AdS–dS Stationary Rotating Black Hole Exact Solution within Einstein–Nonlinear Electrodynamics

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Abstract

In this report the exact rotating charged black hole solution to the Einstein–nonlinear electrodynamics theory with a cosmological constant is presented. This black hole is equipped with mass, rotation parameter, electric and magnetic charges, cosmological constant $\Lambda$, and three parameters due to the nonlinear electrodynamics: $\beta$ is associated to the potential vectors $A_\mu$ and $^*P_\mu$, and two constants, $F_0$ and $G_0$, due to the presence of the invariants $F$ and $G$ in the Lagrangian $L(F(x^\mu), G(x^\mu))$. This solution is of Petrov type D, characterized by the Weyl tensor eigenvalue $\Psi_2$, the traceless Ricci tensor eigenvalue $S = 2\Phi_{(11)}$, and the scalar curvature $R$; it allows for event horizons, exhibits a ring singularity and fulfills the energy conditions. Its Maxwell limit is the de Sitter-Anti–de Sitter–Kerr–Newman black hole solution.

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I. INTRODUCTION

Recently the first solution for a spinning charged black hole within Einstein–nonlinear electrodynamics theory was reported [1]. Now, its generalization with cosmological con-
stant of both signs is under consideration. These solutions are the first exact stationary axisymmetric solutions of the Einstein gravity coupled to nonlinear electrodynamics with a Lagrangian function $L(x^\mu)$ depending on the both invariants $F$ and $G$ existing in any electrodynamics, $L(F, G)$. The Kerr [2]–Newman [3] solution is singled out as the unique stationary axisymmetric black hole solution of Petrov type D [4] in Einstein–Maxwell theory with $L(F)$. The Kerr and Kerr–Newman black hole solutions are quite relevant theoretically and astrophysically. According to the experiments to detect gravitational waves, the collision of two massive black holes caused the emission of the gravitational waves registered by the LIGO experiment [5] in 2015. It is believed that nonlinear electrodynamics ought to play a relevant role in the astrophysics of strongly magnetized objects containing plasma as their constituent and in the behavior of magnestars. Therefore, these new black hole solutions supporting nonlinear electromagnetic fields could open new perspectives on the physics of rotating celestial bodies. From the theoretical point of view these Petrov type D solutions can be studied from various angles, but, what is important to us, is that they may point the way to follow in the search of stationary rotating regular black hole solutions. Having in mind the search of regular solutions (gravitational structure free of singularities) we adopted the nonlinear electrodynamics as source to the Einstein equations. Moreover, if one looks for regular rotating charged black hole solutions one has to proceed further to general Petrov type I metrics.

Two decades ago we reported the first exact regular spherical symmetric back hole solution [6], in the framework of Einstein–NLE equations opening a fruitful and active area of research in the search for exact solutions. Previous to this work, there was known a model—the Bardeen model [7]—whose metric determines an Einstein tensor, which, via the Einstein equations “defines” a matter tensor $T_{ab}$ fulfilling the energy conditions, and therefore, one could associate to it a viable energy matter-field tensor, although in the Bardeen publication, no mention to a possible matter content was mentioned. Only later [8], we succeeded to derive a nonlinear magnetic electrodynamics source for the Bardeen model, since then it acquires the status of exact solution in the NLE frame.

Recently, for static spherical symmetric metrics the general exact solution coupled to NLE [9], with an arbitrary structural metric function, which, via a pair of independent Einstein equations, allows one to derive the single associated field tensor component $\mathcal{E}$, and the Lagrangian–Hamiltonian field function $\mathcal{L}–\mathcal{H}$, which determine the entire solution, should
it be singular or regular.

There exist a geometrical approach to construct “solutions” of the Einstein equations as pointed out in Stephani et al. \cite{10}, pag 20: “Any metric whatsoever is a “solution” of (1.1)–Einstein equations–if no restriction is imposed on the energy–momentum tensor, since (1.1) the becomes just a definition of $T_{ab}$; so we must first make some assumption about the structure of $T_{ab}$...For “exact solution” these authors do not give any precise definition. Referring to exact solutions we adopt the criteria of Hawking an Ellis \cite{11}, pag 117: “We shall mean by an exact solution of the Einstein’s equations, a space–time $(\mathcal{M}, g)$ in which the field equations are satisfied with $T_{ab}$ the energy momentum tensor of some specified form of matter which obeys postulate (a) (‘local causality’) of chapter 3 and some of the energy conditions of §4.3...” In this respect we consider as solutions those fulfilling the HE criteria of exact solution, and reserve the name of models for those results derived by the “metric–defined matter tensor” procedure, even when these models fulfil reasonable (weak–strong) energy conditions. Therefore, in this sense, all the reported until now Kerr–like, see \cite{12} and the references therein, rotating black holes are models, they would become solutions, if someone should be able to determine the corresponding, if any, matter-field energy–momentum tensor $T_{ab}$. In the above paragraph we are using quotation marks although our transcriptions are partial with minor modifications.

This exact solution describes a AdS-dS stationary rotating black hole endowed with several parameters; it fulfils a set of four generalized “Maxwell equations” for the electrodynamics fields $F_{\mu\nu}$ and $P_{\mu\nu}$ and two independent Einstein–NLE equations related with the two independent eigenvalues of the NLE energy–momentum tensor. The NLE is determined by a Lagrangian function $L(x^\alpha)$ constructed on the basis of the two electromagnetic invariants $F(x^\alpha)$ and $G(x^\alpha)$, $L(F,G)$, depending consequently on the coordinates $(x^\alpha)$.

II. REVIEW ON NONLINEAR ELECTRODYNAMICS

To avoid misinterpretations due to the use of different definitions and sign conventions by different research groups, we give a self–contained resume of the NLE we are dealing with. Most of this introductory material is borrowed from the first paragraphs of \cite{1}. It is well known the standard definition of the energy momentum tensor in general relativity:
following Stephani [13], §9.4, from the variational principle upon the action

\[ W = \int \sqrt{-g} \, d^4x \left( R/2 + \kappa L_M \right), \]  

(1)

taking into account \( \delta \sqrt{-g} = \delta g^{\mu\nu} \frac{\partial}{\partial g^{\mu\nu}} \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \), the variation yields

\[ \delta W = -\frac{1}{2} \int \left( R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} - \kappa T^{\mu\nu} \right) \sqrt{-g} \delta g_{\mu\nu} \, d^4x. \]  

(2)

Hence from the vanishing of this variation one arrives at the Einstein equations in the presence of a cosmological constant, coupled to matter and fields described by an energy–momentum tensor \( T^{\mu\nu} \)

\[ R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} - \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}, \quad T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_M)}{\delta g_{\mu\nu}}, \]  

(3)

where \( L_M \) stands for the matter–field Lagrangian \( L_M \); the cosmological constant arises from the contribution in the action of the \( \Lambda \)–term, \( L_\Lambda = -\Lambda/\kappa \), \( \kappa T^{\mu\nu} = -\Lambda g^{\mu\nu} \).

The electrodynamics is described by an antisymmetric electromagnetic field tensor \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \), together with its dual field tensor

\[ \ast F_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} F^{\mu\nu}, \quad \ast F^{\alpha\beta} = \frac{1}{2} \eta^{\alpha\beta\mu\nu} F_{\mu\nu}, \]  

(4)

where the numerical \( \epsilon \)–tensor is associated to the 4–Kronecker tensor.

These tensors, being antisymmetric, allow for two invariants \( F \) and \( G \) (pseudo–scalar):

\[ F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad G = \frac{1}{4} \ast F_{\mu\nu} \ast F^{\mu\nu}, \quad \ast F_{\mu\nu} F^{\nu\sigma} = G \delta^{\nu}_{\mu}. \]  

(5)

In nonlinear electrodynamics the Lagrangian function \( L \) depends on these electromagnetic invariants \( F \) and \( G \), \( L = L(F, G) \). To construct the energy–momentum tensor \( T^{\mu\nu} \) one accomplishes the variation of \( L_M = -L(F, G) \) with respect to \( g_{\mu\nu} \) of the Lagrangian function, which yields

\[ T^{\mu\nu} = -L g^{\mu\nu} + L_F F^{\mu\sigma} F^\nu_\sigma + L_G F^{\mu\sigma} \ast F^\nu_\sigma \]  

\[ =: -L g^{\mu\nu} + F^{\mu\sigma} P^\nu_\sigma, \]  

(6)

where \( P_{\mu\nu} \) is a new electromagnetic field, which always can be introduced in nonlinear electrodynamics, namely

\[ P_{\mu\nu} = 2 \frac{\partial L}{\partial F^{\mu\nu}} = L_F F_{\mu\nu} + L_G \ast F_{\mu\nu}, \]  

(7)
which one identifies as the $P_{\mu\nu}$ field tensor; see $p^{kl}$–field tensor of Born–Infeld [14]. Plebański [15], see [17] too.

