Loop Variables and the Interacting Open String in a Curved Background.

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Abstract

Applying the loop variable proposal to a sigma model (with boundary) in a curved target space, we give a systematic method for writing the gauge and generally covariant interacting equations of motion for the modes of the open string in a curved background. As in the free case described in an earlier paper, the equations are obtained by covariantizing the flat space (gauge invariant) interacting equations and then demanding gauge invariance in the curved background. The resulting equation has the form of a sum of terms that would individually be gauge invariant in flat space or at zero interaction strength, but mix amongst themselves in curved space when interactions are turned on. The new feature is that the loop variables are deformed so that there is a mixing of modes. Unlike the free case, the equations are coupled, and all the modes of the open string are required for gauge invariance.
1 Introduction

In an earlier paper [1] (hereafter “I”) the loop variable proposal [2] was extended to free open strings in curved space. A systematic method of writing down gauge and generally covariant equations for the massive modes of the open string was given. The basic method was to take the loop variable equation in flat space and apply it to curved space by making the loop variables conjugate to Riemann normal coordinates. This then gives a covariantized version of the flat space equation. The problem then is that if one tries to map the gauge transformations of the loop variables to gauge transformations of space time fields in the usual manner, one finds that the space time fields do not have well defined gauge transformations. In the loop variable approach this is a central issue. The equations written in terms of loop variable momenta are always gauge invariant. What is non trivial is the existence of a well defined map from loop variables to space time fields in such a way that gauge transformations are also well defined on the space time fields. If this can be achieved then we have gauge invariant equations in space time.

Thus, if we let \( L \) be the loop variable expression and \( S \) be the corresponding expression in terms of space time fields,

\[
\mathcal{M} : L \rightarrow S
\]

\[
\mathcal{M} : L^g (= L + \delta L) \rightarrow S^g = S + \Delta S
\] (1.1)

where

\[
\mathcal{G} : L \rightarrow L^g
\] (1.2)

is the gauge transformation of \( L \). These two equations define the action \( \mathcal{G} \), gauge transformation on the space time fields: the gauge transformation should be such that

\[
\mathcal{G} : S \rightarrow S + \delta S = S^g = S + \Delta S
\] (1.3)

where \( S^g \) is defined by (1.1) as the map of \( L^g \). In curved space space the map \( \mathcal{M} \) is the one obtained by covariantizing the flat space map using Riemann normal coordinate method mentioned above. General Covariance demands that the gauge transformation of space time fields is the covariantized version of the flat space transformation with possible additions of curvature dependent terms. In the free case as discussed in I, it is easy to see that given these two constraints it is not possible to satisfy (1.3). The solution

\footnote{A review of the loop variable approach is given in [4, 3].}
was to modify the map from loop variables to space time fields in such a way that with the same gauge transformations (1.3) is satisfied. Thus we change \( M \) to \( M' \):

\[
M' : L \rightarrow S' = S + S_1
\]

\[
M' : L^g(= L + \delta L) \rightarrow S'^g = S + S_1 + \Delta S \tag{1.4}
\]

Notice that the map on \( \delta L \) is unchanged. Now we have

\[
G : S'(= S + S_1) \rightarrow S + S_1 + \delta(S + S_1) \tag{1.5}
\]

and we want

\[
\delta(S + S_1) = \Delta S \tag{1.6}
\]

What was shown in [1] was that it is always possible to find \( S_1 \) such that (1.6) is satisfied. The gauge transformations are always just the usual gauge transformations, covariantized for curved space. As an example of this, take the case where

\[
L = k_0^\rho k_1^{\mu} k_1^\nu
\]

\[
S = D_\rho S_{1, 1, 1, \nu}
\]

Then

\[
\delta L = k_0^\rho \lambda_1 k_0(\mu k_1^\nu)
\]

Thus

\[
M : \delta L \rightarrow \Delta S = D_\rho D_{(\mu} A_{1, 1, \nu)} + \frac{1}{3}(R^\alpha_{\nu\rho\mu} + R^\alpha_{\mu\rho\nu})A_{1, 1, \alpha} \tag{1.7}
\]

On the other hand

\[
\delta S = D_\rho(\delta S_{1, 1, \nu\mu}) = D_\rho D_{(\mu} A_{1, 1, \nu}) \tag{1.8}
\]

We see a mismatch between (1.7) and (1.8).

