A stochastic-Lagrangian particle system for the Navier–Stokes equations

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Abstract

This paper is based on a formulation of the Navier–Stokes equations developed by Constantin and the first author (Commun. Pure Appl. Math. at press, arXiv:math.PR/0511067), where the velocity field of a viscous incompressible fluid is written as the expected value of a stochastic process. In this paper, we take $N$ copies of the above process (each based on independent Wiener processes), and replace the expected value with $1/N$ times the sum over these $N$ copies. (We note that our formulation requires one to keep track of $N$ stochastic flows of diffeomorphisms, and not just the motion of $N$ particles.)

We prove that in two dimensions, this system of interacting diffeomorphisms has (time) global solutions with initial data in the space $C^{1,\alpha}$, which consists of differentiable functions whose first derivative is $\alpha$ Hölder continuous (see section 3 for the precise definition). Further, we show that as $N \to \infty$ the system converges to the solution of Navier–Stokes equations on any finite interval $[0, T]$. However for fixed $N$, we prove that this system retains roughly $O(1/N)$ times its original energy as $t \to \infty$. Hence the limit $N \to \infty$ and $T \to \infty$ do not commute. For general flows, we only provide a lower bound to this effect. In the special case of shear flows, we compute the behaviour as $t \to \infty$ explicitly.

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1. Introduction

The Navier–Stokes equations

\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p &= 0, \\
\nabla \cdot u &= 0
\end{align*}

\textsuperscript{(1.1)} \textsuperscript{(1.2)}
describe the evolution of a velocity field of an incompressible fluid with kinematic viscosity \( \nu > 0 \). These equations have been used to model numerous physical problems, for example air flow around an airplane wing, ocean currents and meteorological phenomena to name a few \([4, 15, 18]\). The mathematical theory (existence and regularity \([5, 14]\)) of these equations has been extensively studied and is still one of the outstanding open problems in modern PDEs \([7, 8]\).

The questions addressed in this paper are motivated by a formalism of (1.1)–(1.2) developed in \([9]\) (equations (2.1)–(2.2)). This formalism essentially superimposes Brownian motion onto particle trajectories, and then averages with respect to the Wiener measure. In this paper, we take \( N \) independent copies of the Wiener process and replace the expected value in the above formalism with \( 1/N \) times the sum over these \( N \) independent copies (see equations (2.5)–(2.6) for the exact details). In the original formulation the random trajectory of a particle induced by a single Brownian motion interacts with its own law. This is essentially a self-consistent, mean-field interaction. In this paper, we replace this with \( N \) copies or replicas whose average is used to approximate the interaction with the processes own law. This technique has been extensively used in numerical computation (e.g. \([2, 17, 19]\)). We note that in our formulation, we are required to keep track of \( N \) different stochastic flows of diffeomorphisms, and not just the motion of \( N \) different particles, as is the conventional approach.

We study both the behaviour as \( N \to \infty \) and \( t \to \infty \) of the system obtained. The behaviour as \( N \to \infty \) is as expected: in two dimensions on any finite time interval \([0, T]\), the system converges as \( N \to \infty \) to the solution of the true Navier–Stokes equations roughly at the rate \( O(1/\sqrt{N}) \). In three dimensions, we can only guarantee this if we have certain \textit{a priori} bounds on the solution (theorem 4.1). These \textit{a priori} bounds are of course guaranteed for a short time, but are unknown (in the three-dimensional setting) for a long time \([7, 8]\).

At first glance, the behaviour as \( t \to \infty \) for fixed \( N \) is less intuitive. For the two-dimensional problem, figures 1(a) and (b) show a graph of \( \|\omega^N_t\|_{L^2}^2 \) versus time, with \( N = 2 \) and \( N = 8 \), respectively\(^3\). A little reflection shows that this behaviour is not completely surprising. The dissipation occurs through the averaging of different copies of the flow. With only \( N \) copies, one can only produce dissipation of order \( 1/N \) of the original energy. It is tempting to speak of the inability to represent the correct interaction of small scale structures

\(^3\) These computations were done using a \( 24 \times 24 \) mesh on the periodic box with side length \( 2\pi \). The initial vorticity was randomly chosen, and normalized with \( \|\omega^N_0\|_{L^2} = 1 \). The behaviour depicted in these two figures is however characteristic and insensitive to changes of the mesh size, length or diffusion coefficient.
with such a small number of data. However, we cannot make this precise and since each of the objects being averaged is an entire diffeomorphism with an infinite amount of information it is unclear what this means.

In section 5 we obtain a sharp lower bound to this effect. We show (theorem 5.2) that
\[ \lim_{t \to \infty} \mathbb{E} \| \nabla u_t \|_{L^2}^2 \geq \frac{1}{N L^2} \| u_0 \|_{L^2}^2, \]
where \( L \) is a length scale. Further, we explicitly compute the \( t \to \infty \) behaviour in the special case of shear flows and verify that our lower bound is sharp.

We note that we considered the analogue of the system above for the one-dimensional Burgers equations. As is well known the viscous Burgers equations have global strong solutions. However, preliminary numerical simulations show that the system above forms shocks almost surely, even for very large \( N \). We are currently working on understanding how to continue this system past these shocks, in a manner analogous to the entropy solutions for the inviscid Burgers equations, and studying its behaviour as \( t \to \infty \) and \( N \to \infty \).

We do not propose this particle system as an efficient particle method for numerical computation. Though there may be special cases where it may be useful, in general the computational cost of representing \( N \) entire diffeomorphisms is large. Rather we see it as an interesting and novel regularization which might give useful insight into the structure and role of dissipation in the system.

2. The particle system

In this section we construct a particle system for the Navier–Stokes equations based on stochastic-Lagrangian trajectories. We begin by describing a stochastic-Lagrangian formulation of the Navier–Stokes equations developed in [9, 11].

Let \( W \) be a standard two- or three-dimensional Brownian motion, and \( u_0 \) some given divergence free \( C^{2, \alpha} \) initial data. Let \( \mathbb{E} \) denote the expected value with respect to the Wiener measure and \( P \) be the Leray–Hodge projection onto divergence free vector fields. Consider the system of equations
\begin{align*}
\dot{X}_t(x) &= u_t(X_t(x)) \, dt + \sqrt{2} \nu \, dW_t, \quad X_0(x) = x, \\
u_t &= \mathbb{E} P[(\nabla^* Y_t)(u_0 \circ Y_t)], \quad Y_t = X_t^{-1}. \tag{2.2}
\end{align*}

With a slight abuse of notation, we denote by \( X_t \) the map from initial conditions to the value at time \( t \). Hence \( X_t \) is a stochastic flow of diffeomorphisms with \( X_0 \) equal to the identity and \( Y_t \) the ‘spatial’ inverse. In other words, \( Y_t : X_t(x) \mapsto x \). Also by \( \nabla^* Y_t \) we mean the transpose of the Jacobian of map \( Y_t \). Observe that \( (\nabla^* Y_t)(u_0 \circ Y_t) \) can be viewed as a function of \( x \) where both the Jacobian and the vector field \( u_0 \circ Y_t \) to which it is applied are evaluated at \( x \).

