A2BCD: An Asynchronous Accelerated Block Coordinate Descent Algorithm With Optimal Complexity*

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Abstract

In this paper, we propose the Asynchronous Accelerated Nonuniform Randomized Block Coordinate Descent algorithm (A2BCD), the first asynchronous Nesterov-accelerated algorithm that achieves optimal complexity. This parallel algorithm solves the unconstrained convex minimization problem, using \( p \) computing nodes which compute updates to shared solution vectors, in an asynchronous fashion with no central coordination. Nodes in asynchronous algorithms do not wait for updates from other nodes before starting a new iteration, but simply compute updates using the most recent solution information available. This allows them to complete iterations much faster than traditional ones, especially at scale, by eliminating the costly synchronization penalty of traditional algorithms.

We first prove that A2BCD converges linearly to a solution with a fast accelerated rate that matches the recently proposed NU_ACDM, so long as the maximum delay is not too large. Somewhat surprisingly, A2BCD pays no complexity penalty for using outdated information. We then prove lower complexity bounds for randomized coordinate descent methods, which show that A2BCD (and hence NU_ACDM) has optimal complexity to within a constant factor. We confirm with numerical experiments that A2BCD outperforms NU_ACDM, which is the current fastest coordinate descent algorithm, even at small scale. We also derive and analyze a second-order ordinary differential equation, which is the continuous-time limit of our algorithm, and prove it converges linearly to a solution with a similar accelerated rate.

1 Introduction

In this paper, we propose and prove the convergence of the Asynchronous Accelerated Nonuniform Randomized Block Coordinate Descent algorithm (A2BCD), the first asynchronous Nesterov-accelerated algorithm that achieves optimal complexity. No previous attempts have been able to

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prove a speedup for asynchronous Nesterov acceleration. We aim to find the minimizer \( x^* \) of the unconstrained minimization problem:

\[
\min_{x \in \mathbb{R}^d} f(x) = f(x(1), \ldots, x(n))
\]  

where \( f \) is \( \sigma \)-strongly convex for \( \sigma > 0 \) with \( L \)-Lipschitz gradient \( \nabla f = (\nabla_1 f, \ldots, \nabla_n f) \). \( x \in \mathbb{R}^d \) is composed of coordinate blocks \( x(1), \ldots, x(n) \). The coordinate blocks of the gradient \( \nabla_i f \) are assumed \( L_i \)-Lipschitz with respect to the \( i \)th block. That is, \( \forall x, h \in \mathbb{R}^d: \)

\[
||\nabla_i f(x + P_i h) - \nabla_i f(x)|| \leq L_i ||h||
\]  

where \( P_i \) is the projection onto the \( i \)th block of \( \mathbb{R}^d \). Let \( \bar{L} \equiv \frac{1}{n} \sum_{i=1}^{n} L_i \) be the average block Lipschitz constant. These conditions on \( f \) are assumed throughout this whole paper. Our algorithm can be applied to non-strongly convex objectives (\( \sigma = 0 \)) or non-smooth objectives using the black box reduction techniques proposed in (Allen-Zhu and Hazan, 2016).

Coordinate descent methods, in which a chosen coordinate block \( i_k \) is updated at every iteration, are a popular way to solve (1.1). Randomized block coordinate descent (RBCD, (Y. Nesterov, 2012)) updates a uniformly randomly chosen coordinate block \( i_k \) with a gradient-descent-like step: \( x_{k+1} = x_k - (1/L_i) \nabla_i f(x_k) \). This algorithm decreases the error \( \mathbb{E}(f(x_k) - f(x_*)) \) to \( \epsilon (f(x_0) - f(x_*)) \) in \( K(\epsilon) = \mathcal{O}(n\sqrt{\bar{L}/\sigma} \ln(1/\epsilon)) \) iterations.

Using a series of averaging and extrapolation steps, accelerated RBCD (Y. Nesterov, 2012) improves RBCD’s iteration complexity \( K(\epsilon) \) to \( \mathcal{O}(n\sqrt{\bar{L}/\sigma} \ln(1/\epsilon)) \), which leads to much faster convergence when \( \frac{\bar{L}}{\sigma} \) is large. This rate is optimal when all \( L_i \) are equal (Lan and Zhou, 2017). Finally, using a special probability distribution for the random block index \( i_k \), non-uniform accelerated coordinate descent method (Allen-Zhu, Qu, et al., 2016) (NU_ACDM) can further decrease the complexity to \( \mathcal{O}(\sum_{i=1}^{n} \sqrt{L_i/\sigma} \ln(1/\epsilon)) \), which can be up to \( \sqrt{n} \) times faster than accelerated RBCD, since some \( L_i \) can be significantly smaller than \( L \). NU_ACDM is the current state-of-the-art coordinate descent algorithm for solving (1.1).

Our A2BCD algorithm generalizes NU_ACDM to the asynchronous-parallel case. We solve (1.1) with a collection of \( p \) computing nodes that continually read a shared-access solution vector \( y \) into local memory then compute a block gradient \( \nabla_i f \), which is used to update shared solution vectors \( (x, y, z) \). Proving convergence in the asynchronous case requires extensive new technical machinery.

A traditional synchronous-parallel implementation is organized into rounds of computation: Every computing node must complete an update in order for the next iteration to begin. However, this synchronization process can be extremely costly, since the lateness of a single node can halt the entire system. This becomes increasingly problematic with scale, as differences in node computing speeds, load balancing, random network delays, and bandwidth constraints mean that a synchronous-parallel solver may spend more time waiting than computing a solution.

Computing nodes in an asynchronous solver do not wait for others to complete and share their updates before starting the next iteration, but simply continue to update the solution vectors with the most recent information available, without any central coordination. This eliminates costly idle time, meaning that asynchronous algorithms can be much faster than traditional ones, since they have much faster iterations. For instance, random network delays cause asynchronous algorithms to complete iterations \( \Omega(\ln(p)) \) time faster than synchronous algorithms at scale. This and other factors that influence the speed of iterations are discussed in (Hannah and Yin, 2017a). However, since many iterations may occur between the time that a node reads the solution vector, and the
time that its computed update is applied, effectively the solution vector is being updated with outdated information. At iteration $k$, the block gradient $\nabla_i f$ is computed at a delayed iterate $\hat{y}_k$ defined as:

$$
\hat{y}_k = (y(k-j(k,1)), \ldots, y(k-j(k,n)))
$$

(1.3)

for delay parameters $j(k,1), \ldots, j(k,n) \in \mathbb{N}$. Here $j(k,i)$ denotes how many iterations out of date coordinate block $i$ is at iteration $k$. Different block may be out of date by different amounts, which is known as an inconsistent read. We assume\(^1\) that $j(k,i) \leq \tau$ for some constant $\tau < \infty$.

**Our results:** In this paper, we prove that A2BCD attains NU_ACDM’s state-of-the-art iteration complexity to highest order for solving (1.1), so long as delays are not too large. Hence we prove that there is no significant complexity penalty, despite the use of outdated information. The proof is very different from that of (Allen-Zhu, Qu, et al., 2016), and involves significant technical innovations and formidable complexity related to the analysis of asynchronicity.

Since asynchronous algorithms have much faster iterations, and A2BCD needs essentially the same number of epochs as NU_ACDM to compute a solution of a target accuracy, we expect A2BCD to be faster than all existing coordinate descent algorithms. We confirm this with computational experiments, comparing A2BCD to NU_ACDM, which is the current fastest block coordinate descent algorithm.

We also prove that A2BCD (and hence NU_ACDM) has optimal complexity to within a constant factor over a fairly general class of randomized block coordinate descent algorithms. We do this by proving that any algorithm $A$ in this class must in general complete at least $\Omega \left( \sum_{i=1}^{n} \sqrt{L_i/\sigma} \ln(1/\epsilon) \right)$ random gradient evaluations to decrease the error by a factor of $\epsilon$. This extends results in (Lan and Zhou, 2017) to the case where $L_i$ are not all equal, and the algorithm in question can be asynchronous.

These results are significant, because it was an open question whether Nesterov-type acceleration was compatible with asynchronicity. Not only is this possible, but asynchronous algorithms may even attain optimal complexity in this setting. In light of the above, it also seems plausible that accelerated incremental algorithms for finite sum problems $f \triangleq \sum_{i=1}^{m} f_i(x)$ can be made asynchronous-parallel, too.

We also derive a second-order ordinary differential equation (ODE), which is the continuous-time limit of A2BCD. This extends the ODE found in (Su, Boyd, and Candes, 2014) to an asynchronous accelerated algorithm minimizing a strongly convex function. We prove this ODE linearly converges to a solution with the same rate as A2BCD’s, without needing to resort to the restarting technique employed in (Su, Boyd, and Candes, 2014). We prove this result using techniques that motivate and clarify the our proof strategy of the main result.

## 2 Main results

We define the **condition number** $\kappa = L/\sigma$, and let $L = \min_i L_i$ be the smallest block Lipschitz constant. We should consider functions $f$ where it is efficient to calculate blocks of the gradient, so that coordinate-wise parallelization is efficient. That is, the function should be “coordinate friendly” (Zhimin Peng, Wu, et al., 2016). This turns out to be a rather wide class of algorithms. So for\(^1\)

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\(^1\)This condition can be relaxed however by techniques in (Hannah and Yin, 2017b; Sun, Hannah, and Yin, 2017; Zhimin Peng, Xu, et al., 2017; Hannah and Yin, 2017a)
instance, while the $L^2$-regularized empirical risk minimization problem is not coordinate friendly in general, the equivalent dual problem is, and hence can be solved efficiently by A2BCD (see (Q. Lin, Lu, and Xiao, 2014), and Section 5).

To calculate the $k + 1$th iteration of the algorithm from iteration $k$, we use a block of the gradient $\nabla v_k f$. Finally, we assume that the delays $j(k,i)$ are independent of the block sequence $i_k$, but otherwise arbitrary (This assumption can be relaxed. See (Sun, Hannah, and Yin, 2017; Leblond, Pedregosa, and Lacoste-Julien, 2017; Cannelli et al., 2017)).

**Definition 1.** Asynchronous Accelerated Randomized Block Coordinate Descent (A2BCD).

Let $f$ be $\sigma$-strongly convex, and let its gradient $\nabla f$ be $L$-Lipschitz with block coordinate Lipschitz parameters $L_i$ (1.2). Using these parameters, we sample $i_k$ in an independent and identically distributed (IID) fashion according to

$$P[i_k = j] = \frac{L_{1/2}^{1/2}}{S}, \quad j \in \{1, \ldots, n\},$$

for $S \triangleq \sum_{i=1}^{n} L_i^{1/2}$. (2.1)

Let $\tau$ be the maximum asynchronous delay in the parallel system (or an overestimate of this). We define the dimensionless asynchronicity parameter $\psi$, which is proportional to the maximum delay $\tau$, and quantifies how strongly asynchronicity will affect convergence:

$$\psi = 9 \left( \frac{1}{2} L^{-1/2} L^{1/4} \frac{1}{2} \psi \right) \times \tau$$

We use the above system parameters and $\psi$ to define the coefficients $\alpha, \beta,$ and $\gamma$ via (2.3),(2.4), and (2.5).

$$\alpha \triangleq \left( 1 + (1 + \psi) \sigma^{-1/2} S \right)^{-1}$$

$$\beta \triangleq 1 - (1 - \psi) \sigma^{-1/2} S^{-1}$$

$$h \triangleq 1 - \frac{1}{2} \sigma^{1/2} L^{-1/2} \psi.$$ (2.5)

Hence A2BCD algorithm is hence defined via the iterations: (2.6),(2.7), and (2.8):

$$y_k = \alpha v_k + (1 - \alpha) x_k,$$ (2.6)

$$x_{k+1} = y_k - h L^{-1/2} \nabla i_k f (\hat{y}_k),$$ (2.7)

$$v_{k+1} = \beta v_k + (1 - \beta) y_k - \sigma^{-1/2} L_{1/2}^{-1/2} \nabla i_k f (\hat{y}_k).$$ (2.8)

Here $\sigma$ may be underestimated, and $L, L_1, \ldots, L_n$ may be overestimated if exact values are unavailable. Notice that $x_k$ can be eliminated from the above iteration, and the block gradient $\nabla i_k f (\hat{y}_k)$ only needs to be calculated once per iteration. A larger (or overestimated) maximum delay $\tau$ will cause a larger asynchronicity parameter $\psi$, which leads to more conservative step sizes to compensate.

