r,F} \text{)}\]
\[(3.4) \quad \mathcal{A}_{r,F}'(t) = -(r+1) \int_M \psi \sigma_{r+1} dA_{X_t}.\]

Proof. We have $(F(\nu_t))' = \langle \text{grad}_n F, \nu_t' \rangle$, so by use of Lemma 3.1 Lemma 3.2, (2.2), (2.3), (2.8), (2.26) and Stokes formula, we have
\[
\mathcal{A}_{r,F}'(t) = \int_M (F(\nu_t) \sigma_r + (F(\nu_t))' \sigma_r) dA_{X_t} + F(\nu_t) \sigma_r \partial_t dA_{X_t}
\]
\[= \int_M \{F(\nu_t) \text{div}(T_{r-1} \text{grad } \psi) + F(\nu_t) \psi(T_{r-1} \circ d\nu_t, d\nu_t) + F(\nu_t)(\text{grad } \sigma_r, \xi)
\]
\[+ \langle \sigma_r(\text{grad}_n F) \circ \nu_t, - \text{grad } \psi + d\nu_t(\xi) \rangle + F(\nu_t)(\text{grad } F(\nu_t), \xi)
\]
\[= -nH \sigma_r F(\nu_t) + F(\nu_t) \sigma_r \text{div } \xi dA_{X_t}
\]
\[= \int_M \langle T_{r-1} \text{grad } F(\nu_t), \text{grad } \psi + F(\nu_t) \psi(T_{r-1} \circ d\nu_t, d\nu_t)
\]
\[+ \psi(\text{grad } \sigma_r(\text{grad}_n F) \circ \nu_t) - nH \psi F(\nu_t) \sigma_r dA_{X_t}
\]
\[= \int_M \psi(\text{div } \sigma_r(\text{grad}_n F) \circ \nu_t) + \psi(T_{r-1} \text{grad } F(\nu_t))
\]
\[+ F(\nu_t) \text{tr}(\sigma_r + T_{r-1} \circ d\nu_t + \sigma_r I) dA_{X_t}
\]
\[= -(r+1) \int_M \psi \sigma_{r+1} dA_{X_t}.\]
Remark 3.4. When $F = 1$, Lemma 4.1 and Theorem 3.3 were proved by R. Reilly [14] (also see [2, 3]).

From (1.6), (2.4) and (3.4), we get

**Proposition 3.5.** (the first variational formula). For all variations of $X$ preserving $V$, we have

$$
s_r' (t) = \mathcal{F}_{r,F,\Lambda}(t) = - \int_M \psi \{(r + 1)\sigma_{r+1} - \Lambda\} dA_X. $$

Hence we obtain the Euler-Lagrange equation for such a variation

$$
(r + 1)\sigma_{r+1} - \Lambda = 0.
$$

**Theorem 3.6.** (the second variational formula). Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional closed hypersurface, which satisfies (3.6), then for all variations of $X$ preserving $V$, the second variational formula of $s_r,F$ at $t = 0$ is given by

$$
s''_r(0) = \mathcal{F}''_{r,F,\Lambda}(0) = -(r + 1) \int_M \psi \{L_r \psi + \psi \langle T_r \circ d\nu, d\nu \rangle \} dA_X,
$$

where $\psi$ satisfies

$$
\int_M \psi dA_X = 0.
$$

**Proof.** Differentiating (3.5), we get (3.7) by use of (3.6). □

We call $X : M \to \mathbb{R}^{n+1}$ to be a stable critical point of $s_r,F$ for all variations of $X$ preserving $V$, if it satisfies (3.6) and $s''_r(0) \geq 0$ for all $\psi$ with condition (3.8).

§4. **Proof of Theorem 1.3**

Firstly, we prove that if $X(M)$ is, up to translations and homotheties, the Wulff shape, then $X$ is stable.

