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Flows Excited by Shear Stress in Freestanding Symmetric Smectic C Films

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Abstract: The purpose of this paper is to show some useful routes in describing the shear-driving flows in a freely suspended symmetric smectic C film stretched between two circular frames, the rest outer and rotating inner frames. Our calculations, based on a corresponding extension of the Eriksen–Leslie theory, show that the shear-driving flow in the film, excited by a rotating inner circular frame, causes a twisting rotation of the $\hat{c}$-director around the normal to smectic layers. It is found that the shear flow excited by the rotating frame in a positive sense (counterclockwise) causes a twisting rotation of the $\hat{c}$-director in a negative sense (clockwise) around the normal to smectic film. It is also shown that the twisting of the $\hat{c}$-director’s field has a jumping nature when a large reorientation is carried out in a short space of time. It was also shown that the twisting dynamics of the $\hat{c}$-director field strongly depends on the curvature of the inner rotating frame.

Keywords: liquid crystals; microfluidics; freestanding smectic films

1. Introduction

A unique property of smectic liquid crystals (Sm) is the ability, under appropriate conditions, to form freely suspended smectic films (FSSFs) across an opening. In particular, the FSSFs can be stretched between two circular frames and can be modeled as a stack of smectic layers bounded by air. Thus, we are concerned with the freely suspended smectic A (SmA) or smectic C (SmC) films having layered structures, in which the long axis of molecules is normal or tilted to the planes containing the layers and the interlayers spacing is roughly the length of the molecule. It has been shown, both experimentally and theoretically [1–8], that in such quasi-two dimensional systems, a series of layer-thinning transitions occurs, causing the films to thin in a stepwise manner as the temperature is increased above the bulk SmA(C)-isotropic or nematic transition temperature. Layer-by-layer thinning transitions in FSSFs are one of the most impressive discoveries in liquid crystal physics in recent years. However, along with investigations of structural transitions in FSSFs, the study of dynamic processes in these films is also of great interest. In general, the hydrodynamic equations for SmCs are more complex than for nematics, which makes it difficult to study flow dynamics. Taking into account that the SmC crystals are layered structures in which the director $\hat{n}$ makes an angle $\theta$ with respect to the layer normal, the basic hydrodynamic description of SmC employs two vectors, namely, a unit vector $\hat{a}$ defining the layer normal and a unit vector $\hat{c}$, called the $\hat{c}$-director, which is the unit orthogonal projection of $\hat{n}$ onto the smectic plane as shown in Figure 1a. It should be noted that recently a lot of efforts have been spent on the study of flows excited in freely suspended SmC films (FSSCFs) stretched between rotating inner and resting outer circular...
frames [9], or under the action of shear stress (SS) or a rotating electric field [10,11]. One of the promising principles underlying the movement of micro-sized volumes of LC materials in FSSCFs bounded by two SmC–air interfaces is based on the coupling between the tangential component of the SS $\sigma_{zx}$ and the director field $\hat{n}$ [12].

**Figure 1.** (a) The freely suspended smectic C film confined between fixed outer and rotating inner frames. (b) The vectors $\hat{e}_\phi$, $\hat{e}_r$, and $\hat{e}_z$ are the basis vectors in the cylindrical coordinate system used for description of the SmC film.

All this indicates that certain qualitative and quantitative successes in the hydrodynamic description of relaxation processes in SmC films under the influence of shear stress [9–11,13] have been achieved. Nevertheless, it is still too early to talk about the development of a theory that would make it possible to describe the processes of dissipation and excitation of backflow flow in FSSCFs caused by shear stress.

The outline of this paper is as follows: The system of equations describing the twisting dynamics of the director field and fluid flow in FSSCF is given in Section 2. The numerical results for the number of hydrodynamic regimes are given in Section 3. The conclusions are summarized in Section 4.

### 2. Formulation of the Balance Equations for Smectic C System

We are primarily concerned with the description of the physical mechanism responsible for shear driven flow in FSSCF confined between the fixed outer, with the radius $R_2$, and rotating inner, with the radius $R_1$, circular frames (see Figure 1a). We consider the smectic C film in the form of a stack of circular smectic layers, freely suspended across a circular hole, and shear is applied by rotating inner circular frame, in which the director $\hat{n}(r, t) = \cos \theta \hat{a} + \sin \theta \hat{c}(r, t)$ makes a fixed angle $\theta$ with the layer normal $\hat{a}$ (see Figure 1a). In the chosen coordinate system, the unit vector $\hat{c}$, called $\hat{c}$-director, is the unit orthogonal projection of $\hat{n}$ on the smectic layers, as shown in Figure 1b. In the future, we will assume that the components of the director $\hat{c}(r, t) = \cos \Phi(r, t) \hat{e}_r + \sin \Phi(r, t) \hat{e}_\phi$, as well as all physical quantities, depend only on the coordinate $r$ and on time $t$. Here $\hat{e}_c$, $\hat{e}_r$, and $\hat{e}_\phi = \hat{e}_z \times \hat{e}_r$ are the basis vectors in the cylindrical coordinate system, while the angle $\Phi(r, t)$ is the azimuthal angle between the $\hat{c}$-director and the unit vector $\hat{e}_c$. To investigate the twisting dynamics of the $\hat{c}$-director field excited by the rotation of the inner circular frame when the outer frame is at rest, we consider a set of the balance equations. First is the torque balance equation $\sum_{i=c,a} \left( T_{el}^i + T_{vis}^i \right) = 0$, where $T_{el}^i$ and $T_{vis}^i$ are the elastic and viscous torques with respect to the vectors $\hat{c}$ ($i = c$) and $\hat{a}$ ($i = a$), respectively. The last equation takes the form [12]

$$\left[ \frac{\delta W_{el}}{\delta \hat{c}} + b^c \right] \times \hat{c} + \left[ \frac{\delta W_{el}}{\delta \hat{a}} + b^a \right] \times \hat{a} = 0,$$

(1)
where $W_\ell = \frac{1}{2} \left[ K_2 (\nabla \cdot \mathbf{e})^2 + K_3 (\mathbf{a} \cdot \nabla \times \mathbf{e})^2 \right]$ is the elastic energy density of the SmC system, $K_2$ and $K_3$ are the elastic constants of the SmC phase, $\mathbf{b}^\prime = -2[\lambda_1 \mathbf{D}^\ell + \tau_1 \mathbf{D}^0 + \lambda_4 \mathbf{A}]$ and $\mathbf{b}^\dagger = -2[\lambda_3 \mathbf{C} + \tau_2 \mathbf{e}(\mathbf{e} \cdot \mathbf{D}^0) + \lambda_4 \mathbf{A}]$ are two vectors which provide a convenient formulation for the viscous part of the dissipation function $R^\text{vis} = \frac{1}{2} (\sigma^\text{vis} + (\sigma^\text{vis})^T)$.

