Abstract We show that the octonions are a twisting of the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in the quasitensor category of representations of a quasi-Hopf algebra associated to a group 3-cocycle. We consider general quasi-associative algebras of this type and some general constructions for them, including quasi-linear algebra and representation theory, and an automorphism quasi-Hopf algebra. Other examples include the higher $2^n$-onion Cayley algebras and examples associated to Hadamard matrices.

1 Introduction

In this paper we provide a natural setting for the octonion algebra, namely as an algebra in a quasitensor category. Such categories have a tensor product and associativity isomorphisms $V \otimes (W \otimes Z) \cong (V \otimes W) \otimes Z$ for any three objects, but these need not, however, be the trivial vector space isomorphisms. These categories also arise naturally as the representation categories of quasi-Hopf algebras. Our first result is to identify the correct ‘octonion generating quasi-Hopf algebra’ in the category of representations of which the octonions live. The categorical point of view then provides further general constructions on the octonions. Moreover, the framework has many other interesting quasi-associative algebras beyond these.

Our general construction of the octonions mirrors, for discrete groups, Drinfeld’s construction of the quantum groups $U_q(g)$. Namely, we consider finite group function algebras $k(G)$ ($k$ a field)
regarded trivially as quasi-Hopf algebras \((k(G), \phi)\) where \(\phi\) is a group 3-cocycle on \(G\). However, for the octonions, the cocycle is a coboundary and can be identified as the result of twisting \(k(G)\) by a 2-cochain \(F\). In an extension of Drinfeld’s theory of twisting or ‘gauge equivalence’[1], any algebra on which \(k(G)\) acts also has to be twisted to remain a module-algebra. In this sense the octonions are gauge-equivalent or the twisting of the group algebra of \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) by a 2-cochain, which we provide. This accounts for many of the properties of octonions as gauge-equivalent to properties of a group algebra.

Section 2 recalls preliminaries about quasi-Hopf algebras and quasitensor categories. As a modest result, we give in detail the twisting theory of module algebras associated to the twisting of quasi-Hopf algebras. We then study the case of quasi-Hopf algebras \((k(G), \phi)\) and hence the general constructions behind the paper. For technical reasons we actually prefer to work dually with dual quasi-Hopf algebras \((kG, \phi)\) where \(kG\) is the group algebra, although this is equivalent when \(G\) is finite. We introduce in particular the example \(k_FG\) as the quasi-associative algebra associated to the group algebra of \(G\) by twisting. Its algebraic properties are studied in Section 3. Then Section 4 presents the octonions (and more trivially, the quaternions and the complex numbers) as examples of this type. We show how many of their properties may be understood in terms of the 2-cochain \(F\). Section 5 provides new quasi-associative algebras beyond the octonions. Section 6 introduces a suitable quasi-Hopf algebra of ‘automorphisms’ or comeasurings associated to any quasi-algebra of the type we consider. Finally, Section 7 develops some first steps in quasi-linear algebra, meaning a theory of matrix representations of quasi-algebras.

2 Preliminaries: General constructions

An introduction to quantum groups, including quasitensor categories and quasi-Hopf algebras is in [3], the main notations of which we use here. In fact, the natural setting for us is the dual of Drinfeld’s axioms[1], namely the notion of a dual quasi-Hopf algebra[3][2].

Thus, a dual quasibialgebra is a \((H, \Delta, \epsilon, \phi)\) where the coproduct \(\Delta : H \to H \otimes H\) and counit \(\epsilon : H \to k\) form a coalgebra (the axioms are those of a unital associative algebra with arrows reversed) and are multiplicative with respect to a ‘product’ \(H \otimes H \to H\). This is required to be
an action of the associative algebra $H$.

In the finite-dimensional case a coaction of $H$ and a map $\phi: H \to k$ such that

$$\sum \phi(a_{(1)}, b_{(1)}, c_{(1)}) \phi(a_{(2)}, b_{(2)}, c_{(2)}) = \sum \phi(a_{(1)}, b_{(1)}, c_{(1)})(a_{(2)} \cdot b_{(2)}) \cdot c_{(2)}$$

for all $a, b, c \in H$. Here $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ is a notation and $\phi$ is a unital 3-cocycle in the sense

$$\sum \phi(b_{(1)}, c_{(1)}, d_{(1)}) \phi(a_{(1)}, b_{(2)} c_{(2)}, d_{(2)}) \phi(a_{(2)}, b_{(3)}, c_{(3)}) = \sum \phi(a_{(1)}, b_{(1)}, c_{(1)} d_{(1)}) \phi(a_{(2)} b_{(2)}, c_{(2)}, d_{(2)}),$$

for all $a, b, c, d \in H$, and $\phi(a, 1, b) = \epsilon(a) \epsilon(b)$ for all $a, b \in H$. We also require that $\phi$ is convolution-invertible in the algebra of maps $H \otimes ^3 \to k$, i.e. that there exists $\phi^{-1}: H \otimes ^3 \to k$ such that

$$\sum \phi(a_{(1)}, b_{(1)}, c_{(1)}) \phi^{-1}(a_{(2)}, b_{(2)}, c_{(2)}) = \epsilon(a) \epsilon(b) \epsilon(c) = \sum \phi(a_{(1)}, b_{(1)}, c_{(1)}) \phi^{-1}(a_{(2)}, b_{(2)}, c_{(2)})$$

for all $a, b, c \in H$.

A dual quasibialgebra is a quasi-Hopf algebra if there is a linear map $S: H \to H$ and linear functionals $\alpha, \beta: H \to k$ such that

$$\sum (Sa_{(1)}) a_{(3)} \alpha(a_{(2)}) = 1 \alpha(a), \quad \sum a_{(1)} Sa_{(3)} \beta(a_{(2)}) = 1 \beta(a),$$

$$\sum \phi(a_{(1)}, Sa_{(3)}, a_{(5)}) \beta(a_{(4)}) \alpha(a_{(4)}) = \epsilon(a), \quad \sum \phi^{-1}(Sa_{(1)}, a_{(3)}, Sa_{(5)}) \alpha(a_{(2)}) \beta(a_{(4)}) = \epsilon(a)$$

for all $a \in H$.

Finally, $H$ is called dual quasitriangular if there is a convolution-invertible map $\mathcal{R}: H \otimes H \to k$ such that

$$\mathcal{R}(a \cdot b, c) = \sum \phi(c_{(1)}, a_{(1)}, b_{(1)}) \mathcal{R}(a_{(2)}, c_{(2)}) \phi^{-1}(a_{(3)}, c_{(3)}, b_{(2)}) \mathcal{R}(b_{(3)}, c_{(4)}) \phi(a_{(4)}, b_{(4)}, c_{(5)}),$$

$$\mathcal{R}(a, b \cdot c) = \sum \phi^{-1}(b_{(1)}, c_{(1)}, a_{(1)}) \mathcal{R}(a_{(2)}, c_{(2)}) \phi(b_{(2)}, a_{(3)}, c_{(3)}) \mathcal{R}(a_{(4)}, b_{(4)}, c_{(4)}) \phi^{-1}(a_{(5)}, b_{(5)}, c_{(5)}),$$

$$\sum b_{(1)} \cdot a_{(1)} \mathcal{R}(a_{(2)}, b_{(2)}) = \sum \mathcal{R}(a_{(1)}, b_{(1)}) a_{(2)} \cdot b_{(2)}$$

for all $a, b, c \in H$.

We recall also that a corepresentation or comodule under a coalgebra means vector space $V$ and a map $\beta: V \to V \otimes H$ obeying $(\text{id} \otimes \Delta) \circ \beta = (\beta \otimes \text{id}) \circ \Delta$ and $(\text{id} \otimes \epsilon) \circ \beta = \text{id}$. This is the notion of an action with arrows reversed. In the finite-dimensional case a coaction of $H$ means an action of the associative algebra $H^*$.
A monoidal category is a category \( C \) of objects \( V, W, Z, \ldots \) and a functor \( \otimes : C \times C \to C \) and a natural transformation \( \Phi : ((\otimes) \otimes) \to (\otimes(\otimes)) \) between the two functors \( C \times C \times C \to C \), where \( \Phi \) obeys Mac Lane’s ‘pentagon identity’ for equality of the two obvious isomorphisms \( \sim \).

\[
(((V \otimes W) \otimes Z) \otimes U \sim V \otimes (W \otimes (Z \otimes U)))
\]

built from \( \Phi \) for any four objects \( V, W, Z, U \). A braided or ‘quasitensor’ category is a monoidal one which has, in addition, a natural transformation \( \Psi : \otimes \to \otimes^{\text{op}} \) obeying two ‘hexagon’ coherence conditions, see \cite{1}.

The comodules \( M^H \) over a quasi-Hopf algebra form such a category with

\[
\Phi_{V,W,Z}(v \otimes w \otimes z) = \sum v^{(1)} \otimes (w^{(1)} \otimes z^{(1)}) \phi(v^{(2)}, w^{(2)}, z^{(2)})
\]

for \( v \in V, w \in W, z \in Z \). Here \( \beta(v) = \sum v^{(1)} \otimes v^{(2)} \) is a notation and the tensor product is two comodules is a comodule by composition with the product of \( H \). In the quasitriangular case the category is braided \cite{1}, with \( \Psi_{V,W}(v \otimes w) = \sum w^{(1)} \otimes v^{(1)} R(v^{(2)}, w^{(2)}) \). There is also a conjugate or dual coaction on \( V^* \) made possible by the antipode \( S \). The converse is also true, namely any (braided) monoidal category with duals and with a multiplicative functor to the category of vector spaces (and some finiteness properties) comes from the comodules over a dual (quasitriangular) quasi-Hopf algebra \cite{1}.

If \( H \) is a dual quasi-Hopf algebra then so is \( H_F \) with the new product, \( \Phi, R, \alpha, \beta \) given by

\[
a \cdot_F b = \sum F^{-1}(a_{(1)}, b_{(1)}) a_{(2)} b_{(2)} F(a_{(3)}, b_{(3)})
\]

\[
\phi_F(a, b, c) = \sum F^{-1}(b_{(1)}, c_{(1)}) F^{-1}(a_{(1)}, b_{(2)} c_{(2)}) \phi(a_{(2)}, b_{(3)}, c_{(3)}) F(a_{(3)} b_{(4)}, c_{(4)}) F(a_{(4)}, b_{(5)})
\]

\[
\alpha_F(a) = \sum F(S a_{(1)}, a_{(3)}) \alpha(a_{(2)}), \quad \beta_F(a) = \sum F^{-1}(a_{(1)}, S a_{(3)}) \beta(a_{(2)})
\]

\[
R_F(a, b) = \sum F^{-1}(b_{(1)}, a_{(1)}) R(a_{(2)}, b_{(2)}) F(a_{(3)}, b_{(3)})
\]

for all \( a, b, c \in H \). Here \( F \) is any convolution-invertible map \( F : H \otimes H \to k \) obeying \( F(a, 1) = F(1, a) = \epsilon(a) \) for all \( a \in H \) (a 2-cochain). This is the dual version of the twisting operation or ‘gauge equivalence’ of Drinfeld, so called because it does not change the category of comodules up to monoidal equivalence.

**Definition 2.1** Let \( H \) be a dual quasi-Hopf algebra. An \( H \)-comodule quasialgebra \( A \) is an algebra in the category of \( H \)-comodules. This means an \( H \)-comodule, a product map \( \cdot \) associative.
in the category and equivariant under the action of $H$. Explicitly,

$$(a \cdot b) \cdot c = \sum a^{(1)} \cdot (b^{(1)} \cdot c^{(1)}) \phi(a^{(2)}, b^{(2)}, c^{(2)}), \quad \beta(a \cdot b) = \beta(a) \beta(b), \quad \forall a, b \in H$$

where the last expression uses the tensor product algebra in $A \otimes H$.

