Global symmetric classical and strong solutions of the full compressible Navier-Stokes equations with vacuum and large initial data

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Abstract

First of all, we get the global existence of classical and strong solutions of the full compressible Navier-Stokes equations in three space dimensions with initial data which is large and spherically or cylindrically symmetric. The appearance of vacuum is allowed. In particular, if the initial data is spherically symmetric, the space dimension can be taken not less than two. The analysis is based on some delicate \textit{a priori} estimates globally in time which depend on the assumption $\kappa = O(1 + \theta^q)$ where $q > r$ ($r$ can be zero), which relaxes the condition $q \geq 2 + 2r$ in \cite{14, 29, 42}. This could be viewed as an extensive work of \cite{18} where the equations hold in the sense of distributions in the set where the density is positive with initial data which is large, discontinuous, and spherically or cylindrically symmetric in three space dimension. Finally, with the assumptions that vacuum may appear and that the solutions are not necessarily symmetric, we establish a blow-up criterion in terms of $\|\rho\|_{L^\infty_t L^\infty_x}$ and $\|\rho\theta\|_{L^4_t L^{12}_x}$ for strong solutions.

Key Words: Compressible Navier-Stokes equations, heat conducting fluids, vacuum, global classical and strong solutions, blow-up criterions.

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1 Introduction

The full compressible Navier-Stokes equations can be written in the sense of Eulerian coordinates in \( \Omega \subset \mathbb{R}^N \) as follows:

\[
\begin{cases}
\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \text{div}(T) + \rho \mathbf{f}, \\
(\rho E)_t + \text{div}(\rho E \mathbf{u}) + \text{div}(P \mathbf{u}) = \text{div}(T \mathbf{u}) + \text{div}(\kappa \nabla \theta) + \rho \mathbf{u} \cdot \mathbf{f}.
\end{cases}
\]

Here \( T \) is the stress tensor given by

\[
T = \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})') + \lambda \text{div} \mathbf{u} I_N,
\]

where \( I_N \) is an \( N \times N \) unit matrix; \( \rho = \rho(x, t), \mathbf{u} = \mathbf{u}(x, t) = (u_1, \ldots, u_N)(x, t) \) and \( \theta = \theta(x, t) \) are unknown functions denoting the density, velocity and absolute temperature, respectively; \( P = P(\rho, \theta), E, \mathbf{f} = \mathbf{f}(x, t) = (f_1, \ldots, f_N)(x, t) \) and \( \kappa \) denote respectively pressure, total energy, external forces and coefficient of heat conduction, where \( E = e + \frac{1}{2} \rho \mathbf{u}^2 \) (\( e \) is the internal energy); \( \mu \) and \( \lambda \) are coefficients of viscosity, satisfying the following physical restrictions:

\[
\mu > 0, \ 2\mu + N\lambda \geq 0;
\]

\( P \) and \( e \) satisfy the second principle of thermodynamics:

\[
P = \rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial P}{\partial \theta}.
\]

(1.1) is a well-known model which describes the motion of compressible fluids. There were lots of works on the existence, uniqueness, regularity and asymptotic behavior of the solutions during the last five decades. While, because of the stronger nonlinearity in (1.1) compared with the Navier-Stokes equations for isentropic flow (no temperature equation), many known mathematical results focused on the absence of vacuum (vacuum means \( \rho = 0 \)), refer for instance to [21, 22, 29, 30, 34, 35, 40] for classical solutions. More precisely, the local classical solutions to the Navier-Stokes equations with heat-conducting fluid in Hölder spaces was obtained respectively by Itaya in [21] for Cauchy problem and by Tani in [40] for IBVP with \( \inf \rho_0 > 0 \), where the space dimension \( N = 3 \). Using delicate energy methods in Sobolev spaces, Matsumura and Nishida in [34, 35] showed that the global classical solutions exist provided that the initial data is small in some sense and away from vacuum in three space dimension. For large initial data in one space dimension, Kazhikhov, Shelukhin in [30] (for polytropic perfect gas with \( \mu, \lambda, \kappa = \text{const.} \)) and Kawohl in [29] (for real gas with \( \kappa = \kappa(\rho, \theta), \mu, \lambda = \text{const.} \)) respectively got the global classical solutions to (1.1) in Lagrangian coordinates with \( \inf \rho_0 > 0 \). The internal energy \( e \) and the coefficient of heat conduction \( \kappa \) in [29] satisfy the following assumptions for \( \rho \leq \overline{\rho} \) and \( \theta \geq 0 \) (we translate these conditions in Eulerian coordinates)

\[
\begin{cases}
\rho(\rho, 0) \geq 0, \ \nu(1 + \theta^r) \leq \partial_{\theta} e(\rho, \theta) \leq N(\overline{\rho})(1 + \theta^r), \\
\kappa_0(1 + \theta^q) \leq \kappa(\rho, \theta) \leq \kappa_1(1 + \theta^q), \\
|\partial_{\rho} \kappa(\rho, \theta)| + |\partial_{\rho \rho} \kappa(\rho, \theta)| \leq \kappa_1(1 + \theta^q),
\end{cases}
\]

(1.3)
where $r \in [0,1]$, $q \geq 2 + 2r$, and $\nu$, $N(\overline{\gamma})$, $\kappa_0$ and $\kappa_1$ are positive constants. For the perfect gas (i.e., $P = R \rho \theta$, $e = C_v \theta$ for some constants $R > 0$ and $C_v > 0$) in the domain exterior to a ball in $\mathbb{R}^N$ ($N = 2$ or 3) with $\mu, \lambda, \kappa = \text{const.}$, Jiang in [22] got the existence of global spherically symmetric classical large solutions in Hölder spaces.

In fact, Kawohl in [29] also considered the case of density-dependent viscosity for another boundary condition with $\inf \rho_0 > 0$, where $0 < \mu_0 \leq \mu(\rho) \leq \overline{\mu}$ for any $\rho > 0$, and $\mu_0$ and $\overline{\mu}$ are positive constants. This was generalized to the case $\mu(\rho) = \rho^\alpha$ by Jiang in [23] for $\alpha \in (0, \frac{1}{2})$, and by Qin, Yao in [36] for $\alpha \in (0, \frac{1}{3})$, respectively.

On the existence, asymptotic behavior of the weak solutions of the full compressible Navier-Stokes equations with $\inf \rho_0 > 0$, please refer for instance to [24, 25, 28] for the existence of weak solutions in 1D and for the existence of spherically symmetric weak solutions in $\mathbb{R}^N$ ($N = 2, 3$), and refer to [18] for the existence of spherically and cylindrically symmetric weak solutions in $\mathbb{R}^3$, and refer to [13] for the existence of variational solutions in a bounded domain in $\mathbb{R}^N$ ($N = 2, 3$).

In the presence of vacuum (i.e. $\rho$ may vanish), to our best knowledge, the mathematical results on global well-posedness of the full compressible Navier-Stokes equations are usually limited to the existence of weak solutions with special pressure, viscosity and heat conductivity (see [1, 14]). More precisely, Feireisl in [14] got the existence of so-called variational solutions in dimension $N \geq 2$. The temperature equation in [14] is satisfied only as an inequality in the sense of distributions. Anyhow, Feireisl’s work is the very first attempt towards the existence of weak solutions to the full compressible Navier-Stokes equations in higher dimensions, where the coefficients of viscosity are constants and

\[
\begin{align*}
  &\kappa = \kappa(\theta) \in C^2[0, \infty), \quad \kappa(1 + \theta^q) \leq \kappa(\theta) \leq \overline{\kappa}(1 + \theta^q) \quad \text{for all } \theta \geq 0, \\
  &P = P(\rho, \theta) = \rho e(\rho) + \theta P_\theta(\rho) \quad \text{for all } \rho \geq 0 \text{ and } \theta \geq 0, \\
  &\rho e, \rho P_\theta \in C[0, \infty) \cap C^1(0, \infty); \quad \rho e(0) = 0, \quad \rho P_\theta(0) = 0, \\
  &P'_{e}(\rho) \geq a_1 \rho^{7-1} - b_1 \quad \text{for all } \rho > 0; \quad \rho e(\rho) \leq a_2 \rho^{r_\Gamma} + b_1 \quad \text{for all } \rho \geq 0, \\
  &P_\theta \text{ is non-decreasing in } [0, \infty); \quad P_\theta(\rho) \leq a_3(1 + \rho^{r_\Gamma}) \quad \text{for all } \rho \geq 0,
\end{align*}
\]

where $\Gamma < \frac{7}{3}$ if $N = 2$ and $\Gamma = \frac{7}{3}$ for $N \geq 3$; $q \geq 2$, $r_\Gamma > 1$, and $a_1, a_2, a_3, b_1, \kappa$ and $\overline{\kappa}$ are positive constants. Note that the perfect gas equation of state (i.e. $P = R \rho \theta$ for some constant $R > 0$) is not involved in (1.4). In order that the equations are satisfied as equalities in the sense of distribution, Bresch and Desjardins in [11] proposed some different assumptions from [14], and obtained the existence of global weak solutions to the full compressible Navier-Stokes equations with large initial data in $\mathbb{T}^3$ or $\mathbb{R}^3$. In [11], the viscosity $\mu = \mu(\rho)$ and $\lambda = \lambda(\rho)$ may vanish when vacuum appears, and $\kappa$, $P$ and $e$ are assumed to be satisfied

\[
\begin{align*}
  &\kappa(\rho, \theta) = \kappa_0(\rho, \theta)(\rho + 1)(\theta^q + 1), \\
  &P = R \rho \theta + p_c(\rho), \\
  &e = C_v \theta + e_c(\rho),
\end{align*}
\]

where $q \geq 2$, $R$ and $C_v$ are two positive constants, $p_c(\rho) = O(\rho^{-\ell})$ and $e_c(\rho) = O(\rho^{-\ell-1})$ (for some $\ell > 1$) when $\rho$ is small enough, and $\kappa_0(\rho, \theta)$ is assumed to satisfy

\[
\omega_0 \leq \kappa_0(\rho, \theta) \leq \frac{1}{\omega_0},
\]

for $\omega_0 > 0$. On the local existence and uniqueness of strong solutions for $N = 3$, please see [11] for the perfect gas with $\mu$, $\lambda$, $\kappa = \text{const.}$ While, there are no global smooth solutions to (1.1) for Cauchy problem when the initial density is of nontrivially compact support and $\kappa = 0$ (see [11]).
Except for the Cauchy problems with initial density compactly supported, it is still unknown whether the global strong (or classical) solutions exist or not when vacuum appears (i.e., the density may vanish) until recently. In our previous paper [12], we got existence and uniqueness of global classical solutions to the full compressible Navier-Stokes equations in one dimension with large initial data and vacuum. In [12], the coefficient of conduction $k$ depends on the temperature, growing as $1 + \theta^q$ where $q \geq 2 + 2r$ ($r$ can be zero).

As a first step to study the problems ($1.1$) in high dimensions, we study the problems in high dimensions with some symmetry which reduces the whole system to an one dimensional system with singular source terms. The singularity may be due to $x = 0$, $x = \infty$ or appearance of vacuum, where $x$ is the radius. Our main concern here is to show the existence and uniqueness of global classical and strong solutions to ($1.1$) with vacuum and initial data which is large, and spherically or cylindrically symmetric in three space dimension. In particular, if the initial data is spherically symmetric, the space dimension can be taken not less than two. This extends the results in [18], where the equations hold in the sense of distributions in the set where the density is positive with initial data which is large, discontinuous, and spherically or cylindrically symmetric in three space dimension. Besides, when the solutions are not necessarily symmetric, we shall establish a blow-up criterion for strong solutions with vacuum.

For compressible isentropic Navier-Stokes equations (i.e. no temperature equation), there are so many results about the well-posedness and asymptotic behaviors of the solutions when vacuum appears. Refer to [15, 27, 31, 33] and [17, 32, 41, 46, 47, 48] for global weak solutions with constant viscosity and with density-dependent viscosity, respectively. Refer to [6, 10] and [2, 3, 5, 37] for global strong solutions and for local strong (classical) solutions with constant viscosity, respectively. Recently, Huang, Li, Xin in [20] and Ding, Wen, Yao, Zhu in [8, 7] independently got existence and uniqueness of global classical solutions, where the initial energy in [20] is assumed to be small in $\mathbb{R}^3$ and $\rho - \bar{\rho} \in C([0, T]; H^3(\mathbb{R}^3))$, $u \in C([0, T]; D^1(\mathbb{R}^3) \cap D^2(\mathbb{R}^3)) \cap L^\infty([\tau, T]; D^4(\mathbb{R}^3))$ (for $\tau > 0$) which generalized the results in [3], and the initial data in [8, 7] could be large for dimension $N = 1$ and could be large but spherically symmetric for $N \geq 2$, and $(\rho, u) \in C([0, T]; H^4(I))$ ($I$ is bounded in $\mathbb{S}$, and is bounded or an exterior domain in $\mathbb{T}$).

We would like to give some notations which will be used throughout the paper.

Notations:

(i) $I = [a, b]$; $Q_T = I \times [0, T]$ for $T > 0$.

(ii) $\int_{\Sigma} f = \int_{\Sigma} f \, dx$, for $\Sigma = I$ or $\mathbb{R}^3$.

(iii) For $1 \leq l \leq \infty$, denote the $L^l$ spaces and the standard Sobolev spaces as follows:

\[ L^l = L^l(\Sigma), \quad D^{k,l} = \{ u \in L^1_{\text{loc}}(\Sigma) : \| \nabla^k u \|_{L^l} < \infty \}, \]

\[ W^{k,l} = L^l \cap D^{k,l}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2}; \]

\[ D_0^l = \{ u \in L^6 : \| \nabla u \|_{L^2} < \infty \}, \]

\[ \| u \|_{D^{k,l}} = \| \nabla^k u \|_{L^l}. \]
Throughout Section 2.1 and the proofs of the main theorems in the section, we take $\Sigma = I$ in the Notations.

### 2.1 Global symmetric classical and strong solutions with vacuum

Throughout Section 2.1 and the proofs of the main theorems in the section, we take $\Sigma = I$ in the Notations. For simplicity, we assume that the external force $f = 0$. Assume $\Omega = \{x|a < |x| < b\}$, for $0 < a < b < \infty$. Then for the symmetric cases, (2.1.1) takes the form:

\[
\begin{cases}
\rho_t + (\rho u)_x + \frac{\rho u^2}{x} = 0, & \rho \geq 0, \ a < x < b, \ t > 0, \\
p\mu_t + p\mu_u x - \frac{\rho u^2}{x} + P_x = \beta(u_{xx} + \frac{m\mu_x}{x} - \frac{m\mu}{x^2}), \\
p\nu_t + p\nu_u x = \mu(v_{xx} + \frac{m\nu_x}{x} - \frac{m\nu}{x^2}), \\
p\nu_t + p\nu_u x = \mu(w_{xx} + \frac{m\nu_x}{x}), \\
p\nu_t + p\nu_u x = (\kappa\theta_x)_x + \frac{m\kappa\theta_x}{x} + \varphi,
\end{cases}
\]

where $\beta = 2\mu + \lambda$, $\varphi = \lambda(u_{xx} + \frac{m\mu}{x})^2 + \mu \left( \frac{w_x^2}{x} + 2u_x^2 + (v_x - \frac{m\nu}{x})^2 + 2\frac{m\nu^2}{x^2}\right)$. In the spherically symmetric case, $m = N - 1$, $x = |x|$, $u(x, t) = u(x(t)\frac{t}{x})$ and $v = w = 0$. In the cylindrically symmetric case, $m = 1$, $x = \sqrt{x_1^2 + x_2^2}$, and

\[
u(x, t) = u(x, t) \frac{(x_1, x_2, 0)}{x} + v(x, t) \frac{(-x_2, x_1, 0)}{x} + w(x, t)(0, 0, 1).
\]

We consider the initial and boundary conditions:

\[
(\rho, u, v, w, \theta) \big|_{t=0} = (\rho_0, u_0, v_0, w_0, \theta_0)(x) \text{ in } I,
\]

and

\[
(u, v, w, \theta) \big|_{x=a,b} = 0, \ t \geq 0.
\]

### 2.1.1 Assumptions

$(A_1)$: $\rho_0 \geq 0$, $\int f \rho_0 > 0$.

$(A_2)$: $\mu$ and $\lambda$ are constants. $\mu > 0$, $2\mu + (m+1)\lambda > 0$. $e = C_0 Q(\theta) + e_c(\rho)$, $P = \rho Q(\theta) + P_c(\rho)$, $\kappa = \kappa(\theta)$, for some constant $C_0 > 0$. The constant $C_0$ plays no role in the analysis, we assume...
$C_0 = 1.$

(A$_3$): $P_c(\rho) \geq 0$, $e_c(\rho) \geq 0$, for $\rho \geq 0$; $P_c \in C^2[0, \infty)$; $\rho|\frac{\partial e_c}{\partial \rho}| \leq C_1 e_c(\rho)$, for some constant $C_1 > 0$.

(A$_4$): $Q(\cdot) \in C^2[0, \infty)$ satisfies
\[
\begin{cases}
C_2(\beta + (1 - \beta)\theta + \theta^{1+r}) \leq Q(\theta) \leq C_3(\beta + (1 - \beta)\theta + \theta^{1+r}), \\
C_4(1 + \theta^q) \leq Q'(\theta) \leq C_5(1 + \theta^q),
\end{cases}
\]
for some constants $C_i > 0$ ($i = 2, 3, 4, 5$) and $r \geq 0$, $\beta = 0$ or 1.

(A$_5$): $\kappa \in C^2[0, \infty)$ satisfies
\[C_6(1 + \theta^q) \leq \kappa(\theta) \leq C_7(1 + \theta^q),\]
for $q > r$, and some constants $C_i > 0$ ($i = 6, 7$).

### 2.1.2 Global strong solutions

**Theorem 2.1.1** (Strong solutions) In addition to (A$_1$)-(A$_5$), we assume $\rho_0 \geq 0$, $\rho_0 \in H^2$, $u_0 \in H^2 \cap H^1_0$, $\theta_0 \in H^2$, $\partial_x \theta_0|_{x=0,1} = 0$, and that the following compatibility conditions are valid:
\[
\begin{align*}
\beta(u_{0xx} + \frac{mv_{0x}}{x} - \frac{mv_0}{x^2}) - P_x(\rho_0, \theta_0) &= \sqrt{\rho_0} g_1, \\
\mu(v_{0xx} + \frac{mu_{0x}}{x} - \frac{mu_0}{x^2}) &= \sqrt{\rho_0} g_2, \\
\mu(w_{0xx} + \frac{mw_{0x}}{x}) &= \sqrt{\rho_0} g_3, \\
(\kappa(\theta_0) \theta_{0x})_x + \frac{m\kappa(\theta_0) \theta_{0x}}{x} + \varphi(x,0) &= \sqrt{\rho_0} g_4, \quad x \in I,
\end{align*}
\]
for some $g_i \in L^2$, $i = 1, 2, 3, 4$. Then there exists a unique global solution $(\rho, u, v, w, \theta)$ to (2.1.1)-(2.1.3) such that for any $T > 0$
\[
\rho \in C([0,T]; H^2), \quad (u, v, w, \theta) \in C([0,T]; H^2) \cap L^2([0,T]; H^3),
\]
\[\sqrt{\rho} u_t, \sqrt{\rho} v_t, \sqrt{\rho} w_t, \sqrt{\rho} e_t \in L^\infty([0,T]; L^2), \quad (u_t, v_t, w_t) \in L^2([0,T]; H^1_0), \quad \theta_t \in L^2([0,T]; H^1).
\]

**Remark 2.1.2** From the assumptions (A$_2$)-(A$_4$), we know that the polytropic perfect gas (i.e., $P = R\rho \theta$, $\varepsilon = C_4 \theta$ for some constants $R > 0$ and $C_\varepsilon > 0$) is included if we take $r = \beta = 0$ and $e_c(\rho) = P_c(\rho) = 0$ and $Q = \frac{C_\varepsilon \theta}{C_0}$.

**Remark 2.1.3** The global existence of strong solutions depends on $q > r$ in our analysis. For the polytropic perfect gas, Theorem 2.1.1 works for any $q > 0$, which relaxes the restriction $q \geq 2$ in [2, 23, 42].

**Remark 2.1.4** Some similar compatibility conditions as (2.1.4) can be referred to [4] and references therein. In [4], the local $H^2$-regularity of $u$ and $\theta$ for the polytropic perfect gas was obtained. The detailed reasons why such conditions were needed can be found in [4]. Roughly speaking, $g_1$, $g_2$, $g_3$ and $g_4$ are equivalent to $\sqrt{\rho} u_t$, $\sqrt{\rho} v_t$, $\sqrt{\rho} w_t$ and $\sqrt{\rho} e_t$ at $t = 0$, respectively.

**Remark 2.1.5** From the derivation of the Navier-Stokes equations from the Boltzmann equation through the Chapman-Enskog expansion to the second order (see [23] and references therein), we know that $\mu = \mu(\theta)$, $\lambda = \lambda(\theta)$ and $\kappa = \kappa(\theta)$. As in [14, 29, 42], $\mu$ and $\lambda$ are assumed to be constants here, because of the restrictions of mathematical technique.
2.1.3 Global classical solutions

**Theorem 2.1.6** (Classical solutions) In addition to (A1)-(A5), we assume \( \rho_0 \geq 0, \rho_0 \in H^3, (\sqrt{\rho_0})_x \in L^\infty, u_0 \in H^3 \cap H^1_0, \theta_0 \in H^3, \partial_x \theta_0|x=0 \equiv 0, (Q, P_c, \kappa) \in W^{3, \infty}, and that the compatibility conditions (2.1.4) are satisfied for some \( g_i \in L^2, i = 1, 2, 3, 4, and \sqrt{\rho_0}g_j \in H^1_j, j = 1, 2, 3, \) and \( \sqrt{\rho_0}g_j \in H^2. \) Then there exists a unique global solution \((\rho, u, v, w, \theta)\) to (2.1.1)-(2.1.3) such that for any \( T > 0 \)

\[
\begin{align*}
\rho & \in C([0,T];H^3), (u,v,w,\theta) \in L^\infty([0,T];H^3) \cap L^2([0,T];H^4), \\
(\sqrt{\rho_0}u_t, \sqrt{\rho_0}v_t, \sqrt{\rho_0}w_t) & \in L^\infty([0,T];L^2), (pu_t, pv_t, pw_t) \in L^\infty([0,T];H^1_0), \\
(u_t, v_t, w_t) & \in L^2([0,T];H^1), \theta_t \in L^2([0,T];H^1), ,\rho_0 \gamma_t \in L^\infty([0,T];H^1).
\end{align*}
\]

**Remark 2.1.7** If \( \rho_0 \in H^4 \) and \((u_0, v_0, w_0)\) satisfies a stronger compatibility conditions

\[
\begin{align*}
\beta(u_{0xx} + \frac{mu_{0x}}{x} - \frac{mu_0}{x^2}) - P_2(\rho_0, \theta_0) &= \rho_0 \tilde{g}_1, \\
\mu(v_{0xx} + \frac{mv_{0x}}{x} - \frac{mv_0}{x^2}) &= \rho_0 \tilde{g}_2, \\
\mu(w_{0xx} + \frac{mw_{0x}}{x}) &= \rho_0 \tilde{g}_3, \quad x \in I,
\end{align*}
\]

for some \( \tilde{g}_i \in H^2_1 \) and \((\sqrt{\rho_0}g_1)_x \in L^2 \), we can obtain by using the similar arguments as in (4.2) that \( \rho \in C([0,T];H^4) \) and \((u, v, w) \in C([0,T];H^4) \cap L^2([0,T];H^5) \).