Then, the twisting dynamics of the $\mathbf{e}$-director field in FSSC can be obtained by solving the system of nonlinear partial differential equations, Equations (1) and (2), with the appropriate boundary conditions both for the tangential component of the velocity field

$$u(\tau, t)_{\tau=R_1} = u_{\text{intr}}, \quad u(\tau, t)_{r=R_2} = 0,$$

and for the azimuthal angle $\Phi$

$$\Phi(\tau, t)_{\tau=R_1} = \frac{\pi}{2}, \quad \Phi(\tau, t)_{r=R_2} = 0,$$

with the initial orientation chosen in the form

$$\Phi(\tau, 0) = \Phi^\text{ini}(\tau),$$

where $\Phi^\text{ini}(\tau)$ is obtained from Equation (1) with $u(\tau) = 0$.

Thus, the $\mathbf{e}$-director is strongly anchored to both circles, homeotropically to the inner and planar to the outer circular frames, and the angle $\Phi(\tau, 0)$ has to satisfy the boundary (Equation (6)) and its initial (Equation (7)) conditions.

In order to observe the response of the FSSC stretched between two circular frames, the fixed outer and rotating inner, we consider the dimensionless analog of the torque balance equation. For the $\mathbf{e}(\tau, \tau) = \cos \Phi(\tau, \tau) \mathbf{e}_\tau + \sin \Phi(\tau, \tau) \mathbf{e}_\theta$-director field, which is described by the azimuthal angle $\Phi(\tau, \tau)$, the dimensionless torque balance equation takes the form

$$\Phi_{\tau} = B(\Phi) \left[ \Phi_{\tau\tau} + \frac{\Phi_{\tau}}{r} \right] + \frac{\ell - 1}{2} \sin 2\Phi \left[ \Phi_{\theta\theta} + \frac{1}{r^2} \right] + \left[ \frac{1}{2} \left( 1 - \gamma + \frac{1}{2} \right) \cos 2\Phi - \sin 2\Phi \right] u(\tau, \tau),$$

where $r$ denotes the dimensionless radius, i.e., scaled by $d, d = R_2 - R_1$ is the width of the smectic film bounded by two circular frames on a scale of about ten micrometers, $\tau = K_1 t$ is the dimensionless time, $u$ is the dimensionless tangential component of the velocity field,
which connects the tangential velocity \( u \) which defines the pressure in the FSSCF, and, second, is rotated in the positive (anti-clockwise) dimensionless boundary conditions both for the azimuthal angle the system of nonlinear partial differential Equations (6) and (9), with the appropriate Substituting Equation (6) into the last Equation (9), we have

\[
\mathcal{P}_r = -\mathcal{E} \frac{u^2}{r} + \frac{\mu_3}{\gamma_5} \frac{1}{2} u_r \sin 2\Phi \cos^2 \Phi + \frac{1}{2} \frac{\mu_3}{\gamma_5} \left( \frac{u_r}{r} - \frac{u}{r^2} \right) \left( \sin 4\Phi - 2 \sin 2\Phi \right) + \frac{1}{2} \frac{\mu_3}{\gamma_5} \left( \frac{u_r}{r} - \frac{u}{r^2} \right) \Phi_r \left( 4 \cos 2\Phi \cos^2 \Phi - \sin 4\Phi \right) + \frac{1}{8} \frac{\mu_4}{\gamma_5} \sin 2\Phi \left( u_{rr} - \frac{u_r}{r} + \frac{u}{r^2} \right) - \Phi_r \cos 2\Phi \left( 2\Phi_r + u_r + \frac{u}{r} \right) - \gamma \left( \Phi_{rr} \sin 2\Phi + 2\Phi_r \Phi_r \cos 2\Phi \right) + \frac{1}{2} \gamma \left[ \sin 2\Phi \left( u_{rr} + \frac{3u_r}{r} - 3 \frac{u}{r^2} \right) - \Phi_r \cos 2\Phi \left( u_r - \frac{u}{r} \right) \right],
\]

which defines the pressure in the FSSCF, and, second,

\[
\mathcal{E} u_r = \frac{\mu_0 + \mu_4}{4\gamma_5} \left( u_{rr} + \frac{u_r}{r} - \frac{u}{r^2} \right) + \frac{\mu_3}{8\gamma_5} \left[ \sin^2 2\Phi \left( u_{rr} + \frac{u_r}{r} - \frac{u}{r^2} \right) + \Phi_r \left( u_{rr} - \frac{u}{r^2} \right) \sin 4\Phi \right] + \frac{1}{8} \left[ 4\Phi_{r\tau} - 8\Phi_r \Phi_{\tau \tau} + 2\Phi_r u_r \left( \sin 2\Phi + \cos 2\Phi \right) \right] + \frac{1}{8} \Phi_r \frac{u_r}{r} \left( 3 \sin 2\Phi + 2 \cos 2\Phi \right) + \frac{1}{8} \left[ \sin 2\Phi \left( u_{rr} + \frac{u_r}{r} - \frac{u}{r^2} \right) + 2 \sin^2 \Phi \left( u_{rr} + \frac{u_r}{r} - 13 \frac{u}{r^2} \right) \right] + \gamma \left[ \Phi_{r\tau} \cos 2\Phi - 2\Phi_r \Phi_r \sin 2\Phi + \frac{2\Phi_r}{r} \cos 2\Phi \right] - \gamma \Phi_r \left( 2u_r + \frac{u}{r} \right) \sin 2\Phi + \gamma \left[ \left( u_{rr} + \frac{u_r}{r} + 2 \frac{u}{r^2} \right) \cos 2\Phi - \frac{u}{r^2} \right],
\]

which connects the tangential velocity \( u \) with the azimuthal angle \( \Phi \). Here, \( \mathcal{E} = \frac{\rho K_3}{2\gamma_5} \). Substituting Equation (6) into the last Equation (9), we have

\[
\mathcal{E} u_r = \mathcal{G}_1 u_{rr} + \mathcal{G}_2 \frac{u_r}{r} + \mathcal{G}_3 \frac{u}{r^2} + \mathcal{G}_4,
\]

where the functions \( \mathcal{G}_i \) \( (i = 1, \ldots, 4) \) are defined in the Appendix A.

Thus, the twisting dynamics of the \( \hat{e} \)-director field in FSSCF can be described by the system of nonlinear partial differential Equations (6) and (9), with the appropriate dimensionless boundary conditions both for the azimuthal angle

\[
\Phi(r, \tau)_{r=r_1} = \frac{\pi}{2}, \quad \Phi(r, \tau)_{r=r_2} = 0,
\]

and for the tangential component of the velocity field \( u(r, \tau) \), when the inner circular frame is rotated in the positive (anti-clockwise)
was chosen to be equal to 10

\[ G \]

\( \Phi \) where (phase, the value of the polar angle corresponding to the first time step in the Equation (14), we can calculate the initial distribution of the velocity field \( u \) of \( \hat{u} \) from the Equation (6) using the relaxation method with \( u \) conditions. Calculations were carried out using the relaxation [15] and sweep [16] methods.

Thus, when the \( \hat{v} \)-director is strongly anchored to both frames, homeotropically to the inner and planar to the outer circular frames, the angle \( \Phi (r, \tau) \) has to satisfy the boundary conditions (10) and its initial orientation is chosen to be equal to

\[ \Phi (r, 0) = \Phi_{d}^{0i} (r), \]

where \( \Phi_{d}^{0i} (r) \) is obtained from Equation (6) with \( u (r)_{r_{1} \leq r \leq r_{2}} = 0 \).

