Abstract

Given a triangle $\Delta$, we study the problem of determining the smallest enclosing and largest embedded isosceles triangles of $\Delta$ with respect to area and perimeter. This problem was initially posed by Nandakumar [16,18] and was first studied by Kiss, Pach, and Somlai [12], who showed that if $\Delta'$ is the smallest area isosceles triangle containing $\Delta$, then $\Delta'$ and $\Delta$ share a side and an angle. In the present paper, we prove that for any triangle $\Delta$, every maximum area isosceles triangle embedded in $\Delta$ and every maximum perimeter isosceles triangle embedded in $\Delta$ shares a side and an angle with $\Delta$. Somewhat surprisingly, the case of minimum perimeter enclosing triangles is different: there are infinite families of triangles $\Delta$ whose minimum perimeter isosceles containers do not share a side and an angle with $\Delta$.

1 Introduction

The following classical problem is the starting point of our investigation. Given two convex bodies, $C$ and $C'$ in $\mathbb{R}^d$, decide whether $C$ can be moved into a position where it covers $C'$. One can easily list some necessary conditions, for instance, the volume, the surface area and the diameter of $C$ has to be at least as large as the one of $C'$. However, solving the decision problem can be rather challenging, even in $\mathbb{R}^2$, or for special cases that might seem friendly at first sight.

For instance, consider the setup, where $C'$ is the ‘shadow’ of $C$, that is, $C$ is embedded into $\mathbb{R}^3$ and $C'$ is the orthogonal projection of $C$ onto a 2-dimensional affine subspace. The necessary conditions are clearly satisfied and it looks plausible that there is always a congruent copy of $C$ which covers $C'$. However, the proof of this fact is far from straightforward [4,13], and rather surprisingly, the result does not generalize to higher dimensions: for $d \geq 3$, no convex $d$-polytope embedded in $\mathbb{R}^{d+1}$ can cover all of its shadows [4].

Another special case is where both convex bodies are triangles in $\mathbb{R}^2$: given two triangles $\Delta$ and $\Delta'$, find an efficient way to decide whether $\Delta$ can be brought into a position where it covers $\Delta'$. This is a classical problem posed by Steinhaus [25] in 1964 and an algorithmic solution was proposed only 29 years later by Post [21], who described a set of 18 polynomial inequalities of degree 4 such that a copy of $\Delta$ can cover $\Delta'$ if and only if at least one of these inequalities are satisfied. The key geometric component of Post’s solution is the following.

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Lemma 1.1 (Post [21]). If a triangle $\triangle$ can be moved to a position where it covers another triangle $\triangle'$, then one can also find a covering position of $\triangle$ with a side that contains one side of $\triangle'$.

Results of this kind help us to reduce the number of configurations to consider, and are of both theoretical and practical interest.

1.1 Optimal covers from a class

In the present paper, we study a variant of the covering problem where the body $C$ (or $C'$) is not fixed, but can be chosen from a family of possible objects and we want to find a solution which is in some sense optimal, for example, has minimum area or perimeter.

Several classical problems in geometry can be viewed as covering problems of this kind: finding an optimal enclosing triangle, polygon, or ellipse (Löwner-John ellipse) for a given input set [2, 3, 5–7, 9, 10, 22, 23] as well as their higher dimensional analogues (that is, simplices, polytopes, ellipsoids, [11, 19, 27]). Apart from their theoretical interest, these problems have found applications in various areas of computer science and mathematics (optimization, packing and covering, approximation algorithms, convexity, computational geometry), see [8, 14, 24]. In the past decade, several explicit algorithms were proposed for the case of triangles [2, 15, 20, 26].

Nandakumar [16–18] raised the following two special instances of the above question: given a triangle $\triangle$, determine the minimum area and the minimum perimeter isosceles triangles that contain $\triangle$. In what follows, we answer these questions, together with their ‘dual’ versions: given a triangle $\triangle$, determine the maximum area and the maximum perimeter isosceles triangles embedded (i.e., contained) in $\triangle$.

The case of minimum area isosceles containers has been recently studied by Kiss, Pach, and Somlai [12]: they described all isosceles containers of a given triangle $\Delta$ for which the minimum is attained. Here, we complete the picture: we characterize the optimal solutions of the other three problems stated above. We will conclude that for three of the above problems, the optimum is attained for a ‘trivial’ configuration, where the two triangles share a side and an angle at one end of this side.

Theorem 1.2. Let $\Delta$ be a triangle in $\mathbb{R}^2$ and

(i) let $\Delta'' \supseteq \Delta$ be a minimum area isosceles container of $\Delta$. Then $\Delta''$ and $\Delta$ have a side in common and at one endpoint of this side they also have the same angle [12];

(ii) let $\Delta' \subseteq \Delta$ be a maximum area embedded isosceles triangle in $\Delta$. Then $\Delta'$ and $\Delta$ have a side in common and at one endpoint of this side they also have the same angle;

(iii) let $\Delta' \subseteq \Delta$ be a maximum perimeter embedded isosceles triangle in $\Delta$. Then $\Delta'$ and $\Delta$ have a side in common and at one endpoint of this side they also have the same angle.

Somewhat surprisingly, the analogous statement is false for minimum perimeter containers.

Theorem 1.3. There are infinite families of triangles $\Delta$ such that none of their minimum perimeter isosceles containers shares a side with $\Delta$ (and an angle at the end of this side).

We describe 5 different types of isosceles containers such that any triangle $\Delta$ has a minimum perimeter isosceles container $\Delta'$ belonging to one of these types. Only 3 out of these types will have the property that $\Delta$ and $\Delta'$ share a side and at one of the endpoints of this side they also have the same angle.
Our paper is organized as follows. In Section 2, we fix the notation and list some easy preliminary statements. In Section 3 and Section 4, we present the proofs of Theorem 1.2(ii) and Theorem 1.2(iii), respectively. Finally, Section 5 is dedicated to the description of the 5 types of isosceles containers mentioned in Theorem 1.3 and to the proof of this result.

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2 Preliminaries and notation

In this section, we introduce the notation used in this note and state three easy lemmas (Lemmas 2.1-2.3 and 2.5). Their straightforward proofs are given in the Appendix.

Lemma 2.1. Let $\Delta_1$ and $\Delta_2$ be two triangles.

(i) Any maximum area (resp. perimeter) similar copy $\Delta'_1 \subseteq \Delta_2$ of $\Delta_1$ satisfies the following properties:
   (a) there is a side of $\Delta_2$ that contains a side of $\Delta'_1$;
   (b) every side of $\Delta_2$ contains a vertex of $\Delta'_1$;
   (c) $\Delta'_1$ and $\Delta_2$ have a common vertex.

(ii) Any minimum area (resp. perimeter) similar copy $\Delta'_1 \supseteq \Delta_2$ of $\Delta_1$ satisfies the following properties:
   (a) there is a side of $\Delta'_1$ that contains two vertices of $\Delta_2$;
   (b) every side of $\Delta'_1$ contains a vertex of $\Delta_2$;
   (c) $\Delta'_1$ and $\Delta_2$ have a common vertex.

Optimal isosceles enclosing and embedded triangles satisfy further properties.

Lemma 2.2.

(i) For every triangle $\Delta$ there exists a minimum area (resp. perimeter) isosceles container of $\Delta$, and a maximum area (resp. perimeter) isosceles triangle embedded in $\Delta$.

(ii) If $\Delta_1$ is a maximum area (resp. perimeter) isosceles triangle embedded in $\Delta$, then every vertex of $\Delta_1$ lies on a side of $\Delta$.

(iii) If $\Delta_2$ is a minimum area (resp. perimeter) isosceles container of $\Delta$, then every vertex of $\Delta$ lies on a side of $\Delta_2$.

For any two points, $A$ and $B$, let $AB$ denote the closed segment connecting them, and let $|AB|$ stand for the length of $AB$. To unify the presentation, in the sequel we fix a triangle $ABC$ with side lengths $a = |BC|$, $b = |AC|$, $c = |AB|$. If two sides are of the same length, then $ABC$ is the unique minimum area and perimeter isosceles container (and also maximum area and perimeter embedded isosceles triangle) of itself. Therefore without loss of generality, we assume that $a < b < c$.