From $d\phi = (D^2F + F1) \circ dx$, $d\phi$ is perpendicular to $x$. So $\nu = -x$ is the unit inner normal vector. We have

$$
d\phi = -A_F \circ d\nu = \sum_{ijk} A_{jk} h_{ki} \omega_i e_j.
$$

On the other hand,

$$
d\phi = \sum_i \omega_i e_i,
$$

so we have

$$
s_{ij} = \sum_k A_{ik} h_{kj} = \delta_{ij}.
$$
From this, we easily get $\sigma_r = C_n^r$ and $\sigma_{r+1} = C_n^{r+1}$, thus the Wulff shape satisfies (3.6) with $\Lambda = (r+1)C_n^{r+1}$. Through a direct calculation, we easily know for Wulff shape,

\begin{equation}
\mathcal{A}_r''(0) = -(r+1)C_n^{r-1} \int_M [\text{div}(A_F \text{grad } \psi) + \psi \langle A_F \circ \nu, \nu \rangle] dA_X,
\end{equation}

and $\psi$ satisfies

\begin{equation}
\int_M \psi dA_X = 0.
\end{equation}

From Palmer [13] (also see Winklmann [18]), we know $\mathcal{A}_r''(0) \geq 0$, that is, the Wulff shape is stable.

Next, we prove that if $X$ is stable, then up to translations and homotheties, $X(M)$ is the Wulff shape. We recall the following lemmas:

**Lemma 4.1.** ([7, 8]) For each $r = 0, 1, \cdots, n-1$, the following integral formulas of Minkowski type hold:

\begin{equation}
\int_M (H^F_r F(\nu) + H^F_{r+1}(X, \nu)) dA_X = 0, \quad r = 0, 1, \cdots, n-1.
\end{equation}

**Lemma 4.2.** ([7, 8, 13]) If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \text{const} \neq 0$, then up to translations and homotheties, $X(M)$ is the Wulff shape.

From Lemma 4.1 and (3.8), we can choose $\psi = \alpha F(\nu) + H^F_{r+1}(X, \nu)$ as the test function, where $\alpha = \int_M F(\nu) H^F_r dA_X / \int_M F(\nu) dA_X$. For every smooth function $f: M \to \mathbb{R}$, and each $r$, we define:

\begin{equation}
I_r[f] = L_r f + f \langle T_r \circ \nu, \nu \rangle,
\end{equation}

Then, we have from (3.7)

\begin{equation}
\mathcal{A}_r''(0) = -(r+1) \int_M \psi I_r[\psi] dA_X.
\end{equation}

**Lemma 4.3.** For each $0 \leq r \leq n-1$, we have

\begin{equation}
I_r[F \circ \nu] = -\langle \text{grad } \sigma_{r+1}, (\text{grad}_{S^n} F) \circ \nu \rangle + \sigma_1 \sigma_{r+1} - (r+2) \sigma_{r+2},
\end{equation}

and

\begin{equation}
I_r[(X, \nu)] = -\langle \text{grad } \sigma_{r+1}, X^\top \rangle - (r+1) \sigma_{r+1}.
\end{equation}

**Proof.** From (2.8) and (2.26),

\begin{align*}
I_r[F \circ \nu] &= \text{div}\{T_r \circ \nu(F(\nu))\} + F(\nu) \langle T_r \circ \nu, \nu \rangle \\
&= \text{div}(T_r \circ \nu(\text{grad}_{S^n} F) \circ \nu) + F(\nu) \langle T_r \circ \nu, \nu \rangle \\
&= \text{div}(P_{r+1}(\text{grad}_{S^n} F) \circ \nu + F(\nu) \text{tr}(P_{r+1} d\nu) - (\langle \text{grad } \sigma_{r+1}, (\text{grad}_{S^n} F) \circ \nu \rangle \\
&\quad - \sigma_{r+1} \{\text{div}(P_0(\text{grad}_{S^n} F) \circ \nu) + F(\nu) \text{tr}(P_0 d\nu)\}.
\end{align*}
I_r[⟨X, ν⟩] = \text{div}(T_r\text{grad}(X, ν)) + ⟨X, ν⟩\langle T_r \circ dν, dν⟩
= \text{div}(T_r \circ dνX^\top) + ⟨X, ν⟩\langle T_r \circ dν, dν⟩
= \text{div}(P_{r+1}X^\top) + ⟨X, ν⟩\text{tr}(P_{r+1}dν) - ⟨\text{grad} σ_{r+1}, X^\top⟩

-σ_{r+1}\{\text{div}(P_0X^\top) + ⟨X, ν⟩\text{tr}(P_0dν)\}.

