Variational Properties of the Gauss-Bonnet Curvatures

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Abstract

The Gauss-Bonnet curvature of order $2k$ is a generalization to higher dimensions of the Gauss-Bonnet integrand in dimension $2k$, as the scalar curvature generalizes the two dimensional Gauss-Bonnet integrand.

In this paper, we evaluate the first variation of the integrals of these curvatures seen as functionals on the space of all Riemannian metrics on the manifold under consideration. An important property of this derivative is that it depends only on the curvature tensor and not on its covariant derivatives. We show that the critical points of this functional once restricted to metrics with unit volume are generalized Einstein metrics and once restricted to a pointwise conformal class of metrics are metrics with constant Gauss-Bonnet curvature.

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1 Introduction and Statement of the Results

Let $M$ be a compact smooth (oriented) manifold of dimension $n$, and let $\mathcal{M}$ be the space of smooth Riemannian metrics on $M$ endowed with a natural $L^2$-Sobolev norm. This allows us to speak about differentiable functionals $\mathcal{M} \to \mathbb{R}$. A functional $F : \mathcal{M} \to \mathbb{R}$ is called Riemannian if it is invariant under the action of the diffeomorphism group. We say that $F$ has a gradient at $g$ if there exists a symmetric tensor $a \in \mathcal{C}^1$ such that for every symmetric tensor $h \in \mathcal{C}^1$ we have

$$F'_g h = \frac{d}{dt} |_{t=0} F(g + th) = \langle a, h \rangle,$$
where $C^1$ denotes the space of symmetric tensors in $\Lambda^* M \otimes \Lambda^* M$ and $\langle , \rangle$ is the integral inner product.

A classical Riemannian functional is the total scalar curvature

$$S(g) = \int_M \text{scal} \mu_g,$$

where $\text{scal}$ denotes the scalar curvature function of $g$ and $\mu_g$ is the volume element of $g$. An important point about this functional $S$, is that its critical points, when restricted to $M_1 = \{g \in M : \text{vol}(g) = 1\}$, are Einstein metrics. Also, its gradient is the Einstein tensor, precisely

$$S'_g h = \langle \frac{1}{2} \text{scal} g - \text{Ric}, h \rangle.$$

Where Ric denotes the Ricci tensor.

A natural generalization of the functional $S$ is the Riemannian functional

$$H_{2k}(g) = \int_M h_{2k} \mu_g,$$

where, for each $1 \leq k \leq n/2$, $h_{2k}$ is the Gauss-Bonnet curvature of order $2k$. This curvature is determined by the complete contraction of the Gauss-Kronecker tensor of order $k$. Furthermore, these curvatures coincide with the intrinsic curvature invariants of $(M, g)$ which appear in the well known tube formula of H. Weyl [11], see section 2 below for precise definitions.

For $k = 1$, $H_2 = S/2$ is one half the total scalar curvature functional. Also, if the dimension $n$ of $M$ is even, then $H_n$ does not depend on the metric. It is, up to a constant, the Euler-Poincaré characteristic of $M$.

Marcel Berger proved in [II] that the gradient of $H_4$, like the gradient of $S$, depends only on the curvature tensor $R$ and does not include its covariant derivatives. The expression of the gradient he obtained was complicated and hardly generalizable to higher $H_{2k}$. So he asked the following two questions:

- Does the above phenomena remain true for all higher $H_{2k}$?
- Characterize the critical Riemannian metrics for the functional $H_{2k}$.

In this paper, we completely answer the first question, and give partial answers to the second one. The main result of this paper is the following:
Main Theorem. For every compact $n$-dimensional Riemannian manifold $(M,g)$, and for every integer $k$ such that $2 \leq 2k \leq n$, the functional $H_{2k}$ is differentiable, and at $g$ we have

$$H'_{2k}h = \frac{1}{2} \langle h_{2k}g - \frac{1}{(2k-1)!}c^{2k-1}R^k, h \rangle.$$

Where $R^k$ denotes the exterior product of the Riemann curvature tensor $R$ with itself $k$-times in the ring of curvature structures, $c$ is the contraction map and $\langle , \rangle$ is the integral scalar product.

Remark that, for $k = 1$, the main theorem shows that

$$H'_{2}h = \frac{1}{2} \langle h_{2}g - cR, h \rangle = \frac{1}{2} \langle \frac{\text{scal}}{2}g - \text{Ric}, h \rangle.$$

So that we recover the above formula about the total scalar curvature. Also, in the case $2k = n$, we have (see (10) below):

$$H'_{n}h = \frac{1}{2} \langle h_{n}g - \frac{1}{(n-1)!}c^{n-1}R^k, h \rangle = 0.$$

This is not a surprise, because $H_n$ does not depend on the metric by the Gauss-Bonnet theorem.

Let us note here that the main theorem was established earlier by David Lovelock in a not well known paper to the mathematicians [9]. His proof is based on classical tensor analysis. Our proof of the main theorem is simple and coordinate free. The key point of our proof is that it is possible to write the Gauss-Bonnet curvatures as exterior products of the metric $g$ with the Riemann curvature tensor $R$ (see the definition below for a precise formulation), then one can get the desired derivative using the power rule of differentiation and stokes’ theorem.

The main theorem shall be proved in section 4. In section 2, we recall some useful facts about the ring of curvature structures from [6]. In section 3, we show that many of the classical results of Hodge theory on differential forms can be naturally extended to the context of double forms. We consider here only those results which shall be used later in this paper. Then, we define and study some operators on double forms which will play a key role in the proof of the main theorem.

In section 5, we study the critical metrics of the functional $H_{2k}$, when restricted to a normalized conformal class of some metric, and we prove that
they are metrics with constant \((2k)\)-Gauss-Bonnet curvature. It is then natural to ask whether in each conformal class of a Riemannian metric, on a smooth compact manifold of dimension \(n > 2k\), there exists a metric with constant \((2k)\)-Gauss-Bonnet curvature. That is a natural generalization of the famous Yamabe problem.