2.1 Special embedded isosceles triangles

Given a triangle $ABC$, we describe its special embedded isosceles triangles, that is, all those isosceles triangles contained in $ABC$ that have a common side with $ABC$ and share an angle at one of the endpoints of the common side. Recall that these triangles play a distinguished role in Theorem 1.2.
Special embedded triangles of the first kind. Let \( A' \) be a point of \( AC \) with \( |A'C| = |BC| \) and let \( B' \) and \( A'' \) be two points of \( AB \) such that \( |AB'| = |AC| \) and \( |A''B| = |BC| \) (see Figure 1). We say that \( A'BC, AB'C, \) and \( A''BC \) are the special embedded triangles of the first kind associated with \( ABC \).

Special embedded triangles of the second kind. Let \( C_1 \) be the intersection of the perpendicular bisector of \( AB \) and the segment \( AC \). Analogously, let \( A_1 \) be the intersection of the perpendicular bisector of \( BC \) and \( AC \), and let \( B_1 \) be the intersection of the perpendicular bisector of \( BC \) and the line \( AC \) (see Figure 2). The triangles \( A_1BC, AB_1C, \) and \( ABC_1 \) are the special embedded triangles of the second kind associated with \( ABC \).

Special embedded triangles of the third kind. Let \( \overline{A} \) be a point of \( AB \), where \( |\overline{A}C| = |BC| \). Analogously, let \( \overline{A} \in AC \), and \( \overline{B} \in BC \) such that \( |\overline{AB}| = |BC| \), and \( |\overline{BA}| = |AC| \) (see Figure 3). Note that if \( ABC \) is non-acute, then \( \overline{ABC} \) and \( ABC \) do not exist. \( \overline{ABC}, \overline{ABC}, \) and \( ABC \) are called the special embedded triangles of the third kind associated with \( ABC \).

Figure 1: Special embedded triangles of the first kind.
Figure 2: Special embedded triangles of the second kind.
Figure 3: Special embedded triangles of the third kind.
2.1.1 Basic inequalities for special embedded triangles.

We collect a few inequalities on the area and perimeter of special isosceles embedded triangles. For a triangle \( \Delta \), let \( \text{per}(\Delta) \) and \( \text{area}(\Delta) \) denote the perimeter and the area of \( \Delta \), respectively.

Lemma 2.3. If \( ABC \) satisfies \( a < b < c \), then

(i) \( \text{area}(A''BC) < \text{area}(A'BC) \);
(ii) \( \text{area}(A_1BC) < \text{area}(AB'C) \) and \( \text{area}(AB_1C) < \text{area}(ABC_1) \);
(iii) \( \text{area}(\overline{ABC}) < \text{area}(ABC_1) \), \( \text{area}(\overline{A_2BC}) < \text{area}(\overline{ABC}) \),
and \( \text{area}(A\overline{BC}) < \text{area}(AB'C) \);
(iv) if \( ABC \) is obtuse, then \( \text{area}(A'BC) < \text{area}(ABC_1) \).

Lemma 2.3 imply that only 3 of the special embedded triangles of \( ABC \) can be optimal.

Corollary 2.4. If \( ABC \) satisfies \( a < b < c \), then any maximum area special embedded triangle of \( ABC \) is one of the following triangles: \( A'BC, AB'C, ABC_1 \).

We note that similar results hold for the perimeter function implying that any maximum perimeter special embedded triangle of \( ABC \) is \( AB'C, A_1BC, \) or \( ABC_1 \).

2.2 Special enclosing isosceles triangles

Given a triangle \( ABC \), now we describe its special enclosing isosceles triangles, that is, all those isosceles triangles containing \( ABC \) that have a common side with \( ABC \) and share an angle at one of the endpoints of the common side. Recall that these triangles play a distinguished role in Theorems 1.2 and 1.3.

Special containers of the first kind. Let \( B' \) denote the point on the ray \( \overrightarrow{CB} \), for which \( |B'C| = |AC| \). Analogously, let \( C' \) (and \( C'' \)) denote the points on \( \overrightarrow{AC} \) (resp., \( \overrightarrow{BC} \)) such that \( |AC'| = |AB| \) (resp., \( |BC''| = |AB| \)), see Figure 4. We call the triangles \( AB'C, ABC', \) and \( ABC'' \) special containers of the first kind associated with \( ABC \).

![Figure 4: Special containers of the first kind (AB′C, ABC′, and ABC″) and second kind (AB_1C, ABC_1, and ABC_2).](image)
Special containers of the second kind. Let \( B_1 \) denote the point on the ray \( \overrightarrow{AB} \), different from \( A \), for which \( |B_1C| = |AC| \). Analogously, let \( C_1 \) (resp., \( C_2 \)) denote the point on \( \overrightarrow{AC} \) (resp., \( \overrightarrow{BC} \)) for which \( |BC_1| = |AB| \) and \( C_1 \neq A \) (resp., \( |AC_2| = |AB| \) and \( C_2 \neq B \)), see Figure 4. The triangles \( AB_1C \), \( ABC_1 \), and \( ABC_2 \) are called the special containers of the second kind associated with \( ABC \).

Special containers of the third kind. Let \( \overline{A} \) be the intersection of the perpendicular bisector of \( BC \) and the line \( AC \). Since we have \( b = |AC| < |AB| = c \), the point \( \overline{A} \) lies outside of \( ABC \). Analogously, denote by \( \overline{B} \) (resp., \( \overline{C} \)) the intersection of the perpendicular bisector of \( AC \) (resp. \( AB \)) and the line \( BC \). (If \( ABC \) is non-acute \( \overline{A}BC \) and \( \overline{A}BC \) do not contain \( ABC \) (Figure 5).) The triangles \( \overline{ABC} \), \( \overline{AB}C \), and \( \overline{AB}C \) are called the special containers of the third kind associated with \( ABC \), provided that they contain \( ABC \).

Figure 5: Special containers of the third kind (\( \overline{ABC}, \overline{AB}C, \overline{AB}C \)) in the acute and in the non-acute cases.

2.2.1 Basic inequalities for special containers

Similarly to the case of maximum area embedded triangles, we can show that not all special containers can be of minimum perimeter.

Lemma 2.5. If \( ABC \) satisfies \( a < b < c \), then

(i) \( \text{per} (ABC') < \text{per} (ABC'') \) and \( \text{per} (AB'C) < \text{per} (AB_1C) \);

(ii) \( \text{per} (ABC') < \text{per} (ABC_2) < \text{per} (ABC_1) \);

(iii) \( \text{per} (ABC') < \text{per} (\overline{ABC}) < \text{per} (\overline{AB}C) \).

Lemma 2.5 immediately gives the following corollary.

Corollary 2.6. If \( ABC \) satisfies \( a < b < c \), then any minimum perimeter special container of \( ABC \) is one of the following triangles: \( AB'C \), \( ABC'' \), \( \overline{AB}C \).

Again, we note that similar results hold for the area function implying that a minimum area special container of \( ABC \) is \( AB'C \), \( ABC'' \), or \( AB_1C \).

3 Maximum area embedded isosceles triangles

– Proof of Theorem 1.2(ii)

Let \( ABC \) be a triangle and let \( XYZ \) denote one of its maximum area isosceles embedded triangles. In this section, we prove that \( XYZ \) has to be a special embedded triangle. We use
the notation \( a = |BC|, b = |AC|, c = |AB|, x = |YZ|, y = |XZ|, z = |XY| \), and assume (with no loss of generality) that \( a < b < c \). By Lemmas 2.1 and 2.2, we have the following statements on maximum area embedded isosceles triangles.

**Lemma 3.1.** Let \( XYZ \) be any maximum area isosceles triangle embedded in \( ABC \). Then

(i) a side of \( ABC \) contains a side of \( XYZ \);
(ii) every side of \( ABC \) contains a vertex of \( XYZ \);
(iii) \( ABC \) and \( XYZ \) have a common vertex;
(iv) no vertex of \( XYZ \) lies in the interior of \( ABC \).

If \( XYZ \) has at least two common vertices with \( ABC \), then by Lemma 3.1(iv), \( XYZ \) and \( ABC \) have a common side and a common angle. Therefore, we can assume that \( ABC \) and \( XYZ \) have exactly one common vertex.