In further calculations, we considered the circular FSSCF stretched between the fixed outer frame of radius \( R_{2} = 500 \mu m \), and rotating inner frame of radius \( R_{1} = 10, 20 \) and 50 \( \mu m \), respectively. For the case of \( d = (4-n\text{-decylloxybenzal})-2\text{-chloro}-1,4\text{-phenylene diamine} \) (DOBCP), at the temperature \( \sim 300 K \) and density \( 10^{3} \text{ kg/m}^{3} \), corresponding to the SmC phase, the value of the polar angle \( \theta \) is equal to \( \sim 20^\circ \), the experimental data for elastic constants are: \( [\mu N] \) (14) \( K_{2} \) = 0.64 and \( K_{3} \) = 1.58, whereas the measured data both for the Leslie and rotational viscosity coefficients are \( [\mu A s] \) (14); \( \mu_{0} \sim 0.07, \mu_{3} \sim 0.0001, \mu_{4} \sim 0.002, \gamma_{2} \sim -0.0025, \) and \( \gamma_{5} \sim 0.03, \) respectively.

Considering the fact that the value of \( d = R_{2} - R_{1} \) varies from 490 to 450 \( \mu m \), the value of the scaling time \( t_{s} = \frac{2\pi a^{2}}{K_{2}} \) can be estimated as \( \sim 8749 s \), at \( R_{1} = 20 \mu m \), and \( \sim 7690 s \), at \( R_{1} = 50 \mu m \), respectively. Two parameters, \( \bar{E} = \frac{\bar{\mu} K_{3}}{K_{2}} \) and \( \bar{d} = \frac{\nu a}{K_{2}} \), which are involved in the hydrodynamical equations, are equal to \( \sim 9 \times 10^{-7} \) and \( \sim 3.3 \times 10^{-3} \), respectively.

Whereas \( \bar{E} \ll 1 \), the left side of Equation (9) can be neglected, and this equation can be rewritten in the following form:

\[ G_{1} u_{,r} + G_{2} \frac{u_{,r}}{r} + G_{3} \frac{u}{r^{2}} + G_{4} = 0, \]

where the set of functions \( G_{i} (i = 1, \ldots, 4) \) is given in Appendix A with accounting that \( \delta \ll 1 \).

3. Shear-Driven Flows in SmC Film Induced by Rotating Inner Circular Frame

The twisting dynamics of the \( \hat{v} \)-director field, or evolution of the azimuthal angle \( \Phi (r, \tau) \), and the tangential component of the velocity field \( u (r, \tau) \) between two circular frames, the rest outer and rotating inner circular frames, is described by the Equations (6) and (14), together with the boundary (Equations (10)–(12)) and the initial (Equation (13)) conditions. Calculations were carried out using the relaxation [15] and sweep [16] methods.

The initial distribution of the \( \hat{v} \)-director, or the azimuthal angle \( \Phi (r, \tau) \), was derived from the Equation (6) using the relaxation method with \( u (r)_{r_{1} \leq r \leq r_{2}} = 0 \) and with the boundary condition Equation (10), for the azimuthal angle. Having the initial distribution of \( \hat{v} \)-the director or azimuth angle and the functions \( G_{i} (i = 1, \ldots, 4) \), which are included in the Equation (14), we can calculate the initial distribution of the velocity field \( u (r, \Delta \tau) \), corresponding to the first time step \( \Delta \tau \). The next time step \( \Delta \tau \) for the velocity field and \( \hat{v} \)-director’s distribution over the width of the FSSCF was calculated by the sweep method.

In the calculations, the relaxation criterion \( \epsilon = \frac{|(\Phi_{i}^{m+1} (r, \tau) - \Phi_{i}^{m} (r, \tau))/(\Phi_{i}^{m} (r, \tau))|}{(|(\Phi_{i}^{m+1} (r, \tau) - \Phi_{i}^{m} (r, \tau))/(\Phi_{i}^{m} (r, \tau))| + 1)} \) was chosen to be equal to \( 10^{-4} \), and the numerical procedure was then carried out until
a prescribed accuracy was achieved. Here $m$ is the iteration number. Figure 2a shows the reorientation of the azimuthal angle $\Phi(r, \tau)$ [in rad.] to its stationary distribution $\Phi^{st}(r) \equiv \Phi(r, \tau = \tau_R)$ between two circular frames, the rest outer ($r_2 = 1.0417$ ($R_2 = 500 \mu m$)) and rotating, in the positive sense (anti-clockwise) (case A) with the dimensionless velocity $U_{in} = 2000$ ($u_{in} \sim 0.11 \text{ mm/s}$), the inner ($r_1 = 0.0417$ ($R_1 = 20 \mu m$)) circular frames, for a number of dimensionless times (scaled by $2\gamma d^2/K$) $\tau_1 = 10^{-6}$ (\sim 8.7 ms) [curve 1], $\tau_2 = 0.005 (\sim 43.7 s)$ [curve 2], $\tau_3 = 0.02 (\sim 175 s)$ [curve 3], $\tau_4 = 0.03 (\sim 262 s)$ [curve 4], $\tau_5 = 0.06 (\sim 525 s)$ [curve 5], and $\tau_6 = \tau_R = 0.11 (\sim 962 s)$ [curve 6], respectively. Here $\tau_R$ is the relaxation time.

![Figure 2](image)

**Figure 2.** (a) The reorientation of the angle $\Phi(r, \tau_k)$ ($k = 1, \ldots, 6$) [in rad.] to its stationary distribution $\Phi^{st}(r) (\equiv \Phi(r, \tau = \tau_R))$ between two circular frames, the rest outer ($r_2 = 1.0417$ ($R_2 = 500 \mu m$)) and rotating in the positive sense (anti-clockwise) the inner ($r_1 = 0.0417$ ($R_1 = 20 \mu m$)) frames, for a number of dimensionless times (scaled by $2\gamma d^2/K$) $\tau_1 = 10^{-6}$ [curve 1], $\tau_2 = 0.005$ [curve 2], $\tau_3 = 0.02$ [curve 3], $\tau_4 = 0.03$ [curve 4], $\tau_5 = 0.069$ [curve 5], and $\tau_6 = \tau_R = 0.11$ [curve 6], respectively. (b) The same as in (a), but the reorientation of the angle $\Phi(r_k, \tau)$ [in rad.] to its stationary distribution $\Phi^{st}(r_k)$, at three values of the dimensionless distances (scaled by $d$) counted from the center of the smectic film: $r_k = 0.25$ ($k = 1$, curve 1), 0.50 ($k = 2$, curve 2), and 0.95 ($k = 3$, curve 3), respectively.

It should be noted that this value of the dimensional velocity $u_{in} \sim 0.11 \text{ mm/s}$ corresponds to the value of the angular velocity $\omega_{in} = u_{in}/R_1 \sim 5.5 \text{ s}^{-1}$ (i.e., less than one revolution per second) of rotation of the inner frame of the radius $r_1 = 0.0417$ ($R_1 = 20 \mu m$).

