Controlling the average degree in random power-law networks

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We describe a procedure that allows continuously tuning the average degree \( \langle k \rangle \) of uncorrelated networks with power-law degree distribution \( p(k) \). In order to do this, we modify the low-\( k \) region of \( p(k) \), while preserving the large-\( k \) tail up to a cutoff. Then, we use the modified \( p(k) \) to obtain the degree sequence required to construct networks through the configuration model. We analyze the resulting nearest-neighbor degree and local clustering to verify the absence of \( k \)-dependencies. Finally, a further modification is introduced to eliminate the sample fluctuations in the average degree.

I. INTRODUCTION

Artificial networks are an important substrate for studying the dynamics of many complex systems. In particular scale-free, or more generally power-law networks, have been widely used for that purpose. The power-law exponent that characterizes the decay of the degree distribution is a crucial quantity that can produce drastic changes in the phenomenology of the system. But the average degree can also play an important role as a control parameter promoting critical phenomena (see, for instance, [1–5]). Moreover, it can influence other structural measures of a network, such as average nearest neighbor degree [6]. Hence, the average degree needs to be taken into account for unbiased comparisons [7].

However, it is an often neglected quantity when building networks. As we will see below, in this section, the resulting average connectivity of a network may present significant deviations from the initially proposed value, especially for power-law networks due to their heterogeneity. Then, our purpose is to present a simple procedure that allows to adjust the average degree in random networks that are constructed via the configuration model [8, 9].

Let us consider distributions of degrees with the power-law form

\[
p(k) = \frac{N}{k^\gamma}, \quad \text{for } k_{\text{min}} \leq k \leq k_{\text{max}},
\]

with \( \gamma > 2 \) and where \( N = \sum_{k=k_{\text{min}}}^{k_{\text{max}}} k^{-\gamma} \) is the normalization constant. We will consider that the minimal degree can take values \( k_{\text{min}} \geq 2 \) (hence, excluding only nodes with one link), and \( k_{\text{max}} \)
is a maximal allowed value (cutoff). The natural cutoff is $k_{\text{max}} \propto N^{1/(\gamma - 1)}$ \cite{11}, but a structural cutoff $k_{\text{max}} \propto \sqrt{N}$ has been considered in the literature \cite{12, 13} to reduce $k$-dependencies in such networks.

In order to build a network, after randomly drawing the degree sequence from the distribution \cite{1}, we link the nodes according to the configuration model \cite{8, 9}. When drawing the degrees for each network, the effective maximal degree will be $k_{\text{max}}^{*} \leq k_{\text{max}}$ and the mean degree of the network, $\bar{k}$, will in general differ from the average computed with the distribution \cite{11}

$$\langle k \rangle = \sum_{k=k_{\text{min}}}^{k_{\text{max}}} kp(k). \quad (2)$$

An illustration is given in Fig. 1, for networks with different values of $\gamma$, and two different sizes $N$. In all cases $k_{\text{min}} = 2$ and the prescription $k_{\text{max}} = \sqrt{N}$ was used. For each sample network, $\bar{k}$ was recorded, and the histogram of values over 1000 realizations is shown in Fig. 1. The deviation of $\bar{k}$ around $\langle k \rangle$ decreases with $N$, as $1/\sqrt{N}$, as expected. But, when varying $\gamma$, a large spread of values of $\bar{k}$ emerges. This spread increases with $N$, although the deviation for each value of $\gamma$ decreases with $N$. Moreover, recall that in the infinite network limit, the moments of order $n \geq \gamma - 1$ are divergent.