**Proposition 2.2** If $A$ is an $H$-comodule quasialgebra and $F : H \otimes H \rightarrow k$ a 2-cochain, then $A_F$ with the new product

$$a \cdot_F b = \sum a^{(1)} b^{(1)} F(a^{(2)}, b^{(2)})$$

and unchanged unit, is an $H_F$-comodule quasialgebra.

**Proof** This is elementary and follows from the equivalence of the comodule categories under twisting. See also [6] in the module version. $\square$

There is a parallel theory with all arrows reversed. Thus, $H$ can be a quasi-Hopf algebra, with associative product and $\Delta$ coassociative up to conjugation by an invertible 3-cocycle $\phi \in H \otimes^3 H$. In this case the modules of $H$ form a monoidal category and, in the quasitriangular case a braided one. In this case we work with $H$-module quasialgebras and their twistings by $F \in H \otimes H$.

When the theory is developed in this comodule form, it is an easy matter to specialize to the following class of examples: let $H = kG$ the group algebra of a group. This has coproduct etc.

$$\Delta x = x \otimes x, \quad \epsilon x = 1, \quad Sx = x^{-1}, \quad \forall x \in G$$

forming a Hopf algebra. However, for any point-wise invertible group cocycle $\phi : G \times G \times G \rightarrow k$ in the sense

$$\phi(y, z, w)\phi(x, yz, w)\phi(x, y, z) = \phi(x, y, zw)\phi(xy, z, w), \quad \phi(x, e, y) = 1$$

extended linearly to $kG \otimes^3 k$, we can regard $(kG, \phi)$ as a dual quasi-Hopf algebra. Group inversion provides an antipode with $\alpha = \epsilon, \beta(x) = 1/\phi(x, x^{-1}, x)$. Finally, a dual quasitriangular structure is possible only when $G$ is Abelian and corresponds to invertible $R : G \times G \rightarrow k$ such that

$$R(xy, z) = R(x, z)R(y, z)\frac{\phi(z, x, y)\phi(x, y, z)}{\phi(x, z, y)}, \quad R(x, yz) = R(x, z)R(x, y)\frac{\phi(y, x, z)}{\phi(y, z, x)\phi(x, y, z)}$$

(10)
for all $x, y, z \in G$.

A special case is when $\phi$ is a coboundary

$$
\phi(x, y, z) = \frac{F(x, y)F(xy, z)}{F(y, z)F(x, yz)}, \quad \mathcal{R}(x, y) = \mathcal{R}_0(x, y) \frac{F(x, y)}{F(y, x)}, \quad \beta(x) = \frac{F(x^{-1}, x)}{F(x, x^{-1})}
$$

(11)

for any invertible $F$ obeying $F(x, e) = 1 = F(e, x)$ for all $x \in G$ and any invertible bicharacter $\mathcal{R}_0$. This is the twisting of the group algebra $(kG, \mathcal{R}_0)$ regarded as a dual quasitriangular Hopf algebra $[2]$ with trivial initial $\phi_0$.

Next, a coaction of $kG$ means precisely a $G$-grading, where $\beta(v) = v \otimes |v|$ on homogeneous elements of degree $|v|$. Hence the notion of an $H$-comodule quasialgebra in this case becomes:

**Definition 2.3** A $G$-graded quasialgebra is an $G$-graded vector space $A$, a product map $A \otimes A \to A$ preserving the total degree and associative in the sense

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)\phi(|a|, |b|, |c|), \quad \forall a, b, c \in A$$

(of homogeneous degree), for a 3-cocycle $\phi$. A $G$-graded quasi-algebra is called coboundary if $\phi$ is a coboundary as in (11).

This is the setting which we will use, with the above as the underlying explanation of the constructions. If $G$ is finite we can equally regard its function algebra $k(G)$ with $\phi \in k(G)^{\otimes 3}$ as a quasi-Hopf algebra $k_{\phi}(G)$ as in [4], and then view a $G$-graded quasialgebra as equivalently a $k_{\phi}(G)$-module quasialgebra. Here the action of $h \in k_{\phi}(G)$ is $h.v = vh(|v|)$ on homogeneous elements.

**Corollary 2.4** $k_{\phi}G$ defined as $kG$ with a modified product

$$x \cdot_F y = xyF(x, y), \quad \forall x, y \in G$$

is a coboudary $G$-graded quasialgebra. The degree of $x \in G$ is $x$, and $F$ is any 2-cochain on $G$.

**Proof** This is a special case of the twisting proposition. Here $kG$ coacts on itself by $\beta = \Delta$, i.e. the degree of $x \in G$ is $x$. We now twist $kG$ to the dual quasi-Hopf algebra $(kG, \phi = \partial F)$. In the process, we also twist $kG$ as a comodule algebra to $k_{\phi}G$ as a comodule quasialgebra under this dual quasi-Hopf algebra. $\square$

We note an elementary properties of $k_{\phi}G$. 

6
Proposition 2.5 With trivial initial bicharacter $R_0$, the category of $G$-graded vector spaces is symmetrically braided (in the sense $\Psi^2 = \text{id}$) and $k_F G$ is braided-commutative in the sense

$$a \cdot b = \cdot \circ \Psi(a \otimes b), \forall a, b \in k_F G.$$ 

Proof This is immediate from (11) and the definition of $\Psi$ from $R$. The latter is clearly

$$\Psi(a \otimes b) = b \otimes a^F(a, b)_{F(|a|, |b|)}$$

for elements of homogeneous degree $|a|, |b|$, which is braided but trivially braided in the sense $\Psi^2 = \text{id}$ (i.e. the category of $G$-graded spaces in this case is symmetric monoidal rather than strictly braided.) \qed

3 More about the quasi-algebras $k_F G$

In this section, we will study further properties of the $G$-graded quasialgebras $k_F G$ beyond the general ones arising from their categorical structure in the preceding section. We assume that $G$ is Abelian, that $F$ is a 2-cochain and $\phi$ a 3-cocycle.

First of all, we note that $k_F G$ has a natural symmetric bilinear form whereby the basis of group elements is orthonormal. In general, the associated quadratic function on $k_F G$ will not be multiplicative (a quadratic character).

Proposition 3.1 If $k_F G$ admits a quadratic character then $F^2$ is a coboundary and $\phi^2 = 1$. If the Euclidean norm quadratic function defined by $q(x) = 1$ for all $x \in G$ is multiplicative (making $k_F G$ a composition algebra) then $F^2 = 1$.

Proof (For all discussions of quadratic forms we suppose that $k$ has characteristic not 2). Given a quadratic character $q : k_F G \to k$, we have $q(x \cdot y) = q(F(x, y)xy) = F^2(x, y)q(xy) = q(x)q(y)$ on $x, y \in G$, i.e. $F^2(x, y) = q(x)q(y)/q(xy)$ is a coboundary in the group cohomology. In general, if $F^2 = \partial q$ we still need to specify a bilinear form with diagonal $q$, so the converse is not automatic. If we take the canonical quadratic function associated to $G$ as an orthonormal basis, we will have $q(x \cdot F y) = F^2(x, y) = F^2(x, y)q(x)q(y)$ for all $x, y \in G$, so if this is multiplicative then $F^2 = 1$. \qed

This will be the case for some of the Cayley algebras in the next section, as well as for many other examples, and is the reason that $F, \phi$ typically have values $\pm 1$ in these cases.
Also, we already know from the construction in Corollary 2.4 that
\[(x \cdot y) \cdot z = x \cdot (y \cdot z)\phi(x, y, z), \quad \phi(x, y, z) = \frac{F(x, y)F(xy, z)}{F(x, yz)F(y, z)}\]
and hence that \(k_FG\) is associative iff \(\phi = 1\). Also, recall that \(F(x, e) = F(e, x) = 1\) (where \(e \in G\) is the group identity) is part of the cochain definition, and
\[\phi(e, x, y) = \phi(x, e, y) = \phi(x, y, e) = 1\]
is part of the cocycle definition (the middle one implies the other two), in particular it holds for our coboundary \(\phi\).

Likewise, we know from Proposition 2.5 that \(k_FG\) is braided-commutative with respect the braiding
\[\Psi(x \otimes y) = R(x, y)y \otimes x, \quad R = \frac{F(x, y)}{F(y, x)}, \quad \forall x, y \in G.\]
Hence it is commutative in the usual sense iff \(F\) is symmetric. This is also clear from the form of the product in \(k_FG\) since \(G\) itself is Abelian. More interesting for us,

**Definition 3.2** We say that \(k_FG\) is altercommutative if \(\Psi\) is given by \(R\) of the form
\[R(x, y) = \begin{cases} 1 & \text{if } x = e \text{ or } y = e \text{ or } x = y \\ -1 & \text{otherwise} \end{cases}\]
for all \(x, y \in G\).

Note that \(R(x, x) = R(x, e) = R(e, x) = 1\) for any \(k_FG\), so the content here is the value \(-1\) in the remaining ‘otherwise’ case. Also note that an altercommutative \(k_FG\) can never be commutative unless \(G = \mathbb{Z}_2\). The condition is somewhat similar to the notion of a ‘supercommutative’ algebra. One also has (more familiar) cases for the breakdown of associativity, such as the notion of an alternative algebra. We have,

**Proposition 3.3** \(k_FG\) is an alternative algebra if and only if
\[\phi^{-1}(y, x, z) + R(x, y)\phi^{-1}(x, y, z) = 1 + R(x, y)\]
\[\phi(x, y, z) + R(z, y)\phi(x, z, y) = 1 + R(z, y)\]
for all \(x, y, z \in G\). In this case,
\[\phi(x, x, y) = \phi(x, y, y) = \phi(x, y, x) = 1\]
for all \(x, y \in G\).
Proof It is enough to consider the conditions of an alternative algebra on our basis elements, \(x, y, z \in G\), i.e.

\[
(x \cdot y) \cdot z - x \cdot (y \cdot z) + (y \cdot x) \cdot z - y \cdot (x \cdot z) = 0
\]

\[
(x \cdot y) \cdot z - x \cdot (y \cdot z) + (x \cdot z) \cdot y - x \cdot (z \cdot y) = 0.
\]

This translates at once into the two equations

\[
F(x, y)F(xy, z) + F(y, x)F(yx, z) = F(y, z)F(x, yz) + F(x, z)F(y, xz)
\]

\[
F(x, z)F(xz, y) + F(x, y)F(xy, z) = F(z, y)F(x, zy) + F(y, z)F(x, yz)
\]

for all \(x, y, z \in G\) when we put the product of \(kF G\) in terms of the associative product in \(G\). Dividing through then gives the equations in terms of \(\phi, R\) as stated.

Also, setting \(x = y\) in the first equation gives us (for characteristic of \(k\) not 2) \(\phi(x, x, z) = 1\). Setting \(y = z\) in the second equation likewise gives us \(\phi(x, y, y) = 1\). Given these, setting \(x = z\) in either gives \(\phi(x, y, x) = 1\). Actually, it is known that the condition of being an alternating algebra is equivalent to

\[
(a \cdot a) \cdot b = a \cdot (a \cdot b), \quad (a \cdot b) \cdot b = a \cdot (b \cdot b)
\]

for all \(a, b\) in the algebra, which more immediately implies \(\phi(x, x, y) = \phi(x, y, y) = 1\) on basis elements. (Given these, the same two equations applied to \(a = x + y, b = z\) in the first case and \(a = x, b = y + z\) in the second case provide the full equations for an alternating algebra on basis elements \(x, y, z\), and hence imply that \((a \cdot b) \cdot a = a \cdot (b \cdot a)\) holds as well, as usual.) \(\square\)

Next we consider involutions. Since we have a special basis of \(kF G\) it is natural to consider involutions diagonal in this basis.