Denote the midpoints of the sides \( BC \), \( AC \), and \( AB \) by \( m_A \), \( m_B \), and \( m_C \), respectively. We divide the boundary of \( ABC \) into 3 polylines defined as

\[
\begin{align*}

\widehat{m_A m_B} &= m_A C \cup C m_B, \quad \widehat{m_B m_C} = m_B A \cup A m_C, \quad \widehat{m_C m_A} = m_C B \cup B m_A.
\end{align*}
\]

We get the following constraint on the position of \( X, Y, \) and \( Z \):

**Lemma 3.2.** Let \( XYZ \) be a maximum area embedded isosceles triangle of the triangle \( ABC \). Then each of \( \widehat{m_A m_B}, \widehat{m_B m_C}, \) and \( \widehat{m_C m_A} \) contains exactly one vertex of \( XYZ \).

**Proof.** By Lemma 3.1, \( X, Y, Z \) lies on the boundary of \( ABC \). Assume, without loss of generality, that \( \widehat{m_A m_C} \) contains \( X \) and \( Z \) (see Figure 6).

![Figure 6: Proof of Lemma 3.2](image)

Let \( T_1 = m_A m_C \cap XY \) and \( T_2 = m_A m_C \cap YZ \). Then area \((XT_1T_2Z) \leq \text{area}(Bm_A m_C)\) and by \(|T_1T_2| \leq |m_A m_C|\) we obtain that area \((T_2T_1Y) \leq \text{area}(m_A m_B m_C)\). Thus we have

\[
\text{area}(XYZ) \leq \text{area}(Bm_A m_C) + \text{area}(m_A m_B m_C) = \frac{1}{2} \text{area}(ABC).
\]

On the other hand, since \( c \leq a + b \leq 2b \), the special embedded triangle \( AB'C \) satisfies

\[
\text{area}(AB'C) = \frac{b^2 \sin(<CAB)}{2} > \frac{bc \sin(<CAB)}{4} = \frac{1}{2} \text{area}(ABC).
\]

Hence, \( \text{area}(XYZ) < \text{area}(AB'C) \), which contradicts the maximality of the area of \( XYZ \). \( \square \)
Lemmas 3.1 and 3.2 imply that a maximum area embedded isosceles triangle of $ABC$ is either special or its vertex arrangement corresponds to one of the 9 cases illustrated in Figure 7.

Figure 7: The 9 possible arrangements of the points $X, Y, Z$ in a given triangle $ABC$.

To complete the proof of Theorem 1.2(ii), it remains to prove that none of the arrangements depicted on Figure 7 can be optimal. We prove this for each of the 9 cases, separately. Note that in some instances, we will refer to special embedded triangles using their specific labeling introduced in Section 2.1.

**Case A:** *The common vertex of $ABC$ and $XYZ$ is $A = X$.*

**Subcase A.1:** $Y \in BC$ and $Z \in AC$.

Observe that since $b < c$, the orthogonal projection of $A$ onto $CB$ is contained in $Cm_A$, which implies that $\angle AYB$ is obtuse. Thus, we can rotate $XYZ$ about $X$ such that two of its vertices get to the interior of $ABC$ and so, by Lemma 3.1, $XYZ$ cannot be of maximum area.

**Subcase A.2:** Both $Y$ and $Z$ are in $BC$.

If $y = z$, then we can increase area ($XYZ$) by moving $Z$ towards $C$ and $Y$ towards $B$ while maintaining $|XZ| = |XY|$, since $\alpha = \angle CAB < 90^\circ$. If $ABC$ is acute, then we can do this until the vertices $Z$ and $C$ will coincide, and triangle $XYZ$ will be the same as the special embedded triangle $ABC$. If $ABC$ is non-acute, then $y \neq z$. Clearly, $|AZ| = y > |ZB| > |YZ| = x$. Hence, $x \neq y$. A similar argument shows that $x \neq z$.

**Subcase A.3:** $Y \in AB$ and $Z \in BC$.

Since $a < b$, the orthogonal projection $\tilde{Z}$ of $Z$ to the line segment $AB$ lies in $m_CB$.

If $x = y$, then $|AY| = 2|AZ| > 2|Am_C| = |AB|$, a contradiction to $Y \in AB$. 

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If $x = z$, then the altitude with base $z$ in $XYZ$ is smaller than the altitude with base $c$ in $ABC$. On the other hand, $x = z < a$, if $<ZYB \geq 90^\circ$. In this case, the special embedded triangle $A''BC$ satisfies area $(A''BC) > area (XYZ)$. Otherwise, $x = z < y$ (as $<AYZ > 90^\circ$) and $y < c$. Let $Y'$ be the point in $AB$ that is defined by the equality $|AY'| = |AZ|$. (The existence of $Y' \in AB$ is a consequence of $y < c$.) Then, area $(XYZ) > area (XY'Z)$. In both cases it follows that the area of $XYZ$ cannot be optimal.

If $y = z$, consider the special embedded triangle $AB'C$, define $l$ to be the line parallel to $B'C$ going through $Y$ and let $Z' = l \cap BC$, see Figure 8. Since $Z' \in CZ$, we have

$$area (XYZ) < area (XYZ') = area (AB'C) \cdot \frac{b + |B'Y|}{b} \cdot \frac{c - b - |B'Y|}{c - b}.$$  

The inequality follows from the fact that $Z' \in CZ$. Therefore, the altitude of $XYZ$ with base $z$ is greater than the altitude of $XYZ'$ with base $z$. Thus, it is enough to show that

$$\frac{b + |B'Y|}{b} \cdot \frac{c - b - |B'Y|}{c - b} < 1.$$  

As $b > 0$ and $c - b > 0$, this is equivalent to $|B'Y|(2b - c + |B'Y|) > 0$, which follows from the triangle inequality $c < a + b < 2b$.

Figure 8: Illustration for Subcase A.3.

Case B: The common vertex of $ABC$ and $XYZ$ is $B = Y$.

Subcase B.1: $X \in AC$ and $Z \in BC$.

Since $a < c$, we have that $<AXY > 90^\circ$, and hence, we can rotate the triangle $XYZ$ around $Y$ so that the image of the vertices $X, Z$ will be inside of $ABC$. As in Subcase A.1, this implies that the area of $XYZ$ is not optimal.

Subcase B.2: Both $X$ and $Z$ are in $AC$.

Observe that $b < c$ implies that $A$ and $C$ are on the same side of the perpendicular bisector of $BC$. This implies that $|XY| = z > |XC| > |XZ| = y$. If $x = z$, we can ‘open’ $<XYZ$ as in Subcase A.2 and get that area $(XYZ) < area (A'B'C)$. Hence, we can assume that $x = y$.

If the triangle $ABC$ is non-acute, then consider the special embedded triangle $ABC_1$. Since the altitudes of $ABC_1$ and $XYZ$ from vertex $B = Y$ are equal, and $x = y < |BC_1| = |AC_1|$ (as $<BCA \geq 90^\circ$), we have that area $(XYZ) < area (ABC_1)$.

If $ABC$ is acute, let $\hat{B}$ denote the orthogonal projection of $B$ onto $AC$. If $Z \in A\hat{B}$, then we can slightly rotate $XYZ$ around $Y$ (as $<YXA > <Y'XA > 90^\circ$). Thus, by Lemma 3.1(iv), the area of $XYZ$ is not maximal. Thus, we can assume that $Z \in C\hat{B}$, that is, $<YZA \leq 90^\circ$. Similarly as above, this implies that $x = |YZ| < a = |BC|$ and thus the special embedded triangle $A'BC$ satisfies area $(XYZ) < area (A'BC)$.

Subcase B.3: $X \in AB$ and $Z \in AC$.

If $y = z$, then, since $<CAB < \min(<AXZ, <ZXY)$, we get that $y = |XZ| < |AZ| <
\( b = |AC| \), which immediately implies that the special embedded triangle \( AB'C \) satisfies area \( (XYZ) < \text{area} \ (AB'C) \).

