WALTER TALBOT’S THESIS

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Walter Richard Talbot was the fourth African American to earn a PhD in Mathematics. His doctoral degree is from the University of Pittsburgh in 1934 in geometric group theory. As far as I know, research in Pittsburgh on geometric group theory seems to have started with Professors J. S. Taylor and M. M. Culver at the University of Pittsburgh, who were both academic descendents of Felix Klein. They were both members of Talbot’s thesis committee.

Talbot’s research grew out of Klein’s work on fundamental domains [3] [2]. A contemporary research program was the determination of fundamental domains of finite group actions on complex vector spaces [5], [8], [4]. His thesis is not widely available, and this note gives a brief synopsis of the main results of his thesis, expressed using modern mathematical methods and language, and placed in general context [7], [6].

1. A FUNDAMENTAL DOMAIN FOR THE SYMMETRIC GROUP

Let $W$ be a finite group acting on a a finite-dimensional complex vector space $V$. The general research problem is to give an explicit description of a fundamental domain.

Let $\text{Herm}(V)$ be the space of hermitian forms on $V$; that is, the real vector space of space sesquilinear forms $h : V \times V \to \mathbb{C}$ such that

\[
\overline{h(u,v)} = h(v,u), \quad h(u, \mu v) = \mu h(u,v), \quad u, v \in V, \quad \mu \in \mathbb{C}.
\]

The general linear group $GL(V)$ of $V$ acts on $\text{Herm}(V)$ by

\[
(g \cdot h)(u, v) = h(g^{-1}u, g^{-1}v), \quad g \in GL(V)
\]

and restricts to a representation of $W$ on $\text{Herm}(V)$.

Consider the following special context. Let $W = S_n$, the symmetric group on $n$ letters. Take $V \subset \mathbb{C}^n$ to be the standard (irreducible) $n-1$-dimensional representation
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of $W$ on 

$$V = \{(z_1, \ldots, z_n) \mid z_1 + \cdots + z_n = 0\}$$

occurring in the permutation representation of $W$ on $\mathbb{C}^n$. Each irreducible representation of $S_n$ can be realized over $\mathbb{R}$. Pick a real form $V_\mathbb{R}$ of $V$. The group $W = S_n$ acts on $\text{Herm}(V)$ as above. Its decomposition into irreducibles is given as follows, with conventional notation:

**Lemma 1.**

$$\text{Herm}(V) = 1 \oplus V_\mathbb{R} \oplus V_{n-2,2} \oplus V_{n-1,1,1}.$$  

**Proof.** By breaking $\text{Herm}(V)$ into real and imaginary parts, the representation splits as

$$\text{Sym}^2(V_\mathbb{R}) \oplus i\Lambda^2(V_\mathbb{R}) \cong \text{Sym}^2(V_\mathbb{R}) \oplus \Lambda^2(V_\mathbb{R}) \cong (1 \oplus V_\mathbb{R} \oplus V_{n-2,2}) \oplus (V_{n-1,1,1}) \cong V_\mathbb{R} \otimes V_\mathbb{R},$$

as given in Exercise 4.19 in [1].

If $h$ is a hermitian form, then its zero set

$$Z(h) = \{ u \in V \mid h(u,u) = 0 \}$$

partitions $V$ into positive $h \geq 0$ and negative $h \leq 0$ regions. Given hermitian forms $h_i$, we write $h_i \geq h_j$ to indicate the positive region of $V$ bounded by the zero-set $Z(h_i - h_j)$.

Select Hermitian forms

$$h_1, h_2, \ldots, h_n$$

giving the permutation basis of $1 \oplus V_\mathbb{R} \subset \text{Herm}(V)$. Define chambers $T(g)$ in $V$, for $g \in S_n$, by inequalities

\begin{equation}
    h_{g_1} \geq \cdots \geq h_{g_n}.
\end{equation}

These chambers partition $V$ and have walls given by hypersurfaces $Z(h_i - h_j)$, where $h_i - h_j$ span the standard representation $V_\mathbb{R}$ of $S_n$, realized in the space of Hermitian matrices. Explicitly, the representation $1 \oplus V_\mathbb{R}$ has permutation basis

$$h_i(z, z') = \bar{z}_i z'_i, \quad z, z' \in V \subset \mathbb{C}^n, \quad i = 1, \ldots, n.$$ 

Thus, Talbot’s chambers (1) of $V$ are given by

$$|z_{g_1}|^2 \geq \cdots \geq |z_{g_n}|^2.$$ 

There is an obvious correspondence with the Weyl chambers of the action of $S_n$ on $V_\mathbb{R}$. (For a point of historical comparison, Weyl gave lectures on the structure and representation of continuous groups in Princeton in 1933-34.) More to the point, as
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B. Ion has observed, the proper setting for Talbot’s thesis is the Cartan decomposition for $GL(n, \mathbb{C})/U(n)$, where the space of Hermitian matrices is the Lie algebra complement of the Lie algebra of $U(n)$, and $1 \oplus V_{\mathbb{R}} \subset \text{Herm}(V)$ is the maximal abelian with an action of the Weyl group $W = S_n$.

2. End notes

Talbot works projectively in $\mathbb{P}V$ rather than $V$, but this does not play any role in the results.

Talbot only works with $n = 5$ and in fact with the alternating group $A_5$. When $n = 5$, there is another subspace $1 \oplus V_{3,2} \subset \text{Herm}(V)$ that is (up to the sign character) the permutation representation $1 \oplus V_{2,1,1}$ of $S_5$ on six letters (equivalent to the action of $S_5$ on its six Sylow-5 subgroups). Talbot also considers regions of $V$ cut out by a permutation basis of hermitian forms in this representation. He also analyzes the boundary of chambers.

References

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