One more proof of the Abhyankar-Moh-Suzuki Theorem

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ABSTRACT. We extract the Abhyankar-Moh-Suzuki theorem from the Lin-Zaidenberg theorem.

INTRODUCTION

Let $\Gamma_0$ be the zero locus of a primitive polynomial $p \in \mathbb{C}[x, y]$. Suppose that $\Gamma_0$ is a smooth irreducible simply connected curve (i.e. $\Gamma_0 \simeq \mathbb{C}$). Then the Abhyankar-Moh-Suzuki theorem [14], [16] states that

in a suitable polynomial coordinate system on $\mathbb{C}^2$ the curve $\Gamma_0$ is a coordinate axis, i.e. $p$ may be viewed as a coordinate function.

The Lin-Zaidenberg theorem [9] states that if $\Gamma_0$ is not isomorphic but only homeomorphic to $\mathbb{C}$ as a topological space then

in a suitable polynomial coordinate system on $\mathbb{C}^2$ the polynomial $p(x, y)$ is of the form $x^k + y^l$ where $k, l \geq 2$ are relatively prime.

These theorems are extremely important tools of the modern affine algebraic geometry and it is not a surprise that different methods of proving them were suggested after the original papers. In the case of the Abhyankar-Moh-Suzuki theorem there is at least a dozen of proofs (e.g., see [11], [14], [15], [12], [7], [2], [3], [1], [18], [8], [10], [13]). Some of these papers contain actually simultaneous proofs of the Abhyankar-Moh-Suzuki and Lin-Zaidenberg theorems (e.g., see [12], [8], [13]) but none of them used the beautiful technique developed in [9] (and refined later in [17]). At first glance this technique (based of the theory of Teichm"{u}ller spaces and hyperbolic analysis) does not imply the Abhyankar-Moh-Suzuki theorem but we show below that it is applicable and there is a simple reduction of the Abhyankar-Moh-Suzuki theorem to the Lin-Zaidenberg one.

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1. Reduction

Definition 1.1. Recall that every pair $(k, l)$ of relatively prime natural numbers defines a weighted degree function $d$ on the ring of polynomials in $x$ and $y$ such that for a nonzero polynomial $r(x, y) = \sum a_{ij}x^iy^j$ one has $d(r) = \max_{(i,j) \in I}(li + kj)$ where $I = \{(i,j) \in (\mathbb{Z}_{\geq 0})^2|a_{ij} \neq 0\}$. We let $I' = \{(i,j) \in I|li + kj = d(p)\}$ and we call $\hat{r}(x, y) = \sum_{(i,j) \in I'} a_{ij}x^iy^j$ the leading quasi-homogeneous part of $r$ with respect to the grading induced by $d$. 

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Lemma 1.2. Let the zero locus of a primitive polynomial \( p(x, y) \) be isomorphic to \( \mathbb{C}^2 \). Then there is an automorphism \( \alpha \) of \( \mathbb{C}^2 \) such that either \( q := p \circ \alpha \) is linear or the leading quasi-homogeneous part of \( q \) with respect to some weighted degree function \( d \) is of the form \( \hat{q}(x, y) = (x^k + y^l)^n \) where \( k, l \geq 2 \) are relatively prime.

Proof. For \( d \) as in Definition 1.1 the fact that \( k \) and \( l \) are relatively prime implies that the leading quasi-homogeneous part of any nonzero polynomial (and in particular \( p \)) is automatically of the form

\[
a x^{n_0} \prod_{i=1}^{m} (c_i x^k + y^l)^{n_i}
\]

where \( n_i \geq 0 \), \( a \neq 0 \), \( c_1, \ldots, c_m \in \mathbb{C} \) are distinct, and \( d(p) = n_0 + kl \sum_{i=1}^{m} n_i \). Furthermore, unless \( p \) depends on one variable only (in which case it is automatically linear) the pair \((k, l)\) can be chosen so that \( m \geq 1 \).

However, in this case after the natural embedding of \( \mathbb{C}^2 \) into the weighted projective plane, associated with the grading induced by \( d \), one can see that the irreducible curve \( \Gamma_0 = q^{-1}(0) \) must have \( m + 1 \) (resp. \( m \)) points at infinity for \( n_0 > 0 \) (resp. \( n_0 = 0 \)) and, therefore, at least the same number of punctures. Since \( \Gamma_0 \simeq \mathbb{C} \) has one puncture we conclude that

\[
\hat{p}(x, y) = a(cx^k + y^l)^n
\]

where \( a \) and \( c \neq 0 \). If either \( k \) or \( l \) (say \( l \)) is equal to 1 then applying a de Joinquère’s transformation \((x, y) \rightarrow (x, y + cx^k)\) we can reduce the standard degree of \( p \). Thus, choosing \( \alpha \) so that \( q = p \circ \alpha \) has the minimal possible standard degree we see that unless \( q \) is linear one can suppose that \( \hat{q}(x, y) = (x^k + y^l)^n \) with relatively prime \( k, l \geq 2 \).

\[\Box\]

Proposition 1.3. In Lemma 1.2 one has \( n = 1 \), i.e. unless \( q \) is linear the leading quasi-homogeneous part of \( q \) is \( \hat{q}(x, y) = x^k + y^l \) with relatively prime \( k, l \geq 2 \).