Now we assume that \( x = z \). If \( A \) and \( Z \) lie on the same side of the perpendicular bisector of \( AB \), then we can reflect \( XYZ \) to this perpendicular bisector. We denote this reflection by \( X'Y'Z' \). Clearly, \( X'Y' \in AB \), and \( Z' \) is inside of \( ABC \), which implies that area \( (XYZ) \) is not maximal. If \( Z \) is on the perpendicular bisector of \( AB \), then \( XYZ \) is strictly contained in the special embedded triangle \( ABC_1 \), so area \( (XYZ) < \text{area} \ (ABC_1) \). If \( Z \) and \( C \) are on the same side of the perpendicular bisector of \( AB \), then \( x = z < |AZ| < |AC| = b \), and hence area \( (XYZ) < \text{area} \ (AB'C) \).

It remains to handle the case \( x = y \). We show that area \( (XYZ) < \text{area} \ (ABC_1) \). The condition \( x = y \) implies that \( Z \in C_1C \). Plainly, \( z = c - |AX| \). Denote the lengths of the altitudes from \( C_1 \) in \( ABC_1 \) and from \( Z \) in \( XYZ \) by \( h_{C_1} \) and \( h_Z \), respectively. Clearly, we get

\[
\begin{align*}
\text{area} \ (XYZ) &= \text{area} \ (ABC_1) \frac{c - |AX|}{c} = \text{area} \ (ABC_1) \frac{c^2 - |AX|^2}{c^2} < \text{area} \ (ABC_1).
\end{align*}
\]

**Case C:** The common vertex of \( ABC \) and \( XYZ \) is \( C = Z \).

**Subcase C.1:** \( X \in AC \) and \( Y \in AB \).
If \( Y \) and \( B \) are on the same side of the altitude from \( C \), then we can rotate \( XYZ \) about \( Z \) so that \( X \) and \( Y \) get to the interior of \( ABC \) which by Lemma 3.1(iv) implies that \( XYZ \) is not optimal. If \( Y \) and \( B \) are on different sides of the altitude from \( C \), then \( XYZ \) is strictly contained in the special embedded triangle \( AB'C \).

**Subcase C.2:** Both \( X \) and \( Y \) are contained in \( AB \).
If \( x = y \), then we can open \( \angle YZX \), which increases the area of \( XYZ \), so area \( (XYZ) \) is not maximal. Suppose that \( x = z \). If \( Y \) and \( A \) are on the same side of the altitude from \( C \), then \( XYZ \) is strictly contained in the special embedded triangle \( AB'C \). If \( Y \) and \( A \) lie on different sides of the altitude, then the special embedded triangle \( A''BC \) satisfies area \( (XYZ) < \text{area} \ (A''BC) \). Indeed, their altitudes from \( C \) are the same, and for their bases we have \( x = z < a \). Thus \( XYZ \) is not maximal. Analogously, for \( y = z \) a similar argument shows that area \( (XYZ) < \text{area} \ (AB'C) \).

**Subcase C.3:** \( X \in AB \) and \( Y \in BC \).
We can rotate \( XYZ \) about \( Z \) such that the images of \( X \) and \( Y \) lie in the interior of \( ABC \), and so, by Lemma 3.1(iv), we get that area \( (XYZ) \) is not maximal.

We have shown that none of the triangles \( XYZ \) of the 9 cases in Figure 7 is a maximum area embedded isosceles triangle of \( ABC \), which completes the proof of Theorem 1.2(ii).

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\Box
\]

### 4 Maximum perimeter embedded isosceles triangles

**– Proof of Theorem 1.2(iii)**

In this section, we prove that for any triangle \( ABC \), any maximum perimeter isosceles triangle \( XYZ \) embedded in \( ABC \) shares a vertex and the angle at that vertex with \( ABC \). First we collect the observations in Lemmas 2.1 and 2.2 concerning maximum perimeter embedded isosceles triangles.

**Lemma 4.1.** Let \( XYZ \) be a maximum perimeter isosceles triangle embedded in \( ABC \). Then

(i) each side of \( ABC \) contains a vertex of \( XYZ \);
(ii) no vertex of the triangle XYZ lies in the interior of the triangle ABC;
(iii) there is a side of ABC which contains a side of XYZ;
(iv) ABC and XYZ share a vertex.

We will show that an isosceles triangle embedded in ABC which does not share an angle
with ABC cannot be of minimum perimeter. Notice that if ABC and XYZ share at least
two vertices, then, by Lemma 4.1(ii), they also share an angle, so we are done. Thus, it is
enough to consider those cases where the triangles XYZ and ABC share exactly one vertex,
without loss of generality the common vertex is A. Note that in this section, we do not assume
a special labeling of ABC, in particular, we do not necessarily have \(|BC| < |AC| < |AB|\).

On the other hand, we assume that XYZ is labeled so that \(|XY| = |YZ|\). We consider the
following cases, separately:

Case A: X and Z lie on the same side of ABC.

We can always rotate X or Z (for simplicity, assume it is X) about Y so that the rotated
point \(X'\) lies in the interior of ABC and \(<XYZ < <X'YZ\), see Figure 9. By the Hinge
theorem\(^1\) per (XYZ) < per (X'YZ).

![Figure 9: Illustration for Case A](image)

Case B: X and Z lie on different sides of ABC.

We will make use of the following classical lemma on the perimeter of the Minkowski sum of
convex bodies.

**Lemma 4.2** (see e.g. [28, exercise 4-7]). Let \(K_1\) and \(K_2\) be two convex bodies in the plane
and let \(K = \frac{K_1 + K_2}{2}\) be the Minkowski mean of \(K_1\) and \(K_2\). Then the perimeter of \(K\) is equal
to the arithmetic mean of the perimeters of \(K_1\) and \(K_2\). If \(K_1\) and \(K_2\) are not homothetic
triangles, then \(K\) is a convex polygon with at least four sides.

**Subcase B.1:** The common vertex of ABC and XYZ is A = Y.

If none of X and Z is on the side opposite to Y, then XYZ and ABC have a common angle
at Y. Thus, we can assume that either X or Z is on the side opposite to Y, say it is X.

The idea is to show that the triangle XYZ is strictly contained in the Minkowski mean \(M\)
of two other non-homothetic isosceles triangles embedded in ABC, thus, by Lemma 4.2 one
of these two must have a strictly larger perimeter (by the fact that if \(C_1, C_2\) are two convex
planar sets such that \(C_1 \subseteq C_2\), then per \((C_1) \leq \) per \((C_2)\) [1, 12.10.2]).

Let \(\delta\) be a constant satisfying \(\delta < \min \{|XB|, |XC|\}\). Define the points \(X^1\) and \(X^2\) by
translating X by \(\delta\) towards C and B, respectively. Let \(Z^1\) and \(Z^2\) be such that they are
contained on the side AB with \(|YZ^1| = |YX^1|\) and \(|YZ^2| = |YX^2|\), see Figure 10. Let \(M\)
be the Minkowski mean of \(X^1YZ^1\) and \(X^2YZ^2\). The vertex Y is contained in both triangles,

\(^1\)Hinge theorem: Let XYZ be a triangle and let \(X'Y'Z'\) be another triangle such that \(XY = X'Y', YZ =
Y'Z'\), and \(<XYZ < <X'Y'Z'\). Then per (XYZ) < per (X'Y'Z').
thus it is also contained in $M$. It is also easy to see that $X \in M$ since $X = \frac{1}{2}(X_1 + X^2)$. We show that $Z$ is contained in the segment between $Y$ and $\frac{1}{2}(Z_1 + Z^2)$, which implies $Z \in M$. To this end, observe that the segment $YX$ is a median of the triangle $X_1YX^2$ and thus $|YX| < \frac{1}{2}(|YX^1| + |YX^2|)$, which directly implies that $|YZ| < \frac{1}{2}(|YZ^1| + |YZ^2|)$.

**Subcase B.2:** The common vertex of $ABC$ and $XYZ$ is $A = Z$ and both $X$ and $Y$ are in the interior of the side of $ABC$ opposite to $Z$.

Define the points $X^1$ and $X^2$ by translating $X$ by $\delta$ towards $C$ and $B$, respectively. We choose $\delta$ to be small enough such that there are points $Y^1, Y^2$ in the segment $BC$ with $|Y^1Z| = |Y^1X^1|$ and $|Y^2Z| = |Y^2X^2|$, see Figure 11. Let $M$ be the Minkowski mean of $X^1YZ^1$ and $X^2YZ^2$. As before, it is clear that the vertices $X$ and $Z$ are contained in $M$.