Lemma 3.4 \(kF G\) admits an involution which is diagonal in the basis \(G\) iff \(R = \partial s\) (a group coboundary) for some 1-cochain \(s : G \to k^*\) with \(s^2 = 1\). In this case, one has \(R(x, y) = R(y, x)\) and \(\phi(x, y, z) = \phi(z, y, x)^{-1}\) for all \(x, y, z \in G\).

Proof Consider the endomorphism \(\sigma\) of the vector space \(kF G\) of the form

\[
\sigma(x) = s(x)x, \quad \forall x \in G,
\]

\[
\sigma(x) = s(x)x, \quad \forall x \in G,
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\sigma(x) = s(x)x, \quad \forall x \in G,
\]
extended linearly, for some function $s : G \to k$. For an involution we need $\sigma^2 = \text{id}$, $s(1) = 1$ and $\sigma(a \cdot b) = \sigma(b)\sigma(a)$, $\forall a, b$ in our algebra. It is enough to consider these on the basis elements. Then clearly the first two correspond to

i) $s(e) = 1$ and $s^2(x) = 1$ for all $x \in G$.

For the second condition,

$$\sigma(x \cdot y) = s(xy)x \cdot y = s(xy)F(x, y)xy$$

while,

$$\sigma(y) \cdot \sigma(x) = s(y)y \cdot s(x)x = s(x)s(y)F(y, x)xy.$$ 

Equality for all $x, y$ corresponds to

ii) $\frac{s(x)s(y)}{s(xy)} = \frac{F(x, y)}{F(y, x)}$ for all $x, y \in G$.

We interpret this as stated, where the right hand side is $R$ corresponding to the braiding $\Psi$. In view of i), it implies that $R^2(x, y) = 1$ and hence that $R$ is symmetric. It also implies that $\partial R = 1$ in the group cohomology, which is the condition on $\phi$ stated since $R(x, y) = F(x, y)/F(y, x)$ for all $x, y \in G$. $\square$

In fact, we will be particularly interested in involutions with the property,

$$a + \sigma(a), a \cdot \sigma(a) \in k1$$

(a multiple of the identity) for all $a$ in the algebra. Let us call this a strong involution.  

**Proposition 3.5** $k_FG$ admits a diagonal strong involution $\sigma$ if and only if

i) $G \simeq (\mathbb{Z}_2)^n$ for some $n$,

ii) $\sigma(e) = e$, $\sigma(x) = -x$ for all $x \neq e$,

iii) $k_FG$ is altercommutative in the sense of Definition 3.2.

**Proof** Consider an endomorphism of the diagonal form $\sigma(x) = s(x)x$. A general element $a = \sum_{x \in G} \alpha_x x$, we have of course

$$\sigma(a) = \sum_{x \in G} \alpha_x s(x)x,$$

hence

$$a + \sigma(a) = \sum_{x \in G} \alpha_x(1 + s(x))x \in k1$$
for all coefficients $\alpha_x$ if and only if $s(x) = -1$ for all $x \neq e$. Here the group identity $e \in G$ is the basis element $1 \in k_FG$. Since we also need $\sigma(1) = 1$ for an involution, these two conditions hold iff

ii’) $s(e) = 1$ and $s(x) = -1$ for all $x \neq e$.

Next, consider a basis element $x \in G$, then

$$x \cdot \phi(x) = s(x)x \cdot x = s(x)F(x, x)x^2 \in k1$$

implies that $x^2 = e$, i.e. every element of $G$ has order 2. Hence $G$ (which is a finite Abelian group) is isomorphic to $(\mathbb{Z}_2)^n$ for some $n$. Next, consider $x + y$ for basis elements $x \neq y$. Then

$$(x + y) \cdot \phi(x + y) = (x + y) \cdot (s(x)x + s(y)y)$$

$$= (s(x)F(x, x) + s(y)F(y, y))e + (s(x)F(x, y) + s(y)F(y, x))xy \in k1$$

tells us that

iii’) $s(x)F(y, x) + s(y)F(x, y) = 0$ for all $x \neq y$.

When $x = e$ or $y = e$ this is empty. Otherwise, given ii’) it is equivalent to $\mathcal{R}(x, y) = -1$ for all $x \neq y$, $x \neq e$ and $y \neq e$, which is the altercommutativity condition.

Conversely, given these facts, a general element $a = \sum_{x \in G} \alpha_x x$ obeys

$$a \cdot \sigma(a) = \sum_{x,y \in G} \alpha_x \alpha_y s(y)x \cdot y = \sum_{x,y \in G} \alpha_x \alpha_y F(x, y)s(y)xy$$

$$= \sum_{z \in G} z \left( \sum_{x,y \in G} \delta_{xy,z} \alpha_x \alpha_y s(y)F(x, y) \right) \in k1$$

since the terms with $z \neq e$ have contribution only when $x \neq y$, and in this case the terms cancel pairwise due to condition iii’). Hence if these conditions hold, we have the stated properties for $\sigma$. They clearly imply the ones in the preceding lemma as well. □

On the other hand, given $k_FG$ we have a natural function $s(x) = F(x, x)$ and consider now the particular endomorphism corresponding to this.

**Proposition 3.6** If $\sigma(x) = F(x, x)x$ for all $x \in G$ defines a strong involution and $|G| \neq 2$ in $k$, then the algebra $k_FG$ is simple.
Let $I$ be an ideal of $k_F G$ different from $k_F G$, and $a = \sum_{x \in G} \alpha_x x$ an element of $I$. Then for each $y \neq e \in G$,

$$a \cdot y = \sum_{x \in G} \alpha_x F(x, y)xy \in I, \quad y \cdot a = \sum_{x \in G} \alpha_x F(y, x)xy \in I.$$ 

Adding these together and using Proposition 3.5 in the form of the conditions ii') and iii'), we see that

$$a \cdot y + y \cdot a = \sum_{x \neq y} \alpha_x (F(y, x) + F(x, y))xy + \alpha_y 2F(y, y)e = 2\alpha_e y - 2\alpha_y e \in I.$$

Then $(a \cdot y + y \cdot a) \cdot y = -2(\alpha_e e + \alpha_y y) \in I$ as well. Since this is for all $y \neq e$, we have $\sum_{y \neq e} \alpha_y y + (|G| - 1)\alpha_e e \in I$ and hence $\alpha_e e \in I$ provided $|G| - 2 \neq 0$ in $k$. In this case $\alpha_e = 0$ and $-\alpha_y e + \alpha_e y = -\alpha_y e \in I$ tells us that $\alpha_y = 0$ for all $y$, i.e. $a = 0$. Hence $I = \{0\}$. $\square$

We also see from Proposition 3.5 that $\sigma$ a strong involution restricts us to $G \cong (\mathbb{Z}_2)^n$. In this context we have a partial converse to Proposition 3.1.

**Proposition 3.7** If $G \cong (\mathbb{Z}_2)^n$ then the Euclidean norm quadratic function defined by $q(x) = 1$ for all $x \in G$ makes $k_F G$ a composition algebra if and only if

i) $F^2(x, y) = 1$ for all $x, y \in G$

ii) $F(x, xz)F(y, yz) + F(x, yz)F(y, xz) = 0$, for all $x, y, z \in G$ with $x \neq y$.

In this case the conditions in the Proposition 3.6 hold, i.e. $\sigma(x) = F(x, x)x$ for all $x \in G$ is a strong involution and $k_F G$ is simple if $|G| \neq 2$. Moreover, $k_F G$ is alternative.

**Proof** Suppose that $q$ is multiplicative. $q(x \cdot y) = q(x)q(y)$ on all basis elements $x, y \in G$ is $F^2 = 1$, as we know already from Proposition 3.1. The next case $q((x + y) \cdot z) = q(x + y)q(z)$ on basis elements $x, y, z \in G$ with $x \neq y$ does not yield any new condition (both sides are 2). Now consider the elements $x, y, z, w \in G$ with $x \neq y, z \neq w$ but $xz = yw$. Because every element of $G$ is of order 2, this also means $xw = yz$. Because $G$ is a group, $xz \neq xw$, however. Hence

$$q((x + y) \cdot (z + w)) = q(xz(F(x, z) + F(y, t)) + xw(F(x, w) + F(y, z)))$$

$$= (F(x, z) + F(y, w))^2 + (F(x, w) + F(y, z))^2 = 4 + 2(F(x, z)F(y, w) + F(x, w)F(y, z))$$

while $q(x + y)q(z + w) = 4$. This is the second condition stated after writing $w = xyz$ and renaming $xz$ to $z$. 

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Conversely, assuming these conditions and given \( a = \sum x \alpha x \) and \( b = \sum y \beta y \), we have

\[
a \cdot b = \sum_{z \in G} \left( \sum_{x \in G} \alpha x \beta xz F(x, xz) \right)
\]

since every element of \( G \) has order 2. Hence

\[
q(ab) = \sum_{z} \left( \sum_{x} \alpha x \beta xz F(x, xz) \right)^2 = \sum_{z} \sum_{x} \sum_{y} \alpha x \alpha y \beta xz \beta yz F(x, xz) F(y, yz).
\]

In this sum the diagonal part where \( x = y \) contributes

\[
\sum_{z} \sum_{x} \alpha^2 x \beta^2 xz = q(a)q(b)
\]

since \( F^2 = 1 \), while the remaining contribution from \( x \neq y \) is

\[
\sum_{z} \sum_{x \neq y} \alpha x \alpha y \beta xz \beta yz F(x, xz) F(y, yz), \quad (*)
\]

By condition ii) this is equal to

\[
- \sum_{x \neq y} \sum_{z} \alpha x \alpha y \beta xz \beta yz F(x, yz) F(y, xz) = - \sum_{x \neq y} \sum_{w} \alpha x \alpha y \beta yw \beta xw F(x, xw) F(y, yw)
\]

where we change the order of summation and change variables to \( w = xyz \). But this has the same form as our original expression for (*) but with a minus sign, hence this is zero.

Next, we observe that the condition ii) can be broken down equivalently as the conditions

ii.a) \( F(x, xy) + F(x, y) = 0 \) for all \( x, y \in G \) with \( x \neq e \).

ii.b) \( F(x, yz) F(y, xz) + F(x, z) F(y, z) = 0 \) for all \( x, y, z \in G \) with \( x \neq e, y \neq e \) and \( x \neq y \).

The first of these is ii) in either of the cases \( x = e, y \neq e \) or \( x \neq e, y = e \) (followed by a relabeling), while the second is the remaining case \( x \neq e, y \neq e, x \neq y \) after making use of ii.a) to substitute \( F(x, xz) \) and \( F(y, yz) \).

In this case, ii.a) implies the condition ii’) of Proposition 3.6. On the other hand, \( z = e \) in the original form of the present condition ii) gives us

\[
F(x, x) F(y, y) + F(x, y) F(y, x) = 0, \quad \forall x \neq y
\]

which, given \( F^2 = 1 \), implies the condition iii’) of Proposition 3.6. Hence \( \phi \) is strongly involutive.