2.2 A blow-up criterion in terms of \( \|\rho\|_{L^\infty_t L^1_x} \) and \( \|\rho^\theta\|_{L^4_t L^2_x} \) for strong solutions

Throughout Section 2.2 and the proofs of the main theorems in the section, we take \( \Sigma = \mathbb{R}^3 \) in the **Notations.** In order to establish some sharp blow-up criterions, we only consider that \( \kappa = \text{constant} \), and that the state equations of \( P \) and \( e \) is of ideal polytropic gas type: \( P = a \rho \theta, e = C_0 \theta \), where \( a \) and \( C_0 \) are two positive constants. The constants \( a, C_0 \) and \( \kappa \) in the equations play no roles in the section, we assume \( a = C_0 = \kappa = 1 \). If the solutions are regular enough (such as strong solutions), (1.1) is equivalence to the following system:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla P(\rho, \theta) &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u, \\
\rho \theta_t + \rho u \cdot \nabla \theta + \rho \theta \text{div} u &= \frac{k}{2} \left| \nabla u + (\nabla u)^t \right|^2 + \lambda (\text{div} u)^2 + \Delta \theta, \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

System (2.2.1) is supplemented with initial conditions

\[
(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \quad x \in \mathbb{R}^3,
\]

with

\[
\rho(x, t) \to 0, \quad u(x, t) \to 0, \quad \theta(x, t) \to 0, \quad \text{as } |x| \to \infty, \quad \text{for } t \geq 0.
\]

We give the definition of strong solutions to (2.2.1)-(2.2.3) throughout Section 2.2 and the proofs of main theorem in the section.

**Definition 2.2.1** (Strong solution) For \( T > 0 \), \((\rho, u, \theta)\) is called a strong solution to the compressible Navier-Stokes equations (2.2.1)-(2.2.3) in \( \mathbb{R}^3 \times [0,T] \), if for some \( q \in (3,6], \)

\[
0 \leq \rho \in C([0,T];W^{1,q} \cap H^1 \cap L^1), \rho_t \in C([0,T];L^2 \cap L^q), \\
(u, \theta) \in C([0,T];L^2 \cap D_0^1) \cap L^2(0,T;D^2,q), \quad (u_t, \theta_t) \in L^2(0,T;D_0^1), \\
(\sqrt{\rho u_t}, \sqrt{\rho \theta_t}) \in L^\infty(0,T;L^2),
\]

and \((\rho, u, \theta)\) satisfies (2.2.1) a.e. in \( \mathbb{R}^3 \times (0,T] \).
We present our main theorem, which is on a blow-up criterion for strong solutions to \((2.2.1)-(2.2.3)\), as follows:

**Theorem 2.2.2** Assume \(\rho_0 \geq 0\), \(\rho_0 \in H^1 \cap W^{1,q} \cap L^1\), for some \(q \in (3,6]\), \((u_0, \theta_0) \in D^2 \cap D_0^1\), and the following compatibility conditions are satisfied:

\[
\begin{align*}
\mu \Delta u_0 + (\mu + \lambda) \nabla \text{div} u_0 - \nabla P(\rho_0, \theta_0) &= \sqrt{\rho_0} g_1, \\
\kappa \Delta \theta_0 + \frac{4}{3} |\nabla u_0 + (\nabla u_0)|^2 + \lambda (\text{div} u_0)^2 &= \frac{\sqrt{\rho_0}}{g_2}, \quad x \in \mathbb{R}^3,
\end{align*}
\]

(2.2.4)

for some \(g_i \in L^2, \ i = 1,2\). Let \((\rho, u, \theta)\) be a strong solution to \((2.2.1)-(2.2.3)\) in \(\mathbb{R}^3 \times [0,T]\). If \(0 < T^* < +\infty\) is the maximum time of existence of the strong solution, then

\[
\limsup_{T \nearrow T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\rho \theta\|_{L^4(0,T;L^{\frac{12}{7}})} \right) = \infty,
\]

(2.2.5)

provided \(3\mu > \lambda\).

**Remark 2.2.3** Under the conditions of Theorem 2.2.2, the local existence of the strong solutions was obtained in [4]. Thus, the assumption \(T^* > 0\) makes sense.

**Remark 2.2.4** For the ideal polytropic gas with \(\kappa = \text{constant}\), we noticed recently that Huang and Li in [19] got the global existence of classical and weak solutions to Cauchy problem of \((1.1)\) in \(\mathbb{R}^3\) with small initial energy and non-vacuum state at infinity.

**Remark 2.2.5** Before Theorem 2.2.2, there have been several results on the blow-up criterions for strong solutions to \((2.2.1)\), please refer for instance to [11, 12, 39, 43] and references therein. More precisely,

- **Fan-Jiang-Ou** ([11], 3D)
  \[
  \limsup_{T \nearrow T^*} \left( \|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)} \right) = \infty,
  \]
  (2.2.6)
  provided \(7\mu > \lambda\). Here the appearance of vacuum is allowed.

  It is well-known that the bound of \(\|\nabla u\|_{L^1(0,T;L^\infty)}\) yields that \(\|\rho\|_{L^\infty(0,T;L^\infty)}\) is bounded (see (2.2) in [11]), if the initial density is bounded. When \(\|\nabla u\|_{L^1(0,T;L^\infty)}\) in (2.2.6) is relaxed by the upper bound of the density, the following blow-up criterions were obtained:

  - **Fang-Zi-Zhang** ([12], 2D)
    \[
    \limsup_{T \nearrow T^*} \left( \|\theta\|_{L^\infty(0,T;L^\infty)} + \|\rho\|_{L^\infty(0,T;L^\infty)} \right) = \infty,
    \]
    (2.2.7)
    where the appearance of vacuum is allowed;

  - **Sun-Wang-Zhang** ([39], 3D)
    \[
    \limsup_{T \nearrow T^*} \left( \|\theta\|_{L^\infty(0,T;L^\infty)} + \|\rho\|_{L^\infty(0,T;L^\infty)} + \frac{1}{\rho} \|\theta\|_{L^\infty(0,T;L^\infty)} \right) = \infty,
    \]
    (2.2.8)
    provided \(7\mu > \lambda\). Here the appearance of vacuum is not allowed.

  - **Wen-Zhu** ([43], 3D)
    \[
    \limsup_{T \nearrow T^*} \left( \|\theta\|_{L^\infty(0,T;L^\infty)} + \|\rho\|_{L^\infty(0,T;L^\infty)} \right) = \infty,
    \]
    (2.2.9)
    provided \(3\mu > \lambda\). Here the appearance of vacuum is allowed.
Remark 2.2.6 Theorem 2.2.2 is an extension of our former results in [43] (see (2.2.9)). One of the main ingredients is that the estimates of \( \| \sqrt{\rho \theta} \|_{L^\infty(0,T;L^2)} \) and \( \| \nabla u \|_{L^\infty(0,T;L^2)} \) are done together, i.e.,

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} (\rho|\theta|^2 + |\nabla u|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho|u|^2) \, dx \, dt \leq C.
\]

In [43], \( \| \sqrt{\rho \theta} \|_{L^\infty(0,T;L^2)} \) and \( \| \sqrt{\rho u} \|_{L^\infty(0,T;L^2)} \) were done together, which needed the upper bounds of the temperature and the density.

3 Preliminaries

The lemmas in the section will be useful in the next two sections.

Lemma 3.1 ([42]) Let \( \Omega = [\tilde{a}, \tilde{b}] \) be a bounded domain in \( \mathbb{R} \), and \( \rho \) be a non-negative function such that

\[
0 < M \leq \int_{\Omega} \rho \leq K,
\]

for constants \( M > 0 \) and \( K > 0 \). Then

\[
\| v \|_{L^\infty(\Omega)} \leq \frac{K}{M} \| v_x \|_{L^1(\Omega)} + \frac{1}{M} \left| \int_{\Omega} \rho v \right|,
\]

for any \( v \in H^1(\Omega) \).

Remark 3.2 The version of higher dimensions for Lemma 3.1 can be found in [13] or [14].

Corollary 3.3 ([42]) Consider the same conditions in Lemma 3.1, and in addition assume \( \Omega = I \), and

\[
\| \rho v \|_{L^1(I)} \leq \overline{v}.
\]

Then for any \( l > 0 \), there exists a positive constant \( C = C(M,K,l,\overline{v}) \) such that

\[
\| v^l \|_{L^\infty(I)} \leq C \| (v^l)_x \|_{L^1(I)} + C,
\]

for any \( v^l \in H^1(I) \).

Lemma 3.4 (Poincaré inequality) For any \( v \in H^1_0(I) \), we have

\[
\| v \|_{L^\infty(I)} \leq \| v_x \|_{L^1}.
\]

Lemma 3.5 ([38]). Assume \( X \subset E \subset Y \) are Banach spaces and \( X \hookrightarrow \hookrightarrow E \). Then the following imbedding are compact:

(i) \( \left\{ \varphi : \varphi \in L^q(0,T;X), \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y) \right\} \hookrightarrow \hookrightarrow L^q(0,T;E), \) if \( 1 \leq q \leq \infty \);

(ii) \( \left\{ \varphi : \varphi \in L^\infty(0,T;X), \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y) \right\} \hookrightarrow \hookrightarrow C([0,T];E), \) if \( 1 < r \leq \infty \).
4 Proof of Theorem 2.1.1

In the section, we denote by $C$ a generic constant depending only on $\| (\rho_0, u_0, v_0, w_0, \theta_0) \|_{H^2}$, $\| g_i \|_{L^2} (i = 1, 2, 3, 4)$, $T$, $\lambda$, $\mu$, $a$, $b$, and some other known constants, but independent of the solutions and the lower bounds of the density. We denote by $A \lesssim B$ if there exists a generic constant $C$ such that $A \leq CB$.

The strategies on proving Theorem 2.1.1 are very classical. More precisely, we derive various a priori estimates for strong solutions of the Navier-Stokes equations (2.1.1)-(2.1.3), which are independent of positive lower bounds of the initial density. Then we shall construct a sequence of approximate initial data where the initial density has a lower bound $\varepsilon > 0$. With these a priori estimates uniform for $\varepsilon$, we take the limits $\varepsilon \to 0^+$.

From now on, for any $T > 0$, we shall derive some delicate a priori estimates for the strong solutions $(\rho, u, v, w, \theta)$ as in Theorem 2.1.1 with $\inf_{(x, t) \in Q_T} \rho > 0$. These energy estimates will be finished by five steps.

Step 1: Basic energy inequality

**Lemma 4.1** Under the conditions of Theorem 2.1.1, we have for any $t \in [0, T]$ \[
\int_I x^m \rho (1 + e + u^2 + v^2 + w^2) \leq C.
\]

**Proof.** This bound is standard and follow directly from the equations (1.1)\textsubscript{1}, (1.1)\textsubscript{3} and the boundary conditions. \qed

Step 2: Upper bound of density

**Lemma 4.2** Under the conditions of Theorem 2.1.1, we have \[
\| \rho \|_{L^\infty (Q_T)} \leq C.
\]

**Proof.** The idea of the proof is essentially that of a similar result of Frid, Shelukhin \textsuperscript{[16]} where $m = 1$, but with a slightly modification. We omit it for brevity. \qed

Step 3: $H^1$-estimates of $(\rho, u, v, w)$

The next lemma plays an important role in the paper, whose proofs are improved in contrast with \textsuperscript{[42]}. The condition $q > r$ instead of $q \geq 2 + 2r$ is enough here.

**Lemma 4.3** Under the conditions of Theorem 2.1.1, for $q > r$, and for any $0 < \alpha < \min \{ 1, q - r \}$, assume $2\mu + (m + 1)\lambda > 0$, we have \[
\int_{Q_T} x^m \frac{(1 + \theta^q) \theta_x^2}{\theta^{1+\alpha}} \leq C,
\]
where the generic constant $C$ depends on $\alpha$.

**Remark 4.4** The proofs of this lemma depend on the boundary condition $\theta_x|_{x=a,b} = 0$. 

Proof. From (1.2) and (2.1.1), we get
\[
\rho e_\theta \theta_t + \rho u e_\theta \theta_x + \theta P_\theta (u_x + \frac{mu}{x}) = (\kappa(\theta) \theta_x)_x + \frac{m \kappa \theta_x}{x} + \varphi. \tag{4.1}
\]
Substituting \(e = Q(\theta) + e_c(\rho)\) and \(P = \rho Q(\theta) + P_c(\rho)\) into (4.1), we get
\[
\rho Q'(\theta) \theta_t + \rho u Q'(\theta) \theta_x + \rho \theta Q'(\theta) (u_x + \frac{mu}{x}) = (\kappa(\theta) \theta_x)_x + \frac{m \kappa \theta_x}{x} + \varphi, \tag{4.2}
\]
or
\[
(\rho Q)_t + (\rho u Q)_x + \frac{m \rho u Q}{x} + \rho \theta Q'(\theta) (u_x + \frac{mu}{x}) = (\kappa(\theta) \theta_x)_x + \frac{m \kappa \theta_x}{x} + \varphi. \tag{4.3}
\]
Multiplying (4.2) by \(x^m \theta^{-\alpha}\), and integrating by parts over \(Q_T\), we have
\[
\int_{Q_T} x^m \left( \frac{\alpha \kappa(\theta) \theta_x^2}{\theta_1^{1+\alpha}} + \lambda (u_x + \frac{mu}{x})^2 + \frac{\mu [u_x^2 + 2u_x^2 + (v_x - \mu x)^2 + \frac{2mu^2}{x^2}]}{\theta^\alpha} \right) \, dx \, dt = \int_{Q_T} x^m \rho Q'(\theta) (u_x + \frac{mu}{x}) \, dx \, dt \tag{4.4}
\]
\[
\leq \int_{I_1} x^m \rho (1 + \theta^{1+r}) + \int_{I_1} x^m \rho (1 + \theta^{1+r}) \, dx \, dt \leq \int_{Q_T} x^m \rho \theta^{1-\alpha}(1 + \theta r) \, dx \, dt \tag{4.5}
\]
where we have used (A4) and Young inequality. Since \(\mu > 0\), we have from (4.4), (A2), (A4) and Lemma 4.1
\[
\alpha \int_{Q_T} x^m \frac{\kappa \theta_x^2}{\theta^{1+\alpha}} \leq C - \int_{Q_T} x^m \lambda (u_x + \frac{mu}{x})^2 + 2\mu [u_x^2 + \frac{mu^2}{x^2}] \frac{\theta^\alpha}{x^\alpha} + C \int_{Q_T} x^m \rho \theta^{1-\alpha}(1 + \theta r) \, dx \, dt = I_1 + I_2 + I_3. \tag{4.6}
\]
Without loss of generality, we assume \(\lambda < 0\). In fact, if \(\lambda \geq 0\), \(I_2\) is obviously a good term:
\[
I_2 \leq -2\mu \int_{Q_T} x^m \frac{u_x^2 + \frac{mu^2}{x^2}}{\theta^\alpha}.
\]
For \(\lambda < 0\), we use Cauchy inequality to get
\[
I_2 = -\int_{Q_T} x^m \frac{(2\mu + \lambda)u_x^2 + (2\mu + m\lambda)\frac{mu^2}{x^2} + 2m\lambda u_x}{\theta^\alpha} \tag{4.7}
\]
For \(I_3\), using Cauchy inequality again, we get
\[
I_3 \leq [2\mu + (m + 1)\lambda] \int_{Q_T} x^m \frac{u_x^2 + \frac{mu^2}{x^2}}{\theta^\alpha} + C \int_{Q_T} x^m \rho^2(1 + \theta^{2+2r-\alpha}) \tag{4.8}
\]
\[
\leq [2\mu + (m + 1)\lambda] \int_{Q_T} x^m \frac{u_x^2 + \frac{mu^2}{x^2}}{\theta^\alpha} + C \int_{T} \parallel \theta \parallel_{L^\infty}^{1+r-\alpha} \int_{I} x^m \rho \theta^{1+r} + C \tag{4.9}
\]
\[
\leq [2\mu + (m + 1)\lambda] \int_{Q_T} x^m \frac{u_x^2 + \frac{mu^2}{x^2}}{\theta^\alpha} + C \int_{T} \parallel \theta \parallel_{L^\infty}^{1+r-\alpha} + C.
\]

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Substituting (4.6) and (4.7) into (4.5), we have
\[ \alpha \int_{Q_T} x^m \frac{\kappa \theta^2}{\theta^{1+\alpha}} \leq C \int_0^T \| \theta \|_{L^{1+r-\alpha}}^2 + C. \] (4.8)

Now we estimate the first term of the right hand side of (4.8).

**Case 1:** \( r < q < 1 + 2r - \alpha \)

\[ C \int_0^T \| \theta \|_{L^{1+r-\alpha}}^2 \leq 1 + \int_0^T \| \theta^{r-\alpha} \theta_x \|_{L^2} \]
\[ \leq C + C \int_0^T \left( \int_I \frac{\theta^2 \theta^q}{\theta^{1+\alpha}} \theta^{2r-\alpha-1-q} \right) \frac{1}{2} \]
\[ \leq \frac{1}{4} \alpha \int_{Q_T} x^m \frac{\kappa \theta^2}{\theta^{1+\alpha}} + C \int_0^T \| \theta \|_{L^{1+r-\alpha}} + C \]
\[ \leq \frac{1}{4} \alpha \int_{Q_T} x^m \frac{\kappa \theta^2}{\theta^{1+\alpha}} + \frac{1}{2} C \int_0^T \| \theta \|^2_{L^{1+r-\alpha}} + C, \]
where we have used Corollary 3.3, (A5) and Young inequality.

This gives
\[ C \int_0^T \| \theta \|_{L^{1+r-\alpha}}^2 \leq \frac{1}{2} \alpha \int_{Q_T} x^m \frac{\kappa \theta^2}{\theta^{1+\alpha}} + C. \] (4.9)

**Case 2:** \( q \geq 1 + 2r - \alpha \)

Using Young inequality, we have
\[ C \int_0^T \| \theta \|_{L^{1+r-\alpha}}^2 \leq 1 + \int_0^T \| \theta^{r-\alpha} \theta_x \|_{L^2} \]
\[ \leq 1 + \int_0^T \left( \int_I \frac{\theta^2 \theta^q}{\theta^{1+\alpha}} \right)^{\frac{1}{2}} \]
\[ \leq C + C \int_0^T \left( \int_I \frac{\kappa \theta^2}{\theta^{1+\alpha}} \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{2} \alpha \int_{Q_T} x^m \frac{\kappa \theta^2}{\theta^{1+\alpha}} + C. \] (4.10)

Substituting (4.9) or (4.10) into (4.8), we complete the proof of Lemma 4.3. \(\square\)

The next estimate is a corollary of Lemma 4.3 whose proof can be found in [42] (Corollary 3.1). For completeness, we present the proof.

**Corollary 4.5** Under the conditions of Theorem 2.1.1, we have
\[ \int_0^T \| \theta \|_{L^{q-\alpha+1}}^2 \leq C. \]

**Proof.** By Corollary 3.3 (A5) and Lemma 4.3, we have
\[ \int_0^T \| \theta \|_{L^{q-\alpha+1}}^2 = \int_0^T \| \theta^{\frac{q-\alpha+1}{2}} \|^2_{L^\infty} \]
\[ \leq C \int_0^T \int_I \left( \theta^{\frac{q-\alpha+1}{2}} \theta_x \right)^2 + C \]
\[ = C \int_0^T \int_I \frac{\theta \theta_x^2}{\theta^{\alpha+1}} + C \]
\[ \leq C. \] \(\square\)
Lemma 4.6 Under the conditions of Theorem 2.1, we have
\[ \int_{Q_T} x^m(u_x^2 + v_x^2 + x^{-2}u^2 + x^{-2}v^2) \leq C. \]

Proof. Multiplying (2.1.1)_2 and (2.1.1)_3 by \( x^m u \) and \( x^m v \) respectively, and integrating by parts over \( I \), we have
\[ \frac{1}{2} \frac{d}{dt} \int_I x^m \rho u^2 - \int_I x^{m-1} \rho uv + \int_I x^m u P + \beta \int_I x^m (u_x^2 + mx^{-2}u^2) = 0. \] (4.11)

Adding (4.12) into (4.11), we have
\[ \frac{1}{2} \frac{d}{dt} \int_I x^m \rho v^2 + \int_I x^{m-1} \rho uv + \mu \int_I x^m (v_x^2 + mx^{-2}v^2) = 0. \] (4.12)

Integrating (4.13) over \((0, T)\), and using integration by parts and Cauchy inequality, we have
\[ \beta \int_{Q_T} x^m (u_x^2 + mx^{-2}u^2) + \mu \int_{Q_T} x^m (v_x^2 + mx^{-2}v^2) \leq \frac{1}{2} \beta \int_{Q_T} x^m (u_x^2 + mx^{-2}u^2) + C \int_{Q_T} x^m P^2. \]

This gives
\[ \frac{1}{2} \beta \int_{Q_T} x^m (u_x^2 + mx^{-2}u^2) + \mu \int_{Q_T} x^m (v_x^2 + mx^{-2}v^2) \leq \int_{Q_T} x^m \rho^2 (1 + \theta^{2+2r}) + 1 \]
\[ \leq \int_T \| \theta \|_{L^\infty}^{1+r} \int_I x^m \rho \theta^{1+r} + 1 \]
\[ \leq \int_T \| \theta \|_{L^\infty}^{1+r} + 1, \]

where we have used \((A_2), (A_3), (A_4)\) and Lemmas 4.1, 4.2.

Since \( q > r \) and \( 0 < \alpha < q - r \), we have \( 1 + r < q - \alpha + 1 \). Thus, using Young inequality and Corollary 4.5, we have
\[ \frac{1}{2} \beta \int_{Q_T} x^m (u_x^2 + mx^{-2}u^2) + \mu \int_{Q_T} x^m (v_x^2 + mx^{-2}v^2) \leq \int_0^T \| \theta \|_{L^\infty}^{q-\alpha+1} + 1 \leq C. \]
Lemma 4.7 Under the conditions of Theorem 2.1.1, we have for any \( t \in [0, T] \)
\[
\int_I x^m(v_x^2 + x^{-2}v^2) + \int_{Q_T} x^mpv_t^2 \leq C.
\]

Proof. Multiplying (2.1.1)_3 by \( x^mv_t \), integrating by parts over \( I \), and using Cauchy inequality, we have
\[
\frac{\mu}{2} \frac{d}{dt} \int_I x^m(v_x^2 + mx^{-2}v^2) + \int_I x^mpv_t^2 = -\int_I x^m \rho uv_x v_t - \int_I x^{m-1} \rho u v v_t \leq \frac{1}{2} \int_I x^m \rho v_t^2 + C \int_I x^m \rho v^2 v_x + C \int_I x^{m-2} \rho v^2 v_x.
\]
Thus, we have
\[
\mu \frac{d}{dt} \int_I x^m(v_x^2 + mx^{-2}v^2) + \int_I x^mpv_t^2 \leq \|u\|_{L^\infty} \|\rho\|_{L^\infty} \int_I x^m v_x^2 + \|v\|_{L^\infty}^2 \int_I x^m \rho u^2 v_x^2 + \frac{1}{2} \int_I x^m \rho u^2 v_x^2 + \frac{1}{2} \int_I x^m \rho v_t^2 v_x.
\]
where we have used Lemmas 4.1-4.2 and Poincaré inequality.

By Gronwall inequality and Lemma 4.6, we complete the proof of Lemma 4.7.