In turn, the Figure 2b shows the evolution of the azimuthal angle $\Phi(r_k, \tau)$ ($k = 1, 2, 3$) [in rad.] to its stationary distribution

$$\Phi^{st}(r_k) = \begin{cases} 49.8 \text{ rad, } r_1 = 0.25 \text{ (curve 1)}, \\ 49.5 \text{ rad, } r_2 = 0.50 \text{ (curve 2)}, \\ 12.3 \text{ rad, } r_3 = 0.95 \text{ (curve 3)}, \end{cases}$$

respectively, between both these frames when the inner circular frame is rotated in the positive sense (anti-clockwise) with the dimensionless velocity $U_{in} = 2000$.

Our calculations show that the evolution of the $\Phi$-profile between two circular frames, started with $\tau = \tau_2$, is characterized by convex profiles growing in the positive sense (anti-clockwise), and the highest absolute value of $|\Phi(r, \tau)|$ is reached in the vicinity of the inner rotating circular frame, after time $\tau_4 = 0.03$ (\sim 262 s) (see Figure 2a, curve 4). At the further stage of the evolution process ($\tau \geq \tau_5$), the angle profile $\Phi(r, \tau)$ proceeds in such a way that the central region of the film, enclosed between two circular frames, undergoes accelerated twisting, and, finally, the azimuthal angle $\Phi(0.2 \leq r \leq 0.5, \tau = \tau_R)$ takes the value $\sim 50$ [rad]. Thus, when the inner circular frame rotates in the positive sense with the dimensionless velocity $U_{in} = 2000$, in the central part of the SmC film, the $\hat{e}$-director twists almost 8 times around the normal to the smectic film.

For the case when the inner ($r_1 = 0.0417$ ($R_1 = 20 \mu m$)) circular frame rotates in the negative sense (clockwise) (case B), with the dimensionless velocity $U_{in} = 2000$.
(\(u_{in} \sim 0.11\) mm/s), while the outer frame rests), the reorientation of the azimuthal angle \(\Phi(r, \tau)\) to its stationary distribution \(\Phi^{st}(r) \equiv \Phi(r, \tau = \tau_R)\) between these frames is shown in Figures 3a,b. Figure 3a shows the reorientation of the \(\Phi\)-profile between two circular frames for a number of dimensionless times \(\tau_1 = 10^{-6}\) (~8.7 ms) [curve 1], \(\tau_2 = 0.005\) (~43.7 s) [curve 2], \(\tau_3 = 0.02\) (~175 s) [curve 3], \(\tau_4 = 0.03\) (~262 s) [curve 4], \(\tau_5 = 0.06\) (~525 s) [curve 5], and \(\tau_6 = \tau_R = 0.11\) (~962 s) [curve 6], respectively. Here \(\tau_R\) is the relaxation time. At the same time, the Figure 3b shows the evolution of the azimuthal angle \(\Phi(r_k, \tau)\) \((k = 1, 2, 3)\) [in rad.] to its stationary distribution

\[
\Phi^{st}(r_k) = \begin{cases} 
-22.0 \text{ rad, } r_1 = 0.25 \text{ (curve 1),} \\
-21.9 \text{ rad, } r_2 = 0.50 \text{ (curve 2),} \\
-10.5 \text{ rad, } r_3 = 0.95 \text{ (curve 3),}
\end{cases}
\]

respectively, between both these frames when the inner circular frame is rotated in the negative sense (clockwise) with the dimensionless velocity \(U_{in} = 2000\).

Our calculations show that the evolution of the \(\Phi\)-profile between two circular frames, started with \(\tau = \tau_2\), is characterized by convex profiles growing in the negative sense (clockwise), and the highest absolute value of \(|\Phi(r, \tau)|\) is reached in the vicinity of the outer rest circular frame (see Figure 3a, curve 2). At the further stage of the evolution process \((\tau \geq \tau_3)\), the angle profile \(\Phi(r, \tau)\) proceeds in such a way that the region of the film, close to the rotating inner frame, undergoes accelerated twisting, and, finally, the azimuthal angle \(|\Phi(0.15 \leq r \leq 0.75, \tau = \tau_R)|) takes the value \(~22\) [rad]. Thus, when the inner circular frame rotates in the negative sense with the dimensionless velocity \(U_{in} = 2000\), in the central part of the SmC film, the \(\hat{c}\)-director twists almost 3.5 times around the normal to the smectic film.

Physically, this means that the shear flow, which is initiated by the rotation of the inner frame, both in a positive (anti-clockwise) (case A) and negative (clockwise) (case B) senses, with the same amount of angular velocity (~5.5 s\(^{-1}\)), causes a different twisting of the \(\hat{c}\)-director around the normal to the smectic film. In case A, the twisting of the \(\hat{c}\)-director around the normal to the smectic film is up to \(\Phi^{st}(r) \sim 50\) [rad], while in case B, the maximum twisting of the \(\hat{c}\)-director around the normal to the smectic film is up to \(\Phi^{st}(r) \sim 22\) [rad], approximately 2.3 times less. Since the \(\hat{c}\)-director is planar oriented to the outer frame \((\Phi(r, \tau) = 0)\), in case B, it is necessary to spend more energy on twisting the director’s field, due to working against the elastic force of the FSSCF, than in case A. As a result, the maximum twisting of the \(\hat{c}\)-director around the normal to the smectic film, in the case of A, approximately, 2.3 times higher than in the case of B.

![Figure 3](image-url)

**Figure 3.** (a) Same as in Figure 2a, but the reorientation of the angle \(\Phi(r_k)(k = 1, \ldots, 6)\) [in rad.] to its stationary distribution \(\Phi^{st}(r_k)\) between two circular frames, the rest outer and rotating in the negative sense (clockwise) the inner frames. (b) The same as in (a), but the reorientation of the angle \(\Phi(r_k, \tau)\) \((k = 1, 2, 3)\) [in rad.] to its stationary distribution \(\Phi^{st}(r_k)\) between the rest outer and rotating, in the negative sense (case B), inner circular frames, at three values of \(r_k : r_1 = 0.25\) (curve 1), \(r_2 = 0.50\) (curve 2), and \(r_3 = 0.95\) (curve 3), respectively.
It should be noted that the process of twisting of $\hat{c}$-director’s field has a jumping nature, when the reorientation is carried out by a relatively large amount in a short time (see Figures 2b and 3b, respectively). Moreover, this trend can be clearly traced near the rotating frame ($r = 0.25$ and $0.50$), when jumps up to 10 radians are realized. It should also be noted that in the case of clockwise rotation of the inner frame (case B), these jumps of $\hat{c}$-director’s reorientation are about six radians (almost full circle of rotation relative to the normal to the smectic film), practically, during the entire relaxation process. In both cases A and B, the director’s twisting relative to the normal to the film is bigger near the center of that film than away from it.