Finally, also in this case, the equations of an alternative algebra in terms of \( F \) (see the proof of Proposition 3.3) reduce to the following. If \( x = e \) or \( y = e \), the first equation is trivial.
Otherwise, the case \( x \neq e, y \neq e, x = y \) reduces to \( \phi(x, x, z) = 1 \), which in our present case where \( G \cong (\mathbb{Z}_2)^n \) reduces to
\[
F(x, x) = F(x, x)F(x, z).
\]
This holds because the left hand side is -1 by the condition ii) of Proposition 3.6 and the right hand side is \(-F(x, z)^2 = -1\) by ii.a) and \( F^2 = 1 \). The remaining case is \( x \neq e, y \neq e \) and \( x \neq y \). In this case the altercommutativity property in Proposition 3.6 reduces the first equation for an alternative algebra to
\[
F(y, z)F(x, yz) + F(x, z)F(y, xz) = 0.
\]
Since \( F^2 = 1 \), this is equivalent to ii.b)

On the other hand, under the assumptions of Proposition 3.6, the conditions ii.a)-ii.b) are equivalent to
\[
i.c) \ F(x, y) + F(xy, y) = 0 \text{ for all } x, y \in G \text{ with } y \neq e
\]
\[
i.d) \ F(x, y)F(x, z) + F(xy, z)F(xz, y) = 0 \text{ for all } x, y, z \in G \text{ with } y \neq e, z \neq e \text{ and } y \neq z
\]
(or one can obtain them directly from our original condition ii)). We use these versions in a similar analysis for the content of the second of the conditions in Proposition 3.3 for an alternative algebra. \( \square \)

We have written the proof of the last part of the proposition in a reversible way. Hence we also conclude,

**Corollary 3.8** If \( \sigma(x) = F(x, x)x \) for all \( x \in G \) is a strong involution, and \( F^2 = 1 \), then the following are equivalent,

i) \( k_F \mathbb{G} \) is an alternative algebra,

ii) \( k_F \mathbb{G} \) is a composition algebra.

The conditions in Proposition 3.7 and the corollary are evidently highly restrictive, because if \( k \) has characteristic different from 2 it is known that we have only the following composition algebras with the Euclidean norm: \( k \), the algebra with basis 1, \( v, v^2 = -1 \), the algebra over \( k \) with the product of quaternions and the algebra over \( k \) with the product of octonions. If \( k \) is algebraically closed these algebras are isomorphic to \( k, k \oplus k, M_2(k) \) (2 \( \times \) 2 matrices) and Zorn’s algebra of vectorial matrices \( \mathbb{S} \). Equivalently, one knows that these are the only
simple alternative algebras. On the other hand, the diagonal strong involution conditions in Proposition 3.6 are definitely weaker and hold for the entire family of Cayley algebras, as we will see in the next section.

Finally, for completeness, we include a slight generalisation of Proposition 3.7.

**Proposition 3.9** Let $k_FG$ be an algebra that admits a strong diagonal involution $\sigma(x) = s(x)x$. Then the non degenerate form $n(x) = x \cdot \sigma(x)$ makes $k_FG$ a composition algebra if and only if

1. $s(xy)F(x, y)^2F(xy, xy) = s(x)s(y)F(x, x)F(y, y)$, for all $x, y \in G$.
2. $F(x, xz)F(y, yz)F(z, z)s(z) + F(x, yz)F(y, xz)F(xyz, xyz)s(xyz) = 0$, for all $x, y \in G$ with $x \neq y$.

**Proof** Let $A$ be a quasialgebra $k_FG$ that admits a strong diagonal involution $\sigma(x) = s(x)x$ and let us consider the form $n(x) = x\sigma(x)$. For all $x, y \in G$ if the form $n(x)$ admits composition we have $n(x \cdot y) = n(x)n(y)$ and then $n(x \cdot y) = F^2(x, y)F(xy, xy)s(xy) = n(x)n(y) = s(x)s(y)F(x, x)F(y, y)$. On the other hand for two elements of the algebra $k_FG$, $a = \sum_{x \in G} \alpha_x x$ and $b = \sum_{y \in G} \beta_y y$ we have $n(a) = \sum_{x \in G} \alpha_x^2 s(x)F(x, x)$ and $n(b) = \sum_{y \in G} \beta_y^2 s(y)F(y, y)$. But we know that $a \cdot b = \sum_{z \in G} z(\sum_{x \in G} \alpha_x \beta_{xz} F(x, xz))$ so $n(a \cdot b) = \sum_{x \in G} [\sum_{x \in G} \alpha_x \beta_{xz} F(x, xz)]^2 s(z)F(z, z) = \sum_{x, y, z \in G} \alpha_x \beta_{xz} F(x, xz)\alpha_y \beta_{yz} F(y, yz)F(z, z)$, and like in the proof of Proposition 3.7 the result follows after comparing the last expression with

$$n(a)n(b) = \left(\sum_{x \in G} \alpha_x^2 s(x)F(x, x)\right) \left(\sum_{y \in G} \beta_y^2 s(y)F(y, y)\right).$$

\[\Box\]

4 Cayley algebras

In this section, we show that the ‘complex number’ algebra, the quaternion algebra, the octonion algebra and the higher Cayley algebras are all $G$-graded quasialgebras the form $k_FG$ for suitable $G$ and $F$, which we construct. We recall that these algebras can be constructed inductively by the Cayley-Dickson process; we show that this process is compatible with our quasialgebra approach to nonassociative algebras.

Let $A$ be finite-dimensional (not necessarily associative) algebra with identity element 1 and a strong involution $\sigma$, i.e. an involution such that $a + \sigma(a), a \cdot \sigma(a) \in k1$ for all $a \in A$. We have
studied this condition in the context of our quasialgebras $k_F G$ in Proposition 3.5. The Cayley Dickson process says that we can obtain a new algebra $\tilde{A} = A \oplus vA$ of twice the dimension (i.e. elements are denoted $a, va$ for $a \in A$) and multiplication defined by

$$(a + vb) \cdot (c + vd) = (a \cdot c + \alpha d \cdot \sigma(b)) + v(\sigma(a) \cdot d + c \cdot b)$$

and with a new strong involution $\bar{\sigma}$

$$\bar{\sigma}(a + vb) = \sigma(a) - vb.$$ 

The symbol $v$ here is a notation device to label the second copy of $A$ in $\tilde{A}$. However, $v \cdot v = \alpha 1$ according to the stated product, so one should think of the construction as a generalisation of the idea of complexification when $\alpha = -1$. If $A = k$ and $\alpha = -1$ then $\tilde{A} = k[v]$ modulo the relation $v^2 = -1$ will be called the ‘complex number algebra’ over a general field $k$. As in the preceding section, we suppose $k$ has characteristic not 2.

We start with the cochain version of the Cayley-Dickson construction, motivated by the formulae above.

**Proposition 4.1** Let $G$ be a finite Abelian group $F$ a cochain on it (so $k_F G$ is a $G$-graded quasialgebra). For any $s: G \to k^*$ with $s(e) = 1$ we define $\tilde{G} = G \times \mathbb{Z}_2$ and on it the cochain $\tilde{F}$ and function $\bar{s}$,

$$\tilde{F}(x, y) = F(x, y), \quad \tilde{F}(x, vy) = s(x)F(x, y), \quad \tilde{F}(vx, y) = F(y, x)$$

$$\tilde{F}(vx, vy) = \alpha s(x)F(y, x), \quad \bar{s}(x) = s(x), \quad \bar{s}(vx) = -1$$

for all $x, y \in G$. Here $x \equiv (x, e)$ and $vx \equiv (x, v)$ denote elements of $\tilde{G}$, where $\mathbb{Z}_2 = \{e, \nu\}$ with product $\nu^2 = e$.

If $\sigma(x) = s(x)x$ is a strong involution, then $k_F \tilde{G}$ is the Cayley-Dickson process applied to $k_F G$.

**Proof** The only features to be checked for a cochain are that $\tilde{F}$ should be pointwise invertible (which is clear from invertibility of $s, F$) and $\tilde{F}(e, vx) = s(e)F(e, x) = 1$ and $\tilde{F}(vx, e) = F(e, x) = 1$. Hence we have a new quasi-algebra $k_F \tilde{G}$. 

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This reproduces the product $\cdot$ of the Cayley-Dickson process with respect to $\sigma(x) = s(x)x$, since that is
\[
vx \cdot y = v(y \cdot x) = vF(y, x)yx \\
vx \cdot vy = \alpha ys(x) \cdot x = \alpha s(x) F(y, x)yx \\
x \cdot vy = vs(x)x \cdot y = s(x) F(x, y)xvy
\]
in terms of the product of $\bar{G}$ on the right. Moreover, $\bar{s}$ on $G$ clearly reproduces the $\bar{\sigma}$ in the Cayley-Dickson procedure as well. $\square$

This provides a cochain approach to the Cayley-Dickson process. The last proposition also makes evident that all composition algebras with identity element, over a field of characteristic different from two, are in fact quasialgebras $k_F G$. We know that if we start by a field $k$ with characteristic different from 2 we can construct the sequence of algebras $k(\nu)$ (the Cayley-Dickson extension of $k$ with $\alpha = \nu$), $(k(\nu), \beta)$ (the Cayley-Dickson extension of $k(\nu)$ with $\alpha = \beta$) and $((k(\nu), \beta), \gamma)$ (the Cayley-Dickson extension of $(k(\nu), \beta)$ with $\alpha = \gamma$), in all cases admitting a strong diagonal involution such that the associated form admits composition. If $A$ is a composition algebra with identity, over a field of characteristic different from two, it is isomorphic to one of these 4 classes of algebras. This proves too that the last proposition of the section 3 is a complete characterization of all composition algebras $k_F G$ with identity, in terms of its cochain $F$. We also know from Proposition 3.5 that all these algebras with strong diagonal involution are altercommutative.

As in Section 3, we are particularly interested in the canonical involution defined by $s(x) = F(x, x)$ and in the Cayley-Dickson extension with $\alpha = -1$. In that case $\bar{F}$ is determined from $F$ alone. Let us call this choice the standard Cayley-Dickson process in our cochain approach.

**Proposition 4.2** If $F(x, x) = -1$ for all $x \in G$ and $x \neq e$ then the same holds for $\bar{F}$ under the standard Cayley-Dickson process. Moreover, in this case,

(i) $\bar{s}$ has the standard form on $\bar{G}$.

(ii) If $F^2 = 1$ then $\bar{F}^2 = 1$ as well.

(iii) If $k_F G$ is altercommutative then so is $k_{\bar{F}} G$. 

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Proof We have 
\[ \bar{\sigma}(x) = \bar{F}(x, x)x = F(x, x)x = \sigma(x), \quad \bar{\sigma}(vx) = \bar{F}(vx, vx)vx = -F(x, x)^2vx = -vx \]

since \( F^2(x, x) = 1 \). The first two parts are then immediate. The third part is 
\[ \bar{R}(vx, vy) = \frac{\bar{F}(vx, vy)}{F(vy, vx)} = \frac{F(x, x)}{F(y, y)}\bar{R}(x, y) \]

has the altercommutative form in Definition 3.2 when \( R \) does and when \( F(x, x) = -1 \) for \( x \neq e \). 
\[ \square \]

In particular, the standard complex, quaternion, octonion etc algebras are all of this form given by iterating the standard Cayley-Dickson process. To describe their cochains, we consider the special case where \( G = (\mathbb{Z}_2)^n \) and \( F \) is of the form 
\[ F(x, y) = (-1)^f(x, y) \]
for some \( \mathbb{Z}_2 \)-valued function \( f \) on \( G \times G \) (which is a natural supposition for the class with \( F^2 = 1 \)).