We impose periodic boundary conditions on the displacement \( \lambda_t(y) = X_t(y) - y \), and on the Leray–Hodge projection \( P \).

In [9, 11] it was shown that the system (2.1)–(2.2) is equivalent to the Navier–Stokes equations in the following sense: if the initial data are regular \( (C^{2, \alpha}) \), then the pair \( X, u \) is a solution to the system (2.1)–(2.2) if and only if \( u \) is a (classical) solution to the incompressible Navier–Stokes equations with periodic boundary conditions and initial data \( u_0 \).

We digress briefly and comment on the physical significance of (2.1)–(2.2). Note first that equation (2.2) is \emph{algebraically} equivalent to the equations
\begin{align*}
u_t &= -\Delta^{-1} \nabla \times \omega_t, \\
\omega_t &= \mathbb{E}[(\nabla X_t) u_0] \circ Y_t. \tag{2.3, 2.4}
\end{align*}
This follows by direct computation, and was shown \[6, 9\] for instance. We recall that (2.4) is the usual vorticity transport equation for the Euler equations, and (2.3) is just the Biot–Savart formula.

Thus in particular when \(\nu = 0\), the system (2.1)–(2.2) is exactly the incompressible Euler equations. Hence the system (2.1)–(2.2) essentially does the following: we add Brownian motion to Lagrangian trajectories. Then recover the velocity \(u\) in the same manner as for the Euler equations, but additionally average out the noise.

We note that the system (2.1)–(2.2) is nonlinear in the sense of McKean [20]. The drift of the flow \(X\) depends on its distribution. However in this case, the law of \(X\) alone is not enough to compute the drift \(u\). This is because of the presence of the \(\nabla^* Y\) term in (2.2), which requires knowledge of spatial covariances, in addition to the law of \(X\). In other words, one needs that law of the entire flow of diffeomorphism and not just the law of the one-point motions.

We now motivate our particle system. For the formulation (2.1)–(2.2), the natural numerical scheme would be to use the law of large numbers to compute the expected value. Let \((W^i)\) be a sequence of independent Wiener processes, and consider the system

\[
\begin{align*}
\mathrm{d}X^i_t &= u^N_t(X^i_t) \, \mathrm{d}t + \sqrt{2\nu} \, \mathrm{d}W^i_t, \\
u^N_t &= \frac{1}{N} \sum_{i=1}^{N} P[(\nabla^* Y^i_t)(u^0 \circ Y^i_t)]
\end{align*}
\]

with initial data \(X_0(x) = x\). We impose again periodic boundary conditions on the initial data \(u_0\), the displacement \(\lambda_t(x) = X_t(x) - x\) and the Leray–Hodge projection \(P\).

We note that the algebraic equivalence of (2.2) and (2.3)–(2.4) is still valid in this setting. Thus the system (2.5)–(2.6) could equivalently be formulated by replacing equation (2.6) with the more familiar equations

\[
\omega^N_t = \frac{1}{N} \sum_{i=1}^{N} [(\nabla X^i_t)(\omega_0)] \circ Y^i_t
\]

and

\[
u^N_t = -\Delta^{-1} \nabla \times \omega^N_t.
\]

Finally we clarify our previous remark, stating that the above formulation requires us to keep track of \(N\) stochastic flows, and knowledge of the one-point motions of \(X^i_t\) alone is not sufficient. The standard method of obtaining a solution to the heat equation (assuming the drift \(u\) is time independent) would be to consider the process (2.1), and read off the solution \(\theta_t\) by

\[
\theta_t(a) = \mathbb{E}\theta_0 \circ X_t(a),
\]

where \(\theta_0\) is the given initial temperature distribution. Thus knowing the trajectories (and distribution) of the process \(X\) starting at one particular point \(a\) will be sufficient to determine the solution \(\theta_t\) at that point.

This however is not the case for our representation. The reason is twofold: first, the representation (2.1)–(2.2) involves a non-local singular integral operator. Second our representation involves composing with the spatial inverse of the flow \(X_t\), and then averaging.

If we for a moment ignore the non-locality of the Leray–Hodge projection, determining \(u_t\) at one fixed point \(a\) one would need the law of \(Y_t(a)\), for which the knowledge of \(X_t(a)\) alone is not enough. One needs the entire (spatial) map \(X_t\) to compute the spatial inverse \(Y_t(a)\).
The above is not a serious impediment to a numerical implementation. Given an initial mesh \( \Delta \), we first compute \( X^{i,N}_{t} \) on this mesh. By definition of the inverse, one knows \( Y^{i,N}_{t} \) on the (non-uniform) mesh \( X^{i,N}_{t}(\Delta) \), after which one can interpolate and find \( Y^{i,N}_{t} \) on the mesh \( \Delta \). In two spatial dimensions, global existence and regularity (theorem 3.5) together with incompressibility will show that this mesh does not degenerate in finite time.

This, surprisingly, is not the case for the (one-dimensional) Burgers equations. Numerical computations indicate the mesh \( \Delta \) almost surely degenerates in finite time for non-monotone initial data, and the solution 'shocks' almost surely. Thus, while the system (2.5)–(2.6) appears natural, and convergence as \( N \to \infty \) is to be expected, caution is to be exercised. We suspect that the results (existence, convergence, etc) proven in this paper for the system (2.5)–(2.6) are in fact false for the Burgers equations. This is indeed puzzling as global existence, and regularity for the viscous Burgers equations is well known. It further underlines the fact that the finite \( N \) approximation modifies the dissipation in a different way than other approximations such as a spectral approximation.

In the next section, we show the existence of global solutions to (2.5)–(2.6) in two dimensions. In section 4, we show that the solution to (2.5)–(2.6) converges to the solution of the Navier–Stokes equations as \( N \to \infty \). Finally in section 5 we study the behaviour of the system (2.5)–(2.6) as \( t \to \infty \) (for fixed \( N \)), and partially explain the behaviour shown in figures 1(a) and (b).

3. Global existence of the particle system in two dimensions

In this section we prove that the particle system (2.5)–(2.6) has global solutions in two dimensions. Once we are guaranteed global in time solutions, we are able to study the behaviour as \( t \to \infty \), which we do in section 5. We also note that as a consequence of theorem 3.5 (proved here), our convergence result as \( N \to \infty \) (theorem 4.1) applies on any finite time interval \([0, T]\) in the two-dimensional situation.