Finally, in section 6 we examine some properties of the critical metrics of the functional \(H_{2k}\) in the space of all Riemannian metrics with volume 1. These are generalized Einstein metrics.

## 2 Preliminaries

Let \((M, g)\) be a compact smooth (oriented) Riemannian manifold of dimension \(n\). We denote by \(\Lambda^* M = \bigoplus_{p \geq 0} \Lambda^p M\) the ring of differential forms on \(M\). Considering the tensor product over the ring of smooth functions, we define \(D = \Lambda^* M \otimes \Lambda^* M = \bigoplus_{p,q \geq 0} D^{p,q}\) where \(D^{p,q} = \Lambda^p M \otimes \Lambda^q M\). It is a graded associative ring and it is called the ring of double forms on \(M\). The exterior product in \(D\), sometimes called the Kulkarni-Nomizu product, will be denoted by a dot., this shall be omitted whenever possible.

The ring of curvature structures on \(M\) is the ring \(C = \sum_{p \geq 0} C^p\) where \(C^p\) denotes symmetric elements in \(D^{p,p}\).

The standard inner product <, > on \(\Lambda^p M\) and the Hodge star operator \(*\) extend in a natural way to \(D\) (we assume here that the manifold is orientable). These were used in [6] to study several properties of this ring. In particular, the following relations are proved for all \(\omega, \omega_1, \omega_2 \in D\):

\[
g \omega = \ast c \ast \omega, \tag{1}
\]

\[
< g \omega_1, \omega_2 > = < \omega_1, c \omega_2 >. \tag{2}
\]

Where \(c\) denotes the contraction map.

Also, for all \(\omega, \theta \in D^{p,q}\), we have

\[
< \omega, \theta > = \ast (\omega, \ast \theta) = (-1)^{(p+q)(n-p-q)} \ast (\ast \omega, \theta), \tag{3}
\]

\[
\ast \ast \omega = (-1)^{(p+q)(n-p-q)} \omega. \tag{4}
\]

**Remark.** A double form \(\omega \in D^{p,q}\) can be seen as a symmetric bilinear form acting on \(p\)-vectors. Under this identification one can check easily that

\[
\ast \omega(\cdot, \cdot) = (-1)^{(p+q)(n-p-q)} \omega(\ast \cdot, \ast \cdot).
\]
The minus sign appears in fact because for a usual $p$-form $\theta$ and for the usual Hodge star operator we have $(*\theta)(\cdot) = (-1)^{p(n-p)} \theta(*\cdot)$.

In the following, we shall denote by $\omega^q$, the product of $\omega$ with itself $q$-times in the ring $\mathcal{C}$.

Let us recall the following definitions:

**Definition 2.1** [6] The $(p, q)$-curvature, denoted $s_{(p,q)}$, for $1 \leq q \leq \frac{n}{2}$ and $0 \leq p \leq n - 2q$, is the sectional curvature of the following $(p,q)$-curvature tensor

$$R_{(p,q)} = \frac{1}{(n-2q-p)!} \ast (g^{n-2q-p} R^q) \quad (5)$$

In other words, for a tangent $p$-plane $P$, $s_{(p,q)}(P)$ is the sectional curvature of the tensor $\frac{1}{(n-2q-p)!} g^{n-2q-p} R^q$ at the orthogonal complement of $P$.

Note that the tensors $R_{(p,q)}$ satisfy the first Bianchi identity and they are divergence free.

Here in this paper, we are mainly interested in the following special cases, the $(0,q)$ and $(1,q)$-curvatures:

**Definition 2.2** ([6], [7]) Let $q$ be a positive integer such that $2 \leq 2q \leq n$.

1. The $(2q)$-Gauss-Bonnet curvature, denoted $h_{2q}$, is the $(0,q)$-curvature. That is the function defined on $M$ by

$$h_{2q} = s_{(0,q)} = \frac{1}{(n-2q)!} \ast (g^{n-2q} R^q). \quad (6)$$

2. The $(2q)$-Einstein-Lovelock tensor, denoted $T_{2q}$, is defined to be the $(1,q)$-curvature tensor, that is

$$T_{2q} = \ast \frac{1}{(n-2q-1)!} g^{n-2q-1} R^q. \quad (7)$$

If $2q = n$, we set $T_n = 0$.

Recall the following properties of these curvatures [6]

$$h_{2q} = \frac{1}{(n-2q)!} \ast (g^{n-2q} R^q) = \frac{1}{(2q)!} c^{2q} R^q, \quad (8)$$

that is the complete contraction of $R^q$. Also,

$$T_{2q} = \frac{1}{(2q)!} c^{2q} R^q g - \frac{1}{(2q-1)!} c^{2q-1} R^q = h_{2q} g - \frac{1}{(2q-1)!} c^{2q-1} R^q. \quad (9)$$
For $q = 1$ we recover the usual Einstein tensor $T_2 = \frac{1}{2}c^2 R g - c R$. For $2q = n$, we have $R^q = \frac{\lambda g^n}{n!}$ for some constant $\lambda$, hence

$$h_{2q} g - \frac{c^{2q-1} R^q}{(2q - 1)!} = \frac{1}{n!} c^n (\frac{g^n}{n!}) g - \frac{1}{(n - 1)!} c^{n-1} (\frac{g^n}{n!}) = \lambda g - \lambda g = 0. \quad (10)$$

This justifies our definition for $T_n$.

Note that $c^{2q-1} R^q$ can be considered as a generalization of the Ricci curvature tensor.