The convergence of this algorithm can be stated in terms of a Lyapunov function that we will define shortly. We first introduce the asynchronicity error, a powerful tool for analyzing asynchronous algorithms used in several recent works (Z. Peng et al., 2016; Hannah and Yin, 2017b; Sun, Hannah, and Yin, 2017; Hannah and Yin, 2017a). This error is a weighted sum of the history of the algorithm, with the weights $c_i$ decreasing as one goes further back in time. This error appears naturally in the analysis. Much like a well-chosen basis in linear algorithm, it appears to be a natural quantity to consider when analyzing convergence of asynchronous algorithms.
Definition 2. Asynchronicity error. Using the above parameters, we define:

\[ A_k = \sum_{j=1}^{\tau} c_j \|y_{k+1-j} - y_{k-j}\|^2 \]  
(2.9)

for \( c_i = \frac{6}{5} L^{1/2} \kappa^{3/2} \tau \sum_{j=i}^{\tau} \left(1 - \sigma^{1/2} S^{-1}\right)^{i-j-1} \psi^{-1} \).  
(2.10)

Here we define \( y_k = y_0 \) for all \( k < 0 \).

The determination of the coefficients \( c_i \) is in general a very involved process of trial and error, intuition, and balancing competing requirements. Obtaining a convergence proof and optimal rates relies on the skillful choice of these coefficients \( c_i \). The algorithm doesn’t depend on the coefficients, however; they are only used in the analysis.

We define \( E_k[X] \) as the expectation of \( X \) conditional on conditioned on \( (x_0, \ldots, x_k), (y_0, \ldots, y_k), (z_0, \ldots, z_k), \) and \( (i_0, \ldots, i_{k-1}) \). To simplify notation, we assume that the minimizer \( x^* = 0 \), and that \( f(x^*) = 0 \) with no loss in generality. We define the Lyapunov function:

\[ \rho_k = \|v_k\|^2 + A_k + cf(x_k) \]  
(2.11)

for \( c = 2\sigma^{-1/2} S^{-1} (\beta \alpha^{-1} (1 - \alpha) + 1) \).  
(2.12)

We define the iteration complexity \( K(\epsilon) \) with respect to some error \( E^k \) as the number of iterations \( K \) such that the expected error \( \mathbb{E}[E^K] \) decreases to less than \( \epsilon E^0 \).

We now present this paper’s first main contribution.

Theorem 1. Let \( f \) be \( \sigma \)-strongly convex with a gradient \( \nabla f \) that is \( L \)-Lipschitz with block Lipschitz constants \( \{L_i\} \). Let \( \psi \) defined in (2.2) satisfy \( \psi \leq \frac{3}{2} \) (i.e. \( \tau \leq \frac{1}{2L} S^{1/2} L^{1/2} L^{-3/4} \kappa^{-1/4} \)). Then for \( A2BCD \) we have:

\[ \mathbb{E}_k \left[ \|v_{k+1}\|^2 + A_{k+1} + cf(x_{k+1}) \right] \leq \left(1 - (1 - \psi) \sigma^{1/2} S^{-1}\right) \left(\|v_k\|^2 + A_k + cf(x_k)\right) \]

As \( \sigma^{-1/2} S \to \infty \), the error \( \|v_{k+1}\|^2 + A_{k+1} + cf(x_{k+1}) \) has the corresponding complexity \( K(\epsilon) = K_{A2BCD}(\epsilon) \) for:

\[ K_{A2BCD}(\epsilon) = \left(\sum_{i=1}^{n} \frac{L_i}{\sigma} + O(1)\right) \ln \left(\frac{1}{\epsilon}\right) \frac{1}{1 - \psi} \]

(2.13)

This result is proven in Section A. A stronger result can be proven, however this adds too much to the complexity of the proof. See Section D for a discussion. In practice, asynchronous algorithms are far more resilient to delays than the theory predicts. \( \tau \) can be much larger without negatively affecting the convergence rate and complexity. This is perhaps because we are limited to a worst-case analysis, which is not representative of the average-case performance.

Authors in (Allen-Zhu, Qu, et al., 2016) (Theorem 5.1) obtain a linear convergence rate of \( 1 - 2(1 + 2\sigma^{-1/2} S) \) for \( NU_ACDM \) which leads to the corresponding iteration complexity of \( K_{NU_ACDM}(\epsilon) = \frac{\ln (1/\epsilon)}{1 - \psi} \).
\[
\left(\sigma^{-1/2} \sum_{i=1}^{n} L_i^{1/2} + O(1)\right) \ln(1/\epsilon) \text{ as } \sigma^{-1/2} S \to \infty. \text{ And hence, we have:}
\]
\[
K_{A2BCD}(\epsilon) = \frac{1}{1 - \psi} \left(1 + o(1)\right) K_{\text{NU_ACDM}}(\epsilon)
\]

Hence when \(0 \leq \psi \ll 1\), or equivalently, when \(\tau \ll S^{1/2} \frac{1}{\tilde{\lambda}} L^{-3/4} K^{-1/4}\) the complexity of \(A2BCD\) asymptotically matches that of \(\text{NU_ACDM}\). Hence \(A2BCD\) combines state-of-the-art complexity with the faster iterations and superior scaling that asynchronous iterations allow.

We now present some special cases of the conditions on the maximum delay \(\tau\) required for good complexity.

**Corollary 3.** Let the conditions of Theorem 1 hold. Additionally, let all coordinate-wise Lipschitz constants \(L_i\) be equal (i.e. \(L_i = L_1\), \(\forall i\)). Then we have \(K_{A2BCD}(\epsilon) \sim K_{\text{NU_ACDM}}(\epsilon)\) when \(\tau \ll n^{1/2} K^{-1/4} (L_1/L)^{3/4}\). Further, let us assume all coordinate-wise Lipschitz constants \(L_i\) equal \(L\). Then \(K_{A2BCD}(\epsilon) \sim K_{\text{NU_ACDM}}(\epsilon) = K_{\text{ACDM}}(\epsilon)\), when \(\tau \ll n^{1/2} K^{-1/4}\)

**Remark 1. Reduction to synchronous case.** Notice that when \(\tau = 0\), we have \(\psi = 0\), \(c_i \equiv 0\) and hence \(A_k \equiv 0\). Thus \(A2BCD\) becomes equivalent to \(\text{NU_ACDM}\), the Lyapunov function\(^3\) \(\rho_k\) becomes equivalent to one found in (Allen-Zhu, Qu, et al., 2016)(pg. 9), and Theorem 1 yields the same complexity to highest order.

### 2.1 Optimality

\(\text{NU_ACDM}\) and hence \(A2BCD\) are in fact optimal among a wide class of randomized block gradient algorithms. We consider a class of algorithms slightly wider than that considered in (Lan and Zhou, 2017) to encompass asynchronous algorithms. Their result only apply to the case where all \(L_i\) are equal in this setting. Our result applies for unequal \(L_i\), and potentially asynchronous algorithms. For a subset \(S \in \mathbb{R}^d\), we let \(\text{IC}(S)\) (inconsistent read) denote the set of vectors \(v\) such that \(v = (v_{1,1}, v_{2,2}, \ldots, v_{d,d})\) for some vectors \(v_1, v_2, \ldots, v_d \in S\). Here \(v_{i,j}\) denotes the \(j\)th component of vector \(v_i\). That is, the coordinates of \(v\) are some combination of coordinates of vectors in \(S\). Let \(X_k = \{x_0, \ldots, x_k\}\).

**Definition 4. Asynchronous Randomized Incremental Algorithms Class.** Consider the unconstrained minimization problem (1.1) for \(f\) that is \(\sigma\)-strongly convex with \(L\)-Lipschitz gradient, with block-coordinate-wise Lipschitz constants \(\{L_i\}_{i=1}^{n}\). We define the class \(\mathcal{A}\) as algorithms \(G\) on this problem such that:

1. For each parameter set \((\sigma, L_1, \ldots, L_n, n)\), \(G\) has an associated IID random variable \(i_k\) with some fixed distribution \(\mathbb{P}[i_k] = p_i\) for \(\sum_{i=1}^{n} p_i = 1\).
2. For \(k \geq 0\), the iterates of \(G\) satisfy:

\[
x_{k+1} \in \text{span}\{\text{IC}(X_k), \nabla_{i_0} f(\text{IC}(X_0)), \nabla_{i_1} f(\text{IC}(X_1)), \ldots, \nabla_{i_k} f(\text{IC}(X_k))\}
\]

This is a rather general class: \(x_{k+1}\) can be constructed from any inconsistent reading of past iterates \(\text{IC}(X_k)\), and any past gradient of an inconsistent read \(\nabla_{i_j} f(\text{IC}(X_{j}))\).

\(^3\)Their Lyapunov function is in fact a generalization of a one found in (Y. Nesterov, 2012)
Theorem 2. For any algorithm $G \in A$, and parameter set $(\sigma, L_1, \ldots, L_n, n)$ for the unconstrained minimization problem, and $k \in \mathbb{N}$, there is a dimension $d$, a corresponding function $f$ on $\mathbb{R}^d$, and a starting point $x_0$, such that

$$
E \|x_k - x^*\|^2 / \|x_0 - x^*\|^2 \geq \frac{1}{2} \left( 1 - 4/(\sum_{j=1}^n \sqrt{L_i/\sigma} + 2n) \right)^k
$$

Hence the complexity $I(\epsilon)$ for all algorithms in $A$ satisfies

$$
K(\epsilon) \geq \frac{1}{4} (1 + o(1)) \left( \sum_{j=1}^n \sqrt{L_i/\sigma} + 2n \right) \ln (1/2\epsilon)
$$

Our proof in Section C follows very similar lines to (Lan and Zhou, 2017), which is inspired by Nesterov (Yurii Nesterov, 2013).

3 ODE Analysis

In this section we present and analyze an ODE which is the continuous-time limit of $A2BCD$. This ODE is a strongly convex, and asynchronous version of the ODE found in (Su, Boyd, and Candes, 2014). For simplicity, assume $L_i = L, \forall i$. We rescale $f$ so that the strong convexity modulus $\sigma = 1$, and hence $\kappa = L/\sigma = L$. Taking the discrete limit of synchronous $A2BCD$ (i.e. accelerated RBCD), we can derive the following ODE (see Section (B.1)):

$$
\ddot{Y} + 2n^{-1} \kappa^{-1/2} \dot{Y} + 2n^{-2} \kappa^{-1} \nabla f (Y) = 0 \quad (3.1)
$$

We define the parameter $\eta \triangleq n \kappa^{1/2}$, and the energy:

$$
E(t) = e^{n^{-1} \kappa^{-1/2} t} \left( f(Y(t)) + \frac{1}{4} \|Y(t) + \eta \dot{Y}(t)\|^2 \right) \quad (3.2)
$$

This is very similar to the Lyapunov function discussed in (2.11), with $\frac{1}{4} \|Y(t) + \eta \dot{Y}(t)\|^2$ fulfilling the role of $\|v_k\|^2$, and $A_k = 0$ (since there is no delay yet). Much like the traditional analysis in the proof of Theorem 1, we can derive a linear convergence result that has a similar rate. See Section B.2 for the proof.

Lemma 5. If $Y$ satisfies (3.1), then the energy $E(t)$ satisfies $E'(t) \leq 0$. This implies $E(t) \leq E(0)$, which is equivalent to:

$$
f(Y(t)) + \frac{1}{4} \|Y(t) + n \kappa^{-1/2} \dot{Y}(t)\|^2 \leq \left( f(Y(0)) + \frac{1}{4} \|Y(0) + \eta \dot{Y}(0)\|^2 \right) e^{-n^{-1} \kappa^{-1/2} t}
$$

We may also analyze an asynchronous version of (3.1) to motivate our proof of the main theorem in Section A:

$$
\ddot{Y} + 2n^{-1} \kappa^{-1/2} \dot{Y} + 2n^{-2} \kappa^{-1} \nabla f (\dot{Y}) = 0, \quad (3.3)
$$

\footnote{4I.e. we replace $f(x)$ with $\frac{1}{\sigma} f$.}

\footnote{5For compactness, we have omitted the $(t)$ from time-varying functions $Y(t)$, $\dot{Y}(t)$, $\nabla Y(t)$, etc.}
\[ \dot{Y}(t) \] is a delayed version of \( Y(t) \) defined similarly to (1.3), with the delay bounded by \( \tau \).

Unfortunately, the energy satisfies (see Section (B.4), (B.7)):

\[
e^{-\eta^{-1}}E'(t) \leq -\frac{1}{8}\eta \|\dot{Y}\|^2 + 3\kappa^2\eta^{-1}\tau D(t), \text{ for } D(t) \triangleq \int_{t-\tau}^{t} \|\dot{Y}(s)\|^2 \, ds.
\]

Since the right-hand side can be positive, the energy \( E(t) \) may not be decreasing in general. But, we may add a continuous-time version of the asynchronicity error (see (Sun, Hannah, and Yin, 2017)), much like in Definition 2, to create a decreasing energy. Let \( c_0 \geq 0 \) and \( r > 0 \) be arbitrary constants that will be set later. Define:

\[
A(t) = \int_{t-\tau}^{t} c(t-s) \|\dot{Y}(s)\|^2 \, ds, \text{ for } c(t) \triangleq c_0 \left( e^{-rt} + \frac{e^{-r\tau}}{1-e^{-r\tau}} (e^{-rt} - 1) \right).
\]

**Lemma 6.** When \( r\tau \leq \frac{1}{2} \), the asynchronicity error \( A(t) \) satisfies:

\[
e^{-rt} \frac{d}{dt} (e^{rt} A(t)) \leq c_0 \|\dot{Y}(t)\|^2 - \frac{1}{2}\tau^{-1}c_0 D(t).
\]

See Section B.3 for the proof. Adding this error to the Lyapunov function serves a similar purpose in the continuous-time case as in the proof of Theorem 1 (see Lemma 11). It allows us to negate \( \frac{1}{2}\tau^{-1}c_0 \) units of \( D(t) \) for the cost of creating \( c_0 \) units of \( \|\dot{Y}(t)\|^2 \).