We first establish some notational convention: we let \( L > 0 \) be a length scale, and assume work with the spatial domain is \([0, L]^d\), where \( d \geq 2 \) is the spatial dimension. We define the non-dimensional \( L^p \) and Hölder norms by

\[
\|u\|_{L^p} = \left( \frac{1}{L^d} \int_{[0, L]^d} |u|^p \right)^{\frac{1}{p}},
\]

\[
\|u\|_{L^\infty} = \sup_{x \in [0, L]^d} |u(x)|,
\]

\[
|u|_{L^a} = L^a \sup_{x, y \in [0, L]^d} \frac{|u(x) - u(y)|}{|x - y|^a},
\]

\[
\|u\|_{k,\alpha} = \sum_{|m| = k} L^k \|D^m u\|_{L^\infty} + \sum_{|m| < k} L^{|m|} \|D^m u\|_{L^\infty}.
\]

We denote the Hölder space \( C^{k,a} \) and Lebesgue space \( L^p \) to be the space of functions \( u \) which are periodic on \([0, L]^d\) with \( \|u\|_{k,\alpha} < \infty \) and \( \|u\|_{L^p} < \infty \), respectively.

Let \( W^1, \ldots, W^N \) be \( N \) independent (two-dimensional) Wiener processes, with filtration \( \mathcal{F}_t \).

**Proposition 3.1.** Let \( t_0 \geq 0 \), and \( u^1_{t_0}, \ldots, u^N_{t_0} \) be \( \mathcal{F}_{t_0} \)-measurable, periodic mean 0 functions such that the norms \( \|u^i_{t_0}\|_{1,\alpha} \) are almost surely bounded. Then there exist
\[ T = T(\alpha, \|u^i_{t_0}\|_{1,\alpha}, \ldots, \|u^N_{t_0}\|_{1,\alpha}) \] such that the system

\[ X^i_{t_0,t}(x) = x + \int_{t_0}^{t} u_i(X^i_{t_0,s}(x)) \, ds + \sqrt{2\nu(W^i_{t_0} - W^i_t)}, \quad Y^i_{t_0,t} = (X^i_{t_0,t})^{-1}, \tag{3.1} \]

has a solution on the interval \([t_0, T]\). Further there exists deterministic \(U = U(\alpha, \|u^i_{t_0}\|_{1,\alpha})\) such that

\[ \sup_{t \in [t_0, T]} \|P[(\nabla^* Y^i_{t_0,t})u^i_{t_0} \circ Y^i_{t_0,t}]\|_{1,\alpha} \leq U \tag{3.3} \]

almost surely. Consequently, \(u \in C([t_0, T], C^{1,\alpha})\), and \(\|u\|_{1,\alpha} \leq U\).

Proposition 3.1 is proved in appendix A.

**Definition 3.2.** We call \(X^i, u\) in proposition 3.1 the solution of the system (3.1)–(3.2) with initial data \(u^i_{t_0}\).

The functions \(X^i, Y^i\) are not periodic themselves, but have periodic displacements: namely, if we define

\[ \lambda^i_t(y) = X^i_t(y) - y, \tag{3.4} \]
\[ \mu^i_t(x) = Y^i_t(x) - x, \tag{3.5} \]
then \(\mu^i, \lambda^i\) are periodic.

We note that if \(t_0 = 0\) and the \(\omega^i_{t_0}\) are all equal, then the system (3.1)–(3.2) reduces to (2.5)–(2.6). However when formulated as above, solutions can be continued past time \(T\) by restarting the flows \(X^i\), as in lemma below.

**Lemma 3.3.** Say \(t_0 < t_1 < t_2\), and \(X^i_{t_0,t}, u_s\) solve (3.1)–(3.2) on \([t_0, t_2]\), with initial data \(u^i_{t_0}\). For any \(t > t_0\) define

\[ u^i_t = P[(\nabla^* Y^i_{t_0,t})u^i_{t_0} \circ Y^i_{t_0,t}]. \]

Let \(\tilde{X}^i_{t_1,t}, \tilde{u}_s\) solve (3.1)–(3.2) on \([t_1, t_2]\) with initial data \(u^i_{t_1}\). Then for all \(s \in [t_1, t_2]\) we have \(\tilde{u}^i_s = u^i_s\) and

\[ X^i_{t_0,t_1} = \tilde{X}^i_{t_1,t} \circ X^i_{t_0,t_1} \]
almost surely.

**Proof.** The proof is identical to the proof of proposition 3.3.1 in [11], and we do not provide it here. \(\square\)

For the remainder of this section, we assume without loss of generality that \(t_0 = 0\) (we allow of course \(F_0\) to be non-trivial). For ease of notation, we use \(X^i\) to denote \(X^i_{0,t}\).

We now prove that the system (3.1)–(3.2) has global solutions in two dimensions. This essentially follows from a Beale–Kato–Majda type condition [1], and the two-dimensional vorticity transport.

**Lemma 3.4.** If \(u\) is divergence free and periodic in \(\mathbb{R}^d\), then for any \(\alpha \in (0, 1)\), there exists a constant \(c = c(\alpha, d)\) such that

\[ \|\nabla u\|_{L^\infty} \leq c\|\omega\|_{L^\infty} \left(1 + \ln^+ \left(\frac{\|\omega\|_{L^\infty}}{\|\omega\|_{L^\infty}}\right)\right), \]

where \(\omega = \nabla \times u\).
The lemma is a standard fact about singular integral operators, and we provide a proof in appendix B for completeness.

**Theorem 3.5.** In two dimensions, the system (3.1)–(3.2) has time global solutions provided the initial data $u_1^0, \ldots, u_N^0$ are periodic, $\mathcal{F}_0$-measurable with $\|u^0_i\|_1$ bounded almost surely. In particular we have global solutions to (3.1)–(3.2) in two dimensions, if $u_1^0 = \cdots = u_N^0 = u_0$ is deterministic, Hölder $1 + \alpha$ and periodic.

**Proof.** Taking the curl of (3.2) gives the familiar Cauchy formula [6, 9, 10]

$$\omega_t = \frac{1}{N} \sum_{i=1}^{N} [\nabla X_i'] \omega_i^0 \circ Y'_i,$$

where $\omega_t = \nabla \times u_t$. In two dimensions reduces to

$$\omega_t = \frac{1}{N} \sum_{i=1}^{N} \omega_i^0 \circ Y'_i. \quad (3.7)$$

Taking Hölder norms gives

$$\|\omega_t\|_\alpha \leq c \frac{1}{N} \sum_{i=1}^{N} \|\omega_i^0\|_\alpha (1 + \|\nabla Y'_i\|_{L^\infty}^\alpha) \quad \text{a.s.} \quad (3.8)$$

Now differentiating (3.1) gives

$$\nabla X'_i = I + \int_0^t (\nabla u_s \circ X'_i) \nabla X'_i \, ds \quad \text{a.s.}$$

Taking the $L^\infty$ norm, and applying Gronwall’s lemma shows

$$\|\nabla X'_i\|_{L^\infty} \leq \exp \left( c \int_0^t \|\nabla u_s\|_{L^\infty} \, ds \right) \quad \text{a.s.}$$

