Lattice and quintic nonlinearity induced stripe phase in Bose–Einstein condensate under non-inertial and inertial motion

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Abstract
We consider a parametrically forced Bose–Einstein condensate in the combined presence of an optical lattice and harmonic oscillator potential in the mean field approach. A spatial symmetry broken Bose-condensed phase in non-inertial and inertial frame yields a stripe phase in the presence of both cubic and quintic nonlinearities. We show that the existence of such stripe phase solely depends on the interplay between the quintic nonlinearity and the lattice potential. Furthermore, we observe that a time-dependent harmonic oscillator frequency destroys such stripe ordering. A linear stability analysis of the obtained solution is performed and we found that the solution is stable. In order to gain a better understanding of the underlying physics, we compute the energy, showing nonlinear compression of the condensate in some parameter domain.

1. Introduction

Bose–Einstein condensate (BEC) in a periodic potential has been an area of active research for more than two decades [1, 2]. It is truly an interdisciplinary field, which has connections with many areas of physics: electrons in crystal lattices [3, 4], polarons [5, 6], photons in optical fibers [7], gauge theories [8] and exotic phase transitions [9–12], to mention a few. A major advantage of the Bose-condensed gas in a periodic potential is its tunability over a wide range of parameter domain. For example, the depth and width of an optical lattice can easily be tuned by controlling the intensity and frequency of the laser beam respectively, whereas, the two-body atom-atom interaction strength can be controlled by means of Feshbach resonance. This has opened up the possibility to realize many novel phases of matter, existing in theoretical realm [2, 13–17], many of which have been realized in experiments as well [18–23].

Recently observed supersolid stripe phase has provided a unique platform to explore the novelty of the striped ordering [18]. The existence of the stripe phase is characterized by the spontaneous breaking of the translational symmetry, giving rise to the diagonal long-range order, mimicking a crystalline structure. A number of papers have already studied the properties of the stripe phase in a spin-orbit coupled BEC systems [24–27]. In the weak coupling regime of spin-orbit coupled BEC, the resulting Hamiltonian possesses a spectrum with a doubly degenerate ground state. Therefore, the condensate can reside at each minimum of the single particle energy-momentum dispersion. The resulting interference pattern yields a stripe phase in the system. Stripe phases are also well-known in high-temperature superconductivity, where it arises due to the interplay between antiferromagnetic interactions among the magnetic ions and the Coulomb interactions between the carriers [28]. The stripe phase has also been predicted in Bose–Hubbard model in a triangular optical lattice [29], quantum Hall system [30], core-corona system [31], Harper–Hofstadter–Mott model [32, 33] etc.

Till date, significant research works have been carried out to investigate the dynamics of a BEC in an optical lattice, where only cubic nonlinearity is present. However, BEC with both cubic and quintic nonlinearity in the presence of an optical lattice potential has not received much attention. Quintic nonlinearity is in general considered as the phenomenological manifestation of the three-body interaction. For higher densities of the Bose-condensed gas at absolute zero, the three-body collisions play a crucial role, where it is modelled by a
quintic nonlinearity in a modified Gross-Pitaevskii equation in the mean-field theory. One of the major application of quintic nonlinearity is its role in realization of the Tonks-Girardeau (TG) gas [34–36]. Apart from TG gas, quintic nonlinearity appears in a number of references. Recently, the formation of breathers through spatiotemporal vortex light bullets is discussed in a cubic-quintic nonlinear medium [37]. Oscillation of a dark soliton in a BEC with both cubic and quintic nonlinearities is observed, where the quintic nonlinearity in one dimension arises due to the interplay between the radial and axial degrees of freedom [38]. A classical dynamical phase transition in BEC occurring through the loss of superfluidity in the mean-field regime is also proposed, where both cubic and quintic nonlinearity are considered apart from an optical lattice potential [39]. It is worth mentioning that such quintic term often leads to interesting phenomena through instabilities, e.g., Faraday waves [40], Bose-Nova effect [41] etc. However, the quintic nonlinearity combining with an optical lattice potential can suppress these instabilities, which leads to many interesting physics.

Motivated by the above works, we propose a scheme to observe stripe-ordered phase in BEC, in the presence of both cubic and quintic nonlinearity. In addition to the optical lattice potential, we consider that the system is trapped in a harmonic oscillator potential as well. For the better understanding of the underlying physics, we consider all the associated parameters are time-dependent, however, this makes the system quite complicated. In order to simplify the model, we transform the equation of motion into the center of mass frame. We analyze the following two possible cases: (a) when the center of mass (COM) is moving with a time-dependent velocity. This is also known as the non-inertial frame, where COM is bustling with finite acceleration. It is found that the finite acceleration of the COM solely depends on the frequency of the harmonic trap, also known as the Kohn mode. (b) When the harmonic trap is switched off, the COM is either moving with a constant velocity or becomes static, depending on the chirped profile. This is known as the inertial frame of reference, where the COM has zero acceleration.