To argue that $Y \in M$, we shall show that $Y$ is contained in the segment between $X$ and $\frac{1}{2}(Y^1 + Y^2)$. To simplify the calculations, we move and scale the triangle so that $A = (0, 1), B = (b, 0), C = (c, 0)$ and $X = (x', 0)$ with $b < x' < c$. Note that since $\angle ZXY$ is acute, $x' < 0$. For each $b < x < 0$, let $X_x = (x, 0)$ and $Y_x$ be the point in $BC$ such that $|ZY_x| = |Y_xX_x|$ and define $f(x) = |Y_xX_x|$, see Figure 12. Observe that $Y$ is contained in the segment between $X$ and $\frac{1}{2}(Y^1 + Y^2)$ iff $\frac{1}{2}(f(x') - \delta) + f(x' + \delta)) > f(x')$. Thus, it is sufficient to show that $f(x)$ is a convex function on $(b, 0)$. To find an analytic formula for $f(x)$, we introduce some auxiliary points. Let $O = (0, 0)$ and $P_x$ be the orthogonal projection of $Y_x$ to the segment $X_xZ$. Note that $P_x$ is the midpoint of $XZ$. Then the triangles $X_xP_xY_x$ and $X_xOZ$ are similar, which yields

$$f(x) = |Y_xX_x| = |X_xZ| \cdot \frac{|X_xP_x|}{|X_xO|} = \sqrt{1 + x^2} \cdot \frac{\sqrt{1 + x^2/2}}{-x} = \frac{1 + x^2}{-2x}$$

The second derivative of $f$ is $f''(x) = -1/x^3$, thus $f(x)$ is convex on the interval $(b, 0)$.

**Subcase B.3:** The common vertex of $ABC$ and $XYZ$ is $A = Z$ and $X, Y$ lie in the interior of different sides of $ABC$.

Firstly, since $X$ and $Z$ lie on different sides of $ABC$, we get that $X$ is on the side opposite to $Z$, see Figure 13. If $\angle AXB$ is obtuse, then we can rotate the triangle $XYZ$ about $Z$ and obtain a copy of $XYZ$ which has two vertices in the interior of $ABC$, thus by Lemma 4.1.
Figure 12: Embedding the instance in $\mathbb{R}^2$.

$XYZ$ cannot be of maximum perimeter. Therefore, $\angle AXB$ and consequently $\angle ACB$ are acute.

Define the points $X^1$ and $X^2$ by translating $X$ by $\delta$ towards $C$ and $B$, respectively. We choose an increment $\delta \in (0, 1/c)$ which is small enough such that there are points $Y^1, Y^2$ in the segment $AC$ with $|Y^1 Z| = |Y^1 X^1|$ and $|Y^2 Z| = |Y^2 X^2|$. Let $M$ be the Minkowski mean of $X^1 Y Z^1$ and $X^2 Y Z^2$. The vertices $X$ and $Z$ are clearly contained in $M$.

Figure 13: Illustration for Case B.3.

To prove that $Y \in M$, we shall show that $Y$ is contained in the segment between $Z$ and $\frac{1}{2}(Y^1 + Y^2)$. Again, we translate and scale of the triangle so that $A = (0, 1), B = (b, 0), C = (c, 0)$ and $X = (x', 0)$ with $b < x' < c$. Since $\angle AXB$ is acute, we have $x' \geq 0$. For each $x \in [x' - \delta, x' + \delta]$, let $X_x = (x, 0)$ and $Y_x$ be the point in $ZC$ such that $|ZY_x| = |Y_x X_x|$ and define $f(x) = |ZY_x|$, see Figure 14. Note that since $x' \geq 0$ and $\delta$ is smaller than $1/c$, each $x \in [x' - \delta, x' + \delta]$ satisfies $-1/c < x$. We want to show that the function $f(x)$ is convex, which then directly implies that $Y$ is contained in the segment between $Z$ and $\frac{1}{2}(Y^1 + Y^2)$.

Let $P_x = (p_1(x), p_2(x))$ be a point on $AC$ such that the segment $P_x X_x$ is orthogonal to $AX_x$. Note that $P_x$ satisfies $|ZP_x| = 2f(x)$.

Let $\gamma$ denote the angle $\angle ACX_x$, then $p_2(x) = 1 - 2\sin(\gamma) \cdot f(x)$ which is concave if and only if $f(x)$ is convex. Since $X_x P_x$ is orthogonal to $AX_x$ and $P_x$ is contained in $AC$, we get the following equations on $p_1(x)$ and $p_2(x)$

$$p_1(x) \cdot x - p_2(x) = x^2, \quad p_1(x) + c p_2(x) = c,$$

which gives $p_2(x) = \frac{cx - x^2}{cx + 1}$. Taking the second derivative, we get

$$p''_2(x) = -\frac{2(1 + c^2)}{(1 + cx)^2} < 0 \text{ for all } x \in \left(-\frac{1}{c}, \infty\right).$$
Figure 14: Embedding the instance in $\mathbb{R}^2$.

We proved that none of the triangles of types A.1-A.2 and B.1-B.3 is a maximum perimeter embedded isosceles triangle of $ABC$, which completes the proof of Theorem 1.2(iii).

5 Minimum perimeter enclosing triangles

– Proof of Theorem 1.3

In this section, we prove that any smallest perimeter isosceles container of a triangle is either a special container or one of two non-special containers defined in the next subsection. We also show that this is the shortest possible characterization of isosceles containers, that is, any of the five examples can be realized as a minimum perimeter isosceles container for some triangle $ABC$. Now, we define two non-special isosceles containers that can be optimal.

5.1 Two examples for non-special minimum perimeter containers of a triangle

Let $P$ be a point in $\mathbb{R}^2$ and $l$ a line such that $P \notin l$ and let $m$ denote the distance of $P$ from $l$. Define an isosceles triangle $PRS$ such that $S$ and $R$ lie on $l$ and its apex angle $\gamma$ is in $\mathbb{R}$, see Figure 15. Let $p$ denote the perimeter of $PRS$. Note that $p$ can be considered as a function of $\gamma$.

![Figure 15: Illustration for Proposition 5.1](image)

**Proposition 5.1.** The function $p$ has a unique minimum at

$$\gamma^* = 4 \tan^{-1}\left(\frac{1}{2} \left(1 + \sqrt{5} - \sqrt{2(1 + \sqrt{5})}\right)\right) \approx 76.3466^\circ. \quad (1)$$

**Proof outline.** It is easy to see that $|PR| = |RS| = \frac{m}{\sin \gamma}$ and $|PS| = \frac{m}{\sin(90^\circ - \gamma/2)} = \frac{m}{\cos(\gamma/2)}$. Hence $\text{per}(PRS) = m \left(\frac{2}{\sin \gamma} + \frac{1}{\cos(\gamma/2)}\right)$. Elementary analysis shows that the function $f(x) = \frac{2}{\sin x} + \frac{1}{\cos(x/2)}$ is strictly decreasing in $(0^\circ, \gamma^*)$ and strictly increasing in $[\gamma^*, 180^\circ)$. Thus it has a unique minimum in $0 \leq x \leq 180^\circ$ that is taken at the value specified in Equation (1).
Example 5.2. Let $PRS$ be an isosceles triangle with apex angle $\gamma^*$ which is defined as in Proposition 5.1. Now we take an acute triangle $ABC$ in $PRS$ such that $ABC$ and $PRS$ have exactly one common vertex at $A = P$ and $B, C \in SR$ (see Figure 16). Moreover, the largest angle $\gamma$ of $ABC$ is at $C$ with $\gamma < \gamma^*$ being close to $\gamma^*$ (e.g., $76^\circ$) and $ABC$ is almost isosceles ($|AC| \approx |BC|$).

Figure 16: Illustration for Example 5.2

Claim 5.3. The perimeter of $PRS$ is strictly smaller than the perimeter of any special container of $ABC$.