**Remark.** Let us remark here some analogies between the Einstein-Lovelock tensors and the usual Einstein tensor. Recall that, the former is the main linear combination of the metric tensor $g$ and its Ricci curvature to be divergence free. The same property is true for Einstein-Lovelock tensors if we substitute the generalized Ricci curvatures $c^{2q-1} R^q$ to the usual one.

Another similarity between these tensors can be noticed at the level of their sectional curvatures: The full contraction of the Riemann curvature tensor $R$ in the directions orthogonal to a given direction $v$ produces the sectional curvature of the usual Einstein tensor, that is $T_2(v, v)$. Doing the same operation but for the Gauss-Kronecker tensors $R^q$ one generates the sectional curvatures of the Einstein-Lovelock tensors, that is $T_{2q}(v, v)$.

Finally, let us recall the following property (11) which provides another analogy:

$$\text{trace } T_{2k} = (n - 2q) h_{2q}. \quad (11)$$

### 3 The Second Bianchi Map and other Differential Operators

The second Bianchi sum, denoted $D$, maps $\mathcal{D}^{p,q}$ into $\mathcal{D}^{p+1,q}$. For $\omega \in \mathcal{D}^{p,q}$, we have

$$(D\omega)(x_1 \wedge \ldots \wedge x_{p+1}, y_1 \wedge \ldots \wedge y_q) = \sum_{j=1}^{p+1} (-1)^j \nabla_{x_j} \omega(x_1 \wedge \ldots \wedge \hat{x}_j \wedge \ldots x_{p+1}, y_1 \wedge \ldots \wedge y_q),$$

where $\nabla$ denotes the covariant differentiation with respect to the metric $g$.

If we identify double forms with vector valued differential forms, then $D$ coincides with the operator of exterior differentiation of vector valued differential forms [2]. In particular, the restriction of $D$ to $\mathcal{D}^{p,0}$ coincides with $-d$, where $d$ is the usual exterior derivative on $p$-forms. A second possible extension of $d$ is the adjoint second Bianchi sum:

$$\tilde{D} : \mathcal{D}^{p,q} \rightarrow \mathcal{D}^{p,q+1}$$
defined for $\omega \in D^{p,q}$ by

$$(\tilde{D}\omega)(x_1 \wedge \ldots \wedge x_p, y_1 \wedge \ldots \wedge y_{q+1}) = \sum_{j=1}^{q+1} (-1)^j \nabla_{y_j} \omega(x_1 \wedge \ldots \wedge x_p, y_1 \wedge \ldots \wedge \hat{y}_j \wedge \ldots \wedge y_{q+1})$$

Note that in general we have neither $D^2 = 0$ nor $\tilde{D}^2 = 0$. The composition of these two operators is the operator $D \tilde{D}$:

$$D \tilde{D} : D^{p,q} \rightarrow D^{p+1,q+1}$$

$$(D \tilde{D}\omega)(x_1 \wedge \ldots \wedge x_{p+1}, y_1 \wedge \ldots \wedge y_{q+1}) = \sum_{i=1}^{p+1} \sum_{j=1}^{q+1} (-1)^{i+j} (\nabla^2_{x_i,y_j}) (x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_{p+1}, y_1 \wedge \ldots \wedge \hat{y}_j \wedge \ldots \wedge y_{q+1}).$$

(12)

Remark that the restriction of $D \tilde{D}$ to $D^{0,0}$ is the usual Hessian operator on functions, precisely

$$D \tilde{D}(f)(x,y) = \nabla^2_{x,y} f.$$ 

Also, note that for $h \in D^{1,1}$, we have

$$D \tilde{D}h(x \wedge y, z \wedge u) = \nabla^2_{x,z} h(y, u) - \nabla^2_{x,u} h(y, z) - \nabla^2_{y,z} h(x, u) + \nabla^2_{y,u} h(x, z).$$

(13)

Similarly, one can also consider the differential operator

$$\tilde{D}D : D^{p,q} \rightarrow D^{p+1,q+1}$$

$$(\tilde{D}D\omega)(x_1 \wedge \ldots \wedge x_{p+1}, y_1 \wedge \ldots \wedge y_{q+1}) = \sum_{i=1}^{p+1} \sum_{j=1}^{q+1} (-1)^{i+j} (\nabla^2_{y_i,x_j}) (x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_{p+1}, y_1 \wedge \ldots \wedge \hat{y}_j \wedge \ldots \wedge y_{q+1}).$$

If $\omega$ is a symmetric double form, $D \tilde{D}\omega$ and $\tilde{D}D\omega$ are not necessarily symmetric. Nevertheless, it is true that

$$(\tilde{D}D\omega)(x_1 \wedge \ldots \wedge x_{p+1}, y_1 \wedge \ldots \wedge y_{q+1}) = (D \tilde{D}\omega)(y_1 \wedge \ldots \wedge y_{q+1}, x_1 \wedge \ldots \wedge x_{p+1}).$$

Consequently, the following operator is well defined and will play an important role in the proof of the main theorem

$$D \tilde{D} + \tilde{D}D : C^p \rightarrow C^{p+1}$$

This operator can be considered as a natural generalization, to the ring $C^p$, of the usual Hessian operator on functions.
On the other hand, it is easy to check that for \( \omega \in D^{p,q} \) and \( \theta \in D^{r,s} \) we have

\[
D(\omega \cdot \theta) = D\omega \cdot \theta + (-1)^p \omega \cdot D\theta,
\]

\[
\tilde{D}(\omega \cdot \theta) = \tilde{D}\omega \cdot \theta + (-1)^q \omega \cdot \tilde{D}\theta.
\]

(14)

The operator \( \delta = c\tilde{D} + \tilde{D}c \) was defined by Kulkarni [5] as a natural generalization of the classical \( \delta \) operator. Using the Hodge star operator we shall extend some classical results of Hodge theory to double forms as follows:

**Proposition 3.1** If \( * \) denotes the generalized Hodge star operator on \( \mathcal{D} \), then we have

\[
\delta \omega = (-1)^{(p+q)(n-p-q)} \ast D \ast \omega
\]

for every \((p,q)\)-form \( \omega \) such that \( p \geq 1 \).