Corollary 4.8 Under the conditions of Theorem 2.1.1, we have
\[
\|v\|_{L^\infty(Q_T)} + \int_{Q_T} x^m v_{xx}^2 \leq C.
\]

Proof. This is an immediately result from Lemmas 4.2, 4.6, 4.7, Poincaré inequality and (2.1.1)_3.

Lemma 4.9 Under the conditions of Theorem 2.1.1, we have for any \( t \in [0, T] \)
\[
\int_I x^m w_t^2 + \int_{Q_T} x^mpw_t^2 \leq C.
\]

Proof. Multiplying (2.1.1)_4 by \( x^mw_t \), integrating by parts over \( I \), and using Cauchy inequality, we have
\[
\int_I x^m pw_t^2 + \frac{\mu}{2} \frac{d}{dt} \int_I x^m w_x^2 = -\int_I x^m \rho u w_x w_t \leq \frac{1}{2} \int_I x^m pw_t^2 + \frac{1}{2} \int_I x^m \rho u^2 w_x^2.
\]
Thus, we apply Lemma 4.2 and Poincaré inequality to get
\[
\int_I x^m pw_t^2 + \frac{\mu}{2} \frac{d}{dt} \int_I x^m w_x^2 \leq \|\rho\|_{L^\infty} \|u\|_{L^\infty}^2 \int_I x^m w_x^2 \leq \int_I x^m (u_x^2 + mx^{-2}u^2) \int_I x^m w_x^2.
\]
Using Gronwall inequality and Lemma 4.6, we complete the proof of Lemma 4.9.

Similar to Corollary 4.8, we get the next corollary.
Corollary 4.10  Under the conditions of Theorem 2.1.1, we have

\[ \int_{Q_T} x^m u_{xx}^2 \leq C. \]

Lemma 4.11  Under the conditions of Theorem 2.1.1, we have for any \( t \in [0,T] \)

\[ \int_I x^m (\rho \theta^2 v + u_x^2 + x^{-2} u^2) + \int_{Q_T} x^m (\rho u_t^2 + (1 + \theta^2)^2 \theta^2) \leq C. \]

Proof.  Multiplying (2.1.1)2 by \( x^m u_t \), integrating by parts over \( I \), and using Cauchy inequality, we have

\[ \int_I x^m \rho u_t^2 + \frac{\beta}{2} \frac{d}{dt} \int_I x^m (u_x^2 + mx^{-2} u^2) \]

\[ = - \int_I x^m \rho u u_x u_t + \int_I x^m -1 \rho v^2 u_t - \int_I x^m P_x u_t \]

\[ \leq \frac{\beta}{4} \int_I x^m \rho u_t^2 + C \int_I x^m (u_x^2 + x^{-2} v^4) + \int_I x^m P u x + m \int_I x^m -1 P u_t. \]

This, along with Lemma 4.12 and Poincaré inequality, deduces

\[ \frac{3}{4} \int_I x^m \rho u_t^2 + \frac{\beta}{2} \frac{d}{dt} \int_I x^m (u_x^2 + mx^{-2} u^2) \]

\[ \leq C \| \rho \|_{L^\infty} \| u \|^2_{L^\infty} \int_I x^m u_x^2 + C \int_I x^m -2 \rho v^4 + \frac{d}{dt} \int_I x^m P u x - \int_I x^m P_t u x + m \int_I x^m -1 P u_t \]

\[ \leq C \left( \int_I x^m (u_x^2 + mx^{-2} u^2) \right)^2 + \frac{d}{dt} \int_I x^m P u x + C \int_I x^m -2 \rho v^4 - \int_I x^m P_t u x + m \int_I x^m -1 P u_t \]

\[ = 5 \sum_{i=1}^5 II_i. \]

For \( II_3 \), using Lemma 4.11 and Corollary 4.8, we have

\[ II_3 \leq C \| v \|^2_{L^\infty} \int_I x^m \rho v^2 \leq C. \]  (4.15)

For \( II_4 \), we have

\[ II_4 = - \beta^{-1} \int_I x^m P_t (\beta u_x - P) - \beta^{-1} \int_I x^m P_t P \]

\[ = - \beta^{-1} \int_I x^m (\rho Q)(\beta u_x - P) - \beta^{-1} \int_I x^m P_c (\rho)(\beta u_x - P) - \frac{1}{2} \beta^{-1} \frac{d}{dt} \int_I x^m P^2 \]

\[ = 3 \sum_{i=1}^3 II_{4,i}. \]

For \( II_{4,1} \), using (4.3) and (2.1.1)2, we have

\[ II_{4,1} = - \beta^{-1} \int_I (\beta u_x - P) \left( (x^m \kappa \theta_x) + x^m \beta (x^m \rho u Q) - x^m \rho \theta Q'(\theta)(u_x + \frac{mu}{x}) \right) \]

\[ = \beta^{-1} \int_I x^m (\beta u x - P) (\kappa \theta_x - \rho Q) + \beta^{-1} \int_I x^m (\beta u x - P) \rho \theta Q'(\theta)(u_x + \frac{mu}{x}) \]

\[ - \beta^{-1} \int_I x^m (\beta u x - P) \psi \]

\[ = \beta^{-1} \int_I x^m (\rho u_t + pu u_x - \rho v^2 x - \frac{m \beta u x}{x} + \frac{m \beta u}{x^2})(\kappa \theta_x - \rho Q) \]

\[ + \beta^{-1} \int_I x^m (\beta u x - P) \rho \theta Q'(\theta)(u_x + \frac{mu}{x}) - \beta^{-1} \int_I x^m (\beta u x - P) \psi. \]
Recalling \( \varphi = \lambda(u_x + \frac{m u}{x})^2 + \mu \left( u_x^2 + 2u_x + (v_x - \frac{m u}{x})^2 + \frac{2m u^2}{x^2} \right) \), we have

\[
II_{4,1} \leq \frac{1}{8} \int_I x^m \rho u_t^2 + C \int_I x^m \rho u_t Q^2 + C \int_I x^m (\kappa \theta x)^2 + C \int_I x^m u_x^2 + C \int_I x^m \rho u^4 + C \int_I x^m u_x^2 \\
+ C \int_I x^{m-4} u^2 + C \sup_{x \in I} (1 + \theta^{1+r}) \int_I x^m (u_x^2 + \rho \theta^2 Q^2 + \rho + x^{-2} u^2) \\
+ C \| \beta u_x - P \|_{L^\infty} \int_I x^m (u_x^2 + x^{-2} u^2 + w_x^2 + v_x^2 + x^{-2} v^2) \\
\leq \frac{1}{8} \int_I x^m \rho u_t^2 + C \int_I x^m (u_x^2 + x^{-2} u^2) \int_I x^m \rho (1 + \theta^{2+2r}) + C \int_I x^m (\kappa \theta x)^2 \\
+ C \left( \int_I x^m (u_x^2 + x^{-2} u^2) \right)^2 + C \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \int_I x^m (u_x^2 + \rho (1 + \theta^{2+2r}) + x^{-2} u^2) \\
+ C \| \beta u_x - P \|_{L^\infty} \int_I x^m (u_x^2 + x^{-2} u^2 + w_x^2 + v_x^2 + x^{-2} v^2) + C
\]

where we have used Young inequality, Poincaré inequality, Lemmas \([4.1, 4.2] (A_2), (A_4), (4.15)\), \( q > r \) and \( \alpha < q - r \).

This, along with Sobolev inequality, Cauchy inequality, Lemmas \([4.2, 4.4, 4.9] \text{ and } (2.1.1)\), deduces

\[
II_{4,1} \leq \frac{1}{8} \int_I x^m \rho u_t^2 + C \int_I x^m (\kappa \theta x)^2 + C \left( \int_I x^m (u_x^2 + x^{-2} u^2) \right)^2 \\
+ C \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \left[ 1 + \int_I x^m (u_x^2 + \rho \theta^q + x^{-2} u^2) \right] \\
+ C \| (\beta u_x - P) \|_{L^2} + \| \beta u_x - P \|_{L^2} \left[ 1 + \int_I x^m (u_x^2 + x^{-2} u^2) \right] + C \\
\leq \frac{1}{8} \int_I x^m \rho u_t^2 + C \int_I x^m (\kappa \theta x)^2 + C \left( \int_I x^m (u_x^2 + x^{-2} u^2) \right)^2 \\
+ C \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \left[ 1 + \int_I x^m (u_x^2 + \rho \theta^q + x^{-2} u^2) \right] \\
+ C \left( \| \rho u_t + \rho u u_x - \frac{\rho u^2}{x} - \frac{m \beta u_x}{x^2} + \frac{m \beta u}{x^2} \|_{L^2} + \| \beta u_x - P \|_{L^2} \right) \left[ 1 + \int_I x^m (u_x^2 + x^{-2} u^2) \right] + C \\
\leq \frac{1}{4} \int_I x^m \rho u_t^2 + C \int_I x^m (\kappa \theta x)^2 + C \left( \int_I x^m (u_x^2 + x^{-2} u^2) \right)^2 \\
+ C \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \left[ 1 + \int_I x^m (u_x^2 + \rho \theta^q + x^{-2} u^2) \right] + C.
\]
For $II_{4,2}$, using (2.1.1), integration by parts, Lemma 4.1 (A3) and (A4), we have

$$II_{4,2} \leq -\beta^{-1} \int_I x^m P_t^{\rho}(p_x u + \rho p_x + mx^{-1}\rho)(\beta u_x - P)$$

$$= -\beta^{-1} \int_I x^m P_t u(\beta u_x - P) + \beta^{-1} \int_I x^m(P_t)^{\rho} p_x(\beta u_x - P) + \beta^{-1} \int_I mx^{-1}(P_t)^{\rho} p_x(\beta u_x - P)$$

$$\leq -\beta^{-1} \int_I x^m P_t u(\beta u_x - P) - \beta^{-1} \int_I x^m P_t u(\beta u_x - P)_{x} - \beta^{-1} \int_I mx^{-1}P_t u(\beta u_x - P)$$

$$+ C \int_I x^m(u_x^2 + x^{-2}u^2) + C \int_I x^m \rho(1 + \theta^{2+2r}) + C$$

$$\leq -\beta^{-1} \int_I x^m P_t u(\beta u_x - P) + C \int_I x^m(u_x^2 + x^{-2}u^2) + C \int_I x^m \rho(1 + \theta^{2+2r}) + C.$$

Using (2.1.1), Lemmas 4.1, Corollary 4.8 (A3), Poincaré inequality, Young inequality and $\alpha < q - r$, we have

$$II_{4,2} = -\beta^{-1} \int_I x^m P_t u(\rho u_t + \rho u_x - \frac{\rho v^2}{x} - \frac{m\beta u_x}{x} + \frac{m\beta u}{x^2})$$

$$+ C \int_I x^m(u_x^2 + x^{-2}u^2) + C \int_I x^m \rho(1 + \theta^{2+2r}) + C$$

$$\leq \frac{1}{4} \int_I x^m \rho u_t^2 + C \int_I x^m \rho u^2 + C \|u\|^2_{L^\infty} \|\rho\|_{L^\infty} \int_I x^m u_x^2 + C \|v\|^2_{L^\infty} \int_I x^m \rho v^2$$

$$+ C \int_I x^m(u_x^2 + x^{-2}u^2) + C \int_I x^m \rho(1 + \theta^{2+2r}) + C$$

$$\leq \frac{1}{4} \int_I x^m \rho u_t^2 + C \left( \int_I x^m(u_x^2 + x^{-2}u^2) \right)^2 + C \sup_{x \in I} \theta^{q - \alpha + 1} + C.$$

Putting all the estimates about $II_{4,1}$ and $II_{4,2}$ into $II_4$, we have

$$II_4 \leq -\frac{1}{2\beta} \frac{d}{dt} \int_I x^m P^2 + \frac{1}{2} \int_I x^m \rho u_t^2 + C \int_I x^m(\kappa \theta x)^2 + C \left( \int_I x^m(u_x^2 + x^{-2}u^2) \right)^2$$

$$+ C \sup_{x \in I} (1 + \theta^{q - \alpha + 1}) \left[ 1 + \int_I x^m(u_x^2 + \rho \theta^{q+r+2} + x^{-2}u^2) \right] + C. \quad (4.16)$$

For $II_5$, recalling $P = \rho Q + P_c$, we have

$$II_5 = m \int_I x^{m-1} \rho Q u_t + m \int_I x^{m-1} P_c(\rho) u_t$$

$$\leq \frac{1}{8} \int_I x^m \rho u_t^2 + C \int_I x^m \rho(1 + \theta^{2+2r}) + C \int_I x^m \rho$$

$$\leq \frac{1}{8} \int_I x^m \rho u_t^2 + C \|\theta\|^{q-\alpha+1}_{L^\infty} + 1 \int_I x^m \rho^{1+r} + C$$

$$\leq \frac{1}{8} \int_I x^m \rho u_t^2 + C \|\theta\|^{q-\alpha+1}_{L^\infty} + C, \quad (4.17)$$

where we have used Lemma 4.1, Young inequality and (A4). Note that $P_c(0) = 0$ by (1.2) and (A2).
Putting (4.15), (4.16) and (4.17) into (4.14), we have
\[
\frac{3}{4} \int_I x^m p u_t^2 + \frac{\beta}{2} \frac{d}{dt} \int_I x^m (u_x^2 + m x^2 u^2)
\leq C \left( \int_I x^m (u_x^2 + m x^2 u^2) \right)^2 + \frac{d}{dt} \int_I x^m P u_x - \frac{1}{2\beta} \frac{d}{dt} \int_I x^m P^2 + \frac{5}{8} \int_I x^m \rho u_t^2 + C \int_I x^m (\kappa \theta x)^2
+ C \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \left[ 1 + \int_I x^m (u_x^2 + \rho \theta^{q+r^2} + m x^2 u^2) \right] + C.
\]
Thus,
\[
\frac{1}{8} \int_0^t \int_I x^m p u_t^2 + \frac{\beta}{2} \int_I x^m (u_x^2 + m x^2 u^2)
\leq C \left( \int_I x^m (u_x^2 + m x^2 u^2) \right)^2 + \int_I x^m P u_x + C \int_0^t \int_I x^m (\kappa \theta x)^2
+ C \int_0^t \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \int_I x^m (u_x^2 + \rho \theta^{q+r^2} + m x^2 u^2) + C
\leq C \int_0^t \left( \int_I x^m (u_x^2 + m x^2 u^2) \right)^2 + \frac{\beta}{4} \int_I x^m u_x^2 + C \int_I x^m (\rho^2 Q^2 + P^2_c) + C \int_0^t \int_I x^m (\kappa \theta x)^2
+ C \int_0^t \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \int_I x^m (u_x^2 + \rho \theta^{q+r^2} + m x^2 u^2) + C.
\]
Thus,
\[
\int_0^t \int_I x^m p u_t^2 + \int_I x^m (u_x^2 + m x^2 u^2)
\leq C \int_0^t \left( \int_I x^m (u_x^2 + m x^2 u^2) \right)^2 + C \int_I x^m \rho \theta^{q+r^2} + C \int_0^t \int_I x^m (\kappa \theta x)^2
+ C \int_0^t \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \int_I x^m (u_x^2 + \rho \theta^{q+r^2} + m x^2 u^2) + C.
\]
The next step is to handle the second term and the third one on the right hand side of (4.19).
Corollary 4.13 Under the conditions of Theorem 2.1.1, we have
\[ \|u\|_{L^\infty(Q_T)} \leq C. \]

Corollary 4.12 Under the conditions of Theorem 2.1.1, we have
\[ \int_0^T \| \theta \|_{L^\infty}^{2q+2} \leq C. \]
Proof. By Corollary 4.3 and Cauchy inequality, we have
\[ \int_0^T \sup_{x \in I} \theta^{2q+2} \lesssim \int_0^T \int_I \theta^{2q+1}|\vartheta_x| + 1 \]
\[ \lesssim \frac{1}{2} \int_0^T \sup_{x \in I} \theta^{2q+2} + C \int_{Q_T} \theta^{2q} \theta_x^2. \]
This together with Lemma 4.11 completes the proof of Corollary 4.13. 

\[
\text{Lemma 4.14} \quad \text{Under the conditions of Theorem 2.1.1 we have for any } t \in [0, T]
\]
\[ \int_I x^m(\rho_x^2 + \rho_t^2) + \int_{Q_T} x^m \rho_{xx}^2 \leq C. \]

Proof. Differentiating (2.1.1), we have
\[ \rho_{xt} + \rho_{xx}u + 2\rho_x u_x + \rho u_{xx} + mx^{-1}\rho_x u + mx^{-1}\rho u_x - mx^{-2}\rho u = 0. \] (4.24)

Multiplying (4.24) by \(2x^m\rho_x\), and integrating by parts over \(I\), we have
\[ \frac{d}{dt} \int_I x^m \rho_x^2 = -4 \int_I x^m \rho_x^2 u_x - \int_I x^m (\rho_x^2) u - 2 \int_I x^m \rho_x u_{xx} \]
\[ - 2m \int_I x^m \rho_x^2 u_x - 2m \int_I x^m \rho_x u_x + 2m \int_I x^{m-2} \rho_{xx} u 
\]
\[ = -3 \int_I x^m \rho_x^2 u_x - m \int_I x^{m-1} \rho_x^2 u - 2 \int_I x^m \rho_x u_{xx} \]
\[ - 2m \int_I x^{m-1} \rho_x u_x + 2m \int_I x^{m-2} \rho_{xx} u 
\]
\[ = \sum_{i=1}^5 \text{III}_i. \]

For \(\text{III}_1\), using Sobolev inequality, Young inequality, (A2), (A3), (A4), and Lemmas 4.1, 4.2, 4.11 we have
\[ \text{III}_1 = -3 \beta^{-1} \int_I x^m \rho_x^2 (\beta u_x - P) - 3 \beta^{-1} \int_I x^m \rho_x^2 P 
\]
\[ \lesssim (\|\beta u_x - P\|_{L^\infty} + \|\rho Q + P\|_{L^\infty}) \int_I x^m \rho_x^2 
\]
\[ \lesssim (\|\beta u_x - P\|_{L^2} + \|\beta u_x - P\|_{L^2} + \|1 + \theta^{q+1}\|_{L^\infty}) \int_I x^m \rho_x^2 \]
\[ \lesssim \left(1 + \sup_{x \in I} \theta^{q-\alpha+1} + \|\beta u_x - P\|_{L^2}\right) \int_I x^m \rho_x^2. \] (4.26)

For \(\|\beta u_x - P\|_{L^2}\), using (2.1.1)2, Lemmas 4.2, 4.11 Corollary 4.8 and Corollary 4.12 we have
\[ \|\beta u_x - P\|_{L^2} \lesssim \|\rho u_t\|_{L^2} + \|\rho u u_x\|_{L^2} + \|x^{-1} \rho u^2\|_{L^2} + \|x^{-1} u_x\|_{L^2} + \|x^{-2} u\|_{L^2} \]
\[ \lesssim \sqrt{\rho u_t\|_{L^2} + 1. \] (4.27)

Substituting (4.27) into (4.26), we obtain
\[ \text{III}_1 \lesssim \left(1 + \sup_{x \in I} \theta^{q-\alpha+1} + \sqrt{\rho u_t}\right) \int_I x^m \rho_x^2. \] (4.28)
For $III_i$, $i=2, 3, 4, 5$, we have

$$III_2 \lesssim \|u\|_{L^\infty} \int_I x^m \rho_x^2 \lesssim \int_I x^m \rho_x^2.$$  \hfill (4.29)

$$III_3 \lesssim \int_I x^m \rho_x^2 + \int_I x^m u_{xx}^2.$$  \hfill (4.30)

$$III_4 \lesssim \int_I x^m \rho_x^2 + \int_I x^m u_{xx}^2 \lesssim \int_I x^m \rho_x^2 + 1.$$  \hfill (4.31)

$$III_5 \lesssim \int_I x^m \rho_x^2 + \int_I x^{m-2} u^2 \lesssim \int_I x^m \rho_x^2 + 1.$$  \hfill (4.32)

Putting (4.28)-(4.32) into (4.25), we have

$$\frac{d}{dt} \int_I x^m \rho_x^2 \lesssim \left(1 + \sup_{x \in I} \theta^{q-\alpha+1} + \|\sqrt{\rho u_t}\|_{L^2}\right) \int_I x^m \rho_x^2 + \int_I x^m u_{xx}^2 + 1.$$  \hfill (4.33)

By (4.27), (A2), (A3), (A4) and Lemma 4.2, we have

$$\int_I x^m u_{xx}^2 \lesssim \int_I x^m \rho u_t^2 + \int_I x^m \rho_x^2 Q^2 + \int_I x^m \rho^2 (Q')^2 \theta_x^2 + \int_I x^m \rho_x^2 + 1$$

$$\lesssim \int_I x^m \rho u_t^2 + (1 + \sup_{x \in I} \theta^{2+2q}) \int_I x^m \rho_x^2 + \int_I x^m (1 + \theta^{2+2q}) \theta_x^2 + 1.$$  \hfill (4.34)

Since $q > r$, using Young inequality, we have

$$\int_I x^m u_{xx}^2 \lesssim \int_I x^m \rho u_t^2 + (1 + \sup_{x \in I} \theta^{2+2q}) \int_I x^m \rho_x^2 + \int_I x^m (1 + \theta^q) \theta_x^2 + 1.$$  \hfill (4.35)

Substituting (4.35) into (4.33), and using Corollary 4.5, Lemma 4.11, Corollary 4.13 and Gronwall inequality, we get

$$\int_I x^m \rho_x^2 \leq C.$$  \hfill (4.36)

Substituting (4.36) into (4.34), and using Lemma 4.11 and Corollary 4.13 again, we have

$$\int_{Q_T} x^m u_{xx}^2 \leq C.$$  \hfill (4.37)

The estimate for $\rho_t$ can be obtained easily by (2.1.11). \hfill $\Box$

**Step 4: $H^2$ estimates of $(\rho, u, v, w)$ and $H^1$ estimates of $\theta$**

**Lemma 4.15** Under the conditions of Theorem 2.1.1, we have for any $t \in [0, T]$

$$\int_I x^m \rho u_t^2 + \int_{Q_T} x^m (v_{2t}^2 + x^{-2} v_t^2) \leq C.$$

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**Proof.** Differentiating (2.1.1) with respect to $t$, we have
\[
\rho v_t + \rho v v_t + \rho u v_x + \rho v x v_t + x^{-1} \rho u v + x^{-1} \rho u v + x^{-1} \rho v_t = \mu(v x t + mx^{-1} v_x - mx^{-2} v_t).
\] (4.36)

Multiplying (4.36) by $x^m v_t$, and integrating by parts over $I$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_I x^m \rho v_t^2 + \mu \int_I x^m (v_{xt}^2 + mx^{-2} v_t^2)
= -\frac{1}{2} \int_I x^m \rho v_t^2 - \int_I x^m (\rho t u v_x + \rho u v) v_t - \int_I x^m \left( \rho u v_x + \rho u v + \frac{\rho v t + \rho v v_t}{x} \right) v_t \tag{4.37}
= \sum_{i=1}^3 IV_i.
\]

For $IV_1$, using (2.1.1), integration by parts, Cauchy inequality, Lemma 4.2 and Corollary 4.12, we have
\[
IV_1 = \frac{1}{2} \int_I (x^m \rho u)_x v_t^2
= -\int_I x^m \rho u v v_x - \frac{u_x v + uv x}{x^2} v_t - \int_I x^m \rho u (u v + u v x) v_t
\leq C \int_I x^m \rho u v_t^2 + C \int_I x^m \rho (u^2 u_x^2 + u^4 v^2 + \frac{u^2 u_x^2 v^2 + u^4 v^2}{x^2} + \frac{u^4 v^2}{x^4})
+ \frac{\mu}{6} \int_I x^m v_{xt}^2 + C \int_I x^m \rho^2 u^2 (u^2 v_x^2 + \frac{u^2 v^2}{x^2})
\leq C \int_I x^m \rho u v_t^2 + C \|v_x\|_{L^\infty}^2 \int_I x^m u^2 + C \int_I x^m v_{xx} + C\|u_x\|_{L^\infty}^2 \int_I x^m v_t^2
+ C \int_I x^m v_x^2 + C \int_I x^m v_{xt}^2 + \frac{\mu}{6} \int_I x^m v_{xt}^2
\leq C \int_I x^m \rho u v_t^2 + C \int_I x^m v_x^2 + C \int_I x^m u^2 + \frac{\mu}{6} \int_I x^m v_{xt}^2 + C,
\] (4.39)

where we have used (2.1.1), integration by parts, Cauchy inequality, Lemmas 4.2, 4.7, 4.11, Corollary 4.12, and Sobolev inequality.