Proof outline. By Corollary 2.6, it is enough to show that the special containers $AB'C, ABC'$, and $AB\overline{C}$ have larger perimeter than $PRS$. First observe that, since $a \approx b < c$, $ABC$ is an ‘almost’ isosceles triangle, thus the perimeter per $(AB'C) \approx$ per $(ABC)$ and per $(ABC') > d \cdot$ per $(ABC)$, for a fixed $d > 1$. This implies that per $(AB'C) <$ per $(ABC')$. Now we show that $PRS$ has perimeter smaller than per $(AB'C)$ and per $(ABC)$. Note that, each of $PRS, AB'C$ and $AB\overline{C}$ are isosceles triangles with base vertex $A = P$ and legs on the line $RS$. By Proposition 5.1 the smallest perimeter isosceles triangle under these conditions is $PRS$. Thus, it is enough to guarantee that the triangles $AB'C$ and $AB\overline{C}$ do not coincide with $PRS$ which follows from the fact that $ABC$ and $PRS$ has exactly one common vertex.

Now we turn to our second example. We start by taking the points $A = P = (0, 0), C = (1, v)$ and $S_x = (x, 0)$ and define $R_x$ to be the point on the $S_x\overrightarrow{C}$ ray so that $|PR_x| = |R_xS_x|$. The next claim follows by elementary calculations, its proof is omitted.

Proposition 5.4. For any $x \in (1, 2)$, the perimeter of $PR_xS_x$ can be expressed as

$$\text{per } (PR_xS_x) = f_v(x) = x \left( 1 + \sqrt{1 + \frac{v^2}{(1 - x)^2}} \right).$$

and for any $v \in [0.56, \sqrt{3})$, the function $f_v$ has a unique minimum in $(1, 2)$ denoted by $x_v^*$.\footnote{The formula for $x_v^*$ is given in the Appendix.}

Example 5.5. Consider a triangle $ABC$ that can be embedded in $\mathbb{R}^2$ as $A = (0, 0), C = (1, v)$ and $B = (x_b, 0)$ with $1 < x_b < x_v^*$ (the value $x_v^*$ is defined in Proposition 5.4; see also Figure 17). Let $PRS$ be the an isosceles triangle with $P = A, S = (x_v^*, 0)$, and $R$ defined as the point on the $S_x\overrightarrow{C}$ ray with $|PR| = |RS|$. By definition, $SPR$ is an isosceles container of $ABC$.\footnote{The formula for $x_v^*$ is given in the Appendix.}
Claim 5.6. The perimeter of \(PRS\) is smaller than the perimeter of any special container of \(ABC\).

Proof outline. By Corollary 2.6, we only need to show that \(PRS\) has a smaller perimeter than the special containers \(AB'C', ABC'\), and \(ABC\). Observe that by the choices of \(x^*_v\) and \(x_b\), we have \(\text{per}(ABC) = f_v(x_b) < f_v(x^*_v) = \text{per}(PRS)\).

We verify the remaining cases only for the fixed value \(v = 0.7\). The function \(f_{0.7}(x)\) takes its minimum at \(x^*_{0.7} \approx 1.57517\), and thus \(\text{per}(PRS) = f_{0.7}(x^*_{0.7}) \approx 4.056333\). On the other hand, if we set e.g. \(x_b = 1.57\), we have \(\text{per}(AB'C) \approx 4.229145\) and \(\text{per}(ABC') \approx 4.084007\).

5.2 Proof of Theorem 1.3

We start by proving that every smallest perimeter isosceles container of a triangle \(\Delta = ABC\) is either a special container or one of the two triangles constructed in the Examples 5.2 and 5.5. Later, we will show that each of these five containers is realized as the unique minimum perimeter isosceles container for some triangle \(ABC\). By Lemmas 2.1 and 2.2, we have the following statements on minimum perimeter isosceles containers.

Lemma 5.7. Let \(PRS\) be any minimum area isosceles triangle enclosing \(ABC\). Then

(i) a side of \(PRS\) contains a side of \(ABC\);

(ii) each side of \(PRS\) contains a vertex of \(ABC\);

(iii) \(ABC\) and \(PRS\) share a common vertex;

(iv) no vertex of \(ABC\) lies in the interior of \(PRS\).

In what follows, \(PRS\) is labeled so that \(|PR| = |RS|\). If \(PRS\) shares the vertex \(R\) with \(ABC\), but it does not share the angle at \(R\), then we can get a smaller perimeter container by decreasing \(\angle SRP\) (while keeping \(|PR| = |RS|\) unchanged). Thus without loss of generality, we can assume that \(PRS\) shares the vertex \(P\) with \(ABC\). The above restrictions allow only the following types of minimum perimeter isosceles containers that do not share an angle with \(ABC\) (see also Figure 18)\(^{3}\)

Case 1: If two vertices of \(ABC\) lie in the interior of \(RS\), or one of the vertices of \(ABC\) lies in the interior of the side \(RS\) and one lies in the interior of \(PS\).

The smallest perimeter isosceles containers of these types are precisely the non-special optimal containers shown in Examples 5.2 and 5.5

\[^{3}\text{Note that in this subsection, we do not assume a special labeling of } ABC, \text{ in particular, we do not necessarily have } |BC| < |AC| < |AB|.\]
Case 2: One vertex of $ABC$ is in the interior of $PR$ and one is in the interior of $RS$.

Let $T$ denote the base of the altitude perpendicular to $RS$ and let $B$ denote the vertex in $RS$. If $|SB| \leq |ST|$, then $\angle SBP \geq 90^\circ$, hence we can rotate $ABC$ with center $A = P$ such that the triangle remains in $PRS$ and hence $PRS$ was not minimal, see Figure [19]. Note that this happens if $PRS$ is not acute. From now on, we assume that $\angle SBP < 90^\circ$.

Hence $|AB| < |AR|$ if $B \neq R$.

If $|AC| < |AB|$, then we take $C' \in AR$ such that $|AB| = |AC'| < |AR|$ so $AC' \subset AR$ (Figure [19]). Thus, $ABC'$ is an isosceles container of $ABC$ and $ABC' \subsetneq PRS$. Hence, $PRS$ was not minimal. Therefore, we may assume that $|AC| > |AB|$, as $|AC| = |AB|$ would imply that $ABC$ was isosceles.

If $\angle RAB < \angle BRA$ holds, let $B'$ be the point on the line $AB$ such that $|AC| = |AB'|$ then we have $|AB'| = |AC| < |AR| = |RS|$, and hence per $(AB'C') <$ per $(PRS)$, thus $PRS$ was not minimal. Thus, assume that $\angle BRA \leq \angle RAB$ and as $\angle RAB + \angle BRA = \angle SBP < 90^\circ$, we get that $\angle BRA = \angle PRS < 45^\circ$.

For the remaining part, we embed the configuration in $\mathbb{R}^2$ such that $P = A = (0, 0)$, $R = (x, 0)$ and $B = (1, h)$, where $x > 1$ and $h > 0$, see Figure [20]. Under the assumptions that $|AC| > |AB|$ and $\angle PRS < \min(\angle RAB, 45^\circ)$, we show that per $(PRS)$ as a function of $x$ is increasing. Thus, as $B \neq R, C \neq R$ there exists a smaller perimeter isosceles container of $ABC$ than $PRS$ (e.g. $Pr'S'$ in Figure [20]). The condition $\angle PRS < \angle RAB$ implies that $|PT'| < |RT'|$, where $T'$ is the base of the altitude of $PR$, hence $x > 2$.

Clearly, $|BR| = \sqrt{h^2 + (x-1)^2}$ and $\sin(\angle PRS) = \frac{h}{\sqrt{h^2 + (x-1)^2}}$. Hence per $(PRS) =$
Figure 20: Case 3 in a coordinate system.