Corollary 4.3 If \( G = (\mathbb{Z}_2)^n \) and \( F = (-1)^f \) then the standard Cayley-Dickson process has \( \bar{G} = (\mathbb{Z}_2)^{n+1} \) and \( \bar{F} = (-1)^\bar{f} \). We use a vector notation \( \vec{x} = (x_1, \ldots, x_n) \in (\mathbb{Z}_2)^n \) where \( x_i \in \{0, 1\} \) (and the group \( \mathbb{Z}_2 \) is now written additively). Then 
\[ \bar{f}((\vec{x}, x_{n+1}), (\vec{y}, y_{n+1})) = f(\vec{x}, \vec{y})(1 - x_{n+1}) + f(\vec{y}, \vec{x})x_{n+1} + y_{n+1}f(\vec{x}, \vec{x}) + x_{n+1}y_{n+1}. \]

Proof From the above, we have clearly 
\[ \bar{f}(x, y) = f(x, y), \quad \bar{f}(x, vy) = f(x, y) + f(x, x), \]
\[ \bar{f}(vx, y) = f(y, x), \quad \bar{f}(vx, vy) = 1 + f(x, x) + f(y, x). \]

We then convert this to a vector notation where each copy of \( \mathbb{Z}_2 \) in \( G \) is the additive group of \( \mathbb{Z}_2 \). We then make use of the product in \( \mathbb{Z}_2 \) to express whether a term is included or not (thus \( x_{n+1}y_{n+1} \) contributes 1 iff both \( x_{n+1} = 1 \) and \( y_{n+1} = 1 \), etc.). 
\[ \square \]

Iterating this now generates the \( f \) for the quaternions, octonions etc.:
Proposition 4.4  
i) The ‘complex number’ algebra has this form with

\[ G = \mathbb{Z}_2, \quad f(x, y) = xy, \quad x, y \in \mathbb{Z}_2 \]

where we identify \( G \) as the additive group \( \mathbb{Z}_2 \) but also make use of its product.

ii) The quaternion algebra is of this form with

\[ \bar{G} = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \bar{f}(\vec{x}, \vec{y}) = x_1y_1 + (x_1 + x_2)y_2 \]

where \( \vec{x} = (x_1, x_2) \in \bar{G} \) is a vector notation.

iii) The octonion algebra is of this form with

\[ \bar{\bar{G}} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \bar{\bar{f}}(\vec{x}, \vec{y}) = \sum_{i \leq j} x_iy_j + y_1x_2x_3 + x_1y_2x_3 + x_2x_2y_3. \]

iv) The 16-onion algebra is of the form \( \bar{\bar{\bar{G}}} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and

\[ \bar{\bar{\bar{f}}}(\vec{x}, \vec{y}) = \sum_{i \leq j} x_iy_j + \sum_{i \neq j \neq k \neq i} x_ix_jy_ky_l + \sum_{i \neq j \neq k \neq l} x_iy_jy_kx_4. \]

We are now able to apply our various criteria in the last section for the structure of the algebras of the form \( k_FG \), to this construction and to all these algebras. Note in all these cases (and for the who \( 2^n \)-onion family generated in this way) \( f \) has a bilinear part defined by the bilinear form

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1 \\
\vdots & & \ddots & \\
0 & \cdots & 1 & 1 \\
0 & \cdots & 0 & 1
\end{pmatrix}.
\]

For the complex number and quaternion algebras this is the only part, which implies that \( F \) is a bicharacter and hence, in particular, its group coboundary \( \phi = 1 \), i.e. these algebras are associativity. The \( f \) for the octonions has this bilinear part, which does not change associativity, plus a cubic term which contributes to \( \phi \). The 16-onion has additional cubic an quartic terms, etc. This makes the origin of the breakdown of associativity for the higher members of the family particularly clear.

In the remainder of this section we suppose that the quasialgebra \( k_FG \) is admits a diagonal involution \( \sigma(x) = s(x)x \) in the basis \( G \) and we shall denote by \( k_F\bar{G} \) the generalized Cayley-Dickson extension with respect to this (with general \( \alpha \neq 0 \)). It is easy to see that \( s \) provides a diagonal involution on it.
Proposition 4.5 The associativity cocycle $\phi$ of $k_FG$ and the associativity cocycle $\bar{\phi}$ of $k_{\bar{F}}\bar{G}$ are related by

$$
\bar{\phi}(x, y, z) = \phi(x, y, z), \quad \bar{\phi}(vx, y, z) = R(y, z)\phi(x, y, z) \\
\bar{\phi}(vz, y, z) = R(x, y)\phi(x, y, z), \quad \bar{\phi}(vz, y, vz) = R(x, y)\phi(x, y, z) \\
\bar{\phi}(v, y, vz) = R(x, y)\phi(x, y, z), \quad \bar{\phi}(v, vy, vz) = R(y, z)R(x, y)\phi(x, y, z) \\
\bar{\phi}(x, vy, vz) = R(x, y)\phi(x, y, z), \quad \bar{\phi}(vx, vy, vz) = R(x, y)R(x, y)\phi(x, y, z)
$$

$\forall x, y, z \in G$.

Proof We use the definition of $\bar{\phi}$ as coboundary of $\bar{F}$, the form of this in Proposition 4.1 and $F(x, y) = R(x, y)F(y, x)$ from Definition 3.2. We also use Lemma 3.4 which tells us that $s^2 = 1$, $R(x, y) = s(x)s(y)/s(x, y)$ and $\phi(x, y, z)\phi(z, y, x) = 1$. $\square$

Corollary 4.6 $k_F\bar{G}$ is associative if and only if $k_FG$ is associative and commutative.

Proof We already know that $k_FG$ is commutative iff $F$ is symmetric, which means iff $R = 1$. So in this case if $\phi = 1$ then $\bar{\phi} = 1$. So if $k_FG$ is associative and commutative, $k_F\bar{G}$ is associative. Conversely, if $\bar{\phi} = 1$ then by restriction, $\phi = 1$ so $k_FG$ is associative. Moreover, $1 = \bar{\phi}(x, y, v) = R(x, y)\phi(x, y, e) = R(x, y)$ tells us that $k_FG$ is commutative. $\square$

5 New quasialgebras

It is clear that there are many examples of quasialgebras $k_FG$ according to the group $G$ and the cochain $F$. In this section we will consider examples where $F^2 = 1$; we will see that even in this case we can obtain very different types of algebras.

By Proposition 3.7 we know that if $k_FG$ is an $n$-dimensional composition algebra for the Euclidean norm, the cochain $F$ is defined by an Hadamard matrix $H$ (that is, a matrix such that $H^tH = nI$ where $t$ is transpose and $I$ is the identity matrix). The matrix entries are $H_{x,y} = F(x, y)$ for $x, y \in G$. Motivated by this, we begin by considering more general examples of quasialgebras defined by a cochain given by a normalized Hadamard matrix.

Proposition 5.1 If we consider the symmetric Hadamard matrix
\[ H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \]

as a cochain on the group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) we obtain a nonsimple, associative and commutative algebra.

**Proof**  The algebra defined by \( H \) and the group \( (\mathbb{Z}_2)^2 \) has the multiplication table

\[
\begin{array}{c|cccc}
  & e & x & y & z \\
\hline
  e & e & -z & -y & \\
x & -z & e & -x & \\
y & -y & -x & e & \\
z & & & & \\
\end{array}
\]

By straightforward calculations, one can see that this algebra is commutative and associative.

Moreover, \( I = \langle e + x, y - z \rangle \) is an ideal of this algebra. \( \Box \)

Next we consider \( q = p^r \) where \( p \) is an odd prime number. Let \( \chi(x) \) be the character defined in the finite field \( F_q \) where \( \chi(0) = 0 \), \( \chi(x) = 1 \) if \( x \) is a square, and \( \chi(x) = -1 \) if \( x \) is not a square. Consider the matrix \( Q \) defined by \( Q_{ij} = \chi(e_i - e_j) \) (where \( e_1, e_2, e_3, ... e_q \) are a natural enumeration of the elements of \( F_q \)). Suppose that \( p^r = 3 \pmod{4} \), that is \( p^r + 1 = 4n \) for some \( n \). In this case \( Q \) is skew-symmetric and 

\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
-1 & & & \\
\vdots & Q + I & \\
-1 & & & \\
\end{bmatrix}
\]

is a well-known skew symmetric Hadamard matrix with many applications (eg in the theory of Payley matrices). If we multiply each row except the first by (-1) we obtain a normalized Hadamard matrix of order \( 4n \), which we can use as a cochain. In view of Proposition 3.5 we consider for example the case where \( n \) is a power of 2, so \( p^r + 1 = 2^m \) and \( m \geq 2 \), and we let \( G = (\mathbb{Z}_2)^m \). The octonions and quaternions appear in this family for \( p = 7, 3 \) and \( r = 1 \).

**Proposition 5.2**  The algebras \( k_F \mathbb{Z}_2^m \) where \( m \geq 2 \) and

\[ H = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & & & \\
\vdots & -Q - I & \\
1 & & & \end{bmatrix} \]

is viewed as a cochain, i.e. \( F(x, x) = -1 \) \( (x \neq e = e_0) \) \( F(e_i, e_j) = -\chi(e_i - e_j) \) \( (i > 0) \) \( (j > 0) \) \( (i \neq j) \), are simple and noncommutative.
Proof  By computing $R(x, y) = F(x, y)/F(y, x)$ it is easy to see that the $k_F G$ are altercommutative (and hence noncommutative). From this, the choice $G = (\mathbb{Z}_2)^m$ and $F(x, x) = -1$ for $x \neq e$, we conclude by Proposition 3.5 that the algebras $k_F G$ admit the diagonal strong involution $\sigma(x) = F(x, x)e$, and hence by Proposition 3.6 that they are simple. □

It is known that the Hadamard matrices have only order 2 or 4. However, we can construct another class of algebras that have order 3 or $4n+1$ if we define the cochain by a matrix of the form

$$
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & & & \\
& & \ddots & H \\
1 & & & \\
\end{bmatrix}
$$

where $H$ is an Hadamard matrix. The simplest case is with $H$ an Hadamard matrix of order two:

**Proposition 5.3** Let $k_F \mathbb{Z}_3$ be a quasialgebra which cochain defined by the matrix,

$$
\begin{bmatrix}
1 & 1 & 1 \\
1 & H & \\
1 & & \\
\end{bmatrix}
$$

where $H$ is an Hadamard matrix of order 2. Then $k_F \mathbb{Z}_3$ is a nonassociative simple algebra (commutative or altercommutative).

**Proof** If $H$ is an Hadamard matrix of order 2 we have only two different possibilities for $k_F \mathbb{Z}_3$, with the multiplication table I:

$$
\begin{array}{ccc}
e & x & y \\
x & -y & e \\
y & e & x \\
\end{array}
$$

or the multiplication table II:

$$
\begin{array}{ccc}
e & x & y \\
x & y & -e \\
y & e & x \\
\end{array}
$$

Let $I \neq k_F \mathbb{Z}_3$ be an ideal of $k_F \mathbb{Z}_3$ and let $a = \sum_{x \in \mathbb{Z}_3} \alpha_x x$ an element of $I$. Then in both cases $(a \cdot x) \cdot y \in I$ and if $k_F \mathbb{Z}_3$ is defined by table I this implies that $\alpha_x = 0$ (if not, $x \in I$ and $I = k_F \mathbb{Z}_3$). And if $k_F \mathbb{Z}_3$ is defined by table II this implies that $\alpha_e = 0$ (if not $e \in I$ and $I = k_F \mathbb{Z}_3$).
\( I = k_F\mathbb{Z}_3 \). So \( I \) must be zero. On the other hand it is clear that the algebra defined by table I

is commutative and the one defined by the table II is altercommutative. \( \square \)

Following this idea, one also has

**Proposition 5.4** Let \( k_F\mathbb{Z}_5 \) be a quasialgebra which cochain \( F \) defined by the matrix

\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 \\
\vdots & H \\
1
\end{bmatrix}
\]

where \( H \) is the Hadamard matrix of order 4 with maximal excess \((H = J - 2I \text{ where } J \text{ has order 4 and has all elements equal to } 1)\). Then \( k_F\mathbb{Z}_5 \) is a simple commutative nonassociative algebra.