So the conclusions follow from Lemma 3.2.

As \(H^F_{r+1}\) is a constant, from (4.9) and (4.10), we have

\[
I_r[ψ] = αI_r[F \circ ν] + H^F_{r+1}I_r[⟨X, ν⟩]
\]

(4.11)

\[
= α(σ_1σ_{r+1} - (r + 2)σ_{r+2}) - (r + 1)H^F_{r+1}σ_{r+1}
= C_n^{r+1}\{α[nH^F_{r+1} - (n - r - 1)H^F_{r+2} - (r + 1)(H^F_{r+1})^2}\}.
\]

Therefore we obtain from Lemma 4.1 (recall \(H^F_{r+1}\) is constant and \(\int_M ψ\text{d}A_X = 0\)

\[
\frac{1}{r + 1}σ'^{(r)}_r(0)
= -\int_M ψI_r[ψ]dA_X
= -\int_M ψC_n^{r+1}\{α[nH^F_{r+1} - (n - r - 1)H^F_{r+2} - (r + 1)(H^F_{r+1})^2\}dA_X
= -αC_n^{r+1}\int_M [αF(ν) + H^F_{r+1}⟨X, ν⟩][nH^F_{r+1} - (n - r - 1)H^F_{r+2}]dA_X
= -α^2C_n^{r+1}\int_M F(ν)[nH^F_{r+1} - (n - r - 1)H^F_{r+2}]dA_X
= -α^2C_n^{r+1}\int_M F(ν)[nH^F_{r+1} - (n - r - 1)H^F_{r+2}]dA_X
+αC_n^{r+1}H^F_{r+1}\int_M F(ν)[nH^F_{r+1} - (n - r - 1)H^F_{r+1}]dA_X
= -α^2(n - r - 1)C_n^{r+1}\int_M F(ν)[H^F_{r+1} - H^F_{r+1}]dA_X
- \frac{α(r + 1)C_n^{r+1}(H^F_{r+1})^2}{\int_M F(ν)dA_X}\\int_M F(ν)H^F_{r+1}dA_X \int_M F(ν)H^F_{r+1}dA_X - (\int_M F(ν)dA_X)^2\}.
\]

As \(H^F_{r+1}\) is a constant, it must be positive by the compactness of \(M\). Thus, by Lemma 2.2 \(H^F_1, \ldots, H^F_r\) are all positive. So, from [6] or [19], we have:

(i) for each \(0 ≤ r < n - 1\),

\[
H^F_1H^F_{r+1} - H^F_{r+2} ≥ 0,
\]

with the equality holds if and only if \(λ_1 = \cdots = λ_n\), and

(ii) for each \(1 ≤ r ≤ n - 1\),

\[
\int_M F(ν)H^F_{r+1}dA_X \int_M F(ν)H^F_{r+1}dA_X - (\int_M F(ν)dA_X)^2
≥ \int_M F(ν)H^F_{r+1}dA_X \int_M F(ν)H^F_{r+1}dA_X - (\int_M F(ν)dA_X)^2
≥ 0,
\]

with the equality holds if and only if \(λ_1 = \cdots = λ_n\).

From (4.12) and (4.13), we easily obtain that, for each \(0 ≤ r ≤ n - 1\),

\[
σ'^{(r)}_r(0) ≤ 0,
\]
with the equality holds if and only if $\lambda_1 = \cdots = \lambda_n$. Thus, from Lemma 4.2, up to translations and homotheties, $X(M)$ is the Wulff shape. We complete the proof of Theorem 1.3.

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School of Mathematical Sciences, Shanxi University, Taiyuan 030006, P. R. China.
E-mail address: heyijun@sxu.edu.cn

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China.
E-mail address: hli@math.tsinghua.edu.cn