Determination of the shape of a beam obtained by fused deposition material with general loads

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Abstract. In this paper, we develop a general procedure for the obtaining of the shape of a beam obtained by FDM and having arbitrary loads. The beam is subjected to bending loads and because of its small elasticity modulus the approximate theory presented in the strength of materials cannot be applied. We used the general equation for the deformation of the beam resulting non-linear equations for the bending moment, the shear force, the rotation, the elongation and the deflection. These relations are solved by numerical methods. An example will highlight the theory.

1. Introduction

The bibliography presents only the case of cantilever beam with concentrated of distributed forces normal to the axis of beam and moments [1-6]. For these situations, authors propose various methods for the determination of the shape of the deformed beam, like floating frame of reference [1], weight coefficients of the buckling modes [2], development of the solution into series [3], correlations between mathematical curvature and physical curvature [4], numerical solutions obtained by using iterative procedure [5], consideration of trapezoidal deformation mode, cross-section distortion and absolute nodal coordinate formulation [6]. The most important hypothesis states that the length of the deformed beam is equal to the length of the non-deformed beam, that is, no axial elongation of the beam is taken into consideration [7].

Additive Manufacturing (AM) technology contains different variants as Stereolithography (STL) of a photopolymer liquid, Fused Deposition Modelling (FDM) from plastic filaments, Laminated Object Manufacturing (LEM) from plastic laminations, and Selective Laser Sintering (SLS) from plastic or metal powders. The most used is FDM, also named Fused Deposition Material.

Fused Deposition Material (FDM) is an efficient manufacturing process with the aid of which one may fabricate models with complicate geometry at reasonable prices. The great disadvantage of the method consists in the fact that FDM works only with some particular material like Acrylonitrile Butadiene Styrene (ABS), High Impact PolyStyrene (HIPS), PolyLactic Acid (PLA) etc. Because of the physical properties of these materials, the prototypes manufactured by FDM present small elasticity modulus ($E \approx 2 \times 10^9$ Pa), small rigidity, fragility etc.

In a FDM process, the build material is partially melted and extruded from a nozzle onto a table in an environment, which is temperature controlled. In this way, a part is built layer by layer. The process is an anisotropic one, depending on the orientation of the part, layer thickness, raster angle, part raster width, and raster to raster gap (air gap). Many studies are devoted to the determination of the
mechanical properties of the parts obtained by FDM. In this paper we discuss the shape of the beams manufactured by FDM with general loads, that is, considering the axial deformations of them.

2. The case of a cantilever beam

The situation is captured in figure 1. The cantilever beam OA, having the free end situated at point A is acted by a transversal force $P$ (normal to the axis of the beam), and the axial force $H$. Both forces are considered to be known. In addition, the mechanical and geometric parameters (modulus of elasticity $E$, the dimensions of the transversal section of the beam, that is, the modulus of inertia $I$, the cross section area $A$) are given. We consider that both $E$ and $I$ are piece-wise continuous functions depending on $x$ (the position along the beam). The beam changes its shape, resulting the $OA'$ beam, but the forces $P$ and $H$ do not change their directions and senses. The case studied in this section differs from those presented in the bibliography by the fact that one considers the effect of the axial force $H$. The axial force cannot be neglected in the case of the beams obtained by fused deposition material when the rigidity of the beams is small (the elasticity modulus $E$ has much smaller value).

![Figure 1. Cantilever beam with large deflection and elongation.](image)

If the force $H$ is neglected (the classic case), then the deformed beam is $OA'$, having the same length $L$ with the non-deformed beam. The consideration of the force $H$ leads to a deformed beam of new length $L_f$.

Taking into account the expression of the axial force at an arbitrary point along the beam

$$F_a = P\sin\theta + H\cos\theta$$

one may write the following equations:

$$L_f = \int_0^{L-\Delta} \sqrt{1+(y')^2} \, dx,$$

$$L_f = \int_0^{L-\Delta} \sqrt{1+(y')^2} \, dx,$$
\[
L_f = L + \frac{1}{E(x)A(x)} \left[ P \sin \theta + H \cos \theta \right]_0^L = L + \frac{1}{E(x)A(x)} \int_0^L \left[ \frac{Py' + H}{\sqrt{1 + (y')^2}} \right] \, dx,
\]
(3)

\[
y'' = \frac{P(L - x) - H(h - y)}{E(x)f(x)}\left[1 + (y')^2\right]^{1/2},
\]
(4)

\[
h = y(L - \Delta),
\]
(5)

where \( y = y(x) \) is the equation of the shape of the deformed beam, \( \theta \) is the angle between the tangent at the shape of the deformed beam and the horizontal direction, \( y' = \frac{dy}{dx} = \tan \theta \), \( y'' = \frac{d^2y}{dx^2} \).