Following [1] we let

\[
S_1 = \frac{1}{3}(R^\alpha_{\nu\rho\mu} + R^\alpha_{\mu\rho\nu})(S_{2\mu} - \frac{D_\mu S_2}{2k_0^\nu})
\]

and using \( \delta(S_{2\mu} - \frac{D_\mu S_2}{2k_0^\nu}) = \Lambda_{1, 1, \mu} \) we see that (1.6) is satisfied. Thus we have a well defined map from loop variables to space time fields. When this is done the space time equations are automatically gauge invariant - because they are obtained by a well defined map from an expression that is gauge invariant in terms of loop variables.

\[
^2 \text{The gauge transformations of fields are taken from I}
\]
The above was an outline of the discussion of free open string fields in curved space, that was worked out in I.

In this paper we extend this to the interacting case. In the curved space interacting case, there is a further complication, that makes the flat space technique used for interactions inadequate. This is essentially the same problem that one encounters in the free case, i.e. that of covariant derivatives in curved space making it difficult to define gauge transformations. Thus one has to generalize the technique that was used to define gauge transformations in the flat space interacting case. The technique there was to recursively define the gauge transformation of the higher modes so that the offending terms (i.e. terms involving lower modes that cannot be attributed to gauge transformations of the lower modes), are absorbed into the gauge transformation laws of higher modes, thereby making the map from loop variables to space time always well defined. In the notation used above, the gauge transformation of the highest mode in $S$ is defined so that $\delta S = \Delta S$. It was shown in [3, 4] that there is a systematic recursive way of doing this. We will use the same technique here. However it turns out that if one attempts to do this exactly as in the flat space case, often there is no higher mode that can absorb the offending terms! This is solved by deforming the loop variables so that higher modes always exist. Once this is done, the same procedure that was used in flat space works here also. This deformation is very similar to the deformation that takes us from ordinary derivatives to covariant derivatives. In our case all the loop variables (which are really generalized momenta) have to be deformed.

The above was an outline of the procedure. One sees that the main idea, which is the same in the free case, is to use general covariance and gauge invariance to obtain the equations, starting from the flat space equations. If this procedure is unique then the answer one obtains should be the correct one for string theory. This can be verified by doing explicit curved space sigma model RG calculations. This is an important issue but is not addressed in this paper.

This paper is organized as follows. Section 2 contains a short review of the interacting flat space case [3]. Section 3 describes the problems one encounters when attempts to go to curved space and the resolution of these problems. An outline of a sample calculation is given. The full details have not been worked out because they are very tedious and not particularly illuminating. Section 4 contains a summary and some conclusions. We have not included a review of the loop variable approach. This paper is a follow up of I and should ideally be read in conjunction with I.
2 Review of Interacting Case (Flat Space)

The flat space interacting case was described in [3, 4]. There are two main steps. The first is obtaining the equations for the loop variable momenta, and the second is mapping this to an equation for space time fields and then defining the gauge transformation laws for the space time fields under which these equations are invariant.

**Step 1:** The equations are obtained by first Taylor expanding all the vertex operators about one point on the world sheet (say $z=0$), so that the loop variable looks like that of the free theory. Note that this is legitimate only because of the presence of a world sheet cutoff. The cutoff is kept finite except when we go to the on-shell limit. All the complications are hidden in the $z$-dependent generalized momenta $k_n(t, z(t)) = k_n(t) + a_1 z k_{n-1}(t) + a_2 z^2 k_{n-2}(t) + ...$. We can simplify things by letting $z(t) = t = z$ and using $z$ to label the momenta as well as label the position on the world sheet. Thus $k_n(t, z(t))$ can be denoted by $\bar{k}_n(z)$. Then one applies the same technique as for the free case, which is to perform the operation $\delta \Sigma$ on the loop variable and set it to zero. An example of a free loop variable equation is:

$$(-2k_1.k_0 k_2.k_0 + k_1.k_0 k_1 + 2k_2.k_1 k_0.k_0)i k_0 \mu = 0 \quad (2.9)$$