2x(1 + \sin(\frac{\angle PRS}{2})). As \sin \delta = 2 \sin(\frac{\delta}{2})\sqrt{1 - \sin^2(\frac{\delta}{2})}, we get

\[
\sin\left(\frac{\angle PRS}{2}\right) = \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{x - 1}{h} \sqrt{\frac{1}{1 + \left(\frac{x - 1}{h}\right)^2}}},
\]

where \pm is taken to be \(-\) sign, since \angle PRS < 45°. Therefore,

\[
\text{per}(PRS) = 2x + \sqrt{2x} \sqrt{1 - \frac{x - 1}{h} \sqrt{\frac{1}{1 + \left(\frac{x - 1}{h}\right)^2}}}. \tag{18}
\]

Let \(y = \frac{x - 1}{h}\) and let \(f_h(y) = (1 + hy)\left(1 + \sqrt{\frac{1-y^2}{1+y^2}}\right)\). It follows from our assumptions that \(y > 1/h\). We show that \(f_h(y)\) is strictly increasing in \(y\), which implies that \(PRS\) is not a minimum perimeter isosceles container of \(ABC\). For \(g(y) := 1 + \sqrt{\frac{1-y^2}{1+y^2}}\), we show that \(f'_h(y) = ((1 + hy)g(y))' > 0\), equivalently \(-g'(y) < \frac{hg(y)}{1+hy}\). Simple calculation shows that

\[
-g'(y) = \frac{1}{2\sqrt{2}(1+y^2)} \sqrt{1 + \sqrt{\frac{y^2}{1+y^2}}} < \frac{1}{2(1+y^2)},
\]

where the last inequality holds as \(\frac{y^2}{1+y^2} < 1 \) for all \(y \in \mathbb{R}\). Note that \(g(y) > 1\), hence \(hg(y) > h\).

Thus, it is enough to show that

\[
\frac{1}{2(1+y^2)} < \frac{h}{1+hy} \quad \text{if } y = \frac{x - 1}{h} > 1/h.
\]

This is true if and only if \(0 < 2hy^2 - hy + 2h - 1\). This holds if the roots of this polynomial satisfy \(y_1 < y_2 = \frac{h + \sqrt{-15h^2 + 8h}}{2h} \leq 1/h\). The last inequality is equivalent to \(0 \leq 4h^2 - 3h + 1 = (2h-1)^2 + h\), which is true for \(h > 0\). Therefore, the argument above verifies that in this case \(PRS\) is not minimal. This concludes the proof in Case 2.

**Note on realizability.** Now we briefly discuss that each of the special containers \(AB'C\), \(ABC'\), \(AB\overline{C}\), and triangles constructed in Examples 5.2 and 5.5 can occur as a minimum perimeter container for some \(ABC\). It is easy to find triangles for which one of the special containers is the best among the five options.
To see that the container of Example 5.5 is optimal for some triangles, note that the construction presented in Example 5.2 works only if the special containers of \(ABC\) satisfy \(\gamma^* \in (\angle AB'C, \angle AB'C)\). Now consider the example from the proof of Claim 5.6. It can be easily calculated that under these choices \(\angle (B'C') \approx 78,310868^\circ\). This (together with Claim 5.6) implies that for the example presented in the proof of Claim 5.6, the container described in Example 5.5 is better the one given in Example 5.2 and than any special container.

Finally, we show that the container presented in Example 5.2 is optimal for some triangles. Following the construction in the proof of Claim 5.6, consider the triangle \(ABC\) with \(A = (0, 0)\), and \(C = (1, 0.8)\) and \(B = (0, x^*_0.8)\) such that \(f_0.x^*_{0.8}\) takes its minimum at \(x^*_0.8 \approx 1.62474\). We get that the container constructed in Example 5.5 coincides with the special container \(ABC\) and per \((AB'C) = f_0.x^*_{0.8} \approx 4.264511\). Simple calculation shows that per \((ABC') \approx 4.3250804\), thus per \((AB'C) < \text{per} (ABC')\). Since \(\angle (B'C') \approx 75.974334^\circ < \gamma^* < \angle (BCA) = \angle (B'C'A) \approx 89.327359^\circ\), the construction of Example 5.2 provides smaller perimeter than any of the special containers, indeed if we let \(SPR\) to be the container constructed in Example 5.2 for out choice of \(ABC\), then we get per \((PRS) \approx 4.264431\).

This concludes the proof of Theorem 1.3.

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Appendix - Basic inequalities for embedded triangles

Proof of Lemma 2.1

Observe that if $\Delta$ and $\Delta'$ are similar triangles then $\text{per}(\Delta) < \text{per}(\Delta')$ and $\text{area}(\Delta) < \text{area}(\Delta')$ hold if and only if $\text{diam}(\Delta) < \text{diam}(\Delta')$. Therefore, it is sufficient to prove following the statements:

Claim 5.8. Let $\Delta_1$ be a triangle inside the triangle $\Delta_2$. If $\Delta'_1$ is a maximum diameter triangle similar to $\Delta_1$ contained in $\Delta_2$, then

(A) $\Delta'_1$ has two vertices on a side of $\Delta_2$;
(B) each side of $\Delta_2$ contains a vertex of $\Delta'_1$;
(C) $\Delta'_1$ and $\Delta_2$ have a common vertex.

Proof. Let $\Delta_1 = ABC$, $\Delta'_1 = A'B'C'$, and $\Delta_2 = DEF$.

(A) The statement was proved by Post [21].

(B) Assume indirectly that the side $DE$ does not contain any of the vertices $A', B', C'$ and let $A'$ the closest one to $DE$. Let $l'$ denote the line parallel to $DE$ which contains $A'$. Let $D' = l' \cap DF$ and $E' = l' \cap EF$. Then the triangle $D'E'F \subset DEF$ is similar to $DEF$ and it contains $A'B'C'$. If $S$ is the homothety with center $F$ and ratio $|DF|/|D'F|$, then $S(A'B'C') \subset DEF$ is a similar copy of $A'B'C'$ which has larger diameter than $A'B'C'$, a contradiction. Hence, $DE$ contains a vertex of $A'B'C'$. The argument applies for any other side of $DEF$.

(C) By part (A), $A'B'C'$ has two vertices that are on the same side of $DEF$. If any of these vertices coincide with a vertex of $DEF$, then the second part of the statement holds. Otherwise, the third vertex of $A'B'C'$ must be contained in two sides of $DEF$, thus it is a vertex of $DEF$. \qed

Proof of Lemma 2.2

Let $\Delta = ABC$, $\Delta_1 = DEF$, and $\Delta_2 = PRS$.

(i) We prove that the vertices of minimum area (resp. perimeter) isosceles containers are contained in a compact subset of the plane. (A similar result for the set of embedded triangles is trivial.)

The case of the area was handled in [12]. Concerning the perimeter, it is easy to see that there is a special container whose perimeter is at most twice as large as the one of $ABC$. Thus any minimum perimeter container of $ABC$ is contained in the closed ball $B$ centered at $A$ with radius $2\text{per}(ABC)$.

Thus the collection of those isosceles triangles $\Delta$ satisfies $ABC \subset \Delta \subset B$ can be considered as a compact subset of $\mathbb{R}^6$ with respect to the Euclidean topology. The result simply follows from the fact that both the area and the perimeter are continuous functions on the parameter set.

(iii) The statement for the minimum area has been proved in [12][Lemma 3.2].

Let $PRS$ be a minimum perimeter isosceles container of $ABC$. Then, by Claim 5.8, one side of $PRS$ contains two vertices of $ABC$, and by Lemma 2.1 every side of $PRS$ contains a vertex of $ABC$ and the triangles share a common vertex. Thus either each vertex of $ABC$ is on a side of $PRS$, as we stated, or two vertices of $ABC$ coincide with
two vertices of \( PRS \) (i.e., the triangles share a common side) as in Figure 21. In the latter case we distinguish two subcases when the common side is a leg (Case a.) or a base (Case b.) of \( PRS \). As Figure 21 illustrates, in both subcases we can find a smaller isosceles triangle (green in Figure 21) by shrinking the original triangle \( PRS \) so that the modified isosceles triangle contains \( ABC \) and has smaller area and perimeter.