The novelty of our scheme is to study stripe ordered phase, also known as superstripes, in a combined presence of a harmonic trap and optical lattice in the context of non-inertial and inertial BEC. In order to solve such highly complicated inhomogeneous system, we employ a simple method, the self-similar method, also known as the F-expansion method, where the corresponding excitations are necessarily chirped. We show that a spatially broken symmetry of the condensate leads to a stripe phase in both non-inertial and inertial frame. This is solely due to the interplay between lattice potential and quintic nonlinearity. In the case of pure cubic nonlinearity, the periodicity of the lattice potential induces the same periodicity in the density modulation [14, 15, 17]. However, the periodicity of the density modulation becomes twice to that of the lattice potential, when the quintic nonlinearity is present apart from the cubic nonlinearity. This brings a restriction on the density modulation and hence, the superfluid matter is found to exist only in certain parameter domains, yields a stripe phase. We observe that this stripe phase exists both in the presence and absence of the harmonic trap. A linear stability is performed in order to check the stability of the obtained solution and we found that the solution is stable. On the other hand, the presence of harmonic trap in the system necessitates the presence of a chirped phase, which yields an efficient nonlinear compression in some parameter regime. In order to gain a better understanding of the nonlinear compression, we compute the energy of the system. In presence of a repulsive (regular) harmonic trap, these stripe phases lead to resonances. When the frequency of the chirped pulses is in resonance with the frequency of the harmonic trap, a significant increase in kinetic energy is observed, which gives rise to the nonlinear compression of the condensate. In presence of expulsive harmonic trap (inverted), the energy initially increases and then decreases. In this case, the periodic excitations initially move towards the center of the trap and gains energy. However, due to the instabilities at the top of the expulsive trap, the excitations move downhill, thus the corresponding energy decreases. In the absence of the trap, the lattice potential plays a dominant role. When the lattice moves with a constant velocity, the energy decreases, due to the expansion of the BEC. We observe another resonance behavior when the lattice is static, since the BEC undergoes a rapid nonlinear compression. Such resonance arises when the amplitude of the sinusoidal modulation is comparable with its background density, analogous to the effective pulse compression in nonlinear optical fiber, observed by Moore [42].

2. Model and equation of motion

In this section, we briefly introduce the model and corresponding equation of motion of the system. We consider a BEC, immersed in a time modulated optical lattice potential, where both two- and three-body interactions are present. In the present scenario, an effective three-body interaction appears through a quintic nonlinearity. The nonlinear Schrödinger equation (NLSE) is a generic model, describing the dynamics of Bose–Einstein condensate. We would like to emphasize that quintic nonlinearity appears in nonlinear fiber optics, where the same describes the behavior of the light pulses in an optical fiber [43]. In general, for the pico-second light pulses, the NLSE admits group velocity dispersion, mimicking an interaction, known as `self-phase
modulation’. However, if one introduces a shorter pulse propagation (of the order of femto-second) by increasing the intensity of the incident light, additional nonlinear effects become important. The dynamics of such short pulse propagation can be described by a generalized NLSE, which includes higher order nonlinear terms [43, 44]. Keeping in mind the above consequences, we consider a BEC with cubic and quintic nonlinearities, loaded in a time modulated optical lattice potential. The dynamics of such a system is described by the following generalized NLSE [45–47],

$$i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + U_1|\Psi|^2 + U_2|\Psi|^4 - \nu(t) \right) \Psi. \quad (1)$$

where, $\Psi \equiv \Psi(x, y, z, t)$ is the condensate order parameter, $V_{\text{ext}} \equiv V_{\text{ext}}(x, y, z, t) = V_L(x, y) + V_{\text{z}}(z, t)$ is the external trapping potential and $\nu(t)$ is the time dependent chemical potential. We consider general time-dependent two-body ($U_1 \equiv U_1(t)$) and three-body ($U_2 \equiv U_2(t)$) interactions, which can be tuned by using Feshbach resonance technique [48–51]. In order to have a cigar-shaped BEC, we apply a strong harmonic confinement along the transverse directions ($x$-$y$), $V_L(x, y) = \frac{1}{2}m\omega_z^2(x^2 + y^2)$ with a trapping frequency $\omega_z (\sim 2\pi \times 360 \text{ Hz})$ and harmonic oscillator length scale $a_L = \sqrt{\hbar/m\omega_z}$. In the longitudinal direction, the trapping potential takes the form: $V_{\text{z}}(z, t) = \frac{1}{2}m\omega_{\text{z}}^2(z^2 + V_{\text{ext}}(z) + \omega_{\text{ext}}(t)(\sim 2\pi \times 3.5 \text{ Hz})$ being the longitudinal frequency [52] and $V_{\text{ext}}(z, t)$ is the optical lattice potential. It is worth highlighting that the discrete NLSE (DNLSE) would be useful to use describe the dynamics of the BEC, when the lattice depth is considered to be large. The DNLSE can easily be obtained from NLSE [53, 54]. However, in this manuscript, we consider the regime of shallow optical lattice potential ($V_0 < E_R$ with $E_R$ being the recoil energy), where NLSE is useful to describe the dynamics of the condensate over DNLSE. Due to the symmetry of the system, we make separation of variables along the transverse and longitudinal direction $\Psi = \psi(z, t) \phi(x, y)$, where the transverse component $\phi(x, y)$ is assumed to be a normalized Gaussian function. After a lengthy algebra, one obtains a generalized NLSE in quasi-one dimension in the limit $\omega_z \gg \omega_{\text{ext}}$ [44–55]:

$$i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_L + g_1(t) |\psi|^2 + g_2(t) |\psi|^4 - \nu(t) \right) \psi. \quad (2)$$