One may prove this directly. Related to it is the following general observation:

**Proposition 5.5** Let \( n \geq 4 \) and \( F(x, y) = (-1)^{\delta_{x,y}} \) for \( x, y \neq e \), then \( k_F\mathbb{Z}_n \) is a simple commutative, nonassociative algebra.

**Proof** It’s easy to see that these algebras are commutative and nonassociative. To prove that they are simple we recall that as \( \mathbb{Z}_n \) is a group we know that for \( x, y \in \mathbb{Z}_n \) we have \( L_x \cdot L_y = L_{xy} \) and so \( L_x \cdot L_y \cdot L_{xy} = Id \) if we denote by \( L_x \) the map \( L_x(a) = xa \). So let’s consider an ideal \( I \neq k_F\mathbb{Z}_n \) and let \( a = \sum_{x \in \mathbb{Z}_n} \alpha_x x \) be an element of \( I \). Now let’s consider an element \( y \in \mathbb{Z}_n \) that is different from \( e \) and has order different from 2, 3. Then \( y \cdot a = \sum_{x \in \mathbb{Z}_n} \alpha_x F(y, x)yx = \sum_{x \neq y \in \mathbb{Z}_n} \alpha_x yx - \alpha_y y^2 \) is an element of \( I \). Now if we multiply the last element by \( z = y^2 \) we will have that the element \( z \cdot (\sum_{x \neq y \in \mathbb{Z}_n} \alpha_x yx - \alpha_y y^2) = \sum_{x \neq y \in \mathbb{Z}_n} \alpha_x y^3 x + \alpha_y y^2 \) is an element of \( I \). Finally, we multiply the last element by \( v = y^3 \) and have that the element \( v \cdot (\sum_{x \neq y \in \mathbb{Z}_n} \alpha_x y^3 x + \alpha_y y^2) = -\alpha_e e + \sum_{x \neq e \in \mathbb{Z}_n} \alpha_x x \) is an element of \( I \). So \( \alpha_e e \in I \) and \( \alpha_e = 0 \) and \( I = 0 \) or \( \alpha_e \neq 0 \) and \( I = k_F\mathbb{Z}_n \). \( \square \)

Other natural classes of examples (to be considered elsewhere) include the natural generalisation of the octonions based on Galois sequences in [3, Chap. 2.]. For example, one may take \( G = \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( F \) of the form \( F = e^{\frac{2\pi i}{3}} \), in contrast to the \( F^2 = 1 \) case.

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6 Automorphism quasi-Hopf algebras

We return to the general setting of Section 2, where $G$ is equipped with a 3-cocycle $\phi$ forming a dual quasi-Hopf algebra.

We let $A$ be a finite-dimensional $G$-graded quasi-algebra in the sense of Definition 2.3, and introduce a general construction for its comeasuring or ‘automorphism’ dual quasi-Hopf algebra.

We let \( \{ e_i \} \) be a basis of $A$ with homogeneous degrees, denoted $|e_i| = |i| \in G$. We let $e_i \cdot e_j = \sum_k c_{ij}^k e_k$ define the structure constants of $A$ in this basis. Also, we consider the group $G \times G$ with cocycle $\phi((a, g), (b, h), (c, f)) = \phi(g, h, f) \phi(a, b, c)$.

**Proposition 6.1** Associated to a $G$-graded quasialgebra $A$ is a dual quasi-Hopf algebra $M_1(A)$ (the comeasuring dual quasi-Hopf algebra) defined as the free $G \times G$-quasialgebra generated by \( \{ 1, t_{ij} \} \) where $|t_{ij}| = (|i|, |j|)$ and $|1| = (e, e)$, modulo the additional relations

\[
\sum_a c_{ij}^a t_a^k = \sum_{a,b} c_{ab}^{ka} t_{ai}^a \cdot t_{bj}^b.
\]

We define $\Delta, \epsilon$ as

\[
\Delta t_{ij} = \sum_a t_{ai}^a \otimes t_{bj}^b, \quad \epsilon(t_{ij}) = \delta_{ij}
\]

extended multiplicatively, and extend $\phi$ to a linear functional $\phi : M_1(A)^{\otimes 3} \to k$ by

\[
\phi((t_{i_1 p_1} \cdots t_{i_{\alpha} p_{\alpha}}), (t_{j_1 q_1} \cdots t_{j_{\beta} q_{\beta}}), (t_{k_1 r_1} \cdots t_{i_{\gamma} p_{\gamma}})) = \delta_{i_1 p_1, j_1 q_1} \delta_{i_{\alpha} p_{\alpha}, j_{\beta} q_{\beta}} \delta_{k_1 r_1, i_{\gamma} p_{\gamma}} \phi(|i_1| \cdots |i_{\alpha}|, |j_1| \cdots |j_{\beta}|, |k_1| \cdots |k_{\gamma}|).
\]

**Proof** Since $G \times G$-graded spaces form a monoidal category, we define the free tensor algebra on the vector space spanned by basis $\{ t_{ij} \}$ in the usual way in a monoidal category. This means iterated tensor products in the generators, which we understand as nested to the right. The product is the tensor product composed with the appropriate associativity morphism. The $G \times G$-degree is multiplicative. In our case the result is the algebra generated by $1, t_{ij}$ and the associativity rule

\[
\left( (t_{i_1 p_1} \cdots t_{i_{\alpha} p_{\alpha}}), (t_{j_1 q_1} \cdots t_{j_{\beta} q_{\beta}}), (t_{k_1 r_1} \cdots t_{i_{\gamma} p_{\gamma}}) \right) \phi(|p_1| \cdots |p_{\alpha}|, |q_1| \cdots |q_{\beta}|, |r_1| \cdots |p_{\gamma}|) \phi(|r_1| \cdots |r_{\gamma}|, |j_1| \cdots |j_{\beta}|) \phi(|i_1| \cdots |i_{\alpha}|, |t_{ij}|) = (t_{i_1 p_1} \cdots t_{i_{\alpha} p_{\alpha}}) \cdot ((t_{j_1 q_1} \cdots t_{j_{\beta} q_{\beta}}) \cdot (t_{k_1 r_1} \cdots t_{i_{\gamma} p_{\gamma}})) \phi(|p_1| \cdots |p_{\alpha}|, |q_1| \cdots |q_{\beta}|, |r_1| \cdots |p_{\gamma}|) \phi(|r_1| \cdots |r_{\gamma}|, |j_1| \cdots |j_{\beta}|) \phi(|i_1| \cdots |i_{\alpha}|, |t_{ij}|)
\]
where the degree of \(|t^{i_1}_{p_1} \cdots t^{i_\alpha}_{p_\alpha}| = (|i_1| \cdots |i_\alpha|, |p_1| \cdots |p_\alpha|)\) does not depend on the nesting of the products in the expression.

On this free quasi-associative algebra we define \(\Delta, \epsilon\) as shown. They are extended to products as algebra maps for the non-associative product. It is easy to see that \(\Delta, \epsilon\) are compatible with the quasi-associativity, and that the extended \(\phi\) as shown makes the free quasiassociative algebra into a dual quasi-Hopf algebra \(\hat{M}_1\) in the sense of Section 2. That the extended \(\phi\) is a cocycle reduces to \(\phi\) a group cocycle. The quasi-associativity axiom for a dual quasi-Hopf algebra reduces to the \(G \times G\)-quasiassociativity.

Next, it is easy to verify that the quotient by the relations shown is consistent with the \(G\)-quasiassociativity of our algebra \(A\). In terms of its structure constants, the latter is

\[
\sum_a c_{ij}^a c_{ak}^b = \sum_a c_{jk}^a c_{ia}^b \phi(|i|, |j|, |k|).
\]

Then,

\[
(t^i_p \cdot t^j_q) \cdot t^k_r \phi(|i|, |j|, |k|) c_{jk}^a c_{ia}^b = (t^i_p \cdot t^j_q) \cdot t^k_r c_{ij}^a c_{ak}^b
\]

\[
= c_{pq}^a t^a_c \cdot t^k_r c_{ak}^b = c_{pq}^a c_{cr}^d t^b_d = c_{qr}^e c_{pe}^d \phi(|p|, |q|, |r|) = c_{qr}^e t^i_p \cdot t^a c_{ia}^b \phi(|p|, |q|, |r|)
\]

\[
= t^i_p \cdot (t^j_q \cdot t^k_r) \phi(|p|, |q|, |r|) c_{jk}^a c_{ia}^b
\]

as required. We use the summation convention for upper-lower indices.

Finally, we verify that these relations are compatible with \(\Delta\). Thus,

\[
\Delta(c_{ab}^k t^a_i \cdot t^b_j) = c_{ab}^k t^a_p \cdot t^b_q \otimes t^p_i \cdot t^q_j = c_{pq}^a t^k_b \otimes t^p_i \cdot t^q_j = c_{ij}^a t^k_b \otimes t^b_a = c_{ij}^a \Delta t^k_a
\]

as required. Compatibility with \(\epsilon\) is trivial. It is also clear that \(\phi\) restricts to the quotient in virtue of \(G\)-grading of the algebra \(A\). For example, \(\phi(t^j_qc_{ab}^k, t^r_s) = \delta_j^s \delta^r_s \phi(|i|, |p|, |q|, |r|) c_{pq}^k\) while \(\phi(t^j_qc_{ab}^k, t^r_s) = \delta_j^s \delta^r_s \phi(|i|, |k|, |r|) c_{pq}^k\); but \(c_{pq}^k = 0\) unless \(|p||q| = |r|\). Thus, the quotient of \(\hat{M}_1\) by the relations shown defines a dual quasi-Hopf algebra \(\hat{M}_1\). \(\Box\)

If \(G\) is in addition equipped with a quasi-bicharacter \(\mathcal{R}\) (making \((kG, \phi)\) into a dual quasitriangular quasi-Hopf algebra) then there is a natural braiding in the category of \(G\)-graded spaces as explained in Section 2. In our case, we extend \(\mathcal{R}\) to a quasi-bicharacter on \(G \times G\) by

\[
\mathcal{R}((a, g), (b, h)) = \frac{\mathcal{R}(g, h)}{\mathcal{R}(a, b)}
\]

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**Proposition 6.2** If \( \mathcal{R} \) is a quasi-bicharacter on \( G \), the comeasuring dual quasi-bialgebra \( M_1(A) \) has a natural quotient \( M_1(\mathcal{R}, A) \) with the additional relation of quasi-commutativity as a \( G \times G \)-quasialgebra. Then \( M_1(\mathcal{R}, A) \) is a dual-quasitriangular dual quasi-Hopf algebra with \( \mathcal{R} \) extended as a linear functional on \( M_1(\mathcal{R}, A)^{\otimes 2} \) by

\[
\mathcal{R}(t_{i_1}^{i_1} \cdots t_{i_{\alpha}}^{i_{\alpha}}, t_{j_1}^{(j_1} \cdots t_{j_{\beta}}^{j_{\beta})} = \delta_{i_1}^{i_1} \cdots \delta_{i_{\alpha}}^{i_{\alpha}} \delta_{j_1}^{j_1} \cdots \delta_{j_{\beta}}^{j_{\beta}} \mathcal{R}(|i_1| \cdots |i_{\alpha}|, |j_1| \cdots |j_{\beta}|)
\]

**Proof** Explicitly, quasicommutativity as a \( G \times G \)-graded algebra is

\[
(t_{j_1}^{j_1} \cdots t_{j_{\beta}}^{j_{\beta}}) \cdot (t_{i_1}^{i_1} \cdots t_{i_{\alpha}}^{i_{\alpha}}) \mathcal{R}(|p_1| \cdots |p_{\alpha}|, |q_1| \cdots |q_{\beta}|)
\]

\[
= \mathcal{R}(|i_1| \cdots |i_{\alpha}|, |j_1| \cdots |j_{\beta}|)(t_{i_1}^{i_1} \cdots t_{i_{\alpha}}^{i_{\alpha}}) \cdot (t_{j_1}^{j_1} \cdots t_{j_{\beta}}^{j_{\beta}})
\]

for the braiding \( \Psi \) determined by \( \mathcal{R} \) on \( G \times G \) as shown. This is the quasicommutativity property of a dual-quasitriangular dual quasi-Hopf algebra in the sense of Section 2, with \( \mathcal{R} \) defined as stated. That this \( \mathcal{R} \) is well-defined on the free quasi-associative algebra \( \tilde{M}_1 \) is clear. That it descends to the quotient by the relations of \( M_1 \) follows by the \( G \)-grading of our algebra \( A \) as for \( \phi \) in the preceding proof. That it well-defined on \( M_1(\mathcal{R}, A) \) itself requires repeated use of the quasi-bicharacter property and is omitted \( \square \)

Moreover, both \( M_1(A) \) and hence \( M_1(\mathcal{R}, A) \) coact on \( A 

**Proposition 6.3** \( M_1 \) coacts on \( A \) by \( \beta : e_i \mapsto e_a \otimes t_{a i} \) and \( \beta \) is an algebra map.