![FIG. 1: Normalized histogram of the average connectivity $\bar{k}$ of the samples, computed over 1000 realizations drawn from $p(k)$, for different values of the exponent $\gamma$ given in the legend, and two values of the network size $N$: $10^4$ (filled) and $10^5$ (hollow symbols). We used $k_{\text{min}} = 2$ and the prescription $k_{\text{max}} = \sqrt{N}$ \cite{12}. The solid lines are Gaussian curves centered in $\langle k \rangle$, fitting the width, which decays with the network size as $1/\sqrt{N}$.](image)

Furthermore, if we try to fix $\langle k \rangle$, restrictions emerge. For instance, for $\gamma = 2.5$, and choosing $k_{\text{max}} = \sqrt{N}$ ($\approx 316$, when $N = 10^5$), and $k_{\text{min}} = 2$ (or alternatively 3), then, the average degree
is $\langle k \rangle \simeq 4.39$ (or alternatively $\simeq 6.97$). This means that increasing $k_{\text{min}}$ in one unit produces an increase of about 2.6 in the average degree. More generally, for given $(k_{\text{min}}, k_{\text{max}}, \gamma)$, the outcome of a given value of $\langle k \rangle$, within an allowed tolerance interval, is restricted to a very narrow region in the plane $k_{\text{min}} - k_{\text{max}}$, as can be observed in Fig. 2 for two different values of $\gamma$. Moreover, since $k_{\text{min}}$ is integer, combinations with a low tolerance may be not feasible. Also notice that the average degree becomes insensitive to large enough $k_{\text{max}}$, while the high probability of the small degrees turns $\langle k \rangle$ very sensitive to $k_{\text{min}}$.

![FIG. 2: Regions in the plane $k_{\text{min}} - k_{\text{max}}$ for which the average connectivity $\langle k \rangle$ is within the intervals $k_{\text{max}} > k_{\text{min}}$ (lightgray) [5.5,6.5] (green) and [5.9,6.1] (yellow), for power-law networks with exponent $\gamma = 3.5$ (a) and 2.5 (b). The dashed curves correspond to $\langle k \rangle = 3, 4, \ldots$. Despite the continuous representation, recall that only integer values of $k_{\text{min}}$ and $k_{\text{max}}$ are allowed, which restricts even more the realization of a chosen value of $\langle k \rangle$.](image)

In this context, our goal is to modify the degree distribution to adjust $\langle k \rangle$, given the set of parameters $(k_{\text{min}}, k_{\text{max}}, \gamma)$. In particular, we will take into consideration that, as we have seen in Fig. 2, a limitation comes from the integer character of $k_{\text{min}}$, and also from the high probability that the smallest values of $k$ have in a pure power-law distribution. Therefore, the schemes that we will propose somehow emulate an analytic continuation between consecutive integer values of $k_{\text{min}}$, thus allowing to obtain a continuum of values of $\langle k \rangle$, overcoming the limitations shown in Fig. 2.

The proposed schemes are defined in Section II. In Section III we check that the adjustments of $p(k)$ do not introduce undesired correlations in the networks constructed via the configuration model. In Section IV we describe a procedure to eliminate the fluctuations in the average degree for finite networks. Final considerations are presented in Sec. V.
II. MODIFIED DEGREE DISTRIBUTION

Based on the introductory discussion, we propose two simple schemes to adjust the average degree of a power-law network, with \( p(k) \) given by Eq. (1). These schemes, illustrated in each panel of Fig. 3, consist of modifying the low-degree region of the original \( p(k) \). In case I, only the minimal degree does not follow a power-law. In case II, several points can form a plateau, before the power-law decay. Notice that, by altering the shape of low-\( k \) region, related to the more likely values of \( k \), concomitantly the probability associated with the tail increases, allowing for more highly connected nodes. As we will see, this slight modification allows to adjust the average degree.

![FIG. 3: Schematic representation of schemes I (a) and II (b), in lilac filled symbols. In both cases, \( \gamma = 3.5 \), \( N = 10^5 \), \( \langle k \rangle = 12 \), \( k_{\text{max}} = \sqrt{N} \simeq 316 \). The corresponding pure power-law \( p(k) \) is represented by light-gray hollow symbols.](image)

It is also noteworthy that real degree distributions do not follow a pure power-law but are better represented by the distributions that result from the modified schemes. For example, networks of word co-occurrence, company directors and internet autonomous systems resemble the form of scheme I, while (scientists, film actors) collaboration networks resemble the shape of scheme II [14].