![Figure 21: Illustration for the proof of Lemma 2.2 (iii).](image)

(ii) The proof of this case is analogous to the proof of case [iii] for the perimeter and the proof of [12][Lemma 3.2] for the area.

\[ \square \]

**Proof of Lemma 2.3**

The areas of special embedded triangles of the first, the second and the third type are the following:

\[
\begin{align*}
\text{area} (A'BC) &= \frac{a^2 \sin \gamma}{2}, & \text{area} (AB'C) &= \frac{b^2 \sin \alpha}{2}, & \text{area} (A''BC) &= \frac{a^2 \sin \beta}{2}.
\end{align*}
\]

\[
\begin{align*}
\text{area} (A_1BC) &= \frac{a^2 \tan \beta}{4}, & \text{area} (AB_1C) &= \frac{b^2 \tan \alpha}{4}, & \text{area} (ABC_1) &= \frac{c^2 \tan \alpha}{4}.
\end{align*}
\]

\[
\begin{align*}
\text{area} (\overline{ABC}) &= \frac{a^2 \sin(2\beta)}{2}, & \text{area} (\overline{AB}C) &= \frac{a^2 \sin(2\gamma)}{2}, & \text{area} (\overline{A}BC) &= \frac{b^2 \sin(2\gamma)}{2}.
\end{align*}
\]

1. It follows easily from the previous equalities that

\[
\text{area} (A''BC) = \frac{a^2 \sin \beta}{2} < \frac{a^2 \sin \beta}{2} \frac{c}{b} = \frac{a^2 \sin \beta \sin \gamma}{2} = \text{area} (A'BC).
\]

2. We have seen that area \( (A_1BC) = \frac{a^2 \tan \beta}{4} \) and area \( (AB'C) = \frac{b^2 \sin \alpha}{4} \). Now \( \frac{a^2 \tan \beta}{4} < \frac{b^2 \sin \alpha}{2} \) is equivalent to \( \frac{a^2}{2} = \frac{\sin^2 \alpha}{2\sin^2 \beta} < \frac{\sin \alpha \cos \beta}{\sin \beta} \), by using the law of sines. Reformulating this we have \( \sin \alpha < \sin(2\beta) \), which always holds as \( \alpha < \min\{2\beta, 180 - 2\beta\} \).

The inequality \( \frac{b^2 \tan \alpha}{4} < \frac{a^2 \tan \alpha}{4} \) implies that area \( (AB_1C) < \text{area} (ABC_1) \).

3. We first prove

\[
\text{area} (\overline{ABC}) < \text{area} (ABC_1).
\]
Using the equations given above this is equivalent to
\[
\frac{a^2 \sin(2\beta)}{2} < \frac{c^2 \tan \alpha}{4} \iff a^2 \sin \beta \cos \beta < \frac{c^2 \sin \alpha}{4 \cos \alpha}.
\]
By replacing \( \frac{\sin \alpha}{\sin \beta} \) with \( \frac{a}{b} \) we obtain \( \frac{4ab}{c^2} \cos \alpha \cos \beta < 1 \). Law of cosines gives
\[
\frac{4ab}{c^2} \frac{2bc}{b^2 + c^2 - a^2} < 1.
\]
After a simple rearrangement we obtain
\[
(c^2 + (a^2 - b^2))(c^2 - (a^2 - b^2)) < c^4,
\]
which holds by \( a \neq b \).

The inequality \( \text{area}(ABC) > \text{area}(\overline{ABC}) \) follows from
\[
\text{area}(\overline{ABC}) = \frac{a^2 \sin(2\gamma)}{2} < \frac{b^2 \sin(2\gamma)}{2} = \text{area}(\overline{ABC}).
\]
The length of the legs of the isosceles triangles \( \overline{ABC} \) and \( AB'C \) are equal to \( b \), but the apex angle is greater in the latter triangle. As both apex angles are upper bounded by \( \alpha < 90^\circ \), we get \( \text{area}(\overline{ABC}) < \text{area}(AB'C) \).

4. If \( ABC \) is obtuse, then \( 0^\circ < \alpha < 45^\circ \). The function \( \sin(2\alpha) \) is strictly monotonically increasing on the interval \([0^\circ, 45^\circ]\). Since \( \alpha < \beta \) we have \( \alpha < 180^\circ - \gamma < 90^\circ \). Thus
\[
\sin(2\alpha) < \sin(180 - \gamma) = \sin \gamma,
\]
\[
2 \sin \alpha \cos \alpha < \sin \gamma
\]
\[
\sin \alpha < \frac{\sin \gamma}{2 \cos \alpha}.
\]
We multiply both sides of the inequality with the positive number \( \frac{\sin \alpha \sin \gamma}{2} \):
\[
\frac{\sin^2 \alpha \sin \gamma}{2} < \frac{\sin \alpha \sin^2 \gamma}{4 \cos \alpha}.
\]
By the law of sines this is equivalent to
\[
\frac{a^2 \sin \gamma}{2} < \frac{c^2 \sin \alpha}{4 \cos \alpha},
\]
which implies
\[
\text{area}(A'BC) = \frac{a^2 \sin \gamma}{2} \leq \frac{c^2 \tan \alpha}{4} = \text{area}(ABC_1).
\]
Proof of Lemma 2.5

1. \( \text{per}(\triangle ABC') < \text{per}(\triangle ABC'') \) follows from the Hinge theorem since \( |AB| = |AC'| = |AC''| = c \) and \( \alpha = \angle C'BA < \angle C''BA \Rightarrow \beta < 90^\circ \).

2. Notice that \( |AB| = |AC_1| = |AC_2| = c \). On the other hand \( \angle BAC' = \alpha < \angle BAC_2 = 180^\circ - 2\beta < 180^\circ - 2\alpha \) since \( \alpha + \beta + \gamma = 180^\circ \) and \( \alpha < \beta < \gamma \). Thus the Hinge theorem gives \( \text{per}(\triangle ABC') < \text{per}(\triangle ABC_2) < \text{per}(\triangle ABC_1) \).

3. Similarly to the previous cases, we have \( |AC| = |CB'| = |CB_1| = b \) and \( \angle ACB' < \angle ACB_1 \) so the perimeter of \( \triangle AB'C \) is smaller than that of \( \triangle AB_1C \).

4. First note that the triangles \( \triangle ABC \) and \( \triangle A'BC \) do exist if and only if \( \triangle ABC \) is acute, hence we assume \( \gamma < 90^\circ \). First we show that \( \text{per}(\triangle ABC') > \text{per}(\triangle ABC) \). Indeed, \( \text{per}(\triangle ABC') = 2c(1 + \sin(\alpha/2)) \) and \( \text{per}(\triangle ABC) = a(1 + 1/\cos \gamma) \), thus it is enough to show that

\[
2c(1 + \sin(\alpha/2)) < a(1 + 1/\cos \gamma).
\]

Using the law of sines we obtain

\[
\frac{2(1 + \sin(\alpha/2))}{\sin \alpha} < \frac{a}{\sin \gamma} = \frac{\sin \alpha}{\sin \gamma}.
\]

Equivalently,

\[
\frac{2(1 + \sin(\alpha/2))}{\sin \alpha} < \frac{1 + 1/\cos \gamma}{\sin \gamma} = \frac{2(1 + \cos \gamma)}{\sin(2\gamma)} = \frac{2(1 + \sin(90^\circ - \gamma))}{\sin(180^\circ - 2\gamma)}.
\]

Note that \( (1 + \sin x)/\sin(2x) \) is strictly decreasing on the interval \( (0^\circ, 60^\circ) \). It is clear that \( 90^\circ - \gamma < \alpha \), otherwise \( 90^\circ < \beta < \gamma \), contradicting the assumption that \( \gamma < 90^\circ \). Hence we get that \( \text{per}(\triangle ABC') < \text{per}(\triangle ABC) \).

The inequality \( \text{per}(\triangle ABC) < \text{per}(\triangle A'BC) \) simply follows from that fact that \( \triangle ABC \) and \( \triangle A'BC \) are similar triangles such that the base of \( \triangle ABC \), is of length \( a \) so it shorter than the base of \( \triangle A'BC \), which is of length \( b \).

\[\square\]

Formula for \( x_v^* \)

Let \( \delta_v = \sqrt{48v^6 + 81v^4} - 9v^2 \), then \( x_v^* \) can be expressed as

\[
x_v^* = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{\sqrt{2\delta_v}}{9} - \frac{\sqrt{2\delta_v}^6}{3\delta_v}} + \frac{2 + \frac{\sqrt{2\delta_v}^6}{3\delta_v} - \frac{\sqrt{\delta_v}}{9}}{\sqrt{1 + \frac{\sqrt{2\delta_v}}{9} - \frac{\sqrt{2\delta_v}^6}{3\delta_v}}} \right).
\]