From the antisymmetric tensor field tensor $P_{\mu\nu}$ one constructs its dual field tensor $^*P_{\mu\nu}$ and the invariants\[ ^*P_{\mu\nu} := \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\alpha\beta}P^{\alpha\beta}, \quad ^*P^{\alpha\beta} = -\frac{1}{2}\sqrt{-g}\epsilon^{\alpha\beta\mu\nu}P_{\mu\nu}, \]

\[ P = \frac{1}{4}P_{\mu\nu}P^{\mu\nu}, \quad Q = \frac{1}{4}^*P_{\mu\nu}^*P^{\mu\nu}. \tag{8} \]

The energy–momentum tensor (6) allows for a similar writing in term of the Hamiltonian function $H(P, Q)$, $^*F_{\mu\nu}$, and $^*P_{\mu\nu}$ fields. Replacing in the definition of the energy–momentum tensor (6) the relation

\[ ^*P_{\mu\sigma}^*F^{\sigma\nu} = \frac{3}{4}\delta^{\nu}_{\mu}[\delta^{\lambda}_{\alpha}\delta^{\rho}_{\beta}]P^{\alpha\beta}F_{\lambda\rho} \]

\[ = \frac{1}{4}(4F_{\lambda\mu}P^{\lambda\nu} - 2\delta^{\nu}_{\mu}P_{\alpha\beta}F^{\alpha\beta}) = F_{\lambda\mu}P^{\lambda\nu} - \frac{1}{2}\delta^{\nu}_{\mu}P_{\alpha\beta}F^{\alpha\beta}, \tag{9} \]

one arrives at

\[ T_{\mu\nu} = -\left(L - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\right)g_{\mu\nu} - ^*P_{\mu\alpha}^*F^{\alpha}_{\nu} =: H - ^*P_{\mu\alpha}^*F^{\alpha}_{\nu}. \tag{10} \]

Thus one arrives at the “Hamiltonian function” $H(P, Q)$, in the terminology of Born–Infeld–Plebański, associated with the Lagrangian function $L(F, G)$ via

\[ L(F, G) = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - H(P, Q), \tag{11} \]

which is known as a Legandre transformation, with:

\[ P_{\mu\nu} = 2\frac{\partial L}{\partial F_{\mu\nu}} = L_F F_{\mu\nu} + L_G ^*F_{\mu\nu}, \]

\[ F_{\mu\nu} = 2\frac{\partial H}{\partial P_{\mu\nu}} = H_P P_{\mu\nu} + H_Q ^*P_{\mu\nu}. \tag{12} \]

The Legandre transformation (11) is a consequence of the relation (9) to determine the energy–momentum tensor (10) in term of the Hamiltonian function $H(P, Q)$.

The electrodynamics is determined through the “Faraday–Maxwell” electromagnetic field equations, which in vacuum are

\[ ^*F^{\mu\nu}_{\nu} = 0 \rightarrow (\sqrt{-g}^*F^{\mu\nu})_{\nu} = 0, \tag{13} \]

\[ P^{\mu\nu}_{\nu} = 0 \rightarrow [\sqrt{-g}L_F F^{\mu\nu} + \sqrt{-g}L_G ^*F^{\mu\nu}]_{\nu} = 0, \tag{14} \]
which can be written by means of a closed 2–form \( d\omega = 0 \),

\[
\omega = \frac{1}{2} (F_{\mu\nu} + *P_{\mu\nu}) \, dx^\mu \wedge dx^\nu = \frac{1}{2} (F_{ab} + *P_{ab}) \, e^a \wedge e^b,
\]

since \( F_{\mu\nu} \) and \( *P_{\mu\nu} \) are curls.

Moreover, the Lagrangian function is assumed to be an integrable function of the coordinates \( x^\alpha \) of the 1–form equation

\[
d L(F(x^\alpha), G(x^\alpha)) = \frac{\partial L}{\partial F} \frac{\partial F}{\partial x^\alpha} \, d x^\alpha + \frac{\partial L}{\partial G} \frac{\partial G}{\partial x^\alpha} \, d x^\alpha = \frac{\partial L}{\partial x^\alpha} \, d x^\alpha.
\]  

\( (15) \)

A. Coordinate–dependent nonlinear electrodynamics

In the nonlinear electrodynamics under consideration is assumed the existence of two pair of electromagnetic antisymmetric fields \( F_{\mu\nu} \) and its dual \( *F_{\mu\nu} \) and a nonlinear associated (to be established) second pair of antisymmetric field \( P_{\mu\nu} \) and its dual \( *P_{\mu\nu} \). Each pair allow for two sets of eigenvalues, such that each set is constituted by two pair of different eigenvalues, hence, in the corresponding eigenvector basis the field tensors are symbolically represented as \( (F_{\mu\nu}) = \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2) \) and \( (P_{\mu\nu}) = \text{diag}(\pi_1, \pi_1, \pi_2, \pi_2) \). Consequently, there arise the invariants \( F \) and \( G \) for the field \( F_{\mu\nu} \) and \( P \) and \( G \) a for the field \( P_{\mu\nu} \), see definitions above, with these secondary objects one may build the Lagrangian \( L(F(x^\alpha), G(x^\alpha)) \) or the Hamiltonian \( L(P(x^\alpha), Q(x^\alpha)) \) formulations.

In what follows we focus on the Lagrangian formulation of the nonlinear electrodynamics by fixing the relations between the fields through \( (12) \), \( P_{\mu\nu} = 2 \frac{\partial L}{\partial F_{\mu\nu}} \) and \( F_{\mu\nu} = 2 \frac{\partial L}{\partial P_{\mu\nu}} \). At this level the structure is quite general; the constraints arise by requiring the fulfillment of the “Maxwell” equations \( (14) \)

\[
P_{\mu\nu} ; \nu = 0 \rightarrow [\sqrt{-g} L_F F_{\mu\nu} + \sqrt{-g} L_G *F_{\mu\nu}]_{; \nu} = 0,
\]

stating that \( *P_{\mu\nu} \) is a curl, \( *P_{\mu\nu} = *P_{\nu,\mu} - *P_{\mu,\nu} \); the \( F_{\mu\nu} \) is a curl \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \) and therefore its dual fulfils \( *F_{\mu\nu} ; \nu = 0 \). The above equations allow the determination of the functions \( L_F(x^\alpha) \) and \( L_G(x^\alpha) \), depending on the coordinates, in terms of the field components of \( F_{\mu\nu} \), \( *F_{\mu\nu} \), \( P_{\mu\nu} \), and \( *P_{\mu\nu} \). The determination of the Lagrangian \( L(x^\alpha) \) in terms of the coordinates is achieved by the fulfillment of the closure condition \( d^2 L = 0 \) of the 1–form equation

\[
d L(F(x^\alpha), G(x^\alpha)) = \frac{\partial L}{\partial F} \frac{\partial F}{\partial x^\alpha} \, d x^\alpha + \frac{\partial L}{\partial G} \frac{\partial G}{\partial x^\alpha} \, d x^\alpha = \frac{\partial L}{\partial x^\alpha} \, d x^\alpha.
\]
The energy–momentum tensor for the nonlinear electrodynamics under consideration is derived via the standard definition

\[ T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(-g^{\beta\gamma} L(x^\beta))}{\delta g_{\mu\nu}} = -L(x^\beta) g^{\mu\nu} + F_{\alpha}^{\mu} P^{\nu\alpha}, \]

with the eigenvalue structure \( T^a_b = \text{diag}(\tau_1, \tau_1, \tau_2, \tau_2) \), which is the eigenvalue characteristic of any electrodynamics.

There exist some researchers in electrodynamics in relativity that sustain the point of view that if the Lagrangian of the theory is not constructed in terms of the invariants \( F \) and \( G \) alone the theory is spurious, contrary to the viewpoint that I expose in details above, in which the fundamental cornerstones are the electromagnetic fields \( F^{\mu\nu} \) and \( P^{\mu\nu} \) and their dual ones, which, being solutions of the field equations are coordinate–dependent quantities; the secondary structural functions, i.e., the invariants \( F \) and \( P \) occur to functions of the coordinates, finally, the Lagrangian \( L \) integrates its 1–form equation as a function of coordinates \( x^\alpha \). This wider point of view, although is contained in the Born–Infeld–Plebański formulation via the simple recognition–acceptance of the coordinate description, can be called “Coordinate Dependent Non Linear Electrodynamics.”

This resume explains the constructive way of searching for solutions in nonlinear electrodynamics to be used in this work in the integration of the Einstein equations; one first determines the electromagnetic fields and latter one constructs the invariants and the Lagrangian of the theory to proceed further with the Einstein equations. In this approach the problem of representing the Lagrangian purely in terms of the invariants could become insolvable, because of the appearance of possible transcendent relations of invariants versus coordinates, or polynomials of degree higher than five, Abel–Galois restrictions, among others.