**Proof.** Let \( \omega \in D^{p,q} \), and \((x_i), (y_j)\) orthonormal vector fields about \( m \in M \) such that \((\nabla x_i)_m = (\nabla y_j)_m = 0\). Then at \( m \) we have

\[
(D \ast \omega)(x_1 \wedge ... \wedge x_{n-p+1}, y_1 \wedge ... \wedge y_{n-q})
\]

\[
= \sum_{j=1}^{n-p+1} (-1)^j \nabla_{x_j}(\ast \omega)(x_1 \wedge ... \wedge \hat{x_j} \wedge ... \wedge x_{n-p+1}, y_1 \wedge ... \wedge y_{n-q})
\]

\[
= \sum_{j=1}^{n-p+1} (-1)^j \nabla_{x_j}(\ast \omega)(x_1 \wedge ... \wedge \hat{x_j} \wedge ... \wedge x_{n-p+1}, y_1 \wedge ... \wedge y_{n-q})\]

\[
= \sum_{j=1}^{n-p+1} \sum_{j=1}^{n-p+1} (-1)^{j+j-1+(p+q)(n-p-q)} \nabla_{x_j}(\omega(x_j \wedge \ast(x_1 \wedge ... \wedge x_{n-p+1}), \ast(y_1 \wedge ... \wedge y_{n-q}))
\]

\[
= \sum_{j=1}^{n-p+1} (-1)^{(p+q)(n-p-q)} \delta \omega(x_j \wedge \ast(x_1 \wedge ... \wedge x_{n-p+1}), \ast(y_1 \wedge ... \wedge y_{n-q}))
\]

\[
= (-1)^{(p+q)(n-p-q)} \ast \delta \omega(x_1 \wedge ... \wedge x_{n-p+1}, y_1 \wedge ... \wedge y_{n-q}).
\]

Therefore \( D \ast \omega = (-1)^{n+1} \ast \delta \omega \).

**Proposition 3.2** With respect to the integral scalar product, the operator

\[
(-1)^n + p \delta : D^{p+1,q} \rightarrow D^{p,q}
\]

is the formal adjoint of the operator \( D \).

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Proof. Let $\omega_1 \in D^{p,q}$ and $\omega_2 \in D^{p+1,q}$ then

\[ D(\omega_1 \ast \omega_2) = D\omega_1 \ast \omega_2 + (-1)^p \omega_1.D \ast \omega_2 \]

\[ = D\omega_1 \ast \omega_2 + (-1)^p (-1)^n+1 \omega_1 \ast \delta \omega_2 \]

\[ = \ast \{(D\omega_1) \ast \omega_2 \} + (-1)^{n+p+1} \ast \{\omega_1 \ast \delta \omega_2 \} \]

\[ = \ast \left( \langle D\omega_1, \omega_2 \rangle + (-1)^{n+p+1} \langle \omega_1, \delta \omega_2 \rangle \right). \]

Applying the generalized Hodge operator to both sides of the previous equation we get

\[ -\delta(\ast(\omega_1 \ast \omega_2)) = <D\omega_1, \omega_2> + (-1)^{n+p+1} <\omega_1, \delta \omega_2>. \]

Note that, the integral of the left hand side is zero by Stokes’ theorem. This completes the proof.

\par

In the same way, one can prove without difficulties that if $\delta = cD + Dc$ then for every $(p, q)$-form $\omega$ with $q \geq 1$ we have

\[ \tilde{\delta} \omega = (-1)^{(p+q)(n-p-q)} \ast \tilde{D} \ast \omega, \tag{17} \]

and also that

\[ (-1)^{n+q} \tilde{\delta} : D^{p,q+1} \rightarrow D^{p,q} \tag{18} \]

is the formal adjoint of $\tilde{D}$ with respect to the integral scalar product.

**Corollary 3.3**

1. For $p, q \geq 1$, The operators $\delta \delta$ and $\delta \tilde{\delta}$ send $D^{p,q}$ to $D^{p-1,q-1}$ and respectively satisfy

\[ \tilde{\delta} \delta = (-1)^{(p+q)(n-p-q)} \ast \tilde{D} \ast \ast \delta \]

\[ \delta \tilde{\delta} = (-1)^{(p+q)(n-p-q)} \ast D \tilde{D} \ast. \tag{19} \]

Furthermore, with respect to the integral scalar product, they are respectively the formal adjoints of the operators $(-1)^{p+q}\tilde{D}D$ and $(-1)^{p+q}D\tilde{D}$.  

2. For $p, q \geq 1$, The operator $\tilde{\delta} \delta + \delta \tilde{\delta}$ sends $D^{p,q}$ to $D^{p-1,q-1}$ and satisfies

\[ \tilde{\delta} \delta + \delta \tilde{\delta} = (-1)^{(p+q)(n-p-q)} \ast (\tilde{D} \ast D + D \ast \tilde{D}) \ast. \tag{20} \]

Furthermore, with respect to the integral scalar product, it is the formal adjoint of the operator $(-1)^{p+q}(D\tilde{D} + \tilde{D}D)$.

**Proof.** It is a direct consequence of (16) and (18).
Remark that since $Dg = 0$ and $DR = 0$, then $D(g^p R^q) = 0$. However the tensors $R_{(p,q)}$ do not in general satisfy the second Bianchi identity. Nevertheless, they are divergence free, In fact,

$$\delta(*g^p R^q) = *D(g^p R^q) = 0.$$ 

This fact is used to prove the following proposition:

**Proposition 3.4 (Schur's Theorem)** Let $p, q \geq 1$. If at every point $m \in M$ the $(p, q)$-curvature is a constant (that is on the fiber at $m$), then it is a constant.