Now let us see how the velocity field $u(r, \tau)$ evolves between these two frames, outer rest and rotating inner. Figure 4a shows the evolution of the dimensionless tangential component of the velocity field $u(r, \tau)$ (scaled by $\frac{K_3}{2\mu_0 \alpha}$) to its stationary distribution $u^{st}(r) \equiv u(r, \tau = \tau_R)$ between two circular frames, the rest outer ($r_2 = 1.0417$ ($R_2 = 500 \mu m$)) frame and rotating in the positive sense, (anti-clockwise) (case A) the inner ($r_1 = 0.0417$ ($R_1 = 20 \mu m$)) frame, with the dimensionless velocity $U_{in} = 2000$ ($u_{in} \sim 0.11 \text{ mm/s}$), for a number of dimensionless times (scaled by $\frac{2\gamma_{ps} \alpha^2}{K_3}$) $\tau_1 = 10^{-6}$ ($\sim 8.7 \text{ ms}$) [curve 1], $\tau_2 = 0.005$ ($\sim 43.7 \text{ s}$) [curve 2], $\tau_3 = 0.02$ ($\sim 175 \text{ s}$) [curve 3], $\tau_4 = 0.03$ ($\sim 262 \text{ s}$) [curve 4], $\tau_5 = 0.06$ ($\sim 525 \text{ s}$) [curve 5], and $\tau_6 = \tau_R = 0.11$ ($\sim 962 \text{ s}$) [curve 6], respectively. Our calculations show that the evolution of the $u(r, \tau)$-profile over the width of the FSSCF stretched between these two circular frames is characterized by convex profiles growing in the negative sense during the first four time terms (see curves from 1 to 4, Figure 4a), and the largest absolute value of $|u(r, \tau)| \sim 9500$ ($\sim 0.52 \text{ mm/s}$) is reached near the center of the rotating circular frame, after time $\tau_4 = 0.03$ ($\sim 175 \text{ s}$) (see Figure 4a, curve 4). At the further stage of the evolution process ($\tau \geq \tau_3$), the dimensionless component of the tangential velocity profile $u(r, \tau)$ is proceeded in such a way that only the domain near the rotating circular frame ($0.0417 \leq r \leq 0.14$) undergoes acceleration up to $|u(r \sim 0.1, \tau_R)| \sim 3000$ ($\sim 0.17 \text{ mm/s}$), while the rest part of the SmC film is characterized by almost uniform motion in the negative sense with the velocity of $|u(0.14 \leq r \leq 1.0, \tau_R)| \sim 800$ ($\sim 0.044 \text{ mm/s}$). Figure 4b shows the evolution of the dimensionless tangential component of the velocity field $u(r_k, \tau) (k = 1, 2, 3)$ to its stationary distribution $u^{st}(r_k) \equiv u(r_k, \tau = \tau_R)$ between the rest outer $r_2 = 1.0417$ ($R_2 = 500 \mu m$) and rotating in the positive sense (anti-clockwise) the inner $r_1 = 0.0417$ ($R_1 = 20 \mu m$) frames, at three values of the dimensionless distances (scaled by $d$) counted from the center of the smectic film: $r_k = 0.25$ ($k = 1$, curve 1), 0.50 ($k = 2$, curve 2), and 0.95 ($k = 3$, curve 3), respectively. Our calculations show that the evolution of these $u(r_k, \tau)$ ($k = 1, 2, 3$) profiles at three distances from the center of the smectic film is characterized by the oscillatory behavior of $u(r_k, 0.3 \leq \tau \leq 0.7)$ ($k = 1, 2, 3$) over a short period of time with subsequent relaxations to stationary values, which are equal to

$$|u^{st}(r_k, \tau > 0.07)| = \begin{cases} 
949 \ (\sim 0.052 \text{ mm/s}), & r_1 = 0.25 \ (\text{curve 1}), \\
881 \ (\sim 0.049 \text{ mm/s}), & r_1 = 0.50 \ (\text{curve 2}), \\
184 \ (\sim 0.01 \text{ mm/s}), & r_1 = 0.95 \ (\text{curve 3}), 
\end{cases}$$

respectively. Comparing the results of calculations of both the director (see Figure 2b) and the velocity (see Figure 4b) fields, it can be concluded that the relaxation times of these quantities are almost equal to each other.

Figure 5a shows the evolution of the velocity field $u(r, \tau)$ (scaled by $\frac{K_3}{2\mu_0 \alpha}$) to its stationary distribution $u^{st}(r) \equiv u(r, \tau = \tau_R)$ between two circular frames, the rest outer ($r_2 = 1.0417$ ($R_2 = 500 \mu m$)) frame and rotating in the negative sense (clockwise) (case B) the inner ($r_1 = 0.0417$ ($R_1 = 20 \mu m$)) frame, with the dimensionless velocity $U_{in} = 2000$ ($u_{in} \sim 0.11 \text{ mm/s}$), for a number of dimensionless times (scaled by $\frac{2\gamma_{ps} \alpha^2}{K_3}$) $\tau_1 = 10^{-6}$ ($\sim 8.7 \text{ ms}$) [curve 1], $\tau_2 = 0.005$ ($\sim 43.7 \text{ s}$) [curve 2], $\tau_3 = 0.02$ ($\sim 175 \text{ s}$) [curve 3], $\tau_4 = 0.03$ ($\sim 262 \text{ s}$) [curve 4], $\tau_5 = 0.06$ ($\sim 525 \text{ s}$) [curve 5], and $\tau_6 = \tau_R = 0.11$ ($\sim 962 \text{ s}$)
[curve 6], respectively. Calculations show that the evolution of the \( u(r, \tau) \)-profile over the width of the FSSCF stretched between these two circular frames is characterized by convex profiles growing in the positive sense during the first five time terms (see curves from 1 to 5, the Figure 5a), and the largest absolute value of \(|u(r, \tau)| \sim 4500 (\sim 0.25 \text{ mm/s})\) is reached near the center of the rotating circular frame, after time \( \tau_2 = 0.005 (\sim 0.07 \text{ s}) \) (see Figure 5a, curve 2). At the final stage of the evolution process (\( \tau \sim \tau_R \)), the velocity profile \( u(r, \tau) \) is reduced to \(|u(0.1 \leq r \leq 0.8, \tau_R)| \sim 800 (\sim 0.044 \text{ mm/s})\), while the rest part of the SmC film is characterized by a gradual decrease in speed to 0.

![Figure 4](image-url)

**Figure 4.** (a) The evolution of the velocity field \( u(r, \tau_k) \) (scaled by \( \frac{K_1}{\mu} \)) to its stationary distribution \( u^\text{st}(r) \) between two circular frames, the rest outer \( (r_2 = 1.0417 \ (R_2 = 500 \mu\text{m})) \) and rotating in the positive sense (anti-clockwise) the inner \( (r_1 = 0.0417 \ (R_1 = 20 \mu\text{m})) \) frames, for a number of dimensionless times \( \tau_k \) \( (k = 1, \ldots, 6) \) (scaled by \( \frac{2\gamma_{12}d^2}{\mu} \)), the same as in Figure 2. (b) The same as in (a), but the evolution of the velocity field \( u(r_k, \tau)(k = 1, 2, 3) \) (scaled by \( \frac{K_1}{\mu} \)) to its stationary distribution \( u^\text{st}(r_k) \) between two circular frames, the rest outer and rotating inner (case A), at three values of the dimensionless distances (scaled by \( d \)) counted from the center of the smectic film: \( r_k = 0.25 \) \( (k = 1, \text{ curve 1}) \), \( 0.50 \) \( (k = 2, \text{ curve 2}) \), and \( 0.95 \) \( (k = 3, \text{ curve 3}) \), respectively.

