A note on the classical lower bound for a quantum walk algorithm

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Abstract

A recent paper on quantum walks by Childs et al. [CCD+03] provides an example of a black-box problem for which there is a quantum algorithm with exponential speedup over the best classical randomized algorithm for the problem, but where the quantum algorithm does not involve any use of the quantum Fourier transform. They give an exponential lower bound for a classical randomized algorithm solving the black-box graph traversal problem defined in their paper. In this note we give an improved lower bound for this problem via a straightforward and more complete analysis.

1 Introduction

Almost all the quantum algorithms with exponential speedup over their best known classical counterparts use some type of quantum Fourier transform—even for problems (such as shifted quadratic characters) which are not obvious instances of the Hidden Subgroup problem. However, a recent paper on quantum walks [CCD+03] does not fall into this category. They demonstrate that exponential speedup can be achieved by a different algorithmic technique, the quantum walk. They give a polynomial-time quantum algorithm to solve a black-box graph traversal problem, based on a continuous time quantum walk. Their paper contains three major results: the continuous time quantum walk algorithm itself, its implementation by a circuit with (discrete) quantum gates, and an exponential lower bound for solving the problem classically. We refer to their paper for a detailed description. Here we concentrate on improving their classical lower bound.

They showed that any classical algorithm solving the problem of traversing $G'_n$ (see Figure 2 in [CCD+03]) requires exponential time:

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Theorem 1.1 ([CCD+03, Theorem 6]) Any classical algorithm that makes at most $2^{n/6}$ queries to the oracle finds the EXIT with probability at most $4 \cdot 2^{-n/6}$.

In this note we use a more straightforward approach to analyze this problem. Our analysis also considers some cases Childs et al. neglected. Furthermore, we obtain an improved lower bound, as stated in the following theorem.

Theorem 1.2 Any classical algorithm that makes at most $2^{n/3}$ queries to the oracle finds the EXIT with probability at most $O(n \cdot 2^{-n/3})$.

2 An improved analysis on the classical lower bound

In this section we generally refer to the set-up in Childs et al. [CCD+03]. Our improvement essentially comes from the calculation of the probability that a classical randomized algorithm wins Game 4. We do not need to consider subtrees of $G'_n$ of height $n/2$.

Lemma 2.1 (cf. [CCD+03, Lemma 10]) For any rooted tree $T$ of at most $2^{n/3}$ vertices,

$$\max_T E_G[P^G(T)] = O(n \cdot 2^{-n/3}).$$

Proof. Let $T$ be a tree with $t$ vertices, $t \leq 2^{n/3}$, with image $\pi(T)$ in $G'_n$ under the random embedding $\pi$. For any nonroot node $u \in T$, let $p(u) \in T$ be the parent of $u$.

We assume that in $G'_n$, the ENTRANCE is at level 0 and the EXIT is at level $2n + 1$, and the middle layer is between levels $n$ and $n + 1$. Thus both binary trees have height $n$. To reach the EXIT from the column $n + 1$, $\pi$ has to move right $n$ times in a row, which has probability $2^{-n}$. Since there are at most $t$ tries on each path of $T$, and there are at most $t$ such paths, the probability of finding the EXIT $\Pr[T \text{ exits on } G'_n]$ is bounded by $t^2 \cdot 2^{-n}$.

Now assume that $\pi(T)$ does not exit. It is easy to see that

$$\Pr[\pi \text{ is improper}] = \sum_{a,b \in T} \Pr[\pi(a) = \pi(b) \& \pi(p(a)) \neq \pi(p(b))]$$

(1)

$$= \sum_{a,b \in T} \sum_{u \in G'_n} \Pr[\pi(a) = \pi(b) = u \& \pi(p(a)) \neq \pi(p(b))].$$

(2)

We calculate $\Pr[\pi(a) = \pi(b) = u]$ first for fixed $a, b \in T$, and $u \in G'_n$. Let the height of $u$ be the distance $h$ from $u$ to a leaf of the subtree that $u$ is in. That is, if $u$ is at level $\ell \leq n$, then $h = n - \ell$, and if $u$ is at level $\ell \geq n + 1$, then $h = \ell - n - 1$.

Look at the two paths $p_a$ and $p_b$ in $T$ from the root to $a$ and to $b$, respectively. Either both paths go through the middle layer of $G'_n$ or only one path goes through the middle layer (at least one path must go through the middle layer). Assume first that both paths go through the middle layer.

Assume WLOG that $p_b$ goes through the middle layer at least as many times as $p_a$. From the last time $p_a$ goes through the middle layer, in order to reach $u$ it must first reach some
(possibly improper) ancestor $v$ of $u$. If $v$ has height $h'$, then the probability of $p_a$ getting to $v$ from just before the last middle layer is $2^{-h'} \cdot 2^{h'}/(2^n - t) = 1/(2^n - t)$. Once reaching $v$, the probability of then going to $u$ is then $2^{h-h'}$. So the probability of $p_a$ reaching $u$ is at most
\[ \sum_{h'=h}^{n} \frac{2^{h-h'}}{2^n - t} < \frac{2}{2^n - t}. \]

By our WLOG assumption, there is a final subpath $p'_b$ of $p_b$ not on $p_a$ that goes through the middle layer before reaching $u$. By the same analysis, we get that $p'_b$ reaches $u$ with probability less than $2/(2^n - t)$. The probabilities of both paths reaching $u$ is thus less than $4/(2^n - t)^2$. (The two paths $p_a$ and $p'_b$ are almost independent, but not quite; the dependence only decreases the true probability from the product above, however. With a finer analysis, one can reduce the probability to $3/[2(2^n - t)^2]. )$

Now assume that only one path, say $p_b$ goes through the middle layer, and that $\pi(a) = \pi(b) = u$. Then $a$ is an ancestor of $b$ with depth $\leq n$ in $T$, and $\pi(a)$ is the leftmost point of the cycle, lying in the left half of $G'_n$. The probability that $p_a$ ends with $u$ is thus at most $2^{h-n}$, where $h$ is the height of $u$. Let $p'_b$ be defined as before. Then by an analysis similar to that above, $p'_b$ ends in $u$ with probability $1/(2^n - t)$ (note that $p'_b$ approaches $u$ from the right, i.e., it cannot go through a proper ancestor of $u$ first). By (almost) independence, we then have in this case,

\[ \Pr[\pi(a) = \pi(b) & \pi(p(a)) \neq \pi(p(b))] \leq \frac{2^{h-n}}{2^n - t}. \]

Summing this probability over $u$ in the left half of $G'_n$, we get
\[ \sum_u \frac{2^{h-n}}{2^n - t} = \sum_{h=0}^{n} \frac{2^{h-n}}{2^n - t} = \frac{n+1}{2^n - t}. \]

We can now easily compute the sum [1]. We get that $\pi$ is improper with probability less than
\[ \sum_{a,b \in T} \left( 2^{n+2} \cdot \frac{4}{(2^n - t)^2} + \frac{n+1}{2^n - t} \right) \leq \frac{t^2}{2^n - t} \left( \frac{2^{n+2}}{2^n - t} + n+1 \right) = O(n \cdot 2^{n/3}) \]
if $t \leq 2^{n/3}$.

\[ \square \]

References

[CCD+03] A. M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, and D. A. Spielman. Exponential algorithmic speedup by a quantum walk. In Proceedings of the 35th ACM Symposium on the Theory of Computing. ACM, 2003, [quant-ph/0209131].