An Approximation Algorithm for Active Friending in Online Social Networks

Amo G. Tong, Weili Wu, Ding-Zhu Du

Abstract

In modern online social networks, the relationships between users are typically developed by the sending and accepting invitations. The existing friending services provide friend recommendations but they cannot help the user make the online friend with a target user. In this paper, we investigate the active friending problem which aims at guiding the user to make a friendship with a target user by sending invitations methodically. We consider the prominent linear threshold model and present an $O(\sqrt{n})$-approximation algorithm dealing with general graph structures. The performance of the proposed algorithm is theoretically analyzed and supported by encouraging experimental results.

1 Introduction

Online social systems have gained great popularity as they allow efficient communications. According to (Smith 2018), there are totally 3.03 billion active social media users by the end of April 2018.

Motivation. As one of the essential services of online social platforms, online friending is crucial for both the users and the social network platforms. For online users, by establishing friendships with other users, they can enlarge their social circles and as well as extend their social influence. For social network platforms, encouraging users to build more friendships is one of the major strategies to grow the network. To this end, more and more friending services are now available in online social systems. For example, the “People You May Know” widget is provided by most of the online social systems such as LinkedIn and Facebook. However, the existing friending services merely provide friend recommendations and they cannot help users to develop an online friendship with a target user. When a user $s$ wishes to be an online friend with another user $t$, we call $s$ and $t$ as the initiator and target user, respectively. Our objective is to design strategies which are able to guide the user towards making a friendship with a target user.

Problem Formulation. The friendship between users is typically built via invitations. A major factor affecting the decision of accepting an invitation is the number of the mutual friends. In this paper, we consider a threshold based friending model and assume that whether $t$ accepts the invitation from $s$ depends on the number of common friends of $s$ and $t$. Under this model, in order to make the invitation accepted by $t$, the initiator $s$ may attempt to make friendships with the friends of $t$, and recursively, first be the friend of the friends of $t$. We call the probability that $t$ can accept the invitation from $s$ as the acceptance probability. Since the acceptance probability is positively correlated to the number of invitations, we study the active friending problem which finds an invitation set with the minimum cardinality such that the acceptance probability can reach a given threshold.

Related Work. The existing works primarily focus on the friend recommendation problem. In (Chen et al. 2009), the authors design several people recommendation algorithms to help users find known offline contacts and discover new friends on Beehive. A friend recommendation framework to improve recommending quality by characterizing user interest in several dimensions is studied in (Xie 2010). The work (Agarwal and Bharadwaj 2013) studies the friend recommendation problem from the view of interaction intensity by using the technique of collaborative filtering. The authors of (Hannon, Bennett, and Smyth 2010) also utilize collaborative filtering and consider the problem of recommending twitter users to follow. In (Chu et al. 2013), the authors propose another friend recommendation approach with the consideration of real-life location and dwell time. Different from the above works, this paper considers the active friending problem where we aim at building a friendship between an initiator and a specified target user. The work (Yang et al. 2013) is among the very few works which have considered the active friending problem. In (Yang et al. 2013), the authors consider the independent cascade model and propose a heuristic. In this paper, we consider the friending process under the threshold based model. The linear threshold model has not been considered for the active friending problem, though it has drawn much attention in social network analysis ((Chen, Yuan, and Zhang 2010), Goyal, Lu, and Lakshmanan 2011), (He et al. 2012), (Pathak, Banerjee, and Srivastava 2010), (Lu et al. 2012)). The threshold model has the advantage in modeling the influence of mutual friends on the friending process, which is main reason that we adopt this model. There are two usual formulations of the active friending problem, the minimum version and the maximum version. We consider the minimum version which is to find a

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minimum set of invited users such that the acceptance probability can reach a certain level. The maximum version of the active friending problem has been studied in (Yang et al. 2013) while the minimum version, to the best of our knowledge, has not been studied in the literature. In addition, it is worthy to note that the active friending problem under the linear threshold model is markedly different from that under the independent cascade model. This problem is neither submodular nor supermodular under the independent cascade model (Yang et al. 2013), while it becomes supermodular under the linear threshold model as shown later in Sec. 4.