There is an overall factor of $(\epsilon^2) k_0^{-1}$ multiplying it. If we replace $k_n$ by $\bar{k}_n(z_i)$ we get the interacting loop equation. The first term in this equation after performing the dimensional reduction becomes:

$$(\epsilon^2) k_0^2 (-4) [k_1(z_1).k_0(z_2)k_2(z_3).k_0(z_4)] +$$

$$2\bar{k}_3(z_1).k_0(z_2)k_0^V(z_3)k_0^V(z_4) + k_3^V(z_1)k_0^V(z_2)k_0^V(z_3)k_0^V(z_4)]i k_0 \mu(z) = 0 \quad (2.10)$$

Integrations over all the $z_i$ are understood. Here $\epsilon$ is the world sheet cutoff.

The equations obtained by this procedure are gauge invariant under

$$\bar{k}_n(z) \rightarrow \bar{k}_n(z) + \int dz' \lambda_p(z')\bar{k}_{n-p}(z) \quad (2.11)$$

provided we use the tracelessness constraint on the gauge parameters ($\lambda_p k_n.k_m = 0$ when both $n, m \neq 0$).

**Step 2:** The loop variable expressions are mapped to space time fields using relations of the form:

$$< k_{m\mu}(t_1)k_{n\nu}(t_2) >= S_{m,n\mu\nu} \delta(t_1 - t_2) + S_{m\mu} S_{n,\nu} \quad (2.12)$$
and it’s obvious generalizations. Using these one can work out

\[ \int dz_1 \, dz_2 \, < \tilde{k}_{\mu\nu}(z_1) \tilde{k}_{\mu\nu}(z_2) > \]

\[ = \int dz_1 \, dz_2 \, < (k_{\mu\nu}(z_1) + a_1 z_1 k_{m-1}\mu(z_1) + ...)(k_{\mu\nu}(z_2) + a_1 k_{n-2}\nu(z_2) + ...) > \]

\[ = \int dz_1 \, dz_2 \, [S_{m,n\mu\nu} \delta(z_1 - z_2) + S_{m\mu}S_{n\nu} + a_1 z_1 S_{m-1,n\mu\nu} \delta(z_1 - z_2) + ...] \]

(2.13)

We can now work out the gauge transformations. Take (2.13) as an example. The gauge transformation of the LHS can be worked out using (2.11). The gauge transformation of the fields on the RHS are chosen to satisfy this equation. This is done recursively: The gauge transformation of the highest level field \( S_{m,n\mu\nu} \) is fixed by this equation in terms of the gauge transformations of the lower level field. This gives a systematic way of obtaining the gauge transformation laws of all the fields.

### 3 Interactions in Curved Space

The above method needs to be modified in curved space for the reason mentioned in the introduction. Consider the term

\[ k_{0\rho} k_{\mu\nu} k_{\lambda\rho} \]

(3.14)

When we map to space time fields, the derivative becomes a covariant derivative. Furthermore as in the free case, the gauge transformation of this entire expression will not be given by the covariant derivative of the gauge transformation acting on

\[ < k_{\mu\nu} k_{\mu\nu} > \]

(3.15)

What this means is that (3.14) and (3.15) have to be treated as separate independent entities if gauge transformations are to be consistently defined. But this is inconsistent with the idea of \( k_0 \) being a derivative. In the free case we resolved this problem by changing the map \( \mathcal{M} \). Here that is not possible - the two have to be treated independently and we must give up the idea that \( k_0 \) is a (generally covariant) derivative.