For $IV_2$, using Cauchy inequality, Lemmas 4.2, 4.7, Corollary 4.8, Corollary 4.12, and Sobolev inequality, we get
\[
IV_2 = -\int_I x^m \rho u v_x v_t - \int_I x^m \rho u v v_{xt} v_t - \int_I x^m \rho u v v_t - \int_I x^m \rho u v v_t^2
\leq C \int_I x^m \rho u v_t^2 + C \int_I x^m v_{xx}^2 + \frac{\mu}{6} \int_I x^m v_{xt}^2 + C \int_I x^m \rho^2 u^2 v_t^2 + C \int_I x^m \rho^2 v^2 v_t^2
+ C \int_I x^m \rho v_t^2
\leq C \int_I x^m \rho u v_t^2 + \frac{\mu}{6} \int_I x^m v_{xt}^2 + C(1 + \int_I x^m v_{xx}^2) \int_I x^m \rho v_t^2.
\] (4.40)
Putting (4.38), (4.39) and (4.40) into (4.37), we have
\[
\frac{d}{dt}\int_I x^m \rho \nu_t^2 + \mu \int_I x^m (v^2_{xt} + mx^{-2} v^2_t) \\
\leq C(1 + \int_I x^m v^2_{xx}) \int_I x^m \nu_t^2 + C \int_I x^m v^2_{xx} + C \int_I x^m u^2_{xx} + C \int_I x^m \rho u^2 + C.
\]
(4.41)

It follows from (4.41), Corollary 4.8, Lemmas 4.11, 4.14, the compatibility conditions and Gronwall inequality, we complete the proof of Lemma 4.15.

By (2.1.1)3, Lemmas 4.2, 4.7, 4.11, 4.14, 4.15, Corollary 4.8, Corollary 4.12, and Sobolev inequality, we get the next estimate.

Corollary 4.16 Under the conditions of Theorem 2.1.1, we have for any \(t \in [0, T]\)
\[
\|v\|_{W^{1,\infty}(Q_T)} + \int_I x^m v^2_{xx} + \int_{Q_T} x^m v^4_{xxx} \leq C.
\]

Similar to Lemma 4.15 and Corollary 4.16 we obtain the next lemma and the next corollary.

Lemma 4.17 Under the conditions of Theorem 2.1.1, we have for any \(t \in [0, T]\)
\[
\int_I x^m \rho u^2_t + \int_{Q_T} x^m u^2_{xt} \leq C.
\]

Corollary 4.18 Under the conditions of Theorem 2.1.1, we have for any \(t \in [0, T]\)
\[
\|w\|_{W^{1,\infty}(Q_T)} + \int_I x^m w^2_{xx} + \int_{Q_T} x^m w^2_{xxx} \leq C.
\]

Lemma 4.19 Under the conditions of Theorem 2.1.1, we have for any \(t \in [0, T]\)
\[
\int_I x^m (\rho u^2_t + (1 + \theta^q)^2 \theta^2_t) + \int_{Q_T} x^m (u^2_{xt} + x^{-2} u^2_t + \rho(1 + \theta^q)^2 \theta^2_t) \leq C.
\]

Proof. Differentiating (2.1.1)2 with respect to \(t\), we have
\[
\rho u_{tt} + \rho_t u_t + \rho_t u u_x + \rho u_t u_x + \rho u u_{xt} - \frac{\rho v^2}{x} - \frac{2 \rho u v_t}{x} + P_{xt} = \beta \left( u_{xx} + \frac{mu_{xt}}{x} - \frac{mu_t}{x^2} \right) \quad (4.42)
\]

Multiplying (4.42) by \(x^m u_t\), and integrating by parts over \(I\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_I x^m \rho u^2_t + \beta \int_I x^m (u^2_{xt} + \frac{mu^2}{x^2}) \\
= - \frac{1}{2} \int_I x^m \rho_t u^2_t - \int_I x^m \rho_t u u_x u_t + \int_I x^{m-1} \rho_t v^2 u_t - \int_I x^m \rho (u_t u_x + uu_{xt} - \frac{2vv_t}{x}) u_t \\
+ \int_I P_t (x^m u_{xt} + mx^{m-1} u_t) \\
= \sum_{i=1}^5 V_i.
\]
For $V_1$, using \((2.1.1)\), integration by parts, Cauchy inequality, Lemma 4.2, Corollary 4.12, we have

\[
V_1 = \frac{1}{2} \int_I (x^m \rho u)_x u_t^2
\]
\[
= - \int_I x^m \rho uu_t u_{xt}
\]
\[
\leq \frac{\beta}{8} \int_I x^m u_x^2 + C \int_I x^m \rho u_t^2.
\] (4.44)

For $V_2$ and $V_3$, using \((2.1.1)\), integration by parts and Cauchy inequality again, along with Sobolev inequality, Lemmas 4.2, 4.7, 4.11, Corollary 4.8 and Corollary 4.12, we get

\[
V_2 + V_3 = \int_I (x^m \rho u)_x (u u_x u_t - \frac{v^2 u_t}{x})
\]
\[
= - \int_I x^m \rho (u^2 x_t + u u_{xx} u_t + u u_x u_{xt} - \frac{2v u x u_t + v^2 u_{xt}}{x} + \frac{v^2 u_t}{x^2})
\]
\[
\leq C \int_I x^m \rho u_x^2 u^2 x_t + C \int_I x^m \rho u_x^2 u^2 + C \int_I x^m \rho u_t^2 + C \int_I x^m \rho u_x^2 u_{xt} + \frac{\beta}{8} \int_I x^m u_x^2
\] (4.45)
\[
+ C \int_I x^m \rho^2 u_x^2 u_x^2 + C \int_I x^m \rho u_x^2 v^2 + C \int_I x^m \rho u_t^2 v^4 + C \int_I x^m \rho u_x^2 v_4
\]
\[
\leq C (1 + \int_I u_x^2) \int_I x^m \rho u_t^2 + C \int_I x^m u_x^2 + \frac{\beta}{8} \int_I x^m u_x^2 + C.
\] (4.46)

For $V_4$, we have

\[
V_4 \leq \|u_x\|_{L^\infty} \int_I x^m \rho u_t^2 + \frac{\beta}{8} \int_I x^m u_x^2 + C \int_I x^m \rho^2 u_x^2 u_t^2 + C \int_I x^m \rho u_t^2 + C \int_I x^m \rho^2 v^2 u_t^2
\]
\[
\leq \frac{\beta}{8} \int_I x^m u_x^2 + C (1 + \|u_x\| H^1) \int_I x^m \rho u_t^2 + C;
\] (4.46)

where we have used Sobolev inequality, Lemmas 4.2, 4.11, Corollary 4.8 and Corollary 4.12.

For $V_5$, using Young inequality, \((A_2)\), \((A_3)\), \((A_4)\), and Lemmas 4.2, 4.11, we have

\[
V_5 \leq \frac{\beta}{8} \int_I x^m (u_x^2 + \frac{m u_t^2}{x^2}) + C \int_I x^m P^2
\]
\[
\leq \frac{\beta}{8} \int_I x^m (u_x^2 + \frac{m u_t^2}{x^2}) + C \int_I x^m \rho_t^2 (\partial_P P)^2 + C \int_I x^m \theta_t^2 (\partial_P P)^2
\]
\[
\leq \frac{\beta}{8} \int_I x^m (u_x^2 + \frac{m u_t^2}{x^2}) + C (1 + \sup_{x \in I} \theta^{2+2q}) \int_I x^m \rho_t^2 + C \int_I x^m \rho^2 (1 + \theta^{2q}) \theta_t^2
\]
\[
\leq \frac{\beta}{8} \int_I x^m (u_x^2 + \frac{m u_t^2}{x^2}) + C \sup_{x \in I} \theta^{2+2q} + C \int_I x^m \rho (1 + \theta^{q+r}) \theta_t^2 + C.
\] (4.47)

Putting (4.44), (4.45), (4.46) and (4.47) into (4.43), we have

\[
\frac{d}{dt} \int_I x^m \rho u_t^2 + \beta \int_I x^m (u_x^2 + \frac{m u_t^2}{x^2})
\]
\[
\leq C (1 + \int_I u_{xx}) \int_I x^m \rho u_t^2 + C \int_I x^m u_x^2 + C \sup_{x \in I} \theta^{2+2q} + C \int_I x^m \rho (1 + \theta^{q+r}) \theta_t^2 + C.
\] (4.48)
Integrating (4.48) over $(0,t)$, and using the compatibility conditions, Lemma 4.11 and Corollary 4.13 we obtain

\[
\int_I x^m \rho u_t^2 + \int_0^t \int_I x^m (u_{xt}^2 + \frac{mu^2}{x^2}) \leq C \int_0^t (1 + \int_I u_{xx}^2) \int_I x^m \rho_t^2 + C \int_0^t \int_I x^m \rho(1 + \theta^{q+r}) \theta_t^2 + C. \tag{4.49}
\]

The next step is to estimate the second integral of the right hand side of (4.49).

Multiplying (4.2) by $x^m \left( \int_0^t \kappa(\xi) \, d\xi \right)$ (i.e., $x^m \kappa(\theta_t)$), and integrating by parts over $I$, we have

\[
\int_I x^m \rho Q' \theta_t^2 + \frac{1}{2} \int_I \int x^m \kappa^2 \theta_x^2 \leq -\int_I x^m \rho Q'(u_x + \frac{mu}{x}) \kappa \theta_t - \int_I x^m \rho Q'(u_x + \frac{mu}{x})^2 \left( \int_0^t \kappa \, d\xi \right)_t + \mu \int_I x^m \left( u_x^2 + 2u^2 + (v_x - \frac{mv}{x})^2 + \frac{2mu^2}{x^2} \right) \left( \int_0^t \kappa \, d\xi \right)_t = \sum_{i=1}^4 V I_i. \tag{4.50}
\]

For $V I_1$, using Cauchy inequality, (A4), (A5), Lemma 4.11 and Corollary 4.12 we have

\[
V I_1 \leq \frac{1}{4} \int_I x^m \rho Q' \theta_t^2 + C \int_I x^m \rho Q' \kappa u^2 \theta_x^2 \leq \frac{1}{4} \int_I x^m \rho Q' \theta_t^2 + C \int_I x^m (1 + \theta^q)^2 \theta_x^2. \tag{4.51}
\]

For $V I_2$, using Cauchy inequality, (A4) and (A5) again, along with Lemmas 4.11 4.11 and Corollary 4.12 we have

\[
V I_2 \leq \frac{1}{4} \int_I x^m \rho Q' \theta_t^2 + C \int_I x^m \rho Q' \theta_x^2 (u_x^2 + \frac{u^2}{x^2}) \leq \frac{1}{4} \int_I x^m \rho Q' \theta_t^2 + C (\|u_x\|_{L^\infty} + \|u\|_{L^\infty}) \int_I x^m \rho (1 + \theta^{q+r})^2 \leq \frac{1}{4} \int_I x^m \rho Q' \theta_t^2 + C \int_I x^m u_{xx}^2 + C. \tag{4.52}
\]

For $V I_3$, we have

\[
V I_3 \leq \frac{d}{dt} \left( \lambda \int_I x^m (u_x + \frac{mu}{x})^2 \int_0^t \kappa(\xi) \, d\xi \right) - 2\lambda \int_I x^m (u_x + \frac{mu}{x})(u_{xt} + \frac{mu_t}{x}) \int_0^t \kappa(\xi) \, d\xi \leq \lambda \frac{d}{dt} \left( \int_I x^m (u_x + \frac{mu}{x})^2 \int_0^t \kappa(\xi) \, d\xi \right) + C \|\theta (1 + \theta^q)\|_{L^\infty} \left( \int_I x^m (u_x^2 + \frac{mu^2}{x^2}) \right)^\frac{1}{2} \leq \lambda \frac{d}{dt} \left( \int_I x^m (u_x + \frac{mu}{x})^2 \int_0^t \kappa(\xi) \, d\xi \right) + C (\|\kappa\theta_x\|_{L^2} + 1) \left( \int_I x^m (u_x^2 + \frac{mu^2}{x^2}) \right)^\frac{1}{2}, \tag{4.53}
\]

where we have used Hölder inequality, Lemma 4.11 4.21 and (A5).
For $VI_4$, using Hölder inequality, (4.21) and Lemma 4.11 again, along with Lemmas 4.7, 4.9, we have

$$VI_4 = \mu \frac{d}{dt} \int_I x^m \left( w_x^2 + 2u_x^2 + (v_x - \frac{mv}{x})^2 + \frac{2muu_x}{x^2} \right) \int_0^\theta \kappa(\xi) d\xi$$

$$- 2\mu \int_I x^m \left( w_x w_{xt} + 2u_x u_{xt} + (v_x - \frac{mv}{x})(v_{xt} - \frac{mv}{x}) + \frac{2muu_x}{x^2} \right) \int_0^\theta \kappa(\xi) d\xi$$

$$\leq \mu \frac{d}{dt} \int_I x^m \left( w_x^2 + 2u_x^2 + (v_x - \frac{mv}{x})^2 + \frac{2muu_x}{x^2} \right) \int_0^\theta \kappa(\xi) d\xi$$

$$+ 2\lambda \int_I x^m (u_x + \frac{mu}{x})^2 \int_0^\theta \kappa(\xi) d\xi + C \int_I x^m (1 + \theta^q)^2 \theta_x^2$$

$$+ C(\|\kappa \theta_x\|_{L^2} + 1) \left( \int_I x^m w_{xt}^2 \right)^{\frac{1}{2}} + \left( \int_I x^m (v_{xt}^2 + \frac{v_t^2}{x^2}) \right)^{\frac{1}{2}} + \left( \int_I x^m u_t^2 \right)^{\frac{1}{2}} + C.$$

Integrating (4.55) over $(0, t)$, and using (A4), (A5) and Lemma 4.14 we have

$$\int_0^t \int_I x^m \rho(1 + \theta^{q+r}) \theta_t^2 + \int_I x^m (1 + \theta^q)^2 \theta_x^2$$

$$\leq \int_I x^m \left( w_x^2 + u_x^2 + v_x^2 + \frac{v^2}{x^2} + \frac{u^2}{x^2} \right) \int_0^\theta \kappa(\xi) d\xi + \int_0^t \int_I x^m (1 + \theta^q)^2 \theta_x^2$$

$$+ \int_0^t (\|\kappa \theta_x\|_{L^2} + 1) \left( \int_I x^m w_{xt}^2 \right)^{\frac{1}{2}} + \int_0^t (\|\kappa \theta_x\|_{L^2} + 1) \left( \int_I x^m u_{xt}^2 \right)^{\frac{1}{2}}$$

$$+ \int_0^t (\|\kappa \theta_x\|_{L^2} + 1) \left[ \left( \int_I x^m (v_{xt}^2 + \frac{v_t^2}{x^2}) \right)^{\frac{1}{2}} + \left( \int_I x^m u_t^2 \right)^{\frac{1}{2}} \right] + 1.$$

Using (4.21), (4.56), Cauchy inequality and Lemmas 4.15, 4.17 we have

$$\int_0^t \int_I x^m \rho(1 + \theta^{q+r}) \theta_t^2 + \int_I x^m (1 + \theta^q)^2 \theta_x^2$$

$$\leq C \int_0^t \int_I x^m (1 + \theta^q)^2 \theta_x^2 + C \int_0^t (\|\kappa \theta_x\|_{L^2} + 1) \left[ \left( \int_I x^m u_{xt}^2 \right)^{\frac{1}{2}} + \left( \int_I x^m u_t^2 \right)^{\frac{1}{2}} \right] + C.$$

By (4.49), (4.57), Cauchy inequality and Gronwall inequality, we complete the proof of Lemma 4.19.

By Lemmas 3.1 and 4.19, we get the next corollary.
Corollary 4.20 Under the conditions of Theorem 2.1.1, we have
\[ \|\theta\|_{L^\infty(Q_T)} \leq C. \]

Corollary 4.21 Under the conditions of Theorem 2.1.1, we have for any \( t \in [0, T] \)
\[ \|u\|_{W^{1,\infty}(Q_T)} + \int_I x^m u_{xx}^2 + \int_Q x^m \theta_{xx}^2 \leq C. \]

Proof. By (4.34), Lemma 4.14, Lemma 4.19 and Corollary 4.20, we get
\[ \int_I x^m u_{xx}^2 \leq C. \] (4.58)

It follows from (4.58), Lemma 4.11, Corollary 4.12 and Sobolev inequality, we obtain
\[ \|u\|_{W^{1,\infty}} \leq C. \]

By (4.2), (4.58), (A_4), (A_5), Lemmas 4.2, 4.17, 4.19, Corollaries 4.12, 4.18, 4.20, and Cauchy inequality, we have
\[
\int_I x^m \theta_{xx}^2 \leq \int_I x^m \rho \theta_t^2 + \int_I x^m \theta_t^2 + \int_I x^m (u_x^2 + u_x^2) + \int_I x^m \theta_x^4 + \int_I x^m (u_x^4 + u_x^4) + \int_I x^m (u_x^4 + u_x^4 + \frac{u_x^4}{x^4})
\]
\[
\leq \int_I x^m \rho \theta_t^2 + \int_I x^m \theta_t^2 + 1
\]
\[
\leq \int_I x^m \rho \theta_t^2 + \|\theta_t^2\|_{L^\infty} \int_I x^m \theta_x^2 + 1
\]
\[
\leq \int_I x^m \rho \theta_t^2 + \|\theta \theta_{xx}\|_{L^1} + 1
\]
\[
\leq C \int_I x^m \rho \theta_t^2 + \frac{1}{2} \int_I x^m \theta_{xx}^2 + C.
\]

This deduces
\[ \int_I x^m \theta_{xx}^2 \leq C \int_I x^m \rho \theta_t^2 + C. \] (4.59)

Integrating (4.59) over \((0, T)\), and using Lemma 4.19, we get
\[ \int_Q x^m \theta_{xx}^2 \leq C. \]

Lemma 4.22 Under the conditions of Theorem 2.1.1, we have for any \( t \in [0, T] \)
\[ \int_I x^m \rho_{xx}^2 + \int_Q x^m u_{xxx}^2 \leq C. \]

Proof. Differentiating (4.24) with respect to \( x \), we have
\[
\rho_{xx} = -\rho_{xx} u - 3 \rho_{xx} u_x - 3 \rho_x u_{xx} - \rho u_{xxx} - \frac{m \rho_x u_x}{x} - \frac{2 m \rho_x u}{x^2} + \frac{2 m \rho u_x}{x^2} + \frac{2 m \rho u_{xx}}{x}. \] (4.60)
Multiplying (4.60) by $2x^m \rho_{xx}$, integrating by parts it over $I$, and using Hölder inequality, we have

$$\frac{d}{dt} \int_I x^m \rho_{xx}^2 = -5 \int_I x^m \rho_{xx} u_x - m \int_I x^{m-1} \rho_{xx}^2 u - 6 \int_I x^m \rho_x \rho_{xx} u_x u_x - 2 \int_I x^m \rho u_{xx} u_{xxx}$$

$$-4m \int_I x^{m-1} \rho_x \rho_{xx} u_x + 4m \int_I x^{m-2} \rho_x \rho_{xx} u$$

$$+ 4m \int_I x^{m-2} \rho \rho_{xx} u_x - 4m \int_I x^{m-3} \rho \rho_{xx} u - 2m \int_I x^{m-1} \rho \rho_{xx} u_{xx}$$

$$= \sum_{i=1}^{9} V_{II_i}. \tag{4.61}$$

For $V_{II_i}, i=1, 2, 3$, using Cauchy inequality, Lemma 4.14 and Corollary 4.21 we have

$$V_{II_1} + V_{II_2} + V_{II_3} \lesssim (\|u_x\|_{L^\infty} + \|u\|_{L^\infty}) \int_I x^m \rho_{xx}^2 + \int_I x^m \rho_x^2 u_x^2 + \int_I x^m \rho_x^2$$

$$\lesssim \int_I x^m \rho_{xx}^2 + \|\rho_x\|_{L^\infty}^2 \int_I x^m u_{xx}^2 \tag{4.62}$$

For $V_{II_i}, i=4, 5, 6$, using Cauchy inequality, Lemma 4.14 and Corollary 4.21 again, along with Lemma 4.2 we have

$$V_{II_4} + V_{II_5} + V_{II_6} \lesssim \int_I x^m \rho_{xx}^2 + \int_I x^m u_{xx}^2 + \int_I x^m \rho_x^2 u_x^2 + \int_I x^m \rho_x^2 u_x^2$$

$$\lesssim \int_I x^m \rho_{xx}^2 + \int_I x^m u_{xx}^2 + (\|u_x\|_{L^\infty}^2 + \|u\|_{L^\infty}^2) \int_I x^m \rho_x^2 \tag{4.63}$$

$$\lesssim \int_I x^m \rho_{xx}^2 + \int_I x^m u_{xx}^2 + 1.$$  

For $V_{II_i}, i=7, 8, 9$, using Lemma 4.2 and Corollary 4.21 we have

$$V_{II_7} + V_{II_8} + V_{II_9} \lesssim \int_I x^m \rho_{xx}^2 + \int_I x^m \rho_x^2 u_x^2 + \int_I x^m \rho_x^2 u_x^2$$

$$\lesssim \int_I x^m \rho_{xx}^2 + 1. \tag{4.64}$$

Substituting (4.62), (4.63) and (4.64) into (4.61), we have

$$\frac{d}{dt} \int_I x^m \rho_{xx}^2 \lesssim \int_I x^m \rho_{xx}^2 + \int_I x^m u_{xx}^2 + 1. \tag{4.65}$$

Differentiating (2.1.1) with respect to $x$, we have

$$\beta u_{xxx} = \rho_x u_t + \rho u_{xt} + \rho_x u_{xx} + \rho u_{x}^2 + \rho u_{xx} - \frac{2 \rho u_x}{x} - \frac{\rho_x v}{x} + P_{xx}$$

$$- \frac{m \beta (x u_{xx} - u_x)}{x^2} + \frac{m \beta (x u_x - 2u)}{x^3}. \tag{4.66}$$

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By (4.66), we have

\[
\int_{I} x^{m} u_{xxx}^{2} \lesssim \int_{I} x^{m} \rho_{x}^{2} u_{t}^{2} + \int_{I} x^{m} \rho_{x}^{2} u_{xt}^{2} + \int_{I} x^{m} \rho_{x}^{2} u_{x}^{2} + \int_{I} x^{m} \rho_{x}^{2} u_{x}^{2} + \int_{I} x^{m} \rho_{x}^{2} \rho_{xx}^{2}
\]

\[
+ \int_{I} x^{m} \rho_{x}^{2} v_{x}^{2} + \int_{I} x^{m} \rho_{x}^{2} v_{x}^{2} + \int_{I} x^{m} \rho_{x}^{2} v_{x}^{4} + \int_{I} x^{m} \rho_{x}^{2} v_{x}^{4} + \int_{I} x^{m} P_{xx}^{2} + \int_{I} x^{m} (u_{xx}^{2} + u_{x}^{2} + \frac{u_{x}^{2}}{x^{2}})
\]

\[
\lesssim \int_{I} x^{m} \rho_{x}^{2} u_{t}^{2} + \int_{I} x^{m} \rho_{x}^{2} u_{t}^{2} + \| u \|_{L_{\infty}}^{2} \| u_{x} \|_{L_{\infty}}^{2} \int_{I} x^{m} \rho_{x}^{2} + \| \rho \|_{L_{\infty}}^{2} \| u \|_{L_{\infty}}^{2} \int_{I} x^{m} \rho_{t}^{2}
\]

\[
+ \| \rho \|_{L_{\infty}}^{2} \| u \|_{L_{\infty}}^{2} \int_{I} x^{m} u_{xx}^{2} + \| \rho \|_{L_{\infty}}^{2} \| v \|_{L_{\infty}}^{2} \int_{I} x^{m} v_{xx}^{2} + \| v \|_{L_{\infty}}^{2} \int_{I} x^{m} \rho_{x}^{2}
\]

\[
\lesssim \int_{I} x^{m} \rho_{x}^{2} u_{t}^{2} + \int_{I} x^{m} \rho_{x}^{2} u_{t}^{2} + \int_{I} x^{m} P_{xx}^{2} + 1,
\]

(4.67)

where we have used Lemmas 4.2, 4.14 and Corollaries 4.16, 4.21. We need to control the right hand side of (4.67).