Figure 5b shows the evolution of the velocity field \( u(r_k, \tau)(k = 1, 2, 3) \) to its stationary distribution \( u^\text{st}(r_k) \equiv u(r_k, \tau = \tau_R) \) between two circular frames, the rest outer \( (r_2 = 1.0417 \ (R_2 = 500 \mu\text{m})) \) frame and rotating in the negative sense (clockwise) (case B), with the dimensionless velocity \( U_\text{in} = 2000 \ (u_\text{in} \sim 0.11 \text{ mm/s}) \), the inner \( (r_1 = 0.0417 \ (R_1 = 20 \mu\text{m})) \) frame, at three values of the dimensionless distances (scaled by \( d \)) counted from the center of the smectic film: \( r_k = 0.25 \) \( (k = 1, \text{ curve 1}) \), 0.50 \( (k = 2, \text{ curve 2}) \), and 0.95 \( (k = 3, \text{ curve 3}) \), respectively. Calculations show that the evolution of these \( u(r_k, \tau) \) \( (k = 1, 2, 3) \) profiles at three distances from the center of the smectic film is characterized by the oscillatory behavior of \( u(r_k, 0.3 \leq \tau \leq 0.7) \) \( (k = 1, 2, 3) \) over a short period of time with subsequent relaxations to stationary values, which are equal to

\[
|u^\text{st}(r_k, \tau > 0.095)| = \begin{cases} 
987 \ (\sim 0.054 \text{ mm/s}), & r_1 = 0.25 \ (\text{curve 1}), \\
868 \ (\sim 0.048 \text{ mm/s}), & r_1 = 0.50 \ (\text{curve 2}), \\
275 \ (\sim 0.015 \text{ mm/s}), & r_1 = 0.95 \ (\text{curve 3}), 
\end{cases}
\]

respectively.

Comparing the results of calculations of both the director field (see Figure 3b) and the velocity field (see Figure 5b), it can be concluded that the relaxation time increases slightly with distance from the center of rotation of the smectic C film.

It should be noted that the main part of the smectic C film, both for cases A and B, moves in different directions, but at the same speed \( |u^\text{st}(0.1 \leq r \leq 0.8)| \sim 800 (\sim 0.044 \text{ mm/s}) \). It should also be noted that while the inner circular frame rotates in the positive sense (anti-clockwise) (case A) with the dimensionless velocity \( U_\text{in} = 2000 \ (u_\text{in} \sim 0.11 \text{ mm/s}) \),
at the same time the ε-director’s field performs up to eight revolutions around the normal to the smectic film (see Figure 2a, curve 6).

![Figure 5](image-url)  
**Figure 5.** (a) Same as in Figure 4a, but the evolution of the velocity field $u(r, r_k) (k = 1, ..., 6)$ to its stationary distribution $u^{st}(r)$ between the rest outer and rotating, in the negative sense (case B), inner circular frames. (b) The same as in (a), but the evolution of the velocity field $u(r, r_k)$ to its stationary distribution $u^{st}(r)$ between the rest outer and rotating, in the negative sense (case B), inner circular frames, at three values of $r_k$ : $r_1 = 0.25$ (curve 1), $r_2 = 0.50$ (curve 2), and $r_3 = 0.95$ (curve 3), respectively.

With a change in the radius of the inner rotating frame, there is a change in the evolution of not only the azimuthal angle $\Phi(r, \tau)$, but also the tangential component of the velocity field $u(r, \tau)$ to its stationary distribution $u^{st}(r)$ between the rest outer and rotating the inner circular frames. Figure 6a shows the stationary distribution of the azimuthal angle $\Phi^{st}(r)$ between the rest outer ($R_2 = 500 \, \mu m$) frame and rotating in the positive sense (anti-clockwise) (case A), with the dimensionless velocity $U_{in} = 2000$ ($u_{in} \sim 0.11 \, mm/s$), the inner circular frame, at three values of the inner radius $R_1 = 10 \, \mu m$ (curve 1), 20 $\mu m$ (curve 2), and 50 $\mu m$ (curve 3), respectively. In turn, Figure 6b shows the stationary distribution of the azimuthal angle $\Phi^{st}(r)$ between these frames, when the inner circular frame is rotated in the negative sense (clockwise) (case B), with the same as in case A, the dimensionless velocity $V_{in} = 2000 (u_{in} \sim 0.11 \, mm/s)$. Our calculations show that with an increase in the radius of the inner rotating frame, from $R_1 = 10 \, \mu m$ to 50 $\mu m$, the maximum value of the azimuthal angle $\Phi^{st}(r)$ increases, and in the case of A, it reaches the value $\sim 80 [\text{rad}]$, or almost 13 full circles of rotation the ε-director’s field relative to the normal to the smectic film, while in case B, it reaches the value $\sim 28 [\text{rad}]$, or almost 4.5 full circles of rotation the ε-director’s field relative the same normal. Thus, with an increase in the radius of the inner rotating frame, at the constant velocity $V_{in} = 2000$, the degree of twisting of the FSSCF increases. Figure 7a shows the stationary distribution of the velocity field $u^{st}(r)$ between the rest outer ($R_2 = 500 \, \mu m$) frame and rotating in the positive sense (anti-clockwise) (case A), with the dimensionless velocity $V_{in} = 2000 (u_{in} \sim 0.11 \, mm/s)$, the inner circular frame, at three values of the inner radius $r_1 = 0.02$ ($R_1 = 10 \, \mu m$) (curve 1), $r_1 = 0.0417$ ($R_1 = 20 \, \mu m$) (curve 2), and $r_1 = 0.11$ ($R_1 = 50 \, \mu m$) (curve 3), respectively.

It should be noted that the dimensionless values of the radii of the inner and outer frames are as follows: $r_1 = 0.02$ and $r_2 = 1.02$, for $R_1 = 10 \, \mu m$ and $R_2 = 500 \, \mu m$, $r_1 = 0.0417$ and $r_2 = 1.0417$, for $R_1 = 20 \, \mu m$ and $R_2 = 500 \, \mu m$, and $r_1 = 0.11$ and $r_2 = 1.11$, for $R_1 = 50 \, \mu m$ and $R_2 = 500 \, \mu m$, respectively.
Figure 6. (a) The stationary distribution of the azimuthal angle $\Phi_{st}(r)$ between the rest outer $(R_2 = 500 \, \mu m)$ circular frame and rotating in the positive sense (anti-clockwise) the inner circular frame (case A), at three values of the inner radius $R_1 = 10 \, \mu m$ (curve 1), 20 $\mu m$ (curve 2), and 50 $\mu m$ (curve 3), respectively. (b) Same as in case (a), but the inner circular frame is rotated in the negative sense (clockwise) (case B).