Contribution. We propose a threshold model to model the friending process in online social networks. Based on the proposed model, we formulate the active friending problem as a minimization optimization problem. For the considered problem, we present a randomized approximation algorithm with an approximation ratio of $O(\sqrt{n})$ and a controllable success probability. To the best of our knowledge, this is the first approximation algorithm for active friending problem on general graph structures. As a minor contribution, we show that the considered problem is supermodular. According to the experimental results, the proposed algorithm consistently outperforms the baseline methods, and the produced solution is efficient with respect to the input-output ratio.

The rest of this paper is organized as follows. The preliminaries are provided in Sec. 2. The algorithm and its analysis are shown in Sec. 3. Sec. 4 provides the additional theoretical results. In Sec. 5 we experimentally evaluate the proposed algorithm. Sec. 6 concludes this paper. Due to space limitation, we put unimportant proofs into the supplemental material.

2 Preliminaries

This section provides the preliminaries to the rest of this paper.

Model and Problem

A social network is given by an undirected graph $G = (V, E)$ where $V$ and $E$ denote the user set and the current relation set, respectively. For two users $u$ and $v$, they are online friends iff $(u, v) \in E$. Let $n$ and $m$ be the number of users and edges, respectively. Associated with each ordered pair $(u, v)$ where $u$ and $v$ are friends, there is a weight $w_{uv} \in (0, 1]$ which characterizes the $v$'s familiarity with $u$.

We use $N_u = \{v | (u, v) \in E\}$ to denote the current friends of user $u$. For the pair $u$ and $v$ where $u$ and $v$ are not friends, we explicitly set $w_{uv} = w_{vu} = 0$. For two users $u$ and $v$ who are not yet in a friendship relationship, when they have enough mutual friends, $v$ is willing to accept the invitation from $u$. In particular, each user $v$ holds a threshold $\theta_v$ and $v$ can accept the invitation from $u$ if $\sum_{v' \in V'} w_{v'v} \geq \theta_v$, where $V'$ is the set of the mutual friends of $u$ and $v$. In order to handle the unobserved information,

\[ w_{uv} \neq w_{vu} \]

\[ w_{uv} \text{ is not necessarily equal to } w_{vu} \]

for each user $v$, we consider the $\theta_v$ uniformly selected from $[0, 1]$ and assume $\sum_u w_{uv} \leq 1$ after normalization. As mentioned, we use $s$ and $t$ to denote the initiator and the target user, respectively.

Suppose the set of the current friends of $s$ is $C$. We use $\Phi(C)$ to denote the set of the users who are not a friend of $s$ but are willing to be the friends of $s$, i.e., $\Phi(C) = \{u | u \notin C, \sum_{v \in N_u \cap C} w_{vu} \geq \theta_v\}$. Since we have $w_{vu} = 0$ for the users $u$ and $v$ who are not friends, it is equivalent that

\[ \Phi(C) = \{u | u \notin C, \sum_{v \in C} w_{vu} \geq \theta_u\}. \tag{1} \]

For an invitation set $I$, we call the users in $I$ as the invited users. Note that only the invited users can be the new friend of $s$. Given an invitation set $I \subseteq V$, the friending process goes round by round, shown as follows.

Process 1. Initially, $C_0(I) = N_s$ and the threshold of each user is randomly determined. Repeatedly obtain $C_{i+1}$ by $C_{i+1}(I) = C_i(I) \cup (\Phi(C_i(I)) \cap I)$, until $\Phi(C_i(I)) \cap I$ is empty or $t \in C_{i+1}(I)$. Let $C_{\infty}(I)$ be the $C_i(I)$ when the friending process terminates. $C_{\infty}(I)$ is in fact all the friends of $s$ under $I$, and therefore $t \in C_{\infty}(I)$ means the friending process is successful.

An example for illustration is shown below.

Example 1. Consider a network shown in Fig. 1 where $w_{uv} = 0.1$ for each ordered pair of users and suppose that the threshold of each user is 0.15. The initial friends of $s$ are $v_{11}, v_{12}, v_{13}$ and $v_{14}$. Suppose the invitation set is $\{v_{21}, v_{22}, v_{24}, v_{32}, v_{33}, t\}$. According to the process, $v_{21}$ and $v_{24}$ will be the first new friends of $s$, and, finally $v_{21}, v_{24}, v_{32}, v_{33}$ and $t$ will be the new friends of $s$. Note that $v_{23}$ could be the friend of $s$ but it has not received an invitation, while $v_{22}$ receives an invitation but there are not enough mutual friends of $v_{22}$ and $s$.