**Proof** The definition of the coaction is consistent with the relations of \( A 

\[
\beta(e_i \cdot e_j) = \beta(e_i) \beta(e_j) = e_a \cdot e_b \otimes t_{a i} \cdot t_{b j} = c_{ab}^k e_k \otimes t_{a i} \cdot t_{b j} = c_{ij}^a e_k \otimes t_{k a} = \beta(c_{ij}^a e_a)
\]

in virtue of the relations of \( M_1 \). \( \square \)

In fact, it should be clear from the proof that the relations of \( M \) are the minimum relations such that a coaction of this form extends as an algebra map. In the case where \( \phi \) is trivial we recover in fact the dual (arrows-reversed) version of the measuring bialgebra \( M(A, A) \) in [10], and this is the reason motivation behind our construction. Some further recent applications of this comeasuring bialgebra construction in the associative (not quasi-associative) case appear in [11].

Before turning to examples, we note that \( M_1(A) \) and \( M_1(\mathcal{R}, A) \) have natural further quotients. Thus
Proposition 6.4 The diagonal quotient $M_D$ of $M_1$ by the relations $t^i_j = 0$ when $i \neq j$ is an associative Hopf algebra with generators $t_i$ and relations

$$c_{ij}^k (t_k - t_i t_j) = 0, \quad \Delta(t_i) = t_i \otimes t_i, \quad \epsilon(t_i) = 1.$$ 

It can also be viewed as a dual quasi-Hopf algebra with

$$\phi(t_{i_1} \cdots t_{i_\alpha}, t_{j_1} \cdots t_{j_\beta}, t_{k_1} \cdots t_{k_\gamma}) = \phi(|i_1| \cdots |i_\alpha|, |j_1| \cdots |j_\beta|, |k_1| \cdots |k_\gamma|).$$

The diagonal quotient $M_D(R, A)$ of $M_1(A)$ is the commutative quotient of $M_D$, and can be viewed as a dual quasitriangular dual quasi-Hopf algebra with

$$\mathcal{R}(t_{i_1} \cdots t_{i_\alpha}, t_{j_1} \cdots t_{j_\beta}) = \mathcal{R}(|i_1| \cdots |i_\alpha|, |j_1| \cdots |j_\beta|).$$

Proof This is elementary. The quasi-associativity and quasicommutativity of $M_1(A)$, $M_1(R, A)$ clearly reduce in the diagonal case to usual associativity and commutativity. The coproduct becomes group-like. $\square$

Finally, we have not required yet that $A$ is unital. When it is, we choose our basis so that $e_0 = 1$. In this case we let $\{e_i\}$ denote the remaining basis elements.

Proposition 6.5 When $e_0 = 1$ the unit of $A$, we define $M_0(A)$ as the quotient of $M_1(A)$ by $t^0_0 = 1$ and $t^i_0 = t^0_i = 0$. This forms a dual quasi-Hopf algebra with relations and coproduct

$$c_{ij}^a t^k_a = c_{ab}^k t^a_i t^b_j, \quad c_{ij}^0 c = c_{ab}^0 t^a_i t^b_j, \quad \Delta t^i_j = t^i_a \otimes t^a_j.$$ 

Similarly, $M_0(R, A)$ remains dual quasitriangular and preserves this form.

Proof We set $t^i_0 = 0 = t^0_i$ and denote $t^0_0 = c$. Note that $c_{00}^j = c_{0j}^0 = \delta^j_i$ and $c_{00}^0 = 1$. Therefore, the relations of $M_1(A)$ become (for all labels not 0), the relations as stated, and the additional relations

$$t^i_j c = t^i_j c = c t^i_j, \quad c^2 = c.$$ 

In view of these latter relations, it is natural to set $c = 1$. Moreover, the matrix coproduct and counit are clearly compatible with the quotient. One has $\Delta c = c \otimes c$, so this is consistent with $c = 1$ as well. Moreover, since $|e_0| = e$, the identity in $G$, it is clear that setting $c = 1$ is
consistent with the definition of $\phi, R$ on $M_1$. Likewise, their delta-function form is consistent
with $t_i^0 = 0 = t^0_i$. Hence $M_0(A)$ and $M_0(R, A)$ inherit these structures and are dual quasi-Hopf
algebras.

Finally, the coaction of $M_1$ becomes $\beta(e_0) = e_0 \otimes c$ and $\beta(e_i) = e_a \otimes t^a_i$. The relations of
$M_0$ are such that the bilinear form on the span of $\{e_i\}$ defined by $c_{ij}^0$ is preserved.

Given our basis, we can identify $A/1$ with the span of $\{e_i\}$ for $i \neq 0$, and $B(e_i, e_j) = c_{ij}^0$ is
a natural bilinear form on it. We see that our reduced comeasuring dual quasi-quantum groups
$M_0(A), M_0(R, A)$ preserve this. Also, the two relations for the $t^i_j$ imply that

$$c_{ab}^d c_{dc}^0 (t^a_i \cdot t^b_j) \cdot t^c_k = c_{dc}^0 c_{ij}^a t^d_a \cdot t^c_k = c_{ij}^a c_{ak}^0$$

so that the trilinear form $c_{ij}^a c_{ak}^0$ is also preserved in a certain sense. Finally, we have the
further diagonal quotients of $M_0(A)$ and $M_0(R, A)$.

Corollary 6.6 For $F$ a cochain on $G$, and basis $G$ of $A = k_F G$, the diagonal quotient $M_{D0} =
(kG, \phi)$ as a dual quasi-Hopf algebra.

Proof In this basis $c_{ij}^k = 1$ iff $ij = k$ in $G$ and zero otherwise. Hence the relations of $M_D$
are $t_i t_j = t_k$ for $k = ij$ and empty for $k \neq ij$. In $M_{D0}$ we further identify $t_e = 1$ as in the
group algebra. Finally, $|t_i| = i$ and we obtain $kG, \phi$ as a dual quasi-Hopf algebra. When $G$ is
commutative we obtain a commutative algebra and $M_{D0} = M_{D0}(R, k_F G)$.

We now compute these constructions for the complex numbers and for the quaternions, as
real two and four dimensional algebras. More generally, we work over a general ground field of
characteristic not 2.

Example 6.7 When $A = k[i]$ (where $i^2 = -1$), the comeasuring bialgebra $M_1(k[i])$ is generated
by 1 and a matrix of generators $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with the relations

$$a^2 - c^2 = a = d^2 - b^2, \quad ac + ca = c, \quad -c = bd + db, \quad ab - cd = b = ba - dc, \quad ad + cb = d = bc + da.$$  

This has a natural bialgebra quotient of the form $\begin{pmatrix} c \\ s \\ -s \\ c \end{pmatrix}$ with

$$\Delta c = c \otimes c - s \otimes s, \quad \Delta s = s \otimes c + c \otimes s, \quad c^2 - s^2 = c, \quad sc + cs = s.$$  

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The quotient \( M_0 = M_0(\mathbb{R}) = M_{D0} \) is \( k\mathbb{Z}_2 \) (as generated by \( d \)), and its coaction is \( \beta(1) = 1 \otimes 1, \beta(i) = i \otimes d \).

**Proof** We write out the 8 relations for \( M_0 \) using the structure constants of \( k[i] \). The quotient \( M_0 \) is already diagonal and commutative. Hence by the preceding corollary, it gives \( kG = k\mathbb{Z}_2 \). Its coaction is on \( k[i] \) is the canonical nontrivial one corresponding to the \( G \)-grading. Note that evaluating with the nontrivial character of \( \mathbb{Z}_2 \) gives the canonical automorphism \( i \rightarrow -i \). \( \square \)

The intermediate quotient here is the ‘trigonometric bialgebra’: the coproduct has the same form as the addition rules for the sine and cosine functions. Whereas it is usually considered as a coalgebra\([10]\), we obtain here a natural algebra structure forming a bialgebra. It too coacts on \( k[i] \) by our constructions as the push out of the universal coaction of \( M_1 \).

We also note that when \( A = k\mathbb{Z}_2 \), the comeasuring bialgebra \( M_1(k\mathbb{Z}_2) \) has the same form as in the preceding example but with all minus signs replaces by +. The quotient of the form \( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \) can then be diagonalised as \( g^\pm = a \pm b \) and becomes the bialgebra

\[
g^\pm g^\pm = g^\pm, \quad \Delta g^\pm = g^\pm \otimes g^\pm, \quad \epsilon g^\pm = 1
\]

of two mutually noncommuting projectors \( g^\pm \). This is an infinite-dimensional algebra with every element of the form either \( g^+g^-g^+ \cdots \) or \( g^-g^+g^- \cdots \) (alternating). One may make a similar diagonalisation \( g^\pm = c \pm is \) for the trigonometric bialgebra in the case when \( i = \sqrt{-1} \in k \).

**Proposition 6.8** When \( A = \mathbb{H} \) the quaternion algebra over \( k \), the comeasuring bialgebra \( M_0(\mathbb{H}) \) has generators \( 1 \) and three vectors of generators \( \vec{t}_j = (t^i_{\ j}), \ i = 1, 2, 3 \) and relations

\[
\vec{t}_1 \times \vec{t}_2 = \vec{t}_3, \quad \vec{t}_1 \times \vec{t}_1 = 0, \quad +\text{cyclic}, \quad \vec{t}_i \cdot \vec{t}_j = \delta_{ij}.
\]

Here \( \times \) is the vector cross product and \( \cdot \) is the vector dot product. The quotient \( M_0(\mathbb{R}) \) is defined by the additional relation that the generators commute.

**Proof** We choose the standard basis (where \( e_0 = 1 \) and \( e_i, \ i = 1, 2, 3 \) have the relations \( ee_2 = e_3 \) and \( e^2 = -1 \) and their cyclic permutations). In this case the structure constants are \( c_{ij}^k = \epsilon_{ijk}, \) the totally antisymmetric tensor with \( \epsilon_{123} = 1, \) and \( c_{ij}^0 = -\delta_{ij}, \) the standard Euclidean metric. The relations of \( M_0 \) then become

\[
\delta_{ab} t^a_{\ i} t^b_{\ j} = \delta_{ij}, \quad t^a_{\ i} t^b_{\ j} \epsilon_{abc} = t^c_{\ a} \epsilon_{ij}^a
\]
plus cyclic permutations. For $M_0(\mathcal{R})$ it is enough to note that $\mathcal{R}(|i|, |j|) = -1$ hence that the $t^i_j$ mutually commute. In this case we have $\det(m) = 1$, i.e. $M_0(\mathcal{R})$ is a quotient of $k[SL_3]$. □

The relations in $M_0$ here are asymmetric, a reflection of their role as coacting from the right on $\mathbb{H}$. If we consider also the left-handed versions of our constructions, we have a joint quotient where

$$t^i_at^j_b\delta^{ab} = \delta^{ij}$$

is added. In this case we have a Hopf algebra with $St^i_j = t^j_i$ and the corresponding quotient of $M_0(\mathcal{R})$ is a quotient of the group coordinate ring $k[SO_3]$. Since $M_0(\mathcal{R})$ is universal among commutative bialgebras coacting on $\mathbb{H}$, it follows that, it must project onto the group coordinate ring of the classical automorphism group.