**Proof.** Recall that at each $m \in M$ we have

$$s_{(p,q)} = \lambda \iff R_{(p,q)} = \frac{\lambda g^p}{p!},$$

where $\lambda = \lambda(m)$ is constant at $m$.

Since the tensors $R_{(p,q)}$ are divergence free, then

$$\delta \left( \frac{\lambda g^p}{p!} \right) = 0,$$

and therefore, (see [6])

$$D \left( *\lambda \frac{g^p}{p!} \right) = D \left( \lambda \frac{g^{n-p}}{(n-p)!} \right) = 0.$$ 

Consequently $d\lambda = 0$. This completes the proof. "

Another operator which also will play a central role in the proof of the main theorem is defined as follows:

For each $h \in C^1$, we define the operator $F_h : C^p \to C^p$ as follows. Let $m \in M$ and $\{e_1, ..., e_n\}$ be an orthonormal basis of $T_m M$ diagonalizing $h$, then its value on basis elements is

$$F_h \omega(e_{i_1} \wedge ... \wedge e_{i_p}, e_{j_1} \wedge ... \wedge e_{j_p}) =$$

$$\left( \sum_{k=1}^p h(e_{i_k}, e_{i_k}) + \sum_{k=1}^p h(e_{j_k}, e_{j_k}) \right) \omega(e_{i_1} \wedge ... \wedge e_{i_p}, e_{j_1} \wedge ... \wedge e_{j_p}). \quad (21)$$

It is not difficult to see that if $\omega$ satisfies the first Bianchi identity then so does $F_h(\omega)$. Below we shall prove some useful properties of this operator.
Proposition 3.5 For all $\omega \in C^p$ and $\theta \in C^q$ we have

$$F_h(\omega, \theta) = F_h(\omega) \theta + \omega F_h(\theta).$$

That is, $F_h$ acts by derivations on $C$. In particular we have

$$F_h(\omega^k) = k\omega^{k-1}F_h(\omega).$$  \hspace{1cm} (22)

Proof. Let $m \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_m M$ diagonalizing $h$. Let $\{i_1, \ldots, i_{p+q}\}$ and $\{j_1, \ldots, j_{p+q}\}$ be arbitrary subsets of $\{1, \ldots, n\}$ both with $p + q$ elements, then at $m$ we have

$$\omega F_h(\theta)(e_{i_1} \land \ldots \land e_{i_{p+q}}, e_{j_1} \land \ldots \land e_{j_{p+q}}) =$$

$$\frac{1}{(p!)^2(q!)^2} \left( \sum_{\sigma, \rho \in S_{p+q}} e(\sigma) \epsilon(\rho) \omega(e_{i_{\sigma(p+1)}} \land \ldots \land e_{i_{\sigma(p+q)}}, e_{j_{\rho(1)}} \land \ldots \land e_{j_{\rho(p+q)}}) \right)$$

$$= \frac{1}{(p!)^2(q!)^2} \left( \sum_{\sigma, \rho \in S_{p+q}} e(\sigma) \epsilon(\rho) \omega(e_{i_{\sigma(p+1)}} \land \ldots \land e_{i_{\sigma(p+q)}}, e_{j_{\rho(1)}} \land \ldots \land e_{j_{\rho(p+q)}}) \right)$$

$$= \{F_h(\omega, \theta) - F_h(\omega, \theta)\}(e_{i_1} \land \ldots \land e_{i_{p+q}}, e_{j_1} \land \ldots \land e_{j_{p+q}}).$$

This completes the proof of the proposition. \hfill \blacksquare

Proposition 3.6 The operator $F_h$ is self-adjoint, precisely for all $\omega, \theta \in C$ we have

$$\langle F_h(\omega), \theta \rangle = \langle \omega, F_h(\theta) \rangle.$$  \hspace{1cm} (23)

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis diagonalizing $h$ at $m \in M$, let $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_p\}$ be arbitrary subsets of $\{1, \ldots, n\}$ both with $p$ elements, then at $m$ we have

$$F_h(\omega)(e_{i_1} \land \ldots \land e_{i_p}, e_{j_1} \land \ldots \land e_{j_p}) \theta(e_{i_1} \land \ldots \land e_{i_p}, e_{j_1} \land \ldots \land e_{j_p})$$

$$= \left( \sum_{k=1}^p h(e_{i_k}, e_{i_k}) + h(e_{j_k}, e_{j_k}) \right) \omega(e_{i_1} \land \ldots \land e_{i_p}, e_{j_1} \land \ldots \land e_{j_p})$$

$$\theta(e_{i_1} \land \ldots \land e_{i_p}, e_{j_1} \land \ldots \land e_{j_p})$$

$$= \omega(e_{i_1} \land \ldots \land e_{i_p}, e_{i_1} \land \ldots \land e_{j_p}) F_h(\theta)(e_{i_1} \land \ldots \land e_{i_p}, e_{j_1} \land \ldots \land e_{j_p}).$$
The proposition results immediately after taking the corresponding sums.

The following properties can be checked without difficulties:

**Proposition 3.7** The following are true about the operators $F_h$:

1. If $\omega \in C^p$ then $c^p(F_h(\omega)) = 2p(c^{p-1}\omega, h)$.
2. If $\omega \in C^p$ then $F_g(\omega) = 2p\omega$.
3. If $p \geq 1$, then $F_h(g^p) = 2pg^{p-1}h$.
4. If $h, k \in C^1$ and $\bar{h}, \bar{k}, F_h(k)$ denote respectively the associated linear operators on the tangent space, then
   
   $$F_h(k) = \bar{h}k + k\bar{h}.$$ 

5. If $\omega \in C^2$ and $h \in C^1$, then for all tangent vectors $x, y, z, u$ we have:

   $$F_h(\omega)(x \wedge y, z \wedge u) = h(\omega(x, y)z, u) - h(\omega(x, y)u, z) + h(\omega(z, u)z, y) - h(\omega(z, u)y, x),$$

   where in the right hand side, $\omega$ was considered as a $(3, 1)$-tensor by the mean of the metric $g$.