Minimum Active Friending

Because the threshold of each user is generated at random, we consider the expected acceptance probability over all possible thresholds. Let $f(I)$ be the acceptance probability when $I \subseteq V$ is selected for sending invitations. The maximum value of $f(I)$ may not be 1 because the friending process does not necessarily succeed even if $I = V$. We use $p_{\max}$ to denote the maximum acceptance probability. A high acceptance probability is desirable but the size of the invitation set can be very large if we require that $f(I) = p_{\max}$. Therefore, with respect to the input-output ratio, it is satisfactory to achieve a high percentage $\alpha$ of $p_{\max}$, and we consider the following problem.

Figure 1: An illustration example of friending process.
Problem 1 (Minimum Active Friend ing). Given a ratio \( \alpha \in (0, 1]\), find an invitation set \( I \) with the smallest size such that 
\[
f(I) \geq \alpha \cdot p_{\text{max}}.\]

Our goal is to design an approximation algorithm for solving Problem 1. In the rest of this section, we introduce another two problems that will be used as ingredients to solve Problem 1.

Minimum p-Union (MpU) Problem

It turns out that the active friending problem is closely related to the MpU problem.

Problem 2 (Minimum p-Union (MpU) Problem). Given a set of elements \( V \), a family \( U \) of subsets of \( V \) and an integer \( p \), the MpU problem is to find a subset \( U' \subseteq U \) with \( |U'| = p \) such that \( |\bigcup_{x \in U'} x| \) is minimized.

According to (Chlamtác et al. 2018), there exists a \((2\sqrt{|U|})\)-approximation to the MpU problem. We denote this algorithm as the Chlamtác algorithm, and we will take this algorithm as a subroutine to solve the active friending problem.

Minimum Subset Cover (MSC) Problem

For a set of elements \( V \) and two subsets \( V_1, V_2 \subseteq V \), we say \( V_1 \) is covered by \( V_2 \) if \( V_1 \subseteq V_2 \). The minimum subset cover (MSC) problem is define as follows.

Problem 3 (Minimum Subset Cover (MSC) Problem). Given a set of elements \( V \), a family \( U \) of subsets of \( V \) and an integer \( p \), find a subset \( V^* \) of \( V \) with the minimum cardinality such that at least \( p \) subsets in \( U \) are covered by \( V^* \).

To solve the MSC problem either optimally or approximately, it suffices to consider the subset of \( V \) which is a union of exactly \( p \) subsets of \( E \). Therefore, it is reduced to the MpU problem and the Chlamtác algorithm provides a \((2\sqrt{|U|})\)-approximation for the MSC problem.

3 An Approximation Algorithm

Now we are ready to present the algorithm for solving Problem 1. Our algorithm proceeds with two steps: (1) obtain an unbiased estimator of the objective function by sampling; (2) maximize the obtained estimator by using the Chlamtác algorithm.

An Unbiased Estimator of \( f \)

The concept of realization provides a derandomization of the friending process.

Definition 1 (Realization). For an LT network, a realization is a mapping \( g : V \rightarrow V \) randomly generated as follows. Each user \( v \) randomly selects at most one user among the initial friends where the friend \( u \in N_v \) has the probability

\[
w_{t_I(u,v)}\]

\( t_I(u,v) \) to be selected and with probability \( 1 - \sum_{u \in N_v} w_{t_I(u,v)} \) that \( v \) selects no user. Define that

\[
g(v) = \begin{cases} u & \text{if } v \text{ selects } u \\ \mathcal{N}_0 & \text{if } v \text{ selects no user} \end{cases}
\]

where \( \mathcal{N}_0 \notin V \) is an artificial user introduced for the purpose of analysis and \( \mathcal{N}_0 \) is not a friend of any user.