Only for some simple cases of static spherically symmetric solutions of Einstein–NLE one can construct a posteriori the explicit dependence \( L(F) \). In the stationary axisymmetric case, the Kerr–Newman is the only known until now case where the relation \( L(F) \) can be established, see details in this text. The new NLE solution presented here gives rise to transcendent relations of eight degree and consequently insolvable to establish of the dependence of the coordinates as functions of the invariants, thus a stationary axisymmetric
generalization of the Lagrangian \( L = L(F, G) \) for a NLE generalization of the Kerr–Newman solution is quite difficult to guess. On this respect one can recall the guessing process followed to “derive” the Newman et al. solution [3], which consists in the application of Janis–Newman procedure, see comments on this respect in Plebański et al. [18], pag 458, “Historically, the generalization (of the Kerr solution) for an electric charge was discovered by Newman et al. (1965) by a procedure equally mysterious as the derivation of the Kerr metric itself....” In the Stephani book [?], pag 230, one reads: “Since its mathematical structure is rather complicated, we shall not construct a derivation from the Einstein field equations.” of the Kerr solution. As a by product, we derive here, from results of the coordinate dependent method in NLE, the Kerr–Newman solution determining first the electromagnetic vector potential, next, the Lagrangian function \( L(x) \) and, finally, establishing the relation \( L(F, G) \). Incidentally, in our approach for searching stationary axisymmetric solutions, in the linear Maxwell case \( L = F \), the vector potential fulfills the Laplace equation.

The integration process for a given Lagrangian \( L = L(F, G) \) in terms of \( F \) and \( G \) requires the integration of the “conservation” field equations \( \ast F_{\mu \nu, \nu} = 0 \) and \( P_{\mu \nu, \nu} = 0 \), once this is done, one replaces the electromagnetic fields through the invariants (using the derivatives \( L_F \) and \( L_G \)) into the recently integrated field equations to proceed further with the integration of the vector potentials \( A_\mu \) and \( \ast P_\mu \), which, in many cases, cannot be integrated at all.

### B. Summary on nonlinear electrodynamics in relativity

Although it may sound repetitive, I would insist and emphasize on the eigenvalue structure of the electrodynamics, no matter the names that one coins for its different variants, to differentiate it from the “anisotropic” fluid structure. By extension of the Maxwell field theory from special to general relativity, the electromagnetic field tensor \( F_{\alpha \beta} \) is assumed to be antisymmetric \( F_{\alpha \beta} = A_{\beta, \alpha} - A_{\alpha, \beta} \) depending on a vector potential \( A_\alpha \), thus its tensor matrix allows for two pair of different eigenvalues, symbolically \(( F^{\alpha \beta} ) = \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2)\), associated to \( F_{\alpha \beta} \) and \( \ast F_{\alpha \beta} \). These eigenvalues are related with the field invariants \( F \) and \( G \). In the field formulation of the Maxwell theory, the Lagrangian function \( L \) is assumed to depend on \( F \). In its nonlinear electrodynamics generalization (Born–Infeld–Plebański) the Lagrangian \( L \) depends on both invariants \( F \) and \( G \). There appear in the NLE theory two new field tensors \( P_{\alpha \beta} \) and its dual \( \ast P_{\alpha \beta} \), with their own pair of eigenvalues \( \Lambda_1 \) 

and $\Lambda_2$, such that $(P^{\alpha \beta}) = \text{diag}(\Lambda_1, \Lambda_1, \Lambda_2, \Lambda_2)$, and correspondingly two field invariants $P$ and $Q$.

The energy–momentum tensor $T_{\mu \nu}$ occurs to be given by (6), and as consequence of its structure, its eigenvalue problem solves for two pairs of different eigenvalues such that $(T^{\alpha \beta}) = \text{diag}(\tau_1, \tau_1, \tau_2, \tau_2)$, and consequently belongs to the Segre($((11)(1))$)-Plebański $(2S - 2T)_{(11)}$ class [19], the family of electromagnetic energy–momentum tensors. Thus, the nonlinear electrodynamics can be defined by the invariant eigenvalue properties of its constitutive fields obeying the corresponding field equations (13) and (14), and the corresponding energy–momentum tensor.

On the other hand, taking into account the relation $F_{\mu \sigma} \ast F_{\nu \sigma} = G \delta^\mu_\nu$, one determines the traceless NLE energy–momentum tensor $\Upsilon_{\mu \nu}$ to be

$$\Upsilon^\mu_\nu := T^\mu_\nu - T^\nu_\mu = L_F (F^{\mu \sigma} F_{\nu \sigma} - F \delta^\mu_\nu), \quad T := T^\mu_\mu = -4L + 4L_F F + 4L_G G.$$(16)

Of course, this traceless energy tensor, via the Einstein equations is equivalent to the traceless Ricci tensor $S^\mu_\nu = \kappa \Upsilon^\mu_\nu$. Consequently it falls into the Segre($((11)(1))$)-Plebański $(2S - 2T)_{(11)}$ class of energy tensors, see Stephani et al. [10], Chapter 5. In the linear Maxwell limit, $(T_{ab}) = \text{diag}(\lambda_1, \lambda_1, -\lambda_1, -\lambda_1)$, see [20], pag 20.

C. Definition of exact solution to nonlinear electrodynamics equations in Einstein–NLE theory

By this definition an exact solution means that the tensor field $F_{\alpha \beta}$, and $P_{\alpha \beta}$, and their dual $\ast F_{\alpha \beta}$, and $\ast P_{\alpha \beta}$ fulfil the set of the coordinate “ Faraday–Maxwell” equations (13) and (14), and the Lagrangian 1–form (15) fits the closure condition $d^2 L = 0$. An exact solution to Einstein–NLE theory is a solution that fulfil, besides the electrodynamics equations, the Einstein equations (3) equated to the nonlinear electrodynamics energy–momentum tensor (6).

This definition is wider than the one requiring for the weak Maxwell limit $L \simeq F + O(F^2, G^2)$. The Lagrangian function derivation in terms of coordinates we considered more efficient and constructive than the description at initio through the two invariants $F$ and $G$, this because the field tensor description is done, at the end at day, in the language of coordinates.
III. METRIC AND TETRADS

We present the first stationary axially symmetric exact black hole solution to the Einstein equations coupled to nonlinear electrodynamics in the presence of a (positive or negative) cosmological constant given by the metric

\[
\begin{align*}
    ds^2 &= \frac{a^2 \sin^2 \theta}{\rho^2} \left(1 - \frac{\Lambda}{3} a^2 \cos^2 \theta\right) \left(dt - \frac{a^2 + r^2}{a} d\phi\right)^2 \\
    &+ \frac{\rho^2}{Q(r)} dr^2 + \frac{\rho^2}{1 - \frac{\Lambda}{3} a^2 \cos^2 \theta} d\theta^2 \\
    &- \frac{Q(r)}{\rho^2} \left(dt - a \sin^2 \theta d\phi\right)^2, \quad \rho(\theta, r) := \sqrt{r^2 + a^2 \cos^2 \theta}.
\end{align*}
\]  

(17)

The structural function \(Q(r)\) is the single metric function to be determined by solving the Einstein equations, occasionally we use its representation through the auxiliary function \(K(r)\)

\[
Q(r) = K(r) - 2 m r + r^2 + a^2 + \frac{\Lambda}{3} r^2 (r^2 + a^2).
\]  

(18)

The null tetrad \(e^a\) used is

\[
\begin{align*}
e^1 &= \frac{1}{\sqrt{2}} \left[\frac{\rho}{\Sigma} d\theta \pm i \frac{a \sin \theta \Sigma}{\rho} \left(dt - \frac{a^2 + r^2}{a} d\phi\right)\right], \\
e^2 &= \rho \sqrt{Q} dr, \\
e^3 &= \frac{\sqrt{Q}}{\rho} \left(dt - a \sin^2 \theta d\phi\right) \pm \frac{\rho}{\sqrt{Q}} dr, \\
e^4 &= \frac{\sqrt{Q}}{\rho} \left(dt - a \sin^2 \theta d\phi\right).
\end{align*}
\]  

(20)

Additionally to the null tetrad one may introduce an orthonormal basis \(\{E^a, a = 1, \ldots, 4\} = \{x, y, z, t\}\), where \(t\) is a time–like vector, \(t \cdot t = -1\), such that \(e^1 = (x + iy)/\sqrt{2}, e^2 = (x - iy)/\sqrt{2}, e^3 = (t + z)/\sqrt{2}, e^4 = (t - z)/\sqrt{2}\). In coordinates \(\{\theta, r, \phi, t\}\), as this basis one defines:

\[
\begin{align*}
    E^1 &= \frac{\rho}{\Sigma} d\theta, \quad E^3 = \frac{a \sin \theta \Sigma}{\rho} \left(dt - \frac{a^2 + r^2}{a} d\phi\right), \\
    E^2 &= \frac{\rho}{\sqrt{Q}} dr, \quad E^4 = \frac{\sqrt{Q}}{\rho} \left(dt - a \sin^2 \theta d\phi\right).
\end{align*}
\]  

(20)
These bases are associated to the eigenvector bases; for instance; $F_{\mu\nu} = 2 F_{ab} e^a_{\mu} e^b_{\nu} = 2 F_{12} e^1_{[\mu} e^2_{\nu]} + 2 F_{34} e^3_{[\mu} e^4_{\nu]}$ and $E^{\mu\nu} = E_1^{1\mu} E_1^{1\nu} + E_2^{2\mu} E_2^{2\nu} + E_3^{3\mu} E_3^{3\nu} - E_4^{4\mu} E_4^{4\nu}$.