One modification that achieves this without spoiling the simplicity of the gauge transformation structure is to deform the \( k_n \) to

\[ p_{\mu\nu} = k_{\mu\nu} + g[k_{(n+1)\nu} k_0\nu - k_{(n+1)\nu} k_0\nu] \]

(3.16)
$g$ can be taken proportional to the coupling constant for the simple reason that this deformation is not required when the coupling constant is zero. It is easy to see that $p_n$ has the same gauge transformation as $k_n$. The antisymmetrization with $k_{0\nu}$ is required in order that $p_n$ be invariant under $\lambda_{n+1}$ transformations. Under this deformation $p_{0\mu} = k_{0\mu} + g[k_{1\mu}k_{0\nu} - k_{1\nu}k_{0\mu}]$. This makes $p_0$ similar to a gauge covariant derivative, with $k_{1\mu}$ the gauge field. The strategy is to use the $p_n$'s in place of $k_n$ in the loop variable. Thus we have $\varepsilon^{i}(\sum_{n} p_{n} Y_{n})$ as the loop variable as far as step 1 is concerned. For step 2 we use the same procedure as before and recursively define gauge transformations for the highest level field in each expression. Whenever we have a space time derivative it is accompanied by a higher level field. Thus $k_{n}k_{m}$ becomes $p_{n}p_{m}$ where the highest level field is $S_{n+1,m+1}$. On the other hand $k_{0}k_{n}k_{m}$ becomes $p_{0}p_{n}p_{m}$ where the highest level field is $S_{1,n+1,m+1}$. Thus it becomes an independent expression, not simply the covariant derivative of the other expression.

### 3.1 Example

Consider the following term (we considered this in the free case as well): This term is level 2 and is simpler than the terms occurring in (2.10), even so the space time expression that it is mapped to is very complicated. We will therefore only outline the steps involved in constructing the map to space time fields and the gauge transformation of the fields:

$$L = \int \int \int dz_{1}dz_{2}dz_{3} \ k_{0\rho}(z_{1})\bar{k}_{1\mu}(z_{2})\bar{k}_{1\nu}(z_{3})$$

(3.17)

First we replace it by the deformed version replacing $k$ by $p$:

$$L = \int \int \int dz_{1}dz_{2}dz_{3} \ p_{0\rho}(z_{1})\bar{p}_{1\mu}(z_{2})\bar{p}_{1\nu}(z_{3})$$

(3.18)

We then write

$$\int dz \ \bar{p}_{\mu\nu}(z) = 
\int dz \ \bar{k}_{\mu\nu}(z) + g[\int dz \ \bar{k}_{(n+1)\nu}(z) \int dz' \ k_{0\nu}(z') - \int dz \ \bar{k}_{(n+1)\nu}(z) \int dz' \ k_{0\nu}(z')]$$

(3.19)

and substitute in (3.18). This gives 27 terms and we will not write them all out. We write down some terms to illustrate the procedure. Integrals over all $z$'s are understood:

$$p_{0\rho}(z_{1})\bar{p}_{1\mu}(z_{2})\bar{p}_{1\nu}(z_{3}) = (k_{0\rho}(z_{1}) + g[k_{1\rho}(z_{1})k_{0\nu}(z_{1}') - \bar{k}_{1\nu}(z_{1})k_{0\rho}(z_{1}']])$$

7
\[
(k_{1\mu}(z_2) + g[k_{2\mu}(z_2)k_{0\nu}(z'_2) - k_{2\nu}(z_2)k_{0\mu}(z'_2)])
\]
\[
(k_{1\nu}(z_3) + g[k_{2\nu}(z_3)k_{0\rho}(z'_3) - k_{2\rho}(z_3)k_{0\nu}(z'_3)])
\]
\[
\text{(3.20)}
\]

The leading term amongst the 27 is

\[
k_{0\rho}(z_1)k_{1\mu}(z_2)k_{1\nu}(z_3) = k_{0\rho}(z_1)(k_{1\mu}(z_2) + z_2k_{0\mu}(z_2))(k_{1\nu}(z_3) + z_3k_{0\nu}(z_3))
\]
\[
\text{(3.21)}
\]

