The effect of numerical noises on statistics of a chaotic dynamical system

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Abstract. It is well known that chaotic dynamical systems have the sensitive dependence on initial conditions (SDIC). Unfortunately, numerical noises (such as truncation error and round-off error) always exist in practice. Thus, due to the SDIC, long-term accurate prediction of chaotic dynamical systems is practically impossible. In this paper, using the so-called “clean numerical simulation” (CNS) whose numerical noises might be much smaller even than micro-level physical uncertainty and thus are negligible, we gain accurate prediction of a chaotic dynamical system in a long enough interval of time. Then, based on these reliable simulations, the influence of numerical noises on statistic computations is investigated. It is found that the influence of numerical noises is negligible when statistic results are time-independent. Unfortunately, when the statistics of a chaotic dynamical system are time-dependent, numerical noises have a great influence even on statistic computations.

1. Introduction

It is well known that chaotic dynamical systems have the sensitive dependence on initial conditions (SDIC) [1], which is called the “butterfly-effect”. Unfortunately, numerical noises, i.e. truncation error and round-off error, always exist in practice. Thus, due to the SDIC, long-term accurate prediction of chaotic dynamical systems is practically impossible [1] and therefore numerical simulations of chaos are only mixtures of “true” solution with physical meanings and numerical noises without physical meanings. It is widely believed that turbulence [2, 3, 4, 5] is chaotic. Therefore, statistics are often used in the investigation of turbulence, and the direct numerical simulation (DNS) [6] has played an important role in statistical computation of turbulence. However, due to the SDIC, tiny numerical noises, which grow exponentially in time, may lead to spurious results. Thus, DNS results are in essence the mixtures of “true” solution with physical meaning and numerical noises without physical meanings. Even so, it is widely believed that numerical noises do not affect the statistics of turbulence by means of such kind of simulations. Unfortunately, Yee et al. [7] reported in 1999 that the DNS can produce a spurious solution that is completely different from the physical solution of their considered equations. In addition, Wang et al. [8] demonstrated a spurious evolution of turbulence originated from round-off error in DNS. Therefore, it is necessary to verify the reliability of statistic results for chaotic dynamical systems.
Recently, a numerical method with extremely small numerical noises, namely “clean numerical simulation” (CNS) [9], was proposed to gain reliable simulations of chaotic dynamical systems in a finite but large enough interval of time. The CNS is based on an arbitrary Taylor series method (TSM) [10, 11] and an arbitrary multiple precision [12] for every data, together with a kind of solution verification. By means of the CNS, the numerical noises can be so greatly reduced to be much smaller than the “true” solution that the numerical noises are negligible in a given interval of time even for chaotic dynamic systems. The CNS has been successfully applied in some chaotic dynamical systems, such as the famous Lorenz equation [9, 13, 14, 15] and the three-body problems [16, 17, 18]. Note that, using the traditional Runge-Kutta method and data in double precision, one can gain convergent chaotic results of Lorenz equation only in a few dozens of time interval. However, using the CNS, Liao and Wang [14] successfully obtained a convergent, reliable chaotic solution of the Lorenz equation in an interval [0,10000]. This shows the validity of the CNS for reliable simulations of chaotic dynamical systems.

2. Mathematical equations

It is well known that the famous Lorenz equation [1] is a very simplified model of the Rayleigh-Bénard (RB) flow of viscous fluid. There are exact Navier-Stokes equations for the Rayleigh-Bénard flow. Saltzman [19] deduced a family of highly truncated dynamical systems with different degrees of freedom, and the famous Lorenz equation [1] is only the simplest one among them. For example, in the case of the Prandtl number $\sigma = 10$, the highly truncated dynamical system with seven degrees of freedom (7-DOF) reads

$$
\begin{align*}
\hat{A} &= 23.521BC - 1.500D - 148.046A, \\
\hat{B} &= -22.030AC - 1.589E - 186.429B, \\
\hat{C} &= 1.561AB - 0.185F - 400.276C, \\
\hat{D} &= -16.284CE - 16.284BF - 13.958AG - 1460.631A - 14.805D, \\
\hat{E} &= 16.284CD - 16.284AF - 18.610BG - 1947.508AB - 18.643E, \\
\hat{F} &= 16.284AE + 16.284BD - 486.877AC - 40.028F, \\
\hat{G} &= 27.916AD + 37.220BE - 39.479G,
\end{align*}
$$

(1)

where $\lambda = R/R_c$ is dimensionless Rayleigh number, $R_c$ is the critical Rayleigh number, $A$ and $D$ represent the cellular streamline and thermal fields for the Rayleigh critical mode, and $G$ denotes the departure of the vertical variation, respectively. For details, please refer to Saltzman [19].

These are deterministic equations with chaotic solutions, and are greatly simplified models for the two dimensional Rayleigh-Bénard flow. In physics, the Rayleigh-Bénard flow with a large enough Rayleigh number $R$ is an evolutionary process from an initial equilibrium state to turbulence after a long enough time, mainly because the flow is unstable and besides the micro-level physical uncertainty (such as thermal fluctuation) always exists. Such kind of initial micro-level physical uncertainty due to thermal fluctuation can be expressed by Gaussian random data, as illustrated by Wang et al. [3].