From the metric tensor one evaluates the coordinate components of the Ricci tensor, the scalar curvature, and the Riemann-Weyl curvature tensor. In particular, these curvature quantities, refereed to the above null tetrad, acquire their simplest description—their eigenvector structure. To avoid confusion in the use of indices, we denote the coordinate indices with Greek–Latin symbols $\{x_\mu\} = \{\theta, r, \phi, t\}$, the tetrad components $E^a_b, E_{ab}, a, b = 1, \ldots, 4$ with a prefix $N$ for null tetrad components $NE_{ab}$ and $O$ for orthonormal tetrad components $OE_{ab}$, avoiding in this manner the use of parentheses or tildes.

The Einstein tensor is determined by a diagonal tensor matrix with two pair of eigenvalue components, the corresponding traceless Ricci tensor is described by a diagonal matrix too with two pair of opposite in sign eigenvalues, $\{\lambda_1, \lambda_1, -\lambda_1, -\lambda_1\}$, and hence, according with the Segre–Plebański classification of the matter–field tensors, it can only describes (linear Maxwell and nonlinear) electrodynamics with non–zero invariants.

IV. ALIGNMENT CONDITIONS

The alignment of the tetrad and eigenvectors of the field is, perhaps, the very break point in the derivation of the electromagnetic fields, and consequently of the resolution of the entire problem.

The field tensor $F_{\mu\nu}$ is endowed with four components: $F_{\theta\phi}, F_{\theta t}, F_{r\phi}, F_{rt}$; one looks for the eigenvectors $V^\mu_a, a = 1, \ldots, 4$, of the tensor $F_{\mu\nu}$ by solving the corresponding eigenvalue problem; the alignment of the eigenvectors $V^\nu_a$ along the tetrad basis, or equivalently, aligning the tetrad along the eigenvectors $V^\nu_a$ gives rise to the alignment conditions:

$$ F_{r\phi} = -a \sin^2 \theta F_{rt}, \quad F_{\theta\phi} = -\frac{a^2 + r^2}{a} F_{\theta t}, $$

(21)
	hus two of the field components are independent, say $F_{rt}$ and $F_{\theta t}$, while the remaining two $F_{r\phi}$ and $F_{\theta\phi}$ are determined through the alignment conditions. Since the field tensor $F_{\mu\nu}$ is a curl, it can be determined from its representation in term of the vector potential $A_\mu, F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. The alignment conditions can be integrated for the electromagnetic vector components $A_t$ and $A_\phi$: replacing $F_{\theta\phi} = A_{\phi,\theta}, F_{r\phi} = A_{\phi,r}, F_{rt} = A_{t,r}, F_{\theta t} = A_{t,\theta}$ one
arrives at
\[
[\frac{\partial A_\phi}{\partial r} + a \sin^2 \theta \frac{\partial A_t}{\partial r} = 0, \quad \frac{\partial A_\phi}{\partial \theta} + \frac{a^2 + r^2}{a} \frac{\partial A_t}{\partial \theta} = 0],
\]
(22)
the integrability of \( A_\phi \) leads to
\[
\frac{\partial}{\partial r} \left( \frac{a^2 + r^2}{a} \frac{\partial A_t}{\partial \theta} \right) - \frac{\partial}{\partial \theta} \left( a \sin^2 \theta \frac{\partial A_t}{\partial r} \right) = 0,
\]
i.e., an equation for \( A_t \)
\[
(r^2 + a^2 \cos^2 \theta) \frac{\partial^2 A_t}{\partial \theta \partial r} + 2r \frac{\partial A_t}{\partial \theta} - 2a^2 \sin \theta \cos \theta \frac{\partial A_t}{\partial r} = 0,
\]
(23)
\[
(24)
\]
The general solution is sought in the form \( A_t = Z(\theta, r) \rho^2 \), substituting this expression into the equation (24), one gets
\[
\frac{\partial}{\partial r} \frac{\partial}{\partial \theta} Z(\theta, r) = 0 \rightarrow Z(\theta, r) = Y(\theta) + X(r),
\]
where \( Y(\theta) \) and \( X(r) \), at this level, are arbitrary integration functions, consequently
\[
A_t = Y(\theta) + X(r)
\]
(25)
This solution for \( A_t \) guarantees the integrability of \( A_\phi \), which occurs to be
\[
A_\phi = -\frac{a \sin^2 \theta X(r)}{r^2 + a^2 \cos^2 \theta} - \frac{(a^2 + r^2) Y(\theta)}{a (r^2 + a^2 \cos^2 \theta)}
\]
(26)

V. FIELD EQUATIONS \( (\sqrt{-g_m} P^\mu_\nu)_{,\nu} = 0 \)

The dual tensor \( *P^\mu_\nu \), being a curl, is defined in terms of the vector potential components \( *P_\mu \), similarly as the \( F^\mu_\nu \), namely \( *P^\mu_\nu = *P_\nu,\mu - *P_{\mu,\nu} \). In correspondence with the structure of \( F^\mu_\nu \), as independent dual components of \( *P^\mu_\nu \) one has the following: \( *P_\theta t = *P_{t,\theta}, *P_{r,t} = *P_{t,r} \).

The Maxwell field equations \( (\sqrt{-g_m} P^\mu_\nu)_{,\nu} = 0 \), for \( (\sqrt{-g_m} P^\phi_\theta)_{,\theta} + (\sqrt{-g_m} P^\phi_r)_{,r} = 0 \), where \( g_m \) is the coordinate metric determinant, yield
\[
-a \sin \theta F_{rt} L_G + F_{rt} L_F = \frac{\partial}{\partial r} *P_t(\theta, r),
\]
\[
a \sin \theta F_{rt} L_F + F_{rt} L_G = -\frac{\partial}{\partial \theta} *P_t(\theta, r),
\]
(27)
which can be solved for the derivatives \( L_F \) and \( L_G \), namely
\[
L_F = \frac{a \sin \theta \left( F_{\theta t} \frac{\partial F_{t}}{\partial r} - F_{rt} \frac{\partial F_{\theta}}{\partial r} \right)}{(F_{rt})^2 a^2 \sin^2 \theta + (F_{\theta t})^2},
\]
\[
L_G = -\frac{a^2 F_{rt} \sin^2 \theta \frac{\partial F_{t}}{\partial \theta} + F_{\theta t} \frac{\partial F_{\theta}}{\partial \theta}}{(F_{rt})^2 a^2 \sin^2 \theta + (F_{\theta t})^2},
\]
(28)
which play an important role in the integration of the entire problem. While $(\sqrt{-g_m}P^{\theta\phi})_\theta + (\sqrt{-g_m}P^{t\phi})_t = 0$ gives rise to alignment conditions for the tensor field $*P_{\mu\nu}$,

\[
*P_{r\phi} = -a \sin^2 \theta *P_{rt},
\]
\[
*P_{\theta\phi} = -\frac{a^2 + r^2}{a} *P_{\theta t},
\]

which in terms of the vector potential components $*P_{\mu}$ read

\[
\frac{\partial}{\partial r} *P_{\phi} = -a \sin^2 \theta \frac{\partial}{\partial r} *P_{t},
\]
\[
\frac{\partial}{\partial \theta} *P_{\phi} = -\frac{a^2 + r^2}{a} \frac{\partial}{\partial \theta} *P_{t},
\]

which are similar in all respect to the equations for $A_t$ and $A_\phi$, (22), therefore the solutions for $*P_t$ and $*P_\phi$ are:

\[
*P_t = \frac{A(r) + B(\theta)}{a^2 (\cos \theta)^2 + r^2},
\]
\[
- *P_\phi = \frac{a (\sin \theta)^2 A(r)}{a^2 \cos^2 \theta + r^2} + \frac{(a^2 + r^2) B(\theta)}{a (a^2 \cos^2 \theta + r^2)}.\]

VI. THE KEY EQUATION

The expressions (28) of the derivatives $L_F$ and $L_G$, can be used to calculate the derivatives $L_r$ and $L_\theta$ of the Lagrangian $L$:

\[
\frac{\partial L}{\partial r} = L_F \frac{\partial F}{\partial r} + L_G \frac{\partial G}{\partial r},
\]
\[
\frac{\partial L}{\partial \theta} = L_F \frac{\partial F}{\partial \theta} + L_G \frac{\partial G}{\partial \theta},
\]