Recall $\nabla \cdot u = 0$, and hence $\det(\nabla X') = 1$ almost surely. Thus the entries of $\nabla Y$ are a polynomial (of degree 1) in the entries of $\nabla X$. This immediately gives

$$\|\nabla Y'_i\|_{L^\infty} \leq \exp \left( c \int_0^t \|\nabla u_s\|_{L^\infty} \, ds \right) \quad (3.9)$$

almost surely. Combining this with (3.8) gives us the *a priori* bound

$$\|\omega_t\|_\alpha \leq c \frac{1}{N} \sum_{i=1}^{N} \|\omega_i^0\|_\alpha \exp \left( c \int_0^t \|\nabla u_s\|_{L^\infty} \, ds \right). \quad (3.10)$$

Applying lemma 3.4 gives us

$$\|\nabla u\|_{L^\infty} \leq c \|\omega\|_{L^\infty} \left( 1 + \ln^+ \left( \frac{|\omega|_\alpha}{\|\omega\|_{L^\infty}} \right) \right)$$

$$\leq c \|\omega\|_{L^\infty} \left( 1 + \ln^+ \left( \frac{|\omega|_\alpha}{\|\omega\|_{L^\infty}} \right) + \ln^+ \left( \frac{1}{N} \sum_{i=1}^{N} \|\omega_i^0\|_{L^\infty} \right) \right).$$

Note that the function $x \ln^+(1/x)$ is bounded, so the last term on the right can be bounded above by some constant $c_0$. For the remainder of the proof, we let $c_0 = c_0(\|\omega_i^0\|_\alpha, \alpha)$ denote a constant (with dimensions that of $\omega$) which changes from line to line. Thus

$$\|\nabla u\|_{L^\infty} \leq c_0 + c \|\omega\|_{L^\infty} \left( 1 + \ln^+ \left( \frac{|\omega|_\alpha}{\|\omega\|_{L^\infty}} \right) \right)$$

$$\leq c_0 + c \|\omega\|_{L^\infty} \left( 1 + \ln^+ \left( \frac{1}{N} \sum_{i=1}^{N} \|\omega_i^0\|_{L^\infty} \right) \right).$$
and hence
\[
\|\nabla u_t\|_{L^\infty} \left(1 + \int_0^t \|\nabla u_s\|_{L^\infty} \, ds\right)^{-1} \leq c_0 + c \|\omega_t\|_{L^\infty}.
\]
Integrating gives us the \textit{a priori} bound
\[
\int_0^t \|\nabla u_t\|_{L^\infty} \leq \exp\left(c_0 t + c \int_0^t \|\omega_s\|_{L^\infty} \right) - 1 \leq c_0 + c \|\omega_t\|_{L^\infty}.
\]
(3.11)
since (3.7) implies \(\|\omega_t\|_{L^\infty} \leq N \sum \|\omega_i\|_{L^\infty} \).

Now if we set \(\omega'_t = \omega_0 \circ Y_t \), then (3.9) and the \textit{a priori} bound (3.11) give
\[
\|\omega'_i\|_\alpha \leq \|\omega_0\|_\alpha \exp(c_0 t e^{c_0 t}).
\]
(3.12)
If \(u_i\) is as in lemma 3.3, then (3.12) shows
\[
\|\nabla u_i\|_\alpha \leq c \|\omega'_i\|_\alpha \leq c \|\omega_0\|_\alpha \exp(c_0 t e^{c_0 t}).
\]
(3.13)
Since the mean velocity is a conserved quantity, a bound on \(\|\nabla u_i\|_\alpha\) immediately gives a bound on \(\|u_i\|_{1,\alpha}\), which in conjunction with local existence (proposition 3.1), and lemma 3.3 concludes the proof. \(\square\)

4. Convergence as \(N \to \infty\)

In this section, we fix a time interval \([0, T]\), and show that the particle system (2.5)–(2.6) converges to the solution to the Navier–Stokes equations as \(N \to \infty\). The rate of convergence is \(O(1/\sqrt{N})\), which is comparable to the convergence rate of the random vortex method [3, 16].

As mentioned earlier, the system is intrinsically non-local, and propagation of chaos [20] type estimates are not easy to obtain. Consequently convergence results based on spatially averaged norms are easier to obtain, and we present one such result in this section, under assumptions which are immediately guaranteed by local existence.

**Theorem 4.1.** For each \(i, N\), let \(X_i,N, u^N\) be a solution to the particle system (2.5)–(2.6) with initial data \(u_0\) on some time interval \([0, T]\). Let \(u\) be a solution to the Navier–Stokes equations (with the same initial data) on the interval \([0, T]\). Suppose \(U\) is such that
\[
\sup_{t \in [0,T]} \|\nabla u_t\|_{L^2} \leq U \quad \text{and} \quad \sup_{t \in [0,T]} \|\nabla u^N_t\|_{L^2} \leq U \quad \text{a.s.}
\]
Then \((u^N) \to u\) in the following sense:
\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\|u^N_t - u_t\|_{L^2} = 0.
\]

**Remark.** As can be seen from the proof, we in fact have
\[
\sup_{t \in [0,T]} \mathbb{E}\|u^N_t - u_t\|_{L^2} \leq O\left(\frac{1}{\sqrt{N}}\right).
\]
We note that given \(C^{1,\alpha}\) initial data, local existence (proposition 3.1) guarantees that the conditions of this theorem are satisfied on some small interval \([0, T]\). In two dimensions, theorem 3.5 shows that the conditions of this theorem are satisfied on any finite interval \([0, T]\).
The proof will follow almost immediately from the following lemma.

**Lemma 4.2.** Let $u_{i,N} = P[(∇^* Y_i^N)u_0 ∘ Y_i^N]$ be the $i$th summand in (2.6). Then $u_{i,N}$ satisfies the SPDE

$$
du_{i,N} + [(u_{i,N} \cdot ∇)u_{i,N} - vΔu_{i,N} + (∇^* u_{i,N})u_{i,N} + ∇ p_{i,N}] dt + \sqrt{2ν}∇u_{i,N} dW_t^i = 0 \quad (4.1)
$$

and $u^N$ satisfies the SPDE

$$
du_{i,N} + [(u_{i,N} \cdot ∇)u_{i,N} - vΔu_{i,N} + ∇ p_{i,N}] dt + \sqrt{2ν} \sum_{i=1}^N ∇u_{i,N} dW_t^i = 0. \quad (4.2)
$$

**Remark.** We draw attention to the fact that the pressure term in (4.2) has bounded variation in time.

**Proof.** We first recall a fact from [9, 11] (see also [12]). If $X_t = u_0$, $dt = √2ν dW_t$ and $Y_t = X_t^{-1}$ is the spatial inverse. Then the process $θ_t = f(Y_t)$ satisfies the SPDE

$$
dθ_t + (u_0 \cdot ∇)θ_t dt - vΔθ_t dt + √2ν∇θ_t dW_t = 0. \quad (4.3)
$$

This immediately shows that $Y_i^{i,N}$ and $v_i^{i,N} = u_0 ∘ Y_i^{i,N}$ both satisfy the SPDE (4.3). For notational convenience, we momentarily drop the $N$ as a superscript and use the notation $v_{i,j}$ to denote the $j$th component of $v_i$.