For the first one, we have

\[
\int_{I} x^{m} \rho_{x}^{2} u_{t}^{2} \lesssim \| u_{t} \|_{L_{\infty}}^{2} \int_{I} x^{m} \rho_{x}^{2} \lesssim \int_{I} x^{m} (u_{xx}^{2} + \frac{u_{x}^{2}}{x^{2}}).
\]

(4.68)

For the second one, we have

\[
\int_{I} x^{m} \rho_{x}^{2} u_{t}^{2} \lesssim \| \rho \|_{L_{\infty}}^{2} \int_{I} x^{m} u_{xx}^{2} \lesssim \int_{I} x^{m} u_{xx}^{2}.
\]

(4.69)

For the third one, recalling \( P = \rho Q + P_{c} \), we have

\[
\int_{I} x^{m} P_{xx}^{2} = \int_{I} x^{m} \rho_{xx} Q + 2 \rho_{x} Q' \theta_{x} + \rho Q'' \theta_{x} + \rho Q' \theta_{xx} + P_{c}' \rho_{x}^{2} + P_{c} \rho_{xx}^{2}
\]

\[
\lesssim \int_{I} x^{m} \rho_{xx}^{2} + \int_{I} x^{m} \theta_{xx}^{2} + 1,
\]

(4.70)

where we have used (A3), (A4), Lemmas 4.2, 4.14, 4.19, Corollary 4.20, and Sobolev inequality.

Substituting (4.68), (4.69) and (4.70) into (4.67), we have

\[
\int_{I} x^{m} u_{xxx}^{2} \lesssim \int_{I} x^{m} (u_{xt}^{2} + \frac{u_{x}^{2}}{x^{2}}) + \int_{I} x^{m} \rho_{xx}^{2} + \int_{I} x^{m} \theta_{xx}^{2} + 1.
\]

(4.71)

Putting (4.71) into (4.65), we have

\[
\frac{d}{dt} \int_{I} x^{m} \rho_{xx}^{2} \lesssim \int_{I} x^{m} (u_{xt}^{2} + \frac{u_{x}^{2}}{x^{2}}) + \int_{I} x^{m} \rho_{xx}^{2} + \int_{I} x^{m} \theta_{xx}^{2} + 1.
\]

(4.72)

By (4.72), Lemma 4.19, Corollary 4.21, and Gronwall inequality, we have

\[
\int_{I} x^{m} \rho_{xx}^{2} \leq C.
\]

(4.73)

Using Lemma 4.19 and Corollary 4.21 again, along with (4.71) and (4.73), we get

\[
\int_{Q_{T}} x^{m} u_{xxx}^{2} \leq C.
\]

\( \square \)
Corollary 4.23  Under the conditions of Theorem 2.1.1, we have for any \( t \in [0, T] \)
\[
\int_I x^m \rho_{2t}^2 + \| \rho \|_{W^{1, \infty} (Q_T)} + \| \rho_t \|_{L^\infty (Q_T)} + \int_{Q_T} x^m \rho_{tt}^2 \leq C.
\]

Proof. From (4.24), we have
\[
\int_I x^m \rho_{2t}^2 \lesssim \int_I x^m \rho_{xtu}^2 + \int_I x^m \rho_{2u}^2 + \int_I x^m \rho_{ux}^2 + \int_I x^m \rho_{u2t}^2 + \int_I x^m (\rho^2 u_{2t}^2 + \rho_{tt} u_t^2)
\]
\[
\lesssim \| u \|_{L^\infty}^2 \int_I x^m \rho_{xtu}^2 + \| \rho_u \|_{L^\infty}^2 \int_I x^m \rho_{ux}^2 + \| u_x \|_{L^\infty}^2 \int_I x^m \rho_{u2t}^2 + \| \rho \|_{L^\infty}^2 \int_I x^m u_{2t}^2
\]
\[
+ \| \rho \|_{L^\infty} \int_I x^m \rho_{tu}^2 + \| u \|_{L^\infty}^2 \int_I x^m \rho_{tt}^2
\]
\[
\lesssim \int_I x^m (u_{xt}^2 + u_{tt}^2) + 1,
\]
where we have used Lemmas 4.2, 4.14, Corollary 4.21 and (4.73).

By Sobolev inequality, Lemmas 4.2, 4.14, (4.73), (4.74), we have
\[
\| \rho \|_{W^{1, \infty} (Q_T)} + \| \rho_t \|_{L^\infty (Q_T)} \leq C.
\]

Differentiating (2.1.1) with respect to \( t \), we have
\[
\rho_{tt} = -(\rho_{xt} u + \rho_x u_t + \rho_t u_x + \rho u_{xt}) - \frac{m (\rho u_t + \rho_t u)}{x}.
\]

This deduces
\[
\int_I x^m \rho_{tt}^2 \lesssim \int_I x^m \rho_{xtu}^2 + \int_I x^m \rho_{2u}^2 + \int_I x^m \rho_{ux}^2 + \int_I x^m \rho_{u2t}^2 + \int_I x^m (\rho^2 u_{2t}^2 + \rho_{tt} u_t^2)
\]
\[
\lesssim \| u \|_{L^\infty}^2 \int_I x^m \rho_{xtu}^2 + \| \rho_u \|_{L^\infty}^2 \int_I x^m \rho_{ux}^2 + \| u_x \|_{L^\infty}^2 \int_I x^m \rho_{u2t}^2 + \| \rho \|_{L^\infty}^2 \int_I x^m u_{2t}^2
\]
\[
+ \| \rho \|_{L^\infty} \int_I x^m \rho_{tu}^2 + \| u \|_{L^\infty}^2 \int_I x^m \rho_{tt}^2
\]
\[
\lesssim \int_I x^m (u_{xt}^2 + u_{tt}^2) + 1,
\]
where we have used (4.74), Lemmas 4.2, 4.14, 4.19 and Corollary 4.21.

This combining Lemma 4.19 gives
\[
\int_{Q_T} x^m \rho_{tt}^2 \leq C.
\]
\( \square \)

Step 5: \( H^2 \) estimates of \( \theta \)

Lemma 4.24  Under the conditions of Theorem 2.1.1, we have for any \( t \in [0, T] \)
\[
\int_I x^m \rho \theta_t^2 + \int_{Q_T} x^m |(k \theta_x)_t|^2 \leq C.
\]

Proof. Differentiating (4.12) with respect to \( t \), we have
\[
\rho Q' \theta_{tt} + \rho Q'' \theta_t^2 + \rho_t Q' \theta_t + (\rho u Q' \theta_x)_t + \left( \rho \theta Q'(u_x + \frac{mu}{x}) \right)_t = (k \theta_x)_xt + \frac{m(k \theta_x)_t}{x} + \varphi_t (4.77)
\]
Multiplying (4.77) by $x^m \left(\int_0^\theta \kappa(\xi) \, d\xi\right)_t$ (i.e., $x^m \kappa(\theta)\theta_t$), and integrating by parts over $I$, we have

$$
\frac{1}{2} \frac{d}{dt} \int_I x^m \rho Q' \kappa \theta_t^2 + \int_I x^m |(\kappa \theta_x)_t|^2
= - \frac{1}{2} \int_I x^m \rho_t Q' \kappa \theta_t^2 - \frac{1}{2} \int_I x^m \rho Q'' \theta_t^3 \kappa + \frac{1}{2} \int_I x^m \rho Q' \theta_t^3 - \int_I x^m (\rho u Q' \theta_x)_t \kappa \theta_t
- \int_I x^m \left(\rho Q'(u_x + \frac{mu}{x})\right)_t \kappa \theta_t + \int_I x^m \varphi_t \kappa \theta_t
= \sum_{i=1}^6 \text{VIII}_i.
$$

For $\text{VIII}_1$, using (2.1.1), integration by parts, (A4), (A5), Lemma 4.2, Corollaries 4.12, 4.20 and Cauchy inequality and Poincaré inequality, we have

$$
\text{VIII}_1 = \frac{1}{2} \int_I \left(x^m \rho u\right)_x Q' \kappa \theta_t^2
= - \frac{1}{2} \int_I x^m \rho u Q' \kappa \theta_t^2 - \frac{1}{2} \int_I x^m \rho u Q' \kappa \theta_t \theta_x
\lesssim \|\theta_x\|_{L^\infty} \int_I x^m \rho^2 - \int_I x^m \rho u Q' \theta_t (\kappa \theta_x)_t + \int_I x^m \rho u Q' \theta_t^2 \theta_x
\leq C \|\theta_x\|_{L^\infty} \int_I x^m \rho^2 + \frac{1}{10} \int_I x^m |(\kappa \theta_x)_t|^2 + C \int_I x^m \rho^2
\leq C(1 + \|\theta_{xx}\|_{L^2}) \int_I x^m \rho \theta_t^2 + \frac{1}{10} \int_I x^m |(\kappa \theta_x)_t|^2.
$$

For $\text{VIII}_i$, $i = 2, 3$, we have

$$
\text{VIII}_2 + \text{VIII}_3 \lesssim \|\kappa \theta_t\|_{L^\infty} \int_I x^m \rho \theta_t^2
\lesssim \left(\|\kappa \theta_t\|_{L^2} + \int_I x^m \rho \kappa |\theta_t|\right) \int_I x^m \rho \theta_t^2
\leq \frac{1}{10} \int_I x^m |(\kappa \theta_x)_t|^2 + \left(\int_I x^m \rho \theta_t^2\right)^2 + C,
$$

where we have used (A4), (A5), Lemmas 3.1, 4.1, Corollary 4.20 and Cauchy inequality.

For $\text{VIII}_4$, we have

$$
\text{VIII}_4 = - \int_I x^m \rho_t u Q' \theta_x \kappa \theta_t - \int_I x^m \rho u Q' \theta_x \kappa \theta_t - \int_I x^m \rho u Q'' \theta_x \kappa \theta_t^2 - \int_I x^m \rho u Q' \theta_x \kappa \theta_t
= \sum_{j=1}^4 \text{VIII}_{4,j}.
$$

For $\text{VIII}_{4,1}$, using (2.1.1), (A4), (A5), integration by parts, Cauchy inequality, Lemmas 4.2
along with Lemma 4.2, we have

\[ V III_{4,1} = \int_I (x^m \rho u)x_u Q' \theta_x x \kappa \theta_t \]

\[ = - \int_I x^m \rho u x_u Q' \theta_x x \kappa \theta_t - \int_I x^m \rho u x Q'' \theta_x x \kappa \theta_t - \int_I x^m \rho u x Q' \theta_x x \kappa \theta_t - \int_I x^m \rho u x^2 \theta_x x \kappa \theta_t \]

\[ \leq C \int_I x^m \rho \theta_t^2 + C \int_I x^m \theta_x^2 + \frac{1}{20} \int_I x^m (\kappa \theta_x)_t^2 + C \]

\[ \leq C \left( \int_I x^m \rho \theta_t^2 \right)^2 + C \int_I x^m \theta_x^2 + \frac{1}{20} \int_I x^m (\kappa \theta_x)_t^2 + C. \]  

(4.82)

For \( V III_{4,2} \) and \( V III_{4,3} \), we have

\[ V III_{4,2} + V III_{4,3} \leq \| \theta_x \|_{L^\infty} \int_I x^m \rho u t^2 + \int_I x^m \rho \theta_t^2 + \| u \|_{L^\infty} \| \theta_x \|_{L^\infty} \int_I x^m \rho \theta_t^2 \]

\[ \leq \| \theta_x \|_{L^2}^2 + \| \theta_{xx} \|_{L^2}^2 + (1 + \| \theta_x \|_{L^2} + \| \theta_{xx} \|_{L^2}) \int_I x^m \rho \theta_t^2 \]

\[ \leq \| \theta_{xx} \|_{L^2}^2 + \left( \int_I x^m \rho \theta_t^2 \right)^2 + 1, \]  

(4.83)

where we have used (A4), (A5), Lemma 4.19, Corollaries 4.12, 4.20, and Sobolev inequality.

For \( V III_{4,4} \), using (A4), (A5), Lemma 4.19, Corollaries 4.12, 4.20, and Sobolev inequality again, along with Lemma 4.2, we have

\[ V III_{4,4} = - \int_I x^m \rho u Q' (\kappa \theta_x)_t \theta_t + \int_I x^m \rho u Q' \kappa \theta_x x \kappa \theta_t^2 \]

\[ \leq \frac{1}{20} \int_I x^m (\kappa \theta_x)_t^2 + C (\| \rho \|_{L^\infty} \| u \|_{L^\infty}^2 + \| u \|_{L^\infty} \| \theta_x \|_{L^\infty}) \int_I x^m \rho \theta_t^2 \]

\[ \leq \frac{1}{20} \int_I x^m (\kappa \theta_x)_t^2 + C \int_I x^m \theta_x^2 + C \left( \int_I x^m \rho \theta_t^2 \right)^2 + C. \]

Putting (4.82), (4.83) and (4.84) into (4.81), we have

\[ V III_4 \leq \frac{1}{10} \int_I x^m (\kappa \theta_x)_t^2 + C \left( \int_I x^m \rho \theta_t^2 \right)^2 + C \int_I x^m \theta_x^2 + C. \]  

(4.85)

For \( V III_5 \), we have

\[ V III_5 = - \int_I x^m \rho_t \theta Q' (u_x + \frac{m u}{x}) \kappa \theta_t - \int_I x^m \rho u Q' (u_x + \frac{m u}{x}) \kappa \theta_t^2 \]

\[ - \int_I x^m \rho u Q'' (u_x + \frac{m u}{x}) \kappa \theta_t^2 - \int_I x^m \rho u Q' (u_{xt} + \frac{m u}{x}) \kappa \theta_t \]

\[ = \sum_{j=1}^4 V III_{5,j}. \]
For $VIII_{5,1}$, we have

$$VIII_{5,1} = \int_I (x^m p u_x) \theta' Q(u_x + \frac{mu}{x}) \eta \theta_t$$

$$= -\int_I x^m p u_x \theta ' Q(u_x + \frac{mu}{x}) \eta \theta_t - \int_I x^m p u \theta x Q''(u_x + \frac{mu}{x}) \eta \theta_t$$

$$- \int_I x^m p u \theta x Q'(u_x + \frac{mu}{x}) \eta \theta_t - \int_I x^m p u \theta x Q'(u_x + \frac{mu}{x})(\eta \theta_t)_x$$

$$\leq C \int_I x^m \rho \theta^2_t + C \| u \|_{W^{1,\infty}} \int_I x^m \theta^2_x + C \int_I x^m (u^2_{xx} + u^2_x + \frac{u^2}{x^2}) + \frac{10}{10} \int_I x^m \| (\eta \theta_t)_t \|^2$$

where we have used (2.1.1), integration by parts, Cauchy inequality, $(A_4)$, $(A_5)$, Lemmas 4.2, 4.19, and Corollaries 4.20, 4.21.

For $VIII_{5,2}$ and $VIII_{5,3}$, we have

$$VIII_{5,2} + VIII_{5,3} \leq C \int_I x^m \rho \theta^2_t.$$  

(4.88)

For $VIII_{5,4}$, we have

$$VIII_{5,4} \leq C \int_I x^m \rho \theta^2_t + C \int_I x^m (u^2_{xx} + \frac{u^2}{x^2}).$$

(4.89)

Putting (4.87), (4.88) and (4.89) into (4.86), we have

$$VIII_5 \leq \frac{1}{10} \int_I x^m \| (\kappa \theta_x)_t \|^2 + C \int_I x^m \rho \theta^2_t + C \int_I x^m (u^2_{xx} + \frac{u^2}{x^2}) + C.$$  

(4.90)

For $VIII_6$, recalling $\varphi = \lambda (u_x + \frac{mu}{x})^2 + \mu (u_x^2 + 2u^2 + \frac{v^2}{x^2} - \frac{2mu}{x} + \frac{2m^2}{x^2})$, we have

$$VIII_6 \leq \| \kappa \theta_x \|_{L^\infty} \int_I x^m |\varphi|$$

$$\leq \| \kappa \theta_t \|_{L^\infty} \left( \left( \int_I x^m u^2_{xx} \right)^{\frac{1}{2}} + \left( \int_I x^m \frac{u^2}{x^2} \right)^{\frac{1}{2}} + \left( \int_I x^m (v^2_{xx} + \frac{v^2}{x^2}) \right)^{\frac{1}{2}} \right)$$

$$\leq \frac{1}{10} \int_I x^m \| (\kappa \theta_x)_t \|^2 + C \int_I x^m \rho \theta^2_t + C \int_I x^m u^2_{xx} + C \int_I x^m \frac{u^2}{x^2} + C \int_I x^m \frac{v^2}{x^2}$$

$$+ C \int_I x^m (v^2_{xx} + \frac{v^2}{x^2}),$$

where we have used Lemma 3.1 and Cauchy inequality.

Putting (4.79), (4.80), (4.85), (4.90) and (4.91) into (4.78), we have

$$\frac{d}{dt} \int_I x^m \rho Q' \kappa \theta^2_t + \int_I x^m \| (\kappa \theta_x)_t \|^2$$

$$\leq C \left( \int_I x^m \rho \theta^2_t \right)^2 + C \int_I x^m \theta^2_{xx} + C \int_I x^m (u^2_{xx} + \frac{u^2}{x^2} + w^2_{xx} + v^2_{xxx} + \frac{v^2}{x^2}) + C.$$  

(4.92)

By Gronwall inequality, we complete the proof of Lemma 4.24.
Corollary 4.25 Under the conditions of Theorem 2.1.1, we have
\[ \int_0^T \| \theta_t \|_{L^\infty}^2 \leq C. \]

Proof. By Lemma 3.1 and (A5), we get
\[
\int_0^T \| \theta_t \|_{L^\infty}^2 \lesssim \int_0^T \| \kappa \theta_t \|_{L^\infty}^2
\]
Remark 4.28 The global existence of (\ref{main_eq}) by the standard methods, see for instance \cite{4} and references therein. We end the proof of (\ref{main_eq}) with regularities as in Theorem \ref{thm:main}. The uniqueness of the solutions can be done constructing a sequence of approximate initial data and passing to the limits:

Consider (\ref{main_eq})-(\ref{main_eq_1}) with initial data replaced by \((\rho^\varepsilon, u^\varepsilon, v^\varepsilon, w^\varepsilon, \theta^\varepsilon)\) to (\ref{main_eq})-(\ref{main_eq_1}) for each \(\varepsilon > 0\). These \emph{a priori} estimates in the section are valid for \((\rho^\varepsilon, u^\varepsilon, v^\varepsilon, w^\varepsilon, \theta^\varepsilon)\). Then taking the limits \(\varepsilon \to 0^+\) (taking subsequence if necessary), and using Lemma 3.5 we get a solution denoted by \((\rho, u, v, w, \theta)\) to (\ref{main_eq})-(\ref{main_eq_1}) with regularities as in Theorem \ref{thm:main}. The uniqueness of the solutions can be done by the standard methods, see for instance \cite{4} and references therein. We end the proof of Theorem \ref{thm:main}. \hfill \Box

\noindent \textbf{Remark 4.28} The global existence of \((\rho^\varepsilon, u^\varepsilon, v^\varepsilon, w^\varepsilon, \theta^\varepsilon)\) can be done by local existence (using the similar arguments as in \cite{4})+ some \emph{a priori} estimates globally in time throughout the section.

5 Proof of Theorem \ref{thm:main_new}

In the section, we denote by \(C\) a generic constant depending only on \(\|(\rho_0, u_0, v_0, w_0, \theta_0)\|_{H^3}, \|g_i\|_{L^2}, \|((\sqrt{\rho_0}g_i)_x)\|_{L^2} \) \((i = 1, 2, 3, 4)\), \(T, \lambda, \mu, a, b\), and some other known constants, but independent of the solutions and the lower bounds of the density.

Under the conditions of Theorem \ref{thm:main_new} all those \emph{a priori} estimates in section 3 are also satisfied in this section. Following the quite similar strategies with the proof of Theorem \ref{thm:main} to prove Theorem \ref{thm:main_new} it suffices to get \(H^3\)-\emph{a-priori}-estimates of the classical solutions \((\rho, u, v, w, \theta)\) as in Theorem \ref{thm:main} with \(\inf_{(x,t)\in Q_T} \rho > 0\).

To begin with, we get a \(W^{1, \infty}(Q_T)\)-estimate of \(\sqrt{\rho}\) which was obtained in \cite{8} \cite{7} as a crucial estimate to get \(H^3\)-estimate of \(u\).