Next, using the expressions of the invariant $F$ and $G$ under the alignment conditions (ac) (21) and (29) one obtains

\[
InvF_{ac} = \frac{1}{2} \frac{a^2 \sin^2 \theta F_{\theta t}^2 - F_{rt}^2}{a^2 \sin^2 \theta},
\]
\[
InvG_{ac} = -\frac{F_{\theta t} F_{rt}}{a \sin \theta}
\]

together with the substitution of $L_F$ and $L_G$ from (28) lead to the simple expressions

\[
\frac{\partial}{\partial r} L = \left(\frac{\partial}{\partial r} *P_t\right) \frac{\partial}{\partial r} *F_{\theta t} + \left(\frac{\partial}{\partial r} *P_t\right) \frac{\partial}{\partial r} *F_{rt}
\]

\[
\frac{a}{a \sin \theta}
\]
\[
\frac{\partial}{\partial \theta} L = -\frac{F_{\theta t} \cos \theta (\frac{\partial^* P_t}{\partial r})}{a \sin^2 \theta} + \frac{(\frac{\partial}{\partial r} F_{rt}) \frac{\partial^* P_t}{\partial \theta}}{a \sin \theta} + \frac{(\frac{\partial}{\partial \theta} F_{\theta t}) \frac{\partial^* P_t}{\partial \theta}}{a \sin \theta}.
\]

The integrability of the Lagrangian 1–form equation
\[
dL(\theta, r) = \frac{\partial L}{\partial \theta} d\theta + \frac{\partial L}{\partial r} dr, \tag{37}
\]
is guaranteed by the closure condition \(d^2 L = 0\), i.e., from vanishing of the mixed derivative equation \(\frac{\partial}{\partial \theta} \frac{\partial L}{\partial r} = \frac{\partial}{\partial r} \frac{\partial L}{\partial \theta}\), replacing \(F_{rt} = A_{t, r}, F_{\theta t} = A_{t, \theta}\), one arrives at the key equation
\[
KEY := \frac{\partial^2 A_t}{\partial r \partial r} \cdot \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial^* P_t}{\partial \theta} \right) - \frac{\partial^2 P_t}{\partial r \partial r} \cdot \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial A_t}{\partial \theta} \right) = 0. \tag{38}
\]

In the search for solutions of the Carter–Plebański metric, which will be published elsewhere soon, using the coordinates \(x = a \cos \theta, y = r\) this KEY equation is simply
\[
KEY := \frac{\partial^2 A_t}{\partial y^2} \cdot \frac{\partial^2 P_t}{\partial x^2} - \frac{\partial^2 A_t}{\partial x^2} \cdot \frac{\partial^2 P_t}{\partial y^2} = 0. \tag{39}
\]
The approach to derive solutions of KEY is through the integrals of \(A_t\) and \(P_t\); substituting them into the KEY one arrives at single nonlinear second order equation for the potential functions \(\{Y(\theta), X(r), A(r), B(\theta)\}\), one may proceed further by using a separation of variables procedure. Any solution of this KEY equation ensures the integrability of the Lagrangian function via the equation (37).

**A. Vector potentials of the Kerr–Newman solution**

The simplest solution of the KEY equation corresponds to the Kerr–Newman solution since its vector potential components \(A_t\) and \(P_t\) fulfil the Laplace equation
\[
\frac{\partial^2 A_t}{\partial y^2} + \frac{\partial^2 A_t}{\partial x^2} = 0 \rightarrow \frac{\partial^2 P_t}{\partial x^2} + \frac{\partial^2 P_t}{\partial y^2}, \quad KEY \equiv 0. \tag{40}
\]
Therefore, their solutions are of the form
\[
H_t = H(x + i y) + \bar{H}(x - i y). \tag{41}
\]
For the solution \(2 H(x + i y) = \frac{a_0 + i b_0}{x + i y}\)
\[
2 H_t(x, y) = \frac{a_0 + i b_0}{x + i y} + \frac{a_0 - i b_0}{x - i y} = 2 \frac{a_0 x + b_0 y}{x^2 + y^2} \rightarrow H_t(\theta, r) = \frac{a_0 a \cos \theta + b_0 r}{a^2 \cos^2 \theta + r^2}, \tag{42}
\]
hence

\[ A_t(\theta, r) = \frac{a_0 a \cos \theta + b_0 r}{a^2 \cos^2 \theta + r^2}, \]  

which is just the electromagnetic vector potential \( A_t \) for the Kerr–Newman metric by identifying \( b_0 = e, \ a_0 = g_0 \) which are the electric \( e \) and magnetic \( g_0 \) charges; the field component \( A_\phi \) is obtained from the the alignments conditions (21); it results in

\[ -A_\phi = \frac{a_0 (a^2 + r^2) \cos \theta \ a b_0 r \sin^2 \theta}{a^2 \cos^2 \theta + r^2}. \]  

(44)

For the components \( *P_t \) and \( *P_\phi \) one obtains similar expressions; \( A_t(a_0 \rightarrow f_0, b_0 \rightarrow h_0) \rightarrow *P_t \), and \( A_\phi(a_0 \rightarrow f_0, b_0 \rightarrow h_0) \rightarrow *P_\phi \), where \( f_0 \) and \( h_0 \) are constants related with the charges. The field tensors \( F_{\mu\nu} \) and \( *P_{\mu\nu} \) are derived by differentiation of these vector potentials \( A_\mu \) and \( *P_\mu \).

Of course one could directly derive the electromagnetic fields of the Kerr-Newman solution integrating the (13) and (14) equations, which under the alignment conditions become (27); for the Maxwell electrodynamics \( L = F, L_F = 1, L_G = 0, \) the equations (27) reduces to

\[
\begin{aligned}
\frac{1}{a \sin \theta} F_{\theta t} &= \frac{\partial}{\partial r} *P_t \rightarrow \frac{1}{a \sin \theta} \frac{\partial}{\partial \theta} A_t = \frac{\partial}{\partial r} *P_t \\
 a \sin \theta F_{rt} &= -\frac{\partial}{\partial \theta} *P_t \rightarrow \frac{\partial}{\partial r} A_t = -\frac{1}{a \sin \theta} \frac{\partial}{\partial \theta} *P_t,
\end{aligned}
\]  

(45)

which in terms of the variables \( x = a \sin \theta \) and \( y = r \) rewrite as

\[
\begin{aligned}
\frac{\partial}{\partial x} A_t &= \frac{\partial}{\partial y} *P_t, \quad \frac{\partial}{\partial y} A_t = -\frac{\partial}{\partial x} *P_t,
\end{aligned}
\]  

(46)

with integrability conditions:

\[
\begin{aligned}
A_{t,y,x} &= A_{t,x,x} \rightarrow *P_{t,x,x} + *P_{t,y,y} = 0, \\
*P_{t,y,x} &= *P_{t,x,y} \rightarrow A_{t,x,x} + A_{t,y,y} = 0,
\end{aligned}
\]  

(47)

i.e., the starting point (40) of the integration process we followed above, VI A. It is quite possible that the Kerr–Newman solution is singled out as the unique stationary axisymmetric solution whose electromagnetic potentials are derivable in Einstein–NLE from a Lagrangian function depending explicitly on the \( F \) and \( G \), \( L(F, G) = A_0 F + B_0 G, \) see VIII.

As we show in the forthcoming sections, the “straightforward coordinate dependent method” in nonlinear electrodynamics (SCDM–NLE) permits the determination, from
electromagnetic field tensors $F_{\mu\nu}(x^\alpha)$ and $*P_{\mu\nu}(x^\alpha)$ fulfilling the conservation "Faraday–Maxwell" equations, the corresponding coordinate dependent Lagrangian function $L(x^\alpha)$. In this “method” one does not need to know nor establish the explicit dependence of the Lagrangian on the the invariants $F$ and $G$; these quantities as result of their definition through the electromagnetic field tensor depend on the coordinate too.

VII. THE CUBIC VECTOR POTENTIALS

The cubic potential functions \{\(A(r), B(\theta), X(r), Y(\theta)\)\}, of the form $F(r) = f_0 + f_1 r + f_2 r^2 + f_3 r^3$, and $G(\theta) = g_0 + g_1 \cos \theta + g_2 \cos^2 \theta + g_3 \cos^3 \theta$, substituted in KEY gives rise to a polynomial in $r$ and $\cos \theta$, equating to zero the coefficients of the independent powers $r^s \cos^m \theta$ one gets an algebraic system of equations, whose solutions determines the cubic, in this example, potential functions

\[
A(r) = -r^3 y_1 \beta + y_1 r,
B(\theta) = a^3 x_1 \beta \cos^3 \theta + ax_1 \cos \theta,
X(\theta) = a f_1 \beta \cos^3 \theta + a f_1 \cos \theta,
Y(r) = -r^3 g_1 \beta + g_1 r,
\]

which, in turn, determine via (25) and (26) the vector potential components of the vector field $A_\mu = \{0, 0, A_\phi, A_t\}$:

\[
\rho^2 A_t = (f_1 a^3 \cos^3 \theta - y_1 r^3) \beta + f_1 a \cos \theta + g_1 r,
-\rho^2 A_\phi = a \beta [af_1 (a^2 + r^2) \cos^3 \theta - g_1 r^3 \sin^2 \theta]
+ f_1 (a^2 + r^2) \cos \theta + a g_1 r \sin^2 \theta.
\]