Now we set $u_{i,j} = (∇^* Y_i^j) v_i^j$ and apply Itô’s formula:

\[
\begin{align*}
    du_{i,j} &= d(∂_j Y_i^j) \cdot v_i^j + (∂_j Y_i^j) \cdot dv_i^j + d(∂_j Y_i^{i,k}) v_i^{i,k} \\
    &= (∂_j Y_i^j) \cdot [−(u_{i,N} \cdot ∇)v_i^j + vΔv_i^j] dt - √2ν∂_j Y_i^j \cdot (∇v_i^j dW_t^i) \\
    &+ v_i^j \cdot [−((∂_j u_{i,N} \cdot ∇) Y_i^j − (u_{i,N} \cdot ∇)∂_j Y_i^j) + vΔ∂_j Y_i^j] dt \\
    &− √2νv_i^j \cdot (∇∂_j Y_i^j dW_t^i) + 2ν∂_j^2 Y_i^{i,k}∂_j v_i^{i,k} dt \\
    &= [−(u_{i,N} \cdot ∇)v_i^j + vΔv_i^j − (∇^* u_{i,N}) \cdot v_i^j] dt - √2ν∇v_i^j dW_t^i.
\end{align*}
\]

Restoring the dependence on $N$ to our notation, since $u_{i,N} = P u_{i,N}$ we know that

$$
u_{i,j} = u_{i,j} + ∇q_{i,j}^N
$$

for some function $q_{i,j}^N$. Thus

\[
\begin{align*}
    du_{i,j}^{i,N} &= du_{i,j} + d(∇q_{i,j}^N) \\
    &= [−(u_{i,N} \cdot ∇)u_{i,j}^N + vΔu_{i,j}^N − (∇^* u_{i,N})u_{i,j}^N] dt - √2ν∇u_{i,j}^N dW_t^i \\
    &= [−(u_{i,N} \cdot ∇)u_{i,j}^N + vΔu_{i,j}^N − (∇^* u_{i,N})u_{i,j}^N] dt - √2ν∇u_{i,j}^N dW_t^i \\
    &+ [−(u_{i,N} \cdot ∇)(∇q_{i,j}^N) + vΔ(∇q_{i,j}^N) − (∇^* u_{i,N})(∇q_{i,j}^N)] dt \\
    &− √2ν∇(∇q_{i,j}^N) dW_t^i + d(∇q_{i,j}^N).
\end{align*}
\]

If we define $P_{i,j}^{i,N}$ by

$$
P_{i,j}^{i,N} = \int_0^t [(u_{i,N} \cdot ∇)q_{i,j}^{i,N} − vΔq_{i,j}^{i,N}] ds + \int_0^t √2ν∇q_{i,j}^{i,N} \cdot dW_s^i + q_{i,j}^{i,N}
$$

then we have

$$
du_{i,j}^{i,N} + [(u_{i,N} \cdot ∇)u_{i,j}^N − vΔu_{i,j}^N + (∇^* u_{i,N})u_{i,j}^N] dt + d(∇ P_{i,j}^{i,N}) + √2ν∇u_{i,j}^N dW_t^i = 0. \quad (4.4)
$$
Note that $u^{i,N}$ is divergence free by definition, and thus $\nabla u^{i,N} \, dW^i$ is also divergence free. Thus $d(\nabla p^{i,N})$, the only other term with possibly non-zero quadratic variation, must have a divergence free martingale part. Since the martingale part of $d(\nabla p^{i,N})$ is also a gradient, it must be 0. Thus

$$d(\nabla p^{i,N}) = \nabla p^{i,N} \, dt$$

for some function $p^{i,N}$, which proves (4.1).

The identity (4.2) now follows by summing (4.1) over $i$, dividing by $N$, and defining $p^N$ by

$$p^N_t = \frac{1}{2} \nabla |u^N_t|^2 + \frac{1}{N} \sum_{i=1}^N p^{i,N}_t$$

□

Proof of theorem 4.1. Let $u$ be a solution of the Navier–Stokes equations, with initial data $u_0$, and set $v^N = u^N - u$. Then $v^N$ satisfies the SPDE

$$d(v^N_t + (v^N_t \cdot \nabla) u_t) dt + (u_t \cdot \nabla)v^N_t dt = -\nu \Delta v^N_t dt + \nabla (p^N_t - p_t) dt + \frac{\sqrt{2}v^N_t}{N} \sum_{i=1}^N \nabla u^{i,N}_t \, dW^i_t = 0,$$

where $p$ is the pressure in the Navier–Stokes equations, and $p^N$ the pressure term in (4.2).

Thus by Itô’s formula,

$$\frac{1}{2} d\|v^N_t\|_{L^2}^2 + (v^N_t, (v^N_t \cdot \nabla) u_t) dt + \nu \|\nabla v^N_t\|_{L^2}^2 dt + \frac{2v^N_t}{N} \sum_{i=1}^N v^{i,N}_t \cdot (\nabla u^{i,N}_t \, dW^{i,k}_t)$$

$$= \frac{\nu}{N^2} \sum_{i=1}^N \|\nabla u^{i,N}_t\|_{L^2}^2 \, dt.$$

Here the notation $(f, g)$ denotes the $L^2$ inner product of $f$ and $g$. Taking expected values gives us

$$\frac{1}{2} \partial_t \mathbb{E}\|v^N_t\|_{L^2}^2 \leq \frac{U}{\mathbb{E}\|v^N\|_{L^2}^2} + \frac{\nu U^2}{NL^2}$$

and by Gronwall’s lemma we have

$$\mathbb{E}\|v^N_t\|_{L^2}^2 \leq \frac{2\nu U^2}{L^2N} e^{\frac{Ut}{N}}$$

concluding the proof. □

5. Convergence as $t \to \infty$

In this section, we fix $N$, and consider the behaviour of the system (2.5)–(2.6) as $t \to \infty$. We show that the system (2.5)–(2.6) does not dissipate all its energy as $t \to \infty$. Roughly speaking, we show

$$\limsup_{t \to \infty} \mathbb{E}\|\nabla u_t\|_{L^2}^2 \geq O\left(\frac{1}{N}\right),$$

with constants independent of viscosity. This is in contrast to the true (unforced) Navier–Stokes equations, which dissipate all of its energy as $t \to \infty$ (provided of course the solutions are defined globally in time).

In general we are unable to compute the exact asymptotic behaviour of the system (2.5)–(2.6) as $t \to \infty$. But in the special case of shear flows, we compute this exactly,
and show that the system eventually converges to a constant, retaining exactly $1/N$ times its initial energy.