6. If $\omega \in C^n$, then $F_h(\omega) = 2(tr_g h)\omega$.

**Remark.** The operator $F_h$ can also alternatively be defined by declaring that it acts by derivations on $\mathcal{C}$ and that its restriction to $C^1$ is the symmetric multiplication by $h$ as in the property 4 of the previous proposition.

### 4 Proof of the Main Theorem

The proof is based on the following two lemmas. The first lemma asserts that the directional derivative of the Riemann curvature tensor when considered as a symmetric double form is the sum of an "exact" double form and a term which depends linearly on the curvature:

**Lemma 4.1** The derivative of the Riemann curvature structure $R \in \mathcal{C}^2$ in the direction of $h \in \mathcal{C}^1$ is given by

$$R'h = -\frac{1}{4}(\mathcal{D}\bar{D} + \bar{D}\mathcal{D})(h) + \frac{1}{4}F_h(R).$$  \hspace{1cm} (24)
Proof. Recall that, the directional derivatives $\nabla' h$, $R'h$ at $g$ of the Levi-Civita connection and the $(3,1)$-Riemann curvature tensor are respectively given by [2]

$$g(\nabla' h(x, y), z) = \frac{1}{2}\{\nabla_x h(y, z) + \nabla_y h(x, z) - \nabla_z h(x, y)\},$$

$$R'h(x, y)z = (\nabla_y \nabla' h)(x, z) - (\nabla_x \nabla'h)(y, z).$$

Therefore the derivative of $R$, seen as a double form $R \in \Lambda^2M \otimes \Lambda^2M$, in the direction of $h \in C^1$ is given by

$$R'h(x \wedge y, z \wedge u) = h(R(x, y)z, u) + g(R'h(x, y)z, u)$$

$$= h(\nabla_x h(y, z) + \nabla_y h(x, z) - \nabla_z h(x, y), u)$$

$$= \frac{1}{2}\{\nabla_{yx} h(z, u) + \nabla_{yx} h(x, u) - \nabla_{yu} h(x, z)$$

$$- \nabla_{zy} h(z, u) - \nabla_{zy} h(y, u) + \nabla_{zy} h(y, z)\} + h(R(x, y)z, u)$$

$$= \frac{1}{2}\{\nabla_{yx} h(x, u) + \nabla_{yx} h(y, z) - \nabla_{xz} h(y, u) - \nabla_{yu} h(x, z)$$

$$+ h(R(x, y)z, u) - h(R(x, y)u, z)\}.$$

Where, in the last step, we have used the following identity

$$(\nabla_{x}^2 h - \nabla_{yx}^2 h)(z, u) = h(R(x, y)u, z) + h(R(x, y)z, u).$$

Now using [13], we get

$$R'h(x \wedge y, z \wedge u) = \frac{1}{2}\left\{-D\tilde{D}h(x, y, z, u) + h(R(x, y)z, u) - h(R(x, y)u, z)\right\}.$$ 

To get the derivative of $R$, as a symmetric curvature structure, i.e. in $C^2$, it suffices to take the projection of the previous one, that is

$$R'h(x \wedge y, z \wedge u) = \frac{1}{2}\left\{R'h(x \wedge y, z \wedge u) + R'h(z \wedge u, x \wedge y)\right\}$$

$$= \frac{1}{4}\left\{-D\tilde{D}h(x \wedge y, z \wedge u) - D\tilde{D}h(z \wedge u, x \wedge y) + h(R(x, y)z, u)$$

$$- h(R(x, y)u, z) + h(R(z, u)x, y) - h(R(u, z)x, y)\right\}.$$ 

This completes the proof of the lemma.
Lemma 4.2 Let \((M, g)\) be a Riemannian manifold and \(h \in C^1\), then the differential of \(h_{2k}\) at \(g\), in the direction of \(h\), is given by

\[
h'_{2k}h = \frac{1}{2} < \frac{\epsilon^{2k-1}}{(2k-1)!} R^k, h > - \frac{k}{4} (\delta \tilde{\delta} + \tilde{\delta} \delta) \left( \ast \left( \frac{g^{n-2k}}{(n-2k)!} R^{k-1}h \right) \right).
\]

Proof. Recall that

\[
h_{2k} = \ast \left( \frac{1}{(n-2k)!} g^{n-2k} R^k \right) = \frac{1}{(n-2k)!} g^{n-2k} R^k (\mu_g, \mu_g),
\]

where \(\mu_g\) is considered in the previous formula as an \(n\)-vector. Then using the previous lemma, we have

\[
h'_{2k}h = \frac{1}{(n-2k-1)!} g^{n-2k-1} h R^k (\mu_g, \mu_g) + \frac{1}{(n-2k)!} g^{n-2k} R^{k-1} R' h (\mu_g, \mu_g)
- 2 \frac{1}{(n-2k)!} g^{n-2k} R^k \left( \frac{1}{2} \text{tr}_g h \right) (\mu_g, \mu_g)
= \ast \left( \frac{1}{(n-2k-1)!} g^{n-2k-1} h R^k \right) + \ast \left( \frac{1}{(n-2k)!} g^{n-2k} R^{k-1} R' h \right)
- \ast \left( \frac{1}{(n-2k)!} g^{n-2k} R^k \right) \text{tr}_g h
= < T_{2k} - h_{2k} g, h > + \ast \left\{ \frac{g^{n-2k} R^{k-1}}{4(n-2k)!} (-D \tilde{D} h - \tilde{D} D h + F_h(R)) \right\}.
\]