**Theorem 3.** Let \( c_0 = 6\kappa^2\eta^{-1}\tau^2 \), and \( r = \eta^{-1} \). If \( \tau \leq \frac{1}{\sqrt{48}} n\kappa^{-1/2} \) then we have:

\[
e^{-\eta^{-1}t} \frac{d}{dt} \left( E(t) + e^{\eta^{-1}t} A(t) \right) \leq 0.
\]

Hence \( f(Y(t)) \) convergence linearly to \( f(x_*) \) with rate \( O \left( \exp\left(-t\eta^{-1}\kappa^{-1/2}\right) \right) \).

Notice how this convergence condition is similar to Corollary 3, but a little looser.

**Proof.**

\[
e^{-\eta^{-1}t} \frac{d}{dt} \left( E(t) + e^{\eta^{-1}t} A(t) \right) \leq \left( c_0 - \frac{1}{8}\eta \right) \|\dot{Y}\|^2 + \left( 3\kappa^2\eta^{-1}\tau - \frac{1}{2}\tau^{-1}c_0 \right) D(t)
\]

\[
= 6\eta^{-1}\kappa^2 \left( \tau^2 - \frac{1}{48}n^2\kappa^{-1} \right) \|\dot{Y}\|^2 \leq 0 \quad \Box
\]

The preceding should act as a guide to understanding the convergence of A2BCD. It may make clearer the use of the Lyapunov function to establish convergence in the synchronous case, the function of the asynchronicity error in the proof of the asynchronous case, and may hopefully elucidate the logic and general strategy of the proof in Section A.

### 4 Related work

We now discuss related work that was not addressed in Section 1. Nesterov acceleration is a method for improving an algorithm’s iteration complexity’s dependence the condition number \( \kappa \). Nesterov-accelerated methods have been proposed and discovered in many settings (Yurii Nesterov, 1983; Paul Tseng, 2008; Y. Nesterov, 2012; Q. Lin, Lu, and Xiao, 2014; Lu and Xiao, 2014; Shalev-Shwartz and Zhang, 2016; Allen-Zhu, 2017), including for coordinate descent algorithms (algorithms that use 1 gradient block \( \nabla_i f \) or minimize with respect to 1 coordinate block per iteration), and incremental
algorithms (algorithms for finite sum problems $\frac{1}{n} \sum_{i=1}^{n} f_i(x)$ that use 1 function gradient $\nabla f_i(x)$ per iteration). Such algorithms can often be augmented to solve composite minimization problems (minimization for objective of the form $f(x) + g(x)$, especially for nonsmooth $g$), or include constraints.

Asynchronous algorithms were proposed in (Chazan and Miranker, 1969) to solve linear systems. General convergence results and theory were developed later in (D. P. Bertsekas, 1983; D. P. Bertsekas and J. N. Tsitsiklis, 1997; P. Tseng, D. Bertsekas, and J. Tsitsiklis, 1990; Z. Q. Luo and P. Tseng, 1992; Z.-Q. Luo and Paul Tseng, 1993; P. Tseng, 1991) for partially and totally asynchronous systems, with essentially-cyclic block sequence $i_k$. More recently, there has been renewed interest in asynchronous algorithms with random block coordinate updates. Linear and sublinear convergence results were proven for asynchronous RBBCD (Liu and Wright, 2015; Avron, Druinsky, and Gupta, 2014), and similar was proven for asynchronous SGD (Recht et al., 2011), and variance reduction algorithms (J. Reddi et al., 2015; Leblond, Pedregosa, and Lacoste-Julien, 2017).

In (Z. Peng et al., 2016), authors proposed and analyzed an asynchronous fixed-point algorithm called ARock, that takes proximal algorithms, forward-backward, ADMM, etc. as special cases. (Hannah and Yin, 2017b; Sun, Hannah, and Yin, 2017; Zhimin Peng, Xu, et al., 2017) showed that many of the assumptions used in prior work (such as bounded delay $\tau < \infty$) were unrealistic and unnecessary in general. In (Hannah and Yin, 2017a) the authors showed that asynchronous iterations will complete far more iterations per second, and that a wide class of asynchronous algorithms, including asynchronous RBBCD, have the same iteration complexity as their traditional counterparts. Hence certain asynchronous algorithms can be expected to significantly outperform traditional ones.

In (Xiao et al., 2017) authors propose a novel asynchronous catalyst-accelerated (H. Lin, Mairal, and Harchaoui, 2015) primal-dual algorithmic framework to solve regularized ERM problems. Instead of using outdated information, they structure the parallel updates so that the data that an update depends on is up to date (though the rest of the data may not be). However catalyst acceleration incurs a logarithmic penalty over Nesterov acceleration in general. Authors in (Fang, Huang, and Z. Lin, 2018) skillfully devised accelerated schemes for asynchronous coordinate descent and SVRG using momentum compensation techniques. Although their complexity results have the improved $\sqrt{\kappa}$ dependence on the condition number, they do not prove an asynchronous speedup. Their complexity is $\tau$ times larger than our complexity. Since $\tau$ is necessarily greater than $p$, their results imply that adding more computing nodes will increase running time.

5 Numerical experiments

To investigate the performance of A2BCD, we solve the ridge regression problem. Consider the following primal and corresponding dual objective (see for instance (Q. Lin, Lu, and Xiao, 2014)):

$$
\min_{w \in \mathbb{R}^d} P(w) = \frac{1}{2n} \|A^T w - l\|^2 + \frac{\lambda}{2} \|w\|^2,
$$

(5.1)

$$
\min_{\alpha \in \mathbb{R}^n} D(\alpha) = \frac{1}{2d^2\lambda} \|A\alpha\|^2 + \frac{1}{2d} \|\alpha + l\|^2
$$

(5.2)

where $A \in \mathbb{R}^{d \times n}$ is a matrix of $n$ samples and $d$ features, and $l$ is a label vector. We let $A = [A_1, \ldots, A_m]$ where $A_i$ are the column blocks of $A$. We compare A2BCD (which is asynchronous accelerated), synchronous NU ACDM (which is synchronous accelerated), and asynchronous RBBCD (which is asynchronous non-accelerated). At each iteration, each node randomly selects a coordinate
block according to (2.1), calculates the corresponding block gradient, and uses it to apply an update to the shared solution vectors. Nodes in synchronous NU_ACDM implementation must wait until all nodes apply an update before they can start the next iteration, but the asynchronous algorithms simply compute with the most up-to-date information available.

We use the datasets w1a (47272 samples, 300 features) and rcv1_train.binary (20242 samples, 47236 features) from LIBSVM (Chang and C.-J. Lin, 2011). The algorithm is implemented in a multi-threaded fashion using C++11 and GNU Scientific Library with a shared memory architecture. We use 40 threads on two 2.5GHz 10-core Intel Xeon E5-2670v2 processors.

To estimate \( \psi \), one can first performed a dry run with all coefficient set to 0 to estimate \( \tau \). All function parameters can be calculated exactly for this problem in terms of the data matrix and \( \lambda \). We can then use these parameters and this tau to calculate \( \psi \). \( \psi \) and \( \tau \) merely change the parameters, and do not change execution patterns of the processors. Hence their parameter specification doesn’t affect the observed delay. Through simple tuning though, we found that \( \psi = 0.25 \) resulted in good performance.

In tuning for general problems, there are theoretical reasons why it is difficult to attain acceleration without some prior knowledge of \( \sigma \), the strong convexity modulus (Arjevani, 2017). Ideally \( \sigma \) is pre-specified for instance in a regularization term. If the Lipschitz constants \( L \) cannot be calculated directly (which is rarely the case for the classic dual problem of empirical risk minimization objectives), the line-search method discussed in (Roux, Schmidt, and Bach, 2012) Section 4 can be used.

A critical ingredient in the efficient implementation of A2BCD and NU_ACDM for this problem is the efficient update scheme discussed in (Y. T. Lee and A. Sidford, 2013). In linear regression applications such as this, it is essential to be able to efficiently maintain or recover \( Ay \). This is because calculating block gradients requires the vector \( A^T_i Ay \), and without an efficient way to recover \( Ay \), block gradient evaluations are essentially 50% as expensive as full-gradient calculations. Unfortunately, every accelerated iteration results in dense updates to \( y^k \) because of the averaging step in (2.6). Hence \( Ay \) must be recalculated from scratch.

However (Yin Tat Lee and Aaron Sidford, 2013) introduces a linear transformation that allows for an equivalent iteration that results in sparse updates to new iteration variables \( p \) and \( q \). The original purpose of this transformation was to ensure that the averaging steps (e.g. (2.6)) do not dominate the computational cost for sparse problems. However we find a more important secondary use which applies to both sparse and dense problems. Since the updates to \( p \) and \( q \) are sparse coordinate-block updates, the vectors \( Ap \) and \( Aq \) can be efficiently maintained, and therefore block gradients can be efficiently calculated. Implementation details are discussed in more detail in Appendix E.

In Table 5, we plot the sub-optimality vs. time for decreasing values of \( \lambda \), which corresponds to increasingly large condition numbers \( \kappa \). When \( \kappa \) is small, acceleration doesn’t result in a significantly better convergence rate, and hence A2BCD and async-RBCD both outperform sync-NU_ACDM since they complete faster iterations at similar complexity. Acceleration for low \( \kappa \) has unnecessary overhead, which means async-RBCD can be quite competitive. When \( \kappa \) becomes large, async-RBCD is no longer competitive, since it has a poor convergence rate. We observe that A2BCD and sync-NU_ACDM have essentially the same convergence rate, but A2BCD is up to \( 2 - 3.5 \times \) faster than sync-NU_ACDM because it completes much faster iterations. We observe this advantage despite the fact that we are in an ideal environment for synchronous computation: A small, homogeneous, high-bandwidth, low-latency cluster. In large-scale heterogeneous systems with greater synchronization overhead, bandwidth constraints, and latency, we expect A2BCD’s advantage to be much larger.
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Table 1: Sub-optimality \( f(y^k) - f(x^*) \) (y-axis) vs time in seconds (x-axis) for A2BCD, synchronous NU_ACDM, and asynchronous RBCD for data sets w1a and rcv1_train for various values of \( \lambda \).

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A Proof of the main result

We first recall a couple of inequalities for convex functions.

**Lemma 7.** Let $f$ be $\sigma$-strongly convex with $L$-Lipschitz gradient. Then we have:

\[
\begin{align*}
  f(y) &\leq f(x) + \langle y - x, \nabla f(x) \rangle + \frac{1}{2}L\|y - x\|^2, \forall x, y \quad (A.1) \\
  f(y) &\geq f(x) + \langle y - x, \nabla f(x) \rangle + \frac{1}{2}\sigma\|y - x\|^2, \forall x, y \quad (A.2)
\end{align*}
\]

We also find it convenient to define the norm:

\[
\|s\|_* = \sqrt{\sum_{i=1}^{n} L_i^{-1/2}\|s_i\|^2} 
\]

**A.1 Starting point**

First notice that using the definition (2.8) of $v_{k+1}$ we have:

\[
\begin{align*}
  \|v_{k+1}\|^2 &= \|\beta v_k + (1 - \beta) y_k\|^2 - 2\sigma^{-1/2}L_i^{-1/2} \langle \beta v_k + (1 - \beta) y_k, \nabla_i f(\hat{y}_k) \rangle + \sigma^{-1}L_i^{-1} \|\nabla_i f(\hat{y}_k)\|^2 \\
  \mathbb{E}_k \|v_{k+1}\|^2 &= \underbrace{\|\beta v_k + (1 - \beta) y_k\|^2 - 2\sigma^{-1/2}S^{-1/2} \langle \beta v_k + (1 - \beta) y_k, \nabla f(\hat{y}_k) \rangle}_{A} \\
  &\quad + \underbrace{S^{-1}\sigma^{-1} \sum_{i=1}^{n} L_i^{-1/2}\|\nabla_i f(\hat{y}_k)\|^2}_{B} \\
  &\quad + \underbrace{S^{-1}\sigma^{-1} \sum_{i=1}^{n} L_i^{-1/2}\|\nabla_i f(\hat{y}_k)\|^2}_{C}
\end{align*}
\]

We have the following general identity:

\[
\|\beta x + (1 - \beta) y\|^2 = \beta \|x\|^2 + (1 - \beta) \|y\|^2 - \beta (1 - \beta) \|x - y\|^2, \forall x, y 
\]

(A.5)

It can also easily be verified from (2.6) that we have:

\[
v_k = y_k + \alpha^{-1}(1 - \alpha)(y_k - x_k)
\]

(A.6)

Using (A.5) on term $A$, (A.6) on term $B$, and recalling the definition (A.3) on term $C$, we have from (A.4):

\[
\begin{align*}
  \mathbb{E}_k \|v_{k+1}\|^2 &= \beta \|v_k\|^2 + (1 - \beta) \|y_k\|^2 - (1 - \beta) \|v_k - y_k\|^2 + S^{-1}\sigma^{-1/2}\|\nabla f(\hat{y}_k)\|^2 \\
  &\quad - 2\sigma^{-1/2}S^{-1}\beta\alpha^{-1}(1 - \alpha)\langle y_k - x_k, \nabla f(\hat{y}_k) \rangle - 2\sigma^{-1/2}S^{-1} \langle y_k, \nabla f(\hat{y}_k) \rangle
\end{align*}
\]

This inequality is our starting point. We analyze the terms on the second line in the next section.