The reduced two- and three-body interactions are given by, $g_1 = \frac{m\omega_z}{2\pi k} U_1$ and $g_2 = \frac{m^2\omega_z^2}{2\pi^2} \frac{m}{k^2} U_2$, respectively, where, $k = 2\pi / \lambda$ is the wave vector with $\lambda (\sim 1064 \text{ nm})$ being the wavelength of the laser. The terms inside the bracket in $g_1$ and $g_2$ are due to scaling. The time, spatial coordinate and the wavefunction are scaled as $t \to m/\hbar k^2 t$, $z \to z/k$ and $\psi \to \sqrt{k} \psi$, respectively. The chemical potential $\nu(t)$ and the amplitude of the lattice potential have been normalized in terms of the recoil energy $E_R = \hbar^2 k^2/2m$ [55].

We would like to point out that most of the ultra-cold atomic systems are inhomogeneous. Due to this fact, we now consider a general potential $V(z, t) = V_{\text{HO}}(z, t) + V_{\text{ext}}(z, t)$, comprising of a harmonic trap of the form $V_{\text{HO}}(z, t) = \frac{1}{2}M(t)z^2$, in addition to the time modulated lattice potential $V_{\text{ext}}(z, t) = V_0(t)\cos^2(\xi(t))$, with $\xi \equiv \xi(z, t) = \Theta(t)(z - z_0(t))$ being the center of mass coordinate. Here, the position of the lattice in the COM coordinate is determined by $z_0(t)$ and the $\Theta(t)$, having the dimension of inverse of width, can be controlled by the frequency of the laser beam. The amplitude of the lattice potential $V_0(t)$, can be tuned by the intensity of the laser beam. Both $V_0(t)$ and $V_{\text{HO}}(z, t)$ are scaled by the recoil energy $E_R$ [55].

3. Results: stripe phase

3.1. Methodology

There are various methods that have been employed in order to solve the above equation, e.g., variational approximation and perturbation methods. However, in this paper, we consider a different approach, known as F-expansion method, which has been widely used in nonlinear fiber optics as well as in the context of BEC [56–60]. The F-expansion method is found to be a very useful and direct algebraic method for finding the exact solutions of nonlinear equations [56, 57]. The key point of this method is to transform the nonlinear system from laboratory frame to the center of mass frame (COM) through appropriate transformation, $\xi = \Theta(t)(z - z_0(t))$. In other words, one looks for a traveling wave solution in the center of mass frame and transforms the equation to ordinary differential equation. Without any loss of generality, we set $\hbar = m = 1$. Therefore, we assume the following ansatz in the center of mass frame:

$$\psi(z, t) = \sqrt{\theta(t)\sigma(\xi)} e^{i\theta(t)+\Xi(z,t)}. \quad (3)$$

One must note that $\sigma(\xi)$ refers to the density of the condensate, when the harmonic trap is absent [39]. However, in the presence of the harmonic trap, the condensate density becomes $\theta(t)\sigma(\xi)$. $\Theta(\xi)$ is a nontrivial phase, which is directly related to the velocity of the condensate and $\Xi(z, t)$ is of kinematic origin.
\[ \Xi(z, t) = \varepsilon(t) + p(t)z - \frac{1}{2}c(t)z^2, \] (4)

where, the first term \( \varepsilon(t) \) and \( p(t) \) mimics the energy and the momentum, whereas, \( c(t) \) corresponds to chirped phase. This type of chirped phase regularly arises in nonlinear fiber optics, as an acceleration induced inhomogeneity, which can balance the effect of harmonic trap. The chirped phase regularly appears in fiber optics, as well as nonlinear optics, owing to its origin as acceleration induced inhomogeneity. It often arises due to the interplay between the interaction and harmonic oscillator potential.

From current conservation, amounting to solve the imaginary part of equation (2), we find,
\[ \partial \Theta / \partial \xi = u(1 - \frac{1}{\kappa^2}), \]
where \( c_0 = (\kappa_1^2 - 4\alpha\kappa_2)/16\kappa_1^2 \) is a constant of integration and \( u = \sqrt{2\kappa_1^2 - \kappa_2^2 + 8\kappa_2\mu}/2\kappa_2 \) is the velocity of the soliton for the homogeneous condensate. We observe that the two-body interaction strength needs to be time-dependent: \( g_1(t) = \kappa_1\vartheta(t) \), however, the three body interaction is time-independent; \( g_2 = \kappa_2 \). The amplitude of the lattice potential has been re-scaled as, \( V_0(t) = \alpha \vartheta^2(t) \). Here, \( \kappa_1, \kappa_2 \) and \( \alpha \) are, respectively, the reduced two-body, three-body interaction strength and the amplitude of the lattice potential. The chemical potential of the condensate in the combined presence of the harmonic trap and optical lattice has also been re-scaled as, \( \nu(t) = \mu \vartheta^2(t) \), where \( \mu \) is the chemical potential of the homogeneous condensate.