Proof. Consider the surface \( S \) given by

\[
z^{nkl} q(xz^{-l}, yz^{-k}) = 1
\]

in \( \mathbb{C}^3 \) together with the family of curves \( z|_S : S \rightarrow \mathbb{C} \). Then for \( a \neq 0 \) the curve \( S_a = S \cap \{z = a\} \) is isomorphic to the fiber \( \Gamma_c = \{q(x, y) = c\} \) where \( c = a^{-nkl} \). By \[9\] all nonzero fibers of \( q \) are pairwise isomorphic, which yields a pairwise isomorphism of curves \( S_a \) for \( a \neq 0 \). Note that \( S_0 \) is given by equation \( \hat{q}(x, y) = 1 \) in the plane \( \{z = 0\} \subset \mathbb{C}^3 \) and thus \( S_0 \) consists of \( n \) disjoint components isomorphic to the curve \( x^k + y^l = 1 \). However, in the case of isomorphic irreducible general fibers any degenerate fiber may contain at most one hyperbolic component \[17\] Theorem 5.3. Thus \( n = 1 \).

\[\Box\]

Corollary 1.4. If \( q \) in Proposition 1.3 is not linear then general fibers of \( q \) have negative Euler characteristic.

1More precisely, the smooth part of such a fiber contains at most one hyperbolic component. Furthermore, this component is isomorphic to a quotient of the general fiber with respect to a finite group action.
Proof. Indeed, for relatively prime \( k, l \geq 2 \) the Riemann-Hurwitz formula implies that the Euler characteristic of the curve \( x^k + y^l = 1 \) (and, therefore, \( S_0 \)) is at most \(-1\). The Suzuki formula for Euler characteristic \( [10] \) implies that the same is true for general curves \( S_n \). \( \Box \)

**Lemma 1.5.** Let \( \Delta_r \) be the disc \( \{ x \in \mathbb{C} \mid |x| < r \} \) and \( \pi : \mathbb{C}^2 \to \mathbb{C} \) be the projection to the \( x \)-axis. Then under the assumption of Lemma 1.2 for every \( \delta > 0 \) there exists \( r > 0 \) such that \( \pi \) maps the set \( R = \{ (x, y) \in \mathbb{C}^2 \mid \frac{\partial q(x, y)}{\partial y} = 0 \& |q(x, y)| < \delta \} \) into \( \Delta_r \).

Proof. By Proposition 1.3 \( \hat{q}(x, y) = x^k + y^l \). Hence in the polynomial polyhedron \( \{ (x, y) \in \mathbb{C}^2 \mid |q(x, y)| < \delta \} \) for \( x \) with large absolute value one has \( |x|^l \approx |y|^k \). Furthermore, looking at the quasi-leading part of the polynomial \( \frac{\partial q(x, y)}{\partial y} \) one can see now that for such values of \( x \) the absolute value of \( \frac{\partial q(x, y)}{\partial y} \) must be also large, which yields the desired conclusion. \( \Box \)

**Lemma 1.6.** For \( \delta \) and \( r \) from Lemma 1.5 and any \( c \) with \( 0 < |c| < \delta \) the set \( \Gamma_c \cap \pi^{-1}(\mathbb{C} \setminus \Delta_r) \) is a disjoint union of components biholomorphic to a punctured disc.

Proof. Note that the solutions of the system \( \frac{\partial q(x, y)}{\partial y} = q(x, y) - c = 0 \) are ramification points of the finite morphism \( \pi|_{\Gamma_c} : \Gamma_c \to \mathbb{C} \) that is the restriction of \( \pi \). Hence by Lemma 1.5 \( \Gamma_c \cap \pi^{-1}(\mathbb{C} \setminus \Delta_r) \) is the union of components biholomorphic to a punctured disc. \( \Box \)