**A. Scheme I**

Starting from a pure power-law distribution, we scale the probability of the minimal degree \( k_{\text{min}} \) by a factor \( \alpha \in (0, 1] \) and renormalize the whole distribution, as illustrated in Fig. 3(a). That
is

\[
p(k) = \begin{cases} 
\frac{\alpha k^{-\gamma}}{Z}, & \text{if } k = k_{\text{min}}, \\
\frac{k^{-\gamma}}{Z}, & \text{if } k_{\text{min}} < k \leq k_{\text{max}},
\end{cases}
\]

where \( Z = \alpha k_{\text{min}}^{-\gamma} + \sum_{k=k_{\text{min}}+1}^{k_{\text{max}}} k^{-\gamma} \) is the normalization constant. In order to do that, let us consider the parametrization

\[
k_{\text{min}}(t) = \lfloor t \rfloor, \\
\alpha(t) = \lfloor t \rfloor - t + 1,
\]

where \( t \geq 2 \) and \( \lfloor \cdots \rfloor \) is the floor function. In the particular case \( t = 2 \), we have \( k_{\text{min}} = 2 \) and \( \alpha = 1 \). Increasing \( t \) between consecutive integer values, makes \( \alpha \) decrease from 1 to 0. When \( t \) reaches an integer value, \( \alpha \) is reset to unit, while \( k_{\text{min}} \) increases in one unit. Therefore, when \( t \) is integer, the pure power-law distribution starting from a minimal degree \( k_{\text{min}} = t \) is recovered. For non-integer \( t \), the distribution has the shape illustrated in Fig. 3(a). The plots of the coefficients of the degree distribution and of the average degree, versus the control parameter \( t \), are shown in Fig. 4(a). Note that, for given \( \gamma \) and \( k_{\text{max}} \), tuning \( t \) allows to change continuously the value of the average degree, which is given by

\[
\langle k \rangle(t) = \frac{\alpha k_{\text{min}}^{1-\gamma} + \sum_{k=k_{\text{min}}+1}^{k_{\text{max}}} k^{1-\gamma}}{\alpha k_{\text{min}}^{-\gamma} + \sum_{k=k_{\text{min}}+1}^{k_{\text{max}}} k^{-\gamma}},
\]

where \( k_{\text{min}} \) and \( \alpha \) are given by Eqs. (4).

**FIG. 4:** Parameters of the degree distribution vs. the control parameter \( t \): (a) \( k_{\text{min}} \) and \( \alpha \) in Eq. (4) for scheme I (b) \( L \) and \( \beta \), with \( k_{\text{min}} = 2 \) in Eq. (7) for scheme II. The corresponding value of \( \langle k \rangle \) is also shown.
B. Scheme II

Another modification of the pure power-law distribution is illustrated in Fig. 3(b), where a plateau (meaning uniform distribution for the $L$ lowest degrees) is considered. The modified distribution is

$$p(k) = \begin{cases} 
\beta/Z, & \text{if } k < k_{\text{min}} + L, \\
k_{\text{min}}^\gamma/(k_{\text{min}}^\gamma Z), & \text{otherwise,}
\end{cases}$$

(6)

where $Z = \beta L + \sum_{k=k_{\text{min}}+L}^{k_{\text{max}}} (k_{\text{min}}/k)^\gamma$ is the normalization constant.