The COM motion obeys the following equation,
\[ \frac{\partial \varepsilon_0(t)}{\partial t} + c(t)\varepsilon_0(t) = \vartheta(t)(1 + u), \] (5)
and subsequently, the other parameters are found to emerge from the consistency conditions,
\[ \varepsilon(t) = \left(\mu - \frac{1}{2}\right) \int_0^t \vartheta^2(t') dt', \] (6a)
\[ \frac{\partial \vartheta(t)}{\partial t} = \vartheta(t)c(t), \quad p(t) = \vartheta(t), \] (6b)
\[ \frac{\partial c(t)}{\partial t} = c^2(t) + M(t). \] (6c)

At this point, let us stop for a while and carefully analyze the above equations, which brings interesting physics as well. The COM equation can be written in the form of a linear Schrödinger equation (LSE):
\[ \frac{\partial^2 \varepsilon_0(t)}{\partial t^2} + M(t)\varepsilon_0(t) = 0. \] (7)

The center of mass motion of the sinusoidal excitation, also known as the Kohn mode, is found to depend solely on the frequency of the harmonic trap. It is well known that in the case of a BEC confined in a harmonic oscillator trap, the COM oscillates with the frequency of the harmonic trap. If the trap frequency varies with time, the COM no longer oscillates with the frequency of the harmonic trap. In the present case, we will establish the connection between Kohn mode \([61–64]\) and the existence of the stripe phase, which we discuss in the following section. On the other hand, the equation (6c), known as Reccati equation, can be effectively mapped to LSE. Considering a change of variable, \( c(t) = -\frac{\partial \varphi(t)}{\partial t} \), we obtain \([65]\),
\[ \frac{\partial^2 \varphi(t)}{\partial t^2} + M(t)\varphi(t) = 0. \] (8)

This provides a correspondence between any solvable quantum mechanical potential and quasi-1D BEC and allows one to investigate the dynamics of the cnoidal excitations by engineering the underlying potential through LSE.

Now, the real part of the generalized NLSE reduces to the form of a generalized elliptic equation (for details, see appendix A) in terms of the condensate density \( \sigma(\xi) \) (homogeneous),
\[ \frac{1}{4}\sigma(\xi) \frac{\partial \sigma^2(\xi)}{\partial \xi^2} - \frac{1}{8} \left( \frac{\partial \sigma(\xi)}{\partial \xi} \right)^2 + \frac{1}{2}u^2 - \mu \sigma^2(\xi) + \kappa_1\sigma^3(\xi) + \kappa_2\sigma^4(\xi) + \alpha \cos^2(\xi)\sigma^2(\xi) + \frac{1}{2}\sigma_0 = 0. \] (9)

The first two terms in equation (9), mimic the contribution from the kinetic energy, where third term arises due to the presence of nontrivial phase and chemical potential. The fourth, fifth and sixth terms, respectively, refer to the cubic nonlinearity, quintic nonlinearity and the lattice potential. The last term corresponds to the background contribution.
3.2. Solution
The reduction to a generalized elliptic equation leads to some interesting facts. For constant $\alpha$, one finds a self-similar solution in terms of the periodic cnoidal wave $\sigma(\xi) = A + B \cn(\xi, \tau)$, where $\tau$ is the elliptic modulus parameter. However, in the present case, we are more interested in sinusoidal excitations and hence we choose $\tau = 0$. In this limit, the solution takes the form:

$$\sigma(\xi) = -\frac{\kappa_1}{2\kappa_2} \pm \sqrt{\frac{\alpha}{\kappa_2}} \cos(\xi).$$ (10)

It is worth observing that the obtained solution is quite different from the solutions existing in the literature for BEC with pure cubic nonlinearity, loaded in an optical lattice potential [14, 15, 17], where, the density modulation is same as that of lattice potential. However, due to the presence of the quintic nonlinearity, the density modulation becomes twice to that of the lattice potential [39]. A much deeper analysis reveals that the present solution also exhibits stripe phase in certain parameter regime. It is found that the above solution exists only when the strength of the quintic nonlinearity $\kappa_2$ carries opposite signature to that of both two-body interactions and the scaled amplitude of the lattice potential. Therefore, without any loss of generality, we consider the effective three-body interaction to be attractive, whereas, the two-body interaction is repulsive. Defining $\frac{\kappa_2}{\kappa_c} \sqrt{-\frac{\alpha}{\kappa_2}} = \Gamma$; the solution $\sigma(\xi)$ is found to exist for a particular parameter regime, such that $|\cos(\xi)| > \Gamma$. Hence, the following three cases arise, depending on the value of $\Gamma$:

(a) For $\Gamma < -1$, the solution exists for all $\xi$. In this case, the three body interaction strength ($\kappa_2$) and $\alpha$ must have opposite signature.