**Corollary 1.7.** The Riemann surfaces \( \Gamma_c \) and \( \Gamma_c \cap \pi^{-1}(\Delta_r) \) are diffeomorphic.

1.1. **Proof of the Abhyankar-Moh-Suzuki theorem.** Assume to the contrary that \( q \) in Lemma 1.2 and Proposition 1.3 is not linear. Then the leading quasi-homogeneous part of \( q \) is \( x^k + y^l \) with \( k, l \geq 2 \). Note that for large \( r \) the set \( \Gamma_0 \cap \pi^{-1}(\Delta_r) \) is biholomorphic to a disc. The smoothness of \( \Gamma_0 \) implies that \( \Gamma_c \cap \pi^{-1}(\Delta_r) \) is also biholomorphic to a disc for sufficiently small \( |c| \). By Corollary 1.7 \( \Gamma_c \) is contractible in contradiction with Corollary 1.4. Hence \( q \) must be linear, which concludes the proof. \( \Box \)

**Remark 1.8.** The Abhyankar-Moh-Suzuki theorem is valid over any algebraically closed field \( K \) of characteristic zero while the proof before is done over the field of complex numbers. However, the Lefschetz Principle enables us to reduce the general case to the complex one.

Indeed, the description of the polynomial \( p \) and an isomorphism between \( p^{-1}(0) \) and a line involves a finite number of elements of \( K \). These elements generate an algebraically closed subfield \( K_0 \) of \( K \) which can be also embedded into \( \mathbb{C} \). We can consider \( p \) as a polynomial over \( K_0 \) and, consequently, over \( \mathbb{C} \). If it is known already that every fiber
Γ_c = p^{-1}(c) of p over C is isomorphic to a line then the same is true for the fibers of p over K_0 (and, therefore, over K) since the genus and the number of punctures of any affine curve survive the field extention C : K_0. Hence one can extract the general form of the Abhyankar-Moh-Suzuki theorem from the fact that a polynomial with general fibers isomorphic to a line is a variable in a suitable polynomial coordinate system \[6\].

REFERENCES

[1] S. Abhyankar, T.-T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148-166.

[2] E. Artal-Bartolo, *Une démonstration géométrique du théorème d'Abhyankar-Moh*, J. Reine Angew. Math. **464** (1995), 97-108.

[3] N. A’Campo, M. Oka, *Geometry of plane curves via Tschirnhausen resolution tower*, Osaka J. Math. **33** (1996), no. 4, 1003-1033.

[4] R.V. Gurjar, *A new proof of the Abhyankar-Moh-Suzuki theorem*, Transform. Groups **7** (2002), no. 1, 61-66.

[5] R. V. Gurjar, M. Miyanishi, *On contractible curves in the complex affine plane*, Tohoku Math. J. (2) **48** (1996), no. 3, 459-469.

[6] A. Gutwirth, *An inequality for certain pencils of plane curves*, Proc. Amer. Math. Soc. **12** (1961), 631-638.

[7] M. C. Kang, *On Abhyankar-Moh’s epimorphism theorem*, Amer. J. Math. **113** (1991), no. 3, 399-421.

[8] M. Koras, *On contractible plane curves*, Affine algebraic geometry, 275-288, Osaka Univ. Press, Osaka, 2007.

[9] V.Ya. Lin, M. Zaidenberg, *An irreducible, simply connected algebraic curve in C^2 is equivalent to a quasihomogeneous curve*, (Russian) Dokl. Akad. Nauk SSSR **271** (1983), no. 5, 1048-1052.

[10] L. Makar-Limanov, *A new proof of the Abhyankar-Moh-Suzuki theorem via a new algorithm for the polynomial dependence*, J. Algebra Appl. **14** (2015), no. 9, 1540001, 12 pp.

[11] M. Miyanishi, *Analytic irreducibility of certain curves on a nonsingular affine rational surface*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), 575-587, Kinokuniya Book Store, Tokyo, 1978.

[12] W. D. Neumann, *Complex algebraic plane curves via their links at infinity*, Invent. Math. **98** (1989), no. 3, 445-489.

[13] P. A. Palka, *A new proof of the theorems of Lin-Zaidenberg and Abhyankar-Moh-Suzuki*, J. Algebra Appl. **14** (2015), no. 9, 1540012, 15 pp.

[14] L. Rudolph, *Embeddings of the line in the plane*, J. Reine Angew. Math. **337** (1982), 113-118.

[15] D. R. Richman, *On the computation of minimal polynomials*, J. Algebra **103** (1986), no. 1, 1-17.

[16] M. Suzuki, *Propriétés topologiques des polynomes de deux variables complexes, et automorphismes algébriques de l'espace C^2*, J. Math. Soc. Japan, **26** (1974), 241-257.

[17] M. Zaidenberg, *Isotrivial families of curves on affine surfaces, and the characterization of the affine plane* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 3, 534-567, 688; translation in Math. USSR-Izv. **30** (1988), no. 3, 503-532.

[18] H. A. Zoladek, *A new topological proof of the Abhyankar-Moh theorem*, Math. Z. **244** (2003), no. 4, 689-695.

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