**A.2 The Cross Term**

To analyze these terms, we need a small lemma. This lemma is fundamental in allowing us to deal with asynchronicity.
Lemma 8. Let $\chi, A > 0$. Let the delay be bounded by $\tau$. Then:

$$A \| \hat{y}_k - y_k \| \leq \frac{1}{2} \chi^{-1} A^2 + \frac{1}{2} \chi \tau \sum_{j=1}^{\tau} \| y_{k+1-j} - y_{k-j} \|^2$$

Proof. See (Hannah and Yin, 2017a). \)

Lemma 9. We have:

$$-\langle \nabla f (\hat{y}_k) , y_k \rangle \leq -f(y_k) - \frac{1}{2} \sigma (1 - \psi) \| y_k \|^2 + \frac{1}{2} L \kappa \psi^{-1} \tau \sum_{j=1}^{\tau} \| y_{k+1-j} - y_{k-j} \|^2$$

(A.8)

$$\langle \nabla f (\hat{y}_k) , x_k - y_k \rangle \leq f(x_k) - f(y_k) + \frac{1}{2} L \alpha (1 - \alpha)^{-1} \left( \kappa^{-1} \psi \beta \| v_k - y_k \|^2 + \kappa \psi^{-1} \beta^{-1} \tau \sum_{j=1}^{\tau} \| y_{k+1-j} - y_{k-j} \|^2 \right)$$

(A.9)

The terms in bold in (A.8) and (A.9) are a result of the asynchronicity, and are identically 0 in its absence.

Proof. Our strategy is to separately analyze terms that appear in the traditional analysis of (Y. Nesterov, 2012), and the terms that result from asynchronicity. We first prove (A.8):

$$-\langle \nabla f (\hat{y}_k) , y_k \rangle = -\langle \nabla f (y_k) , y_k \rangle - \langle \nabla f (\hat{y}_k) - \nabla f (y_k) , y_k \rangle$$

$$\leq -f(y_k) - \frac{1}{2} \sigma \| y_k \|^2 + L \| \hat{y}_k - y_k \| \| y_k \|$$

(A.10)

(A.10) follows from strong convexity ((A.2) with $x = y_k$ and $y = x$), and the fact that $\nabla f$ is $L$-Lipschitz. The term due to asynchronicity becomes:

$$L \| \hat{y}_k - y_k \| \| y_k \| \leq \frac{1}{2} L \kappa^{-1} \psi \| y_k \|^2 + \frac{1}{2} L \kappa \psi^{-1} \tau \sum_{j=1}^{\tau} \| y_{k+1-j} - y_{k-j} \|^2$$

using Lemma 8 with $\chi = \kappa \psi^{-1}, A = \| y_k \|$. Combining this with (A.10) completes the proof of (A.8).

We now prove (A.9):

$$\langle \nabla f (\hat{y}_k) , x_k - y_k \rangle$$

$$= \langle \nabla f (y_k) , x_k - y_k \rangle + \langle \nabla f (\hat{y}_k) - \nabla f (y_k) , x_k - y_k \rangle$$

$$\leq f(x_k) - f(y_k) + L \| \hat{y}_k - y_k \| \| x_k - y_k \|$$

$$\leq f(x_k) - f(y_k) + \frac{1}{2} L \alpha (1 - \alpha)^{-1} \left( \kappa^{-1} \psi \beta \| x_k - y_k \|^2 + \kappa \psi^{-1} \beta^{-1} \alpha (1 - \alpha)^{-1} \tau \sum_{j=1}^{\tau} \| y_{k+1-j} - y_{k-j} \|^2 \right)$$

Here the last line follows from Lemma 8 with $\chi = \kappa \psi^{-1} \beta^{-1} \alpha (1 - \alpha)^{-1}, A = nx_k - y_k$. We can complete the proof using the following identity that can be easily obtained from (2.6):

$$y_k - x_k = \alpha (1 - \alpha)^{-1} (v_k - y_k)$$
A.3 Function-value term

Much like (Y. Nesterov, 2012), we need a \( f(x^k) \) term in the Lyapunov function (see the middle of page 357). However we additionally need to consider asynchronicity when analyzing the growth of this term. Again terms due to asynchronicity are emboldened.

**Lemma 10.** We have:

\[
\mathbb{E}_k f(x_{k+1}) \leq f(y_k) - \frac{1}{2} h \left( 2 - h \left( 1 + \frac{1}{2} \sigma^{1/2} L^{-1/2} \psi \right) \right) S^{-1} \|\nabla f(\hat{y}_k)\|_*^2 \\
+ S^{-1} L \sigma^{1/2} \kappa \psi^{-1} \tau \sum_{j=1}^\tau \|y_{k+1-j} - y_{k-j}\|^2
\]

**Proof.** From the definition (2.7) of \( x_{k+1} \), we can see that \( x_{k+1} - y_k \) is supported on block \( i_k \). Since each gradient block \( \nabla_i f \) is \( L_i \) Lipschitz with respect to changes to block \( i \), we can use (A.1) to obtain:

\[
f(x_{k+1}) \leq f(y_k) + \langle \nabla f(y_k) \cdot x_{k+1} - y_k \rangle + \frac{1}{2} L_{i_k} \left\| x_{k+1} - y_k \right\|^2
\]

(from (2.7))

\[
f(y_k) - h L^{-1}_{i_k} \langle \nabla_i f(y_k), \nabla_i f(\hat{y}_k) \rangle + \frac{1}{2} h^2 L^{-1}_{i_k} \|\nabla_i f(\hat{y}_k)\|^2
\]

\[
= f(y_k) - h L^{-1}_{i_k} \langle \nabla_i f(y_k) - \nabla_i f(\hat{y}_k), \nabla_i f(\hat{y}_k) \rangle - \frac{1}{2} h (2 - h) L^{-1}_{i_k} \|\nabla_i f(\hat{y}_k)\|^2
\]

\[
\mathbb{E}_k f(x_{k+1}) \leq f(y_k) - h S^{-1} \sum_{i=1}^n L_i^{-1/2} \langle \nabla_i f(y_k) - \nabla_i f(\hat{y}_k), \nabla_i f(\hat{y}_k) \rangle \tag{A.11}
\]

\[- \frac{1}{2} h (2 - h) S^{-1} \|\nabla f(\hat{y}_k)\|_*^2
\]

Here the last line followed from the definition (A.3) of the norm \( \|\cdot\|_{*1/2} \). We now analyze the middle term:

\[
- \sum_{i=1}^n L_i^{-1/2} \langle \nabla_i f(y_k) - \nabla_i f(\hat{y}_k), \nabla_i f(\hat{y}_k) \rangle
\]

\[
= - \left\| \sum_{i=1}^n L_i^{-1/4} \langle \nabla_i f(y_k) - \nabla_i f(\hat{y}_k) \rangle, \sum_{i=1}^n L_i^{-1/4} \nabla_i f(\hat{y}_k) \right\|
\]

(Cauchy Schwarz)

\[
\leq \left( \sum_{i=1}^n L_i^{-1/4} \|\nabla_i f(y_k) - \nabla_i f(\hat{y}_k)\|^2 \right)^{1/2} \left( \sum_{i=1}^n L_i^{-1/2} \|\nabla_i f(\hat{y}_k)\|^2 \right)^{1/2}
\]

(\( L \leq L_i, \forall i \) and (A.3))

\[
\leq L^{-1/4} \|\nabla f(y_k) - \nabla f(\hat{y}_k)\| \|\nabla f(\hat{y}_k)\|_*
\]

(\( \nabla f \) is \( L \)-Lipschitz)

\[
\leq L^{-1/4} L \|y_k - \hat{y}_k\| \|\nabla f(\hat{y}_k)\|_*
\]
We then apply Lemma 8 to this with \( \chi = 2h^{-1}\sigma^{1/2}L^{-1/4}\kappa\psi^{-1} \), \( A = \|\nabla f(\hat{y}_k)\| \), to yield:

\[
\sum_{i=1}^{m} L_i^{-1/2} (\nabla_i f(y_k) - \nabla_i f(\hat{y}_k)) \leq h^{-1} L\sigma^{1/2}\kappa\psi^{-1} \sum_{j=1}^{\tau} \|y_{k+1-j} - y_{k-j}\|^2 + \frac{1}{4} h\sigma^{1/2}L^{-1/2}\psi \|\nabla f(\hat{y}_k)\|_*^2 \tag{A.12}
\]

Finally to complete the proof, we combine (A.11), with (A.12).

### A.4 Asynchronicity error

The previous inequalities produced difference terms of the form \( \|y_{k+1-j} - y_{k-j}\|^2 \). The following lemma shows how these errors can be incorporated into a Lyapunov function.

**Lemma 11.** Let \( 0 < r < 1 \) and consider the asynchronicity error and corresponding coefficients:

\[
A_k = \sum_{j=1}^{\infty} c_j \|y_{k+1-j} - y_{k-j}\|^2 \\
c_i = \sum_{j=1}^{i} r^{i-j} s_j
\]

This sum satisfies:

\[
\mathbb{E}_k [A_{k+1} - rA_k] = c_1 \mathbb{E}_k \|y_{k+1} - y_k\|^2 - \sum_{j=1}^{\infty} s_j \|y_{k+1-j} - y_{k-j}\|^2
\]

**Remark 2. Interpretation.** This result means that an asynchronicity error term \( A_k \) can negate a series of difference terms \( \sum_{j=1}^{\infty} s_j \|y_{k+1-j} - y_{k-j}\|^2 \) at the cost of producing an additional error \( c_1 \mathbb{E}_k \|y_{k+1} - y_k\|^2 \), while maintaining a convergence rate of \( r \). This essentially converts difference terms, which are hard to deal with, into a \( \|y_{k+1} - y_k\|^2 \) term which can be negated by other terms in the Lyapunov function. The proof is straightforward.