\noindent \textbf{Lemma 5.1} Under the conditions of Theorem \ref{thm:main_new} we have

\[\|(\sqrt{\rho})_x\|_{L^\infty(Q_T)} + \|(\sqrt{\rho})_t\|_{L^\infty(Q_T)} \leq C.\]
Proof. Multiplying \((2.1.1)_1\) by \(\frac{1}{2\sqrt{\rho}}\), we have
\[
(\sqrt{\rho})_{tt} + \frac{mx^{-1}}{2}\sqrt{\rho}u + \frac{1}{2}\sqrt{\rho}u_x + (\sqrt{\rho})_x u = 0. \tag{5.1}
\]
Differentiating \((5.1)\) with respect to \(x\), we get
\[
(\sqrt{\rho})_{xt} + \frac{mx^{-1}}{2}(\sqrt{\rho})_x u + \frac{mx^{-1}}{2}\sqrt{\rho}u_x - \frac{m\sqrt{\rho}u}{2x^2} + \frac{m\sqrt{\rho}u_x}{2x} + \frac{3}{2}(\sqrt{\rho})_{xx} + \frac{1}{2}\sqrt{\rho}u_{xx} + (\sqrt{\rho})_{xx} u = 0.
\]
Denote \(h = (\sqrt{\rho})_x\), we have
\[
h_{tt} + h_x u + h\left(\frac{mx^{-1}u}{2} + \frac{3u_x}{2}\right) + \frac{mx^{-1}}{2}\sqrt{\rho}u_x - \frac{m\sqrt{\rho}u}{2x^2} + \frac{1}{2}\sqrt{\rho}u_{xx} = 0,
\]
which implies
\[
\frac{d}{dt} \left\{ h \exp \left[ \int_0^t \left( \frac{mx^{-1}u}{2} + \frac{3u_x}{2} \right) (x(\tau, y), \tau) d\tau \right] \right\} = - \left( \frac{mx^{-1}\sqrt{\rho}u_x}{2} - \frac{m\sqrt{\rho}u}{2x^2} + \frac{\sqrt{\rho}u_{xx}}{2} \right)
\times \exp \left[ \int_0^t \left( \frac{mx^{-1}u}{2} + \frac{3u_x}{2} \right) (x(\tau, y), \tau) d\tau \right],
\]
where \(x(t, y)\) satisfies
\[
\begin{cases}
\frac{dx(t, y)}{dt} = u(x(t, y), t), & 0 \leq t < s, \\
x(s, y) = y.
\end{cases}
\]
Integrating \((5.2)\) over \((0, s)\), we get
\[
h(y, s) = \exp \left( - \int_0^s \left( \frac{mx^{-1}u}{2} + \frac{3u_x}{2} \right) (x(\tau, y), \tau) d\tau \right) h(x(0, y), 0) - \int_0^s \left[ \left( \frac{mx^{-1}\sqrt{\rho}u_x}{2} - \frac{m\sqrt{\rho}u}{2x^2} + \frac{\sqrt{\rho}u_{xx}}{2} \right)
\times \exp \left( \int_\tau^s \left( \frac{mx^{-1}u}{2} + \frac{3u_x}{2} \right) (x(\tau, y), \tau) d\tau \right) \right] d\tau.
\]
This implies
\[
\| (\sqrt{\rho})_x \|_{L^\infty(Q_T)} \leq C. \tag{5.3}
\]
From \((5.1)\) and \((5.3)\), we get
\[
\| (\sqrt{\rho})_x \|_{L^\infty(Q_T)} \leq C.
\]
The proof of Lemma 5.1 is complete. \(\square\)

Lemma 5.2 Under the conditions of Theorem 2.1.6 we have for any \(t \in [0, T]\)
\[
\int_I x^m \rho^2 |(\kappa\theta_x)_t|^2 + \int_{Q_T} x^m \rho^3 \theta_{tt}^2 \leq C.
\]
Proof. Multiplying (4.77) by \(x^m \rho^a(\kappa \theta_t)_t\) (i.e. \(x^m \rho^a \kappa \theta_{tt} + x^m \rho^a \kappa' \theta_t^2\), where \(\alpha > 0\) is to be decided later), and integrating by parts over \(I\), we have

\[
\int_I x^m \rho^{\alpha+1} \kappa Q' \theta_t^2 + \frac{1}{2} \frac{d}{dt} \int_I x^m \rho^a |(\kappa \theta_x)_t|^2
\]

\[
= \frac{\alpha}{2} \int_I x^m \rho^{\alpha-1} \rho_t |(\kappa \theta_x)_t|^2 - \alpha \int_I x^m \rho^{\alpha-1} \rho_x \kappa \theta_{tt}(\kappa \theta_x)_t - \alpha \int_I x^m \rho^{\alpha-1} \rho_x \kappa' \theta_t^2(\kappa \theta_x)_t
\]

\[+
\int_I x^m \rho^a \phi_{n}(\kappa \theta_t + \kappa' \theta_t^2) - \int_I x^m \rho^{\alpha+1} Q' \kappa' \theta_t^2 - \int_I x^m \rho^a \rho \rho_{Q'} Q^2 + \rho_{Q'} \theta |(\kappa \theta_{tt} + \kappa' \theta_t^2) (5.4)
\]

\[- \int_I x^m \rho^a (\rho u Q'(\theta_x))_t (\kappa \theta_{tt} + \kappa' \theta_t^2) - \int_I x^m \rho^a \left( \rho \theta Q(\theta_x + \frac{nu}{x}) \right)_t (\kappa \theta_{tt} + \kappa' \theta_t^2) = \sum_{i=1}^{8} VIV_i.
\]

For \(VIV_2\), using (A4), (A5), Corollary 4.20 and Cauchy inequality, we have

\[
VIV_2 \leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q' \theta_t^2 + C \int_I x^m \rho^{\alpha-3} \rho_x^2 |(\kappa \theta_x)_t|^2
\]

\[
\leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q' \theta_t^2 + C ||\rho^{\alpha-3} \rho_x^2||_{L^\infty} \int_I x^m |(\kappa \theta_x)_t|^2. \tag{5.5}
\]

For \(VIV_1\) and \(VIV_3\), we have

\[
VIV_1 + VIV_3 \leq C ||\rho^{\alpha-1} \theta_t||_{L^\infty} \int_I x^m |(\kappa \theta_x)_t|^2 + C ||\theta_t||^2_{L^\infty} \int_I x^m \rho^a |(\kappa \theta_x)_t|^2
\]

\[+ C ||\theta_t||^2_{L^\infty} \int_I x^m \rho^{\alpha-2} \rho_x^2. \tag{5.6}
\]

For \(VIV_4\), we have

\[
VIV_4 \leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q' \theta_t^2 + C ||\rho^{\alpha-1}||_{L^\infty} \int_I x^m \phi_t^2 + C \int_I x^m \rho^{\alpha+1} \theta_t^4
\]

\[
\leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q' \theta_t^2 + C ||\rho^{\alpha-1}||_{L^\infty} \int_I x^m (\frac{u^2_{tt}}{x^2} + \frac{u^2_t}{x^2} + \frac{v^2_{tt}}{x^2} + v^2_t + \frac{v^2}{x^2}) + C ||\theta_t||^2_{L^\infty}, \tag{5.7}
\]

where we have used Cauchy inequality, Corollaries 4.16 4.18 4.21 and Lemmas 4.2 4.24.

For \(VIV_5\) and \(VIV_6\), using (A4), (A5), Cauchy inequality, Corollary 4.20 and Lemmas 4.2 4.24 we have

\[
VIV_5 + VIV_6 = - \int_I x^m \rho^{\alpha+1} Q' \kappa' \theta_t^2 \theta_{tt} + \int_I x^m \rho^a [\rho Q'' \theta_t^2 + \rho_t Q' \theta_t] \kappa \theta_{tt}
\]

\[+
\int_I x^m \rho^a \rho \rho_{Q'} Q^2 + \int_I x^m \rho^a \rho_{Q'} \theta |(\kappa \theta_{tt} + \kappa' \theta_t^2) \tag{5.8}
\]

\[\leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q' \theta_t^2 + C \int_I x^m \rho^{\alpha+1} \theta_t^4 + C \int_I x^m \rho^{\alpha-1} \rho_t^2 \theta_t^2
\]

\[\leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q' \theta_t^2 + C ||\theta_t||^2_{L^\infty} (1 + \int_I x^m \rho^{\alpha-1} \rho_t^2) + C.
\]

For \(VIV_7\), using (A4), (A5), Cauchy inequality, and Lemmas 4.2 4.24 again, along with
Corollaries \ref{cor:4.12} \ref{cor:4.27} and Lemma \ref{lem:4.19} we have

\begin{equation}
VIV_7 = - \int_I x^m \rho \rho_t Q'_x \left( \kappa \theta_t + \kappa' \theta_t^2 \right) - \int_I x^m \rho^{\alpha+1} u_t Q'_x \left( \kappa \theta_t + \kappa' \theta_t^2 \right) - \int_I x^m \rho \rho_t u''_t Q'_x \left( \kappa \theta_t + \kappa' \theta_t^2 \right) - \int_I x^m \rho^{\alpha+1} u_q Q''_t \left( \kappa \theta_t + \kappa' \theta_t^2 \right) \leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q'_t \theta_t^2 + C \int_I x^m \rho^{-1} \rho_t^2 + C \int_I x^m \rho^{\alpha+1} \theta_t^4 + C \int_I x^m \rho^{\alpha+1} u_t^2 \tag{5.9}
\end{equation}

Similarly, for \( VIV_8 \), we have

\begin{equation}
VIV_8 = - \int_I x^m \rho \rho_t Q'(u_x + \frac{mu}{x}) \left( \kappa \theta_t + \kappa' \theta_t^2 \right) - \int_I x^m \rho^{\alpha+1} \theta_t Q'(u_x + \frac{mu}{x}) \left( \kappa \theta_t + \kappa' \theta_t^2 \right) - \int_I x^m \rho^{\alpha+1} Q'' \theta_t(u_x + \frac{mu}{x}) \left( \kappa \theta_t + \kappa' \theta_t^2 \right) - \int_I x^m \rho^{\alpha+1} \theta_t Q'(u_{xt} + \frac{mu}{x}) \left( \kappa \theta_t + \kappa' \theta_t^2 \right) \leq \frac{1}{10} \int_I x^m \rho^{\alpha+1} \kappa Q'_t \theta_t^2 + C \int_I x^m \rho^{-1} \rho_t^2 + C \int_I x^m \rho^{\alpha+1} \theta_t^4 + C \int_I x^m (u_{xt}^2 + \frac{u_t^2}{x^2}) \tag{5.10}
\end{equation}

Putting (5.5), (5.6), (5.7), (5.8), (5.9) and (5.10) into (5.4), we have

\begin{equation}
\int_I x^m \rho^{\alpha+1} \kappa Q'_t \theta_t^2 + \frac{d}{dt} \int_I x^m \rho^\alpha |(\kappa \theta_x)_t|^2 \leq C \| \theta_t \|_{L^\infty} \int_I x^m \rho^\alpha |(\kappa \theta_x)_t|^2 + C (\| \rho^{\alpha-3} \rho_t^2 \|_{L^\infty} + \| \rho^{\alpha-1} \rho_t \|_{L^\infty} ) \int_I x^m |(\kappa \theta_x)_t|^2 \tag{5.11}
\end{equation}

\begin{equation}
+ C (1 + \int_I x^m \rho^{-1} \rho_t^2 + \int_I x^m \rho^\alpha \rho_t^2) \| \theta_t \|_{L^\infty}^2 + C \int_I x^m \rho^{\alpha-1} \rho_t^2 + C \int_I x^m \theta_{xt}^2 + C \int_I \rho^\alpha \| x^m \|_{L^\infty} \int_I x^m (u_{xt}^2 + \frac{u_t^2}{x^2}) + C \int_I x^m (u_{xt}^2 + \frac{u_t^2}{x^2}) + C.
\end{equation}

Claim:

\begin{equation}
\int_I x^m \rho^\alpha |(\kappa \theta_x)_t|^2 + \int_{Q_T} x^m \rho^{\alpha+1} \kappa Q'_t \theta_t^2 \leq C, \tag{5.12}
\end{equation}

if

\begin{equation}
\| \rho^{\alpha-3} \rho_t^2 \|_{L^\infty} + \| \rho^{\alpha-1} \rho_t \|_{L^\infty} + \| \rho^{\alpha-1} \|_{L^\infty} \leq C. \tag{5.13}
\end{equation}

In fact, if (5.13) is satisfied, (5.12) will be obtained by using (5.11), Corollaries \ref{cor:4.25} \ref{cor:4.26}, Lemmas \ref{lem:4.15} \ref{lem:4.17} \ref{lem:4.19} \ref{lem:4.24} \ref{lem:2.1.4} and Gronwall inequality.

To ensure that (5.13) is valid, the restriction \( \alpha \geq 3 \) seems necessary. While, to get \( L^\infty H^3_x \) estimate of \( \theta \) (Corollary \ref{cor:5.3}), by \ref{lem:4.2}, \( \alpha \) in (5.12) should satisfy \( \alpha \leq 2 \). Assume \( \alpha \in [1, 2] \), to get (5.13), it suffices to prove

\begin{equation}
\| \rho^{\alpha-3} \rho_t^2 \|_{L^\infty} \leq C.
\end{equation}
In fact,
\[ \| \rho^{\alpha-3} \rho_x^2 \|_{L^\infty} = 4 \| \rho^{\alpha-2} \|_{(\sqrt{\rho})_x^2} \|_{L^\infty}. \]

Assume \( \alpha = 2 \), and use Lemma 5.1, (5.13) is to be obtained. This means that (5.12) is obtained for \( \alpha = 2 \). The proof of Lemma 5.2 is complete. \( \square \)

**Corollary 5.3** Under the conditions of Theorem 2.1.6, we have for any \( t \in [0, T] \)
\[ \int_I x^m (\theta^2_{xxx} + \rho^2 \theta^2_{xt}) \leq C. \]

**Proof.** A direct calculation gives
\[ \rho \kappa \theta_{xt} = \rho (\kappa \theta_x)_t - \rho \kappa' \theta_t \theta_x, \]
which implies
\[ \int_I x^m \rho^2 \theta^2_{xx} \leq C \int_I x^m \rho^2 (\kappa \theta_x)_t^2 + C \| \theta_x \|_{L^\infty}^2 \int_I x^m \rho \theta^2_t \]
\[ \leq C, \]
where we have used \((A_5)\), Lemma 4.2, Lemma 4.24, Lemma 5.2 and Corollary 4.27.

From the first inequality of (4.94), using Lemma 4.24, (5.14) and Lemma 5.1, we obtain
\[ \int_I x^m \theta^2_{xxx} \leq C \int_I x^m \rho^2 \theta^2_{xx} + C \int_I x^m \rho \theta^2_{xt} \]
\[ \leq C, \]
\[ \square \]

**Lemma 5.4** Under the conditions of Theorem 2.1.6, we have for any \( t \in [0, T] \)
\[ \int_I x^m \rho^2 u^2_{xt} + \int_{Q_T} x^m \rho^3 u^2_{tt} \leq C. \]

**Proof.** Similar to Lemma 5.2, multiplying (4.42) by \( x^m \rho^2 u_{tt} \), and integrating by parts over \( I \), we have
\[ \int_I x^m \rho^3 u^2_{xt} + \frac{\beta}{2} \frac{d}{dt} \int_I x^m \rho^2 u^2_{xt} \]
\[ = \beta \int_I x^m \rho \rho_t u^2_{xt} - 2 \beta \int_I x^m \rho \rho_x u_{xt} u_t - \int I x^m \rho^2 u_t (\rho_t u_t + \rho u u_x + \rho u u_x + \rho u u_x - \frac{\rho v^2}{x} - \frac{2 \rho v v_t}{x}) \]
\[ - \beta \int_I x^m \rho^2 u_{xt} \frac{m u_t}{x^2} - \int_I x^m \rho^2 u_{tt} P_{xt} \]
\[ = \sum_{i=1}^5 VV_i. \]

For \( VV_1 \) and \( VV_2 \), using Corollary 4.23, Lemma 5.1 and Cauchy inequality, we have
\[ VV_1 + VV_2 \leq \| \rho \|_{L^\infty} \| \rho_t \|_{L^\infty} \int_I x^m u^2_{xt} - 4 \beta \int_I x^m \rho^2 (\sqrt{\rho})_x u_{xt} u_t \]
\[ \leq C \int_I x^m u^2_{xt} + \frac{1}{8} \int_I x^m \rho^3 u^2_{tt}. \]

(5.16)
For \( VV_3 \), we have

\[
VV_3 \leq \frac{1}{8} \int_I x^m \rho^3 u_{tt}^2 + C \int_I x^m \rho^2 u_t^2 + \int_I x^m \rho^2 u_x^2 + C \int_I x^m \rho^3 u_t^2 u_x^2 \\
+ C \int_I x^m \rho^3 u_{xt}^2 + C \int_I x^m \rho^2 u_t^2 + C \int_I x^m \rho^3 u_{xt}^2 v^2 \\
\leq \frac{1}{8} \int_I x^m \rho^3 u_{tt}^2 + C \| \rho_t \|_{L^\infty} \int_I x^m \rho u_t^2 + C \| \rho \|_{L^\infty} \| \rho_t \|_{L^\infty} \int_I x^m u_x^2 \\
+ C \| u_x \|_{L^\infty} \| \rho \|_{L^\infty} \int_I x^m \rho u_t^2 + C \| \rho \|_{L^\infty} \int_I x^m u_x^2 \\
+ C \| \rho_t \|_{L^\infty} \| v \|_{L^\infty} \int_I x^m \rho v^2 + C \| v \|_{L^\infty} \| \rho \|_{L^\infty} \int_I x^m \rho v_t^2 \\
\leq \frac{1}{8} \int_I x^m \rho^3 u_{tt}^2 + C \int_I x^m u_{xt}^2 + C,
\]

where we have used Cauchy inequality, Corollaries 4.8, 4.21, 4.23, and Lemmas 4.15, 4.19.

For \( VV_4 \), using Cauchy inequality and Lemma 4.19 we have

\[
VV_4 \leq \frac{1}{8} \int_I x^m \rho^3 u_{tt}^2 + C \int_I x^m \rho u_t^2 \leq \frac{1}{8} \int_I x^m \rho^3 u_{tt}^2 + C.
\]

For \( VV_5 \), recalling \( P = \rho Q + P_c \), we have

\[
VV_5 = -\int_I x^m \rho^3 u_{tt} (\rho_x Q + \rho_x Q' \theta_t + \rho_t Q' \theta_x + \rho Q'' \theta_x + \rho Q' \theta_x + P' \rho_{xt} + P'' \rho_t) \\
\leq \frac{1}{8} \int_I x^m \rho^3 u_{tt}^2 + C \int_I x^m \rho u_t^2 + \int_I x^m \rho \rho_x^2 (Q')^2 \theta_t^2 + \int_I x^m \rho \rho_x^2 (Q')^2 \theta_x^2 \\
+ \int_I x^m \rho (Q'')^2 \theta_t^2 + \int_I x^m \rho (P')^2 \rho_{xt}^2 + \int_I x^m \rho (P'')^2 \rho_t^2 \\
\leq \frac{1}{8} \int_I x^m \rho^3 u_{tt} + C,
\]

where we have used Cauchy inequality, (A3), (A4), Corollaries 4.23, 4.27, 5.3 and Lemma 4.24.

Putting (5.16), (5.17), (5.18) and (5.19) into (5.15), we have

\[
\int_I x^m \rho^3 u_{tt}^2 + \beta \frac{d}{dt} \int_I x^m \rho^2 u_{xt}^2 \leq C \int_I x^m u_{xt}^2 + C.
\]

By (5.20), Lemma 4.19 and Gronwall inequality, we complete the proof of Lemma 5.4.

**Corollary 5.5** Under the conditions of Theorem 2.1.6, we have for any \( t \in [0, T] \)

\[
\int_I x^m u_{xxx}^2 \leq C.
\]

**Proof.** By (4.67), (4.70), Lemmas 4.19, 4.22, 5.1, 5.4 Corollaries 4.27 we get

\[
\int_I x^m u_{xxx}^2 \leq C \int_I x^m \rho^2 u_t^2 + C \int_I x^m \rho^2 u_x^2 + C \int_I x^m P_{xx}^2 + C \\
\leq C \int_I x^m \rho (\sqrt[3]{\rho})^2 u_t^2 + C \leq C.
\]
Lemma 5.6 Under the conditions of Theorem 2.1.6 we have for any $t \in [0, T]$
\[
\int_I x^m \rho^2 (v_{xt}^2 + w_{xt}^2) + \int_{Q_T} x^m \rho^3 (v_{tt}^2 + w_{tt}^2) \leq C.
\]

Proof. Multiplying (4.36) by $x^m \rho^2 v_{tt}$, integrating by parts over $I$, we have
\[
\int_I x^m \rho^3 v_{tt}^2 + \frac{\mu}{2} \int_I x^m \rho^2 v_{xt}^2 = \mu \int_I x^m \rho \rho_x v_{xt}^2 - 2\mu \int_I x^m \rho v_x v_{xt} v_{tt} - \int_I x^m \rho^2 v_{tt} (\rho_t v_t + \rho_t u v_x + \rho u v_x + \rho u v_t)
\]
\[
- \int_I x^m \rho^2 v_{tt} (-1) \rho_t u v + (-1) \rho_t v x + (-1) \rho u v_t - \mu \int_I x^m \rho^2 v_{tt} \frac{m v_t}{x^2}
\]
\[
= \sum_{i=1}^5 VVI_i.
\]

For $VVI_1$ and $VVI_2$, using Cauchy inequality, Corollary 4.23 and Lemma 5.1, we have
\[
VVI_1 + VVI_2 \leq \|\rho\|_{L^\infty} \|\rho_t\|_{L^\infty} \int_I x^m v_{xt}^2 - 4\mu \int_I x^m \rho^2 (\sqrt{\rho}) v_{xt} v_{tt}
\]
\[
\leq \frac{1}{8} \int_I x^m \rho^3 v_{tt}^2 + C \int_I x^m v_{xt}^2.
\]

For $VVI_3$, using Cauchy inequality, Corollary 4.23 again, together with Corollaries 4.12, 4.16 and Lemmas 4.15, 4.19 we have
\[
VVI_3 \leq \frac{1}{8} \int_I x^m \rho^3 v_{tt}^2 + C \|\rho_t\|_{L^\infty} \int_I x^m \rho v_t^2 + C \|\rho\|_{L^\infty} \|\rho_t\|_{L^\infty} \|u\|_{L^\infty}^2 \int_I x^m v_{xt}^2
\]
\[
+ C \|v_x\|_{L^\infty} \|\rho\|_{L^\infty}^2 \int_I x^m \rho u_t^2 + C \|\rho\|_{L^\infty} \|u\|_{L^\infty}^2 \int_I x^m v_{xt}^2
\]
\[
\leq \frac{1}{8} \int_I x^m \rho^3 v_{tt}^2 + C \int_I x^m v_{xt}^2 + C.
\]

Similarly, for $VVI_4$ and $VVI_5$, we have
\[
VVI_4 \leq \frac{1}{8} \int_I x^m \rho^3 v_{tt}^2 + C \|\rho\|_{L^\infty} \|u\|_{L^\infty}^2 \|v\|_{L^\infty} \int_I x^m \rho v_t^2 + C \|\rho\|_{L^\infty} \|v\|_{L^\infty}^2 \int_I x^m \rho u_t^2
\]
\[
+ C \|\rho\|_{L^\infty} \|u\|_{L^\infty} \|v\|_{L^\infty} \int_I x^m \rho v_t^2
\]
\[
\leq \frac{1}{8} \int_I x^m \rho^3 v_{tt}^2 + C;
\]
\[
VVI_5 \leq \frac{1}{8} \int_I x^m \rho^3 v_{tt}^2 + C \int_I x^m v_{tt}^2 \leq \frac{1}{8} \int_I x^m \rho^3 v_{tt}^2 + C.
\]

Putting (5.23), (5.24), (5.25) and (5.26) into (5.22), and using Lemma 4.15, Gronwall inequality, we have
\[
\int_I x^m \rho^2 v_{xt}^2 + \int_{Q_T} x^m \rho^3 v_{tt}^2 \leq C.
\]

Similarly, we get
\[
\int_I x^m \rho^2 w_{xt}^2 + \int_{Q_T} x^m \rho^3 w_{tt}^2 \leq C.
\]
\[
\square
\]
Corollary 5.7: Under the conditions of Theorem 2.1.6, we have for any \( t \in [0, T] \)
\[
\int_I x^m(v_{xxx}^2 + w_{xxx}^2) + \int_{Q_T} (v_{xxxx}^2 + w_{xxxx}^2) \leq C.
\]