(b) For $\Gamma > 1$, the solution does not exist.

(c) When $-1 < \Gamma < 1$, the solution exists for $-\pi/2 < -\xi_i < \xi < \xi_i < \pi/2$, where $\xi_i = \cos^{-1}\Gamma$. In this parameter regime, we observe that the superfluid matter wave can only be found in the finite domain, in alternate lattice sites, indicating the existence of a stripe phase. This is a unique characteristic of the present system, where such ordered phase appears purely due to the interplay between quintic nonlinearity and lattice potential. The translational symmetry of the condensate spatial profile is spontaneously broken due to the presence of the lattice. This phenomenon is analogous to the density wave, where the superfluid matter exists in alternative lattice sites.

3.3. Discussions
3.3.1. Non-inertial frame of reference:
In this section, we discuss the existence of stripe phase in different scenarios, when the harmonic trap is present along with a time modulated optical lattice potential. In the present case, we consider $-1 < \Gamma < 1$, where a stripe phase exists due to the interplay between the modulated lattice potential and the interactions, and not due to the harmonic trap. In the following section, we show the existence of the stripe phases when the harmonic trap is switched off. In presence of the harmonic trap, we observe that the COM moves with finite acceleration, and hence such frame of reference is referred as a non-inertial frame. We initially consider the harmonic trap to be a constant, e.g., $M(t) = \gamma^2$, where $\gamma$ refers to the strength of the trap. This represents a regular harmonic trap and the corresponding position in the center of mass frame is $z_0(t) = l_0 \sin(\gamma t)$, where $l_0$ is a constant. The associated chirped phase and the inverse of width follow: $c(t) = \gamma \tan(\gamma t)$ and $\ddot{\theta}(t) = \dot{\theta}_0 \sec(\gamma t)$ with $\dot{\theta}_0$ being a constant. The coupling of the interactions and the lattice potential spontaneously breaks the translational symmetry of the BEC, leading to a stripe phase. The stripe ordering of the condensate is shown in figure 1(a). The edges near $t \sim 1.8$ and $t \sim 4.8$ are the high-density regime, also known as nonlinear compression of the BEC. Such high density arises from the condition when the harmonic trapping frequency is in resonance with the chirped pulses. In the presence of a regular harmonic trap, the COM motion undergoes an oscillation with the trap frequency and thereby satisfies Kohn theorem. Furthermore, the supercurrent takes the following form,

$$J(t) = u(\sigma(\xi) - \sigma_0).$$ (11)

The behavior of supercurrent is shown in figure 2. Figure 2(a) corresponds to $u = 1$. This induces a positive effective net phase in the COM motion (see r.h.s. of equation (5)) and the point of nonlinear compression shifts away from $z = 0$. Figure 2(b) depicts the variations of supercurrent for $u = -1$, where one can see that the point of nonlinear compression lies at $z = 0$, since no net phase is accumulated in COM motion. Furthermore, it is interesting to observe that the supercurrent changes its direction at the point of nonlinear compression, as can be seen in both figures 2(a) and (b).

Inspired by the work [66], we consider another scenario, where the harmonic trap is considered as an inverted trap, i.e., $M(t) = -\gamma^2$, also known as an expulsive oscillator. We also observe the stripe phase, which exists only in the finite domain, with a spatially broken symmetry. The corresponding evolution of the density is
shown in figure 1(b). In this case, the center of mass motion follows a localized profile, $z_0(t) = l_0 \sinh(\gamma t)$. Corresponding chirped pulse and the inverse of the width, both are found to be localized, as expected, $c(t) = \gamma \tanh(\gamma t)$ and $\vartheta(t) = \vartheta_0 \sech(\gamma t)$. As the time increases, the amplitude of the propagating wave decreases, making the BEC spread out. Hence, the competition between time-dependent lattice amplitude and the oscillator trap deforms the simple periodic structure of the matter waves.

However, when the harmonic trapping frequency depends on time, the center of mass motion does not oscillate with the trapping frequency and the stripe phase disappears. This is due to the fact that in presence of a time dependent harmonic trap e.g., $M(t) = e^{-\alpha t}$, the chirped phase is described by the Bessel function and consequently destroys the underlying periodicity of the system. This phenomenon violates the Kohn theorem and hence the stripe phase disappears (for details, see appendix B).
obtained numerically by solving equation where, we set perturbation to the obtained solution, the BEC. In this section, we perform a linear stability analysis of the obtained phases. We apply a small till now, we have focused on the stripe phase that exists through the spontaneously broken spatial symmetry of 4. Linear stability analysis

In this section, we investigate the existence of the superstripes, when the harmonic trap is switched off, i.e., \( \text{CN} \) scheme \([67, 68]\), where \( \delta \psi \) refers to the condensate undergoes a nonlinear compression. This phenomenon of the nonlinear compression is analogous to the effective pulse compression in nonlinear optical fiber and analogous to the findings of Moores \([42]\). Therefore, it is evident that the self-similar sinusoidal excitations can be controlled by means of chirp management, leading to the controlled nonlinear compression and amplification.