The dual vector potential $*P_\mu$ is given, via (31) and (32), by $*P_t$ and $*P_\phi$:

\[
\rho^2 *P_t = \beta (a^3 x_1 \cos^3 \theta - y_1 r^3) + ax_1 \cos \theta + y_1 r,
-\rho^2 *P_\phi = a \beta [a x_1 (a^2 + r^2) \cos^3 \theta - y_1 r^3 \sin^2 \theta]
+ x_1 (r^2 + a^2) \cos \theta + a y_1 r \sin^2 \theta.
\]
Accomplishing the corresponding partial differentiation of the components $A_t$, and $A_\phi$ one gets the field tensor components $F_{\mu\nu}$ and $^*P_{\mu\nu}$:

$$F_{rt} = -r\beta \frac{(2 f_1 a^3 \cos^3 \theta + 3 a^2 g_1 r \cos^2 \theta + g_1 r^3)}{(a^2 \cos^2 \theta + r^2)^2} + g_1 \frac{(a^2 \cos^2 \theta - r^2) - 2 a f_1 r \cos \theta}{(a^2 \cos^2 \theta + r^2)^2},$$

$$F_{\theta t} = \frac{-\beta a^2 \cos \theta \sin \theta [f_1 a \cos \theta (a^2 \cos^2 \theta + 3 r^2) + 2 g_1 r^3]}{(a^2 \cos^2 \theta + r^2)^2} + a \sin \theta [f_1 (a^2 \cos^2 \theta - r^2) + 2 a g_1 r \cos \theta] \frac{a^2 \cos^2 \theta + r^2}{(a^2 \cos^2 \theta + r^2)^2},$$

while the remaining field components are evaluated from the alignment conditions

$$F_{\theta \phi} = -\frac{a^2 + r^2}{a} F_{\theta t},$$

$$F_{r \phi} = -a \sin^2 \theta F_{rt}.$$

On the other hand, accomplishing the corresponding differentiation of the vector potential components $^*P_t$ and $^*P_\phi$ one gets the tensor field components $^*P_{\mu\nu}$:

$$^*P_{rt} = -r\beta \frac{(2 x_1 a^3 \cos^3 \theta + 3 a^2 y_1 r \cos^2 \theta + y_1 r^3)}{(a^2 \cos^2 \theta + r^2)^2} + y_1 \frac{(a^2 \cos^2 \theta - r^2) - 2 a x_1 r \cos \theta}{(a^2 \cos^2 \theta + r^2)^2},$$

$$^*P_{\theta t} = \frac{-\beta a^2 \cos \theta \sin \theta [x_1 a \cos \theta (a^2 \cos^2 \theta + 3 r^2) + 2 y_1 r^3]}{(a^2 \cos^2 \theta + r^2)^2} + a \sin \theta [x_1 (a^2 \cos^2 \theta - r^2) + 2 a y_1 r \cos \theta] \frac{a^2 \cos^2 \theta + r^2}{(a^2 \cos^2 \theta + r^2)^2},$$

the remaining two components are evaluated from the corresponding alignment conditions

$$^*P_{\theta \phi} = -\frac{a^2 + r^2}{a} ^*P_{\theta t},$$

$$^*P_{r \phi} = -a \sin^2 \theta ^*P_{rt}.$$

Notice the symmetry between these tensor field components,

$$\{A_t, A_\phi, F_{\theta \phi}, F_{\theta t}, F_{r \phi}, F_{rt}\} \Leftrightarrow \{^*P_t, ^*P_\phi, ^*P_{\theta \phi}, ^*P_{\theta t}, ^*P_{r \phi}, ^*P_{rt}\},$$

under the replacement $f_1 \Leftrightarrow x_1$ and $g_1 \Leftrightarrow y_1$. 18
VIII. LAGRANGIAN FUNCTION FOR THE CUBIC SOLUTION

To get the Lagrangian function \( L \), one integrates the 1–form \( dL = \frac{\partial L}{\partial r} dr + \frac{\partial L}{\partial \theta} d\theta \), where the derivatives \( \frac{\partial L}{\partial r} \) and \( \frac{\partial L}{\partial \theta} \), (35) and (36), are evaluated for the cubic vector potentials and tensor field components derived in the previous section VII. After a time consuming integration, one can give the Lagrangian function in the form

\[
L = \frac{1}{2 \rho^8} \left( L_{\beta^2} \beta^2 + L_{\beta \beta} + L_{\beta^0} \right) + L_0
\]

where \( L_0 \) is an integration constant and the functions \( L_{\beta^i} \), \( i = 0, 1, 2 \) denote the Lagrangian numerators given correspondingly as

\[
L_{\beta^2} = 4 \cos^7 \theta a^7 f_1 r x_1 + 6 \cos^6 \theta a^6 g_1 r^2 x_1 + 12 \cos^5 \theta a^5 f_1 r^3 x_1
+ 5 \cos^4 \theta a^4 f_1 r^4 y_1 + 23 \cos^3 \theta a^4 g_1 r^4 x_1 + 12 \cos^3 \theta a^3 g_1 r^5 y_1
- 6 \cos^2 \theta a^2 f_1 r^6 y_1 + 12 \cos^2 \theta a^2 g_1 r^6 x_1 + 4 \cos \theta a g_1 r^7 y_1
- 3 f_1 r^8 y_1 + 3 g_1 r^8 x_1,
\]

\[
L_{\beta} = 2 f_1 r^6 y_1 + 6 a^2 \left( 2 f_1 y_1 + g_1 x_1 \right) r^4 \cos^2 \theta
+ 16 a^3 \left( f_1 x_1 - g_1 y_1 \right) r^3 \cos^3 \theta
- 6 a^4 \left( f_1 y_1 + 2 g_1 x_1 \right) r^2 \cos^4 \theta - 2 \cos^6 \theta a^6 g_1 x_1,
\]

\[
L_{\beta^0} = \left( a^4 \cos^4 \theta - 6 a^2 r^2 \cos^2 \theta + r^4 \right) \left( f_1 y_1 + g_1 x_1 \right)
- 4 a r \cos \theta \left( a^2 \cos^2 \theta - r^2 \right) \left( f_1 x_1 - g_1 y_1 \right),
\]

this last expression \( L_{\beta^0} \) corresponds to the numerator of the Maxwell Lagrangian \( L_{\beta^0} = L_{\text{Maxwell}} \).

A. The Lagrangian \( L_{\beta^0} \) in terms of the invariants at zero order of beta

The evaluation of the invariants (34) for the cubic solution at zero degree of \( \beta \) gives:

\[
\text{InvG}_{\beta^0} = \frac{2}{\rho^8} a r \cos \left( \theta \right) \left( a^2 \cos^2 \theta - r^2 \right) \left( f_1^2 - g_1^2 \right)
- \frac{1}{\rho^8} \left( a^4 \cos^4 \theta - 6 a^2 r^2 \cos^2 \theta + r^4 \right) f_1 g_1.
\]
while the invariant $\text{InvF}_{\beta 0}$ occurs to be

$$\text{InvF}_{\beta 0} = \frac{1}{2\rho^8} \left(a^4 \cos^4 \theta - 6 a^2 r^2 \cos^2 \theta + r^4\right) \left(f_i^2 - g_i^2\right)$$
$$+ \frac{4}{\rho^2} a r \cos \theta \left(a^2 \cos^2 \theta - r^2\right) g_i f_i.$$ (62)

Thus one can define the auxiliary functions $Z$ and $W$

$$Z = a^4 \cos^4 \theta - 6 a^2 r^2 \cos^2 \theta + r^4,$$ (63)
$$W = a \cos \theta \left(a^2 \cos^2 \theta - r^2\right),$$ (64)

and build the algebraic system for:

$$I = 2\rho^8 \text{InvF}_{\beta 0}, \ J = \rho^8 \text{InvG}_{\beta 0},$$ (65)

namely

$$I = Z F_1 + 8 W G_1, \ J = 2 W F_1 - Z G_1,$$ (66)

where

$$F_1 = f_i^2 - g_i^2, \ G_1 = g_i f_i.$$ (67)

Solving for $W$ and $Z$ one gets

$$W = \frac{F_1 J + G_1 I}{2(F_1^2 + 4 G_1^2)}, \ Z = \frac{F_1 I - 4 G_1 J}{F_1^2 + 4 G_1^2}. \ (68)$$

The Lagrangian $L_{\beta 0}$, from (60), in terms of $W$ and $Z$ becomes

$$2\rho^8 L_M = Z (f_i y_i + g_i x_i) - 4 W (f_i x_i - g_i y_i),$$ (69)

substituting $W$ and $Z$, from (68), leads to $L_M$ in terms of the invariants

$$2\rho^8 L_M = \frac{F_1 I - 4 G_1 J}{F_1^2 + 4 G_1^2} (f_i y_i + g_i x_i) - 2 \frac{F_1 J + G_1 I}{F_1^2 + 4 G_1^2} (f_i x_i - g_i y_i),$$ (70)

gathering the invariants $I$ and $J$ and substituting $F_1$ and $G_1$, one arrives at

$$2\rho^2 L_M = I \frac{F_0}{f_i^2 + g_i^2} - J \frac{G_0}{f_i^2 + g_i^2} = 2\rho^8 \text{InvF}_{\beta 0} \frac{F_0}{f_i^2 + g_i^2} - \rho^8 \text{InvG}_{\beta 0} \frac{G_0}{f_i^2 + g_i^2},$$ (71)
or

\[ L_{\text{Max}} = 2 \text{Inv} F_{\beta} \frac{F_0}{f_1^2 + g_1^2} - \text{Inv} G_{\beta} \frac{G_0}{f_1^2 + g_1^2}. \]  

(72)

The Maxwell relation \( L = F \) arises for

\[ f_1 = g_0 = y_1 \quad \text{and} \quad g_1 = e = -x_1, \]

then

\[ F_0 := f_1 y_1 - g_1 x_1 \rightarrow e^2 + g_0^2 \quad \text{and} \quad G_0 := f_1 x_1 + g_1 y_1 \rightarrow 0, \]

hence

\[ L_{\beta} = \text{Inv} F_{\beta} = F. \]  

(73)

Therefore, we have demonstrated that the general Lagrangian \( L(\beta) \) in the linear Maxwell limit becomes the Maxwell Lagrangian \( L = F \).