**Proof.** Similar to Corollary 5.5, we get
\[
\int_I x^m(v_{xxx}^2 + w_{xxx}^2) \leq C.
\]
By (4.36), Corollaries 4.12, 4.16, 4.23, and Lemmas 4.15, 4.19, 5.6, we have
\[
\int_{Q_T} \rho v_{xxx}^2 \leq C.
\]
This together with (2.1.1) gives
\[
\int_{Q_T} v_{xxxx}^2 \leq C.
\]
Similarly, we can get
\[
\int_{Q_T} w_{xxxx}^2 \leq C.
\]
\[\square\]

Lemma 5.8: Under the conditions of Theorem 2.1.6, we have for any \( t \in [0, T] \)
\[
\int_I x^m \rho_{xxx}^2 + \int_{Q_T} x^m (u_{xxxx}^2 + \theta_{xxxx}^2) \leq C.
\]

**Proof.** Differentiating (4.66) with respect to \( x \), multiplying it by \( x^m \rho_{xxx} \), and integrating by parts over \( I \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_I x^m \rho_{xxx}^2 = \int_I x^m \rho_{xxx} \left[-mx^{-1} \rho_{xxx} - 3mx^{-1} \rho_x u_{xxx} - 3mx^{-1} \rho_x u_x - \rho_{xxx} u_x + \frac{3m \rho u_{xxx}}{x^2} \right.
\]
\[- \frac{6m \rho u_x}{x^3} - mx^{-1} \rho_{xxx} u + \frac{3m \rho u_x}{x^2} + \frac{6m \rho u_x}{x^2} - \frac{6m \rho u_x}{x^3} + \frac{6m \rho u}{x^4} - 4 \rho_{xxx} u_x - 6 \rho_{xx} u_{xx} - 4 \rho_x u_{xxx} \bigg] + \frac{1}{2} \int_I x^m \rho_{xxx}^2 u_x + \frac{1}{2} \int_I mx^{-1} \rho_{xxx}^2 u - \int_I x^m \rho_{xxx} u_{xxx}. \]
By Sobolev inequality, Cauchy inequality, Corollaries 4.16, 4.21, 4.23, 4.27, 5.3, 5.5, and Lemma 4.22, we get
\[
\frac{d}{dt} \int_I x^m \rho_{xxx}^2 \leq C \int_I x^m \rho_{xxx}^2 + C \int_I x^m u_{xxxx}^2 + C. \quad (5.28)
\]

Differentiating (4.66) with respect to \( x \), we have
\[
\beta u_{xxxx} = \rho_{xx} u_t + 2 \rho_x u_{xt} + u_{xxx} + (\rho_x u_x + \rho u_{xx})_x - \left( \frac{2 \rho v_x}{x} + \frac{\rho_x v^2}{x} - \frac{\rho v^2}{x^2} \right)_x \quad (5.29)
\]
Using Lemma 4.22, Corollaries 4.16, 4.21, 4.23, 4.27, 5.3, 5.5, and (A3), (A4), we obtain
\[
\int_I x^m u_{xxxx}^2 \leq C \int_I x^m |P_{xxx}|^2 + C \int_I x^m (\rho_{xxx}^2 + u_{ox}^2) + C \leq C \int_I x^m \rho_{xxx}^2 + C \int_I x^m (\rho_{xxx}^2 + u_{ox}^2) + C \quad (5.30)
\]
Substituting (5.30) into (5.28), and using (4.42), Lemmas 4.19, 5.4, Gronwall inequality, we obtain
\[ \int_I x^m p_{xxx}^2 \leq C. \] (5.31)

By (5.30) and (5.31), we have
\[ \int_{Q_T} x^m u_{xxx}^2 \leq C. \]

Differentiating (4.93) with respect to \( x \), and using (A4), (A5), Corollaries 4.16, 4.18, 4.21, 4.23, 4.25, 4.26, 4.27, 5.3, and Lemmas 4.22, 5.2, we have
\[ \int_{Q_T} x^m \theta_{xxx}^2 \leq C. \]

The proof of Lemma 5.8 is complete. \( \square \)

6 Proof of Theorem 2.2.2

Let \( 0 < T^* < \infty \) be the maximum time of existence of strong solution \((\rho, u, \theta)\) to (2.2.1)-(2.2.3). Namely, \((\rho, u, \theta)\) is a strong solution to (2.2.1)-(2.2.3) in \( \mathbb{R}^3 \times [0, T] \) for any \( 0 < T < T^* \), but not a strong solution in \( \mathbb{R}^3 \times [0, T^*] \). We shall prove Theorem 2.2.2 by using a contradiction argument. Suppose that (2.2.5) were false, i.e.
\[ M := \limsup_{t \to T^*} (\|\rho(t)\|_{L^\infty} + \int_0^t \|\rho \theta(s)\|_{L^{12}}^4 ds) < \infty. \] (6.1)

The goal is to show that under the assumption (6.1), there is a bound \( C > 0 \) depending only on \( M, \rho_0, u_0, \theta_0, \mu, \lambda, \kappa, \) and \( T^* \) such that
\[ \sup_{0 \leq t < T^*} \|\theta(t)\|_{L^\infty} \leq C. \] (6.2)

With (6.2) and (6.1), we showed in our previous paper [43] that \( T^* \) is not the maximum time, which is the desired contradiction.

Throughout the rest of the section, we denote by \( C \) a generic constant depending only on \( \rho_0, u_0, \theta_0, T^*, M, \lambda, \mu, \kappa \). We denote by
\[ A \lesssim B \]
if there exists a generic constant \( C \) such that \( A \leq CB \).

**Lemma 6.1** Under the conditions of Theorem 2.2.2 and (6.1), it holds that
\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho \leq C, \text{ for any } T \in [0, T^*). \] (6.3)

**Proof.** Integrating (2.2.1)1 over \( \mathbb{R}^3 \times [0, t] \), for \( t < T^* \), and using the assumption \( \rho_0 \in L^1 \), we get (6.3). \( \square \)

**Lemma 6.2** Under the conditions of Theorem 2.2.2 and (6.1), if \( 3\mu > \lambda \), it holds that
\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho|u|^4 + \int_0^T \int_{\mathbb{R}^3} |u|^6 |\nabla u|^2 dx \leq C, \] (6.4)
for any \( T \in (0, T^*) \).
Proof. The detailed proof of Lemma 6.2 could be found in [43], which might be slightly modified.

Lemma 6.3 Under the conditions of Theorem 2.2.2 and (6.1), it holds that for any \( T \in [0,T^*) \)

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} (\rho|\theta|^2 + |\nabla u|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho|u_t|^2) \, dx \, dt \leq C. \tag{6.5}
\]

Proof. Multiplying \( (2.2.1)_2 \) by \( u_t \), and integrating by parts over \( \mathbb{R}^3 \), we have

\[
\int_{\mathbb{R}^3} |u_t|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu|\nabla u|^2 + (\mu + \lambda)|\text{div}u|^2) = \frac{d}{dt} \int_{\mathbb{R}^3} P \text{div}u - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P_t G - \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot u_t \tag{6.6}
\]

where \( G = (2\mu + \lambda)\text{div}u - P \).

Recalling \( P = \rho \theta \), we obtain from \( (2.2.1)_1 \) and \( (2.2.1)_3 \)

\[
P_t = -\text{div}(Pu) - \rho \theta \text{div}u + \mu (\nabla u + (\nabla u)^\top) : \nabla u + \lambda \text{div}\text{div}u + \Delta \theta. \tag{6.7}
\]

Substituting (6.7) into \( VVII_3 \), and using integration by parts and Hölder inequality, we have

\[
VVII_3 = -\frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} Pu \cdot \nabla G + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta \text{div}uG
\]

\[
+ \frac{\mu}{2\mu + \lambda} \int_{\mathbb{R}^3} (\nabla u + (\nabla u)^\top) : (\nabla G \otimes u) + \frac{\lambda}{2\mu + \lambda} \int_{\mathbb{R}^3} \text{div}uu \cdot \nabla G
\]

\[
+ \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} (\mu \Delta u + (\mu + \lambda)\text{div}u) \cdot uG + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \nabla \theta \cdot \nabla G \tag{6.8}
\]

\[
\leq C \|pu\theta\|_{L^2} \|\nabla G\|_{L^2} + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta \text{div}uG + C \|\nabla G\|_{L^2} \|u|\nabla u\|_{L^2}
\]

\[
+ C \|\nabla G\|_{L^2} \|\nabla \theta\|_{L^2} + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} (\mu \Delta u + (\mu + \lambda)\text{div}u) \cdot uG.
\]

Substituting \( (2.2.1)_2 \) into (6.8), and using Sobolev inequality, (6.1) and integration by parts, we have

\[
VVII_3 \leq C \|\nabla G\|_{L^2} \left( \|pu\theta\|_{L^2} + \|u|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} \right) + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho u_t \cdot uG
\]

\[
+ \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot uG - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} Pu \cdot \nabla G
\]

\[
\leq C \|\nabla G\|_{L^2} \left( \|pu\theta\|_{L^2} + \|u|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} \right) + \frac{1}{6} \int_{\mathbb{R}^3} \rho |u_t|^2 \tag{6.9}
\]

\[
+ C \int_{\mathbb{R}^3} \rho |u|^2 |G|^2 + C \|u|\nabla u\|_{L^2}^2
\]

\[
\leq C \|\nabla G\|_{L^2} \left( \|pu\theta\|_{L^2} + \|u|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} \right) + \frac{1}{6} \int_{\mathbb{R}^3} \rho |u_t|^2
\]

\[
+ C \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 + C \int_{\mathbb{R}^3} \rho |u|^2 |\rho \theta|^2.
\]
Taking \( \text{div} \) on both side of (2.2.1), we get
\[
\Delta G = \text{div}(\rho u_t + \rho u \cdot \nabla u).
\]  
(6.10)

By (6.10) and the standard \( L^2 \)-estimates together with (6.1), we get
\[
\|\nabla G\|_{L^2} \lesssim \|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} \lesssim \|\rho u_t\|_{L^2} + \|u\|\|\nabla u\|_{L^2}.
\]  
(6.11)

Substituting (6.11) into (6.9), and using Cauchy inequality, we have
\[
V U I I_3 \leq C\|\rho \theta\|_{L^2}^2 + C\|u\|\|\nabla u\|_{L^2}^2 + C\|\nabla \theta\|_{L^2}^2 + \frac{1}{3} \int_{\mathbb{R}^3} \rho |u_t|^2.
\]  
(6.12)

For the first term of the right hand side of (6.12), using H"older inequality, Sobolev inequality and Cauchy inequality, we have
\[
\|\rho u \theta\|_{L^2} \leq \|u\|_6 \|\rho \theta\|_{L^{\frac{12}{5}}} \leq C\|u\|\|\nabla u\|\|\rho \theta\|_{L^{\frac{12}{5}}}.
\]  
(6.13)

Substituting (6.13) into (6.12), we have
\[
V U I I_3 \leq C\|\rho \theta\|_{L^2}^4 + C\|u\|\|\nabla u\|_{L^2}^2 + C\|\nabla \theta\|_{L^2}^2 + \frac{1}{3} \int_{\mathbb{R}^3} \rho |u_t|^2.
\]  
(6.14)

For \( V U I I_4 \), using Cauchy inequality and (6.1), we have
\[
V U I I_4 \leq \frac{1}{6} \int_{\mathbb{R}^3} \rho |u_t|^2 + C \int_{\mathbb{R}^3} |u|^2|\nabla u|^2.
\]  
(6.15)

Putting (6.14) and (6.15) into (6.6), and integrating it over \([0, t]\), for \( t < T^* \), we have
\[
\int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2)
\leq 2 \int_{\mathbb{R}^3} P \text{div} u + C \int_0^t \|\nabla \theta\|_{L^2}^2 + C
\leq (\mu + \lambda) \int_{\mathbb{R}^3} |\text{div} u|^2 + C \left( \int_{\mathbb{R}^3} \rho \theta^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \right) + C,
\]
where we have used Cauchy inequality, (6.1) and (6.4). Therefore,
\[
\int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 \leq C \left( \int_{\mathbb{R}^3} \rho \theta^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \right) + C.
\]  
(6.16)

Multiplying (2.2.1) by \( \theta \) and integrating by parts over \( \mathbb{R}^3 \), we have
\[
\int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\theta|^2
= - \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u + \int_{\mathbb{R}^3} \frac{\mu}{2} |\nabla u + (\nabla u)\theta|^2 + \int_{\mathbb{R}^3} \lambda |\text{div} u|^2 \theta
\]
\[= \sum_{i=1}^3 V U I I_i,
\]  
(6.17)
For $VVIII_2$ and $VVIII_3$, we have

$$VVIII_2 + VVIII_3 = \int_{\mathbb{R}^3} \mu (\nabla u + (\nabla u)^\prime) : \nabla u \theta + \int_{\mathbb{R}^3} \lambda |\text{div} u|^2 \theta$$

$$= - \int_{\mathbb{R}^3} \mu (\nabla u + \text{div} u) \cdot u \theta - \int_{\mathbb{R}^3} \mu (\nabla u + (\nabla u)^\prime) : (\nabla \theta \otimes u)$$

$$- \int_{\mathbb{R}^3} \lambda u \cdot \text{div} u \theta - \int_{\mathbb{R}^3} \lambda \text{div} uu \cdot \nabla \theta$$

$$= - \int_{\mathbb{R}^3} (\rho u_t + \rho u \cdot \nabla u + \nabla P) \cdot u \theta - \int_{\mathbb{R}^3} \mu (\nabla u + (\nabla u)^\prime) : (\nabla \theta \otimes u)$$

$$- \int_{\mathbb{R}^3} \lambda \text{div} uu \cdot \nabla \theta$$

$$= - \int_{\mathbb{R}^3} \rho u_t \cdot u \theta - \int_{\mathbb{R}^3} \rho (u \cdot \nabla u) \cdot u \theta + \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u + \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla \theta$$

$$- \int_{\mathbb{R}^3} \mu (\nabla u + (\nabla u)^\prime) : (\nabla \theta \otimes u) - \int_{\mathbb{R}^3} \lambda \text{div} uu \cdot \nabla \theta,$$

(6.18)

where we have used integration by parts and (2.2.1)$_2$. Using Hölder inequality, Cauchy inequality, (6.1) and (6.4), we have

$$VVIII_2 + VVIII_3$$

$$\leq \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u\|_{L^4} \|\nabla \theta\|_{L^1} + \|u |\nabla u|\|_{L^2} \|\rho u\|_{L^3} \|\theta\|_{L^6} + \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u$$

$$\leq C \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + C \|u |\nabla u|\|_{L^2} \|\nabla \theta\|_{L^2}$$

$$\leq C \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \rho|\theta|^2 \leq C \int_0^t \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + C \int_0^t \|\nabla u\|_{L^2}^2 + C \|\sqrt{\rho} \theta\|_{L^4}^2.$$

(6.19)

Substituting (6.19) into (6.17), we have

$$\int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \rho|\theta|^2 \leq C \int_0^t \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + C \int_0^t \|\nabla u\|_{L^2}^2 + C \|\sqrt{\rho} \theta\|_{L^4}^2.$$

(6.20)

Integrating (6.20) over $[0,t]$ ($t < T^*$), and using (6.4), we have

$$\int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 + \int_{\mathbb{R}^3} \rho|\theta|^2 \leq C \int_0^t \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + C \int_0^t \|\nabla u\|_{L^2}^2 + C \|\sqrt{\rho} \theta\|_{L^4}^2.$$

(6.21)

Multiplying (6.21) by $2C$, and adding the resulting inequality into (6.16), we have

$$\int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 + \int_{\mathbb{R}^3} \rho|\theta|^2$$

$$\leq \int_0^t \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + \int_0^t \|\sqrt{\rho} \theta\|_{L^4}^2 + 1$$

$$\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 + C \int_0^t \|\sqrt{\rho} \theta\|_{L^2}^2 \|\sqrt{\rho} \theta\|_{L^6}^2 + C$$

$$\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \rho |u_t|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \theta|^2 + C \int_0^t \int_{\mathbb{R}^3} \rho|\theta|^2 + C,$$

where we have used Young inequality. Hölder inequality, Sobolev inequality and (6.1). This, together with Gronwall inequality, gives (6.5).
Lemma 6.4 Under the conditions of Theorem 2.2.2 and (6.1), it holds that for any $t \in (0, T^*)$

$$\int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) + \int_0^t \int_{\mathbb{R}^3} (\rho |\dot{\theta}|^2 + |\nabla \dot{u}|^2) \leq C. \quad (6.22)$$

Proof. By \[43\] (see (4.35) therein), we accurately have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} (\mu |\nabla \dot{u}|^2 + (\mu + \lambda) |\text{div} \dot{u}|^2)
= \int_{\mathbb{R}^3} (P_t \text{div} \dot{u} + u \otimes \nabla P : \nabla \dot{u}) + \mu \int_{\mathbb{R}^3} \left( \text{div} (\Delta u \otimes u) - \Delta (u \cdot \nabla u) \right) \cdot \dot{u}
+ (\mu + \lambda) \int_{\mathbb{R}^3} \left( \text{div} (\nabla \text{div} u \otimes u) - \nabla \text{div} (u \cdot \nabla u) \right) \cdot \dot{u} = \sum_{i=1}^{3} VVIV_i. \quad (6.23)$$

For $VVIV_1$, using \[2.2.1\], integration by parts, \[6.1\] and Hölder inequality, we have

$$VVIV_1 = \int_{\mathbb{R}^3} \left( (\rho \theta) t \text{div} \dot{u} - \rho \theta (\nabla u)^t : \nabla \dot{u} - \rho \theta u \cdot \nabla \text{div} \dot{u} \right)
= \int_{\mathbb{R}^3} \left( \rho \dot{\theta} \text{div} \dot{u} - \rho \theta (\nabla u)^t : \nabla \dot{u} \right)
\leq \|\sqrt{\rho \theta}\|_{L^2} \|\nabla \dot{u}\|_{L^2} + \|\sqrt{\rho \theta}\|_{L^4} \|\nabla u\|_{L^4} \|\nabla \dot{u}\|_{L^2}. \quad (6.24)$$

For $VVIV_2$ and $VVIV_3$, by \[43\] (see (4.37) and (4.38) therein), we have we have

$$VVIV_2 + VVIV_3 \lesssim \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2. \quad (6.25)$$

Substituting \[6.24\] and \[6.25\] into \[6.23\], and using Cauchy inequality and \[6.1\], we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} (\mu |\nabla \dot{u}|^2 + (\mu + \lambda) |\text{div} \dot{u}|^2) \leq \frac{\mu}{2} \|\nabla \dot{u}\|_{L^2}^2 + C \|\sqrt{\rho \theta}\|_{L^2}^2 + C \|\sqrt{\rho \theta}\|_{L^4}^4 + C \|\nabla u\|_{L^4}^4.
$$

Integrating this inequality over $[0, t]$ for $t \in (0, T^*)$, and using \[6.1\], \[6.3\], Hölder inequality and Sobolev inequality, we have

$$\int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \leq C \int_0^t \|\sqrt{\rho \theta}\|_{L^2}^2 + C \int_0^t (\|\nabla u\|_{L^4}^4 + \|\nabla \theta\|_{L^2}^4). \quad (6.26)$$

The next step is to get some estimates for $\theta$. We rewrite \[2.2.1\] as follows:

$$\rho \dot{\theta} + \rho \theta \text{div} u = \frac{\mu}{2} |\nabla u + (\nabla u)^t|^2 + \lambda (\text{div} u)^2 + \Delta \theta. \quad (6.27)$$

Multiplying \[6.27\] by $\dot{\theta}$ and integrating by parts over $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \theta|^2
= - \int_{\mathbb{R}^3} \rho \theta \text{div} u \dot{\theta} + \frac{\mu}{2} \int_{\mathbb{R}^3} (|\nabla u + (\nabla u)^t|^2 + \lambda (\text{div} u)^2) \theta_t
+ \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)^t|^2 + \lambda (\text{div} u)^2 \right) u \cdot \nabla \theta + \int_{\mathbb{R}^3} \Delta \theta u \cdot \nabla \theta \quad (6.28)$$

$$= \sum_{i=1}^{4} VVV_i.$$
For $VVV_1$, using Hölder inequality, (6.1) and Cauchy inequality, we have
\[
V V V_1 \leq C\|\sqrt{\rho \theta}\|_{L^2}\|\nabla u\|_{L^4}\|\rho^{\frac{1}{2}}\theta\|_{L^4} \leq \frac{1}{8}\|\sqrt{\rho \theta}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4 + C\|\nabla \theta\|_{L^2}^4. \tag{6.29}
\]
For $VVV_2$, we have
\[
VVV_2 = \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\text{div} u)^2 \right) \theta - \mu \int_{\mathbb{R}^3} \left( \nabla u + (\nabla u)' : (\nabla u_t + (\nabla u_t)') \right) \theta
- 2\lambda \int_{\mathbb{R}^3} \text{div} \text{div} u \theta
\]
\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\text{div} u)^2 \right) \theta - \mu \int_{\mathbb{R}^3} \left( \nabla u + (\nabla u)' : (\nabla u \cdot \nabla u + (\nabla u \cdot \nabla u)') \right) \theta
+ \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') \cdot (u \cdot \nabla) (\nabla u + (\nabla u)') \theta - 2\lambda \int_{\mathbb{R}^3} \text{div} \text{div} u \theta
+ 2\lambda \int_{\mathbb{R}^3} \text{div}(\nabla u)' : \nabla u \theta + 2\lambda \int_{\mathbb{R}^3} u \cdot \text{div} \text{div} u \theta.
\]
Using integration by parts, we have
\[
VVV_2 = \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\text{div} u)^2 \right) \theta - \mu \int_{\mathbb{R}^3} \left( \nabla u + (\nabla u)' : (\nabla u \cdot \nabla u + (\nabla u \cdot \nabla u)') \right) \theta
+ \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') \cdot (u \cdot \nabla) (\nabla u + (\nabla u)') - \mu \int_{\mathbb{R}^3} \frac{|\nabla u + (\nabla u)'|^2}{2} \text{div} u \theta
- \mu \int_{\mathbb{R}^3} \frac{|\nabla u + (\nabla u)'|^2}{2} u \cdot \nabla \theta - 2\lambda \int_{\mathbb{R}^3} \text{div} \text{div} u \theta + 2\lambda \int_{\mathbb{R}^3} \text{div}(\nabla u)' : \nabla u \theta - 2\lambda \int_{\mathbb{R}^3} \text{div} \text{div} u \theta
+ \lambda \int_{\mathbb{R}^3} (\nabla u)^3 \theta - \lambda \int_{\mathbb{R}^3} |\text{div} u|^2 u \cdot \nabla \theta
= \sum_{i=1}^{9} VVV_{2,i}.
\]
For $VVV_{2,2}$ and $VVV_{2,6}$, using Hölder inequality, Sobolev inequality, we have
\[
VVV_{2,2} + VVV_{2,6} \lesssim \|\nabla \dot{u}\|_{L^2}\|\nabla u\|_{L^3}\|\theta\|_{L^6} \lesssim \|\nabla u\|_{L^2}\|\nabla u\|_{L^4}\|\nabla \theta\|_{L^2}. \tag{6.31}
\]
Since $\nabla u = \nabla \Delta^{-1} (\nabla \text{div} u - \nabla \times \text{curl} u)$, we apply Calderon-Zygmund inequality to get
\[
\|\nabla u\|_{L^3} \lesssim \|\text{curl} u\|_{L^3} + \|\text{div} u\|_{L^3}. \tag{6.32}
\]
Taking curl on both sides of (2.2.1), we have
\[
\mu \Delta (\text{curl} u) = \text{curl}(\rho \dot{u}). \tag{6.33}
\]
By (6.33), the $L^2$-estimates of the elliptic equations and (6.1), we have
\[
\|\nabla \text{curl} u\|_{L^2} \lesssim \|\rho \dot{u}\|_{L^2} \lesssim \|\sqrt{\rho \dot{u}}\|_{L^2}. \tag{6.34}
\]
By (6.10), (6.32) and (6.33), together with Sobolev inequality, we have
\[
\|\nabla u\|_{L^3} \lesssim \|\text{curl} u\|_{L^3} + \|G\|_{L^3} + \|\rho\theta\|_{L^3} \lesssim \|\text{curl} u\|_{H^1} + \|G\|_{H^1} + \|\rho\|_{L^6}\|\theta\|_{L^6}
\lesssim \|\text{curl} u\|_{L^2} + \|\text{div} u\|_{L^2} + \|\nabla (\text{curl} u)\|_{L^2} + \|\nabla G\|_{L^2} + \|\nabla \theta\|_{L^2}
\lesssim \|\sqrt{\rho \dot{u}}\|_{L^2} + \|\nabla \theta\|_{L^2} + 1, \tag{6.35}
\]
where we have used (6.1), (6.3) and (6.5). Substituting (6.35) into (6.31) and using Young inequality, we obtain
\[ VV_{2,2} + VV_{2,6} \lesssim \| \nabla u \|^2_{L^2} (\| \sqrt{\rho} \theta \|_{L^2} + \| \nabla \theta \|_{L^2}^2 + 1) \| \nabla \theta \|_{L^2}. \] (6.36)