4. Linear stability analysis

Till now, we have focused on the stripe phase that exists through the spontaneously broken spatial symmetry of the BEC. In this section, we perform a linear stability analysis of the obtained phases. We apply a small perturbation to the obtained solution, \( \psi(z, t) = \psi_0(z, t) + \delta \psi(z, t) \), where \( \delta \psi \ll \psi_0 \) and \( \psi_0(z, t) \) refers to the analytically obtained solution. Therefore, one can write the equation of motion in terms of the perturbation \( \delta \psi \) neglecting the higher order perturbation, after substituting the above ansatz in equation (2).

\[
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \left( -\frac{1}{2} \frac{\partial^2}{\partial z^2} + V_i - \nu(t) \right) \delta \psi + g_1(2|\psi|^2 \delta \psi + \psi^2 \delta \psi^* ) + g_2(3|\psi|^4 \delta \psi + 2|\psi|^2 \psi^2 \delta \psi^* ).
\]

where, we set \( \hbar = m = 1 \). Figure 4 illustrates the obtained exact analytical solution and the perturbed solutions obtained numerically by solving equation (12). Due to the presence of a harmonic trap and periodic lattice potentials, we use mixed boundary conditions, \( \delta \psi(0, t) = \delta \psi(L, t) = 0 \) and \( \delta \psi(L, t) - i \partial_z \delta \psi(L, t) = 0 \), where \( \partial_z \) refers to the first derivative with respect to \( z \). In our numerical simulation, we use the split-step Crank-Nicholson (CN) scheme \([67, 68]\), where \( dx = 0.01 \) and \( dt = 0.00005 \), and thereby satisfying the CN scheme.
It can be clearly seen from figure 4 that the obtained solution is linearly stable. The blue solid line represents the obtained solution, whereas, the red dotted curve corresponds to the evolution of the perturbed solution \((\psi_0 + \delta \psi)\). Furthermore, we obtained the following quantity \([68–70]\]

\[
\Omega = \frac{\ln \left[ \text{Re}[\bar{\psi}(z, t + \delta t)] \right] - \ln \left[ \text{Re}[\bar{\psi}(z, t)] \right]}{\delta t}
\]  

(13)

at each step of our simulation, which in the limit of \(t \to \infty\) and \(\delta t \to 0\) mimics the largest eigenvalue of the perturbation mode, obtained by solving equation (12). Explicit numerical simulation shows that the eigenvalues correspond to the perturbation mode, obtained by using equations (12) and (13), are negative and hence, the solution is found to be stable. The obtained eigenvalue, calculated at certain intervals of time, is shown in figure 4(b), which clearly shows that the eigenvalue lies in the negative region.

5. Energy spectrum

The analytical calculation of energy with accelerating optical lattice is nontrivial and too big to mention. However, it can be obtained by using the following definition:

\[
E = \frac{\hbar^2}{2m} \int \frac{\partial \psi^* \partial \psi}{\partial z} dz + V_1 \int (\bar{\psi} \psi) dz + g_1(t) \int (\psi^* \psi)^2 dz + g_2(t) \int (\psi^* \psi)^3 dz - \nu(t) \int (\psi^* \psi) dz
\]  

(14)

The energy profile is shown in figures 5 and 6, both in the presence and absence of the harmonic trap. The temporal behavior of energy in this parametrically driven system is studied. In particular, the effect of three-body interaction strength on energy is investigated. In this scenario, the positive-definiteness of the density modulation allows for repulsive two-body and attractive three-body interactions. Figure 5 depicts the time evolution of the energy spectrum in presence of the combined presence of the lattice potential and harmonic
In presence of regular harmonic trap \((M = \gamma^2)\), we observe the nonlinear resonance due to strong nonlinear compression of the condensate. This can be clearly seen from figure 5(a), where the energy spectrum is shown for three different values of \(\kappa_2\). For, \(\kappa_2 = -0.1\), the density of the condensate is found to be high around \(t \sim 1.8\) and \(t \sim 4.8\) (figure 1(a)), subsequently, at the same position, the energy increases abruptly, which gives rise to the nonlinear compression of the condensate. Similar feature is seen for other values of the quintic nonlinearity, \(\kappa_2 = -0.25\) (red curve) and \(-0.8\) (black curve).

For the sake of completeness, we have checked that the potential energy does not exhibit such resonant behavior. The contribution of the supercurrent to the kinetic energy is solely responsible for this phenomenon. It is interesting to mention that in an experiment of BEC in an optical lattice, similar resonant behavior has been observed \([71]\). Figure 5(b) depicts the behavior of the energy for expulsive harmonic trap \((M = -\gamma^2)\). The energy is found to increase initially and then decreases. In this case, the periodic excitations initially move towards to center of the trap, gains its energy. However, for an expulsive trap, the maximum position is unstable and hence, the excitations follow the downstream path and subsequently the corresponding energy decreases. The dynamics become completely different in the absence of the harmonic trap, where the lattice potential plays a dominant role. When the lattice is moving with a constant velocity, the energy is decreasing, due to the expansion of the condensate, as is seen in figure 6(a). We observe a resonance behavior when the lattice is static since the BEC undergoes a nonlinear compression. Such resonance is analogous to the effective pulse compression in nonlinear optical fiber, observed by Moore \([42]\).