A term of \(O(g^3)\) is

\[
g^3 k_{1\rho}(z_1)k_{2\mu}(z_2)k_{2\nu}(z_3)k_{0\nu}(z'_1)k_{0\nu}(z'_2)k_{0\nu}(z'_3)
\]
\[
= -1(k_{1\rho}(z_1) + z_2k_{0\rho}(z_1))(k_{2\mu}(z_2) + z_2k_{1\mu}(z_2) + \frac{z_2^2}{2}k_{1\mu}(z_2))
\]
\[
(k_{2\nu}(z_3) + z_3k_{1\nu}(z_3) + \frac{z_3^2}{2}k_{1\nu}(z_3))
\]
\[
\text{(3.22)}
\]

The value of \(k_{0\nu}\) is always one less than the renormalization group dimension of the vertex operator multiplying \(p_n\) which is \(n - 1\). Thus \(k_{0\nu}\) adds up to -1. This gives the factor -1.

We then map to space time fields using \(\mathcal{M}\).

\[
\mathcal{M} : L \rightarrow S
\]
\[
\text{(3.23)}
\]

The \(z\)-independent term of the sum \(\text{(3.21)}\) maps to

\[
\int_0^a dz_1 \int_0^a dz_2 \int_0^a dz_3 < k_{0\rho}(z_1)k_{1\mu}(z_2)k_{1\nu}(z_3) > = D_\rho(aS_{1,1\mu \nu} + a^2A_{1\mu}A_{1\nu})
\]
\[
\text{(3.24)}
\]

Similarly the \(z\)-independent term in \(\text{(3.22)}\) maps to

\[
\int_0^a dz_1 \int_0^a dz_2 \int_0^a dz_3 < k_{1\rho}(z_1)k_{2\mu}(z_2)k_{2\nu}(z_3) > = aS_{1,2,2\mu \nu} + a^2(S_{1,2\rho \mu}S_{2\nu} + S_{1,2\mu \nu}S_{2\rho} + S_{2,2\mu \nu}A_{1\rho})
\]
\[
+ a^3A_{1,\rho}S_{2,\mu}S_{2,\nu}
\]
\[
\text{(3.25)}
\]

Note that the highest level field \(S_{1,2,2\mu \nu}\) in this expression has no derivative acting on it. Other fields at the same level such as \(S_{1,2,2\mu \nu}\) have derivatives acting on them.

The above illustrates the calculation of \(S\) from \(L\).

The gauge transformation of \(L\) gives \(L^g\), and its map to \(S^g\) can be similarly evaluated. This gives \(\Delta S\) in the notation used for flat space. The gauge transformation of \(S\) will involve \(\delta S_{1,2,2\rho \mu \nu}, \delta S_{1,2,2\rho \mu \nu}, \delta S_{1,2,2\rho \nu \nu}, \delta S_{1,2,2\rho \nu \nu}, \delta S_{1,2,2\nu \nu}, \delta S_{1,2,2\nu \nu}\).
in addition to $\delta S_{2,2\mu\nu}$, $\delta S_{1,2,\rho\mu}$, ... and all the lower fields. This will give $\delta S$. As before, we require that $\Delta S = \delta S$. Thus we can use this to define $\delta S_{1,2,2\rho\mu\nu}$ (which, as pointed out above has no derivatives acting on it) in terms of the other fields. This defines a recursion both in terms of the level of the fields and the number of indices in the non-V directions. Note that $\delta S_{1,2,2VVV}$ will involve only lower level fields, and $\delta S_{1,2,\rho\rhoVV}$ will involve $\delta S_{1,2,2VVV}$ and lower fields, and so on.

This is the same kind of recursive scheme that was used in flat space. The difference is that the derivative terms have been incorporated into a separate independent term in the scheme by means of the deformation.

Finally, instead of using $M$ we can use $M'$ which differs in its action on the free part of the map as explained in the introduction.

To summarize, by means of the deformation \((3.16)\), one is able to write down a system of generally covariant equations and a corresponding set of gauge transformations that leave these equations invariant. These equations have the form of the covariantized flat space equations \((S_0)\) added to which are terms involving higher level modes, \((Sh)\). They can schematically be written as $S_0 + Sh = 0$. These terms are necessary for gauge invariance in curved space, i.e., in curved space $\delta S_0 = -\delta Sh \neq 0$. In flat space, the equations break up into the usual equations plus gauge invariant pieces involving higher level modes, i.e, $\delta S_0 = 0 = \delta Sh$. Presumably the system of equations can be truncated consistently to the original flat space equations. Clearly, as a result of the deformation, the equations obtained here are some linear combinations of the equations obtained in \([3]\) for the flat space case.