We use \( \mathcal{G} \) to denote the set of all possible realizations and let \( \Pr_g \) be the probability that \( g \in \mathcal{G} \) can be generated. In addition, we use \( g \) to denote a random realization generated according to Def. 1.

Process 2. For a realization \( g \) and a subset, we consider a set of nodes constructed step by step as follows. Initially, \( H_0(g, I) = N_s \). Repeatedly obtain \( H_{i+1}(g, I) \) by

\[
H_{i+1}(g, I) = H_i(g, I) \cup (\Psi(H_i(g, I)) \cap I)
\]

where

\[
\Psi(H_i(g, I)) = \{ v \mid v \notin H_i(g, I), g(v) \in H_i(g, I) \},
\]

until \( \Psi(H_i(g, I)) \cap I = 0 \) or \( t \in H_{i+1}(g, I) \). Let \( H_{\infty}(g, I) \) be the set \( H_i(g, I) \) when the process terminates. We use \( f(g, I) \) to indicate that if \( t \) belongs to \( H_{\infty}(g, I) \) where \( f(g, I) \) is define as

\[
f(g, I) = \begin{cases} 1 & \text{if } t \in H_{\infty}(g, I) \\ 0 & \text{else} \end{cases}
\]

Now let us take account of all the possible realizations and consider \( \mathbb{E}[f(g, I)] := \sum_{g \in \mathcal{G}} \Pr_g \cdot f(g, I) \) where the expectation is taken over all the realizations in \( \mathcal{G} \). We use \( H_{\infty}(g, I) \) to denote the random set following the distribution: \( \Pr[H_{\infty}(g, I) = H_{\infty}(g, I)] = \Pr[g] \). Therefore, we have \( \mathbb{E}[f(g, I)] = \Pr[t \in H_{\infty}(g, I)] \).

With an analysis similar to the one given in (Kempe, Kleinberg, and Tardos 2003), we have the following result.

Lemma 1. (Kempe, Kleinberg, and Tardos 2003) \( f(I) = \mathbb{E}[f(g, I)] \).

Proof. The idea is to show that \( C_{\infty}(I) \) and \( H_{\infty}(g, I) \) have the same distribution with respect to the realizations. Please see the supplemental material for a detailed proof.

Motivated by Process 1 and Lemma 1, in order to maximize \( f(I) \), we consider the problem that, given an invitation set \( I \), in which kind of realizations that \( t \) belongs to \( H_{\infty}(g, I) \). It turns out that we do not have to generate the whole set \( H_{\infty}(g, I) \). Instead, it suffices to consider a user set \( t(g) \) identified by Alg. 1. In Alg. 1, we track the user back according to \( g(t) \) starting from the target \( t \), and add the encountered users to \( t(g) \). For each realization \( g \) and invitation set \( I \), we say \( I \) covers \( g \) iff \( t(g) \subseteq I \). The following is a key lemma showing the condition for \( t \) to be a friend of \( s \).

Lemma 2. For each realization \( g \) and invitation set \( I \), \( t \) can be a friend of \( s \) in \( g \) if and only if \( I \) covers \( g \).

Proof. According to Def. 1, it is useful to imagine a realization as a directed graph where \( (u, v) \) exists iff \( g(v) = u \). The users connected to \( t \) forms a path because each user can
Case 1

Case 2

Case 3

Algorithm 1 \( t(g) \)

1. **Input:** \( g, t; \)
2. **Output:** a user set \( t(g); \)
3. \( t(g) \leftarrow \{t\}, u^* \leftarrow t; \)
4. **while** true **do**:
5. \( \text{if } g(u^*) = \emptyset \text{ then } t(g) \leftarrow t(g) \cup \{\emptyset\} \text{ and return } t(g); \)
6. \( \text{if } g(u^*) \not\in t(g) \text{ then } t(g) \leftarrow t(g) \cup \{g(u^*)\} \text{ and return } t(g); \)
7. **if** \( g(u^*) \in N_s \text{ then return } t(g); \)
8. \( t(g) \leftarrow t(g) \cup \{g(u^*)\} \) and \( u^* \leftarrow g(u^*); \)