For $VV_{2,3}, VV_{2,4}, VV_{2,7}$ and $VV_{2,8}$, using Hölder inequality, Sobolev inequality and Calderon-Zygmund inequality, we have
\[ VV_{2,3} + VV_{2,4} + VV_{2,7} + VV_{2,8} \lesssim \int_{\mathbb{R}^3} |\nabla u|^3 |\theta| + \| \nabla u \|^3_{L^{5/3}} \| \theta \|_{L^6} \lesssim \| \nabla u \|^3_{L^{5/3}} \| \nabla \theta \|_{L^2} \] (6.37)
\[ \lesssim \| \nabla u \|^3_{L^{5/3}} \| \nabla \theta \|_{L^2}^3 + \| \div u \|^3_{L^{5/3}} \| \nabla \theta \|_{L^2} \]
\[ \lesssim \| \nabla u \|^3_{L^{5/3}} \| \nabla \theta \|_{L^2}^2 + \| G \|_{L^2} \| \nabla G \|_{L^2}^2 \| \nabla \theta \|_{L^2}^2 + \| \rho \|^3_{L^{3/(3-2)}} \| \theta \|^3_{L^6} \| \nabla \theta \|_{L^2}. \] (6.38)

Using Hölder inequality again, together with (6.1), (6.3), Sobolev inequality, Gagliardo-Nirenberg inequality, (6.5), (6.10) and (6.34), we get
\[ VV_{2,3} + VV_{2,4} + VV_{2,7} + VV_{2,8} \lesssim \| \nabla u \|^2_{L^2} \| \nabla \theta \|_{L^2} + \| \div u \|^2_{L^2} \| \nabla \theta \|_{L^2} + C \| \nabla \theta \|_{L^2}^2 \] (6.39)

For $VV_{2,5}$ and $VV_{2,9}$, using Hölder inequality, Cauchy inequality, Sobolev inequality, Gagliardo-Nirenberg inequality and (6.5), we have
\[ VV_{2,5} + VV_{2,9} \lesssim \int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta| \lesssim \| \nabla u \|^2_{L^2} \| u \|_{L^6} \| \nabla \theta \|_{L^3} \]
\[ \lesssim \| \nabla u \|^2_{L^4} + \| \nabla u \|^2_{L^4} \| \nabla \theta \|_{L^2} \| \nabla \theta \|_{L^2}^2 \]
\[ \leq C \| \nabla u \|^2_{L^4} + C \| \nabla \theta \|_{L^2} \| \nabla \theta \|_{L^2}^2. \] (6.40)

From the standard elliptic estimates and (6.27), we have
\[ \| \nabla^2 \theta \|_{L^2} \lesssim \| \rho \|_{L^2} + \| \rho \|_{L^2} \| \nabla u \|_{L^2} + \| \nabla u \|^2_{L^4} \]
\[ \lesssim \| \sqrt{\rho} \theta \|_{L^2} + \| \rho \|^3_{L^4} \| \nabla u \|_{L^2} + \| \nabla u \|^2_{L^4} \]
\[ \leq C \| \sqrt{\rho} \theta \|_{L^2} + C \| \nabla u \|^2_{L^4} + C \| \nabla \theta \|^2_{L^2}, \] (6.41)

where we have used Hölder inequality, (6.1), (6.3), Sobolev inequality and Cauchy inequality. Substituting (6.40) into (6.39), and using Cauchy inequality, we have
\[ VV_{2,5} + VV_{2,9} \lesssim \frac{1}{8} \| \sqrt{\rho} \theta \|^2_{L^2} + C \| \nabla u \|^4_{L^4} + C \| \nabla \theta \|^4_{L^2} + C. \] (6.42)

Substituting (6.33), (6.38) and (6.41) into (6.30), and using Cauchy inequality, we have
\[ VV_2 \leq \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)^2 + \lambda (\div u)^2 \theta \right) \]
\[ + C \| \nabla u \|^2_{L^2} \left( \| \sqrt{\rho} \theta \|_{L^2} + \| \nabla \theta \|_{L^2} + 1 \right) \| \nabla \theta \|_{L^2} + C \| \sqrt{\rho} \theta \|^2_{L^2} \| \nabla \theta \|_{L^2} \]
\[ + C \| \nabla \theta \|^2_{L^2} + \frac{1}{8} \| \sqrt{\rho} \theta \|^2_{L^2} + C \| \nabla u \|^4_{L^4} + C. \] (6.43)

For $VV_3$, using (6.39) and (6.41), we have
\[ VV_3 \lesssim \int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta| \leq \frac{1}{8} \| \sqrt{\rho} \theta \|^2_{L^2} + C \| \nabla u \|^4_{L^4} + C \| \nabla \theta \|^4_{L^2} + C. \] (6.44)
For VVV₄, using Hölder inequality, Sobolev inequality, Gagliardo-Nirenberg inequality, (6.5), (6.3) and Young inequality, we have

\[
VVV₄ \lesssim \|\Delta \theta\|_{L^2} \|u\|_{L^6} \|\nabla \theta\|_{L^3} \lesssim \|\Delta \theta\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2} ^{\frac{1}{2}} \|\nabla^2 \theta\|_{L^2} ^{\frac{1}{2}}
\]

\[
\lesssim \|\nabla \theta\|_{L^2} ^{\frac{7}{2}} \|\nabla^2 \theta\|_{L^2} \frac{3}{2} \leq \frac{1}{8} \|\nabla \theta\|_{L^2} ^{\frac{7}{2}} + C \|\nabla \theta\|_{L^2} ^{\frac{3}{2}} + C \|\nabla u\|_{L^2} ^{\frac{4}{3}} + C.
\]

(6.44)

Putting (6.29), (6.42), (6.33) and (6.44) into (6.28), integrating the resulting inequality over \([0,t]\) for \(t \in (0, T^*)\), and using Cauchy inequality, (6.1), (6.4) and (6.5), we have

\[
\int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \int_{\mathbb{R}^3} |\nabla \theta|^2 \leq C \int_0^t \|\nabla u\|_{L^4} ^4 + C \int_{\mathbb{R}^3} |\nabla u|^4 \theta |\theta| + \varepsilon \int_0^t \|\nabla \dot{u}\|_{L^2} ^2 + C \varepsilon \int_0^t (\|\sqrt{\rho u}\|_{L^2} ^2 + \|\nabla \theta\|_{L^2} ^2 + 1) \|\nabla \theta\|_{L^2} ^2 + C.
\]

(6.45)

For the second term of the right hand side of (6.45), we have

\[
C \int_{\mathbb{R}^3} |\nabla u|^2 |\theta| \lesssim \|\nabla u\|_{L^4} ^2 \|\theta\|_{L^6} \lesssim \|\nabla \theta\|_{L^2} \|\nabla \theta\|_{L^2} + \|\nabla \theta\|_{L^2} ^2 \|\nabla \theta\|_{L^2}
\]

\[
\lesssim \left( \|\nabla u\|_{L^4} ^2 \|\nabla \theta\|_{L^2} + \|G\|_{L^4} ^2 \|\nabla G\|_{L^2} ^2 \right) \|\nabla \theta\|_{L^2} \|\rho\|_{L^{12}} \|\nabla \theta\|_{L^2} \|\rho\|_{L^{12}} \|\rho\|_{L^{12}} \|\nabla \theta\|_{L^2}
\]

\[
\lesssim \|\sqrt{\rho u}\|_{L^2} ^\frac{1}{2} \|\nabla \theta\|_{L^2} + \|\rho\|_{L^{12}} \|\rho\|_{L^{12}} \|\rho\|_{L^{12}} \|\nabla \theta\|_{L^2}
\]

\[
\lesssim \|\sqrt{\rho u}\|_{L^2} ^\frac{1}{2} \|\nabla \theta\|_{L^2} + \|\nabla \theta\|_{L^2} ^\frac{3}{2} \leq \frac{1}{2} \|\nabla \theta\|_{L^2} ^2 + C \|\sqrt{\rho u}\|_{L^2} ^2 + C,
\]

where we have used Hölder inequality, Calderon-Zygmund inequality, Gagliardo-Nirenberg inequality, (6.1), (6.5), (6.10), Sobolev inequality and Young inequality. Substituting (6.46) into (6.45), we have

\[
\int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \int_{\mathbb{R}^3} |\nabla \theta|^2 \leq C \int_0^t \|\nabla u\|_{L^4} ^4 + C \int_{\mathbb{R}^3} \|\sqrt{\rho u}\|_{L^2} ^2 \|\nabla \theta\|_{L^2} + \varepsilon \int_0^t \|\nabla \dot{u}\|_{L^2} ^2 + C \varepsilon \int_0^t (\|\sqrt{\rho u}\|_{L^2} ^2 + \|\nabla \theta\|_{L^2} ^2 + 1) \|\nabla \theta\|_{L^2} ^2 + C.
\]

(6.47)

Multiplying (6.47) by 2C and adding the resulting inequality into (6.28), we have

\[
C \int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + 2C \int_{\mathbb{R}^3} |\nabla \theta|^2 + \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} \|\nabla \dot{u}\|^2
\]

\[
\leq 2C^2 \int_{\mathbb{R}^3} \|\nabla u\|_{L^2} ^4 + 2C^2 \|\sqrt{\rho u}\|_{L^2} ^2 \|\nabla \theta\|_{L^2} ^2 + 2 \varepsilon C \int_0^t \|\nabla \dot{u}\|_{L^2} ^2
\]

\[
+ 2CC \varepsilon \int_0^t (\|\sqrt{\rho u}\|_{L^2} ^2 \|\nabla \theta\|_{L^2} ^2 + 1) \|\nabla \theta\|_{L^2} ^2 + 2C^2.
\]

Taking \(\varepsilon\) sufficiently small, together with Cauchy inequality, we have

\[
\int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) + \int_0^t \int_{\mathbb{R}^3} (\rho |\dot{\theta}|^2 + |\nabla \dot{u}|^2)
\]

\[
\lesssim \int_0^t \|\nabla u\|_{L^4} ^4 + \int_0^t (\|\sqrt{\rho u}\|_{L^2} ^2 \|\nabla \theta\|_{L^2} ^2 + 1) \|\nabla \theta\|_{L^2} ^2 + 1.
\]

(6.48)
For the first term of the right hand side of (6.48), similar to (6.35), we have
\[
\int_0^t \|\nabla u\|^4_{L^4} \lesssim \int_0^t \|\nabla \theta\|^4_{L^2}
\]
\[
\lesssim \int_0^t \|\nabla \theta\|^2 \|\nabla \theta\|^2_{L^2} + \int_0^t \|G\|^2_{L^2} + \int_0^t \|G\|^2_{L^2} \|\nabla \theta\|^4_{L^2} \tag{6.49}
\]
By (6.48), (6.49) and Cauchy inequality, we have
\[
\int_{R^3} (|\nabla \theta|^2 + \rho |\theta_t|^2) + \int_0^t \int_{R^3} (\rho |\theta|^2 + |\nabla \theta|^2)
\]
\[
\lesssim \int_0^t \|\sqrt{\rho} \theta_t\|^2_{L^2} + \int_0^t \left( \|\sqrt{\rho} \theta_t\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} + 1 \right) \|\nabla \theta\|^2_{L^2} + 1 \tag{6.50}
\]
By (6.1), (6.4), (5.5), we have
\[
\int_0^t (\|\sqrt{\rho} \theta_t\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) \leq C,
\]
for any \( t \in (0, T^*) \). This, together with (6.50) and Gronwall inequality, deduces (6.22). \( \square \)

**Corollary 6.5** Under the conditions of Theorem 2.2.2 and (6.1), it holds that for any \( T \in (0, T^*) \)
\[
\sup_{0 \leq t \leq T} (\|\nabla G\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^6} + [u]_{L^\infty}) + \int_0^T \int_{R^3} |\nabla \theta|^2 \leq C. \tag{6.51}
\]
**Proof.** The detailed proof of the lemma could be found in [43] (see Corollary 4.5 therein). \( \square \)

**Lemma 6.6** Under the conditions of Theorem 2.2.2 and (6.1), it holds that for any \( t \in (0, T^*) \)
\[
\int_{R^3} \rho |\theta_t|^2 + \int_0^t \int_{R^3} |\nabla \theta_t|^2 \leq C. \tag{6.52}
\]
**Proof.** By [43], we have
\[
\frac{1}{2} \frac{d}{dt} \int_{R^3} \rho |\theta_t|^2 + \int_{R^3} |\nabla \theta_t|^2
\]
\[
= - \int_{R^3} \rho \left( \frac{\theta_t}{2} + u \cdot \nabla \theta + \theta \nabla u \right) \theta_t - \int_{R^3} \rho (u_t \cdot \nabla \theta + u \cdot \nabla \theta_t + \theta_t \div u) \theta_t
\]
\[
- \int_{R^3} \rho \div u \theta_t + \mu \int_{R^3} \left( \nabla u + (\nabla u)^t \right) \left( \nabla u_t + (\nabla u_t)^t \right) \theta_t + 2\lambda \int_{R^3} \div u \div u \theta_t \tag{6.53}
\]
\[
= \sum_{i=1}^5 VVV I_i,
\]
and

$$VVVI_1 = -\int_{\mathbb{R}^3} \rho u \cdot \nabla \theta_t \left( \frac{\theta_t}{2} + u \cdot \nabla \theta + \theta \text{div} u \right) - \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta_t \theta_t - \int_{\mathbb{R}^3} \rho u \cdot \nabla (u \cdot \nabla \theta) + u \cdot \nabla \theta \theta_t - \int_{\mathbb{R}^3} \rho u \cdot (\nabla \theta \text{div} u + \theta \nabla \text{div} u) \theta_t$$

$$= \sum_{i=1}^{4} VVVI_{1,i}.$$  

For $VVVI_{1,1}$, we have

$$VVVI_{1,1} \lesssim \frac{1}{24} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C,$$

where we have used Cauchy inequality, (6.1), (6.5), (6.22) and (6.51).

For $VVVI_{1,2}$, using Cauchy inequality, (6.1) and (6.51) again, we have

$$VVVI_{1,2} \lesssim \frac{1}{24} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2.$$

For $VVVI_{1,3}$, by [43] (see (4.65) therein), we have

$$VVVI_{1,3} \lesssim \int_{\mathbb{R}^3} \rho |\theta_t|^2 + \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 + 1.$$

For $VVVI_{1,4}$, integrating by parts, we have

$$VVVI_{1,4} = \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta \text{div} u \theta_t - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla G \theta_t - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla (\rho \theta) \theta_t$$

$$= -\int_{\mathbb{R}^3} \rho u \cdot \nabla \theta \text{div} u \theta_t - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla G \theta_t$$

$$+ \frac{1}{2(2\mu + \lambda)} \int_{\mathbb{R}^3} \rho^2 \theta^2 u \cdot \nabla \theta_t + \frac{1}{2(2\mu + \lambda)} \int_{\mathbb{R}^3} \rho^2 \theta^2 \text{div} u \theta_t.$$

Furthermore, we can get

$$VVVI_{1,4} \lesssim \|\sqrt{\rho} \theta_t\|_{L^2} \|\nabla u\|_{L^6} \|\nabla \theta\|_{L^6} + \|\nabla \theta\|_{L^2} \|\theta_t\|_{L^6} \|\theta\|_{L^6} \|\rho\|_{L^6}$$

$$+ \|\theta\|_{L^2} \|	heta_t\|_{L^6} \|\nabla \theta_t\|_{L^2} + \|\nabla \theta\|_{L^2} \|\theta_t\|_{L^6} \|\theta\|_{L^6} \|\rho\|_{L^6}^2$$

$$\lesssim \frac{1}{24} \|\nabla \theta_t\|_{L^2}^2 + C \|\nabla^2 \theta\|_{L^2}^2 + C \|\sqrt{\rho} \theta_t\|_{L^2}^2 + C,$$

where we have used Hölder inequality, Sobolev inequality, (6.1), (6.3), (6.5), (6.22), (6.51) and Cauchy inequality. Substituting (6.55), (6.56), (6.57) and (6.59) into (6.54), we have

$$VVVI_1 \lesssim \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 + C.$$  

For $VVVI_2$,

$$VVVI_2 = -\int_{\mathbb{R}^3} \rho \tilde{u} \cdot \nabla \theta_t + \int_{\mathbb{R}^3} \rho (u \cdot \nabla) u \cdot \nabla \theta_t - \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta_t \theta_t - \int_{\mathbb{R}^3} \rho |\theta_t|^2 \text{div} u$$

$$\lesssim \|\sqrt{\rho} \tilde{u}\|_{L^2} \|\theta_t\|_{L^6} \|\nabla \theta\|_{L^6}^2 + \|\sqrt{\rho} \theta_t\|_{L^2} \|\nabla u\|_{L^6} \|\nabla \theta\|_{L^3} + \|\sqrt{\rho} \theta_t\|_{L^2} \|\nabla \theta_t\|_{L^2}$$

$$\lesssim \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_{\mathbb{R}^3} \rho |\theta_t|^2 + C \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 + C.$$
where we have used Hölder inequality, Sobolev inequality, (6.1), (6.5), (6.22) and (6.51).

For $VVI_3$, integrating by parts, we have

$$VVI_3 = -\int_{\mathbb{R}^3} \rho \theta \text{div} \theta_t + \int_{\mathbb{R}^3} \rho \theta \text{div}(u \cdot \nabla u) \theta_t$$

$$= -\int_{\mathbb{R}^3} \rho \theta \text{div} \theta_t + \int_{\mathbb{R}^3} \rho \theta \nabla u : (\nabla u) \theta_t + \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla \theta_t$$

$$= -\int_{\mathbb{R}^3} \rho \theta \text{div} \theta_t + \int_{\mathbb{R}^3} \rho \theta \nabla u : (\nabla u) \theta_t + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla G \theta_t$$

$$- \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho^2}{2} \theta^2 \text{div} \theta_t - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \frac{\rho^2}{2} \theta^2 u \cdot \nabla \theta_t. \quad (6.62)$$

Furthermore, using Hölder inequality, Sobolev inequality, (6.1), (6.5), (6.22) and Young inequality, we have

$$VVI_3 \lesssim \| \nabla \dot{u} \|_{L^2} \| \theta_t \|_{L^6} \| \theta \|_{L^6} + \sqrt{\rho} \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^6}^2 \| \theta_t \|_{L^6}$$

$$+ \| \nabla G \|_{L^2} \| \theta_t \|_{L^6} \| \theta \|_{L^6} \| \rho \|_{L^6} + \| \nabla u \|_{L^3} \| \theta_t \|_{L^6} \| \theta \|_{L^6}^2 \| \rho \|_{L^{12}}^2$$

$$+ \| \theta_t \|_{L^6}^2 \| \rho \|_{L^{12}}^2 \| \nabla \theta \|_{L^2} \quad (6.63)$$

$$\lesssim \frac{1}{8} \| \nabla \theta_t \|_{L^2}^2 + C \| \nabla \dot{u} \|_{L^2}^2 + C.$$  

Similar to $VVI_2$, for $VVI_4$ and $VVI_5$, we deduce

$$VVI_4 + VVI_5 \leq C \| \nabla \dot{u} \|_{L^2} \| \nabla u \|_{L^3} \| \theta_t \|_{L^6} + C \int_{\mathbb{R}^3} \| \nabla u \|_{L^3}^3 \| \theta_t \|_{L^6} + C \int_{\mathbb{R}^3} \| \nabla u \|_{L^3}^4 + \frac{1}{16} \int_{\mathbb{R}^3} \| \nabla \theta_t \|_{L^2}^2$$

$$\leq C \| \nabla \dot{u} \|_{L^2} \| \nabla \theta_t \|_{L^2} + C \| \nabla u \|_{L^{12}} \| \nabla \theta_t \|_{L^2} + \frac{1}{16} \int_{\mathbb{R}^3} \| \nabla \theta_t \|_{L^2}^2 + C \quad (6.64)$$

$$\leq \frac{1}{8} \int_{\mathbb{R}^3} \| \nabla \theta_t \|_{L^2}^2 + C \int_{\mathbb{R}^3} \| \nabla \dot{u} \|_{L^2}^2 + C,$$

where we have used Hölder inequality, integration by parts, Cauchy inequality, (6.5), (6.51), the interpolation inequality and Sobolev inequality.

Putting (6.60), (6.61), (6.63) and (6.64) into (6.61), we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho \| \theta_t \|_{L^2}^2 + \int_{\mathbb{R}^3} \| \nabla \theta_t \|_{L^2}^2 \leq C \int_{\mathbb{R}^3} \rho \| \theta_t \|_{L^2}^2 + C \int_{\mathbb{R}^3} (\| \nabla \dot{u} \|_{L^2}^2 + |\nabla^2 \theta|^2) + C. \quad (6.65)$$

By (6.64), (6.22) and Gronwall inequality, we complete the proof of Lemma 6.6 \(\square\)

**Corollary 6.7** Under the conditions of Theorem 2.2.1, 3 and (6.1), it holds that for any \(t \in (0, T^*)\)

$$\int_{\mathbb{R}^3} |\nabla^2 \theta|^2 \leq C. \quad (6.66)$$

**Proof.** It follows from (2.2.1), (6.1), (6.3), (6.5), (6.22), (6.51), (6.52) and the interpolation inequality that

$$\| \nabla^2 \theta \|_{L^2} \lesssim \| \rho \theta_t \|_{L^2} + \| \rho u \cdot \nabla \theta \|_{L^2} + \| \rho \text{div} u \|_{L^2} + \| \nabla u \|_{L^4}^2$$

$$\lesssim \sqrt{\rho} \| \theta_t \|_{L^2} + \| \nabla \theta \|_{L^2} + \| \rho \|_{L^6} \| \theta \|_{L^6} \| \text{div} u \|_{L^6} + 1 \leq C.$$  

\(\square\)

By (6.22), (6.66) and Sobolev inequality, we get the following corollary which is the desired one, i.e., (6.2).
Corollary 6.8 Under the conditions of Theorem 2.2.2 and (6.1), it holds that for any $t \in (0, T^*)$

$$\|\theta\|_{L^\infty(0,t;L^\infty)} \leq C.$$  \hspace{1cm} (6.67)

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