For the remainder of this section, we pick a fixed $N \in \mathbb{N}$ and for notational convenience we omit the superscript $N$. We begin by computing exactly the asymptotic behaviour of the system (2.5)–(2.6) in the special case of shear flows.

**Proposition 5.1.** Suppose the initial data $u_0(x) = (\phi_0(x_2), 0)$ for some $C^{1,\alpha}$ periodic function $\phi$. If $u$ is the velocity field that solves the system (2.5)–(2.6) with initial data $u_0$, then

$$\lim_{t \to \infty} E\omega_t(x)^2 = \frac{1}{N}\|\omega_0\|^2_{L^2},$$

where $\omega = \nabla \times u$ is the vorticity.

**Proof.** Let $X^i, Y^i$ be the flows in the system (2.5)–(2.6), and as before define $u^i$ to be the $i$th summand in (2.6), and $\omega^i = \omega_0 \circ Y^i$.

First note that the SPDEs for $u^i$ and $u$ (equations (4.1) and (4.2)) are all translation invariant. Thus since the initial data are independent of $x_1$, the same must be true for all time. Since $u^i, u$ are divergence free, the second coordinate must necessarily be 0, and the form of the initial data is preserved. Namely,

$$u_t(x) = (\phi_t(x_2), 0) \quad \text{and} \quad u^i_t(x) = (\phi^i_t(x_2), 0)$$

for some $C([0, \infty), C^{1,\alpha})$ periodic functions $\phi^i, \phi$.

Now the definition of $X^i$ shows that

$$X^i_t(y) = \begin{pmatrix} y_1 + \lambda^{i,1}_t(y_2) \\ y_2 + \sqrt{2\nu}W^i_t \end{pmatrix}$$

and hence

$$Y^i_t(x) = \begin{pmatrix} x_1 + \mu^{i,1}_t(x_2) \\ x_2 - \sqrt{2\nu}W^i_t \end{pmatrix}.$$  

Recall $\lambda^i, \mu^i$ are as in (3.4), (3.5), and here the notation $\lambda^{i,1}$ denotes the first coordinate of $\lambda^i$.

This immediately shows

$$\omega^i_t(x) = \omega_0 \circ Y^i_t(x)$$

$$= -\partial_2 \phi_0(x_2 - \sqrt{2\nu}W^i_t),$$

where $W^i$ again denotes the second coordinate of the Brownian motion $W^i$.

Now using standard mixing properties of Brownian motion [13, section 1.3] (or explicitly computing in this case) we know that for every $x \in [0, L]^2$

$$\lim_{t \to \infty} E\omega^i_t(x)^2 = \lim_{t \to \infty} E[-\partial_2 \phi_0(x_2 - \sqrt{2\nu}W^i_t)^2] = \|\omega_0\|^2_{L^2}$$

and

$$\lim_{t \to \infty} E\omega^i_t(x)\omega^j_t(x) = \lim_{t \to \infty} E[\partial_2 \phi_0(x_2 - \sqrt{2\nu}W^i_t)\partial_2 \phi_0(x_2 - \sqrt{2\nu}W^j_t)]$$

$$= \lim_{t \to \infty} [E\partial_2 \phi_0(x_2 - \sqrt{2\nu}W^i_t)] [E\partial_2 \phi_0(x_2 - \sqrt{2\nu}W^j_t)]$$

$$= \left( \frac{1}{L^2} \int_{[0,L]^2} \partial_2 \phi_0 \right)^2$$

$$= 0$$

when $i \neq j$. 

Now by two-dimensional Cauchy formula (3.7)

$$\omega_t = \frac{1}{N} \sum_{i=1}^{N} \omega_0 \circ Y_t^i$$

(since in our case, $\omega_0^1 = \cdots = \omega_0^N = \omega_0$). Thus

$$E\omega_t(x)^2 = \frac{1}{N^2} \sum_{i=1}^{N} E\omega_0 \circ Y_t^i(x)^2 + \frac{2}{N^2} \sum_{i=2}^{N} \sum_{j=1}^{i-1} E\omega_0 \circ Y_t^i(x)\omega_0 \circ Y_t^j(x)$$

(5.4)

and using (5.2) and (5.3) the proof is complete.

We note that all we need for (5.1) to hold is the identities (5.2) and (5.3). Equality (5.2) is guaranteed provided reasonable ergodic properties of the flow $X_t^i$ are known. Equality (5.3) is guaranteed provided the flows $X_t^i$ and $X_t^j$ eventually decorrelate.

While we are unable to guarantee these properties for a more general class of flows, we conclude this section by proving a weaker version of (5.1) for two-dimensional flows with general initial data.

**Theorem 5.2.** Let $X^i, u$ be a solution to the system (2.5)–(2.6) with (spatial) mean zero initial data $u_0 \in C^{1,\alpha}$ and periodic boundary conditions. Suppose further $u \in C([0, \infty), C^{1,\alpha})$. Then

$$\lim_{t \to \infty} \sup E\|\nabla u_t\|_{L^2}^2 \geq \frac{1}{NL^2}\|u_0\|_{L^2}^2. \quad (5.5)$$

Note that the assumption $u \in C([0, \infty), C^{1,\alpha})$ is satisfied in the two-dimensional situation with $C^{1,\alpha}$ initial data (theorem 3.5). The proof we provide below will also work in the three-dimensional situation, as long as global existence and well-posedness of (2.5)–(2.6) is known.

As is standard with the Navier–Stokes equations, the condition that $u_0$ is (spatially) mean zero is not a restriction. By changing coordinates to a frame moving with the mean of the initial velocity, we can arrange that the initial data (in the new frame) have spatial mean 0.

Finally we note that the lower bound in inequality (5.5) is sharp, since in the special case of shear flows we have the equality (5.1). However, we are unable to obtain a bound on $\lim_{t \to \infty} \inf E\|\nabla u_t\|_{L^2}^2$.

**Proof of theorem 5.2.** As before let $u_t^i = P[(\nabla^* Y_t^i)u_0 \circ Y_t^i]$. Using lemma 4.2 and Itô’s formula we have

$$\frac{1}{2} d|u_t^i|^2 + u_t^i \cdot [(u_t \cdot \nabla)u_t^i - \nu \Delta u_t^i + (\nabla^* u)u_t^i + \nabla p^t] dt + \sqrt{2} u_t^i \cdot (\nabla u_t^i \, dW^t) = \nu |\nabla u_t^i|^2 \, dt. \quad (5.6)$$

Note that

$$\int u_t^i \cdot ((\nabla^* u)u_t^i) = \int ((\nabla u)u_t^i) \cdot u_t^i = \int u_t^i \cdot (u_t \cdot \nabla)u_t^i = 0.$$

Thus integrating (5.6) in space, and using the fact that $u_t^i$ is divergence free gives

$$d\|u_t^i\|_{L^2}^2 = 0$$

and hence $\|u_t^i\|_{L^2} = \|u_0^i\|_{L^2}$ almost surely.