Figure 7. (a) The stationary distribution of the dimensionless (scaled by $\frac{R_1}{\mu_1}$) tangential component of the velocity field $u_{st}(r)$ between the rest outer $(R_2 = 500 \, \mu m)$ circular frame and rotating in the positive sense (anti-clockwise) (case A) the inner circular frame, at three values of the inner radius: $r_1 = 0.02$ ($R_1 = 10 \, \mu m$) (curve 1), $r_1 = 0.0417$ ($R_1 = 20 \, \mu m$) (curve 2), and $r_1 = 0.11$ ($R_1 = 50 \, \mu m$) (curve 3), respectively. (b) Same as in case (a), but the inner circular frame is rotated in the negative sense (clockwise) (case B), while the values of the inner radius are: $r_1 = 0.02$ ($R_1 = 10 \, \mu m$) (curve 1), $r_1 = 0.0417$ ($R_1 = 20 \, \mu m$) (curve 2), $r_1 = 0.11$ ($R_1 = 50 \, \mu m$) (curve 3), and $r_1 = 0.25$ ($R_1 = 100 \, \mu m$) (curve 4), respectively.

In turn, the Figure 7b shows the stationary distribution of the velocity field $u_{st}(r)$ between these frames, when the inner circular frame is rotated in the negative sense (clockwise) (case B), with the same as in case A, the dimensionless velocity $V_{in} = 2000 (\mu m \sim 0.11 \, \text{mm/s})$, while the values of the inner radius are: $r_1 = 0.02$ ($R_1 = 10 \, \mu m$) (curve 1), $r_1 = 0.0417$ ($R_1 = 20 \, \mu m$) (curve 2), $r_1 = 0.11$ ($R_1 = 50 \, \mu m$) (curve 3), and $r_1 = 0.25$ ($R_1 = 100 \, \mu m$) (curve 4), respectively.

Our calculations in case A show that with an increase in the radius of the inner rotating frame, from $R_1 = 10 \, \mu m$ to $50 \, \mu m$, the maximum value of the of the stationary velocity field $|u_{st}(r)|$ decreases, from the value $|u_{st}(r \sim 0.04)| \sim 5680 (\sim 0.31 \, \text{mm/s})$, at $r_1 = 0.02$ ($R_1 = 10 \, \mu m$), to $|u_{st}(r \sim 0.072)| \sim 3300 (\sim 0.18 \, \text{mm/s})$, at $r_1 = 0.0417$ ($R_1 = 20 \, \mu m$), and with a further increase to $|u_{st}(r \sim 0.40)| \sim 12400 (\sim 0.68 \, \text{m/s})$, at $r_1 = 0.11$ ($R_1 = 50 \, \mu m$), respectively. In case B, our calculations show that with an increase in the radius of the inner rotating frame from $R_1 = 10 \, \mu m$ to $20 \, \mu m$, the maximum value of the stationary velocity field $|u_{st}(r_1 < r < r_2)|$ is less than the velocity value of the inner circular frame.
\( V_{in} = 2000 \) (\( u_{in} \sim 0.11 \text{ mm/s} \)). As the radius of the inner frame increases, the maximum value of \( |u^d(r_1 < r < r_2)| \) begins to exceed the value of \( V_{in} = 2000 \) (\( u_{in} \sim 0.11 \text{ mm/s} \)) and reaches the maximum value equal to \( |u^d(r = 0.53)| = 3000 \) (\( \sim 0.17 \text{ mm/s} \)), with the value of the radius of the inner rotating frame equal to \( R_1 = 50 \) \( \mu \text{m} \). A further increase in the radius of the inner frame to \( R_1 = 0.25 \) (\( R_1 = 100 \) \( \mu \text{m} \)) leads to a decrease in the maximum value of the velocity \( |u^d(r = 0.60)| = 2300 \) (\( \sim 0.13 \text{ mm/s} \)), which slightly exceeds the rotation speed of the inner frame.

Based on our calculations, we can conclude that with an increase in the radius of the inner rotating frame, a higher degree of twisting of the \( \hat{c} \)-director around the normal to the smectic layers of a freely suspended smectic film C is achieved. Thus, the twisting dynamics of the \( \hat{c} \)-director field depends crucially on the curvature of the inner rotating frame.

4. Conclusions

We numerically investigated some features of the twisting dynamics of a freely suspended smectic C film (FSSCF) stretched between two circular frames, the rest outer and the rotating inner frames. Some numerical advances in predicting the structural and hydrodynamic behavior of shear flow in such the FSSCF excited by a rotating inner frame are based on a corresponding extension of the classical Eriksen–Leslie theory. It is shown that the shear flow in the FSSCF causes a twisting rotation of the \( \hat{c} \)-directors around the normal to the smectic layers. Since the orientation of the \( \hat{c} \)-director is fixed at the rims of the smectic film, shear winds up of the \( \hat{c} \)-director field and the shear flow excited by the rotating frame in a positive sense (counterclockwise) causes the twisting rotation of the \( \hat{c} \)-director in a negative sense (clockwise) around the normal to the smectic film. Our calculations show that the shear stress \( \sigma_z \) produces the tangential component of the velocity field \( u(r, \tau) \) and its effect on the \( \hat{c} \)-director distribution across the FSSCF, stretched between two circular frames, is very strong, so that the middle part of the FSSCF the biggest value of the azimuthal angle \( \Phi(r) \) is equal to 80 [rad], and the director executes, practically, 13 full cycles of rotation.

It is also shown that the twisting of \( \hat{c} \)-director’s field has a jumping nature, when a large reorientation is carried out in a relatively short amount of time. In this regard, an analogy can be drawn with the serious of layer-thinning transitions, causing the SmA films to thin in a stepwise manner as the temperature is increased above the bulk SmA-isotropic or nematic transition temperature. In both cases, we are dealing with a stepwise (jumping) nature of the change in these quantities. Calculations also show that the twisting dynamics of the \( \hat{c} \)-director field depends crucially on the curvature of the inner rotating frame.

A recent experiment was reported to measure elastic distortion and tangential flow in the FSSCF initiated by an electric field [10]. It was shown that the main mechanism responsible for the drive of the \( \hat{c} \)-director is the shear flow \( v(r) \) of the film initiated by rotating electric field. The tangential flow \( v(r) \) in a freely suspended smectic film by probe particles was visualized. The experimentally obtained values of \( v \) range from 5 to 8\( \mu \text{m/s} \), which is consistent with our calculated data for \( u^d(\tau > 0.1) \sim 10 \div 50 \mu \text{m/s} \). It should be noted that the greatest problem when matching experimental data with calculated ones is the lack of reliable viscosity and elasticity coefficients.

This once again shows that the macroscopic description of the nature of the hydrodynamic flow of an anisotropic liquid subtly senses the microscopic structure of the LC material. We believe that the present investigation can shed some light on the problem of hydrodynamic description of the relaxation processes in freely suspended liquid crystal films excited by the shearing stress.

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Appendix A

The dimensionless elastic and viscous components of the torques acting on the vectors \( \hat{c} \) and \( \hat{a} \) are equal to

\[
T_{el}^c = - \left[ \Phi_r + \Phi_r \right] B(\Phi) + \frac{k - 1}{2} \left[ \Phi_r + \Phi_r \right] \sin 2\Phi \right] \hat{e}_z,
\]

\[
T_{el}^a = - \cos \Phi \beta \hat{e}_z,
\]

\[
T_{vis}^c = \left[ \Phi_r - \frac{2\Phi}{2} \left( (1 - \gamma) u_r + (1 + \gamma) \frac{u}{r} \right) + \sin^2 \Phi \frac{u}{r} + \Phi \right] \hat{e}_z,
\]

\[
T_{vis}^a = \left[ \frac{1}{2} \cos 2\Phi \left( (\gamma - 1) u_r - (1 + \gamma) \frac{u}{r} \right) + \sin^2 \Phi \frac{u}{r} + \Phi \right] \hat{e}_z,
\]

respectively, where \( \beta \) is the Lagrange multiplier which has a physical interpretation, as pointed out by Leslie [17].