**Corollary 1.** \( \sum_{g \in G} \Pr[g] \cdot f(g, I) \) is in fact an explicit formula of \( f(I) \). Unfortunately, it is not feasible to directly maximize it because its value cannot be efficiently computed as there are exponential number of realizations in \( G \). Alternatively, we consider a set \( B_I = \{g_1, \ldots, g_l\} \) of \( l \) random realizations each of which is independently generated at random. We partition the realizations in \( B_I \) into two subsets \( B^0_I \) and \( B^1_I \) where \( B^0_I = \{g \in B| y(g) = 0\} \) and \( B^1_I = \{g \in B| y(g) = 1\} \) are the sets of the type-0 realizations and type-1 realizations, respectively. For each set \( B_I \) of realizations and \( I \subseteq V \), define that \( F(B_I, V) = \sum_{g \in B_I} f(g, I) \). Note that \( F(B_I, V) = |B^1_I| \), and we will use \( F(B_I, V) \) and \( |B^1_I| \) interchangeably. According to Corollaries \( \text{[1]} \) and \( \text{[2]} \), \( |F(B_I, V)| \) can be arbitrarily small provided that \( l \) is sufficiently large. As a result, for an invocation set \( I \) satisfying \( F(B_I, I) \geq \alpha \cdot F(B_I, V) \), that \( f(I) \geq \alpha \cdot p_{max} \) should hold with a high probability when \( l \) is sufficiently large. Furthermore, it is desired to find the \( I \) with the minimum cardinality such that \( F(B_I, I) \geq \alpha \cdot F(B_I, V) \). Because type-0 realization cannot be covered by any invocation set, it suffices to consider the type-1 realizations in \( B_I \). Thus, it is equivalent to solve the following problem.

**Problem 4.** Given a collection \( B^1_I \) of type-1 realizations, and an integer \( p \leq |B^1_I| \), find a subset \( V^* \) of \( V \) with the minimum cardinality such that at least \( p \) realizations in \( B^1_I \) are covered.

We can easily check that this problem can be reduced to the MSC problem with the input \( V \setminus \{t(g_1), \ldots, t(g_{|B^1_I|})\} \) and \( p \). Therefore, the Chlamtác algorithm can produce an invitation set \( I^* \) such that

\[
F(B_I, I^*) \geq p,
\]
The following equation system will be sufficient to ensure that the performance is bounded by a provable factor. Throughout this section, we use the following centrality inequalities to analyze the performance. Since $\Pr$ is given in the next lemma.

We use $I_0$ to denote the optimal solution to Problem $1$ associated with the input $V$, and let $I^*$ be the solution produced by Alg. 2. In addition, let $0 < \epsilon < \alpha$ and $N > 0$ be two parameters which are used to control the performance. Since the algorithm is randomized, our goal is to find an invitation set $I^*$ such that, with probability at least $1 - 1/N$, we have $f(I^*) \geq (\alpha - \epsilon) \cdot p_{\text{max}}$ and meanwhile $|I^*|/|I_0|$ can be bounded by a provable factor. Throughout this section, we assume $\epsilon$ and $N$ are fixed.

We use the following centrality inequalities to analyze the accuracy of the estimations. Let $X_i \in [0, 1]$ be i.i.d. random variables where $E(X_i) = \mu$. For each $\delta > 0$, the Chernoff bound (Motwani and Raghavan 2010) states that

$$\Pr \left[ \left| \sum_i X_i - l \cdot \mu \right| \geq \delta \cdot l \cdot \mu \right] \leq 2 \exp(-\frac{l \cdot \mu \cdot \delta^2}{2 + \delta})$$

A Sufficient Condition. Let $\epsilon_0, \epsilon_1 \in (0, 1)$ be some parameters that will be determined later, and $p_{\text{max}}^*$ be an estimate of $p_{\text{max}}$ obtained by Monte Carlo simulation. Suppose a set $B_l = \{g_1, \ldots, g_l\}$ of $l$ random realizations is used in Alg. 2, and let $B_l$ be the set of the type-1 realizations in $B_l$. The following system equation will be sufficient to ensure the desired performance guarantees.