The equations (2), (3), (4), and (5) form a system of four non-linear integral-differential equations with four unknowns \( (y, L_f, \Delta, h) \). The system may be solved by numerical methods and trial procedures. In the case of the cantilever beam, one may add the condition \( y(0) = 0 \), that is, the rotation at the point \( O \) is equal to zero.

The method may be extended to other type of beams, changing the conditions on the frontier.

3. The case of two jointed cantilever beam

The system is described in figure 2 and it consists in two cantilever beams \( OA_1 \) and \( OA_2 \) of initial lengths \( L_{10} \) and \( L_{20} \), modulii of elasticity \( E_1 \) and \( E_2 \), moments of inertia \( I_1 \) and \( I_2 \) (both \( E \) and \( I \) being functions of the position \( x \)). The beams are jointed by a cylindrical joint at the point \( A \). At the point \( A \) acts a transversal force denoted by \( P \). The deformed beams have the lengths \( L_1 \) and \( L_2 \), respectively. A part of the force \( P \), denoted by \( P_1 \), acts upon the first beam, while the rest of the force \( P \), denoted by \( P_2 \), acts upon the second beam. The beams deform, the new position of the point \( A \) being \( A' \).

Proceeding as in the previous paragraph, one may write the equations:

\[
L_1 = L_{10} + \frac{1}{E_1(x_1)A_1(x_1)} \int_0^{L_{10}} \left[ H \cos \theta_1 + P_1 \sin \theta_1 \right] \, dx_1,
\]
(6)

\[
L_2 = L_{20} + \frac{1}{E_2(x_2)A_2(x_2)} \int_0^{L_{20}} \left[ H \cos \theta_2 + P_2 \sin \theta_2 \right] \, dx_2,
\]
(7)

\[
L_1 = L_{10} + \frac{1}{E_1(x_1)A_1(x_1)} \int_0^{L_{10}} \frac{H + P_1 y'_1}{\sqrt{1 + (y'_1)^2}} \, dx_1,
\]
(8)

\[
L_2 = L_{20} + \frac{1}{E_2(x_2)A_2(x_2)} \int_0^{L_{20}} \frac{H + P_2 y'_2}{\sqrt{1 + (y'_2)^2}} \, dx_2,
\]
(9)
\[ P_1 + P_2 = P, \]  
\[ y_1'' = \frac{P_1(L_{10} - x_1) - H(y_1)}{E_1(x_1)I_1(y_1)}, \]  
\[ y_2'' = \frac{P_2(L_{20} - x_2) - H(y_2)}{E_2(x_2)I_2(x_2)}, \]  
\[ h = y_1(L_{10} + d) = y_2(L_{20} - d), \]

where \( y_1' = \frac{dy_1}{dx_1}, \ y_2' = \frac{dy_2}{dx_2}, \ y_1'' = \frac{d^2y_1}{dx_1^2}, \ y_2'' = \frac{d^2y_2}{dx_2^2}. \) The equations (6) – (13) form a system of nonlinear integral differential equations with eight unknowns \( (L_1, L_2, d, H, y_1, y_2, P_1, \text{ and } P_2). \)

**Figure 2.** System of two cantilever beams with large deflections and elongations.

**4. Example**

As example we consider the system in figure 2, for which \( E_1 = E_2 = E = ct, \ I_1 = I_2 = I = ct, \ A_1 = A_2 = A = ct, \ L_{40} = L_{20}. \)

Taking into account the symmetry, the system reduces to:
\[ L_5 = L_{10} + \frac{1}{EA} \int_0^{L_{10}} \frac{H + \frac{P}{2} y'}{\sqrt{1 + (y')^2}} \, dx, \]  
(14)

\[ L_4 = \int_0^{L_{10}} \sqrt{1 + (y')^2} \, dx, \]  
(15)

\[ \frac{y''}{\sqrt{1 + (y')^2}} = \frac{P}{2} (L_{10} - x) - H (h - y) \],
(16)

\[ h = y(L_{10}). \]  
(17)

We assume that the shape of the deformed beam is approximated by a parabola, that is, \( y = a_2 x^2, \) \( y' = 2a_2 x, \) \( y'' = 2a_2. \) The reader may notice that the conditions \( y(0) = 0 \) and \( y'(0) = 0 \) imply the vanishing of the linear component for the equation of the parabola.