In this case, there can be more combinations to obtain a given average degree, since there is an additional parameter. In fact, by choosing $k_{\text{min}}$, the following parametrization controls the length and level of the initial plateau:

$$L(t) = \lfloor t \rfloor + 1 \equiv \lceil t \rceil,$$

$$\beta(t) = \left( \frac{k_{\text{min}}}{k_{\text{min}} + t} \right)^\gamma,$$

(7)

where we can take $t \geq 0$. For $t = 0$, we recover the pure power-law. The behaviors of $L$ and $\beta$ with $t$, for $k_{\text{min}} = 2$, are shown in Fig. 4(b), together with the corresponding average degree

$$\langle k \rangle(t) = \frac{\beta \sum_{k=k_{\text{min}}}^{k_{\text{min}}+L-1} k + \sum_{k=k_{\text{min}}+L}^{k_{\text{max}}} k_{\text{min}}/(k_{\text{min}}/k)^\gamma}{\beta L + \sum_{k=k_{\text{min}}+L}^{k_{\text{max}}} (k_{\text{min}}/k)^\gamma},$$

(8)

where $L$ and $\beta$ are given by Eqs. (7).

III. CORRELATION ANALYSIS

Based on the modified degree distributions, we built networks using the configuration model. Self and multiple connections are not frequent in the cases considered and realizations containing such connections were discarded. Similarly, only networks with giant component size $N$ were considered.

In order to identify degree-degree correlations, we consider the average degree of the neighbors of a given node $i$, $k_{\text{nn},i}$, which in terms of the adjacency matrix $A$, is given by $k_{\text{nn},i} = \frac{1}{k_i} \sum_j k_j A_{ij}$. 
We compute the average over all nodes with the same degree, \( k_i = k \),

\[
k_{nn}(k) = \frac{1}{N_k} \sum_{i \mid k_i = k} k_{nn,i},
\]

where \( N_k \) represents the number of nodes with degree \( k \). This quantity defines the level of assortativity of the network. If \( k_{nn}(k) \) is an increasing (decreasing) function of \( k \), the network is assortative (dissassortative). For uncorrelated networks, it is \( k \)-independent, and given by \[14, 15\]

\[
k_{nn}^{unc} = \frac{\langle k^2 \rangle}{\langle k \rangle}.
\]

We also consider the local clustering coefficient \( c_i \), given by the ratio of the number of existing connections between neighbors of site \( i \), \( e_i = \sum_{j,k} A_{ij}A_{jk}A_{ki} \) (triangles), over its total possible number \( k_i(k_i - 1)/2 \). If \( k_i = 0, 1 \), then \( c_i = 0 \). Grouping the local clustering \( c_i \) of those vertices with the same degree \( k \) (as done in Eq. (9)), we have

\[
c(k) = \frac{1}{N_k} \sum_{i \mid k_i = k} c_i,
\]

which in an uncorrelated network is \( k \)-independent and given by \[9\]

\[
c^{unc} = \frac{(\langle k^2 \rangle - \langle k \rangle)^2}{N\langle k \rangle^3}.
\]

In Fig. 5 we show the theoretical values of \( k_{nn}^{unc} \) and \( c^{unc} \) for degree distributions following
schemes I (filled symbols) and II (hollow symbols), for two different values of \( \langle k \rangle \) and two values of \( \gamma \). Using scheme II, there can be more choices of the minimal degree \( k_{\text{min}} \), since one can tune the length of the plateau, which provides and extra parameter. Notice that the degree distribution of scheme I yields minimal values of \( k_{\text{unc}} \) and \( c_{\text{unc}} \), associated to the reduction of the occurrence of low-\( k \) nodes.

For the constructed networks, we calculated \( k_{\text{nn}}(k) \), the average degree of the neighbors, and the clustering \( c(k) \). The results for scheme I are presented in Fig. 6. The horizontal lines correspond to the values predicted for uncorrelated networks. A very good agreement is observed between measured and theoretical values indicating that correlations are not introduced by the correction scheme, as expected. Same agreement is observed for networks constructed with scheme II (not shown).