Using first (14) and then (20), we have

\[
\ast \left( \frac{g^{n-2k} R^{k-1}}{4(n-2k)!} (-D \tilde{D} h - \tilde{D} D h) \right) = \ast (D \tilde{D} + \tilde{D} D) \left( \frac{-k g^{n-2k} R^{k-1} h}{4(n-2k)!} \right)
= - \frac{k}{4} \left\{ \delta \tilde{\delta} + \tilde{\delta} \delta \right\} \ast \left( \frac{g^{n-2k}}{(n-2k)!} R^{k-1} h \right).
\]

On the other hand, using simultaneously formulas (22), (3), (23), proposition 3.7 and formula (2), we get

\[
\ast \left( \frac{1}{4(n-2k)!} g^{n-2k} R^{k-1} F_h(R) \right)
= \ast \left( \frac{1}{4(n-2k)!} g^{n-2k} F_h(R) \right)
= \frac{1}{4} < F_h(R), \ast \left( \frac{1}{(n-2k)!} g^{n-2k} \right) >= \frac{1}{4} < F_h(R), \frac{1}{(2k)!} g^{2k} >
= \frac{1}{4} < R^k, F_h(\frac{1}{(2k)!} g^{2k}) >= \frac{1}{4} < R^k, \frac{2}{(2k-1)!} g^{2k-1} h >
= \frac{1}{2} \ast \left( \frac{1}{(2k-1)!} g^{2k-1} R^k, h \right).
\]
Recall that $T_{2k} = h_{2k}g - \frac{1}{(2k-1)!}c^{2k-1}R^k$, then finally we have

$$h_{2k}'h = -\frac{1}{2(2k-1)!}c^{2k-1}R^k, h > + \frac{k}{4} (\delta \delta + \tilde{\delta} \tilde{\delta}) \left( * \left( \frac{g^{n-2k}}{(n-2k)!} \right) R^{k-1}h \right).$$

The proof of the lemma is now complete. ■

**Remarks.**

1. In contrast with the volume form, the derivative of $\mu_g$ seen as an $n$-vector is $-\frac{1}{2} \text{tr}_g h \mu_g$ and not just $\frac{1}{2} \text{tr}_g h \mu_g$. This fact was used in the previous proof.

2. The previous two lemmas are of local nature.

We are now ready to prove the main theorem. Note that for $k \in C^1$, both $\delta \tilde{\delta} k$ and $\tilde{\delta} \delta k$ are divergences of some differential 1-form. Therefor their integral is zero by Stokes’ theorem. This applies particularly to $k = * \left( \frac{g^{n-2k}}{(n-2k)!} R^{k-1}h \right) \in C^1$ of the previous formula, so that,

$$H'_{2k}h = \int_M \left( h_{2k}'h + \frac{h_{2k}}{2} \text{tr}_g h \right) \mu_g$$

$$= -\frac{1}{2} < \frac{c^{2k-1}}{(2k-1)!} R^k, h > + \frac{h_{2k}}{2} < g, h >$$

$$= \frac{1}{2} < h_{2k}g - \frac{c^{2k-1}}{(2k-1)!} R^k, h >$$

$$= \frac{1}{2} < T_{2k}, h >,$$

This completes the proof of the main theorem. ■

## 5 A Generalized Yamabe Problem

As a consequence of the main theorem, we have the following result

**Proposition 5.1** For a compact Riemannian manifold $(M, g)$ with dimension $n > 2k$, the $(2k)$-Gauss-Bonnet curvature $h_{2k}$ is constant if and only if $g$ is a critical point for the functional $H_{2k}$ when restricted to the set $\text{Conf}_0(g)$ of metrics pointwise conformal to $g$ and having the same total volume.
**Proof.** We proceed as in the case of the scalar curvature \[2\]. Note that \(g\) is a critical point of \(H_{2k}\) when restricted to Conf\(_0\)(\(g\)) if and only if at \(g\) we have

\[
H'_{2k}.fg = 0,
\]
for all smooth functions \(f\) such that \(\int_M f \mu_g = 0\).

Next using the main theorem, we have

\[
H'_{2k}.fg = < h_{2k}g - \frac{c^{2k-1} R^k}{(2k-1)!}, fg >
\]
\[
= nf h_{2k} - f \frac{c^{2k} R^k}{(2k-1)!}
\]
\[
= (n - 2k) f h_{2k}.
\]

Consequently, for \(\int_M f \mu_g = 0\), \(H'_{2k}.fg = 0\) if and only if the function \(f\) is orthogonal to \(h_{2k}\).

Finally, consider the function

\[
f = h_{2k} - \frac{1}{\text{vol}(g)} \int_M h_{2k} \mu_g,
\]

it is orthogonal to \(h_{2k}\) and to the constants, then it is the zero function. This completes the proof.

It is then natural to ask whether for each \(k\) we have:

**Question:** (Generalized Yamabe problem.)

*In each conformal class of a fixed Riemannian metric on a smooth compact manifold with dimension \(n > 2k\) there exists a metric with \(h_{2k}\) constant.*

A closely related and at the same time parallel problem to the previous one is the \(\sigma_k\)-Yamabe problem. It is at present the subject of intensive researches. We can state it as follows:

*Let \(\sigma_k\) denote the symmetric function of order \(k\) in the eigenvalues of the Schouten tensor. For each \(k\), there exists in each conformal class of a fixed Riemannian metric on a smooth compact manifold a metric with \(\sigma_k\) constant.*

These two problems coincide in the class of a conformally flat metric and when \(k\) is even. In fact, in this case the curvatures \(h_{2k}\) and \(\sigma_{2k}\) differ only by a constant factor, \[6\]. Remark also, that both problems generalize the classical Yamabe problem obtained for \(k = 1\) and \(k = 2\) respectively.
The $\sigma_k$-Yamabe problem was recently proved for $k > n/2$ by Gursky and Viaclovsky [4] with the assumption that the original metric is “admissible”. Then Sheng, Trudinger and Wang [10] completed the proof of the remaining cases where $2 \leq k \leq n/2$ after assuming that the relevant equation is variational.