**Proof.**

\[
\mathbb{E}_k [A_{k+1} - rA_k] = \mathbb{E}_k \sum_{j=0}^{\infty} c_{j+1} \|y_{k+1-j} - y_{k-j}\|^2 - r \mathbb{E}_k \sum_{j=1}^{\infty} c_j \|y_{k+1-j} - y_{k-j}\|^2
\]

\[
= c_1 \mathbb{E}_k \|y_{k+1} - y_k\|^2 + \mathbb{E}_k \sum_{j=1}^{\infty} (c_{j+1} - rc_j) \|y_{k+1-j} - y_{k-j}\|^2
\]

Noting the following completes the proof:

\[
c_{i+1} - rc_i = \sum_{j=i+1}^{\infty} r^{i+1-j-1} s_j - r \sum_{j=i}^{\infty} r^{i-j-1} s_j = -s_i
\]

Given that \( A_k \) allows us to negate difference terms, we now analyze the cost \( c_1 \mathbb{E}_k \|y_{k+1} - y_k\|^2 \) of this negation.
Lemma 12. We have:

$$
\mathbb{E}_k \|y_{k+1} - y_k\|^2 \leq 2\alpha^2 \beta^2 \|v_k - y_k\|^2 + 2S^{-1}L^{-1} \|\nabla f (\hat{y}_k)\|^2
$$

Proof.

$$
y_{k+1} - y_k = (\alpha v_{k+1} + (1 - \alpha) x_{k+1}) - y_k
= \alpha \left( \beta v_k + (1 - \beta) y_k - \sigma^{-1/2}L_{i_k}^{-1/2} \nabla_{i_k} f (\hat{y}_k) \right) + (1 - \alpha) (y_k - hL_{i_k}^{-1} \nabla_{i_k} f (\hat{y}_k)) - y_k
$$

(A.13)

$$
= \alpha \beta v_k + \alpha (1 - \beta) y_k - \alpha \sigma^{-1/2}L_{i_k}^{-1/2} \nabla_{i_k} f (\hat{y}_k) - \sigma^{-1/2}L_{i_k}^{-1/2} \nabla_{i_k} f (\hat{y}_k)
$$

$$
= \alpha \beta (v_k - y_k) - \left( \alpha \sigma^{-1/2}L_{i_k}^{-1/2} + h (1 - \alpha) L_{i_k}^{-1} \right) \nabla_{i_k} f (\hat{y}_k)
$$

$$
\|y_{k+1} - y_k\|^2 \leq 2\alpha^2 \beta^2 \|v_k - y_k\|^2 + 2 \left( \alpha \sigma^{-1/2}L_{i_k}^{-1/2} + h (1 - \alpha) L_{i_k}^{-1} \right)^2 \|\nabla_{i_k} f (\hat{y}_k)\|^2
$$

(A.14)

Here (A.13) following from (2.8), the definition of $v_{k+1}$. (A.14) follows from the inequality $\|x + y\|^2 \leq 2 \|x\|^2 + 2 \|y\|^2$. The rest is simple algebraic manipulation.

$$
\|y_{k+1} - y_k\|^2 \leq 2\alpha^2 \beta^2 \|v_k - y_k\|^2 + 2L_{i_k}^{-1} \left( \alpha \sigma^{-1/2} + h (1 - \alpha) L_{i_k}^{-1/2} \right)^2 \|\nabla_{i_k} f (\hat{y}_k)\|^2
$$

$$
= 2\alpha^2 \beta^2 \|v_k - y_k\|^2 + 2L_{i_k}^{-1} \left( L_{i_k}^{-1/2} \sigma^{-1/2} + h (1 - \alpha) \right)^2 \|\nabla_{i_k} f (\hat{y}_k)\|^2
$$

$$
\mathbb{E} \|y_{k+1} - y_k\|^2 \leq 2\alpha^2 \beta^2 \|v_k - y_k\|^2 + 2S^{-1}L^{-1} \left( L_{i_k}^{-1/2} \sigma^{-1/2} + h (1 - \alpha) \right)^2 \|\nabla f (\hat{y}_k)\|^2
$$

Finally, to complete the proof, we prove $L_{i_k}^{-1/2} \sigma^{-1/2} + h (1 - \alpha) \leq 1$.

$$
L_{i_k}^{-1/2} \sigma^{-1/2} + h (1 - \alpha) = h + \alpha \left( L_{i_k}^{-1/2} \sigma^{-1/2} - h \right)
$$

(definitions of $h$ and $\alpha$: (2.3), and (2.5))

$$
= 1 - \frac{1}{2} \sigma^{1/2}L_{i_k}^{-1/2} \psi + \sigma^{1/2}S^{-1} \left( L_{i_k}^{-1/2} \psi \right)
$$

$$
\leq 1 - \sigma^{1/2}L_{i_k}^{-1/2} \left( \frac{1}{2} \psi - \sigma^{-1/2}S^{-1}L_{i_k}^{-1} \right)
$$

(A.15)

Rearranging the definition of $\psi$, we have:

$$
S^{-1} = \frac{1}{2} \psi \psi \sigma^{1/2}L_{i_k}^{-3/2} \psi \sigma^{-1/2}

(\tau \geq 1 \text{ and } \psi \leq \frac{1}{2}) \leq \frac{1}{18^2} \psi \psi \sigma^{1/2}L_{i_k}^{-3/2} \psi \sigma^{-1/2}
$$

Using this on (A.15), we have:

$$
L_{i_k}^{-1/2} \sigma^{-1/2} + h (1 - \alpha) \leq 1 - \sigma^{1/2}L_{i_k}^{-1/2} \left( \frac{1}{2} \psi - \frac{1}{18^2} \psi \psi \sigma^{-1/2}L_{i_k}^{-3/2} \psi \sigma^{-1/2} \right)
$$

$$
= 1 - \sigma^{1/2}L_{i_k}^{-1/2} \left( \frac{1}{2} \psi - \frac{1}{18^2} \psi \psi \right)^2
$$

$$
(\psi \leq \frac{1}{2}) = 1 - \sigma^{1/2}L_{i_k}^{-1/2} \left( \frac{1}{24} - \frac{1}{18^2} \right) \leq 1.
$$
This completes the proof.

A.5 Master inequality

We are finally in a position to bring together all the previous results together into a master inequality for the Lyapunov function $\rho_k$ (defined in (2.11)). After this lemma is proven, we will prove that the right hand size is negative, which will imply that $\rho_k$ linearly converges to 0 with rate $\beta$.

 Lemma 13. Master inequality. We have:

$$
\mathbb{E}_k [\rho_{k+1} - \beta \rho_k] \leq + \|y_k\|^2 \times \left(1 - \beta - \sigma^{-1/2} S^{-1} \sigma (1 - \psi)\right) \\
+ \|v_k - y_k\|^2 \times \beta \left(2\alpha^2 \beta c_1 + S^{-1} \beta L^{1/2} \kappa^{-1/2} \psi - (1 - \beta)\right) \\
+ f(y_k) \times \left(e - 2\sigma^{-1/2} S^{-1} (\beta \alpha^{-1} (1 - \alpha) + 1)\right) \\
+ f(x_k) \times \beta \left(2\sigma^{-1/2} S^{-1} \alpha^{-1} (1 - \alpha) - c\right) \\
+ \sum_{j=1}^{\tau} \|y_{k+1-j} - y_{k-j}\|^2 \times S^{-1} L \kappa \psi^{-1} \tau \sigma^{1/2} (2\sigma^{-1} + c) - s \\
+ \|\nabla f (\hat{y}_k)\|_2 \times S^{-1} \left(\sigma^{-1} + 2L^{-1} c_1 - \frac{1}{2} c h\left(2 - h \left(1 + \frac{1}{2} \sigma^{1/2} L^{-1/2} \psi\right)\right)\right)
$$

Proof.

$$
\mathbb{E}_k \|v_{k+1}\|^2 - \beta \|v_k\|^2 \\
= (1 - \beta) \|y_k\|^2 - \beta (1 - \beta) \|v_k - y_k\|^2 + S^{-1} \sigma^{-1} \|\nabla f (\hat{y}_k)\|_2^2 - 2\sigma^{-1/2} S^{-1} \langle y_k, \nabla f (\hat{y}_k) \rangle \\
- 2\sigma^{-1/2} S^{-1} \beta \alpha^{-1} (1 - \alpha) \langle y_k - x_k, \nabla f (\hat{y}_k) \rangle \\
\leq (1 - \beta) \|y_k\|^2 - \beta (1 - \beta) \|v_k - y_k\|^2 + S^{-1} \sigma^{-1} \|\nabla f (\hat{y}_k)\|_2^2 \\
+ 2\sigma^{-1/2} S^{-1} \left(-f(y_k) - \frac{1}{2} \sigma (1 - \psi) \|y_k\|^2 + \frac{1}{2} L \kappa \psi^{-1} \tau \sum_{j=1}^{\tau} \|y_{k+1-j} - y_{k-j}\|^2\right) \\
- 2\sigma^{-1/2} S^{-1} \beta \alpha^{-1} (1 - \alpha) (f(x_k) - f(y_k)) \\
+ \sigma^{-1/2} S^{-1} \beta L \left(\kappa^{-1} \psi \beta \|y_k - y_k\|^2 + \kappa \psi^{-1} \beta^{-1} \tau \sum_{j=1}^{\tau} \|y_{k+1-j} - y_{k-j}\|^2\right)
$$

(A.7)
We now collect and organize the similar terms of this inequality.

\[
\begin{align*}
&\leq + \|y_k\|^2 \times \left(1 - \beta - \sigma^{-1/2} S^{-1} \sigma (1 - \psi)\right) \\
&+ \|v_k - y_k\|^2 \times \beta \left(\sigma^{-1/2} S^{-1} \beta L \sigma^{-1} - (1 - \beta)\right) \\
&- f(y_k) \times 2\sigma^{-1/2} S^{-1} (\beta \alpha^{-1} (1 - \alpha) + 1) \\
&+ f(x_k) \times 2\sigma^{-1/2} S^{-1} \beta \alpha^{-1} (1 - \alpha) \\
&+ \sum_{j=1}^\tau \|y_{k+1-j} - y_{k-j}\|^2 \times 2\sigma^{-1/2} S^{-1} L \psi^{-1} \tau \\
&+ \|\nabla f(\hat{y}_k)\|^2 \times \sigma^{-1} S^{-1}
\end{align*}
\]

Now finally, we add the function-value and asynchronicity terms to our analysis. We use Lemma 11 is with \( r = 1 - \sigma^{1/2} S^{-1} \), and

\[
s_i = \begin{cases} 
  s = 6S^{-1} L^{1/2} \kappa^{3/2} \psi^{-1} \tau, & 1 \leq i \leq \tau \\
  0, & i > \tau
\end{cases} \tag{A.18}
\]

Notice that this choice of \( s_i \) will recover the coefficient formula given in (2.9). Hence we have:

\[
E_k [cf(x_k + 1) + A_k + \beta (cf(x_k) + A_k)] \\
\text{(Lemma 10)} \leq cf(y_k) - \frac{1}{2}c h \left(2 - h \left(1 + \frac{1}{2} \sigma^{-1/2} L^{-1/2} \psi\right)\right) S^{-1} \|\nabla f(\hat{y}_k)\|^2 - \sigma^{-1} S^{-1} \kappa \psi^{-1} \tau \\
+ S^{-1} \sigma^{1/2} \kappa \psi^{-1} \tau \sum_{j=1}^\tau \|y_{k+1-j} - y_{k-j}\|^2 \\
\text{(Lemmas 11 and 12)} + c_1 \left(2\alpha^2 \beta^2 \|v_k - y_k\|^2 + 2S^{-1}L^{-1} \|\nabla f(\hat{y}_k)\|^2\right) \\
- \sum_{j=1}^\infty s_j \|y_{k+1-j} - y_{k-j}\|^2 + A_k (r - \beta)
\]

Notice \( A_k (r - \beta) \leq 0 \). Finally, combining (A.17) and (A.19) completes the proof. \( \square \)

In the next section, we will prove that every coefficient on the right hand side of (A.16) is 0 or less, which will complete the proof of Theorem 1.

A.6 Proof of main theorem

**Lemma 14.** The coefficients of \( \|y_k\|^2, f(y_k), \) and \( \sum_{j=1}^\tau \|y_{k+1-j} - y_{k-j}\|^2 \) in Lemma 13 are non-positive.

**Proof.** The coefficient \( 1 - (1 - \psi) \sigma^{1/2} S^{-1} - \beta \) of \( \|y_k\|^2 \) is identically 0 via the definition (2.4) of \( \beta \). The coefficient \( c - 2\sigma^{-1/2} S^{-1} (\beta \alpha^{-1} (1 - \alpha) + 1) \) of \( f(y_k) \) is identically 0 via the definition (2.12) of \( c \).
First notice from the definition (2.12) of $c$:

$$c = 2\sigma^{-1/2}S^{-1} (\beta\alpha^{-1} (1 - \alpha) + 1)$$

(definitions of $\alpha, \beta$)  
$$= 2\sigma^{-1/2}S^{-1} \left( (1 - \sigma^{1/2}S^{-1} (1 - \psi)) (1 + \psi) \sigma^{-1/2}S^{-1} + 1 \right)$$  
$$= 2\sigma^{-1/2}S^{-1} \left( (1 + \psi) \sigma^{-1/2}S + \psi^2 \right)$$  
$$= 2\sigma^{-1} \left((1 + \psi) + \psi^2 \sigma^{1/2}S^{-1}\right)$$  
(\text{A.20})  
$$c \leq 4\sigma^{-1}$$  
(\text{A.21})

Here the last line followed since $\psi \leq \frac{1}{2}$ and $\sigma^{1/2}S^{-1} \leq 1$. We now analyze the coefficient of $\sum_{j=1}^\tau \|y_{k+1-j} - y_{k-j}\|^2$:

$$S^{-1}L_K\psi^{-1}\tau\sigma^{1/2} \left(2\sigma^{-1} + c\right) - s$$  
(\text{A.21})  
$$\leq 6L^{1/2}\kappa^{3/2}\psi^{-1}\tau - s$$

(definition (\text{A.18}) of $s$) \hfill $\Box$

**Lemma 15.** The coefficient $\beta \left(2\sigma^{-1/2}S^{-1}\alpha^{-1} (1 - \alpha) - c\right)$ of $f(y_k)$ in Lemma 13 is non-positive.  
\textit{Proof.}

$$2\sigma^{-1/2}S^{-1}\alpha^{-1} (1 - \alpha) - c$$  
(\text{A.20})  
$$= 2\sigma^{-1/2}S^{-1} (1 + \psi) \sigma^{-1/2}S - 2\sigma^{-1} \left((1 + \psi) + \psi^2 \sigma^{1/2}S^{-1}\right)$$  
$$= 2\sigma^{-1} \left((1 + \psi) - \left((1 + \psi) + \psi^2 \sigma^{1/2}S^{-1}\right)\right)$$  
$$= -2\psi^2\sigma^{-1/2}S^{-1} \leq 0$$ \hfill $\Box$