In relation (16) we denote \( p = y' = \tan \theta, \) and \( \theta_f \) the values of angle \( \theta \) at the point \( A, \) and separate the variables resulting

\[ \int_0^{\theta_f} \cos \theta d\theta = \frac{1}{EL} \int_0^{L_{10}} \left[ \frac{P}{2} (L_{10} - x) - H (a_2 L_{10}^2 - a_2 x^2) \right] \, dx, \]  
(18)

wherefrom

\[ \sin \theta_f = \frac{L_{10}^2}{36EI} (9P - 32a_2 L_{10} H). \]  
(19)

On the other hand, \[ |\sin \theta| = \frac{P}{\sqrt{1 + p^2}}, \] and the last relation becomes

\[ -\frac{2a_2 L_{10}}{\sqrt{1 + 4a_2^2 L_{10}^2}} = \frac{L_{10}^2}{36EI} (9P - 32a_2 L_{10} H). \]  
(20)

Expression (14) leads to

\[ L_7 = L_{10} + \frac{H}{2a_2 EA} \ln \left( 2a_2 L_{10} + \sqrt{1 + 4a_2^2 L_{10}^2} \right) + \frac{P}{4a_2 EA} \left[ \sqrt{1 + 4a_2^2 L_{10}^2} - 1 \right], \]  
(21)

while the relation (15) offers

\[ L_4 = \frac{1}{4a_2} \ln \left( 2a_2 L_{10} + \sqrt{1 + 4a_2^2 L_{10}^2} \right) + \frac{L_{10}}{2} \sqrt{1 + 4a_2^2 L_{10}^2}. \]  
(22)

Equating the expressions (21) and (22), it results

\[ L_{10} + \frac{H}{2a_2 EA} \ln \left( 2a_2 L_{10} + \sqrt{1 + 4a_2^2 L_{10}^2} \right) + \frac{P}{4a_2 EA} \left[ \sqrt{1 + 4a_2^2 L_{10}^2} - 1 \right] \]

\[ = \frac{1}{4a_2} \ln \left( 2a_2 L_{10} + \sqrt{1 + 4a_2^2 L_{10}^2} \right) + \frac{L_{10}}{2} \sqrt{1 + 4a_2^2 L_{10}^2}. \]  
(23)
Equation (19) is a valid one for \(0 < \sin \theta_f < 1\), wherefrom

\[
\frac{32}{9} a_2 L_{10} H < P < \frac{1}{9} \left( \frac{36EI}{L_{10}^2} + 32a_2 L_{30} H \right),
\]

(24)

which can be seen as a condition for the existence of the solution.

The shape of the deformed beam was approximated as a parabola. Other forms of shape can be also selected, but the system becomes more complicated.

The equations (20) and (23) form a system of two nonlinear equations with two unknowns (the parameter \(a_2\) and the axial force \(H\)). This system can be solved by numerical methods.

As a numerical application, we choose the following values: \(E = 2 \times 10^9\) Pa, a square section of dimension \(a = 0.003\) m, and the initial length of bars \(L_{40} = 0.3\) m, \(P = 100\) N. It results \(a_2 = 0.49204\), \(H = 254.487\) N, \(\theta_f = 16.4479^0\), \(h = 0.0443\) m.

5. Conclusions

The paper presents the equations from which one can determine the shape of a cantilever beam with large elongation and deflection. These equations are extended to the case of two cantilever beams jointed by a cylindrical joint. In a similar way, one may determine the equations for a general system of beams.

The obtained system is a highly nonlinear one and its solving is a great challenge. For the particular case described in paragraph 4, the system reduces to two nonlinear equations. Even in this simple case, the determination of the unknowns requires the use of numerical methods.

If the two bars are rigidly connected at the point \(A\), the situation is more complicated, that is, one has to add some frontier conditions. The nonlinear system can now be solved by a variant of point matching method (collocation method) developed by Kantorovich, presented and used for another problem in [8]. The greatest challenge consists in the selection of the initial, which has to approximate the shapes of the beams. Obviously, in this situation, the terms of odd powers has to vanish too, due to the conditions of the cantilever ends of the beams.

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