Dimensionless expressions for the viscous component of the stress tensor have the form

\[
\sigma_{rr}^{vis} = \left( u_r - \frac{u}{r} \right) \left( \frac{\mu_3}{2\lambda_5} \sin 2\Phi \cos^2 \Phi + \frac{\mu_4}{\lambda_5} \sin 2\Phi \right)
- \gamma \sin 2\Phi \left( \Phi_r + \frac{3}{2} \Phi_r + \frac{1}{2} \Phi \right),
\]

\[
\sigma_{\varphi\varphi}^{vis} = \left( u_r - \frac{u}{r} \right) \left( \frac{\mu_3}{2\lambda_5} \sin 2\Phi \sin^2 \Phi + \frac{\mu_4}{\lambda_5} \sin 2\Phi \right)
+ \gamma \sin 2\Phi \left( \Phi_r + \frac{1}{2} \left( \Phi_r + \frac{u}{r} \right) \right),
\]

\[
\sigma_{r\varphi}^{vis} = \left( u_r - \frac{u}{r} \right) \left( \frac{\mu_0}{\lambda_5} + \frac{\mu_3}{2\lambda_5} \sin^2 2\Phi + \frac{\mu_4}{2\lambda_5} \right)
- (\gamma - 1) \Phi_r \cos 2\Phi + \frac{1 - \gamma + \gamma \cos 2\Phi u}{2} \frac{u}{r},
\]

\[
\sigma_{\varphi r}^{vis} = \left( u_r - \frac{u}{r} \right) \left( \frac{\mu_0}{\lambda_5} + \frac{\mu_3}{2\lambda_5} \sin^2 2\Phi + \frac{\mu_4}{2\lambda_5} \right)
+ (\gamma - 1) \Phi_r \cos 2\Phi - \frac{1 - \gamma + \gamma \cos 2\Phi u}{2} \frac{u}{r}.
\]

Expressions for dimensionless components of the vector \( \mathbf{b}^c \) in polar coordinates have the form

\[
b^c = \left[ (1 - \gamma) u_r + (1 + \gamma) \frac{u}{r} + 2\Phi \right] \sin \Phi,
\]
and

\[ b'_p = \left[(1 - \gamma)u_r + (1 + \gamma)\frac{u}{r} - 2\Phi_r\right] \cos \Phi, \]

respectively, while the dimensionless expression for the elastic energy density can be written as

\[ W_{el} = \frac{1}{2} \left[ \kappa \left( \frac{\cos \Phi}{r} - \sin \Phi \Phi_r \right)^2 + \left( \frac{\sin \Phi}{r} + \cos \Phi \Phi_r \right)^2 \right]. \]

All this makes it possible to write the expression for the torque balance equations with respect to both vectors \( \hat{c} \) and \( \hat{a} \) in the forms

\[ \Phi_{r,\tau} = B(\Phi) \left( \Phi_{r,rr} + \frac{\Phi_{r,r}}{r} \right) + \frac{\kappa - 1}{2} \sin 2\Phi \left( \Phi_{r,r}^2 + \frac{1}{r^2} \right) + \left[ \frac{1}{2} \left( 1 - \gamma + \frac{1+\gamma}{r} \right) \cos 2\Phi - \sin 2\Phi \right] u(r, \tau), \]

\[ \cos \Phi \beta_{r,\tau} = \frac{1}{2} \cos 2\Phi \left[ (\gamma - 1)u_r - (1 + \gamma)\frac{u}{r} \right] + \sin^2 \Phi \frac{u}{r} + \Phi_{r,\tau}. \]

The balance equation for linear momentum can be written in the form

\[ \mathcal{E} u_{r,\tau} = \mathcal{G}_1 u_{r,rr} + \mathcal{G}_2 \frac{u_{r,r}}{r} + \mathcal{G}_3 \frac{u_{r,r}}{r^2} + \mathcal{G}_4, \]

where the function \( \mathcal{G}_i \) \((i = 1, \ldots, 4)\) has the following elements:

\[ \mathcal{G}_1 = \frac{1}{8} \left[ 2 + \frac{\mu_0 + \mu_4 + \mu_3 \sin^2 2\Phi}{\lambda_5} + \sin 2\Phi \right] + \frac{1}{4} (1 - \gamma) \cos \Phi (\gamma \cos 2\Phi - 1), \]

\[ \mathcal{G}_2 = \frac{1}{4} \left[ -\frac{\mu_0 + \mu_4 + \mu_3 \sin^2 2\Phi}{\lambda_5} + \sin 2\Phi \right] + \frac{1}{4} \cos 2\Phi (1 + 2\gamma + 2\gamma \cos 2\Phi) - \sin^2 \Phi (1 + (1 + \gamma) \cos 2\Phi) + \frac{1}{4} \Phi_{r,rr} \left[ -5 \sin 2\Phi + \cos 2\Phi - \gamma (2 - \gamma) \sin 4\Phi + \frac{2\mu_3}{\lambda_5} \sin 4\Phi \right], \]

\[ \mathcal{G}_3 = \frac{1}{8} \left[ 2\mu_0 + \mu_4 - 2\lambda_2 + \mu_3 \sin^2 2\Phi \right] - \sin 2\Phi \]

\[ + \frac{1}{8} \cos 2\Phi \left[ 8\gamma + 2(1 + \gamma) (\gamma \cos 2\Phi - 1) - 8 \sin^2 \Phi \right] - \frac{1}{8} \sin^2 \Phi (18 + 8 \cos 2\Phi) - \frac{1}{8} \Phi_{r,rr} \sin 2\Phi \left( 1 + 8\gamma \cos^2 \Phi - 4\frac{2\mu_3}{\lambda_5} \sin 4\Phi \right), \]
\[
G_4 = \left[ \frac{\gamma \cos 2\Phi - \sin^2 \Phi}{r} - \gamma \Phi_r \sin 2\Phi \right] \\
\left[ \Phi_{,rr}B + \frac{\kappa - 1}{2} \Phi_r^2 \sin 2\Phi + \frac{\Phi_r}{r} B + \frac{\kappa - 1 \sin 2\Phi}{2 r^2} \right] + \\
\frac{1 + \gamma \cos 2\Phi}{2} \left[ \Phi_{,rr}B + (\kappa - 1) \left( 2 \Phi_{,rr} \Phi_r + \frac{\Phi_r^2}{r^2} - \frac{1}{r^2} \right) \sin 2\Phi \right] + \\
\frac{1 + \gamma \cos 2\Phi}{2} \left[ (\kappa - 1) \left( \Phi_r^3 + \frac{\Phi_r}{r^2} \right) \cos 2\Phi + B \left( \Phi_{,r} - \frac{\Phi_r}{r^2} \right) \right].
\]

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