6. Conclusion

In summary, we have presented here a detailed study of the parametrically forced Bose–Einstein condensate in the combined presence of the lattice and harmonic potential, where both cubic and quintic nonlinearities are present. We observe a spontaneously broken spatial symmetry of the condensate, mimicking the existence of a stripe phase in this system. It is worth mentioning that such types of solutions are already reported in the literature, however, the novelty of the present work is to study such stripe phase in the combined presence of harmonic trap and lattice potential, in the context of non-inertial and inertial BEC. We show that such stripe phase exists due to the interplay between lattice potential and the quintic nonlinearity. The presence of a lattice potential induces a periodicity in the system. However, due to the presence of the quintic nonlinearity, the periodicity of the density modulation is found to be twice that of the lattice potential. This imposes a restriction on the density modulations and hence, the superfluid matter waves are found to exist only in a finite domain, yielding a stripe phase. A linear stability analysis shows that the obtain solution is stable. Due to the presence of a harmonic trap, we consider a chirped phase in the system, which leads to an efficient nonlinear compression of the condensate in certain parameter regime. In order to gain a better understanding of the nonlinear compression, we compute the energy of the system. The corresponding energy spectrum clearly shows the regimes of the nonlinear resonances, which leads to strong nonlinear compression of the condensate. We discussed that such resonance behavior arises when the harmonic trapping frequency is in resonance with the chirped pulses.
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Appendix A. Methodology

The dynamics of the BEC in a combined presence of a lattice potential and harmonic trap with both cubic and quintic nonlinearity is described by the following generalized NLSE:

\[
i \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_l + g_1(t)|\psi|^2 + g_2(t)|\psi|^4 - \nu(t) \right) \psi.
\]  

(A.1)

In order to solve the above equation, we initially assume the following ansatz:

\[
\psi(z, t) = \rho(z, t)e^{i[\sigma(t) + \Xi(z, t)]}.
\]  

(A.2)

where, \(\rho(z, t) \equiv \rho\) is the amplitude. All other parameters used in this calculation, \(\sigma(\equiv \sigma(\xi))\), \(\vartheta(\equiv \vartheta(t))\), \(\Theta(\equiv \Theta(\xi))\), \(\Xi(\equiv \Xi(z, t))\), \(z_0(\equiv z_0(t))\), \(\epsilon(\equiv \epsilon(t))\), \(p(\equiv p(t))\), \(c(\equiv c(t))\) are defined in the main text. Substitute back this ansatz (with \(h = m = 1\)) into equation (A.1) and separating out the real and imaginary part, we obtain the imaginary part as,

\[
\frac{\partial \rho}{\partial t} = -\frac{1}{2} \left[ \frac{\partial^2 \rho}{\partial z^2} + \rho \left( \frac{\partial^2 \sigma}{\partial z^2} + \frac{\partial \Xi}{\partial z} \right) \right] - \frac{1}{2} \left[ \frac{\partial \rho}{\partial z} \right]^2 + g_1 \rho^3 + g_2 \rho^5 + V_l \rho - \nu \rho.
\]  

(A.3)

and the real part:

\[
-\rho \left( \frac{\partial \sigma}{\partial t} + \frac{\partial \Xi}{\partial t} \right) = \frac{1}{2} \left[ \frac{\partial^2 \rho}{\partial z^2} - \rho \left( \frac{\partial^2 \sigma}{\partial z^2} + \frac{\partial \Xi}{\partial z} \right) \right] + g_1 \rho^3 + g_2 \rho^5 + V_l \rho - \nu \rho.
\]  

(A.4)

Let us first consider the imaginary part and by applying a transformation \(\xi(z, t) = \vartheta(t)(z - z_0(t))\), we transform our equation in the center of mass coordinate. In COM frame, assuming, \(\rho(z, t) = \sqrt{\vartheta(t)} \sigma(\xi)\), we obtain,

\[
\frac{1}{2\sqrt{\vartheta}} \frac{\partial \vartheta}{\partial t} \sqrt{\vartheta} + \sqrt{\vartheta} \frac{\partial \vartheta}{\partial \xi} \left( \frac{\partial \sigma}{\partial t} z - \frac{\partial \sigma}{\partial t} z_0 - \vartheta \frac{\partial z_0}{\partial t} \right) = -\sqrt{\vartheta} \vartheta \frac{\partial \vartheta}{\partial \xi} \frac{\partial \sigma}{\partial \xi} + \sqrt{\vartheta} \frac{\partial \vartheta}{\partial \xi} \frac{\partial^2 \sigma}{\partial \xi^2} \frac{\partial^2 \sigma}{\partial \xi^2} \frac{\partial z_0}{\partial t} + \frac{1}{2\sqrt{\vartheta}} \frac{\partial \vartheta}{\partial \xi}.
\]  