**Lemma 16.** The coefficient $S^{-1} \left(\sigma^{-1} + 2L^{-1}c_1 - \frac{1}{2}ch \left(2 - h \left(1 + 1\sigma^{1/2}L^{-1}2\psi\right)\right)\right)$ of $\|\nabla f(y_k)\|^2$ in Lemma 13 is non-positive.  
\textit{Proof.} We first need to bound $c_1$.

$$\left( (\text{A.18}) \text{ and } (2.9) \right) c_1 = s \sum_{j=1}^\tau \left(1 - \sigma^{1/2}S^{-1}\right)^{-j}$$  
(\text{A.18})  
$$\leq 6S^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau \sum_{j=1}^\tau \left(1 - \sigma^{1/2}S^{-1}\right)^{-j}$$  
$$\leq 6S^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2 \left(1 - \sigma^{1/2}S^{-1}\right)^{-\tau}$$

It can be easily verified that if $x \leq \frac{1}{2}$ and $y \geq 0$, then $(1 - x)^{-y} \leq \exp(2xy)$. Using this fact with $x = \sigma^{1/2}S^{-1}$ and $y = \tau$, we have:

$$\leq 6S^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2 \exp\left(\tau\sigma^{1/2}S^{-1}\right)$$  
(since $\psi \leq 3/7$ and hence $\tau\sigma^{1/2}S^{-1} \leq \frac{1}{7}$)  
$$\leq S^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2 \times 6 \exp\left(\frac{1}{7}\right)$$  
$$c_1 \leq 7S^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2$$  
(\text{A.22)}
We now analyze the coefficient of \(\|\nabla f(\hat{y}_k)\|_2^2\)

\[
\sigma^{-1} + 2L^{-1}c_1 - \frac{1}{2}ch\left(2 - h \left(1 + \frac{1}{2} \sigma^{1/2}L^{-1/2}\psi\right)\right)
\]

(A.22 and 2.5) \(\leq \sigma^{-1} + 14S^{-1}L^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2 - \frac{1}{2}ch\left(1 + \frac{1}{4} \sigma^{1/2}L^{-1}\psi^2\right)\)

\(\leq \sigma^{-1} + 14S^{-1}L^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2 - \frac{1}{2}ch\)

(definition 2.2 of \(\psi\))

\[
= \sigma^{-1} + \frac{14}{81}\sigma^{-1}\psi - \frac{1}{2}ch
\]

(A.20, definition 2.5 of \(h\))

\[
= \sigma^{-1}\left(1 + \frac{14}{81}\psi - \left(1 + \psi + \psi^2\sigma^{1/2}S^{-1}\right)\left(1 - \frac{1}{2}\sigma^{1/2}L^{-1/2}\psi\right)\psi\right)
\]

\((\sigma^{1/2}L^{-1/2} \leq 0 \text{ and } \sigma^{1/2}S^{-1} \leq 1) \leq \sigma^{-1}\left(1 + \frac{14}{81}\psi - (1 + \psi)\left(1 - \frac{1}{2}\psi\right)\right)
\]

\[
= \sigma^{-1}\psi\left(\frac{14}{81} + \frac{1}{2}\psi - \frac{1}{2}\right)
\]

\((\psi \leq \frac{1}{2}) \leq 0 \]

\[\square\]

**Lemma 17.** The coefficient \(\beta \left(2\alpha^2\beta c_1 + S^{-1}\beta L^{1/2}\kappa^{-1/2}\psi - (1 - \beta)\right)\) of \(\|v_k - y_k\|^2\) in 13 is non-positive.

**Proof.**

\[
2\alpha^2\beta c_1 + \sigma^{1/2}S^{-1}\beta\psi - (1 - \psi)\sigma^{1/2}S^{-1}
\]

(A.22) \(\leq 14\alpha^2\beta S^{-1}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2 + \sigma^{1/2}S^{-1}\beta\psi - (1 - \psi)\sigma^{1/2}S^{-1}\)

\(\leq 14\sigma S^{-3}L^{1/2}\kappa^{3/2}\psi^{-1}\tau^2 + \sigma^{1/2}S^{-1}\psi - (1 - \psi)\sigma^{1/2}S^{-1}\)

\(= \sigma^{1/2}S^{-1}\left(14S^{-2}L\kappa^2\psi^{-1} + 2\psi - 1\right)\)

Here the last inequality follows since \(\beta \leq 1\) and \(\alpha \leq \sigma^{1/2}S^{-1}\). We now rearrange the definition of \(\psi\) to yield the identity:

\[
S^{-2}\kappa = \frac{1}{94} L^2 L^{-3} \tau^{-4} \psi^4
\]

Using this, we have:

\[
14S^{-2}L\kappa^2\psi^{-1} + 2\psi - 1
\]

\[
= \frac{14}{94} L^2 L^{-2} \psi^3 \tau^{-2} + 2\psi - 1
\]

\(\leq \frac{14}{94} \left(\frac{3}{7}\right)^3 1^{-2} + \frac{6}{7} - 1 \leq 0\)

Here the last line followed since \(L \leq L\), \(\psi \leq \frac{3}{7}\), and \(\tau \geq 1\). Hence the proof is complete. \(\square\)
**Proof of Theorem 1.** Using the master inequality 13 in combination with the previous Lemmas 14, 15, 16, and 17, we have:

\[ \mathbb{E}_k [\rho_{k+1}] \leq \beta \rho_k = \left(1 - (1 - \psi) \sigma^{1/2} S^{-1}\right) \rho_k \]

When we have:

\[ \left(1 - (1 - \psi) \sigma^{1/2} S^{-1}\right)^k \leq \epsilon \]

then the Lyapunov function \( \rho_k \) has decreased below \( \epsilon \rho_0 \) in expectation. Hence the complexity \( K(\epsilon) \) satisfies:

\[ K(\epsilon) \ln \left(1 - (1 - \psi) \sigma^{1/2} S^{-1}\right) = \ln (\epsilon) \]

\[ K(\epsilon) = \frac{-1}{\ln (1 - (1 - \psi) \sigma^{1/2} S^{-1})} \ln (1/\epsilon) \]

Now it can be shown that for \( 0 < x \leq \frac{1}{2} \), we have:

\[ -1 \leq \ln (1 - x) \leq -1 \]

\[ \frac{1}{x} - 1 \leq \frac{1}{\ln (1 - x)} \leq \frac{1}{x} - \frac{1}{2} \]

Since \( n \geq 2 \), we have \( \sigma^{1/2} S^{-1} \leq \frac{1}{2} \). Hence:

\[ K(\epsilon) = \frac{1}{1 - \psi} \left(\sigma^{-1/2} S + \mathcal{O}(1)\right) \ln (1/\epsilon) \]

An expression for \( K_{\text{NU_ACDM}}(\epsilon) \), the complexity of \( \text{NU_ACDM} \) follows by similar reasoning.

\[ K_{\text{NU_ACDM}}(\epsilon) = \left(\sigma^{-1/2} S + \mathcal{O}(1)\right) \ln (1/\epsilon) \]  

(A.23)

Finally we have:

\[ K(\epsilon) = \frac{1}{1 - \psi} \left(\sigma^{-1/2} S + \mathcal{O}(1)\right) K_{\text{NU_ACDM}}(\epsilon) \]

\[ = \frac{1}{1 - \psi} \left(1 + o(1)\right) K_{\text{NU_ACDM}}(\epsilon) \]

which completes the proof. \( \square \)

**B Ordinary Differential Equation Analysis**

**B.1 Derivation of ODE for synchronous A2BCD**

If we take expectations with respect to \( \mathbb{E}_k \), then synchronous (no delay) A2BCD becomes:

\[ y_k = \alpha v_k + (1 - \alpha) x_k \]

\[ \mathbb{E}_k x_{k+1} = y_k - n^{-1} \kappa^{-1} \nabla f (y_k) \]

\[ \mathbb{E}_k v_{k+1} = \beta v_k + (1 - \beta) y_k - n^{-1} \kappa^{-1/2} \nabla f (y_k) \]
We find it convenient to define $\eta = n \kappa^{1/2}$. Inspired by this, we consider the following iteration:

$$y_k = \alpha v_k + (1 - \alpha) x_k \quad \text{(B.1)}$$
$$x_{k+1} = y_k - s^{1/2} \kappa^{-1/2} \eta^{-1} \nabla f (y_k) \quad \text{(B.2)}$$
$$v_{k+1} = \beta v_k + (1 - \beta) y_k - s^{1/2} \eta^{-1} \nabla f (y_k) \quad \text{(B.3)}$$

for coefficients:

$$\alpha = \left(1 + s^{-1/2} \eta\right)^{-1}$$
$$\beta = 1 - s^{1/2} \eta^{-1}$$

$s$ is a discretization scale parameter that will be sent to 0 to obtain an ODE analogue of synchronous A2BCD. We first use (A.6) to eliminate $v_k$ from (B.3).

$$0 = -v_{k+1} + \beta v_k + (1 - \beta) y_k - s^{1/2} \eta^{-1} \nabla f (y_k)$$
$$0 = -\alpha^{-1} y_{k+1} + \alpha^{-1} (1 - \alpha) x_{k+1}$$
$$+ \beta (\alpha^{-1} y_k - \alpha^{-1} (1 - \alpha) x_k) + (1 - \beta) y_k - s^{1/2} \eta^{-1} \nabla f (y_k)$$

(times by $\alpha$) $$0 = -y_{k+1} + (1 - \alpha) x_{k+1}$$
$$+ \beta (y_k - (1 - \alpha) x_k) + \alpha (1 - \beta) y_k - \alpha s^{1/2} \eta^{-1} \nabla f (y_k)$$
$$= -y_{k+1} + y_k (\beta + \alpha (1 - \beta))$$
$$+ (1 - \alpha) x_{k+1} - x_k \beta (1 - \alpha) - \alpha s^{1/2} \eta^{-1} \nabla f (y_k)$$

We now eliminate $x_k$ using (B.1):

$$0 = -y_{k+1} + y_k (\beta + \alpha (1 - \beta))$$
$$+ (1 - \alpha) \left(y_k - s^{1/2} \eta^{-1} \kappa^{-1/2} \nabla f (y_k)\right) - \left(y_{k-1} - s^{1/2} \eta^{-1} \kappa^{-1/2} \nabla f (y_{k-1})\right) \beta (1 - \alpha)$$
$$- \alpha s^{1/2} \eta^{-1} \nabla f (y_k)$$
$$= -y_{k+1} + y_k (\beta + \alpha (1 - \beta) + (1 - \alpha)) - \beta (1 - \alpha) y_{k-1}$$
$$+ s^{1/2} \eta^{-1} \nabla f (y_{k-1}) (\beta - 1) (1 - \alpha)$$
$$- \alpha s^{1/2} \eta^{-1} \nabla f (y_k)$$
$$= (y_k - y_{k+1}) + \beta (1 - \alpha) (y_k - y_{k-1})$$
$$+ s^{1/2} \eta^{-1} \left(\nabla f (y_{k-1}) (\beta - 1) (1 - \alpha) - \alpha \nabla f (y_k)\right)$$

Now to derive an ODE, we let $y_k = Y\left(k s^{1/2}\right)$. Then $\nabla f (y_{k-1}) = \nabla f (y_k) + \mathcal{O}\left(s^{1/2}\right)$. Hence the above becomes:

$$0 = (y_k - y_{k+1}) + \beta (1 - \alpha) (y_k - y_{k-1})$$
$$+ s^{1/2} \eta^{-1} \left((\beta - 1) (1 - \alpha) - \alpha\right) \nabla f (y_k) + \mathcal{O}\left(s^{3/2}\right)$$
$$0 = \left(-s^{1/2} \tilde{Y} - \frac{1}{2} \tilde{s} \tilde{Y}\right) + \beta (1 - \alpha) \left(s^{1/2} \tilde{Y} - \frac{1}{2} \tilde{s} \tilde{Y}\right)$$
$$+ s^{1/2} \eta^{-1} \left((\beta - 1) (1 - \alpha) - \alpha\right) \nabla f (y_k) + \mathcal{O}\left(s^{3/2}\right) \quad \text{(B.4)}$$
We now look at some of the terms in this equation to find the highest-order dependence on \( s \).