Note also that the $\sigma_k$-Yamabe problem was solved in the conformally flat case by Li-Li [8], and Guan-Wang [3].

Remark. The curvatures $h_{2k}$ are in general different from the symmetric functions in the eigenvalues of the Riemann curvature operator $R$. In fact, they are not even in general symmetric polynomials in the eigenvalues of $R$ as one can check it easily. In fact, for a 4-dimensional Riemannian manifold, suppose the eigenvectors of $R$ are all of rank 1 (decomposed) and let $\lambda_1, ..., \lambda_6$ denote the eigenvalues of $R$. Let us re-arrange so that $\lambda_1, \lambda_2$ (resp. $\lambda_3, \lambda_4$ and $\lambda_5, \lambda_6$) are two eigenvalues corresponding to supplementary 2-planes (eigenvectors), then $h_4$, up to a constant, equals $\lambda_1 \lambda_2 + \lambda_3 \lambda_4 + \lambda_5 \lambda_6$. This is clearly not a symmetric polynomial in $\lambda_1, ..., \lambda_6$.

6 Generalized Einstein Manifolds

The usual Einstein metrics are the critical metrics for the total scalar curvature functional when restricted to those metrics with unit volume. Similarly, considering the critical metrics of the functional associated to the Gauss-Bonnet curvatures we obtain generalized Einstein metrics. Precisely:

**Definition 6.1** For $2 \leq 2k \leq n$, we say that $(M, g)$ is $(2k)$-Einstein if its $(2k)$-Einstein-Lovelock tensor is proportional to the metric, that is

$$T_{2k} = \lambda g.$$

Note that since the tensors $T_{2k}$ are divergence free, the function $\lambda$ is then a constant.

Also, the 2-Einstein manifolds are the usual Einstein manifolds, and when $n = 2k$, we have $T_{2k} = 0$ for any metric on $M$.

The class of $(2k)$-Einstein Riemannian manifolds contains the manifolds with constant sectional curvature, and all isotropy irreducible homogeneous manifolds with their canonical Riemannian metrics.

**Examples.**
Let $M$ be an arbitrary Riemannian manifold with dimension $n \geq 3$, the Riemannian product $M \times \mathbb{R}^q$ is with $T_{2k} = 0$ for $2k \geq n$ but $T_2$ is not proportional to the metric. This example shows that in some sens the generalized Einstein condition is weaker then the usual one. But this is not always true, as shown by the next example:

- Let $M$ be a 4-dimensional Ricci-flat but not flat manifold (for example a K3 surface endowed with the Calabi-Yau metric), and $T^q$ be a flat torus. Then the Riemannian product $M \times T^q$ is Einstein in the usual sens ($T_2 = 0$), but it is not 4-Einstein.

- Let $(M, g)$ be a conformally flat manifold, then it can be shown without difficulties that if $(M, g)$ is $(2k)$-Einstein then the Ricci tensor has at each point at most $k$ distinct eigenvalues. Similar results hold for hypersurfaces of the Euclidean space.

With respect to the orthogonal decomposition into irreducible compo-
nents of the ring of curvature structures [5 [6], the Gauss-Kronecker tensor $R^q$ splits to

$$R^q = \omega_{2q} + g\omega_{2q-1} + ... + g^{2q-1}\omega_1 + g^{2q}\omega_0. \quad (25)$$

Evidently, the tensor $T_{2k}$ is proportional to the metric if and only if the generalized Ricci tensor $c^{2k-1}R^k$ is also proportional to the metric. Therefore it results immediately from lemma 5.7 of [6] the following proposition.

**Proposition 6.2** A Riemannian metric is $(2q)$-Einstein if and only if the component $\omega_1$ of $R^q$ with respect to the irreducible splitting (25) vanishes.

The previous result generalizes a similar well known result about the usual Einstein metrics.

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**References**

[1] Berger, M. *Quelques formules de variation pour une structure riemann-
nienne*, Ann. Scient. Ec. Norm. Sup., série 4, t. 3, 1970, 285-294.

[2] Besse, A. L. *Einstein manifolds*, Springer Verlag.
[3] Guan, P., Wang, G. *A fully nonlinear conformal flow on locally conformally flat manifolds*, J. Reine Angew. Math. 557, 219-238 (2003).

[4] Gursky, M., Viaclovsky, J. *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*, preprint. [arXiv:math.DG/0409187](http://arxiv.org/abs/math.DG/0409187).

[5] Kulkarni, R. S. *On Bianchi Identities* Math. Ann. 199, 175-204 (1972).

[6] Labbi, M. L. *Double forms, curvature structures and the \((p,q)\)-curvatures*, Transactions of the American Mathematical Society, vol. 357, n. 10, 3971-3992 (2005).

[7] Labbi, M. L. *On compact manifolds with positive second Gauss-Bonnet curvature*, Pacific Journal of Math., vol. 227, No. 2, (2006).

[8] Li, A., Li, Y. Y. *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math. 56, 1416-1464 (2003).

[9] Lovelock, D. *The Einstein tensor and its generalizations*, Journal of Mathematical Physics, vol. 12, n. 3, (1971).

[10] Sheng, W., Trudinger, N. S., Wang, X.-J. *The Yamabe problem for higher order curvatures*, preprint. [arXiv:math.DG/0505463](http://arxiv.org/abs/math.DG/0505463).

[11] Weyl, H. *On the volume of tubes*, Amer. J. Math., vol. 61, 461-472 (1939).

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