**Equation System 1.**

$$|p_{\text{max}} - p_{\text{max}}^*| \leq \epsilon_0 \cdot p_{\text{max}}$$

$$|F(B_l, I)/l - f(I)| \leq \epsilon_1 \cdot p_{\text{max}}^*,$$ for each $I \subseteq V$ (10)

$$\beta = \alpha - \epsilon_1 \cdot (1 + \epsilon_0) > 0$$

$$\beta(1 - \epsilon_1 \cdot (1 + \epsilon_0)) - \epsilon_1(1 + \epsilon_0) = \alpha - \epsilon$$

$\epsilon_0, \epsilon_1 > 0$

**Lemma 4.** With Eq System, $f(I^*) \geq (\alpha - \epsilon) \cdot p_{\text{max}}$.

**Proof.** By Eq. (10), we have $f(I^*) \geq F(B_l, I)/l - \epsilon_1 \cdot p_{\text{max}}^*$. By Eq. (9), we have $F(B_l, I) \geq \beta \cdot |B_l| = \beta \cdot F(B_l, V)$. Putting the above together, we have $f(I^*) \geq \beta \cdot F(B_l, V)/l - \epsilon_1 \cdot p_{\text{max}}^*$. Applying Eqs. (10) to $I = V$, we have $F(B_l, V)/l \geq f(V) - \epsilon_1 \cdot p_{\text{max}} = p_{\text{max}} - \epsilon_1 \cdot p_{\text{max}}^*$. Therefore $f(I^*) \geq \beta \cdot (p_{\text{max}} - \epsilon_1 \cdot p_{\text{max}}^*) - \epsilon_1 \cdot p_{\text{max}}$. Furthermore, due to Eq. (9), we have

$$f(I^*) \geq \beta \cdot (p_{\text{max}} - \epsilon_1 \cdot (1 + \epsilon_0) \cdot p_{\text{max}}) - \epsilon_1 \cdot (1 + \epsilon_0) \cdot p_{\text{max}}.$$ Finally, because of Eq. (12), we have $f(I^*) \geq (\alpha - \epsilon) \cdot p_{\text{max}}$.

**Lemma 5.** With Eq System, $\Pr[I^* \subseteq V] \leq 2 \sqrt{\frac{|B_l|}{|I_0|}}$.

**Proof.** Note that $I^*$ is obtained by running the Chlamtác algorithm with the input $V, \{t(g_1), \ldots, t(g_{|B_l|})\}$ and $|\beta \cdot |B_l|\). By Eq. (7), it suffices to show that $F(B_l, I_0) \geq \beta \cdot |B_l|$. Applying Eq. (10) to $I = I_0$, we have $F(B_l, I_0)/l \geq f(I_0) - \epsilon_1 \cdot p_{\text{max}}^*$. Because $I_0$ is the optimal solution to Problem 1, we have $I_0 \geq \alpha \cdot p_{\text{max}}$ and therefore $F(B_l, I_0)/l \geq \alpha \cdot p_{\text{max}} - \epsilon_1 \cdot p_{\text{max}}^*$. Combining Eq. (9), we further have

$$F(B_l, I_0)/l \geq (\alpha - \epsilon_1 \cdot (1 + \epsilon_0)) \cdot p_{\text{max}}.$$ On the other hand, applying Eq. (10) to $I = V$, we have $F(B_l, V)/l - f(V) \leq \epsilon_1 \cdot p_{\text{max}}^* \leq \epsilon_1 \cdot (1 + \epsilon_0) \cdot p_{\text{max}}$, where the last inequality follows from Eq. (13). Since $f(V) = p_{\text{max}}$, we have $p_{\text{max}} \geq \frac{F(B_l, I_0)}{\epsilon_1 \cdot (1 + \epsilon_0) \cdot p_{\text{max}}}$ Combining Eqs. (13) and (11), this implies that $F(B_l, I_0) \geq \beta \cdot |B_l|$. Thus, proved.

According to the above two lemmas, we have the desired performance guarantee provided that Eq System 1 is satisfied. Due to Lemma 5, an estimate $p_{\text{max}}^*$ satisfying Eq. (9) is obtainable. Furthermore, there exist $\epsilon_0$ and $\epsilon_1$ that are able to make Eqs. (11) and (12) satisfied, because the LHS of Eq. (12) approaches to $\alpha$ when $\epsilon_0$ and $\epsilon_1$ approach to 0. In addition, $\beta$ is given by $\epsilon_0$ and $\epsilon_1$. Thus, the only part left to consider is Eq. (10). According to Corollary 1, Eq. (10) can be satisfied if $l$ is sufficiently large. In particular, a threshold is given in the next lemma.