Now suppose that for some $\epsilon > 0$, there exists $t_0$ such that for all $t > t_0$

$$E\|\nabla u_t\|_{L^2}^2 \leq \frac{1}{NL^2}\|u_0\|_{L^2}^2 - \epsilon.$$
Using Itô’s formula and (4.2) gives
\[
\frac{1}{2} d|u|^2 + u \cdot [(u \cdot \nabla)u - v \Delta u + \nabla p] \, dt + \sqrt{\frac{2v}{N}} \sum_{i=1}^{N} u \cdot (\nabla u^i \, dW^i) = \frac{v}{N^2} \sum_{i=1}^{N} |\nabla u^i|^2 \, dt.
\]
Integrating in space, and taking expected values gives
\[
\frac{1}{2} \nu \partial_t E\|u_t\|_{L^2}^2 = E\left[\frac{1}{N^2} \sum_{i=1}^{N} \|\nabla u^i_t\|_{L^2}^2 - \|\nabla u_t\|_{L^2}^2\right]
\geq E\left[\frac{1}{N^2 L^2} \sum_{i=1}^{N} \|u_0^i\|_{L^2}^2 - \|\nabla u_t\|_{L^2}^2\right]
\geq \frac{1}{N L^2} \|u_0\|_{L^2}^2 - \|\nabla u_t\|_{L^2}^2
\geq \epsilon
\]
for \(t \geq t_0\). Here we used the Poincaré inequality to obtain the second inequality above. Note that we have assumed that the initial data have (spatial) mean 0. Since the (spatial) mean is conserved by the system (2.5)–(2.6), \(u_t\) also has (spatial) mean zero, and our application of the Poincaré inequality is valid.

Now, the above inequality immediately implies \(E\|u_t\|_{L^2}^2\) becomes arbitrarily large as \(t \to \infty\). This is a contradiction because
\[
\|u_t\|_{L^2} = \left\| \frac{1}{N} \sum_{i=1}^{N} u^i_t \right\|_{L^2} \leq \frac{1}{N} \sum_{i=1}^{N} \|u^i_t\|_{L^2} = \|u_0\|_{L^2}
\]
holds almost surely.

\[\Box\]

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Appendix A. Local existence
In this appendix we provide the proof of proposition 3.1. A similar proof appeared in [10] (see also [6]), and the proof provided here is based on similar ideas. We present the proof here because we require local existence for \(C^{1,\alpha}\) initial data (the proof in [10] used \(C^{2,\alpha}\)), and to ensure that bounds and existence time therein are independent of \(N\).

Without loss, we assume \(t_0 = 0\), and \(u_0^1, \ldots, u_0^N\) to be \(\mathcal{F}_0\) measurable. Let \(U\) be a large constant and \(T\) a small time, both of which will be specified later.

Define \(\mathcal{U} = \mathcal{U}(T, U)\) to be the set of all time continuous \(\mathcal{F}_t\)-adapted \(C^{1,\alpha}\) valued divergence free and spatially periodic processes \(u\) such that
\[
u_0 = \frac{1}{N} \sum_{i=1}^{N} u_0^i \quad \text{and} \quad \sup_{t \in [0,T]} \|u_t\|_{1,\alpha} \leq U.
\]
hold almost surely. Also, we define \( M = M(T) \) to be the set of all time continuous \( \mathcal{F}_t \)-adapted \( C^{1,\alpha} \) valued spatially periodic processes \( \mu \) such that
\[
\mu_0 = 0 \quad \text{and} \quad \sup_{t \in [0, T]} \| \nabla \mu_t \|_\alpha \leq \frac{1}{2}
\]
hold almost surely.

Now given \( u \in U \) let \( X_{i,u} \) be the flow solving the SDE
\[
dX_{i,u}^t = u_t(X_{i,u}^t) \, dt + \sqrt{2} \, v \, dW^i_t
\]
with initial data \( X_{i,u}^0(y) = y \). As before, define \( Y_{i,u}^t = (X_{i,u}^t)^{-1} \), and define
\[
\lambda_{i,u}^t(y) = X_{i,u}^t(y) - y,
\]
\[
\mu_{i,u}^t(x) = Y_{i,u}^t(x) - x
\]
to be the Eulerian and Lagrangian displacements, respectively.

Finally define the (non-linear) operator \( \mathcal{W} \) by
\[
\mathcal{W}(u)_t = \frac{1}{N} \sum_{i=1}^N P(\nabla^* Y_{i,u}^t(\mu_{0}^i \circ Y_{i,u}^t)).
\]

Clearly a fixed point of \( \mathcal{W} \) will produce a solution to the system (3.1)–(3.2). Thus the proof will be complete if we show that for an appropriate choice of \( T \) and \( U \), \( \mathcal{W} \) maps \( U \) into itself and is a contraction with respect to the weaker norm
\[
\| u \|_U = \sup_{t \in [0, T]} \| u_t \|_u.
\]

We first show \( \mathcal{W} \) maps \( U \) into itself, using the two lemmas below.

**Lemma A.1.** There exists \( c = c(\alpha) \) such that
\[
\| \mathcal{W}(u) \|_{1,\alpha} \leq c \left[ \max_{1 \leq i \leq N} (1 + \| \nabla \mu_{i,u} \|_\alpha)^{2\alpha} \right] \frac{1}{N} \sum_{i=1}^N \| u_{0}^i \|_{1,\alpha} \quad \text{a.s.}
\]

**Proof.** First recall \( P \) vanishes on gradients. Thus
\[
P(\nabla^* Y^t v) = -P(\nabla^* v) Y^t.
\]

(A.1)

Now
\[
\partial_t P(\nabla^* Y^t v) = P(\nabla^* Y^t) \partial_t v + (\nabla^* \partial_t Y^t) v
\]
\[
= P(\nabla^* Y^t) \partial_t v - (\nabla^* v) \partial_t Y^t,
\]
where we used (A.1) for the second term. Note that the right-hand side involves only first order derivatives. Since \( P \) is a standard Calderón–Zygmund singular integral operator, which is bounded on Hölder spaces, we obtain the estimate
\[
\| P(\nabla^* Y^t v) \|_{1,\alpha} \leq c \| \nabla^* Y^t \|_\alpha \| v \|_{1,\alpha}
\]
for some constant \( c = c(\alpha) \).

Applying this estimate to \( \mathcal{W} \), we have
\[
\| \mathcal{W}(u)_t \|_{1,\alpha} \leq c \frac{1}{N} \sum_{i=1}^N \| \nabla^* Y_{i,u}^t(\mu_{0}^i \circ Y_{i,u}^t) \|_{1,\alpha} \quad \text{a.s.}
\]

(A.2)

from which the lemma follows. \( \square \)
Lemma A.2. There exists $T = T(U, \alpha)$ such that $\lambda^{i,u}, \mu^{i,u} \in \mathcal{M}(T)$.