In Fig. 7 for networks based on scheme I, we show the effect of changing \( N \) and the structural cut-off \( k_{\text{max}} \), for two values of \( \gamma \). Notice that larger \( N \) produces the increase of \( k_{\text{nn}} \) and the decrease of \( c \). For both network sizes the cut-off \( k_{\text{max}} = \sqrt{N} \) (squares) allows to keep \( k_{\text{nn}} \) nearly constant for all \( k \). The cut-off \( k_{\text{max}} = \sqrt{\langle k \rangle N} \) (diamonds) is enough to avoid \( k \) dependencies for \( \gamma = 3.5 \) but not in the case \( \gamma = 2.5 \) where the degree distribution has heavy tails. In fact, in the latter case, a noticeable decay of \( k_{\text{nn}} \) and \( c \) with \( k \) is observed, although this effect is reduced by increasing system size. Actually, this dependency which is nearly linear is always present, but becomes negligible for appropriate \( k_{\text{max}} \). This effect is not a consequence of the introduced schemes, but it is also observed when the degree distribution is a pure power-law [12].
FIG. 7: Size effects. (a) $k_{nn}(k)$ and (b) $c(k)$ vs. $k$, for networks built based on scheme I, with $k_{min} = 2$, average degree $\langle k \rangle = 6$ and two values of $\gamma$: 2.5 (blue) and 3.5 (red). The symbols correspond to the average over 100 networks (in this case error bars were not plotted for clarity). Two different network sizes were considered $N = 10^4$ (hollow symbols) and $N = 10^5$ (filled symbols). Moreover, two different values of $k_{max}$ were considered: $\sqrt{N}$ (squares) and $\sqrt{\langle k \rangle N}$ (diamonds). The horizontal lines correspond to the uncorrelated values $k_{nn}^{unc}$ and $c^{unc}$ given by Eq. (12) and Eq. (10), respectively. The insets show the same data for $\gamma = 2.5$ of the main frame using linear scales.

IV. ELIMINATING FLUCTUATIONS IN THE AVERAGE DEGREE

The two schemes proposed above fulfill the function of controlling the average degree of power-law networks. However, as exemplified in figure [1] the sample average $\bar{k}$ can still fluctuate around the prefixed value $\langle k \rangle$, for finite networks. This is because the process by which the sequence of degrees is constructed is purely random. In this subsection, we present a way to eliminate the fluctuations in the average degree. The procedure consists of two steps to chose the sequence of degrees. The first step is deterministic and the second stochastic.

To generate the sequence of degrees, we start from a distribution $p^*(k)$. The expected value for the number of vertices with degree $k$ is $N_k = Np^*(k)$, which in general is not an integer value, as it must be for an individual realization. Then, in order to obtain integer number of vertices, we perform a first deterministic step, where we truncate $N_k$. Then, we add to the list $[Np^*(k)]$ times the degree $k$. This procedure produces fewer vertices and edges than required, which will be corrected in a second stage.

Figure [8](a) shows the intermediate (non-normalized) distribution of degrees $p_0(k) \sim k^{-\gamma}$ after this deterministic process (hence a single realization is shown). For this network we used $N = 10^5$, $\gamma \sim 3.45$ and $\langle k \rangle = 6$, but the number of vertices was reduced to 99926. As $k$ increases, it becomes more evident that $p_0(k) \leq p^*(k)$, returning values below the original (normalized) degree distribution. Notice that, for large enough $k$, we can have $[Np^*(k)] = 0$, while $[Np^*(k)] \gg 1$ for
small $k$.

![Figure 8](image)

FIG. 8: (a) Deterministic non-normalized $p_0(k) = \lfloor Np^*(k) \rfloor / N$ (hollow green symbols), where the original $p^*(k)$ is a pure power-law with exponent $\gamma \sim 3.45$. The number of vertices was initially $N = 10^5$, but after the truncation process, it was reduced to 99926. (b) Final histogram after the stochastic filling step for 1 realization of a network (filled green symbols). The same histogram in logarithmic bin (open black circles). In both panels, the red straight line is a pure power-law with $\gamma \sim 3.45$. The number of missing nodes is $N - \sum_k \lfloor Np^*(k) \rfloor$. The degree sequence needs to be completed, to fulfill the desired network size $N$ and average degree $\langle k \rangle$. The nonnegative fractional part $0 \leq r(k) \equiv Np^*(k) - \lfloor Np^*(k) \rfloor \leq 1$ can be used as a filling probability, to draw the remaining elements of the degree sequence, according to the following algorithm:

1. If $\sum_k \lfloor Np^*(k) \rfloor < N$ and $\sum_k k < \langle k \rangle N$, a number $k$ is uniformly drawn from the interval $[k_{\min}, k_{\max}]$, and we decide if it will be added to the sequence or not with probability $r(k)$. Moreover, each value of $k$ can be selected only once, then if the drawn value has already been used, a new one is drawn.

2. If after some iterations there is only one missing node, then $\langle k \rangle N - \sum_k k = k' \in [k_{\min}, k_{\max}]$. In this case, the degree $k'$ is added to the sequence even if it has been drawn before. The only case in which a repetition is allowed. If the last missing node $k' \notin [k_{\min}, k_{\max}]$, then we start it over from step 1.

Figure 8(b) presents the distribution of degrees $p(k)$, after filling the missing nodes generated in panel (a) for a single realization. The average degree is exactly $\langle k \rangle = 6.0$. In the example the degree distribution is a pure power-law but of course the procedure can be applied to any degree distribution.

It is interesting to note that the probability distribution $r(k)$ to fill the missing values is nearly uniform up to the value of $k$ for which $\lfloor Np^*(k) \rfloor = 0$, in which case $r(k) = Np^*(k)$, hence decaying
as a power law for large $k$ (see inset). Let us mention that $r(k)$ has previously been used to introduce large-degree sites (hubs) via the configuration model in power-law degree distributions [16], although not to control the average degree. Notice that this procedure to eliminate fluctuations is not exclusive of power-law networks and in principle can be applied to any $p(k)$.

V. FINAL REMARKS

We have analyzed two different modifications of the degree distribution that allow us to adjust the average degree $\langle k \rangle$, preserving the power-law character of the distribution. We considered two simple forms of modifying the low-$k$ region, but, of course, other shapes might also be used to produce similar results. Additionally, we have presented a procedure to eliminate fluctuations in the average degree, which can be applied to any degree distribution.

Controlling $\langle k \rangle$ is important since correlations and other structural properties can be affected by the average connectivity [17, 18]. It is particularly relevant when an artificial network is used as a substrate on top of which the dynamics of a complex system evolves. Fixing the average degree in synthetic networks may be also useful for comparisons with real ones, overcoming the difficulties exposed in Ref. [7]. A further advantage of applying the proposed schemes (either I or II) is that they help to eliminate fragments disconnected from the giant component, as soon as the low values of $k$ become less probable. The procedures we proposed to tune the average degree can be directly adapted to adjust the value of other finite moments of the degree distribution.

Let us finally mention that another way to built networks with a power-law degree distribution is using growth techniques. For example, the generalized Barabasi-Albert (GBA) model [19] allows in principle to adjust the degree-distribution exponent $\gamma$ and average degree $\langle k \rangle$, for a given size of the network, however there are restrictions in the values that can be achieved (for instance, when $\gamma = 3.5$, $\langle k \rangle > 4$ is not possible in GBA). For instance in the particular case of a standard Barabasi-Albert network [20], with $\gamma = 3$, only even values of $\langle k \rangle$ are obtained in the limit of large $N$. Moreover, although GBA networks present a negligible Pearson coefficient [21], it is known that these graphs possess $k$-dependencies [10, 22].

We have verified that the modifications introduced in the degree distribution with structural cut-off do not introduce $k$-dependencies in the nearest-neighbor degree $k_{nn}(k)$ and the local clustering coefficient $c(k)$. Although choosing other form of the cutoff may introduce $k$-dependencies, the method to tune $\langle k \rangle$ is still effective.

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