Without loss of generality, let us consider the deterministic 7-DOF equations (1) with the random initial conditions in normal distribution in case of $\lambda = 28$, corresponding to a turbulent flow. Considering the thermal fluctuation, we study here such kind of random initial condition in normal distribution with the mean

$$
\langle A(0) \rangle = 1, \quad \langle B(0) \rangle = \langle C(0) \rangle = \langle D(0) \rangle = \langle E(0) \rangle = \langle F(0) \rangle = \langle G(0) \rangle = 10^{-3}
$$

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and the standard deviation $\sigma_0 = 10^{-30}$. By means of the CNS, we can gain reliable propagation of the micro-level physical uncertainty of a large numbers of random initial conditions. We compare the sample mean and unbiased estimate of standard deviation of $A(t)$.

### 3. Statistical Results

Two thousand samples of reliable numerical simulations of the 7-DOF system (1) are obtained in the time interval $[0,10]$ by means of the CNS using the 80th-order Taylor series ($M = 80$), the 90 decimal-digit precision ($K = 90$) for every data, and the time-step $\delta t = 10^{-3}$. It is found that the numerical errors can be decreased to be much smaller than the micro-level physical uncertainty in the time interval $[0,10]$ under consideration. These numerical simulations are so accurate that we can consider them as the “true” solutions of the chaotic dynamical system (1), which can be used to investigate the influence of numerical noises on statistic computations of chaotic systems. In this way, we successfully distinguish the “true” chaotic solutions with physical meanings from the numerical noises without physical meanings.

![Figure 1](image.png)

**Figure 1.** The influence of the truncation error on the mean of $A(t)$ of the 7-DOF chaotic systems obtained using the time-step $\delta t = 10^{-3}$ and the 90 decimal-digit multiple precision for every data with the negligible round-off error. Solid line in red: the reliable results given by the CNS using the 80th-order Taylor series expansion for $t$ (i.e. $M = 80$); dashed line in blue: the unreliable result given by the 10th-order Taylor series expansion (i.e. $M = 10$).

Obviously, the larger the order $M$ of the Taylor series in the frame of the CNS, the smaller the truncation error. For example, the traditional Runge-Kutta method corresponds to $M = 4$, the 4th-order Taylor series expansion for $t$. Thus, to investigate the influence of the truncation error, we use here the 10th-order Taylor series, i.e. $M = 10$, but retain the 90 decimal-digit multiple precision for every data. Note that our reliable CNS results are gained by means of $M = 80$, i.e. the 80th-order Taylor series, whose truncation errors are negligible in the considered interval of time $t \in [0,10]$. However, when $M = 10$, the truncation error is not negligible in $[0,10]$. In this way, the round-off error is negligible in the considered interval of time $t \in [0,10]$ so that the influence of the truncation error can be investigated independently.

The reliable mean (the red line, given by $M = 80$) of $A(t)$ of the 7-DOF chaotic dynamical system (1) is shown in Figure 1, compared with the unreliable result (the blue dashed line, given by $M = 10$). Note that the reliable mean of $A(t)$ given by the CNS becomes stable when $t > 8.2$, but is time-dependent when $t \leq 8.2$. However, as shown in Figure 1, the unreliable mean of $A(t)$ given by $M = 10$ has a noticeable difference from the reliable CNS result in the time interval $4.2 < t < 8.2$. It suggests that the truncation error has a great influence on the computation of unsteady statistical quantities. Note that the mean of $A(t)$ given by $M = 10$ agrees well with the reliable CNS result when $t > 8.2$. Thus, the truncation error has no influence on the time-independent statistics of chaotic dynamical systems in an equilibrium state.
The scatter diagrams of the probability distribution function (PDF) in the $A-D$ plane of the 7-DOF chaotic dynamical system (1) given by the two numerical schemes (i.e. $M = 10$ and $M = 80$) are as shown in Figure 2, where the PDF is obtained by using the Gaussian kernel density estimator. Note that the PDFs given by the two numerical approaches are almost the same at $t = 8.40$. However, at $t = 8.00$ and $t = 8.10$, the PDFs given by $M = 10$ are quite different from those given by the CNS using $M = 80$. This supports our previous conclusion that the truncation error has a significant influence on the computation of unsteady statistics, but has no influence on steady ones.

Note that the double precision is widely employed in numerical simulations, which leads to round-off errors at every time-step that increase exponentially for a chaotic dynamical system. To simulate the round-off error, we add a random data at each time-step with zero mean and the standard deviation $10^{-16}$, while the 80th-order Taylor expansion ($M = 80$) is still used and all data are expressed in 90 decimal-digit so as to guarantee the negligible truncation error in the considered interval of time $t \in [0; 10]$.

Figure 3 shows the comparison between the reliable CNS statistics (using the 90 decimal-digit multiple precision for every data) and the unreliable results using the double precision. Note that the round-off error has a great influence on the standard deviation of $A(t)$ from the very beginning. At about $t \approx 5$ when the round-off error enlarges exponentially to reach the level of the “true” solution, the unsteady mean of $A(t)$ becomes unreliable. Note that the mean of $A(t)$ given by the double precision becomes time-independent more early, at $t \approx 5$, which is however wrong in physics. As statics of the system truly becomes steady when $t > 8.2$ (that is determined by the reliable CNS result), the round-off error has no influence on the computation of statistics. All of these suggest that the round-off error has a great influence on the unsteady
statistics of chaotic dynamical system, but has a little impact on steady ones.

4. Conclusions
In this paper, we compare the statistic results of a chaotic dynamical model gained by two different approaches, one is the traditional numerical approach, the other is the so-called Clean Numerical Simulation (CNS) with extremely small numerical noises. It is found that the numerical noises have a great influence on the computation of unsteady statistics of chaotic dynamical systems, but have no influence on time-independent statistics for systems.

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