**Lemma 6.** With probability at least $1 - 1/N$, $|F(B_l, I)/l - f(I)| \leq \epsilon_1 \cdot p_{\text{max}}^*$ holds for each $I \subseteq V$, if $|p_{\text{max}}^* - p_{\text{max}}| \leq \epsilon_0 \cdot p_{\text{max}}$ and $l \geq l^*$ where

$$l^* = \left( \frac{\ln 2 + \ln N + n \ln 2}{\epsilon_1^2 \cdot (1 - \epsilon_0)^2 \cdot p_{\text{max}}} \right).$$

**Proof.** For a certain subset $I \subseteq V$, by the Chernoff bound,

$$\Pr \left[ \left| F(B_l, I)/l - f(I) \right| \geq \frac{\epsilon_1 \cdot p_{\text{max}}^*}{l} \cdot f(I) \right] \text{ is no larger than } 2 \exp\left(\frac{-l \cdot \epsilon_1^2 \cdot (p_{\text{max}}^*)^2}{2l} \right).$$

Because $p_{\text{max}}^* \geq (1 - \epsilon_0) \cdot p_{\text{max}}$ and $l \geq l^*$, this probability is no larger than $\frac{2^r}{N}$. Note that there are $2^r$ subsets of $V$. Due to the union bound, with probability at least $1 - 1/N$, $|F(B_l, I)/l - f(I)| \leq \epsilon_1 \cdot p_{\text{max}}^*$ holds simultaneously for all the subsets.

The RAF Algorithm. Given $\epsilon$ and $N$, the whole process consists of three steps: (1) determine $\epsilon_0$ and $\epsilon_1$ such that Eqs. (11) and (12) are satisfied; (2) obtain an estimate $p_{\text{max}}^*$ of...
The Minimum $I$ with $f(I) = p_{max}$

Clearly we have $p_{max} = f(V)$ but we are interested the minimum $I$ such that $f(I) = p_{max}$ but $V_{max} \subseteq V$ be a set of nodes where a node $u$ in $V_{max}$ iff $u$ is on some path from a node in $\{s\} \cup N_s$ to $t$ and $u \notin \{s\} \cup N_s$. It turns out $V_{max}$ is minimum set resulting in the maximum acceptance probability, as shown in the next lemma.

**Lemma 8.** $f(V_{max}) = p_{max}$ and $V_{max}$ is the unique minimum invitation set that achieves $p_{max}$.

The proof of Lemma 8 is given in the supplemental material. One can see that $V_{max}$ can be computed in polynomial time.

### 5 Performance Evaluation

We present the experiments for evaluating the proposed algorithm. Our experiments are performed on a server with a 3.6 GHz quad-core processor running 64-bit JAVA VM.

#### Experimental Setting

We adopt six social network datasets of which the statistics are listed in Table 1. The details of the datasets can be found in J. Leskovec (Leskovec and Krevl 2014). Following the convention (Kempe, Kleinberg, and Tardos 2003), we consider the setting where $w_{u,v} = 1/|N_u|$. In the experiment, we set that $\epsilon = 0.01$ and set $N$ as 100,000 to make the success probability larger than 99.9%. We enumerate $\alpha$ from 0.5 to 0.9. For each dataset, we select the user with the highest degree as the target and report the average result over 1000 random initiators. We compare the RAF algorithm with two popular heuristics, Shortest Path (SP) algorithm and High Degree (HD) algorithm. Given $s$, $t$ and the size of the invitation set, SP fills the invitation by adding the nodes on the shortest paths from $s$ to $t$ until the invitation set reaches the given size. HD selects the nodes with the highest degree as the invited nodes. Consequently, each set of the experiments shows the performance of the algorithms under the same size constraint on the invitation set. In addition, we estimate $p_{max}$ by Monte Carlo simulation for each pair of $s$ and $t$, and compute a subset $V_{max}$ of $V_{max}$ by taking the union of realizations generated during the estimating of $p_{max}$.