We note that the diffusion coefficient is spatially constant, and thus we get the desired (almost sure) control on $\nabla \lambda$. Since $\nabla \cdot u = 0$, $\det(\nabla X^{i,u}) = 1$, giving the desired control on $\nabla \mu$. The details are standard, and we do not provide them here (see for instance [10, 13]).

Now choosing $U = c(\frac{3}{2})^{\frac{2}{3}m}(1/N) \|u_0\|_{1,\alpha}$, and then choosing $T$ as in lemma A.2, lemma A.1 shows that $\mathcal{W}$ maps $\mathcal{U}(U, T)$ into itself. Note that each summand on the right of (A.2) is bounded by $U$, which will prove the bound (3.3). Further, given a uniform (in $i$) bound on $\|u^{i,0}_0\|_{1,\alpha}$, our choice of $U$ can be made independent of $N$.

It remains to show that $\mathcal{W}$ is a contraction. By definition of $\mathcal{W}$ we have

$$\mathcal{W}(u)_t - \mathcal{W}(v)_t = \frac{1}{N} \sum_{i=1}^N P_i (\nabla^* Y^{i,u}_t u^{i,0}_0 \circ Y^{i,u}_t - \nabla^* Y^{i,v}_t u^{i,0}_0 \circ Y^{i,v}_t)$$

$$= \frac{1}{N} \sum_{i=1}^N P_i (\nabla^* Y^{i,u}_t u^{i,0}_0 \circ Y^{i,u}_t - u^{i,0}_0 \circ Y^{i,v}_t)$$

$$+ \frac{1}{N} \sum_{i=1}^N P_i (\nabla^* Y^{i,u}_t - \nabla^* Y^{i,v}_t) u^{i,0}_0 \circ Y^{i,v}_t$$

$$= \frac{1}{N} \sum_{i=1}^N P_i (\nabla^* (u^{i,0}_0 \circ Y^{i,u}_t - u^{i,0}_0 \circ Y^{i,v}_t)),$$

where we used the identity (A.1) to obtain the last equality. Now we recall that $\mu^{i,u}, \mu^{i,v} \in \mathcal{M}$, and take $C^\alpha$ norms. This gives

$$\|\mathcal{W}(u)_t - \mathcal{W}(v)_t\|_{\alpha} \leq \frac{C}{LN} \sum_{i=1}^N \|u^{i,0}_0\|_{1,\alpha} \|Y^{i,u}_t - Y^{i,v}_t\|_{\alpha} \quad \text{a.s.} \quad (A.3)$$

Now from the definition of $Y^{i,u}$ and $Y^{i,v}$ we have

$$Y^{i,u}_t - Y^{i,v}_t = \int_0^t [u_s(Y^{i,u}_s) - v_s(Y^{i,v}_s)] \, ds.$$

Taking $C^\alpha$ norms, and applying Gronwall’s inequality, and absorbing the exponential in time factor into the constant $c$ gives

$$\|Y^{i,u}_t - Y^{i,v}_t\|_{\alpha} \leq c \int_0^t \|u_s - v_s\|_{\alpha} \, ds \quad \text{a.s.}$$

Returning to (A.3) we have

$$\|\mathcal{W}(u)_t - \mathcal{W}(v)_t\|_{\alpha} \leq \frac{C}{L} \sup_{s \leq t} \|u_s - v_s\|_{\alpha} \frac{1}{N} \sum_{i=1}^N \|u^{i,0}_0\|_{1,\alpha} \quad \text{a.s.}$$

Choosing $t$ small enough one can ensure $\mathcal{W}$ is a contraction mapping. A standard iteration now shows the existence of a fixed point of $\mathcal{W}$, concluding the proof of proposition 3.1.
Appendix B. Logarithmic $L^\infty$ bound on singular integral operators

In this appendix we provide a proof of lemma 3.4. We restate it here for the readers’ convenience.

**Lemma (3.4).** If $u$ is divergence free and periodic in $\mathbb{R}^d$, then for any $\alpha \in (0, 1)$, there exists a constant $c = c(\alpha, d)$ such that

$$\|\nabla u\|_{L^\infty} \leq c \|\omega\|_{L^\infty} \left( 1 + \ln^+ \left( \frac{|\omega|}{\|\omega\|_{L^\infty}} \right) \right),$$

where $\omega = \nabla \times u$.

**Proof.** Let $K$ be a standard Calderón–Zygmund periodic kernel, and we define the operator $T$ by

$$T f = K * f.$$

Then we will prove that

$$\|T f\|_{L^\infty} \leq c \|f\|_{L^\infty} \left( 1 + \ln^+ \left( \frac{|f|}{\|f\|_{L^\infty}} \right) \right). \tag{B.1}$$

This immediately implies lemma 3.4 because we know $\nabla u = -\nabla(\Delta^{-1}) \nabla \times \omega$, and $-\nabla(\Delta^{-1}) \nabla \times$ is a Calderón–Zygmund type singular integral operator.

Now we prove (B.1). We assume for convenience that all functions are periodic on the cube $[0, 1]^d$. We also recall that the kernel $K$ satisfies the properties

1. $K(y) \leq c|y|^{-d}$ when $|y| \leq \frac{1}{2}$.
2. $\int_{|y| \leq r} K(y) \, d\sigma(y) = 0$ for any $r \in (0, \frac{1}{2})$.

Pick any $\epsilon \in (0, \frac{1}{2})$. Then

$$T f(x) = \int_{[0,1]^d} K(y) f(x - y) \, dy$$

$$\leq \int_{|y| \leq \epsilon} K(y) f(x - y) \, dy + \int_{|y| > \epsilon} K(y) f(x - y) \, dy. \tag{B.2}$$

Using property 1 about $K$, we bound the second integral by

$$\left| \int_{|y| > \epsilon} K(y) f(x - y) \, dy \right| \leq c \|f\|_{L^\infty} \int_{r=\epsilon}^{1} \frac{1}{r} r^{d-1} \, dr$$

$$\leq c \|f\|_{L^\infty} \ln\left( \frac{1}{\epsilon} \right).$$

Using property 2 about $K$, we bound the first integral in (B.2) by

$$\left| \int_{|y| \leq \epsilon} K(y) f(x - y) \, dy \right| = \left| \int_{|y| \leq \epsilon} K(y)(f(x - y) - f(x)) \, dy \right|$$

$$\leq c |f|_{\alpha} \int_{|y| \leq \epsilon} |y|^{\alpha-d} \, dy$$

$$= c |f|_{\alpha} \epsilon^\alpha.$$

Combining estimates we have

$$\|T f\|_{L^\infty} \leq c \left[ \epsilon^{\alpha} |f|_{\alpha} + \|f\|_{L^\infty} \ln\left( \frac{1}{\epsilon} \right) \right].$$
A stochastic-Lagrangian particle system for Navier–Stokes

Choosing

\[ \epsilon = \min \left\{ \frac{1}{2}, \left( \frac{\|f\|_{L^\infty}}{|f|_a} \right)^{1/\alpha} \right\} \]

finishes the proof.

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