The corresponding computation for the octonions yields for $M_0$ a dual quasi-Hopf algebra with nontrivial $\phi$. Its detailed form is somewhat more complex than the quaternion case, however; on general grounds we know that it projects for example on to the group coordinate ring $k[G_2]$ (the classical automorphism Hopf algebra of the octonions).

### 7 Quasiassociative linear algebra

In this section we use our categorical approach to octonions to provide the natural ‘quasiassociative’ setting for the basic linear algebra associated to them. We define the natural notion of ‘representation’. We also provide the definition of $V^*$ for any finite-dimensional $G$-graded vector space, and the associated endomorphism quasialgebra $V \otimes V^*$. These constructions are the specialization to the $G$-graded quasi-algebra setting of standard constructions for braided categories.

Thus, the notion of representations, indeed of all linear algebra and quantum group constructions, make sense in any braided category, see [12]. One writes all constructions as compositions of morphisms, inserting the associator $\Phi$ as necessary. For example, in the case of the category of $(kG, \phi)$-comodules, we clearly have:

**Definition 7.1** A representation or ‘action’ of a $G$-graded quasialgebra $A$ is a $G$-graded vector space $V$ and a degree-preserving map $\triangleright : A \otimes V \rightarrow V$ such that

$$(ab)\triangleright v = \phi(|a|, |b|, |v|)a\triangleright (b\triangleright v), \quad 1\triangleright v = v$$
on elements of homogeneous degree. Here $|a\triangleright v| = |a||v|$.

This is the obvious polarization of the quasi-associativity of the product of $A$. Clearly, a quasialgebra acts on itself by the product map (the regular representation).

Next, we recall that an object $V$ in a braided category is called ‘rigid’ if there is an object $V^*$ and morphisms

$$
ev : V^* \otimes V \to 1, \quad \text{coev} : 1 \to V \otimes V^*
$$

such that

$$(\text{id} \otimes \text{ev})\Phi_{V,V^*,V}(\pi \otimes \text{id}) = \text{id}, \quad (\text{ev} \otimes \text{id})\Phi_{V^*,V,V^*}^{-1}(\text{id} \otimes \pi) = \text{id}$$

holds. In the case of the comodule category of a dual quasiHopf algebra, these maps exist whenever $V$ is finite-dimensional, see [2] for the explicit formulae, cf [1]. For the dual quasiHopf algebra $kG, \phi$, i.e. for the category of $G$-graded vector spaces, these maps are given by

$$
ev(f^i \otimes e_j) = \delta^j_i, \quad \text{coev}(1) = \sum_i e_i \otimes f^i \phi^{-1}(|i|, |i|^{-1}, |i|)
$$

in terms of a basis of $V$ with degree $|e_i| = |i| \in G$ and its usual dual, i.e. $\text{ev}$ can be taken as the usual evaluation, $V^*$ as the usual dual, but $\text{coev}$ is modified by the group 3-cocycle. Here $|f^i| = |i|^{-1}$ so that $\text{ev}, \text{coev}$ are degree preserving.

**Proposition 7.2** If $V$ is rigid then $\text{End}(V) = V \otimes V^*$ becomes a $G$-graded quasialgebra. The product map is

$$(v \otimes f)(w \otimes h) = v \otimes h < f, w > \frac{\phi(|v|, |f|, |w||h|)}{\phi(|f|, |w|, |h|)}.$$

Moreover, finite-dimensional representations $V$ of a quasialgebra $A$ are in 1-1 correspondence with quasialgebra maps $A \to \text{End}(V)$.

**Proof** This is proven by commuting diagrams exactly as one would prove these statements in linear algebra, only inserting the associator $\Phi$ wherever needed to change bracketing. Thus (in any monoidal category) one finds

$$(\text{id} \otimes (\text{ev} \otimes \text{id})) \circ (\text{id} \otimes \Phi_{V^*,V,V}^{-1}) \circ \Phi_{V;V^*,V \otimes V^*} : (V \otimes V^*) \otimes (V \otimes V^*) \to V \otimes V^*$$

as the natural product on $\text{End}(V) = V \otimes V^*$. Its action on $V$ is the map

$$(\text{id} \otimes \text{ev}) \circ \Phi_{V,V^*,V} : (V \otimes V^*) \otimes V \to V.$$
If $A$ acts on $V$ then

$$\rho = (\triangleright \otimes \text{id}) \circ \Phi_{A,V,V*} \circ (\text{id} \otimes \text{coev}) : A \to V \otimes V^*$$

is an algebra map. Conversely, given $\rho$,

$$(\text{id} \otimes \text{ev}) \circ \Phi_{V,V*,V} \circ (\rho \otimes \text{id}) : A \otimes V \to V$$

is an action on $V$. We then specialize to the case of $G$-graded quasialgebras using the form of $\Phi$ in terms of the 3-cocycle $\phi$. □

In lieu of all the commutative diagrams here, we will prove this more explicitly in a concrete form for our particular setting. First, we identify $V \otimes V^*$ with matrices in the usual way relative to our basis, i.e.

$$\alpha = \sum \alpha^i_j E_i^j, \quad E_i^j = e_i \otimes f^j$$

as a definition of the components of $\alpha \in V \otimes V^*$. Then the preceding proposition translates into the following proposition. We write $n = \dim(V)$ and $|i| \in G$ as the further data provided by $V$, which we also use.

**Proposition 7.3** Let $|i| \in G$ for $i = 1, \cdots, n$ be a choice of grading function. Then the usual $n \times n$ matrices $M_n$ with the new product

$$(\alpha \cdot \beta)^i_j = \sum_k \alpha^i_k \beta^k_j \frac{\phi(|i|, |k|^{-1}, |k||j|^{-1})}{\phi(|k|^{-1}, |k|, |j|^{-1})}, \quad \forall \alpha, \beta \in M_n$$

form a $G$-graded quasialgebra $M_{n,\phi}$, where $|E_i^j| = |i||j|^{-1} \in G$ is the degree of the usual basis element of $M_n$. An action of a $G$-graded quasialgebra in the $n$-dimensional vector space with grading $|i|$ is equivalent to an algebra map $\rho : A \to M_{n,\phi}$.

**Proof** The product suggested by the preceding proposition is

$$E_i^j \cdot E_k^l = \delta^j_l E_i^k \frac{\phi(|i|, |j|^{-1}, |j||l|^{-1})}{\phi(|j|^{-1}, |j|, |l|^{-1})},$$

which yields the formula shown for $\alpha = \sum \alpha^i_j E_i^j$, etc. This product is quasiassociative since

$$(E_i^j \cdot E_k^l) \cdot E_m^n = \delta^i_m \delta^j_l E_i^j \frac{\phi(|i|, |m|^{-1}, |m||n|^{-1})\phi(|j|, |k|^{-1}, |k||l|^{-1})}{\phi(|m||n|^{-1}, |m||n|^{-1})\phi(|k|^{-1}, |k|, |l|^{-1})}$$

$$E_i^j \cdot (E_k^l \cdot E_m^n) \phi(|E_i^j|, |E_k^l|, |E_m^n|)$$

$$= \delta^j_m \delta^i_k \frac{\phi(|i||j|^{-1}, |k||l|^{-1}, |m||n|^{-1})\phi(|k|, |m|^{-1}, |m||n|^{-1})\phi(|i|, |k|^{-1}, |k||n|^{-1})}{\phi(|m||n|^{-1}, |m||n|^{-1})\phi(|k|^{-1}, |k|, |n|^{-1})}$$
which are equal since
\[
\phi([i][k]^{-1}, |k||m|^{-1}, |m||n|^{-1}) = \frac{\phi([k]^{-1}, |k|, |n|^{-1})\phi([i], |k|^{-1}, |k||m|^{-1})\phi([i], |m|^{-1}, |m||n|^{-1})}{\phi([k]^{-1}, |k|, |m|^{-1})\phi([k], |m|^{-1}, |m||n|^{-1})\phi([i], |k|^{-1}, |k||n|^{-1})}
\]
by repeated use of the cocycle property of \(\phi\).

Note also that the grading function \(|i|\) is equivalent to specifying a \(G\)-graded vector space \(V = \{e_i\} \) with grading \(|e_i| = |i|\). An action of a \(G\)-graded quasialgebra \(G\) is equivalent to structure constants \(v_{\alpha j}^i\) such that
\[
c_{\alpha \beta \gamma} v_{\gamma j}^i = v_{\beta j}^k v_{\alpha k}^j \phi(|\alpha|, |\beta|, |i|), \quad v_0^j = \delta_i^j
\]
where \(\{x_\alpha\}\) (say) is a basis of \(A\) with \(x_0 = 1\) and \(x_\alpha e_i = v_{\alpha j}^i e_j\). The corresponding map \(A \to M_{n,\phi}\) is
\[
\rho(x_\alpha)^{i}{}_{j} = \frac{v_{\alpha j}^i}{\phi([j],[j]^{-1},|j|)\phi([i]|j|^{-1},|j|,|j|^{-1})}.
\]
That \(\rho\) is an algebra map is (from the definitions stated) is
\[
c_{\alpha \beta \gamma} v_{\gamma j}^i \frac{1}{\phi([j],[j]^{-1},|j|)|i|^{-1},|j|,|j|^{-1})} = v_{\alpha a}^i v_{\beta j}^a \frac{\phi(|a|^{-1},|a|,|j|^{-1})\phi(|a||a|^{-1},|a|,|a|^{-1})\phi([i]|a|^{-1},|a|,|a|^{-1})\phi([j],[j]^{-1},|j|)\phi([a]|j|^{-1},|j|,|j|^{-1})}{\phi([i]|a|^{-1},|a|,|j|^{-1})\phi([i]|a|^{-1},|a|,|a|^{-1})\phi([j],[j]^{-1},|j|)\phi([a]|j|^{-1},|j|,|j|^{-1})}.
\]
Since the structure maps are degree preserving, we know that \(|a| = |\beta||j|, \ |i| = |\alpha||a|\) for nonzero terms on the right and side. That \(\rho\) is an algebra map is then equivalent to \(v_{\alpha j}^i\) an action in view of the identity
\[
\phi([\alpha]|\beta|,|j|,|j|^{-1})\phi([\alpha]|\beta||j|,|j|^{-1}|\beta|^{-1},|\beta|) = \phi(|\beta|,|j|,|j|^{-1})\phi(|\beta||j|,|\beta|^{-1}|j|^{-1},|\beta||j|\phi([\alpha]|\beta||j|,|\beta|^{-1}|j|^{-1})\phi([\beta]|j|^{-1},|\beta||j|,|j|^{-1})\phi([\alpha]|\beta|,|j|)
\]
which holds by repeated use of the 3-cocycle property. □

For example, the left regular representation of a quasialgebra on itself provides representation \(\rho : A \to M_{n,\phi}\) where \(n = \dim(A)\). For the octonions, for example, we have a representation in \(8 \times 8\) quasimatrizes.

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