\[
\beta (1 - \alpha) = \left( 1 - s^{1/2} \eta^{-1} \right) \left( 1 - \frac{1}{1 + s^{-1/2} \eta} \right)
= \left( 1 - s^{1/2} \eta^{-1} \right) \frac{s^{-1/2} \eta}{1 + s^{-1/2} \eta}
= s^{-1/2} \eta - 1
= \frac{1 - s^{1/2} \eta^{-1}}{1 + s^{1/2} \eta^{-1}}
= 1 - 2s^{1/2} \eta^{-1} + O(s)
\]

We also have:

\[
(\beta - 1)(1 - \alpha) - \alpha = \beta (1 - \alpha) - 1
= -2s^{1/2} \eta^{-1} + O(s)
\]

Hence using these facts on (B.4), we have:

\[
0 = \left(-s^{1/2} \dot{Y} - \frac{1}{2} s \ddot{Y} \right) + \left(1 - 2s^{1/2} \eta^{-1} + O(s)\right) \left(s^{1/2} \dot{Y} - \frac{1}{2} s \ddot{Y} \right)
+ s^{1/2} \eta^{-1} \left(-2s^{1/2} \eta^{-1} + O(s)\right) \nabla f(y_k) + O(s^{3/2})
0 = -s^{1/2} \dot{Y} - \frac{1}{2} s \ddot{Y} + \left(1 - 2s^{1/2} \eta^{-1} + O(s^{3/2})\right)
\]

\[
0 = \left(-2s^{1/2} \eta^{-2} + O(s^{3/2})\right) \nabla f(y_k) + O(s^{3/2})
0 = -s \dot{Y} - 2s \eta^{-1} \dot{Y} - 2s \eta^{-2} \nabla f(y_k) + O(s^{3/2})
0 = -s \ddot{Y} - 2s \eta^{-1} \ddot{Y} - 2s \eta^{-2} \nabla f(y_k) + O(s^{5/2})
\]

Taking the limit as \( s \to 0 \), we obtain the ODE:

\[
\ddot{Y}(t) + 2\eta^{-1} \dot{Y} + 2\eta^{-2} \nabla f(Y) = 0
\]

### B.2 Convergence proof for synchronous ODE

\[
e^{-\eta^{-1}t} E'(t) = \langle \nabla f(Y(t)), \dot{Y}(t) \rangle + \eta^{-1} f(Y(t))
+ \frac{1}{2} \left(\|Y(t) + \eta \dot{Y}(t)\| + \eta \|Y(t)\| \right)
+ \eta^{-1} \frac{1}{4} \|Y(t) + \eta \dot{Y}(t)\|^2
\]

(Strong convexity (A.2)) \leq \langle \nabla f(Y), \dot{Y} \rangle + \eta^{-1} \langle \nabla f(Y), Y \rangle - \frac{1}{2} \eta^{-1} \|Y\|^2
+ \frac{1}{2} \left(\|Y + \eta \dot{Y}, -\dot{Y} - 2\eta^{-1} \nabla f(Y) \right) + \eta^{-1} \frac{1}{4} \|Y(t) + \eta \dot{Y}(t)\|^2
= -\frac{1}{4} \eta^{-1} \|Y\|^2 - \frac{1}{4} \eta \|\dot{Y}\|^2 \leq 0
\]
Hence we have $E'(t) \leq 0$. Therefore $E(t) \leq E(0)$. That is:

$$E(t) = e^{n^{-1}t^{-1/2t}} \left( f(Y) + \frac{1}{4} \|Y + \eta \dot{Y}\|^2 \right) \leq E(0) = f(Y(0)) + \frac{1}{4} \|Y(0) + \eta \dot{Y}(0)\|^2$$  \hspace{1cm} (B.5)

which implies:

$$f(Y(t)) + \frac{1}{4} \|Y(t) + \eta \dot{Y}(t)\|^2 \leq e^{-n^{-1}t^{-1/2t}} \left( f(Y(0)) + \frac{1}{4} \|Y(0) + \eta \dot{Y}(0)\|^2 \right)$$  \hspace{1cm} (B.6)

### B.3 Asynchronicity error lemma

This result is the continuous-time analogue of Lemma 11. First notice that $c(0) = c_0$ and $c(\tau) = 0$. We also have:

$$c'(t) / c_0 = -re^{-rt} - re^{-rt} \frac{e^{-r\tau}}{1 - e^{-r\tau}}$$

$$= -r \left( e^{-rt} + \frac{e^{-r\tau}}{1 - e^{-r\tau}} \right)$$

$$= -r \left( e^{-rt} - e^{-r\tau} \frac{e^{-r\tau}}{1 - e^{-r\tau}} \right)$$

$$= -r e^{-rt} \left( e^{-r\tau} - 1 \right) \frac{e^{-r\tau}}{1 - e^{-r\tau}}$$

$$c'(t) = -rc(t) - re^{-r\tau}$$

Hence using $c(\tau) = 0$:

$$A'(t) = c_0 \|\dot{Y}(t)\|^2 + \int_{t-\tau}^{t} c'(t-s) \|\dot{Y}(s)\|^2 \, ds$$

$$= c_0 \|\dot{Y}(t)\|^2 - rA(t) - re^{-r\tau}D(t)$$

Now when $x \leq \frac{1}{2}$, we have $\frac{e^{-x}}{1 - e^{-x}} \geq \frac{1}{2} x^{-1}$. Hence when $r \tau \leq \frac{1}{2}$, we have:

$$A'(t) \leq c_0 \|\dot{Y}(t)\|^2 - rA(t) - \frac{1}{2} \tau^{-1} c_0 D(t)$$

and the result easily follows.
B.4 Convergence analysis for the asynchronous ODE

We consider the same energy as in the synchronous case (that is, the ODE in (3.1)). Similar to before, we have:

\[
e^{-\eta^{-1}t}E'(t) \leq \langle \nabla f(Y), \dot{Y} \rangle + \eta^{-1} \langle \nabla f(Y), Y \rangle - \frac{1}{2} \eta^{-1} \|Y\|^2 \\
+ \frac{1}{2} \langle Y + \eta \dot{Y}, -\dot{Y} - 2 \eta^{-1} \nabla f(\dot{Y}) \rangle + \eta^{-1} \frac{1}{4} \|Y(t) + \eta \dot{Y}(t)\|^2 \\
= \langle \nabla f(Y), \dot{Y} \rangle + \eta^{-1} \langle \nabla f(Y), Y \rangle - \frac{1}{2} \eta^{-1} \|Y\|^2 \\
+ \frac{1}{2} \langle Y + \eta \dot{Y}, -\dot{Y} - 2 \eta^{-1} \nabla f(\dot{Y}) \rangle + \eta^{-1} \frac{1}{4} \|Y(t) + \eta \dot{Y}(t)\|^2 \\
- \eta^{-1} \langle Y + \eta \dot{Y}, \nabla f(\dot{Y}) \rangle - \nabla f(Y) \\
= -\frac{1}{4} \eta^{-1} \|Y\|^2 - \frac{1}{4} \eta \|\dot{Y}\|^2 - \eta^{-1} \langle Y + \eta \dot{Y}, \nabla f(\dot{Y}) - \nabla f(Y) \rangle
\]

where the final equality follows from the proof in Section B.2. Hence

\[
e^{-\eta^{-1}t}E'(t) \leq -\frac{1}{4} \eta^{-1} \|Y\|^2 - \frac{1}{4} \eta \|\dot{Y}\|^2 + L \eta^{-1} \|Y\| \|\dot{Y} - Y\| + L \|\dot{Y}\| \|\dot{Y} - Y\|
\]

(B.7)

Now we present an inequality that is similar to (8).

**Lemma 18.** Let \( A, \chi > 0 \). Then:

\[
\|Y(t) - \dot{Y}(t)\| A \leq \frac{1}{2} \chi \tau D(t) + \frac{1}{2} \chi^{-1} A^2
\]

**Proof.** Since \( \dot{Y}(t) \) is a delayed version of \( Y(t) \), we have: \( \dot{Y}(t) = Y(t - j(t)) \) for some function \( j(t) \geq 0 \) (though this can be easily generalized to an inconsistent read). Recall that for \( \chi > 0 \), we have \( ab \leq \frac{1}{2} \left( \chi a^2 + \chi^{-1} b^2 \right) \). Hence

\[
X(t) - \dot{X}(t) = \int_{s=t-j(t)}^{t} X'(s) \, ds
\]

\[
\|X(t) - \dot{X}(t)\| A = \left\| \int_{s=t-j(t)}^{t} X'(s) \, ds \right\| A
\leq \frac{1}{2} \chi \left\| \int_{s=t-j(t)}^{t} X'(s) \, ds \right\|^2 + \frac{1}{2} \chi^{-1} A^2
\]

(Holder’s inequality)

\[
\leq \frac{1}{2} \chi \left( \int_{s=t-j(t)}^{t} \|X'(s)\|^2 \, ds \right) \left( \int_{s=t-j(t)}^{t} 1 \, ds \right) + \frac{1}{2} \chi^{-1} A^2
\leq \frac{1}{2} \chi \tau \left( \int_{s=t-j(t)}^{t} \|X'(s)\|^2 \, ds \right) + \frac{1}{2} \chi^{-1} A^2
\]

\( \square \)
We use this lemma twice on \( \|Y\| \|\hat{Y} - Y\| \) and \( \|Y\| \|\hat{Y} - Y\| \) in (B.7) with \( \chi = 2L, A = \|\hat{Y}\| \) and \( \chi = 4L\eta^{-1}, A = \|\hat{Y}\| \) respectively, to yield:

\[
e^{-\eta^{-1}t} e'(t) \leq -\frac{\eta^{-1}}{4} \|Y\|^2 - \frac{\eta}{4} \|\hat{Y}\|^2 + L\eta^{-1} \left( L\tau D(t) + \frac{1}{4} L^{-1} \|Y\|^2 \right) + L \left( 2L\eta^{-1} \tau D(t) + \frac{1}{8} L^{-1} \eta \|\hat{Y}\|^2 \right)\]

\[
= -\frac{1}{8} \eta \|\hat{Y}\|^2 + 3L^2 \eta^{-1} \tau D(t)
\]

The proof of convergence is completed in Section 3.

C Optimality proof

For parameter set \( \sigma, L_1, \ldots, L_n, n \), we construct a block-separable function \( f \) on the space \( \mathbb{R}^{bn} \) (separated into \( n \) blocks of size \( b \)), which will imply this lower bound. Define \( \kappa_i = L_i/\sigma \). We define the matrix \( A_i \in \mathbb{R}^{b \times b} \) via:

\[
A_i \triangleq \begin{pmatrix}
2 & -1 & 0 & \\
-1 & 2 & \ddots & \ddots \\
0 & \ddots & \ddots & -1 & 0 \\
& \ddots & -1 & 2 & -1 \\
& & 0 & -1 & \theta_i
\end{pmatrix}, \text{ for } \theta_i = \frac{\kappa_i^{1/2} + 3}{\kappa_i^{1/2} + 1}.
\]

Hence we define \( f_i \) on \( \mathbb{R}^b \) via:

\[
f_i = \frac{L_i - \sigma}{4} \left( \frac{1}{2} \langle x, A_i x \rangle - \langle e_1, x \rangle \right) + \frac{\sigma}{2} \|x\|^2
\]

which is clearly \( \sigma \)-strongly convex and \( L_i \)-Lipschitz on \( \mathbb{R}^b \). From Lemma 8 of (Lan and Zhou, 2017), we know that this function has unique minimizer

\[
x_{*,(i)} \triangleq \left( q_i, q_i^2, \ldots, q_i^b \right), \text{ for } q = \frac{\kappa_i^{1/2} - 1}{\kappa_i^{1/2} + 1}.
\]

Finally, we define \( f \) via:

\[
f(x) \triangleq \sum_{i=1}^n f_i (x_{(i)}).
\]

Now let \( e(i,j) \) be the \( j \)th unit vector of the \( i \)th block of size \( b \) in \( \mathbb{R}^{bn} \). For \( I_1, \ldots, I_n \in \mathbb{N} \), we define the subspaces

\[
V_i (I) = \text{span}\{ e(i,1), \ldots, e(i,1) \},
\]

\[
V (I_1, \ldots, I_n) = V_1 (I_1) \oplus \ldots \oplus V_n (I_n).
\]

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V (I_1, \ldots, I_n) is the subspace with the first I_1 components of block 1 nonzero, the first I_2 components of block 2 nonzero, etc. First notice that IC (V (I_1, \ldots, I_n)) = V (I_1, \ldots, I_n). Also, clearly, we have:

\[ \nabla_i f (V (I_1, \ldots, I_n)) \subset V (0, \ldots, 0, \min \{ I_i + 1, b \}, 0, \ldots, 0). \]  

(C.1)

\( \nabla_i f \) is supported on the \( i \)th block, hence why all the other indices are 0. The pattern of nonzeros in \( A \) means that the gradient will have at most 1 more nonzero on the \( i \)th block (see (Yurii Nesterov, 2013)).