#### Results

The results are shown in Fig. 3 and Table 2. Each graph in Fig. 3 gives fours curves which plot $p_{max}$ and the results under RAF, HD and SP, respectively. Table 2 shows the details when $\alpha = 0.5$ and $\alpha = 0.8$.

**Observations.** As shown in Fig. 3, the RAF algorithm consistently outperforms other heuristics. The SP algorithm can take account of the length of the path but it fails to consider the weight of the edges. The HD algorithm is able to select high-degree nodes but it cannot maintain a high connectivity from $s$ to $t$. Therefore, the HD algorithm can perform poorly under certain scenarios such as Fig. 3b and 3c. However, SP does not dominate HD, as shown in Fig. 3c. Fig. 3 primarily shows the relative performance of the algorithms and the absolute difference between the curves could...
Table 1: Datasets

|       | Wiki  | HepTh | HepPh | Epinions | Gplus  | Youtube |
|-------|-------|-------|-------|----------|--------|---------|
| nodes #| 7K    | 28K   | 35K   | 75K      | 108K   | 1.1M    |
| edges #| 103K  | 353K  | 421K  | 508K     | 30M    | 6.0M    |

Table 2: Experimental Results

|       | Wiki  | HepTh | HepPh | Epinions | Gplus  | Youtube |
|-------|-------|-------|-------|----------|--------|---------|
| $p_{max}$ | 0.042 | 0.022 | 0.019 | 0.086    | 0.472  | 0.040   |
| $|V_{max}^*|\sqrt{max}$ | 1019  | 1900  | 320    | 9760     | 54063  | 334067  |
| $\alpha$ | 0.5   | 0.8   | 0.5   | 0.8      | 0.5    | 0.8     |
| $f(I^*)$ | 0.023 | 0.034 | 0.011 | 0.017    | 0.012  | 0.015   |
| $|I^*|\sqrt{max}$ | 12.23 | 148.1 | 66.49  | 310.9    | 7.745  | 59.05   |

Table 3: Acceptance Probability

|       | Wiki  | HepTh | HepPh | Epinions | Gplus  | Youtube |
|-------|-------|-------|-------|----------|--------|---------|
| $\alpha$ | 0.5   | 0.8   | 0.5   | 0.8      | 0.5    | 0.8     |
| $f(I^*)$ | 0.023 | 0.034 | 0.011 | 0.017    | 0.012  | 0.015   |
| $|I^*|\sqrt{max}$ | 12.23 | 148.1 | 66.49  | 310.9    | 7.745  | 59.05   |

be more significant with the increase in the scale of $f(I)$. Table 2 shows the details of $V_{max}$ and $I^*$. Because $|V_{max}^*|$ is a lower bound of $|V_{max}|$, it demands at least $|V_{max}^*|$ invited nodes to achieve the $p_{max}$ given in Table 2. If we take ratio($I$) := $f(I)/|I|$ as the input-output ratio of $I$, the $I^*$ produced by the RAF algorithm is more efficient than $V_{max}$. On Youtube when $\alpha$ is 0.8, ratio($I^*$)/ratio($V_{max}$) is more than 1.5. As a extreme example, on Wiki when $\alpha$ is 0.5, ratio($I^*$)/ratio($V_{max}$) is at least 88.6. The observation herein shows that, for the input-output ratio, it is reasonable to achieve an acceptance probability which is close to $p_{max}$ but not exactly $p_{max}$. More experimental results are available in the supplementary material.

6 Concluding Remarks

In this paper, we study the active friending problem in online social networks. We consider the linear-threshold model and design an algorithm with provable performance guarantees. The effectiveness of the proposed algorithm is supported by encouraging experimental results.

One promising future work is to customize the active friending problem for specific social networks, e.g., Facebook, Twitter and LinkedIn. Based on the diffusion model tailored for different social networks, solutions to active friending are expected to have higher practicability and effectiveness. Better solutions for general graphs are possibly obtainable due to the supermodularity discussed in Sec. 4. On the other hand, the approximation hardness of the active friending problem is still open.

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