(A.5)

Now, collecting the self-similar terms, by equating the coefficients of \(\sqrt{\vartheta}\) and \(\frac{\partial \vartheta}{\partial \xi}\), we obtain \(\frac{\partial \vartheta}{\partial \xi} = \vartheta \epsilon\). The remaining terms can be easily written with \(p = \vartheta\) in the following form:

\[
\frac{\partial \sigma}{\partial \xi} \frac{\partial \Theta}{\partial \xi} + \frac{1}{2} \sqrt{\vartheta} \frac{\partial^2 \sigma}{\partial \xi^2} - \frac{u}{\vartheta} \frac{\partial \sigma}{\partial \xi} = 0,
\]  

(A.6)

where, the COM motion obeys the following equation,

\[
\frac{\partial z_0(t)}{\partial t} + c(t) z_0(t) = \vartheta(t)(1 + u),
\]  

(A.7)

Therefore, equation (A.6) can be cast in a compact form:

\[
\frac{\partial \Theta}{\partial \xi} = u(1 - \frac{\sigma_0}{\sigma}),
\]  

(A.8)

where, \(\sigma_0\) is a constant of integration.
In a similar manner, the real equation becomes,
\[
\sqrt{\gamma} \sqrt{\sigma} \frac{\partial \Theta}{\partial \xi} (e_2 \zeta + \frac{\partial e_0}{\partial t} - cz) - \sqrt{\gamma} \sqrt{\sigma} \frac{\partial e}{\partial t} - \sqrt{\gamma} \sqrt{\sigma} \frac{\partial p}{\partial t} z + \frac{1}{2} \sqrt{\gamma} \sqrt{\sigma} \frac{\partial c}{\partial t} z^2
\]
\[
= -\frac{1}{2} \sqrt{\gamma} \sqrt{\sigma} \frac{\partial^2 \sigma}{\partial \xi^2} + \frac{1}{2} \sqrt{\gamma} \sqrt{\sigma} \left[ \left( \frac{\partial^2 \Theta}{\partial \xi^2} \right)^2 + p^2 + c^2 z^2 + 2\partial \Theta^2 \frac{\partial p}{\partial \xi} - 2\partial c \frac{\partial e}{\partial \xi} x - 2pcz \right]
\]
\[
+ g_1 \sigma^{3/2} + g_2 \sigma^{5/2} + \sqrt{\gamma} V_0 \cos^2 \xi \sqrt{\sigma}
\]
(A.9)

Now, equating the coefficient of \( \sqrt{\sigma} z^2 \), we obtained the Recatti equation
\[
\frac{\partial c(t)}{\partial t} = c^2(t) + M(t).
\]
(A.10)

Furthermore, equating the coefficients of other self-similar terms and using equation (A.10), we obtained all the equations mentioned in equation (6). The strength of the cubic nonlinearity, quintic nonlinearity and the depth of the lattice potential are, respectively, then rescaled as, \( g_1 = \kappa_1 \vartheta, g_2 = \kappa_2 \) and \( V_0 = \alpha \vartheta^2 \). The remaining terms can then easily be put in the form of equation (9):
\[
\frac{1}{4} \sigma(\xi) \frac{\partial^2 \sigma^2(\xi)}{\partial \xi^2} = \frac{1}{8} \left( \frac{\partial \sigma}{\partial \xi} \right)^2 + \frac{1}{2} \mu^2 - \mu \sigma^2(\xi) + \kappa_1 \sigma^3(\xi) + \kappa_2 \sigma^4(\xi)
\]
\[
+ \alpha \cos^2(\xi) \sigma^2(\xi) + \frac{1}{2} \sigma_0 = 0.
\]
(A.11)

**Appendix B. Vanishing of the stripe phase**

In the presence of a time dependent harmonic oscillator potential, i.e., \( M(t) = e^{-\gamma t} \), the chirp phase profile can easily be obtained from the following Recatti equation:
\[
\frac{\partial \vartheta(t)}{\partial t} = c^2(t) + M^2(t).
\]
(B.1)

The solution of this equation appears in terms of the Bessel function:
\[
c(t) = \frac{e^{-\vartheta^2 t}}{J_1 \left( \frac{\vartheta^2}{\gamma} \right) Y_1 \left( \frac{1}{\gamma} \right) - J_0 \left( \frac{\vartheta^2}{\gamma} \right) Y_0 \left( \frac{1}{\gamma} \right)}
\]
(B.2)

where, \( J_i \) and \( Y_i \) \( i = 0, 1 \) are the Bessel function of the first and second kind. Subsequently, one can obtain the exact form of the inverse of width \( \vartheta(t) \) and COM of BEC \( z_0(t) \) from equations (6b) and (7), respectively. The stripe ordered phase ceases to exist in this case since the time dependent harmonic potential destroys the underlying periodicity of the system and leads to the violation of the Kohn mode. Below in figure B1, we show the density profile for the time dependent harmonic trap, which clearly shows the vanishing of the stripe phase.

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