Let the initial point \( x_0 \) belong to \( V (I_1, \ldots, I_n) \). Let \( I_{K,i} \) be the number of times we have had \( i_k = i \) for \( k = 0, \ldots, K - 1 \). By induction on (2.14) using (C.1), we have:

\[ x_k \in V (\min \{ \bar{I}_1 + I_{k,1}, b \}, \ldots, \min \{ \bar{I}_n + I_{k,m}, b \}) \]

Hence if \( x_{0,(i)} \in V_i (0) \) and \( k \leq b \), then

\[ \| x_{k,(i)} - x_{*,(i)} \|^2 \geq \min_{x \in V_i(I_{k,i})} \| x - x_{*,(i)} \|^2 = \sum_{j=I_{k,i}+1}^{b} q_i^{2j} = (q_i^{2I_{k,i}+2} - q_i^{2b+2}) / (1 - q_i^2) \]

Therefore for all \( i \), we have:

\[ \mathbb{E} \| x_k - x_* \|^2 \geq \mathbb{E} \| x_{k,(i)} - x_{*,(i)} \|^2 \geq \mathbb{E} \left[ (q_i^{2I_{k,i}+2} - q_i^{2b+2}) / (1 - q_i^2) \right] \]

To evaluate this expectation, we note:

\[ \mathbb{E} q_i^{2I_{k,i}} = \sum_{j=0}^{k} \binom{k}{j} p_i^j (1 - p_i)^{k-j} q_i^{2j} \]

\[ = (1 - p_i)^k \sum_{j=0}^{k} \binom{k}{j} (q_i^2 p_i (1 - p_i)^{-1})^j \]

\[ = (1 - p_i)^k \left( 1 + q_i^2 p_i (1 - p_i)^{-1} \right)^k \]

\[ = (1 - (1 - q_i^2) p_i)^k \]

Hence

\[ \mathbb{E} \| x_k - x_* \|^2 \geq \left( 1 - (1 - q_i^2) p_i \right)^k q_i^2 / (1 - q_i^2) . \]

For any \( i \), we may select the starting iterate \( x_0 \) by defining its block \( j = 1, \ldots, n \) via:

\[ x_{0,(j)} = (1 - \delta_{ij}) x_{*,(j)} \]

For such a choice of \( x_0 \), we have

\[ \| x_0 - x_* \|^2 = \| x_{*,(i)} \|^2 = q_i^2 + \ldots + q_i^{2b} = q_i^2 \frac{1 - q_i^{2b}}{1 - q_i} \]
Hence for this choice of $x_0$:

$$\mathbb{E} \|x_k - x_*\|^2 / \|x_0 - x_*\|^2 \geq \left( 1 - (1 - q_i^2) p_i \right)^k \left( (1 - q_i^2)^{-1} - 1 \right) / (1 - q_i^{2b})$$

Now notice:

$$\left(1 - (1 - q_i^2) p_i \right)^k = (q_i^{-2} - (q_i^{-2} - 1) p_i)^k q_i^{2k} \geq q_i^{2k}$$

Hence

$$\mathbb{E} \|x_k - x_*\|^2 / \|x_0 - x_*\|^2 \geq (1 - (1 - q_i^2) p_i)^k (1 - q_i^{2b-2k}) / (1 - q_i^{2b})$$

Now if we let $b = 2k$, then we have:

$$\mathbb{E} \|x_k - x_*\|^2 / \|x_0 - x_*\|^2 \geq (1 - (1 - q_i^2) p_i)^k (1 - q_i^{4k})$$

$$= (1 - (1 - q_i^2) p_i)^k / (1 + q_i^{2k})$$

$$\mathbb{E} \|x_k - x_*\|^2 / \|x_0 - x_*\|^2 \geq \frac{1}{2} \max_i (1 - (1 - q_i^2) p_i)^k$$

Now let us take the minimum of the right-hand side over the parameters $p_i$, subject to $\sum_{i=1}^n p_i = 1$. The solution to this minimization is clearly:

$$p_i = (1 - q_i^2)^{-1} / \left( \sum_{j=1}^n (1 - q_j^2)^{-1} \right)$$

Hence

$$\mathbb{E} \|x_k - x_*\|^2 / \|x_0 - x_*\|^2 \geq \frac{1}{2} \left( 1 - \left( \sum_{j=1}^n (1 - q_j^2)^{-1} \right)^{-1} \right)^k$$

$$\sum_{j=1}^n (1 - q_j^2)^{-1} = \frac{1}{4} \sum_{j=1}^n (\kappa_i^{1/2} + 2 + \kappa_i^{-1/2})$$

$$\geq \frac{1}{4} \left( \sum_{j=1}^n \kappa_i^{1/2} + 2n \right)$$

$$\mathbb{E} \|x_k - x_*\|^2 / \|x_0 - x_*\|^2 \geq \frac{1}{2} \left( 1 - \frac{4}{\sum_{j=1}^n \kappa_i^{1/2} + 2n} \right)^k$$
Hence the complexity $I(\epsilon)$ satisfies:

$$\epsilon \geq \frac{1}{2} \left(1 - \frac{4}{\sum_{j=1}^{n} \kappa_j^{1/2} + 2n}\right)^{I(\epsilon)}$$

$$I(\epsilon) \geq -\left(\ln \left(1 - \frac{4}{\sum_{j=1}^{n} \kappa_j^{1/2} + 2n}\right)\right)^{-1} \ln(1/2\epsilon)$$

$$= \frac{1}{4} (1 + o(1)) \left(n + \sum_{j=1}^{n} \kappa_j^{1/2}\right) \ln(1/2\epsilon)$$

\section{Extensions}

As mentioned, a stronger result than Theorem 1 is possible. In the case when $L_i = L$ for all $i$, we can consider a slight modification of the coefficients:

$$\alpha \triangleq \left(1 + (1 + \psi) \sigma^{-1/2} S\right)^{-1}$$  \hspace{1cm} (D.1)

$$\beta \triangleq 1 - (1 + \psi)^{-1} \sigma^{1/2} S^{-1}$$  \hspace{1cm} (D.2)

$$h \triangleq \left(1 + \frac{1}{2} \sigma^{1/2} L^{-1/2} \psi\right)^{-1}.$$  \hspace{1cm} (D.3)

for the asynchronicity parameter:

$$\psi = 6n^{1/2} \kappa^{-1} \times \tau$$  \hspace{1cm} (D.4)

This leads to complexity:

$$K(\epsilon) = (1 + \psi) n \kappa^{1/2} \ln(1/\epsilon)$$  \hspace{1cm} (D.5)

Here there is no restriction on $\psi$ as in Theorem 1, and hence there is no restriction on $\tau$. Assuming $\psi \leq 1$ gives optimal complexity to within a constant factor. Notice then that the resulting condition of $\tau$

$$\tau \leq \frac{1}{6} n \kappa^{-1/2}$$  \hspace{1cm} (D.6)

now essentially matches the one in Theorem 3 in Section 3. While this result is stronger, it increases the complexity of the proof substantially. So in the interests of space and simplicity, we do not prove this stronger result.

\section{Efficient Implementation}

As mentioned in Section 5, authors in (Yin Tat Lee and Aaron Sidford, 2013) proposed a linear transformation of an accelerated RBCD scheme that results in sparse coordinate updates. Our
proposed algorithm can be given a similar efficient implementation. We may eliminate \( x^k \) from \( A_{2B}C_D \), and derive the equivalent iteration below:

\[
\begin{pmatrix}
y_{k+1} \\
v_{k+1}
\end{pmatrix} = \begin{pmatrix} 1 - \alpha \beta, & \alpha \beta \\ 1 - \beta, & \beta \end{pmatrix} \begin{pmatrix} y_k \\ v_k \end{pmatrix} - \begin{pmatrix} \alpha \sigma^{-1/2} L^{-1/2}_{ik} \left( 1 - \alpha \right) L^{-1}_{ik} \nabla_{i^k} f \left( \hat{y}^k \right) \\ \sigma^{-1/2} L^{-1/2}_{ik} \nabla_{i^k} f \left( \hat{y}^k \right) \end{pmatrix} \]

\[ \triangleq C \begin{pmatrix} y_k \\ v_k \end{pmatrix} - Q_k \]

where \( C \) and \( Q_k \) are defined in the obvious way. Hence we define auxiliary variables \( p_k, q_k \) defined via:

\[
\begin{pmatrix} y_k \\ v_k \end{pmatrix} = C_k \begin{pmatrix} p_k \\ q_k \end{pmatrix} \tag{E.1}
\]

These clearly follow the iteration:

\[
\begin{pmatrix} p_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} p_k \\ q_k \end{pmatrix} - C^{-(k+1)} Q_k \tag{E.2}
\]

Since the vector \( Q_k \) is sparse, we can evolve variables \( p_k, q_k \) in a sparse manner, and recover the original iteration variables at the end of the algorithm via E.1.

The gradient of the dual function is given by:

\[
\nabla D \left( y \right) = \frac{1}{\lambda d} \left( \frac{1}{d} A^T A y + \lambda \left( y + l \right) \right)
\]

As mentioned before, it is necessary to maintain or recover \( A y^k \) to calculate block gradients. Since \( A y^k \) can be recovered via the linear relation in (E.1), and the gradient is an affine function, we maintain the auxiliary vectors \( A p^k \) and \( A q^k \) instead.

Hence we propose the following efficient implementation in Algorithm 1. We used this to generate the results in Table 5. We also note also that it can improve performance to periodically recover \( v^k \) and \( y^k \), reset the values of \( p^k, q^k, C \) to \( v^k, y^k \), and \( I \) respectively, and restarting the scheme (which can be done cheaply in time \( O \left( d \right) \)).

We let \( B \in \mathbb{R}^{2 \times 2} \) represent \( C^k \), and \( b \) represent \( B^{-1} \). \( \otimes \) is the Kronecker product. Each computing node has local outdated versions of \( p, q, Ap, Aq \) which we denote \( \hat{p}, \hat{q}, \hat{A}p, \hat{A}q \) respectively. We also find it convenient to define:

\[
\begin{bmatrix}
D_{1}^{k} \\
D_{2}^{k}
\end{bmatrix} = \begin{bmatrix} \alpha \sigma^{-1/2} L^{-1/2}_{ik} + h \left( 1 - \alpha \right) L^{-1}_{ik} \\ \sigma^{-1/2} L^{-1/2}_{ik} \end{bmatrix}
\tag{E.3}
\]
Algorithm 1 Shared-memory implementation of A2BCD

1: **Inputs:** Function parameters \( A, \lambda, L, \{ L_i \}_{i=1}^n, n, d \). Delay \( \tau \) (obtained in dry run). Starting vectors \( y, v \).
2: **Shared data:** Solution vectors \( p, q \); auxiliary vectors \( Ap, Aq \); sparsifying matrix \( B \).
3: **Node local data:** Solution vectors \( \hat{p}, \hat{q} \); auxiliary vectors \( \hat{Ap}, \hat{Aq} \); sparsifying matrix \( \hat{B} \).
4: Calculate parameters \( \psi, \alpha, \beta, h \) via 1. Set \( k = 0 \).
5: **Initializations:** \( p \leftarrow y, q \leftarrow v, Ap \leftarrow Ay, Aq \leftarrow Av, B \leftarrow I \).
6: while not converged, each computing node asynchronous do
7: Randomly select block \( i \) via (2.1).
8: Read shared data into local memory: \( \hat{p} \leftarrow p, \hat{q} \leftarrow q, \hat{Ap} \leftarrow Ap, \hat{Aq} \leftarrow Aq, \hat{B} \leftarrow B \).
9: Compute block gradient: \( \nabla_i f(\hat{y}) = \frac{1}{n \lambda} \left( \frac{1}{n} A_i^T \left( \hat{B}_{1,1} \hat{Ap} + \hat{B}_{1,2} \hat{Aq} \right) + \lambda \left( \hat{B}_{1,1} \hat{p} + \hat{B}_{1,2} \hat{q} \right) \right) \)
10: Compute quantity \( g_i = A_i^T \nabla_i f(\hat{y}) \)

Shared memory updates:
11: Update \( B \leftarrow \begin{bmatrix} 1 - \alpha \beta & \alpha \beta \\ 1 - \beta & \beta \end{bmatrix} \times B \), calculate inverse \( b \leftarrow B^{-1} \).
12: \( \begin{bmatrix} p \\ q \end{bmatrix} = b \begin{bmatrix} D_1^k \\ D_2^k \end{bmatrix} \otimes \nabla_i f(\hat{y}) \), \( \begin{bmatrix} Ap \\ Aq \end{bmatrix} = b \begin{bmatrix} D_1^k \\ D_2^k \end{bmatrix} \otimes g_i \)
13: Increase iteration count: \( k \leftarrow k + 1 \)
14: end while
15: Recover original iteration variables: \( \begin{bmatrix} y \\ v \end{bmatrix} \leftarrow B \begin{bmatrix} p \\ q \end{bmatrix} \). Output \( y \).