Since the function \( W \) and \( Z \) are polynomials of third and fourth degree in the coordinates \( x = a \cos \theta \) and \( r \) one can express this coordinates in terms of the Maxwell invariants \( F_M \) and \( G_M \), which are awful enough, as new coordinates and gets the Lagrangian \( L \) as function of \( F_M \) and \( G_M \). The problem of solving for \( r \) and \( x \) analytically in term of the whole invariants \( F_\beta \) and \( G_\beta \) faces the insuperable problem of searching the roots for polynomials (with radicals) of eight degree in \( r \) and \( x \); for the impossibility of a polynomial to be solvable by radicals see N.H. Abel and E. Galois works, thus the remaining alternative is the numerical analysis approach which is widely used in Mathematical Physics.

**IX. INTEGRATION OF THE EINSTEIN EQUATIONS**

The Einstein tensor \( NE^a_b \) possesses two pairs of different eigenvalues, \( NE^2_2 = NULL \), \( NE^1_1 = NE_{12} \) and \( NE^3_3 = NE^4_4 = -NE_{34} \), such that the tensor matrix \( (NE^a_b) = \text{diag}(NE^1_1, NE^1_1, NE^4_4, NE^4_4) \) while the orthonormal tetrad components fulfil \( OE^2_2 = OE^4_4 = OE_{11} \) and \( OE^3_3 = OE^1_1 = OE_{44} \), with matrix tensor \( (OE^a_b) = \text{diag}(OE^1_1, OE^1_4, OE^1_4, OE^4_4); (OE_{ab}) = \text{diag}(OE_{11}, -OE_{44}, OE_{11}, OE_{44}) \). There is a relationship between these tetrad components

\[ NE_{12} = OE_{11}, \quad NE_{34} = OE_{44}. \]  

(74)
We consider as independent Einstein equations those given by the orthonormal components $OE_{11}$ and $OE_{44}$:

\[ OE_{11} = \frac{\Lambda a^4}{3\rho^4} \cos^2 \theta \left(3 \cos^2 \theta - 1\right) \]
\[ + \frac{1}{2\rho^4} \left(\rho^2 Q'' - 2 r Q' + 2Q - 2 a^2 \cos^2 \theta - 2 a^2\right) \]
\[ = \kappa OT_{11} + \Lambda \eta_{11} = \kappa \left(L - \frac{F_\theta (\partial_r P_t)}{a \sin \theta}\right) + \Lambda \]
\[ = \Lambda + \frac{1}{2\rho^4} \left(\rho^2 K'' - 2 r K' + 2K\right), \quad (75) \]

and

\[ OE_{44} = -\frac{\Lambda a^2}{3\rho^4} \left(3 a^2 \cos^4 \theta + 6r^2 \cos^2 \theta - r^2\right) - \frac{1}{\rho^4} \left(r Q' - Q - r^2 + a^2\right) \]
\[ = \kappa OT_{44} + \Lambda \eta_{44} = \kappa \left(-L + \frac{F_r (\partial_r P_t)}{a \sin \theta}\right) - \Lambda \]
\[ = -\Lambda - \frac{1}{\rho^4} (r K' - K). \quad (76) \]

Adding these equations one arrives at the traceless Ricci tensor eigenvalue $S = S_1^1$ equation:

\[ 2 S_1^1 - \kappa (OT_{11} + OT_{44}) = -\frac{\Lambda a^2}{3\rho^4} \left(a^2 \cos^2 \theta + 6r^2 \cos^2 \theta - r^2\right) \]
\[ + \frac{1}{2\rho^4} \left(\rho^2 Q'' - 4 r Q' + 4Q - 2 a^2 \cos^2 \theta - 4 a^2 + 2 r^2\right) \]
\[ - \frac{\kappa}{a \sin \theta} \left(\left(\frac{\partial A_t}{\partial \theta}\right) \frac{\partial^* P_t}{\partial r} - \left(\frac{\partial A_t}{\partial r}\right) \frac{\partial^* P_t}{\partial \theta}\right) =: \frac{1}{6\rho^4} \left(V(r) a^2 \cos^2 \theta + W(r)\right), \quad (77) \]

where the contribution of the electromagnetic field is represented by

\[ OT_{44} + OT_{11} = \frac{\left(\frac{\partial}{\partial r^*} P_t\right) \frac{\partial}{\partial \theta} A_t - \left(\frac{\partial}{\partial \theta} P_t\right) \frac{\partial}{\partial r} A_t}{a \sin (\theta)}, \quad (78) \]

whose evaluation gives

\[ \kappa (OT_{11} + OT_{44}) = \frac{\kappa F_0}{\rho^4} \left(3 \beta^2 r^2 a^2 \cos^2 \theta - a^2 \beta \cos^2 \theta + \beta r^2 + 1\right). \quad (79) \]

Therefore the auxiliary functions $V(r)$, and $W(r)$ which stand for the separable equation terms are:

the $V(r)$ equation is given by

\[ V(r) = 3 \frac{d^2 Q}{dr^2} + 6 \beta \kappa F_0 \left(1 - 3 \beta r^2\right) - 2 a^2 \Lambda - 12 \Lambda r^2 - 6 = 0, \]
\[ F_0 := f \gamma g - g \gamma f, \quad (80) \]
which integrates for “a middle of the road” $Q(r)$ as

$$Q(r) = \frac{1}{2} \beta \kappa r^2 F_0 (\beta r^2 - 2) + \frac{1}{3} \Lambda r^2 (a^2 + r^2) + r^2 - 2m r + C_0, \quad (81)$$

where $m$ and $C_0$ are constant of integration. Substituting it into the second $W$ equation

$$W(r) = 3 r^2 \frac{d^2 Q}{dr^2} - 12 r \frac{d}{dr} Q + 12 Q$$

$$- 6 \kappa F_0 (1 + \beta r^2) + 6 r^2 + 2 \Lambda a^2 r^2 - 12 a^2 = 0, \quad (82)$$

one determines the integration constant $C_0$

$$C_0 = \kappa F_0/2 + a^2. \quad (83)$$

Hence, the structural function $Q(r)$ can be given as

$$Q(r) = \frac{\kappa F_0}{2} (\beta r^2 - 1)^2 F_0 + \frac{\Lambda}{3} r^2 (a^2 + r^2) + r^2 - 2m r + a^2, \quad (84)$$

or, in the representation of $Q(r)$ in terms of the auxiliary function $K(r)$ from (18)

$$K(r) = \frac{\kappa F_0}{2} (1 - \beta r^2)^2, \quad (85)$$

in the case one were integrating the Einstein equations for the function $K(r)$ and its derivatives.

This structure, in certain sense, represents the linear superposition in $Q$ of different contributions to the matter tensor; for vacuum, the mass $m$, and the rotation parameter $a$, $\Lambda$ for the cosmological constant, the parameters $e$ and $g_0$ for the electric and magnetic charges through the constant $F_0$, and finally the $\beta$–parameter responding to the nonlinearity of the electrodynamics. In general electrodynamics $L(F, G)$, beside the constant $F_0 = f_1 y_1 - g_1 x_1 \neq 0$ to have $L_F \neq 0$, there is a second field constant $G_0 := f_1 x_1 + g_1 y_1$ which takes care of the presence of the second invariant $G$ through the non vanishing of $L_G$ even in the case of the linear Maxwell field. Without any loss of generality, one may set $f_1 = g_0, g_1 = e$ and equate

$$x_1 = - \frac{F_0 e - G_0 g_0}{e^2 + g_0^2}, \quad y_1 = \frac{F_0 g_0 + G_0 e}{e^2 + g_0^2}. \quad (86)$$

In the Maxwell case, the Kerr–Newman solution is determined for

$$f_1 = g_0, g_1 = e, y_1 = g_0, x_1 = -e, G_0 = 0, F_0 = e^2 + g_0^2,$$
with \( L(F) = L(F_0) \neq 0 \) and \( L_G = 0 \). Therefore, as a by product, we got the Kerr–Newman solution for a Lagrangian depending on the two invariants \( F \approx E^2 - B^2 \) and \( G \approx E \cdot B \), although it can be determined via duality rotations. 