To the extent that the equations for loop variables are RG equations we have followed string theory prescriptions. The map to space time fields is then fixed by considerations of gauge invariance and general covariance - not the one that one would normally use in string theory (because of the deformation). However if it is true that the final set of equations are unique (given the particle content, gauge invariance and generally covariance), up to field redefinition, then this must be the answer that string theory also gives.

4 Conclusions

In this paper we have given a systematic method of writing down interacting gauge invariant and generally covariant equations for all the modes of the open string. The method is a combination of the methods used in \([1]\) for the free open string in curved space and \([3]\) for interacting open strings in flat space. It consists of the following ingredients:
1. Riemann Normal Coordinates are used for the sigma model in curved space and generally covariant equations are obtained from the loop variable equation as described in [1].

2. The loop variable momenta are deformed as described in [3,16], so the loop variable equation is different from the flat space loop variable equations.

3. The definition of gauge transformation of all fields can be defined unambiguously using the same technique as used in the flat space case. Once the loop variable equations are obtained in terms of $p_n$, we reexpress them in terms of $k_n$ and then proceed exactly as in the flat space case to define gauge transformations recursively for the higher level modes in terms of lower level modes.

We have illustrated the various steps involved by examples. However, the full answer (while reasonably simple in terms of the deformed loop variables), is in even the simplest case, quite complicated when written out in terms of space time fields. It seems unlikely that working with explicit space time field equations will lead to any insights or solutions. We suspect that it is better to work with loop variables directly. This is a problem for the future. Perhaps the most interesting conclusion for the present, is just the fact that there exists an algorithm for writing down gauge and generally covariant interacting equations for all the higher mass and spin modes of the open string in curved space.

It would be very useful to show by explicit computations on the sigma model that these equations are indeed equivalent to the string equations (upto field redefinitions), instead of relying on indirect arguments of gauge invariance and general covariance.

If indeed these equations are equivalent to string theory equations, then this is a step towards a background independent formulation of string theory [3]. The loop variable approach is not tied down to any particular background. It is also important to find out whether there is an action principle here analogous to that discussed in [3]. Some aspects of this action have been discussed in [6].

It would also be interesting to compare these results with other results on higher spin interactions in curved space [7,8,9,10,11,12].

References

[1] B. Sathiapalan, Mod. Phys. Lett. A20 (2005) 227, hep-th/0412033.

[2] B. Sathiapalan, Nucl. Phys. B326, (1989) 376.
[3] B. Sathiapalan, Mod. Phys. Lett. A17 (2002) 1175, hep-th/0201216.

[4] B. Sathiapalan, Int. J. Mod. Phys. A18 (2003) 767, hep-th/0207098.

[5] E. Witten, Phys. Rev. D46 (1992) 5467, hep-th/9208027; Phys. Rev. D47 (1993) 3405, hep-th/9208027.

[6] A. M. Polyakov, "Gauge Fields and Strings", Harwood Academic Publishers, New York, 1987.

[7] C. Aragone and S. Deser, Phys. Lett. B86 (1979) 161.

[8] A. Bengtsson, I. Bengtsson, and L. Brink, Nucl. Phys. B227 (1983) 31, 41.

[9] F. A. Berends, G. J. Burgers and H. van Dam, Z. Phys. C24 (1984) 247.

[10] I. L. Buchbinder, D. M. Gitman, V. A. Krykhtin, V. D. Pershin, Nucl. Phys. B584 (2000) 615, hep-th/9910188.

[11] E. S. Fradkin and M. A. Vasiliev, Phys. Lett. B189 (1987) 89; Nucl. Phys. B291 (1987) 141.

[12] M. A. Vasiliev, hep-th/0409260 and references therein.