It should be pointed out that the equation for the curvature scalar

\[
\frac{1}{2} R = O E_{11} - O E_{44} = \kappa [2 L - \frac{\partial A_t}{\partial \theta} \frac{\partial P_t}{\partial r} - \frac{\partial A_t}{\partial r} \frac{\partial \ast P_t}{\partial \theta} ] + 2 \Lambda = \Lambda + \frac{K''}{2 \rho^2}, \tag{87}
\]

can be used, as a short cut, to derive the Lagrangian \( L \), once the vector potential components have been determined, i.e., the components \( A_t \) and \( \ast P_t \) that satisfy the KEY equation.

**X. CURVATURE QUANTITIES \( \Psi_2, S, R \)**

The evaluation of the curvature quantities yields:

The single Weyl invariant eigenvalue \( \Psi_2 \), in there dependence in \( Q(r) \), becomes

\[
12 \left( a \cos \theta + ir \right) \left( ir - a \cos \theta \right)^3 \Psi_{2Q} \\
= -2 a^2 \left( a^2 \cos^2 \theta + 6 r^2 \cos^2 \theta - r^2 + 4 i a r \cos \theta \right) \Lambda / 3 \\
+ \left( a \cos \theta - ir \right)^2 \frac{d^2}{dr^2} Q + 6 \left( i a \cos \theta + r \right) \frac{d}{dr} Q - 12 Q \\
-8 i a r \cos \theta - 2 a^2 \cos^2 \theta + 12 a^2 + 2 r^2, \tag{88}
\]

while in its dependence in the function \( K(r) \) simplifies considerable

\[
12 \left( a \cos \theta + ir \right) \left( ir - a \cos \theta \right)^3 \Psi_{2K} \\
= 12 \left( -i a \cos \theta + r \right) m + \left( a \cos \theta - ir \right)^2 \frac{d^2}{dr^2} K + 6 \left( i \cos \theta a + r \right) \frac{d}{dr} K - 12 K. \tag{89}
\]

For the cubic solution, this \( \Psi_2 \) quantity becomes

\[
-12 \left( i r - a \cos \theta \right)^3 \left( i r + a \cos \theta \right) \Psi_2(\beta) = 6 \kappa F_\theta \\
-12 m \left( r - i a \cos \theta \right) - 6 a^2 \kappa F_\theta \beta^2 r^2 \cos^2 \theta \\
+ \left( 2 a^2 \cos^2 \theta - 2 r^2 + 8 i a r \cos \theta \right) \kappa F_\theta \beta. \tag{90}
\]

The traceless Ricci tensor eigenvalue \( S = S^1_{11} = 2 \Phi^{(11)} \), in terms of the auxiliary function \( K \), amounts to

\[
S^1_{11} = \frac{1}{2 \rho^4} \left( \rho^2 \frac{d^2}{dr^2} K - 4 r \frac{d}{dr} K + 4 K \right), \tag{91}
\]
which for the solution under consideration results in

\[ 2 \rho^4 S(\beta) = \kappa F_0 \beta \left( 3 a^2 \beta r^2 \cos^2 \theta - a^2 \cos^2 \theta + r^2 \right) + \kappa F_0. \]  

(92)

Finally, the scalar curvature is given by

\[ R(\beta) = -2 \frac{\Lambda a^2}{3 \rho^2} (6 \cos^2 \theta - 1) - \frac{Q'' - 2}{\rho^2} = -4 \Lambda - \frac{K''}{\rho^2} = 2 \frac{\kappa F_0 \beta (1 - 3 \beta r^2)}{a^2 \cos^2 \theta + r^2} - 4 \Lambda. \]  

(93)

**XI. ENERGY CONDITIONS**

Remarkable is the simplicity and the invariant character of the energy conditions which hold for any non linear electrodynamics energy–momentum tensor of the studied class referred to the eigenvector orthonormal tetrad frame;

\[ \mu + p_\theta = \frac{2}{\kappa} S \geq 0, \mu - p_\theta = \frac{1}{2\kappa} R = -\frac{1}{2} T^\alpha_a \geq 0, \]

\[ \mu + p_r = 0, \mu = T_{\mu \nu} u^\mu u^\nu \geq 0. \]  

(94)

On the other hand, the local energy flow vector \( q^a = T^{ab} u_b = [0, 0, 0, -\kappa \mu(\theta, r)] \) is always a timelike vector except when the energy density vanishes, since the norm of the flow vector is \( q^a q_a = -\kappa^2 \mu(\theta, r)^2 \).

For our solution these quantities require the scalar curvature \( R(\beta) \geq 0 \), and as well \( S(\beta) = S^1(\beta) \geq 0 \).

The energy conservation \( T^{\mu \nu} u_\nu = 0 \) is encoded in the Einstein equations. The energy density \( \mu \) measured by an observer with 4–velocity, \( u^\mu, u^\mu u_\mu = -1 \) is defined as \( \mu(x^\alpha) = T_{\nu \mu} u^\mu u^\nu; \) for the energy–tensor refereed to the orthonormal tetrad, it occurs to be:

\[ \kappa \mu(\theta, r) := OE_{44} = -\frac{\Lambda a^2}{3 \rho^4} \left( 3 a^2 \cos^4 \theta + 6 r^2 \cos^2 \theta - r^2 \right) \]

\[ -\frac{1}{\rho^4} \left( r Q' - Q + a^2 - r^2 \right) = -\Lambda - \frac{r K' - K}{\rho^4}. \]  

(95)

Isolating the derivative \( \frac{d}{dr} Q(\theta) \),

\[ \frac{dQ}{dr} = \frac{1}{r} \left( Q + r^2 - a^2 \right) - \frac{\Lambda a^2}{r} \left( 3 a^2 \cos^4 \theta + 6 r^2 \cos^2 \theta - r^2 \right) - \frac{\rho^4}{r} \kappa \mu(\theta, r), \]  

(96)

and substituting it into the equation \( OE_{11} \), one arrives at the relation

\[ OE_{11} = -\frac{\rho^2}{2 r} \frac{\partial}{\partial r} \kappa \mu(\theta, r) - \kappa \mu(\theta, r) \]  

(97)
therefore, identifying $OE_{11} =: \kappa p_\theta$, one has

$$2S = OE_{11} + OE_{44} = \kappa p_\theta (\theta, r) + \kappa \mu (\theta, r) = -\kappa \frac{\rho^2}{2r} \frac{\partial}{\partial r} \mu (\theta, r) \geq 0 \quad (98)$$

On the other hand from the trace of the Einstein tensor one gets

$$E^\mu_\mu = OE_a^a = OE^1_1 + OE^2_2 + OE^3_3 + OE^4_4 = 2OE_{11} - 2OE_{44} = -R = \kappa T + 4\Lambda \quad (99)$$

where it has been taking into account $OE_{22} = -OE_{44}$, and $OE_{33} = OE_{11}$. Thus by the definitions $OE_{11} =: \kappa p_\theta$ and $OE_{44} =: \kappa \mu$ on arrives at

$$\kappa \left( \mu (\theta, r) - p_\theta (\theta, r) \right) = \frac{1}{2} R. \quad (100)$$

For the solution under consideration one gets:

the energy density $\mu(\theta, r)$,

$$\kappa \mu(\theta, r) = -\Lambda - \frac{F_0 \kappa (3 \beta r^2 + 1) (\beta r^2 - 1)}{2\rho^4} \geq 0, \quad (101)$$

the scalar curvature $R$,

$$R = -4\Lambda - 2 \frac{\kappa F_0 \beta (3 \beta r^2 - 1)}{\rho^2} \geq 0, \quad (102)$$

the traceless Ricci tensor eigenvalue $S$,

$$S^I_1 (\theta, r) = \frac{\kappa F_0}{\rho^4} \left( 3 \cos^2 \theta a^2 \beta^2 r^2 - \cos^2 \theta a^2 \beta + r^2 \beta + 1 \right) \geq 0, \quad (103)$$

finally the pressure $p_\theta (\theta, r) \geq 0$,

$$\kappa p_\theta (\theta, r) = OE_{11} = \frac{1}{2\rho^4} \kappa F_0 \left( 6 a^2 \cos^2 \theta \beta^2 r^2 + 3 \beta^2 r^4 - 2 a^2 \cos^2 \theta \beta + 1 \right). \quad (104)$$

**XII. FINAL REMARKS**

In this work we presented in detail the derivation of the stationary axisymmetric black hole solution to the Einstein equations coupled to nonlinear electrodynamics in the presence of a cosmological constant of both signs. This solution possesses mass $m$, rotation parameter $a$, electric and magnetic charges $e$ and $g_0$, nonlinear electrodynamics parameter $\beta$, and two parameters $F_0$ and $G_0$ associated to the presence of the invariants $F$ and $G$ in the Lagrangian $L$, and finally a cosmological constant $\Lambda$ for de Sitter and Anti de Sitter branches of solution.
In the forthcoming works of the series we deal mostly with the physical interpretation of the solution: the study the trajectories of neutral and charged test particles, the trajectories of light rays, the birefringence of light, the black hole properties of the solution; horizons, maximal extension, Penrose diagrams, and thermodynamics, among others.

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